Contents

1. Conventions

We let $\Lambda \in \mathrm{CAlg}_{\mathbb{Z}_l}$.

- On the spectral side, we consider a derived artin stack X, and $\mathrm{QCoh}(-), \mathrm{Coh}(-)$.
- On the automoprhic side, we consider v-stacks, and lisse-sheaves.

When we recall the (de Rham) geometric setting, we let $\Sigma \in \mathrm{SmProj}_{\mathbb{F}}^{\mathrm{cn}}$, where $\mathbb{F} = \mathbb{C}$.

2. Introduction

This note analyzes the Tate period in the setting of Fargues Fontaine curve. Let $(G, X) = \left(\mathrm{GL}_1, \mathbb{A}^1\right)$ with dual pair $\left(\check{G}, \check{X}\right) = (\mathbb{G}_m, \mathbb{A}^1)$. Associated to these datum we can define the period sheaf $\mathcal{P}_X \in \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$ and the *L-sheaf* $\mathcal{L}_{\check{X}} \in \mathrm{IndCoh}(Z^1(W_E, \check{G})/\check{G})$, which categorifie classical notion of period and L-functions respectively.

Recall in [FS24], they constructed a spectral action

$$\mathrm{Perf}(Z^1(W_E, \check{G})/\check{G}) \circlearrowright \mathcal{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)$$

This action is required to be satisfy various properties. [FS24, p. IX], in particular the Hecke action, Equation 2.

Conjecture 2.1. [FS24] Let l be a prime coprime to q. $\Lambda \hookrightarrow L$ ring of integers of algebraic field extension. Fix $\sqrt{q} \in \Lambda$, and a Whittaker datum: fix a borel $B \hookrightarrow G$, and a generic character $\psi : U(E) \to \Lambda^\times$. This gives rise to $\mathrm{cInd}_{U(E)}^{G(E)}\psi$, which in turn yields a sheaf $\mathcal{W}_\psi$ on Bun_G. There is an $\mathrm{Perf}(Z^1(W_E, \check{G})/\check{G})$-linear equivalence

$$\mathbb{L} : \mathrm{Coh}^b_{\mathrm{Nilp}}([Z^1(W_E, \check{G})/\check{G}]) \simeq D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda)^\omega \tag{1}$$

where we have

$$\mathcal{O} \mapsto \mathcal{W}_\psi$$

Example 2.2. When $\check{G}$ is a torus, $\mathrm{Coh}_{\mathrm{Nil}} = \mathrm{Perf}$, so that the spectral action pins down the equivalence, see Section 9.

When G is the Torus, Equation 1 was proved by Zou [Zou24]. Our goal is to show that

$$\mathbb{L}(\mathcal{L}_{\check{X}}) = \mathcal{P}_X$$

Let us now describe two parallel decomposition, in the Geometric de Rham Langlands, following [FW24]. Using decomposition $\mathbb{A}^1 = 0 \bigsqcup \left(\mathbb{A}^1 \backslash 0\right)$ In the automorphic side we have

$$\begin{tikzcd}
Z \arrow[r, hook] \arrow[dr, "="'] & \mathrm{Bun}_G^X \arrow[d] & U \simeq \bigsqcup_{d \geq 0} C^{(d)} \arrow[l] & C^{(d)} \arrow[l] \arrow[d] \\
& \mathrm{Bun}_G \simeq \bigsqcup_{d \in \mathbb{Z}} \mathrm{Bun}_G^d & & \mathrm{Bun}_G^d \arrow[ll]
\end{tikzcd}$$

This induces a short exact sequence on $\mathrm{Dmod}(\mathrm{Bun}_G)$, with

$$\pi_! k_U \longrightarrow \pi_! k_{\mathrm{Bun}_G^X} \longrightarrow \pi_! i_* k_Z \simeq k_Z$$

Note that Dmod does not see the difference between formal completions. We have a similar diagram in the spectral side

$$\begin{tikzcd} Z \arrow[rr, hook] \arrow[dr] & & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d] & U \simeq * \arrow[l] \\ & \mathcal{Z} \arrow[ur] & \mathrm{Loc}_{\check{G}} & \end{tikzcd}$$

Inducing short exact sequence

$$\pi_{\mathcal{Z},*}(\omega_{\mathcal{Z}}) \longrightarrow \pi_* \omega_{\mathrm{Loc}_{\check{G}}^{\check{X}}} \longrightarrow \mathcal{O}_{\mathrm{triv}}$$

where $\mathcal{O}_{\mathrm{triv}}$ is skyscraper sheaf induced from $\mathrm{IndCoh}(*)$.

Proposition 2.3. *$\mathbb{L}(\mathcal{O}_{triv}) = k_Z$ under normalization in Equation 1.*

2.1. **Spectral action.** Fix a finite set I. where Hck^I stack given by

$$\mathrm{Hck}^I := L^+G \backslash \mathrm{Gr}_G^I \to (\mathrm{Div}^1)^I$$

This defines the Hecke operator diagram via the correspondence

$$\begin{tikzcd} & \mathrm{Hck}^I \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \end{tikzcd}$$

One can define a full subcategory

$$\mathrm{Sat}_G^I(\Lambda) \hookrightarrow D_{\text{ét}}(\mathrm{Hck}_G^I, \Lambda)^{\mathrm{bd}} \hookrightarrow D_{\text{ét}}(\mathrm{Hck}_G^I, \Lambda)$$

As a consequence

Theorem 2.4. *[FS24, p. IX.2] For any $V \in \mathrm{Rep}_\Lambda({}^LG^I)$ there is naturally associated functor*

$$T_V : D_{lis}(Bun_G, \Lambda) \to D_{\blacksquare}(Bun_G \times (Div^1)^I)$$

$Div^1 \to [*/W_E]$, and that $\pi_1 Div^1 = W_E$, we have

$$T_V : D_{lis}(Bun_G, \Lambda) \longrightarrow D_{lis}(Bun_G, \Lambda)^{BW_E} \longrightarrow D_{lis}(Bun_G, \Lambda)$$

Further, the action satisfies the following commutative diagram

$$\begin{tikzcd} & D_{\mathrm{lis}}(\mathrm{Hk}^I, \Lambda) \arrow[dr] & \\ \mathrm{Rep}_\Lambda({}^LG) \arrow[ur] \arrow[dr, "\Delta^*"] & & \mathrm{End}(D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \\ & \mathrm{IndPerf}(Z^1(W_E, \hat{G})/\check{G}) \arrow[ur] & \end{tikzcd} \tag{2}$$

2.2. **What the L-sheaf is.**

Example 2.5. Classically, let F be a global field. Then for $V \in \mathrm{Rep}(\check{G(F)})$, every where unramified, and a map $\rho : \Gamma_F \to \check{G}$, a global parameter, we obtain

- Local parameters $\Gamma_{F_v} \to \check{G}$, where $\Gamma_{F_v} \circlearrowright V^{I_v}$.
- Local L-values

$$L_v(V) := \mathrm{tr}\left(\mathrm{Fr}_v, V^{I_v}\right)$$

Combining to give *global L-value*

$$L(V) := \prod_{v \in V} L_v(V)$$

3. Iwasawa-Tate Case on Fargues-Fontaine Curve

The Fargues-Fontaine curve should be a *global object* of dimension 2 under the TQFT dictionary of [BSV].

Proposition 3.1. *(1) The connected components of* Bun_G *are indexed by the integers* $n \in \mathbb{Z}$ *and the component* Bun_G^n *classifies line bundles of degree* n.

(2) Let $\mathcal{BC}(n)$ *be the Banach-Colmez space of the line bundle* $O(n)$*. The relative stack* $\mathrm{Bun}_G^{X,n} = \pi^{-1}(\mathrm{Bun}_G^n)$ *is a* $\mathcal{BC}(n)$*-torsor over* Bun_G^n.

(3) The restriction $\mathcal{P}_X^n = \mathcal{P}_X|_{\mathrm{Bun}_G^n}$ *is described as follows.*

(a) If $n < 0$, $\mathcal{P}_X^n = \Lambda$ *as the trivial character of* $\mathbb{Q}_p^\times$.

(b) If $n > 0$, $\mathcal{P}_X^n = \Lambda[-2n]$.

(c) If $n = 0$, $\mathcal{P}_X^n = C_c^\infty(\mathbb{Q}_p, \Lambda)$.

3.1. The Global Period Conjecture.

In one line, this states that the normalized period sheaf attached to a Hamiltonian G-space should correspond to to the normalized L-sheaf attached to $(\check{G}, \check{M})$.

$$\mathcal{P}^{\mathrm{norm}}$$

By *normalized*, is what we make precise here.

We first briefly recall the construction [FS24], [Far16].

