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Deforming Galois representations and the conjectures of Serre and Fontaine-Mazur

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Introduction

Let p be an odd prime, $\mathbf{k}$ a finite field of characteristic p, and $W(\mathbf{k})$ the ring of Witt vectors of $\mathbf{k}$. The study of Galois representations $\bar{\rho} : G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{k})$ and their deformations to characteristic zero p-adic rings such as $W(\mathbf{k})$ has progressed rapidly in recent years. In much of this work $\bar{\rho}$ has been assumed to be modular; that is, $\bar{\rho}$ is the reduction of a p-adic Galois representation associated to a classical Hecke eigenform.

The assumption of modularity allows one to bring to bear the tools of algebraic geometry (e.g. Jacobians of modular curves, the theory of congruences). The works of [Ri], [W], [TW], [Di2], [CDT], and [BCDT] use these tools to great success. The seminal work of [W] and [TW] is of the following flavor. Let $\bar{\rho} : G_{\mathbf{Q}} = \mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}_2(\mathbf{k})$ be a modular mod p Galois representation. Then all deformations of $\bar{\rho}$ to p-adic rings unramified outside a finite set of primes R whose restrictions to the decomposition groups $G_v = \mathrm{Gal}(\bar{\mathbf{Q}}_{\mathbf{v}}/\mathbf{Q}_{\mathbf{v}})$ for $v \in R$ satisfy certain local conditions are themselves modular. Thus ρ factors through $G_R = \mathrm{Gal}(\mathbf{Q}_{\mathbf{R}}/\mathbf{Q})$ where R contains p and infinity and $\mathbf{Q}_{\mathbf{R}}$ is the maximal extension of $\mathbf{Q}$ unramified outside R. The assumed modularity of $\bar{\rho}$ implies that $\bar{\rho}$ is odd, that is $\det(\bar{\rho}(c)) = -1$ where c is complex conjugation. Serre conjectured that all odd absolutely irreducible mod p two dimensional Galois representations are modular of prescribed weight, level and character.

Fontaine and Mazur have conjectured that all p-adic Galois representations unramified outside a finite set R satisfying the local condition of *potential semistability* (a generalization of the local conditions alluded to above) that are not Tate twists of even representations with finite image arise in the étale cohomology of some smooth variety. See [Fo] for the definition of potentially semistable, semistable, crystalline etc. and [FM] for the precise statement of their conjecture. A consequence of their conjecture (combined with other well-known conjectures) is that all two dimensional p-adic Galois representations satisfying these local properties are modular. (It is known that representations coming from modular forms satisfy their local conditions.) This can be interpreted as a characteristic zero version of Serre's conjecture. It is interesting to note that there are no parity assumptions in the Fontaine-Mazur conjecture. Indeed, part of the conjecture is that *even* two dimensional p-adic Galois representations cannot satisfy their local conditions unless they do so vacuously as the Tate twists of representations with finite image.

Our approach is to start with a mod p Galois representation $\bar\rho$ that is *not* assumed to be modular. Let $\mathrm{Ad}^0\bar\rho$ be the set of all trace zero two-by-two matrices over $\mathbf{k}$ with Galois action through $\bar\rho$ and by conjugation. We may assume, after twisting, that $\bar\rho$ and $\mathrm{Ad}^0\bar\rho$ are ramified at exactly the same primes. Denote by S the union of this set of primes with $\{p,\infty\}$. In [R2] it is proved that, subject to certain conditions, $\bar\rho$ can be deformed to $W(\mathbf{k})$ by allowing ramification at a finite additional set of primes. There we could say nothing about the local at p properties of this deformation. Drawing on the ideas there and some new ones we generalize the results of [R2].

THEOREM 1. *Let $\mathbf{k}$ be a finite field of characteristic $p \geq 7$ and $\bar\rho : G_{\mathbf{Q}} \to \mathrm{GL}_2(\mathbf{k})$. Let χ be the* (mod p) *cyclotomic character.*

a) *If $\bar\rho$ is even and $\bar\rho\,|_{G_p}$ is not twist equivalent to $\begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$ or twist equivalent to the indecomposable representation $\begin{pmatrix} \chi^{p-2} & * \\ 0 & 1 \end{pmatrix}$ then there is a finite set R of primes containing p and infinity and a deformation $\rho : G_R \to \mathrm{GL}_2(W(\mathbf{k}))$ of $\bar\rho$.*

b) *If $\bar\rho$ is odd then there is a finite set R of primes containing p and infinity and a deformation $\rho : G_R \to \mathrm{GL}_2(W(\mathbf{k}))$ of $\bar\rho$. Suppose further that $\bar\rho\,|_{G_p}$ is not twist equivalent to the trivial representation or the indecomposable unramified representation given by $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. If $\bar\rho\,|_{G_p}$ is a twist of an ordinary representation then, up to a twist by a character of finite order, ρ can be taken to be ordinary. If inertia at p acts via fundamental characters of level 2 then, up to a twist by a character of finite order, ρ can be taken to be crystalline.*

We now give some consequences of Theorem 1. Khare has used Theorem 1 above in [Kh2] to prove the compatibility of Serre's conjecture with base change in certain cases. In a beautiful paper Taylor has recently proven in [T1] a version of the Fontaine-Mazur conjecture. A consequence of his result (combined with Theorem 1 above) is that often odd two dimensional mod p representations of $G_{\mathbf{Q}}$ occur in the torsion of GL_2 type abelian varieties. This can be regarded as evidence for Serre's conjecture. We also have the following corollary.

COROLLARY 1. a) *Let* $\bar{\rho} : G_{\mathbf{Q}} \to \mathrm{SL}_2(\mathbf{k})$ *be surjective. There is a finite set of primes* R *and a deformation* $\rho : G_R \to \mathrm{SL}_2(W(\mathbf{k}))$ *of* $\bar{\rho}$ *unramified outside the finite set* R.

b) *There exist infinitely many even surjective representations* $\rho : G_{\mathbf{Q}} \to \mathrm{SL}_2(\mathbf{Z}_7)$. *Each such representation is unramified outside a finite set of primes.*

Proof. For part a), since $\bar{\rho}$ is even by hypotheses, we must check that $\bar{\rho}\mid_{G_p}$ is not an excluded case described in Theorem 1, part a). If this were the case we know $\bar{\rho}\mid_{G_p}$ is a twist by a character ϕ of G_p of a representation in one of the two forms Theorem 1, part a). Taking determinants of the restriction to G_p, and keeping in mind $\det\bar{\rho} = 1$, we see $\chi = \phi^{-2}$ or $\chi = \phi^2$. In either case the mod p cyclotomic character χ must be the square of another character. It is an exercise in local class field theory to see this is not the case. Theorem 1 part a) applies so $\bar{\rho}$ deforms to $W(\mathbf{k})$. If the determinant of the deformation ρ is not 1 we may twist by a character to guarantee it will be 1. (Recall p is odd.) For part b) we recall that Mestre has shown in [Me] that $\mathrm{SL}_2(\mathbf{F_7})$ is a regular extension of $\mathbf{Q}(T)$. Specialization gives us an infinite supply of $\mathrm{SL}_2(\mathbf{F_7})$ extensions of $\mathbf{Q}$. By part a) these deform to $\mathrm{SL}_2(\mathbf{Z_7})$. The surjectivity follows from Chapter IV, Lemma 3 of [Se3].

We remark that the ρ's of Theorem 1 can be made *unique*. That is, when we insist that, up to a twist, $\rho\mid_{G_q} = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ for $q \in R - S$ there will be only one such ρ (subject to local conditions at primes in S described later in this paper). Note that *a priori* $\rho\mid_{G_q}$ could possibly be decomposable. For $\bar{\rho}$ odd, Taylor's theorem and the Weil bounds on eigenvalues of Frobenius imply $\rho\mid_{G_q}$ is indecomposable.

Finally, we would like to acknowledge that our use of the auxiliary sets of primes contained in R are chosen to essentially annihilate a dual Selmer group as in [W] and [TW]. Many of our computations mimic ideas of [W], [TW], [Di1], [CDT], [Boe1] and others. Also, see [T2] for a generalisation of the results in this paper. We are also grateful to Gebhard Böckle for explaining to us the details of the excluded cases in Theorem 1 part a) and we thank Chandrashekhar Khare for his many helpful suggestions.

Notation.

p: A prime ≥ 5.
$\mathbf{k}$: A finite field of characteristic $p \geq 5$.
$\tilde{\mathbf{k}}$: A subfield of $\mathbf{k}$ which is the minimal field of definition of $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$.
$\mathbf{k}(\phi)$: The one dimensional space $\mathbf{k}$ with Galois action by the character ϕ.
$W(\mathbf{k})$: The ring of Witt vectors of $\mathbf{k}$.

S:	The union of $\{p\}$, $\{\infty\}$, and the finite set of places at which $\mathrm{Ad}^0\bar{\rho}$ (or $\bar{\rho}$) is ramified.
Q:	A finite set of primes disjoint from S, whose elements lie in a certain Chebotarev classes.
$\mathbf{Q_v}$:	The completion of $\mathbf{Q}$ at the place v.
$\mathbf{Q_S}$:	The maximal separable extension of $\mathbf{Q}$ unramified outside the places of S.
$G_{\mathbf{Q}}$:	$\mathrm{Gal}(\bar{\mathbf{Q}}/\mathbf{Q})$.
G_S:	$\mathrm{Gal}(\mathbf{Q_S}/\mathbf{Q})$.
G_v:	$\mathrm{Gal}(\bar{\mathbf{Q}}_{\mathbf{v}}/\mathbf{Q_v})$.
I_v:	The inertia subgroup of G_v.
$\mathbf{1}$:	The two-by-two identity matrix over $W(\mathbf{k})$ or $W(\mathbf{k})/p^m$ for suitable m.
χ:	The cyclotomic character, either in characteristic zero or $\bmod p^m$.
$\bar{\rho}$:	A continuous representation from G_S or G_v to $\mathrm{GL}_2(\mathbf{k})$.
$\mathcal{N}_v$:	A subspace of $H^1(G_v, \mathrm{Ad}^0\bar{\rho})$.
$\mathcal{C}_v$:	Deformations of $\bar{\rho}\mid_{G_v}$ to $W(\mathbf{k})$ whose $\bmod p^n$ reductions are stable under the action $\mathcal{N}_v$.
$\mathrm{Ad}^0\bar{\rho}$:	The trace zero two-by-two matrices over $\mathbf{k}$ with G_S action through $\bar{\rho}$ and by conjugation.
$(\mathrm{Ad}^0\bar{\rho})^*$:	The Cartier dual of $\mathrm{Ad}^0\bar{\rho}$.
$\widetilde{\mathrm{Ad}}^0\bar{\rho}$:	A descent of $\mathrm{Ad}^0\bar{\rho}$ to its minimal field of definition $\tilde{\mathbf{k}}$.
$(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$:	A descent of $(\mathrm{Ad}^0\bar{\rho})^*$ to its minimal field of definition $\tilde{\mathbf{k}}$.

Throughout this paper we use **Fact** to denote a previously known result.

Deformation theory. In this section we recall some less well-known facts about deformation theory. We refer the reader to [M1] or [M2] for standard details of the theory. We only remark here that we will *always* fix the determinant of the deformations we consider. This corresponds to the fact that we must study the cohomology of $\mathrm{Ad}^0\bar{\rho}$ as opposed to that of all two-by-two matrices over $\mathbf{k}$. If $\det\bar{\rho} = \phi\chi^a$ where ϕ is unramified at p and $1 \leq a \leq p-1$, then we fix the determinant of our deformations to be $\tilde{\phi}\chi^a$ where $\tilde{\phi}$ is the Teichmüller lift of ϕ.

For the lemmas below we suppose $\bar{\rho} : G \to \mathrm{GL}_2(\mathbf{k})$ is a representation of a profinite group G whose image is in the upper triangular subgroup of $\mathrm{GL}_2(\mathbf{k})$. Let U^0 and U^1 respectively be the subgroups of $\mathrm{Ad}^0\bar{\rho}$ consisting of matrices of the form $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}$. Note that U^0 and U^1 are stable under the action of G.

LEMMA 1. *Suppose for $\bar{\rho}$ as above $H^2(G, U^0) = 0$. Let ϕ_1 and ϕ_2 be the diagonal characters of $\bar{\rho}$ with Teichmüller lifts $\tilde{\phi}_1$ and $\tilde{\phi}_2$. Denote by $\phi_{1,n}$ and $\phi_{2,n}$ the* mod p^n *reductions of $\tilde{\phi}_1$ and $\tilde{\phi}_2$. For $i = 1, 2$ let $\gamma_i : G \to (W(\mathbf{k}))^*$ be* any *characters congruent to* 1 mod p *with* mod p^n *reductions $\gamma_{i,n}$ such that $\tilde{\phi}_1\tilde{\phi}_2 = \tilde{\phi}_1\tilde{\phi}_2\gamma_1\gamma_2$. (The last hypothesis is to fix the determinant.) If $\rho_n : G \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ is given by $\begin{pmatrix} \phi_{1,n}\gamma_{1,n} & * \\ 0 & \phi_{2,n}\gamma_{2,n} \end{pmatrix}$ with* mod p *reduction $\bar{\rho}$, then there is a deformation $\rho_{n+1} : G \to \mathrm{GL}_2(W(\mathbf{k})/p^{n+1})$ with diagonal characters $\phi_{1,n+1}\gamma_{1,n+1}$ and $\phi_{2,n+1}\gamma_{2,n+1}$. In other words, the reducible deformation theory of $\bar{\rho}$ with fixed diagonal characters is unobstructed.*

Proof. See Theorems 6.1 and 6.2 of [R1] for related computations. One easily sees the obstruction to lifting lies in $H^2(G, U^0)$ which is assumed to be trivial.

LEMMA 2. *Suppose $H^2(G, U^1) = 0$. If $\rho_n : G \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ is an upper triangular representation whose* mod p *reduction is $\bar{\rho}$ then there is an upper triangular $\rho_{n+1} : G \to \mathrm{GL}_2(W(\mathbf{k})/p^{n+1})$ whose* mod p^n *reduction is ρ_n. In other words the reducible deformation theory of $\bar{\rho}$ is unobstructed.*

Proof. The obstruction to lifting lies in $H^2(G, U^1)$.

Facts from Galois cohomology. We recall, in our context, a few well-known facts from Galois cohomology that we will need. We refer the reader to [H], [Mi] and [Se2] for details and proofs. Here M is finite p torsion (local or global) Galois module. Let $G_V = \mathrm{Gal}(\mathbf{Q_V}/\mathbf{Q})$ where V is a (possibly infinite) set of primes including p and infinity and $\mathbf{Q_V}$ is the maximal extension of $\mathbf{Q}$ unramified outside V. Also, $G_v = \mathrm{Gal}(\bar{\mathbf{Q}}_\mathbf{v}/\mathbf{Q_v})$. Finally, $M^* = \mathrm{Hom}(M, \mu_p)$ is the Cartier dual of M.

Define $P_V^1(M)$ to be the restricted direct product $\square_{v \in V} H^1(G_v, M)$ with respect to the images of the inflation maps $H^1(G_v/I_v, M) \to H^1(G_v, M)$ for those $v \in V$ at which M is unramified. (So $M^{I_v} = M$ for such v.) We define $P_V^2(M)$ to be the direct sum $\oplus_{v \in V} H^2(G_v, M)$. For $i = 1, 2$ we denote the kernel of the map $H^i(G_V, M) \to P_V^i(M)$ by $Ш_V^i(M)$.

Fact 1 (*Local duality*). For $i = 0, 1, 2$ there is a perfect pairing $H^i(G_v, M) \times H^{2-i}(G_v, M^*) \to H^2(G_v, \mu_p) \simeq \mathbf{Z}/p$.

Fact 2 (*Local Euler-Poincaré characteristic*). Recall $\#M$ is a power of p.

1) For v finite, not equal to p we have $\#H^0(G_v, M) \cdot \#H^2(G_v, M) = \#H^1(G_v, M)$.

2) For $v = p$ we have $\#H^0(G_p, M) \cdot \#H^2(G_p, M) \cdot \#M = \#H^1(G_p, M)$.

Fact 3 (*Global duality*). 1) The groups $Ш^1_V(M^*)$ and $Ш^2_V(M)$ are dual to one another.

2) The images of the restriction maps $H^1(G_V, M) \to P^1_V(M)$ and $H^1(G_V, M^*) \to P^1_V(M^*)$ are exact annihilators of one another.

