Lecture 17 - The Weyl Group

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1 Information on Weyl groups

Theorem 1.1 *Let $\triangle$ be a base of Φ*

a) If $\gamma \in E$ is regular, there exists some $\sigma \in \mathcal{W}$ so that $(\sigma(\gamma), \alpha) > 0$ for all $\alpha \in \triangle$; in particular $\mathcal{W}$ acts transitively on Weyl chambers

b) If $\triangle'$ is any other base, there is an element $\sigma' \in \mathcal{W}$ so that $\sigma'(\triangle') = \triangle$; in particular $\mathcal{W}$ acts transitively on bases

c) If α is any root, there exists some $\sigma \in \mathcal{W}$ so that $\sigma(\alpha) \in \triangle$

d) $\mathcal{W}$ is generated by the σ_α for $\alpha \in \triangle$

e) If $\sigma(\triangle) = \triangle$ for some $\sigma \in \mathcal{W}$, then $\sigma = 1$; in particular $\mathcal{W}$ acts simply transitively on the bases.

Pf. For the time being we'll make a distinction between $\mathcal{W}'$, the group generated by reflection in elements in $\triangle$, and the full Weyl group $\mathcal{W}$.

a) Set $\delta = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. Choose σ so that $\big(\sigma^{-1}(\delta), \gamma\big)$ is as small as possible. But then

$$\big(\sigma^{-1}(\delta), \gamma\big) \;\geq\; \big(\sigma^{-1}(\sigma_\alpha\delta),\, \gamma\big) \;=\; (\delta\,-\,\alpha,\, \sigma\gamma) \;=\; \big(\sigma^{-1}\delta,\, \gamma\big)\,-\,(\alpha,\, \sigma\gamma) \qquad (1)$$

Equality exists only when γ is not regular, which is assumed not to be the case. Thus $(\sigma\gamma, \alpha) > 0$ as promised.

b) We have seen that $\mathcal{W}'$ transitively permutes the Weyl chambers. Since every base determine a chamber, there is an element $\sigma \in \mathcal{W}'$ that takes the chamber $\mathcal{C}$ (corresponding to $\triangle$) to the chamber $\mathcal{C}'$ (corresponding to $\triangle'$). By the indecomposability of $\triangle$ and $\triangle'$, they must now be the same (up to ordering).

c) Find a vector γ so that $\alpha \perp \gamma$ but so that γ is not perpendicular to any other root. Slightly perturbing γ, we can make (α, γ) smaller than any other positive (β, γ). Clearly then α is indecomposable.

d) Given any root β, by (b) and (c) we know there is a transformation $\sigma \in \mathcal{W}'$ so that $\sigma(\beta) \in \triangle$. Then

$$\sigma_\beta \; = \; \sigma^{-1}\sigma_{\sigma^{-1}\beta}\sigma \tag{2}$$

is an element of $\mathcal{W}'$. Thus any generator of $\mathcal{W}$ is an element of $\mathcal{W}'$, so that $\mathcal{W} = \mathcal{W}'$.

e) Any element $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$ can be written minimally. By a previous lemma, σ takes at least one positive element (namely α_t) to a negative element, and therefore does not act on $\triangle$ itself.

□

According to the theorem, given any map $\sigma \in \mathcal{W}$ we can write

$$\sigma \; = \; \sigma_{\alpha_1} \dots \sigma_{\alpha_t} \tag{3}$$

where $\alpha_i \in \triangle$. If t is minimal, we say t is the *length* of the map σ relative to the base $\triangle$, denoted $l(\sigma)$. We also define $l(1) = 0$. Define $n(\sigma)$ to be the number of positive roots $\beta > 0$ for which $\sigma(\beta) < 0$.

Lemma 1.2 *Given any $\sigma \in \mathcal{W}$, we have $n(\sigma) = l(\sigma)$.*

Pf. This is an induction on $l(\sigma)$. Clearly it is true for $l(\sigma) = 0$. Assume the theorem holds for all τ with $l(\tau) \leq t-1$, and let σ be so that $l(\sigma) = t$. Then if $\sigma = \sigma_{\alpha_1} \dots \sigma_{\alpha_t}$ is a minimal expression for σ, we have that σ sends α_t to $-\alpha_t$. Then $\sigma' = \sigma\sigma_{\alpha_t} = \sigma_{\alpha_1} \dots \sigma_{\alpha_{t-1}}$ (which is also minimal) sends α_t to α_t, but since σ_{α_t} sends all positive roots besides α_t to positive roots, we have $n(\sigma') = n(\sigma) - 1$. Since clearly also $l(\sigma') = l(\sigma) - 1$, the theorem is proved by induction. □

Lemma 1.3 *Assume λ, μ are vectors in the closure of $\mathfrak{C}(\triangle)$, and assume some $\sigma \in \mathcal{W}$ has $\sigma\lambda = \mu$. Then σ is a product of simple reflections that fix λ; in particular $\lambda = \mu$.*

Pf. Since σ sends at least one simple root, say α, to a negative root, we have

$$0 \geq (\sigma\alpha,\, \mu) \; = \; \left(\alpha,\, \sigma^{-1}\mu\right) \; = \; (\alpha,\, \lambda) \; \geq \; 0 \tag{4}$$

Therefore equality holds, so $\sigma_\alpha\lambda = \lambda$. But then $\sigma'\lambda = \sigma\sigma_\alpha\lambda = \sigma\lambda = \mu$. However $l(\sigma') = l(\sigma) - 1$, so we can apply induction on $l(\sigma)$ to obtain the result. □

2 Irreducible root systems

A root system Φ is called irreducible if it cannot be partitioned into non-empty subsets, so that the elements of either subset is perpendicular to all vectors in the other.

