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Homotopy colimits in the category of small categories

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In (13), Quillen defines a higher algebraic K-theory by taking homotopy groups of the classifying spaces of certain categories. Certain questions in K-theory then become questions such as when do functors induce a homotopy equivalence of classifying spaces, or when is a square of categories homotopy cartesian? Quillen has given some techniques for answering such questions. F. Waldhausen has extended these ideas in (19), and broadened the range of applications to include geometric topology (20).

On the other hand, classifying spaces of monoidal and symmetric monoidal categories provide the most prominent examples of spaces with the extra structure required to deloop them ((1), (11), (12), (14)).

Thus there has been rising interest in the relation between categories and the homotopy types of their classifying spaces. In this paper, it is shown that when a fundamental homotopy construction, the homotopy colimit, is performed on a diagram of spaces obtained by applying the classifying space functor to a diagram of categories, the result has the homotopy type of the classifying space of a certain category, the Grothendieck construction on the diagram. This theorem may be applied to produce categories which are 'deloopings' and 'group completions' of symmetric monoidal categories, which is of interest in the studies mentioned above. It also motivates the constructions of R. Charney's proof of homological stability of GL_n of a PID (3). Mapping cones, mapping telescopes, double mapping cylinders, suspensions, and geometric realizations of simplicial spaces are all examples of homotopy colimits (2), XII, § 3; the theorem makes these constructions available in CAT.

This paper is a version of parts of my thesis (17), stripped of the more exotic categorical notions and omitting some of the applications.

I will assume the reader is familiar with the basics of the homotopy theory of categories, given in § 1 of (13); and with the homotopy colimit (homotopy direct limit) of Bousfield and Kan, discussed in chapter XII of (2). The applications in § 4 require extensive knowledge of (14). Statements labelled 'Remarks' often require more background: the reader may ignore them if he wishes. Let me fix some conventions. I will use the unbased version of the homotopy colimit, as in (2), XII, 3·7. I use the nerve functor N of (2); thus a p-simplex of $N\mathbf{C}$ is a string of p morphisms $C_0 \leftarrow C_1 \leftarrow \cdots \leftarrow C_p$, and d_0 of this is $C_1 \leftarrow \cdots \leftarrow C_p$. This differs from the nerve N' of (13) in that $N'\mathbf{C} = N\mathbf{C}^{\mathrm{op}}$, where $\mathbf{C}^{\mathrm{op}}$ is the opposite category to $\mathbf{C}$.

The first sections of this paper are written simplicially. TOP enthusiasts should recall that the classifying space functor is geometric realization, $|\ |$, composed with the nerve functor N; and contemplate the relation between simplicial and topological homotopy colimits as in (2), XII, 3·7 and (18), § 8. I say a map $f: X \to Y$ of simplicial sets is a homotopy equivalence if $|f|$ is a homotopy equivalence of spaces; I do not require f to have a simplicial homotopy inverse. I say a functor f is a homotopy equivalence if Nf is.

My final convention is that all categories I deal with are to be small (i.e. with a set, not a class, of objects), unless blatantly otherwise. For example, CAT, the category of small categories is obviously large.

1. *The homotopy colimit theorem*

1·1. *Definition*. Let $F: \mathbf{K} \to \mathrm{CAT}$ be a functor. The Grothendieck construction on F, $\mathbf{K}\int F$, is the category with objects the pairs (K, X) with K an object of $\mathbf{K}$ and X an object of $F(K)$, and with morphisms $(k, x): (K_1, X_1) \to (K_0, X_0)$ given by a morphism $k: K_1 \to K_0$ in $\mathbf{K}$ and an $x: F(k)(X_1) \to X_0$ in $F(K_0)$. Composition is defined by

$$(k, x).(k', x') = (kk', x.F(k)(x')).$$

This construction has been extensively used by category theorists, see (4), (5), (6), (18). For special $\mathbf{K}$ it appears as 'the canonical cofibred category over $\mathbf{K}$ associated to F' in (7), (13).

Note a natural transform $f: F \Rightarrow F'$ of functors $\mathbf{K} \to \mathrm{CAT}$ induces a functor $\mathbf{K}\int f: \mathbf{K}\int F \to \mathbf{K}\int F'$ by $\mathbf{K}\int f(K, X) = (K, f(K)(X))$, $\mathbf{K}\int f(k, x) = (k, f(K_0)(x))$ for $(k, x): (K_1, X_1) \to (K_0, X_0)$. Thus $\mathbf{K}\int$ is a functor from the category of functors $\mathbf{K} \to \mathrm{CAT}$.

1·2. THEOREM. (*Homotopy colimit theorem*). *Let* $F: \mathbf{K} \to \mathrm{CAT}$ *be a functor. There is a natural homotopy equivalence*

$$\eta: \operatorname{hocolim} NF \to N(\mathbf{K}\textstyle\int F)$$

of the homotopy colimit of NF *and the nerve of the Grothendieck construction.*

The proof will follow this outline. First we define η. We then construct a functor $\tilde{F}: \mathbf{K} \to \mathrm{CAT}$ and produce natural homotopy equivalences

$$\operatorname{hocolim} NF \xleftarrow{\lambda_1} \operatorname{hocolim} N\tilde{F} \xrightarrow{\lambda_2} N(\mathbf{K}\textstyle\int F)$$

Finally, we construct a simplicial homotopy $H: \eta.\lambda_1 \simeq \lambda_2$. As λ_1, λ_2 are homotopy equivalences, it will follow that η is. The constructions of this proof are closely related to those of the proof of theorem A in (13).

1·2·1. LEMMA. *There is a natural map* $\eta: \operatorname{hocolim} NF \to N(\mathbf{K}\int F)$.

Proof. By (2), XII, 5·2, one knows hocolim NF is the diagonal of a bisimplicial set $\amalg_* NF$. A (p, q)-simplex of $\amalg_* NF$ is a string of p composible morphisms of $\mathbf{K}$,

$$K_0 \xleftarrow{k_1} K_1 \xleftarrow{k_2} \cdots \xleftarrow{k_p} K_p,$$

with a string of q composible morphisms of $F(K_p)$, $X_0 \xleftarrow{x_1} X_1 \xleftarrow{x_2} \cdots \xleftarrow{x_q} X_q$. The simplicial operators $(d_i, 1)$, $(s_i, 1)$, $(1, d_i)$, $(1, s_i)$ act by composing morphisms, etc., as in N, except

$$\begin{aligned}(d_p, 1)(K_0 &\xleftarrow{k_1} \cdots \xleftarrow{k_p} K_p, X_0 \xleftarrow{x_1} \cdots \xleftarrow{x_q} X_q) \\ &= (K_0 \xleftarrow{k_1} \cdots \xleftarrow{k_{p-1}} K_p, F(k_p)(X_0) \xleftarrow{F(k_p)(x_1)} \cdots \longleftarrow F(k_p)(X_p).\end{aligned}$$

We define η on p-simplices in $(\operatorname{hocolim} NF)_* = \amalg_* NF(p,p)$ by

$$\begin{aligned}\eta_p(K_0 &\xleftarrow{k_1} \cdots \xleftarrow{k_p} K_p, X_0 \xleftarrow{x_1} \cdots \xleftarrow{x_p} X_p) \\ &= (K_0, F(k_1 \ldots k_p)(X_0)) \xleftarrow{(k_1, F_1(k_1 \ldots k_p)(x_1))} \cdots \\ &\qquad (K_1, F(k_2 \ldots k_p)(X_2)) \leftarrow \cdots \xleftarrow{(k_p, F(k_p)(x_p))} (K_p, F(k_p)(X_p)),\end{aligned}$$

where the latter is an element of $N_p(\mathbf{K} \int F)$. One verifies this formula defines a simplicial map η.

1·2·2. *Definition.* For $F: \mathbf{K} \to \mathrm{CAT}$, let $\tilde{F}: \mathbf{K} \to \mathrm{CAT}$ be the functor $\tilde{F}(K) = \pi/K$, for $\pi: \mathbf{K} \int F \to \mathbf{K}$ the functor $\pi(K, X) = K$, $\pi(k, x) = k$. Explicitly, $\tilde{F}(K)$ is the category with objects (l, X), with $l: L \to K$ a morphism in $\mathbf{K}$ and X an object of $F(L)$. A morphism $(k_1, x_1): (l_1, X_1) \to (l_0, X_0)$ consists of a $k_1: L_1 \to L_0$ such that $l_1 = l_0 . k_1$, and an $x_1: F(k_1)(X_1) \to X_0$. Composition is given by

$$(k_1, x_1).(k_2, x_2) = (k_1 k_2, x_1 . F(k_2)(x_2)).$$

A $k: K \to K'$ in $\mathbf{K}$ induces a functor $\tilde{F}(k): \tilde{F}(K) \to \tilde{F}(K')$ given on objects by

$$\tilde{F}(k)(l, X) = (kl, X),$$

and on morphisms by $\tilde{F}(k)(k_1, x_1) = (k_1, x_1)$. Thus $\tilde{F}$ is a functor $\mathbf{K} \to \mathrm{CAT}$.

1·2·3. Lemma. *There is a natural homotopy equivalence*

$$\lambda_1: \operatorname{hocolim} N\tilde{F} \to \operatorname{hocolim} NF.$$

Proof. There is a canonical functor $\tilde{F}(K) \to F(K)$ given by $(l, X) \mapsto F(l)(X)$, $(k, x) \mapsto F(l_0)(x)$ for $(k, x): (l_1, X_1) \to (l_0, X_0)$. This functor has a right adjoint $F(K) \to \tilde{F}(K)$ sending $X \mapsto (1, X)$. By (13), proposition 2, corollary 1, it follows $N\tilde{F}(K) \to NF(K)$ is a homotopy equivalence. As the $\tilde{F}(K) \to F(K)$ give a natural transform $\tilde{F} \Rightarrow F$ of functors $\mathbf{K} \to \mathrm{CAT}$, $N\tilde{F} \to NF$ is a natural equivalence and induces a natural homotopy equivalence λ_1: hocolim $N\tilde{F} \to$ hocolim NF by (2), XII, 4·2.

