Introduction to the Gan-Gross-Prasad and Ichino-Ikeda conjectures I

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Hecke theory

- Let $f \in S_2(N)$ cuspidal modular form of weight 2 level $N \geqslant 2$ i.e. $f : \mathcal{H} = \{Im(z) > 0\} \to \mathbb{C}$ holom st

$$f(\frac{az+b}{cz+d}) = (cz+d)^2 f(z), \forall \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N) = \{\gamma \in SL_2(\mathbb{Z}) \mid \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} [N]\}$$

  and f “vanishes at the cusps". Fourier expansion : $f = \sum_{n \geqslant 1} a_n q^n$, $q = e^{2i\pi z}$.
- $Y_0(N) = \Gamma_0(N) \backslash \mathcal{H} \hookrightarrow X_0(N) = \Gamma_0(N) \backslash \mathcal{H}^*$ where $\mathcal{H}^* = \mathcal{H} \cup \mathbf{P}^1(\mathbb{Q})$ the completed modular curve of level N. Then

$$S_2(N) \simeq \Omega^1(X_0(N)), \;\; f \mapsto \omega_f = f(z)dz.$$

- L-function : $L(s,f) = \sum_{n \geqslant 1} \frac{a_n}{n^s}$ ($\Re(s) > 2$). Then, we have (Hecke)

$$(2\pi)^{-s}\Gamma(s)L(s,f) = \int_0^\infty f(iy)y^s d^\times y.$$

  $\Rightarrow L(s,f)$ has analytic contn to $\mathbb{C}$ and a functional eqn $s \leftrightarrow 2-s$.

Hecke operators and Euler factorization

- Hecke operators : $T_p \curvearrowright S_2(N) = \Omega^1(X_0(N))$, $p \nmid N$ prime, given by the correspondence

$$X_0(N) \xleftarrow[\alpha]{} X_0(pN) \xrightarrow{\beta} X_0(N)$$

$$\omega_{T_p f} = \beta_* \alpha^* \omega_f,$$

where α is induced from identity and β from $z \mapsto \begin{pmatrix} p & \\ & 1 \end{pmatrix} .z = pz$.

- Eulerian product : if f is eigen for all Hecke operators, new (i.e. does not come by pullback from $X_0(M)$, $M \mid N$ $M \neq N$) and normalized ($a_1 = 1$) then

$$L(s,f) = \prod_p (1 - a_p p^{-s} + \mathbf{1}_N(p) p^{1-2s})^{-1}, \quad \Re(s) > 2$$

where $\mathbf{1}_N(p) = 1$ if $p \nmid N$ and 0 otherwise. Moreover, in this case the FE takes the form

$$N^{s/2}(2\pi)^{-s}\Gamma(s)L(s,f) = \varepsilon(f)N^{1-s/2}(2\pi)^{s-2}\Gamma(2-s)L(2-s,f)$$

with $\varepsilon(f) = \pm 1$.

Relation to elliptic curves

- Let $E : y^2 = x^3 + ax + b$, $4a^3 + 27b^2 \neq 0$, an elliptic curve over $\mathbb{Q}$. We define its L-function :

$$L(s,E) = \prod_p (1 - a_p p^{-s} + \mathbf{1}_N(p) p^{1-2s})^{-1}, \quad \Re(s) > 3/2$$

where N is the conductor and $a_p = p - |E(\mathbb{F}_p)|$ for almost all p.

Theorem (Wiles, Taylor-Wiles, BCDT)

There exists a normalized new Hecke-eigenform $f_E \in S_2(N)$ st

$$L(s,f_E) = L(s,E).$$

- Application : $L(s,E)$ has analytic contn and a FE.

- Hecke's integral formula at $s = 1$ (central point) gives

$$L(1, f) = 2\pi \int_0^{\infty} f(iy) y d^\times y = -2i\pi \int_0^{i\infty} \omega_f.$$

- Application (Manin-Drinfeld) : for $E/\mathbb{Q}$ elliptic curve we have

$$\frac{L(E, 1)}{\Omega_E} \in \mathbb{Q}$$

where $\Omega_E = \int_{E(\mathbb{R})} \omega$ for $\omega \in \Omega^1(E/\mathbb{Q})$.

