RESEARCH STATEMENT

MILTON LIN

My research focuses on foundational aspects of pure mathematics and machine learning. In mathematics, I study the geometric Langlands program, with an emphasis on its metaplectic and relative extensions. In machine learning, I explore associative memory models and scaling properties, leveraging algebraic and topological methods to gain deeper insights.

The first three pages of this document summarize my research, followed by a detailed discussion of my work in mathematics. Harvard's unique interdisciplinary environment is ideal for my research. I look forward to the opportunity to interact and collaborate with members from different departments.

Research in Mathematics: p-adic Geometry and the Langlands Program

I split my current and future projects into two categories: **core projects**, where I am primarily focused on advancing mixed characteristic and metaplectic aspects of the Langlands program, and additional **ongoing work in related areas**, including categorical deformations, motivic aspects, and stacky approaches.

Core proejcts. In the geometric Langlands program, my graduate work has focused on extensions in the mixed characteristic setting, [ILZ24], this is joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn Univeristy). Our work applied the framework of Zhu's perfect geometry [Zhu17] to prove the Casselman-Shalika formula [NP01] for mixed characteristics. The Casselman-Shalika formula computes the "fourier coefficients" of automorphic forms and is fundamental to modern works of geometric Langlands, see [FR22]. Moving forward, I will continue this research in two directions:

(1) **Relative aspects of Langlands**, joint project with Yuta Takaya (University of Tokyo), we aim to explore relative aspects of the Langlands program on the Fargues-Fontaine curve, [FS24], recent conjectures of Ben-Zvi. Sakellaridis, and Venkatesh [BSV], particularly the relationship between period sheaves and L-sheaves as in [FW24].

(2) **Metaplectic aspects of Langlands**, joint with Toan Pham (Johns Hopkins Univeristy) I intend to give a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+24].

Related works. In addition to my primary projects in the relative and metaplectic aspects of Langlands, I am equally committed to three other areas of study, each of which contributes to the broader foundation of representation theory:

(1) **Categorical deformations of representation category** This builds upon my current research on the Whittaker category, from the point of view of deformation theory. We will first document a careful proof of Lurie's theorem, [Lur10, Thm 10.10], which describe *formal deformation of categories*, as gerbes see [Lur10, Ch.8-10] for definitions. Then, we will explore deformations of representation of Lusztig's small quantum group, using recent

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advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.

(2) **Motivic aspects of Langlands** Building on [RS20], I aim to define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we prove the same statement in [NP01] in the category of mixed Tate motives. The difficulty is that the original argument requires characteristic 0 coefficients, whilst in positive characteristic, there are problems of extensions.

(3) **Stacky Approaches to Periods** I use recent advancements in formalism of stacks to study periods. I have proven that the unipotent fundamental group associated to a pointed scheme can be recovered via a *stacky approach*, see [GR14], [Toë06], [MR23]: for a given scheme X, there exists a natural stack, $\mathbf{U}(X_{\mathrm{dR}})$ whose fundamental group coincides with the unipotent de Rham fundamental group as studied in [Bro14]. By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [1] We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].

Research in Machine Learning: Geometry of Associative Memory Networks

In machine learning, I am particularly interested [2] in foundational theories of associative memory networks, going back to the work of Hopfield and to modern-day associative memories, [KH16]. These networks serve as a bridge between biological realism and computational efficiency. I hope to bring my background in algebra, category theory, and geometry to give insights to the nature of modern networks.

(1) **Polytopal Decomposition of Memory Networks** in joint work with Chris Hillar (former Redwood Research, currently startup on Algebraic) and Tenzin Chan (Algebraic) we focus on the polytopal decomposition of the weight spaces of memory networks and its relation to network scaling. Similar works include, [ZNL18].

(2) **Study of Homotopy Type:** To further my study with Chris Hillar, we propose to explore *Hopfield networks* using a recent formalism by Manin and Marcolli [MM24], which uses *summing functors* and *Gamma spaces* to model the allocation of resources in neural networks. The formalism allows us to study a *homotopy type* - a mathematical construct at a deeper level than *homology*[3]. Homotopy captures invariants of the network up to continuous deformations. The relationship between memory capacity and homotopical invariants would be the main subject of study.

(3) **Emprical Study of Assocative Memory Models beyond the storage capcity** My joint work with Muhan Gao (Johns Hopkins University) empirically studies modern energy-based memory network transformers, such as [KH16], for language modeling and classification tasks. This was previously studied in the context of vision task in [Hoo+24]. We further explore these model in the regime where the stored memories is beyond the theoretical capacity, see Equation (5) and (6) of [KH16]. This research will also highlight the limitations of synthetic memory networks, especially in their use as proxies for explaining biological networks, see also [KH21].

