Math. Ann. 275, 365–399 (1986)

Mathematische Annalen
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Stable Trace Formula: Elliptic Singular Terms

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Let F be a local field of characteristic 0. For any torus T over F Tate-Nakayama duality yields a canonical isomorphism

$$H^1(F, T) \xrightarrow{\sim} \pi_0(\hat{T}^\Gamma)^D,$$

where $\hat{T}$ is the connected Langlands dual group of T, Γ is the absolute Galois group of F, $\pi_0(\ )$ denotes group of connected components, and $(\ )^D$ denotes Pontryagin dual.

If F is p-adic, this isomorphism can be generalized to a canonical bijection

$$H^1(F, G) \xrightarrow{\sim} \pi_0(Z(\hat{G})^\Gamma)^D,$$

where $Z(\hat{G})$ denotes the center of $\hat{G}$. This was done in [K3], but the proof given there does not extend to the case $F = \mathbb{R}$. In fact, in the real case all we get is a canonical map

$$H^1(\mathbb{R}, G) \to \pi_0(Z(\hat{G})^\Gamma)^D.$$

In 1.2 we construct this map and determine its kernel and image. The construction works for p-adic fields as well. In Sect. 1 we establish a number of basic properties of the map.

Now let F be a number field. For any torus T over F Tate-Nakayama duality has the following three consequences. First, there exists a canonical isomorphism

$$\ker^1(F, T) \xrightarrow{\sim} \ker^1(F, \hat{T})^D,$$

where $\ker^i(F, *)$ denotes the kernel of

$$H^i(F, *) \to \prod_v H^i(F_v, *)$$

(the product is taken over all places v of F). Second, there exists a canonical isomorphism

$$H^1(F, T(\mathbb{A})/T(\bar{F})) \xrightarrow{\sim} \pi_0(\hat{T}^\Gamma)^D,$$

* Partially supported by the National Science Foundation under Grant DMS-8200785

where $\bar{F}$ is an algebraic closure of F and $\mathbb{A}$ is its adele ring. Third, the composition

$$H^1(F, T(\mathbb{A})) \to H^1(F, T(\mathbb{A})/T(\bar{F})) \to \pi_0(\hat{T}^\Gamma)^D$$

is obtained by combining the local isomorphisms

$$H^1(F_v, T(\bar{F}_v)) \tilde{\to} \pi_0(\hat{T}^{\Gamma(v)})^D$$

[$\Gamma(v)$ denotes $\mathrm{Gal}(\bar{F}_v/F_v)$] with the homomorphisms

$$\pi_0(\hat{T}^{\Gamma(v)})^D \to \pi_0(\hat{T}^\Gamma)^D$$

dual to the inclusions $\hat{T}^\Gamma \subset \hat{T}^{\Gamma(v)}$.

All three of these results can be generalized to connected reductive groups G over F. The first result was generalized in [K3, Sect. 4] to a canonical bijection

$$\ker^1(F, G) \tilde{\to} \ker^1(F, Z(\hat{G}))^D.$$

Because the Hasse principle is not known for E_8, it was possible to prove this only for G having no E_8 factors.

The second result generalizes to a canonical map

$$H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to \pi_0(Z(\hat{G})^\Gamma)^D,$$

where Z_G is the center of G. Its kernel is equal to the image of $H^1(F, G_{\mathrm{ad}})$.

The generalization of the third result is that the composition

$$H^1(F, G(\mathbb{A})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to \pi_0(Z(\hat{G})^\Gamma)^D$$

is obtained by putting together the local maps

$$H^1(F, G(\bar{F}_v)) \to \pi_0(Z(\hat{G})^{\Gamma(v)})^D$$

(see 2.5 for a precise statement) and that its kernel is the image of $H^1(F, G)$ (see 2.6). In Sect. 2 we prove all this, along with some related facts. In Sect. 10 we review some facts about elliptic and fundamental tori that are needed in Sects. 1–2.

These local and global cohomological results can be applied to harmonic analysis on reductive groups. Consider the local case first. Let G be a connected reductive group over F. For simplicity we assume that the derived group G_{der} is simply connected. Let γ be a semi-simple element of $G(F)$ and write I for the centralizer G_γ of γ. Since G_{der} is simply connected, I is a connected reductive F-group. The conjugacy classes within the stable conjugacy class of γ [i.e., the $G(\bar{F})$-conjugacy class in this case] are in 1–1 correspondence with the elements of

$$\ker[H^1(F, I) \to H^1(F, G)].$$

If γ is regular, then I is a torus and $H^1(F, I)$ is a group. In [L2, S2] it is explained how to use characters on this group to form κ-orbital integrals, which are used to match functions on G with functions on endoscopic groups of G.

If γ is not regular, then I is not a torus and $H^1(F, I)$ is not a group. Nevertheless we have a map

$$H^1(F, I) \to \pi_0(Z(\hat{I})^\Gamma)^D,$$

and $\pi_0(Z(\hat{I})^\Gamma)^D$ is a group. We use characters of this group – namely, elements of $\pi_0(Z(\hat{I})^\Gamma)$ – to define κ-orbital integrals. In Sect. 5 we conjecture that these

κ-orbital integrals can be matched with stable orbital integrals of semi-simple elements of endoscopic groups.

There are two ways in which the situation is more complicated than in the regular case. First, it is necessary to take into account the signs $e(I)$ of [K2]. Second, in the matching theorem for an endoscopic group H of G, some semi-simple elements of $H(F)$ will be matched with non-semi-simple elements of $G(F)$. We conjecture that the semi-simple elements of $H(F)$ which are matched with semi-simple elements of $G(F)$ are precisely those we call (G, H)-regular: a semi-simple $\gamma_H \in H(F)$ is *(G, H)-regular* if $\alpha(\gamma_H) \neq 1$ for every root α of G that does not come from H. This definition is explained in detail in Sect. 3. If $\gamma_H \in H(F)$ is (G, H)-regular semi-simple and is matched with the semi-simple element $\gamma \in G(F)$, then the (connected) centralizer I_H of γ_H in H is an inner form of the (connected) centralizer I of γ in G.

Now we come to the application of the global cohomological results. It turns out that all three global results are needed for the stabilization of the part of the trace formula for G indexed by elliptic semi-simple conjugacy classes in $G(F)$. The application of the second global result is the most interesting. It can be best understood by considering the case in which G is a simply connected semi-simple group. Choose an inner twisting $\psi: G_0 \to G$ with G_0 quasi-split. Let γ_0 be a semi-simple element of $G_0(F)$, and let I_0 denote the (connected) centralizer of γ_0 in G_0. Let γ be an element of $G(\mathbb{A})$ that is conjugate to $\psi(\gamma_0)$ under $G(\bar{\mathbb{A}})$. Then in Sect. 6 we construct from γ and γ_0 an element of $H^1(F, I_0(\bar{\mathbb{A}})/Z_{I_0}(\bar{F}))$. The second global result then produces an element

$$\mathrm{obs}(\gamma) \in \pi_0(Z(\hat{I}_0)^\Gamma)^D .$$

Assuming that G has no E_8 factors, we show that $\mathrm{obs}(\gamma)$ is trivial if and only if the $G(\mathbb{A})$-conjugacy class of γ contains an element of $G(F)$.

If we assume only that G_{der} is simply connected, the situation becomes slightly more complicated: $\mathrm{obs}(\gamma) \in \mathfrak{K}(I_0/F)^D$, where $\mathfrak{K}(I_0/F)$ is the subgroup of $\pi_0([Z(\hat{I}_0)/Z(\hat{G})]^\Gamma)$ defined in Sect. 4. The global conjecture on transfer factors in Sect. 6 involves $\mathrm{obs}(\gamma)$. For regular elements γ_0 all this was done by Langlands [L2, Ch. VII] and was reinterpreted in [K3].

The stabilization of the elliptic semi-simple part of the trace formula for G is given in Sect. 9. It is necessary to make a number of assumptions (see 9.3). The manipulation of the terms is done in essentially the same way as in [L2]; however, we cannot use Langlands's convergence argument because of the signs $e(I)$ occurring in the singular κ-orbital integrals, and so we are forced to prove the finiteness results of Sects. 7–8. These are enough to show that all the sums we consider have only a finite number of non-zero terms.

I am indebted to J. Arthur for several very helpful conversations about these finiteness questions, which he has already resolved for the ordinary trace formula [A]. The stable trace formula requires slightly more detailed results, which are unfortunately quite technical.

In 7.5 we establish the hypothesis made in [L2, p. 185]. Although the other finiteness results make this one unnecessary for the stabilization in Sect. 9, it may have other applications.

Much of the notation used in the paper has already appeared in this introduction. Other unexplained notation and terminology is taken from [K3]. Throughout the paper we have a field F (sometimes local, sometimes global) and an algebraic closure $\bar{F}$ of F, and we write Γ for $\mathrm{Gal}(\bar{F}/F)$. We often use $A(G)$ as an abbreviation for $\pi_0(Z(\hat{G})^\Gamma)^D$.

1. Local Cohomological Results

In this section F is a local field of characteristic 0.

1.1. For any torus T over F we have a canonical isomorphism

$$H^1(F,T) \xrightarrow{\sim} \pi_0(\hat{T}^\Gamma)^D, \tag{1.1.1}$$

obtained from the Tate-Nakayama isomorphism

$$H^1(F,T) \xrightarrow{\sim} H^1(F, X^*(T))^D$$

and the isomorphism

$$\pi_0(\hat{T}^\Gamma) \xrightarrow{\sim} H^1(F, X_*(\hat{T}))$$

induced by the connecting homomorphism for the exponential sequence

$$1 \to X_*(\hat{T}) \to \mathrm{Lie}(\hat{T}) \to \hat{T} \to 1 .$$

For any connected reductive group G over F we write $A(G)$ for the finite abelian group $\pi_0(Z(\hat{G})^\Gamma)^D$. Now consider the functors $G \mapsto H^1(F,G)$ and $G \mapsto A(G)$ from the category of connected reductive F-groups and normal homomorphisms to the category of pointed sets.

1.2. Theorem. *There is a unique extension of* (1.1.1) *to a morphism of functors*

$$\alpha_G : H^1(F,G) \to A(G) . \tag{1.2.1}$$

For any maximal torus T of G the diagram

$$\begin{array}{ccc} H^1(F,T) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow \\ A(T) & \longrightarrow & A(G) \end{array} \tag{1.2.2}$$

commutes, where $A(T) \to A(G)$ is induced by $Z(\hat{G}) \hookrightarrow \hat{T}$. If F is p-adic, then (1.2.1) *is an isomorphism of functors. If $F = \mathbb{R}$, then*

$$\ker(\alpha_G) = \mathrm{im}[H^1(\mathbb{R}, G_{SC}) \to H^1(\mathbb{R}, G)]$$

and

$$\mathrm{im}(\alpha_G) = \ker[\pi_0(Z(\hat{G})^\Gamma)^D \to \pi_0(Z(\hat{G}))^D] ,$$

where $\pi_0(Z(\hat{G})^\Gamma)^D \to \pi_0(Z(\hat{G}))^D$ is induced by the norm homomorphism $Z(\hat{G}) \to Z(\hat{G})^\Gamma$.

We extend the morphism of functors in two stages. At the first stage we extend it to groups whose derived group is simply connected. Consider such a group G.

and let $D=G/G_{\mathrm{der}}$. Since (1.2.1) must be functorial for the normal homomorphism $G\to D$, and since $Z(\hat{G})=\hat{D}$, we are forced to define α_G as the composed map

$$H^1(F,G)\to H^1(F,D)\xrightarrow{\sim} A(D)=A(G).$$

It is easy to check that this map is functorial in G.

At the second state we extend (1.2.1) to all groups G. Let $g\in H^1(F,G)$. We claim that there exist a z-extension $H\to G$ and an element $h\in H^1(F,H)$ such that $h\mapsto g$. The first step in finding H, h is to choose a finite Galois extension K/F such that K splits G and the image of g in $H^1(K,G)$ is trivial. Next we choose any z-extension $H\to G$ whose kernel Z is isomorphic to $R_{K/F}(Z_0)$, where Z_0 is a split K-torus [we use $R_{K/F}(\ )$ to denote restriction of scalars]. Consider the commutative diagram

$$\begin{array}{ccccccc}
\dots\longrightarrow & H^1(F,H) & \longrightarrow & H^1(F,G) & \longrightarrow & H^2(F,Z) \\
 & \downarrow & & \downarrow & & \downarrow \\
\dots\longrightarrow & H^1(K,H) & \longrightarrow & H^1(K,G) & \longrightarrow & H^2(K,Z).
\end{array}$$

Because Z is induced from Z_0, the restriction map $H^2(F,Z)\to H^2(K,Z)$ is injective, and the diagram shows that g lies in the image of $H^1(F,H)$.

Now it is clear how to define $\alpha_G(g)$. Since (1.2.1) must be functorial for the normal homomorphism $H\to G$, we are forced to define $\alpha_G(g)$ to be the image of $\alpha_H(h)$ under $A(H)\to A(G)$. There is no ambiguity in h, because $H^1(F,Z)$ is trivial, but we do need to show that our definition is independent of the choice of z-extension. Suppose that $H_1\to G$, $H_2\to G$ are two z-extensions of G and that $h_1\mapsto g$, $h_2\mapsto g$ ($h_i\in H^1(F,H_i)$ for $i=1,2$). Let H_3 be the fiber product of H_1, H_2 over G. Then $H_3\to G$ is a z-extension with kernel $Z_3=Z_1\times Z_2$, where $Z_i=\ker(H_i\to G)$ for $i=1,2$. For $i=1,2,3$ let z_i be the image of g under the connecting homomorphism $H^1(F,G)\to H^2(F,Z_i)$. Then $z_3=(z_1,z_2)$, and since z_1,z_2 are trivial, it follows that there exists $h_3\in H^1(F,H_3)$ such that $h_3\mapsto g$. Applying the functoriality in the first stage to the canonical projections $H_3\to H_1$, $H_3\to H_2$, we see that H_1,h_1 and H_2,h_2 yield the same $\alpha_G(g)$ (the uniqueness of h_1,h_2,h_3 implies that $h_3\mapsto h_1$, $h_3\mapsto h_2$).

We still need to show that α_G is functorial in G. Consider a normal homomorphism $G_1\to G_2$. Let $g_1\in H^1(F,G_1)$ and let g_2 be the image of g_1 in $H^1(F,G_2)$. For $i=1,2$ choose z-extensions $H_i\to G_i$ such that g_i belongs to the image of $H^1(F,H_i)$. Let H_3 be the fiber product of H_1 and H_2 over G_2. Then the diagram

$$\begin{array}{ccc}
H_3 & \longrightarrow & H_2 \\
\downarrow & & \downarrow \\
G_1 & \longrightarrow & G_2
\end{array}$$

commutes. Furthermore $H_3\to H_2$ is a normal homomorphism, and $H_3\to G_1$ is a z-extension with kernel $Z_1\times Z_2$, where $Z_i=\ker(H_i\to G_i)$ for $i=1,2$. In the same way as before we see that g_1 belongs to the image of $H^1(F,H_3)$. Applying the functoriality in the first stage to $H_3\to H_2$, we find that $\alpha_{G_1}(g_1)$ maps to $\alpha_{G_2}(g_2)$ under $A(G_1)\to A(G_2)$.

