DAY I, TALK 1. ROAD MAP

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1. What is the geometric Langlands conjecture supposed to say

The goal of this talk is to outline what will be going in this workshop, how the different talks fit together.

More importantly, as some of the talks will create an impression of focusing on "technicalities", we'll now try to explain why these technicalities are essential.

These include DG categories (vs. triangulated categories), derived algebraic geometry (vs. usual algebraic geometry), IndCoh (vs. QCoh).

We'll also try to motivate the discussion of *factorization categories*.

1.1. **The geometric/automorphic side.** Throughout the workshop we will be working over an algebraically closed field k of characteristic 0.

The input into our story is the datum of a (smooth, complete) algebraic curve X and a reductive group G.

1.1.1. The object of study on the geometric/automorphic side is the algebraic stack Bun_G, classifying principal G-bundles on X.

1.1.2. As will be reviewed in Talk I.2, we will be working with various "spaces" in algebraic geometry, where by a "space" we mean an arbitrary prestack, i.e., a contravariant functor from the category $\mathrm{Sch}^{\mathrm{aff}}$ of affine schemes to that of groupoids.

For now, we can keep assuming that $\mathrm{Sch}^{\mathrm{aff}}$ stands for classical (i.e., non-derived) affine schemes, and by a groupoid we mean an ordinary (i.e., not an ∞-) groupoid.

For a prestack $\mathcal{Y}$ and $S \in \mathrm{Sch}^{\mathrm{aff}}$ we shall write

$$\mathrm{Maps}(S, \mathcal{Y})$$

for the corresponding groupoid, i.e., the value of $\mathcal{Y}$ on S.

The Yoneda embedding maps affine schemes into the category PreStk of prestacks, and we can think of $\mathrm{Maps}(S, \mathcal{Y})$ as the groupoid of maps from S to $\mathcal{Y}$, as perstacks.

Note that even in the classical world, PreStk is a 2-category, but one in which all 2-morphisms are invertible. So, Hom between objects isn't a set, but rather a groupoid.

1.1.3. In this language, Bun_G is the prestack that associates to an affine scheme S the groupoid, whose objects are principal G-bundles $\mathcal{P}_G$ on $S \times X$, and where the morphisms are isomorphisms $\mathcal{P}^1_G \to \mathcal{P}^2_G$ between G-bundles.

Date: March 7, 2014.

1.1.4. Geometric Langlands is about an equivalence of certain two categories. The category on the geometric/automorphic side is

$$\text{D-mod}(\text{Bun}_G),$$

i.e., the *DG category of D-modules on* Bun_G.

1.1.5. The first questions to ask is

- Why a DG category and not a triangulated category?
- How is this DG category defined?

It trurns out that the above two questions are tightly linked. To any DG category $\mathbf{C}$ one can attach its homotopy category $\text{Ho}(\mathbf{C})$, which is a triangulated category.

Let us concede that at the end of the day, we are interested in the triangulated category $\text{Ho}(\text{D-mod}(\text{Bun}_G))$. However, the problem is that it's impossible (or at least, very difficult and cumbersome to the point of uselessness) to define $\text{Ho}(\text{Bun}_G)$ without defining $\text{D-mod}(\text{Bun}_G)$ first.

1.1.6. Namely, say we know how to attach to any scheme S the corresponding DG category $\text{D-mod}(S)$, and hence the triangulated category $\text{Ho}(\text{D-mod}(S))$. Given a prestack $\mathcal{Y}$, we want to define $\text{D-mod}(\mathcal{Y})$ (or $\text{Ho}(\text{D-mod}(\mathcal{Y}))$) by *gluing* the categories $\text{D-mod}(S)$ over all schemes (or affine schemes) mapping to $\mathcal{Y}$.

The problem is that the *gluing procedure* alluded to above, is not defined for triangulated categories. (This was the problem that Hartshorne had to confront in his "Residues and duality"; this is why that book is so thick, instead of being just 10 pages long.)

Now, the advatnage of the ∞-category language is that *gluing* can be defined for DG-categories. Namely, we view the assigmment

$$S \mapsto \text{D-mod}(S)$$

as a functor

$$((\text{Sch}^{\text{aff}})_{/\mathcal{Y}})^{\text{op}} \to \text{DGCat},$$

and $\text{D-mod}(\mathcal{Y})$ is defined as the *limit* of this functor.

The point is that for any index category $\mathcal{I}$ and a functor $\Phi : \mathcal{I} \to \mathbf{D}$, where $\mathbf{D}$ is an ∞-category (such as DGCat), there is a well-defined procedure of taking the limit, denoted

$$\lim_{i \in \mathcal{I}} \Phi(i) \in \mathbf{D},$$

under some mild technical conditions on $\mathbf{D}$ (satisfied in our case).

1.1.7. However, in the above procedure there is a hidden difficulty. Namely, we need to define the assigment $S \mapsto \text{D-mod}(S)$ as a functor

$$\text{D-mod}^! : (\text{Sch}^{\text{aff}})^{\text{op}} \to \text{DGCat}\,.$$

This is what Talk 1.2 will try to explain.

The good thing is that we only have to do this once. I.e., having $\text{D-mod}^!$ as a functor will solve the problem of defining $\text{D-mod}(\mathcal{Y})$ for *all prestacks*, once and for all.

Remark 1.1.8. The superscript "!" stands for the fact that for a map $f : S_1 \to S_2$, the corresponding map

$$\text{D-mod}(S_2) \to \text{D-mod}(S_1)$$

in DGCat, i.e., the functor between the corresponding categories, is $f^!$, i.e., the usual D-module pullback.

1.2. **The spectral/Galois side.**

1.2.1. Recall that to a reductive group G there corresponds its Langlands dual $\check{G}$. The object of study on the spectral/Galois side is the stack $\mathrm{LocSys}_{\check{G}}$ of $\check{G}$-local systems on X.