Lemma 2.1 *A root system Φ is irreducible if and only if every base $\triangle$ is irreducible.*

Pf. If a base $\triangle$ is reducible so that $\triangle = \triangle' \cup \triangle''$, then the Weyl group, which is generated by that base, is also reducible. This is due to the fact that if $(\alpha, \beta) = 0$ then $\sigma_\alpha \sigma_\beta = \sigma_\beta \sigma_\alpha$, so $\mathcal{W} = \mathcal{W}' \times \mathcal{W}''$. Every root is conjugate to a simple root, but since $\mathcal{W}'$, $\mathcal{W}''$ fix, respectively, the (orthogonal) subspaces spanned by $\triangle'$, $\triangle''$, any root that is conjugate to an element of $\triangle'$ is not conjugate to an element of $\triangle''$, and vice-versa. Thus since the subspaces $span\triangle'$ and $span\triangle''$ are fixed under $\mathcal{W}$, every root is either in one or the other, so therefore the root system is decomposable. The converse is even easier. $\square$

Lemma 2.2 *Let Φ be an irreducible root system. Relative to the partial ordering $<$ on Φ^+ there is a unique maximal root β (in particular $ht\,\beta > ht\,\alpha$ for all $\alpha \in \Phi$ and $(\beta, \alpha) \geq 0$ for all $\alpha \in \Phi^+$).*

Pf. Choose a β so that β is maximal among all roots that it is comparable to. We first prove that it is comparable to all simple roots. If not, there is some $\alpha \in \triangle$ so that $\beta - \alpha$ is not a root. Thus $(\beta, \alpha) \leq 0$. But if equality holds, then α is orthogonal to all simple roots that comprise β, so that $\triangle$ is partitioned orthogonally, an impossibility. Since $(\beta, \alpha) < 0$, we have that $\beta + \alpha$ is a root, and $\beta + \alpha > \beta$, an impossibility.

Since β is comparable to all simple roots, we can see it is comparable to all positive roots. Specifically, if $\beta - \alpha$ is not a root then $(\beta, \alpha) \leq 0$, but equality cannot hold because both are positive linear combinations of base roots and β involves all base roots, so because $(\beta, \alpha) < 0$ then $\beta + \alpha$ is a root and is comparable to β, contradicting the maximality of β. $\square$

Lemma 2.3 *Let Φ be irreducible. Then $\mathcal{W}$ acts irreducibly on E. In particular the $\mathcal{W}$-orbit of any $\alpha \in \Phi$ spans E.*

Pf. If $\mathcal{W}$ does not act irreducibly, so $E' \subset E$ is a proper subspace preserved by $\mathcal{W}$, then the orthogonal compliment E'' also has an action of $\mathcal{W}$. By reducibility, clearly either $\alpha \in E'$ or else $E' \subset P_\alpha$ and similarly for E''. As a consequence all roots lie in either E' or E'', contradicting the irreducibility of Φ. $\square$

Lemma 2.4 *Let Φ be irreducible. Then at most two root lengths occur in Φ, and all roots of a given length are conjugate under $\mathcal{W}$.*

Pf. Provided $(\alpha, \beta) \neq 0$, the only possible ratios of their length-squares are $\frac{1}{3}, \frac{1}{2}, 1, 2, 3$. Further, for any α, its orbit under $\mathcal{W}$ contains a vector α' so that $(\alpha', \beta) > 0$. This proved the first assertion, since the existence of three root lengths would imply a ratio of $\frac{2}{3}$.

Let α, β be roots of the same root length. We may assume $(\alpha, \beta) > 0$. If they are distinct, then one of $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$ is ± 1, and therefore $\langle \alpha, \beta \rangle = \langle \beta, \alpha \rangle = +1$. Then

$$(\sigma_\alpha \sigma_\beta \sigma_\alpha)(\beta) \;=\; (\sigma_\alpha \sigma_\beta)(\beta - \alpha) \;=\; \sigma_\alpha(-\alpha) \;=\; \alpha \tag{5}$$

□

Lemma 2.5 *Let Φ be irreducible, and have two root lengths. Then the maximal root is long.*

Pf. Let β be the maximal root. Because β is comparable to all positive roots, $(\beta, \alpha) \geq 0$ for all $\alpha \in \Phi^+$, and equality cannot hold, so β is in the fundamental Weyl chamber. Given any α, it is in some chamber, and since the Weyl group transitively permutes chamber, we can assume α is in the fundamental chamber. Then $\beta - \alpha$ is a positive root, and $(\gamma, \beta - \alpha) \geq 0$ for any γ in the closure of the fundamental chamber. Then $|\beta|^2 - (\beta, \alpha) \geq 0$ and $(\alpha, \beta) - |\alpha|^2 \geq 0$, so that $|\beta|^2 > (\alpha, \beta) \geq |\alpha|^2$. □