Waldspurger's formula

- $f \in S_2(N)$ new Hecke-eigenform and $K/\mathbb{Q}$ imaginary quadratic of disc D. We consider the "base-change" L-function

$$L(s, f_K) = L(s, f)L(s, f \times \chi_K)$$

  where $\chi_K : (\mathbb{Z}/D\mathbb{Z})^\times \to \{\pm 1\}$.
- Assume N prod of an odd number of distinct inert primes in $K \Rightarrow \varepsilon(f_K) = +1$.
- B : unique quaternion alg $/\mathbb{Q}$ st $B_\infty := B \otimes \mathbb{R} \simeq \mathbb{H}$, $B_p := B \otimes \mathbb{Q}_p$ division alg for $p \mid N$ and $B_p \simeq M_2(\mathbb{Q}_p)$ otherwise. Let $\mathcal{O}_B \subset B$ max order and $Cl(\mathcal{O}_B)$ its set of right ideal classes.
- Jacquet-Langlands : $\exists f' : Cl(\mathcal{O}_B) \to \mathbb{C}$ with "same Hecke-eigenvalues as f".
- $\exists K \hookrightarrow B$ st $\mathcal{O}_K \subset \mathcal{O}_B \rightsquigarrow Cl(\mathcal{O}_K) \to Cl(\mathcal{O}_B)$, $I \mapsto I\mathcal{O}$. Waldspurger's period :

$$y_K = \sum_{x \in Cl(\mathcal{O}_K)} f'(x).|Aut(x)|$$

  where $Aut(x) = \{\gamma \in B^\times \mid \gamma x = x\}/\mathcal{O}_K^\times$.
- Waldspurger/Gross : $L(1, f_K) = |D|^{-1/2} \dfrac{(f,f)}{(f',f')} |y_K|^2$ where (f,f) and (f',f') are the L^2-scalar products of f and f'.
- Applications : $L(1, f_K) \geqslant 0$, BSD in rank 0, p-adic L-fns...

- y_K and $\int_0^\infty f(iy)y^{k/2}d^\times y$ are classical realizations of "automorphic periods".
- The formulas of Hecke and Waldspurger generalize to higher rank groups :

$$\text{Hecke} \to \text{Rankin-Selberg theory}$$

$$\text{Waldspurger} \to \text{Gan-Gross-Prasad conjecture.}$$

- Here we were dealing with the group $GL(2)$ in the background. When restated in the appropriate adelic/automorphic setting we will see $GL(2)$ in the foreground an the two formulas will look very similar.

Restatement of Hecke's and Waldspurger's formulas

- $\mathbb{A} = \mathbb{R} \times \prod'_p \mathbb{Q}_p = \{(x_\infty, (x_p)_p) \in \mathbb{R} \times \prod'_p \mathbb{Q}_p \mid x_p \in \mathbb{Z}_p \text{ for a.a. } p\}$ the *adèle ring* of $\mathbb{Q}$, we have a diagonal embedding $\mathbb{Q} \hookrightarrow \mathbb{A}$;
- $f \in S_2(N)$ corresponds to a *smooth* function $\varphi : \mathrm{GL}_2(\mathbb{Q}) \backslash \mathrm{GL}_2(\mathbb{A}) \to \mathbb{C}$ which is rapidly decreasing and generating an irreducible $\mathrm{GL}_2(\mathbb{A})$-irreducible repn π by right translations (such function is called a *cusp form* and π a *cuspidal automorphic repn*);
- Similarly $f' : Cl(\mathcal{O}_B) \to \mathbb{C}$ corresponds to a cusp form φ' on $B^\times \backslash B^\times_{\mathbb{A}}$ generating a $B^\times_{\mathbb{A}}$-irred repn π' where $B_{\mathbb{A}} = B \otimes \mathbb{A}$.
- We have

$$\mathrm{GL}_2(\mathbb{A}) = \mathrm{GL}_2(\mathbb{R}) \times \prod_p{}' \mathrm{GL}_2(\mathbb{Q}_p), \; B^\times_{\mathbb{A}} = B^\times_\infty \times \prod_p{}' B^\times_p$$

