# DAY II, TALK 2. FACTORIZATION I

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Contents



## 1. Factorization spaces

1.1. The purpose of this talk is to introduce *factorization*, an important aspect of local geometric Langlands.

1.2. **The affine Grassmannian.** The first place one observes factorization is in the geometry of the *affine Grassmannian*.

Let $G$ be an affine algebraic group, let $x \in X$ be a point in a smooth curve, let $O_x = k[[t_x]]$ be the formal power series based at $x$, and let $K_x = k((t_x))$ the formal Laurent series (here $t_x$ is a coordinate at $x$: its only role is to clarify the notation).

1.3. The affine Grassmannian $\mathrm{Gr}_{G,x}$ is a certain indscheme with $k$-points $\mathrm{Gr}_{G,x}(k)$ the quotient $G(K_x)/G(O_x)$.

Formally, its $S$-points are:

$$\mathrm{Gr}_{G,x}(S) := \Big\{ \ \mathcal{P}_G \text{ a } G\text{-bundle on } \mathcal{D}_{x,S} \text{ with a trivialization on } \overset{o}{\mathcal{D}}_{x,S}. \ \Big\}$$

Here, if $S = \mathrm{Spec}(A)$ then $\mathcal{D}_{x,S} := \mathrm{Spec}(A[[t_x]])$ and $\overset{o}{\mathcal{D}}_{x,S} := \mathrm{Spec}(A((t_x)))$. We note that, although these are schemes highly of infinite type over $k$, the notion of $G$-bundle on any scheme makes perfect sense.

1.4. There is a global description as well, and which has the advantage of avoiding the disc and the punctured disc:

$$\mathrm{Gr}_{G,x}(S) := \Big\{ \ \mathcal{P}_G \text{ a } G\text{-bundle on } X \times S \text{ with a trivialization on } (X \backslash x) \times S. \ \Big\}$$

Indeed, there is a clear restriction map from the latter moduli problem to the former, and it is an equivalence by fpqc descent.

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1.5. **Multi-point version.** Beilinson and Drinfeld observed that the above description occurs as a specialization of various prestacks.

For each integer $n \geqslant 0$, define $\mathrm{Gr}_{G,X^n}$ as the prestack:

$$\mathrm{Gr}_{G,X^n}(S) := \left\{ \begin{array}{l} x = (x_i)_{i=1}^n : S \to X^n,\ \mathcal{P}_G \text{ a } G\text{-bundle on } X \times S \\ \text{with a trivialization on } X \times S \backslash \big( \cup_{i=1}^n \Gamma_{x_i} \big). \end{array} \right\}$$

Here $\Gamma_{x_i} \subseteq X \times S$ is the graph of the map $x_i : S \to X$ and the union is best understood as the sum of effective divisors.

One can show that $\mathrm{Gr}_{G,X^n}$ is an indscheme of ind-finite type, and that it is ind-proper for $G$ reductive. It is equipped with a canonical structure map $\mathrm{Gr}_{G,X^n} \to X^n$, and there is a canonical section to this structure map defined by the triple:

$$(\mathrm{id}_{X^n} : X^n \to X^n, \mathcal{P}_G^{triv}, \text{ the given trivialization}) \in \mathrm{Hom}(X^n, \mathrm{Gr}_{G,X^n}).$$

*Remark* 1.5.1. For a point $x \in X(k)$, the fiber of $\mathrm{Gr}_{G,X^n}$ over the point $(x, \ldots, x) \in X^n$ is exactly $\mathrm{Gr}_{G,x}$.

*Example* 1.5.2. For $G = \mathbb{G}_m$, $\mathrm{Gr}_{G,X^n}$ can be interpreted as the moduli of points $x_1, \ldots, x_n \in X$ plus a divisor on $X$ supported on the $\{x_1, \ldots, x_n\}$.

*Remark* 1.5.3. We can further refine the above construction to descend $\mathrm{Gr}_{G,X^n}$ to a prestack $\mathrm{Gr}_{G,X^n_{dR}}$ over $X^n_{dR}$. Indeed, one takes the $S$-points of this space as:

$$\mathrm{Gr}_{G,X^n_{dR}}(S) := \left\{ \begin{array}{l} x = (x_i)_{i=1}^n : S^{red} \to X^n,\ \mathcal{P}_G \text{ a } G\text{-bundle on } X \times S \\ \text{with a trivialization on } X \times S \backslash \big( \cup_{i=1}^n \Gamma_{x_i} \big). \end{array} \right\}$$

where $\Gamma_{x_i} \subseteq X \times S^{red} \subseteq X \times S$ indicates the graph of the map $x_i : S^{red} \to X$. (The main point is that the complement $X \times S \backslash \big( \cup_{i=1}^n \Gamma_{x_i} \big)$ does not depend on the extension of $x$ to a map $S \to X^n$).

