PERVERSE MOTIVES AND GRADED DERIVED CATEGORY $\mathcal O$

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ABSTRACT. For a variety with a Whitney stratification by affine spaces, we study categories of motivic sheaves which are constant mixed Tate along the strata. We are particularly interested in those cases where the category of mixed Tate motives over a point is equivalent to the category of finite-dimensional bigraded vector spaces. Examples of such situations include rational motives on varieties over finite fields and modules over the spectrum representing the semisimplification of de Rham cohomology for varieties over the complex numbers. We show that our categories of stratified mixed Tate motives have a natural weight structure. Under an additional assumption of pointwise purity for objects of the heart, tilting gives an equivalence between stratified mixed Tate sheaves and the bounded homotopy category of the heart of the weight structure. Specializing to the case of flag varieties, we find natural geometric interpretations of graded category $\mathcal O$ and Koszul duality.

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1. Introduction

In [BG86] Beilinson and Ginzburg laid out a vision how "mixed geometry" should allow to construct graded versions of the BGG-category \mathcal{O} , and why these graded versions should be governed by Koszul rings. Motivated by inversion formulas for Kazhdan–Lusztig polynomials, they also conjectured these Koszul rings to be their own Koszul duals. This was pushed through in [BGS96]; however, the beauty of the original ideas got kind of obscured by difficulties stemming from the fact, that in all

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realizations of mixed geometry available back then, there would be some (unwanted) non-trivial extensions between Tate motives. In the current paper, we want to show how recent advances in constructing triangulated categories of motives and motivic six functor formalisms allow to clear away this difficulty and realize the original vision in its full beauty.

We will define, for a given motivic triangulated category \mathscr{T} and a "Whitney—Tate" stratified variety (X,\mathcal{S}) over an arbitrary field, a triangulated \mathbb{Q} -linear category MTDer $_{\mathcal{S}}(X,\mathscr{T})$ of \mathscr{T} -motives which are constant mixed Tate along the strata, called **stratified mixed Tate motives**. Of particular interest for our applications are those motivic triangulated categories \mathscr{T} where the category of mixed Tate \mathscr{T} -motives over the base field is equivalent to the category of bigraded \mathbb{Q} -vector spaces of finite dimension, viewed as the derived category of the abelian category of finite-dimensional \mathbb{Z} -graded \mathbb{Q} -vector spaces. This happens for motives with rational coefficients over finite fields and for the category of motives associated to the enriched Weil cohomology theory given by semisimplified Hodge realization [Dre13]. These two cases are related to the ℓ -adic and Hodge module approximations to mixed geometry previously available.

For representation-theoretic purposes, the case of X = G/P with the stratification $\mathcal{S} = (B)$ by Borel orbits is particularly interesting. We show in this case, that the category $\mathrm{MTDer}_{(B)}(G/P)$ of stratified mixed Tate motives on the flag variety carries two interesting additional structures: a weight structure whose heart is related to categories of Soergel modules, and a perverse t-structure whose heart is a graded version of category \mathcal{O} . In particular, we construct an equivalence of triangulated \mathbb{Q} -categories

$$\mathrm{MTDer}_{(B)}(G/B) \cong \mathrm{Hot^b}(C\operatorname{-SMod}_{\mathrm{ev}}^{\mathbb{Z}}),$$

where $C = H^*(G/B, \mathbb{Q})$ denotes the cohomology ring with rational coefficients of the complex flag manifold sometimes called the coinvariant algebra, $C\operatorname{-SMod}^{\mathbb{Z}}\subset C\operatorname{-Mod}^{\mathbb{Z}}$ denotes a full subcategory of the category of all graded finite-dimensional $C\operatorname{-modules}$ sometimes called the category of Soergel modules, and $C\operatorname{-SMod}^{\mathbb{Z}}_{\operatorname{ev}}$ denotes the full subcategory of Soergel modules concentrated in even degrees only. Put another way, the category $\operatorname{MTDer}_{(B)}(G/B)$ is, up to adding a root of the Tate twist, equivalent to the bounded derived category of the graded version of the principal block \mathcal{O}_0 of category \mathcal{O} constructed in [BGS96]. The idea of such a geometrical or even motivic construction was already clearly present in the seminal preprint [BG86] of Beilinson and Ginzburg.

Let us discuss the relation of our work to what has been done already. The first geometric realizations of the (not yet derived) graded category \mathcal{O} were constructed in [BGS96]. A geometric realization of the graded derived category \mathcal{O} has been constructed by Achar and Riche [AR11]. Another approach by "winnowing" categories of mixed Hodge modules in the sense of Saito is worked out by Achar and Kitchen in [AK11]. Modular coefficient realizations are discussed in [AR14a, AR14b].

These approaches, using Hodge modules, étale or ℓ -adic sheaves, are technically demanding due to problems with non-semisimplicity of the corresponding categories of sheaves for the one-point flag variety. This is the main motivation for us to suggest yet another realization of the graded derived category \mathcal{O} . While the theory of motives is also built on technically demanding foundations, our point in the present paper is that at least such problems as the non-semisimplicity of Frobenius actions disappear, and the geometric construction of the graded derived category \mathcal{O} is clarified and simplified considerably by using true motives. We hope that our explanations contribute to a better understanding of the original vision laid out in the work [BG86] of Beilinson and Ginzburg. We also expect that the use of mixed

motivic categories will turn out to be fruitful in a lot of other instances where geometric representation theory relies on "mixed geometry". For example, in a joint work in progress with Rahbar Virk, we will discuss how Borel-equivariant motives can be used to establish motivic versions of the results from [Vir13] and construct a very natural geometric categorification of the Hecke algebra.

1.1. Motivic triangulated categories and stratified mixed Tate motives. Let us now outline in more detail the constructions and results to be presented in this work. The most important technical tools used in the paper are the recent works on categories of motives over an arbitrary base and their six-functor formalism: [Ayo07a, Ayo07b], [CD12b] and [Dre13]. With these motivic categories and their six-functor formalism available, many of the standard arguments that have been developed in geometric representation theory can be adapted to the setting of motives, be it étale motives, Beilinson motives or motives with coefficients in enriched mixed Weil cohomology theories. For the applications we have in mind we restrict to categories of mixed Tate motives, which are much better understood, due to the work of Levine [Lev93, Lev10], Wildeshaus [Wil09] and others. We include two sections discussing these technical foundations: Section 2 is a very abridged recollection of basics on triangulated categories of motives, and Section 3 recalls relevant facts about mixed Tate motives.

With these tools in hands, we construct in Section 4, for a motivic triangulated category $\mathscr T$ and a stratified variety $(X,\mathcal S)$, an analogue of the category of sheaves which are constant along strata. This category, denoted by $\operatorname{MTDer}_{\mathcal S}(X,\mathscr T)$, is called the category of stratified mixed Tate $\mathscr T$ -motives, and consists of those motives which are constant mixed Tate along the strata. For this category to be well-behaved, one needs as in [BGS96] or [Wil12] a condition "Whitney–Tate", which ensures that the extension and restriction functors preserve mixed Tate motives. This condition is satisfied in a large number of cases, including, in particular, partial flag varieties stratified by Borel orbits.

While the construction of $\mathrm{MTDer}_{\mathcal{S}}(X)$ works in great generality, we usually work in a more restrictive setting where the underlying motivic triangulated category \mathscr{T} satisfies additional conditions, cf. Convention 4.1:

- (1) one condition is called <u>weight condition</u> and requires the existence of suitably compatible weight structures on the motivic categories,
- (2) the other condition is called grading condition and requires the category $\text{MTDer}(k, \mathcal{T})$ of mixed Tate \mathcal{T} -motives over the base field k to be equivalent to the derived category of the category of \mathbb{Z} -graded vector spaces.

The grading condition implies that the category MTDer_S(X) can be described very explicitly in terms of the combinatorics of the stratification S. These conditions are satisfied in two important cases: rational étale or Beilinson motives over a finite field \mathbb{F}_q , and motives with coefficients in the semisimplified Hodge realization over \mathbb{C} .

1.2. Weight structures. The first block of results in our paper concerns a weight structure on the category of stratified mixed Tate motives. Weight arguments have been used a lot in geometric representation theory, in particular in the framework of mixed geometry. The very recently introduced weight structures alias cot-structures, due independently to Bondarko [Bon10] and Pauksztello, are a convenient framework for formalising such weight arguments. The following result establishes the existence of a weight structure on stratified mixed Tate motives, cf. Proposition 5.1, and Theorem 9.2. It follows rather easily from the existence of weight structures on Beilinson motives, as constructed by Hébert [Héb11] and Bondarko [Bon13].

Theorem 1. Let k be a field, and let \mathscr{T} be a motivic triangulated category over k satisfying the weight condition, cf. Convention 4.1. Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified k-variety in the sense of Definition 4.2 and Definition 4.5.

- (1) The category $MTDer_{\mathcal{S}}(X)$ carries a weight structure in the sense of Bondarko. This weight structure is uniquely determined by the requirement that for the inclusion $j_s: X_s \to X$ of a stratum, the functors j_s^* and $j_s^!$ preserve non-positivity and non-negativity of weights, respectively.
- (2) Assume that \mathcal{T} also satisfies the grading condition of Convention 4.1, that all objects of the heart $\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}$ are pointwise pure in the sense of Definition 6.1, and that $\mathrm{MTDer}_{\mathcal{S}}(X)$ can be embedded as full subcategory of a localization of the derived category of some abelian category. Then the tilting functor of Proposition B.1 induces an equivalence

$$\operatorname{Hot^b}(\operatorname{MTDer}_{\mathcal{S}}(X)_{w=0}) \xrightarrow{\approx} \operatorname{MTDer}_{\mathcal{S}}(X)$$

between the category of stratified mixed Tate motives on X and the bounded homotopy category of the heart of its weight structure.

Remark 1.1. The slightly awkward condition in (2) is one possibility to ensure the applicability of the tilting result in Proposition B.1. This condition is satisfied in the cases which for us are the most interesting: rational motives (étale or Beilinson) over a finite field, and motives with coefficients in the semisimplification of the Hodge realization over $\mathbb C$. The explicit tilting result Proposition B.1 is the only place in the whole paper where we need information beyond the axiomatics of motivic triangulated categories — for the tilting result to hold we need some more information on how the motivic triangulated category at hand is constructed. All other results in the paper only use the axiomatics of motivic triangulated categories.

We also adapt pointwise purity arguments of Springer [Spr84] and a full faithfulness result of Ginzburg [Gin91] to describe the heart of the above weight structure. The following result explicitly describes the category of stratified mixed Tate motives in a case of representation-theoretic interest in terms of Soergel modules, cf. Corollary 6.7, Lemma 6.6, Theorem 8.6 and Corollary 9.4.

Theorem 2. Assume that k is a field and \mathscr{T} is a motivic triangulated category over k satisfying the weight condition. Let $G \supset P \supset B \supset T$ be a split reductive group over k with a parabolic, a Borel and a split maximal torus all defined over k. Let X = G/P be the corresponding partial flag variety with S = (B) its stratification by B-orbits. Then we have:

- (1) The heart $\mathrm{MTDer}_{(B)}(G/P)_{w=0}$ of the weight structure from Theorem 1 is generated, as idempotent complete additive subcategory of $\mathcal{T}(G/P)$, by motives of Bott-Samelson resolutions of Schubert varieties in G/P. These are pointwise pure.
- (2) Let $k = \mathbb{F}_q$ be a finite field and let \mathscr{T} be the motivic triangulated category of (étale or Beilinson) motives with rational coefficients. Then a suitable hypercohomology functor induces an equivalence of categories

$$\mathrm{MTDer}_{(B)}(G/P)_{w=0} \stackrel{\approx}{\to} \mathrm{H}^*(G/P)\operatorname{-SMod}_{\mathrm{ev}}^{\mathbb{Z}}$$

between the heart of the weight structure and the category of even Soergel modules over the cohomology ring of G/P. This equivalence extends to an equivalence

$$\mathrm{MTDer}_{(B)}(G/P) \stackrel{\approx}{\to} \mathrm{Hot^b}(\mathrm{H}^*(G/P)\operatorname{-SMod}_{\mathrm{ev}}^{\mathbb{Z}})$$

between the category of stratified mixed Tate motives and the bounded homotopy category of complexes of even Soergel modules.

(3) Take $k = \mathbb{C}$ and \mathscr{T} the category of modules over the semisimplified Hodge cohomology. Then Hodge (hyper-)cohomology induces an equivalence of categories

$$\mathrm{MTDer}_{(B)}(G/P)_{w=0} \stackrel{\approx}{\to} \mathrm{H}^*(G/P)\operatorname{-SMod}_{\mathrm{ev}}^{\mathbb{Z}}$$

between the heart of the weight structure in (1) and the category of even Soergel modules over the cohomology ring of G/P. This equivalence extends to an equivalence

$$\mathrm{MTDer}_{(B)}(G/P) \stackrel{\approx}{\to} \mathrm{Hot^b}(\mathrm{H}^*(G/P)\operatorname{-SMod}_{\mathrm{ev}}^{\mathbb{Z}}).$$

1.3. Perverse t-structures. The second block of results to be proved in this paper concerns a perverse t-structure on the category of stratified mixed Tate motives. While the existence of motivic t-structures is a very difficult problem, there are some situations where the Beilinson-Soulé vanishing conjectures and hence the existence of a motivic t-structure on mixed Tate motives are known. Alternatively, over C, it is possible to work in a category of motives with coefficients in the semisimplification of the Hodge realization; in this case, the category of mixed Tate motives over the point is the derived category of graded vector spaces and therefore has a natural "motivic" t-structure. In situations as above, we can use the perverse formalism of [BBD82] to equip the category of stratified mixed Tate motives with a perverse t-structure, for any perversity function $p: \mathcal{S} \to \mathbb{Z}$. Its heart is an abelian category $MTPer_{\mathcal{S}}(X)$ of perverse mixed Tate motives. The following results are combinations of the results of Section 10 and Section 11, more precisely Theorem 10.3, Theorem 11.10 and Theorem 11.9; everything is specialized to the two cases of interest (related to ℓ -adic resp. Hodge realizations). We suppress the underlying motivic triangulated category in our notation $MTPer_{\mathcal{S}}(X)$ for perverse motives, to underline that our methods give a uniform proof for both theorems below:

Theorem 3. Let k be a finite field and \mathscr{T} be the motivic triangulated category of (Beilinson or étale) motives with rational coefficients, and let (X, \mathcal{S}) be an affinely Whitney-Tate stratified k-variety in the sense of Definition 4.2 and Definition 4.5.

(1) The category $MTPer_{\mathcal{S}}(X)$ has enough projectives and the tilting functor of Proposition B.1 induces an equivalence of categories

$$\mathrm{Der}^{\mathrm{b}}(\mathrm{MTPer}_{\mathcal{S}}(X)) \stackrel{\approx}{\to} \mathrm{MTDer}_{\mathcal{S}}(X)$$

between the bounded derived category of the abelian category of perverse mixed Tate motives and the triangulated category of stratified mixed Tate motives.

(2) If we consider motives with \mathbb{Q}_{ℓ} -coefficients where ℓ is a prime different from the characteristic of k, the ℓ -adic realization

$$\mathrm{MTPer}_{\mathcal{S}}(X;\mathbb{Q}_{\ell}) \to \mathrm{Perv}_{\mathcal{S}}(X \times_{k} \bar{k};\mathbb{Q}_{\ell})$$

is a degrading functor in the sense of [BGS96].

Theorem 4. Let (X, S) be an affinely Whitney-Tate stratified variety over \mathbb{C} and \mathcal{T} the motivic triangulated category of modules over the semisimplified Hodge realization.

(1) The category $MTPer_{\mathcal{S}}(X)$ has enough projectives and the tilting functor of Proposition B.1 induces an equivalence of categories

$$\operatorname{Der^b}(\operatorname{MTPer}_{\mathcal{S}}(X)) \stackrel{\approx}{\to} \operatorname{MTDer}_{\mathcal{S}}(X)$$

between the bounded derived category of the abelian category of perverse mixed Tate motives and the triangulated category of stratified mixed Tate motives.

(2) Combining in the case of the full flag variety G/B the Hodge realization with the algebraic Riemann–Hilbert correspondence and Beilinson–Bernstein localization is a degrading functor

$$\mathrm{MTPer}_{(B)}(G/B) \to \mathcal{O}_0.$$

Remark 1.2. The algebraic Riemann–Hilbert correspondence of [Dre13] allows to explicitly relate the category MTDer_S(X) in the Hodge situation to a suitable derived category of holonomic \mathscr{D} -modules on X. In particular, while in Theorem 3 the link from the graded version to actual representations was rather weak, the Hodge situation with its Riemann–Hilbert correspondence actually provides a direct relation between the above motivic graded categories MTPer_(B)(G/B) with category \mathcal{O}_0 , more precisely the category of finitely generated \mathfrak{g} -modules locally finite under a Borel subalgebra and annihilated by the central annihilator of the trivial one-dimensional representation, the latter realized via Beilinson–Bernstein localization as a category of \mathscr{D} -modules on the flag variety.

- Remark 1.3. This result recovers part of the results of [AK11]. Eventually, our construction of a graded version of category \mathcal{O} also boils down to split some unwanted extensions. However, using the motivic framework developed thus far allows to shift all the technical difficulties into the notions of motivic triangulated categories and representing sheaves of spectra for enriched Weil cohomology theories. We hope this makes the actual construction of graded versions of category \mathcal{O} more transparent.
- 1.4. Koszul duality remarks. In Corollary 11.11 we investigate the interaction of weights and perversity in somewhat more detail. Suppose we are in the situation of Theorem 1. Then for an affinely Whitney–Tate stratified variety (X, \mathcal{S}) satisfying the pointwise purity conditions the above results put together provide equivalences of categories

$$\operatorname{Der}^{\mathrm{b}}(\operatorname{MTPer}_{\mathcal{S}}(X)) \xrightarrow{\approx} \operatorname{MTDer}_{\mathcal{S}}(X) \xleftarrow{\approx} \operatorname{Hot}^{\mathrm{b}}(\operatorname{MTDer}_{\mathcal{S}}(X)_{w=0}).$$

In this case, we sketch in 11.12, why the corresponding category of perverse sheaves is governed by a Koszul ring. A special case, in which all the above conditions are satisfied, is the case of a partial flag variety X = G/P over $\mathbb C$ with $\mathcal S = (B)$ the stratification by Borel orbits. In this case, the same arguments as in [BGS96] exhibit MTPer_(B)(G/P) as a graded version of the principal block of parabolic category $\mathcal O$, up to formally adding a root of the Tate twist. Summing up, given a partial flag variety X = G/P over $\mathbb C$ the above results provide equivalences of categories

$$\mathrm{Der}^{\mathrm{b}}(\mathrm{MTPer}_{(B)}(G/P)) \xrightarrow{\approx} \mathrm{MTDer}_{(B)}(G/P) \xleftarrow{\approx} \mathrm{Hot}^{\mathrm{b}}(\mathrm{MTDer}_{(B)}(G/P)_{w=0}).$$

The right hand side in turn is equivalent to $\operatorname{Hot}^b(H^*(G/P)\operatorname{-SModf}_{\operatorname{ev}}^{\mathbb{Z}})$ and can be identified as in [Soe90], up to formally adding a root of the Tate twist, with the bounded derived category of some graded version of some block of some category \mathcal{O} for the Langlands dual Lie algebra, which is more or less singular depending on our parabolic. On the other hand, $\operatorname{MTPer}_{(B)}(G/P)$ can be identified as in Theorem 4, up to formally adding a root of the Tate twist, with some graded version of a block of parabolic category \mathcal{O} . Putting all this together, our above results allow to reconstruct the parabolic-singular duality of [BGS96] in a slightly more concrete way. In particular, the Koszul self-duality for the principal block of category \mathcal{O} can be interpreted as a completely canonical equivalence of triangulated \mathbb{Q} -categories

$$K: \mathrm{MTDer}_{(B)}(G/B) \ \stackrel{\approx}{\longrightarrow} \ \mathrm{MTDer}_{(B^{\vee})}(G^{\vee}/B^{\vee})$$

with the property K(M(n)) = (KM)(-n)[-2n] transforming indecomposable injective perverse objects to simple perverse objects, simple perverse objects to projective perverse objects, and perverse ∇ -sheaves to perverse Δ -sheaves, by the way

turning their Weyl group parameters upside down. In [BY13] such an equivalence is established with similar arguments in the setting of mixed ℓ -adic sheaves. It would be very interesting to have a geometric construction of such a functor.

