CRYSTALS VIA THE AFFINE GRASSMANNIAN

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Abstract. Let G be a connected reductive group over $\mathbb{C}$ and let $\mathfrak{g}^\vee$ be the Langlands dual Lie algebra. Crystals for $\mathfrak{g}^\vee$ are combinatorial objects, that were introduced by Kashiwara (cf. for example [5]) as certain "combinatorial skeletons" of finite-dimensional representations of $\mathfrak{g}^\vee$. For every dominant weight λ of $\mathfrak{g}^\vee$ Kashiwara constructed a crystal $\mathbf{B}(\lambda)$ by considering the corresponding finite-dimensional representation of the quantum group $U_q(\mathfrak{g}^\vee)$ and then specializing it to $q = 0$. Other (independent) constructions of $\mathbf{B}(\lambda)$ were given by Lusztig (cf. [8]) using the combinatorics of root systems and by Littelmann (cf. [6]) using the "Littelmann path model". It was also shown in [4] that the family of crystals $\mathbf{B}(\lambda)$ is unique if certain reasonable conditions are imposed (cf. Theorem 1.1).

The purpose of this paper is to give another (rather simple) construction of the crystals $\mathbf{B}(\lambda)$ using the geometry of the *affine grassmannian* $\mathcal{G}_G = G(\mathcal{K})/G(\mathcal{O})$ of the group G, where $\mathcal{K} = \mathbb{C}((t))$ is the field of Laurent power series and $\mathcal{O} = \mathbb{C}[[t]]$ is the ring of Taylor series. We then check that the family $\mathbf{B}(\lambda)$ satisfies the conditions of the uniqueness theorem from [4], which shows that our crystals coincide with those constructed in *loc. cit.* It would be interesting to find these isomorphisms directly (cf., however, [9]).

1. Basic results about crystals

1.1. **Notation.** Let G be a connected reductive group over $\mathbb{C}$ and let $G^\vee$ be the Langlands dual group; let $\mathfrak{g}^\vee$ denote the Lie algebra of $G^\vee$. Let also $\mathrm{Rep}(G^\vee)$ denote the category of finite-dimensional representations of the group $G^\vee$.

Let Λ_G denote the *coweight* lattice of G, which is the same as the weight lattice of $G^\vee$. Let $\Lambda_G^\vee$ denote the dual lattice, i.e. $\Lambda_G^\vee$ is the *weight* lattice of G; let $\langle , \rangle$ be the canonical pairing between Λ_G and $\Lambda_G^\vee$. We will denote by Λ_G^+ the semi-group of dominant coweights. Let I denote the set of vertices of the Dynkin diagram corresponding to G. For $i \in I$ we will denote by $\alpha_i \in \Lambda_G$ the corresponding simple coroot and by $\alpha_i^\vee \in \Lambda_G^\vee$ the corresponding simple root. Let $2\rho_G^\vee \in \Lambda_G^\vee$ be the sum of all positive roots of G. For $\lambda_1, \lambda_2 \in \Lambda_G$, we will write $\lambda_1 \underset{G}{\geq} \lambda_2$ if $\lambda_1 - \lambda_2$ is a linear combination of the α_i with non-negative coefficients.

Let E_i, F_i (for $i \in I$) denote the Chevalley generators of $\mathfrak{g}^\vee$. For every $\lambda \in \Lambda_G^+$ we will denote by $V(\lambda)$ the irreducible representation of $\mathfrak{g}^\vee$ with highest weight λ and for $\mu \in \Lambda_G$, $V(\lambda)_\mu$ will denote the corresponding weight subspace of $V(\lambda)$.

1.2. **Definition.** A crystal is a set $\mathbf{B}$ together with maps

1. $wt : \mathbf{B} \to \Lambda_G,\ \varepsilon_i, \phi_i : \mathbf{B} \to \mathbb{Z}$,
2. $e_i, f_i : \mathbf{B} \to \mathbf{B} \cup \{0\}$,

for each $i \in I$, satisfying the following axioms:

A) For any $\mathbf{b} \in \mathbf{B}$ one has $\phi_i(\mathbf{b}) = \varepsilon_i(\mathbf{b}) + \langle wt(\mathbf{b}), \alpha_i^\vee \rangle$

B) Let $\mathbf{b} \in \mathbf{B}$. If $e_i \cdot \mathbf{b} \in \mathbf{B}$ for some i. Then

$$wt(e_i \cdot \mathbf{b}) = wt(\mathbf{b}) + \alpha_i,\ \varepsilon_i(e_i \cdot \mathbf{b}) = \varepsilon_i(\mathbf{b}) - 1,\ \phi_i(e_i \cdot b) = \phi_i(b) + 1.$$

If $f_i \cdot \mathbf{b} \in \mathbf{B}$ for some i then

$$wt(f_i \cdot \mathbf{b}) = wt(\mathbf{b}) - \alpha_i,\ \varepsilon_i(f_i \cdot \mathbf{b}) = \varepsilon_i(\mathbf{b}) + 1,\ \phi_i(f_i \cdot \mathbf{b}) = \phi_i(\mathbf{b}) - 1.$$

C) For all $\mathbf{b}, \mathbf{b}' \in \mathbf{B}$ one has $\mathbf{b}' = e_i \cdot \mathbf{b}$ if an only if $\mathbf{b} = f_i \cdot \mathbf{b}'$.

Remark. In [4] a more general definition of crystals is considered, where the maps ε_i and ϕ_i are allowed to assume infinite values. However, such crystals will never appear in this paper.

A crystal is called *normal* if one has

$$\varepsilon_i(\mathbf{b}) = \max\{n|\ e_i^n \cdot \mathbf{b} \neq 0\}, \quad \phi_i(\mathbf{b}) = \max\{n|\ f_i^n \cdot \mathbf{b} \neq 0\} \tag{1.1}$$

From now on we will consider only normal crystals. Thus, the maps ε_i and ϕ_i will be uniquely recovered from wt, e_i and f_i.

