# Preuve d'une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen

This is a personal note of the paper [**4**], which proves a conjecture of Frenkel-Gaitsgory-Kazhdan-Vilonen on some exponential sums related to the geometric Langlands correspondence. The main ingredients are the resolution of Lusztig scheme of lattices introduced by Laumon and the decomposition theorem of Beilinson-Bernstein-Deligne-Gabber.

## 1. L'énconcé

Let $k = \mathbb{F}_q$. $\mathcal{O} := k[[\varpi]], K := k[[\varpi]][\frac{1}{\varpi}]$ the field of formal power series in one variable over the field of fractions. Let $d$ and $n$ be two natural numbers. Following [**3**]:

- Fix the lattice $\mathcal{O}^n$.

$$X := \{\text{lattice } L \subset \mathcal{O}[1/\varpi]^n = K^n : L \text{ is contained in } \mathcal{O}^n \}$$

- $X_d$ be the scheme of finite type over $k$ whose set of $k$ points consists of the rings $\mathcal{R} \hookrightarrow \mathcal{O}^n$ st.

$$\dim_k(\mathcal{O}^n/\mathcal{R}) = d$$

Observe that $\mathrm{GL}(\mathcal{O}^n) \circlearrowright X_d$ and we have the further identification

$$X_d(k) \simeq \left\{\text{codim-}d \text{ subspace } \bar{\mathcal{R}} \text{ in } \mathcal{O}^n/\varpi^{nd}\mathcal{O}^n\right\}$$

$$\mathcal{R} \mapsto \bar{\mathcal{R}}$$

[1]

**Example**

For instance, consider the lattice given by

$$\mathcal{R} := \varpi\mathcal{O} \times \varpi\mathcal{O} \times \cdots \varpi\mathcal{O} \times \mathcal{O} \times \cdots \times \mathcal{O}$$

with $d$ copies of $\varpi\mathcal{O}$. Then $\dim_k(\mathcal{O}^n/\mathcal{R}) = d$.

[1] That the map is indeed surjective can be as follows: wlog suppose the lattice is of the form

$$\varpi^{d_1}\mathcal{O} \times \cdots \times \varpi^{d_n}\mathcal{O}$$

Then $0 \leq d_i \leq d$ for all $1 \leq i \leq n$ and that $\sum d_i = i$.

The action of $\mathrm{GL}_n(\mathcal{O})$ can be regarded as the action of an algebraic group $G_d$ with $G_d(k) \simeq \mathrm{GL}_n(\mathcal{O}/\varpi^d\mathcal{O})$, over $X_d$. The orbits of this action are given by a finite number. For each $n$-partition $\lambda$ of $d$, i.e. $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n \geq 0)$ with $|\lambda| = \sum_{i=1}^n \lambda_i = d$, we denote $\mathrm{Gr}^\lambda$ the orbit of $G_d$ acting on the lattice $\varpi^\lambda\mathcal{O}^n$ where $\varpi^\lambda$ is the diagonal matrix $\mathrm{dia}(\varpi^{\lambda_1}, \ldots, \varpi^{\lambda_n})$. We have a stratification by locally closed subsets

$$X \simeq \bigcup_{|\lambda|=n} \mathrm{Gr}^\lambda$$

which is reflected by the Cartan decomposition

$$\mathrm{GL}_n(F) \simeq \bigsqcup_{\lambda_1\geq\cdots\geq\lambda_n} \mathrm{GL}_n(\mathcal{O})\varpi^\lambda\mathrm{GL}_n(\mathcal{O})$$

Indeed, we have [2]

$$\mathrm{Gr}^\lambda(k) \simeq \mathrm{GL}_n(\mathcal{O})\varpi^\lambda a\mathrm{GL}_n(\mathcal{O})/\mathrm{GL}_n(\mathcal{O})$$

For each $\lambda$, let $\overline{\mathrm{Gr}^\lambda}$ denote the closure of orbit $\mathrm{Gr}^\lambda$ in $X_d$. Recall that $\mathrm{Gr}^\mu \hookrightarrow \overline{\mathrm{Gr}^\lambda}$ iff $\mu \leq \lambda$, where the partial order is given by the natural order on $\{n \text{ partitions of } d\ \}$

$$\mu \leq d \text{ iff } \sum_{j=1}^{i} \mu_j \leq \sum_{j=1}^{i} \lambda_j \text{ for } i-1, \ldots, n-1.$$

