Monoids and L-functions

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Throughout, let

$$\begin{aligned}
\mathfrak{k} &= \text{a } \mathfrak{p}\text{-adic field} \\
\mathfrak{o} &= \text{the ring of integers in } \mathfrak{k} \\
\mathfrak{p} &= \text{the prime ideal of } \mathfrak{o} \\
\mathfrak{p}^\times &= \mathfrak{p} - \mathfrak{p}^2 \\
\varpi &= \text{ a generator of } \mathfrak{p} \\
\mathbb{F}_q &= \mathfrak{o}/\mathfrak{p}\,.
\end{aligned}$$

In addition, for the moment, let

$$\begin{aligned}
G &= \mathrm{GL}_n(\mathfrak{k}) \\
P &= \text{ subgroup of upper triangular matrices} \\
U &= \text{ unipotent radical of } P \\
A &= \text{ diagonal matrices of } P \\
M &= M_n(\mathfrak{k})\,.
\end{aligned}$$

Thus projection identifies A with P/U. Fix Haar measures $d^\times x$ on G and dx on M which assign measure 1 to $\mathrm{GL}_n(\mathfrak{o})$ and $M_n(\mathfrak{o})$.

If X is any $\mathfrak{k}$-manifold, let $\mathcal{S}(X)$ be $C_c^\infty(X)$. I use this notation to emphasize significant parallels between $\mathfrak{k}$ and $\mathbb{R}$.

If χ is an unramified character of P (lifted from one of $P/N = A$), let (π_χ, I_χ) be the corresponding representation of G induced from P to G, and $\Phi_\chi(g)$ the associated spherical function. It is not difficult to see that for f in $\mathcal{S}(\mathrm{M}_n(\mathfrak{k}))$ the integral

$$\langle \Phi_{\chi|\det|^s}, f\rangle = \int_G \Phi_\chi(g) f(g)\,|\det(g)|^s\,dg$$

converges for $\mathrm{RE}(s) \gg 0$. It can in fact be meromorphically continued over the whole s-plane, and for certain f can be explicitly evaluated. Characters of P factor through A. Unramified characters of A are characterized by the formula

$$\chi\colon\ \mathrm{diag}(\varpi^{n_i}) \longmapsto \prod\nolimits_{i=1}^n \alpha_i^{n_i} \quad (\alpha_i \in \mathbb{C}^\times)$$

and are therefore parametrized by $(\mathbb{C}^\times)^n$.

Theorem. (Tamagawa) *If f is the characteristic function $\mathfrak{char}(\mathrm{M}_n(\mathfrak{o}) \cap G)$ and χ corresponds to (α_i), then*

$$\langle \Phi_{\chi|\det|^s}, f\rangle = \frac{1}{\prod_i \big(1 - q^{(n-1)/2}\alpha_i q^{-s}\big)}\ .$$

The right hand side of this equation is Langlands' L-function $L(s-(n-1)/2, \pi_\chi, \rho)$ for ρ the identity map from $\mathrm{GL}_n(\mathbb{C})$ to itself. This is a local factor of global L-functions on $\mathrm{GL}_n(\mathbb{A})$, which are among the few Langlands' L-functions known to possess conjectured good analytic behaviour. This calculation of the local factor (which I'll recall briefly in this paper) was first found by Tsuneo Tamagawa, and good behaviour for the related global L-functions was first proved in [Godement-Jacquet:1972] by analyzing the Fourier transform on the Schwartz spaces of matrix algebras, following roughly the path laid out by Tate's thesis.

Right from the early days of the modern theory of automorphic forms the natural question arose, *can similar techniques be found for other reductive groups and other L-functions?* That this was not a straightforward matter was also realized early on, as I'll explain later. The difficulties that arose in a few simple cases seem to have discouraged all investigation on this topic for many years.

However, recently several people have re-examined this question, and new approaches have been suggested. In this essay I hope to give some idea of these recent conjectures relating Langlands' local L-functions to analysis on some interesting $\mathfrak{p}$-adic spaces made by Ngô, based in turn to some extent on earlier work of Braverman and Kazhdan. This is also related to work of Sakellaridis on spherical varieties.

These notes originated with lectures given at the Morningside Center in Beijing in the Spring and Fall of 2013. Perhaps influenced to some extent by these, Wen-Wei Li has been able to refine and in some cases prove some rather vague conjectures I made then. In a future version, I hope to explain some of his work, and to apply his ideas to a few examples he has not looked at.

Contents

I'll begin by recalling the relevant parts of Tate's thesis. Then I'll look at what happens for GL_2 in two ways: one as done by Tamagawa, the other by applying Macdonald's formula for the Satake transform. After that, I'll discuss the Satake transform for arbitrary split reductive groups and its relevance to L-functions.

1. Tate's distributions

Suppose χ to be a continuous character of $\mathfrak{k}^\times$ (i.e. one whose kernel is an open subgroup). Define formally the associated distribution on $\mathcal{S}(\mathfrak{k})$:

$$\langle \chi, f\rangle = \int_{\mathfrak{k}^\times} f(x)\chi(x)\, d^\times x\,.$$

The integral converges if $|\chi(\varpi)| < 1$.

If $f(0) = 0$ or, equivalently, f lies in $\mathcal{S}(\mathfrak{k}^\times)$, this is a finite sum. Otherwise, we may express

$$f = f(0)\cdot \mathfrak{char}(\mathfrak{o}) + f_1$$

with $f_1(0) = 0$, and understanding the distribution χ reduces to examining

$$\langle \chi, \mathfrak{char}(\mathfrak{o})\rangle = \sum_{n\ge 0} \int_{\mathfrak{p}^n - \mathfrak{p}^{n+1}} \chi(x)\, d^\times x = \sum_{n\ge 0} \chi(\varpi^n) \int_{\mathfrak{o}^\times} \chi(u)\, d^\times x\,,$$

since we may write every x in $\mathfrak{p}^n - \mathfrak{p}^{n+1}$ uniquely as $u\varpi^n$ with u in $\mathfrak{o}^\times$. If $\chi|\mathfrak{o}^\times \neq 1$ the integral vanishes and the distribution χ is still well defined, but otherwise we may write $\chi(x) = |x|^s$ and this expression becomes

$$\sum_{n \geq 0} q^{-ns} = \frac{1}{1 - q^{-s}} .$$

In this case the distribution χ has a simple pole at $s = 0$, and it is easy to see that its residue there is

$$\frac{1}{\log q} \cdot \delta_0 .$$

The group $\mathfrak{k}^\times$ acts on $\mathcal{S}(\mathfrak{k})$ and its dual according to the formulas

$$[R_a f](x) = f(xa), \quad \langle R_a F, f\rangle = \langle F, R_{1/a} f\rangle .$$

The distribution χ satisfies on $\mathcal{S}(\mathfrak{k}^\times)$ the equivariance condition

[chi-equivariance] **(1.1)**
$$R_a \chi = \chi(a)\chi ,$$

and for $\chi \neq 1$ extends uniquely to one on $\mathcal{S}(\mathfrak{k})$ satisfying the same condition. But:

♡ [dirac-uniqueness] **1.2. Proposition.** *For $\chi = 1$, the Dirac δ_0 spans the space of all distributions satisfying (1.1).*

In other words, up to scalar the Dirac δ_0 is the unique $\mathfrak{k}^\times$-invariant distribution on $\mathfrak{k}$. Integration over $\mathfrak{k}^\times$ against 1 does not extend invariantly.

Proof. We have an exact sequence

$$1 \longrightarrow \mathcal{S}(\mathfrak{k}^\times) \longrightarrow \mathcal{S}(\mathfrak{k}) \xrightarrow{f \mapsto f(0)} \mathbb{C} \longrightarrow 0 .$$

The associated long exact dual sequence of $\mathfrak{k}^\times$-invariants shows that the dimension of the space of $\mathfrak{k}^\times$-invariants is at most 2, and is equal to 2 only if integration over $\mathfrak{k}^\times$ extends to a $\mathfrak{k}^\times$-invariant distribution. But if $f = \mathfrak{char}(\mathfrak{o})$ then $f - R_{1/\varpi} f$ is equal to the characteristic function of $\mathfrak{o}_\times$ and if Φ is any distribution extending integration on $\mathfrak{k}^\times$

$$\begin{aligned} \langle \Phi - R_\varpi \Phi, f\rangle &= \langle \Phi, f - R_{1/\varpi} f\rangle \\ &= \int_{\mathfrak{o}_\times} d^\times x \\ &= 1 \end{aligned}$$

so that Φ cannot be $\mathfrak{k}^\times$-invariant. ▯

These facts are important in establishing local and global functional equations of L-functions, but I'll not say anything more about that here.

There is another way to interpret this result. Suppose G to be any locally compact group with Haar measure dg. If π is a continuous representation of G, and f in $\mathcal{S}(G)$ then we may define an operator

$$\pi(f)\colon\ v \longmapsto \int_G f(g)\pi(g)v\, dg .$$

Here, we take $G = \mathfrak{k}^\times$ and π to be the one-dimensional representation associated to the character χ, in which case $\pi(f)$ will amount to scalar multiplication by

$$\chi(f) = \int_{\mathfrak{k}^\times} f(x)\chi(x)\, d^\times x .$$

The computation exhibited above shows that $\chi(f)$ may in fact be defined for arbitrary functions f in $\mathcal{S}(\mathfrak{k})$, which are not necessarily of compact support when restricted to $\mathfrak{k}^\times$, as long as $\chi \neq 1$. For example,

$$\mathfrak{char}(\mathfrak{o}) = \sum_{n \geq 0} \mathfrak{char}(\mathfrak{p}^n - \mathfrak{p}^{n+1}),$$

with each summand a function in $\mathcal{S}(\mathfrak{k}^\times)$, but the whole sum not. If $f = \mathfrak{char}(\mathfrak{o})$ and $\chi = |\bullet|^s$ then

$$\chi(f) = \frac{1}{1 - q^{-s}},$$

which is the local ζ-function of $\mathfrak{k}$.

Ngô's idea is to do something similar for other $\mathfrak{p}$-adic reductive groups G and admissible representations π of G—to define $\pi(f)$ for certain smooth functions f on G not necessarily of compact support, and to relate such operators to local L-functions as well as, ideally, to the geometry of certain possibly singular $\mathfrak{p}$-adic spaces.

Remark. *Expressing an unramified character as $|x|^s$ is not perfectly satisfactory, although useful in a global context, since s and $s + 2\pi i/\log q$ determine the same character. Instead, express it as*

$$\chi_z\colon \varpi^n \longmapsto z^n$$

for some z in $\mathbb{C}^\times$ (i.e. take $z = q^{-s}$). Thus

$$\langle \chi_z, \mathfrak{char}(\mathfrak{o}) \rangle = \frac{1}{1 - z}.$$

Exercise. *Prove that the map*

$$f \longmapsto \widehat{f}(z) = \langle \chi_z, f \rangle = \sum_n f(\varpi^n) z^n$$

is an isomorphism of $\mathcal{S}(\mathfrak{k}^\times)^{\mathfrak{o}^\times}$ with the space $\mathbb{C}[z^{\pm 1}]$ of Laurent polynomials $P(z)$. (Hint. This is an elementary exercise, but you might find help in the discussion in one of the appendices.)

Exercise. *Prove that if f has z-transform $\widehat{f}$, as defined in the previous exercise, then $R_\varpi f$ has z-transform $z\widehat{f}$.*

Exercise. *Prove that the map*

$$f \longmapsto \langle \chi_z, f \rangle,$$

which converges for $|z| < 1$, is an isomorphism of $\mathcal{S}(\mathfrak{k})^{\mathfrak{o}^\times}$ with the space of all rational functions

$$\frac{P(z)}{1 - z}$$

with $P(z)$ a Laurent polynomial in $\mathbb{C}[z^{\pm 1}]$.

