Loop group actions on categories and Whittaker invariants

by

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Abstract

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We develop some aspects of the theory of $\mathfrak{D}$-modules on schemes and indschemes of pro-finite type. These notions are used to define $\mathfrak{D}$-modules on (algebraic) loop groups and, consequently, actions of loop groups on DG categories. We also extend the Fourier-Deligne transform to Tate vector spaces.

Let N be the maximal unipotent subgroup of a reductive group G. For a non-degenerate character $\chi : N(\!(t)\!) \to \mathbb{G}_a$, and a category $\mathcal{C}$ acted upon by $N(\!(t)\!)$, there are two possible notions of the category of $(N(\!(t)\!), \chi)$-objects: the invariant category $\mathcal{C}^{N(\!(t)\!),\chi}$ and the coinvariant category $\mathcal{C}_{N(\!(t)\!),\chi}$. These are the Whittaker categories of $\mathcal{C}$, which are in general not equivalent. However, there is always a family of functors $\mathsf{T}_k : \mathcal{C}_{N(\!(t)\!),\chi} \to \mathcal{C}^{N(\!(t)\!),\chi}$, parametrized by $k \in \mathbb{N}$.

We conjecture that each T_k is an equivalence, provided that the $N(\!(t)\!)$-action on $\mathcal{C}$ is the restriction of a $G(\!(t)\!)$-action. We prove this conjecture for $G = GL_n$ and show that the Whittaker categories can be obtained by taking invariants of $\mathcal{C}$ with respect to a very explicit pro-unipotent group subscheme (not indscheme) of $G(\!(t)\!)$.

To my family

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Chapter 1

Introduction

Categorical (or higher) representation theory is the study of symmetries of categories. In mathematical terms, such symmetries are encoded by the notion of *group action* on a category.

To clarify our geometric context, we work within the world of complex algebraic geometry, so that our groups are group prestacks over $\mathbb{C}$. As for the categorical context, we only consider actions of such groups on stable presentable $\mathbb{C}$-linear ∞-categories. In other words, our categories are co-complete differential graded (DG) categories and functors among them are required to be continuous (i.e. colimit preserving). The foundational basis of these notions is contained in the books [L0], [L1]. For succinct reviews, we recommend [G0] and [BZFN].

1.1 Some higher algebra

The above set-up, i.e. the ∞-category DGCat of co-complete DG categories and continuous functors, is extremely convenient for performing algebraic operations on categories, directly generalizing standard operations of classical algebra. For instance,

1. one can form colimits and limits of categories.

2. there is a tensor product that makes DGCat into a symmetric monoidal ∞-category. It commutes with colimits separately in each variable and the monoidal unit is Vect, the category of (complexes of) $\mathbb{C}$-vector spaces.

3. the monoidal structure is closed, i.e. there exists an "internal hom functor", $\mathrm{Hom}(\mathcal{C}, \mathcal{D})$: the category of continuous functors from $\mathcal{C}$ to $\mathcal{D}$.

4. there is a notion of dualizable category; if $\mathcal{C}$ is dualizable, its dual $\mathcal{C}^\vee$ is equivalent to $\mathrm{Hom}(\mathcal{C}, \mathrm{Vect})$.

5. monoidal categories are precisely algebra objects in the symmetric monoidal ∞-category DGCat. They admit ∞-categories of modules: if $\mathcal{A}$ is a monoidal category, we denote by $\mathcal{A}$-**mod** the ∞-category of its (left) modules. Roughly speaking, this consists of categories $\mathcal{M}$ equipped with a functor $\mathcal{A} \otimes \mathcal{M} \to \mathcal{M}$ satisfying the natural compatibility conditions. Right $\mathcal{A}$-modules are defined similarly.

6. combining (3) and (5), given two $\mathcal{A}$-modules $\mathcal{M}$ and $\mathcal{N}$, one can form the category of $\mathcal{A}$-linear functors $\mathrm{Hom}_{\mathcal{A}}(\mathcal{M}, \mathcal{N})$.

7. there is a well-defined notion of *relative* tensor product: one can tensor a right and a left $\mathcal{A}$-module *over* $\mathcal{A}$; the output is a plain DG category.

Coming back to higher representation theory, we would like to add one more item to the above list: given an affine algebraic group G of finite type, we wish to define G-representations on objects of the ∞-category DGCat. To do so, let us mimic the classical setting: the structure of a G-representation on a vector space V consists of a coaction on V of the coalgebra $\Gamma(G, \mathcal{O}_G)$ of functions on G with convolution coproduct (i.e., pull-back along the multiplication).

In our context, one has to replace functions of the group with sheaves: there are thus at least two possible notions of group action, accounting for the two standard sheaf-theoretic contexts of *quasi-coherent sheaves* and $\mathfrak{D}$-*modules* on G, both equipped with convolution.

Thus, categories with a *weak* action are comodules for $(\mathrm{QCoh}(G), m^*)$, whereas categories with a *strong* action (or infinitesimally trivialized action) are comodules for $(\mathfrak{D}(G), m^!)$. We occasionally denote them by $G\text{-}\mathbf{rep}^{weak}$ and $G\text{-}\mathbf{rep}$, respectively. Here m^* (resp., $m^!$) is the quasi-coherent (resp., $\mathfrak{D}$-module) pull-back along the multiplication $m : G \times G \to G$.

However, for G of finite type, $\mathrm{QCoh}(G)$ and $\mathfrak{D}(G)$ are canonically self-dual; under this duality, the coproducts specified above get sent to the convolution products in the context of quasi-coherent sheaves or $\mathfrak{D}$-modules. Thus, we have

$$G\text{-}\mathbf{rep}^{weak} \simeq (\mathrm{QCoh}(G), \star)\text{-}\mathbf{mod} \qquad G\text{-}\mathbf{rep} \simeq (\mathfrak{D}(G), \star)\text{-}\mathbf{mod}.$$

The main source of examples of categorical G-representations comes from geometry. Indeed, let X be a scheme of finite type with an action of G; then $\mathfrak{D}(X)$ is a module category for $(\mathfrak{D}(G), \star)$, via push-forward along the action map $G \times X \to X$. Likewise, $\mathrm{QCoh}(X)$ carries a weak G-action.

Several standard operations with ordinary G-representations generalize immediately to this categorical context.

First, $\mathrm{Rep}(G)$ is a symmetric monoidal category: this follows from the fact that $\Gamma(G, \mathcal{O}_G)$ is a co-commutative Hopf algebra. The same holds true for $G\text{-}\mathbf{rep}^{weak}$ and $G\text{-}\mathbf{rep}$, since $\mathrm{QCoh}(G)$ and $\mathfrak{D}(G)$ are co-commutative *Hopf monoidal* categories.

Secondly, given an ordinary G-representation V, we can take its invariants and coinvariants, V^G and V_G. By definition, $V^G = \mathrm{Hom}_G(\mathbb{C}, V)$ and $V_G = \mathbb{C} \otimes_G V$. In our categorical framework, we have the notions of weak invariants and coinvariants for $\mathcal{C} \in G\text{-}\mathbf{rep}^{weak}$,

$$\mathcal{C}^{G,w} := \mathrm{Hom}_{\mathrm{QCoh}(G)}(\mathrm{Vect}, \mathcal{C}), \quad \mathcal{C}_{G,w} := \mathrm{Vect} \underset{\mathrm{QCoh}(G)}{\otimes} \mathcal{C},$$

as well as strong invariant and coinvariants, for $\mathcal{C} \in G\text{-}\mathbf{rep}$,

$$\mathcal{C}^G := \mathrm{Hom}_{\mathfrak{D}(G)}(\mathrm{Vect}, \mathcal{C}), \quad \mathcal{C}_G := \mathrm{Vect} \underset{\mathfrak{D}(G)}{\otimes} \mathcal{C}.$$

The difference between weak and strong invariants is controlled by the monoidal category of *Harish-Chandra* bimodules, the category of endomorphisms of $\mathfrak{g}$-mod as a strong G-representation (the action of $\mathfrak{D}(G)$ on $\mathfrak{g}$-mod is induced by the adjoint action of G on $\mathfrak{g}$):

$$\mathsf{HC} := \mathrm{Hom}_{\mathfrak{D}(G)}(\mathfrak{g}\text{-mod}, \mathfrak{g}\text{-mod}).$$

Precisely, if G is an affine algebraic complex group and $\mathcal{C} \in G\text{-}\mathbf{rep}$, then $\mathcal{C}^{G,w}$ carries an action of HC and the invariant category $\mathcal{C}^G$ is the category of HC-linear functors from $\mathrm{Rep}(G)$ to $\mathcal{C}^{G,w}$.

If X is a G-scheme of finite type, then $\mathfrak{D}(X)^G \simeq \mathfrak{D}(X/G)$, the category of $\mathfrak{D}$-modules on the quotient stack X/G. This follows immediately from fppf descent for $\mathfrak{D}$-modules. It turns out that $\mathfrak{D}(X)_G \simeq \mathfrak{D}(X/G)$ as well. (The very same statements hold for QCoh in place of $\mathfrak{D}$.)

1.2 Sheaves of categories

It is worthwhile to introduce one more generalization of a classical algebro-geometric concept to the categorical world, that of quasi-coherent sheaf.

First of all, recall that quasi-coherent sheaves are defined on any prestack. A prestack is the most general "space" of algebraic-geometry, that is, an arbitrary contravariant functor from affine DG schemes to ∞-groupoids. Given a prestack $\mathcal{Y}$, we have

$$\mathrm{QCoh}(\mathcal{Y}) := \lim_{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S).$$

Informally, a quasi-coherent sheaf on $\mathcal{Y}$ is a collection of a $\Gamma(S, \mathcal{O}_S)$-module M_S, for any affine DG scheme $S \to \mathcal{Y}$, together with compatibilities along restrictions $S \to S' \to \mathcal{Y}$.

In striking analogy with the above, we introduce the notion of *quasi-coherent sheaf of categories* over a prestack $\mathcal{Y}$. We will omit the adjective "quasi-coherent" for brevity. Thus, the ∞-category of sheaves of categories over a prestack $\mathcal{Y}$ is given by

$$\mathrm{ShvCat}(\mathcal{Y}) := \lim_{S \in (\mathrm{DGSch}^{\mathrm{aff}}_{/\mathcal{Y}})^{\mathrm{op}}} \mathrm{QCoh}(S)\text{-}\mathbf{mod}.$$

Informally, a sheaf of categories on $\mathcal{Y}$ is a collection, for any affine DG scheme $S \to \mathcal{Y}$, of a $\mathrm{QCoh}(S)$-module category $\mathcal{M}_S$, together with compatibilities along restrictions $S \to S' \to \mathcal{Y}$.

We say that a prestack $\mathcal{Y}$ is 1-*affine* if the functor of global sections

$$\mathbf{\Gamma} : \mathrm{ShvCat}(\mathcal{Y}) \to \mathrm{QCoh}(\mathcal{Y})\text{-}\mathbf{mod}$$

is an equivalence. In other words, sheaves of categories over a 1-affine $\mathcal{Y}$ can be reconstructed from their global sections.

Ordinary G-representations are quasi-coherent sheaves over $BG = \mathrm{pt}/G$, the stack classifying G-bundles. Likewise, categorical G-representations ought to be sheaves of categories over BG: in other words, a weak action of G on $\mathcal{C}$ is the datum of a sheaf of categories $\widetilde{\mathcal{C}}$ over BG, whose restriction along $\mathrm{pt} \to BG$ is the original $\mathcal{C}$.

If G is an affine group of finite type, then the two definitions coincide ([G7]). Furthermore, D. Gaitsgory shows that BG is 1-affine, so that

$$G\text{-}\mathbf{rep}^{weak} \simeq \mathrm{Rep}(G)\text{-}\mathbf{mod}.$$

In the same optic, a strong G-action on $\mathcal{C}$ ought to be given by a sheaf of categories over BG_{dR} that restrict to $\mathcal{C}$ along the inclusion $\mathrm{pt} \to BG$. As above, for groups of finite type, this matches the definition given above in terms of the group algebra (*loc. cit.*). However, one easily shows that BG_{dR} is not 1-affine.

1.3 $\mathfrak{D}$-modules on loop groups

In this paper, we adopt the group-algebra definition, as we are interested in strong actions by the loop group $G(\!(t)\!)$ of a reductive group. This is an ind-scheme of infinite type; for such prestacks, it is not clear at the time of writing what the appropriate notion of de Rham functor is. The definition of the DG category $\mathfrak{D}(G(\!(t)\!))$ has not been fully discussed either. The two issues are related, as the interpretation of $\mathfrak{D}$-modules as crystals prescribes that $\mathfrak{D}(\mathcal{Y}) := \mathrm{QCoh}(\mathcal{Y}_{\mathrm{dR}})$ for any prestack $\mathcal{Y}$.

More precisely, the de Rham functor is defined on any prestack ([GR0]), but it may need to be adapted to capture the *correct* notion of $\mathfrak{D}$-modules on loop groups. In fact, $\mathfrak{D}$-modules on $G(\!(t)\!)$ can be defined in an ad hoc way *directly* from the finite-type case.

