

A GEOMETRIC JACQUET-LANGLANDS TRANSFER FOR AUTOMORPHIC FORMS OF HIGHER WEIGHTS

JIZE YU

ABSTRACT. In this paper, we give a geometric construction of the Jacquet-Langlands transfer for automorphic forms of higher weights by studying the geometry of the mod p fibres of different Hodge type Shimura varieties which satisfy a mild assumption and the cohomological correspondences between them.

CONTENTS

1. Introduction	2
1.1. Main theorem	2
1.2. Plan of the paper	3
1.3. Notations	4
1.4. Acknowledgements	4
2. The Integral Coefficient Geometric Satake Equivalence in Mixed Characteristic	4
2.1. The geometry of mixed characteristic affine Grassmannians	5
2.2. The Satake category	5
2.3. More on the convolution Grassmannians	6
2.4. The geometric Satake equivalence	7
3. Local Hecke Stacks	8
3.1. Torsors over the local Hecke stacks	8
3.2. Perverse sheaves on the moduli of local Hecke stacks	9
4. Moduli of Local Shtukas	9
4.1. Moduli of restricted local Shtukas	11
4.2. Perverse sheaves on the moduli of local Shtukas	12
5. Key theorem	13
5.1. Set up	13
5.2. More on local Hecke stacks	13
5.3. More on moduli of local Shtukas	14
5.4. The category of coherent sheaves on the stack of unramified Langlands parameters	14
5.5. Key theorem	15
6. Cohomological correspondences between Shimura varieties	19
6.1. Preliminaries	19
6.2. Étale local systems	19
6.3. Main theorem	20
6.4. Non-vanishing of the geometric Jacquet-Langlands transfer	23
References	24

1991 *Mathematics Subject Classification.* Primary 14G35, 11G18; Secondary 14D24, 20C20.

Key words and phrases. geometric Satake equivalence, Hodge type Shimura varieties, Jacquet-Langlands Transfer, moduli of local Shtukas.

The author is very grateful to Xinwen Zhu for introducing the beautiful theory of moduli of Shtukas to him, and providing numerous helpful comments and discussions while he was working on this project.

1. INTRODUCTION

Let G and G' be two algebraic groups over \mathbb{Q} . We assume that they are isomorphic at all finite places of \mathbb{Q} , but not necessarily at infinity. Roughly speaking, the Jacquet-Langlands correspondence predicts, in many cases, the following phenomenon: there exists a natural map between the set of automorphic representations of G and that of G' such that if π' is the automorphic representation of G' which corresponds to an automorphic representation π of G , then π_ν is isomorphic to π'_ν at all finite places ν . It is considered as one of the first examples of the Langlands philosophy that maps between L -groups induce maps between automorphic representations.

The classical way of establishing this correspondence is via a comparison of the trace formulas for G and G' . This approach allows us to conclude a map between suitable spaces of automorphic forms for G and G' as abstract representations. However, the resulting map is not canonical, and we, therefore, hope to have a more natural way of understanding this correspondence. Alternative geometric approaches of establishing the Jacquet-Langlands correspondence were first noticed by Ribet [12] and Serre [13], and later followed by Ghitza [2], [3], and Helm [5] [4].

In the recent work of Xiao-Zhu [15], a more general strategy is developed to relate the geometry and (ℓ -adic) cohomology of the mod p fibers of different Shimura varieties, which unifies and generalizes some aspects of the above mentioned works. In this paper, we adapt the machinery developed in *loc.cit* and apply the integral coefficient geometric Satake equivalence in mixed characteristic obtained by the author [17]¹ to establish a Jacquet-Langlands transfer for automorphic forms with higher weights.

1.1. Main theorem.

1.1.1. *Statement of the Main Theorem.* Let (G_1, X_1) and (G_2, X_2) be two Hodge type Shimura data equipped with an isomorphism $\theta : G_{1, \mathbb{A}_f} \simeq G_{2, \mathbb{A}_f}$. Assume that there exists an inner twist $\Psi : G_1 \rightarrow G_2$ which is compatible with θ . We assume that $K_i \subset G(\mathbb{A}_f)$ to be sufficiently small such that $\theta K_1 = K_2$. In addition, we assume that p is an unramified prime. Then $K_{1,p}$ (and therefore $K_{2,p}$) is hyperspecial. Let \underline{G}_i be the integral model of G_{i, \mathbb{Q}_p} over \mathbb{Z}_p determined by $K_{i,p}$. Then $\underline{G}_1 \simeq \underline{G}_2$, and we can thus identify their Langlands dual groups, which we denote by $\hat{G}_{\mathbb{Q}_\ell}$. We further fix a pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ of \hat{G} . Choose an isomorphism $\iota : \mathbb{C} \simeq \bar{\mathbb{Q}}_p$. Let $\{\mu_i\}$ denote the conjugacy class of Hodge cocharacters determined by X_i .

Let $V_i := V_{\mu_i}$ denote the irreducible representation of $\hat{G}_{\mathbb{Q}_\ell}$ of highest weight μ_i . Let $\nu \mid p$ be a place of the compositum of reflex field E_1, E_2 of $(G_1, X_1), (G_2, X_2)$ (determined by our choice of isomorphism ι). Write k_ν for the residue field of $E_1 E_2$ at ν . Results of Kisin [6] and Vasiu [14] state that there exists a canonical smooth integral model of $\mathrm{Sh}_K(G_i, X_i)$ over $\mathcal{O}_{E, (\nu)}$. Let $d_i = \dim \mathrm{Sh}_K(G_i, X_i)$ and Sh_{μ_i} denote the mod p fiber of this canonical integral model, base changed to \bar{k}_ν . Our assumption on p implies that the action of the Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ factors through some finite quotient $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ for some finite field \mathbb{F}_{p^n} which contains k_ν . Write $\sigma \in \mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ for the arithmetic Frobenius. Consider the conjugation action of $\hat{G}_{\mathbb{Q}_\ell}$ on the (non-neutral) component $\hat{G}_{\mathbb{Q}_\ell} \sigma \subset \hat{G}_{\mathbb{Q}_\ell} \rtimes \langle \sigma \rangle$. Denote by $\mathrm{Coh}^{\hat{G}}(\hat{G} \sigma)$ the abelian category of coherent sheaves on the quotient stack $[\hat{G}_{\mathbb{Q}_\ell} \sigma / \hat{G}_{\mathbb{Q}_\ell}]$.

To each representation W of $G_{\mathbb{Q}_\ell}$, we can attach an ℓ -adic étale local system $\mathcal{L}_{i, W, \mathbb{Q}_\ell}$ on Sh_{μ_i} by varying the level structure at ℓ (see §6.2). The natural projection $[\hat{G}_{\mathbb{Q}_\ell} \sigma / \hat{G}_{\mathbb{Q}_\ell}] \rightarrow \mathbb{B} \hat{G}_{\mathbb{Q}_\ell}$ attaches to each representation V of $\hat{G}_{\mathbb{Q}_\ell}$ a vector bundle \tilde{V} on $[\hat{G}_{\mathbb{Q}_\ell} \sigma / \hat{G}_{\mathbb{Q}_\ell}]$. Denote the global section of the structure sheaf on the quotient stack $[\hat{G} \sigma / \hat{G}]$ by \mathcal{J} , and the prime-to- p Hecke algebra by \mathcal{H}^p . Fix a half Tate twist $\mathbb{Q}_\ell(1/2)$.

We state our main theorem.

Theorem 1.1. *Let $\mathcal{L}_i := \mathcal{L}_{i, W, \mathbb{Q}_\ell}[d_i](d_i/2)$. Under a mild assumption,*

- (1) *there exists a map*

$$(1.1) \quad \mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}}(\hat{G} \sigma)}(\tilde{V}_1, \tilde{V}_2) \rightarrow \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_1), \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_2)),$$

which is compatible with compositions in the source and target.

- (2) *the ring of endomorphisms $\mathrm{End}_{[\hat{G} \sigma / \hat{G}]}(\mathcal{O}_{[\hat{G} \sigma / \hat{G}]})$ acts on the compactly supported cohomology $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_i)$ via Spc and this action can be identified with the classical Satake isomorphism if $\mathrm{Sh}_K(G_1, X_1) = \mathrm{Sh}_K(G_2, X_2)$ is a Shimura set.*

¹We mention that Peter Scholze announced the same result using the perfectoid approach.

This is a Jacquet-Langlands transfer for automorphic forms of higher weights which generalizes a previous construction given in [15]. We briefly discuss the proof of Theorem 1.1.

1.1.2. Strategy of Proof. Let $\Lambda = \mathbb{Z}_\ell, \mathbb{F}_\ell$. The following theorem plays an essential role in the proof of the main result.

Theorem 1.2. *For any projective Λ -modules $\Lambda_1, \Lambda_2 \in \text{Rep}_\Lambda(\hat{G}_\Lambda)$, we choose appropriate integers (m_1, n_1, m_2, n_2) and a dominant coweight λ of G , and consider the Hecke correspondence of moduli of local Shtukas*

$$(1.2) \quad \text{Sht}_{\Lambda_1}^{\text{loc}(m_1, n_1)} \xleftarrow{h_{\Lambda_1}^-} \text{Sht}_{\Lambda_1 | \Lambda_2}^{\lambda, \text{loc}(m_1, n_1)} \xrightarrow{h_{\Lambda_2}^+} \text{Sht}_{\Lambda_2}^{\text{loc}(m_2, n_2)}.$$

Then there exists the following map

$$(1.3) \quad \mathcal{S}_{\Lambda_1, \Lambda_2} : \text{Hom}_{\text{Coh}_{\hat{G}_\Lambda}(\hat{G}_\Lambda \sigma)}(\widetilde{\Lambda_1}, \widetilde{\Lambda_2}) \longrightarrow \text{Hom}_{\text{D}(\text{Sht}_{\Lambda_1 | \Lambda_2}^{\text{loc}(m_1, n_1)})}((h_{\Lambda_1}^-)^* S(\widetilde{\Lambda_1}), (h_{\Lambda_2}^+)^! S(\widetilde{\Lambda_2})),$$

which is independent of auxiliary choices.

We remark that the target of $\mathcal{S}_{\Lambda_1, \Lambda_2}$ can be understood as limits of the spaces of *cohomological correspondences* between $(\text{Sht}_{\Lambda_1}^{\text{loc}(m_1, n_1)}, S(\widetilde{\Lambda_1}))$ and $(\text{Sht}_{\Lambda_2}^{\text{loc}(m_2, n_2)}, S(\widetilde{\Lambda_2}))$ which are supported on the Hecke correspondence (1.2). In [15], Xiao-Zhu construct the maps \mathcal{S}_{V_1, V_2} for \mathbb{Q}_ℓ -representations $\hat{G}_{\mathbb{Q}_\ell}$ in a categorical way. In fact, they define the category $\text{P}^{\text{Hk}}(\text{Sht}^{\text{loc}}, \mathbb{Q}_\ell)$. Its objects are perverse sheaves supported on finite dimensional subspace of $\text{Sht}_{\mathbb{F}_q}^{\text{loc}}$, and its spaces of morphisms are given by (limits of) cohomological correspondences supported on Hecke correspondences of *restricted* local Shtukas. It receives a canonical functor from the 'normal' category of perverse sheaves $\text{P}(\text{Sht}^{\text{loc}}, \mathbb{Q}_\ell)$, and this functor has a σ -twisted trace structure. Hence, the universal property of categorical traces asserts that it admits a functor from the σ -twisted categorical trace $\text{Tr}_\sigma(\text{Rep}(\hat{G}_{\mathbb{Q}_\ell})) \cong \text{Coh}_{\text{fr}}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell} \sigma)$ in the sense of [20]. Here, the latter category is the full subcategory of $\text{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell} \sigma)$ generated by objects coming from $\text{Rep}_{\mathbb{Q}_\ell}(\hat{G}_{\mathbb{Q}_\ell})$. This idea is made more explicit in *loc.cit.*

The above strategy does not carry over in our situation. First of all, it is not clear how to a suitable notion of categorical trace of To calculate the left σ -twisted categorical trace of $\text{Rep}_{\mathbb{F}_\ell}(\hat{G}_{\mathbb{F}_\ell})$. To accomplish this, we need to appeal to a more general construction of the tensor product for finitely cocomplete categories. In addition, the correspondence category $\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}}, \mathbb{Z}_\ell)$ is not finitely cocomplete any more and the desired maps \mathcal{S} therefore cannot be obtained by the universal property of the categorical trace. One possible way to overcome this difficulty is to upgrade $\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}}, \mathbb{Z}_\ell)$ to a higher category.

Instead of pursuing this idea, we take a more concrete approach. We note that there is a natural isomorphism

$$\text{Hom}_{\text{Coh}_{\hat{G}_\Lambda}(\hat{G}_\Lambda \sigma)}(\widetilde{\Lambda_1}, \widetilde{\Lambda_2}) \cong \text{Hom}_{\hat{G}_\Lambda}(\Lambda_1, \mathcal{O}_G \otimes \Lambda_2),$$

where \mathcal{O}_G denotes the regular representation of \hat{G}_Λ . By the Peter-Weyl theorem, \mathcal{O}_G admits a filtration with associated graded $\bigoplus_\lambda W(\lambda) \otimes S(\lambda^*)$ where $W(\lambda)$ denotes the Schur module of \hat{G}_Λ . For any $W \in \text{Rep}(\hat{G})$, and any $\mathbf{a} \in \text{Hom}_{\hat{G}}(V, W \otimes S(\lambda^*) \otimes W)$, we use the integral coefficient geometric Satake equivalence discussed in §2.4 to construct a cohomological correspondence on restricted local Hecke stacks of $G \times G$. The maps $\mathcal{S}_{V, W}$ are constructed by first pulling this cohomological correspondence back to a cohomological correspondence on restricted local Hecke stacks and then pulling it back to a cohomological correspondence on restricted local Shtukas.

1.2. Plan of the paper. We briefly discuss the organization of the paper. In §2, we recall the geometric properties of mixed characteristic affine Grassmannians and the integral coefficient geometric Satake equivalence following [19] and [17]. In §3 and §4, we summarize the geometry of local Hecke stacks and moduli of local Shtukas. In addition, we define categories of perverse sheaves on these two stacks and study the cohomological correspondences in these categories. Results appeared in these two sections are proved in [15], and we refer to *loc.cit* for detailed proofs. Section 5 is devoted to study the cohomological correspondences between perverse sheaves on the stacks of restricted local Shtukas which come from the integral geometric Satake equivalence. We derive the key ingredient of the main theorem. We apply the results in previous sections to study the cohomological correspondences between mod p fibres of different Hodge type Shimura varieties and prove our main theorem in §6.

1.3. Notations. We fix notations for later use.

Let F be a mixed characteristic local ring with ring of integers \mathcal{O} and residue field $k = \mathbb{F}_q$. We write σ for the arithmetic Frobenius of \mathbb{F}_q . For any k -algebra R , its ring of Witt vectors is denoted by

$$W(R) = \{(r_0, r_1, \dots) \mid r_i \in R\}.$$

We denote by $W_h(R)$ the ring of truncated Witt vectors of length h . Write $\mathbf{Aff}_k^{\text{pf}}$ for the category of perfect k -algebras. For any $R \in \mathbf{Aff}_k^{\text{pf}}$, we know that $W_h(R) = W(R)/p^h W(R)$. We define the ring of Witt vectors in R with coefficient in \mathcal{O} as

$$W_{\mathcal{O}}(R) := W(R) \hat{\otimes}_{W(k)} \mathcal{O} := \varprojlim_n W_{\mathcal{O},n}(R), \text{ and } W_{\mathcal{O},n}(R) = W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n.$$

We define the n -th unit disk, formal unit disk, and formal punctured unit disk to be

$$D_{n,R} := \text{Spec} W_{\mathcal{O},n}(R), \quad D_R := \text{Spec} W_{\mathcal{O}}(R), \quad D_R^{\times} := \text{Spec} W_{\mathcal{O}}(R)[1/\varpi],$$

respectively.

Let L be the completion of the maximal unramified extension of F . Denote by \mathcal{O}_L its ring of integers, and we fix a uniformizer $\varpi \in \mathcal{O}_L$. We will assume G to be an unramified reductive group scheme over \mathcal{O} . We denote by T the abstract Cartan subgroup of G . Let $S \subset T$ denote the maximal split subtorus. In the case where G is a split reductive group, we will choose a Borel subgroup $B \subset G$ over \mathcal{O} and a split maximal torus $T \subset B$. When we need to embed T (or S) into G as a (split) maximal torus, we will state it explicitly.

Let \mathbb{X}_{\bullet} denote the coweight lattice of T and \mathbb{X}^{\bullet} the weight lattice. Let $\Delta \subset \mathbb{X}^{\bullet}$ (resp. $\Delta^{\vee} \subset \mathbb{X}_{\bullet}$) the set of roots (resp. coroots). A choice of the Borel subgroup $B \subset G$ determines the semi-group of dominant coweights $\mathbb{X}_{\bullet}^+ \subset \mathbb{X}_{\bullet}$ and the set of positive roots $\Delta_+ \subset \Delta$. In fact, \mathbb{X}_{\bullet}^+ and Δ_+ are both independent of the choice of B . The q -power (arithmetic) Frobenius σ acts on $(\mathbb{X}^{\bullet}, \Delta, \mathbb{X}_{\bullet}, \Delta^{\vee})$ preserving \mathbb{X}_{\bullet}^+ . The Langlands dual group of G is denoted by \hat{G} .

Let $2\rho \in \mathbb{X}^{\bullet}$ be the sum of all positive roots. Define the partial order “ \leq ” on \mathbb{X}_{\bullet} to be such that $\lambda \leq \mu$ if and only if $\mu - \lambda$ equals a non-negative integral linear combination of positive coroots. For any $\mu \in \mathbb{X}_{\bullet}$, denote ϖ^{μ} by the image of μ under the composition of maps

$$\mathbb{G}_m \rightarrow T \subset G.$$

Let H be a reductive group over a field K . We write H_{ad} for its adjoint group, H_{der} for its derived group, and H_{sc} for the simply connected cover of H_{ad} . Let $T \subset H$ be a maximal torus, we denote by T_{ad} its image in the quotient G_{ad} . We also write T_{sc} for the preimage of T in G_{sc} .