Exercise. *Use the result of the previous exercise to deduce that there is no $\mathfrak{k}^\times$-invariant functional on $\mathcal{S}(\mathfrak{k})$ extending the invariant integral on $\mathcal{S}(\mathfrak{k}^\times)$. (Hint. Call this space $\mathcal{L}$. What is $\mathcal{L}/(z-1)\mathcal{L}$? Why is this relevant?)*

Exercise. *Let ψ be any non-trivial character of the additive group $\mathfrak{k}$. The Fourier transform on $\mathcal{S}(\mathfrak{k})$ takes f to*

$$[\mathcal{F}f](y) = \int_{\mathfrak{k}} f(x)\psi(xy)\,dx.$$

The Fourier transform on the dual space of distributions on $\mathfrak{k}$ is defined by the formula

$$\langle \mathcal{F}D, f \rangle = \langle D, \mathcal{F}f \rangle.$$

This is compatible with the identification of functions in $\mathcal{S}$ as distributions. Prove that the Fourier transform of the distribution χ is a scalar multiple of $|\bullet|\chi^{-1}$. (Hint. What is the Fourier transform of $R_x f$ in terms of that of f?)

2. Tamagawa's theorem

What happens if we replace $\mathfrak{k}$ by $M = \mathrm{M}_n(\mathfrak{k})$ and $\mathfrak{k}^\times$ by $G = \mathrm{GL}_n(\mathfrak{k})$?

As before, there are two slightly different problems to be considered. The first is whether matrix coefficients of admissible representations π of G can be integrated against functions in $\mathcal{S}(M)$, and the second is whether $\pi(f)$ can be defined for f in $\mathcal{S}(M)$. As before, these turn out to be equivalent problems.

MATRIX COEFFICIENTS. Suppose (π, V) to be an irreducible admissible representation of G, $(\widetilde{\pi}, \widetilde{V})$ its admissible dual. There is a canonical map from $\widetilde{V} \otimes V$ to the ring of endomorphisms of V:

$$\Pi_{\tilde{v}\otimes v}\colon\ u \longmapsto \langle \widetilde{v}, u\rangle\, v\,.$$

Let $\mathcal{E}(V)$ be the ring generated by these. The endomorphisms in $\mathcal{E}(V)$ are those that factor through projection $V \to V^K$ onto the space of vectors fixed by some compact open subgroup K. They are of finite rank.

The **matrix coefficient** of $\widetilde{v} \otimes v$ is the function

$$\Phi_{\tilde{v}\otimes v}(g) = \langle \widetilde{v}, \pi(g)v\rangle = \langle \widetilde{v}, \widetilde{\pi}(g)v\rangle = \langle \widetilde{\pi}(g)^{-1}\widetilde{v}, v\rangle\,.$$

If F lies in $\mathcal{E}(V)$, associated to it is the function

$$\Phi_F(g) = \mathrm{trace}\,\pi(g)F\,.$$

These assignments are compatible, since

$$\mathrm{trace}\,\pi(g)\,\Pi_{\tilde{v}\otimes v} = \langle \widetilde{v}, \pi(g)v\rangle\,.$$

It is hence reasonable to call Φ_F the matrix coefficient of F.

Suppose f to lie in $\mathcal{S}(G)$. On the one hand, associated to it is the operator

$$\pi(f) = \int_G f(x)\pi(x)\, d^\times x$$

in $\mathcal{E}(V)$. On the other, if F lies in $\mathcal{E}(V)$ then

$$\langle \Phi_F, f\rangle = \int_G f(x)\Phi_F(x)\, d^\times x$$

is well defined. This determines a map from $\mathcal{S}(G)$ to the admissible linear dual of $\mathcal{E}(V)$. It is the same as the map determined by the pairing $\langle F, f\rangle = \mathrm{trace}\,\pi(f)F$, as a simple computation shows.

The map $f \mapsto \pi(f)$ is $G \times G$-equivariant:

$$\pi(R_g f) = \pi(f)\pi(g)^{-1}, \quad \pi(L_g f) = \pi(g)\pi(f)\,.$$

The space $\mathcal{S}(G)$ is embedded in $\mathcal{S}(M)$. Does the map $f \mapsto \pi(f)$ extend equivariantly to a map defined on $\mathcal{S}(M)$? Not quite. Let $\pi_s = \pi \cdot |\det|^s$. For $\mathrm{RE}(s) \gg 0$ the operator $\pi_s(f)$ is defined and holomorphic in s. It turns out that it extends meromorphically to all of $\mathbb{C}$, and that at a pole its leading term spans the one-dimensional space of equivariant maps from $\mathcal{S}(M)$ to $\mathcal{E}(V)$. In combination with an analysis of how this map relates to the additive Fourier transformation, it implies a version of Tate's local functional equation that can be used in proving a global functional equation for L-functions. I shall not prove here the claim about meromorphicity in full generality. However, I shall say more in an important case.

PRINCIPAL SERIES REPRESENTATIONS. Assume now, primarily in order to make notation simpler, that $n = 2$. Further let

$$\begin{aligned} \delta &= \text{modulus character of } P \\ K &= \mathrm{GL}_n(\mathfrak{o}) . \end{aligned}$$

Thus

$$\delta = |\det \mathrm{Ad}_{\mathfrak{n}}| \colon \begin{bmatrix} x & * \\ 0 & y \end{bmatrix} \longmapsto |x/y| , \quad \begin{bmatrix} \varpi^m & * \\ 0 & \varpi^n \end{bmatrix} \longmapsto q^{-(m-n)} .$$

The non-trivial element of the Weyl group of A in G is represented by the matrix

$$w = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

Conjugation by w swaps diagonal entries in A.

We have already assigned G a Haar measure such that K has measure 1. Similiarly, fix Haar measures on P, N, and A with the property that the subgroups of integral points have measure 1. The group K acts transitively on $\mathbb{P}^2(\mathfrak{k}) \cong P\backslash G$, which implies that $G = PK$, and integration on G is often usefully done by applying the formula

[int-pk] **(2.1)**
$$\int_G f(g)\, dg = \int_K \left(\int_A \delta^{-1}(a) \left(\int_N f(nak)\, dn \right) da \right) dk .$$

For any character χ of A let

$$I_\chi = \left\{ f \in C^\infty(G) \,\middle|\, f(nag) = \chi(a)\delta^{1/2}(a) f(g) \right\} .$$

The group G acts on this space from the right, and this gives rise to an admissible representation of G. For example, if $\chi = \delta^{-1/2}$ then I_χ may be identified with the space $C^\infty(\mathbb{P}^1(\mathfrak{k}))$. If $\chi = \delta^{1/2}$ then I_χ may be identified with the space of smooth one-densities on $P\backslash G$, for which the G-invariant integral can be taken to be

$$f \longmapsto \int_K f(k)\, dk .$$

The space $I_{\chi^{-1}}$ may be identified with the admissible dual of I_χ, with the pairing

$$\langle f_{\chi^{-1}}, f_\chi \rangle = \int_K f_{\chi^{-1}}(k) f_\chi(k)]\, dk .$$

This justifies the choice of normalizing factor $\delta^{1/2}$ in the definition of I_χ.

Functions in $\mathcal{S}(G)$ act as integral operators on I_χ:

$$[\pi_\chi(f)\varphi](g) = \int_G f(x)\varphi(gx)\, dx .$$

♡ [int-pk] By (2.1) we have

$$\begin{aligned} [\pi_\chi(f)\varphi](g) &= \int_G f(x)\varphi(gx)\, dx \\ &= \int_G f(g^{-1}x)\varphi(x)\, dx \\ &= \int_K \left(\int_A \delta^{-1}(a) \left(\int_N \varphi(nak) f(g^{-1}nak)\, dn \right) da \right) dk \\ &= \int_K \varphi(k) \left(\int_A \chi(a)\delta^{-1/2}(a) \left(\int_N f(g^{-1}nak)\, dn \right) da \right) dk . \end{aligned}$$

Again because $G = PK$, we may identify I_χ as a representation of K by restriction to K. The operator $\pi_\chi(f)$ now becomes an integral operator, with kernel

$$K_f(k,\ell) = \int_A \chi(a)\delta^{-1/2}(a) \left(\int_N f(\ell^{-1}nak)\, dn \right) da\,.$$

Its trace is therefore

$$\int_K \left(\int_A \chi(a)\delta^{-1/2}(a) \left(\int_N f(k^{-1}nak)\, dn \right) da \right) dk\,.$$

To make this slightly more intelligible, let

$$\overline{f}(g) = \int_K f(k^{-1}gk)\, dk\,.$$

This is again in $\mathcal{S}(G)$. Also, for any f in $\mathcal{S}(G)$ define

$$f_P(a) = \delta^{-1/2}(a) \int_N f(na)\, dn\,.$$

In summary:

[character-ps] **2.2. Proposition.** *The trace of $\pi_\chi(f)$ on I_χ is*

$$\int_A \chi(a)\overline{f}_P(a)\, da\,.$$

UNRAMIFIED REPRESENTATIONS. Most phenomena already occur for the simplest case, the unramified representations of $\mathrm{GL}_2(\mathfrak{k})$.

The character χ is called **unramified** if it is trivial on $A(\mathfrak{o})$, in which case it factors through the quotient $A/A(\mathfrak{o})$, isomorphic to a free module over $\mathbb{Z}$. In this case, one is most interested in functions f in the Hecke algebra $\mathcal{H}(G)$ of functions in $\mathcal{S}(G)$ bi-invariant under K. For such an f, $\overline{f} = f$ and f_P becomes a function on $A/A(\mathfrak{o})$ of finite support. The formula for the trace of $\pi_\chi(f)$ then becomes

$$\sum_{A/A(\mathfrak{o})} \chi(a) f_P(a)\,.$$

If we identify A with $(\mathbb{G}_m)^2$, then we may choose a basis $\{\alpha, \beta\}$ for the group of unramified characters of A, and the expression in this formula becomes a Laurent polynomial in the $\alpha^{\pm 1}$, $\beta^{\pm 1}$.

Assume χ to be unramified. Since $A/A(\mathfrak{o})$ is isomorphic to $\mathbb{Z}^2$, it is then of the form

$$\chi_{s,t}\colon \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \longmapsto |x|^s |y|^t$$

for some s, t in $\mathbb{C}$. Such an expression is not unique since, for example, s and $s + 2\pi i/\log q$ determine the same character, but it is useful in global expressions. When working only locally, it is often more convenient to express it as

$$\begin{bmatrix} u\varpi^m & 0 \\ 0 & v\varpi^n \end{bmatrix} \longmapsto \alpha^m \beta^n$$

with $\alpha = q^{-s}$, $\beta = q^{-t}$ in $\mathbb{C}^\times$. For example, if $\chi = \delta^{-1/2}$, then $\alpha = q^{1/2}$ and $\beta = q^{-1/2}$. The group K acts transitively on $\mathbb{P}^1(\mathfrak{k})$, which may be identified with $P\backslash G$, so that $G = PK$. The restriction of I_χ to K may therefore be identified with $C_c^\infty(K\cap P\backslash K)$, and consequently the subspace of K-invariant vectors in I_χ has dimension one. Define v_χ to be the vector in I_χ^K with

$$v_\chi(pk) = \delta^{1/2}(p)\chi(p)\,.$$

It is the unique function v in I_χ^K with $v(1)=1$.

