SEMIINFINITE FLAGS. I. CASE OF GLOBAL CURVE $\mathbb{P}^1$.

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1. Introduction

1.1. We learnt of the *Semiinfinite Flag Space* from B.Feigin and E.Frenkel in the late 80-s. Since then we tried to understand this remarkable object. It appears that it was essentially constructed, but under different disguises, by V.Drinfeld and G.Lusztig in the early 80-s. Another recent discovery (*Beilinson-Drinfeld Grassmannian*) turned out to conceal a new incarnation of Semiinfinite Flags. We write down these and other results scattered in the folklore.

1.2. Let $\mathbf{G}$ be an almost simple simply-connected group with a Cartan datum $(I,\cdot)$ and a simply-connected simple root datum $(Y,X,\dots)$ of finite type as in [L4], 2.2. We fix a Borel subgroup $\mathbf{B}\subset\mathbf{G}$, with a Cartan subgroup $\mathbf{H}\subset\mathbf{B}$, and the unipotent radical $\mathbf{N}$. B.Feigin and E.Frenkel define the Semiinfinite Flag Space $\mathcal{Z}$ as the quotient of $\mathbf{G}((z))$ modulo the connected component of $\mathbf{B}((z))$ (see [FF]). Then they study the category $\mathcal{PS}$ of perverse sheaves on $\mathcal{Z}$ equivariant with respect to the Iwahori subgroup $\mathbf{I}\subset\mathbf{G}[[z]]$.

In the first two chapters we are trying to make sense of this definition. We encounter a number of versions of this space. In order to give it a structure of an ind-scheme, we define the (local) semiinfinite flag space as $\widetilde{\mathbf{Q}}=\mathbf{G}((z))/\mathbf{HN}((z))$ (see section 4). The (global) semiinfinite space attached to a smooth complete curve C is the system of varieties $\mathcal{Q}^\alpha$ of "quasimaps" from C to the flag variety of $\mathbf{G}$ — the Drinfeld compactifications of the degree α maps. In the present work we restrict ourselves to the case $C=\mathbb{P}^1$.

The main incarnation of the semiinfinite flag space in this paper is a collection $\mathcal{Z}$ (for *zastava*) of (affine irreducible finite dimensional) algebraic varieties $\mathcal{Z}^\alpha_\chi\subseteq\mathcal{Q}^\alpha$, together with certain closed embeddings and *factorizations*. Our definition of $\mathcal{Z}$ follows the scheme suggested by G.Lusztig in [L3], §11: we approximate the "closures" of Iwahori orbits by their intersections with the transversal orbits of the opposite Iwahori subgroup. However, since the set-theoretic intersections of the above "closures" with the opposite Iwahori orbits can not be equipped with the structure of algebraic varieties, we postulate $\mathcal{Z}^\alpha_\chi$ for the "correct" substitutes of such intersections.

Having got the collection of $\mathcal{Z}^\alpha_\chi$ with factorizations, we imitate the construction of [FS] to define the category $\mathcal{PS}$ (for *polubeskrajni snopovi*) of certain collections of perverse sheaves with $\mathbb{C}$-coefficients on $\mathcal{Z}^\alpha_\chi$ with *factorization isomorphisms*. It is defined in chapter 2; this category is the main character of the present work.

1.3. If $\mathbf{G}$ is of type A,D,E we set $d=1$; if $\mathbf{G}$ is of type B,C,F we set $d=2$; if $\mathbf{G}$ is of type G_2 we set $d=3$. Let q be a root of unity of sufficiently large degree ℓ divisible by $2d$. Let $\mathfrak{u}$ be the small (finite-dimensional) quantum group associated to q and the root datum $(Y,X,\dots)$ as in [L4]. Let $\mathcal{C}$ be the category of X-graded $\mathfrak{u}$-modules as defined in [AJS]. Let $\mathcal{C}^0$ be the block of $\mathcal{C}$ containing the trivial $\mathfrak{u}$-module. B.Feigin and G.Lusztig conjectured (independently) that the category $\mathcal{C}^0$ is equivalent to $\mathcal{PS}$.

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Let $\mathfrak{U} \supset \mathfrak{u}$ be the quantum group with divided powers associted to q and the root datum $(Y, X, \dots)$ as in [L4]. Let $\mathfrak{C}$ be the category of X-graded finite dimensional $\mathfrak{U}$-modules, and let $\mathfrak{C}^0$ be the block of $\mathfrak{C}$ containing the trivial $\mathfrak{U}$-module. The works [KL], [L5] and [KT2] establish an equivalence of $\mathfrak{C}^0$ and the category $\mathcal{P}(\mathcal{G}, \mathbf{I})$. Here $\mathcal{G}$ denotes the affine Grassmannian $\mathbf{G}((z))/\mathbf{G}[[z]]$, and $\mathcal{P}(\mathcal{G}, \mathbf{I})$ stands for the category of perverse sheaves on $\mathcal{G}$ with finite-dimensional support constant along the orbits of $\mathbf{I}$.

1.4. The chapter 3 is devoted to the construction of the *convolution* functor $\mathbf{c}_{\mathcal{Z}} : \mathcal{P}(\mathcal{G}, \mathbf{I}) \longrightarrow \mathcal{PS}$ which is the geometric counterpart of the restriction functor from $\mathfrak{C}^0$ to $\mathcal{C}^0$, as suggested by V.Ginzburg (cf. [GK] §4). One of the main results of this chapter is the Theorem 13.2 which is the sheaf-theoretic version of the classical Satake isomorphism. Recall that one has a *Frobenius homomorphism* $\mathfrak{U} \longrightarrow U(\mathfrak{g}^L)$ (see [L4]) where $U(\mathfrak{g}^L)$ stands for the universal enveloping algebra of the Langlands dual Lie algebra $\mathfrak{g}^L$. Thus the category of finite dimensional $\mathbf{G}^L$-modules is naturally embedded into $\mathfrak{C}$ (and in fact, into $\mathfrak{C}^0$). On the geometric level this corresponds to the embedding $\mathcal{P}(\mathcal{G}, \mathbf{G}[[z]]) \subset \mathcal{P}(\mathcal{G}, \mathbf{I})$. The Theorem 13.2 gives a natural interpretation (suggested by V.Ginzburg) of the weight spaces of $\mathbf{G}^L$-modules in terms of the composition

$$\mathbf{G}^L - mod \simeq \mathcal{P}(\mathcal{G}, \mathbf{G}[[z]]) \subset \mathcal{P}(\mathcal{G}, \mathbf{I}) \xrightarrow{\mathbf{c}_{\mathcal{Z}}} \mathcal{PS}.$$

1.5. Let us also mention here the following conjecture which might be known to specialists (characteristic p analogue of conjecture in 1.3). Let $\mathbf{G}^L$ stand for the Langlands dual Lie group. Let p be a prime number bigger than the Coxeter number of $\mathfrak{g}^L$, and let $\overline{\mathbb{F}}_p$ be the algebraic closure of finite field $\mathbb{F}_p$. Let $\mathfrak{C}_p$ be the category of algebraic $\mathbf{G}^L(\overline{\mathbb{F}}_p)$-modules, and let $\mathfrak{C}_p^0$ be the block of $\mathfrak{C}_p$ containing the trivial module. Let $\mathcal{C}_p$ be the category of graded modules over the Frobenius kernel of $\mathbf{G}^L(\overline{\mathbb{F}}_p)$, and let $\mathcal{C}_p^0$ be the block of $\mathcal{C}_p$ containing the trivial module (see [AJS]). Finally, let $\mathcal{PS}_p$ be the category of snops *with coefficients in* $\overline{\mathbb{F}}_p$, and let $\mathcal{P}(\mathcal{G}, \mathbf{I})_p$ be the category of perverse sheaves on $\mathcal{G}$ constant along $\mathbf{I}$-orbits *with coefficients in* $\overline{\mathbb{F}}_p$. Then the categories $\mathcal{C}_p^0$ and $\mathcal{PS}_p$ are equivalent, the categories $\mathfrak{C}_p^0$ and $\mathcal{P}(\mathcal{G}, \mathbf{I})_p$ are equivalent, and under these equivalences the restriction functor $\mathfrak{C}_p^0 \longrightarrow \mathcal{C}_p^0$ corresponds to the convolution functor $\mathcal{P}(\mathcal{G}, \mathbf{I})_p \longrightarrow \mathcal{PS}_p$ (cf. 1.4). The equivalence $\mathcal{P}(\mathcal{G}, \mathbf{I})_p \xrightarrow{\sim} \mathfrak{C}_p^0$ should be an extension of the equivalence between $\mathcal{P}(\mathcal{G}, \mathbf{G}[[z]])_p \subset \mathcal{P}(\mathcal{G}, \mathbf{I})_p$ and the subcategory of $\mathfrak{C}_p^0$ formed by the $\mathbf{G}^L(\overline{\mathbb{F}}_p)$-modules which factor through the Frobenius homomorphism $Fr : \mathbf{G}^L(\overline{\mathbb{F}}_p) \longrightarrow \mathbf{G}^L(\overline{\mathbb{F}}_p)$. The latter equivalence is the subject of forthcoming paper of K.Vilonen and the second author.

1.6. The Zastava space $\mathcal{Z}$ organizing all the "transversal slices" $\mathcal{Z}_\chi^\alpha$ may seem cumbersome. At any rate the existence of various models of the slices $\mathcal{Z}_\chi^\alpha$ (chapter 1), is undoubtedly beautiful by itself. Some of the wonderful properties of $\mathcal{Q}^\alpha$ and $\mathcal{Z}_\chi^\alpha$ are demonstrated in [Ku], [FK], [FKM] in the case $\mathbf{G} = SL_n$. We expect all these properties to hold for the general $\mathbf{G}$.

1.7. To guide the patient reader through the notation, let us list the key points of this paper. The Theorem 7.3 identifies the different models of $\mathcal{Z}_\chi^\alpha$ (all essentially due to V.Drinfeld) and states the factorization property. The exactness of the convolution functor $\mathbf{c}_{\mathcal{Z}} : \mathcal{P}(\mathcal{G}, \mathbf{I}) \longrightarrow \mathcal{PS}$ is proved in the Theorem 12.12 and Corollary 12.14. The Theorem 13.2 computes the value of the convolution functor on $\mathbf{G}[[z]]$-equivariant sheaves modulo the parity vanishing conjecture 10.7.4.

1.8. In the next parts we plan to study D-modules on the local variety $\widetilde{\mathbf{Q}}$ (local construction of the category $\mathcal{PS}$, global sections as modules over affine Lie algebra $\hat{\mathfrak{g}}$, action of the affine Weyl group by Fourier transforms), the relation of the local and global varieties (local and global Whittaker sheaves, a version of the convolution functor twisted by a character of $N((z))$), and the sheaves on Drinfeld compactifications of maps into partial flag varieties.

1.9. The present work owes its very existence to V.Drinfeld. It could not have appeared without the generous help of many people who shared their ideas with the authors. Thus, the idea of *factorization* (section 9) is due to V.Schechtman. A.Beilinson and V.Drinfeld taught us the *Plücker* picture of the (Beilinson-Drinfeld) affine Grassmannian (sections 6 and 10). G.Lusztig has computed the local singularities of the Schubert strata closures in the spaces $\mathcal{Z}_\chi^\alpha$ (unpublished, cf [L1]). B.Feigin and V.Ginzburg taught us their understanding of the Semiinfinite Flags for many years (in fact, we learnt of Drinfeld's Quasimaps' spaces from V.Ginzburg in the Summer 1995). A.Kuznetsov was always ready to help us whenever we were stuck in the geometric problems (in fact, for historical reasons, the section 3 has a lot in common with [Ku] §1). We have also benefited from the discussions with R.Bezrukavnikov and M.Kapranov. Parts of this work were done while the authors were enjoying the hospitality and support of the University of Massachusetts at Amherst, the Independent Moscow University and the Sveučilište u Zagrebu. It is a great pleasure to thank these institutions.

2. Notations

2.1. **Group G and its Weyl group $\mathcal{W}_f$.** We fix a Cartan datum $(I,\cdot)$ and a simply-connected simple root datum $(Y,X,\dots)$ of finite type as in [L4], 2.2.

Let $\mathbf{G}$ be the corresponding simply-connected almost simple Lie group with the Cartan subgroup $\mathbf{H}$ and the Borel subgroup $\mathbf{B}\supset\mathbf{H}$ corresponding to the set of simple roots $I\subset X$. We will denote by $\mathcal{R}^+\subset X$ the set of positive roots. We will denote by $2\rho\in X$ the sum of all positive roots.

Let $\mathbf{B}_+=\mathbf{B}$ and let $\mathbf{B}_-\supset\mathbf{H}$ be the opposite Borel subgroup. Let $\mathbf{N}$ (resp. $\mathbf{N}_-$) be the radical of $\mathbf{B}$ (resp. $\mathbf{B}_-$). Let $\mathbf{H}_a=\mathbf{B}/\mathbf{N}=\mathbf{B}_-/\mathbf{N}_-$ be the abstract Cartan group. The corresponding Lie algebras are denoted, respectively, by $\mathfrak{b},\mathfrak{b}_-,\mathfrak{n},\mathfrak{n}_-,\mathfrak{h}$.

Let $\mathbf{X}$ be the flag manifold $\mathbf{G}/\mathbf{B}$, and let $\mathbf{A}=\mathbf{G}/\mathbf{N}$ be the principal affine space. We have canonically $H_2(\mathbf{X},\mathbb{Z})=Y;\ H^2(\mathbf{X},\mathbb{Z})=X$.

For $\nu\in X$ let $\mathbf{L}_\nu$ denote the corresponding $\mathbf{G}$-equivariant line bundle on $\mathbf{X}$.

Let $\mathcal{W}_f$ be the Weyl group of $\mathbf{G}$. We have a canonical bijection $\mathbf{X}^\mathbf{H}=\mathcal{W}_f$ such that the neutral element $e\in\mathcal{W}_f=\mathbf{X}^\mathbf{H}\subset\mathbf{X}$ forms a single $\mathbf{B}$-orbit.

We have a Schubert stratification of $\mathbf{X}$ by $\mathbf{N}$- (resp. $\mathbf{N}_-$-)orbits: $\mathbf{X}=\sqcup_{w\in\mathcal{W}_f}\mathbf{X}_w$ (resp. $\mathbf{X}=\sqcup_{w\in\mathcal{W}_f}\mathbf{X}^w$) such that for $w\in\mathcal{W}_f=\mathbf{X}^\mathbf{H}\subset\mathbf{X}$ we have $\mathbf{X}^w\cap\mathbf{X}_w=\{w\}$.

We denote by $\overline{\mathbf{X}}_w$ (resp. $\overline{\mathbf{X}}^w$) the Schubert variety — the closure of $\mathbf{X}_w$ (resp. $\mathbf{X}^w$). Note that $\overline{\mathbf{X}}_w=\sqcup_{y\le w}\mathbf{X}_y$ while $\overline{\mathbf{X}}^w=\sqcup_{z\ge w}\mathbf{X}^z$ where $\le$ denotes the standard Bruhat order on $\mathcal{W}_f$.

Let $e\in\mathcal{W}_f$ be the shortest element (neutral element), let $w_0\in\mathcal{W}_f$ be the longest element, and let $s_i,\ i\in I$, be the simple reflections in $\mathcal{W}_f$.

2.2. **Irreducible representations of G.** We denote by X^+ the cone of positive weights (highest weights of finite dimensional $\mathbf{G}$-modules). The fundamental weights $\omega_i:\ \langle i,\omega_j\rangle=\delta_{ij}$ form the basis of X^+.

For $\lambda\in X^+$ we denote by V_λ the finite dimensional irreducible representation of $\mathbf{G}$ with highest weight λ.

We denote by $V_\lambda^\vee$ the representation dual to V_λ; the pairing: $V_\lambda^\vee\times V_\lambda\longrightarrow\mathbb{C}$ is denoted by $\langle,\rangle$.

For each $\lambda\in X^+$ we choose a nonzero vector $y_\lambda\in V_\lambda^{\mathbf{N}_-}$. We also choose a nonzero vector $x_\lambda\in(V_\lambda^\vee)^\mathbf{N}$ such that $\langle x_\lambda,y_\lambda\rangle=1$.

2.3. **Configurations of I-colored divisors.** Let us fix $\alpha \in \mathbb{N}[I] \subset Y,\ \alpha = \sum_{i \in I} a_i i$. Given a curve C we consider the configuration space $C^\alpha \stackrel{\text{def}}{=} \prod_{i \in I} C^{(a_i)}$ of colored effective divisors of multidegree α (the set of colors is I). The dimension of C^α is equal to the length $|\alpha| = \sum_{i \in I} a_i$.

Multisubsets of a set S are defined as elements of some symmetric power $S^{(k)}$ and we denote the image of $(s_1, ..., s_k) \in S^k$ in $S^{(k)}$ by $\{\{s_1, ..., s_k\}\}$. We denote by $\mathfrak{P}(\alpha)$ the set of all partitions of α, i.e multisubsets $\Gamma = \{\{\gamma_1, ..., \gamma_k\}\}$ of $\mathbb{N}[I]$ with $\gamma_r \neq 0$ and $\sum_{r=1}^k \gamma_i = \alpha$.

For $\Gamma \in \mathfrak{P}(\alpha)$ the corresponding stratum C^α_Γ is defined as follows. It is formed by configurations which can be subdivided into m groups of points, the r-th group containing γ_r points; all the points in one group equal to each other, the different groups being disjoint. For example, the main diagonal in C^α is the closed stratum given by partition $\alpha = \alpha$, while the complement to all diagonals in C^α is the open stratum given by partition

$$\alpha = \sum_{i \in I} (\underbrace{i_k + i_k + \ldots + i_k}_{a_k \text{ times}})$$

Evidently, $C^\alpha = \bigsqcup_{\Gamma \in \mathfrak{P}(\alpha)} C^\alpha_\Gamma$.

CHAPTER 1. The spaces Q and Z

3. Quasimaps from a curve to a flag manifold

3.1. We fix a smooth projective curve C and $\alpha \in \mathbb{N}[I]$.

3.1.1. *Definition.* An algebraic map $f:\ C \longrightarrow \mathbf{X}$ has degree α if the following equivalent conditions hold:

a) For the fundamental class $[C] \in H_2(C, \mathbb{Z})$ we have $f_*[C] = \alpha \in Y = H_2(\mathbf{X}, \mathbb{Z})$;

b) For any $\nu \in X$ the line bundle $f^*\mathbf{L}_\nu$ on C has degree $\langle \alpha, \nu \rangle$.

3.2. The Plücker embedding of the flag manifold $\mathbf{X}$ gives rise to the following interpretation of algebraic maps of degree α.

For any irreducible V_λ we consider the trivial vector bundle $\mathcal{V}_\lambda = V_\lambda \otimes \mathcal{O}$ over C.

For any $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ we denote by the same letter the induced morphism $\phi: \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$.

Then a map of degree α is a collection of *line subbundles* $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda,\ \lambda \in X^+$ such that:

a) $\deg \mathfrak{L}_\lambda = -\langle \alpha, \lambda \rangle$;

b) For any surjective $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have $\phi(\mathfrak{L}_\lambda \otimes \mathfrak{L}_\mu) = \mathfrak{L}_\nu$;

c) For any $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(\mathfrak{L}_\lambda \otimes \mathfrak{L}_\mu) = 0$.

Since the surjections $V_\lambda \otimes V_\mu \to V_{\lambda+\mu}$ form one $\mathbb{C}^*$-orbit, systems $\mathcal{L}_\lambda$ satisfying (b) are determined by a choice of $\mathfrak{L}_{\omega_i}$ for the fundamental weights $\omega_i,\ i \in I$.

If we replace the curve C by a point, we get the Plücker description of the flag variety $\mathbf{X}$ as the set of collections of lines $L_\lambda \subseteq V_\lambda$ satisfying conditions of type (b) and (c). Here, a Borel subgroup B in $\mathbf{X}$ corresponds to a system of lines $(L_\lambda,\ \lambda \in X^+)$ if the lines are the fixed points of the unipotent radical N of B, $L_\lambda = (V_\lambda)^N$, or equivalently, if N is the common stabilizer for all lines $N = \bigcap_{\lambda \in X^+} G_{L_\lambda}$.

The space of degree α quasimaps from C to $\mathbf{X}$ will be denoted by $\overset{\circ}{\mathcal{Q}}{}^{\alpha}$.

3.3. **Definition.** (V.Drinfeld) The space $\mathcal{Q}^{\alpha} = \mathcal{Q}^{\alpha}_{C}$ of *quasimaps* of degree α from C to $\mathbf{X}$ is the space of collections of *invertible subsheaves* $\mathfrak{L}_{\lambda} \subset \mathcal{V}_{\lambda},\ \lambda \in X^{+}$ such that:

a) $\deg \mathfrak{L}_{\lambda} = -\langle \alpha, \lambda \rangle$;

b) For any surjective $\mathbf{G}$-morphism $\phi:\ V_{\lambda} \otimes V_{\mu} \longrightarrow V_{\nu}$ such that $\nu = \lambda + \mu$ we have $\phi(\mathfrak{L}_{\lambda} \otimes \mathfrak{L}_{\mu}) = \mathfrak{L}_{\nu}$;

c) For any $\mathbf{G}$-morphism $\phi:\ V_{\lambda} \otimes V_{\mu} \longrightarrow V_{\nu}$ such that $\nu < \lambda + \mu$ we have $\phi(\mathfrak{L}_{\lambda} \otimes \mathfrak{L}_{\mu}) = 0$.

3.3.1. *Lemma.* a) The evident inclusion $\overset{\circ}{\mathcal{Q}}{}^{\alpha} \subset \mathcal{Q}^{\alpha}$ is an open embedding;

b) $\mathcal{Q}^{\alpha}$ is a projective variety.

Proof. Obvious. $\Box$

3.3.2. Here is another version of the Definition, also due to V.Drinfeld. The principal affine space $\mathbf{A} = \mathbf{G}/\mathbf{N}$ is an $\mathbf{H}_a$-torsor over $\mathbf{X}$. We consider its affine closure $\overline{\mathbf{A}}$, that is, the spectrum of the ring of functions on $\mathbf{A}$. Recall that $\overline{\mathbf{A}}$ is the space of collections of vectors $v_{\lambda} \in V_{\lambda},\ \lambda \in X^{+}$, satisfying the following Plücker relations:

a) For any surjective $\mathbf{G}$-morphism $\phi:\ V_{\lambda} \otimes V_{\mu} \longrightarrow V_{\nu}$ such that $\nu = \lambda + \mu$, and $\phi(y_{\lambda} \otimes y_{\mu}) = y_{\nu}$, we have $\phi(v_{\lambda} \otimes v_{\mu}) = v_{\nu}$;

b) For any $\mathbf{G}$-morphism $\phi:\ V_{\lambda} \otimes V_{\mu} \longrightarrow V_{\nu}$ such that $\nu < \lambda + \mu$ we have $\phi(v_{\lambda} \otimes v_{\mu}) = 0$.

