A Physical Origin for Singular Support Conditions in Geometric Langlands

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Abstract

We explain how the nilpotent singular support condition introduced to the geometric Langlands conjecture by Arinkin and Gaitsgory arises naturally from the point of view of $N = 4$ supersymmetric gauge theory. We define what it means in topological quantum field theory to localize a category of boundary conditions at a choice of vacuum, both in functorial field theory and in the language of factorization algebras. For twisted $N = 4$ gauge theories with gauge group G, the moduli of vacua is equivalent to $\mathfrak{h}^*/W$, and the nilpotent singular support condition arises by localizing at 0. We then investigate the categories obtained by localizing at points in larger strata, and conjecture that these categories are equivalent to the geometric Langlands categories with gauge symmetry broken to a Levi subgroup, and furthermore that by assembling such for the groups GL_n with $n \geq 1$ one finds a hidden factorization structure for the geometric Langlands theory.

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1 Introduction

In their pioneering 2007 paper [KW07], Kapustin and Witten realized that a form of the geometric Langlands conjecture arises as a twisted form of S-duality for $N = 4$ gauge theories. In this work we'll explain how the refinement to the geometric Langlands conjecture introduced by Arinkin and Gaitsgory [AG15] – the restriction to sheaves with nilpotent singular support – naturally arises in 4-dimensional gauge theory by taking into account the action of 4d local operators; that is, restricting to the category of boundary conditions compatible with a fixed choice of vacuum. The data of this choice of vacuum suggests the existence of a refinement to the geometric Langlands conjecture that takes this further data into account – an equivalence compatible with an additional structure coming from interpreting the moduli space of vacua in terms of the motion of D3-branes; this resembles a factorization structure as appearing in the work of Kontsevich–Soibelman [KS11]. We'll conclude by discussing some partial results in that direction.

From the gauge theoretic point of view, we see Arinkin–Gaitsgory singular support conditions by considering the *localization* of the category of boundary conditions in Kapustin–Witten theory at a point with respect to the action of 4d local operators. We can think of this concept in a much more general n-dimensional field theory, splitting the theory into sectors indexed by the spectrum of the ring of local operators – the moduli space of vacua. For instance, in a fully extended 2-dimensional topological theory viewed functorially, dualizability implies that the state space $Z(S^1)$ is necessarily finite-dimensional, so has discrete spectrum in general (viewed as an $\mathbb{E}_2$-algebra). In this case, localization splits this theory into discrete sectors, which are much simpler, indexed by points in this space. Often – for instance in fully extended functorial topological field theories – the moduli of vacua is a single point and hence there is no non-trivial localization. However topological twists of supersymmetric gauge theories typically admit continuous moduli of vacua, for which the structure of localization appears to lead to interesting new features.

We'll mainly consider the example of the Kapustin–Witten B-twisted $N = 4$ theory. In this example the algebra of local operators is equivalent to $\mathcal{O}(\mathfrak{g}^*[2]/G)$ – the space of functions on the shifted coadjoint quotient– and so the moduli space of vacua is $\mathfrak{h}^*[2]/W$, the affinization of $\mathfrak{g}^*[2]/G$. This algebra naturally acts on the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ – the "naïve" category of boundary conditions suggested by (categorified) geometric quantization – and we'll show that the localization of this category at the point $0 \in \mathfrak{h}^*[2]/W$ recovers the Arinkin–Gaitsgory category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$ of sheaves with nilpotent singular support.

By remembering the origin of this theory as a topological twist, we can promote the action of the algebra $\mathcal{O}(\mathfrak{g}^*[2]/G)$ to an action of the algebra $\mathcal{O}(\mathfrak{g}^*[2]/G)(\!(t)\!)$ of "twisted" local operators, where t is a formal parameter of degree 2. This parameter also appears in some of the key constructions of [AG15]: the definition of singular support involves a map from $\mathcal{O}((T^*[-1]X)^{\mathrm{cl}}) \to \mathrm{HH}^{2\bullet}(X)$ where X is a quasi-smooth derived scheme. By Hochschild–Kostant–Rosenberg the algebra $\mathrm{HC}^\bullet(X)$ of Hochschild cochains admits a filtration whose associated graded is $\mathcal{O}(T^*[1]X)$, so this map naturally involves a degree shift by 2. One can introduce this degree-shifting construction for general twisted theories by considering the action of the full algebra of twisted local operators and considering support conditions in the spectrum of the classical part of this algebra. One can sometimes think of this in terms of modifying the R-symmetry $U(1)$-action on the twisted theory in order to see a larger space of vacua (with a residual action of the original R-symmetry circle).

From the point of view of geometric representation theory this perspective on singular support conditions suggests novel structures on the categories arising as categories of boundary conditions in twisted supersymmetric gauge theories. In the example of the B-twisted $N = 4$ theory for the group GL_n we conjecture that varying the vacuum compatibility condition yields a hidden factorization structure on the category of boundary conditions – or at least on its Hochschild cochains. This is natural from the point of view of string theory, where motion in the moduli space of vacua arises from separating a stack of n D3 branes in type IIB string theory (more precisely to see a factorization algebra we must consider all values of n at once to obtain a family over the Ran space). This conjectural factorization structure is closely related to factorization structures on cohomological Hall algebras constructed by Kontsevich and Soibelman [KS11].

This work fits into the context of the extensive literature on supersymmetric boundary conditions for twisted $N = 4$ theories and their behaviour under S-duality. After Kapustin and Witten's original paper (which discussed magnetic eigenbranes as duals to Dirichlet boundary conditions on the B-side) a large family of boundary conditions was constructed by Gaiotto and Witten [GW09b] using a connection between Nahm's equations and $N = 4$ gauge theory demonstrated by Diaconescu [Dia97]. These boundary conditions are closely related to moduli spaces of opers for a Levi subgroup of the gauge group (see [GW12, Wit11]). It would be interesting to investigate the precise support conditions satisfied by these Nahm pole boundary conditions in the moduli space of vacua as one modifies the Nahm pole condition, and how the behaviour of these boundary conditions under S-duality as investigated in [GW09a, Gai16] relates to the well-known role of the space of opers in the geometric Langlands correspondence as prominently studied by Beilinson and Drinfeld [BD97].

1.1 Algebraic Moduli Stacks in Kapustin–Witten Gauge Theory

Let us briefly recall some aspects of Kapustin and Witten's approach. Firstly, for a 4-dimensional $N = 4$ supersymmetric Yang–Mills theory with gauge group G, Kapustin and Witten identified a $\mathbb{CP}^1$-family of topologically twisted theories such that S-duality acts antipodally on the $\mathbb{CP}^1$ parameter Ψ (a rational combination of the twisting supercharge and the coupling constant). We'll restrict attention to the duality between the theory with gauge group G and parameter $\Psi = 0$ and the dual theory with Langlands dual gauge group $G^\vee$ and parameter $\Psi = \infty$. After compactification along a compact Riemann surface Σ, these theories become the A-model with target $\mathrm{Higgs}_G(\Sigma)$ and the B-model with the target $\mathrm{Flat}_{G^\vee}(\Sigma)$. Kapustin and Witten argue that the category of boundary conditions in the A-model is equivalent to the category $\mathrm{D}(\mathrm{Bun}_G(\Sigma))$ of D-modules on $\mathrm{Bun}_G(\Sigma)$ and the corresponding category for the B-model is equivalent to the category $\mathrm{QC}(\mathrm{Flat}_{G^\vee}(\Sigma))$, so S-duality leads to a version of the geometric Langlands correspondence by identifying the categories of boundary conditions in these two dual theories.

Remark 1.1. Throughout this paper we use the notation $\mathrm{Flat}_G(\Sigma)$ for the derived moduli stack of flat G-bundles on Σ. In the geometric representation theory literature this is usually denoted by $\mathrm{LocSys}_G(\Sigma)$ or $\mathrm{Loc}_G(\Sigma)$, a notation we prefer to reserve for the moduli stack of G-local systems (which is analytically equivalent but algebraically quite different).

If one is motivated by connections to geometric representation theory, one important drawback of Kapustin and Witten's perspective is that there is no visible dependence on the algebraic structure of Σ. For instance, in their description the stack $\mathrm{Flat}_{G^\vee}(\Sigma)$ is understood as the character stack, rather than as the algebraic moduli stack of flat bundles. In particular, taken at face value, the physical approach can only lead to a topological version of the geometric Langlands conjecture, which doesn't see the algebraic structure of Σ. A topological conjecture of this form has recently been proposed by Ben-Zvi–Brochier–Jordan [BZBJ15] and Ben-Zvi–Nadler [BZN16], which they call "Betti Langlands" (see also recent work by Nadler and Yun [NY16]). In our previous work [EY15], we explained how to perform the topological twists introduced by Kapustin and Witten in a way that remembers the algebraic structure on the relevant moduli stacks. We identified a $\mathbb{CP}^1$-family of topological twists and computed the derived spaces of solutions to the classical equations of motion as algebraic derived stacks for the twisting parameters 0 and ∞, which we refer to as the A-twist and the B-twist respectively. This identification proceeded through an intermediate "holomorphic" twist, which both the A- and the B-twists factor through. This step was crucial in identifying the algebraic structures of the two twists, since the holomorphic twist is the minimal twist of $N = 4$ super Yang–Mills theory which admits a purely algebraic description in four dimensions.

Let us state the main results of our previous paper [EY15]. For a derived stack $\mathcal{X}$, one defines $\mathcal{X}_{\mathrm{Dol}} = T_{\mathrm{form}}[1]\mathcal{X}$, where $T_{\mathrm{form}}[1]\mathcal{X}$ is the formal completion of the shifted tangent space $T[1]\mathcal{X}$ along its zero section $\mathcal{X}$. The moduli space of solutions to the equations of motion in the holomorphic twist takes the following form.

Theorem 1.2 ([EY15, Theorem 4.2])**.** For a smooth proper complex algebraic surface X, one has

$$\mathrm{EOM}_{G,\mathrm{hol}}(X) = T^*_{\mathrm{form}}[-1]\mathrm{Higgs}_G(X),$$

where $\mathrm{Higgs}_G(X) := \underline{\mathrm{Map}}(X_{\mathrm{Dol}}, BG)$ is the moduli stack of Higgs bundles on X.

Since it's possible to rewrite $T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}(X_{\mathrm{Dol}}, BG) = T_{\mathrm{form}}[1]\underline{\mathrm{Map}}(X_{\mathrm{Dol}}, BG)$ using the AKSZ-PTVV formalism [PTVV13], one can rewrite $\mathrm{EOM}_{G,\mathrm{hol}}(X)$ as

$$\mathrm{EOM}_{G,\mathrm{hol}}(X) = \underline{\mathrm{Map}}(X_{\mathrm{Dol}}, BG)_{\mathrm{Dol}}.$$

The next main theorem shows that the two natural deformations of $\mathrm{EOM}_{\mathrm{hol}}(X)$, as in the context of non-abelian Hodge theory, precisely correspond to the two further twists: the A-twist and B-twist.

Theorem 1.3 ([EY15, Theorem 4.9, Proposition 4.18])**.** For a smooth proper complex algebraic surface X, one has

1. $\mathrm{EOM}_{G,A}(X) = \underline{\mathrm{Map}}(X_{\mathrm{Dol}}, BG)_{\mathrm{dR}}$ and
2. $\mathrm{EOM}_{G^\vee,B}(X) = \underline{\mathrm{Map}}(X_{\mathrm{dR}}, BG^\vee)_{\mathrm{Dol}} = T^*_{\mathrm{form}}[-1]\,\mathrm{Flat}_{G^\vee}(X)$.

Note that $\mathrm{Flat}_{G^\vee}(X)$, being realized as a deformation of $\mathrm{Higgs}_{G^\vee}(X)$, is the moduli stack of de Rham local systems, which is the space with the relevant algebraic structure appearing in the geometric Langlands conjecture.

According to Kapustin–Witten, one has to compactify the four-dimensional twisted theory along a smooth proper curve Σ to proceed. In our language, this amounts to setting $X = C \times \Sigma$ and rewriting the equations of motion in a way that the dependence on Σ is rigidified. With a little extra work we show the following.

Theorem 1.4 ([EY15, Corollary 4.11, Theorem 4.21])**.** If X is of the form $X = C \times \Sigma$ for smooth proper curves C and Σ, one has

1. $\mathrm{EOM}_{G,A}(C \times \Sigma) = T^*_{\mathrm{form}}[-1]\left(\underline{\mathrm{Map}}\,(C, \mathrm{Higgs}_G(\Sigma))_{\mathrm{dR}}\right)$ and
2. $\mathrm{EOM}_{G^\vee,B}(C \times \Sigma) = T^*_{\mathrm{form}}[-1]\underline{\mathrm{Map}}\,(C_{\mathrm{dR}}, \mathrm{Flat}_{G^\vee}(\Sigma))$.

Remark 1.5. Costello [Cos13] identified the space of the solutions to the equations of motion for the A-model with target X as the cotangent theory to the de Rham stack of the moduli space of holomorphic maps from C to X and for B-model as the cotangent theory to the moduli space of maps from C_{dR} to X, in a smooth category and at the formal level (as opposed to as a global derived stack). Thus, one can read our result as simultaneously achieving an algebraization and a globalization of his description and confirming Kapustin–Witten's description – at the classical level – in a way that captures the relevant algebraic structures for the geometric Langlands conjecture.

1.2 The Geometric Langlands Conjecture

The original Langlands program consists of a tantalizing set of conjectures which relate harmonic analysis, representation theory, algebraic geometry, and algebraic number theory. In particular, the reciprocity conjecture, in its simplest form, expects a close relationship between certain automorphic representations of $G(\mathbb{A}_{\mathbb{Q}})$ and homomorphisms from the Galois group $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ to the Langlands dual group $G^\vee$.

Through Weil's well-known "Rosetta stone" [Wei79], there is an analogy between number fields and function fields of curves over a finite field, from which one can make a corresponding version of the Langlands reciprocity conjecture for function fields. Namely, one replaces automorphic functions of $G(\mathbb{A}_F)$ for a number field F by automorphic sheaves over $\mathrm{Bun}_G(\Sigma)$ via Grothendieck's function-sheaf correspondence, and $G^\vee$-representations of $\pi_1^{\text{ét}}(F) = \mathrm{Gal}(\overline{F}/F)$ by $G^\vee$-local systems on Σ. It is a form of this conjecture that Drinfeld proved for $G = \mathrm{GL}_2$ [Dri83], after which Laumon formulated a precise conjecture for GL_n [Lau87]. Later the work of Frenkel, Gaitsgory, and Vilonen [FGV02] combined with a result of Gaitsgory [Gai04] explicitly constructed a cuspidal automorphic sheaf associated to an irreducible local system in unramified cases.

Once formulated in such geometric terms, one can also make such a conjecture for the function field of a curve Σ over the field $\mathbb{C}$ of complex numbers, rather than over a finite field, and directly translate the results discussed above to this complex algebraic context. On the other hand, since the geometry becomes easier, one can hope to prove a much stronger result: instead of just a set-theoretic relationship between cuspidal sheaves on $\mathrm{Bun}_G(\Sigma)$ and irreducible local systems on Σ, one would like to construct an equivalence of *categories* in which those objects naturally sit.

In the complex geometric setting, an algebraic local system can be best approximated by a de Rham local system, since – as we emphasized above – a Betti local system wouldn't be sensitive to the algebraic structure of Σ. For instance, $G^\vee$-local systems can be thought of as the skyscraper sheaves on $\mathrm{Flat}_{G^\vee}(\Sigma)$. On the other side, an automorphic sheaf can be described as a D-module on $\mathrm{Bun}_G(\Sigma)$: the category of constructible sheaves makes sense over $\mathbb{C}$, and can then be embedded into the larger, more algebraic, category of D-modules by the Riemann-Hilbert correspondence. With this in mind, Beilinson and Drinfeld imagined the following beautiful guiding picture.

Meta-Conjecture 1.6 ("Best Hope" Conjecture)**.** There is an equivalence of dg categories

$$\mathbb{L}_G \colon \mathrm{D}(\mathrm{Bun}_G(\Sigma)) \simeq \mathrm{QC}(\mathrm{Flat}_{G^\vee}(\Sigma))$$

compatible with the actions of natural symmetries on both sides.

This statement should be viewed as motivational but imprecise. When G is abelian, the conjecture is literally true, and was proved independently by Laumon [Lau96] and Rothstein [Rot96] using a twisted Fourier–Mukai transform adapted to send D-modules on an abelian variety A to quasi-coherent sheaves on a twisted form of $T^*A^\vee$. However, as soon as G is non-abelian the two categories, as stated above, fail to be equivalent. It turns out that the category on the A-side is "too big"; that there are D-modules which correspond to no sheaf on the B-side. An example of this was verified by V. Lafforgue [Laf09] for the curve $\mathbb{P}^1$, who demonstrated an equivalence between $\mathrm{QC}(\mathrm{Flat}_{G^\vee}(\mathbb{P}^1))$ and a proper subcategory of $\mathrm{D}(\mathrm{Bun}_G(\mathbb{P}^1))$. As such, in order to rescue the geometric Langlands conjecture we need to either enlarge the category on the B-side, or shrink the category on the A-side.

Arinkin and Gaitsgory [AG15] proposed a category enlarging $\mathrm{QC}(\mathrm{Flat}_{G^\vee}(\Sigma))$ that has the potential to repair the geometric Langlands conjecture. One natural approach is to declare that coherent objects, as opposed to perfect ones, should compactly generate the category, which leads one to consider $\mathrm{IndCoh}(\mathrm{Flat}_{G^\vee}(\Sigma))$. However,

$\mathrm{IndCoh}(\mathrm{Flat}_{G^\vee}(\Sigma))$ turns out to be too big. The best way to see this, which incidentally also makes the conjecture of Arinkin and Gaitsgory plausible, is to consider functoriality for the group G. Namely, for a parabolic subgroup P of G with its Levi quotient L, one has a geometric Eisenstein series functor $\mathrm{Eis}_P \colon \mathrm{D}(\mathrm{Bun}_L(\Sigma)) \to \mathrm{D}(\mathrm{Bun}_G(\Sigma))$ and expects a corresponding functor $\mathrm{Eis}^{\mathrm{spec}}_{P^\vee} \colon \mathcal{S}(\mathrm{Flat}_{L^\vee}(\Sigma)) \to \mathcal{S}(\mathrm{Flat}_{G^\vee}(\Sigma))$ for a fixed sheaf theory $\mathcal{S}$, so that the conjectural Langlands duality functors make the following diagram commute.

$$\begin{tikzcd}
\mathrm{D}(\mathrm{Bun}_L(\Sigma)) \arrow[r, "\mathbb{L}_L"] \arrow[d, "\mathrm{Eis}_P"'] & \mathcal{S}(\mathrm{Flat}_{L^\vee}(\Sigma)) \arrow[d, "\mathrm{Eis}^{\mathrm{spec}}_{P^\vee}"] \\
\mathrm{D}(\mathrm{Bun}_G(\Sigma)) \arrow[r, "\mathbb{L}_G"] & \mathcal{S}(\mathrm{Flat}_{G^\vee}(\Sigma)).
\end{tikzcd}$$

However, it turns out if we take either $\mathcal{S} = \mathrm{QC}$ or $\mathcal{S} = \mathrm{IndCoh}$ then one can't form such a commutative square. The main technical ingredient in the paper of Arinkin and Gaitsgory is to introduce the theory of singular supports for a coherent sheaf on a quasi-smooth stack, like $\mathrm{Flat}_{G^\vee}(\Sigma)$, so that one has a range of sheaf theories interpolating between these two extremes. Moreover, the main theorem of their work is to identify the category $\mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$ consisting of objects satisfying the nilpotent singular support condition as the minimal category which contains the images of the (spectral) geometric Eisenstein series functor from $\mathrm{QC}(\mathrm{Flat}_{L^\vee}(\Sigma))$ for all parabolic subgroups $P^\vee$.

This leads to the following corrected form of the best hope conjecture.

Conjecture 1.7 (Arinkin–Gaitsgory)**.** There is an equivalence of dg categories

$$\mathbb{L}_G \colon \mathrm{D}(\mathrm{Bun}_G(\Sigma)) \simeq \mathrm{IndCoh}_{\mathcal{N}_{G^\vee}}(\mathrm{Flat}_{G^\vee}(\Sigma))$$

compatible with the actions of natural symmetries on both sides.

Remark 1.8. We'll describe some of the natural symmetries briefly here. A large family of natural symmetries is again motivated by the original Langlands correspondence. Namely, a flat $G^\vee$-bundle, which corresponds to a Galois representation to $G^\vee$, gives rise to a skyscraper sheaf at a point in $\mathrm{Flat}_{G^\vee}(\Sigma)$, which is an eigenobject with respect to an operator modelling tensoring with a vector bundle (in a suitable sense). The conjectural equivalence is required to send these eigenobjects to eigenobjects for the *Hecke operators* on the dual side. By the compatibility with the natural symmetries, we mean that the equivalence is equivariant for the action of the monoidal category of Hecke operators (which is identified with the monoidal category of tensoring operators by the geometric Satake theorem). For a more detailed discussion of the aspects of this equivalence that will play a role in the present work, see Section 3.2.2.

1.3 Vacuum Conditions

Now, the goal of this present work is to understand how the refinements to the geometric Langlands conjecture introduced by Arinkin and Gaitsgory should be understood from the point of view of of gauge theory, as in the approach of Kapustin and Witten. We will interpret the refined conjecture as a consequence of S-duality on the category of boundary conditions of the 4-dimensional gauge theory – specifically the idea that duality should relate boundary conditions compatible with a particular choice of vacuum. This idea was also discussed in a recent paper of Balasubramanian [Bal16], and in the Betti context by Ben-Zvi and Nadler [BZN16, Section 3.6].

There are at least two different ways of thinking about the moduli space of vacua in topological field theories, depending on which formalism we use to model the theory itself. On the one hand we could take an algebraic point of view and model the *observables* of the theory. For instance, Costello and Gwilliam [CG16, CG17] developed a machine by which one can construct observables in quantum field theories as a *factorization algebra*. In the case where the theory is topological this recovers a more familiar notion from homotopical algebra: the notion of an $\mathbb{E}_n$-*algebra*. On the other hand we could use a functorial model for topological quantum field theory, via the Atiyah–Segal axioms. An n-dimensional functorial field theory also produces an $\mathbb{E}_n$-algebra as the space of states on S^{n-1}, where the multiplication is given by configurations of little n-balls viewed as n-dimensional bordisms.

