Moduli spaces of local Langlands parameters

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Abstract. This is a survey on motivations, foundations, and main results around the so-called "moduli spaces of local Langlands parameters", with special emphasis on the description of the coarse moduli spaces, and their expected role in a local Langlands correspondence "in families".

Contents

Motivations : categorical and "in families" LLC

Spectacular recent developments in the local Langlands program [FS21] [Zhu20] [BZCHN20] [Hel23] suggest that the parametrization of irreducible representations of p-adic groups by "Galois representation-theoretic" data, as initially envisionned by Langlands and made increasingly more precise by many other authors, is the tip of an iceberg that should relate much more general objects on both sides, and in a more "categorical" way. On the parameters side, these more general objects are (some elaborated form of) coherent sheaves on the *moduli space of local Langlands parameters.*

These moduli spaces are thus becoming central objects in the field, being the venue where one side of the expected categorification of the local Langlands correspondence (CLLC) is played. However, their first occurrence goes back to the 2016 preprint version of [Hel20b], where they are introduced in the particular case of GL_n. There, the motivation was to reformulate a conjecture stated in [EH14] about the existence of a "local Langlands correspondence in families" (LLIF). This conjecture was about assigning suitable "families" of representations of $\mathrm{GL}_n(F)$ to n-dimensional representations of the Galois group $\Gamma_F = \mathrm{Gal}(\overline{F}/F)$ over complete local noetherian rings with residue field of characteristic ℓ different from p. One avatar of this conjecture involved comparing universal deformation rings of Galois $\overline{\mathbb{F}}_\ell$-representations with completion of the integral Bernstein center at certain $\overline{\mathbb{F}}_\ell$-points. Formulated in this way, it essentially remained in the well-established realm of *deformation theory* of Galois representations. However, one remarkable insight of Helm in [**Hel20b**] has been to algebraize the whole setting. First, he defined an algebraic moduli space of n-dimensional Weil group representations over $\mathbb{Z}_\ell$, whose formal neighborhoods of points recover the usual deformation rings in Galois representation theory. Second, he reduced the conjecture in [**EH14**] to showing the existence of an isomorphism between the ring of functions on this moduli space and the endomorphism ring of the space of $\mathbb{Z}_\ell$-valued Whittaker functions on $\mathrm{GL}_n(F)$ (which is incidentally isomorphic to the integral Bernstein center, in this GL_n-case). This isomorphism was finally obtained in [**HM18**].

An a priori huge difference between CLLC and LLIF is that, while the first one crucially uses the stacky nature of the fine moduli space, the second one only sees the coarse moduli space through its ring of functions. In order to give a more precise yet informal glimpse of the relation between CLLC and LLIF, let us introduce some notation.

- F denotes a local non-archimedean field of residue characteristic p
- Γ_F denotes "its" Galois group and $W_F \subset \Gamma_F$ its Weil group.
- G denotes a reductive group over F, and $\hat{G}$ denotes its dual split reductive group scheme over $\operatorname{Spec}\mathbb{Z}$. From the F-rational structure on G, we get a pinning-preserving action of W_F on $\hat{G}$ following Langlands' classical construction.
- Λ denotes a coefficient ring. It will mainly be one of $\mathbb{Z}_\ell$, $\mathbb{Q}_\ell$, $\mathbb{F}_\ell$ or $\mathbb{Z}[\frac{1}{p}]$, or any integral extension of one of these rings. In any case, p will always be invertible in Λ.
- $\mathrm{Rep}_\Lambda(G(F))$ denotes the category of smooth $\Lambda G(F)$-modules.

Recall that Langlands parameters are 1-cocycles $W_F \longrightarrow \hat{G}$ up to $\hat{G}$-conjugation (more on parameters below). Accordingly, the moduli stack of Langlands parameters for G will have the form $Z^1/\hat{G}$, with Z^1 an affine Λ-scheme that parametrizes 1-cocycles, endowed with the natural action of $\hat{G}$. One prediction of CLLC is the existence of a certain fully faithful embedding

$$\mathrm{Rep}_\Lambda(G(F)) \longrightarrow \mathrm{Qcoh}(Z^1/\hat{G})$$

of $\mathrm{Rep}_\Lambda(G(F))$ into the (derived ∞) category of quasi-coherent sheaves on the stack $Z^1/\hat{G}$. Moreover, when G is quasi-split, such an embedding should only depend on the choice of a Whittaker datum (U,ψ). More precisely, the embedding attached to (U,ψ) should map the Whittaker representation $\mathcal{W} = \mathcal{W}_{(U,\psi)} \in \mathrm{Rep}(G(F))$ to the structural sheaf of $Z^1/\hat{G}$. Being a full embedding, this would in particular imply an isomorphism

$$\mathrm{End}(\mathcal{O}_{Z^1/\hat{G}}) = \Gamma(Z^1/\hat{G},\mathcal{O}) = \Gamma(Z^1,\mathcal{O})^{\hat{G}} \xrightarrow{\sim} \mathrm{End}(\mathcal{W}).$$

But the LHS here is nothing but the ring of global functions on the coarse moduli space $Z^1 /\!\!/ \hat{G}$, and the above isomorphism is thus the main prediction of LLIF, at least in Helm's version.

So LLIF is a consequence of CLLC, and since the passage from fine to coarse moduli space is highly destructive, LLIF is a priori only a remote approximation of CLLC. However, it is interesting to note that a conjecture of Hellmann suggests how to get CLLC from LLIF plus a bit of representation theory, at least in the case of $G = GL_n$ and coefficients $\Lambda = \overline{\mathbb{Q}}_\ell$. Indeed, observe that LLIF provides a functor. Namely, it provides a quasicoherent sheaf $\tilde{\mathcal{W}} := \mathcal{W} \otimes_{\mathcal{O}(Z^1)^{\hat{G}}} \mathcal{O}(Z^1)$ on $Z^1/\hat{G}$ with an action of $G(F)$, and thus a functor $V \mapsto V \otimes^L_{G(F)} \tilde{\mathcal{W}}$. This is certainly not the right functor, but the conjecture of Hellmann [Hel23, §4] is that CLLC should have the form $V \mapsto V \otimes^L_{G(F)} \tilde{\mathcal{V}}$ where $\tilde{\mathcal{V}}$ is a certain quotient of $\tilde{\mathcal{W}}$ defined by Emerton and Helm in their work on LLIF. Note however that the most naive generalization of this recipe won't be sufficient for more general groups, if only because of the existence of non generic cuspidal representations.

The aim of these notes is to explain the construction and main properties of the moduli space $Z^1/\hat{G}$, with special emphasis on the coarse moduli space $Z^1 /\!\!/ \hat{G}$. Our first task is to sort out what kind of Langlands parameters we want to use.

1. Langlands parameters. What parameters ?

Let ${}^LG := \hat{G} \rtimes W_F$ denote the Langlands dual of G. An *L-homomorphism* $W_F \longrightarrow {}^LG(\Lambda)$ is a group homomorphism that is a section of the second projection. So it has the form ${}^L\varphi(w) = (\varphi(w), w)$ for a certain 1-cocycle $W_F \longrightarrow \hat{G}(\mathbb{C})$. The two items ${}^L\varphi$ and φ determine each other uniquely.

1.1. The complex story. The original classical local Langlands correspondence predicts a finite-to-one map $\pi \mapsto \varphi_\pi$ from irreducible objects of $\mathrm{Rep}_{\mathbb{C}}(G(F))$ to a certain set of "Langlands parameters". The latter are $\hat{G}(\mathbb{C})$-conjugacy classes of certain objects that appear in the literature in two different flavors :

(1) maps $\rho : W_F \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG(\mathbb{C})$ whose restriction to the first factor is an L-homomorphism with open kernel and whose restriction to the second factor is algebraic, or
(2) pairs (r, N), where $r : W_F \to {}^LG(\mathbb{C})$ is an L-homomorphism with open kernel and $N \in \mathrm{Lie}(\hat{G}_{\mathbb{C}})$ a nilpotent element, such that $\mathrm{Ad}_r(w)N = |w|N$.

Actually, we are missing one important requirement here, known in the literature as *Frobenius semi-simplicity.* This asks that for any $w \in W_F$, the element $\rho(w, 1)$, resp $r(w)$, be semisimple in ${}^LG(\mathbb{C})$. The reason why we omitted it is that this condition does not fit well in families, and would prevent us from defining a nice variety parametrizing these objects (think of the set of semisimple elements in $\hat{G}(\mathbb{C})$, which forms a constructible subset, but not a locally closed subvariety). Instead, semi-simplicity will be recovered when we take the coarse moduli space (think of the Chevalley-Steinberg isomorphism $\hat{G} /\!\!/ \hat{G} = \hat{T} /\!\!/ W$).

Let us fix an open compact subgroup $K \subset W_F$. It is easy to see that, for $i = 1, 2$, objects of type (i) that are trivial on K are parametrized by an affine complex variety $Z^1_{(i)}$. Moreover, there is a map $\rho \mapsto (r, N)$ from objects of type (1) to objects of type (2), given by

$$r(w) = \rho\left(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix}\right), N = d\rho \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right).$$

This map induces a morphism of varieties $Z^1_{(1)} \longrightarrow Z^1_{(2)}$.

By a well known application of the Jacobson-Morozov theorem, the above map induces a bijection on $\hat{G}(\mathbb{C})$-*conjugacy classes of Frobenius-semi-simple objects*, making sure that the two corresponding notions of Langlands parameters coincide.

However, this map is not injective on general $\hat{G}(\mathbb{C})$-conjugacy classes, and actually not even surjective (see [MIY22, Example 3.5]). Accordingly, the geometry of $Z^1_{(1)}$ and $Z^1_{(2)}$ are very different.

To get an idea of what is going on, look at $G = \mathrm{GL}_2$ and $K = I_F$. Then $Z^1_{(1)}$ is the disjoint union $\mathrm{GL}_2 \sqcup (\mathbb{C}^\times \times \mathrm{PGL}_2)$ of two 4-dimensional smooth components corresponding to $\rho_{|\mathrm{SL}_2}$ being trivial or not. In contrast, $Z^1_{(2)}$ is connected with two 4-dimensional irreducible components given by the closed locus where $N = 0$ (isomorphic to GL_2) and the closure of the open locus where $N \neq 0$ (this open locus is isomorphic to $\mathbb{C}^\times \times \mathrm{PGL}_2$). These two components intersect along the 3-dimensional closed subvariety of points of the form $\{(F, 0)\}$ with F having two eigenvalues of ratio q. So in this example, $Z^1_{(1)}$ appears as an open subvariety of the normalization of $Z^1_{(2)}$. More generally, it is proven in [MIY22] that the map $Z^1_{(1)} \longrightarrow Z^1_{(2)}$ separates irreducible components and is "weakly birationnal".

It is interesting to look at the map $Z^1_{(1)} /\!\!/ \hat{G} \longrightarrow Z^1_{(2)} /\!\!/ \hat{G}$ induced on the corresponding coarse moduli spaces. As a general principle, complex points of $Z^1_{(i)} /\!\!/ \hat{G}$ correspond to closed orbits of $Z^1_{(i)}(\mathbb{C})$ (see section 5), which are orbits of "semistable" objects in $Z^1_{(i)}(\mathbb{C})$. An application of the Hilbert-Mumford criterion shows that the semistable points of $Z^1_{(1)}(\mathbb{C})$ are precisely the Frobenius-semisimple objects, while the semistable points of $Z^1_{(2)}(\mathbb{C})$ are the points of the form $(r, 0)$ with r Frobenius-semisimple. In other words, $(Z^1_{(1)} /\!\!/ \hat{G})(\mathbb{C})$ is precisely the set of Langlands parameters, while $(Z^1_{(2)} /\!\!/ \hat{G})(\mathbb{C})$ is the set of so-called "infinitesimal characters" (in Vogan's terminology), and the map $Z^1_{(1)} /\!\!/ \hat{G} \longrightarrow Z^1_{(2)} /\!\!/ \hat{G}$ takes a Langlands parameter to its infinitesimal character.

The upshot of this discussion is that, in the case of coefficients $\Lambda = \mathbb{C}$, we have a priori two natural candidates for the moduli space $Z^1/\hat{G}$ to be used in a putative CLLC. Although $Z^1_{(1)}$ seems closer to the classical notion of Langlands parameter, it is not a reasonable candidate for CLLC. Indeed, the desired functor would have to send the trivial and Steinberg representations to certain indecomposable sheaves. Being indecomposable, these sheaves should be supported on a single component. By compatibility with classical LLC, the sheaf associated to Steinberg would be on the component where $\rho_{|SL_2}$ is non trivial, and the one associated to the trivial representation on the other component. But these sheaves have no non-trivial extension, unlike these two representations, so this is incompatible with full faithfulness of the desired functor. Indeed, the moduli space that shows up in both [Hel23] and [BZCHN20] is $Z^1_{(2)}/\hat{G}$. Accordingly, the coarse moduli space that is relevant for LLIF is $Z^1_{(2)} /\!\!/ \hat{G}$.

1.2. The ℓ-adic story. Let us transport the above discussion from $\mathbb{C}$ to $\overline{\mathbb{Q}}_\ell$, either by repeating the definitions or by choosing an isomorphism of fields $\mathbb{C} \simeq \overline{\mathbb{Q}}_\ell$. The original LLIF problem in [Hel20b] takes place over coefficients Λ that are finite extensions of $\mathbb{Z}_\ell$, and similarly for the putative CLLC of [Zhu20] and [FS21]. This means we need a moduli space $Z^1/\hat{G}$ over $\mathbb{Z}_\ell$.

Note that the objects (r, N) of type (2) above make perfect sense over any $\mathbb{Z}_\ell$-algebra R in place of $\mathbb{C}$ and are parametrized by an affine scheme over $\mathbb{Z}_\ell$, providing an integral model of $Z^1_{(2)}$. However, one requirement of LLIF is that the geometric special fiber $Z^1(\overline{\mathbb{F}}_\ell)$ should parametrize continuous Galois representations (ie L-homomorphisms) $\bar\rho : \Gamma_F \longrightarrow {}^LG(\overline{\mathbb{F}}_\ell)$. So one would need a recipe to go from a pair (r, N) to such a $\bar\rho$. The only recipe known to the author involves an exponential map from nilpotent elements in Lie $\hat G_{\overline{\mathbb{F}}_\ell}$ to unipotent elements in $\hat G(\overline{\mathbb{F}}_\ell)$, so the first issue is that such an exponential map may not exist when ℓ is small (e.g. when $\ell < n$ for GL_n). Anyway, assuming ℓ large enough to have the exponential map, the recipe goes as follows; fix a trivialization of $\mathbb{Z}_\ell(1) := \varprojlim \mu_{\ell^n} \xrightarrow{\sim} \mathbb{Z}_\ell$ and a lift of Frobenius Fr in W_F. We then get a morphism $t_\ell : I_F \longrightarrow \mathbb{Z}_\ell$ and a retraction $W_F \longrightarrow I_F$, $w \mapsto i_w$. Then the map $w \mapsto \bar\rho(w) := r(w)\exp(\overline{t_\ell(i_w)}N)$ defines a L-homomorphism $\Gamma_F \longrightarrow {}^LG(\overline{\mathbb{F}}_\ell)$ as desired, and this sets up a bijection between $\hat G(\overline{\mathbb{F}}_\ell)$-conjugacy classes. The same recipe actually works over any complete local $\mathbb{Z}_\ell$-algebra and allows one to compare the deformation spaces of (r, N) and $\bar\rho$. This is where we run into more trouble, as the following example shows.

Example 1.1. Suppose $G = \mathrm{GL}_2$, $\ell > 2$ and $q \equiv -1 \,(\mathrm{mod}\ \ell)$. Pick $\zeta \in \mu_\ell(\overline{\mathbb{Q}}_\ell)$ and define a character $\chi : I_F.\mathrm{Fr}^{2\mathbb{Z}} \longrightarrow \overline{\mathbb{Z}}_\ell^\times$ by $\chi(i) = \zeta^{t_\ell(i)}$ and $\chi(\mathrm{Fr}^2) = 1$. Then the 2-dimensional irreducible $\overline{\mathbb{Q}}_\ell$-representation $\rho := \mathrm{ind}_{I_F.\mathrm{Fr}^{2\mathbb{Z}}}^{W_F}\chi$ of W_F is irreducible, and its semisimplified reduction is isomorphic to the direct sum $r = 1 \oplus \varepsilon$ of the trivial character and the quadratic unramified character over $\overline{\mathbb{F}}_\ell$. Since ρ is irreducible, it contains an invariant $\overline{\mathbb{Z}}_\ell$-lattice L_ρ whose reduction $\bar\rho$ is a non-trivial extension of ε by 1. Thus, the Weil-Deligne representation corresponding to $\bar\rho$ has the form (r, N) with non-zero N. But any lift of (r, N) must have non-trivial N and we see that no lift of (r, N) matches L_ρ, that is, the deformation spaces of (r, N) and $\bar\rho$ are different.

So we see that objects of type (2) do not provide the correct moduli space over $\mathbb{Z}_\ell$ in general. However, the recipe above works over coefficients $\mathbb{Q}_\ell$ for any ℓ and takes objects of type (2) above to objects of type

(3) L-homomorphisms ${}^L\varphi : W_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$ that are ℓ-adically continuous.

Here "ℓ-adically continuous" means "continuous" when we endow $\hat G(\overline{\mathbb{Q}}_\ell)$ with the topology inherited from the natural topology on $\overline{\mathbb{Q}}_\ell$. Moreover, Grothendieck's quasi-unipotence theorem shows that this induces a bijection on the corresponding $\hat G(\overline{\mathbb{Q}}_\ell)$ conjugacy classes.

Similarly we can impose ℓ-adic continuity for an L-homomorphism ${}^L\varphi : W_F \to {}^LG(R)$ if R is a finite extension of $\mathbb{Z}_\ell$ or, more generally, a complete local algebra with residue field contained in $\overline{\mathbb{F}}_\ell$. In the case where $R = \overline{\mathbb{F}}_\ell$, a continuous ${}^L\varphi$ will uniquely extend to Γ_F. Therefore, if we were able to extend the notion of "ℓ-adically continuous" to arbitrary $\mathbb{Z}_\ell$-algebras R, and show that the corresponding functor

$$R \mapsto \{L\text{-homomorphisms } {}^L\varphi : W_F \to {}^LG(R) \text{ that are } \ell\text{-adically continuous}\}$$

is representable by an affine $\mathbb{Z}_\ell$-scheme Z^1, then certainly the $\overline{\mathbb{F}}_\ell$-points of Z^1 would parametrize Galois representations (L-parameters) and their completed local rings would be their usual framed deformation rings, as required for the LLIF program.

