Poisson geometry of the moduli of local systems on smooth varieties

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To Masaki Kashiwara on his 70th birthday

Abstract

We study the moduli of G-local systems on smooth but not necessarily proper complex algebraic varieties. We show that, when considered as derived algebraic stacks, they carry natural Poisson structures, generalizing the well known case of curves. We also construct symplectic leaves of this Poisson structure by fixing local monodromies at infinity, and show that a new feature, called *strictness*, appears as soon as the divisor at infinity has non-trivial crossings.

Contents

Introduction

For a smooth complex algebraic curve X and a reductive group G, it is well known that the moduli space $M_G(X)$ of G-local systems[1] carries a canonical Poisson structure (see [Fo-Ro, GHJW, Gol, Gu-Ra]). Moreover, the symplectic leaves of this Poisson structure can be identified with moduli of G-local systems having fixed conjugacy classes of monodromies at infinity. This topological picture also has algebraic counterparts for which local systems are replaced by flat bundles or Higgs bundles possibly with irregular singularities, and is known to be compatible with the comparison isomorphisms between these different incarnations of the moduli problem (see for instance [Bo]). However, as far as the authors are aware, very little is known about the Poisson geometry of local systems moduli on higher dimensional varieties outside of the proper case.

The purpose of this note is to explore the moduli of G-local systems on higher dimensional smooth open varieties, with a particular focus on their Poisson geometry. For us, the results presented in this work represent a first step towards an understanding of moduli of local systems on higher dimensional varieties, with a long term goal to extend Simpson's non-abelian Hodge theory to the non-proper case.

As a first comment, derived algebraic geometry is useful, and probably unavoidable, for this project. Indeed, for a higher dimensional compact oriented manifold M, it is known (see [PTVV, To1]) that the moduli of G-local systems on M carries canonical symplectic structure provided that

1. the moduli is considered as a *derived algebraic stack* and not simply as a scheme or a stack.

2. the symplectic structures involve a *cohomological shift* by $2 - d$, where d is the dimension of M.

In this work, we consider the derived moduli stack $Loc_G(X)$ of G-local systems on a complex smooth algebraic variety X of complexe dimension d. We establish two principal results that can be summarized as follows:

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[1]In this paper, 'G-local systems' are representations of $\pi_1(X)$ into G

Theorem A *(see Theorem 4.9)*

1. *The derived stack $Loc_G(X)$ carries a canonical $(2-2d)$-shifted Poisson structure. This recovers the usual Poisson structure when $d = 1$.*

2. *The $(2-2d)$-Poisson structure on $Loc_G(X)$ has generalized symplectic leaves. In particular $Loc_G(X, \lambda_\bullet)$, the derived moduli of G-local systems with fixed conjugacy classes λ_i of monodromies at infinity is such a generalized symplectic leaf.*

Before describing the content of this work we need to add a couple of comments concerning the previous result. The Poisson structure on $Loc_G(X)$ will be constructed by using a very specific topological property of smooth complex algebraic varieties, namely that their *boundary at infinity* is a compact manifold (of real dimension $2d - 1$ if X is of dimension d). As a consequence, there is a natural map $Loc_G(X) \to Loc_G(\partial X)$ sending a G-local system on X to its restriction to the boundary. By the work of Calaque [Ca] it is known that such restriction maps come equiped with a canonical Lagrangian structure, and by the work of Melani and Safronov [Me-Sa I, Me-Sa II] it is known that Lagrangian structures induce Poisson structures. This roughly explains why statement (1) is true. Statement (2) is subtler, mainly because one has to make precise what 'fixing the monodromies at infinity' means. This is particularly important in the derived setting where the fixing of the local monodromies involves higher homotopy coherence conditions. Moreover, we only prove (2) under the restrictive condition that the divisor at infinity of X can be chosen to have at most double intersections. We will see that even in this simple case a new feature appears, and that we have to impose an additional condition on the local monodromies at infinity that we call *strictness* (see Definition 4.6). This condition is invisible on the non-derived moduli space, but is required in order to construct symplectic leaves in the full derived moduli stack.

The paper is organized as follows. In section 1 we start with a short reminder on the derived moduli of G-local systems on a space, and the various ways in which one could describe this derived moduli in concrete algebraic terms. In sections 2 and 3 we briefly recall shifted symplectic and Poisson structures, and introduce the notion of *generalized symplectic leaves* in this context. In section 4 we focus on the case of complex smooth algebraic varieties. We examine their structure at infinity and deduce the existence of the shifted Poisson structure on the derived moduli of local systems. We first analyze the special case of a smooth divisor at infinity and show that the construction works in essentially the same manner as in the case of curves. Finally we study the case of a divisor with two smooth intersecting components and show how the strictness condition appears naturally when one tries to construct symplectic leaves. We also provide families of examples of strict pairs. In section 5 we collect some ideas indicating how the statements of this paper can be generalized to the de Rham setting in which local systems are replaced by bundles with flat connections.

Notation and conventions

k - a field of characteristic zero.

$\mathbf{cdga}_k^{\leq 0}$ - the ∞-category of non-positively graded commutative dg-algebras over k.

$\mathbb{T}$ - the ∞-category of spaces, or equivalently the ∞-category of simplicial sets.

$\mathrm{Pro}(\mathbb{T})$ - the ∞-category of pro-simplicial sets.

G - a reductive group over k.

$[G/G]$ - the stack quotient of G by the conjugation action of G on itself.

λ - an element in G.

$\mathbf{C}_\lambda$ - the conjugacy class of λ in G.

$_\alpha\widetilde{[G/G]} \to D$ - locally constant stack with fiber $[G/G]$ obtained by twisting the constant stack with a circle bundle classified by $\alpha : D \to BS^1$.

Γ - a finitely presentable discrete group.

$R_G(\Gamma)$ - the affine k-scheme parametrizing group homomorphisms $\Gamma \to G$.

$M_G(X)$ - the G-character scheme of X.

$\mathcal{M}_G(X)$ - the stack of G-local systems on X.

$Loc_G(X)$ - the derived stack of G-local systems on X.

$Loc_G(X, \{\lambda_i\})$ - the derived stack of G-local systems with fixed conjugacy classes of monodromies at infinity.

$\Gamma(F, \mathrm{Sym}_{\mathcal{O}}(\mathbb{T}_F[-n-1]))[n+1]$ - the complex of n-shifted polyvectors on a derived Artin stack F.

1 The moduli of local systems as a derived stack

In this section we review the basic constructions of character schemes, the stack of local systems, as well as the derived stack of local systems, associated to a connected finite CW complex X. We explain how to understand the derived structure on the moduli stack of local systems by means of a *free resolutions of the space* X. We also discuss the basics of differential calculus on this derived stack by presenting an explicit model for computing algebraic de Rham cohomology. Most of the material here is well known or at least part of the folklore.

1.1 The character scheme and the stack of local systems

Let k be a field of characteristic zero, X a connected finite CW complex, and G a reductive group over k. We consider G-local systems on X which are by definition locally constant principal G-bundles on X. If we fix a base point $x \in X$ we can equivalently view G-local systems as G-valued representations of the discrete group $\Gamma := \pi_1(X, x)$.

The **moduli of G-local systems** can then be defined by

$$M_G(X) := R_G(\Gamma)/G := \mathrm{Hom}_{\mathsf{grp}}(\Gamma, G)/G.$$

This formula can have several interpretations, depending on how we view its terms. In the most straightforward interpretation (see e.g. [Lu-Ma]) $R_G(\Gamma) = \mathrm{Hom}_{\mathsf{grp}}(\Gamma, G)$ is an affine scheme over k, classifying group homomorphisms $\Gamma \to G$. It can be constructed explicitly as a closed subscheme in G^p where p is the number of a chosen set of generators for Γ and the ideal cutting out $R_G(\Gamma)$ is given by the relations among these generators. Alternatively we can define $R_G(\Gamma)$ as the affine k-scheme which represents the functor sending a commutative k-algebra A to the set $\mathrm{Hom}_{\mathsf{grp}}(\Gamma, G(A))$ of group homomorphisms from Γ to the group of A-points $G(A)$. The group G acts on $R_G(\Gamma)$ by conjugation. The quotient of $R_G(\Gamma)$ by G can itself be interpreted as an affine GIT quotient. Thus $M_G(X)$ is an affine scheme over k whose ring of functions is the ring of G-invariant functions on $R_G(\Gamma)$. The set of k-points of $M_G(X)$ is in one-to-one correspondence with the set of isomorphism classes of semi-simple locally constant principal G-bundles on X. The scheme $M_G(X)$ is also often called the G-character scheme of X.

A less naive viewpoint is to consider the quotient stack of $R_G(\Gamma)$ by the action of the group G. This stack, denoted by $\mathcal{M}_G(X) = [R_G(\Gamma)/G]$, is called the **stack of G-local systems** on X. The k-points of $\mathcal{M}_G(X)$ form a groupoid equivalent to the groupoid of all G-local systems on X. The stack $\mathcal{M}_G(X)$ is an algebraic stack in the sense of Artin and comes equipped with a structure morphism $\mathcal{M}_G(X) \longrightarrow M_G(X)$ which is universal for morphism to schemes. In other words the character variety $M_G(X)$ is a coarse moduli space for the stack $\mathcal{M}_G(X)$.

1.2 Simplicial resolutions and the derived stack of local systems

In this work we will need a slightly more refined version of the stack of local systems called the ***derived stack of local systems***. The derived stack of local systems arises naturally both as a way of encoding the algebraic complexity of the relations defining Γ and as a device for repairing singularities in $\mathcal{M}_G(X)$.

The scheme $R_G(\Gamma)$ and hence the stack $\mathcal{M}_G(X)$ can in general be very singular. However, when the group Γ happens to be free of rank p, $R_G(\Gamma)$ is isomorphic to G^p and is thus smooth over k. When Γ is not free we can consider [May, Chapter VI], [Go-Ja, Chapter 5] a simplicial free resolution $B\Gamma_\bullet \simeq X$ of the space X. More precisely, given a base point $x \in X$, consider a simplicial group model for the loop group $\Omega_x(X)$. This simplicial group can be resolved by free groups, i.e. replaced by a weakly equivalent simplicial group $\Gamma_\bullet$ where each Γ_n is free on a finite number of generators. Note that the geometric realization of the simplicial space $B\Gamma_\bullet$ is homotopy equivalent to X, and we can thus view $\Gamma_\bullet$ as a free resolution of the pointed space (X, x). Note also that this resolution depends on X and not just on the group Γ (except when X is itself a $K(\Gamma, 1)$ in which case $\Gamma_\bullet$ is a free resolution of the group Γ). Applying $R_G(-)$ to $\Gamma_\bullet$ yields a cosimplicial affine scheme $R_G(\Gamma_\bullet)$, or equivalently a simplicial commutative k-algebra $\mathcal{O}(R_G(\Gamma_\bullet))$. The passage to normalized chains defines a commutative dg-algebra, whose quasi-isomorphism type does not depend on the choice of the resolution $\Gamma_\bullet$ of X. In other words we get a commutative dg-algebra $\mathscr{A}_G(X)$ which, up to quasi-isomorphism, only depends on the homotopy type of X.