A detailed construction will be given in the main body of the text, where we also discuss the theory of $\mathfrak{D}$-modules on schemes of pro-finite type. Nevertheless, let us briefly hint at it here: $\mathbf{G} := G(\!(t)\!)$ comes with the canonical "decreasing" sequence of congruence subgroups $\mathbf{G}^r$ shrinking to the identity element. For each r, let $\pi : \mathbf{G}/\mathbf{G}^{r+1} \to \mathbf{G}/\mathbf{G}^r$ be the projection. Since $\mathbf{G}/\mathbf{G}^r$ are ind-schemes on ind-finite type, $\mathfrak{D}$-modules on them make sense according to the theory developed in the published literature. Then, one puts

$$\mathfrak{D}^*(\mathbf{G}) := \lim_{r,\pi_*} \mathfrak{D}(\mathbf{G}/\mathbf{G}^r) \simeq \operatorname*{colim}_{r,\pi^*} \mathfrak{D}(\mathbf{G}/\mathbf{G}^r).$$

Remark 1.3.1. One of the reasons why we privilege strong actions comes from our interest in the local Langlands correspondence (reviewed later), which explicitly involves strong actions of loop groups on categories. Hereafter, unless specified otherwise, by the term "action" we shall mean "strong action".

1.4 Fourier transform

In our effort to generalize classical notions of algebra and representation theory to the categorical world, let us revisit the classical Fourier transform.

An ordinary representation of a vector group A can be decomposed according to characters of A, which are by definition linear maps $\chi : A \to \mathbb{C}$, that is elements of $A^\vee$. More fundamentally, once the appropriate function-theoretic context is assumed, Fourier transform gives an equivalence

$$\mathsf{FT} : (\mathrm{Fun}(A), \star) \xrightarrow{\simeq} (\mathrm{Fun}(A^\vee), \cdot)$$

between the algebra of functions on A with convolution and the algebra of functions of $A^\vee$ with pointwise multiplication. The spectral decomposition of an A-representation cited above follows by looking at categories of modules for both algebras:

$$A\text{-rep} := (\mathrm{Fun}(A), \star)\text{-mod} \xrightarrow{\simeq} (\mathrm{Fun}(A^\vee), \cdot)\text{-mod}.$$

Moreover, if V is an A-representation, the vector space of its invariant V^A is equivalent to the fiber of V at $0 \in A^\vee$, where V is now, via Fourier transform, a sheaf over $A^\vee$.

The fiber of V over a nonzero character $\chi \in A^\vee$ corresponds to $V^{A,\chi}$: the vector space of A-invariants *against* the character χ. Explicitly, if plain A-invariants are those elements $v \in V$ for which $a \cdot v = v$, then (A, χ)-invariants are those $v \in V$ for which

$$a \cdot v = e^{\chi(a)} v.$$

A categorification of the above situation, in case $A = \mathbb{A}^n$ is the n-dimensional affine space, is provided by the Deligne-Fourier transform, [La]: an equivalence of monoidal categories

$$\mathsf{FT} : (\mathfrak{D}(A), \star) \xrightarrow{\simeq} (\mathfrak{D}(A^\vee), \overset{!}{\otimes}), \tag{1.4.1}$$

which yields

$$A\text{-}\mathbf{rep} = (\mathfrak{D}(A), \star)\text{-}\mathbf{mod} \xrightarrow{\simeq} (\mathfrak{D}(A^\vee), \overset{!}{\otimes})\text{-}\mathbf{mod}. \tag{1.4.2}$$

The kernel of this transform is no longer a function on $A \times A^\vee$, but a sheaf: precisely, the pull-back of $exp \in \mathfrak{D}(\mathbb{G}_a)$ via the duality pairing $Q : A \times A^\vee \to \mathbb{G}_a$. The substitute of "exponential function" exp is the $\mathfrak{D}$-module encoded by the defining differential equation for exp. Its homomorphism property corresponds to the isomorphism $m^!(exp) \simeq exp \boxtimes exp$.

Invariants and coinvariants of $\mathcal{C} \in A$-**rep**, correspond to fiber and cofiber of $\mathcal{C}$ over $0 \hookrightarrow A^\vee$, respectively. The fiber at non-zero $\chi \in A^\vee$ is identified with the category of (A, χ)-invariants of $\mathcal{C}$: objects $c \in \mathcal{C}$ for which the *coaction* (dual to the action) is isomorphic to

$$\mathsf{coact}(c) \simeq \chi^!(exp) \otimes c \in \mathfrak{D}(A) \otimes \mathcal{C}.$$

In the main body of the paper, we extend (1.4.1) and thus (1.4.2) to $A = \mathbb{A}^n(\!(t)\!)$ (the ind-scheme of formal loops in $\mathbb{A}^n$), or, more generally, to an ind-pro-finite dimensional vector space.

1.5 Local geometric Langlands duality

The importance of categorical representation theory in mathematics became first evident in the framework of the *local geometric Langlands correspondence*, a conjecture put forward by E. Frenkel and D. Gaitsgory in the course of several papers (primarily, [FG0], [FG1], [FG2]). See also the book [F] for a review.

The local Langlands conjecture predicts the existence of an equivalence

$$\mathfrak{D}(G(\!(t)\!))\textbf{-mod} \simeq \mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)),$$

where $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$ is the stack of local systems on the punctured disk $\mathcal{D}^\times = \mathrm{Spec}(\mathbb{C}(\!(t)\!))$, for the *Langlands dual group* $\check{G}$.

This is, philosophically, a kind of Fourier transform: assuming that $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$ is 1-affine, so that

$$\mathrm{ShvCat}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)) \simeq \mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times))\textbf{-mod},$$

categorical representations of $G(\!(t)\!)$ would admit a spectral decomposition over the stack $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$.

According to [FG0], such a correspondence ought to be implemented by a universal kernel $\mathcal{C}_{univ}$: a category lying over $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$ and acted on by $G(\!(t)\!)$ fiberwise. In *loc. cit.*, the authors did not identify $\mathcal{C}_{univ}$, but described its "base-change" along various maps $\mathcal{Y} \to \mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$. For instance, one of the main results of [FG0] determines the base-change of $\mathcal{C}_{univ}$ along the forgetful map from *opers* $\mathrm{Op}_{\check{G}}$ on the punctured disk to $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$:

$$\mathcal{C}_{univ} \underset{\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times))}{\otimes} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}) \simeq \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}.$$

In [FG1], the category of $G[[t]]$-invariant objects of $\mathcal{C}_{univ}$ is argued to be equivalent to $\mathrm{Rep}(\check{G})$. This actually gives a clue as to what $\mathcal{C}_{univ}$ is: we will identify a conjectural candidate below, after introducing the procedure $\mathcal{C} \rightsquigarrow \mathcal{C}^{N(\!(t)\!),\chi}$ of *Whittaker invariants*.

1.6 Whittaker actions

Let $G = GL_2$ and $\mathcal{C}$ be a category with an action of $G(\!(t)\!)$. Since $N = \mathbb{G}_a$ is abelian, $N(\!(t)\!)$-invariants of $\mathcal{C}$ can be studied via Fourier transform, as explained above.

The dual of $\mathbb{G}_a(\!(t)\!)$ is itself, the duality being the residue pairing. In particular, the canonical element $1 \in \mathbb{G}_a(\!(t)\!)$ yields the residue character

$$\mathrm{Res} : \mathbb{G}_a(\!(t)\!) \to \mathbb{G}_a, \quad f(t) = \sum f_n t^n \mapsto f_{-1}.$$

The *Whittaker invariant* category of $\mathcal{C}$ is, by definition $\mathrm{Whit}(\mathcal{C}) := \mathcal{C}|_1$ the fiber of $\mathcal{C}$ at $1 \in \mathbb{G}_a(\!(t)\!)^\vee$. Alternatively,

$$\mathrm{Whit}(\mathcal{C}) := \mathcal{C}^{\mathbb{G}_a(\!(t)\!),\mathrm{Res}}.$$

It is the latter expression that admits a generalization to any reductive group G. Indeed, if G has rank greater than one, the maximal unipotent subgroup $N \subset G$ is no longer abelian, hence Fourier transform is not available. However, $N(\!(t)\!)$ still admits a non-degenerate character χ, defined as follows. Denote by $\{\alpha_1, \ldots, \alpha_r\}$ be the simple roots of G, thought of as maps $N \to \mathbb{G}_a$. We let

$$\chi : N(\!(t)\!) \to \mathbb{G}_a \quad n(t) \mapsto \mathrm{Res}\Big(\sum_{j=1}^{r} \alpha_j(n(t))\Big). \tag{1.6.1}$$

Such character is non-degenerate (i.e., nonzero on any root space) and of conductor zero (i.e., null on $N[[t]]$).

If $\mathcal{C}$ is acted upon by $G(\!(t)\!)$, we set the Whittaker invariant category of $\mathcal{C}$ to be

$$\mathrm{Whit}(\mathcal{C}) := \mathcal{C}^{N(\!(t)\!),\chi}.$$

We conjecture that $\mathcal{C}_{univ} \simeq \mathrm{Whit}(\mathfrak{D}(G(\!(t)\!)))$. It is immediately clear that the latter $G(\!(t)\!)$ admits an action of $G(\!(t)\!)$. However, the very fact that the proposed candidate for $\mathcal{C}_{univ}$ lives over $\mathrm{LocSys}_{\check{G}}(\mathcal{D}^\times)$ has not been yet estabilished. We will partially address this question in another publication.

There is a parallel theory of coinvariants and Whittaker coinvariants. One could propose that $\mathcal{C}_{univ} \simeq \mathfrak{D}(G(\!(t)\!))_{N(\!(t)\!),\chi}$, the coinvariant Whittaker category of $\mathfrak{D}(G(\!(t)\!))$, leading to a different local geometric Langlands correspondence. After Gaitsgory, we re-propose the following conjecture:

Conjecture 1.6.1. *If $\mathcal{C}$ is a category acted on by $G(\!(t)\!)$, then $\mathcal{C}^{N(\!(t)\!),\chi} \simeq \mathcal{C}_{N(\!(t)\!),\chi}$.*

The statement makes sense if $\mathcal{C}$ is endowed just with an $N(\!(t)\!)$-action, and not with a $G(\!(t)\!)$-action. However, in this case the conjecture is false. It is also false if χ is degenerate, e.g. $\chi = 0$; in other words, the conjecture is special to the Whittaker categories.

1.7 Overview of the results

In this paper we prove the above conjecture for $G = GL_n$. Actually, we do a little more. Following ideas of Gaitsgory, we construct a functor, for any G,

$$\mathsf{T} : \mathcal{C}_{N(\!(t)\!),\chi} \longrightarrow \mathcal{C}^{N(\!(t)\!),\chi}.$$

(Rather, we construct a sequence of such, indexed by an integer $k \geq 1$.) Our main theorem reads:

Theorem 1.7.1. *If $G = GL_n$, the natural functor $\mathsf{T} : \mathcal{C}_{N(\!(t)\!),\chi} \to \mathcal{C}^{N(\!(t)\!),\chi}$ is an equivalence.*

To prove this, we introduce an explicit group-scheme $\mathbf{H}_k$, depending on $k \geq 1$, endowed with a character ψ. For $n = 2$ and $n = 3$, $\mathbf{H}_k$ looks like

$$\mathbf{H}_k = \begin{pmatrix} 1 + t^k\mathcal{O} & t^{-k}\mathcal{O} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{H}_k = \begin{pmatrix} 1 + t^k\mathcal{O} & t^{-k}\mathcal{O} & t^{-2k}\mathcal{O} \\ t^{2k}\mathcal{O} & 1 + t^k\mathcal{O} & t^{-k}\mathcal{O} \\ 0 & 0 & 1 \end{pmatrix},$$

and the generalization to any n is straightforward (here $\mathcal{O}$ denotes the ring of formal Taylor series). The character ψ computes the sum of the residues of the entries in the over-diagonal.

We shall relate $(N(\!(t)\!), \chi)$-invariants and coinvariants to $(\mathbf{H}_k, \psi)$-invariants. Namely, we first produce natural functors

$$\mathcal{C}_{N(\!(t)\!),\chi} \longleftarrow \mathcal{C}^{\mathbf{H}_k,\psi} \longleftarrow \mathcal{C}^{N(\!(t)\!),\chi}.$$

Secondly, we show that both functors are equivalences. The proof of that uses induction on n and Fourier transform: indeed, N is the a semi-direct product $N' \ltimes \mathbb{A}^{n-1}$, where N' refers to the maximal unipotent subgroup of GL_{n-1}.

Thirdly, the composition of the inverse functors is shown to be our functor T, for the chosen value of k.

1.8 Notation and detailed contents

Let G be a complex reductive group, B a chosen Borel subgroup and $N \subseteq B$ its unipotent radical. We indicate by $\mathbf{G} := G(\!(t)\!)$ the loop group of G and, for $r \geq 0$, by $\mathbf{G}^r \subseteq G[[t]]$ the congruence subgroups. Similar notations hold for B and N. As noted above, $\mathbf{N}$ is equipped with an additive character, χ.