1.6. **Factorization of the affine Grassmannian.** As we vary $n$, the spaces $\mathrm{Gr}_{G,X^n}$ satisfy various combinatorial patterns. We have already observed one: $\mathrm{Gr}_{G,X^n} \times_{X^n} X \simeq \mathrm{Gr}_{G,X}$, where $X \to X^n$ is the diagonal map.

It is convenient in describing these patterns to change our indexing slightly: for $I$ a finite set, we let $\mathrm{Gr}_{G,X^I} \to X^I$ be defined in the obvious way (for a specified total ordering of $I$, it identifies with $\mathrm{Gr}_{G,X^{|I|}}$).

Note that a surjection $f : I \twoheadrightarrow J$ induces a *diagonal map* $\Delta_f : X^J = \mathrm{Hom}(J, X) \to X^I = \mathrm{Hom}(I, X)$ (i.e., $\Delta_f : (x_j)_{j \in J} \mapsto (x_{f(i)})_{i \in I}$), which is a closed embedding.

Then we have two basic combinatorial structures:

**Ran's condition:** For every $I \twoheadrightarrow J$ we have an isomorphism:

$$\mathrm{Gr}_{G,X^I} \underset{X^I}{\times} X^J \simeq \mathrm{Gr}_{G,X^J}$$

**Factorization:** For every decomposition $I = I_1 \coprod I_2$, we have an isomorphism:

$$\mathrm{Gr}_{G,X^I} \underset{X^I}{\times} [X^{I_1} \times X^{I_2}]_{disj} \xrightarrow{\simeq} \Big( \mathrm{Gr}_{G,X^{I_1}} \times \mathrm{Gr}_{G,X^{I_2}} \Big) \underset{X^I}{\times} [X^{I_1} \times X^{I_2}]_{disj}. \quad (1.6.1)$$

Here $[X^{I_1} \times X^{I_2}]_{disj} \subseteq X^{I_1} \times X^{I_2} = X^I$ is the open subscheme whose $k$-points are $(x_i)_{i \in I}$ such that for every $i_1 \in I_1$ and $i_2 \in I_2$ we have $x_{i_1} \neq x_{i_2}$.

We observe that Ran's compatibility here is a complete tautology.

*Proof of factorization.* Suppose that we are given a test scheme $S$ and a map $x = (x_i)_{i \in I} : S \to [X^{I_1} \times X^{I_2}]_{disj}$.

Define Zariski opens $U_1$ and $U_2$ of $X \times S$ by:

$$U_1 := (X \times S) \setminus \cup_{i \in I_1} \Gamma_{x_i}$$
$$U_2 := (X \times S) \setminus \cup_{i \in I_2} \Gamma_{x_i}.$$

Observe that $U_1 \cup U_2 = X \times S$ because $x$ was assumed to map through the disjoint locus, and we have $U_1 \cap U_2 = U_1 := (X \times S) \setminus \cup_{i \in I} \Gamma_{x_i}$. We obtain:

$$\{G\text{–bundles } \mathcal{P}_{G,1}, \mathcal{P}_{G,2} \text{ on } X \text{ with trivialization of } \mathcal{P}_{G,1} \text{ on } U_2 \text{ and trivialization of } \mathcal{P}_{G,2} \text{ on } U_1\} \xrightarrow{\simeq}$$
$$\{G\text{–bundles } \overset{o}{\mathcal{P}}_{G,1} \text{ on } U_1 \text{ and } \overset{o}{\mathcal{P}}_{G,2} \text{ on } U_2 \text{ each with trivialization on } U_1 \cap U_2\} \simeq$$
$$\{\text{A } G\text{–bundle } \mathcal{P}_G \text{ on } X \times S \text{ with a trivialization on } (X \times S) \setminus \cup_{i \in I} \Gamma_{x_i}\}.$$

Here the first map is induced by restriction, and the second map is given by gluing. This gives the isomorphism (1.6.1).

□

*Example* 1.6.1. For $I = \{1, 2\}$, this says that the fiber of $\mathrm{Gr}_{G,X^2}$ over a point $(x, y)$, $x \neq y$, is the product $\mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,y}$, and the fiber over a point $(x, x)$ is $\mathrm{Gr}_{G,x}$.

