Representations of p-adic groups for the modularity seminar.

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Abstract

These are lecture notes, for a "modularity seminar", and I make no claim to originality. I have attempted to give references, but these references do not necessarily reflect the history (I might reference one source for a proof of a theorem, when the theorem was first proven by another). Please send corrections to Marty Weissman at `weissman.marty@gmail.com`.

1 *Notation*

k will always denote a nonarchimedean local field. It will not hurt to assume that $k = \mathbb{Q}_p$.

The valuation on k will be normalized in such a way that $val(k^\times) = \mathbb{Z}$.

$\mathcal{O}$ will always denote the valuation ring of k.

The letter ϖ will always denote a uniformizing element of k, i.e., $val(\varpi) = 1$.

We write $\mathbb{F}_q$ for the residue field of k: $\mathbb{F}_q = \mathcal{O}/\varpi\mathcal{O}$. Here $q = p^f$ for some positive integer f and some prime number p.

We use boldface letters, like $\mathbf{X}$ to denote varieties over k. We use ordinary letters, like X, to denote the k-points of such varieties (with their natural topology).

We often use the language of categories, functors, and natural transformations. In these notes, we typically define functors only half-way: we describe a functor on objects, and leave it to the reader to determine the functor on morphisms when we say something like "For every object X, $F(X)$ is... F extends to a functor from..."'

2 *ℓ-spaces and groups*

Definition 2.1 (Bernstein) *An ℓ-space[1] is a locally compact Haus-*

[1] J. Bernstein. *Represenations of p-adic groups*. Harvard University, 1992. Lectures by Joseph Bernstein. Written by Karl E. Rumelhart.

dorff topological space, in which every point has a basis of open compact neighborhoods. Let $\mathfrak{Sp}_\ell$ be the category of ℓ-spaces and continuous maps.

When X is an ℓ-space, the space of "*smooth*" functions on X is defined to be:

Of course, there is nothing special about $\mathbb{C}$ here – its topology is not being used. Everything we discuss will go through, as long as $\mathbb{C}$ denotes an uncountable, algebraically closed field of characteristic zero.

$$C^\infty(X) = \{f : X \to \mathbb{C} \colon f \text{ is locally constant}\}.$$

The subspace $C_c^\infty(X)$ consists of *compactly supported* smooth functions.

Proposition 2.2 *Let X be an ℓ-space, and U an open subset of X with complement $Z = X - U$. Then the linear maps "extension by zero" and "restriction to Z" yield a short exact sequence of complex vector spaces:*

$$0 \to C_c^\infty(U) \to C_c^\infty(X) \to C_c^\infty(Z) \to 0.$$

Example 2.3 *Let $X = k$, where k is a nonarchimedean local field. Let $U = k^\times$ be the open subset of nonzero elements. Then "extension by zero" and "evaluation at zero" yield a short exact sequence of complex vector spaces:*

$$0 \to C_c^\infty(k^\times) \to C_c^\infty(k) \to \mathbb{C} \to 0.$$

Compare and contrast this with the archimedean case – there one should work with Schwarz functions, where one finds that "extension by zero" and "Taylor expansion at 0" yield a short exact sequence of complex vector spaces:[2]

$$0 \to S(\mathbb{R}^\times) \to S(\mathbb{R}) \to \mathbb{C}[[T]] \to 0.$$

[2] Émile Borel. *Sur quelques points de la théorie des fonctions*. Paris., 1894. Original from Columbia University.

The following fact is discussed properly in Chapter 3.1 of Platonov and Rapinchuk[3]:

[3] Vladimir Platonov and Andrei Rapinchuk. *Algebraic groups and number theory*, volume 139 of *Pure and Applied Mathematics*. Academic Press Inc., Boston, MA, 1994. Translated from the 1991 Russian original by Rachel Rowen.

Fact 2.4 *Let $\mathbf{X}$ be an algebraic variety over a nonarchimedean local field k. There is a "natural" topology on $X = \mathbf{X}(k)$ for which X is an ℓ-space. In other words, there is a functor from the category of varieties over k (and regular maps) to the category of ℓ-spaces (and continuous maps), which equals the functor of k-points after forgetting the topology.*

In particular, $\mathbf{GL}_n(k)$ is an ℓ-space, $\mathbf{P}^1(k)$ is an ℓ-space, etc..

In fact, this functor can be uniquely characterized by just a few properties; in unpublished notes[4], Brian Conrad proves:

[4] Brian Conrad. Weil and Grothendieck approaches to adelic points. Unpublished notes, available online.

Theorem 2.5 *Let R be a topological ring. There is a unique functor $\mathbf{X} \mapsto \mathbf{X}(R)$ from the category of affine finite-type R-schemes to the category of topological spaces, such that*

1. *Forgetting the topology yields the functor of R-points.*
2. *The functor is compatible with the formation of fibre products.*

3. *The functor carries closed immersions to topological embeddings.*
4. *The functor applied to $\mathbf{X} = Spec(R[T])$ yields the given topology on $\mathbf{X}(R) = R$.*

Furthermore, when R is Hausdorff, closed immersions of schemes yield closed embeddings of topological spaces and when R is locally compact, $\mathbf{X}(R)$ is locally compact for all $\mathbf{X}$.

It should be noted that Conrad extends this further, removing the affine hypothesis under the hypothesis that $R^\times$ is open in R, and inversion is continuous on $R^\times$ – these conditions are satisfied when R is a local field.[5]

[5] The situation is more subtle when R is the ring of adeles for a global field; such a ring is locally compact and Hausdorff, but $R^\times$ is no longer open in R.

Definition 2.6 *An ℓ-*group *is a group in the category of ℓ-spaces. In other words, an ℓ-group is a group G, endowed with a topology for which G is an ℓ-space and the unit, inverse, and composition maps:*

$$pt \to G, \quad G \to G, \quad G \times G \to G$$

are continuous.

Proposition 2.7 *Let G be a topological group. Then G is an ℓ-group[6] if and only if the identity element has a basis of neighborhoods consisting of open compact subgroups of G.*

[6] This is given by some authors as the definition of an ℓ-group. I find it more natural to think about groups in a category and prove the equivalence.

Proof: If G has a neighborhood basis around the identity consisting of open compact subgroups, then translation of these open compact subgroups gives a neighborhood basis around any point in G. It follows quickly that G is an ℓ-space.

Conversely, if G is an ℓ-space and a topological group, then there is a neighborhood basis of the identity consisting of open compact *subsets* of G. Let V be such a compact open subset containing the identity of G. Define

$$K_V = \{x \in G\colon xV \subset V \text{ and } x^{-1}V \subset V\}.$$

Then K_V is a subgroup of G, and a subset of V. It is the intersection of compact sets, hence compact. The proof that K_V is open is a bit tricky, and we refer to the notes of Paul Garrett [7].

[7] P. Garrett. Smooth representations of totally disconnected groups. Introductory notes, available online. Updated July 8, 2005.

Q.E.D

Corollary 2.8 *If $\mathbf{G}$ is an algebraic group over a nonarchimedean local field k, then $G = \mathbf{G}(k)$ is naturally a ℓ-group.*

Here are a few examples of ℓ-groups arising as $\mathbf{G}(k)$, and open compact neighborhoods of the identity.

Example 2.9 *Let $\mathbf{G}_a$ denote the additive group over k. Thus $G_a = (k, +)$ is the additive group of the field k. Let $val : k^\times \to \mathbb{Z}$ denote the valuation on k, normalized to be surjective. For $m \in \mathbb{Z}$, define a compact open subgroup of G_a:*

$$K_m = \{x \in k \colon val(x) \geq m\}.$$

Then

$$G_a = \bigcup_{m \in \mathbb{Z}} K_m, \quad \{0\} = \bigcap_{m \in \mathbb{Z}} K_m.$$

Note above that the additive group G_a is the union of its compact open subgroups. This is not typical, for ℓ-groups. But it does hold for groups $G = \mathbf{G}(k)$, whenever $\mathbf{G}$ is a unipotent group over a p-adic field k. This plays a very important role for harmonic analysis on unipotent p-adic groups.

Example 2.10 *Let $\mathbf{G}_m$ denote the multiplicative group over k. Thus $G_m = k^\times$ is the multiplicative group of the field k. A choice of uniformizing element $\varpi \in k^\times$ determines a decomposition of topological groups:*

$$k^\times \cong \mathcal{O}^\times \times \mathbb{Z}.$$

The compact open subgroups

$$U_m = \{x \in k^\times \colon val(x - 1) \geq m\},$$

for $m \geq 1$, form a neighborhood basis at the identity in $k^\times$.