Let us describe how this paper is organized. In Sect. 2, we discuss some foundations of the theory of $\mathfrak{D}$-modules on schemes (and ind-schemes) of pro-finite type. The main examples of such are $\mathbf{G}$, $\mathbf{N}$ and variations thereof. There are two categories of $\mathfrak{D}$-modules on $\mathbf{G}$, dual to each other. The first, $\mathfrak{D}^*(\mathbf{G})$, carries a convolution monoidal structure; its dual $\mathfrak{D}^!(\mathbf{G})$ is hence comonoidal and also carries a symmetric monoidal structure via the diagonal. It turns out that $\mathfrak{D}^*(\mathbf{G}) \simeq \mathfrak{D}^!(\mathbf{G})$, but we will never use this fact.

We proceed in Sect. 3 to define loop group actions on categories and the concept of invariants and coinvariants. Since $\mathbf{N}$ is exhausted by its compact open group sub-schemes, we analyze group actions by pro-unipotent group schemes in great detail. For instance, we define and study natural functors among the original category, the invariant category and the coinvariant category. We show that the latter two are equivalent. A noteworthy feature of out set-up is that any category with a $\mathbf{G}$-action is *smoothly generated*, that is, generated by objects each of which is invariant for a high congruence subgroup $\mathbf{G}^r \subset \mathbf{G}$.

In Sect. 4, we take up Whittaker actions of $\mathbf{N}$ on categories: this is a special case of the above theory that accounts for the presence of the character $\chi : \mathbf{N} \to \mathbb{G}_a$. For any category acted on by $\mathbf{N}$, we construct a functor (denoted by T, as above) from Whittaker coinvariants to Whittaker invariants. We conjecture that this functor is an equivalence, provided that the $\mathbf{N}$-action is the restriction of a $\mathbf{G}$-action.

We discuss the abelian theory in Sect. 5. We first review the theory of the Fourier-Deligne transform for finite dimensional vector spaces (in schemes) and then extend it to $\mathfrak{D}$-modules on $\mathbb{A}^n(\!(t)\!)$ (more generally, to $\mathfrak{D}$-modules on an ind-pro-vector space). We prove that it still gives a monoidal equivalence. Finally, we re-interpret the concepts of the previous sections (invariants, coinvariants, averaging) in Fourier-transformed terms.

The material of Sect. 6 is technical: we first analyze how actions by a semi-direct product $H \ltimes K$ can be understood in terms of the actions of H and K. Secondly, we discuss categories fibering over a K-space X and acted on by the group K in a compatible fashion. In this situation, we study how the operations of taking the fiber of $\mathcal{C}$ at $x \in X$ and taking K-invariants interact.

In Sect. 7, we take up the proof of Theorem 1.7.1. We discuss some combinatorics of GL_n and define some group schemes of $GL_n(\!(t)\!)$ that will play a central role. Our proofs are on induction on n, for which the tools of Sect 6 are essential.

Chapter 2

$\mathfrak{D}$-modules on schemes and ind-schemes of infinite type

Leaving aside the notion of sheaves of categories, we wish to say that a group prestack $(\mathcal{G}, m)$ acts strongly on a category $\mathcal{C}$ if the latter is endowed with an action of the monoidal category $(\mathfrak{D}(\mathcal{G}), \star)$. Similarly, we say that $\mathcal{G}$ acts weakly on $\mathcal{C}$ is the latter is a module for $(\mathrm{QCoh}(\mathcal{G}), \star)$. We mostly focus on strong actions. For such, the above definition makes sense whenever we can provide a construction of $\mathfrak{D}(\mathcal{G})$ as a *dualizable* category endowed with the convolution monoidal functor m_*.

The ultimate goal of this section is to supply this definition in our cases of interest: $\mathcal{G} = G(\!(t)\!)$ and $\mathcal{G} = N(\!(t)\!)$. To address this, we proceed in two steps. First, we identify the kind of algebraic structure that $G(\!(t)\!)$ and $N(\!(t)\!)$ possess: the answer is that they are *ind-pro schemes*. Roughly speaking, these are (0-)prestacks constructed from schemes of finite type out of affine smooth projections and closed embeddings. Secondly, we develop the theory of $\mathfrak{D}$-modules on ind-pro schemes. However, in the treatment below we present these two steps in opposite order, namely we start from the abstract theory.

2.1 $\mathfrak{D}$-modules on pro-schemes

We adopt the convention of [GR1]: a (classical) *ind-scheme* is a prestack that can be presented as a filtered colimit of quasi-compact and quasi-separated schemes along closed embeddings. A frequently used subcategory of such prestacks is that of ind-schemes of *ind-finite type*: it consists of those ind-schemes formed out of schemes of finite type.

We shall need the intermediate set-up of *ind-pro-schemes*[1]: ind-schemes written as colimits of schemes, each of which can be further written as a filtered limit of schemes of finite type

[1] These definitions appeared in the seminar note [Ba].

under affine, smooth and surjective maps. Later we shall see that $G((t))$ and $N((t))$ fall under this rubric and discuss two dual (and equivalent) theories of $\mathfrak{D}$-modules on ind-pro-schemes.

Let $\mathrm{Sch}^{qc,qs}$ be the ordinary category of quasi-compact quasi-separated schemes and $\mathrm{Sch}^{\mathrm{ft}}$ its full subcategory of schemes of finite type. Between the two lies the ordinary category $\mathrm{Sch}^{pro} := \mathrm{Pro}^{\mathrm{aff,sm,s}}\,\mathrm{Sch}^{\mathrm{ft}}$ of schemes of *pro-finite type*: schemes that can be written as filtered limits of schemes of finite type along affine smooth surjective maps. To shorten the terminology, we refer to objects of Sch^{pro} just as *pro-schemes*. The existence of the embedding $\mathrm{Sch}^{pro} \hookrightarrow \mathrm{Sch}^{qc,qs}$ is shown in Appendix C of [TT].[2]

We define $\mathfrak{D}^*$-modules on pro-schemes as follows: we let

$$\mathfrak{D}^* : \mathrm{Sch}^{pro} \to \mathrm{DGCat}$$

be the right Kan extension of $\mathfrak{D} : \mathrm{Sch}^{\mathrm{ft}} \to \mathrm{DGCat}$ along the inclusion $\mathrm{Sch}^{\mathrm{ft}} \hookrightarrow \mathrm{Sch}^{pro}$. Here, $\mathfrak{D} : \mathrm{Sch}^{\mathrm{ft}} \to \mathrm{DGCat}$ is the usual functor that assigns $S \rightsquigarrow \mathfrak{D}(S)$ and $f \rightsquigarrow f_*$. So, $\mathfrak{D}^*$ is covariant by construction: for any morphism $f : X \to Y$ in Sch^{pro}, we denote by f_* the corresponding functor $\mathfrak{D}^*(X) \to \mathfrak{D}^*(Y)$.

Explicitly, suppose $Z \in \mathrm{Sch}^{pro}$ be presented as

$$Z \simeq \lim_{r \in \mathcal{R}} Z^r, \tag{2.1.1}$$

where $\mathcal{R}^{op}$ is a filtered category. (In most cases of interest, $\mathcal{R} = (\mathbb{N}, <)^{op}$.) For any arrow $s \to r$ in $\mathcal{R}^{op}$, let $\pi_{s \to r} : Z^s \to Z^r$ be the corresponding projection. Then,

$$\mathfrak{D}^*(Z) \simeq \lim_{r \in \mathcal{R}, \pi_*} \mathfrak{D}(Z^r), \tag{2.1.2}$$

the limit being taken with respect to the pushforwards $(\pi_{s \to r})_*$.

In several occasions, we will make use of the following categorical fact, due to J. Lurie (cf. [G0]). Let $\mathcal{C}_\bullet : \mathcal{I} \to \mathrm{DGCat}$ be a diagram of categories, where, for each $\gamma : i \to j$, the corresponding functor $F_\gamma : \mathcal{C}_i \to \mathcal{C}_j$ admits a right adjoint G_γ. Then, there is an equivalence of categories

$$\mathsf{LC} : \lim_{i \in \mathcal{I}^{\mathrm{op}}, G} \mathcal{C}_i \xrightarrow{\simeq} \operatorname*{colim}_{i \in \mathcal{I}, F} \mathcal{C}_i.$$

Moreover, under LC, the tautological functors of "insertion" and "evaluation"

$$\mathsf{ins}_i : \mathcal{C}_i \to \operatorname*{colim}_{i \in \mathcal{I}, F} \mathcal{C}_i \qquad \mathsf{ev}_i : \lim_{i \in \mathcal{I}^{\mathrm{op}}, G} \mathcal{C}_i \to \mathcal{C}_i$$

form an adjoint pair.

[2] This reference was pointed out by S. Raskin, who has independently developed the theory of $\mathfrak{D}$-modules on schemes of infinite type.

In the case at hand, smoothness of each $\pi_{s\to r}$ implies the existence of the left adjoints $(\pi_{s\to r})^*$, so that

$$\mathfrak{D}^*(Z) \simeq \lim_{\mathcal{R},\pi_*} \mathfrak{D}(Z^r) \simeq \operatorname*{colim}_{\mathcal{R}^{op},\pi^*} \mathfrak{D}(Z^r).$$

The latter expression explains the notation $\mathfrak{D}^*$, which is meant to indicate a colimit along $*$-pullback functors. It also shows that $\mathfrak{D}^*(Z)$ is compactly generated, hence dualizable. We denote the insertion

$$\mathsf{ins}_s : \mathfrak{D}(Z^s) \to \operatorname*{colim}_{\mathcal{R}^{op},\pi^*} \mathfrak{D}(Z^r)$$

formally by $(\pi_{\infty\to s})^*$. Analogously, the evaluation functor will be formally denoted by $(\pi_{\infty\to s})_*$. Compact objects of $\mathfrak{D}^*(Z)$ are those of the form $(\pi_{\infty\to s})^*(M)$, for M compact in $\mathfrak{D}(Z^s)$.

Thanks to the continuity of the push-forward functors π_* and the filteredness of $\mathcal{R}^{\mathrm{op}}$, the isomorphism $\lim_{\mathcal{R},\pi_*}\mathfrak{D}(Z^r) \to \operatorname*{colim}_{\mathcal{R}^{op},\pi^*}\mathfrak{D}(Z^r)$ can be made explicit ([G0]). Namely,

$$\mathsf{LC} : M = \{M^r\}_r \mapsto \operatorname*{colim}_{r\in\mathcal{R}^{op}} \Big((\pi_{\infty\to r})^*(M^r) \Big), \tag{2.1.3}$$

while the inverse equivalence is completely determined by the assignment:

$$\mathsf{LC}^{-1} : (\pi_{\infty\to r})^*(M^r) \mapsto \left\{ \operatorname*{colim}_{k\,:\,k\to r,\,k\to s} (\pi_{k\to s})_*(\pi_{k\to r})^*(M^r) \right\}_s . \tag{2.1.4}$$

Clearly, the latter expression greatly simplifies if each pull-back $(\pi_{r_1\to r_2})^*$ is fully faithful: in that case, the colimit in (2.1.4) is independent of k. Thus, we propose the following definition: we say that a pro-scheme Z is *pseudo-contractible* if it admits a presentation as in (2.1.1) with transition maps giving rise to fully faithful $*$-pullback functors. By smoothness, the !-pullbacks are also fully faithful.

The dual of $\mathfrak{D}^*(Z)$ is easily computed and it is by the definition the category of $\mathfrak{D}^!$-modules on Z:

$$\mathfrak{D}^!(Z) := \big(\mathfrak{D}^*(Z)\big)^\vee \simeq \operatorname*{colim}_{\mathcal{R}^{op},\pi^!} \mathfrak{D}(Z^r).$$

This is a consequence of the fact that $\mathfrak{D}(S)^\vee \simeq \mathfrak{D}(S)$ via the classical Verdier duality, and that $(\pi_*)^\vee \simeq \pi^!$ under this self-duality.

The assigment $Z \rightsquigarrow \mathfrak{D}^!(Z)$ upgrades to a contravariant functor mapping $f : X \to Y$ to

$$f^! := (f_*)^\vee : \mathfrak{D}^!(Y) \to \mathfrak{D}^!(X).$$

As before, let

$$(\pi_{\infty\to s})^! : \mathfrak{D}(Z^s) \to \operatorname*{colim}_{\mathcal{R}^{op},\pi^!} \mathfrak{D}(Z^r)$$

symbolize the tautological insertion map. Compact objects of $\mathfrak{D}^!(Z)$ are those objects of the form $(\pi_{\infty\to s})^!(M)$, for all $s \in \mathcal{R}$ and all M compact in $\mathfrak{D}(Z^s)$.