2. How to deal with continuity

Here are several possible definitions for an L-homomorphism ${}^L\varphi : W_F \longrightarrow {}^LG(R)$ to be "ℓ-adically continuous", when R is any $\mathbb{Z}_\ell$-algebra.

2.1. Helm's original approach. Note that if R is ℓ-adically separated, there is a natural separated group topology on $G(R)$ generated by kernels of reduction maps $\hat{G}(R) \longrightarrow \hat{G}(R/\ell^m R)$. So, concretely in this case, ${}^L\varphi$ is said to be ℓ-adically continuous if for each m the composed map $W_F \longrightarrow \hat{G}(R/\ell^m) \rtimes W_F$ is continuous for the discrete topology on $\hat{G}(R/\ell^m)$ and the natural one on W_F. Following [Hel20b] and [DHKM20], for an arbitrary $\mathbb{Z}_\ell$-algebra, we may then declare ${}^L\varphi$ to be ℓ-adically continuous if it comes by pushforward $R_0 \longrightarrow R$ from an ℓ-adically continuous L-homomorphism over a separated $\mathbb{Z}_\ell$-algebra R_0.

The obvious drawback of such a definition, is that it is not clear at all why the functor $R \mapsto \{\ell\text{-adically continuous } {}^L\varphi\}$ should be representable, or even define a fppf sheaf. In the case at hand, it will quite miraculously turn out to be so, thanks to the fairly simple struture of W_F.

2.2. Condensed Mathematics. Another approach is taken in [FS21]. There, R is considered as a condensed $\mathbb{Z}_\ell$-algebra $S \mapsto R(S)$ with, for any profinite set S, the ring $R(S)$ being the ring of maps $S \longrightarrow R$ that factor over a finitely generated $\mathbb{Z}_\ell$-submodule and are continuous for the ℓ-adic topology on this submodule. In turn, $\hat{G}(R)$ becomes a condensed group. On the other hand, the topological group W_F can also be considered as a condensed group, and the idea of [FS21] is to consider L-homomorphisms of condensed groups. Actually, this is the notion that appears naturally from their cohomological setting regarding Hecke operators on the Fargues-Fontaine curve.

This definition can also be expressed in the usual topological language. Endow R with the finest topology which restricts to the natural ℓ-adic topology on all finitely generated $\mathbb{Z}_\ell$-submodules (in [Zhu20], this topology is called "ind-ℓ-adic"). This induces a $\mathbb{Z}_\ell$-linear topology on any free R-modules of finite rank. Then, declare that ${}^L\varphi$ is continuous if, for any (equivalently, one faithful) algebraic representation $\hat{G} \longrightarrow \mathrm{GL}(V)$, the induced action of the kernel of $W_F \longrightarrow \mathrm{Aut}(\hat{G})$ on $V \otimes R$ is continuous. This continuity means that any $v \in V \otimes R$ is contained in a finitely generated $\mathbb{Z}_\ell$-module that is stable under an open subgroup of W_F, with the induced action being continuous.

It is not clear a priori that this notion recovers the former one. Namely, the ind-ℓ-adic topology of an ℓ-adically separated algebra R is generally finer than the ℓ-adic topology. However, again due to the specific structure of W_F, both notions will give rise to the same moduli space. The main reason is the magic of Frobenius and its interplay with inertia.

2.3. Discretization of tame inertia. Since $\ell \neq p$, whatever definition of "ℓ-adically continuous" we take for ${}^L\varphi$, the restriction of φ to the wild inertia subgroup P_F will factor over a finite quotient. So the only topological difficulties here come from the tame inertia quotient I_F/P_F. This is a pro-cyclic group, so let us choose a generator s. Let us also choose a lift of Frobenius Fr in W_F/P_F. These satisfy the relation $\mathrm{Fr} s \mathrm{Fr}^{-1} = s^q$. We then consider the subgroup $\langle \mathrm{Fr}, s\rangle = s^{\mathbb{Z}[\frac{1}{q}]} \rtimes \mathrm{Fr}^{\mathbb{Z}}$ of W_F/P_F, we denote by W_F^0 its inverse image in W_F, and we endow it with the topology that extends the profinite topology of P_F and induces the discrete topology on $\langle \mathrm{Fr}, s\rangle$. So W_F^0 is a dense subgroup of Γ_F and Γ_F identifies to the profinite completion of W_F^0, as any finite quotient of W_F^0 is the Galois group of a finite extension of F. Note that (in contrast to the subgroup W_F of G_F), the subgroup W_F^0 of W_F very much depends on the choices of Fr and s.

Let us now declare an L-homomorphism ${}^L\varphi : W_F^0 \longrightarrow {}^LG(R)$ to be *continuous* if it is continuous for the discrete topology on ${}^LG(R)$ and the above topology on W_F^0. This is equivalent to asking that the associated 1-cocycle $\varphi : W_F^0 \longrightarrow \hat{G}(R)$ is trivial on some open subgroup of P_F.

2.4. A candidate moduli space. For a fixed normal open subgroup $P \subset P_F$ that acts trivially on $\hat{G}$, it is easy to see that the functor on $\mathbb{Z}[\frac{1}{p}]$-algebras

$$R \mapsto \left\{\text{1-cocycles } \varphi : W_F^0/P \longrightarrow \hat{G}(R)\right\}$$

is representable by a finitely generated $\mathbb{Z}[\frac{1}{p}]$-algebra $R_{\hat{G},1_P}$. We then put

$$Z^1 = Z^1(W_F^0/P, \hat{G}) := \operatorname{Spec}(R_{\hat{G},1_P}).$$

Concretely, the discrete group W_F^0/P is finitely generated, say by n generators, and Z^1 is the closed subscheme of $(\hat{G})^n$ defined by the relations between these generators and the cocycle condition.

Although Z^1 a priori depends on the chosen discretization W_F^0 of W_F, there are a few tests we can do to see whether it is a reasonable candidate for LLIF or CLLC.

(1) When $R = \overline{\mathbb{F}}_\ell$ or R is any noetherian complete local algebra with residue field $\overline{\mathbb{F}}_\ell$, a continuous ${}^L\varphi : W_F^0 \to {}^LG(R)$ can be uniquely continuously extended to the profinite completion of W_F^0, which is Γ_F. So Z^1 fulfills the requirement that its $\overline{\mathbb{F}}_\ell$-points parametrize ${}^LG(\overline{\mathbb{F}}_\ell)$-valued Galois representations, and their formal neighbourhoods coincide with the usual framed deformation spaces of these representations.

(2) When $R = \overline{\mathbb{Q}}_\ell$, we cannot argue in the same way, but we claim that any continuous ${}^L\varphi : W_F^0 \to {}^LG(\overline{\mathbb{Q}}_\ell)$ extends uniquely continuously to W_F, so that $\overline{\mathbb{Q}}_\ell$-points of Z^1 parametrize objects of type (3) above, as desired. Note that, using a continuous splitting $W_F \simeq P_F \rtimes (W_F/P_F)$ as in [**Iwa55**], it suffices to check this when $P = P_F$. After choosing a faithful representation of ${}^LG$, wa may assume that $G = \mathrm{GL}_n$, so that ${}^L\varphi$ is given by a morphism $\varphi : W_F^0/P_F = \langle \mathrm{Fr}, s\rangle \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Then the relation $\mathrm{Fr}s\mathrm{Fr}^{-1} = s^q$ shows that the eigenvalues of $\varphi(s)$ are roots of unity, hence the semisimple part $\varphi(s)_{\mathrm{ss}}$ has finite order prime to p. On the other hand, writing the unipotent part as $\varphi(s)_{\mathrm{u}} = \exp(N)$, we see that φ extends to W_F/P_F by the formula on I_F/P_F given by $\varphi(s^i) = (\varphi(s)_{\mathrm{ss}})^i \exp(t_\ell(i)N)$ for $i \in \widehat{\mathbb{Z}[\frac{1}{q}]}$.

For a more general $\mathbb{Z}_\ell$-algebra R, the following theorem implies that any continuous L-homomorphism $W_F^0 \longrightarrow {}^LG(R)$ can actually be extended to a unique ℓ-adically continuous L-homomorphism $W_F \longrightarrow {}^LG(R)$.

Theorem 2.1. *Fix an open subgroup $P \subset P_F$ as above and let ${}^L\varphi_{\hat{G},1_P}$ be the universal L-homomorphism $W_F^0 \longrightarrow {}^LG(R_{\hat{G},1_P})$.*

(1) *The ring $R_{\hat{G},1_P}$ is ℓ-adically separated.*

(2) *The universal ${}^L\varphi_{\hat{G},1_P}$ extends uniquely to an ℓ-adically continuous L-homomorphism ${}^L\varphi_{\hat{G},1_P,\ell} : W_F \longrightarrow {}^LG(R_{\hat{G},1_P} \otimes \mathbb{Z}_\ell)$, which is universal for ℓ-adically continuous L-homomorphisms on W_F that are trivial on P.*

(3) *The L-homomorphism ${}^L\varphi_{\hat{G},1_P,\ell}$ is also ind-ℓ-adically continuous (as in 2.2) and is universal for ind-ℓ-adically continuous L-homomorphisms on W_F that are trivial on P.*

One consequence of (2) is that the base change $Z^1(W_F^0/P,\hat{G})_{\mathbb{Z}_\ell}$ does not depend[1] on our choice of discretization W_F^0. The upshot of (3) is that this base change is the moduli space of condensed L-homomorphisms considered in [**FS21**, Chapter VIII]. We will explain the proof of this theorem in the next sections.

2.5. Remark on derived structures. While the affine scheme Z^1 defined above is perfectly suitable for LLIF, it is not a priori clear whether it is sufficient for CLLC. This is because the coherent side of CLLC, namely the (derived ∞) category of quasi-coherent modules on Z^1, is quite sensitive to a possible non-trivial derived structure on Z^1. Such a non-trivial derived struture may arise if, for example, the closed immersion $Z^1 \hookrightarrow (\hat{G})^n$ that one gets after choosing a set of n generators of W_F^0/P turns out to be not regular. Technically, one should work in some "derived category" of $\mathbb{Z}[\frac{1}{p}]$-algebras, namely the ∞-category of "animated" $\mathbb{Z}[\frac{1}{p}]$-algebras, and consider the same functor $R \mapsto \left\{\text{1-cocycles } \varphi : W_F^0/P \longrightarrow \hat{G}(R)\right\}$ as before, with a correct ∞-categorical notion of 1-cocycle. We refer to [**Zhu20**, §2.2] for more details and a proof that such a functor is indeed representable by an affine derived scheme $\tilde{Z}^1$, whose classical underlying scheme is Z^1.

While this derived setting may be necessary when studying moduli spaces of representations of global Galois groups, it follows from [**Zhu20**, Prop. 2.2.13] and the first statement of the following theorem that, in our case, we have $\tilde{Z}^1 = Z^1$.

THEOREM 2.2. *The affine scheme $Z^1(W_F/P,\hat{G})$ is a reduced flat local complete intersection of dimension $\dim \hat{G}$ over $\mathbb{Z}[\frac{1}{p}]$.*

Both theorems above imply that any irreducible component of $Z^1(W_F/P,\hat{G})$ is faithfully flat over $\mathbb{Z}[\frac{1}{p}]$. The proof of these theorems is given in the next two sections, together with a parametrization of the connected components of $Z^1(W_F/P,\hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]}$ and a description of these components in terms of smaller groups.

3. A toy model over $\overline{\mathbb{Z}}[\frac{1}{p}]$: tame parameters

In this section, we focus on the case where $P = P_F$ and the group G is tamely ramified, so that the action of P_F on $\hat{G}$ is trivial.

Actually, we will consider a slightly different setting. We denote by Λ a finite flat integral extension of $\mathbb{Z}[\frac{1}{p}]$ and let H be a group scheme over Λ. We then consider the affine Λ-scheme

$$X_{H,1} := \underline{\mathrm{Hom}}(W_t^0, H)$$

that parametrizes morphisms from the tame quotient $W_t^0 := W_F^0/P_F = \langle \mathrm{Fr}, s\rangle$ of W_F^0 to H. Explicitly it is the closed subscheme of $H \times H$ that represents the functor on Λ-algebras

$$R \mapsto \mathrm{Hom}(W_t^0, H(R)) = \{(F,\sigma) \in H(R) \times H(R),\ F\sigma F^{-1} = \sigma^q\}.$$

[1]It is not yet known whether the full scheme $Z^1(W_F^0/P,\hat{G})$ nor the stack $Z^1(W_F^0/P,\hat{G})/\hat{G}$ really depends on the choice of discretization, but at least it is true that the coarse quotient scheme $Z^1(W_F^0/P,\hat{G}) /\!\!/ \hat{G}$ is independent of this choice by [**DHKM20**, Thm. 4.18].

The main case of interest for us is when $H = \hat{G} \rtimes W$ where W is a finite quotient of W_t^0 through which the action of W_F on $\hat{G}$ factors. In this case, the scheme $Z^1(W_t^0, \hat{G})$ which we are ultimately interested in, is a direct summand of the scheme $X_{H,1}$ above. However, it will be useful to consider a larger class of group schemes H.

3.1. Generalized reductive group schemes. Recall that a group scheme H over a base S is called *reductive* if it is smooth over S and all its geometric fibers are reductive and connected. It will come in handy to have a slight generalization where some non-connectedness is allowed. To this aim, recall that for any smooth group scheme over S, there is a unique open subgroup scheme H^0 of H whose geometric fibers are the neutral components of the geometric fibers of H. Further, under the hypothesis that H^0 is reductive, it is known [**Con14**, Prop. 3.1.3] that the quotient sheaf $\pi_0(H) := H/H^0$ is representable by a separated étale group scheme over S. We will say that H is *generalized reductive* if it satisfies the following properties.

- H is a smooth affine group scheme over S
- H^0 is a reductive group scheme over S.
- $\pi_0(H) := H/H^0$ is a finite (étale) group scheme over S.

Moreover, we will say that such an H is *split* if H^0 is a split reductive group scheme and $\pi_0(H)$ is constant. The main properties we will need about these objects are summarized in the following two lemmas.

LEMMA 3.1. *Suppose H is split over S and let (B, T, X) be a pinning of H^0.*

(1) *The normalizers $N_H(T)$, resp $N_H(T, B)$, of the torus T, resp of the Borel pair (B, T), are split generalized reductive with neutral component T.*
(2) *If the center $Z(H^0)$ of H^0 is smooth over S, the normalizer $N_H(B, T, X)$ of the pinning is generalized reductive with neutral component $Z(H^0)^0$.*

Note that for each one of the group schemes N in the above lemma, one checks on geometric fibers that the natural morphism $N \longrightarrow \pi_0(H)$ is surjective (eg étale-sheaf theoretically) and the kernel $N \cap H^0$ pertains to the theory of reductive group schemes, for which an efficient reference is [**Con14**].

LEMMA 3.2. *Suppose $S = \operatorname{Spec} R$ with R a normal subring of a number field.*

(1) *Any generalized reductive H over S splits over a finite integral extension.*
(2) *If H is generalized reductive over S and $P \subset H(S)$ is a solvable finite subgroup with order invertible on S, then the centralizer $C_H(P)$ and the normalizer $N_H(P)$ of P are generalized reductive, with $C_H(P)^0 = N_H(P)^0$.*

We refer to [**DHKM20**], Theorem 1.13 for a proof of (1) and Theorem A.12 for (2).

3.2. Flatness and complete intersection. In the sequel, we always assume that the group scheme H over Λ is "generalized reductive".

PROPOSITION 3.3. *$X_{H,1}$ is syntomic (i.e. flat and local complete intersection) over S of the same relative dimension as H over $S := \operatorname{Spec} \Lambda$.*

PROOF. We have a cartesian diagram

$$\begin{tikzcd} X_{H,1} \arrow[r] \arrow[d] & H \times_S H \arrow[d] \\ S \arrow[r] & H \end{tikzcd}$$

where the RHS map is $H \times_S H \longrightarrow H$, $(F,\sigma) \mapsto F\sigma F^{-1}\sigma^{-q}$. This shows that the closed subscheme $X_{H,1}$ of the smooth S-scheme $H \times_S H$ is defined by the vanishing of $\dim_S H$ functions. So, in order to prove that the map $X_{H,1} \longrightarrow S$ is syntomic (and even a global complete intersection), it is enough to show that all its geometric fibers have dimension equal to $\dim_S(H \times_S H) - \dim_S H = \dim_S H$ (where $\dim_S$ denotes relative dimension over S).

So let L be an algebraically closed field over Λ, and let us write $X := X_{H,1}$ to lighten the notation. Since X_L is the fiber at 1 of the morphism $H_L \times H_L \longrightarrow H_L$, $(F,\sigma) \mapsto F\sigma F^{-1}\sigma^{-q}$, each irreducible component of X_L has dimension $\geqslant \dim H_L = \dim_S H$. On the other hand, let $\sigma \in H(L)$ be in the image of the second projection $\pi_2 : X \longrightarrow H$. Then the fiber $\pi_2^{-1}(\sigma) \subset X$ is a torsor under the centralizer H_σ of σ in H_L, hence it has dimension $\dim H_\sigma$. It follows that the preimage $\pi_2^{-1}(\tilde{\sigma})$ of the H_L^0-conjugacy class $\tilde{\sigma}$ of σ in H_L has dimension $\dim H_\sigma + (\dim H_L - \dim H_\sigma) = \dim H_L = \dim_S H$. So, the desired equidimensionality follows from the following claim : $\pi_2(X(L))$ *consists of finitely many conjugacy classes in* $H(L)$.