Let $\mathcal{H}(G)$ be the ring of functions in $\mathcal{S}(G)$ bi-invariant with respect to K. If f lies in $\mathcal{H}(G)$ then $\pi_\chi(f)$ takes

♡ [character-ps] I_χ^K to itself, hence acts by a constant. The constant is the trace of R_f on I_χ, which by Proposition 2.2 is

$$c_\chi(f) = \int_A \chi(a)\delta^{-1/2}(a)\left(\int_N f(na)\,dn\right)\,da$$

In this formula, we may take $f=\mathfrak{char}(M_2(\mathfrak{o}))$ as long as χ is in a suitable range. The integral may be explicitly evaluated:

$$\begin{aligned}
c_\chi(f) &= \int_A \delta^{-1/2}(a)\chi(a)\,\mathrm{meas}\big(N\cap M_2(\mathfrak{o})a^{-1}\big)\,da \\
&= \int_{\mathfrak{k}^\times\times\mathfrak{k}^\times} |x|^{-1/2}|y|^{1/2}\mathrm{meas}(\mathfrak{o}y^{-1})|x|^s|y|^t\,d^\times x\,d^\times y \\
&= \int_{\mathfrak{k}^\times\times\mathfrak{k}^\times} |x|^{-1/2}|y|^{1/2}|y|^{-1}|x|^s|y|^t\,d^\times x\,d^\times y \\
&= \left(\int_{\mathfrak{k}^\times\cap\mathfrak{o}} \chi_{q^{1/2}\alpha}(x)\,d^\times x\right)\left(\int_{\mathfrak{k}^\times\cap\mathfrak{o}} \chi_{q^{1/2}\beta}(x)\,d^\times x\right) \\
&= \frac{1}{(1-q^{1/2}\alpha)(1-q^{1/2}\beta)}\,.
\end{aligned}$$

If we replace χ by $\chi\cdot|\det|^s$, this becomes

$$\frac{1}{(1-q^{1/2}\alpha q^{-s})(1-q^{1/2}\beta q^{-s})}\,,$$

Tamagawa's formula for the local L-function of I_χ.

This calculation is the exact analogue of Tate's, with the matrix algebra replacing $\mathfrak{k}$.

Exercise. *Verify Tamagawa's formula for $n>2$.*

3. The Satake transform and Tamagawa's theorem

The method used in the previous section to deduce a formula for local L-functions associated to matrix algebras does not generalize to other classical groups such as GSp, as we shall see in a later section. In this section I'll look again at $\mathrm{M}_2(\mathfrak{k})$ with this in mind.

I'll begin be recalling an elementary fact. First define A^{--} to be the set of matrices

$$\begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$$

with $|a_1/a_2|\le 1$.

[volume-ktk] **3.1. Lemma.** *For a in A^{--}*

$$\mathrm{meas}(KaK) = \begin{cases} 1 & \text{if } a \text{ is scalar} \\ (1+1/q)\cdot\delta^{-1}(a) & \text{otherwise.} \end{cases}$$

Proof. The case in which a is a scalar is trivial. Otherwise we may suppose that

$$a = \begin{bmatrix} \varpi^m & 0 \\ 0 & 1 \end{bmatrix}$$

with $m \geq 1$. We want to show that $\mathrm{meas}(KaK) = (1 + 1/q)q^m = (q + 1)q^{m-1}$. But

$$\begin{aligned} \mathrm{meas}(KaK) &= KaK/K \\ &= K/K \cap aKa^{-1}\,. \end{aligned}$$

The subgroup $K \cap aKa^{-1}$ is made up of matrices

$$\begin{bmatrix} p & \varpi^m q \\ r & s \end{bmatrix}$$

with p, q, r, s in $\mathfrak{o}$. The map

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \longmapsto \begin{bmatrix} p \\ q \end{bmatrix} \mod \mathfrak{p}$$

induces a surjection from $K/K \cap tKt^{-1}$ to $\mathbb{P}^1(\mathbb{F}_q)$, which is of size $q + 1$, with fibres of size q^{m-1}. ▯

Remark. The set $K/K \cap tKt^{-1}$ is isomorphic to the finite projective space $\mathbb{P}^1(\mathfrak{o}/\mathfrak{p}^n)$.

Define the matrix coefficient

$$\Phi_\chi(g) = \langle \widetilde{v}_{\chi^{-1}}, R_g v_\chi \rangle = \int_K v_\chi(kg)\, dk\,.$$

For example, if $\chi = \delta^{-1/2}$ then Φ_χ is the constant 1. For traditional reasons, it is called the **spherical function** parametrized by χ.

The function Φ_χ is right- and left-invariant under multipication by K. Since G is the disjoint union of double cosets

$$Ka_{m,n}K \qquad \left(a_{m,n} = \begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix} \right)$$

with $m \geq n$, the function Φ_χ is determined by its restriction to the diagonal matrices $a_{m,n}$. There is a formula due to Ian Macdonald that specifies $\Phi_\chi(a_{m,n})$ explicitly. Let

$$a^\vee = \begin{bmatrix} \varpi & 0 \\ 0 & 1/\varpi \end{bmatrix}.$$

Recall that w is the non-trivial element of the Weyl group. The following gives an explicit formula for Φ_χ.

[macdonald] **3.2. Proposition.** (Macdonald's formula) *For $a = a_{m,n}$ with $m \geq n$*

$$\Phi_\chi(a) = \frac{\delta^{1/2}(a)}{1 + 1/q} \cdot \left(\frac{1 - q^{-1}\chi^{-1}(a^\vee)}{1 - \chi^{-1}(a^\vee)} \cdot \chi(a) + \frac{1 - q^{-1}[w\chi]^{-1}(a^\vee)}{1 - [w\chi]^{-1}(a^\vee)} \cdot [w\chi](a) \right).$$

This is a relatively simple case of Macdonald's formula, which gives very generally a formula for unramified matrix coefficients of arbitrary reductive groups. We'll see shortly a more explicit version of this one when $\chi = \chi_{\alpha,\beta}$.

Although it is well known, I'll recall a proof of this formula in an appendix. The basic point is that there is an asymptotic formula for the matrix coefficients of arbitrary admissible representations of a reductive $\mathfrak{p}$-adic group. For unramified spherical functions, the asymptotic formula happens to be valid on all of G.

INTEGRAL OPERATORS. I recall that if (π, V) is any admissible representation of G and f lies in $\mathcal{S}(G)$ then

$$\pi(f) = \int_G f(g)\pi(g)\,dg\,.$$

I recall also that $\mathcal{H}$ is the convolution algebra of functions of compact support on G that are right- and left-invariant with respect to K. Convolution by functions in $\mathcal{H}$ takes the one-dimensional space of vectors in I_χ fixed by K to itself, therefore acts through a homomorphism c_χ from $\mathcal{H}$ to $\mathbb{C}$. In other words,

$$\pi_\chi(f)v_\chi = c_\chi(f)\,v_\chi\,.$$

For a fixed f this gives rise to a function on the space of all characters χ, which is isomorphic to $(\mathbb{C}^\times)^2$. For reasons already mentioned and recalled in an appendix, this is a polynomial in $\mathbb{C}[\alpha^{\pm 1}, \beta^{\pm 1}]$, which I call the Fourier (or sometimes the Mellin) transform of f, since it is a very special case of a much more general Fourier transform. It is also frequently called the Satake transform in the literature, because it was first discussed for a large class of groups in [Satake:1963].

Explicitly, $\mathcal{H}$ has as basis the functions

$$f_{m,n} = \mathfrak{char}(K a_{m,n} K)$$

with $m \ge n$.

♡ [volume-ktk] If $\chi = \delta^{-1/2}$ then $\pi(\mathfrak{char}(KaK))$ is multiplication by the volume of KaK, for which Lemma 3.1 gives us a formula. This is a special case of:

[satake-macdonald] **3.3. Lemma.** *Let $f = \mathfrak{char}(KaK)$. For any unramified character of A*

$$c_\chi(f) = \langle \Phi_\chi, f\rangle = \mathrm{meas}(KaK)\,\Phi_\chi(a)\,.$$

Proof. This is straightforward. Let $f = \mathfrak{char}(KaK)$. Then

$$\begin{aligned}
\pi(f)v_\chi &= \int_G f(g)R_g v_\chi\,dg \\
&= \sum_{x\in KaK/K} R_x v_\chi \\
&= \sum_{k\in K/K\cap aKa^{-1}} R_{ka} v_\chi \\
c_\chi(f) &= [KaK/K]\int_K v_\chi(ka)\,dk \\
&= [KaK/K]\int_K v_{\chi^{-1}}(k)v_\chi(ka)\,dk \\
&= [KaK/K]\,\langle v_{\chi^{-1}}, R_a v_\chi\rangle \\
&= \mathrm{meas}(KaK)\,\Phi(a)\,.
\end{aligned}$$

[satake-formula] **3.4. Theorem.** *Suppose $f = f_{m,n}$ and $\chi = \chi_{\alpha,\beta}$. If $m = n$ then*

$$c_\chi(f) = \alpha^n\beta^n\,.$$

Otherwise

$$
\begin{aligned}
c_\chi(f) = q^{(m-n)/2}&\big(\alpha^m\beta^n + \alpha^{m-1}\beta^{n+1} + \cdots + \alpha^n\beta^m\big) \\
&\qquad - q^{((m-1)-(n+1))/2}\big(\alpha^{m-1}\beta^{n+1} - \alpha^{n+1}\beta^{m-1}\big)\,.
\end{aligned}
$$

Proof. The case $m = n$ is immediate. Otherwise $m \geq n+1$—say $m = n+p$ with $p \geq 1$—and

$$
\begin{aligned}
c_\chi(f) &= q^{(m-n)/2}\left(\frac{1-q^{-1}(\beta/\alpha)}{1-\beta/\alpha}\cdot\alpha^m\beta^n + \frac{1-q^{-1}(\alpha/\beta)}{1-\alpha/\beta}\cdot\alpha^n\beta^m\right) \\
&= q^{(m-n)/2}\left(\frac{\alpha-q^{-1}\beta}{\alpha-\beta}\cdot\alpha^m\beta^n - \frac{\beta-q^{-1}\alpha}{\alpha-\beta}\cdot\alpha^n\beta^m\right) \\
&= q^{(m-n)/2}\left(\left(\frac{\alpha^{m+1}\beta^n - \alpha^n\beta^{m+1}}{\alpha-\beta}\right) - q^{-1}\left(\frac{\alpha^m\beta^{n+1} - \alpha^{n+1}\beta^m}{\alpha-\beta}\right)\right) \\
&= q^{(m-n)/2}\big(\alpha^m\beta^n + \alpha^{m-1}\beta^{n+1} + \cdots + \alpha^n\beta^m\big) \\
&\qquad - q^{((m-1)-(n+1))/2}\big(\alpha^{m-1}\beta^{n+1} - \alpha^{n+1}\beta^{m-1} \\
&= q^{p/2}\alpha^n\beta^n\left(\frac{\alpha^{p+1}-\beta^{p+1}}{\alpha-\beta}\right) - q^{(p-2)/2}\alpha^{n+1}\beta^{n+1}\left(\frac{\alpha^{p-1}-\beta^{p-1}}{\alpha-\beta}\right)\,.
\end{aligned}
$$

Expanding, we recover the formula we want. ∎

The expressions in these formulas are symmetric polynomials in $\alpha^{\pm 1}$, $\beta^{\pm 1}$, and it is easy to deduce from them that the Satake homomorphism is a surjection from $\mathcal{H}$ to the ring $\mathbb{C}[\alpha+\beta, (\alpha\beta)^{\pm 1}]$ of all symmetric polynomials. It is a bit harder to show that it is an isomorphism, and I shall not do so. As a consequence, $\mathcal{H}$ may be identified with this ring, which I shall do from now on.

Let $T_{m,n}$ be the image of $f_{m,n}$ under the Satake homoorphism.