We denote by \mathcal{E}^0 the trivial G -torsor. For any perfect k -algebra R , the arithmetic Frobenius σ induces an automorphism $\sigma \otimes \text{id}$ of the D_R . Let \mathcal{E} be a G -torsor over D_R , and we denote the G -torsor $(\sigma \otimes \text{id})^* \mathcal{E}$ by ${}^{\sigma} \mathcal{E}$.

Let X be an algebraic space over k . We write σ_X for the absolute Frobenius morphism of X . We denote by $X^{p^{\infty}} := \varprojlim_{\sigma_X} X$ the perfection of X .

Let $\ell \neq p$ be a prime number. Fix a half Tate twist $\mathbb{Z}_{\ell}(1/2)$. We write $\langle d \rangle := [d](d/2)$. Let X and Y be two algebraic stacks which are perfectly of finitely presentation in the opposite category of perfect k -algebras. For each $f : X \rightarrow Y$ being a perfectly smooth morphism of relative dimension d , we write $f^{\star} := f^{\star} \langle d \rangle$.

Throughout the paper, we write Λ for \mathbb{Z}_{ℓ} and \mathbb{F}_{ℓ} and E for \mathbb{Z}_{ℓ} unless otherwise stated. For stacks X_1, X_2 , and perverse sheaves $\mathcal{F}_i \in \text{P}(X_i, \Lambda)$, we sometimes write the space of cohomological correspondences $\text{Corr}_X((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2))$ as $\text{Corr}_X(\mathcal{F}_1, \mathcal{F}_2)$ for simplicity.

Let X be a stack, we denote by $\omega_X \in D_b^c(X, E)$ the dualizing sheaf of X in the bounded derived category of sheaves on X . For a perfect pfp algebraic space (cf.[15, A.1.7]) X over k , we denote by $H_i^{BM}(X_{\bar{k}}) := H^{-i}(X_{\bar{k}}, \omega_x(-i/2))$ the i th Borel-Moore homology of $X_{\bar{k}}$.

1.4. Acknowledgements. I am very grateful to my advisor Xinwen Zhu for introducing me to the beautiful theory of moduli of Shtukas to me and providing numerous helpful comments and discussions while I was working on this project.

2. THE INTEGRAL COEFFICIENT GEOMETRIC SATAKE EQUIVALENCE IN MIXED CHARACTERISTIC

We briefly recall the geometric properties of the mixed characteristic affine Grassmannians and the integral coefficient geometric Satake equivalence. We refer to [19] and [17] for more details.

2.1. The geometry of mixed characteristic affine Grassmannians.

Definition 2.1. The mixed characteristic *affine Grassmannian* Gr_G of G over k is defined as the presheaf over $\mathbf{Aff}^{\mathrm{pf}}_k$

$$\mathrm{Gr}_G(R) = \left\{ (\mathcal{E}, \phi) \left| \begin{array}{l} \mathcal{E} \rightarrow D_R \text{ is a } G\text{-torsor, and} \\ \phi : \mathcal{E} \mid_{D_R^\times} \simeq \mathcal{E}^0 \mid_{D_R^\times} \end{array} \right. \right\}.$$

In addition to this moduli interpretation, it can be understood as a homogeneous space. We define the p -adic *jet group* (resp. *loop group*) as the presheaf $L^+G(R) := G(W_{\mathcal{O}}(R))$ (resp. $LG := G(W_{\mathcal{O}}(R[1/\varpi]))$) over the $\mathbf{Aff}^{\mathrm{pf}}_k$. We define $L^nG(R) := G(W_{\mathcal{O},n}(R))$ to be the n -th *jet group*. It is represented by the perfection of an algebraic k -group and we have $L^+G = \varprojlim_n L^nG$. We define the n -th *congruence group* $L^+G^{(n)}(R) := \ker(L^+G \rightarrow L^nG(R))$. Then Gr_G can be realized as the fpqc quotient space $[LG/L^+G]$. It is clear that $LG \rightarrow \mathrm{Gr}_G$ is an L^+G -torsor and L^+G naturally acts on Gr_G . We can form the twisted product which we also call the *convolution product* of affine Grassmannians

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G := LG \times^{L^+G} \mathrm{Gr}_G := [LG \times \mathrm{Gr}_G / L^+G],$$

where L^+G acts on $LG \times \mathrm{Gr}_G$ anti-diagonally as $g^+ \cdot ([g_1], [g_2]) := ([g_1(g^+)^{-1}], [g^+g_2])$. It also admits a moduli interpretation as follows

$$\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G(R) = \left\{ (\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \left| \begin{array}{l} \mathcal{E}_1, \mathcal{E}_2 \text{ are } G\text{-torsors on } D_R, \text{ and} \\ \beta_1 : \mathcal{E}_1 \mid_{D_R^\times} \simeq \mathcal{E}^0 \mid_{D_R^\times}, \beta_2 : \mathcal{E}_2 \mid_{D_R^\times} \simeq \mathcal{E}_1 \mid_{D_R^\times} \end{array} \right. \right\}.$$

We define the convolution morphism

$$m : \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \longrightarrow \mathrm{Gr}_G,$$

such that

$$(\mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2) \longmapsto (\mathcal{E}_2, \beta_1\beta_2).$$

The celebrated theorem of Bhatt-Scholze [1] shows that the functor Gr_G is representable by an inductive limit of perfections of projective varieties.

For each algebraically closed field K containing \bar{k} , the moduli interpretation of affine Grassmannians assigns to each $(\mathcal{E}, \phi) \in \mathrm{Gr}_G(K)$ an element in $G(W_{\mathcal{O}}(K)) \backslash G(\mathcal{O}(K)[1/\varpi]) / G(\mathcal{O}(K)) \simeq \mathbb{X}_{\bullet}^+$ by the Cartan decomposition. We denote the corresponding dominant coweight by $\mathrm{Inv}(\phi)$ and call it the *relative position* of (\mathcal{E}, ϕ) .

Definition 2.2. For each $\mu \in \mathbb{X}_{\bullet}^+$, we define

- (1) the (spherical) *Schubert variety*

$$\mathrm{Gr}_{\mu} := \{(\mathcal{E}, \phi) \in \mathrm{Gr}_G \mid \mathrm{Inv}(\phi) \leq \mu\},$$

- (2) the *Schubert cell*

$$\mathring{\mathrm{Gr}}_{\mu} := \{(\mathcal{E}, \phi) \in \mathrm{Gr}_G \mid \mathrm{Inv}(\phi) = \mu\}.$$

The Schubert cell $\mathring{\mathrm{Gr}}_{\mu}$ is the orbit of ϖ^{μ} in Gr_G under the L^+G -action. It is the perfection of a quasi-projective smooth variety of dimension $(2\rho, \mu)$, and L^+G acts on it via a finite type quotient L^mG . Taking the Zariski closure of $\mathring{\mathrm{Gr}}_{\mu}$ gives Gr_{μ} . Moreover, for any $\mu, \nu \in \mathbb{X}_{\bullet}^+$, $\mu \leq \nu$ if and only if $\mathrm{Gr}_{\mu} \subseteq \mathrm{Gr}_{\nu}$.

2.2. The Satake category. We know that $\pi_0(\mathrm{Gr}_G \otimes \bar{k}) \simeq \pi_1(G)$ (cf. [19, Proposition 1.21]). The affine Grassmannian $\mathrm{Gr}_G \otimes \bar{k}$ has the decomposition into connected components

$$\mathrm{Gr}_G \otimes \bar{k} = \sqcup_{\xi \in \pi_1(G)} \mathrm{Gr}_{\xi}.$$

Recall our discussion in §2.1. We have

$$\mathrm{Gr}_{\xi} = \varinjlim_{\mu \in \xi} \mathrm{Gr}_{\mu},$$

where $\mu \in \xi$ means that the natural map $\mathbb{X}_{\bullet} \rightarrow \pi_1(G)$ sends μ to ξ . The connecting morphism $i_{\mu, \nu} : \mathrm{Gr}_{\mu} \rightarrow \mathrm{Gr}_{\nu}$ is a closed embedding if $\mu \leq \nu$. For $m \leq m'$ be integers such that L^+G acts on Gr_{μ} through L^mG and $L^{m'}G$, there is a canonical equivalence

$$P_{L^mG \otimes \bar{k}}(\mathrm{Gr}_{\mu}, \Lambda) \cong P_{L^{m'}G \otimes \bar{k}}(\mathrm{Gr}_{\mu}, \Lambda).$$

We define the category of $L^+G \otimes \bar{k}$ -equivariant Λ -coefficient perverse sheaves on $\mathrm{Gr}_G \otimes \bar{k}$ as

$$\begin{aligned} P_{L^+G \otimes \bar{k}}(\mathrm{Gr}_G \otimes \bar{k}, \Lambda) &:= \bigoplus_{\xi \in \pi_1(G)} P_{L^+G \otimes \bar{k}}(\mathrm{Gr}_\xi, \Lambda), \\ P_{L^+G \otimes \bar{k}}(\mathrm{Gr}_\xi, \Lambda) &:= \varinjlim_{(\mu, m)} P_{L^m G \otimes \bar{k}}(\mathrm{Gr}_\mu, \Lambda). \end{aligned}$$

Here, the limit is taken over the pairs $\{(\mu, m) \mid \mu \in \xi, m \text{ is large enough}\}$ with partial order given by $(\mu, m) \leq (\mu', m)$ if $\mu \leq \mu'$ and $m \leq m'$. The connecting morphism is the composition

$$P_{L^m G \otimes \bar{k}}(\mathrm{Gr}_\mu, \Lambda) \cong P_{L^{m'} G \otimes \bar{k}}(\mathrm{Gr}_\mu, \Lambda) \xrightarrow{i_{\mu, \mu'}^*} P_{L^{m'} G \otimes \bar{k}}(\mathrm{Gr}_{\mu'}, \Lambda)$$

which is a fully faithful embedding. We also call this category the *Satake category* and sometimes denote it by $\mathrm{Sat}_{G, \Lambda}$ for simplicity. We denote by IC_μ for each $\mu \in \mathbb{X}_\bullet^+$ the *intersection cohomology sheaf* on $\mathrm{Gr}_{\leq \mu}$. Its restriction to each open strata Gr_μ is constant and in particular, $\mathrm{IC}_\mu|_{\mathrm{Gr}_\mu} \simeq \Lambda[(2\rho, \mu)]$.

We recall the monoidal structure in $\mathrm{Sat}_{G, \Lambda}$ defined by Lusztig's convolution of sheaves. Consider the following diagram

$$\mathrm{Gr}_G \times \mathrm{Gr}_G \xleftarrow{p} LG \times \mathrm{Gr}_G \xrightarrow{q} \mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G \xrightarrow{m} \mathrm{Gr}_G,$$

where p and q are natural projections. We define for any $\mathcal{A}_1, \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr}_G, \Lambda)$,

$$\mathcal{A} \star \mathcal{A}_2 := Rm_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2),$$

where $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr}_G \tilde{\times} \mathrm{Gr}_G, \Lambda)$ is the unique object such that

$$q^*(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2) \simeq p^*({}^p H^0(\mathcal{A}_1 \boxtimes \mathcal{A}_2)).$$

We remark that unlike the construction in $P_{L^+G}(\mathrm{Gr}_G, \bar{\mathbb{Q}}_\ell)$, taking the 0-th perverse cohomology in the above definition is necessary. This is because when we work with \mathbb{Z}_ℓ -sheaves, the sheaf $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ may not be perverse. In fact, $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ is perverse if one of $H^*(\mathcal{A}_i)$ is a flat \mathbb{Z}_ℓ -module. For more details, please refer to [11, Lemma 4.1]. A priori, the convolution product $\mathcal{A}_1 \star \mathcal{A}_2$ is a complex of L^+G equivariant sheaves on Gr_G . It is the semi-smallness of the convolution morphism m (cf.[19]) that guarantees $\mathcal{F}_1 \star \mathcal{F}_2$ to be a perverse sheaf.

2.3. More on the convolution Grassmannians. We recall the definition and geometry of the convolution product of multiple copies of affine Grassmannians and related constructions. A detailed exposition maybe found in [19, §3.1].

Let $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$ be a sequence of dominant coweights.

Definition 2.3. We define the convolution product of $\mathrm{Gr}_{\mu_1}, \mathrm{Gr}_{\mu_2}, \dots, \mathrm{Gr}_{\mu_n}$ as the presheaf $\mathrm{Gr}_{\mu_\bullet}$ which classifies isomorphism classes of modifications of G -torsors over D_R

$$(2.1) \quad \mathcal{E}_n \xrightarrow{\beta_1} \mathcal{E}_{n-1} \xrightarrow{\beta_{n-1}} \dots \xrightarrow{\beta_1} \mathcal{E}_0 = \mathcal{E}^0,$$

where $\mathrm{Inv}(\beta_i) \leq \mu_i$.

We define $\mathrm{Gr}_{\leq \mu_\bullet}^{(\infty)}$ as the L^+G -torsor over $\mathrm{Gr}_{\leq \mu_\bullet}$ which classifies a point in $\mathrm{Gr}_{\leq \mu_\bullet}$ as (2.1) together with an isomorphism $\mathcal{E}_n \simeq \mathcal{E}^0$. For any integer n , we define $\mathrm{Gr}_{\leq \mu_\bullet}^{(n)}$ to be the $L^n G$ -torsor over $\mathrm{Gr}_{\leq \mu_\bullet}$ which classifies a point in $\mathrm{Gr}_{\leq \mu_\bullet}$ together with an isomorphism $\mathcal{E}_n|_{D_{n,R}} \simeq \mathcal{E}^0|_{D_{n,R}}$. For any $m < n$, there is an isomorphism

$$\mathrm{Gr}_{\leq \mu_\bullet} \simeq \mathrm{Gr}_{\leq \mu_1, \dots, \mu_m} \tilde{\times} \mathrm{Gr}_{\leq \mu_{m+1}, \dots, \mu_n} := \mathrm{Gr}_{\leq \mu_1, \dots, \mu_m}^{(\infty)} \times^{L^+G} \mathrm{Gr}_{\leq \mu_{m+1}, \dots, \mu_n}.$$

Many of the constructions in later sections make use of the following lemma [19, Lemma 3.1.7]:

Lemma 2.4. *For any sequence of dominant coweights $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$, there exists a non-negative integer m , such that for any non-negative integer n , the action of $L^{m+n}G$ on $\mathrm{Gr}_{\leq \mu_\bullet}$ is trivial. In other words, the natural action of L^+G on $\mathrm{Gr}_{\leq \mu_\bullet}$ factors through the finite type quotient $L^{m+n}G$.*

We will call such an integer m a μ_\bullet -large integer. We also call a pair of non-negative integers (m, n) ($m = \infty$ allowed) to be μ_\bullet -large if $m - n$ is a μ_\bullet -large integer.

Replacing $\mathrm{Gr}_{\leq \mu_i}$ by Gr_{μ_i} , we can similarly define $\mathrm{Gr}_{\mu_\bullet} := \mathrm{Gr}_{\mu_1} \tilde{\times} \dots \tilde{\times} \mathrm{Gr}_{\mu_n}$. It admits a stratification

$$(2.2) \quad \mathrm{Gr}_{\leq \mu_\bullet} = \cup_{\mu'_\bullet \leq \mu_\bullet} \mathrm{Gr}_{\mu'_\bullet},$$

where $\mu'_\bullet \leq \mu_\bullet$ means $\mu'_i \leq \mu_i$ for each i .

As in [19], we let $|\mu_\bullet| := \sum \mu_i$. Then the convolution map induces the following morphism

$$m : \mathrm{Gr}_{\leq \mu_\bullet} \longrightarrow \mathrm{Gr}_{\leq |\mu_\bullet|},$$

such that

$$(\mathcal{E}_i, \beta_i) \longmapsto (\mathcal{E}_n, \beta_1 \cdots \beta_n).$$

Let ν_\bullet be another sequence of dominant coweights. We define the following stack

$$\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0 := \mathrm{Gr}_{\leq \mu_\bullet} \times_{m_{\mu_\bullet}, \mathrm{Gr}_G, m_{\nu_\bullet}} \mathrm{Gr}_{\leq \nu_\bullet}.$$

Write the natural projections from $\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0$ to $\mathrm{Gr}_{\leq \mu_\bullet}$ and $\mathrm{Gr}_{\leq \nu_\bullet}$ as $h_{\mu_\bullet}^\leftarrow$ and $h_{\nu_\bullet}^\rightarrow$, respectively. We call the following diagram

$$(2.3) \quad \mathrm{Gr}_{\leq \mu_\bullet} \xleftarrow{h_{\mu_\bullet}^\leftarrow} \mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0 \xrightarrow{h_{\nu_\bullet}^\rightarrow} \mathrm{Gr}_{\leq \nu_\bullet}$$

the *Satake correspondence*.

Definition 2.5. An irreducible component of $\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0$ of dimension $(\rho, |\mu_\bullet| + |\nu_\bullet|)$ is called a *Satake cycle*. Denote the set of Satake cycles of $\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0$ by $\mathbb{S}_{\mu_\bullet|\nu_\bullet}$. For $\mathbf{a} \in \mathbb{S}_{\mu_\bullet|\nu_\bullet}$, write $\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^{0, \mathbf{a}}$ for the Satake cycle labelled by \mathbf{a} .

It is clear that $\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0 \cong \mathrm{Gr}_{\nu_\bullet|\mu_\bullet}^0$, and we conclude that

$$(2.4) \quad \mathbb{S}_{\mu_\bullet|\nu_\bullet} = \mathbb{S}_{\nu_\bullet|\mu_\bullet}.$$

2.4. The geometric Satake equivalence. We state the geometric Satake equivalence [17] which is an indispensable ingredient of our construction of the Jacquet-Langlands correspondence.