Of course, $\mathbf{G}_m = \mathbf{GL}_1$, and the above example generalizes to $\mathbf{GL}_n$ without much difficulty.

Example 2.11 *Let $\mathbf{GL}_n$ be the algebraic group of n by n invertible matrices. Let ϖ be a uniformizing element of k. A neighborhood basis of the identity in $GL_n = \mathbf{GL}_n(k)$, consisting of compact open subgroups, is given by:*

$$K_m = \{g \in GL_n(\mathcal{O}_k) \colon g \equiv 1, \text{ modulo } \varpi^n \mathcal{O}_k\}.$$

3 Representations

Smooth representations

Let G be an ℓ-group. Nothing will really be lost if one takes $G = \mathbf{GL}_n(\mathbb{Q}_p)$ in what follows.

Definition 3.1 *A* representation *of G is a pair (π, V), where V is a complex vector space (often infinite-dimensional!) and $\pi : G \to Aut_{\mathbb{C}}(V)$ is an action of G on V by $\mathbb{C}$-linear automorphisms. Let $\mathfrak{Rep}_G$ be the category of representations of G and G-intertwining $\mathbb{C}$-linear maps.*

Let $Op(G)$ be the set of open subgroups of G – recall that $Op(G)$ is a basis of neighborhoods of the identity in G. For any subgroup $H \subset G$, and any representation (π, V) of G, we write V^H for the H-invariant subspace of V. We write V_H for the H co-invariant quotient[8] of V, i.e.,

$$V_H = {}^{V}/_{[H-1]V}, \quad [H-1]V = Span_{\mathbb{C}}\{\pi(h)v - v\}_{v \in V, h \in H}.$$

[8] Let W be a vector space upon which H acts trivially. Then every H-intertwining map from W to V factors uniquely through V^H. Every H-intertwining map from V to W factors uniquely through V_H.

Definition 3.2 *When (π, V) is a representation of G, the subspace V^∞ of smooth vectors is defined by:*

$$V^\infty = \bigcup_{H \in Op(G)} V^H.$$

A representation (π, V) of G is called smooth *if $V = V^\infty$. Let $\mathfrak{Rep}_G^\infty$ denote the category*[9] *of smooth representations of G and G-intertwining $\mathbb{C}$-linear maps.*

[9] The category $\mathfrak{Rep}_G^\infty$ is an abelian category with enough injectives and arbitrary direct sums.

Proposition 3.3 *If (π, V) is a representation of G, then (π, V^∞) is a subrepresentation*[10] *of (π, V), and (π, V^∞) is smooth. This defines a functor from $\mathfrak{Rep}_G$ to $\mathfrak{Rep}_G^\infty$. If (σ, W) is any smooth representation of G, and $\phi : W \to V$ is a morphism in $\mathfrak{Rep}_G$, then ϕ factors uniquely through the inclusion $V^\infty \hookrightarrow V$.*

[10] A subrepresentation of (π, V) is just a G-stable subspace.

Proof: The proof is not difficult, and is left to the reader.

Q.E.D

The category $\mathfrak{Rep}_G^\infty$ is usually not semisimple. However, for compact groups the category is semisimple and we discuss this a bit further.

Let K be a compact ℓ-group. Let $\hat{K}$ be a set of representatives for the isomorphism classes of irreducible smooth representations (abbreviated *irrep* hereafter) of K – in other words, if τ is an irrep of K then there exists a unique $\rho \in \hat{K}$ such that $\tau \cong \rho$.

Lemma 3.4 *Every irrep τ of K is finite-dimensional and factors through a finite quotient of K.*

Proof: Let (τ, W) be an irrep of K, and let w be a nonzero vector in W. Let $H \subset K$ be an open subgroup such that $w \in W^H$. By compactness of K, we find that $\#(K/H) < \infty$. Choosing representatives $k_1, \ldots, k_d$ for K/H, we find that

$$\mathrm{Span}_{\mathbb{C}}\{\tau(k)w\}_{k \in K} = \mathrm{Span}_{\mathbb{C}}\{\tau(k_i)w\}_{1 \le i \le d}.$$

By irreducibility, the left side is all of W. The right side is finite-dimensional, and so $dim(W) \le d = \#(K/H)$.

Now, each vector $\tau(k_i)w$ is fixed by the open subgroup $k_i H k_i^{-1}$. Hence we find that

$$\tau(k_i)w \in W^N, \text{ where } N = \bigcap_{i=1}^{d} k_i H k_i^{-1}.$$

Observe that N is an open normal subgroup of K, so K/N is a finite quotient of K, and τ factors through this quotient.

Q.E.D

Definition 3.5 *Let (π, V) be a smooth representation of K, and $(\tau, W) \in \hat{K}$. The τ-isotypic subrepresentation of V is the image V_τ of the natural injective K-intertwining operator:*

$$W \otimes_{\mathbb{C}} Hom_K(W, V) \to V.$$

The τ-isotypic subrepresentations of a smooth representation (π, V) of K are certainly semisimple – they are isomorphic to a direct sum of copies of τ.

Theorem 3.6 *Let (π, V) be a smooth representation of K. Then the inclusions of isotypic subrepresentations yield an isomorphism*

$$\bigoplus_{\tau \in \hat{K}} V_\tau \cong V.$$

Proof: Schur's orthogonality (for finite groups) implies that the distinct isotypic subrepresentations of V have zero intersection. Thus it remains to prove that every vector $v \in V$ can be expressed as a finite sum of vectors in isotypic subrepresentations.

But if $v \in V$, then $v \in V^H$ for some open subgroup $H \subset K$. With the techniques of the previous lemma, we find that $v \in V^N$ for some open normal subgroup $N \subset H \subset K$. Let $W \subset V$ be the smallest subrepresentation of K containing v. We find that W is finite-dimensional, and the representation of K on W factors through the quotient K/N.

From the complete decomposability of representations of finite groups, we find that W decomposes into a finite number of K/N-isotypic components. Pulling back, we find that W decomposes into a finite number of K-isotypic components. In particular, v can be expressed as a finite sum of vectors from isotypic subrepresentations of V.

Q.E.D

It is important to contrast the case of compact ℓ-groups (which are really no more difficult than finite groups) with noncompact ℓ-groups. The simplest example of a noncompact ℓ-group is $\mathbb{Z}$ –

every representation of $\mathbb{Z}$ is smooth. The category of representations of $\mathbb{Z}$ is isomorphic to the category of $\mathbb{C}[T^{\pm 1}]$-modules, transferring the action of $n \in \mathbb{Z}$ to the action of $T^n \in \mathbb{C}[T^{\pm 1}]$.

There are plenty of examples of non-semisimple representations of $\mathbb{Z}$; one may take $(\pi, \mathbb{C}^2)$ for example, where

$$\pi(1) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

There is a short exact sequence of $\mathbb{Z}$-representations:

$$0 \to \mathbb{C} \to (\pi, \mathbb{C}^2) \to \mathbb{C} \to 0,$$

where we write $\mathbb{C}$ here for the trivial representation. This is essentially the best we can do for "decomposing" the representation π into irreducibles.

One might also consider an infinite-dimensional representation, like the space $V = C_c^\infty(\mathbb{Z})$ of compactly (finitely) supported $\mathbb{C}$-valued functions on $\mathbb{Z}$, on which $\mathbb{Z}$ acts by translation π:

$$[\pi(n) f](x) = f(x+n).$$

Then (π, V) has *no irreducible subrepresentation*, though it has infinitely many irreducible quotients. Indeed, summation yields a trivial irreducible quotient

$$\Sigma : V \to \mathbb{C}, \quad \Sigma(f) = \sum_{n \in \mathbb{Z}} f(n).$$

In fact, one can show

Theorem 3.7 *Let (π, V) be a representation of $\mathbb{Z}$. If V is finitely-generated as a $\mathbb{C}[T^{\pm 1}]$-module, then there exists an irreducible quotient of V.*

PROOF: Consider V as a $\mathbb{C}[T^{\pm 1}]$-module. Every irreducible representation of $\mathbb{Z}$ is a character (one-dimensional) $\chi_z : \mathbb{Z} \to \mathbb{C}^\times$ (this will follow from Schur's lemma, proven a bit later), for some $z \in \mathbb{C}^\times$, where we define

$$\chi_z(n) = z^n.$$

If $Hom(V, \chi_z) = 0$, then we find that $V / m_z V = 0$ for every maximal ideal $m_z = \langle T - z \rangle$ of $\mathbb{C}[T^{\pm 1}]$. From Nakayama's lemma, it follows that $V = 0$.