The action of $\mathbf{H}_a$ extends to $\overline{\mathbf{A}}$ but it is not free anymore. Consider the quotient stack $\hat{\mathbf{X}} = \overline{\mathbf{A}}/\mathbf{H}_a$. The flag variety $\mathbf{X}$ is an open substack in $\hat{\mathbf{X}}$. A map $\hat{\phi}:\ C \to \hat{\mathbf{X}}$ is nothing else than an $\mathbf{H}_a$-torsor Φ over C along with an $\mathbf{H}_a$-equivariant morphism $f:\ \Phi \to \overline{\mathbf{A}}$. The degree of this map is defined as follows.

Let $\lambda:\ \mathbf{H}_a \to \mathbb{C}^*$ be the character of $\mathbf{H}_a$ corresponding to a weight $\lambda \in X$. Let $\mathbf{H}_{\lambda} \subset \mathbf{H}_a$ be the kernel of the morphism λ. Consider the induced $\mathbb{C}^*$-torsor $\Phi_{\lambda} = \Phi/\mathbf{H}_{\lambda}$ over C. The map $\hat{\phi}$ has degree $\alpha \in \mathbb{N}[I]$ if

$$\text{for any } \lambda \in X \quad \text{we have} \quad \deg(\Phi_{\lambda}) = \langle \lambda, \alpha \rangle.$$

Definition. The space $\mathcal{Q}^{\alpha}$ is the space of maps $\hat{\phi}:\ C \to \hat{\mathbf{X}}$ of degree α such that the generic point of C maps into $\mathbf{X} \subset \hat{\mathbf{X}}$.

The equivalence of the two versions of Definition follows by comparing their Plücker descriptions.

3.4. In this subsection we describe a stratification of $\mathcal{Q}^{\alpha}$ according to the singularities of quasimaps.

3.4.1. Given $\beta, \gamma \in \mathbb{N}[I]$ such that $\beta + \gamma = \alpha$, we define the proper map $\sigma_{\beta,\gamma}:\ \mathcal{Q}^{\beta} \times C^{\gamma} \longrightarrow \mathcal{Q}^{\alpha}$.

Namely, let $f = (\mathfrak{L}_{\lambda})_{\lambda \in X^+} \in\ \mathcal{Q}^{\beta}$ be a quasimap of degree β; and let $D = \sum_{i \in I} D_i \cdot i$ be an effective colored divisor of multidegree $\gamma = \sum_{i \in I} d_i i$, that is, $\deg(D_i) = d_i$. We define $\sigma_{\beta,\gamma}(f, D) \stackrel{\mathrm{def}}{=} f(-D) \stackrel{\mathrm{def}}{=} (\mathfrak{L}_{\lambda}(-\langle D, \lambda \rangle))_{\lambda \in X^+} \in \mathcal{Q}^{\alpha}$, where we use the pairing $Div^I(C) \underset{\mathbb{Z}}{\otimes} X \to Div(C)$ given by $\langle D, \lambda \rangle = \sum_{i \in I} \langle i, \lambda \rangle \cdot D_i$.

3.4.2. **Theorem.** $\mathcal{Q}^\alpha = \bigsqcup_{0\le\beta\le\alpha} \sigma_{\beta,\alpha-\beta}(\overset{\circ}{\mathcal{Q}}{}^\beta \times C^{\alpha-\beta})$

Proof. Any invertible subsheaf $\mathfrak{L}_\lambda \subseteq \mathcal{V}_\lambda$ lies in a unique line subbundle $\tilde{\mathfrak{L}}_\lambda \subseteq \mathcal{V}_\lambda$ called the *normalization* of $\mathfrak{L}$. So any quasimap $\mathfrak{L}$ defines a map $\tilde{\mathfrak{L}}$ (called the *normalization* of $\mathfrak{L}$) of degree $\beta \le \alpha$ and an I-colored effective divisor D (called the *defect* of $\mathfrak{L}$) corresponding to the torsion sheaf $\tilde{\mathfrak{L}}/\mathfrak{L}$, such that $\mathfrak{L} = \tilde{\mathfrak{L}}(-D)$. □

3.4.3. *Definition.* Given a quasimap $f = (\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \mathcal{Q}^\alpha$, its *domain of definition* $U(f)$ is the maximal Zariski open $U(f) \subset C$ such that for any λ the invertible subsheaf $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda$ restricted to $U(f)$ is actually a line subbundle.

3.4.4. *Corollary.* For a quasimap $f = (\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \mathcal{Q}^\alpha$ of degree α the complement $C - U(f)$ of its domain of definition consists of at most $|\alpha|$ points. □

3.5. From now on, unless explicitly stated otherwise, $C = \mathbb{P}^1$.

Proposition. (V.Drinfeld) $\overset{\circ}{\mathcal{Q}}{}^\alpha$ is a smooth manifold of dimension $2|\alpha| + \dim(\mathbf{X})$.

Proof. We have to check that at a map $f \in \overset{\circ}{\mathcal{Q}}{}^\alpha$ the first cohomology $H^1(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X})$ vanishes (where $\mathcal{T}\mathbf{X}$ stands for the tangent bundle of $\mathbf{X}$), and then the tangent space $\Theta_f \overset{\circ}{\mathcal{Q}}{}^\alpha$ equals $H^0(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X})$. As $\mathcal{T}\mathbf{X}$ is generated by the global sections, $f^*\mathcal{T}\mathbf{X}$ is generated by global sections as well, hence $H^1(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X}) = 0$. To compute the dimension of $\Theta_f \overset{\circ}{\mathcal{Q}}{}^\alpha = H^0(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X})$ it remains to compute the Euler characteristic $\chi(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X})$. To this end we may replace $\mathcal{T}\mathbf{X}$ with its associated graded bundle $\oplus_{\theta\in\mathcal{R}^+}\mathbf{L}_\theta$. Then

$$\chi(\mathbb{P}^1, f^*(\bigoplus_{\theta\in\mathcal{R}^+} \mathbf{L}_\theta)) = \sum_{\theta\in\mathcal{R}^+} (\langle\alpha,\theta\rangle + 1) = \langle\alpha, 2\rho\rangle + \sharp\mathcal{R}^+ = 2|\alpha| + \dim \mathbf{X}$$

□

3.6. Now we are able to introduce our main character. First we consider the open subspace $U^\alpha \subset \mathcal{Q}^\alpha$ formed by the quasimaps containing $\infty \in \mathbb{P}^1$ in their domain of definition (see 3.4.3). Next we define the closed subspace $\mathcal{Z}^\alpha \subset U^\alpha$ formed by quasimaps with value at ∞ equal to $\mathbf{B}_- \in \mathbf{X}$:

$$\mathcal{Z}^\alpha \overset{\mathrm{def}}{=} \{f \in U^\alpha | f(\infty) = \mathbf{B}_-\}$$

We will see below that $\mathcal{Z}^\alpha$ is an affine algebraic variety.

3.6.1. It follows from Proposition 3.5 that $\dim \mathcal{Z}^\alpha = 2|\alpha|$.

4. Local Flag space

In this section we define a version of $\mathcal{Q}^\alpha$ where one replaces the global curve C by the formal neighbourhood of a point.

4.1. We set $\mathcal{O}=\mathbb{C}[[z]]\xrightarrow{p_n}\mathcal{O}_n=\mathbb{C}[[z]]/z^n,\mathcal{K}=\mathbb{C}((z))$.

We define the scheme $\overline{\mathbf{A}}(\mathcal{O})$ (of infinite type): its points are the collections of vectors $v_\lambda\in V_\lambda\otimes\mathcal{O},\ \lambda\in X^+$, satisfying the Plücker equations like in 3.3.2. It is a closed subscheme of $\prod_{i\in I}V_{\omega_i}\otimes\mathcal{O}$. We define the open subscheme $\overline{\mathbf{A}}(\mathcal{O})_n\subset\overline{\mathbf{A}}(\mathcal{O})$: it is formed by the collections $(v_\lambda)_{\lambda\in X^+}$ such that $p_n(v_{\omega_i})\neq 0$ for all $i\in I$. Evidently, for $0\le n\le m$, one has $\overline{\mathbf{A}}(\mathcal{O})_n\subset\overline{\mathbf{A}}(\mathcal{O})_m$.

We define the open subscheme $\mathcal{S}\subset\overline{\mathbf{A}}(\mathcal{O})$ as the union $\bigcup_{n\ge 0}\overline{\mathbf{A}}(\mathcal{O})_n$. One has $\mathcal{S}=\mathbf{A}(\mathcal{O})$.

The scheme $\mathcal{S}$ is equipped with the free action of $\mathbf{H}_a:\ h(v_\lambda)_{\lambda\in X^+}=(\lambda(h)v_\lambda)_{\lambda\in X^+}$. The quotient scheme $\mathbf{Q}=\mathcal{S}/\mathbf{H}_a$ is a closed subscheme in $\prod_{i\in I}\mathbb{P}(V_{\omega_i}\otimes\mathcal{O})$. It is formed by the collections of lines satisfying the Plücker equations. We denote the natural projection $\mathcal{S}\longrightarrow\mathbf{Q}$ by pr.

4.2. For $\eta\in\mathbb{N}[I]$ we define the closed subscheme $\mathcal{S}^{-\eta}\subset\mathcal{S}$ formed by the collections $(v_\lambda)_{\lambda\in X^+}$ such that $v_\lambda=0\mod z^{\langle\eta,\lambda\rangle}$. We have the natural isomorphism $\mathcal{S}\xrightarrow{\sim}\mathcal{S}^{-\eta},\ (v_\lambda)_{\lambda\in X^+}\mapsto(z^{\langle\eta,\lambda\rangle}v_\lambda)_{\lambda\in X^+}$.

Now we can extend the definition of $\mathcal{S}^\chi$ to arbitrary $\chi\in Y$. Namely, we define $\mathcal{S}^\chi$ to be formed by the collections $(v_\lambda\in V_\lambda\otimes\mathcal{K})_{\lambda\in X^+}$ such that $(z^{\langle\chi,\lambda\rangle}v_\lambda)_{\lambda\in X^+}\in\mathcal{S}$. Evidently, $\mathcal{S}^\chi\subset\mathcal{S}^\eta$ iff $\chi\le\eta$, and then the inclusion is the closed embedding. The open subscheme $\mathcal{S}^\eta-\bigcup_{\chi<\eta}\mathcal{S}^\chi\subset\mathcal{S}^\eta$ will be denoted by $\overset{\bullet}{\mathcal{S}}{}^\eta\subset\mathcal{S}^\eta$. The ind-scheme $\bigcup_{\eta\in Y}\mathcal{S}^\eta$ will be denoted by $\widetilde{\mathcal{S}}$. The ind-scheme $\widetilde{\mathcal{S}}$ is equipped with the natural action of the proalgebraic group $\mathbf{G}(\mathcal{O})$ (coming from the action on $\prod_{i\in I}V_{\omega_i}\otimes\mathcal{K}$), and the orbits are exactly $\overset{\bullet}{\mathcal{S}}{}^\eta,\ \eta\in Y$.

4.3. All the above (ind-)schemes are equipped with the free action of $\mathbf{H}_a$, and taking quotients we obtain the schemes $\mathbf{Q}^\eta=\mathcal{S}^\eta/\mathbf{H}_a,\ \eta\in Y$. They are all closed subschemes of the ind-scheme $\prod_{i\in I}\mathbb{P}(V_{\omega_i}\otimes\mathcal{K})$. We have $\mathbf{Q}^\chi\subset\mathbf{Q}^\eta$ iff $\chi\le\eta$, and then the inclusion is the closed embedding. The ind-scheme $\widetilde{\mathbf{Q}}=\widetilde{\mathcal{S}}/\mathbf{H}_a$ is the union $\widetilde{\mathbf{Q}}=\bigcup_{\eta\in Y}\mathbf{Q}^\eta$. The ind-scheme $\widetilde{\mathbf{Q}}$ is equipped with the natural action of the proalgebraic group $\mathbf{G}(\mathcal{O})$ (coming from the action on $\prod_{i\in I}\mathbb{P}(V_{\omega_i}\otimes\mathcal{K})$), and the orbits are exactly $\overset{\bullet}{\mathbf{Q}}{}^\eta=\overset{\bullet}{\mathcal{S}}{}^\eta/\mathbf{H}_a,\ \eta\in Y$.

4.4. We consider $C=\mathbb{P}^1$ with two marked points $0,\infty\in C$. We choose a coordinate z on C such that $z(0)=0,z(\infty)=\infty$.

4.4.1. For $\alpha\in\mathbb{N}[I]$ we define the space $\widehat{\mathcal{Q}}^\alpha\xrightarrow{pr}\mathcal{Q}^\alpha$ formed by the collections $(v_\lambda\in\mathfrak{L}_\lambda\subset\mathcal{V}_\lambda)_{\lambda\in X^+}$ such that

a) $(\mathfrak{L}_\lambda\subset\mathcal{V}_\lambda)_{\lambda\in X^+}\in\mathcal{Q}^\alpha$;

b) v_λ is a regular nonvanishing section of $\mathfrak{L}_\lambda$ on $\mathbb{A}^1=\mathbb{P}^1-\infty$;

c) $(v_\lambda)_{\lambda\in X^+}$ satisfy the Plücker equations like in 3.3.2.

It is easy to see that $\widehat{\mathcal{Q}}^\alpha\xrightarrow{pr}\mathcal{Q}^\alpha$ is a $\mathbf{H}_a$-torsor: $h(v_\lambda,\mathfrak{L}_\lambda)=(\lambda(h)v_\lambda,\mathfrak{L}_\lambda)$.

4.4.2. Taking a formal expansion at $0\in C$ we obtain the closed embedding $\mathfrak{s}_\alpha:\ \widehat{\mathcal{Q}}^\alpha\hookrightarrow\mathcal{S}$. Evidently, $\mathfrak{s}_\alpha$ is compatible with the $\mathbf{H}_a$-action, so it descends to the same named closed embedding $\mathfrak{s}_\alpha:\ \mathcal{Q}^\alpha\hookrightarrow\mathbf{Q}$.

4.4.3. *Lemma.* Let $\beta\in\mathbb{N}[I]$. Then $\operatorname{codim}_{\mathbf{Q}}\mathbf{Q}^{-\beta}\ge 2|\beta|$.

Proof. Choose $\alpha\ge\beta$, and consider the closed embedding $\mathfrak{s}_\alpha:\ \mathcal{Q}^\alpha\hookrightarrow\mathbf{Q}$. Then $\mathfrak{s}_\alpha^{-1}(\mathbf{Q}^{-\beta})=\mathcal{Q}^{\alpha-\beta}$ embedded into $\mathcal{Q}^\alpha$ as follows: $(\mathfrak{L}_\lambda\subset\mathcal{V}_\lambda)_{\lambda\in X^+}\mapsto(\mathfrak{L}_\lambda(-\langle\beta,\lambda\rangle 0)\subset\mathcal{V}_\lambda)_{\lambda\in X^+}$. Now $\operatorname{codim}_{\mathbf{Q}}\mathbf{Q}^{-\beta}\ge\operatorname{codim}_{\mathcal{Q}^\alpha}\mathcal{Q}^{\alpha-\beta}=2|\beta|$. $\square$

5. Plücker sections

In this section we describe another model of the space $\mathcal{Z}^\alpha$ introduced in 3.6.

5.1. We fix a coordinate z on the affine line $\mathbb{A}^1 = \mathbb{P}^1 - \infty$. We will also view the configuration space $\mathbb{A}^\alpha \overset{\text{def}}{=} (\mathbb{A}^1)^\alpha$ (see 2.3) as the space of collections of unitary polynomials $(Q_\lambda)_{\lambda\in X^+}$ in z, such that (a) $\deg(Q_\lambda) = \langle\alpha, \lambda\rangle$, and (b) $Q_{\lambda+\mu} = Q_\lambda Q_\mu$.

5.2. Recall the notations of 2.2. For each $\lambda \in X^+$ we will use the decomposition $V_\lambda = \mathbb{C}y_\lambda \oplus (\mathrm{Ker}x_\lambda) = (V_\lambda)^{\mathbf{N}} \oplus \mathfrak{n}_-V_\lambda$, compatible with the action of $\mathfrak{h} = \mathfrak{b}_- \cap \mathfrak{b}$, i.e., with the weight decomposition. For a section $v_\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda) = V_\lambda \otimes \mathbb{C}[z] \overset{\text{def}}{=} V_\lambda[z]$, we will use a polynomial $Q_\lambda \overset{\text{def}}{=} \langle x_\lambda, v_\lambda\rangle \in \mathbb{C}[z]$, to write down the decomposition $v_\lambda = Q_\lambda \cdot y_\lambda \oplus v''{}_\lambda \in \mathbb{C}[z]\cdot y_\lambda \oplus (\mathrm{Ker}x_\lambda)[z] = V_\lambda[z]$.

Definition. (V.Drinfeld) The space Z^α of *Plücker sections* of degree α is the space of collections of sections $v_\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda) = V_\lambda \otimes \mathbb{C}[z] \overset{\text{def}}{=} V_\lambda[z],\ \lambda \in X^+$; such that for $v_\lambda = Q_\lambda \cdot y_\lambda \oplus v''{}_\lambda \in \mathbb{C}[z]\cdot y_\lambda \oplus (\mathrm{Ker}x_\lambda)[z]$, one has

a) Polynomial Q_λ is unitary of degree $\langle\alpha, \lambda\rangle$;

b) Component $v''{}_\lambda$ of v_λ in $(\mathrm{Ker}x_\lambda)[z]$ has degree strictly less than $\langle\alpha, \lambda\rangle$;

c) For any $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ and $\phi^\vee(x_\nu) = x_\lambda \otimes x_\mu$ we have $\phi(v_\lambda \otimes v_\mu) = v_\nu$;

d) For any $\mathbf{G}$-morphism $\phi:\ V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(v_\lambda \otimes v_\mu) = 0$.

5.2.1. Collections $(v_\lambda)_{\lambda\in X^+}$ that satisfy (c), are determined by a choice of $v_{\omega_i},\ i \in I$. Hence Z^α is an affine algebraic variety.

5.2.2. Due to the properties a),c) above, the collection of polynomials Q_λ defined in a) satisfies the conditions of 5.1. Hence we have the map

$$\pi_\alpha :\ \mathsf{Z}^\alpha \longrightarrow \mathbb{A}^\alpha$$

6. Beilinson-Drinfeld Grassmannian

In this section we describe yet another model of the space $\mathcal{Z}^\alpha$ introduced in 3.6.

6.1. Let C be an arbitrary smooth projective curve; let $\mathcal{T}$ be a left $\mathbf{G}$-torsor over C, and let τ be a section of $\mathcal{T}$ defined over a Zariski open subset $U \subset C$, i.e., a trivialization of $\mathcal{T}$ over U. We will define a $\mathbf{B}$- (resp. $\mathbf{B}_-$-) type $d(\tau)$ (resp. $d_-(\tau)$): a measure of singularity of τ at $C - U$.

6.1.1. Section τ defines a $\mathbf{B}$-subtorsor $\mathbf{B}\cdot\tau \subseteq \mathcal{T}$. This reduction of $\mathcal{T}$ to $\mathbf{B}$ over U is the same as a section of $\mathbf{B}\backslash\mathcal{T}$ over U. Since $\mathbf{G}/\mathbf{B}$ is proper, this reduction (i.e. section), extends uniquely to the whole C. Thus we obtain a $\mathbf{B}$-subtorsor $\overline{\mathbf{B}\cdot\tau} \subseteq \mathcal{T}$ (the closure of $\mathbf{B}\cdot\tau \subseteq \mathcal{T}|U$ in $\mathcal{T}$), equipped with a section τ defined over U.

Using the projection $\mathbf{B} \longrightarrow \mathbf{H}_a$ we can induce $\overline{\mathbf{B}\cdot\tau}$ to a torsor over C for the abstract Cartan group $\mathbf{H}_a \cong \mathbf{B}/\mathbf{N}$ of $\mathbf{G}$; namely, $\mathcal{T}_{\tau,\mathbf{B}} \overset{\text{def}}{=} \mathbf{N}\backslash\overline{\mathbf{B}\cdot\tau}$, equipped with a section $\tau_{\mathbf{B}}$ defined over U.

The choice of simple coroots (cocharacters of $\mathbf{H}_a$) $I \subset Y$ identifies $\mathbf{H}_a$ with $(\mathbb{C}^*)^I$. Thus the section $\tau_{\mathbf{B}}$ of $\mathcal{T}_{\tau,\mathbf{B}}$ produces an I-colored divisor $d(\tau)$ supported at $C - U$. We will call $d(\tau)$ the $\mathbf{B}$-*type* of τ.

Replacing $\mathbf{B}$ by $\mathbf{B}_-$ in the above construction we define the $\mathbf{B}_-$-*type* $d_-(\tau)$.

6.2. Recall that A.Beilinson and V.Drinfeld have introduced the *relative Grassmannian* $\mathcal{G}_C^{(n)}$ over C^n for any $n \in \mathbb{N}$ (see [B]): its fiber $p_n^{-1}(x_1, \dots, x_n)$ over an n-tuple $(x_1, \dots, x_n) \in C^n$ is the space of isomorphism classes of $\mathbf{G}$-torsors $\mathcal{T}$ equipped with a section τ defined over $C - \{x_1, \dots, x_n\}$.

We will consider a certain finite-dimensional subspace of a partialy symmetrized version of the relative Grassmannian.

Definition. (A.Beilinson and V.Drinfeld) $\mathbf{Z}^\alpha$ is the space of isomorphism classes of the following data:

a) an I-colored effective divisor $D \in \mathbb{A}^\alpha$;

b) $\mathbf{G}$-torsor $\mathcal{T}$ over $\mathbb{P}^1$ equipped with a section τ defined over $\mathbb{P}^1 - supp(D)$ such that:

i) $\mathbf{B}$-type $d(\tau) = 0$;

ii) $\mathbf{B}_-$-type $d_-(\tau)$ is a negative divisor (opposite to effective) such that $d_-(\tau) + D$ is effective.

