Résolution de Demazure affines et formule de Casselman-Shalika

This is a note on [**11**].

1. Introduction

Let $G \in \mathrm{AlgGrp}_k^{\mathrm{cn.red.split}}$, $k = \mathbb{F}_q$. For each $\lambda \in X_\bullet(T)_+$, it is possible to construct a projective k-scheme $\bar{\mathrm{Gr}}_\lambda$, whose set of k points is

$$\overline{\mathrm{Gr}^\lambda}(k) := \bigsqcup_{\lambda' \leq \lambda} K\varpi^{\lambda'} K/K$$

of which the group K, viewed as an algebra group over k of infinite dimension, acts through a quotient of finite type. The action induces a stratification of open orbits

$$\overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\lambda' \leq \lambda} \mathrm{Gr}^{\lambda'}$$

The scheme $\overline{\mathrm{Gr}^\lambda}$ is not smooth in general, for a prime $l \neq \mathrm{char}\, k$, it is natural do consider the l-adic IC complex

$$\mathcal{A}_\lambda := \mathrm{IC}(\overline{\mathrm{Gr}^\lambda}, \bar{\mathbb{Q}}_\lambda)$$

which is K-equivariant. The associated function from Frobenius trace:

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

is defined on the set of k points of $\overline{\mathrm{Gr}^\lambda}$.

Let $\check{G}$ be the group defined over $\bar{\mathbb{Q}}_l$ whose roots is dual to that of G. In [Sat63], Satake constructed a canonical isomorphic of the hecke algebra $\mathcal{H}$ with the algebra of regular functions on $\check{G}$, which are $\mathrm{Ad}(\check{G})$ equivariant.

The constant terms which re the Fourier coefficients of the functions A_λ are remarkably simple. Let $B := TU$ the a subgroup of Borel of G and ρ the half sum of roots of T in $\mathrm{Lie}(U)$. After Lustzig and Kato, the constant integral term is equal to

$$\int_{U(F)} A_\lambda(x\varpi^\nu)\, dx = (-1)^{2\langle \rho, \nu \rangle} q^{\langle \rho, \nu \rangle} m_\lambda(\nu)$$

where $m_\lambda(\nu)$ is the dimension of the weight space ν in $V(\lambda)$.

Example:

The principle object of this paper is to prove the gometric statement of the above result. For each $\nu \in X_\bullet(T)$ there is a well defined subscheme $S_\nu \subset \mathrm{Gr}$ such that

$$S_\nu(k) := U(F)\varpi^\nu G(\mathcal{O})/G(\mathcal{O})$$

We show that the complex

$$R\Gamma_c(S_\nu \otimes_k \bar{k}, \mathcal{A}_\lambda)$$

is concentrated in degree $2\langle \rho, \nu \rangle$ and that the Frobenius endomoprhism acts on $H^{2\langle \rho, \nu \rangle}$ as multiplication by $q^{2\langle \rho, \nu \rangle}$....

When ν is dominant, we can define a moprhism $h : S_\nu \to \mathbb{G}_a$ such that $\theta(x) = \psi(h(x))$, where $\psi : k \to \bar{\mathbb{Q}}_l^\times$ is a nontrivial additive character on k. We show that the complex

$$R\Gamma_c(S_\nu \otimes_k \bar{k}, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi)$$

Here is the organization of the article. After recalling in 3, known results on affine Grassmanian, we state the principle theorems in 4.1 and 4.2 in **??**. The proof of the theorem occupies the rest of the article. This is based on the study of the geometry of certain resolutions from the simplest $\overline{\mathrm{Gr}}^\lambda$, which corresponds to when λ is minuscule or quasi-minuscule. This strategy is used in [**10**], where the conjecture of [**5**] is proved for GL_n.

In 5 and 6, we prove geometric properties of the intersection $S_\nu \cap \overline{\mathrm{Gr}}^\lambda$, which were probably well known but cannot be found in the literature. 6.2 allows us to show the statements 4.1, 4.2 in the case ν is conjugated by λ by anb element of the Weyl group. We remark on passing, the statement ...

We then study 7, ... , the geometry of $\overline{\mathrm{Gr}}^\lambda$ in the most simple case, that is, when λ is minuscule 7, or when it is quasiminuscule. If λ is minuscule, then $\overline{\mathrm{Gr}}^\lambda$ is equal to Gr^λ and is isomorphic to the scheme G/P of subgroups of G which are conjugate to some parabolic P, further, only the ν which are conjugate to λ are involved, so that 4.1 and 4.2 follows as in the case from 6.2. In section

1.0.1. *Highest weight theory of reductive groups.* To motivate: consider G is of *multiplicative type.* This an extension of *Cartier duality*

$$\mathrm{Comm}(\mathrm{FinSch}_k) \xrightarrow{\ \simeq\ } \mathrm{Comm}(\mathrm{FinSch}_k)^{\mathrm{op}}$$

This an enlargement of the torus equivalence [**1**, 14.1]

$$\mathrm{Mod}^{\mathrm{fin.\ gen,op}} \xrightarrow{\ \simeq\ } \mathrm{AlgGrp}_k^{\mathrm{diag}}$$

For a triplet (T, B, G),

Theorem 1.1. [**1**, 32.8]

   (1) Every irreducible representation has a highest weight, which is dominant.
   (2) For all $\lambda \in X_\bullet(T)$, exist as unique $V := V^\lambda \in \mathrm{Rep}_k(G)$ with highest weight λ.

