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Spherical Functions and a q-Analogue of Kostant's Weight Multiplicity Formula

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The purpose of this paper is to give a q-analogue of Kostant's weight multiplicity formula for irreducible representations of complex semisimple Lie algebras, conjectured by Lusztig [6] quite recently. To prove this, we use the theory of spherical functions on p-adic groups (or on Hecke algebras) developed by Satake, Macdonald et al. extensively. In the course of the proof of the above result, we give a short proof of the theorem of Lusztig [6] which describes the weight multiplicities in terms of intersection homology (or Kazhdan-Lusztig polynomials $P_{y,w}$).

1. Statement of Results

Let V be a finite dimensional real vector space with a positive definite inner product $\langle\ ,\ \rangle$. Let R be a root system in V. We fix R^+, a set of positive roots of R (with respect to some ordering). Denote by P the weight lattice of R, i.e., $P=\{x\in V\,|\,\langle x,\alpha^\vee\rangle\in\mathbb{Z}\ (\forall\alpha\in R)\}$ where $\alpha^\vee=2\alpha/\langle\alpha,\alpha\rangle$. We say $\lambda\geqq\mu$ ($\lambda,\mu\in P$) if $\lambda-\mu=\sum_{\alpha>0} n_\alpha\cdot\alpha$ ($n_\alpha\in\mathbb{Z}_+=\mathbb{N}\cup\{0\}$). This '$\geqq$' is a partial order on P. Put $P^{++}=\{\lambda\in P\,|\,\langle\lambda,\alpha^\vee\rangle\geqq 0\ (\forall\alpha\in R^+)\}$ (the set of dominant weights). Since we shall mainly argue in $\mathbb{Z}[P]$, the group ring of P over $\mathbb{Z}$, we use a multiplicative notation: $\mathbb{Z}[P]=\bigoplus_{\lambda\in P}\mathbb{Z}\cdot e^\lambda$ and $e^\lambda e^\mu=e^{\lambda+\mu}$. Let W be the Weyl group of R. Then W naturally acts on R, P and V.

Let q be an indeterminate. Following Lusztig [6], we define a q-analogue of Kostant's partition function $\hat{\mathscr{P}}$:

$$\hat{\mathscr{P}}(\kappa;q)=\sum_{\substack{(n_1,\ldots,n_N)\in\mathbb{Z}_+^N\\ n_1\alpha_1+\ldots+n_N\alpha_N=\kappa}} q^{n_1+\ldots+n_N}\qquad(\kappa\in P)\tag{1.1}$$

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where $\{\alpha_1, \dots, \alpha_N\} = R^+$. For $\lambda, \mu \in P$, we put

$$K_{\lambda,\mu}(q) = \sum_{w \in W} \operatorname{sgn}(w) \hat{\mathscr{P}}(w(\lambda+\rho)-(\mu+\rho); q) \tag{1.2}$$

where $2\rho = \sum_{\alpha>0} \alpha$ and sgn: $W \to \{\pm 1\}$ is the sign character.

Let χ_λ $(\lambda \in P^{++})$ be the 'irreducible character with highest weight λ' of a complex semisimple Lie algebra, say $\mathfrak{g}$, of type R, i.e.,

$$\chi_\lambda = \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)} / \sum_{w \in W} \operatorname{sgn}(w) e^{w\rho}.$$

For $\mu \in P^{++}$, we put $W_\mu(q) = \sum_{w \in W_\mu} q^{\ell(w)}$, the Poincaré polynomial of W_μ (the stabilizer of μ in W). Now we can state the following

Theorem 1.3. *For $\lambda \in P^{++}$, we have*

$$\chi_\lambda = \sum_{\substack{\mu \in P^{++} \\ \mu \leqq \lambda}} K_{\lambda,\mu}(q) W_\mu(q)^{-1} \sum_{w \in W} e^{w\mu} \prod_{\alpha>0} \frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}. \tag{1.4}$$

If we put $q=1$ in the above, then (1.4) becomes Kostant's weight multiplicity formula. On the other hand, if we put $q=0$, then (1.4) becomes a well known identity

$$\chi_\lambda = \sum_{w \in W} e^{w\lambda} \prod_{\alpha>0} (1-e^{-w\alpha})^{-1}.$$

When R is of type A_n, the identity (1.4) is proved in [8; p. 131]. The proof of Theorem 1.3 requires Macdonald's formulas for spherical functions and Plancherel measures, and is given in Sect. 3.