Fact 4 (*Global Euler-Poincaré characteristic*). For V finite we have

$$\frac{\#H^0(G_V,M)\cdot\#H^2(G_V,M)}{\#H^1(G_V,M)} = \frac{\#H^0(G_\infty,M)}{\#M}.$$

Fact 5. Let $\hat{\mathbf{Z}}$ be the completion of $\mathbf{Z}$. Then if M is a $\hat{\mathbf{Z}}$ module we have $\#H^0(\hat{\mathbf{Z}}, M) = \#H^1(\hat{\mathbf{Z}}, M)$.

LEMMA 3. *Let M be a finite dimensional* $\mathbf{k}$ *vector space with G_p action. Then*

$$\dim_{\mathbf{k}} H^1(G_p/I_p, M^{I_p}) = \dim_{\mathbf{k}} H^0(G_p, M).$$

Proof. Since $G_p/I_p \simeq \hat{\mathbf{Z}}$ we see

$$\dim_{\mathbf{k}} H^1(G_p/I_p, M^{I_p}) = \dim_{\mathbf{k}} H^0(G_p/I_p, M^{I_p}) = \dim_{\mathbf{k}} H^0(G_p, M).$$

Local at $l \neq p$ considerations. In this section we analyze, for $l \neq p$, local representations $\bar{\rho} : G_l \to \mathrm{GL}_2(\mathbf{k})$. We recall Diamond's classification of two dimensional mod p representations of G_l. Diamond has four cases that he calls P, S, V and H. We recall these below under the hypotheses that $\mathrm{Ad}^0\bar{\rho}$ is ramified. We do not give all of Diamond's equivalent formulations of each case and note that we interchange his roles of p and l. Let I_l be inertia at l and let $\tilde{I}_l$ and $\tilde{G}_l$ denote the images of I_l and G_l in the projective representation associated to $\bar{\rho}$. Note $\tilde{I}_l$ and $\tilde{G}_l$ act faithfully on $\mathrm{Ad}^0\bar{\rho}$. Here we assume $p \geq 5$.

- P: $\bar{\rho}$ is twist equivalent to $\begin{pmatrix} \phi & 0 \\ 0 & 1 \end{pmatrix}$ for some ramified character ϕ of G_l.

- S: a) $l \equiv 1 \bmod p$ and $\bar{\rho}$ is twist equivalent to $\begin{pmatrix} 1 & \phi \\ 0 & 1 \end{pmatrix}$ where ϕ is a ramified additive character of G_l, or

  b) $l \not\equiv 1 \bmod p$ and $\bar{\rho}$ is twist equivalent to the indecomposable representation $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ where χ is the mod p cyclotomic character.

- V: $\bar{\rho}$ is twist equivalent to $\mathrm{Ind}_{G_{\mathbf{E}}}^{G_l}\xi$ where $\mathbf{E}$ is the unramified quadratic extension of $\mathbf{Q_l}$ and ξ is a character of $G_{\mathbf{E}} = \mathrm{Gal}(\bar{\mathbf{E}}/\mathbf{E})$ that is not equal to its conjugate under the action of $\mathrm{Gal}(\mathbf{E}/\mathbf{Q_l})$.

- H: a) l is odd and $\bar{\rho}$ is twist equivalent to $\mathrm{Ind}_{G_{\mathbf{E}}}^{G_l}\xi$ where $\mathbf{E}$ is a ramified quadratic extension of $\mathbf{Q_l}$ and ξ is a character of $G_{\mathbf{E}} = \mathrm{Gal}(\bar{\mathbf{E}}/\mathbf{E})$ whose restriction to $I_{\mathbf{E}}$ is not equal to its conjugate under the action of $\mathrm{Gal}(\mathbf{E}/\mathbf{Q_l})$.

  b) $l = 2$, $\tilde{I}_l$ (respectively $\tilde{G}_l$) is isomorphic to D_4 (respectively A_4), A_4 (respectively A_4), or A_4 (respectively S_4).

We remark that here and throughout the rest of this paper we feel free to twist our $\bmod p$ Galois representations, both local and global, by characters of finite order. Twisting does *not* affect the cohomology of $\mathrm{Ad}^0\bar{\rho}$ and hence does *not* affect the deformation theory. Because ultimately we are interested in deformations to $W(\mathbf{k})$ and we can always "untwist" these deformations by the Teichmüller lift of these characters of finite order.

Recall the facts below from [R2]. The precise references in [R2] are included parenthetically.

Fact 6. $H^2(G_v, \mathrm{Ad}^0\bar{\rho}) \neq 0$ if and only if $\mathrm{Ad}^0\bar{\rho}$ has a one dimensional quotient by a two dimensional G_v stable subspace on which G_v acts by χ, the cyclotomic character. This results holds for all finite v including $v = p$. (Lemma 3 of [R2])

Fact 7. If $l \not\equiv \pm 1 \bmod p$ then $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$. (Proposition 2 of [R2].)

Fact 8. Suppose $\bar{\rho} : G \to \mathrm{GL}_2(\mathbf{k})$ where G is any profinite group. Let $\rho_n : G \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ be a deformation of $\bar{\rho}$ to $W(\mathbf{k})/p^n$. Suppose $\rho_{n+1,1}$ and $\rho_{n+1,2}$ are two deformations of ρ_n to $W(\mathbf{k})/p^{n+1}$. Then there is an element $g \in H^1(G, \mathrm{Ad}^0\bar{\rho})$ such that $(\mathbf{1} + p^n g).\rho_{n+1,1} = \rho_{n+1,2}$. (The remarks following the Fact in the Deformation Theory section of [R2])

LEMMA 4. *Let $\mathbf{E}/\mathbf{Q_l}$ be a quadratic extension. Suppose $\bar{\rho}$ is twist equivalent to $\mathrm{Ind}_{G_{\mathbf{E}}}^{G_l}\xi$ where ξ is a character of $G_{\mathbf{E}}$ that is not equal to its conjugate under the action of $\mathrm{Gal}(\mathbf{E}/\mathbf{Q_l})$. Then $\mathrm{Ad}^0\bar{\rho} \simeq A_1 \oplus A_2$ where A_i is an absolutely irreducible $\mathbf{k}[G_l]$ module with $\dim_{\mathbf{k}} A_i = i$ and $H^0(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$. Furthermore $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$ unless $\mathbf{E}/\mathbf{Q_l}$ is unramified and $l \equiv -1 \bmod p$ in which case $H^2(G_l, \mathrm{Ad}^0\bar{\rho})$ is one dimensional.*

Proof. We see, that up to a twist, $\bar{\rho}\,|_{G_{\mathbf{E}}} = \begin{pmatrix} \xi & 0 \\ 0 & \xi^\sigma \end{pmatrix}$ where σ generates $\mathrm{Gal}(\mathbf{E}/\mathbf{Q_l})$. Then as a $\mathbf{k}[G_{\mathbf{E}}]$ module we see $\mathrm{Ad}^0\bar{\rho} \simeq \mathbf{k} \oplus \mathbf{k}(\xi/\xi^\sigma) \oplus \mathbf{k}(\xi^\sigma/\xi)$. Observe $\mathrm{Gal}(\mathbf{E}/\mathbf{Q_l})$ acts nontrivially on $\mathbf{k}$ and permutes $\mathbf{k}(\xi/\xi^\sigma)$ and $\mathbf{k}(\xi^\sigma/\xi)$. We take $A_1 = \mathbf{k}$ and $A_2 = \mathbf{k}(\xi/\xi^\sigma) \oplus \mathbf{k}(\xi^\sigma/\xi)$ and the absolute irreducibility results are clear. That $H^0(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$ is also clear. As for the H^2 result, by Fact 6 we have $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) \neq 0$ if and only if $\mathrm{Ad}^0\bar{\rho}$ has a one dimensional quotient on which G_l acts via χ. But A_1 is the only one dimensional quotient of $\mathrm{Ad}^0\bar{\rho}$ and G_l acts via $\mathrm{Gal}(\mathbf{E}/\mathbf{Q_l})$. This is the action of χ precisely when $\mathbf{E}$ is unramified over $\mathbf{Q_l}$ and $l \equiv -1 \bmod p$. In this case, since A_1 is the only one dimensional quotient of $\mathrm{Ad}^0\bar{\rho}$ we have $H^2(G_l, \mathrm{Ad}^0\bar{\rho})$ is one dimensional.

PROPOSITION 1. *There is a class $\mathcal{C}_l$ of deformations of $\bar{\rho}$ to $W(\mathbf{k})$ and a subspace $\mathcal{N}_l$ of $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ of codimension $\dim_{\mathbf{k}} H^2(G_l, \mathrm{Ad}^0\bar{\rho})$ such that if we fix any $h \in H^1(G_l, \mathrm{Ad}^0\bar{\rho})$, $h \notin \mathcal{N}_l$ then $\bar{\rho}$ can be successively deformed to an element of $\mathcal{C}_l$ by deforming from $W(\mathbf{k})/p^n$ to $W(\mathbf{k})/p^{n+1}$ with adjustments at each step made only by a multiple of h.*

Proof. If $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$ then *all* deformation problems are unobstructed and we take $\mathcal{N}_l = H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ and $\mathcal{C}_l$ to be all deformations of $\bar{\rho}$ to $W(\mathbf{k})$. We do this because after deforming from $\bmod p^n$ to $\bmod p^{n+1}$ there is no need to adjust the deformation by an element of $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ so that we can deform to $W(\mathbf{k})/p^{n+2}$. Henceforth we need only consider cases where $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) \neq 0$. Thus by Fact 7 we may assume $l \equiv \pm 1 \bmod p$.

We treat Diamond's four cases separately. In each case we follow the outline below. First we must exhibit our class $\mathcal{C}_l$ of deformations of $\bar{\rho}$ to $W(\mathbf{k})$. If ρ_n is the $\bmod p^n$ reduction of an element in this class we will abuse notation and write $\rho_n \in \mathcal{C}_l$. Then we will find a one dimensional space of cohomology classes spanned by $g \in H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ such that for all $a \in \mathbf{k}$ we have $(\mathbf{1} + ap^{n-1}g).\rho_n \in \mathcal{C}_l$. Thus altering the $\bmod p^n$ reduction of an element in $\mathcal{C}_l$ by a multiple of g gives the $\bmod p^n$ reduction of another element in $\mathcal{C}_l$. We say that $\mathcal{N}_l$ preserves $\mathcal{C}_l$. We let $\mathcal{N}_l$ be the space spanned by g. ($\mathcal{N}_l$ will always be trivial or one dimensional in this section.) If $\mathcal{N}_l$ is codimension one in $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ (as it will be) we may choose h to be *any* element in $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ not in $\mathcal{N}_l$. Clearly ρ_n has a deformation ρ_{n+1} to $W(\mathbf{k})/p^{n+1}$ that is in $\mathcal{C}_l$. By Fact 8 *any* deformation τ_{n+1} of ρ_n to $W(\mathbf{k})/p^{n+1}$ differs from ρ_{n+1} by an element of $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$. Since h and g span $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ we see $\tau_{n+1} = (\mathbf{1} + (ah + bg)p^n).\rho_{n+1}$ for some $a, b \in \mathbf{k}$. Thus $(\mathbf{1} - ah).\tau_{n+1} = (\mathbf{1} + bgp^n).\rho_{n+1} \in \mathcal{C}_l$. The crucial point in the argument is that $\mathcal{N}_l$ is codimension one.

We apply the outline to the four cases.

In case P, we see $\mathrm{Ad}^0\bar{\rho} \simeq \mathbf{k}(\phi) \oplus \mathbf{k} \oplus \mathbf{k}(\phi^{-1})$, where ϕ is ramified. Since $l \neq p$ we see, since the cyclotomic character χ is unramified, that $\mathrm{Ad}^0\bar{\rho}$ has a quotient as in Fact 6 if and only if $l \equiv 1 \bmod p$. In this case using local duality we have $\dim_{\mathbf{k}} H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 1$. It is easy to see $\dim_{\mathbf{k}} H^0(G_l, \mathrm{Ad}^0\bar{\rho}) = 1$. Using the local Euler-Poincaré characteristic (Fact 2) we have $\dim_{\mathbf{k}} H^1(G_l, \mathrm{Ad}^0\bar{\rho}) = 2$. Let $\tilde{\phi}$ be the Teichmüller lift of ϕ. Let $\mathcal{C}_l$ be the set of deformations ρ of $\bar{\rho}$ to $W(\mathbf{k})$ that are given by $\begin{pmatrix} \tilde{\phi}\gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$ where $\gamma : G_l \to W(\mathbf{k})^*$ is *any* unramified character congruent to 1 mod p. (Recall we are fixing the determinant of all deformations we consider.) For $\sigma \in G_l$ let n_σ be its image under the composition $G_l \to G_l/I_l \simeq \hat{\mathbf{Z}} \to \mathbf{Z}/p$. Consider $g \in H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ given by $g(\sigma) = \begin{pmatrix} n_\sigma & 0 \\ 0 & -n_\sigma \end{pmatrix}$. For $\rho \in \mathcal{C}_l$ we study ρ_n. For $a \in \mathbf{k}$ we see that

$$(\mathbf{1} + ap^{n-1}g).\rho_n(\sigma) = \begin{pmatrix} \tilde{\phi}(\sigma)\gamma(\sigma)(1 + an_\sigma p^{n-1}) & 0 \\ 0 & \gamma^{-1}(\sigma)(1 - an_\sigma p^{n-1}) \end{pmatrix}.$$

This is the reduction mod p^n of an element of $\mathcal{C}_l$ with the character γ' chosen to be *any* unramified lift to $W(\mathbf{k})^*$ of the unramified mod p^n character $\gamma(1 + ap^{n-1}g)$. (It is easy to see such lifts exist.) Thus in case P we choose $\mathcal{N}_l$ to be the space spanned by g. If we choose $h \in H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ so that $h \notin \mathcal{N}_l$ the proposition is proved in this case.

In both parts of case S observe $\bar{\rho}$ cuts out a tamely ramified extension of $\mathbf{Q_l}$ so any deformation to $W(\mathbf{k})$ factors through the Galois group of the maximal tamely ramified extension of $\mathbf{Q_l}$. This quotient of $\mathrm{Gal}(\bar{\mathbf{Q}}_\mathbf{l}/\mathbf{Q_l})$ is well understood. It is topologically generated by σ_l and τ_l where τ_l topologically generates inertia and the relation $\sigma_l \tau_l \sigma_l^{-1} = \tau_l^l$ holds. In both cases the class $\mathcal{C}_l$ is defined to be deformations of $\bar{\rho}$ to $W(\mathbf{k})$ given by $\tau_l \mapsto \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}$ and $\sigma_l \mapsto \begin{pmatrix} l & x \\ 0 & 1 \end{pmatrix}$ for any suitable $x, y \in W(\mathbf{k})$. Note any mod p^n deformation of $\bar{\rho}$ given as above deforms to $W(\mathbf{k})$. One need only lift x and y from $W(\mathbf{k})/p^n$ to $W(\mathbf{k})$.

In case S, part a) one sees, using that $l \equiv 1 \bmod p$ and local duality (Fact 1), that $\dim_\mathbf{k} H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 1$ and $\dim_\mathbf{k} H^0(G_l, \mathrm{Ad}^0\bar{\rho}) = 1$. By the local Euler-Poincaré characteristic we see $\dim_\mathbf{k} H^1(G_l, \mathrm{Ad}^0\bar{\rho}) = 2$. Consider the element $g \in H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ given by $g(\tau_l) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $g(\sigma_l) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We see that if ρ_n is the mod p^n reduction of an element of $\mathcal{C}_l$ and $a \in \mathbf{k}$ we have that

$$(\mathbf{1} + ap^{n-1}g).\rho_n(\tau_l) = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix}, \quad (\mathbf{1} + ap^{n-1}g).\rho_n(\sigma_l) = \begin{pmatrix} l & x + ap^{n-1} \\ 0 & 1 \end{pmatrix}.$$

Thus $(\mathbf{1} + ap^{n-1}g).\rho_n \in \mathcal{C}_l$ with a different value for x. In case S part a) we let $\mathcal{N}_l$ be the space of codimension one in $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ spanned by g and we choose h to be any element of $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ not in $\mathcal{N}_l$.

We consider case S, part b). By the remark at the end of the first paragraph of this proof we may assume $l \equiv -1 \bmod p$. Note that in this case $\chi = \chi^{-1}$. One sees that $\mathrm{Ad}^0\bar{\rho}$ has a unique one dimensional quotient on which

G_l acts via $\chi = \chi^{-1}$. Using Facts 1 and 2 one shows both $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ and $H^2(G_l, \mathrm{Ad}^0\bar{\rho})$ are one dimensional and we choose $\mathcal{N}_l = \{0\}$ and h any nonzero element of $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$. We have now proved the proposition for case S.

