RELATIVE LANGLANDS

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This is based on joint work with Yiannis Sakellaridis and Akshay Venkatesh. The general plan is to explain a connection between physics and number theory which goes through the intermediary: extended topological field theory (TFT). The moral is that boundary conditions for $\mathcal{N}=4$ super Yang-Mills (SYM) lead to something about periods of automorphic forms.

Slogan: the relative Langlands program can be explained via relative TFT.

1. Periods of automorphic forms on $\mathbb{H}$

First we provide some background from number theory. Recall we can picture the upper-half-space $\mathbb{H}$ as in fig. 1.

We are thinking of a modular form φ as a holomorphic function on $\mathbb{H}$ which transforms under the modular group $\mathrm{SL}_2(\mathbb{Z})$, or in general some congruent subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$, like a $k/2$-form (differential form) and is holomorphic at ∞.

We will consider some natural measurements of φ. In particular, we can "measure it" on the red and blue lines in fig. 1. Note that we can also think of $\mathbb{H}$ as in fig. 2, where the red and blue lines are drawn as well.

Since φ is invariant under $\mathrm{SL}_2(\mathbb{Z})$, it is really a periodic function on the circle, so it has a Fourier series. The niceness at ∞ condition tells us that it starts at 0, so we get:

$$\varphi = \sum_{n\geq 0} a_n q^n \tag{1}$$

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Notes by: Jackson Van Dyke, all errors introduced are my own.

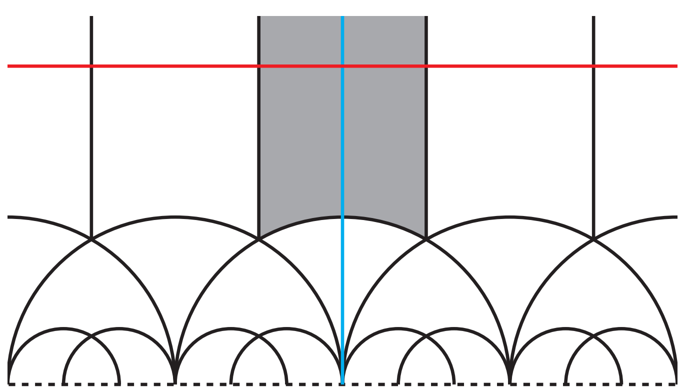

Figure 1. Fundamental domain for the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$ in gray.

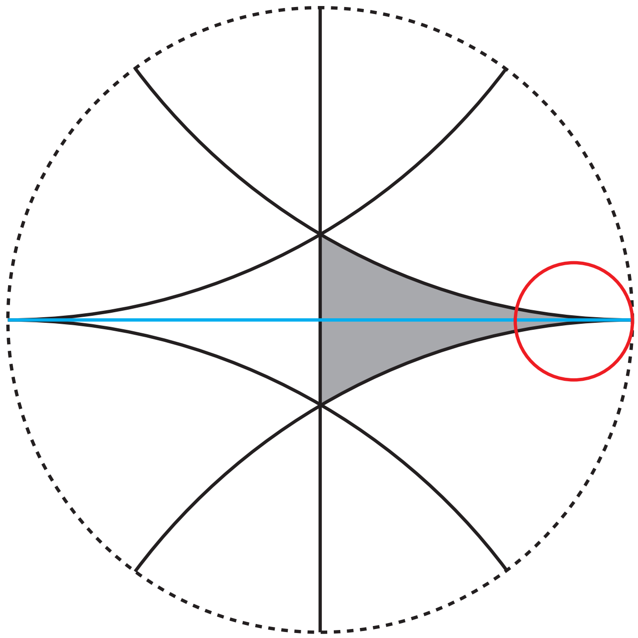

FIGURE 2. The upper half plane viewed as the disk, where the fundamental domain is still in gray. The red and blue lines are the same as in fig. 1.

where q is the exponential of the coordinate on $\mathbb{H}$.

We will consider three measurements.

1. The Eisenstein period (or G/N period) is

$$\int_{\text{red}} \varphi \, dt = a_0 \ . \tag{2}$$

2. The Whittaker period (or $(G/N, \psi)$ period) is the Fourier coefficient

$$a_1 = \int \varphi e^{\psi(t)} \, dt \tag{3}$$

   for the character ψ. When φ is a cusp form,[1] we can rescale such that $a_1 = 1$. This is called the Whittaker normalization condition.[2]

3. The Hecke period (or G/T period) is as follows. The idea is to integrate over the blue curve. This converges if φ is a cusp form. Slightly more general, we can take

$$\int_0^\infty \varphi\,(iy)\, y^s \frac{dy}{y} \ . \tag{4}$$

[1]This means $a_0 = 0$.

[2]This is important for matching forms with Galois representations.

This is the definition of the L-function:

$$\frac{\Gamma(s)}{(2\pi)^s} L(\varphi, s) = \int_0^\infty \varphi(iy)\, y^s \frac{dy}{y} \tag{5}$$

$$= \sum \frac{a_n}{n^s} \tag{6}$$

where these are the same Fourier coefficients as before. This only converges for s in some right-half plane, then this integral representation can be used to show this has an analytic continuation to all values of s, and has a functional equation in s. When we talk about L-functions we really care about values of the L function. This puts this on the same footing as the previous two periods. A particularly interesting value is the average:

$$L(\varphi, 1) = 2\pi \int_0^\infty \varphi(iy)\ dy\ . \tag{7}$$

We are really interested in these modular forms because they can be matched to Galois representations. To do this, we need the extra condition that φ is an eigenfunction for the Hecke operator. In this case, it can be associated to a two-dimensional representation

$$\rho\colon\ \mathrm{Gal}\left(\overline{\mathbb{Q}}, \mathbb{Q}\right) \to \mathrm{GL}_2\left(\overline{\mathbb{Q}}\right)\ . \tag{8}$$

Then the upshot is that these three periods are meaningful measurements of this representation. In particular we get the following.

