RESEARCH STATEMENT

MILTON LIN

Personal Experiences

My research in pure mathematics has centered on bridging discrete structures (number theory) with continuous spaces (topology) through algebraic formalism. I have attached them to the end of the document for those of interest. Building on my background, I aim to explore how **algebraic and categorical methods** can reveal qualitative information on the learning dynamics of modern language models and neural circuits. This involves examining **biological realism** in computational frameworks.

My Motivation from Biology. Network models, ranging from binary threshold neurons to Hodgkin-Huxley-type models, provide diverse insights into biological neural circuits, offering computational interpretations of intricate connectivity and dynamics [AK16; BS23]. My interest is particularly sparked by the pairwise Maximum Entropy (MaxEnt) model, well known for predicting neural spike trains [Alj+16].

Spike trains in continuous time can be discretized by dividing time into intervals. For a time interval $[t_k, t_k + \delta_k)$ and neuron i, the activity is represented as:

$$x_i(t_k) = \begin{cases} 1 & \text{if neuron } i \text{ spikes in } [t_k, t_k + \delta_k), \\ 0 & \text{if neuron } i \text{ is silent in } [t_k, t_k + \delta_k). \end{cases}$$

The MaxEnt model assigns probabilities to binary vectors x according to:

$$p_\theta(x) = \frac{1}{Z} \exp(-E_\theta(x)),$$

where $E_\theta(x)$ is the energy function parameterized by θ, and Z is the partition function. In the pairwise MaxEnt model, $E_\theta(x) = x^\top \theta x$, where θ is a symmetric matrix. Remarkably, any MaxEnt model of neural activity can be transformed into an associative memory model by updating the binary vector x iteratively to minimize $E(x)$. For the pairwise MaxEnt model, this recovers the Hopfield network [Hop84].

Initially, I was drawn to Minimum Probability Flow [SBD20], a biologically plausible credit assignment mechanism [Oro+24] for learning MaxEnt parameters, which has also been applied to associative memory models [HMK14; HSK15], which sparked a deeper interest in developing algorithms for fitting statistical models to neural spike train data.

Despite these advances, the theory of associative memory networks lacks a coherent framework akin to classical statistical learning theory, particularly for understanding learning dynamics in high-dimensional parameter spaces. This led to my collaborative project with Chris Hillar. My research employs algebraic and geometric tools to address these gaps. In Section 1, I outline my collaborative work with Hillar, which hopes to address variations of memory networks [KH16; BF23]. These models are significant as they:

(1) Capture fundamental aspects of memory storage and retrieval [KH21],

(2) Connect to modern deep learning architectures like transformers [Ram+21; Niu+24], and

(3) Are energy-based, offering interpretability and tools from statistical mechanics [Car24].

In Section 2, I extend this work to study modern deep-learning networks through the lens of dense associative memories. Collectively, these projects aim to explore the limitations of synthetic memory networks and provide an algebraic framework for coherently analyzing various families of models.

In Section 3, I discuss two future projects I wish to embark on that leverage my background in mathematics: Section 3.1 discuss categorical frameworks to study scaling properties, and Section 3.2 discuss reinforcement learning algorithm development in the baby context of mathematical research problems.

Research to be Conducted at NITMB. I believe my background in geometric representation theory, number theory, and algebraic combinatorics brings a different perspective to the study; my ongoing exploration of machine learning applications, developed just this last year, also complements the program's aim to integrate computational perspectives into these areas.

By fall 2025, I will complete my study on parameter spaces of memory networks with Chris Hillar, see Section 2, and experiments with Muhan Gao on experiments involving memory networks in language modeling. Over the spring, I plan to extend my research on parameter spaces to stochastic and multi-agent models [ZYB19], and begin two of my future projects in Section 3.

