Lecture 15 - Root System Axiomatics

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In this lecture we examine root systems from an axiomatic point of view.

1 Reflections

If $v \in \mathbb{R}^n$, then it determines a hyperplane, denoted P_v, through the origin. Reflection about this hyperplane, denoted σ_v, is just

$$\sigma_v(x) \;=\; x \,-\, \frac{2\,(x,\,v)}{(v,\,v)}\,v. \tag{1}$$

We make the general definition

$$\langle\, w,\; v\,\rangle \;\triangleq\; \frac{2\,(w,\,v)}{(v,\,v)} \tag{2}$$

meaning

$$\sigma_v(x) \;=\; x \,-\, \langle x,\,v\rangle\; v \tag{3}$$

Lemma 1.1 *Let $\Phi \subset \mathbb{R}^n$ be a finite set which spans $\mathbb{R}^n$, and also has the property that $v \in \Phi$ implies that σ_v leaves Φ invariant. Assume $\sigma \in GL(\mathbb{R}^n)$ leaves Φ invariant, fixes a hyperplane $P \subset \mathbb{R}^n$ in the pointwise sense, and sends some vector $v \in \mathbb{R}^n$ to its negative. Then $\sigma = \sigma_v$ and $P = P_v$.*

Pf. Set $\tau = \sigma \circ \sigma_v$, so obviously $\tau : \Phi \to \Phi$. Note that $\tau : \mathbb{R}v \to \mathbb{R}v$ acts as the identity. We can identify $\mathbb{R}^n \approx P \oplus \mathbb{R}v$ so $w \in \mathbb{R}^n$ is $w = w_P + w_v$, $w_P \in P$ and $w_v \in \mathbb{R}v$. We have $\sigma_v(w_P + w_v) = w_P - \frac{2(w_P,v)}{(v,v)}v - w_v$ so $\tau(w_P) = w_P + \frac{2(w_P,v)}{(v,v)}v + w_v$. Thus τ acts on $\mathbb{R}^n/\mathbb{R}v \approx P$, and acts as the identity. This means τ has only unity eigenvalues, so its minimal polynomial divides $(T-1)^n$. If $w \in \Phi$ then $\{w, \tau(w), \dots, \tau^k(w)\}$ cannot all be independent if $k \geq Card(\Phi)$. Thus there is some number l so that τ^l acts as the identity on Φ and therefore on $\mathbb{R}^n$ (l will be the l.c.m. of the various k for the various $w \in \Phi$). Thus also the minimal polynomial of τ divides $T^l - 1$. However $gcd\left(T^k - 1, (T-1)^n\right) = T - 1$, so the minimal polynomial is $T - 1$. Thus $\tau = Id$. $\square$

2 Root System Axiomatics

We say a subset Φ of $E = \mathbb{R}^n$ (with its Euclidean metric) is a *root system* provided

R1) Φ is finite, spans E, and does not contain 0

R2) If $\alpha \in \Phi$, then $c\alpha \in \Phi$ if only if $c = \pm 1$

R3) If $\alpha \in \Phi$, then σ_α leaves Φ invariant

R4) If $\alpha, \beta \in \Phi$, then $\langle \beta, \alpha \rangle \in \mathbb{Z}$

The dimension of $\mathbb{R}^n$ is called the *rank* of the root system.

Given a root system $\Phi \in E$, denote by $\mathcal{W} \subset GL(E)$ the group of transformations generated by σ_α for all $\alpha \in \Phi$. This is called the *Weyl group* for the root system.