4. Iwasawa Tate on the $\mathcal{A}$-side.

There is a stack for the proétale topology

$$\mathrm{Bun}_G : \mathrm{Pftd}_{\mathbb{F}_q} \to \mathcal{S}$$

$$S \mapsto \mathrm{Tors}_G(X_{S,E}) \quad X_{S,E} \in \mathrm{Adic}_E$$

Bun_G has a stratification $\bigsqcup_{n\in\mathbb{Z}} \mathrm{Bun}_G^n$, which we describe in

Proposition 4.1. *[FS24, Ch. II]*

$$Bun_G^{n,X} \simeq \begin{cases} \mathcal{BC}(n)/\underline{\mathbb{Q}_p^\times} & n > 0 \\ [\underline{\mathbb{Q}_p}/\underline{\mathbb{Q}_p}^\times] & n = 0 \\ [*/\underline{\mathbb{Q}_p}^\times] & n < 0 \end{cases}$$

The fiber of $\mathrm{Bun}_G^X \to \mathrm{Bun}_G$ over a G bundle L is again a vector space. In fact, let $\mathcal{E} := \mathrm{map}(\mathcal{O}_X, \mathcal{E}^{\mathrm{univ}} \otimes \Omega_C^{1/2})$ be the perfect complex, such that

$$\begin{tikzcd} E \simeq \Gamma(X_S, \mathcal{L}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{V}(\mathcal{E}) \arrow[d] \\ S \arrow[r, "\mathcal{L}"] & \mathrm{Bun}_G \end{tikzcd}$$

where we recall the construction of $\mathbb{V}$, in Section 8.1.

In the work of [Ans21, Cor 3.10], there is a Fourier transform,

$$R\tau_* : D(X_S) \to D(S_v, \underline{E})$$

which satisfies a *relative Serre duality.* i.e.

$$(R\tau^* K^\vee) \simeq (R\tau^* K)^\vee[1] \quad K \in D(X_S)$$

We may allow $S = \mathrm{Bun}_G$ or any v-stack. Let $\mathcal{E}^{\mathrm{univ}}$ be the universal complex on X_{Bun_G}. Thus we study $R\tau_* \mathcal{E}^{\mathrm{univ}}$ which, supposedly, associated v-stack is , Bun_G^X, is the moduli stack of section of $\Gamma_{\mathrm{dR}}(X, \mathcal{E}^{\mathrm{univ}} \otimes \Omega_X^{1/2})$.

Let us now discuss the Fourier transform in more detail. Recall Serre Duality suppose S is a derived Artin stack, where E, E' are vector bundles, $E^\vee \simeq E'[1]$, which is used in [FW24]. The Fourier vector bundle theory has a *somewhat* acceptable theory. For a choice of $\psi : \mathbb{Q}_p \to \Lambda^\times$, we can define Fourier transform

$$\mathrm{FT}_E : D_{\text{ét}}(E, \Lambda) \to D_{\text{ét}}(E^\vee, \Lambda)$$

where the functoriality of the shift is almost the same.

In the work of [BSV] on one consider the constructed

4.1. Bun_G^X in mixed characteristic setting.

Lemma 4.2.

$$Bun_G^X \simeq \bigoplus_{n\in\mathbb{Z}} \mathcal{BC}(n)/\underline{E}^\times$$

Proposition 4.3. *(1) The connected components of* Bun_G *are indexed by the integers* $n \in \mathbb{Z}$ *and the component* Bun_G^n *classifies line bundles of degree* n*.*

(2) Let $\mathcal{BC}(n)$ *be the Banach-Colmez space of the line bundle* $O(n)$*. The relative stack* $\mathrm{Bun}_G^{X,n} = \pi^{-1}(\mathrm{Bun}_G^n)$ *is a* $\mathcal{BC}(n)$*-torsor over* Bun_G^n*.*

(3) The restriction $\mathcal{P}_X^n = \mathcal{P}_X|_{\mathrm{Bun}_G^n}$ *is described as follows.*

(a) If $n < 0$*,* $\mathcal{P}_X^n = \Lambda$ *as the trivial character of* $E^\times$*.*

(b) If $n > 0$*,* $\mathcal{P}_X^n = \nu[-2n]$ *with* ν *the valuation character on* $E^\times$*.*

(c) If $n = 0$*,* $\mathcal{P}_X^n = C_c^\infty(E, \Lambda)$*.*

Proof. (1) and (2) is immediate [ADD REFERENCE]. We verify (3). First, (a) follows from the description that $\mathrm{Bun}_G^{X,n} = \mathrm{Bun}_G^n$. For (b), we have $\mathcal{BC}(1) \cong \mathrm{Spd}\,\mathbb{F}_q[\![T^{1/p^\infty}]\!]$ by [FS24, Proposition II.2.2]. Moreover, if we take a pullback along a geometric point $\mathrm{Spa}(C, O_C) \to \mathrm{Bun}_G^n$ with an untilt $C^\sharp$ of characteristic 0, we have an exact sequence $0 \to \mathcal{BC}(n)_C \to \mathcal{BC}(n+1)_C \to \mathbb{G}_{a,C^\sharp} \to 0$ by [**SW20**]. Thus, we can prove that $\mathcal{P}_X^n|_C = \Lambda[-2n]$ via an induction on n. [We will later determine the action of $E^\times$.] For (c), we see from the description that $\mathrm{Bun}_G^{X,0} = [E/E^\times] \to [*/E^\times]$. $\square$

Definition 4.4. The *relative curve associated to* $S = \mathrm{Spa}(R, R^+) \in Pftd_{\mathbb{F}_q}$ is given by

$$Y_{S,E} := \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \backslash V(\pi[\varpi])$$

where π is uniformizer of E, and ϖ is uniformizer of R.

$$X_{S,E} := Y_{S,E}/\varphi^{\mathbb{Z}}$$

Any point of the base, $S = \mathrm{Spa}(R, R^+) \to \mathrm{Spa}\, E^\diamond$, induces a Cartier divisor

$$S^\sharp \hookrightarrow X_S$$

The formal completion along this divisor is $\mathrm{Spf} B^+_{\mathrm{dR}}(R^\sharp)$. [1] We work over the algebraic closure. [2]

4.2. **The structure of Bun_G.**

Theorem 4.5. *Consider the v-topology on $Pftd_{\bar{\mathbb{F}}_p}$.* [3]

- *$Bun_{G,\bar{\mathbb{F}}_q}$ is an artin v-stack (l-cohomogically)*[4] *smooth of dimension* 0*.*
- *Let $\breve{E} := \widehat{E^{nr}}$, note that $X_{S,E} = Y_{S,E}/\varphi^{\mathbb{Z}}$, where $Y_S \to \mathrm{Spa}(\breve{E})$. Let $B(G) := G(\breve{E})/\sigma$-cjg, $b \sim gbg^{-\sigma}$, the Kottwitz sets of Iso crystal.*

$$B(G) \simeq |Bun_{G,\bar{\mathbb{F}}_q}|$$

The geometry of $\mathrm{Bun}_{G,\bar{\mathbb{F}}_q}$ is nice.

Definition 4.6. Let G_b denote the automorphism group of G-isocrystal attached to b.

Example 4.7. If G is quasisplit

- G_b is an inner form of a Levi subgroup of G
- G_b is an inner form of G iff b is basic.

Theorem 4.8.
- $\pi_0(Bun_{G,\bar{\mathbb{F}}_q}) \simeq \pi_1(G)_\Gamma$.
- *There is a nice Harder Narasimhan stratification. In particular, there is an open substack*

$$Bun^{ss}_{G,\bar{\mathbb{F}}_p} \hookrightarrow Bun_{G,\bar{\mathbb{F}}_p}$$

With the following stratification

$$Bun^{ss}_{G,\bar{\mathbb{F}}_p} \simeq \bigsqcup_{[b]\ basic} [*/\underline{G_b(E)}]$$

where G_b is inner form of G, for example $G_1 = G$.

Example 4.9. When $E = \mathbb{F}_q((\varpi))$, we suppose $\mathrm{Spa}(E)^\diamond = \mathrm{Spa}\, E$. Then the B_{dR} Grassmanian is a proétale sheaf in Pftd_E. If $S = \mathrm{Spa}(R, R^+) \in \mathrm{Pftd}_E$, then

$$\mathrm{Gr}^{B_{\mathrm{dR}}}(S) = \{\mathcal{F}, \xi\} / \simeq \quad \mathcal{F} \in \mathrm{Tors}_G(\mathrm{Spec}\, B^+_{\mathrm{dR}}(R)), \xi \text{ is trivialization at } B_{\mathrm{dR}}(R)$$

Note that we may replace bundles on $\mathrm{Spa}(R, R^+)$ with proétale bundles on $\mathrm{Spec}\, R$ due to the result of Kedlaya and Liu.

[FS24] has defined a 5 functor formalism of solid sheaves, with $f_\natural$ taking the place of $Rf_!$.

Proposition 4.10. *[FS24, Prop. VII.7.1]* $\mathcal{D}_{lis}(Bun^b_G.\Lambda) \simeq \mathcal{D}(G_b(E)\text{-}Mod_\Lambda)$.