6.2.1. By the definition, the space $\mathbf{Z}^\alpha$ is equipped with a projection p_α to $\mathbb{A}^\alpha$: $(D, \mathcal{T}, \tau) \mapsto D$. For a subset $U \subset \mathbb{A}^1$ we will denote by $\mathbf{Z}_U^\alpha$ the preimage $p_\alpha^{-1}(U)$.

6.2.2. The reader may find another realization of $\mathbf{Z}^\alpha$ in 10.5 below.

6.3. In this subsection we will formulate the crucial *factorization* property of $\mathbf{Z}^\alpha$.

6.3.1. Recall the following property of the Beilinson-Drinfeld relative Grassmannian $\mathcal{G}_C^{(n)} \xrightarrow{p_n} C^n$ (see [B]). Suppose an n-tuple $(x_1, \dots, x_n) \in C^n$ is represented as a union of an m-tuple $(y_1, \dots, y_m) \in C^m$ and a k-tuple $(z_1, \dots, z_k) \in C^k$, $k + m = n$, such that all the points of the m-tuple are disjoint from all the points of the k-tuple. Then $p_n^{-1}(x_1, \dots, x_n)$ is canonically isomorphic to the product $p_m^{-1}(y_1, \dots, y_m) \times p_k^{-1}(z_1, \dots, z_k)$

6.3.2. Suppose we are given a decomposition $\alpha = \beta + \gamma$, $\beta, \gamma \in \mathbb{N}[I]$ and two disjoint subsets $U, \Upsilon \subset \mathbb{A}^1$. Then $U^\beta \times \Upsilon^\gamma$ lies in $\mathbb{A}^\alpha$, and we will denote the preimage $p_\alpha^{-1}(U^\beta \times \Upsilon^\gamma)$ in $\mathbf{Z}^\alpha$ by $\mathbf{Z}_{U,\Upsilon}^{\beta,\gamma} = \mathbf{Z}^\alpha|_{(U^\beta \times \Upsilon^\gamma)}$ (cf. 6.2.1).

The above property of relative Grassmannian immediately implies the following

Factorization property. There is a canonical factorization isomorphism $\mathbf{Z}_{U,\Upsilon}^{\beta,\gamma} \cong \mathbf{Z}_U^\beta \times \mathbf{Z}_\Upsilon^\gamma$, i.e.,

$$\mathbf{Z}^\alpha|_{(U^\beta \times \Upsilon^\gamma)} \cong \mathbf{Z}^\beta|_{U^\beta} \times \mathbf{Z}^\gamma|_{\Upsilon^\gamma}.$$

6.4. **Remark.** Let us describe the fibers of p_α in terms of the normal slices to the semiinfinite Schubert cells in the loop Grassmannian.

6.4.1. Let $\mathcal{G}$ be the usual affine Grassmannian $\mathbf{G}((z))/\mathbf{G}[[z]]$. It is naturally identified with the fiber of $\mathcal{G}_{\mathbb{P}^1}^{(1)}$ over the point $0 \in \mathbb{P}^1$. Due to the Iwasawa decomposition in p-adic groups, there is a natural bijection between Y and the set of orbits of the group $\mathbf{N}((z))$ (resp. $\mathbf{N}_-((z))$) in $\mathcal{G}$; for $\gamma \in Y$ we will denote the corresponding orbit by S_γ (resp. T_γ). We will denote by $\overline{T}_\gamma$ the "closure" of T_γ, that is, the union $\cup_{\beta \geq \gamma} T_\gamma$.

It is proved in [MV] that the intersection $\overline{T}_\gamma \cap S_\beta$ is not empty iff $\gamma \leq \beta$. Then it is an affine algebraic variety, a kind of a normal slice to T_β in $\overline{T}_\gamma$. Let us call it $TS_{\gamma,\beta} \stackrel{\text{def}}{=} \overline{T}_\gamma \cap S_\beta$ for short. If $\text{rank}(\mathbf{G}) > 1$

then $TS_{\gamma,\beta} = \overline{T}_\gamma \cap S_\beta$ is not necessarily irreducible. But it is always equidimensional of dimension $|\beta-\gamma|$. There is a natural bijection between the set of irreducible components of $TS_{\gamma,\beta} = \overline{T}_\gamma \cap S_\beta$ and the canonical basis of $U^+_{\beta-\gamma}$ (the weight $\beta-\gamma$ component of the quantum universal enveloping algebra of $\mathfrak{n}$) (see [L4] for the definition of canonical basis of U^+).

6.4.2. Recall the diagonal stratification of $\mathbb{A}^\alpha$ defined in 2.3 and the map $p_\alpha : \mathbf{Z}^\alpha \to \mathbb{A}^\alpha$. We consider a partition $\Gamma:\ \alpha = \sum_{k=1}^m \gamma_k$ and a divisor D in the stratum $\mathbb{A}^\alpha_\Gamma$. The interested reader will check readily the following

Claim. $p_\alpha^{-1}(D)$ is isomorphic to the product $\prod_{k=1}^m TS_{-\gamma_k,0} = \prod_{k=1}^m \overline{T}_{-\gamma_k} \cap S_0 \cong \prod_{k=1}^m \overline{T}_0 \cap S_{\gamma_k}$.

In particular, the fiber over a point in the closed stratum is isomorphic to $TS_{-\alpha,0} = \overline{T}_{-\alpha} \cap S_0 \cong \overline{T}_0 \cap S_\alpha$, while the fiber over a generic point is isomorphic to the product of affine lines $TS_{-i,0} \cong \overline{T}_0 \cap S_{-i} \cong \mathbb{A}^1$, that is, the affine space $\mathbb{A}^{|\alpha|}$.

6.4.3. *Corollary.* $\mathbf{Z}^\alpha$ is irreducible.

7. Equivalence of the three constructions

7.1. In this subsection we construct an isomorphism $\varpi :\ \mathcal{Z}^\alpha \xrightarrow{\sim} \mathsf{Z}^\alpha$, i.e., from the subsheaves $\mathfrak{L}_\lambda \subseteq \mathcal{V}_\lambda$ we construct the sections $v_\lambda \in \Gamma(\mathbb{A}^1, \mathcal{V}_\lambda)$.

7.1.1. Let $f \in \mathcal{Z}^\alpha$ be a quasimap given by a collection $(\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda = V_\lambda \otimes \mathcal{O}_{\mathbb{P}^1})_{\lambda\in X^+}$. Since $\mathfrak{L}_\lambda|_{\mathbb{A}^1}$ is trivial, it has a unique up to proportionality section v_λ generating it over $\mathbb{A}^1$.

We claim that the pairing $\langle x_\lambda, v_\lambda\rangle$ does not vanish identically. In effect, since $\deg(f) = \alpha$, the meromorphic section $\frac{v_\lambda}{z^{\langle\alpha,\lambda\rangle}}$ of $\mathcal{V}_\lambda$ is regular nonvanishing at $\infty \in \mathbb{P}^1$. Moreover, since $f(\infty) = \mathbf{B}_-$, we have $\frac{v_\lambda}{z^{\langle\alpha,\lambda\rangle}}(\infty) \in V_\lambda^{\mathbf{N}_-}$. Thus, $\langle x_\lambda, \frac{v_\lambda}{z^{\langle\alpha,\lambda\rangle}}\rangle(\infty) \neq 0$.

Now we can normalize v_λ (so far defined up to a multiplication by a constant) by the condition that $\langle x_\lambda, v_\lambda\rangle$ is a unitary polynomial. Let us denote this polynomial by Q_λ. It has degree $d_\lambda \le \langle\alpha,\lambda\rangle$ since $\deg(f) = \alpha$. Since $\frac{v_\lambda}{z^{\langle\alpha,\lambda\rangle}}(\infty) \in V_\lambda^{\mathbf{N}_-}$, we see that $\deg\langle e, v_\lambda\rangle < d_\lambda$ for any $e \perp y_\lambda$. Moreover, since $\deg(f) = \alpha$ we must then have $d_\lambda = \langle\alpha,\lambda\rangle$.

Thus we have checked that the collection $(v_\lambda)_{\lambda\in X^+}$ satisfies the conditions a),b) of the Definition 5.2. The conditions c),d) of *loc. cit.* follow from the conditions b),c) of the Definition 3.3. In other words, we have defined the Plücker section

$$\varpi(f) \overset{\text{def}}{=} (v_\lambda)_{\lambda\in X^+} \in \mathsf{Z}^\alpha$$

7.1.2. Here is the inverse construction. Given a Plücker section $(v_\lambda)_{\lambda\in X^+} \in \mathsf{Z}^\alpha$ we define the corresponding quasimap $f = (\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \mathcal{Z}^\alpha$ as follows.

We can view v_λ as a regular section of $\mathcal{V}_\lambda(\langle\alpha,\lambda\rangle\infty)$ over the whole $\mathbb{P}^1$. It generates an invertible subsheaf $\mathfrak{L}'_\lambda \subset \mathcal{V}_\lambda(\langle\alpha,\lambda\rangle\infty)$. We define

$$\mathfrak{L}_\lambda \overset{\text{def}}{=} \mathfrak{L}'_\lambda(-\langle\alpha,\lambda\rangle\infty) \subset \mathcal{V}_\lambda$$

7.1.3. It is immediate to see that the above constructions are inverse to each other, so that $\varpi :\ \mathcal{Z}^\alpha \longrightarrow \mathsf{Z}^\alpha$ is an isomorphism.

7.1.4. *Remark.* Note that the definition of the space $\mathcal{Z}^\alpha$ depends only on the choice of Borel subgroup $\mathbf{B}_- \subset \mathbf{G}$, while the definition of Z^α depends also on the choice of the opposite Borel subgroup $\mathbf{B} \subset \mathbf{G}$ or, equivalently, on the choice of the Cartan subgroup $\mathbf{H} \subset \mathbf{B}_-$.

We want to stress that the projection $\pi_\alpha : \mathcal{Z}^\alpha = \mathsf{Z}^\alpha \longrightarrow \mathbb{A}^\alpha$ *does depend* on the choice of $\mathbf{B}$. Let us describe $\pi_\alpha \varpi(f)$ for a genuine map (as opposed to quasimap) $f \in \mathcal{Z}^\alpha$. To this end recall (see 2.1) that the $\mathbf{B}$-invariant Schubert varieties $\overline{\mathbf{X}}_{s_i w_0}$, $i \in I$, are divisors in $\mathbf{X}$. Their formal sum may be viewed as an I-colored divisor $\mathfrak{D}$ in $\mathbf{X}$. Then $f^*\mathfrak{D}$ is a well defined I-colored divisor on $\mathbb{P}^1$ since $f(\mathbb{P}^1) \not\subset \mathfrak{D}$ since $f(\infty) = \mathbf{B}_- \in \mathbf{X}_{w_0}$. For the same reason the point ∞ does not lie in $f^*\mathfrak{D}$, so $f^*\mathfrak{D}$ is really a divisor in $\mathbb{A}^1$. It is easy to see that $f^*\mathfrak{D} \in \mathbb{A}^\alpha$ and $f^*\mathfrak{D} = \pi_\alpha \varpi(f)$.

7.2. In this subsection we construct an isomorphism $\xi : \mathsf{Z}^\alpha \xrightarrow{\sim} \mathbf{Z}^\alpha$, so from a system of sections v_λ we construct a $\mathbf{G}$-torsor $\mathcal{T}$ with a section τ and an I-colored divisor D.

7.2.1. *Lemma.* (*The Plücker picture of* $\mathbf{G}$.) The map $\psi : g \mapsto (gx_\lambda, gy_\lambda)_{\lambda \in X^+}$ is a bijection between $\mathbf{G}$ and the space of collections $\{(u_\lambda \in V_\lambda^\vee, v_\lambda \in V_\lambda)_{\lambda \in X^+})\}$ satisfying the following conditions:

a) For any $\mathbf{G}$-morphism $\phi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ and $\phi^\vee(x_\nu) = x_\lambda \otimes x_\mu$ we have $\phi(v_\lambda \otimes v_\mu) = v_\nu$;

b) For any $\mathbf{G}$-morphism $\phi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(v_\lambda \otimes v_\mu) = 0$;

c) For any $\mathbf{G}$-morphism $\varphi : V_\lambda^\vee \otimes V_\mu^\vee \longrightarrow V_\nu^\vee$ such that $\nu = \lambda + \mu$ and $\varphi(x_\lambda \otimes x_\mu) = x_\nu$ we have $\varphi(u_\lambda \otimes u_\mu) = u_\nu$;

d) For any $\mathbf{G}$-morphism $\varphi : V_\lambda^\vee \otimes V_\mu^\vee \longrightarrow V_\nu^\vee$ such that $\nu < \lambda + \mu$ we have $\varphi(u_\lambda \otimes u_\mu) = 0$;

e) $\langle u_\lambda, v_\lambda \rangle = 1$.

Proof. We are considering the systems $(v, u) = (v_\lambda \in V_\lambda,\ u_\lambda \in V_\lambda^\vee,\ \lambda \in X^+)$ such that both v and u are Plücker sections and $\langle v, u \rangle = 1$, i.e., $\langle v_\lambda, u_\lambda \rangle = 1$ for each λ.

These form a $\mathbf{G}$-torsor and we have fixed its element (y, x), which we will use to think of this torsor as a Plücker picture of $\mathbf{G}$.

The stabilizers $\mathbf{G}_v$ and $\mathbf{G}_u$ are the unipotent radicals of the opposite Borel subgroups, for instance $\mathbf{G}_x = \mathbf{N}$ and $\mathbf{G}_y = \mathbf{N}_-$. So this torsor canonicaly maps into the open $\mathbf{G}$-orbit in $\mathbf{X} \times \mathbf{X}$ and the fiber at $(\mathbf{B}', \mathbf{B}'')$ is a torsor for a Cartan subgroup $\mathbf{B}' \cap \mathbf{B}''$. □

7.2.2. Given a Plücker section $(v_\lambda)_{\lambda \in X^+}$, the collection of meromorphic sections $(x_\lambda \in \mathcal{V}_\lambda^\vee, \frac{v_\lambda}{Q_\lambda} \in \mathcal{V}_\lambda)$ evidently satisfies the conditions a)–e) of the above Lemma, and hence defines a meromorphic function $g : \mathbb{A}^1 \longrightarrow \mathbf{G}$. Let us list the properties of this function.

a) By the definition 7.2.1 of the isomorphism ψ, since g fixes the Plücker section x the function g actually takes values in $\mathbf{N} \subset \mathbf{G}$;

b) The argument similar to that used in 7.1.1 shows that g can be extended to $\mathbb{P}^1$, is regular at ∞, and $g(\infty) = 1 \in \mathbf{N}$ (since $\frac{v_\lambda}{Q_\lambda}(\infty) = y_\lambda$, $g(\infty)$ stabilizes x_λ and y_λ so it lies in $\mathbf{N} \cap \mathbf{N}_-$);

c) Let $D = \pi_\alpha(v_\lambda)$ (see 5.2.2) be the I-colored divisor supported at the roots of Q_λ. Then g is regular on $\mathbb{P}^1 - D$.

7.2.3. We define $\xi(v_\lambda) = (D, \mathcal{T}, \tau) \in \mathbf{Z}^\alpha$ as follows: $D = \pi_\alpha(v_\lambda)$; $\mathcal{T}$ is the trivial $\mathbf{G}$-torsor; the section τ is given by the function g. Let us describe the corresponding $\mathbf{H}_a$-torsor $\mathcal{T}_{\tau, \mathbf{B}_-}$ with meromorphic section $\tau_{\mathbf{B}_-}$. To describe an $\mathbf{H}_a$-torsor $\mathfrak{L}$ with a section s it suffices to describe the induced $\mathbb{C}^*$-torsors $\mathfrak{L}_\lambda$ with sections s_λ for all characters $\lambda:\ \mathbf{H}_a \longrightarrow \mathbb{C}^*$. In fact, it suffices to consider only $\lambda \in X^+$. Then $\mathfrak{L}_\lambda$ is given by the construction of 7.1.2, and $s_\lambda = \frac{v_\lambda}{Q_\lambda}$.

Thus, the conditions i),ii) of the Definition 6.2 are evidently satisfied.

7.2.4. To proceed with the inverse construction, we will need the following

Lemma. Suppose $(D, \mathcal{T}, \tau) \in \mathbf{Z}^\alpha$. Then $\mathcal{T}$ is trivial and has a canonical section ς.

Proof. By the construction 6.1.1, $\mathcal{T}$ is induced from the $\mathbf{B}$-torsor $\overline{\mathbf{B}\cdot\tau}$. By the Definition 6.2, the induced $\mathbf{H}_a$-torsor $\mathcal{T}_{\tau,\mathbf{B}}$ is trivial, that is, $\overline{\mathbf{B}\cdot\tau}$ can be further reduced to an $\mathbf{N}$-torsor. But any $\mathbf{N}$-torsor over $\mathbb{P}^1$ is trivial since $H^1(\mathbb{P}^1, \mathbf{V}) = 0$ for any unipotent group $\mathbf{V}$ (induction in the lower central series). □

7.2.5. According to the above Lemma, we can find a unique section ς of $\mathcal{T}$ defined over the whole $\mathbb{P}^1$ and such that $\varsigma(\infty) = \tau(\infty)$. Hence a triple $(D, \mathcal{T}, \tau) \in \mathbf{Z}^\alpha$ canonically defines a meromorphic function

$$g \overset{\text{def}}{=} \tau\varsigma^{-1}:\ \mathbb{P}^1 \longrightarrow \mathbf{G},$$

i.e., $g(x)\cdot\varsigma(x) = \tau(x),\ x \in \mathbb{P}^1$. One sees immediately that g enjoys the properties 7.2.2a)–c). Now we can apply the Lemma 7.2.1 in the opposite direction and obtain from g a collection $\psi^{-1}(g) = (x_\lambda, \tilde{v}_\lambda)_{\lambda\in X^+}$ with $\tilde{v}_\lambda$ a certain meromorphic sections of $\mathcal{V}_\lambda$. According to 5.1, the divisor D defines a collection of unitary polynomials $(Q_\lambda)_{\lambda\in X^+}$, and we can define $v_\lambda \overset{\text{def}}{=} Q_\lambda \tilde{v}_\lambda$. One checks easily that $(v_\lambda) \in \mathsf{Z}^\alpha$, and $(D, \mathcal{T}, \tau) = \xi(v_\lambda)$.

In particular, $\xi:\ \mathsf{Z}^\alpha \longrightarrow \mathbf{Z}^\alpha$ is an isomorphism.

7.3. We conclude that $\mathcal{Z}^\alpha, \mathsf{Z}^\alpha, \mathbf{Z}^\alpha$ are all the same and all maps to $\mathbb{A}^\alpha$ coincide. We preserve the notation $\mathcal{Z}^\alpha$ for this *Zastava* space, and π_α for its projection onto $\mathbb{A}^\alpha$. We combine the properties 3.6.1, 5.2.1, 6.3.2, 6.4.3 into the following

Theorem. a) $\mathcal{Z}^\alpha$ is an irreducible affine algebraic variety of dimension $2|\alpha|$;

b) For any decomposition $\alpha = \beta + \gamma,\ \beta, \gamma \in \mathbb{N}[I]$, and a pair of disjoint subsets $U, \Upsilon \subset \mathbb{A}^1$, we have the *factorization property* (notations of 6.2.1 and 6.3.2):

$$\mathcal{Z}^{\beta,\gamma}_{U,\Upsilon} = \mathcal{Z}^\beta_U \times \mathcal{Z}^\gamma_\Upsilon$$

CHAPTER 2. The category $\mathcal{PS}$

8. Schubert stratification

8.1. We will stratify $\mathcal{Z}^\alpha$ in stages. We denote by $\mathcal{Q}^\alpha \supseteq \overset{\bullet}{\mathcal{Q}}{}^\alpha \supseteq \overset{\circ}{\mathcal{Q}}{}^\alpha$, respectively the variety of all quasimaps of degree α and the subvarieties of the quasimaps defined at 0 and of genuine maps. In the same way we denote the varieties of based quasimaps $\mathcal{Z}^\alpha \supseteq \overset{\bullet}{\mathcal{Z}}{}^\alpha \overset{\text{def}}{=} \mathcal{Z}^\alpha \cap \overset{\bullet}{\mathcal{Q}}{}^\alpha \supseteq \overset{\circ}{\mathcal{Z}}{}^\alpha \overset{\text{def}}{=} \mathcal{Z}^\alpha \cap \overset{\circ}{\mathcal{Q}}{}^\alpha =$ based maps of degree α.

Recall (see 3.4.1) the map $\sigma_{\beta,\gamma}:\ \mathcal{Q}^\beta \times C^{(\gamma)} \to \mathcal{Q}^{\beta+\gamma},\ \sigma_{\beta,\gamma}(f, D) = f(-D)$. For $\beta \le \alpha (= \beta + \gamma)$, it restricts to the embedding $\mathcal{Q}^\beta \hookrightarrow \mathcal{Q}^\alpha,\ f \mapsto f(-(\alpha-\beta)\cdot 0) = f(\ (\beta-\alpha)\cdot 0)$, and in particular $\mathcal{Z}^\beta \hookrightarrow \mathcal{Z}^\alpha$.

8.2. In the first step we stratify $\mathcal{Z}^\alpha$ according to the singularity at 0. It follows immediately from the Theorem 3.4.2 that

$$\mathcal{Z}^\alpha \cong \bigsqcup_{0\le\beta\le\alpha} \overset{\bullet}{\mathcal{Z}}{}^\beta.$$

The closed embedding of a stratum $\overset{\bullet}{\mathcal{Z}}{}^\beta$ into $\mathcal{Z}^\alpha$ will be denoted by $\sigma_{\beta,\alpha-\beta}$.

8.3. Next, we stratify the quasimaps $\overset{\bullet}{\mathcal{Z}}{}^\alpha$ defined at 0, according to the singularity on $\mathbb{C}^*$. Again, it follows immediately from the Theorem 3.4.2 that

$$\overset{\bullet}{\mathcal{Z}}{}^\alpha \cong \bigsqcup_{0\le\beta\le\alpha} \overset{\circ}{\mathcal{Z}}{}^\beta \times (\mathbb{C}^*)^{\alpha-\beta}.$$

8.4. One more refinement comes from the decomposition of the flag variety $\mathbf{X}$ into the $\mathbf{B}$-invariant Schubert cells. Given an element w in the Weyl group $\mathcal{W}_f$, we define the locally closed subvarieties (*Schubert strata*) $\overset{\bullet}{\mathcal{Z}}{}^\alpha_w \subset \overset{\bullet}{\mathcal{Z}}{}^\alpha$ and $\overset{\circ}{\mathcal{Z}}{}^\alpha_w \subset \overset{\circ}{\mathcal{Z}}{}^\alpha$, as the sets of quasimaps f such that $f(0) \in \mathbf{X}_w$. The closure of $\overset{\bullet}{\mathcal{Z}}{}^\alpha_w$ in $\mathcal{Z}^\alpha$ will be denoted by $\overline{\mathcal{Z}}{}^\alpha_w$. Evidently,

$$\overset{\bullet}{\mathcal{Z}}{}^\alpha = \bigsqcup_{w\in\mathcal{W}_f} \overset{\bullet}{\mathcal{Z}}{}^\alpha_w \text{ and } \overset{\circ}{\mathcal{Z}}{}^\alpha = \bigsqcup_{w\in\mathcal{W}_f} \overset{\circ}{\mathcal{Z}}{}^\alpha_w.$$

Beware that $\overset{\bullet}{\mathcal{Z}}{}^\alpha_w$ may happen to be empty: e.g. for $\alpha = 0, w \ne w_0$.