Fix a prime $l$ distinct to characteristic $p$ of $k$. Let $\bar{\mathbb{Q}}_l$ be algebriac closure of $\mathbb{Q}_l$.
Let $\mathcal{A}_\lambda$ denote the $l$-adic intersection complex of $\overline{\mathrm{Gr}^\lambda}$.
For each $\alpha \in \mathbb{N}^n$ st. $|\alpha| = d$. Let $S_\alpha$ be the locally closed subset of $X_d$ whose set of $k$ points is the ring $\mathcal{R} \hookrightarrow \mathcal{O}^n$ st. for all $i$.

$$\left(\mathcal{R} \cap \bigoplus_{j=1}^{i} e_j\mathcal{O}\right) \Big/ \left(\mathcal{R} \cap \bigoplus_{j=1}^{i-1} e_j\mathcal{O}\right) \simeq \left(\bigoplus_{j=1}^{i-1} e_j\mathcal{O} + \varpi^{\alpha_i} e_i\mathcal{O}\right) \Big/ \left(\bigoplus_{j=1}^{i-1} e_j\mathcal{O}\right) \tag{1}$$

The point of this moduli description: is that it is $N(F)$ invariant.

[2]Though, $\lambda$s more generally for Grassmanian are allowed to be nonnegative.

**Example**

- Consider the $\mathcal{O}$-submodule of $\mathcal{O}^2$ spanned by
$$\mathcal{R} := \left(\varpi^2 e_1, \varpi e_1 + \varpi e_2\right)$$
This corresponds to standard lattice acted by the matrice.
$$\begin{pmatrix} \varpi^2 & \varpi \\ 0 & \varpi \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^2 & 0 \\ 0 & \varpi \end{pmatrix}$$
If I left multiply this matrix by any unipotent natrx $N \in N(F)$, then the resulting induced lattice also satisfies, 1.
- More generally,
$$\mathcal{R} = \varpi^{\alpha_1} e_1 \mathcal{O} \oplus \left(\varpi^{\beta} e_1 + \varpi^{\alpha_2} e_2\right) \mathcal{O}$$
as $\mathcal{O}$-basis. We can depict this as matrix
$$\begin{pmatrix} 1 & \varpi^{\beta-\alpha_2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \varpi^{\alpha_1} & 0 \\ 0 & \varpi^{\alpha_2} \end{pmatrix} = \begin{pmatrix} \varpi^{\alpha_1} & \varpi^{\beta} \\ & \varpi^{\alpha_2} \end{pmatrix}$$

where $(e_i)$ is the standard basis of $\mathcal{O}^n$. The stratification
$$X_d := \bigcup_{|\alpha|=d} S_\alpha$$
is reflected from the decomposition
$$\mathrm{GL}_n(F) \simeq \bigsqcup_{\alpha \in \Lambda} N(F) \varpi^\alpha \mathrm{GL}_n(\mathcal{O})$$
where $N$ is the subgroup of upper triangular matrices in $\mathrm{GL}_n$. Indeed
$$S_\alpha(k) \simeq N(F) \varpi^\alpha \mathrm{GL}_n(\mathcal{O}) / \mathrm{GL}_n(\mathcal{O})$$
The Frobenius trace of $\mathcal{A}_\lambda$ is naturally identified with the function $A_\lambda$ with compact support in $\mathrm{GL}_n(F)$ which is bi $\mathrm{GL}_n(\mathcal{O})$ invariant. Fix a nontrivial additive character $\psi : k \to \bar{\mathbb{Q}}_l^\times$, and denote $\theta : N(F) \to \bar{\mathbb{Q}}_l^\times$.
$$\theta(n) := \psi\left(\sum_{i=1}^{n-1} \mathrm{res}(n_{i,i+1} d\varpi)\right)$$
Consider the integral
$$I(\varpi^\alpha, \mathcal{A}_\lambda) := \int_{N(F)} A_\lambda(n\varpi^\alpha)\theta(n)\, dn$$
where the Haar measure is normalized on $dn$ of $N(F)$ is defined so that $N(\mathcal{O})$ has measure 1.
Note that one considered such integral more classically in the context of Satake isomorphism: there is a Satake transform map
$$\mathcal{S} : \mathcal{H}_G \to \mathcal{H}_T \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}]$$

$$\mathcal{S}(f) := t \mapsto \delta_{B(F)}(t)^{1/2} \int_{N(F)} f(tn)\, dn \quad t \in T(F)$$