The second statement of the theorem does not follow from the functoriality of α, since $T\to G$ is not a normal homomorphism unless G is a torus. However, if G_{der}

is simply connected, we let $D = G/G_{\text{der}}$, and obtain the desired commutativity from the functoriality of α for $T \to D$ and $G \to D$. Then, in the general case, we use z-extensions $H \to G$ to reduce to the case just treated.

Now we consider the remaining statements of the theorem. First we show that

$$\ker(\alpha_G) = \operatorname{im}[H^1(F, G_{\text{SC}}) \to H^1(F, G)].$$

If G_{der} is simply connected, this equality is obvious from the definition of α_G. In the general case the functoriality of α for $G_{\text{SC}} \to G$ shows that $\operatorname{im}[H^1(F, G_{\text{SC}}) \to H^1(F, G)]$ is contained in $\ker(\alpha_G)$. Now let $g \in \ker(\alpha_G)$ and choose a z-extension

$$1 \to Z \to H \to G \to 1$$

for which there exists $h \in H^1(F, H)$ with $h \mapsto g$. Consider the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^1(F, Z) & \longrightarrow & H^1(F, H) & \longrightarrow & H^1(F, G) \\ \downarrow & & \downarrow & & \downarrow \\ A(Z) & \longrightarrow & A(H) & \longrightarrow & A(G). \end{array}$$

Since Z is an induced torus, $H^1(F, Z)$ and $A(Z)$ are trivial. Therefore $\alpha_H(h) = 1$, and

$$h \in \operatorname{im}[H^1(F, H_{\text{SC}}) \to H^1(F, H)]$$

by the case already treated; since $H_{\text{SC}} = G_{\text{SC}}$, this shows that $g \in \operatorname{im}[H^1(F, G_{\text{SC}}) \to H^1(F, G)]$. Thus we have proved the statement about $\ker(\alpha_G)$ in case $F = \mathbb{R}$. If F is p-adic, Kneser's vanishing theorem for $H^1(F, G_{\text{SC}})$ shows that $\ker(\alpha_G)$ is trivial, and a twisting argument (see 1.4) shows that α_G is injective.

Our last task is to prove the statements about $\operatorname{im}(\alpha_G)$ in the real and p-adic cases.

If F is real (respectively p-adic), let T be a fundamental (respectively elliptic) maximal F-torus of G. Then

$$H^1(F, T) \to H^1(F, G)$$

is surjective (see Sect. 10) and α_T is an isomorphism; therefore the second statement of the theorem implies that $\operatorname{im}(\alpha_G)$ is equal to $\operatorname{im}[A(T) \to A(G)]$. Let U denote the inverse image of T in G_{SC}. The exact sequence

$$1 \to Z(\hat{G}) \to \hat{T} \to \hat{U} \to 1$$

yields [K3, 2.3] an exact sequence

$$\ldots \to X_*(\hat{U})^\Gamma \to A(G)^D \to A(T)^D \to A(U)^D \to \ldots.$$

In the p-adic case $X_*(\hat{U})^\Gamma$ is trivial, and therefore $A(T) \to A(G)$ is surjective, which in turn implies that α_G is surjective. In the real case we have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} \ldots & \longrightarrow & X_*(\hat{U}) & \longrightarrow & \pi_0(Z(\hat{G})) & \longrightarrow & \pi_0(\hat{T}) & \longrightarrow & \ldots \\ & & \downarrow & & \downarrow & & \downarrow & & \\ \ldots & \longrightarrow & X_*(\hat{U})^\Gamma & \longrightarrow & \pi_0(Z(\hat{G})^\Gamma) & \longrightarrow & \pi_0(\hat{T}^\Gamma) & \longrightarrow & \ldots, \end{array}$$

in which all three vertical arrows are norm homomorphisms. In proving the commutativity of the left square one uses the following description of the connecting homomorphism

$$X_*(D_3)^\Gamma \to \pi_0(D_1^\Gamma) \tag{1.2.3}$$

of [K3, 2.3].

Let $\mu \in X_*(D_3)^\Gamma = \mathrm{Hom}_\Gamma(X^*(D_3), \mathbb{Z})$. Let μ_1 be the image of μ in $\mathrm{Hom}_\Gamma(X^*(D_3), \mathbb{C})$ and choose

$$\mu_2 \in \mathrm{Hom}_\Gamma(X^*(D_2), \mathbb{C})$$

such that $\mu_2 \mapsto \mu_1$. Let μ_3 be the image of μ_2 in $\mathrm{Hom}_\Gamma(X^*(D_2), \mathbb{C}^\times)$; then μ_3 comes from a (unique) element $\mu_4 \in \mathrm{Hom}_\Gamma(X^*(D_1), \mathbb{C}^\times) = D_1^\Gamma$. Changing μ_2 by an element of $\mathrm{Hom}_\Gamma(X^*(D_1), \mathbb{C})$ changes μ_4 by an element of $(D_1^\Gamma)^0$. The image of μ under (1.2.3) is equal to the image of μ_4 in $\pi_0(D_1^\Gamma)$.

Now we return to our commutative diagram. The map $X_*(\hat{U}) \to \pi_0(Z(\hat{G}))$ is surjective, since $\pi_0(\hat{T}) = \{1\}$. Furthermore the norm map $X_*(\hat{U}) \to X_*(\hat{U})^\Gamma$ is surjective [in other words, $\hat{H}^0(\mathbb{R}, X_*(\hat{U})) = \{0\}$] since U is a fundamental torus in a simply connected semi-simple group, and such a torus is necessarily of the form $T_a \times T_i$ with T_a anisotropic and T_i induced from $\mathbb{C}$ (see Sect. 10). At this point we know that $X_*(\hat{U})^\Gamma \to \pi_0(Z(\hat{G})^\Gamma)$ and the norm map $\pi_0(Z(\hat{G})) \to \pi_0(Z(\hat{G})^\Gamma)$ have the same image in $\pi_0(Z(\hat{G})^\Gamma)$. We also know that an element of $A(G)$ [i.e., a character on $\pi_0(Z(\hat{G})^\Gamma)$] is in the image of α_G if and only if it vanishes on

$$\mathrm{im}[X_*(\hat{U})^\Gamma \to \pi_0(Z(\hat{G})^\Gamma)].$$

The last statement of the theorem is now clear.

1.3. Let a_σ be a 1-cocycle of Γ in $G(\bar{F})$ and let $^*G$ be the inner twist of G by a_σ:

$$^*G(\bar{F}) = G(\bar{F}),$$

$$^*\sigma = \mathrm{Int}(a_\sigma) \circ \sigma.$$

The center of the dual group of $^*G$ is the same as that of G, which allows us to identify $A(^*G)$ with $A(G)$ and to regard the map (1.2.1) for $^*G$ as a map

$$H^1(F, {}^*G) \to A(G).$$

Let b_σ be a 1-cocycle of Γ in $^*G(\bar{F})$. Then $c_\sigma := b_\sigma a_\sigma$ is a 1-cocycle of Γ in $G(\bar{F})$. Denote by a', b', c' the images of $a_\sigma, b_\sigma, c_\sigma$ in $A(G)$.

1.4. Lemma. *The element c' is the product of a' and b'.*

If G_{der} is simply connected, the result is easy to prove, using that $A(G) = H^1(F, D)$, where $D = G/G_{\mathrm{der}} = {}^*G/{}^*G_{\mathrm{der}}$.

Now we do the general case. Choose a finite Galois extension K/F such that K splits G and $a_\sigma, b_\sigma, c_\sigma$ have trivial restriction to $\mathrm{Gal}(\bar{F}/K)$. Choose a z-extension $H \to G$ whose kernel is of the form $R_{K/F}(Z_0)$ for a split K-torus Z_0.

Choose a 1-cocycle A_σ of Γ in $H(\bar{F})$ such that $A_\sigma \mapsto a_\sigma$. Use A_σ to get $^*H$. Choose a 1-cocycle B_σ of Γ in $^*H(\bar{F})$ such that $B_\sigma \mapsto b_\sigma$. Let $C_\sigma = B_\sigma A_\sigma$; then $C_\sigma \mapsto c_\sigma$. The general case now follows from the special case, applied to $A_\sigma, B_\sigma, C_\sigma$.

1.5. Langlands [L1] (see also [B, Sect. 10]) associates to any element of $H^1(F, Z(\hat{G}))$ a character on $G(F)$ [more generally, he associates to any element of $H^1(W_F, Z(\hat{G}))$ a quasicharacter on $G(F)$]. Equivalently, there is a homomorphism

$$G(F) \to H^1(F, Z(\hat{G}))^D.$$

Let $1 \to G \to H \to I \to 1$ be an exact sequence of connected reductive F-groups. We get a diagram

$$\begin{array}{ccccccccccccc} 1 & \longrightarrow & G(F) & \longrightarrow & H(F) & \longrightarrow & I(F) & \longrightarrow & H^1(G) & \longrightarrow & H^1(H) & \longrightarrow & H^1(I) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \ldots & \longrightarrow & C(G) & \longrightarrow & C(H) & \longrightarrow & C(I) & \longrightarrow & A(G) & \longrightarrow & A(H) & \longrightarrow & A(I) \end{array}$$

in which we have used $H^1(G)$ [respectively $C(G)$] as an abbreviation for $H^1(F, G)$ [respectively $H^1(F, Z(\hat{G}))^D$]. The bottom row comes from applying $(\ )^D$ to the long exact sequence [K3, 2.3] attached to

$$1 \to Z(\hat{I}) \to Z(\hat{H}) \to Z(\hat{G}) \to 1.$$

1.6. Lemma. *The diagram above is commutative.*

The only non-obvious point is the commutativity of

$$\begin{array}{ccc} I(F) & \longrightarrow & H^1(F, G) \\ \downarrow & & \downarrow \\ H^1(F, Z(\hat{I}))^D & \longrightarrow & A(G). \end{array}$$

If G, H, I are tori, the bottom row of the commutative diagram can be rewritten as part of the dual of the long exact cohomology sequence for

$$1 \to X^*(I) \to X^*(H) \to X^*(G) \to 1$$

(see [K3, 2.2]), and the vertical maps are then induced by the Tate-Nakayama pairings

$$H^i(F, T) \times H^{2-i}(F, X^*(T)) \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^\times$$

(see [K3, 3.1–3.3]). Therefore the square above is commutative or anticommutative, depending on the order in which the cup-product is taken for $i=1$. In this lemma we normalize the pairing by requiring that the square be commutative.

If the derived groups of G, H, I are simply connected, we write $B = G/G_{\mathrm{der}}$, $C = H/H_{\mathrm{der}}$, $D = I/I_{\mathrm{der}}$, and reduce to the case of tori by considering the diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G & \longrightarrow & H & \longrightarrow & I & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & 1. \end{array}$$

In the next step we assume only that G_{der} is simply connected. Choose a z-extension $J \to I$ and let K be the fiber product of H and J over I. We get a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & G \arrow[r] \arrow[d, equal] & K \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & 1 . \end{tikzcd}$$

The derived groups of G, J are simply connected; it follows that the same is true of K. Since $J(F) \to I(F)$ is surjective, the result follows from the previous step, applied to

$$1 \to G \to K \to J \to 1 .$$

Finally we do the general case. Choose a z-extension $J \to H$, and let K be the inverse image of G under $J \to H$. Then K_{der} is simply connected (it is a normal subgroup of J_{der}), and we have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & K \arrow[r] \arrow[d] & J \arrow[r] \arrow[d] & I \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & G \arrow[r] & H \arrow[r] & I \arrow[r] & 1 . \end{tikzcd}$$

We finish by applying the previous step to the top row of the diagram.

1.7. Suppose that G is semi-simple, and let C denote the kernel of $G_{\mathrm{SC}} \to G$. The exact sequence

$$1 \to C \to G_{\mathrm{SC}} \to G \to 1$$

induces a map

$$H^1(F, G) \to H^2(F, C) .$$

Duality for finite groups gives us an isomorphism

$$H^2(F, C) \to H^0(F, X^*(C))^D .$$

Furthermore $X^*(C)$ is equal to $Z(\hat{G})$. Putting all this together, we get a map

$$H^1(F, G) \to A(G) .$$

1.8. **Lemma.** *This map is equal to* α_G.

The proof uses z-extensions as in the proof of Theorem 1.2 and then proceeds in the same way as the proof of Remark 6.5 in [K3].

2. Global Cohomological Results

In this section F is a number field.

2.1. For any torus T over F we have a canonical isomorphism

$$H^1(F, T(\mathbb{A})/T(\bar{F})) \tilde{\to} \pi_0(\hat{T}^\Gamma)^D , \tag{2.1.1}$$

obtained from the Tate-Nakayama isomorphism

$$H^1(F, T(\mathbb{A})/T(\bar{F})) \tilde{\to} H^1(F, X^*(T))^D$$

and the isomorphism $\pi_0(\hat{T}^\Gamma) \tilde{\to} H^1(F, X_*(\hat{T}))$ mentioned in 1.1.

Let G be a connected reductive group over F. As in Sect. 1 we write $A(G)$ instead of $\pi_0(Z(\hat{G})^\Gamma)^D$. We write Z_G for the center of G. Consider the two functors $G \mapsto H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$ and $G \mapsto A(G)$ from the category of connected reductive F-groups and normal homomorphisms to the category of pointed sets.

2.2. Theorem. *There is a unique extension of* (2.1.1) *to a morphism of functors*

$$\beta_G : H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to A(G). \tag{2.2.1}$$

The kernel of β_G *is equal to the image of*

$$H^1(F, G(\bar{F})/Z_G(\bar{F})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})).$$

The proof of existence and uniqueness of (2.2.1) is analogous to the proof of the corresponding part of Theorem 1.2. If G_{der} is simply connected, we write D for G/G_{der} and obtain β_G as the composition

$$H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to H^1(F, D(\mathbb{A})/D(\bar{F})) \tilde{\to} A(D) = A(G).$$

In the general case we need to use z-extensions

$$1 \to Z \to H \to G \to 1.$$

We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 1 \to & Z(\mathbb{A}) & \to & H(\mathbb{A}) & \to & G(\mathbb{A}) & \to 1 \\ & \cup & & \cup & & \cup & \\ 1 \to & Z(\bar{F}) & \to & Z_H(\bar{F}) & \to & Z_G(\bar{F}) & \to 1. \end{array}$$

The surjectivity of $H(\mathbb{A}) \to G(\mathbb{A})$ follows from the discussion of unramified z-extensions in the proof of Proposition 7.1. We get from the exact sequence

$$1 \to Z(\mathbb{A})/Z(\bar{F}) \to H(\mathbb{A})/Z_H(\bar{F}) \to G(\mathbb{A})/Z_G(\bar{F}) \to 1$$

an exact cohomology sequence, and because Z is an induced torus, the last part of this sequence reduces to

$$1 \to H^1(F, H(\mathbb{A})/Z_H(\bar{F})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to H^2(F, Z(\mathbb{A})/Z(\bar{F})).$$

Let $g \in H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$. Just as in the proof of Theorem 1.2 we can choose the z-extension so that there exists a (unique) $h \in H^1(F, H(\mathbb{A})/Z_H(\bar{F}))$ such that $h \mapsto g$. We are forced to define $\beta_G(g)$ to be the image of $\beta_H(h)$ under $A(H) \to A(G)$. The proof that this definition is independent of the choice of z-extension and is functorial in G goes the same way as for Theorem 1.2.