2. Notation

Let k be a finite field of q elements of characteristic p, with algebraic closure $\bar{k}$. Let T be split maximal torus of G and B, B^- be the Borel subgroups such that $B \cap B^- = T$. We denote $\langle -, - \rangle$ the natural paring $X, X^\vee := \operatorname{Hom}(\mathbb{G}_m, T)$. Let $R \hookrightarrow X$ be the system of roots associated to (G, T) and R_+ the roots corresponding to B (resp. B^-) and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. For each $\alpha \in \Phi$, we denote U_α the the root subgroup of G corresponding to α. Let $\Phi^\vee \hookrightarrow X_\bullet$ be the dual roots provided by the bijection

$$\Phi \to \Phi^\vee \quad \alpha \mapsto \alpha^\vee$$

Denote by $\Phi^\vee_+$ the set of positive coroots. Let W be the Weyl group of (G, T). [1] Let

$$\rho := (1/2) \sum_{\alpha \in R_+} \alpha$$

the half sum of positive roots. For each simple root, we have

$$\langle \rho, \alpha^\vee \rangle = 1$$

We denote $Q^\vee := \mathbb{Z}\Phi^\vee$ (resp. $Q^\vee_+ := \mathbb{N}_{\geq 0}\Phi^\vee_+$). We denote by $X_{\bullet,+}$ the cone of dominant cocharacter

$$X_{\bullet,+} := \{\lambda \in X_\bullet : \langle \alpha, \lambda \rangle \geq 0 \forall \alpha \in \Phi_+\}$$

We consider the partial order on $X_\bullet$ as follows: $\nu \geq \nu'$ if and only if $\nu - \nu' \in Q^\vee_+$. We denote $\check{G}$ the dual group over $\bar{\mathbb{Q}}_l$. It is provided with $\check{T} \hookrightarrow \check{B}$. For each $\lambda \in X_{\bullet,+}$ We denote

$$\Omega(\lambda) := \{\nu \in X_\bullet : \forall w \in W \quad w\nu \leq \lambda\}$$

This is the set of weight of $\check{T}$ in V_λ, the $\check{G}$-simple $\bar{\mathbb{Q}}_l$ module of highest weight λ. We denote M the set of minimal elements[2] in $X_{\bullet,+} \backslash \{0\}$.

Proposition 2.1. Let $\mu \in M$. We have the following equivalent:

(1) If $\langle \alpha, \mu \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$, and μ is a minimal element in $X_{\bullet,+}$, then $\Omega(\mu) = W\mu$. In this case, we say that μ is minuscule cocharacter. [3]

(2) Otherwise,[4] there exists a unique root such that $\langle \gamma, \mu \rangle \geq 2$; its a maximal positive root, and we have $\mu = \gamma^\vee$ and $\Omega(\mu) = W\mu \cup \{0\}$. In this case, we say that μ is *quasi-miniscule*.

PROOF. The first [**2**, Chap. VI, Ex. 1.24]. We prove the second. Let $\gamma \in \Phi$ such that $\langle \gamma, \mu \rangle \geq 2$.   □

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[1] The Weyl group is given by $N_G(T)/Z_G(T)$. Typical example to keep in mind is $s :=$ $\begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, see [**1**, 26]

[2] The condition of being minimal: is that there does not exists such that

[3] Take $\mu = (1, 0)$.

[4] In GL_2 there is only *one* positive root. Thus, this criteria simply says that as long as (a, b) satisfies $a \geq b + 2$, then it is not minuscule.

2.0.1. *Remarks on the the sets appearing here.* If $\lambda \in X_{\bullet,+}$ then

$$W\lambda = \left\{ \mu \in X_\bullet \,:\, L_\mu \in \mathrm{Gr}_G^\lambda \right\}$$

using the Cartan decompostion. Further by the closure relation of the orbits

$$\left\{ \mu \in X_\bullet \,:\, L_\mu \in \overline{\mathrm{Gr}^\lambda} \right\} = \left\{ \mu \in X_\bullet \,:\, \mu^+ \leq \lambda \right\}$$

where μ^+ is unique W-conjugate of μ which is dominant.

2.0.2. *Remarks on minuscule condition.* Minuscule representations occur in the study of cohomology of flag varieties [6] and the classification of Shimura datum, [4, 1.2].

- lie theoretic point of view: if one considers the induced adjoint representation of $\lambda : \mathbb{G}_m \to T \circlearrowleft \mathfrak{g}$, we have a decomposition

$$\mathfrak{g} \simeq \bigoplus \mathfrak{g}_\lambda(i) \quad \mathfrak{g}_\lambda(i) := \left\{ X \in \mathfrak{g} \,:\, \mathrm{Ad}\lambda(a)X = a^i \cdot X \right\}$$

  λ is minuscule implies $\mathfrak{g}_\lambda(i) = 0$ for $|i| \geq 2$.
- representation theory: when all weights are conjugate under the Weyl group.[5] being minuscule also implies for the highest weight representation V^λ of $\check{G}$, all weights in V^λ have multiplicity 1.
- context of Shimura varieties

$$\mathrm{Hom}^*(\mathbb{S}, G_\mathbb{R})/G^{\mathrm{ad}}(\mathbb{R}) \simeq \mathrm{Hom}^{*'}(\mathbb{G}_m, G_\mathbb{C})/G(\mathbb{C})$$