For any $p \geq 0$ it is immediate that $T_{p+n,n} = \alpha^n\beta^n T_{p,0}$. A few more examples with $p > 0$ small allow one to see a pattern:

$$
\begin{aligned}
T_{0,0} &= 1 \\
T_{1,0} &= q^{1/2}(\alpha+\beta) \\
T_{2,0} &= q(\alpha^2+\alpha\beta+\beta^2) - \alpha\beta \\
T_{3,0} &= q^{3/2}(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3) - q^{1/2}\alpha\beta(\alpha+\beta)\,.
\end{aligned}
$$

These formulas can be inverted. Let

$$
T_m = \begin{cases} 1 & \text{if } m = 0 \\ q^{1/2}(\alpha+\beta) & \text{if } m = 1 \\ q(\alpha^2+\alpha\beta+\beta^2) & \text{if } m = 2 \\ q^{m/2}(\alpha^m + \alpha^{m-1}\beta + \cdots + \alpha\beta^{m-1} + \beta^m) & \text{otherwise.} \end{cases}
$$

The formulas at hand can be rewritten as

$$
\begin{aligned}
T_{0,0} &= T_0 \\
T_{1,0} &= T_1 \\
T_{2,0} &= T_2 - T_{1,1} \\
T_{3,0} &= T_3 - T_{2,1} \\
&\ \ldots
\end{aligned}
$$

and then solved by recursion to get

$$\begin{aligned} T_0 &= T_{0,0} \\ T_1 &= T_{1,0} \\ T_2 &= T_{2,0} + T_{1,1} \\ T_3 &= T_{3,0} + T_{2,1} \\ T_4 &= T_{4,0} + T_{3,1} + T_{2,2} \\ &\ldots \\ T_m &= \sum_{\substack{k+\ell=m \\ k\ge\ell\ge 0}} T_{k,\ell}\,. \end{aligned}$$

Since

$$\begin{aligned} \frac{1}{1-\alpha} &= 1 + \alpha + \alpha^2 + \alpha^3 + \cdots \\ \frac{1}{1-\beta} &= 1 + \beta + \beta^2 + \beta^3 + \cdots \end{aligned}$$

we have

$$\frac{1}{(1-q^{1/2}\alpha)(1-q^{1/2}\beta)} = 1 + q^{1/2}(\alpha+\beta) + q(\alpha^2+\alpha\beta+\beta^2) + q^{3/2}(\alpha^3+\alpha^2\beta+\alpha\beta^2+\beta^3+\cdots$$

If we replace χ by $|\det|^s$ we replace α, β by αq^{-s}, βq^{-s}. The previous formula now gives us the classical formula of Hecke

$$\sum_{n\ge 0} T_n q^{-ns} = \frac{1}{I - T_{1,0}q^{-s} + qT_{1,1}q^{-2s}}\,.$$

The intersection $M_2(\mathfrak{o}) \cap \mathrm{GL}_2(\mathfrak{k})$ is the union of the double cosets $Ka_{k,\ell}K$ with $k \ge \ell \ge 0$, so the left hand side here is, at least formally, $\pi(\mathfrak{char}(M_2(\mathfrak{o}))$. It converges for $\mathrm{RE}(s) \gg 0$, and since the right hand side is meromorphic in s we finally deduce a slightly different form for our L-function:

[tamagawas-formula] **3.5. Corollary.** *Define* $\pi_s = \pi \cdot |\det|^s$. *Then*

$$\pi_s(\mathfrak{char}(M_2(\mathfrak{o})) = \mathrm{L}(s - {}^1\!/_2, \pi, I)\,.$$

Here I is the standard representation of GL_n. This relation between a local L-function and Hecke operators originated with an observation of Ramanujan about a certain automorphic form. This in turn was interpreted more generally by Louis Mordell and subsequently more generally yet by Erich Hecke. The first interpretation of a purely local nature is apparently due to Tamagawa, and published in [Tamagawa:1963]. In fact he proved a similar result valid for $G = \mathrm{GL}_n(\mathfrak{k})$, $M = \mathrm{M}_n(\mathfrak{k})$. In that case, too, $\pi(\mathfrak{char}(\mathrm{M}_n(\mathfrak{o}))$ is a local L-factor.

Exercise. *Can you find a characterization of the Mellin transforms of all of* $\mathcal{S}(M_2(\mathfrak{o}))^K$*? (This is not easy. Note that there are three* $G \times G$*-orbits on* $M_2(\mathfrak{k})$*, the matrices of ranks* 0*,* 1*, and* 2*. This gives rise to a filtration of* $\mathcal{S}(M)$ *by* $G \times G$*-stable spaces. What are the corresponding isotropy subgroups? The corresponding images under the Mellin transform?)*

4. L-functions and the Hecke algebra

In this section, I'll begin to look at how the techniques of the previous section might apply to give information about more general L-functions.

Let

$$
\begin{aligned}
G &= \text{ a split reductive group defined overe } \mathfrak{k} \\
P &= \text{ a Borel subgroup of } GF \\
A &= \text{ a maximal split torus in } P \\
U &= \text{ unipotent radical of } P \\
\Sigma &= \text{ roots of } (G, A) \\
\Delta &= \text{ simple roots associated to the choice of } P \\
W &= \text{ Weyl group} \\
L &= X^*(A) \\
L^\vee &= X_*(A) \\
K &= G(\mathfrak{o}) \\
\mathcal{H} &= C_c^\infty(K\backslash G/K, \mathbb{C})\,.
\end{aligned}
$$

The based root datum of G, P, A is $\mathcal{L} = (L, \Delta, L^\vee, \Delta^\vee)$. The Langlands dual group $G^\vee$ is the complex reductive group with based datum $\mathcal{L}^\vee = (L^\vee, \Delta^\vee, L, \Delta)$.

The dual torus $A^\vee$ is that with $X^*(A^\vee) = X_*(A)$ and $X_*(A^\vee) = X^*(A)$. Points of $A^\vee$ parametrize unramified characters of A. The Weyl group for $G^\vee$ may be identified with that of G, and acts compatibly on these. The quotient $A^\vee/W$ may be identified with semi-simple conjugacy classes of $G^\vee$.

Let I_χ be the normalized principal series representation of G determined by the unramified character χ. The Hecke algebra $\mathcal{H}$ acts on the one-dimensional space I_χ^K through a homomorphism c_χ. The Hecke algebra is thus identified with the ring of affine functions on $G^\vee$ invariant under conjugation. It π is an unramified representation of G, let $g_\pi^\vee$ be a representative of the corresponding semi-simple conjugacy class.

[lspirho-symn] **4.1. Lemma.** *If π is an unramified representation of G and (σ, V) a representation of $G^\vee$, then for $\mathrm{RE}(s) \gg 0$*

$$
L(s, \pi, \sigma) = \frac{1}{\det(I - \sigma(g_\pi^\vee) q^{-s})} = \sum_{n \ge 0} \text{trace } [\mathrm{Sym}^n \sigma](g_\pi^\vee)\, q^{-ns}\,.
$$

Here $\mathrm{Sym}^n \sigma$ is the representation on the space $\mathrm{Sym}^n V$ of symmetric n-tensors.

Proof. This is a special case of the more basic equation in formal series

$$
\frac{1}{\det(I - Tx)} = \sum_{n \ge 0} \text{trace } [\mathrm{Sym}^n T]\, x^n\,,
$$

in which T is an arbitrary linear transformation of a space V. If T is diagonalizable, this just reduces to a product of geometric series

$$
\frac{1}{\prod (1 - \alpha_i x)} = \prod (1 + \alpha_i x + \alpha_i^2 x^2 + \cdots)\,.
$$

But the diagonalizable matrices are Zariski-dense in the space of all matrices. ∎

In the last section, we used a special case of this to deduce a formula for $L(s, \pi, \sigma)$, when $G = \mathrm{GL}_2$ and σ the identity map from GL_2 to itself, as the Satake transform of the characteristic function of $\mathrm{M}_2(\mathfrak{o})$. In this section I ask, *how can this be generalized?*

Each term in the sum on the right is, for fixed s, a conjugation-invariant affine function on $G^\vee$, hence corresponds to a function $f_{\sigma,k}$ in the Hecke algebra. But how to take into account the factor q^{-s}? What about the sum?

In Langlands' original conjecture the factor q^{-s} has a natural origin, one which is hidden in the usual accounts. In most of these, the dual group is taken to be $G^\vee$. But the real dual group ought to be Langlands' L-group, which is a semi-direct product of some form of Galois group and the complex group $G^\vee$. If G is split, this is not immediately evident, since the action of the Galois group on $G^\vee$ is trivial, whereas in general it acts by outer automorphisms. This happens already for unramified quasi-split groups. In that case, the form of the Galois group concerned is the group generated by the Frobenius automorphism $\mathfrak{F}$ of the maximal unramified extension of the $\mathfrak{p}$-adic field. The unramified principal series are parametrized by conjugacy classes in $G^\vee \times \mathfrak{F}$, which is not generally the same as the conjugacy classes of $G^\vee$ itself. The factor q^{-s} comes from $\mathfrak{F}$.

However, in all cases in which a functional equation has been proved, the factor q^{-s} has another origin. For example, suppose G to be GL_n and σ to be the standard representation. Any representation π lies in the family of representations $\pi_s = \pi \cdot |\det|^s$, and $g^\vee_{\pi_s} = g^\vee_\pi\, q^{-s}$. Therefore

$$\sigma(g^\vee_\pi)\, q^{-s} = \sigma(g^\vee_{\pi_s})\,.$$

There is no need in this case to have the variable s occurring independently in the definition of the L-function—one just uses the fact that

$$\mathrm{L}(s,\pi,\sigma) = \mathrm{L}(0,\pi_s,\sigma)\,,$$

and works with a slightly simpler $\mathrm{L}(\pi,\sigma)$.

One can try to generalize what happens for GL_n. We want to postulate the existence of something like a determinant map on G. I do this by a condition on the dual group:

> *Assume that we are given an identification of the center $Z^\vee$ of $G^\vee$ with $\mathbb{C}^\times$ and that $\sigma(z) = z \cdot I$ for z in $Z^\vee$.*

Typical cases are the standard representations of GL_n and GSp_{2n}.

The given embedding of $\mathbb{C}^\times$ into $G^\vee$ corresponds to an algebraic character of G:

$$\det\colon\ G \to \mathfrak{k}^\times\,.$$

It seems reasonable to define $\det^\vee$ to be the original embedding of $\mathbb{C}^\times$ into $G^\vee$. With this notation, if π is any unramified admissible representation of G corresponding by Langlands' functoriality to a semi-simple conjugacy class in $G^\vee$ represented by $g^\vee$ in $G^\vee$, then $\pi_s = \pi \cdot |\det|^s$ corresponds to $g^\vee_s = g^\vee \cdot \det^\vee(|\varpi|^s)$. Now

$$\mathrm{L}(s,\pi,\sigma) = \frac{1}{\det(I - \sigma(g^\vee_s))}\,.$$

[satake-ngo] **4.2. Theorem.** *With the assumption above, there exists a unique function f_σ bi-invariant with respect to K on G such that*

$$\pi_s(f_\sigma) = L(s - {}^{\langle\lambda,\rho^\vee\rangle}/_2, \pi, \sigma)\,.$$

The term $\langle\lambda,\rho^\vee\rangle/2$ is chosen so as to agree with the term $(n-1)/2$ in the case of GL_n, since in that case $\lambda = \varepsilon_1$ and the fundamental weights are

$$\begin{aligned} &\varepsilon_1 \\ &\varepsilon_1 + \varepsilon_2 \\ &\dots \\ &\varepsilon_1 + \dots + \varepsilon_{n-1}\,. \end{aligned}$$

This is somewhat arbitrary, of course, but the example of GL_n suggests that this normalization might have geometric significance.