8.4.1. Finally, the last refinement comes from the diagonal stratification of the configuration space $(\mathbb{C}^*)^\delta = \bigsqcup_{\Gamma\in\mathfrak{P}(\delta)} (\mathbb{C}^*)^\delta_\Gamma$. Altogether, we obtain the following stratifications of $\mathcal{Z}^\alpha$:

$$\begin{aligned}\mathcal{Z}^\alpha &\cong \bigsqcup_{\alpha\ge\beta} \overset{\bullet}{\mathcal{Z}}{}^\beta \quad (\textit{coarse stratification})\\ &\cong \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma} \overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma \quad (\textit{fine stratification})\\ &\cong \bigsqcup_{w\in\mathcal{W}_f,\ \Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma} \overset{\circ}{\mathcal{Z}}{}^\gamma_w \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma \quad (\textit{fine Schubert stratification}).\end{aligned}$$

Similarly, we have the *fine stratification* (resp. *fine Schubert stratification*) of $\mathcal{Q}^\alpha$:

$$\mathcal{Q}^\alpha = \bigsqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma} \overset{\circ}{\mathcal{Q}}{}^\gamma \times (\mathbb{P}^1 - 0)^{\beta-\gamma}_\Gamma = \bigsqcup_{w\in\mathcal{W}_f,\ \Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\ge\beta\ge\gamma} \overset{\circ}{\mathcal{Q}}{}^\gamma_w \times (\mathbb{P}^1 - 0)^{\beta-\gamma}_\Gamma$$

Here $\overset{\circ}{\mathcal{Q}}{}^\gamma_w \subset \overset{\circ}{\mathcal{Q}}{}^\gamma$ denotes the locally closed subspace of maps $\mathbb{P}^1 \to \mathbf{X}$ taking value in $\mathbf{X}_w \subset \mathbf{X}$ at $0 \in \mathbb{P}^1$. The strata $\overset{\circ}{\mathcal{Q}}{}^\gamma_w \times (\mathbb{P}^1 - 0)^{\beta-\gamma}_\Gamma$ are evidently smooth.

Note that the strata $\overset{\bullet}{\mathcal{Z}}{}^\alpha_w$ are not necessarily smooth in general, e.g. for $\mathbf{G} = SL_3$, α the sum of simple coroots, $w = w_0$. To understand the "fine Schubert strata" $\overset{\circ}{\mathcal{Z}}{}^\gamma_w \times (\mathbb{C}^*)^\beta_\Gamma$ we need to understand the varieties $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$.

8.5. **Conjecture.** For $\gamma \in \mathbb{N}[I], w \in \mathcal{W}_f$ the variety $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$ is smooth. Hence the "fine Schubert stratification" is really a stratification.

8.5.1. *Lemma.* For γ sufficiently dominant (i.e. $\langle \gamma, i' \rangle > 10$) and arbitrary $w \in \mathcal{W}_f$ the variety $\overset{\circ}{\mathcal{Z}}{}^\gamma_w$ is smooth.

Proof. Let us consider the map $\varrho_\gamma : \overset{\circ}{\mathcal{Q}}{}^\gamma \longrightarrow \mathbf{X} \times \mathbf{X},\ f \mapsto (f(0), f(\infty))$. We have $\overset{\circ}{\mathcal{Z}}{}^\gamma = \varrho_\gamma^{-1}(\mathbf{X}_w, \mathbf{B}_-)$. It suffices to prove that ϱ_γ is smooth and surjective. Recall that the tangent space Θ_f to $\overset{\circ}{\mathcal{Q}}{}^\gamma$ at $f \in \overset{\circ}{\mathcal{Q}}{}^\gamma$ is canonically isomorphic to $H^0(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X})$. Let us interpret $\mathbf{X}$ as a variety of Borel subalgebras of $\mathfrak{g}$. We denote $f(0)$ by $\mathfrak{b}_0$, and $f(\infty)$ by $\mathfrak{b}_\infty$. So we have to prove that the canonical map $H^0(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X}) \longrightarrow \mathcal{T}_{\mathfrak{b}_0}\mathbf{X} \oplus \mathcal{T}_{\mathfrak{b}_\infty}\mathbf{X}$ is surjective. To this end it is enough to have $H^1(\mathbb{P}^1, f^*\mathcal{T}\mathbf{X}(-0-\infty)) = 0$. This in turn holds whenever γ is sufficiently dominant. $\Box$

8.5.2. *Lemma.* For γ sufficiently dominant we have $\dim \overset{\circ}{\mathcal{Z}}{}^\gamma_w = 2|\gamma| - \dim \mathbf{X} + \dim \mathbf{X}_w$.

Proof. The same as the proof of 8.5.1. $\Box$

8.5.3. *Remark.* Unfortunately, one cannot prove the conjecture 8.5 for arbitrary γ the same way as the Lemma 8.5.1: for arbitrary γ the map ϱ_γ is not smooth. The simplest example occurs for $\mathbf{G} = SL_4$ when γ is twice the sum of simple coroots. This example was found by A.Kuznetsov.

9. Factorization

This section follows closely §4 of [FS].

9.1. Now we replace the maps into the flag variety $\mathbf{X}$ with the maps from $\mathbb{P}^1$ to the product $\mathbf{X} \times Y = \sqcup_{\chi \in Y} \mathbf{X}_\chi$. So for arbitrary $\chi \in Y$ and $\alpha \in \mathbb{N}[I]$ we obtain the spaces $\mathcal{Z}^\alpha_\chi$ of based maps into $\mathbf{X}_\chi$ and it makes sense now to add the subscript χ to all the strata (coarse, Schubert, fine) defined in the previous section.

We will consider a system $\mathcal{Z}$ of varieties $\mathcal{Z}^\alpha_\chi,\ \alpha, \gamma \in Y$, together with two kinds of maps defined for any $\beta, \gamma \in \mathbb{N}[I]$:

a) closed embeddings,

$$\sigma^{\beta,\gamma}_\chi :\ \mathcal{Z}^\beta_\chi \hookrightarrow \mathcal{Z}^{\beta+\gamma}_{\chi+\gamma},$$

b) factorization identifications

$$\mathcal{Z}^{\beta,\gamma}_{\chi, U_\varepsilon, \Upsilon_\varepsilon} = \mathcal{Z}^\beta_{\chi, U_\varepsilon} \times \mathcal{Z}^\gamma_{\chi-\beta, \Upsilon_\varepsilon}$$

defined for $\varepsilon > 0$ and $U_\varepsilon \overset{\text{def}}{=} \{z \in \mathbb{C},\ |z| < \varepsilon\}$, and $\Upsilon_\varepsilon \overset{\text{def}}{=} \{z \in \mathbb{C},\ |z| > \varepsilon\}$.

Of course, without the subscript these are the factorizations from 6.3.2 and the embeddings from 8.1.

9.2. We will denote by $\mathcal{IC}^\alpha_\chi$ the perverse IC-extension of the constant sheaf at the generic point of $\mathcal{Z}^\alpha_\chi$.

The following definition makes sense only modulo the validity of conjecture 8.5.

Definition. A *snop* $\mathcal{K}$ is the following collection of data:

a) $\chi = \chi(\mathcal{K}) \in Y$, called the *support estimate* of $\mathcal{K}$;

b) For any $\alpha \in \mathbb{N}[I]$, a perverse sheaf $\mathcal{K}^\alpha_\chi$ on $\mathcal{Z}^\alpha_\chi$ smooth along the fine Schubert stratification;

c) For any $\beta,\gamma\in\mathbb{N}[I],\ \varepsilon>0$, a *factorization isomorphism*

$$\mathcal{K}_\chi^{\beta+\gamma}|_{\mathcal{Z}^{\beta,\gamma}_{\chi,U_\varepsilon,\Upsilon_\varepsilon}}\xrightarrow{\sim}\mathcal{K}_\chi^\beta|_{\mathcal{Z}^\beta_{\chi,U_\varepsilon}}\boxtimes\mathcal{IC}^\gamma_{\chi-\beta}|_{\mathcal{Z}^\gamma_{\chi-\beta,\Upsilon_\varepsilon}}$$

satisfying the *associativity constraints* as in [FS], §§3,4. We spare the reader the explicit formulation of these constraints.

9.3. Since at the moment the conjecture 8.5 is unavailable we will provide an ugly provisional substitute of the Definition 9.2. Namely, recall that $\mathcal{Z}^\alpha=\sqcup_{\alpha\geq\beta\geq\gamma}\overset{\circ}{\mathcal{Z}}{}^\gamma\times(\mathbb{C}^*)^{\beta-\gamma}$. We introduce an open subvariety

$$\ddot{\mathcal{Z}}^\alpha=\bigsqcup_{\alpha\geq\beta\geq\gamma\gg0}\overset{\circ}{\mathcal{Z}}{}^\gamma\times(\mathbb{C}^*)^{\beta-\gamma}$$

The union is taken over sufficiently dominant γ, i.e. such that $\langle\gamma,i'\rangle>10$ for any $i\in I$. Certainly, if α itself is not sufficiently dominant, $\ddot{\mathcal{Z}}^\alpha$ may happen to be empty. We have the fine Schubert stratification

$$\ddot{\mathcal{Z}}^\alpha=\bigsqcup_{w\in\mathcal{W}_f,\ \Gamma\in\mathfrak{P}(\beta-\gamma)}^{\alpha\geq\beta\geq\gamma\gg0}\overset{\circ}{\mathcal{Z}}{}^\gamma_w\times(\mathbb{C}^*)^{\beta-\gamma}_\Gamma$$

with smooth strata (see the Lemma 8.5.1).

Now we can repeat the Definition 9.2 replacing $\mathcal{Z}^\alpha_\chi$ by $\ddot{\mathcal{Z}}^\alpha_\chi$. Thus in 9.2 b) we have to restrict ourselves to sufficiently dominant α, and in 9.2 c) β has to be sufficiently dominant as well.

9.3.1. In what follows we use the Definition 9.2. The reader unwilling to believe in the Conjecture 8.5 will readily substitute the conjectural Definition 9.2 with the provisional working Definition 9.3.

9.4. **Examples.** We define the *irreducible* and *standard* snops.

9.4.1. Let us describe a snop $\mathcal{L}(w,\chi)$ for $\chi\in Y,\ w\in\mathcal{W}_f$.

a) The support of $\mathcal{L}(w,\chi)$ is χ.

b) $\mathcal{L}(w,\chi)^\alpha_\chi$ is the irreducible IC-extension $\mathcal{IC}(\overline{\mathcal{Z}}{}^\alpha_{w,\chi})=j_{!*}\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi})$ of the perverse IC-sheaf on the Schubert stratum $\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi}\subset\overset{\bullet}{\mathcal{Z}}{}^\alpha_\chi$. Here j stands for the affine open embedding $\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi}\hookrightarrow\overline{\mathcal{Z}}{}^\alpha_w$.

In particular, $\mathcal{IC}(\overline{\mathcal{Z}}{}^\alpha_{w_0,\chi})=\mathcal{IC}^\alpha_\chi$.

c) Evidently, $\overline{\mathcal{Z}}{}^\beta_{w,\chi,U_\varepsilon}$ (resp. $\overline{\mathcal{Z}}{}^{\beta,\gamma}_{w,\chi,U_\varepsilon,\Upsilon_\varepsilon}$) is open in $\overline{\mathcal{Z}}{}^\beta_{w,\chi}$ (resp. $\overline{\mathcal{Z}}{}^\alpha_{w,\chi}$) for any $\beta+\gamma=\alpha$. Moreover, $\overline{\mathcal{Z}}{}^{\beta,\gamma}_{w,\chi,U_\varepsilon,\Upsilon_\varepsilon}=\overline{\mathcal{Z}}{}^\beta_{w,\chi,U_\varepsilon}\times\mathcal{Z}^\gamma_{\chi-\beta,\Upsilon_\varepsilon}$. This induces the desired factorization isomorphism.

9.4.2. If we replace in 9.4.1b) above $j_{!*}\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi})$ by $j_!\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi})=:\mathcal{M}(w,\chi)^\alpha_\chi$ (resp. $j_*\mathcal{IC}(\overset{\bullet}{\mathcal{Z}}{}^\alpha_{w,\chi})=:\mathcal{DM}(w,\chi)^\alpha_\chi$) we obtain the snop $\mathcal{M}(w,\chi)$ (resp. $\mathcal{DM}(w,\chi)$).

9.5. Given a snop $\mathcal{K}$ with support χ, and $\eta\geq\chi,\ \alpha\in\mathbb{N}[I]$, we define a sheaf $'\mathcal{K}^\alpha_\eta$ on $\mathcal{Z}^\alpha_\chi$ as follows. We set $\gamma\overset{\text{def}}{=}\eta-\chi$. If $\alpha\geq\gamma$ we set

$$'\mathcal{K}^\alpha_\eta\overset{\text{def}}{=}(\sigma_\chi^{\alpha-\gamma,\gamma})_*\mathcal{K}^{\alpha-\gamma}_\chi$$

(for the definition of σ see 9.1). Otherwise we set $'\mathcal{K}^\alpha_\eta\overset{\text{def}}{=}0$.

It is easy to see that the factorization isomorphisms for $\mathcal{K}$ induce similar isomorphisms for $'\mathcal{K}$, and thus we obtain a snop $'\mathcal{K}$ with support $\eta\geq\chi$.

9.6. We define the category $\widetilde{\mathcal{PS}}$ of snops.

9.6.1. Given two snops $\mathcal{F},\mathcal{K}$ we will define the morphisms $\mathrm{Hom}(\mathcal{F},\mathcal{K})$ as follows. Let $\eta\in Y$ be such that $\eta\geq\chi(\mathcal{F}),\chi(\mathcal{K})$. For $\alpha=\beta+\gamma\in\mathbb{N}[I]$ we consider the following composition:

$$\vartheta_\eta^{\beta,\gamma}:\ \mathrm{Hom}_{\mathcal{Z}_\eta^\alpha}({}'\mathcal{F}_\eta^\alpha,{}'\mathcal{K}_\eta^\alpha)\longrightarrow\mathrm{Hom}_{\mathcal{Z}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}}({}'\mathcal{F}_\eta^\alpha|_{\mathcal{Z}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}},{}'\mathcal{K}_\eta^\alpha|_{\mathcal{Z}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}})\xrightarrow{\sim}$$

$$\mathrm{Hom}_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}\times\mathcal{Z}^\gamma_{\eta-\beta,\Upsilon_\varepsilon}}({}'\mathcal{F}_\eta^\beta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}}\boxtimes\mathcal{IC}^\gamma_{\eta-\beta}|_{\mathcal{Z}^\gamma_{\eta-\beta,\Upsilon_\varepsilon}},{}'\mathcal{K}_\eta^\beta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}}\boxtimes\mathcal{IC}^\gamma_{\eta-\beta}|_{\mathcal{Z}^\gamma_{\eta-\beta,\Upsilon_\varepsilon}})=\mathrm{Hom}_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}}({}'\mathcal{F}_\eta^\beta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}},{}'\mathcal{K}_\eta^\beta|_{\mathcal{Z}^\beta_{\eta,U_\varepsilon}})$$

(the second isomorphism is induced by the factorization isomorphisms for ${}'\mathcal{F}$ and ${}'\mathcal{K}$, and the third equality is just Künneth formula).

Now we define

$$\mathrm{Hom}(\mathcal{F},\mathcal{K})\stackrel{\mathrm{def}}{=}\varinjlim_\eta\varprojlim_\alpha\mathrm{Hom}_{\mathcal{Z}_\eta^\alpha}({}'\mathcal{F}_\eta^\alpha,{}'\mathcal{K}_\eta^\alpha)$$

Here the inverse limit is taken over $\alpha\in\mathbb{N}[I]$, the transition maps being $\vartheta_\eta^{\beta,\alpha-\beta}$, and the direct limit is taken over $\eta\in Y$ such that $\eta\geq\chi(\mathcal{F}),\chi(\mathcal{K})$, the transition maps being induced by the obvious isomorphisms $\mathrm{Hom}_{\mathcal{Z}_\eta^\alpha}({}'\mathcal{F}_\eta^\alpha,{}'\mathcal{K}_\eta^\alpha)=\mathrm{Hom}_{\mathcal{Z}_{\eta+\gamma}^{\alpha+\gamma}}({}'\mathcal{F}_{\eta+\gamma}^{\alpha+\gamma},{}'\mathcal{K}_{\eta+\gamma}^{\alpha+\gamma})$.

9.6.2. With the above definition of morphisms and obvious composition, the snops form a category which we will denote by $\widetilde{\mathcal{PS}}$.

9.7. Evidently, the snops $\mathcal{L}(w,\chi)$ are irreducible objects of $\widetilde{\mathcal{PS}}$. It is easy to see that any irreducible object of $\widetilde{\mathcal{PS}}$ is isomorphic to some $\mathcal{L}(w,\chi)$.

We define the category $\mathcal{PS}$ of *finite snops* as the full subcategory of $\widetilde{\mathcal{PS}}$ formed by the snops of finite length. It is an abelian category. We will see later that $\mathcal{M}(w,\chi)$ and $\mathcal{DM}(w,\chi)$ (see 9.4.2) lie in $\mathcal{PS}$ for any w,χ.

One can prove the following very useful technical lemma exactly as in [FS], 4.7.

9.7.1. *Lemma.* Let $\mathcal{F},\mathcal{K}$ be two finite snops. Let $\eta\geq\chi(\mathcal{F}),\chi(\mathcal{K})$. There exists $\beta\in\mathbb{N}[I]$ such that for any $\alpha\geq\beta$ the canonical maps $\mathrm{Hom}(\mathcal{F},\mathcal{K})\longrightarrow\mathrm{Hom}_{\mathcal{Z}_\eta^\alpha}({}'\mathcal{F}_\eta^\alpha,{}'\mathcal{K}_\eta^\alpha)$ are all isomorphisms. □

CHAPTER 3. Convolution with affine Grassmannian

10. Plücker model of affine Grassmannian

10.1. Let $\mathcal{G}$ be the usual affine Grassmannian $\mathbf{G}((z))/\mathbf{G}[[z]]$. It is the ind-scheme representing the functor of isomorphism classes of pairs $(\mathcal{T},\tau)$ where $\mathcal{T}$ is a $\mathbf{G}$-torsor on $\mathbb{P}^1$, and τ is its section (trivialization) defined off 0 (see e.g. [MV]). It is equipped with a natural action of proalgebraic group $\mathbf{G}[[z]]$, and we are going to describe the orbits of this action. It is known (see e.g. *loc. cit.*) that these orbits are numbered by dominant cocharacters $\eta\in Y^+\subset Y$.

Here $Y^+\subset Y$ stands for the set of cocharacters η such that $\langle\eta,i'\rangle\geq0$ for any $i\in I$. For $\eta\in Y^+$ we denote the corresponding $\mathbf{G}[[z]]$-orbit in $\mathcal{G}$ by $\mathcal{G}_\eta$, and we denote its closure by $\overline{\mathcal{G}}_\eta$.

Recall that for a dominant character $\lambda\in X^+$ we denote by V_λ the corresponding irreducible $\mathbf{G}$-module, and we denote by $\mathcal{V}_\lambda$ the trivial vector bundle $V_\lambda\otimes\mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^1$.

10.2. **Proposition.** The orbit closure $\overline{\mathcal{G}}_\eta \subset \mathcal{G}$ is the space of collections $(\mathcal{U}_\lambda)_{\lambda\in X^+}$ of vector bundles on $\mathbb{P}^1$ such that

a) $\mathcal{V}_\lambda(-\langle\eta,\lambda\rangle 0) \subset \mathcal{U}_\lambda \subset \mathcal{V}_\lambda(\langle\eta,\lambda\rangle 0)$, or equivalently, $\mathcal{U}_\lambda(-\langle\eta,\lambda\rangle 0) \subset \mathcal{V}_\lambda \subset \mathcal{U}_\lambda(\langle\eta,\lambda\rangle 0)$;

b) $\deg \mathcal{U}_\lambda = \deg \mathcal{V}_\lambda = 0$, or in other words, $\dim \mathcal{V}_\lambda(\langle\eta,\lambda\rangle 0)/\mathcal{U}_\lambda = \langle\eta,\lambda\rangle \dim V_\lambda$;

c) For any surjective G-morphism $\phi:\ V_\lambda\otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi:\ \mathcal{V}_\lambda\otimes\mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$ (hence $\phi:\ \mathcal{V}_\lambda(\langle\eta,\lambda\rangle 0)\otimes\mathcal{V}_\mu(\langle\eta,\mu\rangle 0) \longrightarrow \mathcal{V}_\nu(\langle\eta,\lambda+\mu\rangle 0))$ we have $\phi(\mathcal{U}_\lambda\otimes\mathcal{U}_\mu) = \mathcal{U}_\nu$.

Proof. **G**-torsor on a curve C is the same as a tensor functor from the category of **G**-modules to the category of vector bundles on C. □

10.3. Let us give a local version of the above Proposition. Recall that $\mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K} = \mathbb{C}((z))$. For a finite-dimensional vector space V, a *lattice* $\mathfrak{V}$ in $V\otimes\mathcal{K}$ is an $\mathcal{O}$-submodule of $V\otimes\mathcal{K}$ *commeasurable* with $V\otimes\mathcal{O}$, that is, such that $(V\otimes\mathcal{O})\cap\mathfrak{V}$ is of finite codimension in both $V\otimes\mathcal{O}$ and $\mathfrak{V}$.