1. $a_0 \neq 0$ corresponds to ρ being reducible.
2. When $a_0 = 0$, and we normalize to $a_1 = 1$, after adding the appropriate adjectives we should get a bijection between such automorphic forms and representations.
3. We get an equality $L(\varphi, s) = L(\rho, s)$, where the L function associated to ρ is something like

$$L(\rho, s) \text{ “}=\text{''} \prod_p \frac{1}{\det\left(1 - p^{-s}\rho(F)\right)} \tag{9}$$

where F is the Frobenius conjugacy class.

Recall that 1 over the characteristic polynomial of an operator $F \in \mathrm{End}\, V$:

$$\frac{1}{\det(1 - tF)} \tag{10}$$

can be thought of as the graded trace of F on $\mathrm{Sym}^\bullet V$. The characteristic polynomial itself can be thought of as the graded trace on the exterior algebra. So these factors in $L(\rho, s)$ somehow come in as characters of symmetric algebras. This L function encodes a huge amount of information.

Example 1. If we have weight two modular forms, which correspond to elliptic curves, this particular value of the L function, $L(\varphi, 1)$, has to do with the Birch-Swinnerton-Dyer conjectures. In particular, the vanishing of this tells us whether we have infinitely many vanishing points on an elliptic curve.

2. General automorphic forms

2.1. Arithmetic locally symmetric spaces. What we have been dealing with so far has been $\mathbb{H} = \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2$, modded out by $\mathrm{SL}_2 \mathbb{Z}$:

$$\mathrm{SL}_2 \mathbb{Z} \backslash \mathbb{H} = \mathrm{SL}_2 \mathbb{Z} \backslash \mathrm{SL}_2(\mathbb{R}) / \mathrm{SO}_2 \ . \tag{11}$$

This is an example of an arithmetic locally symmetric space. Now we can generalize this to a group G, defined over some field F, say $F = \mathbb{Q}$, which is split and reductive, e.g. $\mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_n$. To any such G we can attach a version of this arithmetic locally symmetric space, and this is the subject of the theory of automorphic forms. In particular we get a space

$$[G]_K = G(F) \backslash G(\mathbb{A}) / K \tag{12}$$

where we think of F as a number field (e.g. $\mathbb{Q}$), $\mathbb{A}$ is the adeles of F, and K is some compact subgroup. So $G(\mathbb{A})$ is some kind of restricted product of $G(F_v)$, and then inside of each $G(F_v)$ we have $G(\mathcal{O}_v)$. For almost all v, K is going to be this subgroup. So the claim is that if we do this for SL_2, and the maximal version of K, we get our space $\mathrm{SL}_2(\mathbb{Z}) \backslash \mathbb{H}$ from before.

2.2. General automorphic forms. Now we have a notion of an automorphic form, which is a generalization of modular form. These are functions $\varphi \in L^2([G]_K)$ which are Hecke eigenfunctions.

Remark 1. Fourier theory on $\mathbb{R}_x$ concerns itself with decomposing L^2 functions on $\mathbb{R}$ in terms of these special characters e^{itx}, indexed by this parameter t, which are eigenfunctions for differentiation. So we should think of automorphic forms as being analogous to these characters.

The Langlands philosophy tells us that these forms φ should correspond to

$$\rho : \mathrm{Gal}\left(\overline{\mathbb{Q}}/\mathbb{Q}\right) \to G^\vee\left(\overline{\mathbb{Q}}_l\right) \ . \tag{13}$$

In particular, the data of the eigenvalues is encoded in conjugacy classes of $G^\vee$. So this tells is where Frobenius classes should go.

2.3. Periods of automorphic forms. Given a subgroup $H \subset G$, we get this locally-symmetric space

$$[H] \subset [G] \ . \tag{14}$$

The idea is, that given a modular form φ, we get an H-period:

$$\mathcal{P}_H(\varphi) = \int_{[H]} \varphi \ . \tag{15}$$

Example 2. For SL_2, the Eisenstein period a_0 was given by:

$$H = N = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \ . \tag{16}$$

Then the Whittaker period a_1 is given by upper triangular matrices twisted by some character ψ. Then the Hecke period is given by:

$$H = \mathbb{G}_m = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \ . \tag{17}$$

2.4. **Relationship between periods and L-functions.** These periods should be thought of as measurements of the automorphic forms, and they tend to have a lot of meaning (on both sides of Langlands).

On the other hand, Langlands tells us that we have a notion of an automorphic L-function, which is as follows. It is labelled by an automorphic form φ, a representation $V \in \mathbf{Rep}(G^\vee)$, and a variable s (which we will usually suppress). It is given by

$$L(\varphi, V, s) = \prod \frac{1}{\text{char. poly. of conj. class in } G^\vee \circlearrowright V} . \tag{18}$$

Recall this conjugacy class tells us the Hecke eigenvalues. Again, this will match with an L-function coming from a Galois representation:

$$L(\rho, V, s) . \tag{19}$$

It makes sense these will match since the Hecke eigenvalues correspond to conjugacy classes of $V^\vee$, so these are built from the same data.

But when we talked about an L function of a modular form, we defined it as a period, i.e. as an integral. In fact, there is a general principle:

good properties of L-functions come from realizations as periods.

The immediate problem is which period we want it to be realized as. Periods come from subgroups $H \subset G$, and L functions come from representation of $G^\vee$. There is no way to line these things up. They are somehow two completely different classes of data. There are plenty of examples where particular periods are related to particular L-functions, but we're trying to understand a systematic idea of which period should relate to which kind of L-function.

The punchline will be as follows. In the world of automorphic forms, we have periods (subgroups $H \subset G$) and in the world of Galois representations, we have L-functions ($V \in \mathbf{Rep}(G^\vee)$). Then we want to expand our interest to a bigger class of objects on both sides so that these things live in bijection with one another.