Then one can analyze $\mathcal{S}(c_\lambda)$, with $\{c_\lambda\}_{\lambda \in X_{\bullet,+}} \in \mathcal{H}_G$ are the naïve basis, via intersection of orbits $N(F)\varpi^\mu \cap G(\mathcal{O})\varpi^\lambda G(\mathcal{O})$, [**5**, 4.3].
In [**2**], they proved

**Theorem 1.1.** If $\alpha \neq \lambda$,

$$I(\varpi^\alpha, A_\lambda) = 0$$

If $\alpha = \lambda$, then

$$I(\varpi^\lambda, A_\lambda) = q^{\langle \lambda, \delta \rangle}$$

where

$$\delta = \frac{1}{2}(n-1, n-3, \ldots, 1-n)$$

is the half sum of positive roots. Thus,

$$(\lambda, \delta) = \sum_{i=1}^{n} \lambda_i \delta_i$$

In the case when $\alpha = (\alpha_1, \ldots, \alpha_n)$, is nondecreasing, we can find $n' \in N(F) \cap \varpi^\alpha \mathrm{GL}_n(\mathcal{O})\varpi^{-\alpha}$ st. $\theta(n') \neq 1$. As $A_\lambda$ is $\mathrm{GL}_n(\mathcal{O})$ bi-invariant we have

$$\begin{aligned}\int_{N(F)} A_\lambda(n\varpi^\alpha)\theta(n)\, dn &= \int_{N(F)} A_\lambda(nn'\varpi^\alpha)\theta(n)\, dn \\ &= \theta(n')^{-1} \int_{N(F)} A_\lambda(n\varpi^\alpha)\theta(n)\, dn\end{aligned}$$

thus $I(\varpi^\alpha, A_\lambda) = 0$. The interesting case is therefore when

$$N(F) \cap \varpi^\alpha \mathrm{GL}(n, \mathcal{O})\varpi^{-\alpha} \hookrightarrow N(\mathcal{O})$$

in this case, the character

$$h : N(F) \to k \quad n \mapsto \sum_{i=1}^{n-1} \mathrm{res}(n_{i,i+1} d\varpi)$$

induces a morphism

$$h_\alpha : S_\alpha \to \mathbb{G}_a$$

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[3]Indeed, consider the map

$$N(F)\varpi^\alpha \mathrm{GL}_n(\mathcal{O})/\mathrm{GL}_n(\mathcal{O}) \to \mathbb{G}_a(k)$$
$$n\varpi^\alpha g \mapsto h(n)$$

we need to show this is well defined. Suppose we have two representatives

$$n\varpi^\alpha = n'\varpi^\alpha g' \quad g' \in \mathrm{GL}_n(\mathcal{O})$$

Then

$$(n')^{-1}n = \varpi^\alpha g' \varpi^{-\alpha} \in N(\mathcal{O}) \subset \ker(N(F) \xrightarrow{h} k)$$

Alternatively we can think of $h$ as a map on $N(F)\varpi^\alpha G(\mathcal{O})/G(\mathcal{O}) \to k$, which factors the stabilizer of the action $N(F) \circlearrowleft \varpi^\alpha G(\mathcal{O})/G(\mathcal{O})$.

**Theorem 1.2.** If $\alpha \neq \lambda$

$$R\Gamma_c(S_\alpha \otimes_k \bar{k}, \mathcal{A}_\lambda \otimes h_\alpha^* \mathcal{L}_\psi) = 0$$

If $\alpha = \lambda$ we have

$$R\Gamma_c(S_\alpha \otimes_k \bar{k}, \mathcal{A}_\lambda \otimes h_\alpha^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_l[-2\langle \lambda, \delta \rangle](-\langle \lambda, \delta \rangle)$$

Here $\bar{k}$ is an algebraic closure of $k$ and $\mathcal{L}_\psi$ is the Artin Schrier sheaf of $\mathbb{G}_{a,k}$ associated to a character $\psi$.

