A short story

How to read this? There is much more to fit in two pages, so I will provide references and omit details. I leave the technicalities of my research to the research statement.

For me, mathematics plays a role closer to novels or poetry. Many aspects of it are fictional, requiring imagination and effort to appreciate. Nonetheless, they often deeply reflect the authors' emotions and the greater community.

1. Duality

I am interested in a particular symmetry known as *duality*: when there are two related ways to see the same concept. For example, there is Maxwell's equations in physics. In the absence of charge and currents, it can be written as

$$\nabla \cdot E = 0 \quad \nabla \cdot B = 0$$
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \quad \partial \times B = \frac{1}{c}\frac{\partial E}{\partial t}$$

By changing the roles of (E, B) to $(-B, E)$, the equations are the same!
The *Langland's program* hopes to uncover a duality between *number theory* and *harmonic analysis*. The modern approach to finding such a bridge is by reformulating the problem in terms of *geometry*, which broadly construed is the study of shapes.

$$\begin{tikzcd}
 & \{\text{Geometric Langlands}\} & \\
\{\text{Harmonic Analysis}\} \arrow[ur, bend left] \arrow[rr, dashed, leftrightarrow, "?" description] & & \{\text{Number Theory}\} \arrow[ul, bend right]
\end{tikzcd}$$

Let us see how Number theory is related to *geometry*. Number theory studies Diophantine equations, the integer solution set, $X(\mathbb{Z})$, to a set of polynomial equations

$$X := \{f_i \in \mathbb{Z}[t_1, \ldots, t_n] \quad i = 1, \ldots, r\}$$

One can consider the same problem of describing solution set $X(R)$, for different "coefficient" rings R. An insight from A. Grothendieck, is that the aggregate of all this data is described by a geometric object called *scheme*.
Further reading: An overview of various dualites, [**31**]. A slightly more techincal introduction to Langlands, [**4**].
To go from geometry to analysis, one packages this data using a generating function. For instance, the *Zeta function*

$$\exp\left(\sum_{n>0}\frac{|X(\mathbb{F}_{p^n})|}{n}t^n\right)=:\zeta_X(t)$$

Already, this says something about the distribution of primes using Euler's discovery on prime factorization![1]

How does this appear in analysis? One looks at harmonic analysis to a particular class of spaces referred as *locally symmetric spaces*. A family of examples can be constructed as follows. Let Γ be a *congruence subgroup*[2] of $\mathrm{SL}_2(\mathbb{Z})$,

$$\mathcal{M}_\Gamma := \Gamma\backslash\mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2 = \Gamma\backslash\mathbb{H}$$

where $\mathbb{H} := \{z = x+iy : x, y, \in \mathbb{R}, y > 0\}$ is the complex upper half plane. A function f [3] on such space, is *modular*: they have many symmetries. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}$$

Thus, if $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, then f is 1-periodic: $f(z) = f(z+1)$, hence admitting Fourier expansion. The collection of a special subclass of such modular functions, admit natural symmetries called *Hecke operators*. The joint eigenvalues of the operators encode the same data as the Zeta function!

Further reading: More details is in [**29**, 24]. This is related to Fermat's last theorem. Modular forms and its relation to cryptography, [**32**].

2. Modern approaches: higher category theory

To express the duality, it is indispensable to use the language of *higher categories*. Homotopy theory studies a world where *deformations* are natural. *Category theory* studies families of objects formally. Higher category theory is the modern language which combine these two concepts. It has been a recent success of J. Lurie, [**19**] , E. Riehl and D. Verity, [**27**] - and many more - to set up a concrete foundation.

[1]The uniqueness and existence of prime factorization.
[2]In particular, we can have $\Gamma = \mathrm{SL}_2(\mathbb{Z})$
[3]Here we would be consider *weight* 0 *modular forms level* $\mathrm{SL}_2(\mathbb{Z})$.

