$\mathbb{E}_n$-Algebras (Lecture 22)

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Let k be an algebraically closed field, ℓ a prime number which is invertible in k, X an algebraic curve over k, and G a parahoric group scheme over X whose generic fiber is split reductive. In the previous lecture, we described a sheaf $\mathcal{A}$ on the Ran space $\mathrm{Ran}(X)$, whose stalk at a point $\mu : S \to X(k)$ with image $\{x_1, \ldots, x_m\}$ is given by

$$\mathcal{A}_\mu = \bigotimes_{1 \le i \le m} C^*(\mathrm{Gr}_{G,x_i}; \mathbf{Q}_\ell).$$

In order to calculate the trace of Frobenius on the cohomology of $\mathrm{Bun}_G(X)$ (in the special case where X and G are defined over a finite field $\mathbf{F}_q$), we would like to understand the Verdier dual of $\mathcal{A}$ (or, more accurately, of a modified version of $\mathcal{A}$ which we will discuss later). Our goal in this lecture is to give some idea of what one might expect $\mathbf{D}(\mathcal{A})$ to look like.

We begin with some general remarks in the setting of topology. Let Y be a topological space equipped with a base point $y \in Y$, and let ΩY denote its based loop space

$$\Omega Y = \{p : [0,1] \to Y : p(0) = p(1) = y\}.$$

Then ΩY is equipped with a "concatenation" operation $\star : \Omega Y \times \Omega Y \to \Omega Y$

$$(p \star q)(t) = \begin{cases} p(2t) & \text{if } 0 \le t \le \frac{1}{2} \\ q(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

Concatenation of paths satisfies an associative law up to homotopy, and equips the set $\pi_0(\Omega Y) = \pi_1(Y, y)$ with the structure of a group. However, it is not associative on the nose: the loops $(p \star q) \star r$ and $p \star (q \star r)$ are parametrized differently. Nevertheless, concatenation is associative "up to coherent homotopy," and for many purposes one can treat ΩY as if it were a topological group. Let us begin by describing one way to make this "homotopy coherence" more explicit. Roughly speaking, the idea is to think of ΩY as equipped with several different multiplication operations, one for each embedding of the disjoint union $(0,1) \amalg (0,1)$ into $(0,1)$.

Definition 1. An $\mathbb{E}_1$-*space* A consists of the following data:

(1) For every open set $U \subseteq \mathbb{R}^1$ which is homeomorphic to a disk, a topological space $A(U)$.

(2) For every collection of disjoint open disks $U_1, \ldots U_m$ contained in an open disk V, a map $\mu : A(U_1) \times \cdots \times A(U_m) \to A(V)$.

The maps μ appearing in (2) are required to be compatible with composition (in the obvious sense). Moreover, we also require the following:

(3) If $U \subseteq V$ are open disks in $\mathbb{R}^1$, then the multiplication map $\mu : A(U) \to A(V)$ is a homotopy equivalence.

Let A be an arbitrary $\mathbb{E}_1$-space. It follows from axiom (3) that for any pair of open disks U and V, the spaces $A(U)$ and $A(V)$ are related by a chain of homotopy equivalences

$$A(U) \to A(\mathbb{R}) \leftarrow A(V).$$

We may therefore imagine roughly that each $A(U)$ is an incarnation of the same space X, and that the multiplication maps $\mu : A(U_1) \times \cdots \times A(U_m) \to A(V)$ give maps $X^m \to X$, satisfying an associative law.

Example 2. Let (Y, y) be a pointed topological space. For each open disk $U \subseteq \mathbb{R}^1$, let U^+ denote the one-point compactification of U, and let $*$ denote the base point of U^+. We define an $\mathbb{E}_1$-space A by the formula $A(U) = \mathrm{Map}((U^+, *), (Y, y))$. Each U^+ is homeomorphic to a circle, so each of the spaces $A(U)$ can be identified (noncanonically) with the based loop space ΩY.

Note that an inclusion of disjoint disks $U_1 \cup U_2 \cup \cdots \cup U_m \hookrightarrow V$ induces a "collapse map" $V^+ \to U_1^+ \vee \cdots \vee U_m^+$, which determines a multiplication

$$\mu : A(U_1) \times \cdots \times A(U_m) \to A(V).$$

When $m = 1$, these maps are homotopy equivalences.

