STACKY APPROACH TO MOTIVIC PERIODS

MILTON LIN

Contents

Notations and conventions

- **Any sections with *to be written/added* are only sketched.**
- $\mathcal{S}$ denotes the ∞-category of anima. For $\mathcal{C}$ an ∞-category, and $X, Y \in \mathcal{C}$ we let $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathcal{S}$ denote the mapping space.

Fix base R, a discrete commutative ring. We consider the following homotopy rings:

- $\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$, the ∞-category of $\mathbb{E}_\infty$ algebra. This has two full subcategories, the *coconnective* and *connective* algebras.

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{ccn}} := \mathrm{CAlg}_{\leq 0} \arrow[r, hook, shift right] \arrow[d, shift left] & \mathrm{CAlg} \arrow[l, shift right, "\tau_{\leq 0}"'] \arrow[r, shift right, "\tau_{\geq 0}"'] \arrow[d] & \mathrm{CAlg}_{\geq 0} := \mathrm{CAlg}^{\mathrm{cn}} \arrow[l, hook', shift right] \arrow[d, shift left] \\
\mathrm{Mod}_{\leq 0} \arrow[u, shift left, "\mathrm{Sym}^{\mathrm{co}}"] \arrow[r, hook, shift right] & \mathrm{Mod} \arrow[l, shift right, "\tau_{\leq 0}"'] \arrow[r, shift right, "\tau_{\geq 0}"'] & \mathrm{Mod}_{\geq 0} \arrow[l, hook', shift right] \arrow[u, shift left, "\mathrm{Sym}"]
\end{tikzcd}$$

Date: April 1, 2024.

- SCR the ∞-category of simplicial commutative rings. This is the sifted completion of the category of polynomial algebra $\mathrm{Poly}_{\mathbb{Z}}$. Dually, we have coSCR, the ∞-category of co-simplicial rings. Importantly, there is are Dold Kan and co dual Dold-Kan inducing
$$\theta : \mathrm{SCR} \to \mathrm{CAlg}^{\mathrm{cn}}$$
$$\mathrm{co}\theta : \mathrm{coSCR} \to \mathrm{CAlg}^{\mathrm{ccn}}$$
these are equivalences when we consider relative over a base field k of characteristic zero.

- All three categories have the ordinary category of discrete rings, $\mathrm{CAlg}^{\heartsuit}$ embedding to it. We let $\mathrm{Aff}_R^{\heartsuit} := (\mathrm{CAlg}_R^{\mathrm{op}})^{\heartsuit} \hookrightarrow \mathrm{Aff}_R$ be the ordinary category of affine schemes over R.

- $\mathrm{CAlg}_R^{\mathrm{aug}} := (\mathrm{CAlg}_R^{\mathrm{cn}})_{/R}$, be the ∞-category of augmented R-algebra.

- $\mathrm{Stk}_R := \mathrm{Shv}_R := \mathrm{Shv}_{\mathcal{S}}(\mathrm{CAlg}_R^{\heartsuit}, \tau) \hookrightarrow \mathrm{PStk}_R$ denotes the ∞- category of *stacks*[1] and prestacks over R. Unless stated otherwise, τ is the fpqc-topology. Let the category of *pointed* stacks be denoted as $(\mathrm{Stk}_R)_* := (\mathrm{Shv}_R)_*$.

- $\mathrm{dStk}_R := \mathrm{Shv}_{\mathcal{S}}(\mathrm{Aff}_R, \tau)$, the category of *derived stacks*.

Remark 0.1. One can formulate a similar theory for Cdga, the ∞-category of commutative differential graded algebra (we use cohomological grading, as per convention here).

If $R \in \mathrm{Cdga}^{\heartsuit}$, there are equivalences $\mathrm{Cdga}_R \simeq \mathrm{CAlg}_R$, with the ∞-category of $\mathbb{E}_\infty$-algebras over R. We will freely interchange between the variations in this case. Cdga_R is not as useful outside of characteristic zero, as there does not exist model categories.

1. Introduction

Let $X \in \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{sm,proj}}$, and $X^\circ : X \backslash D$, where D is a divisor with normal crossing.

Example 1.1. $X^\circ = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, studied in [Del89].