2.0.3. *Highest weight á la Bourbaki.* Let us keep in mind the following example of $\mathfrak{sl}_2(k)$. [2, n°2, VIII]. It has three distinct elements,

$$X_+, X_-, H$$

Our goal is study $V \in \mathrm{Rep}_e(\mathfrak{sl}_2(k))$, $H \circlearrowleft V$ is diagonalizable. The first representation is the adjoint representation.

$$\mathfrak{sl}_2(k) \circlearrowleft \mathfrak{sl}_2(k) \quad g \cdot x := [g, x]$$

In fact, $\mathfrak{h}$, the e-span of H, acts on $\mathfrak{sl}_2(k)$ by commuting operator. This yields the general decomposition

$$\mathfrak{sl}_2(k) \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi \subset \mathfrak{h}^\vee} \mathfrak{sl}_{2,\alpha}$$

(1) It has an abelian subalgebra of semisimple elements.

This shows the strategy to understand a simple Lie algebra L is

(1) Find an abelian subalgebra H

For Δ a commutative monoid, let $\mathrm{Fun}(\Delta_{\mathrm{disc}}, \mathrm{Mod}_k)$ of Δ-indexed categories. Then via the composition $\dim : \mathrm{Mod}_k \to \mathbb{Z}$, we obtain

$$\mathrm{ch} : \mathrm{Mod}_k^{\Delta_{\mathrm{disc}}} \to \mathbb{Z}^{\Delta_{\mathrm{disc}}}$$

dim is an additive functor, see [2, n°6, Exmple, Ch. VIII]. If $\Delta = \mathfrak{h}^\vee$.

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[5]These are "most" of the small representations. For type A_n: the minuscule representations are the exterior powers. of a group.

3. La Grassmannienne affine

Recall the construction, [**7**]. As *loc. cit.* call a k-space, resp. k-group a sheaf of set, resp. of group over the Alg_k with respect to fppf topology. Consider a the k-group LG and the K-subgroup $L^{\geq 0}G$.

It is clear that $L^{\geq 0}G$. is represented by the projective limit of schemes of finite type

$$R \mapsto G(R[[\varpi]]/\varpi^n)$$

Denote by $L^{(N)}G(R)$ the set of $g \in LG(R)$ such that both the order of the poles of $\rho(g)$ and $\rho(g^{-1})$ does not exceed N. After *loc. cit.* $L^{(N)}(G)$ is representable by a scheme and

$$\mathrm{Gr} \simeq \varinjlim \mathrm{Gr}^{(N)}$$

where $\mathrm{Gr}^{(N)} = L^{(N)}G/L^{\geq 0}G$. Denote $L^{\leq 0}G$ the k group $R \mapsto G(R[\varpi^{-1}])$ and let

$$L^{<0}G := \ker(L^{\leq 0}G \xrightarrow{\varpi^{-1} \mapsto 0} G)$$

This is a subgroup of LG.

Proposition 3.1. The morphism

$$L^{<0}G \times L^{\geq 0}G \to LG$$

is an open immersion.

We identify $L^{<0}G$ with the open $L^{<0}Ge_0$ where e_0 is a fixed based point of Gr. The Grassmanin Gr is covered by the opentralsates $gL^{<0}Ge_0$. These are easy to study for the local geometry of Gr. For example $L^{<0}G$ is not reduced in general, neither is Gr.

The group $L^{\geq 0}G$ acts naturally on Gr. For all $\lambda \in X_\bullet$ denote e_λ the point $\varpi^\lambda e_0$ of Gr. For $\lambda \in X_{\bullet,+}$ denote Gr^λ the $L^{\geq 0}G$ orbit of e_λ. Denote $\overline{\mathrm{Gr}^\lambda}$ the closure of Gr^λ. Also

$$L^{\geq \lambda}G := \mathrm{ad}\varpi^\lambda L^{\geq 0}G, \quad L^{<\lambda G} := \mathrm{ad}\varpi^\lambda L^{<0}G$$

Denote J the prieimage of $U \hookrightarrow B$ under the homomoprhism $L^{\geq 0}G \to G$ deinfed by $\varpi \mapsto 0$. This is a projective limit of unipotent groups. Denote by

$$J^{\geq \lambda} := J \cap L^{\geq \lambda}G$$

$$J^\lambda := J \cap L^{<\lambda}G$$

Example

$G = \mathrm{GL}_2$, then

$$J(k) = \begin{pmatrix} 1 + tk[[t]] & k[[t]] \\ tk[[t]] & 1 + tk[[t]] \end{pmatrix} = \begin{pmatrix} 1 + t\mathcal{O} & \mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix}$$

$$J^{(1,0)}(k) = k[\frac{1}{t}] \cap k[[t]] = k$$

Don't confuse this with LU!

Let $\alpha \in R$, $i \in \mathbb{Z}$, let $U_{\alpha,i}$ be the image of the homomorphism

$$\mathbb{G}_a \to LG$$

$$x \mapsto U_\alpha(\varpi^i x)$$

The multiplication defines an isomoprhism

$$\prod_{\alpha \in R_+, \langle \alpha, \lambda \rangle > 0} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{\alpha,i} \to J^\lambda$$

where we made a choice of total order on the set of factors. In particular J^λ is isomorphic to an affine space of dimension $2 \langle \rho, \lambda \rangle$.