- 1.5. Structure of the paper: We begin with a short recollection on triangulated categories of motives in Section 2, and a recollection on mixed Tate motives in Section 3. The Whitney—Tate condition and the description of the category of stratified mixed Tate motives is recalled in Section 4, some more detailed discussion of the Whitney—Tate condition is deferred to Appendix A. In Section 5, we explain the weight structure on stratified mixed Tate motives. Pointwise purity and its relevance for the study of the heart is discussed in Sections 6 and 7. In Section 8, we reformulate Ginzburg's full faithfulness result in the motivic setting. The latter result is used in Section 9 to prove a tilting result identifying stratified mixed Tate motives with the homotopy category of Soergel modules. Some background on tilting can be found in Appendix B. In Section 10, we discuss the perverse t-structure on stratified mixed Tate motives, and in Section 11 we show how ℓ -adic and Hodge realization functors of perverse mixed Tate motives provide a grading on category \mathcal{O} .
- 1.6. Conventions: In a category \mathcal{C} , we denote by $\mathcal{C}(A,B)$ the set of morphisms from A to B. The symbol Hom is reserved for "inner hom". Homotopy categories are typically denoted by Hot, derived categories by Der. For an object X of a category with a final object *, we denote by $\operatorname{fin} = \operatorname{fin}_X : X \to *$ the unique morphism. For an S-scheme X, we denote in particular by $\operatorname{fin} : X \to S$ the structure morphism. Most of the time, we work with the category Sch/k of schemes which are separated and of finite type over a base field k. We occasionally might refer to those objects as varieties.
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2. Triangulated categories of motives (d'après Ayoub, Cisinski-Déglise,...)

In this section, we provide a recollection of the construction and properties of triangulated categories of motives and the corresponding six-functor formalism. The general idea of motives and the six functors as a formalization of cohomological properties of algebraic varieties goes back to the development of étale cohomology by Grothendieck and his collaborators in the SGA volumes. While the construction of an abelian category of motives depends on difficult open conjectures, there are now reasonably good triangulated categories of motives available. This is based on work of Voevodsky [FSV00] who defined triangulated categories of motives over a field. One possible approach for establishing the existence and properties of the six functors in motivic settings was proposed by Voevodsky and worked out in detail in the thesis of Ayoub [Ayo07a, Ayo07b]. Building on this, constructions of triangulated categories of motives over rather general base schemes together with constructions of the relating six functors were also given in [CD12b].

We will start the recollection with a discussion of the notion of a motivic triangulated category, a framework for a motivic six-functor formalism from [CD12b]. Then we will recall two examples of motivic triangulated categories, namely étale

motives [Ayo14] and Beilinson motives [CD12b]. After that, we discuss a third example of motivic triangulated categories, namely the ones associated to enriched mixed Weil cohomology theories [Dre13]. The results in our paper will be formulated for a general motivic triangulated category, but most of the representation-theoretic applications will additionally require the motivic triangulated category to satisfy the grading and weight conditions 4.1.

2.1. Motivic triangulated categories. As mentioned, there are several ways of encoding the properties of a motivic six functor formalism. One possibility is the notion of homotopical stable algebraic derivator of Ayoub [Ayo07a], and another is the notion of motivic triangulated categories of Cisinski–Déglise [CD12b]. We are going to list the relevant properties of motivic triangulated categories which we will need for the constructions in the paper. For details, the reader is referred to [CD12b].

Definition 2.1. Let \mathscr{S} be a category called the "base category" together with a class \mathscr{P} of morphisms called " \mathscr{P} -morphisms", which is stable under composition and base change and contains all isomorphisms.

(1) A 2-functor $\mathscr{M}: \mathscr{S}^{\mathrm{op}} \to \mathscr{C}at$ is called \mathscr{P} -fibred if for any morphism p in \mathscr{P} the functor p^* has a left adjoint p_{\sharp} and for any cartesian square

$$Y \xrightarrow{q} X$$

$$\downarrow f$$

$$T \xrightarrow{p} S$$

with p a \mathscr{P} -morphism the natural exchange transformation is an isomorphism $q_{\sharp}g^* \stackrel{\sim}{\to} f^*p_{\sharp}$, cf. [CD12b, Definitions 1.1.1, 1.1.2 and 1.1.10]. (2) Let $\mathscr{C}at^{\otimes}$ denote the 2-category of symmetric monoidal categories. A \mathscr{P} -

- (2) Let $\mathscr{C}at^{\otimes}$ denote the 2-category of symmetric monoidal categories. A \mathscr{P} fibred 2-functor $\mathscr{M}: \mathscr{S}^{\mathrm{op}} \to \mathscr{C}at^{\otimes}$ is called monoidal if for any \mathscr{P} morphism $f: T \to S$ and all M and N the natural exchange transformation
 is an isomorphism $Ex(f_{\sharp}, \otimes_T): f_{\sharp}(M \otimes_T f^*N) \stackrel{\sim}{\to} f_{\sharp}M \otimes_S N$, cf. [CD12b,
 Definitions 1.1.21 and 1.1.27].
- (3) Let $\mathscr{T}ri^{\otimes}$ denote the 2-category of triangulated monoidal categories. A \mathscr{P} -fibred monoidal 2-functor $\mathscr{M}:\mathscr{S}^{\mathrm{op}}\to\mathscr{T}ri^{\otimes}$ is called a \mathscr{P} -premotivic triangulated category if all pull-back functors f^* admit triangulated right adjoints f_* and all $M\otimes_S(-)$ admit right adjoints $\mathrm{Hom}_S(M,-)$, cf. [CD12b, Definition 1.4.2].

Remark 2.2. From now on, we will only consider the base category $\mathscr{S} = \mathscr{S}_k$ of separated schemes of finite type over some field k with \mathscr{P} the class of smooth morphisms of finite type. A \mathscr{P} -premotivic category will henceforth just be called a premotivic category over k or, in case the ground field is fixed anyhow, a premotivic category.

Next we recall from [CD12b, Section 2] further properties that make a premotivic category motivic.

Definition 2.3. Let k be a fixed ground field.

- (1) A premotivic triangulated category $\mathscr T$ satisfies the <u>homotopy property</u> if for any scheme $S \in \mathscr S$ the counit of the adjunction associated to the projection $p: \mathbb A^1_S \to S$ is an isomorphism $1 \stackrel{\sim}{\to} p_* p^*$.
- (2) A premotivic triangulated category $\mathscr T$ satisfies the <u>stability property</u> if for every $S\in\mathscr S$ and every smooth S-scheme $f:X\to S$ with section $s:S\to X$ in $\mathscr S$, the associated Thom transformation $f_\sharp s_*$ is an equivalence of categories.

- (3) A premotivic triangulated category \mathscr{T} satisfies the <u>localization property</u> if $\mathscr{T}(\emptyset) = 0$ and for each closed immersion $i: Z \to S$ with open complement $j: U \to S$, the pair (j^*, i^*) is conservative and the counit $i^*i_* \to 1$ is an isomorphism.
- (4) A premotivic triangulated category satisfies the <u>adjoint property</u> if for any proper morphism f in \mathscr{S} , the functor f_* admits a right adjoint $f^!$.
- (5) A motivic triangulated category over k is a premotivic triangulated category which satisfies the homotopy, stability, localization and adjoint properties, cf. [CD12b, Definition 2.4.45].

Remark 2.4. Via the procedure in [CD12b, 1.1.34] one can associate motives in $\mathscr{T}(S)$ to smooth morphisms of varieties $X \to S$. This uses the fact that $p^*: \mathscr{T}(S) \to \mathscr{T}(X)$ is required to have a left adjoint for smooth p, and one defines the motive $M_S(X) = p_\sharp(\mathbb{Q}_X) \in \mathscr{T}(S)$. Here \mathbb{Q}_X denotes the tensor unit in $\mathscr{T}(X)$ which will also be denoted by \underline{X} later on. Sometimes we use the abbreviation $\mathscr{T}(X) = \mathscr{T}_X$ and often we use the notation $\mathscr{T}_X(\mathcal{F},\mathcal{G})$ for spaces of morphisms in $\mathscr{T}(X)$. From the homotopy, stability and localization properties, one gets a computation of the motive of \mathbb{P}^1_S as $M_S(\mathbb{P}^1) = \mathbb{Q}_S \oplus \mathbb{Q}_S(1)[2]$, where $\mathbb{Q}(1)$ is \otimes -invertible. This motive is called the Tate motive, and tensoring with it is called Tate twist.

For a motivic triangulated category \mathcal{T} , the following properties hold, as proved in [CD12b]. The list below is a variant of the dix leçons in [Héb11], where we omitted those statements that are contained in the definition above or are specific to Beilinson motives.

(1) For any morphism $f:Y\to X$ in \mathscr{S} , the adjunctions lead to natural isomorphisms

$$\operatorname{Hom}_X(M, f_*N) \cong f_* \operatorname{Hom}_Y(f^*M, N),$$

cf. [CD12b, 1.1.33].

(2) For any morphism $f: Y \to X$ in \mathscr{S} , one can construct a further pair of adjoint functors, the **exceptional functors**

$$f_!: \mathscr{T}(Y) \leftrightarrows \mathscr{T}(X): f^!$$

which fit together to form a covariant (resp. contravariant) 2-functor $f \mapsto f_!$ (resp. $f \mapsto f^!$). There exists a natural transformation $\alpha_f : f_! \to f_*$ which is an isomorphism when f is proper. Moreover, α is a morphism of 2-functors.

(3) For any cartesian square

$$X' \xrightarrow{g'} X$$

$$f' \downarrow \qquad \qquad \downarrow f$$

$$Y' \xrightarrow{g} Y$$

there exist natural isomorphisms of functors

$$g^* f_! \xrightarrow{\sim} f'_! g'^*, \qquad g'_* f'^! \xrightarrow{\sim} f^! g_*,$$

cf. [CD12b, Theorem 2.2.14]

(4) For any morphism $f: Y \to X$ in \mathscr{S} , there exist natural isomorphisms

$$Ex(f_!^*, \otimes) : (f_!K) \otimes_X L \xrightarrow{\sim} f_!(K \otimes_Y f^*L),$$

$$\operatorname{Hom}_X(f_!L, K) \xrightarrow{\sim} f_* \operatorname{Hom}_Y(L, f^!K),$$

$$f^! \operatorname{Hom}_X(L, M) \xrightarrow{\sim} \operatorname{Hom}_Y(f^*L, f^!M),$$
cf. [CD12b, Theorem 2.2.14].

(5) For $f:X\to Y$ a smooth morphism of relative dimension d, there are canonical natural isomorphisms

$$\mathfrak{p}_f: f_{\sharp} \to f_!(d)[2d], \qquad \mathfrak{p}_f': f^* \to f^!(-d)[-2d],$$

cf. [CD12b, Theorem 2.4.50].

(6) An alternative formulation of the **localization property**, cf. [CD12b, Definition 2.3.2]: for $i: Z \to X$ a closed immersion with open complement $j: U \to X$, there are distinguished triangles of natural transformations

$$j_! j^! \to 1 \to i_* i^* \to j_! j^! [1]$$

 $i_* i^! \to 1 \to j_* j^* \to i_* i^! [1]$

where the first and second maps are the counits and units of the respective adjunctions, cf. [CD12b, Proposition 2.3.3, Theorem 2.2.14].

- (7) For any closed immersion $i: Z \to S$ of pure codimension n between regular schemes in \mathscr{S} , the standard map $\mathrm{M}_Z(Z) \to i^! \mathrm{M}_S(S)(n)[2n]$ is an isomorphism, cf. [CD12b, Theorem 14.4.1].
- (8) Define the subcategory of constructible objects $\mathcal{T}^c(S) \subset \mathcal{T}(S)$ to be the thick full subcategory generated by $M_S(X)(n)$ for $n \in \mathbb{Z}$ and $X \to S$ smooth. This subcategory coincides with the full subcategory of compact objects if motives of smooth varieties are compact, cf. [CD12b, Proposition 1.4.11]. Under some conditions satisfied in all cases we consider, the motives of smooth varieties are compact and the six functors preserve compact objects, cf. [CD12b, Section 4.2] and [CD12b, Theorem 15.2.1] for the statement for Beilinson motives.
- (9) For $f: X \to S = \operatorname{Spec} k$ a morphism in \mathscr{S} , the motive $f^!(\operatorname{M}_S(S))$ is a dualizing object, i.e., setting $D_X(M) = \operatorname{Hom}(M, f^!(\operatorname{M}_S(S)))$ the natural map $M \to D_X(D_X(M))$ is an isomorphism for all $M \in \mathscr{T}^c(X)$. For all $M, N \in \mathscr{T}^c(X)$, there is a canonical duality isomorphism

$$D_X(M \otimes D_X(N)) \simeq \operatorname{Hom}_X(M,N).$$

Furthermore, for any morphism $f: Y \to X$ in \mathscr{S} and any $M \in \mathscr{T}^c(X)$ and $N \in \mathscr{T}^c(Y)$, there are natural isomorphisms

$$D_Y(f^*(M)) \simeq f^!(D_X(M)), \qquad f^*(D_X(M)) \simeq D_Y(f^!(M))$$

 $D_X(f_!(N)) \simeq f_*(D_Y(M)), \qquad f_!(D_Y(N)) \simeq D_X(f_*(N)),$
cf. [CD12b, Theorem 15.2.4].

2.2. Examples: étale motives, Beilinson motives. Next, we recall two instances of motivic triangulated categories, namely étale motives [Ayo14] and Beilinson motives [CD12b]. For étale motives, the construction of the categories is carried out in detail in [Ayo07b, Section 4.5], and an overview of the construction is given in [Ayo14, Section 2.3]. Beilinson motives are constructed in [CD12b, Section 14]. In the case of rational coefficients, which is the only relevant case for our work, both constructions turn out to lead to the same result and the reader can choose the construction they prefer.

Let S be a separated scheme of finite type over a field k. The category Sm/S of smooth schemes of finite type over S admits among others the Nisnevich topology and the étale topology. For $\tau \in \{\operatorname{Nis}, \operatorname{\acute{e}t}\}$, we denote by $\operatorname{Sh}_{\tau}(\operatorname{Sm}/S, \mathbb{Q})$ the category of τ -sheaves of \mathbb{Q} -vector spaces on Sm/S , cf. [CD12b, Example 5.1.4]. There is a model structure on the category of unbounded complexes in $\operatorname{Sh}_{\tau}(\operatorname{Sm}/S, \mathbb{Q})$ whose weak equivalences are the quasi-isomorphisms, its homotopy category is the derived category $\operatorname{Der}(\operatorname{Sh}_{\tau}(\operatorname{Sm}/S, \mathbb{Q}))$. For $X \in \operatorname{Sm}/S$ a smooth S-scheme, $\mathbb{Q}(X)$ denotes the "representable" sheaf associating to $U \in \operatorname{Sm}/S$ the \mathbb{Q} -vector space freely generated by $\operatorname{Sch}_S(U,X)$. One can then use a Bousfield localization (on the model category

level) or a Verdier quotient (on the derived category level) to enforce \mathbb{A}^1 -invariance, i.e., to turn the natural projection $\mathbb{Q}(X \times \mathbb{A}^1) \to \mathbb{Q}(X)$ into a quasi-isomorphism for any smooth S-scheme X, cf. [CD12b, 5.2.b]. The result is the **effective** \mathbb{A}^1 -**derived category**, denoted by $\mathrm{Der}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\tau}(\mathrm{Sm}/S,\mathbb{Q}))$, cf. [CD12b, Example 5.2.17].

The monomorphism $1: S \to \mathbb{G}_{m,S}$ in the category Sm/S gives rise to a morphism $1: \mathbb{Q}(S) \to \mathbb{Q}(\mathbb{G}_{m,S})$ of representable sheaves, viewed as complexes concentrated in degree 0. The cone of this morphism in $\operatorname{Der}_{\mathbb{A}^1}^{\operatorname{eff}}(\operatorname{Sh}_{\tau}(\operatorname{Sm}/S,\mathbb{Q}))$ is called suspended Tate S-premotive $\mathbb{Q}_S(1)[1]$. One can then use the formalism of symmetric spectra, cf. [Hov01], to invert tensoring with the suspended Tate S-premotive, cf. [CD12b, Section 5.3]. The homotopy category of the corresponding model structure on symmetric spectra in the effective \mathbb{A}^1 -derived category is called the **stable** \mathbb{A}^1 -**derived category** $\operatorname{Der}_{\mathbb{A}^1,\tau}(S,\mathbb{Q})$, cf. [CD12b, Example 5.3.31].

At this point, for $\tau = \text{\'et}$, we can define the category of \'etale motives with rational coefficients to be the \'etale stable \mathbb{A}^1 -derived category with rational coefficients, $\mathrm{Der}_{\mathbb{A}^1, \text{\'et}}(S, \mathbb{Q})$.

On the other hand, for $\tau = \text{Nis}$, Beilinson motives are constructed as a category of modules over the 0-th graded part of rational K-theory: following the work of Voevodsky, Riou and Panin–Pimenov–Röndigs, there exists for each scheme S a spectrum KGL_S representing Weibel's homotopy invariant K-theory in the stable homotopy category SH(S). With rational coefficients, the ring spectrum $\text{KGL}_{\mathbb{Q},S}$ decomposes as a direct sum of Adams eigenspaces $\text{KGL}_S^{(i)}$. The zeroth eigenspace $\text{KGL}_S^{(0)}$ is called the **Beilinson motivic cohomology spectrum** $H_{E,S}$, cf. [CD12b, Definition 14.1.2]. The category $\text{DM}_E(S)$ of Beilinson motives over S is then defined to be the Verdier quotient of the \mathbb{A}^1 -derived category $\text{Der}_{\mathbb{A}^1,\text{Nis}}(S,\mathbb{Q})$ by the subcategory of H_E -acyclic objects. Alternatively (glossing over the difficulties making $H_{E,S}$ a strict commutative ring spectrum), one can construct $\text{DM}_E(S)$ as homotopy category of a model structure on a category of $H_{E,S}$ -modules. For $X \in \text{Sm}/S$ a smooth S-scheme, the image of $\mathbb{Q}(X)$ in $\text{DM}_E(S)$ is defined to be the motive $M_S(X)$ of X.

Finally, we need to explain how morphisms between motives as above are computed. For our applications, we will only need to compute morphisms between Tate motives. Working over a base field k and with rational coefficients, the result is easy enough to state, cf. [Ayo14, Theorem 4.12] for the étale case and [CD12b, Corollary 14.2.14] for the Beilinson case:

$$\mathbf{D}\mathbf{A}^{\text{\'et}}_{\operatorname{Spec}(k)}(\mathbb{Q},\mathbb{Q}(p)[q]) \cong \operatorname{DM}_{\operatorname{B}}(\mathbb{Q},\mathbb{Q}(p)[q]) \cong \operatorname{gr}_{\gamma}^{p} \operatorname{K}_{2p-q}(k)_{\mathbb{Q}}.$$

This means that in both cases the morphisms between Tate motives are given in terms of graded pieces of the γ -filtration on rational algebraic K-groups. The basic reason for this coincidence is the fact that rationally algebraic K-theory and étale K-theory are isomorphic. In particular, for mixed Tate motives with rational coefficients discussed later, it does not matter which of the above categories we are working in.

2.3. Realization functors and mixed Weil cohomology theories. Next, we discuss realization functors on the category of Beilinson motives, cf. [CD12b, Section 17].

Fix a coefficient field \mathbb{K} of characteristic 0. For a sheaf E of commutative differential graded \mathbb{K} -algebras on Sm /k, there is an associated cohomology theory

$$\mathrm{H}^n(X, E) := \mathrm{Der}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}\,/k, \mathbb{Q}))(\mathbb{Q}(X), E[n]).$$

In the above, $X \in \text{Sm}/k$ is a smooth scheme. This cohomology theory is called a **mixed Weil cohomology theory**, if it satisfies the following axioms, cf. [CD12b, 17.2.1]:

(1)
$$\mathrm{H}^0(\operatorname{Spec} k, E) \cong \mathbb{K}$$
 and $\mathrm{H}^i(\operatorname{Spec} k, E) \cong 0$ for $i \neq 0$.

(2)
$$\dim_{\mathbb{K}} H^{i}(\mathbb{G}_{m}, E) = \begin{cases} 1 & i = 0 \text{ or } i = 1; \\ 0 & \text{otherwise.} \end{cases}$$

(3) For any two smooth k-schemes X and Y, the Künneth formula holds:

$$\bigoplus_{p+q=n} \operatorname{H}^{p}(X, E) \otimes_{\mathbb{K}} \operatorname{H}^{q}(Y, E) \cong \operatorname{H}^{n}(X \times_{k} Y, E).$$

By [CD12b, Proposition 17.2.4], any mixed Weil cohomology theory E is representable by a commutative ring spectrum \mathcal{E} in $\mathrm{DM}_{\mathrm{B}}(k)$. In [CD12b, 17.2.5], realization functors on the category of Beilinson motives are defined by considering the homotopy category of \mathcal{E} -modules over X and taking the realization functor to be

$$\mathrm{DM}_{\mathrm{B}}(X) \to \mathrm{Der}(X,\mathcal{E}) : M \mapsto \mathcal{E}_X \otimes_X^L M.$$

In the above, the category $Der(X, \mathcal{E})$ is the homotopy category of a model structure on the category of \mathcal{E} -modules in $DM_{\mathcal{E}}(X)$. These realization functors preserve compact objects, cf. [CD12b, 17.2.18], hence we obtain realization functors

$$\mathrm{DM}_{\mathrm{B},c}(X) \to \mathrm{Der}_c(X,\mathcal{E}) : M \mapsto \mathcal{E}_X \otimes_X^L M.$$

Both these realization functors commute with the six functor formalism. Moreover, for any field extension L/k, there is an equivalence of symmetric monoidal triangulated categories

$$Der(L, \mathcal{E}) \cong Der(\mathbb{K}\text{-mod})$$

between the \mathcal{E} -modules over L and the derived category of \mathbb{K} -modules. This equivalence restricts to an equivalence $\mathrm{Der}_c(L,\mathcal{E})\cong\mathrm{Der}^\mathrm{b}(\mathbb{K}$ -modf) between the compact \mathcal{E} -modules over L and the bounded derived category of finitely generated \mathbb{K} -modules.