By construction $H^0(\mathscr{A}_G(X))$ is naturally isomorphic to $\mathcal{O}(R_G(\Gamma))$ and $H^i(\mathscr{A}_G(X))$ vanish for $i > 0$. The other cohomologies $H^i(\mathscr{A}_G(X))$ for $i < 0$ can be non-zero. When $X = K(\Gamma, 1)$, the cohomology $H^\bullet(\mathscr{A}_G(X))$ is the so called ***representation homology*** of the group ring $k[\Gamma]$ in the sense of [Be-Kh-Ra] and codifies many interesting invariants of the group Γ. For an arbitrary CW complex X the k-vector spaces $H^i(\mathscr{A}_G(X))$ are invariants of the space X and may be non-trivial even when X is simply connected (see Example 1.1).

As explained in [To1] the non-positively graded cdga $\mathscr{A}_G(X)$ has a spectrum $Spec\,\mathscr{A}_G(X)$ which is a ***derived affine scheme***, that is - an affine k-scheme equipped with a sheaf of cdga. The conjugation action of G on the various $R_G(\Gamma_n)$ gives rise to an action on the commutative dg-algebra $\mathscr{A}_G(X)$ and hence G acts on its spectrum. The quotient stack

$$Loc_G(X) := [Spec\,\mathscr{A}_G(X)/G]$$

is the ***derived stack of G-local systems on X*** of [To-Ve2]. We refer the reader to [To-Ve] for the formalism of derived schemes and derived stacks, in particular we will not explain in this work how to formally construct the ∞-category of derived stacks and how to define the above quotient.

Note that, as explained in [To-Ve2], $Loc_G(X)$ can also be considered as an $(\infty-)$functor

$$Loc_G(X) : \mathbf{cdga}_k^{\leq 0} \longrightarrow \mathbb{T}$$

on the ∞-category $\mathbf{cdga}_k^{\leq 0}$ of non-positively graded commutative k-linear dg-algebras. This functor sends a dg-algebra A to the simplicial set $Map(S(X), BG(A))$ of maps from the singular simplicies in X to the simplicial set of A-points of the stack BG (see [To-Ve2] for details). In the special case when $G = GL_n$ the simplicial set $Loc_G(X)(A)$ also admits an alternative sheaf theoretic description. Consider the category whose objects are sheaves of A-dg-modules on X that are locally quasi-isomorphic to the constant sheaf $A^{\oplus n}$, and whose morhpisms are quasi-isomorphisms between such sheaves. The nerve of this category is naturally equivalent to $Loc_G(X)(A)$ (see [To-Ve2]).

Example 1.1 There is another useful description of the derived stack $Loc_G(X)$ which instead of a free resolution $\mathbf{\Gamma_\bullet}$ of the space X uses a cell decomposition of X as follows.

Let us assume that we have fixed a cell decomposition of X:

$$\varnothing = X_0 \hookrightarrow \cdots \hookrightarrow X_k \hookrightarrow X_{k+1} \hookrightarrow \cdots \hookrightarrow X_n = X,$$

where each inclusion $X_k \hookrightarrow X_{k+1}$ is obtained by a push-out

$$
\begin{array}{ccc}
X_k & \hookrightarrow & X_{k+1} \\
\uparrow & & \uparrow \\
S^{n_k} & \hookrightarrow & B^{n_k+1}
\end{array}
$$

adding a $(n_k + 1)$-dimensional cell. The derived stack $Loc_G(X)$ itself decomposes as a tower of maps

$$Loc_G(X) \longrightarrow \cdots \longrightarrow Loc_G(X_{k+1}) \longrightarrow Loc_G(X_k) \longrightarrow \cdots \longrightarrow Loc_G(\varnothing) = *,$$

where each map is a part of a pull-back square

$$
\begin{array}{ccc}
Loc_G(X_{k+1}) & \longrightarrow & Loc_G(X_k) \\
\downarrow & & \downarrow \\
Loc_G(B^{n_k+1}) = BG & \longrightarrow & Loc_G(S^{n_k}).
\end{array}
$$

Moreover, for an m-dimensional sphere S^m, the derived stack $Loc_G(S^m)$ can be computed explicitly, for instance by induction on m using the cell decomposition $S^m = B^m \sqcup_{S^{m-1}} B^m$. We have

$$Loc_G(S^0) \simeq BG \times BG \qquad Loc_G(S^1) \simeq [G/G]$$

and for any $m > 1$

$$Loc_G(S^m) \simeq [Spec\, \mathscr{A}_G(S^m)/G]$$

with $\mathscr{A}_G(S^m) \simeq Sym_k(\mathfrak{g}^\vee[m-1])$, and where $\mathfrak{g}^\vee$ is the k-linear dual of the Lie algebra of G.

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1.3 Cotangent complexes and differential forms

The derived stack $Loc_G(X)$, being a quotient of a derived affine scheme by an algebraic group, is a derived Artin stack. In particular it has a cotangent complex $\mathbb{L}_{Loc_G(X)}$ which is a quasi-coherent complex on $Loc_G(X)$. This can be described explicitly in terms of the G-equivariant dg-algebra $\mathscr{A}_G(X)$ as follows.

First note that the ∞-category of quasi-coherent complexes on $[Spec\,\mathscr{A}_G(X)/G]$ is naturally equivalent to the ∞-category of G-equivariant $\mathscr{A}_G(X)$-dg-modules. The derivative of the G-action on $\mathscr{A}_G(X)$ induces a morphism of $\mathscr{A}_G(X)$-dg-modules

$$a : \mathbb{L} \longrightarrow \mathfrak{g}^\vee \otimes_k \mathscr{A}_G(X),$$

where $\mathbb{L}$ is the cotangent complex of the commutative dg-algebra $\mathscr{A}_G(X)$, and $\mathfrak{g}^\vee$ is the dual of the Lie algebra of G. The homotopy fiber of a is a well defined $\mathscr{A}_G(X)$-module which, considered as a quasi-coherent module on $Spec\,\mathscr{A}_G(X)$, is naturally equivalent to the pull-back of the cotangent complex of $Loc_G(X)$ by the atlas map $Spec\,\mathscr{A}_G(X) \to Loc_G(X)$. This homotopy fiber carries a natural G-equivariant structure, making it into a quasi-coherent module on $Loc_G(X)$, which is the cotangent complex of $Loc_G(X)$.

Global sections are easier to understand in this setting, and are simply obtained by taking G-invariants:

$$a^G : \mathbb{L}^G \longrightarrow (\mathfrak{g}^\vee \otimes_k \mathscr{A}_G(X))^G.$$

The homotopy fiber of the map a^G computes $\Gamma(Loc_G(X), \mathbb{L}_{Loc_G(X)})$, the complex of global sections of the cotangent complex.

More generally, as explained in [PTVV, To1], we can talk about differential forms and the whole de Rham complex (endowed with its natural Hodge filtration) on the derived Artin stack $Loc_G(X)$. This is a complex $\mathcal{A}^\bullet(Loc_G(X))$, filtered by subcomplexes $F^p\mathcal{A}^\bullet(Loc_G(X)) \subset \mathcal{A}^\bullet(Loc_G(X))$. The complex $\mathcal{A}^\bullet(Loc_G(X))$ computes de Rham cohomology of $Loc_G(X)$, while the complex

$$\mathcal{A}^{p,cl}(Loc_G(X)) := F^p A^\bullet(Loc_G(X))[p]$$

is called the ***complex of closed p-forms on*** $Loc_G(X)$. In our setting, these complexes can be described explicitly as follows. With the same notations as above, we form the graded k-module

$$C := (\mathrm{Sym}_{\mathscr{A}_G(X)}(\mathbb{L}[-1]) \otimes_k \mathrm{Sym}_k(\mathfrak{g}^\vee[-2]))^G.$$

This graded module comes equipped with a differential which is the sum of three different terms: the internal cohomological differential of $\mathscr{A}_G(X)$, the differential induced by the coaction map $a : \mathbb{L} \to \mathfrak{g}^\vee \otimes_k \mathscr{A}_G(X)$, and the de Rham differential on $\mathrm{Sym}_{\mathscr{A}_G(X)}(\mathbb{L}[-1])$. This makes C into a complex of k-modules. Moreover, C comes equipped with a natural Hodge filtration, i.e. the stupid filtration for the natural grading on $\mathrm{Sym}_{\mathscr{A}_G(X)}(\mathbb{L}[-1]) \otimes_k \mathrm{Sym}_k(\mathfrak{g}^\vee[-2]))$. The complex C with this filtration is a model for the filtered complex $\mathcal{A}^\bullet(Loc_G(X))$.

2 Symplectic and Lagrangian structures

Recall from [PTVV, CPTVV] the notions of shifted symplectic and Poisson structures on derived Artin stacks. As we noted above, for a derived Artin stack F we have a complex of closed 2-forms $\mathcal{A}^{2,cl}(F)$, defined as the second layer in the Hodge filtration on its de Rham complex (shifted by 2). An n-cocyle in the complex $\mathcal{A}^{2,cl}(F)$ is called a **closed 2-form of degree** n (see [PTVV]). Such a form is furthermore **non-degenerate** if the contraction with the induced element in $H^n(F, \wedge \mathbb{L}_F) = H^n(\mathcal{A}^{2,cl}(F))$ gives a quasi-isomorphism of quasi-coherent complexes $\omega^\flat : \mathbb{T}_F \overset{\sim}{\to} \mathbb{L}_F[n]$. A non-degenerate closed 2-form of degree n on F is called an n-**shifted symplectic structure**. This notion of symplectic structure can be extended to the relative setting and gives rise to the notion of a Lagrangian structure. For a morphism $f : F \to F'$ between derived Artin stacks, an $(n-1)$-**shifted isotropic structure on** f consists by definition of a pair (ω, h), where ω is an n-shifted symplectic structure on F', and h is a homotopy between $f^*(\omega)$ and 0 inside the complex $\mathcal{A}^{2,cl}(F)$, i.e. h is a degree $(n-1)$ cochain in $\mathcal{A}^{2,cl}(X)$ with coboundary $f^*(\omega)$. Such an isotropic structure is a $(n-1)$-**shifted Lagrangian structure** if moreover the induced canonical morphism $h^\flat : \mathbb{T}_f \overset{\sim}{\to} \mathbb{L}_F[n-1]$ from the relative tangent complex $\mathbb{T}_f$ of f to the shifted cotangent complex of F is a quasi-isomorphism.

As shown in [PTVV], when X is a compact oriented manifold of dimension d, the derived stack $Loc_G(X)$ has a natural $(2-d)$-shifted symplectic structure. This structure is canonical up to a choice of a non-degenerate element in $(\mathrm{Sym}^2 \mathfrak{g}^\vee)^G$ which always exists since G is assumed to be reductive. This statement can be extended to a compact oriented manifold X with non-empty boundary ∂X. By [Ca], the induced restriction map

$$Loc_G(X) \longrightarrow Loc_G(\partial X)$$

carries a canonical $(2-d)$-shifted Lagrangian structure for which the $3-d = 2-(d-1)$-shifted symplectic structure on the target is the one discussed above. When $\partial X = \varnothing$ we have that $Loc_G(\partial X) = Loc_G(\varnothing) = *$ is a point and the Lagrangian structure on $Loc_G(X) \to *$ recovers the $(2-d)$-shifted symplectic structure on $Loc_G(X)$.