Proposition 3.2. The natural morphism

$$J^\lambda \to \mathrm{Gr}^\lambda$$

$$j \mapsto j e_\lambda$$

is an open immersion.

PROOF. It is clear that multiplication induces an isomorphism

$$J^\lambda \times J^{\geq \lambda} \xrightarrow{\simeq} J$$

It is also clear that the multiplication induces an open immersion

$$J \times B^- \to L^{\geq 0} G$$

Moreover, $J^{\geq \lambda}$ and B^- are subgroups of $L^{\geq \lambda} G$ which fixes e_λ. The lemma follows. $\square$

It follows from 3.2 that Gr^λ is smooth irreducible and of dimension $2 \langle \rho, \lambda \rangle$. There exists an embedding $\mathrm{Gr}^\lambda \hookrightarrow \mathrm{Gr}^{(N)}$ for N sufficiently large, hence the closure $\overline{\mathrm{Gr}^\lambda}$ is a porjective scheme, irreducible and stable by the action of $L^{\geq 0} G$. It is well known, see [8, 11], that $\overline{\mathrm{Gr}^\lambda}$ is the union of orbits $\mathrm{Gr}^{\lambda'}$ such that $\lambda' \leq \lambda$. In particular, if μ is minuscule [6], then Gr^μ is a smooth projective scheme.
Let [7]

$$L^{>0} G := \ker \left(L^{\geq 0} G \to G \right)$$

This is a projective limit of unipotent groups. It is clear that for $\lambda \in X_{\bullet,+}$ the moprhism

$$L^{>0} G \cap L^{\geq \lambda G} \times L^{>0} G \cap L^{<\lambda} G \to L^{>0} G$$

is an isomorphism and that [8]

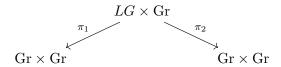$$L^{>0} G \cap L^{<\lambda} G = \prod_{\alpha \in \Phi_+, \langle \alpha, \lambda \rangle > 1} U_{\alpha,i}$$

Let P_λ be the parabolic subgroup generated by B^- and by the radical subgroups with $\langle \alpha, \lambda \rangle = 0$.

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[6]don't we only need being minimal in $X_{\bullet,+}$?

[7]Loops with formal series with no constant terms.

[8]Taking $\lambda = (1,0)$, whose that the only term that matters is in the top right.

Following Lusztig, Ginzburg, Mkirkovic and Vilonen, we define the convolution product $\mathcal{A}_{\lambda_1} * \mathcal{A}_{\lambda_2}$ for $\lambda_1, \lambda_2 \in X_{\bullet,+}$. Consider the moprhisms

$$LG \times \mathrm{Gr}$$

$$\pi_1 \swarrow \qquad \searrow \pi_2$$

$$\mathrm{Gr} \times \mathrm{Gr} \qquad\qquad \mathrm{Gr} \times \mathrm{Gr}$$

$$\pi_1(g, x) = (ge_0, x) \quad \pi_2(g, x) = (ge_0, gx)$$

The moprhism $\pi - 1$ is the quotient[9] morphism for the action $L^{\geq 0}G$ on $LG \times \mathrm{Gr}$ defined by

$$\alpha_1(h)(g, x) = (gh^{-1}, x)$$

whilst $\pi - 2$ is the quotient morphism of the action of $L^{\geq 0}G$ on $LG \times \mathrm{Gr}$ deinfed by

$$\alpha_2(h)(g, x) = (gh^{-1}, hx)$$

For $\lambda_1, \lambda_2 \in X_{\bullet,+}$ let

$$\overline{\mathrm{Gr}^{\lambda_1}} \,\overline{\times}\, \overline{\mathrm{Gr}_2^{\lambda}}$$

be the quotinet of $\pi_1^{-1}(\overline{\mathrm{Gr}^{\lambda_1}} \times \overline{\mathrm{Gr}_2^{\lambda}})$ by $\alpha_2(L^{\geq 0}G)$. The existsence of this question is guaranteed by the local triviality of the moprhism $LG \to \mathrm{Gr}$. More precisely, as the open s of $\overline{\mathrm{Gr}^{\lambda}}$, of the form

$$gL^{<0}Ge_0 \cap \overline{\mathrm{Gr}^{\lambda_1}}$$

the schemes

$$\overline{\mathrm{Gr}^{\lambda_1}} \,\overline{\times}\, \overline{\mathrm{Gr}^{\lambda_2}}$$

and

$$\overline{\mathrm{Gr}^{\lambda_1}} \times \overline{\mathrm{Gr}^{\lambda^2}}$$

are isomorphic. Further, these isomoprhisms are clearly compatible with the stratification of $\overline{\mathrm{Gr}_1^{\lambda}} \times \overline{\mathrm{Gr}^{\lambda_2}}$ by the locally closed subsets $\mathrm{Gr}^{\lambda_1'} \times \mathrm{Gr}^{\lambda_2'}$. The projection on second factor defines a moprhism

$$m : \overline{\mathrm{Gr}^{\lambda_1}} \,\overline{\times}\, \overline{\mathrm{Gr}^{\lambda_2}} \to \overline{\mathrm{Gr}^{\lambda_1 + \lambda_2}}$$

3.0.1. *Some remarks on the twisted products.* Whenever we have

3.0.2. *Examples of parabolics.* Let $\lambda = (\lambda_1, \lambda_2)$. Generating from roots. For a root α, we can construct

$$\langle B, M_\alpha \rangle$$

where $M_\alpha := Z(T_\alpha), T_\alpha := \ker(T \xrightarrow{\alpha} \mathbb{G}_m)$.