1·2·4. Lemma. *There is a natural equivalence* $\lambda_2: \operatorname{hocolim} N\tilde{F} \to N(\mathbf{K} \int F)$.

Proof. We note that a p-simplex of $N\tilde{F}(K)$ may be identified with a string of p morphisms in $\mathbf{K} \int F$,

$$(L_0, X_0) \xleftarrow{(l_1, x_1)} (L_1, X_1) \leftarrow \cdots \leftarrow (L_q, X_q),$$

together with a map $l: L_0 \to K$; this corresponds to the p-simplex given by the string of p morphisms in $\tilde{F}(K)$.

$$(l, X_0) \xleftarrow{(l_1, x_1)} (ll_1, X_1) \leftarrow \cdots \leftarrow (ll_1 \ldots l_q, X_q).$$

Thus the bisimplicial set $\amalg_* N\tilde{F}$ has (p,q)-simplices the data

$$K_0 \xleftarrow{k_1} \cdots \xleftarrow{k_p} K_p,\ K_p \xleftarrow{l} L_0,\ (L_0, X_0) \leftarrow \cdots \leftarrow (L_q, X_q).$$

The map λ_2: $\operatorname{diag} \amalg_* N\tilde{F} \to N(\mathbf{K} \int F)$ sends such a (q,q)-simplex to the q-simplex $(L_0, X_0) \leftarrow \cdots \leftarrow (L_q, X_q)$ of $N(\mathbf{K} \int F)$.

It remains to show λ_2 is a homotopy equivalence. We consider $N(\mathbf{K} \int F)$ as a bisimplicial set with (p,q)-simplices $N(\mathbf{K} \int F)(p,q) = N(\mathbf{K} \int F)_q$; i.e. it is constant in the p direction. Clearly $N(\mathbf{K} \int F)_* = \operatorname{diag} N(\mathbf{K} \int F)_{**}$. The map λ_2 is the diagonalization of an obvious bisimplicial map Λ: $\amalg_* N\tilde{F} \to N(\mathbf{K} \int F)_{**}$. We recall two facts about bisimplicial sets: first, there is a natural isomorphism of the spaces resulting from realization $|p \to \operatorname{diag} T_p| \cong |q \to |p \to T_{p,q}||$ for a bisimplicial T_{**}. Second, given a bisimplicial f_{**}: $T_{**} \to S_{**}$, if for all q, $|p \to f_{p,q}|: |p \to T_{p,q}| \to |p \to S_{p,q}|$ is a homotopy equivalence, then $|\operatorname{diag} f|$ is. The first fact is proved in (13), and the second follows from the first by (11), A 4 (ii) or (14), A 1 (also (13)). Thus we are reduced to showing for each q that $\Lambda(*, q)$: $\amalg_* N\tilde{F}(*, q) \to N(\mathbf{K} \int F)_{*q}$ is a homotopy equivalence. But $\Lambda(*, q)$ is the coproduct over all q-simplices $(L_0, X_0) \leftarrow \cdots \leftarrow (L_q, X_q)$ of $N(\mathbf{K} \int F)$ of the map

$$|p \to \{(K_0 \leftarrow K_1 \leftarrow \cdots \leftarrow K_p, K_p \leftarrow L_0)\}| \to |p \to a \text{ point}| = a \text{ point}.$$

The former is the classifying space of the category $L_0 \backslash \mathbf{K}$, (13), which is contractible as $L_0 \backslash \mathbf{K}$ has an initial object. Thus $\Lambda(*, q)$, and so λ_2, are homotopy equivalences.

1·2·5. LEMMA. *There is a simplicial homotopy* H: $(\operatorname{hocolim} N\tilde{F}) \times \Delta[1] \to N(\mathbf{K} \int F)$ *from* $\eta . \lambda_1$ *to* λ_2.

Proof. Note $\Delta[1]$ is $N\mathbf{1}$ for the category with objects 0, 1, and one non-identity map, $1 \to 0$. A p-simplex of hocolim $N\tilde{F} \times \Delta[1]$ is thus given by

$$\begin{aligned}(K_0 \xleftarrow{k_1} \cdots \xleftarrow{k_p} K_p,\ K_p \xleftarrow{l} L_0,\ (L_0, X_0) \xleftarrow{(l_1, x_1)} \cdots \leftarrow (L_p, X_p))\\ \times (0 \leftarrow 0 \leftarrow \cdots \leftarrow 0 \leftarrow 1 \leftarrow \cdots \leftarrow 1),\end{aligned}$$

where there are i 0's and $p+1-i$ 1's with $0 \leqslant i \leqslant p+1$. H sends this simplex to

$$\begin{aligned}(K_0, F(k_1 \ldots k_p l)(X_0)) &\xleftarrow{(k_1, F(k_1 \ldots k_p l)(x_1))} (K_1, F(k_2 \ldots k_p l l_1)(X_1)) \leftarrow \cdots \\ &\leftarrow (K_{i-1}, F(k_{i-1} \ldots l_{i-1})(X_{i-1})) \xleftarrow{(k_i \ldots l_i, F(k_i \ldots l_{i-1})(x_i))} (L_i, X_i) \xleftarrow{(l_{i+1}, x_{i+1})} \cdots \\ &\leftarrow (L_{i+1}, X_{i+1}) \leftarrow \cdots \leftarrow (L_p, X_p)\end{aligned}$$

with the understanding that if $i = 0, p+1$, we apply this formula and then lop off all terms involving nonsensical K_{-1}, L_{p+1}, etc. One verifies the appropriate simplicial identities.

This completes the proof of the theorem.

1·3. *Remarks.* The nerve functor has a left adjoint C, the categorization functor. To a simplicial set X, C associates a category CX whose objects are the 0-simplices of X. One forms CX by considering the 0-simplices and 1-simplices of X as a graph in the sense of (10), II. 7, taking the free category on this graph, and then modding out the relations $d_2x.d_0x = d_1x$ for every 2-simplex x of X, and $s_0x = 1_x$ for every 0-simplex ((10), II, 8). The adjunction $\epsilon\colon CN \to Id$ is an isomorphism, and N is a full embedding of CAT in the category of simplicial sets.

For $F\colon \mathbf{K} \to \mathrm{CAT}$, one sees there is a natural isomorphism $C(\operatorname{hocolim} NF) \cong \mathbf{K}\int F$, as $\mathbf{K}\int F$ has the appropriate universal mapping property. This isomorphism identifies our canonical $\eta\colon \operatorname{hocolim} NF \to N(\mathbf{K}\int F)$ with the adjunction $Id \to NC$.

The universal mapping property of $\mathbf{K}\int F$ is the following:

1·3·1. Proposition. *There is a natural bijection between the set of functors $g\colon \mathbf{K}\int F \to \mathbf{C}$ and the set of data consisting of*

(1) *for each object K of $\mathbf{K}$, a functor $g(K)\colon F(K) \to \mathbf{C}$,*

(2) *for each morphism $k\colon K \to K'$ in $\mathbf{K}$, a natural transform*

$$g(k)\colon g(K) \Rightarrow g(K').F(k)$$

such that $g(1_K) = 1\colon g(K) \Rightarrow g(K)$; and for $K'' \xleftarrow{k'} K' \xleftarrow{k} K$, $g(k'k) = g(k')g(k)$.

Proof. Given data as above, define $g\colon K\int F \to \mathbf{C}$ by $g(K,X) = g(K)(X)$ on objects, and on the morphisms $(k,x)\colon (K,X) \to (K',X')$, $g(k,x)$ is

$$g(K)(X) \xrightarrow{g(k)(X)} g(K')(F(k)(X)) \xrightarrow{g(K')(x)} g(K')(X').$$

Conversely, given $g\colon \mathbf{K}\int F \to \mathbf{C}$, we specify data by defining the functor $g(K)$ by $g(K)(X) = g(K,X)$, $g(K)(x) = g(1_K, x)$, and the natural transform $g(k)$ to have components $g(k)(X) = g(k, 1_X)$.

1·3·2. We reinterpret this result to show how it says $\mathbf{K}\int F$ has the mapping property of a homotopy colimit.

We have a functor $\Delta \to \mathrm{CAT}$ from the skeletal category of finite ordered sets. This functor sends the ordered set $\{0, 1, \dots, p\}$ to the category $\mathbf{p}$ with objects the elements of the set, and with a unique morphism $i \to j$ if and only if $j \leqslant i$. The standard maps in Δ induce coface and codegeneracy functors, $\boldsymbol{\delta}^i\colon \mathbf{p-1} \to \mathbf{p}$, $\boldsymbol{\sigma}^i\colon \mathbf{p+1} \to \mathbf{p}$.

A functor $g(K)\colon F(K) \to \mathbf{C}$ corresponds to a $g[K]\colon F(K) \times \mathbf{0} \to \mathbf{C}$, as $F(K) \cong F(K) \times \mathbf{0}$. A natural transform $g(k)$ corresponds to a functor $g[K' \xleftarrow{k} K]\colon F(K) \times \mathbf{1} \to \mathbf{C}$. Similarly, for a string of morphisms in $\mathbf{K}$, $K_0 \xleftarrow{k_1} K_1 \leftarrow \cdots \leftarrow K_n$, the string of natural transforms $g(k_i)$ corresponds to a functor $g[K_0 \xleftarrow{k_1} \cdots \xleftarrow{k_n} K_n]\colon F(K_n) \times \mathbf{n} \to \mathbf{C}$.