But we can calculate the coefficients $K_{\lambda,\mu}(q)$ in (1.4) in another way: Let E be the affine space whose underlying vector space is V. For $\lambda \in P$, we denote by t_λ the translation by λ on E. We put $T = \{t_\lambda \mid \lambda \in P\}$ and $\tilde{W} = W \ltimes T$ (semidirect product of W by T). Both are subgroups of the affine transformation group of E. Though $\tilde{W}$ is not a Coxeter group in general, we can define the length function ℓ, Bruhat ordering $\geqq$ on $\tilde{W}$, and Kazhdan-Lusztig polynomials $P_{y,w}(q) \in \mathbb{Z}[q]$ ([4]) for $y, w \in \tilde{W}$ (see Sect. 2 and 4). For $\lambda \in P^{++}$, let w_λ be the longest element in $W t_\lambda W$. Put $2\rho^\vee = \sum_{\alpha>0} \alpha^\vee$.

Theorem 1.5.[1] *For $\lambda \in P^{++}$, we have*

$$\chi_\lambda = \sum_{\substack{\mu \in P^{++} \\ \mu \leqq \lambda}} q^{\langle \lambda-\mu, \rho^\vee \rangle} P_{w_\mu, w_\lambda}(q^{-1}) W_\mu(q)^{-1} \sum_{w \in W} e^{w\mu} \prod_{\alpha>0} \frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}} \tag{1.6}$$

When R is of type A_n, this formula (1.6) is due to Lusztig [5; Theorem 2]. If we put $q=1$, then (1.6) shows

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[1] After writing this paper, I was informed by G. Lusztig that the way of my proof of Theorem 1.5 is similar to his method used in the first version of [6], and that this theorem can be obtained easily from the results of [6]. I am grateful to G. Lusztig for his comments.

Corollary 1.7. ([6; Theorem 6.1]). *If $\lambda, \mu \in P^{++}$ with $\lambda \geqq \mu$,*

$P_{w_\mu, w_\lambda}(1)=$ *the multiplicity of μ in the irreducible representation of $\mathfrak{g}$ with highest weight λ.*

Moreover, the uniqueness of the expression like (1.4) and (1.6) (see (2.7)) implies that Lusztig's conjecture [6; (9.4)] is true. Namely we get

Theorem 1.8. *For $\lambda, \mu \in P^{++}$ with $\lambda \geqq \mu$, we have*

$$K_{\lambda,\mu}(q)=q^{\langle \lambda-\mu, \rho^\vee \rangle} P_{w_\mu, w_\lambda}(q^{-1}).$$

The proof of Theorem 1.5 goes along similar lines as the proof of [5; Theorem 2] (we use spherical functions (resp. irreducible characters) instead of Hall-Littlewood functions (resp. Schur functions)) and will be given in Sect. 4.

2. Spherical Functions on Hecke Algebras

2.1. Let $\tilde{W}=W \ltimes T$ be the group defined in Sect. 1. (This is the modified affine Weyl group of a simply-connected complex semisimple group of type R, in the sense of [3].) For simplicity, we henceforth assume that R is irreducible. Let S be the set of simple reflections of W (relative to R^+). Let $\tilde{\alpha}$ be the element of R such that $-\tilde{\alpha}^\vee \in R^\vee$ is maximal, and put $s_0=w_{\tilde{\alpha}} t_{\tilde{\alpha}}$ where $w_{\tilde{\alpha}}$ is the reflection corresponding to $\tilde{\alpha}$. The element s_0 is a reflection and its reflection hyperplane is given by $\{x \in E \mid \langle x, \tilde{\alpha}^\vee \rangle + 1 = 0\}$. We denote by T_{root} the subgroup of T generated by t_α $(\alpha \in R)$. The subgroup of $\tilde{W}$, $W_{\mathrm{aff}}=W \cdot T_{\mathrm{root}}$ (affine Weyl group of type $R^\vee$) is generated by $S_{\mathrm{aff}}=S \cup \{s_0\}$ as a Coxeter group. Let Ω be the normalizer of S_{aff} in $\tilde{W}$. Then $\tilde{W}$ is the semidirect product of Ω by W_{aff}. We extend the length function $\ell\colon W_{\mathrm{aff}} \to \mathbb{Z}_+$ (with respect to S_{aff}) to $\tilde{W}$ by $\ell(xw)=\ell(w)$ $(x \in \Omega, w \in W_{\mathrm{aff}})$.