[1] Already, one sees that the various formal completions are distinct.
[2] note that $* := \mathrm{Spa}(\bar{\mathbb{F}}_q)$ is not representable.
[3] This is analogous to the fpqc topology of schemes, which is finer than the pro-étale topology
[4] Notions in v-stacks are to be made sense l-cohoomogically.

Example 4.11. $G = T$ is a torus over E. Then $B(T) = X_*(T)_\Gamma$. Thus, all b are basic. We have a semi-infinite orthogonal decomposition

$$D_{\mathrm{lis}}(\mathrm{Bun}_T, \Lambda) \simeq \prod_{[\chi] \in X_*(T)_\Gamma} \mathcal{D}(T(E)\text{-}\mathrm{Mod}_\Lambda)$$

Example 4.12. $G = \mathrm{GL}_1$. Everything is semistable, so

$$\mathrm{Pic} := \mathrm{Bun}_{\mathrm{GL}_1} \simeq \bigsqcup_{\mathbb{Z}} [*/\underline{E^\times}]$$

5. Iwasawa Tate on the $\mathcal{B}$-side

For $\Lambda \in \mathrm{Alg}_{\mathbb{Z}_l}$ algebra made a condensed ring via $\Lambda := \Lambda^{\mathrm{disc}} \otimes_{\mathbb{Z}_l^{\mathrm{disc}}} \mathbb{Z}_l$.

Theorem 5.1. *Let $Z^1(W_E, \hat{G}) \in Fun(Aff_{\mathbb{Z}_l}, \mathcal{S})$ be the functor sending*

$$\Lambda \mapsto \mathrm{Map}_{cts}(W_E, \hat{G}(\Lambda))$$

This is represented by a flat locally complete intersection scheme.

We can define a zero dimensional lci algebraic stack over $\mathbb{Z}_l$.

Definition 5.2. The *stack of l-adically continuous L-parameters over* Λ

$$\mathrm{Loc}_{\hat{G},\Lambda} := [Z^1(W_E, \hat{G})_\Lambda / \hat{G}_\Lambda]$$

This definition works well for any reductive groups.

Example 5.3. When $\Lambda = \overline{\mathbb{Q}}_l$, $\mathrm{Par}_{\hat{G}}$ parameterized isomorphism class of l-adically continuous L-parameters

$$\phi : W_E \to {}^L G(\bar{\mathbb{Q}}_l) := \hat{G}(\bar{\mathbb{Q}}_l) \rtimes W_E$$

Proposition 5.4. *When $G = T$ is a torus, $D^{b,qc}_{Coh,Nilp}(Par_{\widehat{T}}) \simeq Perf^{qc}$*

6. $\mathcal{B}$-side Brief Recollection on the proof of Feng and Wang

Note that we would like to compute $\mathrm{Loc}_{\check{G}}$ in the *de Rham* context. In this case $\mathbb{F} = k = \mathbb{C}$. Although it is in the *étale* context $\mathbb{F} = \bar{\mathbb{F}}_q$, for which we can do function sheaf dictionary. Though in [BSV], they discussed the computation in *any* context. The first goal is to understand the diagram

$$\begin{tikzcd} \mathrm{fib}\rho \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d] \\ * \arrow[r, "\rho"] & \mathrm{Loc}_{\check{G}} \end{tikzcd}$$

Then this induces a localization sequence

$$\begin{tikzcd} Z \arrow[r] & \mathrm{Loc}_{\check{G}}^{\check{X}} \arrow[d, "\pi"] & U \simeq * \simeq \mathrm{Map}(C_{\mathrm{dR}}, \mathbb{G}_m/\mathbb{G}_m) \arrow[l, hook'] \arrow[d, "\pi"] \\ & \mathrm{Loc}_{\check{G}} & */\mathbb{G}_m \arrow[l, hook', "i"] \end{tikzcd}$$

This induces a short exact sequence

$$\hat{\pi}_{Z,*}(\omega_{\mathrm{Loc}^{\check{X}}_{\check{G}}}) \to \pi_* \left(\omega_{\mathrm{Loc}^{\check{X}}\check{G}}\right) \longrightarrow \mathcal{O}_{\mathrm{triv}}$$

where $\mathcal{O}_{\mathrm{triv}} := i_* \pi_* \mathrm{triv}$. The strategy is that first one identifies the nonunital part. For a symmetric monoidal category, $(\mathcal{C}, \otimes, 1)$

$$\mathrm{CAlg}^{\mathrm{nu}}(\mathcal{C}) \simeq \mathrm{CAlg}^{\mathrm{aug}}(\mathcal{C})$$

$$A \mapsto 1 \oplus A$$

$$\bar{A} := \ker \varepsilon \leftarrow\!\shortmid (A, \varepsilon)$$

The first claim is then that $\overline{\hat{\pi}_{Z,*}\left(\omega_{\mathrm{Loc}^{\check{X}}}\right)}$ is identified with factorization algebra associated to $\mathrm{std} \in \mathrm{Rep}(\check{\mathrm{GL}}_1)$. We then describe how to identify extension class in.

7. Identification of graded algebra

7.1. Definition of Ran space and divisor version.

Definition 7.1. Let $\mathrm{Ran}(X) := \mathrm{colim}_{\mathrm{FinSurj}^{\mathrm{op}}} X^I$.

As explained in [GL]

$$\mathrm{Shv}^!(\mathrm{Ran}(\Sigma)) \simeq \mathrm{Fun}_{\mathrm{Sch}_k}(\mathrm{Fin}^{\mathrm{Surj}}, \mathrm{Shv}^!)$$

Thus we can regard an object as a family

$$\left\{\mathcal{F}^{(T)} \in \mathrm{Shv}(C^T)\right\}_{T \in \mathrm{Fin}}$$

$\mathrm{Ran}(\Sigma)$ has the structure of a commutative semigroup. Yielding the following adjunction

$$\begin{array}{c} \left(\mathrm{Shv}(C), \otimes^!\right) \\ \big\downarrow \\ (\mathrm{Shv}(\mathrm{Ran}(C)), \otimes^*) \end{array}$$

where under the functorially perspective $\otimes^*$ corresponds to the day convolution. This yields the commutative diagram

$$\mathrm{Shv}(C) \longrightarrow \mathrm{CAlg}^{\mathrm{nu}}(\mathrm{Shv}(\mathrm{Ran}(C)), \otimes^!)$$

There is an almost equivalent version: using divisors.

Finally there is the ranification map that allows us to transport a factorization algebra in the classical setting to the divisor version.

$$\mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C)) \xrightarrow{\simeq} \mathrm{Shv}(\mathrm{Ran}(C), \otimes^*)$$

Example 7.2. Let $\Lambda^+ := \mathbb{Z}_{>0}$.

7.2. **Ran version of a stack.** Let $\mathcal{Y} \in \mathrm{PShv}_{\mathcal{S}}(\mathrm{Aff})$. Then we can always define the *Ran* version

$$\begin{tikzcd}
\mathcal{Y}^I \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathcal{Y}_{\mathrm{Ran}(X)} \arrow[d] \\
* \arrow[r] & \mathrm{Ran}(X)
\end{tikzcd}$$

where $\mathcal{Y}_{\mathrm{Ran}(C)}(R)$, consists of $x \in Ran(C)$, and a map, with $S = \operatorname{Spec} R$,

$$(D_x)_{\mathrm{dR}} \times_{S_{\mathrm{dR}}} S \to \mathcal{Y}$$

The diagram is a pullback given that $* \to \mathrm{Ran}(X)$ picks out $(x_1, \dots, x_I) \subseteq X(R)$. Since

$$(D_x)_{\mathrm{dR}} = \bigcup_{i=1}^{I} (D_{x_i})_{\mathrm{dR}} \simeq \bigsqcup_{i=1}^{I} *$$

7.3. **Recollection on Ran spaces.** For a $\check{G}$ eqivaraint factorization algebra $\mathcal{A} \in \mathrm{Shv}(\mathrm{pt}/\check{G})_{\mathrm{Div}^{\Lambda^+}(C)}$ it has an underlying structure. $\mathrm{oblv}(\mathcal{A}) \in \mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C))$. In particular

$$\begin{tikzcd}
\mathrm{Fact}_{\check{G}}(\mathrm{Div}^+(C)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Shv}((\mathrm{pt}/\check{G})_{\mathrm{Div}^{\Lambda^+}(C)}) \arrow[d] \\
\mathrm{Fact}(\mathrm{Div}^+(C)) \arrow[r] & \mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C)) \simeq \prod_{\lambda \in \Lambda^+} \mathrm{Shv}(C^{(\lambda)})
\end{tikzcd}$$

The the notation $\mathrm{Div}^{\Lambda^+}(C) \to (*/\check{G})$, is the formal completion of the images of the divisors.