To prove the claim, let $\sigma \in \pi_2(X(L))$ and consider its Jordan decomposition $\sigma = \sigma_{\mathrm{ss}}\sigma_{\mathrm{u}}$. We will show that σ_{ss} has finite order bounded independently of σ (and L). Since there are only finitely many conjugacy classes of semisimple elements of bounded finite order, and finitely many conjugacy classes of unipotent elements in the centralizer of such an element [FG12], this will prove the claim. So let e be the order of $\pi_0(H_L)$. Then $(\sigma_{\mathrm{ss}})^e \in H_L^0$ belongs to a maximal torus T of H_L. Since $(\sigma_{\mathrm{ss}})^{qe}$ is conjugate to $(\sigma_{\mathrm{ss}})^e$ in $H(L)$ and also belongs to T, there is $n \in N_{H(L)}(T)$ such that $n(\sigma_{\mathrm{ss}})^e n^{-1} = (\sigma_{\mathrm{ss}})^{qe}$. Denoting by ω the order of the Weyl group of H_L^0, we get $(\sigma_{\mathrm{ss}})^e = n^{e\omega}(\sigma_{\mathrm{ss}})^e n^{-e\omega} = (\sigma_{\mathrm{ss}})^{q^{e\omega}e}$, and it follows that $(\sigma_{\mathrm{ss}})^{e(q^{e\omega}-1)} = 1$. ☐

REMARK 3.4. It follows from the proof that any irreducible component of $(X_{H,1})_L$ is the closure of the preimage of a H^0-conjugacy class in $\pi_2(X_{H,1}(L))$. In other words, any irreducible component is the closure of the image of the map

$$H_L^0 \times (H_\sigma)^0 \longrightarrow (X_{H,1})_L,\ (h,k) \mapsto (hFkh^{-1}, h\sigma h^{-1})$$

for some $(F,\sigma) \in X(L)$.

REMARK 3.5. The fact that H^0 is reductive is crucial when using the finiteness of the set of unipotent classes. Without this hypothesis, the moduli space may fail to be l.c.i. and therefore its *derived* analogue (see subsection 2.5) may have non-trivial higher structure. This happens for example when H is a Borel subgroup, see [Hel23, Example 2.3].

3.3. Unobstructed points and generic smoothness. Let L be an algebraically closed field over Λ and let $x \in X_{H,1}(L)$ be an L-point of $X_{H,1}$. We compute the tangent space $T_x(X_{H,1})_L$ and its dimension as an L-vector space. To lighten the notation a bit, we will simply write X for $X_{H,1}$ in this subsection.

LEMMA 3.6. *Let* $\varphi : W_t^0 \longrightarrow H(L)$ *denote the morphism associated to* x*, and let* W_t^0 *act on the Lie algebra* $\operatorname{Lie} H$ *through* φ *composed with the adjoint action.*

(1) $T_x X_L \simeq Z^1(W_t^0, \operatorname{Lie} H_L)$
(2) $\dim_L T_x X_L = \dim H_L + \dim_L H^0(W_t^0, (\operatorname{Lie} H_L)^* \otimes \omega)$

where ω is the "cyclotomic character" that takes Fr *to q and s to* 1. *In particular, x is a smooth point of X_L if and only if $H^0(W_t^0, (\operatorname{Lie} H_L)^* \otimes \omega) = 0$.*

Proof. (1) It is convenient here to see X as the closed subscheme of $H^{W_t^0}$ defined by conditions $\varphi(ww')\varphi(w)^{-1}\varphi(w')^{-1} = 1$. Then $T_x X_L$ identifies to the common kernel of the corresponding linear maps $\prod_w T_{\varphi(w)} H_L \longrightarrow T_1 H_L = \operatorname{Lie} H_L$. Identifying $T_h H_L$ to $\operatorname{Lie} H_L$ via right translation under h, this exhibits $T_x X_L$ as the common kernel of certain linear maps $\prod_w \operatorname{Lie} H_L \longrightarrow \operatorname{Lie} H_L$. In order to compute these maps, recall that the tangent space $T_x X_L$ can be seen as the fiber of the map $X(L[\varepsilon]) \xrightarrow{\pi_*} X(L)$ over x. So an element $\tilde{x} \in T_x X_L$ is given by some homomorphism $\tilde{\varphi} : W_t^0 \longrightarrow H(L[\varepsilon])$ such that $\tilde{\varphi}(w) \in T_{\varphi(w)} H_L$ for any w. Under right translations as above, this corresponds to the element $(\tilde{h}_w := \tilde{\varphi}(w)\varphi(w)^{-1})_{w \in W}$ in $\prod_w \operatorname{Lie} H_L$. The condition $\tilde{\varphi}(ww')\tilde{\varphi}(w)^{-1}\tilde{\varphi}(w')^{-1} = 1$ then The condition $\tilde{\varphi}(ww')\tilde{\varphi}(w)^{-1}\tilde{\varphi}(w')^{-1} = 1$ then translates into the equality

$$\tilde{h}_{ww'} = \tilde{\varphi}(ww')\varphi(ww')^{-1} = \tilde{\varphi}(w)\tilde{\varphi}(w')\varphi(w')^{-1}\varphi(w)^{-1} = h_w \varphi(w) h_{w'} \tilde{\varphi}(w)^{-1}.$$

In additive notation, this means $\tilde{h}_{ww'} = h_w + \operatorname{Ad}_{\varphi(w)}(h_{w'})$, which is the cocycle condition of statement (1).

(2) This can be proved by explicit computation as in [**DHKM20**, Lemma 5.1]. Here is an alternative argument. We have an exact sequence

$$0 \longrightarrow (\operatorname{Lie} H_L)^{\varphi(W_t^0)} \longrightarrow \operatorname{Lie} H_L \xrightarrow{\partial} Z^1(W_t^0, \operatorname{Lie} H_L) \longrightarrow H^1(W_t^0, \operatorname{Lie} H_L) \longrightarrow 0$$

where $\partial \tilde{h} = (w \mapsto \operatorname{Ad}_{\varphi(w)} \tilde{h} - \tilde{h})$. This shows that

$$\dim_L T_x X_L = \dim H_L + \dim_L H^1(W_t^0, \operatorname{Lie} H_L) - \dim_L H^0(W_t^0, \operatorname{Lie} H_L).$$

Now, the dévissage $s^{\mathbb{Z}[\frac{1}{q}]} \hookrightarrow W_t^0 \twoheadrightarrow \mathrm{Fr}^{\mathbb{Z}}$ allows one to compute cohomology of W_t^0 in stages, via a spectral sequence. Since $\mathbb{Z}$ and $\mathbb{Z}[\frac{1}{q}]$ have cohomological dimension 1 on finite dimensional L-vector spaces and satisfy vanishing of Euler-Poincaré dimension, we see that W_t^0 has cohomological dimension 2 and satisfies vanishing of Euler-Poincaré dimension. This implies that

$$\dim_L T_x X_L = \dim H_L + \dim_L H^2(W_t^0, \operatorname{Lie} H_L).$$

Now, to compute $H^2(W_t^0, \operatorname{Lie} H_L) = H^1(\langle \mathrm{Fr} \rangle, H^1(s^{\mathbb{Z}[\frac{1}{q}]}, \operatorname{Lie} H_L))$, we first note that the canonical map $H^1(s^{\mathbb{Z}}, \operatorname{Lie} H_L) = (\operatorname{Lie} H_L)_\sigma \longrightarrow H^1(s^{\mathbb{Z}[\frac{1}{q}]}, \operatorname{Lie} H_L)$ is an isomorphism (here $\sigma = \varphi(s)$ and the index σ denotes σ-coinvariants). Indeed, since σ_{ss} has finite order, all $\varphi(s^{q^{-n}})$ belong to the Zariski closure of $\langle \sigma \rangle$ in $\operatorname{Aut}_L(\operatorname{Lie} H_L)$. Via this isomorphism, the action of Fr on $H^1(\sigma^{\mathbb{Z}[\frac{1}{q}]}, \operatorname{Lie} H_L)$ corresponds to the natural action of $F = \varphi(\mathrm{Fr})$ on $(\operatorname{Lie} H_L)_\sigma$ multiplied by q^{-1}. So we get an isomorphism $H^2(W_t^0, \operatorname{Lie} H_L) \simeq \left(\omega^{-1} \otimes (\operatorname{Lie} H_L)_\sigma\right)_F$ and, after taking duals, we see that $H^2(W_t^0, \operatorname{Lie} H_L)^* \simeq H^0(W_t^0, (\operatorname{Lie} H_L)^* \otimes \omega)$. $\square$

Now, we would like to show that X_L is generically smooth, at least if the characteristic of L is 0 or is sufficiently large. In the next lemma we deal with the characteristic 0 case. We refer to [**DHKM20**, §5] for a discussion of when the result is still true in positive characteristic.

Proposition 3.7. *If* $\operatorname{Char} L = 0$*, any irreducible component of* X_L *contains a smooth point. Consequently,* $X_{H,1}$ *is generically smooth and, in particular, reduced.*

Proof. From Remark 3.4, it suffices to show that, for any $(F, \sigma) \in X(L)$, there is $c \in (H_\sigma)^0(L)$ such that (Fc, σ) is a smooth point. By the last lemma, it suffices to find $c \in (H_\sigma)^0(L)$ such that Fc has no fixed vector in $((\operatorname{Lie} H_L)^* \otimes \omega)^\sigma$, i.e. such that q^{-1} is not an eigenvalue of Fc on $(\operatorname{Lie} H_L)^{*,\sigma}$.

We first assume that σ is unipotent (and thus belongs to $H^0(L)$). By the Jacobson-Morozov theorem, we can find a cocharacter $\lambda : \mathbb{G}_m \longrightarrow H$ such that $\log \sigma$ has weight 2 under $\operatorname{Ad}_\lambda$ acting on $\operatorname{Lie} H$. Moreover such a λ is unique up to conjugacy by an element of the unipotent radical $R_u(H_\sigma)$. So there is $c_1 \in R_u(H_\sigma)(L)$ such that Fc_1 centralizes λ. Now put $F_\lambda := \lambda(q^{\frac{1}{2}}) \in H(L)$. We have $F_\lambda \sigma F_\lambda^{-1} = \sigma^q$, hence $F_\lambda^{-1} F c_1 \in H_\sigma \cap H_\lambda$. The algebraic group $H_\sigma \cap H_\lambda$ is known to be a Levi complement of H_σ (see e.g. [**Jan04**, Prop. 5.10]). In particular, it is reductive, so we may fix a pinning of its neutral component $(H_\sigma \cap H_\lambda)^0$ and find $c_2 \in (H_\sigma \cap H_\lambda)^0$ such that $F_\lambda^{-1} F c_1 c_2$ normalizes this pinning. Putting $m := |\pi_0(H_\sigma \cap H_\lambda)|$, this implies that $(F_\lambda^{-1} F c_1 c_2)^m$ belongs to the center $Z(H_\sigma \cap H_\lambda)$, and even to the subgroup $Z(H_\sigma \cap H_\lambda)^{F_\lambda^{-1} F c_1 c_2}$ fixed under conjugacy by $F_\lambda^{-1} F c_1 c_2$. If m' denotes the order of $\pi_0(Z(H_\sigma \cap H_\lambda)^{F_\lambda^{-1} F c_1 c_2})$, we see that $(F_\lambda^{-1} F c_1 c_2)^{mm'}$ is an element of the torus $(Z(H_\sigma \cap H_\lambda)^{F_\lambda^{-1} F c_1 c_2})^0$. Since a torus is a divisible group, there is an element $c_3 \in (Z(H_\sigma \cap H_\lambda)^{F_\lambda^{-1} F c_1 c_2})^0$ such that $(F_\lambda^{-1} F c_1 c_2 c_3)^{mm'} = 1$. Putting $c := c_1 c_2 c_3 \in (H_\sigma)^0$ and $c' := F_\lambda^{-1} F c \in H_\sigma \cap H_\lambda$, we see that $Fc = F_\lambda c' = c' F_\lambda$, so the eigenvalues of Fc on $(\operatorname{Lie} H_L)^{*,\sigma}$ are products of eigenvalues of c' and of F_λ. The eigenvalues of c' are roots of unity. On the other hand, we have a decomposition $(\operatorname{Lie} H_L)^* = \operatorname{Lie}(Z(H)_L)^* \oplus (\operatorname{Lie} H_{\mathrm{der},L})^*$. The eigenvalues of F_λ on $\operatorname{Lie}(Z(H)_L)^*$ are again roots of unity, and we know that λ has non-negative weights on $\operatorname{Lie} H_\sigma$. Since the Killing form identifies $(\operatorname{Lie} H_{\mathrm{der},L})^*$ and $\operatorname{Lie} H_{\mathrm{der},L}$, we see that q^{-1} is not an eigenvalue of Fc on $(\operatorname{Lie} H_L)^{*,\sigma} = \operatorname{Lie} H_\sigma$.

We now reduce the general case to the unipotent case. So let $\sigma = \sigma_{\mathrm{ss}} \sigma_{\mathrm{u}}$ be the Jordan decomposition of σ. Since F normalizes the reductive algebraic subgroup $(H_{\sigma_{\mathrm{ss}}})^0$, we may repeat the above argument to find an element $h \in (H_{\sigma_{\mathrm{ss}}})^0$ such that Fh normalizes a pinning of this group and has finite order. This means that the Zariski closure H' of the subgroup generated by $H_{\sigma_{\mathrm{ss}}}$ and F has neutral component $(H_{\sigma_{\mathrm{ss}}})^0$. By the unipotent case, there is $c \in (H'_{\sigma_{\mathrm{u}}})^0 = (H_\sigma)^0$ such that q^{-1} is not an eigenvalue of Fc on $\operatorname{Lie} H'_{\sigma_{\mathrm{u}}} = \operatorname{Lie} H_\sigma$, as desired. □

3.4. Faithful flatness and ℓ-adic continuity. The next statement implies that all irreducible components of $X_{H,1}$ (endowed with their reduced subscheme structure) are faithfully flat over Λ, and that the ring of functions $R_{H,1}$ of $X_{H,1}$ is ℓ-adically separated for all primes $\ell \neq p$.

Lemma 3.8. *Any irreducible component of* $X_{H,1}$ *contains a* $\overline{\mathbb{Z}}_\ell$*-point, for any prime* $\ell \neq p$*. In particular the ring of functions* $R_{H,1}$ *of* $X_{H,1}$ *is* ℓ*-adically separated.*

Proof. Let Y be an irreducible component of $X = X_{H,1}$, and choose an embedding $\Lambda \hookrightarrow \overline{\mathbb{Z}}_\ell$. Since $X_{H,1}$ is flat over Λ, the base change $Y_{\overline{\mathbb{Q}}_\ell}$ is non-empty, and is an irreducible component of $X_{\overline{\mathbb{Q}}_\ell}$. In the proof of the last proposition, we showed that $Y_{\overline{\mathbb{Q}}_\ell}$ contains a point (F, σ) with F of the form $F = F_\lambda c' = c' F_\lambda$ where $F_\lambda = \lambda(q^{\frac{1}{2}})$ for some cocharacter λ of $H_{\overline{\mathbb{Q}}_\ell}$, and c' has finite order. Up to conjugating this point under $H^0(\overline{\mathbb{Q}}_\ell)$, we may assume that λ factors through a maximal torus of $H_{\overline{\mathbb{Q}}_\ell}$ that comes from a maximal torus of $H_{\overline{\mathbb{Z}}_\ell}$. Then λ itself has to be defined over $\overline{\mathbb{Z}}_\ell$. Since q is an ℓ-adic unit, it follows that the closure $\overline{\langle F\rangle}$ in $H(\overline{\mathbb{Q}}_\ell)$ of the group generated by F for the analytic topology, is compact. On the other hand σ itself is the product $\sigma = \sigma_{\rm ss}\sigma_{\rm u}$ of a finite order element and a unipotent element, hence $\overline{\langle\sigma\rangle}$ is compact and, finally, $\overline{\langle F,\sigma\rangle}$, which is a quotient of $\overline{\langle F\rangle} \ltimes \overline{\langle\sigma\rangle}$, is compact too. In particular it is contained in $H(L)$ for some finite subextension L of $\mathbb{Q}_\ell$ in $\overline{\mathbb{Q}}_\ell$, and it stabilizes a facet of the building of H^0 over L. After maybe extending L, we may assume that it stabilizes an hyperspecial vertex, and that this vertex is conjugate, under $H^0(L)$ to the "origin" vertex associated to the integral model $H_{\mathbb{Z}_\ell}$. But the stabilizer of the origin vertex is $H(\mathcal{O}_L)Z(H)(L)$ and the maximal bounded subgroup therein is $H(\mathcal{O}_L)$. So, after conjugation under $H^0(L)$, we get a point $(F,\sigma) \in Y(\overline{\mathbb{Q}}_\ell)$ such that $\langle F,\sigma\rangle \subset H(\overline{\mathbb{Z}}_\ell)$. This means $(F,\sigma) \in Y(\overline{\mathbb{Z}}_\ell)$, as desired.

For the remaining statement, we leave it as an exercise to show that for a noetherian integral ring A, either ℓ is invertible in A, or A is ℓ-adically separated. In our case, the existence of a $\overline{\mathbb{Z}}_\ell$-point shows that ℓ is not invertible in the ring $\mathcal{O}(Y)$, hence the latter is ℓ-adically separated. Since $R_{H,1}$ is reduced, the map $R_{H,1} \longrightarrow \prod_Y \mathcal{O}(Y)$ is injective, so $R_{H,1}$ is ℓ-adically separated too. $\square$

Now we wish to extend the universal morphism $\varphi_{H,1} : W_t^0 \longrightarrow H(R_{H,1})$ to the non-discretized tame Weil group $W_t = W_F/P_F$.