We will give the definition of $\mathrm{LocSys}_{\check{G}}$ shortly. But let us say right away that the category that we will consider on the spectral/Galois side is $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, or rather its small modification.

The fact that we are considering $\mathrm{QCoh}(-)$ rather than $\mathrm{D}\text{-mod}(-)$ creates a whole new set of difficulties. It is now that derived algebraic geometry makes its appearence.

1.2.2. Let A be a commutative k-algebra. By definition, for $S = \mathrm{Spec}(A)$,

$$\mathrm{QCoh}(S) = A\text{-mod},$$

where in both cases we consider the corresponding DG categories, rather than the abelian ones.

Now, let A' be a commutative DG algebra, concentrated in non-positive cohomological degrees. By definition $S' := \mathrm{Spec}(A')$ is a derived affine scheme, and

$$\mathrm{QCoh}(S') := A'\text{-mod}.$$

Suppose that $A = H^0(A')$. Then the DG categories (and the underlying triangulated categories)

$$A\text{-mod and } A'\text{-mod}$$

are completely different (even though the corresponding abelian categories are equivalent).

1.2.3. The passage $A' \rightsquigarrow A$ and the corresponding assigment $S' \rightsquigarrow S$ is the procedure of associating to a derived scheme the underlying classical scheme. As we have just seen, this procedure changes the category of quasi-coherent sheaves.

Now, it is an empirical fact that most of the important categories that arise in nature in the form $\mathrm{QCoh}(\mathcal{Y})$, where $\mathcal{Y}$ is a prestack, need to be understood in the sense of regarding $\mathcal{Y}$ as a derived prestack (we shall explain what a derived prestack is shortly).

That is to say that if our moduli problem arises naturally on derived affine schemes, we need to keep $\mathcal{Y}$ in this world, rather than passing to the underlying classical prestack, because the latter will produce the wrong QCoh category.

This will be the case of $\mathrm{LocSys}_{\check{G}}$. We shall define it as a derived prestack, and $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ needs to be understood in this sense. If one replaces $\mathrm{LocSys}_{\check{G}}$ by the underlying classical prestack, we will get a different category for QCoh, and the latter will fail the geometric Langlands equivalence, for example, for G being a torus.

That said, the category $\mathrm{D}\text{-mod}(-)$, unlike $\mathrm{QCoh}(-)$ only depends on the underlying classical prestack. This is why in our discussion of the geometric side we could work with classical schemes and prestacks.

1.2.4. We shall now explain what a derived prestack is. From now on $\mathrm{Sch}^{\mathrm{aff}}$ will denote the category of *derived* affine schemes, which, by definition, is the opposite category of commutative DG algebras concentrated in non-positive cohomological degrees.

A derived prestack is an arbitrary contravariant functor from $\mathrm{Sch}^{\mathrm{aff}}$ to the category of groupoids. From now on we shall write PreStk for the category of derived prestacks. [1]

However, we must now allow not just ordinary groupoids, but ∞-groupoids. The reason is that the most basic prestacks take values in ∞-groupoids. Namely, the prestacks arising from affine schemes via the Yoneda embedding already take values in ∞-groupoids. This is because $\mathrm{Sch}^{\mathrm{aff}}$ is an ∞-category rather than an ordinary one.

1.2.5. We can summarize the above discussion as follows: the nature of the problem dictates to us that we need to consider derived prestacks. The latter necessitates enlarging our world of prestacks from functors with values in ordinary groupoids to one with values in ∞-groupoids.

This plunges us completely into the world of higher categories. However, we were in this world already when we wrote the formula

$$\text{D-mod}(\mathcal{Y}) = \lim_{S\in((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}} \text{D-mod}(S).$$

Only then our index category (i.e., $({}^{cl}\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}}$) was an ordinary one, while the target category (i.e., DGCat) was an ∞-category. In the derived setting, both the sourceand the target will be ∞-categories.

1.2.6. Given a (derived!) prestack $\mathcal{Y}$ we define

$$\mathrm{QCoh}(\mathcal{Y}) = \lim_{S\in((\mathrm{Sch}^{\mathrm{aff}})_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S).$$

Here our input is the functor

$$\mathrm{QCoh}^* : (\mathrm{Sch}^{\mathrm{aff}})^{\mathrm{op}} \to \mathrm{DGCat},$$

that associates to $S = \mathrm{Spec}(A) \in \mathrm{Sch}^{\mathrm{aff}}$ the category $\mathrm{QCoh}(S) = A\text{-mod}$, and for a map $f : S_1 \to S_2$ the functor

$$f^* : \mathrm{QCoh}(S_2) \to \mathrm{QCoh}(S_1).$$

1.2.7. Finally, let us define LocSys_H as a derived prestack, where H is an algebraic group (we will need for $H = \check{G}$). For this, we need to give three more definitions.

First, we define $\mathrm{pt}/H \in \mathrm{PreStk}$ to be the prestack that associates to $S \in \mathrm{Sch}^{\mathrm{aff}}$ the category of principal H-bundles on S.

1.2.8. Given two prestacks $\mathcal{Y}_1$ and $\mathcal{Y}_2$ we let

$$\mathbf{Maps}(\mathcal{Y}_1, \mathcal{Y}_2)$$

be the prestack defined by

$$\mathrm{Maps}(S, \mathbf{Maps}(\mathcal{Y}_1, \mathcal{Y}_2)) = \mathrm{Maps}(S \times \mathcal{Y}_1, \mathcal{Y}_2).$$

E.g., the prestack Bun_G introduced earlier was by definition $\mathbf{Maps}(X, \mathrm{pt}/H)$.

[1]From now on we will use the word scheme/prestack to mean a derived scheme/prestack; we will use the adjective "classical" to denote the corresponding classical objects.