We list some examples of mixed Weil cohomology theories to which the above results can be applied, cf. [CD12a, Section 3]:

- (1) Algebraic de Rham cohomology is a mixed Weil cohomology with associated commutative ring spectrum \mathcal{E}_{dR} , cf. [CD12a, Section 3.1].
- (2) Rigid cohomology is a mixed Weil cohomology theory with associated commutative ring spectrum \mathcal{E}_{rig} , cf. [CD12a, Section 3.2].
- (3) ℓ -adic cohomology is a mixed Weil cohomology theory with associated commutative ring spectrum $\mathcal{E}_{\mathrm{et},\ell}$, cf. [CD12a, Section 3.3].

In particular, the ℓ -adic and de Rham realization functors will be relevant for our discussion.

2.4. Enriched mixed Weil cohomology theories. There is one other example of motivic triangulated category which we will consider. It arises from an enriched refinement of the mixed Weil cohomology theories discussed above. The existence of such motivic triangulated categories and their properties were worked out in the thesis of Drew, cf. [Dre13].

Recall from [Dre13, Definition 2.1.1] the following definition of a mixed Weil cohomology theory enriched in a Tannakian category.

Definition 2.5. Let S be a noetherian scheme of finite Krull dimension, and let \mathcal{T}_0 be a Tannakian category of finite Ext-dimension, and denote $\mathcal{T} = \text{Ind-}\mathcal{T}_0$. A mixed Weil cohomology theory enriched in \mathcal{T} is a presheaf E_S of commutative differential graded algebras in \mathcal{T} on the category of smooth affine S-schemes, satisfying the following axioms:

- (W1) descent for Nisnevich hypercoverings,
- (W2) \mathbb{A}^1 -invariance,
- (W3) normalization, i.e., $E_S(S)$ is contractible,

- (W4) for $\sigma_1: S \mapsto \mathbb{G}_{m,S}$ the unit section, the object $\mathbb{Q}_{\mathcal{T}}(-1) := \ker(\mathbb{E}_S(\sigma_1)[1])$ belongs to the heart of the natural t-structure of $\operatorname{Der}(\mathcal{T})$ and induces an autoequivalence $\mathbb{Q}_{\mathcal{T}}(-1) \otimes^{\mathbb{L}}_{\mathcal{T}}(-)$ of $\operatorname{Der}(\mathcal{T})$.
- (W5) Künneth formula, i.e., for any smooth affine schemes X, Y over S, the canonical morphism $E_S(X) \otimes_{\mathcal{T}}^{\mathbf{L}} E_S(Y) \to E_S(X \times_S Y)$ is a weak equivalence.

Example 2.6 ([Dre13], Theorem 2.1.8). Associating to a smooth \mathbb{C} -scheme the singular cohomology of its associated complex manifold, equipped with its polarisable mixed Hodge structure, yields a mixed Weil cohomology theory with coefficients in $\mathcal{MHS}^{\mathrm{pol}}_{\mathbb{Q}}$.

Proposition 2.7 ([Dre13], Theorem 2.1.4). Let S be a noetherian scheme of finite Krull dimension, and let \mathcal{T}_0 be a Tannakian category of finite Ext-dimension, and denote $\mathcal{T} = \text{Ind-}\mathcal{T}_0$. For a mixed Weil cohomology theory E_S enriched in \mathcal{T} , there exists a commutative ring spectrum \mathcal{E}_S in the category $\mathcal{SH}(S,\mathcal{T})$ of symmetric $\mathbb{Q}_{\mathcal{T}}(1)$ -spectra over S with values in \mathcal{T} -complexes which represents E_S .

Proposition 2.8 ([Dre13], Proposition 2.2.1). The assignment $X \mapsto \operatorname{Mod}(\mathcal{E}_X)$ extends to a monoidal motivic triangulated category $X \mapsto \operatorname{Der}(\mathcal{E}_X)$. In particular, the full six functor formalism (including the duality statements for the compact objects) applies to $\operatorname{Der}(\mathcal{E}_X)$.

The particular enriched mixed Weil cohomology theory relevant for us is a simplification of the abovementioned Hodge realization, which we want to discuss now. Before, we shortly recall some statements concerning real mixed Hodge structure from [Del94]:

Definition 2.9. Let V be a finite-dimensional \mathbb{R} -vector space. A real mixed Hodge structure on V is given by

- (1) a finite ascending weight filtration $W^{\leq n}$ on V,
- (2) a finite descending Hodge filtration $F^{\geq p}$ on $V \otimes_{\mathbb{R}} \mathbb{C}$,

such that for $p+q \neq n$ and $\overline{F}^{\geq p}$ the conjugate filtration to $F^{\geq p}$, we have

$$\operatorname{Gr}_F^p \operatorname{Gr}_{\overline{F}}^q \operatorname{Gr}_W^n(V) = 0.$$

Proposition 2.10. The category of real mixed Hodge structures is an abelian rigid tensor category. The functor sending a mixed Hodge structure M to its underlying real vector space is a fiber functor making the category of real mixed Hodge structures a neutral Tannakian category.

Proposition 2.11. The functor

$$\mathrm{Gr}^W: \mathcal{MHS}^{\mathrm{pol}}_{\mathbb{Q}} o \mathcal{HS}^{\mathrm{pol},\mathbb{Z}}_{\mathbb{C}}: A \mapsto \bigoplus_{n \in \mathbb{Z}} \mathrm{Gr}^W_n(A \otimes_{\mathbb{Q}} \mathbb{C}),$$

which sends a mixed Hodge structure to the Hodge structure given by the direct sum of the subquotients of the weight filtration is an exact tensor functor.

Assume $\Phi: \mathcal{T}_0 \to \mathcal{T}_0'$ is an exact tensor functor of Tannakian categories, and E is a mixed Weil cohomology theory with values in \mathcal{T}_0 . By [Dre13, Lemma 2.1.3], $\Phi \circ E$ is a mixed Weil cohomology theory with values in \mathcal{T}_0' . We apply this to $\mathrm{Gr}^W: \mathcal{MHS}_\mathbb{Q}^\mathrm{pol} \to \mathcal{HS}_\mathbb{C}^\mathrm{pol}$ and the refined Betti cohomology E_{Hodge} of [Dre13, Theorem 2.1.8]. We get a mixed Weil cohomology theory E_{GrH} with coefficients in graded pure Hodge structures. We get an associated motivic category $\mathrm{Der}(\mathcal{E}_{\mathrm{GrH}})$ with a full six-functor formalism, weight structures and all, by the above-cited results of [Dre13]. Moreover, by [Dre13, Theorem 2.2.7], restricting to mixed Tate objects, there is an equivalence

$$\mathrm{MTDer}(\mathcal{E}_{\mathrm{GrH}})(\mathrm{Spec}\,\mathbb{C})\cong\mathrm{Der}^{\mathrm{b}}(\mathrm{Modf}^{\mathbb{Z}}(\mathbb{C})),$$

and this equivalence respects t-structures, weight structures and compact objects. Here $\mathrm{Modf}^{\mathbb{Z}}(\mathbb{C})$ denotes the category of finitely generated \mathbb{Z} -graded complex vector spaces.

2.5. Weight structures. Finally, we have to discuss weight structures on categories of motives. We first recall, for the reader's convenience, the definition of weight structures from [Bon10, Definition 1.1.1]. Note, however, that our sign convention for the weight is opposite to the one of loc.cit. We follow the sign convention used in most other works on weight structures, such as [Wil12] and [Héb11].

Definition 2.12. Let \mathcal{C} be a triangulated category. A weight structure on \mathcal{C} is a pair $w = (\mathcal{C}_{w \leq 0}, \mathcal{C}_{w \geq 0})$ of full subcategories of \mathcal{C} such that with the notations $\mathcal{C}_{w \leq n} := \mathcal{C}_{w \leq 0}[n]$ and $\mathcal{C}_{w \geq n} := \mathcal{C}_{w \geq 0}[n]$ the following conditions are satisfied:

- (1) the categories $C_{w\leq 0}$ and $C_{w\geq 0}$ are closed under taking direct summands;
- (2) $C_{w \leq 0} \subset C_{w \leq 1}$ and $C_{w \geq 1} \subset C_{w \geq 0}$;
- (3) for any pair of objects $X \in \mathcal{C}_{w \leq 0}, Y \in \mathcal{C}_{w \geq 1}$, we have $\mathcal{C}(X,Y) = 0$;
- (4) for any object $X \in \mathcal{C}$ there is a distinguished triangle $A \to X \to B \to A[1]$ with $A \in \mathcal{C}_{w \leq 0}$ and $B \in \mathcal{C}_{w \geq 1}$.

The full subcategory $C_{w=0} = C_{w \leq 0} \cap C_{w \geq 0}$ is called the **heart of the weight** structure w.

Hébert has constructed weight structures on the categories of Beilinson motives. The result is the following, cf. [Héb11, Theorems 3.3 and 3.8]:

Theorem 2.13. Let k be a field. For any separated scheme X of finite type over k, there is a canonical weight structure w on $\mathrm{DM}_{\mathrm{B},c}(X)$. The family of these weight structures on $\mathrm{DM}_{\mathrm{B},c}$ is characterized uniquely by the following three properties:

- (1) if X is regular, then $\mathbb{Q}_X(n)[2n] \in \mathrm{DM}_{\mathrm{B},c}(X)_{w=0}$ for all $n \in \mathbb{Z}$;
- (2) for any separated finite type morphism $f: X \to Y$, the functors f^* , $f_!$ (and f_{\sharp} for f smooth) are w-left exact, i.e., they preserve non-positivity of weights;
- (3) for any separated finite type morphism $f: X \to Y$, the functors f_* , $f^!$ (and f^* for f smooth) are w-right exact, i.e., they preserve non-negativity of weights.

By [Dre13, Theorem 2.3.2-2.3.4], there are \mathcal{E}_X -analogues of Hébert's theorems on weight structures for Beilinson motives provided the following axiom is satisfied:

(W6) for all smooth affine schemes X over the base and $r, s \in \mathbb{Z}$ with 2r < s we have

$$\operatorname{Der}(\mathcal{T})(\mathbb{Q}_{\mathcal{T}}, \operatorname{E}_{S}(X)(r)[s]) = 0,$$

Whenever this axiom is satisfied, then there is, for each S-scheme X, a weight structure on $\mathrm{Der}(\mathcal{E}_X)$ whose heart is generated by \mathcal{E}_X -motives of Y for $f:Y\to X$ projective and Y regular. This in particular applies to the motivic categories associated to the Hodge realization with values in $\mathcal{MHS}^{\mathrm{pol}}_{\mathbb{Q}}$, cf. the remark on p.8 of [Dre13]. Since $\mathrm{E_{GrH}}$ arises from taking the associated graded of the Hodge realization, validity of axiom (W6) for $\mathrm{E_{Hodge}}$ implies the validity of axiom (W6) for $\mathrm{E_{GrH}}$ so that the categories of $\mathrm{E_{GrH}}$ -modules will have weight structures satisfying the statement of Theorem 2.13.

3. MIXED TATE MOTIVES

In this section, we discuss triangulated categories of mixed Tate motives as well as weight and t-structures on them. We mainly follow [Lev93], [Lev10] and [Wil09] for the t-structures and [Héb11], [Wil08] for the weight structures. While triangulated categories of mixed Tate motives and weight structures on them can be defined

in a rather general setup, the existence of a non-degenerate t-structure and a corresponding abelian category of mixed Tate motives depends on the Beilinson–Soulé vanishing conjectures. For our purposes, this will suffice as we are mostly interested in the existence of mixed Tate motives over smooth varieties with an \mathbb{A}^n -filtration over fields where the Beilinson–Soulé conjectures are known to hold.

3.1. **Triangulated mixed Tate motives.** Fix a motivic triangulated category \mathscr{T} . Recall that for a scheme S, the suspended Tate motive $\mathbb{Q}_S(1)[1]$ is defined as the cone of $\mathrm{M}_S(S) \to \mathrm{M}_S(\mathbb{G}_{\mathrm{m},S})$ in $\mathscr{T}(S)$, cf. Section 2 above or [CD12b, 5.3.15]. Then one sets $\mathbb{Q}_S(n) = \mathbb{Q}_S(1)^{\otimes n}$. The following is [Lev10, Definition 3.14].

Definition 3.1. For each smooth k-scheme S, we define the **triangulated category of mixed Tate motives over** S, denoted by $\mathrm{MTDer}(S) = \mathrm{MTDer}(\mathscr{T}, S)$, to be the strictly full triangulated subcategory of $\mathscr{T}(S)$ generated by the objects $\mathbb{Q}_S(n)$.

In the following, whenever we consider a category of mixed Tate motives over a scheme S, this scheme will always be smooth. There are some direct consequences of this definition, cf. [Lev10, Proposition 3.15].

Proposition 3.2. The category MTDer(S) is a tensor triangulated category. Its objects are compact, i.e., there is an inclusion $MTDer(S) \subset \mathcal{F}^c(S)$.

Proof. The first follows from $\mathbb{Q}_S(n) \otimes \mathbb{Q}_S(m) \cong \mathbb{Q}_S(n+m)$. The second follows from [CD12b, Section 4.2] and the definition of $\mathbb{Q}_S(1)$ which only uses the smooth S-schemes S and $\mathbb{G}_{m,S}$.

Remark 3.3. If k is a field, then by [Wil08, Theorem 2.5] the restriction of Hébert's weight structure w on $\mathrm{DM}_{\mathrm{B,c}}(k)$ is a weight structure on $\mathrm{MTDer}(k)$.

3.2. Weight t-structure on mixed Tate motives (d'après Levine). In the following, we recall [Lev10, Definition 3.16 and Theorem 3.19]. It provides an approach to defining weights for mixed Tate motives different from the weight structures discussed above. Levine's approach uses a suitable t-structure on the triangulated category of mixed Tate motives, and produces different weights related to the weights defined above via décalage.

In the following, we consider categories MTDer(S) which are triangulated categories of Tate type, in the sense of [Lev93, Definition 1.1]. This means, in addition to Proposition 3.2, that the following conditions on morphisms in \mathcal{I}_S are satisfied:

- (1) $\mathscr{T}_S(\mathbb{Q}_S(n)[a], \mathbb{Q}_S(m)[b]) = 0 \text{ if } n > m.$
- (2) $\mathscr{T}_S(\mathbb{Q}_S(n)[a], \mathbb{Q}_S(n)[b]) = 0 \text{ if } a \neq b.$
- (3) $\mathscr{T}_S(\mathbb{Q}_S(n), \mathbb{Q}_S(n)) = \mathbb{Q} \cdot \mathrm{id}.$

We will only need the weight t-structure to define the t-structure on mixed Tate motives below. Therefore, it suffices to say that these conditions are in particular satisfied whenever S satisfies the Beilinson–Soulé vanishing conditions. However, the conditions hold in greater generality: if S is the spectrum of a field, vanishing results in K-theory show that MTDer(S) is of Tate type, cf. [Lev93, Theorem 4.1].

Definition 3.4. Denote by W_n MTDer(S) the strictly full triangulated subcategory of MTDer(S) generated by the Tate motives $\mathbb{Q}_S(-a)$ with $a \leq n$. Denote by $W_{[n,m]}$ MTDer(S) the strictly full triangulated subcategory of MTDer(S) generated by the Tate motives $\mathbb{Q}(-a)$ with $n \leq a \leq m$. Denote by $W^{>n}$ MTDer(S) the strictly full triangulated subcategory of MTDer(S) generated by the Tate motives $\mathbb{Q}(-a)$ with a > n.

The following recalls [Lev10, Theorem 3.19].

Proposition 3.5. Assume MTDer(S) is a tensor triangulated category of Tate type. Then the following statements are true:

- (1) $(W_n \operatorname{MTDer}(S), W^{>n-1} \operatorname{MTDer}(S))$ is a t-structure on $\operatorname{MTDer}(S)$ with heart $W_{[0,0]} \operatorname{MTDer}(S)$.
- (2) The truncation functors

$$W_n: \mathrm{MTDer}(S) \to W_n \, \mathrm{MTDer}(S), W^{>n}: \mathrm{MTDer}(S) \to W^{>n} \, \mathrm{MTDer}(S)$$

are exact, W_n is right adjoint to the corresponding inclusion and $W^{>n}$ is left adjoint to the corresponding inclusion.

(3) For each n < m there is an exact functor

$$W_{[n+1,m]}: \mathrm{MTDer}(S) \to W_{[n+1,m]} \, \mathrm{MTDer}(S)$$

and a natural distinguished triangle

$$W_n \to W_m \to W_{[n+1,m]} \to W_n[1].$$

(4)
$$\operatorname{MTDer}(S) = \bigcup_{n \in \mathbb{Z}} W_n \operatorname{MTDer}(S) = \bigcup_{n \in \mathbb{Z}} W^{>n} \operatorname{MTDer}(S).$$

We denote by $\operatorname{gr}_n^W:\operatorname{MTDer}(S)\to W_{[n,n]}\operatorname{MTDer}(S)$ the corresponding composition of truncation functors, it assigns to a mixed Tate motive the n-th subquotient of the weight filtration.

If $\mathrm{MTDer}(S)$ is a category of Tate type, then the category $W_{[n,n]}$ $\mathrm{MTDer}(S)$ can be identified with the derived category $\mathrm{Der}^{\mathrm{b}}(\mathbb{Q}\operatorname{-mod}f)$ of finite-dimensional $\mathbb{Q}\operatorname{-vector}$ spaces.

3.3. **t-structure on mixed Tate motives (d'après Levine).** Now we recall the existence of abelian categories of mixed Tate motives under the assumption of the Beilinson–Soulé vanishing conjectures, cf. [Lev10, Definition 3.21, Theorem 3.22], cf. also the field case [Lev93, Theorem 1.4, Proposition 2.1, Theorem 4.2].

Definition 3.6. Fix a motivic triangulated category \mathscr{T} . We say that a separated smooth finite type k-scheme S satisfies **Beilinson–Soulé vanishing** for \mathscr{T} if for m < 0, we have

$$\mathscr{T}_k(M_k(S), \mathbb{Q}_k(n)[m]) = 0.$$

As mentioned above, we identify $W_{[n,n]}$ MTDer (\mathcal{T},S) with Der $^{\mathrm{b}}(\mathbb{Q}\operatorname{-modf})$, and this allows to define for each mixed Tate motive $M\in\mathrm{MTDer}(\mathcal{T},S)$ the $\mathbb{Q}\operatorname{-vector}$ space $\mathrm{H}^m(\operatorname{gr}_n^W M)$, cf. [Lev10, Remark 3.20]. Let MTDer $(\mathcal{T},S)^{\leq 0}$ be the full subcategory of those $M\in\mathrm{MTDer}(\mathcal{T},S)$ such

Let $\mathrm{MTDer}(\mathcal{T},S)^{\leq 0}$ be the full subcategory of those $M\in\mathrm{MTDer}(\mathcal{T},S)$ such that

$$\mathrm{H}^m(\mathrm{gr}_n^W M) = 0$$
 for all $m > 0$ and $n \in \mathbb{Z}$.

Let $\mathrm{MTDer}(\mathscr{T},S)^{\geq 0}$ be the full subcategory of those $M\in\mathrm{MTDer}(\mathscr{T},S)$ such that

$$H^m(\operatorname{gr}_n^W M) = 0$$
 for all $m < 0$ and $n \in \mathbb{Z}$.

Finally, we set $MT(\mathcal{T}, S) = MTDer(\mathcal{T}, S)^{\leq 0} \cap MTDer(\mathcal{T}, S)^{\geq 0}$.

Theorem 3.7. Suppose the smooth scheme S satisfies the Beilinson–Soulé vanishing conjectures for \mathscr{T} .

- (1) $(\mathrm{MTDer}(\mathcal{T}, S)^{\leq 0}, \mathrm{MTDer}(\mathcal{T}, S)^{\geq 0})$ is a non-degenerate t-structure on the category $\mathrm{MTDer}(\mathcal{T}, S)$ with heart $\mathrm{MT}(\mathcal{T}, S)$ containing the Tate motives $\mathbb{Q}_S(n), n \in \mathbb{Z}$.
- (2) The category $MT(\mathcal{T}, S)$ is a rigid \mathbb{Q} -linear abelian tensor category.

Remark 3.8. Recall that for \mathcal{T} given by étale or Beilinson motives the homomorphisms in $\mathrm{MTDer}(k)$ can be computed from rational K-theory as

$$MTDer(k)(\mathbb{Q}(n), \mathbb{Q}(n+q)[p]) \cong K_{2q-p}(k)^{(q)}.$$

In the case of global fields and finite fields, there is also a precise relation between Ext-groups in the abelian category of mixed Tate motives and rational K-theory. More precisely, there are natural isomorphisms, cf. [Lev93, Corollary 4.3]:

$$\operatorname{Ext}_{\mathrm{MT}(k)}^p(M,N) \stackrel{\sim}{\to} \mathrm{MTDer}(k)(M,N[p]).$$

In particular, the vanishing of rational K-theory for finite fields and global function fields implies that for such k, there are no extensions between objects in MT(k).

