GEOMETRIZATION OF THE LOCAL LANGLANDS CORRESPONDENCE

LECTURES BY LAURENT FARGUES

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1. Introduction

These are notes from the Fall 2015 lectures by Laurent Fargues on "Geometrization of the Local Langlands Correspondence" delivered at the University of Chicago in the Geometric Langlands Seminar. The notes were typed by Sean Howe who apologizes for any errors introduced in the transcription. Corrections and suggestions are welcome, and can be sent to seanpkh@gmail.com.

VERSION NOTES (v0.5) This is the first version contains all lectures, but the end of the final lecture is still a rough draft. Here are some changes pending in the upcoming versions.

- Clean up Lecture 9
- Section headings
- Add (more) internal references
- Add (more) external citations
- Add a full-page map of the curve incorporating various diagrams

2. Lecture 2015-10-08

2.1. **Context.** Let E be a discretely valued non-archimedean field with uniformizer π and finite residue field $\mathbb{F}_q = \mathcal{O}_E/\pi$. Let $F/\mathbb{F}_q$ be a perfectoid field.

In joint work with Fontaine, we attach to this data a curve (in a generalized sense) $X_{F,E}/E$. More generally, if $S/\mathbb{F}_q$ is perfectoid then we can define $X_{S,E}$, which can be thought of as a "family of curves"

$$X_S = \left(X_{k(s)}\right)_{s \in |S|}.$$

If G is a reductive group over E one can define a stack

$$\mathrm{Bun}_G : S \to \{G\text{-bundles on } X_S\}$$

We will be interested in perverse l-adic sheaves on Bun_G. We begin by explaining the construction of the curve.

2.2. **Holomorphic functions of the variable** p**.** We take E as before. There are two cases: either $E = \mathbb{F}_q((\pi))$ or $[E : \mathbb{Q}_p] < \infty$. We take $F/\mathbb{F}_q$ to be a perfectoid field.

Definition 1. A *perfectoid field in characteristic* p is a perfect field F of characteristic p complete with respect to a non-trivial absolute value $|\cdot| : F \to \mathbb{R}_+$.

As before, to this data we attach a curve X. It has two "incarnations":

- X^{ad} = "compact p-adic Riemann surface"
- X = "complete algebraic curve," where by algebraic curve we mean a one-dimensional noetherian regular scheme over E.

The algebraic curve was discovered first, but it is easier to define X^{ad}, so we begin there.

There is a space Y, which can be thought of as a punctured open disk in variable π with coefficients in F such that

$$X^{\mathrm{ad}} = Y/\phi^{\mathbb{Z}}$$

where ϕ is the Frobenius. We now explain the construction of Y.

Definition 2. $\mathbb{A}$ ($= \mathbb{A}^{\mathrm{inf}}$ elsewhere) is the unique π-adically complete π-torsion free lift of $\mathcal{O}_F$ as an $\mathbb{F}_q$-algebra. So, $\mathbb{A}$ is an $\mathcal{O}_E$-algebra that is π-adically complete such that $A/\pi = \mathcal{O}_F$. It is unique up to unique isomorphism.

There is a unique Teichmuller multiplicative lift $[-] : \mathcal{O}_F \to \mathbb{A}$, and

$$\mathbb{A} = \left\{ \sum_{n \geqslant 0} [x_n]\pi^n | x_n \in \mathcal{O}_F \right\}$$

That is, every element of $\mathbb{A}$ can be expressed this way and the expression is unique.

For each of the two possibilites for E we can give an explicit description of $\mathbb{A}$:

- $E = \mathbb{F}_q((\pi))$: In this case $[-]$ is also additive so
$$[-] : \mathcal{O}_F \hookrightarrow \mathbb{A}$$
is an algebra morphism and
$$\mathbb{A} = \mathcal{O}_F[[\pi]].$$
- $[E : \mathbb{Q}_p] < \infty$: In this case
$$\mathbb{A} = \mathbb{W}_{\mathcal{O}_E}(\mathcal{O}_F),$$
where $\mathbb{W}_{\mathcal{O}_E}$ denotes the ramified Witt vectors. For $E = \mathbb{Q}_p$ these are the standard Witt vectors, otherwise they are as defined by Drinfeld.

  Interjection. ***Drinfeld explains:*** *In the regular formation of Witt vectors, p appears in two places, in the exponents and in the coefficients of the formulas. To obtain the ramified Witt vectors we must replace it in exponents with q, the degree of the residue field extension, and in the coefficients with* π*, the uniformizer).*

In this case, we can also describe the ramified Witt vectors by

$$\mathbb{W}_{\mathcal{O}_E}(\mathcal{O}_F) = W(\mathcal{O}_F) \otimes_{\mathcal{O}_{E_0}} \mathcal{O}_E$$

where $E_0/\mathbb{Q}_p$ is the maximal unramified subextension of $E/\mathbb{Q}_p$ so that $\mathcal{O}_{E_0} = W(\mathbb{F}_q)$ (recall $\mathbb{F}_q$ is the residue field of E).

The addition and multiplication in $\mathbb{A}$ of elements

$$\sum_{n\geqslant 0} [x_n]\pi^n \text{ and } \sum_{n\geqslant 0} [y_n]\pi^n$$

is given in Teichmuller coordinates by universal generalized polynomials in $\mathbb{F}_q[x_i^{1/p^\infty}, y_j^{1/p^\infty}]_{i,j\geqslant 0}$.

At this point we fix a choice of $\varpi_F \in F$ such that $0 < |\varpi_F| < 1$. We will use ϖ_F in some constructions, but the resulting objects will not depend on our choice.

We equip $\mathbb{A}$ with the $(\pi, [\varpi_F])$-adic topology. This is the topology of weak (term by term) convergence in Teichmuller coordinates.

We now define a space $\mathcal{Y}$ which will contain Y.

Definition 3. $\mathcal{Y} = \mathrm{Spa}(\mathbb{A})_a = \mathrm{Spa}(\mathbb{A})\backslash V(\pi) \cup \mathrm{Spa}(\mathbb{A})\backslash V([\varpi_F])$.

We explain some of the notation: $\mathrm{Spa}(\mathbb{A})$, from the language of Huber's adic spaces, is the topological space of continuous valuations on $\mathbb{A}$ plus a structural presheaf of rings. Here a valuation is in the most general sense (in particular it is not necessarily of rank 1), and can be thought of as a map $\nu : \mathbb{A} \to \Gamma \cup +\infty$ for Γ a totally ordered abelian group or in terms of valuation rings. As a reference for adic spaces, one can consult Huber - *A generalization of formal schemes and rigid analytic varieties*, or the notes from Scholze's course on p-adic geometry.

The subscript a on $\mathrm{Spa}(\mathbb{A})_a$ indicates we take the analytic points – that is we must throw away the valuations with open support. In this case, the valuations we remove are those factoring through

$$(\mathbb{A}/(\pi, [\varpi_F]))_{\mathrm{red}} = k_F$$

where k_F is the residue field of F.

We note that $\mathcal{Y} = \mathrm{Spa}(\mathbb{A})_a$ is not affinoid.

Definition 4. $Y = \mathrm{Spa}(\mathbb{A})\backslash V(\pi[\varpi_F])$.

We have that $\mathcal{Y}$ is an adic space over $\mathcal{O}_E$ and Y is an adic space over E (one must verify that the structural presheaf is actually a sheaf, but we will return to this point later).

2.2.1. *What are these things?*

We first examine the case $E = \mathbb{F}_q((\pi))$.

In this case, $Y = \mathbb{D}_F^*$, the rigid analytic punctured disk of radius 1 over F with coordinate π. That is, $\mathbb{D}_F^* \subset \mathbb{A}_F^1$ defined by $0 < |\pi| < 1$.

We have natural maps

$$\begin{tikzcd}
 & \mathbb{D}_F^* \arrow[dl] \arrow[dr] & \\
\mathrm{Spa}(F) & & \mathbb{D}_{\mathbb{F}_q}^* = \mathrm{Spa}(\mathbb{F}_q((\pi))) = \mathrm{Spa}(E)
\end{tikzcd}$$

The map on the left to $\mathrm{Spa}(F)$ is the usual structure morphism and is locally of finite type. However, we are interested in Y as a space over E (via the map on the right), where the structural morphism is *not* locally of finite type.

We can write down the ring of functions explicitly as

$$\mathcal{O}(\mathbb{D}_F^*) = \left\{ \sum_{n\in\mathbb{Z}} a_n \pi^n \mid a_n \in F \text{ and } \forall \rho \in (0,1),\ \lim_{n\to\infty} |a_n|\rho^n = 0 \right\}$$

The space $\mathbb{D}_F^*$ is Stein, so it can be understood completely in terms of its ring of functions. On the level of functions, the map to $\mathbb{D}_{\mathbb{F}_q}^*$ that we are interested in is given by the inclusion of $\mathbb{F}_q((\pi))$ into the ring of power series $\mathcal{O}(\mathbb{D}_F^*)$ as described above.

In this description, we have

$$\mathcal{Y} = \mathbb{D}_F \cup \{\zeta\}$$

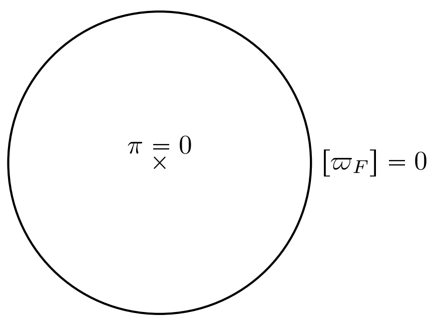
where $\mathbb{D}_F$ is the open disk $\{0 \leqslant \pi < 1\}$ and the valuation ζ is of rank 1, given by

$$\text{For } f = \sum_{n\geqslant 0} [x_n]\pi^n,\ |f(\zeta)| = \begin{cases} 0 \text{ if } \forall n,\ |x_n| < 1 \\ p^{-n_0},\ n_0 = \inf\{n \mid |x_n| = 1\} \end{cases}$$

Interjection. ***Drinfeld asks****: Another understandable story is if we take $\mathbb{A}^2$ and look at the completed local ring at 0, then remove the closed point. Is it related to this story?*
Answer*: Yes, if we remove perfect hypothesis and take $F = \mathbb{F}_q((T))$ and repeat above, then $\mathbb{A} = \mathbb{F}_q[[\pi, T]]$, and*

$$\text{“}\mathcal{Y} = \mathrm{Spec}(\mathbb{F}_q[[\pi,T]] \backslash V(\pi,T))^{\mathrm{an}}\text{”}$$

We can visualize $\mathcal{Y}$ as a unit disk, where the center point is $\pi = 0$ and the boundary is $[\varpi_F] = 0$; when both are removed we have Y.



It is useful to think of a function being holomorphic on $[\varpi_F] = 0$ as equivalent to being bounded near it.

We now return to the case of any E (i.e. either $E = \mathbb{F}_q((\pi))$ or $[E : \mathbb{Q}_p] < \infty$).

Consider the ring

$$\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi_F]}\right] = \left\{ \sum_{n>>-\infty} [x_n]\pi^n \mid x_n \in F,\ \sup_n |x_n| < \infty \right\}.$$

It is the ring of holomorphic functions on Y meromorphic at $\pi = 0$ (corresponding to the condition that the sum is over $n >> -\infty$) and at $[\varpi_F] = 0$ (corresponding to the condition that $\sup_n |x_n| < \infty$).

For $\rho \in (0,1]$ and

$$f = \sum_{n>>-\infty} [x_n]\pi^n \in \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi_F]}\right]$$

set

$$|f|_\rho = \sup_n |x_n|\rho^n.$$

Set also

$$|f|_0 = q^{-\mathrm{ord}_\pi f}.$$

For $\rho \in (0,1)$, $|\cdot|_\rho$ is the Gauss supremum norm on the annulus $|\pi| = \rho$, $|\cdot|_0$ is the order of vanishing at $\pi = 0$, and $|\cdot|_1$ is the order of vanishing at $[\varpi_F] = 0$.

There is a radius function

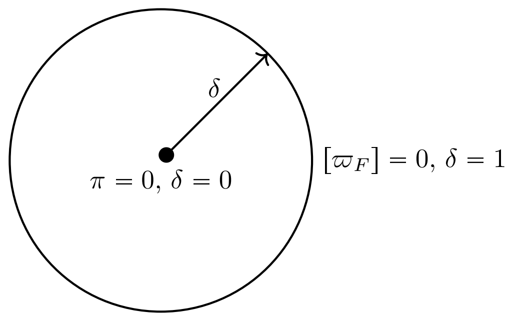
$$\begin{aligned} \delta : |\mathcal{Y}| &\longrightarrow [0,1] \\ y &\longmapsto |\pi\left(y^{\max}\right)| \\ &\quad \text{(fake formula)} \end{aligned}$$

The real formula is

$$\delta(y) = q^{-\frac{\log|\pi(y^{\max})|}{\log|[\varpi_F](y^{\max})|}}$$

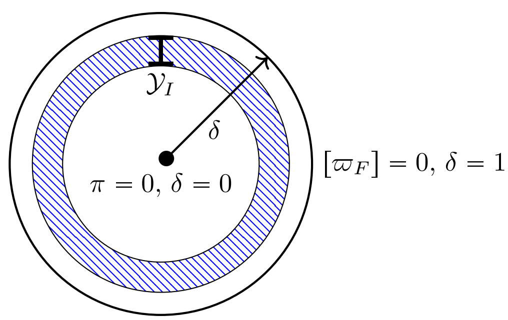
Where $y^{\max}$ is the unique maximal rank 1 (i.e. with values in $\mathbb{R}$) generization of y.

In our picture for $\mathcal{Y}$, δ is the distance from the center:



Let $I \subset [0,1]$ be an interval with $I \neq \varnothing$, 0, 1 and with extremities in $|F|^{1/\infty}$. Then,

$$\mathcal{Y}_I = \text{annulus } \mathrm{Int}(\{y \in \mathcal{Y} \mid \delta(y) \in I\})$$



Example 2.1. $Y = \mathcal{Y}_{(0,1)}$

We have $\mathcal{O}(\mathcal{Y}) = \mathbb{A}$, and

$$\mathcal{O}(\mathcal{Y}_I) = \text{completion w.r.t. } (|\cdot|_\rho)_{\rho \in I}$$

$$\text{of } \begin{cases} \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi_F]}\right] \text{ if } I \subset (0,1) \\ \mathbb{A}\left[\frac{1}{\pi}\right] \text{ if } I \subset (0,1] \text{ and } 1 \in I \\ \mathbb{A}\left[\frac{1}{[\varpi_F]}\right] \text{ if } I \subset [0,1) \text{ and } 0 \in I \end{cases}$$

This is a Frechet algebra. It is a Banach algebra if I is compact (by the maximum modulus principal, $|\cdot|_\rho \leqslant \sup\{|\cdot|_{\rho_1}, |\cdot|_{\rho_2}\}$ if $\rho \in [\rho_1, \rho_2]$).

We define $B = \mathcal{O}(Y)$, a Frechet algebra.

Warning. If $E = \mathbb{F}_q((\pi))$,

$$B = \left\{ \sum_{n \in \mathbb{Z}} [x_n]\pi^n \mid x_n \in F \text{ and } \forall \rho \in (0,1),\ \lim_{n \to \infty} |x_n|\rho^n = 0 \right\}.$$

However, if $E/\mathbb{Q}_p$, there may exist $f \in B$ that do not have such a Laurent expansion at $\pi = 0$; this is one of the big difficulties of p-adic Hodge theory. For example, the elements of the form above are in B, but their sums and products may not be of this form. The reason is that for $f = \sum_{n>>\infty}[x_n]\pi^n$ and $g = \sum_{n>>\infty}[y_n]\pi^n$ two elements of $\mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi_F]}\right]$, there is no simple formula for

$$|f-g|_\rho$$

whereas when $E = \mathbb{F}_q((\pi))$,

$$|f-g|_\rho = \sup_n |x_n - y_n|\rho^n.$$

For another example, the periods of p-divisible groups are of this form, but periods of higher Dieudonné-Manin slope cannot be expressed in this way.

More precisely, let us consider a slope (in the sense of Dieudonné-Manin) $\lambda \in \mathbb{Q}_{>0}, \lambda = d/h$ with $(d,h) = 1$. There is a crystalline Frobenius φ acting on B (as we will see later). If $E = \mathbb{F}_q((\pi))$ then there is a bijection

$$\begin{aligned} \mathfrak{m}_F^d &\longrightarrow B^{\varphi^h = \pi^d} \\ (x_0, ..., x_{d-1}) &\longmapsto \textstyle\sum_{i=0}^{d-1} \sum_{n\in\mathbb{Z}} [x_i^{q^{-nh}}]\pi^{nd+i}. \end{aligned}$$

The induced E-vector space structure on $\mathfrak{m}_F^d$ via this bijection is given by the identification of $\mathfrak{m}_F^d$ with the $\mathcal{O}_F$-points of a slope λ Drinfeld module.

If $E/\mathbb{Q}_p$, and $\lambda \in (0,1]$ there is again such a bijection where the E-vector space structure on $\mathfrak{m}_F^d$ is now identified with the $\mathcal{O}_F$-points of a formal π-divisible $\mathcal{O}_E$-module (a p-divisible group when $E = \mathbb{Q}_p$) of slope λ. But for $\lambda > 1$, contrary to the function field case, there is no geometric object generalizing p-divisible groups whose $\mathcal{O}_F$-points would give $B^{\varphi^h=\pi^d}$ and the elements contained in it may not have a Laurent expansion around $\pi = 0$. The Banach space $B^{\varphi^h=\pi^d}$ shows up as crystalline periods, i.e. elements of $\mathrm{Hom}_\varphi(H^\bullet_{\text{cris}}, B)$ where $H^\bullet_{\text{cris}}$ is the crystalline cohomology of a the special fiber of a proper smooth scheme over the p-adic integers. For the H^1_{cris} only periods of the p-divisible groups attached to the Picard variety show up, but for H^i_{cris} with $i > 1$, Dieudonné-Manin slopes that are not in $[0,1]$ may appear.

3. Lecture 2015-10-15

3.1. **Perfectoid Fields.** In this section, K is a complete field with respect to a non-trivial valuation $|\cdot| : K \to \mathbb{R}_+$.

Definition 5. K is perfectoid if

(1) The valuation of K is not discrete, and
(2) $\exists \varpi_K \in K$, $|p| \leqslant |\varpi_K| < 1$ s.t.
$$\mathcal{O}_K/\varpi_K \xrightarrow{\text{Frob}} \mathcal{O}_K/\varpi_K$$
is surjective.

In fact we have (2) $\iff$ (2′) $\mathcal{O}_K \to \mathcal{O}_K/p$ is surjective.