2.2. We introduce an indeterminate $q^{1/2}$ satisfying $(q^{1/2})^2=q$. Let $H(\tilde{W},q)$ be the Hecke algebra of $\tilde{W}$ over $\mathbb{Q}(q^{1/2})$. Namely, $H(\tilde{W},q)$ is a $\mathbb{Q}(q^{1/2})$-vector space with a basis $\{T_w\}_{w \in \tilde{W}}$ and their multiplication law is given by

$$\begin{aligned} &T_s^2=(q-1)\,T_s+q\,T_e \qquad (s \in S_{\mathrm{aff}});\\ &T_w T_{w'}=T_{ww'} \qquad (\ell(w)+\ell(w')=\ell(ww')). \end{aligned}$$

Let $\phi_0=W(q)^{-1} \sum_{w \in W} T_w$ be an idempotent of $H(\tilde{W},q)$. Here $W(q)=W_0(q) = \sum_{w \in W} q^{\ell(w)}$. We define the subalgebra of $H(\tilde{W},q)$ by $H(\tilde{W},q;W) = \phi_0 \cdot H(\tilde{W},q) \cdot \phi_0$ (with unit element ϕ_0). Put $\phi_\lambda = W(q)^{-1} \sum_{w \in Wt_{-\lambda}W} T_w$ for $\lambda \in P^{++}$. Since $W=\coprod_{\lambda \in P^{++}} Wt_{-\lambda}W$ (disjoint union), the set $\{\phi_\lambda\}_{\lambda \in P^{++}}$ forms a basis of $H(\tilde{W},q;W)$, i.e., $H(\tilde{W},q;W)=\bigoplus_{\lambda \in P^{++}} \mathbb{Q}(q^{1/2}) \cdot \phi_\lambda$. It is known that $H(\tilde{W},q;W)$ is semisimple and commutative. More precisely, as will be seen below, $H(\tilde{W},q;W)$ is isomorphic to $\mathbb{Q}(q^{1/2})[P]^W=\mathbb{Q}(q^{1/2})\otimes\mathbb{Z}[P]^W$ under the Satake isomorphism, where $\mathbb{Z}[P]^W$ is the subalgebra of $\mathbb{Z}[P]$ consisting of all W-invariants: We define $\delta^{1/2}\in\mathrm{Hom}(T,\mathbb{Q}(q^{1/2})^\times)$ by $\delta^{1/2}(t_\lambda)=q^{\langle\lambda,\rho^\vee\rangle}$ ($\lambda\in P$; recall $2\rho^\vee=\sum_{\alpha>0}\alpha^\vee$). Let $\eta\in\mathrm{Hom}(T,\mathbb{Q}(q^{1/2})[P]^\times)$ be the 'identity' map, i.e., $\eta(t_\lambda)=e^\lambda$ ($\lambda\in P$). Incidentally we note here that W canonically acts on $\mathrm{Hom}(T,\mathbb{Q}(q^{1/2})[P]^\times)$ (e.g., we have $(w\cdot\eta)(t_\lambda)=e^{w^{-1}\lambda}$ for $w\in W$, $\lambda\in P$). Now we define M_η the space of 'generic' principal series representation of $H(\tilde{W},q)$ over $\mathbb{Q}(q^{1/2})[P]$ as in [3; 2.8]. That is,

$$M_\eta=\{f\colon \tilde{W}\to\mathbb{Q}(q^{1/2})[P]\,|\,f(wt)=(\eta\delta^{1/2})(t)f(w)\ (w\in\tilde{W},\ t\in T)\}$$

and $H(\tilde{W},q)$ acts on M_η exactly as in [3, 9]. Using this representation, we can state the following well-known result [9, 10].

Theorem 2.3 (Satake). *The map $\gamma\colon H(\tilde{W},q;W)\to\mathbb{Q}(q^{1/2})[P]$ defined by $\gamma(\phi)=\mathrm{Tr}(\phi|M_\eta)$ ($\phi\in H(\tilde{W},q;W)$) induces an isomorphism of $H(\tilde{W},q;W)$ onto $\mathbb{Q}(q^{1/2})[P]^W$.*

We denote the above isomorphism (called the Satake isomorphism) by the same letter γ.