[1] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

[2] Details: https://cwlin4916.github.io/Trees/Application/Postdoc/ResearchA.pdf

[3] which is common in topological data analysis (TDA), see [Cur16] for a short survey.

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RESEARCH STATEMENT IN PURE MATHEMATICS

MILTON LIN

Introduction

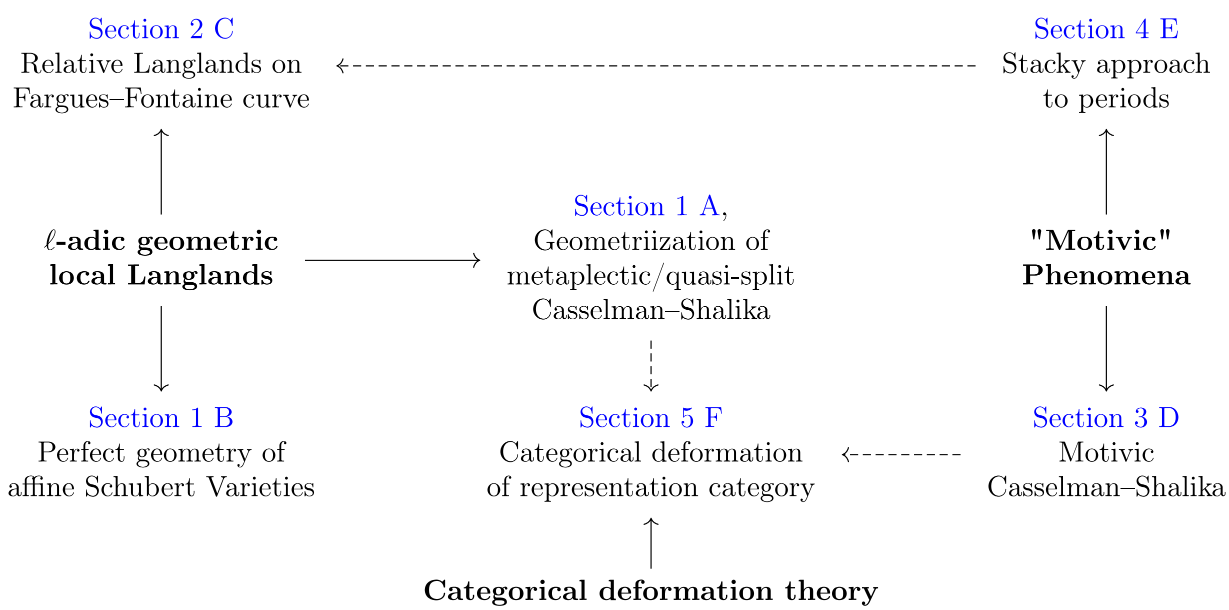
My areas of interest in mathematics are:

(1) The **Langlands correspondence**, particularly, the various incarnations of the **Casselman–Shalika formula**, A, and **relative aspects of the ℓ-adic geometric local Langlands correspondence**, C.

(2) **Stable homotopy theory**, through the lens of **categorical deformation theory**, F.

(3) **Motivic phenomena**, where I hope to explore the motivic version of Casselman–Shalika formula, D, and a stacky approach to periods, E.

Majority of the research presented here originates from my study of the Casselman–Shalika formula in the mixed characteristic setting, as outlined in Section 1. These areas of research are interconnected, as shown in the following diagram.



The priority of research is listed in the following order[1],

A=C=F>D>E> B.

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[1]The alphabet links to the goal rather than the section.

Notations. Theorems stated have full proof written by either me or joint with collaborators, *unless* it is annotated with: ***conjecture*** – no proofs have been written down but is believed to hold, or ***in progress*** – where we have partial progress. We use freely the language of higher categories and higher algebra, [Lur09], [Lur18].

1. Mixed Characteristic Geometry and the Casselman–Shalika formula

Let G be a connected reductive group over a nonarchimedean local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld [AG15], [Gai14]. One fundamental aspect of the program is to understand the *Whittaker Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, as described *op.cit.*, this corresponds to finding the Fourier coefficients of automorphic functions:

Example 1.1. Let $G = \mathrm{PGL}_2$ be the projective linear group over $\mathbb{Q}$. A modular function, f, has an adelic formulation, $\widetilde{f}$ on $G(\mathbb{A}_\mathbb{Q})$. Up to a *normalizing factor*, the Fourier coefficients coincide with the Whittaker coefficients (Jacquet period),

$$a_m \sim \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \widetilde{f}(n\alpha_m)\psi(-n)\, dn \quad \text{for } m \geq 1$$

where $\alpha_m \in T(\mathbb{A}_\mathbb{Q}^{\mathrm{fin}})$ is m considered as a finite idèle and ψ is a standard character on $N(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_\mathbb{Q}$, where N is the radical of standard Borel. For more details, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which we aim to replicate in the mixed characteristic setting. In joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn University) [ILZ24], we proved a variation of this problem: the *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian Gr_G, analogous to the equi-characteristic geometrization carried out in [NP01].