We can deduce this theorem from Grothendieck trace formula. Here are the main steps of theorem, [**4**, 2]. We consider the easy case when $\alpha = \lambda$. We prove that if $\mu < \lambda$, the intersection $S_\lambda \cap \mathrm{Gr}^\mu$ is empty, so that the support oif $\mathcal{A}_\lambda$ lies in $\mathrm{Gr}^\lambda$. We also prove that $S_\lambda \cap \mathrm{Gr}^\lambda$ is an affine space such that the homomorphism $h_\alpha$ restricted to $S_\lambda \cap \mathrm{Gr}^\lambda$ is constant with value 0, which gives the result in the case when $\alpha = \lambda$. This is the content of 2.
To prove the asserstion in the case $\alpha \neq \lambda$, we utilize the resolution of the scheme $X_d$. This is resolution is introduced by Laumon in a slightly different context. Let $\tilde{X}_d$, be the scheme of finite type who set of $k$ points consists of flags of lattices

$$\mathcal{O}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \mathcal{R}_d = \mathcal{R}$$

st. $\dim(\mathcal{R}_{i-1}/\mathcal{R}_i) = 1$. The morphism

$$\pi : \tilde{X}_d \to X_d$$

$$(\mathcal{R}_0 \supset \cdots \mathcal{R}_n) \mapsto \mathcal{R}_n$$

is a semismall[4] resolution in the sense of Goresky and MacPherson. Furthermore this is equivariant wrt to the action of $G_d$, so we have

$$\pi_* \bar{\mathbb{Q}}_l[\dim(X_d)]\left(\frac{1}{2}\dim X_d\right) = \bigoplus_\lambda \mathcal{A}_\lambda \boxtimes V_\lambda$$

where the $V_\lambda$ are $\bar{\mathbb{Q}}_l$ vs, thanks to the decomposition thoerem and the the subgroup of stabilizers in $G_d$ are geometrically connected.
By comparison with the construction of Lusztig in Springer correspondence, we see that $V_\lambda$ is the space of representation of the symmetric group $\mathfrak{S}_d$ corresponding to the young partion $\lambda$ of $d$, [**3**]. We will use only the fact that the dimension of $V_\lambda$ is equal to the number of standard $\lambda$-tables. It suffices to show

$$R\Gamma_c(S_\lambda \otimes_k \bar{k}, R\pi_* \bar{\mathbb{Q}}_l \otimes h_\lambda^* \mathcal{L}_\psi) = V_\lambda[-2\langle \lambda, \delta \rangle - d(n-1)][-\langle \lambda, \delta \rangle - \frac{1}{2}d(n-1)]$$

For this, we study the geometry of $\tilde{S}_\lambda = S_\lambda \times_{X_d} \tilde{X}_d$. We have

$$R\Gamma_c(S_\lambda \otimes_k \bar{k}, R\pi_* \bar{\mathbb{Q}}_l \otimes h_\lambda^* \mathcal{L}_\psi) = R\Gamma_c(\tilde{S}_\lambda \otimes_k \bar{k}, \tilde{h}_\lambda^* \mathcal{L}_\psi)$$

where $\tilde{h}_\lambda$ is the composition $h_\lambda \circ \left(\pi\Big|_{\tilde{S}_\lambda}\right)$.

[4]

We prove that $\tilde{S}_\lambda$ is a dsjoint union of locally closed subsets $\tilde{S}_\tau$ which are affine of the same dimension

$$\langle \lambda, \delta \rangle + \frac{1}{2} d(n-1) = \langle \lambda, (n-1, \dots, 1, ) \rangle$$

where $\tau$ consists of the set of elements $(\alpha^i)_{i=1}^d$, with $\alpha^i = (\alpha^i_j)_{j=1}^n \in \mathbb{N}^n$ satisfying

- $\alpha^{i-1}_j \leq \alpha^i_j$.
- $\sum_{j=1}^n \alpha^i_j = i$.
- $\alpha^d = \lambda$.