Proposition 3.9. Let k be a field satisfying the Beilinson-Soulé vanishing conjectures for \mathscr{T} . Assume S is smooth and $M_k(S)$ is in MTDer(k). Then S also satisfies the Beilinson-Soulé vanishing conjectures for \mathscr{T} .

Proof. As in the proof of Proposition 3.5, we have (together with Beilinson–Soulé for the base field)

$$\operatorname{Hom}_{\mathscr{T}(k)}(\mathbb{Q}_k(a), \mathbb{Q}_k(b)[m]) \cong \operatorname{Hom}_{\mathscr{T}(k)}(\operatorname{M}_k(k), \mathbb{Q}_k(b-a)[m]) = 0$$

for m < 0 (or in the stronger version for $m \leq 0$ and $b \neq a$). By definition, every object M of MTDer(\mathcal{T}, k) can be constructed from $\mathbb{Q}_k(n)$, $n \in \mathbb{Z}$ using triangles. The corresponding long exact sequences then yield the claim.

Most of the time, we will apply the above result to flag varieties G/B, Bruhat cells $BxB/B \cong \mathbb{A}^n$ in flag varieties or Bott–Samelson resolutions of Schubert varieties. These varieties have motivic cell structures, hence their motives are mixed Tate.

Remark 3.10. The Beilinson–Soulé vanishing conjecture holds for finite fields by the K-theory computations of Quillen, cf. [Qui72]. The Beilinson–Soulé vanishing holds for global fields - for number fields by the K-theory computations of Borel, and for function fields by the group homology computations of Harder, cf. [Har77].

Remark 3.11. Finally, we want to remark that the Beilinson–Soulé condition for $\mathcal{T}(\mathbb{C})$ given by semisimplified Hodge realization is a triviality. This holds more generally whenever \mathcal{T} satisfies the grading condition - the motivic t-structure is then the natural t-structure on the category of \mathbb{Z} -graded \mathbb{C} -vector spaces. This is one of the reasons why the motives with coefficients in semisimplified Hodge realization are useful: we get results over the base field \mathbb{C} where the Beilinson–Soulé conjectures for étale or Beilinson motives are not known.

3.4. Summary of structures. Let k be a field satisfying the Beilinson–Soulé vanishing conjectures. The category MTDer(k) is a tensor triangulated category, equipped with a weight structure and a t-structure.

The results of Wildeshaus [Wil09, Théorème 1.1, Corollaire 1.4] imply that there is an exact functor

$$\operatorname{real}:\operatorname{Der^b}(\operatorname{MT}(k))\to\operatorname{MTDer}(k)$$

which is an equivalence of categories and induces the identity on MT(k). Note that the result in loc.cit. is stated for number fields, but all that is required for the proof is the Beilinson–Soulé vanishing.

The above-mentioned identification $W_{[n,n]}$ MTDer $(k) \approx \text{Der}^{b}(\mathbb{Q}\text{-mod}f)$ restricts to an equivalence $W_{[n,n]}$ MT $(k) \approx \mathbb{Q}\text{-mod}f$. By [Lev93, Theorem 4.2], this equivalence provides an exact faithful tensor functor

$$\bigoplus_{i\in\mathbb{Z}}\operatorname{gr}_i^W:\operatorname{MT}(k)\to\mathbb{Q}\operatorname{-modf}^{\mathbb{Z}}$$

from the category of mixed Tate motives to the category of finite-dimensional graded \mathbb{Q} -vector spaces. In the special cases where k is a finite field or a global function field, the vanishing of rational K-theory allows to identify $\mathrm{MT}(k)$ with the category of finite-dimensional graded \mathbb{Q} -vector spaces. The result is an identification of $\mathrm{MTDer}(k)$ with the bounded derived category of finite-dimensional graded \mathbb{Q} -vector spaces.

In the case of a field k, there are now two ways of defining weights for mixed Tate motives. The comparison between these two is given by [Wil08, Theorem 3.8]: a mixed Tate motive M is in $\mathrm{MTDer}(k)_{w=0}$ if and only if $\mathrm{H}^i(\mathrm{gr}_j^W M) = 0$ for $i \neq 2j$. The weight structure w assigns weight q-2p to the motives $\mathbb{Q}(p)[q]$, the weight t-structure W assigns weight p.

In the case of a finite field (again using vanishing of rational K-theory), the motives of w-weight 0 form a tilting collection. This provides another equivalence of triangulated categories $\operatorname{Der}^b(\operatorname{MTDer}(k)_{w=0}) \cong \operatorname{MTDer}(k)$. The result is an easy version of Koszul duality that "interchanges the weight and t-structure." It is the unique triangulated self-equivalence that maps $\mathbb{Q}(n)$ to $\mathbb{Q}(-n)[-2n]$: the first object has cohomological degree 0 and weight -2n, the latter has cohomological degree -2n and weight 0. The results of our paper can be interpreted as saying that the Koszul duality of [BGS96] for stratified mixed Tate motives over partial flag varieties is essentially obtained by perverse glueing from this toy example.

It is interesting to note that in the case of a number fields, the hearts of the weight and t-structure are not equivalent. The heart of the weight structure is semi-simple, while the heart of the t-structure has a lot of interesting arithmetic extensions of Tate motives. A functor as above still exists and embeds the heart of the t-structure into the heart of the weight structure, splitting the extensions. It is hence not exactly clear if the above Koszul duality functor can have a "geometric construction". We thank Jörg Wildeshaus for discussions on this point.

4. Stratified mixed Tate motives

In the following section, we consider categories of motives over stratified varieties. We want to study motives which are constant mixed Tate along the strata. For this, we need a condition which, in analogy with the case of sheaves on topological spaces, we call **Whitney-Tate**. This condition is in particular satisfied for partial flag varieties with the stratification by Schubert cells. A further discussion of Whitney-Tate stratifications is deferred to Appendix A.

Notational convention 4.1. From this moment on, we will consider motivic triangulated categories \mathscr{T} over $\mathscr{S} = \operatorname{Sch}/k$, i.e., we will only work over schemes separated and of finite type over some field. All the constructions will take place in the motivic triangulated category \mathscr{T} , which will sometimes be suppressed from the notation. In particular, whenever we speak of motives, we are referring to objects in some category $\mathscr{T}(X)$ where hopefully the exact nature of \mathscr{T} will be clear from context.

For the representation-theoretic applications, we will usually consider a more restricted setting in which the following two additional conditions are satisfied:

(weight condition): A motivic triangulated category \mathscr{T} over Sch/k is said to satisfy the <u>weight condition</u> if for each scheme X there is a weight structure on $\mathscr{T}(X)$, such that this collection of weight structures satisfies the conclusion of Hébert's theorem 2.13.

(grading condition): A motivic triangulated category \mathscr{T} (with coefficients in a field \mathbb{K} of characteristic 0) over Sch/k is said to satisfy the grading condition if $\operatorname{MTDer}(\mathscr{T}, k)$ is equivalent (as tensor-triangulated category)

to the bounded derived category of finite-dimensional $\mathbb{Z}\text{-graded}$ $\mathbb{K}\text{-vector}$ spaces.

From the discussion in Section 2 and Section 3, these two conditions are satisfied for rational motives over finite fields and for \mathcal{E}_{GrH} -motives over \mathbb{C} . These are the situations of interest for our representation-theoretic applications.

Definition 4.2. By a **stratification** of a variety we mean a finite partition

$$X = \bigsqcup_{s \in \mathcal{S}} X_s$$

of X into locally closed smooth subvarieties, called the **strata** of our stratification, such that the closure of each stratum is again a union of strata. If all strata are isomorphic to affine spaces \mathbb{A}^{n_s} of some dimension depending on $s \in \mathcal{S}$, we speak of a **stratification by affine spaces** or of an **affinely stratified variety**.

4.3. Given a stratified variety (X, \mathcal{S}) we consider the full triangulated subcategories

$$\mathrm{MTDer}_{\mathcal{S}}^*(X), \mathrm{MTDer}_{\mathcal{S}}^!(X) \subset \mathscr{T}(X)$$

of all motives M such that for each inclusion $j_s: X_s \hookrightarrow X$ of a stratum j_s^*M respectively $j_s^!M$ belongs to $\mathrm{MTDer}(X_s)$.

Lemma 4.4. Given a stratified variety (X, S) the category $MTDer_S^*(X)$ is generated as a triangulated category by the objects $j_{s!}M$ for $s \in S$ and $M \in MTDer(X_s)$. Similarly $MTDer_S^!(X)$ is generated by the objects $j_{s*}M$.

Proof. We prove the first statement, the second is similar. We argue by induction on the number of strata, the case of no stratum being obvious. Let $j_s: X_s \hookrightarrow X$ be the inclusion of an open stratum and $i: Z \hookrightarrow X$ the inclusion of its complement. For $M \in \mathrm{MTDer}_{\mathcal{S}}^*(X)$ consider the "Gysin" or "localization" triangle

$$j_{s!}j_s^*M \to M \to i_!i^*M \to j_{s!}j_s^*M[1].$$

Obviously, $j_s^*M \in \mathrm{MTDer}(X_s)$, and so the first term is of the required form. On the other hand, $i^*M \in \mathrm{MTDer}_{\mathcal{S}}^*(Z)$ and the induction hypothesis implies that i^*M is built from motives $k_t!N$ with $k_t: Z_t \hookrightarrow Z$ a stratum of Z and $N \in \mathrm{MTDer}(Z_t)$. Hence $i_!i^*M$ is of the required form, and the claim is proved.

Definition 4.5. A stratified variety (X, S) is called **Whitney-Tate** if and only if for all $s, t \in S$ and $M \in \mathrm{MTDer}(X_s)$ we have $j_t^* j_{s*} M \in \mathrm{MTDer}(X_t)$.

Remark 4.6. By Verdier duality, this condition is equivalent to asking $j_t^! j_{s!} M \in \mathrm{MTDer}(X_t)$. Using Lemma 4.4 we deduce in this case the equality $\mathrm{MTDer}_{\mathcal{S}}^!(X) = \mathrm{MTDer}_{\mathcal{S}}^*(X)$.

Definition 4.7. Given a Whitney–Tate stratified variety, the category

$$\mathrm{MTDer}_{\mathcal{S}}^{!}(X) = \mathrm{MTDer}_{\mathcal{S}}^{*}(X) = \mathrm{MTDer}_{\mathcal{S}}(X) \subset \mathscr{T}(X)$$

is called the category of stratified mixed Tate motives.

Remark 4.8. Similar categories have appeared before, in the setting of ℓ -adic sheaves in [BGS96, Section 4.4], and in the setting of Tate motives in [Wil12, Section 4, Theorem 4.4]. In particular, [Wil12, Theorem 4.4] states that an affinely stratified variety is Whitney–Tate if the orbit closures are regular.

Remark 4.9. Let us recall some facts concerning the Bruhat decomposition of a split reductive group G. If $T \subset B$ is a maximal torus in a Borel subgroup of G, multiplication gives an isomorphism of varieties $T \times B_{\mathbf{u}} \stackrel{\sim}{\to} B$, where $B_{\mathbf{u}}$ is the unipotent radical of B. If $U_{\alpha} \subset G$ are the root subgroups for T, which are all isomorphic to the additive group, and R^+ is the system of positive roots of T in

B, multiplication gives an isomorphism of varieties $\prod_{\alpha \in R^+} U_\alpha \stackrel{\sim}{\to} B_u$ and an open embedding $\prod_{\alpha \in R^+} U_{-\alpha} \times B \hookrightarrow G$ for the products taken in an arbitrary but fixed order. For $W \supset W_P$ the Weyl groups of G and a parabolic subgroup P, respectively, we have the Bruhat decompositions

$$G = \bigsqcup_{x \in W} BxB$$
, $P = \bigsqcup_{x \in W_P} BxB$ and $G = \bigsqcup_{\bar{y} \in W/W_P} B\bar{y}P$.

The double cosets can be described quite explicitly by isomorphisms

$$\prod_{\alpha \in R^+ \backslash yR^+} U_\alpha \times \{\dot{y}\} \times P \overset{\sim}{\to} B\bar{y}P$$

given again by multiplication with $y \in W$ the shortest representative of \bar{y} and \dot{y} representing y. In particular, $G \to G/P$ and also all $G/Q \to G/P$ for inclusions $Q \subset P$ of parabolic subgroups are trivial fibre bundles over any cell $B\bar{y}P/P$. Moreover, the preimage in G/Q of the cell $B\bar{y}P/P$ is the union of cells $B\bar{x}Q/Q$ for $\bar{x} \in W/W_Q$ satisfying $\bar{x}W_P = \bar{y}$. The maps induced between these cells are trivial fibre bundles with affine spaces as fibres. For more details, one might consult [Bor91].

Proposition 4.10. Let G be a connected split reductive algebraic group over the field k. Let $T \subset B \subset P \subset G$ be a choice of split maximal torus T, a Borel subgroup B and a parabolic subgroup P. Then the stratification of G/P by B-orbits is Whitney—Tate

Proof. Let us first concentrate on the case P = B.

Recall that Bruhat cells are parametrized by the elements of the Weyl group W of G. Given an element t in the Weyl group, let $X_t = BtB/B$ be the corresponding Bruhat cell. For every simple reflection s in W, corresponding to a simple positive root α w.r.t. B, denote by P_s the parabolic corresponding to the set $\{\alpha\}$, i.e., the subgroup generated by the Borel and the root subgroup for $-\alpha$. The inclusion $B \subset P_s$ induces a projection $\pi_s : G/B \to G/P_s$ from G/B onto the partial flag variety for the parabolic P_s . The projection π_s is Zariski-locally trivial with fiber $P_s/B \cong \mathbb{P}^1$. Now consider the pullback square

$$X_t \sqcup X_{ts} = Y \xrightarrow{u} G/B$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \pi_s$$

$$D \xrightarrow{v} G/P_s$$

where D is a B-orbit in G/P_s . By what was said above, $Y \cong \mathbb{P}^1 \times D$ and it decomposes into two B-orbits as shown. Without loss of generality, the B-orbits correspond to the Weyl group elements t and ts with t < ts in the Bruhat order, and with this choice we have $X_{ts} \cong \mathbb{A}^1 \times D$ and $X_t \cong \{\infty\} \times D$. We denote the open immersion $j: X_{ts} \hookrightarrow Y$ and the closed immersion $i: X_t \hookrightarrow Y$. In the following, we denote $j_t: X_t \to G/B$ the inclusion of cells in G/B. The projection induces an isomorphism $p: X_t \stackrel{\sim}{\to} D$, thus we get $p^*p_*X_t \cong i^*Y$ and applying $u_!$ and base change we get $\pi_s^*\pi_{s*}j_{t!}X_t \cong u_!Y$. On the other hand we have a triangle $j_!j^*Y \to Y \to i_!i^*Y \to j_!j^*Y[1]$ and with $u_!$ a triangle

$$j_{ts!}\underline{X}_{ts} \to \pi_s^* \pi_{s*} j_{t!}\underline{X}_t \to j_{t!}\underline{X}_t \to j_{ts!}\underline{X}_{ts}[1]$$

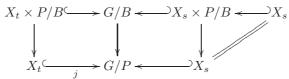
on G/B. This shows that any triangulated subcategory of $\mathcal{T}(G/B)$ stable under all $\pi_s^* \pi_{s*}$ for all simple reflections s and containing the skyscraper $j_{e!} \underline{X}_e$ at the one-point cell X_e has to contain $j_{r!} \underline{X}_r$ for all Bruhat cells BrB/B. Now it is sufficient to see that our triangulated subcategory MTDer $^!_{(B)}(G/B)$ from

4.3 has all these properties, since then all $j_{r!}\underline{X}_r$ belong to it and our stratification is indeed Whitney–Tate. Given $M \in \mathrm{MTDer}^!_{(B)}(G/B)$ we thus need to show $\pi_s^*\pi_{s*}M \in \mathrm{MTDer}^!_{(B)}(G/B)$, i.e., the !-restriction of $u^!\pi_s^*\pi_{s*}M$ to X_t resp. X_{ts} is constant mixed Tate. Since π_s is smooth, we can exchange $u^!\pi_s^* \cong p^*v^!$, and base change allows to exchange $v^!\pi_{s*} \cong p_*u^!$. Hence, it will be sufficient to show $p^*p_*u^!M \in \mathrm{MTDer}^!_{(B)}(Y)$. Now sure enough we have $N := u^!M \in \mathrm{MTDer}^!_{(B)}(Y)$. We consider the push-forward of the localization triangle:

$$p_*i_*i^!N \to p_*N \to p_*j_*j^!N \to p_*i_*i^!N[1].$$

Here $p \circ i$ is an isomorphism and $p \circ j$ is the projection $\mathbb{A}^1 \times D \to D$, as remarked above. By the homotopy property we deduce $p_*N \in \mathrm{MTDer}(D)$, and then $p^*p_*N \in \mathrm{MTDer}_{(B)}^!(Y)$ follows easily.

The case of a general parabolic P can be deduced from the Borel case above as follows: the inclusion $B \subset P$ induces a projection $\pi: G/B \to G/P$ which is smooth and projective; actually, it is a Zariski-locally trivial bundle with fiber P/B and for any cell $X_t = BtP/P \subset G/P$, we have $\pi^{-1}(X_t) \cong X_t \times P/B$. To determine the restriction $j_t^*j_{s*}M$ from a cell X_t to a cell X_s of G/P, we can use the following diagram



Moreover, as in the proof for the case B above, there is a Bruhat-cell of $X_s \times P/B$ which is isomorphic to X_s . To compute $j_t^*j_{s*}M$, it suffices, by base change, to pull back M to $X_t \times P/B$, extend to G/B and restrict to $X_s \subset X_s \times P/B$. Since G/B is Whitney–Tate, the result is a constant mixed Tate motive on X_s , and we are done.

Remark 4.11. Further conditions for a stratification to be Whitney–Tate can be found in Appendix A. These conditions allow another proof of the above Proposition 4.10, using that fibres of Bott–Samelson resolutions of Schubert cells have mixed Tate motives, cf. Proposition A.3.

Example 4.12. Let G be a connected split reductive algebraic group over the field k. Let $T \subset B \subset P \subset G$ be a choice of split maximal torus T, a Borel subgroup B and a parabolic subgroup P. It is well-known that the partial flag varieties G/P are affinely stratified by the B-orbits alias Schubert cells. By a **paraboloid** B-variety, we mean a B-variety Y which is isomorphic to a locally closed B-stable subset of a partial flag variety G/P. Plainly, these are affinely stratified by B-orbits as well. In this case we denote the stratification by (B) and call the objects of $\operatorname{MTDer}_{(B)}(Y)$ **Bruhat**—**Tate sheaves**. By the arguments in Section A, the Bruhat stratifications of paraboloid B-varieties are also Whitney—Tate.

4.13. Other examples of affinely stratified varieties can be found among smooth projective spherical varieties, Hessenberg varieties and symmetric spaces. In all these cases, locally closed cells arise from the Białynicki-Birula decomposition associated to suitably chosen \mathbb{G}_{m} -actions and in most cases of interest, these also give rise to stratifications.

5. Weight structure for stratified mixed Tate motives

In the following section, we discuss the existence and properties of a weight structure on the category $\mathrm{MTDer}_{\mathcal{S}}(X)$ of stratified mixed Tate motives for a Whitney–Tate stratified variety (X,\mathcal{S}) . We fix a motivic triangulated category \mathcal{T} , which is

required to satisfy the weight condition - so that we can talk about weight structures. For the later results on combinatorial models for the heart, we will additionally need the grading condition, but this is not required for the definition of the weight structure.

Proposition 5.1. Let (X, S) be an affinely Whitney-Tate stratified variety. Then on the category $MTDer_{S}(X)$ of stratified mixed Tate motives, cf. Definition 4.7, we obtain a weight structure w by setting

$$\mathrm{MTDer}_{\mathcal{S}}(X)_{w < 0} := \{ M \mid j_s^* M \in \mathrm{MTDer}(X_s)_{w < 0} \text{ for all strata } s \in \mathcal{S} \}$$

$$\mathrm{MTDer}_{\mathcal{S}}(X)_{w\geq 0} := \{ M \mid j_s^! M \in \mathrm{MTDer}(X_s)_{w\geq 0} \text{ for all strata } s \in \mathcal{S} \}$$

This weight structure coincides with the restriction of Hébert's weight structure on $\mathcal{T}(X)$ to $\mathrm{MTDer}_{\mathcal{S}}(X)$.

