Honors Single Variable Calculus 110.113

November 28, 2023

Contents

1 The natural numbers

Lecture 1, Monday, August 28th, Last updated: 01/09/23, dmy.
Reading: [13, Ch.2-3]

We assume the notion of *set*, 2, and take it as a primitive notion to mean a "collection of distinct objects."

Learning Objectives

Next eight lectures:

- To construct the objects:

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}$$

  and define the notion of *sets*, 2.

- To prove properties and reason with these objects. In the process, you will learn various proof techniques. Most importantly, *proof by induction* and *proof by contradiction*.

This lecture:

- how to define the natural numbers, $\mathbb{N}$, and appreciate the role of *definitions*.

- how to apply induction. In particular, we would see that even proving statements as associativity of natural numbers is nontrivial!

Pedagogy

1. $\mathbb{N}$ is presented differently in distinct foundations, such as ZFC or type theory. Our presentation is to be *agnostic* of the foundation. From a working mathematician point of view, it *does not matter*, how the natural numbers are constructed, as long as they obey the properties of the axioms, 1.1.

2. We take the point of view that in mathematics, there are various type of objects. Among all objects studied, some happened to be *sets*. Some presentation of mathematics[a] will regard all objects as sets.

The various types of mathematics are more or less equivalent in our context.

[a] such as ZFC

Why should we delve into the foundations? Two reasons:

1. Foundational language is how many mathematicians do new mathematics. One defines new axioms and explores the possibilities.

2. How can we even discuss mathematics without having a rigorous understanding of our objects?

Discussion

A *natural (counting) number*[a], as we conceived informally is an element of

$$\mathbb{N} := \{0, 1, 2, \ldots\}$$

What is ambiguous about this?

- What does "$\cdots$" mean? How are we sure that the list does not cycle back?

- How does one perform operations?

- What *exactly* is a natural number? What happens if I say

$$\{0, A, AA, AAA, AAAA, \ldots\}$$

  are the numbers?

We will answer these questions over the course.

[a]It does not matter if we regard 0 as a natural number or not. This is a convention.

Forget about the natural numbers we love and know. If one were to define the *numbers*, one might conclude that the numbers are about a concept.

Axioms 1.1. The *Peano Axioms*: [1] Guiseppe Peano, 1858-1932.

1. 0 is a natural number.
$$0 \in \mathbb{N}$$

2. if n is a natural number then we have a natural number, called the *successor of* n, denoted $S(n)$.
$$\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$$

[1]In 1900, Peano met Russell in the mathematical congress. The methods laid the foundation of *Principa Mathematica*

3. 0 is not the successor of any natural number.

$$\forall n \in \mathbb{N}, S(n) \neq 0$$

4. If $S(n) = S(m)$ then $n = m$.

$$\forall n, m \in \mathbb{N}, S(n) = S(m) \Rightarrow n = m$$

5. Principle of induction. Let $P(n)$ be any *property* on the natural number n. Suppose that

   a. $P(0)$ is true.
   b. When ever $P(n)$ is true, so is $P(S(n))$.

   Then $P(n)$ is true for all n natural numbers.

Discussion

What could be meant by a *property?* The principle of induction is in fact an *axiom schema*, consisting of a collection of axioms.

- "n is a prime".
- "$n^2 + 1 = 3$".

We have not yet shown that any collection of object would satisfy the axioms. This will be a topic of later lectures. So we will assume this for know.

Axiom 1.2. There exists a set $\mathbb{N}$, whose elements are the *natural numbers*, for which 1.1 are satisfied.

There can be many such systems, but they are all equivalent for doing mathematics.

Discussion

With only up to axiom 4: This can be *not* so satisfying. What have we done? We said we have a collection of objects that satisfy some concept F="natural numbers". But how do we know, Julius Ceasar does not belong to this concept?

Definition 1.3. We define the following natural numbers:

$$1 := S(0), 2 := S(1) = S(S(0)), 3 := S(2) = S(S(S(0)))$$

$$4 := S(3), 5 := S(4)$$

Intuitively, we want to continue the above process and say that whatever created iteratively by the above process are the *natural numbers*.

Discussion

- Give a set that satisfies axioms 1 and 2 but not 3.
- Give a set that satisfies axioms 1,2 and 3 but not 4.
- Give a set satisfying axioms 1,2,3 and 4, but not 5.

$$\{n/2 : n \in \mathbb{N}\} = \{0, 0.5, 1, 1.5, 2, 2.5, \cdots\}$$

Proposition 1.4. 1 is not 0.

Proof. Use axiom 3. □

Proposition 1.5. 3 is not equal to 0.

Proof. $3 = S(2)$ by definition, 1.3. If $S(2) = 0$, then we have a contradiction with Axiom 2, 1.1. □

1.1 Addition

Definition 1.6. (Left) Addition. Let $m \in \mathbb{N}$.

$$0 + m := m$$

Suppose, by induction, we have defined $n + m$. Then we define

$$S(n) + m := S(n + m)$$

In the context of 1.13, for each n, our function is $f_n := S : \mathbb{N} \to \mathbb{N}$ is $a_{S(n)} := S(a_n)$ with $a_0 = m$.

Proposition 1.7. For $n \in \mathbb{N}$, $n + 0 = n$.

Proof. Warning: we cannot use the definition 1.6. We will use the principle of induction. What is the *property* here in Axiom 5 of 1.1? The property $P(n)$ is "$0 + n = n$" for each $n \in \mathbb{N}$. We will also have to check the two conditions 5a. and 5b.

a "$P(0)$ *is true.*". People refer to this as the "base case $n = 0$": $0 + 0 = 0$, by 1.6.

b "*If $P(m)$ is true then $P(m+1)$ is true*". The statement "*Suppose $P(m)$ is true*" is often called the "inductive hypothesis". Suppose that $m + 0 = m$. We need to show that $P(S(m))$ *is true*, which is

$$S(m) + 0 = S(m)$$

By def, 1.6,

$$S(m) + 0 = S(m + 0)$$

By hypothesis,

$$S(m + 0) = S(m)$$

By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Such proof format is the typical example for writing inductions, although in practice we will often leave out the italicized part.

Example

Prove by induction

$$\sum_{i=1}^{n} i^2 := 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

We observed that we have successfully shown *right* addition with respect to 0 behaves as expected.

Discussion

What should we expect $n + S(m)$ to be?

- Why can't we use 1.6?
- Where would we use 1.7?

Proof is hw.

Proposition 1.8. Prove that for $n, m \in \mathbb{N}$, $n + S(m) = S(n + m)$.

Proof. We induct on n. Base case: $m = 0$.

5b. Suppose $n + S(m) = S(n + m)$. We now prove the statement for

$$S(n) + S(m) = S(S(n) + m)$$

by definition of 1.6,

$$S(n) + S(m) = S(n + S(m))$$

which equals to the right hand side by hypothesis.

□

Proposition 1.9. Addiction is commutative. Prove that for all $n, m \in \mathbb{N}$,

$$n + m = m + n$$

Proof. We prove by induction on n. With m fixed. We leave the base case away. □

Proposition 1.10. Associativity of addition. For all $a, b, c \in \mathbb{N}$, we have

$$(a + b) + c = a + (b + c)$$

Proof. hw. □

Discussion

Can we define "+" on any collection of things? What are examples of operations which are noncommutative and associative? For example, the collection of words?

$$+ : (\text{Seq. English words}) \times (\text{Seq. English words}) \to (\text{Seq. English words})$$

$$\text{"a"}, \text{"b"} \mapsto \text{"ab"}$$

This can be a meaningless operation. Let us restrict to the collection of *interpreable* outcomes. This can be formalized in the setting of *context free grammars.*

In the following examples, there is *structural ambiguity.*

1. (Ice) (cream latte)
2. (British) ((Left) (Waffles on the Falkland Islands))
3. (Local HS Dropouts) (Cut) (in Half)
4. (I ride) (the) (elephant in (my pajamas))
5. (We) ((saw) (the) (Eiffel tower flying to Paris.))

2,3 are actual news title. Parsing sentences have a rich history, see [6].

What use is there for addition? We can define the notion of *order* on $\mathbb{N}$. We will see later that this is a *relation* on $\mathbb{N}$.

Definition 1.11. Ordering of $\mathbb{N}$. Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \leq n$ iff there is $a \in \mathbb{N}$, such that $n = m + a$.

1.2 Multiplication

Now that we have addition, we are ready to define multiplication as 1.6.

Definition 1.12.

$$0 \cdot m := 0$$

$$S(n) \cdot m := (n \cdot m) + m$$

1.3 Recursive definition

What does the induction axiom bring us? Please ignore the following theorem on first read.

Theorem 1.13. Recursion theorem. Suppose we have for each $n \in \mathbb{N}$,

$$f_n : \mathbb{N} \to \mathbb{N}$$

Let $c \in \mathbb{N}$. Then we can assign a natural number a_n for each $n \in \mathbb{N}$ such that

$$a_0 = c \quad a_{S(n)} = f_n(a_n) \forall n \in \mathbb{N}$$

Discussion

The theorem seems intuitively clear, but there can be pitfalls.

- When defining $a_0 = c$, how are we sure this is *not* redefined after some future steps? This is Axiom 3. of 1.1

- When defining $a_{S(n)}$ how are we sure this is not redefined? This uses Axiom 4. of 1.1.

- One rigorous proof is in [5, p48], but requires more set theory.

Proof. The property $P(n)$ of 1.1 is "{ a_n is well-defined}". Start with $a_0 = c$.

- Inductive hypothesis. Suppose we have defined a_n - meaning that there is only one value!

- We can now define $a_{S(n)} := f_n(a_n)$.

□

1.4 References and additional reading

- Nice lecture notes by Robert.

- Russell's book [10, 1,2] for an informal introduction to cardinals.

2 Naïve set theory: the axioms

Week 1, Wednesday, August 30th

As in the construction of $\mathbb{N}$, we will define a *set* via axioms.

Discussion

Why put a foundation of sets?

- This is to make rigorous the notion of a "collection of mathematical objects".
- This has its roots in cardinality. How can you "count" a set without knowing how to define a collection?
- The concept of a set can be used - and is till used in practice - as a practical foundation of mathematics. This forms the basis of *classical mathematics*.

Learning Objectives

In this lecture:

- We discuss *set* in detail. We will need this to construct the integers, $\mathbb{Z}$.
- We illustrate what one *can* and *can not* do with sets.

Pedagogy

Again, we don't say what they *are*. This approach is often taken, such as [5].

Discussion

What object can be called a *set*?

A *set* should be

- determined by a *description of the objects* [a] For example, we can consider

$$E := \text{"The set of all even numbers"}$$

$$P := \text{"The set of all primes"}$$

- If x is an object and A is a set, then we can ask whether $x \in A$ or $x \notin A$. *Belonging* is a primitive concept in sets.

[a] this set consists of all objects satisfying this description and *only those objects*.

In this lecture we will discuss some axioms.

Axiom 2.1. If A is a *set* then A is also a *object*.

Axiom 2.2. Axiom of extension. Two sets A, B are equal if and only if (for all objects x , $(x \in A \Leftrightarrow x \in B)$)

Axiom 2.3. There exist a set $\emptyset$ with no elements. I.e. for any object x, $x \notin \emptyset$.

Proposition 2.4 (Single choice)**.** Let A be nonempty. There exists an object x such that $x \in A$.

Proof. Prove by contradiction. Suppose the statement is false. Then for all objects x, $x \notin A$. By axiom of extension, $A = \emptyset$. □

Discussion

How did we use the axiom of extension? Colloquially, some mathematicians would say "trivially true".

2.1 Subcollections

Definition 2.5. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1, 2\} \subset \{1, 2, 3\}$.

2.2 Comprehension axiom

Definition 2.6. Axiom of Comprehension.

Definition 2.7. *General* comprehension principle. (The paradox leading one). For any property φ, one may form the set of all x with property $P(x)$, we denote this set as

$$\{x \mid P(x)\}$$

Proposition 2.8. Russell, 1901. The general comprehension principle cannot work.

Proof. Let

$$R := \{x : x \text{ is a set and } x \notin x\}$$

This is a set. Then

$$R \in R \Leftrightarrow R \notin R$$

$\square$

Discussion

How is this different from the axiom of specification?

Discussion

How can it even be the case that $x \in x$, for a set? Can this hold for any set x below?

- $\emptyset$
- The set of all primes.
- The set of natural numbers.

The latter two shows that : this set itself is *not even a number*! Indeed, In Zermelo-Frankel set theory foundations it will be proved that $x \notin x$ for all set x. So the set R in 2.8 is the *set of all sets.*

2.3 References

- A nice introduction to set theory is Saltzman's notes [11].
- The relevant section in Tao's notes, [13, 3].
- For the axioms of set theory, an elementary introduction is [5], and also notes by Asaf, [8].

3 Homework for week 1

[2]

In these exercises: our goal is to get familiar with

- manipulating axioms in a definition.
- the notion of the principle of induction.

Problems:

1. Prove 5 is not equal to 2.
2. (*) Prove 1.8.
3. (*) Prove 1.9, assuming 1.8 if necessary.
4. (*) Prove 1.10 assuming 1.8, 1.9 if necessary.
5. (*) $n \in \mathbb{N}$ is *positive* if and only if $n \neq 0$. Prove that if $a, b \in \mathbb{N}$, *a is positive,* then $a + b$ is positive.
6. (***) Let M be a set with 2023 elements. Let N be a positive integer, $0 \leq N \leq 2^{2023}$. Prove that it is possible to color each subset of S so that
   (a) The union of two white subsets is white.
   (b) The union of two black subsets is black.
   (c) There are exactly N white subsets.
7. (**) Integers 1 to n are written ordered in a line. We have the following algorithm:
   - If the first number is k then reverse order of the first k numbers.

   Prove that 1 appears first in the line after a finite number of steps.
8. (**) We defined $\leq$ of natural numbers in 1.11. A finite sequence $(a_i)_{i=1}^n := \{a_1, \ldots, a_n\}$ of natural numbers is *bounded*, if there exists some other natural number M, such that $a_i \leq M$ for all $1 \leq i \leq n$. Show that every finite sequence of natural numbers, $a_1, \ldots, a_n$, is bounded.

[2] *Due: Week 2, Write the numbering of the three questions to be graded clearly on the top of the page. Each unstarred problem worth 12 points. Each star is an extra 5 points.*

Hints for problems

1: prove using Peano's axioms. First prove 3 is not equal to 0.

6: The number 2023 is irrelevant. Induct on the size of the set M. What happens when $M = 1$? For the inductive argument: suppose the statement is true when M has size n. In the case when M has size $n+1$, consider when

- $0 \leq N \leq 2^n$. Use the hypothesis on the first n elements.
- $N \geq 2^n$. Use symmetry here that there was nothing special about "white".

7: Let us consider the inductive scenario. If $n+1$ were in the first position, we are done by induction. Thus, let us suppose $n+1$ never appears in the first position, *and* it is not in the last position, which is given by number $k \neq n+1$.

- Would the story be the same if we switch the position of k and $n+1$?

Discussion

As one observes, both 6 and 7 uses a natural *symmetry* in the problem.

4 Power set construction

Lecture 3: will miss one class due to Labor day.
Reading: [13, Ch.3.1-4], [9, 2].

Learning Objectives

In last lectures, we

- Defined $\mathbb{N}$ axiomatically using the Peano axioms.
- Used induction to prove properties of operations as $+$ and $\times$ on $\mathbb{N}$.

In the next two lectures

- Discuss the remaining axioms of set theory. We begin by discussing new notions: *subsets*, 2.1, We end with the construction of the power set.
- Discuss *equivalence relation*, 7, and *ordered pairs*, 7.1. which constructs the integers and the rationals

4.1 Remaining axioms of set theory

Week 2
In this section we continue from previous lecture and discuss remaining axioms from what is known as the *Zermelo-Fraenkel (ZF) axioms of set theory*, due to Ernest Zermelo and Abraham Fraenkel.

Axiom 4.1. Singleton set axiom. If a is an object. There is a set $\{a\}$ consists of just one element.

Axiom 4.2. Axiom of pairwise union. Given any two sets A, B there exists a set $A \cup B$ whose elements which belong to either A or B or both.

Often we would require a stronger version.

Axiom 4.3. Axiom of union. Let A be a set of sets. Then there exists a set

$$\bigcup A$$

whose objects are precisely the elements of the set.

Example

Let

- $A = \{\{1,2\}, \{1\}\}$
- $A = \{\{1,2,3\}, \{9\}\}$

Discussion

Using the axioms, can we get from $\{1,3,4\}$ to $\{2,4,5\}$?

We will now state the power set axiom for completeness but revisit again.

Axiom 4.4. Axiom of power set. Let X, Y be sets. Then there exists a set Y^X consists of all functions $f : X \to Y$.

We will review definition of function later, 4.11.

4.2 Replacement

If you are an ordinary mathematician, you will probably never use replacement.

Axiom 4.5. Axiom of replacement. For all $x \in A$, and y any object, suppose there is a statement $P(x,y)$ pertaining to x and y. $P(x,y)$ satisfies the property for a given x, there is a *unique* y. There is a set

$$\{y : P(x,y) \text{ is true for some } x \in A\}$$

Discussion

This can intuitively be thought of as the set

$$\{y : y = f(x) \text{ some } x \in A\}$$

That is, *if* we can define a function, then the range of that function is a set. However, $P(x,y)$ described may *not* be a function, see [4, 4.39].

Example

- Assume, we have the set $S := \{-3, -2, -1, 0, 1, 2, 3, \dots\}$, $P(x, y)$ be the property that $y = 2x$. Then we can construct the set

$$S' := \{-6, -4, -2, 0, 2, 4, 6, \dots\}$$

- If x is a set, then so is $\{\{y\} : y \in x\}$. Indeed, we let

$$P(x, y) : "y = \{x\}"$$

Again, this is a *schema* as described previously in axiom of comprehension 2.6.

Proposition 4.6. The axiom of comprehension 2.6 follows from axiom of replacement 4.5.

Proof. Let ϕ be a property pertaining to the elements of the set X. We can define the property [3]

$$\psi(x, y) : \begin{cases} y = \{x\} & \text{if } \phi(x) \text{ is true} \\ \emptyset & \text{if } \phi(x) \text{ is false} \end{cases}$$

Let

$$\mathcal{A} := \{y : \exists x, \quad \psi(x, y) \text{ is true}\}$$

be the collection of sets defined by axiom of replacement. Then by union axiom

$$\bigcup \mathcal{A} := \{x \in X : \phi(x) \text{ is true}\}$$

$\square$

4.3 Axiom of regularity (well-founded)

As a first read, you can skip directly and read 4.9. For a set S, and a binary relation, $<$ on S, we can ask if it is *well-founded.* It is well founded when we can do *induction.*

Definition 4.7. A subset A of S is $<$*-inductive* if for all $x \in S$,

$$(\forall t \in S, t < x) \Rightarrow x \in A$$

Definition 4.8. Let X, Y we denote the *intersection of* X *and* Y[4] as

$$X \cap Y := \{x \in X : x \in X \text{ and } x \in Y\} = \{y \in Y : y \in X \text{ and } y \in Y\}$$

X and Y are *disjoint* if $X \cap Y = \emptyset$.

[3] This can be written in the language of "property" via $(\phi(x) \to y = \{x\}) \wedge (\neg\phi(x) \to y = \emptyset)$

[4] which exists, thanks to axiom of comprehension.

One would ould ask the $\in$ relation on all sets to be inductive. Then what would be required for that $A \notin A$?