In case V, by Lemma 4 we have that $H^0(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$ and that if $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) \neq 0$ its dimension is one. By the local Euler-Poincaré characteristic we see $\dim_{\mathbf{k}} H^1(G_l, \mathrm{Ad}^0\bar{\rho}) = 1$. Thus if $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) \neq 0$ we take $\mathcal{N}_l = 0$ and $\mathcal{C}_l$ to be the Teichmüller deformation of $\bar{\rho}$. (The order of the image of $\bar{\rho}$ is prime to p as here $\bar{\rho}$ is a two dimensional representation induced from a character with values in the multiplicative group of a finite field of characteristic p.)

In case H, if l is odd we have, since $\mathbf{E}/\mathbf{Q_l}$ is ramified, that $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$ by Lemma 4. If $l = 2$, then keeping in mind $p \geq 5$ so the order of the image of $\bar{\rho}$ is prime to p, we see $\mathrm{Ad}^0\bar{\rho}$ is semisimple as a $\mathbf{k}[G_l]$ module. In all the possibilities $\tilde{G}_l$ contains A_4 as a subgroup that acts faithfully on $\mathrm{Ad}^0\bar{\rho}$. But from the representation theory of A_4 in characteristics other than 2 or 3 one sees that $\mathrm{Ad}^0\bar{\rho}$ must be absolutely irreducible and thus $\mathrm{Ad}^0\bar{\rho}$ has no one dimensional quotients by G_l stable $\mathbf{k}$ subspaces. Thus by Fact 6 we see $H^2(G_l, \mathrm{Ad}^0\bar{\rho}) = 0$ for $l = 2$ as well.

Remark. In all cases we find a *smooth* quotient of the local deformation ring in $\dim H^1$-$\dim H^2$ variables. In each case $\mathcal{N}_l$ is the subspace of $H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ one gets from the surjective map from the deformation ring to this smooth quotient by dualizing the map on tangent spaces.

Local at p considerations. Recall we assume $p \geq 5$. In this section we carry out an analysis of the possible $\bar{\rho} : G_p \to \mathrm{GL}_2(\mathbf{k})$ and their deformations to $W(\mathbf{k})$. Note that here we do *not* assume that $\bar{\rho}$ and $\mathrm{Ad}^0\bar{\rho}$ are ramified at p. We determine, in most cases two kinds of $\mathcal{N}_p$ and $\mathcal{C}_p$, one for the odd case and one for the even case. We will insist that

$$\dim_{\mathbf{k}} H^1(G_p, \mathrm{Ad}^0\bar{\rho}) - \dim_{\mathbf{k}} H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = \dim_{\mathbf{k}} \mathcal{N}_p + \delta$$

where $\delta = 0$ or 2 as our global situation is even or odd. (Actually, in the even case we will require $H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = 0$. This assumption is simply a shortcoming of our method.) Our purpose, as in the previous section, is to show $\bar{\rho}$ can be deformed to an element of $\mathcal{C}_p$ by deforming from $W(\mathbf{k})/p^m$ to $W(\mathbf{k})/p^{m+1}$ with adjustments at each step made only by an element of a subspace of $H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ complementary to $\mathcal{N}_p$.

We refer the reader to the end of this section for an explanation of the reasons why some cases provide difficulties. We advise the reader, during a first reading, to accept the values in the tables of this section and that $\mathcal{N}_p$ preserves $\mathcal{C}_p$ and proceed to the Recollections section. Finally, we note that the results here are due to Böckle and can be found in [Boe2]. We have recast his results in a language more suitable for our application.

We treat the cases where inertia acts via fundamental characters of level one and two separately.

First suppose that inertia acts through $\bar{\rho}$ via fundamental characters of level two. Following Section 2.2 of [Se1], we may assume (after possibly extending scalars) that $\bar{\rho}\mid_{I_p}=\begin{pmatrix}\psi^{u+pv} & 0\\ 0 & \psi^{v+pu}\end{pmatrix}$ where ψ is a fundamental character of level two. After twisting by a suitable power of $\chi=\psi^{p+1}$ we may assume $\bar{\rho}\mid_{I_p}=\begin{pmatrix}\psi^{a} & 0\\ 0 & \psi^{pa}\end{pmatrix}$ where $1\le a\le p-1$. We suppose $\det(\bar{\rho})=\phi\chi^a$ where ϕ is an unramified character of G_p. We only consider deformations with determinant $\tilde{\phi}\chi^a$ where $\tilde{\phi}$ is the Teichmüller lift of ϕ.

It is shown in Lemma 5 of [R2] that $H^2(G_p,\mathrm{Ad}^0\bar{\rho})=0$ in the above case. Thus *all* deformation problems are unobstructed when inertia at p acts via fundamental characters of level two. The argument of Lemma 4 shows $H^0(G_p,\mathrm{Ad}^0\bar{\rho})=0$ so $\dim_{\mathbf{k}}H^1(G_p,\mathrm{Ad}^0\bar{\rho})=3$ by the local Euler-Poincaré characteristic. The universal deformation ring is a power series ring over $W(\mathbf{k})$ in three variables. By Theorem $B2$ of [FM] we have that $\bar{\rho}$ is crystalline and the universal crystalline deformation ring (since the determinant is fixed) is a power series ring in one variable over $W(\mathbf{k})$. (Actually, the result of [FM] is for $\bar{\rho}:G_p\to \mathrm{GL}_2(\mathbf{F_p})$ and deformations to $\mathbf{Z_p}$. It is easy to see that their results extend to $\mathbf{k}$ and $W(\mathbf{k})$. Alternatively, the arguments of Theorem 3.1 and Lemma 4.4 of [R1] are easily generalized to this situation. Also, [FM] computes the universal crystalline deformation ring with arbitrary determinant which is smooth in two variables. The determinant restriction eliminates one of these variables.) In the odd case it is this family of crystalline deformations to $W(\mathbf{k})$ with det $=\tilde{\phi}\chi^a$ that we take to be $\mathcal{C}_p$. In the even case we let $\mathcal{C}_p$ be the family of *all* deformations of $\bar{\rho}$ to $W(\mathbf{k})$. In the odd case we take $\mathcal{N}_p$ to be $H^1_{cr}(G_p,\mathrm{Ad}^0\bar{\rho})$, the one dimensional subspace of $H^1(G_p,\mathrm{Ad}^0\bar{\rho})$ that is the tangent space of the crystalline ring. If $\bar{\rho}$ is even we choose $\mathcal{N}_p$ to be *all* of $H^1(G_p,\mathrm{Ad}^0\bar{\rho})$. In tables below $h^i=\dim_{\mathbf{k}}H^i(G_p,\mathrm{Ad}^0\bar{\rho})$.

Table 1, $\bar{\rho}\mid_{I_p}=\begin{pmatrix}\psi^{a} & 0\\ 0 & \psi^{pa}\end{pmatrix}$

$\mathcal{C}_p$, odd	$\mathcal{N}_p$, odd, dim	$\mathcal{C}_p$, even	$\mathcal{N}_p$, even	h^0	h^1	h^2
Crystalline defs	H^1_{cr}, 1	All	All	0	3	0

The word "All" refers to $\mathcal{C}_p$ being all deformations to $W(\mathbf{k})$ and $\mathcal{N}_p$ being all of $H^1(G_p,\mathrm{Ad}^0\bar{\rho})$. Recall the statement "$\mathcal{N}_p$ preserves $\mathcal{C}_p$" means that if $\rho\in\mathcal{C}_p$ with $\bmod p^n$ reduction ρ_n then $(\mathbf{1}+p^{n-1}g).\rho_n$ is the $\bmod p^n$ reduction of an element of $\mathcal{C}_p$ for all $g\in\mathcal{N}_p$. That $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in the odd case follows from the existence and smoothness of the crystalline deformation ring. That $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in the even case follows from the existence and smoothness of the unrestricted deformation ring. We have thus proved the following proposition.

PROPOSITION 2. *Suppose inertia acts through $\bar{\rho}$ via fundamental characters of level* 2. *Then the data in Table* 1 *above is correct, and in both the odd and even cases $\mathcal{N}_p$ preserves $\mathcal{C}_p$.*

It remains to consider the case when inertia acts on the semisimplification of $\bar{\rho}$ via fundamental characters of level one. After twisting by a character of order prime to p we may assume that $\bar{\rho} = \begin{pmatrix} \phi\chi^a & 0 \\ 0 & 1 \end{pmatrix}$ or $\bar{\rho} = \begin{pmatrix} \phi\chi^a & * \\ 0 & 1 \end{pmatrix}$ and is indecomposable. Here $1 \leq a \leq p-1$, and $\phi : G_p \to \mathbf{k}^*$ is an unramified character.

We fix the determinant of all deformations we consider to be $\tilde{\phi}\chi^a$ where $\tilde{\phi}$ is the Teichmüller lift of ϕ. Each of the two cases above breaks into several subcases. In most subcases, we find, for $\bar{\rho}$ both even and odd, classes $\mathcal{C}_p$ of deformations of $\bar{\rho}$ to $W(\mathbf{k})$ and subspaces $\mathcal{N}_p$ of $H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ such that the action of an element of $\mathcal{N}_p$ preserves $\mathcal{C}_p$.

In Tables 2 and 3 below, for the split and indecomposable cases, we give the relevant data. We again take h^i to be $\dim_{\mathbf{k}} H^i(G_p, \mathrm{Ad}^0\bar{\rho})$. Note that by the local Euler-Poincaré characteristic (Fact 2) we always have $h^1 = 3 + h^0 + h^2$. Let us now explain (with proof) all the information in the Tables 2 and 3 except the values of $\mathcal{N}_p$ and $\mathcal{C}_p$ in the odd case. The sets of dashes indicate the two (even) cases that we cannot produce a (global) deformation of $\bar{\rho}$ to $W(\mathbf{k})$. By Fact 6 we know $h^2 = 0$ unless $\mathrm{Ad}^0\bar{\rho}$ has a one dimensional quotient (by a two dimensional G_p stable subspace) on which G_p acts via χ. In the split case this happens only when $\phi\chi^a = \chi$ or χ^{p-2}. But these cases are twists of each other and are treated together. Recall $U^1 \subset \mathrm{Ad}^0\bar{\rho}$ is the set of all two-by-two trace zero matrices over $\mathbf{k}$ of the form $\begin{pmatrix} x & y \\ 0 & -x \end{pmatrix}$. Then for $\bar{\rho}$ indecomposable U^1 is the unique two dimensional G_p stable subspace of $\mathrm{Ad}^0\bar{\rho}$. One sees that G_p acts on the quotient $\mathrm{Ad}^0\bar{\rho}/U^1$ via $\phi^{-1}\chi^{-a}$. Thus $h^2 \neq 0$ exactly when $\phi\chi^a = \chi^{-1} = \chi^{p-2}$. In either the split or indecomposable case we see $h^2 = 1$. Note that these situations are excluded in Theorem 1, part a). The h^0 computations are easy and left to the reader. The h^1 results follow immediately. When $h^2 = 0$, for $\bar{\rho}$ even, we choose $\mathcal{C}_p$ to be all deformations of $\bar{\rho}$ to $W(\mathbf{k})$ and $\mathcal{N}_p$ to be $H^1(G_p, \mathrm{Ad}^0\bar{\rho})$. Since $H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = 0$ the deformation ring is smooth and it is clear that $\mathcal{N}_p$ preserves $\mathcal{C}_p$. As remarked earlier, when $H^2(G_p, \mathrm{Ad}^0\bar{\rho}) \neq 0$ we do not know how to proceed in the even case.

The dimension of $\mathcal{N}_p$ and the fact that $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in the odd case with inertia acting via fundamental characters of level one are justified in the rest of this section. We also need to prove that the various $\mathcal{C}_p$ are nonempty!

Roughly speaking we are showing the relevant (restricted) deformation ring is smooth. This will complete our verification of the tables.

Suppose that $\bar{\rho}$ is *flat*, that is the Galois action of $\bar{\rho}$ on the two dimensional $\mathbf{k}$ vector space is the generic fiber of a finite flat group scheme over $\mathbf{Z_p}$. This notion corresponds to $\bar{\rho}$ being weight 2 of level prime to p in the (global) modular case. A flat deformation ring exists. (See [R1].) If $\bar{\rho}$ is indecomposable then it is established in [R1] that the deformation ring is universal and smooth in one variable. (In [R1] arbitrary determinants are allowed, so the ring there is smooth in two variables.) We take the $W(\mathbf{k})$ valued points of this deformation ring to be $\mathcal{C}_p$ and the tangent space of this ring, H^1_{flat}, to be $\mathcal{N}_p$. (Note $\mathcal{C}_p$ is nonempty!) The existence and smoothness of this deformation ring implies $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in this case. If $\bar{\rho}$ is flat and split, i.e. $\bar{\rho} = \begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$, then the flat deformation is only versal. One easily computes the tangent space to be two dimensional (keeping in mind we are fixing the determinant) and one sees that $\mathcal{N}_p$ preserves $\mathcal{C}_p$ by the following argument. If $\rho \in \mathcal{C}_p$ with $\operatorname{mod} p^n$ reduction ρ_n consider, for $g \in \mathcal{N}_p$, $(\mathbf{1} + p^{n-1}g).\rho_n$. The underlying Galois module is the generic fiber of a finite flat group scheme over $\mathbf{Z_p}$ and one sees the corresponding Fontaine-Laffaille module (or Honda system if one prefers) lifts to a p-dvisible group. Thus $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in the case where $\bar{\rho}$ is flat and split. This argument justifies all the values in Tables 2 and 3 for $\bar{\rho}$ flat in the odd case.

In most of the remaining cases we take $\mathcal{N}_p$ to be $H^1_{\text{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$, the *ordinary* cohomology classes in $H^1(G_p, \mathrm{Ad}^0\bar{\rho})$. Recall U^0 is the subgroup of $\mathrm{Ad}^0\bar{\rho}$ consisting of matrices of the form $\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}$. Then we define $H^1_{\text{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ to be the kernel of the map

$$H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))^{G_p/I_p}.$$

We first compute the dimensions of those $\mathcal{N}_p$ that are not taken to be $H^1_{\text{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ before proceeding to the ordinary cases. Afterwards we will establish that $\mathcal{N}_p$ preserves the nonempty set $\mathcal{C}_p$. We remark here that the values in Table 4 are established by routine computations in Galois cohomology so no proofs are given.

PROPOSITION 3. *Suppose* $\bar{\rho} = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ *is indecomposable and not flat and we are in the odd case. Then* $\mathcal{N}_p$ *is one dimensional.*

Proof. In this case we define $\mathcal{N}_p$ to be the kernel of the map $H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho}/U^0)$. We use the symbol $H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$ for this $\mathcal{N}_p$. The result follows from taking G_p cohomology of the short exact sequence

$$0 \to U^0 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^0 \to 0$$

of G_p modules. We see $(U^0)^{G_p}$ and $(\mathrm{Ad}^0\bar\rho)^{G_p}$ are trivial so we have the sequence

$$0 \to (\mathrm{Ad}^0\bar\rho/U^0)^{G_p} \to H^1(G_p, U^0) \to H^1(G_p, \mathrm{Ad}^0\bar\rho) \to H^1(G_p, (\mathrm{Ad}^0\bar\rho/U^0)).$$

One easily sees the first three terms have dimension 1, 2 and 3. The result follows.

In the tables below we give the relevant data for the local at p deformation problems in the even and odd cases. The entries $\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ mean all upper triangular deformations of $\bar\rho$ to $W(\mathbf{k})$. The term $\overline{H^1(G_p, U^1)}$ means the image of $H^1(G_p, U^1)$ under the map $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar\rho)$. In the tables $\gamma : G_p \to W(\mathbf{k})^*$ is *any* unramified character congruent to 1 mod p. Recall $\tilde\phi : G_p \to W(\mathbf{k})^*$ is the Teichmüller lift of the unramified character $\phi : G_p \to \mathbf{k}^*$.