Proposition 3.9. *Fix a prime $\ell \neq p$. There is a unique ℓ-adically continuous extension ${}^\ell\varphi_{H,1} : W_t \longrightarrow H(R_{H,1}\otimes\mathbb{Z}_\ell)$ of $\varphi_{H,1}$, and it is universal among ℓ-adically continuous morphisms $W_t \longrightarrow H(R)$ when R runs over $\Lambda\otimes\mathbb{Z}_\ell$-algebras. Moreover, it is also continuous and universal for the ind-ℓ-adic topology (see 2.2).*

Proof. Put $\sigma := \varphi_{H,1}(s) \in H(R_{H,1})$. In the course of the proof of Proposition 3.3, we showed the existence of a prime-to-p number M such that σ^M is fibrewise unipotent, i.e. has unipotent image in $H(L)$ for any algebraically closed field L over $R_{H,1}$. In particular, for all $m \in \mathbb{N}$, the image of σ^M in $H(R_{H,1}/\ell^m R_{H,1})$ has finite ℓ-power order, hence $\varphi_{H,1,\mathrm{mod}\,\ell^m} : W_t^0 \longrightarrow H(R_{H,1}/\ell^m R_{H,1})$ extends uniquely to W_t and its restriction to $I_F/P_F = s^{\hat{\mathbb{Z}}^p}$ factors through $\hat{\mathbb{Z}}^p := \widehat{\mathbb{Z}[\frac{1}{p}]} \longrightarrow \mathbb{Z}/M\mathbb{Z}\times\mathbb{Z}_\ell$. Going to the limit, we get a morphism ${}^\ell\hat\varphi_{H,1} : W_t \longrightarrow H((R_{H,1}\otimes\mathbb{Z}_\ell)^\wedge)$ where $\wedge$ denotes ℓ-adic completion. We need to show that this extension factors through $H(R_{H,1}\otimes\mathbb{Z}_\ell)$, which is a subgroup of $H((R_{H,1}\otimes\mathbb{Z}_\ell)^\wedge)$ since $R_{H,1}$ is ℓ-adically separated. It is enough to prove it after composing with a faithful representation $\iota : H \hookrightarrow \mathrm{GL}_N$ of H over Λ. But there, the extension of $\iota\circ{}^\ell\hat\varphi_{H,1}$ is explicitly given on s^z, for $z \in \hat{\mathbb{Z}}^p$, by the formula

$$\iota\sigma^z = \iota\sigma^a\iota\sigma^{Mz'} = \iota\sigma^a\left(1+(\iota\sigma^M-1)\right)^{z'} = \iota\sigma^a\sum_{k=0}^{N-1}\binom{k}{z'_\ell}(\iota\sigma^M-1)^k$$
$$\text{for } z = a+Mz' \in \widehat{\mathbb{Z}}^p \text{ with } a\in\{0,\cdots,M-1\} \text{ and } z'_\ell \text{ the image of } z' \text{ in } \mathbb{Z}_\ell$$

since $\iota\sigma^M$ is a unipotent matrix. Note that the generalized binomial coefficient $\binom{k}{z'_\ell}$ lies in $\mathbb{Z}_\ell$, and the formula provides a matrix in $\mathrm{GL}_N(R_{H,1}\otimes\mathbb{Z}_\ell)$, as desired. The verification of the universality of ${}^\ell\varphi_{H,1}$ is quite formal and left to the reader. Finally, to prove continuity of ${}^\ell\varphi_{H,1}$ for the ind-ℓ-adic topology, it is enough to show that for any vector $v \in (R_{H,1}\otimes\mathbb{Z}_\ell)^N$, the $\mathbb{Z}_\ell$-submodule generated by the set

$\{\iota({}^{\ell}\varphi_{H,1}(s^z))v, z \in M\hat{\mathbb{Z}}^p\}$ is finitely generated. But the above formula shows it is contained in the $\mathbb{Z}_\ell$-submodule generated by $\{(1-\iota\sigma^{Mk})(v), k = 0, \cdots, N-1\}$. $\square$

3.5. Connected components and spaces of 1-cocycles. At the beginning of this section, we noted that the scheme of tame cocycles $Z^1(W_t^0, \hat{G})$, which we are ultimately interested in, is a summand of the scheme $X_{H,1} = \underline{\mathrm{Hom}}(W_t^0, H)$ for the group $H = \hat{G} \rtimes W$, where W is a finite quotient of W_t through which the action of W_t on $\hat{G}$ factors. Conversely, we will now see that $X_{H,1}$ decomposes as a sum of spaces of 1-cocycles, each of which turns out to be connected.

To simplify the discussion, we will work over $\overline{\mathbb{Z}}[\frac{1}{p}]$ (but everything below holds over a sufficiently large finitely generated subring there, by standard limit arguments). In this setting, we know that H^0 is split and $\pi_0(H)$ is constant. By pushing forward homomorphisms, we get a morphism

$$X_{H,1} = \underline{\mathrm{Hom}}(W_t^0, H) \xrightarrow{\pi} \mathrm{Hom}(W_t^0, \pi_0(H))$$

from $X_{H,1}$ to a finite discrete scheme, whence a decomposition

$$X_{H,1} = \textstyle\bigsqcup_{\bar{\varphi}} X_{H,1,\bar{\varphi}} \text{ with } X_{H,1,\bar{\varphi}} = \pi^{-1}(\{\bar{\varphi}\}).$$

The following proposition shows in particular that each summand is non-empty.

Proposition 3.10. *For each $\bar{\varphi} : W_t^0 \longrightarrow \pi_0(H)$, there is an element $\varphi \in X_{H,1,\bar{\varphi}}(\overline{\mathbb{Z}}[\frac{1}{p}])$ such that $\varphi(W_t^0)$ is finite and normalizes a Borel pair of H^0. Moreover, if the center $Z(H^0)$ is smooth over $\overline{\mathbb{Z}}[\frac{1}{p}]$, then one can choose φ such that $\varphi(W_t^0)$ normalizes a pinning of H^0.*

Proof. Let us fix a pinning $\varepsilon = (B, T, (X_\alpha)_{\alpha\in\Delta})$ of H^0, and consider the normalizer $\mathcal{T} := N_H(B, T)$ of the underlying Borel pair in H. By Lemma 3.1, this is a split generalized reductive group with neutral component T and $\pi_0(\mathcal{T}) = \pi_0(H)$. So we may apply our results so far to $X_{\mathcal{T},1} := \underline{\mathrm{Hom}}(W_t^0, \mathcal{T})$ and to its summand $X_{\mathcal{T},1,\bar{\varphi}}$, which is a closed subscheme of $X_{H,1,\bar{\varphi}}$.

Let us show that $X_{\mathcal{T},1,\bar{\varphi}}(\overline{\mathbb{Z}}[\frac{1}{p}]) \neq \emptyset$. To this aim, we first note that the map $\mathcal{T}(\overline{\mathbb{Z}}[\frac{1}{p}]) \longrightarrow \pi_0(H)$ is surjective. Indeed, this follows from [DHKM24, Lemma 2.1] and the fact that the morphism $\mathcal{T} \longrightarrow \pi_0(H)$ is a T-torsor. So, the obstruction to lift $\bar{\varphi}$ to a morphism $\varphi : W_t^0 \longrightarrow \mathcal{T}(\overline{\mathbb{Z}}[\frac{1}{p}])$ lies in $H^2_{\mathrm{Ad}_{\bar{\varphi}}}(W_t^0, T(\overline{\mathbb{Z}}[\frac{1}{p}]))$. Here $\mathrm{Ad}_{\bar{\varphi}}$ denotes the action of W_t^0 obtained by composition of $\bar{\varphi}$ with the conjugation action of $\mathcal{T}$ on $T = \mathcal{T}^0$, which factors through $\pi_0(\mathcal{T}) = \pi_0(H)$. Now, as already noted in the proof of Lemma 3.6, we have $H^2(W_t^0, T(\overline{\mathbb{Z}}[\frac{1}{p}])) = [(T(\overline{\mathbb{Z}}[\frac{1}{p}]))_{\mathrm{Ad}_{\bar{\varphi}(s)}}]_{q^{-1}\mathrm{Ad}_{\bar{\varphi}(\mathrm{Fr})}}$. But this group vanishes because $\mathrm{Ad}_{\bar{\varphi}(\mathrm{Fr})}$ is a finite order automorphism and $T(\overline{\mathbb{Z}}[\frac{1}{p}])$ is p'-divisible. Hence $X_{\mathcal{T},1,\bar{\varphi}}(\overline{\mathbb{Z}}[\frac{1}{p}]) \neq \emptyset$.

At this point we have almost proven the first claim of the proposition, except for the finiteness of the image. However, observe that for any $\varphi \in X_{\mathcal{T},1,\bar{\varphi}}(\overline{\mathbb{Z}}[\frac{1}{p}])$, the element $\sigma = \varphi(s)$ has finite prime to p order, since a prime to p power of it is unipotent in $\mathcal{T}(\overline{\mathbb{Q}})$ hence equal to 1. So, writing $F := \varphi(\mathrm{Fr})$ as usual, it will suffice to show the existence of an element $c \in T(\overline{\mathbb{Z}}[\frac{1}{p}])^\sigma$ such that Fc has finite order, in which case the pair (Fc, σ) defines a homomorphism as desired. To this aim, observe that there is n such that F^n belongs to $(T^{\mathrm{Ad}_{\bar{\varphi}}})^0(\overline{\mathbb{Z}}[\frac{1}{p}])$. But since the latter abelian group is divisible, there is $c \in (T^{\mathrm{Ad}_{\bar{\varphi}}})^0(\overline{\mathbb{Z}}[\frac{1}{p}])$ such that $c^n = F^n$, hence $(Fc)^n = 1$, as desired.

For the second claim of the proposition, we proceed exactly as above, replacing $\mathcal{T}$ by the normalizer $\mathcal{Z}$ of the fixed pinning. In this case $H^2_{\mathrm{Ad}_{\bar\varphi}}(W_t^0, Z(\overline{\mathbb{Z}}[\frac{1}{p}]))$ vanishes because $Z^0(\overline{\mathbb{Z}}[\frac{1}{p}])$ is divisible and $\pi_0(Z)$ is a p-group (by our smoothness assumption), hence is p'-divisible. □

COROLLARY 3.11. *Let $\bar\varphi : W_t^0 \longrightarrow \pi_0(H)$, and fix an element φ as in the last proposition. Denote by Ad_φ the action of W_t on H^0 that we get via conjugation inside H. Then we get an isomorphism*

$$Z^1_{\mathrm{Ad}_\varphi}(W_t^0, H^0) \xrightarrow{\sim} X_{H,1,\bar\varphi},\ \eta \mapsto \eta\cdot\varphi,$$

This shows that $X_{H,1,\bar\varphi}$ is a space of 1-cocycles for a nice finite and Borel-pair-preserving action of W_t^0 on H^0. In the smooth center case, this shows that $X_{H,1,\bar\varphi}$ is the space of tame Langlands parameters for "the" quasi-split group over F whose Langlands dual is $H^0 \rtimes_{\mathrm{Ad}\varphi} W_F$.

In [DHKM20] and in the first version of these notes, the following result was proved under the hypothesis that the center of H^0 is smooth over $\overline{\mathbb{Z}}[\frac{1}{p}]$, and conjectured in general. During the review process of these notes, the general case was solved by Sean Cotner.

THEOREM 3.12 ([Cot24]). *Each non empty summand $X_{H,1,\bar\varphi}$ is connected.*

Cotner's approach rests on the last corollary, as did the original approach of [DHKM20]. His proof of the connectedness of the space of cocycles in turn is close to, and improves on, the proof of the following special case (where the action is required to fix a pinning, and not only a Borel pair) given in section 5.4 (see Remark 5.14 in particular).

THEOREM 3.13. *Suppose G is a tamely ramified reductive p-adic group. Then the $\overline{\mathbb{Z}}[\frac{1}{p}]$-scheme of tame Langlands parameters $Z^1(W_t^0, \hat G)$ is connected.*

This special case is enough to get Theorem 3.12 under the additional assumption that the center of H^0 is smooth, thanks to the second part of Proposition 3.10.

EXAMPLE 3.14. Suppose $G = T$ is a torus that splits over a tamely ramified extension. Then $Z^1(W_t^0, \hat T)$ identifies to the kernel of

$$\hat T \times \hat T \longrightarrow \hat T,\ (F,\sigma) \mapsto F \cdot {}^{\mathrm{Fr}}\sigma \cdot ({}^{s^q}F)^{-1} \cdot \left(\sigma \cdot {}^{s}\sigma \cdots {}^{s^{q-1}}\sigma\right)^{-1},$$

hence it has the structure of a diagonalizable group scheme over $\overline{\mathbb{Z}}[\frac{1}{p}]$. Such a scheme is connected over $\overline{\mathbb{Z}}[\frac{1}{p}]$ if and only if the order of the torsion of its character group is prime to p. If the action of s on $\hat T$ is trivial (ie T is unramified), then $Z^1(W_t^0, \hat T) \simeq \hat T \times \hat T^{\mathrm{Fr}=(\cdot)^q}$, and its character group has torsion subgroup $X^*(\hat T)/(q-\mathrm{Fr})X^*(\hat T)$ which has prime to p order as desired. On the other hand, if $L_s : t \mapsto t({}^s t)^{-1}$ is an isogeny of $\hat T$, then the map $Z^1(W_t^0, \hat T) \xrightarrow{\varphi \mapsto \varphi(s)} \hat T$ is surjective and its kernel is isomorphic to the kernel of the isogeny $L_{s^q} : t \mapsto t({}^{s^q}t)^{-1}$, which has prime to p degree again. For the general case, consider the prime-to-p degree isogeny $\hat T \twoheadrightarrow \hat T_s \times (\hat T/\hat T^s)$. It induces a morphism $Z^1(W_t^0, \hat T) \longrightarrow Z^1(W_t^0, \hat T_s) \times Z^1(W_t^0, \hat T/\hat T^s)$ whose kernel and cokernel are finite with prime-to-p order. So the general case follows from the two special cases above.

4. Reduction to the tame case and connected components

We now take over the study of $Z^1(W_F^0/P, \hat{G})$ for an open subgroup P of P_F that is distinguished in W_F and acts trivially on $\hat{G}$.

Again, we will study a slightly more general setting where we fix a generalized reductive group scheme H over Λ and consider the affine scheme

$$X_{H,1_P} := \underline{\mathrm{Hom}}(W_F^0/P, H)$$

that represents the functor on R-algebras $R \mapsto \mathrm{Hom}\left(W_F^0/P, H(R)\right)$. Thanks to Iwasawa theorem [Iwa55], there is a (non-canonical) semi-direct product decomposition $W_F^0/P \simeq P_F/P \rtimes W_t^0$. Hence we will first study the affine scheme

$$Y_{H,1_P} := \underline{\mathrm{Hom}}(P_F/P, H)$$

that represents the functor $R \mapsto \mathrm{Hom}\left(P_F/P, H(R)\right)$, and then we will use the restriction map $X_{H,1_P} \longrightarrow Y_{H,1_P}$.

4.1. Homomorphisms from a finite group. In this paragraph, we look more generally at the scheme $Y := \underline{\mathrm{Hom}}(\Gamma, H)$ where H is a group scheme over a noetherian affine base $S = \mathrm{Spec}\, R$ and Γ is a finite group. Conjugation induces an action of the group scheme H on Y relatively over S, given by a morphism

$$H \times_S Y \longrightarrow Y,\ (h, \phi) \mapsto \mathrm{Ad}_h \circ \phi,$$

where Ad denotes conjugation. The "universal orbit morphism" associated to this action is

$$\alpha :\ H \times_S Y \longrightarrow Y \times_S Y,\ (h, \phi) \mapsto (\mathrm{Ad}_h \circ \phi, \phi).$$

This morphism controls the usual orbit morphisms and the centralizers and transporters. Namely :

- for $\phi \in \mathrm{Hom}(\Gamma, H(R'))$, the corresponding orbit morphism $H_{R'} \longrightarrow Y_{R'}$ is the base change of the Y-morphism α along the corresponding R'-point $\mathrm{Spec}\, R' \longrightarrow Y$,
- for $\phi, \phi' \in \mathrm{Hom}(\Gamma, H(R'))$, the transporter from ϕ to ϕ' is the $S' := \mathrm{Spec}(R')$-scheme obtained by pullback of α along the corresponding map $S' \longrightarrow Y \times_S Y$.

Lemma 4.1. *Assume that H is smooth and Γ has invertible order on R. Then $\underline{\mathrm{Hom}}(\Gamma, H)$ is smooth, and all orbit morphisms and transporters are smooth.*

Proof. We refer to [DHKM20, A.1] for details. Since all schemes are of finite presentation over S, it suffices to verify the corresponding infinitesimal lifting properties. So let R' be an R-algebra with ideal I of square 0. To prove smoothness of Y over S, we need to show that any $\phi_0 : \Gamma \longrightarrow H(R'/I)$ can be lifted to a $\phi : \Gamma \longrightarrow H(R')$. But an explicit computation shows that the obstruction for doing so lies in $H^2_{\mathrm{Ad}_{\phi_0}}(\Gamma, \mathrm{Lie}(H) \otimes_R I)$, which vanishes since Γ has invertible order in R. On the other hand, to prove smoothness of the universal orbit morphism, we need to show that for any pair of homomorphisms $\phi, \phi' : \Gamma \longrightarrow H(R')$ whose reductions $\phi_0, \phi'_0 : \Gamma \longrightarrow H(R'/I)$ are conjugate under some $h_0 \in H(R'/I)$, there is a lifting $h \in H(R')$ of h_0 which conjugates ϕ to ϕ'. Again, an explicit computation shows that the obstruction lies in $H^1_{\mathrm{Ad}_{\phi_0}}(\Gamma, \mathrm{Lie}(H) \otimes_R I)$, which vanishes. $\square$

Since orbit morphisms are smooth, they are in particular open. In the case where R is an algebraically closed field, we can pick a set $\Phi \subset Y(R)$ of representatives of $H(R)$-orbits. Then we get a decomposition $Y = \coprod_{\phi\in\Phi} H\cdot\phi$, with $H\cdot\phi$ the image of the orbit morphism through ϕ. In particular, Φ has to be finite. Moreover, the orbit morphism induces an isomorphism $H/C_H(\phi) \xrightarrow{\sim} H\cdot\phi$, where $C_H(\phi)$ is the centralizer.

As far as we know, there is no general result along these lines over a general base ring R in the literature. However, here is a result sufficient for our purposes.

THEOREM 4.2. *Suppose that H is generalized reductive, that R is a normal subring of a number field, and that Γ has order invertible in R.*

1. *The categorical quotient $Y /\!\!/ H^0 = \mathrm{Spec}\, (\mathcal{O}_Y)^{H^0}$ is finite étale over R and represents the quotient sheaf Y/H^0 for the étale topology.*
2. *For any $\phi \in Y(R')$, the quotient sheaf $H^0_{R'}/C_{H^0}(\phi)$ for the étale topology is representable by an R'-scheme, and the orbit morphism $H^0_{R'} \longrightarrow Y_{R'}$ identifies it to an open and closed subscheme of $Y_{R'}$.*
3. *There is a finite normal extension R'/R and a finite set $\Phi \subset Y(R')$ such that the orbit maps induce an isomorphism $\coprod_{\phi\in\Phi} H^0_{R'}/C_{H^0}(\phi) \xrightarrow{\sim} Y_{R'}$.*

PROOF. For (1), we refer to [DHKM20, Thm. A.7] and content ourselves with indicating two main ingredients. The key point is to prove that when R is a strictly Henselian d.v.r., the canonical map from the constant sheaf associated to $Y(R)/H^0(R)$ to the quotient sheaf Y/H^0 is an isomorphism. In other words, given any strictly Henselian local R-algebra R', the map $Y(R)/H^0(R) \longrightarrow Y(R')/H^0(R')$ should be a bijection. If the map $R \longrightarrow R'$ is local, then we have a factorization $Y(R)/H^0(R) \xrightarrow{\sim} Y(k_R)/H^0(k_R) \longrightarrow Y(k_{R'})/H^0(k_{R'}) \simeq Y(R')/H^0(R')$ where the isomorphisms come from smoothness of transporters and Artin approximation. This leaves us with the case of a field extension, which can be dealt with by using V. Lafforgue's pseudocharacters. Otherwise we have a factorization $Y(R)/H^0(R) \longrightarrow Y(\overline{K})/H^0(\overline{K}) \longrightarrow Y(k_{R'})/H^0(k_{R'}) \simeq Y(R')/H^0(R')$ with $\overline{K}$ an algebraic closure of the fraction field of R, and we use Bruhat-Tits theory to get bijectivity of the first map. This is the part of the argument where we use that R is regular of dimension 1.