Theorem 1.2. *[ILZ24] The geometric Casselman–Shalika formula holds over the Witt vector Grassmanian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori–Satake equivalence, [Bez+19] without using nearby cycles.

Theorem 1.3 (I.-Lin-Z., in progress)**.** *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [2]

As of writing, D. Hansen, L. Hamman, and L. Mann have announced significant advancements to the global version of the formula. Building on all these progresses, I have developed a strong interest in exploring the geometrization of the Casselman–Shalika formula for covering groups [GGW18].

Research Goal A. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula over equal-characteristic local fields. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+20; Bru+24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

[2] In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

Lastly, we would like to highlight that in our recent work [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\le\mu}$, for minuscule and quasi-minuscule μ. In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal B. Geometry of general perfect Schubert variety *Prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhats-Tits building of G, we have an associated group scheme $\mathcal{G}$ over $\mathcal{O}$.*

Conjecture 1.4. The smooth locus of $\mathrm{Gr}_{\mathcal{G},\le\mathfrak{s}}$ is $\mathrm{Gr}_{\mathcal{G},\mathfrak{s}}$ in perfect geometry, in the sense of [Zhu17].

2. Relative langlands on the Fargues Fontaine curve

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and L-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with $(\hat{G}, \hat{X})$ of hyperspherical varieties. Let $\Lambda = \bar{\mathbb{Q}}_l$. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_!\Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \pi_*\omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. Bun_G has a Hardar-Narasimhan straification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G)$. In our case, $G = \mathbb{G}_m$, $\mathrm{Bun}_{\mathbb{G}_m}$ is stratified by $\mathrm{Bun}_{\mathbb{G}_m}^n$ for $n \in \mathbb{Z} = B(T)$. Interesting phenomena occurs for $n > 0$, and the study of period sheaves reduces to the study of Bun_G^X restricted to Bun_G^n. This corresponds to the Abel-Jacobi map previously studied by Fargues [Far20] and Hansen [Han21]. On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles, [FW24, Ch. 6] recently developed by Le-Bras et al.

Conjecture 2.1. [Lin-T., in progress] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[3] $\mathcal{L}_{\hat{X}}$.

Research Goal C. Relative Langlands on the Fargues Fontaine curve. *Complete Conjecture 2.1 as a first step and then the Hecke case, which classically corresponds to Hecke's integral representation of standard L-function for GL_2. Lastly, one can ask whether on the $\mathcal{B}$-side, the same constructions of [BSV, Ch. 11] works for the p-adic (Emerton-Gee) L-parameter stacks, which potentially give new interpretations to p-adic L-functions.*

3. Motivic Phenomena

In this section S denotes a base scheme satisfying the Beilinson-Soulé vanishing conjecture, such as $\mathrm{Spec}\,\mathbb{Z}$, finite field. R denotes a regular coherent coefficient ring. I discuss one side project I have done, and one area of research, D, which extends my joint work [ILZ24].

In Grothendieck's quest towards the Weil conjectures, he reduced to problems of motives. Beilinson proposed an extension of the notion of pure motives to mixed motives, which was realized through the works of Voevodsky, Levine for a field, and extended by Ayoub and Cisikinki-Déglise, [CD19].

[3] One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on X which we do not discuss.

For a finite type S-scheme $X \in \mathrm{Sch}_S^{\mathrm{ft}}$, we can construct the derived categories of motives over X, $\mathrm{DM}(X, R)$. If X is smooth then its Ext-groups,

$$\pi_{-m} \operatorname{Map}_{\mathrm{DM}(X,R)}(1_X, 1_X(n)) \simeq \mathrm{CH}^n(X, 2n-m)_R$$

are Bloch's higher Chow groups. Motivic categories and the Chow groups are difficult to explicitly work with due to the lack of motivic t-structures. One method is due to the work of Beauville [Bea83], using Fourier transform. In my joint work, [Has+24] we extended the work of Beauville,

Theorem 3.1 (Lin et al.)**.** *Let $X \to Y$ be an abelian scheme, whose base Y is smooth and quasi projective over a field. There is an explicit N, such that one obtains a Beauville decomposition*

$$CH^i(X)_{\mathbb{Z}[1/N]} \simeq \bigoplus_s CH^i_{(s)}(X)_{\mathbb{Z}[1/N]}$$

where $CH^i_{(s)}(X)_R := \left\{x \in CH^i(X)_R : [n]_X^* x = n^{2i-s}x \quad \forall n \in \mathbb{Z}\right\}$ and $[n]$ is the multiplication of an abelian scheme.