  and the repns π, π' decompose as (restricted) tensor products

$$\pi = \pi_\infty \otimes \bigotimes_p{}' \pi_p = \bigotimes_v{}' \pi_v, \quad \pi' = \bigotimes_v{}' \pi'_v$$

  where π_v, π'_v are irred repns of the local groups $\mathrm{GL}_2(\mathbb{Q}_v)$, $B^\times_v$ (here $\mathbb{Q}_\infty = \mathbb{R}$).
- The condition that f and f' have the same Hecke eigenvalues now reads $\pi'_p \simeq \pi_p$ for $p \nmid N$ (where $B^\times_p \simeq \mathrm{GL}_2(\mathbb{Q}_p)$).

- Let $A := \begin{pmatrix} \star & \\ & \star \end{pmatrix} \simeq \mathbb{G}_m^2 \hookrightarrow \mathrm{GL}_2$ (*split torus*), $T := R_{K/\mathbb{Q}}\mathbb{G}_m \hookrightarrow B^\times$ ou GL_2 (*non-split torus* assoc to K, the embedding coming from $M_2(\mathbb{Q}) \hookleftarrow K \hookrightarrow B$).
- Periods :

$$\text{Hecke period } \mathcal{P}_A(\varphi) = \int_{A(\mathbb{Q})\backslash A(\mathbb{A})/Z(\mathbb{A})} \varphi(a)da, \ \ \varphi \in \pi;$$

$$\text{Waldspurger period } \mathcal{P}_T(\varphi) = \int_{T(\mathbb{Q})\backslash T(\mathbb{A})/Z(\mathbb{A})} \varphi(t)dt, \ \ \varphi \in \pi \text{ ou } \pi'.$$

- In this adelic context Hecke's and Waldspurger's formulas become :

$$\mathcal{P}_A(\varphi) \sim L(\frac{1}{2},\pi)\prod_v \mathcal{P}_{A_v}(\varphi_v), \ \varphi = \otimes'_v\varphi_v \in \pi = \bigotimes_v{}' \pi_v;$$

$$|\mathcal{P}_T(\varphi)|^2 \sim \frac{L(\frac{1}{2},\pi_K)}{L(1,\pi,\mathrm{Ad})}\prod_v \mathcal{P}_{T_v}(\varphi_v,\varphi_v), \ \varphi = \otimes'_v\varphi_v \in \pi'.$$

- Here $L(s,\pi)$, $L(s,\pi_K)$ and $L(s,\pi,\mathrm{Ad})$ are (completed) automorphic L-functions (always normalized so that $1/2$ is the center of symmetry) and $\mathcal{P}_{A_v}$, $\mathcal{P}_{T_v}$ are "local periods" which are $A_v = A(\mathbb{Q}_v)$- and $T_v = T(\mathbb{Q}_v)$-invariant forms on π_v, π'_v.
- Once again the first formula admits a deformation to all $s \in \mathbb{C}$ giving an integral repn of the L-function. This is not true in the 2nd case.

Local periods and ε-factors

$$\mathcal{P}_A(\varphi) \sim L(\frac{1}{2},\pi)\prod_v \mathcal{P}_{A_v}(\varphi_v),\ \varphi = \otimes'_v\varphi_v \in \pi;$$

$$|\mathcal{P}_T(\varphi)|^2 \sim \frac{L(\frac{1}{2},\pi_K)}{L(1,\pi,\mathrm{Ad})}\prod_v \mathcal{P}_{T_v}(\varphi_v,\varphi_v),\ \varphi = \otimes'_v\varphi_v \in \pi'.$$

- The local periods $\mathcal{P}_{A_v}$ are always non-vanishing (for some choice of φ_v). Therefore

$$L(\frac{1}{2},\pi) \neq 0 \Leftrightarrow \mathcal{P}_A\,|_\pi \not\equiv 0.$$

- This is not true for $\mathcal{P}_{T_v}$. Instead we have

$$\mathcal{P}_{T_v} \not\equiv 0 \Leftrightarrow \mathrm{Hom}_{T_v}(\pi'_v,\mathbb{C}) \neq 0.$$

- $L(s,\pi_K)$ has a FE $s \leftrightarrow 1-s$ involving a global sign $\varepsilon(\pi_K) = \prod_v \varepsilon(\pi_{K,v})$.