1.2.9. Given a prestack $\mathcal{Y}$ we introduce a new prestack $\mathcal{Y}_{\mathrm{dR}}$ (called the de Rham prestack of $\mathcal{Y}$) by

$$\mathrm{Maps}(S, \mathcal{Y}_{\mathrm{dR}}) := \mathrm{Maps}(S_{\mathrm{red}}, \mathcal{Y}),$$

wheee S is the classical reduced affine scheme corresponding to S.

1.2.10. Finally, for a prestack $\mathcal{Y}$, we define:

$$\mathrm{LocSys}_H(\mathcal{Y}) := \mathbf{Maps}(\mathcal{Y}_{\mathrm{dR}}, \mathrm{pt}/H).$$

We will use a short-hand notation LocSys_H for $\mathcal{Y} = X$ (our fixed curve).

1.2.11. The above gives a definition of LocSys_H. However, there is one crucial thing that we lose when we define things this way:

Namely, we lose the way to approach local systems via connection forms. This loss is not complete: one can recover this (see [AG, Sect. 9]), but this is not tautological.

The difficulty is that in the world of derived algebraic geometry, the formula

$$\nabla = \nabla_0 + \alpha$$

does not a priori make sense.

1.2.12. In fact, in derived algebraic geometry, as in higher category theory, one cannot define things by explicit formulas.

I.e., in practical situations, we almost never can specify an ∞-category by spelling out who are the objects, what are the morphisms, and how they compose. This is because, the datum of an ∞-category contains an infinite data of compatibilities for higher compositions.

Instead, most of the ∞-categories and functors between them (as well as objects within a given ∞-category, or morphisms between two given objects) that arise in practice are obtained by applying some universal procedures to a few of the basic pre-existing ones.

1.3. **Rough statement of the geometric Langlands conjecture.**

1.3.1. A very rough statement of the conjecture would say that the categories $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ and $\text{D-mod}(\mathrm{Bun}_G)$ are equivalent.

However, this is simply not correct (unless G is a torus). We need to introduce a correction that slightly modifies the spectral side.

1.3.2. The correction has to do with rather fine aspects of homological algebra (which is probably the reason why it was overlooked for a long time).

First, we need to specify what kind of DG categories we are working with. Namely, unless specified otherwise, all our DG categories are assumed *cocomplete*, i.e., contain infinite direct sums.

A functor between two such DG categories is said to be *continuous* if it commutes with infinite direct sums. Unless specified otherwise, all our functors (between any two given cocomplete DG categories) will be assumed continuous.

In a cocomplete DG category $\mathbf{C}$ it makes sense to talk about *compact* objects. Namely, an object $\mathbf{c} \in \mathbf{C}$ is said to be compact if the functor

$$\mathcal{H}om_{\mathbf{C}}(\mathbf{c}, -)$$

commutes with direct sums. We let $\mathbf{C}^c \subset \mathbf{C}$ the full (but not cocomplete) DG subcategory that consists of compact objects of $\mathbf{C}$.

A DG category $\mathbf{C}$ is said to be compactly generated if it does not contain a proper DG subcategory, closed under infinite direct sums, that contains all of its compact objects.

Now, if $\mathbf{C}^0$ is a *non-cocomplete* DG category, one can universally produce a cocomplete compactly generated one, denoted $\mathrm{Ind}(\mathbf{C}^0)$, that received a fully faithful functor

$$\mathbf{C}^0 \to (\mathrm{Ind}(\mathbf{C}^0))^c,$$

and such that every compact object of $\mathrm{Ind}(\mathbf{C}^0)$ is a direct summand of one in the image $\mathbf{C}^0$.

The category $\mathrm{Ind}(\mathbf{C}^0)$ is constructed by taking formal infinite direct sums of objects of $\mathbf{C}^0$, then addiging all the cones, etc; it is called the *ind-completion* of $\mathbf{C}^0$.

The univeresal property of $\mathrm{Ind}(\mathbf{C}^0)$ is restriction along $\mathbf{C}^0 \to \mathrm{Ind}(\mathbf{C}^0)$ defines a bijection (or, rather, an equivalence) between continuous functors $\mathrm{Ind}(\mathbf{C}^0) \to \mathbf{C}$, where $\mathbf{C}$ is a cocomplete DG category, and just functors $\mathbf{C}^0 \to \mathbf{C}$.

Finally, for a compactly generated category $\mathbf{C}$, one shows that the functor $\mathrm{Ind}(\mathbf{C}^c) \to \mathbf{C}$, obtained from the univeresal property, is an equivalence.

1.3.3. Let $S = \mathrm{Spec}(A)$ be an affine DG scheme. The category

$$\mathrm{QCoh}(S) = A\text{-mod}$$

is cocomplete, and its subcategory $\mathrm{QCoh}(S)^c$ equals that of *perfect complexes*. I.e., it is those objects that can be realized as finite complexes consisting of projective, finitely generated A-modules.

Suppose now that S is *almost of finite type*. This means that $H^0(A)$ is a finitely generated algebra over k and each $H^i(A)$ is finitely generated as an $H^0(A)$-module.

Then inside $\mathrm{QCoh}(S)$ one can single out another full (but not cocomplete) subcategory, denoted

$$\mathrm{Coh}(S).$$

Its objects are those $\mathcal{F} \in \mathrm{QCoh}(S) = A$-mod that have only finitely many non-zero cohomology groups, and such that each $H^i(\mathcal{F})$ is finitely generated as a module over $H^0(A)$.

We denote

$$\mathrm{IndCoh}(S) := \mathrm{Ind}(\mathrm{Coh}(S)).$$

The universal property of the ind-completion produces a functor

$$\Psi_S : \mathrm{IndCoh}(S) \to \mathrm{QCoh}(S),$$

obtained by ind-extending the tautological embedding $\mathrm{Coh}(S) \hookrightarrow \mathrm{QCoh}(S)$.

