Bun_G minicourse: introduction

RAMpAGe seminar

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The main theorem of [FS]

Fargues-Scholze gives a purely geometric construction of the automorphic-to-Galois direction of local Langlands:

Theorem (Fargues-Scholze)

Let $F/\mathbb{Q}_p$ be a finite extension, and let G/F be a reductive group. To every irreducible smooth representation π of $G(F)$ (with coefficients in $\overline{\mathbb{Q}}_\ell$) there is an associated semisimple L-parameter $\varphi \colon W_F \to \hat{G}(\overline{\mathbb{Q}}_\ell)$.

The construction mirrors that of V. Lafforgue in the function field setting. The goal of this talk is to review the ideas leading up to this theorem.

Local class field theory

The Langlands program begins with class field theory.

For a global field K, this refers to the reciprocity map $\mathbf{A}_K^\times/K^\times \to \mathrm{Gal}(\overline{K}/K)^{\mathrm{ab}}$.

For a p-adic field F, this refers to the isomorphism $F^\times \to W_F^{\mathrm{ab}}$.

The latter can be constructed via a *Lubin-Tate formal group* $H/\mathcal{O}_{\breve{F}}$: this is the unique connected p-divisible $\mathcal{O}_F$-module of height 1 and dimension 1. Adjoining the torsion of H to $\breve{F}$ produces its maximal abelian extension.

(If $F = \mathbb{Q}_p$, then $H = \mu_{p^\infty}$ and we recover the local Kronecker-Weber theorem.)

Local Langlands for GL_n

Theorem (Harris-Taylor 2002)

There is a bijection $\pi \mapsto \varphi_\pi$ from smooth irreducible representations of $\mathrm{GL}_n(F)$ to Frobenius-semisimple Weil-Deligne representations, compatible with L- and ε-factors of pairs.

A Weil-Deligne representation is a continuous homomorphism $W_F \to \mathrm{GL}_n(\mathbb{C})$ together with a monodromy operator; these can be packaged together as an L-parameter $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{C})$.

The proof uses geometry: one studies the deformation space M of a connected p-divisible $\mathcal{O}_F$-module of dimension 1 and height n, together with some crucial global input from unitary Shimura varieties.

M is *Lubin-Tate space*; it is an example of a Rapoport-Zink space.

Local Langlands for general G

Let G/F be a reductive group. Let $\hat{G}$ be the (complex) dual group: this has the dual root datum as G.

Conjecture

There is a finite-to-one surjective map $\pi \mapsto \varphi_\pi$ from irreducible admissible representations of $G(F)$ to L-parameters $W_F \times \mathrm{SL}_2(\mathbb{C}) \to \hat{G}$, satisfying various desiderata.

The conjecture is known in many cases (see Tasho's talk), but one may still ask for a geometric construction following Harris-Taylor which is uniform in G. It is not immediately clear what the analogue of Lubin-Tate space should be in general.

Interlude: Geometric Langlands

Let X be a curve over a finite field k, with function field K. (Level 1) automorphic forms on K are functions on

$$\prod_v \mathrm{GL}_n(\mathcal{O}_{K_v}) \backslash \mathrm{GL}_n(\mathbf{A}_K) / \mathrm{GL}_n(K),$$

but this set has a meaning: it classifies rank n vector bundles on X.

This suggests we can *geometrize* the theory by replacing the set with the Artin stack $\mathrm{Bun}_{\mathrm{GL}_n}$, which assigns to a scheme S/k the groupoid of rank n vector bundles on $X \times_k S$.

In this geometrization, automorphic forms become complexes of sheaves on $\mathrm{Bun}_{\mathrm{GL}_n}$, that is, objects of $D(\mathrm{Bun}_{\mathrm{GL}_n})$.

Hecke operators

We can also geometrize Hecke operators. For each $i = 1, \dots, n$ we have a correspondence

$$\begin{tikzcd} & \mathrm{Hecke}_i \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_{\mathrm{GL}_n} & & X \times \mathrm{Bun}_{\mathrm{GL}_n} \end{tikzcd}$$

classifying *modifications* $\mathcal{E} \to \mathcal{E}'$ whose cokernel is a skyscraper sheaf supported at a point $x \in X$ and isomorphic to $\mathcal{O}_x^{\oplus i}$. Then h_2 is a $\mathrm{Grass}(n, i)$-bundle.