Theorem (Tunnell, Saito)

$$\mathrm{Hom}_{T_v}(\pi'_v,\mathbb{C}) \neq 0 \Leftrightarrow \varepsilon(\pi_{K,v}) = \begin{cases} +1 & \text{if } B_v \simeq M_2(\mathbb{Q}_v) \\ -1 & \text{if } B_v \text{ is a div alg.} \end{cases}$$

Thus :

$$\mathcal{P}_{T_v} \not\equiv 0 \Leftrightarrow \mathrm{Hom}_{T_v}(\pi'_v, \mathbb{C}) \neq 0 \Leftrightarrow \varepsilon(\pi_{K,v}) = \begin{cases} +1 & \text{if } B_v \simeq M_2(\mathbb{Q}_v) \\ -1 & \text{if } B_v \text{ is a div alg.} \end{cases}$$

- Starting with π on $GL_2(\mathbb{A})$ we now look for a quaternion alg $B/\mathbb{Q}$ st $\varepsilon(\pi_{K,v}) = -1$ precisely when $B_v \not\simeq M_2(\mathbb{Q}_v)$.
- By class field theory this is possible iff the number of such v is even i.e. iff

$$\varepsilon(\pi_K) = \prod_v \varepsilon(\pi_{K,v}) = +1.$$

- If this is the case, by Jacquet-Langlands theory we can also transfer π to π' on $B_{\mathbb{A}}^{\times}$ (that is $\pi'_p \simeq \pi_p$ for a.a. p).
- If $\varepsilon(\pi_K) = -1$ then $L(\frac{1}{2}, \pi_K) = 0$ by the FE.
- All in all we obtain

$$L(\frac{1}{2}, \pi_K) \neq 0 \Leftrightarrow \exists (B, \pi') \text{ “as before" st } \mathcal{P}_T \mid_{\pi'} \neq 0$$

Moreover the pair (B, π') if it exists is unique.

Generalizations :

$$A/Z \simeq \mathrm{GL}_1 \hookrightarrow \mathrm{GL}_2 \rightsquigarrow \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{n+1} \text{ (Rankin-Selberg)}$$

$$(T/Z \hookrightarrow B^\times/Z) \simeq (SO(2) \hookrightarrow SO(3)) \text{ or } (U(1) \hookrightarrow PU(2))$$

$$\rightsquigarrow (SO(n) \hookrightarrow SO(n+1)) \text{ or } (U(n) \hookrightarrow U(n+1)) \text{ (Gan-Gross-Prasad).}$$

Aside on automorphic L-functions

- $G/\mathbb{Q}$ (or any number field) connected reductive gp, $\pi = \bigotimes'_v \pi_v \hookrightarrow \mathcal{A}_{(cusp)}(G(\mathbb{Q})\backslash G(\mathbb{A}))$ irred (cuspidal) autom repn ;
- ${}^L G = \widehat{G} \rtimes Gal(K/\mathbb{Q})$ Langlands L-group (here $\widehat{G}$ is a conn red gp/$\mathbb{C}$). ex : ${}^L\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C})$, ${}^L U(n, K/\mathbb{Q}) = \mathrm{GL}_n(\mathbb{C}) \rtimes Gal(K/\mathbb{Q})$ where $K/\mathbb{Q}$ quad,
$$Gal(K/\mathbb{Q}) = \{1, c\} \text{ and } cgc^{-1} = J^t g^{-1} J^{-1} \text{ where } J = \begin{pmatrix} & & 1 \\ & \iddots & \\ (-1)^{n-1} & & \end{pmatrix};$$
- For a.a. p, π_p is "unramified" (i.e. $\pi_p^{G(\mathbb{Z}_p)} \neq 0$ for a "good" model $G/\mathbb{Z}_p$) and classified by a conj class $\mathcal{L}(\pi_p) \subset {}^L G$ (the *Langlands-Satake parameter*) ;
- To $r : {}^L G \to \mathrm{GL}_n(\mathbb{C})$ alg repn and S finite set of places st π_p unr for $p \notin S$, we associate a partial automorphic L-function
$$L^S(s, \pi, r) = \prod_{p \notin S} \det(1 - \mathcal{L}(\pi_p) p^{-s})^{-1} = \prod_{p \notin S} L(s, \pi_p, r), \ \Re(s) \gg 0;$$
- It's expected that $L^S(s, \pi, r)$ has analytic contn and, when completed by suitable Euler factors at places in S, a FE ;
- ex : $r = St : {}^L\mathrm{GL}_n = \mathrm{GL}_n(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$ yields $L^S(s, \pi) = L^S(s, \pi, St)$, $r = \otimes : {}^L(\mathrm{GL}_n \times \mathrm{GL}_m) = \mathrm{GL}_n(\mathbb{C}) \times \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_{nm}(\mathbb{C})$ yields $L^S(s, \pi \times \sigma) = L^S(s, \pi \boxtimes \sigma, \otimes)$;