Proof. To prove the existence of such a weight structure we proceed by induction on the number of strata. If there is no stratum, the claim is correct. Otherwise, decompose X as the disjoint union of an open stratum $j: X_s \hookrightarrow X$ and its closed complement $i: Z \hookrightarrow X$. Using Bondarko's result [Bon13, Proposition 1.7 (13), (15)] on glueing weight structures, we obtain a weight structure on $\mathrm{MTDer}_{\mathcal{S}}(X)$ by setting

$$\operatorname{MTDer}_{\mathcal{S}}(X)_{w \leq 0} := \{ M \mid i^*M \in \operatorname{MTDer}(Z)_{w \leq 0}, \ j^*M \in \operatorname{MTDer}(X_s)_{w \leq 0} \}$$

$$\operatorname{MTDer}_{\mathcal{S}}(X)_{w \geq 0} := \{ M \mid i^!M \in \operatorname{MTDer}(Z)_{w \geq 0}, \ j^!M \in \operatorname{MTDer}(X_s)_{w \geq 0} \}$$

Now recall that for any separated finite type morphism f, the functors f^* and $f^!$ are left and right weight-exact, respectively, for Hébert's weight structure. This implies that objects of weight ≤ 0 for Hébert's weight structure are also of weight ≤ 0 for our weight structure, and similarly for ≥ 0 . For the reverse inclusions, we use the same induction. Assume the result is established for Z. By [Bon13, Proposition 1.7 (13)], the $(w \leq 0)$ -part of the glued weight structure on MTDer $_{\mathcal{S}}(X)$ is generated by $j_!$ MTDer $_{\mathcal{S}}(X)_{w\leq 0}$ and i_* MTDer $_{\mathcal{S}}(Z)_{w\leq 0}$. This implies all its objects also belong to the $(w \leq 0)$ -part of Hébert's weight structure. A dual argument takes care of the $(w \geq 0)$ -part of the weight structures. Finally, it also follows directly from the above arguments that the weight structure constructed this way has the description claimed in the statement of the proposition.

Remark 5.2. This generalizes [Wil12, Corollary 4.12] to some cases where closures of strata are not necessarily regular.

6. Pointwise purity, Bott-Samelson motives and the heart

In the next section, we investigate the heart of the weight structure defined in Section 5, in the special case of flag varieties. We show that motives of Bott–Samelson resolutions of Schubert cells satisfy an additional property called **pointwise purity** and deduce that the heart of the weight structure is generated by motives of Bott–Samelson resolutions.

Definition 6.1. Let (X, S) be an affinely Whitney–Tate stratified variety. A stratified Tate motive $M \in \mathrm{MTDer}_{S}(X)_{w=0}$ is called **pointwise *-pure** if for each inclusion $i_s: X_s \to X$ of a stratum, we have $i_s^*M \in \mathrm{MTDer}(X_s)_{w=0}$. Similarly, we define the concept **pointwise !-pure**. If both conditions are satisfied, the motive is called **pointwise pure**.

Proposition 6.2. Let (X,S) be an affinely Whitney-Tate stratified variety, and denote by $\text{fin}: X \to \text{pt}$ the structure morphism. For any pointwise *-pure stratified Tate motive $M \in \text{MTDer}_{\mathcal{S}}(X)$, the object $\text{fin}_! M$ is pure Tate of weight 0, in formulas $\text{fin}_! M \in \text{MTDer}(\text{pt})_{w=0}$.

Proof. The statement is proved by induction on the number of strata. If there is no stratum, the claim is evident. For the inductive step, consider the embedding $j: X_s \to X$ of an open stratum and let $i: Z \hookrightarrow X$ be the embedding of its complement. We have the localization sequence

$$j_!j^*M \cong j_!j^!M \to M \to i_!i^*M \cong i_*i^*M \to j_!j^*M[1].$$

After proper pushforward, this sequence becomes

$$\operatorname{fin}_! j^*M \to \operatorname{fin}_! M \to \operatorname{fin}_! i^*M \to \operatorname{fin}_! j^*M[1].$$

By induction we may assume $\text{fin}_! i^*M \in \text{MTDer}(\text{pt})_{w=0}$. On the other hand, the homotopy property implies that since $X_s \cong \mathbb{A}^n$, the pushforward $\text{fin}_* : \text{MTDer}(X_s) \to \text{MTDer}(\text{pt})$ is in fact an equivalence which is compatible with the weight structures and duality. Therefore, $(\text{fin}_s)_! j^*M \in \text{MTDer}(\text{pt})_{w=0}$. By [Bon13, Proposition 1.7(2)], hearts of weight structures are extension-stable, so $\text{fin}_! M \in \text{MTDer}(\text{pt})_{w=0}$.

Corollary 6.3. Let (X, S) be an affinely Whitney-Tate stratified variety. Given $M, N \in \mathrm{MTDer}_{S}(X)$ with M pointwise *-pure and N pointwise !-pure we have $\mathscr{T}_{X}(M, N[a]) = 0$ for any a > 0.

Proof. We first note, that point (iv) of [Héb11, Théorème 3.7] has a Hom-analogue, and implies that on the category MTDer(k) of mixed Tate motives, the functor Hom is in fact weight-exact. Using this, we see that for any stratum, $j_s^! \operatorname{Hom}(M,N) \cong \operatorname{Hom}(j_s^*M,j_s^!N)$ is pure of weight zero by the assumption, thus $\operatorname{Hom}(M,N)$ is pointwise !-pure. By Proposition 6.2 we deduce that its direct image $\operatorname{fin}_* \operatorname{Hom}(M,N)$ is pure of weight zero. This in turn means

$$\mathscr{T}_{\mathrm{pt}}(\underline{\mathrm{pt}}, \mathrm{fin}_* \operatorname{Hom}(M,N)[a]) = 0$$

for a>0 by the definition of a weight structure. But this is just another way to write the space we claim to vanish. \Box

Definition 6.4. Let k be a field and let $G \supset B \supset T$ be a split reductive group with a choice of maximal torus T and Borel subgroup B. We define the collection of full subcategories

$$\mathrm{MTDer}^{\mathrm{bs}}_{(B)}(G/Q) \subset \mathrm{MTDer}_{(B)}(G/Q)$$

for all standard parabolic subgroups $B\subset Q\subset G$ to be the smallest collection with the following properties:

- (1) the collection contains the skyscraper at the one-point-cell of G/B, i.e., for $j_e : \operatorname{pt} \hookrightarrow G/B$ the embedding of the B-orbit B/B, we have $(j_e)_* \operatorname{\underline{pt}} \in \operatorname{MTDer}_{(B)}^{\operatorname{bs}}(G/B)$,
- (2) the collection is stable under $M \mapsto M(n)[2n]$ and direct summands,
- (3) the collection is extension-stable in the sense that for a distinguished triangle $A \to B \to C \to A[1]$ with A and C in the subcategory, B is also in the subcategory, and
- (4) if $\pi: G/P \to G/Q$ is a projection for standard parabolic subgroups $P \subset Q$, then we have

$$M \in \mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P) \Rightarrow \pi_! M, \pi_* M \in \mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/Q)$$

 $M \in \mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/Q) \Rightarrow \pi^! M, \pi^* M \in \mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P)$

6.5. It is not difficult to see that the direct images of the constant motives on Bott–Samelson resolutions all belong to $\mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P)$. We will call the objects of this category the **Bott–Samelson motives**.

Lemma 6.6. Bott-Samelson motives are pointwise pure.

Proof. Pointwise purity is obviously satisfied for $(j_e)_*\underline{\mathrm{pt}} \in \mathrm{MTDer}_{(B)}^\mathrm{bs}(G/B)$ and is stable under $M \mapsto M(n)[2n]$ and direct summands. It is also extension-stable, because the heart of the weight structure on $\mathrm{MTDer}(\mathrm{pt})$ is extension-stable. It then suffices to show that pointwise purity is stable under push-forwards and pullbacks along projections $\pi: G/P \to G/Q$ where $P \subset Q$ are any two parabolic subgroups of G. Recall that π is a Zariski-locally trivial fiber bundle with fiber Q/P.

For pullbacks this is more or less evident: let $M \in \mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/Q)$ and assume that for each stratum $j_s: X_s \to G/Q$, we have $j_s^*M, j_s^!M \in \mathrm{MTDer}(X_s)_{w=0}$. We want to show that for each stratum $j_t: X_t \to G/P$, we have $j_t^*\pi^*M \in \mathrm{MTDer}(X_t)_{w=0}$. The projection is B-equivariant, and the fiber $X_s \times Q/P$ of π over X_s is a union of B-orbits in G/P. From the evident commutative diagram and the fact that $\mathrm{MTDer}(X_s) \cong \mathrm{MTDer}(k)$ because X_s is an affine space, we find that $j_t^*\pi^*M$ is the restriction of a motive from $\mathrm{MTDer}(X_s \times Q/P)_{w=0}$. Evidently, $j_t^*\pi^*M$ is pure of weight 0 for every stratum $j_t: X_t \to G/P$. By relative purity applied to the smooth projection $\pi: G/P \to G/Q$, the same statement also holds for $\pi^!M$.

We next consider the direct image functors. The inclusion of a B-orbit $j:D\hookrightarrow G/Q$ can be embedded into a commutative diagram

$$Q/P \longleftrightarrow Y \xrightarrow{j'} G/P$$

$$\downarrow \qquad \qquad \downarrow \pi$$

$$pt \longleftrightarrow D \xrightarrow{j} G/Q$$

in which both squares are pullback squares. As noted above, Q/P is the fiber of π over any point of D, and $Y \cong \pi^{-1}(D)$. By the B-equivariance of π , the B-orbits in Y are precisely the inverse images of the B-orbits in Q/P. By base change, we have $j^*\pi_*M\cong (\pi')_*(j')^*M$, and, because π is smooth and projective, similar statements for the other pullbacks and push-forwards. Since D is an affine space, we can identify MTDer(pt) \cong MTDer(D) and MTDer(Q/P) \cong MTDer(D), hence we are reduced to the case Q/P projecting to a point. But by Proposition 6.2 we know that fine $M \in$ MTDer(pt) $_{w=0}$ for any pointwise *-pure motive $M \in$ MTDer(D). This shows that pointwise purity is stable under direct image functors and finishes the proof.

Corollary 6.7. There is an equality of full subcategories

$$\mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P) = \mathrm{MTDer}_{(B)}(G/P)_{w=0}.$$

Proof. From Corollary 6.3, we find that MTDer^{bs} := MTDer^{bs}_(B)(G/P) is negative in the sense of [Bon13]. An induction on the dimension of the partial flag varieties shows that the smallest triangulated idempotent complete subcategory of MTDer_(B)(G/P) which contains MTDer^{bs} is MTDer_(B)(G/P) itself. From [Bon13, Proposition 1.7(6)], there is a unique weight structure on MTDer_(B)(G/P) such that MTDer^{bs} is pure of weight 0. By Proposition 5.1, the weight structure defined by MTDer^{bs} has to coincide with the weight structure of Hébert. The heart of this weight structure is then MTDer^{bs}, by [Bon13, Proposition 1.7(6)].

7. Pointwise purity via equivariance

In this section, we discuss another way of establishing the condition of pointwise purity that was so crucial in identifying the objects of the heart in Section 6. We adapt an argument of Springer [Spr84] to the motivic setting, showing that suitably equivariant motives on locally \mathbb{A}^1 -contractible G-varieties are pointwise pure.

Definition 7.1. Given a variety X with an action of an algebraic group G, a motive $M \in \mathcal{T}(X)$ is called **weakly** G-equivariant if and only if there exists an isomorphism

$$\operatorname{act}^* M \cong \operatorname{pr}^* M$$

of motives in $\mathcal{T}(G \times X)$. Here act, pr : $G \times X \to X$ denote the action and projection map, respectively.

Remark 7.2. We want to stress that the isomorphism in Definition 7.1 is **not** part of the data, nor do we require any compatibilities for it. Therefore, the condition of weak G-equivariance is indeed quite weak. By proper and smooth base change, we see easily that weak equivariance is preserved under f^* , f_* , $f^!$, $f_!$ for any G-equivariant morphism f. In particular, the Bott–Samelson motives of Section 6 are weakly G-equivariant.

The following is a straightforward translation of arguments of Springer [Spr84, Proposition 1 and Corollaries], repeating [Soe89, 1.3]. This leads to an alternative proof that Bott–Samelson motives are pointwise pure.

Proposition 7.3. Let X be a variety, let $Z \subset X$ be a closed subvariety, and assume that there exists an action $\mathbb{G}_m \times X \to X$ which contracts X onto Z. Let $M \in \mathcal{T}(X)$ be weakly \mathbb{G}_m -equivariant. Let $a: Z \hookrightarrow X$ denote the inclusion and $p: X \to Z$ the morphism mapping each point to its limit. Then in $\mathcal{T}(Z)$ there exists an isomorphism

$$p_*M \cong a^*M$$
.

Proof. We will prove the stronger claim that the adjunction map $M \to a_* a^* M$ becomes an isomorphism after applying p_* . To prove this, let $b: U \hookrightarrow X$ be the open embedding of the complement of Z. By the localization sequence

$$b_!b^!M \to M \to a_*a^*M \to b_!b^!M[1]$$

it will be sufficient to show $p_*b_!b^!M=0$.

In fact, we will show $p_*N=0$ for any weakly equivariant $N\in \mathcal{T}(X)$ with $a^*N=0$. The strategy is to construct an automorphism of p_*N that factors through zero. For a \mathbb{G}_{m} -action to contract to a subvariety Z means that the action $\mathbb{G}_{\mathrm{m}}\times X\to X$ can be extended to a morphism act : $\mathbb{A}^1\times X\to X$ such that we have act $\circ\kappa=a\circ p$ where $\kappa:X\to\mathbb{A}^1\times X:x\mapsto (0,x)$ is the 0-section, and $p:X\to Z$ is the morphism mapping each point to its limit. Consider now the morphism

$$\begin{array}{cccc} \tau: & \mathbb{A}^1 \times X & \to & \mathbb{A}^1 \times X \\ & (t,x) & \mapsto & (t, \operatorname{act}(t,x)) \end{array}$$

To make the notation more transparent, let us consider the commutative diagram

in which all morphisms except u are the product with suitable identities, so all squares are cartesian. The morphism u is the unit section $z \mapsto (1,z)$. By weak equivariance, there exists an isomorphism $\nu^*\pi^*N \cong \nu^*\tau^*\pi^*N$. On the other hand, we have $\kappa^*\tau^*\pi^*N = p^*a^*N = 0$ by assumption. The localization sequence for κ and ν thus gives us the third isomorphism of a chain of morphisms

$$\pi^*N \leftarrow \nu_! \nu^* \pi^* N \cong \nu_! \nu^* \tau^* \pi^* N \xrightarrow{\sim} \tau^* \pi^* N$$

with adjunction morphisms at the beginning. Clearly all these morphisms pull back to isomorphisms under ν^* . Applying q_* , we get a morphism $\alpha: q_*\tau^*\pi^*N \to q_*\pi^*N$, and base change shows $\mu^*(\alpha)$ is an isomorphism. On the other hand, the adjunction

morphism $\pi^*N \to \tau_*\tau^*\pi^*N$ also pulls back under ν to an isomorphism $\nu^*\pi^*N \stackrel{\sim}{\to} \nu^*\tau_*\tau^*\pi^*N$, and thus for the induced morphism $\beta: q_*\pi^*N \to q_*\tau_*\tau^*\pi^*N \stackrel{\sim}{\to} q_*\tau^*\pi^*N$ by the same argument $\mu^*(\beta)$ is an isomorphism. We have thus constructed a morphism

$$\alpha \circ \beta : q_*\pi^*N \to q_*\pi^*N$$

with the property, that $\mu^*(\alpha \circ \beta)$ is an isomorphism. Thus $u^*\mu^*(\alpha \circ \beta)$ has to be an isomorphism as well. Next we show $\omega_*(\alpha \circ \beta) = 0$. Since this factors through $\omega_*q_*\tau^*\pi^*N \cong p_*\pi_*\nu_!\nu^*\tau^*\pi^*N \cong p_*\pi_*\nu_!\nu^*\pi^*N$, it is sufficient to show that the latter object is zero. For this consider the localization triangle

$$\pi_* \nu_! \nu^* \pi^* N \to \pi_* \pi^* N \to \pi_* \kappa_* \kappa^* \pi^* N \to \pi_* \nu_! \nu^* \pi^* N[1]$$

and remark that its second arrow has to be an isomorphism, so the first term has to be zero. However by smooth base change, we get a canonical isomorphism $\omega^* p_* N \cong q_* \pi^* N$. Thus we may apply Lemma 7.4 below to our morphism $\alpha \circ \beta$ and deduce that, since $u^* \mu^* (\alpha \circ \beta)$ is an isomorphism, $\omega_* (\alpha \circ \beta)$ has to be an isomorphism, too. This however implies $0 = \omega_* q_* \pi^* N \cong \omega_* \omega^* p_* N \cong p_* N$ as claimed.

Lemma 7.4. Let Y be a variety and $\omega : \mathbb{A}^n \times Y \to Y$ the projection. Given $M \in \mathcal{T}(Y)$, the adjunction map is an isomorphism $\alpha_M : M \xrightarrow{\sim} \omega_* \omega^* M$. If in addition $s: Y \to \mathbb{A}^n \times Y$ is any section of the projection, in other words a morphism with $\omega \circ s = \mathrm{id}_Y$, then for any two objects $M, N \in \mathcal{T}(Y)$ and any morphism $f \in \mathcal{T}_{\mathbb{A}^n \times Y}(\omega^* M, \omega^* N)$ the obvious morphisms form a commutative diagram

8. Full faithfulness and combinatorial models

In this section, we adapt the arguments of Ginzburg [Gin91] to the motivic setting. We establish a full faithfulness result which allows to compute morphisms between pure stratified Tate motives in terms of maps between their bigraded motivic cohomology rings. This full faithfulness result will allow us to identify the category $MTDer_{(B)}(G/B)$ in terms of a homotopy category of Soergel modules.

8.1. The full faithfulness result now requires that we work in a motivic triangulated category \mathscr{T} which satisfies both the weight and grading conditions. In particular, the grading condition implies that $\mathscr{T}_{\mathrm{pt}}(\underline{\mathrm{pt}},\underline{\mathrm{pt}}(p)[q]) \neq 0$ only for p=q=0, in which case this is a one-dimensional vector space over \mathbb{Q} generated by the identity morphism of $\underline{\mathrm{pt}}$. For \mathbb{Q} -Modf $\mathbb{Z}^{\times\mathbb{Z}}$ the category of finite dimensional $(\mathbb{Z}\times\mathbb{Z})$ -graded \mathbb{Q} -vector spaces, we thus get an equivalence of \mathbb{Q} -linear monoidal categories

$$\mathbb{Q}\operatorname{-Modf}^{\mathbb{Z}\times\mathbb{Z}}\stackrel{\approx}{\to}\mathrm{MTDer}(\mathrm{pt})$$

mapping \mathbb{Q} sitting in bidegree (p,q) to the motive pt(p)[q].

Definition 8.2. Let (X, \mathcal{S}) be an affinely Whitney–Tate stratified variety. Given stratified mixed Tate motives $M, N \in \mathrm{MTDer}_{\mathcal{S}}(X)$ we define the bigraded vector space

$$\overline{\mathrm{MTDer}}_{\mathcal{S}}(M,N) := \bigoplus_{(i,j) \in \mathbb{Z} \times \mathbb{Z}} \mathscr{T}_X(M,N(i)[j])$$

This can also be interpreted as the bigraded vector space corresponding to the motive $\operatorname{fin}_*\operatorname{Hom}_X(M,N)$ under the equivalence 8.1.

We consider the bigraded ring

$$HX := \overline{\mathrm{MTDer}}_{\mathcal{S}}(X, X)$$

and the hypercohomology functor

$$\mathbb{H}: \mathrm{MTDer}_{\mathcal{S}}(X) \to \mathrm{Mod}_{\mathrm{H}X}: M \mapsto \overline{\mathrm{MTDer}}_{\mathcal{S}}(\underline{X}, M).$$

In the following, we will bootstrap Ginzburg's arguments from [Gin91] in the setting of motives, and the above bigraded cohomology rings. We fix some terminology to be used throughout the section. For a stratum X_s , we denote by $\nu_s: \overline{X_s} \to X$ the inclusion of its closure, by $j_s: X_s \to \overline{X_s}$ the inclusion of the stratum into its closure and by $i_s: \overline{X_s} \setminus X_s \to \overline{X_s}$ the inclusion of the closed complement. We use the notation fin rather freely, for all sorts of structure morphisms of k-varieties, trusting the readers to figure out on their own the variety belonging to the structure morphism.

We first establish an exact sequence as in [Gin91, Proposition 3.6].

Proposition 8.3. Let $L, M \in \mathrm{MTDer}_{\mathcal{S}}(X)$ be stratified mixed Tate motives, such that L is pointwise *-pure and M is pointwise !-pure. We set $L_s = \nu_s^* L$, $M_s = \nu_s^! M$. Then there is an exact sequence

$$0 \to \overline{\mathrm{MTDer}}_{\mathcal{S}}(i^*L_s, i^!M_s) \to \overline{\mathrm{MTDer}}_{\mathcal{S}}(L_s, M_s) \to \overline{\mathrm{MTDer}}_{\mathcal{S}}(j^*L_s, j^*M_s) \to 0.$$

Proof. By an induction on the number of strata, we can assume that $\overline{X}_s = X$. This allows to simplify notation to $j = j_s$ the inclusion of the open stratum and $i = i_s$ it closed complement.