A *Hecke eigensheaf* is an object $A \in D(\mathrm{Bun}_{\mathrm{GL}_n})$ satisfying

$$Rh_{2*} h_1^* A \cong \bigwedge^i \varphi \boxtimes A[i(n-i)]$$

for $i = 1, \dots, n$. Here the eigenvalue φ is a local system of rank n.

Geometric Langlands for GL_n

A *Hecke eigensheaf* is an object $A \in D(\mathrm{Bun}_{\mathrm{GL}_n})$ satisfying

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Theorem (Frenkel-Gaitsgory-Vilonen 2001)

Given an irreducible rank n local system φ on X, there exists a perverse sheaf A on $\mathrm{Bun}_{\mathrm{GL}_n}$ which is a Hecke eigensheaf with eigenvalue φ.

This is a geometrization of the result of L. Lafforgue (the global Langlands correspondence between automorphic representations π and Galois representations φ). The Hecke eigensheaf property corresponds to the equality of L-factors

$$L_v(\pi, s) = L_v(\varphi, s)$$

for each place $v \in |X|$.

More general groups, and the Satake isomorphism

The whole discussion can be generalized to the setting of a (split) reductive group G. Unramified automorphic forms are functions on

$$\prod_v G(\mathcal{O}_{K_v})\backslash G(\mathbf{A}_K)/G(K).$$

For each $v \in |X|$ the *spherical Hecke algebra*

$$\mathcal{H}_v = C_c(G(\mathcal{O}_{K_v})\backslash G(K_v)/G(\mathcal{O}_{K_v}))$$

acts on automorphic forms; the eigenvectors for these correspond to automorphic representations of G.

Theorem (Satake isomorphism)

There is an isomorphism

$$\mathcal{H}_v \to \mathrm{Rep}\,\hat{G}$$

onto the representation ring of $\hat{G}$.

More general groups, and the Satake isomorphism

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Example for GL_2: the indicator functions for the double cosets of $\mathrm{diag}(\pi_v, 1)$ and $\mathrm{diag}(\pi_v, \pi_v)$ correspond (up to some scalar factors) to the standard representation of GL_2 and its determinant, respectively.

Geometric Satake

Let Bun_G classify G-torsors on X. Let us define one (big) Hecke correspondence

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{tikzcd}$$

classifying all modifications $\mathcal{E} \to \mathcal{E}'$ at $x \in X$. Then the fiber of h_2 over each $v \in |X|$ is $G(K_v)/G(\mathcal{O}_{K_v})$. This is the set of k-points of the *affine Grassmannian* $\mathrm{Gr}_G = LG/L^+G$, an ind-scheme.

Theorem (Geometric Satake, Mirkovic-Vilonen)

There is an equivalence $V \mapsto \mathcal{S}_V$ of symmetric monoidal categories between:

- *Representations of $\hat{G}$, and*
- *L^+G-equivariant perverse sheaves on Gr_G.*

Geometric Satake

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- *Representations of $\hat{G}$, and*
- *L^+G-equivariant perverse sheaves on Gr_G.*

We might even think of this theorem as a *canonical model* for $\hat{G}$: The symmetric monoidal category of L^+G-equivariant perverse sheaves on Gr_G determines under the Tannakian formalism an algebraic group, which is exactly our $\hat{G}$.

The geometric Langlands conjecture

Let X be a curve, let G be a split reductive group, and consider

$$\begin{tikzcd}[column sep=small] & \text{Hecke} \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\ \mathrm{Bun}_G & & X \times \mathrm{Bun}_G \end{tikzcd}$$

Conjecture

Let φ be an irreducible $\hat{G}$-local system on X (that is, $\varphi\colon \pi_1(X) \to \hat{G}$). There exists a nonzero object A_φ of $D(\mathrm{Bun}_G)$ satisfying the eigensheaf property:

$$Rh_{2*}(h_1^* A_\varphi \otimes \mathcal{S}_V) \cong (r \circ \varphi) \boxtimes A_\varphi$$

for each representation $r\colon \hat{G} \to \mathrm{Aut}\, V$.

Back to the p-adic world

How to carry these ideas from a curve X to a p-adic field F?

We could perhaps set $X =$ "Spa $\breve{F}/\phi$", where $\breve{F}$ is the completion of the max unramified extension of F, and ϕ is the Frobenius. The idea here is that representations of $\pi_1(X) = W_F$ appear in local Langlands (rather than $\mathrm{Gal}(\overline{F}/F)$).