Proposition. The orbit closure $\overline{\mathcal{G}}_\eta \subset \mathcal{G}$ is the space of collections $(\mathfrak{V}_\lambda)_{\lambda\in X^+}$ of lattices in $V_\lambda\otimes\mathcal{K}$ such that

a) $z^{\langle\eta,\lambda\rangle}(V_\lambda\otimes\mathcal{O}) \subset \mathfrak{V}_\lambda \subset z^{-\langle\eta,\lambda\rangle}(V_\lambda\otimes\mathcal{O})$, or equivalently, $z^{\langle\eta,\lambda\rangle}\mathfrak{V}_\lambda \subset V_\lambda\otimes\mathcal{O} \subset z^{-\langle\eta,\lambda\rangle}\mathfrak{V}_\lambda$;

b) $\dim(z^{-\langle\eta,\lambda\rangle}(V_\lambda\otimes\mathcal{O})/\mathfrak{V}_\lambda) = \langle\eta,\lambda\rangle \dim V_\lambda$;

c) For any surjective G-morphism $\phi:\ V_\lambda\otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi:\ (V_\lambda\otimes\mathcal{O})\otimes(V_\mu\otimes\mathcal{O}) \longrightarrow (V_\nu\otimes\mathcal{O})$ (hence $\phi:\ z^{-\langle\eta,\lambda\rangle}(V_\lambda\otimes\mathcal{O})\otimes z^{-\langle\eta,\mu\rangle}(V_\mu\otimes\mathcal{O}) \longrightarrow z^{-\langle\eta,\lambda+\mu\rangle}(V_\nu\otimes\mathcal{O})$), we have $\phi(\mathfrak{V}_\lambda\otimes\mathfrak{V}_\mu) = \mathfrak{V}_\nu$.

□

10.4. Let $\mathbf{I}\subset\mathbf{G}[[z]]$ be the Iwahori subgroup; it is formed by all $g(z)\in\mathbf{G}[[z]]$ such that $g(0)\in\mathbf{B}\subset\mathbf{G}$. We will denote by $\mathcal{P}(\mathcal{G},\mathbf{I})$ the category of perverse sheaves on $\mathcal{G}$ with finite-dimensional support, constant along **I**-orbits. The stratification of $\mathcal{G}$ by **I**-orbits is a certain refinement of the stratification $\mathcal{G} = \sqcup_{\eta\in Y^+}\mathcal{G}_\eta$. Namely, each $\mathcal{G}_\eta$ decomposes into **I**-orbits numbered by $\mathcal{W}_f/\mathcal{W}_\eta$ where $\mathcal{W}_\eta$ stands for the stabilizer of η in $\mathcal{W}_f$. For $w\in\mathcal{W}_f/\mathcal{W}_\eta$ we will denote the corresponding **I**-orbit by $\mathcal{G}_{w,\eta}$. Let us introduce a Plücker model of $\mathcal{G}_{w,\eta}$.

10.4.1. For $\eta\in Y^+$ let $I_\eta\subset I$ be the set of all i such that $\langle\eta,i'\rangle = 0$ (thus for $i\notin I_\eta$ we have $\langle\eta,i'\rangle > 0$). Then $\mathcal{W}_\eta$ is generated by the simple reflections $\{s_i, i\in I_\eta\}$. Let $\mathbf{P}(I_\eta)$ be the corresponding parabolic subgroup (e.g. for $I_\eta = \emptyset$ we have $\mathbf{P}(I_\eta) = \mathbf{B}$, while for $I_\eta = I$ we have $\mathbf{P}(I_\eta) = \mathbf{G}$). Let $\mathbf{X}(I_\eta) = \mathbf{G}/\mathbf{P}(I_\eta)$ be the corresponding partial flag variety. The **B**-orbits on $\mathbf{X}(I_\eta)$ are naturally numbered by $\mathcal{W}_f/\mathcal{W}_\eta$: $\mathbf{X}(I_\eta) = \sqcup_{w\in\mathcal{W}_f/\mathcal{W}_\eta}\mathbf{X}(I_\eta)_w$. The Plücker embedding realizes $\mathbf{X}$ as a closed subvariety in $\prod_{i\in I}\mathbb{P}(V_{\omega_i})$. Its image under the projection $\prod_{i\in I}\mathbb{P}(V_{\omega_i}) \longrightarrow \prod_{i\notin I_\eta}\mathbb{P}(V_{\omega_i})$ exactly coincides with $\mathbf{X}(I_\eta)$.

10.4.2. *Lemma-Definition.* a) For $\eta = \sum_{i\in I} n_i i$, and $(\mathcal{U}_\lambda)_{\lambda\in X^+}\in\mathcal{G}_\eta$ we have $\dim(\mathcal{U}_{\omega_i} + \mathcal{V}_{\omega_i}((n_i-1)\cdot 0)/\mathcal{V}_{\omega_i}((n_i-1)\cdot 0)) = \dim V_{\omega_i}^{\mathbf{U}(I_\eta)}$ where $\mathbf{U}(I_\eta)$ is the unipotent radical of $\mathbf{P}(I_\eta)$.

b) Thus $\mathcal{U}_{\omega_i}, i\in I$, defines a subspace K_i in $\mathcal{V}_{\omega_i}(n_i\cdot 0)/\mathcal{V}_{\omega_i}((n_i-1)\cdot 0) = V_{\omega_i}$. This collection of subspaces $(K_i)_{i\in I}\in\prod_{i\in I}\mathrm{Gr}(V_{\omega_i})$ satisfies the Plücker relations and thus gives a point in $\mathbf{X}(I_\eta)$;

c) We will denote by $\mathbf{r}$ the map $\mathcal{G}_\eta \longrightarrow \mathbf{X}(I_\eta)$ defined in b);

d) For $w\in\mathcal{W}_f/\mathcal{W}_\eta$ we have $\mathcal{G}_{w,\eta} = \mathbf{r}^{-1}(\mathbf{X}(I_\eta)_w)$. □

We are obliged to D.Gaitsgory who pointed out a mistake in the earlier version of the above Lemma.

10.4.3. For $\theta \in Y$ we consider the corresponding homomorphism $\theta : \mathbb{C}^* \longrightarrow \mathbf{H} \subset \mathbf{G}$ as a formal loop $\theta(z) \in \mathbf{G}((z))$. It projects to the same named point $\theta(z) \in \mathbf{G}((z))/\mathbf{G}[[z]] = \mathcal{G}$. There is a natural bijection between the set of $\theta(z)$, $\theta \in Y$, and the set of Iwahori orbits: each Iwahori orbit $\mathcal{G}_{w,\eta}$ contains exactly one of the above points, namely, the point $w\eta(z)$.

10.5. Recall the Beilinson-Drinfeld avatar $\mathbf{Z}^\alpha$ of the Zastava space $\mathcal{Z}^\alpha$ (see 6.2). In this subsection we will give a Plücker model of $\mathbf{Z}^\alpha$.

Proposition. $\mathbf{Z}^\alpha$ is the the space of pairs $(D, (\mathfrak{U}_\lambda)_{\lambda \in X^+})$ where $D \in \mathbb{A}^\alpha$ is an I-colored effective divisor, and $(\mathfrak{U}_\lambda)_{\lambda \in X^+}$ is a collection of vector bundles on $\mathbb{P}^1$ such that

a) $\mathcal{V}_\lambda(-\infty D) \subset \mathfrak{U}_\lambda \subset \mathcal{V}_\lambda(+\infty D)$;

b) $\mathcal{V}_\lambda^{\mathbf{N}} \subset \mathfrak{U}_\lambda \supset \mathcal{V}_\lambda^{\mathbf{N}_-}(-\langle D, \lambda\rangle)$ (notations of 3.4.1), the first inclusion being a *line subbundle* (and the second an invertible subsheaf);

c) $\deg \mathfrak{U}_\lambda = 0$;

d) For any surjective G-morphism $\phi : V_\lambda \otimes V_\mu \longrightarrow V_\nu$ and the corresponding morphism $\phi : \mathcal{V}_\lambda \otimes \mathcal{V}_\mu \longrightarrow \mathcal{V}_\nu$ (hence $\phi : \mathcal{V}_\lambda(+\infty D) \otimes \mathcal{V}_\mu(+\infty D) \longrightarrow \mathcal{V}_\nu(+\infty D)$) we have $\phi(\mathfrak{U}_\lambda \otimes \mathfrak{U}_\mu) = \mathfrak{U}_\nu$.

Proof. Obvious. □

10.5.1. *Remark.* Recall the isomorphism $\varpi^{-1}\xi : \mathbf{Z}^\alpha \xrightarrow{\sim} \mathcal{Z}^\alpha$ constructed in section 7. Let us describe it in terms of 10.5. The Lemma 7.2.4 says that there is a unique system of isomorphisms $\iota_\lambda : \mathfrak{U}_\lambda \xrightarrow{\sim} \mathcal{V}_\lambda$, $\lambda \in X^+$, identical at $\infty \in \mathbb{P}^1$ and compatible with tensor multiplication. Then $\varpi^{-1}\xi(D, (\mathfrak{U}_\lambda)_{\lambda \in X^+}) = (\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda)_{\lambda \in X^+}$ where $\mathfrak{L}_\lambda = \iota_\lambda(\mathcal{V}_\lambda^{\mathbf{N}_-}(-\langle D, \lambda\rangle))$.

10.6. Let $\mathfrak{M}$ be the scheme representing the functor of isomorphism classes of $\mathbf{G}$-torsors on $\mathbb{P}^1$ equipped with trivialization in the formal neighbourhood of $\infty \in \mathbb{P}^1$ (see [Ka] and [KT1]).

10.6.1. The scheme $\mathfrak{M}$ is stratified by the locally closed subschemes $\mathfrak{M}_\eta : \mathfrak{M} = \sqcup_{\eta \in Y^+} \mathfrak{M}_\eta$ according to the isomorphism types of $\mathbf{G}$-torsors. Namely, due to Riemann's classification, for a $\mathbf{G}$-torsor $\mathcal{T}$ and any $\lambda \in X^+$ the associated vector bundle $\mathcal{V}_\lambda^{\mathcal{T}}$ decomposes as a direct sum of line bundles $\mathcal{O}(r_k^\lambda)$ of well-defined degrees $r_1^\lambda \geq \ldots \geq r_{\dim V_\lambda}^\lambda$. Then $\mathcal{T}$ lies in the stratum $\mathfrak{M}_\eta$ iff $r_1^\lambda = \langle \eta, \lambda\rangle$.

For any $\eta \in Y^+$ the union of strata $\mathfrak{M}^\eta := \sqcup_{Y^+ \ni \chi \leq \eta} \mathfrak{M}_\chi$ forms an open subscheme of $\mathfrak{M}$. This subscheme is a projective limit of schemes of finite type, all the maps in projective system being fibrations with affine fibers. Moreover, $\mathfrak{M}^\eta$ is equipped with a free action of a prounipotent group $\mathbf{G}^\eta$ (a congruence subgroup in $\mathbf{G}[[z^{-1}]]$) such that the quotient $\underline{\mathfrak{M}}^\eta$ is a smooth scheme of finite type. The theory of perverse sheaves on $\mathfrak{M}$ smooth along the stratification by $\mathfrak{M}_\eta$ is developed in [KT1]. We will refer the reader to this work, and will freely use such perverse sheaves, e.g. $\mathcal{IC}(\mathfrak{M}_\eta)$.

10.6.2. Restricting a trivialization of a $\mathbf{G}$-torsor from $\mathbb{P}^1 - 0$ to the formal neighbourhood of $\infty \in \mathbb{P}^1$ we obtain the closed embedding $\mathbf{i} : \mathcal{G} \hookrightarrow \mathfrak{M}$. The intersection of $\mathfrak{M}_\eta$ and $\mathcal{G}_\chi$ is nonempty iff $\eta \leq \chi$, and then it is transversal. Thus, $\overline{\mathcal{G}}_\eta \subset \mathfrak{M}^\eta$. According to [KT2], the composition $\overline{\mathcal{G}}_\eta \hookrightarrow \mathfrak{M}^\eta \longrightarrow \underline{\mathfrak{M}}^\eta$ is a closed embedding.

10.6.3. For a $\mathbf{G}$-torsor $\mathcal{T}$ and an irreducible $\mathbf{G}$-module V_λ we denote by $\mathcal{V}_\lambda^{\mathcal{T}}$ the associated vector bundle. Following 3.2 and 3.3 we define for *arbitrary* $\alpha \in Y$ the scheme $\overset{\circ}{\mathfrak{Q}}{}^\alpha$ (resp. $\mathfrak{Q}^\alpha$) representing the functor of isomorphism classes of pairs $(\mathcal{T}, (\mathfrak{L}_\lambda)_{\lambda \in X^+})$ where $\mathcal{T}$ is a $\mathbf{G}$-torsor trivialized in the formal neighbourhood of $\infty \in \mathbb{P}^1$, and $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda^{\mathcal{T}},\ \lambda \in X^+$, is a collection of line subbundles (resp. invertible subsheaves) of degree $\langle -\alpha, \lambda \rangle$ satisfying the Plücker conditions (cf. *loc. cit.*). The evident projection $\overset{\circ}{\mathfrak{Q}}{}^\alpha \longrightarrow \mathfrak{M}$ (resp. $\mathfrak{Q}^\alpha \longrightarrow \mathfrak{M}$) will be denoted by $\overset{\circ}{\mathbf{p}}$ (resp. $\mathbf{p}$). The open embedding $\overset{\circ}{\mathfrak{Q}}{}^\alpha \hookrightarrow \mathfrak{Q}^\alpha$ will be denoted by $\mathbf{j}$. Clearly, $\mathbf{p}$ is projective, and $\overset{\circ}{\mathbf{p}} = \mathbf{p} \circ \mathbf{j}$.

The free action of prounipotent group $\mathbf{G}^\eta$ on $\mathfrak{M}^\eta$ lifts to the free action of $\mathbf{G}^\eta$ on the open subscheme $\mathbf{p}^{-1}(\mathfrak{M}^\eta) \subset \mathfrak{Q}^\alpha$. The quotient is a scheme of finite type $\underline{\mathfrak{Q}}^{\alpha,\eta}$ equipped with the projective morphism $\mathbf{p}$ to $\underline{\mathfrak{M}}^\eta$. There exists a $\mathbf{G}[[z^{-1}]]$-invariant stratification $\mathfrak{S}$ of $\mathfrak{Q}^\alpha$ such that $\mathbf{p}$ is stratified with respect to $\mathfrak{S}$ and the stratification $\mathfrak{M} = \sqcup_{\eta \in Y^+} \mathfrak{M}_\eta$. One can define perverse sheaves on $\mathfrak{Q}^\alpha$ smooth along $\mathfrak{S}$ following the lines of [KT1]. In particular, we have the irreducible Goresky-Macpherson sheaf $\mathcal{IC}(\mathfrak{Q}^\alpha)$.

Following 3.4.2 we introduce a decomposition of $\mathfrak{Q}^\alpha$ into a disjoint union of locally closed subschemes according to the isomorphism types of $\mathbf{G}$-torsors and defects of invertible subsheaves:

$$\mathfrak{Q}^\alpha = \bigsqcup_{\beta \leq \alpha}^{\eta \in Y^+} \overset{\circ}{\mathfrak{Q}}{}^\beta_\eta \times C^{\alpha - \beta}$$

where $C = \mathbb{P}^1$ and $\overset{\circ}{\mathfrak{Q}}{}^\beta_\eta = \overset{\circ}{\mathbf{p}}{}^{-1}(\mathfrak{M}_\eta) \subset \overset{\circ}{\mathfrak{Q}}{}^\beta$.

10.7. IC sheaves.

10.7.1. The Goresky-MacPherson sheaf $\mathcal{IC}^\alpha$ on $\mathcal{Z}^\alpha$ is smooth along stratification

$$\mathcal{Z}^\alpha = \bigsqcup_{\Gamma \in \mathfrak{P}(\beta - \gamma)}^{\alpha \geq \beta \geq \gamma \geq 0} \overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta - \gamma}_\Gamma$$

(cf. 8.4). It is evidently constant along strata, so its stalk at a point in $\overset{\circ}{\mathcal{Z}}{}^\gamma \times (\mathbb{C}^*)^{\beta-\gamma}_\Gamma$ depends on the stratum only. Moreover, due to factorization property, it depends not on $\alpha \geq \beta$ but only on their difference $\alpha - \beta \in \mathbb{N}[I]$. We will denote it by $\mathcal{IC}^{\alpha-\beta}_\Gamma$. In case $\mathbf{G} = SL_n$ these stalks were computed in [Ku].

10.7.2. Recall (see 3.4.2) that $\mathcal{Q}^\beta,\ \beta \in \mathbb{N}[I]$, is stratified by the type of defect:

$$\mathcal{Q}^\beta = \bigsqcup_{\Gamma \in \mathfrak{P}(\beta - \gamma)}^{\beta \geq \gamma \geq 0} \overset{\circ}{\mathcal{Q}}{}^\gamma \times C^{\beta - \gamma}_\Gamma$$

The Goresky-Macpherson sheaf $\mathcal{IC}(\mathcal{Q}^\beta)$ on $\mathcal{Q}^\beta$ is constant along the strata. It is immediate to see that its stalk at any point in the stratum $\overset{\circ}{\mathcal{Q}}{}^\gamma \times C^{\beta-\gamma}_\Gamma$ is isomorphic, up to a shift, to $\mathcal{IC}^0_\Gamma$. In particular, it depends on the defect only.

10.7.3. **Proposition.** a) The Goresky-Macpherson sheaf $\mathcal{IC}(\mathfrak{Q}^\beta)$ on $\mathfrak{Q}^\beta,\ \beta \in Y$, is constant along the locally closed subschemes

$$\mathfrak{Q}^\beta = \bigsqcup_{\Gamma \in \mathfrak{P}(\beta - \gamma)}^{\beta \geq \gamma} \overset{\circ}{\mathfrak{Q}}{}^\gamma \times C^{\beta - \gamma}_\Gamma$$

b) The stalk of $\mathcal{IC}(\mathfrak{Q}^\beta)$ at any point in the $\overset{\circ}{\mathfrak{Q}}{}^\gamma \times C_\Gamma^{\beta-\gamma}$ is isomorphic, up to a shift, to $\mathcal{IC}_\Gamma^0$.

Proof. Will be given in 12.7. □

10.7.4. Let $\phi \in \mathfrak{Q}^\beta$. The stalk $\mathcal{IC}(\mathfrak{Q}^\beta)_\phi$ is a graded vector space.

Conjecture. *(Parity vanishing)* Nonzero graded parts of $\mathcal{IC}(\mathfrak{Q}^\beta)_\phi$ appear in cohomological degrees of the same parity.

10.7.5. *Remark.* In case $\mathbf{G} = SL_n$ the conjecture follows from the Proposition 10.7.3 and [Ku] 2.5.2. In general case the conjecture follows from the unpublished results of G.Lusztig. We plan to prove it in the next part.

11. Convolution diagram

11.1. **Definition.** For $\alpha \in Y, \eta \in Y^+$ we define the *convolution diagram* $\mathcal{GQ}_\eta^\alpha$ as the space of collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda\in X^+}$ of vector bundles with invertible subsheaves such that

a) $(\mathcal{U}_\lambda)_{\lambda\in X^+} \in \overline{\mathcal{G}}_\eta$, or in other words, $(\mathcal{U}_\lambda)_{\lambda\in X^+}$ satisfies the conditions 10.2 a)-c);

b) $\mathfrak{L}_\lambda \subset \mathcal{U}_\lambda$ has degree $-\langle\alpha, \lambda\rangle$;

c) For any surjective $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu = \lambda + \mu$ we have (cf. 10.2 c) $\phi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = \mathcal{L}_\nu$;

d) For any $\mathbf{G}$-morphism $\phi: V_\lambda \otimes V_\mu \longrightarrow V_\nu$ such that $\nu < \lambda + \mu$ we have $\phi(\mathcal{L}_\lambda \otimes \mathcal{L}_\mu) = 0$.

11.1.1. Let us denote by $\overset{\circ}{\mathcal{G}}\mathcal{Q}_\eta^\alpha$ the open subvariety in $\mathcal{GQ}_\eta^\alpha$ formed by all the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)$ such that $\mathfrak{L}_\lambda$ is a *line subbundle* in $\mathcal{U}_\lambda$ for any $\lambda \in X^+$. The open embedding $\overset{\circ}{\mathcal{G}}\mathcal{Q}_\eta^\alpha \hookrightarrow \mathcal{GQ}_\eta^\alpha$ will be denoted by $\mathbf{j}$.

11.2. **Definition.** a) We define the projection $\mathbf{p}: \mathcal{GQ}_\eta^\alpha \longrightarrow \overline{\mathcal{G}}_\eta$ as $\mathbf{p}(\mathcal{U}_\lambda, \mathfrak{L}_\lambda) = (\mathcal{U}_\lambda)$;

b) We define the map $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\alpha+\eta}$ as follows:

$$\mathbf{q}(\mathcal{U}_\lambda, \mathfrak{L}_\lambda) = (\mathfrak{L}_\lambda(-\langle\eta, \lambda\rangle 0) \subset \mathcal{U}_\lambda(-\langle\eta, \lambda\rangle 0) \subset \mathcal{V}_\lambda)$$

(cf. 10.2 a) and the Definition 3.3).

11.2.1. We will denote by $\overset{\circ}{\mathbf{p}}$ the restriction of $\mathbf{p}$ to the open subvariety $\overset{\circ}{\mathcal{G}}\mathcal{Q}_\eta^\alpha \overset{\mathbf{j}}{\hookrightarrow} \mathcal{GQ}_\eta^\alpha$.

11.2.2. *Remark.* Note that $\mathcal{GQ}_0^\alpha = \mathcal{Q}^\alpha$, $\overset{\circ}{\mathcal{G}}\mathcal{Q}_0^\alpha = \overset{\circ}{\mathcal{Q}}{}^\alpha$, $\mathbf{q} = \mathrm{id}$, and $\mathbf{p}$ is the projection to the point $\overline{\mathcal{G}}_0$.

Note also that while in the Definition 3.3 we imposed the positivity condition $\alpha \in \mathbb{N}[I]$ (otherwise $\mathcal{Q}^\alpha$ would be empty) here we allow arbitrary $\alpha \in Y$. It is easy to see that $\mathcal{GQ}_\eta^\alpha$ (as well as $\overset{\circ}{\mathcal{G}}\mathcal{Q}_\eta^\alpha$) is nonempty iff $\eta + \alpha \in \mathbb{N}[I]$.