In this case, the factorization isomorphism follows from covering $X$ by the opens $X \setminus x$ and $X \setminus y$.

*Example* 1.6.2. For the case $G = \mathbb{G}_m$ of Example 1.5.2, the factorization isomorphisms are given by sum of divisors.

The Ran conditions and factorization satisfy various natural compatibilities that we choose to omit here: roughly, any two isomorphisms that the axioms force between two $D$-modules on open subsets of $X^I$ should be required to be the same isomorphism (plus higher categorical versions of this).

We note that these isomorphisms occur already for the corresponding objects defined over $X^I_{dR}$.

1.7. To summarize, this means that $I \mapsto \mathrm{Gr}_{G,X^I_{dR}}$ is a *factorization space*:

*Definition* 1.7.1. A *factorization space* $\mathcal{S}$ over $X$ is a rule:

$$I \mapsto \mathcal{S}_{X^I_{dR}} \in \mathsf{PreStk}_{/X^I_{dR}}$$

defined on non-empty finite sets $I$ equipped with compatible (e.g. associative) isomorphisms of Ran type and factorization isomorphisms.

*Notation* 1.7.2. For a factorization space as above, we will use the notation $\mathcal{S}_{X^I}$ for the base change of $\mathcal{S}_{X^I_{dR}}$ from $X^I_{dR}$ to $X^I$.

*Example* 1.7.3. A morphism $H \to G$ of algebraic groups induces a map $\mathrm{Gr}_H \to \mathrm{Gr}_G$ of factorization spaces, defined by inducing an $H$-bundle to a $G$-bundle. For $H$ the trivial group, we call this map the *unit map*.

## 2. Factorization spaces coming from discs

2.1. A general, intentionally vague principle says that geometry that only has to do with the disc and punctured disc will factorize. We will make this principle more precise below, and give some examples of it.

The reader familiar with $D$-schemes and multijets may safely skip this section.

This material is not needed for understanding the axiomatics of §3, and therefore was relegated to the tutorial. The reader may skip it on a first pass.

2.2. Here is a brief account of the material of this section.

We want to construct various factorization spaces out of the geometry of the disc and the punctured disc. For example, for $Y$ an affine scheme, we want a factorization space whose fiber at a point $x$ is the scheme (resp. ind-scheme) $Y(O_x)$ (resp. $Y(K_x)$), i.e., the moduli of maps from the disc (resp. punctured disc) to $Y$. (We remark that these words will be made precise in what follows).

The first problem one must grapple with is that there are two definitions of the disc at $x$. Given a coordinate $t$ at $x$, we have the *formal disc* $\widehat{\mathcal{D}}_x := \mathrm{Spf}(k[[t]]) = \mathrm{colim}\,\mathrm{Spec}(k[t]/t^n)$, and the *adic disc* $\mathcal{D}_x := \mathrm{Spec}(k[[t]])$.

There is only one option for the punctured disc: we can only take $\mathrm{Spec}(k((t)))$, i.e., the complement of the closed point of $\mathcal{D}_x$.[1]

The formal disc has the advantage that its definition only involves finite type geometry, but it is necessary to use the adic disc when one also wants to work with the punctured disc.

We will give versions of these discs over the curve in §2.3. We then define the factorization spaces of local horizontal sections of a $D$-space and mulitjets of an affine scheme in §2.6 and §2.8 respectively.

2.3. **Parametrized discs.** Suppose that $X$ is a smooth affine curve.

Let $S$ be an affine test scheme and let $x = (x_i)_{i \in I} : S \to X^I$ be a map.

We define the *formal disc* $\widehat{\mathcal{D}}_x$ at $x$ to be the formal completion of $X \times S$ along $\Gamma_x$ (we remind that $\Gamma_x$ is the union of the graphs of the maps $x_i$). Note that $\widehat{\mathcal{D}}_x$ is an ind-affine indscheme.

We define the *adic disc* $\mathcal{D}_x \in \mathsf{AffSch}$ to be the value of the partially defined left adjoint of the functor $\mathsf{AffSch} \hookrightarrow \mathsf{PreStk}$ evaluated on $\widehat{\mathcal{D}}_x$. Note that ind-affineness of $\widehat{\mathcal{D}}_x$ implies that this functor is defined here: it is the spectrum of the limit of the corresponding commutative rings.