2.5. **Spherical varieties.** In the SL_2 case we only considered certain periods given by integrating over certain things. We want a generic way of understanding when a subgroup H is good to integrate over in the sense that it gives us a useful period. In [14–17], Sakellaridis and Venkatesh complete a systematic study of periods. The key property of the subgroup H is as follows. Consider the homogeneous space $X = G/H$. So X is a variety with an action of G. Then the key property is that we want X to be a spherical variety. In this case we say H is a spherical subgroup. This is some kind of strong finiteness property on X. This is the nonabelian version of a toric variety.

Example 3. If G is the torus, this is just asking for a toric variety.

Example 4. For $G = \mathbb{G}_m$, the toric variety which appears is $\mathbb{A}^1$. So this is the study of functions on $\mathbb{A}^1$, and decomposing them under the $\mathbb{G}_m$ action. This was Tate's thesis.

The definition of a spherical variety either asks for an open orbit for the Borel subgroup, or it has a multiplicity 1 property for algebraic representations. In the case of G/H, this asks for every irreducible to appear at least once. So this is a nice finiteness condition.

Examples of spherical varieties are given by the following.

- Toric varieties $G = T$
- Flag varieties G/B
- Symmetric spaces G/K
- "Group case" $G = (H \times H) \circlearrowright H$
- Whittaker G/N twisted by character ψ
- $\mathrm{SL}_n \circlearrowright \mathbb{A}^n$, $\mathrm{GL}_n \times \mathrm{GL}_n \circlearrowright \mathrm{Mat}_{n\times n}$
- Branching problems (Gan-Gross-Prasad)
  - $\mathrm{GL}_{n+1} \times \mathrm{GL}_n \circlearrowright \mathrm{GL}_{n+1}$
  - $\mathrm{SO}_{n+1} \times \mathrm{SO}_n \circlearrowright \mathrm{SO}_{n+1}$
  - $\mathrm{U}_{n+1} \times \mathrm{U}_n \circlearrowright \mathrm{U}_{n+1}$

And a bunch more.

Building off of [6, 10, 11], Sakellaridis and Venkatesh take $G \circlearrowright X$ and attach certain Langlands dual data. Then they use this data to describe and control the theory of periods of automorphic forms. For example, we might wonder when

$$\mathcal{P}_X(\varphi) \neq 0 \ . \tag{20}$$

This is called X distinction.

3. Hamiltonian G-spaces

Recall we wanted to understand some larger class of objects which contained periods, and some larger class of objects which contains L-functions. The proposal is that, in both cases, the expanded class of objects is the collection of boundary condition/boundary theories for a TFT.

On the side containing periods, these theories will correspond to Hamiltonian group actions of G, and on the side of L-functions these will correspond to Hamiltonian actions of $G^\vee$:

$$\begin{tikzcd} \text{Periods} \arrow[d, hook] & L-\text{functions} \arrow[d, hook] \\ \text{Hamiltonian } G \text{ spaces} \arrow[d, hook] & \text{Hamiltonian } G^\vee \text{ spaces} \arrow[d, hook] \\ \text{Boundary theories} & \text{Boundary theories} \end{tikzcd} \ . \tag{21}$$

The idea is to consider

$$H \quad \rightsquigarrow \quad X = G/H \quad \rightsquigarrow \quad M = T^*X \ . \tag{22}$$

Them M has a G action, but it also has a moment map

$$M \to \mathfrak{g}^* \tag{23}$$

which is equivariant and generates this action infinitesimally. Once we say this in terms of Hamiltonian G actions, some things start to pop out.

Remark 2. There are examples which are not cotangent bundles, so it is worthwhile to pass to this generality. The theory of θ-correspondence in number theory doesn't fit into the homogeneous space picture, but does fit into the Hamiltonian G space picture.

3.1. **Gaiotto-Witten.** In [5] the authors study boundary conditions/theories for $d = 4$ $\mathcal{N} = 4$ (maximally supersymmetric) SYM. In particular, they find that for every Hamiltonian G space there is a boundary condition of this theory. The "most important thing" about $d = 4$ SYM is electromagnetic duality. It was explained in [8] that this is related to the geometric program Langlands. The great thing about this theory is that the associated to G is equivalent to the theory associated to $G^\vee$. This tells us that the boundaries for these two groups should correspond. So we might hope that for our favorite Hamiltonian G space, that its dual is a Hamiltonian $G^\vee$ space. In particular, for $H \subset G$ and $V \in \mathbf{Rep}\left(G^\vee\right)$, we have

$$\begin{tikzcd} \text{boundaries}_G \arrow[d, leftrightarrow] & \text{Hamiltonian } G \text{ spaces} \arrow[l, hook'] \arrow[d, dashed, leftrightarrow] & \left\{T^*\left(G/H\right)\right\} \arrow[l, hook'] \\ \text{boundaries}_{G^\vee} & \text{Hamiltonian } G^\vee \text{ spaces} \arrow[l, hook'] & \left\{T^*V\right\} \arrow[l, hook'] \end{tikzcd} \quad . \tag{24}$$

Recall V was the datum used to get an L-function, so now we are just viewing this datum as a Hamiltonian $G^\vee$ space. So $H \subset G$ and $V \in \mathbf{Rep}\left(G^\vee\right)$ both somehow sit inside of boundaries of their respective theories, and we can use this interpretation to explain the relationship between periods and L-functions.