We still need to determine the kernel of β_G. We start with the special case in which G_{der} is simply connected. Let $D = G/G_{\text{der}}$. The kernel of β_G is equal to the kernel of

$$H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to H^1(F, D(\mathbb{A})/D(\bar{F})), \tag{2.2.2}$$

and it is clear that this set contains the image of $H^1(F, G(\bar{F})/Z_G(\bar{F}))$. To prove the opposite inclusion we consider an element g in the kernel of (2.2.2). Since $G(\mathbb{A}) \to D(\mathbb{A})$ and $Z_G(\bar{F}) \to D(\bar{F})$ are surjective, the sequence

$$1 \to G_{SC}(\mathbb{A})/Z_{G_{SC}}(\bar{F}) \to G(\mathbb{A})/Z_G(\bar{F}) \to D(\mathbb{A})/D(\bar{F}) \to 1$$

is exact, and we see that there exists $g_{SC} \in H^1(F, G_{SC}(\mathbb{A})/Z_{G_{SC}}(\bar{F}))$ such that $g_{SC} \mapsto g$. This reduces us to the case in which G is semi-simple and simply connected. We must show that

$$H^1(F, G(\bar{F})/Z_G(\bar{F})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$$

is surjective in this case.

To simplify the notation we temporarily write Z instead of Z_G. The commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & Z(\bar{F}) \arrow[r] & G(\mathbb{A}) \arrow[r] & G(\mathbb{A})/Z(\bar{F}) \arrow[r] & 1 \\ 1 \arrow[r] & Z(\bar{F}) \arrow[r] \arrow[u, equal] & G(\bar{F}) \arrow[r] \arrow[u] & G(\bar{F})/Z(\bar{F}) \arrow[r] \arrow[u] & 1 \end{tikzcd}$$

gives us a commutative diagram with exact rows

$$\begin{tikzcd} \dots \arrow[r] & H^1(F, G(\mathbb{A})) \arrow[r] & H^1(F, G(\mathbb{A})/Z(\bar{F})) \arrow[r] & H^2(F, Z) \\ \dots \arrow[r] & H^1(F, G) \arrow[r] \arrow[u] & H^1(F, G_{ad}) \arrow[r] \arrow[u] & H^2(F, Z). \arrow[u, equal] \end{tikzcd}$$

We claim that $H^1(F, G_{ad}) \to H^2(F, Z)$ is surjective. Let $z \in H^2(F, Z)$. At any finite place of F for which some 2-cocycle representing z has trivial restriction to the inertia subgroup at v, the image of z in $H^2(F_v, Z)$ is trivial. Therefore there exists a finite set V of places of F outside of which z is locally trivial. There exists a maximal F-torus T of G such that T is anisotropic at every finite place in V and fundamental at every real place in V; for this T we have $H^2(F_v, T) = \{0\}$ for all $v \in V$ (see Sect. 10) and also $\ker^2(F, T) = \{0\}$ (so long as V is non-empty, which we may as well assume). Using all this, we see that the image of z in $H^2(F, T)$ is trivial. Therefore $z \in \operatorname{im}[H^1(F, T_{ad}) \to H^2(F, Z)]$, from which it is obvious that z belongs to the image of $H^1(F, G_{ad})$.

Now we continue with the proof that

$$H^1(F, G_{ad}) \to H^1(F, G(\mathbb{A})/Z(\bar{F}))$$

is surjective. Let c_σ be a 1-cocycle of Γ in $G(\mathbb{A})/Z(\bar{F})$. Now look at the last commutative diagram we wrote down. The claim that we just proved shows that there exists a 1-cocycle g_σ of Γ in $G_{ad}(\bar{F})$ such that c_σ, g_σ have the same image in $H^2(F, Z)$. We can use g_σ to twist G and $G(\mathbb{A})$, obtaining $^*G$ and $^*(G(\mathbb{A}))$. Note that $(^*G)(\mathbb{A}) = {}^*(G(\mathbb{A}))$. There exists a 1-cocycle h_σ of Γ in $(^*G)(\mathbb{A})$ such that $c_\sigma = \bar{h}_\sigma g_\sigma$, where $\bar{h}_\sigma$ denotes the image of h_σ in $(^*G)(\mathbb{A})/Z(\bar{F})$. Since $^*G$ is simply connected, the map

$$H^1(F, {}^*G) \to H^1(F, (^*G)(\mathbb{A}))$$

is surjective. Therefore c_σ is cohomologous to a 1-cocycle of the form $h'_\sigma g_\sigma$, where h'_σ is a 1-cocycle in ${}^*G_{\mathrm{ad}}(\bar{F})$. This proves that the class of c_σ lies in the image of $H^1(F, G_{\mathrm{ad}})$.

Now we need to determine $\ker(\beta_G)$ in the general case. Let $g \in H^1(F, G(\mathbb{A})/Z_G(\bar{F}))$. As before we choose a z-extension $H \to G$ for which there exists a (unique) $h \in H^1(F, H(\mathbb{A})/Z_H(\bar{F}))$ such that $h \mapsto g$. Suppose that $g \in \ker(\beta_G)$. The homomorphism $A(H) \to A(G)$ is injective because Z is an induced torus; therefore $h \in \ker(\beta_H)$. By what we have already proved h belongs to the image of $H^1(F, H_{\mathrm{ad}})$. Therefore g belongs to the image of $H^1(F, G_{\mathrm{ad}})$.

Now suppose that g belongs to the image of $H^1(F, G_{\mathrm{ad}})$. Then h belongs to the image of $H^1(F, H_{\mathrm{ad}}) = H^1(F, G_{\mathrm{ad}})$. By what we have already proved h belongs to $\ker(\beta_H)$. Therefore g belongs to $\ker(\beta_G)$.

2.3. Let a_σ be a 1-cocycle of Γ in $G(\mathbb{A})/Z_G(\bar{F})$. Let d_σ be the image of a_σ under $G(\mathbb{A})/Z_G(\bar{F}) \to G_{\mathrm{ad}}(\mathbb{A})$. As in 1.3 we get a twist ${}^*(G(\mathbb{A}))$ of $G(\mathbb{A})$. For each place v of F choose a place w of $\bar{F}$ over v, and let $d_\sigma(v)$ be the 1-cocycle of $\Gamma(v) := \mathrm{Gal}(\bar{F}_w/F_v) \subset \Gamma$ in $G_{\mathrm{ad}}(\bar{F}_w)$ obtained by taking the component of d_σ at w. Let ${}^*G_v$ be the twist of $G_v := G_{F_v}$ by $d_\sigma(v)$. Then

$$H^1(F, {}^*(G(\mathbb{A}))) \xrightarrow{\sim} \bigoplus_v H^1(F_v, {}^*G_v),$$

where $\bigoplus_v$ denotes the subset of the direct product consisting of (x_v) such that $x_v = 1$ for all but a finite number of v. The local maps of Sect. 1 give us

$$\bigoplus_v H^1(F_v, {}^*G_v) \to \bigoplus_v A({}^*G_v).$$

The center of the dual group of ${}^*G_v$ is $Z(\hat{G})$, and thus the obvious inclusion $Z(\hat{G})^{\Gamma(v)} \supset Z(\hat{G})^\Gamma$ induces a homomorphism $A({}^*G_v) \to A(G)$ for each v. The sum of these homomorphisms is a homomorphism

$$\bigoplus_v A({}^*G_v) \to A(G).$$

Putting all this together, we get a map

$$H^1(F, {}^*(G(\mathbb{A}))) \to A(G). \tag{2.3.1}$$

Let b_σ be a 1-cocycle of Γ in ${}^*(G(\mathbb{A}))$. Write $\bar{b}_\sigma$ for the image of b_σ in ${}^*(G(\mathbb{A})/Z_G(\bar{F}))$, and let $c_\sigma = \bar{b}_\sigma a_\sigma$; then c_σ is 1-cocycle of Γ in $G(\mathbb{A})/Z_G(\bar{F})$. Denote by a', b', c' the images of $a_\sigma, b_\sigma, c_\sigma$ in $A(G)$ [apply (2.2.1) to a_σ, c_σ and (2.3.1) to b_σ].

2.4. Lemma. *The element c' is the product of a' and b'.*

If G is a torus, the result is part of the global Tate-Nakayama theory (see [K3, 3.4.3]). The rest of the proof parallels that of Lemma 1.4.

2.5. Corollary. *The composition*

$$H^1(F, G(\mathbb{A})) \to H^1(F, G(\mathbb{A})/Z_G(\bar{F})) \to A(G)$$

is equal to the composition of

(i) $H^1(F, G(\mathbb{A})) \tilde{\to} \bigoplus_v H^1(F_v, G)$,

(ii) $\bigoplus_v H^1(F_v, G) \to \bigoplus_v A(G_v)$,

(iii) $\bigoplus_v A(G_v) \to A(G)$.

To prove this take $a_\sigma = 1$ in previous lemma.

2.6. Proposition. *The kernel of the map $H^1(F, G(\mathbb{A})) \to A(G)$ of Corollary 2.5 is equal to the image of*

$$H^1(F, G) \to H^1(F, G(\mathbb{A}))\,.$$

We temporarily write Z instead of Z_G. We have a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
\ldots \longrightarrow & H^1(F,Z) & \longrightarrow & H^1(F,G(\mathbb{A})) & \longrightarrow & H^1(F,G(\mathbb{A})/Z(\bar{F})) & \longrightarrow & H^2(F,Z) & \\
& \| & & \uparrow & & \uparrow & & \| & \\
\ldots \longrightarrow & H^1(F,Z) & \longrightarrow & H^1(F,G) & \longrightarrow & H^1(F,G_{\mathrm{ad}}) & \longrightarrow & H^2(F,Z)\,, &
\end{array}$$

which we used in the proof of Theorem 2.2. Let $x \in H^1(F, G(\mathbb{A}))$ and let y denote the image of x in $H^1(F, G(\mathbb{A})/Z(\bar{F}))$. In view of the statement about $\ker(\beta_G)$ given in Theorem 2.2, what we must show is that x belongs to the image of $H^1(F, G)$ if and only if y belongs to the image of $H^1(F, G_{\mathrm{ad}})$. This is obvious from the diagram.

2.7. Let a_σ be a 1-cocycle of Γ in $G(\mathbb{A})/Z_G(\bar{F})$ and let b_σ be a 1-cocycle of Γ in $G(\bar{F})/Z_G(\bar{F})$. Then $c_\sigma = a_\sigma b_\sigma^{-1}$ is a 1-cocycle of Γ in $^*G(\mathbb{A})/Z(^*G)(\bar{F})$, where $^*G$ is the inner twist of G by b_σ:

$$^*G(\bar{F}) = G(\bar{F})\,,$$

$$^*\sigma = \mathrm{Int}(b_\sigma) \circ \sigma\,.$$

We can identify $A(^*G)$ with $A(G)$. Denote by a', c' the images of a_σ, c_σ in $A(G)$.

2.8. Lemma. *The elements a', c' are equal.*

The proof follows the usual pattern. If G_{der} is simply connected, we use $D = G/G_{\mathrm{der}}$. Then we use z-extensions to reduce the general case to the case just treated.

3. (G, H)-Regular Elements in an Endoscopic Group H for G

In this section F is a local or global field of characteristic 0, G is a connected reductive group over F, and (H, s, η) is an endoscopic triple for G [K3, Sect. 7].

3.1. For the moment we work over $\bar{F}$. Let γ_H be a semi-simple element of H. Choose a maximal torus T_H of H containing γ_H. There is a canonical G-conjugacy class of embeddings $j: T_H \to G$; choose one and let $\gamma = j(\gamma_H)$. The conjugacy class of γ is independent of the choice of T_H and j; thus $\gamma_H \mapsto \gamma$ induces a Γ-equivariant map from the set of semi-simple conjugacy classes in H to the set of semi-simple conjugacy classes in G.

Let $T=j(T_H)$ and use j to identify T_H, T. Let R (respectively R_H) denote the set of roots of T in G (respectively H). We have $R_H \subset R \subset X^*(T)$. We say that γ_H is (G,H)-regular if $\alpha(\gamma_H) \neq 1$ for every root α of G that does not come from a root of H. The (G,H)-regularity of γ_H depends only on γ_H, not on the choice of T_H and j.

Let I (respectively I_H) denote the identity component of the centralizer of γ in G (respectively, γ_H in H). The set $R(\gamma)$ of roots of T in I is equal to $\{\alpha \in R | \alpha(\gamma)=1\}$. The set $R_H(\gamma)$ of roots of T in I_H is equal to $\{\alpha \in R_H | \alpha(\gamma)=1\}$. Therefore $R_H(\gamma) \subset R(\gamma)$ and the two sets are equal if and only if γ_H is (G,H)-regular.

Now assume that γ_H is (G,H)-regular. Then $R_H(\gamma)=R(\gamma)$ and the same is true for the coroots of T in I_H, I. The theory of root data for reductive groups shows that $j: T_H \tilde{\to} T$ extends to an isomorphism $j_1: I_H \to I$, unique up to inner automorphisms coming from T. If $\gamma_H \in H(F)$ and $\gamma \in G(F)$, then I_H, I are defined over F and j_1 is an inner twisting. In particular we have $Z(\hat{I}_H)=Z(\hat{I})$.

3.2. Lemma. *Let γ_H, γ be as above and assume that γ_H is (G,H)-regular. If the centralizer of γ in G is connected, then so is the centralizer of γ_H in H.*

Let h belong to the centralizer of γ_H in H. Let T_H, j, T, I, I_H be as above. We want to show that $h \in I_H$. Using the conjugacy of maximal tori in I_H, we may modify h by an element of I_H and assume that h normalizes T_H. Let ω be the corresponding element of the Weyl group $\Omega(T_H, H)$. The isomorphism j allows us to regard $\Omega(T_H, H)$ as a subgroup of $\Omega(T, G)$. Since G_γ is connected and ω fixes γ, we must have $\omega \in \Omega(T, I) \subset \Omega(T, G)$. We also have $\Omega(T,I)=\Omega(T_H, I_H)$, because γ is (G,H)-regular. Therefore $\omega \in \Omega(T_H, I_H)$, which implies that $h \in I_H$.

4. Definitions of $\mathfrak{D}$, $\mathfrak{E}$, $\mathfrak{K}$

In this section F is a local or global field of characteristic 0 and G is a connected reductive group over F. Let γ be a semi-simple element of $G(F)$, and let I denote the identity component of G_γ. As before we write $A(G)$ instead of $\pi_0(Z(\hat{G})^\Gamma)^D$.

4.1. We define $\mathfrak{D}(I/F)$ to be $\ker[H^1(F,I) \to H^1(F,G)]$. If G_γ is connected, then $\mathfrak{D}(I/F)$ is in 1–1 correspondence with the conjugacy classes in the stable conjugacy class of γ. The 1–1 correspondence arises as follows. Let $\gamma' \in G(F)$ be a stable conjugate of γ. Then $\{g \in G(\bar{F}) | g\gamma g^{-1}=\gamma'\}$ is an F-torsor under I.

4.2. There is a canonical Γ-equivariant injection of $Z(\hat{G})$ into $Z(\hat{I})$. To construct it we choose a maximal F-torus T of I. Then T is also a maximal F-torus of G. We can regard $Z(\hat{G})$, $Z(\hat{I})$ as subgroups of $\hat{T}$. It is easy to see that $Z(\hat{G}) \subset Z(\hat{I})$ and that this injection does not depend on the choice of T.