Table 2, $\bar\rho = \begin{pmatrix} \phi\chi^a & 0 \\ 0 & 1 \end{pmatrix}$

$\phi\chi^a$	$\mathcal{C}_p$, odd	$\mathcal{N}_p$, odd, dim	$\mathcal{C}_p$, even	$\mathcal{N}_p$, even	h^0	h^1	h^2
$\phi\chi^a = \mathrm{Id}$	$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$	$\overline{H^1(G_p, U^1)}, 4$	All	All	3	6	0
$\phi\chi^a = \chi, \chi^{p-2}$	Flat defs	H^1_{flat} , 2	—	—	1	5	1
$\phi\chi^a \neq \mathrm{Id}; \chi, \chi^{p-2}$	$\rho = \begin{pmatrix} \chi^a\gamma & * \\ 0 & \gamma^{-1} \end{pmatrix}$	H^1_{ord}, 2	All	All	1	4	0

Table 3, $\bar\rho = \begin{pmatrix} \phi\chi^a & * \\ 0 & 1 \end{pmatrix}$

$\phi\chi^a$	$\mathcal{C}_p$, odd	$\mathcal{N}_p$, odd,dim	$\mathcal{C}_p$, even	$\mathcal{N}_p$, even	h^0	h^1	h^2
$\phi\chi^a = \mathrm{Id}$, unramified	$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$	$\overline{H^1(G_p, U^1)}, 2$	All	All	1	4	0
$\phi\chi^a =$ Id, ramified	$\begin{pmatrix} \chi^{p-1}\gamma & * \\ 0 & \gamma^{-1} \end{pmatrix}$	H^1_{ord}, 2	All	All	1	4	0
$\phi\chi^a = \chi$, Flat	Flat Defs	H^1_{flat} , 1	All	All	0	3	0
$\phi\chi^a = \chi$, Not Flat	$\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$	$H^1_{\mathrm{nf}}, 1$	All	All	0	3	0
$\phi\chi^a = \chi^{p-2}$	$\begin{pmatrix} \chi^{p-2}\gamma & * \\ 0 & \gamma^{-1} \end{pmatrix}$	H^1_{ord}, 1	—	—	0	4	1
$\phi\chi^a \neq \mathrm{Id}, \chi, \chi^{p-2}$	$\begin{pmatrix} \chi^a\gamma & * \\ 0 & \gamma^{-1} \end{pmatrix}$	H^1_{ord}, 1	All	All	0	3	0

We compute $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ via some exact sequences. Associated to the short exact sequence

$$0 \to U^0 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^0 \to 0$$

of G_p modules we have the long exact sequence of group cohomology. The part we are interested in is

$$\cdots \to H^1(G_p, U^0) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to H^1(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) \to H^2(G_p, U^0).$$

We also have the inflation restriction sequence

$$\begin{aligned} 0 &\to H^1(G_p/I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p}) \to H^1(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) \\ &\to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))^{G_p/I_p} \to 0. \end{aligned}$$

The last term in this sequence is in fact $H^2(G_p/I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p})$ which is trivial as $G_p/I_p \simeq \hat{\mathbf{Z}}$, a group of cohomological dimension 1. Gluing these sequences together as below we have

$$\begin{tikzcd} & 0 \arrow[d] & \\ & H^1(G_p/I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p}) \arrow[d] & \\ H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \arrow[r] & H^1(G_p, \mathrm{Ad}^0\bar{\rho}/U^0) \arrow[r] \arrow[d] & H^2(G_p, U^0) \\ & H^1(I_p, \mathrm{Ad}^0\bar{\rho}/U^0)^{G_p/I_p} \arrow[d] & \\ & 0 & \end{tikzcd}$$

LEMMA 5. *$H^2(G_p, U^0) \neq 0$ exactly when ϕ is trivial and $a = 1$ (or $a = p - 2$ in the split case which is twist equivalent to the case when $a = 1$). If $H^2(G_p, U^0) = 0$ then*

$$\dim_{\mathbf{k}} H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho}) = 1 + h^0 + h^2 - \dim_{\mathbf{k}} H^2(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)).$$

Proof. Recall $h^i = \dim_{\mathbf{k}} H^i(G_p, \mathrm{Ad}^0\bar{\rho})$ and G_p acts on U^0 via $\phi\chi^a$. But $H^2(G_p, U^0) \neq 0$ if and only if this action equals that of χ. If $H^2(G_p, U^0) = 0$ then a simple diagram chase shows

$$H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to H^1(I_p, \mathrm{Ad}^0\bar{\rho}/U^0)^{G_p/I_p}$$

is surjective. Since $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ is the kernel of this map we have

$$\begin{aligned} \dim_{\mathbf{k}} H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho}) &= h^1 - \dim_{\mathbf{k}} H^1(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) \\ &\quad + \dim_{\mathbf{k}} H^1(G_p/I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p}). \end{aligned}$$

By Lemma 3

$$\dim_{\mathbf{k}} H^1(G_p/I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p}) = \dim_{\mathbf{k}} H^0(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)).$$

The result finally follows by using the local Euler-Poincaré characteristic to reduce all H^1's to H^0's and H^2's.

We compute the dimensions of the $\mathcal{N}_p$ for $\bar{\rho}$ odd and I_p acting via fundamental characters of level one. Suppose $\bar{\rho}$ is split.

- Suppose $\phi\chi^a = \mathrm{Id}$. Since $\bar{\rho}$ is trivial the sequence
$$0 \to U^1 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0$$
of $\mathbf{k}[G_p]$ modules splits. Taking G_p cohomology we see the map β : $H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ is injective. One finds $\dim_{\mathbf{k}} H^1(G_p, U^1) = 4$ so $\dim_{\mathbf{k}} \overline{H^1(G_p, U^1)} = 4$ and we are done.
- Suppose $\phi\chi^a = \chi$ or χ^{p-2}. In these cases $\bar{\rho}$ is twist equivalent to $\begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$. Since $\bar{\rho}$ is flat we see from the remarks prior to Proposition 3 that the flat tangent space is two dimensional.
- $\phi\chi^a \neq \mathrm{Id}, \chi, \chi^{p-2}$. By Lemma 5 we see $H^2(G_p, U^0) = 0$. We compute $h^0 = 1$ and $h^2 = 0$. By the latter part of Lemma 5 we have $H^2(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) = 0$ so $\dim_{\mathbf{k}} H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho}) = 2$.

Suppose $\bar{\rho}$ is indecomposable. This case also splits into several parts. Before addressing these we include Table 4 below which will be useful for calculations throughout the rest of this section. The numbers represent the $\mathbf{k}$ dimension of the cohomology group when $\bar{\rho}$ is indecomposable as described. Note that $\bar{\rho}$ flat is omitted from Table 4 since we have already addressed this case. The values in the table below are easily verified and therefore considered proved. The H^2 values are found via minor variant of Fact 6, the H^0 values are easy to compute, and the H^1's are computed by the local Euler-Poincaré characteristic (Fact 2). We use Table 4 to verify the values in Tables 2 and 3 above for $\bar{\rho}$ indecomposable and to show (in the rest of this section) that $\mathcal{N}_p$ preserves $\mathcal{C}_p$.

Table 4, $\bar{\rho} = \begin{pmatrix} \phi\chi^a & * \\ 0 & 1 \end{pmatrix}$

$\phi\chi^a$	$H^1(G_p, U^1)$	$H^1(G_p, (U^1/U^0))$	$H^2(G_p, U^0)$	$H^2(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0))$
$\phi\chi^a = \mathrm{Id}$, unramified	3	2	0	0
$\phi\chi^a = \mathrm{Id}$, ramified	3	2	0	0
$\phi\chi^a = \chi$, Not Flat	2	2	1	0
$\phi\chi^a = \chi^{p-2}$	2	2	0	1
$\phi\chi^a \neq \mathrm{Id}, \chi, \chi^{p-2}$	2	2	0	0

Let us now turn to verifying the dimensions of $\mathcal{N}_p$ in the odd indecomposable cases that remain.

- $\phi\chi^a = \mathrm{Id}$ and $\bar{\rho}$ is unramified. Consider the short exact sequence

$$0 \to U^1 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0$$

and take G_p cohomology. We get the long exact sequence

$$\begin{aligned} 0 &\to (U^1)^{G_p} \to (\mathrm{Ad}^0\bar{\rho})^{G_p} \to (\mathrm{Ad}^0\bar{\rho}/U^1)^{G_p} \to H^1(G_p, U^1) \\ &\xrightarrow{\beta} H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to \cdots \end{aligned}$$

whose first three terms are easily seen to be one dimensional. Consulting Table 4 the next term is three dimensional. Thus the map $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ has a two dimensional image so

$$\dim_{\mathbf{k}} \overline{H^1(G_p, U^1)} = 2$$

and we are done.

- $\phi\chi^a = \mathrm{Id}$ and $\bar{\rho}$ is ramified. As $H^2(G_p, U^0) = 0$, Lemma 5 applies. Since $h^0 = 1$, $h^2 = 0$ and $\dim_{\mathbf{k}} H^2(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) = 0$ we see $\dim_{\mathbf{k}}\mathcal{N}_p = \dim_{\mathbf{k}} H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho}) = 2$.

- $\phi\chi^a = \chi$ and $\bar{\rho}$ is flat. This case has been addressed. $\mathcal{N}_p = H^1_f$ is one dimensional.

- $\phi\chi^a = \chi$ and $\bar{\rho}$ is not flat. This case has been addressed by Proposition 3.

- $\phi\chi^a = \chi^{p-2}$. Since $H^2(G_p, U^0) = 0$ and $h^0 = 0$, $h^2 = 1$, and

$$\dim_{\mathbf{k}} H^2(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) = 1$$

we see by Lemma 5

$$\dim_{\mathbf{k}} H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho}) = 1.$$

- $\phi\chi^a \neq \mathrm{Id}$, χ, χ^{p-2}. Since $H^2(G_p, U^0) = 0$, $h^0 = 0$, $h^2 = 0$ and $\dim_{\mathbf{k}} H^2(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) = 0$ we see $\dim_{\mathbf{k}} H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho}) = 1$ by Lemma 5.

We have established the correctness of the values in Tables 2 and 3. We now turn to the questions of showing, when $\bar{\rho}$ is odd and inertia acts via fundamental characters of level one, the various $\mathcal{C}_p$'s are nonempty and preserved by their respective $\mathcal{N}_p$'s. Note that we have already established this for $\bar{\rho}$ flat and odd. We need not worry about the even cases as there we have taken $\mathcal{C}_p$ to be *all* deformations to the smooth deformation ring and it is clear $\mathcal{N}_p$ preserves $\mathcal{C}_p$.

First we show the $\mathcal{C}_p$ are nonempty.

LEMMA 6. *Suppose we are in the odd case and $\bar{\rho}$ is not an indecomposable representation $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$. Then there is a deformation ρ of $\bar{\rho}$ to $W(\mathbf{k})$ of the form $\begin{pmatrix} \tilde{\phi}\chi^a\gamma & * \\ 0 & \gamma^{-1} \end{pmatrix}$ where $\gamma : G_p \to W(\mathbf{k})^*$ is an unramified character congruent to* 1 mod p. *Thus $\mathcal{C}_p$ is nonempty.*

Proof. We show $\bar{\rho}$ deforms to a characteristic zero representation of the form $\rho = \begin{pmatrix} \tilde{\phi}\chi^a\gamma & * \\ 0 & \gamma^{-1} \end{pmatrix}$ regardless of whether $\mathcal{N}_p = H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ or $\overline{H^1(G_p, U^1)}$. Observe in either case $\rho \in \mathcal{C}_p$.

Note $\bar{\rho}$ may or may not be decomposable. By hypothesis $\phi\chi^a \neq \chi$. From Table 4 we have $H^2(G_p, U^0) = 0$. Let $\gamma : G_p \to W(\mathbf{k})^*$ be any unramified character congruent to 1 mod p. By Lemma 1 the deformation theory for $\bar{\rho}$ where we only consider reducible deformations with diagonal characters $\tilde{\phi}\chi^a\gamma$ and γ^{-1} is unobstructed. Thus $\rho \in \mathcal{C}_p$ as desired.

LEMMA 7. *Suppose $\bar{\rho} = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ and is indecomposable. $\mathcal{C}_p$ is nonempty in this case.*

Proof. It suffices to show $\bar{\rho}$ lifts to a characteristic zero representation of the form $\begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$. Note the kernel of $\bar{\rho}$ fixes the field $\mathbf{Q_p}(\mu_p, a^{1/p})$ where μ_p are the p^{th} roots of unity and $a \in \mathbf{Q_p}(\mu_p)$. One can find a family of compatible deformations ρ_n of $\bar{\rho}$ to $W(\mathbf{k})/p^n$ that factors through $\mathrm{Gal}(\mathbf{Q_p}(\mu_{p^n}, a^{1/p^n})/\mathbf{Q_p})$ by Kummer theory. This completes the proof.

We have now shown $\mathcal{C}_p$ is nonempty for all $\bar{\rho}$ in the odd case. It remains to prove $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in the odd cases where inertia acts via fundamental characters of level one.

PROPOSITION 4. *Suppose $\bar{\rho}$ is odd and split. Then $\mathcal{N}_p$ preserves $\mathcal{C}_p$.*

Proof. If $\phi\chi^a = \chi$ then $\bar{\rho}$ is flat and we have already addressed this case.

If $\phi\chi^a = \mathrm{Id}$ then $\mathcal{C}_p$ is all upper triangular deformations and $\mathcal{N}_p$ is $\overline{H^1(G_p, U^1)}$. As $H^2(G_p, U^1) = 0$ in this case the versal upper triangular deformation ring is smooth in this case so $\mathcal{N}_p$ preserves $\mathcal{C}_p$.

In the remaining cases we will exhibit a basis for $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ that clearly preserves $\mathcal{C}_p$. This will prove the proposition. Observe $\mathrm{Ad}^0\bar{\rho} \simeq \mathbf{k} \oplus \mathbf{k}(\phi\chi^a) \oplus \mathbf{k}(\phi^{-1}\chi^{-a})$ where $\mathbf{k}(\varphi)$ is the one dimensional $\mathbf{k}$ vector space with G_p

action via the character $\varphi : G_p \to \mathbf{k}^*$. Recalling that cohomology commutes with direct sums we see $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ is spanned by a nonzero element of the one dimensional space $H^1(G_p, \mathbf{k}(\phi\chi^a))$ and nonzero element of the unramified one dimensional subspace in the two dimensional space $H^1(G_p, \mathbf{k})$. Suppose $\rho_n \in \mathcal{C}_p$. Then $\rho_n : G_p \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ is given by $\begin{pmatrix} \tilde{\phi}\chi^a\gamma_n & * \\ 0 & \gamma_n^{-1} \end{pmatrix}$ is a deformation of $\bar{\rho}$ to $W(\mathbf{k})/p^n$. For any $g \in H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ we see $(\mathbf{1} + p^{n-1}g).\rho_n = \begin{pmatrix} \tilde{\phi}\chi^a\gamma_n' & *' \\ 0 & \gamma_n'^{-1} \end{pmatrix}$ where γ_n' is a (possibly) different unramified character of G_p congruent to 1 mod p.

If we can show that $(1+p^{n-1}g).\rho_n \in \mathcal{C}_p$ we will have that $\mathcal{N}_p$ preserves $\mathcal{C}_p$. Since $\phi\chi^a \neq \chi$ we see $H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = 0$ by Lemma 5. By Lemma 1 the reducible deformation theory of $\bar{\rho}$ with fixed diagonal characters is unobstructed. Choose the diagonal characters to be $\tilde{\phi}\chi^a\gamma'$ and γ'^{-1} where γ' is *any* characteristic zero lift of γ'_n. The proof is complete.

We now prove an analog of Proposition 4 in the indecomposable case. We first deal with the cases where $\mathcal{N}_p = H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$. The primary difficulty is to make sure that the elements of $\mathcal{N}_p$ are "upper triangular", that is their action on upper triangular deformations gives upper triangular deformations. Recall U^1 is the subgroup of $\mathrm{Ad}^0\bar{\rho}$ of upper triangular matrices. The short exact sequence

$$0 \to U^1 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0$$

gives the long exact G_p cohomology sequence. We are interested in the map

$$\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho}).$$

In the cases where we have taken $\mathcal{N}_p$ to be $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ we will see $\mathcal{N}_p$ is contained in the image of β.