Example 3.1. The following are perfectoid fields (remember K here is always complete with respect to a non-trivial valuation):

- K algebraically closed is perfectoid.
- If char$K = p$ then K is perfectoid if and only if it's perfect.
- Fontaine and Wintenberger give the following criterion: Suppose $L/\mathbb{Q}_p$ is a complete d.v.r with perfect residue field and L'/L is an infinite degree algebraic Galois extension such that $\forall s \in \mathbb{R}$, $\mathrm{Gal}(L'/L)^s$ is open in $\mathrm{Gal}(L'/L)$, where $\mathrm{Gal}(L'/L)^s$ denotes a term in the higher ramification filtration (which, in general, is only closed, and thus is open if and only if it is of finite index). Such an extension is called arithmetically profinite (because the topology induced by the ramification filtration is the same as the profinite topology). Then, $\widehat{L}'$ is perfectoid. They also give a similar criterion for algebraic but not Galois extensions.
- $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ is perfectoid by the criterion of Fontaine-Wintenberger above.
- $\widehat{\mathbb{Q}_p(p^{1/p^\infty})} = \bigcup_{n \geqslant 0} \widehat{\mathbb{Q}_p(p^{1/p^n})}$ is perfectoid, as can be verified by hand.

3.1.1. *Tilting.* Let K be a perfectoid field. We set
$$K^\flat = \left\{ (x^{(n)})_{n \geqslant 0} \mid x^{(n)} \in K,\, (x^{(n+1)})^p = x^{(n)} \right\}$$
For $x, y \in K^\flat$, we define multiplication by
$$(xy)^{(n)} = x^{(n)} y^{(n)}$$
and addition by
$$(x+y)^{(n)} = \lim_{b \to \infty} \left(x^{(n+b)} + y^{(n+b)} \right)^{p^b}$$
We also define a valuation by
$$|x| = |x^0|$$
Then, $K^\flat$ is a perfectoid field of characteristic p and $|K| = |K^\flat| \subset \mathbb{R}_+$. Note that if K is of characteristic p then $K = K^\flat$.

Moreover,
$$\begin{aligned} \mathcal{O}_{K^\flat} &\longrightarrow \varprojlim_{\text{Frob}} \mathcal{O}_K/p \\ (x^{(n)})_{n \geqslant 0} &\longmapsto (x^{(n)} \mod p)_{n \geqslant 0} \end{aligned}$$
is a bijection whose inverse is given by
$$(y_n)_{n \geqslant 0} \mapsto \left(\lim_{b \to \infty} \widehat{y_{n+b}}^{p^b} \right)_{n \geqslant 0}$$
where $\widehat{y_{n+b}}$ is any lift of y_{n+b}. This explains the formula for addition on $K^\flat$ given above, and we could have alternatively defined $\mathcal{O}_{K^\flat}$ this way and then defined $K^\flat$ by $K^\flat = \mathrm{Frac}\mathcal{O}_{K^\flat}$.

Example 3.2. In the setting of Fontaine-Wintenberger from the earlier example, let $K = \widehat{L}'$ for L'/L arithmetically profinite. Then

$$K^\flat \simeq k_{L'}((T^{1/p^\infty})).$$

For example:

- If $K = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$, $\epsilon = (\zeta_{p^n})_{n\geqslant 0} \in K^\flat$ and $\pi_\epsilon = \epsilon - 1 \in K^\flat$, then
$$K^\flat = \mathbb{F}_p((\pi_\epsilon^{1/p^\infty})).$$
This example gives rise to the theory of (ϕ, Γ)-modules.

- If we take $K = \widehat{\mathbb{Q}_p(p^{1/p^\infty})}$ and $\pi = (p^{1/p^n})_{n\geqslant 0} \in K^\flat$ then
$$K^\flat = \mathbb{F}_p((\pi^{1/p^\infty}))$$

Theorem 3.1 (Purity theorem)**.** *Let K be a perfectoid field. Then:*

(1) If L is a finite degree extension of K then L is perfectoid and $[L^\flat : K^\flat] = [L : K]$.

(2) $\mathcal{O}_L/\mathcal{O}_K$ is almost étale in the sense that if $n = [L : K]$, $\forall 0 < \epsilon < 1$, $\exists e_1, ..., e_n \in \mathcal{O}_L$ such that
$$\epsilon \leqslant \left|\mathrm{disc}\left(\mathrm{Tr}_{L/K}(e_i e_j)\right)_{1\leqslant i,j\leqslant n}\right| \leqslant 1$$

(3) $(-)^\flat$ induces an equivalence
$$\textit{Finite étale } K\textit{-algebras} \leftrightarrow \textit{Finite étale } K^\flat\textit{-algebras}$$

Remark 1. There is a useful MathOverflow post explaining why this is called a purity theorem: [1].

Corollary 3.1.

(1) K is algebraically closed if and only if $K^\flat$ is algebraically closed.

(2) $\mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{K^\flat}/K)$ *where*
$$\overline{K^\flat} = \bigcup_{K\subset L\subset \overline{K}\ \textit{finite}} L^\flat$$
.

Remark 2. Note that in part (2) of the Purity theorem, $\mathcal{O}_L/\mathcal{O}_K$ is not in general finite. This type of almost étale statement showed up already in Tate's paper on p-divisible groups.

3.2. **Back to Y.** Recall that E is $\mathbb{F}_q((\pi))$ or a finite extension of $\mathbb{Q}_p$, and $F/\mathbb{F}_q$ is perfectoid. We defined

$$\mathbb{A} = \begin{cases} \mathcal{O}_F[[\pi]] \text{ if } E = \mathbb{F}_q((\pi)) \\ \mathbb{W}_{\mathcal{O}_E}(\mathcal{O}_F) \text{ if } [E : \mathbb{Q}_p] < \infty \end{cases}$$

Then

$$Y = \mathrm{Spa}(\mathbb{A})\backslash V(\pi, [\varpi_F])$$

for some $0 < |\varpi_F| < 1$, and

$$Y \subset \mathcal{Y} = \mathrm{Spa}(\mathbb{A})\backslash V(\pi) \cup \mathrm{Spa}(\mathbb{A})\backslash V([\varpi_F]).$$

3.2.1. *Classical points of Y.*

Definition 6. $f = \sum_{n\geqslant 0} \pi^n \in \mathbb{A}$ is *primitive* if $x_0 \neq 0$ and $\exists d$ such that $|x|_d = 1$. For a primitve f, set

$$\deg(f) = \text{smallest d s.t. } |x|_d = 1.$$

An f that is primitive is *irreducible* if $\deg(f) > 0$ and $\nexists g, h \in \mathbb{A}$ primitive of degree > 0 such that $f = gh$.

For g, h primitive, $\deg(gh) = \deg(g) + \deg(h)$ so the set of primitive elements is a graded monoid (graded by deg).

Example 3.3.

- $\{\text{primitive of deg } 0\} = \mathbb{A}^\times$
- For $a \in F, 0 < |a| < 1$, $\pi - [a]$ is primitive of degree 1.

Notation: $\mathrm{Irred}/\sim = \{\text{irreducible primitive}\}/\mathbb{A}^\times$. So, $\mathrm{Irred}/\sim \hookrightarrow \mathrm{Spec}(\mathbb{A})$ via the ideal generated.

Example 3.4. If $E = \mathbb{F}_q((\pi))$, then by the Weierstrass preparation theorem,

$$\mathrm{Irred}/\sim = \{P \in \mathcal{O}_F[\pi] \mid P \text{ is a monic irreducible and } 0 < |P(0)| < 1\} = |\mathbb{D}_F^*|^{\mathrm{cl}}$$

where $|\mathbb{D}_F^*|^{\mathrm{cl}}$ are the classical Tate points of the punctured disk $\mathbb{D}_F^*$ (recall that with respect to the structure morphism to F this is a Tate rigid analytic space.)

Let $B = \mathcal{O}(Y)$ as in the first lecture.

Theorem 3.2. *For $y = (f) \in \mathrm{Irred}/\sim$, set $k(y) = B/f$ and $\theta_y : B \twoheadrightarrow B/f = k(y)$. Then,*

(1) $k(y)$ is a perfectoid field over E.

(2) The map

$$\begin{aligned} F &\longrightarrow k(y)^\flat \\ a &\longmapsto (\theta_y([a^{p^{-n}}]))_{n\geqslant 0} \end{aligned}$$

defines an extension $k(y)^\flat/F$ such that

$$[k(y)^\flat : F] = \deg y.$$

(3) This defines a bijection

$$\mathrm{Irred}^{\deg=1}/\sim \xrightarrow{\sim} \{(K, i) \mid K/E \textit{ perfectoid and } i : F \tilde{\to} K^\flat\}/\sim$$

(4) If F is algebraically closed then $\deg y = 1$ and $\exists a \in F, 0 < |a| < 1$ s.t. $y = (\pi - [a])$.

This is one of the most difficult theorems in the work with Fontaine.

Point (4) tells us that if F is algebraically closed and $g \in \mathbb{A}$ is primitive of degree d, then $g = u \cdot (\pi - [a_1])...(\pi - [a_d])$ for $u \in \mathbb{A}^\times$. This gives a type of Weierstrass factoriation for inequal characteristic.

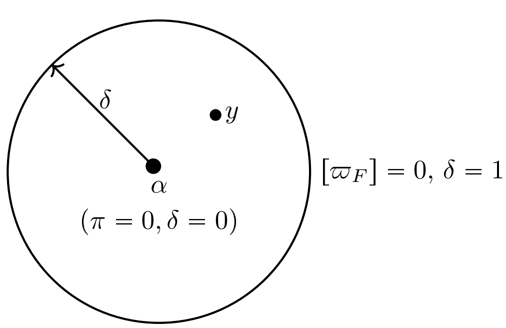
Warning. The $[a_1]$, ..., $[a_d]$ appearing in such a factorization are not unique if $E/\mathbb{Q}_p$.

Indeed, suppose F is algebraically closed, $a \in F$, $0 < |a| < 1$ and $y = (\pi - [a])$. Then $k(y)$ is algebraically closed and $[a] = \pi$ in $k(y)$. By the identification of F with $k(y)^\flat$, the map $a \mapsto \theta_y([a])$ is the map $a \to a^0$. Thus, $(\pi - [a']) = (\pi - [a])$ exactly when $a'^0 = a^0 = \pi$. The set

$$\{(x^{(n)})_{n\geqslant 0} \in k(y)^\flat \mid x^{(0)} = \pi\}$$

is a $\mathbb{Z}_p(1)$-torsor in $k(y)$, and thus there is a $\mathbb{Z}_p(1)$ worth of ambiguity in the choice of a giving the point $y = (x - [a])$.

3.3. **An updated picture.** We denote by $|\mathcal{Y}|^{\mathrm{cl}}$ the classical Tate points, which we can just define to be $\mathrm{Irred}/\sim$. Then $|\mathcal{Y}|^{\mathrm{cl}} \subset |Y|$. If we fix a point $y \in |\mathcal{Y}|^{\mathrm{cl}}$ then in our picture from before we can visualize it as lying in the interior of the disk:



Note we have named the point $\pi = 0$ from before α. The local ring at α of $\mathcal{Y}$ is

$$\mathcal{O}_{\mathcal{Y},\alpha} = \lim_{\rho \to 0} \mathcal{O}(\mathcal{Y}_{[0,\rho]}).$$

In p-adic Hodge theory it is also denoted by

$$\mathcal{O}_{\mathcal{Y},\alpha} = \mathcal{O}_{\mathcal{E}^\dagger}.$$

Here the dagger means "overconvergent", or convergent over a small neighborhood. It is a Henselian DVR with uniformizer π, and

$$\widehat{\mathcal{O}_{\mathcal{E}^\dagger}} = \mathcal{O}_{\mathcal{E}}$$

where

$$\mathcal{O}_{\mathcal{E}} = \begin{cases} F[[\pi]] \text{ if } E = \mathbb{F}_q((\pi)) \\ \mathbb{W}_{\mathcal{O}_E}(F) \text{ if } [E : /\mathbb{Q}_p] < \infty. \end{cases}$$

As we have seen, the residue field at y, $k(y)$, is perfectoid over E and an untilt of a finite extension of F of degree $\deg(y)$. We can think of these untilts degenerating to F itself as $\delta(y) \to 0$. We have

$$\widehat{\mathcal{O}_{Y,y}} = \mathbb{B}^+_{dR}(k(y)),$$

where the construction $\mathbb{B}^+_{dR}(k(y))$ is one of Fontaine's rings in p-adic Hodge theory. It is the f-adic completion of $\mathbb{A}[1/\pi]$ where $y = (f)$ (this is essentially Fontaine's original definition).

Remark 3. Fontaine's original point of view was to start with a perfectoid field K of characteristic zero, then take $F = K^\flat$ and construct $\mathbb{B}^+_{dR}$ by completing the corresponding $\mathbb{A}$. This automatically gives a point y on Y since K is canonically an untilt of $F = K^\flat$.

More precisely, for $K/\mathbb{Q}_p$ perfectoid, $F = K^\flat$, Fontaine constructs

$$\begin{aligned} \theta : W(\mathcal{O}_F) &\twoheadrightarrow \mathcal{O}_K \\ \textstyle\sum_{n \geqslant 0} [x_n] p^n &\longmapsto \textstyle\sum_{n \geqslant 0} x_n^{(0)} p^n \end{aligned}$$

and shows that $\ker \theta[1/p]$ is a principal ideal in $\mathbb{A}[1/p]$ generated by a degree 1 primitive element.

Once can ask whether it is possible to give genuinely different untilts of perfectoid fields in characteristic zero. There is an action of $\mathrm{Aut}_{\mathrm{cont}}(F)$ on $|\mathcal{Y}_F|^{\mathrm{cl}}$ by acting on the identification of the tilt with F, and the question can be restated, for F algebraically closed, as asking whether or not this action is transitive. If F is spherically complete, this action has been known to be transitive.

By recent work of Kedlaya, it is also transitive for $F = \mathbb{C}_p^\flat = \widehat{\overline{\mathbb{F}_p((T))}}$, as was conjectured by Fargues-Fontaine. He proves that $\mathrm{Aut}_{\mathrm{cont}}(F)$ acts transitively on $\mathfrak{m}_F \backslash \{0\}$. This result implies that if $K^\flat \simeq \mathbb{C}_p^\flat$, then $K \simeq \mathbb{C}_p$.

4. Lecture 2015-10-22

Recall that E is $\mathbb{F}_q((\pi))$ or a finite extension of $\mathbb{Q}_p$, and $F/\mathbb{F}_q$ is perfectoid. We defined

$$\mathbb{A} = \begin{cases} \mathcal{O}_F[[\pi]] \text{ if } E = \mathbb{F}_q((\pi)) \\ \mathbb{W}_{\mathcal{O}_E}(\mathcal{O}_F) \text{ if } [E : \mathbb{Q}_p] < \infty \end{cases}$$

Then

$$Y = \mathrm{Spa}(\mathbb{A}) \backslash V(\pi, [\varpi_F])$$

for some $0 < |\varpi_F| < 1$, and

$$Y \subset \mathcal{Y} = \mathrm{Spa}(\mathbb{A}) \backslash V(\pi) \cup \mathrm{Spa}(\mathbb{A}) \backslash V([\varpi_F]).$$

Last time we defined the "classical Tate points"

$$|Y|^{\mathrm{cl}} = \{\text{zeroes of primitive elements in } \mathbb{A}\}.$$

4.1. **The geometry of Y.** At the end of the last lecture we discussed the structure of Y near a classical point $y \in |Y|^{\mathrm{cl}}$. Recall that the residue field at y, $k(y)$, is a perfectoid extension of E with $[k(y)^\flat : F] = \deg(y)$, and, writing $y = V(f)$ for f a primitive irreducible element of $\mathbb{A}$, we have

$$\widehat{\mathcal{O}_{Y,y}} = \mathbb{B}_{\mathrm{dR}}^+(k(y)) = \text{the } f-\text{adic completion of } \mathbb{A}\left[\frac{1}{\pi}\right].$$

We also began to discusss the structure near $\pi = 0$. Recall

$$\mathcal{O}_{\mathcal{E}^+} = \mathcal{O}_{\mathcal{Y},\pi=0} = \lim_{\rho \to 0^+} \mathcal{O}(Yc_{[0,\rho]})$$

where $\mathcal{Y}_{[0,\rho]}$ is the "annulus" $0 \leqslant \delta \leqslant \rho$. It is a Henselian DVR with uniformizer π, and its completion is

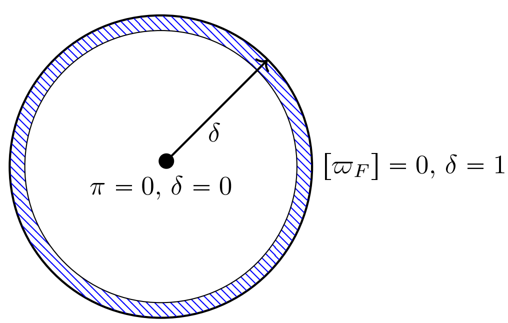
$$\mathcal{O}_{\mathcal{E}} = \widehat{\mathcal{O}_{\mathcal{E}^+}} = \begin{cases} F[[\pi]] \text{ if } E = \mathbb{F}_q((\pi)) \\ \mathbb{W}_{\mathcal{O}_E}(F) \text{ if } [E : \mathbb{Q}_p] < \infty. \end{cases}$$

In addition, we define the Robba ring by

$$\mathcal{R} = \lim_{\rho \to 0^+} \mathcal{O}(\mathcal{Y}_{(0,\rho]})$$

where, because of the open inequality, we are allowing an essential singularity at $\pi = 0$.

We will now look at the structure near $[\varpi_F] = 0$. In our picture, we are looking near the boundary of the disk:



To simplify the discussion, we restrict to the case $E = \mathbb{Q}_p$.

Remark 4. To adapt the following discussion to a general finite extension $E/\mathbb{Q}_p$, one must use π-divided powers. These are defined in [3] (see also [4, Appendix B]).

Let $\rho = |\varpi_F| \in (0, 1)$.

$$\begin{aligned} B_{\mathrm{cris},\rho}^+ &:= H^0(\mathrm{Spec}(\mathcal{O}_F/\varpi_F)/\mathrm{Spec}(\mathbb{Z}_p), \mathcal{O}_{\mathrm{cris}})[1/p] \\ &= \widehat{\mathbb{W}(\mathcal{O}_F)\left[\frac{[\varpi_F^n]}{n!}\right]_{n\geqslant 1}}[1/p] \end{aligned}$$

The thing inside of the completion is the divided power envelope of $([\varpi_F])$, which we can view as living inside of $\mathbb{W}(\mathcal{O}_F)[1/p]$. This p-adic completion is flat so we do not lose any information. If we did not invert p, we would have the ring known as $A_{\mathrm{cris},\rho}$.

We have

$$\mathcal{O}\left(\mathcal{Y}_{[\rho,1]}\right) \subset B^+_{\mathrm{cris},\rho} \subset \mathcal{O}\left(\mathcal{Y}_{[\rho^{\frac{1}{p-1}},1]}\right)$$

where we note the closed inequality at 1 should be interpreted as taking functions holomorphic at $[\varpi_F]=0$.

Remark 5. This formula is not valid as written for $p=2$; in that case we must replace 2 with 4 as is the case in much of p-adic Hodge theory for the prime 2. Here and elsewhere we write formulas valid for odd p.

