Simplicial sets and rings

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Outline

1 – Simplicial objects

Δ = full subcategory of **Cats** spanned by the categories

$$[n] = (\bullet \leftarrow \bullet \leftarrow \cdots \leftarrow \bullet \leftarrow \bullet)$$

with $n+1$ objects for $n \geqslant 0$.

$[0] = (\bullet)$

$[1] = (\bullet \leftarrow \bullet)$

$[2] = (\bullet \leftarrow \bullet \leftarrow \bullet)$

$[3] = (\bullet \leftarrow \bullet \leftarrow \bullet \leftarrow \bullet)$

The categories [2] and [3] and the face functors $\mathbf{d}_j$

The **face functors**

$$\mathrm{d}_j \colon [n-1] \to [n]$$

hit all objects except the j–th (for $j = 0, \ldots, n$).

Categories of simplicial objects

$\mathbf{C}$ = a category

$\mathbf{sC}$ = (simplicial objects in $\mathbf{C}$) = $\mathrm{Fun}(\Delta^{\mathrm{op}},\mathbf{C})$

Examples. $\mathbf{C}$ = **Sets**, **Abel**, and **Ring**

2 – Simplicial sets

$$X_\bullet \colon \Delta^{\mathrm{op}} \longrightarrow \mathbf{Sets}$$

For the representable examples

$$\Delta^n_\bullet \colon [m] \mapsto \mathrm{Fun}([m],[n])$$

the Yoneda lemma gives $\mathbf{sSets}(\Delta^n_\bullet, X_\bullet) = X_n$.

More generally, the **nerve** $\mathrm{N}\mathbf{C}_\bullet$ of a category $\mathbf{C}$ is

$$[m] \mapsto \mathrm{Fun}([m],\mathbf{C}),$$

so that $\Delta^n_\bullet = \mathrm{N}[n]_\bullet$.

Geometric realization

The cosimplicial topological space

$$[n] \mapsto \Delta^n_{\mathrm{top}} = \{\, x \in \mathbb{R}^{n+1} \,|\, x_j \geqslant 0, \sum_j x_j = 1 \,\}$$

defines an adjunction

$$|\quad| : \mathbf{sSets} \Longleftrightarrow (\text{topological spaces}) : \mathrm{Sing}_\bullet$$

with left adjoint $\qquad |X_\bullet| = \left(\coprod_n X_n \times \Delta^n_{\mathrm{top}} \right) / \sim$

and right adjoint $\qquad \mathrm{Sing}_\bullet(Y) = \{\Delta^\bullet_{\mathrm{top}} \to Y \text{ continuous}\}.$

The homotopy theory of simplicial sets I

The **homotopy groups** of a pointed simplicial set are defined as the homotopy groups of its geometric realzation:

$$\pi_*(X_\bullet, x_0) = \pi_*(|X_\bullet|, x_0).$$

Weak **equivalences** between simplicial sets induce bijections between the π_0's and isos between all higher homotopy groups for all basepoints.

The homotopy theory of simplicial sets II

cofibrations = degree-wise injections

$\Longrightarrow$ All objects are cofibrant.

fibrations = determined by the equivalences and cofibrations

The fibrant objects are the **Kan sets** or ∞–**groupoids**.

Mapping spaces

The set $\mathbf{sSets}(X_\bullet, Y_\bullet)$ of morphisms $X_\bullet \to Y_\bullet$ is the set of 0–simplices of a simplicial set

$$\mathbf{sSets}(X_\bullet, Y_\bullet)_\bullet$$

with set of n–simplices gives by

$$\mathbf{sSets}(X_\bullet \times \Delta^n_\bullet, Y_\bullet) = \mathbf{sSets}(X_\bullet, Y_\bullet^{\Delta^n_\bullet}).$$

These mapping spaces are only well-behaved if ($X_\bullet$ is cofibrant and) $Y_\bullet$ is fibrant.

3 – Simplicial abelian groups

There is an equivalence

$$\mathbf{sAbel} \longleftrightarrow \text{(non-negative chain complexes)}$$

of categories (Dold–Kan).

It sends a simplicial abelian group $M_\bullet$ to the chain complex

$$(\mathrm{N}M_\bullet = \bigcap_{j>0} \mathrm{Ker}(\mathrm{d}_j), \mathrm{d}_0)$$

that has

$$\mathrm{H}_n(\mathrm{N}M_\bullet) = \pi_n(M_\bullet, 0).$$

The homotopy theory of chain complexes

equivalences $=$ homology isomorphisms

fibrations $=$ epis in positive degrees

$\Longrightarrow$ All objects are fibrant.

cofibrations $=$ monos with cokernels degree-wise projective

4 – Simplicial (commutative) rings (with unit)

The **tensor product** $M_\bullet \otimes N_\bullet$ of simplicial abelian groups $M_\bullet$ and $N_\bullet$ has

$$[n] \longmapsto M_n \otimes N_n.$$

Simplicial rings $R_\bullet \colon [n] \mapsto R_n$ are simplicial abelian groups with a multiplication

$$m\colon R_\bullet \otimes R_\bullet \longrightarrow R_\bullet$$

satisfying the usual axioms. All R_n are rings.