Automorphic L-functions continued

- In these cases completed L-functions $L(s,\pi)$ and $L(s,\pi\times\sigma)$ have been defined and studied by Godement-Jacquet and Jacquet-Piatetski-Shapiro-Shalika / Shahidi.
- Other example : $r = \mathrm{Ad} : {}^LG \to \mathrm{GL}(\mathrm{Lie}({}^LG)) = \mathrm{GL}(\mathrm{Lie}(\widehat{G}))$ (or modulo the center). By recent work of Arthur, Mok and Kaletha-Minguez-Shin-White +Shahidi, when G is classical (unitary, special orthogonal or symplectic) there is a completed L-function $L(s,\pi,\mathrm{Ad})$ with analytic contn and FE.

Rankin-Selberg theory (Jacquet, Piatetski-Shapiro, Shalika)

- Let $\sigma \hookrightarrow \mathcal{A}_{cusp}(GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A}))$, $\pi \hookrightarrow \mathcal{A}_{cusp}(GL_{n+1}(\mathbb{Q})\backslash GL_{n+1}(\mathbb{A}))$ be cuspidal autom repns ;
- Rankin-Selberg period :

$$\varphi \otimes \varphi' \in \sigma \otimes \pi \mapsto \mathcal{P}_{GL_n}(\varphi \otimes \varphi') := \int_{GL_n(\mathbb{Q})\backslash GL_n(\mathbb{A})} \varphi(h)\varphi'(h)dh.$$

- Jacquet-Piatetski-Shapiro-Shalika : for $\varphi = \otimes'_v \varphi_v \in \sigma$ and $\varphi' = \otimes'_v \varphi'_v \in \pi$ we have

$$\mathcal{P}_{GL_n}(\varphi \otimes \varphi') = L(\frac{1}{2}, \sigma \times \pi) \prod_v \mathcal{P}_{GL_n,v}(\varphi_v \otimes \varphi'_v)$$

  for some local periods $\mathcal{P}_{GL_n,v}$ and where $L(s, \sigma \times \pi)$ is the (completed) tensor L-function.
- This identity actually comes from an integral representation of $L(s, \sigma \times \pi)$ which is also used to define Euler factors at bad primes and to show analytic contn.
- We recover Hecke's formula when $n = 1$ and $\sigma = \mathbf{1}$.
- Once again $\mathcal{P}_{\mathrm{GL}_n,v} \not\equiv 0$ for all v and therefore

$$L(1/2, \sigma \times \pi) \neq 0 \Leftrightarrow \mathcal{P}_{\mathrm{GL}_n} \mid_{\sigma \otimes \pi} \not\equiv 0.$$

Gan-Gross-Prasad conjecture for unitary groups

- Let $K/\mathbb{Q}$ quad ext (not nec. imaginary), (V,h) : n-diml Hermitian sp over K (i.e. the form h is non-deg, linear in the 1st variable and st $h(v,v') = \overline{h(v',v)}$).
- Set $V' := V \oplus^{\perp} K.e$ with $h(e,e) = 1$.
- $U(V) = \{g \in \mathrm{GL}_K(V) \mid h(gv,gv') = h(v,v')\}$ and $U(V')$ the corresp. unitary gps (seen as algebraic gps $/\mathbb{Q}$). Then $U(V) \hookrightarrow U(V')$.
- Let $\sigma \hookrightarrow \mathcal{A}_{cusp}(U(V)(\mathbb{Q})\backslash U(V)(\mathbb{A}))$ and $\pi \hookrightarrow \mathcal{A}_{cusp}(U(V')(\mathbb{Q})\backslash U(V')(\mathbb{A}))$ be cuspidal autom repns.
- GGP period :