Proof. This is elementary, but I have to recall some formalities. The restriction of $\mathrm{Sym}^n\sigma$ to the centre of $G^\vee$ is may be identified with the character $z \mapsto z^n$ of $\mathbb{C}^\times$. If f in $\mathcal{H}$ maps to a component of $\mathrm{Sym}^n\sigma$ then its support lies in the inverse image of q^{-n}, and if $m \neq n$ then the inverse images have no overlap. So the infinite sum defining $\mathrm{L}(s,\pi,\sigma)$ is locally finite. ∎

The natural question now is, *can we find an explicit formula for the function* f_σ*?* If $G = \mathrm{GL}_n$ and σ the tautological representation, it is the characteristic function of $\mathrm{GL}_n(\mathfrak{k}) \cap \mathrm{M}_n(\mathfrak{o})$, but it was pointed out already in §3 of Appendix I of [Satake:1963] that nothing so simple occurs in general. More precisely, the group GSp_4 is contained in the cone MSp_4, and it was shown by Satake that the transform of $\mathrm{MSp}_4(\mathfrak{o})$ is not equal to the associated Langlands L-function. Recently Ngô has suggested that this is related to the fact that MSp_4 is a singular cone, whereas $\mathrm{M}_n(\mathfrak{k})$ is a non-singular variety in which GL_n embeds. He has further conjectured that the function f_σ is related by the sheaf-function correspondence of Grothendieck to a perverse sheaf on an infinite-dimensional variety whose $\mathbb{F}_p$-rational points may be identified with $G(\mathfrak{k})/G(\mathfrak{o})$. But considerations of harmonic analysis suggest that there should be a simpler description of the asymptotic behaviour of f_σ. Furthermore, as far as I know there is no conjecture relating this asymptotic behaviour to the nature of the singularity of MSp_4, which is what one might expect and hope for.

It seems that in general the right space to replace M_n is a monoid M_σ associated by Vinberg to the representation σ. It remains to figure out the asymptotic behaviour of f_σ on G, and also its relation, if any, with perverse sheaves on M_σ. This is also suggested by a more general question about the specification of a natural space containing f_σ on which one can do interesting harmonic analysis, as one can with the action of $\mathrm{GL}_n(\mathfrak{k})$ on $\mathcal{S}(\mathrm{M}_n(\mathfrak{k}))$.

This requires understanding better the Satake isomorphism, which is what I am going to attempt in the next section.

But before that, I want to say more about the assumption regarding $Z^\vee$ and σ. How can one construct a large family of examples? Suppose $G^\vee$ to be a semi-simple complex group, σ an irreducible representation of $G^\vee$ that is injective on its center. Define

$$G^\vee_\sigma = \frac{G^\vee \times \mathbb{C}^\times}{\{(1/z, \sigma(z))\}}$$

(with z running over the center of $G^\vee$). The canonical map from $\mathbb{C}^\times$ to $G^\vee_\sigma$ is an isomorphism with the center of $G^\vee_\sigma$, and σ extends to a unique representation of $G^\vee_\sigma$ such that $\sigma(z) = z \cdot I$. for z in $\mathbb{C}^\times$. The canonical map from $G^\vee$ is also an embedding that identifies $G^\vee$ as the derived group of $G^\vee_\sigma$.

5. The fine structure of the Satake transform

Continue the notation of the previous section. For any λ in $X_*(A)$ let ϖ^λ be the image in A of ϖ with respect to it. The map taking λ to ϖ^λ induces an isomorphism of $X_*(A)$ with $A/A(\mathfrak{o})$. The functions $\mathfrak{char}(K\varpi^\lambda K)$ as λ varies over the positive cone in $X_*(T)$ make up a basis of $\mathcal{H}$. The Satake homomorphism is given explicitly in terms of this basis by a formula proved first by Ian Macdonald.

The lattice $X_*(A)$ is the same as $X^*(A^\vee)$. The character of $A^\vee$ corresponding to λ will be e^λ. The positive cone in $X_*(A)$ is the same as the cone of dominant weights in $X^*(A^\vee)$. Each dominant weight in $X^*(A^\vee)$ gives rise to an irreducible representation π^λ of $G^\vee$ with highest weight λ. In the Grothendieck group of $A^\vee$ it will be

$$\pi^\lambda = \sum_\mu m^\lambda_\mu\, e^\mu$$

in which the sum is over the weights μ of the representation with their multiplicities.

Let χ be an unramified character of A. The original form of Macdonald's formula is for the spherical function Φ_χ, which is completely determined by its values on A^{--}.

Let I be the Iwahori subgroup of K, the inverse image of the Borel subgroup in $G(\mathbb{F}_q)$ corresponding to P. Define

$$\mu_G = \frac{\mathrm{meas}(Iw_\ell I)}{\mathrm{meas}(K)} = \frac{1}{\sum_{w\in W} q^{-\ell(w)}}\,.$$

This is the proportion of points in $G(\mathbb{F}_q)$ contained in the largest open Bruhat cell.

There are a few variants of Macdonald's formula, describing the asymptotic behaviour of matrix coefficients in terms of Jacquet modules. The only version referred to here is an explicit formula for unramified spherical functions, in which case the asymptotic behaviour is valid everywhere.

[macdonalds-g] **5.1. Theorem.** (Macdonald's formula for the unramified spherical function) *For χ an unramified character of A and $f = \mathfrak{char}(KaK)$ with $a \in A^{--}$*

$$\Phi_\chi(f) = \mu_G \cdot \delta^{1/2}(a) \left(\sum_{w\in W} \frac{\prod_{\gamma>0}\big(1 - q^{-1}[w\chi](\varpi^{-\gamma})\big)}{\prod_{\gamma>0}\big(1 - [w\chi](\varpi^{-\gamma})\big)} \cdot [w\chi](a) \right)$$

The relationship of the Satake homomorphism with Macdonald's formula is very simple:

[macd-sat] **5.2. Lemma.** *For $f = \mathfrak{char}(KaK)$ with t in A^{--}*

$$c_\chi(f) = |KaK/K|\,\Phi_\chi(a)\,.$$

♡ [satake-macdonald] The proof of this is exactly the same as that of Lemma 3.3.

[kak-volume] **5.3. Lemma.** *For a in A^{--}*

$$|KaK/K| = \frac{\mu_{M_a}}{\mu_G} \cdot \delta^{-1}(a)\,,$$

where M_a is the Levi component centralizing a.

Here a corresponds to an element $\lambda^\vee$ of $X_*(A)$ and $M_a = M_\Theta = M_\lambda$ where

$$\Theta = \big\{\alpha \in \Delta \,\big|\, \langle \alpha, \lambda^\vee\rangle = 0\big\}\,.$$

♡ [volume-ktk] The proof is similar to that of Lemma 3.1. It fibres $K/K \cap tKt^{-1}$ over the flag variety $G(\mathbb{F}_q)/P_\Theta(\mathbb{F}_q)$.

[satake-explicit] **5.4. Theorem.** *For $f = \mathfrak{char}(K\varpi^\lambda K)$ with λ in $X_*^{++}(A)$*

$$c_\chi(f) = \mu_{M_\lambda} \cdot \delta^{-1/2}(\varpi^\lambda) \left(\sum_{w\in W} \frac{\prod_{\gamma>0}\big(1 - q^{-1}[w\chi](\varpi^{-\gamma})\big)}{\prod_{\gamma>0}\big(1 - [w\chi](\varpi^{-\gamma})\big)} \cdot [w\chi](\varpi^\lambda) \right)$$

Macdonald's formula is at first sight an assertion about spherical functions on the group G. But it can also be interpreted as a statement about certain functions on the dual group. As we shall see in a moment, somewhat hidden in this formula are instances of Weyl's character formula for representations of $G^\vee$. The basic point is very simple—the image of $\mathfrak{char}(K\varpi^\lambda K)$ in $\mathcal{H}$ with respect to the Satake transform, considered as a function of χ, is a conjugation-invariant function of semi-simple classes in $G^\vee$, and in fact the Satake homomorphism is an isomorphism of $\mathcal{H}$ with $\mathbb{C} \otimes R_{G^\vee}$. The characters of finite-dimensional representations of $G^\vee$ are a basis of the representation ring $R_{G^\vee}$. This may be interpreted also as the ring of functions on $G^\vee$ that are invariant under conjugation.

It is a natural question to ask what the relationship between these two bases is. The first step is to identify the set $A^{--}/A(\mathfrak{o})$ with the set of dominant weights of $A^\vee$. For example, if $m \ge n$ then

$$\begin{bmatrix} \varpi^m & 0 \\ 0 & \varpi^n \end{bmatrix}$$

lies in A^{--}.

Thus $\chi(\varpi^\lambda)$ may be interpreted as $\lambda(a_\chi^\vee)$, the evaluation of the character λ on the semi-simple element $a_\chi^\vee$ in $A^\vee$. The right hand side of Macdonald's formula can now be interpreted as an identity of functions on $A^\vee$:

$$c(f_\lambda) = \mu_{M_\lambda} \cdot q^{\langle \lambda, \rho^\vee \rangle} \left(\sum_{w \in W} \frac{\prod_{\gamma > 0} \big(1 - q^{-1} e^{-w\gamma}\big)}{\prod_{\gamma > 0} \big(1 - e^{-w\gamma}\big)} \cdot e^{w\lambda} \right)$$

with $f_\lambda = \mathfrak{char}(K \varpi^\lambda K)$.

I now expand the product and invert the order of sums.

$$c(f_\lambda) = \mu_{M_\lambda} q^{\langle \lambda, \rho^\vee \rangle} \left(\sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \sum_W \frac{e^{w(\lambda - \gamma_S)}}{\prod_{\gamma > 0} (1 - e^{-w\gamma})} \right) .$$

Here, for $S \subseteq \Sigma^{++}$

$$\gamma_S = \sum_{\gamma \in S} \gamma \,.$$

One form of Weyl's character formula tells us that for λ dominant

$$\pi^\lambda = \sum_W \frac{e^{w\lambda}}{\prod_{\gamma > 0} (1 - e^{-w\gamma})} \,.$$

I therefore write the previous formula as

$$c(f_\lambda) = \mu_{M_\lambda} \, q^{\langle \lambda, \rho^\vee \rangle} \sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \pi^{\lambda - \gamma_S} \,.$$

There is a small problem with this, since even when λ is dominant it may well happen that $\lambda - \gamma_S$ is not. It is important to take this into consideration at the same time as a matter of symmetry.

Another form of Weyl's formula asserts that

$$\pi^\lambda = \frac{\sum_w \operatorname{sgn}(w) \, \chi^{w(\lambda + \rho)}}{\sum_w \operatorname{sgn}(w) \, \chi^{w\rho}} \,.$$

This form possesses a certain symmetry that allows us to evaluate it even when λ is not dominant. It tells us that $\pi^\lambda = \operatorname{sgn}(w) \pi^\mu$ whenever $w(\lambda + \rho) = \mu + \rho$. This suggests that I define

$$\Pi^\lambda = \pi^{\lambda - \rho}, \quad \text{equivalently} \quad \Pi^{\lambda + \rho} = \pi^\lambda \,.$$

The symmetry now becomes $\Pi^{w\lambda} = \operatorname{sgn}(w) \, \Pi^\lambda$. With this new notation, the formula becomes

$$c(f_\lambda) = \mu_{M_\lambda} \, q^{\langle \lambda, \rho^\vee \rangle} \sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \Pi^{\lambda + (\rho - \gamma_S)} = \mu_{M_\lambda} \, q^{\langle \lambda, \rho^\vee \rangle} \sum_{S \subseteq \Sigma^+} (-q)^{-|S|} \Pi^{\lambda + \rho_S}$$

with $\rho_S = \rho - \gamma_S$. Thus, for example, $\rho_\emptyset = \rho$ and $\rho_{\Sigma^+} = -\rho$. In fact, every $w\rho$ is one of the ρ_S, since

$$\begin{aligned} 2\rho &= \sum_{\gamma>0} \gamma \\ 2w\rho &= \sum_{\gamma>0} w\gamma \\ &= \sum_{\gamma>0, w^{-1}\gamma>0} \gamma \; - \sum_{\gamma>0, w^{-1}\gamma>0} \gamma \\ &= \sum_{\gamma>0} \gamma \; - \; 2 \sum_{\gamma>0, w^{-1}\gamma<0} \gamma \\ &= 2(\rho - S_w) \end{aligned}$$

if

$$S_w = \{\gamma > 0 \,|\, w^{-1}\gamma < 0\} \, .$$

The set $\mathcal{C}_\rho$ is the set of weights of the irreducible representation π^ρ of highest weight ρ. It plays an important role in proving Weyl's character formula. I'll say more about it in the next section.