The $\mathbb{E}_1$-space A of Example 2 is one way of encoding the "homotopy coherent" multiplication on the based loop space ΩY. To appreciate this, it is useful to place Definition 1 in a larger context.

Definition 3. Let $\mathcal{C}$ be a symmetric monoidal ∞-category (that is, an ∞-category equipped with a tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is commutative and associative up to coherent homotopy), and let $n \geq 0$ be an integer. An *$\mathbb{E}_n$-algebra object of* $\mathcal{C}$ consists of the following data:

(1) For each open set $U \subseteq \mathbb{R}^n$ which is homeomorphic to a disk, an object $A(U) \in \mathcal{C}$.

(2) For every collection of disjoint disks $U_1, \ldots, U_m$ contained in a larger disk V, a multiplication map

$$\mu : \bigotimes_{1 \leq i \leq m} A(U_i) \to A(V).$$

We require that the multiplications maps μ be compatible with composition, and that μ is an equivalence in $\mathcal{C}$ in the special case where $m = 1$.

The collection of $\mathbb{E}_n$-algebras in $\mathcal{C}$ can be organized into another (symmetric monoidal) ∞-category, which we will denote by $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$.

Remark 4. If A is an $\mathbb{E}_n$-algebra object of $\mathcal{C}$, we will generally abuse notation by writing A to denote the object $A(\mathbb{R}^n) \in \mathcal{C}$. Note that each $A(U)$ is (canonically) equivalent to $A(\mathbb{R}^n)$ via the inclusion $U \hookrightarrow \mathbb{R}^n$.

Example 5. Let $\mathcal{C}$ be an ordinary category, and let A be an $\mathbb{E}_n$-algebra object of $\mathcal{C}$. Then each inclusion $j : U_1 \amalg \cdots \amalg U_m \to V$ induces a map $A^{\otimes m} \to A$. One can show that this map depends only on the isotopy class of the embedding j: in other words, it depends only on an ordering of the set $\{1, \ldots, m\}$ in the case $n = 1$, and is independent of j if $n > 1$. Consequently, the category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is equivalent to the category of associative algebras in $\mathcal{C}$ when $n = 1$, and to the category of commutative algebras in the case $n > 1$.

Remark 6. Let $F : \mathcal{C} \to \mathcal{D}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories. Then F carries $\mathbb{E}_n$-algebras in $\mathcal{C}$ to $\mathbb{E}_n$-algebras in $\mathcal{D}$. In particular, if A is an $\mathbb{E}_n$-algebra in the ∞-category $\mathcal{S}$ of spaces, then $\pi_0 A$ is an $\mathbb{E}_n$-algebra in the ordinary category of sets: that is, we can regard $\pi_0 A$ as a monoid if $n = 1$, and as a commutative monoid if $n > 1$. We say that an $\mathbb{E}_n$-algebra A in $\mathcal{S}$ is *group-like* if $\pi_0 A$ is a group.

Example 7. Let (Y, y) be a pointed topological space. The construction $U \mapsto \mathrm{Map}((U^+, *), (Y, y))$ determines an $\mathbb{E}_n$-algebra in the ∞-category $\mathcal{S}$ of spaces, whose underlying space is the iterated loop space $\Omega^n Y$. This construction determines a functor from the ∞-category $\mathcal{S}_*$ of pointed spaces to the ∞-category $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{S})$ of $\mathbb{E}_n$-spaces. It follows from a theorem of May that this functor is close to being invertible: more precisely, it restricts to an equivalence from the ∞-category of $(n-1)$-connected spaces to the ∞-category of grouplike $\mathbb{E}_n$-spaces (for $n > 0$).

Example 8. For every commutative ring R, the construction $X \mapsto C_*(X; R)$ determines a symmetric monoidal functor from the ∞-category $\mathcal{S}$ of spaces to the ∞-category $\mathrm{Mod}_{\mathbb{R}}$ of chain complexes of R-modules. Consequently, for any topological space Y, we can regard $C_*(\Omega^n(Y); R)$ as an $\mathbb{E}_n$-algebra in Mod_R.