If the sequence $\alpha^i$ is nonincreasing we show (as in the case with $\lambda$ is nonincreadsing) that

$$R\Gamma_c(S_\tau \otimes_k \bar{k}, \tilde{h}^*_\lambda \mathcal{L}_\psi) = 0$$

Thoe $\tau$ whose $\alpha^i$ has decreasing subscripts, corresponds bijectively to standard Young $\lambda$-tables, see 3

## 2. Etude de $S_\alpha$

For $\alpha \in \mathbb{N}^n$, $S_\alpha$ is isomorphic to an affine space whose coordinates can be explicitly constructed using the uniformizer $\varpi$. We denote $\bar{\mathcal{O}} := \mathcal{O} \otimes_k \bar{k}$ and $\bar{F} := F \otimes_k \bar{k}$.

**Proposition 2.1.** For each $\mathcal{R} \in S_\alpha(\bar{k})$. There exists a unique upper triangular matrix of the form

$$x = \begin{pmatrix} \varpi^{\alpha_1} & x_{1,2} & \cdots & x_{1,n} \\ & \varpi^{\alpha_2} & \cdots & x_{2,n} \\ & & \ddots & \vdots \\ & & & \varpi^{\alpha_n} \end{pmatrix}$$

where the $x_{i,j}$ are polynomials in $\varpi$ with coeffficients in $\bar{k}$ of degree strictly smaller thann $\alpha_i$, st. $\mathcal{R} = x\bar{\mathcal{O}}^n$.

Proof. Let $\mathcal{R} \in S_\alpha(\bar{k})$, it decomposes as

$$\mathcal{R} = \mathcal{R}' \oplus (\varpi^{\alpha_n} e_n + y)\, \bar{\mathcal{O}}$$

where

$$\mathcal{R}' := \mathcal{R} \cap \bigoplus_{j=1}^{n-1} e_j \bar{\mathcal{O}} \in S_{\alpha'}(\bar{k})$$

[5] where $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ and $y \in \bigoplus_{j=1}^{n-1} e_j \bar{\mathcal{O}}$. The proof should be done - see footnote. □

As a corollary

**Proposition 2.2.** $S_\alpha$ is an affine space of dimension

$$\langle \alpha, (n-1, \dots, 0) \rangle$$

[5] I am not so clear of the original proof. Aren't we done from here? By induction. For any choice of $y' \in \{y + r', r' \in \mathcal{R}'\}$, we get the same lattice $\mathcal{R}$. We also know that $r' = \varpi^{\alpha_{n-1}} e_{n-1}\mathcal{O} + y'$. In otherwords, we can guarantee $x_{n-1,n}$ has degree $< \alpha_{n-1}$ in $\varpi$.

Our next goal is to undersand

$$S_\lambda \cap \overline{\mathrm{Gr}^\lambda}$$

**Proposition 2.3.** (1) Let $\mu$ and $\lambda$ be partition of $d$ st. $\mu < \lambda$ we have

$$S_\lambda \cap \mathrm{Gr}^\mu = \emptyset$$

(2) The intersection $S_\lambda \cap \mathrm{Gr}^\lambda$ is an affine space of dimension $2\langle \lambda, \delta\rangle$.
(3) The restriction of $h_\lambda$ to $S_\lambda \cap \mathrm{Gr}^\lambda$ is constant of value 0.

Proof. (1) Let $\mathcal{R} = x\bar{\mathcal{O}}^n \in S_\lambda \cap \mathrm{Gr}^\mu(\bar{k})$ where $x$ is a matrix as in **??**. For all minors[6] of order $i \geq 1$ of $x$ are divisible by $\varpi^{\mu_{n-i+1}+\cdots+\mu_n}$. [7] If we consider submatrices of the last $i$ colomns, we obtain inequality we obtain that