Since each $\pi_{s\to r}$ is smooth, $(\pi_{s\to r})^! \simeq (\pi_{s\to r})^*[2d_{rs}]$, where d_{rs} is the dimension of $\pi_{s\to r}$. Thus, the right adjoint of $(\pi_{s\to r})^!$ is isomorphic to $(\pi_{s\to r})_*[-2d_{rs}]$ (hence, it is continuous). We can realize $\mathfrak{D}^!(Z)$ as a limit:

$$\mathfrak{D}^!(Z) \simeq \lim_{r\in\mathcal{R}, \pi_*[-2d_\pi]} \mathfrak{D}(Z^r). \tag{2.1.5}$$

Suppose that each Z^r has a well-defined dimension. Then, comparing this formula with (2.1.2), we construct the canonical equivalence

$$\eta_Z : \mathfrak{D}^!(Z) \xrightarrow{\simeq} \mathfrak{D}^*(Z), \tag{2.1.6}$$

induced by the inverse family of shift functors $\mathrm{id}[-2\dim(Z^r)] : \mathfrak{D}(Z^r) \to \mathfrak{D}(Z^r)$.

By ([G0]), the duality pairing ε_Z between $\mathfrak{D}^*(Z)$ and $\mathfrak{D}^!(Z)$ consists of the assignment

$$\varepsilon_Z(\{M^s\}_s, (\pi_{\infty\to r})^! N^r) \simeq \varepsilon_{Z^r}(M^r, N^r). \tag{2.1.7}$$

In other words, $(\pi_{\infty\to r})^!$ is dual to the functor $\mathsf{ev}_r = (\pi_{\infty\to r})_* : \mathfrak{D}^*(Z) \to \mathfrak{D}(Z^r)$. Alternatively, by (2.1.4), we obtain

$$\begin{aligned} \varepsilon_Z\big((\pi_{\infty\to r})^* M^r, (\pi_{\infty\to r})^! N^r\big) &= \operatorname*{colim}_{s\in\mathcal{R}_{/r}} \varepsilon_{Z^s}\big((\pi_{s\to r})^* M^r, (\pi_{s\to r})^! N^r\big) \\ &= \operatorname*{colim}_{s\in\mathcal{R}_{/r}} \Gamma_{\mathrm{dR}}\big(Z^s, (\pi_{s\to r})^*(M^r \otimes N^r)\big). \end{aligned} \tag{2.1.8}$$

Remark 2.1.1. If Z is pseudo-contractible, the latter formula simplifies as

$$\varepsilon_Z((\pi_{\infty\to r})^* M^r, (\pi_{\infty\to r})^! N^r) \simeq \Gamma_{\mathrm{dR}}(Z^r, M^r \otimes N^r).$$

Lemma 2.1.2. *For any $M^r \in \mathfrak{D}(Z^r)_{\mathrm{cpt}}$, Verdier duality $\mathbb{D}_Z : \mathfrak{D}^!(Z) \xrightarrow{\simeq} \mathfrak{D}^*(Z)$ is computed "component-wise", i.e., it sends*

$$\mathbb{D}_Z : (\pi_{\infty\to r})^!(M^r) \mapsto (\pi_{\infty\to r})^*(\mathbb{D}_{Z^r}(M^r)).$$

Proof. By definition, it suffices to exhibit a canonical equivalence

$$\mathrm{Hom}_{\mathfrak{D}^*(Z)}\Big((\pi_{\infty\to r})^*(\mathbb{D}_{Z^r}(M^r)), P\Big) \simeq \varepsilon_Z\big((\pi_{\infty\to r})^!(M^r), P\big), \tag{2.1.9}$$

for any $P \in \mathfrak{D}^*(Z)$ pulled back from a finite type quotient. This follows immediately from the adjunction $\big((\pi_{\infty\to r})^*, (\pi_{\infty\to r})_*\big)$ for $\mathfrak{D}^*$-modules, combined with the duality between $(\pi_{\infty\to r})^!$ and $(\pi_{\infty\to r})_*$. ∎

2.2 Basic functoriality

In this subsection we work out part of the theory of $\mathfrak{D}^*$ and $\mathfrak{D}^!$-modules on pro-schemes: we discuss various push-forward and pull-back functors, the tensor products, the projection and base-change formulas.

By definition of right Kan extension, if $f : X \to Y$ is a morphism of pro-schemes, the functor $f_* : \mathfrak{D}^*(X) \to \mathfrak{D}^*(Y)$ is determined by the formula

$$\mathsf{ev}_r^Y \circ f_* \simeq (f^r)_* \circ \mathsf{ev}_r^X \tag{2.2.1}$$

for any morphism $f^r : X^r \to Y^r$ of schemes of finite type covered by f.

In practice, if X and Y are given presentations compatible with f, meaning that f is the limit of morphisms $f^r : X^r \to Y^r$, then f_* is defined as the limit of the $(f^r)_*$'s. The general case is reduced to this, by reindexing a sub-presentation of X: indeed, any map $X \to Y^r$ factors as $X \to X^s \to Y^r$, for some s mapping to r in $\mathcal{R}$.

For instance, *de Rham cohomology* Γ_{dR} of $X \simeq \lim_r X^r$ is defined as

$$\Gamma_{\mathrm{dR}} := (p_X)_* : \mathfrak{D}^*(X) \to \mathrm{Vect},$$

where $p_X : X \to \mathrm{pt}$ is the tautological map. Explicitly, if $M \in \mathfrak{D}^*(X)$ is represented by the collection $\{M^r\}_r$, with $M^r \in \mathfrak{D}(X^r)$ and isomorphims $(\pi_{s\to r})_*(M^s) \simeq M^r$, then

$$\Gamma_{\mathrm{dR}}(M) := \Gamma_{\mathrm{dR}}(X^r, M^r),$$

the RHS being independent of r.

If $i_x : \mathrm{pt} \hookrightarrow X$ is a closed point, we define the "delta" $\mathfrak{D}$-module at x according to the usual formula: $\delta_{x,X} := (i_x)_*(\mathbb{C}) \in \mathfrak{D}^*(X)$. Explicitly, in the realization of $\mathfrak{D}^*(X)$ as a limit, δ_x is represented by the collection of $\delta_{x^r,X^r} \in \mathfrak{D}(X^r)$, where x^r is the image of x under the projection $X \to X^r$.

Contrarily to the finite-type case, $(i_x)_*$ is not proper: as pointed out before, compact $\mathfrak{D}^*$-modules on X are (in particular) $*$-pulled back along some projection $X \to X^r$ and $\delta_{x,X}$ is not such. As a consequence, $(i_x)_*$ does not admit a continuous right adjoint.

However, $(i_x)_*$ is fully faithful, being the limit of the embeddings $(i_{x^r})_*$ (Kashiwara's lemma). More generally, if $f : Y \hookrightarrow X$ is *q-closed* (which means that f can be written as a limit of a filtered family of closed embeddings between schemes of finite type), then f_* is fully faithful.

Consider again an arbitrary map $f : X \to Y$ of pro-schemes. The left adjoint to f_*, denoted f^*, is only partially defined. We say that f^* is defined on $M \in \mathfrak{D}^*(Y)$ if the functor $\mathrm{Hom}(M, f_*(-))$ is corepresentable (by an object that we denote as $f^*(M)$). A sufficient condition for f^* to be defined on all of $\mathfrak{D}^*(Y)$ is that each f^r be smooth. More generally:

Lemma 2.2.1. *Let $N^r \in \mathfrak{D}(Y^r)$ be an object in the domain of definition of $(f^r)^*$. Then $f^*((\pi^Y_{\infty\to r})^*(N^r))$ is well-defined and computed by*

$$f^*\big((\pi^Y_{\infty\to r})^*(N^r)\big) = (\pi^X_{\infty\to r})^*\big((f^r)^*(N^r)\big).$$

Proof. This is obtained from (2.2.1) after passing to left adjoints. □

In particular, whenever X^r admits a constant sheaf k_{X^r}, the functor of de Rham cohomology $p_* : \mathfrak{D}^*(X) \to \mathrm{Vect}$ is corepresented by the *constant $\mathfrak{D}^*$-module*

$$k_X := (\pi_{\infty\to r})^*(k_{X^r}), \tag{2.2.2}$$

where the RHS is independent of r. If X is pseudo-contractible, k_X is easily written as a limit: $k_X \simeq \{k_{X^r}\}_r$.

Given $f : X \to Y$ as above, let us discuss the functor $f^! : \mathfrak{D}^!(Y) \to \mathfrak{D}^!(X)$, dual to f_*. To compute it, one represents f as a compatible family of maps $f^r : X^r \to Y^r$; then, by (2.1.7), one readily gets

$$f^! \simeq \operatorname*{colim}_{r\in\mathcal{R}^{op}} (\pi_{\infty\to r})^! \circ (f^r)^!. \tag{2.2.3}$$

Here are two basic examples.

First, the case of $i_x : \mathrm{pt} \to X$. If $M = \{M^r\}_r \in \lim_{(\pi^!)^R} \mathfrak{D}(X^r) \simeq \mathfrak{D}^!(X)$, then $(i_x)^!$ sends M to the vector space

$$(i_x)^!(M) \simeq \mathrm{colim}_r(M^r|_{x^r}).$$

The transition maps in the RHS are as follows: for $s \to r$, letting $\pi = \pi_{s\to r}$,

$$M^r|_{x^r} \simeq \pi^!(M^r)\big|_{x^s} \simeq \pi^! \circ (\pi^!)^R(M^s)\big|_{x^s} \xrightarrow{\mathsf{counit}} M^s|_{x^s}.$$

Secondly, the case of $p : X \to \mathrm{pt}$. Then $p^! : \mathrm{Vect} \to \mathfrak{D}^!(X)$ gives the *dualizing sheaf*, $\omega_X := p^!(\mathbb{C})$. Explicitly,

$$\omega_X := (\pi_{\infty\to r})^!(\omega_{X^r}),$$

being clear that the RHS does not depend on r. If X is pseudo-contractible, ω_X is computed naively in the realization of $\mathfrak{D}^!(X)$ as a limit (2.1.5):

$$\omega_X \simeq \{\omega_{X^r}\}_r.$$

The functor $f_!$, left adjoint to $f^!$, is only partially defined. For instance, here is a typically infinite dimensional phenomenon.

Remark 2.2.2. If X is infinite dimensional, $(i_x)^!$ does not have a left adjoint. Indeed, the value of this hypothetical left adjoint on $\mathbb{C}$ would have to be compact, hence of the form $(\pi_{\infty\to r})^!(F)$ for some r and some $F \in \mathfrak{D}(X^r)_{\mathrm{cpt}}$. It is easy to see that this causes a contradiction. For simplicity, assume that $\mathcal{R} = (\mathbb{N}, <)^{op}$ and that all $\pi^!$ are fully faithful. For any $s \to r$, adjunction forces

$$\mathrm{Hom}_{\mathfrak{D}^!(X)}\left((\pi_{\infty\to r})^!(F), (\pi_{\infty\to s})^!(-)\right) \simeq (i_{x^s})^!$$

as functors from $\mathfrak{D}(X^s)$ to Vect. It follows that $(\pi_{s\to r})^! F \simeq \delta_{x^s}$, which is absurd if $\pi_{s\to r}$ is of positive dimension.

This example shows that $f_!$ may not be defined even if all $(f^r)_!$ are. However, in this paper we will only need to deal with $p_! : \mathfrak{D}^!(X) \to \mathrm{Vect}$ for pseudo-contractible X. Then, we have

$$(p^r)^! \simeq \mathsf{ev}_{r,!} \circ p^!$$

by contruction and fully faithfulness of $(\pi_{\infty\to r})^!$; here, $\mathsf{ev}_{r,!}$ denotes the evaluation functor $\mathfrak{D}^!(X) \to \mathfrak{D}(X^r)$. Hence, by passing to the left adjoints, we obtain

$$(p^r)_! \simeq p_! \circ (\pi_{\infty\to r})^!,$$

which implies that

$$p_!(M) \simeq \operatorname*{colim}_{r\in\mathcal{R}^{\mathrm{op}}} (p^r)_!(M^r)$$

for $M = \{M^r\}_r \in \mathfrak{D}^!(X) = \lim_{(\pi^!)^R} \mathfrak{D}(X^r)$, provided that $(p^r)_!$ is defined on M^r (e.g., if M^r is holonomic).