Theorem 2.6. *There is an equivalence of monoidal categories*

$$(2.5) \quad \mathrm{Sat}_G : \mathrm{Rep}_\Lambda(\hat{G}_\Lambda) \simeq \mathrm{Sat}_{G, \Lambda},$$

where $\mathrm{Rep}_\Lambda(\hat{G}_\Lambda)$ denotes the category of Λ -representations of \hat{G}_Λ which are finitely generated over Λ . In particular, for any sequence μ_\bullet of dominant coweights, the Satake equivalence assigns V_{μ_\bullet} the tensor product of highest weight representations V_{μ_i} corresponding to μ_i to the convolution product $\mathrm{IC}_{\mu_\bullet}$ of intersection cohomology sheaf IC_{μ_i} .

Remark 2.7. As explained in [18, §5.5], the Galois group $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ acts on the Satake category $\mathrm{Sat}_{G, \Lambda}$ by tensor auto-equivalences. It in turn induces an action of $\mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$ on \hat{G} which preserves $(\hat{G}, \hat{B}, \hat{T})$. Let $V \in \mathrm{Rep}_E(\hat{G})$ and $\gamma \in \mathrm{Gal}(\bar{\mathbb{F}}_p/\mathbb{F}_p)$. We write ${}^\gamma V$ for the representation

$$\hat{G} \xrightarrow{\gamma^{-1}} \hat{G} \rightarrow \mathrm{GL}_E(V)$$

of \hat{G} .

For three sequences of dominant weight $\mu_{1\bullet}$, $\mu_{2\bullet}$, and $\mu_{3\bullet}$ the following lemma is an immediate consequence of Theorem 2.6.

Corollary 2.8. We have the following natural isomorphism

$$(2.6) \quad \mathrm{Hom}_{\hat{G}}(V_{\mu_{i\bullet}}, V_{\mu_{j\bullet}}) \cong \mathrm{Corr}_{\mathrm{Gr}_{\mu_{i\bullet}|\mu_{j\bullet}}^0}((\mathrm{Gr}_{\leq \mu_{i\bullet}}, \mathrm{IC}_{\mu_{i\bullet}}), (\mathrm{Gr}_{\leq \mu_{j\bullet}}, \mathrm{IC}_{\mu_{j\bullet}})),$$

such that the natural composition on the left hand

$$\mathrm{Hom}_{\hat{G}}(V_{\mu_{1\bullet}}, V_{\mu_{2\bullet}}) \otimes \mathrm{Hom}_{\hat{G}}(V_{\mu_{2\bullet}}, V_{\mu_{3\bullet}}) \rightarrow \mathrm{Hom}_{\hat{G}}(V_{\mu_{1\bullet}}, V_{\mu_{3\bullet}})$$

is compatible with the composition of cohomological correspondences on the right hand side

$$\begin{aligned} & \mathrm{Corr}_{\mathrm{Gr}_{\mu_{1\bullet}|\mu_{2\bullet}}^0}((\mathrm{Gr}_{\leq \mu_{1\bullet}}, \mathrm{IC}_{\mu_{1\bullet}}), (\mathrm{Gr}_{\leq \mu_{2\bullet}}, \mathrm{IC}_{\mu_{2\bullet}})) \otimes \mathrm{Corr}_{\mathrm{Gr}_{\mu_{2\bullet}|\mu_{3\bullet}}^0}((\mathrm{Gr}_{\leq \mu_{2\bullet}}, \mathrm{IC}_{\mu_{2\bullet}}), (\mathrm{Gr}_{\leq \mu_{3\bullet}}, \mathrm{IC}_{\mu_{3\bullet}})) \\ & \rightarrow \mathrm{Corr}_{\mathrm{Gr}_{\mu_{1\bullet}|\mu_{3\bullet}}^0}((\mathrm{Gr}_{\leq \mu_{1\bullet}}, \mathrm{IC}_{\mu_{1\bullet}}), (\mathrm{Gr}_{\leq \mu_{3\bullet}}, \mathrm{IC}_{\mu_{3\bullet}})) \end{aligned}$$

which is obtained by pushing forward the cohomological correspondences along the natural map

$$Gr_{\mu_{1\bullet}|\mu_{2\bullet}}^0 \times_{\mathrm{Gr}_{\mu_{2\bullet}}} Gr_{\mu_{2\bullet}|\mu_{3\bullet}}^0 \rightarrow Gr_{\mu_{1\bullet}|\mu_{3\bullet}}^0.$$

In addition, there is a canonical isomorphism

$$(2.7) \quad \mathrm{Hom}_{\mathrm{P}(\mathrm{Gr}_G)}(m_{\mu_\bullet} \mathrm{IC}_{\mu_\bullet}, m_{\nu_\bullet} \mathrm{IC}_{\nu_\bullet}) \cong H_{(2\rho, |\mu_\bullet| + |\nu_\bullet|)}^{\mathrm{BM}}(\mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0).$$

Proof. The lemma can be proved exactly as [19, Corollary 3.4.4], and we refer to *loc.cit* for details of the proof. \square

3. LOCAL HECKE STACKS

We review the definition of local Hecke stacks and study their geometric properties following [15].

Definition 3.1. Let $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$ be a sequence of dominant coweights of G . The *local Hecke stack* $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}}$ is defined as the moduli problem which assigns to each perfect k -algebra R the groupoid of chains of modifications of G -torsors

$$(3.1) \quad \mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \dots \dashrightarrow \mathcal{E}_0$$

over D_R of relative positions $\leq \mu_n, \dots, \leq \mu_1$, respectively.

It may also be understood as the homogeneous space $[L^+G \backslash \mathrm{Gr}_{\mu_\bullet}^0]$. Similarly, we define

$$\mathrm{Hk}_{\mu_\bullet|\nu_\bullet}^{0,\mathrm{loc}} := [L^+G \backslash \mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0]$$

as the stack which classifies for each perfect k -algebra R the rectangles of modifications

$$\begin{array}{ccc} \mathcal{E}_n & \dashrightarrow & \dots \dashrightarrow \mathcal{E}_0 \\ \parallel & & \parallel \\ \mathcal{E}'_n & \dashrightarrow & \dots \dashrightarrow \mathcal{E}'_0, \end{array}$$

of G -torsors over D_R with modifications in the upper (resp. lower) row bounded by μ_\bullet (resp. ν_\bullet).

Taking quotient of the Satake correspondence (2.3) by L^+G , we get the *Satake correspondence* for local Hecke stacks,

$$(3.2) \quad \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}} \xleftarrow{h_{\mu_\bullet}^\leftarrow} \mathrm{Hk}_{\mu_\bullet|\nu_\bullet}^{0,\mathrm{loc}} \xrightarrow{h_{\nu_\bullet}^\rightarrow} \mathrm{Hk}_{\nu_\bullet}^{\mathrm{loc}}.$$

It is clear from the definition that these stacks are not of finite type, thus we need their finite dimensional quotient to apply the ℓ -adic formalism.

Definition 3.2. For a sequence of dominant coweights $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$, choose a μ_\bullet -large integer m , and we define the *m -restricted local Hecke stack* to be the stack

$$\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)} := [L^mG \backslash \mathrm{Gr}_{\mu_\bullet}^{\leq m}].$$

Similarly, choose m large enough for μ_\bullet and ν_\bullet , for example, m is taken to be $(\mu_\bullet, \nu_\bullet)$ -large, and we define $\mathrm{Hk}_{\mu_\bullet|\nu_\bullet}^{0,\mathrm{loc}(m)} := [L^mG \backslash \mathrm{Gr}_{\mu_\bullet|\nu_\bullet}^0]$. We have the *Satake correspondence* on restricted local Hecke stacks,

$$(3.3) \quad \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)} \xleftarrow{h_{\mu_\bullet}^\leftarrow} \mathrm{Hk}_{\mu_\bullet|\nu_\bullet}^{0,\mathrm{loc}(m)} \xrightarrow{h_{\nu_\bullet}^\rightarrow} \mathrm{Hk}_{\nu_\bullet}^{\mathrm{loc}(m)}.$$

3.1. Torsors over the local Hecke stacks. Let $\mathbb{B}L^+G$ (resp. BL^mG for $m \in \mathbb{Z}_{\geq 0}$) denote the moduli stack which classifies for every perfect k -algebra R the groupoid of G -torsors over \bar{D}_R (resp. $D_{m,R}$). For non-negative integers $m_1 \leq m_2$, the natural quotient maps

$$L^+G \xrightarrow{\mathrm{res}_{m_2} := \mathrm{res}_{m_2}^\infty} L^{m_2}G \xrightarrow{\mathrm{res}_{m_1}^{m_2}} L^{m_1}G$$

induce restriction maps between stacks

$$(3.4) \quad \mathbb{B}L^+G \xrightarrow{\mathrm{res}_{m_2} := \mathrm{res}_{m_2}^\infty} \mathbb{B}L^{m_2}G \xrightarrow{\mathrm{res}_{m_1}^{m_2}} \mathbb{B}L^{m_1}G.$$

Clearly, for any non-negative integers $m_1 \leq m_2 \leq m_3$, we have $\mathrm{res}_{m_1}^{m_2} \circ \mathrm{res}_{m_2}^{m_3} = \mathrm{res}_{m_1}^{m_3}$, where m_3 can be taken to be ∞ .

Let $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$ be a sequence of dominant coweights. We have natural morphisms

$$t_\leftarrow, t_\rightarrow : \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}} \rightarrow \mathbb{B}L^+G$$

which send (3.1) to the torsors \mathcal{E}_n and \mathcal{E}_0 , respectively.

For restricted local Hecke stacks, we choose a pair of μ_\bullet -large integers (m, n) . Then the natural maps

$$\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)} \simeq [L^m G \backslash \mathrm{Gr}_{\leq \mu_\bullet}^{(n)} / L^n G] \xrightarrow{t_\leftarrow \times t_\rightarrow} \mathbb{B} L^n G \times \mathbb{B} L^m G$$

induce the $L^m G$ -torsor

$$(3.5) \quad \mathrm{Gr}_{\leq \mu_\bullet} \rightarrow \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)},$$

and the $L^n G$ -torsor

$$(3.6) \quad [L^m G \backslash \mathrm{Gr}_{\leq \mu_\bullet}] \rightarrow [L^{m+n} G \backslash \mathrm{Gr}_{\leq \mu_\bullet}].$$

Following the notations in [15], we denote the two torsors by \mathcal{E}_\leftarrow and \mathcal{E}_\rightarrow , respectively. For any pairs of μ_\bullet -large integers (m_1, n_1) and (m_2, n_2) such that $m_1 \leq m_2$ and $n_1 \leq n_2$, denote the natural restriction map of restricted local Hecke stacks as

$$(3.7) \quad \mathrm{res}_{m_1}^{m_2} : \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2)} \rightarrow \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_1)}.$$

It is compatible with the restriction maps in (3.4) in the sense that the following diagram is commutative

$$\begin{array}{ccccc} \mathbb{B} L^{n_2} G & \xleftarrow{t_\leftarrow} & \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2)} & \xrightarrow{t_\rightarrow} & \mathbb{B} L^{m_2} G \\ \downarrow \mathrm{res}_{n_1}^{n_2} & & \downarrow \mathrm{res}_{m_1}^{m_2} & & \downarrow \mathrm{res}_{m_1}^{m_2} \\ \mathbb{B} L^{n_1} G & \xleftarrow{t_\leftarrow} & \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_1)} & \xrightarrow{t_\rightarrow} & \mathbb{B} L^{m_1} G. \end{array}$$

Let μ_\bullet and ν_\bullet be two sequences of dominant coweights. Choose non-negative integers m_1, m_2, n such that (m_1, m_2) is μ_\bullet -large and (m_2, n) is ν_\bullet -large. Then we have the following isomorphism

$$(3.8) \quad [L^{m_1} G \backslash \mathrm{Gr}_{\leq \mu_\bullet, \nu_\bullet}^{(n)}] \cong \mathrm{Hk}_{\mu_\bullet, \nu_\bullet}^{\mathrm{loc}(m_1)} \cong \mathrm{Hk}_{\nu_\bullet}^{\mathrm{loc}(m_2)} \times_{t_\rightarrow, \mathbb{B} L^{m_2} G, \mathrm{res}_{m_2}^{m_1} \circ t_\leftarrow} \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_1)},$$

which induces the following perfectly smooth morphisms

$$(3.9) \quad \mathrm{Hk}_{\mu_\bullet, \nu_\bullet}^{\mathrm{loc}(m_1)} \rightarrow \mathrm{Hk}_{\nu_\bullet}^{\mathrm{loc}(m_2)} \times \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_1)} \xrightarrow{\mathrm{id} \times \mathrm{res}_{m_1}^{m_2}} \mathrm{Hk}_{\nu_\bullet}^{\mathrm{loc}(m_2)} \times \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2 - m_1)}.$$

3.2. Perverse sheaves on the moduli of local Hecke stacks. Let $m_1 \leq m_2$ be two μ_\bullet -large integers. The natural (twisted) pullback functor

$$\mathrm{res}_{m_1}^{m_2} := (\mathrm{res}_{m_1}^{m_2})^* := (\mathrm{res}_{m_1}^{m_2})^*[d](d/2) : \mathrm{P}(\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_1)}) \rightarrow \mathrm{P}(\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2)})$$

is an equivalence of categories. We define the category of perverse sheaves on the local Hecke stack as

$$\mathrm{P}(\mathrm{Hk}_k^{\mathrm{loc}}, \Lambda) := \bigoplus_{\xi \in \pi_1(G)} \mathrm{P}(\mathrm{Hk}_\xi^{\mathrm{loc}}, \Lambda), \quad \mathrm{P}(\mathrm{Hk}_\xi^{\mathrm{loc}}, \Lambda) := \varinjlim_{(\mu, m) \in \xi \times \mathbb{Z}_{\geq 0}} \mathrm{P}(\mathrm{Hk}_\mu^{\mathrm{loc}(m)}, \Lambda).$$

Here the connecting morphism in the definition of $\mathrm{P}(\mathrm{Hk}^{\mathrm{loc}}, \Lambda)$ is the fully faithful embedding

$$\mathrm{P}(\mathrm{Hk}_{\mu_1}^{\mathrm{loc}(m_1)}, \Lambda) \xrightarrow{\mathrm{res}_{m_1}^{m_2}} \mathrm{P}(\mathrm{Hk}_{\mu_1}^{\mathrm{loc}(m_2)}, \Lambda) \xrightarrow{i_{\mu_1, \mu_2}^*} \mathrm{P}(\mathrm{Hk}_{\mu_2}^{\mathrm{loc}(m_2)}, \Lambda).$$

Finally, via descent, there is a natural equivalence of categories $\mathrm{P}(\mathrm{Hk}_\mu^{\mathrm{loc}(m)}, \Lambda) \cong \mathrm{P}_{L^m G}(\mathrm{Gr}_{\leq \mu}, \Lambda)$, which induces an equivalence $\mathrm{P}(\mathrm{Hk}_k^{\mathrm{loc}}, \Lambda) \cong \mathrm{P}_{L^+ G \otimes k}(\mathrm{Gr}_G \otimes k, \Lambda)$.

4. MODULI OF LOCAL SHTUKAS

In this section, we define moduli of local Shtukas, and its restricted version, on which we also construct various correspondences. Via these constructions, we define the category of Λ -coefficient perverse sheaves on the moduli of local Shtukas and their cohomological correspondences. In the rest of this thesis, we will make use of the theory of cohomological correspondences between perfect schemes and perfect pfp algebraic spaces. We refer to [15, Appendix A] for a detailed discussion.

Definition 4.1. Let $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$ be a sequence of dominant coweights. The *moduli of local Shtukas* $\text{Sht}_{\mu_\bullet}^{\text{loc}}$ classifies for each perfect k -algebra R sequences of modifications of G -torsors

$$\mathcal{E}_n \dashrightarrow \mathcal{E}_{n-1} \dashrightarrow \dots \dashrightarrow \mathcal{E}_0 \cong {}^\sigma \mathcal{E}_n$$

over D_R of relative positions $\leq \mu_n, \dots, \leq \mu_1$ respectively.

It follows from the definition that

$$\text{Sht}_{\mu_\bullet}^{\text{loc}} \cong \text{Hk}_{\mu_\bullet}^{\text{loc}} \times_{t_{\leftarrow} \times t_{\rightarrow}, \mathbb{B}L^+G \times \mathbb{B}L^+G, \text{id} \times \sigma} \mathbb{B}L^+G.$$

There is a natural forgetful map $\psi^{\text{loc}} : \text{Sht}_{\mu_\bullet}^{\text{loc}} \rightarrow \text{Hk}_{\mu_\bullet}^{\text{loc}}$ which forgets the isomorphisms $\mathcal{E}_0 \cong {}^\sigma \mathcal{E}_n$. One can define the stack

$$\text{Sht}_{\mu_\bullet | \nu_\bullet}^{0, \text{loc}}$$

which classifies for each perfect k -algebra R the following rectangle of modifications

$$\begin{array}{ccc} \mathcal{E}_n & \dashrightarrow & \dots \dashrightarrow \mathcal{E}_0 \cong {}^\sigma \mathcal{E}_n \\ \parallel & & \parallel \\ \mathcal{E}'_m & \dashrightarrow & \dots \dashrightarrow \mathcal{E}'_0 \cong {}^\sigma \mathcal{E}'_m, \end{array}$$

of G -torsors over D_R with modifications in the upper (resp. lower) row bounded by μ_\bullet (resp. ν_\bullet). We get the *Satake correspondence* for moduli of local Shtukas

$$\text{Sht}_{\mu_\bullet}^{\text{loc}} \xleftarrow{s_{\mu_\bullet}^{\leftarrow}} \text{Sht}_{\mu_\bullet | \nu_\bullet}^{0, \text{loc}} \xrightarrow{s_{\mu_\bullet}^{\rightarrow}} \text{Sht}_{\nu_\bullet}^{\text{loc}}.$$

We introduce the partial Frobenius morphism between the moduli of local Shtukas which will play an important role in later constructions.