11.3. Let us give a local version of the convolution diagram. Recall the notations of 10.3 and 4.2.

Definition. For $\eta \in Y^+$ we define the *extended local convolution diagram* $\mathcal{GS}_\eta$ as the ind-scheme formed by the collections $(\mathfrak{V}_\lambda, v_\lambda)_{\lambda\in X^+}$ of lattices in $V_\lambda \otimes \mathcal{K}$ and vectors in $V_\lambda \otimes \mathcal{K}$ such that

a) $(\mathfrak{V}_\lambda)_{\lambda\in X^+} \in \overline{\mathcal{G}}_\eta$, or in other words, $(\mathfrak{V}_\lambda)_{\lambda\in X^+}$ satisfies the condition 10.3a)-c);

b) $v_\lambda \in \mathfrak{V}_\lambda$;

c) $(v_\lambda)_{\lambda\in X^+} \in \widetilde{\mathcal{S}}$ (see 4.2).

11.4. **Definition.** a) We define the projection $\mathbf{p}: \ \mathcal{GS}_\eta \longrightarrow \overline{\mathcal{G}}_\eta$ as $\mathbf{p}(\mathfrak{V}_\lambda, v_\lambda) = (\mathfrak{V}_\lambda)$;

b) We define the map $\mathbf{q}: \ \mathcal{GS}_\eta \longrightarrow \mathcal{S}$ as follows (cf. 10.3a):

$$\mathbf{q}(\mathfrak{V}_\lambda, v_\lambda)_{\lambda\in X^+} = (z^{\langle\eta,\lambda\rangle} v_\lambda \in z^{\langle\eta,\lambda\rangle}\mathfrak{V}_\lambda \subset V_\lambda \otimes \mathcal{O})_{\lambda\in X^+}$$

11.5. The torus $\mathbf{H}_a$ acts in a natural way on $\mathcal{GS}_\eta: \ h(\mathfrak{V}_\lambda, v_\lambda) = (\mathfrak{V}_\lambda, \lambda(h)v_\lambda)$. The action is evidently free, and we denote the quotient by $\mathcal{G}\mathbf{Q}_\eta$, the *local convolution diagram.* The map $\mathbf{p}: \ \mathcal{GS}_\eta \longrightarrow \overline{\mathcal{G}}_\eta$ commutes with the action of $\mathbf{H}_a$ (trivial on $\overline{\mathcal{G}}_\eta$), so it descends to the same named map $\mathbf{p}: \ \mathcal{G}\mathbf{Q}_\eta \longrightarrow \overline{\mathcal{G}}_\eta$. The map $\mathbf{q}: \ \mathcal{GS}_\eta \longrightarrow \mathcal{S}$ commutes with the action of $\mathbf{H}_a$ (for the action on $\mathcal{S}$ see 4.1), so it descends to the same named map $\mathbf{q}: \ \mathcal{G}\mathbf{Q}_\eta \longrightarrow \mathbf{Q}$.

The proalgebraic group $\mathbf{G}(\mathcal{O})$ acts on $\mathcal{G}\mathbf{Q}_\eta$ and on $\mathbf{Q}$, and the map $\mathbf{q}$ is equivariant with respect to this action.

11.6. Let us compare the local convolution diagram with the global one. For $\alpha \in Y$, taking formal expansion at $0 \in C$ as in 4.4.2, we obtain the closed embedding $\mathfrak{s}: \ \mathcal{GQ}^\alpha_\eta \hookrightarrow \mathcal{G}\mathbf{Q}_\eta$. It is easy to see that the following diagram is cartesian:

$$\begin{array}{ccc}
\mathcal{GQ}^\alpha_\eta & \xrightarrow{\mathfrak{s}} & \mathcal{G}\mathbf{Q}_\eta \\
{\scriptstyle\mathbf{q}}\downarrow & & {\scriptstyle\mathbf{q}}\downarrow \\
\mathcal{Q}^{\eta+\alpha} & \xrightarrow{\mathfrak{s}} & \mathbf{Q}
\end{array}$$

11.7. Recall the locally closed embedding $\mathcal{Z}^\alpha \subset \mathcal{Q}^\alpha$.

Definition. We define the *restricted convolution diagram* $\mathcal{GZ}^\alpha_\eta \subset \mathcal{GQ}^\alpha_\eta$ as the preimage $\mathbf{q}^{-1}(\mathcal{Z}^{\eta+\alpha})$. The open subvariety $\mathcal{GZ}^\alpha_\eta \cap \mathcal{G}\overset{\circ}{\mathcal{Q}}{}^\alpha_\eta$ will be denoted by $\mathcal{G}\overset{\circ}{\mathcal{Z}}{}^\alpha_\eta$. We will preserve the notations $\mathbf{p}, \mathbf{q}$ (resp. $\overset{\circ}{\mathbf{p}}, \mathbf{j}$) for the restrictions of these morphisms to $\mathcal{GZ}^\alpha_\eta$ (resp. $\mathcal{G}\overset{\circ}{\mathcal{Z}}{}^\alpha_\eta$).

11.8. We construct the *Beilinson-Drinfeld* avatar $\mathcal{G}\mathbf{Z}^\alpha_\eta$ of the restricted convolution diagram $\mathcal{GZ}^\alpha_\eta$.

Definition. $\mathcal{G}\mathbf{Z}^\alpha_\eta$ is the space of triples $(D, (\mathcal{U}_\lambda)_{\lambda\in X^+}, (\mathfrak{U}_\lambda)_{\lambda\in X^+})$ where $D \in \mathbb{A}^{\eta+\alpha}$ is an effective I-colored divisor, and $(\mathcal{U}_\lambda)_{\lambda\in X^+}, (\mathfrak{U}_\lambda)_{\lambda\in X^+}$ are the collections of vector bundles on $\mathbb{P}^1$ such that

a) $(\mathcal{U}_\lambda)_{\lambda\in X^+} \in \overline{\mathcal{G}}_\eta$, or in other words, $(\mathcal{U}_\lambda)_{\lambda\in X^+}$ satisfies the conditions 10.2 a)-c);

b) $(D, (\mathfrak{U}_\lambda)_{\lambda\in X^+}) \in \mathbf{Z}^{\eta+\alpha}$, or in other words, $(D, (\mathfrak{U}_\lambda)_{\lambda\in X^+})$ satisfies the conditions 10.5 a)-d);

c) $\iota_\lambda(\mathcal{V}^{\mathbf{N}_-}_\lambda(-\langle D, \lambda\rangle)) \subset \mathcal{U}_\lambda(-\langle\eta, \lambda\rangle 0)$ (notations of 10.5.1).

11.8.1. The identification $\mathcal{GZ}_\eta^\alpha = \mathcal{G}\mathbf{Z}_\eta^\alpha$ easily follows from 10.5. Under this identification, for $(D, (\mathcal{U}_\lambda)_{\lambda\in X^+}, (\mathfrak{U}_\lambda)_{\lambda\in X^+}) \in \mathcal{G}\mathbf{Z}_\eta^\alpha$, we have

$$\mathbf{p}(D, (\mathcal{U}_\lambda)_{\lambda\in X^+}, (\mathfrak{U}_\lambda)_{\lambda\in X^+}) = (\mathcal{U}_\lambda)_{\lambda\in X^+}$$

and

$$\mathbf{q}(D, (\mathcal{U}_\lambda)_{\lambda\in X^+}, (\mathfrak{U}_\lambda)_{\lambda\in X^+}) = (D, (\mathfrak{U}_\lambda)_{\lambda\in X^+})$$

11.9. We will introduce the *fine* stratifications of $\mathcal{GQ}_\eta^\alpha$ and $\mathcal{GZ}_\eta^\alpha$ following the section 7.

11.9.1. *Fine stratification of* $\mathcal{GQ}_\eta^\alpha$. We have

$$\mathcal{GQ}_\eta^\alpha = \bigsqcup \overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \times (\mathbb{P}^1 - 0)_\Gamma^{\beta-\gamma}$$

Here the union is taken over dominant $\chi \leq \eta$ in Y^+, arbitrary $\gamma \leq \beta \leq \alpha \in Y$, and partitions $\Gamma \in \mathfrak{P}(\beta-\gamma)$. Furthermore, $\overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \subset \overset{\circ}{\mathcal{GQ}}{}_\chi^\gamma$ is an open subvariety formed by all the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda\in X^+}$ such that $(\mathcal{U}_\lambda) \in \mathcal{G}_\chi \subset \overline{\mathcal{G}}_\chi$, and $\mathfrak{L}_\lambda$ is a *line subbundle* in $\mathcal{U}_\lambda$.

The stratum $\overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \times (\mathbb{P}^1 - 0)_\Gamma^{\beta-\gamma}$ is formed by the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda\in X^+}$ such that $\mathbf{p}(\mathcal{U}_\lambda, \mathfrak{L}_\lambda) \in \mathcal{G}_\chi \subset \overline{\mathcal{G}}_\eta$; the normalization (see 3.4.2) of $\mathfrak{L}$ in $\mathcal{U}$ has degree γ; and the defect D (see *loc. cit.*) of $\mathfrak{L}$ in $\mathcal{U}$ equals $(\alpha-\beta)0 + D'$ where $D' \in (\mathbb{P}^1 - 0)_\Gamma^{\beta-\gamma}$.

Proposition. $\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha$ is smooth for arbitrary α, η, i.e. the fine strata are smooth.

Proof will be given in 12.4.

11.9.2. *Fine Schubert stratification of* $\mathcal{GQ}_\eta^\alpha$. We have

$$\mathcal{GQ}_\eta^\alpha = \bigsqcup \overset{\circ\circ}{\mathcal{GQ}}{}_{w,\chi}^\gamma \times (\mathbb{P}^1 - 0)_\Gamma^{\beta-\gamma}$$

Here the union is taken over dominant $\chi \leq \eta$ in Y^+, representatives $w \in \mathcal{W}_f/\mathcal{W}_\chi$, arbitrary $\gamma \leq \beta \leq \alpha \in Y$, and partitions $\Gamma \in \mathfrak{P}(\beta-\gamma)$. Furthermore, $\overset{\circ\circ}{\mathcal{GQ}}{}_{w,\chi}^\gamma \subset \overset{\circ}{\mathcal{GQ}}{}_\chi^\gamma$ is an open subvariety formed by all the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda\in X^+}$ such that $(\mathcal{U}_\lambda) \in \mathcal{G}_{w,\chi} \subset \overline{\mathcal{G}}_\chi$, and $\mathfrak{L}_\lambda$ is a *line subbundle* in $\mathcal{U}_\lambda$.

The stratum $\overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \times (\mathbb{P}^1 - 0)_\Gamma^{\beta-\gamma}$ is formed by the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda\in X^+}$ such that $\mathbf{p}(\mathcal{U}_\lambda, \mathfrak{L}_\lambda) \in \mathcal{G}_{w,\chi} \subset \overline{\mathcal{G}}_\eta$; the normalization of $\mathfrak{L}$ in $\mathcal{U}$ has degree γ; and the defect D (see *loc. cit.*) of $\mathfrak{L}$ in $\mathcal{U}$ equals $(\alpha-\beta)0+D'$ where $D' \in (\mathbb{P}^1 - 0)_\Gamma^{\beta-\gamma}$.

Proposition. $\overset{\circ\circ}{\mathcal{GQ}}{}_{w,\eta}^\alpha$ is smooth for arbitrary α, η, w, i.e. the fine Schubert strata are smooth.

Proof will be given in 12.4.

11.9.3. *Fine Schubert stratification of* $\mathcal{GZ}_\eta^\alpha$. Similarly, we have

$$\mathcal{GZ}_\eta^\alpha = \bigsqcup \overset{\circ}{\mathcal{GZ}}{}_{w,\chi}^\gamma \times (\mathbb{C}^*)_\Gamma^{\beta-\gamma}$$

Here the union is taken over dominant $\chi \leq \eta$ in Y^+, representatives $w \in \mathcal{W}_f/\mathcal{W}_\chi$, arbitrary $\gamma \leq \beta \leq \alpha \in Y$, and partitions $\Gamma \in \mathfrak{P}(\beta-\gamma)$. Furthermore, $\overset{\circ}{\mathcal{GZ}}{}_{w,\chi}^\gamma \subset \overset{\circ}{\mathcal{GZ}}{}_\chi^\gamma$ is an open subvariety formed by all the collections $(\mathcal{U}_\lambda, \mathfrak{L}_\lambda)_{\lambda\in X^+}$ such that $(\mathcal{U}_\lambda) \in \mathcal{G}_{w,\chi} \subset \overline{\mathcal{G}}_\chi$, and $\mathfrak{L}_\lambda$ is a *line subbundle* in $\mathcal{U}_\lambda$.

The stratum $\overset{\circ}{\mathcal{G}\mathcal{Z}}{}^{\gamma}_{w,\chi}\times(\mathbb{C}^*)^{\beta-\gamma}_{\Gamma}$ is formed by the collections $(\mathcal{U}_\lambda,\mathfrak{L}_\lambda)_{\lambda\in X^+}$ such that $\mathbf{p}(\mathcal{U}_\lambda,\mathfrak{L}_\lambda)\in\mathcal{G}_{w,\chi}\subset\overline{\mathcal{G}}_\eta$; the normalization of $\mathfrak{L}$ in $\mathcal{U}$ has degree γ; and the defect D of $\mathfrak{L}$ in $\mathcal{U}$ equals $(\alpha-\beta)0+D'$ where $D'\in(\mathbb{C}^*)^{\beta-\gamma}_{\Gamma}$.

11.9.4. The reader unwilling to believe that $\overset{\circ}{\mathcal{G}\mathcal{Z}}{}^{\gamma}_{w,\chi}$ is smooth for arbitrary $\gamma\in Y$ may repeat the trick of 9.3. Namely, one can replace $\mathcal{G}\mathcal{Z}^\alpha_\eta$ with an open subvariety $\ddot{\mathcal{G}\mathcal{Z}}{}^\alpha_\eta$ formed by the union of the above strata for sufficiently dominant γ; they are easily seen to be smooth. Moreover, $\mathbf{q}(\ddot{\mathcal{G}\mathcal{Z}}{}^\alpha_\eta)\supset\ddot{\mathcal{Z}}^{\eta+\alpha}$, and $\ddot{\mathcal{G}\mathcal{Z}}{}^\alpha_\eta\supset\mathbf{q}^{-1}(\ddot{\mathcal{Z}}^{\eta+\alpha})$.

11.10. Let us introduce the *fine Schubert stratification* of the local convolution diagram (see 11.5). Iwahori subgroup $\mathbf{I}\subset\mathbf{G}(\mathcal{O})$ acts on $\widetilde{\mathbf{Q}}$, and defines the *fine Schubert stratification* of $\widetilde{\mathbf{Q}}$ by Iwahori orbits $\dot{\mathbf{Q}}^\alpha_w\subset\dot{\mathbf{Q}}^\alpha,\ \alpha\in Y, w\in\mathcal{W}_f$. Furthermore, we have

$$\mathcal{G}\mathbf{Q}_\eta=\bigsqcup\mathcal{G}\dot{\mathbf{Q}}^{-\alpha}_{w,\chi}$$

Here the union is taken over dominant $\chi\le\eta$ in Y^+, representatives $w\in\mathcal{W}_f/\mathcal{W}_\chi$, and $\alpha\in\mathbb{N}[I]$. The stratum $\mathcal{G}\dot{\mathbf{Q}}^{-\alpha}_{w,\chi}$ consists of collections $(\mathfrak{V}_\lambda,v_\lambda)_{\lambda\in X^+}$ (vectors $v_\lambda\in\mathfrak{V}_\lambda$ are defined up to multiplication by $\mathbb{C}^*$) such that

a) $(\mathfrak{V}_\lambda)_{\lambda\in X^+}\in\mathcal{G}_{w,\chi}\subset\overline{\mathcal{G}}_\eta$;

b) $z^{-\langle\alpha,\lambda\rangle}v_\lambda\in\mathfrak{V}_\lambda$ for all λ, but $z^{-\langle\alpha,\lambda\rangle-1}v_\lambda\notin\mathfrak{V}_\lambda$ for some λ.

12. Convolution

12.1. Let $\mathfrak{A},\mathfrak{B}$ be smooth varieties, and let $\mathfrak{p}:\ \mathfrak{A}\longrightarrow\mathfrak{B}$ be a map. Suppose $\mathfrak{A}$ (resp. $\mathfrak{B}$) is equipped with a stratification $\mathfrak{S}$ (resp. $\mathfrak{T}$), and $\mathfrak{p}$ is stratified with respect to the stratifications. Let $\mathfrak{R}$ be another stratification of $\mathfrak{B}$, transversal to $\mathfrak{T}$. Let $\mathcal{B}$ (resp. $\mathcal{A}$) be a perverse sheaf on $\mathfrak{B}$ (resp. $\mathfrak{A}$) smooth along $\mathfrak{R}$ (resp. $\mathfrak{S}$). Let $b=\dim\mathfrak{B}$.

Lemma. a) $\mathcal{A}\otimes\mathfrak{p}^*\mathcal{B}[-b]$ is perverse;

b) $\mathcal{A}\otimes\mathfrak{p}^*\mathcal{B}[-b]=\mathcal{A}\overset{!}{\otimes}\mathfrak{p}^!\mathcal{B}[b]$;

c) Let $\overline{R}$ (resp. $\overline{S}$) be the closure of a stratum R in $\mathfrak{R}$ (resp. S in $\mathfrak{S}$). Then $\mathcal{IC}(\overline{S})\otimes\mathfrak{p}^*\mathcal{IC}(\overline{R})[-b]=\mathcal{IC}(\overline{S}\cap\mathfrak{p}^{-1}\overline{R})$;

d) Let R be a stratum of stratification $\mathfrak{R}$. Then $\mathfrak{p}^{-1}R$ is smooth.

Proof. a,b) Let $g:\ \mathfrak{G}\longrightarrow\mathfrak{A}\times\mathfrak{B}$ denote the closed embedding of the graph of $\mathfrak{p}$. The perverse sheaf $\mathcal{A}\boxtimes\mathcal{B}$ on $\mathfrak{A}\times\mathfrak{B}$ is smooth along the product stratification $\mathfrak{S}\times\mathfrak{R}$. The transversality of $\mathfrak{R}$ and $\mathfrak{T}$ implies that the embedding g is noncharacteristic with respect to $\mathcal{A}\boxtimes\mathcal{B}$ (see [KS], Definition 5.4.12). Furthermore, by definition, $\mathcal{A}\otimes\mathfrak{p}^*\mathcal{B}=g^*(\mathcal{A}\boxtimes\mathcal{B})$, and $\mathcal{A}\overset{!}{\otimes}\mathfrak{p}^!\mathcal{B}=g^!(\mathcal{A}\boxtimes\mathcal{B})$. Now a) is nothing else than [KS], Corollary 10.3.16(iii), while b) is nothing else than [KS], Proposition 5.4.13(ii). □

c) We consider $\overline{S}\times\overline{R}$ as a subvariety of $\mathfrak{A}\times\mathfrak{B}$. Since $g:\ \mathfrak{G}\longrightarrow\mathfrak{A}\times\mathfrak{B}$ is noncharacteristic with respect to $\mathcal{IC}(\overline{S})\boxtimes\mathcal{IC}(\overline{R})=\mathcal{IC}(\overline{S}\times\overline{R})$, we conclude that $g^*\mathcal{IC}(\overline{S}\times\overline{R})[-b]=\mathcal{IC}(\mathfrak{G}\cap(\overline{S}\times\overline{R}))$. It remains to note that $\mathfrak{G}\cap(\overline{S}\times\overline{R})=\overline{S}\cap\mathfrak{p}^{-1}\overline{R}$, and $g^*\mathcal{IC}(\overline{S}\times\overline{R})=\mathcal{IC}(\overline{S})\otimes\mathfrak{p}^*\mathcal{IC}(\overline{R})$. □

d) We will view $\mathfrak{p}^{-1}R$ as a subscheme of $\mathfrak{A}$ (scheme-theoretic fiber over R). Let $a\in\mathfrak{p}^{-1}R\subset\mathfrak{A}$. The Zariski tangent space $T_a(\mathfrak{p}^{-1}R)$ equals $d\mathfrak{p}_a^{-1}(T_bR)$ where $b=\mathfrak{p}(a)$, and $d\mathfrak{p}_a:\ T_a\mathfrak{A}\longrightarrow T_b\mathfrak{B}$ stands for the differential of $\mathfrak{p}$ at a. Let $\mathcal{T}$ be a stratum of $\mathfrak{T}$ containing $b=\mathfrak{p}(a)$. Then $T_b\mathcal{T}$ is contained in $d\mathfrak{p}_a(T_a\mathfrak{A})$ since $\mathfrak{p}$ is stratified with respect to $\mathfrak{T}$. Furthermore, $T_bR+T_b\mathcal{T}=T_b\mathfrak{B}$ due to the transversality of $\mathcal{T}$ and R. Hence $T_bR+d\mathfrak{p}_a(T_a\mathfrak{A})=T_b\mathfrak{B}$. Hence $\dim(d\mathfrak{p}_a^{-1}(T_bR))=\dim\mathfrak{A}-\dim\mathfrak{B}+\dim R$. We conclude that the dimension of the Zariski tangent space $T_a(\mathfrak{p}^{-1}R)$ is independent of $a\in\mathfrak{p}^{-1}R$, and thus $\mathfrak{p}^{-1}R$ is smooth. □

12.2. Consider the following cartesian diagram:

$$\begin{array}{ccc} \mathcal{G}\mathcal{Q}^\alpha_\eta & \xrightarrow{\mathbf{i}} & \mathfrak{Q}^\alpha \\ \mathbf{p}\downarrow & & \mathbf{p}\downarrow \\ \overline{\mathcal{G}}_\eta & \xrightarrow{\mathbf{i}} & \mathfrak{M} \end{array}$$

Proposition. a) For a $\mathbf{G}[[z]]$-equivariant perverse sheaf $\mathcal{F}$ on $\overline{\mathcal{G}}_\eta$ the sheaf $\mathcal{IC}(\mathfrak{Q}^\alpha)\otimes\mathbf{p}^*\mathbf{i}_*\mathcal{F}[-\dim\underline{\mathfrak{M}}^\eta]$ is supported on $\mathcal{G}\mathcal{Q}^\alpha_\eta$ and is perverse;

b) $\mathcal{IC}(\mathcal{G}\mathcal{Q}^\alpha_\eta)=\mathcal{IC}(\mathfrak{Q}^\alpha)\otimes\mathbf{p}^*\mathbf{i}_*\mathcal{IC}(\overline{\mathcal{G}}_\eta)[-\dim\underline{\mathfrak{M}}^\eta]$.