1.1. **Periods.** Periods are classically integrals of rational differential forms:
$$\log(2) = \int_{1 \le z \le 2} \frac{dz}{z}, \quad \zeta(2) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2}$$
More generally, they are the matrix coefficient from Grothendieck's comparison theorem
$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^*_{\mathrm{Betti}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$
with respect to the $\mathbb{Q}$ structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to* X. These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"[2] which should govern the arithmetic structure of periods.

[1] we simply refer sheaves as stacks, which is not the convention. Often these require some *geometric context*, see [Toë06], [Lur11a].

[2] For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

1.2. **Goal.** We will study the *pro unipotent homotopy groups*

$$\pi_1^{U,?}(X^\circ, x)$$

in various realization $? \in \{\text{ét}, \text{Betti}, \text{dR}, \text{cris}\}$. We discuss the de Rham version in Section 1.3.

1.3. **The de Rham unipotent homotopy groups.** In this section k would be a field of characteristic 0. What should be $\pi_1^{U,\mathrm{dR}}$? We propose a definition in Definition 1.2. In Proposition 1.1 we prove its equivalence to a classical definition. For a stack X one can associate its unipotent homotopy type $\mathbf{U}(X)$ see Section 3.1.

Definition 1.1. Denote the *de Rham complex functor*

$$\begin{tikzcd} \mathrm{CAlg}_k^{\heartsuit,\mathrm{sm}} \arrow[r, "\mathrm{dR}"] \arrow[d] & \mathrm{CAlg}_k^{\mathrm{ccn}} \\ \mathrm{CAlg}_k^{\mathrm{cn}} \arrow[ur] & \end{tikzcd}$$

where on smooth discrete algebras,

$$\mathrm{dR}(A) = \Omega^*_{A/k}$$

is the algebraic de Rham complex. This is then left Kan extended to $\mathrm{CAlg}_k^{\mathrm{cn}}$.

Lemma 1.1. *$A \in Poly_k \hookrightarrow CAlg_k^\heartsuit$, a finitely generated polynomial algebra over k. then*

$$(\operatorname{Spec} A)^{dR} \simeq \operatorname{Spec} dRA$$

where

$$(-)^{dR} : \mathrm{Shv}_k \to \mathrm{Shv}_k$$

is the associated endo fucntor of de Rham stack functor in Example 3.1, and Spec *is the Yoneda embedding, see Proposition 3.1.*

Proof. This is [Mon22, Lem 2.0.5] combined with [Mon22, Thm 2.0.1]. □

Definition 1.2. For a pointed cohomologically connected scheme $X \in (\mathrm{Sch}_k)_*$, we let

$$\pi_1^{u,\mathrm{dR}}(X) := \pi_1(\mathbf{U}(X^{\mathrm{dR}}), *)$$

be its unipotent de Rham fundamental group scheme.

Proposition 1.1. *If X° is $\left(Aff_{\mathbb{Q}}^{\heartsuit,ft}\right)_*$, a finite type pointed, connected, affine scheme over $\mathbb{Q}$, then*

$$\pi_1^{u,dR}(X^\circ) \simeq \operatorname{Spec} H^0(B(dR(X^\circ)))$$

in $Grp(\mathrm{Sch}_{\mathbb{Q}})$, where the right hand side is the Bar complex construction definition of Haine [Hai87] the right-hand object being what is classically used to define the unipotent de Rham homotopy group, [Bro14].

Proof. Let $A := \mathrm{dR}(X^\circ)$. Consider the homotopy sheaf Definition 3.3,

$$\big(\pi_1 \operatorname{Spec} A : R \mapsto \pi_1(\mathrm{Map}_{\mathrm{CAlg}_k}(A, R), *)\big) \in \mathrm{Shv}_{\mathrm{Set}}(\mathrm{CAlg}_k^{\heartsuit}, \mathrm{fpqc})$$

By Toën's representability Theorem 3.1 and hypercompleteness of affine stacks Proposition 3.2 this sheaf is representable by pro-unipotent group scheme, $\pi_1^u(\operatorname{Spec} A, *) \in \mathrm{Grp}(\mathrm{Aff}_k)$. By [Ols16, Ch.6] we have

$$\pi_1(\mathbf{U}(\operatorname{Spec} A), *) \simeq \pi_1^u \operatorname{Spec} A \simeq \operatorname{Spec} H^0 BA$$

where first equivalence is by Definition 3.2. Lastly,

$$\operatorname{Spec} A \simeq (X^\circ)^{\mathrm{dR}}$$

by Lemma 1.1. □

Remark 1.1. We would hope the proof generalize to schemes with log structures. A definition of de Rham homotopy of scheme with log structures can be found in [Shi00].