Remark 6. Fontaine originally began with $\mathbb{C}_p = \widehat{\overline{\mathbb{Q}_p}}$ and $F = \mathbb{C}_p^\flat$ (which in our language means he has fixed a classical point on Y corresponding to this untilt). He then took $\varpi_F = \mathfrak{p}$ where $\mathfrak{p}$ is an element of $\mathbb{C}_p^\flat$ such that $\mathfrak{p}^{(0)} = p$ and defined

$$A_{\mathrm{cris}} = \widehat{\mathbb{W}(\mathcal{O}_{\mathbb{C}_p^\flat})\left[\frac{[\varpi_F^n]}{n!}\right]_{n\geqslant 1}}$$

and

$$B^+_{\mathrm{cris}} = A_{\mathrm{cris}}[1/p].$$

4.2. **Plenty of holomorphic analysis results.** There are many holomorphic analysis results that are "easy" for $E = \mathbb{F}_q((\pi))$ but more difficult for $E/\mathbb{Q}_p$. Such results for $\mathbb{F}_q((\pi))$ can be found in [11]. For $E/\mathbb{Q}_p$ they are in [6], [8], and [7].

The main difficulty in the mixed characteristic case is that the Weierstrass factorization is not unique. For example, because of this there is no canonical Euclidean division.

Here are two of the main results:

Theorem 4.1. *Let $I \subset (0,1)$ be compact and nonempty with extremities in $p^{\mathbb{Q}}$. Then $\mathcal{O}(\mathcal{Y}_I)$ is a PID with maximal spectrum equal to $|\mathcal{Y}_I|^{\mathrm{cl}}$.*

Theorem 4.2. *Let $f \in B = \mathcal{O}(Y)$. Then,*

$$\{\text{ Slopes of the Newton polygon of } f \ \} = \{-\log_q \delta(y) \mid y \in |Y|^{\mathrm{cl}} \text{ s.t. } f(y)=0\}$$

with multiplicities.

Here if

$$f = \sum_{n \gg -\infty} [x_n]\pi^n \in \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi_F]}\right]$$

then

$$\mathrm{Newt}(f) = \text{ the decreasing convex hull of } \{(n, v(x_n))\}_{n\in\mathbb{Z}}.$$

Such elements are dense in B, but as we have seen in the last lecture, not every element of B has a Laurent expansion. To define the Newton polygon for an arbitrary element, we first introduce some notation. For $\rho \in [0,1]$, we define v_r by

$$q^{-v_r(\cdot)} = |\cdot|_\rho.$$

Then for $f \in B$, $\mathrm{Newt}(f)$ is defined to be the inverse Legendre transform of the function

$$\begin{aligned}(0,\infty) &\longrightarrow \mathbb{R}\\ r &\longmapsto v_r(f).\end{aligned}$$

If $f \in \mathbb{A}\left[\frac{1}{\pi}, \frac{1}{[\varpi_F]}\right]$, $f = \sum_{n\gg-\infty}[x_n]\pi^n$, then $v_r(f) = \inf v(x_n) + nr_n \in \mathbb{Z}$ for $r \in [0,\infty)$.

Interjection. *Drinfeld: To a holomorphic function f on a non-Archimedean disk one can associate 3 types of data:*

- *the Newton polygon, which is defined in terms of the coefficients of f,*
- *the collection of norms $|f|_\rho$ (maximum modulus on the circle of radius ρ) for all $\rho \in [0,1]$,*

- *and the absolute values of the zeros of f.*

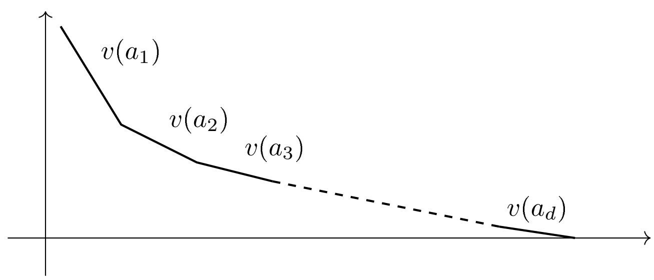
The theory of non-Archimedean holomorphic functions tells us that the 3 types of data are essentially equivalent: if you know one of them you can express the other two in terms of it. In the Archimedean case the situation is more complicated: instead of equalities relating the three types of data one only gets inequalities. So Hadamard might see the non-Archimedean case as childish games.

Fargues: Yes, the corresponding result to convexity of the Newton polygon for v_r in the archimedean case is Hadamard's 3 circles theorem, which tells us that $r \mapsto v_r(f)$ is convex. The fact that the slopes of Newt(f) *are the valuations of the zeroes of f corresponds to Jensen's theorem, but Jensen's theorem only gives an inequality rather than equality. On the other hand, some tools are missing in the p-adic world, e.g. Blaschke products, which also make it more difficult in some sense. More about this is explained in [7].*

One important point for these results is that $\forall \rho \in [0,1]$, $|\cdot|_\rho$ is multiplicative, i.e., $\forall r \in [0,+\infty)$, v_r is a valuation. This implies that if $f, g \in B$, then $\mathrm{Newt}(f \cdot g) = \mathrm{Newt}(f) \star \mathrm{Newt}(g)$ where $\star$ is tropical convolution (which can be described as slope-wise concatenation of the polygons).

For example, this gives us that

$$\mathrm{Newt}((\pi - [a_1]) \cdot \ldots \cdot (\pi - [a_d])) = \mathrm{Newt}(\pi - [a_1]) \star \ldots \star \mathrm{Newt}(\pi - [a_d]) =$$



Where the slopes are exactly the valuations of the roots with multiplicity. If you instead expand out the coefficients of this element using the addition and multiplication laws for Witt vectors it becomes very difficult to compute the Newton polygon. In general, there are no useful Laurent expansions (at $\pi = 0$, $[\varpi_F] = 0$, or at a classical point) for computing the Newton polygon of an arbitrary element of B.

4.3. **The adic curve.** There is a Frobenius map $\varphi : \mathbb{A} \to \mathbb{A}$

$$\textstyle\sum_{n\geqslant 0}[x_n]\pi^n \overset{\varphi}{\longmapsto} \sum_{n\geqslant 0}[x_n^q]\pi^n$$

Here, if we think of π as the variable, then this is an arithmetic Frobenius since it is taking the coefficient to the qth power rather than the variable (which would give a geometric Frobenius).

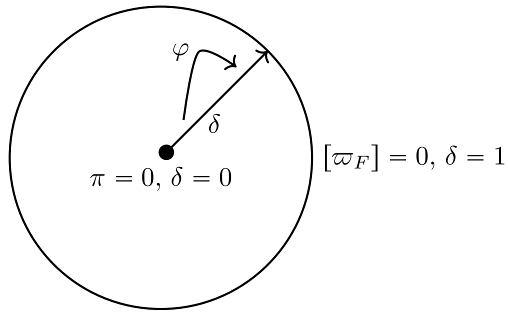
This induces an automorphism φ of $\mathcal{Y}$. An easy check shows it satisfies

$$\delta(\phi(y)) = \delta(y)^{1/q}$$

where

$$\delta : |\mathcal{Y}| \to [0,1]$$

is the map we defined before. In our picture, φ fixes the center ($\pi = 0$) and boundary ($[\varpi_F] = 0$) of the circle while expanding from the center towards the boundary:



The map φ stabilizes the divisors (π) and $([\varpi_F])$ and acts properly discontinuously on $Y = \mathcal{Y}_{(0,1)}$.

Definition 7. $X^{\mathrm{ad}} = Y/\varphi^{\mathbb{Z}}$.

X^{ad} is a quasi-compact, partially proper E-adic space (the condition "partially proper" is similar to the valuative criterion for schemes; recall that for a finite type adic space over E, being quasi-compact and partially proper would be equivalent to being proper). It is an adic space over Spa$E$ but it is *not* locally of finite type in any sense. We have

$$H^0(X^{\mathrm{ad}}, \mathcal{O}) = E = B^{\phi=\mathrm{Id}}.$$

Interjection. *Drinfeld: It may be useful to think of the Tate curve, defined as* $\mathbb{G}_m/q^{\mathbb{Z}}$ *for* $|q| < 1$, *as a simpler classical construction that is in some ways similar.*

Example 4.1. If $E = \mathbb{F}_q((\pi))$ then $Y = \mathbb{D}^*_F$. The map to Spa(F) is *not* Frobenius equivariant so does not descend to X, but the map to Spa$(E) = \mathbb{D}^*_{\mathbb{F}_q}$ is, and so X^{ad} is an adic space over Spa(E), but it is not of locally finite type.

4.4. **The schematic curve.** This section was first called "the algebraic curve", but it was pointed out in the lecture that this could be misleading since the object constructed will not be locally of finite type.

We first construct a line bundle $\mathcal{O}(1)$ on X^{ad}. Its geometric realization is given by

$$Y \times_{\varphi^{\mathbb{Z}}} \mathbb{A}^1_E \to Y/\varphi^{\mathbb{Z}}$$

where φ acts by π^{-1} on $\mathbb{A}^1_E$. Another way to say this is that $\mathcal{O}(1)$ corresponds to the φ-equivariant line bundle on Y given by the trivial bundle $\mathcal{O}_Y$ with φ-equivariant structure $f \mapsto \pi^{-1}\varphi(f)$ for $f \in \mathcal{O}_Y$.

Definition 8. We define the graded ring

$$\begin{aligned} P &= \oplus_{d\geqslant 0} H^0(X^{\mathrm{ad}}, \mathcal{O}(d)) \\ &= \oplus_{d\geqslant 0} B^{\varphi=\pi^d} \end{aligned}$$

where

$$B^{\varphi=\pi^d} = \{f \in B \mid \varphi(f) = \pi^d f\}.$$

Then,

$$X = \mathrm{Proj}(P) = \text{ an } E\text{-scheme not locally of finite type.}$$

Remark 7. If $E = \mathbb{F}_q((\pi))$ then for $d > 0$, $B^{\varphi=\pi^d}$ is isomorphic to $\mathfrak{m}_F^d$ via the bijection

$$(x_0, \ldots, x_{d-1}) \longmapsto \sum_{i=0}^{d-1} \sum_{n\in\mathbb{Z}} [x_i^{q^{-n}}]\pi^{nd+i}.$$

These are the periods of a (local) Drinfeld module of height 1 and dimension d. This is a formal $\mathcal{O}_E$-module, and by periods in this case we mean the $\mathcal{O}_F$ points of this module.

If $E/\mathbb{Q}_p$, this formula is true only if $d = 1$, and there is no explicit formula for $B^{\varphi=\pi^d}$ for $d > 1$. To expand on this, let $a \in \mathfrak{m}_F$, $a \neq 0$. Then

$$\sum_{n\in\mathbb{Z}} [a^{q^{-n}}]\pi^n = \text{“}2i\pi\text{” in } k(y)$$

where $y = V(\pi - [a])$ and the π in "$2i\pi$" is not the uniformizer of E but the analog of 3.14... in this setting. Then sections of $\mathcal{O}(1)$ are families of "$2i\pi$"s. In the case of $E = \mathbb{Q}_p$, for $\epsilon \in 1 + \mathfrak{m}_F \backslash \{1\}$, let

$$u_\epsilon = 1 + [\epsilon^{1/p}] + ... + [\epsilon^{(p-1)/p}].$$

Then u_ϵ is primitive of degree 1 and we let $y_\epsilon = V(u_\epsilon)$ be the corresponding classical point of Y. Attached to y_ϵ we have the perfectoid extension $k(y_\epsilon)/E$ and we can define $t_\epsilon = \log([\epsilon]) \in B^{\phi = p}$. This is the "$2i\pi$" associated to y_ϵ – note it is only determined up to $\mathbb{Z}_p^\times \cdot t_\epsilon$. This is a choice of orientation, just like the classical complex $2i\pi$ is only determined after fixing an i, so up to ± 1.

We don't understand the additive structure on P_d for $d > 1$ when $E/\mathbb{Q}_p$, but we understand the multiplicative structure well:

Theorem 4.3. *If F is algebraically closed, then $\forall x \in P_d$, $x \neq 0$ and $d \geqslant 1$, $\exists z_1, ..., z_d \in P_1$ such that*

$$x = z_1 \cdot ... \cdot z_d.$$

In fact, P is "graded factorial" with irreducible elements of degree 1, i.e., the monoid

$$\bigcup_{d \geqslant 0} (P_d \backslash \{0\}) / E^\times$$

is free on

$$(P_1 \backslash \{0\}) / E^\times.$$

This is a tough theorem, and uses the Weierstrass factorization result and the results on analytic functions discussed earlier.

5. Lecture 2015-10-26

5.1. **Properties of the schematic curve.** Recall that E is $\mathbb{F}_q((\pi))$ or a finite extension of $\mathbb{Q}_p$ and F is a perfectoid field over $\mathbb{F}_q$. From this we defined an adic space Y over E with a frobenius φ, and in the last lecture we introduced the adic curve $X^{\mathrm{ad}} = Y/\varphi^{\mathbb{Z}}$ over E and a line bundle $\mathcal{O}(1)$ defined by the automorphy factor (1-cocycle) $\varphi \mapsto \pi^{-1}$.

From this we defined the graded ring

$$\begin{aligned} P &= \oplus_{d\geqslant 0} H^0(X^{\mathrm{ad}}, \mathcal{O}(d)) \\ &= \oplus_{d\geqslant 0} B^{\varphi=\pi^d} \end{aligned}$$

and the schematic curve $X = \mathrm{Proj}(P)$, a scheme over $\mathrm{Spec} E$ (that was discovered in the work of Fargues and Fontaine before X^{ad}).

Theorem 5.1.

(1) *X is a noetherian regular scheme of dimension 1.*

(2) *There is a morphism of ringed spaces $X^{\mathrm{ad}} \to X$ such that*

$$|X^{\mathrm{ad}}|^{\mathrm{cl}} \xrightarrow{\sim} |X|$$

is a bijection (where the lefthand side is $|Y|^{\mathrm{cl}}/\varphi^{\mathbb{Z}}$ and the right hand side is the set of closed points of X), and, if

$$|X^{\mathrm{ad}}|^{\mathrm{cl}} \ni x^{\mathrm{ad}} \mapsto x \in |X|$$

then

$$\widehat{\mathcal{O}_{X,x}} \xrightarrow{\sim} \widehat{\mathcal{O}_{X^{\mathrm{ad}},x^{\mathrm{ad}}}} = B^+_{\mathrm{dR}}(k(x^{\mathrm{ad}})).$$

(3) *If F is algebraically closed then*

$$\begin{aligned} P_1\backslash\{0\}/E^\times &\xrightarrow{\quad\sim\quad} |X| \\ t &\longmapsto \infty_t \ (\textit{where } V^+(t) = \infty_t) \end{aligned}$$

(4) *For $x \in |X|$, set $\deg(x) = [k(x)^\flat : F]$ ($= 1$ if F is algebraically closed). Then, for all $f \in E(X)^\times$, where $E(X)$ is the field of rational functions of X (the stalk of $\mathcal{O}$ at the generic point), $\deg(div(f)) = 0$. Thus, in this sense, "X is complete."*

Remark 8. Combining (2) and (3), for F algebraically closed we get a diagram

$$\begin{tikzcd} {|X^{\mathrm{ad}}|^{\mathrm{cl}} = |Y|^{\mathrm{cl}}/\varphi^{\mathbb{Z}}} \arrow[r, "\sim"] \arrow[dr, dashed, leftrightarrow] & {|X|} \\ & P_1\backslash\{0\}/E^\times \arrow[u, "\wr"'] \end{tikzcd}$$

We can describe the dotted arrow as follows: $P_1\backslash\{0\} = B^{\varphi=\pi}$, and thus given $t \in P_1\backslash\{0\}$, we can look at its divisor $\mathrm{div}(t)$ on Y. Because t satisfies the functional equation $\phi(t) = \pi(t)$, its divisor satisfies the functional equation $\phi^*\mathrm{div}(t) = \mathrm{div}(t)$ (since π is invertible on Y). In this case we find that $\mathrm{div}(t)$ is a single orbit of ϕ, and fixing a point y in that orbit we write that it as $\varphi^{\mathbb{Z}}(y)$. This orbit corresponds to $\infty_t \in |X|$, and in the above diagram we we have

$$\begin{tikzcd} \varphi^{\mathbb{Z}}(y) \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \infty_t \arrow[d, leftrightarrow] \\ & t \end{tikzcd}$$

We now describe the Picard group of X when F is algebraically closed. Given $t \in P_1\backslash\{0\}$ corresponding to $\infty_t \in |X|$, we can form the ring

$$B_e = B[1/t]^{\phi=\mathrm{Id}} = P[1/t]_0$$

which appears in p-adic Hodge theory. It depends on t, even though t is not in the notation.

Remark 9. We observe that when taking φ-invariants one can work with many different rings because of the functional equation. For example, one has $B[1/t]^{\phi=\mathrm{Id}} = B_{\mathrm{cris},\rho}^{\varphi=\mathrm{Id}}$ where $B_{\mathrm{cris},\rho} = B^+_{\mathrm{cris},\rho}[1/t]$. To see this, first we observe that

$$\bigcap_{n\geqslant 0} \varphi^n(B^+_{\mathrm{cris},\rho}) = \mathcal{O}(\mathcal{Y}_{(0,1]}) =: B^+$$

because of the expanding nature of φ and because

$$\mathcal{O}\left(\mathcal{Y}_{(\rho,1]}\right) \subset B^+_{\mathrm{cris},\rho} \subset \mathcal{O}\left(\mathcal{Y}_{(\rho^{\frac{1}{p-1}},1]}\right).$$

Then, solving the functional equation for Newton polygon shows that for all $d \geqslant 0$,

$$(B^+)^{\varphi=\pi^d} = B^{\varphi=\pi^d}$$

i.e. a function on Y satisfying this φ-invariance extends holomorphically to the boundary of the disk, $[\varpi_F] = 0$.

We have

$$X\backslash\{\infty_t\} = \mathrm{Spec}(B_e),$$

and B_e is a PID (here we are using F algebraically closed). To see it is a PID, one uses that

$$(B_e, -\mathrm{ord}_{\infty_t})$$

is an almost Euclidean ring that is not Euclidean (the function $-\mathrm{ord}_{\infty_t}$ gives a degree function or, in the language of Bourbaki, a "stathme"). By almost Euclidean, it is meant that

$$\forall x, y \neq 0, \exists a, b \text{ s.t. } x = ay + b \text{ and } \deg(b) \leqslant \deg(y).$$

It would be Euclidean if the inequality on degrees of b and y were strict. Using that B_e is almost Euclidean and a bit more, one can show that B_e is a PID. Thus, $\mathrm{Pic}^0(X)$ vanishes and $\mathrm{Pic}(X) = \mathbb{Z}$ via degree.