1. Geometric Structures in Parameter Spaces

Given a model architecture $\mathcal{A}$, such as transformers, CNN, multilayer perceptron, designed to interpolate a task $\mathcal{T}$, we investigate the encoded information within the parameter space $\mathrm{Par}_{\mathcal{A}}$. We define the mapping:

$$\mathcal{A}_{(-)} : \mathrm{Par}_{\mathcal{A}} \to \mathcal{T} \quad \Theta \mapsto \mathcal{A}_{\Theta}$$

which assigns each parameter Θ, $\mathcal{A}_{\Theta} \in \mathcal{T}$, an object of architecture $\mathcal{A}$, designed for a task, $\mathcal{T}$. For instance, let $\mathcal{A} := \mathrm{FF}\,[n, \sigma]$ be the class of L-layer feedforward neural network with hyperparameters: *width* $n = (n_i)_{i=1}^{L+1}$ and collection of activation functions $(\sigma_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{i+1}})_{i=1}^{L}$. For a set of parameter

$$\Theta := \{A_i, b_i\}_{i=1}^{L} \in \mathbb{R}^{\sum_{i=1}^{L} n_i(n_i+1)}$$

we can associate a function

$$\mathcal{A}_{\Theta} := f_L \circ \cdots \circ f_1 : \mathbb{R}^{n_1} \to \mathbb{R}^{n_{L+1}}$$

where for each $i = 1, \ldots, L$, and $f_i : \mathbb{R}^{n_i} \to \mathbb{R}^{n_{i+1}}$ is a linear function given by

$$f_i := \sigma_i \left(A_i x_i + b_i\right),$$

This induces our desired map, from space of parameters to the set of functions from $\mathbb{R}^{n_1}$ to $\mathbb{R}^{n_{L+1}}$.

$$\mathcal{A}_{(-)} : \mathrm{Par}_{\mathcal{A}} \to \mathrm{Fct}(\mathbb{R}^{n_1}, \mathbb{R}^{n_{L+1}})$$
$$\Theta \mapsto \mathcal{A}_{\Theta}$$

The pair $(\mathcal{A}(-), \mathcal{T})$, partitions the collection of parameters according to those which induces the same object $f \in \mathcal{T}$. We obtain a collection of subsets of parameter space

$$\Sigma_{\mathcal{T}}[\mathrm{Par}_{\mathcal{A}}] = \{\mathrm{Par}_f\}_{f \in I(\mathcal{D})} \quad \mathrm{Par}_f = \{\Theta \in \mathrm{Par}_{\mathcal{A}} \,:\, \mathcal{A}_{\Theta} = f\}$$

by considering regions inducing the same object under the mapping $\mathcal{A}_{(-)}$. Often, $\Sigma_{\mathcal{T}}[\mathrm{Par}_{\mathcal{A}}]$ is more than a *set*, but is a set with *structure*. For deep feedforward neural networks, this structure manifests as the face poset of hyperplane arrangement, encapsulating both model expressivity and decision boundaries.

We studied the case when $\mathcal{A}$ is a Hopfield network, and the number of top dimensional faces in $\Sigma_{\mathcal{T}}[\mathcal{A}]$ reduces to a problem of counting regions induced by hyperplane arrangements, and is explicitly given by Zaslavsky's theorem, [Sta07]. A simple corollary is:

Corollary 1.1 (Hillar, Lin)**.** *A Hopfield network of two nodes with asynchronous (or synchronous) updates cannot express XOR functions.*

We are currently focusing on parameter estimation. How is the dynamics of parameter estimation reflected in the parameter space? [Koh+22] has studied a similar story for linear convolutional networks. Our particular estimation method of interest in *minimum probability flow*, [SBD20], which has been used by joint author Chris Hillar in the case of Hopfield networks, [HMK14]. We summarize here where the future research would focus on:

(1) Understanding a suitable notion of "equivalence class of networks". What can we say about pairs $(\mathcal{A}, \mathcal{T}), (\mathcal{A}', \mathcal{T}')$ and the differences in their induced face posets, $\Sigma_{\mathcal{T}}[\mathrm{Par}_{\mathcal{A}}], \Sigma_{\mathcal{T}'}[\mathrm{Par}_{\mathcal{A}'}]$ Similarly, can one describe distributions on the $\mathrm{Par}_{\mathcal{A}}$ that corresponds to networks with certain properties?[1] We hope this gives a coarse-grained comparisons of various networks.