Suppose that $f : X \to Y$ can be presented as the limit of maps $f^r : X^r \to Y^r$ such that all the squares

$$\begin{tikzcd} X^s \arrow[r, "\pi"] \arrow[d, "f^s"] & X^r \arrow[d, "f^r"] \\ Y^s \arrow[r, "\pi"] & Y^r \end{tikzcd} \tag{2.2.4}$$

are Cartesian (equivalently, f is finitely presented, [R]). In this case, base-change yields commutative squares

$$\begin{tikzcd} \mathfrak{D}(X^s) \arrow[r, "\pi_*"] & \mathfrak{D}(X^r) \\ \mathfrak{D}(Y^s) \arrow[u, "(f^s)^!"] \arrow[r, "\pi_*"] & \mathfrak{D}(Y^r) \arrow[u, "(f^r)^!"] \end{tikzcd} \qquad \begin{tikzcd} \mathfrak{D}(X^s) \arrow[d, "(f^s)_*"'] & \mathfrak{D}(X^r) \arrow[l, "\pi^!"'] \arrow[d, "(f^r)_*"] \\ \mathfrak{D}(Y^s) & \mathfrak{D}(Y^r), \arrow[l, "\pi^!"] \end{tikzcd} \tag{2.2.5}$$

which allow to define the functors

$$f^{\mathrm{i}} : \mathfrak{D}^*(Y) \to \mathfrak{D}^*(X), \quad f^{\mathrm{i}}(\{N^r\}_r) := \{(f^r)^!(N^r)\}_r$$

$$f_\bullet : \mathfrak{D}^!(X) \to \mathfrak{D}^!(Y), \quad f_\bullet(\{M^r\}_r) := \mathrm{colim}_r (\pi_{\infty\to r})^!(f^r)_*(M^r).$$

It follows from the finite-type case that the pairs of functors (f_*, g^{i}) and $(f_\bullet, g^!)$ satisfy the base-change formula (see [R] for a thorough treatment).

We will not use these functors in the present paper, but will often use the *renormalized* push-forward

$$f_*^{ren} : \mathfrak{D}^!(X) \to \mathfrak{D}^!(Y), \ \ f_*^{ren} := \eta_Y^{-1} \circ f_* \circ \eta_X.$$

Obviously, the (ordinary) category Sch^{pro} admits products. Moreover, for two pro-schemes X and Y, there are canonical equivalences

$$\begin{aligned} \mathfrak{D}^*(X) \otimes \mathfrak{D}^*(Y) &\xrightarrow{\boxtimes} \mathfrak{D}^*(X \times Y) \\ \mathfrak{D}^!(X) \otimes \mathfrak{D}^!(Y) &\xrightarrow{\boxtimes} \mathfrak{D}^!(X \times Y), \end{aligned} \tag{2.2.6}$$

which follow at once from dualizability of each $\mathfrak{D}(X^r)$ and the fact that $\mathfrak{D}^*$ and $\mathfrak{D}^!$ can be represented as colimits.

Remark 2.2.3. It is easy to see that Sch^{pro} admits fiber products. Indeed, let $X \to Z \leftarrow Y$ be a diagram of pro-schemes. Let $Z = \lim Z^r$ be a presentation of Z. For each $r \in \mathcal{R}$, the composition $X \to Z \twoheadrightarrow Z^r$ factors through a projection $X \twoheadrightarrow X'$, with X' of finite type. We let $X^r := X'$. In this way we contruct compatible pro-scheme presentations of X and Y Moreover, $\lim_r(X^r \times_{Z^r} Y^r)$ is a presentation of $X \times_Z Y$,

Note that $\mathfrak{D}^!(X)$ is symmetric monoidal under the "pointwise" tensor product, which is defined as usual:

$$- \underset{X}{\otimes} - : \mathfrak{D}^!(X) \otimes \mathfrak{D}^!(X) \xrightarrow{\boxtimes} \mathfrak{D}^!(X \times X) \xrightarrow{\Delta^!} \mathfrak{D}^!(X),$$

where $\Delta : X \to X \times X$ is the diagonal. The fact that this operation endows $\mathfrak{D}^!(X)$ with a symmetric monoidal structure is a consequence of the same fact in the finite dimensional case, combined with the observation that the functors $(\pi_{s\to r})^!$ are monoidal. This tensor product is completely determined by

$$(\pi_{\infty\to r})^!(M) \otimes_X (\pi_{\infty\to r})^!(M') = (\pi_{\infty\to r})^!(M \otimes_{X^r} M').$$

For any $f : X \to Z$, the functor $f^! : \mathfrak{D}^!(Z) \to \mathfrak{D}^!(X)$ is symmetric monoidal, hence $(\mathfrak{D}^!(Z), \otimes)$ acts on $\mathfrak{D}^!(X)$. The relative analogue of (2.2.6) holds true as well:

Lemma 2.2.4. *Let $X \to Z \leftarrow Y$ be a diagram of pro-schemes. There are canonical equivalences*

$$\mathfrak{D}^!(X) \underset{\mathfrak{D}^!(Z)}{\otimes} \mathfrak{D}^!(Y) \simeq \mathfrak{D}^!(X \times_Z Y). \tag{2.2.7}$$

Proof. The canonical functor is induced by pullback along $X \times_Z Y \to X \times Y$. To prove it is an equivalence, we reduce it to the finite dimensional case, where it is true by a result of [BZN]. By fixing compatible presentations of X, Y, Z and $X \times_Z Y$ as in Remark 2.2.3, we obtain

$$\mathfrak{D}^!(X \times_Z Y) \simeq \operatorname*{colim}_{r \in \mathcal{R}^{\mathrm{op}}} \mathfrak{D}(X^r \times_{Z^r} Y^r) \simeq \operatorname*{colim}_{r \in \mathcal{R}^{\mathrm{op}}} \left(\mathfrak{D}^!(X^r) \underset{\mathfrak{D}^!(Z^r)}{\otimes} \mathfrak{D}^!(Y^r)\right) \simeq \mathfrak{D}^!(X) \underset{\mathfrak{D}^!(Z)}{\otimes} \mathfrak{D}^!(Y).$$

$\square$

Remark 2.2.5. Using the functor η_X of (2.1.6), one can define a symmetric monoidal structure on $\mathfrak{D}^*(Z)$ making $\mathfrak{D}^*(X)$ and $\mathfrak{D}^*(Y)$ into $\mathfrak{D}^*(Z)$-modules and such that

$$\mathfrak{D}^*(X) \underset{\mathfrak{D}^*(Z)}{\otimes} \mathfrak{D}^*(Y) \simeq \mathfrak{D}^*(X \times_Z Y).$$

There is an action $\overset{!*}{\otimes}$ of $(\mathfrak{D}^!(X), \otimes)$ on $\mathfrak{D}^*(X)$, induced by the usual tensor product of $\mathfrak{D}(X)$ for $X \in \mathrm{Sch}^{\mathrm{ft}}$:

$$(\pi_{\infty \to r})^!(M) \overset{!*}{\otimes} (\pi_{\infty \to r})^*(P) = (\pi_{\infty \to r})^*(M \otimes_{X^r} P).$$

This follows from the functorial equivalence

$$\pi^!(M) \otimes_X \pi^*(P) \simeq \pi^*(M \otimes_Y P),$$

valid for any smooth map $\pi : X \to Y$ between schemes of finite type. Written more compactly, the action is

$$M \overset{!*}{\otimes} N \simeq \eta_X(M \otimes \eta_X^{-1}(N)).$$

When we consider the above as a right action, we write it as $\overset{*!}{\otimes}$, for clarity. The following lemma is the *projection formula* in this context:

Lemma 2.2.6. *For $f : X \to Y$, the functor f_* is linear with respect to the action of $\mathfrak{D}^!(Y)$, i.e. there is a canonical isomorphism*

$$f_*(M) \overset{*!}{\otimes} N \simeq f_*(M \overset{*!}{\otimes} f^!(N)). \tag{2.2.8}$$

Proof. Note that $M \overset{!*}{\otimes} - : \mathfrak{D}^*(X) \to \mathfrak{D}^*(X)$ is the dual functor of $M \otimes - : \mathfrak{D}^!(X) \to \mathfrak{D}^!(X)$. Hence, for $P \in \mathfrak{D}^!(Y)$ arbitrary, we have

$$\begin{aligned} \varepsilon_Y\big(f_*(M) \overset{*!}{\otimes} N, P\big) &\simeq \varepsilon_Y\big(f_*(M), N \otimes P\big) \simeq \varepsilon_Y\big(M, f^!(N) \otimes f^!(P)\big) \\ &\simeq \varepsilon_Y\big(M \overset{*!}{\otimes} f^!(N), f^!(P)\big) \simeq \varepsilon_Y\big(f_*(M \overset{*!}{\otimes} f^!(N)), P\big), \end{aligned}$$

which concludes the proof. □

2.3 $\mathfrak{D}$-modules on ind-pro-schemes

Before proceeding with the definition of the two theories of $\mathfrak{D}$-modules on ind-pro-schemes, let us recall the definition of the category of $\mathfrak{D}$-modules on ind-schemes of ind-finite type. Let $\mathrm{IndSch}^{\mathrm{ft}} := \mathrm{Ind}^{cl}\,\mathrm{Sch}^{\mathrm{ft}}$ denote the category of such. The functor

$$\mathfrak{D} : \mathrm{IndSch}^{\mathrm{ft}} \longrightarrow \mathrm{DGCat}$$

is defined to be the left Kan extension of $\mathfrak{D} : \mathrm{Sch}^{\mathrm{ft}} \to \mathrm{DGCat}$ along the inclusion $\mathrm{Sch}^{\mathrm{ft}} \hookrightarrow \mathrm{IndSch}^{\mathrm{ft}}$. Explicitly, if $\mathcal{Y} \in \mathrm{IndSch}^{\mathrm{ft}}$ is written as a filtered colimit $\mathcal{Y} = \mathrm{colim}_{\mathcal{I}}\, Y_i$, with $Y_i \in \mathrm{Sch}^{\mathrm{ft}}$ and closed embeddings $\iota_{i \to j} : Y_i \hookrightarrow Y_j$, then $\mathfrak{D}(\mathcal{Y}) \simeq \mathrm{colim}_{\mathcal{I}}\, \mathfrak{D}(Y_i)$, with respect to the pushforward morphisms $(\iota_{i \to j})_*$.

We repeat the same process for ind-pro-schemes. Namely, we define the ordinary category $\mathrm{IndSch}^{pro} := \mathrm{Ind}^{cl}\,\mathrm{Sch}^{pro}$ of *ind-pro-schemes* to be the one comprising ind-schemes that can be formed as colimits of pro-schemes under closed embeddings. Then, the functor

$$\mathfrak{D}^* : \mathrm{IndSch}^{pro} \to \mathrm{DGCat}$$

is defined as the left Kan extension of $\mathfrak{D}^* : \mathrm{Sch}^{pro} \to \mathrm{DGCat}$ along the inclusion $\mathrm{Sch}^{pro} \hookrightarrow \mathrm{IndSch}^{pro}$.

Analogously, the functor

$$\mathfrak{D}^! : (\mathrm{IndSch}^{pro})^{\mathrm{op}} \to \mathrm{DGCat}$$

is defined as the right Kan extension of $\mathfrak{D}^! : (\mathrm{Sch}^{pro})^{\mathrm{op}} \to \mathrm{DGCat}$ along the inclusion $(\mathrm{Sch}^{pro})^{\mathrm{op}} \hookrightarrow (\mathrm{IndSch}^{pro})^{\mathrm{op}}$. For $f : X \to Y$ a map in IndSch^{pro}, we continue to denote by $f_* : \mathfrak{D}^*(X) \to \mathfrak{D}^*(Y)$ and $f^! : \mathfrak{D}^!(Y) \to \mathfrak{D}^!(X)$ the induced functors.

Lemma 2.3.1. *The duality $\mathfrak{D}^! \simeq (\mathfrak{D}^*)^\vee$ and the isomorphism $(f_*)^\vee \simeq f^!$ continue to hold for ind-pro-schemes.*

Proof. Let $\mathcal{Y} = \operatorname{colim}_{j\in\mathcal{J}} Y_i$ be a presentation of $\mathcal{Y}$ as an ind-pro-scheme as above. The two categories $\mathfrak{D}^!(\mathcal{Y})$ and $\mathfrak{D}^*(\mathcal{Y})$ are, by construction,

$$\mathfrak{D}^!(\mathcal{Y}) \simeq \lim_{j\in\mathcal{J}^{\mathrm{op}},\iota^!} \mathfrak{D}^!(Y_j) \qquad \mathfrak{D}^*(\mathcal{Y}) \simeq \operatorname*{colim}_{j\in\mathcal{J},\iota_*} \mathfrak{D}^*(Y_j).$$

They are evidently dual to each other, thanks to the validity of the lemma for pro-schemes. The duality $(f_*)^\vee \simeq f^!$ is a formal consequence of this. □

Proposition 2.3.2. *The ordinary category* IndSch^{pro} *is symmetric monoidal via Cartesian product and the functor* $\mathfrak{D}^! : (\mathrm{IndSch}^{pro})^{\mathrm{op}} \to \mathrm{DGCat}$ *is symmetric monoidal.*

Proof. The first assertion is obvious. As for the second, recall that $\mathfrak{D}^!$ is symmetric monoidal as a functor $(\mathrm{Sch}^{pro})^{\mathrm{op}} \to \mathrm{DGCat}$. Furthermore, for any map of pro-schemes $f : X \to Y$, the pull-back $f^! : \mathfrak{D}^!(Y) \to \mathfrak{D}^!(X)$ is symmetric monoidal. The combination of these two facts yields the assertion. □

2.4 $\mathfrak{D}$-modules on $G(\!(t)\!)$ and $N(\!(t)\!)$

Finally, let us take up the case of loop groups. The following construction is well-known:

Lemma 2.4.1. *The loop group* $\mathbf{G} := G(\!(t)\!)$ *is an ind-pro-scheme.*

Proof. Consider the (schematic) quotient map to the affine Grassmannian $\mathfrak{q} : \mathbf{G} \to \mathrm{Gr}$ and choose a presentation of Gr as an ind-scheme of ind-finite type: $\mathrm{Gr} \simeq \operatorname{colim}_{n,\iota} Z_n$. Pulling-back each Z_n along $\mathfrak{q}$, we obtain an ind-scheme presentation of $\mathbf{G}$:

$$\mathbf{G} \simeq \operatorname*{colim}_{n\in\mathbb{N}} \mathfrak{q}^{-1}(Z_n).$$

Of course, each $\mathfrak{q}^{-1}(Z_n) = Z_n \underset{\mathrm{Gr}}{\times} \mathbf{G}$ is of infinite type. However, as $\mathfrak{q}$ factors through $\mathbf{G} \to \mathbf{G}^r$ for any $r \in \mathbb{N}$, we can write:

$$\mathfrak{q}^{-1}(Z_n) \simeq \lim_{r\in\mathbb{N}} \big(Z_n \underset{\mathrm{Gr}}{\times} \mathbf{G}/\mathbf{G}^r\big),$$

where the limit is taken along the maps induced by the projections $\pi_{s\to r} : \mathbf{G}/\mathbf{G}^s \to \mathbf{G}/\mathbf{G}^r$. This is a presentation of $\mathfrak{q}^{-1}(Z_n)$ as a pro-scheme, and the proof in concluded. □

We now justify the definitions given in the introduction.