Example 5.1. We give another example of an almost Euclidean ring: Let $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ be the Severi-Brauer variety over $\mathbb{R}$ associated to $\mathbb{H}$, which we can describe as

$$\widetilde{\mathbb{P}^1_{\mathbb{R}}} = \mathbb{P}^1_{\mathbb{C}}/z \sim \frac{1}{\bar{z}}$$

or as a quadric with no real points

$$\widetilde{\mathbb{P}^1_{\mathbb{R}}} = \{[x:y:z] \in \mathbb{P}^2_{\mathbb{R}} \,|\, x^2 + y^2 + z^2 = 0\}.$$

Taking the closed point $\infty \in \widetilde{\mathbb{P}^1_{\mathbb{R}}}$ corresponding to the points $0, \infty \in \mathbb{P}^1_{\mathbb{C}}$ via the first description of $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$, we obtain a ring with a degree function

$$(\Gamma(\widetilde{\mathbb{P}^1_{\mathbb{R}}}\backslash\{\infty\}, \mathcal{O}), -\mathrm{ord}_\infty)$$

which is almost Euclidean but not Euclidean. Note this example was not chosen at random — it is pretty clear now that there is a relation between vector bundles on the curve and Simpson's twistors.

5.2. **Classification of vector bundles on X or X^{ad}.** We assume F is algebraically closed and $\overline{\mathbb{F}_q}$ is the algebraic closure of $\mathbb{F}_q$ in F. For $h \geqslant 1$, we define E_h to be the degree h unramified extension of E.

Warning. If you replace E by E_h in all constructions before then you get

$$\mathbb{A}_{F,E} = \mathbb{A}_{F,E_h}$$

but the Frobenius is changed: $\varphi_{E_h} = \varphi_E^h$.

Now, because $\overline{\mathbb{F}_q} \subset F$, the structure map from Y to $\mathrm{Spa}(E)$ factors as

$$\begin{tikzcd} Y \arrow[d] \arrow[dr] & \\ \mathrm{Spa}(E) & \mathrm{Spa}(\widehat{E^{\mathrm{ur}}}) \arrow[l] \end{tikzcd}$$

Thus we can make an identification

$$Y_E \otimes_E E_h = \bigsqcup_{\mathrm{Gal}(E_h/E)} Y_E,$$

and the action of $\varphi_E \otimes \mathrm{Id}$ on the right hand side is given by cycling the components and mapping via φ_E between them. So,

$$\begin{aligned} X_E^{\mathrm{ad}} \otimes_E E_h &= (\textstyle\coprod_{\mathrm{Gal}(E_h/E)} Y_E)/\text{cycling action of } \varphi_E \otimes \mathrm{Id} \\ &= Y_E/\varphi_E^{h\mathbb{Z}} = Y_E/\varphi_{E_h}^{\mathbb{Z}} = X_{E_h}^{\mathrm{ad}} \end{aligned}$$

and the covering

$$X_{E_h} \to X_E$$

is the unfolding covering

$$Y/\varphi^{h\mathbb{Z}} \to Y/\varphi^{\mathbb{Z}}.$$

Remark 10. As a sidenote, $|X \otimes_E \widehat{E^{\mathrm{ur}}}|$ has the homotopy type of the solenoid $(\mathbb{R} \times \widehat{\mathbb{Z}})/\mathbb{Z}$.

Definition 9. For $\lambda = \frac{d}{h} \in \mathbb{Q}$, $(d,h) = 1$, set

$$\mathcal{O}_{X_E}(\lambda) = \text{ pushfoward of } \mathcal{O}_{X_{E_h}}(d) \text{ via } X_{E_h} \to X_E.$$

Remark 11. Since $\forall f \in E(X)^\times$, $\deg(\mathrm{div} f) = 0$, $\exists \deg : \mathrm{Pic}(X) = \mathrm{Div}(X)/\sim \to \mathbb{Z}$. Furthermore, since F is algebraically closed, we have $\mathrm{Pic}^0(X) = 0$ and

$$\begin{aligned} \mathrm{Pic}(X) &\xrightarrow{\sim} \mathbb{Z} \\ [\mathcal{O}(d)] &\longleftrightarrow d. \end{aligned}$$

That $\mathrm{Pic}^0(X) = 0$ is equivalent to $\mathrm{cl}(B_e) = 0$, which is true since, as stated in the previous section, F algebraically closed implies B_e is a PID.

We have that $\deg(\mathcal{O}(\lambda)) = d$, $\mathrm{rank}(\mathcal{O}(\lambda)) = h$, and

$$\mu(\mathcal{O}(\lambda)) = \lambda$$

where $\mu = \frac{\deg}{\mathrm{rank}}$ (a Harder-Narasimhan slope).

Remark 12. For any F, $\pi_1^{\mathrm{geo}}(X_{F,E}) = \mathrm{Gal}(\overline{F}/F)$. There is a correspondence

$$\{\text{Slope 0 semistable vector bundles on } X_{F,E}\} \leftrightarrow \mathrm{Rep}_{\mathbb{Q}_p}(\mathrm{Gal}(\overline{F}/F))$$

which is a type of p-adic Narasimhan-Seshadri theorem.

$\mathcal{O}(\lambda)$ is a stable vector bundle of slope λ, and we have the following cohomological computations:

$$H^0(\mathcal{O}(\lambda)) = \begin{cases} B^{\phi^h = \pi^d} \text{ if } \lambda \geqslant 0 \\ E \text{ if } \lambda = 0 \\ 0 \text{ if } \lambda < 0 \end{cases}$$

$$H^1(\mathcal{O}(\lambda)) = 0 \text{ if } \lambda \geqslant 0$$

and in particular, $H^1(\mathcal{O}) = 0$ so the curve has "arithmetic genus 0." (Note there is no canonical divisor on the curve so cannot find its geometric genus). In this sense the curve is like $\mathbb{P}^1$, however, contrary to $\mathbb{P}^1$, $H^1(\mathcal{O}(-1)) \neq 0$.

Remark 13. $H^1(\mathcal{O}) = 0$ is equivalent to $(B_e, \deg)$ being almost Euclidean because $H^1(\mathcal{O})$ can be computed by the Cech cohomology of the covering of X by $\mathrm{Spec} B_e$ and $\mathrm{Spec} B_{\mathrm{dR}}^+$ (a formal neighborhood of ∞_t) to give

$$H^1(\mathcal{O}) = B_e \backslash B_{\mathrm{dR}}/B_{\mathrm{dR}}^+$$

which can then be shown to contain a single element using that B_e is almost Euclidean. The fact that $H^1(\mathcal{O}(-1)) \neq 0$ is similarly equivalent to $(B_e, \deg)$ not being Euclidean. In fact, we find $H^1(\mathcal{O}(-1)) \cong k(\infty)$, which is infinite dimensional over E (for experts in p-adic Hodge theory, we note that it is a finite dimensional vector space in the sense of Colmez).

Theorem 5.2.

(1) *Any slope λ semistable vector bundle on X is isomorphic to a finite sum of $\mathcal{O}(\lambda)$.*

(2) *The Harder-Narasimhan filtration of a vector bundle is split.*

(3) *$Bun_X/\sim \simeq \{\lambda_1 \geqslant \ldots \geqslant \lambda_n | n \in \mathbb{N}, \lambda_i \in \mathbb{Q}\}$ via $(\lambda_i) \leftrightarrow \bigoplus \mathcal{O}(\lambda_i)$.*

Here (3) is a direct consequence of (1) and (2), and (2) is a direct consequence of (1) and the homological computation since

$$\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = H^1(\mathcal{O}(\mu) \otimes \mathcal{O}(-\lambda))$$

and the latter is equal to 0 if $\lambda \leqslant \mu$ since $\mathcal{O}(\mu) \otimes \mathcal{O}(-\lambda)$ can be decomposed into a finite sum of $\mathcal{O}(\mu - \lambda)$ (it is semistable of slope $\mu - \lambda$).

Remark 14. Earlier Drinfeld pointed out that for classical algebraic curves, the theory of vector bundles was first described explicitly for the projective line (Grothendieck) and for elliptic curves (Atiyah), each of which is simpler than the general theory for higher genus algebraic curves. We might say that the complexity of the theory of vector bundles on the curve lies somewhere inbetween the theory for $\mathbb{P}^1$ and the theory for an elliptic curve. In any case, this is a very difficult theorem.

Remark 15. We make some historical remarks about Theorem 5.2.

- For $E = \mathbb{F}_q(\pi))$ and X^{ad}, this is due to Hartl-Pink [9]. (They did not introduce X^{ad}, however; instead they classified φ-equivariant vector bundles on $\mathbb{D}^*_F$.)
- Recall the Robba ring $\mathcal{R} = \lim_{\rho \to 0^+} \mathcal{O}(\mathcal{Y}_{(0,\rho]})$, which is a Bezout ring. Kedlaya [10] classified φ-modules over $\mathcal{R}$, and the "expanding" property of Frobenius implies
$$\varphi - \mathrm{Mod}_{\mathcal{R}} = \text{Vector Bundles on } X^{\mathrm{ad}}.$$
So, Kedlaya's theorem is equivalent to the classical theorem on X^{ad}. (In fact, Kedlaya's work allowed for a general characteristic p field F, e.g. $F = \mathbb{F}_q((\pi))$, after replacing $\mathbb{W}$ by a Cohen ring. His applications focused on F a characteristic p local field).
- For X the schematic curve, Fargues and Fontaine did the classification. At the beginning of the work they did not have X^{ad}; they had constructed $|Y|^{\mathrm{cl}}$ as a set plus a bijection $|Y|^{\mathrm{cl}}/\varphi^{\mathbb{Z}} \xrightarrow{\sim} |X|$, but did not know the sheaf property for the structure sheaf. Scholze's work on perfectoid spaces later provided this.

From the adic viewpoint developed here, the proof of Theorem 5.2 depends principally on the following two results:

Theorem 5.3.

(1) (Lafaille/Gross-Hopkins). If
$$0 \to \mathcal{E} \to \mathcal{O}\left(\frac{1}{n}\right) \to i_{\infty *}k(\infty) \to 0$$
is a degree -1 modification of $\mathcal{O}\left(\frac{1}{n}\right)$*, then* $\mathcal{E} \cong \mathcal{O}^n$*.*

(2) (Drinfeld) If
$$0 \to \mathcal{O}^n \to \mathcal{E} \to i_{\infty *}k(\infty) \to 0$$
then $\exists r \in \{1, .., n\}$ *such that* $\mathcal{E} \simeq \mathcal{O}^{n-r} \oplus \mathcal{O}\left(\frac{1}{r}\right)$.

Admitting these two results, the proof can be explained in an hour. The details for the rank 2 case, where all the ideas are already present, are contained in [7]. There is a complete proof in the main article about the curve [5] (which is a very long article). It is also described in the notes from a course given by Fargues at Jussieu in Spring 2014 (available at http://webusers.imj-prg.fr/~laurent.fargues/Notes.html).

6. Lecture 2015-11-05 – Proof of the classification of vector bundles

In this lecture we discuss the proof of the classification theorem for vector bundles on the curve.

We first fix some notation:
Let $E/\mathbb{Q}_p$ be a finite extension and let $F/\mathbb{F}_q$ be algebraically closed. Let $\overline{\mathbb{F}_q}$ the algebraic closure of $\mathbb{F}_q$ in F. Let X be the curve over E. For every $n \geqslant 1$, we take E_n/E to be the degree n unramified extension of E, and then

$$X_n = X \otimes_E E_n = X_{E_n}.$$

As discussed in the previous lecture, we have a tower

$$\begin{array}{c} (X_n)_{n\geqslant 1} \\ \Big\downarrow{\scriptstyle \widehat{\mathbb{Z}}-\text{procyclic}} \\ X = X_1 \end{array}$$

where the maps to the base are the unfolding maps

$$\begin{array}{ccc} X_n^{\mathrm{ad}} & = & Y/\varphi^{n\mathbb{Z}} \\ \big\downarrow & & \big\downarrow \\ X^{\mathrm{ad}} & = & Y/\varphi^{\mathbb{Z}} \end{array}$$

The proof of the theorem proceeds simultaneously for all n, and involves moving between different levels of the tower.

Theorem 6.1. *For all $\mathcal{E} \in \mathrm{Bun}_X, \exists (\lambda_i)_i$ such that $\mathcal{E} \cong \bigoplus \mathcal{O}(\lambda_i)$.*

Here $\mathcal{O}(\lambda_i)$ are the stable vector bundles of slope λ_i introduced before. The proof begins with a series of dévissages to reduce to a statement about modifications of vector bundles which can be understood using p-adic periods.

6.1. Dévissage.

First dévissage: $\mathrm{Ext}^1(\mathcal{O}(\lambda), \mathcal{O}(\mu)) = 0$ if $\lambda \leqslant \mu$, so we are reduced to proving that for all λ and for all $\mathcal{E} \in \mathrm{Bun}_X^{\mathrm{ss},\lambda}$, $\mathcal{E} \cong \mathcal{O}_X(\lambda)^n$ for some $n \in \mathbb{N}$.

Second dévissage: Denoting by π_n the map $X_n \to X$, for $\mathcal{E}$ a vector bundle over X we have

$$\mu(\pi_n^* \mathcal{E}) = n \cdot \mu(\mathcal{E}).$$

By using descent along π_n, we can thus reduce to $\lambda \in \mathbb{Z}$. Then, by twisting and using $\mu(\mathcal{E}(d)) = \mu(\mathcal{E}) + d$ for all $d \in \mathbb{Z}$, we reduce to $\lambda = 0$, i.e.

$$\mathcal{E} \in \mathrm{Bun}_X^{\mathrm{ss},0}.$$

Third dévissage: Given $\mathcal{E} \in \mathrm{Bun}_X^{\mathrm{ss},0}$, which is an abelian category, it suffices to prove that $H^0(X, \mathcal{E}) \neq 0$. This is because if there is a nonzero global section, then we obtain a short exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{E}' \to 0$$

where $\mathcal{E}'$ is also semistable of slope zero. Arguing by induction on rank, we obtain that $\mathcal{E}' \cong \mathcal{O}_X^i$, and using $\mathrm{Ext}^1(\mathcal{O}, \mathcal{O}) = 0$ (because $H^1(\mathcal{O}_X) = 0$), we conclude.

Fourth dévissage:

Theorem 6.2. *The classification theorem is equivalent to the statement that $\forall n \geqslant 1$, if*

$$0 \to \mathcal{O}_X(-\frac{1}{n}) \to \mathcal{E} \to \mathcal{O}_X(1) \to 0$$

where $\mathcal{E}$ is of degree 0 (and rank $n+1$ since $\mathcal{O}_X(-\frac{1}{n})$ is rank n) then $H^0(X, \mathcal{E}) \neq 0$.

Proof. Starting with the classification theorem, $\mathcal{E} = \oplus_i \mathcal{O}_X(\lambda_i)$, and since $\deg \mathcal{E} = 0$, there must be an i such that $\lambda_i \geqslant 0$. Thus, $H^0(X, \mathcal{E}) \neq 0$ (because $H^0(X, \mathcal{O}(\lambda)) \neq 0$ if $\lambda \geqslant 0$).

In the other direction, we proceed by induction on the rank of $\mathcal{E}$ to prove that $H^0(X, \mathcal{E}) \neq 0$: Let $\mathrm{rank}(\mathcal{E}) = n+1$, and let $\mathcal{L} \subset \pi_n^* \mathcal{E}$ be a sub line bundle of maximal degree $d = \deg(\mathcal{L})$. By the computation of the Picard group of X, we have $\mathcal{L} \cong \mathcal{O}_{X_n}(d)$. Because $\pi_n^*\mathcal{E}$ is semistable of slope 0, $d \leqslant 0$. Let $\mathcal{E}' = \pi_n^*\mathcal{E}/\mathcal{L}$, so that we have a short exact sequence

$$0 \to \mathcal{O}_{X_n}(d) \to \pi_n^*\mathcal{E} \to \mathcal{E}' \to 0.$$

We now break up into cases based on the value of d:

- If $d = 0$ then $\mathcal{O}_{X_n} \subset \pi_n^*\mathcal{E}$. Thus, $H^0(X_n, \pi_n^*\mathcal{E}) \neq 0$. This is equal to $H^0(X, \mathcal{E}) \otimes_E E_n$, and thus $H^0(X, \mathcal{E}) \neq 0$.
- If $d \leqslant -2$ then $\deg(\mathcal{E}') \geqslant 0$ (because the degree is additive in short exact sequences and the middle term has degree 0). Thus, by the induction hypothesis applied to $\mathcal{E}'$, $H^0(X_n, \mathcal{E}') \neq 0$. Taking a non-zero section of $\mathcal{E}'$ and composing it with a nontrivial map $\mathcal{O}_{X_n}(d+2) \to \mathcal{O}_{X_n}$ (which exists because $d+2 \leqslant 0$), we obtain a nontrivial map
$$v : \mathcal{O}_{X_n}(d+2) \to \mathcal{E}'.$$
We pullback via v to get a new short exact sequence
$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{X_n}(d) \arrow[r] \arrow[d] & \mathcal{E}'' \arrow[r] \arrow[d, hook] & \mathcal{O}_{X_n}(d+2) \arrow[r] \arrow[d, "v"] & 0 \\ 0 \arrow[r] & \mathcal{O}_{X_n}(d) \arrow[r] & \pi_n^*\mathcal{E} \arrow[r] & \mathcal{E}' \arrow[r] & 0 \end{tikzcd}$$
Twisting the top row, we obtain
$$0 \to \mathcal{O}_{X_n}(-1) \to \mathcal{E}''(-d-1) \to \mathcal{O}_{X_n}(1) \to 0$$
and thus by the statement we have assumed, $H^0(X_n, \mathcal{E}''(-d-1)) \neq 0$. Taking a non-zero section we get a map
$$\mathcal{O}_{X_n}(d+1) \hookrightarrow \mathcal{E}'' \hookrightarrow \pi_n^*\mathcal{E}.$$
But this contradicts the maximality of d among degrees of sub line bundles of $\pi_n^*\mathcal{E}$, thus this case ($d \leqslant -2$) cannot occur.
- If $d = -1$, then our sequence reads
$$0 \to \mathcal{O}_{X_n}(-1) \to \pi_n^*\mathcal{E} \to \mathcal{E}' \to 0.$$
Using that $\pi_n^* = \pi_n^!$ (since π_n is étale), by adjunction we get a nonzero map
$$\nu : \mathcal{O}_X(-\frac{1}{n}) \to \mathcal{E}.$$
We take $\mathrm{Im}\nu \subset \mathcal{E}$ to be the saturation of the image of ν (so that $\mathcal{E}/\mathrm{Im}\nu$ is a vector bundle, i.e. there is no torsion). Because $\mathrm{Im}\nu \subset \mathcal{E}$ and $\mathcal{E}$ is semistable of slope 0, $\mu(\mathrm{Im}\nu) \leqslant 0$. On the other hand, the image of ν is a quotient of $\mathcal{O}_X(-\frac{1}{n})$ and thus has slope $\geqslant -\frac{1}{n}$ (by semistability of $\mathcal{O}_X(-\frac{1}{n})$) and since taking the saturation can only raise degree, we also have $-\frac{1}{n} \leqslant \mu(\mathrm{Im}\nu)$. Thus
$$-\frac{1}{n} \leqslant \mu(\mathrm{Im}\nu) \leqslant 0.$$
Since $\mathrm{Im}\nu$ has rank n, the slope is a rational number with denominator n, and we see that $\mu(\mathrm{Im}\nu)$ must be either $-\frac{1}{n}$ or 0. We consider each possibility:
  (a) $\mu(\mathrm{Im}\nu) = 0$. In this case, $\mathrm{Im}\nu$ is semistable of slope zero and rank n, and so, by the inductive hypothesis $H^0(X, \mathrm{Im}\nu) \neq 0$, and thus $H^0(X, \mathcal{E}) \neq 0$.
  (b) $\mu(\mathrm{Im}\nu) = \mathcal{O}_X(-\frac{1}{n})$. In this case, ν is an isomorphism onto $\mathrm{Im}\nu$ because they have equal rank and degree. Thus, we have an exact sequence
  $$0 \to \mathcal{O}_X(-\frac{1}{n}) \to \mathcal{E} \to \mathcal{E}/\mathrm{Im}\nu \to 0$$
  and, by considering the degrees and ranks of the first two terms, we find that $\mathcal{E}/\mathrm{Im}\nu$ has rank 1 and degree 1, and thus is isomorphic to $\mathcal{O}_X(1)$. So, we can write our sequence as
  $$0 \to \mathcal{O}_X(-\frac{1}{n}) \to \mathcal{E} \to \mathcal{O}_X(1) \to 0$$

and, by the statement we have assumed, we conclude $H^0(X, \mathcal{E}) \neq 0$.