Proof. a) Let us restrict the right column of the above diagram to the open subscheme $\mathfrak{M}^\eta\subset\mathfrak{M}$, and take its quotient by $\mathbf{G}^\eta$. We obtain the cartesian diagram

$$\begin{array}{ccc} \mathcal{G}\mathcal{Q}^\alpha_\eta & \xrightarrow{\mathbf{i}} & \underline{\mathfrak{Q}}^{\alpha,\eta} \\ \mathbf{p}\downarrow & & \mathbf{p}\downarrow \\ \overline{\mathcal{G}}_\eta & \xrightarrow{\mathbf{i}} & \underline{\mathfrak{M}}^\eta \end{array}$$

of schemes of finite type. Here the rows are closed embeddings, and $\underline{\mathfrak{M}}^\eta$ is smooth. The stratification $\mathfrak{S}$ of $\mathfrak{Q}^\alpha$ is invariant under the action of $\mathbf{G}^\eta$, and descents to the same named quotient stratification of $\underline{\mathfrak{Q}}^{\alpha,\eta}$. Similarly, the stratification of $\mathfrak{M}$ by the isomorphism type of $\mathbf{G}$-torsors descents to the stratification $\mathfrak{T}$ of $\underline{\mathfrak{M}}^\eta$. The sheaf $\mathbf{i}_*\mathcal{F}$ on $\underline{\mathfrak{M}}^\eta$ is smooth along the stratification $\mathfrak{R}$ transversal to $\mathfrak{T}$. We have to prove that $\mathcal{IC}(\underline{\mathfrak{Q}}^{\alpha,\eta})\otimes\mathbf{p}^*\mathbf{i}_*\mathcal{F}[-\dim\underline{\mathfrak{M}}^\eta]$ is perverse. In order to apply the Lemma 12.1a) we only have to find an embedding $\underline{\mathfrak{Q}}^{\alpha,\eta}\overset{u}{\hookrightarrow}\mathfrak{A}$ into a smooth scheme such that the map $\mathbf{p}$ and stratification $\mathfrak{S}$ extend to $\mathfrak{A}$. Then we apply the Lemma 12.1a) to the sheaf $\mathcal{A}=u_*\mathcal{IC}(\underline{\mathfrak{Q}}^{\alpha,\eta})$ on $\mathfrak{A}$.

We will construct $\mathfrak{A}$ as a projective bundle over $\underline{\mathfrak{M}}^\eta$. The points of $\underline{\mathfrak{M}}^\eta$ are the $\mathbf{G}$-torsors $\mathcal{T}$ over C trivialized in some infinitesimal neighbourhood of $\infty\in C$. The points of $\underline{\mathfrak{Q}}^{\alpha,\eta}$ are the $\mathbf{G}$-torsors $\mathcal{T}$ over C trivialized in some infinitesimal neighbourhood of $\infty\in C$ along with collections of invertible subsheaves $\mathfrak{L}_\lambda\subset\mathcal{V}^\mathcal{T}_\lambda,\lambda\in X^+$, satisfying Plücker relations.

Now, if $\alpha=\sum_{i\in I}a_ii$ is dominant enough, the fiber of $\mathfrak{A}$ over $\mathcal{T}\in\underline{\mathfrak{M}}^\eta$ is $\prod_{i\in I}\mathbb{P}(\Gamma(C,\mathcal{V}^\mathcal{T}_{\omega_i}(a_i)))$. The map u sends $(\mathcal{T},(\mathfrak{L}_\lambda\subset\mathcal{V}^\mathcal{T}_\lambda)_{\lambda\in X^+})$ to $(\mathfrak{L}_{\omega_i}(a_i))\in\prod_{i\in I}\mathbb{P}(\Gamma(C,\mathcal{V}^\mathcal{T}_{\omega_i}(a_i)))$.

If α is not dominant enough, we first embed $\underline{\mathfrak{Q}}^{\alpha,\eta}$ into $\underline{\mathfrak{Q}}^{\beta+\alpha,\eta}$ for dominant enough β as follows: $(\mathcal{T},(\mathfrak{L}_\lambda\subset\mathcal{V}^\mathcal{T}_\lambda)_{\lambda\in X^+})\mapsto(\mathcal{T},(\mathfrak{L}_\lambda(-\langle\beta,\lambda\rangle 0)\subset\mathcal{V}^\mathcal{T}_\lambda)_{\lambda\in X^+})$. Then we compose with the above projective embedding of $\underline{\mathfrak{Q}}^{\beta+\alpha,\eta}$.

This completes the proof of a). □

b) We apply the Lemma 12.1c) to $\overline{R}=\overline{\mathcal{G}}_\eta,\ \overline{S}=\underline{\mathfrak{Q}}^{\alpha,\eta}$. □

12.3. **Proposition.** a) For a perverse sheaf $\mathcal{F}$ in $\mathcal{P}(\overline{\mathcal{G}}_\eta,\mathbf{I})$ the sheaf $\mathcal{IC}(\mathfrak{Q}^\alpha)\otimes\mathbf{p}^*\mathbf{i}_*\mathcal{F}[-\dim\underline{\mathfrak{M}}^\eta]$ is supported on $\mathcal{G}\mathcal{Q}^\alpha_\eta$ and is perverse;

b) $\mathcal{IC}(\mathcal{GQ}^\alpha_{w,\eta}) = \mathcal{IC}(\mathfrak{Q}^\alpha) \otimes \mathbf{p}^*\mathbf{i}_*\mathcal{IC}(\overline{\mathcal{G}}_{w,\eta})[-\dim \underline{\mathfrak{M}}^\eta]$.

Proof. The same as the proof of the Proposition 12.2; we need only to find a refinement $\mathfrak{W}$ of the stratification $\mathfrak{M} = \bigsqcup_{\eta\in Y^+} \mathfrak{M}_\eta$ which would be transversal to the Iwahori orbits $\mathcal{G}_{w,\chi}$ in $\mathcal{G}$. Now $\mathfrak{M} = \bigsqcup_{\eta\in Y^+} \mathfrak{M}_\eta$ is the stratification by the orbits of proalgebraic group $\mathbf{G}[[z^{-1}]]$ acting naturally on $\mathfrak{M}$. The desired refinement is the stratification by the orbits of subgroup $\mathbf{I}_- \subset \mathbf{G}[[z^{-1}]]$ formed by the formal loops $g(z) \in \mathbf{G}[[z^{-1}]]$ such that $g(\infty) \in \mathbf{B}_-$. □

12.3.1. Recall the notations of 11.2.1.

Conjecture. a) The map $\overset{\circ}{\mathbf{p}} : \overset{\circ}{\mathcal{GQ}}{}^\alpha_\eta \longrightarrow \overline{\mathcal{G}}_\eta$ is smooth onto its image;

b) Up to a shift, $\mathcal{IC}(\mathfrak{Q}^\alpha) \otimes \mathbf{p}^*\mathbf{i}_*\mathcal{F}[-\dim \underline{\mathfrak{M}}^\eta] = \mathbf{j}_{!*}\overset{\circ}{\mathbf{p}}{}^*\mathcal{F}$ for any perverse sheaf $\mathcal{F}$ in $\mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$.

12.4. **Proof of the Propositions 11.9.1 and 11.9.2.** We apply the Lemma 12.1d) to the following situation: $\mathfrak{A} = \overset{\circ}{\underline{\mathfrak{Q}}}{}^{\alpha,\eta} \subset \underline{\mathfrak{Q}}^{\alpha,\eta}$, $\mathfrak{B} = \underline{\mathfrak{M}}^\eta$, $R = \mathcal{G}_{w,\eta}$, $\mathfrak{p} = \overset{\circ}{\mathbf{p}}$. The stratification $\mathfrak{T}$ is defined as follows. Recall the stratification $\mathfrak{W}$ of $\mathfrak{M}$ by $\mathbf{I}_-$-orbits introduced in the proof of 12.3. It is invariant under the action of $\mathbf{G}^\eta$ and descends to the desired stratification $\mathfrak{T}$ of $\underline{\mathfrak{M}}^\eta$ transversal to R.

It remains to note that $\mathfrak{A} = \overset{\circ}{\underline{\mathfrak{Q}}}{}^{\alpha,\eta} = \overset{\circ}{\mathbf{p}}{}^{-1}(\mathfrak{M}^\eta)/\mathbf{G}^\eta$ is smooth being a quotient by the free group action of an open subscheme $\overset{\circ}{\mathbf{p}}{}^{-1}(\mathfrak{M}^\eta)$ of the smooth scheme $\overset{\circ}{\mathfrak{Q}}{}^\alpha$. Thus the assumptions of 12.1d) are in force, and we conclude that $\overset{\circ\circ}{\mathcal{GQ}}{}^\alpha_{w,\eta} = \overset{\circ}{\mathbf{p}}{}^{-1}(\mathcal{G}_{w,\eta})$ is smooth.

The proof of smoothness of $\overset{\circ\circ}{\mathcal{GQ}}{}^\alpha_\eta$ is absolutely similar. □

12.5. We define the following locally closed subscheme $\mathfrak{Z}^\alpha \subset \mathfrak{Q}^\alpha$. Its points are the $\mathbf{G}$-torsors $\mathcal{T}$ over C trivialized in the formal neighbourhood of $\infty \in C$ along with collections of invertible subsheaves $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda^\mathcal{T}, \lambda \in X^+$, satisfying Plücker relations plus two more conditions:

a) in some neighbourhood of $\infty \in C$ the invertible subsheaves $\mathfrak{L}_\lambda \subset \mathcal{V}_\lambda^\mathcal{T}$ are line subbundles. Thus they may be viewed as a reduction of $\mathcal{T}$ to $\mathbf{B} \subset \mathbf{G}$ in this neighbourhood. Since $\mathcal{T}$ is trivialized in the formal neighbourhood of $\infty \in C$, we obtain a map from this neighbourhood to the flag manifold $\mathbf{X}$.

b) The value of the above map at $\infty \in C$ equals $\mathbf{B}_- \in \mathbf{X}$.

We have the following cartesian diagram:

$$\begin{array}{ccc}
\mathcal{GZ}^\alpha_\eta & \xrightarrow{\mathbf{i}} & \mathfrak{Z}^\alpha \\
{\scriptstyle\mathbf{p}}\downarrow & & {\scriptstyle\mathbf{p}}\downarrow \\
\overline{\mathcal{G}}_\eta & \xrightarrow{\mathbf{i}} & \mathfrak{M}
\end{array}$$

Proposition. For a perverse sheaf $\mathcal{F} \in \mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$ the sheaf $\mathcal{IC}(\mathfrak{Z}^\alpha) \otimes \mathbf{p}^*\mathbf{i}_*\mathcal{F}[-\dim \underline{\mathfrak{M}}^\eta]$ is supported on $\mathcal{GZ}^\alpha_\eta$ and is perverse.

Proof. Similar to the proof of the Proposition 12.2. □

12.5.1. *Remark.* Let us denote the embedding of $\mathfrak{Z}^\alpha$ into $\mathfrak{Q}^\alpha$ by s. One can easily check that $\mathcal{IC}(\mathfrak{Z}^\alpha) = s^*\mathcal{IC}(\mathfrak{Q}^\alpha)[-\dim \mathbf{X}]$.

12.6. Recall the notations of 8.1.

Proposition. a) $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$ (resp. $\mathbf{q}: \mathcal{GZ}_\eta^\alpha \longrightarrow \mathcal{Z}^{\eta+\alpha}$) is proper;

b) Restriction of $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$ (resp. $\mathbf{q}: \mathcal{GZ}_\eta^\alpha \longrightarrow \mathcal{Z}^{\eta+\alpha}$) to $\overset{\bullet}{\mathcal{Q}}{}^{\eta+\alpha} \subset \mathcal{Q}^{\eta+\alpha}$ (resp. to $\overset{\bullet}{\mathcal{Z}}{}^{\eta+\alpha} \subset \mathcal{Z}^{\eta+\alpha}$) is an isomorphism.

Proof. a) is evident.

b) It suffices to consider the case of $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$. Let $(\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \mathcal{Q}^{\eta+\alpha}$. Then $(\mathcal{U}_\lambda, \mathfrak{L}'_\lambda)_{\lambda\in X^+} \in \mathbf{q}^{-1}(\mathfrak{L}_\lambda)_{\lambda\in X^+}$ iff for any $\lambda \in X^+$ we have $\mathfrak{L}'_\lambda = \mathfrak{L}_\lambda(\langle\eta,\lambda\rangle 0)$, and $\mathcal{U}_\lambda \supset \mathcal{V}_\lambda(-\langle\eta,\lambda\rangle 0) + \mathfrak{L}'_\lambda$.

Consider $\mathfrak{L}_\lambda = \mathcal{V}_\lambda^{\mathbf{N}_-}(-\langle\eta+\alpha,\lambda\rangle\infty)$, so that $\varphi = (\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \overset{\bullet}{\mathcal{Q}}{}^{\eta+\alpha}$. Then $(\mathcal{U}_\lambda, \mathfrak{L}'_\lambda)_{\lambda\in X^+} \in \mathbf{q}^{-1}(\mathfrak{L}_\lambda)_{\lambda\in X^+}$ iff for any $\lambda \in X^+$ we have $\mathfrak{L}'_\lambda = \mathfrak{L}_\lambda(\langle\eta,\lambda\rangle 0) = \mathcal{V}_\lambda^{\mathbf{N}_-}(\langle\eta,\lambda\rangle 0 - \langle\eta+\alpha,\lambda\rangle\infty)$, and $\mathcal{U}_\lambda \supset \mathcal{V}_\lambda(-\langle\eta,\lambda\rangle 0) + \mathfrak{L}'_\lambda = \mathcal{V}_\lambda(-\langle\eta,\lambda\rangle 0) + \mathcal{V}_\lambda^{\mathbf{N}_-}(\langle\eta,\lambda\rangle 0)$.

In other words, $(\mathcal{U}_\lambda)_{\lambda\in X^+}$ lies in the intersection of $\overline{\mathcal{G}}_\eta$ with the semiinfinite orbit T_η (see 6.4.1). This intersection consists exactly of one point (see [MV]). Thus $\mathbf{q}^{-1}(\varphi)$ consists of one point.

Recall the cartesian diagram 11.6. We have $\mathfrak{s}(\overset{\bullet}{\mathcal{Q}}{}^{\eta+\alpha}) \subset \overset{\bullet}{\mathbf{Q}}{}^\circ$ (notations of 4.3), in particular, $\mathfrak{s}(\varphi) \in \overset{\bullet}{\mathbf{Q}}{}^\circ$. Since the map $\mathbf{q}: \mathcal{G}\mathbf{Q}_\eta \longrightarrow \mathbf{Q}$ is $\mathbf{G}(\mathcal{O})$-equivariant, and its fiber over $\mathfrak{s}(\varphi)$ consists of one point, we conclude that $\mathbf{q}$ is isomorphism over the $\mathbf{G}(\mathcal{O})$-orbit $\overset{\bullet}{\mathbf{Q}}{}^0$. Since $\overset{\bullet}{\mathcal{Q}}{}^{\eta+\alpha} = \mathfrak{s}^{-1}(\overset{\bullet}{\mathbf{Q}}{}^\circ)$, applying the cartesian diagram 11.6, we deduce that $\mathbf{q}$ is isomorphism over $\overset{\bullet}{\mathcal{Q}}{}^{\eta+\alpha}$. □

12.7. **Proof of the Proposition 10.7.3.** We are interested in the stalk of $\mathcal{IC}(\mathfrak{Q}^\alpha)$ at a point $(\mathcal{T}, (\mathfrak{L}_\lambda)_{\lambda\in X^+}) \in \overset{\circ}{\mathfrak{Q}}{}^\gamma \times C_\Gamma^{\beta-\gamma} \subset \mathfrak{Q}^\alpha$. Suppose that the isomorphism class of $\mathbf{G}$-torsor $\mathcal{T}$ equals $\eta \in Y^+$, i.e. $\mathcal{T} \in \mathfrak{M}_\eta$. The stalk in question evidently does not depend on a choice of $\mathcal{T} \in \mathfrak{M}_\eta$ and the defect $D \in C_\Gamma^{\beta-\gamma}$. In particular, we may (and will) suppose that $\mathcal{T} \in \mathbf{i}(\mathcal{G}_\eta)$, and $D \in (C-0)_\Gamma^{\beta-\gamma}$. Then one can see easily that the stalk in question is isomorphic, up to a shift, to the stalk of Goresky-MacPherson sheaf $\mathcal{IC}(\mathcal{GQ}_\eta^\alpha)$ at the point $(\mathcal{T}, (\mathfrak{L}_\lambda)_{\lambda\in X^+}) \in \mathcal{GQ}_\eta^\alpha$.

On the other hand, according to the Proposition 12.6 b), the latter stalk is isomorphic to the stalk of $\mathcal{IC}(\mathcal{Q}^{\eta+\alpha})$ at the point $\mathbf{q}(\mathcal{T}, (\mathfrak{L}_\lambda)_{\lambda\in X^+})$. This point has the same defect D. Applying 10.7.2 we complete the proof of the Proposition 10.7.3. □

12.8. Recall that a map $\pi: \mathcal{X} \longrightarrow \mathcal{Y}$ is called *dimensionally semismall* if the following condition holds: let $\mathcal{Y}_m$ be the set of all points $y \in \mathcal{Y}$ such that $\dim(\pi^{-1}y) \geq m$, then for $m > 0$ we have $\mathrm{codim}_\mathcal{Y}\mathcal{Y}_m \geq 2m$. Let us define $\mathcal{X}_m = \pi^{-1}\mathcal{Y}_m$. Then we can formulate an equivalent condition of semismallness as follows: for any $m \geq 0$ we have $\mathrm{codim}_\mathcal{X}\mathcal{X}_m \geq m$.

Suppose $\mathcal{X}$ (resp. $\mathcal{Y}$) is equipped with a stratification $\mathfrak{S}$ (resp. $\mathfrak{T}$), and π is stratified with respect to $\mathfrak{S}$ and $\mathfrak{T}$. Then π is called *stratified semismall* (see [MV]) if π is proper, and the restriction $\pi|_S$ to any stratum in $\mathfrak{S}$ is dimensionally semismall. In this case $\pi_* = \pi_!$ takes perverse sheaves on $\mathcal{X}$ smooth along $\mathfrak{S}$ to perverse sheaves on $\mathcal{Y}$ smooth along $\mathfrak{T}$ (see *loc. cit.*).

12.9. Recall the fine Schubert stratifications of $\mathcal{Q}^{\eta+\alpha}$ (resp. $\mathcal{GQ}_\eta^\alpha$) introduced in 8.4.1 (resp. 11.9.2). The map $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$ is stratified with respect to these stratifications.

Proposition. $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$ is stratified semismall.

Proof will use a few Lemmas.

12.9.1. *Lemma.* $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$ is dimensionally semismall.

Proof. Recall that we have $\mathcal{Q}^{\eta+\alpha} = \sqcup_{\beta\leq\eta+\alpha}\overset{\bullet}{\mathcal{Q}}{}^\beta$. It is enough to prove that for $(\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \overset{\bullet}{\mathcal{Q}}{}^\beta$ we have $\dim \mathbf{q}^{-1}(\mathfrak{L}_\lambda)_{\lambda\in X^+} \leq |\eta+\alpha-\beta|$.

Let us start with the case $\mathfrak{L}_\lambda = \mathcal{V}_\lambda^{\mathbf{N}_-}(\langle\beta-\eta-\alpha,\lambda\rangle 0 - \langle\beta,\lambda\rangle\infty)$. Then, like in the Proposition 12.6, we have $\mathbf{q}^{-1}(\mathfrak{L}_\lambda)_{\lambda\in X^+} = \overline{\mathcal{G}}_\eta \cap \overline{T}_{\beta-\alpha}$, and according to [MV], we have $\dim(\overline{\mathcal{G}}_\eta \cap \overline{T}_{\beta-\alpha}) \leq |\eta+\alpha-\beta|$.

Now $\overset{\bullet}{\mathcal{Q}}{}^\beta$ is stratified by the defect: $\overset{\bullet}{\mathcal{Q}}{}^\beta = \sqcup_{\Gamma\in\mathfrak{P}(\beta-\gamma)}^{\beta\geq\gamma\geq 0}\overset{\circ}{\mathcal{Q}}{}^\gamma \times (C-0)_\Gamma^{\beta-\gamma}$, and $\mathbf{q}$ is evidently stratified with respect to this stratification. The point $(\mathcal{V}_\lambda^{\mathbf{N}_-}(\langle\beta-\eta-\alpha,\lambda\rangle 0-\langle\beta,\lambda\rangle\infty))_{\lambda\in X^+}$ lies in the smallest (closed) stratum $\gamma=0, \Gamma=\{\{\beta\}\}$. Since the dimension of preimage is a lower semicontinuous function on $\overset{\bullet}{\mathcal{Q}}{}^\beta$, we conclude that for any point $(\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \overset{\bullet}{\mathcal{Q}}{}^\beta$ we have $\dim \mathbf{q}^{-1}(\mathfrak{L}_\lambda)_{\lambda\in X^+} \leq |\eta+\alpha-\beta|$. □

12.9.2. Recall the fine stratification of $\mathcal{GQ}_\eta^\alpha$ (resp. of $\mathcal{Q}^{\eta+\alpha}$) introduced in 11.9.1 (resp. in 8.4.1).

Lemma. $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$ is stratified semismall with respect to fine stratifications.

Proof. We consider a fine stratum $\overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \times (\mathbb{P}^1-0)_\Gamma^{\beta-\gamma} \subset \mathcal{GQ}_\eta^\alpha$ (see 11.9.1). Temporarily we will write $\mathbf{q}_\eta^\alpha$ for $\mathbf{q}$ to stress its dependence on η and α.

The restriction of $\mathbf{q}_\eta^\alpha$ to the stratum $\overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \times (\mathbb{P}^1-0)_\Gamma^{\beta-\gamma}$ decomposes into the following composition of morphisms:

$$\overset{\circ\circ}{\mathcal{GQ}}{}_\chi^\gamma \times (\mathbb{P}^1-0)_\Gamma^{\beta-\gamma} \overset{a\times\mathrm{id}}{\hookrightarrow} \mathcal{GQ}_\chi^\gamma \times (\mathbb{P}^1-0)_\Gamma^{\beta-\gamma} \xrightarrow{\mathbf{q}_\chi^\gamma\times\mathrm{id}} \mathcal{Q}^{\chi+\gamma} \times (\mathbb{P}^1-0)_\Gamma^{\beta-\gamma} \overset{b}{\hookrightarrow} \mathcal{Q}^{\eta+\alpha}$$

Here a is the open inclusion; and $b(\mathfrak{L}, D') = \mathfrak{L}((\beta-\alpha+\chi-\eta)0 - D')$.