4.3. Lemma. *Assume that F is local. Then the diagram*

$$\begin{array}{ccc} H^1(F,I) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow \\ A(I) & \longrightarrow & A(G) \end{array}$$

commutes, where the bottom arrow is induced by the injection $Z(\hat{G}) \to Z(\hat{I})$ of 4.2.

Since $I \to G$ is not a normal homomorphism, we cannot apply the functoriality of $H^1(F, G) \to A(G)$ directly. If F is complex, the result is trivial. If F is p-adic (respectively real), we choose an elliptic (respectively fundamental) maximal F-torus T of I. Now consider the diagram

$$\begin{array}{ccccc} H^1(F,T) & \longrightarrow & H^1(F,I) & \longrightarrow & H^1(F,G) \\ \downarrow & & \downarrow & & \downarrow \\ A(T) & \longrightarrow & A(I) & \longrightarrow & A(G). \end{array}$$

The lemma follows from the surjectivity of $H^1(F,T) \to H^1(F,I)$ (see Sect. 10) and the second statement of Theorem 1.2.

4.4. Assume that F is local. We define $\mathfrak{E}(I/F)$ to be the finite abelian group $\ker[A(I) \to A(G)]$. Lemma 4.3 gives us a canonical map $\mathfrak{D}(I/F) \to \mathfrak{E}(I/F)$, which is bijective in the p-adic case.

4.5. Assume that F is global. We define $\mathfrak{E}(I/\mathbb{A})$ to be $\bigoplus_v \mathfrak{E}(I/F_v)$, where v runs through the set of places of F.

4.6. The exact sequence

$$1 \to Z(\hat{G}) \to Z(\hat{I}) \to Z(\hat{I})/Z(\hat{G}) \to 1$$

gives us a homomorphism [K3, Corollary 2.3]

$$\pi_0([Z(\hat{I})/Z(\hat{G})]^\Gamma) \to H^1(F, Z(\hat{G})).$$

We define $\mathfrak{K}(I/F)$ to be the subgroup of $\pi_0([Z(\hat{I})/Z(\hat{G})]^\Gamma)$ consisting of all elements whose image in $H^1(F, Z(\hat{G}))$ is

(a) trivial if F is local,

(b) locally trivial if F is global.

If F is local, then

$$\mathfrak{K}(I/F) = \operatorname{cok}[A(G)^D \to A(I)^D] = \mathfrak{E}(I/F)^D.$$

5. Local Conjectures (Orbital Integrals)

In this section F is a local field of characteristic 0 and G is a connected reductive group over F. To keep the discussion as simple as possible we assume that G_{der} is simply connected, so that G_γ is connected for all semi-simple $\gamma \in G$, and stable conjugacy and $G(\bar{F})$-conjugacy are the same [K1]. The general case is no harder, but leads to more awkward statements.

5.1. We need to recall that there is a sign $e(I) = \pm 1$ attached to any connected reductive group I over F [K2].

5.2. Let γ be a semi-simple element of $G(F)$ and let $I = G_\gamma = G_\gamma^0$. Choose Haar measures dg, di on $G(F), I(F)$ respectively, and let O_γ denote the linear form on

$C_c^\infty(G(F))$ given by

$$O_\gamma(f) = \int_{I(F)\backslash G(F)} f(g^{-1}\gamma g)\frac{dg}{di}.$$

For any stable conjugate $\gamma' \in G(F)$ of γ the group $I' = G_{\gamma'}$ is an inner twist of I and in the usual way di gives us a Haar measure di' on $I'(F)$. We use dg, di' to form $O_{\gamma'}$, and we define a linear form SO_γ on $C_c^\infty(G(F))$ by the formula

$$SO_\gamma = \sum_{\gamma'} e(I')O_{\gamma'},$$

where γ' runs over a set of representatives for the conjugacy classes in the stable conjugacy class of γ. For G such that G_{der} is not simply connected it is necessary to modify the definition of SO_γ by including the factor $|\ker[H^1(F, I') \to H^1(F, G_{\gamma'})]|$ in the summand indexed by γ' [as usual, $I' = (G_{\gamma'})^0$].

5.3. Conjecture. *The distribution SO_γ is stable (see* [L2] *for the definition of stable distribution).*

5.4. Let (H, s, η) be an endoscopic triple for G. Choose an extension of $\eta: \hat{H} \to \hat{G}$ to an L-homomorphism $\eta': {}^LH \to {}^LG$. One expects to have a correspondence (f, f^H) [L2, S2] between functions $f \in C_c^\infty(G(F))$, $f^H \in C_c^\infty(H(F))$, such that

$$SO_{\gamma_H}(f^H) = \sum_\gamma \Delta(\gamma_H, \gamma)O_\gamma(f)$$

for every G-regular semi-simple element $\gamma_H \in H(F)$. The sum runs over a set of representatives for the conjugacy classes in $G(F)$ belonging to the $G(\bar{F})$-conjugacy class associated to γ_H (3.1); if this $G(\bar{F})$-conjugacy class contains no element of $G(F)$, than the sum is empty and the right side of the equation is 0. The complex numbers $\Delta(\gamma_H, \gamma)$ are called transfer factors; at the moment these have a complete definition only in the archimedean case [S2]. Presumably the final definition, whatever it turns out to be, will only specify the function $\Delta(\cdot, \cdot)$ up to multiplication by a non-zero scalar. Of course changing $\Delta(\cdot, \cdot)$ by a scalar causes the correspondence (f, f^H) to change by that scalar. The correspondence and the transfer factors depend on η'.

5.5. Conjecture. The function $\Delta(\cdot, \cdot)$ can be extended (continuously, in all likelihood) to all pairs (γ_H, γ) consisting of a (G, H)-regular semi-simple element $\gamma_H \in H(F)$ and a corresponding element $\gamma \in G(F)$, in such a way that

$$SO_{\gamma_H}(f^H) = \sum_\gamma \Delta(\gamma_H, \gamma)e(G_\gamma)O_\gamma(f).$$

We are again using compatible measures on I_{γ_H}, G_γ as in 5.2 [the two groups are inner twists of each other since γ_H is (G, H)-regular].

5.6. Note that $e(T) = 1$ for any torus T, so that 5.5 is compatible with 5.4 in case γ_H is G-regular.

The relation between $\Delta(\gamma_H, \gamma)$, $\Delta(\gamma_H, \gamma')$ for $\gamma' \in G(F)$ in the stable conjugacy class of γ should be

$$\Delta(\gamma_H, \gamma') = \Delta(\gamma_H, \gamma)\langle \mathrm{inv}(\gamma, \gamma'), \kappa\rangle.$$

Here $\mathrm{inv}(\gamma, \gamma')$ is the element of $\mathfrak{E}(I/F)$ obtained as the image under $\mathfrak{D}(I/F) \to \mathfrak{E}(I/F)$ of the element of $\mathfrak{D}(I/F)$ that measures the difference between γ, γ' (see 4.1). The element $\kappa \in \mathfrak{K}(I/F)$ comes from s *via*

$$Z(\hat{H}) \hookrightarrow Z(\hat{I}_H) \xrightarrow{\sim} Z(\hat{I}),$$

and $\langle \cdot , \cdot \rangle$ denotes the duality pairing between $\mathfrak{E}(I/F)$ and $\mathfrak{K}(I/F)$ (see 4.6). Thus we could also write the equation in 5.5 as

$$SO_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma) O_\gamma^\kappa(f),$$

where

$$O_\gamma^\kappa(f) = \sum_{\gamma'} \langle \mathrm{inv}(\gamma, \gamma'), \kappa \rangle e(G_{\gamma'}) O_{\gamma'}(f).$$

6. Global Conjecture

In this section F is a number field and G is a connected reductive group over F. We assume that G_{der} is simply connected. Consider an inner twisting $\psi : G_0 \to G$ with G_0 quasi-split. Let γ_0 be a semi-simple element of $G_0(F)$, and let I_0 denote the centralizer of γ_0 in G_0.

6.1. Let γ be an element of $G(\mathbb{A})$ that is conjugate to $\psi(\gamma_0)$ under $G(\bar{\mathbb{A}})$. Our aim is to construct an obstruction $\mathrm{obs}(\gamma) \in \mathfrak{K}(I_0/F)^D$ to the existence of an element of $G(F)$ in the $G(\mathbb{A})$-conjugacy class of γ. We must assume that G_{SC} has no E_8 factors, since the construction uses the Hasse principle for G_{SC}. For regular semi-simple γ_0 such an obstruction was obtained by Langlands in [L2, Chap. VII]. The method used here, however, is based on [K3, Sect. 9].

6.2. The first step is to get an obstruction $\mathrm{obs}_1(\gamma)$ to the existence of an element of $G(F)$ in the $G_{\mathrm{SC}}(\mathbb{A})$-conjugacy class of γ. This obstruction lies in $A(I_1)$, where I_1 is the centralizer of γ_0 in $(G_0)_{\mathrm{SC}}$ [with $A(\cdot)$ as in Sect. 2]. Let X_0 denote the set of pairs (i, g) satisfying

(a) $g \in G_{\mathrm{SC}}(\mathbb{A})$,

(b) $i : I_0 \to G$ is conjugate to $\psi|_{I_0}$ under $G(\bar{F})$,

(c) $i(\gamma_0) = g\gamma g^{-1}$.

It is not hard to see that X_0 is non-empty. We take i to be $\psi|_{I_0}$ and look for $g \in G_{\mathrm{SC}}(\bar{\mathbb{A}})$ such that $\psi(\gamma_0) = g\gamma g^{-1}$. By hypothesis such a g exists in $G(\bar{\mathbb{A}})$. Now we just need to note that

$$(G_0)_{\mathrm{SC}}(\bar{\mathbb{A}}) \cdot I_0(\bar{\mathbb{A}}) = G_0(\bar{\mathbb{A}}),$$

the point being that

$$(G_0)_{\mathrm{SC}}(\mathfrak{o}_{E_w}) \cdot I_0(\mathfrak{o}_{E_w}) = G_0(\mathfrak{o}_{E_w})$$

for any finite extension E of F that splits G_0, I_0 and any finite place w of E for which $I_0 \to G_0$ is defined over $\mathfrak{o}_{E_w}$ (see [K4, 3.3.4]).

The three groups Γ, $G_{\mathrm{SC}}(\bar{F})$, $I_1(\bar{\mathbb{A}})$ all act on X_0. Let $(i, g) \in X_0$, $\sigma \in \Gamma$, $h \in G_{\mathrm{SC}}(\bar{F})$, $t \in I_1(\bar{\mathbb{A}})$. The actions are given by:

(i) $\sigma(i, g)=(\sigma(i), \sigma(g))$,
(ii) $h \cdot (i, g)=(\mathrm{Int}(h) \circ i, hg)$,
(iii) $(i, g) \cdot t=(i, i(t^{-1})g)$.

The actions of $G_{\mathrm{SC}}(\bar{F})$ and $I_1(\bar{\mathbb{A}})$ commute, and for $x \in X_0$, $h \in G_{\mathrm{SC}}(\bar{F})$, $t \in I_1(\bar{\mathbb{A}})$ we have $\sigma(h \cdot x)=\sigma(h) \cdot \sigma(x)$ and $\sigma(x \cdot t)=\sigma(x) \cdot \sigma(t)$.

Let X denote the quotient of X_0 by $G_{\mathrm{SC}}(\bar{F})$. It is not hard to check that X is an F-torsor under $I_1(\bar{\mathbb{A}})/Z(I_1)(\bar{F})$. Assuming that G_{SC} has no E_8 factors, so that it satisfies the Hasse principle, one sees easily that γ is $G_{\mathrm{SC}}(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if $[X/I_1(\bar{F})]^{\Gamma}$ is non-empty. Let c be the element of $H^1(F, I_1(\bar{\mathbb{A}})/Z(I_1)(\bar{F}))$ determined by X. It is immediate that $[X/I_1(\bar{F})]^{\Gamma}$ is non-empty if and only if c lies in the image of $H^1(F, I_1(\bar{F})/Z(I_1)(\bar{F}))$. Define $\mathrm{obs}_1(\gamma)$ to be the image of c under β_{I_1} (see 2.2). Theorem 2.2 gives us the following result.

6.3. Lemma. *Assume that G_{SC} has no E_8 factors. Then γ is $G_{\mathrm{SC}}(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if* $\mathrm{obs}_1(\gamma)$ *is trivial.*

6.4. Let $\gamma' \in G(\mathbb{A})$ and suppose that γ' is $G(\bar{\mathbb{A}})$-conjugate to γ. Let $Y=\{h \in G_{\mathrm{SC}}(\bar{\mathbb{A}}) \mid h\gamma h^{-1}=\gamma'\}$. As in 6.2 we see that Y is non-empty; therefore Y is an F-torsor under $G_{\mathrm{SC}}(\bar{\mathbb{A}})_{\gamma}$. Let X (respectively X') denote the F-torsor under $I_1(\bar{\mathbb{A}})/Z(I_1)(\bar{F})$ obtained from γ (respectively γ'). By considering the map $Y \times X \to X'$ defined by $(h, (i, g)) \mapsto (i, gh^{-1})$, we see that we are in the situation of 2.3, 2.4. Therefore $\mathrm{obs}_1(\gamma')=\mathrm{obs}_1(\gamma) \cdot \mathrm{inv}_1(\gamma, \gamma')$, where $\mathrm{inv}_1(\gamma, \gamma')$ denotes the image under

$$H^1(F, G_{\mathrm{SC}}(\bar{\mathbb{A}})_{\gamma}) \to A(I_1) \tag{6.4.1}$$

of the class of Y [(6.4.1) comes from (2.3.1)].

6.5. Now we get $\mathrm{obs}(\gamma)$ from $\mathrm{obs}_1(\gamma)$. Let $D=G/G_{\mathrm{der}}$. Then we have exact sequences

$$1 \to G_{\mathrm{SC}} \to G \to D \to 1\,,$$

$$1 \to (G_0)_{\mathrm{SC}} \to G_0 \to D \to 1\,,$$

$$1 \to I_1 \to I_0 \to D \to 1\,.$$

In particular $Z(\hat{I}_1)$ is equal to $Z(\hat{I}_0)/Z(\hat{G})$, which implies that $\mathfrak{K}(I_0/F)$ can be regarded as a subgroup of $\pi_0(Z(\hat{I}_1)^{\Gamma})$. By duality we get a homomorphism

$$A(I_1) \to \mathfrak{K}(I_0/F)^D, \tag{6.5.1}$$

and we define $\mathrm{obs}(\gamma)$ to be the image of $\mathrm{obs}_1(\gamma)$ under (6.5.1).