In fact, in order to get a Lagrangian structure on a map between moduli of local systems it is not necessary for the map to be induced from restricting to an actual boundary. Indeed, for a continuous map between finite CW complexes $f : Y \to X$, there is a notion of an **orientation of dimension** d on f. By definition such an orientation is given by of a morphism of complexes $\mathsf{or} : C^\bullet(Y, X) \longrightarrow k[1-d]$, where $C^\bullet(Y, X)$ is the cofiber of the pull-back map $f^* C^\bullet(X) \to C^\bullet(Y)$ on singular cochains with coefficients in k. The morphism or is also assumed to satisfy a non-degeneracy condition that ensures Poincaré duality between $H^*(X)$ and $H^*(X, Y)$. Concretely, the composition of the cup-product on $C^\bullet(X)$ with the orientation map produces a well defined pairing

$$C^\bullet(X) \otimes C^\bullet(X, Y) \longrightarrow k[1-d]$$

and we require that this pairing is non-degenerate on cohomology and induces a quasi-isomorphism $C^\bullet(Y, X) \simeq C^\bullet(X)^*[1 - d]$.

By [Ca], when $f : Y \to X$ is endowed with an orientation of dimension d, the pullback map on the derived stacks of local systems $f^* : Loc_G(Y) \longrightarrow Loc_G(X)$ carries a canonical $(2-d)$-shifted Lagrangian structure (again up to a choice of a non-degenerate element in $\mathrm{Sym}^2(\mathfrak{g}^\vee)^G$).

Example 2.1 In the special case where X is a Riemann surface with boundary ∂X, we expect this to match the well known symplectic structures on moduli of G-local systems on X with prescribed monodromies at infinity which are usually constructed by quasi-Hamiltonian reduction (see [Al-Ma-Me]). Indeed, here $Y = \partial X$ is a disjoint union of oriented circles, and we thus have $Loc_G(Y) \simeq \prod [G/G]$ where $[G/G]$ denotes the stack quotient of G by its conjugation self action. The stack $Loc_G(S^1) = [G/G]$ carries a canonical symplectic structure of degree 1. Moreover, for any element $\lambda \in G$ with conjugacy class $\mathbf{C}_\lambda$, and centralizer G_λ the inclusion $\mathbf{C}_\lambda \subset G$ produces a canonical Lagrangian structure $BG_\lambda \simeq [\mathbf{C}_\lambda/G] \subset [G/G]$. Therefore, by choosing a family of elements $\lambda_i \in G$, we have two 0-shifted Lagrangian morphisms

$$\prod BG_{\lambda_i} \qquad\qquad Loc_G(X).$$
$$\searrow \qquad\qquad \swarrow$$
$$\prod [G/G]$$

By [PTVV] the fiber product of these two maps therefore comes equipped with a 0-shifted symplectic structure. This fiber product, denoted by $Loc_G(X, \{\lambda_i\})$ is the derived stack of G-local systems on X whose local monodromies at infinity are required to belong to the conjugacy classes $\mathbf{C}_{\lambda_i}$. This should recover the symplectic structures of [Al-Ma-Me, Fo-Ro, GHJW, Gol, Gu-Ra]. However, the precise analysis of this comparison has not been carried out in the literature.

3   Poisson structure and generalized symplectic leaves

There is a notion of a shifted Poisson structure, generalizing the notion of a shifted symplectic structure. The definitions can be found in [CPTVV, Pri, PV] but are long and technical and will not be discussed here. We only recall that for any derived Artin stack F we can form the complex of n-**shifted polyvectors** $\Gamma(F, \mathrm{Sym}_\mathcal{O}(\mathbb{T}_F[-n - 1]))[n + 1]$ which carries a canonical Lie bracket making it into a graded dg-Lie algebra (see [CPTVV] for details). By definition, an n-**shifted Poisson structure** on F consists of a morphism in the ∞-category of graded dg-Lie algebras

$$p : k[-1](2) \longrightarrow \Gamma(F, \mathrm{Sym}_\mathcal{O}(\mathbb{T}_F[-n - 1]))[n + 1],$$

where $k[-1](2)$ is the graded dg-Lie algebra which is just k placed in homological degree 1 and grading degree 2, equipped with the zero Lie bracket.

One of the main comparison results of [CPTVV, Pri] states that the space of all n-shifted symplectic structures on a derived stack F is equivalent to the space of all non-degenerate n-shifted Poisson structures. This result was recently generalized [Me-Sa I, Me-Sa II] to Lagrangian structures: the space of all Lagrangian structures on a morphism $F \to F'$ is equivalent to the space of all non-degenerate coisotropic structures. In particular, an n-shifted Lagrangian morphism of derived Artin stacks $F \to F'$ always induces an n-shifted Poisson structure on F. Moreover, it is expected that all n-shifted Poisson structures arise this way as soon as one allows F' to be a formal derived stack rather than a derived Artin stack. Here the formal derived stack F' is the quotient of F by a derived Lie algebroid defined by the Poisson structure. Although the quotient F' can be given a precise meaning (see for instance [Nu]), the fact this can be enhanced to an equivalence between shifted Poisson structures on F and Lagrangian maps out of F has been announced by Costello-Rozenblyum but has not been written yet. At any rate, this suggests that we can view an n-shifted Poisson structure on a given derived stack F as an *equivalence class of n-shifted Lagrangian morphisms* $F \to F'$ with F' possibly formal derived Artin stack. Here two such morphisms $F \to F'$ and $F \to F''$ are declared to be equivalent if there is a third one $F \to G$ and a commutative diagram

$$
\begin{array}{ccc}
 & & F' \\
 & \nearrow & \uparrow a \\
F & \longrightarrow & G \\
 & \searrow & \downarrow b \\
 & & F''
\end{array}
$$

with a and b formally étale and compatible with the Lagrangian structures. In fact any morphism $f : F \to F'$ can be factored as $F \longrightarrow \widehat{F} \longrightarrow F'$ where $\widehat{F}$ is the formal completion of the morphism f and $\widehat{F} \to F$ is étale. The derived stack $\widehat{F}$ is only a formal stack in general, and can be obtained as the quotient of F by the action of the Lie algebroid induced from the morphism f, i.e. the Lie algebroid corresponding to the relative tangent complex $\mathbb{T}_f$ of f. Intuitively this quotient contracts infinitesimally all fibers of f. Therefore, the derived stack G in the above diagram can always be taken to be $\widehat{F}$ (for one of the two morphisms). Conversely, given an n-shifted Poisson structure on F, one can define from it a symplectic Lie algebroid in the sense of [Py-Sa], whose quotient is expected to recover a Lagrangian map $F \longrightarrow F'$ that induces back the original Poisson structure on F.

Even though this point of view can not be extracted completely from the currently existing literature, it will be adopted in this paper, and we will only deal with class of Poisson structures arising from Lagrangian morphisms. Thus for us an n-*shifted Poisson structure on* F will be defined as an equivalence class of n-shifted Lagrangian maps $F \to F'$ where F' is a formal derived stack (in fact in most of the examples we consider F' will be a derived Artin stack). The typical example is thus the
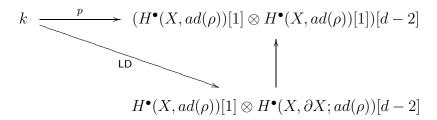
restriction map

$$Loc_G(X) \longrightarrow Loc_G(\partial X) \tag{1}$$

where X is a compact oriented manifold of dimension d with boundary ∂X. By [Ca] this is Lagrangian map and so by the discussion above it can be considered as a $(2-d)$-shifted Poisson structure on $Loc_G(X)$. When X is a Riemann surface this recovers the Poisson structure of [Fo-Ro, GHJW, Gol, Gu-Ra]. In general the bivector underlying the shifted Poisson bracket given by (1) can be understood explicitly as follows. The tangent complex of $Loc_G(X)$ at a given G-local system ρ is $H^*(X, ad(\rho))[1]$. By Lefschetz duality we have a natural quasi-isomorphism $(H^\bullet(X, ad(\rho))[1])^\vee \simeq H^\bullet(X, \partial X, ad(\rho))[d-2]$, and thus a natural element

$$k \xrightarrow{\quad LD \quad} H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, \partial X; ad(\rho))[d-2].$$

We can compose this with the boundary map $H^\bullet(X, \partial X; ad(\rho)) \to H^\bullet(X, ad(\rho))[1]$ to obtain a map

$$k \xrightarrow{\quad p \quad} (H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, ad(\rho))[1])[d-2]$$

$$\text{LD} \searrow \qquad \uparrow$$

$$H^\bullet(X, ad(\rho))[1] \otimes H^\bullet(X, \partial X; ad(\rho))[d-2]$$

This morphism p is the underlying bivector of the $(2-d)$-shifted Poisson structure on $Loc_G(X)$.

Classically a Poisson structure on a smooth variety induces a foliation of the variety by symplectic leaves. In our setting, for an n-shifted Poisson structure on a derived stack F given by a Lagrangian map $f : F \to F'$, the symplectic leaves are the appropriately interpreted fibers of f. Here we need the qualifier 'appropriately interpreted' because we must consider the fibers in the sense of symplectic geometry, that is - as fiber products of Lagrangians in F'. Note that specifying a Lagrangian morphism $\Lambda \to F'$ is the same thing as specifying a morphism $* \to F'$ in the category of Lagrangian correspondences, and thus is a "point" in this sense. We are therefore led to the following notion.

Definition 3.1 *Let F be a derived Artin stack with an n-shifted Poisson structure given by an n-shifted Lagrangian morphism $f : F \to F'$. A **generalized symplectic leaf of** F is a derived stack of the form $F \times_{F'} \Lambda$ for any n-shifted Lagrangian morphism $\Lambda \to F'$*

By [PTVV] a generalized symplectic leaf carries a canonical n-shifted symplectic structure. However, the above definition is a bit awkward as it depends on the choice of f. We will not try to refine this definition and will take it as a model of several constructions appearing in the sequel of this work.

Again, the typical example is given by a compact Riemann surface with boundary X. The restriction map $Loc_G(X) \longrightarrow Loc_G(\partial X) = \prod [G/G]$ carries a 0-shifted Lagrangian structure and thus corresponds

to a 0-shifted Poisson structure on $Loc_G(X)$. As we have already seen, among the generalized symplectic leaves of $Loc_G(X)$ we have $Loc_G(X, \{\lambda_i\})$, the derived moduli stack of G-local systems on X whose monodromies at infinity are fixed to be conjugate to the given elements $\lambda_i \in G$.

As a final note, it is instructive to point out that the above notion of generalized symplectic leaves is a rather flabby notion. For instance, when the n-shifted Poisson structure on F is non-degenerate (i.e. comes from an n-shifted symplectic structure) then the generalized symplectic leaves are all n-shifted symplectic derived stacks of the form $F \times R$ for some other n-shifted symplectic derived stack R. Another example is given by the Poisson structure on $Loc_G(X)$ induced by the restriction map $Loc_G(X) \longrightarrow Loc_G(\partial X)$, for an oriented manifold with boundary. Assume that Y is another oriented manifold with an identification $\partial Y \simeq \partial X$, then $Loc_G(M)$ becomes a generalized symplectic leaf, when $M = Y \sqcup_{\partial X} X$. This provides a lot of generalized symplectic leaves, all given by the different possible ways to complete X to an oriented manifold without boundary.