From either point of view, the *moduli space of vacua* is the spectrum of the cohomology of this $\mathbb{E}_n$-algebra. This is the universal space on which this algebra of either observables or states (depending on the model) acts. Furthermore, given a choice of boundary condition $\mathcal{F}$ along a codimension 2 manifold Y, the endomorphism algebra $\mathrm{End}_{Z(Y)}(\mathcal{F})$ in the category of boundary conditions *also* lives over the moduli space of vacua – this is automatic from the functorial point of view, but data that must be specified by hand from the factorization algebra point of view. One can therefore consider the localization of this algebra at a point in the moduli space of vacua. This admits a physical interpretation: we think of this localization as controlling states in the bulk boundary system in a neighbourhood of the vacuum, and say that a boundary condition is *compatible* with a choice of vacuum if this localization does not vanish. From this point of view we can define a localized *category* of boundary conditions as the full subcategory of boundary conditions compatible with a vacuum (note that in the main body of the paper we'll use a somewhat more technical construction of the same idea).

Remark 1.9. One of the most important aspects of the Langlands program as a whole is the compatibility with respect to the action of the symmetries discussed in Remark 1.8. In the categorical geometric Langlands correspondence, these symmetries are realized as actions of monoidal categories which Kapustin–Witten interpreted as actions of line operators. From our 4d perspective, these categories naturally have the structure of factorizable monoidal categories.

Our main result is the following, which gives a gauge theoretic explanation to the nilpotent singular support condition in geometric Langlands.

Theorem 1.10 (Theorem 3.21)**.** The moduli space of vacua in the B-twisted $N = 4$ supersymmetric Yang–Mills theory with gauge group G is equivalent to $\mathfrak{h}^*/W$, where $\mathfrak{h}$ is the Cartan algebra and W is the Weyl group. The algebra $\mathcal{O}(\mathfrak{h}^*/W)$ canonically acts on the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$, and the localization at the point 0 is equivalent to the category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$ of sheaves with nilpotent singular support.

It's natural to ask what happens for other points in $\mathfrak{h}^*/W$. We can described the categories compatible with more general vacua in terms of singular support conditions, and we conjecture that the categories obtained from this admit a very natural, symmetrical description. An analysis of the geometry of the relevant singular support conditions motivates the following conjectural description.

Conjecture 1.11. The localization of the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ at a point $v \in \mathfrak{h}^*/W$ is equivalent to the category $\mathrm{IndCoh}_{\mathcal{N}_L}(\mathrm{Flat}_L(\Sigma))$ where L is the stabilizer of a semisimple lift of v to $\mathfrak{g}^*$ in G.

One might ask in what sense the category of boundary conditions can be defined as a category *over* the stratified space $\mathfrak{h}^*/W$, and how exactly the fibers are related. We consider this question in the final section of this article, Section 4.2.

1.4 Summary of the Paper

We begin with Section 2 where we provide the necessary background and state the basic definitions used in the rest of the paper. The first subsection, Section 2.1, contains the background material we'll use concerning topological field theory, from the factorization and the functorial points of view. In particular we explain how homotopical algebra connects with such models for topological field theory. We then review the main ideas of localization of triangulated and dg categories from Benson–Iyengar–Krause [BIK08] and Arinkin–Gaitsgory [AG15] in Section 2.2. We particularly emphasise Arinkin and Gaitsgory's conditions for global complete intersection stacks, in which case singular support conditions admit a particularly nice description which we explain in Section 2.2.2. Finally in Section 2.3 we introduce the moduli space of vacua and explain what it means to localize a category at a point in this moduli space. We'll use the factorization algebra model for our main examples, but we also explain an alternative construction using functorial field theories (which is less general but provides additional motivation).

In Section 3 we apply these ideas to the examples of Kapustin–Witten twisted $N = 4$ gauge theory, with a focus on the B-twisted theory. We begin by reviewing our construction of these theories [EY15] in Section 3.1, by the general procedure of the topological twist. In Section 3.2 we recall from [AG15] important facts about the moduli stack of flat G-bundles on a curve and the geometric Satake equivalence. Then in Sections 3.3 and 3.4 we compute the moduli of vacua in the B-twisted theory, and calculate the localization of the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ at the point 0 in this moduli space, recovering the category of sheaves with nilpotent singular support. This requires taking some care – the relationship between the calculation of the moduli space of vacua and the twist itself is somewhat subtle. We explain this issue in Section 3.4.

We conclude with Section 4, in which we investigate what happens when one moves away from 0 in the moduli space of vacua. In Section 4.1 we conjecture a natural answer – that one obtains the nilpotent singular support category but with the gauge symmetry group broken to a Levi subgroup – and provide some evidence for this conjecture. Then finally in Section 4.2 we conjecture that these categories should satisfy a natural factorization condition on the moduli space of vacua, and describe how this is related to factorization structures on cohomological Hall algebras as investigated by Kontsevich and Soibelman.

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2 Boundary Conditions and Vacua

In this section we'll give an abstract definition, motivated by a construction in terms of locally constant factorization algebras and a parallel construction in functorial topological quantum field theory, of a support condition for a dg-category $\mathcal{C}$ which we understand as the category of boundary conditions. Physically speaking, the support condition corresponds to compatibility with a choice of a point in the moduli space of vacua: each point in the moduli space yields an interesting full subcategory of $\mathcal{C}$ by picking out boundary conditions compatible with such a choice.

We'll begin by recalling the formalism of extended functorial QFTs, and the factorization algebra model for quantum field theories developed by Costello and Gwilliam [CG16, CG17]. After an exposition on these ideas, with an emphasis on the topological case, we'll then introduce the moduli space of vacua in each of these contexts, and explain what it means to localize a dg-category at a point in this space.

Remark 2.1. We should emphasise that when we refer to "topological" quantum field theory we don't necessarily mean something equivalent to a functorial TQFT; we'll often have in mind something more general. For instance, topological twists of supersymmetric field theories need not be topological in the strict mathematical sense, but can depend on additional geometric structure (see the example of Donaldson-Witten theory as discussed by Moore and Witten [MW98], and Kapustin–Witten twisted $N = 4$ gauge theories as we discussed in our previous paper [EY15]).

2.1 Factorization Algebras and Functorial Field Theory

In this subsection, we will describe the main mathematical models for quantum field theories that appear in the literature, with a focus towards the topological case. We'll begin with reviewing the classical BV formalism for a Lagrangian description of a classical field theory from the modern perspective of derived geometry. Then we provide an exposition of functorial topological field theory and how one should think of it as arising from the data provided by the classical BV formalism. We also explain the formalism of factorization algebras and mention a result of Costello and Gwilliam on constructing factorization algebras from such data. Finally we discuss how these two constructions are supposed to be compared for topological field theories.

2.1.1 The Classical BV Formalism

We begin with some abstract motivation. In the classical BV formalism, the derived moduli space of solutions $\mathrm{EOM}(M)$ to the equations of motion for an n-dimensional field theory on an n-dimensional closed manifold M is always (-1)-shifted symplectic in the sense of Pantev, Toën, Vaquié, and Vezzosi [PTVV13]. Indeed, if S is an action functional on a space Φ of fields on M, the space $\mathrm{EOM}(M)$ is defined as the derived intersection

$$\mathrm{EOM}(M) = \Phi \cap_{T^*\Phi} \mathrm{Graph}(\mathrm{d}S)$$

of the graph of $\mathrm{d}S$ with the zero section in the 0-shifted symplectic space $T^*\Phi$. It's a general fact that the derived intersection of two Lagrangians in a k-shifted symplectic space is always $(k-1)$-shifted symplectic, which induces a (-1)-shifted symplectic structure on the space of derived solutions to the equations of motion in any classical field theory.

A classical field theory is not just given by a single (-1)-shifted symplectic space that captures the moduli space of solutions to the equations of motion. It is expected that for a codimension k closed submanifold Y, there is a sense in which the moduli space of germs of solutions near the submanifold, which we denote by $\mathrm{EOM}(Y)$, has a canonical $(k-1)$-shifted symplectic structure. In fact, for a topological field theory, one expect this to be literally true.

An important class of examples of topological field theories capturing this additional data can be succinctly described by the AKSZ construction.

Example 2.2. (1) (Topological classical mechanics) Let X be a symplectic space. Then topological classical mechanics with the phase space X is described by the mapping stack $\mathrm{EOM}_{\mathrm{TCM}}(Y) = \underline{\mathrm{Map}}(Y_B, X)$.

(2) (B-model) Let X be a Calabi–Yau variety. The B-model with target X is described by $\mathrm{EOM}_B(Y) = \underline{\mathrm{Map}}(Y_B, T^*_{\mathrm{form}}[1]X)$ where $T^*_{\mathrm{form}}[1]X$ is the formal completion of the 0-section in the 1-shited cotangent space.

(3) (Chern–Simons theory) Let G be a reductive group. Then Chern–Simons theory with gauge group G is $\mathrm{EOM}_{\mathrm{CS}}(Y) = \underline{\mathrm{Map}}(Y_B, BG)$.

In the three examples, $(d) = (1)$, (2) and (3) indicates a d-dimensional classical field theory, where one can consider any closed oriented manifold of dimension $\leq d$ as the source space Y. For instance, for the B-model, $Y = \Sigma$, $Y = S^1$, and $Y = \mathrm{pt}$ yield (-1)-shifted, 0-shifted, and 1-shifted symplectic spaces, respectively.

We exhibited nontrivial examples of this phenomenon for topological field theories which are not necessarily of AKSZ type in our previous work [EY15]. We review some examples in Section 3.1.

In fact, more is expected. A classical field theory on M is supposed to define a sheaf of (-1)-shifted symplectic stacks on M given by $U \mapsto \mathrm{EOM}(U)$. However, it is not easy to construct this sheaf in practice, because the notion of (-1)-shifted symplectic structure is defined with respect to Verdier duality, rather than linear duality. This sheaf structure can however be realized in the context of perturbative field theories, as described in [CG17].

For future reference, we also record the structure for the algebra of functions on a shifted symplectic space. Just as the algebra of functions on a symplectic manifold has a canonical Poisson structure, the algebra of functions on an n-shifted symplectic space is a $\mathbb{P}_{n+1}$-algebra. Here a $\mathbb{P}_{n+1}$-algebra is a commutative differential graded algebra together with a Poisson bracket of cohomological degree $-n$. The numbering is made in such a way that $\mathbb{P}_d$-algebras are related to d-dimensional classical field theories. In particular, a $\mathbb{P}_1$-algebra recovers the usual notion of a Poisson algebra relevant to classical mechanics understood as 1-dimensional classical field theory.

2.1.2 Factorization Algebras

The principal approach to understanding quantum field theory taken in this article is motivated by the Heisenberg picture of quantum mechanics, where one has the algebra of operators as the main character. This is the subject of factorization algebras as developed by Costello and Gwilliam [CG16, CG17]. Although the formalism of factorization algebras doesn't a priori encode the information of the Hilbert space of a quantum field theory, it works well in the generality of any quantum field theory admitting a Lagrangian description, contrary to the model of functorial field theory which doesn't yet have as firm a constructive foundation for non-topological field theories. In this section we'll summarise some basic concepts from the theory of factorization algebras, referring the reader to Costello and Gwilliam for details.

For a manifold M, a *prefactorization algebra* $\mathcal{F}$ on M is an assignment of a cochain complex $\mathcal{F}(U)$ to every open subset $U \subset M$ together with some structure maps: if $U_1, \cdots, U_n \subset V$ are disjoint open subsets of V for an open subset $V \subset M$, then we have a map $\mathcal{F}(U_1) \otimes \cdots \otimes \mathcal{F}(U_n) \to \mathcal{F}(V)$ satisfying a natural associativity condition that for $\amalg_{i\in I} U_i \subset \amalg_{j \in J} V_j \subset W$, the associated diagram

$$\begin{tikzcd} \bigotimes_{i\in I} \mathcal{F}(U_i) \arrow[r] \arrow[dr] & \bigotimes_{j\in J} \mathcal{F}(V_j) \arrow[d] \\ & \mathcal{F}(W) \end{tikzcd}$$

commutes. A *factorization algebra* is a prefactorization algebra satisfying a descent axiom, just like a sheaf is a presheaf with the gluing axiom satisfied. As we only need the notion for the sake motivation, we won't articulate the technical definition here but take for granted that those prefactorization algebras arising from field theory are indeed factorization algebras.

One can construct factorization algebra structures from the data of a classical field theory by a version of deformation quantization. Suppose we have a classical field theory given by an assignment $U \mapsto \mathrm{EOM}(U)$ for open subsets $U \subset M$. Then a collection of observations that only depend on behavior of those solutions to the Euler–Lagrange equations on $U \subset M$ is by definition a function on the space $\mathrm{EOM}(U)$, which we denote by $\mathrm{Obs}^{\mathrm{cl}}(U)$ and refer to as the space of classical observables on U. As $\mathrm{EOM}(U)$ is (-1)-shifted symplectic, $\mathrm{Obs}^{\mathrm{cl}}(U)$ has the structure of a $\mathbb{P}_0$-algebra, namely, a commutative algebra with a Poisson bracket of cohomological degree 1. The assignment $U \mapsto \mathrm{Obs}^{\mathrm{cl}}(U)$ is a cosheaf of $\mathbb{P}_0$-algebras. In particular, this forms a factorization algebra valued in $\mathbb{P}_0$-algebras.

What's more interesting is the algebra of quantum observables. Given an open subset of U a spacetime manifold M, the Heisenberg uncertainty principle that one cannot make measurements concurrently translates into the statement that the space $\mathrm{Obs}^{\mathrm{q}}(U)$ of quantum observables is never a commutative algebra. However, one can still make observations happening at *different points* of spacetime, leading to the structure maps of a factorization algebra $\mathrm{Obs}^{\mathrm{q}}(U_1) \otimes \cdots \otimes \mathrm{Obs}^{\mathrm{q}}(U_n) \to \mathrm{Obs}^{\mathrm{q}}(V)$ for *disjoint* open subsets $U_1, \cdots, U_n \subset V$ for an open subset $V \subset M$. These are encoding the *operator product* of observables. We'll write the map

$$\mathrm{Obs}^{\mathrm{q}}(U_1) \otimes \mathrm{Obs}^{\mathrm{q}}(U_2) \to \mathrm{Obs}^{\mathrm{q}}(V)$$

as $O_1 \otimes O_2 \mapsto O_1 * O_2$.

One has to further specify the relation between $\mathrm{Obs}^{\mathrm{cl}}$ and $\mathrm{Obs}^{\mathrm{q}}$ in the quantization process. Recall the usual deformation quantization problem that the observables of classical mechanics form a Poisson algebra, or a $\mathbb{P}_1$-algebra, $(A, \{\ ,\ \})$ and we want to quantize it to obtain the observables of quantum mechanics realized as an associative algebra, or an $\mathbb{E}_1$-algebra, $(\widetilde{A}, *)$ over $\mathbb{R}[\![\hbar]\!]$ such that there is an isomorphism $\widetilde{A} \otimes_{\mathbb{R}[\![\hbar]\!]} \mathbb{R} \simeq A$ as associative algebras and for $a, b \in A$, one has

$$\lim_{\hbar \to 0} \frac{a * b - b * a}{\hbar} = \{a, b\}.$$

Likewise, a quantization of $\mathbb{P}_0$-algebra A is an $\mathbb{E}_0$-algebra $\widetilde{A}$ over $\mathbb{R}[\![\hbar]\!]$ such that $\widetilde{A} \otimes_{\mathbb{R}[\![\hbar]\!]} \mathbb{R} \simeq A$ and the induced bracket from the $\mathbb{E}_0$ structure coincides with the given $\mathbb{P}_0$ structure (for more detail and background, see [CG17, Chapter 2]).

Theorem 2.3 (Costello–Gwilliam)**.** [CG17] For a classical field theory $\mathrm{Obs}^{\mathrm{cl}}$ and a choice of BV quantization, there exists a factorization algebra $\mathrm{Obs}^{\mathrm{q}}$ over $\mathbb{R}[\![\hbar]\!]$ so that for every open subset $U \subset M$, $\mathrm{Obs}^{\mathrm{q}}(U)$ is a quantization of the $\mathbb{P}_0$ algebra $\mathrm{Obs}^{\mathrm{cl}}(U)$ in the above sense.

On constructs such a factorization algebra quantizing a classical field theory by studying the obstruction-deformation complex associated to the given action functional.

Remark 2.4. This theorem is formulated and proved in the context of perturbative field theory, which is why we have $\hbar$ as a formal parameter as opposed to a number. Accordingly, what we write as EOM should be understood as a sheaf (on the spacetime manifold M) of (-1)-shifted symplectic formal moduli spaces and $\mathrm{Obs}^{\mathrm{cl}}$ as a cosheaf of certain Chevalley–Eilenberg cochain complexes. Moreover, there are technical issues to be dealt with coming from the infinite-dimensionality of the spaces involved, which can be done by using the Wilsonian effective field theory philosophy and renormalization techniques following Costello [Cos11].

Now we want to understand some examples of factorization algebras.

The easiest class of examples is the class of locally constant factorization algebras. A factorization algebra $\mathcal{F}$ is called *locally constant* if, for every inclusion of disks $D \hookrightarrow D'$, the induced structure map $\mathcal{F}(D) \to \mathcal{F}(D')$ is a quasi-isomorphism. A theorem due to Lurie [Lur09] classifies such factorization algebras.

Theorem 2.5 (Lurie)**.** A locally constant factorization algebra on $\mathbb{R}^n$ has a natural structure of $\mathbb{E}_n$-algebra.

In particular, as the factorization algebra of a topological field theory is by definition locally constant, an n-dimensional topological field theory yields an $\mathbb{E}_n$-algebra as its algebra of observables.

The formalism of factorization algebras can capture another interesting class of examples. If we have a factorization algebra $\mathcal{F}$ on $\mathbb{C}$ which is holomorphically translation-invariant, that is, we suppose that $\mathcal{F}(D(z, r)) \simeq \mathcal{F}(D(0, r))$ holds for all $z \in \mathbb{C}$ and $r \in \mathbb{R}_{>0}$ and that the vector field $\frac{\partial}{\partial \bar{z}}$ acts homotopically trivially on $\mathcal{F}$, then under certain technical conditions the cohomology of $\mathcal{F}$ yields a vertex algebra (for more detail, see [Gwi12]) and [CG16, Chapter 5]. For instance, nicely blending this framework with Gelfand–Kazhdan formal geometry, Gorbounov, Gwilliam, and Williams constructed a sheaf of vertex algebras known as chiral differential operators on X from the classical field theory data of the curved $\beta\gamma$-system with target X [GGW16].

2.1.3 Functorial Field Theories

An alternative approach to the formalisation of quantum field theory is given by generalizing the Schrödinger picture of quantum mechanics, where we have the Hilbert space and time evolution operator as the main characters, by identifying quantum mechanics as the one-dimensional case of quantum field theory. That is, one assigns to a codimension one submanifold of spacetime its space of states, and to a bordism between such submanifolds the time evolution operator. Realizing the idea rigorously in complete generality is a daunting task, but there have been extensive developments for those theories that depend only on the topology of the spacetime: a definition of functorial TQFT was first proposed by Atiyah [Ati88], following Segal's study on conformal field theory [Seg88], and later extended to incorporate boundary conditions for states along codimension two manifolds and to higher order boundary conditions, notably by Lurie [Lur09] following ideas developed by Freed, Quinn [FQ93, Fre93, Fre94] and Baez–Dolan [BD95].

We'll consider the following definition of an n-dimensional functorial TQFT incorporating categories of boundary conditions

Definition 2.6. A 2-extended *n-dimensional TQFT* is a symmetric monoidal functor of 2-categories

$$Z\colon \mathrm{Bord}_n^2 \to \mathrm{dgCat}$$

where Bord_n^2 is a 2-category of n-dimensional bordisms (originally defined by Schommer-Pries [SP09] as a symmetric monoidal bicategory; Later Calaque and Scheimbauer [CS15] constructed a fully extended bordism category as an (∞, n)-category whose truncation to an $(\infty, 2)$-category is homotopy equivalent to Schommer-Pries's construction [CS15, Proposition 8.17]).

Remark 2.7. The *cobordism hypothesis* as formulated by Lurie [Lur09] says in particular that in the case where $n = 2$, a (framed) 2-extended TQFT is completely determined by the category assigned to the point, which is necessarily *2-dualizable* – it has left and right duals and the evaluation and coevaluation morphisms have left and right adjoints. It is possible to weaken this notion to that of a "non-compact" field theory, where one restricts to bordisms with non-empty incoming boundary in every connected component. Such theories correspond instead to *Calabi–Yau categories* (see Costello [Cos07] and Lurie [Lur09, Section 4.2]).

Remark 2.8. It will be useful to briefly recall the motivation for this functorial definition. First of all, according to this motivation, an n-dimensional TQFT Z assigns to a closed $(n-1)$-manifold M a cochain complex [1] which we regard as the space of quantum states on M. Now, the claim is that for a closed $(n-2)$-manifold Y, the assigned dg category $Z(Y)$ can naturally be interpreted as the category of boundary conditions along Y. In order to justify this, note that for a spacetime of form $M = Y \times [0, 1]$, one cannot make sense of $Z(M)$ directly as the cochain complex of quantum states on M as M is not closed, but it does admit an interpretation as the space of quantum states on M that satisfy *boundary conditions* along the two boundary components $Y \times \{0\}$ and $Y \times \{1\}$. Namely, for such a pair of boundary conditions $\mathcal{F}_1$ and $\mathcal{F}_2$ (i.e. objects of the category $Z(Y)$), the space of compatible quantum states, which we write as $\mathcal{H}_{\mathcal{F}_1,\mathcal{F}_2} := \mathrm{Hom}_{Z(Y)}(\mathcal{F}_1, \mathcal{F}_2)$, is a cochain complex. Moreover, the composition map $\mathcal{H}_{\mathcal{F}_1,\mathcal{F}_2} \otimes \mathcal{H}_{\mathcal{F}_2,\mathcal{F}_3} \to \mathcal{H}_{\mathcal{F}_1,\mathcal{F}_3}$, which is a cochain map by the axioms defining a dg category, tells us how to glue a state in $\mathcal{H}_{\mathcal{F}_1,\mathcal{F}_2}$ and a state in $\mathcal{H}_{\mathcal{F}_2,\mathcal{F}_3}$ together to obtain a state on the manifold $Y \times [0, 2]$ satisfying boundary conditions $\mathcal{F}_1$ and $\mathcal{F}_3$, using the fact that the spaces of states on $Y \times [0, 1]$ and on $Y \times [0, 2]$ are canonically equivalent in a TQFT. Furthermore, the axioms tell us that in each space of states $\mathcal{H}_{\mathcal{F}_1,\mathcal{F}_1}$ there's a "unit state", so that gluing it onto a state in $\mathcal{H}_{\mathcal{F}_1,\mathcal{F}_2}$ has no effect.