Definition 4.2. Let $\mu_\bullet = (\mu_1, \mu_2, \dots, \mu_n)$ be a sequence of dominant coweights of G . We define the *partial Frobenius morphism* to be

$$(4.1) \quad F_{\mu_\bullet} : \text{Sht}_{\mu_1, \dots, \mu_n}^{\text{loc}} \longrightarrow \text{Sht}_{\sigma(\mu_n), \mu_1, \dots, \mu_{n-1}}^{\text{loc}}$$

$$(\mathcal{E}_n \dashrightarrow \dots \dashrightarrow \mathcal{E}_0 \cong {}^\sigma \mathcal{E}_n) \mapsto (\mathcal{E}_{n-1} \dashrightarrow \dots \dashrightarrow {}^\sigma \mathcal{E}_n \dashrightarrow {}^\sigma \mathcal{E}_{n-1}).$$

Definition 4.3. Let μ_\bullet and ν_\bullet be two sequences of dominant coweights. For each perfect k -algebra R , the prestack $\text{Sht}_{\mu_\bullet | \nu_\bullet}^{\text{loc}}$ classifies the following commutative diagram of modifications of G -torsors over D_R

$$\begin{array}{ccc} \mathcal{E}_n & \dashrightarrow & \dots \dashrightarrow \mathcal{E}_0 \cong {}^\sigma \mathcal{E}_n \\ \downarrow \beta & & \downarrow \beta^\sigma \\ \mathcal{E}'_m & \dashrightarrow & \dots \dashrightarrow \mathcal{E}'_0 \cong {}^\sigma \mathcal{E}'_m, \end{array}$$

where the top (resp. bottom) row defines an R -point of $\text{Sht}_{\mu_\bullet}^{\text{loc}}$ (resp. $\text{Sht}_{\nu_\bullet}^{\text{loc}}$). Let $\overleftarrow{h}_{\mu_\bullet}^{\text{loc}}$ (reps. $\overrightarrow{h}_{\nu_\bullet}^{\text{loc}}$) denote the morphism which maps the above commutative rectangle to its upper (resp. lower) row. We define the *Hecke correspondence* of local Shtukas to be to following diagram

$$(4.2) \quad \text{Sht}_{\mu_\bullet}^{\text{loc}} \xleftarrow{\overleftarrow{h}_{\mu_\bullet}^{\text{loc}}} \text{Sht}_{\mu_\bullet | \nu_\bullet}^{\text{loc}} \xrightarrow{\overrightarrow{h}_{\nu_\bullet}^{\text{loc}}} \text{Sht}_{\nu_\bullet}^{\text{loc}}.$$

If in addition, the relative position of β is bounded by λ , we get a closed sub-prestack $\text{Sht}_{\mu_\bullet | \nu_\bullet}^{\lambda, \text{loc}}$. In particular, if $\lambda = 0$, the Hecke correspondence (4.2) reduces to the Satake correspondence.

The Hecke correspondence can be considered as the composition of two Satake correspondences and the cohomological correspondence given by the partial Frobenius morphism. More precisely, we recall [15, Lemma 5.2.14].

Lemma 4.4. *Let μ_\bullet and ν_\bullet be two sequences of dominant coweights. Choose λ to be a dominant coweight such that $\lambda \geq |\mu_\bullet| + \sigma(\lambda)$ or $\lambda \geq |\mu_\bullet| + \nu$, then we have the following commutative diagram of prestacks*

$$\begin{array}{ccccc}
 & & \text{Sht}_{\mu_\bullet|\nu_\bullet}^{\theta, \text{loc}} & & \\
 & \swarrow & & \searrow & \\
 \text{Sht}_{\mu_\bullet|(\sigma(\theta^*), \lambda)}^{0, \text{loc}} & & & & \text{Sht}_{(\lambda, \theta^*)|\nu_\bullet}^{0, \text{loc}} \\
 \swarrow s_{\mu_\bullet}^{\leftarrow} \quad \downarrow s_{\sigma(\theta^*), \lambda}^{\rightarrow} & & & & \downarrow s_{\lambda, \theta^*}^{\leftarrow} \quad \searrow s_{\nu_\bullet}^{\rightarrow} \\
 \text{Sht}_{\mu_\bullet}^{\text{loc}} & \xrightarrow{F_{\lambda, \theta^*}^{-1}} & \text{Sht}_{\sigma(\theta^*), \lambda}^{\text{loc}} & \xrightarrow{\quad} & \text{Sht}_{\lambda, \theta^*}^{\text{loc}} & \xrightarrow{\quad} & \text{Sht}_{\nu_\bullet}^{\text{loc}}
 \end{array}$$

In addition, the pentagon in the middle is a Cartesian square when composing F_{λ, θ^*} with $s_{\sigma(\theta^*), \lambda}^{\rightarrow}$.

4.1. Moduli of restricted local Shtukas. Let $\mu_\bullet = 0$ or more generally, a central cocharacter, $\text{Sht}_{\mu_\bullet}^{\text{loc}} \simeq \mathbb{B}G(\mathcal{O})$ which is not perfectly of finite presentation as a prestack. Thus to apply the ℓ -adic formalism, it is desirable to study the following finite type approximation of $\text{Sht}_{\mu_\bullet}^{\text{loc}}$.

Definition 4.5. Let μ_\bullet be a sequence of dominant coweights and (m, n) a pair of μ_\bullet -large integers. We define the moduli stack $\text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)}$ of (m, n) -restricted local iterated shtukas as the stack that classifies for every perfect k -algebra R ,

- (1) an R -point of $\text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$,
- (2) an isomorphism

$$\Psi : \sigma(\mathcal{E}_{\leftarrow} |_{D_{n, R}}) \simeq (\mathcal{E}_{\rightarrow} |_{D_{m, R}}) |_{D_{n, R}}$$

of $L^n G$ -torsors over $\text{Spec} R$, where \mathcal{E}_{\leftarrow} and $\mathcal{E}_{\rightarrow}$ are defined in (3.5) and (3.6), respectively.

The above definition gives a canonical isomorphism

$$\text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)} \cong \text{Hk}_{\mu_\bullet}^{\text{loc}(m)} \times_{t_{\leftarrow} \times \text{res}_{n_1}^m \circ t_{\rightarrow}, \mathbb{B}L^n G \times \mathbb{B}L^n G, \text{id} \times \sigma} \mathbb{B}L^n G.$$

The natural forgetful morphism $\psi^{\text{loc}(m, n)} : \text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)} \rightarrow \text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$ is a perfectly smooth morphism of relative dimension $n \dim G$. For two sequences of dominant coweights μ_\bullet, ν_\bullet , we define $\text{Sht}_{\mu_\bullet|\nu_\bullet}^{\text{loc}(m, n)}$ to be the stack which classifies for each perfect k -algebra R , an R -point of $\text{Hk}_{\mu_\bullet|\nu_\bullet}^{\text{loc}(m, n)}$ together with an isomorphism $\sigma(\mathcal{E}_{\leftarrow} |_{D_{n, R}}) \simeq (\mathcal{E}_{\rightarrow} |_{D_{m, R}}) |_{D_{n, R}}$. Let (m_1, n_1) and (m_2, n_2) be two pairs of μ_\bullet -large integers such that $m_1 \leq m_2$ and $n_1 \leq n_2$. We define the restriction morphism

$$(4.3) \quad \text{res}_{m_1, n_1}^{m_2, n_2} : \text{Sht}_{\mu_\bullet}^{\text{loc}(m_2, n_2)} \rightarrow \text{Sht}_{\mu_\bullet}^{\text{loc}(m_1, n_1)}$$

as the composition of the following morphisms

$$\begin{aligned}
 \text{res}_{m_1, n_1}^{m_2, n_2} : \text{Sht}_{\mu_\bullet}^{\text{loc}(m_2, n_2)} &\cong \text{Hk}_{\mu_\bullet}^{\text{loc}(m_2)} \times_{t_{\leftarrow} \times \text{res}_{n_2}^{m_2} \circ t_{\rightarrow}, \mathbb{B}L^{n_2} G \times \mathbb{B}L^{n_2} G, \text{id} \times \sigma} \mathbb{B}L^{n_2} G \\
 &\xrightarrow{\text{res}_{m_1}^{m_2} \times \text{res}_{n_1}^{n_2}} \text{Hk}_{\mu_\bullet}^{\text{loc}(m_1)} \times_{t_{\leftarrow} \times \text{res}_{n_1}^{m_1} \circ t_{\rightarrow}, \mathbb{B}L^{n_1} G \times \mathbb{B}L^{n_1} G, \text{id} \times \sigma} \mathbb{B}L^{n_1} G \\
 &\cong \text{Sht}_{\mu_\bullet}^{\text{loc}(m_1, n_1)}.
 \end{aligned}$$

For $(m_2, n_2) = (\infty, \infty)$, we write $\text{res}_{m_1, n_1}^{m_2, n_2}$ as res_{m_1, n_1} for simplicity. For three pairs of μ_\bullet -large integers (m_i, n_i) such that $m_1 \leq m_2 \leq m_3$ and $n_1 \leq n_2 \leq n_3$, we have

$$(4.4) \quad \text{res}_{m_1, n_1}^{m_2, n_2} \circ \text{res}_{m_2, n_2}^{m_3, n_3} = \text{res}_{m_1, n_1}^{m_3, n_3}.$$

The Satake correspondences for restricted local Hecke stacks and the Satake correspondences for restricted local Shtukas are related by the restriction and forgetful morphisms. We summarize this relation in following

diagram

(4.5)

$$\begin{array}{ccccc}
\mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_2, n_2)} & \xleftarrow{\mathrm{res}_{m_1, n_1}^{m_2, n_2}} & \mathrm{Sht}_{\mu_{\bullet}|\nu_{\bullet}}^{0, \mathrm{loc}(m_2, n_2)} & \xrightarrow{\mathrm{res}_{m_1, n_1}^{m_2, n_2}} & \mathrm{Sht}_{\nu_{\bullet}}^{\mathrm{loc}(m_2, n_2)} \\
\downarrow \psi^{\mathrm{loc}(m_2, n_2)} & \searrow & \downarrow \psi^{\mathrm{loc}(m_2, n_2)} & \searrow & \downarrow \psi^{\mathrm{loc}(m_2, n_2)} \\
& \mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_1, n_1)} & \xleftarrow{\mathrm{res}_{m_1, n_1}^{0, \mathrm{loc}(m_1, n_1)}} & \mathrm{Sht}_{\mu_{\bullet}|\nu_{\bullet}}^{0, \mathrm{loc}(m_1, n_1)} & \xrightarrow{\mathrm{res}_{m_1, n_1}^{0, \mathrm{loc}(m_1, n_1)}} & \mathrm{Sht}_{\nu_{\bullet}}^{\mathrm{loc}(m_1, n_1)} \\
& \downarrow \psi^{\mathrm{loc}(m_1, n_1)} & & \downarrow \psi^{\mathrm{loc}(m_1, n_1)} & & \downarrow \psi^{\mathrm{loc}(m_1, n_1)} \\
\mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m_2)} & \xleftarrow{\quad} & \mathrm{Hk}_{\mu_{\bullet}|\nu_{\bullet}}^{0, \mathrm{loc}(m_2)} & \xrightarrow{\quad} & \mathrm{Hk}_{\nu_{\bullet}}^{\mathrm{loc}(m_2)} \\
& \searrow & \downarrow \psi^{\mathrm{loc}(m_1, n_1)} & \searrow & \downarrow \psi^{\mathrm{loc}(m_1, n_1)} \\
& \mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m_1)} & \xleftarrow{\quad} & \mathrm{Hk}_{\mu_{\bullet}|\nu_{\bullet}}^{0, \mathrm{loc}(m_1)} & \xrightarrow{\quad} & \mathrm{Hk}_{\nu_{\bullet}}^{\mathrm{loc}(m_1)}
\end{array}$$

where

- (1) all rectangles are commutative,
- (2) all rectangles are Cartesian except for the two on the left and right side of the cuboid.

Let $\mu_{\bullet} = (\mu_1, \dots, \mu_n)$ be a sequence of dominant cocharacters. We call a quadruple of non-negative integers (m_1, n_1, m_2, n_2) μ_{\bullet} -acceptable if

- (1) $m_1 - m_2 = n_1 - n_2$ are μ_n -large (or equivalently $\sigma(\mu_n)$ -large),
- (2) $m_2 - n_1$ is μ_{\bullet} -large.

We can define the partial Frobenius morphism

$$(4.6) \quad F_{\mu_{\bullet}}^{-1} : \mathrm{Sht}_{\sigma(\mu_n), \mu_1, \dots, \mu_{n-1}}^{\mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{\mu_1, \dots, \mu_2}^{\mathrm{loc}(m_2, n_2)}$$

for restricted local Shtukas. The construction of $F_{\mu_{\bullet}}^{-1}$ is technical and we refer to [15, Construction 5.3.12] for detailed discussion.

4.2. Perverse sheaves on the moduli of local Shtukas. Let μ_{\bullet} be a sequence of dominant coweights and $(m_1, n_1), (m_2, n_2)$ be two pairs of μ_{\bullet} -large integers such that $m_1 \leq m_2, n_1 \leq n_2$, and $m_2 \neq \infty$. Define the functor

$$(4.7) \quad \mathrm{res}_{m_1, n_1}^{m_2, n_2} := (\mathrm{res}_{m_1, n_1}^{m_2, n_2})^* : \mathrm{P}(\mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_2, n_2)}, \Lambda) \rightarrow \mathrm{P}(\mathrm{Sht}_{\mu_{\bullet}, \Lambda}^{\mathrm{loc}(m_1, n_1)}).$$

Then (4.4) yields

$$(4.8) \quad \mathrm{res}_{m_1, n_1}^{m_2, n_2} \circ \mathrm{res}_{m_2, n_3}^{m_3, n_3} = \mathrm{res}_{m_1, n_1}^{m_3, n_3}.$$

Like res_n^m , the functor $\mathrm{res}_{m_j, n_j}^{m_i, n_i}$ is also an equivalence of categories if $m_j > 1$.

We define the category of perverse sheaves on the moduli of local Shtukas as

$$(4.9) \quad \mathrm{P}(\mathrm{Sht}_k^{\mathrm{loc}}, \Lambda) := \bigoplus_{\xi \in \pi_1(G)} \mathrm{P}(\mathrm{Sht}_{\xi}^{\mathrm{loc}}, \Lambda), \quad \mathrm{P}(\mathrm{Sht}_{\xi}^{\mathrm{loc}}, \Lambda) := \varinjlim_{(m, n, \mu)} \mathrm{P}(\mathrm{Sht}_{\mu}^{\mathrm{loc}(m, n)}, \Lambda)$$

where the limit is taken over the triples $\{(m, n, \mu) \in \mathbb{Z}^2 \times \xi \mid (m, n) \text{ is } \mu \text{ large}\}$ with the product partial order. As in [15], we call objects in $\mathrm{P}(\mathrm{Sht}_{\xi}^{\mathrm{loc}}, \Lambda)$ *connected objects*. The connecting morphism is given by the composite of fully faithful functor

$$\mathrm{P}(\mathrm{Sht}_{\mu_1}^{\mathrm{loc}(m_1, n_1)}, \Lambda) \xrightarrow{\mathrm{res}_{m_1, n_1}^{m_2, n_2}} \mathrm{P}(\mathrm{Sht}_{\mu_1}^{\mathrm{loc}(m_2, n_2)}, \Lambda) \xrightarrow{i_{\mu_1, \mu'_1}} \mathrm{P}(\mathrm{Sht}_{\mu'_1}^{\mathrm{loc}(m_2, n_2)}, \Lambda).$$

For each dominant coweight μ and a pair of μ -large integers (m, n) , we define the natural pullback functor

$$(4.10) \quad \Psi^{\mathrm{loc}(m, n)} := \mathrm{res}_{m, 0}^{m, n} : \mathrm{P}(\mathrm{Hk}_{\mu}^{\mathrm{loc}(m)}, \Lambda) \rightarrow \mathrm{P}(\mathrm{Sht}_{\mu}^{\mathrm{loc}(m, n)}, \Lambda).$$

We observe that $\Psi^{\mathrm{loc}(m, n)}$ commutes with the connecting morphism in (4.9) by (4.8) and the proper smooth base change. Then we can take the limit and direct sum of $\Psi^{\mathrm{loc}(m, n)}$ and derive the following well-defined functor

$$(4.11) \quad \Psi^{\mathrm{loc}} : \mathrm{P}(\mathrm{Hk}_k^{\mathrm{loc}}, \Lambda) \rightarrow \mathrm{P}(\mathrm{Sht}_k^{\mathrm{loc}}, \Lambda).$$

Let $\mathcal{F}_i \in \mathcal{P}(\text{Sht}_{\xi_i}^{\text{loc}}, \Lambda)$ be connected objects. It is realized as $\mathcal{F}_{i, \mu_i}^{(m_i, n_i)} \in \mathcal{P}(\text{Sht}_{\mu_i}^{\text{loc}(m_i, n_i)}, \Lambda)$ for some μ_i and some pair of μ_i -large integers (m_i, n_i) . We define the set of cohomological correspondences between \mathcal{F}_1 and \mathcal{F}_2 as

$$\begin{aligned} & \text{Corr}_{\text{Sht}^{\text{loc}}}(\mathcal{F}_1, \mathcal{F}_2) \\ &:= \bigoplus_{\xi \in \pi_1(G)} \varinjlim_{\mu_1 | \mu_2} \text{Corr}_{\text{Sht}_{\mu_1 | \mu_2}^{\lambda, \text{loc}(m_1, n_1)}} \left((\text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)}, \mathcal{F}_{1, \mu_1}^{(m_1, n_1)}), (\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}, \mathcal{F}_{2, \mu_2}^{(m_2, n_2)}) \right), \end{aligned}$$

where the limit is taken over all partially ordered sextuples $(\mu_1, \mu_2, \lambda, m_1, n_1, m_2, n_2)$ such that

- (m_1, n_1, m_2, n_2) is $(\mu_1 + \lambda, \lambda)$ and $(\mu_2 + \lambda, \lambda)$ -acceptable,
- $\mu_i \in \xi_i$, for some $\xi_i \in \pi_1(G)$,
- $\lambda \in \xi$.