(2) Extend our analysis of architectures *higher order networks*, which includes transformer networks, and simplicial hopfield network [BF23]. $\mathrm{Par}_{\mathcal{A}}$ decomposes into semi-algebraic sets, and one may approach with the theory of splines, [LLL24].

2. Understanding modern networks through memory networks

Recent developments in associative memory networks have significantly advanced these models along two fronts: i) *Improved storage capacity*, progressing from polynomial [KH16], to exponential [Dem+17], and in other point of views, [HT14] ii) *Integration into modern deep learning architectures*, such as attention mechanisms [Ram+21], energy-based transformers [Hoo+23], and higher-order models like simplicial Hopfield networks [BF23]. Their relations with, and their potential to explain, modern transformer-based decoder models are under explored.

Joint with Chris Hillar, Muhan Gao (Johns Hopkins University), and Tenzin Chan (Algebraic) we evaluate dense assocative memories, [KH16] beyond the theoretical memory capacity, see Equation (5) and (6) of *op. cit.*. While much effort has been focused on designing networks that extends the memory capacity, there is little work on studying such regimes. Our first empirical results show that storage capacity is not a hard constraint to task performance. Such insensitivity to memory capacity echoes trends seen in scaling laws of deep learning. Moving forward, we are exploring

(1) *Generalization and catastrophic forgetting:* The behavior of stored memory patterns appears highly sensitive to the nature of the task. How does task variability influence memory retrieval, and could this sensitivity offer insights into *catastrophic forgetting*? Understanding this phenomenon, especially in the context of continual learning, could bridge memory networks with advances in lifelong machine learning [Kem+17].

(2) *Correlated data and memory convergence* Experimental evidence shows that correlated datasets significantly alter convergence behavior to stored memory patterns. Can these observations be formalized theoretically? A deeper understanding of how data structure impacts memory retrieval could inform both theoretical bounds and practical applications.

The end goal is to provide both empirical and theoretical comparison with modern networks; works along these lines include, [ND21], [Niu+24], and [CDB24].

[1] A similar question was asked in [Mon+14].

3. Future research projects

3.1. Categorical Models and Homotopy Theory. The following project extends previous project Section 1. Categorical approaches have gained momentum as a systematic framework for studying network structures [Gav+24]. This has been particularly successful in the field of *geometric deep learning* [Bro+21], where abstract mathematical structures help describe complex neural networks. We propose to explore memory networks using a recent formalism by Manin et al. [MM24], which uses summing functors and Gamma spaces to model the allocation of resources in neural networks. These concepts will allow us to understand how the complexity of memory networks scales as network size increases. The formalism allows us to study a *homotopy type* - a mathematical construct at a deeper level than *homology*[2]. Homotopy captures invariants of network up to continuous deformations. Previous studies have shown that stimulus space can be reconstructed up to homotopy [Man15].

Specifically, we will examine how *memory capacity* correlates with *homotopical invariants* like *Betti numbers* (which measure the number of independent cycles in a space) and *simplicial complexes* (which provide a higher-dimensional generalization of networks). Burns and Fukai have already done early work in this direction [BF23], but much remains to be explored.

3.2. Reinforcement Learning Approaches in Mathematical Research. Building on a recent study of applications of reinforcement learning to mathematics - a research problem in combinatorial group theory, [She+24] - I hope to continue further and develop algorithms that expand the action space dynamically. As an initial step, I am implementing a variant of the options framework combined with hindsight experience replay [Lev+19] in the controlled experimental setting of *op. cit.*. The options framework of Markov decision processes stands as a first step in temporal planning, [SB18]. Future approaches could also include various methods of multi-agent reinforcement learning.

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RESEARCH STATEMENT IN PURE MATHEMATICS

MILTON LIN

Introduction

My areas of interest in mathematics are:

(1) The **Langlands correspondence**, particularly, the various incarnations of the **Casselman–Shalika formula**, A, and **relative aspects of the ℓ-adic geometric local Langlands correspondence**, C.

(2) **Stable homotopy theory**, through the lens of **categorical deformation theory**, F.