Example 9. Let R be a commutative ring. The ∞-category $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}_R)$ admits inverse limits, which are computed "pointwise" (in other words, which are compatible with the forgetful functor $\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Mod}_R) \to \mathrm{Mod}_R$). In particular, we can regard the unit object $R \in \mathrm{Mod}_R$ as an $\mathbb{E}_n$-algebra, so that the cochain complex

$$C^*(Y; R) \simeq \varprojlim_{y \in Y} R$$

also admits the structure of an $\mathbb{E}_n$-algebra (for any value of n: it is actually an example of an $\mathbb{E}_\infty$-algebra over R).

Remark 10 (Additivity Theorem)**.** Let $\mathcal{C}$ be an symmetric monoidal ∞-category. Then $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ is also a symmetric monoidal ∞-category, so one can consider $\mathbb{E}_m$-algebras in $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$. There is a forgetful functor

$$\Phi : \mathrm{Alg}_{\mathbb{E}_{m+n}}(\mathcal{C}) \to \mathrm{Alg}_{\mathbb{E}_m}(\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})),$$

given on objects by the formula

$$(\Phi A(U))(V) = A(U \times V).$$

One can show that this functor is an equivalence of ∞-categories. In particular, we have an equivalence

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C}) \simeq \mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Alg}_{\mathbb{E}_1}(\cdots(\mathcal{C}))).$$

Roughly speaking, an $\mathbb{E}_n$-algebra object of $\mathcal{C}$ can be regarded as an object of $\mathcal{C}$ equipped with n different (but compatible) associative algebra structures.

Variant 11. Let $\mathcal{C}$ be a symmetric monoidal ∞-category. A *nonunital* $\mathbb{E}_n$*-algebra in* $\mathcal{C}$ consists of a collection of objects $A(U) \in \mathcal{C}$ (indexed by open sets $U \subseteq \mathbb{R}^n$ which are homeomorphic to $\mathbb{R}^n$) together with multiplication maps

$$\mu : A(U_1) \otimes \cdots \otimes A(U_m) \to A(V)$$

where we require $m > 0$. One can show that the ∞-category of nonunital $\mathbb{E}_n$-algebras is equivalent to a subcategory of $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$ (in other words, if a nonunital $\mathbb{E}_n$-algebra admits a unit, then that unit is unique up to a contractible space of choices).

Construction 12. For every collection of disjoint open disks $U_1, \ldots, U_m \subseteq \mathbb{R}^n$, let $\mathrm{Ran}(U_1, \ldots, U_m) \subseteq \mathrm{Ran}(\mathbb{R}^n)$ be the collection of nonempty finite subsets of $\mathbb{R}^n$ which are contained in $U_1 \cup \cdots \cup U_m$ and have nontrivial intersection with each U_i. Then the open sets $\mathrm{Ran}(U_1, \ldots, U_m)$ form a basis for the topology on $\mathrm{Ran}(\mathbb{R}^n)$. Let $\mathcal{U}$ denote the collection of open subsets of $\mathrm{Ran}(\mathbb{R}^n)$ which belong to this basis. If A is a nonunital $\mathbb{E}_n$-algebra in an ∞-category $\mathcal{C}$, then we can define a functor $F_A : \mathcal{U} \to \mathcal{C}$ by the formula

$$F_A(\mathrm{Ran}(U_1, \ldots, U_m)) = A(U_1) \otimes \cdots \otimes A(U_m).$$

Assuming that the ∞-category $\mathcal{C}$ admits colimits (and that the tensor product on $\mathcal{C}$ distributes over colimits), one can show that F_A extends to a $\mathcal{C}$-valued cosheaf $\mathcal{F}_A$ on $\mathrm{Ran}(\mathbb{R}^n)$. The stalk of $\mathcal{F}_A$ at a point $S \in \mathrm{Ran}(\mathbb{R}^n)$ can be identified with the tensor product $A^{\otimes S}$. The cosheaf $\mathcal{F}_A$ is locally constant along each (open) stratum of $\mathrm{Ran}(\mathbb{R}^n)$, and its behavior when passing from one stratum to another is one way of encoding the multiplication on the algebra A.

References

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