Example 2.9. Consider a specific 2-dimensional quantum field theory: the B-model with the target X a Calabi–Yau manifold. A standard model for the category of boundary conditions in the B-model is $Z(\mathrm{pt}) = \mathrm{Coh}(X)$ – the derived category of coherent sheaves on X [Sha99]. The cochain complex of quantum states on an interval I with boundary conditions $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Coh}(X)$ is $\mathcal{H}_{\mathcal{F}_1,\mathcal{F}_2} = \mathrm{Hom}(\mathcal{F}_1, \mathcal{F}_2)$. This is also known as the space of open string states with the given boundary conditions.

In particular we can consider the category such a functorial TQFT assigns to the $(n-2)$-sphere. This category $Z(S^{n-2})$ is an $\mathbb{E}_{n-1}$-algebra object in dgCat by applying the functor to n-dimensional pairs of pants. To be more explicit, consider an $(n-1)$-manifold of form $M = B_1(0) \setminus (B_\varepsilon(x_1) \sqcup \cdots \sqcup B_\varepsilon(x_k))$, where $B_r(x)$ is an $(n-1)$-dimensional ball of radius r around the point $x \in \mathbb{R}^{n-1}$ and ε is sufficiently small. By definition, it defines a functor

$$Z(M)\colon \mathcal{L}^{\otimes k} \to \mathcal{L}.$$

As the definition of this functor depended on a point in a configuration space of $(n-1)$-balls, we obtain a $\mathbb{E}_{n-1}$-algebra structure. We call the $\mathbb{E}_{n-1}$-monoidal category $\mathcal{L} = Z(S^{n-2})$ the category of *line operators* of the theory Z.

One important feature possessed by the line operators is that they act on the category of boundary conditions along an $(n-2)$-manifold Y. Indeed, the bordism $(Y \times [0, 1]) \setminus B_\varepsilon(y, t)$ defines a functor $Z(S^{n-2}) \otimes Z(Y) \to Z(Y)$, or equivalently a monoidal functor $\mathcal{L} \to \mathrm{Fun}(Z(Y), Z(Y))$, where $\mathrm{Fun}(\mathcal{C}, \mathcal{C})$ is the category of functors from $\mathcal{C}$ to itself with the composition monoidal structure. Indeed, the embedding of the ball $B_\varepsilon(y, t)$ into $Y \times [0, 1]$ picks out a particular $\mathbb{E}_1$-category, i.e. monoidal category, structure on $\mathcal{L}$ in the direction of $[0, 1]$ so that $Z(Y)$ is made into a module for $\mathcal{L}$ as a monoidal category.

[1] We can think of the differential graded structure here as arising from the BRST formalism: the space of states is graded by ghost number and the BRST operator acts as a differential.

Remark 2.10. The category of line operators can become interesting for $n \geq 3$; if $n = 2$, the monoidal action just amounts to

$$\mathrm{id}\colon \mathrm{Fun}(Z(\mathrm{pt}), Z(\mathrm{pt})) \to \mathrm{Fun}(Z(\mathrm{pt}), Z(\mathrm{pt}))$$

for $Z(S^0) = Z(\mathrm{pt} \amalg \mathrm{pt}^*) = \mathrm{Fun}(Z(\mathrm{pt}), Z(\mathrm{pt}))$ and $Y = \mathrm{pt}$.

An even more fundamental feature of a quantum field theory is its the algebra of local operators. The *local operators* in a TQFT Z are elements of the cochain complex $Z(S^{n-1})$. By n-dimensional bordisms analogous to those we used to define the $\mathbb{E}_{n-1}$ structure on the category of line operators, this complex becomes an $\mathbb{E}_n$-algebra.

In the case $n = 2$, there is a whistle (open-closed) bordism from S^1 to I: for any choice of a boundary condition, say $B \in Z(\mathrm{pt})$, the whistle gives rise to a morphism $Z(S^1) \to \mathrm{Hom}_{Z(\mathrm{pt})}(B, B)$. Then, in arbitrary dimension, the product of the whistle with an $(n-2)$-manifold Y gives a morphism $Z(S^1 \times Y) \to \mathrm{Hom}_{Z(Y)}(\mathcal{F}, \mathcal{F})$ for each $\mathcal{F} \in Z(Y)$. Composing with the complement of a point $\{0\} \times \{y\} \in B^2 \times Y$ understood as a bordism from S^{n-1} to $S^1 \times Y$, an n-dimensional TQFT provides us with an operator of the form

$$\alpha_{y,\mathcal{F}}\colon Z(S^{n-1}) \to \mathrm{Hom}_{Z(Y)}(\mathcal{F}, \mathcal{F})$$

for each choice of a point $y \in Y$ and a boundary condition $\mathcal{F}$ along Y. This makes the space $\mathrm{Hom}_{Z(Y)}(\mathcal{F}, \mathcal{F})$ into a module for $Z(S^{n-1})$. The physical interpretation of this structure is that the local operators in the bulk theory, i.e. operators supported away from the boundary, act on states on $Y \times [0, 1]$ satisfying a fixed boundary condition without altering it. From the construction, the action is through a particular $\mathbb{E}_2$-algebra structure which a priori depends on the choice $y \in Y$.

Remark 2.11. This is a heuristic remark. Given the monoidal category of line operators $\mathcal{L} = Z(S^{n-2})$, a line operator $L \in \mathcal{L}$ gives a functor $L\colon Z(Y) \to Z(Y)$. Then $\mathrm{End}_{\mathcal{L}}(L, L)$ is the algebra of local operators living on the line operator L. In particular, for the trivial line operator $1 \in \mathcal{L}$, which corresponds to the unit object of the monoidal category $\mathcal{L}$, one obtains $\mathrm{End}_{\mathcal{L}}(1, 1)$, which is understood as the algebra of local operators without any restriction. In other words, one expects

$$Z(S^{n-1}) = Z(B^{n-1} \amalg_{S^{n-2}} B^{n-1}) = \mathrm{End}_{Z(S^{n-2})}(1, 1),$$

since the $(n-1)$-ball B^{n-1} as a bordism from the empty set to S^{n-2} is thought of as corresponding to the unit object of $\mathcal{L}$. In this sense, the category of line operators in TQFT "knows about" the algebra of local operators in the theory.

Finally let us explain one perspective on functorial field theory from the perspective of the classical BV formalism (not the only possible perspective, we should mention for instance the work of Cattaneo, Mnëv, and Reshetikhin [CMR15], which takes a different point of view). In one sentence, one obtains a functorial field theory through (categorified) geometric quantization.

Recall that in the classical BV formalism for a codimension 1 submanifold, we would obtain a (0-shifted) symplectic structure, as expected for a phase space. The Hilbert space is given by the geometric quantization of this space. Geometric quantization is not well-defined in this generality, and without specifying additional data, but when $\mathrm{EOM}(N) = T^*\mathcal{X}_N$ is the cotangent bundle it admits a canonical geometric quantization by $\mathcal{O}(\mathcal{X}_N)$. Similarly, for a codimension 2 manifold Y, if we had $\mathrm{EOM}(Y) = T^*[1]\mathcal{X}_Y$, then its categorified geometric quantization can be taken to be $\mathrm{Coh}(\mathcal{X}_Y)$, or better, its ind-completion $\mathrm{IndCoh}(\mathcal{X}_Y)$ as the *Hilbert* category (see Wallbridge [Wal16], and Calaque [Cal15] in the case of AKSZ type theories with Betti source).

Example 2.12 (B-model)**.** Recall that B-model with target Calabi–Yau variety X is described by $\mathrm{EOM}_B(Y) = \underline{\mathrm{Map}}(Y_B, T^*_{\mathrm{form}}[1]X)$. For a codimension 1 manifold S^1, we obtain $\mathrm{EOM}_B(S^1) = \mathcal{L}(T^*_{\mathrm{form}}[1]X) = T^*_{\mathrm{form}}(\mathcal{L}X)$, where $\mathcal{L}X$ is the derived loop space: the derived fiber product $X \times_{X \times X} X$. The Hilbert space is given by its geometric quantization, which is $\mathcal{O}(\mathcal{L}X) = \mathrm{PV}(X)$, the algebra of polyvector fields on X. For a codimension 2 manifold pt, we have $\mathrm{EOM}_B(\mathrm{pt}) = T^*_{\mathrm{form}}[1]X$ which yields $Z(\mathrm{pt}) = \mathrm{Coh}(X)$. Note that by the cobordism hypothesis $Z(\mathrm{pt})$ determines $Z(S^1) = \mathrm{HC}^\bullet(X)$ as the Hochschild cochains, but the formality identifies it with the Hilbert space $\mathrm{PV}(X)$ we computed in a different way. This example explains the fundamental relationships between loop spaces, polyvector fields, and Hochschild (co)chains in terms of physics.

Remark 2.13. While this is the most convenient formalism in which to understand the category of boundary conditions and the action of $Z(S^{n-1})$ on it, we'll principally use the factorization algebra formalism for our examples. The reason is that this formalism is not general enough to capture the Kapustin–Witten twisted theories. For instance, the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ is not 2-dualizable – or even Calabi–Yau – because $\mathrm{Flat}_G(\Sigma)$ is neither smooth nor proper. Therefore it doesn't generate a fully extended 2d TQFT, or even a non-compact TQFT. Perhaps more profoundly the Kapustin–Witten theories depend, even at the classical level, on a choice of complex structure (that is, the dependence on the source is not Betti). Even if one manages to define a sense in which 2d extended theory which assigns $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ or $D(\mathrm{Bun}_G(\Sigma))$ to the point, the theory one obtains will be sensitive to the choice of complex structure on the curve Σ.

2.1.4 Comparisons for Topological Field Theory

Now we would like to compare the two formalisms for topological field theory starting from the Lagrangian data defining a classical field theory. An issue is that for constructing a functorial field theory it is essential to have a nonperturbative description of the moduli space of solutions to the classical equations of motion, whereas the formalism of factorization algebras has been developed only within the context of perturbative field theory. However, we claim that in the context of topological field theories, one can indeed find a shortcut for constructing nonperturbative quantizations in the context of factorization algebras.

We compare the two perspectives using the *state-operator correspondence*. The idea is that for an n-dimensional classical topological field theory on M, its associated functorial field theory $Z\colon \mathrm{Bord}_n \to \mathrm{Vect}_n$ and factorization algebra $\mathrm{Obs}^{\mathrm{q}}$ satisfy the relation $\mathrm{Obs}^{\mathrm{q}}(U) \simeq Z(\partial U)$ for an open set U of M, understood as an n-manifold. In order ro say something more precise we will recall the following general definition following Costello and Gwilliam [CG16, Section 4.9].

Definition 2.14. Let $\mathrm{Obs}^{\mathrm{q}}$ be a translation-invariant factorization algebra on $\mathbb{R}^n$, over a ring R (for instance, $R = k$ or $k[\![\hbar]\!]$ for $k = \mathbb{R}$ or $\mathbb{C}$). A *state* of $\mathrm{Obs}^{\mathrm{q}}$ is a linear map $\langle - \rangle \colon \mathrm{Obs}^{\mathrm{q}}(\mathbb{R}^n) \to R$.

For a topological field theory one can model $\mathbb{R}^n$ as an open ball B^n. As $Z(S^{n-1})$ is the space of states on S^{n-1} (which we imagine to be of infinite radius as we are working in a topological theory), a more natural identification would be $\mathrm{Obs}^{\mathrm{q}}(U) \cong Z(\partial U)^*$. On the other hand, the space of states is a Hilbert space and in the topological field theory, it is finite-dimensional, which allows us to choose an identification $\mathrm{Obs}^{\mathrm{q}}(U) \simeq Z(\partial U)$.

Having this in mind, let us discuss what we should do for finding local quantum observables starting from the data of a nonperturbative classical field theory $U \mapsto \mathrm{EOM}(U)$. The space of classical local observables is just the space of functions on the global moduli space $\mathrm{EOM}(\mathbb{R}^n)$, which forms a $\mathbb{P}_0$-factorization algebra. If it is an n-dimensional topological field theory, then its factorization structure is that of an $\mathbb{E}_n$-algebra from the local constancy. From the Poisson additivity result $\mathbb{P}_0 \otimes \mathbb{E}_n = \mathbb{P}_n$ (first announced by Rozenblyum, but not yet publically available – a different proof is due to Safronov [Saf16]), this structure can be described by a $\mathbb{P}_n$-algebra.

Now the process of BV quantization promotes the $\mathbb{P}_0$-algebra structure on $\mathrm{Obs}^{\mathrm{cl}}(U)$ to an $\mathbb{E}_0$-algebra structure on $\mathrm{Obs}^{\mathrm{q}}(U)$ for each open set $U \subset M$. Accordingly, for an n-dimensional topological field theory, from the Dunn additivity result $\mathbb{E}_0 \otimes \mathbb{E}_n = \mathbb{E}_n$ [Dun88, Lur14] , the factorization algebra is described by an $\mathbb{E}_n$-algebra. Conceptually speaking, this $\mathbb{E}_n$-algebra structure is the one coming from the identification $Z(S^{n-1})^* = \mathrm{Obs}^{\mathrm{q}}(B^n)$ where B^n is an n-ball.

Here an $\mathbb{E}_n$-algebra is a homotopy commutative algebra parametrized by configurations of n-disks. In particular an $\mathbb{E}_1$-algebra is a (homotopy) associative algebra, also known as an A_∞-algebra. On the other hand, $\mathbb{E}_n$-algebras are commutative up to homotopy for $n \geq 2$; there is a sense that $\mathbb{E}_n$-algebra becomes more and more commutative as n gets larger. Taking the cohomology of an $\mathbb{E}_n$-algebra always gives a commutative algebra.

In this sense, one should be able to translate a BV quantization problem for (nonperturbative) n-dimensional topological field theory just from the quantization problem going from a $\mathbb{P}_n$-algebra to an $\mathbb{E}_n$-algebra. In order to elucidate this translation, let us try to discuss their relationship.

For an $\mathbb{E}_n$-algebra A, by definition we have a map $\mathrm{Emb}(\coprod_I B^n, B^n) \times A^I \to A$. For $I = \{1,2\}$, we have a map $S^{n-1} \to \mathrm{Emb}(B^n \coprod B^n, B^n)$ by considering the first disk fixed at the origin. This gives a map $S^{n-1} \times A^2 \to A$. Taking cohomology, we get a map $\mathrm{H}^\bullet(S^{n-1}) \otimes \mathrm{H}^\bullet(A)^{\otimes 2} \to \mathrm{H}^\bullet(A)$. Thinking of the nontrivial class in $\mathrm{H}^{n-1}(S^{n-1})$, we have a map $\mathrm{H}^\bullet(A)^{\otimes 2}[n-1] \to \mathrm{H}^\bullet(A)$, or $(\mathrm{H}^\bullet(A)[n-1])^{\otimes 2} \to \mathrm{H}^\bullet(A)[n-1]$.

Theorem 2.15 (Cohen [Coh76])**.** Let A be an $\mathbb{E}_n$-algebra. Then the above map on $\mathrm{H}^\bullet(A)$ induces a Lie bracket of degree $1-n$ on $\mathrm{H}^\bullet(A)$. Moreover, if $n > 1$, then $\mathrm{H}^\bullet(A)$ is a $\mathbb{P}_n$-algebra.

Surprisingly, this process of going from an $\mathbb{E}_n$-algebra to a $\mathbb{P}_n$-algebra by taking cohomology doesn't lose any information for $n \geq 2$. This is articulated by the following formality result of the $\mathbb{E}_n$ operad.

Theorem 2.16. [Toë13, Corollary 5.4] For $n \geq 0$, if $X = \operatorname{Spec} A$, then the dg Lie algebra $C^{\mathbb{E}_{n+1}}(X)[n+1]$ is non-canonically quasi-isomorphic to the dg Lie algebra $\mathrm{Pol}(X,n)[n+1]$. The quasi-isomorphism depends on the choice of a Drinfeld associator.

Here $C^{\mathbb{E}_{n+1}}(X)$ is the $\mathbb{E}_{n+1}$-Hochschild cochain complex, which is an $\mathbb{E}_{n+2}$-algebra, and hence $C^{\mathbb{E}_{n+1}}[n+1]$ has the structure of a dg Lie algebra controlling the deformations of the $\mathbb{E}_{n+1}$-algebra structure on A. For the right-hand side, an n-shifted symplectic structure on X defines an n-shifted Poisson structure on $\mathcal{O}(X)$, or a $\mathbb{P}_{n+1}$-algebra, understood as an element of $\mathrm{Pol}(X,n) = \mathcal{O}(T^*[n+1]X)$, which itself is a $\mathbb{P}_{n+2}$-algebra and hence $\mathrm{Pol}(X,n)[n+1]$ has the structure of a dg Lie algebra controlling the deformations of $\mathbb{P}_{n+1}$-algebra structure on A.

By a choice of the formality isomorphism, an $\mathbb{E}_{n+1}$-algebra structure on A can be encoded by the $\mathbb{P}_{n+1}$-algebra structure on $\mathrm{H}^\bullet(A)$. In particular, given an $\mathbb{P}_{n+1}$-algebra, if it does not admit any deformations as $\mathbb{P}_{n+1}$-algebra, then it doesn't admit any $\mathbb{E}_{n+1}$-algebra deformations either, so the desired $\mathbb{E}_{n+1}$-algebra structure is necessarily the trivial one coming from the commutative algebra stucture.

One upshot of this argument is that if one is given an $\mathbb{E}_{n+1}$-algebra A^{q} quantizing a $\mathbb{P}_{n+1}$-algebra A^{cl} which does not admit any $\mathbb{P}_{n+1}$-algebra deformations, then as commutative algebras, $\mathrm{H}^\bullet(A^{\mathrm{q}})$ and A^{cl} are necessarily equivalent.

Example 2.17 (Quantum mechanics and deformation quantization)**.** Observables of (not necessarily topological) classical mechanics form a locally constant $\mathbb{P}_0$-factorization algebra on the real line, which yield an $\mathbb{E}_1$-algebra in $\mathbb{P}_0$-algebras. By the Poisson additivity theorem, this is a $\mathbb{P}_1$-algebra or an ordinary Poisson algebra. The deformation quantization problem asks if one can promote this to the algebra of observables in quantum mechanics, which is an associative algebra, or $\mathbb{E}_1$-algebra. The formality isomorphism in this case realizes the identification between deformations of $\mathbb{P}_1$-algebras and $\mathbb{E}_1$-algebras. In other words, whenever we have a $\mathbb{P}_1$-algebra, it corresponds to a $\mathbb{E}_1$-algebra upon choice of the formality isomorphism, which proves the deformation quantization. Indeed, this is the strategy exploited in Tamarkin's fundamental work [Tam98].

In sum, from the data of a classical field theory $U \mapsto \mathrm{EOM}(U)$, we can construct the algebra of local operators as well as the category of boundary conditions. That is, functions on EOM(pt) form a $\mathbb{P}_n$-algebra and its quantization forms an $\mathbb{E}_n$-algebra capturing the factorization algebra structure of local quantum observables. In codimension 2, $\mathrm{EOM}(Y^{n-2})$ is 1-shifted symplectic, and we obtain the category of boundary conditions by categorified geometric quantization (in suitable examples). Our main examples – topologically twisted $N=4$ super Yang–Mills – yield examples of categories with $\mathbb{E}_2$-algebra actions by exactly this procedure.

2.2 Support and Localization in dg Categories

In this section we'll recall some of the main constructions in the work of Benson, Iyengar, and Krause [BIK08] and of Arinkin and Gaitsgory [AG15]. We'll explain what it means for an $\mathbb{E}_2$-algebra to act on a dg category, and how to think about the support of an object – or equivalently how to think about localizations of the category. We'll begin by recalling some facts about Hochschild cochains which will be useful for our definitions, introduce the notion of support, and finally explain Arinkin–Gaitsgory's notion of the *scheme of singularities* of a derived stack.

2.2.1 Localization and Singular Support

We'll begin by recalling the construction of the Hochschild cochains of a dg category following [AG15, Appendix E].

Proposition 2.18. There is an adjunction

$$L\colon \{\mathbb{E}_2\text{-algebras}\} \rightleftarrows \{\text{monoidal dg categories}\}\colon R$$

where – as an $\mathbb{E}_1$-algebra – $R(\mathcal{L}) = \mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}})$, and where – as a dg category – $L(\mathcal{A}) = \mathcal{A}^{\mathrm{op}}\text{-mod}$.

The additional product on the $\mathbb{E}_1$-algebra $\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}})$ is inherited from the monoidal structure on $\mathcal{L}$. Indeed, as Arinkin and Gaitsgory explain, the above adjunction should be thought of as a specific example of an adjunction that makes sense internally in any ∞-category, applied here internally to $\mathbb{E}_1$-algebras. Using this adjunction we can define Hochschild cochains as an $\mathbb{E}_2$-algebra.

Definition 2.19. The *Hochschild cochains* $\mathrm{HC}^\bullet(\mathcal{C})$ of a dg category $\mathcal{C}$ are the $\mathbb{E}_2$-algebra $R(\mathrm{End}_{\mathrm{dgCat}_{\mathrm{cont}}}(\mathcal{C}))$ obtained from Proposition 2.18 – that is, the algebra $\mathrm{End}_{\mathrm{End}_{\mathrm{dgCat}_{\mathrm{cont}}}(\mathcal{C})}(\mathrm{id}_{\mathcal{C}})$. The cohomology of this $\mathbb{E}_2$-algebra is the *Hochschild cohomology* $\mathrm{HH}^\bullet(\mathcal{C})$ of $\mathcal{C}$.

For the rest of this section, $\mathcal{C}$ will denote a dg category, and A will denote an $\mathbb{E}_2$-algebra. The idea of singular support is that one can consider the support of a sheaf under a natural action of the Hochschild cochains. In order to make sense of this, we'll need to explain what "support" means in a general context. This formalism was originally developed by Benson, Iyengar, and Krause [BIK08] for triangulated categories.

Definition 2.20. A (left) *action* of an $\mathbb{E}_2$-algebra A on $\mathcal{C}$ is an action of the monoidal dg category A-mod on $\mathcal{C}$, that is a monoidal functor $A\text{-mod} \to \mathrm{End}_{\mathrm{dgCat}_{\mathrm{cont}}}(\mathcal{C})$.