(3) **Motivic phenomena**, where I hope to explore the motivic version of Casselman–Shalika formula, D, and a stacky approach to periods, E.

Majority of the research presented here originates from my study of the Casselman–Shalika formula in the mixed characteristic setting, as outlined in Section 1. These areas of research are interconnected, as shown in the following diagram.

$$\begin{tikzcd}
\begin{array}{c}\text{Section 2 C}\\ \text{Relative Langlands on}\\ \text{Fargues–Fontaine curve}\end{array} & & \begin{array}{c}\text{Section 4 E}\\ \text{Stacky approach}\\ \text{to periods}\end{array} \arrow[ll, dashed] \\
\begin{array}{c}\textbf{ℓ-adic geometric}\\ \textbf{local Langlands}\end{array} \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c}\text{Section 1 A,}\\ \text{Geometriization of}\\ \text{metaplectic/quasi-split}\\ \text{Casselman–Shalika}\end{array} \arrow[d, dashed] & \begin{array}{c}\textbf{"Motivic"}\\ \textbf{Phenomena}\end{array} \arrow[u] \arrow[d] \\
\begin{array}{c}\text{Section 1 B}\\ \text{Perfect geometry of}\\ \text{affine Schubert Varieties}\end{array} & \begin{array}{c}\text{Section 5 F}\\ \text{Categorical deformation}\\ \text{of representation category}\end{array} & \begin{array}{c}\text{Section 3 D}\\ \text{Motivic}\\ \text{Casselman–Shalika}\end{array} \arrow[l, dashed] \\
& \textbf{Categorical deformation theory} \arrow[u] &
\end{tikzcd}$$

The priority of research is listed in the following order[1],

A=C=F>D>E> B.

Date: November 30, 2024.

[1]The alphabet links to the goal rather than the section.

Notations. Theorems stated have full proof written by either me or joint with collaborators, *unless* it is annotated with: ***conjecture*** – no proofs have been written down but is believed to hold, or ***in progress*** – where we have partial progress. We use freely the language of higher categories and higher algebra, [Lur09], [Lur18].

1. Mixed Characteristic Geometry and the Casselman–Shalika formula

Let G be a connected reductive group over a nonarchimedian local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld [AG15], [Gai14]. One fundamental aspect of the program is to understand the *Whittaker Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, as described *op.cit.*, this corresponds to finding the Fourier coefficients of automorphic functions:

Example 1.1. Let $G = \mathrm{PGL}_2$ be the projective linear group over $\mathbb{Q}$. A modular function, f, has an adelic formulation, $\widetilde{f}$ on $G(\mathbb{A}_\mathbb{Q})$. Up to a *normalizing factor*, the Fourier coefficients coincide with the Whittaker coefficients (Jacquet period),

$$a_m \sim \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \widetilde{f}(n\alpha_m)\psi(-n)\,dn \quad \text{for } m \geq 1$$

where $\alpha_m \in T(\mathbb{A}_\mathbb{Q}^{\mathrm{fin}})$ is m considered as a finite idèle and ψ is a standard character on $N(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_\mathbb{Q}$, where N is the radical of standard Borel. For more details, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which joint Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn University) [ILZ24], we replicated a variation: the *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian Gr_G, analogous to the equi-characteristic geometrization carried out in [NP01].

Theorem 1.2. *[ILZ24] The geometric Casselman–Shalika formula holds over the Witt vector Grassmanian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori–Satake equivalence, [Bez+19] without using nearby cycles.

Theorem 1.3 (I.-Lin-Z., in progress)**.** *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [2]

As of writing, D. Hansen, L. Hamman, and L. Mann have announced significant advancements to the global version of the formula. Building on all these progresses, I have developed a strong interest in exploring the geometrization of the Casselman–Shalika formula for covering groups [GGW18].

Research Goal A. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula over equal-characteristic local fields. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+20; Bru+24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

[2] In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

Lastly, we would like to highlight that in our recent work [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\leq\mu}$, for minuscule and quasi-minuscule μ. In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal B. Geometry of general perfect Schubert variety *Prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhats-Tits building of G, we have an associated group scheme $\mathcal{G}$ over $\mathcal{O}$.*

Conjecture 1.4. The smooth locus of $\mathrm{Gr}_{\mathcal{G},\leq\mathfrak{s}}$ is $\mathrm{Gr}_{\mathcal{G},\mathfrak{s}}$ in perfect geometry, in the sense of [Zhu17].