Axiom 4.9. Axiom of foundation (regularity) The $\in$ relation is "well-founded". That is for all nonempty sets x, there exists $y \in x$ such that either

- y is not a set.
- or if y is a set, $x \cap y = \emptyset$.

An alternative way to reformulate, is that y is a *minimal element* under $\in$ relation of sets.

Example

- $\{\{1\}, \{1,3\}, \{\{1\}, 2, \{1,3\}\}\}$. What are the $\in$-minimal elements?
- Can I say that there is a "set of all sets"? No, see how.

One can use axiom of foundation that we cannot have an infinite descending sequence:

Proposition 4.10. There are no infinite descent $\in$-chains. Suppose that (x_n) is a sequence of nonempty sets. Then we cannot have

$$\cdots \in x_{n+1} \in x_n \cdots \in x_1 \in x_0$$

Similarly one can use axiom of replacement for the product, at p32.

4.4 Function

Discussion

How would you intuitively define a function?

Definition 4.11. Let X, Y be two sets. Let

$$P(x, y)$$

be a *property* pertaining to $x \in X$ and $y \in Y$, such that for all $x \in X$, there *exists a unique* $y \in Y$ such that $P(x, y)$ is true. A *function associated to* P is an object

$$f_P : X \to Y$$

such that for each $x \in X$ assigns an output $f_P(x) \in Y$, to be the unique object such that $P(x, f_P(x))$ is true. [5]

[5] We will often omit the subscript of P.

- X is called the *domain*
- Y is called the *codomain*.

Definition 4.12. The *image*...?

Discussion

What kind of properties P does not satisfy the condition of being function?

- "$y^2 = x$".
- "$y = x^2$".

5 The various sizes of infinity

Lecture 4: for competition. We will use our defined notion of, "counting numbers" or "inductive numbers", $\mathbb{N}$ to *count* other sets. This is *cardinality*. In this section, we fix sets X, Y.

Definition 5.1. A function $f : X \to Y$ is

- *injective* if for all $a, b \in X$, $f(a) = f(b)$ implies $a = b$.
- *surjective* if for all $b \in Y$, exists $a \in X$ st. $f(a) = b$.
- *bijective* if f is both injective and bijective.

Example

- the map from $\emptyset \to X$ an injection. The conditions for injectivity vacuously holds.
- $\mathbb{N}$ is in bijection with the set of even numbers,

$$\mathbb{E} := \{n \in \mathbb{N}\,;\, \exists k \in \mathbb{N} \,:\, n = 2k\}$$

- there is no bijection from an empty set to a nonempty set.

Definition 5.2. Two sets X, Y have *equal cardinality* if there is a bijection

$$X \simeq Y$$

- A set is said to have *cardinality* n if

$$\{i \in \mathbb{N} \,:\, 1 \leq i \leq n\} \simeq X$$

  In this case, we say X is *finite*. Otherwise, X is *infinite*.

- A set X is *countably infinite*[6] if it has same cardinality with $\mathbb{N}$.

Definition 5.3. We denote the *cardinality of a set* X by $|X|$. [7]

[6] Or *countable*. Sometimes countable means (finite *and* countably infinite).

[7] This definition does *not make sense yet!*. What if a set has two cardinality? Let us assume this is well-defined first. See question 2.

Discussion

To think about infinity is an interesting problem. Consider Hilbert's Grand Hotel.

- One new guest.
- 1000 guest.
- Hilbert Hotel 2 move over.
- Hilbert chain. Directs customer m in hotel n to position $3^n \times 5^m$. (This shows that $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$.)

Historically, some take *cardinal numbers* as i.e. the equivalence class of bijective sets as the primitive notion.

Definition 5.4. Let X, Y be sets: We denote

- $|X| \leq |Y|$ if there is an injection from X to Y.
- $|X| = |Y|$ if there is a bijection between X and Y.
- $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

One of the beautiful results in Set theory is the Schroeder Bernstein theorem.

Theorem 5.5. The $\leq$ relation on cardinality, is reflexive: if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$. [8]

Without axiom of choice, one cannot say the following: for all sets X and Y, either $|Y| \leq |X|$ or $|X| \leq |Y|$.

[8] Why is this not obvious? Challenge: google and try to understand the proof.

6 Homework for week 2

Due: Week 3, Friday. All questions in 6.1, Boolean algebra is compulsory. Select 3 other questions to be graded.

Reading: A nice reference in set theory, [3, 4]. We collectively refer to the axioms of set theory we have discussed thus far as the ZF axioms. We did not discuss the axiom of replacement, [13, 3.5] and regularity. This will be left as required reading for certain problems.

Problems

1. Let A, B, C be sets.

   (a) Prove set inclusion, is reflexive and transitive, i.e.

   $$(A \subseteq B \wedge B \subseteq A) \Rightarrow A = B$$

   $$(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$$

   the notation $\wedge$ here reads "and".

   (b) Prove that the union operation $\cup$ on sets 4.2, is associative and commutative:

   $$A \cup (B \cup C) = (A \cup B) \cup C$$
   $$A \cup B = B \cup A$$

2. (**) Let I be a set and that for all $\alpha \in I$, I have a set A_α. [9] Read about the axiom of replacement; see [13, Axiom 3.5] or 4.5.

   (a) Prove that under the ZF axioms, one can form the union of the collection:

   $$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha \,:\, \alpha \in I\}$$

   In particular, explain why the following two objects

   i.
   $$\{A_\alpha \,:\, \alpha \in I\}$$

[9] For example, if $I = \{a, b, c\}$, then I have three sets

$$A_a, A_b, A_c$$

ii.
$$\bigcup \{A_\alpha : \alpha \in I\}$$
are sets.

(b) Give a one line explanation briefly describing why axiom of union 4.3 is insufficient to construct the set $\bigcup_{\alpha \in I} A_\alpha$.

3. The *axiom of regularity* states

Axiom 6.1. [13, 3.9] If A is a nonempty set, then there is at least one element $x \in A$ which is either not a set or, (if it is a set) disjoint from A.

Prove (with singleton set axiom) that for all sets $A, A \notin A$.

4. (***) Let A, B, C, D be sets. This exercise shows that we can actually construct *ordered pairs* using the ZF axioms.[10] Prove

   - We can construct the following set [11]
   $$\langle A, B\rangle := \{A, \{A, B\}\}$$
   from the axioms of set theory.
   - $\langle A, B\rangle = \langle C, D\rangle$ if and only if $A = C, B = D$. For this part you will require the *axiom of regularity.* in problem 3. You are free to use the results there.

5. This is a variation of problem 4 [12]. Suppose for two sets A, B we define
$$[A, B] = \{\{A\}, \{A, B\}\}$$
In this case, the problem is a lot easier. Prove $[A, B] = [C, D]$ if and only if $A = C, B = D$.

6. (***) Show that the collection
$$\{Y : Y \text{ is a subset } X\}$$
is a set using the ZF axioms. We denote this as the power set 2^X, where 2 is regarded as the two elements set $\{0, 1\}$. You will need to use the axiom of replacement.
Here are two important remarks on possible false solutions:

[10] Another definition is discussed in or [13, 3.5.1], where they assume this as an axiom.
[11] RIP. So another model of this is $\langle A, B\rangle := \{\{A\}, \{A, B\}\}$
[12] which is what I should have written

(a) (Ryan's) if your property for axiom of replacement $P(x,y) =$ "y is a subset of x" then this is *not correct.* The condition for replacement is that *there is at most one* y, [13, 3.6].

(b) (Kauí's) You cannot use axiom of comprehension, this is similar to Russell's paradox!

As a hint: $\{0,1\}^X$ is a set, by 4.4. For $Y \subseteq X$, $f \in \{0,1\}^X$, let $P(f,Y)$ be the property that

$$Y = f^{-1}(1) := \{x \in X \,:\, f(x) = 1\}$$

6.1 Boolean algebras

This section is compulsory. Boolean algebras form the foundation of probability theory. We will need this later when we get to the projects.

Reading: For some overview of the context, see [2, 1-3], [7, 1], or Tao's Lecture 0 on probability theory.

Definition 6.2. Let Ω be a set. A *Boolean algebra* in Ω is a set $\mathcal{E}$ of subsets of Ω (equivalently, $\mathcal{E} \subseteq 2^\Omega$) satisfying

1. $\emptyset \in \mathcal{E}$
2. closed under unions and intersections.
$$E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$$
$$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$$
3. closed under complements.

A *σ-algebra in* Ω is a Boolean algebra in Ω such that it satisfies

4. Countable[13] closure. If $A_i \in \mathcal{E}$ for $i \in \mathbb{N}$, then $\bigcup A_i \in \mathcal{E}$.

Problems

1. Prove that $\mathcal{E} := \{\emptyset, \Omega\}$ is a σ-algebra.
2. Prove that $2^\Omega := \{E \,:\, E \subseteq \Omega\}$ is a σ-algebra.
3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Hints for problems

3. There are 3 cases. What happens $A = \emptyset$ or $A = \Omega$? Now consider the case $A \neq \emptyset$ and $A \neq \Omega$.

[13] A set X is countable if it is in bijection with $\mathbb{N}$. We will explore this word in further detail in the future.

Solutions to Week 2

Featured solutions: Solutions to Q2, by Yvette, Q4, by Sri, Q6, by Tyler, Boolen algebra, by Granger.

Q2:

2) Let I be a set & $\forall \alpha \in I$, there is a set A_α.

a) Prove under ZF axioms, one can form the union of the collection:

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\}$$

$\forall \alpha$ in I, there is a property connecting $\alpha \to A_\alpha$ $\to$ Axiom of Replacement

$\bigcup_{\alpha \in I} A_\alpha$ contains the image of the property on $\forall \alpha \in I$ $\to$ Axiom of Union/Collection

By Axiom of Replacement, $\forall \alpha \in I$, $\exists A_\alpha$ s.t. $\exists P(\alpha, A_\alpha)$ pertaining to α & A_α.

Then there is a set s.t. $\bigcup_{i=0}^{\infty} A_\alpha = \bigcup\{A_\alpha : \alpha \in I\}$

Q4

(4) (a) Proposition: we can construct the set $\langle A, B \rangle := \{A, \{A, B\}\}$

Proof: A and B are sets.

sets are objects. by axiom

$\therefore \exists$ Singleton sets: $\{A\}$, $\{B\}$ by singleton set axiom

$\exists$ Pair set: $\{A, B\}$ by pair set axiom

Treating $\{A, B\}$ as an object,

$\exists$ singleton set: $\{\{A, B\}\}$

$\{A\} \cup \{\{A, B\}\} = \{A, \{A, B\}\}$ by pairwise union axiom

$\therefore$ we have constructed a set $\{A, \{A, B\}\}$

1b) Proposition: $\langle A, B\rangle = \langle C, D\rangle$ iff $A = C \wedge B = D$

Proof: $\langle A, B\rangle := \{A, \{A, B\}\}$, $\langle C, D\rangle := \{C, \{C, D\}\}$

$\{A, \{A, B\}\} = \{C, \{C, D\}\}$

$A = C$ or $A = \{C, D\}$

Suppose $A = \{C, D\}$, then $C = \{A, B\}$.

A and C are sets.

Sets are objects. by axiom

$\therefore \exists$ pair set: $\{A, C\}$ by pair set axiom

By axiom of regularity, either A or C is disjoint from $\{A, C\}$.

Case #1: If A is disjoint,

$C \in A$, but $C \in \{A, C\}$.

This is a contradiction.

Case #2: If C is disjoint,

$A \in C$, but $A \in \{A, C\}$.

This is a contradiction.

We have proven that $A \neq \{C, D\}$.

$\therefore \underline{A = C}$

Then, $\{A, B\} = \{C, D\}$.

$\{C, B\} = \{C, D\}$. from A=C

$C = C \Rightarrow \underline{B = D}$ or $C = D$, $B = C$

$\Rightarrow B = C = D$

$\Rightarrow \underline{B = D}$

We have proven that $\underline{\langle A, B\rangle = \langle C, D\rangle \Rightarrow A = C \wedge B = D}$.

If $A = C \wedge B = D$,

$LHS = \{A, \{A, B\}\} \overset{\text{from } A=C}{=} \{C, \{C, B\}\} \overset{\text{from } B=D}{=} \{C, \{C, D\}\} = RHS$

We have proven that $\underline{A = C \wedge B = D \Rightarrow \langle A, B\rangle = \langle C, D\rangle}$.

$\therefore$ $\underline{\langle A, B\rangle = \langle C, D\rangle \text{ iff } A = C \wedge B = D}$. <proven>

Q6:

6 Prop. the collection $\{Y : Y \text{ is a subset of } X\}$ is a set.

Pf: • X is a set.

• $0, 1$ are objects, by Pair set Axiom $\exists \{0,1\}$

• by Powerset Ax. $\exists \{0,1\}^X$, which is a set of functions that map X to $\{0,1\}$

B— each one of these functions, f, map a different group of X, or a subset, to 1 and the rest of the elements to zero.

• by the axiom of replacement, we can replace each of these elements with a Y such that $Y = f^{-1}(\{1\})$ for that f, replacing each f with the subset that f uniquely correlates to 1.

$$f \in \{0,1\}^X$$
$$p(f, Y) : Y = f^{-1}(\{1\}) := \{x \in X : f(x) = 1\}$$

– by relplacement $\exists \{Y : Y = f^{-1}(\{1\}) \text{ for some } f \text{ in } \{0,1\}^X\}$

• and by replacing each f, with each f mapping a subgroup to 1, with the subgroup they map, we get a set that is all subgroups of X

$\therefore$ or, $\{Y : Y = f^{-1}(\{1\}) \text{ for some } f \text{ in } \{0,1\}^X\} = \{Y : Y \text{ is a subset of } X\}$

$\therefore$ $\{Y : Y \text{ is a subset of } X\}$ is a set.

1. Prove that $\varepsilon := \{\varnothing, \Omega\}$ is a $\sigma - algebra$.

   1. $$\varnothing \in \Omega$$

   2. Closed under unions and intersections.

      a. $$\varnothing \cup \Omega = \Omega \in \varepsilon$$

---

      b. $$\varnothing \cap \Omega = \Omega \in \varepsilon$$

   3. Closed under complements.

      a. $$\varnothing^c = \Omega \in \varepsilon$$

      b. $$\Omega^c = \varnothing \in \varepsilon$$

   4. Closed under countable closure.

      a. $\forall i \in N,\ A_i = \varnothing$, then $\bigcup_{i=0}^{\infty} A_i = \varnothing \in \varepsilon$

      b. $\exists i \in N,\ A_i = \Omega$ and for all the rest $A_j = \varnothing$, then $\bigcup_{i=0}^{\infty} A_i = \Omega \in \varepsilon$

Therefore $\varepsilon := \{\varnothing, \Omega\}$ is a $\sigma - algebra$.

2. Prove that $2^{\Omega} := \{E : E \subseteq \Omega\}$ is a $\sigma - algebra$.

   1. $\emptyset \in 2^{\Omega}$ since $\emptyset$ is defined to contain no elements and thus vacuously satisfies as a subset for any set according to the definition of a subset.
   2. Closed under pairwise unions and intersections.
      a. If we can prove the union of two subsets of any set is another subset of that set, then, since 2^{Ω} contains all subsets of Ω, the union of any two subsets in 2^{Ω} will be closed in 2^{Ω}.

         Accessory Proof 1

         $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

         We assume by definition, if $x \in A \rightarrow x \in C$ and $x \in B \rightarrow x \in C$.

         By definition, if $x \in A$ or $x \in B \rightarrow x \in A \cup B$.

         Therefore, by our original assumption $x \in A \cup B \rightarrow x \in C$.

      b. If we can prove the intersection of two subsets of any set is another subset of that set, then, since 2^{Ω} contains all subsets of Ω, the intersection of any two subsets in 2^{Ω} will be closed in 2^{Ω}.

         Accessory Proof 2

         $A \subseteq C$ and $B \subseteq C$, then $A \cap B \subseteq C$.

         We assume by definition, if $x \in A \rightarrow x \in C$ and $x \in B \rightarrow x \in C$.

         By definition, if $x \in A$ and $x \in B \rightarrow x \in A \cap B$.

         Therefore, by our original assumption $x \in A \cap B \rightarrow x \in C$.

3. Closed under complements.

    a. Since all subsets of Ω are contained in 2^{Ω} by definition and all complements of an arbitrary subset, A, of Ω will just generate another subset, A^c, of Ω by the definition of complements.

4. Closed under countable union.

    a. If we can prove the countable union of subsets of any set yields another subset of the same set, then, since 2^{Ω} contains all subsets of Ω, the countable union of any combination of subsets in 2^{Ω} will be closed in 2^{Ω}.

Accessory Proof 3 (Prop: $\forall i \in N,\ A_i \subseteq B \rightarrow \bigcup_{i=0}^{\infty} A_i \subseteq B$).

We assume, by definition subset $\forall i \in N, x \in A_i \rightarrow x \in B$.

By definition of union $\forall i \in N, x \in A_i \rightarrow x \in \bigcup_{i=0}^{\infty} A_i$.

Therefore $x \in \bigcup_{i=0}^{\infty} A_i \rightarrow\ x \in B$.

3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Assume the set ε is the smallest σ-algebra in Ω containing A.

Therefore, to satisfy part 1 of the Boolean algebra, $\emptyset \in \varepsilon$.

Also, to satisfy closed under complement $\emptyset^c = \Omega \in \varepsilon$ and $A^c \in \varepsilon$.

Therefore $\varepsilon = \{\emptyset, A, A^c, \Omega\}$, (Special case where $A = \emptyset$ and $A^c = \Omega$ then $\varepsilon = \{\emptyset, \Omega\}$)

1. $$\emptyset \in \varepsilon$$

2. Closed under pairwise unions and intersections.

   a. $\emptyset \cup A = A \in \varepsilon, \emptyset \cup \emptyset = \emptyset \in \varepsilon, \emptyset \cup A^c = A^c \in \varepsilon,$

   $\emptyset \cup \Omega = \Omega \in \varepsilon, A \cup A = A \in \varepsilon, A \cup A^c = \Omega \in \varepsilon,$

   $A \cup \Omega = \Omega \in \varepsilon, A^c \cup A^c = A^c \in \varepsilon, \mathrm{A}^c \cup \Omega = \Omega \in \varepsilon,$

   $\Omega \cup \Omega = \Omega \in \varepsilon.$

   b. $\emptyset \cap A = \emptyset \in \varepsilon, \emptyset \cap \emptyset = \emptyset \in \varepsilon, \emptyset \cap A^c = \emptyset \in \varepsilon,$

   $\emptyset \cap \Omega = \emptyset \in \varepsilon, A \cap A = A \in \varepsilon, A \cap A^c = \emptyset \in \varepsilon,$

$A \cap \Omega = \mathrm{A} \in \varepsilon, A^c \cap A^c = A^c \in \varepsilon, \mathrm{A^c} \cap \Omega = \mathrm{A^c} \in \varepsilon,$

$\Omega \cap \Omega = \Omega \in \varepsilon.$

3. Closed under complements.
   a. $\varnothing^c = \Omega \in \varepsilon, \Omega^c = \varnothing \in \varepsilon, (A)^c = A^c \in \varepsilon, (A^c)^c = A \in \varepsilon$
4. Closed under countable union.
   a. $\forall i, A_i = \varnothing \rightarrow \cup_{i=0}^{\infty} = \varnothing \in \varepsilon,\ \forall i, A_i = A \rightarrow \cup_{i=0}^{\infty} = A \in \varepsilon,$

   $\forall i, A_i = A^c \rightarrow \cup_{i=0}^{\infty} = A^c \in \varepsilon,\ \forall i, A_i = \Omega \rightarrow \cup_{i=0}^{\infty} = \Omega \in \varepsilon,$

   For all following cases $\forall i \in N$.