Lemma 2.4.2.

$$\mathfrak{D}^*(\mathbf{G}) \simeq \operatorname*{colim}_{r,\pi^*} \mathfrak{D}(\mathbf{G}/\mathbf{G}^r), \quad \mathfrak{D}^!(\mathbf{G}) \simeq \operatorname*{colim}_{r,\pi^!} \mathfrak{D}(\mathbf{G}/\mathbf{G}^r).$$

Proof. We only prove the first formula, the proof of the second being completely analogous. Let $Z_n^r := Z_n \underset{\mathrm{Gr}}{\times} \mathbf{G}/\mathbf{G}^r$, so that

$$\mathbf{G} = \underset{n,\iota}{\mathrm{colim}}\ \lim_{r,\pi} Z_n^r \qquad \mathbf{G}/\mathbf{G}^r = \mathrm{colim}_{n,\iota}\, Z_n^r$$

are presentations of $\mathbf{G}$ and $\mathbf{G}/\mathbf{G}^r$ as an ind-pro-scheme and an ind-scheme, respectively. For each n and r, consider the evidently Cartesian square:

$$\begin{tikzcd}
Z_n \times_{\mathrm{Gr}} \mathbf{G}/\mathbf{G}^{r+1} \arrow[r, "\pi"] \arrow[d, "\iota"'] & Z_n \times_{\mathrm{Gr}} \mathbf{G}/\mathbf{G}^r \arrow[d, "\iota"'] \\
Z_{n+1} \times_{\mathrm{Gr}} \mathbf{G}/\mathbf{G}^{r+1} \arrow[r, "\pi"] & Z_{n+1} \times_{\mathrm{Gr}} \mathbf{G}/\mathbf{G}^r.
\end{tikzcd} \tag{2.4.1}$$

Consequently, the category of $\mathfrak{D}^*$-modules on $\mathbf{G}$ is expressed as follows:

$$\mathfrak{D}^*(\mathbf{G}) \simeq \underset{n,\iota_*}{\mathrm{colim}}\ \lim_{r,\pi_*} \mathfrak{D}(Z_n^r) \simeq \underset{n,\iota_*}{\mathrm{colim}}\ \underset{r,\pi^*}{\mathrm{colim}}\, \mathfrak{D}(Z_n^r) \simeq \underset{r,\pi^*}{\mathrm{colim}}\ \underset{n,\iota_*}{\mathrm{colim}}\, \mathfrak{D}(Z_n^r) \simeq \underset{r,\pi^*}{\mathrm{colim}}\, \mathfrak{D}(\mathbf{G}/\mathbf{G}^r),$$

where the switch of colimits in the third equivalence is a consequence of base-change along the above square. □

We now prove that $\mathfrak{D}^*(\mathbf{G})$ has a convolution monoidal structure. This will enable us to provide well-defined notion of categorical $\mathbf{G}$-actions.

Lemma 2.4.3. *The functor $m_* : \mathfrak{D}^*(\mathbf{G} \times \mathbf{G}) \to \mathfrak{D}^*(\mathbf{G})$, together with the equivalence*

$$\mathfrak{D}^*(\mathbf{G}) \otimes \mathfrak{D}^*(\mathbf{G}) \xrightarrow{\boxtimes} \mathfrak{D}^*(\mathbf{G} \times \mathbf{G}) \tag{2.4.2}$$

endows $\mathfrak{D}^*(\mathbf{G})$ with a monoidal structure.

Proof. The functor $\mathfrak{D}^! : (\mathrm{IndSch}^{pro})^{\mathrm{op}} \to \mathrm{DGCat}$ is contravariant and, by Proposition 2.3.2, symmetric monoidal. Hence, it sends algebras in IndSch^{pro} to comonoidal categories: in particular, $\big(\mathfrak{D}^!(\mathbf{G}), m^!\big)$ is comonoidal. By duality, we obtain the required statement. □

Let us now discuss *self-duality* of $\mathfrak{D}^*(\mathbf{G})$. Note that each projection $\mathbf{G}/\mathbf{G}^{r+1} \twoheadrightarrow \mathbf{G}/\mathbf{G}^r$ is smooth of dimension, say, n_r. A *trivialization of the dimension torsor* of $\mathbf{G}$ is, by definition, a sequence of integers $\{d_k\}_{k\in\mathbb{N}}$ such that $d_{r+1} - d_r = n_r$.

Lemma 2.4.4. *To any trivialization $\{d_k\}_k$ of the dimension torsor of $\mathbf{G}$ there corresponds a self duality of $\mathfrak{D}^*(\mathbf{G})$, that is, an equivalence $\mathfrak{D}^*(\mathbf{G}) \xrightarrow{\simeq} \mathfrak{D}^!(\mathbf{G})$.*

Proof. It was explained before that $\mathfrak{D}^!(\mathbf{G})$ can be written as a limit:

$$\mathfrak{D}^!(\mathbf{G}) \simeq \lim_{\pi_*[-2n_r]} \mathfrak{D}(\mathbf{G}/\mathbf{G}^r).$$

Thinking of $\mathfrak{D}^*(\mathbf{G})$ and $\mathfrak{D}^!(\mathbf{G})$ in their realizations as limits, the isomorphism $\mathfrak{D}^*(\mathbf{G}) \xrightarrow{\simeq} \mathfrak{D}^!(\mathbf{G})$ is given by the inverse system of functors $\mathrm{id}[2d_r] : \mathfrak{D}(\mathbf{G}/\mathbf{G}^r) \to \mathfrak{D}(\mathbf{G}/\mathbf{G}^r)$. $\square$

Remark 2.4.5. The proof of Lemma 2.4.3 shows that it is useful to keep $\mathfrak{D}^*(\mathbf{G})$ is naturally monoidal, while $\mathfrak{D}^!(\mathbf{G})$ is naturally comonoidal. Hence, we will never use the result of Lemma 2.4.4.

Unlike $\mathbf{G}$, the ind-scheme $\mathbf{N}$ is exhausted by its compact open subgroups, hence it is an ind-object in the category of group-schemes. We choose a cofinal sequence $\mathbf{N}_k$ of such groups: let $t^{-\rho} \in \mathbf{T}$ be the loop corresponding to ρ, the sum of the fundamental weights. For all $k \geq 1$, we set $\mathbf{N}_k := \mathrm{Ad}_{t^{-k\rho}}(N[[t]])$, so that

$$\mathbf{N} \simeq \mathrm{colim}_k \, \mathbf{N}_k.$$

Notational warning: for us $k \geq 0$ always. The reader should not confuse $\mathbf{N}_k$ (just defined) with $\mathbf{N}^k$ (a congruence subgroup). The former contains $N(\mathcal{O})$, while the latter is contained in $N(\mathcal{O})$; however, they both coincide with $N(\mathcal{O})$ for $k = 0$.

Lemma 2.4.6. *The loop group* $\mathbf{N}$ *is an ind-pro-scheme.*

Proof. Each $\mathbf{N}_k$ is a pro-scheme: indeed, writing $\mathbf{N}_k \simeq \lim_r \mathbf{N}_k/\mathbf{N}^r$, the conclusion follows immediately. $\square$

Example: $G = GL_2$

When $G = GL_2$, we obtain that $N(\!(t)\!) \simeq \mathbb{A}^1(\!(t)\!)$ is an ind-pro-scheme. Slightly more generally, $\mathbf{A} := \mathbb{A}^n(\!(t)\!)$ is an ind-pro-scheme for any $n \in \mathbb{N}$. Indeed, setting $\mathbf{A}_i := t^{-i} \cdot \mathbb{A}^m[[t]]$ we can present $\mathbf{A}$ as an ind-scheme:

$$\mathbf{A} \simeq \mathrm{colim}_i \, \mathbf{A}_i.$$

Furthermore, each $\mathbf{A}_i$ is a pro-scheme with presentation, say, $\mathbf{A}_i \simeq \lim_j A_{i,j}$, where $A_{i,j} := \mathbf{A}_i/t^j$. This combines to

$$\mathbf{A} \simeq \mathrm{colim}_i \, \mathbf{A}_i \simeq \mathrm{colim}_i \left(\lim_j A_{i,j}\right).$$

Inclusions and projections fit into an $\mathbb{N} \times \mathbb{N}$-family of Cartesian diagrams:

$$\begin{tikzcd} A_{i,j} \arrow[r, hook, "\iota"] \arrow[d, "\pi"'] & A_{i+1,j} \arrow[d, "\pi"] \\ A_{i,j-1} \arrow[r, hook, "\iota"] & A_{i+1,j-1}. \end{tikzcd} \tag{2.4.3}$$

By construction, $\mathfrak{D}^*(\mathbf{A})$ is expressed as

$$\mathfrak{D}^*(\mathbf{A}) \simeq \operatorname*{colim}_{i\in\mathbb{N},\mathrm{i}_*} \mathfrak{D}^*(A_i) \simeq \operatorname*{colim}_{i,\iota_*} \lim_{j,\pi_*} \mathfrak{D}^*(A_{i,j}). \tag{2.4.4}$$

Recall that $\mathfrak{D}^*(\mathbf{A})$ is monoidal under convolution product. This structure comes from the analogous structure for each $\mathfrak{D}^*(A_i)$, together with the fact that ι_* is monoidal.

We record the following facts, which repeat verbatim the analogous results for $\mathbf{G}$:

Lemma 2.4.7.

$$\mathfrak{D}^*(\mathbf{N}) \simeq \lim_{r,\pi_*} \mathfrak{D}(\mathbf{N}/\mathbf{N}^r) \simeq \operatorname*{colim}_{r,\pi^*} \mathfrak{D}(\mathbf{N}/\mathbf{N}^r) \quad \textit{and} \quad \mathfrak{D}^!(\mathbf{N}) \simeq \operatorname*{colim}_{r,\pi^!} \mathfrak{D}(\mathbf{N}/\mathbf{N}^r).$$

Lemma 2.4.8. *Group multiplication endows $\mathfrak{D}^*(\mathbf{N})$ and $\mathfrak{D}^*(\mathbf{N}_k)$ with compatible convolution monoidal structures.*

The compatibility between the monoidal structures of $\mathfrak{D}^*(\mathbf{N}_k)$ and $\mathfrak{D}^*(\mathbf{N})$ is expressed by the fact that $i_* : \mathfrak{D}^*(\mathbf{N}_k) \to \mathfrak{D}^*(\mathbf{N})$ is monoidal, which in turn follows from the canonical isomorphism $i \circ m_{\mathbf{N}_k} \simeq m_{\mathbf{N}} \circ i$.

Lemma 2.4.9. *To any trivialization $\{d_k\}_k$ of the dimension torsor of $\mathbf{N}$ there corresponds an equivalence $\mathfrak{D}^*(\mathbf{N}) \simeq \mathfrak{D}^!(\mathbf{N})$.*

Chapter 3

Loop group actions on categories

In this Section we discuss actions of loop groups on categories. By definition, $\mathcal{C} \in \mathrm{DGCat}$ is acted on by $\mathbf{G}$ if it is endowed with an action of the monoidal category $(\mathfrak{D}^*(\mathbf{G}), m_*)$ (see Lemma 2.4.3). The totality of categories with $\mathbf{G}$ action forms an ∞-category, denoted by $\mathbf{G}\text{-}\mathbf{rep}$. The Hopf monoidal structure of $\mathfrak{D}^*(\mathbf{G})$ ensures that $\mathbf{G}\text{-}\mathbf{rep}$ is monoidal: we provide the relevant constructions in the Sect. 3.1.