4. Topological field theory

The first thing to note is that we are using field theory as some kind of metaphor. So the examples we will see won't quite satisfy all of the axioms of a fully extended topological field theory, but there are a lot of structures which do fit in. Let Z be an extended $4d$ TFT. These are things which output data as in table 1. Then there are relations among these things. For example, when we write 3-manifold, we really mean a closed 3-manifold. Then the way the 3-dimensional things talk to 2-dimensional things is via boundaries.[3] So if we have a 3-manifold M with boundary a surface $\Sigma = \partial M$, we really attach an element

$$Z\left(M\right) \in Z\left(\partial M\right) \ . \tag{25}$$

And if we have a surface $\widetilde{\Sigma}$ with n punctures, i.e. n boundary component homeomorphic to S^1, we get an object

$$Z\left(\widetilde{\Sigma}\right) \in \otimes_{i=1}^n Z\left(S^1\right) \ . \tag{26}$$

Another useful thing that the field theory has, which will be useful, is structure coming from "defects". So, for example, if we have $Z\left(M^3\right)$, this comes with a big commutative algebra of operators, called loop operators (or line defects). The idea is that

$$Z\left(M \times I\right) : M \to M \tag{27}$$

and if we carve out a loop from the interior we get a commutative algebra acting on M. The point is that this has become a cobordism from the disjoint union of M with the thing we carved out, and Z turns disjoint unions into tensor products. So we get an action of whatever Z assigned to the thing we carved out to $Z\left(M\right)$. The point is that this is where the Hecke operators will come from.

[3] Really Z is a functor defined on a bordism category.

Table 1. Output of a four-dimensional topological field theory.

Dimension	Output
4	Number $\in \mathbb{C}$ (rarely well-defined algebraically, requires analysis)
3	(dg) vector space
2	(dg) category
1	$(\infty, 2)$-category
0	$(\infty, 3)$-category? (rarely understood)

4.1. **Boundary theory.** A boundary theory is a codimension 1 defect. This is somehow the "richest" of all defects. One way to say this formally, is that it is a morphism

$$B\colon \mathbf{1} \to Z \tag{28}$$

where $\mathbf{1}$ is the trivial theory. One way to think about this is a map from

$$B : 1 \to Z(\bullet)\ . \tag{29}$$

So this can be thought of as an object of the higher category $Z(\bullet)$.

A crude phrasing of the cobordism hypothesis is saying that $Z(\bullet)$ determines the whole theory. So if we think of the objects of $Z(\bullet)$ as boundary conditions, this is saying that boundary conditions determine the theory. So morphisms B as above, or objects of $Z(\bullet)$ are the boundary conditions of Z, or alternatively the theories determined by these objects, i.e. *boundary theories*. So the boundary theory is somehow a TFT of one dimension lower.

The point is that for any M, the boundary condition gives me an element

$$B_M : 1 \to Z(M)\ . \tag{30}$$

So it is a rule for choosing an element of what Z assigns to something. There's even more to them. They enable you to talk about manifolds which have a boundary component labeled by B. This should be thought of as a new class of "closed manifolds".

It is nice to think of these boundary theories as elements, but with enough duality, we can think of the dual version, which is a functional:

$$Z(M) \to 1\ . \tag{31}$$

This is how we will realize periods. Recall these were functionals on a space of automorphic forms, so we want to think of that functional as coming from a boundary condition.

5. Langlands program: an arithmetic TFT perspective

5.1. **Three-dimensional setting.** Recall André Weil's Rosetta stone tells us that number fields, i.e. finite $F/\mathbb{Q}$, are nicely analogous to function fields. Explicitly this is an analogy between Spec of the associated ring of integers of the number

Table 2. We will view objects relevant for the settings in the right column as manifolds of the corresponding dimension in the left column, insofar as they are getting plugged into a TFT.

Dimension	Settings	
4	Periods, L-functions, trace formula	
3	<u>Global arithmetic</u> number fields, curves $C/\mathbb{F}_q$	
2	<u>Local arithmetic</u> local fields, e.g. $\mathbb{Q}_p$, $\mathbb{F}_q((t))$	<u>Global geometric</u> curves $\overline{C}/\overline{\mathbb{F}}_q$
1		<u>local geometric</u> punctured disks $\overline{\mathbb{F}}_q((t))$, $\mathbb{C}((t))$

field and smooth projective curves $C/\mathbb{F}_q$. Then there is another analogy between both of these and Riemann surfaces. All together we have:

$$\begin{tikzcd} \text{number fields } F/\mathbb{Q} \arrow[rr, leftrightarrow] \arrow[dr, leftrightarrow] & & \text{function fields } C/\mathbb{F}_q \arrow[dl, leftrightarrow] \\ & \Sigma \text{ Riemann surface} & \end{tikzcd} \quad . \tag{32}$$

Now we want to take the point of view that there was a chip missing from this Rosetta stone. And what the original text had a Σ bundle over S^1 in place of Σ. Now in the case of a curve $C/\mathbb{F}_q$, we could have looked at $\overline{C}/\overline{\mathbb{F}}_q$. Then we recover this curve, by thinking of fixed points of the Frobenius automorphism σ. There isn't really an analogue of Frobenius here, but we can equip Σ with an diffeomorphism σ and then consider the mapping torus

$$\Sigma \times I/\left((x,0) \sim (\sigma(x), 1)\right) \tag{33}$$

which is of course a Σ bundle over S^1. The point being that we want to think of a Riemann surfaace as being analogous to a curve over an algebraically closed field of positive characteristic, rather than over a finite field.

This fits well with the analogy between number fields (Spec of rings of integers) as 3-manifolds. This is known as the knots and primes analogy. This can be attributed to many people such as Mazur [12], Manin, Morishita [13], Kapranov [7], and Reznikov. The recent work [3, 9] of Minhyong Kim plays a central role. This work deals with an arithmetic analogue of Chern-Simons theory as a way of encoding arithmetic information. So in the Rosetta stone, we really want to think of a number field as a three-manifold.

The upshot is that we are thinking of all three objects in the Rosetta stone as three-manifolds. For example we will view $\operatorname{Spec}\mathbb{Z}$ as some kind of three-manifold missing some point at ∞. Similarly a curve over a finite field will be a three-manifold, and a Σ-bundle over S^1 is of course a three-manifold. This fits with table 2 in the sense that number fields and curves over function fields are global arithmetic objects which we are thinking of as three-manifolds.