Vector bundles on this X would be $\breve{F}$-vector spaces equipped with a Frobenius-linear automorphism; ie, *isocrystals*. Already some of these (those with slopes in $[0, 1]$) correspond to p-divisible $\mathcal{O}_F$-modules over $\overline{k}$ (k=residue field of F).

The object $X = \mathrm{Spa}\,\breve{F}/\phi$

We want to define a stack Bun_G which associates to a test object S the groupoid of G-torsors on $X \times S$. But what sort of object is our S, and what does $X \times S$ mean?

Answer: S is drawn from the category Perf of *perfectoid spaces over* $\overline{\mathbf{F}}_q$.

When $F = \mathbf{F}_q((t))$, we can have $S = \mathrm{Spa}(R, R^+)$ be an adic space over $\overline{\mathbf{F}}_q$, and then $X \times S$ gets interpreted literally: it the quotient of the open disc over S modulo the Frobenius automorphism of S (cf. Hartl-Pink).

When F is p-adic, "$X \times S$" no longer makes literal sense as a fiber product. Instead, it must be interpreted as the *Fargues-Fontaine curve*.

The Fargues-Fontaine curve

Let F be a p-adic field with uniformizer π, and let $S = \mathrm{Spa}(R, R^+)$ be a perfectoid space over $\overline{\mathbf{F}}_q$, with pseudo-uniformizer ϖ. Let

$$\text{“}\mathrm{Spa}\, R^+ \times \mathrm{Spa}\, \mathcal{O}_F\text{''} := \mathrm{Spa}\, W_{\mathcal{O}_F}(R^+)$$

and let

$$Y_S = \text{“}S \times \mathrm{Spa}\, F\text{''} \subset \text{“}\mathrm{Spa}\, R^+ \times \mathrm{Spa}\, \mathcal{O}_F\text{''}$$

be the open subset defined by $|\pi[\varpi]| \neq 0$. The *Fargues-Fontaine curve* is the adic space

$$X_S = Y_S/\phi_S$$

where ϕ_S is the Frobenius automorphism on S.

Warning: *There is no morphism $X_S \to S$.*

The absolute Fargues-Fontaine curve

$$X_S = \text{“}S \times \mathrm{Spa}\, F\text{”}/\varphi_S$$

When $S = \mathrm{Spa}\, C$ for C algebraically closed, the curve $X_S = X_C$ has a schematic counterpart X_C^{sch} which (despite not being finite type over a field) is very nice: X_C^{sch} less one point is the spectrum of a PIDs, and total degrees of meromorphic functions are 0. There is a morphism $X_C \to X_C^{\mathrm{sch}}$ satisfying a GAGA theorem (Fargues).

Closed points of X_C^{sch} (="classical points of X_C") correspond to Frobenius-classes of untilts of C to $C^\sharp/F$.

Vector bundles on the absolute curve

$$X_C = \text{“}\mathrm{Spa}\, C \times \mathrm{Spa}\, F\text{”}/\varphi_S$$

Let (V, σ_V) be an F-isocrystal: this means a $\breve{F}$-vector space together with a Frobenius-linear automorphism σ_V.

Since $\mathcal{O}_{\breve{F}} \subset W_{\mathcal{O}_F}(\mathcal{O}_C)$, there is a morphism "Spa $C \times$ Spa F" $\to$ Spa $\breve{F}$, so we can use V to construct a (trivial) vector bundle on Y_C, which we can then descend to X_C using σ_V.
Thus we can talk about vector bundles $\mathcal{O}(q)$ for any $q \in \mathbb{Q}$.

Theorem (Fargues-Fontaine, Fargues)

Every vector bundle on X_C is isomorphic to a direct sum of $\mathcal{O}(q)$s. More generally, isomorphism classes of G-torsors on X_C are in bijection with the Kottwitz set

$$B(G) = G(\breve{F})/\sigma\text{-conjugacy}$$

The relative Fargues-Fontaine curve, and Bun_G

More generally, if $S \in \mathrm{Perf}$, we have the relative curve X_S. Degree 1 divisors on X_S correspond to Frobenius-classes of untilts of S over F.