For each μ in $\mathcal{C}_\rho$ define

$$P_\mu(x) = \sum_{S \,|\, \rho_S = \mu} x^{|S|} \, .$$

Thus $P_\rho(x) = 1$, $P_{w\rho}(x) = x^{\ell(w)}$, and if we set $x = 1$ this becomes the same as the multiplicity of the weight μ in π^ρ. Our formula now becomes

$$c(f_\lambda) = \mu_{M_\lambda} \, q^{\langle \lambda, \rho^\vee \rangle} \sum_{\mu \in \mathcal{C}_\rho} P_\mu(-q^{-1}) \Pi^{\lambda+\mu} \, .$$

There is still one more modification to come. In calculating with this formula, any Π^λ can be transformed to some $\pm\Pi^\mu$ with μ in X^{++}, by applying the familiar algorithm of W-reduction to the positive chamber. And then either μ is of the form $\nu + \rho$ with ν dominant, in which case $\Pi^\mu = \pi^\nu$, or it is singular and $\Pi^\mu = 0$.

There is one significant case in which one can use these observations to improve the formula we have so far. Suppose $\Theta \subseteq \Delta$ to be the subset of α with $\langle \lambda, \alpha^\vee \rangle = 0$. Then $w\lambda = \lambda$ for w in W_Θ, and $w(\lambda + \mu) = \lambda + w\mu$ for w in W_Θ. If μ is singular with respect to Θ then $\Pi^{\lambda+\mu} = 0$, so in the previous formula one can restrict to the μ that are not singular in this sense. Every such μ in $\mathcal{C}_\rho$ is equal to the W_Θ-transform of some unique ν in the subset

$$[W_\Theta \backslash \mathcal{C}_\rho] = \big\{ \nu \in \mathcal{C}_\rho \,\big|\, \langle \nu, \alpha^\vee \rangle > 0 \text{ for } \alpha \in \Theta \big\} \, ,$$

Therefore the previous formula can be rewritten once again:

[macd-satake] **5.5. Theorem.** (Macdonald's formula for the Satake transform) *Suppose λ dominant, and let Θ be the subset of simple roots α such that $\langle \lambda, \alpha^\vee \rangle = 0$. Then*

$$q^{-\langle \lambda, \rho^\vee \rangle} c(f_\lambda) = \sum_{\mu \in [W_\Theta \backslash \mathcal{C}_\rho]} \left(\frac{\sum_{W_\Theta} \operatorname{sgn}(w) \, P_{w\mu}(-q^{-1})}{\sum_{W_\Theta} q^{-\ell(w)}} \right) \Pi^{\lambda+\mu} \, .$$

The denominator can be expressed as

$$\textstyle\sum_{W_\Theta} \operatorname{sgn}(w) \, P_{w\rho}(-q^{-1}) \, ,$$

and this guarantees that the matrix of the Satake transformation is unipotent. We can now do a simple reality check. If $\lambda = 0$ then $f_\lambda = \mathfrak{char}(K)$ and $c(f_0) = 1$ identically, for trivial reasons. What does the formula tell us? If $\lambda = 0$ then $M_\lambda = G$ and the μ-term is $\sum q^{-\ell(w)}$. A basic property of the set $\mathcal{C}_\rho$ is that all weights in it are singular except the $w\rho$—the weight ρ is the smallest of the regular dominant weights. This is because if λ is in X^{++}, it must be either in $\rho + X^{++}$ or fixed by some s_α, since the intervening band has width 1. So all terms in the sum over $\mathcal{C}_\rho$ vanish except those for the $w\rho$, and for one of those $\Pi^{w\rho} = \mathrm{sgn}(w)\,\Pi^\rho = \mathrm{sgn}(w)$. So there is no contradiction with the trivial evaluation.

There is one thing about this formula that is not *a priori* evident. The denominator μ_{M_λ} implies that the coefficient of $\Pi^{\lambda-\mu}$ is at least a formal series in q^{-1}. In fact:

[quotient-poly] **5.6. Lemma.** *Suppose $\Theta \subseteq \Delta$ and μ to be an element of $[W_\Theta \backslash \mathcal{C}_\rho]$. Then the quotient*

$$\mathcal{M}_\mu(x) = \frac{\sum_{W_\Theta} \mathrm{sgn}(w)\, P_{w\mu}(-x)}{\sum_{w\in W_\Theta} x^{\ell(w)}}$$

is a polynomial in x.

For example, if $\Theta = \emptyset$ there is nothing to prove, and we have already seen that the quotient is 1 if $\Theta = \Delta$. The proof of the last claim involved interpreting $c(f_0)$ in terms of representation theory, and I imagine something similar would work in general by looking at the asymptotic behaviour of matrix coefficients in different directions at infinity. But I'll not pursue this here.

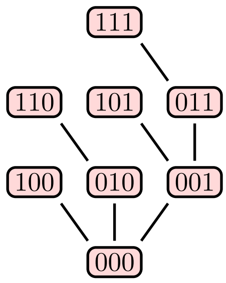
A proof was apparently first published in §3.3.8 of [Matsumoto:1977]. Other proofs appear in the discussion after the statement of Theorem 6.6 in [Lusztig:1983] and at the end of [Haines et al.:2009]. However, the proof I'd like to see reduces to a question about how the action of W on $\mathcal{C}_\rho$ interacts with the polynomials $P_\mu(x)$. I do not presently see anything simple along these lines. I'll say something about this in a later section.

6. Practical computation

The Satake transform is a bit hard to compute explicitly, but not impossible. First one assembles data concerning only the set $\mathcal{C}_\rho$.

1. Compute the polynomials $P_\mu(x)$ for μ in $\mathcal{C}_\rho$. This is done by scanning through the subsets $S \subseteq \Delta$, in order of set size. This can be done very efficiently by considering subsets as integers in the range $[0, 2^N)$, with N equal to the number of positive roots. We build a tree of such subsets by assigning as predecessor of n the number obtained from n by deleting its lowest bit. It is easy to traverse this tree by means of a queue. As we add bit i to S to get T, we add root i to λ_S to get λ_T, and $x^{|T|}$ to P_{ρ_T}. This is the bottleneck in any algorithm, since 2^N can be astronomically large even for systems of small rank. For example, for F_4 there are 2^{24} subsets.

Here, for example, is the tree of subsets of $\{0, 1, 2\}$:



images/settree.eps

2. Then for each $\Theta \subseteq \Delta$ we construct the tables recording $\mathcal{M}_\mu(x)$ for μ in $[W_\Theta \backslash \mathcal{C}_\rho]$. We do this by scanning through $\mathcal{C}_\rho$, for each μ in it finding $w \in W_\Theta$ such that $w\mu$ lies in $[W_\Theta \backslash \mathcal{C}_\rho]$. We find w as the product of the s_α

reducing a coordinate $\langle \mu, \alpha^\vee \rangle$—i.e. we don't have to deal here directly with W_Θ. We don't even have to deal with it explicitly in ordr to compute the Poincaré series

$$\sum\nolimits_{W_\Theta} x^{\ell(w)} ,$$

since this can be found from data attached to the W_Θ-orbit of $-\rho$ in $\mathcal{C}_\rho$.

3. Then for a dominant weight λ we compute the Satake transform of $q^{-\langle \mu, \rho^\vee \rangle} \mathfrak{char}(K\varpi^\mu K)$ for every μ in $\lambda - L^+_\Delta \cap X^{++}$, using the observation of Moody-Patera that we do not have to leave X^{++} to scan it. We then linearly order this set, so as to give us a unipotent matrix. In this process, it may happen that we need to evaluate Π^ν for ν not dominant. Find $w\nu$ that is dominant, and use formula $\Pi^{w\nu} = \mathrm{sgn}(w)\,\Pi^\nu$.

4. Invert this unipotent matrix. At the end we get an expression

$$\pi^\lambda = \sum\nolimits_{\substack{\mu \le \lambda \\ \mu \gg 0}} q^{-\langle \mu, \rho^\vee \rangle} K_{\mu,\lambda}(q^{-1}) \mathfrak{char}(K\varpi^\mu K) .$$

Here $\mu \le \lambda$ means that μ lies in $\lambda - L^+_\Delta$, and $\mu \gg 0$ that μ is dominant.

Considering how complicated an explicit evaluation of the Satake transform is, it might be surprising to learn that there are several interesting ways to interpret the polynomials $K_{\lambda,\mu}(x)$, as we shall see later.

7. A small puzzle

This section will be a more or less self-contained introduction to the set $\mathcal{C}_\rho$.

I recall that for $S \subseteq \Sigma^+$

$$\gamma_S = \sum_{\gamma \in S} \gamma, \quad \rho_S = \rho - \gamma_S ,$$

and that

$$w\rho = \rho_{S_w} \ \text{ with } \ S_w = \{\gamma > 0 \,|\, w^{-1}\gamma < 0\} .$$

[wgammas] **7.1. Lemma.** *For $S \subseteq \Sigma^+$, w in W*

$$w(\gamma_S) = \gamma_T$$

with

$$T = \big\{\gamma > 0 \,\big|\, w^{-1}\gamma \in S \text{ or } w^{-1}\gamma \in -(\Sigma^+ - S)\big\} .$$

Proof. From the formula

[plusorminus] **(7.2)**
$$\rho_S = {}^1\!/_2 \Big(\sum_{\gamma > 0, \gamma \notin S} \gamma - \sum_{\gamma > 0, \gamma \in S} \gamma \Big) .$$

0

In particular, if α is a simple root then $s_\alpha(\gamma_S) = \gamma_T$ with

$$T = \begin{cases} s_\alpha(S - \{\alpha\}) & \text{for } \alpha \in S \\ s_\alpha(S) \cup \{\alpha\} & \text{otherwise.} \end{cases}$$

(Keep in mind that s_α permutes the complement of α in the set of positive roots.) This generalizes how s_α acts on W itself, incrementing or decrementing length.