TABLE 3. Langlands dual groups

G	$G^\vee$
GL_n	GL_n
SL_n	PGL_n
SO_{2n}	SO_{2n}
SO_{2n+1}	Sp_{2n}

5.2. **Two-dimensional setting.** As is indicated in table 2, in the world of 2-manifolds, there will be two classes involved. Local arithmetic ones, and global geometric ones. The local arithmetic ones are as follows. For a local field such as $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$ we can think of them as 2-manifolds. Formally, this is a result of Mazur [12], based on work of Artin and Verdier, saying that they have Étale cohomological dimension 2. We can think of $\mathbb{Q}_p$ as living in the boundary at p of:

$$\operatorname{Spec} \mathbb{Z}\left[\frac{1}{p}\right] . \tag{34}$$

We can think of this as removing a knot corresponding to the prime p, and then the boundary of the tubular neighborhood of that knot is this 2-manifold.

The global geometric 2-manifolds will be more honest 2-manifolds such as $\overline{C}/\overline{\mathbb{F}}_q$. In the Weil dictionary this is the analogue of a compact Riemann surface.

5.3. **One-dimensional setting.** Following this dictionary, there is a local geometric setting which gives 1-manifolds. Namely, if we consider something like a punctured disk over $\overline{\mathbb{F}}_q$, or over $\mathbb{C}$:

$$\operatorname{Spec} \overline{\mathbb{F}}_q((t)) \qquad\qquad \operatorname{Spec} \mathbb{C}((t)) \tag{35}$$

we get 1-manifolds.

5.4. **Statement of Langlands.** So now we can apply a TFT to these global arithmetic three-manifolds, these two types of 2-manifolds, and these local geometric 1-manifolds. Then the top row of table 2, we should get number for things like periods and L-functions. This is kind of the hardest part to make sense of, but this is where we would like to eventually get.

Now we can say what the Langlands program actually asserts. We start with G, with Langlands dual $G^\vee$. We should think of G as living over $\mathbb{Q}$, and $G^\vee$ as living over $\overline{\mathbb{Q}}_l$ or $\mathbb{C}$. Examples are in table 3. Now that we have the dictionary developed above, we we can phrase geometric and classical Langlands in the same way. At least schematically, the Langlands program is an equivalence of four-dimensional TFT's:

$$\mathcal{A}_G \simeq \mathcal{B}_{G^\vee} . \tag{36}$$

$\mathcal{A}_G$ is called the automorphic theory, and $\mathcal{B}_{G^\vee}$ is called the spectral theory. $\mathcal{A}_G$ has something to do with the moduli space of G-bundles, Bun_G, and $\mathcal{B}_{G^\vee}$ has something to do with the moduli space of $G^\vee$ local systems, $\mathbf{Loc}_{G^\vee}$.

5.4.1. *Galois side.* From the number theory point of view, the Galois (or spectral or B) side is very complicated, and there is a sense in which this is all about answering questions about Galois groups. But from the point of view of the structure of the field theories, the B-side is easier to explain. The idea is that $\mathcal{B}_{G^\vee}$ is linearize spaces of Langlands parameters, i.e.

$$\mathbf{Loc}_{G^\vee}(M) \ . \tag{37}$$

These are also known as character stacks or representation spaces of these groups. In particular this consists of continuous representations of the Galois group, i.e. Étale fundamental groups into $G^\vee\left(\overline{\mathbb{Q}}_l\right)$. This is what local systems are all about: representations of π_1. And in number theory, the goal is to study Galois groups by studying their continuous l-adic representations.

Warning 1. One has to be extremely careful about what is going on at $l = p$, and at ∞. But we will sweep this under the rug. This can be dealt with by saying the correct adjectives.

A more serious issue, outside of the Riemann surface situation, is that for general M, these spaces $\mathbf{Loc}_{G^\vee}(M)$ don't exist.[4] All people really have access to are points of these spaces, and deformations of them.

We mean linearization in the following sense. For a three-manifold, i.e. one of these global arithmetic objects, we are supposed to attach a vector space of functions on $\mathbf{Loc}_{G^\vee}(M)$. To a 2-manifold , e.g. a local field, we attach a (derived) category of (quasi-)coherent sheaves on $\mathbf{Loc}_{G^\vee}(M)$.

5.5. **Automorphic side.** Recall we generalized fig. 2 to general locally symmetric spaces $[G]_K$. For K maximal compact (the unramified version) we write $[G]_{\mathrm{unr}}$. This is what we will study on the automorphic side. Recall automorphic forms live in

$$L^2\left([G]_K\right) = L^2\left(G(F)\backslash G(\mathbb{A})/K\right) \ . \tag{38}$$

Notice, at a very coarse level, that this is a vector space, which took in the data of a number field. In particular, we want to think of this as the vector space which $\mathcal{A}_G$ assigns to this arithmetic 3-manifold F.

This beast seems kind of unfriendly, but André Weil provides us with a nice realization. In the case that F is not a number field, but rather a function field of a curve, this quotient space has a beautiful description. In particular, for $C/\mathbb{F}_q$ we get

$$[G]_{\mathrm{unr}} = \mathrm{Bun}_G(C)(\mathbb{F}_q) \ . \tag{39}$$

So we have this space $[G]_{\mathrm{unr}}$, which we were thinking of as an analogue of $\mathbb{H}$, and now we want to think of them as Bun_G of the associated three-manifold. The idea is that this space is playing the role of the $\mathbb{F}_q$ points of Bun_G, which is enough to get this vector space of functions.

The takeaway is that:

$$\mathcal{A}_G\,(\text{global arithmetic}) = L^2\left([G]_{\mathrm{unr}}\right) \ . \tag{40}$$

Then the claim is that Hecke operators come naturally from the field theory.