In the next subsections, we discuss invariant and coinvariants, especially for unipotent group schemes. For such groups, we show that the invariant and the coinvariant categories are naturally equivalent. This will be of fundamental importance for the study of Whittaker categories and for the proof of our main theorem.

3.1 Hopf algebras and Hopf monoidal categories

Given a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$, the ∞-category $\mathrm{Coalg}(\mathcal{C}^{\otimes})$ of its coalgebra objects inherits a symmetric monoidal structure, compatible with the forgetful functor $\mathrm{Coalg}(\mathcal{C}^{\otimes}) \to \mathcal{C}$. A *Hopf algebra* in $\mathcal{C}$ is, by definition, an object in $\mathrm{Alg}(\mathrm{Coalg}(\mathcal{C}))$. It easy to check that we can switch "Alg" with "Coalg", so that

$$\mathrm{HopfAlg}(\mathcal{C}) := \mathrm{Alg}(\mathrm{Coalg}(\mathcal{C})) \simeq \mathrm{Coalg}(\mathrm{Alg}(\mathcal{C})). \tag{3.1.1}$$

Consequently, the subcategory $\mathrm{HopfAlg}(\mathcal{C})^{dualizable} \subseteq \mathrm{HopfAlg}(\mathcal{C})$ spanned by Hopf algebras in $\mathcal{C}$ that are dualizable as objects of $\mathcal{C}$ is closed under duality.

If $\mathcal{C} = \mathrm{Vect}^{\heartsuit}$, we recover ordinary *bialgebras*: to get to ordinary Hopf algebras, one needs to specify an antipode map. Since the latter concept will not be needed in this paper, we have omitted it from the definition of Hopf category. However, the antipode is evident in any example considered.

For $\mathcal{C} = \mathrm{DGCat}$, we obtain the concept of *Hopf monoidal* category (or just Hopf category, for short): in other words, $\mathcal{H} \in \mathrm{DGCat}$ is Hopf if it is both comonoidal and monoidal, in a compatible manner.

Any group object (G, m) in $\mathcal{C} = \mathrm{Set}$ (or Sch, IndSch, IndSch^{pro} etc.) in a Hopf algebra in $\mathcal{C}$, with multiplication being m and comultiplication being $\Delta : G \to G \times G$. The compatibility between the two sctructures follows at once from commutativity of the diagram

$$\begin{tikzcd} G \times G \arrow[r, "\Delta\times\Delta"] \arrow[d, "m"] & G \times G \times G \times G \arrow[d, "m_{13}\times m_{24}"] \\ G \arrow[r, "\Delta"] & G \times G, \end{tikzcd} \tag{3.1.2}$$

which shows that m is a morphism of coalgebras (or that Δ is a morphism of algebras).

The only example of Hopf category we shall consider is the following. For a group ind-pro-scheme G, we claim that $\mathcal{H} = \mathfrak{D}^!(G)$ can be naturally endowed with the structure of a Hopf category.

Recall that $\mathfrak{D}^!$ on ind-pro-schemes is symmetric monoidal and contravariant, and that any such functor maps algebras to coalgebras and viceversa. Hence, by (3.1.1), $\mathfrak{D}^!(G)$ is indeed Hopf, with comultiplication induced by $m^!$ and multiplication by $\Delta^!$.

Remark 3.1.1. The mentioned antipode $\mathcal{H} \to \mathcal{H}$ in this case is pull-back along the inversion automorphism $\mathrm{inv} : G \to G$.

The coalgebra structure on a Hopf category $\mathcal{H}$ allows to form the ∞-category $\mathcal{H}\text{-}\mathbf{comod} := \mathrm{Comod}_{\mathcal{H}}(\mathrm{DGCat})$ of comodules categories for $\mathcal{H}$. The rest of the structure of a Hopf category endows $\mathcal{H}\text{-}\mathbf{comod}$ with a monoidal structure compatible with the tensor product of the underlying DG categories: informally, given $\mathcal{C}$, $\mathcal{E} \in \mathcal{H}\text{-}\mathbf{comod}$, their product $\mathcal{C} \otimes \mathcal{E}$ has the following $\mathcal{H}$-comodule structure

$$\mathcal{C} \otimes \mathcal{E} \xrightarrow{\mathsf{coact}_{\mathcal{C}} \otimes \mathsf{coact}_{\mathcal{E}}} \mathcal{C} \otimes \mathcal{E} \otimes \mathcal{H} \otimes \mathcal{H} \xrightarrow{\mathrm{mult}_{\mathcal{H}}} (\mathcal{C} \otimes \mathcal{E}) \otimes \mathcal{H}.$$

Hence, we can consider algebra objects in $\mathcal{H}\text{-}\mathbf{comod}$, that is, monoidal categories with a compatible coaction of $\mathcal{H}$.

Let $\mathcal{H}$ be a Hopf category, which is dualizable as a plain category. Then, as pointed out before, $\mathcal{H}^\vee$ (the dual of $\mathcal{H}$ as a plain category) is naturally a Hopf category. Given an object $\mathcal{B} \in \mathrm{Alg}(\mathcal{H}^\vee\text{-}\mathbf{comod})$, we shall form another monoidal category $\mathcal{H} \ltimes \mathcal{B}$, called the *crossed product algebra* of $\mathcal{H}$ and $\mathcal{B}$.

Consider the ∞-category $\mathcal{B}\text{-}\mathbf{mod}(\mathcal{H}^\vee\text{-}\mathbf{comod})$. There is an adjuction

$$\mathrm{DGCat} \rightleftarrows \mathcal{B}\text{-}\mathbf{mod}(\mathcal{H}^\vee\text{-}\mathbf{comod}),$$

the left adjoint being the composition of the free functors and the right adjoint being the forgetful functor. Both functors are continuous and the right adjoint is conservative. Thus, $\mathcal{B}\text{-}\mathbf{mod}(\mathcal{H}^\vee\text{-}\mathbf{comod})$ concides with the category of modules for a monad whose underlying functor is $\mathcal{C} \mapsto \mathcal{B} \otimes (\mathcal{H}^\vee)^\vee \otimes \mathcal{C}$. The monad structure endows $\mathcal{B} \otimes \mathcal{H}$ with an algebra structure: the resulting monoidal category is by definition $\mathcal{H} \ltimes \mathcal{B}$.

Example

Given an ind-pro group $\mathcal{G}$, consider the Hopf category $\mathcal{H} = \mathfrak{D}^*(\mathcal{G})$. Let X an ind-pro-scheme acted upon by $\mathcal{G}$. We claim that $\mathcal{B} = \mathfrak{D}^!(X)$, equipped with the point-wise tensor product, belongs to $\mathrm{Alg}(\mathcal{H}^\vee\text{-}\mathbf{comod})$. In fact, the datum of the action $\mathcal{G} \times X \xrightarrow{\mathsf{act}} X$ yields the coaction of $\mathcal{H}^\vee$ on $\mathfrak{D}^!(X)$, and the required compatibility arises from the commutative diagram

$$\begin{tikzcd} \mathcal{G} \times X \arrow[r, "\mathsf{act}"] \arrow[d, "\Delta_{\mathcal{G}} \times \Delta_X"'] & X \arrow[d, "\Delta_X"'] \\ \mathcal{G} \times \mathcal{G} \times X \times X \arrow[r, "\mathsf{act} \times \mathsf{act}"] & X \times X. \end{tikzcd} \tag{3.1.3}$$

Thus, we have a well-defined category $\mathfrak{D}^*(\mathcal{G}) \ltimes \mathfrak{D}^!(X)$. This example will be crucial in the sequel.

3.2 Invariant and coinvariant categories

Let $\mathcal{C}$ be an arbitrary (co-complete DG) category equipped with a (strong) action of $\mathbf{G}$, by which we mean that $\mathcal{C}$ is a left module category for $(\mathfrak{D}^*(\mathbf{G}), \star)$. Equivalently, the same datum can be encoded by a coaction of the comonoidal category $(\mathfrak{D}^!(\mathbf{G}), m^!)$ on $\mathcal{C}$. In fact, this is the following general result in category theory:

Lemma 3.2.1. *Let $\mathcal{C}^\otimes$ be a symmetric monoidal ∞-category, and $A \in \mathcal{C}$ an algebra object, with dual $A^\vee$ (automatically a coalgebra object). Then $A\text{-mod}(\mathcal{C}) \simeq A^\vee\text{-comod}(\mathcal{C})$.*

The previous section shows that $\mathbf{G}\text{-}\mathbf{rep}$ is monoidal: in terms of the coaction, for $\mathcal{C}, \mathcal{E} \in \mathbf{G}\text{-}\mathbf{rep}$, the coaction of $\mathfrak{D}^!(\mathbf{G})$ on $\mathcal{C} \otimes \mathcal{E}$ given by

$$c \otimes e \mapsto \Delta^!_{\mathbf{G}}\big(\mathsf{coact}(c) \boxtimes \mathsf{coact}(e)\big). \tag{3.2.1}$$

The universal example of a category with a $\mathbf{G}$-action is the regular representation: i.e., $\mathfrak{D}^*(\mathbf{G})$, considered as a module over itself. In analogy with this, we sometimes denote the action map $\mathfrak{D}^*(\mathbf{G}) \otimes \mathcal{C} \to \mathcal{C}$ by the convolution symbol, $\star$.

The *trivial representation* is Vect, the category of complexes of $\mathbb{C}$-vector spaces, endowed with the $\mathbf{G}$-action specified by the monoidal functor $\Gamma_{\mathrm{dR}} : \mathfrak{D}^*(\mathbf{G}) \to \mathrm{Vect}$. This is called the *trivial action* of $\mathbf{G}$ on Vect. More generally, we say that a $\mathfrak{D}^*(\mathbf{G})$-module $\mathcal{C}$ is acted on *trivially* by $\mathbf{G}$ if the action $\mathfrak{D}^*(\mathbf{G}) \otimes \mathcal{C} \to \mathcal{C}$ is prescribed by $M \otimes c \mapsto \Gamma_{\mathrm{dR}}(\mathbf{G}, M) \otimes c$.

We define the *coinvariant category*:

$$\mathcal{C}_{\mathbf{G}} := \mathrm{Vect} \underset{\mathfrak{D}^*(\mathbf{G})}{\otimes} \mathcal{C}.$$

This is computed by the Bar resolution of the relative tensor product, i.e. the simplicial category

$$\cdots\ \mathfrak{D}^*(\mathbf{G}) \otimes \mathfrak{D}^*(\mathbf{G}) \otimes \mathcal{C} \rightrightarrows \mathfrak{D}^*(\mathbf{G}) \otimes \mathcal{C} \rightrightarrows \mathcal{C}, \tag{3.2.2}$$

where the maps are given by action, multiplication and trivial action, according to the usual pattern. To show this, one uses the "free" resolution of Vect as a right $\mathfrak{D}^*(\mathbf{G})$-module and invokes the fact that relative tensor product commutes with colimits in each variable.

We define the *invariant category*:

$$\mathcal{C}^{\mathbf{G}} := \mathrm{Hom}_{\mathfrak{D}^*(\mathbf{G})}(\mathrm{Vect}, \mathcal{C}).$$

Again, using the free resolution of Vect as a left $\mathfrak{D}^*(\mathbf{G})$-module, $\mathcal{C}^{\mathbf{G}}$ can be computed by the totalization of

$$\mathcal{C} \rightrightarrows \mathrm{Hom}(\mathfrak{D}^*(\mathbf{G}), \mathcal{C}) \rightrightarrows \mathrm{Hom}(\mathfrak{D}^*(\mathbf{G}) \otimes \mathfrak{D}^*(\mathbf{G}), \mathcal{C})\ \cdots .$$

Moreover, as $\mathfrak{D}^*(\mathbf{G})$ and $\mathfrak{D}^!(\mathbf{G})$ are in duality, the latter becomes

$$\mathcal{C} \rightrightarrows \mathfrak{D}^!(\mathbf{G}) \otimes \mathcal{C} \rightrightarrows \mathfrak{D}^!(\mathbf{G}) \otimes \mathfrak{D}^!(\mathbf{G}) \otimes \mathcal{C}\ \cdots . \tag{3.2.3}$$

If $\mathcal{C}$ is a dualizable category, its dual $\mathcal{C}^\vee = \mathrm{Hom}(\mathcal{C}, \mathrm{Vect})$ inherits a *right* action of $\mathbf{G}$, described informally by

$$\mathcal{C}^\vee \otimes \mathfrak{D}^*(\mathbf{G}) \to \mathcal{C}^\vee, \quad \phi(-) \otimes M \mapsto \phi(M \star -).$$

With this structure,

Lemma 3.2.2. *If $\mathcal{C}$ is dualizable, then $(\mathcal{C}_{\mathbf{G}})^\vee \simeq (\mathcal{C}^\vee)^{\mathbf{G}}$.*

Proof. The dual of (3.2.2) is the cosimplicial category (3.2.3) for $\mathcal{C}^\vee$. □

The very same definitions apply to $\mathbf{N}$: a strong action of $\mathbf{N}$ on $\mathcal{C}$ means a left action of the monoidal category $(\mathfrak{D}^*(\mathbf{N}), \star)$ on $\mathcal{C}$. Moreover, in view of the extra property of $\mathbf{N}$ of being a colimit of groups, the above action corresponds to a family of compatible actions of $(\mathfrak{D}^*(\mathbf{N}_k), \star)$ on $\mathcal{C}$.