Let $(\mu_1, \mu_2, \lambda, m_1, n_1, m_2, n_2) \leq (\mu'_1, \mu'_2, \lambda', m'_1, n'_1, m'_2, n'_2)$ be another such sextuple. The connecting morphism between the cohomological correspondences

$$(4.12) \quad \text{Corr}_{\text{Sht}_{\mu_1 | \mu_2}^{\lambda, \text{loc}(m_1, n_1)}} \left((\text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)}, \mathcal{F}_{1, \mu_1}^{(m_1, n_1)}), (\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}, \mathcal{F}_{2, \mu_2}^{(m_2, n_2)}) \right)$$

and

$$(4.13) \quad \text{Corr}_{\text{Sht}_{\mu'_1 | \mu'_2}^{\lambda', \text{loc}(m'_1, n'_1)}} \left((\text{Sht}_{\mu'_1}^{\text{loc}(m'_1, n'_1)}, \mathcal{F}_{1, \mu'_1}^{(m'_1, n'_1)}), (\text{Sht}_{\mu'_2}^{\text{loc}(m'_2, n'_2)}, \mathcal{F}_{2, \mu'_2}^{(m'_2, n'_2)}) \right)$$

is given by first pulling back (4.13) to the Hecke correspondence

$$\text{Sht}_{\mu_1}^{\text{loc}(m'_1, n'_1)} \longleftarrow \text{Sht}_{\mu_1 | \mu_2}^{\lambda, \text{loc}(m'_1, n'_1)} \longrightarrow \text{Sht}_{\mu_2}^{\text{loc}(m'_2, n'_2)},$$

along the restriction morphism, then pushing it forward to the Hecke correspondence

$$\text{Sht}_{\mu'_1}^{\text{loc}(m'_1, n'_1)} \longleftarrow \text{Sht}_{\mu'_1 | \mu'_2}^{\lambda', \text{loc}(m'_1, n'_1)} \longrightarrow \text{Sht}_{\mu'_2}^{\text{loc}(m'_2, n'_2)}.$$

The connecting morphism is well-defined and can be composed. We refer to [15, §5.4.1] for more discussions.

5. KEY THEOREM

In this section, we state and prove the key theorem for our construction of the Jacquet-Langlands transfer. We will make use of the theory of the cohomological correspondences throughout this chapter. Instead of explaining all the details, we refer to [15, Appendix A.2] for a nice discussion.

5.1. Set up. Fix a half Tate twist $\Lambda(1/2)$. Recall notations $\langle d \rangle$ and f^* introduced in §1.3. Throughout this section, we consider the Langlands dual group scheme \hat{G}_Λ over Λ of G and its Λ -representations. The subscripts Λ will be omitted for simplicity. We generalize a few notions introduced in previous sections for the sake of stating the key theorem.

5.2. More on local Hecke stacks. Let $V_\bullet := V_1 \boxtimes V_2 \boxtimes \cdots \boxtimes V_s \in \text{Rep}(\hat{G}^s)$ and assume that for each i , V_i has the Jordan-Hölder factors $\{V_{\mu_{ij}}\}_j$.

The integral geometric Satake equivalence (cf. Theorem 2.5) Sat_{G^s} sends V_\bullet to an $(L^+G \otimes \bar{k})^s$ -equivariant perverse sheaf $\text{Sat}_{G^s}(V_\bullet)$ on $(\text{Gr}_G \otimes \bar{k})^s$. We write Gr_{V_\bullet} for the support of the external tensor product $\text{Sat}(V_1) \boxtimes \text{Sat}(V_2) \boxtimes \cdots \boxtimes \text{Sat}(V_s)$. Let m be a non-negative integer. We call it V_i -large if m is μ_{ij} -large for each j , and we call it V_\bullet -large if $m = m_1 + m_2 + \cdots + m_s$ such that m_i is V_i -large for each i . For a V_\bullet -large integer m , $\text{Sat}_{G^n}(V_\bullet)$ descends to a perverse sheaf supported on $\text{Hk}_{V_\bullet}^{\text{loc}(m)} := [L^m G \backslash \text{Gr}_{V_\bullet}]$. We write $S(V)^{\text{loc}(m)}$ for the twist of this perverse sheaf by $\langle m \dim G \rangle$. Note that $S(V_\bullet)^{\text{loc}(m)}$ is isomorphic to the “ \star ”-pullback of $S(V_1)^{\text{loc}(m_1)} \boxtimes S(V_2)^{\text{loc}(m_2)} \boxtimes \cdots \boxtimes S(V_s)^{\text{loc}(m_s)}$ along the perfectly smooth morphism $\text{Hk}_{V_\bullet}^{\text{loc}(m)} \rightarrow \prod_i \text{Hk}_{V_i}^{\text{loc}(m_i)}$ constructed in (3.9).

In the case $s = 1$, we have

$$\text{Gr}_{V_1} = \cup_j \text{Gr}_{\mu_{1j}}, \quad \text{Hk}_{V_1}^{\text{loc}(m)} = \cup_j \text{Hk}_{\mu_{1j}}^{\text{loc} m}.$$

In general, $\mathrm{Hk}_{V_\bullet}^{\mathrm{loc}(m)}$ is of the form $\cup_{\mu_\bullet} \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)}$. Via descent, Corollary 2.7 gives the following natural isomorphism:

$$(5.1) \quad \mathrm{Hom}_{\hat{G}}(V_\bullet, W_\bullet) \cong \mathrm{Corr}_{\mathrm{Hk}_{V_\bullet|W_\bullet}^{0,\mathrm{loc}(m)}}((\mathrm{Hk}_{V_\bullet}^{\mathrm{loc}(m)}, \mathrm{Sat}_G^{\mathrm{loc}(m)}(V_\bullet)), (\mathrm{Hk}_{W_\bullet}^{\mathrm{loc}(m)}, \mathrm{Sat}_G^{\mathrm{loc}(m)}(W_\bullet))).$$

Here and below, we regard V_\bullet and W_\bullet as representations of \hat{G} via the diagonal embedding $\hat{G} \hookrightarrow \hat{G}^s$.

Let V_\bullet and W_\bullet be two representations of \hat{G}^s . We can similarly define $\mathrm{Gr}_{V_\bullet|W_\bullet}^0 := \mathrm{Gr}_{V_\bullet} \times_{\mathrm{Gr}_G} \mathrm{Gr}_{W_\bullet}$ and $\mathrm{Hk}_{V_\bullet|W_\bullet}^{0,\mathrm{loc}(m)} = [L^m G \backslash \mathrm{Gr}_{V_\bullet|W_\bullet}^0]$, and Satake correspondences supported on them.

5.3. More on moduli of local Shtukas. Let $V_\bullet \in \mathrm{Rep}(\hat{G}^s)$. For a pair of non-negative integers (m, n) , we can generalize the notion of μ_\bullet -large and define the notion of V_\bullet -large. Let (m, n) be a pair of V_\bullet - and W_\bullet -large integers, we can define the moduli of restricted local Shtukas $\mathrm{Sht}_{V_\bullet}^{\mathrm{loc}(m,n)}$ and $\mathrm{Sht}_{V_\bullet|W_\bullet}^{\mathrm{loc}(m,n)}$. Similar to $\mathrm{Hk}_{V_\bullet}^{\mathrm{loc}}$, the stacks $\mathrm{Sht}_{V_\bullet}^{\mathrm{loc}(m,n)}$ and $\mathrm{Sht}_{V_\bullet|W_\bullet}^{\mathrm{loc}(m,n)}$ can be regarded as unions of $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)}$ and unions of $\mathrm{Sht}_{\mu_\bullet|\nu_\bullet}^{\mathrm{loc}(m,n)}$. We have the natural forgetful map

$$(5.2) \quad \psi^{\mathrm{loc}(m,n)} : \mathrm{Sht}_{V_\bullet}^{\mathrm{loc}(m,n)} \rightarrow \mathrm{Hk}_{V_\bullet}^{\mathrm{loc}(m)}.$$

Choose a pair of V_\bullet -large integers (m, n) such that $n > 0$. Write

$$S(\widetilde{V}_\bullet)^{\mathrm{loc}(m,n)} := \Psi^{\mathrm{loc}(m,n)}(\mathrm{Sat}(V_\bullet)^{\mathrm{loc}(m)}) \in \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}(m,n)}, \Lambda)$$

for the pullback of $\mathrm{Sat}(V)^{\mathrm{loc}(m)}$ along the morphism $\psi^{\mathrm{loc}(m,n)}$ (up to a shift and twist). For $s = 1$, $S(\widetilde{V})^{\mathrm{loc}(m,n)}$ represents the perverse sheaf $S(\widetilde{V}) := \Psi(\mathrm{Sat}_G(V)) \in \mathrm{P}(\mathrm{Sht}_k^{\mathrm{loc}}, \Lambda)$.

Consider the front face of the diagram (4.5). The middle and right vertical maps are perfectly smooth. Pulling back the cohomological correspondence on the right hand side of (5.1) to the upper edge and pre-composing it with (5.1), we get the map

$$(5.3) \quad \mathcal{C}^{\mathrm{loc}(m,n)} : \mathrm{Hom}_{\hat{G}}(V_\bullet, W_\bullet) \rightarrow \mathrm{Corr}_{\mathrm{Sht}_{V_\bullet|W_\bullet}^{0,\mathrm{loc}(m,n)}}(S(\widetilde{V}_\bullet)^{\mathrm{loc}(m,n)}, S(\widetilde{W}_\bullet)^{\mathrm{loc}(m,n)}).$$

The map $\mathcal{C}^{\mathrm{loc}(m,n)}$ is compatible with the compositions at the source and target, and we refer to [15, Lemma 6.1.8] for the proof.

Let $V_\bullet \in \mathrm{Rep}(\hat{G}^s)$ and $W \in \mathrm{Rep}(\hat{G})$. We call a quadruple of non-negative integers (m_1, n_2, m_1, n_1) $V_\bullet \boxtimes W$ -acceptable if

- $m_1 - m_2 = n_1 - n_2$ is W -large,
- (m_2, n_1) is V_\bullet -large.

For a quadruple of $V_\bullet \boxtimes W$ -acceptable integers (m_1, n_2, m_1, n_1) , we can construct the partial Frobenius morphism

$$(5.4) \quad F_{V_\bullet \boxtimes W}^{-1} : \mathrm{Sht}_{\sigma W \boxtimes V_\bullet}^{\mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{V_\bullet \boxtimes W}^{\mathrm{loc}(m_2, n_2)}$$

similar to (4.1). Here, σW is the Frobenius twist of W as in Remark 2.6.

Let $V_1, V_2 \in \mathrm{Rep}(\hat{G})$. For any projective object $W \in \mathrm{Rep}(\hat{G})$, choose a quadruple of $((V_1 \otimes V_2 \otimes W) \boxtimes W^*)$ -acceptable integers (m_1, n_1, m_2, n_2) . We define the following stack

$$(5.5) \quad \mathrm{Sht}_{V_1|V_2}^{W, \mathrm{loc}(m_1, n_1)} := \mathrm{Sht}_{V_1|\sigma W^* \boxtimes (\sigma W \otimes V_1)}^{\mathrm{loc}(m_1, n_1)} \times_{\mathrm{Sht}_{(\sigma W \otimes V_1) \boxtimes W^*}^{\mathrm{loc}(m_2, n_2)}} \mathrm{Sht}_{(\sigma W \otimes V_1) \boxtimes W^*|V_2}^{\mathrm{loc}(m_1, n_1)}.$$

5.4. The category of coherent sheaves on the stack of unramified Langlands parameters. Recall from Remark 2.6 that the Langlands dual group \hat{G} is naturally equipped with an action of the arithmetic Frobenius σ . Consider the σ -twisted conjugation action of \hat{G} on \hat{G} . We denote by $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ the abelian category of \hat{G} -equivariant coherent sheaves on the (non-neutral) component $\hat{G}\sigma \subset \hat{G} \rtimes \sigma$. Equivalently, $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ can be regarded as the abelian category of coherent sheaves on the quotient stack $[\hat{G}\sigma/\hat{G}]$ where \hat{G} acts on $\hat{G}\sigma$ by the usual conjugation action.

Let $V \in \mathrm{Rep}(\hat{G})$ be an algebraic representation of \hat{G} . There is an associated vector bundle on $\hat{G}\sigma$ with global section $\mathcal{O}_{\hat{G}} \otimes V$. Consider the following action of \hat{G} on $\mathcal{O}_{\hat{G}} \otimes V$. For any $g \in \hat{G}$ and $(f, v) \in \mathcal{O}_{\hat{G}} \otimes V$, $g \cdot (f, v) := (gf\sigma^{-1}(g), gv)$. The associated vector bundle thus gives an object $\tilde{V} \in \mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$.

5.5. Key theorem. The following theorem is an analogue of [15, Theorem 6.0.1].

Theorem 5.1. *Let $V_1, V_2 \in \text{Rep}_\Lambda(\hat{G}_\Lambda)$ be two projective Λ -modules. Then there exists the following map*

$$(5.6) \quad \mathcal{S}_{V_1, V_2} : \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \longrightarrow \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_2)),$$

which is compatible with the natural composition maps in the source and target.

We prove this theorem in the rest of this section.

5.5.1. We give an explicit construction of \mathcal{S}_{V_1, V_2} . Consider the following canonical isomorphisms

$$(5.7) \quad \begin{aligned} & \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \\ & \cong \text{Hom}_{\mathcal{O}_{\hat{G}\sigma}}(\mathcal{O}_{\hat{G}\sigma} \otimes V_1, \mathcal{O}_{\hat{G}\sigma} \otimes V_2)^{\hat{G}} \\ & \cong \text{Hom}(V_1, \mathcal{O}_{\hat{G}\sigma} \otimes V_2)^{\hat{G}} \\ & \cong (V_1^* \otimes \mathcal{O}_{\hat{G}\sigma} \otimes V_2)^{\hat{G}}. \end{aligned}$$

Let $W \in \text{Rep}_\Lambda(\hat{G}_\Lambda)$ be a projective Λ -module with Λ -basis $\{e_i\}_i$ and dual basis $\{e_i^*\}_i$. We construct the map

$$\Theta_W : \text{Hom}_{\hat{G}_\Lambda}(V_1, \sigma W^* \otimes V_2 \otimes W) \cong \text{Hom}_{\hat{G}}(V_1, \text{Hom}(\sigma W \otimes W^*, V_2)) \rightarrow \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2),$$

by sending $\mathbf{a} \in \text{Hom}_{\hat{G}_\Lambda}(V_1, \sigma W^* \otimes V_2 \otimes W)$ to the $V_1^* \otimes V_2$ -valued function $\Theta_W(\mathbf{a})$ on $\hat{G}\sigma$ defined by

$$(\Theta_W(\mathbf{a})(g))(v_1) := \sum_i (\mathbf{a}(v_1))(ge_i \otimes e_i^*).$$

It suffices to construct the map

$$\mathcal{C}_W : \text{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes W \otimes V_2) \rightarrow \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_2)).$$

for every $W \in \text{Rep}_\Lambda(\hat{G}_\Lambda)$. Let $\mathbf{a} \in \text{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes W \otimes V_2)$. We have the following coevaluation and evaluation maps:

$$\delta_{\sigma W} : \mathbf{1} \rightarrow \sigma W^* \otimes \sigma W, e_W : W \otimes W^* \rightarrow \mathbf{1}.$$

Choose a quadruple (m_1, n_1, m_2, n_2) of $(V_1 \otimes V_2 \otimes W) \boxtimes W^*$ -large integers. Then the map $\mathcal{C}^{\text{loc}(m_1, n_1)}$ defined in (5.3) sends \mathbf{a} to the cohomological correspondence

$$(5.8) \quad \mathcal{C}^{\text{loc}(m_1, n_1)}(\mathbf{a}) : S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)} \longrightarrow S(\sigma \widetilde{W}^* \boxtimes (\widetilde{V}_2 \otimes \widetilde{W}))^{\text{loc}(m_1, n_1)}.$$

The partial Frobenius morphism (5.4) gives rise to the cohomological correspondence (cf.[15, A.2.3])

$$(5.9) \quad \mathbb{D}\Gamma_{F(W \otimes V_2) \boxtimes W^*}^{-1} : S(\sigma \widetilde{W}^* \boxtimes (\widetilde{V}_2 \otimes \widetilde{W}))^{\text{loc}(m_1, n_1)} \longrightarrow S((\widetilde{V}_2 \otimes \widetilde{W}) \boxtimes \widetilde{W}^*)^{\text{loc}(m_2, n_2)}.$$

Finally, $\mathcal{C}^{\text{loc}(m_2, n_2)}$ sends $\text{id} \otimes e_W$ to the cohomological correspondence

$$(5.10) \quad \mathcal{C}^{\text{loc}(m_2, n_2)}(\text{id} \otimes e_W) : S((\widetilde{V}_2 \otimes \widetilde{W}) \boxtimes \widetilde{W}^*)^{\text{loc}(m_2, n_2)} \longrightarrow S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}.$$

The composition of cohomological correspondences (5.8), (5.9), and (5.10) yields a cohomological correspondence

$$\mathcal{C}_W(\mathbf{a}) \in \text{Corr}_{\text{Sht}_{V_1|V_2}^{W, \text{loc}(m_1, n_1)}}(S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)}, S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}).$$

The construction of the map \mathcal{S}_{V_1, V_2} can be summarized in the following diagram

$$\begin{array}{ccc} \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) & \xrightarrow{\mathcal{S}_{V_1, V_2}} & \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_2)) \\ & \nwarrow \Theta_W \quad \nearrow \mathcal{C}_W & \\ & \text{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes V_2 \otimes W) & \end{array} .$$

5.5.2. We prove that the cohomological correspondence constructed in the previous section is well-defined and can be composed.

Let \mathbf{a}' denote the image of \mathbf{a} under the canonical isomorphism $\mathrm{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes V_2 \otimes W) \cong \mathrm{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W^*, V_2)$.