Conjecture 1.1. *(1) There exists X^{an} which is the (analytic) Betti stack of X, see Section 4, such that the unipotent Betti homotopy group $\pi_1^{u,Betti}(X(\mathbb{C}), x)$ as defined in [Bro17] is isomorphic to $\pi_1(\boldsymbol{U}(X^{an}))$.*

(2) A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,dR}(X, x) \simeq \pi_1^{u,Betti}(X(\mathbb{C}), x) \otimes \mathbb{C}$$

Remark 1.2. this is a little different to the comparison theorem as suggested [Toë06, Ch. 3.5]. In this case X° is only smooth, but *not projective.*

1.4. **Further directions.** By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [3] We collect a few examples below that suggests avenues with a view towards p-adic cohomology theories, such as $X^{\Delta}, X^{\mathrm{crys}}$ and X^{dR}. (prismatic, crsytalline and de Rham stack, respectively). We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].

In the examples below, let V be a complete discrete valuation ring with a perfect residue field of characteristic $p > 0$ and fraction field K, $K_0 := \mathrm{Frac} W(k) \hookrightarrow K$. $X \in \mathrm{Sch}_V^{\mathrm{sm,prop}}$.

Example 1.2. $\pi_1^{u,\mathrm{crys}}$ has a Tannakian description as given in Shiho's [Shi00]. Part of the strategy is formal: for one Tannakian category when can consider the *nilpotent* part. In *op.cit. Ch.5* one constructs a unipotent crystalline de-Rham comparison map,

$$\pi_1^{u,\mathrm{crys}}(X_V^\circ, x) \otimes_{K_0} K \simeq \pi_1^{u,\mathrm{dR}}(X_K^\circ, x)$$

which has been shown in the case of cohomologies by Berthelot and Ogus.

[3] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

2. Recollection on Chen's theorem

Morgan showed that the homotopy Lie algebra of a smooth complex algebraic variety has a mixed Hodge structure by using Sullivan's minimal models. Haine [Hai87] generalized this result to arbitrary complex variety, the key result was using Chen's theorem, Theorem 2.1. We begin by discussing an interpretation of Chen's theorem, Section 2.1.

2.1. Differential forms on loop space.

2.1. Differential forms on loop space. To be written. References: [Che73]. Our goal is to briefly review the proof of the following theorem.

Theorem 2.1. *Let $x, y \in X^\circ(\mathbb{C})$. For all integer $N \geq 0$,*

$$\mathcal{O}(\pi_{1,N}^{uni,dR}(BdR(X^\circ)) \otimes \mathbb{C} \simeq \mathcal{O}(\pi_{1,N}^{Betti}(X^\circ, x, y)) \otimes \mathbb{C} \tag{1}$$

where

$$\mathcal{O}(\pi_{1,N}^{uni,dR}(BdR(X^\circ)) := L_N B(dR(X^\circ))$$

L_N being the length filtration on the bar complex. taking colimit along N, we induce

$$\mathcal{O}(\pi_1^{uni,dR}(BdR(X^\circ)) \otimes \mathbb{C} \simeq \mathcal{O}(\pi_1^{uni,Betti}(X^\circ, x, y)) \otimes \mathbb{C}$$

The proof follows by using a combinatorial presentation of relative cohomology.

2.2. Relation to Malcev–Lie algebra.

3. Recollection on Stacks Approach

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R)$$
$$X \mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?})$$

Example 3.1. The *de Rham stack* X^{dR} over $\mathbb{Q}$, has points given by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A (cf. [GR14]).

This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques.

3.1. Stacks approach to unipotent group scheme. We recall the work of [MR23]. Let $(X, x) \in (\mathrm{Sch}_k)_*$ such that it is cohomologically connected [4] A classical homotopical invariant for schemes is the étale fundamental group introduced by Grothendieck. Nori, upgraded this definition to that of a *group scheme*, $\pi_1^N(X, x)$, which is constructed using Tannakian methods. One can associate a unipotent homotopy type $\mathbf{U}(X)$, which recovers Nori's unipotent homotopy group scheme, Definition 3.2,

$$\pi_1(\mathbf{U}(X), x) \xrightarrow{\simeq} \pi_1^{U,N}(X, x) \tag{2}$$

This is proved in [MR23, §3.1.].