*Example* 2.3.1. Suppose $S = \mathrm{Spec}(k)$, so a map $x : \mathrm{Spec}(k) \to X^I$ is equivalent to the data of $x_i \in X(k)$, $i \in I$. We enumerate these points without redundancy as $x_1, \ldots, x_m$ with $x_k \neq x_\ell$ for $1 \leqslant k < \ell \leqslant m$. Then $\widehat{\mathcal{D}}_x = \coprod_{k=1}^m \widehat{\mathcal{D}}_{x_i}$. Moreover, each $\widehat{\mathcal{D}}_{x_k}$ identifies with:[2]

$$\mathrm{Spf}(k[[t_{x_k}]]) := \operatorname*{colim}_n \mathrm{Spec}(k[t_{x_k}]/t_{x_k}^n)$$

for $t_{x_k}$ a uniformizer at $x_k$. For the adic disc: the phrase "spectrum of the limit of the corresponding commutative rings" then plays out here as saying:

$$\mathcal{D}_x := \mathrm{Spec}(\lim_n \prod_{k=1}^m k[t_{x_k}]/t_{x_k}^n) = \mathrm{Spec}(\prod_{k=1}^m k[[t_{x_k}]]) = \coprod_{k=1}^m \mathrm{Spec}(k[[t_{x_k}]]).$$

[1]We advise the reader unfamiliar with these issues and with this claim to try to find another definition.

[2]At the derived level, this identification is proved in [GR]. It may serve as a nice exercise for the reader still becoming accustomed to derived algebraic geometry.

*Remark* 2.3.2. These constructions only depend on the underlying map $S^{red} \to X^I$, so we will sometimes write e.g. $\widehat{\mathcal{D}}_x \to X_{dR} \times S$ even when $x$ is only defined as a map $S^{red} \to X$.

Observe that formation of $\widehat{\mathcal{D}}_x$ is étale local on $X$ in the natural sense.

2.4. We define the *punctured disc* $\overset{o}{\mathcal{D}}_x \in \mathsf{Sch}$ at $x$ as:

$$\overset{o}{\mathcal{D}}_x := \mathcal{D}_x \backslash \Gamma_x.$$

These constructions organize into the diagram:

$$\begin{tikzcd}
\Gamma_x \arrow[r] & \widehat{\mathcal{D}}_x \arrow[r] & \mathcal{D}_x \arrow[d] & \overset{o}{\mathcal{D}}_x \arrow[l] \\
 & & X \times S. &
\end{tikzcd}$$

2.5. **Factorization of the discs.** Let $I = I_1 \cup I_2$. Suppose $x : S \to [X^{I_1} \times X^{I_2}]_{disj}$ with projections $x_1 : S \to X^{I_1}$ and $x_2 : S \to X^{I_2}$.

Then one easily finds:

$$\widehat{\mathcal{D}}_x = \widehat{\mathcal{D}}_{x_1} \coprod \widehat{\mathcal{D}}_{x_2}$$
$$\mathcal{D}_x = \mathcal{D}_{x_1} \coprod \mathcal{D}_{x_2}$$
$$\overset{o}{\mathcal{D}}_x = \overset{o}{\mathcal{D}}_{x_1} \coprod \overset{o}{\mathcal{D}}_{x_2}.$$

These isomorphisms are responsible for the factorization of the spaces constructed below.

2.6. **Factorization spaces associated with $D$-spaces.** Let $\mathcal{Y}_{X_{dR}}$ be a prestack with a map to $X_{dR}$. In this situation, we can associate to $\mathcal{Y}_{X_{dR}}$ a factorization space $\mathcal{Y}$ over $X$ by the formula:

$$\mathcal{Y}_{X^I_{dR}}(S) := \left\{ \begin{array}{l} x = (x_i) : S^{red} \to X^I, \\ \Gamma_{x,dR} \times_{S_{dR}} S \to \mathcal{Y}_{X_{dR}} \text{ a morphism over } X_{dR}. \end{array} \right\} \tag{2.6.1}$$

Here $\Gamma_x \subseteq X \times S^{red}$ indicates the union of the graphs of the maps $x_i : S^{red} \to X$. One easily verifies that this defines a factorization space. Note that for $I = *$, the above formula recovers $\mathcal{Y}_{X_{dR}}$.

2.7. Now suppose that the structure map $\mathcal{Y}_{X_{dR}} \to X_{dR}$ is affine (in particular, schematic). We summarize the situation in saying that $Y := \mathcal{Y}_{X_{dR}} \times_{X_{dR}} X$ is an *affine $D_X$-scheme.*

One can show in this case that $\mathcal{Y}_{X^I_{dR}}$ is an affine $D_{X^I}$-scheme, i.e., the structure map to $X^I_{dR}$ is affine.