1.3. **Tensor product of crystals.** Let $\mathbf{B}_1$ and $\mathbf{B}_2$ be two crystals. Following Kashiwara ([5]) we define their tensor product $\mathbf{B}_1 \otimes \mathbf{B}_2$ as follows. As a set $\mathbf{B}_1 \otimes \mathbf{B}_2$ is just equal to $\mathbf{B}_1 \times \mathbf{B}_2$. The corresponding maps are defined in the following way. Let $\mathbf{b}_1 \in \mathbf{B}_1, \mathbf{b}_2 \in \mathbf{B}_2$. We will denote by $\mathbf{b}_1 \otimes \mathbf{b}_2$ be the corresponding element in $\mathbf{B}_1 \times \mathbf{B}_2$. Then we set

$$wt(\mathbf{b}_1 \otimes \mathbf{b}_2) = wt(\mathbf{b}_1) + wt(\mathbf{b}_2),$$

$$e_i \cdot (\mathbf{b}_1 \otimes \mathbf{b}_2) = \begin{cases} e_i \cdot \mathbf{b}_1 \otimes \mathbf{b}_2, & \text{if} \quad \varepsilon_i(\mathbf{b}_1) > \phi_i(\mathbf{b}_2) \\ \mathbf{b}_1 \otimes e_i \cdot \mathbf{b}_2, & \text{otherwise} \end{cases}$$

$$f_i \cdot (\mathbf{b}_1 \otimes \mathbf{b}_2) = \begin{cases} f_i \cdot \mathbf{b}_1 \otimes \mathbf{b}_2, & \text{if} \quad \varepsilon_i(\mathbf{b}_1) \geq \phi_i(\mathbf{b}_2) \\ \mathbf{b}_1 \otimes f_i \cdot \mathbf{b}_2, & \text{otherwise} \end{cases}$$

$$\varepsilon_i(\mathbf{b}_1 \otimes \mathbf{b}_2) = \max\{\varepsilon_i(\mathbf{b}_2), \varepsilon_i(\mathbf{b}_1) - \phi_i(\mathbf{b}_2) + \varepsilon_i(\mathbf{b}_2)\}$$

$$\phi_i(\mathbf{b}_1 \otimes \mathbf{b}_2) = \max\{\phi_i(\mathbf{b}_1), \phi_i(\mathbf{b}_2) - \varepsilon_i(\mathbf{b}_1) + \phi_i(\mathbf{b}_1)\}.$$

It is known (cf. [4]) that $\mathbf{B}_1 \otimes \mathbf{B}_2$ is crystal and that $\otimes$ is an associative operation on crystals. Moreover, if $\mathbf{B}_1$ and $\mathbf{B}_2$ are normal then $\mathbf{B}_1 \otimes \mathbf{B}_2$ is normal as well.

1.4. **Highest weight crystals.** Let $\mathbf{B}$ be a crystal. We say that $\mathbf{B}$ is *a highest weight crystal of weight* $\lambda \in \Lambda_G$ if there exists an element $\mathbf{b}_\lambda \in \mathbf{B}$, such that

1. $wt(\mathbf{b}_\lambda) = \lambda$.
2. $e_i \cdot \mathbf{b}_\lambda = 0$ for every $i \in I$.
3. $\mathbf{B}$ is generated by all the f_i acting on $\mathbf{b}_\lambda$.

It is clear from (1.1) that if $\mathbf{B}$ is a normal crystal, then one necessarily has $\lambda \in \Lambda_G^+$. The following lemma gives a useful reformulation of the definition of a highest weight crystal.

Lemma 1.1. *A crystal* $\mathbf{B}$ *is a highest weight crystal of highest weight* λ *if and only if there exists an element* $\mathbf{b}_\lambda \in \mathbf{B}$*, such that*

1. $wt(\mathbf{b}_\lambda) = \lambda$ *and* $wt(\mathbf{b}) < \lambda$ *for every* $\mathbf{b} \in \mathbf{B} - \mathbf{b}_\lambda$.
2. $e_i \cdot \mathbf{b}_\lambda = 0$ *for every* $i \in I$.
3. *For every* $\mathbf{b} \in \mathbf{B} - \mathbf{b}_\lambda$ *there exists* $i \in I$ *such that* $e_i \cdot \mathbf{b} \neq 0$.

1.5. **Closed families of crystals.** Assume that for every $\lambda \in \Lambda_G^+$ we are given a normal crystal $\mathbf{B}(\lambda)$ of highest weight λ. We say that the $\mathbf{B}(\lambda)$ form a closed family of crystals if for every $\lambda, \mu \in \Lambda_G^+$ there exists an embedding $\mathbf{B}(\lambda+\mu) \hookrightarrow \mathbf{B}(\lambda) \otimes \mathbf{B}(\mu)$ (which necessarily sends $\mathbf{b}_{\lambda+\mu}$ to $\mathbf{b}_\lambda \otimes \mathbf{b}_\mu$).

Theorem 1.1. *(cf.* [4]*, 6.4.21) Assume that G is of adjoint type. Then there exists a unique closed family of crystals $\mathbf{B}(\lambda)$.*

Different constructions of closed families of crystals were given by Kashiwara ([5]) using quantum groups and by Lusztig ([8]) and Littelmann ([6]) using the combinatorics of the root systems. The main goal of this paper is to give another construction of the closed family $\mathbf{B}(\lambda)$, using the geometry of the affine Grassmannian.

2. Basic results about affine Grassmannian

2.1. **Definition.** Let $\mathcal{K} = \mathbb{C}((t))$, $\mathcal{O} = \mathbb{C}[[t]]$. By the *affine Grassmannian* of G we will mean the quotient $\mathcal{G}_G = G(\mathcal{K})/G(\mathcal{O})$. It is known (cf. [1]) that $\mathcal{G}_G$ is the set of $\mathbb{C}$-points of an ind-scheme over $\mathbb{C}$, which we will denote by the same symbol.

The orbits of the group $G(\mathcal{O})$ on $\mathcal{G}_G$ can be described as follows. One can identify the lattice Λ_G with the quotient $T(\mathcal{K})/T(\mathcal{O})$. Fix $\lambda \in \Lambda_G^+$ and let $\lambda(t)$ denote any lift of λ to $T(\mathcal{K})$. Let $\mathcal{G}_G^\lambda$ denote the $G(\mathcal{O})$-orbit of $\lambda(t)$ (which clearly does not depend on the choice of $\lambda(t)$). Then it is well-known (cf. [7]) that

$$\mathcal{G}_G = \bigsqcup_{\lambda \in \Lambda_G^+} \mathcal{G}_G^\lambda.$$

Moreover, for every $\lambda \in \Lambda_G^+$ the orbit $\mathcal{G}_G^\lambda$ is finite-dimensional and its dimension is equal to $\langle \lambda, 2\rho_G^\vee \rangle$.