The compatibility conditions on the $g(k)$'s and the determinations of the

$$g[K_0 \leftarrow \cdots \leftarrow K_n]\text{'s}$$

by the $g(k) = g[K_i \leftarrow K_{i+1}]$'s then amount to the commutativity of the diagrams below and on the following page

$$\begin{tikzcd}
F(K_p) \times \mathbf{p} \arrow[dr, "{g[K_0 \leftarrow \dots \leftarrow K_p]}"] & \\
& \mathbf{C} \\
F(K_p) \times \mathbf{p-1} \arrow[uu, "{i < p \quad 1 \times \delta^i}"] \arrow[ur, "{g[d_i(K_0 \leftarrow \dots \leftarrow K_p)]}"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
F(K_p) \times \mathbf{p} \arrow[dr, "{g[K_0 \leftarrow \dots \leftarrow K_p]}"] & \\
& \mathbf{C} \\
f(K_p) \times \mathbf{p+1} \arrow[uu, "1 \times \sigma^i"] \arrow[ur, "{g[s_i(K_0 \leftarrow \dots \leftarrow K_p)]}"'] &
\end{tikzcd}$$

where we regard $(K_0 \leftarrow \cdots \leftarrow K_p)$ as a p-simplex of $N\mathbf{K}$ and define d_i, s_i accordingly. Conversely, given such data, it is induced by unique data of the form in 1·3·1.

$$
\begin{tikzcd}
 & F(K_p)\times \mathbf{p} \arrow[dr, "{g[K_0 \leftarrow \cdots \xleftarrow{k_p} K_p]}"] & \\
F(K_p)\times \mathbf{p-1} \arrow[ur, "1\times\delta^p"] \arrow[dr, "F(k_p)\times 1"'] & & \mathbf{C} \\
 & F(K_{p-1})\times \mathbf{p-1} \arrow[ur, "{g[d_p(K_0 \leftarrow \cdots \leftarrow K_p) = (K_0 \leftarrow \cdots \leftarrow K_{p-1})]}"'] &
\end{tikzcd}
$$

Applying the functor N to these diagrams, and noting $N\mathbf{p} = \Delta[p]$, we get precisely the data required by (2), XII, 2·3, to define a simplicial map hocolim $NF \to N\mathbf{C}$. Conversely, as N is fully faithful, given the data for such a simplicial map, it is induced by unique data for defining a functor $\mathbf{K}\int F \to \mathbf{C}$. Thus simplicial maps hocolim $NF \to N\mathbf{C}$ naturally bijectively correspond to functors $\mathbf{K}\int F \to \mathbf{C}$. As C is the left adjoint to N, it follows that C (hocolim NF) is isomorphic to $\mathbf{K}\int F$ as claimed.

We also note the universal mapping property of 1·3·1 may be generalized to $\mathbf{K}\int F$ for op-lax functors F as described in § 3. In the terminology of Section 3, the generalization says $\mathbf{K}\int$: Op-lax ($\mathbf{K}$, CAT) $\to$ CAT is left adjoint to the functor which sends each category $\mathbf{C}$ to the constant op-lax functor on $\mathbf{K}$ with value $\mathbf{C}$; that is, $\mathbf{K}\int$ sends an op-lax functor to its op-lax colimit over $\mathbf{K}$. This theorem was originally proved by Gray (4), § 8.

The universal mapping property of $\mathbf{K}\int F$ may be used to give a proof of 1·2·5, as follows:

Simplicial maps H: hocolim $N\tilde{F} \times \Delta[1] \to N(\mathbf{K}\int F)$ correspond bijectively to functors $\mathbf{K}\int F \times \mathbf{1} = C$ (hocolim $N\tilde{F} \times \Delta[1]) \to \mathbf{K}\int F$, and those in turn correspond to a pair of functors Λ_0, Λ_1: $\mathbf{K}\int \tilde{F} \to \mathbf{K}\int F$ with a natural transform $\Lambda_1 \Rightarrow \Lambda_0$. To obtain the H of 1·2·5 we take Λ_0, Λ_1 such that for the object $(K, K \xleftarrow{l} L, X)$ of $\mathbf{K}\int \tilde{F}$ we have $\Lambda_0(K, l, X) = (K, F(l)(X))$, $\Lambda_1(K, k, X) = (L, X)$, with the obvious extension to morphisms. The natural transform $\Lambda_1 \Rightarrow \Lambda_0$ has components at (K, l, X) given by $(l, 1)$: $(L, X) \to (K, F(l)(X))$. This approach offers a more conceptual proof of 1·2·5, avoiding tedious computation.

2. *Applications: delooping monoidal categories*

2·1·1. *Definition.* A strict monoidal category is a monoid object in CAT; i.e. a category $\mathbf{C}$ with a functor $\oplus$: $\mathbf{C} \times \mathbf{C} \to \mathbf{C}$ and an object 0 such that

$$C_1 \oplus (C_2 \oplus C_1) = (C_1 \oplus C_2) \oplus C_3 \quad \text{and} \quad 0 \oplus C_1 = C_1 = C_1 \oplus 0$$

naturally for objects C_1, C_2, C_3 of $\mathbf{C}$.

More common in practice are (not strict) monoidal categories, where the above identites are replaced by natural isomorphisms satisfying coherence conditions. Examples of such are small additive categories with $\oplus$ the direct sum, the subcategory of isomorphisms in a small additive category, and any small category with finite products or co-products. It is known that any monoidal category is 'additively' equivalent to a strict monoidal one ((8), 1·2, (11), 4·2). Thus there is no loss of generality in dealing only with strict monoidal categories. On the other hand, general monoidal categories may be handled by the extension of our techniques given in Sections 3 and 4.

2·1·2. *Definition.* Let C be a monoid object in a category $\mathscr{K}$. We define the reduced bar construction on C, a simplicial object in $\mathscr{K}$, i.e. a functor $\overline{W}C\colon \Delta^{\mathrm{op}} \to \mathscr{K}$ by

$$\overline{W}C(\mathbf{n}) = \prod_1^n C.$$

For $0 < i < n$, $d_i\colon \mathbf{n} \to \mathbf{n-1}$,

$$\overline{W}C(d_i) = \prod_1^{i-1} 1 \times m \times \prod_{i+2}^{n} 1\colon \prod_1^n C \to \prod_1^{n-1} C,$$

where $m\colon C \times C \to C$ is the multiplication. $\overline{W}C(d_0)$. $\overline{W}C(d_n)$ project away the first and last factors of $\Pi_1^n C$ respectively. Finally, for $s_i\colon \mathbf{n} \to \mathbf{n+1}$,

$$\overline{W}C(s_i) = \prod_1^{i} 1 \times j \times \prod_{i+1}^{n} 1\colon \prod_1^n C \to \prod_1^{n+1} C$$

where $j\colon 0 \to C$ is the unit of the monoid C.

Note if $F\colon \mathscr{K} \to \mathscr{L}$ is any functor preserving products and 0, and if C is a monoid in $\mathscr{K}$, then FC is a monoid in $\mathscr{L}$ and $\overline{W}(FC) = F \circ \overline{W}C\colon \Delta^{\mathrm{op}} \to \mathscr{K} \to \mathscr{L}$. Recall the nerve functor N, geometric realization $|\ |$, and the classifying space functor B all satisfy this condition.

Applying all this to a strict monoidal category $\mathbf{C}$, we get $\overline{W}\mathbf{C}\colon \mathbf{\Delta}^{\mathrm{op}} \to$ CAT. On the other hand, $N\mathbf{C}$ is a simplicial monoid, and we have $\overline{W}N\mathbf{C}\colon \mathbf{\Delta}^{\mathrm{op}} \to$ SIMPLICIAL SETS, which we consider as a bisimplicial set, with (p,q)-simplices $\Pi_1^p N_q\mathbf{C}$. Now

$$|\mathrm{diag}\, \overline{W}N\mathbf{C}| \cong |p \to |\, q \to \prod_1^p N_q \mathbf{C}|\,| = |p \to \prod_1^p B\mathbf{C}| = |\overline{W}B\mathbf{C}|,$$

which is the usual classifying space of the topological monoid $B\mathbf{C}$ (15).

2·1·3. Proposition. *For* $\mathbf{C}$ *a strict monoidal category, there are natural homotopy equivalences*

$$N(\Delta^{\mathrm{op}} \textstyle\int \overline{W}\mathbf{C}) \xleftarrow{\eta} \mathrm{hocolim}\, N\overline{W}\mathbf{C} \to \mathrm{diag}\, N\overline{W}\mathbf{C} \cong \mathrm{diag}\, \overline{W}N\mathbf{C}.$$

Thus $B(\mathbf{\Delta}^{\mathrm{op}} \int \overline{W}\mathbf{C})$ *is naturally homotopy equivalent to the classifying space of the topological monoid* $B\mathbf{C}$.

Proof. The second statement follows from the first by the above remarks. The first map is an equivalence by 1·2, the second map is the homotopy equivalence of (2), XII, 3·4, and the last is induced by the isomorphism $N \circ \overline{W}\mathbf{C} \cong \overline{W}N\mathbf{C}$ remarked above.

This proposition furnishes us with a categorical model for the classifying space of a monoid. To avoid confusion with the functor $B = |N(\ )|$, we will refer to this process as 'delooping'. If $\mathbf{C}$ is a strict monoidal category such that $\pi_0 B\mathbf{C}$ is a group, then in fact $B\mathbf{C} \simeq \Omega\,|\overline{W}B\mathbf{C}| \simeq \Omega B(\mathbf{\Delta}^{\mathrm{op}} \int \overline{W}\mathbf{C})$.