Lemma 2.1 *If $\sigma \in GL(E)$ leaves Φ invariant, then $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$ and $\langle \beta, \alpha \rangle = \langle \sigma(\beta), \sigma(\alpha) \rangle$ for all $\alpha, \beta \in \Phi$.*

Pf. There is some $\gamma \in \Phi$ so that $\sigma^{-1}(\beta) = \gamma$. Thus

$$\begin{aligned} \sigma\sigma_\alpha\sigma^{-1}(\beta) &= \sigma\sigma_\alpha(\gamma) \\ &= \sigma\left(\gamma - \langle \gamma, \alpha \rangle \alpha\right) \end{aligned} \tag{4}$$

and since $\gamma - \langle \gamma, \alpha \rangle \alpha$ is a root, so is $\sigma(\gamma - \langle \gamma, \alpha \rangle \alpha)$. Thus $\sigma\sigma_\alpha\sigma^{-1}$ leaves Φ invariant. Also, $\sigma\sigma_\alpha\sigma^{-1}$ sends $\sigma(\alpha)$ to $-\sigma(\alpha)$, and fixes the plane $\sigma(P_\alpha)$. By the lemma we have $\sigma\sigma_\alpha\sigma^{-1} = \sigma_{\sigma(\alpha)}$.

For the second assertion,

$$\begin{aligned} \sigma\left(\beta - \langle \sigma(\beta), \sigma(\alpha) \rangle \alpha\right) &= \sigma(\beta) - \langle \sigma(\beta), \sigma(\alpha) \rangle \sigma(\alpha) \\ &= \sigma_{\sigma(\alpha)}(\sigma(\beta)) \\ &= \sigma\sigma_\alpha\sigma^{-1}\sigma(\beta) \\ &= \sigma\sigma_\alpha(\beta) \\ &= \sigma\left(\beta - \langle \beta, \alpha \rangle \alpha\right). \end{aligned} \tag{5}$$

Taking σ^{-1} we get the assertion. $\square$

If $\alpha \in \Phi$ define its "dual" $\hat{\alpha}$ to be

$$\hat{\alpha} = \frac{2\alpha}{(\alpha, \alpha)} \tag{6}$$

and define $\hat{\Phi} = \{\hat{\alpha} \,|\, \alpha \in \Phi\}$. This is also a root system, and the Weyl groups of Φ and $\hat{\Phi}$ are canonically isomorphic. In the Lie algebra situation, $\hat{\alpha}$ is the metric dual (under the Killing form) to h_α.

Proposition 2.2 *Assume $\alpha, \beta \in \Phi$. We have $\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \geq 0$. The only possibilities for $\langle \alpha, \beta \rangle$ are $0, \pm 1. \pm 2, \pm 3$. If θ is the angle between α and β, then (up to sign) θ can only take on the values $0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$. Finally, one of $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$ is ± 1.*

Pf. Since $\cos\theta = \frac{(\alpha,\beta)}{|\alpha||\beta|}$ so

$$\langle \alpha,\, \beta \rangle \langle \beta, \alpha \rangle \;=\; 4\cos^2(\theta). \tag{7}$$

Since $0 \leq 4\cos^2\theta \leq 4$, and because $4\cos^2\theta = 4$ only when $\theta = 0, \pi$, the first two assertions hold. Obviously $\cos\theta = 0, \pm 1, \pm\frac{1}{\sqrt{2}}, \pm\frac{\sqrt{3}}{2}$, so the third assertion also holds. The final assertion is obvious from the fact that $\theta \neq \pi$. $\square$

Proposition 2.3 *Let $\alpha, \beta \in \Phi$ be non-proportional roots. If $(\alpha, \beta) > 0$ (the angle is strictly acute) then $\alpha - \beta$ is a root. If $(\alpha, \beta) < 0$ (the angle is strictly obtuse) then $\alpha + \beta$ is a root.*

Pf. The second assertion follows from the first, by replacing α by $-\alpha$. Both $\langle \alpha, \beta \rangle, \langle \beta, \alpha \rangle$ are positive, so by the previous proposition, one of them equals 1. If $\langle \alpha, \beta \rangle = 1$ then

$$\sigma_\beta(\alpha) \;=\; \alpha \,-\, \langle \alpha, \beta \rangle\, \beta \;=\; \alpha - \beta \tag{8}$$

is in Φ. If $\langle \beta, \alpha \rangle = 1$ then

$$\sigma_\alpha(\beta) \;=\; \beta \,-\, \langle \beta, \alpha \rangle\, \alpha \;=\; \beta - \alpha \tag{9}$$

is in Φ, so also $\alpha - \beta \in \Phi$. $\square$

The set of all roots of the form $\beta + i\alpha$ for $i \in \mathbb{Z}$ is called the *α-string through β*.