[4] In the sense that for a Riemann surface we get a nice algebraic stack.

5.6. **Langlands reciprocity.** Very coarsely, what Langlands reciprocity is trying to say, is that

$$\mathcal{A}_G(M) \leftrightarrow \mathcal{B}_{G^\vee}(M) \tag{41}$$

compatibly with Hecke operators. So before we had these automorphic forms which are eigenfunctions of the Hecke operators, and on the $\mathcal{B}$-side, the Hecke operators just act via multiplication, but eigenfunctions for multiplication are like functions on a point. So points on the right correspond to eigenfunctions on the left, i.e. Galois representations correspond to automorphic forms as we expect.

In number theory these things only make sense at the level of deformation spaces, and they go under the name of the $R = T$ theorems.[5]

5.7. **Geometric global.** This is the subject of the geometric Langlands program as it is usually formulated. So we start with either a Riemann surface $\Sigma/\mathbb{C}$ or a curve $\overline{C}/\overline{\mathbb{F}}_q$. We know $\mathcal{A}_G$ somehow concerns Bun_G. We can take $\mathrm{Bun}_G\,\Sigma$, but we're supposed to get a category, so we take the category of sheaves on it:

$$\mathcal{A}_G(\Sigma) = \mathbf{Shv}(\mathrm{Bun}_G\,\Sigma)\ . \tag{42}$$

These are some type of constructible l-adic sheaves. E.g. in the de Rham version this is a category of $\mathcal{D}$-modules. The point being that this isn't a category of quasi-coherent sheaves, it's a more topological version. Then the geometric Langlands conjecture identifies this with the $\mathcal{B}$-side assignment, which is a category of quasi-coherent sheaves on $\mathbf{Loc}_{G^\vee}\Sigma$. In symbols, the duality is:

$$\mathcal{A}_G(\Sigma) = \mathbf{Shv}(\mathrm{Bun}_G\,\Sigma) = \mathbf{QC}^!_{\mathcal{N}}(\mathbf{Loc}_{G^\vee}\Sigma) = \mathcal{B}_{G^\vee}(\Sigma)\ . \tag{43}$$

The shriek and $\mathcal{N}$ indicate that this category is, strictly speaking, a category of ind-coherent sheaves with nilpotent singular support. This equivalence should also respect Hecke operators.

So far, we have only dealt with closed manifolds. Instead, we can consider manifolds with boundary, which then corresponds to the ramified version of the geometric Langlands conjecture. In the language of field theory, the data which determines these missing points is what is attached to the circle. So the 2-category attached to the circle, is the kind of data which deals with ramification.

5.8. **Arithmetic local.** Consider a local field $\mathbb{F}_v$, e.g. $\mathbb{Q}_p$ or $\mathbb{F}_q((t))$. Then we are supposed to construct a category $\mathcal{A}_G(F_v)$. Inside of this we have the category of smooth representations:

$$\mathcal{A}_G(F_v) \supset \mathbf{Rep}^{\mathrm{sm}}(G(F_v))\ . \tag{44}$$

On the other side, we have

$$\mathcal{B}_{G^\vee}(F_v) = \mathbf{QC}(\mathbf{Loc}_{G^\vee})\ . \tag{45}$$

Then the local Langlands correspondence is an embedding:

$$\mathbf{Rep}^{\mathrm{sm}}(G(F_v)) \hookrightarrow \mathbf{QC}(\mathbf{Loc}_{G^\vee})\ . \tag{46}$$

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[5] One is supposed to write $R = T$, but really R goes on the RHS of (41) and the T goes on the LHS of (41).

Even for torus these are not the same. There is a nice thing we can write which is an equality:

$$\begin{tikzcd} \mathbf{Shv}\left(\mathrm{Bun}_g\left(FF\right)\right) \arrow[r, equal] & \mathcal{A}_G\left(F_v\right) & \mathbf{Rep}^{\mathrm{sm}}\left(G\left(F_v\right)\right) \arrow[l, hook'] \\ & \mathcal{B}_{G^\vee}\left(F_v\right) \arrow[r, equal] & \mathbf{QC}\left(\mathbf{Loc}_{G^\vee}\right) \end{tikzcd} \tag{47}$$

where FF is the Fargues-Fontaine curve. This is the content of Fargues' conjecture, a restatement of the Langlands conjecture. This is kind of the modern version of the local Langlands conjecture.

The reason people study local Langlands is because, globally, there is always ramification (i.e. you're never really looking at $\operatorname{Spec}\mathbb{Z}$ with no primes removed, or in terms of modular forms, you're never looking at $\mathrm{SL}_2\left(\mathbb{Z}\right)$, you're looking at a congruent subgroup) and then you need to specify some local data. The local data is given by the 2-manifold which is the boundary of the 3-manifold, so you need the local Langlands correspondence.

6. Local Langlands

What we want to talk about are boundary theories. So boundary conditions for both the $\mathcal{A}$ and $\mathcal{B}$ theories. The observation, with Sakellaridis and Venkatesh, is that the data of a spherical variety (or the Hamiltonian G-space $M = T^*X$) should be thought of as defining a boundary theory $\mathcal{P}_X$ for the automorphic theory $\mathcal{A}_G$. And that the data of the dual group $G^\vee$ acting on the dual variety $G^\vee \circlearrowright X^\vee$ (or the Hamiltonian $G^\vee$-space[6] $M^\vee = T^*X^\vee$) defines a boundary theory $\mathcal{L}_{X^\vee}$ for $\mathcal{B}_{G^\vee}$. Then we want to match those up.

The data that is attached to a spherical variety by Sakellaridis and Venkatesh attach to a spherical variety X, can be reinterpreted as giving the data of a Hamiltonian $G^\vee$-space. As a result, there is an explicit dictionary which, given a spherical variety, tells us which space we should be looking at. So we get a long list of predicted duals in table 4.