Let $\overline{\mathcal{G}_G}^\lambda$ denote the closure of $\mathcal{G}_G^\lambda$ in $\mathcal{G}_G$; this is an irreducible projective algebraic variety. We will denote by IC^λ the intersection cohomology complex on $\overline{\mathcal{G}_G}^\lambda$. Let $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G)$ denote the category of $G(\mathcal{O})$-equivariant perverse sheaves on $\mathcal{G}_G$. It is known that every object of this category is a direct sum of the IC^λ.

2.2. **The convolution.** Define the ind-scheme $\mathcal{G}_G \star \mathcal{G}_G$ to be $G(\mathcal{K}) \underset{G(\mathcal{O})}{\times} \mathcal{G}_G$. Let

$$\pi : G(\mathcal{K}) \times \mathcal{G}_G \to \mathcal{G}_G \star \mathcal{G}_G$$

denote the natural projection. One has the natural maps $p_1, p_2 : G(\mathcal{K}) \times \mathcal{G}_G \to \mathcal{G}_G$ and $m : \mathcal{G}_G \star \mathcal{G}_G \to \mathcal{G}_G$ defined as follows. Let $g \in G(\mathcal{K}), x \in \mathcal{G}_G$. Then

$$p_1(g,x) = g \bmod G(\mathcal{O}); \quad p_2(g,x) = x; \quad m(g,x) = g \cdot x.$$

For $\lambda_1, \lambda_2 \in \Lambda_G^+$ we set $\mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2} = \pi(p_1^{-1}(\mathcal{G}_G^{\lambda_1}) \cap p_2^{-1}(\mathcal{G}_G^{\lambda_2}))$. In addition, we define

$$(\mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2})^{\lambda_3} = m^{-1}(\mathcal{G}_G^{\lambda_3}) \cap \mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2}$$

It is known (cf. [7]) that

$$\dim((\mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2})^{\lambda_3}) = \langle \lambda_1 + \lambda_2 + \lambda_3, \rho_G^\vee \rangle. \tag{2.1}$$

(It is easy to see that although $\rho_G^\vee \in \frac{1}{2}\Lambda_G^\vee$, the RHS of (2.1) is an integer whenever the above intersection is non-empty.)

For any $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G)$ we define the convolution $\mathcal{S}_1 \star \mathcal{S}_2$ as follows. Consider $p_1^*\mathcal{S}_1 \otimes p_2^*\mathcal{S}_2$. Then due to the fact that $\mathcal{S}_1$ is $G(\mathcal{O})$-equivariant, there exists a canonical perverse sheaf $\mathcal{S}_1 \widetilde{\otimes} \mathcal{S}_2$ on $\mathcal{G}_G \star \mathcal{G}_G$ such that $\pi^*(\mathcal{S}_1 \widetilde{\otimes} \mathcal{S}_2) \simeq p_1^*\mathcal{S}_1 \otimes p_2^*\mathcal{S}_2$.

We define
$$\mathcal{S}_1 \star \mathcal{S}_2 = m_!(\mathcal{S}_1 \widetilde{\otimes} \mathcal{S}_2).$$

Theorem 2.1. *(cf.* [7],[3] *and* [10]*)*

1. *Let* $\mathcal{S}_1, \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G)$. *Then* $\mathcal{S}_1 \star \mathcal{S}_2 \in \mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G)$.
2. *The convolution* $\star$ *extends to a structure of a tensor category on* $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G)$*, which is equivalent to the category Rep*$(G^\vee)$.

2.3. **Restriction functors to Levi subgroups.** Let P be a Borel subgroup in G and let N_P be its unipotent radical. Let $M = P/N_P$ be the corresponding Levi factor. Let $P^\vee$ and $M^\vee$ be the corresponding parabolic and Levi subgroups of $G^\vee$. We have the restriction functor $\mathrm{Res}^{G^\vee}_{M^\vee} : \mathrm{Rep}(G^\vee) \to \mathrm{Rep}(M^\vee)$. Let us explain how to represent this functor geometrically, i.e. as a functor $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G) \to \mathrm{Perv}_{M(\mathcal{O})}(\mathcal{G}_M)$.

Let $\Lambda_{G,P}$ denote the lattice of characters of the torus $Z(M^\vee)$ (the center of $M^\vee$). There is a natural surjection $\alpha_{G,P} : \Lambda_G \to \Lambda_{G,P}$. One can identify $\Lambda_{G,P}$ with the set of connected components of $\mathcal{G}_M$.

One can also identify $\Lambda_{G,P}$ with the set of orbits of the group $[P,P](\mathcal{K}) \cdot M(\mathcal{O})$ on $\mathcal{G}_G$. This is done in the following way. Let $\theta \in \Lambda_{G,P}$. Fix a lift $\widetilde{\theta}$ of θ to Λ_G. Let S^θ_P denote the $[P,P](\mathcal{K}) \cdot M(\mathcal{O})$-orbit of the element $\widetilde{\theta}(t) \in T(\mathcal{K})$ (cf. Sect. 2.1). It is easy to see that S^θ_P depends only on θ (and not on the choice of $\widetilde{\theta}(t)$).

Lemma 2.1. *The following hold:*

1. *One has* $\mathcal{G}_G = \bigsqcup\limits_{\theta \in \Lambda_{G,P}} S^\theta_P$.
2. *Let* $\mathcal{G}^\theta_M$ *denote the connected component of* $\mathcal{G}_M$ *corresponding to* θ. *Then there exists a canonical* $[P,P](\mathcal{K}) \cdot M(\mathcal{O})$*–equivariant map* $\mathfrak{t}^\theta_P : S^\theta_P \to \mathcal{G}^\theta_M$ *which is equal to identity on the set*
$$\{\nu \in \Lambda_G = T(\mathcal{K})/T(\mathcal{O}) |\ \alpha_{G,P}(\nu) = \theta\}.$$
(Note that this set is naturally embedded into both S^θ_P *and* $\mathcal{G}^\theta_M$ *due to the fact that* T *is embedded in both* G *and* M*).*

Let $\nu \in \Lambda^+_M \subset \Lambda_G$ and let $\theta = \alpha_{G,P}(\nu)$. Let us denote by S^ν_P the pre-image $(\mathfrak{t}^\theta_P)^{-1}(\mathcal{G}^\nu_M) \subset S^\theta_P$. The schemes S^ν_P are nothing but orbits of the group $N_P(\mathcal{K}) \cdot M(\mathcal{O})$ on $\mathcal{G}_G$. We will denote by $\mathfrak{t}^\nu_P$ the restriction of $\mathfrak{t}^\theta_P$ to S^ν_P.