Remark. In (17), this proposition is used to prove Quillen's comparison theorem, '$+ = \Omega BQ$'. For Let $\mathbf{A}$ be a small additive category, strictly monoidal under direct sum $\oplus$, and $\mathrm{Is}\,\mathbf{A}$ the subcategory of isomorphisms. One constructs a homotopy equivalence $\mathbf{\Delta}^{\mathrm{op}}\int \overline{W}\,\mathrm{Is}\,\mathbf{A} \to Q\mathbf{A}$, and applies the above proposition to conclude

$$\Omega BQ\mathbf{A} \simeq \Omega B(\mathbf{\Delta}^{\mathrm{op}}\textstyle\int \overline{W}\,\mathrm{Is}\,\mathbf{A}) \simeq \Omega\,|\overline{W}B\,\mathrm{Is}\,\mathbf{A}|$$

is a group completion $B\,\mathrm{Is}\,\mathbf{A}^+$ of the topological monoid $B\,\mathrm{Is}\,\mathbf{A}$. There is unfortunately an error in the proof given in (17) of the critical Charney's lemma, IV, 5·2. To show $f\colon Q^{\mathrm{se}}\mathbf{A} \to Q\mathbf{A}$ is homotopy equivalence, one considers f/A_* for each $A_p \rightarrowtail\!\!\!\to \dots \rightarrowtail\!\!\!\to A_0$. Pullback over A_i induces a functor $\perp : f/A_* \times f/A_* \to f/A_*$ satisfying the conditions of the proof of the theorem on (7), p. 227, and one concludes f/A_* is contractible, which implies the result as in (17).

2·2·1. *Definition.* A Γ-category is a functor $\mathbf{C}\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathrm{CAT}$, for $\mathbf{\Gamma}^{\mathrm{op}}$ the category of finite based sets, or equivalently the opposite category of the $\mathbf{\Gamma}$ of (14). There are n based surjections $\{0, 1, \dots, n\} = \mathbf{n} \to \{0, 1\} = \mathbf{1}$ of sets (with basepoint 0) that send all but one element of $\mathbf{n}$ to 0 in $\mathbf{1}$, and we require that these induce a homotopy equivalence

$$\mathbf{C}(\mathbf{n}) \to \prod_1^n \mathbf{C}(\mathbf{1}),$$

and that $\mathbf{C}(\mathbf{0})$ is homotopy equivalent to the category with one morphism $\mathbf{0}$.

Note this differs from the definition of Γ-category in (14) in that we require only that

$$\mathbf{C}(\mathbf{n}) \to \prod_1^n \mathbf{C}(\mathbf{1})$$

be a homotopy equivalence, not an equivalence of categories. For such a $\mathbf{C}$,

$$B\mathbf{C} = |N\mathbf{C}| \colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathrm{TOP}$$

is a Γ-space in the sense of (14). We now establish categorical models for the iterated deloopings of (14).

Recall the equivalence of finite based sets with the $\mathbf{\Gamma}^{\mathrm{op}}$ of (14) sends $f\colon \{0, 1, \dots, n\} \to \{0, 1, \dots, k\}$ to the map $\{1, \dots, k\} \to$ Power set $\{1, \dots, n\}$ sending i $f^{-1}(i)$.

2·2·2. *Definition.* For $\mathbf{C}\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathrm{CAT}$ a Γ-category, let $\overline{W}^n\mathbf{C}\colon \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{CAT}$ be the functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \prod_1^n \mathbf{\Delta}^{\mathrm{op}} \to \prod_1^n \mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\Lambda^n} \mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\mathbf{C}} \mathrm{CAT},$$

where $\mathbf{\Delta}^{\mathrm{op}} \to \Pi_1^n \mathbf{\Delta}^{\mathrm{op}}$ is the diagonal, $\Pi_1^n \mathbf{\Delta}^{\mathrm{op}} \to \Pi_1^n \mathbf{\Gamma}^{\mathrm{op}}$ is the product of the canonical map $\mathbf{\Delta}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ of (14), 1·2, and $\Lambda^n\colon \Pi_1^n \mathbf{\Gamma}^{\mathrm{op}} \to \mathbf{\Gamma}^{\mathrm{op}}$ is the functor

$$(\mathbf{m}_1, \dots, \mathbf{m}_n) \to (\mathbf{m}_1 \times \mathbf{m}_2 \times \dots \times \mathbf{m}_n)$$

of (14), 1·3, corresponding to the smash product of n based sets $\mathbf{m}_i = \{0, 1, \dots, m_i\}$ with basepoint 0.

We warn the reader to beware of notational confusion. In (14), $\mathbf{B}^k$ means the k-delooped Γ-space of a Γ-space, and $|\;|$ means the classifying space of a category. We reserve B for the classifying space of a category, and use $|p \to A(p \times m)|$ for the delooping of a Γ-space A.

2·2·3. Proposition. *$B(\mathbf{\Delta}^{\mathrm{op}}\int \overline{W}^n\mathrm{C})$ is naturally homotopy equivalent to Segal's nth delooping of the Γ-space $B\mathrm{C}$.*

Proof. We have by 1·2 and (2), XII, 3·4 natural homotopy equivalences

$$N(\mathbf{\Delta}^{\mathrm{op}}\int \overline{W}^n\mathrm{C}) \xleftarrow{\sim} \operatorname{hocolim} N\,\overline{W}^n\mathrm{C} \xrightarrow{\sim} \operatorname{diag}(p,q\to N_q\,\overline{W}^n\mathrm{C}(p)).$$

But $$|\operatorname{diag}(p,q\to N_q\,\overline{W}^n\mathrm{C}(p))| \cong |p\to|q\to N_q\,\overline{W}^n\mathrm{C}(p)|\,| \cong |p\to B\mathrm{C}(\prod^n p)|$$
$$\cong |p_1\mapsto|p_2\to\cdots|p_n\to B\mathrm{C}(p_1\times\ldots\times p_n)|\ldots|$$

is Segal's nth delooping of $B\mathrm{C}$. Note all simplicial spaces arising from realization of a multisimplicial set are good in the sense of (14), so there is no need to use the thickened realization.

2·3·4. *Remark.* One can build categorical models of the fibration sequences exhibiting the $(n-1)$th delooping as the loop space on the nth. For let $W^n\mathrm{C}\colon \mathbf{\Delta}^{\mathrm{op}}\to\mathrm{CAT}$ be the functor

$$\mathbf{\Delta}^{\mathrm{op}}\to\prod_1^n\mathbf{\Delta}^{\mathrm{op}} \xrightarrow{P\times\prod_2^n 1} \prod_1^n\mathbf{\Delta}^{\mathrm{op}}\to\prod_1^n\mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\wedge^n} \mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\mathrm{C}} \mathrm{CAT},$$

where $P\colon\mathbf{\Delta}^{\mathrm{op}}\to\mathbf{\Delta}^{\mathrm{op}}$ is as in (14), 1·5. The natural transform $d_0\colon P\Rightarrow Id$ induces $W^n\mathrm{C}\Rightarrow\overline{W}^n\mathrm{C}$ and the induced $B(\mathbf{\Delta}^{\mathrm{op}}\int W^n\mathrm{C})\to B(\mathbf{\Delta}^{\mathrm{op}}\int\overline{W}^n\mathrm{C})$ is identified up to homotopy with

$$|p_1,p_2,\ldots,p_n\to B\mathrm{C}(Pp_1\times p_2\times\ldots\times p_n)|\to|p_1,\ldots,p_n\to B\mathrm{C}(p_1\times p_2\times\ldots\times p_n)|,$$

which is Segal's path space fibration over the nth delooping. For $n\geqslant 2$, there is a natural $\overline{W}^{n-1}\mathrm{C}\Rightarrow W^n\mathrm{C}$ such that $B(\mathbf{\Delta}^{\mathrm{op}}\int\overline{W}^{n-1}\mathrm{C})\to B(\mathbf{\Delta}^{\mathrm{op}}\int W^n\mathrm{C})$ is identified to the inclusion of the homotopy fibre of $B(\mathbf{\Delta}^{\mathrm{op}}\int W^n\mathrm{C})\to B(\mathbf{\Delta}^{\mathrm{op}}\int\overline{W}^n\mathrm{C})$, obtained by mimicking Segal's construction to work for functors $\mathbf{\Gamma}^{\mathrm{op}}\to\mathrm{CAT}$.

3. Op-lax *functors and left* op-lax *natural transforms*

It is well known that one may form $\mathbf{K}\int F$ for $F\colon\mathbf{K}\to\mathrm{CAT}$ something less than a functor. In Section 3 we study this more general situation, which we apply in Section 3 to greatly extend our techniques.

3·1·1. An op-lax functor $F\colon\mathbf{K}\to\mathrm{CAT}$ consists of functions assigning:

(1) to each object K of $\mathbf{K}$, a category $F(K)$,

(2) to each morphism $k\colon K_1\to K_0$ in $\mathbf{K}$, a functor $F(k)\colon F(K_1)\to F(K_0)$,

(3) to each composible pair of morphisms $K_2\xrightarrow{k_2}K_1\xrightarrow{k_1}K_0$ is $\mathbf{K}$, a natural transform $f(k_1,k_2)\colon F(k_1k_2)\Rightarrow F(k_1)F(k_2)$,

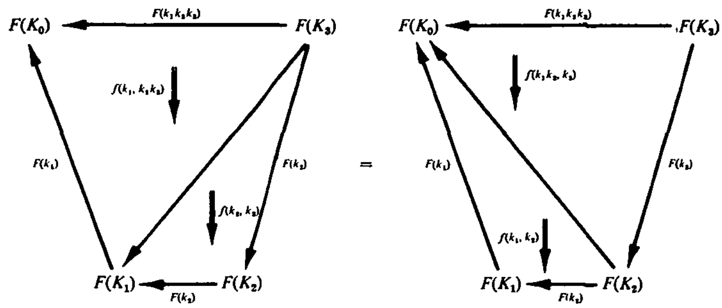
(4) to each object K of $\mathbf{K}$, a natural transform $f(K)\colon F(1_K)\Rightarrow 1_{F(K)}$.