□

Fifth dévissage:

Theorem 6.3. *The classification theorem is equivalent to the combination of the following two statements:*

(1) If

$$0 \to \mathcal{E} \to \mathcal{O}\left(\frac{1}{n}\right) \to \mathcal{F} \to 0$$

with $\mathcal{F}$ a degree 1 torsion sheaf (i.e. $\mathcal{F} = i_{x*}k(x)$ for a closed point x) then $\mathcal{E} \simeq \mathcal{O}_X^n$.

(2) If

$$0 \to \mathcal{O}^n \to \mathcal{E} \to \mathcal{F} \to 0$$

with $\mathcal{F}$ a degree 1 torsion sheaf then $\exists r \in \{1, .., n\}$ such that $\mathcal{E} \simeq \mathcal{O}^{n-r} \oplus \mathcal{O}\left(\frac{1}{r}\right)$.

Proof. The proof that classification implies these two statements is left as an exercise. In the other direction, we give here the proof for rank 2 vector bundles. We will proceed by proving the equivalent statement of the fourth dévissage. So, suppose $\mathcal{E}$ is of degree 0 and there is a short exact sequence

$$0 \to \mathcal{O}_X(-\frac{1}{n}) \to \mathcal{E} \to \mathcal{O}_X(1) \to 0.$$

We will show that $H^0(X, \mathcal{E}) \neq 0$.

We begin by choosing an embedding of $\mathcal{O}_X(-1)$ into $\mathcal{O}_X(1)$ and then forming the pushout diagram to obtain a new bundle $\mathcal{E}'$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-1) \arrow[r] \arrow[d, hook] & \mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{O}_X(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X(1) \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{O}_X(1) \arrow[r] & 0
\end{tikzcd}$$

□

We have that $\mathcal{E}'$ is isomorphic to $\mathcal{O}_X(1)^2$ since $\mathrm{Ext}^1(\mathcal{O}(1), \mathcal{O}(1)) = 0$ (X has arithmetic genus 0). We denote by $\mathcal{F}$ the cokernel of $\mathcal{E} \to \mathcal{E}'$, which, because it is also the cokernel of $\mathcal{O}_X(-1) \to \mathcal{O}_X(1)$, is a degree 2 torsion sheaf. We have an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(1)^2 \longrightarrow \mathcal{F} \longrightarrow 0.$$

We now choose $\mathcal{F}' \subset \mathcal{F}$ of degree 1, and define a new $\mathcal{E}'$ via pullback along the inclusion $\mathcal{F}' \subset \mathcal{F}$:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{E}' \arrow[r] \arrow[d, hook] & \mathcal{F}' \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathcal{E} \arrow[r] & \mathcal{O}_X(1)^2 \arrow[r] & \mathcal{F} \arrow[r] & 0.
\end{tikzcd}$$

Now, $\mathcal{F}/\mathcal{F}'$ is torsion of degree 1, and we have an exact sequence

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}_X(1)^2 \longrightarrow \mathcal{F}/\mathcal{F}' \longrightarrow 0\ .$$

Taking duals and twisting by 1, we obtain

$$0 \longrightarrow \mathcal{O}_X^2 \longrightarrow \mathcal{E}'^\vee(1) \longrightarrow (\mathcal{F}/\mathcal{F}')^\vee \longrightarrow 0$$

where here $(\mathcal{F}/\mathcal{F}')^\vee = \mathrm{Ext}(\mathcal{F}/\mathcal{F}', \mathcal{O}_X)$ is the Pontryagin dual of the torsion sheaf $\mathcal{F}/\mathcal{F}'$, and is again of degree 1. Now, applying hypothesis (2) to $\mathcal{E}'^\vee(1)$ then taking duals and untwisting, we find two possibilities for $\mathcal{E}'$:

$$\mathcal{E}' = \begin{cases} \mathcal{O}_X \oplus \mathcal{O}_X(1) \text{ or} \\ \mathcal{O}_X(1/2) \end{cases}$$

We consider each of these cases separately. For the first, we have an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X \oplus \mathcal{O}_X(1) \longrightarrow \mathcal{F}'' \longrightarrow 0$$

where $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ is torsion of degree 1, and we can consider $\ker(\mathcal{O}_X(1) \to \mathcal{F}'') \subset \mathcal{E}$. Because $\mathcal{F}''$ has degree 1, this must be either $\mathcal{O}_X(1)$ or $\mathcal{O}_X$, and either way it has a non-zero section, so $H^0(X, \mathcal{E}) \neq 0$.

In the second case we have an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(1/2) \longrightarrow \mathcal{F}'' \longrightarrow 0$$

where $\mathcal{F}'' = \mathcal{F}/\mathcal{F}'$ is torsion of degree 1, and applying hypothesis (1) we see see that $\mathcal{E} \cong \mathcal{O}_X^2$ and thus $H^0(X, \mathcal{E}) \neq 0$.

This completes the proof of the fifth dévissage in the rank 2 case. For any rank, we proceed similarly by starting with

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_X(-1/n) \arrow[r] \arrow[d, hook] & \mathcal{E} \arrow[r] \arrow[d, hook] & \mathcal{O}_X(1)^n \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathcal{O}_X(1) \arrow[r] & \mathcal{E}' \arrow[r] & \mathcal{O}_X(1) \arrow[r] & 0
\end{tikzcd}$$

which gives $\mathcal{E}' \cong \mathcal{O}_X(1)^{n+1}$ and thus a modification

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(1)^{n+1} \longrightarrow \mathcal{F} \longrightarrow 0$$

where $\mathcal{F}$ is torsion of degree $n+1$. We then write this as a sequence of modifications until we obtain the desired result, using (1) or (2) at each step.

6.2. **Construction of modifications via p-adic periods.** Let $L = \widehat{E^{\mathrm{un}}}$ be the completion of the maximal unramified extension of E. The residue field of L is $\overline{\mathbb{F}_q}$. We recall F is algebraically closed so $F/\overline{\mathbb{F}_q}/\mathbb{F}_q$. Because of this, the structure morphism $Y \to \mathrm{Spa}E$ factors as

$$\begin{tikzcd}
Y \arrow[r] \arrow[dr] & \mathrm{Spa}E \\
& \mathrm{Spa}L \arrow[u]
\end{tikzcd}$$

We denote by $\varphi - \mathrm{Mod}_L$ the category of isocrystals over L in the sense of Dieudonné-Manin), that is of pairs (D, φ) with D a finite dimensional L vector space and φ a semilinear automorphism of D.

We have the functor $\varphi - \mathrm{Mod}_L \to \mathrm{Bun}_{X^{\mathrm{ad}}}$, $(D, \varphi) \mapsto \mathcal{E}(D, \phi)^{\mathrm{ad}}$, where the geometric realization of $\mathcal{E}(D, \phi)^{\mathrm{ad}}$ is as

$$Y \times_{\varphi^{\mathbb{Z}}} D \to Y/\varphi^{\mathbb{Z}},$$

i.e. $\mathcal{E}(D, \varphi)^{\mathrm{ad}}$ corresponds to the φ-equivariant vector bundle on Y

$$(D \otimes_L \mathcal{O}_Y, \varphi \otimes \varphi).$$

Via GAGA, we have $\mathrm{Bun}_X \simeq \mathrm{Bun}_X^{\mathrm{ad}}$, and so there is a corresponding algebraic vector bundle $\mathcal{E}(D, \varphi)$ over X. We can describe $\mathcal{E}(D, \varphi)$ concretely as

$$\mathcal{E}(D, \varphi) = \widetilde{\bigoplus_{d \geqslant 0} H^0(X^{\mathrm{ad}}, \mathcal{E}(D, \varphi)^{\mathrm{ad}}(d))}$$

or equivalently

$$\mathcal{E}(D, \varphi) = \widetilde{\bigoplus_{d \geqslant 0} (D \otimes_L B)^{\varphi \otimes \varphi = \pi^d}}$$

where $B = \mathcal{O}(Y)$. To make sense of this, recall that we have defined

$$X = \mathrm{Proj} \bigoplus_{d \geqslant 0} H^0(X^{\mathrm{ad}}, \mathcal{O}_X^{\mathrm{ad}}(d)) = \mathrm{Proj} \bigoplus_{d \geqslant 0} B^{\varphi = \pi^d}.$$

If (D, φ) is simple of slope λ, then $\mathcal{E}(D, \varphi) \cong \mathcal{O}_X(-\lambda)$. Here to define the slope of an isocrystal we set $\mathrm{ht}(D, \phi) = \dim_L D$ and $\dim(D, \varphi)$ to be the terminal point of the Newton polygon $\mathrm{Newt}(D, \varphi)$ (note also that $\det(D, \varphi) \cong L(\dim(D, \varphi)))$, and then the slope is the dimension divided by the height. Then $\deg(\mathcal{E}(D, \varphi)) = -\dim(D, \varphi)$ and $\mathrm{rank}(\mathcal{E}(D, \varphi)) = \mathrm{ht}(D, \varphi)$, which accounts for the change of sign in the slope as one passes from isocrystal to vector bundle.

Suppose now we have fixed a point $\infty \in |X|$ with residue field C/E (so $C^\flat = F$). As before, we denote

$$B_{\mathrm{dR}}^+ = B_{\mathrm{dR}}^+(C) = \widehat{\mathcal{O}_{X,\infty}}$$

which is a discrete valuation ring with uniformizer $t \in B^{\varphi=\pi}$ such that $\{\infty\} = V^+(t)$. Then,

$$\widehat{\mathcal{E}(D,\varphi)_\infty} = D \otimes_L B_{\mathrm{dR}}^+,$$

as a B_{dR}^+-module and

$$\Gamma(X \backslash \{\infty\}, \mathcal{E}(D,\varphi)) = (D \otimes_L B[\frac{1}{t}])^{\varphi = \mathrm{Id}}$$

as a $B_e = B[\frac{1}{t}]^{\varphi=\mathrm{Id}}$-module.

We recall that B_e is a PID, and thus $\mathcal{E}(D,\varphi)$ is determined by $D \otimes_L B_{\mathrm{dR}}^+$, $(D \otimes_L B[\frac{1}{t}])^{\varphi=\mathrm{Id}}$, and gluing datum, which is the isomorphism

$$(D \otimes_L B[\frac{1}{t}])^{\varphi=\mathrm{Id}} \otimes_{B_e} B_{\mathrm{dR}} = D \otimes_L B_{\mathrm{dR}}.$$

Thus, as sets,

$$\mathrm{Bun}_X \simeq \{(M, W, u) \mid M \text{ a free } B_e-\text{module}, \ W \text{ a free } B_{\mathrm{dR}}^+-\text{module}, u : M \otimes_{B_e} B_{\mathrm{dR}} \xrightarrow{\sim} W[\frac{1}{t}]\}/\sim$$

and

$$\mathrm{Bun}_X^{\mathrm{rank}=n} \simeq GL_n(B_e) \backslash GL_n(B_{\mathrm{dR}})/GL_n(B_{\mathrm{dR}}^+).$$

We define

$$\mathrm{Gr}(D) = \{B_{\mathrm{dR}}^+ - \text{lattices in } D \otimes_L B_{\mathrm{dR}}\} = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$$

where $G = GL(D)$. If $\Lambda \in \mathrm{Gr}(D)$, then $\mathcal{E}(D,\varphi,\Lambda)$ is defined to be the modification of $\mathcal{E}(D,\varphi)$ at ∞ such that

$$\widehat{\mathcal{E}(D,\varphi,\Lambda)_\infty} = \Lambda.$$

Thus, we have that $\deg(\mathcal{E}(D,\varphi,\Lambda)) = -\dim(D,\varphi) + [\Lambda : D \otimes B_{\mathrm{dR}}^+]$. Because the comparison theorem for algebraic varieties gives us isocrystals and lattices like this (this is one way to think of periods) with the corresponding underlying vector bundles trivial (with basis given by the étale cohomology), enough varieties will give us enough periods to find all modifications and thus prove the theorem, as we will see.

It is interesting to consider this construction of modifications further in two specific cases – the "$U(1)$-equivariant" case and the "minuscule case". We discuss the $U(1)$-equivariant case now, which we will return to again next lecture along with the minuscule case.

The "$U(1)$-equivariant" case:
Suppose $C = \hat{\overline{K}}$ for K/E discretely valued with perfect residue field, and let $G_K = \mathrm{Gal}(\overline{K}/K)$, which we will think of as being a non-archimedean version of $U(1)$. Then G_K acts on X stabilizing ∞: for our t such that $V^+(t) = \{\infty\}$, $\forall \sigma \in G_K$, $t^\sigma = \chi_{\mathrm{cyc}}(\sigma) t$ where χ_{cyc} is the cyclotomic character $G_K \to \mathbb{Z}_p^\times$. In fact, and $G_K \subset \mathrm{Aut}(F)$ is equal to the stabilizer of ∞.

We will explain in a moment (and expand on it in the next lecture) that G_K-equivariant modifications can be encoded in terms of filtrations on the isocrystal. Before doing so, we recall the Archimedean analog:

Let Gr be the usual affine Grassmannian associated to $G/\mathbb{C}$, so that

$$\mathrm{Gr}(\mathbb{C}) = G(\mathbb{C}((t)))/G(\mathbb{C}[[t]]).$$

There is an action of $\mathbb{C}^\times$ on Gr via $t \mapsto \lambda t$. Using this action one can decompose Gr into affine Schubert cells

$$\mathrm{Gr} = \bigcup_{\mu \in X_*(T)_+} Gr_\mu.$$

Each affine Schubert cell has a natural morphism to a corresponding flag variety

$$\mathrm{Gr}_\mu \to G/P_\mu$$

which is an affine fibration. The left hand side has a $\mathbb{C}^\times$ action as described above, and the map is $\mathbb{C}^\times$-equivariant for the trivial action on G/P_μ. It induces an isomorphism

$$\mathrm{Gr}_\mu^{\mathbb{C}^\times} \xrightarrow{\sim} G/P_\mu$$

and, if μ is minuscule, $\mathrm{Gr}_\mu^{\mathbb{C}^\times} = Gr_\mu$. In the next lecture we will refine this picture for the twistor space $\tilde{\mathbb{P}^1_{\mathbb{R}}}$, which does not carry an action of all of $\mathbb{C}^\times$ but only of $U(1)$ inside of $\mathbb{C}^\times$, which is why we call this the $U(1)$-equivariant case.

Going back to the non-archimedean world, the analog of the affine Grassmannian with its action of $\mathbb{C}^\times$ (or $U(1)$) is

$$\mathrm{Gr} = G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+)$$

with the action of G_K (where G is again $GL(D)$).

We note that there is a unique G_K equivariant map $K \hookrightarrow B_{\mathrm{dR}}^+$ that is a section of $\theta : B_{\mathrm{dR}}^+ \to C$. If $K/K_0/E$, where K_0 is the maximal unramified extension, and $D \in \varphi - \mathrm{Mod}_{K_0}$, then

$$Gr_\mu^{G_K} \xrightarrow{\sim} (G/P_\mu)(K).$$

There is a bijection

$$\begin{array}{ccc} \{\text{finite decreasing filtrations of } D \otimes_{K_0} K\} & \xrightarrow{\sim} & \{G_K - \text{invariant lattices in } D \otimes_{K_0} B_{\mathrm{dR}}\} \\ \mathrm{Fil}^\bullet D_K & \longmapsto & \mathrm{Fil}^0(D_K \otimes_K B_{\mathrm{dR}}) = \sum_{i \in \mathbb{Z}} \mathrm{Fil}^i_{D_K} \otimes t^{-i} B_{\mathrm{dR}}^+ \end{array}$$

Now, $\mathcal{E}(D, \varphi)$ is a G_K-equivariant vector bundle on X for $(D, \varphi) \in \varphi - \mathrm{Mod}_{K_0}$, and, by this discussion, to give a G_K-equivariant modification of $\mathcal{E}(D, \varphi)$ is the same as to give a finite decreasing filtration of $D \otimes_{K_0} K$. So, Fontaine's filtered φ-modules are the same as G_K-equivariant modifications at ∞ of $\mathcal{E}(D, \varphi)$.

7. Lecture 2015-11-09 – Proof of the classification of vector bundles (continued)

Recall that we had reduced the proof of classification to two statements about modifications of vector bundles on X:

(1) If
$$0 \to \mathcal{E} \to \mathcal{O}\left(\frac{1}{n}\right) \to \mathcal{F} \to 0$$
with $\mathcal{F}$ a degree 1 torsion sheaf (i.e. $\mathcal{F} = i_{x*}k(x)$ for a closed point x) then $\mathcal{E} \simeq \mathcal{O}_X^n$.

(2) If
$$0 \to \mathcal{O}^n \to \mathcal{E} \to \mathcal{F} \to 0$$
with $\mathcal{F}$ a degree 1 torsion sheaf then $\exists r \in \{1, .., n\}$ such that $\mathcal{E} \simeq \mathcal{O}^{n-r} \oplus \mathcal{O}\left(\frac{1}{r}\right)$.

We now return to our discussion of periods and modifications.

7.1. **Periods give modifications of vector bundles.** As in the last lecture, let $L = \widehat{E^{\mathrm{un}}}$ be the completion of the maximal unramified extension of E. We again consider $\varphi-\mathrm{Mod}_L$ the category of isocrystals over L in the sense of Dieudonné-Manin), that is of pairs (D, φ) with D a finite dimensional L vector space and φ a semilinear automorphism of D. As before, there is a map from isocrystals to vector bundles on X sending (D, φ) to $\mathcal{E}(D, \varphi)$.