Proposition 2.21. There is a canonical action of $\mathrm{HC}^\bullet(\mathcal{C})^{\mathrm{op}}$ on $\mathcal{C}$, where here the superscript op indicates the opposite $\mathbb{E}_1$-algebra [2].

Proof. Given any monoidal dg category $\mathcal{L}$ the counit of the adjunction from 2.18 is a monoidal functor

$$\varepsilon_{LR}\colon \mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}})^{\mathrm{op}}\text{-mod} \to \mathcal{L}.$$

□

From this point of view, an alternative characterization of the action of A on $\mathcal{C}$ is as an $\mathbb{E}_2$-homomorphism $A \to \mathrm{HC}^\bullet(\mathcal{C})^{\mathrm{op}}$, or as a monoidal functor $A^{\mathrm{op}}\text{-mod} \to \mathrm{HC}^\bullet(\mathcal{C})\text{-mod}$. We can use this to give a suitable definition of localization of a dg category. In what follows we'll write $\mathrm{H}^{2\bullet}(A)$ for the even cohomology of A viewed as a commutative algebra upon forgetting the grading. In particular, after passing to the even cohomology we can ignore the superscript "op".

Definition 2.22. Let A be an $\mathbb{E}_2$-algebra, and let Y be a closed subset of $\mathrm{Spec}\,\mathrm{H}^{2\bullet}(A)$. The *localization* $A\text{-mod}_Y$ at Y is the full subcategory of A-mod consisting of modules M where M, or rather its image in the homotopy category, is set-theoretically supported on Y as a $\mathrm{H}^{2\bullet}(A)$-module.

Remark 2.23. For this to be the correct definition Y should be *conical*: invariant for the $\mathbb{C}^\times$ on A coming from the grading. This is because we forgot the grading on $\mathrm{H}^{2\bullet}(A)$; it really only makes sense if we localize on a closed subset in the graded sense, i.e. on a homogeneous ideal. For comparison, see the discussion in [AG15, Section 3.6].

Remark 2.24. We define localization in terms of set-theoretic support, but in principle we could equally well consider a stricter notion where we restrict the scheme-theoretic support of modules. This scheme theoretic notion, however, does not appear to produce non-trivial localized categories in natural examples: see Remark 2.39 for a

[2] Arinkin and Gaitsgory write this as $A^{\mathrm{int\text{-}op}}$, where "int" stands for "internal", they write $A^{\mathrm{ext\text{-}op}}$ for the opposite using the other ("external") $\mathbb{E}_1$-structure.

more detailed discussion. One could define more general localization conditions by taking the scheme-theoretic support conditions with respect to ideals which are not maximal, having the effect of allowing deformations in some but not all derived directions. While this may be mathematically interesting in some examples, we don't currently have a physically meaningful reason to consider such constructions.

Definition 2.25. Suppose A is an $\mathbb{E}_2$-algebra acting on a dg category $\mathcal{C}$, and let Y be a closed subset of $\operatorname{Spec} \mathrm{H}^{2\bullet}(A)$. The *localization* $\mathcal{C}_Y$ of $\mathcal{C}$ at Y is the tensor product

$$\mathcal{C}_Y = A\text{-mod}_Y \otimes_{A\text{-mod}} \mathcal{C}$$

with respect to the action of A-mod on $\mathcal{C}$.

Definition 2.26. The *support* of an object $c \in \mathcal{C}$ in the affine scheme $\operatorname{Spec} \mathrm{H}^{2\bullet}(A)$ is the minimal closed subset Y of $\operatorname{Spec} \mathrm{H}^{2\bullet}(A)$ such that c lies in $\mathcal{C}_Y$.

The localization at Y is a full subcategory of $\mathcal{C}$ – we think of it as follows. The Hochschild cohomology $\mathrm{HH}^\bullet(\mathcal{C})$ maps to the endomorphism algebra $\mathrm{End}_{\mathrm{Ho}(\mathcal{C})}(c)$ for each object c in $\mathcal{C}$ and its image in the homotopy category which we'll abuse notation to write as $c \in \mathrm{Ho}(\mathcal{C})$. By the universal property of Hochschild cochains – where A is an $\mathbb{E}_2$-algebra acting on $\mathcal{C}$– we obtain a map $\mathrm{H}^{2\bullet}(A) \to \mathrm{HH}^\bullet(\mathcal{C})$. Then the localization is the full subcategory spanned by objects c such that $\mathrm{End}_{\mathrm{Ho}(\mathcal{C})}(c)$ has support in Y with respect to the composition.

Now, let's consider the special case where $\mathcal{C} = \mathrm{IndCoh}(X)$ for a derived stack X. We'll take the universal $\mathbb{E}_2$-algebra acting on $\mathrm{IndCoh}(X)$, namely $A = \mathrm{HC}^\bullet(\mathrm{IndCoh}(X))$ – the Hochschild cochains of the category $\mathrm{IndCoh}(X)$. As we discuss below, the singular support of a sheaf on X is essentially going to be the support with respect to the action of the even Hochschild cohomology.

Remark 2.27. From now on, when X is a derived stack we'll simply write $\mathrm{HC}^\bullet(X)$ for the complex $\mathrm{HC}^\bullet(\mathrm{IndCoh}(X))$.

There's a natural interpretation of the even Hochschild cohomology of a derived stack X in terms of the geometry of the shifted cotangent space of X up to degree shifting. Let us first introduce the relevant geometric object.

Definition 2.28. The *scheme of singularities* $\mathrm{Sing}(X)$ of X is the classical truncation of the (-1)-shifted cotangent space, that is, $\mathrm{Sing}(X) := (T^*[-1]X)^{\mathrm{cl}}$.

While this definition makes sense generally, it's most interesting in the situation where X is quasi-smooth. A derived stack X is called *quasi-smooth* if its tangent complex is concentrated in degrees ≤ 1. In this case, one can write $\mathrm{Sing}(X) = \mathrm{Spec}_{X^{\mathrm{cl}}} \mathrm{Sym}(H^0(\mathbb{T}_X[1]))$ and there is a natural projection map $\mathrm{Sing}(X) \to X$.

As mentioned, the Hochschild cochains of X are closely connected to its shifted cotangent space, which – we'll see – introduces a connection between the even Hochschild cohomology and the scheme of singularities. We'll make a construction for affine derived schemes and then extend it to derived stacks using an atlas. The following is a version of the HKR theorem.

Lemma 2.29 ([AG15, Corollary G.2.7])**.** If Z is an eventually coconnective affine derived scheme then there is a canonical isomorphism of associative algebras

$$\Gamma(Z; U_{\mathcal{O}_Z}(\mathbb{T}_Z[-1])) \to \mathrm{HC}^\bullet(Z)$$

from the derived global sections of the universal enveloping algebra of the Lie algebra object $\mathbb{T}_Z[-1]$ to the Hochschild cochains of Z.

This induces, for any (eventually coconnective) affine derived scheme Z, an isomorphism

$$\mathrm{H}^\bullet(Z; U_{\mathcal{O}_Z}(\mathbb{T}_Z[-1])) \to \mathrm{HH}^\bullet(Z),$$

therefore, in particular, a map $\Gamma(Z; \mathcal{O}_{Z^{\mathrm{cl}}}) \to \mathrm{HH}^0(Z)$ of commutative algebras, and a map $\Gamma(Z; \mathrm{H}^0(\mathbb{T}_Z[1])) \cong \Gamma(Z; H^1(\mathbb{T}_Z)) \to \mathrm{HH}^2(Z)$ of $\Gamma(Z; \mathcal{O}_{Z^{\mathrm{cl}}})$-modules. Taking the symmetric algebra, this defines a canonical map of commutative algebras

$$\Gamma(\mathrm{Sing}(Z); \mathcal{O}_{\mathrm{Sing}(Z)}) \to \mathrm{HH}^{2\bullet}(Z).$$

Therefore we can define singular support in the following way.

Definition 2.30. The *singular support* of a sheaf $\mathcal{F} \in \mathrm{IndCoh}(Z)$ for a quasi-smooth affine derived scheme Z is a closed subspace of $\mathrm{Sing}(Z)$ defined as the support of $\mathrm{End}(\mathcal{F})$ as a module over $\Gamma(\mathrm{Sing}(Z); \mathcal{O}_{\mathrm{Sing}(Z)})$ with respect to the composition. The category of sheaves with singular support in $Y \subseteq \mathrm{Sing}(Z)$ is denoted by $\mathrm{IndCoh}_Y(Z)$.

Now, suppose X is a quasi-smooth derived stack. We can define the singular support of a sheaf $\mathcal{F} \in \mathrm{IndCoh}(X)$ using a smooth atlas of affine derived schemes.

Definition 2.31. If Y is a closed subspace of $\mathrm{Sing}(X)$, we define the category $\mathrm{IndCoh}_Y(X)$ to be the limit

$$\mathrm{IndCoh}_Y(X) = \lim_{Z \to X} \mathrm{IndCoh}_{Y \times_X Z}(Z)$$

over smooth maps from affine derived schemes to X. Here the inclusion $Y \hookrightarrow \mathrm{Sing}(X)$ defines a map $Y \to X$ since X is quasi-smooth.

2.2.2 Singular Support for Global Complete Intersection Stacks

In this section we'll review some results of Arinkin and Gaitsgory about singular support for a special sort of stack which is amenable to computation. This material closely follows [AG15, Section 9] but we include it here for ease of reference and for the sake of including a slight modification that will be important in Section 3.4: the inclusion of a formal invertible degree 2 parameter. We'll endeavour to keep our notation consistent with the notation used by Arinkin and Gaitsgory.

Definition 2.32. A derived stack $\mathcal{Z}$ is a *global complete intersection* if it can be obtained as a fiber product of smooth stacks

$$\begin{tikzcd} \mathcal{Z} \arrow[r] \arrow[d] & \mathcal{U} \arrow[d] \\ \mathcal{X} \arrow[r, shift left] & \mathcal{V} \arrow[l, shift left] \end{tikzcd}$$

where $\mathcal{X} \to \mathcal{V}$ is a section of a smooth schematic map $\mathcal{V} \to \mathcal{X}$. One can associate to such a global complete intersection a pair of groupoids, namely $\mathcal{G}_{\mathcal{Z}/\mathcal{U}} := \mathcal{Z} \times_{\mathcal{U}} \mathcal{Z}$ as a derived group scheme over $\mathcal{Z}$ and $\mathcal{G}_{\mathcal{X}/\mathcal{V}} := \mathcal{X} \times_{\mathcal{V}} \mathcal{X}$, a derived group scheme over $\mathcal{X}$.

According to [AG15, Section 9.2], in this context we can understand singular support conditions for $\mathrm{IndCoh}(\mathcal{Z})$ in terms of an action of the monoidal category $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$. Indeed, the groupoid $\mathcal{G}_{\mathcal{Z}/\mathcal{U}}$ acts on $\mathcal{Z}$, and so the category $\mathrm{IndCoh}(\mathcal{Z})$ is a module over the monoidal category $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{Z}/\mathcal{U}})$. There's a natural monoidal pullback functor

$$\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{X})} \mathrm{QC}(\mathcal{U}) \to \mathrm{IndCoh}(\mathcal{G}_{\mathcal{Z}/\mathcal{U}})$$

which makes $\mathrm{IndCoh}(\mathcal{Z})$ into a module over $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{IndCoh}(\mathcal{X})} \mathrm{IndCoh}(\mathcal{U})$, or indeed over $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$. In order to understand singular support in terms of this action, we first identify the category on the left hand side as

$$\begin{aligned} \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{X})} \mathrm{QC}(\mathcal{U}) &\cong \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})} \left(\mathrm{QC}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{X})} \mathrm{QC}(\mathcal{U})\right) \\ &\cong \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})} \mathrm{QC}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}) \\ &\cong \mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}). \end{aligned}$$

By [AG15, Lemma 9.2.6] the category $\mathrm{IndCoh}_Y(\mathcal{Z})$ is equivalent to the localization

$$\mathrm{IndCoh}(\mathcal{Z}) \otimes_{\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U})} \mathrm{IndCoh}_{f(Y)}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U})$$

where $f \colon \mathrm{Sing}(\mathcal{Z}) \to \mathrm{Sing}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}) \cong V^* \times_{\mathcal{X}} \mathcal{U}$ is the morphism obtained by first embedding $\mathrm{Sing}(\mathcal{Z})$ into $V^* \times_{\mathcal{X}} \mathcal{Z}$ and then composing with the defining map $\mathcal{Z} \to \mathcal{U}$. We'll want a slightly modified version of this statement.

Let t be a degree 2 parameter, and write $\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!)$ for the tensor product $\mathrm{IndCoh}(\mathcal{Z}) \otimes \mathbb{C}(\!(t)\!)$-mod. We'll describe the localization $\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!)$ associated to a closed subset $Y \subseteq \mathrm{Sing}(\mathcal{Z})$, given by localizing at the closed subset generated by Y under the action of $\mathbb{C}(\!(t)\!)$. This localization is defined as follows.

Definition 2.33. If $\mathcal{Z}$ is a quasi-smooth affine derived scheme we define $\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!)$ to be the localization of $\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!)$ at the ideal $\mathcal{I}_Y(\!(t)\!)$ in $\mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!)$ generated by $\mathcal{I}_Y$ under the $\mathbb{C}(\!(t)\!)$ action, using the natural composite morphism $\mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!) \to \mathrm{HC}^\bullet(\mathrm{IndCoh}(\mathcal{Z}))(\!(t)\!) \to \mathrm{HC}^\bullet(\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!))$. We'll often refer heuristically to the "closed set $Y(\!(t)\!)$" even though this does not literally make sense since $\mathbb{C}(\!(t)\!)$ is not connective. If $\mathcal{Z}$ is a more general quasi-smooth derived stack we define $\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!)$ as the limit

$$\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!) = \lim_{Z \to \mathcal{Z}} \mathrm{IndCoh}_{(Y \times_{\mathcal{Z}} Z)(\!(t)\!)}(Z)(\!(t)\!)$$

over smooth affine charts $Z \to \mathcal{Z}$.

Remark 2.34. In Arinkin and Gaitsgory's setup the supports of objects are necessarily conical (i.e. $\mathbb{C}^\times$-invariant). As we mentioned above in Remark 2.23 this is because $\mathrm{HH}^{2\bullet}(\mathcal{Z})$ is naturally a graded object and it's important to remember that grading, and consider only homogeneous ideals. Upon introducing the parameter t this still applies, but now $\mathbb{C}^\times$ acts not only on the Hochschild cohomology $\mathrm{HH}^{2\bullet}(\mathcal{Z})$ but also on the parameter t with weight 2.

This means that we can consider support conditions for any conical subset of $\mathrm{Sing}(\mathcal{Z})(\!(t)\!)$ (speaking heuristically – more precisely we work with a smooth affine cover and locally specify homogenous ideals in $\mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!)$), and one can obtain such a subset from *any* closed subset of the classical scheme of singularities $Y \subseteq \mathrm{Sing}(\mathcal{Z})$ provided $\mathbb{C}^\times$ acts with even weights. Indeed, let A be a dg commutative algebra with $\mathbb{C}^\times$-action and an ideal I, and choose generators $\{a_i\}$ where a_i has weight $2w_i$. Embed A into $A(\!(t)\!)$ by sending a_i to the weight zero element $a_i t^{-w_i}$. The image of I under this map is $\mathbb{C}^\times$-invariant, or equivalently generates a homogenous ideal in $A(\!(t)\!)$. This ideal recovers I when we set the parameter t to 1.

The localizations we obtain by this procedure are somewhat more general than the localizations at closed sets $Y(\!(t)\!)$ described above. However, using the modified embedding $A \hookrightarrow A(\!(t)\!)$ giving the generators weight zero – in the case of singular support a modified embedding $\mathcal{O}(\mathrm{Sing}(\mathcal{Z})) \to \mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!)$ – we can make our homogeneous ideals into ideals of the form $\mathcal{I}_Y(\!(t)\!)$, and therefore find ourselves in the situation described in definition 2.33 once more. Essentially, we modified the action of the algebra $\mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!)$ using an automorphism that becomes trivial after composition with the evaluation map $t \mapsto 1$.

Having set up the basic definitions,we claim that there is an equivalence

$$\mathrm{IndCoh}_{f^{-1}(Y)(\!(t)\!)}(\mathcal{Z})(\!(t)\!) \cong \mathrm{IndCoh}(\mathcal{Z})(\!(t)\!) \otimes_{\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U})(\!(t)\!)} \mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U})(\!(t)\!)$$

where now $Y \subseteq \mathrm{Sing}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U})$ is a Zariski-closed subset.

This is proven in exactly the same way as [AG15, Lemma 9.2.6]. As they argue, it suffices to prove the claim in the case where $\mathcal{Z}$ is affine. This argument is not affected by the introduction of the parameter t since the descent takes place independently of this parameter (heuristically we're performing descent for maps $Z \times \mathrm{Spec}\,\mathbb{C}(\!(t)\!) \to \mathcal{Z} \times \mathrm{Spec}\,\mathbb{C}(\!(t)\!)$ which are constant in the second factor). The map $f\colon \mathrm{Sing}(\mathcal{Z}) \to V^* \times_{\mathcal{X}} \mathcal{U}$ induces a map $f^*\colon \mathrm{Sym}(V)(\!(t)\!) \otimes_{\mathcal{O}(\mathcal{X})} \mathcal{O}(\mathcal{U}) \to \mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!)$. We then note that the triangle

$$\begin{tikzcd}
\mathrm{Sym}(V)(\!(t)\!) \otimes_{\mathcal{O}(\mathcal{X})} \mathcal{O}(\mathcal{U}) \arrow[rr, "f^*"] \arrow[dr] & & \mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!) \arrow[dl] \\
& \mathrm{HH}^{2\bullet}(\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!)) &
\end{tikzcd} \qquad (*)$$

defined using this map and the action of $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times \mathcal{U})(\!(t)\!)$ on $\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!)$ commutes. This implies the claim that we want since we can equivalently take the localization at $f^{-1}(Y)(\!(t)\!)$ with respect to the $\mathcal{O}(\mathrm{Sing}(\mathcal{Z}))(\!(t)\!)$-action, or the localization at its preimage under f^*, i.e. the localization at $f(f^{-1}(Y))(\!(t)\!) = Y(\!(t)\!)$ with respect to the $\mathrm{Sym}(V)(\!(t)\!) \otimes \mathcal{O}(\mathcal{U})$-action.

In particular, suppose $Y \subseteq V^* \times_{\mathcal{X}} \mathcal{Z}$ is a subset of the form $\widetilde{Y} \times_{\mathcal{X}} \mathcal{Z}$ for some closed subset $\widetilde{Y} \subseteq V^*$. Then according to [AG15, Proposition 8.4.14] there is an equivalence

$$\mathrm{IndCoh}_Y(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U}) \cong \mathrm{IndCoh}_{\widetilde{Y}}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{X})} \mathrm{QC}(\mathcal{U}).$$

We can make the same argument identically including the parameter t in order to say that there is an equivalence

$$\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{G}_{\mathcal{X}/\mathcal{V}} \times_{\mathcal{X}} \mathcal{U})(\!(t)\!) \cong \mathrm{IndCoh}_{\widetilde{Y}(\!(t)\!)}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})(\!(t)\!) \otimes_{\mathrm{QC}(\mathcal{X})(\!(t)\!)} \mathrm{QC}(\mathcal{U})(\!(t)\!).$$

The upshot of all this is that if $Y \subseteq \mathrm{Sing}(\mathcal{Z})$ is a closed subset of the form $f^{-1}(\widetilde{Y} \times_{\mathcal{X}} \mathcal{U})$ then we can understand the localization $\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!)$ as the localization along $\widetilde{Y}(\!(t)\!)$ of $\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!)$ with respect to the action of $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})(\!(t)\!)$. Even better, by Koszul duality we can identify $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ with the category $\mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}}$-mod and thus [AG15, Corollary 9.1.7] we are able to identify $\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!)$ as the localization along $\widetilde{Y}(\!(t)\!)$ of $\mathrm{IndCoh}(\mathcal{Z})(\!(t)\!)$ with respect to the action of $\mathrm{HC}(\mathcal{X}/\mathcal{V})(\!(t)\!)^{\mathrm{op}}$-mod. Specifically here we're using Koszul duality *before* taking the tensor product with $\mathbb{C}(\!(t)\!)$-mod; it's still true that this induces an equivalence of categories

$$\mathrm{HC}(\mathcal{X}/\mathcal{V})(\!(t)\!)^{\mathrm{op}}\text{-mod}_{Y(\!(t)\!)} \cong \mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})(\!(t)\!)$$

with specified support conditions. It suffices to check this smoothly locally, so we can assume $\mathcal{X}$ is affine in which case it follows from commutativity of the triangle (*).

We can summarize the result we'll need – whose proof we described above – as follows.

Proposition 2.35. If $Y \subseteq \mathrm{Sing}(\mathcal{Z})$ is a closed subset of the form $f^{-1}(\widetilde{Y} \times_{\mathcal{X}} \mathcal{U})$, then there is a canonical equivalence

$$\mathrm{IndCoh}_{Y(\!(t)\!)}(\mathcal{Z})(\!(t)\!) \cong \mathrm{HC}(\mathcal{X}/\mathcal{V})(\!(t)\!)^{\mathrm{op}}\text{-mod}_{\widetilde{Y}(\!(t)\!)} \otimes_{\mathrm{HC}(\mathcal{X}/\mathcal{V})(\!(t)\!)^{\mathrm{op}}\text{-mod}} \mathrm{IndCoh}(\mathcal{Z})(\!(t)\!),$$

where the category $\mathrm{HC}(\mathcal{X}/\mathcal{V})^{\mathrm{op}}$-mod acts on $\mathrm{IndCoh}(Z)$ by first using Koszul duality to identify it with $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$, then using the monoidal functor

$$\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}}) \otimes_{\mathrm{QC}(\mathcal{X})} \mathrm{QC}(\mathcal{U}) \to \mathrm{IndCoh}(\mathcal{G}_{\mathcal{Z}/\mathcal{U}})$$

to define an action of $\mathrm{IndCoh}(\mathcal{G}_{\mathcal{X}/\mathcal{V}})$ on $\mathrm{IndCoh}(\mathcal{Z})$.