Consider the internal Hom motive $\operatorname{Hom}(L,M)$ (in motives over X) and form the localization triangle

$$i_!i^!\operatorname{Hom}(L,M) \to \operatorname{Hom}(L,M) \to j_*j^*\operatorname{Hom}(L,M) \to [1]$$

By standard isomorphisms it can be transformed to a distinguished triangle

$$i_* \operatorname{Hom}(i^*L, i^!M) \to \operatorname{Hom}(L, M) \to i_* \operatorname{Hom}(j^*L, j^!M) \to [1]$$

Now as in the proof of Corollary 6.3 the object $\operatorname{Hom}(L,M)$ and its exceptional pullbacks $i^!\operatorname{Hom}(L,M)$ and $j^!\operatorname{Hom}(L,M)$ are pointwise !-pure. Applying fin_* will thus lead to a triangle of motives on the point, which are all pure of weight zero, so that the degree-one morphism has to vanish. Applying Definition 8.2 this establishes the required short exact sequence.

Theorem 8.4 (Full faithfulness of cohomology). Let (X, S) be an affinely Whitney-Tate stratified proper variety and let $L, M \in \mathrm{MTDer}_S(X)$ be pointwise *-pure and pointwise !-pure, respectively. Assume in addition that, for each embedding j of a stratum, $\mathbb{H}L \to \mathbb{H}j_*j^*L$ is surjective and $\mathbb{H}j_!j^!M \to \mathbb{H}M$ is injective. Then the hypercohomology functor induces a bijection

$$\overline{\mathrm{MTDer}}_{\mathcal{S}}(L,M) \overset{\sim}{\to} \mathrm{Mod}_{\mathrm{H}X}(\mathbb{H}L,\mathbb{H}M).$$

8.5. In the above, $\mathrm{Mod}_{\mathrm{H}X}$ means the vector space of all homomorphisms of $\mathrm{H}X$ modules, ignoring any gradings. Requiring the grading to be respected, we have
under the same conditions a bijection

$$\mathrm{MTDer}_{\mathcal{S}}(L,M) \overset{\sim}{\to} \mathrm{Mod}_{\mathrm{H}X}^{\mathbb{Z} \times \mathbb{Z}}(\mathbb{H}L,\mathbb{H}M)$$

between morphisms of stratified mixed Tate motives and morphisms of bigraded HX-modules which are homogeneous of bidegree (0,0). We discuss in Remark 8.9 below why the conditions of the theorem are satisfied for Bott–Samelson sheaves. In this special case, there is also an alternative proof comparing dimensions of the homomorphism spaces involved.

Proof. We first note that the morphism is simply given by applying hypercohomology: an element $f \in \overline{\mathrm{MTDer}}_{\mathcal{S}}(L, M)$ is a map $f : L \to M(i)[j]$, and the image of f in $\mathrm{Mod}_{\mathrm{H}X}(\mathbb{H}L, \mathbb{H}M)$ is $\mathbb{H}(f)$.

The proof is due to Ginzburg [Gin91], whose arguments we repeat. Let $u:D\hookrightarrow X$ be the embedding of an open stratum and $i:Z\hookrightarrow X$ the embedding of its closed complement. The proof consists of embedding our morphism as middle vertical in a commutative diagram

$$\begin{array}{ccccc} \overline{\mathrm{MTDer}}_{\mathcal{S}}(i^*L,i^!M) & \hookrightarrow & \overline{\mathrm{MTDer}}_{\mathcal{S}}(L,M) & \twoheadrightarrow & \overline{\mathrm{MTDer}}_{\mathcal{S}}(u^*L,u^*M) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Mod}_{\mathrm{HZ}}(\mathbb{H}i^*L,\mathbb{H}i^!M) & \hookrightarrow & \mathrm{Mod}_{\mathrm{HX}}(\mathbb{H}L,\mathbb{H}M) & \to & \mathrm{Mod}_{\mathrm{HD}}(\mathbb{H}u^*L,\mathbb{H}u^*M) \end{array}$$

with the upper row short exact, the lower row left exact, and all vertical maps given by the corresponding hypercohomology functors. Once this is established, the left vertical is an isomorphism by an induction on the number of strata, for the right vertical this is clear anyhow, and by a diagram chase we are done. So the problem is to construct the horizontal maps and show the required exactness of the horizontal sequences. The upper sequence is established in Proposition 8.3.

To discuss the lower horizontal, let $c \in HX$ be the class corresponding to the open cell D. This class is given by the following composition, where $z : \operatorname{pt} \hookrightarrow X$ denotes the inclusion of any point of the cell D and d denotes the dimension of the cell $D \cong \mathbb{A}^d$:

$$\underline{X} \to z_* z^* \underline{X} \stackrel{\sim}{\to} z_! z^! \underline{X}(d)[2d] \to \underline{X}(d)[2d]$$

Note that, given a cohomology class $\gamma: \underline{X} \to \underline{X}(q)[p]$, its effect on the hypercohomology of a motive \mathcal{F} can be realized as hypercohomology of the morphism $\gamma \otimes \mathrm{id}: \underline{X} \otimes \mathcal{F} \to \underline{X}(q)[p] \otimes \mathcal{F}$ up to the natural identification $\underline{X} \otimes \mathcal{F} \cong \mathcal{F}$. In our case, the action of the class c on the hypercohomology of a stratified motive \mathcal{F} is thus induced from the composition

$$\mathcal{F} \to z_* z^* \mathcal{F} \stackrel{\sim}{\to} z_! z^! \mathcal{F}(d)[2d] \to \mathcal{F}(d)[2d]$$

where the first and third maps are units and counits of the respective adjunctions and the middle isomorphism is due to \mathcal{F} being constant on the open cell D. The outer morphisms admit natural factorizations as $\mathcal{F} \to u_* u^* \mathcal{F} \to z_* z^* \mathcal{F}$ and $z_! z^! \mathcal{F}(d)[2d] \to u_! u^! \mathcal{F}(d)[2d] \to \mathcal{F}(d)[2d]$, respectively. On the total bigraded hypercohomology, these induce isomorphisms

$$\mathbb{H}(u_*u^*\mathcal{F}) \stackrel{\sim}{\to} \mathbb{H}(z_*z^*\mathcal{F}) \text{ and } \mathbb{H}(z_!z^!\mathcal{F}(d)[2d]) \stackrel{\sim}{\to} \mathbb{H}(u_!u^!\mathcal{F}(d)[2d]),$$

since X is assumed to be proper. Applying this to the motives M and N, we get commutative diagrams

The upper surjection in the upper diagram and the lower injection in the lower diagram are from the assumptions. The horizontal exact sequences are obtained as in the proof of Proposition 8.3, using our pointwise purity assumptions on L and M. For the upper diagram, we need to dualize, which is ok because X is proper and hence $\operatorname{fin}_* \cong \operatorname{fin}_!$.

These diagrams lead to isomorphisms (im $c: \mathbb{H}L \to \mathbb{H}L$) $\stackrel{\sim}{\to} \mathbb{H}u_*u^*L \stackrel{\sim}{\to} \mathbb{H}u^*L$ and (im $c: \mathbb{H}M \to \mathbb{H}M$) $\stackrel{\sim}{\to} \mathbb{H}u_*u^*M \stackrel{\sim}{\to} \mathbb{H}u^*M$. For the lower right horizontal in our diagram from the beginning of the proof we then just take the map restricting a module homomorphism to the induced homomorphism on im c.

We have to check that the right square commutes. The map $\overline{\text{MTDer}}_{\mathcal{S}}(L, M) \to \overline{\text{MTDer}}_{\mathcal{S}}(u^*L, u^*M)$ comes from u^* -restriction of the inner Hom. Since the map c

in the diagram is similarly defined via the restriction functors, the right square of the diagram commutes. The lower left horizontal in our diagram from the beginning of the proof comes from the natural morphisms $L \to i_* i^* L$ and $i_! i^! M \to M$ and is an injection, since we have $\mathbb{H}L \to \mathbb{H}i_* i^* L$ and $\mathbb{H}i_! i^! M \hookrightarrow \mathbb{H}M$ by the above. The composition in the lower horizontal is clearly zero. The only thing left to show is that in the middle each element in the kernel also belongs to the image. Now if $f: \mathbb{H}L \to \mathbb{H}M$ goes to zero, it will obviously factor as $\mathbb{H}L \to (\operatorname{cok} c) \to (\ker c) \subset \mathbb{H}M$. But the left diagram above gives us a natural isomorphism $(\operatorname{cok} c) \overset{\sim}{\to} \mathbb{H}i_* i^* L$ and the right diagram above shows that $(\ker c)$ is the image of $\mathbb{H}i_! i^! M \hookrightarrow \mathbb{H}M$. Thus f will actually come from some $\tilde{f}: \mathbb{H}i_* i^* L \to \mathbb{H}i_! i^! M$ as claimed.

Theorem 8.6. Let $G \supset P \supset B$ be a reductive algebraic group over k with a choice of Borel subgroup B and parabolic subgroup P. Then on the heart of the weight structure $\mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P) = \mathrm{MTDer}_{(B)}(G/P)_{w=0}$ from Corollary 6.7 the hypercohomology functor $\mathbb H$ from Definition 8.2 restricts to a fully faithful functor

$$\mathbb{H}: \mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P) \overset{\sim}{\hookrightarrow} \mathrm{Mod}_{\mathrm{H}(G/P)}^{\mathbb{Z} \times \mathbb{Z}}$$

8.7. Since fin! = fin* has to preserve weights, it is clear that the modules in the image of our functor will only live in bidegrees (2j, j). If we just keep the first, i.e., the cohomological grading, the category of graded modules over the cohomology ring $H^*(G/P)$ of the flag variety forming the essential image of our functor will be denoted $H^*(G/P)$ -SMod $_{\text{ev}}^{\mathbb{Z}}$. It consists of the modules with even grading in a category of graded modules sometimes called "Soergel modules".

Proof. We apply Theorem 8.4 on the full faithfulness of hypercohomology and have to check that the conditions needed are satisfied. We already know from Lemma 6.6 or alternatively Proposition 7.3 that Bott–Samelson sheaves are pointwise pure. The remaining conditions are easily deduced from Remark 8.9 below. \Box

Proposition 8.8. Let X be a proper variety and let $M \in \mathcal{F}^c(X)_{w=0}$ be pure. Let $v: V \hookrightarrow X$ be the embedding of an open subset and suppose there is an action of \mathbb{G}_{m} on V contracting V to a fixed point $x \in V$, for which M is weakly equivariant. Then for the inclusion $i: x \hookrightarrow X$ the obvious map is a surjection

$$\mathbb{H}M \to \mathbb{H}i_*i^*M.$$

Proof. This is due to Ginzburg [Gin91], whose arguments we repeat. By Proposition 7.3 the contraction induces an isomorphism $\operatorname{fin}_* v^*M \stackrel{\sim}{\to} i^*M$ and both sides are pure of weight zero. If we now let r be the embedding of the complement of V, we get a distinguished triangle

$$\operatorname{fin}_* r_! r^! M \to \operatorname{fin}_* M \to \operatorname{fin}_* v_* v^* M \to \operatorname{fin}_* r_! r^! M[1].$$

Here the degree one morphism has to vanish, since both $r^!$ and $r_! = r_*$ never make weights smaller, so we get a short exact sequence

$$0 \to \mathbb{H} r_! r^! M \hookrightarrow \mathbb{H} M \to \mathbb{H} v_* v^* M \to 0$$

and in particular a surjection $\mathbb{H}M \to \mathbb{H}i_*i^*M$.

Remark 8.9. If in addition the inclusion of the point x factors over the inclusion of an affine space $j:D\hookrightarrow X$ and $j^*M\in \mathrm{MTDer}(D)$ is mixed Tate, then our surjection factors as $\mathbb{H}M\to \mathbb{H}j_*j^*M\stackrel{\sim}{\to} \mathbb{H}i_*i^*M$ and thus the first map has to be a surjection as well. Dual arguments show the injectivity $\mathbb{H}i_!i^!M\hookrightarrow \mathbb{H}M$ and of $\mathbb{H}j_!j^!M\hookrightarrow \mathbb{H}M$ under the dual assumptions.

Remark 8.10. Similar results hold under more general assumptions. For instance, if k is a number field and $\mathcal{T} = \mathrm{DM_B}$ or $\mathcal{T} = \mathbf{DA_{\mathrm{\acute{e}t}}}$, then the grading condition is not satisfied. Nevertheless, a full faithfulness result as above remains true. It expresses morphisms between stratified mixed Tate motives in terms of morphisms between the associated motivic cohomology rings. However, the actual description of the motivic cohomology of the flag variety is more complicated. It combines the motivic cohomology of the base field (which is quite nontrivial) with the cell structure information of the flag variety. Since it is not clear if full faithfulness in such more general situations can be useful, we chose not to spell out the details.

9. Tilting for motives

9.1. In this section, we require that the motivic triangulated category \mathscr{T} is one of the following: $\mathscr{T} = \mathrm{DM_B}$, $\mathscr{T} = \mathrm{DA_{\acute{e}t}}$ or $\mathscr{T} = \mathcal{E}_{\mathrm{GrH}}$. All these categories satisfy both the weight and grading condition. However, for the tilting results in this section, we need additional information on how the motivic triangulated categories are constructed. All the above categories are constructed as suitable localizations of abelian categories of (symmetric spectra in) complexes of sheaves on a site. In this situation, we can apply the tilting result Proposition B.1. This is the only place of the paper where we need such explicit information on the construction, and can not make do with the axiomatics of motivic triangulated categories. Sorry.

Theorem 9.2. Let (X, S) be a Whitney-Tate affinely stratified variety. Assume the setting laid out in 9.1. Assume furthermore that all objects of $MTDer_S(X)_{w=0}$ are pointwise pure. Then the tilting functor, cf. Proposition B.1, induces an equivalence

$$\operatorname{Hot^b}(\operatorname{MTDer}_{\mathcal{S}}(X)_{w=0}) \xrightarrow{\approx} \operatorname{MTDer}_{\mathcal{S}}(X)$$

between the category of stratified mixed Tate motives on X and the bounded homotopy category of the heart of the weight structure.

9.3. Remark that by [Bon13, Proposition 1.7(6)] two weight structures on an idempotent complete triangulated category with the same heart are equal, if this heart already generates the whole triangulated category in question. Now the obvious embedding $\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0} \hookrightarrow \mathrm{Hot}^{\mathrm{b}}(\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0})$ as complexes concentrated in degree zero induces an equivalence with the heart of the obvious weight structure on $\mathrm{Hot}^{\mathrm{b}}$. On the other hand, its composition with the equivalence of the theorem also induces an equivalence with the heart of the motivic weight structure on MTDer, since by construction this composition is isomorphic to the embedding of the heart of the weight structure. Thus under the equivalence of the theorem the obvious weight structure on $\mathrm{Hot}^{\mathrm{b}}$ coincides with the motivic weight structure on MTDer.

Proof. This is a special case of the general tilting equivalence from Proposition B.1. Repeating the proof of Corollary 6.3, for any two pointwise pure stratified mixed Tate motives $M, N \in \mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}$, we deduce $\mathrm{MTDer}_{\mathcal{S}}(M, N[a]) = 0$ for $a \neq 0$ from the grading condition and thus by 8.1 there are no nonzero morphisms between objects of different weight in $\mathrm{MTDer}(\mathrm{pt})$. Now, in the situation fixed in 9.1, $\mathcal{T}(X)$ is constructed from $\mathrm{Der}(\mathrm{Sh}_{\tau}(\mathrm{Sm}/S,\mathbb{Q}))$ by \mathbb{A}^1 -localization, stabilization via symmetric spectra and possibly a further Bousfield localization at $\mathrm{H_B}$ or $\mathscr{E}_{\mathrm{GrH}}$. In particular, $\mathcal{T}(X)$ can be embedded as a full subcategory of the derived category of an abelian category: the abelian category is the one of symmetric sequences in $\mathrm{Sh}_{\tau}(\mathrm{Sm}/S,\mathbb{Q})$. Finally, $\mathrm{MTDer}_{\mathcal{S}}(X)$ embeds by definition as full subcategory of $\mathcal{T}(X)$. Using this embedding, it is possible to choose injective resolutions for the objects of $\mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}$. These form a tilting collection satisfying all the conditions necessary to apply Proposition B.1. This implies the existence of a fully

faithful functor

$$\operatorname{Hot^b}(\operatorname{MTDer}_{\mathcal{S}}(X)_{w=0}) \stackrel{\sim}{\hookrightarrow} \operatorname{MTDer}_{\mathcal{S}}(X)$$

The heart of the weight structure on $\mathrm{MTDer}_{\mathcal{S}}(X)$ generates the category, therefore the functor is also essentially surjective.

Corollary 9.4. Let $G \supset P \supset B$ be a split reductive algebraic group over the field k, with a choice of Borel subgroup B and parabolic subgroup P, and let Y be a paraboloid B-variety. Then the tilting functor of Proposition B.1 provides an equivalence of categories

$$\operatorname{Hot^b}(\operatorname{MTDer}_{(B)}(Y)_{w=0}) \stackrel{\approx}{\to} \operatorname{MTDer}_{(B)}(Y)$$

between the bounded homotopy category of the additive category of pure Bruhat-Tate sheaves and the triangulated category of all Bruhat-Tate sheaves. For Y = G/P, we obtain an equivalence of triangulated categories

$$\operatorname{Hot^b}(\operatorname{H}^*(G/P)\operatorname{-SModf}_{\operatorname{ev}}^{\mathbb{Z}})\stackrel{\approx}{\to} \operatorname{MTDer}_{(B)}(G/P)$$

between the bounded homotopy category of even Soergel modules and stratified mixed Tate motives over G/P.

Proof. By Corollary 6.7 we have $\mathrm{MTDer}_{(B)}^{\mathrm{bs}}(G/P) = \mathrm{MTDer}_{(B)}(G/P)_{w=0}$ and by Lemma 6.6 all objects of this category are pointwise pure. The same statements follow easily for any paraboloid B-variety, and thus the first equivalence is a special case of Theorem 9.2. The second equivalence follows using the faithfulness Theorem 8.6 in conjunction with the definition of Soergel modules from 8.7.

10. Perverse Tate motives

In this section, we describe a t-structure on the category $\operatorname{MTDer}_{\mathcal{S}}(X)$ of stratified mixed Tate motives, for (X,\mathcal{S}) an affinely Whitney–Tate stratified variety. The t-structure is obtained via the BBD-glueing formalism [BBD82] from the t-structure on mixed Tate motives $\operatorname{MTDer}(k)$, which exists for base fields satisfying the Beilinson–Soulé vanishing conjectures. The heart of the t-structure is an abelian category of perverse mixed Tate motives. In the next section, we will show that the perverse mixed Tate motives provide a grading on category \mathcal{O} .

- 10.1. In this section, we assume that the motivic triangulated category satisfies the grading condition. Alternatively, working with étale or Beilinson motives, the results also work if we assume that the ground field k satisfies the Beilinson–Soulé vanishing conjectures.
- 10.2. Using the work of Levine [Lev93], this assumption implies that the categories MTDer(X_s) of mixed Tate motives on the strata $X_s \cong \mathbb{A}^{n_s}$ have non-degenerate t-structures. For a more detailed recollection of the motivic t-structures and abelian categories of mixed Tate motives, see Section 3.

Theorem 10.3. Let (X, S) be an affinely Whitney-Tate stratified variety. For any perversity function $p: S \to \mathbb{Z}$ the following subcategories define a t-structure on $\mathrm{MTDer}_S(X)$:

$$\mathrm{MTDer}_{\mathcal{S}}(X)^{\leq 0} := \left\{ M \mid j_s^*M \in \mathrm{MTDer}(X_s)^{\leq p(s)} \text{ for all strata } s \in \mathcal{S} \right\}$$

$$\mathrm{MTDer}_{\mathcal{S}}(X)^{\geq 0} := \{ M \mid j_s^! M \in \mathrm{MTDer}(X_s)^{\geq p(s)} \text{ for all strata } s \in \mathcal{S} \}$$

Proof. The proof proceeds by induction on the number of strata. For the base case, we can use the t-structure given by Theorem 3.7.

Otherwise choose an open stratum $j:U\hookrightarrow X$ and its closed complement $i:Z\hookrightarrow X$. By inductive assumption, we have a non-degenerate t-structure on MTDer_S($\overline{X_s}$ \

 X_s). On the open stratum U, we have a t-structure on MTDer_S(Z), again from Theorem 3.7.

We want to glue these two t-structures to obtain a t-structure on $\mathrm{MTDer}_{\mathcal{S}}(X)$ with

$$\operatorname{MTDer}_{\mathcal{S}}(X)^{\leq 0} := \left\{ M \mid i^*M \in \operatorname{MTDer}(Z)^{\leq 0}, \ j_t^*M \in \operatorname{MTDer}(X_t)^{\leq p(t)} \right\}$$

$$\operatorname{MTDer}_{\mathcal{S}}(X)^{\geq 0} := \left\{ M \mid i^!M \in \operatorname{MTDer}(Z)^{\geq 0}, \ j_t^!M \in \operatorname{MTDer}(X_t)^{\geq p(t)} \right\}$$

The claim that this is indeed a non-degenerate t-structure on MTDer_S($\overline{X_s}$) is a consequence of [BBD82, Theorem 1.4.10] once we verify the axioms [BBD82, 1.4.3].