2.8. **Multi-jets.** Let $Y \to X$ be an affine morphism. We associate to $Y$ factorization spaces $\mathfrak{J}ets(Y)$ and $\mathfrak{J}ets^{mer}(Y)$.

For $I$ a finite set, we define $\mathfrak{J}ets_{X^I_{dR}}(Y)$ as:

$$\mathfrak{J}ets_{X^I_{dR}}(Y)(S) = \{\ x = (x_i) : S \to X^I_{dR},\ \widehat{\mathcal{D}}_x \to Y \text{ a morphism over } X.\ \}$$

This obviously defines a factorization space. One can show that $\mathfrak{J}ets_{X^I}(Y) := \mathfrak{J}ets_{X^I_{dR}}(Y) \times_{X^I_{dR}} X^I$ is a scheme (of infinite-type).

*Remark* 2.8.1. The above is actually a special case of the material of §2.6. Indeed, the restriction functor:

$$\{\text{Affine } D_X\text{-schemes}\} \to \{\text{Affine } X\text{-schemes}\}$$

admits a right adjoint of restriction of scalars (alias: Weil restriction); applying this right adjoint to $Y$ and taking horizontal sections produces multijets.

2.9. We can obviously replace $\widehat{\mathcal{D}}_x$ by $\mathcal{D}_x$ in the above formula. Therefore, the following definition is reasonable:

$$\mathfrak{J}ets^{mer}_{X^I_{dR}}(Y)(S) = \{\ x = (x_i) : S \to X^I_{dR},\ \overset{o}{\mathcal{D}}_x \to Y \text{ a morphism over } X.\ \}$$

(where we emphasize that $Y \to X$ is assumed affine).

2.10. **Notational convention.** To avoid overburdening the notation, we institute the following convention: for $Y$ an affine scheme, we write $\mathfrak{J}ets_{X^I_{dR}}(Y)$ in place of $\mathfrak{J}ets_{X^I_{dR}}(Y \times X)$, and similarly for $\mathfrak{J}ets^{mer}$. This convention is particularly used for $Y = G$.

2.11. **The affine Grassmannian revisited.** By fpqc descent, we can identify $\mathrm{Gr}_{G,X^I_{dR}}$ with the prestack:

$$S \mapsto \left\{ \begin{array}{l} x = (x_i)_{i \in I} : S^{red} \to X^I,\ \mathcal{P}_G \text{ a } G\text{-bundle on } \mathcal{D}_x \\ \text{with a trivialization on } \overset{o}{\mathcal{D}}_x. \end{array} \right\}$$

This realization makes factorization of $\mathrm{Gr}_G$ doubly-obvious.

We observe that we have an action of $\mathfrak{J}ets^{mer}_{X^I_{dR}}(G)$ on $\mathrm{Gr}_{G,X^I_{dR}}$ by changing the trivialization of the bundle. This action is compatible with factorization in the obvious sense.

## 3. Factorization algebras

3.1. **Informal remarks.** The reader turned off by informality may safely skip this material.

At first approximation, a factorization algebra is a rule that assigns a vector space $\mathcal{A}_{x_1,\ldots,x_n} \in \mathsf{Vect}$ to every subset $\{x_1, \ldots, x_n\}$ in a way "continuous" (and with a connection) with respect to moving points and allowing points to collide; we require the factorization condition that we have isomorphisms:

$$\mathcal{A}_{x_1,\ldots,x_n} \simeq \mathcal{A}_{x_1,\ldots,x_m} \otimes \mathcal{A}_{x_{m+1},\ldots,x_n}$$

for every decomposition $\{x_1, \ldots, x_n\} = \{x_1, \ldots, x_m\} \coprod \{x_{m+1}, \ldots, x_n\}$, these isomorphisms themselves being appropriately transitive and "continuous" with respect to the above operations.

We observe that if we know the fibers of $\mathcal{A}$ at points of $X$, then we know the fibers for every subset of $X$.

Therefore, we can interpret the factorization structure as a sheaf $\mathcal{A}_X$ on $X$ plus a way to glue the above fibers to extend this sheaf to the moduli space of finite subsets of $X$.

One can understand $\mathcal{A}_X$ as a "local" invariant (e.g., encoding the geometry of the disc or punctured disc), and the extension to a factorization sheaf — that specification of behavior as points in $X$ collide — as the extra structure needed to pass from local to global. This factorization structure is impossible in number theory, so it does not have such a good description in the language of usual harmonic analysis.