2.3 Localization at a Vacuum

2.3.1 Localization in the Factorization Algebra Context

In the functorial context, the notion of boundary conditions admits a precise mathematical model *if* one is genuinely working in a purely topological theory. However, many theories that one wishes to study in practice are not actually topological in the sense of Atiyah–Segal. For instance, an n-dimensional topological quantum field theory in the sense of Atiyah–Segal would assign a number as a partition function to an n-dimensional closed manifold, but in general there is not sufficient data to produce a number unless we first choose a vacuum state. Accordingly, once we make a choice of vacuum state then only the boundary conditions compatible with the choice would be relevant, leading to a more refined category of boundary conditions. The connections to geometric representation theory that we have in mind can be captured by the following slogan:

Support conditions for categories of boundary conditions in a TQFT arise by demanding compatibility with a choice of vacuum state.

To put it another way, our main aim in this paper is to explain a physical motivation for certain kinds of condition that one can impose whenever one interprets a category as the category of boundary conditions in a TQFT. This will have nontrivial content because not every TQFT arising in physics is of Atiyah–Segal type: we will make the idea more precise. In this subsection we'll describe the notion of a "vacuum state" in a TQFT. In the following subsection we'll observe that in a genuine 2-extended TQFT this notion is not very interesting, because the moduli of vacua is necessarily discrete.

Let $A = \mathrm{Obs}^{\mathrm{q}}(\mathbb{R}^n)$ be a locally constant factorization algebra on $\mathbb{R}^n$, or equivalently an $\mathbb{E}_n$-algebra. Let $\mathcal{B}$ be a dg category such that A acts on $\mathcal{B}$, i.e. there exists an $\mathbb{E}_2$-algebra map $A^{\mathrm{op}} \to \mathrm{HC}^\bullet(\mathcal{B})$. We'll usually construct such a setup by quantizing a *classical* field theory: A will quantize the local classical observables and $\mathcal{B}$ will arise by categorical geometric quantization. That is, $\mathcal{B}$ will be a category quantizing L, where $L \to \mathrm{EOM}(Y)$ is a chosen Lagrangian in the space of solutions to the equations of motion on an $(n-2)$-manifold Y. Often $\mathrm{EOM}(Y)$ will be a 1-shifted cotangent bundle $T^*[1]\mathcal{X}$, and there's a natural choice of Lagrangian given by the zero-section $\mathcal{X}$ so that $\mathcal{B} = \mathrm{IndCoh}(\mathcal{X})$; in particular this will be the case for our B-twisted theory.

We begin by introducing the moduli space of vacua – the universal space with an action of the algebra of local observables.

Definition 2.36. Recall from Definition 2.14 that the space of *states* in a quantum field theory is the space of functionals $\phi\colon \mathrm{Obs}^{\mathrm{q}}(B^n) \to \mathbb{R}$. A state ϕ is called a *vacuum state* if it is translation invariant and satisfies the *cluster decomposition property.* That is, for all local observables $\mathcal{O}_1$ on $B_{r_1}(0)$ and $\mathcal{O}_2$ on $B_{r_2}(0)$, we have

$$(\mathcal{O}_1 * \tau_x(\mathcal{O}_2))(\phi) - \mathcal{O}_1(\phi)\mathcal{O}_2(\phi) \to 0 \text{ as } x \to \infty$$

where τ_x denotes the translation of an observable by $x \in \mathbb{R}^n$. In the case where our theory is topological the condition of translation invariance is automatic, and the cluster decomposition property simply says that ϕ induces a ring homomorphism from the cohomology of local observables to $\mathbb{R}$, or equivalently a point in mSpec of the cohomology of local observables. We broaden this definition slightly, and say that the *moduli space $\mathcal{V}$ of vacua* is the space $\mathrm{Spec}\,\mathrm{H}^\bullet(\mathrm{Obs}^{\mathrm{q}}(B^n))$.

One can now define the localization $\mathrm{H}^\bullet(A)\text{-mod}_{\{v\}}$ at a choice $v \in \mathcal{V}$ of vacuum following the formalism developed above, i.e. via Definition 2.25.

Definition 2.37. The *localization* of the category $\mathcal{B}$ at a point $v \in \mathcal{V}$ is the tensor product $\mathcal{B} \otimes_{A\text{-mod}} A\text{-mod}_v$. As usual this can be thought of as the full subcategory of objects $\mathcal{F} \in \mathcal{B}$ such that $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ is supported at v with respect to the action of A through Hochschild cohomology.

Remark 2.38. We motivate this definition in the following way. Objects in the localization of the category $\mathcal{B}$ of boundary conditions at v are objects $\mathcal{F}$ of $\mathcal{B}$ so that $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ is supported at v, just as in the previous subsection. What does this mean from the point of view of vacua? Well, we should think of the space $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ as the *phase space* of our topological field theory coupled to the boundary condition $\mathcal{F}$. Indeed, in general, the hom space $\mathrm{Hom}_{\mathcal{B}}(\mathcal{F}_1, \mathcal{F}_2)$ can be interpreted as the space of states on a strip $Y^{n-2} \times [0,1]$ with boundary conditions $\mathcal{F}_1$ and $\mathcal{F}_2$ on the two boundary components. In particular we obtain $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ by putting the boundary condition $\mathcal{F}$ at both boundary components. We view this as the phase space on $Y \times [0,1]$ in the theory coupled to the boundary condition $\mathcal{F}$. Alternatively (assuming we're working in a topological context) we can view this as the space of observables on Y times a half disk $D = \{(x,y)\colon x^2+y^2 \leq 1, x > 0\}$, with boundary condition $\mathcal{F}$ on the closed edge (see Figure 1). This is a version of the state-operator correspondence.

From this point of view there's clearly a map from the algebra A of observables in the bulk to the algebra $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ of observables in this coupled bulk-boundary system by the inclusion of those observables supported at a small ball in the interior of $D \times Y$. In particular using this inclusion one can evaluate elements of $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ at vacua in $\mathcal{V} = \mathrm{Spec}\,A$. So, from this point of view, an object $\mathcal{F}$ survives the localization if the bulk-boundary phase space $\mathrm{End}_{\mathcal{B}}(\mathcal{F})$ is supported at v, in other words if the bulk-boundary system is acted on non-trivially by those observables that only depend on a small neighbourhood of $v \in \mathcal{V}$. So objects of the localization of $\mathcal{B}$ are those boundary conditions that can "see" the vacuum state v.

Remark 2.39. From this physical point of view, using specifically a set-theoretic support condition to define localization at a vacuum is well-motivated. Since vacuum conditions involve an asymptotic constraint (the cluster decomposition property), it is natural to allow compatible boundary conditions to depend on an infinitesimal neighborhood of a fixed choice of vacuum state, which corresponds to fixing the support in the set-theoretic sense.

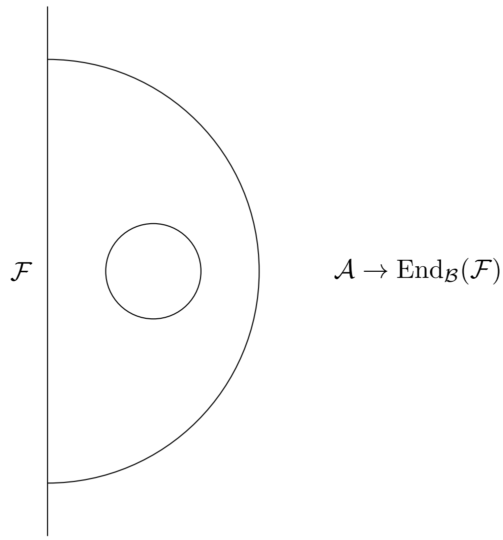

Figure 1: The action of $\mathcal{A}$ on $\mathcal{F}$ is mediated by a whistle cobordism with the given boundary condition.

Moreover, and from a purely mathematical perspective, if we had instead used a scheme-theoretical support condition then the category of boundary conditions compatible with the vacuum often turns out to be trivial. We'll explain this in a family of examples in a way which we hope is suggestive of a more general story.

For simplicity we'll consider the case where A is a local ring, so $\operatorname{Spec} A$ consists of a single closed point (but of course is generally non-trivial as an affine scheme). For instance, consider the topological B-model with target a compact Calabi–Yau manifold X, so $\mathcal{B} = \mathrm{Coh}(X)$ is the category of boundary conditions and $A = \mathrm{HC}^\bullet(X)$ is a finite-dimensional but not semi-simple ring. Then the category of boundary conditions is always compatible with the unique choice of a vacuum by definition. On the other hand, to understand the meaning of being scheme-theoretically supported at the point, let us investigate the map $\mathrm{HH}^2(\mathrm{Coh}(X)) \to \mathrm{Ext}^2_{\mathrm{Coh}(X)}(\mathcal{F}, \mathcal{F})$ for $\mathcal{F} \in \mathrm{Coh}(X)$ and try to see for which $\mathcal{F}$ this map vanishes. It is possible to proceed rigorously, but it will be enough to provide a heuristic, but conceptual, explanation for this remark.

If $\mathcal{B}$ is a dg-category, there is an exact triangle

$$\mathrm{Def}_{\mathcal{B}}(\mathcal{F}) \to \mathrm{Def}(\mathcal{B}, \mathcal{F}) \to \mathrm{Def}(\mathcal{B})$$

of DGLAs governing the corresponding deformation theories (see [BKP17, Proposition 4.3] for a statement in the context of spectral geometry), where $\mathrm{Def}_{\mathcal{B}}(\mathcal{F})$ is the shifted tangent complex $\mathbb{T}_{\mathcal{F}}[-1]\mathcal{M}_{\mathcal{B}} = \mathrm{End}(\mathcal{F})$ to the moduli $\mathcal{M}_{\mathcal{B}}$ of objects in $\mathcal{B}$ (see for instance [TV07]), $\mathrm{Def}(\mathcal{B})$ is $\mathrm{HC}^\bullet(\mathcal{B})[1]$ [BKP17, Section 3.3], and $\mathrm{Def}(\mathcal{B}, \mathcal{F})$ is their extension governing deformations of the category $\mathcal{B}$ together with the object $\mathcal{F}$. In particular, when $\mathcal{B} = \mathrm{Coh}(X)$, we find the boundary map $\mathrm{HH}^2(X) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$, where $\mathrm{HH}^2(X)$ describes the deformation of X and $\mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ describes the obstruction of $\mathrm{Def}(\mathcal{F})$. Now by HKR theorem we have $\mathrm{HH}^2(\mathrm{Coh}(X)) = H^0(X, \wedge^2 \mathbb{T}_X) \oplus H^1(X, \mathbb{T}_X) \oplus H^2(X, \mathcal{O}_X)$; from the perspective of deformation theory, these terms correspond to a noncommutative deformation of $\mathcal{O}_X$, a complex structure deformation of X, and a gerbal deformation of $\mathrm{Coh}(X)$, respectively. Then the map $\mathrm{HH}^2(X) \to \mathrm{Ext}^2(\mathcal{F}, \mathcal{F})$ would measure if for a given element of $\mathrm{HH}^2(X)$, deforming a given category, the object $\mathcal{F}$ remains to be an object of the deformed category. In particular, looking for an object scheme theoretically supported on the unique point amounts to finding $\mathcal{F}$ which survives for any such deformation of $\mathrm{Coh}(X)$, which is an extremely strong constraint.

For instance, consider a K3 surface X. No curve class survives a complex structure deformation. The structure sheaf doesn't survive a gerbal deformation. Finally, any sheaf on points does not survive a noncommutative deformation as the Weyl algebra doesn't admit a finite-dimensional representation. In sum, the category of boundary conditions on K3 surfaces scheme-theoretically supported on the unique point is trivial.

Remark 2.40. If the theory is semi-simple, which corresponds to the case where $\operatorname{Spec} A$ consists of a finite number of points, then one can decompose the category $\mathcal{C}$ into the corresponding finite number of simpler subcategories living over each vacuum. For instance, this happens when one considers the A-model or B-model with target a Fano manifold, or a Landau–Ginzburg theory. One can think of the subcategory as describing the category of boundary conditions in the IR limit with the given vacuum (for a related discussion and more for the example of Fukaya–Seidel type categories, one should refer to the work of Gaiotto, Moore, and Witten [GMW15]).

2.3.2 Motivation from Functorial Field Theory

In this section we'll explain an alternative construction of the localized category of boundary conditions in a topological field theory, from the functorial point of view. Given some assumptions on the compatibility between the algebraic and functorial perspectives on quantum field theory we expect this definition to be a special case of definition 2.37 in a precise sense, but for the present work we won't need a precise result of this nature; this section should be thought of as providing additional motivation for our constructions.

Let Z be a 2-extended n-dimensional TQFT as defined in Section 2.1.3. If Z comes from the restriction of a fully extended TQFT then full dualizability forces the algebra $Z(S^{n-1})$ to be finite-dimensional, and therefore to have discrete spectrum, but if we consider for instance a non-compact TQFT: one that is only defined on n-dimensional bordisms with non-empty outgoing boundary in each connected component, then this is no longer necessarily the case. In many interesting examples the spectrum of the algebra $Z(S^{n-1})$ consists of a single point, but more generally we can try to investigate a procedure to *localize* the theory Z at a point in this spectrum, thus splitting our topological field theory into sectors of this form.

More concretely, if we choose a compact $(n-2)$-manifold Y we can consider the dimensional reduction of Z along Y, that is, the fully extended 2-dimensional TQFT Z_Y defined by $Z_Y(X) = Z(Y \times X)$. We can now consider the localization of this theory with respect to the action of $Z(S^{n-1})$. As we discussed in Section 2.1.3 there is an $\mathbb{E}_2$-action of $Z(S^{n-1})$ on the dg category $Z(Y)$, where the $\mathbb{E}_n$-algebra $Z(S^{n-1})$ is broken down to $\mathbb{E}_2$ by the reduction along the $(n-2)$-manifold Y. We can therefore apply the definitions of Section 2.2, where now the role of the space of vacua is played by $\mathrm{H}^{2\bullet}(Z(S^{n-1}))$.

Definition 2.41. If v is a point in $\mathrm{H}^{2\bullet}(Z(S^{n-1}))$, the space of boundary conditions along Y *compatible* with the point v is defined to be the localized category $Z(Y)_v$.

As we noted above, in the fully extended context the algebra $Z(S^{n-1})$ is finite-dimensional, so the moduli space $\mathrm{H}^{2\bullet}(Z(S^{n-1}))$ is always discrete. In a less extended context this need no longer be true.

Remark 2.42. According to the state-operator correspondence, we anticipate an $\mathbb{E}_n$-equivalence between $Z(S^{n-1})$ and the local quantum observables $\mathrm{Obs}^{\mathrm{q}}(B^n)$ from the factorization algebra point of view in any context where both can be defined (as discussed in Section 2.1.4). As such, the above definition should be viewed as a special case of the definitions of the previous section, in the case where we have a topological functorial description of our theory.

Remark 2.43. One can define these categories separately for each $(n-2)$-manifold Y; it's natural to ask whether there's a localized 2-extended n-dimensional functorial field theory Z_v which assigns $Z(Y)_v$ to each Y. We hope to return to this question in future work.

3 Vacua and Singular Supports

Having described the abstract idea of localization in a few different ways, we'll explain how to implement the definition for our main example, which is the B-twisted $N = 4$ gauge theory we constructed in [EY15]. We'll comment on how the definitions work for the A-twisted theory, but there are still a number of unanswered questions that must be addressed before that example can be treated in depth.

3.1 Twisted $N = 4$ Gauge Theories

According to the definitions given in Section 2.3.1, we have to describe a locally constant factorization algebra of quantum observables, along with the category of boundary conditions on a curve Σ acted upon by the algebra of observables. We'll motivate these objects in the B-twisted theory using our construction of the classical B-twisted theory, along with some formality results specifying the quantization. The model we'll derive in this section will need a somewhat subtle correction, which we'll address in Section 3.4 below.

Let's begin by recalling what it means to twist a supersymmetric field theory, and our construction of the classical Kapustin–Witten twisted theories. Suppose we're given a perturbative classical field theory E with an action of the super group $H = \mathbb{C}^\times \ltimes \Pi\mathbb{C}$ which we denote by (α, Q), where α denotes the action of $\mathbb{C}^\times$ and Q denotes the action of $\Pi\mathbb{C}$. Following Costello [Cos13], we can give a construction of the *twisted theory* as a family of perturbative classical field theories parametrized by $\mathbb{C}(\!(t)\!)$ – where t is a fermionic parameter of cohomological degree 1 – by

$$E^{\Pi\mathbb{C}} \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!).$$

Here we note that $E^{\Pi\mathbb{C}}$ lives over the classifying stack $B(\Pi\mathbb{C})$, so it has the structure of a module over $\mathcal{O}(B(\Pi\mathbb{C})) = \mathbb{C}[\![t]\!]$.

Definition 3.1. The theory *twisted* by the action (α, Q) is the "generic fiber" of this $\mathbb{C}(\!(t)\!)$ -family, that is, it's the space $E^Q = (E^{\Pi\mathbb{C}} \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!))^{\mathbb{C}^\times}$ of $\mathbb{C}^\times$-invariants after inverting the formal parameter t, or heuristically restricting to the odd shifted formal punctured disk.

Remark 3.2. We'll return to this definition in Section 3.4, when we discuss exactly how to think about the moduli space of vacua in such a twisted theory.

In [EY15], we extended this definition to define twists *globally*, meaning twists of (-1)-shifted symplectic derived stacks with an action of the super group H. Using this extended definition, we calculated the following key examples.

Example 3.3. The *holomorphically twisted* $N = 4$ gauge theory on a smooth proper complex algebraic surface X has the following (-1)-shifted symplectic moduli stack of solutions to its equations of motion

$$\mathrm{EOM}_{\mathrm{hol}}(X) = T^*_{\mathrm{form}}[-1]\,\mathrm{Higgs}^{\mathrm{fer}}_G(X)$$

where $\mathrm{Higgs}^{\mathrm{fer}}_G(X)$ is the moduli stack of G-Higgs bundles on X where the Higgs field is fermionic: that is, the mapping stack $\underline{\mathrm{Map}}(\Pi TX, BG)$. There are two natural actions of the supergroup H on this moduli stack: $\mathbb{C}^\times$ acts on the Higgs field with weight one, and the supertranslation either acts on the base or the cotangent fiber. We proved that the corresponding global twists are given by

$$\begin{aligned} \mathrm{EOM}_B(X) &= T^*_{\mathrm{form}}[-1]\mathrm{Flat}_G(X) \\ \text{and } \mathrm{EOM}_A(X) &= \mathrm{Higgs}_G(X)_{\mathrm{dR}} \end{aligned}$$

respectively.

One can also compute the twist of the space of solutions to the equations of motion on a smooth proper curve Σ. The stacks that arise are now 1-shifted symplectic: one finds

$$\begin{aligned} \mathrm{EOM}_B(\Sigma) &= T^*_{\mathrm{form}}[1]\mathrm{Flat}_G(\Sigma) \\ \text{and } \mathrm{EOM}_A(\Sigma) &= \mathrm{Higgs}_G(\Sigma)_{\mathrm{dR}} = T^*_{\mathrm{form}}[1](\mathrm{Bun}_G(\Sigma)_{\mathrm{dR}}). \end{aligned}$$

One might note that the last identification can be further reduced to $\mathrm{Bun}_G(\Sigma)_{\mathrm{dR}}$ since the tangent complex of a de Rham stack is always trivial. However, this above identification has nontrivial content since it extends to an identification compatible with the realization of the de Rham stack as a specialization of the Hodge stack; for a further discussion on this point see [EY15, Section 4.3].

Finally, one can compute the twists of the space of solutions to the equations of motion on a point. In this case one obtains 3-shifted symplectic stacks:

$$\begin{aligned} \mathrm{EOM}_B(\mathrm{pt}) &= T^*[3]BG \cong \mathfrak{g}^*[2]/G \\ \text{and } \mathrm{EOM}_A(\mathrm{pt}) &= (BG)_{\mathrm{dR}}. \end{aligned}$$

The latter statement follows becayse $\mathrm{EOM}_A(\mathrm{pt}) = \underline{\mathrm{Map}}(\mathbb{D}^2, BG)_{\mathrm{dR}} = \underline{\mathrm{Map}}(\mathbb{D}^2, BG_{\mathrm{dR}})$. Since $\mathbb{D}^2 = \operatorname{colim} \mathbb{D}^2_{(n)}$, where $\mathbb{D}^2_{(n)}$ is the n^{th} infinitesimal neighborhood of the origin, this moduli space is equivalent to

$$\lim_{n\to\infty} \underline{\mathrm{Map}}(\mathbb{D}^2_{(n)}, BG_{\mathrm{dR}}) \cong \lim_{n\to\infty} \underline{\mathrm{Map}}(\mathrm{pt}, BG_{\mathrm{dR}}) = \underline{\mathrm{Map}}(\mathrm{pt}, BG_{\mathrm{dR}}).$$

The factorization algebras of local observables are the $\mathbb{P}_4$-algebras of functions on these stacks. Note that

$$\mathcal{O}(\mathrm{EOM}_{A,G^\vee}(\mathrm{pt})) = \mathrm{H}^\bullet_{\mathrm{dR}}(BG^\vee) = \mathrm{Sym}((\mathfrak{h}^\vee)^*[-2])^W = \mathrm{Sym}(\mathfrak{h}[-2])^W = \mathcal{O}(\mathfrak{h}^*[2]/W)$$

which is the same as $\mathcal{O}(\mathrm{EOM}_{B,G}(\mathrm{pt})) = \mathcal{O}(\mathfrak{h}^*[2]/W)$ at the classical level. This should be thought of as a coincidence – by considering classical observables in the purely algebraic sense we lost a lot of information about the stacks of local solutions to the equations of motion which aren't captured by the algebras of global functions. However, as we will see in Proposition 3.13 these algebras of local classical observables do not admit any quantum corrections, so in fact we have an equivalence between the algebras of *quantum* observables in the dual theories. This should, in contrast, not be thought of as a pure coincidence, but rather as part of the duality.

3.2 Moduli Space of Flat Bundles and Hecke Symmetry

In this section we'll recall some key properties of the derived stack $\mathrm{Flat}_G(\Sigma)$ of flat G-bundles on a smooth complex curve Σ. In the local case, where we consider a formal bubble $\mathbb{B} = \mathbb{D} \amalg_{\mathbb{D}^\times} \mathbb{D}$ instead of Σ, we'll also recall the derived statement of the geometric Satake correspondence due to Bezrukavnikov and Finkelberg [BF08]. Our main reference will be Sections 10 to 12 of [AG15].