   For any case where $\exists i : A_i = \Omega \rightarrow \cup_{i=0}^{\infty} = \Omega \in \varepsilon.$

   For any case where $\exists i : A_i = A^c$ and $\exists j : A_j = A \rightarrow \cup_{i=0}^{\infty} = \Omega \in \varepsilon.$

   For any case where $\forall i : A_i \neq A^c \wedge \Omega$ and $\exists j : A_j = A \rightarrow \cup_{i=0}^{\infty} = \mathrm{A} \in \varepsilon.$

   For any case where $\forall i : A_i \neq A \wedge \Omega$ and $\exists j : A_j = A^c \rightarrow \cup_{i=0}^{\infty} = \mathrm{A^c} \in$.

7 Equivalence Relation

Week 3 Reading: [13, Ch.3.5, Ch.4], *On the construction of* $\mathbb{Q}$*, see* [4, 2.4].

Learning Objectives

Last few lectures:

- Defined the natural numbers and sets axiomatically.
- Discussed how *cardinality* came up from "counting" sets.

This and next lecture:

- discuss equivalence relation.
- construct $\mathbb{Z}, \mathbb{Q}$. Extend addition and multiplication in this context.

7.1 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 7.1. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1, 2\} = \{2, 1\}$

In ordered pairs

- $(1, 2) \neq (2, 1)$

Definition 7.2. Let X, Y be two sets. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Currently, we can either put the existence of such a set as an axiom, or use the axioms of set theory, this is in hw.

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [5, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 7.3. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$
$$a \sim a$$
- *Transitive*. For all $a, b, c \in A$,
$$a \sim b, b \sim c \Rightarrow a \sim c$$
- *Symmetric*. For all $a, b \in A$,
$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Often times, people do not describe the subset R, but describe it a relation *equivalently* as a rule: saying $a, b \in A$ are related if some property $P(a, b)$ is true. In short hand, one writes

$$a \sim b \text{ iff } \ldots$$

Definition 7.4. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as [14]

$$[x] := \{y \in A \,:\, x \sim y\}$$

An element in such an equivalence is called a *representative* of that class.

Definition 7.5. Quotient set. Given an equivalence relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

Example

Consider $\mathbb{N}$ and the equivalence relation that $a \sim b$ iff a and b have the same parity. [a]

- There are two equivalence classes: the odds and evens.
- For the odd class, a *representative*, or an element in the equivalence class, is 3.

[a] i.e. both or odd or even.

There is a relation between equivalence and partition of sets.

Definition 7.6. A *partition* of a set X is a collection ???

7.2 Integers

What are the integers? It consists of the natural numbers and the negative numbers. What is *subtraction*? We do not know yet. Can we define *negative* numbers without referencing minus sign? Yes, we can. Say

$$-1\text{is "}0 - 1\text{" is } (0, 1)$$

Discussion

Let us say we define the integers as pairs (a, b) where $a, b \in \mathbb{N}$. Would this be our desired

$$\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$$

- How many -1s are there?

But we have a problem, there are multiple ways to express -1. Our system cannot have multiple -1s. What are other ways We can also have $1 - 2$, or the pair $(1, 2)$.

[14] It does not matter if we write $\{y \in A \,:\, y \sim x\}$ by symmetry condition.

Discussion

Now that we have our $\mathbb{Z}$, how do we define addition? [a]Can we leverage our understanding?

[a]What is addition abstractly? It is an operation $+ : X \times X \to X$.

Intuitively, the *integers* is an expression [15] of non-negative integers, (a, b), thought of as $a - b$. Two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 7.7. Let

$$R \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

consists of all pairs (a, b) and (c, d) such that $a + d = b + c$. Equivalently,

$$R := \{(a, b), (c, d) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \,:\, a + d = b + c\}$$

The *integers* is the set

$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

Definition 7.8. Addition, multiplication. [13, 4.1.2] .

We can now finally define negation.

Definition 7.9. [13, 4.1.4].

Proposition 7.10. Algebraic properties. Let $x, y, z \in \mathbb{Z}$.

- Addition
  - Symmetric $x + y = y + x$.
  - Admits identity element.

7.3 Rational nunbers

Reading: [4, 2.4]. Be careful of the notation used! See 7.11.

Definition 7.11. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

$$\mathbb{Z} \setminus \{0\} := \{n \in \mathbb{Z} \,:\, n \neq 0\}$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote *the equivalence class* of pair (a, b) by $[a/b]$

[15]Rather than a pair, as an expression has multiple ways of presentation

Again, we need the notion of addition, multiplication, and negation.

Definition 7.12. Let $[a/b], [c/d] \in \mathbb{Q}$. Then

1. Addition:
$$[a/b] + [c/d] := [(ad + bc)/bd]$$

2. Multiplication
$$[a/b] \cdot [c/d] := [(ac)/(bd)]$$

3. Negation.
$$-[a/b] := [(-a)/b]$$

7.3.1 Is addition well-defined?

This subsection gives an extensive discussion of well-definess. The notation we use here is from 7.11. In 1. we *want* to define a function:

$$+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

which takes as input two equivalence class and outputs a new one. Let us consider two equivalence class

$$x := \left\{a'/b' \,:\, a'/b' \sim a/b\right\} \in \mathbb{Q}$$
$$y := \left\{c'/d' \,:\, c'/d' \sim c/d\right\} \in \mathbb{Q}$$

To add these two classes, we proceeded as follows:

1. We pick two representatives from each class, let us say a/b of x and c/d of y.

2. We define
$$x + y := [(ad + bc)/bd]$$

Why can't we say this is the definition of addition, yet? In the above description, $x + y$ can take *more than one possible value* - which is not a function!
For example, one could have chosen other pair of representatives, a'/b', and c'/d', and obtained $x + y$ as

$$[(a'd' + b'c')/b'd']$$

Thus, we have to check that

$$[(a'd' + b'c')/b'd'] = [(ad + bc)/bd]$$

To check this: by definition, this means we have to show:

$$bd(a'd' + b'c') = (ad + bc)b'd'$$

which is

$$bda'd' + bdb'c' = adb'd' + bcb'd' \tag{1}$$

Now $a'/b' \sim a/b$ and $c/d \sim c'/d'$ means $a'b = ab'$ and $cd' = c'd$, Now using commutativity in $\mathbb{Z}$, and the required two equalities for Eq. 1

$$bda'd' = a'bdd' =^{(a'b=ab')} ab'dd' = adb'd'$$

$$bdb'c' = c'dbb' =^{(cd'=c'd)} cd'bb' = bcb'd'$$

7.4 Order relation

Similarly, we can define also define order relation.

Definition 7.13. Let $x \in \mathbb{Q}$,

- x is *positive* iff $x = [a/b]$ where a, b are positive integers, we often denote positive integers as $\mathbb{Z}_{>0}$.
- x is *negative* iff $x = -y$ where y is some positive rational.

With the notion of positive rationals[16] from def. 7.13, we can define order relation $<, \leq$ on $\mathbb{Q}$.

Definition 7.14. Let $x, y \in \mathbb{Q}$, then we denote

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is zero or positive.

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 7.15. $\sqrt{2}$ is not rational.

Proof. ??? □

[16] The same trick is used to define order in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

8 Homework for week 3

Due: Week 4, Saturday. You will select 3 problems to be graded.
Problems 1-3 are on cardinality. Problem 4 is on a general construction of equivalence relations. Problems 5-7 is about addition, multiplication, and division on $\mathbb{Z}$ and $\mathbb{Q}$.

1. Show that the relation $\leq$ is transitive, i.e. $|X| \leq |Y|, |Y| \leq |Z|$ then $|X| \leq |Z|$.

2. (**) Prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite. [17] Prove that $\mathbb{Q}$ is countably infinite. *You are free to use results from previous problems and theorems stated in lectures.*

3. (**) Let X be any set. Prove that there is no surjection (hence, bijection) between X and $\{0,1\}^X$. Deduce that $\{0,1\}^{\mathbb{N}}$ is uncountable. Argue the first part by contradiction: suppose there exists a surjection
$$f : X \to \{0,1\}^X$$
   - Consider the set
   $$A = \{x \in X \,:\, x \notin f(x)\}$$
   - As f is a surjection (write the general definition) there must exists $a \in X$ such that $f(a) = A$. Do case work on whether $a \in A$ or $a \notin A$.

4. (**) Let X be any set. Recall that a binary relation on X, is any subset $R \subseteq X \times X$. We define $R^{(n)}$ as follows
   - For $n = 0$,
   $$R^{(0)} = \{(x,x) \,:\, x \in X\}$$
   - Suppose $R^{(n)}$ has been defined.
   $$R^{(n+1)} := \left\{(x,y) \in X \times X \,:\, \exists z \in X, (x,z) \in R^{(n)}, (z,y) \in R\right\}$$

   (a) Show that
   $$R^t := \bigcup_{n \geq 1} R^{(n)} = R^{(1)} \cup R^{(2)} \cup \cdots$$
   defines a *smallest* transitive relation on X containing R. i.e. if Y is any other transitive relation on X containing R, then $R^t \subseteq Y$.

[17] Knowing the Cartesian product is required for this problem, skip 5. and 6. if unfamiliar.

(b) Show that

$$R^{tr} := \bigcup_{n \geq 0} R^{(n)} = R^{(0)} \cup R^{(1)} \cdots$$

is the *smallest* reflexive and transitive relation on X. i.e. if Y is any other transitive and reflexive relation on X containing R, then $R^{tr} \subseteq Y$.

5. (***) Show that addition, product, and negation are well-defined for rational numbers; see def. 7.11 or [13, 4.2]. You are free to use any facts and properties you know about $\mathbb{Z}$, such as the cancellation law.

6. (*) Let $x, y, z \in \mathbb{Z}$. Use the definition of addition and multiplication from 7.8, or [13, 4.1], show :

   (a) $x(y + z) = xy + xz$.

   (b) $x(yz) = (xy)z$.

   You are free to use any facts and properties you know about $\mathbb{N}$.

7. Let $x, y \in \mathbb{Z}$. You are free to use any facts you know about $\mathbb{N}$, in particular, it would be helpful to use the following the result: [13, 2.3.3]: *Let $n, m \in \mathbb{N}$. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero.* Show that if $xy = 0$ then $x = 0$ or $y = 0$.

8.1 Tri-weekly diary

8. (**) Write a 800-1000 words diary or story. Pen down a diary on your experiences with the course topics and experiences so far, focusing particularly on:

   - Concepts or ideas that you initially found challenging or confusing. For example, the axioms of natural numbers $\mathbb{N}$, set theory, etc.
   - Topics that have piqued (if any, XD) your curiosity.
   - Topics that you wanted to be covered, and why.
   - Topics that you would like further elaboration.
   - People you find fun to be with (or scared of)!

   + (*) points for the best diary.

9 The real numbers

Week 3, Reading: [13, 5]*, notes by Todd, Cauchy's construction. Goldrei's textbook gives another construction of* $\mathbb{R}$ *using Dedekind cuts,* [4, 2.2].

Learning Objectives

We have defined $\mathbb{Q}$. To define $\mathbb{R}$.

- We use Cauchy sequence.

Pedagogy

We can define real numbers geometrically, adopted by Euclid, and mostly between 1500-1850, or as presented in [12]

- This ultimately leads to Dedekind's picture of how an irrational number sits among the rational.

9.1 Characterizing properties of $\mathbb{R}$: the completeness properrty

As with construction of $\mathbb{N}$, ultimately for $\mathbb{R}$, we are interested in the structural properties they have. The essential properties of $\mathbb{R}$ can be described by Thm. 9.1. If you have learned any algebra, this is also known as a complete ordered field.

Theorem 9.1. Properties of $\mathbb{R}$, this is a rehash of the list in [4, 2.3]. $\mathbb{R}$ is a set with

- operations $+$ and $\cdot$
- relations $=$ and $\leq$
- special elements $0, 1$ with $0 \neq 1$.

such that

1. $\leq$ is a reflexive and transitive relation.
2. $\leq$ behaves well under addition and multiplication : If $x \leq y$ and $z \geq 0$.
    - then $x + z \leq y + z$
    - $x \cdot z \leq y \cdot z$.
3. The operation $+$, def. is commutative and associative, admits inverses and admits identity 0. In other words:

- Associativity: for all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
- Commutativity: for all $x, y \in \mathbb{R}$, $x + y = y + x$.
- Admits inverse: for all $x \in \mathbb{R}$, there exists y such that

$$x + y = y + x = 0$$

- Admits identity 0: for all $x \in \mathbb{R}$,

$$x + 0 = 0 + x = x$$

4. The operation $\cdot$ is commutative and associative, admits inverses and identity 1:

5. Completeness: for any $A \subseteq \mathbb{R}$, $A \neq \emptyset$ which is bounded above has a least in upper bound *in* $\mathbb{R}$.

Proof. Properties of $+$ is left as homework. $\square$

Worthy of distinction is the last axiom.

Definition 9.2. A *partial order* on a set X, is a relation $\leq$ on X which is

- reflexive
- transitive: for all $a, b, c \in X$, if $a \leq b$, $b \leq c$, then $a \leq c$.
- antisymmetric: for all $a, b \in X$, $a \leq b$ and $b \leq a$ implies $a = b$.

Example

$(\mathbb{N}, \leq), (\mathbb{Q}, \leq), (\mathbb{Z}, \leq)$ are all partial orders. However $<$ is *not*.

We will apply the following definitions to the case of $X = \mathbb{R}$.

Definition 9.3. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation.

- $M \in X$ is a *upper bound* iff for all $x \in E$, $x \leq M$.
- $M \in X$ is a *lower bound* iff for all $x \in E$ $x \geq M$.

Definition 9.4. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation. $M \in X$ is a *least upper bound for E* if

1. M is an upper bound for E.
2. any other upper bound M' on E must satisfy $M \leq M'$.

Definition 9.5. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation. $M \in X$ is a *least upper bound for* E if

1. M is anlower bound for E.

2. any other lowerbound M' on E must satisfy $M \geq M'$.

Example

Let us consider $(\mathbb{Q}, \leq)$. What is the order relation here? see **??**. Discuss the upper bound and least upper bound for the following sets.

- $E := \{x \in \mathbb{Q} : x > 0\}$.
- $E := \left\{x \in \mathbb{Q} : x^2 < 2\right\}$
- $E := \emptyset$

Definition 9.6. Upper bounds in reals. This is the case of 9.3 with $(X, \leq) = (\mathbb{R}, \leq)$.

1. if $E \neq \emptyset$ and has an upper bound, we denote

$$\sup E$$

as its least upper bound. This exists in $\mathbb{R}$ by 16.12.

2. If $E \neq \emptyset$ it has no upper bound, we write

$$\sup E := +\infty$$

3. If $E = \emptyset$ we set.

$$\sup E = -\infty$$

we have a similar definition for greatest lower bound. We can easily extend this to the case for extended reals, see def. 16.9.

Example

What is

$$\sup E := \left\{x \in \mathbb{R} : x^2 \leq 2\right\}?$$

9.2 Cauchy sequences

Let us start by constructing $\sqrt{2}$ using $\mathbb{Q}$. The idea is to represent such a number using sequence. All inequalities and numbers discussed in this section will be rationals.

Discussion

- If a "real" number x grows continually, but is bounded, does it approach a limiting value?

Definition 9.7. Let $m \in \mathbb{Z}$. A *sequence of rational numbers* denoted $(a_n)_{n=m}^{\infty}$ is a function

$$\{n \in \mathbb{Z} : n \geq m\} \to \mathbb{Q}$$

This is a sequence starting from m. This is equivalent to the data of a function in

$$\mathrm{Fct}(\mathbb{Z}_{\geq m}, \mathbb{Q})$$

where $\mathbb{Z}_{\geq m} := \{n \in \mathbb{Z} : n \geq m\}$. The collection of all sequences in $\mathbb{Q}$, is defined as the union of all sequences starting from any $m \in \mathbb{Z}$.

$$\mathrm{Seq}(\mathbb{Q}) := \bigcup_{m \in \mathbb{Z}} \mathrm{Fct}(\mathbb{Z}_{\geq m}, \mathbb{Q})$$

The same definition works if we replace any appearance of $\mathbb{Q}$ above with $\mathbb{R}$, see def 10.3.
We can slowly increase our level of "closeness" of *a sequence* to *a point* via these three definitions.

Definition 9.8. We can slowly increase our level of "closeness" of *a sequence* to *a point* via these three definitions. Let $x \in \mathbb{Q}$, a sequence $(a_n)_{n=0}^{\infty}$ of rationals

1. Let $\varepsilon \in \mathbb{Q}_{>0}$. $(a_n)_{n=0}^{\infty} = \{a_0, a_1, \ldots\}$ is *ε-adherent* to x if exists $N \in \mathbb{N}$ st. $|a_N - x| < \varepsilon$.
2. Let $\varepsilon \in \mathbb{Q}_{>0}$ we say $(a_n)_{n=0}^{\infty}$, is *ε-close to x* if $|a_n - x| < \varepsilon$ for all $n \geq 0$.
3. Let $\varepsilon \in \mathbb{Q}_{>0}$ we say $(a_n)_{n=0}^{\infty}$ is *eventually ε-close to x* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|a_n - x| < \varepsilon$$
4. *converges to x* iff it is eventually ε-close to x for all $\varepsilon \in \mathbb{Q}_{>0}$.

We will give the same define for real sequences, see **??**

Definition 9.9. A sequence is $(x_n)_0^\infty$,

- *eventually ε-steady*, if exists some N such that for all $n, m \geq N$,

$$|x_n - x_m| < \varepsilon$$

- a *Cauchy sequence* iff for all $\varepsilon > 0$, $(x_n)_{n=0}^\infty$ is eventually ε-steady.

Example

Proofs using quantifiers. Prove for all positive rationals, ε, there exists a positive rational δ such that $\delta < \varepsilon$.
Mathematicians often translate this to notation

$$\forall \varepsilon \in \mathbb{Q}_{>0}, (\exists \delta \in \mathbb{Q}_{>0}, \delta < \varepsilon)$$

but this is up to taste.

Proof. ??? □

Proposition 9.10. Prove that $(a_n)_{n=1}^\infty := (1/n)_{n=1}^\infty$ is a Cauchy sequence.

Proof. See text [13]. □

Example

- $(n)_{n=0}^\infty, (\sqrt{n})_{n=0}^\infty$ are not Cauchy.

Discussion

We want to use a Cauchy sequence to represent the real numbers. However, two sequences can represent the same number. Consider

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

$$1.5, 1.42, 1.4143, 1.41422, \ldots$$

Definition 9.11. Two sequences $(x_n)_{n=0}^\infty, (y_n)_{n=0}^\infty$ are *eventually ε-close.* if there exists some N, such that for all $n \geq N$,

$$|a_n - b_n| < \varepsilon$$

Discussion

Are the following two sequences Cauchy equivalent?

- $(10^{10}, 10^{1}000, 1, 1, \ldots)$ and $(1, 1, \ldots,)$

Definition 9.12. Let $\mathcal{C}$ denote the set of cauchy sequences.[18] Then we set

$$\mathbb{R} := \mathcal{C}/\sim$$

where $\sim$ is the equivalence relation that

$$(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty} \text{ if and only if } (x_n)_{n=0}^{\infty} \text{ and } (y_n)_{n=0}^{\infty} \text{ are eventually } \varepsilon\text{-close}$$

We denote the equivalence of $(x_n)_{n=0}^{\infty}$ as $[(x_n)]$. Note that in [13], Tao denotes the class as $\mathrm{LIM}_{n\to\infty} x_n$.