1.3.4. If S is such that A itself has only finitely many cohomologies (in which case we say that S is *eventually coconnective*), we have

$$\mathrm{QCoh}(S)^c \subset \mathrm{Coh}(S).$$

It follows that in this case, the above functor Ψ_S admits a (fully faithful) left adjoint, denoted Ξ_S, obtained by ind-extending the functor

$$\mathrm{QCoh}(S)^c \hookrightarrow \mathrm{Coh}(S) \hookrightarrow \mathrm{IndCoh}(S).$$

Now, it is easy to show that the functors Ψ_S and Ξ_S are mutually inverse equivalences if and only if S is a *smooth classical scheme* (i.e., if $H^i(A) = 0$ for $i \neq 0$ and $H^0(A)$ is a smooth k-algebra in the classical sense).

Thus, if S is not smooth, the categories $\mathrm{QCoh}(S)$ and $\mathrm{IndCoh}(S)$ are different. The difference between $\mathrm{Coh}(S)$ and $\mathrm{QCoh}(S)^c$ may appear negligible: for example, for any object $\mathcal{F} \in \mathrm{Coh}$ and any $n \in \mathbb{Z}$, there exists an approximation

$$\mathcal{F}_n \to \mathcal{F}, \quad \mathcal{F}_n \in \mathrm{QCoh}(S)^c, \quad \mathrm{Cone}(\mathcal{F}_n \to \mathcal{F}) \in \mathrm{Coh}(S)^{\leq -n}.$$

And yet this difference is significant. In a parallel context of representations of pro p-groups, it gives rise to the phenomenon of Tate cohomology.

And it is this difference that is responsible for the correction to $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ that appears in geometric Langlands.

1.3.5. Namely, in Talks I.3 and II.2 it will be explained that the category IndCoh makes sense for $\mathrm{LocSys}_{\check{G}}$. Moreover, a certain particular subcategory, denoted

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \subset \mathrm{IndCoh}(\mathrm{LocSys}_{\check{G}})$$

will be introduced, and it is this category that is conjecturally equivalent to $\text{D-mod}(\mathrm{Bun}_G)$.

In what follows, we will denote by

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \text{D-mod}(\mathrm{Bun}_G)$$

the conjectural geometric Langlands functor.

2. Characterization of the geometric Langlands equivalence

The bare statement that the categories $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$ and $\text{D-mod}(\mathrm{Bun}_G)$ is neither very interesting, nor does is suggest any ways to approach the proof.

The goal of this section is to specify (some of) the properties that the functor $\mathbb{L}_G$ is supposed to satisfy. These properties will eventually pave way to the construction of this functor.

2.1. **The classical Whittaker model.** The most basic requirement on $\mathbb{L}_G$ is the property that it is supposed to satisfy vis-à-vis the *Whittaker model*, explained in the next section.

This property goes back to the Gelfand-Piatetskii-Shapiro construction of automorphic functions from local data. We shall first consider the function-theoretic set-up, which motivates our geometric constructions.

2.1.1. Let us recall that the groupoid of k-points of Bun_G identifies with

$$G(\mathbb{O})\backslash G(\mathbb{A})/G(K),$$

where $K = k(X)$ is the field of rational functions on X, and $\mathbb{A}$ is the ring of adèles and $\mathbb{O} \subset \mathbb{A}$ is the subring of integral adèles.

Let us for a moment assume that instead of our field k (assumed algebraically closed and of characteristic 0), we are working over a finite field $\mathbb{F}_q$. Then $G(\mathbb{A})$ is a locally compact group, $G(\mathbb{O}) \subset G(\mathbb{A})$ is its maximal compact subgroup and $G(K) \subset G(\mathbb{A})$ is a discrete subgroup.

We will be considering complex-valued functions on the double quotient $G(\mathbb{O})\backslash G(\mathbb{A})/G(K)$. This is the space of automorphic functions, of interest in number theory (or, rather, to the analog of number theory, where insted of the global field being a finite extension of $\mathbb{Q}$, we are dealing with K).

2.1.2. Let N be the unipotent radical of the Borel of G. Recall that $N/[N,N]$ canonically identifies with $\mathbb{G}_a^{\oplus r}$, where r is the semi-simple rank of G. Fix a non-zero character

$$\mathbb{G}_a(\mathbb{A})/\mathbb{G}_a(K) = \mathbb{A}/K \to \mathbb{C}^*.$$

And consider the corresponding character, denoted χ on $N(\mathbb{A})$:

$$N(\mathbb{A}) \to N/[N,N](\mathbb{A}) \simeq \mathbb{G}_a(\mathbb{A})^{\oplus r} \overset{\text{sum}}{\longrightarrow} \mathbb{G}_a(\mathbb{A}) \to \mathbb{C}^*.$$

By construction, this character is trivial on $N(K) \subset N(\mathbb{A})$.

The space of (unramified) Whittaker functions, denoted $\text{Whit}(G)_{\text{glob}}$, is by definition

$$f \in \text{Funct}(G(\mathbb{O})\backslash G(\mathbb{A})), \quad f(g \cdot n) = \chi(n) \cdot f(g), \quad n \in N(\mathbb{A}). \tag{2.1}$$

By definition, this is a subspace inside the space of functions on the double quotient

$$G(\mathbb{O})\backslash G(\mathbb{A})/N(K).$$

The importance of $\text{Whit}(G)_{\text{glob}}$ stems from the combination of the following two facts:

2.1.3. First, $\text{Whit}(G)_{\text{glob}}$ is *a lot* simpler than the original space of functions on the original double quotient $G(\mathbb{O})\backslash G(\mathbb{A})/G(K)$.