Some proofs of parts of the axioms are deferred to the following subsection. The first two axioms, 1.4.3.1 and 1.4.3.2, are satisfied by the assumption and Proposition 10.5. The axioms 1.4.3.3 and 1.4.3.5 are easy to see, using basic properties of the six-functor formalism for motives. With all the functors restricting to the subcategories MTDer_S, the localization sequence of the motivic triangulated category \mathscr{T} also restricts to the triangulated subcategories MTDer_S, hence we also have axiom 1.4.3.4.

It is then clear that this t-structure can also be described by the non-inductive formulas given in the proposition. \Box

10.4. We are only interested in the case of the so-called middle perversity given by $p(s) = -\dim X_s$. For this perversity, we denote the heart of the corresponding t-structure by

$$\mathrm{MTPer}_{\mathcal{S}}(X)$$

and call its objects perverse mixed Tate motives on X.

Proposition 10.5. Let (X, S) be an affinely Whitney–Tate stratified variety. Then we have the following:

(1) The functor $j^*: \mathcal{T}(\overline{X_s}) \hookrightarrow \mathcal{T}(X_s)$ restricts to a functor

$$j^*: \mathrm{MTDer}_{\mathcal{S}}(\overline{X_s}) \to \mathrm{MTDer}(X_s).$$

(2) The functors $i^*, i^! : \mathcal{T}(\overline{X_s}) \hookrightarrow \mathcal{T}(\overline{X_s} \setminus X_s)$ restrict to functors

$$i^*, i^! : \mathrm{MTDer}_{\mathcal{S}}(\overline{X_s}) \to \mathrm{MTDer}_{\mathcal{S}}(\overline{X_s} \setminus X_s).$$

In particular, the adjunction conditions [BBD82, 1.4.3.1 and 1.4.3.2] are satisfied.

- *Proof.* (1) By the definition of MTDer_S($\overline{X_s}$) as triangulated subcategory generated by the images of $i_* = i_!$, j_* and $j_!$ and the fact that j^* is a triangulated functor, it suffices to prove the assertion for these generators. For elements of the form j_*M and $j_!M$, the claim follows from the well-known identifications $j^*j_* \cong \operatorname{id} \cong j^*j_!$ and $i^*i_* \cong \operatorname{id} \cong i^!i_*$. For $M \in \mathrm{MTDer}_S(Z)$, we have $j^*i_* = 0$. Hence the claim follows.
- (2) Now let $M \in \mathrm{MTDer}(U)$. We want to prove that the images of the functors i^*j_* and $i^!j_!$ lie in $\mathrm{MTDer}_{\mathcal{S}}(\overline{X_s} \setminus X_s)$. Since i^*j_* is dual to $i^!j_!$ and the motivic duality restricts to $\mathrm{MTDer}_{\mathcal{S}}(\overline{X_s} \setminus X_s)$, it suffices to prove one of the assertions.

We prove by induction that for each stratum X_t in $\overline{X_s} \setminus X_s$ with inclusion $i_t: \overline{X_t} \hookrightarrow \overline{X_s}$ we have $i_t^*j_*M \in \mathrm{MTDer}_{\mathcal{S}}(\overline{X_t})$. This will in particular prove the claim. For a closed stratum X_t in $\overline{X_s}$, this follows from the assumption that (X,\mathcal{S}) is Whitney–Tate. For the inductive step, we use the localization sequence of the motivic triangulated category \mathscr{T} on $\mathscr{T}(\overline{X_t})$. We denote by $i_Y:Y=\overline{X_t}\setminus X_t\hookrightarrow \overline{X_t}$ the closed immersion and by $j_t:X_t\hookrightarrow \overline{X_t}$ its open complement. The inductive assumption is that the claim is true for Y, i.e., $i_Y^*i_t^*j_*M\in\mathrm{MTDer}_{\mathcal{S}}(Y)$. Then, by assumption that (X,\mathcal{S}) is Whitney–Tate again, we also have $j_t^*i_t^*j_*M\in\mathrm{MTDer}(X_t)$. The localization sequence decomposes $i_t^*j_*M$ as

$$(j_t)_! j_t^* i_t^* j_* M \to i_t^* j_* M \to (i_Y)_* i_Y^* i_t^* j_* M \to (j_t)_! j_t^* i_t^* j_* M$$
[1].

By what was said above, the first and third term are in MTDer_S($\overline{X_t}$), which proves the claim.

Finally, the Axioms 1.4.3.1 and 1.4.3.2 follow since all the six functors restrict to the categories MTDer_S. Since these are full subcategories, the corresponding adjunctions between the functors also restrict to MTDer_S. \Box

We list some of the further consequences of the glueing formalism for t-structures from [BBD82, Section 1.4].

First of all, we note that there are modified versions of the six functors. For a stratified scheme X, a stratum X_s and the inclusions $i: \overline{X_s} \setminus X_s \hookrightarrow \overline{X_s}$ and $j: X_s \hookrightarrow \overline{X_s}$, we can define the following functors:

$${}^{p}j_{!}, {}^{p}j_{*}: \mathrm{MT}(X_{s}) \leftrightarrows \mathrm{MTPer}_{\mathcal{S}}(\overline{X_{s}}): {}^{p}j^{!} = {}^{p}j^{*}$$
 ${}^{p}i_{!} = {}^{p}i_{*}: \mathrm{MTPer}_{\mathcal{S}}(\overline{X_{s}} \setminus X_{s}) \leftrightarrows \mathrm{MTPer}_{\mathcal{S}}(\overline{X_{s}}): {}^{p}i^{!}, {}^{p}i^{*}.$

These form adjunctions ${}^p j_! \dashv {}^p j^* \dashv {}^p j_*$ and ${}^p i^* \dashv {}^p i_* \dashv {}^p i^!$, cf. [BBD82, Proposition 1.4.16]. There is also a modified analogue of the localization sequences: for each perverse mixed Tate motive $M \in \mathrm{MTPer}_{\mathcal{S}}(\overline{X_w})$, there are by [BBD82, Lemma 1.4.19] exact sequences

$$0 \to {}^{p}i_{*}\mathcal{H}^{-1}i^{*}M \to {}^{p}j_{!}{}^{p}j^{*}M \to M \to {}^{p}i_{*}{}^{p}i^{*}M \to 0$$
$$0 \to {}^{p}i_{*}{}^{p}i^{!}M \to M \to {}^{p}i_{*}{}^{p}j^{*}M \to {}^{p}i_{*}\mathcal{H}^{1}i^{!}M \to 0.$$

As in [BBD82, Definition 1.4.22], we can define a functor "intermediate extension" as

$$j_{!*}: \mathrm{MT}(X_s) \to \mathrm{MTPer}_{\mathcal{S}}(\overline{X_s}): M \mapsto \mathrm{Im}\left({}^p j_! M \to {}^p j_* M\right).$$

Note that an intermediate extension of Chow motives has already been considered in [Wil12] also in situations where the motivic t-structure is not available.

Finally, [BBD82, Proposition 1.4.26] characterizes the simple perverse mixed Tate motives in MTPer_S(\overline{X}_s) as those of the form $p_{i_*}M$ for M a simple perverse mixed Tate motive in MTPer_S($\overline{X}_s \setminus X_s$) and those of the form $j_{!*}\mathbb{Q}(a)[-p(s)]$, $a \in \mathbb{Z}$. The representation-theoretic significance of these objects, the intersection complexes, will be discussed in the next section.

11. MOTIVIC GRADED VERSIONS

In this section, we discuss graded versions of category \mathcal{O} arising from motivic triangulated categories. We consider two versions, an ℓ -adic and a Hodge version.

11.1. For the ℓ -adic version, let $k = \mathbb{F}_q$ be a finite field, and consider the motivic triangulated categories $\mathscr{T} = \mathrm{DM}_{\mathrm{B}}$ or $\mathscr{T} = \mathbf{DA}_{\mathrm{\acute{e}t}}$ over Sch/k . Let us consider a prime ℓ different from the characteristic of k. For a k-variety X we consider the derived category $\mathrm{Der}(X \times_k \bar{k}; \mathbb{Q}_\ell)$ of the category of ℓ -adic sheaves on $X \times_k \bar{k}$. The ℓ -adic realization of [CD12b] followed by pulling back to the geometric situation gives triangulated functors

$$\operatorname{Real}_{\ell}: \operatorname{DM}_{\operatorname{B},c}(X; \mathbb{Q}_{\ell}) \to \operatorname{Der}^{\operatorname{b}}(X \times_{k} \bar{k}; \mathbb{Q}_{\ell})$$

compatible with all six functors of Grothendieck, where we take motives with \mathbb{Q}_{ℓ} -coefficients for better compatibility. For an affinely Whitney–Tate stratified variety (X, \mathcal{S}) , we denote by $\mathrm{Der}_{\mathcal{S}}(X \times_k \bar{k}; \mathbb{Q}_{\ell}) \subset \mathrm{Der}^{\mathrm{b}}(X \times_k \bar{k}; \mathbb{Q}_{\ell})$ the full triangulated subcategory of all complexes whose restrictions to all strata are constant of finite rank. Then the above realizations induce triangulated functors

$$\operatorname{Real}_{\ell}: \operatorname{MTDer}_{\mathcal{S}}(X; \mathbb{Q}_{\ell}) \to \operatorname{Der}_{\mathcal{S}}(X \times_{k} \bar{k}; \mathbb{Q}_{\ell})$$

Any choice of an isomorphism $\operatorname{Real}_{\ell}(\underline{\operatorname{pt}}(1)) \cong \operatorname{Real}_{\ell}(\underline{\operatorname{pt}})$ leads to natural isomorphisms $\operatorname{Real}_{\ell} \mathcal{F}(n) \xrightarrow{\sim} \operatorname{Real}_{\ell} \mathcal{F}$.

11.2. For the Hodge version, let $k = \mathbb{C}$ and consider the motivic triangulated categories $\mathscr{T} = \mathcal{E}_{GrH}$ over Sch /k. For a complex variety X we consider the derived category $Der(X;\mathbb{C})$ of the category of sheaves of \mathbb{C} -vector spaces on X. The Hodge realization of [Dre13] gives triangulated functors

$$\operatorname{Real}_H: \mathscr{T}(X;\mathbb{C}) \to \operatorname{Der}^{\mathrm{b}}(X;\mathbb{C})$$

compatible with all six functors of Grothendieck. For an affinely Whitney–Tate stratified variety (X, \mathcal{S}) , we denote by $\mathrm{Der}_{\mathcal{S}}(X; \mathbb{C}) \subset \mathrm{Der}^{\mathrm{b}}(X; \mathbb{C})$ the full triangulated subcategory of all complexes whose restrictions to all strata are constant of finite rank. Then the above realizations induce triangulated functors

$$\operatorname{Real}_H: \operatorname{MTDer}_{\mathcal{S}}(X; \mathbb{C}) \to \operatorname{Der}_{\mathcal{S}}(X; \mathbb{C})$$

Any choice of an isomorphism $\operatorname{Real}_H(\underline{\operatorname{pt}}(1)) \cong \operatorname{Real}_H(\underline{\operatorname{pt}})$ leads to natural isomorphisms $\operatorname{Real}_H \mathcal{F}(n) \xrightarrow{\sim} \operatorname{Real}_H \mathcal{F}.$

Note that in both these cases, the weight and grading condition on the motivic triangulated category \mathcal{T} are satisfied, so that all the previously established results are applicable.

Theorem 11.3. Let (X, S) be an affinely Whitney-Tate stratified variety.

(1) In the situation of 11.1, for any $\mathcal{F}, \mathcal{G} \in \mathrm{MTDer}_{\mathcal{S}}(X; \mathbb{Q}_{\ell})$, the realization functor together with the isomorphisms in 11.1 above leads to isomorphisms

$$\bigoplus_{n\in\mathbb{Z}}\mathrm{MTDer}_{\mathcal{S}}(\mathcal{F},\mathcal{G}(n))\overset{\sim}{\to}\mathrm{Der}_{\mathcal{S}}(\mathrm{Real}_{\ell}\,\mathcal{F},\mathrm{Real}_{\ell}\,\mathcal{G}).$$

(2) In the situation of 11.2, for any $\mathcal{F}, \mathcal{G} \in \mathrm{MTDer}_{\mathcal{S}}(X;\mathbb{C})$, the realization functor together with the isomorphisms in 11.2 above leads to isomorphisms

$$\bigoplus_{n\in\mathbb{Z}} \mathrm{MTDer}_{\mathcal{S}}(\mathcal{F},\mathcal{G}(n)) \xrightarrow{\sim} \mathrm{Der}_{\mathcal{S}}(\mathrm{Real}_H \,\mathcal{F},\mathrm{Real}_H \,\mathcal{G})$$

Proof. We give the proof of (1), the proof of (2) is similar. We know from Section 4 that MTDer_S is generated as a triangulated category by the shifted twisted costandard objects $j_{s!}\underline{X}_s(n)$ as well as by the shifted twisted standard objects $j_{s*}\underline{X}_s(m)$. By devissage, it is sufficient to check the claim for \mathcal{F} costandard and \mathcal{G} standard. In this case however, we can use base change to switch to the case of a single stratum, which follows from 8.1: the identification of morphisms in MTDer(\underline{pt}) with Adams eigenspaces of Quillen K-theory implies that $\mathrm{DM}_{\mathbb{F}}(\underline{pt},\underline{pt}(p)[q]) \neq 0$ only for p=q=0, in which case this is a one-dimensional vector space over \mathbb{Q} generated by the identity morphism of \underline{pt} , and then the claim follows from homotopy invariance.

11.4. Let (X, S) be an affinely Whitney–Tate stratified variety. By compatibility with the six functors, the realization from 11.1 or 11.2 induces an exact functor between the corresponding categories of perverse sheaves

$$\operatorname{Real}_{\ell}: \operatorname{MTPer}_{\mathcal{S}}(X; \mathbb{Q}_{\ell}) \to \operatorname{Perv}_{\mathcal{S}}(X \times_{k} \bar{k}; \mathbb{Q}_{\ell})$$

$$\operatorname{Real}_H: \operatorname{MTPer}_{\mathcal{S}}(X;\mathbb{C}) \to \operatorname{Perv}_{\mathcal{S}}(X;\mathbb{C})$$

Clearly, a stratified mixed Tate motive in MTDer_S(X) is perverse if and only if its realization is perverse. We deduce from [BBD82, 4.1.3] that the costandard objects $\Delta_s := j_{s!} \underline{X}_s [\dim X_s]$ as well as the standard objects $\nabla_s := j_{s*} \underline{X}_s [\dim X_s]$ are actually perverse motives, i.e., they belong to MTPer_S(X). As an aside, let us remark that the last statement even follows with \mathbb{Q} -coefficients.

Remark 11.5. It would be much more satisfying to have a "motivic" proof that the standard and costandard objects are perverse, without having to resort to checking it on étale realization. However, this would require a version of Artin vanishing in the motivic setting, which at the moment does not seem to be known. We thank Rahbar Virk for discussions about this point. Actually, in the case of $\mathscr{T} = \mathcal{E}_{GrH}$, it might actually be possible to translate the statements known in complex geometry to the "motivic setting", but we have not checked that.

Lemma 11.6. Assume the situation in 11.1 or 11.2, and let (X,S) be an affinely Whitney-Tate stratified variety. Consider the category $MTPer_S(X)$, i.e., we take perverse motives for the middle perversity. Let $j: U \to X$ be an open stratum of dimension d.

- (1) The object $j_{!*}\mathbb{Q}[d]$ is simple.
- (2) The object $j_!\mathbb{Q}[d]$ is the projective cover of $j_{!*}\mathbb{Q}[d]$.
- (3) The object $j_*\mathbb{Q}[d]$ is the injective hull of $j_{!*}\mathbb{Q}[d]$.

Proof. We have seen in 11.4 above (using ℓ -adic realization) that all the objects appearing are indeed perverse motives, i.e., that $j_{!*}\mathbb{Q}[d]$, $j_{!}\mathbb{Q}[d]$ and $j_{*}\mathbb{Q}[d]$ are in MTPer_S(X).

As mentioned earlier, (1) is a consequence of [BBD82, Proposition 1.4.26]. The statements (2) and (3) are dual, we only prove (2).

We first note that $\mathbb{Q}[d]$ is a projective object in MTDer(k): by assumption, we can identify MTDer(k) with the bounded derived category of graded \mathbb{Q} -vector spaces (with homogeneous maps). The category MT(k)[d] then consists of graded vector spaces, considered as complexes concentrated in degree d. In that case, projectivity of \mathbb{Q} is obvious.

Now we discuss projectivity of $j_!\mathbb{Q}[d]$. We are grateful to Rahbar Virk for pointing out the following argument. To prove projectivity it suffices to show vanishing of $\operatorname{Ext}^1(j_!\mathbb{Q}[d], M) = 0$, where Ext^1 is to be interpreted as morphisms in the derived category $\operatorname{Der}_{\operatorname{MTPer}_{\mathcal{S}}(X)}(j_!\mathbb{Q}[d], M[1])$. The latter can be identified with Yoneda Ext^1 , and via [BBD82, Corollary 1.1.10, Theorem 1.3.6] with Ext^1 in the category $\operatorname{MTDer}_{\mathcal{S}}(X)$. Using the adjunctions of the six-functor formalism and the vanishing of $j^*i_!$ and $i^*j_!$, we find $\operatorname{Ext}^1(j_!\mathbb{Q}[d],(i_s)_{!*}\mathbb{Q}[1]) = 0$ for any stratum other than the open. On the open stratum, the vanishing of the Ext^1 follows from $\mathbb{Q}[d]$ being projective in $\operatorname{MTDer}(k)$ and homotopy invariance.

To see that $j_!\mathbb{Q}[d] \to j_{!*}\mathbb{Q}[d]$ is the projective cover, we also use an adjunction argument. It follows from short exact sequences before [BBD82, Corollaire 1.4.24] that the kernel of the surjection $j_!\mathbb{Q}[d] \to j_{!*}\mathbb{Q}[d]$ is supported on the complement of U. Any submodule M of $j_!\mathbb{Q}[d]$ whose sum with the kernel equals $j_!\mathbb{Q}[d]$ then has to be $j_!\mathbb{Q}[d]$, so $j_!\mathbb{Q}[d]$ is in fact the projective cover of $j_{!*}\mathbb{Q}[d]$.

Proposition 11.7. Assume the situation in 11.1 or 11.2, and let (X, S) be an affinely Whitney-Tate stratified variety. Then the abelian category MTPer_S(X) has finite homological dimension and enough projective objects and each of those has a finite filtration with subquotients of the form $\Delta_s(\nu)$ for $s \in S$ and $\nu \in \mathbb{Z}$. Similarly, it has enough injective objects and each of those has a finite filtration with subquotients of the form $\nabla_s(\nu)$.

Proof. We want to apply [BGS96, Theorem 3.2.1]. We note that a version of this results is true where (2) is replaced by the requirement that the partial order in (3) satisfies the descending chain condition. This is necessary because condition (2) is not satisfied in our situation: for each stratum X_s of dimension d_s with $j: X_s \to \overline{X_s}$ and $i: \overline{X_s} \to X$, we have that all $i_* \circ j_{!*}\mathbb{Q}(a)[d_s]$, $a \in \mathbb{Z}$, are simple.

The strengthened condition (3) is then still true, the partial order is given by the inclusion of support of $M \in \mathrm{MTPer}_{(B)}(Y)$ and the descending chain condition follows since there are only finitely many strata in Y.

Condition (1) is satisfied, i.e., $\operatorname{MTPer}_{(B)}(Y)$ is an artinian category, every object has finite length: the functors pi_* etc. are defined by applying i_* and then truncating. Therefore, these functors preserve finite length of objects. We can use the exact sequences of [BBD82, Lemme 1.4.19] to inductively reduce the finite length assertion to artinianness of $\operatorname{MT}(k)$. The latter is clear since $\operatorname{MT}(k)$ obviously is equivalent to the category of finite-dimensional graded vector spaces.

Condition (4) is established in Lemma 11.6. As mentioned above, the short exact sequences before [BBD82, Corollaire 1.4.24] imply that the kernel of $j_!\mathbb{Q}[d] \to j_{!*}\mathbb{Q}[d]$ and the cokernel of $j_!*\mathbb{Q}[d]$ are supported on the complement of U, whence Condition (5).

Finally, [BBD82, Lemma 3.2.4] allows to reduce Condition (6) to the vanishing of Lemma 11.8 below. $\hfill\Box$

Lemma 11.8. Assume the situation in 11.1 or 11.2, and let (X, S) be an affinely Whitney-Tate stratified variety. For any two strata $j_t: X_t \to X$ and $j_s: X_s \to X$ and $(n, a) \neq (0, 0)$, we have

$$\mathrm{MTDer}_{\mathcal{S}}(j_{t!}X_t, j_{s*}X_s(a)[n]) = 0.$$

Proof. If $X_t \neq X_s$, then $j_t^! j_{s*} = 0$ implies the vanishing directly. If $X_t = X_s$, then remark first $j_t^! j_{t*} = \text{id}$, since this holds as well for an open as for a closed embedding. Thus we are reduced to showing MTDer $(\underline{D}, \underline{D}(a)[n]) = 0$ for $(n, a) \neq (0, 0)$ and D an affine space, and this follows from homotopy invariance and 8.1.

Theorem 11.9. Let (X, S) be an affinely Whitney-Tate stratified variety.