For (2), it suffices to prove it for the universal homomorphism ϕ_{univ} corresponding to the identity of Y. Its orbit morphism is $\alpha : H^0 \times_S Y \longrightarrow Y \times_S Y$. Let $C_{H^0}(\phi_{\mathrm{univ}}) \longrightarrow Y$ be the universal centralizer over Y, i.e. the pullback of α along the diagonal. Then α induces an isomorphism $(H^0 \times_S Y)/C_{H^0}(\phi_{\mathrm{univ}}) \xrightarrow{\sim} Y \times_{Y/H^0} Y$, where all quotients are sheaf-theoretic for the étale topology. By (1) we know that the target is representable by an affine scheme, hence the source is representable too. Moreover, the morphism $Y \times_{Y/H^0} Y \longrightarrow Y \times_S Y$ is a base change of $Y/H^0 = Y/H^0 \times_{Y/H^0} Y/H^0 \longrightarrow Y/H^0 \times_S Y/H^0$ which is an open and closed immersion since Y/H^0 is finite étale over S.

For (3), we refer to [DHKM20, Thm. A.9] and content ourselves with explaining the problem and the main tool to solve it. We may first use Lemma 3.2 to replace R by a finite extension so that H splits over R. Using (1), we then may replace R by a finite étale extension that splits $\mathcal{O}(Y)^{H^0}$. Then the problem is to find R' such that any point $[\phi]$ of $(Y/H^0)(R)$ lifts to some representative $\phi \in Y(R')$. By smoothness of $Y \longrightarrow Y/H^0$, we certainly can find R' faithfully étale over R such that this property holds. So, up to replacing R by a finite extension, we can find R' such that $\operatorname{Spec} R' \longrightarrow \operatorname{Spec} R$ is a Zariski covering, i.e. $R' = \prod_{i=1}^{r} R_i$ with each R_i a localization of R. So the problem is to pass from this Zariski covering to some finite flat covering. In *loc.cit.*, this is done via the strong approximation theorem. $\square$

REMARK 4.3. If H is assumed to be reductive, it splits over a faithfully étale R' over R, and the split $H_{R'}$ descends to $\operatorname{Spec} \mathbb{Z}$ by classification. In this case, we can use the theorem to prove that its conclusions still hold true over any ring R that either is flat over $\mathbb{Z}$ or contains a field.

4.2. Restriction to P_F. We now take our study of $X_{H,1_P} = \underline{\mathrm{Hom}}(W_F^0/P, H)$ up again, and we revert to the notation at the beginning of this section. Recall our base ring Λ is a finite integral extension of $\mathbb{Z}[\frac{1}{p}]$, so that Theorem 4.2 above applies to $Y_{H,1_P} = \underline{\mathrm{Hom}}(P_F/P, H)$. After maybe enlarging Λ, we may assume that there is a finite set $\Phi \subset Y_{H,1_P}(\Lambda)$ such that $Y_{H,1_P}$ decomposes as a sum of orbits

$$Y_{H,1_P} = \coprod_{\phi\in\Phi} Y_{H,[\phi]} \text{ with } H^0/C_{H^0}(\phi) \xrightarrow{\sim} H^0\cdot\phi =: Y_{H,[\phi]}.$$

Pulling back by the restriction morphism $X_{H,1_P} \longrightarrow Y_{H,1_P}$, we then get a decomposition

$$X_{H,1_P} = \coprod_{\phi\in\Phi} X_{H,[\phi]}\,, \text{ with } \; H^0 \times^{C_{H^0}(\phi)} X_{H,\phi} \xrightarrow{\sim} X_{H,[\phi]}$$

and where

$$X_{H,\phi} = \underline{\mathrm{Hom}}(W_F^0, H)_\phi := \{\varphi \in X_{H,1_P}, \varphi_{|P_F} = \phi\}.$$

The contracted product above denotes the quotient of $H^0 \times X_{H,\phi}$ by the diagonal action of $C_{H^0}(\phi)$. This quotient is defined first as a sheaf for the étale topology, and the isomorphism above (quite formal in the sheaf setting) shows it is indeed representable. Of course, $X_{H,\phi}$ may be empty. We will say ϕ is *admissible* if $X_{H,\phi} \neq \emptyset$.

For $\phi \in \Phi$, we denote by $H_\phi := N_H(\phi(P_F))$ the scheme-theoretic normalizer in H of the image of ϕ in $H(\Lambda)$. According to Lemma 3.2, this is again a generalized reductive group scheme over Λ. Note that $H_\phi(\Lambda)$ contains $\phi(P_F)$, so we have a Λ-scheme $X_{H_\phi,\phi} = \underline{\mathrm{Hom}}(W_F^0/P, H_\phi)_\phi$ as above. Moreover, since P_F is distinguished in W_F, any $\varphi : W_F^0/P \longrightarrow H(R)$ factors through $H_\phi(R)$, so the closed immersion $H_\phi \hookrightarrow H$ induces an isomorphism

$$X_{H_\phi,\phi} = \underline{\mathrm{Hom}}(W_F^0/P, H_\phi)_\phi \xrightarrow{\sim} \underline{\mathrm{Hom}}(W_F^0/P, H)_\phi = X_{H,\phi}.$$

Let us now fix a splitting $W_F^0/P = W_t^0 \ltimes P_F/P$ as in [Iwa55]. This provides us with a restriction map

$$X_{H_\phi,\phi} = \underline{\mathrm{Hom}}(W_F^0/P, H_\phi)_\phi \longrightarrow \underline{\mathrm{Hom}}(W_t^0, H_\phi) = X_{H_\phi,1}.$$

LEMMA 4.4. *The restriction map above is an open and closed embedding.*

PROOF. Consider the morphism $X_{H_\phi,1} \longrightarrow Y_{H_\phi,1_P} \times_\Lambda Y_{H_\phi,1_P}$ which takes (F,σ) to $({}^F\phi, {}^\sigma\phi)$, where ${}^F\phi = \mathrm{Ad}_F \circ \phi \circ \mathrm{Ad}_{\mathrm{Fr}}^{-1}$ and ${}^\sigma\phi = \mathrm{Ad}_\sigma \circ \phi \circ \mathrm{Ad}_s^{-1}$. Then $X_{H_\phi,\phi}$ identifies to the pullback of this morphism along the diagonal embedding $\operatorname{Spec} \Lambda \xrightarrow{(\phi,\phi)} Y_{H_\phi,1_P} \times_\Lambda Y_{H_\phi,1_P}$, so it suffices to see that this diagonal embedding is an open and closed immersion. But the map $\operatorname{Spec} \Lambda \xrightarrow{\phi} Y_{H_\phi,1_P}$ induces an isomorphism $\operatorname{Spec} \Lambda \xrightarrow{\sim} Y_{H_\phi,[\phi]}$ because $(H_\phi)^0 = C_H(\phi)^0$. Since $Y_{H_\phi,[\phi]}$ is open and closed in $Y_{H_\phi,1_P}$, this shows that ϕ is an open and closed immersion, hence so is (ϕ,ϕ). $\square$

4.3. Proof of Theorems 2.1 and 2.2. Both these theorems are a consequence of the following one.

THEOREM 4.5. *The Λ-scheme $X_{H,1_P} = \underline{\mathrm{Hom}}(W_F^0/P, H)$ is reduced, flat and l.c.i. of relative dimension* $\dim H$*. For any prime $\ell \neq p$, its ring of functions $R_{H,1_P}$ is ℓ-adically separated and the universal homomorphism $\varphi_{H,1_P} : W_F^0/P \longrightarrow R_{H,1_P}$ extends uniquely to a universal ℓ-adically and ind-ℓ-adically continuous homomorphism ${}^\ell\varphi_{H,1_P} : W_F/P \longrightarrow H(R_{H,1_P} \otimes \mathbb{Z}_\ell)$.*

PROOF. For an admissible $\phi \in \Phi$, the Λ-scheme $X_{H,\phi}$ is a summand of $X_{H_\phi,1}$, hence it is flat and l.c.i. of relative dimension $\dim H_\phi$ by Proposition 3.3, and its ring of functions $R_{H,\phi}$ is ℓ-adically separated by Lemma 3.8. The universal homomorphism $\varphi_{H,\phi} : W_F^0/P_F \longrightarrow H(R_{H,\phi})$ has the form $\varphi_{H_\phi,1} \ltimes \phi$ with respect to the splitting $W_F/P = W_t \ltimes (P_F/P)$ chosen for the restriction map $X_{H_\phi,\phi} \longrightarrow X_{H_\phi,1}$. Therefore, with the notation of Proposition 3.9, we see that ${}^\ell\varphi_{H,\phi} := {}^\ell\varphi_{H_\phi,1} \ltimes \phi$ is a universal continuous extension for $\varphi_{H,\phi}$.

Now, it follows from the isomorphism $X_{H,[\phi]} \simeq H^0 \times^{C_{H^0}(\phi)} X_{H,\phi}$ that the Λ-scheme $X_{H,[\phi]}$ is flat and l.c.i. of relative dimension $\dim H$, and that its ring of functions $R_{H,[\phi]} = (\mathcal{O}_{H^0} \otimes_\Lambda R_{H,\phi})^{C_{H^0}(\phi)}$ is ℓ-adically separated. Moreover, the universal homomorphism $\varphi_{H,[\phi]}$ is given by the formula

$$\varphi_{H,[\phi]}(w) : \mathcal{O}_H \longrightarrow \mathcal{O}_{H^0} \otimes_\Lambda \mathcal{O}_H \xrightarrow{\mathrm{Id}\otimes\varphi_{H,\phi}(w)} \mathcal{O}_{H^0} \otimes_\Lambda R_{H,\phi}, \ \ \forall w \in W_F$$

where the first map is dual to $(g,h) \mapsto ghg^{-1}$. Base changing everything to $\mathbb{Z}_\ell$ and replacing $\varphi_{H,\phi}$ by ${}^\ell\varphi_{H,\phi}$ provides us with the desired ${}^\ell\varphi_{H,[\phi]}$. It only remains to use the decomposition $X_{H,1_P} = \bigsqcup_{\phi\in\Phi} X_{H,[\phi]}$ to conclude the proof of the theorem. $\square$

We note that the proof given here is noticeably simpler than that of [DHKM20]. The simplification comes from the systematic use of Iwasawa's splitting (like in the previous lemma), which is borrowed from Section 3.1 of [Zhu20].

4.4. Connected components over $\overline{\mathbb{Z}}[\frac{1}{p}]$ and spaces of tame 1-cocycles. We keep the same notation Φ and $X_{H,\phi} = \underline{\mathrm{Hom}}(W_F^0/P, H)_\phi$ from above and we base change everything to $\overline{\mathbb{Z}}[\frac{1}{p}]$ for convenience. As in subsection 3.5, we can use the morphism

$$X_{H,\phi} = \underline{\mathrm{Hom}}(W_F^0, H)_\phi = \underline{\mathrm{Hom}}(W_F^0, H_\phi)_\phi \longrightarrow \mathrm{Hom}(W_F^0, \pi_0(H_\phi))_\phi$$

to get a finer decomposition

$$X_{H,\phi} = \bigsqcup_{\bar\varphi} X_{H,\phi,\bar\varphi}$$

with $\bar\varphi$ running in the finite set $\mathrm{Hom}(W_F^0, \pi_0(H_\phi))_\phi$. Again, we say the pair $(\phi,\bar\varphi)$ is *admissible* if $X_{H,\phi,\bar\varphi} \neq \emptyset$.

THEOREM 4.6. *For each admissible pair $(\phi,\bar\varphi)$, the scheme $X_{H,\phi,\bar\varphi}$ is connected. Moreover, there is $\varphi \in X_{H,\phi,\bar\varphi}(\overline{\mathbb{Z}}[\frac{1}{p}])$ such that $\varphi(W_F^0)$ is finite and normalizes a Borel pair of $(H_\phi)^0 = C_H(\phi)^0$. If the center of $(H_\phi)^0$ is smooth over $\overline{\mathbb{Z}}[\frac{1}{p}]$, then one can even find φ such that it preserves a pinning of $(H_\phi)^0$.*

Note that for any $(\phi,\bar\varphi)$ and φ as in the proposition, we get an isomorphism

$$Z^1_{\mathrm{Ad}_\varphi}(W_t^0, (H_\phi)^0) \xrightarrow{\sim} X_{H,\phi,\bar\varphi}, \ \eta \mapsto \eta\cdot\varphi, \tag{4.1}$$

that we can think of as a "reduction to tame parameters".

Proof. We know by Lemma 4.4 that, after choosing a splitting $W_F^0/P_F = W_t^0 \ltimes (P_F/P)$, the restriction map $X_{H,\phi,\bar{\varphi}} \longrightarrow X_{H_\phi,1,\bar{\varphi}}$ is an open and closed embedding. By Cotner's Theorem 3.12, we know that $X_{H_\phi,1,\bar{\varphi}}$ is connected. Since $X_{H,\phi,\bar{\varphi}} \neq \emptyset$, this restricion map is an isomorphism and $X_{H,\phi,\bar{\varphi}}$ is connected. Since $\phi(P_F)$ centralizes $(H_\phi)^0$, the remaining assertions follow from Proposition 3.10. $\square$

Remark 4.7. The existence of φ in the above theorem is Theorem 3.4 of [**DHKM20**], whose proof is arguably much more complicated than the five lines argument above. Again, one simplification here comes from Lemma 4.4, which was overlooked in [**DHKM20**]. The other "simplification" is that we have used Cotner's connectedness result here. Note that the main result of [**Cot24**] rests on reduction to tame parameters, but using the tame version, namely Theorem 4.7 of [**Cot24**], we can avoid any circular argument.

Here is a more self-contained proof of the existence of φ, that avoids Cotner's connectedness theorem (whose proof spreads over 8 pages) but still streamlines the original proof.

Alternative proof of reduction to tame parameters. With the notation of the previous proof, let us follow the first steps of the proof of proposition 3.10 ; let us fix a pinning $\varepsilon = (B, T, (X_\alpha)_{\alpha\in\Delta})$ of $(H_\phi)^0$, and let us consider the normalizer $\mathcal{T} := N_{H_\phi}(B, T)$ of the underlying Borel pair in $(H_\phi)^0$. Note that $\phi(P_F)$ is contained and normal in $\mathcal{T}(\Lambda)$, so that $\mathcal{T} = \mathcal{T}_\phi$. Moreover, we have $\pi_0(\mathcal{T}) = \pi_0(H_\phi)$. Hence the restriction map $X_{\mathcal{T},\phi,\bar{\varphi}} \longrightarrow X_{\mathcal{T},1,\bar{\varphi}}$ is an open and closed embedding, and we know from Example 3.14 that $X_{\mathcal{T},1,\bar{\varphi}}$ is connected. So it only remains to prove that the scheme $X_{\mathcal{T},\phi,\bar{\varphi}}$ is not empty. To this aim, pick an algebraically closed field L such that $X_{H,\phi,\bar{\varphi}}(L) \neq \emptyset$, and pick a $\varphi_0 \in X_{H,\phi,\bar{\varphi}}(L)$. For any $w \in W_F^0$, there is a $h_w \in (H_\phi)^0(L)$ such that $\mathrm{Ad}_{h_w} \circ \mathrm{Ad}_{\varphi_0(w)}$ belongs to $\mathcal{T}(L)$. The h_w's are only defined up to multiplication by $T(L)$, and the obstruction to finding a family $(h_w)_w$ such that $w \mapsto h_w\varphi_0(w)$ is a homomorphism lies in $H^2_{\mathrm{Ad}_{\bar{\varphi}}}(W_t^0, T(L))$ (see [**DHKM20**, Prop. 3.7] for details). As in the proof of Proposition 3.10, this H^2 vanishes because $T(L)$ is a divisible group. In the case where $Z((H_\phi)^0)$ is smooth, one argues similarly with the normalizer $\mathcal{Z}$ of the pinning instead of $\mathcal{T}$. $\square$

Remark 4.8. The connectedness of $X_{H,\phi,\bar{\varphi}}$ was proven in [**DHKM20**] under the hypothesis that $Z(H^0)$ is smooth over $\overline{\mathbb{Z}}[\frac{1}{p}]$. Indeed, Lemma 3.11 of loc. cit. shows that $Z((H_\phi)^0)$ is also smooth over $\overline{\mathbb{Z}}[\frac{1}{p}]$ in this case. Hence, as in the above alternative proof, there is φ so that Ad_φ preserves a pinning of $(H_\phi)^0$, and the connectedness of $X_{H,\phi,\bar{\varphi}}$ then follows from Theorem 3.13.

Remark 4.9. While Cotner was able to by-pass the existence of pinning-preserving cocycles, it is still an open question whether it is always possible to reduce to this case when one starts with H of the form $H = {}^LG$ and stick to $\bar{\varphi}$ compatible with the canonical map $W_F \longrightarrow \pi_0({}^LG)$.

In order to parametrize the connected components of $X_{H,1_P}$, one has to take into account the possible disconnectedness of the centralizers $C_{H^0}(\phi)$. This group scheme acts on the set $\pi_\phi := \mathrm{Hom}(W_F^0, \pi_0(H_\phi))_\phi$ by conjugation. Denote by

$C_{H^0}(\phi)_{\bar\varphi}$ the stabilizer of an element $\bar\varphi$, and by $\bar\pi_\phi$ a set of representatives of orbits of $\pi_0(C_{H^0}(\phi))$ on $\mathrm{Hom}(W_F^0,\pi_0(H_\phi))_\phi$. Then we get a decomposition

$$X_{H,1_P}=\coprod_{\phi\in\Phi}\coprod_{\bar\varphi\in\bar\pi_\phi} H^0\times^{C_{H^0}(\phi)_{\bar\varphi}} X_{H,\phi,\bar\varphi}. \tag{4.2}$$

By the above this gives the decomposition of $X_{H,1_P}$ into connected components.