$$\lambda_{n-i+1} + \cdots + \lambda_n \geq \mu_{n-i+1} + \cdots + \mu_n$$

Thus $\lambda \geq \mu$.
(2) Suppose now that $\mu = \lambda$. Consider the $(i+1)\times(i+1)$ submatrix inncluding the coefficient $x_{j,n-i}$ with $j < n-i$ and includes the last $i$ .... If the coefficients $x_{j,k}$ are divisible by $\varpi^{\lambda_k}$ for all $j < k$... It follows that $S_\lambda \cap \mathrm{Gr}^\lambda$ is isomorphic of dimension

$$\begin{aligned}\langle \lambda, (n-1, \ldots, 0)\rangle &- \langle \lambda, (0, \ldots, n-1)\rangle \\ &= \langle \lambda, (n-1, n-3, \ldots, 1-n)\rangle\end{aligned}$$

(3) We have shown that

$$S_\lambda \cap \mathrm{Gr}^\lambda(\bar{k})$$

□

**Proposition 2.4.** We have an isomorphism

$$R\Gamma_c(S_\lambda \otimes_k \bar{k}, \mathcal{A}_\lambda \otimes h_\lambda^* \mathcal{L}_\psi) \simeq \bar{\mathbb{Q}}_l[-2\langle \lambda, \delta\rangle](-\langle \lambda, \delta\rangle)$$

Proof. We have shown in 2.3 that

$$S_\lambda \cap \overline{\mathrm{Gr}^\lambda} = S_\lambda \cap \mathrm{Gr}^\lambda$$

By definition, $\mathcal{A}_\lambda\Big|_{S_\lambda} \simeq \bar{\mathbb{Q}}_l[2\langle \lambda, \delta\rangle](\langle \lambda, \delta\rangle)$ is supported on the affine space $S_\lambda \cap \mathrm{Gr}^\lambda$ of dimension $2\langle \lambda, \delta\rangle$. As mentioned after 1.2, $h_\lambda$ is the zero map, hence the pullback of $\mathcal{L}_\psi$ is the constant sheaf. □

[6] The determinant of $i \times i$ submatrix.
[7] Indeed, one can see this by considering a general element $\mathrm{GL}_n(\mathcal{O})\varpi^\mu\mathrm{GL}_n(\mathcal{O})$

## 3. Etude de $\tilde{S}_\lambda$

Denote $\tilde{S}_\lambda := S_\lambda \times_{X_d} \tilde{X}_d$. The set of $\bar{k}$ points of $\tilde{S}_\lambda$ is the flag of lattices

$$\bar{\mathcal{O}}^n = \mathcal{R}_0 \supset \mathcal{R}_1 \supset \cdots \supset \mathcal{R}_d = \mathcal{R}$$

where $\dim_{\bar{k}}(\mathcal{R}_{i-1}/\mathcal{R}_i) = 1$ and $\mathcal{R} \in S_\lambda(\bar{k})$. For a fixed flag, for each $i = 0, \dots, d$ there exists $\alpha^i \in \mathbb{N}^n$ with $|\alpha^i| = i$ such that $\mathcal{R}_i \in S_{\alpha_i}(\bar{k})$. The scheme $\tilde{S}_\alpha$ is thus straified with respect to the matrix

$$\tau := (\alpha^i_j)^{0\leq i\leq d}_{1\leq j\leq n} \in \mathbb{N}^{(d+1)n}$$

such that

(1) $\alpha^{i-1}_j \leq \alpha^i_j$. [8]
(2) $\sum_{j=1}^n \alpha^i_j = i$. [9]
(3) $\alpha^d = \lambda$.

Let $S_\tau$ denote the corresponding stratification. Denote by $\tilde{h}_\lambda$ the restriction of $h_\lambda \circ \pi_{\tilde{S}_\lambda}$ to $S_\tau$.

**Proposition 3.1.** If exists a $d'$ with $1 \leq d' \leq d-1$ such that the $(\alpha^{d'}_j)^{0\leq i\leq d'}_{1\leq j\leq n}$ is non decreasing then we have

$$R\Gamma_c(S_\tau \otimes_k \bar{k}, \tilde{h}^*_\lambda \mathcal{L}_\psi) = 0$$

**Proposition 3.2.** If $\alpha^{d'}$ is non decreasing, for all $\mathcal{R}' \in S_{\alpha^{d'}}(\bar{k})$ there exists a subgroup $\mathbb{G}_{a,\bar{k}} \hookrightarrow \mathrm{GL}(\mathcal{R}') \cap N(\bar{F})$ such that the reduction $N(\bar{F}) \to \mathbb{G}_{a,\bar{k}}$ defined by

$$n \mapsto \operatorname{res}\left(\sum_{i=1}^{n-1} n_{i,i+1} d\varpi\right)$$

is the identity on inclusion of $\mathbb{G}_{a,\bar{k}}$.