Higher category theory has proven successful in various areas of physics, [2], algebraic topology, [20], particularly algebraic K-theory. In context of Langlands duality, the pioneering work started from D.Gaitsgory et al, [34], which generalized works of field medalist, V. Drinfeld back in the 80s.

p-adic geometry is a deeply studied field that studies a *different* geometry that we are accustomed to: most mathematics working with $\mathbb{R}$ or $\mathbb{C}$ work with the equlidean square-norm. In p-adic geometry, the norm is *non-archimedian*, i.e.

$$|x + y| \leq \max |x|, |y|$$

This lead to both similar and surprising phenomena, for instance, the recent work of *prismatic cohomology*, [6]. The recent work of field medalist P. Scholze, [14] extended geometric Langlands duality in the context of p-adic geometry.
My interests thus revolves around the following two flow

$$\begin{tikzcd}
\{\text{Higher Category Theory}\} \arrow[r, bend left] & \{\text{p-adic Geometric Langlands}\} \arrow[l, bend left]
\end{tikzcd}$$

Can we use higher category to better explain duality phenomena? Conversely, could duality in Langlands reflect deeper structures in other fields of mathematics?

Research statement, Chen-wei (Milton) Lin

2.1. An overview of my research. My research lie broadly around *categorical deformation theory*, in the sense of [**21**, 16], with its applications towards integral and p-adic aspects of the Langlands program. A large aspect of it is motivated by the theme of *categorical Langlands*, which has incarnations in different settings, see [**16**], [**14**], [**33**], and [**11**].

My first inspiration is from the *de Rham geometric Langlands* of D. Arinkin and D. Gaitsgory. Let X be a smooth projective connected curve over a base field $k = \mathbb{C}$,

Conjecture 2.1. [**1**, 1.1.6] For each $x \in X$, we have a commuting diagram

$$\begin{array}{cccc} \text{[Spectral]} & & \text{[Automorphic]} & \\ & & & \\ \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{sHk}^{\mathrm{loc}}_{\check{G},x}) & \simeq & \mathrm{Dmod}(\mathrm{aHk}^{\mathrm{loc}}_{G,x}) & \text{[local symmetries]} \\ \circlearrowleft & & \circlearrowleft & \\ \mathrm{IndCoh}_{\mathcal{N}}(\mathrm{LS}_{\check{G}}(X)) & \simeq & \mathrm{Dmod}(\mathrm{Bun}_G(X)) & \text{[global objects]} \end{array}$$

where the global objects are

- $\mathrm{Bun}_G(X)$: the moduli stack of G-bundles on X.
- $\mathrm{LS}_{\check{G}}(X)$: the moduli stack of $\check{G}$ local systems with flat connections.

and the local objects are

- The automorphic (spectral) Hecke stack, $\mathrm{aHk}^{\mathrm{loc}}_{G,x}$ ($\mathrm{sHk}^{\mathrm{loc}}_{\check{G},x}$): the moduli stack for two G bundles (with flat connections) around a disk and an isomorphism restricted to punctured disk.

Conj 2.1 motivates two projects which I hope to explore

??) An integral version from the point of view of Whittaker categories, [**15**], and deformation of categories.

3) A p-adic version, from the point of view of derived deformation rings, [**12**]. This is joint work with Ashwin Iyengar. We are also exploring spectral actions in the sense of [**14**, X].

2.2. My background in stable homotopy. In my first two years of studies - inspired by my supervisor, David Gepner - I was interested in the formalities of higher category theory, such as the work of E. Riehl, D. Verity [27], and J. Lurie's higher topos theory, [19, 6].

I studied applications of higher topos theory towards

- K-theoretic computations: from trace methods and topological Hochschild homology, [24] and [5], to étale K-theory, [7].
- Elliptic cohomology, or aspects of *spectral algebraic geometry*, in the sense of [21].

An ambitious goal of mine is to bring aspects of stable homotopy theory to Langlands.

Notations and conventions:

2.2. Language. We will freely use the language of higher categories, as in the works of J. Lurie, [19], and [20].

- One may freely exchange the terms *stable* and *triangulated*.
- By *category* we mean $(\infty, 1)$-categories. This subsumes ordinary category.
- $\mathcal{S}$ will denote the category of ∞-groupoids, or animated sets.
- Sp denotes the category of *spectra*, see [20, 1]. It has a unit object $\mathbb{S}$.

2.3. Algebraic groups.

- $G_{\mathbb{Z}}$ a Chevalley group over $\mathbb{Z}$.[4] We denote $G_e := G_{\mathbb{Z}} \times_{\mathbb{Z}} e$.
- $\mathrm{Rep}_e(G_e)$ denotes the *algebraic representations with coefficients in* e of G_e, this is a stable category with t-structure whose heart, $\mathrm{Rep}_e(G_e)^{\heartsuit}$ is the ordinary category of algebraic representations.