4.5. Expected mirror properties on the representation theory side. Let G be a quasi-split reductive group over F with L-group ${}^LG=\hat G\rtimes W_F$. Working over $\overline{\mathbb{Z}}[\frac1p]$, we have obtained a decomposition

$$Z^1(W_F^0,\hat G):=\varinjlim_P Z^1(W_F^0/P,\hat G)=\prod_{\phi,\varphi}\hat G\times^{C_{\hat G}(\phi)_{\bar\varphi}} Z^1_{\mathrm{Ad}_\varphi}(W_t^0,C_{\hat G}(\phi)^0) \tag{4.3}$$

where

- ϕ runs over a set $\Phi\subset Z^1(P_F,\hat G(\overline{\mathbb{Z}}[\frac1p]))$ of representatives of $\hat G$-orbits of "wild inertia parameters"
- For ϕ given, φ runs over a finite set of cocycles $W_F\longrightarrow \hat G(\overline{\mathbb{Z}}[\frac1p])$ that extend ϕ and normalize a Borel pair of $C_{\hat G}(\phi)^0$.

Recall we say that the pair (ϕ,φ) is "admissible" if the corresponding summand is non-zero, in which case this summand is now known to be connected. In the case where $\hat G$ has smooth center over $\overline{\mathbb{Z}}[\frac1p]$, e.g. when G_{der} is simply connected, then one can choose φ that normalizes a pinning of $C_{\hat G}(\phi)^0$. Let $G_{\phi,\varphi}$ be the quasi-split tamely ramified group whose dual $\widehat{G_{\phi,\varphi}}$ is $C_{\hat G}(\phi)^0$ endowed with the action Ad_φ. Then φ induces an embedding

$$\zeta_{\phi,\varphi}:\ {}^LG_{\phi,\varphi}=C_{\hat G}(\phi)^0\rtimes W_F\longrightarrow {}^LG=\hat G\rtimes W_F$$

compatible with projections on W_F, i.e. a morphism of L-groups.

These facts suggest the following mirror properties on the abelian category $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac1p]}(G(F))$ of smooth $\overline{\mathbb{Z}}[\frac1p]G(F)$-modules.

(1) There should exist a decomposition as a product of full subcategories

$$\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac1p]}G(F)=\prod_{\phi,\varphi}\mathrm{Rep}_{\phi,\varphi}G(F)$$

characterized by the property that a $\pi\in\mathrm{Irr}_{\mathbb{C}}G(F)$ belongs to $\mathrm{Rep}_{\phi,\varphi}G(F)$ if and only if the Langlands parameter φ_π satisfies $(\varphi_\pi)_{|P_F}\sim\phi$ and $\overline{\varphi_\pi}\sim\overline{\varphi}$. In other words, the idempotent $e_{\phi,\varphi}$ of the Bernstein center $\mathfrak{Z}_{\mathbb{C}}(G(F))$ cut out by these representations should belong to $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac1p]}(G(F))$. Moreover, each factor $\mathrm{Rep}_{\phi,\varphi}G(F)$ should be a "stable block" of $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac1p]}(G(F))$ in the sense that the idempotent $e_{\phi,\varphi}$ should be primitive in the ring $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac1p]}(G(F))\cap\mathfrak{Z}^{st}_{\mathbb{C}}(G(F))$, where $\mathfrak{Z}^{st}_{\mathbb{C}}(G(F))$ denotes the subring of $\mathfrak{Z}_{\mathbb{C}}(G(F))$ of elements that act by the same scalar on any two irreducible representations in the same (putative) L-packet.

(2) When $C_{\hat G}(\phi)$ is connected, there should exist a faithful embedding

$$\zeta^*_{\phi,\varphi}:\mathrm{Rep}_1(G_{\phi,\varphi}(F))\hookrightarrow\mathrm{Rep}_{\phi,\varphi}G(F)$$

that, after choosing Whittaker data on each side, induces the Langlands transfer map associated to $\zeta_{\phi,\varphi}$ on complex irreductible representations,

and takes the depth 0 factor of the Whittaker space of $G_{\phi,\varphi}(F)$ to the (ϕ,φ)-factor of the Whittaker space of G.
In the case where $C_{\hat{G}}(\phi)^0$ is disconnected, the source of the embedding $\zeta^*_{\phi,\varphi}$ should be $\mathrm{Rep}_1(G_{\phi,\varphi}(F)')$ where $G_{\phi,\varphi}(F)'$ is a suitable extension of a quotient of $\pi_0(C_{\hat{G}}(\phi)_{\bar{\varphi}})$ by $G_{\phi,\varphi}(F)$ (see section 2.1.3 and example 2.1.4 of [**Dat17**] for a discussion in a slightly different context).

Moreover, the formalism of the extended LLC suggests the following refinement.

(3) Let $S_{\phi,\varphi}$ denote the abelian group of characters of the finite abelian group $\pi_0(Z(C_{\hat{G}}(\phi)^0)^{\varphi(W_F)}/Z(\hat{G})^{W_F})$. Then there should exist a further decomposition

$$\mathrm{Rep}_{\phi,\varphi} G(F) = \coprod_{\alpha\in S_{\phi,\varphi}} \mathrm{Rep}^{\alpha}_{\phi,\varphi} G(F)$$

characterized by the property that a $\pi \in \mathrm{Irr}_{\mathbb{C}} G(F)$ belongs to $\mathrm{Rep}^{\alpha}_{\phi,\varphi} G(F)$ if and only if the extended Langlands parameter $(\varphi_\pi, \varepsilon_\pi)$ satisfies that $(\varphi_\pi)_{|P_F} \sim \phi$, $\overline{\varphi_\pi} \sim \overline{\varphi}$ and $(\varepsilon_\pi)_{|S_{\phi,\varphi}}$ is α-isotypic (here ε_π is an irreducible representation of the group $S_{\varphi_\pi} := \pi_0(C_{\hat{G}}(\varphi_\pi)/Z(\hat{G})^{W_F})$ and the restriction is taken along the map $S_{\phi,\varphi} \longrightarrow S_{\varphi_\pi}$ induced by the inclusion $Z(C_{\hat{G}}(\phi)^0)^{\varphi(W_F)} \subset C_{\hat{G}}(\varphi_\pi)$). In other words, the idempotent $e^{\alpha}_{\phi,\varphi}$ of the Bernstein center $\mathfrak{Z}_{\mathbb{C}}(G(F))$ cut out by these representations should belong to $\mathfrak{Z}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G(F))$. Moreover, each factor $\mathrm{Rep}^{\alpha}_{\phi,\varphi} G(F)$ should be a *block* of $\mathrm{Rep}_{\overline{\mathbb{Z}}[\frac{1}{p}]}(G(F))$.

(4) When $C_{\hat{G}}(\phi)$ is connected, there should exist an equivalence of categories

$$\mathrm{Rep}_1(G^{\alpha}_{\phi,\varphi}(F)) \xrightarrow{\sim} \mathrm{Rep}^{\alpha}_{\phi,\varphi} G(F)$$

that induces the usual transfer of irreducible representations. Here, $\alpha \in S_{\phi,\varphi}$ is seen as an element of the kernel of the map $H^1(F, G_{\phi,\varphi}) \longrightarrow H^1(F,G)$ deduced from the inclusion $Z(C_{\hat{G}}(\phi)^0)^{\varphi(W_F)} \supset Z(\hat{G})^{W_F}$ via Kottwitz' isomorphism, and $G^{\alpha}_{\phi,\varphi}$ is the associated pure inner form of $G_{\phi,\varphi}$.
In the case where $C_{\hat{G}}(\phi)$ is not connected, the group $G^{\alpha}_{\phi,\varphi}(F)$ should be replaced by a suitable extension of a quotient of $\pi_0(C_{\hat{G}}(\phi)_{\bar{\varphi}})$ by this group.

In items (2) and (4), Rep_1 corresponds to the pair $(1,1)$ and, because of item (1), should be the *depth* 0 *category* of $G^{\alpha}_{\phi,\varphi}(F)$. In other words these predictions are about a functorial process to reduce the representation theory of $G(F)$ to the depth 0 representation theory of auxiliary tamely ramified groups.

Remark 4.10. Even when $\hat{G}$ does not have smooth center, we expect the decomposition to exist, as well as the equivalences of categories, albeit maybe involving some twists. Regarding connectedness, note that $C_{\hat{G}}(\phi)$ is always connected for classical groups if $p > 2$.

Here is what is known so far. The predicted decomposition (1) and (3) are constructed in [**Dat18**] under the assumption that G is "very tame" (i.e. all maximal tori of G are tamely ramified), in which case $C_{\hat{G}}(\phi)$ is connected and is actually a Levi subgroup of $\hat{G}$. The construction is much inspired by Kaletha's construction of supercuspidal packets for this kind of groups [**Kal19**]. In particular, it is compatible with his proposed Langlands correspondence. The main tools are therefore

Yu's theory of generic characters [Yu01] and Fintzen's exhaustion results on Yu's constructions [Fin21]. The fact that the depth 0 category is indecomposable is proved in [DL22] for quasi-split tamely ramified groups. Regarding the existence of equivalences of categories, the only known case is GL_n and it follows from work of Chinello [Chi18] in types theory. However, the compatibility of these equivalences with LLC is far from clear.

REMARK 4.11. These predictions are clearly compatible with the CLLC philosophy, except for our coefficients $\overline{\mathbb{Z}}[\frac{1}{p}]$ and for our insistence on working with abelian categories. More on this in subsection 4.7.

4.6. Further decomposition over $\overline{\mathbb{Z}}_\ell$. We now fix a prime $\ell \neq p$ and work over $\overline{\mathbb{Z}}_\ell$ for simplicity. Again, as long as we fix a depth $P \subset P_F$, everything holds over a sufficiently big extension of $\mathbb{Z}_\ell$. We denote by I_F^ℓ the kernel of the map $I_F \xrightarrow{t_\ell} \mathbb{Z}_\ell(1)$. This is the maximal pro-ℓ'-subgroup of I_F. It will play the role played by P_F so far.

We have seen that the universal 1-cocycle on $Z^1(W_F^0/P, \hat{G})_{\overline{\mathbb{Z}}_\ell}$ has a unique ℓ-adically continuous extension ${}^\ell\varphi_{\hat{G},1_P}$ to W_F. We claim that the restriction of ${}^\ell\varphi_{\hat{G},1_P}$ to I_F^ℓ factors over a finite quotient of I_F^ℓ, depending on P. This would be clear if we had defined Z^1 in terms of condensed parameters. In our setting, we can reduce to the tame setting, evaluate at each geometric generic point, and use the fact that the semisimple part of ${}^\ell\varphi_{\hat{G},1}(s)$ at such a point has finite order, as in the proof of Proposition 3.3. In any case, by restriction, we thus get a morphism

$$Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell} \longrightarrow Z^1(I_F^\ell, \hat{G})_{\overline{\mathbb{Z}}_\ell},$$

where the RHS denotes the colimit of spaces of cocycles for finite quotients of I_F^ℓ. Since such finite quotients have invertible order in $\overline{\mathbb{Z}}_\ell$, we can apply the results of subsection 4.1. So let $\Phi_\ell \subset Z^1(I_F^\ell, \hat{G}(\overline{\mathbb{Z}}_\ell))$ be a set of representatives of $\hat{G}(\overline{\mathbb{Z}}_\ell)$-conjugacy classes. For each $\phi_\ell \in \Phi_\ell$, denote by $Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell,\phi_\ell}$ the closed locus where $({}^\ell\varphi_{\hat{G}})_{|I_F^\ell} = \phi_\ell$. By pull back, we get a decomposition

$$Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell} = \coprod_{\phi_\ell \in \Phi_\ell} \hat{G} \times^{C_{\hat{G}}(\phi_\ell)} Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell,\phi_\ell}.$$

Arguing as in the last subsections, we can refine this decomposition as follows. For each ϕ_ℓ, there is a finite subset $\Sigma_{\phi_\ell} \subset Z^1(W_F^0, \hat{G}(\overline{\mathbb{Z}}_\ell))_{\phi_\ell}$ such that each $\varphi \in \Sigma_{\phi_\ell}$ has finite image and normalizes a Borel pair of $C_{\hat{G}}(\phi)^0$, and we have

$$Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell} = \coprod_{\phi_\ell \in \Phi_\ell} \coprod_{\varphi \in \Sigma_{\phi_\ell}} \hat{G} \times^{C_{\hat{G}}(\phi_\ell)_\varphi} Z^1_{\mathrm{Ad}_\varphi}(W_t^0, C_{\hat{G}}(\phi_\ell)^0)_1 \tag{4.4}$$

with notation similar to the end of subsection 4.4. Here, observe that the action Ad_φ of W_F on $C_{\hat{G}}(\phi)^0$ is "tamely ℓ-ramified" in the sense that it is tamely ramified and inertia acts through a ℓ-group quotient. The index 1 means we are looking at the closed locus associated to $\phi_\ell = 1_{I_F^\ell}$. Again, in the case where the center $Z(\hat{G})$ is smooth over $\mathbb{Z}_\ell$, we can choose Σ_{ϕ_ℓ} such that each $\varphi \in \Sigma_{\phi_\ell}$ normalizes a pinning of $C_{\hat{G}}(\phi)^0$. So, informally, this decomposition shows how $Z^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}_\ell}$ is built out of spaces of tamely ℓ-ramified parameters for tamely ℓ-ramified groups. The following theorem, to be proved in Section 5.5, shows that the last decomposition above is the decomposition in connected components.

Theorem 4.12. *Assume $\hat{G}$ is endowed with a Borel pair-preserving finite tamely ℓ-ramified action. Then $Z^1(W_t^0, \hat{G})_{\overline{\mathbb{Z}}_\ell,1}$ is connected.*

4.7. Mirror properties over $\overline{\mathbb{Z}}_\ell$ coefficients. The above pattern again suggests a corresponding mirror pattern about the category $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G(F))$ for a quasi-split reductive group G over F, along the same items (1)–(4) as in subsection 4.5. We won't repeat them explicitly but we mention an important difference in (3) : we shouldn't expect that the factor category $\mathrm{Rep}^{\alpha}_{\overline{\mathbb{Z}}_\ell,(\phi_\ell,\varphi)}(G(F))$ be indecomposable. This has to do with the existence of "unstable" primitive idempotents in the Bernstein center, meaning that these idempotents kill a proper subset of representations in a certain L-packet. As an example, think of $G = \mathrm{GSp}_4$ and $\phi_\ell = 1$ (hence also $\varphi = 1$ and $\alpha = 1$). For large ℓ, the associated factor $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell,1}(G(F))$ consists of unipotent representations and decomposes as in characteristic 0, as the sum of the Iwahori block and the block generated by the unipotent cuspidal representation.

More generally for G tamely ℓ-ramified, the factor $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell,1}(G(F))$ associated to $\phi_\ell = 1$ is usually called the "unipotent ℓ-factor". Although it may not be indecomposable, as in the above example, the pattern is about showing how $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell} G(F)$ decomposes as a product of full subcategories, each of which is equivalent to the unipotent ℓ-factor of some auxiliary tamely ℓ-ramified group.

Here is what is known at the moment : the expected decomposition (1) of the category $\mathrm{Rep}_{\overline{\mathbb{Z}}_\ell}(G(F))$ follows from Theorem IX.5.2 of [**FS21**], where Fargues and Scholze construct a map from the ring of $\hat{G}$-invariant functions on $Z^1(W_F^0, \hat{G})_{\mathbb{Z}_\ell}$ to the center of the category $\mathrm{Rep}_{\mathbb{Z}_\ell}(G(F))$. Also, their Conjecture X.1.4 together with the decomposition (4.4) certainly would imply the existence of functors as expected, at least at the derived level, and their compatibility with Langlands functoriality would be "built-in".

Before these developments, the expected decompositions (1) and (3) had been constructed quite explicitly by Lanard [**Lan18**] [**Lan21**] in depth 0, compatibly with all known correspondences (DeBacker-Reeder, or Arthur for classical groups). The expected equivalences of categories have also been constructed for $G = GL_n$ (still in depth 0) in [**Dat18**], although their compatibility with LLC is not known.

5. Coarse moduli spaces and applications

In this section, we study the coarse moduli space $Z^1(W_F^0/P, \hat{G}) /\!\!/ \hat{G}$, i.e. the quotient in the category of affine schemes, also called GIT quotient. Concretely, it is the spectrum of invariant functions, i.e. if $Z^1(W_F^0/P, \hat{G}) = \mathrm{Spec}\, R_{\hat{G},1_P}$ then $Z^1(W_F^0/P, \hat{G}) /\!\!/ \hat{G} = \mathrm{Spec}\,(R_{\hat{G},1_P})^{\hat{G}}$.

5.1. Geometric invariant theory. Let Λ be a coefficient ring as before, and let $X = \mathrm{Spec}\, R$ be a Λ-scheme endowed with an action of a reductive group Λ-scheme H. We put $X /\!\!/ H := \mathrm{Spec}\, R^H$. Here, the H-invariants are in the sense of representations of group schemes : the action corresponds to a comodule structure $R \xrightarrow{\rho} R \otimes_\Lambda \mathcal{O}(H)$ and R^H is the equalizer of $\mathrm{Id} \otimes 1$ and ρ. Clearly $X /\!\!/ H$ has the expected universal property of a quotient in the category of affine Λ-schemes.

Here are some standard facts on these objects (recall H is reductive here).

(1) R^H is a finitely generated Λ-algebra (due to Mumford if Λ is a field of characteristic 0, to Haboush in positive characteristic, to Seshadri and Thomason [**Tho87**] if $\Lambda = \mathbb{Z}[\frac{1}{p}]$).

(2) The formation of $X /\!\!/ H$ commutes with flat base change, but not with arbitrary base change in general. However, if L is a field over Λ, the canonical map $X_L /\!\!/ H_L \longrightarrow (X /\!\!/ H)_L$ is a universal homeomorphism [Alp14].
(3) Let $\Lambda = L$ be an algebraically closed field and let $\pi : X \longrightarrow X /\!\!/ H$ be the canonical map. For any $x \in (X /\!\!/ H)(L)$, the fiber $\pi^{-1}(x) \subset X(L)$ contains a unique closed $H(L)$-orbit. This sets up a bijection between closed orbits in $X(L)$ and L-points of $X /\!\!/ H$.
(4) The Hilbert-Mumford criterion states that the orbit of a point $x \in X(L)$ is closed if, and only if, for any cocharacter $\lambda : \mathbb{G}_m \longrightarrow H$ such that the morphism $t \mapsto \lambda(t) \cdot x$ extends to $\mathbb{A}^1$, the value at $t = 0$ of this extension (usually denoted by $\lim_{t \mapsto 0} \lambda(t) \cdot v$) belongs to the orbit of x.
(5) Suppose X is reduced and Y is a H-stable closed subscheme of X such that $Y(L)$ contains all closed $H(L)$-orbits of $X(L)$ for any algebraically closed field over Λ. Then the natural map $Y /\!\!/ H \longrightarrow X /\!\!/ H$ is a universal homeomorphism and is an isomorphism after tensoring with $\mathbb{Q}$. [This follows from [Alp14] as explained in [DHKM20, Prop. 4.17]].