Theorem 2.4 (Macdonald). *For $\lambda\in P^{++}$, we have*

$$\gamma(\phi_\lambda)=\frac{q^{\langle\lambda,\rho^\vee\rangle}}{W_\lambda(q^{-1})}\sum_{w\in W}e^{w\lambda}\prod_{\alpha>0}\frac{1-q^{-1}e^{-w\alpha}}{1-e^{-w\alpha}} \tag{2.5}$$

For our later use, we sketch the proof of the above theorem along the lines as [1].

2.6. *Proof of Theorem 2.4.* First we note that

$$\phi_\lambda=W(q)^{-1}\sum_{w\in Wt_{-\lambda}W}T_w=\frac{W(q^{-1})}{W_\lambda(q^{-1})}\phi_0\cdot T_{t_{-\lambda}}\cdot\phi_0.$$

Let 1_η be the element of M_η defined by $1_\eta(w)=1$ for all $w\in W$. Then $\phi_0\cdot M_\eta=\mathbb{Q}(q^{1/2})[P]\cdot 1_\eta$ and $\phi_0\cdot 1_\eta=1_\eta$. On the other hand, by [3; Proposition 2.9], we have

$$1_\eta=\sum_{w\in W}\prod_{\substack{\alpha>0\\ w\alpha>0}}\left(\frac{1-q^{-1}e^{-\alpha}}{1-e^{-\alpha}}\right)A(w^{-1},w\cdot\eta)f_{w\cdot\eta}$$

where $A(w^{-1},w\cdot\eta)\in\mathrm{Hom}_{H(\tilde{W},q)}(M_{w\cdot\eta},M_\eta)\otimes\mathbb{Q}(q^{1/2})(P)$ ($\mathbb{Q}(q^{1/2})(P)$ is the quotient field of $\mathbb{Q}(q^{1/2})[P]$) and $f_{w\cdot\eta}\in M_{w\cdot\eta}$ with $f_{w\cdot\eta}(e)=1$ and $f_{w\cdot\eta}(y)=0$ ($y\in W$; $y\neq e$). Since

$$\begin{aligned} T_{t_{-\lambda}}\cdot f_{w\cdot\eta}&=(w\cdot\eta)(t_\lambda)\,\delta^{1/2}(t_\lambda)f_{w\cdot\eta}\\ &=e^{w^{-1}\lambda}q^{\langle\lambda,\rho^\vee\rangle}f_{w\cdot\eta} &&([9; (4.1.9)]);\\ \phi_0\cdot f_{w\cdot\eta}&=W(q^{-1})^{-1}1_{w\cdot\eta} &&([9; (4.4.2)]);\end{aligned}$$

and

$$A(w^{-1},w\cdot\eta)1_{w\cdot\eta}=\prod_{\substack{\alpha>0\\ w^{-1}\alpha<0}}\left(\frac{1-q^{-1}e^{-w^{-1}\alpha}}{1-e^{-w^{-1}\alpha}}\right)1_\eta \qquad \text{(cf. [3; (1.20.2)])},$$

we finally obtain

$$\phi_0 \cdot T_{t_{-\lambda}} \cdot 1_\eta = \frac{q^{\langle \lambda, \rho^\vee \rangle}}{W(q^{-1})} \sum_{w \in W} e^{w\lambda} \prod_{\alpha > 0} \left(\frac{1 - q^{-1} e^{-w\alpha}}{1 - e^{-w\alpha}} \right) 1_\eta .$$

Hence we get (2.5).

Let χ_λ be the 'irreducible character with highest weight λ' for $\lambda \in P^{++}$ as in Sect. 1. Then it is well-known that the set $\{\chi_\lambda\}_{\lambda \in P^{++}}$ forms a basis of $\mathbb{Q}(q^{1/2})[P]^W$. As γ: $H(\tilde{W}, q; W) \xrightarrow{\sim} \mathbb{Q}(q^{1/2})[P]^W$, we can expand χ_λ by $\{\gamma(\phi_\mu)\}_{\mu \in P^{++}}$ in a unique way.

Lemma 2.7. *There exist $L_{\lambda,\mu} \in \mathbb{Q}(q^{1/2})$ for $\lambda, \mu \in P^{++}$ with $\lambda \geqq \mu$ such that*

$$\chi_\lambda = \sum_{\substack{\mu \in P^{++} \\ \mu \leqq \lambda}} L_{\lambda,\mu} \gamma(\phi_\mu),$$

and above $L_{\lambda,\mu}$ are uniquely determined.