Example 7.3. We construct an object in $\mathrm{Fact}\,\mathrm{Sym}^*\,\mathrm{std} \in \mathrm{Shv}(\mathrm{Loc}_{\check{G}})$. To do this we use the localization map

$$\mathrm{Loc}^{\mathrm{spec}} : \mathrm{Shv}(*/\check{G}_{\mathrm{Div}^{\Lambda^+}}) \to \mathrm{Shv}(\mathrm{Loc}_{\check{G}})$$

This is induced fro the evaluation map, i.e. that there is a diagram

$$\begin{tikzcd}
\mathrm{Div}^{\Lambda^+}(C) \times \mathrm{Loc}_{\check{G}} \arrow[r] \arrow[d] & (\mathrm{pt}/\check{G})_{\mathrm{Div}^{\Lambda^+}(C)} \\
\mathrm{Loc}_{\check{G}} &
\end{tikzcd}$$

Which intuitively - as the definition goes through $\mathrm{Ran}(C)$ - is the evaluation sends a pair,

$$D \in \mathrm{Div}^{\Lambda^+}(C) \text{ and a } \check{G}\text{-bundle } L : C_{\mathrm{dR}} \to */\check{G}$$

to the precomposition. Thus, we in fact only have to determine object in $\mathrm{Shv}(*/\check{G}_{\mathrm{Div}^{\Lambda^+}}) \to \mathrm{Shv}(\mathrm{Div}^{\Lambda^+}(C))$. Thus, we have that $\overline{\mathrm{Sym}}^*\mathrm{std}$ is a constant cocommutative coalgebra on $\mathrm{Shv}(C)^{\mathbb{Z}_{\geq 0}}$. Then we have

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^*\mathrm{std}) \in \mathrm{Fact}(\mathrm{Div}^{\mathbb{Z}_{\geq 0}}(C))$$

In fact this has a very simple description, formally by adjunction

$$\mathrm{Fact}(\overline{\mathrm{Sym}}^*\mathrm{std}) \simeq \{\pi_{n!}(\mathrm{std}^{\boxtimes n})_{\Sigma_n}\}$$

where $\pi_n : C^n \to C^{(n)}$ is the quotient map.

7.4. **Discussion of Betti setting.**

7.5. **Identifying the extension class.** On the other we have

$$\begin{aligned}\mathrm{Map}_{\mathcal{C}}(\mathcal{O}_{\mathrm{triv}}, \hat{\pi}_{Z,*}(\omega_{\mathrm{Loc}^{\check{X}}})) &\simeq \mathrm{Map}_{\mathcal{C}}(i_* q_* \mathcal{O}_{\mathrm{pt}}, -) \\ &\simeq \mathrm{Map}_{\mathrm{QCoh}(B\mathbb{G}_m)}(q_*\mathcal{O}_{\mathrm{pt}}, i^! \pi_{\mathcal{Z},*}\omega_{\mathcal{Z}}) \\ &\simeq \mathrm{Map}_{\mathrm{Rep}(\mathbb{G}_m)}\end{aligned}$$

where this $\Re(\mathbb{G}_m)$ is from the following adjunction

$$\begin{tikzcd} \mathrm{QCoh}(B\mathbb{G}_m) \arrow[r] & \mathrm{Rep}(\mathbb{G}_m) \arrow[d] \arrow[r, phantom, "\simeq"] & \mathrm{coMod}_{k[t^{\pm 1}]} \\ & \mathrm{QCoh}(*) \arrow[r] & \mathrm{Mod}_k \end{tikzcd}$$

Now to identify the $i^!$, we have the following diagram

$$\begin{tikzcd} \mathcal{V} \arrow[r] & \mathcal{V}/\mathbb{G}_m \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] \\ * \arrow[r] & B\mathbb{G}_m \arrow[r] & \mathrm{Loc} \end{tikzcd}$$

induced from the diagram of formal completion.

$$\begin{tikzcd} & \mathcal{V}/\mathbb{G}_m \arrow[r] \arrow[d] & \mathcal{Z} \arrow[d] & Z \arrow[l] \arrow[dl] \\ V \arrow[r] \arrow[d] & V/\mathbb{G}_m \arrow[r] \arrow[d] & \mathrm{Loc}^{\check{X}} \arrow[d] & \\ * \arrow[r] & */\mathbb{G}_m \arrow[r] & \mathrm{Loc} & \end{tikzcd}$$

where each of the squares are fiber pullbacks. We have that

$$i^! \pi_{\mathcal{Z},*}\omega_{\mathcal{Z}} \simeq \pi_{\mathcal{V}/\mathbb{G}_m}(\varpi_{\mathcal{V}/\mathbb{G}_m})$$

This allows us to What have the equivalence

$$\mathrm{Map}_{\mathrm{IndCoh}(B\mathbb{G}_m)}(\mathcal{O}(\mathbb{G}_m), \mathrm{Sym}\,\mathcal{E})) \simeq \prod_{n\in\mathbb{N}} \mathrm{Map}_{\mathrm{QCoh}(*)}(k, \mathrm{Sym}^n_k \mathcal{E}) \simeq \prod_{n\in\mathbb{N}} \Omega^\infty \mathrm{Sym}^n_k \mathcal{E}$$

Note that to compute homotopy orbits of $E \in \mathrm{Fun}(BG, \mathrm{Sp})$, for G a finite group, of a spectrum we use the Whitehead tower, giving fiber sequence

$$\tau_{\geq n+1}E \to \tau_{\geq n}E \to \Sigma^n H\pi_n E$$

these induces [Lur09, Ch.1]

$$E^2_{p,q} \simeq H_p(\Sigma_n, \pi_q E) \Rightarrow \pi_{p+q}(E_{hG})$$

As k is a projective $k[\Sigma_n]$ my module when k is of characteristic 0, where $H_p(\Sigma_n, \pi_q E) \simeq \mathrm{Ext}^p_{k[\Sigma_n]}(k, \pi_q E)$ is concentrated only for degrees $p = 0$. We deduce

$$\pi_0 \mathrm{Sym}^n \mathcal{E} \simeq \mathrm{Sym}^n_k \pi_0\mathcal{E} \simeq H^0(\Sigma, k)$$

Definition 7.4. Let $\mathrm{Mod}_A^{\mathrm{free}}$, where $A \simeq \mathbb{Z}[x_1, \dots, x_m]$, is a free polynomial ring. Then one can define the derived symmetric powers

$$\begin{tikzcd}
\mathrm{Mod}_A^{\mathrm{free}} \arrow[d] \arrow[dr, "M \mapsto \pi_0(M \otimes_A M \otimes_A \cdots \otimes_A M)_{\Sigma_n}"] & \\
\mathrm{Mod}_A^{\mathrm{cn}} \arrow[r] & \mathrm{CAlg}_A^{\mathrm{cn}}
\end{tikzcd}$$

Remark 7.5. For an algebraic group $G \in \mathrm{Grp}(\mathrm{Sch}_k)$, $\mathrm{QCoh}(BG) \simeq \mathrm{Rep}(G) \in \mathrm{Pr}^L_{\mathrm{st},\omega}$, thus

$$\mathrm{IndCoh}(BG) \simeq \mathrm{QCoh}(BG)$$

Note that if G were a finite group, where we regard

$$\mathrm{Grp}(\mathrm{Set}) \to \mathrm{Grp}(\mathrm{Sch}_K)$$

$$X \mapsto \coprod_{x \in X} \mathrm{Spec}\, k =: \underline{X}$$

where we note that $\coprod_X \times \coprod_Y \simeq \coprod_{X \times Y}$, indeed the cardinality of right hand side is $|X \times Y|$, whilst that of left hand side is $|X| \cdot |Y|$. The map from (x, y) component to right hand side is to $*_x \to \coprod_x$, and $*_y \to \coprod_Y$. This induces a bijection on the level of sets. We abusively denote for a finite group G, then

$$\mathrm{QCoh}(BG) \simeq \mathrm{Fun}(|BG|, \mathrm{Mod})$$

Indeed to see this: take a Čech resolution of the land side

$$\varprojlim \left(\mathrm{QCoh}(*) \to \mathrm{QCoh}(G) \otimes \mathrm{QCoh}(G) \to \cdots \right)$$

then this reduces to the observation that

$$\mathrm{Fun}(G, \mathrm{Mod}) \simeq \prod_{g \in G} \mathrm{Mod} \simeq \mathrm{QCoh}\left(\coprod_{g \in G} * \right)$$

using the fact that $\mathrm{QCoh}(-)$ satisfies fpqc-descent.