6.6. Theorem. *Assume that G_{SC} has no E_8 factors. Then γ is $G(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if* $\mathrm{obs}(\gamma)$ *is trivial.*

For each place v of F we choose a place w of $\bar{F}$ lying over F and let $\Gamma(v) = \mathrm{Gal}(\bar{F}_w/F_v) \subset \Gamma$. Consider the obvious maps

$$\lambda : \bigoplus_v \pi_0(Z(\hat{I}_1)^{\Gamma(v)})^D \to \pi_0(Z(\hat{I}_1)^{\Gamma})^D$$

$$\mu : \bigoplus_v \pi_0(Z(\hat{I}_1)^{\Gamma(v)})^D \to \bigoplus_v \pi_0(Z(\hat{I}_0)^{\Gamma(v)})^D$$

as well as their duals λ^D, μ^D. On the one hand, by duality and the definition of $\mathfrak{K}(I_0/F)$, the obstruction $\mathrm{obs}(\gamma)$ is trivial if and only if $\mathrm{obs}_1(\gamma)$ belongs to the subgroup

$$\lambda(\ker(\mu)) \tag{6.6.1}$$

of $A(I_1)$. On the other hand, by 6.3 and 6.4 the element γ is $G(\mathbb{A})$-conjugate to an element of $G(F)$ if and only if $\mathrm{obs}_1(\gamma)^{-1}$ belongs to the image under (6.4.1) of the subset S of $H^1(F, G_{SC}(\mathbb{A})_\gamma)$ obtained by intersecting the kernels of

$$H^1(F, G_{SC}(\mathbb{A})_\gamma) \to H^1(F, G(\mathbb{A})_\gamma), \tag{6.6.2}$$

$$H^1(F, G_{SC}(\mathbb{A})_\gamma) \to H^1(F, G_{SC}(\mathbb{A})). \tag{6.6.3}$$

To finish the proof it is enough to show that the image S' of S under (6.4.1) is equal to the subgroup (6.6.1). It is obvious that S' is contained in the subgroup (6.6.1). We now verify the opposite inclusion. To simplify notation we assume that $F = \mathbb{Q}$, so that there is only one infinite place, denoted ∞. The general case is no harder. Since $H^1(F, G_{SC}(\mathbb{A})) \tilde{\to} H^1(\mathbb{R}, G_{SC}(\mathbb{R}))$ by Kneser's vanishing theorem, the set S contains the set

$$T = \bigoplus_p \ker[H^1(\mathbb{Q}_p, G_{SC}(\mathbb{Q}_p)_{\gamma_p}) \to H^1(\mathbb{Q}_p, G(\mathbb{Q}_p)_{\gamma_p})]$$

of $H^1(\mathbb{Q}, G_{SC}(\mathbb{A})_\gamma)$, where p runs over the set of finite places of $\mathbb{Q}$ and γ_p is the component of γ at p. Let $\mu(v)$ denote the homomorphism

$$\pi_0(Z(\hat{I}_1)^{\Gamma(v)})^D \to \pi_0(Z(\hat{I}_0)^{\Gamma(v)})^D.$$

Using the bijectivity of (1.2.1) in the p-adic case, we see that S' contains

$$T' = \lambda\Bigl(\bigoplus_p \ker \mu(p)\Bigr).$$

It is clear that μ is equal to $\bigoplus_v \mu(v)$, where the sum is over all places v of $\mathbb{Q}$.

To finish the proof it is enough to show that T' is equal to the subgroup (6.6.1), which is equivalent to showing that the natural map

$$\ker \mu \cap \ker \lambda \to \ker \mu(\infty)$$

is surjective. This in turn follows from the surjectivity of

$$\ker[H^1(\mathbb{Q}, I_1) \to H^1(\mathbb{Q}, I_0)] \to \ker \mu(\infty)$$

(use Proposition 2.6); it remains to verify this surjectivity.

Recall that D denotes G/G_{der}. From 1.6, 1.7 and the exact sequence

$$1 \to I_1 \to I_0 \to D \to 1$$

we get a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & D(\mathbb{Q}) & \longrightarrow & H^1(\mathbb{Q}, I_1) & \longrightarrow & H^1(\mathbb{Q}, I_0) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & D(\mathbb{R}) & \longrightarrow & H^1(\mathbb{R}, I_1) & \longrightarrow & H^1(\mathbb{R}, I_0) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longrightarrow & C(D) & \longrightarrow & A(I_{1,\infty}) & \longrightarrow & A(I_{0,\infty}) & \longrightarrow & \cdots,
\end{array}$$

where $C(D)=H^1(F,\hat{D})^D=D(\mathbb{R})/N_{\mathbb{C}/\mathbb{R}}D(\mathbb{C})$. Since $D(\mathbb{Q})$ is dense in $D(\mathbb{R})$, it maps onto $D(\mathbb{R})/N_{\mathbb{C}/\mathbb{R}}D(\mathbb{C})$. Therefore

$$\ker[H^1(\mathbb{Q},I_1)\to H^1(\mathbb{Q},I_0)]$$

maps onto

$$\ker[A(I_{1,\infty})\to A(I_{0,\infty})]=\ker\mu(\infty).$$

This finishes the proof.

6.7. Let γ', $\mathrm{inv}_1(\gamma,\gamma')$ be as in 6.4. Let $\mathrm{inv}(\gamma,\gamma')$ be the image of $\mathrm{inv}_1(\gamma,\gamma')$ under the homomorphism (6.5.1)

$$A(I_1)\to\mathfrak{K}(I_0/F)^D.$$

Then $\mathrm{obs}(\gamma')=\mathrm{obs}(\gamma)\cdot\mathrm{inv}(\gamma,\gamma')$.

6.8. Let $\gamma_0'\in G_0(F)$ and suppose that γ_0' is stably conjugate to γ_0. Using γ_0' rather than γ_0, we get

$$\mathrm{obs}(\gamma)'\in\mathfrak{K}(I_0'/F)^D,$$

where I_0' is the centralizer of γ_0' in G_0. There is an inner twist $I_0'\to I_0$, canonical up to conjugation by an element of $I_0(\bar{F})$. This allows us to identify $\mathfrak{K}(I_0'/F)^D$ with $\mathfrak{K}(I_0/F)^D$; with this identification we have $\mathrm{obs}(\gamma)'=\mathrm{obs}(\gamma)$. This follows from Lemma 2.8.

6.9. Now we are ready to state a global conjecture, which generalizes the global hypothesis in [L2, Chap. VII, Sect. 7]. Let (H,s,η) be an endoscopic triple for G, and choose an extension of $\eta:\hat{H}\to\hat{G}$ to an L-homomorphism $\eta':{}^LH\to{}^LG$. We assume that the local Conjecture 5.5 holds at every place of F. We write $\Delta_v(\cdot,\cdot)$ for the transfer factors at the place v. Recall that $\Delta_v(\cdot,\cdot)$ can be replaced by $c\cdot\Delta_v(\cdot,\cdot)$ for any $c\in\mathbb{C}^\times$.

6.10. Conjecture. For a suitable normalization of the local transfer factors the following statements hold.

(a) For any (G,H)-regular semi-simple $\gamma_H\in H(F)$ and any $\gamma\in G(\mathbb{A})$ coming from γ_H the expression $\prod_v \Delta_v(\gamma_H,\gamma)$ has only a finite number of terms $\neq 1$ and hence has a well-defined product, which we denote by $\Delta(\gamma_H,\gamma)$.

(b) Let γ_H,γ be as in (a). Choose an inner twisting $\psi:G_0\to G$ and $\gamma_0\in G_0(F)$ such that γ_0 comes from γ_H (γ_0 exists by [K1, Theorem 4.4]). Let I_0 denote the centralizer of γ_0 in G_0, and let $\mathrm{obs}(\gamma)\in\mathfrak{K}(I_0/F)^D$ be the obstruction of 6.5. Then

$$\Delta(\gamma_H,\gamma)=\langle\mathrm{obs}(\gamma),\kappa\rangle, \tag{6.10.1}$$

where $\kappa\in\mathfrak{K}(I_0/F)$ is obtained from s via

$$Z(\hat{H})\hookrightarrow Z(\hat{I}_H)\xrightarrow{\sim}Z(\hat{I}_0)$$

[as usual, $I_H=H_{\gamma_H}=(H_{\gamma_H})^0$].

6.11. Remark. We see from 6.8 that the right side of (6.10.1) is independent of the choice of γ_0. We see from 6.7 that the left and right sides are multiplied by the same factor if γ is replaced by a $G(\mathbb{A})$-conjugate γ'.

7. Local Finiteness Results

In this section F is a p-adic field, $\mathfrak{o}$ is the valuation ring of F, k is the residue field of $\mathfrak{o}$, $\bar{k}$ is an algebraic closure of k, and G is an unramified connected reductive group over F. Let x_0 be a hyperspecial point in the building of G, and let $\mathbf{G}$ be the corresponding extension of G to a group scheme over $\mathfrak{o}$ [T]. We write K for the hyperspecial maximal compact subgroup $\mathbf{G}(\mathfrak{o})=\mathrm{Stab}_{G(F)}(x_0)$ of $G(F)$.

7.1. Proposition. *Let γ be a semi-simple element of K such that $1-\alpha(\gamma)\in\mathfrak{o}_{\bar{F}}$ is either 0 or a unit for every root α of G, and let $I=G_\gamma^0$. Then I is unramified and $I(F)\cap K$ is a hyperspecial maximal compact subgroup of $I(F)$. Furthermore,*

$$\ker[H^1(F,I)\to H^1(F,G_\gamma)]$$

is trivial. Finally, if $\gamma'\in K$ is stably conjugate to γ, then γ' is conjugate to γ under K.

During the first part of the proof we assume that G_{der} is simply connected, so that $I=G_\gamma$. For the moment we consider the following special case. We assume that G is split over F; that $\gamma\in\mathbf{A}(\mathfrak{o})$, where $\mathbf{A}$ is a split maximal $\mathfrak{o}$-torus of $\mathbf{G}$; and that γ' is conjugate to γ under $G(F)$.

Let $\mathbf{G}_\gamma$ be the closed subgroup scheme of $\mathbf{G}$ whose points in any $\mathfrak{o}$-algebra R are given by

$$\mathbf{G}_\gamma(R)=\{g\in\mathbf{G}(R)|g\gamma g^{-1}=\gamma\}.$$

The image of γ in $\mathbf{G}(k)$ is semi-simple (it belongs to $\mathbf{A}(k)$), and the derived group of $\mathbf{G}_k$ is simply connected; thus we see that the special fiber of $\mathbf{G}_\gamma$ is a connected reductive group. The hypothesis about the values at γ of the roots α of G implies that special and general fibers of $\mathbf{G}_\gamma$ have the same dimension. Therefore $\mathbf{G}_\gamma$ is smooth over $\mathfrak{o}$ [SGA 3 VI_B 4.4] with connected reductive fibers. In particular G_γ is unramified and $\mathbf{G}_\gamma(\mathfrak{o})=G_\gamma(F)\cap K$ is a hyperspecial maximal compact subgroup of $G_\gamma(F)$ [T].

The statement about γ' can be proved in the same way as Lemma 19 of [H]. We have assumed that γ' is conjugate to γ under $G(F)$. Choose a Borel $\mathfrak{o}$-subgroup $\mathbf{B}$ of $\mathbf{G}$ containing $\mathbf{A}$ and let $\mathbf{N}$ be the unipotent radical of $\mathbf{B}$. Then $G(F)=K\cdot N(F)\cdot A(F)$, and without loss of generality we may assume that $\gamma'=n\gamma n^{-1}$ for some $n\in N(F)$.

To finish this step of the proof it is enough to show that $n\in N(\mathfrak{o})N_\gamma(F)$, where $N_\gamma=G_\gamma\cap N$. Choose a total order $\alpha_1<\ldots<\alpha_r$ of the usual kind on the set Δ of B-positive roots of A, and use this order to identify $\mathbf{N}$ with $\prod_{i=1}^{r}\mathbb{G}_a$ over $\mathfrak{o}$. This identification does not respect the multiplication law for $\mathbf{N}$; nevertheless $\prod_{i=j}^{r}\mathbb{G}_a$ is identified with a subgroup $\mathbf{N}_j$ of $\mathbf{N}$ $(1\leqq j\leqq r)$, and the projection $\mathbf{N}_j\to\mathbb{G}_a$ on the factor indexed by j is a homomorphism. Write n as $(x_1,\ldots,x_r)\in\prod_{i=1}^{r}\mathbb{G}_a$. Then

$n\gamma^{-1}n^{-1}\gamma^{-1}$ belongs to $\mathbf{N}(\mathfrak{o})$ and its first coordinate is $(1-\alpha_1(\gamma))x_1$. If $1-\alpha_1(\gamma)=0$, then α_1 is a root of G_γ, and by multiplying n on the right by an element of $N_\gamma(F)$, we may assume that $x_1=0$. If $1-\alpha_1(\gamma)\neq 0$, then by hypothesis $1-\alpha_1(\gamma)$ is a unit, and therefore $x_1\in\mathfrak{o}$. Multiplying n on the left by an element of $\mathbf{N}(\mathfrak{o})$, we may again assume that $x_1=0$. Then $n\in N_2$ and the same argument can be applied to x_2. Continuing with this until all the positive roots have been used, we see that $n\in\mathbf{N}(\mathfrak{o})\cdot N_\gamma(F)$.

Now we proceed with the next step. We continue to assume that G_{der} is simply connected, but drop the rest of the assumptions. Choose a maximal F-torus T of G containing γ, and choose a finite Galois extension E/F such that

(i) T splits over E,

(ii) γ,γ' are conjugate under $G(E)$.

Choose a maximal split $\mathfrak{o}_E$-torus $\mathbf{A}$ of $\mathbf{G}$ [(i) implies that G is split over E], and choose an element $\gamma''\in\mathbf{A}(\mathfrak{o}_E)$ that is conjugate under $G(E)$ to γ and γ'. Our previous work shows that γ,γ',γ'' are all conjugate under $G(\mathfrak{o}_E)$ and that $\mathbf{G}_{\gamma''}$ is smooth over $\mathfrak{o}_E$. Since $(\mathbf{G}_\gamma)_{\mathfrak{o}_E}$ is isomorphic to $\mathbf{G}_{\gamma''}$ and $\mathfrak{o}_E$ is faithfully flat over $\mathfrak{o}$, the group $\mathbf{G}_\gamma$ is smooth over $\mathfrak{o}$ and its fibers are connected reductive groups. In particular G_γ is unramified and $\mathbf{G}_\gamma(\mathfrak{o})=G_\gamma(F)\cap K$ is a hyperspecial maximal compact subgroup of $G_\gamma(F)$.

Now consider the closed subscheme $\mathbf{Y}$ of $\mathbf{G}$ whose points in any $\mathfrak{o}$-algebra R are given by

$$\mathbf{Y}(R)=\{g\in\mathbf{G}(R)|g\gamma g^{-1}=\gamma'\}.$$

Of course $\mathbf{Y}$ is smooth over $\mathfrak{o}$, since it becomes isomorphic to $\mathbf{G}_\gamma$ over $\mathfrak{o}_E$. Let $\tilde{\gamma},\tilde{\gamma}'$ denote the images of γ,γ' in $\mathbf{G}(k)$. The conjugacy of γ,γ' in $\mathbf{G}(\mathfrak{o}_E)$ implies the stable conjugacy of $\tilde{\gamma},\tilde{\gamma}'$. Furthermore we have seen that the special fiber of $\mathbf{G}_\gamma$ is connected. Therefore $\tilde{\gamma},\tilde{\gamma}'$ are conjugate under $\mathbf{G}(k)$. In other words, $\mathbf{Y}(k)$ is non-empty, and now the smoothness of $\mathbf{Y}$ over $\mathfrak{o}$ implies that $\mathbf{Y}(\mathfrak{o})$ is non-empty. This finishes the proof in the case that G_{der} is simply connected.