Definition

Bun_G *is the stack assigning to* S *the groupoid of* G*-torsors on* X_S*.*

Theorem (FS)

Bun_G *has the following properties.*

1. Bun_G *is a smooth Artin v-stack of dimension 0.*
2. $|\mathrm{Bun}_G| \cong B(G)$. *(Viehmann $\implies$ homeomorphism.)*
3. *The semistable locus* $\mathrm{Bun}_G^{\mathrm{ss}}$ *is dense and open, and given by*

$$\mathrm{Bun}_G^{\mathrm{ss}} = \coprod_{b \in B(G)_{\mathrm{ss}}} [*/\underline{G_b(F)}].$$

Bun_G and local Langlands

Ultimately, [FS] uses the geometry of Bun_G to construct $\pi \mapsto \varphi_\pi$.

Leaving the construction itself to the later talks, we will now discuss two directions related to local Langlands:

1. A (very general) Kottwitz conjecture describing the cohomology of spaces generalizing the Rapoport-Zink spaces,
2. Fargues' conjecture, asserting the existence of an eigensheaf A_φ on Bun_G for each L-parameter φ.

Example with $F = \mathbb{Q}_p$ and GL_2

Every rank 2 vector bundle on X_C is either $\mathcal{O}(m) \oplus \mathcal{O}(n)$ or else $\mathcal{O}(k/2)$ for k odd.

There is one basic vector bundle in each degree: the $\mathcal{O}(m)^{\oplus 2}$ and the $\mathcal{O}(k/2)$. Their automorphism groups are $\mathrm{GL}_2(\mathbb{Q}_p)$ and $D^\times$ respectively. $D =$quaternion algebra).

Thus the degree 0 part of Bun_G has dense open $[*/\underline{\mathrm{GL}_2(F)}]$ corresponding to the trivial vector bundle, $\mathcal{O}^{\oplus 2}$ which can degenerate to $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, $\mathcal{O}(2) \oplus \mathcal{O}(-2)$, etc.

The degree 1 part of Bun_G has dense open subset $[*/\underline{D^\times}]$ corresponding to $\mathcal{O}(1/2)$, which can degenerate to $\mathcal{O}(1) \oplus \mathcal{O}$, etc.

Contact with p-divisible groups

Continuing with this example, for $S = \mathrm{Spa}(R, R^+)$ we have

$$H^0(X_S, \mathcal{O}(1/2)) \cong B^+_{\mathrm{cris}}(R)^{\phi^2=p} \cong \tilde{H}_0(R^\circ)$$

where $H_0/\overline{\mathbf{F}}_q$ is the formal group of height $1/2$ and dimension 1 (Fontaine), and $\tilde{H}_0$ means $\varprojlim_p H_0$.

If $S^\sharp$ is an untilt of S, corresponding to a degree 1 divisor $D_{S^\sharp} \subset X_S$, and H is a deformation of H_0 to $R^{\sharp\circ}$ then we get an isomorphism

$$\tilde{H}(R^{\sharp\circ}) \cong \tilde{H}_0(R^\circ).$$

The logarithm exact sequence

$$0 \to V_pH \to \tilde{H}(R^{\sharp\circ}) \stackrel{\log}{\to} R^\sharp \to 0$$

is the result of applying H^0 to

$$0 \to V_pH \otimes \mathcal{O} \to \mathcal{O}(1/2) \to i_{D_{S^\sharp}*}\mathcal{O}_{S^\sharp} \to 0.$$

Contact with p-divisible groups

So each time we have a deformation of H_0 to H, we get an exact sequence

$$0 \to V_p H \otimes \mathcal{O} \to \mathcal{O}(1/2) \to i_{D_{S^\sharp}*}\mathcal{O}_{S^\sharp} \to 0.$$

Theorem (Scholze-W.)

Let $H_0/\overline{\mathbf{F}}_q$ be a p-divisible group, and let $\mathcal{O}(H_0)$ be the vector bundle associated to its isocrystal. Given a perfectoid space S in characteristic 0, the following categories are equivalent:

- *Deformations of H_0 to S up to isogeny,*
- *Modifications on $X_{S^\flat}$ at the divisor D_S:*

$$0 \to \mathcal{T} \to \mathcal{O}(H_0) \to i_{D_S*}\mathrm{Lie}(H_0) \to 0$$

where $\mathcal{T}$ is semistable of slope 0 (ie, trivial over every geometric point).

Generalization to spaces of shtukas

The Hecke correspondence in the context of Bun_G is:

$$\begin{tikzcd}
 & \text{Hecke} \arrow[dl, "h_1"'] \arrow[dr, "p \times h_2"] & \\
\mathrm{Bun}_G & & \mathrm{Div}^1 \times \mathrm{Bun}_G
\end{tikzcd}$$

It parametrizes modifications of G-torsors along a divisor on the curve; those are parametrized by $\mathrm{Div}^1 = \mathrm{Spd}\, \breve{F}/\varphi$.