Q.E.D

Thus the moral is: smooth representations of noncompact groups often do not have irreducible subrepresentations; but usually (assuming a finite-type hypothesis) have irreducible quotients. Another example of this phenomenon is given by the following

Example 3.8 *Let $V = C_c^\infty(k)$ be the space of smooth (i.e., locally constant) compactly supported functions on k, viewed as a representation of $k = \mathbf{G}_a(k)$ by translation:*

$$[\pi(g)f](x) = f(x+g), \text{ for all } g \in k, x \in k, f \in V.$$

Then V has no irreducible subrepresentation. Indeed, we will see that all irreducible subrepresentations are characters – but if translation of a function acts as a character, the function cannot be compactly supported. However, every irreducible smooth representation of k occurs as a quotient; if $(\psi, \mathbb{C})$ is a smooth character of k then the following gives a nontrivial k-intertwining map from (π, V) to $(\psi, \mathbb{C})$:

$$f \mapsto \int_k f(x)\overline{\psi(x)}dx,$$

where we fix the Haar measure for which $\mathcal{O}$ has measure 1.

Contragredience, admissibility

When (π, V) is a smooth representation of G, the linear dual space $V' = Hom_{\mathbb{C}}(V, \mathbb{C})$ is a representation of G via:

$$[\pi'(g)\lambda)](v) = \lambda(\pi(g^{-1})v) \text{ for all } \lambda \in V', v \in V.$$

But this representation is rarely smooth:

Definition 3.9 *If (π, V) is a smooth representation of G, define $\tilde{V} = (V')^\infty$ – the space of smooth vectors in the linear dual of V. Let $\tilde{\pi}$ denote the resulting representation of G on $\tilde{V}$. The representation $(\tilde{\pi}, \tilde{V})$ is called the* contragredient *representation of (π, V). The contragredient defines a contravariant functor from $\mathfrak{Rep}_G^\infty$ to itself.*

It is very important to note that the contragredient does *not* define a duality – there is a natural transfomation of functors from the identity functor to the double-contragredient, but this is not a natural isomorphism. The contragredient functor does define a duality for *admissible representations*:

Definition 3.10 *A representation (π, V) of G is called* admissible *if it is smooth and for all $H \in Op(G)$, $dim(V^H) < \infty$.*

We may characterize admissible representations also as follows:

Proposition 3.11 *Let (π, V) be a smooth representation of G. Let K be a compact open subgroup of G. Then (π, V) is admissible if and only if for every $\tau \in \hat{K}$, the (K, τ)-isotypic component V_τ is finite-dimensional.*

Proof: Suppose first that every (K, τ)-isotypic component of V is finite-dimensional. Let H be an open subgroup of G. Let $H' = H \cap K$; then K/H' is finite. Define

$$J = \bigcap_{k \in K/H'} kH'k^{-1}.$$

Then we find that J is a normal subgroup of K, J is open compact, and $J \subset H$.

It follows that $V^H \subset V^J$ and:

$$V^J = \bigoplus_{\tau} (V_\tau)^J.$$

But there are only finitely many isomorphism classes of irreducible smooth representations of K for which $V^J \neq 0$, since there are only finitely many isomorphism classes of irreducible representations of the quotient group K/J. Hence V^J is a finite direct sum, of finite-dimensional spaces. Hence V^J is finite-dimensional, and so V^H is finite-dimensional. Hence V is admissible.

The converse is similar, and left to the reader.

Q.E.D

Proposition 3.12 *Let (π, V) be an smooth representation of G. Then (π, V) is admissible if and only if the natural homomorphism $V \to \tilde{\tilde{V}}$ is an isomorphism.*

Proof: If (π, V) is admissible, one may choose an open compact subgroup $K \subset G$, and decompose V into its isotypic components:

$$V = \bigoplus_{\tau \in \hat{K}} V_\tau.$$

The linear dual of V is then a direct product of finite-dimensional spaces:

$$V' = \prod_{\tau \in \hat{K}} Hom(V_\tau, \mathbb{C}).$$

One may check that the smooth vectors in V' are now:

$$\tilde{V} = \bigoplus_{\tau \in \hat{K}} V'_\tau.$$

It follows that $\tilde{V}$ is admissible.

The details and other converse are left to the reader.

Q.E.D

The following theorem is much deeper.

Theorem 3.13 (Jacquet) [11] *If* $\mathbf{G}$ *is a connected reductive algebraic group over a local nonarchimedean field, and* (π, V) *is an irreducible smooth representation of* $G = \mathbf{G}(k)$*, then* (π, V) *is admissible.*

[11] Hervé Jacquet. Sur les représentations des groupes réductifs *p*-adiques. *C. R. Acad. Sci. Paris Sér. A-B*, 280:Aii, A1271–A1272, 1975.

Corollary 3.14 *If* $\mathbf{G}$ *is a connected reductive algebraic group over a local nonarchimedean field, and* (π, V) *is an irreducible smooth representation of G (or a representation of finite length), then* (π, V) *is admissible and V is isomorphic to its double contragredient.*

Schur's lemma

Theorem 3.15 (Jacquet) [12] *Suppose that G has a countable basis for its topology. Let* (π, V) *be an irreducible smooth representation of G. Then the dimension of V is countable and* $End_G(V) = \mathbb{C}$*.*

[12] Hervé Jacquet. Sur les représentations des groupes réductifs *p*-adiques. *C. R. Acad. Sci. Paris Sér. A-B*, 280:Aii, A1271–A1272, 1975.

Proof: (We have followed DeBacker's notes[13]) Let $0 \neq v \in V$, and let K be a compact open subgroup of G for which $v \in V^K$. Then G/K is a countable set (since G has a countable basis for its topology) and we may choose representatives $g_1, g_2, \ldots$ for this countable set of cosets. We find that

[13] S. DeBacker. *Some notes on the representation theory of reductive p-adic groups*. Available online.

$$\mathrm{Span}_{\mathbb{C}}\{\pi(g)v\}_{g \in G} = \mathrm{Span}_{\mathbb{C}}\{\pi(g_i)v\}_{i=1,2,\ldots}.$$

The left side is a nonzero subrepresentation of V, hence equals V by irreducibility. The right side is a countable-dimensional vector space, and the first assertion is proven.

For the second assertion, consider any $e \in End_G(V)$, and a nonzero vector $v \in V$ again. The operator e is uniquely determined by $e(v)$, since $e(\pi(g)v) = \pi(g)e(v)$, and the vectors $\pi(g)v$ span V as a complex vector space.

It follows that the map $e \mapsto e(v)$ is an injective $\mathbb{C}$-linear map from $End_G(V)$ to V. Hence $End_G(V)$ has countable dimension. But since V is an irreducible representation of G, we know that $End_G(V)$ is a skew-field. Consider the (commutative) subfield:

$$\mathbb{C}(e) \subset End_G(V).$$

If $\mathbb{C} \neq \mathbb{C}(e)$ – i.e., if e is not a scalar endomorphism of V – then e must be transcendental over $\mathbb{C}$. But note that $\mathbb{C}(e)$ is uncountable-dimensional as a $\mathbb{C}$-vector space since the set

$$\{(e-c)^{-1} \colon c \in \mathbb{C}\}$$

is uncountable and $\mathbb{C}$-linearly independent. This is a contradiction.

Hence $\mathbb{C} = \mathbb{C}(e)$ – every element of $End_G(V)$ is a scalar endomorphism.

Q.E.D

This adaptation of Schur's lemma has the usual consequences:

Corollary 3.16 *If G is an abelian ℓ-group with countable basis for its topology, then every irreducible representation of G is one-dimensional.*[14]

[14] We call a one-dimensional smooth representation a *character*.

Corollary 3.17 *Let G be an ℓ-group with countable basis for its topology. Let (π, V) be an irreducible smooth representation of G. Let Z be the center of G. Then there exists a smooth character $\chi : Z \to \mathbb{C}^\times$ such that*

$$\pi(z)v = \chi(z) \cdot v \text{ for all } z \in Z, v \in V.$$

When G is an ℓ-group with countable basis for its topology, and center Z, it is often convenient to consider not the category $\mathfrak{Rep}_G^\infty$, but rather the full subcategory consisting of representations with a given central character. If $\chi : Z \to \mathbb{C}^\times$ is a character of Z, and (π, V) is any smooth representation of G, we say that (π, V) has central character χ if $\pi(z)v = \chi(z) \cdot v$ for all $z \in Z$. Of course, not all smooth representations of G have a central character (though irreps do). We define $\mathfrak{Rep}_{G,\chi}^\infty$ to be the full subcategory of $\mathfrak{Rep}_G^\infty$, whose objects are those smooth representations with central character χ.