Proof. □

*Proof of 3.1*: Denote $Z := \pi'^{-1}(\mathcal{R}_\bullet)$ and $h$ the restriction of $\tilde{h}_\lambda$ to $Z$.
The proposition then follows from a general formula in [**1**].

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[8] This is depicted in the inclusion

$$\mathcal{R}_{i-1} \begin{pmatrix} \ddots & & \\ & \varpi^{\alpha^{i-1}_j} & \\ & & \ddots \end{pmatrix} \supset \begin{pmatrix} \ddots & & \\ & \varpi^{\alpha^i_j} & \\ & & \ddots \end{pmatrix} = \mathcal{R}_i$$

[9] This is forced by the condition $\dim_{\bar{k}}(\mathcal{R}_{i-1}/\mathcal{R}_i) = 1$. Thus, a 2 chain condition is given by

$$\mathcal{R}_0 = \bar{\mathcal{O}}^2 \supset \begin{pmatrix} \varpi & \\ & \bar{\mathcal{O}} \end{pmatrix} \supset \begin{pmatrix} \varpi & \\ & \varpi \end{pmatrix} = \mathcal{R}_2$$

where $d = 2$.

**Proposition 3.3.** Let $Z$ be a scheme of finite type[10] over $\bar{k}$, provided ewith ana ction $\xi : \mathbb{G}_a \times Z \to Z$. Let $F$ be a complex on $Z$ provided with an isomorphism $\xi^* F = \mathcal{L}_\psi \boxtimes F$, then $R\Gamma_c(Z, F)$.

**Proposition 3.4.** If for all $i = 0, \ldots, d$, $\alpha^i$ is non decreasing then we have an isomorphism

$$R\Gamma_c(S_\tau \otimes_k \bar{k}, \tilde{h}_\lambda^* \mathcal{L}_\psi)) \simeq \bar{\mathbb{Q}}_l[-2\langle \lambda, (n-1, \ldots, 1, 0)\rangle](-1\langle \lambda, (n-1, \ldots, 0)\rangle)$$

*Final proof of theorem* [**2**, 2]: For finishing the proof, it suffices to show that the matrices $\tau$ satisfying

- $\alpha_j^{i-1} \leq \alpha_j^i$.
- $\sum_{j=1}^n \alpha_j^i = i$.
- $\alpha^d = \lambda$.
- $\alpha_{j-1}^i \geq \alpha_j^i$.

are in one to correspondence with standard[11] $\lambda$-tablues. We see that $\tau$ satisfying the first three conditions but not the fourth are maps

$$\tau : \{1, \ldots, d\} \to \{1, \ldots, n\}$$

such that for all $j$, $|\tau^{-1}(j)| = \lambda_j$. Given such a map, we can succesively writein $1, \ldots, d$ in the $\lambda$-Young diagram. The number $i$ is assigned into the first empty cell in [12] the $j = \tau(i)$th line. [13] s Such a table is standard if and only if

$$\alpha_{j-1}^i \geq \alpha_j^i \quad \forall i = 1, \ldots, d,\ j = 1, \ldots, n$$

We can also reason more directly as follows. The space $V_\lambda$ admits a basis indexed by the components of the fiber of $\pi : \tilde{X}_d \to X_d$ over a geometric point, which are irreducible and of maximal dimension. For example $\varpi^\lambda \bar{\mathcal{O}}_n \in \mathrm{Gr}^\lambda(\bar{k})$. Using 3.2...

[10] Ths is the geometrization of the classical statement discussed in **??**

[11] If entries of each row and each column in the young diagram are increasing.

[12] Note totally sure what this is meant

[13] For example:

# Bibliography

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