Let $b \in G(F)_{\mathrm{bas}}$. Let $\mathrm{Sht}_{G,b} \to \mathrm{Div}^1$ be the pullback through $h_1 \times h_2$ of the map $* \to \mathrm{Bun}_G \times \mathrm{Bun}_G$ corresponding to $(\mathcal{E}_1, \mathcal{E}_b)$.

Thus if $S/\breve{F}$ is a perfectoid space, then $\mathrm{Sht}_{G,b}(S)$ parametrizes modifications $\mathcal{E}_1 \dashrightarrow \mathcal{E}_b$ along D_S. Then $\mathrm{Sht}_{G,b}$ has an action of $G(F) \times G_b(F)$.

The Kottwitz conjecture for shtuka spaces

The cohomology of $\mathrm{Sht}_{G,b}$ suggests a means of transfering representations from $G_b(F)$ to $G(F) \times W_F$.

Let $V \in \mathrm{Rep}\,\hat{G}$, and let $\mathcal{S}_V$ be the corresponding $\mathbb{Z}_\ell$-sheaf on $\mathrm{Sht}_{G,b}$: this is equivariant for all actions.

Definition

Let ρ be a smooth irreducible representation of $G_b(F)$ with coefficients in $\overline{\mathbb{Q}}_\ell$. Define

$$R\Gamma(G,b,V)[\rho] = \varinjlim_{K \subset G(F)} R\mathrm{Hom}_{G_b(F)}(R\Gamma_c(\mathrm{Sht}_{G,b}/K, \mathcal{S}_V) \otimes \overline{\mathbb{Q}}_\ell, \rho),$$

a derived representation of $G(F)$ carrying an action of W_F.

Theorem (FS)

$R\Gamma(G,b,V)[\rho]$ is admissible as a $G(F)$-module.

The Kottwitz conjecture for shtuka spaces

Given G, b, V and a smooth irreducible representation ρ of $G_b(F)$, we have constructed a $G(F)$-admissible representation $R\Gamma(G, b, V)[\rho]$ carrying an action of W_F. How does this interact with the (conjectural) Langlands correspondence?

Conjecture

Assume the local Langlands conjecture (as Tasho will describe it). Suppose ρ has supercuspidal L-parameter φ. Let S_φ be the centralizer of φ in $\hat{G}$. As classes in the Grothendieck group of $G(F) \times W_F$, we have

$$R\Gamma(G, b, V)[\rho] = \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_V \circ \varphi).$$

Here $\Pi_\varphi(G)$ is the L-packet of representations of G belonging to φ, and $\delta_{\pi,\rho}$ is the relative position.

The Kottwitz conjecture for shtuka spaces

Conjecture

For ρ in a supercuspidal L-packet:

$$R\Gamma(G, b, V)[\rho] = \sum_{\pi \in \Pi_\varphi(G)} \pi \boxtimes \mathrm{Hom}_{S_\varphi}(\delta_{\pi,\rho}, r_V \circ \varphi).$$

Theorem (Hansen-Kaletha-W.)

Assume the local Langlands conjecture. The equality in the conjecture is true, up to ignoring the W_F-action.

The proof will be explained in the 3rd and 4th talks. It reduces everything to a Lefschetz trace formula.

$D(\mathrm{Bun}_G)$ and Fargues' conjecture

Fargues' original article on geometrization put forward a conjecture about Bun_G which is exactly in line with geometric Langlands:

Conjecture (Fargues)

(Choose a Whittaker datum.) Let φ be a discrete L-parameter. There exists an object $A_\varphi \in D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ carrying an action of S_φ (=centralizer of φ in $\hat{G}$) which is a Hecke eigensheaf with eigenvalue φ. The restriction of A_φ to $[*/\underline{G_b(F)}]$ decomposes as a direct sum of π_b over the L-packet $\Pi_{G_b}(\varphi)$.

Stating this conjecture precisely requires defining $D(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ correctly, which is highly nontrivial (and requires condensed mathematics).

Actually [FS] constructs a candidate for A_φ using the *spectral action*, but it is not clear that $A_\varphi \neq 0$!

$D(\mathrm{Bun}_G)$ and Fargues' conjecture

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Theorem (Anschütz-le Bras)

The conjecture is true for $G = \mathrm{GL}_n$ and irreducible φ.