Corollary 3.18 *If (π, V) and (σ, W) are two irreducible smooth representations of G – an ℓ-group with countable basis for its topology – then $Hom_G(V, W)$ is either zero or one-dimensional.*

Induction, Compact Induction

Our treatment of smooth induction follows Bernstein [15], to some extent. Let H be a closed subgroup of an ℓ-group G. Let (π, V) be a smooth representation of G, and let (σ, W) be a smooth representation of H. Restriction of representations is quite simple:

[15] J. Bernstein. *Represenations of p-adic groups*. Harvard University, 1992. Lectures by Joseph Bernstein. Written by Karl E. Rumelhart.

Definition 3.19 *Define*[16] *$Res_H^G \pi$ to be the restriction of π to H. This extends to a functor, Res_H^G from $\mathfrak{Rep}_G^\infty$ to $\mathfrak{Rep}_H^\infty$.*

[16] We always put the smaller group below, and larger group above, in our notation for induction and restriction.

Induction of representations, as usual, is not as simple.

Definition 3.20 *Define $\mathbb{C}[[H \backslash_\sigma G, W]]$ to be the vector space of functions $f : G \to W$ such that:*

$$f(hx) = \sigma(h)(f(x)), \text{ for all } x \in G, h \in H.$$

This is a representation of G by right translation:

$$[gf](x) = f(xg) \text{ for all } x, g \in G.$$

Define $Ind_H^G W$ to be the subspace $\mathbb{C}[[H \backslash_\sigma G, W]]^\infty$ of smooth vectors for this action. This extends to a functor, Ind_H^G from $\mathfrak{Rep}_H^\infty$ to $\mathfrak{Rep}_G^\infty$.

More concretely, an element of $Ind_H^G W$ is a function $f : G \to W$ which satisfies the following conditions:

1. $f(hx) = \sigma(h)(f(x))$ for all $x \in G$, $h \in H$.
2. There exists an open subgroup $K \subset G$ such that $f(xk) = f(x)$ for all $x \in G$. In other words, f is *uniformly*[17] *locally constant*.

[17] The uniformity is that K can be chosen independently of x.

There is an important *subfunctor* of Ind_H^G, called *compact induction*:

Definition 3.21 *Define $ind_H^G W$ to be the subspace of $Ind_H^G W$, consisting of those functions $f \in Ind_H^G W$ satisfying the additional condition:*

> *There exists a compact subset $X \subset G$ such that $f(g) = 0$ unless $g \in H \cdot X$. In other words, f is compactly supported, modulo H.*

Then $ind_H^G W$ is a G-subrepresentation of $Ind_H^G W$; it yields a subfunctor $ind_H^G \subset Ind_H^G$.

Compact induction is simpler in many ways; for example, the condition of uniform local constancy simplifies to the condition of local constancy. Of course, if $H\backslash G$ is a compact space, then the functors ind_H^G and Ind_H^G coincide. Less trivially,

Proposition 3.22 *If (σ, W) is an admissible representation of H, and $H\backslash G$ is compact, then $Ind_H^G W$ is an admissible representation of G.*

Proof: We leave the proof as an exercise. This can be found in Proposition 9 of Bernstein's notes[18] as well.

[18] J. Bernstein. *Represenations of p-adic groups*. Harvard University, 1992. Lectures by Joseph Bernstein. Written by Karl E. Rumelhart.

Q.E.D

Frobenius reciprocity can now be formulated in the smooth setting:

Theorem 3.23 *Let (π, V) be a smooth representation of G, and (σ, W) a smooth representation of H, a closed subgroup of G. Then there is a natural isomorphism:*

$$Hom_G(V, Ind_H^G W) \cong Hom_H(Res_H^G V, W).$$

This identifies Ind_H^G as a functor which is right adjoint to the functor Res_H^G. Both functors are exact.

Most typically, the functor ind_H^G of compact induction is used when H is a closed and open (clopen) subgroup of G; in this case, $H\backslash G$ is a discrete space. It follows that

Lemma 3.24 *Let H be a clopen subgroup of G. Let (σ, W) be a smooth representation of H. Then there is a natural isomorphism of representations of G:*

$$ind_H^G W \cong W \otimes_{\mathbb{C}[H]} \mathbb{C}[G].$$

From the adjointness of ring extension and pullback, we find that

Theorem 3.25 (p. 125 of Cartier) [19] *Let H be a clopen subgroup of G. Let (π, V) be a smooth representation of G, and (σ, W) a smooth representation of H, a closed subgroup of G. Then there is a natural isomorphism:*

$$Hom_G(ind_H^G W, V) \cong Hom_H(W, Res_H^G V,).$$

This identifies ind_H^G as a functor which is left adjoint to the functor Res_H^G. Both functors are exact.

[19] P. Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

Pullback, corestriction

Suppose now that $B = T \ltimes U$, where T, U are closed subgroups of an ℓ-group B. Let $p : B \to T$ be the projection map. There is a functor given by *pullback*:

Definition 3.26 *Let (η, Y) be a smooth representation of T. Define $p^*\eta : B \to Aut_{\mathbb{C}}(V)$ by*

$$p^*\eta(b) = \eta(p(b)).$$

Then $(p^*\eta, Y)$ is a smooth representation of B, and p^* extends to a functor from $\mathfrak{Rep}_T^\infty$ to $\mathfrak{Rep}_B^\infty$.

Of course, one may introduce the general pullback of smooth representations, including restriction to a subgroup as well as the above pullback as special cases. The *pushforward* functor is defined by coinvariants:

Definition 3.27 *Let (σ, W) be a smooth representation of B. Define $p_* W = W_U = W/[U-1]W$ to be the space of U-coinvariants of W. Then, since T normalizes U, it follows that $\sigma(T)$ stabilizes $[U-1]W$ and hence the action σ of T on W descends to an action $p_*\sigma$ of T on $W_U = p_* W$. This extends to a functor p_* from $\mathfrak{Rep}_B^\infty$ to $\mathfrak{Rep}_T^\infty$.*

In this situation, we have the following adjointness theorem.

Theorem 3.28 (p. 125 of Cartier) [20] *Let (η, Y) be a smooth representation of T, and (σ, W) be a smooth representation of B. Then there is a natural isomorphism:*

$$Hom_B(p_* W, Y) \cong Hom_T(W, p^* Y).$$

This makes p_* a left adjoint to p^*.

[20] P. Cartier. Representations of p-adic groups: a survey. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 111–155. Amer. Math. Soc., Providence, R.I., 1979.

Indeed, the coinvariants W_U can be naturally identified with a module obtained by extension of scalars:

$$W_U \cong W \otimes_{\mathbb{C}[B]} \mathbb{C}[T],$$

where $\mathbb{C}[T]$ is viewed as a $\mathbb{C}[B]$-module via the trivial action of U. The result follows from adjointness of ring-extension and pull-back, suitably interpreted.

4 *Representations of GL_2, external theory*

Hereafter, we let $G = \mathbf{GL}_2(k)$, where k is a nonarchimedean local field; very little will be lost by taking $k = \mathbb{Q}_p$. As usual, we study the representations of a complicated group G, by understanding the representations of "easier" subgroups, and the functors of restriction and induction.

In addition, we drop the adjective "smooth" hereafter; all groups will be ℓ-groups, and all representations will be smooth. By "irrep", we mean an irreducible smooth representation.

By the *external theory*, we focus our attention on subgroups H of G which arise as $H = \mathbf{H}(k)$ for *algebraic* subgroups $\mathbf{H} \subset \mathbf{G}$. The primary subgroups of interest are:

$$B = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a, d \in k^\times, b \in k \right\},$$

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} : a, d \in k^\times \right\} \cong k^\times \times k^\times,$$

$$U = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in k \right\} \cong k.$$

$$Z = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} : a \in k^\times \right\} = Z(G) \cong k^\times.$$

These subgroups arise as the k-points of algebraic subgroups $\mathbf{B} = \mathbf{TU} \subset \mathbf{G}$. At the level of k-points, one has a semidirect product decomposition $B = T \ltimes U$. We write $p : B \to T$ for the canonical projection.

Representation theory of T

Corresponding to the obvious isomorphism $\mathbf{T} \cong \mathbf{G}_m \times \mathbf{G}_m$, there is an isomorphism of ℓ-groups: $T \cong k^\times \times k^\times$. The algebraic characters and cocharacters of $\mathbf{T}$ are:

Perhaps this treatment of the torus T is excessive in notation, for such a simple case. The advantage is that everything here generalizes easily to split tori of any rank.

$$X^\bullet(\mathbf{T}) = Hom(\mathbf{T}, \mathbf{G}_m) \cong \mathbb{Z}^2,$$

$$X_\bullet = X_\bullet(\mathbf{T}) = Hom(\mathbf{G}_m, \mathbf{T}) \cong \mathbb{Z}^2.$$

There is a canonical perfect pairing:

$$X_\bullet \times X^\bullet \to \mathbb{Z},$$

given by the identification $Hom(\mathbf{G}_m, \mathbf{G}_m) \cong \mathbb{Z}$.