[4] $H^0(X, k) \simeq k$.

Proposition 3.1. *We have an adjunction [Toë06, Cor. 2.2.4]*

$$\begin{tikzcd} CAlg_k^{\heartsuit} \arrow[d, hook] \arrow[dr, hook] & \\ (CAlg_k^{ccn})^{op} \arrow[r, shift left, hook, "\mathrm{Spec}"] & PShv(CAlg_k^{\heartsuit}) \arrow[l, shift left, "\boldsymbol{U}"] \end{tikzcd}$$

The right adjoint is alternatively denoted as $\Gamma(-, \mathcal{O})$, the *global sections* functor.

Definition 3.1. An object in the essential image of Spec in Proposition 3.1 is an *affine stack*, and the right adjoint $\mathbf{U}$ is called the *affinization*.

From the results of [Toë06], discussed in Section 3.3, we can define the homotopy groups:

Definition 3.2. Let $X \in (\mathrm{Sch}_k)_*$ which is cohomologically connected. Define

$$\pi_i^u(X) := \pi_i(\mathbf{U}(X), *) \in \mathrm{Grp}(\mathrm{Aff}_k)$$

as the *unipotent homotopy groups of* X, where $\mathbf{U}$ is as defined in Proposition 3.1.

Remark 3.1. The unipotent type can be defined for $(\mathrm{Stk}_R)_*$. But they are not necessarily representable, see [Mon22, p. 5].

Proposition 3.2. Spec *factors through* $\mathrm{Shv}_k^{\wedge}$.

Proof. By faithfully flat descent, Spec factors through Shv_k. For hypercompleteness see [Lur11b, Appendix D]. □

Example 3.2. $K(\mathbb{G}_a, i) := \mathrm{Spec}\,\mathrm{Sym}_k^{\mathrm{co}}\, k[-i]$ for $i > 0$ are affine stacks.

Example 3.3. Zero truncated quasi-affine stacks are *not* affine.

3.2. **Homotopy and hypercomplete sheaves.** Let $X \in \mathrm{Stk}_k$, $R \in \mathrm{CAlg}^{\heartsuit}$ in this section. In this paper, we would only be considering hypercomplete sheaves.

Definition 3.3. Let $n \geq 0$, then

$$\pi_n(X, *) \in \mathrm{Shv}_{\mathrm{Set}}(\mathrm{CAlg}_R^{\heartsuit}, \mathrm{fpqc})$$

is the sheafification of the presheaf

$$A \mapsto \pi_n(X(A), *)$$

We will be interested in hypercomplete sheaves, see [CM21] for a discusscusion in the prestable setting.

Definition 3.4. A morphism $f : X \to Y$ in an ∞-topos $\mathfrak{X}$ is ∞-*connective* if

(1) it is an effective epimorphism.

(2) $\pi_k f = *$ for $k \geq 0$.

Definition 3.5. $X \in \mathfrak{X}$ is *hypercomplete* iff it is local to ∞-connective morphism. We denote the hypercomplete objects as $\mathfrak{X}^\wedge$, fitting into an adjunction

$$\mathfrak{X}^\wedge \rightleftarrows \mathfrak{X}$$

Hypercompleteness can also be characterized by hypercoverings.

Example 3.4. Let $(\mathcal{C}, \tau)$ be an ∞-stie, [Lur09, Ch.6]. Let $\mathcal{D}$ be an $(n+1,1)$ category for $n \geq 0$, [Lur09, 2.3.4]. Then $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, D)$ satisfies descent for coverings iff it satisfies descent for hypercovering. In particular, this is useful when $(\mathcal{C}, \tau)$ is an ordinary category as the representables factors through $\mathrm{Set} \hookrightarrow \mathcal{S}$.

3.3. **Representability results of Toën.**

Theorem 3.1. *[Toë06, Thm. 2.4.1, 2.4.5] Let $X \in (\mathrm{Shv}_k^\wedge)_*$, such that $\pi_0 X \simeq *$, then X is an affine stack iff $\pi_i(X, *)$ is representable by an affine group scheme $\pi_i^u X$ for all $i > 0$.*

Remark 3.2. [Toë06, Thm. 2.4.], if $H^0(B) \simeq k$, for $B \in (\mathrm{CAlg}_k^{\mathrm{ccn}})_*$, then $\mathrm{Spec}\, B$ is pointed connected.

3.4. **Nori's unipotent scheme.** To be written.