2.4. Operadic notions, see [20, 2].

- Let $m \geq n$, be two non-negative integers. For a $\mathbb{E}_m$-category, $\mathcal{M}$. $\mathbb{E}_n(\mathcal{M})$ denotes the $\mathbb{E}_n$-algebra objects in $\mathcal{M}$.
- For $R \in \mathbb{E}_n(\mathrm{Mod}_k)$, let LCat_R denote R-linear categories. [5]

2.5. For each project I will describe

- The main question and my (our) current approach.
- What I (we) have attempted. This can be safely omitted.

[4] split reductive group scheme over $\mathbb{Z}$, in the sense of [9].

[5] The category LMod_R is an $\mathbb{E}_{n-1}$ algebra object and LCat_R is $\mathbb{E}_{n-2}$ algebra object, see [20, 6].

3. p-adic spectral Hecke algebra

Working jointly with Ashwin Iyengar we are exploring evidences of the p-adic categorical Langlands correspondence, [11]. The original motivation is from T. Feng's work, [12], whose notation we follow for this exposition. Let $\mathcal{O} = W(k)/\mathbb{Z}_p$ be a finite extension of $\mathbb{Z}_p$, with finite residue field k of characteristic p. We have[6]

Theorem 3.1. [12, 6.3] Let Λ be a finite quotient of $\mathcal{O}$.

$$\begin{array}{cccc} \text{[Spectral]} & & \text{[Automorphic]} & \\ & & & \\ \pi_*(\mathcal{S}_q^{\mathrm{Hk}} \otimes_{\mathcal{O}} \Lambda)^\vee & \xrightarrow{\simeq} & \mathcal{H}_q(\Lambda) & \text{[local symmetries]} \\ \circlearrowleft & & \circlearrowleft & \\ \pi_*(\mathcal{R}_S \otimes_{\mathcal{O}} \Lambda)^\vee & \simeq & \widetilde{\mathbb{T}}_{\mathfrak{m}} \otimes \Lambda & \text{[global objects]} \end{array}$$

The global objects are:

- $(\widetilde{\mathbb{T}}_U)$, [30, 1,2]: a subalgebra of the endomorphism of cohomology of a locally symmetric space with level structure U.
- We localize at a maximal ideal $\mathfrak{m}$, which conjecturally corresponds to a global Galois representation $\bar{\rho}$.
- $\mathcal{R}_S$, [18]: the formal derived global moduli of Galois representations unramified outside a set S which is crystalline at p, and reduces to $\bar{\rho}$.

The local symmetries are defined at *Taylor-Wiles prime q for $\bar{\rho}$*, [12, 3.2.2]. In this case, $\bar{\rho}$ is unramified at q and $q = 1 \in k$. The *spectral Hecke algebra* is

$$\mathcal{S}_q^{\mathrm{Hk}} := \mathcal{S}_q^{\mathrm{nr}} \otimes_{\mathcal{S}_q} \mathcal{S}_q^{\mathrm{nr}}$$

with $\mathcal{S}_q$ ($\mathcal{S}_q^{\mathrm{nr}}$) the framed local (unramified) deformation ring, [12, 3.2.3].

3.1. Question: would a similar picture hold in p-adic Langlands? The case when $q = p$ is still a mystery. Following the story of Feng, one expects:

$$\mathcal{S}_q^{\mathrm{crys},\underline{h}} \otimes_{\mathcal{S}_q} \mathcal{S}_q^{\mathrm{crys},\underline{h}} \simeq \mathcal{H}_q(\Lambda)$$

We are currently aiming to

- relate our objects of interest as appearing from geometry, as suggested in the conjectures of categorical Langlands, [10, 6], [33, 4.3.1]: in particular, relating the spectral side to *crystalline substacks (of EG stacks)*, [10],
$$\mathcal{X}_d^{\mathrm{crys},\underline{h},\tau}$$
for $\underline{h}$ fixed Hodge Tate weight, and inertial type τ.
- Understand a possible spectral action, in the sense of [25], in the p-adic setting.