2. Relative langlands on the Fargues Fontaine curve

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and L-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with $(\hat{G}, \hat{X})$ of hyperspherical varieties. Let $\Lambda = \bar{\mathbb{Q}}_l$. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_!\Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \hat{\pi}_*\omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. On the automorphic side, Bun_G has a Hardar-Narasimhan straification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G) \simeq \mathbb{Z}$. Interesting phenomena occurs for $n \in \mathbb{Z}_{\geq 0} \subset \mathbb{Z} \simeq B(\mathbb{G}_m)$, and the study of period sheaves reduces to the study of the Abel-Jacobi map previously studied by Fargues [Far20] and Hansen [Han21]. On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles, [FW24, Ch. 6] recently developed by Anschütz and Le Bras [AL21].

Conjecture 2.1. [Lin-T., in progress] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[3] $\mathcal{L}_{\hat{X}}$.

Research Goal C. Relative Langlands on the Fargues Fontaine curve. *Complete Conjecture 2.1 as a first step and then the Hecke case, which classically corresponds to Hecke's integral representation of standard L-function for GL_2. Lastly, one can ask whether on the $\mathcal{B}$-side, the same constructions of [BSV, Ch. 11] works for the p-adic (Emerton-Gee) L-parameter stacks, which potentially give new interpretations to p-adic L-functions.*

3. Motivic Phenomena

In this section S denotes a base scheme satisfying the Beilinson-Soulé vanishing conjecture, such as $\mathrm{Spec}\,\mathbb{Z}$, finite field. R denotes a regular coherent coefficient ring. I discuss one side project I have done, and one area of research, D, which extends my joint work [ILZ24].

In Grothendieck's quest towards the Weil conjectures, he reduced to problems of motives. Beilinson proposed an extension of the notion of pure motives to mixed motives, which was realized through the works of Voevodsky, Levine for a field, and extended by Ayoub and Cisikinki-Déglise, [CD19].

[3] One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on X which we do not discuss.

For a finite type S-scheme $X \in \mathrm{Sch}_S^{\mathrm{ft}}$, we can construct the derived categories of motives over X, $\mathrm{DM}(X, R)$. If X is smooth then its Ext-groups,

$$\pi_{-m} \operatorname{Map}_{\mathrm{DM}(X,R)}(1_X, 1_X(n)) \simeq \mathrm{CH}^n(X, 2n-m)_R$$

are Bloch's higher Chow groups. Motivic categories and the Chow groups are difficult to explicitly work with due to the lack of motivic t-structures. One method is due to the work of Beauville [Bea83], using Fourier transform. In my joint work, [Has+24] we extended the work of Beauville,

Theorem 3.1 (Lin et al.)**.** *Let $X \to Y$ be an abelian scheme, whose base Y is smooth and quasi projective over a field. There is an explicit N, such that one obtains a Beauville decomposition*

$$CH^i(X)_{\mathbb{Z}[1/N]} \simeq \bigoplus_s CH^i_{(s)}(X)_{\mathbb{Z}[1/N]}$$

where $CH^i_{(s)}(X)_R := \left\{x \in CH^i(X)_R \,:\, [n]_X^* x = n^{2i-s}x \quad \forall n \in \mathbb{Z}\right\}$ and $[n]$ is the multiplication of an abelian scheme.

This extends to a $\mathfrak{sl}_2$ action, which we discussed in *op.cit*; and if S were an algebraically closed field, this implies various structural results. The key ingredient was using G. Pappas' version of integral Grothendieck–Riemann–Roch, [Pap07].

Returning to motivic t-structures, it was shown by Levine that t-structures exist on a nice subcategory $\mathrm{DTM}(X) \hookrightarrow \mathrm{DM}(X)$ of *mixed Tate movies* for nice schemes X. This was extended to schemes with cellular Whitney-Tate stratification by Soergel and Wendt, and to prestacks in [RS20].