Remark 7.6. The coalgebra structure of $\mathbb{G}_m$. We imagine $\mathcal{O}(G)$, as a family of function

$$\{f_R : G(R) \to R\}_{R \in \mathrm{Alg}_k}$$

Then Δf is the unique element on $\mathcal{O}(G) \otimes \mathcal{O}(G)$, such tha t

$$\Delta f(a, b) = f(a, b)$$

In the case of $\mathbb{G}_m$, this element would be that linearly extended from $\mathcal{O}(G) \to \mathcal{O}(G) \otimes \mathcal{O}(G)$ by $t \mapsto t \otimes t$. We can also analyze how $\mathbb{G}_m$ acts on $\mathcal{O}(G)$. Let $t^n \in \mathcal{O}(G)$.

$$r \cdot t^n(r') := (r \cdot r')^n = r^n (r')^n$$

Thus $r \in \mathbb{G}_m(R)$ acts by character n.

Remark 7.7.
- Vezossi has a cheat sheaf graded modules.

8. Linear Stacks: recollection

Let us work in the category Sch_S, where $S = \mathrm{Spec}\, k$. If G is reductive algebraic group, then we can construct the GIT quotient $\mathrm{Spec}\, A^G \simeq X/G$. Naïvely, if

Theorem 8.1. *Luna's étale slice theorme. Let $G \in AlgGrp_k^{red}$. Let $x \in X$ be a closed point such that $Gx \hookrightarrow X$ is closed.*

8.1. **Relative notion of vector bundle.** By stack I mean either $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}^{\heartsuit}, \mathcal{S})$, or $\mathrm{Shv}_{\mathrm{fpqc}}(\mathrm{Aff}, \mathcal{S})$.

Definition 8.2. Given $Y \in \mathrm{Stack}_X$, we can construct the stack of sections.

$$\mathrm{Sect}(X, Y)$$

In otherwords, when we mean V, G we will implicitly mean V as a vector bundle on X, and $G \in \mathrm{Grp}(\mathrm{Stack}_X)$.

Example 8.3. We set

$$\mathrm{Sect}(X, V/G)$$

- $\mathcal{F} \to X$ a G-bundle.
- $\mathcal{F} \to V$ a G-equivariant map. This is equivalent to the datum of associated to data of section. One way to see this is via the "cube".

Thus, initively the fiber is $H^0(X_\kappa, \mathcal{F} \times^G V)$. In fact we will argue that that this is $R\Gamma(X_k, F \times^G V)$, a perfect connective cochain complex. Thus its dual is connective perfect chain complex. Then we claim the fibre at a point $k \xrightarrow{\mathcal{F}} \mathrm{Sect}(X, X/G)$, corresponds to

$$\mathbb{V}(R\Gamma(X_k, \mathcal{F} \times^G V)^\vee) \in \mathrm{Aff}_k$$

8.2. **Construction of symmetric bundle.** References, [FYZ24]. We are at the *de Rham setting*. That is our sheaves are over $\mathbb{C}$, with $\mathbb{C}$ coefficient theory. Let $\mathcal{E} \in \mathrm{QCoh}(X)$. We will construct the associated linear stack, [Mon21]. In this set up, we will consider $\mathrm{Stack}_X := \mathrm{Shv}_{\text{ét}}(\mathrm{CAlg}^{\heartsuit}, \mathcal{S})$.

Definition 8.4.

$$\mathbb{V} : \mathrm{QCoh}(X) \to \mathrm{Stack}_X$$

$$\mathbb{V}(\mathcal{E})(T \xrightarrow{f} X) := \mathrm{Map}_{\mathrm{QCoh}(T)}(f^*(\mathcal{E}), \mathcal{O}_T) \in \mathcal{S}$$

Proposition 8.5. *If $\mathcal{E} \in QCoh(X)^{cn}$, then*

$$\mathbb{V}(\mathcal{E}) \simeq \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}_X}(\mathcal{E})$$

Proof.

$$\begin{aligned}\mathrm{Mod}_{\mathrm{QCoh}(X)}(\mathcal{E}, f_*\mathcal{O}_T) &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X))}\left(\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}, f_*\mathcal{O}_T\right) \\ &\simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})}(\mathrm{Sym}_{\mathcal{O}_X} \mathcal{E}, f_*\mathcal{O}_Y) \\ &\simeq \mathrm{Map}_{\mathrm{Stack}_X}(T, \mathrm{Spec}_X \mathrm{Sym}_{\mathcal{O}} \mathcal{E})\end{aligned}$$

Where we note that Spec_X is only defined for *connective* spectrum, **??**. □

Proposition 8.6. *Let $Stack_X^{aff}$ be the full subcategory of of Delign Munford Stacks spanned by affine morphisms. There is an equivalence*

$$\left(Stack_X^{aff}\right)^{op} \xrightarrow{\simeq} CAlg(QCoh(X)^{cn})$$

$$(Y \to X) \mapsto f_*\mathcal{O}_Y$$

The (left adjoint) inverse of this functor is Spec_X. *In otherwords,*

$$\mathrm{Map}_{Stack_X}(Y, \mathrm{Spec}_X \mathcal{F}) \simeq \mathrm{Map}_{CAlg(QCoh(X))}(\mathcal{F}, f_*\mathcal{O}_Y)$$

Proof. We can do descent, and assume $X = \operatorname{Spec} A$. Then we have

$$\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}}) \simeq \mathrm{CAlg}_A^{\mathrm{cn}} \xrightarrow{\simeq} \mathrm{Stack}_{\operatorname{Spec} A}^{\mathrm{aff}}$$

$\square$

Remark 8.7. Note that $\mathbb{V}$ factors through $\mathbb{G}_m$-*Stack*$_X$. This can be seen through the points: for $\operatorname{Spec} B \to X = \operatorname{Spec} A$, then [5]

$$\mathbb{G}_{m,X}(B) \simeq \mathrm{Map}_{\mathrm{Mod}_B}^{\simeq}(B, B) \circlearrowright \mathrm{Map}_{\mathrm{Mod}_A}(\mathcal{E}, B)$$

Example 8.8. Let $\mathcal{E} = \mathcal{O}_X$, $X = \operatorname{Spec} A$, then for every A-algebra B,

$$\mathbb{V}(\mathcal{O}_X^\vee)(B) \simeq \mathrm{Map}_{\mathrm{Mod}_A}(A^\vee, B) \simeq B$$

[6] And the $\mathbb{G}_m$-action is the natural action on the space.

Example 8.9. Let $S := \operatorname{Spec} A$ be an affine scheme, then $\mathrm{QCoh}(S) \simeq \mathrm{Mod}_A$. $E = \mathbb{V}(\mathcal{O}_S^\vee) \to S$, thus $\mathcal{O}_E = \mathrm{Sym}_{\mathcal{O}_S} \mathcal{O}_S^\vee$. This is naturally an $\mathcal{O}_S$-module, so that

$$\pi_* : \mathrm{QCoh}(E) \to \mathrm{QCoh}(S)$$

satisfies

$$\pi_* \mathcal{O}_E = \mathrm{Sym}_{\mathcal{O}_S} \mathcal{O}_S^\vee \simeq \mathbb{A}_S^1 \to S$$

Remark 8.10. Why dual? Consider $\mathrm{Sym}_k V^\vee$, regarded as an algebra.

$$\mathrm{Map}_{\mathrm{Alg}_k}(\mathrm{Sym}_k V^\vee, B) \simeq \mathrm{Map}_{\mathrm{Mod}_k^\heartsuit}(V^\vee, B)$$

but we know that

$$V \otimes_k B \simeq \mathrm{map}_{\mathrm{Mod}_k^\heartsuit}(V^\vee, B)$$

By property of dualizable objects.

Definition 8.11. $\mathcal{L} \in \mathrm{Mod}_{\mathcal{O}_X}$ is invertible if $\mathcal{L} \otimes - : \mathrm{Mod}_{\mathcal{O}_X} \to \mathrm{Mod}_{\mathcal{O}_X}$ is an equivalence.

Example 8.12. If $X = \operatorname{Spec} R$, then $\mathcal{L}$ is invertible iff it corresponds to a rank one projective module over R.

It may be slightly unintuitive but we have the equivalence

$$\mathrm{Mod}_{\mathcal{O}_X}^{\text{loc.free rk 1}} \simeq \mathrm{Mod}_{\mathcal{O}_X}^{\text{invertible}}$$

and line bundles, those locally free isomorphic that $L \times \mathbb{A}_X^1 \to X$, has globalls ections which are locally free of rank 1.

Example 8.13. $\mathbb{G}_m$ bundle. Let $X = \operatorname{Spec} B$, then

$$X \to B\mathbb{G}_m$$

corresponds to a $\mathbb{G}_m$ torsor hence line bundle. The associated $\mathbb{G}_m$torsor can all be constructed as follows

$$\mathrm{Spec}_X \left(\mathrm{Sym}_B L[L^{-1}]\right)$$

For instance if we take the rank 1 module B, then we are considering $B[x][x^{-1}]$.

where Spec_X is as in **??**.