Theorem 2.2. *(*[1]*, cf. also* [2] *and* [10]*)*

1. *Let* ν *(resp.,* λ*) be a dominant integral coweight of* M *(resp., of* G*) Then the intersection* $S^\nu_P \cap \mathcal{G}^\lambda_G$ *has dimension* $\leq \langle \nu + \lambda, \rho^\vee_G \rangle$ *and hence the fibers of the projection*
$$\mathfrak{t}^\nu_P : S^\nu_P \cap \mathcal{G}^\lambda_G \to \mathcal{G}^\nu_M$$
are of dimension $\leq \langle \nu + \lambda, \rho^\vee_G \rangle - \langle \nu, 2\rho^\vee_M \rangle$.
2. *Let* $\mathrm{IC}^\lambda|_{S^\theta_P}$ *denote the* $*$*-restriction of* IC^λ *to* S^θ_P. *Then for* $\lambda \in \Lambda^+_G$ *and* $\theta \in \Lambda_{G,P}$*, the direct image*
$$\mathfrak{t}^\theta_{P!}(\mathrm{IC}^\lambda|_{S^\theta_P})[\langle \theta, 2(\rho^\vee_G - \rho^\vee_M)\rangle]$$
lives in the cohomological degrees ≤ 0 *(in the perverse t-structure). (In the above formula we have used the fact that* $2(\rho^\vee_G - \rho^\vee_M)$ *naturally belongs to the dual lattice of* $\Lambda_{G,P}$*.)*
3. *The functor* $\mathrm{Perv}_{G(\mathcal{O})}(\mathcal{G}_G) \to \mathrm{Perv}_{M(\mathcal{O})}(\mathcal{G}_M)$ *given by*
$$\mathcal{S} \mapsto \bigoplus_\theta H^0(\mathfrak{t}^\theta_{P!}(\mathcal{S}|_{S^\theta_P})[\langle \theta, 2(\rho^\vee_G - \rho^\vee_M)\rangle]$$

has a structure of a tensor functor and under the equivalence of Theorem 2.1 it is naturally isomorphic to Res^G_M.

If B is a Borel subgroup of G then one has $\Lambda_G = \Lambda_{G,P}$. In this case for every $\mu \in \Lambda_G$ we will write S^μ instead of S^μ_B. It is clear that for any parabolic P, S^μ lies inside $S_P^{\alpha_{G,P}(\mu)}$.

3. The construction of $\mathbf{B}^G(\lambda)$

In this section we will state our two main theorems. Their proofs will be given in the next two sections.

3.1. **The set $\mathbf{B}^G(\lambda)$.** Let M be as in Sect. 2.3. For $\lambda \in \Lambda^+_G$ and $\nu \in \Lambda^+_M$ we let $\mathbf{B}^G_M(\lambda)_\nu$ denote the set of irreducible components of the intersection $S^\nu_P \cap \mathcal{G}^\lambda_G$ of dimension $\langle \nu+\lambda, \rho^\vee_G\rangle$. Since the variety $\mathcal{G}^\nu_M$ is connected and simply connected, it follows, that $\mathbf{B}^G_M(\lambda)_\nu$ can also be identified with the set of irreducible components of any fiber of the map $\mathfrak{t}^\nu_P : S^\nu_P \cap \mathcal{G}^\lambda_G \to \mathcal{G}^\nu_M$ of dimension $\langle \nu+\lambda, \rho^\vee_G\rangle - \langle \nu, 2\rho^\vee_M\rangle$.

For $\mu \in \Lambda_G$ we will denote $\mathbf{B}^G_T(\lambda)_\mu$ just by $\mathbf{B}^G(\lambda)_\mu$ and we set

$$\mathbf{B}^G(\lambda) := \bigcup_{\mu\in\Lambda_G} \mathbf{B}^G(\lambda)_\mu.$$

Thus, $\mathbf{B}^G(\lambda)$ is a finite set, endowed with a map $wt : \mathbf{B}^G(\lambda) \to \Lambda_G$ (by definition, $wt(\mathbf{b}) = \mu$ for $\mathbf{b} \in \mathbf{B}^G(\lambda)_\mu$).

3.2. **Decomposition with respect to a parabolic.** We would like now to extend the map $wt : \mathbf{B}^G(\lambda) \to \Lambda_G$ to a structure of a normal crystal on $\mathbf{B}^G(\lambda)$, i.e. we need to define the operations e_i and f_i.

Let P be any parabolic subgroup in G.

Proposition 3.1. *For every* $\lambda \in \Lambda^+_G, \mu \in \Lambda_G$ *there is a canonical bijection*

$$\mathbf{d}^G_M : \bigsqcup_{\nu\in\Lambda^+_M} \mathbf{B}^G_M(\lambda)_\nu \times \mathbf{B}^M(\nu)_\mu \simeq \mathbf{B}^G(\lambda)_\mu.$$

This bijection can be uniquely characterized as follows: one has $\mathbf{d}(\mathbf{b}_1, \mathbf{b}_2) = \mathbf{b}$ *for* $\mathbf{b}_1 \in \mathbf{B}^G_M(\lambda)_\nu, \mathbf{b}_2 \in \mathbf{B}^M(\nu)_\mu$ *if and only if the following conditions hold.*

1. $\theta := \alpha_{G,P}(\mu) = \alpha_{G,P}(\nu)$.
2. $\mathbf{b}_2$ *is a dense subset of* $\mathfrak{t}^\theta_P(\mathbf{b})$.
3. $(\mathfrak{t}^\nu_P)^{-1}(\mathbf{b}_2) \cap \mathbf{b}_1$ *is a dense subset of* $\mathbf{b}$.