This follows from [9; (4.4.9)]. We note here that

$$W_\mu(q^{-1})^{-1} \sum_{w \in W} e^{w\mu} \prod_{\alpha > 0} \frac{1 - q^{-1} e^{-w\alpha}}{1 - e^{-w\alpha}} \in \mathbb{Z}[q^{-1}][P]^W \quad \text{(cf. [9; (3.3.8)(iii)])}.$$

3. Proof of Theorem 1.3

3.1. We first recall Macdonald's result on the Plancherel measure. Let $\hat{P} = \mathrm{Hom}(P, U(1))$, the Pontrjagin dual of P. Here $U(1) = \{z \in \mathbb{C} \mid |z| = 1\}$. Let ds be the normalized Haar measure on $\hat{P}$, Regard e^λ $(\lambda \in P)$ as a character on $\hat{P}$ by $e^\lambda(s) = s(\lambda)$. Hence $\int_{\hat{P}} e^\lambda(s) \overline{e^\mu(s)}\, ds = \delta_{\lambda,\mu}$ (Kronecker's delta) for $\lambda, \mu \in P$, where $\overline{e^\mu}$ is the complex conjugate of e^μ (i.e., $\overline{e^\mu(s)} = e^{-\mu}(s)$ for $s \in \hat{P}$).

From now on, we shall assume $q \in \mathbb{R}$ and $q > 1$ in the rest of this section. We define the measure $d\mu(s)$ on $\hat{P}$ by

$$d\mu(s) = \frac{W(q^{-1})}{|W|} \prod_{\alpha > 0} \left| \frac{1 - e^\alpha(s)}{1 - q^{-1} e^\alpha(s)} \right|^2 ds = \frac{W(q^{-1})}{|W|} \prod_{\alpha \in R} \frac{1 - e^\alpha(s)}{1 - q^{-1} e^\alpha(s)} ds.$$

Let us define the inner product on $C(\hat{P})$, the space of continuous functions on $\hat{P}$ by

$$\langle f, g \rangle = \int_{\hat{P}} f(s) \overline{g(s)}\, d\mu(s) \qquad (f, g \in C(\hat{P})).$$

The following is a reformulation of [7; (5.1.2)].

Theorem 3.2 (Macdonald). *For $\lambda, \mu \in P^{++}$, we have*

$$\langle \gamma(\phi_\lambda), \gamma(\phi_\mu) \rangle = \begin{cases} q^{2\langle \lambda, \rho^\vee \rangle} W(q^{-1}) / W_\lambda(q^{-1}) & \text{if } \lambda = \mu; \\ 0 & \text{otherwise.} \end{cases}$$

Now we calculate $L_{\lambda,\mu}$ in (2.7) by using (3.2). For $\lambda, \mu \in P^{++}$ with $\lambda \geqq \mu$, (3.2) asserts that

$$\langle \chi_\lambda, \gamma(\phi_\mu)\rangle = L_{\lambda,\mu} q^{2\langle \mu, \rho^\vee\rangle} W(q^{-1})/W_\mu(q^{-1}). \tag{3.3}$$

On the other hand,

$$\langle \chi_\lambda, \gamma(\phi_\mu)\rangle = \frac{W(q^{-1})}{|W|} \int_{\hat{P}} \chi_\lambda(s)\overline{\gamma(\phi_\mu)(s)} \prod_{\alpha\in R} \frac{1-e^\alpha(s)}{1-q^{-1}e^\alpha(s)} ds.$$

But, by the well-known identity

$$\chi_\lambda = \sum_{w\in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)-\rho} / \prod_{\alpha>0} (1-e^{-\alpha})$$

and by (2.5),

$$\begin{aligned}\langle \chi_\lambda, \gamma(\phi_\mu)\rangle &= \frac{q^{\langle \mu,\rho^\vee\rangle} W(q^{-1})}{|W|\, W_\mu(q^{-1})} \int_{\hat{P}} \Big(\sum_{w\in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)-\rho}(s)\Big) \\ &\quad \cdot \Big(\sum_{\in W} \operatorname{sgn}(y) e^{-y(\mu+\rho)+\rho}(s) \prod_{\alpha>0} (1-q^{-1}e^{-y\alpha}(s))^{-1}\Big) ds \\ &= q^{\langle \mu,\rho^\vee\rangle} \frac{W(q^{-1})}{W_\mu(q^{-1})} \int_{\hat{P}} \sum_{w\in W} \operatorname{sgn}(w) e^{w(\lambda+\rho)-(\mu+\rho)}(s) \\ &\quad \cdot \prod_{\alpha>0} (1-q^{-1}e^{-\alpha}(s))^{-1} ds.\end{aligned}$$