Now we consider the general case. We choose an unramified z-extension $\alpha: H\to G$ of G. The kernel Z of α is an unramified torus lying in the center of H. The building of H is the product of the buildings of H and Z. Let y_0 be a hyperspecial point in the building of H that projects to x_0 and let $\mathbf{H}$ be the corresponding extension of H to a group scheme over $\mathfrak{o}$. Let $\mathbf{Z}$ be the unique (up to isomorphism) extension of Z to a torus over $\mathfrak{o}$. There is an exact sequence

$$1\to\mathbf{Z}\to\mathbf{H}\to\mathbf{G}\to 1$$

and the morphism $\mathbf{H}\to\mathbf{G}$ is smooth. Furthermore, $\mathbf{H}(k)\to\mathbf{G}(k)$ is surjective, since $\mathbf{Z}_k$ is connected. Therefore $\mathbf{H}(\mathfrak{o})\to\mathbf{G}(\mathfrak{o})$ is surjective. Choose $\delta\in\mathbf{H}(\mathfrak{o})$ such that $\delta\mapsto\gamma$. Let $\mathbf{I}$ be the quotient of $\mathbf{H}_\delta$ by $\mathbf{Z}$ [SGA 3 XXII 4.3.2]. Then $\mathbf{I}$ is an extension of I to a smooth group scheme over $\mathfrak{o}$ with connected reductive fibers. Using that $\mathbf{H}(\mathfrak{o})\to\mathbf{G}(\mathfrak{o})$ and $\mathbf{H}_\delta(\mathfrak{o})\to\mathbf{I}(\mathfrak{o})$ are surjective, we see that $I(F)\cap K$ is equal to $\mathbf{I}(\mathfrak{o})$ and is therefore a hyperspecial maximal compact subgroup of $I(F)$.

Now we prove the statement about γ'. Since γ',γ are stably conjugate there exist $\delta'\in H(F)$ and $h\in H(\bar{F})$ such that $\delta'\mapsto\gamma'$ and $\delta'=h\delta h^{-1}$ [K1]. We claim that $\delta'\in\mathbf{H}(\mathfrak{o})$. Since $\delta'\mapsto\gamma'\in\mathbf{G}(\mathfrak{o})$, it is enough to check that δ' acts trivially on the building of Z, and this is obvious from the equation $\delta'=h\delta h^{-1}$. Our previous work shows that δ,δ' are conjugate under $\mathbf{H}(\mathfrak{o})$. Therefore γ,γ' are conjugate under $\mathbf{G}(\mathfrak{o})$.

Finally, we prove that

$$\ker[H^1(F,I)\to H^1(F,G_\gamma)]$$

is trivial. It is easy to translate this into the following statement about H [with $\delta\in\mathbf{H}(\mathfrak{o})$, $\delta\mapsto\gamma$ as above]: if $\delta'\in H(F)$ is stably conjugate to δ and if $\delta'\mapsto\gamma$, then δ' is conjugate to δ. Just as above we see that $\delta'\in\mathbf{H}(\mathfrak{o})$ and hence that δ' is indeed conjugate to δ.

7.2. Assume that $\gamma\in K$ satisfies the hypothesis of 7.1. Let f_K denote the characteristic function of the subset K of $G(F)$. Let dg (respectively di) be a Haar measure on $G(F)$ [respectively $I(F)$] that gives measure 1 to K [respectively $I(F)\cap K$]. Let γ' be a stable conjugate of γ and form the orbital integral $O_{\gamma'}(f_K)$ using dg/di', where di' is the measure on $I'(F)$ obtained from di.

7.3. Corollary. *The orbital integral $O_{\gamma'}(f_K)$ vanishes unless γ' is conjugate to γ, in which case it equals* 1.

If γ' is not conjugate to γ, then by 7.1 the orbit of γ' does not meet K, and clearly $O_{\gamma'}(f_K)=0$. It remains to show that $O_\gamma(f_K)=1$. We have $O_\gamma(f_K)=\operatorname{meas} I(F)\backslash X$ where

$$X=\{g\in G(F)\,|\,g^{-1}\gamma g\in K\}\,.$$

If G_{der} is simply connected, 7.1 shows directly that $X=I(F)\cdot K$. Using an unramified z-extension $H\to G$ as in the proof of 7.1, we see that $X=I(F)\cdot K$ in general. Therefore $O_\gamma(f_K)=\operatorname{meas}(K)\cdot\operatorname{meas}(I(F)\cap K)^{-1}=1$.

7.4. For simplicity we assume that G_{der} is simply connected for the rest of Sect. 7. Let (H,s,η) be an endoscopic triple for G. Let γ_H be a semi-simple (G,H)-regular element of $H(F)$ and let γ be a corresponding element of $G(F)$, as in 3.1. Let $I=G_\gamma$ and let κ be the element of $\mathfrak{K}(I/F)$ obtained from s as in 5.6. Then we can form κ-orbital integrals O_γ^κ (again see 5.6).

7.5. Proposition. *Assume that H is not an unramified group. Let f belong to the Hecke algebra $\mathfrak{H}(G(F),K)$. Then $O_\gamma^\kappa(f)=0$.*

Let Γ_0 denote the inertia subgroup of Γ. Since G is unramified, Γ_0 acts trivially on $X^*(Z(G))$, and therefore $Z(G)$ can be embedded in an unramified F-torus C'. Let $C=C'/Z(G)$. We form an unramified group G_1 by taking the quotient of $G\times C'$ by $Z(G)$, with $Z(G)$ embedded diagonally in $G\times C'$. The center of G_1 is connected (it is isomorphic to C') and there is an obvious exact sequence

$$1\to G\to G_1\to C\to 1\,,$$

which yields a dual exact sequence

$$1\to\hat{C}\to\hat{G}_1\to\hat{G}\to 1\,.$$

Pulling back *via* $\eta:\hat{H}\to\hat{G}$, we get the following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{H}_1 & \longrightarrow & \hat{H} & \longrightarrow & 1\\
 & & \Big\| & & \Big\downarrow{\scriptstyle\eta_1} & & \Big\downarrow{\scriptstyle\eta} & & \\
1 & \longrightarrow & \hat{C} & \longrightarrow & \hat{G}_1 & \longrightarrow & \hat{G} & \longrightarrow & 1\,,
\end{array}$$

where $\hat{H}_1$ denotes the fiber product of $\hat{G}_1, \hat{H}$ over $\hat{G}$. Note that $\hat{H}_1$ is a connected reductive group over $\mathbb{C}$.

We now define an action of Γ on $\hat{H}_1$. The fiber product construction does not automatically give such an action, since η need not be a Γ-map. However, the $\hat{G}$-conjugacy class of η is fixed by Γ, which means that for each $\sigma \in \Gamma$ there exists $g_\sigma \in \hat{G}$ such that

$$\eta \circ \sigma = \mathrm{Int}(g_\sigma) \circ \sigma \circ \eta . \tag{7.5.1}$$

For each $\sigma \in \Gamma$ choose $x_\sigma \in \hat{G}_1$ such that $x_\sigma \mapsto g_\sigma$. Then the restriction of $\mathrm{Int}(x_\sigma)$ to $\sigma(\eta_1(\hat{H}_1))$ is independent of the choices of g_σ and x_σ. We let Γ act on $\hat{H}_1$ in the unique way for which

$$\eta_1 \circ \sigma = \mathrm{Int}(x_\sigma) \circ \sigma \circ \eta_1 . \tag{7.5.2}$$

We have a commutative diagram with exact rows

$$\begin{tikzcd} 1 \arrow[r] & \hat{C} \arrow[r] & Z(\hat{H}_1) \arrow[r] & Z(\hat{H}) \arrow[r] & 1 \\ 1 \arrow[r] & \hat{C} \arrow[r] \arrow[u, equal] & Z(\hat{G}_1) \arrow[r] \arrow[u] & Z(\hat{G}) \arrow[r] \arrow[u] & 1 . \end{tikzcd}$$

Our next step is to show that s does not belong to the image of $Z(\hat{H}_1)^{\Gamma_0}$. Let s_1 be any element of $Z(\hat{H}_1)$ that maps to s. Since H is not unramified, we may choose $\sigma \in \Gamma_0$ such that σ acts non-trivially on $\hat{H}$. The group G, however, is unramified, and thus σ acts trivially on $\hat{G}$; (7.5.1) now shows that $g_\sigma \notin \eta(\hat{H})$ (otherwise σ would act on $\hat{H}$ by an inner automorphism, which would necessarily be trivial since σ preserves some splitting of $\hat{H}$). Therefore $x_\sigma \notin \eta_1(\hat{H}_1)$. But $\eta_1(\hat{H}_1)$ is the identity component of the centralizer of s_1 in $\hat{G}_1$, and we have arranged that the derived group of $\hat{G}_1$ be simply connected, so that this centralizer is in fact connected. We conclude that x_σ does not centralize s_1 and hence that $\sigma(s_1) \neq s_1$.

Because F is local there is no harm in assuming that $s \in Z(\hat{H})^\Gamma$. The connecting homomorphism for the exact sequence

$$1 \to \hat{C} \to Z(\hat{H}_1) \to Z(\hat{H}) \to 1$$

sends s to an element $\alpha \in H^1(F, \hat{C})$, which is the Langlands parameter for a character χ of $C(F)$. We have seen that s does not belong to the image of $Z(\hat{H}_1)^{\Gamma_0}$. Therefore α is a ramified Langlands parameter for the unramified torus C, which shows that χ is non-trivial on $\mathbf{C}(\mathfrak{o})$ for the unique extension $\mathbf{C}$ of C to a torus over $\mathfrak{o}$.

Let H_1 be a quasi-split connected reductive group over F whose dual is $\hat{H}_1$. Choose an embedding $H \to H_1$ over F dual to $\hat{H}_1 \to \hat{H}_1$. Although (H_1, s_1, η_1) is not an endoscopic triple for G_1 [since $s_1 \notin Z(\hat{G}_1) \cdot Z(\hat{H}_1)^\Gamma$], the results of Sect. 3 extend to H_1, G_1. Let I_1 (respectively I_{H_1}) denote the connected centralizer of γ in G_1 (respectively γ_H in H_1). Then γ is (G_1, H_1)-regular and we have

$$Z(\hat{H}_1) \hookrightarrow Z(\hat{I}_{H_1}) \xrightarrow{\sim} Z(\hat{I}_1) .$$

We also have a commutative diagram

$$\begin{tikzcd} 1 \arrow[r] & \hat{C} \arrow[r] \arrow[d, equal] & Z(\hat{H}_1) \arrow[r] \arrow[d] & Z(\hat{H}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \hat{C} \arrow[r] & Z(\hat{I}_1) \arrow[r] & Z(\hat{I}) \arrow[r] & 1 . \end{tikzcd}$$

Let $g_1 \in G_1(F)$, let c be the image of g_1 in $C(F)$, and let $\gamma' = g_1 \gamma g_1^{-1}$. Then $\gamma' \in G(F)$ is a stable conjugate of γ. We claim that

$$\langle \mathrm{inv}(\gamma, \gamma'), \kappa \rangle = \chi(c^{-1}). \tag{7.5.3}$$

Let κ_0 denote the image of s in $Z(\hat{I})^\Gamma$. From the commutative diagram above we see that the connecting homomorphism for

$$1 \to \hat{C} \to Z(\hat{I}_1) \to Z(\hat{I}) \to 1$$

maps κ_0 to α. Lemma 1.6 tells us that $\chi(c^{-1}) = \langle d, \kappa_0 \rangle$, where d is the image of c^{-1} under the connecting homomorphism

$$C(F) \to H^1(F, I)$$

coming from

$$1 \to I \to I_1 \to C \to 1 .$$

To finish proving the claim it is enough to check that $d = \mathrm{inv}(\gamma, \gamma')$. Choose $g_2 \in I_1(\bar{F})$ such that $g_2 \mapsto c^{-1}$. Then $g_1 = g g_2^{-1}$ for some $g \in G(\bar{F})$. It is clear that the 1-cocycle

$$(g^{-1} \cdot \tau(g)) = (g_2^{-1} \cdot \tau(g_2)) \quad (\tau \in \Gamma)$$

represents the classes of both $\mathrm{inv}(\gamma, \gamma')$ and d.

With g_1, c as above we use g_1 and f to obtain $f_1 \in C_c^\infty(G(F))$, where

$$f_1(x) = f(g_1 x g_1^{-1}).$$

From (7.5.3) we see that

$$O_\gamma^\kappa(f_1) = \chi(c) \cdot O_\gamma^\kappa(f).$$

Here we used the fact that $e(G_\gamma) = e(G_{g_1 \gamma g_1^{-1}})$, which is obvious since $\mathrm{Int}(g_1)$ defines an F-isomorphism between the two groups.

Choose a maximal F-torus T in G whose split component is as large as possible and whose apartment contains x_0. We have an exact sequence of unramified tori

$$1 \to T \to T_1 \to C \to 1 ,$$

where T_1 is the centralizer of T in G_1. Let $\mathbf{T}, \mathbf{T}_1$ denote the unique extensions of T, T_1 to tori over $\mathfrak{o}$. Then

$$1 \to \mathbf{T}(\mathfrak{o}) \to \mathbf{T}_1(\mathfrak{o}) \to \mathbf{C}(\mathfrak{o}) \to 1$$

is exact.

We have seen that χ is non-trivial on $\mathbf{C}(\mathfrak{o})$. Thus we may choose $g_1 \in \mathbf{T}_1(\mathfrak{o})$ such that $\chi(c) \neq 1$, where c denotes the image of g_1 in $\mathbf{C}(\mathfrak{o})$. With f_1 as before we have

$$O_\gamma^\kappa(f_1) = \chi(c) \cdot O_\gamma^\kappa(f).$$

We claim that $f_1 = f$. Since $f \in \mathfrak{H}(G(F), K)$, it is enough to check that $g_1^{-1} K a K g_1 = KaK$ for any $a \in T(F)$. This is obvious, since g_1 normalizes K and commutes with a. We conclude that

$$O_\gamma^\kappa(f) = \chi(c) \cdot O_\gamma^\kappa(f),$$

and hence that $O_\gamma^\kappa(f) = 0$.

8. Global Finiteness Result

In this section F is a number field, G is a connected reductive group over F, and $\psi: G_0 \to G$ is an inner twisting with G_0 quasi-split over F.

8.1. The inner twisting ψ induces maps

$$\{\text{stable classes in } G(F_v)\} \to \{\text{stable classes in } G_0(F_v)\} \tag{8.1.1}$$

(see [K1]) for each place v of F. We say that the $G(\mathbb{A})$-conjugacy class of $\gamma \in G(\mathbb{A})$ *comes from* $\gamma_0 \in G_0(F)$ if every local component of γ maps under (8.1.1) to the stable class of γ_0.

8.2. Proposition. *Let C be a compact subset of $G(\mathbb{A})$. Then there are only finitely many $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that meet C and come from some semi-simple element of $G_0(F)$.*

First we do the case in which G_{der} is simply connected. Choose a faithful representation $G \to GL_n$. The coefficients of the characteristic polynomial of an $n \times n$ matrix give us a continuous map $GL_n(\mathbb{A}) \to \mathbb{A}^n$. Let C' be the image of C under the composed map

$$G(\mathbb{A}) \to GL_n(\mathbb{A}) \to \mathbb{A}^n.$$

Then $C' \cap F^n$ is both discrete and compact and is therefore finite. Let γ be a semi-simple element of $G(\mathbb{A})$ whose $G(\mathbb{A})$-conjugacy class meets C and comes from some semi-simple element of $G_0(F)$. Then the image of γ in $\mathbb{A}^n$ belongs to $C' \cap \bar{F}^n \cap \mathbb{A}^n = C' \cap F^n$. Thus, in proving the finiteness statement it is enough to consider conjugacy classes in $G(\mathbb{A})$ mapping to a fixed element of $\mathbb{A}^n$. Furthermore, there are only a finite number of semi-simple conjugacy classes in $G_0(\bar{F})$ whose image under

$$G_0(\bar{F}) \xrightarrow{\psi} G(\bar{F}) \to GL_n(\bar{F}) \to \bar{F}^n$$

is a fixed element of $\bar{F}^n$, which means that it is enough to consider conjugacy classes in $G(\mathbb{A})$ that come from a fixed semi-simple $\gamma_0 \in G_0(F)$.