Proof. For $\mathbf{b}_2 \in \mathbf{B}^M(\nu)_\mu$ consider the variety $(\mathfrak{t}^\nu_P)^{-1}(\mathbf{b}_2) \cap \mathcal{G}^\lambda_G \subset S^\mu \cap \mathcal{G}^\lambda_G$. It follows from Sect. 3.1 that the set of its irreducible components of dimension $\langle \mu+\lambda, \rho^\vee_G\rangle$ is in a bijection with $\mathbf{B}^G_M(\lambda)_\nu$.

Thus, for $\mathbf{b}_1 \in \mathbf{B}^G_M(\lambda)_\nu$, we set $\mathbf{d}^G_M(\mathbf{b}_1 \times \mathbf{b}_2)$ to be the closure in $S^\mu \cap \mathcal{G}^\lambda_G$ of the corresponding irreducible component of $(\mathfrak{t}^\nu_P)^{-1}(\mathbf{b}_2) \cap \mathcal{G}^\lambda_G$.

The fact that this map is a bijection satisfying all the required properties is straightforward. □

3.3. **Operations e_i and f_i.** Fix now any $i \in I$. Let P_i be the corresponding "sub-minimal" parabolic subgroup of G (by definition, P_i is the parabolic subgroup of G, whose unipotent radical contains all simple roots except for $\alpha^\vee_i$). Let also M_i be the corresponding Levi factor and $\mathfrak{m}^\vee_i$ the dual Lie algebra.

Consider the decomposition of Proposition 3.1 for $M = M_i$. Since $\mathfrak{m}^\vee_i$ is a reductive Lie algebra, whose semi-simple part is isomorphic to $\mathbf{sl}(2)$, it follows from Theorem 2.2(3) and the representation theory of $\mathbf{sl}(2)$ that for every $\mathbf{b}_2 \in \mathbf{B}^{M_i}(\nu)_\mu$ there exists no more than one $\mathbf{b}_2' \in \mathbf{B}^{M_i}(\nu)_{\mu+\alpha_i}$ (resp. $\mathbf{b}_2'' \in \mathbf{B}^{M_i}(\nu)_{\mu-\alpha_i}$).

Let now $\mathbf{b} \in \mathbf{B}^G(\lambda)_\mu$. Assume that $\mathbf{b} = \mathbf{d}^G_M(\mathbf{b}_1 \times \mathbf{b}_2)$. Thus we define

$$e_i \cdot \mathbf{b} = \begin{cases} \mathbf{d}^G_M(\mathbf{b}_1 \times \mathbf{b}_2') & \text{if there exists } \mathbf{b}_2' \in \mathbf{B}^{M_i}(\nu)_{\mu+\alpha_i} \\ 0 \quad \text{otherwise} \end{cases}$$

and

$$f_i \cdot \mathbf{b} = \begin{cases} \mathbf{d}^G_M(\mathbf{b}_1 \times \mathbf{b}_2'') & \text{if there exists } \mathbf{b}_2'' \in \mathbf{B}^{M_i}(\nu)_{\mu-\alpha_i} \\ 0 \quad \text{otherwise} \end{cases}$$

Theorem 3.1. 1. *The maps e_i, f_i and wt define a structure of a normal crystal on $\mathbf{B}^G(\lambda)$.*
2. *The crystal $\mathbf{B}^G(\lambda)$ defined above is a highest weight crystal of highest weight λ.*
3. *The crystals $\mathbf{B}^G(\lambda)$ defined above form a closed family (in the sense of Sect. 1.5).*

The first point of this theorem follows readily from the representation theory of $\mathbf{sl}(2)$. The geometric content of the second point of Theorem 3.1 is summarized in the next corollary:

Let w_0 denote the longest element of the Weyl group of G and for $i \in I$ let $\mathfrak{s}_i$ be the corresponding simple reflection. Let λ, μ be a pair of elements of Λ_G with $\lambda \in \Lambda^+_G$. Let $\mathbf{b}$ be an irreducible component of dimension $\langle \lambda + \mu, \rho^\vee_G \rangle$ of $S^\mu \cap \mathcal{G}^\lambda_G$.

Corollary 3.1. *Assume that $\mu \neq \lambda$ (resp., $w_0(\mu) \neq \lambda$). Then one can find $i \in I$ and $\nu \in \Lambda^+_{M_i}$ with $\mu \neq \nu$ (resp., $\mathfrak{s}_i(\mu) \neq \nu$) such that the map $\mathfrak{t}^\nu_{P_i} : (\mathbf{b} \cap S^\nu_{P_i}) \to S^\mu_{M_i} \cap \mathcal{G}^\nu_{M_i}$ is dominant.*

Finally, we note that the third point of Theorem 3.1 combined with Theorem 1.1 implies that our crystals $\mathbf{B}^G(\lambda)$ are isomorphic to those constructed in [5], [8] and [6]. Indeed, when G is adjoint this is immediate and, in general, if G and G' are isogenous, the corresponding crystals $\mathbf{B}^G(\lambda)$ and $\mathbf{B}^{G'}(\lambda)$ are isomorphic for $\lambda \in \Lambda^+_G \cap \Lambda^+_{G'}$.

3.4. **Refinement.** Here we would like to refine the statement of Theorem 3.1(3). Namely, we want to describe the crystal $\mathbf{B}^G(\lambda_1) \otimes \mathbf{B}^G(\lambda_2)$ in geometric terms.

For $\lambda_1, \lambda_2, \lambda_3 \in \Lambda^+_G$ let $\mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3}$ be the set of all irreducible components of dimension $\langle \lambda_1 + \lambda_2 + \lambda_3, \rho^\vee_G \rangle$ of the variety $(\mathcal{G}^{\lambda_1}_G \star \mathcal{G}^{\lambda_2}_G)^{\lambda_3}$.

Theorem 3.2. *One has a canonical isomorphism of crystals*

$$\mathbf{B}^G(\lambda_1) \otimes \mathbf{B}^G(\lambda_2) = \bigsqcup_{\lambda_3 \in \Lambda^+_G} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}^G(\lambda_3)$$

where the crystal structure on the right hand side comes from the second multiple.