We write $T_\circ$ for the maximal compact subgroup of T; there is a unique maximal compact subgroup, and $T_\circ$ is isomorphic to $\mathcal{O}^\times \times \mathcal{O}^\times$. While this isomorphism is non-canonical, there is a canonical isomorphism:

$$X_\bullet \cong T/T_\circ,$$

given by sending $\alpha \in X_\bullet$ to $\alpha(\varpi) \in T/T_\circ$; the choice of uniformizing element ϖ does not affect the $T_\circ$-coset of $\alpha(\varpi)$. The complex dual torus of T is defined by:

$$\hat{T} = Hom(X_\bullet, \mathbb{C}^\times) = X^\bullet \otimes_{\mathbb{Z}} \mathbb{C}^\times \cong \mathbb{C}^\times \times \mathbb{C}^\times.$$

Since T is abelian, the irreps of $T \cong k^\times \times k^\times$ are one-dimensional – they are given by a pair $\chi = (\chi_1, \chi_2)$ of (smooth) characters

$$\chi_1, \chi_2 : k^\times \to \mathbb{C}^\times.$$

We will pay particular attention to the *unramified* characters of T – these are given by pairs (χ_1, χ_2) of characters, which are both trivial on $\mathcal{O}^\times$. Writing $T_\circ = \mathcal{O}^\times \times \mathcal{O}^\times$, the unramified characters are just $Hom(T/T_\circ, \mathbb{C}^\times)$. Thus the unramified characters of T are described easily by the dual torus:

$$Hom_{unr}(T, \mathbb{C}^\times) = Hom(T/T_\circ, \mathbb{C}^\times) \cong \hat{T} = Hom(\mathbb{Z}, \hat{T}).$$

Much more generally, local class field theory implies that

$$Hom_{cont}(T, \mathbb{C}^\times) \cong Hom_{cont}(W_k, \hat{T}).$$

The unramified characters correspond to those continuous homomorphisms from W_k to $\hat{T}$ that factor through the quotient $W_k^{unr} \cong \mathbb{Z}$. We follow the convention that the unramified character of T corresponding to $t \in \hat{T}$ should correspond to the unramified character of W_k which sends a *geometric* Frobenius element to t.

This is known as the local Langlands corresponence for T, and was generalized by Langlands to arbitrary tori in an article that took thirty years to publish (finally in Pac. J. of Math.[21]).

[21] .

Jacquet functor, supercuspidals

For the classification of irreps of $G = \mathbf{GL}_2(k)$, and more generally in the classification of irreps of reductive p-adic groups, the most important method is parabolic induction and Harish-Chandra's theory of cuspidal representations.[22]

[22] This is the local analogue of the dichotomy between Eisenstein series and cuspforms.

Definition 4.1 *Let (π, V) be a representation of G. The* Jacquet functor *is*

$$J_B^G = p_* \circ Res_B^G : \mathfrak{Rep}_G^\infty \to \mathfrak{Rep}_T^\infty.$$

In particular, $J_B^G V = V_U$ is the space of U-coinvariants of V, viewed as a smooth representation of T.

Definition 4.2 *Let (η, Y) be a representation of T. The* functor of parabolic induction *is*

$$I_B^G = Ind_B^G \circ p^* : \mathfrak{Rep}_T^\infty \to \mathfrak{Rep}_G^\infty.$$

In particular, $I_B^G Y$ consists of uniformly locally constant functions $f : G \to Y$ which satisfy

$$f(tux) = \eta(t)(f(x) \text{ for all } t \in T, u \in U, x \in G,$$

and G acts on this space of functions by right translation.

Theorem 4.3 *The functor J_B^G is left adjoint to I_B^G; for a representation (π, V) of G and a representation (η, Y) of T, there is a natural isomorphism:*

$$Hom_G(V, I_B^G Y) \cong Hom_T(J_B^G V, Y).$$

Proof: Adjointness of Res_B^G and Ind_B^G implies

$$Hom_G(V, I_B^G Y) = Hom_G(V, Ind_B^G p^* Y) \cong Hom_B(Res_B^G V, p^* Y).$$

Adjointness of p^* and p_* implies

$$Hom_B(Res_B^G V, p^* Y) \cong Hom_T(p_* Res_B^G V, Y) = Hom_T(J_B^G V, Y).$$

The naturality of these isomorphisms, i.e., the adjointness of functors, implies the adjointness of I_B^G and J_B^G as required.

Q.E.D

In what follows, it will be more convenient to use the *normalized* parabolic induction and Jacquet functor. Let $\delta : T \to \mathbb{R}_{>0}^\times$ be the character[23] given by:

$$\delta \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = |a/d|.$$

Viewing characters of T as pairs of characters of $k^\times$, we find that

$$\delta = (|\cdot|, |\cdot|^{-1}).$$

We write $I_B^G \delta^{1/2}$ for the functor which on objects sends a representation η of T to $I_B^G(\eta \otimes \delta^{1/2})$. Similarly, we write $\delta^{-1/2} J_B^G$ for the functor which sends a representation π of G to $\delta^{-1/2} \otimes J_B^G \pi$.

[23] This is usually called the *modular character*. It describes the effect of T-conjugation on a Haar measure on U. Something like it should be used whenever carrying out induction and restriction involving non-unimodular groups (like B).

One advantage of this normalization is that *unitarizability* is preserved; if χ is a unitary character of T (it has values in the unit circle in the complex plane), then there is a natural Hermitian inner product on $I_B^G \delta^{1/2}\chi$; this implies that subrepresentations of $I_B^G \delta^{1/2}\chi$ have complements – it eventually yields complete reducibility of $I_B^G \delta^{1/2}\chi$.

The adjointness of J_B^G and I_B^G implies adjointness of the normalized functors; in particular,

$$Hom_G(V, I_B^G \delta^{1/2} Y) \cong Hom_T(\delta^{-1/2} J_B^G V, Y).$$

The following result makes the representation theory of p-adic groups much easier, in some ways, than the representation theory of real Lie groups:

Proposition 4.4 *The functors I_B^G and J_B^G are exact. Same for the functors $I_B^G \delta^{1/2}$ and $\delta^{-1/2} J_B^G$.*

Proof: (Sketch) Exactness of the functor I_B^G is easy, as is left-exactness of J_B^G. To demonstrate the right-exactness of J_B^G, it suffices to demonstrate the right-exactness of the "U-coinvariant functor" p_*. This follows from the fact that U is the union of compact subgroups – the functor of coinvariants for a compact group is exact (a basic result in group homology with coefficients in a vector space over a characteristic zero field) – and the exactness of direct limits.

For the normalized functors, the result follows by exactness of twisting, which is trivial to check.

Q.E.D

A useful basic result is that I_B^G and J_B^G are compatible with twisting and central characters, in a simple way.

Proposition 4.5 *Let $\chi = (\chi_1, \chi_2)$ be a character of T. Then $I_B^G \chi$ has central character $\chi_1 \chi_2$. Furthermore, let χ_0 be a character of $k^\times$ and write $\chi_0 \chi$ for the character $(\chi_0 \chi_1, \chi_0 \chi_2)$ of T; then there is a natural isomorphism of representations of G:*

$$I_B^G (\chi_0 \chi) \cong (\chi_0 \circ det) \otimes I_B^G \chi.$$

Proof: The proof is straightforward and left to the reader.

Q.E.D

The Jacquet functor gives an initial classification of irreps of $G = GL_2(k)$:

Definition 4.6 *A representation (π, V) of G is called* supercuspidal *if $J_B^G V = 0$.*

This would not be such an interesting definition if it were not for the following nontrivial theorem, due in various parts, and somewhat independently, to various authors (Bernstein [24], Casselman[25], Adler and Roche[26], among possible others):

[24] J. Bernstein. *Represenations of p-adic groups*. Harvard University, 1992. Lectures by Joseph Bernstein. Written by Karl E. Rumelhart.

[25] W. Casselman. *Introduction to the theory of admissible representations of p-adic reductive groups*. 1974. Unpublished manuscript, available online.

[26] Jeffrey D. Adler and Alan Roche. Injectivity, projectivity and supercuspidal representations. *J. London Math. Soc. (2)*, 70(2):356–368, 2004.