This principle is realized in a precise way by *chiral homology*, introduced below. We draw the reader's attention to Example 3.6.2, which gives a clear feeling for the local-to-global character of the resulting invariants.

3.2. Factorization algebras occur through linearization of factorization spaces, and the definition is designed for exactly this purpose.

*Definition* 3.2.1. A *factorization algebra* $\mathcal{A}$ on $X$ is a rule that attaches to each non-empty finite set $I$ a $D$-module $\mathcal{A}_{X^I} \in D(X^I)$ equipped with:

**Ran isomorphisms:** For $f : I \twoheadrightarrow J$, we have an isomorphism:

$$\Delta_f^!(\mathcal{A}_{X^I}) \simeq \mathcal{A}_{X^J}.$$

**Factorization isomorphisms:** For every decomposition $I = I_1 \coprod I_2$, we have isomorphisms:

$$\mathcal{A}_{X^I}|_{[X^{I_1}\times X^{I_2}]_{disj}} \simeq \mathcal{A}_{X^{I_1}} \boxtimes \mathcal{A}_{X^{I_2}}|_{[X^{I_1}\times X^{I_2}]_{disj}}.$$

These isomorphisms are required to satisfy the natural compatibilities (the appropriate words to say to spell this out precisely in the derived setting are given in [FG]).

*Example* 3.2.2. $I \mapsto \omega_{X^I} \in D(X^I)$ defines a factorization algebra.

*Example* 3.2.3 (Borel-Moore homology). If $\mathcal{S}$ is a "nice" factorization space with structure maps $p_I : \mathcal{S}_{X^I} \to X^I$, then $p_{I,*,dR}(\omega_{\mathcal{S}_{X^I_{dR}}})$ defines a factorization algebra. Here by *nice*, we mean e.g. that $\mathcal{S}_{X^I}$ is an indscheme of ind-finite type for each $I$.

In particular, the homology of the affine Grassmannian for reductive $G$ forms a factorization algebra.

*Example* 3.2.4 (Kac-Moody algebra). Let $\mathrm{Gr}^{\wedge}_{G,X^I_{dR}}$ denote the formal completion of $\mathrm{Gr}_{G,X^I_{dR}}$ along $X^I_{dR}$, which we consider mapping to $\mathrm{Gr}_{G,X^I_{dR}}$ along the unit map. Let $p_I$ denote the structure map $\mathrm{Gr}^{\wedge}_{G,X^I_{dR}} \to X^I_{dR}$.

Because $\mathrm{Gr}^{\wedge}_{G,X^I}$ is an ind-finite ind-scheme, we have adjoint functors:

$$\mathsf{IndCoh}(\mathrm{Gr}^{\wedge}_{G,X^I_{dR}}) \underset{p_I^!}{\overset{p_{I,*}^{\mathsf{IndCoh}}}{\rightleftarrows}} \mathsf{IndCoh}(X^I_{dR}) = D(X^I).$$

Then $I \mapsto \mathcal{A}^{KM}_{X^I} := p_{I,!}(\omega_{\mathrm{Gr}^{\wedge}_{G,X^I_{dR}}})$ defines a factorization algebra, called the *Kac-Moody* algebra for $G$.

One can compute its fibers explicitly as the "vacuum representation" $U(\mathfrak{g}((t_x)))/U(\mathfrak{g}((t_x))) \cdot \mathfrak{g}[[t_x]]$. Indeed, one knows that for $H_1 \subseteq H_2$ algebraic groups, $\Gamma(H_2/H_1, \delta_1) \simeq U(\mathfrak{h}_2)/U(\mathfrak{h}_2) \cdot \mathfrak{h}_1$; our claim is an infinite-dimensional analogue with $(H_1 \subseteq H_2) = (G(O_x) \subseteq G(K_x))$.

This example will be discussed in more detail later in the conference.

*Remark* 3.2.5. There is a variant of Example 3.2.4 involving the central extension of the loop algebra; one incorporates an appropriate twist involving the determinant line bundle of the affine Grassmannian.

*Example* 3.2.6. If $Y \to X$ is an affine $D$-scheme as in §2.7, then $I \mapsto p_{I,*}(\mathcal{O}_{\mathcal{Y}_{X^I_{dR}}})$ defines a factorization algebra, where $p_I$ is the structure map $\mathcal{Y}_{X^I_{dR}} \to X^I_{dR}$.

The factorization algebras occurring in this way are called *commutative factorization algebras*.