Lemma 5.2. *Let $X, Y, W_1, W_2, W'_1, W'_2$ be representations of \hat{G} , and $f_1 \otimes f_2 : W_1 \otimes W_2 \rightarrow W'_1 \otimes W'_2$ be a $\hat{G} \times \hat{G}$ -module homomorphism. Let $\mathbf{b} \in \mathrm{Hom}_{\hat{G}}(X, \sigma W_1 \otimes Y \otimes W_2)$ and $\mathbf{b}' \in \mathrm{Hom}_{\hat{G}}(Y \otimes W'_2 \otimes W'_1, Y)$. We omit choosing appropriate integers (m_i, n_i) for simplicity. Then we have*

$$(5.11) \quad \mathcal{C}(\mathbf{b}' \circ (\mathrm{id} \otimes f_2 \otimes f_1)) \circ \mathbb{D}\Gamma_{F^{-1}}^* \circ \mathcal{C}(\mathbf{b}) = \mathcal{C}(\mathbf{b}') \circ \mathbb{D}\Gamma_{F^{-1}}^* \circ \mathcal{C}((\sigma f_1 \circ \mathrm{id} \otimes f_2) \circ \mathbf{b}).$$

In particular, the cohomological correspondence $\mathcal{S}_{V_1, V_2}(\mathbf{a})$ equals to the composition of the following cohomological correspondences:

$$\begin{aligned} \mathcal{C}(\delta_{\sigma W} \otimes \mathrm{id}_{V_1}) : S(\widetilde{V}_1) &\longrightarrow S(\sigma \widetilde{W}^* \boxtimes (\sigma \widetilde{W} \otimes \widetilde{V}_1)), \\ \mathbb{D}\Gamma_{F^{-1}}^* : S(\sigma \widetilde{W}^* \boxtimes (\sigma \widetilde{W} \otimes \widetilde{V}_1)) &\longrightarrow S((\sigma \widetilde{W} \otimes \widetilde{V}_1) \boxtimes \widetilde{W}^*) \\ \mathcal{C}(\mathbf{a}') : S((\sigma \widetilde{W} \otimes \widetilde{V}_1) \boxtimes \widetilde{W}^*) &\longrightarrow S(\widetilde{V}_2). \end{aligned}$$

Proof. Consider the following diagram

$$(5.12) \quad \begin{array}{ccccc} S(\widetilde{X}) & \xrightarrow{\mathcal{C}(\mathbf{b})} & S(\sigma \widetilde{W}_1 \otimes \widetilde{Y} \otimes \widetilde{W}_2) & \xrightarrow{\mathbb{D}\Gamma_{F^{-1}}} & S(\widetilde{Y} \otimes \widetilde{W}_2 \otimes \widetilde{W}_1) & \xrightarrow{\mathcal{C}(\mathbf{b}' \circ (\mathrm{id} \otimes f_2 \otimes f_1))} \\ & \searrow \mathcal{C}((\sigma f_2 \circ \mathrm{id} \otimes f_1) \circ \mathbf{b}) & \downarrow \mathcal{C}(\sigma f_1 \circ \mathrm{id} \otimes f_2) & & \downarrow \mathcal{C}(\mathrm{id} \otimes f_2 \otimes f_1) & \searrow \\ & & S(\sigma \widetilde{W}'_1 \otimes \widetilde{Y} \otimes \widetilde{W}'_2) & \xrightarrow{\mathbb{D}\Gamma_{F^{-1}}} & S(\widetilde{Y} \otimes \widetilde{W}'_2 \otimes \widetilde{W}'_1) & \xrightarrow{\mathcal{C}(\mathbf{b}')} S(\widetilde{Y}) \end{array}$$

The bent triangles on the left and right are clearly commutative by Corollary 2.7. It suffices to prove that the rectangle in the middle is commutative. But this is a direct consequence of [15, Lemma 6.1.13].

Let $X = V_1, Y = \mathbf{1}, W_1 = W'_1 = W^*, W_2 = \sigma W \otimes V_1, W'_2 = W \otimes V_2$. Write \mathbf{a}'' for the image of \mathbf{a} under the canonical isomorphism $\mathrm{Hom}(\sigma W \otimes V_1 \otimes W^*, V_2) \cong \mathrm{Hom}(\sigma W \otimes V_1, W \otimes V_2)$. Take $\mathbf{b} = \delta_{\sigma W} \otimes \mathrm{id}$, $f_1 = \mathrm{id}$, and $f_2 = \mathbf{a}''$. Then the second assertion follows from the above commutative diagram. \square

Lemma 5.3. *For any $\alpha \in \mathrm{Hom}_{\hat{G}}(\widetilde{V}_1, \widetilde{V}_2)$, the construction of \mathcal{S}_{V_1, V_2} is independent from the choice of*

- (1) projective Λ -modules $W \in \mathrm{Rep}_{\Lambda}(\hat{G}_{\Lambda})$,
- (2) $\mathbf{a} \in \mathrm{Hom}_{\hat{G}}(V_1, \sigma W^* \otimes V_2 \otimes W)$, such that $\Theta_W(\mathbf{a}) = \alpha$,
- (3) $(V_1 \otimes V_2) \otimes W \boxtimes W^*$ -acceptable integers (m_1, n_1, m_2, n_2) .

Proof. The proof is completely similar to that of [15, Lemma 6.2.5], and we briefly discuss it here.

We start by proving the independence of (3). Choose another quadruple of $(V_1 \otimes V_2) \otimes W \boxtimes W^*$ -acceptable integers $(m'_1, n'_1, m'_2, n'_2) \geq (m_1, n_1, m_2, n_2)$. We have the following diagram of Hecke correspondences

$$\begin{array}{ccccc} \mathrm{Sht}_{V_1}^{\mathrm{loc}(m'_1, n'_1)} & \longleftarrow & \mathrm{Sht}_{V_1|V_2}^{\lambda, \mathrm{loc}(m'_1, n'_1)} & \longrightarrow & \mathrm{Sht}_{V_2}^{\mathrm{loc}(m'_2, n'_2)} \\ \downarrow \mathrm{res}_{m'_1, n'_1}^{m'_1, n'_1} & & \downarrow \mathrm{res}_{m'_1, n'_1}^{m'_1, n'_1} & & \downarrow \mathrm{res}_{m'_2, n'_2}^{m'_2, n'_2} \\ \mathrm{Sht}_{V_1}^{\mathrm{loc}(m_1, n_1)} & \longleftarrow & \mathrm{Sht}_{V_1|V_2}^{\lambda, \mathrm{loc}(m_1, n_1)} & \longrightarrow & \mathrm{Sht}_{V_2}^{\mathrm{loc}(m_2, n_2)}. \end{array}$$

This is the upper face of diagram (4.5). As we discussed in §3, all the vertical maps are smooth, the two squares are commutative, and the left square is Cartesian. Then $\mathcal{C}_{S(\lambda)}^{\mathrm{loc}(m'_1, n'_1)}(\mathbf{a})$ equals the pullback of $\mathcal{C}_{S(\lambda)}^{\mathrm{loc}(m_1, n_1)}(\mathbf{a})$ along the vertical maps.

Next, we prove the independence of (1) and (2) simultaneously. Consider that \hat{G} acts on the filtration of \mathcal{O}_G by right regular representation. Then \mathcal{O}_G is realized as an ind-object in $\mathrm{Rep}_{\Lambda}(\hat{G})$. Let $X \in \mathrm{Rep}_{\Lambda}(\hat{G})$ be

a projective object and we denote by \underline{X} the underlying Λ -module of X equipped with the trivial \hat{G} -action. Consider the following \hat{G}_Λ -equivariant maps

$$a_X : X \rightarrow \mathcal{O}_G \otimes \underline{X}, \quad x \mapsto a_X(x)(g) := gx,$$

$$m_X : \underline{X}^* \otimes X \rightarrow \mathcal{O}_G, \quad (x^*, x) \mapsto m_X(x^*, x)(g) := x^*(gx),$$

where we identify $\mathcal{O}_G \otimes \underline{X}$ as the space of \underline{X} -valued functions on \hat{G} in the definition of a_X and m_X . Taking $X = W$, we have the following $\hat{G} \times \hat{G}$ -module maps

$$\sigma W^* \otimes V_2 \otimes W \xrightarrow{a_{\sigma W^*}} \underline{W}^* \otimes \sigma \mathcal{O}_G \otimes V_2 \otimes W \xrightarrow{m_W} \sigma \mathcal{O}_G \otimes V_2 \otimes \mathcal{O}_G.$$

The map $\hat{G} \times \hat{G} \rightarrow \hat{G}\sigma$, $(g_1, g_2) \mapsto \sigma(g_1)^{-1}\sigma(g_2)\sigma$ induces a natural map $d_\sigma : \Lambda[\hat{G}\sigma] \rightarrow \sigma \mathcal{O}_G \otimes \mathcal{O}_G$ which intertwines the σ -twisted conjugation action on $\Lambda[\hat{G}\sigma]$ and the diagonal action of \hat{G} on $\sigma \mathcal{O}_G \otimes \mathcal{O}_G$. For any $\alpha \in \text{Hom}_{\hat{G}}(V_1, \mathcal{O}_G \otimes V_2)$, denote by α' the image of α under the following map

$$\text{Hom}_{\hat{G}}(V_1, \mathcal{O}_G \otimes V_2) \xrightarrow{d_\sigma} \text{Hom}_{\hat{G}}(V_1, \sigma \mathcal{O}_G \otimes V_2 \otimes \mathcal{O}_G).$$

Direct computation yields the followings

$$(m_W \circ a_{\sigma W^*}) \circ \mathbf{a}' = d_\sigma(\alpha') : V_1 \rightarrow \sigma \mathcal{O}_G \otimes V_2 \otimes \mathcal{O}_G,$$

and

$$\text{id}_{V_2} \otimes e_W = \text{ev}_{(1,1)} \circ (m_W \circ a_{W^*}) : V_2 \otimes W \otimes W^* \rightarrow V_2,$$

where $\text{ev}_{(1,1)}$ denotes the evaluation at $(1, 1) \in \hat{G} \times \hat{G}$. In Lemma 5.2, let $W_1 \otimes W_2 := W \otimes W^*$, $W'_1 \otimes W'_2 := \mathcal{O}_G \otimes \mathcal{O}_G$, $f_1 \otimes f_2 := m_W \circ a_{W^*}$, $\mathbf{b} := \mathbf{a}'$, and $\mathbf{b}' := \text{ev}_{(1,1)}$. Then we have

$$\begin{aligned} \mathcal{C}_{S(\lambda)}(\mathbf{a}) &= \mathcal{C}(\text{id}_{V_2} \otimes e_W) \circ \mathbb{D}\Gamma_{F_{(V_2 \otimes S(\lambda)) \boxtimes S(\lambda)}^*}^{-1} \circ \mathcal{C}(\mathbf{a}') \\ &= \mathcal{C}(\text{ev}_{(1,1)}) \circ \mathbb{D}\Gamma_{F_{(V_2 \otimes \mathcal{O}_G) \boxtimes \mathcal{O}_G}^*}^{-1} \circ \mathcal{C}(d_\sigma(\alpha')). \end{aligned}$$

We see from the last equality in the above that $\mathcal{C}_{S(\lambda)}(\mathbf{a})$ depends only on α and the lemma is thus proved. \square

We claim that our construction of \mathcal{S}_{V_1, V_2} is compatible with the composition of morphisms. More precisely, we have the following lemma.

Lemma 5.4. *For any $V_1, V_2, V_3 \in \text{Rep}_\Lambda(\hat{G}_\Lambda)$ which are projective Λ -modules and $S_1, S_2 \in \text{Rep}_\Lambda(\hat{G}_\Lambda)$, we have the following commutative diagram*

$$(5.13) \quad \begin{array}{ccc} \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \otimes \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_2, \widetilde{V}_3) & \xrightarrow{\phi} & \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_3) \\ \uparrow & & \uparrow \\ \text{Hom}_{\hat{G}}(\sigma S_1 \otimes V_1 \otimes S_1^*, V_2) \otimes \text{Hom}_{\hat{G}}(\sigma S_2 \otimes V_2 \otimes S_2^*, V_3) & \xrightarrow{\phi'} & \text{Hom}_{\hat{G}}(\sigma S_2 \otimes \sigma S_1 \otimes V_1 \otimes S_1^* \otimes S_2^*, V_3) \\ \downarrow \mathcal{C}_{S_1} \otimes \mathcal{C}_{S_2} & & \downarrow \mathcal{C}_{S_1 \otimes S_2} \\ \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_2)) \otimes \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{V}_2), S(\widetilde{V}_3)) & \xrightarrow{\phi''} & \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_3)), \end{array}$$

where

- the unlabelled vertical arrows are given by the Peter-Weyl theorem
- ϕ is the compositions of morphisms in $\text{Coh}^{\hat{G}}(\hat{G}\sigma)$
- ϕ'' is the composition described in §3.2
- $\phi'(\mathbf{a}_1 \otimes \mathbf{a}_2)$ is defined to be the homomorphism

$$\sigma S_2 \otimes \sigma S_1 \otimes V_1 \otimes S_1^* \otimes S_2^* \xrightarrow{\text{id}_{\sigma S_2} \otimes \mathbf{a}_1 \otimes \text{id}_{S_2^*}} \sigma S_2 \otimes V_2 \otimes S_2^* \xrightarrow{\mathbf{a}_2} V_3.$$

Proof. The lemma can be proved following the same idea in the proof of [15, Lemma 6.2.7]. \square

5.5.3. We study the endomorphism ring of the unit object in $P(\text{Sht}_k^{\text{loc}}, \Lambda)$. This will be used to prove the "S = T" theorem for Shimura sets in §12.3.

Let δ_1 denote the intersection cohomology sheaf IC_0 on $\text{Sht}_0^{\text{loc}(m,n)}$. The group theoretic description of the moduli of restricted local Shtukas (cf. [15, §5.3.2]) implies that $\text{Sht}_0^{\text{loc}(m,n)}$ is perfectly smooth. Thus δ_1 may be realized as

$$\delta_1^{m,n} := \Lambda\langle(m-n)\dim G\rangle \in P(\text{Sht}_0^{\text{loc}(m,n)}, \Lambda)$$

for every $m \geq n$. Fix a square root $q^{1/2}$. We write E for \mathbb{Z}_ℓ

Corollary 5.5. (1) There is a natural isomorphism

$$\text{Corr}_{\text{Sht}^{\text{loc}}}(\delta_1, \delta_1) \simeq \mathcal{H}_{G,E}$$

where $\mathcal{H}_{G,E}$ denotes the Hecke algebra $C_c^\infty(G(\mathcal{O}) \backslash G(F)/G(\mathcal{O}), E)$.

(2) We denote the map

$$\mathcal{S}_{\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}, \mathcal{O}_{[\hat{G}\sigma/\hat{G}]}} : \text{End}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}) \rightarrow \text{Corr}_{\text{Sht}^{\text{loc}}}(\delta_1, \delta_1)$$

by $\mathcal{S}_{\mathcal{O}}$ for simplicity. Under the isomorphism in (1), the map $\mathcal{S}_{\mathcal{O}} \otimes \text{id}_{E[q^{-1/2}, q^{1/2}]}$ coincides with the classical Satake isomorphism.

Proof. Recall the definition of the Borel-Moore homology $H_i^{\text{BM}}(X)$ for a perfect pfp algebraic space which is defined over an algebraically closed field (cf. [15, A.1.3]). Assume X_1 and X_2 to be perfectly smooth algebraic spaces of pure dimension. Let $X_1 \leftarrow C \rightarrow X_2$ be a correspondence. Then

$$\begin{aligned} (5.14) \quad & \text{Corr}_C((X_1, E\langle d_1 \rangle), (X_2, E\langle d_2 \rangle)) \\ &= \text{Hom}_{D_b^{\hat{G}}(C, E)}(E\langle d_1 \rangle, \omega_C\langle d_2 - 2\dim X_2 \rangle) \\ &= H_{2\dim X_2 + d_1 - d_2}^{\text{BM}}(C). \end{aligned}$$

Then if $2\dim C = 2\dim X_2 + d_1 - d_2$, the cohomological correspondences from $(X_1, E\langle d_1 \rangle)$ to $(X_2, E\langle d_2 \rangle)$ can be identified as the set of irreducible components of C of maximal dimension.

For a perfect pfp algebraic space X of dimension d , define I to be the set of top-dimensional irreducible components of X . Then $H_d^{\text{BM}}(I)$ is the free E -module generated by the d -dimensional irreducible components of X , and thus can be identified with the space $C(I, E)$ of E -valued functions on I . The map $f \mapsto \sum_{C_i \in I} f(C_i)[C_i]$ establishes a bijection

$$(5.15) \quad C(I, E) = H_d^{\text{BM}}(X).$$

With the above preparations, we get an isomorphism

$$(5.16) \quad \mathcal{H}_{G,E} \simeq \text{Corr}_{\text{Sht}^{\text{loc}}}(\delta_1, \delta_1),$$

via a similar argument as for [15, Proposition 5.4.4], and we finish the proof of (1).