Definition 9.13. Let $x, y \in \mathbb{R}$. Choose two representatives[19], say $(x_n)_{n=0}^{\infty} \in x$ and $(y_n)_{n=0}^{\infty} \in y$, then

- the sum of x and y is defined as

$$x + y := [(x_n + y_n)_{n=0}^{\infty}]$$

  Addition is well-defined. [13, 5.3.6, 5.3.7].

- the product of x and y is defined as

$$x \cdot y := [(x_n \cdot y_n)_{n=0}^{\infty}]$$

Now we can define the order relation on $\mathbb{R}$, compare to def. 7.13

Definition 9.14. $x \in \mathbb{R}$ is

- *positive* iff there exists a positive rational $c \in \mathbb{Q}_{>0}$, and $(x_n)_{n=0}^{\infty} \in x$ such that $x_n \geq c$ for all $n \geq 1$.
- *negative* iff $-(x_n)_{n=0}^{\infty} := (-x_n)_{n=0}^{\infty}$ is positive.

Definition 9.15. Let $x, y \in \mathbb{R}$, we say

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is positive or $x = y$.

[18] This is a subset of $\mathbb{Q}^{\mathbb{N}}$.

[19] an element of the equivalence class

10 More on Sequences

Reading: [13, 6].
Previously, we have worked with Cauchy sequences of rational numbers, see def 9.9, these were used to define $\mathbb{R}$. Now let us work with Cauchy sequences of real numbers:

Definition 10.1. A sequence $(x_n)_0^\infty$ of real numbers, i.e. a map $\mathbb{N} \to \mathbb{R}$, is

- *eventually ε-steady*, if exists some N such that for all $n, m \geq N$,

$$|x_n - x_m| < \varepsilon$$

- a *Cauchy sequence* iff for all $\varepsilon > 0$, $(x_n)_{n=0}^\infty$ is eventually ε-steady.

Learning Objectives

- Understand the notion of supremum and infima.
- Note that all convergent sequence is bounded, but is the bounded sequences convergent? This is the monotone convergence theorem. [13, 6.3.8].

We have the following hierarchy of sequences in reals:

$$\{\text{Convergent Seq in } \mathbb{R}\} \subseteq \{\text{Cauchy Seq in } \mathbb{R}\} \subseteq \{\text{Bounded Seq in } \mathbb{R}\}$$

which we will short hand denote as

$$\mathrm{CvgSeq}(\mathbb{R}) \subseteq \mathrm{CcSeq}(\mathbb{R}) \subseteq \mathrm{BddSeq}(\mathbb{R})$$

We may ask: what bounded sequence are convergent?

Theorem 10.2. Let $(a_n)_{n=0}^\infty$

Now that we have defined $\mathbb{R}$, we will review again the notion of convergence.

Definition 10.3. Sequences of real numbers. Same as 9.7 but with $\mathbb{R}$ instead of $\mathbb{Q}$. We write it for completeness A *sequence of real numbers* denoted $(a_n)_{n=m}^\infty$ is a function

$$\{n \in \mathbb{Z} : n \geq m\} \to \mathbb{R}$$

This is a sequence starting from m. This is equivalent to the data of a function in

$$\mathrm{Fct}(\mathbb{Z}_{\geq m}, \mathbb{R})$$

where $\mathbb{Z}_{\geq m} := \{n \in \mathbb{Z} : n \geq m\}$. The collection of all sequences in $\mathbb{R}$, is denoted as $\mathrm{Seq}(\mathbb{R})$.

Definition 10.4. Same as 9.8 but with real sequences and converging to real number. Let $x \in \mathbb{R}$.

1. Let $\varepsilon \in \mathbb{R}_{>0}$. $(a_n)_{n=0}^{\infty} = \{a_0, a_1, \ldots\}$ is ε-*adherent* to x if exists $N \in \mathbb{N}$ st. $|a_N - x| < \varepsilon$.

2. Let $\varepsilon \in \mathbb{R}_{>0}$ we say $(a_n)_{n=0}^{\infty}$, is ε-*close to* x if $|a_n - x| < \varepsilon$ for all $n \geq 0$.

3. Let $\varepsilon \in \mathbb{R}_{>0}$ we say $(a_n)_{n=0}^{\infty}$ is *eventually* ε-*close to* x if there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|a_n - x| < \varepsilon$$

4. *converges to* x iff it is eventually ε-close to x for all $\varepsilon \in \mathbb{R}_{>0}$. IN this case we denote
$$\lim_{n \to \infty} (a_n) = a$$

Discussion

Consider our favourite sequence of 1.

$$0.9, 0.99, 0.999$$

- What are choices of x that satisfy 1?

Discussion

- In 1. what if $n = 0$? For instance
$$1, 0, 0, 0, 0, 0, \ldots$$
is ε close to 1. This wouldn't be a nice definition of the sequence "converging to x".

- In 2. This may be too much of demand? What about the sequence
$$1, 1/2, 1/3, \ldots, 1/n, \ldots$$

Proposition 10.5. Uniqueness of limits of sequences. [13, 6.1.7]. Let (a_n) be a sequence of real numbers. Let $L \neq L'$ be distinct real numbers. Such that we cannot have both
$$\lim_{n \to \infty} a_n = L \text{ and } \lim_{n \to \infty} a_n = L'$$
The notation means that $\lim_n a_n = L$ means "a_n converges to L "

The limit operation behaves well for convergent sequences.

11 Homework for week 4

Due: Week 5, Wednesday. You will select 3 problems to be graded.
References: [4, 2], [13, 5].

You are free to assume anything you know about $\mathbb{Q}$. The problem on Dedekind construction is one problem it self. It has extended number of points not because of its difficulty, but because of its length.

Problems

1. (**) Prove that the relation defined in def. 9.12, is an equivalence relation.

2. Review the definition of addition on $\mathbb{R}$, 9.13. Prove that addition, $+$, on $\mathbb{R}$ satisfies properties from 9.1. That is, prove :

   - Associativity: for all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
   - Commutativity: for all $x, y \in \mathbb{R}$, $x + y = y + x$.
   - Admits identity 0: for all $x \in \mathbb{R}$,

   $$x + 0 = 0 + x = x$$

3. (*) Review the definition of multiplication on $\mathbb{R}$, def. 9.13. Prove that any $x \in \mathbb{R}$ where $x \neq 0$ [20] admits a multiplicative inverse y, i.e. exists $y \in \mathbb{R}$ such that
   $$x \cdot y = y \cdot x = 1$$

4. Let $E \subseteq \mathbb{Q}$. Prove that under the order relation $\leq$, least upper bound is unique if exists

5. (**) Here we discuss some conditions to see whether a sequence of rationals $(a_n)_{n=0}^{\infty}$ is Cauchy:

   (a) Suppose that for all $n \in \mathbb{N}$,
   $$|a_{n+1} - a_n| < 2^{-n}$$
   prove that (a_n) is Cauchy.

   (b) if we replace the condition in a. as
   $$|a_{n+1} - a_n| < 1/(n + 1)$$
   for all $n \in \mathbb{N}$, give an example where (a_n) is not Cauchy.

[20] here $0 := (0)_{n=0}^{\infty}$ is the Cauchy sequence consisting of 0s

6. (***) How can we construct $\sqrt{2}$ using Cauchy sequence? Consider the following three sequence $(a_n), (b_n), (x_n)$ defined as follows

$$a_0 = 1, b_0 = 2$$

for each $n \geq 0$,

$$x_n = \frac{1}{2}(a_n + b_n)$$

$$a_{n+1} = \begin{cases} x_n & x_n^2 < 2 \\ a_n & \text{otherwise} \end{cases}$$

$$b_{n+1} = \begin{cases} b_n & x_n^2 < 2 \\ x_n & \text{otherwise} \end{cases}$$

(a) Prove that all sequences are Cauchy.

(b) Prove that all sequences are Cauchy equivalent.

(c) Prove $[(a_n)_{n=0}^{\infty}] \cdot [(a_n)_{n=0}^{\infty}] = 2$.

7. Show that a Cauchy sequence is bounded.

Week 4 solutions

Remaining available upon request
Featured solutions: Q1, by Ethan, Q3, by Kauí, Q5a, by Ethan.
Q1:

1. Prove the relation $\mathbb{R} := \sim$ is an equivalence relation

NTS: if $\forall \varepsilon \in \mathbb{Q}_{>0}, \exists N_\varepsilon\ \forall n \geq N_\varepsilon, |a_n - b_n| < \varepsilon$ is equivalent

Reflexsive: $\{a_n\} \sim \{a_n\}$

$|a_n - a_n| < \varepsilon$

$a_n - a_n = 0$

b/c $\varepsilon \in \mathbb{Q}_{>0}$, $\varepsilon > 0$

Therefore $|a_n - a_n| < \varepsilon$

Symmetric: $\{a_n\} \sim \{b_n\} \Longrightarrow \{b_n\} \sim \{a_n\}$

$|a_n - b_n| < \varepsilon \Rightarrow |b_n - a_n| < \varepsilon$

$a_n - b_n = -(b_n - a_n)$

proving

$|a_n - b_n| = |b_n - a_n| < \varepsilon$

transitive: $\{a_n\} \sim \{b_n\}, \{b_n\} \sim \{c_n\} \Rightarrow \{a_n\} \sim \{c_n\}$

$\forall \varepsilon \in \mathbb{Q}_{>0}, \exists N, \forall n \geq N_1, |a_n - b_n| < \varepsilon/2$

$\forall \varepsilon \in \mathbb{Q}_{>0}, \exists N, \forall n \geq N_2, |b_n - c_n| < \varepsilon/2$

$\{a_n\} \sim \{c_n\} \rightarrow \forall \varepsilon \in \mathbb{Q}_{>0}, \exists N = N_1 + N_2\ \forall n \geq N$

$|a_n - c_n| = |a_n - b_n + b_n - c_n|$

$\Downarrow$ by triangle inequality def'n

$|a_n - b_n| + |b_n - c_n|$

$\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2} = \varepsilon$

Therefore $|a_n - c_n| < \varepsilon$.

Q3:

Question 3: Prove that any $x \in R$ where $x \neq 0$ admits a multiplicative inverse y.

We need to show $\forall [X] \in R$, I can find $[Y] \in R$ such that a $Xn \in [X]$ multiplied by a $Yn \in [Y]$ will be equivalent to [1].
So, we need to show that $|Xn*Yn - 1| < \varepsilon$, $\forall \varepsilon \in Q_{>0}$.

3.1 Let (Xn) be a Cauchy sequence such that Xn = f(n), where f(n) is a function f: N -> $Q_{\neq 0}$.
The function f(n) is defined as follows: choose a representative $(An) \in [X]$. Xn = f(n) equals:

1. An, if $An \neq 0$.
2. 2^{-n}, if An = 0. .

We now have to show that such a function really yields $(Xn) \in [X]$. For that, we have to see if they are Cauchy equivalent:
$|An - Xn| < \varepsilon$, $\forall \varepsilon \in Q_{>0}$.
We have two options:

1. Xn = An, in which case we have $0 < \varepsilon$ being true.
2. Xn - An = 2^{-n}. As 2^{-n} is eventually smaller than any ε, we see that we can find an N such that $\forall n > N$, $|An - Xn| < \varepsilon$

Thus, (Xn) such as Xn = f(n) is equivalent to An, meaning $Xn \in [X]$.

Since we have $Xn \neq 0$, there exists the inverse of Xn, 1/Xn = 1/f(n).
Then, let $(Yn) \in [Y]$ be a Cauchy sequence such that Yn = g(n), where g(n) is such that:
g: N -> $Q_{\neq 0}$, such that g(n) = 1/f(n).
By properties of rationals, we know Xn*Yn = f(n)*(1/f(n)) = 1.

3.2 Now, we can show $|Xn*Yn - 1| < \varepsilon$, $\forall \varepsilon \in Q_{>0}$.
Since Xn*Yn = 1, we have $|Xn*Yn - 1| = 0 < \varepsilon$, $\forall \varepsilon \in Q_{>0}$.
Thus, we prove that all $x \in R$ where $x \neq 0$ admits a multiplicative inverse.

Q5a:

5. a) for $\forall n \in \mathbb{N}$, $|a_{n+1} - a_n| < 2^{-n}$
Prove that (a_n) is cauchy.

by defn $|a_{n+1} - a_n|$ is cauchy if it is $< \varepsilon$

Fix $\varepsilon \in \mathbb{Q}_{>0}$, Then we show (a_n) is eventual ε-steady

$\exists S$ st. $\forall N, M \geq S$ where $|a_M - a_N| < \varepsilon$

$|a_M - a_N| = |a_M - a_{M-1} + a_{M+1} - a_{M-2} + \ldots + a_{n+1} - a_n|$

With the triangle inequality

$|a_M - a_N| \leq |a_M - a_{M-1}| + |a_{M+1} - a_{M-2}| + \ldots |a_{n+1} - a_n| < \varepsilon$

If we sub 2^{-n} back in...

$|a_M - a_N| < 2^{-n} \rightarrow < 2^{(-(M-1))} + 2^{(-(M-2))} + \ldots + 2^{(-N)}$ converges to 2

$|a_M - a_N| < 2^{(-N)} \cdot 2 = 2^{1-N}$ which must $< \varepsilon$

let $a, b \in \mathbb{N}_{>0}$, $2^{1-N} < \varepsilon = \frac{a}{b}$

$2 \cdot 2^{-N} < \frac{a}{b} \Rightarrow \frac{2}{2^N} < \frac{a}{b}$ let $N = b$

$\frac{2}{2^b} < \frac{1}{b} < \frac{a}{b}$

Proving $|a_{n+1} - a_n| < 2^{-n}$, (a_n) is cauchy.

12 Continuity

Week5, Reading [13, 9.3].
Previously we have been dealing with sequences, 10.

Learning Objectives

In the next two lectures:

- Understand the underlying algebra
- State the Intermediate Value Theorem.

Define the exponential function exp, or $e^{(-)}$. To do this we need.

- Continuity.
- Formal power series.

12.1 Subsets in analysis

Reading: [13, 9.1].
In analysis, we often work with certain subsets of $\mathbb{R}$. To define these, we need to know the partial order $\leq$ on $\mathbb{R}$, see def. 9.15.

Definition 12.1. Let $a, b \in \mathbb{R}$. We can construct

- We define the closed interval.

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

- The *half open* intervals

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\} \quad (a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

- The open intervals

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

  We will also let $\mathbb{R}$ be an open interval as a special case.

Lastly, in any of the above cases we call:

- a, b to be the boundary points.
- any point in (a, b) as an *interior point.*

We will revise the above definition once we have defined extended reals, def. 16.7.

Example

What is

- $(2, 2)$
- $[2, 2)$
- $(4, 3)$.
- $[3, 3]$.

12.2 Working with real valued functions

In this section we study real valued functions

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto f(x)$$

Example

1. Characteristic functions. Important for measure theory.
2. Polynomial functions.

We will denote the collection of functions from $\mathbb{R}$ to $\mathbb{R}$ as

$$\mathrm{Fct}(\mathbb{R}, \mathbb{R})$$

Throughout, we will attempt to understand the following types of functions

$$C^{\infty}(\mathbb{R}, \mathbb{R}) \subseteq C^{k}(\mathbb{R}, \mathbb{R}) \subseteq \mathrm{Cts}(\mathbb{R}, \mathbb{R}) \subseteq \mathrm{Fct}(\mathbb{R}, \mathbb{R})$$

Whenever you have a collection of objects you can always ask what structure/operations it has.

Definition 12.2. [13, 9.2.1] Structure on $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$. This is what algebraist refer as *composition rings.*

1. Composition.
2. Multiplication.
3. Addition.
4. Negation.

Except the compositional structure, all such structures exist on *function algebras.* These are sets of the form $\mathrm{Fct}(X, \mathbb{R})$ for X any set. For example, when $X = \mathbb{N}$,

$$\mathrm{Fct}(\mathbb{N}, \mathbb{R}) = \{(x_n)_{n=0}^{\infty} : x_n \in \mathbb{R}\}$$

This set of functions is the set of real sequences starting at 0. The goal now is to study $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$ generalizing what we know about $\mathrm{Fct}(\mathbb{N}, \mathbb{R})$

Discussion

Which of the following are true?

1. $(f+g)\circ h = (f\circ h) + (g\circ h)$.
2. $(f+g)\cdot h = (f\cdot h) + (g\cdot h)$.

History

In the realm of geometry, there is a duality between spaces and their algebra of functions, [1].

In the context of sequences, we were able to make sense of "limit" to a point, "∞"

$$\lim_{n\to\infty} x_n = L$$

[21] Similarly, in the context $\mathrm{Fct}(\mathbb{R},\mathbb{R})$ we would like to consider limit to points $a \in \mathbb{R}$, writing

$$\lim_{x\to a} f(x) = L$$

We first introduction a new notion:

Definition 12.3. The restriction operation: let $E \subseteq X \subseteq \mathbb{R}$ be subsets of $\mathbb{R}$. The restriction map is defined as

$$\mathrm{Fct}(X,\mathbb{R}) \to \mathrm{Fct}(E,\mathbb{R})$$

$$f \mapsto f|_E$$

where $f|_E(x) := f(x)$.

[21] in fact, this is the limit of $\mathbb{N}$, when phrased correctly.

12.3 Limiting value of functions

Reading, [13, 9.3]. We know what it means for a sequence to converge. Now we understand what it means for a function defined on an *interval* to converge.

Definition 12.4. Converging function.

1. Let $X \subseteq \mathbb{R}$ be an interval. $f \in \mathrm{Fct}(X, \mathbb{R})$ *is ε close to L* if for all $x \in X$,

$$|f(x) - L| < \varepsilon$$

2. [13, 9.3.3]. Let $X \subseteq \mathbb{R}$ be an interval. $f \in \mathrm{Fct}(X, \mathbb{R})$ is *local ε-close to L at a* iff there exists $\delta > 0$ such that
   (a) $(a - \delta, a + \delta) \subseteq X$ [22]
   (b) $f|_{(a-\delta,a+\delta)}$ is ε-close to L.

3. Let $L \in \mathbb{R}$, and $a \in X$, then we say $f(x)$ *converges to L as x approaches a* or f *converges to L at a*, iff for all $\varepsilon \in \mathbb{R}_{>0}$, f is local ε-close to L at a. In which case we denote

$$\lim_{x \to a} f(x) = L$$

Example

In 1. Let $f(x) = x^2$.

- 4-close to 2?
- 1-close to 1?

$g(x) = x^3$. $g_1 := g|_{[0,1]}$ and $g_2 := g|_{[1,2]}$.

- 4-close to 2?
- 1-close to 1?

It is *not necessary* that X is an interval and that $a \in X$. The definition can easily be generalized

	Sequences (x_n)	f converging to L at a.
Domain	$\mathbb{N}$	$X \subset \mathbb{R}$ contains a
ε-close	$\forall n \in \mathbb{N}\ |x_n - L| < \varepsilon$.	$\forall x \in X, |f(x) - L| < \varepsilon$.
eventually ε-close local ε-close at a	$\exists N$, for all $n \geq N\ |x_n - L| < \varepsilon$	$\exists \delta > 0, \forall x \in (a - \delta, a + \delta)$. $|f(x) - L| < \varepsilon$,
Converges	$\forall \varepsilon > 0$, (x_n) is ev' ε-close	$\forall \varepsilon > 0$, (x_n) is local ε-close

[22] Note that replacing any of the brackets here with a squared one yields the same definition.

Convergence can in fact be replaced by sequential convergence.