Explicitly, the Langlands correspondence is an equivalence of TFT's, and our conjecture says that the boundary theories $\mathcal{P}_X$ and $\mathcal{L}_{X^\vee}$ should match, where these varieties are related in some specific way:

$$\begin{array}{ccc} \mathcal{A}_G & \simeq & \mathcal{B}_{G^\vee} \\ \Cup & & \Cup \\ \mathcal{P}_X & \longleftrightarrow & \mathcal{L}_{X^\vee} \end{array} . \tag{48}$$

The important thing is, that we want an equivalence of field theories which matches boundary conditions. So this kind of informal statement encodes a lot of more precise conjectures in each of the settings we had. This is what we want to call the relative Langlands duality.

6.1. $\mathcal{A}$-side.

[6]One needs some extra structure here which is weaker than a cotangent bundle, but is still some extra polarization data. Some kind of metaplectic type structure.

TABLE 4. Dual spherical varieties

	G	$G^\vee$
Usual (non-relative) Langlands	Group	Group
Tate's thesis	$\mathbb{G}_m \circlearrowright \mathbb{A}^1$	$\mathbb{G}_m \circlearrowright \mathbb{A}^1$
	$\mathrm{PGL}_2 / \mathbb{G}_m$ (Hecke)	$\mathrm{SL}_2 \circlearrowright \mathbb{A}^2$ (Standard L-function)
Tamagawa #	Point (Neumann)	Whittaker (Nahm pole)
Whittaker normalization / Frenkel-Ngô [4]	Whittaker	point
	G/B (Eisenstein)	$G^\vee / B^\vee$ (Eisenstein)
	$\mathrm{SO}_{2n} \times \mathrm{SO}_{2n+1} / \mathrm{SO}_{2n+1}$ (Gan-Gross-Prasad)	$\mathrm{SO}_{2n} \times \mathrm{Sp}_{2n}, \mathrm{std} \otimes \mathrm{std}$ (θ-correspondence)

6.1.1. *Global arithmetic setting.* Let $G \circlearrowright X$ be a spherical variety. Then we want to think about this as a correspondence:

$$\begin{tikzcd} & X/G \arrow[dl] \arrow[dr] & \\ \bullet/G & & \bullet \end{tikzcd} \tag{49}$$

So roughly thinking we are thinking of $\bigcup_G$ as maps into $\bullet/G$. So this gives us some kind of functional where we pull push along this correspondence. So now define, in the global setting: $\mathcal{P}_X(\varphi)$ to to be the X-period of an automorphic form. As before, when $X = G/H$ this is restriction of the form to Bun_H, but in general we can define it to be this push pull operation.

So we are thinking of this boundary condition as a functional. So for every manifold input, we can think of it as a functional on that space. So when we take $\mathcal{P}_X(\varphi)$, we are really thinking of this as the partition function:

$$\mathcal{P}_X(\varphi) = \mathcal{A}_G\left(M^2 \times I\right) \tag{50}$$

where on one end of I we put φ, and on the other end we put X. So periods are partition functions on a 4-manifold.

6.1.2. *Global geometric setting.* We can write down a sheaf

$$\mathcal{P}_X \in \mathbf{Shv}\left(\mathrm{Bun}_G \overline{C}\right) \tag{51}$$

by taking the constant sheaf on the point, and pulling and pushing under

$$\begin{tikzcd} & \mathrm{Map}\left(\overline{C}, X/G\right) \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G = \mathrm{Map}\left(C, \bullet/G\right) & & \bullet \end{tikzcd} \quad . \tag{52}$$

6.1.3. *Local arithmetic.* Given our variety X, we can look at the local field F_v (or just $\mathbb{Q}_p$ for concreteness) and then $X(\mathbb{Q}_p)$ is a space with a $G(\mathbb{Q}_p)$ action. Then we can look at

$$L^2\left(X\left(\mathbb{Q}_p\right)\right) \in \mathbf{Rep}\left(G\left(\mathbb{Q}_p\right)\right) \ . \tag{53}$$

So in the local situation, to a spherical variety (or any G variety), we get a particular representation (a large one, very far from irreducible) then we can ask harmonic analysis questions about this such as which representations appear in this decomposition. These are the kinds of questions which appear in the relative Langlands program. This is the object we attach in the $\mathcal{A}$-theory. So this is $\mathcal{P}_X$ in the surface case.

6.1.4. *Local geometric.* For a 1-manifold, we can look at the category of sheaves on the loop space:

$$\mathbf{Shv}\left(LX\right) \circlearrowleft LG \tag{54}$$

where the loop group $\mathcal{L}G$ is acting. So this is an object of some 2-category.

6.2. **$\mathcal{B}$-side.** Now we have $G^\vee \circlearrowright X^\vee$.

Remark 3. On the $\mathcal{B}$-side, we can actually extend the TFT all the way down to a point. So this is a nice link to reality. But back away from reality:

6.2.1. *Global geometric.* We can define a sheaf in a similar way as before. We have

$$\begin{tikzcd} & \mathrm{Map}_{\mathrm{loc}}\left(\Sigma, X^\vee/G^\vee\right) \arrow[dl] \arrow[dr] & \\ \bullet & & \mathrm{Map}_{\mathrm{loc}}\left(\Sigma, \bullet/G^\vee\right) = \mathbf{Loc}_{G^\vee}\left(\Sigma\right) \end{tikzcd} \tag{55}$$

where loc indicates that we are looking at locally constant maps, or the Betti version of maps. So somehow this only depends on the homotopy type of Σ. And we can push pull, to get:

$$\mathcal{L}_{X^\vee} \in \mathbf{QC}\left(\mathbf{Loc}_{G^\vee}\right) \ . \tag{56}$$