LEMMA 8. *Suppose $\bar{\rho}$ is odd, indecomposable, and that we are in a case where $\mathcal{N}_p$ is taken to be $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ or $H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$. If the image of the map $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ contains $\mathcal{N}_p$ then $\mathcal{N}_p$ preserves $\mathcal{C}_p$.*

Proof. Let $\rho_n \in \mathcal{C}_p$ with diagonal characters $\tilde{\phi}\chi^a\gamma_n$ and γ_n^{-1} where γ_n is unramified. Let $g \in \mathcal{N}_p$. By hypothesis there exists a $\tilde{g} \in H^1(G_p, U^1)$ be such that $\beta(\tilde{g}) = g$. Since $\rho_n \in \mathcal{C}_p$ is reducible we see $(\mathbf{1} + p^{n-1}\tilde{g}).\rho_n$ is reducible. Thus $(\mathbf{1} + p^{n-1}g).\rho_n$ is also reducible, and since $g \in \mathcal{N}_p$ we see that the diagonal characters of $(\mathbf{1} + p^{n-1}g).\rho_n$, are $\tilde{\phi}\chi^a\gamma'_n$ and γ'^{-1}_n or χ and 1 as $\mathcal{N}_p = H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ or $H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$ where γ'_n is unramified. When $\mathcal{N}_p = H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ we see by the argument in the proof of Lemma 6 that $(1 + p^{n-1}g).\rho_n$ deforms to a characteristic zero element of $\mathcal{C}_p$. When

$\mathcal{N}_p = H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$ the argument in the proof of Lemma 7 applies. Thus in either case $\mathcal{N}_p$ preserves $\mathcal{C}_p$.

It now suffices to show the hypothesis of Lemma 8 is satisfied in each of the nonflat indecomposable cases.

PROPOSITION 5. *Suppose* $\bar{\rho} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$ *is ramified and we are in the odd case. The image of* $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ *contains* $\mathcal{N}_p = H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$*. Thus* $\mathcal{N}_p$ *preserves* $\mathcal{C}_p$ *in this case.*

Proof. Consider the commutative diagram below. (N.B. The columns are not exact.)

$$\begin{tikzcd}
H^1(G_p, U^1) \arrow[r, "\beta"] \arrow[d] & H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \arrow[d] \\
H^1(G_p, U^1/U^0) \arrow[r] \arrow[d] & H^1(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) \arrow[d] \\
H^1(I_p, U^1/U^0)^{G_p/I_p} \arrow[r] & H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))^{G_p/I_p}
\end{tikzcd}$$

Since $G_p/I_p = \hat{\mathbf{Z}}$ has trivial H^2's we have the exact sequence

$$0 \to H^1(G_p/I_p, (U^1/U^0)^{I_p}) \to H^1(G_p, (U^1/U^0)) \to H^1(I_p, (U^1/U^0))^{G_p/I_p} \to 0.$$

By Lemma 3

$$\dim_{\mathbf{k}} H^1(G_p/I_p, (U^1/U^0)^{I_p}) = \dim_{\mathbf{k}} H^0(G_p, (U^1/U^0)) = 1.$$

Since from Table 4 we have $\dim_{\mathbf{k}} H^1(G_p, (U^1/U^0)) = 2$ we see

$$\dim_{\mathbf{k}} H^1(I_p, (U^1/U^0))^{G_p/I_p} = 1.$$

We will show the bottom horizontal map is trivial. Thus the map

$$H^1(G_p, U^1) \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))^{G_p/I_p}$$

by going down twice and then to the right is trivial. Since the diagram commutes we see the map $H^1(G_p, U^1) \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))^{G_p/I_p}$ by going to the right and then down twice is trivial. Thus we will have shown that the image of $H^1(G_p, U^1)$ under the map $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ lies in $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$, the kernel of the composite of the two right vertical maps.

A dimension counting argument establishes equality and the desired reverse containment.

We consider the I_p cohomology sequence of the short exact sequence

$$0 \to U^1/U^0 \to \mathrm{Ad}^0\bar{\rho}/U^0 \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0.$$

We get the long exact sequence

$$\begin{aligned} 0 &\to (U^1/U^0)^{I_p} \to (\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p} \to (\mathrm{Ad}^0\bar{\rho}/U^1)^{I_p} \\ &\to H^1(I_p, (U^1/U^0)) \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) \to \cdots \end{aligned}$$

whose first three terms are all easily seen to be one dimensional. (N.B. That $(\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p}$ is one dimensional follows from the fact that $\bar{\rho}$ is *ramified.* If $\bar{\rho}$ had been indecomposable and unramified $(\mathrm{Ad}^0\bar{\rho}/U^0)^{I_p}$ would have been two dimensional. This one dimensionality is the crucial step in establishing the triviality of the bottom horizontal map in our diagram.) Thus the map $H^1(I_p, (U^1/U^0)) \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))$ has one dimensional kernel. We have the left exact sequence

$$0 \to (\mathrm{Ad}^0\bar{\rho}/U^1)^{I_p} \to H^1(I_p, (U^1/U^0)) \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0)).$$

Since taking G_p/I_p invariants is left exact we get the the exact sequence

$$0 \to \left((\mathrm{Ad}^0\bar{\rho}/U^1)^{I_p}\right)^{G_p/I_p} \to H^1(I_p, (U^1/U^0))^{G_p/I_p} \to H^1(I_p, (\mathrm{Ad}^0\bar{\rho}/U^0))^{G_p/I_p}.$$

But the first term is simply $(\mathrm{Ad}^0\bar{\rho}/U^1)^{G_p}$ which is one dimensional. Since $\dim_{\mathbf{k}} H^1(I_p, (U^1/U^0))^{G_p/I_p} = 1$ the bottom horizontal map in our diagram is trivial.

PROPOSITION 6. *Suppose* $\bar{\rho} = \begin{pmatrix} \chi & * \\ 0 & 1 \end{pmatrix}$ *is not flat. Then*

$$\mathcal{N}_p = H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$$

preserves $\mathcal{C}_p$ *in the odd case.*

Proof. As in Lemma 8, it suffices to show that the image of the map $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ contains $\mathcal{N}_p = H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$ which is the kernel of the right vertical map in the diagram below.

$$\begin{tikzcd} H^1(G_p, U^1) \arrow[r, "\beta"] \arrow[d] & H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \arrow[d] \\ H^1(G_p, (U^1/U^0)) \arrow[r] & H^1(G_p, (\mathrm{Ad}^0\bar{\rho}/U^0)) \end{tikzcd}$$

We know $\dim_{\mathbf{k}} H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho}) = 1$. We will show the horizontal maps are injective and the left vertical map has a one dimensional kernel. The commutativity of the diagram then shows $\beta(H^1(G_p, U^1)) \supset H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$ and we are done.

Consider the exact sequence

$$0 \to U^1 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0$$

and take G_p cohomology. Since $(\mathrm{Ad}^0\bar{\rho}/U^1)^{G_p} = 0$ we have

$$\cdots \to 0 \to H^1(G_p, U^1) \xrightarrow{\beta} H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to \cdots$$

so the top horizontal map is injective. The same argument applied to the exact sequence

$$0 \to U^1/U^0 \to \mathrm{Ad}^0\bar{\rho}/U^0 \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0$$

gives the injectivity of the bottom horizontal map.

Finally, consider the short exact sequence

$$0 \to U^0 \to U^1 \to U^1/U^0 \to 0$$

and take G_p cohomology. We have

$$\begin{aligned} 0 &\to (U^0)^{G_p} \to (U^1)^{G_p} \to (U^1/U^0)^{G_p} \\ &\to H^1(G_p, U^0) \to H^1(G_p, U^1) \to H^1(G_p, (U^1/U^0)). \end{aligned}$$

We find the first two terms are trivial, the third is one dimensional, and the last three terms are two dimensional. Thus the map

$$H^1(G_p, U^1) \to H^1(G_p, (U^1/U^0))$$

has one dimensional kernel so the the image of β contains $H^1_{nf}(G_p, \mathrm{Ad}^0\bar{\rho})$. By Lemma 8 the proposition is proved.

Proposition 7. *Suppose* $\bar{\rho} = \begin{pmatrix} \phi\chi^a & * \\ 0 & 1 \end{pmatrix}$ *is indecomposable where* $\phi\chi^a \neq \chi$ *or the identity character. Then* $\mathcal{N}_p = H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ *preserves* $\mathcal{C}_p$ *in the odd case.*

Proof. Again we must show the image of the map $\beta : H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ contains the one dimensional subspace $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ of $H^1(G_p, \mathrm{Ad}^0\bar{\rho})$. We refer the reader to the diagram of Proposition 5. Taking the G_p cohomology of the short exact sequence

$$0 \to U^1 \to \mathrm{Ad}^0\bar{\rho} \to \mathrm{Ad}^0\bar{\rho}/U^1 \to 0$$

we see (using $p \geq 5$ when $\phi\chi^a = \chi^{p-2}$) that $(\mathrm{Ad}^0\bar{\rho}/U^1)^{G_p} = 0$ so we have

$$\cdots \to 0 \to H^1(G_p, U^1) \xrightarrow{\beta} H^1(G_p, \mathrm{Ad}^0\bar{\rho}) \to \cdots$$

so β is injective. To finish the proof, using the commutativity of the diagram, it suffices to show that the composed map of going down twice and then to the right has nontrivial kernel. For then the composition of going to the right and then down twice has nontrivial kernel. Since the kernel of the composition of the two right vertical maps is the one dimensional space $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ and β is injective we see the image of β would then contain $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$.

Considering the short exact sequence

$$0 \to U^0 \to U^1 \to U^1/U^0 \to 0$$

and taking G_p cohomology we get

$$\cdots \to H^1(G_p, U^1) \to H^1(G_p, (U^1/U^0)) \to H^2(G_p, U^0).$$

Since $\phi\chi^a \neq \chi$ we have $H^2(G_p, U^0) = 0$ by Lemma 5. The top left vertical map in our diagram is then surjective. The kernel of the bottom left vertical map is the image under inflation of $H^1(G_p/I_p, (U^1/U^0)^{I_p})$. By Lemma 3 the dimension of this cohomology group over $\mathbf{k}$ is simply $\dim_{\mathbf{k}} H^0(G_p, (U^1/U^0)) = 1$. Thus the composition of the two left vertical maps has nontrivial kernel and the proposition is proved.

PROPOSITION 8. *If we are in either of the two cases where* $(\mathcal{C}_p, \mathrm{odd}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ *then* $\mathcal{N}_p$ *preserves* $\mathcal{C}_p$.

Proof. In both cases we see $H^2(G_p, U^1) = 0$ so the upper triangular deformation theory of $\bar{\rho}$ is unobstructed. Thus $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in these cases.

We have now verified all the results in the tables and shown $\mathcal{N}_p$ preserves $\mathcal{C}_p$ in all cases.

PROPOSITION 9. *With the exception of the two cases where* $\mathcal{C}_p = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ *and* $\mathcal{N}_p = \overline{H^1(G_p, U^1)}$, *all deformations to* $W(\mathbf{k})$ *in* $(\mathcal{C}_p, \mathrm{odd})$ *above are semistable.*

Proof. When inertia acts via fundamental characters of level two the elements of $(\mathcal{C}_p, \mathrm{odd})$ are all crystalline and known to be semistable. (See [Fo].)

When inertia acts via fundamental characters of level one then when $(\mathcal{C}_p, \mathrm{odd}) \neq \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$ they are semistable by the proposition of Section 3.1 of [PR].

We remark that we have felt free to twist $\bar{\rho}$ by a character of finite order and replace $\bar{\rho}$ with the twisted version. This affects Proposition 9 only up to the character of finite order. In particular, if one does not twist our characteristic zero representations are only *potentially* semistable or crystalline.

What is the "difference" between the cases we can handle and those we cannot? In the even case, if $H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = 0$ there are no difficulties. Otherwise $\dim_{\mathbf{k}} H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = 1$. Let $d = \dim_{\mathbf{k}} H^1(G_p, \mathrm{Ad}^0\bar{\rho})$. Our requirements for $\mathcal{N}_p$ and $\mathcal{C}_p$ amount to insisting the deformation ring associated to $\bar{\rho}$ has $W(\mathbf{k})[[T_1, \dots T_{d-1}]]$ as a quotient and insisting that $\mathcal{C}_p$ consist exactly of the $W(\mathbf{k})$ valued points of this quotient. G. Böckle has kindly pointed out to us that no such quotients exist when $H^2(G_p, \mathrm{Ad}^0\bar{\rho}) \neq 0$.

In the odd case, after twisting, the deformation ring must have a smooth quotient in $r - 2 - \dim_{\mathbf{k}} H^2(G_p, \mathrm{Ad}^0\bar{\rho})$ variables, all of whose $W(\mathbf{k})$ valued points are semistable. We do not know how to show such a ring exists in the two cases where $(\mathcal{C}_p, \text{odd}) = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$. In these cases $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ is of the appropriate codimension in $H^1(G_p, \mathrm{Ad}^0\bar{\rho})$, but $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ is *not* contained in the image of the map $H^1(G_p, U^1) \to H^1(G_p, \mathrm{Ad}^0\bar{\rho})$, that is the elements of $H^1_{\mathrm{ord}}(G_p, \mathrm{Ad}^0\bar{\rho})$ are *not* upper triangular. It might perhaps be that we are limited in that we are trying to only construct *ordinary* representations. A more refined semistable theory could settle these cases.

Recollections. In this section we recall a number of facts that we will need from [R2].

We have $\bar{\rho} : G_S \to \mathrm{GL}_2(\mathbf{k})$ and $\mathrm{Ad}^0\bar{\rho}$ is the additive group of two-by-two trace zero matrices over $\mathbf{k}$ with G_S action through $\bar{\rho}$ and by conjugation. We assume S contains p and infinity even if $\bar{\rho}$ is unramified at these places. Let $\mathbf{Q}(\bar{\rho})$ be the field fixed by the kernel of $\bar{\rho}$. Let $(\mathrm{Ad}^0\bar{\rho})^*$ be the Cartier dual of $\mathrm{Ad}^0\bar{\rho}$. We assume that $\bar{\rho}$ has minimal conductor among its twists. In particular we may assume $\mathbf{Q}(\bar{\rho})/\mathbf{Q}$ and $\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})/\mathbf{Q}$ are ramified at exactly the same finite primes. Let N be the kernel of the action of G_S on $\mathrm{Ad}^0\bar{\rho}$ and let N^d be the kernel of the action of G_S on $(\mathrm{Ad}^0\bar{\rho})^*$. Let $\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})$ be the field fixed by N and let $\mathbf{D} = \mathbf{Q}(\mathrm{Ad}^0\bar{\rho}) \cap \mathbf{Q}(\mu_p)$ and $\mathbf{K} = \mathbf{Q}(\mathrm{Ad}^0\bar{\rho})\mathbf{Q}(\mu_p)$. Clearly $N^d \supseteq \mathrm{Gal}(\mathbf{Q_S}/\mathbf{K})$.

$$\begin{tikzcd}
\mathbf{Q}(\bar{\rho}) \arrow[dd, dash] & & \\
& \mathbf{K} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{Q}(\mathrm{Ad}^0\bar{\rho}) \arrow[dr, dash] & & \mathbf{Q}(\mu_p) \arrow[dl, dash] \\
& \mathbf{D} \arrow[d, dash] & \\
& \mathbf{Q} &
\end{tikzcd}$$

Our hypotheses on $\bar{\rho}$ are as follows:

- $\bar{\rho}$ and $\mathrm{Ad}^0\bar{\rho}$ are absolutely irreducible representations of G_S.
- $H^1(G_S/N, \mathrm{Ad}^0\bar{\rho}) = 0$.
- $H^1(G_S/N^d, (\mathrm{Ad}^0\bar{\rho})^*) = 0$.
- There is an element

  $$c = a \times b \in \mathrm{Gal}(\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})/\mathbf{D}) \times \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{D}) \simeq \mathrm{Gal}(\mathbf{K}/\mathbf{D}) \subseteq \mathrm{Gal}(\mathbf{K}/\mathbf{Q})$$

  such that under the map $\mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{Q}) \to (\mathbf{Z}/p)^*$ we have that b maps to $t \neq \pm 1$ and a lifts to an element of $\mathrm{Gal}(\mathbf{Q}(\bar{\rho})/\mathbf{Q})$ with necessarily distinct eigenvalues of ratio t. (Note that the hypothesis on t immediately implies $p \geq 5$.)

The absolute irreducibility of $\mathrm{Ad}^0\bar{\rho}$ implies that of $\bar{\rho}$.