[6] Analogous to the geometric Langlands, 2.1

3.2. What else have we attempted? We tried applying the ideas in the work of T. Feng. It did not work as two complications arise

(1) the Satake homomorphism, behaves not as expected, see [**28**, 3]
(2) the lack of nice presentation of the crystalline deformation rings. As was crucial to the argument, see [**12**, 4].

mixed characteristic Casselman-Shalika formula and Whittaker categories

Let G be a split connected reductive algebraic group over the finite field $\mathbb{F}_q$. Let $\mathrm{Sph}_{G,e}^{\heartsuit} := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, e)$ be the *spherical category* of G, or the category of L^+G equivariant perverse sheaves on Gr_G with coefficients in e. For e a field, this is a *highest weight* category, with standard and costandard objects,

$$j_!(\lambda, e) := \pi_0 j_!^{\lambda} k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle] \text{ and } j_*(\lambda, e) := \pi_0 j_*^{\lambda} k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle]$$

If e is of characteristic 0, the category is semisimple, with simple objects

$$\mathcal{A}_\lambda := j_{!*}(\lambda, e)$$

and isomorphic to $\mathrm{Rep}(\widehat{G}, e)$, algebraic representations of the dual group of G with coefficients in e, [**22**]. The reader is welcome to skip from here to the statement of geometric Casselman-Shalika, 3.4.

3.3. The associated function from Frobenius trace.

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

defined on the set of k points of $\overline{\mathrm{Gr}^\lambda}$, can be viewed as a function of the unramified Hecke algebra [**17**], $\mathcal{H}_G$[7]. The constant term map

$$\mathcal{H}_G \to \mathcal{H}_T, f \mapsto f^B$$

has formula given by

$$f^B(t) := \delta_{B(K)}^{1/2}(t) \int_{N(K)} f(tu)\, du$$

The obvious basis elements $\{f_\lambda\}_{\lambda \in \Lambda_+}$ defined as indicator functions of double cosets, has a surprisingly simple formula, [**23**], under the constant term map

$$f_\lambda^B(t) = \int_{U(K)} A_\lambda(x\varpi^\nu)\, dx = (-1)^{2\langle \rho, \nu\rangle} q^{\langle \rho, \nu \rangle} m_\lambda(\nu)$$

[7]compactly supported functions in $G(K)$ this is bi-equivariant with respect to $G(\mathcal{O})$

3.4. The geometric Casselman-Shalika formula. The equal characteristic *geometric* Casselman -Shalika states

Theorem 3.2. [**13**, 8.1.2]

$$H_c^i(S^\mu, j_{!*}(\lambda, e)\Big|_{S^\mu} \otimes_e \chi_\mu^*(\mathcal{L}_\psi)) = \begin{cases} e & \text{if } \lambda = \mu \text{ and } \langle 2\check{\rho}, \lambda\rangle = i \\ 0 & \text{otherwise.} \end{cases}$$

This is a geometrization of the classical Casselman-Shalika formula described in 3.3. The first goal of the project is therefore to give a mixed characteristic (of the geometry) version. This will make extensive use of recent of results of Fargues and Scholze, [**14**].

The project's second goal is to set up the foundations of Whittaker category in mixed characteristic, by understanding it as a left module over the spherical Hecke category. This is important in setting up geometric Langlands in the mixed characteristic setting, see 3.5.

By generalizing, suggests a fundamental property of the representation theory of reductive groups over local non-archimedean fields and allows one to import further arithmetic information.

3.5. Related works. Beyond its applications in the original paper. [**13**], the geometric CS formula in equal characteristic has been applied in recent work [**3**] to give an *Iwahori-Whittaker model* of the Satake category.

The implication of such a geometric model is twofold. Firstly, it gives a geometric description of the representation category.

$$D^b_{\mathrm{IW}}(\mathrm{Gr}_G, e) \simeq D^b(\mathrm{Rep}_e(\check{G})^\heartsuit)$$

But further shows the derived category is *abelian*, which is much more easy to control.

Secondly, this result fits in the framework of *fundamental local equivalence* (FLE), a program initiated by D. Gaitsgory, [**16**]. The equivalence is present in [**8**, Thm. 3]. The Iwahori-Whittaker model is what the Whittaker filtration stabilizes to, see [**26**].

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