[5] Or more generally, given that that the geometric morphism $X \to \mathrm{pt}$ is connected

[6] It should be the case that

8.3. **Symmetric algebra construction.** Note we always have a functor

$$\mathrm{Mod}_{R,\geq 0} \to \mathrm{Mod}_{R,\geq 0}$$
$$M \mapsto \mathrm{Sym}^n_R(M) := \pi_0(M \otimes_A M \cdots \otimes_A M)_{\Sigma_n}$$

Example 8.14. If M were rank 1.

9. Nilpotent support condition

Let us recall the definition of nilpotent cone. Classically, one constructs the nilpotent cone as the pullback.

$$\begin{tikzcd} \mathcal{N} \arrow[rr] \arrow[d] \arrow[drr, phantom, "\lrcorner"] & & \check{\mathfrak{g}} \arrow[d] \\ 0 \arrow[r] & \check{\mathfrak{g}}//\check{G} := \mathrm{Spec}(\mathrm{Sym}(\check{\mathfrak{g}})^{\check{G}}) & \end{tikzcd}$$

In [AG15], they constructed global nilpotent cone, as

$$\mathrm{Arth}_{\check{G}} = \mathrm{Sing}(\mathrm{LocSys}_{\check{G}})$$

Theorem 9.1.

10. L-parameters

References: for an introduction, see [Sch21], [Che23]. The main papers of discussions are [Zhu21], [FS24]. Further notes, Bvchurch.

Local Langlands correspondence predicts [7]

$$\pi_0 \, \mathrm{Rep}_{\mathbb{C}}^{\mathrm{Irr,sm}}(G(E)) \to \{L\text{-parameters}\}$$

(1) How does one define the arithmetic version of L-parameters?

(2) What is the geometric version of parameters? How does it relate to the construction in [FS24].

Example 10.1 (Harris-Taylor)**.** They proved the map to be an isomorphism in the case of $G = \mathrm{GL}_n$.

10.1. Breaking down field extensions.

10.2. **Discretizing the unramified part.** As a first approximation, one replaces $\Gamma_E := \mathrm{Gal}(\bar{E}/E)$ with W_E. Every finite extension K/E is still a local field with ring of integers $\mathcal{O}_K$. There is a a canonical extension of valuation $v : K \to \mathbb{Z}$ extending that of E.

Definition 10.2. An algebraic extension K/E is *unramified* if $e_{E/F} := v(\varpi_K)/v(\varpi_E)$, is 1.

These are extensions extensions of the finite field, $\mathbb{F}_q$,

Example 10.3. Unramified extensions of function field. $\mathbb{F}_{q^n}((\varpi))/\mathbb{F}_q((\varpi))$.

Note that $|\mathbb{F}_{q^n} : \mathbb{F}_q| = q^{n-1}$. From Example 10.3, we see that one obtains unramified extensions by adjoining all roots of unity of order coprime to 1, see Example 10.4.

Example 10.4. $\bar{\mathbb{F}}_p$ obtained from $\mathbb{F}_p$.

[7] Usually, work with $\mathbb{C}$ coefficients. Because you work with $\mathbb{C}$-coefficeinnts, there's a canonical square root of $\sqrt{q}$. This is actually implicit in the assignment.

We have the following diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & I_E & \longrightarrow & W_E & \longrightarrow & \mathbb{Z} & \longrightarrow & 1 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 1 & \longrightarrow & I_E & \longrightarrow & \Gamma_E & \longrightarrow & \Gamma_{\mathbb{F}_q} \simeq \widehat{\mathbb{Z}} & \longrightarrow & 1 \end{array}$$

But what does this mean?

10.3. **Discretizing the ramified part.** Fix a compatible system roots of unity $\{\zeta_n\}_{p \neq n}$.

$$K^t := \bigcup_{p \neq n} K^{\mathrm{nr}}(\varpi_E^{1/n})$$

This further fits in the following diagram

$$1 \longrightarrow W_E/P_E \longrightarrow I_E/P_E \longrightarrow W_E/I_E \longrightarrow 1$$

Note for the inertia group we have the following sequence which denote as the projection

$$I_E \longrightarrow I_E^t := I_E/P_E \simeq \prod_{l' \neq p} \mathbb{Z}_{l'} \longrightarrow \mathbb{Z}_l$$

In fact one can understand quite well the normalizing action of $\Gamma_E \circlearrowleft I_E$ under the maps t and t_l.

Proposition 10.5. *[HC]*

10.4. **Various definition of L-parameters.** There are at less three different definition of Langlands parameter of $\overline{\mathbb{Q}}_\ell$.

(1) Paris (r, N).

(2) ${}^L\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ which are l-adically continuous, see Definition 10.8.

The approach taken by, [DHKM24] is the third, this follows by "discreteizing" the tame inertia group I_E/P_E, as explained in [..]

10.5. **Langlands dual group.** References, [GH22]. Let $G \in \mathrm{AlgGrp}_{\mathbb{F}}^{\text{spl, red}}$. If one records the data of torus, and denote $\mathrm{Spl}_{\mathbb{F}}$, then we have the following commutative diagram

$$\begin{tikzcd}
\mathrm{Spl}_{\mathbb{F}} \arrow[rr, "\simeq"] \arrow[dr, "\simeq"] & & \mathrm{Spl}_{\mathbb{F}'} \arrow[dl, "\simeq"] \\
& \mathrm{RRD} &
\end{tikzcd}$$

Where the map sends

$$(G, T) \mapsto \Psi(G, T)$$

From $G \in \mathrm{AlgGrp}_{E}^{\text{spl,red}}$ this arises the dual grp $\widehat{G}/\mathbb{Z}$. This has an action of $\Gamma_E \twoheadrightarrow Q$, which factors through a finite quotient.

Next we briefly recalling the construction of L-group. Is that there is an short exact sequence [GH22, Prop. 7.3.3]

$$1 \to \mathrm{Inn}(G) \to \mathrm{Aut}(G) \to \mathrm{Aut}(\Psi) \to 1$$

Example 10.6. Consider a torus over $T/\mathbb{Q}$. $\Gamma_{\mathbb{Q}}$.

Definition 10.7. ${}^L G := \widehat{G} \rtimes Q$

This seems awkward: it seems dependent with Q, but should not matter.

10.6. **Comparison of various definitions.**

Definition 10.8. An L-parameter over $\mathbb{C}$ is a continuous map

$$\begin{tikzcd}
W_E \arrow[rr] \arrow[dr] & & {}^L G(\mathbb{C}) \arrow[dl] \\
& W_E &
\end{tikzcd}$$

Equivalenty, this is a continuous 1 cocycle [8]

$$W_E \to \widehat{G}(\mathbb{C})$$

These are the kind of parameters we can attach to representations. In fact, this does not really matter if we change $\mathbb{C}$, to any Λ a $\mathbb{Z}_l$-algebra.

Proposition 10.9. *continuity iff factours thorugh a discrete quotient W_E/I', $I' \hookrightarrow I_E$ open finite index subgroup.*

Proof. The topology of complex numbers is incompatible with the inertia subgroup. □

Deligne: it is better to also keep track of a monodromy operator. N.

10.10. Some questions.

- are requiring the image of frobenius to be ss. in $\widehat{G}$? A: I might do that but for the moment I don't wannt to .

[8]Recall that if $G \circlearrowright^{\varphi} A$, then a continuous cocycle f is the condition that $f(gh)=f(g) + \varphi(g)f(h)$ for $g, h \in G$.

- Last time you explained what naturally arises form geometric Satake is a semidirect product, where the Weil group actionn is tiwsted. Don't you want a similar twist?

10.11. Take 2. Definitions.

Definition 10.12. A L-parameter $/\mathbb{C}$ is a pair (φ, N) where

$$\phi : W_E \to {}^L G(\mathbb{C})$$

cts grp homomorphism, $N \in \mathrm{Lie}\widehat{\mathfrak{g}} \otimes \mathbb{C}$ st. for all $w \in W_E$,

$$\mathrm{Ad}(\varphi(w))(N) = q^{|w|} N$$

For $G = \mathrm{GL}_n$ these are also called the Weil Deligne representations. I will discuss this later.

10.13. There is also a further refinement that does not play a role. Take 3.

Definition 10.14. A L-parameter $/\mathbb{C}$ is (φ, r) where

$$\varphi : W_E \to {}^L G(\mathbb{C})$$

cts grp homomorphism st.

$$r : \mathrm{SL}_2 \to \widehat{G}/\mathbb{C}$$

st (r, φ) commute $W_E \times \mathrm{SL}_2 \to {}^L G$.

Then

$$\varphi'(w) := \varphi(w) r(\mathrm{dia}(q^{|w|/2}, q^{-|w|/2}))$$

with $N = \mathrm{Lie}\, r \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

The reason I mentioned take 3 is this is what appears in the modern discussion. All takes give rise to a variety.

10.15. Q: Do you need to say that monodromy operator in Take 2 is nilptoent? A: good question, *No.* The condition in fact implies all ev. of N are 0.

10.16. Parameters in sense of Take 2 and 3 are up to $\widehat{G}(\mathbb{C})$ cjg, in bijection, but scheme structures are different.