Namely, it has a *local nature*, i.e., it is comprised of spaces that are attached to closed points of x (=places of the global field K). More precisely,

$$\text{Whit}(G)_{\text{glob}} = \bigotimes_{x \in |X|}{}' \text{Whit}_x(G),$$

where $\otimes'$ is the restricted tensor product (like in the definition of automorphic representations), and

$$\text{Whit}_x(G) = \{f \in \text{Funct}(G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)), \quad f(g \cdot n) = \chi(n) \cdot f(g), \quad n \in N(\mathcal{K}_x),\} \tag{2.2}$$

where $\mathcal{O}_x \subset \mathcal{K}_x$ are the local ring and the local field of x, respectively.

2.1.4. Second, $\text{Whit}(G)_{\text{glob}}$ can be related to the space of functions on $G(\mathbb{O})\backslash G(\mathbb{A})/G(K)$ by the following operator (from the latter to the former):

$$f \mapsto \text{Coeff}(f), \quad \text{Coeff}(f)(g) := \int\limits_{n \in N(\mathbb{A})/N(K)} \chi(n^{-1}) \cdot f(g \cdot n).$$

Although the above operator Coeff is not injective, it does retain a lot of information.

2.2. **The geometric Whittaker model.** We will now indicate a construction, similar to the one above, but taking place in the world of D-modules (rather than spaces functions).

2.2.1. First, we need to find a category, denoted by analogy with Sect. 2.1.2, $\text{Whit}(G)_{\text{glob}}$ that will be the geometric replacement of the space of (unramified) Whittaker functions.

Whatever this category is, it is supposed to map to the category of D-modules on a prestack, whose k-points are given by the double quotient $G(\mathbb{O})\backslash G(\mathbb{A})/N(K)$.

This is the prestack that, informally speaking, classifies G-bundles on X, equipped with a *generic* reduction to N. The precise definition of this prestack will be the subject of Talk II.3; we denote it $\text{Bun}_G^{N\text{-gen}}$.

Having the prestack $\text{Bun}_G^{N\text{-gen}}$, we can consider the category

$$\text{D-mod}(\text{Bun}_G^{N\text{-gen}}).$$

Now, $\mathrm{Whit}(G)_{\mathrm{glob}}$ will be a full subcategory in $\text{D-mod}(\mathrm{Bun}_G^{N\text{-gen}})$, defined by a condition, which is a geometric analog of the condition (2.1). The exact meaning of this procedure will be explained in Talk IV.2.

The category $\mathrm{Whit}(G)_{\mathrm{glob}}$ will be related to the original category $\text{D-mod}(\mathrm{Bun}_G)$ by a functor, denoted Coeff, which is a geometric analog of the operator in Sect. 2.1.4. This will also be explained in Talk IV.2.

Now, as in the function-theoretic set-up, the functor Coeff loses information (i.e., is not fully faithful). But this can be rectified by a procedure which will be explained in Sect. 2.4.

2.2.2. The reason for considering the category $\mathrm{Whit}(G)_{\mathrm{glob}}$ is that it enjoys the *locality* property with respect to X, analogous to the corresponding property in Sect. 2.1.3.

This brings us to the idea of *factorization category*, which will be the subject of Talks II.2 and III.1.

Roughly speaking, a factorization category is an assignment to each finite subset of points $I \subset X$ of a DG category $\mathbf{C}_I$, and a system of equivalences

$$(I_1 \cap I_2 = \emptyset) \rightsquigarrow \mathbf{C}_{I_1 \sqcup I_2} \simeq \mathbf{C}_{I_1} \otimes \mathbf{C}_{I_2}.$$

Here $\otimes$ is the tensor product operation on DG categories, that will be explained in the tutorial of Talk I.2. (Having this operation is another reason we need to consider DG categories rather than triangulated ones.)

Having a factorization category, there is an inductive limit procedure (analogous to the operation of restricted product) that produces a single DG category

$$\mathbf{C}_{\mathrm{Ran}(X)}.$$

This procedure will be explained in Talk III.1

2.2.3. Now, the claim is that the category $\mathrm{Whit}(G)_{\mathrm{glob}}$ arises in this way from a datum of a factorization category

$$I \rightsquigarrow \mathrm{Whit}(G)_I.$$

That is

$$\mathrm{Whit}(G)_{\mathrm{glob}} = \mathrm{Whit}(G)_{\mathrm{Ran}(X)}.$$

The corresponding categories $\mathrm{Whit}(G)_I$ will be defined in a way analogous to (2.2), using the *affine Grassmannian* of G.

2.3. **The spectral side.**

2.3.1. Recall that our goal is to compare the geometric side (i.e., the category $\text{D-mod}(\mathrm{Bun}_G)$) with the spectral side (i.e., the category $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$). It will turn out that the category $\mathrm{Whit}(G)_{\mathrm{glob}} = \mathrm{Whit}(G)_{\mathrm{Ran}(X)}$ will help us do that.

This is not altogether surprising, based on the analogy with the function-theoretic situation. There it is also known that for an automorphic function f, the corresponding Whiitaker function $\mathrm{Coeff}(f)$ can be recovered from the Galois-theoretic information (in the guise of Frobenius eigenvalues).

But in geometry, the connection will be more robust.

2.3.2. We begin by introducing the spectral counterpart of $\mathrm{Whit}(G)_{\mathrm{glob}}$. We start with the symmetric monoidal category $\mathrm{Rep}(\check{G})$, i.e., the DG category of representations of $\check{G}$.