Proposition 12.5. Let $X \subseteq \mathbb{R}$ be an interval. Let $a \in X$ be an interior point as def 12.1. [23] Let $L \in \mathbb{R}$. Then the following are equivalent:

1. f converges to L at a.

2. For every sequence $(a_n)_{n=0}^{\infty}$ where $a_n \in X$ where $\lim_{n\to\infty}(a_n) = a$, def 10.4, we have
$$\lim_{n\to\infty} (f(a_n))_{n=0}^{\infty} = L$$

Proof. Exercise. Proof of $2 \Rightarrow 1$. We prove the *contrapositive* of this statement, i.e. $\neg 1 \Rightarrow \neg 2$. If 1 is false, this means:

$$\exists \varepsilon > 0, \forall \delta > 0, \exists x \in [a - \delta, a + \delta], |f(x) - L| \geq \varepsilon$$

To show 2 is false, we need to construct a sequence (a_n) such that

- $\lim_n a_n = a$
- $\lim_n f(a_n) \neq L$.

As 1 is false, for all $n \in \mathbb{N}_{\geq 1}$, we can find δ_n such that

(a) δ_n is a strictly decreasing sequence to 0.

(b) There is some $a_n \in [a - \delta_n, a + \delta_n]$,
$$|f(a_n) - L| \geq \varepsilon$$

(a) guarantees that $\lim a_n = a$ and (b) guarantees that $\lim f(a_n) \neq L$. $\square$

Many results on continuity can thus be reduced to the case of sequences.

[23] This is so that we don't have to discuss boundary cases.

12.4 Continuous functions

Definition 12.6. Let $X \subset \mathbb{R}$ be an interval, let $a \in X$ be an interior point, def. 12.1. f is *continuous at a* if f is converges to $f(a)$ as x approaches a.

Example

Continuous functions:

1. Polynomial functions.
2. Linear functions.
3. The constant function $f(x) = c$ for some $c \in \mathbb{R}$, is continuous everywhere.

1. sgn is not continuous at 0.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Definition 12.7. Let $X \subset \mathbb{R}$ be an open interval. $f : X \to \mathbb{R}$ is continuous at $x_0 \in X$ if for all $\varepsilon \in \mathbb{R}_{>0}$ f is local ε-close to $f(x_0)$. Explicitly, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \forall x \in (x_0 - \delta_\varepsilon, x_0 + \delta_\varepsilon)$$

where $(x_0 - \delta_\varepsilon, x_0 + \delta_\varepsilon) \subseteq X$.

Definition 12.8. If f is continuous at all $x \in X$, we say f is *continuous*. We denote the set of all continuous functions on X as

$$\mathrm{Cts}(X, \mathbb{R})$$

We may ask which structures/operations, as 12.2, on $\mathrm{Fct}(X, \mathbb{R})$ which extends to $\mathrm{Cts}(X, \mathbb{R})$.

Proposition 12.9. Let $X \subseteq \mathbb{R}$ be an open set. $f, g : X \to \mathbb{R}$ are functions which are continuous at $a \in X$. Then the following functions are continuous at a:

1. $f + g$
2. $f \cdot g$
3. $\max(f, g)$

4. $\min(f, g)$

Example

1. $f(x) = |x|$ is continuous on $\mathbb{R}$. We will later see that this is not differentiable at 0.

13 Homework for week 5

Due: Week 6, Wednesday. You will select 3 problems to be graded. Q2: you are free to use any properties of limits of sequences as long as you state them correctly. In Q4 you will need the maximum principle 15.2. In Q6c, you are free to use any previous results stated and results from class.

1. Which of the following are true on $\mathrm{Fct}(\mathbb{R},\mathbb{R})$: let $f, g, h \in \mathrm{Fct}(\mathbb{R},\mathbb{R})$:

   (a) Composition $\circ$ is associativity :
   $$f \circ (g \circ h) = (f \circ g) \circ h$$

   (b) Composition distributes over multiplications:
   $$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

   (c) Composition distributes over addition:
   $$(f + g) \circ h = f \circ h + g \circ h$$

2. (**) Let $(x_n)_{n=0}^{\infty}$ be a sequence of real numbers, *assume that x_n converges to some real number L*. Let $x_1 = 2$,
   $$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$$
   show that $L^2 = 2$. Note that the definition of convergence here is the same as rational number, 10.4. (***) Extra credit if you can show that $(x_n)_{n=0}^{\infty}$ converges.

3. (***) Prove 12.5.

4. (**) In set up of 12.9 prove

   (a) $f \cdot g$

   (b) $\max(f, g)$

   are continuous at a.

5. (*) In set up of 12.9 with $X = \mathbb{R}$, prove $f \circ g$, is continuous at a.

6. (***) Show by definition that the following functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous:

   (a) $f(x) = c$ for all $x \in \mathbb{R}$.

   (b) $f(x) = x$ for all $x \in \mathbb{R}$.

   (c) $f(x) = \sum_{i=0}^{n} c_i x^i$, where c_i is a real number for $i = 1, \ldots, n$. E.g. $f(x) = x^2 + x + \sqrt{2}$. You are free to use any results stated in notes or in previous problems.

14 Solutions

Question 3: Let $X \subseteq R$ be an interval. Let $a \in X$ be an interior point. Let $L \in R$. Prove the following are equivalent:
1. f converges to L at a.
2. For every sequence $(a_n)\infty_{n=0}$ where $a_n \in X$ where $\lim_{n\to\infty}(a_n) = a$, $\lim_{n\to\infty} (f(a_n))\infty_{n=0} = L$

3.1: Proof that 1 ⇒ 2:
Since f converges to L at a, we have:
$\forall \varepsilon > 0$, $\exists \delta>0$ such that $\forall x \in [a - \delta, a + \delta]$, $|f(x) - L| < \varepsilon$.

Also, since we are considering sequences where $\lim_{n\to\infty}(a_n) = a$, we have:
$\forall \varepsilon > 0$, $\exists N$ such that $\forall n \geq N$, $|a_n - a| < \varepsilon$.

So, we can simply choose N such that $\forall n \geq N$, $|a_n - a| < \delta$, meaning $|f(a_n) - L| < \varepsilon$ $\forall n \geq N$.
Thus, 1 ⇒ 2.

3.2: Proof 2 ⇒ 1.
Proof by contradiction: assume 2 is true and 1 is not true.

If 1 is not true, we have:
$\exists \varepsilon > 0$ such that $\forall \delta>0$, $\exists x \in [a - \delta, a + \delta]$ such that $|f(x) - L| < \varepsilon$.

So, we should be able to construct the following sequence:
$(S_n)^{\infty}_{n=0} = \{S_n \in [a - \delta_n, a + \delta_n], |f(S_n) - L| < \varepsilon\}$, with $\delta_n = 1/n$.
This sequence, however, disrespects 2, as S_n approaches a but $f(S_n)$ does not approach L.
Thus, we arrive at a contradiction, meaning 2 ⇒ 1.

15 Main results of continuity

We will consider two fundamental results in continuity of functions, [12, 7].

1. Maximum principle, thm 15.2, see also [13, 9.6]. For this, we would have to review the notion of limsup.
2. Intermediate value theorem.

Together these two results would imply that if f is a continuous function on $[a, b]$, then

$$f([a, b]) = [e, f] \quad a, b, e, f \in \mathbb{R}$$

The notation means the image of f.

Definition 15.1. Let $X \subset \mathbb{R}$ be any subset. Then the *image* of f,

$$\text{im} f := f(X) := \{y \in \mathbb{R} : \exists x \in X f(x) = y\}$$

15.1 Maximum principle

Theorem 15.2. Let $a < b$ be real numbers. Let f be a continuous function on an open interval containing $[a, b]$. $f : [a, b] \to \mathbb{R}$, then f attains its maximum at some point.

The proof of maximum principle, thm. 15.2, breaks down into the following steps:

1. Show that f is bounded, def 15.4. Suppose not, then exists a sequence $(x_n)_{n=0}^{\infty}$ such that $f(x_n) \to +\infty$. Each x_n lies in the same bounded interval. By Bolzano-Weirstrass, 15.6, we can find a convergent subsequence, this is a contradiction. (Why?)

2. Let $E := \sup f(X) \in \mathbb{R}$ by part 1 and completeness property of reals. We find a sequence of elements $x_n \in X$ such that $f(x_n) \to E$.

3. We find a converging subsequence $(x_{n_k})_{k=0}^{\infty}$, def 15.5, of (x_n), using Bolzano-Weirestrass, them. 15.6, such that $\lim_k (x_{n_k}) = x_{\max}$. Then by definition of continuity
$$f(x_{\max}) = L$$

Proposition 15.3. Let $a < b$ be real numbers. Let f be a continuous function on an open interval containing $[a, b]$. $f : [a, b] \to \mathbb{R}$, then f is bounded.

Proof. □

Definition 15.4. Let $X \subseteq \mathbb{R}$ be a subset, $f : X \to \mathbb{R}$ be any function. f is *bounded* if exists $M \in \mathbb{R}$
$$|f(x)| \leq M$$

The definition function bounded above (below and bounded) is a special case of 9.3. It is equivalent to saying that the image of f, def 4.12, is bounded above (below and bounded.)

Example

Which of the functions are bounded?

- $f(x) = 1/x - a$. $X = \mathbb{R} \setminus \{0\}$.
- $f(x) \in \operatorname{Poly}(X, \mathbb{R})$ with $X = \mathbb{R}$ and $X = (0, 1)$.

15.2 Bolzano-Weierstrass Theorem

We will now study a new collection of sequences: those sequences which have converging subsequences. They fit in the following hierarchy:

$$\mathrm{CvgSeq}(\mathbb{R}) \subseteq \mathrm{CcSeq}(\mathbb{R}) \subseteq \mathrm{BddSeq}(\mathbb{R}) \subseteq \mathrm{CvgSubSeq}(\mathbb{R})$$

Definition 15.5. Subsequence. Let (a_n) be a sequence of reals. Then *a subsequence of* (a_n) is a sequence $(b_k) = (a_{f(k)})_{k=0}^{\infty}$ given by the datum of a function

$$f : \mathbb{N} \to \mathbb{N}$$

which is strictly increasing : for all $i, j \in \mathbb{N}$

$$f(i) > f(j) \text{ if } i > j$$

Often times, people don't say the function f and write instead

$$(b_k)_{k=0}^{\infty} = (a_{n_k})_{k=0}^{\infty}$$

Example

Consider

- $f(x) = 2^x$.
- $f(x) = x + 1$
- $f(x) = x^2$.

What are the subsequences associated to these functions when $(a_n) = ((-1)^n)_{n=0}^{\infty}$.

We begin with the following famous theorem, which is equivalent to the completeness property (or axiom) of the real numbers.

Theorem 15.6. Bolzano-Weirestrass. Let (a_n) be a bounded sequences, then there is at least one subsequence (a_n) which converges.

Recall the definition of limit points, 18.2.

Proof. The conditions implies that $\limsup a_n < \infty$ of which is a limit point. □

Example

Consider the following sequences:

- $(x_n)_{n=0}^{\infty} = (-1)^{n+1}n$. Does this have a converging subsequence.?
- $(x_n)_{n=0}^{\infty} = (n \bmod 5)$. Does this have a converging subsequence?
- $(x_n)_{n=1}^{\infty}$ be a sequence such that for each $x_n \in [0, 100]$: does this have a convergent subsequence?

16 limsup and liminf

Reading: [13, 6.4].

Learning Objectives

- Limit points.
- Find the sup and inf of he set of all limit points of a sequence (a_n). i.e. For all $L \in \text{LimitPoint}(a_n)$,

$$\liminf a_n \leq L \leq \limsup a_n$$

see Prop. 16.6.

In general sequences we consider do not converge.

Definition 16.1. $L \in \mathbb{R}$ is a *limit point* of $(a_n)_{n=0}^{\infty}$ if there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.

An equivalent characterization is

Definition 16.2. $L \in \mathbb{R}$ is a *limit point* of (a_n) if for every $\varepsilon > 0$, for every N, there exists $n \geq N$, such that $|a_n - L| < \varepsilon$.

Proposition 16.3. The two definitions 16.1 and 18.2 are equivalent.

We have many limit points. Consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \ldots \quad (*) \tag{2}$$

This is the sequence $((-1)^n(1 + 10^{-n})_{n=1}^{\infty}$.
What two limits do you see? It is a combination of two sequences:

- $1.1, 1.001, 1.0001, 1.00001, \ldots$.
- $-1.01, -1.00001, -1.000001, \ldots$.

Example

Let (a_n) be a sequence which converges to L. What are its limit points?

Definition 16.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence. Then for : all $N \in \mathbb{N}$ we set

$$a_N^+ := \sup_n \left((a_n)_{n=N}^{\infty}\right)$$

$$a_N^- := \inf_n (a_n)_{n=N}^{\infty}$$

With the new sequence $(a_N^+)_{N=0}^{\infty}, (a_N^-)_{N=0}^{\infty}$, define

$$\limsup_n a_n := \inf_N \left((a_N^+)_{N=0}^{\infty}\right)$$

$$\liminf a_n = \sup_N \left((a_N^-)_{N=0}^{\infty}\right)$$

where sup is the least upper bound 9.4, and inf, is the greatest lower bound.

This definition is not ideal. Some of the terms may be undefined if we used the definition of sup for reals, def. 9.4.

For example, if $(a_n) := (n) = (0, 1, 2, 3, \ldots)$, then for all N,

$$a_N^+ = +\infty$$

We would then have to compute

$$\sup \{+\infty\}$$

$+\infty$ *not* a number in $\mathbb{R}$, and we previously have only defined sup on subsets $E \subseteq \mathbb{R}$, def 9.4. The convention is to formally include $+\infty$ as a new symbol into our number system, as def. 16.7. We need rules on how to work with taking supremum of sets containing $-\infty, +\infty$. 16.9. Under this convention

$$\sup \{+\infty\} = +\infty$$

$$\sup \{-\infty\} = \sup \{-\infty\} \setminus \{-\infty\} = \sup \emptyset = -\infty$$

Example

If the sequence (a_n) where bounded below, and at least one $(a_N^+) \in \mathbb{R}$, then by MCT, the sequence limits to some real number.

Definition 16.5.

Example

In (*) of sequence

- $(a_n^+) = (a_0^+, a_1^+, \dots)$ is the sequence

$$(1.1, 1.01, 1.001)$$

- $\inf(a_n^+) = 1$.

Proposition 16.6. [13, 6.4.12] Properties of limsup and liminf. Let $L^+ := \limsup_n a_n, L^- := \liminf_n a_n$.

1. Elements of (a_n) are eventually less than x for every $x > L^+$: i.e.
2. if c is any limit point of $(a_n)_{n=0}^{\infty}$ we have $L^- \le c \le L^+$.
3. Suppose $L^+ < \infty$, then it is a limit point.

Proof. 3. Let us show that L^+ is a limit point. Fix $\varepsilon > 0$ and N. Then $\limsup = \inf(\sup a_N^+)$, choose $N_1 \ge N$ such that $L^+ \le \sup(a_n)_{n=N_1}^{\infty} \le L^+ + \varepsilon$. Then choose $N_2 \ge N_1$[24] such that

$$L^+ - \varepsilon < a_{N_2} < L^+ + \varepsilon$$

□

16.1 Extending the number system

We will begin by defining the *suprema* and *infima* of sets. We may or may not work with an extended number system. But we include it here to show how one could extend a number system.

Definition 16.7. The *extended number system* consists of

$$\bar{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

Let $x, y, z \in \bar{\mathbb{R}}$. Define the order relation, 7.3 $x \le y$ if and only if one of the following holds.

1. If $x, y \in \mathbb{R}$, $x \le y$.
2. $x = -\infty$

[24] why can you do this?

3. $y = \infty$.

Thus, we have artificially add in new terms.

- We do not include any operations. This can be dangerous. Of course, this can be done. We can set

$$c + (+\infty) = (+\infty) + c := +\infty \quad \forall c \in \mathbb{R}$$

$$c + (-\infty) = (-\infty) + c =: -\infty \quad \forall c \in \mathbb{R}$$

  but requires a lot of care.

- We can define order and negation.

This is a common practice for mathematics, in order for one to make better statements.

Definition 16.8. Negation of extended reals.

Example

What is the supremum of the set

- $$\{0, 1, 2, 3, 4, 5, \ldots\}$$

- $$\{1 - 2, 3, -4, 5, -6, \ldots\}$$

- $(a_n) = ((-1)^n 1/n)$.

Definition 16.9. Least upper bound. See the usual definition of upper bound 9.3. Let $E \subseteq \bar{\mathbb{R}}$. Then $\sup E$, the least upper bound [13, 6.2.6] is defined by the following rule:

- Let $E \subseteq \mathbb{R}$. So $\infty, -\infty \notin E$. Then $\sup E$ is as 9.6.

- If $\infty \in E$, then $\sup(E) = \infty$.

We can define the infimum without the use of another definition. [25]

Definition 16.10. We let

$$\inf E := -\sup(-E)$$

$$-E := \{-x : x \in E\}$$

[25] although, in practice, we *think* of inf as we did for defining 9.4.

Example

Let E be negative integers.

$$\inf(E) = -\sup(-E) = -\infty$$

16.2 Completness axiom

Leet us recall one of the key properties of real numbers, the least upper bound property. 9.3. To prove this we use the explicit model of Cauchy sequences we have previously used.

Proposition 16.11. Let $(a_n)_{n=1}^{\infty}$ be a Cauchy sequence of rationals, and $x \in \mathbb{R}$. Show if $a_n \leq x$ for all $n \geq 1$, then the real number $[(a_n)] \leq x$. Hint: you are free to use the fact that for any two real numbers $x < y$ there exists $q \in \mathbb{Q}$, such that $x < q < y$.

Proof. ??? □

Theorem 16.12. Completeness axiom. [13, 5.5.9]. Let E be a nonempty subset of $\mathbb{R}$. If E has an upper bound, then it has *exactly one least upper bound* in $\mathbb{R}$. Thus, $\sup E \in \mathbb{R}$.

Proof. The hard part is *existence*. Uniqueness is not hard (Exercise). Let M be an upper bound of E. We break the proof in 3 steps.

1. We construct a sequence of $\mathbb{Q}$.

$$\{m_n\}$$

   which has the following property

   - $\frac{m_n}{n}$ are upper bounds
   - $\frac{m_{n-1}}{n}$ are not upper bound.

2. Sequence in 1 is a Cauchy sequence. Let the real number be $s := [(\frac{m_n}{n})]$. This is the same as the class $[\left(\frac{m_n - 1}{n}\right)_{n=1}^{\infty}]$.

3. s is indeed a least upperbound.

   - It is an upperbound because for any $x \in E$, $x \leq \frac{m_n}{n}$ for all $n \geq 1$. It follows $x \leq s$, see Prop. 16.11.
   - It is a least upper bound because if y is any upper bound, then $y \geq \frac{m_n - 1}{n}$ for all $n \geq 1$. So $y \geq s$.

To prove step 1, we will invoke the following result: if $L < K$ are integers such that L/n is an upper bound for E and K/n is, then there exists an integer $L < m \leq K$ such that m/n is upper bound by $(m-1)/n$ is not. The proof is by induction on the size of $|K - L| = r$. If $r = 1$ is true then we are done. Suppose statement holds when $r = k$. We show it holds for $r = k + 1$. $\square$

An example in $\mathbb{Q}$ which does not satisfy this property.

Proposition 16.13. Let

$$E := \left\{q \in \mathbb{Q} \,:\, q^2 < 2\right\}$$

This set is bounded, but it has no *least* upper bound.