(1) In the situation of 11.1, the realization functor

$$\operatorname{Real}_{\ell}: \operatorname{MTPer}_{\mathcal{S}}(X; \mathbb{Q}_{\ell}) \to \operatorname{Perv}_{\mathcal{S}}(X \times_{k} \bar{k}; \mathbb{Q}_{\ell})$$

considered in 11.4 is a degrading functor in the sense of [BGS96].

(2) In the situation of 11.2, the realization functor

$$\operatorname{Real}_H: \operatorname{MTPer}_{\mathcal{S}}(X; \mathbb{C}) \to \operatorname{Perv}_{\mathcal{S}}(X; \mathbb{C})$$

considered in 11.4 is a degrading functor in the sense of [BGS96].

Proof. Again, we only consider the ℓ -adic realization, the Hodge realization argument being similar. We need to show that the induced functor

$$\operatorname{Real}_{\ell} : \operatorname{Der^{b}}(\operatorname{MTPer}_{\mathcal{S}}(X; \mathbb{Q}_{\ell})) \to \operatorname{Der^{b}}(\operatorname{Perv}_{\mathcal{S}}(X \times_{k} \bar{k}; \mathbb{Q}_{\ell}))$$

induces isomorphisms

$$\bigoplus_{n\in\mathbb{Z}}\operatorname{Der}(P,Q(n))\overset{\sim}{\to}\operatorname{Der}(\operatorname{Real}P,\operatorname{Real}Q)$$

for any complexes P,Q. But since there are enough projectives, by Proposition 11.7, and these clearly go to projectives, we just need to show the analogous statement for the functor Real: $\operatorname{Hot}^{\mathrm{b}}(p\operatorname{MTPer}_{\mathcal{S}}(X;\mathbb{Q}_{\ell})) \to \operatorname{Hot}^{\mathrm{b}}(p\operatorname{Perv}_{\mathcal{S}}(X\times_{k}\bar{k};\mathbb{Q}_{\ell}))$ on the bounded homotopy category of projective objects. For single projective objects however we already know it from Theorem 11.3, and from there the extension to the bounded homotopy categories is immediate.

Theorem 11.10. Assume the situation in 11.1 or 11.2. Let (X, \mathcal{S}) be an affinely Whitney-Tate stratified variety and let $p \operatorname{MTPer}_{\mathcal{S}}(X)$ be the additive category of projective perverse objects. Then we get equivalences of categories

$$\operatorname{Der^{b}}(\operatorname{MTPer}_{\mathcal{S}}(X)) \stackrel{\approx}{\leftarrow} \operatorname{Hot^{b}}(p \operatorname{MTPer}_{\mathcal{S}}(X)) \stackrel{\approx}{\rightarrow} \operatorname{MTDer}_{\mathcal{S}}(X)$$

by the obvious functor towards the left and a tilting functor as in Proposition B.1 towards the right.

Proof. The equivalence to the left follows easily from 11.7. To obtain the tilting equivalence, it will be sufficient to show $\operatorname{MTDer}_{\mathcal{S}}(P,M[n])=0$ for $P,M\in\operatorname{MTPer}_{\mathcal{S}}(X)$ with P projective and $n\neq 0$. By an induction on a Δ -flag of our projective P we deduce $\operatorname{MTDer}_{\mathcal{S}}(P,\nabla_t[n])=0$ for $n\neq 0$. By an induction on a ∇ -flag, we get $\operatorname{MTDer}_{\mathcal{S}}(P,I[n])=0$ for $n\neq 0$ and any injective object $I\in\operatorname{MTPer}_{\mathcal{S}}(X)$. Now remember we needed to show $\operatorname{MTDer}_{\mathcal{S}}(P,M[n])=0$ for $P,M\in\operatorname{MTPer}_{\mathcal{S}}(X)$ with P projective and $n\neq 0$. For n<0 or n=1 this is clear anyhow. Thus if it is ok for two terms of a short exact sequence, it is also ok for the third term. Thus if it is ok for all terms of a finite resolution of a given object, it will also be ok for the given object itself. But by Lemma 11.8 it is ok for injective objects, and every object has a finite injective resolution.

Corollary 11.11. Assume the situation in 11.1 or 11.2. Let (X, S) be an affinely Whitney-Tate stratified variety. Then:

- (1) All simple perverse motives $\mathcal{L} \in \mathrm{MTPer}_{\mathcal{S}}(X)$ are up to a shift in the heart of the weight structure, in formulas $\mathcal{L} \in (\mathrm{MTDer}_{\mathcal{S}}(X))_{w=p}$ for some $p \in \mathbb{Z}$;
- (2) All perverse motives $\mathcal{L} \in \mathrm{MTPer}_{\mathcal{S}}(X)$, which are pure of a given weight, are semisimple;
- (3) All pure motives $\mathcal{L} \in \mathrm{MTDer}_{\mathcal{S}}(X)_{w=p}$ are isomorphic to the direct sum of their perverse cohomology objects, which in turn are perverse semisimple.

Proof. The first two points follow using Theorem 11.9 from the analogous result for ℓ -adic sheaves in [BBD82] by applying a suitable realization functor. The same argument shows that all perverse cohomology objects of a pure object are pure, more precisely given $\mathcal{F} \in \mathrm{MTDer}_{\mathcal{S}}(X)_{w=0}$ we have ${}^{p}\mathcal{H}^{n}\mathcal{F} \in \mathrm{MTDer}_{\mathcal{S}}(X)_{w=-n}$ for the n-th perverse cohomology object. This in turn says by the definition of a weight structure, that the triangles inductively putting together the object \mathcal{F} from its perverse cohomology objects all have the relevant map zero and so the object \mathcal{F} has to be the direct sum of its perverse cohomology objects.

11.12. If (X, S) is an affinely Whitney–Tate stratified variety, whose pure objects are even pointwise pure, we deduce that the category of perverse motives $\operatorname{MTPer}_{S}(X)$ has the "Koszul property": Given two simple objects \mathcal{N}, \mathcal{M} of weights n, m, the only nonzero extensions from the first to the second with respect to the abelian category $\operatorname{MTPer}_{S}(X)$ are in $\operatorname{Ext}^{n-m}(\mathcal{N}, \mathcal{M})$. To see this, one may use Theorem 11.10 to identify the Ext-group in question with $\mathcal{T}(\mathcal{N}, \mathcal{M}[n-m])$ and then compute it by the spectral sequence explained in [BGS96, 3.4.1]. In particular $\operatorname{MTPer}_{S}(X)$ is then, up to formally adding a square root of the Tate twist, equivalent to the category of finite dimensional modules over a Koszul ring of finite dimension over \mathbb{Q} . All this is, up to the very satisfying interpretation by true motives, already contained in [BGS96]. The details are left to the reader.

APPENDIX A. STRATIFICATIONS AND SINGULARITIES WITH MIXED TATE RESOLUTIONS

In the following section, we provide another criterion for a stratified variety (X, S) to be Whitney–Tate. This will also provide another proof of Proposition 4.10. In the following section, we fix a motivic triangulated category \mathcal{T} .

We first recall [Wil12, Theorem 4.4]:

Proposition A.1. Let (X, S) be a smooth stratified scheme such that for each stratum X_s , the closure $\overline{X_s}$ is also smooth. Then for each pair of strata X_s and X_t with $i_t: X_t \hookrightarrow \overline{X_s}$, the compositions $i_t^* \circ j_*, i_t^! \circ j_! : \mathscr{T}(X_s) \to \mathscr{T}(X_t)$ preserve the

 $triangulated\ subcategories\ of\ mixed\ Tate\ motives.$ In particular, the stratification is Whitney-Tate.

Next, we provide a criterion for being Whitney—Tate which covers some cases where closures of strata are not regular. In this more general setting, we can still argue as in the smooth case after a suitable resolution of singularities. This, however, requires that there is a resolution such that the motives of the fibers of the resolution are mixed Tate. The condition is a motivic version of condition (*) in [BGS96, Section 1.4].

Proposition A.2. Let (X, S) be a stratified scheme. Assume that for each stratum X_s of X there exists a resolution of singularities $\rho_s : \widetilde{X}_s \to \overline{X}_s$ with the following properties:

- (1) ρ_s is surjective and proper,
- (2) \widetilde{X}_s is smooth,
- (3) for each stratum $X_t \hookrightarrow \overline{X_s}$, the restriction $\widetilde{X_s} \times_{\overline{X_t}} X_t$ satisfies

$$M_{X_t}\left(\widetilde{X_s} \times_{\overline{X_s}} X_t\right) \in MTDer(X_t).$$

Then for each pair of strata X_s and X_t with $i_t: X_t \hookrightarrow \overline{X_s}$, the compositions

$$i_t^* \circ j_*, i_t^! \circ j_! : \mathscr{T}(X_s) \to \mathscr{T}(X_t)$$

preserve the triangulated subcategories of mixed Tate motives. In particular, the stratification is Whitney-Tate.

Proof. Since $i_t^*j_*$ is dual to $i_t^!j_!$ and the motivic duality restricts to MTDer (X_t) , it suffices to prove one of the assertions. Since $i_t^*j_*$ is compatible with Tate twists, it suffices to prove that $i_t^*j_*\mathbb{Q} \in \mathrm{MTDer}(X_t)$.

For the proof, we now fix X_t . Without loss of generality, we can assume that X_t is closed in X. If not, we consider the scheme $X \setminus (\overline{X_t} \setminus X_t)$. This still satisfies the conditions, and the closed complement $\overline{X_t} \setminus X_t$ does not enter the computations of the compositions of functors.

The proof that $i_t^* \circ (j_s)_* \mathbb{Q}$ is mixed Tate now proceeds by induction on the dimension of X_s . Therefore, assume that for all X_r with $X_r \hookrightarrow \overline{X_s}$, the claim is satisfied, i.e. $i_t^* \circ (j_r)_* \mathbb{Q}$ is mixed Tate. Now consider the following diagram:

$$\widetilde{X}_{s} \times_{\overline{X}_{s}} X_{t} \xrightarrow{i'_{t}} \widetilde{X}_{s} \xleftarrow{j'} \widetilde{X}_{s} \times_{\overline{X}_{s}} X_{s}$$

$$\downarrow p \qquad \qquad \downarrow p_{s}$$

$$X_{t} \xrightarrow{i_{t}} \overline{X}_{s} \xleftarrow{j} X_{s}$$

The arrow p in the middle is the resolution of singularities provided by the assumption. The rest of the diagram consists of restricting p to the strata X_s and X_t .

Proper base change for the left square states $i_t^* p_* M \cong (p_t)_* (i_t')^* M$. Then it suffices to show that $(i_t')^* p^* j_*$ is in $\mathrm{MTDer}(\widetilde{X_s} \times_{\overline{X_s}} X_t)$, and that $(p_t)_*$ preserves mixed Tate motives.

The fact that $(p_t)_*$ preserves mixed Tate motives follows from part (3) of our assumption: as $(p_t)_*$ commutes with Tate twists, it suffices to show that $(p_t)_*\mathbb{Q}$ is contained in MTDer (X_t) . But by assumption,

$$(p_t)_* \mathbb{Q} \cong \mathrm{M}_{X_t}(\widetilde{X_s} \times_{\overline{X_s}} X_t) \in \mathrm{MTDer}(X_t).$$

To prove that $(i'_t)^*p^*j_*$ is in MTDer $(\widetilde{X_s} \times_{\overline{X_s}} X_t)$, we employ the localization sequence in the situation $X = \widetilde{X_s}$, $i'_t : Z = \widetilde{X_s} \times_{\overline{X_s}} X_t \hookrightarrow X$ and $j' : U = X \setminus Z \hookrightarrow X$. In that situation, the localization sequence for $\mathbb{Q} \in \mathcal{F}(\widetilde{X_s})$ has the form

$$(i'_t)_*(i'_t)^! \mathbb{Q} \to \mathbb{Q} \to (j')_*(j')^* \mathbb{Q} \to (i'_t)_*(i'_t)^! \mathbb{Q}[1].$$

But $(j')^*\mathbb{Q}_X \cong \mathbb{Q}_U$ and $(i'_t)^*\mathbb{Q}_X \cong \mathbb{Q}_Z$. By absolute purity, we have $(i'_t)^!\mathbb{Q}_Z \cong \mathbb{Q}(-d)[-2d]$ with d the codimension of Z in X. Restricting this sequence using $(i'_t)^*$ provides the following triangle in $\mathcal{F}(Z)$:

$$\mathbb{Q}_Z(-d)[-2d] \to \mathbb{Q}_Z \to (i'_t)^*(j')_* \mathbb{Q}_U \to \mathbb{Q}_Z(-d)[-2d+1].$$

It suffices to show that the difference between the motives $(p_t)_*(i_t')^*(j')_*\mathbb{Q}_U$ and $(p_t)_*(i_t')^*p^*(j_s)_*\mathbb{Q}_{X_s}$ is mixed Tate. The preimages $p^{-1}(X_r)$ provide a stratification of \widetilde{X}_s by part (2) of the assumption. Inductively applying a localization argument similar to the one used in [Wil12, Theorem 4.4] to $U = \widetilde{X}_s \setminus Z$ by taking out smooth closed strata, we see that the difference between $j_*\mathbb{Q}_U$ and $p^*(j_s)_*\mathbb{Q}_{X_s}$ is given by extensions of mixed Tate motives on the strata $p^{-1}(X_r)$. Therefore, it suffices to show that for each stratum X_r in \overline{X}_s , the functor $(p_t)_* \circ (i_t')^* \circ (j_r')_* : \mathscr{T}(\widetilde{X}_s \times_{\overline{X}_s} X_r) \to \mathscr{T}(X_t)$ preserves mixed Tate motives. By the inductive assumption, this is true for $i_t^* \circ (j_r)_* : \mathscr{T}(X_r) \to \mathscr{T}(X_t)$, and by part (3) of our assumption, it is also true for $i_t^* \circ (j_r)_* \circ (p_r)_*$. Obviously $(j_r)_* \circ (p_r)_* \cong p_* \circ (j_r')_*$. By proper base change, $i_t^* \circ p_* \cong (p_t)_* \circ (i_t')^*$. Combining these, we find that $(p_t)_* \circ (i_t')^* \circ (j_r')_* \cong i_t^* \circ (j_r)_* \circ (p_r)_*$. Hence, this latter composition preserves mixed Tate motives, which finishes the proof.

Proposition A.3. Let G be a split reductive group, let $B \subset G$ be a Borel subgroup, and denote by (B) the stratification of G/B by Schubert cells. Then for each $w \in W$, the Bott-Samelson resolution $\rho_w : BS(w) \to \overline{X_w}$ of Demazure-Hansen has the following properties:

- (1) ρ_w is surjective and proper,
- (2) BS(w) is smooth,
- (3) for each $v \in W$ with $X_v \in \overline{X_w}$, the restriction $BS(w) \times_{\overline{X_w}} X_v$ satisfies

$$M_{X_v}\left(BS(w)\times_{\overline{X_w}}X_v\right)\in \mathrm{MTDer}(X_v).$$

In particular, the Bruhat stratification of a flag variety is Whitney-Tate.

Proof. Properties (1) and (2) are well known. Property (3) follows by iterative use of the localization sequence once we can show that for each point x of \overline{X}_w , the fibre of the Bott–Samelson resolution $\rho_w^{-1}(x)$ has a paving by affine spaces. This is the case, as discussed in [Hai].

APPENDIX B. GENERAL TILTING EQUIVALENCES

Here we formulate a general tilting-type theorem. Possible sources for statements of this type are [Ric89, Kel94]. We will sketch a proof and discuss the tilting functor.

Proposition B.1. Let \mathcal{A} be an abelian category and $(T_i)_{i\in I}$ a family of complexes in $\operatorname{Hot}(\mathcal{A})$ such that for all $i, j \in I$ and $n \in \mathbb{Z}$ we have $\operatorname{Hot}_{\mathcal{A}}(T_i, T_j[n]) \xrightarrow{\sim} \operatorname{Der}_{\mathcal{A}}(T_i, T_j[n])$ and

$$\operatorname{Der}_{\mathcal{A}}(T_i, T_i[n]) \neq 0 \Rightarrow n = 0$$

Then the embedding of the full additive subcategory of Der(A) generated by the objects T_i can be extended to a fully faithful triangulated functor

$$\operatorname{Hot^b}\left(\operatorname{add}(T_i\mid i\in I)\right)\stackrel{\sim}{\hookrightarrow}\operatorname{Der}(\mathcal{A})$$

Proof. For simplicity let us first consider the case of a finite family of objects T_1, \ldots, T_r . Let us consider the complex $T = \bigoplus_i T_i$. Its endomorphism complex

$$E:=\mathrm{End}(T):=\bigoplus_n \mathcal{A}(T,T[n])$$

has a natural structure of a dg-ring with idempotents $1_i \in E$ given by the projection to each factor. Then the localization functor induces by devissage an equivalence between the full triangulated subcategories

$$\langle T_1, \ldots, T_r \rangle_{\Delta}$$

generated by the objects T_i in $\text{Hot}(\mathcal{A})$ and $\text{Der}(\mathcal{A})$ respectively. On the other hand the functor $\text{Hom}(T,\)$ induces an equivalence from the first of these triangulated categories to the full triangulated subcategory

$$\langle 1_1 E, \dots, 1_r E \rangle_{\Delta} \subset \operatorname{dgDer-} E$$

generated by the right dg-modules 1_iE in the localization dgDer-E of the category of right dg-modules over E by quasi-isomorphisms. Now recall that for any quasi-isomorphism $D \xrightarrow{\sim} E$ of dg-rings the restriction induces an equivalence of triangulated categories

$$dgDer-E \stackrel{\approx}{\to} dgDer-D$$

Up to this point we did not need the condition $\operatorname{Der}_{\mathcal{A}}(T_i, T_j[n]) \neq 0 \Rightarrow n = 0$. This additional assumption however implies that the cohomology $\mathcal{H}E$ of E is concentrated in degree zero. We therefore have quasiisomorphisms

$$\mathcal{H}E \quad \stackrel{\sim}{\longleftarrow} \quad \mathcal{Z}^0E \oplus E^{<0} \quad \stackrel{\sim}{\longrightarrow} \quad E$$

of dg-rings. Let us abbreviate $H := \mathcal{H}E$. Under the equivalence of triangulated categories

$$\operatorname{Der}(\operatorname{mod-} H) = \operatorname{dgDer-} H \stackrel{\approx}{\to} \operatorname{dgDer-} E$$

defined by our quasi-isomorphisms the objects 1_iH will correspond to 1_iE . In addition, the localization functor $\text{Hot}(\text{mod-}H) \to \text{Der}(\text{mod-}H)$ induces, again by devissage, an equivalence between the full triangulated subcategories generated by the right modules 1_iH in both of these triangulated categories. The first of these triangulated categories in turn coincides with the homotopy category

$$\operatorname{Hot}^{\operatorname{b}}(\operatorname{add}(1_1H,\ldots,1_rH))$$

of the full additive subcategory add $(1_1H,\ldots,1_rH)\subset \operatorname{mod-}H$ generated by our right H-modules. Now sure enough the obvious maps give isomorphisms $1_iH1_j\stackrel{\sim}{\to} \operatorname{Mod}_H(1_jH,1_iH)$ to the space of homomorphisms of right H-modules and $1_iH1_j\stackrel{\sim}{\to} \operatorname{Der}_A(T_i,T_i)$. This gives us an equivalence

$$\operatorname{add}(1_1H,\ldots,1_rH) \stackrel{\approx}{\to} \operatorname{add}(T_1,\ldots,T_r)$$

of additive categories and finishes the proof of the proposition in the case of a finite family of objects.

The general case follows similar lines. Instead of a single generator, we have to consider categories enriched in abelian groups. Objects like these are called ringoids or rings with many objects in the literature. The usual definitions of modules still apply to rings with many objects, and the above proof works in that setting. More details can be found in [Kel94].

B.2. Given objects $\bar{T}_i \in \text{Der}(\mathcal{A})$ with $(\text{Der}_{\mathcal{A}}(\bar{T}_i, \bar{T}_j[n]) \neq 0 \Rightarrow n = 0)$ we can quite often find representatives $T_i \in \text{Hot}(\mathcal{A})$ with the properties required in the Proposition by choosing some kind of projective or injective resolutions.

B.3. In the setting of Proposition B.1, suppose in addition that Hot \mathcal{A} and Der \mathcal{A} admit countable direct sums, that the localization functor preserves those, and that all the objects T_i are compact in Hot \mathcal{A} and Der \mathcal{A} . Then the embedding $\operatorname{add}^{\infty}(T_i \mid i \in I) \subset \operatorname{Der}(\mathcal{A})$ of the full additive subcategory consisting of all countable direct sums of copies of objects among the T_i can be extended to a fully faithful triangulated functor

$$\operatorname{Hot}^{\operatorname{b}}\left(\operatorname{add}^{\infty}(T_{i}\mid i\in I)\right)\overset{\sim}{\hookrightarrow}\operatorname{Der}(\mathcal{A})$$

The argument stays essentially the same. The conditions that $\operatorname{Hot} \mathcal{A}$ and $\operatorname{Der} \mathcal{A}$ admit countable direct sums and that the localization functor preserves those are satisfied for example in the case where \mathcal{A} is a category of sheaves, since in this case a right adjoint for the localization functor can be obtained by choosing K-injective resolutions following [Spa88].

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