In Talk III.1 it will be explained that to any symmetric monoidal category one can associate a datum of factorization category. Applying this to $\mathrm{Rep}(\check{G})$, we obtain the corresponding factorization category

$$I \rightsquigarrow \mathrm{Rep}(\check{G})_I.$$

In particular, we obtain the "restricted product category"

$$\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}.$$

Now, in Talks III.2 and IV.1 it will be explained that the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ is related to the category $\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$ by a pair of mutually adjoint functors

$$\mathrm{Loc}_{\check{G},\mathrm{spec}} : \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \rightleftarrows \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) : \text{co-}\mathrm{Loc}_{\check{G},\mathrm{spec}},$$

with the right adjoint $\text{co-}\mathrm{Loc}_{\check{G},\mathrm{spec}}$ being fully faithful.

Note that above we considering the category $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$ itself, and not its modification $\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}})$.

2.3.3. We have the following basic result:

Theorem 2.3.4. *The factorization categories*

$$I \rightsquigarrow \mathrm{Whit}(G)_I \text{ and } I \rightsquigarrow \mathrm{Rep}(\check{G})_I$$

are canonically equivalent.

This theorem will be explained in Talk V.1.

In particuar, we obtain an equivalence

$$\mathrm{Whit}(G)_{\mathrm{Ran}(X)} \simeq \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}. \tag{2.3}$$

2.3.5. We are now ready to state the Whittaker compatibility property of the geometric Langlands equivalence $\mathbb{L}_G$:

Conjecture 2.3.6. *The functor $\mathbb{L}_G$ makes the following diagram commutative*

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \arrow[r, "(2.3)"] & \mathrm{Whit}(G)_{\mathrm{Ran}(X)} \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[u, "\text{co-}\mathrm{Loc}_{\check{G},\mathrm{spec}}"] & \\
\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \arrow[u] \arrow[r, "\mathbb{L}_G"] & \text{D-mod}(\mathrm{Bun}_G), \arrow[uu, "\mathrm{Coeff}"']
\end{tikzcd}$$

where the lower-left vertical arrow is the right adjoint to the emedding

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \hookrightarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}).$$

2.4. **The extended Whittaker model.**

2.4.1. If we belive in Conjecture 2.3.6, then its statement explains the mechanism for the failure of the functor Coeff to be fully faithful. Namely, it is reflected on the spectral side by the presence of the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}),$$

right adjoint to the tautological embedding.

That is to say, that if we did not introduce the modification

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \rightsquigarrow \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}),$$

we would have to expect the functor Coeff to be fully faithful, which is absurd: for example it annihilates the constant D-module on Bun_G.

We will now explain a modification of Conjecture 2.3.6, in which the vertical arrows are fully faithful.

2.4.2. We begin with the geometric/automorphic side. Recall that the category $\mathrm{Whit}(G)_{\mathrm{glob}}$ was defined by analogy with the function-theoretic situation, while the latter used a *non-degenerate* character χ.

I.e., it was a character of $N/[N,N](\mathbb{A}) \simeq \mathbb{G}_a^{\oplus r}(\mathbb{A})$ that was non-trivial on each $\mathbb{G}_a(\mathbb{A})$ factor. However, one can consider other types of characters (2^r in total), allowing the character to be zero on some specified factors. The above set of 2^r elements is in bijection with the set of conjugacy classes of parabolics in G.

In the geometric situation, to each standard parabolic P we will attach a certain category $\mathrm{Whit}(G,P)_{\mathrm{glob}}$, so that $\mathrm{Whit}(G)_{\mathrm{glob}} = \mathrm{Whit}(G,G)_{\mathrm{glob}}$.

Furthermore, the categories $\mathrm{Whit}(G,P)_{\mathrm{glob}}$ will naturally glue to a single category, denoted $\mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}}$. This will be explained in Talk IV.2.

For each P we will have the corresponding functor

$$\mathrm{Coeff}(P) : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G,P)_{\mathrm{glob}},$$

and we'll also have the corresponding glued functor

$$\mathrm{Coeff}^{\mathrm{ext}} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}}.$$

We have the following key assertion:

Conjecture 2.4.3. *The functor* $\mathrm{Coeff}^{\mathrm{ext}}$ *is fully faithful.*

This conjecture is in fact a theorem for $G = GL_n$.

Remark 2.4.4. One can say that the entire geometric Langlands project hinges on Conjecture 2.4.3. Whatever other gaps there remain in the proof of the Langlands conjecture, they are of technical nature. It is Conjecture 2.4.3 that is the real stumbling block.

2.4.5. We now pass to the discussion of the spectral side. Here too, for each parabolic we will construct a category, denoted

$$\mathrm{I}(\check{G}, \check{P}, \mathrm{spec}).$$

The assignment

$$P \rightsquigarrow \mathrm{I}(\check{G}, \check{P}, \mathrm{spec})$$

will come equipped with a *gluing datum*, which will allow to construct a category

$$\mathrm{I}(\check{G}, \mathrm{spec})^{\mathrm{ext}}.$$

This will be done in Talk V.3. In addition, it will be explained thay the functor

$$\text{co-Loc}_{\check{G},\mathrm{spec}} : \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)}$$

naturally extends to a functor

$$\text{co-Loc}^{\mathrm{ext}}_{\check{G},\mathrm{spec}} : \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{I}(\check{G}, \mathrm{spec})^{\mathrm{ext}},$$

and the latter functor is also fully faithful.