4 Symplectic leaves in the moduli of G-local systems on smooth complex varieties

In this section we fix a smooth (separated, quasi-compact and connected) complex algebraic variety Z of complex dimension d. We denote by $X := Z(\mathbb{C})$ the underlying topological space of $\mathbb{C}$-points of X endowed with the Eucledian topology. We also keep the notation k for a given field of characteristic zero and we fix a reductive group G over k with a chosen non-degenerate element in $Sym^2(\mathfrak{g}^\vee)^G$. The derived stack $Loc_G(X)$ is then a derived Artin stack of finite type over k and we are interested in the following problem:

Problem 4.1 *Show that $Loc_G(X)$ carries a natural $(2 - 2d)$-shifted Poisson structure and describe its generalized symplectic leaves.*

As we noted before, there are way too many generalized symplectic leaves according to our definition 3.1. To make the problem more manageable we will focus on a class of generalized symplectic leaves that is geometrically meaningful. We also want to keep in mind the case of curves, and when Z is of dimension 1 we want our description to recover the symplectic derived stacks $Loc_G(X, \{\lambda_i\})$, of G-local systems with prescribed monodromy at infinity.

In the discussion below we will propose a first answer to Problem 4.1. However, we will restrict ourselves to varieties Z with nice behavior at infinity. As we will see the problem has a rather direct and easy answer when the divisor at infinity for Z can be chosen to be smooth. We will also provide a solution when this divisor can be chosen to be simple normal crossings with two components where

already some new phenomena arise. We have not analyzed more complicated behaviors but we are convinced that one can indeed extend our result to any variety Z.

4.1 The boundary at infinity of a smooth variety

We start by a general discussion of the notion of boundary at infinity of a space, and study the specific case of complex algebraic varieties. These results are not new and we do not claim any originality, but we record them here for the lack of an adequate reference.

Definition 4.2 *The* **boundary of a topological space** Y *is by definition the pro-homotopy type*

$$\partial Y := \mathop{\mathfrak{Lim}}_{K \subset Y}(Y - K) \in \mathrm{Pro}(\mathbb{T}),$$

where $\mathfrak{Lim}$ is the limit taken in the ∞-category $\mathrm{Pro}(\mathbb{T})$ of pro-homotopy types and over the opposite category of compact subsets $K \subset Y$.

The pro-object ∂Y is in general not constant and can be extremely complicated. However, when $Y = X = Z(\mathbb{C})$ is the underlying space of a smooth variety Z then ∂Y is equivalent to a constant pro-object. In fact, more is true:

Proposition 4.3 *For a smooth n-dimensional complex algebraic variety Z with underlying topological space $X = Z(\mathbb{C})$, the pro-object ∂X is equivalent to a constant pro-object in $\mathbb{T}$ which has the homotopy type of a compact oriented topological manifold of dimension $2n - 1$.*

Proof: Let $Z \subset \mathfrak{Z}$ be a smooth compactification such that $D = \mathfrak{Z} - Z$ is a divisor with simple normal crossing. Fix a Riemannian metric on the C^∞ manifold underlying $\mathfrak{Z}$ and for any $\epsilon > 0$ consider the compact subsets

$$K_\epsilon := \{ x \in \mathfrak{Z} \,|\, d(x, D) \geq \epsilon \} \subset X.$$

The system of compact subset $\{K_\epsilon\}_{\epsilon \in \mathbb{R}_{>0}}$ is cofinal in the system of all compact subsets of X. We mean here cofinal in the sense of ∞-category theory, and the important consequence is that the two pro-objects $\mathop{\mathfrak{Lim}}_{K \subset Y}(Y - K)$ and $\mathop{\mathfrak{Lim}}_{\epsilon > 0}(Y - K_\epsilon)$ are equivalent in the ∞-category $\mathrm{Pro}(\mathbb{T})$. Moreover, the sets $D_\epsilon = \mathfrak{Z} - K_\epsilon$ of points of distance $< \epsilon$ from D satisfy:

- For $\epsilon_1 < \epsilon_2$ small enough, the inclusion $D_{\epsilon_1} \subset D_{\epsilon_2}$ is a homotopy equivalence;

- For small enough ϵ the tubular neighborhood D_ϵ retracts to D.

This is clear near the smooth points of D. But near a singular point D is given by the local equation $z_1 z_2 \cdots z_k = 0$ for some local complex analytic coordinates $z_1, \ldots, z_n$ on $\mathfrak{Z}$. In this case the function

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$|z_1 z_2 \cdots z_k|^2$ on $\mathfrak{Z}$ has a non vanishing gradient and the gradient flow gives the desired retraction and homotopy equivalence.

Restricting the retraction and homotopy equivalence to the corresponding punctured tubular neighborhoods $D_\epsilon - D = X - K_\epsilon$ we then get that the opens $X - K_\epsilon \subset X$ satisfy:

- For $\epsilon_1 < \epsilon_2$ small enough the inclusion $X - K_{\epsilon_1} \subset X - K_{\epsilon_2}$ is a homotopy equivalence;

- For ϵ small enough, $X - K_\epsilon$ retracts to $\{x \in \mathfrak{Z} \,|\, d(x, D) = \epsilon\}$.

This shows that the pro-object ∂X is equivalent to the constant pro-object $X - K_\epsilon$ for ϵ small enough and that this constant pro-object is given by $\{x \in \mathfrak{Z} \,|\, d(x, D) = \epsilon\}$. But $\{x \in \mathfrak{Z} \,|\, d(x, D) = \epsilon\}$ is a compact submanifold of X of dimension $2n - 1$ as this can be checked locally. Indeed if D is given by the local equation $z_1 \cdots z_k = 0$, then locally the exponenial map on $\mathfrak{Z}$ gives a an identification of $\{x \in \mathfrak{Z} \,|\, d(x, D) = \epsilon\}$ with the closed subset in $\mathbb{C}^n$ given by the equation $|z_1 \cdots z_k| = \epsilon$. It comes, moreover, equiped with a canonical orientation coming from the complex structure of X. $\qquad\square$

Remark 4.4 In the setup of the proof of the previous proposition it is instructive to compare the constant pro-object ∂X with the boundary of the real oriented blowup of $\mathfrak{Z}$ along the normal crossings divisor D. Recall [Gil] that given a strict normal crossings divisor $D \subset \mathfrak{Z}$ in a smooth complex algebraic variety, we can form a new topological space - the real oriented blowup $\mathrm{Bl}_D(\mathfrak{Z})$ of $\mathfrak{Z}$ along D. The space $\mathrm{Bl}_D(\mathfrak{Z})$ comes with a natural continuous map $\pi : \mathrm{Bl}_D(\mathfrak{Z}) \to \mathfrak{Z}$ and is uniquely characterized (see [Gil]) by the properties:

(a) $\pi : \mathrm{Bl}_D(\mathfrak{Z}) - \pi^{-1}(D) \to \mathfrak{Z} - D$ is a homeomorphism.

(b) If $(U, z_1, \ldots, z_n)$ is an analytic chart of Z, such that $U \cap D$ is given by the equation $z_1 \cdots z_k = 0$, and if $S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$ is the unit circle, then

$$\pi^{-1}(U) \cong \left\{ (z, t) \in U \times (S^1)^k \mid z_1 = t_1 |z_1|, \ldots, z_k = t_k |z_k| \right\},$$

and in this identification the projection π is given by $\pi(z, t) = z$.

From (a) and (b) it is clear that $\pi : \mathrm{Bl}_D(\mathfrak{Z}) \to \mathfrak{Z}$ is defined in the C^∞ category and as a C^∞ object $\mathrm{Bl}_D(\mathfrak{Z})$ is a manifold with corners. As a topological space $\mathrm{Bl}_D(\mathfrak{Z})$ is just a topological manifold with boundary δ given by the preimage of D. The topological manifold with boundary $\mathrm{Bl}_D(\mathfrak{Z})$ is homotopy equivalent to its interior $\mathrm{Bl}_D(\mathfrak{Z}) - \delta = \mathfrak{Z} - D$ and the pair $(\mathrm{Bl}_D(\mathfrak{Z}), \delta)$ is homotopy equivalent to the pair $(X, \partial X)$. Thus $\delta \cong \pi^{-1}(D)$ provides another model for the constant pro-object ∂X.

Note however that the structure of δ as a C^∞ manifold with corners or even as a stratified topological manifold depends on the good compactification $\mathfrak{Z}$ of X. Indeed if we replace $\mathfrak{Z}$ by the usual complex

blow up $\widehat{\mathfrak{Z}}$ of a point p in $\mathfrak{Z}$ which is a smooth point in D, then $\widehat{\mathfrak{Z}}$ is a new good compactification of X whose boundary divisor $\widehat{D}$ has an extra component. The real oriented blow up $\mathrm{Bl}_{\widehat{D}}(\widehat{\mathfrak{Z}})$ will have an extra corner and so will have a boundary $\widehat{\delta}$ which is the same as a topological manifold but is different as a c^∞ manifold with corners.

This is the reason of why we only view ∂X as the homotopy type of a topological manifold and not as the isotopy type of a stratified manifold or a manifold with corners: we need an notion which is intrinsically associated to X, and does not depend on a particular good compactification.

By construction both X and ∂X have the homotopy type of a finite CW complex, and thus the derived stacks $Loc_G(X)$ and $Loc_G(\partial X)$ discussed in the previous section are derived Artin stacks of finite presentation. Moreover, the canonical map $\partial X \longrightarrow X$ induces a restriction morphism of derived Artin stacks

$$r : Loc_G(X) \longrightarrow Loc_G(\partial X).$$

Since the constant pro-object ∂X can be identified with the topological submanifold $\{x \in \mathfrak{Z} \mid d(x, D) = \epsilon\}$ of the complex manifold Z, we see that ∂X inherits a canonical orientation of dimension $2n - 1$. Thus by [PTVV] the derived stack $Loc_G(\partial X)$ carries a canonical $(3 - 2n)$-shifted symplectic structure which depends only on this canonical orientation and on the chosen non-degenerate G-invariant bilinear form on $\mathfrak{g}$. In fact more is true: the morphism $\partial X \longrightarrow X$ has the homotopy type of the inclusion of the boundary of an oriented $2n$-dimensional manifold. By [Ca] this implies that the restriction morphism

$$r : Loc_G(X) \longrightarrow Loc_G(\partial X)$$

carries a canonical Lagrangian structure with respect to the canonical shifted symplectic structure on $Loc_G(\partial X)$ we just described. On the level of tangent complexes, this Lagrangian structure reflects Poincaré-Lefschetz duality of the manifold with boundary $(X, \partial X)$. For a given G-local system ρ on X, the Lagrangian structure provides a natural quasi-isomorphism of complexes

$$\mathbb{T}_{Loc_G(X), \rho} \simeq \mathbb{L}_{Loc_G(X)/Loc_G(\partial X), \rho}[2 - 2n]$$

which on cohomology spaces induces the Poincaré duality isomorphism on $(X, \partial X)$ with coefficients in $ad(\rho)$:

$$H^i(X, ad(\rho)) \simeq H^{2n-i}(X, \partial X; ad(\rho))^\vee.$$

As explained in the previous section the Lagrangian morphism $r : Loc_G(X) \longrightarrow Loc_G(\partial X)$ defines a canonical $(2 - 2n)$-shifted Poisson structure on the derived Artin stack $Loc_G(X)$ and so our problem 4.1 reduces to the problem of describing the **generalized symplectic leaves** of $Loc_G(X)$.