*Remark* 3.2.7. A factorization algebra $\mathcal{A}$ is said to be *classical* if $\mathcal{A}_{X^I}[-|I|] \in D(X^I)^{\heartsuit}$ (it is important here that $X$ be a smooth curve). These are the class of factorization algebras principally studied in [BD]. For example, the Kac-Moody factorization algebra is classical.

3.3. **Ran space.** It order to package the factorization axioms more succinctly, we introduce *Ran's space.*

Let $\mathsf{fSet}$ denote the category of non-empty finite sets under surjective maps. We define:

$$\mathrm{Ran}_X := \operatorname*{colim}_{I \in \mathsf{fSet}^{op}} X^I \in \mathsf{PreStk}.$$

As for any prestack, we have the category $D(\mathrm{Ran}_X) := \mathsf{QCoh}(\mathrm{Ran}_{X,dR})$ of $D$-modules on $\mathrm{Ran}_X$.

Observing that:

$$\mathrm{Ran}_{X,dR} = \operatorname*{colim}_{I \in \mathsf{fSet}^{op}} X^I_{dR}$$

we see that an object of $D(\mathrm{Ran}_X)$ is the same datum as a compatible family of objects of the DG categories $D(X^I)$, i.e., we have:

$$D(\mathrm{Ran}_X) \simeq \lim_{I \in \mathsf{fSet}} D(X^I).$$

*Remark* 3.3.1. Although the structure maps involved in forming $\mathrm{Ran}_X$ are closed embeddings, $\mathrm{Ran}_X$ is not an indscheme because $\mathsf{fSet}^{op}$ is not filtered.

Still, because these structure maps are closed embeddings (in particular: proper), we have a canonical identification:

$$D(\mathrm{Ran}_X) := \operatorname*{colim}_{I \in \mathsf{fSet}^{op}} D(X^I) \in \mathsf{DGCat}_{cont} \tag{3.3.1}$$

where the structure maps are de Rham pushforward functors (recall that the procedure for such identifications has been explained in Tutorial II.2).

*Remark* 3.3.2. Any factorization algebra $\mathcal{A}$ on $X$ has an underlying object $\mathcal{A}_{\mathrm{Ran}_X} \in D(\mathrm{Ran}_X)$: this is a reformulation of Ran's condition (i.e., it doesn't use factorization at all).

*Remark* 3.3.3. Given a factorization space $I \mapsto Y_{X^I_{dR}}$, we obtain a space $Y_{\mathrm{Ran}_{X_{dR}}}$ by passing to the colimit of the spaces $Y_{X^I_{dR}}$.

In particular, we have a space $\mathrm{Gr}_{G,\mathrm{Ran}_{X_{dR}}}$ that, for $X$ proper, is equipped with a canonical map to $\mathrm{Bun}_{G,dR}(X)$.

*Remark* 3.3.4. One can show that for $S$ a classical scheme, one has:

$$\mathrm{Ran}_X(S) = \{\text{a non-empty finite set of maps } S \to X\}.$$

Note that this gives a complete description of $\mathrm{Ran}_{X_{dR}}$.

Therefore, we can consider $\mathrm{Ran}_X$ as the moduli space of non-empty finite subsets of $X$.

3.4. **Chiral homology.** Chiral homology is an important procedure for producing global invariants from factorization algebras.

The identification $D(\mathrm{Ran}_X) \simeq \operatorname{colim}_I D(X^I)$ of (3.3.1) defines a functor $H^*_{dR}(\mathrm{Ran}_X, -) : D(\mathrm{Ran}_X) \to \mathsf{Vect}$ induced by the compatible system of functors $H^*_{dR}(X^I, -)$.

3.5. We now suppose that $X$ is proper. Then $H^*_{dR}(\mathrm{Ran}_X, -)$ is the left adjoint to the upper-! functor $\mathsf{Vect} \to D(\mathrm{Ran}_X)$, $k \mapsto \omega_{\mathrm{Ran}_X}$; indeed, this is an obvious consequence of the limit/colimit formalism giving rise to (3.3.1).

*Definition* 3.5.1. For $\mathcal{A}$ a factorization algebra on $X$, we define the *chiral homology* $H_{ch,*}(\mathcal{A}) \in \mathsf{Vect}$ of $\mathcal{A}$ as $H^*_{dR}(\mathrm{Ran}_X, \mathcal{A}_{\mathrm{Ran}_X})$.