4. Proof of Theorem 3.1(2)

4.1. **Notation.** Let Z be a complex algebraic variety of dimension d and let $X \subset Z$ be a d–dimensional irreducible component of Z. Then we can define an element $\mathrm{v}(X) \in H^{2d}_c(Z, \mathbb{C})$ as follows. Let $Y_1, ..., Y_n$ be other irreducible components of Z and let

$$X^0 = X - \bigcup_{k=1}^n X \cap Y_k.$$

Denote by i the embedding of X^0 into Z. Consider the complex $i_!\mathbb{C}$ on Z. Then, one has a natural map $i_!\mathbb{C} \to \mathbb{C}$ of (complexes of) sheaves on Z and, therefore, a map

$$H^{2d}_c(Z, i_!\mathbb{C}) \to H^{2d}_c(Z, \mathbb{C}).$$

Now, since X^0 is irreducible, one has

$$H_c^{2d}(Z, i_!\mathbb{C}) = H_c^{2d}(X^0, \mathbb{C}) \simeq \mathbb{C}.$$

Thus, by composing the above two maps, we get an element $\mathrm{v}(X) \in H_c^{2d}(Z, \mathbb{C})$. Moreover, the collection of elements $\mathrm{v}(X)$ (for all irreducible components X of Z of the top dimension) is a basis of $H_c^{2d}(Z, \mathbb{C})$.

4.2. **The basis in $\mathrm{Hom}_M(U(\nu), V(\lambda))$.** Let as before M be a Levi subgroup of G. For $\nu \in \Lambda_M^+$ we will denote by $U(\nu)$ the irreducible representation of M with highest weight ν. We would like now to construct a basis in the vector space $\mathrm{Hom}_M(U(\nu), V(\lambda))$, parametrized by the set $\mathbf{B}_M^G(\lambda)_\nu$. This is done in the following way.

By Theorem 2.2 one can identify $\mathrm{Hom}_M(U_M(\nu), V(\lambda))$ with

$$H_c^{2(\langle \lambda+\nu, \rho_G^\vee\rangle - \langle \nu, 2\rho_M^\vee\rangle)}((\mathfrak{t}_P^\nu)^{-1}(x) \cap \mathcal{G}_G^\lambda, \mathbb{C})$$

for any $x \in \mathcal{G}_M^\nu$. Recall that $\mathbf{B}_M^G(\lambda)_\nu$ can be naturally identified with the set of irreducible components of $(\mathfrak{t}_P^\nu)^{-1}(x) \cap \mathcal{G}_G^\lambda$ of dimension $2(\langle \lambda + \nu, \rho_G^\vee\rangle - \langle \nu, 2\rho_M^\vee\rangle)$. Hence, the construction of Sect. 4.1 yields a basis $\mathrm{v}_M^G(\mathbf{b})$, $\mathbf{b} \in \mathbf{B}_M^G(\lambda)_\nu$ in $\mathrm{Hom}_M(U(\nu), V(\lambda))$.

4.3. **Compatibility of bases.** Fix a weight $\mu \in \Lambda_G$ and consider the vector space $V(\lambda)_\mu$. Fix also a parabolic subgroup P with a Levi subgroup M as before. Then from Sect. 4.2 one constructs two bases in $V(\lambda)_\mu$, parametrized by $\mathbf{B}^G(\lambda)_\mu$: the first one is $\{\mathrm{v}_T^G(\mathbf{b})\}_{\mathbf{b} \in B(\lambda)_\mu}$ and the other one is equal to

$$\bigsqcup_{\nu \in \Lambda_M^+} \{\mathrm{v}_M^G(\mathbf{b}_1) \otimes \mathrm{v}_T^M(\mathbf{b}_2) | \text{ for } \mathbf{b}_1 \in \mathbf{B}_M^G(\lambda)_\nu \text{ and } \mathbf{b}_2 \in \mathbf{B}^M(\nu)_\mu\}.$$

Let us now investigate the connection between these two bases. Let $F^\nu V(\lambda)$ denote the direct sum of all M-isotypic components of $V(\lambda)$ of the form $U_M(\nu')$, where $\nu' \underset{M}{\geq} \nu$. Set $G^\nu V(\lambda) = F^\nu V(\lambda) / \sum\limits_{\nu' \underset{M}{>} \nu} F^{\nu'} V(\lambda)$.

Proposition 4.1. *Let $\mathbf{b} \in \mathbf{B}^G(\lambda)_\mu$. Assume that $\mathbf{b} = \mathbf{d}_M^G(\mathbf{b}_1 \times \mathbf{b}_2)$ where $\mathbf{b}_1 \in \mathbf{B}_M^G(\lambda)_\nu$ and $\mathbf{b}_2 \in \mathbf{B}^M(\nu)_\mu$. Then*

1. $\mathrm{v}_T^G(\mathbf{b}) \in F^\nu V(\lambda)$.
2. *The images of $\mathrm{v}_T^G(\mathbf{b})$ and $\mathrm{v}_M^G(\mathbf{b}_1) \otimes \mathrm{v}_T^M(\mathbf{b}_2)$ in $G^\nu V(\lambda)$ coincide.*

Proof. The filtration $F^\nu V(\lambda)$ is compatible with the direct sum decomposition $V(\lambda) = \underset{\mu}{\oplus} V(\lambda)_\mu$. Let $F^\nu V(\lambda)_\mu$ (resp., $G^\nu V(\lambda)_\mu$) denote the corresponding subspace (resp., sub-quotient) of $V(\lambda)_\mu$.

By Theorem 2.2, we can identify $V(\lambda)_\mu$ with the cohomology $H_c^{\langle \lambda+\mu, \rho_G^\vee\rangle}(S^\mu \cap \mathcal{G}_G^\lambda, \mathbb{C})$. In addition, the filtration $F^\nu V(\lambda)_\mu$ on $V(\lambda)_\mu$ coincides with the filtration on the compactly supported cohomology induced by the decreasing sequence of open subsets in S^μ:

$$\bigsqcup_{\nu' \underset{M}{\geq} \nu} S^\mu \cap S_P^{\nu'}.$$

Therefore, $G^\nu V(\lambda)_\mu \simeq H_c^{\langle \lambda+\mu, \rho_G^\vee\rangle}(S^\mu \cap S_P^\nu \cap \mathcal{G}_G^\lambda, \mathbb{C})$.