We may as well assume that there exists $\gamma \in G(\mathbb{A})$ whose conjugacy class comes from γ_0 (otherwise there is nothing to prove). Choose a compact open subgroup K of $G(\mathbb{A}_f)$. Choose a finite set V of places of F, including all infinite places, such that

(a) for all $v \notin V$ the group G is unramified at v and K can be written as $K_v K^v$, where K_v is a hyperspecial maximal compact subgroup of $G(F_v)$ and K^v is a compact open subgroup of $G(\mathbb{A}_f^v)$ ($\mathbb{A}_f^v$ is the ring of finite adeles with trivial v-component),

(b) for all $v \notin V$ the v-component γ_v of γ belongs to K_v and $1-\alpha(\gamma_v)$ is either 0 or a unit in $\mathfrak{o}_{\bar{F}_v}$ for every root α of G,

(c) C is contained in

$$\prod_{v \in V} G(F_v) \cdot \prod_{v \notin V} K_v.$$

It is possible to achieve (b) since γ comes from γ_0 and $1-\alpha(\gamma_0) \in \bar{F}$ is either 0 or a unit locally almost everywhere.

Let X be the set of $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that meet C and come from γ_0. We want to show that X is finite. At any place v of F the stable conjugacy class of γ_v contains only finitely many conjugacy classes. Therefore it is enough to show that any conjugacy class in X contains an element γ' such that $\gamma'_v = \gamma_v$ for all $v \notin V$, and this follows immediately from 7.1.

Now we do the general case. Choose a z-extension $\alpha: H \to G$ with kernel Z. Let Y be the set of $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that meet C and come from some semi-simple element of $G_0(F)$. We want to prove that Y is finite. Let $G(\mathbb{A})^1$ denote the intersection of the kernels of the characters $|\lambda|$, where λ runs through the set of homomorphisms $G \to \mathbb{G}_m$ over F. Any conjugacy class in Y is contained in $G(\mathbb{A})^1$, and thus by replacing C by $C \cap G(\mathbb{A})^1$, we may as well suppose that C is contained in $G(\mathbb{A})^1$. It is easy to see that α maps $H(\mathbb{A})^1$ onto $G(\mathbb{A})^1$. Because $H(\mathbb{A})^1$ is locally compact, we can find a compact subset C_H of $H(\mathbb{A})^1$ such that $\alpha(C_H) = C$. Furthermore, the compactness of $Z(F)\backslash Z(\mathbb{A})^1$ implies the existence of a compact subset C_Z of $Z(\mathbb{A})^1$ such that $Z(F) \cdot C_Z = Z(\mathbb{A})^1$. Replacing C_H by $C_Z \cdot C_H$, we may assume that $Z(F) \cdot C_H = \alpha^{-1}(C) \cap H(\mathbb{A})^1$.

We twist the z-extension $H \to G$ as in Sect. 5 of [K1], obtaining a commutative diagram

$$\begin{array}{ccc} H_0 & \xrightarrow{\psi_H} & H \\ {\scriptstyle \alpha_0}\downarrow & & \downarrow{\scriptstyle \alpha} \\ G_0 & \xrightarrow{\psi} & G, \end{array}$$

where $\alpha_0: H_0 \to G_0$ is a z-extension with kernel Z and ψ_H is an inner twisting. Let Y_H be the set of $H(\mathbb{A})$-conjugacy classes in $H(\mathbb{A})$ that meet C_H and come from some semi-simple element of $H_0(F)$. From our previous work we know that Y_H is finite, and we will finish the proof by showing that the natural map $Y_H \to Y$ is surjective.

Consider a conjugacy class in Y and choose an element $\gamma \in C$ in that conjugacy class. Choose a semi-simple element $\gamma_0 \in G_0(F)$ such that γ comes from γ_0. Choose $\delta_0 \in H_0(F)$ such that $\delta_0 \mapsto \gamma_0$. For every place v of F there exists $\delta_v \in H(F_v)$ such that δ_v, δ_0 are stably conjugate and $\delta_v \mapsto \gamma_v$, where γ_v denotes the v-component of γ. Let δ be the element of $\prod_v H(F_v)$ whose v-components are the elements δ_v. It is easy to see that $\delta \in H(\mathbb{A})$ (see the part of the proof of Proposition 7.1 that uses $H \to G$). Since $\delta \in H(\mathbb{A})^1$ and $\alpha(\delta) = \gamma \in C$, we have $\delta \in zC_H$ for some $z \in Z(F)$. Then the $H(\mathbb{A})$-conjugacy class of $z^{-1}\delta$ belongs to Y_H and maps to the $G(\mathbb{A})$-conjugacy class of γ.

9. Stabilization of the Elliptic Terms in the Trace Formula

In this section F is a number field and G is a connected reductive group over F.

9.1. We use the canonical Haar measure dg on $G(\mathbb{A})$ (the one used in the definition of the Tamagawa number of G). For $f \in C_c^\infty(G(\mathbb{A}))$ we write $T_e(f)$ for the elliptic part of the trace formula [A] for f:

$$T_e(f) = \sum_{\gamma \in E} |G_\gamma(F)/I(F)|^{-1} \tau(I) O_\gamma(f),$$

where E is a set of representatives for the elliptic semi-simple conjugacy classes in $G(F)$, I is the identity component of G_γ, $\tau(I)$ is the Tamagawa number of I, and $O_\gamma(f)$ is the adelic orbital integral

$$\int_{I(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g)\frac{dg}{di}$$

[di is the canonical Haar measure on $I(\mathbb{A})$].

What does it mean for a semi-simple $\gamma \in G(F)$ to be elliptic? Over a general field F there are two reasonable definitions. The first is that γ is elliptic if and only if γ belongs to some elliptic maximal F-torus of G. The second is that γ is elliptic if and only if $Z(I)^0/Z(G)^0$ is anisotropic. If every connected reductive F-group possesses elliptic maximal F-tori (e.g., if F is a number field or a p-adic field), then the two definitions are equivalent, but in general they are not (e.g., $F=\mathbb{R}$). In this section we are working over a number field, and it makes no difference which definition we use.

9.2. For quasi-split G we define the stable analogue $ST_e(f)$ of $T_e(f)$ by the formula

$$ST_e(f)=\sum_{\gamma\in E_{st}} |(G_\gamma/I)(F)|^{-1}\tau(G)SO_\gamma(f),$$

where E_{st} is a set of representatives for the elliptic semi-simple stable conjugacy classes in $G(F)$, and $SO_\gamma(f)$ is the stable adelic orbital integral

$$SO_\gamma(f)=\sum_i e(\gamma_i)O_{\gamma_i}(f).$$

Here the sum runs over

$$i\in\ker[H^1(F,I(\mathbb{A}))\to H^1(F,G(\mathbb{A}))];$$

as usual i determines an element $\gamma_i\in G(\mathbb{A})$ [up to $G(\mathbb{A})$-conjugacy] whose local components are all stably conjugate to γ. The number $e(\gamma_i)$ is defined by

$$e(\gamma_i)=\prod_v e(I_{i,v}),$$

where $I_{i,v}$ is the connected centralizer in G_v of the component of γ_i at the place v of F (see 5.1). Since we always use canonical measures in defining our adelic orbital integrals O_{γ_i}, it is automatic that these measures satisfy the usual consistency requirement.

The finiteness results of Sects. 7 and 8 show that the sums defining $SO_\gamma(f)$ and $ST_e(f)$ have only a finite number of non-zero terms. Corollary 7.3 shows that the integral defining $O_{\gamma_i}(f)$ is convergent. It is easy to see that $SO_\gamma(f)$ and $|(G_\gamma/I)(F)|$ depend only on the stable conjugacy class of γ in $G(F)$. Thus the definition of $ST_e(f)$ makes sense.

9.3. From now on we simplify the discussion by assuming that G_{der} is simply connected. We choose an inner twisting $\psi: G_0\to G$ with G_0 quasi-split over F. We also choose a set $\mathfrak{E}$ of representatives for the isomorphism classes of elliptic endoscopic triples (H,s,η) for G [K3], and for each $(H,s,\eta)\in\mathfrak{E}$ we choose an L-homomorphism $\eta': {}^LH\to{}^LG$ extending η. Our goal is to stabilize $T_e(f)$, but to do this we must make some assumptions. First of all we assume the local and global conjectures in Sects. 5 and 6. We also assume that the "fundamental lemma" [L2, Chap. III] on spherical functions holds for G, H at places v where both groups are unramified (for our purposes it is enough to have the fundamental lemma for the unit element in the Hecke algebra of G). With these assumptions the local function correspondences of Sect. 5 yield functions $f^H \in C_c^\infty(H(\mathbb{A}))$ (as usual we write f^H even though f^H also depends on s, η'), and these functions have the property that

$$SO_{\gamma_H}(f^H) = \sum_{\gamma} \langle \mathrm{obs}(\gamma), \kappa \rangle e(\gamma) O_\gamma(f) \tag{9.3.1}$$

for any (G, H)-regular semi-simple $\gamma_H \in H(F)$. In the sum γ runs over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that come from γ_H. The number $\langle \mathrm{obs}(\gamma), \kappa \rangle$ is explained in 6.9, and the number $e(\gamma)$ is given by

$$e(\gamma) = \prod_v e(I_v),$$

where I_v is the (connected) centralizer in G_v of the v-component of γ.

There are still more assumptions. We will need the Hasse principle and therefore we assume that G has no E_8 factors. Finally, we assume that Weil's conjecture on Tamagawa numbers is true for all groups I whose dimension is less than that of G. Recall that the conjecture for the group I states that $\tau(I_{\mathrm{SC}}) = 1$. This conjecture is true for quasi-split groups [La], but it remains to be shown that $\tau(I)$ is invariant under inner twists. However, one hopes [J-L, L3] that this last statement can be proved during the stabilization of the full trace formula for I, and thus our last assumption is quite reasonable, since the stabilization of the full trace formula for G will undoubtedly be done by induction on $\dim G$.

9.4. To any elliptic endoscopic triple (H, s, η) of G there is associated a number $\iota(G, H)$ [L2]. We need to recall the following formula for $\iota(G, H)$, given in Sect. 8 of [K3]:

$$\iota(G, H) = \tau_1(G) \cdot \tau_1(H)^{-1} \cdot \lambda^{-1}.$$

Here $\tau_1(G)$ denotes the relative Tamagawa number $\tau(G)/\tau(G_{\mathrm{SC}})$ of G, and λ denotes the cardinality of the group

$$\mathrm{Aut}(H, s, \eta)/H_{\mathrm{ad}}(F)$$

(see [K3, 7.5] for the definition of $\mathrm{Aut}(H, s, \eta)$).

9.5. We define $T_e^*(f)$ by the sum used to define $T_e(f)$, omitting all terms indexed by central elements γ of $G(F)$. Similarly we define $ST_e^{**}(f^H)$ by omitting certain terms from the sum defining $ST_e(f^H)$. If H is a quasi-split inner form of G, we omit all terms indexed by central elements γ_H of $H(F)$. If H is not a quasi-split inner form of G, then we omit all terms indexed by elements $\gamma_H \in H(F)$ that are not (G, H)-regular.

9.6. Theorem. *Under the assumptions of 9.3 we have*

$$T_e^*(f) = \sum_{(H, s, \eta) \in \mathfrak{E}} \iota(G, H) ST_e^{**}(f^H).$$

Since we have assumed that G_{der} is simply connected, we have $G_\gamma = I$. By Lemma 3.2 we also have $H_{\gamma_H} = I_H$ for (G, H)-regular semi-simple $\gamma_H \in H(F)$, where I_H denotes the identity component of H_{γ_H}. Therefore the numbers $|G_\gamma(F)/I(F)|$ and $|(H_{\gamma_H}/I_H)(F)|$ appearing in the sums defining $T_e^*(f)$ and $ST_e^{**}(f^H)$ are equal to 1.

Choose a set E_0^* of representatives for the non-central elliptic semi-simple stable conjugacy classes in $G_0(F)$. Then

$$T_e^*(f) = \sum_{\gamma_0 \in E_0^*} \tau(I_0) \sum_{\gamma} O_\gamma(f) \tag{9.6.1}$$

where I_0 is the (connected) centralizer of γ_0 in G_0 and the second sum runs over a set of representatives γ for the $G(F)$-conjugacy classes in $G(F)$ contained in the $G(\bar{F})$-conjugacy class of $\psi(\gamma_0)$. Note that I_0 is an inner twist of I and that $\dim(I) < \dim(G)$, so that our assumption on Tamagawa numbers implies that $\tau(I) = \tau(I_0)$ (this follows from Weil's conjecture for I, since one knows that the relative Tamagawa numbers of I, I_0 are equal).

The orbital integral $O_\gamma(f)$ depends only on the $G(\mathbb{A})$-conjugacy class of γ in $G(\mathbb{A})$. Consider a fixed γ contributing to the sum (9.6.1). Let I be the connected centralizer of γ in G. Then the number of terms in the second sum in (9.6.1) that are indexed by $G(\mathbb{A})$-conjugates of γ is equal to

$$|\ker[\ker^1(F, I) \to \ker^1(F, G)]| \tag{9.6.2}$$

with $\ker^1(F, *)$ as in Sect. 4 of [K3]. We are assuming that G has no E_8 factors. The same is then true of I. From Sect. 4 of [K3] we get a diagram

$$\begin{array}{ccc} \ker^1(F, I) & \longrightarrow & \ker^1(F, G) \\ \downarrow & & \downarrow \\ \ker^1(F, Z(\hat{I}))^D & \longrightarrow & \ker^1(F, Z(\hat{G}))^D \end{array}$$

in which the vertical maps are bijections and the bottom horizontal map is induced by the natural injection $Z(\hat{G}) \to Z(\hat{I})$ of 4.2.

This diagram commutes. We are assuming that G_{der} is simply connected; to prove the commutativity in the general case we would use a z-extension of G. Let $D = G/G_{\mathrm{der}}$. Then $\ker^1(F, G) \tilde{\to} \ker^1(F, D)$ and $\ker^1(F, Z(\hat{G}))^D \tilde{\to} \ker^1(F, \hat{D})^D$ by Lemma 4.3.1 of [K3]. Therefore the desired commutativity follows from the functoriality of

$$\ker^1(F, G) \to \ker^1(F, Z(\hat{G}))^D$$

for the normal homomorphisms $G \to D$ and $I \to D$.