Research Goal D. Motivic Whittaker categories. *Define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we can prove the same statement in [NP01] in the category of mixed Tate motives, in which all the relevant objects are well defined. The difficulty, however, is that the original argument requires characteristic* 0 *coefficients, whilst in positive characteristic, there are problems of extensions.*

4. Stacky approaches and periods

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R)$$
$$X \mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?})$$

For instance, the *de Rham stack* X^{dR} over $\mathbb{Q}$, has points given by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A (cf. [GR14]). This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques. I explored this concept in basic constructions of F. Brown's work on *motivic periods*, [Bro14]. If X were a smooth variety over $\mathbb{Q}$, the matrix coefficient from Grothendieck's comparison theorem

$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^*_{\mathrm{Betti}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

with respect to the $\mathbb{Q}$ structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to* X. These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"[4] which should govern the arithmetic structure of periods.

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the projective space minus three points over $\mathbb{Q}$. The period associated is studied through Chen's de Rham comparison theorem, [Che76], which relates to iterated integrals

[4] For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

[Che73] and multiple zeta values [Bro14]. To give a "stacky" perspective of this period one uses unipotent types of stacks, which is defined in Toën, [Toë06], and recently in [MR23]. This is an endofunctor $\mathbf{U}$ on stacks, sending a stack X to its unipotent homotopy type. My first result is:

Theorem 4.1. *(Lin) Unipotent de Rham fundamental group, $\pi_1^{u,dR}(X,x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{dR}))$.*

Research Goal E. A stacky approach to motivic periods. *This research aims to use geometric techniques in the study of periods. As a proof of concept, we will recover de Rham comparison theorem through the Riemann-Hilbert correspondence between the analytic Betti stack and the de Rham stack. $\mathbb{P}^1 \setminus \{0,1,\infty\}$, is not proper, which requires us to incorporate log structures. I expect to prove:*

Conjecture 4.2. There exists X^{Betti} such that the unipotent Betti group $\pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}),x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{\mathrm{Betti}}))$. A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,\mathrm{dR}}(X,x) \simeq \pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}),x) \otimes \mathbb{C}$$

By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [5] *We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].*

5. Deformation theory and the sphere spectrum $\mathbb{S}$

Let $\mathcal{S}$ denote the ∞-category of ∞-groupoids/anima. The *stabilization* of $\mathcal{S}$ is Sp, the ∞-category of spectra. This is the natural category to study cohomological invariants. Within Sp, lies the universal cohomology theory, $\mathbb{S}$, the *sphere spectrum* By Chevalley's works, connected reductive groups over $\mathbb{C}$ have a canonical split $\mathbb{Z}$-form $G_{\mathbb{Z}}$, see [Con15]. One can analogously ask: *is there a $\mathbb{S}$-form for algebraic groups*? A first approximation is the existence of an algebraic category $\mathrm{Rep}_{\mathbb{S}}(G_{\mathbb{S}})$, which *deforms* to $\mathrm{Rep}_{\mathbb{Z}}(G_{\mathbb{Z}})$. To study, we begin with *formal deformation of categories*, which we briefly recall.

Let $\mathcal{C}$ be a symmetric monoidal category. There is a natural hierarchy of commutativity fitting in the diagram

$$\cdots \to \mathbb{E}_n(\mathcal{C}) \to \mathbb{E}_{n-1}(\mathcal{C}) \to \cdots \to \mathbb{E}_1(\mathcal{C})$$

where $\mathrm{CAlg}(\mathcal{C}) := \mathbb{E}_\infty(\mathcal{C}) := \varprojlim \mathbb{E}_n(\mathcal{C})$ of symmetric algebra objects can be identified with the limit. One can formalize the notion of $\mathbb{E}_n$-algebra objects via *disk operads*, or *little cubes operads*.

Example 5.1. Let $\mathcal{C} = (\mathcal{S}, \times)$. Let $Y \in \mathcal{S}_*$ is a pointed ∞-groupoid, its *k-fold based loop spaces*, $\Omega_*^k Y$ is a classical example of $\mathbb{E}_k$ algebra object in $(\mathcal{S}, \times)$.