These must satisfy the conditions that for

$$K_0\xleftarrow{k_1}K_1\xleftarrow{k_2}K_2\xleftarrow{k_3}K_3,$$

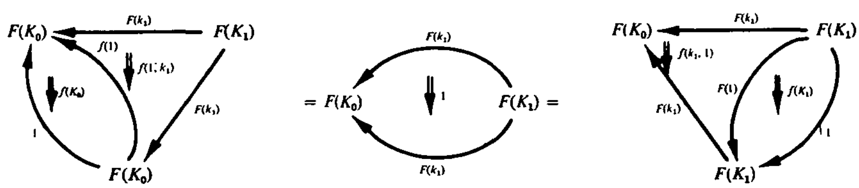
$$(F(k_1)f(k_2,k_3)).f(k_1,k_2k_3) = (f(k_1,k_2)F(k_3)).f(k_1k_2,k_3);$$

that is,



and that for $K_0 \xleftarrow{k_1} K_1$,

$$(f(K_0)\,F(K_1))\,.\,f(1, k_1) = 1_{F(k_1)} = f(k_1, 1)\,.\,(F(k_1)f(K_1));$$



Note a functor $F\colon \mathbf{K} \to \mathrm{CAT}$ is an op-lax functor with $f(k_0, k_1) = 1$, $f(K) = 1$.

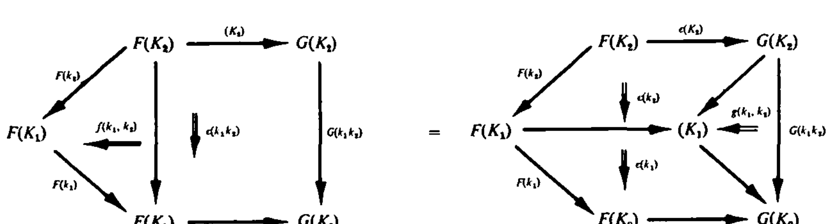
3·1·2. *Definition.* For $F\colon \mathbf{K} \to \mathrm{CAT}$ an op-lax functor, the *Grothendieck construction* $\mathbf{K}\int F$ is the category with objects and morphisms as in Definition 1·1, but with the composition of $(k_1, x_1)\colon (K_1, X_1) \to (K_0, X_0)$ and $(k_2, x_2)\colon (K_2, X_2) \to (K_1, X_1)$ given by $(k_1, x_1)\,.\,(k_2, x_2) = (k_1 k_2, x_1 F(k_1)(x_2)\,.\,f(k_1, k_2))$. Note the presence of $f(k_1, k_2)$. The identity map of (K, X) is $(1_K, f(K)(X))$. One checks that the category axioms hold as the conditions on f in Definition 3·1·1 yield associativity of composition and existence of left and right identity morphisms.

3·1·3. *Definition.* Given F, G, op-lax functors $\mathbf{K} \to \mathrm{CAT}$, a *left op-lax natural transform*, or *lont*, $\epsilon\colon F \Rightarrow G$ consists of functions assigning

(1) to each K an object of $\mathbf{K}$, a functor $\epsilon(K)\colon F(K) \to G(K)$,

(2) to each $K_1 \xrightarrow{k_1} K_0$ in $\mathbf{K}$, a natural transform $\epsilon(k_1)\colon G(k_1)\,\epsilon(K_1) \Rightarrow \epsilon(K_0)\,F(k_1)$

such that for $K_2 \xrightarrow{k_2} K_1 \xrightarrow{k_1} K_0$



and for K

$$\begin{tikzcd}
F(K) \arrow[r, "\epsilon(K)"] \arrow[d, bend right=50, "1"'] \arrow[d, bend left=50, "F(1)"] \arrow[d, phantom, "\overset{f(K)}{\Leftarrow}"] & G(K) \arrow[d, bend left=50, "G(1)"] \arrow[dl, phantom, "\Downarrow \epsilon(1)"] & & F(K) \arrow[r, "\epsilon(K)"] \arrow[d, bend right=50, "1"'] & G(K) \arrow[d, bend right=50, "1"'] \arrow[d, bend left=50, "G(1)"] \arrow[d, phantom, "\Leftarrow"] \\
F(K) \arrow[r, "\epsilon(K)"'] & G(K) & = & F(K) \arrow[r, "\epsilon(K)"'] & G(K)
\end{tikzcd}$$

Note a natural transform $\epsilon\colon F \Rightarrow G$ of functions is a lont with $\epsilon(k) = 1$.

Given lonts $F \Rightarrow G$, $G \Rightarrow H$ there is an obvious composite lont $F \Rightarrow H$. Thus for fixed $\mathbf{K}$ we have a category of op-lax functors $\mathbf{K} \to \mathrm{CAT}$ and lonts between such, Oplax $(\mathbf{K}, \mathrm{CAT})$.

3·1·4. *Definition.* Let $F, G\colon \mathbf{K} \to \mathrm{CAT}$ be op-lax functors, $\epsilon\colon F \Rightarrow G$ a lont. We define a functor $\mathbf{K}\int\epsilon\colon \mathbf{K}\int F \to \mathbf{K}\int G$ on objects by $\mathbf{K}\int\epsilon(K, X) = (K, \epsilon(K)(X))$. For a morphism in $\mathbf{K}\int F$, $(k, x)\colon (K_1, X_1) \to (K_0, X_0)$, we have a morphism in $G(K_0)$,

$$G(k)\,\epsilon(K_1)\,(X_1) \xrightarrow{\epsilon(k)(X)} \epsilon(K_0)\,F(k)\,(X_1) \xrightarrow{\epsilon(K_0)(x)} \epsilon(K_0)\,(X_0).$$

We set $\mathbf{K}\int\epsilon(k, x) = (k, \epsilon(K_0)\,(x)\,.\,\epsilon(k)\,(X))$.

One notes 3·1·2 and 3·1·4 determine a functor $\mathbf{K}\int\colon \mathrm{Oplax}\,(\mathbf{K}, \mathrm{CAT}) \to \mathrm{CAT}$.

3·2. We do not have a hocolim NF defined by (2) for F an op-lax functor, so we cannot compare this to $N(\mathbf{K}\int F)$. Instead, we will naturally associate a functor $\tilde{F}\colon \mathbf{K} \to \mathrm{CAT}$ to the op-lax functor F, and compare $N(\mathbf{K}\int F)$ to $N(\mathbf{K}\int\tilde{F})$ and hocolim $N\tilde{F}$. The construction of $\tilde{F}$ is given by R. Street in (16), and is referred to as the Kleisli rectification of, or Street's first construction on F in (17). Explicit formulae for $\tilde{F}$ may be found in detail in (16) we sketch a more conceptual approach.

3·2·1. *Definition.* Let $\phi\colon \mathbf{L} \to \mathbf{K}$ be any functor. We form the comma category $(\phi, \mathbf{K})$ to have objects triples $(K, K \xleftarrow{a} \phi L, L)$ with K an object of $\mathbf{K}$, L an object of $\mathbf{L}$, and a a morphism in $\mathbf{K}$. A morphism $(k, l)\colon (K_1, a_1, L_1) \to (K_0, a_0, L_0)$ is a pair of morphisms $k\colon K_1 \to K_0$, $l\colon L_1 \to L_0$ such that $k\,.\,a_1 = a_0\,.\,\phi(l)$.

We have projections $p_1\colon (\phi, \mathbf{K}) \to \mathbf{K}$, $p_2\colon (\phi, \mathbf{K}) \to \mathbf{L}$ given by $p_1(K, a, L) = K$, $p_2(K, a, L) = L$. There is an inclusion $\iota\colon \mathbf{L} \to (\phi, \mathbf{K})$ with $\iota(L) = (\phi L, 1, L)$. This ι is left adjoint to the projection p_2, so by (13), proposition 2, corollary 1, $N\iota$ and Np_2 are homotopy equivalences $N\mathbf{L} \simeq N(\phi, \mathbf{K})$.

The construction $(\mathbf{L} \xrightarrow{\phi} \mathbf{K}) \mapsto ((\phi, \mathbf{K}) \xrightarrow{p_1} \mathbf{K})$ is a functor from the category $\mathrm{CAT}/\mathbf{K}$ of categories augmented over $\mathbf{K}$, into itself, and $\iota\colon \mathbf{L} \to (\phi, \mathbf{K})$ is a natural transform in $\mathrm{CAT}/\mathbf{K}$ from Id to this functor.

3·2·2. From $(\phi, \mathbf{K})$ we construct a functor $\Phi\colon \mathbf{K} \to \mathrm{CAT}$. $\Phi(K)$ is the subcategory of $(\phi, \mathbf{K})$ consisting of all morphisms which project to 1_K under p_1. Thus $\Phi(K)$ has objects (K_0, a_0, L_0) with $K_0 = K$, and morphisms $(1, l)$. For k_1: $k_1\colon K_1 \to K_0$ we get

$$\Phi(k_1)\colon \Phi(K_1) \to \Phi(K_0)$$

by
$$\Phi(k_1)\,(K_1, a, L) = (K_0, k_1\,.\,a, L),\ \Phi(k_1)\,(1, l) = (1, l).$$

We have an isomorphism $\mathbf{K}\int\Phi \xrightarrow{\cong} (\phi, \mathbf{K})$ given on objects by

$$(K,(K,a,L)) \mapsto (K,a,L),$$

and on morphisms by $(k,(1,l)) \mapsto (k,l)$. The construction of Φ and this isomorphism is functorial in $\phi\colon \mathbf{L} \to \mathbf{K}$ of $\mathrm{CAT}/\mathbf{K}$.