Theorem 4.7 *The following conditions are equivalent, for an irrep (π, V) of G, whose central character is $\chi : Z \to \mathbb{C}^\times$:*

1. *(π, V) is supercuspidal – $J_B^G V = 0$.*
2. *For all $v \in V$, and $\lambda \in \tilde{V} = (V')^\infty$, the matrix coefficient $m_{v,\lambda}$ is compactly supported, modulo Z; here $m_{v,\lambda} \in C^\infty(G)$ is defined by*
$$m_{v,\lambda}(g) = \lambda(\pi(g)v).$$
3. *There exists $v \in V$ and $\lambda \in V^\infty$, such that $m_{v,\lambda} \neq 0$ and $m_{v,\lambda}$ is compactly supported, modulo Z.*
4. *(π, V) is injective in the category $\mathfrak{Rep}^\infty_{G,\chi}$.*
5. *(π, V) is projective in the category $\mathfrak{Rep}^\infty_{G,\chi}$.*

In particular, if (π, V) is a smooth representation of G which possesses a central character, there are subrepresentations V^{sc}, V^{ind} such that V^{sc} is supercuspidal, and V^{ind} has no supercuspidal subrepresentation (nor quotient), and $V = V^{sc} \oplus V^{ind}$.

The description of supercuspidal representations of G is beyond the scope of these notes; let us just say that all such representations arise via compact induction, from irreducible representations of compact-modulo-Z subgroups of G, e.g., $Z \cdot GL_2(\mathcal{O})$. We refer to the excellent recent book of Bushnell-Henniart [27] for more.

[27] Colin J. Bushnell and Guy Henniart. *The local Langlands conjecture for* GL(2), volume 335 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 2006.

Geometric Decomposition

Consider now an irrep (π, V) of G which is not supercuspidal; that is, $J_B^G V \neq 0$. A priori, $J_B^G V$ is just a smooth represenation of T.

Lemma 4.8 *The representation $J_B^G V$ is finitely-generated as a T-module.*

Proof: Let v be a nonzero vector in V, and let H be an open subgroup of G fixing V. The compactness of $B \backslash G \cong \mathbf{P}^1(k)$ implies that there are a finite number of double cosets in $B \backslash G / H$. Choosing representatives $g_1, \ldots, g_d$ for these cosets, we find that V is generated – as a B-module – by the finite set $\{\pi(g_i)v\}_{1 \leq i \leq d}$. Thus, since U acts trivially on V_U, we find that $V_U = J_B^G V$ is generated – as a T-module – by the projections of the vectors $\pi(g_i)v$ for $1 \leq i \leq d$.

Q.E.D

It follows that

Lemma 4.9 *The representation $J_B^G V$ has an irreducible quotient.*

Proof: The proof is somewhat difficult, and we just sketch the idea. Note that the choice of uniformizing element ϖ yields a decomposition $T \cong X_\bullet \times T_\circ$ of ℓ-groups, where $X_\bullet$ is (non-canonically) isomorphic to $\mathbb{Z}^2$. One may first decompose $J_B^G V$ as a representation of the compact ℓ-group $T_\circ$:

$$J_B^G V = \bigoplus_{\phi \in \hat{T}_\circ} (J_B^G V)_\phi.$$

Each $T_\circ$-isotypic component is then a representation of $X_\bullet \cong \mathbb{Z}^2$. In other words, each $T_\circ$-isotypic component is a $\mathbb{C}[X_\bullet]$-module. Since $J_B^G V$ is nonzero, there exists a $\phi \in \hat{T}_\circ$ such that $(J_B^G)_\phi \neq 0$.

Thus to check that $J_B^G V$ has an irreducible quotient, it suffices to check that $(J_B^G V)_\phi$ has an irreducible quotient as a $\mathbb{C}[X_\bullet]$-module. From our previous study of the representations of $\mathbb{Z}$, it suffices (by Nakayama's lemma) to check that $(J_B^G V)_\phi$ is finitely-generated as a $\mathbb{C}[X_\bullet]$-module. But this follows from the fact that $J_B^G V$ is finitely-generated as a T-module, and $T_\circ$ acts via a character on $(J_B^G V)_\phi$.[28]

Q.E.D

[28] In fact, just knowing that $J_B^G V$ is finitely generated as a T-module is enough to show that it has an irreducible quotient, using a Zorn's lemma argument. It is not really necessary to use the $T_\circ$-isotypic components.

When (π, V) is an irrep of G, we find that $\delta^{-1/2} J_B^G V$ has an irreducible quotient – a character χ of T:

$$Hom_T(\delta^{-1/2} J_B^G V, \chi) \neq 0.$$

We choose to use the normalized functors, for reasons that will become clear. It follows that

$$Hom_G(V, I_B^G \delta^{1/2} \chi) \neq 0,$$

and so V is a subrepresentation of $I_B^G \delta^{1/2} \chi$. Thus the non-supercuspidal representations arise as subrepresentations of *principal series* – representations parabolically induced from characters of tori.

For this reason (and since we are not prepared to discuss supercuspidal representations here), we study the representations $I_B^G \delta^{1/2} \chi$ – the principal series representations of G. The key to studying these representations is the Bruhat decomposition:

$$G = B \sqcup BwB, \quad w = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Here BwB is an open subset of G, and B is its closed complement. The short exact sequence of $\mathbb{C}$-modules:

$$0 \to C_c^\infty(BwB) \to C_c^\infty(G) \to C_c^\infty(B) \to 0$$

is in fact a short exact sequence of smooth representations of $B \times B$ by left and right translation. From this (and a little work to check right-exactness) we obtain a short exact sequence of B-representations (by right-translation):

$$0 \to I^w(\delta^{1/2}\chi) \to I_B^G \delta^{1/2}\chi \to I^1(\delta^{1/2}\chi) \to 0,$$

where:

$$I^w(\chi) = \{f \in C_c^\infty(BwB), \text{ such that } f(tux) = \chi(t)\delta^{1/2}(x)f(x)\},$$

$$I^1(\chi) = \{f : B \to \mathbb{C}, \text{ such that } f(tux) = \chi(t)\delta^{1/2}f(x)\} \cong \mathbb{C}.$$

An explicit and nontrivial[29] computation demonstrates that:

$$\delta^{-1/2}J_B^G(I^w(\delta^{1/2}\chi)) \cong \chi^w, \quad \delta^{-1/2}J_B^G(I^1(\delta^{1/2}\chi)) \cong \chi.$$

To summarize, there is a short exact sequence of T-representations

$$0 \to \chi^w \to \delta^{-1/2}J_B^G I_B^G \delta^{1/2}\chi \to \chi \to 0. \quad (1)$$

Here,

$$\chi = (\chi_1, \chi_2), \quad \chi^w = (\chi_2, \chi_1).$$

If χ and χ^w are distinct characters of T, then the short exact sequence splits and:

$$\delta^{-1/2}J_B^G I_B^G \delta^{1/2}\chi \cong \chi \oplus \chi^w.$$

Lemma 4.10 *If W is any subquotient of $I_B^G\delta^{1/2}\chi$, then $J_B^G W \neq 0$.*

Proof: If $J_B^G W = 0$, then W is supercuspidal. It follows, from injectivity and projectivity of supercuspidals[30], and the fact that $I_B^G\delta^{1/2}\chi$ has a central character $\delta^{1/2}\chi_1\chi_2$, that the subquotient W of $I_B^G\delta^{1/2}\chi$ also arises as a submodule. Hence

$$Hom(W, I_B^G\delta^{1/2}\chi) \neq 0.$$

By adjointness,

$$Hom(J_B^G W, \delta^{1/2}\chi) \neq 0.$$

This contradicts the fact that W is supercuspidal.

Q.E.D

Corollary 4.11 *The representation $I_B^G\delta^{1/2}\chi$ has length at most two.*

Proof: The exactness of the functor J_B^G, the previous lemma, and the fact that $\delta^{-1/2}J_B^G I_B^G\delta^{1/2}\chi$ is two-dimensional implies this corollary.

[29] A geometric argument – that $\mathbf{B}w\mathbf{B}$ is isomorphic to $\mathbf{B} \times \mathbf{U}$ as a k-variety – implies that $I^w(\delta^{1/2}\chi)$ is one-dimensional. Seeing that $I^1(\delta^{1/2}\chi)$ is one-dimensional is easier. One identifies the projection of $I^w(\delta^{1/2}\chi)$ onto its U-coinvariants with an integral over U – tracking through the T-action proves the result.