As we saw in Example 2.1, when Z is of complex dimension one the generalized symplectic leaves are obtained by quasi-Hamiltonian reduction. Recall that in the language of derived algebraic geometry the

pertinent reductions were constructed as Lagrangian intersections. Indeed if $\dim_{\mathbb{C}} Z = 1$, the boundary ∂X has the homotopy type of a disjoint union of oriented circles, and so the restriction map r can be identified with a map

$$r : Loc_G(X) \longrightarrow \prod_i [G/G],$$

where the product is taken over the points of $\mathfrak{Z} - Z$ for some smooth compactification $\mathfrak{Z}$ of Z. The Lagrangian structure on this map is equivalent to the data of a quasi-Hamiltonian system, i.e. of the data of an equivariant group-valued moment map (see [Ca, Sa] for details). Fix elements $\lambda_i \in G$ for each point $i \in \mathfrak{Z} - Z$, and consider the centralizers $G_{\lambda_i} \subset G$ of the elements λ_i. We have canonical maps $BG_{\lambda_i} \longrightarrow [G/G]$ which are the residual gerbes of each points λ_i in $[G/G]$. For the canonical 1-shifted symplectic structure on $[G/G]$ each of the maps $BG_{\lambda_i} \longrightarrow [G/G]$ comes equiped with a canonical Lagrangian structure (for degree reasons the space of Lagrangian structures on this map is a contractible space). As a result, we can form the Lagrangian intersection

$$Loc_G(X, \{\lambda_i\}) := Loc_G(X) \underset{\prod_i [G/G].}{\times} \prod_i BG_{\lambda_i}$$

This is the derived Artin stack of G-local systems on Z with local monodromy around the point $i \in \mathfrak{Z} - Z$ fixed to be in the conjugacy class of λ_i. Being a Lagrangian intersection of 1-shifted Lagrangian structures this derived stack carries a canonical 0-shifted symplectic structure, which on the smooth locus recovers the well known symplectic structure on symplectic leaves in character varieties.

Going back to the general case where Z is not necessarily a curve anymore we again would like to realize the generalized symplectic leaves of the shifted Poisson derived stack $Loc_G(X)$ by an appropriate quasi-Hamiltonian reduction construction. For this we start by fixing a good smooth compactification $\mathfrak{Z}$ of Z, i.e. a smooth proper complex variety $\mathfrak{Z}$, containing Z as a Zariski open subset and such that $D = \mathfrak{Z} - Z$ is a simple normal crossing divisor.

The idea is to again construct another Lagrangian map $Loc_G(\partial X, \{\lambda_i\}) \longrightarrow Loc_G(\partial X)$, where the λ_i are elements in G but now i labels the irreducible components D_i of $D = \mathfrak{Z} - Z$. In the presence of intersections of the components of D the construction of the Lagrangian $Loc_G(\partial X, \{\lambda_i\}) \rightarrow Loc_G(\partial X)$ appears to be quite complicated. However, we analyse below two special cases: the case of a smooth divisor at infinity and the case where D has only two irreducible components, or more generaly has no more than double points (which is enough for the case of dimension 2). We believe that the general case can be handled using similar ideas but we have not pursued this direction.

4.2   The smooth divisor case

First we consider the simplest case where D is a smooth divisor - a disjoint union of connected components D_i. In this case ∂X has the homotopy type of an oriented circle bundle over $\bigsqcup_i D_i$, which is

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classified by a collection $\{\alpha_i\}$ where $\alpha_i \in H^2(D_i, \mathbb{Z})$ is the first Chern class of the normal bundle of $D_i \subset \mathfrak{Z}$. Let us fix as above elements $\lambda_i \in G$ with centralizers $G_{\lambda_i} \subset G$. The group S^1 acts on the stack $[G/G] = \mathbf{Map}(S^1, BG)$ by loop rotations and this action and the cohomology classes α_i can be used to define twisted versions $_{\alpha_i}\widetilde{[G/G]}$ of $[G/G]$ on each D_i.

To understand this properly we first need to discuss the notion of a locally constant family of derived stacks over a space. For this, we recall that any space T can be viewed as a constant derived stack $T \in \mathbf{dSt}_k$ over k. By definition, a **family of derived stacks over** T is a derived stack F together with a map $F \longrightarrow T$. If T is connected, all the fibers of $F \longrightarrow T$ are abstractly equivalent as objects in $\mathbf{dSt}_k$. We say that the family has fiber F_0 if all its fibers are (non-canonically) equivalent to F_0. Since $\mathbf{dSt}_k$ is an ∞-topos, there is an equivalence between H-equivariant derived stacks and derived stacks over BH. Below we apply this systematically to the case when $H = S^1 = B\mathbb{Z}$.

Write the classes α_i as continuous maps

$$\alpha_i : D_i \longrightarrow BS^1.$$

As the group S^1 acts on the stack $[G/G]$ we can form the quotient $[[G/G]/S^1]$ which is a stack over BS^1. Using α_i we can pull back $[[G/G]/S^1]$ to D_i by α_i to get a locally constant family of stacks on D_i, whose fibers are $[G/G]$. We denote this family by $_{\alpha_i}\widetilde{[G/G]} \to D_i$ and we write

$$\widetilde{[G/G]} = \bigsqcup_i {}_{\alpha_i}\widetilde{[G/G]} \longrightarrow \bigsqcup_i D_i = D$$

for the corresponding locally constant family over all of D.

Alternatively we can construct $\widetilde{[G/G]}$ as follows. The class α_i defines a circle bundle $\widetilde{D}_i \to D_i$, and so the collection $\{\alpha_i\}$ defines a circle bundle $p : \widetilde{D} = \bigsqcup_i \widetilde{D}_i \to \bigsqcup_i D_i = D$ over all of D. In terms of this projection we have $\widetilde{[G/G]} \simeq p_*(BG)$ as derived stacks over D.

Next observe that for each i, the group S^1 also acts on the classifying stack BG_{λ_i}, by means of the central element $\lambda_i \in Z(G_{\lambda_i}) = \pi_1(\mathsf{aut}(BG_{\lambda_i}), \mathsf{id})$. Moreover, for each i the canonical 1-shifted Lagrangian map $BG_{\lambda_i} \longrightarrow [G/G]$ comes equipped with a natural S^1-equivariant structure for the S^1 actions on BG_{λ_i} and $[G/G]$. Twisting the source and target of this Lagrangian map by using α_i we get locally constant families of stacks $_{\alpha_i}\widetilde{BG}_{\lambda_i} \to D_i$ and $_{\alpha_i}\widetilde{[G/G]} \to D_i$, and a 1-shifted Lagrangian morphism

$$_{\alpha_i}\widetilde{BG}_{\lambda_i} \longrightarrow {}_{\alpha_i}\widetilde{[G/G]} \tag{2}$$

inside the ∞-category of locally constant families of derived Artin stacks over D_i. Since each D_i is a compact topological manifold endowed with a canonical orientation, the map (2) induces on derived stack of global sections induces a $(3 - 2n)$-shifted Lagrangian morphism of derived Artin stacks

$$r_i : Loc_{G_{\lambda_i}, \alpha_i}(D_i) \longrightarrow Loc_G(\partial_i X).$$

This result is a consequence of the slight generalization of the main theorem of [PTVV] for which the mapping stacks are replaced by global sections of locally constant derived stacks. This slight generalization is proven the exact same way as the case of constant coefficients, and we will freely use it in this paper.

Here, $\partial_i X$ is the connected component of ∂X lying over D_i, and by definition $Loc_{G_{\lambda_i}, \alpha_i}(D_i)$ is the derived stack of α_i-twisted principal G_{λ_i}-bundles on D_i. Combining all the r_i we get the desired $(3 - 2n)$-shifted Lagrangian morphism

$$r = \prod_i r_i : \prod_i Loc_{G_{\lambda_i}, \alpha_i}(D_i) \longrightarrow \prod_i Loc_G(\partial_i X) = Loc_G(\partial X).$$

By the Lagrangian intersection theorem of [PTVV] we thus have that the fiber product of derived stacks

$$Loc_G(X, \{\lambda_i\}) := \left(\prod_i Loc_{G_{\lambda_i}, \alpha_i}(D_i) \right) \underset{Loc_G(\partial X)}{\times} Loc_G(X)$$

carries a canonical $(2 - 2n)$-shifted symplectic structure. By construction/definition, $Loc_G(X, \{\lambda_i\})$ is the derived stack of locally constant G-bundles on X whose local monodromy around D_i is fixed to be in the conjugacy class $\mathbf{C}_{\lambda_i}$ of λ_i. Also by construction the natural projection

$$Loc_G(X, \{\lambda_i\}) \longrightarrow Loc_G(X)$$

exhibits $Loc_G(X, \{\lambda_i\})$ as a **symplectic leaf** of the $(2 - 2n)$-shifted Poisson structure on $Loc_G(X)$.

Remark 4.5 Note that the derived stack $Loc_{G_{\lambda_i}, \alpha_i}(D_i)$ may be empty. Indeed, the groupoid of k-points of this stack is the groupoid of G-local systems on $\partial_i X$ whose local monodromy around D_i is conjugate to λ_i. These k-points can also be described as follows. Let $Z(G_{\lambda_i})$ be the center of G_{λ_i}. Any $G_{\lambda_i}/Z(G_{\lambda_i})$-local system on D_i determines a class in $H^2(D_i, Z(G_{\lambda_i}))$, which is the obstruction to lifting this local system to a G_{λ_i}-local system. For $Loc_{G_{\lambda_i}, \alpha_i}(D_i)$ to be non-empty one needs to have a $G_{\lambda_i}/Z(G_{\lambda_i})$-local system on D_i whose obstruction class matches with the image of α_i under the map $H^2(D_i, \mathbb{Z}) \to H^2(D_i, Z(G_{\lambda_i}))$ given by $\lambda_i : \mathbb{Z} \to Z(G_{\lambda_i})$. Given α_i and λ_i the existence of such a local system is a subtle question, closely related to existence of Azumaya algebras. For instance when λ_i is a regular semi-simple element then G_{λ_i} is a maximal torus in G (assume G simple and k algebraicaly closed), and thus we see that the image of α_i in $H^2(D_i, G_{\lambda_i})$ must be zero. For instance, if in this situation λ_i is of infinte order, this forces α_i to be a torsion class in $H^2(D_i, \mathbb{Z})$.