[singular] **7.3. Lemma.** *If μ lies in $\mathcal{C}_\rho$ and is not in the W-orbit of ρ, then it is singular.*

Proof. We may assume that μ is in the positive Weyl chamber X^{++}. But then if it is not in $\rho + X^{++}$ we must have $0 \le \langle \mu, \alpha^\vee \rangle < 1$ for some simple α. ∎

For every μ in $\mathcal{C}_\rho$, let

$$P_\mu(x) = \sum_{S \mid \rho_S = \mu} x^{|S|} .$$

[pirho] **7.4. Lemma.** *The set $\mathcal{C}_\rho$ is contained in the convex hull of the W-orbit of ρ. Every μ in $\mathcal{C}_\rho$ is a weight of the irreducible representation and its multiplicity is $P_\mu(1)$.*

♡ [plusorminus] *Proof.* The first claim follows from (7.2), the second from Weyl's character formula. ∎

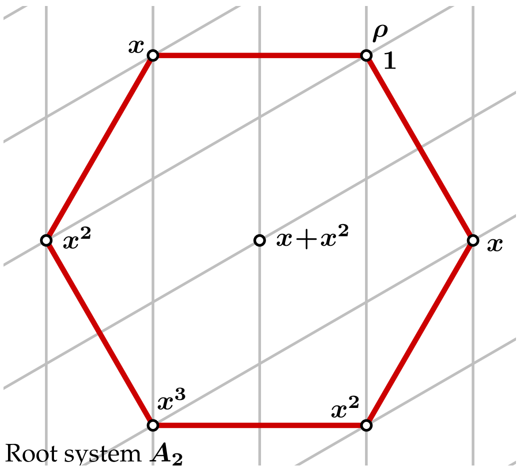
Now I come back to the question as to whether the quotient

$$\frac{\sum_{W_\Theta} \operatorname{sgn}(w)\, P_{w\mu}(-q^{-1})}{\sum_{w \in W_\Theta} q^{-\ell(w)}}$$

is a polynomial, for every μ in $[W_\Theta \backslash \mathcal{C}_\rho]$. Of course it is equivalent, and sometimes more convenient, to make a similar claim about

$$\frac{\sum_{W_\Theta} \operatorname{sgn}(w)\, P_{w\mu}(x)}{\sum_{w \in W_\Theta} (-x)^{\ell(w)}}$$

To guide the discussion, I include below the diagrams for root systems of rank two. In all cases, the set of simple roots is a pair a, b, in which a is a horizontal vector. The lines drawn have equations $\langle \gamma, \alpha^\vee \rangle = k$ for simple roots α. One thing all diagrams have in common is that each $w\rho$ is a unique sum of positive roots, those in S_w, and that the qiestion has an affirmative answer for these is evident. Therefore, only those weights not in this orbit are of real interest.

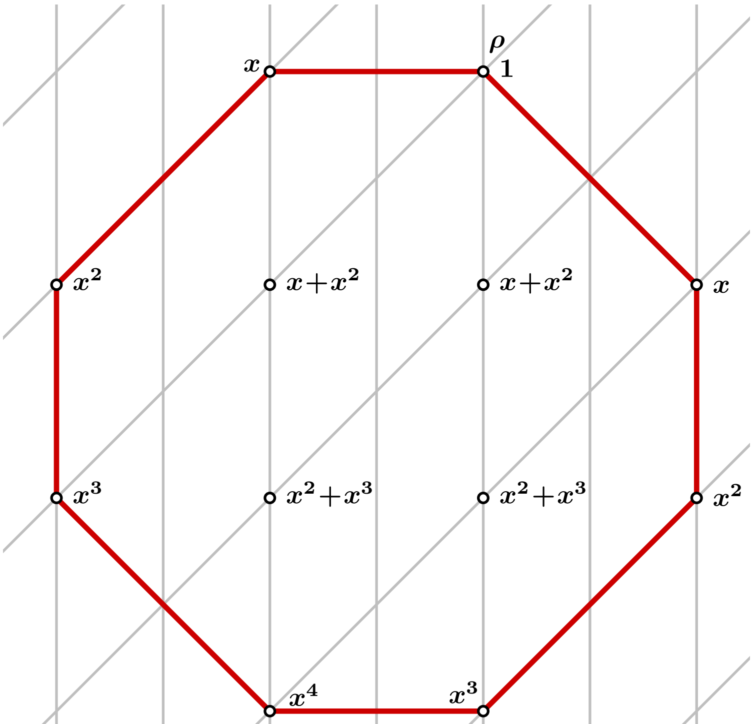


images/a2rho-0.eps Root system A_2

SYSTEM A2. Here $\rho = c = a + b$. Then

$$\begin{aligned} 0 &= \rho - c \\ &= \rho - a - b \end{aligned}$$

so that $P_0(x) = x + x^2$. Since 0 is singular, the question is vacuous.



images/c2rho-0.eps

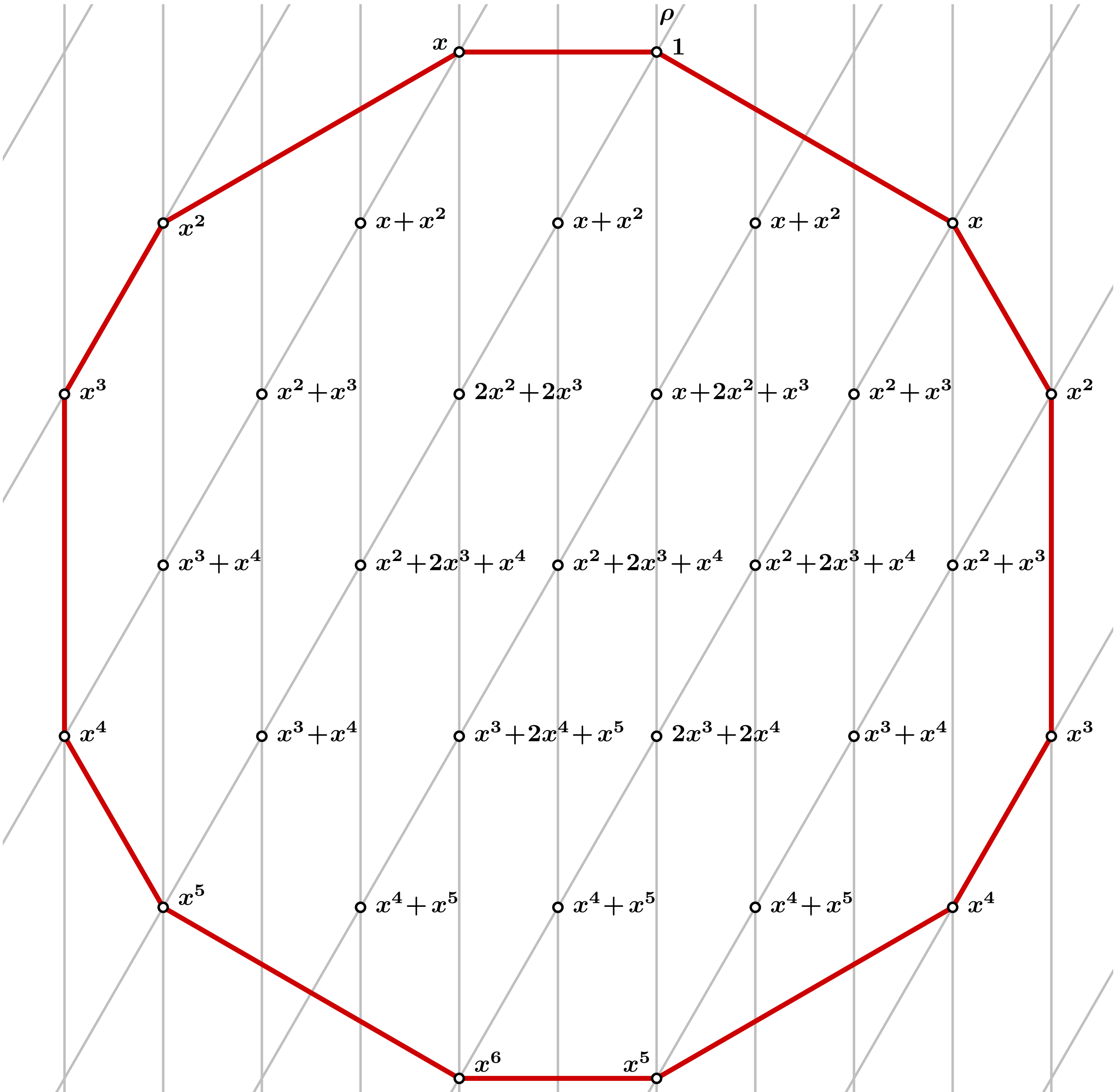
Root system C_2

SYSTEM C2. This is slightly more interesting, but the question is almost trivial here. Each simple reflection takes a weight μ into one with the same multiplicity. Therefore $P_{s_\alpha \mu}(1) = P_\mu(1)$, and of course $1-x$ divides the difference. This observation remains valid in all cases for all Θ containing a single simple root. There is still something of interest in the diagram, however. Let $c = b + a$, $d = b + 2a$. If $\mu = b + a$ then $b + a$ has root partitions

$$b + a$$

$$c$$

and hence $P_\mu(x) = x + x^2$. Consider the effect of s_α. It swaps $b + a$ and $b + 2a$. Both these points have $P(x) = x + x^2$. But this is because it swaps the sum $b + a$, c into the sums d, $c + a$ according to the recipe laid out earlier. In other words, it raises the degree of one term and lowers that of the other. In general, it is very difficult to see what the pattern governing this process is.



images/g2rho-0.eps Root system G_2

SYSTEM G2. This illustrates even better the last remark. Let

$$
\begin{aligned}
c &= b + a \\
d &= b + 2a \\
e &= b + 3a \\
f &= 2b + 3a\,.
\end{aligned}
$$

The reflection s_α swaps $b + a$ and $b + 2a$. For the first, $P = x + 2x^2 + x^3$, but for the second $2x^2 + 2x^3$. I leave it as an exercise to see how this results from the effect on root partitions.

SYSTEM C3. It's too bad I can't draw an intelligible diagram for this. In fact, what happens is just complicated enough that I had to write a computer program to see what was going on. I did this to test what I thought was a plausible conjecture, but in fact it turned out to be false.

Very generally, suppose $\Theta \subseteq \Delta$, and suppose μ to be in $[W_\Theta \backslash \mathcal{C}_\rho]$. This means that $w \mapsto w\mu$ is a bijection of W_Θ with the orbit of μ. The action of W_Θ on this orbit is covered by an action on root partitions of the weights. That is to say, if $\mu = \rho_S$ for $S \subseteq \Sigma^{++}$ then $w\rho_{w(S)}$ is a root partition of $w\mu$. So of course each W_Θ-orbit among the partitions of μ is also in bijection with W_Θ.

The simplest example of how this works is $\mu = \rho$. Here each $w\mu$ is associated to exactly one root partition, and whenever $s_\alpha w > w$ the degree of the polynomial $P_{w\mu}$ is incremented. This suggests the possibility that whenever μ is a Θ-regular weight in $\mathcal{C}_\rho$ and $\mu = \rho_S$, then the sum

$$\sum_{W_\Theta} \operatorname{sgn}(w)\,(-x)^{|w(S)|}$$

will be divisible by $\sum_{W_\Theta} x^{\ell(w)}$. If this were true, it would certainly imply an affirmative answer to the original question.

Well, this seems to happen often, but it does not happen always. There is a counter example for the root system C_2, with an orbit of partitions having the relevant polynomial $2x^2 - 4x^3 + 2x^4$, whereas the denominator
♡ **[quotient-poly]** is $1 + 2x + 2x^2 + 2x^3 + x^4$. The conclusion of Lemma 5.6 remains valid, however, because there is another orbit with polynomial $-2x^2 + 4x^3 - 2x^4$ with opposite parity, and in summing these partitions cancel. I do not know how to understand these phenomena, or even whether they are important.

DETAILS. Let γ_0, γ_1, γ_2 be the simple roots of the system C_3, with γ_2 the long root. There are nine positive roots:

$$\begin{aligned}
\gamma_0 &= (\ \ 2, -1, \ \ 0) \\
\gamma_1 &= (-1, \ \ 2, -1) \\
\gamma_2 &= (\ \ 0, -2, \ \ 2) \\
\gamma_3 &= (\ \ 1, \ \ 1, -1) \\
\gamma_4 &= (-1, \ \ 0, \ \ 1) \\
\gamma_5 &= (\ \ 1, -1, \ \ 1) \\
\gamma_6 &= (-2, \ \ 2, \ \ 0) \\
\gamma_7 &= (\ \ 0, \ \ 1, \ \ 0) \\
\gamma_8 &= (\ \ 2, \ \ 0, \ \ 0)
\end{aligned}$$

The coordinates are the $\langle \gamma, \alpha^\vee \rangle$ for α in Δ. Equivalently, γ is the corresponding linear combination of fundamental weights. The way in which the Weyl group acts on the roots is encoded in the following table of simple reflections. Only the indices of positive roots are recorded. A $-$ sign indicates the result is negative, which happens only when $s_i\gamma_i = -\gamma_i$.