- In take 2, $N \neq 0$ can degenerate to $N = 0$.
- In take 3, SL_2 has "rigid" representations.

We would *want* these degenerations. So take 2 is the correct one.

10.17. Deligne's motivation. Fix $\mathbb{C} \simeq \bar{\mathbb{Q}}_l$. Take 2'.

Definition 10.18. An L-parameter over $\bar{\mathbb{Q}}_l$ is a continuous group homomorphism

$$\begin{tikzcd} \varphi : W_E \arrow[rr] \arrow[dr] & & {}^L G(\bar{\mathbb{Q}}_l) \arrow[dl] \\ & Q & \end{tikzcd}$$

Equivlently a continuous 1-cocycle $W_E \to \widehat{G}(\mathbb{C})$.

Theorem 10.19. *Grothendieck Deligne. Take 2 and Take 2' are equivalent: Fix a trivialization of $\mathbb{Z}_l(1) \simeq \mathbb{Z}_l$ and a Frobenius element $\Phi \in W_E$. Once you made these choices, we get a retract from the short exact sequence.*

$$t_l : \; W_E \longrightarrow I_E \longrightarrow \mathbb{Z}_l(1) \simeq \mathbb{Z}_l$$

Then any continuous grp homomorphism φ_l. is of the form

$$\varphi_l(w) = \varphi(w) \exp(t_l(w) \cdot N)$$

for a *unique* L-parameter (φ, N) in the sense of Take 2. The key point:

$$W_E \to \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$$

need not be trivial on an open subgro $I' \hookrightarrow I_E$, can only find I' such it factors over $I' \twoheadrightarrow \mathbb{Z}_l$, then $\mathrm{Hom}(\mathbb{Z}_l, \mathrm{GL}_2(\bar{\mathbb{Q}}_l)$ Then

$$\mathrm{Hom}(\mathbb{Z}_l, \mathrm{GL}_2(\mathbb{Q}_l))$$

are, on an open open subgrp:

$$x \mapsto \exp(xN)$$

where N is uniptoent matrix. Thus, we are looking at matrices N st... [32:22]

10.20. Note: to go from l-adic parameter to a WD type parameter, is *not canonical*. Upto iso, some how not matter. For me now what is most canonical is the cts group homomorphism. We will now adopt Take 2' as the definition.

- For this reason, we are forced to work over $\mathbb{Z}_l$.

Goal: construct a moduli space of L-parameters. i.e. scheme locally of finite type.

$$Z^1(W_E, \widehat{G})/\mathbb{Z}_l$$

st. A valued points are the continuous grp homomorphisms. i.e. its 1- cocyles.

$$\begin{tikzcd} W_E \arrow[rr] \arrow[dr] & & {}^L G(A) \arrow[dl] \\ & Q & \end{tikzcd}$$

We *definitely* want: nontrivial topology. Any $\mathbb{Z}_l$-module M can be endowed with the filtered colimit topology. i.e.

$$M \simeq \varinjlim_{M' \hookrightarrow M, \text{fin.gen}} M'$$

equivalently, something is open iff its restriction to any of the M' is open. But this is a mismatch.

10.21. In the Language of condensed mathematics, there is always a fully faithful embedding [CS19, Prop 1.7]

$$\mathrm{TopSpc}^{\kappa \mathrm{cg}} \hookrightarrow \mathrm{CondSet}$$

from κ-compactly generated spaces to κ condensed sets which preserves products. In other words, this induces a map. the correspondence, condensed group:

$$\underline{M} = M_{\mathrm{disc}} \otimes_{\mathbb{Z}_{l,\mathrm{disc}}} \mathbb{Z}_l$$

why is this the same?

- All operations commutes with filtered colimits.

- What one has to check is that the map factors through.

10.22. So a priori: this might be a derived scheme. If it would be a derived scheme, then the usual topological framework is not so good to talk. You would have to mix topology and homotopy.

- If you stick a dg-algebra A result does turn out to be classical.

10.23.

Theorem 10.24. *There is a scheme $Z^1(W_E, \widehat{G})/\mathbb{Z}_l$ of L-parameters for G - it is a disjoint union of affine scheme of finite type over $\mathbb{Z}_l$,that are flat, complete intersections and of dimension $\dim G = . \dim \widehat{G}$. In the usual Langlands, this is the local systems.*

Note:

- can divide by cjg of $\widehat{G}$ and get an Artin stack $\mathrm{LS}_{\widehat{G}}$.
- The natural extension to animated $\mathbb{Z}_l$-algebras is the same moduli space.
- Usual this scheme is only affine.
- I will explain the index set. The connected components are much more subtle.

Proof. Any cts 1 -cocycle:

$$\varphi : W_E \to \widehat{G}(A)$$

is trivial on an open subgrp P of wild inertia. This implies alreadyw e have the union

$$Z^1(W_E, \widehat{G}) = \bigcup_P Z^1(W_E/P, \widehat{G})$$

- The transion maps are open and closed immersions. Why is this ? To understand $W_E/P, W_E/P'$.
- We somehow look at the locus of elements of order p in side $\widehat{G}$. [51:35]

□

Convention.

- Let E be narc. local field, G/E red. grp.
- Let $l \neq p$, $\widehat{G}/\mathbb{Z}_l$ dual grp, this is canonically split.
- This comes equipped with an action W_E, there's an algebraic one, which factors through a finite quotient, the other one related to cyclotomic twist.
- Fix $\sqrt{q}$. So for all occurences $\mathbb{Z}_l$ relplace it with $\mathbb{Z}_l[\sqrt{q}]$.

11. Recollection of stacks of local system

References. [Zhu21]. Underlying the following story, there is the Hidden smoothness philosophy. In this section we denote $\mathrm{Stk}_k := \mathrm{Shv}(\mathcal{S})$.

Example 11.1. The circle. $\mathrm{Loc}_G(S^1) = G/G$.

11.1. Local systems in geometric Langlands.

[BSV, Appendix C.2]

There are three different stacks of local system:

(1) de Rham stacks
(2) Betti context: $\mathbb{F} = \mathbb{C}$, k is algebraically closed of characteristic 0.
(3) stack of restricted local system $\mathrm{Loc}^{\text{ét}}_{\hat{G}}$.

11.2. Representation schemes. Let Γ be a grp. M an affine group scheme over a ring k. We can define teh representation stack $\mathcal{R}_{\Gamma,M}$ i.e. group homomorphisms $\Gamma \to M(A)$.

Proposition 11.2. *Assumptions: k is Noetherian.*

- *M sm. affine group scheme of dim d.*
- *Γ fg. of time $FP_\infty(k)$. Meaning, there exists a resolution of k $P^\bullet \to k$ by finite projective modules $k[\Gamma]$,*

11.3. Betti moduli stack. If $C \in \mathrm{SmProj}_{\mathbb{C}}$, then for $c_0 \in C$, we have a presentation

Proposition 11.3. *Let C be smooth orientable genus g curve,*

$$\pi_1(C, c_0) \simeq \left\langle a_1, \dots, a_g, b_1, \dots, b_g \;:, \prod_{i=1}^{g} [a_i, b_i] = 1 \right\rangle$$

Hence, this impose some compactness conditions on $\pi_1(C)$ - being finitely presented.

Example 11.4. Fundamental group of $T^2 \simeq \mathbb{R}^2/\mathbb{Z}^2$ two torus. here.

We define the derived fiber product

$$e \longrightarrow \mathrm{GL}_n$$

Example 11.5. $C = \mathbb{CP}^n$. Then we have that

$$\pi_k \mathbb{CP}^n = \begin{cases} * & k = 0 \\ 1 & k = 2 \end{cases}$$

Indeed we can use the Hopf fibration, with $S^{2n+1} \hookrightarrow \mathbb{C}^{n+1}$, inducing

$$\begin{tikzcd} U(1) \simeq S^1 \arrow[r] & S^{2n+1} \arrow[d, two heads] \\ & \mathbb{CP}^n \end{tikzcd}$$

for all $n \geq 1$. THe details of computations are here. Thus we have

$$\mathbb{Z} \longrightarrow \pi_1(S^{2n+1}) \longrightarrow \pi_1(\mathbb{CP}^n) \longrightarrow 0$$

11.4. **de Rham moduli stack.**

Definition 11.6. Let $G \in \mathrm{AffGrp}_{\mathbb{F}}$. We set

$$\mathrm{LS}_G^{\mathrm{dR}}(S) := \mathrm{Map}(\Sigma_{\mathrm{dR}} \times S, BG) \simeq$$

11.5. **étale case.** This is apparently referred as the étale case. Our coefficient is $e = \bar{\mathbb{Q}}_l$. Suppose X has genus $g > 0$. Naı̈vely this should copy the definition of $\mathrm{LS}_G^{\mathrm{Betti}}$.