LEMMA. *Let $r = \dim_{\mathbf{k}} Ш^1_S((\mathrm{Ad}^0\bar{\rho})^*)$, s be the number of $v \in S$ such that $H^2(G_v, \mathrm{Ad}^0\bar{\rho}) \neq 0$, and $\delta = 0$ or 2 as $\bar{\rho}$ is even or odd. Then*

$$\dim_{\mathbf{k}} H^1(G_S, \mathrm{Ad}^0\bar{\rho}) = r + s + \delta.$$

Proof. By the proof of Proposition 1 and the tables in the local at p considerations section we know that $\dim_{\mathbf{k}} H^2(G_v, \mathrm{Ad}^0\bar{\rho}) = 0$ or 1 for all $v \in S$. We then have, using Theorem 4.10 of Chapter I of [Mi] the exact sequence

$$0 \to Ш^2_S(\mathrm{Ad}^0\bar{\rho}) \to H^2(G_S, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v \in S} H^2(G_v, \mathrm{Ad}^0\bar{\rho}) \to H^0(G_S, (\mathrm{Ad}^0\bar{\rho})^*)^*.$$

The last term is trivial as $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$ are assumed absolutely irreducible. By Fact 3 we have $Ш^2_S(\mathrm{Ad}^0\bar{\rho})$ and $Ш^1_S((\mathrm{Ad}^0\bar{\rho})^*)$ are dual so

$$\dim_{\mathbf{k}} H^2(G_S, \mathrm{Ad}^0\bar{\rho}) = r + s.$$

The result follows using the global Euler-Poincaré characteristic (Fact 4) keeping in mind $\dim_{\mathbf{k}} H^0(G_\infty, \mathrm{Ad}^0\bar{\rho}) = 1 + \delta$.

LEMMA 10. *If $p \geq 7$ and the projective image of $\bar{\rho} : G_S \to \mathrm{GL}_2(\mathbf{k})$ equals a conjugate of $\mathrm{PSL}_2(\mathbf{k}')$ or $\mathrm{PGL}_2(\mathbf{k}')$ over the quadratic extension of $\mathbf{k}$ for some subfield $\mathbf{k}'$ of $\mathbf{k}$, then the four hypotheses on $\bar{\rho}$ above are satisfied.*

Proof. The proofs of Lemmas 17 through 21 of [R2] guarantee the first three conditions hold. The existence of the element c follows from the proof of Theorem 2 of [R2]. All of this follows as we only need consider the *projective* image of $\bar{\rho}$ to understand $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$. The results of [R2] only require the projective image of $\bar{\rho}$ contain $\mathrm{PSL}_2(\mathbf{k}')$ for some field $\mathbf{k}'$ of characteristic greater than or equal to 7.

It is the lemmas of [R2] that require $p \geq 7$ and this is why we have this hypothesis here and in Theorem 1.

The following facts are proved or cited in Sections 6 and 7 of [R2].

Fact 9. A finite dimensional absolutely irreducible mod p representation of a profinite group has a minimal field of definition. If the image of $\bar{\rho}$ contains $\mathrm{SL}_2(\mathbf{k})$ the minimal field of definition of $\bar{\rho}$ and $\mathrm{Ad}^0\bar{\rho}$ is $\mathbf{k}$. (The fact of the Descents section of [R2] and Lemma 17 of that paper.)

Remark. The minimal fields of definition of $\bar{\rho}$ and $\mathrm{Ad}^0\bar{\rho}$ need not be the same. In this case we denote the minimal field of definition of $\mathrm{Ad}^0\bar{\rho}$ by $\tilde{\mathbf{k}}$ and refer to the descended representation space by $\widetilde{\mathrm{Ad}}^0\bar{\rho}$. If $\tilde{\mathbf{k}}$ is the minimal field of definition of $\mathrm{Ad}^0\bar{\rho}$, then it is easy to see that $\tilde{\mathbf{k}}$ is the minimal field of definition of $(\mathrm{Ad}^0\bar{\rho})^*$. We refer to its descent by $(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$. The existence of these descents is used in this paper only insofar as we recall various facts from [R2] that depend on the descents. Ultimately we need the descents to use Chebotarev's theorem to provide us with primes satisfying certain properties such as the set Q in Fact 16 of this section and the various sets $T, T_i, \tilde{T_i}$ of the next section. Since our primary interest is in the applications of Chebotarev's theorem we only mention $\widetilde{\mathrm{Ad}}^0\bar{\rho}$ and $(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$ in a few key places.

$$\begin{tikzcd}
 & L\tilde{\mathbf{M}}1 \arrow[dl, dash] \arrow[ddr, dash] & \\
\mathbf{L} \arrow[d, dash] & & \\
\mathbf{L_i} \arrow[dr, dash] & & \tilde{\mathbf{M}}1 \arrow[dl, dash] \\
 & \mathbf{K} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{Q}(\mathrm{Ad}^0\bar{\rho}) \arrow[dr, dash] & & \mathbf{Q}(\mu_p) \arrow[dl, dash] \\
 & \mathbf{D} \arrow[d, dash] & \\
 & \mathbf{Q} &
\end{tikzcd}$$

Fact 10. Let $\{f_1, \ldots, f_{r+s+\delta}\}$ be a basis of $H^1(G_S, \widetilde{\mathrm{Ad}}^0\bar{\rho})$. Let

$$g \in H^1(G_S, (\widetilde{\mathrm{Ad}}^0\bar{\rho})^*)$$

be nonzero.

1) There correspond to the f_i extensions $\mathbf{L_i}$, Galois over $\mathbf{Q}$, with $\mathrm{Gal}(\mathbf{L_i}/\mathbf{K}) \simeq \widetilde{\mathrm{Ad}}^0\bar{\rho}$ as $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$ modules where $\mathrm{Gal}(\mathbf{L_i}/\mathbf{K})$ inherits a $\tilde{\mathbf{k}}$ structure from f_i. The exact sequence

$$1 \to \mathrm{Gal}(\mathbf{L_i}/\mathbf{K}) \to \mathrm{Gal}(\mathbf{L_i}/\mathbf{Q}) \to \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \to 1$$

splits. Each $\mathbf{L_i}$ is linearly disjoint over $\mathbf{K}$ with the composite of the $\mathbf{L_j}$, $j \neq i$. Let $\mathbf{L}$ denote the composite of the $\mathbf{L_i}$, $1 \leq i \leq r+s+\delta$. Then $\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \simeq (\widetilde{\mathrm{Ad}}^0\bar{\rho})^{r+s+\delta}$ (with inherited $\tilde{\mathbf{k}}$ structure) and the exact sequence

$$1 \to \mathrm{Gal}(\mathbf{L}/\mathbf{K}) \to \mathrm{Gal}(\mathbf{L}/\mathbf{Q}) \to \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \to 1$$

splits.

2) There corresponds to g an extension $\tilde{\mathbf{M}}1$, Galois over $\mathbf{Q}$, of $\mathbf{K}$ with $\mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K}) \simeq (\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$ as $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$ modules where the $\tilde{\mathbf{k}}$ structure is inherited from g. The exact sequence

$$1 \to \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K}) \to \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{Q}) \to \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \to 1$$

splits.

3) $\tilde{\mathbf{M}}1$ is linearly disjoint from $\mathbf{L}$ over $\mathbf{K}$ and $\mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{K}) \simeq \mathrm{Gal}(\mathbf{L}/\mathbf{K}) \times \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})$.

(Lemmas 8, 11, and 12 of [R2])

Fact 11. The composite $\mathbf{L}\tilde{\mathbf{M}}1$ is Galois over $\mathbf{Q}$ and the exact sequence

$$1 \to \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{K}) \to \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q}) \to \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \to 1$$

splits. Thus $\mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q}) \simeq \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{K}) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{Q})$. (Lemma 13 of [R2])

Consider the element

$$c = a \times b \in \mathrm{Gal}(\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})/\mathbf{D}) \times \mathrm{Gal}(\mathbf{Q}(\mu_p)/\mathbf{D}) \simeq \mathrm{Gal}(\mathbf{K}/\mathbf{D}) \subseteq \mathrm{Gal}(\mathbf{K}/\mathbf{Q})$$

as in the hypotheses on $\bar{\rho}$ and its action on $\widetilde{\mathrm{Ad}}^0\bar{\rho}$ and $(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$. In [R2] it was observed that $\widetilde{\mathrm{Ad}}^0\bar{\rho}$ decomposes under the action of c into $\mathbf{F_p}$ eigenspaces with distinct eigenvalues t, 1 and $1/t$. Recall $t \neq \pm 1 \bmod p$. By duality $(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$, keeping in mind the b component of c acts on $(\mathrm{Ad}^0\bar{\rho})^*$ by multiplication by t, decomposes into $\mathbf{F_p}$ eigenspaces with eigenvalues 1, t, and t^2.

Fact 12. $\mathrm{Gal}(\mathbf{L_i}/\mathbf{K})$ contains a nontrivial $\mathbf{F_p}$ subspace on which c acts trivially. $\mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})$ contains an $\mathbf{F_p}$ subspace on which c acts trivially. (Lemma 14 of [R2])

Let $\alpha_i \in \mathrm{Gal}(\mathbf{L}/\mathbf{K}) \simeq (\widetilde{\mathrm{Ad}}^0\bar{\rho})^{r+s+\delta}$ be an element that is zero at all entries except the i^{th} at which it is an element of $\widetilde{\mathrm{Ad}}^0\bar{\rho}$ on which c acts trivially. Let ζ be a nonzero element of $\mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})$ on which c acts trivially. Consider $\eta_i = (\alpha_i \times \zeta) \rtimes c$ in

$$\begin{aligned}\Big(\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \times \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})\Big) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{D}) \;&\subseteq\; \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{K}) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{Q})\\ &\simeq\; \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q}).\end{aligned}$$

We need a few local facts. (These will apply to primes with Frobenius in the conjugacy class of $\eta_i \in \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q})$.)

Fact 13. Let $t \neq \pm 1 \in \mathbf{F}_{\mathbf{p}}^*$. Let $q \equiv t \bmod p$ be a prime and $\bar{\rho} : G_q \to \mathrm{GL}_2(\mathbf{k})$ be unramified. Suppose that $\bar{\rho}(\mathrm{Frob}_q)$ has distinct eigenvalues with ratio t. Then $\dim_{\mathbf{k}} H^0(G_q, \mathrm{Ad}^0\bar{\rho}) = 1$, $\dim_{\mathbf{k}} H^1(G_q, \mathrm{Ad}^0\bar{\rho}) = 2$, and $\dim_{\mathbf{k}} H^2(G_q, \mathrm{Ad}^0\bar{\rho}) = 1$. (Lemma 1 of [R2])

Consider $\bar{\rho} : G_q \to \mathrm{GL}_2(\mathbf{k})$ with q as above. Observe $\bar{\rho}$ is unramified so any deformation to $W(\mathbf{k})$ is tamely ramified and thus factors through the Galois group over $\mathbf{Q_q}$ of the maximal tamely ramified extension. Let σ_q and τ_q be topological generators of the tame Galois group where τ_q generates inertia. Then, up to a twist, $\bar{\rho}$ is given by $\sigma_q \mapsto \begin{pmatrix} q & 0 \\ 0 & 1 \end{pmatrix}$ and $\tau_q \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Define $\mathcal{C}_q$ to be all deformations given by $\sigma_q \mapsto \begin{pmatrix} q & px \\ 0 & 1 \end{pmatrix}$ and $\tau_q \mapsto \begin{pmatrix} 1 & py \\ 0 & 1 \end{pmatrix}$. Define $\mathcal{N}_q$ to be space spanned by $g \in H^1(G_q, \mathrm{Ad}^0\bar{\rho})$ where $g(\tau_q) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $g(\sigma_q) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Fact 14. Let $\bar{\rho} : G_q \to \mathrm{GL}_2(\mathbf{k})$ be unramified with $\bar{\rho}(\mathrm{Frob}_q)$ having distinct eigenvalues with ratio $t \in \mathbf{F_p}$, $t \neq \pm 1$ and $q \equiv t \bmod p$. Let $\mathcal{N}_q$ and $\mathcal{C}_q$ be as above and let $h \in H^1(G_q, \mathrm{Ad}^0\bar{\rho})$ be such that $h \notin \mathcal{N}_q$. Then $\bar{\rho} : G_q \to \mathrm{GL}_2(\mathbf{k})$ can be deformed to an element of $\mathcal{C}_q$ one step at a time with adjustments at each step made only by a multiple of h. Thus $\mathcal{N}_q$ preserves $\mathcal{C}_q$. (Proposition 1 of [R2])

Fact 15. $\eta_i = (\alpha_i \times \zeta) \rtimes c$ has order pd (where d is the order of t in $\mathbf{F}_{\mathbf{p}}^*$) in $\mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q})$. Let u_i be a prime of $\mathbf{Q}$ unramified in $\mathbf{L}\tilde{\mathbf{M}}1$ with Frobenius in the conjugacy class of η_i. Note u_i is a prime as in Fact 13. Then $f_i\,|_{G_{u_i}} \notin \mathcal{N}_{u_i}$. For $i \neq j$ we have $f_i\,|_{G_{u_j}} = 0$. For $g \in H^1(G_S, (\mathrm{Ad}^0\bar{\rho})^*)$ corresponding to $\tilde{\mathbf{M}}1$ we have $g\,|_{G_{u_i}} \neq 0$. (Corollaries 1 and 2 of [R2])

Fact 16. There is a set $Q = \{q_1, q_2, \ldots, q_r\}$ of primes, $q_i \notin S$ such that

1) q_i is unramified in $\mathbf{K}/\mathbf{Q}$ and Frob_{q_i} is in the conjugacy class of c in $\mathrm{Gal}(\mathbf{K}/\mathbf{Q})$. Also, $\bar{\rho}(\mathrm{Frob}_{q_i})$ has distinct eigenvalues with ratio $t \neq \pm 1$ in $\mathbf{F_p}$ and $q_i \equiv t \bmod p$.

2) $Ш^1_{S\cup Q}((\mathrm{Ad}^0\bar{\rho})^*) = 0$ and $Ш^2_{S\cup Q}(\mathrm{Ad}^0\bar{\rho}) = 0$.

3) $f_i \mid_{G_{q_j}} = 0$ for $i \neq j$, $1 \leq i \leq r + s + \delta$ and $1 \leq j \leq r$.

4) $f_j \mid_{G_{q_j}} \notin \mathcal{N}_{q_j}$ for $1 \leq j \leq r$.

5) The inflation map $H^1(G_S, \mathrm{Ad}^0\bar{\rho}) \to H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho})$ is an isomorphism. (Proposition 4 of [R2] and its proof)

Proof. Parts 1) through 4), proved in [R2], use the existence of the element $c \in \mathrm{Gal}(\mathbf{K}/\mathbf{Q})$ in our hypotheses on $\bar{\rho}$. Only part 5 is not proved in [R2]. Recall $r = \dim_{\mathbf{k}} Ш^2_S(\mathrm{Ad}^0\bar{\rho}) = \dim_{\mathbf{k}} Ш^1_S((\mathrm{Ad}^0\bar{\rho})^*)$. Proposition 1.6 of [W] becomes, after taking $L_v = H^1(G_v, \mathrm{Ad}^0\bar{\rho})$ for all $v \in S \cup Q$,

$$\frac{\#H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho})}{\#Ш^1_{S\cup Q}((\mathrm{Ad}^0\bar{\rho})^*)} = h_\infty \prod_{v \in S\cup Q, v\neq\infty} h_v,$$

where

$$h_\infty = \frac{\#H^0(G_\infty, (\mathrm{Ad}^0\bar{\rho})^*) \cdot \#H^0(G_S, \mathrm{Ad}^0\bar{\rho})}{\#H^0(G_S, (\mathrm{Ad}^0\bar{\rho})^*)}$$

and

$$h_v = \frac{\#H^2(G_v, \mathrm{Ad}^0\bar{\rho})}{\#[H^1(G_v, \mathrm{Ad}^0\bar{\rho}) : L_v]}.$$

Since $\dim_{\mathbf{k}} H^2(G_v, \mathrm{Ad}^0\bar{\rho}) = 1$ for the r primes in Q and $\dim_{\mathbf{k}} H^2(G_v, \mathrm{Ad}^0\bar{\rho}) = 1$ for s primes of S and $\dim_{\mathbf{k}} H^2(G_v, \mathrm{Ad}^0\bar{\rho}) = 0$ for the remaining primes of S we see $\prod_{v\in S\cup Q, v\neq\infty} h_v = (\#\mathbf{k})^{r+s}$. As $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$ are assumed to be absolutely irreducible we see $H^0(G_S, \mathrm{Ad}^0\bar{\rho})$ and $H^0(G_S, (\mathrm{Ad}^0\bar{\rho})^*)$ are trivial. Finally we see $\dim_{\mathbf{k}} H^0(G_\infty, (\mathrm{Ad}^0\bar{\rho})^*) = \delta$ where we recall $\delta = 0$ or 2 as $\bar{\rho}$ is even or odd so $h_\infty = (\#\mathbf{k})^\delta$ as $\bar{\rho}$ is even or odd. Assembling all this and keeping in mind that we already know $Ш^1_{S\cup Q}((\mathrm{Ad}^0\bar{\rho})^*)$ is trivial we see $\#H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho}) = (\#\mathbf{k})^{r+s+\delta}$. Thus the (necessarily injective) inflation map $H^1(G_S, \mathrm{Ad}^0\bar{\rho}) \to H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho})$ is an isomorphism.