Now $\mathbf{q}_\chi^\gamma \times \mathrm{id}$ is semismall according to the Lemma 12.9.1. This completes the proof of the Lemma. □

12.9.3. *Lemma.* The restriction of $\mathbf{q}$ to the fine Schubert stratum $\overset{\circ\circ}{\mathcal{GQ}}{}_{w,\eta}^\alpha \subset \mathcal{GQ}_\eta^\alpha$ is dimensionally semismall for any $w \in \mathcal{W}_f/\mathcal{W}_\eta$.

Proof. Let $\mathbf{K} \subset \mathbf{I} \subset \mathbf{G}(\mathcal{O})$ denote the first congruence subgroup formed by the loops $g(z) \in \mathbf{G}(\mathcal{O})$ such that $g(0) = 1$. The point $\eta(z) \in \mathcal{G}_{e,\eta} \subset \mathcal{G}_\eta$ was introduced in 10.4. For a positive integer m the subset $(\mathcal{GQ}_\eta^\alpha)_m$ (with respect to $\mathbf{q}: \mathcal{GQ}_\eta^\alpha \longrightarrow \mathcal{Q}^{\eta+\alpha}$) was introduced in 12.8.

Claim 1. $\operatorname{codim}_{\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha}(\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha)_m = \operatorname{codim}_{\overset{\circ}{\mathbf{p}}{}^{-1}(g\cdot\mathbf{K}\cdot\eta(z))}[\overset{\circ}{\mathbf{p}}{}^{-1}(g\cdot\mathbf{K}\cdot\eta(z)) \cap (\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha)_m]$ for any $m \geq 0$ and $g \in \mathbf{G}$.

In effect, due to the $\mathbf{G}$-equivariance of $\overset{\circ}{\mathbf{p}}$ and $\mathbf{q}$, the RHS does not depend on a choice of $g \in \mathbf{G}$, so it suffices to consider $g = e$.

The stabilizer of $\eta(z)$ in $\mathbf{G}$ is nothing else than the parabolic subgroup $\mathbf{P}(I_\eta)$ introduced in 10.4. We have

$$\mathcal{G}_\eta = \mathbf{G} \times_{\mathbf{P}(I_\eta)} [\mathbf{K}\cdot\eta(z)];\ \overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha = \mathbf{G} \times_{\mathbf{P}(I_\eta)} [\overset{\circ}{\mathbf{p}}{}^{-1}(\mathbf{K}\cdot\eta(z))];\ (\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha)_m = \mathbf{G} \times_{\mathbf{P}(I_\eta)} [\overset{\circ}{\mathbf{p}}{}^{-1}(\mathbf{K}\cdot\eta(z)) \cap (\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha)_m]$$

The Claim follows.

Claim 2. $\operatorname{codim}_{\overset{\circ}{\mathbf{p}}{}^{-1}(g\cdot\mathbf{K}\cdot\eta(z))}[\overset{\circ}{\mathbf{p}}{}^{-1}(g\cdot\mathbf{K}\cdot\eta(z)) \cap (\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha)_m] = \operatorname{codim}_{\overset{\circ\circ}{\mathcal{GQ}}{}_{w,\eta}^\alpha}[\overset{\circ\circ}{\mathcal{GQ}}{}_{w,\eta}^\alpha \cap (\overset{\circ\circ}{\mathcal{GQ}}{}_\eta^\alpha)_m]$ for any $m \geq 0$, $g \in \mathbf{G}$, and $w \in \mathcal{W}_f/\mathcal{W}_\eta$.

In effect, let us choose g in the normalizer of $\mathbf{H}$ representing w. Let us denote by $\mathbf{P}_w$ the intersection $\mathbf{P}(I_\eta)\cap g\mathbf{P}(I_\eta)g^{-1}$. Then we have

$$\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_{w,\eta}=\mathbf{P}(I_\eta)\times_{\mathbf{P}_w}[\overset{\circ}{\mathbf{p}}{}^{-1}(g\cdot\mathbf{K}\cdot\eta(z))];\ \overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_{w,\eta}\cap(\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_\eta)_m=\mathbf{P}(I_\eta)\times_{\mathbf{P}_w}[\overset{\circ}{\mathbf{p}}{}^{-1}(g\cdot\mathbf{K}\cdot\eta(z))\cap(\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_\eta)_m]$$

The Claim follows.

Comparing the two Claims we obtain

$$\operatorname{codim}_{\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_{w,\eta}}(\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_{w,\eta})_m\geq\operatorname{codim}_{\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_{w,\eta}}[\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_{w,\eta}\cap(\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_\eta)_m]=\operatorname{codim}_{\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_\eta}(\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}^\alpha_\eta)_m\geq m$$

The last inequality holds by the virtue of the Lemma 12.9.1. This completes the proof of the Lemma. □

12.9.4. Now we are ready to finish the proof of the Proposition. It remains to show that the restriction of $\mathbf{q}$ to any fine Schubert stratum is dimensionally semismall. It follows from the Lemma 12.9.3 in the same way as the Lemma 12.9.2 followed from the Lemma 12.9.1 ("twisting by defect"). This completes the proof of the Proposition. □

12.10. **Corollary.** The functor $\mathbf{q}_*=\mathbf{q}_!$ takes perverse sheaves on $\mathcal{G}\mathcal{Q}^\alpha_\eta$ smooth along the fine Schubert stratification to perverse sheaves on $\mathcal{Q}^{\eta+\alpha}$ smooth along the fine Schubert stratification. □

12.11. A few remarks are in order.

12.11.1. *Remark.* Recall the fine Schubert stratification of the local convolution diagram introduced in 11.5 and 11.10. The arguments used in the proof of the Proposition 12.9 along with the Lemma 4.4.3 show that the map $\mathbf{q}:\ \mathcal{G}\mathbf{Q}_\eta\longrightarrow\mathbf{Q}$ is stratified semismall with respect to the fine Schubert stratification.

12.11.2. *Remark.* The same arguments as in the proof of Lemma 12.9.3 show that the convolution $\mathcal{A}*\mathcal{B}$ of perverse sheaves on $\mathcal{G}$ (see [L2], or [G], [MV]) is perverse if $\mathcal{B}$ is $\mathbf{G}(\mathcal{O})$-equivariant. In the particular case $\mathcal{A}\in\mathcal{P}(\mathcal{G},\mathbf{I})$ this was also proved by G.Lusztig in [L6] using calculations in the affine Hecke algebra.

12.12. **Theorem.** Let $\eta\in Y^+,\alpha\in\mathbb{N}[I]$. Consider the following diagram:

$$\begin{tikzcd}
\overline{\mathcal{G}}_\eta \arrow[d, hook, "\mathbf{i}"'] & \mathcal{G}\mathcal{Q}^\alpha_\eta \arrow[l, "\mathbf{p}"'] \arrow[r, "\mathbf{q}"] \arrow[d, hook, "\mathbf{i}"'] & \mathcal{Q}^{\eta+\alpha} \\
\mathfrak{M} & \mathfrak{Q}^\alpha \arrow[l, "\mathbf{p}"'] &
\end{tikzcd}$$

For a perverse sheaf $\mathcal{F}$ in $\mathcal{P}(\overline{\mathcal{G}}_\eta,\mathbf{I})$, the sheaf $\mathbf{c}^\alpha_{\mathcal{Q}}(\mathcal{F}):=\mathbf{q}_*(\mathcal{IC}(\mathfrak{Q}^\alpha)\otimes\mathbf{p}^*\mathcal{F})[-\dim\underline{\mathfrak{M}}^\eta]$ on $\mathcal{Q}^{\eta+\alpha}$ is perverse and smooth along the fine Schubert stratification.

Proof. By the virtue of 12.3 we know that $\mathcal{IC}(\mathfrak{Q}^\alpha)\otimes\mathbf{p}^*\mathcal{F}[-\dim\underline{\mathfrak{M}}^\eta]$ is a perverse sheaf on $\mathcal{G}\mathcal{Q}^\alpha_\eta$. In order to apply the Corollary 12.10 we have to check that $\mathcal{IC}(\mathfrak{Q}^\alpha)\otimes\mathbf{p}^*\mathcal{F}[-\dim\underline{\mathfrak{M}}^\eta]$ is smooth along the fine Schubert stratification. The sheaf $\mathbf{p}^*\mathcal{F}$ is evidently smooth along the fine Schubert stratification. The sheaf $\mathcal{IC}(\mathfrak{Q}^\alpha)$ is constant along the stratification by defect (see 10.7.3), hence $\mathbf{i}^*\mathcal{IC}(\mathfrak{Q}^\alpha)$ is smooth along the fine Schubert stratification. This completes the proof of the Theorem. □

12.13. **Conjecture.** Let $\eta \in Y^+, \alpha \in Y$ be such that $\eta + \alpha \in \mathbb{N}[I]$. Consider the following diagram:

$$\begin{tikzcd} \overline{\mathcal{G}}_\eta \arrow[d, hook, "\mathbf{i}"'] & \mathcal{G}\mathcal{Q}^\alpha_\eta \arrow[l, "\mathbf{p}"'] \arrow[r, "\mathbf{q}"] \arrow[d, hook, "\mathbf{i}"'] & \mathcal{Q}^{\eta+\alpha} \\ \mathfrak{M} & \mathfrak{Q}^\alpha \arrow[l, "\mathbf{p}"'] & \end{tikzcd}$$

For a perverse sheaf $\mathcal{F}$ in $\mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$, the sheaf $\mathbf{q}_*(\mathcal{IC}(\mathfrak{Q}^\alpha) \otimes \mathbf{p}^*\mathcal{F})[-\dim \underline{\mathfrak{M}}^\eta]$ on $\mathcal{Q}^{\eta+\alpha}$ is perverse and smooth along the fine Schubert stratification.

12.14. **Corollary.** Let $\eta \in Y^+, \alpha \gg 0$. Consider the following diagram:

$$\begin{tikzcd} \overline{\mathcal{G}}_\eta \arrow[d, hook, "\mathbf{i}"'] & \mathcal{G}\ddot{\mathcal{Z}}^\alpha_\eta \arrow[l, "\mathbf{p}"'] \arrow[r, "\mathbf{q}"] \arrow[d, hook, "\mathbf{i}"'] & \ddot{\mathcal{Z}}^{\eta+\alpha} \\ \mathfrak{M} & \mathfrak{Z}^\alpha \arrow[l, "\mathbf{p}"'] & \end{tikzcd}$$

(notations of 11.9.4 and 12.5).

For a perverse sheaf $\mathcal{F} \in \mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$, the sheaf $\mathbf{c}^\alpha_{\mathcal{Z}}(\mathcal{F}) := \mathbf{q}_*(\mathcal{IC}(\mathfrak{Z}^\alpha) \otimes \mathbf{p}^*\mathcal{F})[-\dim \underline{\mathfrak{M}}^\eta]$ on $\ddot{\mathcal{Z}}^{\eta+\alpha}$ is perverse and smooth along the fine Schubert stratification.

Proof. Let us denote by s the locally closed embedding $\ddot{\mathcal{Z}}^{\eta+\alpha} \overset{s}{\hookrightarrow} \mathcal{Q}^{\eta+\alpha}$. Also, temporarily, let us denote the maps $\mathbf{p}$ and $\mathbf{q}$ from the diagram 12.12 (resp. 12.14) by $\mathbf{p}_{\mathcal{Q}}$ and $\mathbf{q}^{\mathcal{Q}}$ (resp. $\mathbf{p}_{\mathcal{Z}}$ and $\mathbf{q}^{\mathcal{Z}}$) to stress their difference.

Then we have $\mathbf{q}^{\mathcal{Z}}_*(\mathcal{IC}(\mathfrak{Z}^\alpha) \otimes \mathbf{p}^*_{\mathcal{Z}}\mathcal{F})[-\dim \underline{\mathfrak{M}}^\eta] = s^*\mathbf{q}^{\mathcal{Q}}_*(\mathcal{IC}(\mathfrak{Q}^\alpha) \otimes \mathbf{p}^*_{\mathcal{Q}}\mathcal{F})[-\dim \underline{\mathfrak{M}}^\eta - \dim \mathbf{X}]$ (cf. 12.5.1). We also have $\operatorname{codim}_{\mathcal{Q}^{\eta+\alpha}} \ddot{\mathcal{Z}}^{\eta+\alpha} = \dim \mathbf{X}$, and the fine Schubert strata in $\ddot{\mathcal{Z}}^{\eta+\alpha}$ are intersections of $\ddot{\mathcal{Z}}^{\eta+\alpha}$ with the fine Schubert strata in $\mathcal{Q}^{\eta+\alpha}$. One can check readily that the functor $s^*[-\dim \mathbf{X}]$ takes perverse sheaves on $\mathcal{Q}^{\eta+\alpha}$ smooth along the fine Schubert stratification to perverse sheaves on $\ddot{\mathcal{Z}}^{\eta+\alpha}$ smooth along the fine Schubert stratification. The application of 12.12 completes the proof of the Corollary. $\Box$

12.15. **Conjecture.** Let $\eta \in Y^+, \alpha \in Y$ be such that $\eta + \alpha \in \mathbb{N}[I]$. Consider the following diagram:

$$\begin{tikzcd} \overline{\mathcal{G}}_\eta \arrow[d, hook, "\mathbf{i}"'] & \mathcal{G}\mathcal{Z}^\alpha_\eta \arrow[l, "\mathbf{p}"'] \arrow[r, "\mathbf{q}"] \arrow[d, hook, "\mathbf{i}"'] & \mathcal{Z}^{\eta+\alpha} \\ \mathfrak{M} & \mathfrak{Z}^\alpha \arrow[l, "\mathbf{p}"'] & \end{tikzcd}$$

For a perverse sheaf $\mathcal{F}$ in $\mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$, the sheaf $\mathbf{q}_*(\mathcal{IC}(\mathfrak{Z}^\alpha) \otimes \mathbf{p}^*\mathcal{F})[-\dim \underline{\mathfrak{M}}^\eta]$ on $\mathcal{Z}^{\eta+\alpha}$ is perverse and smooth along the fine Schubert stratification.

12.16. Now we will compare $\mathbf{c}^\alpha_{\mathcal{Z}}(\mathcal{F})$ for a fixed $\mathcal{F} \in \mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$ and various α. Recall the notations of 9.2 and 9.3.

Proposition. For any $\beta, \gamma \in \mathbb{N}[I], \varepsilon > 0$, there is a factorization isomorphism

$$\mathbf{c}^{\beta+\gamma-\eta}_{\mathcal{Z}}\mathcal{F}|_{\ddot{\mathcal{Z}}^{\beta,\gamma}_{U_\varepsilon,\Upsilon_\varepsilon}} \xrightarrow{\sim} \mathbf{c}^{\beta-\eta}_{\mathcal{Z}}\mathcal{F}|_{\ddot{\mathcal{Z}}^{\beta}_{U_\varepsilon}} \boxtimes \mathcal{IC}^\gamma|_{\ddot{\mathcal{Z}}^{\gamma}_{\Upsilon_\varepsilon}}$$

Proof. Follows easily from 11.8. $\Box$

12.17. The above Proposition shows that we can organize the collection $(\mathbf{c}_{\mathcal{Z}}^{\alpha-\eta}\mathcal{F})$ for $\alpha \in \mathbb{N}[I]$ into a snop $\mathbf{c}_{\mathcal{Z}}\mathcal{F}$. Namely, we set the support estimate $\chi(\mathbf{c}_{\mathcal{Z}}\mathcal{F}) = \eta$, $(\mathbf{c}_{\mathcal{Z}}\mathcal{F})_\eta^\alpha = \mathbf{c}_{\mathcal{Z}}^{\alpha-\eta}\mathcal{F}$. This way we obtain an exact functor $\mathbf{c}_{\mathcal{Z}}: \mathcal{P}(\mathcal{G}, \mathbf{I}) \longrightarrow \mathcal{PS}$.

13. Examples of convolution

13.1. Let $\mathcal{F}$ be a perverse sheaf in $\mathcal{P}(\overline{\mathcal{G}}_\eta, \mathbf{I})$. For $\chi \leq \eta, w \in \mathcal{W}_f/\mathcal{W}_\chi$ we have $\mathcal{G}_{w,\chi} \subset \overline{\mathcal{G}}_\eta$. The sheaf $\mathcal{F}$ is constant along $\mathcal{G}_{w,\chi}$, and we denote by $\mathcal{F}_{w,\chi}$ its stalk at any point in $\mathcal{G}_{w,\chi}$.

Lemma. The stalk of $\mathcal{IC}(\mathfrak{Q}^\alpha) \otimes \mathbf{p}^*\mathcal{F}$ at any point in a fine Schubert stratum $\overset{\circ\circ}{\mathcal{G}\mathcal{Q}}{}_{w,\chi}^\gamma \times (\mathbb{C}^*)_\Gamma^{\beta-\gamma} \subset \mathcal{G}\mathcal{Q}_\eta^\alpha$ (see 11.9.2) equals $\mathcal{F}_{w,\chi} \otimes \mathcal{IC}_\Gamma^{\alpha-\beta}$ (see 10.7.1).

Proof. Follows immediately from the Proposition 10.7.3. □

13.2. Let $\mathbf{G}^L$ be the Langlands dual group. Its character lattice coincides with Y, and the dominant characters are exactly Y^+. For $\eta \in Y^+$ we denote by W_η the irreducible $\mathbf{G}^L$-module with the highest weight η. For $\chi \in Y$ we denote by ${}_{(\chi)}W_\eta$ the weight χ-subspace of W_η.

Let $\mathcal{IC}(\overline{\mathcal{G}}_\eta)$ denote the Goresky-MacPherson sheaf of $\overline{\mathcal{G}}_\eta$. A natural isomorphism $H^\bullet(\overline{\mathcal{G}}_\eta, \mathcal{IC}(\overline{\mathcal{G}}_\eta)) \overset{\sim}{\longrightarrow} W_\eta$ is constructed in [MV].

Recall that for $\chi \in Y$, $w \in \mathcal{W}_f$, the irreducible snop $\mathcal{L}(w,\chi)$ was introduced in 9.4.1. The following result was suggested by V.Ginzburg.

Theorem. There is a natural isomorphism of snops:

$$\mathbf{c}_{\mathcal{Z}}\mathcal{IC}(\overline{\mathcal{G}}_\eta) \overset{\sim}{\longrightarrow} \bigoplus_{\chi\in Y} {}_{(\chi)}W_\eta \otimes \mathcal{L}(w_0,\chi)$$

Proof. It is a reformulation of the main result of [MV]. In effect, by the Proposition 12.2b) we know that $\mathcal{IC}(\mathfrak{Q}^\alpha) \otimes \mathbf{p}^*\mathcal{IC}(\overline{\mathcal{G}}_\eta)[-\dim \underline{\mathfrak{M}}^\eta] = \mathcal{IC}(\mathcal{G}\mathcal{Q}_\eta^\alpha)$. So we have to prove that $\mathbf{q}_*\mathcal{IC}(\mathcal{G}\mathcal{Q}_\eta^\alpha) = \bigoplus_{0\leq\beta\leq\eta+\alpha} {}_{(\beta-\alpha)}W_\eta \otimes \mathcal{IC}(\mathcal{Q}^\beta)$. Here we make use of the filtration $\mathcal{Q}^{\eta+\alpha} = \bigcup_{0\leq\beta\leq\eta+\alpha} \mathcal{Q}^\beta$ subject to the stratification $\mathcal{Q}^{\eta+\alpha} = \bigsqcup_{0\leq\beta\leq\eta+\alpha} \overset{\bullet}{\mathcal{Q}}{}^\beta$ by the defect at $0 \in C$.

We know that $\mathbf{q}$ is proper, semismall, and stratified with respect to the above stratification. By the Decomposition Theorem (see [BBD]), we have *a priori* $\mathbf{q}_*\mathcal{IC}(\mathcal{G}\mathcal{Q}_\eta^\alpha) = \bigoplus_{0\leq\beta\leq\eta+\alpha} L_\beta \otimes \mathcal{IC}(\mathcal{Q}^\beta)$ for some vector spaces L_β. To identify L_β with ${}_{(\beta-\alpha)}W_\eta$ it suffices to compute the stalks at $\phi = (\mathfrak{L}_\lambda)_{\lambda\in X^+} \in \overset{\bullet}{\mathcal{Q}}{}^\beta$ where $\mathfrak{L}_\lambda = \mathcal{V}_\lambda^{\mathbf{N}_-}(\langle\beta-\eta-\alpha,\lambda\rangle \cdot 0 - \langle D,\lambda\rangle)$ for some $D \in (\mathbb{P}^1 - 0)_\Gamma^\beta$.

As in the proof of 12.9.1 we have $\mathbf{q}^{-1}(\phi) = \overline{\mathcal{G}}_\eta \cap \overline{T}_{\beta-\alpha} = \bigsqcup_{\gamma\geq 0}(\overline{\mathcal{G}}_\eta \cap T_{\beta-\alpha+\gamma})$. According to the Lemma 13.1, we have $\mathcal{IC}(\mathcal{G}\mathcal{Q}_\eta^\alpha)|_{\overline{\mathcal{G}}_\eta\cap T_{\beta-\alpha+\gamma}} = \mathcal{IC}(\overline{\mathcal{G}}_\eta)|_{\overline{\mathcal{G}}_\eta\cap T_{\beta-\alpha+\gamma}} \otimes \mathcal{IC}_\Gamma^\gamma$. According to [MV], we have $H_c^\bullet(\overline{\mathcal{G}}_\eta \cap T_{\beta-\alpha+\gamma}, \mathcal{IC}(\overline{\mathcal{G}}_\eta)) = {}_{(\beta-\alpha+\gamma)}W_\eta$. Due to the parity vanishing (see *loc. cit.* and 10.7.4), the spectral sequence computing $H^\bullet(\overline{\mathcal{G}}_\eta \cap \overline{T}_{\beta-\alpha}, \mathcal{IC}(\mathcal{G}\mathcal{Q}_\eta^\alpha))$ collapses and gives $H^\bullet(\overline{\mathcal{G}}_\eta \cap \overline{T}_{\beta-\alpha}, \mathcal{IC}(\mathcal{G}\mathcal{Q}_\eta^\alpha)) = \bigoplus_{0\leq\gamma\leq\eta+\alpha-\beta} {}_{(\beta-\alpha+\gamma)}W_\eta \otimes \mathcal{IC}_{\{\{\gamma\}\}}^0 = \bigoplus_{0\leq\gamma\leq\eta+\alpha-\beta} {}_{(\beta-\alpha+\gamma)}W_\eta \otimes \mathcal{IC}(\mathcal{Q}^{\beta+\gamma})_\phi$.

This completes the proof of the Theorem. □

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