3·2·3. *Definition.* For $F\colon K \to \mathrm{CAT}$ an op-lax functor, we define $\tilde{F}\colon \mathbf{K} \to \mathrm{CAT}$ to be the Φ associated by 3·2·2 to $\pi\colon \mathbf{K}\int F \to \mathbf{K}$ with $\pi(K,X) = K$. For $\epsilon\colon F \Rightarrow G$ a lont, $\tilde{\epsilon}\colon \tilde{F} \Rightarrow \tilde{G}$ is the natural transform of functors induced by $\mathbf{K}\int \epsilon\colon \mathbf{K}\int F \to \mathbf{K}\int G$, a morphism of $\mathrm{CAT}/\mathbf{K}$, using the functoriality of the construction of Φ.

Note $\tilde{F}(K) = \pi/K$ is the category whose objects are (k,K',X'), $k\colon K' \to K$ a morphism in $\mathbf{K}$ and X' an object of $F(K')$. A morphism $(k_1.x_1)\colon (k,K',X') \to (k',K'',X'')$ is a $k_1\colon K' \to K''$ such that $k'.k_1 = k$, and a $x_1\colon F(k_1)(X') \to X''$.

3·2·4. *Definition.* The lont $j\colon F \Rightarrow \tilde{F}$ is determined by the formula

$$j(K)\colon F(K) \to \tilde{F}(K)$$

and is the functor sending X to $(1_K, K, X)$ and $x_1\colon X \to X'$ to $(1_K, f(K)(X')).F(1_K)(x_1)$. For $k\colon K_1 \to K_0$, $j(k)\colon \tilde{F}(k_1).j(K_1) \Rightarrow j(K_0).F(k_1)$ is the natural transform with components given at X by $(k,1)\colon (k_1,K_1,X) \to (1,K_0,F(k)(X))$.

These $j\colon F \to \tilde{F}$ are such that for any lont $\epsilon\colon F \Rightarrow G$

$$\begin{tikzcd} F \arrow[r, Rightarrow, "j"] \arrow[d, Rightarrow, "\epsilon"'] & \tilde{F} \arrow[d, Rightarrow, "\tilde{\epsilon}"] \\ G \arrow[r, Rightarrow, "j"'] & \tilde{G} \end{tikzcd}$$

commutes.

3·2·5. LEMMA. *Under the isomorphism* $\mathbf{K}\int\tilde{F} \cong (\pi, \mathbf{K})$ *of* 3·2·2,

$$\begin{tikzcd} & \mathbf{K}\int\tilde{F} \arrow[dd, equal, "\wr"] \\ \mathbf{K}\int F \arrow[ur, "\mathbf{K}\int j"] \arrow[dr] & \\ & (\pi, \mathbf{K}) \end{tikzcd}$$

commutes. Thus $\mathbf{K}\int j$ *is a homotopy equivalence.*

Proof. That the diagram commutes is easily checked. It follows $\mathbf{K}\int j$ is a homotopy equivalence as ι is by 3·2·1.

3·2·6. LEMMA. *For each* K, $j(K)\colon F(K) \to \tilde{F}(K)$ *is a homotopy equivalence.*

Proof. This $j(K)$ has a left adjoint $\tilde{F}(K) \to F(K)$, sending (k,K',X') to $F(k)(X')$ and $(k_1,x)\colon (k,K',X') \to (k',K'',X'')$ to

$$F(k)(X') = F(k'.k_1)(X') \xrightarrow{f(k',\,k_1)} F(k')F(k_1)(X') \xrightarrow{F(k')(x)} F(k')(X'').$$

The result follows by (13), proposition 2, corollary 1.

3·3. THEOREM. *For* $F\colon \mathbf{K} \to \mathrm{CAT}$ *an* op-lax *functor, we have a diagram of natural homotopy equivalences*

$$N(\mathbf{K} \textstyle\int F) \xrightarrow{N(\mathbf{K}} N(\mathbf{K} \textstyle\int \tilde{F}) \xleftarrow{\eta} \operatorname{hocolim} N\tilde{F}$$

Proof. This is just Theorem 1·2 and Lemma 3·2·5.

3·3·1. COROLLARY. *Let* $F, G\colon \mathbf{K} \to \mathrm{CAT}$ *be* op-lax *functors,* $\epsilon\colon F \Rightarrow G$ *a lont. If for all* K *in* $\mathbf{K}$, $\epsilon(K)\colon F(K) \to G(K)$ *is a homotopy equivalence, then* $\mathbf{K}\int\epsilon\colon \mathbf{K}\int F \to \mathbf{K}\int G$ *is a homotopy equivalence.*

Proof. By 3·2·4's commutative diagram and Lemma 3·2·6, we see $\tilde{\epsilon}(K)\colon \tilde{F}(K) \to \tilde{F}(K)$ is a homotopy equivalence if $\epsilon(K)$ is. Now consider the commutative diagram, where the horizontal maps are homotopy equivalences by Theorem 3·3

$$\begin{tikzcd}
N(\mathbf{K}\int F) \arrow[r, "N(\mathbf{K}\int i)"] \arrow[d, "N(\mathbf{K}\int\epsilon)"] & N(\mathbf{K}\int\tilde{F}) \arrow[d, "N(\mathbf{K}\int\tilde{\epsilon})"] & \operatorname{hocolim} N\tilde{F} \arrow[l, "\eta"'] \arrow[d, "\operatorname{hocolim} N\tilde{\epsilon}"] \\
N(\mathbf{K}\int G) \arrow[r] & N(\mathbf{K}\int\tilde{G}) & \operatorname{hocolim} N\tilde{G} \arrow[l, "\eta"']
\end{tikzcd}$$

As each $N\tilde{\epsilon}(K)$ is a homotopy equivalence, $\operatorname{hocolim} N\tilde{\epsilon}$ is by (2), XII, 4·2. It follows $N(\mathbf{K}\int\tilde{\epsilon})$ and $N(\mathbf{K}\int\epsilon)$ are.

Remark. We have $(\mathbf{K}\int\tilde{F}) \simeq (\pi, \mathbf{K}) \xrightarrow{p_2} \mathbf{K}\int F$, a homotopy equivalence by 3·2·1. For F a functor, $\operatorname{hocolim} N\tilde{F} \xrightarrow{\eta} (\mathbf{K}\int\tilde{F}) \to \mathbf{K}\int F$ is just the λ_2 of Lemma 1·2·4.

It is also curious that $\mathbf{K}\int F \simeq \operatorname*{colim}_{\mathbf{K}} \tilde{F}$.

4. *Segal's machine in* CAT: *delooping and group completing symmetric monoidal categories.*

We proceed to provide ourselves with examples of op-lax functors.

4·1·1. *Definition. A symmetric monoidal category* $\mathbf{S}$ is a small category together with a functor $\oplus\colon \mathbf{S} \times \mathbf{S} \to \mathbf{S}$, an object 0, and natural isomorphisms

$$\alpha\colon (S_1 \oplus S_2) \oplus S_3 \xrightarrow{\cong} S_1 \oplus (S_2 \oplus S_3), \quad \lambda\colon 0 \oplus S_1 \xrightarrow{\cong} S_1,$$

and
$$\gamma\colon S_1 \oplus S_2 \xrightarrow{\cong} S_2 \oplus S_1$$

satisfying the coherence conditions that $\gamma^2 = 1$, and that

$$\begin{tikzcd}
((S_1 \oplus S_2) \oplus S_3) \oplus S_4 \arrow[r, "\alpha"'] \arrow[d, "\alpha \oplus S_4"] & (S_1 \oplus S_2) \oplus (S_3 \oplus S_4) \arrow[r, "\alpha"'] & S_1 \oplus (S_2 \oplus (S_3 \oplus S_4)) \\
(S_1 \oplus (S_2 \oplus S_3)) \oplus S_4 \arrow[rr, "\alpha"] & & S_1 \oplus ((S_2 \oplus S_3) \oplus S_4) \arrow[u, "S_1 \oplus \alpha"']
\end{tikzcd}$$

$$\begin{tikzcd}
(S_1 \oplus S_2) \oplus S_3 \arrow[r, "\alpha"] \arrow[d, "\gamma \oplus S_3"'] & S_1 \oplus (S_2 \oplus S_3) \arrow[r, "\gamma"] & (S_2 \oplus S_3) \oplus S_1 \arrow[d, "\alpha"] \\
(S_2 \oplus S_1) \oplus S_3 \arrow[r, "\alpha"] & S_2 \oplus (S_1 \oplus S_3) \arrow[r, "S_2 \oplus \gamma"] & S_2 \oplus (S_3 \oplus S_1)
\end{tikzcd}$$

$$\begin{tikzcd}
(O \oplus S_1) \oplus S_2 \arrow[r, "\gamma \oplus S_2"] \arrow[dr, "\lambda \oplus S_2"'] & (S_1 \oplus O) \oplus S_2 \arrow[r, "\alpha"] & S_1 \oplus (O \oplus S_2) \arrow[dl, "S_1 \oplus \lambda"] \\
& S_1 \oplus S_2 &
\end{tikzcd}$$

commute.

For example, any small category with product or coproduct $\oplus$ is a symmetric monoidal.