To prove part (2), we first note that the statement holds for $E = \mathbb{Q}_\ell$ by [15, Theorem 6.0.1(2)]. We sketch the proof here. Let μ be a central minuscule dominant coweight, and ν be a dominant coweight such that $\sigma(\nu) = \nu$. Choose (m_1, n_1, m_2, n_2) to be $(\nu + \mu, \nu)$ -acceptable. Take $\mathbf{a} \in \text{Hom}_{\hat{G}}(V_\nu \otimes V_\mu \otimes V_{\nu^*}, V_\mu)$ to be the map induced by the evaluation map $\mathbf{e}_\nu : V_\nu \otimes V_{\nu^*} \rightarrow \mathbf{1}$. Consider the following diagram

$$\text{pt} \longleftarrow \text{Gr}_{\leq \nu^*} \xrightarrow{\Delta} \text{Gr}_{\leq \nu^*} \times \text{Gr}_{\leq \nu^*} \xleftarrow{\sigma \times \text{id}} \text{Gr}_{\mu^*} \times \text{Gr}_{\leq \mu^*} \xleftarrow{\Delta} \text{Gr}_{\leq \nu^*} \longrightarrow \text{pt}.$$

Recall the cohomological correspondences $\delta_{\text{IC}_{\nu^*}}$ and $e_{\text{IC}_{\nu^*}}$ defined in [15, §A.2.3.4]. Then

$$\mathcal{C}_{V_\nu}^{\text{loc}(m_1, n_1)}(\mathbf{a}) = \delta_{\text{IC}_{\nu^*}} \circ \Gamma_{\sigma \times \text{id}}^* \circ e_{\text{IC}_{\nu^*}} \in H_0^{\text{BM}}(\text{Gr}_{\nu^*}(k)),$$

and the cohomological correspondence $\mathcal{C}_{V_\nu}^{\text{loc}(m_1, n_1)}(\mathbf{a})$ can be identified with the function f on $\text{Gr}_{\nu^*}(k)$ whose value at $x \in \text{Gr}_{\nu^*}(k)$ is given by $\text{tr}(\phi_x \mid \text{Sat}(V_{\nu^*})_{\bar{x}})$. Then up to a choice of $q^{1/2}$, the map $\mathcal{S}_{\mathcal{O}, \mathcal{O}} \otimes_{\mathbb{Q}_\ell} \text{id}_{\mathbb{Q}_\ell[q^{1/2}, q^{-1/2}]}$ coincides with the classical Satake isomorphism.

Now we come back to the case $E = \mathbb{Z}_\ell$. Write Q for $\mathbb{Q}_\ell[q^{1/2}, q^{-1/2}]$. The above argument shows that

$$\mathcal{S}_{\mathcal{O}} \otimes Q : \text{End}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}) \otimes_{\mathbb{Z}_\ell} Q \rightarrow \text{Corr}_{\text{Sht}_k^{\text{loc}}}(\delta_1, \delta_1) \otimes_{\mathbb{Z}_\ell} Q$$

coincide with the classical Satake isomorphism. Note that

$$\mathrm{End}_{\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)}(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]} \otimes_{\mathbb{Z}_\ell} Q \simeq \mathbb{Z}_\ell[\hat{G}]^{\hat{G}} \otimes_{\mathbb{Z}_\ell} Q,$$

where \hat{G} acts on \hat{G} by the σ -twisted conjugation. Considering the Satake transfer of the image of \mathbb{Z}_ℓ -basis of $\mathbb{Z}_\ell[\hat{G}]^{(\hat{G})}$ in $\mathbb{Z}_\ell[\hat{G}]^{(\hat{G})} \otimes_{\mathbb{Z}_\ell} Q$, we conclude the proof of (2). \square

6. COHOMOLOGICAL CORRESPONDENCES BETWEEN SHIMURA VARIETIES

In this section, we adapt the machinery developed in previous sections and apply it to the study of the cohomological correspondences between different Hodge type Shimura varieties following the idea of [15].

6.1. Preliminaries. Let (G, X) be a Shimura datum. For the rest of this paper, E will denote the reflex field of (G, X) (cf.[10]). Let $K \subset G(\mathbb{A}_f)$ be a (sufficiently small) open compact subgroup and denote by $\mathrm{Sh}_K(G, X)$ the corresponding Shimura variety defined over E . Fix a prime $p > 2$ such that K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$. We write \underline{G} for the reductive group which extends G to $\mathbb{Z}_{(p)}$ and such that $\underline{G}(\mathbb{Z}_p) = K_p$. Choose ν to be a place of E lying over p . We write $\mathcal{O}_{E,(\nu)}$ for the localization of \mathcal{O}_E at ν . Results of Kisin [6] and Vasiu [14] state that for any Hodge type Shimura datum (G, X) , there is a smooth integral canonical model $\mathcal{S}_K(G, X)$ of $\mathrm{Sh}_K(G, X)$, which is defined over $\mathcal{O}_{E,(\nu)}$. Let k_ν denote the residue field of $\mathcal{O}_{E,(\nu)}$ and fix an algebraic closure \bar{k}_ν of k_ν . We denote by $\mathrm{Sh}_{\mu,K} := (\mathcal{S}_K(G, X) \otimes k_\nu)^{\mathrm{pf}}$ the perfection of the special fiber of $\mathcal{S}_K(G, X)$. The perfection of mod p fibre of Shimura varieties and moduli of local Shtukas are related by a map $\mathrm{loc}_p : \mathrm{Sh}_{\mu,K} \rightarrow \mathrm{Sht}_\mu^{\mathrm{loc}}$. The construction of loc_p is via a \underline{G} -torsor over the crystalline site $(\mathcal{S}_{K,k_\nu}/\mathcal{O}_{E,(\nu)})_{\mathrm{CRIS}}$ and we refer to [15, §7.2.1] for a detailed discussion. In the Siegel case, it may be understood as the perfection of the morphism sending an abelian variety to its underlying p -divisible group. We need the following result of Xiao-Zhu [15, Proposition 7.2.4] for our proof of the main theorem.

Proposition 6.1. Let (m, n) be a pair of μ -large integers. The morphism

$$\mathrm{loc}_p(m, n) := \mathrm{res}_{m,n} \circ \mathrm{loc}_p : \mathrm{Sh}_\mu \rightarrow \mathrm{Sht}_\mu^{\mathrm{loc}(m,n)}$$

is perfectly smooth.

6.2. Étale local systems. Let $\ell \neq p$ be a prime number. Assume that $\rho : G \rightarrow GL_{\mathbb{Q}_\ell}(W)$ is a \mathbb{Q}_ℓ -representation of G . If $K \subset G(\mathbb{A}_f)$ is sufficiently small, we associate an étale local system $\mathcal{L}_{\ell,W}$ on $\mathrm{Sh}_{\mu,K}$ to W following the idea of [8, §4] and [9, §III.6] as follows.

Write $K = K_\ell K^\ell$ with $K_\ell \subset G(\mathbb{Q}_\ell)$ and $K^\ell \subset G(\mathbb{A}_f^\ell)$. The representation ρ restricts to a representation

$$\rho_{K_\ell} : K(\mathbb{Q}_\ell) \rightarrow G(\mathbb{Q}_\ell) \rightarrow GL(W_{\mathbb{Q}_\ell}).$$

Note that $K(\mathbb{Q}_\ell)$ is compact, and there exists a lattice $\Lambda_{W,\ell} \subset W_{\mathbb{Q}_\ell}$ fixed by $K(\mathbb{Q}_\ell)$. Now we vary the levels at ℓ . Define

$$K_\ell^{(n)} := K_\ell \cap \rho_{K(\mathbb{Q}_\ell)}^{-1}(\{g \in GL(\Lambda_{W,\ell}) \mid g \equiv 1 \pmod{\ell^n}\}).$$

Then we get a system of open neighborhoods of $1 \in G(\mathbb{Q}_\ell)$. For each n , the construction of $K_\ell^{(n)}$ gives rise to a representation

$$\rho_{K_\ell}^n : K_\ell/K_\ell^{(n)} \rightarrow GL(\Lambda_{W,\ell}/\ell^n \Lambda_{W,\ell}).$$

The natural projection map $\mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell} \rightarrow \mathrm{Sh}_{\mu,K_\ell K^\ell}$ is a finite étale cover with the group of deck transformations being $K_\ell/K_\ell^{(n)}$. Then the trivial étale $\mathbb{Z}/\ell^n\mathbb{Z}$ -local system $\mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell} \times \Lambda_{W,\ell}/\ell^n \Lambda_{W,\ell}$ on $\mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell}$ gives rise to the étale $\mathbb{Z}/\ell^n\mathbb{Z}$ -local system

$$\mathcal{L}_{W,\ell,n} := \mathrm{Sh}_{\mu,K_\ell^{(n)}K^\ell} \times^{K_\ell/K_\ell^{(n)}} \Lambda_{W,\ell}/\ell^n \Lambda_{W,\ell}.$$

Let

$$(6.1) \quad \mathcal{L}_{W,\mathbb{Z}_\ell} := \varprojlim_n \mathcal{L}_{W,\ell,n}.$$

This is an étale \mathbb{Z}_ℓ -local system on $\mathrm{Sh}_{\mu,K}$. It can be checked that $\mathcal{L}_{W,\mathbb{Q}_\ell} := \mathcal{L}_{W,\mathbb{Z}_\ell} \otimes \mathbb{Q}$ is an étale \mathbb{Q}_ℓ -local system on $\mathrm{Sh}_{\mu,K}$ which is independent of the choice of Λ_ℓ .

6.3. Main theorem. Let (G_1, X_1) and (G_2, X_2) be two Hodge type Shimura data (cf. [10]) equipped with an isomorphism $\theta : G_{1, \mathbb{A}_f} \simeq G_{2, \mathbb{A}_f}$. Let $\{\mu_i\}$ denote the conjugacy class of Hodge cocharacters determined by X_i and consider them as dominant characters of \hat{T} . In particular, μ_1 and μ_2 are both minuscule. Then [15, Corollary 2.1.5] implies that there is a canonical inner twist $\Psi_{\mathbb{R}} : G_1 \rightarrow G_2$ over \mathbb{C} . Recall notations in §1.3. We define $\mu_{i, \text{ad}}$ to be the composition of μ_i with the quotient $G \rightarrow G_{\text{ad}}$ and consider it as a character of \hat{T}_{sc} . We assume that

$$\mu_{1, \text{ad}} \mid_{Z(\hat{G}_{\text{sc}}^{\Gamma_{\mathbb{Q}}})} = \mu_{2, \text{ad}} \mid_{Z(\hat{G}_{\text{sc}}^{\Gamma_{\mathbb{Q}}})}.$$

It follows from [15, Corollary 2.1.6] that $\Psi_{\mathbb{R}}$ comes from a unique global inner twist $\Psi : G_{1, \mathbb{Q}} \rightarrow G_{2, \mathbb{Q}}$ such that $\Psi = \text{Int}(h) \circ \theta$, for some $\theta : G_{1, \mathbb{A}_f} \simeq G_{2, \mathbb{A}_f}$ and $h \in G_{2, \text{ad}}(\bar{\mathbb{A}}_f)$.

We assume that $K_i \subset G(\mathbb{A}_f)$ to be sufficiently small such that $\theta K_1 = K_2$. Choose a prime p such that $K_{1, p}$ (and therefore $K_{2, p}$) is hyperspecial. Let \underline{G}_i be the integral model of G_{i, \mathbb{Q}_p} over \mathbb{Z}_p determined by $K_{i, p}$. Then $\underline{G}_1 \simeq \underline{G}_2$, and we can thus identify their Langlands dual groups $(\hat{G}, \hat{B}, \hat{T})$. Choose an isomorphism $\iota : \mathbb{C} \simeq \mathbb{Q}_p$. Let $\nu \mid p$ be a place of the compositum of reflex fields of (G_i, X_i) determined by our choice of isomorphism ι . We write Sh_{μ_i} for the mod p fibre of the canonical integral model of $\text{Sh}_{K_i}(G_i, X_i)$ base change to k_{ν} . We make the following assumption

$$(6.2) \quad \mu_1 \mid_{Z(\hat{G}^{\Gamma_{\mathbb{Q}_p}})} = \mu_2 \mid_{Z(\hat{G}^{\Gamma_{\mathbb{Q}_p}})}.$$

The assumption guarantees the existence of the ind-scheme $\text{Sh}_{\mu_1 | \mu_2}$ which fits into the following commutative diagram

$$(6.3) \quad \begin{array}{ccccc} \text{Sh}_{\mu_1, K_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \text{Sh}_{\mu_1 | \mu_2} & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \text{Sh}_{\mu_2, K_2} \\ \downarrow \text{loc}_p & & \downarrow & & \downarrow \text{loc}_p \\ \text{Sht}_{\mu_1}^{\text{loc}} & \xleftarrow{\overleftarrow{h}_{\mu_1}^{\text{loc}}} & \text{Sht}_{\mu_1 | \mu_2}^{\text{loc}} & \xrightarrow{\overrightarrow{h}_{\mu_2}^{\text{loc}}} & \text{Sht}_{\mu_2}^{\text{loc}} \end{array},$$

making both squares to be Cartesian.

Remark 6.2. In the case that $(G_1, X_1) = (G_2, X_2)$, $\text{Sh}_{\mu_1 | \mu_2}$ is the perfection of the mod p fibre of a natural integral model of some Hecke correspondence. If $(G_1, X_1) \neq (G_2, X_2)$, then $\text{Sh}_{\mu_1 | \mu_2}$ can be regarded as exotic Hecke correspondences between mod p fibres of different Shimura varieties. We refer to [15, §7.3.3, §7.3.4] for a detailed discussion.

Let (G_i, X_i) $i = 1, 2, 3$ be three Hodge type Shimura data, together with the isomorphisms $\theta_{i, j} : G_{i, \mathbb{A}_f} \simeq G_{j, \mathbb{A}_f}$ satisfying the natural cocycle condition. Choose a common level K using the isomorphism $\theta_{i, j}$. Let p be an unramified prime, such that the assumption (12.2) holds for each pair of $((G_i, X_i), (G_j, X_j))$. Choose a half Tate twist $\mathbb{Q}_{\ell}(1/2)$.

Let $V_i := V_{\mu_i}$ be the highest weight representation of $\hat{G}_{\mathbb{Q}_{\ell}}$ of highest weight μ_i . Write $\tilde{V}_i \in \text{Coh}^{\hat{G}_{\mathbb{Q}_{\ell}}}(\hat{G}_{\mathbb{Q}_{\ell}} \sigma)$ for the vector bundle associated to V_i analogous to §5.1.3. Recall from §6.1 that, to each representation W of $G_{\mathbb{Q}_{\ell}}$, we can attach the étale local system $\mathcal{L}_{W, \mathbb{Q}_{\ell}}$ on Sh_{μ_i} . Let $d_i = \langle 2\rho, \mu_i \rangle = \dim \text{Sh}_K(G_i, X_i)$. Denote the global section of the structure sheaf on the quotient stack $[\hat{G} \sigma / \hat{G}]$ by \mathcal{J} , and the prime-to- p Hecke algebra by \mathcal{H}^p .

Theorem 6.3. *There exists a map*

$$(6.4) \quad \text{Spc} : \text{Hom}_{\text{Coh}^{\hat{G}_{\mathbb{Q}_{\ell}}}(\hat{G}_{\mathbb{Q}_{\ell}} \sigma)}(\tilde{V}_1, \tilde{V}_2) \rightarrow \text{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(H_c^*(\text{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_{\ell}} \langle d_1 \rangle), H_c^*(\text{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_{\ell}} \langle d_2 \rangle)),$$

which is compatible with compositions on the source and target.

Proof. Choose a lattice $\Lambda_i \in \text{Rep}_{\mathbb{Z}_\ell}(\hat{G}_{\mathbb{Z}_\ell})$ in V_i . We denote by $\widetilde{\Lambda}_i \in \text{Coh}^{\hat{G}_{\mathbb{Z}_\ell}}(\hat{G}_{\mathbb{Z}_\ell}\sigma)$ the coherent sheaf which corresponds to Λ_i as in §5.1. Then

$$\begin{aligned}
 (6.5) \quad \text{Hom}_{\text{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) &\simeq \text{Hom}_{\hat{G}_{\mathbb{Q}_\ell}}(V_1, V_2 \otimes \mathbb{Q}_\ell[\hat{G}]) \\
 &\simeq \text{Hom}_{\hat{G}_{\mathbb{Q}_\ell}}(\Lambda_1 \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell, (\Lambda_2 \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[\hat{G}]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell) \\
 &\simeq \text{Hom}_{\hat{G}_{\mathbb{Z}_\ell}}(\Lambda_1, \Lambda_2 \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[\hat{G}]) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \\
 &\simeq \text{Hom}_{\text{Coh}^{\hat{G}_{\mathbb{Z}_\ell}}(\hat{G}_{\mathbb{Z}_\ell}\sigma)}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.
 \end{aligned}$$

By Theorem 5.1, we get a map

$$(6.6) \quad \mathcal{S}_{\Lambda_1, \Lambda_2} : \text{Hom}_{\text{Coh}^{\hat{G}_{\mathbb{Z}_\ell}}(\hat{G}_{\mathbb{Z}_\ell}\sigma)}(\widetilde{\Lambda}_1, \widetilde{\Lambda}_2) \rightarrow \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{\Lambda}_1), S(\widetilde{\Lambda}_2)).$$

Combining (6.5) with (6.6), we get the following map

$$(6.7) \quad \text{Hom}_{\text{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \rightarrow \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{\Lambda}_1), S(\widetilde{\Lambda}_2)) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell.$$

Choose a dominant coweight ν and a quadruple (m_1, n_1, m_2, n_2) that is $(\mu_1 + \nu, \nu)$ -acceptable and $(\mu_2 + \nu, \nu)$ -acceptable. We have the following diagram

$$\begin{array}{ccccc}
 \text{Sh}_{\mu_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \text{Sh}_{\mu_1|\mu_2}^\nu & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \text{Sh}_{\mu_2} \\
 \downarrow \text{loc}_p & & \downarrow \text{loc}_p^\nu & & \downarrow \text{loc}_p \\
 \text{Sht}_{\mu_1}^{\text{loc}} & \xleftarrow{\overleftarrow{h}_{\mu_1}^{\text{loc}}} & \text{Sht}_{\mu_1|\mu_2}^{\nu, \text{loc}} & \xrightarrow{\overrightarrow{h}_{\mu_2}^{\text{loc}}} & \text{Sht}_{\mu_2}^{\text{loc}} \\
 \downarrow \text{res}_{m_1, n_1} & & \downarrow \text{res}_{m_1, n_1}^\nu & & \downarrow \text{res}_{m_2, n_2} \\
 \text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)} & \xleftarrow{\overleftarrow{h}_{\mu_1}^{\text{loc}(m_1, n_1)}} & \text{Sht}_{\mu_1|\mu_2}^{\nu, \text{loc}(m_1, n_1)} & \xrightarrow{\overrightarrow{h}_{\mu_2}^{\text{loc}(m_2, n_2)}} & \text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}
 \end{array}, \quad (6.8)$$

where

- all squares are commutative (discussions on diagram (4.5) and diagram (6.3),
- except for the square at the down right corner, and the other three squares are Cartesian (discussions on diagram (6.3) and diagram (4.5),
- the morphism $\overleftarrow{h}_{\mu_1}$ is perfectly proper ([15, Lemma 5.2.12]),
- the morphisms $\text{loc}_p(m_i, n_i)$ are perfectly smooth (Proposition 6.1).