This extends to a $\mathfrak{sl}_2$ action, which we discussed in *op.cit*; and if S were an algebraically closed field, this implies various structural results. The key ingredient was using G. Pappas' version of integral Grothendieck–Riemann–Roch, [Pap07].

Returning to motivic t-structures, it was shown by Levine that t-structures exist on a nice subcategory $\mathrm{DTM}(X) \hookrightarrow \mathrm{DM}(X)$ of *mixed Tate movies* for nice schemes X. This was extended to schemes with cellular Whitney-Tate stratification by Soergel and Wendt, and to prestacks in [RS20].

Research Goal D. Motivic Whittaker categories. *Define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we can prove the same statement in [NP01] in the category of mixed Tate motives, in which all the relevant objects are well defined. The difficulty, however, is that the original argument requires characteristic 0 coefficients, whilst in positive characteristic, there are problems of extensions.*

4. Stacky approaches and periods

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R)$$
$$X \mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?})$$

For instance, the *de Rham stack* X^{dR} over $\mathbb{Q}$, has points given by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A (cf. [GR14]). This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques. I explored this concept in basic constructions of F. Brown's work on *motivic periods*, [Bro14]. If X were a smooth variety over $\mathbb{Q}$, the matrix coefficient from Grothendieck's comparison theorem

$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^*_{\mathrm{Betti}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

with respect to the $\mathbb{Q}$ structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to X*. These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"[4] which should govern the arithmetic structure of periods.

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the projective space minus three points over $\mathbb{Q}$. The period associated is studied through Chen's de Rham comparison theorem, [Che76], which relates to iterated integrals

[4]For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

[Che73] and multiple zeta values [Bro14]. To give a "stacky" perspective of this period one uses unipotent types of stacks, which is defined in Toën, [Toë06], and recently in [MR23]. This is an endofunctor $\mathbf{U}$ on stacks, sending a stack X to its unipotent homotopy type. My first result is:

Theorem 4.1. *(Lin) Unipotent de Rham fundamental group, $\pi_1^{u,dR}(X,x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{dR}))$.*

Research Goal E. A stacky approach to motivic periods. *This research aims to use geometric techniques in the study of periods. As a proof of concept, we will recover de Rham comparison theorem through the Riemann-Hilbert correspondence between the analytic Betti stack and the de Rham stack. $\mathbb{P}^1\setminus\{0,1,\infty\}$, is not proper, which requires us to incorporate log structures. I expect to prove:*

Conjecture 4.2. There exists X^{Betti} such that the unipotent Betti group $\pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}),x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{\mathrm{Betti}}))$. A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,\mathrm{dR}}(X,x) \simeq \pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}),x)\otimes\mathbb{C}$$

By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [5] *We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].*

5. Deformation theory and the sphere spectrum $\mathbb{S}$

Let $\mathcal{S}$ denote the ∞-category of ∞-groupoids/anima. The *stabilization* of $\mathcal{S}$ is Sp, the ∞-category of spectra. This is the natural category to study cohomological invariants. Within Sp, lies the universal cohomology theory, $\mathbb{S}$, the *sphere spectrum* By Chevalley's works, connected reductive groups over $\mathbb{C}$ have a canonical split $\mathbb{Z}$-form $G_{\mathbb{Z}}$, see [Con15]. One can analogously ask: *is there a $\mathbb{S}$-form for algebraic groups*? A first approximation is the existence of an algebraic category $\mathrm{Rep}_{\mathbb{S}}(G_{\mathbb{S}})$, which *deforms* to $\mathrm{Rep}_{\mathbb{Z}}(G_{\mathbb{Z}})$. To study, we begin with *formal deformation of categories*, which we briefly recall.

Let $\mathcal{C}$ be a symmetric monoidal category. There is a natural hierarchy of commutativity fitting in the diagram

$$\cdots\to\mathbb{E}_n(\mathcal{C})\to\mathbb{E}_{n-1}(\mathcal{C})\to\cdots\to\mathbb{E}_1(\mathcal{C})$$

where $\mathrm{CAlg}(\mathcal{C}):=\mathbb{E}_\infty(\mathcal{C}):=\varprojlim\mathbb{E}_n(\mathcal{C})$ of symmetric algebra objects can be identified with the limit. One can formalize the notion of $\mathbb{E}_n$-algebra objects via *disk operads*, or *little cubes operads*.