This relates to L-functions (and is called the L-sheaf) as follows. This is similar to what is done in [4]. Now we can ask what this sheaf looks like. Suppose we just have a representation $X^\vee = V^\vee \circlearrowleft G^\vee$. Then we get a sheaf downstairs which looks like functions on the fibers. In particular:

$$\mathcal{L}_{X^\vee}\big|_\rho = \mathrm{Sym}\, H^\bullet\left(\rho, V\right) \ . \tag{57}$$

Recall that L functions are these inverses of characteristic polynomials. These are exactly what give traces on symmetric algebras, which are now appearing. And now if our curve was really over $\overline{\mathbb{F}}_q$, then this sheaf is equivariant for Frobenius, and so we can take the trace of Frobenius on this sheaf. So we are using the Atiyah-Bott-Lefschetz fixed point formula, which tells us we get a function on the σ fixed points, which which are local systems on C over the finite field. So we get the L function of $L(\rho, V)$ with a correction factor:

$$\frac{L(\rho, V)}{L(\rho, \mathrm{ad})}\,. \tag{58}$$

So this ratio of L-functions just appear directly by taking trace of the Frobenius. Now the same calculation in general says that we can think of the L-function of $X^\vee$ as a generalization of the L function. I.e. for our three-manifold $C/\mathbb{F}_q$, we get

$$\mathcal{G}_{G^\vee}(C/\mathbb{F}_q) = \mathcal{O}\left(\mathbf{Loc}_{G^\vee}\mathbb{F}_q\right) \tag{59}$$

and inside of this we have

$$\mathcal{L}_{X^\vee} \tag{60}$$

which, as a function, corresponds exactly to the L function of $X^\vee$.

In other words, on $M^3 \times I$, if we put a Galois representation ρ on one end, and $\mathcal{L}_{X^\vee}$ on the other, then

$$\mathcal{B}_{X^\vee}\left(M^3 \times I\right) \tag{61}$$

is the L function. So this L function is appearing as the partition function on a 4-manifold.

This tells us now how to match periods with L-functions. Periods are the $\mathcal{A}$ theory, and L-functions are the $\mathcal{B}$ theory with a dual boundary condition. So in order to understand their relationship, we need to understand the relationship between the boundary conditions.

7. Why Hamiltonian actions

This is very close to the theory of Coulomb branches developed in [2]. The relationship to S-duality is being studied in an upcoming paper of Justin Hilburn and Philsang Yoo. We are told from the physics to expect a duality between boundary theories for the $\mathcal{A}$-side and boundary theories for the $\mathcal{B}$-side. What we know, is that inside of the $\mathcal{B}$-side we have explicit examples coming from $G^\vee$-actions or Hamiltonian $G^\vee$ actions. The point is that we have an inclusion of affine Hamiltonian $G^\vee$ spaces into the boundary theories for $\mathcal{B}$. Then the claim is that this inclusion has a kind of left adjoint, given by some kind of affinization. So just from the abstract nonsense of field theory, we can construct an affine Hamiltonian $G^\vee$ space from a boundary theory. But in fact, it's even better. So given the data of a spherical variety for G, we can build, not the true honest theory, but some shadow of the theory, which is its affinization:

$$\begin{tikzcd}
\text{boundary theories for } \mathcal{A} \arrow[r, leftrightarrow] \arrow[dr] & \text{boundary theories for } \mathcal{B} \arrow[d, shift right] \\
 & \text{affine Hamiltonian } G^\vee \text{ spaces} \arrow[u, shift right]
\end{tikzcd}\,. \tag{62}$$

This is built using the local, unramified, geometric Langlands conjecture. So this is kind of the most interesting one.

Local geometric means that we are looking at the level of the circle (punctured disk). Unramified means we are going to fill this in to a disk. So we are looking at the value of the field theory on the disk with boundary condition given by the value on the circle. This is a 2-manifold, so we get a category. But it also has an action of the Hecke operators. When we cross this disk with an interval and remove a little 2-disk, the Hecke operators live in the category attached to S^2, acting on the category of interest. This translates to the following conjecture. On the $\mathcal{A}$-side we get $\mathbf{Shv}\left(LX/LG_+\right)$ where we have quotiented by positive loops. This has this action by the Hecke operators, which have to do with double cosets $LG_+\backslash LG/LG_+$. Now the Geometric Satake correspondence, in particular the version from [1], says that the Hecke operators are some version of the dual group. This is the fundamental link between the two sides of the Langlands program. In particular:

$$\mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) \simeq \mathbf{QC}\left(\mathfrak{g}^*/G^\vee\right) \ . \tag{63}$$

So now we want to fill in this square, i.e. what sits on the $\mathcal{B}$-side, with an action of $\mathbf{QC}\left(\mathfrak{g}^*/G^\vee\right)$. Well if we have an action of $G^\vee$, and a moment map, and a map from the quotient:

$$M^\vee/G^\vee \to \mathfrak{g}^{\vee *}/G^\vee \tag{64}$$

then we can consider

$$\mathbf{QC}\left(M^\vee/G^\vee\right) \ . \tag{65}$$

So we fill up the square:

$$\begin{array}{ccc} \mathbf{Shv}\left(LX/LG_+\right) & \simeq & \mathbf{QC}\left(M^\vee/G^\vee\right) \\ \circlearrowleft & & \circlearrowleft \\ \mathbf{Shv}\left(LG_+\backslash LG/LG_+\right) & \simeq & \mathbf{QC}\left(\mathfrak{g}^{\vee *}/G^\vee\right) \end{array} \tag{66}$$

This is why Hamiltonian G spaces come into the picture. Here we are thinking of $M^\vee = T^*X^\vee$, but these are not always cotangent bundles. The claim is that if you believe in some kind of conjecture like this, it can be used to produce the affinization of $M^\vee$, the restriction to the regular locus, etc. Then you can check that this lines up with various theorems. Then one of the key features of spherical varieties, is the set of orbits on the LHS is discrete. So this is closer to topology than one would hope for, and this is exactly why the answer is so nice on the RHS.

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