The assertion of the proposition follows now from properties 1–3 of the bijection $\mathbf{d}_M^G$. ∎

4.4. **Proof of Theorem 3.1(2).** Let us explain how Proposition 4.1 implies Theorem 3.1(2). Conditions 1 and 2 of Lemma 1.1 follow from the well–known fact that the intersection $S^\mu \cap \mathcal{G}_G^\lambda$ is empty unless $\lambda \geq \mu$ and for $\mu = \lambda$, the above intersection is dense in $\mathcal{G}_G^\lambda$ and hence is irreducible. Thus, we just need to prove that $\mathbf{B}^G(\lambda)$ satisfies the third condition of Lemma 1.1.

Let $\mathbf{b} = \mathbf{d}_M^G(\mathbf{b}_1 \times \mathbf{b}_2)$ with $\mathbf{b}_1 \in \mathbf{B}_M^G(\lambda)_\nu$ and $\mathbf{b}_2 \in \mathbf{B}^M(\nu)_\mu$. Consider the element $\mathrm{v} := \mathrm{v}_T^G(\mathbf{b}) \in V(\lambda)_\mu$. Since $\mu < \lambda$, there exists $i \in I$ and a vector $\mathrm{v}_1 \in V(\lambda)$ such that $F_i(\mathrm{v}_1) = \mathrm{v}$. We claim that this implies that $e_i \cdot \mathbf{b} \neq 0$.

Indeed, let us denote by v' the element $\mathrm{v}_{M_i}^G(\mathbf{b}_1) \otimes \mathrm{v}_T^{M_i}(\mathbf{b}_2)$. By definition, it is sufficient to show that $E_i(\mathrm{v}') \neq 0$.

We have the canonical M_i–invariant projection $V(\lambda) \to G^\nu V(\lambda)$ and let w and w' be the images under this projection of v and v', respectively. Now, Proposition 4.1 implies that $\mathrm{w} = \mathrm{w}'$. Hence, if w_1 denotes the projection of v_1, we obtain that $F_i(\mathrm{w}_1) = \mathrm{w}'$. But this means that $E_i(\mathrm{w}') \neq 0$ and hence $E_i(\mathrm{v}') \neq 0$.

5. Proof of Theorem 3.2

5.1. **Theorem 3.2 on the level of sets.** We will prove a more general assertion. Namely, for a parabolic subgroup P with the Levi factor M and $\lambda_1, \lambda_2 \in \Lambda_G^+$ and $\nu \in \Lambda_M^+$, we will establish a canonical bijection

$$\mathbf{e}_M^G : \bigsqcup_{\lambda_3 \in \Lambda_G^+} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}_M^G(\lambda_3)_\nu \simeq \bigsqcup_{\nu_1, \nu_2 \in \Lambda_M^+} \mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2} \times \mathbf{C}^M(\nu_1, \nu_2)_\nu \tag{5.1}$$

Proof. Consider the variety

$$m^{-1}(S_P^\nu) \cap (\mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2}).$$

According to (2.1) and Theorem 2.2(1), its set of irreducible components of dimension $\langle \lambda_1 + \lambda_2 + \nu, \rho_G^\vee \rangle$ can be identified with the LHS of (5.1).

Now, for $\theta_1, \theta_2 \in \Lambda_{G,P}$, let us denote by $S_P^{\theta_1} \star S_P^{\theta_2}$ the following scheme:

$$S_P^{\theta_1} \star S_P^{\theta_2} := [P,P](\mathcal{K}) M(\mathcal{O}) \cdot \widetilde{\theta}_1(t) \underset{P(\mathcal{O})}{\times} S_P^{\theta_2},$$

where $\widetilde{\theta}_1(t)$ is as in Sect. 2.3. It is easy to see that the natural map $S_P^{\theta_1} \star S_P^{\theta_2} \to \mathcal{G}_G \star \mathcal{G}_G$ is a locally closed embedding.

Similarly, for $\nu_1, \nu_2 \in \Lambda_M^+$ and $\lambda_1, \lambda_2 \in \Lambda_G^+$ we define the sub-scheme $(S_P^{\nu_1} \cap \mathcal{G}_G^{\lambda_1}) \star (S_P^{\nu_2} \cap \mathcal{G}_G^{\lambda_2})$ of $\mathcal{G}_G \star \mathcal{G}_G$ as $S_P^{\theta_1} \star S_P^{\theta_2} \cap \mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2}$.

We have a commutative diagram

$$\begin{CD}
S_P^{\theta_1} \star S_P^{\theta_2} @>{m}>> S_P^{\theta_1+\theta_2} \\
@V{\mathfrak{t}_P^{\theta_1} \star \mathfrak{t}_P^{\theta_2}}VV @VV{\mathfrak{t}_P^{\theta_1+\theta_2}}V \\
\mathcal{G}_M^{\theta_1} \star \mathcal{G}_M^{\theta_2} @>{m}>> \mathcal{G}_M^{\theta_1+\theta_2}.
\end{CD}$$

Therefore, to each element of the set $\mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2} \times \mathbf{C}^M(\nu_1, \nu_2)_\nu$ we can attach an irreducible component of dimension $\langle \lambda_1 + \lambda_2 + \nu, \rho_G^\vee \rangle$ in $(S_P^{\nu_1} \cap \mathcal{G}_G^{\lambda_1}) \star (S_P^{\nu_2} \cap \mathcal{G}_G^{\lambda_2})$. By taking its closure in $m^{-1}(S_P^\nu) \cap (\mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2})$ we obtain an irreducible component of $m^{-1}(S_P^\nu) \cap (\mathcal{G}_G^{\lambda_1} \star \mathcal{G}_G^{\lambda_2})$ and it is easy to see that the map we have just described is a bijection.

This proves our assertion.