We fix a point $\infty \in |X|$ with residue field C/E, and denote
$$B_{\mathrm{dR}}^+ = \widehat{\mathcal{O}_{X,\infty}},$$
and, for $t \in B^{\varphi=\pi}$ such that $V^+(t) = \{\infty\}$,
$$B_{\mathrm{dR}} = \mathrm{Frac}\widehat{\mathcal{O}_{X,\infty}} = B_{\mathrm{dR}}^+[\frac{1}{t}].$$

For any B_{dR}^+-lattice
$$\Lambda \subset D \otimes_L B_{\mathrm{dR}}^+ = \mathcal{E}\widehat{(D, \varphi)}_\infty[1/t],$$
we associate the bundle $\mathcal{E}(D, \varphi, \Lambda)$ which is the modification of $\mathcal{E}(D, \varphi)$ at ∞ with $\mathcal{E}\widehat{(D, \varphi, \Lambda)} = \Lambda$.

The $U(1)$-equivariant case:
Suppose K/E is a discrete valuation ring with perfect residue field, and K_0 is the maximal unramified subextension $(K/K_0/E)$, $C = \widehat{\overline{K}}$, and $G_K = \mathrm{Gal}(\overline{K}/K$. Then G_K acts on X stabilizing ∞ (the action of G_K on B_{dR} is the same as that defined by Fontaine a long time ago). We denote
$$\varphi - \mathrm{Mod}\ \mathrm{Fil}_{K/K_0} = \{(D, \varphi), \mathrm{Fil}^\bullet \,|(D, \varphi) \text{ an isocrystal over } K_0 \text{ and } \mathrm{Fil}^\bullet \text{ a filtration of } D \otimes_{K_0} K\}.$$

Then, as discussed last time, to give an object of $\varphi - \mathrm{Mod}\ \mathrm{Fil}_{K/K_0}$ is the same as to give an isocrystal (D, φ) and a G_K-invariant lattice in $D \otimes_{K_0} B_{\mathrm{dR}}$, where a lattice Λ is obtained as
$$\Lambda = \sum_{i \in \mathbb{Z}} \mathrm{Fil}^i D_K \otimes_K t^{-i} B_{\mathrm{dR}}^+$$
and we use the canonical G_K-invariant section of θ from K to B_{dR}^+ to form the tensor product. In geometric terms, there is an equivalence between $\varphi - \mathrm{Mod}\ \mathrm{Fil}_{K/K_0}$ and G_K equivariant modifications of vector bundles $\mathcal{E}(D, \varphi)$.

Theorem 7.1 (reformulation of comparison results by many people in p-adic Hodge theory, the most general of which is due to Tsuji)**.** *Let $E/\mathbb{Q}_p$, let $\mathfrak{X}/\mathcal{O}_K$ be proper and smooth, and let $i \in \mathbb{N}$. Denoting by (D, φ) the isocrystal*
$$(D, \varphi) = (H^i_{\mathrm{cris}}(\mathfrak{X}_{k_K}/W(k_K))[1/p], \textit{crystalline Frobenius})$$
where k_K is the residue field of K, and by $\mathrm{Fil}^\bullet D_K$ the Hodge filtration on
$$H^i_{\mathrm{cris}} \otimes K = H^i_{\mathrm{dR}}(\mathfrak{X}_K/K),$$
there is an isomorphism of G_K-equivariant vector bundles on X
$$\mathcal{E}(D, \varphi, \mathrm{Fil}^\bullet D_K) = H^i_{\acute{e}t}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_p) \otimes \mathcal{O}_X.$$

A classical crystalline comparison theorem is obtained by taking global sections which gives (when $E = \mathbb{Q}_p$)

$$H^i_{\text{ét}}(\mathfrak{X}_{\overline{K}}, \mathbb{Q}_p) = \mathrm{Fil}^0(D_{K_0} \otimes B_{\text{cris}})^{\varphi = Id}$$

as G_K-modules.

We now discuss the Archimedean analog. We consider the twistor $\mathbb{P}^1$,

$$\widetilde{\mathbb{P}^1_{\mathbb{R}}} = \mathbb{P}^1_{\mathbb{C}} / \left(z \sim \frac{-1}{\overline{z}} \right)$$

and the $\mathbb{Z}/2\mathbb{Z}$ covering $\pi : \mathbb{P}^1_{\mathbb{C}} \to \widetilde{\mathbb{P}^1_{\mathbb{R}}}$ which is the analog of the $\hat{\mathbb{Z}}$ covering $(X_n)_{n \geqslant 1}$ of $X = X_1$ (recall the maps $X_n \to X$ are, in terms of adic spaces, the unfolding covering $Y/\varphi^{n\mathbb{Z}} \to Y/\varphi^{\mathbb{Z}}$). We denote by $\infty \in \widetilde{\mathbb{P}^1_{\mathbb{R}}}$ the point such that

$$\pi^{-1}(\infty) = \{\infty, 0\}$$

and set $t = \frac{1}{z}$ to be the local coordinate at ∞ (for $\mathbb{P}^1_{\mathbb{C}}$). There is an action of $\mathbb{C}^\times$ on $\mathbb{P}^1_{\mathbb{C}}$ given by $\lambda \cdot t = \lambda t$ and the action of $U(1) \subset \mathbb{C}^\times$ descends to an action of $U(1)$ on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$.

For each $\lambda \in \frac{1}{2}\mathbb{Z}$ we can define a vector bundle $\mathcal{O}(\lambda)$ on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$ such that

(1) If $\lambda \in \mathbb{Z}$, $\mathcal{O}(\lambda)$ is a line bundle such that $\pi^*(\mathcal{O}(\lambda)) = \mathcal{O}(2\lambda)$

(2) If $\lambda \in \frac{1}{2}\mathbb{Z} \backslash \mathbb{Z}$, $\mathcal{O}(\lambda)$ is a rank 2 vector bundle given by

$$\mathcal{O}(\lambda) = \pi_* \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(2\lambda).$$

We can classify the vector bundles on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$: any vector bundle is a direct sum of $\mathcal{O}(\lambda)$, $\lambda \in \frac{1}{2}\mathbb{Z}$. Now, if V is a finite dimensional real vector space, then we have a bijection

$$\{\text{Filtrations of } V_{\mathbb{C}}\} \xrightarrow{\sim} \{U(1) - \text{equivariant modifications of } V \otimes_{\mathbb{R}} \mathcal{O}_{\widetilde{\mathbb{P}^1_{\mathbb{R}}}}\}$$

Here we should think of $U(1)$ as playing the same role as G_K in the non-Archimedean case.

Moreover, we have that

$$(V, \mathrm{Fil}^\bullet V_{\mathbb{C}})$$

is a weight w real Hodge structure if and only if

$$\mathcal{E}(V, \mathrm{Fil}^\bullet V_{\mathbb{C}}),$$

the associated vector bundle on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$, is semistable of slope $w/2$, i.e. isomorphic to $\mathcal{O}(\frac{w}{2})^n$ for some $n \in \mathbb{N}$.

Thus, we can reformulate Hodge theory as the following result:

Theorem 7.2. *If $X/\mathbb{C}$ is proper and smooth then for all $i \in \mathbb{N}$ the $U(1)$-equivariant modification of $H^i(X(\mathbb{C}), \mathbb{R}) \otimes \mathcal{O}_{\widetilde{\mathbb{P}^1_{\mathbb{R}}}}$ coming from the Hodge filtration is a slope $i/2$ vector bundle on $\widetilde{\mathbb{P}^1_{\mathbb{R}}}$.*

The minuscule case:
We now discuss the second interesting case. For us, minuscule will mean we are looking at, for a fixed $(D, \varphi) \in \varphi - \mathrm{Mod}_L$, lattices Λ such that

$$D \otimes tB^+_{\text{dR}} \subset \Lambda \subset D \otimes B^+_{\text{dR}}.$$

Giving such a lattice is equivalent to giving a subspace $\mathrm{Fil}D_C$ of D_C via $\Lambda = \theta^{-1}(\mathrm{Fil}D_C)$, where θ here is the map

$$\theta : D \otimes_L B^+_{\text{dR}} \to D_C$$

deduced from

$$\theta : B^+_{\text{dR}} \to C.$$

We will now discuss the comparison theorem for p-divisible groups in this context.

Let $G/\mathcal{O}_C$ be a p-divisible group, for example, $G = A[p^\infty]$ for $A/\mathcal{O}_C$ an abelian scheme. To such a G we can attach an isocrystal (D,φ), the rational covariant Dieudonné module of $G \otimes_{\mathcal{O}_C} k_C$, where k_C is the residue field of C, and a subspace $\mathrm{Fil}D_C \subset D_C$, the Hodge filtration. We have

$$D_C/\mathrm{Fil}D_C = \mathrm{Lie}G[1/p]$$

and

$$\mathrm{Fil}D_C = \omega_{G^D}[1/p]$$

where G^D denotes the Cartier dual of G. The comparison theorem in this case is an exact sequence of sheaves on X

$$0 \to V_p(G) \otimes_{\mathbb{Q}_p} \mathcal{O}_X \to \mathcal{E}(D, p^{-1}\varphi) \to i_{\infty *}\mathrm{Lie}G[1/p] \to 0$$

where $V_p(G)$ is the rational Tate module and the p^{-1} shows up in the central term as a matter of normalization.

In particular, if $\mathrm{Lie}G[1/p]$ is one dimensional, we obtain a degree 1 modification of $\mathcal{E}(D, p^{-1}\varphi)$ that gives a trivial vector bundle. If we can show every modification arises in this way, then we will be able to deduce (1) from the start of the lecture. We now focus on this case.

The Lubin-Tate space:
let $\mathbb{G}$ be the 1-dimensional formal p-divisible group over $\overline{\mathbb{F}_p}$ of height n. Let $\mathfrak{X}$ be the formal deformation space of $\mathbb{G}$. After choosing coordinates, we have

$$\mathfrak{X} \simeq \mathrm{Spf}(W(\overline{\mathbb{F}_p})[[x_1, ..., x_{n-1}]]).$$

There is a Gross-Hopkins period map

$$\pi_{\mathrm{dR}} : \mathfrak{X}_\eta \xrightarrow{\text{étale}} \mathbb{P}^{n-1}$$

where $\mathfrak{X}_\eta$ is the rigid analytic generic fiber, which is isomorphic (via our choice of coordinates) to the open unit ball $\overset{\circ}{\mathbb{B}}{}^{n-1}$.

Remark 16. In the case of a p-divisible group attached to an ordinary elliptic curve, the analogous map is $q \mapsto \tau = \log q$, where q is the Serre-Tate coordinate.

The map π_{dR} is such that if

$$(G,\rho) \in \mathfrak{X}_\eta(C) = \mathfrak{X}(\mathcal{O}_C)$$

where G is a p-divisible group over $\mathcal{O}_C$ and $\rho : \mathbb{G} \otimes_{\overline{\mathbb{F}_p}} k_C \xrightarrow{\sim} G \otimes_{\mathcal{O}_C} k_C$, then

$$\pi_{\mathrm{dR}}(G,\rho) = \rho_*^{-1}(\mathrm{Fil}\mathbb{D}(\mathbb{G}_{k_C})_C) \in \mathbb{P}(\mathbb{D}(\mathbb{G})_{\mathbb{Q}})(C)$$

where ρ_* is the induced isomorphism $\mathbb{D}(\mathbb{G})_{\mathbb{Q}} \otimes W(k_C)_{\mathbb{Q}} \to \mathbb{D}(G \otimes_{\mathcal{O}_C} k_C)_{\mathbb{Q}}$ and $\mathrm{Fil}(\mathbb{D}(G_{k_C})_C)$ is the Hodge filtration, which is a subspace of dimension $n-1$.

Theorem 7.3 (Lafaille, Gross-Hopkins)**.** *π_{dR} is surjective on C points.*

Thus, any codimension 1 filtration of $\mathbb{D}(G)_C$ is the Hodge filtration of a p-divisible group that is a lift of $\mathbb{G}$. Now,

$$\mathcal{E}(\mathbb{D}(\mathbb{G}), p^{-1}\varphi) = \mathcal{O}_X(1/n)$$

and so the exact sequence of vector bundles on X from before gives

$$0 \to V_p(G) \otimes_{\mathbb{Q}_p} \mathcal{O}_X \to \mathcal{O}_X(1/n) \to i_{\infty *}\mathrm{Lie}G[1/p] \to 0$$

and the surjectivity of π_{dR} on C points (for every choice of ∞) implies that any degree 1 modification can be realized in this way, thus every such modification is a trivial vector bundle.

To finish the lecture, we now explain how the theorem of Lafaille/Gross-Hopkins can be given a very concrete formulation.

Let

$$f(T) = \sum_{k \geqslant 0} \frac{T^{p^{kn}}}{p^k} = T + \frac{T^{p^n}}{p} + \frac{T^{p^{2n}}}{p^2} + ... \in \mathbb{Q}_p[[T]]$$

(if $n = 1$ then $\exp(f(T))$ is the Artin-Hasse exponential in $\mathbb{Z}_p[[T]]$). Then

$$F(X,Y) = f^{-1}(f(X) + f(Y)) \in \mathbb{Z}_p[[X,Y]]$$

and thus F is a 1-dimensional formal group law over $\mathbb{Z}_p$ with $f = \log_F$. It has height n.

We define the module of quasilogarithms

$$\mathrm{Qlog}_F = \{g \in T\mathbb{Q}_p[[T]] \mid g(X +_F Y) - g(X) - g(Y) \in \mathbb{Z}_p[[X,Y]][1/p] \text{ (i.e. is bounded)}\}$$

Then, the contravariant Dieudonné module of $F \otimes_{\mathbb{Z}_p} \mathbb{F}_p$ can be described as

$$\mathbb{D} = \mathrm{Qlog}_F/\sim = \mathrm{Qlog}_F/T\mathbb{Z}_p[[T]][1/p].$$

(The standard definition for the contravariant Dieudonné modules of a p-divisible group G is $\mathbb{D}^{\mathrm{Contra}}(G) = \mathrm{Hom}(G, C\mathbb{W})$ where $C\mathbb{W}$ is the Witt covectors, $C\mathbb{W} = \varinjlim_n \mathbb{W}_n$.)

There is a Frobenius φ on $\mathrm{Qlog}_F/\sim$ given by $\varphi(g) = g(T^p)$. Then $\mathrm{Qlog}_F/\sim$ is n-dimensional with basis

$$\mathrm{Qlog}_F/\sim = \langle \log_F, \varphi(\log_F), ..., \varphi^{n-1}(\log_F)\rangle.$$

We have $\varphi^n(\log_F) \sim p\log_F$ so in this basis we can write

$$\varphi = \begin{bmatrix} 0 & 0 & ... & ... & p \\ 1 & 0 & ... & ... & 0 \\ 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 1 & 0 \end{bmatrix}$$

and $\langle \log_F \rangle \subset Qlog_F/\sim$ is the Hodge filtration defined by the lift F of $F \otimes \mathbb{F}_p$. Then Laffaille/Gross-Hopkins says that for any $(\lambda_0, ..., \lambda_{n-1}) \in C^n \backslash (0, ..., 0)$, there exists $\mu \in C^\times$ such that

$$\sum_{i=0}^{n-1} \lambda_i \varphi^i(\log_F) = \mu g$$

with $g \in T \cdot C[[T]]$ the logarithm of a formal group over $\mathcal{O}_C$ (i.e. such that $g^{-1}(g(x) + g(y)) \in \mathcal{O}_C[[X,Y]]$). Thus we obtain a completely concrete formulation in terms of power series.

8. Lecture 2015-11-12

8.1. The space $Y_{S,E}$.

Let E be a complete valued field with finite residue field $\mathbb{F}_q$. If E is a discretely valued, then as before either $E = \mathbb{F}_q((\pi))$ or E is a finite extension of $\mathbb{Q}_p$. We will also keep in mind the case where E is perfectoid, in particular the cases $E = \mathbb{F}_q((\pi^{1/p^\infty}))$ and $E = \widehat{E'_\infty}$ where E' is a finite extension of $\mathbb{Q}_p$ and E'_∞/E' is generated by the torsion points of a Lubin-Tate formal group over $\mathcal{O}'_E$.

Let ϖ_E be a pseudo-uniformizer of E such that $\varpi_E | p$ (we take ϖ_E to be a uniformizer π if E is discretely valued).

Let R be a perfectoid $\mathbb{F}_q$ algebra. In characteristic p, a perfectoid algebra is a perfect algebra complete with respect to a power-multiplicative norm $||\cdot||$ and such that there exists a topologically nilpotent unit, i.e. such that $R^\times \cap R^{00} \neq \varnothing$ where R^{00} is the set of topologically nilpotent elements, $R^{00} \subset R^0 = \{x \in R \mid ||x|| \leqslant 1\}$.

Given such an E and R, there is a unique ϖ_E-adic flat $\mathcal{O}_E$-algebra $\mathbb{A}$ such that

$$\mathbb{A}/\varpi_E = R^0 \otimes_{\mathbb{F}_q} \mathcal{O}_E/\varpi_E.$$

Example 8.1. We describe $\mathbb{A}$ explicitly for the E above:

- If $E = \mathbb{F}_q((\pi))$ then $\mathbb{A} = R^0[[\pi]]$.
- If $[E : \mathbb{Q}_p] < \infty$ then
$$\mathbb{A} = \mathbb{W}_{\mathcal{O}_E}(R^0) = \{\sum_{n \geqslant 0} [x_n]\pi^n \mid x_n \in R^0\}.$$
- If $E = \mathbb{F}_q((\pi^{1/p^\infty}))$ then $\mathbb{A} = R^0[[\pi^{1/p^\infty}]]$.
- If $E = \widehat{E'_\infty}$ then
$$\mathbb{A} = W_{\mathcal{O}_{E'}}(R_0) \hat{\otimes}_{\mathcal{O}_{E'}} \widehat{\mathcal{O}_{E'_\infty}}$$

We equip $\mathbb{A}$ with the $(\varpi_E, [\varpi_R])$-adic topology, where $\varpi_R \in R^{00} \cap R^\times$ is a topologically nilpotent unit. We set

$$Y_{R,E} = \mathrm{Spa}(\mathbb{A}) \backslash V(\varpi_E[\varpi_R])$$

This construction for affinoid perfectoid spaces over $\mathbb{F}_q$ glues to give a functor

$$\begin{aligned} \mathrm{Perf}_{\mathbb{F}_q} &\longrightarrow \text{Analytic adic spaces over } E \\ S &\longmapsto Y_{S,E} \end{aligned}$$

such that

$$Y_{\mathrm{Spa}(R),E} = Y_{R,E}$$

for R as above. The construction satisfies:

(1) If $\mathrm{char} E = p$ then $Y_{S,E} = S \times_{\mathrm{Spa}\mathbb{F}_q} \mathrm{Spa}E$.