3.2.1 Geometry of Flat_G

We'll begin by describing the tangent and cotangent complexes of $\mathrm{Flat}_G(\Sigma)$. Let $\operatorname{Spec} R$ be an affine derived scheme. The R-points of $\mathrm{Flat}_G(\Sigma)$ are given by the space of algebraic G-bundles on $\operatorname{Spec} R \times \Sigma$ with a partial flat connection in the direction of Σ. The *tangent complex* at such an R point (P, ∇) can be identified as the complex

$$\begin{aligned}\mathbb{T}_{(P,\nabla)}\mathrm{Flat}_G(\Sigma) &= \Gamma(\operatorname{Spec} R \times \Sigma_{\mathrm{dR}}; \mathfrak{g}_P)[1] \\ &\cong (\Omega^\bullet(\operatorname{Spec} R \times \Sigma; \mathfrak{g}_P)[1], \mathrm{d}_\nabla)\end{aligned}$$

where d_∇ is the covariant derivative in the direction of Σ. Similarly by Verdier duality in the direction of Σ, the *cotangent complex* can be described as

$$\mathbb{L}_{(P,\nabla)}\mathrm{Flat}_G(\Sigma) \cong (\Omega^\bullet(\operatorname{Spec} R \times \Sigma; \mathfrak{g}_P^*)[1], \mathrm{d}_\nabla).$$

Since Σ is smooth and proper and BG is 2-shifted symplectic, the derived stack $\mathrm{Flat}_G(\Sigma)$ is AKSZ 0-shifted symplectic. The shifted symplectic structure is easy to identify from this point of view: it's the isomorphism between the tangent and cotangent complexes induced from the Killing form isomorphism $\mathfrak{g} \cong \mathfrak{g}^*$. We can also easily see the following, since for any R the cohomology of the tangent complex is concentrated in degrees less than or equal to 1.

Proposition 3.4. The stack $\mathrm{Flat}_G(\Sigma)$ is quasi-smooth.

Let's now consider the shifted cotangent *spaces* $T^*[k]\mathrm{Flat}_G(\Sigma)$, especially in the cases where $k = \pm 1$. If $\operatorname{Spec} R$ is a *classical* affine scheme, an R-point of the classical part $(T^*[k]\mathrm{Flat}_G(\Sigma))^{\mathrm{cl}}$ is an R-point of $\mathrm{Flat}_G(\Sigma)$ – i.e. a G-bundle P on $\Sigma \times \operatorname{Spec} R$ with a partial flat connection Δ in the Σ direction – along with an element of $H^{1+k}_\nabla(\operatorname{Spec} R \times \Sigma; \mathfrak{g}_P^*)$. If $k = -1$, this space is nothing but the *scheme of singularities* that we defined in Definition 2.28. The scheme of singularities of $\mathrm{Flat}_G(\Sigma)$ has another name.

Definition 3.5. The space of *Arthur parameters* for the group G and curve Σ is the classical stack $(T^*[-1]\mathrm{Flat}_G(\Sigma))^{\mathrm{cl}}$, whose R-points consist of a G bundle P on $\Sigma \times \operatorname{Spec} R$ with a partial flat connection in the Σ direction and a flat section ϕ of the coadjoint bundle $\mathfrak{g}_P^*$.

Remark 3.6. In the original Langlands program, Langlands reciprocity involves not just a Galois representation but also an action of the so-called "Arthur SL_2". The notation is supposed to be suggestive of this additional data, and Arinkin and Gaitsgory's formulation of the geometric Langlands conjecture is supposed to suggest restricting to a subspace of representations where the Arthur SL_2 acts in a particular way.

The refined statement of the geometric Langlands conjecture involves sheaves with singular support in the following substack of $\mathrm{Arth}_G(\Sigma)$.

Definition 3.7. The *global nilpotent cone* is the subspace of $\mathrm{Arth}_G(\Sigma)$ where the Arthur parameter ϕ is *nilpotent*. In other words, there is a map $\mathrm{Arth}_G(\Sigma) \to \mathfrak{g}^*/G$ given by evaluating the flat section ϕ at a point $x \in \Sigma$, and one defines the global nilpotent cone as the fiber product

$$\begin{aligned} \mathcal{N}_G &= \mathrm{Arth}_G(\Sigma) \times_{\mathfrak{g}^*/G} \mathrm{Nilp}/G \\ &\cong \mathrm{Arth}_G(\Sigma) \times_{\mathfrak{h}^*/W} \{0\} \end{aligned}$$

where Nilp is the nilpotent cone in $\mathfrak{g}^*$.

Note that this is independent of the choice of a point $x \in \Sigma$ as ϕ is a flat section.

We'll conclude this subsection by noting that $\mathrm{Flat}_G(\Sigma)$ forms a global complete intersection stack as in Section 2.2.2. Consider the pullback diagram

$$\begin{tikzcd} \mathrm{Flat}_G(\Sigma) \arrow[r] \arrow[d] & \mathrm{Flat}_G^{\mathrm{RS}}(\Sigma) \arrow[d] \\ BG \arrow[r, shift left] & \mathfrak{g}/G \arrow[l, shift left] \end{tikzcd}$$

after choosing a point $x \in \Sigma$ (see also diagram [AG15, (12.11)]). Here the stack $\mathrm{Flat}_G^{\mathrm{RS}}(\Sigma)$ isn't quite smooth, but is smooth in a Zariski neighbourhood of $\mathrm{Flat}_G(\Sigma)$ [AG15, Proposition 10.6.6]. The groupoid associated to this diagram is the groupoid $\mathrm{Hecke}_G^{\mathrm{spec}} = BG \times_{\mathfrak{g}/G} BG$. We'll now discuss the action of this groupoid on $\mathrm{Flat}_G(\Sigma)$ – and the action of the category $\mathrm{IndCoh}(\mathrm{Hecke}_G^{\mathrm{spec}})$ – in the context of the geometric Satake correspondence.

3.2.2 Geometric Satake Equivalence

This is one of the few parts of the paper where the group G and its Langlands dual group $G^\vee$ can appear in the same expression. In our notation we'll usually just write G, and write $G^\vee$ for the automorphic side (A-side), and G for the Galois side (B-side) only when they are compared. Note that this notation is in contrast to the choice most often appearing in the geometric Langlands literature, (for instance, [AG15, Section 12]), but we prefer this choice because of our extensive focus on the Galois side (aka B-side).

Consider the affine Grassmannian $\mathrm{Gr}_{G,x} = G(\mathcal{K}_x)/G(\mathcal{O}_x)$ for a point $x \in \Sigma$. One of the main characters of the geometric Langlands theory is the *spherical Hecke category* at a point $x \in \Sigma$ defined by $\mathrm{Sph}_{G,x} = \mathrm{D}(\mathrm{Bun}_G(\mathbb{B}))$, where $\mathbb{B} = \mathbb{D} \amalg_{\mathbb{D}^\times} \mathbb{D}$ for a formal disk $\mathbb{D}$ and a formal punctured disk $\mathbb{D}^\times$. More precisely, we think of $\mathrm{Bun}_G(\mathbb{B})$ as the double quotient stack $G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x)$ and $\mathrm{D}(\mathrm{Bun}_G(\mathbb{B})) = \mathrm{D}_{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})$ as the category of $G(\mathcal{O}_x)$-equivariant D-modules on the affine Grassmannian $\mathrm{Gr}_{G,x}$. Its monoidal structure is given by the pull-push diagram

$$\begin{tikzcd} & G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x) & \\ & G(\mathcal{O}_x)\backslash G(\mathcal{K}_x) \times_{G(\mathcal{O}_x)} G(\mathcal{K}_x)/G(\mathcal{O}_x) \arrow[u, "m"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x) & & G(\mathcal{O}_x)\backslash G(\mathcal{K}_x)/G(\mathcal{O}_x), \end{tikzcd}$$

that is by $M_1 * M_2 = m_*(p_1^* M_1 \otimes p_2^* M_2)$. There are some technical issues involved in working with D-modules on general infinite-dimensional spaces, but one can make sense of them in this case by writing the affine Grassmannian as an ind-finite type ind-scheme which is $G(\mathcal{O}_x)$-invariant at each level (for instance this guarantees that the spherical Hecke category is compactly generated).

For the naïve geometric Satake equivalence, we consider the heart $\mathrm{Sph}_{G,x}^{\heartsuit} := \mathrm{D}_{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x})^{\heartsuit}$ of the natural t-structure which is the abelian category of $G(\mathcal{O}_x)$-equivariant D-modules on $\mathrm{Gr}_{G,x}$. It is a theorem of Lusztig [Lus83] that the monoidal structure described above preserves the heart $\mathrm{Sph}_{G,x}^{\heartsuit}$. The geometric Satake theorem in this context refers to the following fundamental theorem relating the group G and its Langlands dual group $G^\vee$.

Theorem 3.8 (Ginzburg [Gin95], Mirkovic–Vilonen [MV07])**.** There exists a canonical equivalence of monoidal categories

$$\mathrm{Sat}_x^{\heartsuit} \colon \mathrm{Sph}_{G^\vee,x}^{\heartsuit} \simeq \mathrm{Rep}(G)^{\heartsuit}.$$

However, one should not immediately try to promote this to an equivalence of derived categories. For the unit object δ_1 given by the delta function at $1 \in \mathrm{Gr}_{G,x}$, one computes its endomorphisms in the derived category to be

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Sph}_{G,x}}(\delta_1, \delta_1) &= \mathrm{End}_{\mathrm{D}(\mathrm{pt})^{G(\mathcal{O}_x)}}(\delta, \delta) \\ &\cong \mathrm{H}^\bullet(BG(\mathcal{O}_x)) \\ &\cong \mathrm{H}^\bullet(BG). \end{aligned}$$

On the other hand, since $\mathrm{Rep}(G)$ for a reductive group G is a semisimple category, one has $\mathrm{Hom}_{\mathrm{Rep}(G)}(1,1) = \mathbb{C}$. Therefore in order to sensibly promote the geometric Satake equivalence to an equivalence of derived categories one needs to enhance the right-hand side, so that one has an equivalence encoding all the higher Ext information on the left-hand side.

In order to do that, it is convenient to renormalize $\mathrm{Sph}_{G,x}$. We consider the full subcategory $\mathrm{Sph}_{G,x}^{\mathrm{loc,c}} \hookrightarrow \mathrm{Sph}_{G,x}$ of locally compact objects, that is, those which become compact under the forgetful functor $\mathrm{D}_{G(\mathcal{O}_x)}(\mathrm{Gr}_{G,x}) \to \mathrm{D}(\mathrm{Gr}_{G,x})$. Note that it is closed under monoidal operations. Then we define the renormalized spherical Hecke category $\mathrm{Sph}_{G,x}^{\mathrm{ren}}$ to be the ind-completion of $\mathrm{Sph}_{G,x}^{\mathrm{loc,c}}$.

On the other side one considers $\mathrm{IndCoh}(\mathrm{Hecke}_G^{\mathrm{spec}})$, where $\mathrm{Hecke}_G^{\mathrm{spec}} \cong \mathrm{Flat}_G(\mathbb{B})$ is just the derived fiber product $BG \times_{\mathfrak{g}/G} BG$ (the groupoid mentioned in the previous subsection). This category has a convolution monoidal structure whose unit is the skyscraper sheaf at the trivial flat bundle.

Here is what is called the derived geometric Satake equivalence.

Theorem 3.9 (Bezrukavnikov–Finkelberg [BF08])**.** There is an equivalence of monoidal categories

$$\mathrm{Sph}_{G^\vee,x}^{\mathrm{ren}} \simeq \mathrm{IndCoh}(\mathrm{Hecke}_G^{\mathrm{spec}}).$$

The importance of these monoidal categories is that they act on the category on any curve Σ as follows. Let us explain this point for the Galois side (the corresponding action on the automorphic side is defined verbatim). If we choose a point $x \in \Sigma$, then there is a natural action of the monoidal category $\mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B}))$ on the category $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ by pull-tensor-push along the natural maps

$$\begin{tikzcd} & \mathrm{Flat}_G(\mathbb{B}) & \\ & \mathrm{Flat}_G(\Sigma \underset{\mathbb{D}^\times}{\sqcup} \mathbb{D}_x) \arrow[u, "q_x"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathrm{Flat}_G(\Sigma) & & \mathrm{Flat}_G(\Sigma), \end{tikzcd}$$

all of which are induced from the natural inclusion maps (note that there are two different inclusion maps from Σ to $\Sigma \sqcup_{\mathbb{D}^\times} \mathbb{D}_x$).

Remark 3.10. Indeed, this action preserves the subcategory $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma)) \subset \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ and one version of the geometric Langlands correspondence asserts an equivalence of two dg categories

$$D(\mathrm{Bun}_{G^\vee}(\Sigma)) \simeq \mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$$

as module categories over those monoidal categories.

3.3 Vacua for Twisted $N = 4$ Theories

Now we'd like to apply the constructions of Section 2.3 to examples modelling the Kapustin–Witten twisted 4d gauge theories described in Section 3.1. According to the recipe, we need to specify an $\mathbb{E}_4$-algebra A of 4d local observables and a dg category $\mathcal{B}$ with an action of A modelling *boundary conditions* along the curve Σ. We'll build this data starting from a dg category $\mathcal{B}$ quantizing the space of solutions $\mathrm{EOM}(\Sigma)$ along a curve in the twisted theory, which is a module for a monoidal dg category $\mathcal{L}$ quantizing the local space $\mathrm{EOM}(\mathbb{B})$, so that $\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}})$ quantizes the factorization algebra of local classical observables.

For most of this section we'll focus on the B-twisted theory (we'll discuss what is known in the A-twisted case in Remark 3.22). Set

$$\mathcal{B} = \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$$

viewed as the (naïve) categorical geometric quantization of the moduli space $\mathrm{EOM}_B(\Sigma) = T^*_{\mathrm{form}}[1]\mathrm{Flat}_G(\Sigma)$ as discussed in Section 2.1.3. Likewise set $\mathcal{L} = \mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B}))$, which acts on $\mathcal{B}$ after choosing a point $x \in \Sigma$ as discussed above.

The action of $\mathcal{L}$ on $\mathcal{B}$ leads to a map of $\mathbb{E}_2$-algebras $\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}}) \to \mathrm{HC}^\bullet(\mathcal{B})$ as in Section 2.2. For the unit δ of the monoidal category $\mathcal{L}$, one can compute $\mathrm{Hom}_{\mathcal{L}}(\delta,\delta) \cong \mathcal{O}(\mathfrak{g}^*[2]/G)$ as $\mathbb{E}_2$-algebras (see [AG15, Section 12.4]). Therefore there's a natural map from $\mathrm{End}_{\mathcal{L}}(1_{\mathcal{L}}) \cong \mathcal{O}(\mathfrak{h}^*[2]/W)$ to the Hochschild cochains $\mathrm{HC}^\bullet(\mathcal{B})$.

Remark 3.11. Heuristically, if $\mathrm{Flat}_G(\Sigma)$ were a smooth scheme, then the HKR theorem would identify its Hochschild cohomology with $\mathcal{O}(T^*[1]\mathrm{Flat}_G(\Sigma))$. In these terms the action of $\mathcal{O}(\mathfrak{h}^*[2]/W)$ would be given by pullback along the composite map

$$T^*[1]\mathrm{Flat}_G(\Sigma) \xrightarrow{\mathrm{ev}_x} \mathfrak{g}^*[2]/G \longrightarrow \mathfrak{h}^*[2]/W,$$

defined by evaluating the section at a point x in Σ to obtain an element of $\mathfrak{g}^*$ well-defined up to conjugation, where the degree -2 part of the cotangent fiber parametrizes sections of the coadjoint bundle.

Remark 3.12. The categories above, for instance $\mathcal{B} = \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ are not 2-dualizable – or even Calabi–Yau – since the stack $\mathrm{Flat}_G(\Sigma)$ is neither smooth nor proper. This forces us to use the factorization algebra model for localization rather than the model coming from extended functorial field theory. One might imagine that these categories model extended 2d TQFTs which are non-compact in both directions as in Remark 2.7, but lacking a satisfactory formalism for such theories we won't attempt to take this point of view.

We need to quantize the $\mathbb{P}_4$-algebra of classical observables to an $\mathbb{E}_4$-algebra. We'll do this using a very simple example of a general body of theory about $\mathbb{E}_n$-deformations. We refer to Toën's ICM address [Toë14] and Francis [Fra13] for details.

Proposition 3.13. The algebra $\mathcal{O}(\mathfrak{h}^*[2]/W)$ doesn't admit any $\mathbb{P}_4$ deformations.

Proof. The $\mathbb{P}_4$ deformations of the given $\mathbb{P}_4$-algebra structure on $\mathcal{O}(\mathfrak{h}^*[2]/W)$ correspond to Maurer–Cartan elements in the dg Lie algebra whose underlying graded vector space is the homotopy fiber of the natural map $\mathrm{Pol}(\mathfrak{h}^*[2]/W,3)[4] \to \mathcal{O}(\mathfrak{h}^*[2]/W)[4]$. As $\mathrm{Pol}(\mathfrak{h}^*[2]/W,3)[4] = \mathcal{O}(T^*[4](\mathfrak{h}^*[2]/W))$ is a symmetric algebra freely generated by elements of cohomological degree 2 and the natural map surjects onto $\mathcal{O}(\mathfrak{h}^*[2]/W)[4]$, there are no elements in degree 1 in the corresponding dg Lie algebra, so the claim immediately follows. □

Therefore by Theorem 2.16 and the discussion immediately following it, we should take $A = \mathcal{O}(\mathfrak{h}^*[2]/W)$ with the commutative algebra structure as the $\mathbb{E}_4$-algebra of quantum local observables of our theory. Before defining the localization, however, we'll need to perform this calculation in a slightly more sophisticated fashion which remembers the origin of the theory in question as a topological twist.

Remark 3.14. There's a natural interpretation of this action in terms of *surface operators* for the B-twisted theory. Indeed, the classical B-twisted theory assigns to a closed curve Σ the 1-shifted symplectic stack $T^*[1]\mathrm{Flat}_G(\Sigma)$, or alternatively the (sheaf of) $\mathbb{P}_2$-algebras $\mathcal{O}(T^*[1]\mathrm{Flat}_G(\Sigma))$. At the classical level, there's a map of algebras $\mathcal{O}(\mathfrak{g}^*[2]/G) \to \mathcal{O}(T^*[1]\mathrm{Flat}_G(\Sigma))$ for every point $x \in \Sigma$ which we think of as the inclusion of those operators that depend only on a small neighbourhood of x. We know that the algebra $\mathcal{O}(\mathfrak{g}^*[2]/G)$ doesn't receive any quantum corrections. Let's suppose that this is also true for $\mathcal{O}(T^*[1]\mathrm{Flat}_G(\Sigma))$, so this map is exact at the quantum level. We claim that this map is precisely the map into the Hochschild cohomology described above. Indeed, if we assume a state-operator correspondence then in the 2d theory obtained from reduction along Σ the local operators are precisely the Hochschild cochains of the category assigned to the point (since these are states on the circle with an appropriate framing).

3.4 Degree-Shifting Correction

We still need to modify this definition before we can connect compatibility with a vacuum and Arinkin–Gaitsgory's singular support condition. In fact, this modification is a meaningful process in a much bigger generality, but we mostly work with our main example.

In order to motivate this modification we'll return to the construction of the twisted theory. Recall that in the perturbative setting (and hence in the nonperturbative setting as well) we took $\mathbb{C}^\times$-invariants of the family

$$E_{\mathrm{hol}}^{\Pi\mathbb{C}} \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!)$$

built from the B-supercharge, where t is a fermionic parameter of degree 1. This process was taken as the definition of a twist; it picks out the B-twist as a well-defined theory. Based on this definition, we have seen that our moduli space of vacua is $\mathfrak{h}^*[2]/W$, as opposed to the underived space $\mathfrak{h}^*/W$ physicists might more naturally consider. This is not a mistake, as our twisting process necessarily involves a meaningful degree shift; for instance, the correct cohomological grading of Hochschild cohomology (or polyvector fields) as the local operators of the B-model topological open string theory is introduced precisely by the topological twisting construction for a 2-dimensional supersymmetric σ-model. Without this degree shift one would not obtain the natural cohomological grading.

On the other hand, it is possible to recover the space physicists think of as the moduli space of vacua. The idea is that for physicists, taking the $\mathbb{C}^\times$-invariants for a particular choice of R-symmetry $\mathbb{C}^\times$ is not a natural operation. Accordingly, we should keep track of the whole twisted family before taking $\mathbb{C}^\times$-invariants. Indeed, because the family is locally constant in the t direction, everything is living over $\mathbb{C}(\!(t)\!)$. For example, one has

$$\mathrm{Obs}^{\mathrm{q}}_t(B^4) := \mathrm{Obs}^{\mathrm{q}}(B^4)^{\Pi\mathbb{C}} \otimes_{\mathbb{C}[\![t]\!]} \mathbb{C}(\!(t)\!)) \cong \mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)$$

as the full algebra of quantum observables and one can similarly understand the full category of boundary conditions along Σ as

$$\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))(\!(t)\!) := \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma)) \otimes_{\mathrm{Vect}} \mathbb{C}(\!(t)\!)\text{-mod}.$$

The action of local observables in this family is the same as in the previous section. After choosing a point $x \in \Sigma$ the spectral Hecke category $\mathcal{L}(\!(t)\!) = \mathrm{IndCoh}(\mathrm{Flat}_G(\mathbb{B}))(\!(t)\!)$ after adjoining the parameter t acts on $\mathcal{B}(\!(t)\!) = \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))(\!(t)\!)$, which induces a map from $\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)$ to the Hochschild cochains $\mathrm{HC}^\bullet(\mathcal{B}(\!(t)\!))$ by taking endomorphisms of the unit. Heuristically we can think of this as an evaluation map at the chosen point x. That is, there is an evaluation map

$$\mathrm{ev}_x \colon T^*[1]\mathrm{Flat}_G(\Sigma) \to \mathfrak{h}^*[2]/W$$

which induces algebra maps

$$\begin{aligned} \mathrm{ev}_x^* \colon \mathcal{O}(\mathfrak{h}^*[2]/W) &\to \mathcal{O}(T^*[1]\mathrm{Flat}_G(\Sigma)) \\ \text{and hence } \mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!) &\to \mathcal{O}(T^*[1]\mathrm{Flat}_G(\Sigma))(\!(t)\!)[s] \end{aligned}$$

where now s is a fermionic parameter of cohomological degree -2. As before we should only think of this description as a heuristic since we can't literally compare $\mathcal{O}(T^*[1]\mathcal{X})$ and $\mathrm{HC}^\bullet(X)$ when X is a stack and we don't have a version of the HKR theorem available. For a precise definition of the action of local observables we'll use the description via the action of the spectral Hecke category.