□

Note now that for the torus T, $\mathbf{C}^T(\mu_1, \mu_2)_\mu = \emptyset$ unless $\mu_1 + \mu_2 = \mu$ and in the latter case this is the set of one element. Therefore, for $M = T$ (5.1) yields the needed isomorphism

$$\mathbf{B}^G(\lambda_1) \times \mathbf{B}^G(\lambda_2) \overset{\mathbf{e}_T^G}{\simeq} \bigsqcup_{\lambda_3 \in \Lambda_G^+} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}^G(\lambda_3).$$

5.2. **Compatibility of decompositions.** Consider the set $\mathbf{B}^G(\lambda_1) \times \mathbf{B}^G(\lambda_2)$ which, as we have seen above, can be canonically identified with $\bigsqcup_{\lambda_3 \in \Lambda_G^+} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}^G(\lambda_3)$.

There are *a priori* two different ways to identify this set with

$$\bigsqcup_{\nu_1, \nu_2 \in \Lambda_M^+} \mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2} \times \mathbf{B}^M(\nu_1) \times \mathbf{B}^M(\nu_2) :$$

One is

$$\mathbf{B}^G(\lambda_1) \times \mathbf{B}^G(\lambda_2) \xrightarrow{\mathbf{d}_M^G \times \mathbf{d}_M^G} \bigsqcup_{\nu_1, \nu_2 \in \Lambda_M^+} \mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2} \times \mathbf{B}^M(\nu_1) \times \mathbf{B}^M(\nu_2).$$

The other one is the composition

$$\bigsqcup_{\lambda_3 \in \Lambda_G^+} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}^G(\lambda_3) \overset{\mathbf{d}_T^M}{\simeq} \bigsqcup_{\lambda_3 \in \Lambda_G^+; \nu \in \Lambda_M^+} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}_M^G(\lambda_3)_\nu \times \mathbf{B}^M(\nu) \overset{\mathbf{e}_M^G}{\simeq}$$
$$\bigsqcup_{\nu_1, \nu_2 \in \Lambda_M^+} \mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2} \times \mathbf{C}^M(\nu_1, \nu_2)_\nu \times \mathbf{B}^M(\nu) \overset{\mathbf{e}_T^M}{\simeq}$$
$$\bigsqcup_{\nu_1, \nu_2 \in \Lambda_M^+} \mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2} \times \mathbf{B}^M(\nu_1) \times \mathbf{B}^M(\nu_1).$$

However, it is easy to see from the construction that these two identifications coincide.

5.3. **Reduction to $PGL(2)$.** We have established the isomorphism of sets

$$\bigsqcup_{\lambda_3 \in \Lambda_G^+} \mathbf{C}^G(\lambda_1, \lambda_2)_{\lambda_3} \times \mathbf{B}^G(\lambda_3) \simeq \mathbf{B}^G(\lambda_1) \times \mathbf{B}^G(\lambda_2)$$

and we must show that the e_i and f_i operations on both sides coincide.

For $i \in I$ consider the corresponding parabolic P_i. We decompose the LHS as

$$\bigsqcup_{\nu_1, \nu_2 \in \Lambda_{M_i}^+} (\mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2}) \times (\mathbf{C}^M(\nu_1, \nu_2)_\nu \times \mathbf{B}^M(\nu))$$

and the RHS as

$$\bigsqcup_{\nu_1, \nu_2 \in \Lambda_{M_i}^+} (\mathbf{B}_M^G(\lambda_1)_{\nu_1} \times \mathbf{B}_M^G(\lambda_2)_{\nu_2}) \times (\mathbf{B}^M(\nu_1) \times \mathbf{B}^M(\nu_2)).$$

According to Sect. 5.2, these decompositions are compatible. By definition, in both cases, the e_i and f_i operations preserve these decompositions and act "along" the second multiple.

This observation reduces the assertion of Theorem 3.2 from G to M_i. In addition, it is easy to see that we can replace M_i by its adjoint group, i.e. it remains to analyze the case of $G = PGL(2)$.

5.4. **Proof of Theorem 3.2 for $PGL(2)$.** For $G = PGL(2)$ we will identify Λ_G (resp., Λ_G^+) with $\mathbb{Z}$ (resp., with $\mathbb{Z}^+$). The positive root $\alpha \in \Lambda_G$ corresponds to $2 \in \mathbb{Z}$.

Let l_1, l_2 be two elements of $\mathbb{Z}^+$. The action of e and f breaks $\mathbf{B}^G(l_1) \otimes \mathbf{B}^G(l_2)$ into orbits and it is sufficient to show that this decomposition coincides with

$$\mathbf{B}^G(l_1) \otimes \mathbf{B}^G(l_2) \simeq \bigsqcup_{l \in \mathbb{Z}^+} \mathbf{C}^G(l_1, l_2)_l \times \mathbf{B}^G(l)$$

(note that in this case each $\mathbf{C}^G(l_1, l_2)_l$ has at most one element.)

For that end, it is sufficient to show that for $m_1, m_2 \in \mathbb{Z}$ a generic point in $(S^{m_1} \cap \mathcal{G}_G^{l_1}) \star (S^{m_2} \cap \mathcal{G}_G^{l_2})$ projects under the map $m : \mathcal{G}_G \star \mathcal{G}_G \to \mathcal{G}_G$ to S^n, where

$$n = \max\{l_1 - m_2, m_1 + l_2\}.$$

For $l \in \mathbb{Z}^+$ and $m \in \mathbb{Z}$, the intersection $S^m \cap \mathcal{G}_G^l$ is non–empty if only if $l - m \in 2\mathbb{Z}^+$, $l \geq |m|$ and in the latter case it consists of cosets of the form

$$\begin{pmatrix} t^m & t^{(m-l)/2} \cdot p(t) \\ 0 & 1 \end{pmatrix} \cdot PGL(2, \mathcal{O}) \mid p(t) \in \mathbb{C}[[t]],\, p(0) \neq 0.$$

Therefore, the image of $(S^{m_1} \cap \mathcal{G}_G^{l_1}) \star (S^{m_2} \cap \mathcal{G}_G^{l_2})$ under m consists of cosets of the form

$$\begin{pmatrix} t^{m_1+m_2} & t^{\max\{l_1-m_2, m_1+l_2\}} \cdot p(t) \\ 0 & 1 \end{pmatrix} \cdot PGL(2, \mathcal{O}) \mid p(t) \in \mathbb{C}[[t]],\, p(0) \neq 0.$$

This finishes the proof of Theorem 3.2.

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