Proof. We know that there are no rationals q, $q^2 = 2$. Now suppose $s \in \mathbb{Q}$ is a least upperbound. Then

- If $s < \sqrt{2}$, we can always find rational s' such that

  $$s < s' < \sqrt{2}$$

  So s cannot be an upper bound.

- If $s > \sqrt{2}$, we can find a rational s' such that

  $$s > s' > \sqrt{2}$$

  and s' is an upper bound for E.

$\square$

What are the consequences? It says something about convergence of sequences.

Proposition 16.14. Least upper bound. [13, 6.3.6].

Proof. This boils down to [13, 5.5.9]. $\square$

Proposition 16.15. MCT [13, 6.3.8]. Every monotone bounded sequence converges. Let (a_n) be a bounded sequence of real numbers, which is also increasing. Then limit exists and

$$\lim a_n = \sup(a_n)_{n=0}^{\infty} \leq M$$

Proof. By 16.12, $\sup a_n$ exists and is unique. Let us pick an $\varepsilon > 0$. Then by definition there exist n...??? $\square$

17 Homework for week 6

Due: Week 7, Friday. You will select 3 problems to be graded. Note that when we are arguing with the real numbers, we forget that they are a "Cauchy equivalence class of rationals" and work with them by their properties, see subsec. 9.1. Q1-5, Q9 reviews class content on Bolzano-Weierestrass. Q6-8 are problems on limsup and liminf.

1. Prove 16.3.
2. (****) Write a proof of completeness property Thm. 16.12. This involves completing the unfinished details of the theorem referenced.
3. (**) Write out the full proof Bolzano-Weierestrass's Thm. 15.6. You have to write proofs of any result used.
4. Deduce the boundedness of functions on closed interval Prop. 15.3, using Bolzano-Weierestrass's Thm. 15.6.
5. Sandwiching sequences. Suppose that $(a_n), (b_n), (c_n)$ are three sequences of real numbers satisfying
$$a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$$
and $a_n \to l$ and $c_n \to l$. Show that $b_n \to l$. (**) Give an alternative prove using the comparison principle for liminf and limsup.
6. (**) Using the completeness property 16.12, show that there exists a positive real number x such that $x^2 = 2$ by considering the set
$$E := \left\{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 2\right\}$$
7. (**) Comparison principle for liminf and limsup. Let $(a_n), (b_n)$ be two sequences of reals, such that
$$a_n \leq b_n \forall n \in \mathbb{N}$$
Show that
$$\liminf a_n \leq \limsup b_n$$
To begin: fix $N, M \in \mathbb{N}$. Show that
$$a_N^- \leq b_M^+$$
Hence, by varying M, show that
$$a_N^- \leq \inf_M (b_M^+)_{M=0}^{\infty} = \limsup b_n$$
8. (***) Let (a_n) be a sequence of real numbers. Then (a_n) converges iff it is bounded and $\liminf a_n = \limsup a_n$.
9. (*) Show that every cauchy sequence converges using Bolzano Weirestrass, Thm. 15.6.

18 Extending definition of continuity

We have defined the limit, $\lim_{x\to a} f(x)$ of a function $f : X \to \mathbb{R}$ (if exists) at a point $a \in X$, def. 12.6. But this generalizes when a is *not* in the interval. This is particularly useful when a is *not* in the domain.

Definition 18.1. Let $X \subseteq \mathbb{R}$. $a \in \mathbb{R}$, is *adherent point* iff for all $\delta > 0$, $(a-\delta, a+\delta) \cap X \neq \emptyset$.

This is slightly different to the notion of *limit points*[26].

Definition 18.2. Let $X \subseteq \mathbb{R}$. $a \in \mathbb{R}$, is *limit point* iff for all $\delta > 0$, $(a-\delta, a+\delta) \cap (X \setminus \{a\}) \neq \emptyset$.

This means every δ-neighborhood[27] of a contains an element in X with out a.

Example

What are limit points of the following subsets?

- $\{3\}$
- $(1,2) \cup \{3\}$?
- $\mathbb{Q}$.

This is the case when we want to define differentiation.

Example

Sample computation.

- $f(x) = x : \mathbb{R} \to \mathbb{R}$, what is $\lim_{x\to 0} f(x)$.
- $f(x) = \frac{\sin x}{x} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. Can we make sense of $\lim_{x\to 0} f(x)$?

The definition for continuity to a point which is *not* in the domain works verbatim, 12.4.

Definition 18.3. Converging function on arbitrary domain. Let $f \in \mathrm{Fct}(X, \mathbb{R})$, and a be a limit point of X in $\mathbb{R}$, 18.2,

1. Let $X \subseteq \mathbb{R}$ be a subset. $f \in \mathrm{Fct}(X, \mathbb{R})$ *is ε close to L* if for all $x \in X$,

$$|f(x) - L| < \varepsilon$$

[26] Although in the literature they may confuse these two.

[27] For us, this is just a colloquial way to say interval, there is a precise meaning in a more general context of topology.

2. Let $X \subseteq \mathbb{R}$ be an interval. $f \in \mathrm{Fct}(X, \mathbb{R})$ is *local ε-close to L at a* iff there exists $\delta > 0$ such that
$$f|_{(a-\delta, a+\delta)\cap X}$$
is ε-close to L.

3. Let $L \in \mathbb{R}$, $f(x)$ *converges to L as x approaches a* or f *converges to L at a*, iff for all $\varepsilon \in \mathbb{R}_{>0}$, f is local ε-close to L at a. In which case we denote
$$\lim_{x \to a, x \in X} f(x) = L$$

Definition at 2 would be vacuously true if a is a point where for some $\delta > 0$,
$$(a - \delta, a + \delta) \cap X = \emptyset$$

This is why we need a to be a limit point. Thus, there are certain choices of a we would like to focus. These are called *limit points* of the set X, this is similar to that for infinite sequences, def. 18.2.

Example

Consider $f(x) := \frac{|x|}{x} : \mathbb{R} \setminus \{0\} \to \mathbb{R}$. Does the limit exists ?
- Firs compute $|x|/x$ for small values of x.
- Notice this gives different values depending on x.

18.0.1 Domain of definition

Definition 18.4. A function, $f : X \to Y$, def. 4.11, with codomain Y is determined by a property $P(x, y)$ for $x \in X, y \in Y$. The *domain of definition* is the set of all $x \in X$ for which there is a *unique* y such that $P(x, y)$ is true.

Colloquially, we write this as

Definition 18.5. Let $f : X \to Y$ be a function.
$$\{x \in Y \;:\; f(x) \text{ is well-defined as a function with codomain } Y\}$$
[28] is the *domain of definition.*

[28]

Example

We would consider function:

1. $f(x) = \frac{1}{x-a}$, where $a \in \mathbb{R}$. The domain of definition is

$$\mathbb{R} \setminus \{a\} := \{x \in \mathbb{R} : x \neq a\}$$

2. $f(x) = +\sqrt{x}$, has domain of definition

$$\{x \in \mathbb{R} : x \geq 0\} =: \mathbb{R}_{\geq 0} =: [0, +\infty)$$

19 Differentiation

Week 8 Reading: [13, 10.1-10.3].

Learning Objectives

- As warm up, for a function $f : X \to \mathbb{R}$, we extend the definition of limit of f at a point $a \in X$, def 18.3 limit of f at a *limit point* a of X, 18.
- First goal is to prove Rolle's theorem.

Now we will consider a smaller class of functions compared to continuous functions.

$$C^1(\mathbb{R},\mathbb{R}) \subseteq \mathrm{Diff}(\mathbb{R},\mathbb{R}) \subseteq \mathrm{Cts}(\mathbb{R},\mathbb{R})$$

Definition 19.1. Differentiability at point. Let $X \subseteq \mathbb{R}$, and let $a \in X$, be an element of X a limit point. If the limit

$$\lim_{x\to a, x\in X\setminus\{a\}} \frac{f(x)-f(a)}{x-a}$$

converges to some real number L. We write

$$\frac{df}{dx}(a) := f'(a) := L$$

To make a differentiability definition on the whole domain, we extend as we did for the definition continuity, def. 12.8.

Definition 19.2. Let $X \subseteq \mathbb{R}$. We say that f is *differentiable on* X if for all limit points $a \in X$, f is differentiable at a.

The operation of differentiation

$$\frac{d}{dx} : \mathrm{Diff}(\mathbb{R},\mathbb{R}) \to \mathrm{Cts}(\mathbb{R},\mathbb{R})$$
$$f \mapsto f'(x) := \frac{df}{dx}(x)$$

satisfies various properties.

- linearity: [29] $(cf)'(x) = c \cdot f'(x)$.
- ??

[29] Recall given $f : X \to \mathbb{R}$, $c \in \mathbb{R}$, we defined the scalar multiple of a function $cf : X \to \mathbb{R}$ as point wise multiplication, $(cf)(x) = c \cdot f(x)$.

19.1 Local minima and maxima

Definition 19.3. Let $X \subseteq \mathbb{R}$, $f : X \to \mathbb{R}$ a function and $a \in X$. f *attains a local maxima at* a if there exists $\delta > 0$ such that

$$f(a) \geq f(x) \quad \forall x \in (a-\delta, a+\delta) \cap X$$

[30]

Recall from the section on maximal principle,

Proposition 19.4. Local extrema are stationary. [13, 10.2.6]. Suppose $f : (a', b') \to \mathbb{R}$ attains a local maximum (or minimum) at a, and is differentiable at a, then $f'(a) = 0$.

Proof. HW. Hint:

$$f'(a) := \lim_{x \to a, X \setminus \{a\}} \frac{f(x) - f(a)}{x - a} =: L$$

exists. Let us spell this out. This means if we fix some $\varepsilon > 0$, there exists $\delta > 0$, such that

$$L - \varepsilon < \frac{f(x) - f(a)}{x - a} \leq L + \varepsilon \quad x \in (a-\delta, a+\delta) \subseteq (a', b')$$

Let us also choose δ such that $f(a)$ is a local maximum in $(a-\delta, a+\delta)$. For $x \in (a, a-\delta)$, we have

$$\frac{f(x) - f(a)}{x - a} \geq 0$$

and for $x \in (a, a+\delta)$,

$$\frac{f(x) - f(a)}{x - a} \leq 0$$

If the limit L were positive, then we can choose $L - \varepsilon < 0$. This yields a contradiction. □

Example

Consider $f : (-1, 1) \to \mathbb{R}$:

- $f(x) = -|x|$ is a function which is continuous and attains global maximum at 0, but is not differentiable at 0. [a]
- $f(x) = x^3$ is a function continuous, $f'(x) = 0$ but does not have global maxim or minina at a.

[a] Global maximum is not sufficient!

[30] This is what it means more generally for a *property* $\mathbf{P}$ of a function f to be local. That $\mathbf{P}$(f) is true iff exists some restriction $f|_U$, $\mathbf{P}(f|_U)$ is true. The choice of U depends on the context.

Theorem 19.5. Rolle's theorem. [13, 10.2.7]. Let $a < b$, let $g : [a,b] \to \mathbb{R}$ be continuous, which is differentiable on (a,b), then exists $x \in (a,b)$ such that $g'(x) = 0$.

Proof. We use the maximal principle 15.2. g attains maximum value at $c \in [a,b]$. This is both a local and global maximum. $\square$

Proposition 19.6. Mean value theorem. Let $a < b$ and $f : [a,b] \to \mathbb{R}$ be continuous function which is differnetiable on (a,b). Then exists $x \in (a,b)$ such that

$$f'(x) = \frac{f(b) - f(a)}{b - a}$$

Proof. We well transform [31] our function so that it satisfies 19.5. In otherwords, we solve for $f(x) - cx$ satisfying the conditions that

$$f(a) - ca = f(b) - cb$$

We deduce that

$$c = \frac{f(a) - f(b)}{a - b}$$

Now apply Thm. 19.5. $\square$

19.2 Basic properties of derivatives

We have discussed the definition of derivative. As one can see, it is rather hard to compute.

Theorem 19.7. [13, 10.1.13] Let $X \subset \mathbb{R}$ be an open interval and $f, g \in \mathrm{Fct}(X, \mathbb{R})$. Let f, g be differentiable at a. Then

1. $f' = 0$ if f is a constant function.

2. If g is differentiable at a, g is nonzero on X, then $1/g$ is also differentiable at a. Then
$$\left(\frac{1}{g}\right)'(a) = -\frac{g'(a)}{g(a)^2}$$

[31] add a linear function

Proof. This is in exercise. We discuss derivative of $1/g$.

$$\begin{aligned}\left(\frac{1}{g}\right)'(a) &= \lim_{x\to a, x\in X\setminus a} \frac{\frac{1}{g(x)} - \frac{1}{g(a)}}{x-a} \\ &= \lim_{x\to a, X\setminus a}\left(\frac{g(x)-g(a)}{x-a}\cdot\frac{1}{g(x)g(a)}\right) \\ &= -g'(a)\cdot\lim_{x\to a}\left(\frac{1}{g(x)g(a)}\right) \\ &= -\frac{g'(a)}{g(a)^2}\end{aligned}$$

where we used the fact that

$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x)\cdot\lim_{x\to a} g(x)$$

Provided both limits on the right hand side exists. Note also the negative sign in third equality. $\square$

Example

For each of the examples below, justify why the functions are differentiable.

- Let $f : \mathbb{R}\setminus\{1\} \to \mathbb{R}$, $f(x) = \frac{1}{x-1}$. Compute

$$\lim_{x\to a} f(x) = f'(a)$$

  for $a \neq 1$.

- $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$. Compute
  - $f'(9)$.
  - $f'(a^2)$ for $a \in \mathbb{R}$.
  -
- If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{Z}$, then $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

Theorem 19.8. Chain rule. Let $X, Y \subseteq \mathbb{R}$, $a \in X$

Proof. Prove that $g \circ f$ is differentiable at a:

$$\lim_{x\to a, X\setminus a} \frac{gf(x) - gf(a)}{x-a} =$$

$$\lim_{x\to a, X\setminus a} \frac{g(f(x)) - g(f(a))}{f(x)-f(a)}\frac{f(x)-f(a)}{x-a}$$

But this is bad! $\square$

Proposition 19.9. Newton's approximation. Let $X \subseteq \mathbb{R}$ and $a \in X$ be limit point of X.

19.3 Monotone functions

Definition 19.10. Let $X \subset \mathbb{R}$ be any subset. f is *monotone increasing* iff for all $x, y \in X$, $y > x$ $f(y) \geq f(x)$.

Discussion

What is a function which

- monotone but not continuous.
- continuous but not monotone

Example

Let $b \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be the constant function. Prove that $f'(x) = 0$ for all $x \in \mathbb{R}$.

20 Homework for week 7

Due: Week 8, Friday. You will select 3 problems to be graded. Q1 and Q2 are about the nature of limit points. For Q3,4, review def. 19.1. These are about properties of differentiation. [32]

1. Let $X \subseteq \mathbb{R}$. What are the adherent points, 18.1, and limit points, 18.2, of the following sets?
   (a) $X := \left\{1, \frac{1}{2}, 2, \frac{1}{3}, 3, \frac{1}{4}, 4, \ldots\right\}$.
   (b) $(0, +\infty) := \{x \in \mathbb{R} : x > 0\}$.
   (c) $[a, b]$ for $a, b \in \mathbb{R}$.

2. (****) Let $A_0 = [0, 1]$. For $k, l \in \mathbb{R}$, A a set, define

$$kA := \{kx : x \in A\}$$

$$l + A := \{l + x : x \in A\}$$

   Now define inductively

$$A_{n+1} := \frac{1}{3}A_n \cup \left(\frac{2}{3} + \frac{1}{3}A_n\right)$$

   Consider the following set:

$$C := \bigcap_{n \in \mathbb{N}_{\geq 0}} A_n$$

   Show that every point in the set is a limit point of the set. This a typical example of *perfect set*: this is a subset whose adherent points (its closure) are precisely itself, and are all limit points.

3. (*) (Diff $\Rightarrow$ Cts.) Let $X = (x, y) \subseteq \mathbb{R}$, let $a \in X$. Let $f : X \to \mathbb{R}$ be a function. Write the definition of
   - f being continuous at a.
   - f being differentiable at a.

   Show that if f is differentiable at a, then f is also continuous at a, def. 12.6.

4. Let $X = (x, y) \subseteq \mathbb{R}$. If f is the constant function, i.e. $f(a) = c$ for all $a \in X$, for some $c \in \mathbb{R}$. Show that
   (a) For all $a \in X$, f is differentiable at a.

[32] The statements have more generally holds if we replace "Let $X = (x, y)$ and $a \in X$ with "X be any set and a a limit point of X."

(b) $f'(a) = 0$.

5. (*) Let $X = (x, y) \subseteq \mathbb{R}$ be an open interval and $a \in X$. Let $f, g \in \mathrm{Fct}(X, \mathbb{R})$. Let f, g be functions differentiable at a. Show that

   (a) $f + g$ is also differentiable and
   $$(f+g)'(a) = f'(a) + g'(a)$$

   (b) $f \cdot g$ is also differentiable and
   $$(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$$

   (c) Suppose further that g is nonzero on X[33]. Prove that f/g is also differentiable at a and
   $$\left(\frac{f}{g}\right)'(a) = \frac{f'(a)g(a) - f(a)g'(a)}{g(a)^2}$$

6. (*) If $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^n$, $n \in \mathbb{N}_{\geq 0}$. Then prove that

   - $f(x)$ is differentiable, def 19.2. Hint: induction.
   - $f'(a) = na^{n-1}$ for all $a \in \mathbb{R}$.

[33]for all $x \in X, g(x) \neq 0$

20.1 Hints for question 2:

What is happening? We can think of A_1 as

$$\left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right]$$

What is A_2? This is

$$\left(\left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{3}{9}\right]\right) \cup \left(\left[\frac{6}{9}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]\right)$$

In otherwords, for each interval in I_1 we subdivide it into three parts and take the middle part. In general A_n has 2^n intervals each of length $1/3^n$.
Let $x \in C$. To show that this is a limit point of C, we show that for any $\varepsilon > 0$,

$$(x - \varepsilon, x + \varepsilon) \cap (C \setminus \{x\}) \neq \emptyset$$

Let us fix an $\varepsilon > 0$. (the same argument holds for any $\varepsilon > 0$).

1. Prove there exists some $N \in \mathbb{N}$, such that $\frac{1}{3^N} < \varepsilon$ (Archimedean property). Show that there is some integer $m \in \mathbb{N}$, such that

   $$x \in \left[\frac{m}{3^n}, \frac{m+1}{3^n}\right]$$

   Thus $x \in A_m$.

2. In constructing A_{m+1} we subdivide the interval $\left[\frac{m}{3^N}, \frac{m+1}{3^N}\right]$ so that we have two cases

   (a) $x \in \left[\frac{3m}{3^{N+1}}, \frac{3m+1}{3^{N+1}}\right]$. First show that there exists $c \in \left[\frac{3m+2}{3^N}, \frac{3m+3}{3^{N+1}}\right] \cap C$. What does this say about $(x - \varepsilon, x + \varepsilon) \cap C \setminus \{x\}$?

   (b) $x \in \left[\frac{3m+2}{3^N}, \frac{3m+3}{3^{N+1}}\right]$. Do a similar analysis as (a).

Combine your thought on 1 and 2 to show that x is indeed a limit point.

21 Integration

Learning Objectives

We will

- Understand for what class of functions integration makes sense. This will include continuous functions.