Example 5.1 (Chevalley-Steinberg). Let $X = H$ and let H act on itself by conjugation. Suppose H is split and let T be a maximal torus with normalizer N and Weyl group Ω. Then the Chevalley-Steinberg theorem says that the map $T \subset H$ induces an isomorphism

$$T /\!\!/ N = T /\!\!/ \Omega \xrightarrow{\sim} H /\!\!/ H.$$

The fact that it is a bijection on L-points is easy to deduce from the Hilbert-Mumford criteron in point (4) above. However, showing it is true over any ring Λ requires more work on the algebras of invariants and involves the combinatorics of root systems. Also, this is an example where taking invariants turns out to commute with any base change.

Example 5.2 (Twisted Chevalley-Steinberg). Let $X = H$ and H act on itself by θ-conjugation $(g, h) \mapsto gh\theta(g)^{-1}$ for some automorphism θ. Equivalently, we are looking at ordinary conjugation of H on the coset $H \rtimes \theta$. Suppose θ normalizes a Borel pair (B, T) of H (over an algebraically closed field, one can always reduce to this case). In particular, θ stabilizes N and acts on $\Omega = N/T$. Let $N_\theta \subset H$ be the stabilizer of $T \rtimes \theta$ in $H \rtimes \theta$. This is also the inverse image in N of the fixed points subgroup Ω^θ. The twisted Chevalley-Steinberg theorem says that the map $T \subset H$ induces an isomorphism

$$T \rtimes \theta /\!\!/ N_\theta = T_\theta /\!\!/ \Omega^\theta \xrightarrow{\sim} H \rtimes \theta /\!\!/ H.$$

Here $T_\theta = T/(\mathrm{Id} - \theta)T$ is the θ-coinvariant quotient torus of T.

5.2. Parabolic subgroups. As in the previous sections, H will again denote a "generalized reductive" group scheme and all actions will utimately come from the conjugation action of H^0 on H. Given a cocharacter $\lambda : \mathbb{G}_m \longrightarrow H^0$ defined over Λ, there is a smooth closed subgroup scheme $P_\lambda \subset H$ whose points are those $x \in H$ for which the orbit morphism $\mathbb{G}_m \longrightarrow H$, $z \mapsto \lambda(z)x\lambda(z)^{-1}$ extends to $\mathbb{A}^1$. Moreover, there is a projection morphism $P_\lambda \xrightarrow{\pi_\lambda} M_\lambda := C_H(\lambda)$ that takes x to $\lim_{t \mapsto 0} \lambda(t)x\lambda(t)^{-1}$. Such a subgroup scheme P_λ, resp. M_λ, is called a R-parabolic subgroup, resp. a R-Levi subgroup, of H. Their relevance stems from the Hilbert-Mumford criterion.

REMARK 5.3. The intersection $P_\lambda \cap H^0$ is a parabolic subgroup scheme of H^0 and $M_\lambda \cap H^0$ is a Levi component of $P_\lambda \cap H^0$. Moreover, it is known that any parabolic subgroup or Levi subgroup of H^0 is of this form. So the construction above offers a notion of parabolic/Levi subgroups for disconnected reductive group schemes, which is the one used e.g. by Richardson in [**Ric88**]. However, other notions appear in the literature. For example, in the case of an L-group $H = \hat{G} \rtimes W$, Borel in [**Bor79**] defines a parabolic subgroup of H to be the normalizer of a parabolic subgroup of H^0 *that projects onto* $\pi_0(H) = W$. In [**DM94**], Digne and Michel define a parabolic subgroup of H to be the normalizer of a parabolic subgroup of H^0. Over an algebraically closed field, Borel-parabolic are DM-parabolic, which are Richardson-parabolic ("R-parabolic" for short), but both converse implications may fail.

EXAMPLE 5.4. The normalizer of a Borel subgroup in H is a parabolic subgroup in the R, B, or DM sense. Similarly, the normalizer of a Borel pair is a Levi subgroup in each sense.

5.3. Semisimple homomorphisms. Suppose Γ is an abstract finitely generated group and H is a generalized reductive group scheme over a ring of coefficients Λ. We are interested in the action of H^0 by conjugation on $X := \underline{\mathrm{Hom}}(\Gamma, H)$.

THEOREM 5.5. *Let L be an algebraically closed field over Λ and $\varphi : \Gamma \longrightarrow H(L)$ be a homomorphism. The following properties are equivalent :*

1. *The orbit of φ under $H^0(L)$ is closed in $X(L)$.*
2. *For any R-parabolic subgroup P such that $\varphi(\Gamma) \subset P(L)$, there is an R-Levi component M of P such that $\varphi(\Gamma) \subset M(L)$.*
3. *For any maximal torus S of the centralizer $C_H(\varphi)$, the image $\varphi(\Gamma)$ is not contained in a proper R-parabolic subgroup of $C_H(S)$.*

If these properties hold true, we will say that φ is semisimple. *In this case, the connected centralizer $C_H(\varphi)^0$ and the connected normalizer $N_H(\varphi(\Gamma))^0$ are reductive.*

PROOF. Property (2) means that $\varphi(\Gamma)$ is *H-completely reducible* in the sense of Serre. By [**BMR05**, Thm. 3.1 and §6], this is equivalent to property (3), which means that the Zariski closure $\overline{\varphi(\Gamma)}$ is a *strongly reductive* subgroup of H in the sense of [**Ric88**]. Now, pick generators $\gamma_1, \cdots, \gamma_n$ of Γ, so that X is a closed subscheme of H^n. By [**Ric88**, Thm. 16.7], property (3) is equivalent to the H-orbit $\mathcal{O}$ of the element $(\varphi(\gamma_1), \cdots \varphi(\gamma_n)) \in H(L)^n$ for the diagonal action of H by conjugation being closed in H^n. But the H-orbit of φ is the intersection $\mathcal{O} \cap X$, which is thus closed in X. The last statement is Proposition 3.12 of [**BMR05**]. $\square$

EXAMPLE 5.6. If Γ is finite and has order invertible in L, then any $\varphi : \Gamma \longrightarrow H(L)$ is semisimple, since its H^0-orbit is closed (and also open, as we saw).

EXAMPLE 5.7. Let $\Gamma = s^{\mathbb{Z}}$ and $\sigma := \varphi(s)$. If σ is a semisimple element of $H(L)$, then φ is semisimple in the above sense. The converse is true if $\pi_0(H)$ has order invertible in L, but may fail in general, e.g. if $H = \mathbb{Z}/\ell\mathbb{Z}$ and L has characteristic ℓ. In general, if φ is semisimple, then any Borel subgroup of H^0 normalized by σ contains a maximal torus normalized by σ (note that, by a result of Steinberg, any element of $H(L)$ normalizes a Borel subgroup of H^0).

EXAMPLE 5.8. If $H = GL_n$, then φ is semisimple as a homomorphism if and only if the corresponding n-dimensional representation is semisimple in the usual sense.

It is well known that the restriction of a semisimple representation to a normal subgroup remains semisimple. The following result is a generalization.

THEOREM 5.9. [BMR05, Thm. 3.10] *Suppose Δ is a normal subgroup of Γ and $\varphi : \Gamma \longrightarrow H(L)$ is semisimple. Then $\varphi_{|\Delta}$ is semisimple.*

The following corollary will be useful in our context.

COROLLARY 5.10. *Suppose $\Gamma = \Delta \rtimes \Sigma$, and let $\varphi : \Gamma \longrightarrow H(L)$ be a homomorphism such that $\varphi(\Delta)$ is finite. Put $H_{\varphi_{|\Delta}} := N_H(\varphi(\Delta))$. Then the following are equivalent :*

(1) *φ is semisimple*
(2) *$\varphi_{|\Delta}$ is semisimple and $(\varphi_{|\Sigma})^{|H_{\varphi_{|\Delta}}} : \Sigma \longrightarrow H_{\varphi_{|\Delta}}(L)$ is semisimple.*

Beware that, when φ is semisimple, $\varphi_{|\Sigma}$ need not be semisimple as a homomorphism $\Sigma \longrightarrow H(L)$. For example, the restriction of the two-dimensional irreducible $\overline{\mathbb{F}}_2$-representation of $S_3 = \mathbb{Z}/3\mathbb{Z} \rtimes \{\pm 1\}$ to $\{\pm 1\}$ is not semisimple.

PROOF. To lighten the notation, we will simply write H for $H(L)$ and similarly for all other closed algebraic subgroups of H_L. Also, let us write simply $\phi := \varphi_{|\Delta}$. Our hypothesis that $\varphi(\Delta)$ is finite implies that $(H_\phi)^0 = C_{H^0}(\phi)^0$. Let us put

$$\begin{cases} \mathrm{Hom}(\Gamma, H)_\phi := \{\theta \in \mathrm{Hom}(\Gamma, H),\, \theta_{|\Delta} = \phi\} \\ \mathrm{Hom}(\Gamma, H)_{[\phi]} := \{\theta \in \mathrm{Hom}(\Gamma, H),\, \theta_{|\Delta} \text{ is } H^0\text{-conjugate to } \phi\} \end{cases}$$

Denoting by $\rho : \mathrm{Hom}(\Gamma, H) \longrightarrow \mathrm{Hom}(\Delta, H)$ the restriction map and by $H^0 \cdot \phi$ the H^0-orbit of ϕ, we then have $\mathrm{Hom}(\Gamma, H)_{[\phi]} = \rho^{-1}(H^0 \cdot \phi)$. In particular, we see that

$$\phi \text{ semisimple} \Rightarrow \mathrm{Hom}(\Gamma, H)_{[\phi]} \text{ closed in } \mathrm{Hom}(\Gamma, H).$$

On the other hand, we have $\mathrm{Hom}(\Gamma, H)_\phi = \mathrm{Hom}(\Gamma, H_\phi)_\phi$, and the restriction map $\mathrm{Hom}(\Gamma, H_\phi)_\phi \longrightarrow \mathrm{Hom}(\Sigma, H_\phi)$ is an open and closed immersion (as in Lemma 4.4). So we see that $(\varphi_{|\Sigma})^{|H_\phi}$ is semisimple if and only if $\varphi^{|H_\phi}$ is semisimple, or equivalently, the orbit $C_{H^0}(\phi) \cdot \varphi$ is closed in $\mathrm{Hom}(\Gamma, H)_\phi$. Using that $\mathrm{Hom}(\Gamma, H)_{[\phi]} = H^0 \times^{C_{H^0}(\phi)} \mathrm{Hom}(\Gamma, H)_\phi$, we infer that

$$(\varphi_{|\Sigma})^{|H_\phi} \text{ semisimple } \Leftrightarrow H^0 \cdot \varphi \text{ closed in } \mathrm{Hom}(\Gamma, H)_{[\phi]}.$$

At this point, we have proved (2)⇒(1). The other implication follows from the last theorem. □

We now turn to our case of interest.

PROPOSITION 5.11. *Let $\Gamma = W_t^0 = \langle \mathrm{Fr}, s\rangle$ and let $\varphi : W_t^0 \longrightarrow H(L)$ be semisimple. Put $F = \varphi(\mathrm{Fr})$ and $\sigma = \varphi(s)$ as before. Then the following holds :*

(1) *σ normalizes a Borel pair of H^0, and F normalizes a Borel pair of $C_H(\sigma)^0$.*
(2) *σ has finite order bounded independently of L.*

PROOF. (1) Follows from the previous corollary applied to $\Delta = s^{\mathbb{Z}[1/p]}$ and $\Sigma = \mathrm{Fr}^{\mathbb{Z}}$, and Example 5.7. For (2), observe that, since σ normalizes a Borel pair of H^0, the element $\sigma^{|\pi_0(H)|}$ belongs to the torus of this Borel pair, i.e. is semisimple. But in the proof of Proposition 3.3, we have already worked out a bound, depending only on q and H, for the order of a semisimple element conjugate to its q^{th}-power. □

Note that item (2) implies in particular that any semisimple $\varphi : W_t^0 \longrightarrow H(L)$ extends uniquely continuously to W_t. More precisely and more generally, we have :

COROLLARY 5.12. *For an open subgroup $P \subset P_F$ distinguished in W_F, there is an open subgroup $I \subset I_F$ such that W_F^0/P surjects onto W_F/I and, for any algebraically closed field L, every semisimple homomorphism $\varphi : W_F^0/P \longrightarrow H(L)$ factors through W_F/I. As a consequence, the morphism induced by restriction*

$$\underline{\mathrm{Hom}}(W_F/I, H) /\!\!/ H^0 \longrightarrow \underline{\mathrm{Hom}}(W_F^0/P, H) /\!\!/ H^0$$

is a universal homeomorphism and is an isomorphism after extending scalars to $\mathbb{Q}$.

PROOF. This follows from the process of reduction to the tame case and from (5) in 5.1. $\square$

COROLLARY 5.13. *If L has characteristic 0, a $\varphi : W_F^0 \longrightarrow H(L)$ is semisimple if and only if $\varphi(I_F^0)$ is finite and $\varphi(\mathrm{Fr})$ is semisimple.*

It follows in particular that $\mathbb{C}$-points of $Z^1(W_F^0, \hat{G}) /\!\!/ \hat{G}$ are the "infinitesimal characters" of usual local Langlands parameters.

5.4. Application to connectedness over $\overline{\mathbb{Z}}[\frac{1}{p}]$. Here, coefficients are $\Lambda = \overline{\mathbb{Z}}[\frac{1}{p}]$ and we shall sketch a proof of Theorem 3.13 different from that in [DHKM20, §4.6]. Let us more generally start with a reductive $\hat{G}$ over $\mathbb{Z}[\frac{1}{p}]$ endowed with a finite action of W_t^0 that preserves a Borel pair $(\hat{B}, \hat{T})$.

We first explain why $Z^1 := Z^1(W_t^0, \hat{G})$ is connected, under the simplifying assumption that the action of W_t^0 on $\hat{G}$ is unramified. The first principle we use is that the map

$$Z^1(W_t^0, \hat{G}) \longrightarrow Z^1(W_t^0, \hat{G}) /\!\!/ \hat{G}$$

induces a bijection on π_0, since $\hat{G}$ is connected. So, it suffices to prove that $Z^1(W_t^0, \hat{G}) /\!\!/ \hat{G}$ is connected. To this aim, we introduce the closed subschemes A and B of $Z^1(W_t^0, \hat{G})$ defined by :

$$A := \Big((N_{\hat{G}}(\hat{T}) \rtimes \mathrm{Fr}) \times \hat{T}\Big) \cap Z^1(W_t^0, \hat{G}) \subset (\hat{G} \rtimes \mathrm{Fr}) \times \hat{G}$$

$$B := (\hat{G} \rtimes \mathrm{Fr}) \times \{1\} \subset Z^1(W_t^0, \hat{G}).$$

Proposition 5.11 (1) implies that, for any algebraically closed field L over Λ, each closed $\hat{G}(L)$-orbit of $Z^1(W_t^0, \hat{G}(L))$ has a representative in $A(L)$. Therefore, the composed map $A \longrightarrow Z^1 /\!\!/ \hat{G}$ is surjective, and it will suffice to prove that all connected components of A are mapped into the same component of $Z^1 /\!\!/ \hat{G}$. To this aim, observe that A is a closed subgroup $N_{\hat{G}}(\hat{T})$-scheme of $N_{\hat{G}}(\hat{T}) \times \hat{T}$. Namely, A is the kernel of the isogeny $t \mapsto n\mathrm{Fr}(t)n^{-1}t^{-q}$ over $N_{\hat{G}}(\hat{T})$. In particular, A is a finite flat diagonalizable group scheme over $N_{\hat{G}}(\hat{T})$ of order prime to p. Since p is the only invertible prime on any component of $N_{\hat{G}}(\hat{T})$, we infer that $\pi_0(A) \xrightarrow{\sim} \pi_0(N_{\hat{G}}(\hat{T})) = \Omega$ and that all components of A intersect $A \cap B = (N_{\hat{G}}(\hat{T}) \rtimes \mathrm{Fr}) \times \{1\}$. But B is connected, so it is mapped to a certain connected component C of $Z^1 /\!\!/ \hat{G}$. It follows that all components of A are also mapped into C and, therefore, $Z^1 /\!\!/ \hat{G}$ is connected.

We now return to the general case where the action of s on $\hat{G}$ is non-trivial (but still normalizes $(\hat{B},\hat{T})$). We can modify the definition of A and B as follows :

$$A := (N_s \rtimes \mathrm{Fr}) \times (\hat{T}^{s,0} \rtimes s) \cap Z^1(W_t^0, \hat{G}),$$

$$B := (\hat{G}^s \rtimes \mathrm{Fr}) \times \{1 \rtimes s\} \subset Z^1(W_t^0, \hat{G}).$$

Here $\hat{T}^{s,0}$ denotes the maximal torus in the fixed point subgroup scheme $\hat{T}^s$, and $N_s = \{n \in N_{\hat{G}}(\hat{T}), n.s^q(n)^{-1} \in \hat{T}^{s,0}\}$. It follows again from Proposition 5.11 (1) that A surjects onto $Z^1 /\!\!/ \hat{G}$ (see Lemma 2.9 in [**DHKM24**] for details). In this case, Lemma 2.8 of [**DHKM24**] shows that A is the preimage of the section $N_s \longrightarrow T^{s,0}$, $n \mapsto s^q(n)n^{-1}$ by the endo-isogeny $t \mapsto n\mathrm{Fr}(t)n^{-1}t^{-q}$ of the N_s-torus $N_s \times T^{s,0}$. So, it is still a torsor over a finite flat group N_s-scheme of prime-to-p order. Moreover, Lemma 2.7 of *loc.cit.* shows that $\pi_0(N_s) = \Omega^s$ and that p is the only invertible prime on the components of N_s. Hence, as above, it follows that $\pi_0(A) \xrightarrow{\sim} \pi_0(N_s) = \Omega^s$. Thus, in order to finish the argument along the same lines as in the simplified setting above, it would be enough to know that

(1) the fixed point subgroup scheme $\hat{G}^s$ is connected
(2) an element of Ω^s can be lifted to a point of $A \cap B = N_{\hat{G}}(\hat{T})^s$.