Then the morphism $\text{loc}_p^\nu(m_1, n_1) := \text{res}_{m_1, n_1}^\nu \circ \text{loc}_p^\nu$ is also perfectly proper. Thus we can pullback the cohomological correspondences (cf. [15, A.2.11]) on the right hand side of (6.6) along $\text{loc}_p^\nu(m_1, n_1)$ to obtain a map

$$\text{loc}_p^\nu(m_1, n_1)^\star : \text{Corr}_{\text{Sht}^{\text{loc}}}(S(\widetilde{\Lambda}_1), S(\widetilde{\Lambda}_2)) \rightarrow \text{Corr}_{\text{Sh}_{\mu_1|\mu_2}^\nu}(\text{loc}_p(m_1, n_1)^\star S(\widetilde{\Lambda}_1), \text{loc}_p(m_2, n_2)^\star (S(\widetilde{\Lambda}_2))).$$

Note that μ_i are minuscule, then the \star -pullback of $S(\widetilde{\Lambda}_i)$ along $\text{loc}_p(m_i, n_i)$ equals $\mathbb{Z}_\ell\langle d_i \rangle$. Next, we construct a natural map

$$(6.9) \quad \mathfrak{C}_W : \text{Corr}_{\text{Sh}_{\mu_1|\mu_2}^\nu}((\text{Sh}_{\mu_1}, \mathbb{Z}_\ell\langle d_1 \rangle), (\text{Sh}_{\mu_2}, \mathbb{Z}_\ell\langle d_2 \rangle)) \rightarrow \text{Corr}_{\text{Sh}_{\mu_1|\mu_2}^\nu}((\text{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Z}_\ell}\langle d_1 \rangle), (\text{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Z}_\ell}\langle d_2 \rangle)).$$

For each $n \in \mathbb{Z}^+$, we note that there exists an ind-scheme $\mathrm{Sh}_{\mu_1|\mu_2}^{(n)}$ which fits into the following commutative diagram such that both squares are Cartesian

$$\begin{array}{ccccc} \mathrm{Sh}_{\mu_1, K_\ell^{(n)} K^\ell} & \xleftarrow{\overleftarrow{h}_{\mu_1}^{(n)}} & \mathrm{Sh}_{\mu_1|\mu_2}^{\nu, (n)} & \xrightarrow{\overrightarrow{h}_{\mu_2}^{(n)}} & \mathrm{Sh}_{\mu_2, K_\ell^{(n)} K^\ell} \\ \downarrow p_1^n & & \downarrow p^n & & \downarrow p_2^n \\ \mathrm{Sh}_{\mu_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \mathrm{Sh}_{\mu_1|\mu_2}^\nu & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \mathrm{Sh}_{\mu_2} \end{array}$$

Here the three vertical maps are the natural quotients by the finite group K_ℓ/K_ℓ^n and are thus étale.

Let $(f_n)_n : (\overleftarrow{h}_{\mu_1})^*(\mathbb{Z}/\ell^n \mathbb{Z} \langle d_1 \rangle)_n \rightarrow (\overrightarrow{h}_{\mu_2})^!(\mathbb{Z}/\ell^n \mathbb{Z} \langle d_2 \rangle)_n$ be a cohomological correspondence in $\mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^\nu}((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell \langle d_2 \rangle))$. For each $n \in \mathbb{Z}^+$, the shifted pullback (cf. [15, A.2.12]) of f_n gives rise to a cohomological correspondence

$$\tilde{f}_n : (\overleftarrow{h}_{\mu_1}^{(n)})^*(\mathbb{Z}/\ell^n \mathbb{Z} \langle d_1 \rangle) \rightarrow (\overrightarrow{h}_{\mu_2}^{(n)})^!(\mathbb{Z}/\ell^n \mathbb{Z} \langle d_2 \rangle)$$

in $\mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^{\nu, (n)}}((\mathrm{Sh}_{\mu_1, K_\ell^{(n)} K^\ell}, \mathbb{Z}/\ell^n \mathbb{Z} \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2, K_\ell^{(n)} K^\ell}, \mathbb{Z}/\ell^n \mathbb{Z} \langle d_2 \rangle))$. For any representation W of $G_{\mathbb{Q}_\ell}$, recall the $\mathbb{Z}/\ell^n \mathbb{Z}$ module $\Lambda_{W, \ell}/\ell^n \Lambda_{W, \ell}$ constructed in §6.2. The cohomological correspondence \tilde{f}_n gives rise to a cohomological correspondence

$$\tilde{g}_n \in \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^{\nu, (n)}}(\mathrm{Sh}_{\mu_1, K_\ell^{(n)} K^\ell} \times \Lambda_{W, \ell}/\ell^n \Lambda_{W, \ell} \langle d_1 \rangle, \mathrm{Sh}_{\mu_2, K_\ell^{(n)} K^\ell} \times \Lambda_{W, \ell}/\ell^n \Lambda_{W, \ell} \langle d_2 \rangle).$$

In addition, the cohomological correspondence \tilde{f}_n is $K_\ell/K_\ell^{(n)}$ -equivariant. Then it follows that the cohomological correspondence \tilde{g}_n is also $K_\ell/K_\ell^{(n)}$ -equivariant and descends to a cohomological correspondence

$$g_n \in \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^\nu}((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \ell, n} \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \ell, n} \langle d_2 \rangle)).$$

Defining $\mathfrak{C}_W((f_n)_n) := (g_n)_n$ completes the construction of \mathfrak{C}_W .

Compose the maps we previously construct,

$$\begin{aligned} (6.10) \quad & \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^{\nu, \mathrm{loc}(m_1, n_1)}}((\mathrm{Sht}_{\mu_1}^{\mathrm{loc}(m_1, n_1)}, S(\widetilde{\Lambda}_1)^{\mathrm{loc}(m_1, n_1)}), (\mathrm{Sht}_{\mu_1}^{\mathrm{loc}(m_2, n_2)}, S(\widetilde{\Lambda}_2)^{\mathrm{loc}(m_2, n_2)})) \\ & \xrightarrow{\mathrm{loc}_p^{\nu, (m_1, n_1)} *} \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^\nu}((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell \langle d_2 \rangle)) \\ & \xrightarrow{\mathfrak{C}_W} \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^\nu}((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_2 \rangle)) \\ & \xrightarrow{H_c^*} \mathrm{Hom}_{\mathcal{H}^p}(H_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_1 \rangle), H_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_2 \rangle)). \end{aligned}$$

We justify that the composition of maps in (6.10) factors through $\mathrm{Corr}_{\mathrm{Sht}^{\mathrm{loc}}}(S(\widetilde{V}_1), S(\widetilde{V}_2))$. Note that the proof of Lemma 5.3.(3) and the definition of $\mathrm{loc}_p^{\nu, (m_1, n_1)} *$ imply that for a quadruple (m'_1, n'_1, m'_2, n'_2) of $(\mu_1 + \nu, \nu)$ -acceptable and $(\mu_2 + \nu, \nu)$ -acceptable integers, the functor $\mathrm{loc}_p^{\nu, (m_1, n_1)} *$ commutes with the connecting morphism in (4.12) (with μ_1, μ_2, λ fixed). Let $\nu \leq \nu'$ and (m'_1, n'_1, m'_2, n'_2) be a quadruple of non-negative integers satisfying appropriate acceptance conditions. The proper smooth base change shows that $\mathrm{loc}_p^{\nu, (m'_1, n'_1)} *$ commutes with enlarging ν to ν' . In addition, the proper smooth base change together with the construction of \mathfrak{C}_W show that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^\nu}((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell \langle d_2 \rangle)) & \xrightarrow{\mathfrak{C}_W} & \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^\nu}((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_2 \rangle)) \\ \downarrow i_* & & \downarrow i_* \\ \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^{\nu'}}((\mathrm{Sh}_{\mu_1}, \mathbb{Z}_\ell \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathbb{Z}_\ell \langle d_2 \rangle)) & \xrightarrow{\mathfrak{C}_W} & \mathrm{Corr}_{\mathrm{Sh}_{\mu_1|\mu_2}^{\nu'}}((\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_1 \rangle), (\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Z}_\ell} \langle d_2 \rangle)). \end{array}$$

Thus the map \mathfrak{C}_W is compatible with the enlargement of ν . Finally, by [15, Lemma A.2.8], the composition of maps $H_c^* \circ \mathfrak{C}_W$ commutes with enlarging ν to ν' . We complete the proof of the statement at the beginning of this paragraph.

Composing (6.7) with (6.11), we get a canonical map

$$(6.11) \quad \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \rightarrow \mathrm{Hom}_{\mathcal{H}^p}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_1 \rangle), \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_2 \rangle)).$$

The fact that (6.9) is compatible with the compositions of the source and target can be proved in an analogous way as [15, Lemma 7.3.12], and we omit the details. Then the action of \mathcal{J} naturally translates to the right hand side of (6.9) and upgrades it to our desired map

$$\mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}(\sigma)}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \rightarrow \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_1 \rangle), \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_2 \rangle)).$$

□

As discussed in *loc.cit*, the action of \mathcal{J} on $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_i \rangle)$ is expected to coincide with the usual Hecke algebra action, which may be understood as the Shimura variety analogue of V. Lafforgue's $S = T$ theorem (cf. [7]). We prove this in the case of Shimura sets.

Proposition 6.4. Let $\mathrm{Sh}_K(G, X)$ be a zero-dimensional Shimura variety. Then the action of \mathcal{J} on $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_i \rangle)$ is given by the classical Satake isomorphism.

Proof. Let $f \in \mathcal{J}$. Since the Shimura variety we consider is zero-dimensional, it follows from [15, A.2.3(5)] that the cohomological correspondence $\mathrm{loc}_p^*(\mathcal{S}_\mathcal{O}(f))$ can be identified with a \mathbb{Z}_ℓ -valued function on $\mathrm{Sh}_{\mu|\mu}$. By our construction of the map Spc , this function is given by the pullback of a function f' on $\mathrm{Sht}_{\mu|\mu}^{\mathrm{loc}} = G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)$. Corollary 5.5(2) thus implies that the function f' is exactly the function $\mathcal{S}_\mathcal{O}(f) \in H_{G, [\mathbb{Z}_\ell[p^{-1/2}, p^{1/2}]]}$ which is the image of f under the classical Satake isomorphism.

For any $n \in \mathbb{Z}^+$, take $W = \mathbb{Z}_\ell^n$. Recall our construction of \mathfrak{C}_W , the cohomological correspondence \tilde{f}_n is given by a finite direct sum of the function $\mathrm{loc}_p^*(\mathcal{S}_\mathcal{O}(f))$ since the Shimura variety we consider is a set of discrete points. Then the action of $\mathrm{Spc}(f)$ on $\mathrm{H}_c^*(\mathrm{Sh}_{\mu_i}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_i \rangle)$ is given by the classical Satake isomorphism. For general W , we take resolutions of it as in (6.10), and the statement follows from the case $W = \mathbb{Z}_\ell^n$. □

6.4. Non-vanishing of the geometric Jacquet-Langlands transfer. In Theorem 6.2.2, we constructed the geometric Jacquet-Langlands transfer

$$\mathrm{Spc} : \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \rightarrow \mathrm{Hom}_{\mathcal{H}^p \otimes \mathcal{J}}(\mathrm{H}_c^*(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_1 \rangle), \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_2 \rangle)).$$

It is natural to ask when this transfer map is nonzero. We discuss this issue in this section. The idea essentially follows from the discussion in [15, §7.4], and we briefly sketch it here.

Assume that $\mathrm{Sh}_{\mu_1, K_1}(G_1, X_1)$ is a zero dimensional Shimura variety. The Jacquet-Langlands transfer map induces the following map

$$\mathrm{JL}_{1,2}(\mathbf{a}) : \mathrm{H}_c^0(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell}) \rightarrow \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_2 \rangle),$$

for $\mathbf{a} \in \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_1, \widetilde{V}_2)$. Let $\mathbf{a}' \in \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_2, \widetilde{V}_1)$ be the morphism such that the induced map

$$\mathrm{JL}_{2,1}(\mathbf{a}') : \mathrm{H}_c^*(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_\ell}\langle d_2 \rangle) \rightarrow \mathrm{H}_c^0(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell})$$

is dual to $\mathrm{JL}_{1,2}(\mathbf{a})$ when viewing it as a cohomological correspondence (cf. [15, §A.2.18]).

The composition map $\mathrm{JL}_{2,1}(\mathbf{a}') \circ \mathrm{JL}_{1,2}(\mathbf{a})$ gives rise to an endomorphism of $\widetilde{V}_{\mu_1} \in \mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)$. By [15, Theorem 1.4.1], the hom spaces $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_1, \widetilde{V}_2)$ and $\mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma)}(\widetilde{V}_2, \widetilde{V}_1)$ are both finite projective \mathcal{J} -modules. Thus it makes sense to consider the determinant of the pairing

$$(6.12) \quad \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_2, \widetilde{V}_1) \otimes \mathrm{Hom}_{\mathrm{Coh}^{\hat{G}_{\mathbb{Q}_\ell}}(\hat{G}_{\mathbb{Q}_\ell}\sigma_p)}(\widetilde{V}_1, \widetilde{V}_2) \rightarrow \mathcal{J}.$$

In particular, this determinant can be regarded as a regular function on the stack $[\hat{G}_{\mathbb{Q}_\ell}\sigma_p / \hat{G}_{\mathbb{Q}_\ell}]$; for a detailed discussion on the pairing (6.12), see [16].

By Theorem 6.1.2 in *loc.cit*, we conclude the following result:

Theorem 6.5. *Let π_f be an irreducible \mathcal{H}_K -module, and let*

$$H_c^0(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell})[\pi_f] := \mathrm{Hom}_{\mathcal{H}_K}(\pi_f, H_c^0(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell})) \otimes \pi_f$$

denote the π_f -isotypical component. Then, the map

$$\mathrm{JL}_{1,2}(\mathbf{a}) : H_c^0(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell}) \rightarrow H_c^d(\mathrm{Sh}_{\mu_2}, \mathcal{L}_{W, \mathbb{Q}_\ell})$$

restricted to $H_c^0(\mathrm{Sh}_{\mu_1}, \mathcal{L}_{W, \mathbb{Q}_\ell})[\pi_f]$ is injective if the Satake parameters of π_f is general with respect to V_{μ_2} in the sense of [15].

REFERENCES

- [1] Bhargav Bhatt and Peter Scholze, *Projectivity of the witt vector affine grassmannian*, *Inventiones mathematicae* **209** (2017), no. 2, 329–423.
- [2] Alexandru Ghitza, *Siegel modular forms (mod p) and algebraic modular forms*, arXiv preprint math/0306224 (2003).
- [3] ———, *All siegel hecke eigensystems (mod p) are cuspidal*, arXiv preprint math/0511728 (2005).
- [4] David Helm, *A geometric jacquet-langlands correspondence for $u(2)$ shimura varieties*, *Israel Journal of Mathematics* **187** (2012), no. 1, 37–80.
- [5] David Helm et al., *Towards a geometric jacquet-langlands correspondence for unitary shimura varieties*, *Duke Mathematical Journal* **155** (2010), no. 3, 483–518.
- [6] Mark Kisin, *Integral models for shimura varieties of abelian type*, *Journal of the American Mathematical Society* (2010).
- [7] Vincent Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de langlands globale*, *Journal of the American Mathematical Society* **31** (2018), no. 3, 719–891.
- [8] Ruochuan Liu and Xinwen Zhu, *Rigidity and a riemann-hilbert correspondence for p -adic local systems*, *Inventiones mathematicae* **207** (2017), no. 1, 291–343.
- [9] James S Milne, *Canonical models of (mixed) shimura varieties and automorphic vector bundles*, *Automorphic forms, Shimura varieties, and L-functions* **1** (1990), 283–414.
- [10] ———, *Introduction to shimura varieties*, *Harmonic analysis, the trace formula, and Shimura varieties* **4** (2005), 265–378.
- [11] Ivan Mirković and Kari Vilonen, *Geometric langlands duality and representations of algebraic groups over commutative rings*, *Annals of mathematics* **166** (2007), no. 1, 95–143.
- [12] Kenneth Ribet et al., *Bimodules and abelian surfaces*, *Algebraic number theory* **17** (1989), 359–407.
- [13] Jean-Pierre Serre and R Livné, *Two letters on quaternions and modular forms (mod p)*, *Israel Journal of Mathematics* **95** (1996), no. 1, 281–293.
- [14] Adrian Vasiu, *Good reductions of shimura varieties of hodge type in arbitrary unramified mixed characteristic, part i*, arXiv preprint arXiv:0707.1668 (2007).
- [15] Liang Xiao and Xinwen Zhu, *Cycles on shimura varieties via geometric satake*, arXiv preprint arXiv:1707.05700 (2017).
- [16] ———, *On vector-valued twisted conjugate invariant functions on a group*, arXiv preprint arXiv:1802.05299 (2018).
- [17] Jize Yu, *The integral geometric satake equivalence in mixed characteristic*, arXiv preprint arXiv:1903.11132 (2019).
- [18] Xinwen Zhu, *An introduction to affine grassmannians and the geometric satake equivalence*, arXiv preprint arXiv:1603.05593 (2016).
- [19] ———, *Affine grassmannians and the geometric satake in mixed characteristic*, *Annals of Mathematics* (2017), 403–492.
- [20] ———, *Geometric satake, categorical traces, and arithmetic of shimura varieties*, arXiv preprint arXiv:184.07375 (2018).

INSTITUTE FOR ADVANCED STUDY, SCHOOL OF MATHEMATICS, 1 EINSTEIN DRIVE, PRINCETON, NJ 08540

Current address: Department of Mathematics, Caltech, 1200 East California Boulevard, Pasadena, CA 91125

E-mail address: jize.yu.math@gmail.com