2.4.6. Furthermore, in Talks V.2 and V.3 it will be explained that the validity of geometric Langlands conjecture for proper Levi subgroups of G will supply fully faithful functors

$$\mathrm{I}(\check{G}, \check{P}, \mathrm{spec}) \hookrightarrow \mathrm{Whit}(G, P)_{\mathrm{glob}},$$

that combine to a fully faithful functor

$$\mathrm{I}(\check{G}, \mathrm{spec})^{\mathrm{ext}} \to \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}}.$$

2.4.7. Finally, we are ready to state the following sharpening of Conjecture 2.3.6, which characterizes the functor $\mathbb{L}_G$ uniquely (modulo Conjecture 2.4.3):

Conjecture 2.4.8. *There exists a unquely defined equivalence*

$$\mathbb{L}_G : \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \text{D-mod}(\mathrm{Bun}_G)$$

that makes the diagram

$$\begin{tikzcd} \mathrm{I}(\check{G}, \mathrm{spec})^{\mathrm{ext}} \arrow[r] & \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \arrow[u, "\text{co-Loc}^{\mathrm{ext}}_{\check{G},\mathrm{spec}}"] \arrow[r, "\mathbb{L}_G"] & \text{D-mod}(\mathrm{Bun}_G) \arrow[u, "\mathrm{Coeff}^{\mathrm{ext}}"'] \end{tikzcd} \tag{2.4}$$

commutative.

2.4.9. Let us assume Conjecture 2.4.3 and assess what we have so far. What we have is the diagram

$$\begin{tikzcd} \mathrm{I}(\check{G}, \mathrm{spec})^{\mathrm{ext}} \arrow[r] & \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}} \\ \mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \arrow[u, "\text{co-Loc}^{\mathrm{ext}}_{\check{G},\mathrm{spec}}"] & \text{D-mod}(\mathrm{Bun}_G), \arrow[u, "\mathrm{Coeff}^{\mathrm{ext}}"'] \end{tikzcd} \tag{2.5}$$

with the vertical arrows fully faithful.

We would like to complete it to the diagram (2.4). The existence of the functor $\mathbb{L}_G$ is equivalent to the assertion that the essential image of the composed functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}}$$

is contained in the essential image of the functor

$$\mathrm{Coeff}^{\mathrm{ext}} : \text{D-mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}}.$$

We will prove this by "hitting" the diagram (2.5) from below by another diagram, see Theorem 3.3.5. What this other diagram is will be explained in the next section.

3. Compatibility with Kac-Moody localization

The geometric Langlands equivalence is expected to satisfy another set of compatibilities, and this time it is one that does not have any analog in the function-theoretic set-up.

These compatibilities are behind the construction in [BD], and in pur approach constitute another ingredient that would eventually allow to prove Conjecture 2.4.8 for GL_2 (and presumably, for GL_n with a bit more technical work).

3.1. **Kac-Moody modules.**

3.1.1. Starting from Talk 4.3 we will begin introducing another set of ingredients needed for the proof of Conjecture 2.4.8.

Namely, for every point $x \in X$ we will consider the category

$$\mathrm{KL}(G, \mathrm{crit})_x := \widehat{\mathfrak{g}}_x\text{-mod}_{\mathrm{crit}}^{G(\mathcal{O}_x)}.$$

Here $\widehat{\mathfrak{g}}_x$ denotes the Kac-Moody Lie algebra, associated to the local field $\mathcal{K}_x$, and $\widehat{\mathfrak{g}}_x\text{-mod}_{\mathrm{crit}}$ denotes the category of $\widehat{\mathfrak{g}}_x$-modules at the critical level. Finally,

$$\widehat{\mathfrak{g}}_x\text{-mod}_{\mathrm{crit}}^{G(\mathcal{O}_x)}$$

denotes the category of $\widehat{\mathfrak{g}}_x$-modules at the critical level, which are integrable with respect to $G(\mathcal{O}_x)$. These notions will be introduced in Talk IV.3.

Furthermore, the assignment

$$x \rightsquigarrow \mathrm{KL}(G, \mathrm{crit})_x$$

upgrades to a datum of factorization category

$$I \rightsquigarrow \mathrm{KL}(G, \mathrm{crit})_I.$$

We will consider the corresponding category

$$\mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)}.$$

3.1.2. The reason for the appearence of the category $\mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)}$ is that it is endowed with a localization functor

$$\mathrm{Loc}_{G,\mathrm{crit}} : \mathrm{KL}(G, \mathrm{crit})_{\mathrm{Ran}(X)} \to \text{D-mod}(\mathrm{Bun}_G).$$

The construction of this localization functor will be the subject of Talks IV.3 and V.4. Here we will just say that it is a loop group analog of the Beilinson-Bernstein localization functor from $\mathfrak{g}$-modules to D-modules on the flag variety.

Remark 3.1.3. The functor $\mathrm{Loc}_{G,\mathrm{crit}}$ is more naturally defined to have as target the category

$$\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$$

of critically twisted D-modules. However, the categories $\text{D-mod}(\mathrm{Bun}_G)$ and $\text{D-mod}_{\mathrm{crit}}(\mathrm{Bun}_G)$ are equivalent, using a choice of a *critical line bundle* on Bun_G. The above choice can be made canonical, once we choose a square root $\omega_X^{\frac{1}{2}}$ of the canonical line bundle ω_X on X itself.

Remark 3.1.4. The functor $\mathrm{Loc}_{G,\mathrm{crit}}$ is a close relative of the functor $\mathrm{Loc}_{\check{G},\mathrm{spec}}$, mentioned in Sect. 2.3.2. In fact, for a given reductive group H, the functor $\mathrm{Loc}_{H,\mathrm{crit}}$ is the value at the critical level of the family of functors

$$\mathrm{Loc}_{H,\kappa} : \mathrm{KL}(H,\kappa)_{\mathrm{Ran}(X)} \to \text{D-mod}_\kappa(\mathrm{Bun}_G)$$

that depends on the level κ. The functor $\mathrm{Loc}_{H,\mathrm{spec}}$ can be regarded as the limiting case when $\kappa \to \infty$.

3.1.5. The functor $\mathrm{Loc}_{G,\mathrm{crit}}$ has the following feature: its essential image *almost* generates the category $\text{D-mod}(\mathrm{Bun}_G)$.