By the definition (1.1) of $\hat{\mathscr{P}}$,

$$e^{w(\lambda+\rho)-(\mu+\rho)}(s) \prod_{\alpha>0} (1-q^{-1}e^{-\alpha}(s))^{-1} = \sum_{\kappa\in P} \hat{\mathscr{P}}(\kappa; q^{-1}) e^{w(\lambda+\rho)-(\mu+\rho)-\kappa}(s)$$

(the right hand side converges by virtue of the assumption on q). Therefore we have

$$\langle \chi_\lambda, \gamma(\phi_\mu)\rangle = q^{\langle \mu,\rho^\vee\rangle} \frac{W(q^{-1})}{W_\mu(q^{-1})} K_{\lambda,\mu}(q^{-1}). \tag{3.4}$$

Hence (3.3) and (3.4) show that

$$L_{\lambda,\mu} = q^{-\langle \mu,\rho^\vee\rangle} K_{\lambda,\mu}(q^{-1}). \tag{3.5}$$

Combining (2.5), (2.7), (3.5) and replacing q^{-1} by q, we get (1.4) for $0<q<1$. This completes the proof of Theorem 1.3.

4. Proof of Theorem 1.5

4.1. Let $\leqq$ denote the Bruhat ordering on W_{aff} with respect to S_{aff}. We extend this ordering to $\tilde{W}$ by

$$xy \leqq x'w \Leftrightarrow x = x' \quad \text{and} \quad y \leqq w \qquad (x, x' \in \Omega,\ y, w \in W_{\mathrm{aff}}).$$

Let $P_{y,w}(q)\in\mathbb{Z}[q]$ $(y, w\in\tilde{W};\ y\leqq w)$ be Kazhdan-Lusztig polynomials [4] of $\tilde{W}$. (In [4], only Coxeter groups are considered, but we extend them to our case by $P_{xy,xw}(q)=P_{y,w}(q)$ for $x\in\Omega$ and $y, w\in W_{\mathrm{aff}}$ with $y\leqq w$; see [6].) There is a unique ring automorphism $\psi\mapsto\bar{\psi}$ $(\psi\in H(\tilde{W},q))$ of $H(\tilde{W},q)$ such that $\overline{q^{1/2}}=q^{-1/2}$ and $\overline{T_w}=(T_{w^{-1}})^{-1}$. Note that $H(\tilde{W},q;W)$ is stable under '—'-operation because $\bar{\phi}_0=\phi_0$. The polynomials $P_{y,w}(q)$ are characterized by the following properties (4.2)-(4.4):

(4.2) $\overline{q^{-\ell(w)/2}\sum_{y\leqq w} P_{y,w}(q)\,T_y}=q^{-\ell(w)/2}\sum_{y\leqq w} P_{y,w}(q)\,T_y.$

(4.3) $P_{y,w}(q)$ is a polynomial in q of degree $(\ell(w)-\ell(y)-1)/2$ if $y\leqq w$.

(4.4) $P_{w,w}(q)=1.$

We let i be the automorphism of $\tilde{W}$ defined by $i(wt_\lambda)=w_0ww_0t_{-w_0(\lambda)}$ $(w\in W,\ \lambda\in P)$. Here w_0 is the longest element of W. Then i stabilizes S and $\{s_0\}$. Hence we have

(4.5) $$P_{i(y),i(w)}(q)=P_{y,w}(q)\qquad (y, w\in\tilde{W};\ y\leqq w).$$

Let w_λ be the longest element of $Wt_\lambda W$ $(\lambda\in P^{++})$. It is known that $w_\lambda=w_0t_\lambda$ and $\ell(w_\lambda)=\ell(w_0)+2\langle\lambda,\rho^\vee\rangle$ (see [2; Proposition 1.23]). We note the following (which is easily checked):