Proposition 2.4 *Facts about strings:*

- *The α-string through β is unbroken.*
- *σ_α reverses all α-strings.*
- *Strings have length at most 4.*

Pf. Let $q, r \in \mathbb{Z}^{\geq 0}$ be integers so $\beta - r\beta \in \Phi$ but $\beta - r'\alpha \notin \Phi$ for any $r' > r$, and so $\beta + q\alpha \in \Phi$ by $\beta + q'\alpha \notin \Phi$ for any $q' > q$.

Now assume p, s can be found with $-r \leq p < p+1 \leq s-1 < s \leq q$ so that $\beta + p\alpha \in \Phi$ and $\beta + s\alpha \in \Phi$, by $\beta + p'\alpha \notin \Phi$ for all $p+1 \leq p' \leq s-1$. Then according to the lemma, we have both $(\alpha, \beta) + p|\alpha|^2 = (\alpha, \beta + p\alpha) > 0$ and $(\alpha, \beta) + s|\alpha|^2 = (\alpha, \beta + s\alpha) < 0$. Thus $p < -\frac{(\alpha,\beta)}{|\alpha|^2} < s$ which is impossible by assumption.

Because σ_α reverses the string, we have

$$\sigma_\alpha(\beta + q\alpha) \ = \ \beta - r\alpha \tag{10}$$

so that

$$\begin{aligned} \beta \ - \ \langle \beta, \alpha \rangle \, \alpha \ - \ q\alpha \ &= \ \beta - r\alpha \\ - \ \langle \beta, \alpha \rangle \, \alpha \ &= \ (q - r)\alpha \end{aligned} \tag{11}$$

Therefore $|q - r|$ is at most 4. $\square$

3 Bases and the Weyl group

We have seen that new roots can be constructed from old, via strings. The question naturally arises, can we find some minimal set of roots that generates all the others?

A subset $\triangle \subset \Phi$ is called a *base* if

B1) $\triangle$ is a basis of E

B2) Each root β can be written as a sum $\beta = \sum c_\alpha \alpha$ where $\alpha \in \triangle$ and $k_\alpha \in \mathbb{Z}$ where all k_α are either all positive or all negative (or zero).

If a base for Φ exists, we can say a root is positive or negative if its coefficients are all positive or negative. We can also define the *height* of a root by $ht\beta = \sum k_\alpha$ where $\beta = \sum k_\alpha \alpha$. Of course if a root system has more than one base, a root might have different heights relative to these bases. However because any given base is a basis, a root has a well-defined height relative to a given base. Finally we have an ordering of roots. Given roots β, α we have

$$\alpha \ > \ \beta \tag{12}$$

provided $\alpha - \beta$ is a root and is positive. Obviously this is only a partial ordering.

Lemma 3.1 *If $\triangle$ is a base of Φ, then $(\alpha, \beta) \leq 0$ for $\alpha \neq \beta$ for all $\alpha, \beta \in \triangle$, and $\alpha - \beta$ is not a root.*

Pf. Easy. $\square$

Let $\gamma \in E$ be any vector. We call γ *regular* if it does not lie in P_α for any $\alpha \in \Phi$. Define $\Phi^+(\gamma)$ to be

$$\Phi^+(\gamma) \ = \ \left\{ \alpha \in \Phi \ \middle| \ (\alpha, \gamma) \ > \ 0 \right\}. \tag{13}$$

Clearly if γ is regular, then $\Phi = \Phi^+(\gamma) \cup -\Phi^+(\gamma)$. An element $\alpha \in \Phi$ is called *indecomposable* provided $\alpha = \beta_1 + \beta_2$ implies that either β_1 or β_2 is not in $\Phi^+(\gamma)$.

Theorem 3.2 *Let $\gamma \in E$ be regular. Then the set $\triangle(\gamma)$ of indecomposable roots in $\Phi^+(\gamma)$ is a base. Further, every base $\triangle$ is a $\triangle(\gamma)$ for some regular γ.*

Pf. Next time. $\square$