21.1 Integration on piecewise constant functions

Definition 21.1. Let I be a bdd interval. A *finite partiion* of I, is a *finite* set , $\mathbf{P}$ of bounded intervals $\{P \subseteq I\}_{P \in \mathbf{P}}$, such that

- their union is I: $\bigcup_{P \in \mathbf{B}} = I$.
- they are pairwise disjoint: $P \cap P' = \emptyset$ for all $P, P' \in \mathbf{P}$.

Definition 21.2. Let $I \subseteq \mathbb{R}$. Let $E \subseteq I$ be any subset. Then we denote

$$1_E : X \to \mathbb{R}$$

as the indicator function of E.

$$1_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

for $x \in I$.

Definition 21.3. Let $I \subseteq \mathbb{R}$. Let $f \in \mathrm{Fct}(I, \mathbb{R})$. f is *piecewise constant* (pc) if it can be written of the form

$$f := \sum_{P \in \mathbf{P}} c_P 1_P$$

for some partition $\mathbf{P}$ of I. In this case, we say that f is *piecewise constant with respect to* $\mathbf{P}$.
We let

$$\mathrm{PwCst}(I, \mathbb{R})$$

be the set of piecewise constant functions

Example

Let $f : [1, 4] \to \mathbb{R}$.

$$f = 2 \cdot 1_{[1,3)} + 4 \cdot 1_{\{3\}} + 6 \cdot 1_{(3,4]}$$

Definition 21.4. Integration with respect to a partition. Let I be a bounded interval. Let $\mathbf{P}$ be partition of I, and $f : I \to \mathbb{R}$ a function pc with respect to $\mathbf{P}$. Then

$$\int_{\mathbf{P}} f := \sum_{P \in \mathbf{P}} c_P |P|$$

Example

[13, 11.2.12].

To make sense of this formula we need to know how to define the *length* (or *measure*) of bounded interval.

Definition 21.5. Let I be a bounded interval. We now define a function

$$| \quad | : \text{Bounded intervals of } \mathbb{R} \to \mathbb{R}$$

We define this by cases:

- If $I = \emptyset$, then $|I| = \emptyset$.
- If $a < b$, I is $[a, b], (a, b], [a, b)$ or (a, b), then
$$|I| = b - a$$
- If $a = b$, and $I = [a] = \{a\}$, then
$$|I| = 0$$

Warning: this is *not* to be confused with *cardinality*, **??**.

Let I be bdd interval. $f : I \to \mathbb{R}$ be pc function on I. We *want* to define

$$\int_I f := \int_{\mathbf{P}} f$$

for *any* choice $\mathbf{P}$ of partition. What if for two different partitions, we get different values?

Proposition 21.6. Def. 21.7 Let I be be a bounded interval and $f : I \to \mathbb{R}$ be a function. Suppose that $\mathbf{P}$, $\mathbf{P'}$ are partitions of I such that f is piecewise constant with respect to both, then

$$\int_{\mathbf{P}} f = \int_{\mathbf{P'}} f$$

Prop. 21.6 shows that Def. 21.7 below is well-defined.

Definition 21.7. Integration with respect to *any* partition. . Let I be bdd interval. $f : I \to \mathbb{R}$ be pc function on I. We define

$$\int_I f := \int_{\mathbf{P}} f$$

for *any* choice $\mathbf{P}$ of partition.

The crucial step is that one can always find a refinement of a partition.

Definition 21.8. Let I be a bounded interval. Let $\mathbf{P}, \mathbf{P}'$ be two partition. We say $\mathbf{P}'$ is a *refinement* of $\mathbf{P}$ if for all $K \in \mathbf{P}'$ there exists $J \in \mathbf{P}$ such that $K \subseteq J$.

22 Riemann-Integrability

Reading: [13, 11].

$$\begin{array}{ccccc} \mathrm{Uni}^{\mathrm{bdd}}(I,\mathbb{R}) & \subseteq & \mathrm{RInt}(I,\mathbb{R}) & \subseteq & \mathrm{Fct}^{\mathrm{bdd}}(I,\mathbb{R}) \\ =^{\mathrm{cl}} & & \cup\!\mid & & \\ \mathrm{Cts}^{\mathrm{bdd}}(I,\mathbb{R}) & & \mathrm{PwCst}(I,\mathbb{R}) & & \end{array}$$

The $=^{\mathrm{cl}}$ is to signify this only holds when I is a closed bounded interval.
We will only consider bounded functions. We will consider unbounded integrals next in [...]. So far we have *only* defined integration on piecewise constant functions, def. 21.7.

$$\int_I (-) : \mathrm{PwCst}(I,\mathbb{R}) \to \mathbb{R}$$

We can define a partial order on functions.

Definition 22.1. Let $f, g \in \mathrm{Fct}(I,\mathbb{R})$. Then $f \leq g$ iff $f(x) \leq g(x)$ for all $x \in I$.

Definition 22.2. Let $f : I \to \mathbb{R}$ be a bounded function defined on a bounded interval. We set

$$\overline{\int}_I f := \inf \left\{ \int_I g \; : \; g \in \mathrm{PwCst}(I,\mathbb{R}), g \geq f \right\}$$

Proposition 22.3. [13, 11.3.3] Let $f : I \to \mathbb{R}$ be function on a bounded interval I which is bounded by some real number.

Proof.

$$\overline{\int}_I f \leq M|I|$$

$\square$

Definition 22.4. Let $f \in \mathrm{Bdd}(I, \mathbb{R})$. If

$$\underline{\int}_I f = \overline{\int}_I f$$

then f is *Riemann integrable on* I.

Theorem 22.5. [13, 11.5.1] Let I be a bounded interval, and let f be a function which is uniformly continuous on I, then $f \in \mathrm{RInt}(I)$.

What is the relation of this and classical notion of Riemann sums?
By definition we have

Proposition 22.6. [13, 11.5.11]. Let I be bdd interval. $f \in \mathrm{Uni}(I, \mathbb{R})$. Then $f \in \mathrm{RInt}(I, \mathbb{R})$.

Proof. □

To leverage this to continuous functions on a closed interval, we will need

Theorem 22.7. [13, 9.9.16] Let $a < b$, and let $f \in \mathrm{Cts}([a, b], \mathbb{R})$. Then $f \in \mathrm{Uni}([a, b], \mathbb{R})$.

To prove this we begin with the notion of *uniform continuity.* [to be continued]

Example

Consider $f(x) = 1/x$. For what values of δ is it sufficient that

$$|f(x) - f(1)| < 0.1$$

On the other hand, for values of δ is it required that

$$|f(x) - f(0.1)| < 0.1$$

? Now ask the same question for $g(x) = 2x$.

Definition 22.8. [13, 9.9.2]

Proposition 22.9. [13, 9.9.15] Let X be a subset of $\mathbb{R}$, $f : X \to \mathbb{R}$. Suppose that $E \subseteq X$ is bounded, then $f(E)$ is also bounded.

23 Fundamental Theorem of Calculus

Theorem 23.1. First Fundamental theorem calculus. Let $a < b$ be real numbers. $f \in \mathrm{RInt}([a, b], \mathbb{R})$. Let

$$F(x) := \int_{[a,x]} f$$

Then F is continuous. If f is continuous at c in (a, b), then F is differentiable at c, $F'(c) = f(c)$.

Example

Polynomial functions.

Definition 23.2.

23.1 Differential notation convention

We will often use dx and dt to signify the variables involved.

24 Improper Integral

Reading: [12, p283].
Let $a \in \bar{\mathbb{R}}$, see 16.7, $G : (a, +\infty) \to \mathbb{R}$ be a function.

$$\lim_{t \to +\infty} G$$

exists if there exists $L \in \bar{\mathbb{R}}$, such that

1. If finite: for all $\varepsilon > 0$, there exists $N_\varepsilon > 0$, such that for all $t > N_\varepsilon$,

$$|L - G(t)| < \varepsilon \quad \forall t > N_\varepsilon$$

2. If $+\infty$ ...

This allows us to define the limit of integral.

Definition 24.1. For all $t > a$, if

$$G(t) := \int_a^t f(x)\, dx$$

exists. We define

$$\int_a^\infty f(x)\, dx := \lim_{t \to +\infty} \int_a^t f(x),\, dx$$

if it exists and is *finite*. We say the integral $\int_a^\infty f(x),\, dx$ is *convergent* and *divergent* otherwise.

Example

1. $\int_1^\infty \frac{1}{x}\,dx$. Recall

25 Homework for Week 8

Due: Week 8, Wednesday. You will select 3 problems to be graded. Q1-2 are regarding continuity. Q3-4 are regarding the proof of the well-definedness of the integral, def 21.7. Q5-7 are exercises on integrating piecewise constant functions.

1. Let $f = 1_{\mathbb{Q}} : \mathbb{R} \to \mathbb{R}$ be the indicator function on $\mathbb{Q}$, i.e.

$$1_{\mathbb{Q}}(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

   . Show that f is not continuous at any $x \in \mathbb{R}$.

2. Let $f : [1, 4] \to \mathbb{R}$.

$$f = 2 \cdot 1_{[1,3)} + 4 \cdot 1_{\{3\}} + 6 \cdot 1_{(3,4]}$$

   show that the function is not continuous at 3.

3. (*) Let I be a bounded interval. Given any two partition of I, $\mathbf{P}, \mathbf{P}'$ one can find a *common* refinement, Def. 21.8, $\mathbf{P}''$. That is $\mathbf{P}''$ is both a refinement of $\mathbf{P}$ and $\mathbf{P}'$.

4. (**) Prove 21.6. You may use any results in the homework before this proposition. Hint: suppose we have two partitions $\mathbf{P}$,$\mathbf{P}'$, with a common refinement $\mathbf{P}''$, show that

$$\int_{\mathbf{P}} f = \int_{\mathbf{P}''} f = \int_{\mathbf{P}'} f$$

5. Let $I = [a, b]$ be a bounded interval, where $a < b$. Let $f \in \text{PwCst}(I, \mathbb{R})$, Show that

   (a) The restrictions of the function

$$f|_{(a,b]}, f|_{[a,b)}, \text{ and } f|_{(a,b)}$$

   are all piecewise constant.

   (b) Show that the integrals, by def 21.7, below

$$\int_{[a,b]} f = \int_{(a,b]} f|_{(a,b]} = \int_{[a,b)} f|_{[a,b)} = \int_{(a,b)} f|_{(a,b)}$$

   all coincide.

6. (*) Let $\text{PwCst}(I, \mathbb{R})$ be the set of piecewise constant functions on a bounded interval $\mathbb{R}$. Prove that the set of piecewise onstant functions are closed under addition, multiplication, and constant multiplication. I.e. if $f, g \in \text{PwCst}(I, \mathbb{R})$, then $f + g, f \cdot g, c \cdot f \in \text{PwCst}(I, \mathbb{R})$.

7. (**) Prove the following properties of integration: let I be a bounded interval. You are free to assume 6, and the well-definedness of the integral. We will again ask about the properties of the function

$$\int_I (-) : \mathrm{PwCst}(I, \mathbb{R}) \to \mathbb{R}$$

Show that for all $f, g \in \mathrm{PwCst}(I, \mathbb{R})$

(a) $\int_I (f + g) = \int_I f + \int_I g$.

(b) If $f(x) \geq 0$ for all $x \in I$, then $\int_I f \geq 0$.

(c) If $f(x) \geq g(x)$ for all $x \in I$, then $\int_I f \geq \int_I g$.

(d) For any $c \in \mathbb{R}$, $\int_I (c \cdot f) = c \cdot \int_I f$.

26 Homework for week 9

Due: Week 10, Wednesday. Select 3 problems to be graded.

1. Give an example of a function $f : (-1, 1) \to \mathbb{R}$ such that

   (a) f attains global maximum at $1/2$ but is not differentiable at $1/2$. Explain why this does *not* contradict Prop. 19.4.

   (b) f is differentiable, with 0 derivative at $1/2$, bu such that at $1/2$ it is neither a global minima or maxima. Explain why this does *not* contradict Prop. 19.4.

2. Write the definition of uniform continuity. Show that

   (a) $1/x$ is *not* uniformly continuous on $(0, 1)$.

   (b) $1/x$ is uniformly continuous on $[1, 2]$.

3. (*) Show that if $f : [a, \infty) \to \mathbb{R}$

   - is a differentiable function , def 19.2,
   - has a bounded derivative, i.e. there exists $M \in \mathbb{R}$, such that for all $x \in [a, \infty)$ for which $f'(x)$ exists, $f'(x) \leq M$.

   Then f is uniformly continuous on its domain. Hint: use Prop. 19.6.

4. (**) (Newton's Approximation) Let X be a subset of $\mathbb{R}$, let $x_0 \in X$ be a limit point of X, let $f : X \to \mathbb{R}$ be a function, and let $L \in \mathbb{R}$. Then the following statements are logically equivalent:

   (a) f is differentiable at x_0 on X with derivative L.

   (b) For every $\varepsilon > 0$, there exists a $\delta > 0$ such that $f(x)$ is $\varepsilon \cdot |x - x_0|$-close to $f(x_0) + L(x - x_0)$ whenever $x \in X$ is δ -close to x_0, i.e.,

   $$|f(x) - (f(x_0) + L(x - x_0))| \leq \varepsilon \cdot |x - x_0|$$

   whenever $x \in X$ and $|x - x_0| \leq \delta$.

5. (**) Show that if $f : I \to \mathbb{R}$ is piecewise constant, def. 21.3, then

   (a) f is Riemann integrable, def 22.4.

   (b) The new definition of *Riemeann integral.* def 22.4, [34] yield the same value as the old one, def. 21.7.

6. (*) Prove [13, 11.3.11]

7. (*) Prove [13, 11.3.12]

8. (*) Prove [13, 11.9.5]

[34]is *not clear* by definition

27 Power Series

27.1 Series

Definition 27.1. A *formal series* is an expression of the form $\sum_{n=m}^{\infty} a_n$, where $a_n \in \mathbb{R}, m \in \mathbb{Z}$. The set of all formal series starting is $\mathrm{Fs}(\mathbb{R})$.

Note that

$$\mathrm{Fs}_m(\mathbb{R}) := \left\{ \sum_{n=m}^{\infty} a_n \, : \, a_n \in \mathbb{R} \right\} \simeq \mathrm{Fct}(\mathbb{Z}_{\geq m}, \mathbb{R}) = \mathrm{Seq}_m(\mathbb{R}) := \{(a_n)_{n=m}^{\infty} \, : \, a_n \mathbb{R}\}$$

where $\mathrm{Seq}(\mathbb{R})$, is def. 10.3. That is the set of formal series starting at m is same as sequences starting at m, $\{a_n\}_{n=m}^{\infty}$ The notation $\simeq$ means bijection, def 5.1.

Definition 27.2. [13, 7.2]. Let $\sum_{n=0}^{\infty} a_n$ be a formal series.[35] The *kth partial sums* form a sequence of real numbers

$$\left\{ s_k := \sum_{n=0}^{k} a_n \right\}_{k=0}^{\infty}$$

- $\sum_{n=0}^{\infty} a_n$ *converges* if $\lim_{k\to\infty} s_k$ exists in $\mathbb{R}$. If not, we say it *diverges*. We let $\mathrm{CvgFs}(\mathbb{R})$, denote the set of *convergent formal series.*
- $\sum_{n=0}^{\infty} a_n$ *absolutely converges* if $\sum_{n=0}^{\infty} |a_n|$ converges.

Example

- $\sum \frac{1}{n}$ is divergent.
- $\sum (-1)^n$.
- $\sum \frac{1}{n^2-1}$. This can be written as $\frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$

This is conceptually the same as a sequence of real numbers. But we ask different questions. The first step is to reduce to positive real formal power series.

$$\begin{tikzcd} \mathrm{AbsCvgFs}(\mathbb{R}) \arrow[r, hook] & \mathrm{CvgFs}(\mathbb{R}_{>}0) \arrow[r, hook, shift right] & \mathrm{CvgFs}(\mathbb{R}) \arrow[l, dashed, shift right] \end{tikzcd}$$

[35] One can simply generalize these definitions to other formal series beginning with other indices.

This reduction is the absolute convergence test,

$$\begin{array}{c}\{\text{Monotone convergence, 27.6} + \text{Cauchy convergence, 27.4}\}\\ \downarrow \\ \{\text{Comparison Test}\}\\ \downarrow \\ \{\text{Root Test}\}\end{array}$$

Proposition 27.3. [13, 6.4.8]

Proposition 27.4. Let $\sum a_n$ be fs. Then $\sum a_n$ is converges iff and only if for every $\varepsilon > 0$, there exists N such that

$$|\sum_{n=p}^{q} a_n| \leq \varepsilon \quad p, q \geq N$$

Proof. Use the fact that a sequence converges iff it is Cauchy. □

As a direct corollary is the zero test 27.9.

Proposition 27.5. [13, 7.2.9] Absolute convergence. Let $\sum a_n$ be formal power series of real numbers. If the series is absolutely convergent, then it is also convergent.

Proof. We break this down into two steps

1. if $\lim s_k = \alpha \in \mathbb{R}$ exists then $|\lim s_k| = \lim |s_k|$. This is by continuity of $|\quad|$.
2. $$\lim |s_k| \leq \lim \sum |a_k|$$

□

Proposition 27.6. Basic comparison. $\sum a_n$ be formal power seris of positive integers. This series is convergetn if and only if exists M such that

$$\sum_{n=0}^{N} a_n \leq M$$

for all M.

Proof. $S_N := \sum_{n=0}^{N} a_n$ is a bounded monotone sequence. Thus the limit exists. □

Proposition 27.7. Comparison test. Let $\sum a_n, \sum b_n \in \mathrm{Fps}(\mathbb{R})$. Suppose that $|a_n| \leq b_n$ for all n. Then if $\sum b_n$ is convergent then $\sum a_n$ is absolutely convergent.

Proof. Use 27.4. □

Example

Determine whether each of the following are convergent or divergent. You are free to use the result of [13, 7.3.7].

- $\sum \frac{n}{n^2-\cos^2(n)}$ Note: $\sum \frac{1}{n}$ diverges.
- $\sum \frac{1}{3^n-n}$.

Key example of convergence series:

Proposition 27.8. Geometric series are absolutely convergent. Let $x \in \mathbb{R}$,

- If $|x| \geq 1$, then the series $\sum x^n$ is divergent.
- if $|x| < 1$, then $\sum x^n = \frac{1}{1-x}$.

Example

Consider $\sum n(-2)^n(x-3)^n$.

$$\frac{1}{\limsup_n |n(-2)^n|^{1/n}}$$

Proposition 27.9. Zero test. Let $\sum a_n \in \mathrm{CvgFs}(\mathbb{R})$. Then $\lim a_n = 0$.

The collection of convergent formal series satisfies basic nice properties.

Proposition 27.10. [13, 7.2.13].

1. Convergent of formal series only depends on tail. Let $\sum_{n=0}^{\infty} a_n \in \mathrm{Fs}(\mathbb{R})$. Then for any $m, k \in \mathbb{Z}$,

$$\sum_{n=m}^{\infty} a_n \in \mathrm{CvgFs}(\mathbb{R}) \Leftrightarrow \sum_{n=k}^{\infty} a_n \mathrm{CvgFs}(\mathbb{R})$$

Theorem 27.11. Root test. Let $\sum a_n$ Let $\alpha := \limsup_n |a_n|^{1/n}$. [36]

1. If $\alpha < 1$, then $\sum a_n$ is absolutely convergent.
2. If $\alpha > 1$.

[36] In most texts, they assume this limit exists.

3. If $\alpha = 1$ we cannot deduce anything.

Proof. We prove 1. Let $\alpha < 1$. We also know that $0 \leq \alpha$, as $|a_n|^{1/n} \geq 0$ for all n. Our strategy is to bound the sequence $\sum a_n$ by a geometric series.