(4.6) $$y\leqq w_\lambda \Leftrightarrow y\in Wt_\mu W\quad (\exists\,\mu\in P^{++};\ \mu\leqq\lambda).$$

As in [5], we consider the element

$$q^{-\langle\lambda,\rho^\vee\rangle}\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}} P_{w_\mu,w_\lambda}(q)\,\phi_\mu.$$

By virtue of (4.5), (4.6) and the fact $P_{yw_\mu z,w_\lambda}(q)=P_{w_\mu,w_\lambda}(q)$ for all $y, z\in W$ ([4; (2.3.g)]), (4.2) implies

(4.7) $$\overline{q^{-\langle\lambda,\rho^\vee\rangle}\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}} P_{w_\mu,w_\lambda}(q)\,\phi_\mu}=q^{-\langle\lambda,\rho^\vee\rangle}\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}} P_{w_\mu,w_\lambda}(q)\,\phi_\mu.$$

Let us apply the Satake isomorphism γ on the both sides of (4.7). Then we have

(4.8) $$q^{\langle\lambda,\rho^\vee\rangle}\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}} P_{w_\mu,w_\lambda}(q^{-1})\,\gamma(\overline{\phi_\mu})=q^{-\langle\lambda,\rho^\vee\rangle}\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}} P_{w_\mu,w_\lambda}(q)\,\gamma(\phi_\mu).$$

Now we calculate the value $\gamma(\overline{\phi_\mu})=\mathrm{Tr}(\overline{\phi_\mu}\,|\,M_\eta)$. As

$$\phi_\mu=(W(q^{-1})/W_\mu(q^{-1}))\,\phi_0\cdot T_{t_{-\mu}}\cdot\phi_0,$$

we have

$$\overline{\phi_\mu}=(W(q)/W_\mu(q))\,\phi_0\cdot T^{-1}_{t_{w_0(\mu)}}\cdot\phi_0.$$

Since $-w_0(\mu)\in P^{++}$, the same argument as in (2.6) shows that

$$\mathrm{Tr}(\phi_0\cdot T^{-1}_{t_{w_0(\mu)}}\cdot\phi_0|M_\eta)=W(q^{-1})^{-1}q^{-\langle\mu,\rho^\vee\rangle}\sum_{w\in W}e^{ww_0\mu}\prod_{\alpha>0}\frac{1-q^{-1}e^{-w\alpha}}{1-e^{-w\alpha}}$$
$$=W(q)^{-1}q^{-\langle\mu,\rho^\vee\rangle}\sum_{w\in W}e^{w\mu}\prod_{\alpha>0}\frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}.$$

Thus we have

$$\gamma(\overline{\phi_\mu})=q^{-\langle\mu,\rho^\vee\rangle}W_\mu(q)^{-1}\sum_{w\in W}e^{w\mu}\prod_{\alpha>0}\frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}. \tag{4.9}$$

Applying (2.5) and (4.9) to (4.8), we get

$$\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}}q^{\langle\lambda-\mu,\rho^\vee\rangle}P_{w_\mu,w_\lambda}(q^{-1})\,W_\mu(q)^{-1}\sum_{w\in W}e^{w\mu}\prod_{\alpha>0}\frac{1-qe^{-w\alpha}}{1-e^{-w\alpha}}$$
$$=\sum_{\substack{\mu\in P^{++}\\ \mu\leqq\lambda}}q^{-\langle\lambda-\mu,\rho^\vee\rangle}P_{w_\mu,w_\lambda}(q)\,W_\mu(q^{-1})^{-1}\sum_{w\in W}e^{w\mu}\prod_{\alpha>0}\frac{1-q^{-1}e^{-w\alpha}}{1-e^{-w\alpha}}.$$

Since $\deg P_{w_\mu,w_\lambda}(q)\leqq(\ell(w_\lambda)-\ell(w_\mu)-1)/2=\langle\lambda-\mu,\rho^\vee\rangle-1/2$ if $\mu<\lambda$, the right hand side is a polynomial (with coefficients in $\mathbb{Z}[P]^W$) in q^{-1} and the left hand side is a polynomial in q. The constant terms are equal to

$$\sum_{w\in W}e^{w\lambda}\prod_{\alpha>0}(1-e^{-w\alpha})^{-1}=\chi_\lambda.$$

Therefore we have proved (1.6).

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