4.3   The case of two components

We now assume that $D = D_1 \cup D_2$ is the union of two smooth irreducible components meeting transversally at a smooth codimension two subvariety $D_{12} = D_1 \cap D_2$. Since the local fundamental group of

$3 - D$ is abelian we fix two commuting elements λ_1, λ_2 in G. Our goal is to construct a derived moduli stack $Loc_G(X, \{\lambda_1, \lambda_2\})$ of G-bundles on X with fixed monodromy λ_1 around D_1 and fixed monodromy λ_2 on around D_2 and to realize this stack as a generalized symplectic leaf of $Loc_G(X)$.

To set up the problem we need to introduce some notation and auxiliary stacks. In this setting the homotopy type ∂X can be represented (see Remark 4.4) as a homotopy push-out

$$\partial X \simeq \partial_1 X \bigsqcup_{\partial_{12}X} \partial_2 X.$$

Here $\partial_i X$ is an oriented circle bundle over $D_i^o = D_i - D_{12}$, and $\partial_{12}X$ is an oriented $S^1 \times S^1$-bundle over D_{12}. These circle bundles are the restrictions of the natural circle bundles in $\mathcal{O}_3(D_i)$ or equivalently of the natural circle bundles in the normal bundles of D_i in 3. The space $\partial_{12}X$ has the homotopy type of an oriented compact manifold of dimension $2n - 2$, and each component $\partial_i X$ has the homotopy type of an oriented compact manifold of dimension $2n - 1$ with boundary canonically identified with $\partial_{12}X$. In the same manner each boundary $\partial(D_i^o)$ is naturally identified with an oriented S^1-fibration over D_{12}.

For each D_i^o we have a $\mathbb{Z}$-gerbe on D_i^o given by restriction of $\alpha_i \in H^2(D_i, \mathbb{Z})$, which is the restriction of the first Chern class of the normal bundle of D_i inside 3. As before, we can form the α_i-twisted Lagrangian maps

$$\alpha_i \widetilde{BG}_{\lambda_i} \longrightarrow \alpha_i \widetilde{[G/G]},$$

of locally constant derived stacks on D_i^o. We now use the mapping theorem for manifolds with boundary of [Ca] (see also [To1]) applied to the manifold with boundary D_i^o and the Lagrangian map above. Unfolding the definitions we get a Lagrangian map of derived Artin stacks

$$\Gamma(D_i^o;\ \alpha_i \widetilde{BG}_{\lambda_i}) \longrightarrow \Gamma(\partial(D_i^o);\ \alpha_i \widetilde{BG}_{\lambda_i}) \bigtimes_{\Gamma(\partial(D_i^o);\ \alpha_i \widetilde{[G/G]})} \Gamma(D_i^o;\ \alpha_i \widetilde{[G/G]}),$$

where Γ here denotes the derived stack of global sections[2]. By construction, we have

$$\Gamma(D_i^o;\ \alpha_i \widetilde{[G/G]}) \simeq Loc_G(\partial_i X) \quad \text{and} \quad \Gamma(\partial(D_i^o);\ \alpha_i \widetilde{[G/G]}) \simeq Loc_G(\partial_{12}X).$$

We write

$$Loc_G(\partial_i X, \lambda_i) := \Gamma(D_i^o;\ \alpha_i \widetilde{BG}_{\lambda_i}),$$

for the ***derived stack of G-bundles on $\partial_i X$ with monodromy λ_i around D_i^o***. Similarly we write

$$Loc_G(\partial_{12}X, \lambda_i) := \Gamma(\partial(D_i^o);\ \alpha_i \widetilde{BG}_{\lambda_i}),$$

[2]As explained above derived stacks of global sections are the twisted version of derived mapping stacks and can be defined formally as being direct images of derived stacks.

for the **derived stack of G-bundles on $\partial_{12}X$ with monodromy λ_i around D_i.** We can thus rewrite the above Lagrangian maps as

$$\ell_i : Loc_G(\partial_i X, \lambda_i) \longrightarrow Loc_G(\partial_{12}X, \lambda_i) \underset{Loc_G(\partial_{12}X)}{\times} Loc_G(\partial_i X).$$

For $i = 1, 2$ these are two Lagrangian maps towards an $(3 - n)$-shifted symplectic target. We can consider the direct product, which is still a Lagrangian morphism

$$\ell := \ell_1 \times \ell_2 : Loc_G(\partial_1 X, \lambda_1) \times Loc_G(\partial_2 X, \lambda_2) \longrightarrow \prod_{i=1,2} Loc_G(\partial_{12}X, \lambda_i) \underset{Loc_G(\partial_{12}X)}{\times} Loc_G(\partial_i X).$$

Here we think of ℓ as a Lagrangian correspondence between two Lagrangians in $Loc_G(\partial_{12}X) \times Loc_G(\partial_{12}X)$, namely

$$Loc_G(\partial_{12}X, \lambda_1) \times Loc_G(\partial_{12}X, \lambda_2) \longrightarrow Loc_G(\partial_{12}X) \times Loc_G(\partial_{12}X)$$

and

$$Loc_G(\partial_1 X) \times Loc_G(\partial_2 X) \longrightarrow Loc_G(\partial_{12}X) \times Loc_G(\partial_{12}X).$$

Pulling back everything to the diagonal of $Loc_G(\partial_{12}X) \times Loc_G(\partial_{12}X)$ we get a Lagrangian morphism

$$\ell : Loc_G(\partial_1 X, \lambda_1) \underset{Loc_G(\partial_{12}X)}{\times} Loc_G(\partial_2 X, \lambda_2) \longrightarrow Loc_G(\partial X) \times Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\}),$$

where we use the short cut notation

$$Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\}) := Loc_G(\partial_{12}X, \lambda_1) \underset{Loc_G(\partial_{12}X)}{\times} Loc_G(\partial_{12}X, \lambda_2).$$

In contrast with the smooth divisor case this setting has an important new feature, namely the extra term $Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\})$, which does not appear when the smooth components of D do not intersect. Thus, in order to get a Lagrangian map towards $Loc_G(\partial X)$ alone we need to find an extra Lagrangian mapping to $Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\})$. It is not clear to us that such a Lagrangian always exists, but there is a natural candidate for it that we will now describe.

We let $G_{(\lambda_1, \lambda-2)} = G_{\lambda_1} \cap G_{\lambda_2}$ be the centralizer of the pair (λ_1, λ_2). On D_{12}, we have a natural $\mathbb{Z}^2$-gerbe, i.e. the external sum $\alpha_1 \boxplus \alpha_2$ of the restrictions of the two gerbes over D_i. It corresponds to $\partial_{12}X$ as a principal $S^1 \times S^1$-bundle over D_{12}. The group $S^1 \times S^1$ acts on the stack $BG_{(\lambda_1, \lambda_2)}$, by the canonical map $\mathbb{Z}^2 \to \pi_1(\mathsf{aut}(BG_{(\lambda_1, \lambda_2)}), \mathsf{id}) = Z(G_{(\lambda_1, \lambda_2)})$ given by the pair (λ_1, λ_2). This provides a twist $_{\alpha_1 \boxplus \alpha_2}\widetilde{BG}_{(\lambda_1, \lambda_2)}$ of $BG_{(\lambda_1, \lambda_2)}$ on D_{12}. In the same way, we can define a twist of the double loop stack $\mathbf{Map}(S^1 \times S^1, BG) = [G * G/G]$, where $G * G$ is the derived subscheme of commuting elements in $G \times G$. That is we have a twisted form $_{\alpha_1 \boxplus \alpha_2}\widetilde{[G * G/G]}$ of $[G * G/G]$ over D_{12}. The two elements λ_i, provide natural inclusion maps

$$G_{\lambda_2} \times \{\lambda_2\} \longrightarrow G * G, \qquad \{\lambda_1\} \times G_{\lambda_1} \longrightarrow G * G.$$

These induce two inclusion maps on quotient stacks $[G_{\lambda_i}/Z_{\lambda_i}] \to [G*G/G]$, which are naturally $S^1 \times S^1$-equivariant. We thus get maps of twisted stacks on D_{12}, i.e. $_{\alpha_i}[\widetilde{G_{\lambda_i}/G_{\lambda_i}}] \longrightarrow {}_{\alpha_1 \boxplus \alpha_2}[\widetilde{G*G/G}]$. Denote the fiber product of these two maps by $\mathcal{F}_{12}$. By definition, this is a locally constant family of derived Artin stacks furnished with a fiberwise (-1)-shifted symplectic structures, and so we get an equivalence of derived Artin stacks equipped with $(3-n)$-shifted symplectic structures

$$\Gamma(D_{12}, \mathcal{F}_{12}) \simeq Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\}).$$

There is a canonical point (λ_1, λ_2) inside $[G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$ whose stabilizer is $G_{(\lambda_1, \lambda_2)}$. This induces a morphism $BG_{(\lambda_1, \lambda_2)} \longrightarrow [G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$, which is is $S^1 \times S^1$-equivariant in a natural way. We therefore get a twisted version of this map $_{\alpha_1 \boxplus \alpha_2}\widetilde{BG}_{(\lambda_1, \lambda_2)} \longrightarrow \mathcal{F}_{12}$. This map has a canonical isotropic structure, and to be more precise the space of isotropic structures on the above map is a contractible space for degree reasons. By taking global sections we thus obtain an isotropic map

$$\ell_{12} : Loc_{G_{(\lambda_1, \lambda_2)}, \boldsymbol{\alpha}}(D_{12}) \longrightarrow Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\})),$$

where $Loc_{G_{(\lambda_1, \lambda_2)}, \boldsymbol{\alpha}}(D_{12})$ is defined to be $\Gamma(D_{12}; {}_{\alpha_1 \boxplus \alpha_2}\widetilde{BG}_{(\lambda_1, \lambda_2)})$.

The question now reduces to understanding whether the isotropic map ℓ_{12} is Lagrangian. This is the case when the map of derived stacks

$$BG_{(\lambda_1, \lambda_2)} \longrightarrow [G_{\lambda_1}/G_{\lambda_1}] \underset{[G*G/G]}{\times} [G_{\lambda_2}/G_{\lambda_2}]$$

is Lagrangian. A simple examination of the amplitudes of the tangent complexes shows that this map is Lagrangian if and only if the tangent complex of $[G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$ at the canonical point (λ_1, λ_2) is cohomologically concentrated in the two extremal degrees -1 and 2. This leads to the following notion.

Definition 4.6 *A pair of elements $(\lambda_1, \lambda_2) \in G \times G$ is called* **strict** *if it is a commuting pair and if the morphism*

$$BG_{(\lambda_1, \lambda_2)} \longrightarrow [G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$$

is Lagrangian (for its canonical isotropic structure).

Assume that (λ_1, λ_2) is a strict pair. We now have a new Lagrangian

$$\ell_{12} : Loc_{G_{(\lambda_1, \lambda_2)}, \boldsymbol{\alpha}}(D_{12}) \longrightarrow Loc_G(\partial_{12}X).$$

By composing with the Lagrangian ℓ constructed above, we get the desired Lagrangian map

$$Loc_G(\partial_1 X, \lambda_1) \underset{Loc_G(\partial_{12}X)}{\times} Loc_G(\partial_2 X, \lambda_2) \underset{Loc_G(\partial_{12}X, \{\lambda_1, \lambda_2\}))}{\times} Loc_{G_{(\lambda_1, \lambda_2)}, \alpha}(D_{12}) \longrightarrow Loc_G(\partial X).$$

The pull-back of this morphism along the restriction map $Loc_G(X) \longrightarrow Loc_G(\partial X)$ is thus a derived Artin stack with a $(2 - 2n)$-shifted symplectic structure, and its projection to $Loc_G(X)$ can be thought of as a symplectic leaf of the Poisson structure on $Loc_G(X)$. We denote this symplectic leaf by $Loc_G(X, \{\lambda_1, \lambda_2\})$. We have therefore proven the following result.