Choose $\varepsilon > 0$, such that

$$0 < \alpha + \varepsilon < 1$$

Then by definition of $\inf \sup \left\{|a_n|^{1/n}\right\}$ there exists N such that [37]

$$|a_n|^{1/n} \leq \alpha + \varepsilon \quad \forall n \geq N$$

This is equivalent to

$$\sum_{n=N}^{\infty} |a_n| \leq \sum_{n=N}^{\infty} (\alpha + \varepsilon)^n$$

□

Root test is hard to use often. For instance, for exponentials.

Proposition 27.12. [13, 7.5.2] The ratio test: Let (c_n) be sequence of positive. Then

$$\liminf \frac{c_{n+1}}{c_n} \leq \liminf c_n^{1/n} \leq \limsup c_n^{1/n} \leq \limsup \frac{c_{n+1}}{n}$$

Proof. □

As a corollary:

Proposition 27.13. [13, 7..5.3] Ratio test.

Proof. □

Example

Determine whether the following are convergent or divergent. You will need the fact that $\lim_{n\to\infty} n^{1/n} = 1$. [a]

1. $\sum_{n=1}^{\infty} \frac{n^n}{3^{1+2n}}$. This is divergent.
2. $\sum_{n=1}^{\infty} \frac{(-12)^n}{n}$.

[a] One way to prove this is using continuity of $\lim \frac{\log n}{n} = 0$. There are elementary ways to show why this limit is 0.

[37] Note that we use the same argument if $\lim a_n$ exists.

28 Formal Power Series

What we will show is that there is a map

$$\begin{tikzcd}
\mathrm{Fps}(\mathbb{R}) \arrow[r, dashed] & \mathrm{Fct}(\mathbb{R},\mathbb{R}) \\
\mathrm{CvgFps}(\mathbb{R}) \arrow[u] \arrow[r] \arrow[d] & \mathrm{Fct}(\mathbb{R},\mathbb{R}) \arrow[u] \arrow[d] \\
\mathrm{CvgFps}^R(\mathbb{R}) \arrow[r] & \mathrm{Fct}((-R,R),\mathbb{R})
\end{tikzcd}$$

Definition 28.1. Let $a \in \mathbb{R}$. A *formal power series centered at a* is any series written in the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n \quad \forall n \in \mathbb{N}, \quad c_n \in \mathbb{R}$$

Definition 28.2. Let $\alpha = \limsup_{n\to\infty} |c_n|^{\frac{1}{n}}$. The *radius of converengence* is the quantity

$$R := \begin{cases} \frac{1}{\alpha} & \alpha \in \mathbb{R} \\ 0 & \alpha = +\infty \\ +\infty & \alpha = 0 \end{cases}$$

Where did the $1/n$ come from? This comes from the root test, 27.11.

Theorem 28.3. Let $\sum c_n(x-a)^n$ be formal power series. R its radius of convergence.

1. Divergence.

$$\sum c_n(x-a)^n \text{ is divergent for }, \quad |x-a| > R$$

   is divergent

2. Convergence.

$$\sum c_n(x-a)^n \text{ is convergent for } |x-a| < R$$

Proof. Consider

$$\limsup |c_n(x-a)^n|^{\frac{1}{n}} = \limsup |c_n|^{\frac{1}{n}} |x-a| = \frac{1}{R}|x-a|$$

if $\limsup |c_n|^{\frac{1}{n}} \in \mathbb{R}$. We leave it to the reader to check the special case, when $\limsup |c_n|^{\frac{1}{n}} = +\infty$. Thus if $|x-a| < R$, then the series is absolutely convergent by 27.11. □

29 Expoential and logarithm

Example

The exponential.

$$\exp(x) := \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

1. The series has radius of convergence $R = \infty$.

In practice computing the limit should be $\lim_n$. Recall in 16.6.

Definition 29.1. The *natural logarithm*

$$\log : (0, \infty) \to \mathbb{R}$$

is the inverse of $\exp : \mathbb{R} \to (0, \infty)$.

30 Differentiation power seris

But why consider $\mathrm{CvgFps}^R(\mathbb{R})$? It is because differentiation and integration behave the way we want them to be, 30.2.

Definition 30.1. Let $E \subset \mathbb{R}$. $f : E \to \mathbb{R}$ be an open interval. f is *real analytic* if for all $a \in E$

Discussion

Is

Theorem 30.2. [13, 4.1.6]

1. Uniform convergence on bounded. For $0 < r < R$, $\sum_{n=0}^{\infty} c_n(x-a)^n$ converges uniformly to f on $[a-r, a+r]$. Hence, f is continuous on $(a-R, a+R)$.
2. Differentiation of power series. The function f is differentiable on $(a-R, a+R)$ and for any $0 < r < R$ the series
$$\sum_{n=0}^{\infty} nc_n(x-a)^{n-1}$$
converges *uniformly* to f' on the interval $[a-r, a+r]$.
3. Integration of power series. For any closed bounded interval $[y, z] \subseteq (a-R, a+R)$ we gave
$$\int_{[y,z]} f = \sum_{n=0}^{\infty} c_n \frac{(z-a)^{n+1} - (y-a)^{n+1}}{n+1}$$

Now we will discuss the notion of uniform convergence. This extends the notion of sequence convergence, which was an assignment

$$\begin{tikzcd}
\mathrm{Seq}(\mathbb{R}) \arrow[r, dashed] & \mathbb{R} \\
\mathrm{CvgSeq}(\mathbb{R}) \arrow[u, phantom, "\cup\!\mid"] \arrow[r, "\lim_{n\to\infty}"'] & \mathbb{R} \arrow[u, phantom, "\parallel"]
\end{tikzcd}$$

- This is conceptually important: we always approximate functions with known functions. Often we use polynomial functions, [14, 3.8].

Definition 30.3. Let $X \subseteq \mathbb{R}$, Let $(f_n : X \to \mathbb{R})_{n=1}^{\infty}$ be a sequence of functions. Let $f : X \to \mathbb{R}$. We say $f_n \to f$ *pointwise* on X if for all $x \in X$ we have

$$\lim_n f_n(x) = f(x)$$

Example

1. $f_n : \mathbb{R} \to \mathbb{R}, f_n(x) = x/n$. Then $f_n \to 0$ pointwise.
2. Non preservation of continuity. Let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = x^n$. Let $f(x) := 1_{[1]} : [0,1] \to \mathbb{R}$. Then
$$f_n \xrightarrow{\text{pt}} f$$
3. Does not preserve integration. Let $f_n : [0,1] \to \mathbb{R}$, $f_n := 2n1_{[\frac{1}{2n},\frac{1}{n}]}$. Then
$$f_n \xrightarrow{\text{pt}} 0$$
but
$$\int_{[0,1]} f_n = 1$$
for all n. Thus $\int$ does not commutae with taking pointwise limit. OIr
$$\lim_n \left(\int f_n \right) \neq \int \left(\lim_n f_n \right)$$

These examples show that we do not have the following commutativity

$$\begin{tikzcd}
\mathrm{Seq}(\mathrm{Fct}^*(X,\mathbb{R})) \arrow[r] \arrow[d] & \mathrm{Fct}(X,\mathbb{R}) \arrow[d] \\
\mathrm{Seq}(\mathrm{Fct}^*(X,\mathbb{R})) \arrow[r] & \mathrm{Fct}(X,\mathbb{R})
\end{tikzcd}$$

for $\mathrm{Fct}^*(X,\mathbb{R}) := \mathrm{Diff}(X,\mathbb{R})$.

$$\begin{tikzcd}
\mathrm{Seq}(\mathrm{Fct}^*(X,\mathbb{R})) \arrow[r] \arrow[d, "\mathrm{Seq}(\int)"] & \mathrm{Fct}(X,\mathbb{R}) \arrow[d, "\int"] \\
\mathrm{Seq}(\mathbb{R}) \arrow[r] & \mathbb{R}
\end{tikzcd}$$

Definition 30.4. [14, 3.2.7]. Let $f_n : X \to \mathbb{R}$ be a sequence of functions. If for all $\varepsilon > 0$, there exists N, such that for all $n \geq N$,
$$|f_n(x) - f(x)| < \infty$$
for all $x \in X$. In which case we denote $f_n \xrightarrow{u} f$.

Note that the condition of "$|f_n(x) - f(x)| < \infty$ for all $x \in X$" in def 30.4 has a short hand notation using the notion of ∞-norm.

Definition 30.5. let $g : X \to \mathbb{R}$ be a bounded real functions, where X is nonempty. Then 'the *supremum norm of* g is the real value

$$|g|_\infty := \sup \{g(x) \,:\, x \in X\} \in \mathbb{R}$$

Example

Let $f_n : X \to \mathbb{R}$ be $f_n(x) := \frac{x}{n}$. This converges uniformly. Fix any $\varepsilon > 0$, by archimedean property, choose $N_\varepsilon \in \mathbb{N}$ such that $\frac{1}{N_\varepsilon} < \varepsilon$, then

$$|f_n - f|_\infty < \varepsilon$$

Theorem 30.6. Let $X, Y \subseteq \mathbb{R}$. Let $f_n : X \to \mathbb{R}$ be sequence of functions such that $f_n \xrightarrow{u} f$, def. 30.4, then f is continuous at $a \in X$, if f_n are continuous at a for all n.

Proof. This is the standard triple bound: fix $\varepsilon > 0$. Choose N , such that

$$|f_n - f|_\infty < \frac{1}{3}\varepsilon$$

for all $n \geq N$. Pick any such n and fix it. Let us choose $n = N$. Then as f_N is continuous at a, there exists $\delta_\varepsilon > 0$ such that for all $x \in X \cap (a - \delta_\varepsilon, a + \delta_\varepsilon)$,

$$|f_N(x) - f(a)| < \frac{1}{3}\varepsilon$$

$$\begin{aligned}
|f(x) - f(a)| &\leq |f_N(x) - f(x)| + |f_N(x) - f_N(a)| + |f_N(a) - f(a)| \\
&\leq 2 \cdot |f_N - f|_\infty + \frac{1}{3}\varepsilon \\
&\leq \varepsilon
\end{aligned}$$

□

Theorem 30.7. [14, 3.6.1] Uniform convergence implies integral. Let $(f_n)_{n=1}^{\infty} \in \mathrm{RInt}([a,b], \mathbb{R})$ be a sequence of Riemann integrable function. Suppose $f_n \xrightarrow{u} f$, then $f \in \mathrm{RInt}([a,b], \mathbb{R})$.

Proof. We first show that f is Riemann integrable. Hw. □

Example

$$f_n : (-1, 1) \to \mathbb{R} \quad f_n(x) = x^n$$

we know that

$$s_N(x) \xrightarrow{\text{pt}} \frac{1}{1-x}$$

does

$$s_N(x) := \sum_{n=0}^{N} x^n \xrightarrow{u} \frac{1}{1-x}$$

Recall that the $s_N(x) = \frac{1-x^{N+1}}{1-x}$. Thus

$$\left| s_N(x) - \frac{1}{1-x} \right| = \left| -\frac{x^{N+1}}{1-x} \right|$$

However, the series is uniformly convergent on r for $0 < r < 1$. This uses the M-test, 30.9.

Is there a way to test uniform convergence of functions? Often, we use sequences of function to approximate functions. In fact, we use *series* of functions, to approximate values just like we did in formal series. This is the Weierstrass M-test.

Definition 30.8. [14, 3.5.2] Let $(f_n)_{n=1}^{\infty} \in \mathrm{Seq}(\mathrm{Fct}(X, \mathbb{R}))$ and $s \in \mathrm{Fct}(X, \mathbb{R})$. Let $s_N := \sum_{n=1}^{N} f_n$. Then we write

$$\sum_{n=1}^{\infty} f_n \xrightarrow{\text{pt}} s$$

if

$$s_N \xrightarrow{\text{pt}} s$$

to de note $\sum_{n=1}^{\infty} f_n$ *converges pointwise to* s. and

Theorem 30.9. Weierstrass M-test. Let $X \subseteq \mathbb{R}$. Let $f_n : X \to \mathbb{R}$, be a sequence of bounded functions such that $\sum_{n=0}^{\infty} |f_n|_{\infty} \in \mathrm{CvgFs}(\mathbb{R})$. [38]Then $\sum_{n=0}^{\infty} f_n \xrightarrow{u} f$.

Proof. Let us note that the notion of uniformly convergence has an equivalence Cauchy formulation: if forall $\varepsilon > 0$, there exists N, such that for all $n, m \geq N$,

$$|s_n - s_m|_{\infty} < \varepsilon$$

the $s_n \xrightarrow{u} s$, for some function s. Indeed:

[38] This is a series of just functions

- Let $s(x) := \lim_n s_n(x)$ which exists pointwise. Why?
- Then fix any choice ε. Choose N such that

$$|s_n - s_m|_\infty < \frac{1}{2}\varepsilon$$

for $n, m \geq N$. Thus, for any choice of x,

$$|s(x) - s_n(x)| = \lim_m |s_n(x) - s_m(x)| < \varepsilon$$

This (why?) show that $s_n \xrightarrow{u} s$. [39] It now suffices to show that $\sum f_n$ is uniformly Cauchy. $\square$

[39] In otherwords, uniformly Cauchy functions, converges uniformly.

31 Homework for Week 10

Due: Wednesday Friday December 1st Pick 4 questions to do. Q4 is a side recollection of differentiation.

1. Prove the zero test. 27.9.

2. (*) Let $(a_n)_{n=1}^{\infty}$ be a decreasing sequence, where $a_n \geq 0$ Then $\sum_{n=1}^{\infty} a_n \in \mathrm{CvgFs}(\mathbb{R})$ iff $\sum_{k=0}^{\infty} 2^k a_{2^k} \in \mathrm{CvgFs}(\mathbb{R})$.

3. (**) Prove the alternating series test: let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers which are nonnegative and decreasing. i.e. $a_n \in \mathbb{R}_{\geq 0}$ and $a_{n+1} \leq a_n$. The $\sum_{n=0}^{\infty}(-1)^n a_n \in \mathrm{CvgFs}(\mathbb{R})$ iff $\lim_{n\to\infty} a_n = 0$. Here is the guideline for the hard direction:

   (a) Let $s_k := \sum_{n=0}^{k} a_n(-1)^n$ show that

   $$\begin{cases} s_{k+2} \geq s_k & \text{k odd} \\ s_{k+2} \leq s_k & \text{k even} \end{cases}$$

   conclude by induction that for all $m \in \mathbb{N}$,

   $$\begin{cases} s_{k+2m} \geq s_k & \text{k odd} \\ s_{k+2m} \leq s_k & \text{k even} \end{cases}$$

   (b) Now *fix* a choice of even k. We show that we will be able to bound for any $l \geq k$, that

   $$s_k - a_{k+1} \leq s_l \leq s_k$$

   - Show that for all $m \in \mathbb{N}_{\geq 0}$

     $$s_{k+2m+1} \geq s_{k+1} = s_k - a_{k+1}$$

     for any choice of m.
   - Show that for all $m \in \mathbb{N}_{\geq 0}$

     $$s_{k+2m+1} \leq s_{k+2m} \leq s_k$$

   Thus, show that $(s_j := \sum_{n=0}^{j} a_n(-1)^n)_{j\geq 0}$ is eventually a_{k+1} steady. i.e. exists N such that for all $p, q \geq N$,

   $$|s_p - s_q| < a_{k+1}$$

   (c) Use the fact that $\lim_{k\to\infty} a_k = 0$ to conclude that $(s_j)_{j\geq 0}$ converges.

4. (*) Let $I = [a, b]$ and $x \in (a, b)$. Suppose that f is monotone increasing and differentiable on I, ($f(c) \geq f(d)$ if $c \geq d$ for all $c, d \in [a, b]$), then $f'(x) \geq 0$.

5. You are free to use any previous results, provided you state them fully. Determine (with proof) whether

$$\sum_{n=0}^{\infty} \frac{(-1)^{n-3}\sqrt{n}}{n+3}$$

is convergent or divergent.[40]

6. (**) Prove (treat carefully of the cases when $\alpha = 0$ or $\alpha = +\infty$) root test 27.11 In the cases when

   (a) $\alpha > 1$.

   (b) $\alpha = 1$:

   Give one example each for which $\sum a_n$ diverge and converge.

7. Show that for $x \in \mathbb{R}$, such that $|x| < 1$,

$$\sum (-1)^{n+1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

is convergent, def 27.2. You are free to use any test, provided you state them fully.

[40] To show a sequence is decreasing in this case is not hard.

32 Additional Problems

Here is a selection of problems that can be submitted by Dec 19th, replacing the previously submitted homework problem. For any problems - clearly state any results you wish to use. This file will be uploaded now and then.

1. Recall the definition of partial order on $\mathbb{Q}$, see def. 7.13. Let $a, b, c, d \in \mathbb{Q}$. Prove that if
   - $a < b$ and $c < d$ then $a + c < b + d$.
   - if $a < b$ and $c > 0$, then $ac < bc$.

2. Prove
$$1 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$
by induction.

3. Prove that
$$\sum_{k=0}^{l} \binom{n}{k}\binom{m}{l-k} = \binom{n+m}{l}$$
using a counting argument, by applying binomial theorem $(1+x)^n(1+x)^m$.

4. Let $f : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ be the function
$$x \mapsto \frac{1}{1+x}$$
What is the domain, see def. 18.5, of f and $f \circ f$ be the composition of f by itself, i .e. it is the function that
$$x \mapsto \frac{1}{1 + \frac{1}{1+x}}$$

5. In the following cases, find a δ such that $|f(x) - l| < \varepsilon$ for all x satisfying $0 < |x - a| < \delta$.
   (a) $f(x) = x^4$, $l = a^4$.
   (b) $f(x) = \frac{1}{x}$, $a = 1$, $l = 1$.

6. Let $f : X \to \mathbb{R}$ be a function which is continuous at a, 12.6. Show that
$$\lim_{x \to a} f(x) = l$$
iff
$$\lim_{x \to a} [f(x) - l] = 0$$

7. (*) For $k \in \mathbb{Z}_{<0}$, consider a sequence $\{c_n\}_{n\in\mathbb{Z}}$ satisfying
$$c_{n+k} = c_n^{1/2}$$
for all $n \in \mathbb{Z}$. If $\lim_{n\to\infty} c_n = 0$, show that $c_n = 0$ for all n.

8. [13, 9.8.3] Let $a < b$ be real numbers and $f : [a, b] \to \mathbb{R}$ be a function which is both continuous and strictly monotone increasing. f is bijection $[a, b]$ to $[f(a), f(b)]$, with inverse $f^{-1} : [f(a), f(b)] \to [a, b]$. Prove this is also continuous and strictly monotone increasing.

9. Suppose f is a function satisfying $|f(x)| \leq |x|$, for all x, show that f is continuous at 0.

10. For previous question, give an example of f which is not continuous at any $a \neq 0$ in its domain.

11. Prove that there does not exists $q \in \mathbb{Q}$ such that $q^2 = 2$. Hence, show that
$$E := \left\{q \in \mathbb{Q} \,:\, q^2 < 2\right\}$$
has no least upper bound.

12. Determine whether each of the following functions is bounded above or below in the indicated interval.
    (a) $f : (-1, 1) \to \mathbb{R}$, $f(x) = x^2$.
    (b) $f : (-1, 1) \to \mathbb{R}$, $f(x) = x^3$.
    (c) $f : (-1, 1) \to \mathbb{R}$, $f(x) = x^2$.on $\mathbb{R}$.

13. Suppose that f is continuous on $[a, b]$ and $f(x)$ is always rational - what can we say about f?

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