There is a coherence theorem for symmetric monoidal categories, which says that for any two functors $\Pi_1^n \mathbf{S} \to \mathbf{S}$ written as the iterated composite of $\oplus$'s, permutation of factors, and applications of $? \mapsto 0 \oplus ?$ and $? \mapsto ? \oplus 0$, there is a unique canonical natural isomorphism from one functor to the other, made up of λ', α's, and γ's. A precise correct statement requires some machinery; the reader is referred to (8), § 1, and the references listed there for technical details, and also to the volume in which (9) appears. Coherence theory was initiated by MacLane, but early work was slightly imprecise. We will appeal to coherence theory to claim certain diagrams commute; the reader willing to have faith or to restrict himself to situations where he can concretely check this (e.g. $\oplus$ is a direct sum) may blissfully ignore these technicalities.

4·1·2. *Definition.* Let S be a symmetric monoidal category. We define an op-lax functor $S\colon \mathbf{\Gamma}^{\mathrm{op}} \to \mathrm{CAT}$. On objects,

$$S(n) = \prod_1^n \mathbf{S}.$$

For $\phi\colon \mathbf{n} = \{0, \ldots, n\} \to \mathbf{k}$ a map of finite based sets with base point 0,

$$S(\phi)\colon \prod_1^n \mathbf{S} \to \prod_1^k \mathbf{S}$$

sends $(S_1, \ldots, S_n)$ to

$$\Big(\bigoplus_{j \in \phi^{-1}(1)} S_j, \ldots, \bigoplus_{j \in \phi^{-1}(n)} S_j\Big).$$

Here for $J \subseteq \{1, \ldots, n\}$, $\bigoplus_{j \in J} S_j$ is defined inductively by

$$\bigoplus_{j \in \{\,\}} S_j = 0, \qquad \bigoplus_{j \in \{i\}} S_j = S_i,$$

and

$$\bigoplus_{j \in J} S_j = \Big(\bigoplus_{j \in J'} S_j\Big) \oplus S_i$$

if $J' = J - (i) \neq \varnothing$ $(i \in J)$ and $\forall j \in J'$ $(j < i)$.

For $\psi\colon \mathbf{m} \to \mathbf{n}$, $\phi\colon \mathbf{n} \to \mathbf{k}$, coherence theory gives a canonical natural isomorphism

$$\bigoplus_{l \in (\phi\psi)^{-1}(i)} S_j \xrightarrow{\cong} \bigoplus_{j \in \phi^{-1}(i)} \Big(\bigoplus_{l \in \psi^{-1}(j)} S_l \Big)$$

as

$$\coprod_{j \in \phi^{-1}(i)} \psi^{-1}(j) = (\phi\psi)^{-1}(i).$$

These give for $i = 1, 2, \ldots, k$ the components of a natural transform $S(\phi\psi) \Rightarrow S(\phi)S(\psi)$. Also $S(1) = 1$. Take these as the structure transforms of the op-lax functor S, the compatibility conditions of Definition 3·1·1 hold by coherence.

4·1·3. *Definition.* A symmetric monoidal functor $F\colon \mathbf{S} \to \mathbf{T}$ between two symmetric monoidal categories is a functor together with natural transforms

$$f\colon FS_1 \oplus FS_2 \to F(S_1 \oplus S_2), \quad \bar{f}\colon 0 \to F0$$

subject to the coherence conditions that

$$\begin{tikzcd}
(FS_1 \oplus FS_2) \oplus FS_3 \arrow[r, "f \oplus FS_3"] \arrow[d, "\alpha"] & F(S_1 \oplus S_2) \oplus FS_3 \arrow[r, "f"] & F((S_1 \oplus S_2) \oplus S_3) \arrow[d, "F\alpha"] \\
FS_1 \oplus (FS_2 \oplus FS_3) \arrow[r, "FS_1 \oplus f"] & FS_1 \oplus F(S_2 \oplus S_3) \arrow[r, "f"] & F(S_1 \oplus (S_2 \oplus S_3))
\end{tikzcd}$$

$$\begin{tikzcd}
FS_1 \oplus FS_2 \arrow[r, "f"] \arrow[d, "\gamma"] & F(S_1 \oplus S_2) \arrow[d, "F(\gamma)"] \\
FS_2 \oplus FS_1 \arrow[r, "f"] & F(S_2 \oplus S_1)
\end{tikzcd}
\qquad
\begin{tikzcd}
FO \oplus FS \arrow[r, "f"] & F(O \oplus S) \arrow[d, "F\lambda"] \\
O \oplus FS \arrow[u, "\bar{f} \oplus FS"] \arrow[r, "\lambda"] & FS
\end{tikzcd}$$

all commute.

4·1·4. *Definition.* For $F\colon \mathbf{S} \to \mathbf{T}$ a symmetric monoidal functor, we define a lont: $F\colon S \Rightarrow T$ of the associated op-lax functors $\mathbf{\Gamma}^{\mathrm{op}} \to \mathrm{CAT}$. Set

$$F(\mathbf{n}) = \prod_1^n F\colon \prod_1^n \mathbf{S} \to \prod_1^n \mathbf{T}.$$

For $\phi\colon \mathbf{n} \to \mathbf{k}$, the natural transform $F(\phi)\colon T(\phi).F(\mathbf{n}) \Rightarrow F(\mathbf{k}).S(\phi)$ has components at $(S_1, \ldots, S_n)$, $(f_1, \ldots, f_k)$, where

$$f_j\colon \bigoplus_{i \in \phi^{-1}(j)} FS_i \to F\Big(\bigoplus_{i \in \phi^{-1}(j)} S_i \Big)$$

is the unique canonical natural transform given by coherence theory (9), § 2, § 3; or the 'cheap result' of (8), 1·5. The conditions of Definition 3·1·3 hold by coherence.

One may check we have produced in 4·1 a functor from the category of symmetric monoidal categories and functors, SYM MON, to Oplax $(\mathbf{\Gamma}^{\mathrm{op}}, \mathrm{CAT})$.

4·2·1. Proposition. *There is a functor from* SYM MON *into the category of spectra, and a natural group completion from* $B\mathbf{S}$, *to the zeroth space of the associated spectrum.*

Proof. We constructed a functor SYM MON $\to$ Oplax ($\mathbf{\Gamma}^{\mathrm{op}}$, CAT) above, sending $\mathbf{S}$ to S. Applying the construction '$\sim$' of 3·2·3, we get a functor from Oplax ($\mathbf{\Gamma}^{\mathrm{op}}$, CAT) into the category of functors $\mathbf{\Gamma}^{\mathrm{op}} \to$ CAT. This sends S to $\tilde{S}$. As

$$S(n) = \prod_1^n \mathbf{S} = S(1),$$

and $j(n)\colon S(n) \to \tilde{S}(n)$ is a homotopy equivalence (3·2·6), we see $\tilde{S}(n)$ is homotopy equivalent to $\Pi_1^n \tilde{S}(1)$. Thus the composite functor sending $\mathbf{S}$ to $\tilde{S}$ is a functor from SYM MON to the category of Γ:categories (2·2·1). As we saw in 2·2·1, $B\tilde{S}\colon \mathbf{\Gamma}^{\mathrm{op}} \to$ TOP is a Γ-space in the sense of (14), group completing it by (14), § 4 and then feeding the result into the machine of (14), § 3 produces a functor from Γ-categories into spectra. The composite is the functor we seek. We have by 3·2·6, a natural homotopy equivalence $B\mathbf{S} = BS(1) \xrightarrow{Bj(1)} B\tilde{S}(1)$, by (14), $B\tilde{S}(1)$ maps to the zeroth space of the associated spectrum by a natural group completion. The composite is the group completion we seek.

Remark. (1), (11), (12), (14) have given versions of this theorem; but they are all restricted to the subcategory of SYM MON of symmetric monoidal functors F with the f, $\bar{f}$ of Definition 4·1·3 required to be isomorphisms, not just natural transforms. A similar generalization saying that monoidal functors (preserving $\oplus$ up to natural transform, not isomorphism) produce 1-fold loop maps is proved similarly. In (17) this is applied to show the second Adams operation, $\psi^2\colon BO \to BO$, $BU \to BU$, deloops once without localizing the spaces away from 2.

The techniques of (12) immediately generalize to yield the uniqueness of any functor from SYM MON into connective spectra as in Proposition 4·2·1 up to natural weak homotopy equivalence of spectra.

4·2·2. We now build categorical models of these deloopings.

Definition. Let S be a symmetric monoidal category. Then $\overline{W}^n S\colon \mathbf{\Delta}^{\mathrm{op}} \to$ CAT is the op-lax functor

$$\mathbf{\Delta}^{\mathrm{op}} \to \prod_1^n \mathbf{\Delta}^{\mathrm{op}} \to \prod_1^n \mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{\Lambda^n} \mathbf{\Gamma}^{\mathrm{op}} \xrightarrow{S} \mathrm{CAT},$$

where S is as in Definition 4·1·2, and the other functors are as in Definition 2·2·2. We note we may compose functors and op-lax functors as shown to get an op-lax functor (figure this out or read (6), I, 4·5).

4·2·3. PROPOSITION. *$B(\mathbf{\Delta}^{\mathrm{op}} \int \overline{W}^n S)$ is naturally homotopy equivalent to the n-fold 'delooping' of $B\mathbf{S}$ provided by the nth space of the spectrum associated to $\mathbf{S}$ by Proposition* 4·2·1.