Remark 3.15. Geometrically we can think of the algebra $\mathcal{O}(\mathfrak{h}^*[2]/W)((t))$ of twisted local observables as the algebra of functions on a space of *twisted vacua* of the form $\mathfrak{h}^*[2]/W \times \Pi\mathbb{D}_1^\times$, where $\Pi\mathbb{D}_1^\times = \mathrm{Spec}\,\mathbb{C}((t))$ is a "shifted fermionic formal punctured disk". Of course, this doesn't literally make sense because the algebra $\mathbb{C}((t))$ is not connective and hence does not define an affine derived scheme. Nevertheless we think this is a useful perspective for understanding the twisting procedure – we think of the twist as the restriction of a family of theories over a shifted fermionic formal disk to the shifted fermionic formal punctured disk.

In order to reconnect with geometry and with the vacua story from Section 2.3 we'd like to replace our nonexistent space $\mathfrak{h}^*[2]/W \times \Pi\mathbb{D}_1^\times$ of twisted vacua with something more meaningful. The naïve thing to do from the point of view of the usual twisting story would be to take spec of the $\mathbb{C}^\times$-invariants in our algebra of twisted local observables: in this case t has weight 1 and $\mathcal{O}(\mathfrak{h}^*[2]/W)$ has weight zero, so this recovers the naïve space $\mathfrak{h}^*[2]/W$ of vacua from the previous section. Another natural option is to make the following definition.

Definition 3.16. The *refined space of vacua* in a twisted field theory is the classical affine scheme

$$\mathcal{V}^{\mathrm{ref}} = \mathrm{Spec}\,\mathrm{H}^0(\mathrm{Obs}_t^{\mathrm{q}}(B^4))$$

obtained by taking the degree zero part of the total algebra of twisted observables as a $\mathbb{C}((t))$-module.

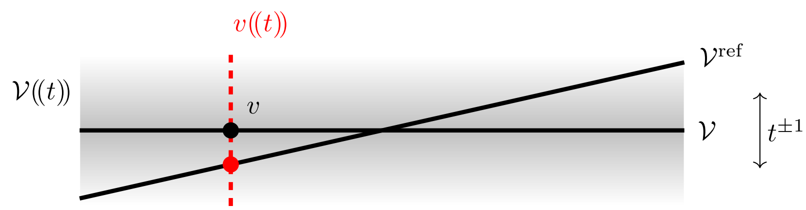

Figure 2: The refined moduli space of vacua $\mathcal{V}^{\mathrm{ref}}$ is viewed geometrically as the classical part of the thickened space $\mathcal{V}((t))$ of twisted vacua. Here we've drawn $\mathcal{V}$ horizontally and the formal degree 2 direction vertically. We've indicated the closed subset $v((t))$ inside of $\mathcal{V}((t))$ associated to a point v of $\mathcal{V}$ by a red dashed line.

Note that in our main example, one can see that $\mathcal{V}^{\mathrm{ref}}$ is $\mathfrak{h}^*/W$ as expected. That is, if $\{x_i\}$ is the set of linear functions on $\mathfrak{h}^*[2]/W$, now the ideal $(x_i t^{-2})$ defines a corresponding actual point in $\mathfrak{h}^*/W$ (in Fig. 2 this is the red point).

Now, we can use the action of the algebra $\mathrm{Obs}_t^{\mathrm{q}}(B^4)$ of twisted local observables on the category $\mathcal{B}((t))$ of twisted boundary conditions to define localizations for the category $\mathcal{B}$ *without* the parameter t in a simple way.

Definition 3.17. The localized category of boundary conditions $\mathcal{B}_v$ with respect to a point $v \in \mathcal{V}^{\mathrm{ref}}$ is defined as the tensor product

$$\mathrm{Vect} \otimes_{\mathbb{C}((t))\text{-mod}} \mathcal{B}((t))_{\mathfrak{m}_v}$$

where we first localize at the maximal ideal $\mathfrak{m}_v$ in the algebra $\mathrm{Obs}_t^{\mathrm{q}}(B^4)$ of twisted local operators generated by the point v, then forget the action of t. In the notation of Section 2.2.2, this can be expanded as

$$\mathrm{Vect} \otimes_{\mathbb{C}((t))\text{-mod}} \mathcal{B}((t)) \otimes_{\mathrm{Obs}_t^{\mathrm{q}}(B^4)\text{-mod}} \mathrm{Obs}_t^{\mathrm{q}}(B^4)\text{-mod}_{v((t))}.$$

Remark 3.18. It's possible to motivate this construction – considering support in the refined moduli space of vacua – in terms of the action of the R-symmetry group. In the holomorphically twisted $N = 4$ theory the residual R-symmetry group P is a parabolic subgroup of $\mathrm{SL}_4(\mathbb{C})$ with Levi isomorphic to $\mathrm{SL}_3(\mathbb{C})$. In particular, its maximal torus has dimension two. There is a preferred $\mathbb{C}^\times$ in this maximal torus, which is the unique copy of $\mathbb{C}^\times$ which yields a $\mathbb{Z}$-graded theory when we use it to construct the twist by the B-supercharge. We can alternatively choose a complementary $\mathbb{C}^\times$-action – so that $\mathfrak{h}^*[2]/W$ has weight -2, matching its cohomological shift. If we take invariants in the twisted moduli space of vacua with respect to this R-symmetry circle we obtain the classical part $\mathfrak{h}^*/W$ of the twisted space of vacua. By doing this the original R-symmetry $\mathbb{C}^\times$-action survives as a weight one action.

Remark 3.19. From the physical point of view why should we define the localization of a twisted theory in this way, instead of first taking the usual $\mathbb{C}^\times$-invariants, then defining the category compatible with a vacuum as we did in the previous section? This more naïve procedure restricts attention to only those vacua which are R-symmetry invariant, and in topologically twisted gauge theories we don't necessarily want to make this restriction. Indeed, one perspective on these twisted $N = 4$ theories (in type A_n) is as the worldsheet theory of n D3 branes in type IIB string theory, with the choice of vacuum arising as the position of these branes in the remaining six dimensions. Only a two dimensional space of vacua is invariant for the A or B topological twist, but even in this twist the residual $\mathbb{C}^\times$ R-symmetry acts non-trivially by rotating these two directions. As such taking the full R-symmetry invariants is unnatural.

We might summarise the difference between the invariants for this new R-symmetry $\mathbb{C}^\times$-action and the original $\mathbb{C}^\times$-action as follows:

	B-twist $\mathbb{C}^\times$-invariants	new $\mathbb{C}^\times$-invariants
vacua	$\mathcal{V}$	$\mathcal{V}^{\mathrm{ref}}$
AKSZ target	$T^*[1]X$	$\widetilde{\mathrm{Sing}}(X)$

Table 1: A table comparing the result of taking invariants for two different R-symmetry circles.

In the left-hand column $\mathbb{C}^\times$ acts trivially on $T^*[1]X$, and in the right-hand column $\mathbb{C}^\times$ acts with weight -2 on the fiber of $T^*[1]X$. We're writing $\widetilde{\mathrm{Sing}}(X) := \mathrm{Sym}_X(\mathbb{T}_X[1])^{\mathrm{cl}}$, which is generally slightly larger than $\mathrm{Sing}(X) = (T^*[-1]X)^{\mathrm{cl}} = \mathrm{Sym}_{X^{\mathrm{cl}}}(\mathbb{T}_X[1])^{\mathrm{cl}}$. To produce exactly this space we should assume that X has tangent complex concentrated in degrees $-1, 0$, and 1. Physically it's not necessary to take $\mathbb{C}^\times$-invariants at all, but taking invariants for the new R-symmetry $\mathbb{C}^\times$ allows direct comparison to the constructions made by Arinkin and Gaitsgory.

From a mathematical point of view, a powerful motivation for this setup is that we recover singular support conditions for ind-coherent sheaves. The twisting procedure, and the difference in degree between the naïve $\mathbb{C}^\times$-invariants $\mathfrak{h}^*[2]/W$ and the modified $\mathbb{C}^\times$-invariants $\mathfrak{h}^*/W$, encodes the degree shift by two in the definition of singular support using the scheme of singularities $(T^*[-1]X)^{\mathrm{cl}}$.

Remark 3.20. In the arguments we made in Section 2.2.2 the parameter t was a bosonic parameter of cohomological degree 2 rather than a fermionic parameter of cohomological degree 1. We can pass to this purely bosonic situation by taking the bosonic part of the algebra of twisted local operators. If t is fermionic of degree 1 then the bosonic part of $\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)$ is nothing but $\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t^2)\!)$ where now t^2 is a bosonic parameter of degree 2. From now on we'll restrict the action of the full algebra of twisted local operators to the action of its bosonic subalgebra, and we'll abuse notation by using t to denote a bosonic degree 2 parameter from now on (so we'll call t what would've been t^2 in the notation above).

Theorem 3.21. The localization $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))_0$ of the category of boundary condition at the vacuum $0 \in \mathfrak{h}^*/W$ is equivalent to the category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$ of sheaves with nilpotent singular support.

Proof. We'll prove this using Proposition 2.35. Consider the tensor product

$$\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))(\!(t)\!) \otimes_{\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)\text{-mod}} \mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)\text{-mod}_{\mathfrak{m}_{v(\!(t)\!)}}$$

defining the category of boundary conditions compatible with the vacuum v. Here the action of the algebra $\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)$ comes from the action of the spectral Hecke category by identifying $\mathcal{O}(\mathfrak{h}^*[2]/W)$ with the endomorphism algebra for the unit object of the spectral Hecke category (and t acts by multiplication).

We identify $\mathrm{Flat}_G(\Sigma)$ as a global complete intersection stack as in Section 3.2. The action of the spectral Hecke category precisely comes from the action of the groupoid $\mathrm{Hecke}_G^{\mathrm{spec}}$. We are therefore in the setting of Proposition 2.35. Applying the proposition with $V = \mathfrak{h}^*[2]/W$ and $\widetilde{Y} = \{0\}$, so $Y = \mathcal{N}_G$, gives us an equivalence of the form

$$\begin{aligned}
&\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))(\!(t)\!) \otimes_{\mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)\text{-mod}} \mathcal{O}(\mathfrak{h}^*[2]/W)(\!(t)\!)\text{-mod}_{\mathfrak{m}_{v(\!(t)\!)}} \\
&\cong \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))(\!(t)\!) \otimes_{\mathrm{HC}^\bullet(\mathrm{Flat}_G(\Sigma))(\!(t)\!)\text{-mod}} \mathrm{HC}^\bullet(\mathrm{Flat}_G(\Sigma))(\!(t)\!)\text{-mod}_{\mathcal{N}_G(\!(t)\!)} \\
&\cong \mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma)(\!(t)\!)).
\end{aligned}$$

Forgetting the action of the parameter t we recover the desired result. $\square$

Remark 3.22. What can we say for the A-twisted theory? All our definitions, and many of our calculations still make sense in that context, for instance one can still check that the algebra of local observables in the A-twisted theory with gauge group G is equivalent to $\mathcal{O}(\mathfrak{h}[2]/W)$ (this is easiest to see as the endomorphism algebra of the unit in the category $\mathcal{L}$ of line operators, which is the spherical Hecke category $D_{G[\![z]\!]}(\mathrm{Gr}_G)$ of equivariant D-modules on the affine Grassmannian. The degree shifting trick still makes sense, and one can still abstractly define the localization of the category of boundary conditions at a point in $\mathfrak{h}^*/W$. However, there's now a technical issue that one must deal with: the ambient category $\mathcal{B}$ one works in on the A-side – the category of boundary conditions without imposing a vacuum condition – should be equivalent (under geometric Langlands) to the *full* category of ind-coherent sheaves on the B-side. In particular it should be strictly larger than the category of D-modules on $\mathrm{Bun}_G(\Sigma)$.

In the local story – that of geometric Satake – the category one considers is the ind-completion of the category of equivariant D-modules on Gr_G which are coherent after forgetting the equivariant structure (see Bezrukavnikov–Finkelberg [BF08]). In Arinkin–Gaitsgory this is called the “renormalized” spherical Hecke category. There are several ways one might try to globalize this definition, and it's not clear which globalization is correct.

Another difficulty is that, whatever choice one makes for this renormalization, even for ordinary D-modules there isn't a nice description of the Hochschild cohomology $\mathrm{HH}^\bullet(D(\mathrm{Bun}_G(\Sigma)))$ as there was on the B-side (if G is non-abelian – in the abelian case one can use a result of Koppensteiner [Kop15]). One can abstractly define the localization at a vacuum using the Hecke action, but actually computing it is more complicated without this explicit description. We hope in the future to make some sensible calculations at least for the vacuum $0 \in \mathfrak{h}^*/W$, where we should recover the usual category of D-modules on $\mathrm{Bun}_G(\Sigma)$ as the localization.

4 Consequences

4.1 Gauge Symmetry Breaking Conjecture

So we've argued that requiring compatibility with the vacuum $0 \in \mathfrak{h}^*/W$ in the B-twisted $N = 4$ theory naturally gives us the category $\mathrm{IndCoh}_{\mathcal{N}_G}(\mathrm{Flat}_G(\Sigma))$ considered by Arinkin and Gaitsgory. It's obvious to ask next what happens if we require compatibility with a different vacuum.

We can make sense of this because of Remark 2.34 – while a non-zero point in $\mathfrak{h}^*/W$ is not a conical subset, it does correspond to a conical subset in the full “space of vacua” $\mathfrak{h}^*[2]/W \times \Pi\mathbb{D}_1^\times$ (or rather a homogenous ideal in the algebra of twisted local operators). The result is that we can define a category of boundary conditions compatible with the single point v, and analyse it using Proposition 2.35. Conjecturally at least this category has a very clean interpretation – the gauge symmetry is broken to a Levi subgroup depending on v.

Conjecture 4.1. The category $\mathcal{B}_v$ obtained by localizing $\mathcal{B} = \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ at the vacuum $v \in \mathfrak{h}^*/W$ is equivalent to $\mathrm{IndCoh}_{\mathcal{N}_L}(\mathrm{Flat}_L(\Sigma))$, where L is the centralizer of its semisimple lift x in $\mathfrak{g}^*$.

This sort of gauge symmetry breaking according to strata in $\mathfrak{h}^*/W$ has been described from a more physical perspective in work of Balasubramanian [Bal16].

Example 4.2. As an easy special case, the equivalence of Theorem 3.21 will hold more generally if $v \in \mathfrak{h}^*/W$ is the image of any central element $g \in \mathfrak{g}^*$ under the map $\mathfrak{g}^*/G \to \mathfrak{h}^*/W$. The singular support of a sheaf compatible with the vacuum v will now be contained in the derived pullback $\{v\} \times_{\mathfrak{h}^*/W} \mathrm{Arth}_G(\Sigma)$ which is isomorphic to the global nilpotent cone $\mathcal{N}_G$ by addition of the constant section of the coadjoint bundle with value g.

There's a nice point of view motivating this conjecture coming from string theory in the case where $G = \mathrm{GL}_n$. In that case we view the space $\mathfrak{h}^*/W$ of vacua as the configuration space of n points in $\mathbb{C}$. A string-theoretic origin for the full $N = 4$ supersymmetric gauge theory for GL_n is as the effective theory of n coincident D3 branes in type IIB string theory. The space of vacua in this theory has real dimension 6, corresponding to the positions of those branes in the 6 orthogonal directions, but in the holomorphically or Kapustin–Witten twisted theories the space of vacua is broken down to a 2 real- or 1 complex-dimensional subspace (the space of those vacua which are invariant for the twisting supercharge). Turning on a vev in this space corresponds to moving these n D3 branes away from the origin in this complex plane, which in particular breaks the gauge symmetry group to a Levi subgroup (which one depends on which branes are still coincident after this motion). So we expect that motion away from 0 in the moduli of vacua corresponds to this gauge symmetry breaking, motivating our conjecture at the level of categories of boundary conditions.

From a different perspective, let's explain some mathematical evidence for this claim. To begin with, we'll explain what we can say for sure about the localization at v.

Definition 4.3. We write $\mathrm{Arth}^v_G(\Sigma) \to \mathrm{Arth}_G(\Sigma)$ for the pullback $\mathrm{Arth}_G(\Sigma) \times_{\mathfrak{h}^*/W} \{v\}$ (so in particular if $v = 0$ we recover the global nilpotent cone).

Remark 4.4. We'll see in just a moment that the subspace $\mathrm{Arth}^v_G(\Sigma)$ is not so geometrically mysterious. It's supported over the locus in $\mathrm{Flat}_G(\Sigma)$ consisting of G-bundles that break to the Levi subgroup L centralizing v. For example, in the generic case where v is a *regular* element in $\mathfrak{h}^*/W$, the only flat G-bundle admitting a flat section conjugate to v is the trivial bundle (that is, the unique completely reducible flat bundle, where the gauge symmetry breaks to the maximal torus H). Over this locus, $\mathrm{Arth}^v_G(\Sigma)$ is just a copy of the global nilpotent cone for L translated by the constant section v, viewing v as an element of the center of the Lie algebra of L.

By the arguments of the previous section, we know that the localization of the category $\mathcal{B} = \mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ at the vacuum v is equivalent to the category $\mathrm{IndCoh}_{\mathrm{Arth}^v_G(\Sigma)}(\mathrm{Flat}_G(\Sigma))$ of sheaves with singular support in $\mathrm{Arth}^v_G(\Sigma)$. What isn't yet known is whether this category is equivalent to the category of sheaves with singular support in $\mathcal{N}_L$. We have some geometric evidence for this conjecture however. We know that singular support is defined as support within the space of singularities, in this case $\mathrm{Arth}_G(\Sigma)$. In fact the loci $\mathrm{Arth}^v_G(\Sigma)$ and $\mathcal{N}_L$ inside the space $\mathrm{Arth}_G(\Sigma)$ are not just equivalent, but so are their formal neighbourhoods.

Proposition 4.5. The formal completions $\mathrm{Arth}_G(\Sigma)^\wedge_{\mathrm{Arth}^v_G(\Sigma)}$ and $\mathrm{Arth}_L(\Sigma)^\wedge_{\mathcal{N}_L}$ are equivalent, where L is the centralizer of a semisimple lift x of v in $\mathfrak{g}^*$.

Proof. We use a result from derived deformation theory (see Gaitsgory–Rozenblyum [GR17, IV.1 1.6.4, IV.3 3.1.4]) which identifies pointed formal moduli problems over a fixed base stack with sheaves of dg Lie algebras on that stack. Concretely, if $\pi : \mathcal{Y} \leftrightarrows \mathcal{X} : \sigma$ is a pointed formal moduli problem over $\mathcal{X}$, the associated sheaf of dg Lie algebras over $\mathcal{X}$ is the relative restricted shifted tangent complex $\sigma^*\mathbb{T}_{\mathcal{Y}/\mathcal{X}}[-1]$.

We need to show that there exists an identification $\mathcal{N}_L \cong \mathrm{Arth}^v_G(\Sigma)$ so that the two formal completions can be compared. First of all, we claim $\mathcal{N}_L \cong \mathrm{Arth}^v_L(\Sigma)$. Using the fact that a central element v in the Lie algebra $\mathfrak{l}$ of L (identifying $\mathfrak{l}$ with $\mathfrak{l}^*$ using an invariant pairing) gives a section of the map $\mathrm{Arth}_L(\Sigma) \to \mathrm{Flat}_L(\Sigma)$, there is a canonical isomorphism $\mathrm{Arth}^v_L(\Sigma) \cong \mathcal{N}_L$, by translating the Arthur parameter by v.

Moreover, if L is the centralizer of v then the map $\mathrm{Arth}^v_L(\Sigma) \to \mathrm{Arth}^v_G(\Sigma)$ is an equivalence. For this, we observe that for a group scheme H acting on a derived stack X and the orbit $H \cdot Y$ of a substack $Y \hookrightarrow X$, there is a natural isomorphism

$$H \cdot Y \cong H \times Y / H_Y$$

where H_Y is the stabilizer of Y in H, and where H_Y acts on $H \times Y$ by $h \cdot (g, y) = (gh^{-1}, h \cdot y)$. Now by choosing a point $p \in \Sigma$, we view $\mathrm{Arth}^v_G(\Sigma)$ as the quotient $\mathrm{Arth}^{v,p}_G(\Sigma)/L$, where $\mathrm{Arth}^{v,p}_G(\Sigma)$ is the derived scheme of Arthur parameters equipped with a framing at the point p. This is directly analogous to the presentation of moduli stacks of flat bundles as a global quotient due to Simpson [Sim94, Theorem 9.10]. Using the observation with $X = \mathrm{Arth}^p_G$, $Y = \mathrm{Arth}^{v,p}_L$, and $H = G$, so that $H \cdot Y = \mathrm{Arth}^{v,p}_G$ holds, we obtain the equivalence we want as follows:

$$\mathrm{Arth}^{v,p}_G(\Sigma)/G \cong L \backslash (\mathrm{Arth}^{v,p}_L(\Sigma) \times G)/G \cong L \backslash \mathrm{Arth}^{v,p}_L(\Sigma).$$

Now it remains to compare the (-1)-shifted relative shifted tangent complexes. It suffices to work with the full spaces $T^*[-1]\,\mathrm{Flat}_G(\Sigma)$ and $T^*[-1]\,\mathrm{Flat}_L(\Sigma)$ instead of the Arthur stacks. First of all, the shifted tangent complex at a closed point $(P,\nabla,\phi)\in T^*[-1]\mathrm{Flat}_G(\Sigma)$ is equivalent to

$$\Omega^\bullet(\Sigma;\mathfrak{m}_P)\oplus\Omega^\bullet(\Sigma;\mathfrak{m}_P^*)[-1]$$

where $\mathfrak{m}$ is the Lie algebra centralizing the value of the section ϕ at any point (independent of the choice of point by flatness). In particular, if $(P,\nabla,\phi)\in\mathrm{Arth}^v_G(\Sigma)$ (which should correspond to $(P_L,\nabla_L,\phi_L)\in\mathcal{N}_L$) so that P lands in the image of $\mathrm{Flat}_L(\Sigma)\to\mathrm{Flat}(G)$, then the shifted tangent complex at a point $(P,\nabla,\phi)\in T^*[-1]\,\mathrm{Flat}_G(\Sigma)$ and $(P_L,\nabla_L,\phi_L)\in T^*[-1]\,\mathrm{Flat}_L(\Sigma)$ coincide. This yields the desired equivalence of dg Lie algebras, which establishes the equivalence of formal completions. □

At a heuristic level therefore, where the singular support of an ind-coherent sheaf on X is thought of in terms of the ordinary support of an associated object living over $\mathrm{Sing}(X)$, this suggests that categories of sheaves on $\mathrm{Flat}_G(\Sigma)$ with singular support in $\mathrm{Arth}^v_G(\Sigma)$ and of sheaves on $\mathrm{Flat}_L(\Sigma)$ with singular support in $\mathcal{N}_L$ should be closely related.