Differential graded rings

If $R_\bullet$ is a simplicial ring, then $\mathrm{N}R_\bullet$ is a differential graded ring.

$$
\begin{tikzcd}
\mathrm{N}(R_\bullet) \otimes \mathrm{N}(R_\bullet) \arrow[r, "\mathrm{EZ}", "\sim"'] \arrow[dr] & \mathrm{N}(R_\bullet \otimes R_\bullet) \arrow[r, "\mathrm{AW}", "\sim"'] \arrow[d, "\mathrm{N}(m)"] & \mathrm{N}(R_\bullet) \otimes \mathrm{N}(R_\bullet) \\
& \mathrm{N}(R_\bullet) &
\end{tikzcd}
$$

$\Longrightarrow \pi_\star(R_\bullet, 0) = \mathrm{H}_\star(\mathrm{N}R_\bullet)$ is a graded ring

Warning: EZ is symmetric, but AW is not.

Examples

$X_\bullet =$ a simplicial set

The simplicial ring $\mathbb{Z}[X_\bullet]$ with

$$[n] \mapsto \mathbb{Z}[X_n]$$

is the **polynomial ring** on $X_\bullet$.

The functor $X_\bullet \mapsto \mathbb{Z}[X_\bullet]$ is left-adjoint to the forgetful functor $R_\bullet \mapsto R_\bullet$ from simplicial rings to simplicial sets:

$$\mathbf{sRing}(\mathbb{Z}[X_\bullet], R_\bullet) = \mathbf{sSets}(X_\bullet, R_\bullet)$$

The homotopy theory of simplicial rings

equivalences, **fibrations** = as for simplicial abelian groups

cofibrations can be obtained by attaching cells:

$$\begin{array}{ccc} \mathbb{Z}[\partial\Delta^n_\bullet] & \longrightarrow & R_\bullet \\ \downarrow & & \downarrow \\ \mathbb{Z}[\Delta^n_\bullet] & \longrightarrow & S_\bullet = R_\bullet \otimes_{\mathbb{Z}[\partial\Delta^n_\bullet]} \mathbb{Z}[\Delta^n_\bullet] \end{array}$$

The functor $X_\bullet \mapsto \mathbb{Z}[X_\bullet]$ preserves cofibrations. All $\mathbb{Z}[X_\bullet]$ are cofibrant. If $X_\bullet \to Y_\bullet$ is an injection, then $\mathbb{Z}[X_\bullet] \to \mathbb{Z}[Y_\bullet]$ is a cofibration.

5 – Representable functors

$$
\begin{aligned}
& F\colon \mathbf{Ring} \longrightarrow \mathbf{Sets} \text{ representable} \\
\iff & F(A) \cong \mathbf{Ring}(R, A) \text{ for some ring } R \text{ (natural bijection)} \\
\iff & F \text{ is an affine scheme}
\end{aligned}
$$

For functors with values in simplicial sets, replace 'natural bijections' with 'zigzags of natural equivalences.'

$$
\begin{aligned}
& F_\bullet\colon \mathbf{sRing} \longrightarrow \mathbf{sSets} \text{ representable} \\
\iff & F_\bullet(A_\bullet) \simeq \mathbf{sRing}(R_\bullet, A_\bullet)_\bullet \text{ for some cofibrant } R_\bullet
\end{aligned}
$$

Examples

The forgetful functor **sRing** $\to$ **sSets** is representable, unsurprisingly by the polynomial in one variable $\Delta^0_\bullet$:

$$\begin{aligned}\mathbf{sRing}(\mathbb{Z}[\Delta^0_\bullet], A_\bullet)_n &= \mathbf{sSets}(\Delta^0_\bullet, A_\bullet)_n \\ &= \mathbf{sSets}(\Delta^n_\bullet, A_\bullet) \\ &= A_n\end{aligned}$$

More generally, the simplicial ring $\mathbb{Z}[X_\bullet]$ represents the functor

$$A_\bullet \longrightarrow A_\bullet^{X_\bullet}.$$

Homotopy invariance

A functor $F_\bullet$: **sRing** $\to$ **sSets** is **homotopy invariant** if equivalences $A_\bullet \to B_\bullet$ induce equivalences $F_\bullet(A_\bullet) \to F_\bullet(B_\bullet)$.

Lemma. Representable functors are homotopy invariant.

We have assumed the representing object to be cofibrant and the simplicial rings $A_\bullet$ and $B_\bullet$ are automatically fibrant. This implies that the mapping spaces are well-behaved.

Simplicial enrichment

A **simplicial enrichment** of a functor $F_\bullet \colon \mathbf{sRing} \to \mathbf{sSets}$ is given by maps

$$\mathbf{sRing}(A_\bullet, B_\bullet)_\bullet \longrightarrow \mathbf{sSets}(F_\bullet(A_\bullet), F_\bullet(B_\bullet))_\bullet$$

that agree with the functor on 0–simplices and are compatible with composition.

Lemma. A functor is homotopy invariant if and only if it is equivalent to a Kan valued functor that admits a simplicial enrichment.