Theorem 4.7 *With the notation above.*

1. *The derived Artin stack $Loc_G(X)$ carries a canonical $(2 - 2n)$-shifted Poisson structure, which is realized by the Lagrangian map $Loc_G(X) \longrightarrow Loc_G(\partial X)$.*

2. *Let $\mathfrak{Z}$ be a smooth compactification of Z, and assume that $\mathfrak{Z} - Z = D$ is smooth with connected components D_i. Then, for any choice of elements $\lambda_i \in G$, the derived Artin stack $Loc_G(X, \{\lambda_i\})$, of principal G-bundles on X whose monodromies around D_i in $\mathbf{C}_{\lambda_i}$, carries a natural $(2 - 2n)$-shifted symplectic structure and is a symplectic leaf of $Loc_G(X)$.*

3. *Let $\mathfrak{Z}$ be a smooth compactification of Z, and assume that $\mathfrak{Z} - Z = D_1 \cup D_2$ is a strcit normal crossings divisor with D_i smooth and connected. Then for any commuting pair of elements $(\lambda_1, \lambda_2) \in G \times G$ the natural map*

$$Loc_G(\partial_1 X, \lambda_1) \underset{Loc_G(\partial_{12} X)}{\times} Loc_G(\partial_2 X, \lambda_2) \longrightarrow Loc_G(\partial X) \times Loc_G(\partial_{12} X, \{\lambda_1, \lambda_2\})$$

comes equipped with a natural Lagrangian structure.

4. *If moreover the pair (λ_1, λ_2) is strict then the derived Artin stack*

$$Loc_G(X, \{\lambda_1, \lambda_2\})$$

comes equipped with a natural $(2 - 2n)$-shifted symplectic structure which is a symplectic leaf of $Loc_G(X)$.

Remark 4.8 In order to better understand this proposition it is instructive to examine the situation on the truncated stacks involved. To start with, the truncation of $Loc_G(\partial X)$ is the underived stack of G-local systems on ∂X. This can be described as a quotient stack $[\mathrm{Hom}_{\mathsf{grp}}(\pi_1(\partial X), G)/G]$ (assuming ∂X is connected). The truncation of $Loc_G(\partial_1 X, \lambda_1) \times_{Loc_G(\partial_{12} X)} Loc_G(\partial_2 X, \lambda_2)$ is then the full sub-stack consisting of all G-local systems on ∂X for which the local monodromy around D_i is conjugate to $\lambda_i \in G$. The truncation of the stack $Loc_G(\partial_{12} X, \{\lambda_1, \lambda_2\})$ is the full sub-stack of the stack of G-local systems on $\partial_{12} X$ whose local monodromies around D_i are conjugate to λ_i. Finally, $Loc_{G_{(\lambda_1, \lambda_2)}, \alpha}(D_{12})$ is the full sub-stack of $Loc_G(\partial_{12} X, \{\lambda_1, \lambda_2\})$ whose local monodromy at points of D_{12} is conjugate to the pair $(\lambda_1, \lambda_2) \in G \times G$.

As a consequence, the truncation of the derived stack $Loc_G(X, \{\lambda_1, \lambda_2\})$ is naturally equivalent to the full sub-stack of $Loc_G(\partial X)$ consisting of G-local systems whose local monodromies around D_i are conjugate to λ_i but also whose local monodromy at points in $D_1 \cap D_2$ is conjugate to the pair (λ_1, λ_2). Therefore statement (4) above can be interpretted as the claim that this stack admits a natural derived structure for which it carries a natural $(2 - 2n)$-shifted symplectic structure.

4.4 Strict pairs

The following result provides many examples of strict pairs.

Proposition 4.9 *Let* (λ_1, λ_2) *be a commuting pair of elements in* G, *and* $u := \mathsf{Id} - \mathsf{ad}(\lambda_1)$ *and* $v := \mathsf{Id} - \mathsf{ad}(\lambda_2)$ *be the corresponding endormorphisms of* $\mathfrak{g}$ *induced by the adjoint representation. Then the pair* (λ_1, λ_2) *is strict if and only* u *is strict with respect to the kernel of* v, *i.e. we have*

$$\mathrm{Im}(v_{|\ker(u)}) = \mathrm{Im}(v) \cap \ker(u).$$

Proof: We use the notation introduced above. Consider the derived stack $[G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$ where $G_{\lambda_i} \subset G$ is the centralizer of λ_i. The derived stack $[G * G/G] = Loc_G(S^1 \times S^1)$ carries a canonial 0-shifted symplectic structure, and each map $[G_{\lambda_i}/G_{\lambda_i}] \longrightarrow [G*G/G]$ is Lagrangian. Therefore $[G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$ carries a canonical (-1)-shifted symplectic structure. For degree reasons the isotropic map

$$BG_{(\lambda_1, \lambda_2)} \longrightarrow [G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$$

is Lagrangian if and only if the tangent complex $\mathbb{T}$ of $[G_{\lambda_1}/G_{\lambda_1}] \times_{[G*G/G]} [G_{\lambda_2}/G_{\lambda_2}]$ taken at the canonical point (λ_1, λ_2) is such that

$$H^0(\mathbb{T}) = H^1(\mathbb{T}) = 0.$$

As $\mathbb{T}$ is equipped with a (-1)-shifted symplectic form, we have that $H^0(\mathbb{T}) = 0$ if and only if $H^1(\mathbb{T}) = 0$. Therefore, the pair (λ_2, λ_1) is strict if and only if $H^0(\mathbb{T}) = 0$.

Let $x := \mathsf{ad}(\lambda_1)$ and $y := \mathsf{ad}(\lambda_2)$. The space $H^0(\mathbb{T})$ sits in an five term exact sequence

$$H^0_y(H^0_x(\mathfrak{g})) \oplus H^0_x(H^0_y(\mathfrak{g})) \longrightarrow H^0_{x,y}(\mathfrak{g}) \longrightarrow H^0(\mathbb{T}) \longrightarrow H^1_y(H^0_x(\mathfrak{g})) \oplus H^1_x(H^0_y(\mathfrak{g})) \longrightarrow H^1_{x,y}(\mathfrak{g}) \longrightarrow \dots$$

Here x and y are considered as actions of $\mathbb{Z}$ on $\mathfrak{g}$, and $H^\bullet_x$ and $H^\bullet_y$ denote group cohomology of $\mathbb{Z}$ with coefficients in $\mathfrak{g}$. In the same way $H^\bullet_{x,y}$ denotes group cohomology of $\mathbb{Z}^2$ with coefficients in $\mathfrak{g}$.

We have canonical isomorphisms $H^0_x(H^0_y) \simeq H^0_y(H^0_x) \simeq H^0_{x,y}$ and the first map above is isomorphic to the sum map on $H^0_{x,y}(\mathfrak{g})$, and therefore is surjective. This implies that $H^0(\mathbb{T}) = 0$ if and only if the last morphism

$$\phi : H^1_y(H^0_x(\mathfrak{g})) \oplus H^1_x(H^0_y(\mathfrak{g})) \longrightarrow H^1_{x,y}(\mathfrak{g})$$

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is an injective map. Using the Serre spectral sequence for the projection to the first factor $\mathbb{Z}^2 \longrightarrow \mathbb{Z}$ we get a short exact sequence

$$0 \longrightarrow H_x^1(H_y^0(\mathfrak{g})) \longrightarrow H_{x,y}^1(\mathfrak{g}) \longrightarrow H_x^0(H_y^1(\mathfrak{g})) \longrightarrow 0.$$

The morphism ϕ above is compatible with this short exact sequence and provides a commutative diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & H_x^1(H_y^0(\mathfrak{g})) & \longrightarrow & H_{x,y}^1(\mathfrak{g}) & \longrightarrow & H_x^0(H_y^1(\mathfrak{g})) & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & H_x^1(H_y^0(\mathfrak{g})) & \longrightarrow & H_y^1(H_x^0(\mathfrak{g})) \oplus H_x^1(H_y^0(\mathfrak{g})) & \longrightarrow & H_y^1(H_x^0(\mathfrak{g})) & \longrightarrow & 0.
\end{array}
$$

The map on the left hand side is an identity, and it thus we see that $H^0(\mathbb{T}) = 0$ if and only if the natural morphism

$$H_y^1(H_x^0(\mathfrak{g})) \longrightarrow H_x^0(H_y^1(\mathfrak{g}))$$

is injective. Unfolding the definition we find the strictness condition of the proposition. $\qquad \square$

Note that since the strictness condition on a pair of elements in G is symmetric by tdefinition, the condition derived in Proposition 4.9 must be symmetric as well. In particular the role of u and v in the statement of Proposition 4.9 can be exchanged, and so both conditions are equivalent to each other and equivalent to strictness. We can use proposition 4.9 to produce the following interesting examples of strict pairs.

Corollary 4.10 *Let (λ_1, λ_2) be a commuting pair of elements in G.*

1. *If at least one of the λ_i is semi-simple then the pair (λ_1, λ_2) is strict.*

2. *Assume that λ_1 and λ_2 are unipotent elements in G, and let*

$$j_i : SL_2 \hookrightarrow G$$

   be group embeddings sending $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ to λ_i. If the two copies of SL_2 in G commute (i.e. the j_1 and j_2 combine into a group homorphism $j_1 \times j_2 : SL_2 \times SL_2 \longrightarrow G$) then the pair (λ_1, λ_2) is strict.

Proof: (1) If λ_1 is semi-simple it defines a grading on $\mathfrak{g}$ which is preserved by $v = \mathsf{Id} - \mathsf{ad}(\lambda_2)$. If $u = \mathsf{Id} - \mathsf{ad}(\lambda_1)$ then $\ker(u)$ is the graded component of degree 0, and this obviously implies that strictness hold $\mathrm{Im}(v_{|\ker(u)}) = \mathrm{Im}(v) \cap \ker(u)$.

(2) The morphism $SL_2 \times SL_2 \longrightarrow G$ induces an $SL_2 \times SL_2$-action on the Lie algebra $\mathfrak{g}$. This action defines a decomposition $\mathfrak{g} = \oplus_{p,q}\mathfrak{g}_{p,q}$ of $\mathfrak{g}$, for which the weights p and q are integers. Moreover with

respect to this decomposition, u acts with bidegree $(1,0)$ and v acts with bidegree $(0,1)$. Finally, the Lefschetz property is satisfied:

$$u_{p,q} : \mathfrak{g}_{p,q} \longrightarrow \mathfrak{g}_{p+1,q}$$

is injective for $p < 0$ and surjective for $p \geq 0$, and similarly

$$v_{p,q} : \mathfrak{g}_{p,q} \longrightarrow \mathfrak{g}_{p,q+1}$$

is injective for $q < 0$ and surjective for $q \geq 0$. Moreover, we have that the map

$$v_{p,q} : \ker(u_{p,q}) \longrightarrow \ker(u_{p,q+1})$$

is surjective for all $q \geq 0$.