Proof. Consider $j\colon S \to \tilde{S}$ as in Definition 3·2·4. This induces a lont $\overline{W}^n j\colon \overline{W}^n S \to \overline{W}^n \tilde{S}$, which for $\mathbf{k} \in \mathbf{\Gamma}^{\mathrm{op}}$ is a homotopy equivalence $\overline{W}^n j(\mathbf{k}) = j(\mathbf{k}^n)$ by 3·2·6. By Corollary 3·3·1 we have then a natural homotopy equivalence $\mathbf{\Delta}^{\mathrm{op}} \int \overline{W}^n j\colon \mathbf{\Delta}^{\mathrm{op}} \int \overline{W}^n S \to \mathbf{\Delta}^{\mathrm{op}} \int \overline{W}^n \tilde{S}$. But by Proposition 2·2·3, $B(\mathbf{\Delta}^{\mathrm{op}} \int \overline{W}^n \tilde{S})$ is naturally homotopy equivalent to the nth delooping of $B\tilde{S}(1)$, i.e. to the nth space of the associated spectrum to the Γ-space $B\tilde{S}$, and this is homotopy equivalent to the nth space of the spectrum for the group completion of the Γ-space $B\tilde{S}$ by (14), which finishes the proof.

Remark. Now just as in 2·2·4, we may define an op-lax functor $W^nS: \mathbf{\Delta}^{op} \to \mathrm{CAT}$ by mimicking Segal's construction for functors $\mathbf{\Gamma}^{op} \to \mathrm{TOP}$ with a corresponding construction for op-lax functors $\mathbf{\Gamma}^{op} \to \mathrm{CAT}$, and produce for $n \geqslant 2$ a sequence

$$\mathbf{\Delta}^{op} \int \overline{W}^{n-1}S \to \mathbf{\Delta}^{op} \int W^nS \to \mathbf{\Delta}^{op} \int \overline{W}^nS,$$

which is identified, after applying B, to Segal's fibration sequence of (14), 1·5 exhibiting the $n-1$st delooping as the loops on the nth.

There is one subtle point in this: for a general op-lax functor S: $\mathbf{\Gamma}^{op} \to \mathrm{CAT}$, and a natural transform $\delta: P \Rightarrow F$ of functor $\mathbf{\Delta}^{op} \to \mathbf{\Gamma}^{op}$, we do *not* generally get a lont $S\delta$: $SP \Rightarrow SF$. However, for our S we do have such an induced lont as all the structure natural transforms $S(\phi\psi) \Rightarrow S(\phi)\, S(\psi)$ of the op-lax functor S are in fact ISOMORPHISMS. For a discussion of this point and other subtleties involving compositions of lonts and op-lax functors, see (6), I, 4·21. The terminology of (6) is quite different from that of this paper, the interested reader will have to translate.

4·3. We will now give a categorical model for group completion. To avoid the difficulties cited in the above remark, we assume we are dealing with a permutative category $\mathbf{S}$, which is just a symmetric monoidal one with the α and λ of 4·1·1 being identity maps. This is no real loss of generality, as every symmetric monoidal category is 'additively' equivalent to a permutative one by (8), 1·2 or (11), 4·2.

4·3·1. *Definition.* For $\mathbf{S}$ a permutative category, let S^+: $\mathbf{\Delta}^{op} \to \mathrm{CAT}$ be the functor

$$S^+(\mathbf{n}) = \prod_1^{n+2} \mathbf{S},$$

with

$$S^+(d_i): \prod_1^{n+2} \mathbf{S} \to \prod_1^{n+2} \mathbf{S}$$

given by

$$\begin{aligned}
0 < i < n, \quad S^+(d_i)\,(S_0, S_0'; S_1, \ldots, S_n) &= (S_0, S_0'; S_1, \ldots, S_i \oplus S_{i+1}, \ldots, S_n),\\
S^+(d_0)\,(S_0, S_0'; S_1, \ldots, S_n) &= (S_0 \oplus S_1, S_0' \oplus S_1; S_2, \ldots, S_n),\\
S^+(d_n)\,(S_0, S_0'; S_1, \ldots, S_n) &= (S_0, S_0'; S_1, \ldots, S_{n-1}),\\
S^+(s_i)\,(S_0, S_0'; S_1, \ldots, S_n) &= (S_0, S_0'; S_1, \ldots, S_i, 0, S_{i+1}, \ldots, S_n).
\end{aligned}$$

We have $\mathbf{S} \to S^+(\mathbf{0})$ sending $S \mapsto (S, 0)$; with the inclusion $S^+(\mathbf{0}) \hookrightarrow \mathbf{\Delta}^{op} \int S^+$ this induces a canonical $\mathbf{S} \to \mathbf{\Delta}^{op} \int S^+$.

4·3·2. PROPOSITION. *For* $\mathbf{S}$ *a permutative category,* $B\mathbf{S} \to B(\mathbf{\Delta}^{op} \int S^+)$ *is a group completion; i.e. is identified up to homotopy equivalence with* $B\mathbf{S} \to \Omega\,|\overline{W}B\mathbf{S}|$.

Proof. Consider (14), §4. From the functor P: $\mathbf{\Delta}^{op} \to \mathbf{\Delta}^{op}$ and natural transform $P \Rightarrow \mathrm{Id}$ of (14), 1·5, we have an induced natural transform of functors $\mathbf{\Delta}^{op} \to \mathrm{CAT}$, $\overline{W}^1S.P \Rightarrow \overline{W}^1S$. We form the pull-back

$$\overline{W}^1S.P \underset{\overline{W}^1S}{\times} \overline{W}^1S.P,$$

this is a functor $\Delta^{op} \to \mathrm{CAT}$ which is just our S^+. From $j\colon S \to \tilde{S}$, we get a lont

$$\overline{W}^1S.P \underset{\overline{W}^1S}{\times} \overline{W}^1S.P \to \overline{W}^1\tilde{S}.P \underset{\overline{W}^1\tilde{S}}{\times} \overline{W}^1_y\tilde{S}.P,$$

which is for each $\mathbf{k}$ in $\mathbf{\Delta}^{op}$ a homotopy equivalence $j(k+2)\colon S(k+2) \to \tilde{S}(k+2)$. Thus by Corollary 3·3·1, Theorem 1·2, and (2), XII, 3·4 we have homotopy equivalences

$$\begin{aligned} B(\mathbf{\Delta}^{op}\textstyle\int S^+) &= B(\mathbf{\Delta}^{op}\textstyle\int \overline{W}^1S.P \underset{\overline{W}^1S}{\times} \overline{W}^1S.P) \xrightarrow{\sim} B(\mathbf{\Delta}^{op}\textstyle\int \overline{W}^1\tilde{S}.P \underset{\overline{W}^1\tilde{S}}{\times} \overline{W}^1\tilde{S}.P) \\ &\xleftarrow{\sim} |\mathrm{hocolim}\, N\overline{W}^1\tilde{S}.P \underset{N\overline{W}^1\tilde{S}}{\times} N\overline{W}^1\tilde{S}.P| \xrightarrow{\sim} |\mathrm{diag}\,(N\overline{W}^1\tilde{S}.P \underset{N\overline{W}^1\tilde{S}}{\times} N\overline{W}^1\tilde{S}.P)| \\ &= |p \to B\tilde{S}(Pp) \underset{B\tilde{S}(p)}{\times} B\tilde{S}(Pp)|. \end{aligned}$$

But the last term is just the underlying space of the group completion of the Γ-space $B\tilde{S}$, as produced in (14), § 4; which is there identified to $\Omega\,|\overline{W}B\tilde{S}| \simeq \Omega\,|\overline{W}BS|$. Chasing the diagrams and considering (14) reveals our canonical map $\mathbf{S} \to \mathbf{\Delta}^{op}\int S^+$ is identified *to the canonical inclusion in the group completion.*

4·3·3. COROLLARY. (i) *Let* $\mathbf{S} = \coprod_n \Sigma_n$ *be the permutative category of isomorphisms of finite sets, with* $\oplus$ *the disjoint union functor. Then* $B(\mathbf{\Delta}^{op}\int S^+) = \Omega^\infty S^\infty$.

(ii) *Let* $\mathbf{S} = \coprod_n GL_n(R)$ *be the permutative category with* $\oplus$ *direct sum of free R-modules. Then* $\pi_i(B(\mathbf{\Delta}^{op}\int S^+)) = K_i(R)$ *for* $i > 0$.

Proof. This follows from Proposition 4·3·2 and (14).

4·3·4. *Remark.* Quillen has constructed (7) for $\mathbf{S}$ a symmetric monoidal category, a new symmetric monoidal category $\mathbf{S}^{-1}\mathbf{S}$ of virtual objects, and shown under certain conditions on $\mathbf{S}$, $B(\mathbf{S}^{-1}\mathbf{S})$ is a group completion of $B\mathbf{S}$.

For $\mathbf{S}$ satisfying these conditions, and permutative, we have a functor

$$\mathbf{\Delta}^{op}\textstyle\int S^+ \to \mathbf{S}^{-1}\mathbf{S},$$

given on objects by $(S_0, S_0', S_1, \ldots, S_n) \to (S_0, S_0')$. We define the functor on enough morphisms to generate $\mathbf{\Delta}^{op}\int S^+$ by the formulae

$$(1, (s_0, s_0';\, s_1, \ldots, s_n)) \mapsto (0, s_0, s_0'), \quad (s_i, 1) \mapsto (0, 1, 1) \quad \text{and} \quad (d_i, 1) \mapsto (0, 1, 1)$$

for $i > 0$, and

$$(d_0, 1)\colon (S_0, S_0';\, S_1, \ldots) \to (S_0 \oplus S_1, S_0' \oplus S_1;\, S_2, \ldots)$$

goes to $(S_0, 1, 1)$. Under the conditions Quillen gives for $\mathbf{S}^{-1}\mathbf{S}$ to be a group completion, one can show this functor has contractible fibres and so is a homotopy equivalence by (13), theorem A. Details may be found in (17).

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