[30] Really, it is deceptive to utilize injectivity and projectivity of supercuspidals for this sort of result. The proof of injectivity and projectivity of supercuspidals relies on results like this lemma, to my recollection. It is much better to prove this lemma using Jacquet's lemma, and compact subgroups with Iwahori decomposition.

Q.E.D

The precise conditions for reducibility of $I_B^G \delta^{1/2}\chi$ are given by the following

Theorem 4.12 *The representation $I_B^G \delta^{1/2}\chi$ is reducible if and only if*

$$\chi_1 = |\cdot|\chi_2, \text{ or } \chi_1 = |\cdot|^{-1}\chi_2.$$

Equivalently, $I_B^G \delta^{1/2}\chi$ is reducible if and only if

$$\chi = \delta^{\pm 1}\chi^w.$$

This theorem requires a lot of work – we refer to the exposition of Tadic for a nice treatment. Partial results follow from Frobenius reciprocity and the short exact sequence (**??**): we find that

$$End_G(I_B^G \delta^{1/2}\chi) \cong Hom_T(\delta^{-1/2} J_B^G I_B^G \delta^{1/2}\chi, \chi).$$

We find two cases:

1. The space $End_G(I_B^G \delta^{1/2}\chi)$ is one-dimensional, if $\chi \neq \chi^w$, or if $\chi = \chi^w$ and the extension $\delta^{-1/2} J_B^G I_B^G \delta^{1/2}\chi$ of χ by itself is nontrivial.
2. The space $End_G(I_B^G \delta^{1/2}\chi)$ is two-dimensional if $\chi = \chi^w$ and the extension $\delta^{-1/2} J_B^G I_B^G \delta^{1/2}\chi$ of χ by itself splits.

By Schur's lemma, if $End_G(I_B^G \delta^{1/2}\chi)$ is two-dimensional, then $I_B^G \delta^{1/2}\chi$ is reducible; but the above observation implies that $\chi = \chi^w$, and Theorem **??** implies that there is no reducibility when $\chi = \chi^w$ (only when $\chi = \delta^{\pm 1}\chi^w$). Hence we find that

Corollary 4.13 *The representation $I_B^G \delta^{1/2}\chi$ is either irreducible, or else is a nonsplit extension of one irreducible representation of G by another irreducible representation of G.*

Proof: If $I_B^G \delta^{1/2}\chi$ is reducible, we find that its G-endomorphisms form a one-dimensional space. Hence it cannot be decomposed into the direct sum of irreducible representations. Since it has length at most two, the result follows immediately.

Q.E.D

One example is particularly easy to see, and important for applications:

Example 4.14 *Considering $\chi = \delta^{-1/2}$, we find that $I_B^G \delta^{1/2}\chi = I_B^G \mathbb{C}$ is a reducible representation of G, of length two. There is a short exact sequence of smooth representations of G:*

$$0 \to \mathbb{C} \to I_B^G \mathbb{C} \to St \to 0.$$

The embedding of $\mathbb{C}$ into $I_B^G\mathbb{C}$ takes a complex number to the corresponding constant function on G. Since its image is clearly one-dimensional, and $I_B^G\mathbb{C}$ is infinite-dimensional, there must be a nontrivial quotient. This quotient is called the Steinberg *representation.*

The symmetry between χ_1 and χ_2 manifests in a rational family (rational, in the parameter $\chi \in Hom_{cont}(T, \mathbb{C}^\times)$) of intertwining operators, from $I_B^G\delta^{1/2}\chi$ to $I_B^G\delta^{1/2}\chi^w$.

Proposition 4.15 *Suppose that $\chi \neq \chi^w$. Then $I_B^G\delta^{1/2}\chi$ is isomorphic to $I_B^G\delta^{1/2}\chi^w$.*

Proof: By Frobenius reciprocity, there is a natural $\mathbb{C}$-linear isomorphism

$$Hom_G(I_B^G\delta^{1/2}\chi, I_B^G\delta^{1/2}\chi^w) \cong Hom_T(\delta^{-1/2}J_B^G I_B^G\delta^{1/2}\chi, \chi^w).$$

Recall the short exact sequence of representations of T **??**:

$$0 \to \chi^w \to \delta^{-1/2}J_B^G I_B^G\delta^{1/2}\chi \to \chi \to 0.$$

It follows that if $\chi \neq \chi^w$, then the above sequence splits, $I_B^G\chi$ and $I_B^G\chi^w$ are irreducible, and hence are isomorphic to each other.

Q.E.D

In fact, the intertwining operators, which exist by Frobenius reciprocity, form a complex algebraic family over (a Zariski-dense subset of) the variety $Hom_{cont}(T, \mathbb{C}^\times)$. However, these operators have zeros and poles, which correspond to the reducibility points of the principal series representations.

Unramified principal series

Especially important for global applications are the unramified principal series; these are the representations $I_B^G\delta^{1/2}\chi$, when $\chi : T/T_\circ \to \mathbb{C}^\times$ is an unramified character of T. In particular,

$$\chi = (\chi_1, \chi_2), \quad \chi_i(x) = s_i^{val(x)},$$

for some nonzero complex numbers s_1, s_2. The pair (s_1, s_2) can be thought of as an element of $\hat{T}$, if one wishes to be canonical. For simplicity, we define

$$I(s_1, s_2) = I_B^G\delta^{1/2}\chi, \text{ when } \chi\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} = (s_1^{val(a)}, s_2^{val(d)}).$$

From Proposition **??**, when $s_1 \neq s_2$, there is an isomorphism:

$$I(s_1, s_2) \cong I(s_2, s_1).$$

We find a reducibility point when $\chi_1 = |\cdot|^{\pm 1}\chi_2$. In other words,

Proposition 4.16 *The unramified principal series $I(s_1, s_2)$ is reducible if and only if $s_1 = q^{\pm 1} s_2$. Here, we recall that $q = \#(\mathcal{O}/\varpi)$ is the order of the residue field of k.*

Proof: This follows from the previous result on reducibility of principal series representations, and changing notation.

Q.E.D

If $s_1 = q^{-1} s_2$, then we find that

$$\delta^{1/2} \cdot (\chi_1, \chi_2) = (|\cdot|^{1/2}\chi_1, |\cdot|^{-1/2}\chi_2) = (|\cdot|^{-1/2}\chi_2, |\cdot|^{-1/2}\chi_2).$$

It follows that

$$I_B^G \delta^{1/2}\chi \cong |det|^{-1/2} s_2^{val(det)} \otimes I_B^G \mathbb{C}.$$

In this case, $I_B^G \delta^{1/2}\chi$ has an irreducible subrepresentation and irreducible quotient:

$$0 \to |\cdot|^{-1/2} s^{val(det)} \to I(s_1, s_2) \to |\cdot|^{-1/2} s^{val(det)} \otimes St \to 0.$$

If $s_1 = qs_2$, then one finds a similar short exact sequence, with a twisted trivial representation as a quotient, and twisted Steinberg representation as a subrepresentation.

To summarize, we have a two-dimensional complex algebraic variety[31] $\hat{T} = MSpec(\mathbb{C}[s_1^{\pm 1}, s_2^{\pm 1}])$, acted upon by a finite group $W = \{1, w\}$, where w switches s_1 and s_2. There's a W-stable subvariety $\hat{T}_{red}$ cut out by the equations $s_1 = q^{\pm 1} s_2$.

[31] We identify complex algebraic varieties with their $\mathbb{C}$-points here.

There is a complex algebraic family (see Bernstein[32] for the precise meaning) of representations $I(s_1, s_2)$ of G, parameterized by $(s_1, s_2) \in \hat{T}$, which is generically irreducible, and everywhere satisfies the conclusion of Schur's lemma. The group $W = \{1, w\}$ acts on $\hat{T}$, and on the Zariski-open irreducible locus $\hat{T} - \hat{T}_{red}$. Intertwining operators make this complex algebraic family of representations into a W-equivariant sheaf, when pulled back to $\hat{T} - \hat{T}_{red}$.

[32] J. N. Bernstein. Le "centre" de Bernstein. In *Representations of reductive groups over a local field*, Travaux en Cours, pages 1–32. Hermann, Paris, 1984. Edited by P. Deligne.

In any case, we find that the irreducible constituents of unramified principal series representations are parameterized by the following data:

1. An unordered pair $\{s_1, s_2\}$ of nonzero complex numbers, such that $s_1 \neq q^{\pm 1} s_2$ or...
2. An ordered pair (s_1, s_2) of nonzero complex numbers, such that $s_1 = q^{-1} s_2$ and an additional "bit of information" encoding whether one takes the twisted trivial subrepresentation or twisted Steinberg quotient representation.