Example 5.1. Let $\mathcal{C}=(\mathcal{S},\times)$. Let $Y\in\mathcal{S}_*$ is a pointed ∞-groupoid, its *k-fold based loop spaces*, Ω^k_*Y is a classical example of $\mathbb{E}_k$ algebra object in $(\mathcal{S},\times)$.

Let $R\in\mathbb{E}_n(\mathrm{Sp})$ be an $\mathbb{E}_n$ ring, and consider LMod_R, the derived category of left R-modules, as an $\mathbb{E}_1$ object in $\mathrm{Pr}^{\mathrm{st}}$, the category of presentable stable categories. This defines $\mathrm{RMod}_{\mathrm{LMod}_R}(\mathrm{Pr}^{\mathrm{st}})$, the category of presentable stable (right) R-linear categories, [Lur18, Appendix D]. Set $\mathrm{Pr}^{\mathrm{st,cg}}_R$, as the full subcategory spanned by those whose underlying category is compactly generated. [6] For G a connected reductive group over a field k, $D^b(\mathrm{Rep}^{\mathrm{fd}}_k(G))$, the bounded derived category of finite dimensional algebraic representations with k coefficients lies in $\mathrm{Pr}^{\mathrm{st,cg}}_R$.

From now on, $k=\mathbb{C}$. Let $\mathrm{Art}^{(n)}_k$ denote the category of $\mathbb{E}_n$ *artinian* ring spectrum, for $n\geq 0$ over k. We refer to [Lur18, Ch. 15] for definition. This extends the classical definition of Artinian local

[5] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

[6] The compact generation is only a smallness condition for our version.

ring, in particular, $R \in \mathrm{Art}_k^{(n)}$ admits an augmentation map $\epsilon : R \to k$. One defines the $\mathbb{E}_{n+2}$-formal moduli problem,

$$\begin{gathered}\mathrm{Cat}\mathcal{D}\mathrm{ef}^{(n)}(\mathcal{C}) : \mathrm{Art}_k^{(n+2)} \to \hat{\mathcal{S}} \\ R \mapsto |\,\{\mathcal{C}\} \times_{\mathrm{Pr}_R^{\mathrm{st,cg}}} \mathrm{Pr}_k^{\mathrm{st,cg}}|\end{gathered}$$

where $|\quad|$ is the underlying Kan complex of the ∞-category. An object consists of: a $\mathcal{C}_B$ right stable R-linear category, and an equivalence $\mathcal{C}_B \otimes_{\mathrm{LMod}_B} \mathrm{LMod}_k \simeq \mathcal{C}$. Our $\mathbb{E}_4$-moduli problem is when $n = 2$ and $\mathcal{C} = D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$. The geometric Casselman–Shalika [FGV01], which is the $\mathbb{E}_2$-algebra equivalence of $\mathcal{C}$ with the *Whittaker sheaves* on the *affine Grassmanian* $\mathrm{Gr}_{\hat{G}}$, describes this moduli problem. Consider moduli of functor of $\mathbb{G}_m$*-gerbes over* X

$$\mathrm{Ge}_{\mathbb{G}_m}(X) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(X, B^2 R^\times) \quad R \in \mathrm{Art}_k^{(4)}$$

where $R^\times \subset \Omega^\infty R$ are the invertible elements of the underlying space of R [7] and B^2 is the second deloop. It was stated without proof in [Lur10]

Theorem 5.2 (Lurie)**.** *There is an equivalence of formal* $\mathbb{E}_4$*-moduli problems*

$$\widehat{Ge_{\mathbb{G}_m}}(\mathrm{Gr}_{\hat{G}}) \xrightarrow{\simeq} Cat\mathcal{D}ef^{(2)}(\mathrm{Rep}_k^{fd} G)$$

where $\widehat{-}$ *is the formal completion of the moduli functor at a base point.*

Research Goal F. Categorical deformations of the representation category *We will first document carefully Lurie's theorem, Theorem 5.2. Then, we will explore deformations of representation of Lusztig's small quantum group, as suggested in op.cit. Remark 10.12, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.*

[7] is the union of the connected components of invertibles in the $\pi_0 R$ of the 0th space of R and is equivalent to the nth loop space of some space, $R^\times \simeq \Omega^n Z$, $n \geq 4$, hence the deloop $B^2 R^\times \in \mathbb{E}_2(\mathcal{S})$

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