(2) If $E = \mathbb{F}_q((\pi))$,
$$Y_{S,E} = \mathbb{D}^*_S = \{0 < |\pi| < 1\} \subset \mathbb{A}^1_S,$$
the punctured open disk over S, and if $E = \mathbb{F}_q((\pi^{1/p^\infty}))$ then
$$Y_{S,E} = \mathbb{D}^{*1/p^\infty}_S = \varprojlim_{\mathrm{Frob}} \mathbb{D}^*_S,$$
the perfectoid punctured open disk over S.

(3) If E'/E then $Y_{S,E'} = Y_{S,E} \hat{\otimes} E'$.

(4) If E is perfectoid, then $Y_{S,E}$ is perfectoid, and $Y^\flat_{S,E} = Y_{S,E^\flat}$.

We define the relative curve by $X_{S,E} = Y_{S,E}/\varphi^{\mathbb{Z}}$ where $\varphi([x]) = [x^q]$.

Example 8.2. If $\mathrm{char}E = p$ then
$$X_{S,E} = (S \times \mathrm{Spa}(E))/\mathrm{Frob}_S^{\mathbb{Z}} \times \mathrm{Id}.$$
If $E = \mathbb{F}_q((\pi))$, $X_{S,E} = \mathbb{D}_S^*/\varphi^{\mathbb{Z}}$.

The same properties as above for the relative Y hold for the relative curve:

- $X_{S,E}\hat{\otimes}_E E' = X_{S,E'}$
- E perfectoid implies $X_{S,E}$ is perfectoid and $X_{S,E}^\flat = X_{S,E^\flat}$.

8.2. **Diamonds.**

8.2.1. $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$. Consider $\mathrm{Perf}_{\mathbb{F}_q}$ equipped with the pro-étale topology. We define $\mathrm{Spa}(\mathbb{Q}_p)^\diamond \in \widetilde{\mathrm{Perf}_{\mathbb{F}_q}}$ (where this denotes the category of sheaves for the pro-étale topology) by
$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond(S) = \{(S^\#, \iota) \mid S^\# \text{ a perfectoid space over } \mathbb{Q}_p \text{ and } \iota : S \xrightarrow{\sim} (S^\#)^\flat\}/\sim$$
i.e. $\mathrm{Spa}(\mathbb{Q}_p)^\diamond(S)$ is the set of isomorphisms classes of untilts of S by perfectoid spaces over $\mathbb{Q}_p$. (Note that a pair (S, ι) as above has no automorphisms, so it is ok to define this as a set instead of a groupoid).

This is a sheaf thanks to Scholze's purity theorem which says that for any untilt $S^\#$ of S, there is an equivalence
$$(-)^\flat : (\mathrm{Perf}/S^\#)_{\text{proét}} \xrightarrow{\sim} (\mathrm{Perf}/S)_{\text{proét}}.$$

For $\mathcal{F} \in \widetilde{\mathrm{Perf}_{\mathbb{Q}_p}}$, define $\mathcal{F}^\diamond \in \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}$ by
$$\mathcal{F}^\diamond(S) = \{(S^\#, s) \mid S^\# \in \mathrm{Spa}(\mathbb{Q}_p)^\diamond(S) \text{ and } s \in \mathcal{F}(S)\}.$$

Example 8.3. If $X \in \mathrm{Perf}_{\mathbb{Q}_p}$ then, viewing X as its functor of points on $\mathrm{Perf}_{\mathbb{Q}_p}$, $X^\diamond$ is represented by $X^\flat$.

The same argument as for $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ will show $F^\diamond$ is a sheaf.

This defines an equivalence of topoi
$$(-)^\diamond : \widetilde{\mathrm{Perf}_{\mathbb{Q}_p}} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}/\mathrm{Spa}(\mathbb{Q}_p)^\diamond$$
where the right hand side is the localized topos of objects equipped with a morphism to $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$.

Let X be a rigid analytic space over $\mathbb{Q}_p$. Then X has a pro-étale cover.

Example 8.4. $\mathrm{Spa}(\mathbb{C}_p\langle T_1^{\pm 1/p^\infty}, ..., T_d^{\pm 1/p^\infty}\rangle) \to \mathrm{Spa}(\mathbb{C}_p\langle T_1^{\pm 1}, ..., T_d^{\pm 1}\rangle)$ is a pro-étale cover of the d-dimensional annulus $|T_1| = ... = |T_d| = 1$ by a perfectoid space. This can be used to deduce the existence of a pro-etale cover for any smooth rigid analytic variety.

Moreover, the functor
$$\text{Normal rigid analytic spaces}/\mathbb{Q}_p \to \widetilde{\mathrm{Perf}_{\mathbb{Q}_p}}$$
is fully faithful. The proof uses the fact that rigid analytic varieties have pro-étale covers, plus something else that uses the normal hypothesis.

Thus, we have an embedding
$$\begin{array}{ccc} \text{Normal rigid analytic spaces}/\mathbb{Q}_p & \hookrightarrow & \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}/\mathrm{Spa}(\mathbb{Q}_p)^\diamond \\ X & \longmapsto & X^\diamond \end{array}$$

If $Y \to X$ is a pro-étale cover with Y perfectoid over $\mathbb{Q}_p$, then
$$X^\diamond = Y^\diamond/R^\flat$$
where $R = Y \times_X Y \subset Y \times Y$ is a perfectoid equivalence relation that can be tilted. Thus, $X^\diamond$ is an alebraic space for $\mathrm{Perf}_{\mathbb{F}_p}$ in the pro-étale topology.

For the pro-étale topology and some examples, one can consult Scholze's Berkeley notes, which use a different site than before that however still gives the same topos.

Definition 10. A diamond is an algebraic space for the pro-étale topology on $\mathrm{Perf}_{\mathbb{F}_p}$.

We now give another description of $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ in this light: Let $\mathbb{Q}_p^{\mathrm{cyc}} = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$. Then $\mathbb{Z}_p^\times$ acts on $\mathbb{Q}_p^{\mathrm{cyc}}$ as the continuous automorphisms (coming from the Galois action on $\mathbb{Q}_p(\zeta_{p^\infty})$) and it is a fact that

$$\mathbb{Q}_p^{\mathrm{cyc},\flat} = \mathbb{F}_p((T^{1/p^\infty})).$$

Thus we obtain an action of $\mathbb{Z}_p^\times$ on $\mathbb{F}_p((T^{1/p^\infty}))$ which is given by $a \in \mathbb{Z}_p^\times$ acting by

$$a : T \mapsto (1+T)^a - 1 = \sum_{k \geqslant 1} \binom{a}{k} T^k$$

Then,

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\mathbb{F}_p((T^{1/p^\infty}))/\mathbb{Z}_p^\times.$$

Another example of a diamond.

Definition 11. An adic space X/E is pre-perfectoid if for all E'/E with E' perfectoid, $X \hat{\otimes}_E E'$ is perfectoid.

Example 8.5. $X = \mathrm{Spa}(\mathbb{Q}_p\langle\langle T^{1/p^\infty}\rangle\rangle)$ is preperfectoid.

If X is pre-perfectoid, then $X^\diamond$ is a diamond:

$$X^\diamond = (X \hat{\otimes}_E \hat{\overline{E}})^\flat / \mathrm{Gal}(\overline{E}/E).$$

Thus for $S \in \mathrm{Perf}_{\mathbb{F}_q}$, $Y_{S,E}$ and $X_{S,E}$ are pre-perfectoid and

$$Y_{S,E}^\diamond = S \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(E)^\diamond,$$

$$X_{S,E}^\diamond = S \times_{\mathrm{Spa}(\mathbb{F}_q)} \mathrm{Spa}(E)^\diamond / \mathrm{Frob}_S^{\mathbb{Z}} \times \mathrm{Id}.$$

Example 8.6. Take $[E : \mathbb{Q}_p] < \infty$ and E_∞/E generated by the torsion points of a Lubin-Tate formal group over $\mathcal{O}_E$ so that $\mathrm{Gal}(E_\infty/E) \cong \mathcal{O}_E^\times$. Then

$$Y_{S,E}^\diamond = (Y_{S,\widehat{E_\infty}})^\flat / \mathcal{O}_E^\times = Y_{S,\widehat{E_\infty}^\flat} / \mathcal{O}_E^\times = \mathbb{D}_S^{*,1/p^\infty} / \mathcal{O}_E^\times.$$

When $E = \mathbb{Q}_p$ and T is the variable on $\mathbb{D}_S^*$ the action of $a \in \mathbb{Z}_p^\times$ is described as before by

$$a : T \mapsto (1+T)^a - 1.$$

8.3. **Untilts as degree 1 Cartier divisors on X_S.** Let us work in the case $[E : \mathbb{Q}_p] < \infty$, let R be a perfectoid $\mathbb{F}_q$-algebra, and fix an untilt $R^\#$ of R.

Fontaine's map

$$\begin{aligned} \theta : \mathbb{W}_{\mathcal{O}_E}(R^0) &\twoheadrightarrow R^{\#,0} \\ \textstyle\sum_{n \geqslant 0} [x_n]\pi^n &\longmapsto \textstyle\sum_{n \geqslant 0} x_n^\# \pi^n \end{aligned}$$

is surjective, where we define $x^\#$ to be y^0 where our untilting isomorphism $R \cong R^{\#,\flat}$ is written $x \mapsto (y^{(n)})$ (recall the latter is a system such that $(y^{(n+1)})^p = y^{(n)}$).

Furthermore, $\ker\theta$ is generated by a degree 1 primitive element

$$f = \sum_{n \geqslant 0} [x_n]\pi^n \in \mathbb{W}_{\mathcal{O}_E}(R_0)$$

(by degree 1 primitive we mean $x_0 \in R^\times \cap R^{00}$ and $x_1 \in R^\times$). This defines a bijection

$$\{\text{Untilts of R}\} \xrightarrow{\sim} \{\text{degree 1 primitive elements}\}/\mathbb{W}_{\mathcal{O}_E}(R^0)^\times.$$

Let $S \in \mathrm{Perf}_{\mathbb{F}_q}$, and let $S^\sharp$ be a untilt of S given by a morphism $S \to \mathrm{Spa}(E)^\diamond$. This gives a section of the map

$$Y_{S,E}^\diamond = S \times \mathrm{Spa}(E)^\diamond \to S$$

and thus we have a commuting diagram

$$\begin{tikzcd} S^{\#\diamond} \arrow[rr] \arrow[dr] & & Y_{S,E}^{\diamond} \arrow[dl] \\ & \mathrm{Spa}(E)^{\diamond} & \end{tikzcd}$$

and by full faithfulness, we obtain a map

$$S^{\#} \to Y_{S,E}$$

which is a closed immersion given locally by $V(f)$ as above.

Mapping down to X we obtain a closed immersion

$$S^{\#} \to X_{S,E}$$

and from this a map from untilts to degree 1 Cartier divisors on $X_{S,E}$. We knew this before Scholze when S was the spectrum of a field (the Fontaine-Fargues curve).

When $S = \mathrm{Spa}(R)$, $S^{\#} = \mathrm{Spa}(R^{\#})$, the formal completion of $X_{S,E}$ along $S^{\#}$ can be described as

$$(X_{S,E}/S^{\#})^{\wedge} = \mathrm{Spf}(B_{\mathrm{dR}}^{+}R^{\#}).$$

9. Lecture 2015-11-23

9.1. **Extension of the classification theorem.** Let $E/\mathbb{Q}_p$ be a finite extension with residue field $\mathbb{F}_q$, let $F/\mathbb{F}_q$ algebraically closed, and let G be a reductive group over E. Let $\overline{\mathbb{F}_q}$ be the algebraic closure of $\mathbb{F}_q$ in F, and let $L = \widehat{E^{\mathrm{un}}}$ with residue field $\overline{\mathbb{F}_q}$. We denote by σ the Frobenius on L.

Let $B(G) = G(L)/\sigma -$conjugacy as studied by Kottwitz. The set $B(G)$ classifies G-isocrystals. From a Tannakian viewpoint, to $b \in G(L)$ representing a σ-conjugacy class in $B(G)$ we obtain a functor

$$\begin{aligned} \operatorname{Rep} G &\longrightarrow \varphi - \mathrm{Mod}_L \\ (V,\rho) &\longmapsto (V \otimes_E L, \rho(b)\sigma) \end{aligned}.$$

Let X/E be the schematical curve (recall this depends on the choice of both E and F), then by composition we obtain a functor

$$\begin{tikzcd} \mathrm{Rep}G \arrow[r] \arrow[rr, bend right] & \varphi - \mathrm{Mod}_L \arrow[r, "\mathcal{E}(-)"] & \mathrm{Bun}_X \end{tikzcd}$$

Here we view a G-bundle in the Tannakian sense as a vector bundle with G-structure. It is equivalent to give a G-torsor on X, and given $b \in G(L)$ we denote the corresponding G-torsor by $\mathcal{E}_b$.

Theorem 9.1.

$$\begin{aligned} B(G) &\xrightarrow{\sim} H^1_{\text{ét}}(X,G) \\ b &\longmapsto \mathcal{E}_b \end{aligned}$$

For G a torus this is an isomorphism of abelian groups; in general it is a bijection of pointed sets. The classification theorem for vector bundles is equivalent to this statement for GL_n, and it is used to proved the general case.

This isomorphism satisfies some further nice properties:

- The unit root (slope 0) G-isocrystals are classified by $H^1(E,G) \subset B(G)$. Viewing these as G-bundles on X, this inclusion is given by pullback of vector bundles via $X \to \mathrm{Spec}E$.
- There is a dictionary between Kottwitz's theory and the generalization of Harder-Narasimhan to G-bundles. For example, we have

$$b \in B(G) \text{ is basic (the slope morphism } \nu_b \text{ is central)} \leftrightarrow \mathcal{E}_b \text{ is semistable.}$$

- Denote by J_b the σ-centralizer of b, which is an algebraic group over E. Then b is basic if and only if J_b is an inner form of G. Given $[b] \in H^1(E,G) \subset B(G)$, J_b is a pure inner form of G in the sense of Vogan (i.e. coming from the map $H^1(E,G) \to H^1(E,G_{\mathrm{ad}})$). Note that, for example, for GL_n there is only one pure inner form by Hilbert's theorem 90. In general the groups J_b for b basic are called extended pure inner forms of G, and Kottwitz formulates local Langlands for extended pure inner forms in the quasi-split case.

  Now, if b is basic, $J_b \times X$ is a pure inner form of $G \times X$ (obtained by twisting by $[\mathcal{E}_b] \in H^1(X,G)$), so all extended pure inner forms become pure inner forms after pullback to the curve. For example, if $D^\times_{1/n}$ is the multiplicative group of the division algebra of invariant $1/n$ over E, then $D^\times_{1/n} \times X = \underline{\mathrm{Aut}}(\mathcal{O}_X(1/n))$.

9.2. **The stack** Bun_G**.** Recall that for $S \in \mathrm{Perf}_{\mathbb{F}_q}$ we have defined X_S, an adic space over E (the relative curve). Then Bun_G is a stack on $\mathrm{Perf}_{\mathbb{F}_q}$ for the pro-étale topology defined by

$$\mathrm{Bun}_G(S) = \{G - \text{bundles on } X_S\}$$

where a G-bundle on X_S can be taken in the Tannakian sense as a tensor functor $\mathrm{Rep}G \to \mathrm{Bun}_{X_S}$ and the right hand side is a groupoid.

The classification result for G-bundles implies $B(G) = |\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q}|$ where the right hand side can be defined as

$$|\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q}| = \left(\bigsqcup_{F/\overline{\mathbb{F}_q} \text{ perfectoid}} \mathrm{Bun}_G(F) \right) / \sim$$

where the equivalence relation is given by isomorphism after pullback to a common perfectoid field extension.

We equip $|\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q}|$ with the topology whose open subsets are $|\mathcal{U}|$ where $\mathcal{U} \subset \mathrm{Bun}_G \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ is an open substack.

We now discuss the connected components of Bun_G: Kottwitz defines a map $\kappa : B(G) \to (\pi_1(G)_\Gamma)$ where $\Gamma = \mathrm{Gal}(\overline{E}/E)$ and $\pi_1(G)$ is Borovoi's π_1.

Theorem 9.2. *κ is locally constant.*

We can think of κ as a G-equivariant first Chern class c_1^G.

Example 9.1. For $G = GL_n$, κ is the degree of a vector bundle.

Conjecture 9.1. *The fibers of κ are connected.*

If the conjecture holds, then combined with the theorem that κ is locally constant this implies κ cuts out the connected components.

We now discuss the Harder-Narasimhan stratification when G is quasi-split. Consider a Borel B containing a maximal torus T and a maximal split torus $A \subset T$. The map

$$\mathrm{HN} : |\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q}| \to X_*(A)^+_{\mathbb{Q}}$$

is semicontinuous. The semistable locus (i.e. the basic locus in Kottwitz' language) is open.

Theorem 9.3 (Kottwitz)**.** $\kappa : B(G)_{basic} \xrightarrow{\sim} \pi_1(G)_\Gamma$

This is variant of the Dieudonné-Manin classification for GL_n, which gives that an isoclinic isocrystal is determined by its right endpoint.

For us, this implies that for each connected component $\kappa^{-1}(\alpha)$, $\alpha \in \pi_1(G)_\Gamma$,

$$|(\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q})^{\alpha,\mathrm{ss}}| = \{\mathrm{pt}\}$$

where the superscript α means we look at $\kappa^{-1}(\alpha)$ and the superscript ss stands for the semistable/open locus.

If $[b] \in B(G)_{\mathrm{basic}}$, $\kappa(b) = \alpha$ then there exists a natural morphism

$$\mathrm{Spa}(\overline{\mathbb{F}_q}) \xrightarrow{x_b} \mathrm{Bun}_G \otimes \overline{\mathbb{F}_q}$$

defined by the construction for any perfectoid $S/\overline{\mathbb{F}_q}$ of $\mathcal{E}_b$.

Theorem 9.4. *The map x_b induces an isomorphism*

$$\left[\mathrm{Spa}(\overline{\mathbb{F}_q})/J_b(E)\right] \xrightarrow{\sim} (\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q})^{\alpha,\mathrm{ss}}$$

The space on the left is the classifying stack of pro-étale $J_b(E)$ torsors.

This is a difficult theorem. There is a big difference here between the study of G-bundles on the curve and G-bundles on the projective line: here automorphisms are given by the *topological* group $J_b(E)$ instead of by an algebraic group.

Now, a $\overline{\mathbb{Q}_l}$-local system on $[\mathrm{Spa}(\overline{\mathbb{F}_q}/J_b(E)]$, suitably interpreted, is a smooth admissible l-adic representation of $J_b(E)$.