4. Betti Analytic Stack

To be written. Such stacks was discussed in [KpT08], [PY16]. The name *Betti analytic stack* can be misleading. This is stack X^{an} is designed so that

$$\pi_1(X^{\mathrm{an}}, *) \simeq \pi_1(|X(\mathbb{C})|, *)$$

where $X(\mathbb{C})$ is given the analytic topology. [5] But here we recall the construction. The natural map $\pi : \mathrm{Aff}_{\mathbb{C}} \to *$, induces a geometric morphism

$$\mathrm{Stk}_{\mathbb{C}} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathcal{S}$$

Definition 4.1. $(\mathrm{Stn}_{\mathbb{C}}, \tau_{\mathrm{an}})$, denote the category of Stein complex analytic spaces with the analytic topology: this consists coverings $\{U_i \to X\}_{i \in I}$, where $U_i \hookrightarrow_{\mathrm{open}} X$ are open immersions, and $\coprod_{i \in I} U_i \to X$ is a surjection. [6] Let $\mathrm{AnStk}_{\mathbb{C}} := \mathrm{Shv}_{\mathcal{S}}(\mathrm{Stn}_{\mathbb{C}}, \tau_{\text{ét}})$.

Proposition 4.1. *There is an analytification functor*

$$(-)^{an} : \left(\mathit{Aff}_{\mathbb{C}}^{dfp}, \tau_{\acute{e}t}\right) \to (\mathit{Stn}_{\mathbb{C}}, \tau_{an})$$

Proof. See [Lur11a], [Por18]. □

[5] The reason for this (bad) choice is also not to be confused with recent works, [SC23].

[6] One can also consider with respect to the $\tau_{\text{ét}}$ étale topology, i.e. $U_i \to X$

In particular there is a well defined functor

$$\mathrm{Stk}_{\mathbb{C}} \to \mathcal{S}$$
$$X \mapsto |X(\mathbb{C})|$$

sending a stack to its underlying analytic topology.

Definition 4.2. we let $X^{\mathrm{an}} := \pi^*(|X(\mathbb{C})|) \in \mathrm{Stk}_{\mathbb{C}}$ be the *Betti analytic stack*.

Lemma 4.1. *(1)* $*^{an} \simeq \operatorname{Spec} \mathbb{C}$.

(2) For $\operatorname{Spec} A \in Stk_{\mathbb{C}}$, $QCoh(X^{an} \times \operatorname{Spec} A) \simeq Fun(|X(\mathbb{C})|, Mod_A)$.

Proof. (1) π^* is left exact, so it preserves the terminal object.

(2) This is by induction. Write $|X(\mathbb{C})|$ as the colimit of a tower cells,

$$|X(\mathbb{C})| \simeq \mathrm{colim}_{n\in\mathbb{N}} X_n$$

Then use that $\pi^*(-)$, $\mathrm{QCoh}(-)$, and $\mathrm{Fun}(-, \mathrm{Mod}_A)$ commutes with colimits in their variables

□

Note that $\mathrm{Fun}(|X(\mathbb{C})|, \mathrm{Mod}_A)$ identify with the locally constant sheaves on $\mathrm{Op}(|X(\mathbb{C})|)$, the site of open subsets of $X(\mathbb{C})$. This implies that $\mathcal{O}_{X^{\mathrm{an}}}$ corresponds to the constant sheaf. In particular $\pi_*\mathcal{O}_{X^{\mathrm{an}}} \simeq R\Gamma(|X(\mathbb{C})|, \mathbb{C})$, where $\pi_* : \mathrm{QCoh}(X^{\mathrm{an}}) \to \mathrm{QCoh}(*) \simeq \mathrm{Mod}_{\mathbb{C}}$.

References

[Bha23] Bhatt, Bhargav. *Prismatic F-gauges*. 2023 (cit. on pp. 4, 5).

[Bro14] Brown, Francis. *Motivic periods and the projective line minus three points*. 2014. arXiv: 1407.5165 [math.NT]. URL: https://arxiv.org/abs/1407.5165 (cit. on p. 3).

[Bro17] Brown, Francis. "Notes on motivic periods". In: *Commun. Number Theory Phys.* 11.3 (2017), pp. 557–655. ISSN: 1931-4523,1931-4531. URL: https://doi.org/10.4310/CNTP.2017.v11.n3.a2 (cit. on p. 4).