Let $R \in \mathbb{E}_n(\mathrm{Sp})$ be an $\mathbb{E}_n$ ring, and consider LMod_R, the derived category of left R-modules, as an $\mathbb{E}_1$ object in $\mathrm{Pr}^{\mathrm{st}}$, the category of presentable stable categories. This defines $\mathrm{RMod}_{\mathrm{LMod}_R}(\mathrm{Pr}^{\mathrm{st}})$, the category of presentable stable (right) R-linear categories, [Lur18, Appendix D]. Set $\mathrm{Pr}_R^{\mathrm{st,cg}}$, as the full subcategory spanned by those whose underlying category is compactly generated. [6] For G a connected reductive group over a field k, $D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$, the bounded derived category of finite dimensional algebraic representations with k coefficients lies in $\mathrm{Pr}_R^{\mathrm{st,cg}}$.

From now on, $k = \mathbb{C}$. Let $\mathrm{Art}_k^{(n)}$ denote the category of $\mathbb{E}_n$ *artinian* ring spectrum, for $n \geq 0$ over k. We refer to [Lur18, Ch. 15] for definition. This extends the classical definition of Artinian local

[5] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

[6] The compact generation is only a smallness condition for our version.

ring, in particular, $R \in \mathrm{Art}_k^{(n)}$ admits an augmentation map $\epsilon : R \to k$. One defines the $\mathbb{E}_{n+2}$-formal moduli problem,

$$\begin{aligned} \mathrm{Cat}\mathcal{D}\mathrm{ef}^{(n)}(\mathcal{C}) : \mathrm{Art}_k^{(n+2)} &\to \hat{\mathcal{S}} \\ R &\mapsto |\{\mathcal{C}\} \times_{\mathrm{Pr}_R^{\mathrm{st,cg}}} \mathrm{Pr}_k^{\mathrm{st,cg}}| \end{aligned}$$

where $|\quad|$ is the underlying Kan complex of the ∞-category. An object consists of: a $\mathcal{C}_B$ right stable R-linear category, and an equivalence $\mathcal{C}_B \otimes_{\mathrm{LMod}_B} \mathrm{LMod}_k \simeq \mathcal{C}$. Our $\mathbb{E}_4$-moduli problem is when $n = 2$ and $\mathcal{C} = D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$. The geometric Casselman–Shalika [FGV01], which is the $\mathbb{E}_2$-algebra equivalence of $\mathcal{C}$ with the *Whittaker sheaves* on the *affine Grassmanian* $\mathrm{Gr}_{\hat{G}}$, describes this moduli problem. Consider moduli of functor of $\mathbb{G}_m$*-gerbes over* X

$$\mathrm{Ge}_{\mathbb{G}_m}(X) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(X, B^2R^\times) \quad R \in \mathrm{Art}_k^{(4)}$$

where $R^\times \subset \Omega^\infty R$ are the invertible elements of the underlying space of R [7] and B^2 is the second deloop. It was stated without proof in [Lur10]

Theorem 5.2 (Lurie)**.** *There is an equivalence of formal* $\mathbb{E}_4$*-moduli problems*

$$\widehat{Ge_{\mathbb{G}_m}}(\mathrm{Gr}_{\hat{G}}) \xrightarrow{\simeq} Cat\mathcal{D}ef^{(2)}(\mathrm{Rep}_k^{fd} G)$$

where $\widehat{-}$ *is the formal completion of the moduli functor at a base point.*

Research Goal F. Categorical deformations of the representation category *We will first document carefully Lurie's theorem, Theorem 5.2. Then, we will explore deformations of representation of Lusztig's small quantum group, as suggested in op.cit. Remark 10.12, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.*

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[7]is the union of the connected components of invertibles in the $\pi_0 R$ of the 0th space of R and is equivalent to the nth loop space of some space, $R^\times \simeq \Omega^n Z$, $n \geq 4$, hence the deloop $B^2R^\times \in \mathbb{E}_2(\mathcal{S})$

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