We now discuss the Hecke correspondences: for $\mu \in X_*(A)^+$, we have a diagram

$$\begin{tikzcd} & \mathrm{Hecke}_{\leqslant \mu} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Spa} E)^\diamond \end{tikzcd} .$$

Here

$$\mathrm{Hecke}_\mu(S) = \{(\mathcal{E}_1, \mathcal{E}_2, u, S^\#)\}$$

where $\mathcal{E}_1$ and $\mathcal{E}_2$ are in $\mathrm{Bun}_G(S)$, $S^\#$ is an untilt of S (so an element of $\mathrm{Spa}(E)^\diamond(S)$) which can be viewed as a "degree 1 Cartier divisor" $S^\# \hookrightarrow X_S$, and $u : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$ is a modification along $S^\# \to X_S$ bounded by μ. The maps giving the correspondence are

$$\overleftarrow{h}((\mathcal{E}_1, \mathcal{E}_2, u, S^\#)) = \mathcal{E}_2$$

$$\overrightarrow{h}((\mathcal{E}_1, \mathcal{E}_2, u, S^\#)) = (\mathcal{E}_1, S^\#).$$

To understand properly Bun_G and the modification in the Hecke correspondence we must introduce the B_{dR}-affine grassmannian. One of the main points of Scholze's Berkeley lectures was to prove this space is a diamond for $G = GL_n$.

We define the B_{dR}-affine Grassmannian $\mathrm{Gr} \to \mathrm{Spa}(E)^\diamond$ as a sheaf on $\mathrm{Perf}_{\mathbb{F}_q}$ by

$$\mathrm{Gr}(R^\#) = \{G\text{-torsors } T \text{ on } \mathrm{Spec}(B^+_{\mathrm{dR}}(R^\#)) \text{ with a trivialization of } T \otimes_{B^+_{\mathrm{dR}}(R^\#)} B_{\mathrm{dR}}(R^\#)\}/\sim$$

where $R^\#/E$ is an untilt of a perfectoid affinoid $\mathbb{F}_q$-algebra R (recall that to give an R point of $\mathrm{Spa}(E)^\diamond$ is to give such an untilt.)

Alternatively, Gr is the sheaf associated to

$$R^\# \mapsto G(B_{\mathrm{dR}}(R^\#))/G(B^+_{\mathrm{dR}}(R^\#)).$$

Inside of Gr we have the closed Schubert cells $\mathrm{Gr}^{\leqslant\mu}$, and $\mathrm{Gr} = \bigcup_\mu Gr^{\leqslant\mu}$.

Theorem 9.5 (Scholze)**.** *For all $\mu \in X_*(A)^+$, $\mathrm{Gr}^{\leqslant\mu}$ is a diamond.*

Note that for μ minuscule, $\mathrm{Gr}^{\leqslant\mu} = \mathrm{Gr}^\mu = \mathcal{F}l_\mu^\diamond$ where $\mathcal{F}l_\mu = G/P_\mu$ is the associated flag variety.

Interjection. ***Drinfeld****: In the equal characteristic case, local Langlands in general is still open, and morally the unequal characteristics case is part of the equal characteristic case. What does the program do in this case, where many complications might disappear?*

Fargues*: In the equal characteristic case one studies bundles on $\mathbb{D}_S^{*,\mathrm{perf}}/\varphi^{\mathbb{Z}}$ and thus the pro-étale site and perfectoid spaces are still necessary, but you can remove diamonds from the picture. One reason to prefer characteristic zero is that in characteristic p there is a technical obstruction to deducing a Harder-Narasimhan theory for arbitrary G from a Harder-Narasimhan theory for GL_n.*

One should be able to define a perverse sheaf IC_μ on $\mathrm{Gr}^{\leqslant\mu}$ such that for μ minuscule, $\mathrm{IC}_\mu = \overline{\mathbb{Q}_l}(\langle\rho,\mu\rangle)[\langle 2\rho,\mu\rangle]$, and one expects to have a geometric Satake correspondence in this setting between perverse sheaves on Gr and representations of ${}^LG$ such that

$$\mathrm{IC}_\mu \leftrightarrow r_\mu$$

where $r_\mu|_{\widehat{G}}$ has highest weight μ.

Finally, because

$$\overrightarrow{h} : \mathrm{Hecke}_{\leqslant}\mu \to \mathrm{Bun}_G \times (\mathrm{Spa}E)^\diamond$$

is a locally trivial fibration with fiber $\mathrm{Gr}^{\leqslant\mu}$, one can define IC_μ on $\mathrm{Hecke}_{\leqslant}\mu$.

9.3. **The conjecture.** Let G/E be quasi-split, and let ${}^LG$ be the l-adic Langlands dual. Let $\Gamma = \mathrm{Gal}(\overline{E}/E)$.

If $\varphi : W_E \to^L G$ is a Langlands parameter, we define

$$S_\varphi = \{g \in \widehat{G} | g\varphi g^{-1} = \varphi\}.$$

Then $Z(\widehat{G})^\Gamma \subset S_\varphi$, and we say φ is discrete if $S_\varphi/Z(\widehat{G})^\Gamma$ is finite (for $G = GL_n$, this means indecomposable).

Conjecture 9.2. *Let φ be a discrete Langlands parameter, and fix a Whittaker datum (this is nothing if $G = GL_n$). Then, there exists a Weil perverse l-adic sheaf $\mathcal{F}_l$ on $\mathrm{Bun}_G \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}$ equipped with an action of S_φ such that*

(1) For all $\alpha \in \pi_1(G)_\Gamma = X^*(Z(\widehat{G})^\Gamma)$, the action of S_φ on $\mathcal{F}_\varphi|_{\mathrm{Bun}_G \otimes_{\mathbb{F}_q} \overline{\mathbb{F}_q}}$ restricted to $Z(\widehat{G})^\Gamma$ is given by α.

(2) *If φ is cuspidal ($\varphi(I_E)$ is finite; this should correspond to supercuspidal L-packets), and*

$$j : (\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q})^{ss} \hookrightarrow \mathrm{Bun}_G \otimes \overline{\mathbb{F}_q},$$

then $j^*\mathcal{F}_\varphi$ is a local system and

$$\mathcal{F}_\varphi = j_! j^* \mathcal{F}_\varphi.$$

(3) *For all $[b] \in B(G)_{basic}$, recall the map*

$$\begin{tikzcd} \left[\mathrm{Spa}(\overline{\mathbb{F}_q})/J_b(E)\right] \arrow[r, "\sim"] \arrow[dr, "x_b"] & (\mathrm{Bun}_G \otimes \overline{\mathbb{F}_q})^{\kappa(b),\mathrm{ss}} \arrow[d] \\ & \mathrm{Bun}_G \otimes \overline{\mathbb{F}_q} \end{tikzcd}$$

Then writing

$$x_b^* \mathcal{F}_\varphi = \bigoplus_{\rho \in \widehat{S_\varphi} \ s.t.\ \rho|_{Z(\widehat{G})} = \kappa(b)} (x_b^* \mathcal{F}_\varphi)_\rho$$

(note that $x_b^*\mathcal{F}_\varphi$ is a representation of $J_b(E) \times S_\varphi$),

$$\{(x_b^* \mathcal{F}_\varphi)_\rho\}_\rho$$

is an L-packet for the local Langlands correspondence for J_b (which is an inner form of G.)
Furthermore, for $b = 1$ so that $J_b = G$, $(x_1^*\mathcal{F})_{trivial}$ is the generic representation associated to the choice of Whittaker datum.

(4) *Hecke Property: For $\mu \in X_*(A)^+$ we have the Hecke correspondence*

$$\begin{tikzcd} & \mathrm{Hecke}_{\leqslant \mu} \otimes \overline{\mathbb{F}_q} \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] & \\ \mathrm{Bun}_G \otimes \overline{\mathbb{F}_q} & & \mathrm{Bun}_G \times (\mathrm{Spa}\widehat{E^{\mathrm{un}}})^\diamond \end{tikzcd}$$

where we use that $\mathrm{Spa}(\widehat{E^{\mathrm{un}}})^\diamond = \mathrm{Spa}(E)^\diamond \otimes \overline{\mathbb{F}_q}$. We also have an l-adic representation $r_\mu \circ \varphi$ of W_E, where r_μ is the representation of ${}^LG$ attached to IC_μ via geometric Satake in this setting as described above, which can be thought of as an l-adic Weil local system on $\mathrm{Spa}(\widehat{E^{\mathrm{un}}})^\diamond$. Then,

$$\overrightarrow{h}_!(\overleftarrow{h}^* \mathcal{F}_\varphi \otimes IC_\mu) \simeq \mathcal{F}_\varphi \boxtimes r_\mu \circ \varphi$$

compatibly with the S_φ action (which is diagonal on the right).

Further, a factorization sheaf property holds when you compose Hecke correspondences.

(5) *A local-global compatibility holds with Caraiani-Scholze [2] (to be explained in the next lecture).*

10. Lecture 2015-11-24

Today we will discuss the local-global compatibility part of Conjecture 9.2. We begin by explaining a specific example of Conjecture 9.2.

Let $G = \mathrm{GL}_n/E$, $\varphi : W_E \to \mathrm{GL}_n(\overline{\mathbb{Q}_l})$ irreducible, and $\mu(z) = (z, 1, ..., 1)$.

Let

$$x_1 : [\mathrm{Spa}(\overline{\mathbb{F}_q})/\mathrm{GL}_n(E)] \to \mathrm{Bun}_{G,\overline{\mathbb{F}_q}}$$

be the classifying map for the trivial vector bundle of rank n. Then there is a Cartesian diagram

$$\begin{tikzcd} {[(\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}})^\diamond/\mathrm{GL}_n(E)]} \arrow[r] \arrow[d] & \mathrm{Hecke}_{\mu,\overline{\mathbb{F}_q}} \arrow[d, "\overrightarrow{h}"] \\ {[\mathrm{Spa}(\overline{\mathbb{F}_q})/\mathrm{GL}_n(E)] \times \mathrm{Spa}E^\diamond} \arrow[r, "(x_1,\mathrm{Id})"] & \mathrm{Bun}_{\mathrm{GL}_n,\overline{\mathbb{F}_q}} \times \mathrm{Spa}(\widehat{E^{\mathrm{un}}})^\diamond. \end{tikzcd}$$

Here $(\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}})^\diamond$ parameterizes modifications

$$u : \mathcal{O}^n \hookrightarrow \mathcal{E}$$

with coker$u$ a length 1 torsion coherent sheaf: an S-point of $(\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}})^\diamond$ gives an untilt $S^\sharp$ of S and a map $S^\sharp \to \mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}}$, i.e. a line bundle $\mathcal{L}$ on $S^\sharp$ and a surjection $\mathcal{O}^n_{S^\sharp} \twoheadrightarrow \mathcal{L}$. This data gives the desired modification of $\mathcal{O}^n_{X_S}$ along the closed embedding $S^\sharp \to X_S$.

The action of $\mathrm{GL}_n(E)$ is by automorphism of $\mathcal{O}^n$ – in fact, $\underline{\mathrm{GL}_n}(E) = \underline{\mathrm{Aut}}(\mathcal{O}^n)$. This is because $\Gamma(X_S, \mathcal{O}_S) = \mathcal{C}(|S|, E)$.

Now, we can define a $\overline{\mathbb{F}_l}$ perverse sheaf on $[\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}}/GL_n(E)]$: first, we define a smooth sheaf.

Consider $\mathcal{F}$ a $GL_n(E)$-equivariant étale sheaf on $(\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}})^{\mathrm{ad}}$. By a result of Berkovich, if $U \to \mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}}$ is an étale map and U is quasi-compact, then there exists a compact open $K \subset \mathrm{GL}_n(E)$ such that K lifts to an action on U (Elkik's approximation/Krasner lemma). Furthermore, the germ of lifts of this action on sufficiently small K is unique, i.e. for any two lifts to two potentially different compact opens, there is a smaller compact open subgroup contained in both where the lifts agree.

Definition 12.

(1) $\mathcal{F}$ is *smooth* if for all U as above, the action of K on $\mathcal{F}(U)$ is discrete.

(2) Let $\mathcal{A}$ be the category of smooth $\mathrm{GL}_n(E)$-equivariant sheaves in $\overline{\mathbb{F}_l}$-vector spaces on $(\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}})_{\text{ét}}$. Then $\mathcal{F} \in D^b(\mathcal{A})$ is *perverse* if for all $C/\widehat{E^{\mathrm{un}}}$ complete and algebraically closed, $U \to \mathbb{P}^{n-1}_C$ étale, $\mathcal{U}$ a p-adic integral model of U, and $K \subset GL_n(E)$ a sufficiently small compact open, we have

$$R\psi(\mathcal{F}|_U)^K \in \mathrm{Perv}(\mathcal{U}, \overline{\mathbb{F}_l})$$

where $R\psi$ is the nearby cycles.

We return to our cartesian diagram and give a name to the top horizontal arrow:

$$\begin{tikzcd} {[(\mathbb{P}^{n-1}_{\widehat{E^{\mathrm{un}}}})^\diamond/\mathrm{GL}_n(E)]} \arrow[r, "f"] \arrow[d] & \mathrm{Hecke}_{\mu,\overline{\mathbb{F}_q}} \arrow[d, "\overrightarrow{h}"] \\ {[\mathrm{Spa}(\overline{\mathbb{F}_q})/\mathrm{GL}_n(E)] \times \mathrm{Spa}E^\diamond} \arrow[r, "(x_1,\mathrm{Id})"] & \mathrm{Bun}_{\mathrm{GL}_n,\overline{\mathbb{F}_q}} \times \mathrm{Spa}(\widehat{E^{\mathrm{un}}})^\diamond. \end{tikzcd}$$

In this setting, Conjecture 9.2 - (4) says (using that r_μ is the standard representation and that $\mathrm{IC}_\mu = \overline{\mathbb{Q}_l}\left(\frac{n-1}{2}\right)[n-1]$)

$$\overrightarrow{h}_!(\overleftarrow{h}^*\mathcal{F}_\varphi\left(\frac{n-1}{2}\right)[n-1]) = \mathcal{F}_\varphi \boxtimes \varphi$$

Then, proper base change applied to our diagram gives

$$R\Gamma(\mathbb{P}^{n-1}_{\mathbb{C}_p}, f^*\overleftarrow{h}^*\mathcal{F}_\varphi)\left(\frac{n-1}{2}\right)[n-1] = \pi\otimes\varphi$$

for the $GL_n(E)\times W_E$ action, where $\pi = x_1^*\mathcal{F}_\varphi$ is the irreducible representation of $\mathrm{GL}_n(E)$ associated to φ from (3) of Conjecture 9.2.

Now, what is $f^*\overleftarrow{h}^*\mathcal{F}_\varphi$? Recall (2) of Conjecture 9.2 says that $\mathcal{F}_\varphi = j_!j^*\mathcal{F}_\varphi$ where $j : (\mathrm{Bun}_G\otimes\overline{\mathbb{F}_q})^{ss}\hookrightarrow \mathrm{Bun}_G\otimes\overline{\mathbb{F}_q}$. Then, in the diagram

$$\begin{tikzcd} {[\mathbb{P}^{n-1\diamond}/GL_n(E)]} \arrow[r, "f"] \arrow[dr] & \mathrm{Hecke}_\mu \arrow[d, "\overleftarrow{h}"] \\ & \mathrm{Bun}_G \end{tikzcd}$$

we have

$$(\overleftarrow{h}\circ f)^{-1}(\mathrm{Bun}_G^{ss}) = [\Omega^\diamond/GL_n(E)],$$

i.e. $\Omega^\diamond$ is the locus where the modification

$$0\to\mathcal{O}_X^n\to\mathcal{E}\to\mathcal{F}\to 0$$

is such that $\mathcal{E}$ is semistable. (Here $\Omega = \mathbb{P}_E^{n-1}\backslash\bigcup_{H\in\mathbb{P}^{\vee n-1}(E)} H^{\mathrm{ad}}$ is the generalization of the Drinfeld upper half plane $\mathbb{P}^1\backslash\mathbb{P}^1(E)$ for $n=2$). Now, $\mathcal{E}$ is supersingular if and only if $\mathcal{E}$ is locally isomorphic for the pro-etale topology to $\mathcal{O}(1/n)$. Then, since $\underline{\mathrm{Aut}}(\mathcal{O}(1/n)) = \underline{D_{1/n}^\times}$, $\mathrm{Isom}(\mathcal{O}(1/n),\mathcal{E})\to\Omega^\diamond$ is a $\underline{D_{1/n}^\times}$ - torsor which in fact is given by the Drinfeld covering. We have $x_b^*\mathcal{F}_\varphi = \rho$, a smooth irreducible representation of $D_{1/n}^\times$, where $\rho\leftrightarrow\varphi$ by local Langlands if and only if $JL(\rho)=\pi$. Then, $f^*\overleftarrow{h}^*\mathcal{F}_\varphi$ is the extension by 0 of the local system associated to ρ via Drinfeld's covering. It is a known result that

$$R\Gamma_c(\Omega_{\mathbb{C}_p}\mathrm{tower}, \overline{\mathbb{Q}_l})[\rho]\left(\frac{n-1}{2}\right)[n-1] = \pi\otimes\varphi$$

and thus this special case was a strong motivation for the conjecture.

10.1. **Local-global compatibility.**

TYPESETTER'S WARNING – This section needs clarification (my fault) and some things might not be right; this will be addressed in a future update

Here we explain the local-global compatibility in the simple case $G = GL_2$. In the setting of Caraiani-Scholze, there is a perfectoid Shimura variety $Sh_{\infty K^p}$ where ∞ refers to the level at p and $K^p\subset G(\mathbb{A}_f^{(p)})$ is a compact open. In this setting there is a Hodge-Tate period map

$$\pi_{\mathrm{HT}} : \mathrm{Sh}_{\infty K^p}\to\mathbb{P}^1\subset\mathrm{Hecke}_\mu.$$

It can be described in the following way: on the curve for $\mathrm{Sh}^\flat_{\infty K^p}$ there is a modification

$$\mathcal{O}^2\hookrightarrow\mathcal{E}$$

where $\mathcal{E}$ is the bundle attached to the F-isocrystal of the universal p-divisible group [TYPESETTER'S NOTE – I'm not quite sure what is meant by the F-isocrystal of the universal p-divisible group in this context; to be clarified in a later update]. The modification is along the divisor corresponding to the untilt $\mathrm{Sh}_{\infty K^p}$, coming from a surjection

$$V_p\otimes\mathcal{O}\to\omega$$

which can be deduced from Scholze's relative comparison theorem and which can be thought of as a surjection

$$\mathcal{O}^2\to\omega$$

by using the canonical trivialization of V_p coming from the infinite level structure at p. This data induces a map $\mathrm{Sh}_{\infty K^p}\to\mathbb{P}^1$, which is the Hodge-Tate period map.

Scholze and Caraiani consider $R\pi_{\mathrm{HT}*}\overline{\mathbb{Q}_l}$ and show that it is perverse in the preceding sense. The conjecture then predicts that for an automorphic π with π_p supercuspidal corresponding to φ,

$$R\pi_{\mathrm{HT}*}\overline{\mathbb{Q}_l}[\pi^p] = \mathcal{F}_\varphi^m$$

(for a multiplicity m determined by the prime to p level).

The conjecture applies more generally to Hodge-type Shimura Varieties where Scholze and Caraiani also prove perversity.

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