Example 11.7. Continuous homomorphism $\mathrm{Map}_{\mathrm{GrpCts}}(\widehat{\mathbb{Z}}^{2g}, e^\times) \simeq (\mathcal{O}_e^\times)^{2g}$.

In general, if some one wish to find a scheme over e whose e points $\mathcal{O}_E^\times$ it is slightly hard.

12. Scholze's L-parameter stack

References. Lecture 12-02-21.

There are two sides to Langland's correspondence.

(1) Automoprhic side: $D(G(E), \mathbb{Z}_l)$, the cat of sm. $G(E)$-repns. This embeds ff.

$$D(G(E), \mathbb{Z}_l) \hookrightarrow D_{lis}(\mathrm{Bun}_G, \mathbb{Z}_l)$$

This is a variant of $D_{\text{ét}}$ that works for $\mathbb{Z}_l$-algebra Λ, uses solid 6-functor formalism

On the Galois side, we have the Artin stack of L-parameters.

$$Z^1(W_E, \widehat{G})/\widehat{G}$$

of L-parameters.

12.1. What is classically done:

$$\text{irreducible object} \mapsto \text{point}$$

$$\pi \mapsto \varphi_\pi$$

but this should vary algebraically.

12.2.

Definition 12.3. The Bernstien center of G is the *commutative* algebra of endomorphisms of the identity functor on $\mathrm{Rep}^{\mathrm{sm}}(G(E))$.

- For each π, we give

$$f(\pi) : \pi \to \pi$$

In particular, if $f \in Z(G)$, $\pi \in \mathrm{Irr}_{\bar{\mathbb{Q}}_l}(G)$. Schur's is true in this setting, $\mathrm{End}(\pi) \simeq \bar{\mathbb{Q}}_l$.

we get scalar $f(\pi) \in \bar{\mathbb{Q}}_l$. So we get a function

$$Z(G)_{\bar{\mathbb{Q}}_l} \to \left\{\text{functions on } \mathrm{Irr}_{\bar{\mathbb{Q}}_l}(G)\right\}$$

In some sense: this should be thought as "the algebraic functions on the set $\mathrm{Irr}_{\bar{\mathbb{Q}}_l}(G)$.

- Bernstein center is not quite of finnite type.

12.4. We want: for any $f \in \mathcal{O}(Z^1(W_E, \widehat{G})^{\widehat{G}}$ the map

$$\pi \mapsto f(\varphi_\pi)$$

Definition 12.5. The *Spectral Bernstein center* is

$$Z^{\text{spec}}(G) := \mathcal{O}(Z^1(W_E, \widehat{G})^{\widehat{G}}$$

also consider

$$Z^{\text{geom}}(G) = \text{"bernstein center} D_{lis}(\text{Bun}_G, \mathbb{Z}_l) = \text{End}(\text{id}) \to Z(G)$$

Theorem 12.6 (Fargues-S)**.** *There exists a canonical map*

$$\psi : Z^{\text{Spec}}(G) \to Z^{geom}(G)/\mathbb{Z}_l$$

12.7. What does this mean concretely? For each $A \in D_{\text{lisse}}(\text{Bun}_G, L)$, $L/\mathbb{Z}_l$ ac. closed filed $\text{End}(A) = L$. There exists unique upto conjguation L-parameter

$$\varphi_A : W_E \to \widehat{G}(L)$$

ss. st. for all $f \in Z^{\text{Spec}}(G)$.

$$f(\varphi_A) = \psi(f)(A) \in L$$

...[36:44]

12.1. Construction of $\psi : Z^{\mathbf{spec}}(G) \to Z^{\mathbf{geom}}(G)$.

12.8. We have the following, for any ∞-cat. $C := D_{lis}(\text{Bun}_G, \mathbb{Z}_l)$, for any set I an exact monoidal functor

$$W_E \twoheadrightarrow Q \circlearrowleft \widehat{G}$$

$$\text{Rep}_{\mathbb{Z}_l}(\widehat{G} \rtimes Q)^I \to \text{End}(\mathcal{C})^{W_E^I}$$

- I.e. when ever you have a representation, you can always build an action to C, the "Hecke action", which W_E^I. This is linear over $\text{Rep}_{\mathbb{Z}_l}(Q^I)$, functorially in I.
- This comes from the Hecke action.

We will only need this abstract data. This is also the same kind of formal structure you get in Betti geometric Langlands.

12.9.

Proposition 12.10. *For any $P \in \mathcal{C}^\omega$, exists $p \hookrightarrow W_E$ open in wild inertia, st. for all I $V \in \text{Rep}(\widehat{G} \rtimes Q)^I$, the W_E^I action on $T_V(A)$ factors over W_E/P^I.*

This basically means we can replace W_E by W_E/P. Then, as last time, by discretization $W \hookrightarrow W_E/P$.

12.11. Last time: we can compute invariant functions

Theorem 12.12.

$$\varinjlim_{n, F_n \to W} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \xrightarrow{\simeq} \mathcal{O}(Z^1(W, \widehat{G})^{\widehat{G}} \dashrightarrow Z^{geom}(G) = \text{End}(id_G)$$

The theorem tells us that to establish our goal $\dashrightarrow$ it is suffiicent to construct the map from the colimit.

Definition 12.13. An <u>excursion datum</u> is a tuple $(I, V \in \mathrm{Rep}(\widehat{G} \rtimes Q)^I$, $\alpha : 1 \to V|_{\Delta\widehat{G}}, \beta : V|_{\Delta\widehat{G}} \to 1, (\gamma_i \in W)_{i \in I}$.

Given excursion data, the excursion operators is the following element of $\mathrm{End}(\mathrm{id}_{\mathcal{C}})$. For any $A \in \mathcal{C}$.

$$A = T_1(A) \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i \in I}} T_V(A) \xrightarrow{\beta} T_1(A) = A$$

Note: $T_V(A)$ has *a lot of endomorphisms*. A priori A does not. This is because we have this equivariance result. There's a bit of translation to do here, but

Proposition 12.14. *The excursion operators define a map*

$$colim_{n, F_n \to W} \mathcal{O}(\widehat{G}^n)^{\widehat{G}} \to \mathrm{End}(id_{\mathcal{C}})$$

12.15. Corollary: the L-parameters are characterized as follows: for all excursion data: the scalar

$$L \xrightarrow{\alpha} V \xrightarrow{\varphi_A(\gamma_i)} V \xrightarrow{\beta} L$$

Agrees with the scalar - Here we assume Schur irreducibility.

$$A \xrightarrow{\alpha} T_V(A) \xrightarrow{(\gamma_i)_{i \in I}} T_V(A) \xrightarrow{\beta} A$$

13. Spectral Action

13.1.

Theorem 13.2 (Nadler Yun, GKRV)**.** *The data above is equivalent to an actin of*

$$Perf(Z^1(W_E, \check{G}/\check{G}) \circlearrowleft D_{lis}(Bun, \mathbb{Z}_l)$$

The previous works over $\mathbb{Q}_l$.

13.3. Let us assume you have no Q. One has a map

$$\begin{tikzcd} \mathrm{Rep}(\widehat{G} \rtimes Q)^I \arrow[r] & \mathrm{Perf}(Z^1(W_E, \widehat{G})/\widehat{G})^{W_E^I} \arrow[d] \\ & \mathrm{End}(\mathcal{C})^{W_E^I} \end{tikzcd}$$

13.4. What does this mean for "elliptic L-parameters? Assume for simplicity G ss. coefficient $\bar{\mathbb{Q}}_l$. We say that φ is elliptic if it defines an isolated component of $Z^1(W_E, \widehat{G})/\widehat{G}$. [1:04:00]

24th June 2021 04-06-21

This is joint work with David Hansen, for a finite type map $f : X \to S$. There is a good notion of *pervisity over S*

Definition 13.5. $A \in D_{\text{ét}}(X, \Lambda)$ is *perverse over S* iff for all geometric fibers $X_{\bar{s}}$, $\bar{s} \to S$ a geometric point $A\Big|_{X_{\bar{s}}} \in D_{\text{ét}}(X_{\bar{s}}, \Lambda)$ is perverse.

This interacts very will *ULA* sheaves[9]. That is

[9] For coherent sheaves there's a notion of flat family. This is what ULA sheaves roughly corresponds to.

Proposition 13.6. *$A \in D_{ét}(X, \Lambda)$ is ULA, iff ${}^{p/S}\mathcal{H}^i(A)$ is ULA and $\mathbb{D}_{X/S}(A)$ is ULA.*

Proposition 13.7. *A is ULA + perverse over S implies all subquotients of A are ULA and $\mathbb{D}_{X/S}(A)$ is ULA and perverse over S.*

All the proofs are based on two key ingredients:

(1) v-descent: this allows us to reduce all question to case when $S = \operatorname{Spec} V$, where V is a valuation ring with fraction field K, algebraically closed. [10]

(2) theory of nearby cycles

[10] These are pretty big rings with very nice properties.