*Remark* 3.5.2. More concretely, we have:

$$H_{ch,*}(\mathcal{A}) \simeq \operatorname*{colim}_I H^*_{dR}(X^I, \mathcal{A}_{X^I}).$$

3.6. **Some computations of chiral homology.** Below we give some examples in which chiral homology can be explicitly computed. The reader may skip this material on a first reading.

*Example* 3.6.1 (Homological contractibility of the Ran space). For $\mathcal{A} = \omega$ (i.e., the factorization algebra $I \mapsto \omega_{X^I}$), we have $H_{ch,*}(\omega) = k$. In other words, $H_*(\mathrm{Ran}_X, k) \simeq k$.[3] Note that this was stated but not proved during the lecture.

Indeed, consider the map $\mathrm{add} : \mathrm{Ran}_X \times \mathrm{Ran}_X \to \mathrm{Ran}_X$ encoding the union of subsets of $X$.

For $n > 0$ the minimal non-vanishing homology degree of $\mathrm{Ran}_X$, we have $H_n(\mathrm{Ran}_X \times \mathrm{Ran}_X, k) \simeq H_n(\mathrm{Ran}_X, k) \oplus H_n(\mathrm{Ran}_X, k)$ induced by the projection maps $\mathrm{Ran}_X \times \mathrm{Ran}_X \to \mathrm{Ran}_X$.

Therefore, for $x \in X$ a fixed point, the two inclusion maps $H_n(\mathrm{Ran}_X, k) \hookrightarrow H_n(\mathrm{Ran}_X \times \mathrm{Ran}_X, k)$ are induced by the two maps $\mathrm{Ran}_X \to \mathrm{Ran}_X \times \mathrm{Ran}_X$ as $s \mapsto (s, x)$ and $s \mapsto (x, s)$.

We also deduce that the diagonal map $\mathrm{Ran}_X \to \mathrm{Ran}_X \times \mathrm{Ran}_X$ induces $\kappa \mapsto (\kappa, \kappa)$ on $n$th homology.

Combining these two claims, we see that:

$$\kappa = \mathrm{add}\,\Delta(\kappa) = \mathrm{add}(\kappa, \kappa) = 2 i_x(\kappa)$$

where $i_x : \mathrm{Ran}_X \to \mathrm{Ran}_X$ is the map of taking disjoint union with $\{x\}$.

But we could have applied our earlier reasoning with $n$-fold products instead to see that $\kappa = n i_x(\kappa)$ for all $n > 1$, therefore giving the result.

*Example* 3.6.2. For a commutative factorization algebra $\mathcal{A}$ associated with an affine $D_X$-scheme $Y$ as in Example 3.2.6, let $\mathrm{Sect}_\nabla(X, Y)$ denote the moduli space of sections of the map $\mathcal{Y}_{X_{dR}} \to X_{dR}$:

$$\mathrm{Sect}_\nabla(X, Y)(S) := \mathrm{Hom}_{/X_{dR}}(X_{dR} \times S, \mathcal{Y}_{X_{dR}}).$$

We recall that the notation $\mathcal{Y}_{X_{dR}} \to X_{dR}$ was introduced in §2.7.

Following [BD] §4, there is an identification:

$$H_{ch,*}(X, \mathcal{A}) \simeq \Gamma(\mathrm{Sect}_\nabla(X, Y), \mathcal{O}_{\mathrm{Sect}_\nabla(X,Y)}).$$

We indicate the proof here.

To construct the map in one direction, observe that are tautological and schematic maps:

$$\mathrm{Sect}_\nabla(X, Y) \times X_{dR} \to Y_{X^I_{dR}}$$

over $X^I_{dR}$. This gives a map:

$$\mathcal{A}_{X^I_{dR}} \to \Gamma(\mathrm{Sect}_\nabla(X, Y), \mathcal{O}_{\mathrm{Sect}_\nabla(X,Y)}) \otimes \omega_{X^I}.$$

[3]Here homology $H_*(\mathcal{Y}, k)$ of a prestack $\mathcal{Y}$ is defined as $\pi_! \pi^!(k)$ for $\pi$ the structure map $\mathcal{Y} \to \mathrm{Spec}(k)$, whenever the left adjoint $\pi_!$ is defined on $\pi^!(k)$.

Passing to the colimit over $I$, applying de Rham cohomology, and applying Example 3.6.1, we obtain the map:

$$H_{ch,*}(X, \mathcal{A}) \to \Gamma(\mathrm{Sect}_\nabla(X, Y), \mathcal{O}_{\mathrm{Sect}_\nabla(X,Y)}).$$

One verifies that it is an equivalence using an appropriate dévissage argument.

## References

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