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[9]The terminology is unclear here. Should edit.

Example

$G = \mathrm{GL}_n$. Let $\lambda = \left(\lambda_1 = \cdots \lambda_{m_1} > \cdots > \lambda_{m_{k-1}+1} = \cdots = \lambda_{m_k}\right)$. The parabolic is of the form:

$$P_\lambda := \begin{pmatrix} \boxed{\mathrm{GL}_{m_1}} & * & * \\ & \ddots & * \\ 0 & & \boxed{\mathrm{GL}_{m_k}} \end{pmatrix}$$

We may consider $\mathrm{ev}_0^{-1}(P_\lambda)$.

Proposition 3.3. [**12**, 2.3.10]

$$\mathrm{ev}_0^{-1}(P_\lambda) \simeq L^{\geq 0}G \cap L^{\geq \lambda}G$$

PROOF. Let us consider the $\mathbb{C}$-points. It would be easy to consider the function $\tilde{\lambda}_{(-)} : \{1, \ldots, n\} \to \mathbb{Z}$ as a function given by

$$\tilde{\lambda}_x = \lambda_i \text{ if } 1 \leq x \leq \lambda_{m_i}$$

Then

$$L^{\geq 0}G(\mathbb{C}) \cap L^{\geq \lambda}G(\mathbb{C}) = \left\{ t^{\tilde{\lambda}_i - \tilde{\lambda}_j} a_{ij} \in G(\mathbb{C}[[t]]) \,:\, a_{ij} \in G(\mathbb{C}[t]]) \right\}$$

$\square$

4. Les énconcés principaux

Recall that U denotes the unipotent radical of B associated to R_+. We define LU,

$$L^{\geq 0}U := LU \cap L^{\geq 0}G, \quad L^{\leq 0}U := LU \cap L^{<0}G$$

For each $\nu \in X_\bullet(T)$ we also denote

$$L^{\geq \nu}U := \varpi^\nu L^{\geq 0}U \varpi^{-\nu}, L^{<\nu}U := \varpi^\nu L^{<0}U \varpi^{-\nu}$$

Example

$G = \mathrm{GL}_2.$ $\lambda := (1,0) \in X_{\bullet,+}.$ Then

$$L^{\geq \lambda}U = \begin{pmatrix} 1 & tk[[t]] \\ & 1 \end{pmatrix}, L^{<\lambda}U = \begin{pmatrix} 1 & t(1/t)k[1/t] \\ & 1 \end{pmatrix}$$

For each $\nu \in X_\bullet$, $L^{<\nu}U$ is a closed subgroup of $L^{<\nu}G$ so we can define $L^{<\nu}Ue_\nu$ as a closed subset of the open set $\varpi^\nu L^{<0}Ge_0$. In particular for all $\lambda \in X_{\bullet,+}$ and $\nu \in X_\bullet$, $S_\nu \cap \overline{\mathrm{Gr}}_\lambda$ is a locally closed subscheme, possibly empty, of $\overline{\mathrm{Gr}}_\lambda$. By the Iwaswa decomposition, this yields a stratification of $\overline{\mathrm{Gr}}_\lambda$. We will give a new proof of the following theorem due to Mirkovic and Vilonen in the case $k = \mathbb{C}$, [**9**].

Theorem 4.1. For each $\lambda \in X_{\bullet,+}$, and $\nu \in X_\bullet$ the complex $R\Gamma_c(S_\nu, \mathcal{A}_\lambda)$ is concentrated in degree $2\langle\rho,\nu\rangle$. Further, the endomoprhism Fr_q acts on $H_c^{2\langle\rho,\nu\rangle}(S_\nu, \mathcal{A}_\lambda)$ as $q^{\langle\rho,\nu\rangle}$.

In the previous statement we wrote $R\Gamma_c(S_\nu, \mathcal{A}_\lambda)$ instead of

$$R\Gamma_c((S_\nu \cap \overline{\mathrm{Gr}^\lambda}) \otimes_k \bar{k}, \mathcal{A}_\lambda)$$

for simplicity. We use this notation systematically in the following and does not cause any ambiguity.
For each $\nu \in X_{\bullet,+}, \nu' \in X_\bullet$, choose a total orer of the positive roots and we have an isomoprhism

$$\prod_{\alpha \in R_+} \prod_{\langle\alpha,\nu'\rangle \leq i < \langle\alpha,\nu\rangle} U_{\alpha,i} = L^{<\nu}U \cap L^{\geq\nu'}U$$

For ν fixed ν' more and more anitdominant , this group forms an inductive system for the limit $L^\nu U$.