Using that $Z(\hat{I}_0) = Z(\hat{I})$, we now see that the number of terms in the second sum in (9.6.1) that are indexed by $G(\mathbb{A})$-conjugates of γ is equal to

$$|\mathrm{cok}[\ker^1(F, Z(\hat{G})) \to \ker^1(F, Z(\hat{I}_0))]| \,. \tag{9.6.3}$$

Now consider an element $\gamma_0 \in E_0^*$ and an element $\gamma \in G(\mathbb{A})$ in the $G(\bar{\mathbb{A}})$-conjugacy class of $\psi(\gamma_0)$. The construction of 6.5 gives us

$$\mathrm{obs}(\gamma) \in \mathfrak{K}(I_0/F)^D \,.$$

Theorem 6.6 implies that

$$|\mathfrak{R}(I_0/F)|^{-1}\sum_{\kappa}\langle \mathrm{obs}(\gamma),\kappa\rangle, \tag{9.6.4}$$

where κ runs over $\mathfrak{R}(I_0/F)$, is equal to 1 if the $G(\mathbb{A})$-conjugacy class of γ contains an element of $G(F)$ and is equal to 0 otherwise.

We define $e(\gamma)$ by the formula

$$e(\gamma)=\prod_{v} e(I_v),$$

where I_v is the (connected) centralizer in G_v of the v-component of γ. If γ is $G(\mathbb{A})$-conjugate to an element of $G(F)$, then the last result in [K2] says that $e(\gamma)=1$.

Before continuing with the manipulation of the elliptic terms, we need to observe that the quotient of the number (9.6.3) by $|\mathfrak{R}(I_0/F)|$ is equal to $\tau_1(G)\cdot\tau_1(I_0)^{-1}$. This can be proved the same way as Lemma 8.3.2 of [K3], using the exact sequence

$$1\to\pi_0(Z(\hat{G})^{\Gamma})\to\pi_0(Z(\hat{I}_0)^{\Gamma})\to\mathfrak{R}(I_0/F)\to\ker^1(F,Z(\hat{G}))\to\ker^1(F,Z(\hat{I}_0)).$$

The 1 at the beginning of this sequence is there because γ_0 is elliptic, which means that $X_*(Z(\hat{I}_0)/Z(\hat{G}))^{\Gamma}$ is trivial.

These remarks show that $T_e^*(f)$ is given by

$$\sum_{\gamma_0\in E_0^*}\tau_1(G)\sum_{\gamma}\sum_{\kappa}\langle \mathrm{obs}(\gamma),\kappa\rangle e(\gamma)O_\gamma(f), \tag{9.6.5}$$

where γ now runs over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ contained in the $G(\bar{\mathbb{A}})$-conjugacy class of $\psi(\gamma_0)$, and κ runs over $\mathfrak{R}(I_0/F)$. The finiteness results of Sects. 7–8 show that this triple sum has only finitely many non-zero terms and hence that it can be rearranged in any way we like.

Now we look at the right side of the equality stated in Theorem 9.6. Since the groups H are quasi-split, we have $\tau(H)=\tau_1(H)$, and therefore

$$\sum_{\mathfrak{E}}\imath(G,H)ST_e^{**}(f^H)$$

is equal to

$$\sum_{\mathfrak{E}}\tau_1(G)\cdot\lambda^{-1}\sum_{\gamma_H\in E_H^{**}}SO_{\gamma_H}(f^H),$$

where E_H^{**} is as follows: if H is a quasi-split inner form of G, then E_H^{**} is a set of representatives for the non-central elliptic semi-simple stable conjugacy classes in $H(F)$, and if H is not a quasi-split inner form of G, then E_H^{**} is a set of representatives for the (G,H)-regular elliptic semi-simple stable conjugacy classes in $H(F)$.

From (9.3.1) we now see that the right side of the equality stated in the theorem is equal to

$$\sum_{\mathfrak{E}}\tau_1(G)\cdot\lambda^{-1}\sum_{\gamma_H\in E_H^{**}}\sum_{\gamma}\langle \mathrm{obs}(\gamma),\kappa\rangle e(\gamma)O_\gamma(f), \tag{9.6.6}$$

where γ runs over a set of representatives for the $G(\mathbb{A})$-conjugacy classes in $G(\mathbb{A})$ that come from γ_H.

We need to prove that (9.6.5) and (9.6.6) are equal. Given $(H, s, \eta) \in \mathfrak{E}$ and $\gamma_H \in E_H^{**}$, we get $\gamma_0 \in G_0(F)$ (up to stable conjugacy) and $\kappa \in \mathfrak{R}(I_0/F)$, where $I_0 = (G_0)_{\gamma_0}$ (see 6.10 for the construction of γ_0, κ). To indicate that (γ_0, κ) are obtained from (H, s, η, γ_H) in this way we write $(H, s, \eta, \gamma_H) \to (\gamma_0, \kappa)$. The following lemma finishes the proof of the theorem.

9.7. Lemma. *Let γ_0 be any elliptic semi-simple element of $G_0(F)$ and let $\kappa \in \mathfrak{R}(I_0/F)$, where $I_0 = (G_0)_{\gamma_0}$. Then there exist $(H, s, \eta) \in \mathfrak{E}$ and a (G, H)-regular semi-simple element γ_H of $H(F)$ such that $(H, s, \eta, \gamma_H) \to (\gamma_0, \kappa)$. Moreover, $(H_1, s_1, \eta_1, \gamma_{H_1}) \to (\gamma_0, \kappa)$ also holds if and only if there exists an isomorphism*

$$(H, s, \eta) \to (H_1, s_1, \eta_1)$$

carrying γ_H into a stable conjugate of γ_{H_1}, and such an isomorphism is unique up to composition with an element of $H_{\mathrm{ad}}(F)$.

Since we will use only the quasi-split form of G in the proof, we may as well simplify notation by dropping the subscripts from G_0, γ_0, I_0.

First we prove the existence of (H, s, η) and γ_H such that $(H, s, \eta, \gamma_H) \to (\gamma, \kappa)$. Choose an elliptic maximal F-torus T of G containing γ. Then T is also an elliptic maximal F-torus in I, so that we have a canonical embedding $Z(\hat{I}) \to \hat{T}$. Choose an embedding $\hat{T} \to \hat{G}$ in the canonical $\hat{G}$-conjugacy class. Using

$$Z(\hat{I}) \to \hat{T} \to \hat{G},$$

we get from κ an element $s \in \hat{G}$, well-defined up to $Z(\hat{G})$. Let $\hat{H} = (\hat{G})_s^0$. Since $s \in [\hat{T}/Z(\hat{G})]^\Gamma$, the Galois action on $\hat{T}$ preserves the root system of $\hat{H}$ and gives us a homomorphism $\varrho : \Gamma \to \mathrm{Out}(\hat{H})$. Let $\sigma \in \Gamma$. The $\hat{G}$-conjugacy class of $\hat{T} \to \hat{G}$ is fixed by σ, which means that there exists $g \in {}^L G$, projecting onto $\sigma \in \Gamma$, such that $\mathrm{Int}(g)$ preserves $\hat{T}$ and acts by σ_T on that group (we write σ_T to distinguish the action of σ on $\hat{T}$ from the action of σ on $\hat{G}$). Using this one checks easily that (s, ϱ) is an elliptic endoscopic datum for G [K3]. Consider the corresponding elliptic endoscopic triple $(H, s, \eta) \in \mathfrak{E}$.

There exists a maximal F-torus T_H of H such that T_H transfers to T. We choose an F-isomorphism $j : T_H \xrightarrow{\sim} T$ such that $T_H \xrightarrow{j} T \hookrightarrow G$ belongs to the canonical $G(\bar{F})$-conjugacy class. Let $\gamma_H = j^{-1}(\gamma)$ and let I_H be the connected centralizer of γ_H in H. We claim that γ_H is (G, H)-regular. Use j to identify T_H and T. What we need to check is that the set $R(\gamma)$ of roots of T in I is equal to the set $R_H(\gamma)$ of roots of T in I_H. Let R be the set of roots of T in G. Then we have

$$R(\gamma) = \{\alpha \in R \mid \alpha(\gamma) = 1\},$$
$$R_H(\gamma) = \{\alpha \in R \mid \alpha(\gamma) = 1 \text{ and } \alpha^\vee(s) = 1\}.$$

But if $\alpha(\gamma) = 1$, then $\alpha^\vee$ is a root of $\hat{T}$ in $\hat{I}$, and therefore $\alpha^\vee(s) = 1$ since $s \in Z(\hat{I})$. This shows that γ_H is indeed (G, H)-regular. It is obvious that $(H, s, \eta, \gamma_H) \to (\gamma, \kappa)$.

Next we suppose that we also have $(H_1, s_1, \eta_1, \gamma_{H_1}) \to (\gamma, \kappa)$, and show that there exists an isomorphism $(H, s, \eta) \to (H_1, s_1, \eta_1)$ carrying γ_H into a stable conjugate of γ_{H_1} (the converse is obvious).

Choose a maximal F-torus T_{H_1} of H_1 containing γ_{H_1} and choose an embedding $j_1 : T_{H_1} \to G$ in the canonical $G(\bar{F})$-conjugacy class, such that $j_1(\gamma_{H_1}) = \gamma$. After conjugating j_1 by an element of $I(\bar{F})$, we may suppose that $j_1(T_{H_1})=T$ and use j_1 to identify T_{H_1} with T over $\bar{F}$. The roots and coroots of T in H_1 are determined by κ, and are therefore the same as for H. Therefore there exists an isomorphism $\alpha_0: H \to H_1$ over $\bar{F}$ which extends $j_1^{-1} \circ j: T_H \to T_{H_1}$. Let $\sigma \in \Gamma$. Then $\sigma(j_1)=\mathrm{Int}(g) \circ j_1$ for some $g \in G(\bar{F})$, and since $\sigma(j_1)(\gamma_{H_1})=\gamma$ we have $g \in I(\bar{F})$. Now using that γ_{H_1} is (G, H_1)-regular, we see that there exists $h \in I_{H_1}(\bar{F})$ such that $\sigma(j_1)=j_1 \circ \mathrm{Int}(h)$. Therefore $\sigma(j_1^{-1} \circ j)$ is conjugate to $j_1^{-1} \circ j$ under $I_{H_1}(\bar{F})$, and we conclude that the $H_1(\bar{F})$-conjugacy class of $\alpha_0: H \to H_1$ is defined over F. Since H, H_1 are quasi-split there exists an $H_1(\bar{F})$-conjugate α of α_0 such that α is defined over F. It is not hard to see that $\alpha: H \to H_1$ defines an isomorphism $(H, s, \eta) \to (H_1, s_1, \eta_1)$. We have assumed that G_{der} is simply connected, and thus 3.2 guarantees that stable conjugacy and $H_1(\bar{F})$-conjugacy coincide for γ_{H_1}. It is now clear that $\alpha(\gamma_H)$ is stably conjugate to γ_{H_1}.

Finally we need to prove the uniqueness assertion about α. In other words, we need to show that if $\alpha \in \mathrm{Aut}(H, s, \eta)$ carries γ_H into a stable conjugate of itself, then $\alpha \in H_{\mathrm{ad}}(F) \subset \mathrm{Aut}(H, s, \eta)$. Choose $h \in H(\bar{F})$ such that $\alpha_0 := \mathrm{Int}(h) \circ \alpha$ carries γ_H into itself. Then α_0 preserves I_H and hence carries T_H into an $I_H(\bar{F})$-conjugate of itself. Thus we may as well suppose that $\alpha_0(T_H)=T_H$. As before we use $j: T_H \to T$ to identify T_H with T, and we also use j to identify the Weyl group $\Omega(T_H, H)$ with a subgroup of $\Omega(T, G)$. The restriction of α_0 to T_H is given by some $\omega \in \Omega(T, G)$. Since G_γ is connected and ω fixes γ, we have

$$\omega \in \Omega(T, I) = \Omega(T_H, I_H) \subset \Omega(T_H, H).$$

Choose a representative h_1 for ω^{-1} in $\mathrm{Norm}_H(T_H)$. Then $\mathrm{Int}(h_1) \circ \alpha_0$ fixes T_H pointwise and must therefore be inner. This shows that α is inner (over $\bar{F}$).

10. Elliptic and Fundamental Tori (Review)

In this section F is either p-adic or real, and G is a connected reductive group over F. In the real case a maximal $\mathbb{R}$-torus T of G is said to be *fundamental* if the dimension of its split component is as small as possible. In order to have uniform statements we adopt the same terminology in the p-adic case; in this case fundamental tori are elliptic [Kn; II, p. 271].

10.1. Lemma. *Let T be a maximal F-torus of G. If T transfers to every inner form of G, then $H^1(F, T) \to H^1(F, G)$ is surjective. If G is an adjoint group, then the converse is true.*

This lemma is true for any field F. For adjoint groups G, the lemma is almost a tautology. The first statement for general G follows from the adjoint case and the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H^1(Z) & \longrightarrow & H^1(G) & \longrightarrow & H^1(G_{\mathrm{ad}}) & \longrightarrow & H^2(Z) \\
 & \| & & \uparrow & & \uparrow & & \| \\
\cdots \longrightarrow & H^1(Z) & \longrightarrow & H^1(T) & \longrightarrow & H^1(T_{\mathrm{ad}}) & \longrightarrow & H^2(Z).
\end{array}$$

10.2. Lemma. *Let T be a fundamental torus of G. Then $H^1(F, T) \to H^1(F, G)$ is surjective.*

It follows from Lemma 2.8 of [S1] that in the real case T transfers to the fundamental torus of any inner form G, and we are done by the previous lemma. In the p-adic case we use the previous lemma to reduce to the adjoint case. Let C be the kernel of $G_{SC} \to G$; then Kneser [Kn] has shown that $H^1(F, G) \to H^2(F, C)$ is an isomorphism, and thus it suffices to prove that $H^1(F, T) \to H^2(F, C)$ is surjective. This follows from the triviality of $H^2(F, T_{SC})$ (use Tate-Nakayama duality; T_{SC} is anisotropic).

10.3. The following description of the fundamental torus in a quasi-split real group G will be used in 10.4. Let B_0 be a Borel subgroup of G defined over $\mathbb{R}$ and let T_0 be a maximal $\mathbb{R}$-torus of B_0. Let ω_0 be the longest element of the Weyl group Ω of T_0. We have $\omega_0 \in \Omega(\mathbb{R})$, since B_0 is defined over $\mathbb{R}$, and we also have $\omega_0^2 = 1$. Therefore ω_0 can be used to twist T_0. The twisted torus T appears in G since G is quasi-split. It is clear that T has no real roots; therefore T is fundamental.

10.4. Lemma. *Assume that G is a simply connected semi-simple group. Let T be a fundamental torus of G. Then $H^2(F, T)$ is trivial.*

In the p-adic case T is anisotropic and the result follows from Tate-Nakayama duality. Now consider the real case. We claim that T is a product $T_a \times T_i$, where T_a is anisotropic and T_i is of the form $R_{\mathbb{C}/\mathbb{R}}(S)$ for a $\mathbb{C}$-torus S.

In proving the claim we may as well transfer T to the quasi-split form and thus reduce to the quasi-split case. It is enough to find a basis for $X_*(T)$ that is permuted by $-\sigma$, where σ denotes complex conjugation. By 10.3 this is the same as finding a basis for $X_*(T_0)$ that is permuted by $(-\omega_0) \circ \sigma$. The set of simple coroots of T_0 does the job.

It is clear that $H^2(\mathbb{R}, T_a)$ and $H^2(\mathbb{R}, T_i)$ are both trivial. This finishes the proof.

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Received April 22, 1985