To such data, we associate the following Langlands parameters:

1. The $GL_2(\mathbb{C})$-conjugacy class containing the semisimple element $\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$. (Note that this only depends on, and uniquely determines, the unordered pair $\{s_1, s_2\}$ of nonzero complex numbers.

2. The $GL_2(\mathbb{C})$-conjugacy class of the pair (t, N), where t is the semisimple element $\begin{pmatrix} s_1 & 0 \\ 0 & s_2 \end{pmatrix}$ (with $s_1 = q^{-1}s_2$), and N is a nilpotent element of $M_2(\mathbb{C})$ satisfying $tNt^{-1} = qN$; for any such (s_1, s_2), there are two such conjugacy classes of pairs: one contains $(t, 0)$ and the other contains (t, N) with $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

The first case can also be thought of as a conjugacy class of pairs (t, N) with $tNt^{-1} = qN$; but when $s_1 \neq q^{\pm 1}s_2$, the only nilpotent N satisfying that identity is zero. In the second case, the extra "bit of information" given by whether $N = 0$ or $N \neq 0$ corresponds to the extra "bit of information" given by whether one chooses the twisted trivial representation of the twisted Steinberg representation, respectively.

If an irreducible constituent of an unramified principal series representation (π, V) corresponds to a parameter (t, N) as above (t semisimple in $GL_2(\mathbb{C})$ and N nilpotent in $M_2(\mathbb{C})$), then the standard (degree 2) L-function of (π, V) is:

$$L(\pi, Stand) = det(1 - tX | Ker(N)).$$

5 *Representations of GL_2, internal theory*

Let K be an open compact subgroup of $G = \mathbf{GL}_2(k)$. It is important to study representations with K-fixed vectors; in order to have a good *category* of representations, we define $\mathfrak{Rep}_G^K$ to be the category of smooth representations of G which are generated (as G-represenations) by their K-fixed vectors. These are called K-spherical representations. For general K, this category is not stable under subquotients!

Let $H(G, K)$ be the Hecke algebra of compactly supported, K-bi-invariant functions on G:

$$H(G, K) = C_c^\infty(K \backslash G / K).$$

If (π, V) is a K-spherical representation, then V^K is naturally an $H(G, K)$-module, via

$$\pi(f)v = \int_G f(g)\pi(g)v dg.$$

If $f_1, f_2 \in H(G,K)$, then

$$\pi(f_1)\pi(f_2)v = \pi(f_1 * f_2)v,$$

where the convolution is defined by

$$[f_1 * f_2](g) = \int_G f_1(h) f_2(h^{-1}g) dh.$$

In fact, this gives an equivalence of categories, from the category of modules over the *convolution* algebra $H(G,K)$ and the category of K-spherical representations.

These categories are somewhat mysterious in general, but when $K = \mathbf{GL}_2(\mathcal{O})$, we have the category of *unramified representations*. These are well-understood; moreover in the factorization of automorphic representations, irreducible unramified representations occur for almost all primes.

Unramified representations

Hereafter, let $K = \mathbf{GL}_2(\mathcal{O})$. The remarkable theorem about the spherical Hecke algebra is the following:

Theorem 5.1 *Define, for $f \in H(G,K)$, the* Satake transform $Sf \in C_c^\infty(T)$

$$[Sf](t) = \delta(t)^{-1/2} \int_U f(ut) du = \delta(t)^{1/2} \int_U f(tu) du.$$

Then

$$Sf \in H(T, T_\circ) = C_c^\infty(T/T_\circ)^W = \mathbb{C}[X_\bullet]^W,$$

where $W = \{1, w\}$. Moreover, S determines an isomorphism of algebras:

$$H(G,K) \cong \mathbb{C}[X_\bullet]^W.$$

In particular, this theorem implies that $H(G,K)$ is a *commutative* $\mathbb{C}$-algebra! Highest weight theory, for the algebraic representations of $GL_2(\mathbb{C})$, implies that

$$\mathbb{C}[X_\bullet(\mathbf{T})]^W = \mathbb{C}[X^\bullet(\hat{\mathbf{T}})]^W \cong Rep(GL_2(\mathbb{C})),$$

where $Rep(GL_2(\mathbb{C}))$ is the complexification of K_0 of the category of finite-dimensional algebraic representations of $GL_2(\mathbb{C})$ – i.e., the complexified representation ring of $GL_2(\mathbb{C})$.

As the category of spherical representations of G is equivalent to the category of $H(G,K)$-modules, which is equivalent to the category of $\mathbb{C}[X_1^{\pm 1}, X_2^{\pm 1}]^W$-modules. It follows that an *irreducible* unramified representation of G is one-dimensional – determined by two nonzero complex numbers (s_1, s_2), modulo switching;

there is a natural bijection between the isomorphism classes of irreducible unramified representations of G and *unordered pairs* $\{s_1, s_2\}$ of nonzero complex numbers.

More canonically, there is a natural bijection between the set of isomorphism classes of irreducible unramified representations of G and the W-orbits on $\hat{T}$.

Connection to unramified principal series

Let $I \subset K$ be the *Iwahori subgroup*, consisting of matrices in $\mathbf{GL}_2(\mathcal{O})$ whose lower-left entry is in $\varpi\mathcal{O}$. Recall that $T_\circ = \mathbf{T}(\mathcal{O}) \cong \mathcal{O}^\times \times \mathcal{O}^\times$. The following is a fundamental theorem of Borel and Matsumoto:

Theorem 5.2 *Let (π, V) be an admissible (smooth and finite-length certainly suffices) representation of G. Consider the natural projection map $V \to V_U$ from V onto the space of $J_B^G V$. This projection map induces an isomorphism of complex vector spaces:*

$$V^I \to (V_U)^{T_\circ}.$$

A corollary of this result is the following:

Corollary 5.3 *If (π, V) is a K-spherical admissible representation of G, then $J_B^G V \neq 0$. If moreover, (π, V) is an irreducible unramified representation of G, then $J_B^G V$ has an unramified character of T as a subquotient.*

By adjointness, and what we know about unramified principal series, we find that

Corollary 5.4 *If (π, V) is an irreducible unramified representation of G, then (π, V) occurs as a subquotient in an unramified principal series representation $I_B^G \delta^{1/2}\chi$, where $\chi : T/T_\circ \to \mathbb{C}^\times$ is uniquely determined by V up to the action of W.*

From this result, we find that an irreducible unramified representation (π, V) of G yields two pairs of complex numbers:

1. Since (π, V) is associated to an irreducible $H(G, K)$-module, we obtain two "Hecke eigenvalues" s_1, s_2 (up to switching). These are called the Satake parameters of (π, V), since they arise from the Satake isomorphism from $H(G, K)$ to $H(T, T_\circ)$.

2. Since (π, V) occurs in an unramified principal series representation, we find that (π, V) is a subquotient of $I(t_1, t_2)$, for nonzero complex numbers t_1, t_2, uniquely determined, up to switching.

Furthermore, although the unramified principal series representation $I_B^G \delta^{1/2}\chi$ may be reducible, it has a unique unramified subrepresentation – the twisted trivial representation (with unramified twist, of course) is always unramified, and the twisted Steinberg representation is never unramified (has no K-fixed vectors).

The connection between these is the following significant theorem:

Theorem 5.5 *The unordered pair $\{s_1, s_2\}$ equals the unordered pair $\{t_1, t_2\}$.*

Let π_{s_1,s_2} denote the irreducible spherical representation of $GL_2(k)$ with parameters $s_1, s_2 \in \mathbb{C}^\times$.

The impact of this theorem, for the theory of modular forms, is the following: Let f be a classical modular form for a congruence subgroup $\Gamma_0(N)$; suppose that f is a cuspidal newform, of some Nebentypus, for good measure. Then one associates to f an automorphic representation $\Pi = \bigotimes' \pi_v$, where the (restricted) tensor product is over all places v of $\mathbb{Q}$. At all primes p not dividing N, the representation π_p is irreducible and unramified.

The previous theorem tells us that the eigenvalue of the T_p operator (and the Nebentypus character), which determines the Hecke eigenvalue and hence the Satake parameter for the representation π_p, also determines the isomorphism class of the representation π_p. The representation π_p is precisely the irreducible unramified constituent of the unramified principal series $I(s_1, s_2)$, where (s_1, s_2) is the Satake parameter deduced from the Hecke eigenvalue of T_p.

Slightly more generally, if p divides N, but p^2 does not divide N, the representation π_p ends up being isomorphic to a twist of the Steinberg representation; proving this requires some analysis of the Iwahori Hecke algebra $H(G, I)$ instead of $H(G, K)$.

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