Property (1) holds true if $\hat{G}$ is simply connected thanks to a well known result of Steinberg. Moreover, lemmas 4.27 and 4.28 of [**DHKM20**] explain how to reduce to this case via some variant of Langlands' "z-extensions" trick. Property (2) is just not true in general, but is true if s stabilizes a pinning, by [**DM94**, Thm. 1.15 iii)'] since then s is "quasi-central". So we have proved Theorem 3.13.

Remark 5.14. Actually, for the above argument, it is sufficient to be able to lift an element of Ω^s to an $\overline{\mathbb{F}}_\ell$-point of $A \cap B$ for some prime ℓ, and for this it suffices that s stabilizes a pinning of $\hat{G}_{\overline{\mathbb{F}}_\ell}$. One observation of Cotner is that, when the center of $\hat{G}$ becomes smooth after inverting ℓ, one can reduce to this setting. This is enough to prove his Theorem 3.12 when $\hat{G}$ has no component of type A. On the other hand, the case of simple groups of type A requires a detailed study and a more involved argument.

5.5. Connectedness over $\overline{\mathbb{Z}}_\ell$. Here, coefficients are $\overline{\mathbb{Z}}_\ell$ (and soon $\overline{\mathbb{F}}_\ell$) and we shall prove Theorem 4.12. So we keep the notation of the last subsection and we assume further that the action of s on $\hat{G}$ has order a power of ℓ. Since all components of $Z^1(W_t^0, \hat{G})_{\overline{\mathbb{Z}}_\ell,1}$ are faithfully flat over $\mathbb{Z}_\ell$, it suffices to prove that the special fiber $Z^1_{\overline{\mathbb{F}}_\ell,1} = Z^1(W_t^0, \hat{G})_{\overline{\mathbb{F}}_\ell,1}$ is connected or, equivalently, that the quotient $Z^1_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$ is connected. As in the previous subsection, we have a surjective map $A_{\overline{\mathbb{F}}_\ell} \longrightarrow Z^1_{\overline{\mathbb{F}}_\ell} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$, and it induces a surjective map $A_{\overline{\mathbb{F}}_\ell,1} \longrightarrow Z^1_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$ where the index 1 means the locus where σ has ℓ-power order. But $\hat{T}^{s,0}$ is a torus, so $\hat{T}^{s,0}(\overline{\mathbb{F}}_\ell)$ has no ℓ-subgroup, hence $A_{\overline{\mathbb{F}}_\ell,1} \subset B_{\overline{\mathbb{F}}_\ell}$. Moreover, since s has ℓ-power order, $(\hat{G}_{\overline{\mathbb{F}}_\ell})^s$ is always connected regardless of its simple-connectedness [**DM94**, Cor. 1.33]. Hence $B_{\overline{\mathbb{F}}_\ell} \simeq \hat{G}^s_{\overline{\mathbb{F}}_\ell} \rtimes \mathrm{Fr}$ is connected, and so is $Z^1_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$.

Actually, we can push this argument a bit further to get an explicit description of $Z^1_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$ up to homeomorphism. Namely, since $B_{\overline{\mathbb{F}}_\ell}$ is stable under $\hat{G}^s_{\overline{\mathbb{F}}_\ell}$, the inclusion $B \subset Z^1$ induces a morphism

$$B_{\overline{\mathbb{F}}_\ell} /\!\!/ \hat{G}^s_{\overline{\mathbb{F}}_\ell} = (\hat{G}^s_{\overline{\mathbb{F}}_\ell} \rtimes \mathrm{Fr}) /\!\!/ \hat{G}^s_{\overline{\mathbb{F}}_\ell} = Z^1(\langle \mathrm{Fr}\rangle, \hat{G}^s_{\overline{\mathbb{F}}_\ell}) /\!\!/ \hat{G}^s_{\overline{\mathbb{F}}_\ell} \longrightarrow Z^1(W_t^0, \hat{G})_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$$

We already know that this morphism is surjective, and it is also injective since an element of $\hat{G}(\overline{\mathbb{F}}_\ell)$ that conjugates an element of $B(\overline{\mathbb{F}}_\ell)$ to another one has to be in $\hat{G}(\overline{\mathbb{F}}_\ell)^s$. Using the twisted Chevalley-Steinberg theorem, we get a bijective morphism

$$(\hat{T}^s_{\overline{\mathbb{F}}_\ell})_{\mathrm{Fr}} /\!\!/ \Omega^{s,\mathrm{Fr}} = (\hat{T}^s_{\overline{\mathbb{F}}_\ell} \rtimes \mathrm{Fr}) /\!\!/ (N^s_{\overline{\mathbb{F}}_\ell})_{\mathrm{Fr}} \longrightarrow Z^1(W_t^0, \hat{G})_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell} \tag{5.1}$$

This morphism is also finite, as follows from Theorem 5.17 or Theorem 5.18 below, so it is a homeomorphism (see the proof of [DHKM20, Prop. 4.19 (2)] for an alternative argument).

5.6. Description of $Z^1(W_F^0, \hat{G}) /\!\!/ \hat{G}$ over $\overline{\mathbb{F}}_\ell$, up to homeomorphism. For a reductivegroup G over F, Equation (4.4) yields a decomposition

$$(Z^1(W_F^0, \hat{G}) /\!\!/ \hat{G})_{\overline{\mathbb{Z}}_\ell} = \coprod_{\phi_\ell \in \Phi_\ell} \coprod_{\varphi \in \Sigma_{\phi_\ell}} Z^1_{\mathrm{Ad}_\varphi}(W_t^0, C_{\hat{G}}(\phi_\ell)^0)_1 /\!\!/ C_{\hat{G}}(\phi_\ell)_\varphi.$$

This reduces us to describing $Z^1(W_t^0, \hat{G})_{\overline{\mathbb{F}}_\ell,1} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}$ for a tamely ℓ-ramified finite action of W_t that preserves a Borel pair of $\hat{G}$, which is exactly what we did right above. As explained in [DHKM20, Cor. 4.21], a possibly nicer way to index the components is to consider a set $\Psi(\overline{\mathbb{F}}_\ell)$ of representatives of $\hat{G}(\overline{\mathbb{F}}_\ell)$-conjugacy classes of pairs (ϕ, β) consisting of

- a *semisimple* continuous 1-cocycle $\phi : I_F \longrightarrow \hat{G}(\overline{\mathbb{F}}_\ell)$, and
- an element $\beta \in \mathrm{Trans}_{\hat{G}\rtimes\mathrm{Fr}}(\phi, {}^{\mathrm{Fr}}\phi)/C_{\hat{G}}(\phi)^0$.

Here Fr denotes a lift of Frobenius in W_F. Note that for any lift $\tilde{\beta}$ of β in $\mathrm{Trans}_{\hat{G}\rtimes\mathrm{Fr}}(\phi, {}^{\mathrm{Fr}}\phi)$, there is a unique continuous 1-cocycle $\varphi_{\tilde{\beta}} : W_F \longrightarrow \hat{G}(\overline{\mathbb{F}}_\ell)$ that extends ϕ and such that ${}^L\varphi_{\tilde{\beta}}(\mathrm{Fr}) = \tilde{\beta}$.

For each pair (ϕ, β) in $\Psi(\overline{\mathbb{F}}_\ell)$, choose a Borel pair $(\hat{B}_\phi, \hat{T}_\phi)$ in $C_{\hat{G}}(\phi)^0$ and a lift $\tilde{\beta}$ that normalizes this Borel pair. Then we get a map $\hat{T}_\phi \longrightarrow Z^1(W_F^0, \hat{G})_{\overline{\mathbb{F}}_\ell}$, $t \mapsto \varphi_{t\tilde{\beta}}$. Using (5.1) and decomposition (4.4), we get :

Theorem 5.15. *The collection of all maps described above induces an homeomorphism*

$$\coprod_{(\phi,\beta)\in\Psi(\overline{\mathbb{F}}_\ell)} (\hat{T}_\phi)_\beta /\!\!/ (\Omega_\phi)^\beta \xrightarrow{\sim} Z^1(W_F^0, \hat{G})_{\overline{\mathbb{F}}_\ell} /\!\!/ \hat{G}_{\overline{\mathbb{F}}_\ell}.$$

In this statement, $(\hat{T}_\phi)_\beta$ denotes the $\tilde{\beta}$-coinvariants of $\hat{T}_\phi$ (which do not depend on the choice of $\tilde{\beta}$), and Ω_ϕ denotes the Weyl group of $\hat{T}_\phi$ in the (possibly non-connected) full centralizer $C_{\hat{G}}(\phi)$.

Remark 5.16. In its current state, representation theory is still unable to describe the Bernstein center of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(G(F))$. However, the Fargues-Scholze morphism [FS21, Thm. IX.5.2] together with the above topological description suggest that the reduced Bernstein center of $\mathrm{Rep}_{\overline{\mathbb{F}}_\ell}(G(F))$ has quite a similar description as the classical Bernstein center of of $\mathrm{Rep}_{\mathbb{C}}(G(F))$

5.7. A finiteness property for Hom schemes. Suppose Γ is a finitely generated discrete group and H is a "generalized" reductive group scheme over a noetherian base S (in the sense of 3.1), containing a closed generalized reductive subgroup scheme H' over S. The closed immersion $\underline{\mathrm{Hom}}(\Gamma, H') \hookrightarrow \underline{\mathrm{Hom}}(\Gamma, H)$ induces a morphism $\underline{\mathrm{Hom}}(\Gamma, H') /\!\!/ H' \longrightarrow \underline{\mathrm{Hom}}(\Gamma, H) /\!\!/ H$, that will rarely be a closed immersion. However, Cotner recently proved the following theorem, solving a conjecture made in a previous version of these notes.

THEOREM 5.17 ([**Cot23**]). *The map* $\underline{\mathrm{Hom}}(\Gamma, H') /\!\!/ H' \longrightarrow \underline{\mathrm{Hom}}(\Gamma, H) /\!\!/ H$ *is a finite morphism.*

For example, the case where $\Gamma = \mathbb{Z}$ follows from the (possibly twisted) Steinberg-Chevalley theorem. Moreover, when S is an algebraically closed field, this statement was essentially proved by Vinberg [**Vin96**] in characteristic 0, and Martin [**Mar03**] in positive characteristics.

Our case of interest in these notes is when $S = \mathrm{Spec}(\overline{\mathbb{Z}}[\frac{1}{p}])$ and $\Gamma = W_F^0/P$ where P is an open subgroup of P_F that is distinguished in W_F. This case was first proved in [**DHKM24**], as a consequence of the following result, which may be of independent interest, and the case where $\Gamma = \mathbb{Z}$ (twisted Steinberg-Chevalley).

THEOREM 5.18. *Let* Fr *be any lift of Frobenius in* W_F^0/P*. The evaluation map* $\varphi \mapsto \varphi(\mathrm{Fr})$ *induces a* finite *morphism*

$$\underline{\mathrm{Hom}}(W_F^0/P, H) /\!\!/ H^0 \longrightarrow H /\!\!/ H^0.$$

PROOF. We sketch a proof and refer to [**DHKM24**, Section 2] for more details. Using decomposition (4.3) we can reduce to showing that the evaluation map $\varphi \mapsto {}^L\varphi(\mathrm{Fr})$ induces a finite morphism

$$Z^1(W_t^0, \hat{G}) /\!\!/ \hat{G} \longrightarrow (\hat{G} \rtimes \mathrm{Fr}) /\!\!/ \hat{G},$$

for a reductive group scheme $\hat{G}$ with a finite action of W_t^0 that preserves a Borel pair. In this case, recall the notation of subsection 5.4, and in particular the closed subscheme $A = (N_s \rtimes \mathrm{Fr}) \times (\hat{T}^{s,0} \rtimes s) \cap Z^1(W_t^0, \hat{G})$. We then have a commutative square

$$\begin{tikzcd}
Z^1(W_t^0, \hat{G}) /\!\!/ \hat{G} \arrow[r] & \hat{G} \rtimes \mathrm{Fr} /\!\!/ \hat{G} \\
A /\!\!/ \hat{T}^{s,0} \arrow[r] \arrow[u] & N_s \rtimes \mathrm{Fr} /\!\!/ \hat{T}^{s,0} \arrow[u]
\end{tikzcd}$$

in which the bottom map is finite since A is a torsor under a finite flat group scheme over N_s, and the right vertical map is finite by the Chevalley-Steinberg theorem (see Lemma 2.2 of *loc.cit.*). It follows that the composed map $A /\!\!/ \hat{T}^{s,0} \longrightarrow \hat{G} \rtimes \mathrm{Fr} /\!\!/ \hat{G}$ is finite. On the other hand, the left vertical map is surjective, and since $Z^1(W_t^0, \hat{G}) /\!\!/ \hat{G}$ is reduced, it follows that $\mathcal{O}(Z^1(W_t^0, \hat{G}))^{\hat{G}}$ injects into $\mathcal{O}(A)^{\hat{T}^{s,0}}$. By noetherianity, $\mathcal{O}(Z^1(W_t^0, \hat{G}))^{\hat{G}}$ is thus a finitely generated module over $\mathcal{O}(\hat{G} \rtimes \mathrm{Fr})^{\hat{G}}$, as desired. $\square$

5.8. The excursion algebra and the Fargues-Scholze comparison. Let us start here with a generalized reductive group H over an affine base S. When Γ is a finitely presented discrete group, choosing a presentation allows one to see

$\underline{\mathrm{Hom}}(\Gamma, H)$ as a limit (equalizer) of smooth affine schemes. There is a more canonical way to do this, by considering the category FF_Γ of all pairs (F,γ) consisting of a finitely generated free group F with a homomorphism $F \xrightarrow{\gamma} \Gamma$. We then have a presentation

$$\mathrm{colim}_{(F,\gamma)\in \mathrm{FF}_\Gamma} \mathcal{O}(\underline{\mathrm{Hom}}(F,H)) \xrightarrow{\sim} \mathcal{O}(\underline{\mathrm{Hom}}(\Gamma, H).$$

Choosing for each integer n a free group over n letters F_n, we can write this colimit in a psychologically better way :

$$\mathrm{colim}_{(n,F_n \xrightarrow{\gamma} \Gamma)} \mathcal{O}(H^n) \xrightarrow{\sim} \mathcal{O}(\underline{\mathrm{Hom}}(\Gamma, H)).$$

This presentation is particularly relevant for endowing the scheme $\underline{\mathrm{Hom}}(\Gamma, H)$ with a derived structure. However, as we already mentioned, such a derived structure would be trivial in our case of interest $\Gamma = W_F^0/P$. The main motivation for introducing it in this case actually comes from the formalism of excursion operators à la V. Lafforgue in the work of Fargues and Scholze [FS21, Ch. IX].

Observe that taking H-invariants on both sides yields a map

$$\mathrm{colim}_{(n,F_n \xrightarrow{\gamma} \Gamma)} \mathcal{O}(H^n)^H \longrightarrow \mathcal{O}(\underline{\mathrm{Hom}}(\Gamma, H))^H. \tag{5.2}$$

The spectrum of the left hand side is sometimes called the *space of pseudorepresentations* of H. When S contains a field of characteristic 0, the map (5.2) is an isomorphism because taking invariants under H is an exact functor on H-modules, and the above colimit of H-algebras is *sifted*, hence commutes with the forgetful functor to H-modules. In general, it induces a universal homeomorphism on spectra, but need not be an isomorphism.

Suppose now $H = {}^LG$ over $S = \overline{\mathbb{Z}}[\frac{1}{p}]$ is the Langlands dual of a reductive group G over F and $\Gamma = W_F^0/P$. The variant of (5.2) where we impose the usual compatibility with the projection ${}^LG \longrightarrow W_F^0$ is called the *Excursion algebra* of G, and we will denote it by $\mathrm{Exc}(W_F^0/P, \hat{G})$. Then the map (5.2) reads

$$\mathrm{Exc}(W_F^0/P, \hat{G}) \longrightarrow \mathcal{O}(Z^1(W_F^0/P, \hat{G}))^{\hat{G}}. \tag{5.3}$$

In this setting, one of the main results of [FS21] is the construction of a collection of compatible maps (for varying P)

$$\mathrm{Exc}(W_F^0/P, \hat{G})_{\overline{\mathbb{Z}}_\ell} \longrightarrow \mathfrak{Z}_{\overline{\mathbb{Z}}_\ell}(G(F)) \tag{5.4}$$

satisfying a number of very nice properties. Here, the right hand side denotes the Bernstein center, as before. This gives a clear motivation to determine when the map (5.3) is an isomorphism. This question was addressed in a spectacular way by Fargues and Scholze in Section VIII.5 of [FS21]. In particular, they prove the following.

THEOREM 5.19 (Thm. VIII.3.6 of [FS21]). *The map (5.3) is an isomorphism whenever ℓ does not divide the order of $\pi_1(\hat{G})_{\mathrm{tors}}$.*

5.9. An application of the Fargues-Scholze map to finiteness for smooth representations. Here, G is a reductive group over F. In [DHKM24], the Fargues-Scholze maps (5.4) are used to prove the following result of pure representation theory.

THEOREM 5.20. *For any open compact subgroup H of G, the Hecke algebra $\overline{\mathbb{Z}}_\ell[H\backslash G/H]$ is a finitely generated module over its center, which is a finitely generated commutative $\overline{\mathbb{Z}}_\ell$-algebra.*

This result has several consequences such as the so-called "second adjointness" between parabolic induction and restriction for smooth $\overline{\mathbb{Z}}_\ell$-representations. These consequences, and also the noetherian property of Hecke algebras, have been known for some time under additional hypothesis, such as G being a classical group or p being big enough, and obtained by pure representation theory (including types theory), see [**Dat09**]. However, nothing was known about the center except for the work of Helm [**Hel16**] for GL_n. What makes the Fargues-Scholze maps a powerful tool in this context is that they provide a rich supply of elements in the Bernstein center. Indeed, although not much is known about these maps in general, the "simple" fact that they are known for tori, compatible with parabolic induction, and compatible with unramified twisting is sufficient to imply that the Bernstein center is "sufficiently big".

We refer to [**DHKM24**] for the proof of the above theorem, which is quite short. We only stress here that a crucial input is the finiteness of the natural maps of algebras $\mathrm{Exc}(W_F^0, \hat{G}) \longrightarrow \mathrm{Exc}(W_F^0, \hat{M})$, when M is a Levi subgroup of G. This finiteness can be directly deduced from Theorem 5.18 and, in particular, does not require Theorem 5.19, nor Theorem 5.17.

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