$$\varphi \otimes \varphi' \in \sigma \otimes \pi \mapsto \mathcal{P}_{U(V)}(\varphi \otimes \varphi') := \int_{U(V)(\mathbb{Q})\backslash U(V)(\mathbb{A})} \varphi(h)\varphi'(h)dh.$$

- As in the GL_2 case we need to vary the gps and representations : for V_0 another n-diml Herm space we have $V_p := V \otimes \mathbb{Q}_p \simeq V_{0,p}$ for a.a. p and we say that $\sigma_0 \hookrightarrow \mathcal{A}_{cusp}(U(V_0))$ is *in the same L-packet* as σ if $\sigma_p \simeq \sigma_{0,p}$ for a.a. p.
- This applies equality well to repns $\pi \hookrightarrow \mathcal{A}_{cusp}(U(V'))$ and $\pi_0 \hookrightarrow \mathcal{A}_{cusp}(U(V_0'))$ where $V_0' = V_0 \oplus^{\perp} K.e$.
- Repns in the same L-packet share the same (automorphic) L-functions (as the Euler factors are the same for a.a. p).

- Let σ_K, π_K be the base-change of σ and π (Mok, Kaletha-Minguez-Shin-White) : these are automorphic repns on $\mathrm{GL}_n(K)\backslash \mathrm{GL}_n(\mathbb{A}_K)$ and $\mathrm{GL}_{n+1}(K)\backslash \mathrm{GL}_{n+1}(\mathbb{A}_K)$ whose components at a.a. p are given by an explicit recipe.
- Form their tensor L-function $L(s, \sigma_K \times \pi_K)$.

Conjecture (Gan-Gross-Prasad)

Assume that σ_K and π_K are 'generic'. The following are equivalent :

1. $L(1/2, \sigma_K \times \pi_K) \neq 0$ *;*
2. *There exist a n-diml Herm space V_0 and $\sigma_0 \hookrightarrow \mathcal{A}_{cusp}(U(V_0))$, $\pi_0 \hookrightarrow \mathcal{A}_{cusp}(U(V_0'))$ in the same L-packets as σ and π st $\mathcal{P}_{U(V_0)}\,|_{\sigma_0 \otimes \pi_0} \not\equiv 0$.*

Moreover the triple (V_0, σ_0, π_0) if it exists is unique.

Conjecture (Ichino-Ikeda, N.Harris)

For $\varphi = \otimes_v' \varphi_v \in \sigma$ and $\varphi' = \otimes_v' \varphi_v' \in \pi$, there is a 'precise' formula

$$|\mathcal{P}_{U(V)}(\varphi \otimes \varphi')|^2 \sim \frac{L(1/2, \sigma_K \times \pi_K)}{L(1, \sigma, \mathrm{Ad}) L(1, \pi, \mathrm{Ad})} \prod_v \mathcal{P}_{U(V)_v}(\varphi_v \otimes \varphi_v', \varphi_v \otimes \varphi_v')$$

for some explicit local periods $\mathcal{P}_{U(V)_v}$ and where $\sim$ means up to a rational factor and some abelian L-values.

Status

- The Gan-Gross-Prasad and Ichino-Ikeda conjectures are known under the restriction that (at least one of) $\sigma_{K,p}$ and $\pi_{K,p}$ are 'supercuspidal' for some prime p (W. Zhang, Z.Yun, H. Xue, B.-P.) ;
- Jiang-L.Zhang (after Ginzburg-Jiang-Rallis) have proved the implication $\mathcal{P}_{U(V)} \mid_{\sigma\otimes\pi} \neq 0 \Rightarrow L(1/2, \sigma_K \times \pi_K) \neq 0$ in general ;
- Grobner-Lin : when K is imaginary, π, σ are tempered cohomological+other assumptions, the Ichino-Ikeda formula is true up to an algebraic number (which however might be zero).