[Che73] Chen, Kuo-tsai. "Iterated integrals of differential forms and loop space homology". In: *Ann. of Math. (2)* 97 (1973), pp. 217–246. ISSN: 0003-486X. URL: https://doi.org/10.2307/1970846 (cit. on p. 5).

[CM21] Clausen, Dustin and Mathew, Akhil. "Hyperdescent and étale K-theory". In: 225.3 (2021) (cit. on p. 6).

[Del89] Deligne, Par P. "Le groupe fondamental de la droite projective moins trois points". In: *Galois Groups over $\mathbb{Q}$ Proceedings of a Workshop Held March 23–27, 1987*. Springer. 1989, pp. 79–297 (cit. on p. 2).

[Dri22] Drinfeld, Vladimir. *A stacky approach to crystals*. 2022. arXiv: 1810.11853 [math.AG]. URL: https://arxiv.org/abs/1810.11853 (cit. on p. 5).

[GR14] Gaitsgory, Dennis and Rozenblyum, Nick. *Crystals and D-modules*. 2014. arXiv: 1111.2087 [math.AG]. URL: https://arxiv.org/abs/1111.2087 (cit. on p. 5).

[Hai01] Hain, Richard. *Iterated Integrals and Algebraic Cycles: Examples and Prospects*. 2001. arXiv: math/0109204 [math.AG]. URL: https://arxiv.org/abs/math/0109204 (cit. on p. 4).

[Hai87] Hain, Richard M. "The de Rham homotopy theory of complex algebraic varieties. I". In: *K-Theory* 1.3 (1987), pp. 271–324. ISSN: 0920-3036. URL: https://doi.org/10.1007/BF00533825 (cit. on pp. 3, 5).

[KpT08] Katzarkov, L., pantev, T., and Toën, B. "Schematic homotopy types and non-abelian Hodge theory". In: (2008) (cit. on p. 7).

[Lur09] Lurie, Jacob. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. URL: https://doi.org/10.1515/9781400830558 (cit. on p. 7).

[Lur11a] Lurie, Jacob. "DAGV: Structured Spaces". In: (2011) (cit. on pp. 2, 7).

[Lur11b] Lurie, Jacob. "Spectral Algebraic Geometry". In: (2011) (cit. on p. 6).

[Mon22] Mondal, Shubhodip. "Reconstruction of the stacky approach to de Rham cohomology". In: *Math. Z.* 302.2 (2022), pp. 687–693. ISSN: 0025-5874,1432-1823. URL: https://doi.org/10.1007/s00209-022-03082-9 (cit. on pp. 3, 6).

[MR23] Mondal, Shubhodip and Reinecke, Emanuel. *Unipotent homotopy theory of schemes*. 2023. arXiv: 2302.10703 [math.AG] (cit. on p. 5).

[Ols16] Olsson, Martin. "The Bar Construction and Affine Stacks". In: *Communications in Algebra* 44 (2016), pp. 3088–3121. URL: https://api.semanticscholar.org/CorpusID:783338 (cit. on p. 4).

[Por18] Porta, Mauro. *Derived complex analytic geometry I: GAGA theorems*. 2018. arXiv: 1506.09042 [math.AG]. URL: https://arxiv.org/abs/1506.09042 (cit. on p. 7).

[PY16] Porta, Mauro and Yu, Tony Yue. *Higher analytic stacks and GAGA theorems*. 2016. arXiv: 1412.5166 [math.AG]. URL: https://arxiv.org/abs/1412.5166 (cit. on p. 7).

[SC23] Scholze, Peter and Clausen, Dustin. "Youtube lectures on Analytic Stacks". In: (2023) (cit. on p. 7).

[Shi00] Shiho, Atsushi. "Crystalline fundamental groups. I. Isocrystals on log crystalline site and log convergent site". In: *J. Math. Sci. Univ. Tokyo* 7.4 (2000), pp. 509–656. ISSN: 1340-5705 (cit. on p. 4).

[Toë06] Toën, Bertrand. "Champs affines". In: *Selecta Math. (N.S.)* 12.1 (2006), pp. 39–135. ISSN: 1022-1824,1420-9020. URL: https://doi.org/10.1007/s00029-006-0019-z (cit. on pp. 2, 4, 6, 7).

[Vol01] Vologodsky, Vadim. *Hodge structure on the fundamental group and its application to p-adic integration*. 2001. arXiv: math/0108109 [math.AG] (cit. on p. 4).