Remark 4.6. Arinkin and Gaitsgory motivated the choice of the nilpotent singular support condition in part by proving that the category of ind-coherent sheaves with nilpotent singular support was the minimal subcategory of $\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$ where the pullback functors induced from the projection $P\to L$ from a parabolic subgroup to its Levi preserve compact objects. From the point of view of the geometric Langlands program this condition is important because one expects the geometric Langlands functor to commute with the so-called *geometric Eisenstein series* functors. If $P\subseteq G$ is a parabolic subgroup with Levi L then the (spectral) geometric Eisenstein series functor $\mathrm{Eis}_P^{\mathrm{spec}}$ is the pull-push functor $p_*q^!$ on ind-coherent sheaves induced from the span

$$\begin{tikzcd} & \mathrm{Flat}_P(\Sigma) \arrow[dl, "q"'] \arrow[dr, "p"] & \\ \mathrm{Flat}_L(\Sigma) & & \mathrm{Flat}_G(\Sigma). \end{tikzcd}$$

This functor preserves the condition of having nilpotent singular support. One can define a pull-push functor of D-modules identically on the automorphic side, and the geometric Langlands equivalence is required to intertwine these functors (possibly up to tensoring with a line bundle).

It's natural to ask what the relationship is between the gauge symmetry breaking we've discussed here and these geometric Eisenstein series functors, since they both involve a relationship between the Arinkin–Gaitsgory category at a group G and a Levi subgroup L. Physically these two relationships have quite different origins: on the one hand by motion in the moduli of vacua and on the other by the existence of a parabolic domain wall [3]. Conjecture 4.1 implies that these two different notions will be compatible in the following sense.

Corollary 4.7. Let $P\subseteq G$ be a parabolic subgroup with Levi L, and let $H_L=L\cap H$ be the maximal torus in L associated to a choice of maximal torus H in G. Choose a point $v\in\mathfrak{h}_L^*/W_L\subseteq\mathfrak{h}^*/W$ with stabilizer M in G and $M\cap L$ in L. Then the geometric Eisenstein series functor

$$\mathrm{Eis}_P^{\mathrm{spec}}\colon \mathrm{IndCoh}(\mathrm{Flat}_L(\Sigma))\to\mathrm{IndCoh}(\mathrm{Flat}_G(\Sigma))$$

preserves the full subcategory of objects compatible with the vacuum v. That is, it restricts to the geometric Eisenstein series functor

$$\mathrm{Eis}_{P\cap M}^{\mathrm{spec}}\colon \mathrm{IndCoh}_{\mathcal{N}_{L\cap M}}(\mathrm{Flat}_{L\cap M}(\Sigma))\to\mathrm{IndCoh}_{\mathcal{N}_M}(\mathrm{Flat}_M(\Sigma))$$

on the subcategory $\mathrm{IndCoh}_{\mathrm{Arth}^v_L(\Sigma)}(\mathrm{Flat}_L(\Sigma))\cong\mathrm{IndCoh}_{\mathcal{N}_{L\cap M}}(\mathrm{Flat}_{L\cap M}(\Sigma))$.

The fact that the geometric Eisenstein series functor preserves the condition of having singular support in Arth^v is straightforward, by an argument completely identical to that of [AG15, Lemma 13.2.5, Proposition 13.2.6]: one argues verbatim, replacing the condition of nilpotence with the condition of having image v under the evaluation map $\mathrm{Arth}_K(\Sigma)\to\mathfrak{h}_K^*/W$ (where $K=G$ or L) and including an invertible degree 2 parameter in order to make sense of the relevant support conditions. One obtains the conjecture by combining this fact with Conjecture 4.1, along with the additional claim that the equivalence of Conjecture 4.1 itself commutes with the geometric Eisenstein series functor.

[3] We should remark that the question of why the relationship of boundary conditions on either side of these parabolic domain walls should be preserved by S-duality is beyond the scope of this paper.

4.2 Factorization Conjecture

As a consequence of Conjecture 4.1, the geometric Langlands categories appear to admit an additional structure: that of *factorization.* For the gauge group $G = \mathrm{GL}(n)$, since we can identify $x \in \mathfrak{h}^*/W$ as a point $\{x_1, \cdots, x_n\}$ in the n-th symmetric power $\mathrm{Sym}^n(\mathbb{C})$ of $\mathbb{C}$, the category of boundary conditions compatible with the vacuum x can be factorized as

$$\mathrm{IndCoh}_{\mathcal{N}_{L_{\underline{x}_{1\cdots n}}}}(\mathrm{Flat}_{L_{\underline{x}_{1\cdots n}}}(\Sigma)) \simeq \mathrm{IndCoh}_{\mathcal{N}_{L_{\underline{x}_{1\cdots m}}}}(\mathrm{Flat}_{L_{\underline{x}_{1\cdots m}}}(\Sigma)) \otimes \mathrm{IndCoh}_{\mathcal{N}_{L_{\underline{x}_{m+1\cdots n}}}}(\mathrm{Flat}_{L_{\underline{x}_{m+1\cdots n}}}(\Sigma))$$

for every decomposition $\{x_1, \cdots, x_n\} = \{x_1, \cdots, x_m\} \amalg \{x_{m+1}, \cdots, x_n\}$ with $x_i \neq x_j$ for $x_i \in \{x_1, \cdots, x_m\}$ and $x_j \in \{x_{m+1}, \cdots, x_n\}$, which we denote by $\underline{x}_{1\cdots n} = \underline{x}_{1\cdots m} \amalg \underline{x}_{m+1\cdots n}$.

In fact, by considering a system

$$\mathrm{GL}(1) \hookrightarrow \mathrm{GL}(2) \hookrightarrow \cdots \hookrightarrow \mathrm{GL}(n) \hookrightarrow \cdots$$

over the base $\mathrm{Sym}(\mathbb{C}) = \bigoplus_{n\geq 1} \mathrm{Sym}^n(\mathbb{C})$, we should be able to assemble the $\mathrm{GL}(k)$ geometric Langlands category into a *factorization category*, or rather, what one might call a “local category”, following Mirkovic’s notion of local spaces [Mir14], since the base space $\mathrm{Sym}(\mathbb{C})$ is not just the Ran space $\mathrm{Ran}(\mathbb{C})$, but remembers the multiplicity of each point. A definition of a factorization category is available, due to Raskin [Ras15], but the idea of a local category has yet to be developed.

Remark 4.8. As we discussed in remark 3.19, type IIB string theory on the background $T^*X \times \mathbb{C}$ for a complex surface X with n D3 branes along $X \times \{x_1, \cdots, x_n\}$ is responsible for the supersymmetric gauge theory with gauge group $\mathrm{GL}(n)$, where having the cotangent direction is necessary to obtain $\mathrm{GL}(n)$ as opposed to $U(n)$. Therefore, the belief that the categorical geometric Langlands correspondence factorizes in the above sense corresponds to the motion of configurations of D3 branes in its string theoretic origin. This is natural from the string theoretic point of view, because D branes are dynamic objects and in particular once we introduce D branes to the stage, the most natural thing to do is to let the number of them vary without any restriction, leading to our configuration.

In order to formulate a precise conjecture, we’ll keep the technical details manageable by first thinking of the geometric Langlands categories purely as factorization categories by composition with the map $\mathrm{Sym}(\mathbb{C}) \to \mathrm{Ran}(\mathbb{C})$ and second by considering an algebraic rather than categorical version of the statement by taking Hochschild cochains of the categories to obtain *bona fide* factorization algebras.

Conjecture 4.9. There exists a factorization algebra $\mathcal{C}$ over $\mathbb{C}$ with the fiber over a point $x \in \mathrm{Ran}(\mathbb{C})$ being

$$\mathcal{C}_x = \bigoplus_{n\geq 1,\ \tilde{x}\in \mathrm{Sym}^n(\mathbb{C})} \mathrm{HC}^\bullet(\mathrm{IndCoh}_{\mathcal{N}_{L_{\tilde{x}}}}(\mathrm{Flat}_{L_{\tilde{x}}}(\Sigma))),$$

where $\tilde{x} \in \mathrm{Sym}^n(\mathbb{C})$ is a lift of $x \in \mathrm{Ran}(\mathbb{C})$.

The operator product expansion here would carry fundamental information on how to relate geometric Langlands correspondences for different gauge groups. We hope to further investigate this conjecture and its variants in future work.

We’ll conclude with some motivation for this conjecture coming from factorization structures appearing in a related construction: the context of cohomological Hall algebras in the work of Kontsevich and Soibelman [KS11]. We claim that the underlying reason why this factorization structure appears is similar to that of our setup. To explain this, let’s consider type IIA string theory on a background of the form $X \times \mathbb{R}^2 \times \mathbb{C}$, where X is a Calabi–Yau 3-fold. We think of X as defining a Calabi–Yau 3-category. On the other hand, a quiver with superpotential yields a Calabi–Yau 3-category of representations of the quiver, where an edge corresponds to a class in Ext^1 and Serre duality determines the other ext groups. Indeed, for some Calabi–Yau 3-folds, by thinking of exceptional sequences, one can explicitly construct such a quiver. Then the dimension vector of a given quiver is precisely the number of corresponding branes in the configuration.

Example 4.10. The Calabi–Yau 3-fold $X = \mathbb{C}^3$ corresponds to a quiver with a single vertex, three self-loops denoted by x, y, z, and a nontrivial superpotential $W = xyz - xzy$. Here having a single vertex represents the fact that there is a unique compact cycle in $\mathbb{C}^3$, namely, a point. On the other hand, from the perspective of string theory, this point corresponds to a position $\mathrm{pt} \times \mathbb{R}$ on which a D0 brane lives. For a fixed number of coincident D0 branes, say N, the Hilbert space of the 1-dimensional worldline theory is the equivariant cohomology $\mathrm{H}^\bullet_{\mathrm{GL}(N)}(\mathrm{Crit}(W), \mathcal{P})$, where $\mathcal{P}$ is a sheaf of vanishing cycles.

Now the theorem of Kontsevich and Soibelman asserts that $\bigoplus_{N \geq 0} \mathrm{H}^\bullet_{\mathrm{GL}(N)}(\mathrm{Crit}(W), \mathcal{P})$ forms a factorization system over $\bigoplus_{N \geq 0} \mathrm{H}^\bullet_{\mathrm{GL}(N)}(\mathrm{pt})$. From our interpretation, we expect such a factorization structure to appear by varying the positions of the D0 branes, identified as $\mathrm{H}^\bullet_{\mathrm{GL}(N)}(\mathrm{pt}) = \mathcal{O}(\mathrm{Sym}^N(\mathbb{C}))$, completely analogously to the story for D3 branes.

References

[AG15] Dima Arinkin and Dennis Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Mathematica*, 21(1):1–199, 2015.

[Ati88] Michael F Atiyah. Topological quantum field theory. *Publications Mathématiques de l'IHÉS*, 68:175–186, 1988.

[BD95] John C Baez and James Dolan. Higher-dimensional algebra and topological quantum field theory. *Journal of Mathematical Physics*, 36(11):6073–6105, 1995.

[Bal16] Aswin Balasubramanian. Phases of $\mathcal{N} = 4$ SYM, S-duality and nilpotent cones. *arXiv preprint arXiv:1609.09320*, 2016.

[BD97] Alexander Beilinson and Vladimir Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. Available at author's website: `http://www.math.uchicago.edu/~mitya/langlands/hitchin/BD-hitchin.pdf`, 1997.

[BZBJ15] David Ben-Zvi, Adrien Brochier, and David Jordan. Integrating quantum groups over surfaces: quantum character varieties and topological field theory. *arXiv preprint arXiv:1501.04652*, 2015.

[BZN16] David Ben-Zvi and David Nadler. Betti geometric Langlands. *arXiv preprint arXiv:1606.08523*, 2016.

[BIK08] Dave Benson, Srikanth Iyengar, and Henning Krause. Local cohomology and support for triangulated categories. In *Annales scientifiques de l'ÉNS*, volume 41, pages 575–621, 2008.

[BF08] Roman Bezrukavnikov and Michael Finkelberg. Equivariant Satake category and Kostant–Whittaker reduction. *Mosc. Math. J*, 8(1):39–72, 2008.

[BKP17] Anthony Blanc, Ludmil Katzarkov, and Pranav Pandit. Generators in formal deformations of categories. *arXiv preprint arXiv:1705.00655*, 2017.

[Cal15] Damien Calaque. Lagrangian structures on mapping stacks and semi-classical TFTs. In *Stacks and Categories in Geometry, Topology, and Algebra*, volume 643, pages 1–24. American Mathematical Soc., 2015.

[CS15] Damien Calaque and Claudia Scheimbauer. A note on the (∞, n)-category of cobordisms. *arXiv preprint arXiv:1509.08906*, 2015.

[CMR15] Alberto S Cattaneo, Pavel Mnev, and Nicolai Reshetikhin. Perturbative quantum gauge theories on manifolds with boundary. *arXiv preprint arXiv:1507.01221*, 2015.

[Coh76] Fred Cohen. The homology of C_{n+1}-spaces, $n \geq 0$. In *The homology of iterated loop spaces*, pages 207–351. Springer, 1976.

[Cos07] Kevin Costello. Topological conformal field theories and Calabi–Yau categories. *Advances in Mathematics*, 210(1):165–214, 2007.

[Cos11] Kevin Costello. *Renormalization and Effective Field Theory*, volume 170. AMS, 2011.

[Cos13] Kevin Costello. Notes on supersymmetric and holomorphic field theories in dimensions 2 and 4. In *Special Issue: In Honor of Dennis Sullivan*, volume 9 of *Pure and Applied Mathematics Quarterly*. 2013.

[CG16] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory, Volume I*. Number 31 in New Mathematical Monographs. Cambridge University Press, 2016.

[CG17] Kevin Costello and Owen Gwilliam. *Factorization Algebras in Quantum Field Theory, Volume II*. 2017. book in progress, available at `http://people.mpim-bonn.mpg.de/gwilliam/`.

[Dia97] Duiliu-Emanuel Diaconescu. D-branes, monopoles and Nahm equations. *Nuclear Physics B*, 503(1-2):220–238, 1997.

[Dri83] Vladimir Drinfeld. Two-dimensional ℓ-adic representations of the fundamental group of a curve over a finite field and automorphic forms on GL(2). *American Journal of Mathematics*, 105(1):85–114, 1983.

[Dun88] Gerald Dunn. Tensor product of operads and iterated loop spaces. *Journal of Pure and Applied Algebra*, 50(3):237–258, 1988.

[EY15] Chris Elliott and Philsang Yoo. Geometric Langlands twists of $N = 4$ gauge theory from derived algebraic geometry. *arXiv preprint arXiv:1507.03048*, 2015.

[Fra13] John Francis. The tangent complex and Hochschild cohomology of $\mathcal{E}_n$-rings. *Compositio Mathematica*, 2:24–25, 2013.

[Fre93] Daniel S Freed. Extended structures in topological quantum field theory. In *Quantum Topology*, pages 162–173. World Scientific, 1993.

[Fre94] Daniel S Freed. Higher algebraic structures and quantization. *Communications in Mathematical Physics*, 159(2):343–398, 1994.

[FQ93] Daniel S Freed and Frank Quinn. Chern-Simons theory with finite gauge group. *Communications in Mathematical Physics*, 156(3):435–472, 1993.

[FGV02] Edward Frenkel, Dennis Gaitsgory, and Kari Vilonen. On the geometric Langlands conjecture. *Journal of the American Mathematical Society*, 15(2):367–417, 2002.

[Gai16] Davide Gaiotto. S-duality of boundary conditions and the Geometric Langlands program. *arXiv preprint arXiv:1609.09030*, 2016.

[GMW15] Davide Gaiotto, Gregory W Moore, and Edward Witten. Algebra of the infrared: String field theoretic structures in massive $\mathcal{N} = (2, 2)$ field theory in two dimensions. *arXiv preprint arXiv:1506.04087*, 2015.

[GW09a] Davide Gaiotto and Edward Witten. S-duality of boundary conditions in $N = 4$ super Yang-Mills theory. *Advances in Theoretical and Mathematical Physics*, 13(3):721–896, 2009.

[GW09b] Davide Gaiotto and Edward Witten. Supersymmetric boundary conditions in $\mathcal{N} = 4$ super Yang-Mills theory. *Journal of Statistical Physics*, 135(5-6):789–855, 2009.

[GW12] Davide Gaiotto and Edward Witten. Knot invariants from four-dimensional gauge theory. *Advances in Theoretical and Mathematical Physics*, 16(3):935–1086, 2012.

[Gai04] Dennis Gaitsgory. On a vanishing conjecture appearing in the geometric Langlands correspondence. *Annals of Mathematics*, 160(2):617–682, 2004.

[GR17] Dennis Gaitsgory and Nick Rozenblyum. A study in derived algebraic geometry. Available at author's website `http://www.math.harvard.edu/~gaitsgde/GL`, accessed June 2nd, 2017.

[Gin95] Victor Ginzburg. Perverse sheaves on a loop group and Langlands' duality. *arXiv preprint alg-geom/9511007*, 1995.

[GGW16] Vassily Gorbounov, Owen Gwilliam, and Brian Williams. Chiral differential operators via Batalin-Vilkovisky quantization. *arXiv preprint arXiv:1610.09657*, 2016.

[Gwi12] Owen Gwilliam. *Factorization Algebras and Free Field Theory*. PhD thesis, Northwestern University, 2012.

[KW07] Anton Kapustin and Edward Witten. Electric-magnetic duality and the geometric Langlands program. *Communications in Number Theory and Physics*, 1:1–236, 2007.

[KS11] Maxim Kontsevich and Yan Soibelman. Cohomological Hall algebra, exponential Hodge structures and motivic Donaldson–Thomas invariants. *Communications in Number Theory and Physics*, (5):231–352, 2011.

[Kop15] Clemens Koppensteiner. Hochschild cohomology of torus-equivariant D-modules. *arXiv preprint arXiv:1506.07370*, 2015.

[Laf09] Vincent Lafforgue. Quelques calculs reliés à la correspondance de Langlands géométrique pour $\mathbb{P}^1$

(version provisoire). Available at author's website: `http://www.math.jussieu.fr/~vlafforg`, 2009.

[Lau87] Gérard Laumon. Correspondance de Langlands Géométrique pour les corps de fonctions. *Duke Mathematical Journal*, 54:309–359, 1987.

[Lau96] Gérard Laumon. Transformation de Fourier généralisée. *arXiv preprint alg-geom/9603004*, 1996.

[Lur09] Jacob Lurie. On the classification of topological field theories. *Current developments in mathematics*, 2008:129–280, 2009.

[Lur14] Jacob Lurie. Higher algebra, 2014. Available from author's website at `http://www.math.harvard.edu/~lurie/papers/HigherAlgebra.pdf`.

[Lus83] George Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. *Astérisque*, 101(102):208–229, 1983.

[Mir14] Ivan Mirković. Loop Grassmannians in the framework of local spaces over a curve. In *Recent Advances in Representation Theory, Quantum Groups, Algebraic Geometry, and Related Topics*, volume 623 of *Contemporary Mathematics*, pages 215–226. 2014.

[MV07] Ivan Mirković and Kari Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *The Annals of Mathematics*, 166(1):95–143, 2007.

[MW98] Gregory Moore and Edward Witten. Integration over the u-plane in Donaldson theory. *Advances in Theoretical and Mathematical Physics*, 1:298–387, 1998.

[NY16] David Nadler and Zhiwei Yun. Spectral action in Betti geometric Langlands. *arXiv preprint arXiv:1611.04078*, 2016.

[PTVV13] Tony Pantev, Bertrand Toën, Michel Vaquié, and Gabriele Vezzosi. Shifted symplectic structures. *Publications mathématiques de l'IHÉS*, 117(1):271–328, 2013.

[Ras15] Sam Raskin. Chiral categories, 2015. Available from author's website at `http://math.mit.edu/~sraskin/chiralcats.pdf`.

[Rot96] Mitchell Rothstein. Sheaves with connection on abelian varieties. *Duke Mathematical Journal*, 84(3):565–598, 1996.

[Saf16] Pavel Safronov. Braces and Poisson additivity. *arXiv preprint arXiv:1611.09668*, 2016.

[SP09] Chris Schommer-Pries. *The Classification of Two-Dimensional Extended Topological Field Theories*. PhD thesis, University of California Berkeley, 2009. Available as an arXiv preprint: arXiv:1112.1000.

[Seg88] Graeme Segal. The definition of conformal field theory. In *Differential geometrical methods in theoretical physics*, pages 165–171. Springer, 1988.

[Sha99] Eric Sharpe. D-branes, derived categories, and Grothendieck groups. *Nuclear Physics B*, 561(3):433–450, 1999.

[Sim94] Carlos T Simpson. Moduli of representations of the fundamental group of a smooth projective variety. II. *Publications Mathématiques de l'Institut des Hautes Études Scientifiques*, 80(1):5–79, 1994.

[Tam98] Dmitry E Tamarkin. Another proof of M. Kontsevich formality theorem for $\mathbb{R}^n$. *arXiv preprint math.QA/9803025*, 24, 1998.

[Toë13] Bertrand Toën. Operations on derived moduli spaces of branes. *arXiv preprint arXiv:1307.0405*, 2013.

[Toë14] Bertrand Toën. Derived algebraic geometry and deformation quantization. *arXiv preprint arXiv:1403.6995*, 2014.

[TV07] Bertrand Toën and Michel Vaquié. Moduli of objects in dg-categories. In *Annales Scientifiques de l'École Normale Supérieure*, volume 40, pages 387–444. Elsevier, 2007.

[Wal16] James Wallbridge. Derived smooth stacks and prequantum categories. *arXiv preprint arXiv:1610.00441*, 2016.

[Wei79] André Weil. Œuvres scientifiques. collected papers. vol. i (1926–1951). 1979.

[Wit11] Edward Witten. Fivebranes and knots. *arXiv preprint arXiv:1101.3216*, 2011.

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