Let $x \in \mathrm{Im}(v) \cap \ker(u)$. We can decompose $x = \sum_{p,q} x_{p,q}$ according to the bigrading $\mathfrak{g} = \oplus_{p,q}\mathfrak{g}_{p,q}$, and by the properties above we have $x_{p,q} = 0$ for $p < 0$. As x lies in the image of u, there are $y_{p,q-1}$ such that $v(y_{p,q-1}) = x_{p,q}$. Moreover, for $q \geq 1$ we can chose $y_{p,q-1} \in \ker(u)$. But if $q \leq 0$, we have $vu(y_{p,q-1}) = u(x_{p,q}) = 0$, and because $v_{p,q-1}$ is injective we have $u(y_{p,q-1}) = 0$. This shows that $y = \sum_{p,q} y_{p,q}$ is such that $u(y) = 0$ and $v(y) = x$. Therefore, $\mathrm{Im}(v) \cap \ker(u) = \mathrm{Im}(v_{|\ker u})$ and strictness holds. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

Remark 4.11 • The hypothesis postulating the existence of commuting SL_2's in part (2) of the previous corollary is a special case of the notion of a Jordan-Lefschetz pair defined and studied by Looijenga and Lunts in [Lo-Lu].

• From the proof of part (2) of the above statement, we see that a stronger result holds: the pair (λ_1, λ_2) is strict if a bigrading $\mathfrak{g} = \oplus_{p,q}\mathfrak{g}_{p,q}$ as in proof exists. Such gradings exist for instance in the setting of principal nilpotent pairs of [Gi].

• Even though the strictness condition is local in nature there can be global Hodge theoretic reasons that enforce stricness. For instance the strictness condition will hold automatically for the local monodromies of any polarized variation of pure complex Hodge structures on X. In fact by the Lefchetz package for the tame non-abelian Hodge correspondence, the strictness condition will hold automatically for the local monodromie of any tame locl system on X which has semismiple global monodromy. We thank Takuro Mochizuki for this remark.

Remark 4.12 Finally we note that strictness is a non-trivial condition. For instance, if λ is any non-trivial unipotent element in G, then the pair (λ, λ) does not satisfy the strictness condition of

proposition 4.9 and thus is not a strict pair. Indeed in this case u is a non-zero nilpotent endomorphism of $\mathfrak{g}$ and thus $\ker(u) \cap \mathrm{Im}(u) \neq 0$, but $\mathrm{Im}(u_{|\ker(u)}) = 0$).

4.5   The case of at most double intersection

The discussion above for a divisor at infinity with at most two smooth components can be easily extended to the case of any components with the condition that at most two components intersect at a given point. This is for instance automatic when Z is a surface.

Assume that we have chosen a compactification $\mathfrak{Z}$ such that $D = \mathfrak{Z} - Z$ can be written a union of smooth connected components $D = \cup D_i$ for $i = 1, \ldots, p$. We moreover assume that $D_i \cap D_j$ is connected when non-empty, and we will denote it by D_{ij} (we always assume $i < j$ here). Finally we assume that $D_i \cap D_j \cap D_k = \varnothing$ for any three distinct labels i, j, k. As usual we denote by D_i^o the open in D_i consisting of smooth points of D inside D_i. The boundary ∂X is now (see Remark 4.4) a union of $\partial_i X$ (S^1-fibrations over D_i^o) glued together along components of their boundaries $\partial_{ij} X$ ($S^1 \times S^1$-fibrations over D_{ij}).

For any i we fix an element $\lambda_i \in G$. We assume that (λ_i, λ_j) is a strict pair in the sense of definition 4.6 as soon as $D_{ij} \neq \varnothing$. Let $G_{\lambda_i} \subset G$ be the centralizer of λ_i in G. We have a category $\mathcal{C}_D$, whose objects are the D_i and the D_{ij} as sub-varieties in $\mathfrak{Z}$, and whose morphisms are the inclusions. There is an ∞-functor

$$F : \mathcal{C}_D^{op} \longrightarrow \mathbf{dSt}_k$$

sending each D_i to $Loc_G(\partial_i X, \lambda_i)$, the derived stack of G-local systems on $\partial_i X$ whose local monodromy along D_i is conjugate to λ_i. By definition the ∞-functor F sends D_{ij} to $Loc_G(\partial_{ij} X)$, where $\partial_{ij} X$ is the part of ∂X sitting over D_{ij} as an $S^1 \times S^1$-bundle. The transition morphisms for the ∞-functor F are defined by restriction.

Let $\mathcal{F}$ be the limit of F inside derived stacks. It has a natural projection to the product

$$\mathcal{F} \longrightarrow \prod_{i<j} Loc_G(\partial_{ij} X, \{\lambda_i, \lambda_j\}),$$

where $Loc_G(\partial_{ij} X, \{\lambda_i, \lambda_j\})$ is defined as before. For each $i < j$ we have a canonical morphism

$$Loc_{G_{(\lambda_i, \lambda_j)}, \alpha}(D_{ij}) \longrightarrow Loc_G(\partial_{ij} X, \{\lambda_i, \lambda_j\}),$$

where $Loc_{G_{(\lambda_i, \lambda_j)}, \alpha}(D_{ij})$ is the derived stack of twisted $G_{(\lambda_i, \lambda_j)}$-local systems on D_{ij} as defined before. The pull-back possesses a natural morphism towards $Loc_G(\partial X)$

$$\mathcal{F} \times_{\prod_{i<j} Loc_G(\partial_{ij} X, \{\lambda_i, \lambda_j\})} \prod_{i<j} Loc_{G_{(\lambda_i, \lambda_j)}, \alpha}(D_{ij}) \longrightarrow Loc_G(\partial X).$$

This proves the following

Proposition 4.13 *Under the above assumptions there exists a natural Lagrangian structure on the morphism*

$$\mathcal{F} \underset{\prod_{i<j} Loc_G(\partial_{ij} X, \{\lambda_i, \lambda_j\})}{\times} \prod_{i<j} Loc_{G_{(\lambda_i, \lambda_j)}, \boldsymbol{\alpha}}(D_{ij}) \longrightarrow Loc_G(\partial X).$$

We can define the derived stack $Loc_G(X, \{\lambda_1, \ldots, \lambda_p\})$ as the pull-back of the Lagrangian in this proposition by the restriction map $Loc_G(X) \to Loc_G(\partial X)$. As a corollary $Loc_G(X, \{\lambda_1, \ldots, \lambda_p\})$ carries a natural $(2 - 2n)$-shifted symplectic structure. As before, the truncation of $Loc_G(X, \{\lambda_1, \ldots, \lambda_p\})$ is the full sub-stack of $Loc_G(X)$ consisting of all G-local systems on X whose local monodromies around D_i is conjugate to λ_i, and whose local monodromies at D_{ij} is conjugate to the strict pair (λ_i, λ_j).

5 Towards a Poisson moduli of connections

We would like to finish this manuscript with some ideas of how to extend the present results when local systems are replace by bundles with flat connexions. To start with, for a smooth complex algebraic variety X, it is not possible anymore to use the boundary ∂X, as this would only make sense in the holomorphic category. Moreover, when X is defined over a smaller field $K \subset \mathbb{C}$ we also want the moduli of flat bundles on X to be defined over K. As a consequence, if we want to generalize theorem 4.9 to the case of flat bundles a first step is to find an algebraic counterpart of ∂X.

As far as we know there is no algebraic version of ∂X, however several authors have been studying in the recent years a formal analogue denoted by $\widehat{\partial} X$ (see [Be-Te, Ef, He-Po-Ve]). For a good compactification $\mathfrak{Z}$ of X, with divisor $D = \mathfrak{Z} - X$, the *formal boundary at infinity of X* is morally defined as $\widehat{D} - D$, where $\widehat{D}$ is the formal completion of $\mathfrak{Z}$ along D. This is only a moral definition as $\widehat{D} - D$ does not actually make sense (it is an empty space when considered in the sense of formal schemes), but several possible incarnations of this object have been introduced in [Be-Te, Ef, He-Po-Ve]. For us, we follow the approach of [Ef] and [He-Po-Ve], which do not define $\widehat{\partial} X$ as an object in its own, but define categories and stacks of sheaves of perfect complexes $Perf(\widehat{\partial} X)$. Using the same line of ideas it is possible to define the derived stack of vector bundles on $\widehat{\partial} X$ endowed with flat connections $\mathsf{Vect}^{\nabla}(\widehat{\partial} X)$. One key result, proved in [Ef], is that $\mathsf{Vect}^{\nabla}(\widehat{\partial} X)$ depends on X alone and not of the chosen compactification $\mathfrak{Z}$ used to define it. The derived stack $\mathsf{Vect}^{\nabla}(\widehat{\partial} X)$ is our algebraic analogue of $Loc_G(\partial X)$ studied in this work. It is then possible to prove statements analogue to the results mentioned in this work. As an example we state here a result that will appear in [PT].

Theorem 5.1 *Let X be a smooth algebraic variety over k of dimension d and $\mathsf{Vect}^{\nabla}(X)$ the derived stack of vector bundles with flat connections on X.*

1. *There is a restriction map $r : \mathsf{Vect}^{\nabla}(X) \to \mathsf{Vect}^{\nabla}(\widehat{\partial} X)$. This map is endowed with a canonical Lagrangian structure of degree $2 - 2d$.*

2. The fibers of r are representable by derived quasi-algebraic stacks locally of finite presentation.

Some comments about the previous statement. First of all, we do not impose any regularity assumption on the connections, and $\mathsf{Vect}^\nabla(X)$ is the derived stack of all connections. In contrast to the case of local systems, the derived stacks $\mathsf{Vect}^\nabla(X)$ and $\mathsf{Vect}^\nabla(\widehat{\partial}X)$ are not representable as they can have infinite dimensional deformation spaces over general ring valued points. The meaning of statement (1) is thus subtle as one has to work with notions such as symplectic and Lagrangian structures on non-representable objects. Moreover, the object $\widehat{\partial}X$ does not exist on its own, so the usual constructions methods for symplectic structures of [Ca, PTVV] do not apply as these are based on evaluation maps which do not exist here. We overcome this difficulty by using a completely different construction method, based on rigid tensor categories and explained in the note [To2]. The consequence of (1) is of course that $\mathsf{Vect}^\nabla(X)$ carries a canonical Poisson structure. Finally, the representability statement (2) states that the derived moduli of flat connections whose formal structures are fixed at infinity is representable.

We also beleive that symplectic leaves of the Poisson structure on $\mathsf{Vect}^\nabla(X)$ can be defined and studied in a similar fashion as what we have done in the topological setting. We expect (2) above to ensure that these symplectic leaves are indeed representable by actual derived algebraic stacks of finite type. Hopefully, the two results 4.9 and 5.1 can then be related by means of the Riemann-Hilbert correspondence. Ultimately one also has to study derived moduli of Higgs bundles in a similar fashion, and relate to three kind of moduli spaces by means of the non-abelian Hodge correspondence of T. Mochizuki.

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