Global considerations. We have $\bar{\rho} : G_S \to \mathrm{GL}_2(\mathbf{k})$ where $\mathbf{k}$ is a finite field of characteristic $p \geq 7$. For the time being we will assume the projective image of $\bar{\rho}$ is conjugate over the quadratic extension of $\mathbf{k}$ to $\mathrm{PSL}_2(\mathbf{k}')$ or $\mathrm{PGL}_2(\mathbf{k}')$ where $\mathbf{k}'$ is a subfield of the quadratic extension of $\mathbf{k}$. Thus Lemma 10 applies and the four hypotheses at the beginning of the Recollections section hold.

Recall that for $l \in S$, $l \neq p$ we have defined $\mathcal{N}_l \subseteq H^1(G_l, \mathrm{Ad}^0\bar{\rho})$ where

$$\dim_{\mathbf{k}} \left(H^1(G_l, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_l\right) = \dim_{\mathbf{k}} H^2(G_l, \mathrm{Ad}^0\bar{\rho}) \leq 1.$$

We have also defined a family $\mathcal{C}_l$ of deformations of $\bar{\rho}\mid_{G_l}$ to $W(\mathbf{k})$. The key property is that $\mathcal{N}_l$ preserves $\mathcal{C}_l$, that is if $a \in \mathbf{k}$, $g \in \mathcal{N}_l$ and ρ_n is the $\bmod p^n$ reduction of an element of $\mathcal{C}_p$ then $(\mathbf{1}+ap^{n-1}g).\rho_n$ is also the $\bmod p^n$ reduction of an element of $\mathcal{C}_l$.We have similarly defined $\mathcal{C}_q$ and $\mathcal{N}_q$ for unramified primes q as in Facts 13 and 14. Also, recall for the prime p and most $\bar{\rho}$ that we have a subspace $\mathcal{N}_p \subseteq H^1(G_p, \mathrm{Ad}^0\bar{\rho})$ with

$$\dim_{\mathbf{k}} \left(H^1(G_p, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_p\right) = \delta + \dim_{\mathbf{k}} H^2(G_p, \mathrm{Ad}^0\bar{\rho})$$

where $\delta = 0$ or 2 as $\bar{\rho}$ is even or odd. The $\bar{\rho}$ for which we do *not* have such an $\mathcal{N}_p$ are those we cannot deform to $W(\mathbf{k})$. If $\bar{\rho} : G_S \to \mathrm{GL}_2(\mathbf{k})$ is even and $\bar{\rho}\mid_{G_p}$ is an excluded case we do not know how to proceed. Otherwise take $\mathcal{N}_p$ and $\mathcal{C}_p$ as in the tables for $\bar{\rho}$ is even or odd and recall $\mathcal{N}_p$ preserves $\mathcal{C}_p$.

Our aim is to find two finite sets of primes Q and T as in Facts 13 and 14. The set Q will be, as in Fact 16, such that the map $H^2(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S\cup Q} H^2(G_v, \mathrm{Ad}^0\bar{\rho})$ is injective, that is $\text{Ш}^1_{S\cup Q}((\mathrm{Ad}^0\bar{\rho})^*)$, and by duality $\text{Ш}^2_{S\cup Q}(\mathrm{Ad}^0\bar{\rho})$ are trivial. The set T will be such that the map

$$H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} \left(H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right)$$

is *surjective.* We will eventually have that the map

$$H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S\cup Q\cup T} \left(H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right)$$

is surjective and that $\text{Ш}^1_{S\cup Q\cup T}((\mathrm{Ad}^0\bar{\rho})^*)$ and its dual $\text{Ш}^2_{S\cup Q\cup T}(\mathrm{Ad}^0\bar{\rho})$ are trivial. Thus all obstructions to deformation problems (unramified outside $S \cup Q \cup T$) will be local and we will have enough global 1 cohomology classes to remove these local obstructions.

We may think of Q as annihilating a dual Selmer group and T as annihilating the cokernel of a Selmer group map.

Consider $\oplus_{v\in S}\mathcal{N}_v \subseteq \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})$.

LEMMA 11. $\mathrm{Dim}_{\mathbf{k}} \left(\oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right) = s + \delta.$

Proof. For $l \neq p$ we have $\dim_{\mathbf{k}}(H^1(G_l, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_l) = \dim_{\mathbf{k}} H^2(G_l, \mathrm{Ad}^0\bar{\rho})$ which is 0 or 1. For p we see $\dim_{\mathbf{k}}(H^1(G_p, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_p) = \delta + \dim_{\mathbf{k}} H^2(G_p, \mathrm{Ad}^0\bar{\rho})$ where $\dim_{\mathbf{k}} H^2(G_p, \mathrm{Ad}^0\bar{\rho}) = 0$ or 1. Recalling that $\oplus_{v\in S} H^2(G_v, \mathrm{Ad}^0\bar{\rho})$ has dimension s (essentially by definition) the codimension result follows.

LEMMA 12. *Recall* $r = \dim_{\mathbf{k}} Ш^1_S((\mathrm{Ad}^0\bar{\rho})^*)$. *Consider the restriction map*

$$\mathrm{Res} : H^1(G_S, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho}).$$

Let $d = \dim_{\mathbf{k}} \mathrm{Res}^{-1}(\oplus_{v\in S}\mathcal{N}_v)$. *Then* $d \geq r$.

Proof. Res induces an injection

$$\widetilde{\mathrm{Res}} : H^1(G_S, \mathrm{Ad}^0\bar{\rho})/\mathrm{Res}^{-1}(\oplus_{v\in S}\mathcal{N}_v) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v.$$

Since $(\oplus_{v\in S}\mathcal{N}_v)$ is of codimension $s+\delta$ in $\oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})$ by Lemma 11 and $\dim_{\mathbf{k}} H^1(G_S, \mathrm{Ad}^0\bar{\rho}) = r+s+\delta$ by Lemma 9 we have $r+s+\delta-d \leq s+\delta$ so $d \geq r$.

Let $\{f_1, f_2, \ldots, f_d\}$ be a basis of $\mathrm{Res}^{-1}(\oplus_{v\in S}\mathcal{N}_v)$. We augment this to a basis $\{f_1, f_2, \ldots, f_{r+s+\delta}\}$ of $H^1(G_S, \mathrm{Ad}^0\bar{\rho})$.

We now refer the reader to the diagram at the beginning of the Recollections section. Recall $\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})$ is the field fixed by N, the maximal subgroup of G_S that acts trivially on $\mathrm{Ad}^0\bar{\rho}$, that $\mathbf{D} = \mathbf{Q}(\mathrm{Ad}^0\bar{\rho}) \cap \mathbf{Q}(\mu_p)$ and $\mathbf{K} = \mathbf{Q}(\mathrm{Ad}^0\bar{\rho})\mathbf{Q}(\mu_p)$, that N^d is the maximal subgroup of G_S that acts trivially on $(\mathrm{Ad}^0\bar{\rho})^*$, and that $N^d \supseteq \mathrm{Gal}(\mathbf{Q_S}/\mathbf{K})$. Also recall the four hypotheses on $\bar{\rho}$ at the beginning of the Recollections section. Since we are assuming the projective image of $\bar{\rho}$ contains $\mathrm{PSL}_2(\mathbf{k}')$ for a suitable subfield $\mathbf{k}'$ of $\mathbf{k}$ there is, by Lemma 10 an element

$$c = a \times b \in \mathrm{Gal}(\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})/\mathbf{D}) \times Gal(\mathbf{Q}(\mu_p)/\mathbf{D}) \simeq \mathrm{Gal}(\mathbf{K}/\mathbf{D})$$

with properties given at the beginning of the Recollections section.

For $i = 1, 2, \ldots, r+s+\delta$ recall $\mathbf{L_i}$ is the extension of $\mathbf{K}$ arising from f_i as in Fact 10, and $\mathbf{L}$ is the composite of all the $\mathbf{L_i}$. Recall also each $\mathbf{L_i}$ is disjoint over $\mathbf{K}$ from the composite of the $\mathbf{L_j}$ for $j \neq i$. Then $\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \simeq (\widetilde{\mathrm{Ad}}^0\bar{\rho})^{r+s+\delta}$ as $\tilde{\mathbf{k}}[G_S]$ modules. (Recall that $\tilde{\mathbf{k}}$ is the minimal field of definition of $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$ and $\widetilde{\mathrm{Ad}}^0\bar{\rho}$ and $(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$ are descents of $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$ to $\tilde{\mathbf{k}}$.) Let Q be the set of r primes whose existence and properties are described in Fact 16.

For $r+1 \leq i \leq d$ take α_i to be an element of $\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \simeq (\widetilde{\mathrm{Ad}}^0\bar{\rho})^{r+s+\delta}$ all of whose entries are 0 except the i^{th} which is a nonzero element on which c acts trivially. Such an element exists by Fact 12. Let T_i be those primes not in $S \cup Q$ with Frobenius in the conjugacy class of

$$\alpha_i \rtimes c \in \mathrm{Gal}(\mathbf{L}/\mathbf{K}) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{D}) \subseteq \mathrm{Gal}(\mathbf{L}/\mathbf{K}) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \simeq \mathrm{Gal}(\mathbf{L}/\mathbf{Q}).$$

The following proposition is the key technical ingredient of this paper that we were not aware of in [R2]. It is motivated by Proposition 2 of Section 8.4 of [H] which is due to Neukirch.

PROPOSITION 10. *For $i = r+1, \ldots, d$ the restriction map*

$$\theta : H^1(G_{S\cup Q\cup T_i}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})$$

is surjective.

Proof. Recall that for a finite Galois module M and a (possibly infinite) set W we define $P^1_W(M) = \square_{v\in W} H^1(G_v, M)$ to be the restricted direct product with respect to the images $H^1_{nr}(G_v, M)$ of the of the inflation maps $H^1(\mathrm{Gal}(\mathbf{Q_v^{nr}}/\mathbf{Q_v}), M) \to H^1(G_v, M)$ at those v for which M is unramified. By local duality $P^1_W(M)$ is dual to $P^1_W(M^*)$ where M^* is the Cartier dual of M. Here W need not contain p.

Consider the restriction map

$$\tilde{\theta} : H^1(G_{S\cup Q\cup T_i}, \mathrm{Ad}^0\bar{\rho}) \to P^1_{S\cup Q\cup T_i}(\mathrm{Ad}^0\bar{\rho}).$$

By global Tate-Poitou duality (Fact 3) the image of $\tilde{\theta}$ is the exact annihilator of the image of the restriction map

$$\tilde{\vartheta} : H^1(G_{S\cup Q\cup T_i}, (\mathrm{Ad}^0\bar{\rho})^*) \to P^1_{S\cup Q\cup T_i}((\mathrm{Ad}^0\bar{\rho})^*).$$

The annihilator in $P^1_{S\cup Q\cup T_i}((\mathrm{Ad}^0\bar{\rho})^*)$ of $\left(P^1_{Q\cup T_i}(\mathrm{Ad}^0\bar{\rho}) + \mathrm{Image}(\tilde{\theta})\right)$ is the intersection of the annihilators of these two summands. (All spaces are closed.) This consists of the set $\{\tilde{\vartheta}(x)\}$for $x \in H^1(G_{S\cup Q\cup T_i}, (\mathrm{Ad}^0\bar{\rho})^*)$ such that $x\,|_{G_v} = 0$ for all $v \in Q \cup T_i$.

We show all such x are trivial. Suppose there exists a nonzero such x. As in part 2) of Fact 10, we see that x gives rise to an extension $\tilde{\mathbf{M}}1/\mathbf{K}$ with $\mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K}) \simeq (\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$ as $\tilde{\mathbf{k}}[G_S]$ modules. (The $\tilde{\mathbf{k}}$ structure is inherited from $(\widetilde{\mathrm{Ad}}^0\bar{\rho})^*$ as in Fact 10 and we are using the hypothesis on $\bar{\rho}$ that $H^1(G_S/N^d, (\mathrm{Ad}^0\bar{\rho})^*) = 0$.) By Fact 10 we have $\tilde{\mathbf{M}}1 \cap \mathbf{L} = \mathbf{K}$ and the exact sequence

$$1 \to \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K}) \to \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{Q}) \to \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \to 1$$

splits so

$$\mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q}) \simeq \left(\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \times \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})\right) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{Q}).$$

$$\begin{tikzcd}[column sep=small]
 & \mathbf{L}\tilde{\mathbf{M}}1 \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{L} \arrow[dr, dash] & & \tilde{\mathbf{M}}1 \arrow[dl, dash] \\
 & \mathbf{K} \arrow[d, dash] & \\
 & \mathbf{Q} &
\end{tikzcd}$$

A nonzero $\zeta \in \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})$ on which $c \in \mathrm{Gal}(\mathbf{K}/\mathbf{Q})$ acts trivially exists by Fact 12. Let $\tilde{T}_i$ be those primes not in $S \cup Q$ with Frobenius in the conjugacy class of $(\alpha_i \times \zeta) \rtimes c$ in

$$\begin{aligned}\Big(\mathrm{Gal}(\mathbf{L}/\mathbf{K}) \times \mathrm{Gal}(\tilde{\mathbf{M}}1/\mathbf{K})\Big) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) &\simeq \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{K}) \rtimes \mathrm{Gal}(\mathbf{K}/\mathbf{Q}) \\ &\simeq \mathrm{Gal}(\mathbf{L}\tilde{\mathbf{M}}1/\mathbf{Q}).\end{aligned}$$

Note that $(\alpha_i \times \zeta) \rtimes c$ projects to $\alpha_i \rtimes c \in \mathrm{Gal}(\mathbf{L}/\mathbf{Q})$ so $\tilde{T}_i$ is a nonempty (infinite) subset of T_i and that for $v \in \tilde{T}_i$ we have $x\ |_{G_v} \neq 0$ by Fact 15. But $x \in H^1(G_{S \cup Q \cup T_i}, (\mathrm{Ad}^0\bar{\rho})^*)$ was chosen so $x\ |_{G_v} = 0$ for all $v \in Q \cup T_i$. This contradiction shows $x = 0$ so the annihilator in $P^1_{S \cup Q \cup T_i}((\mathrm{Ad}^0\bar{\rho})^*)$ of $\Big(P^1_{Q \cup T_i}(\mathrm{Ad}^0\bar{\rho}) \oplus \mathrm{Image}(\tilde{\theta})\Big)$ is trivial. Thus we have

$$\Big(P^1_{Q \cup T_i}(\mathrm{Ad}^0\bar{\rho}) + \mathrm{Image}(\tilde{\theta})\Big) = P^1_{S \cup Q \cup T_i}(\mathrm{Ad}^0\bar{\rho}).$$

Since the cokernel of θ is the cokernel of the composite map

$$\begin{aligned}H^1(G_{S \cup Q \cup T_i}, \mathrm{Ad}^0\bar{\rho}) &\xrightarrow{\tilde{\theta}} P^1_{S \cup Q \cup T_i}(\mathrm{Ad}^0\bar{\rho}) \\ &\rightarrow \frac{P^1_{S \cup Q \cup T_i}(\mathrm{Ad}^0\bar{\rho})}{P^1_{Q \cup T_i}(\mathrm{Ad}^0\bar{\rho})} = \oplus_{v \in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})\end{aligned}$$

the cokernel of θ is

$$\frac{P^1_{S \cup Q \cup T_i}(\mathrm{Ad}^0\bar{\rho})}{P^1_{Q \cup T_i}(\mathrm{Ad}^0\bar{\rho}) + \mathrm{Image}(\tilde{\theta})} = 0.$$

We see θ is surjective.