More precisely, the essential image of $\mathrm{Loc}_{G,\mathrm{crit}}$, together with the essential images of the *Eisenstein series* functors from proper Levi subgroups, generate $\text{D-mod}(\mathrm{Bun}_G)$.

This observation will play a key role in Talk VI.2, for our proof of Conjecture 2.4.8.

3.2. **Opers.** We will now discuss the counterpart of the pair

$$(\mathrm{KL}(G,\mathrm{crit})_{\mathrm{Ran}(X)}, \mathrm{Loc}_{G,\mathrm{crit}})$$

on the spectral side.

3.2.1. In talk VI.1 we will introduce a factorization scheme

$$I \rightsquigarrow \mathrm{Op}(\check{G})_I$$

of $\check{G}$-opers.

In particular, the assignment

$$I \rightsquigarrow \mathrm{QCoh}(\mathrm{Op}(\check{G})_I)$$

forms a factorization category.

We will consider the corresponding category

$$\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}).$$

3.2.2. The key feature of $\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}$ is the existence of the functor

$$\mathrm{Poinc}_{\check{G},\mathrm{spec}} : \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}) \to \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}).$$

Remark 3.2.3. Recall the category $\mathrm{Whit}(G)_{\mathrm{Ran}(X)}$ and the functor Coeff. For any reductive group H, one can introduce the category $\mathrm{Whit}_\kappa(H)_{\mathrm{Ran}(X)}$ and the functor

$$\mathrm{Coeff}_\kappa : \text{D-mod}_\kappa(\mathrm{Bun}_H) \to \mathrm{Whit}_\kappa(H)_{\mathrm{Ran}(X)}$$

for any value of the level κ. The original functor Coeff should be thought of as the value of Coeff_κ at $\kappa = \mathrm{crit}$.

Now, the category $\mathrm{QCoh}(\mathrm{Op}(H))_{\mathrm{Ran}(X)}$ should be thought of as the limiting case of $\mathrm{Whit}_\kappa(H)_{\mathrm{Ran}(X)}$ for $\kappa \to \infty$. The functor $\mathrm{Poinc}_{\check{G},\mathrm{spec}}$ should be thought of as the limiting case of the *dual* functor of Coeff_κ.

3.3. **The fundamental diagram.**

3.3.1. As will be explained in Talk VI.1, we have a canonically defined functor between factorization categories

$$\mathrm{KL}(G,\mathrm{crit})_I \to \mathrm{QCoh}(\mathrm{Op}(\check{G})_I).$$

Conjecturally, this functor is an equivalence.

In particular, we obtain a functor

$$\mathrm{KL}(G,\mathrm{crit})_{\mathrm{Ran}(X)} \to \mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}). \tag{3.1}$$

3.3.2. The following is the statement of compatibility of the geometric Langlands functor with Kac-Moody localization:

Conjecture 3.3.3. *The functor $\mathbb{L}_G$ of Conjecture 2.4.8 makes the following digaram commute:*

$$\begin{tikzcd}
\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \arrow[r, "\mathbb{L}_G"] & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[u] & \\
\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}) \arrow[u, "\mathrm{Poinc}_{\check{G},\mathrm{spec}}"] & \mathrm{KL}(G,\mathrm{crit})_{\mathrm{Ran}(X)}. \arrow[l, "(3.1)"'] \arrow[uu, "\mathrm{Loc}_{G,\mathrm{crit}}"']
\end{tikzcd}$$

3.3.4. Let us finally return to the strategy of the proof of Conjecture 2.4.8. What did the functors $\mathrm{Poinc}_{\check{G},\mathrm{spec}}$ and $\mathrm{Loc}_{G,\mathrm{crit}}$ "buy" us?

The answer is given by the following statement (which we refer to as the "fundamental diagram"):

Theorem 3.3.5. *The following diagram commutes unconditonally:*

$$\begin{tikzcd}
\mathrm{Rep}(\check{G})_{\mathrm{Ran}(X)} \arrow[r, "(2.3)"] & \mathrm{Whit}(G)_{\mathrm{Ran}(X)} \\
\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}) \arrow[u, "\mathrm{co\text{-}Loc}_{\check{G},\mathrm{spec}}"] & \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \arrow[u, "\mathrm{Coeff}"'] \\
\mathrm{QCoh}(\mathrm{Op}(\check{G})_{\mathrm{Ran}(X)}) \arrow[u, "\mathrm{Poinc}_{\check{G},\mathrm{spec}}"] & \mathrm{KL}(G,\mathrm{crit})_{\mathrm{Ran}(X)}. \arrow[l, "(3.1)"'] \arrow[u, "\mathrm{Loc}_{G,\mathrm{crit}}"']
\end{tikzcd}$$

It will be explained in Talk VI.2 how this theorem allows to show that the essential image of the functor

$$\mathrm{IndCoh}_{\mathrm{Nilp}_{\mathrm{glob}}}(\mathrm{LocSys}_{\check{G}}) \to \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}}$$

is contained in the essential image of the functor

$$\mathrm{Coeff}^{\mathrm{ext}} : \mathrm{D\text{-}mod}(\mathrm{Bun}_G) \to \mathrm{Whit}(G)^{\mathrm{ext}}_{\mathrm{glob}},$$

see Sect. 2.4.9, thereby establishing the extsience of the functor $\mathbb{L}_G$.

Further, in Talk VI.2 it will be explained how Theorem 3.3.5 eventually leads to the proof of the fact that $\mathbb{L}_G$ is an equivalence.

References

[AG] D. Arinkin, D. Gaitsgory, *Singular support of coherent sheaves and the geometric Langlands conjecture*, arXiv:1201.6343v1.

[BD] A. Beilinson and V. Drinfeld, *Quantization of Hitchins integrable system and Hecke eigensheaves*, available at http://math.uchicago.edu/ mitya/langlands.html.