γ:	0	1	2	3	4	5	6	7	8
$s_0\gamma$:	$-$	3	2	1	5	4	8	7	6
$s_1\gamma$:	3	$-$	6	0	4	7	2	5	8
$s_2\gamma$:	0	4	$-$	5	1	3	6	7	8

I am interested in the orbits of $W_{1,2}$ on the weights in $\mathcal{C}_\rho$ and on the associated root partitions. In particular I shall look at the orbit

$$\begin{array}{ccccccc}
(-3,\ \ 1,\ \ 1) & \xrightarrow{s_1} & (-2,-1,\ \ 2) & \xrightarrow{s_2} & (-2,\ \ 3,-2) & \xrightarrow{s_1} & (\ \ 1,-3,\ \ 1) \\
\uparrow s_2 & & & & & & \downarrow s_2 \\
(-3,\ \ 3,-1) & \xleftarrow{s_1} & (\ \ 0,-3,\ \ 2) & \xleftarrow{s_2} & (\ \ 0,\ \ 1,-2) & \xleftarrow{s_1} & (\ \ 1,-1,-1)
\end{array}$$

The first element $(-3, \quad 1, \quad 1)$ in this orbit possesses the root partitions

$$[1,2,3,4,5,6],\ [0,1,2,4,6,7],\ [2,3,4,6,7],\ [1,2,5,6,7],\ [4,5,6,7],\ [1,2,4,6,8]\,.$$

The notation is used to condense data, and ought to be self-evident. For example, the first array indicates that this weight possesses the root partition

$$\gamma_1+\gamma_2+\gamma_3+\gamma_4+\gamma_5+\gamma_6\,.$$

Over the orbit of weights lies the orbits of partitions. For example:

$$\begin{array}{ccccccc}
[1,2,3,4,5,6] & \xrightarrow{s_1} & [0,2,4,6,7] & \xrightarrow{s_2} & [0,1,6,7] & \xrightarrow{s_1} & [2,3,5] \\
\uparrow s_2 & & & & & & \downarrow s_2 \\
[1,3,4,5,6] & \xleftarrow{s_1} & [0,2,4,7] & \xleftarrow{s_2} & [0,1,7] & \xleftarrow{s_1} & [3,5]
\end{array}$$

This is what I think of as a normal orbit of partitions, with Poincaré polynomial $x^2(1+2x+2x^2+2x^3+x^4)$. But here is a more curious one:

$$\begin{array}{ccccccc}
[1,2,5,6,7] & \xrightarrow{s_1} & [2,5,6,7] & \xrightarrow{s_2} & [3,6,7] & \xrightarrow{s_1} & [0,1,2,5] \\
\uparrow s_2 & & & & & & \downarrow s_2 \\
[3,4,6,7] & \xleftarrow{s_1} & [0,1,2,4,5] & \xleftarrow{s_2} & [0,1,3,4] & \xleftarrow{s_1} & [0,3,4]
\end{array}$$

Its Poincaré polynomial is $2x^3(1+2x+x^2)$. *A priori* this might be expected to cause some trouble, but the orbit of $[4,5,6,7]$, with the same Poincaré polynomial and different parity, cancels it out.

I began this investigation because I had in mind computing some examples of the Satake transform for groups of low rank. I had hoped that I would find some pattern in the evaluation of the coefficients of the $\Pi^{\lambda+\mu}$ that would enable me to get around scanning through the huge collection of subsets of positive roots. But this no longer seems very plausible.

8. GSp(4)

To start with, let $G = \mathrm{GSp}_{2n}$, the group of symplectic similitudes of a non-degenerate alternating form in $2n$ dimensions. As is well known and easy to verify, all choices of a specific form give isomorphic groups G. Here, I take

$$\omega = \omega_n = \begin{bmatrix} 0 & \ldots & 1 \\ & \ldots & \\ 1 & \ldots & 0 \end{bmatrix}, \quad J = J_{2n} = \begin{bmatrix} 0 & -\omega_n \\ \omega_n & 0 \end{bmatrix},$$

and

$$G = \mathrm{GSp}(2n) = \{X \mid {}^tXJX = c\,J\}$$

for some non-zero scalar $c = \mu(X)$. We can rewrite this equation as

[gsp-defn] **(8.1)**
$$X^*X = {}^*XX = c\,I, \quad {}^*X = J^{-1}\,{}^tX\,J\,.$$

This demonstrates that it is a Zariski-closed algebraic subgroup of GL_{2n}, since the condition to be a scalar amounts to a set of polynomial identities.

Remark. *A more common choice of alternating matrix is*

$$\begin{bmatrix} 0 & -I_n \\ I_n & 0 \end{bmatrix}.$$

This would be convenient in many circumstances, but the choice I have made has one great virtue—the upper triangular matrices in G form a Borel subgroup.

THE ROOT DATUM. The diagonal matrices form in G a maximal torus, made up of the matrices

$$\begin{bmatrix} a_1 & 0 & \ldots & 0 & 0 \\ 0 & a_2 & \ldots & 0 & 0 \\ & & \ldots & & \\ 0 & 0 & \ldots & a_{2n-1} & 0 \\ 0 & 0 & \ldots & 0 & a_{2n} \end{bmatrix}$$

♡ [gsp-defn] with $a_1a_{2n} = a_2a_{2n-1} = \ldots$. This constant product is in fact the constant c of the defining equation (8.1). I'll call this the character μ.

The group G also contains the matrices

$$\begin{bmatrix} X & 0 \\ 0 & \omega\,{}^tX^{-1}\omega^{-1} \end{bmatrix}, \quad \begin{bmatrix} I & X \\ 0 & I \end{bmatrix},$$

with X an arbitrary invertible matrix in the first, and $\omega\,{}^tX\omega^{-1} = X$ (symmetric with respect to the NW–SE axis) in the second.

The group $\mathrm{GSp}(2n)$ embeds into GL_{2n}, and inherits the characters ε_i from it. If $\nu_i = \varepsilon_i - \varepsilon_{2n-i}$, the lattice $X^*(T)$ is the quotient of the lattice spanned by the $\nu_i - \nu_j$. Certain linear combinations of the duals $\varepsilon_i^\vee$ have image in $\mathrm{GSp}(2n)$. Explicitly, define the coroots

$$\alpha_i^\vee = \begin{cases} (\varepsilon_i^\vee - \varepsilon_{i+1}^\vee) - (\varepsilon_{2n-i+1}^\vee - \varepsilon_{2n-i}^\vee) & 1 \le i < n \\ \varepsilon_i^\vee - \varepsilon_{i+1}^\vee & i = n. \end{cases}$$

and define

$$\gamma^\vee = \varepsilon_1^\vee + \cdots + \varepsilon_n^\vee\,.$$

These make up a basis of the lattice of cocharacters of T. The cocharacter

$$\mu^\vee(x) = x\, I_{2n}$$

is an isomorphism of $\mathbb{G}_m$ with the center of G.

Thus if $n = 2$, these are (in slightly different notation)

$$\begin{aligned} \alpha^\vee &= \widehat{\varepsilon}_1 - \widehat{\varepsilon}_2 + \widehat{\varepsilon}_3 - \widehat{\varepsilon}_4 \\ \beta^\vee &= \widehat{\varepsilon}_2 - \widehat{\varepsilon}_3 \\ \gamma^\vee &= \widehat{\varepsilon}_1 + \widehat{\varepsilon}_2 \\ \mu^\vee &= \widehat{\varepsilon}_1 + \widehat{\varepsilon}_2 + \widehat{\varepsilon}_3 + \widehat{\varepsilon}_4 \end{aligned}$$

and in multiplicative notation:

$$\alpha^\vee\colon\ x \longmapsto \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & 1/x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 1/x \end{bmatrix}$$

$$\beta^\vee\colon\ x \longmapsto \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1/x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\gamma^\vee\colon\ x \longmapsto \begin{bmatrix} x & 0 & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The simple roots are

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (1 \le i \le n)\,.$$

or for $n = 2$:

$$\begin{aligned} \alpha &= \varepsilon_1 - \varepsilon_2 \\ \beta &= \varepsilon_2 - \varepsilon_3\,. \end{aligned}$$

The weights have as basis the simple roots together with the dominant weight

$$\gamma = \varepsilon_1\,.$$

This is the highest weight of the defining representation of G, its embedding into GL_{2n}. In the case $n = 2$, which we'll examine in detail, the other weights of this representation are

$$\begin{aligned} \varepsilon_2 &= \gamma - \alpha \\ \varepsilon_3 &= \gamma - \alpha - \beta \\ \varepsilon_4 &= \gamma - 2\alpha - \beta\,. \end{aligned}$$

This highest weight is **minuscule**-i.e. all weights are in a single Weyl orbit.

Exercise. *Find a characterization of the nilpotent upper triangular matrices in G.*

Exercise. *Find a basis of root vectors for the Lie algebra $\mathfrak{n}$ these nilpotent matrices span.*

Exercise. *Find all positive roots.*

Exercise. *Draw a picture of the root system and this minuscule weight.*

THE LANGLANDS DUAL. If the root datum associated to G is $\mathcal{L} = (L, \Delta, L^\vee, \Delta^\vee)$, that of the Langlands dual is $\mathcal{L}^\vee = (L^\vee, \Delta^\vee, L, \Delta)$.

[langlands-dual] **8.2. Proposition.** *The groups* $\mathrm{GSp}(2n)$ *and* $\mathrm{GSpin}(2n+1)$ *are Langlands' duals.*

It is immediate, once one sees Proposition 2.4 of [Asgari:2002]. To go with it, the Langlands dual of $\mathrm{GSpin}(2n)$ is $\mathrm{GO}(2n)$.

In particular, because of a low-dimensional accident:

[dual-gsp4] **8.3. Proposition.** *The group dual to* GSp_4 *is also* GSp_4.

Proof. This is not quite immediate. We need a suitable isomorphism τ of $X^*(T)$ with $X_*(T)$. It must take Δ to $\Delta^\vee$, as must also its transpose ${}^tT$. The transpose ${}^tT$ is defined by the equation

$$\langle {}^t\tau(u), v\rangle = \langle u, \tau(v)\rangle \,.$$

But the pairing matrix is

$$\begin{bmatrix} \langle\alpha,\alpha^\vee\rangle & \langle\alpha,\beta^\vee\rangle & \langle\alpha,\gamma^\vee\rangle \\ \langle\beta,\alpha^\vee\rangle & \langle\beta,\beta^\vee\rangle & \langle\beta,\gamma^\vee\rangle \\ \langle\gamma,\alpha^\vee\rangle & \langle\gamma,\beta^\vee\rangle & \langle\gamma,\gamma^\vee\rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix},$$

so the map

$$\begin{aligned} \tau\colon \alpha &\longmapsto \beta^\vee \\ \beta &\longmapsto \alpha^\vee \\ \gamma &\longmapsto \gamma^\vee \,. \end{aligned}$$

is the one we are looking for. ▢

The isomorphism is not unique, since there exists an involution of GSp_{2n} acting as I in Sp_{2n} but taking μ to $-\mu$:

$$X \longmapsto {}^*X^{-1} \,.$$

Therefore there are two possible identifications of the dual of GSp_4 with GSp_4, both taking B to B, T to T. (This caused me some confusion in verifying the Proposition.)

Exercise. *What is the image of* μ*?*

Let MSp_4 be the set of all 4×4 matrices X such that ${}^*XX$ is scalar. It contains GSp_4 as an open subvariety, and the left and right multiplications by G extend to it—it is a **G-monoid**. It is a cone in $\mathfrak{k}^3$, hence in particular a singular algebraic variety.