LEMMA 13. *For $i = r+1, \ldots, d$ let $\wp_i \in T_i$. Then for $r+1 \leq m \leq d$*

$$\dim_{\mathbf{k}} H^1(G_{S \cup Q \cup \{\wp_{r+1}, \ldots, \wp_m\}}, \mathrm{Ad}^0\bar{\rho}) = m - r + r + s + \delta = m + s + \delta.$$

Proof. We mimic the proof of Fact 16, part 5). The roles there of S and $S \cup Q$ are played here by $S \cup Q$ and $S \cup Q \cup \{\wp_{r+1}, \ldots, \wp_m\}$ respectively. Since $Ш^1_{S \cup Q}((\mathrm{Ad}^0\bar{\rho})^*)$ is trivial it follows that $Ш^1_{S \cup Q \cup \{\wp_1, \ldots, \wp_m\}}((\mathrm{Ad}^0\bar{\rho})^*) = 0$. The rest of the proof is identical to that of Fact 16, part 5). We use that $\wp_i$ is a prime satisfying the hypotheses of Fact 13 (see Fact 16) so $\#H^2(G_{\wp_i}, \mathrm{Ad}^0\bar{\rho}) = \#\mathbf{k}$.

LEMMA 14. *For $i = r+1, \ldots, d$ there exist primes $\wp_i \in T_i$ such that the map*

$$H^1(G_{S \cup Q \cup \{\wp_{r+1}, \ldots, \wp_d\}}, \mathrm{Ad}^0\bar{\rho}) \rightarrow \oplus_{v \in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$$

is onto.

Proof. Throughout we assume $d > r$. If $d = r$ then the statements are vacuously true. Recall that the inflation map $H^1(G_S, \mathrm{Ad}^0\bar{\rho}) \to H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho})$ is an isomorphism by Fact 16, part 5) and the these cohomology groups have dimension $r + s + \delta$. Consider the restriction map

$$H^1(G_S, \mathrm{Ad}^0\bar{\rho}) = H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v.$$

By definition the kernel is dimension d so the image has dimension $r+s+\delta-d$. The target space has dimension $s + \delta$ by Lemma 11.

By Proposition 10 the map

$$H^1(G_{S\cup Q\cup T_{r+1}}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$$

is onto. Since $\left(\oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right)$ is finite dimensional, there is a finite subset D_{r+1} of T_{r+1} such that the map

$$H^1(G_{S\cup Q\cup D_{r+1}}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$$

is onto. There is a (nonunique) prime, call it $\wp_{r+1}$, in D_{r+1} such that the image of

$$H^1(G_{S\cup Q\cup\{\wp_{r+1}\}}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$$

is of dimension $r + s + \delta - d + 1$. That the increase in the dimension of the image is exactly one follows from Lemma 13. If no such prime existed then the image of $H^1(G_{S\cup Q\cup D_{r+1}}, \mathrm{Ad}^0\bar{\rho})$ in $\oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$ would be $r+s+\delta-d$ dimensional. This contradicts the surjectivity.

Now consider the map

$$H^1(G_{S\cup Q\cup\{\wp_{r+1}\}\cup T_{r+2}}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v.$$

It is also surjective by Proposition 10. As above we can find a prime $\wp_{r+2} \in T_{r+2}$ such that the dimension of the image of $H^1(G_{S\cup Q\cup\{\wp_{r+1},\wp_{r+2}\}}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$ is $r + s + \delta - d + 2$.

Continuing in this fashion the lemma is proved.

Denote by T the set $\{\wp_{r+1}, \dots, \wp_d\}$ of Lemma 14. Augment the basis $\{f_1, \dots, f_{r+s+\delta}\}$ of

$$H^1(G_S, \mathrm{Ad}^0\bar{\rho}) = H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho})$$

to a basis $\{f_1, \dots, f_{d+s+\delta}\}$ of $H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho})$.

LEMMA 15. *Recall from Lemma* 12 *that* $d \geq r$ *and* $\{f_1, \dots, f_d\}$ *is a basis of* $\mathrm{Res}^{-1}(\oplus_{v\in S}\mathcal{N}_v)$.

1) *For* $1 \leq i \leq r$ *we have* $f_i\mid_{G_v} = 0$ *for all* $v \in Q \cup T$ except $v = q_i \in Q$. *Also* $f_i \mid_{G_{q_i}} \notin \mathcal{N}_{q_i}$.

2) *For* $r+1 \leq i \leq d$ *we have* $f_i \mid_{G_v} = 0$ *for all* $v \in Q \cup T$ except $v = \wp_i \in T$. *Also* $f_i \mid_{G_{\wp_i}} \notin \mathcal{N}_{\wp_i}$.

3) *Let X be the $\mathbf{k}$ vector space spanned by $\{f_{d+1}, \dots, f_{d+s+\delta}\}$. Then the restriction map*

$$X \to \oplus_{v \in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$$

is an isomorphism.

Proof. The first two parts follow from Fact 15. Recall in the comments following Lemma 12 the kernel of the map

$$H^1(G_S, \mathrm{Ad}^0\bar{\rho}) = H^1(G_{S\cup Q}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v \in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$$

is the space spanned by $\{f_1, \dots, f_d\}$. Part 3) now follows by Lemma 14 and counting dimensions.

LEMMA 16. *The map*

$$H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v \in S\cup Q\cup T} (H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v)$$

is an isomorphism.

Proof. Thus we may say the cokernel of the 'Selmer group map' is trivial. Recall $\{f_1, \dots, f_{d+s+\delta}\}$ is a basis of $H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho})$. We separate this basis into three groups, some of which may be empty.

- Group I - $\{f_1, \dots, f_r\}$
- Group II - $\{f_{r+1}, \dots, f_d\}$
- Group III - $\{f_{d+1}, \dots, f_{d+s+\delta}\}$

By Lemma 15, part 3) the span of group III elements under restriction projects isomorphically to $\oplus_{v \in S} H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v$.

By Lemma 15, part 2) the span of group II elements maps under restriction isomorphically to $\oplus_{v \in T} \left(H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right)$. This is because the $v \in T$ are as in Fact 14. Thus $\dim_{\mathbf{k}}(H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v) = 1$, and for all i we have $f_i \mid_{G_v} = 0$ for all $v \in Q \cup T$ except $v = \wp_i$. Furthermore for $r+1 \le i \le d$ we have $f_i \mid_{G_{\wp_i}} \notin \mathcal{N}_{\wp_i}$.

Similarly, the span of group I elements maps isomorphically to

$$\oplus_{v \in Q} \left(H^1(G_v, \mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right)$$

by Lemma 15, part 1). Thus the map in question has trivial kernel. By counting dimensions we are done.

Recall that N and N^d are the maximal subgroups of G_S that act trivially on $\mathrm{Ad}^0\bar{\rho}$ and $(\mathrm{Ad}^0\bar{\rho})^*$ respectively. Thus N fixes $\mathbf{Q}(\mathrm{Ad}^0\bar{\rho})$ and $N^d \supseteq \mathrm{Gal}(\mathbf{Q_S}/\mathbf{K})$. Recall that if

$$\dim_{\mathbf{k}} Ш^2_S(\mathrm{Ad}^0\bar{\rho}) = r = \dim_{\mathbf{k}} Ш^1_S((\mathrm{Ad}^0\bar{\rho})^*)$$

then there is a set Q of primes $\{q_1, \ldots, q_r\}$ such that $\dim_{\mathbf{k}} Ш^1_{S \cup Q}((\mathrm{Ad}^0 \bar{\rho})^*) = 0$ with the q_i as in Facts 13 and 16. We have the field diagram below.

$$\begin{tikzcd}
\mathbf{Q}(\bar{\rho}) \arrow[dd, dash] & & \\
& \mathbf{K} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathbf{Q}(\mathrm{Ad}^0 \bar{\rho}) \arrow[dr, dash] & & \mathbf{Q}(\mu_p) \arrow[dl, dash] \\
& \mathbf{D} \arrow[d, dash] & \\
& \mathbf{Q} &
\end{tikzcd}$$

THEOREM 1. *Let $\mathbf{k}$ be a finite field of characteristic $p \geq 7$ and $\bar{\rho} : G_{\mathbf{Q}} \to$ $\mathrm{GL}_2(\mathbf{k})$. Let χ be the cyclotomic character.*

a) *If $\bar{\rho}$ is even and $\bar{\rho}\mid_{G_p}$ is not twist equivalent to $\begin{pmatrix} \chi & 0 \\ 0 & 1 \end{pmatrix}$ or twist equivalent to the indecomposable representation $\begin{pmatrix} \chi^{p-2} & * \\ 0 & 1 \end{pmatrix}$ then there is a finite set R of primes containing p and infinity and a deformation $\rho : G_R \to \mathrm{GL}_2(W(\mathbf{k}))$ of $\bar{\rho}$.*

b) *If $\bar{\rho}$ is odd then there is a finite set R of primes containing p and infinity and a deformation $\rho : G_R \to \mathrm{GL}_2(W(\mathbf{k}))$ of $\bar{\rho}$. Suppose further that $\bar{\rho}\mid_{G_p}$ is not twist equivalent to the trivial representation or the indecomposable unramified representation given by $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}$. If $\bar{\rho}\mid_{G_p}$ is a twist of an ordinary representation then, up to a twist by a character of finite order, ρ can be taken to be ordinary. If inertia at p acts via fundamental characters of level 2 then, up to a twist by a character of finite order, ρ can be taken to be crystalline.*

Proof. If the order of the image of $\bar{\rho}$ is prime to p we may take ρ to be the Teichmüller lift of $\bar{\rho}$. This is independent of the parity of $\bar{\rho}$ or whether we are in an excluded case. In this case ρ is ramified at exactly the same primes as $\bar{\rho}$ so we may take $R = S$.

In our local computations we have taken the liberty of twisting $\bar{\rho}\mid_{G_v}$ by a character of finite order. This only affects us insofar as the ρ we construct (in the odd cases where $\mathcal{C}_p$ consists of ordinary at p or crystalline at p deformations of $\bar{\rho}$ to $W(\mathbf{k})$) is ordinary or crystalline only after twisting by a character of finite order. Thus in these cases ρ will be *potentially ordinary or crystalline.* In the even cases that we can handle ρ is simply some deformation of $\bar{\rho}$ to $W(\mathbf{k})$.

If the image of $\bar{\rho}$ contains an element of order p, then since $p \geq 7$ we see from Section 260 of [D] that the image of $\bar{\rho}$ is either contained in a Borel subgroup of $\mathrm{GL}_2(\mathbf{k})$ or the projective image of $\bar{\rho}$ is conjugate over the quadratic extension of $\mathbf{k}$ to either $\mathrm{PGL}_2(\mathbf{k}')$ or $\mathrm{PSL}_2(\mathbf{k}')$ for some subfield $\mathbf{k}'$ of the quadratic extension of $\mathbf{k}$. In the Borel case Theorem 2 of [Kh1] provides us with a reducible potentially semistable deformation of $\bar{\rho}$ to $W(\mathbf{k})$ independent of the parity of $\bar{\rho}$ or whether we are in an excluded case. This ρ is unramified at all primes outside of a finite set R containing S.

In the remaining cases Lemma 10 applies and the four conditions at the beginning of the Recollections section hold. We are in a position to use the results we have developed in this section. Here we will take $R = S \cup Q \cup T$ where Q and T are as in Lemmas 15 and 16.

Note that since $Ш^1_{S\cup Q}((\mathrm{Ad}^0\bar{\rho})^*) = 0$ we have $Ш^1_{S\cup Q\cup T}((\mathrm{Ad}^0\bar{\rho})^*) = 0$ so $Ш^2_{S\cup Q\cup T}(\mathrm{Ad}^0\bar{\rho}) = 0$ by Tate-Poitou duality. (The set Q of primes was chosen to annihilate a 'dual Selmer group'.) By Chapter I, Theorem 4.10 of [Mi], using that $H^0(G_S, (\mathrm{Ad}^0\bar{\rho})^*) = 0$ by absolute irreducibility of $(\mathrm{Ad}^0\bar{\rho})^*$, we see the map $H^2(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho}) \to \oplus_{v\in S\cup Q\cup T} H^2(G_v, \mathrm{Ad}^0\bar{\rho})$ is an isomorphism.

We prove the theorem by induction as in Theorem 1 of [R2]. Recall, for $\bar{\rho}$ odd or even, we have classes $\mathcal{C}_v$ of deformations of $\bar{\rho}$ to $W(\mathbf{k})$ for primes $v \in S$. We also have classes $\mathcal{C}_v$ of deformations to $W(\mathbf{k})$ for primes $v \in Q \cup T$. See the discussion following Fact 13. Note that $\bar{\rho}$ is (trivially) the $\bmod\, p$ reduction of an element of $\mathcal{C}_v$ for *all* primes $v \in S \cup Q \cup T$. This is the base case of our induction.

Suppose then we have a deformation $\rho_n : G_{S\cup Q\cup T} \to \mathrm{GL}_2(W(\mathbf{k})/p^n)$ of $\bar{\rho}$ such that $\rho_n\mid_{G_v}$ is the $\bmod\, p^n$ reduction of an element of $\mathcal{C}_v$ for all $v \in S\cup Q\cup T$, that is $\rho_n\mid_{G_v} \in \mathcal{C}_v$ for all $v \in S \cup Q \cup T$. To complete the induction we must find a deformation $\rho_{n+1} : G_{S\cup Q\cup T} \to \mathrm{GL}_2(W(\mathbf{k})/p^{n+1})$ of ρ_n such that $\rho_{n+1}\mid_{G_v} \in \mathcal{C}_v$ for all $v \in S \cup Q \cup T$. Since ρ_n is the $\bmod\, p^n$ reduction of an element of $\mathcal{C}_v$ for all $v \in S\cup Q\cup T$ we see the local obstructions to deforming to $W(\mathbf{k})/p^{n+1}$ all vanish. As $Ш^2_{S\cup Q\cup T}(\mathrm{Ad}^0\bar{\rho}) = 0$ the global obstruction vanishes and we see there exists a deformation $\tilde{\rho} : G_{S\cup Q\cup T} \to \mathrm{GL}_2(W(\mathbf{k})/p^{n+1})$ of ρ_n to $W(\mathbf{k})/p^{n+1}$. It remains to adjust $\tilde{\rho}$ by a suitable element of $H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho})$ so that this adjustment is the $\bmod\, p^{n+1}$ reduction of an element of $\mathcal{C}_v$ for all $v \in S \cup Q \cup T$.

For any $v \in S \cup Q \cup T$ we know there exist $f_v \in H^1(G_v, \mathrm{Ad}^0\bar{\rho})$ such that $(\mathbf{1} + p^n f_v).\tilde{\rho}\mid_{G_v} \in \mathcal{C}_v$ where f_v is defined up to an element of $\mathcal{N}_v$. By Lemma 16 there is an $f \in H^1(G_{S\cup Q\cup T}, \mathrm{Ad}^0\bar{\rho})$ such that f and $\oplus_{v\in S\cup Q\cup T} f_v$ have the same image in $\oplus_{v\in S\cup Q\cup T}\left(H^1(G_v,\mathrm{Ad}^0\bar{\rho}/\mathcal{N}_v\right)$. Thus $(\mathbf{1}+p^n f).\tilde{\rho}\mid_{G_v}\in\mathcal{C}_v$ for all $v\in S\cup Q\cup T$. The induction is complete.

We have thus constructed $\rho: G_R\to \mathrm{GL}_2(W(\mathbf{k}))$, a deformation of $\bar{\rho}$. If $\bar{\rho}$ is odd and is not an excluded case as in part b) of the theorem, then $\rho\mid_{G_p}\in\mathcal{C}_p$ is, up to a finite twist, ordinary or crystalline as described. In all these nonexcluded cases ρ is potentially semistable at p by Proposition 9.

In the introduction we remarked that the ρ of Theorem 1 can be made unique, that is when we insist that, up to a twist, $\rho\mid_{G_q}=\begin{pmatrix}\chi & * \\ 0 & 1\end{pmatrix}$ for $q\in R-S$ then ρ is unique. (Note that $\rho\mid_{G_q}$ may be decomposable.) This uniqueness follows immediately from the two isomorphisms

$$H^2(G_R,\mathrm{Ad}^0\bar{\rho})\to\oplus_{v\in R}H^2(G_v,\mathrm{Ad}^0\bar{\rho})$$

and

$$H^1(G_R,\mathrm{Ad}^0\bar{\rho})\to\oplus_{v\in R}\left(H^1(G_v,\mathrm{Ad}^0\bar{\rho})/\mathcal{N}_v\right).$$

Indeed, the kernel of the latter map is the (reduced) tangent space for the deformation functor where we insist our deformation of $\bar{\rho}$ be unramified outside R, *minimally ramified* in the sense of [Di1] at primes $l\in S$, $l\neq p$, and of the form described at the beginning of this paragraph for $q\in R-S$. Finally, for p we insist our deformation be of type $\mathcal{C}_p$. (Note that this is no restriction at all if $\bar{\rho}$ is even.) Since the kernel of this map is trivial our restricted deformation ring is simply $W(\mathbf{k})$.

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