Example

Use $G = \mathrm{GL}_2$, $\nu_1 = (1,0)$. Let $\nu_n' := -(n,-n)$, then

$$L^{\geq\nu'}U = \begin{pmatrix} 1 & t^{-2n}k[[t]] \\ & 1 \end{pmatrix}$$

It is then clear that

$$L^{<\nu} = \varinjlim L^{<\nu}U \cap L^{\geq\nu_n'}U$$

For each simple root $\alpha \in \Delta$, denote $u_{\alpha,i}$ the projection over the factor $U_{\alpha,i}$ and

$$h : L^{<\nu}U \cap L^{\geq\nu'}U \to \mathbb{G}_a$$

$$h(x) := \sum_{\alpha \in \Delta} u_{\alpha,-1}(x)$$

Fix a nontrivial additibev character, $\psi : k \to \bar{\mathbb{Q}}_l^\times$, and denote $\mathcal{L}_\psi$ the Artin-Schreier sheaf over $\mathbb{G}_a$ associated to ψ. The character $\theta : U(F) \to \bar{\mathbb{Q}}_l$ considered in introduction is the character $x \mapsto \psi(h(x))$. The following statement was a conjecture of [5]

Theorem 4.2. For $\nu \neq \lambda$ in $X_{\bullet,+}$ the complex $R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi)$ is zero. For $\nu = \lambda$ the complex is isomorphic to $\bar{\mathbb{Q}}_l$ provided with the action of Frobenius by $q^{\langle\rho,\lambda\rangle}$, at degree $2\langle\rho,\lambda\rangle$.

5. L'action du tore T

The torus T normalizes these subgroups $L^{\geq 0}G, L^{<0}G, L^{<\nu}G, \ldots$ of LG so that it acts on all the geometric objects we considered. This action provides a valuable

tool to study their geometry. Choose once and for all a strictly dominant cocharacter $\phi : \mathbb{G}_m \to T$. The $\mathbb{G}_m$ action we consider follows from the following compositions

$$\mathbb{G}_m \hookrightarrow L^{\geq 0}\mathbb{G}_m \xrightarrow{L^{\geq 0}\phi} L^{\geq 0}G \circlearrowright \mathrm{Gr}$$

Proposition 5.1. For all $\nu \in X_\bullet$ the point e_ν is the fixed point of the action $\mathbb{G}_m \circlearrowright S_\nu$. Furthermore, it is the attractive fixed point.

PROOF. For all $x \in L^{<\nu}U(\bar{k})$ is of the form

$$x = \prod_{\alpha \in \Phi_+} \prod_{i < \langle \alpha, \nu \rangle} U_{\alpha,i}(x_{\alpha,i})$$

where $x_{\alpha,i} \in \bar{k}$ are zero for all but a finite number. Thus, for all $z \in \bar{k}^\times$, we have

$$\phi(z)xe_\nu = \prod_{\alpha \in \Phi_+} \prod_{i < \langle \alpha, \nu \rangle} U_{\alpha,i}(z^{\langle \alpha,i \rangle} x_{\alpha,i})e_\nu$$

$\square$

This lemma shows that e_ν are the only fixed points of the action $\mathbb{G}_m \circlearrowright \mathrm{Gr}$. Further, it implies following statement

Proposition 5.2. If the intersection $S_\nu \cap \overline{\mathrm{Gr}^\lambda}$ is nonempty, ν belongs $\Omega(\lambda)$.

PROOF. $\square$

Proposition 5.3. The Euler-Poincaré characteristic $\chi_c(S_\nu \cap \mathcal{Q}_\lambda)$ is equal to 1 if ν is conjugate to λ by an element of W and 0 otherwise.

This statement can be considered as a geometric interpretation of result of Lusztig, [8, 6.1]. Let us use the notation of introduction. Let c_λ be the element of hecke algebra $\mathcal{H}$ defined

$$c_\lambda = (-1)^{2\langle \rho, \lambda \rangle} q^{-\langle \rho, \lambda \rangle} 1_\lambda$$

where 1_λ is the characteristic function of $K\varpi^\lambda K$. We know that

$$(c_\lambda) = (K_{\lambda,\mu}(q))^{-1}(A_\lambda)$$

where $K_{\lambda,\mu}(q)$ is the triangular matrices formed the Kazhdan-Lusztig polynomials. The constant terms of the normalizing constants

$$(-1)^{2\langle \rho, \nu \rangle} q^{-\langle \rho, \nu \rangle} \int_{U(F)} c_\lambda(x\varpi^\mu) \, dx$$

6. Les intersections $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$

For all $\lambda \in X_{\bullet,+}$ we considered

$$J^\lambda = \prod_{\alpha \in \Phi_+} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{\alpha,i}$$

which is clearly a subgroup of $L^{\geq 0}U$. We also prove that the morphism $J^\lambda \to \overline{\mathrm{Gr}^\lambda}$ is an open immersion. A distinct argument of the content of this section is given in [**3**, 5.2].

Proposition 6.1. Let $\lambda \in X_{\bullet,+}$ induces an isomorphism of J^λ with the open subset $\varpi^\lambda L^{<0} Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$ of $\overline{\mathrm{Gr}^\lambda}$.

PROOF. The image of J^λ is contained in $\varpi^\lambda L^{<0} Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$. By 3.2, it is thus a dense open subset of $\varpi^\lambda L^{<0} Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$. $\qquad\square$

Proposition 6.2. Let $\lambda \in X_{\bullet,+}$ for $w \in W$ the moprhism

$$wJ^\lambda w^{-1} \cap LU \to S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$$

defined by

$$j \mapsto je_{w\lambda}$$

is an isomorphism. As a consequence $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is isomorphic to an affine space of dimension $\langle \rho, \lambda + w\lambda \rangle$

PROOF. For $w = 1$, the result follows from the 6.1 due to the following inclusion [10]

$$J^\lambda e_\lambda \subset S^\lambda \cap \overline{\mathrm{Gr}^\lambda} \subset \varpi^\lambda L^{<0} Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$$

$\qquad\square$

We can deduce 4.1 in the case $\nu = w\lambda$ and 4.2 in the case $\nu = \lambda$. Indeed the inclusion

$$wJ^\lambda w^{-1} \cap LU \hookrightarrow L^{\geq 0}U \hookrightarrow L^{\geq 0}G$$

implies that $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is contained in the open orbit Gr^λ. Thus the restriction of $\mathcal{A}_\lambda$ to $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is equal to:

$$\mathcal{A}_\lambda\Big|_{S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}} = \bar{\mathbb{Q}}_l[\langle \rho, 2\lambda \rangle](\langle \rho, \lambda \rangle)$$

The inclusion $J^\lambda \subset L^{\geq 0}U$ implies that the restriction of h to J^λ is zero. Then 4.2 is true in the case $\nu = \lambda$.

The more general statement below will be needed later. For each $\sigma \in X_{\bullet,+}$ denote

(1) $$h_\sigma : LU \to \mathbb{G}_a$$

the morphism

$$had(\sigma) : x \mapsto h(\varpi^\sigma x \varpi^{-\sigma})$$

and also the induced homomorphism $h_\sigma : S_\lambda \to \mathbb{G}_a$. Since σ is dominant, the restriction of h_σ to $L^{\geq 0}U$, and a fortiori to J^λ is zero. We thus also have the following

Proposition 6.3. For all $\lambda, \sigma \in X_{\bullet,+}$ we have

$$R\Gamma_c(S_\lambda, \mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_l[-2\langle \rho, \lambda \rangle](-\langle \rho, \lambda \rangle)$$

_____

[10] If $\lambda = (1,0)$, $J^\lambda(k) = \begin{pmatrix} 1 + t\mathcal{O} & \mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \frac{1}{t} k[1/t] & t \cdot \frac{1}{t} k[1/t] \\ \frac{1}{t} \cdot \frac{1}{t} k[1/t] & k[1/t] \end{pmatrix}$. For sake

7. Minuscules

We utilized the notations fixed in 2. Let μ be nonzero minimal [11] element of $X_{\bullet,+}$. By 2.1, we have the following statement

Proposition 7.1. Let μ be minuscule. We have $\Omega(\mu) = W\mu$. For $\alpha \in R$, we have

$$\langle \alpha, \mu \rangle \in \{0, \pm 1\}$$

For example, in the case of GL_n the minuscule ones are precisely those of the form

$$(l+1, l+1, , \ldots, l+1, l, \ldots, l) \quad l \in \mathbb{Z}$$

If μ is minuscule, by minimality, this implies the orbit Gr^μ is closed. Since for all elements ν of $\Omega(\mu)$ is conjugate to μ by anaction of W for 4.1, 4.2 it suffices to verify for the case $\lambda = \mu$ and $\nu \in \Omega(\mu)$.

Proposition 7.2. We have a canonical isomorphism $\mathrm{Gr}^\mu \to G/P$ st.

$$S^{w\mu} \cap \mathrm{Gr}^\mu \simeq UwP/P$$

Proof. Given **??** and the two assertions of 7.1, we have that $L^{\geq 0}G \cap L^{\geq \mu}G$ is the inverse image of of P under the homomorphism $\mathrm{ev}_0 : L^{\geq 0}G \to G$. For example, see 3.3.

$$\mathrm{Gr}^\mu = L^{\geq 0}G/(L^{\geq 0}G \cap L^{\geq \mu}G) \simeq G/P$$

Given, again, 7.1 we knw that $J^\mu = U_\mu^+ = \prod_{\langle \alpha, \mu \rangle = 1} U_\alpha$, which is the unipotent subgroup of the opposite parabolic of P. As a consequence

$$wJ^\mu w^{-1} \cap LU = wU_\mu^+ w^{-1} \cap U$$

The second assersion follow from 6.2. ☐

8. Quasi-minuscules:étude géométrique

See also exercise of Zhu. Let μ is a quasi-minuscule weight, i.e. a minimal element of $X_{\bullet,+} \setminus \{0\}$, smaller than 0. Recall, that by 2.1 we have

Proposition 8.1. Let μ be quasiminuscule. Then μ is equal to a cocharacter $\gamma^\vee$ associated to a positive maximal root γ.[12] We have $\Omega(\mu) = W\mu \cup \{0\}$. For each rot $\alpha \in \Phi \setminus \{\pm\gamma\}$ we have $\langle \alpha, \mu \rangle \in \{0, \pm 1\}$.

Since 0 is a dominant cocharactere which is smaller smaller than μ, $\overline{\mathrm{Gr}^\mu}$ is the union of Gr^μ and the base point e_0. Denote by P the parabolic subgroup of G generated by T and the subgroup of radical roots U_α such that $\langle \alpha, \gamma^\vee \rangle \leq 0$. Denote

$$V := \mathfrak{h} \oplus \bigoplus_{\alpha \in R \setminus \{\gamma\}} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Lie algebra of T and where $\mathfrak{g}_\alpha$ are the subspaces of weight α of $\mathfrak{g}$.

---

[11] why was this necessary again?

[12] To have an example, consider the root $(1, -1)$.

9. Quasi-minuscules:étude cohomologique

The notation are as the 8. In particular $\mu = \gamma^\vee$ is quasi-minuscule. The resolution

$$\pi_\gamma : \mathbb{P}_\gamma \to \overline{\mathrm{Gr}^\mu}$$

allows us to compute the local intersection cohomology of A_ν at an isolated singularity e_0. The following statement is due to Kazhdan and Lusztig.
Indeed, in the following situation, the hypothesis is much weaker, and their argument applies. We detail the proof for the convenience of the reader.

Proposition 9.1. Let $d = \langle 2\rho, \mu \rangle$ the dimension $\overline{\mathrm{Gr}^\mu}$. For $i \geq 0$, the group $H^i(\mathcal{A}_\mu)_{e_0}$ is trivial. For $i < 0$, we have the short exact sequence

(2) $\qquad 0 \to H^{i+d-2}(G/P)(d/2-1) \to H^{i+d}(G/P)(d/2) \to H^i(\mathcal{A}_\mu)_{e_0} \to 0$

PROOF. Let $\overline{\mathrm{Gr}_\mu}'$ be th eopen of $\overline{\mathrm{Gr}^\mu}$

$$\overline{\mathrm{Gr}^\mu}' :=$$

we have $\pi_\gamma^{-1}(\overline{\mathrm{Gr}^\mu}') = \mathbb{L}_{-\gamma}$. Denote $\mathcal{A}'_\mu$ the restriction of $\mathcal{A}_\mu$ to this open. Denote the inclusion of the closed point $i : \{e_0\} \to \overline{\mathcal{A}}'_\mu$. The natural morphism

$$\mathcal{A}'_\mu \to i_* i^* \mathcal{A}'_\mu$$

induces a restriction of morphism of cohomology (without support())

$$i^* : R\Gamma(\overline{\mathrm{Gr}^\mu}', \mathcal{A}'_\mu) \to (\mathcal{A}'_\mu)_{e_0}$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Proposition 9.2. Let $\mathcal{C}$ be the factor supported by e_0 in the decomposition

$$R\pi_{\gamma*}\bar{\mathbb{Q}}_l[d](d/2) = \mathcal{A}_\mu \oplus \mathcal{C}$$

For $i < 0$, we have

$$H^i(\mathcal{C}) = H^{i+d-2}(G/P)(d/2-1)$$

For $i \geq 0$ we have

$$H^i(\mathcal{C}) = H^{i+d}(G/P)(d/2)$$

Proposition 9.3. We have isomorphisms

$$R\Gamma_c(S_0, \mathcal{A}_\mu) \simeq \bar{\mathbb{Q}}_l^{|\Delta_\gamma|}$$

PROOF. By the theorem for base change of proper morphism, we have

$$R\Gamma_c(\pi_\gamma^{-1}(S_0 \cap \overline{\mathrm{Gr}^\mu}, \bar{\mathbb{Q}}_l)[d](d/2)$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let us now prove statemet 4.2 in the case $\nu = 0$ and $\lambda = \mu$ quasi-minuscule. We actually prove something more general. Recall that for each $\sigma \in X_\bullet$, we defined a morphism $h_\sigma : S_0 \to \mathbb{G}_a$ see Eq. 1.

Proposition 9.4. For each $\sigma \in X_{\bullet,+}$ we have the isomoprhism

$$R\Gamma_c(S_0, \mathcal{A}_\mu \otimes h_\sigma^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_l^{|\Delta_\gamma^\sigma|}$$

where Δ_γ^σ is the set of $\alpha \in \Delta_\gamma$ such that $\langle \alpha, \sigma \rangle > 0$.

The proof of 9.4 is the same as 9.3

9.1. Recollection of the work of Kazhdan Lusztig. We refer to [**13**] for a nice introduction. Recall we have the *Bruaht decomposition:*

$$G = \bigsqcup_W B\dot{w}B$$

arising fom the action

$$B \times B \circlearrowright G$$

Example: SL_n. Quotients $X = G/B$ are those referred to as *flag varieties*. Again, similar to affine Grassmanian, one has a $T \circlearrowright X$.

- $X_w := \mathrm{im}(B\dot{w}B \to G/B)$. These are the B orbits on X.
- The *Schubert varieties* are $S_w := \overline{X_w} \simeq (X_v)_{v \leq w}$.

Now we can construct another action

$$G \circlearrowright X \times X$$

- The orbits are $\mathcal{O}_w$.

10. Convolution

Recall that M is the set of elements in $X_{\bullet,+}\backslash 0$.

Recall that M is the minimal cocahracters in $X_{\bullet,+}$. For each $\mu_\bullet = (\mu_1, \ldots, \mu_n)$ of elements in M, we can construct the projective scheme

$$\overline{\mathrm{Gr}^{\mu_\bullet}} = \overline{\mathrm{Gr}^{\mu_1}} \tilde{\times} \cdots \tilde{\times} \overline{\mathrm{Gr}^{\mu_n}}$$

The projection of the lass factors of Gr^n defines a proper morphism

$$\overline{\mathrm{Gr}^{\mu_\bullet}} \xrightarrow{m_{\mu_\bullet}} \overline{\mathrm{Gr}^{|\mu_\bullet|}}$$

11. Fin des démonstrations

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