It follows that

$$\mathcal{C}_{\mathbf{N}} \simeq \underset{k \in \mathbb{N}}{\mathrm{colim}}\, \mathcal{C}_{\mathbf{N}_k},$$

where the maps in the directed system

$$\mathrm{Vect} \underset{\mathfrak{D}^*(\mathbf{N}_k)}{\otimes} \mathcal{C} \to \mathrm{Vect} \underset{\mathfrak{D}^*(\mathbf{N}_{k+1})}{\otimes} \mathcal{C}$$

are induced by the push-forward functors $i_* : \mathfrak{D}^*(\mathbf{N}_k) \to \mathfrak{D}^*(\mathbf{N}_{k+1})$.

Likewise,

$$\mathcal{C}^{\mathbf{N}} \simeq \lim_{k \in \mathbb{N}} \mathcal{C}^{\mathbf{N}_k},$$

the limit taken with respect to the forgetful maps

$$\mathrm{Hom}_{\mathfrak{D}^*(\mathbf{N}_{k+1})}(\mathrm{Vect}, \mathcal{C}) \to \mathrm{Hom}_{\mathfrak{D}^*(\mathbf{N}_k)}(\mathrm{Vect}, \mathcal{C}).$$

See Proposition 4.1.1 below for a more explicit description of the transition maps in both cases.

3.3 Actions by pro-unipotent group schemes

The above formulas show that, in order to better understand $\mathbf{N}$-actions on categories, one should first discuss $\mathbf{N}_k$-invariants and coinvariants. More generally, in this section we shall discuss the action of a pro-unipotent group scheme $H \subseteq \mathbf{G}$ (e.g. $H = \mathbf{N}_k$) on a category $\mathcal{C}$. To handle $\mathfrak{D}^*(H)$, we choose at first the most naive presentation of H as a pro-scheme:[1]

$$H \simeq \lim_{r,\pi} Q^r, \quad \text{where } Q^r := H/(H \cap \mathbf{G}^r).$$

The essential feature of this situation is that H is pseudo-contractible (see Sect. 2.1): this follows from the contractibility of the fibers of each projection $\pi_{s \to r}$.

A reminder on notation: we indicate by p the map $p : H \to \mathrm{pt}$ and we set $d_r = \dim Q^r$, so that $\omega_{Q^r} \simeq k_{Q^r}[2d_r]$.

We remind the reader that almost contractibility ensures that the sheaves k_H and ω_H can be computed "naively" in the realization of $\mathfrak{D}^*(H)$ and $\mathfrak{D}^!(H)$ as a limits:

$$k_H \simeq \left\{k_{Q^r}\right\}_r \in \lim_{r,\pi_*} \mathfrak{D}(Q^r) \simeq \mathfrak{D}^*(H)$$

$$\omega_H \simeq \left\{\omega_{Q^r}\right\}_r \in \lim_{r,\pi_*[2d_r - 2d_{r+1}]} \mathfrak{D}(Q^r) \simeq \mathfrak{D}^!(H).$$

Consequently, the self-duality $\eta_H : \mathfrak{D}^!(H) \xrightarrow{\simeq} \mathfrak{D}^*(H)$ sends $\omega_H \mapsto k_H$.

We shall see later that convolution with the constant sheaf is a significant operation (Lemma 3.3.4). Here we prove:

[1] In Sect. 3.4 and also in the next lemma, we will need a more clever choice of the quotients Q^r.

Lemma 3.3.1. *For any $M \in \mathfrak{D}^*(H)$, we have*

$$k_H \star M \simeq \Gamma_{\mathrm{dR}}(H, M) \otimes k_H.$$

Proof. Let us first recall the computation in the finite dimensional case. To prove the formula, one introduces the automorphism $\xi : H \times H \to H \times H$ that sends $(x, y) \mapsto (xy, y)$ and computes

$$k_H \star M \simeq m_*(p_2^!(M))[-2d_H] \simeq (p_1)_* \xi_* \xi^!(p_2)^!(M)[-2d_H] \simeq \Gamma_{\mathrm{dR}}(M) \otimes k_H.$$

In the pro-case, we repeat the same computation at each level r. For this to work, we need to alter the presentation of H to make sure that $H/\widetilde{H}^r$ are groups for each r. This can always be achieved, by pro-unipotence of H (see Sect. 3.4).

Thus, for arbitrary $M = \{M^r\}_r \in \mathfrak{D}^*(H)$, we obtain

$$k_H \star M \simeq \left\{k_{Q^r} \star M^r\right\}_r \simeq \left\{\Gamma_{\mathrm{dR}}(Q^r, M^r) \otimes k_{Q^r}\right\}_r.$$

Since $\Gamma_{\mathrm{dR}}(Q^r, M^r) \simeq \Gamma_{\mathrm{dR}}(H, M)$ is independent of r, the desired formula follows. □

The invariant category $\mathcal{C}^H$ for H,

$$\mathcal{C}^H := \mathrm{Hom}_{\mathfrak{D}^*(H)}(\mathrm{Vect}, \mathcal{C}) \simeq \mathrm{Tot}\big(\mathcal{C} \rightrightarrows \mathfrak{D}^!(H) \otimes \mathcal{C} \substack{\to\\[-1ex]\to\\[-1ex]\to} \mathfrak{D}^!(H) \otimes \mathfrak{D}^!(H) \otimes \mathcal{C} \ \cdots\big),$$

comes equipped with a tautological forgetful functor $\mathsf{oblv}^H : \mathcal{C}^H \to \mathcal{C}$. By its very nature, oblv^H is conservative. It admits a right adjoint, called the $*$-averaging functor, $\mathsf{Av}_*^H : \mathcal{C} \to \mathcal{C}^H$. We shall prove later that Av_*^H is continuous. A left adjoint to oblv^H is only partially defined: when it is so, we denote it by $\mathsf{Av}_!^H$.

As explained before for $\mathbf{G}$ and $\mathbf{N}$, the geometric realization of the simplicial category

$$\cdots\ \mathfrak{D}^*(H) \otimes \mathfrak{D}^*(H) \otimes \mathcal{C} \substack{\to\\[-1ex]\to\\[-1ex]\to} \mathfrak{D}^*(H) \otimes \mathcal{C} \rightrightarrows \mathcal{C} \tag{3.3.1}$$

computes the coinvariant category $\mathcal{C}_H$. The latter comes equipped with a tautological functor $\mathrm{pr} : \mathcal{C} \to \mathcal{C}_H$, which admits a right adjoint $\mathbf{i}$, discontinuous in general.

Remark 3.3.2. $\mathcal{C}_H$ can also be computed as the *limit* of the cosimplicial category obtained from (3.3.1) by substituting all the arrows with their (possibly non-continuous) right adjoints. It follows that $\mathbf{i}$ is conservative.

Recall that ε_H, the duality pairing between $\mathfrak{D}^!(H)$ and $\mathfrak{D}^*(H)$, is the functor

$$\varepsilon_H : \mathfrak{D}^*(H) \otimes \mathfrak{D}^!(H) \to \mathrm{Vect}, \quad M \otimes M' \mapsto \Gamma_{\mathrm{dR}}(H; M \overset{*!}{\otimes} M').$$

By construction, the action of $\mathfrak{D}^*(H)$ on $\mathcal{C}$ is related to the coaction $\mathsf{coact} : \mathcal{C} \to \mathfrak{D}^!(H) \otimes \mathcal{C}$ as follows

$$\mathsf{act} : \mathfrak{D}^*(H) \otimes \mathcal{C} \xrightarrow{\mathrm{id}_H \otimes \mathsf{coact}} \mathfrak{D}^*(H) \otimes \mathfrak{D}^!(H) \otimes \mathcal{C} \xrightarrow{\varepsilon_H} \mathcal{C}. \tag{3.3.2}$$

Lemma 3.3.3. *The functor $k_H \star - : \mathcal{C} \to \mathcal{C}$ of convolution with the constant sheaf is canonically isomorphic to*

$$k_H \star - \simeq p_*^{ren} \circ \mathsf{coact}. \tag{3.3.3}$$

Proof. Indeed, by the construction of the duality pairing,

$$k_H \star c \simeq p_*(k_H \overset{*!}{\otimes} \mathsf{coact}(c)) \simeq p_*^{ren}\big(\eta_H^{-1}(k_H) \otimes \mathsf{coact}(c)\big)$$

and $\eta_H(\omega_H) \simeq k_H$. □

We now express the $*$-averaging operation in terms of convolution:

Lemma 3.3.4. *The composition $\mathsf{oblv}^H \circ \mathsf{Av}_*^H$ coincides with the functor $k_H \star - : \mathcal{C} \to \mathcal{C}$. Consequently, Av_*^H is continuous.*

Proof. First, we show that $(p^!)^R : \mathfrak{D}^!(H) \to \mathrm{Vect}$ coincides with the composition

$$(p^!)^R \simeq p_*^{ren} : \mathfrak{D}^!(H) \xrightarrow{\eta_H} \mathfrak{D}^*(H) \xrightarrow{p_*} \mathrm{Vect}\,. \tag{3.3.4}$$

This amounts to an easy verification:

$$\mathrm{Hom}_{\mathfrak{D}^!(H)}(\omega_H, M) \simeq \mathrm{Hom}_{\mathfrak{D}^*(H)}(k_H, \eta_H(M)) \simeq \Gamma_{\mathrm{dR}}(H, \eta_H(M)).$$

Next, it is routine to check that the Beck-Chevalley condition holds for the cosimplicial complex

$$\mathcal{C} \rightrightarrows \mathfrak{D}^!(H) \otimes \mathcal{C} \rightrightarrows\!\!\!\rightarrow \cdots,$$

thus implying

$$\mathsf{oblv}^H \circ \mathsf{Av}_*^H \simeq (p^!)^R \circ \mathsf{coact}. \tag{3.3.5}$$

Thanks to this, the first assertion follows from Lemma 3.3.3. Continuity of Av_*^H is now obvious, as oblv^H is conservative and $k_H \star -$ is continuous. □

Corollary 3.3.5. *The functor oblv^H is fully faithful.*

Proof. Since oblv^H is always conservative, it suffices to prove that $\mathsf{oblv}^H \circ \mathsf{Av}_*^H \circ \mathsf{oblv}^H \simeq \mathsf{oblv}^H$, as functors $\mathcal{C}^H \to \mathcal{C}$. Using (3.3.4) and (3.3.5), we get

$$\mathsf{oblv}^H \circ \mathsf{Av}_*^H \circ \mathsf{oblv}^H \simeq (p^!)^R \circ \mathsf{coact} \circ \mathsf{oblv}^H \simeq (p^!)^R \circ p^! \circ \mathsf{oblv}^H \simeq \Gamma_{\mathrm{dR}}(H, k_H) \otimes \mathsf{oblv}^H,$$

and the conclusion follows by the contractibility of H. □

In view of the above corollary, we regard $\mathcal{C}^H$ as a subcategory of $\mathcal{C}$ and Av_*^H as an endofunctor of $\mathcal{C}$.

Remark 3.3.6. If $H \subset K$ are pro-unipotent group schemes, the embedding $\mathsf{oblv}^K : \mathcal{C}^K \hookrightarrow \mathcal{C}$ factors through $\mathsf{oblv}^H : \mathcal{C}^H \hookrightarrow \mathcal{C}$. We denote by $\mathsf{oblv}^{K\to H}$ (or oblv^{rel} or oblv when the context is clear), the resulting inclusion $\mathcal{C}^K \hookrightarrow \mathcal{C}^H$. This functor always admits a right adjoint, $\mathsf{Av}_*^{H\to K}$, or just Av_*^{rel} when no confusion is likely to occur. By contractibility, the composition $\mathsf{oblv}^{rel} \circ \mathsf{Av}_*^{rel}$ is given by convolution with k_K.

Let us deduce two properties of the !-averaging functor $\mathsf{Av}_!^H$.

Lemma 3.3.7. *The functor* $\mathsf{oblv}^H \circ \mathsf{Av}_!^H$ *is canonically isomorphic to* $p_! \circ \mathsf{coact}$*, whenever the latter is defined.*

Proof. It suffices to check that the right adjoint to $p_! \circ \mathsf{coact}$ is isomorphic to $\mathsf{oblv}^H \circ \mathsf{Av}_*^H$:

$$(p_! \circ \mathsf{coact})^R \simeq \mathsf{coact}^R \circ p^! \simeq (p^!)^R \circ \mathsf{coact},$$

and the claim follows from (3.3.5). The last isomorphism in the above formula is a consequence of the fact that $p^!$ and coact can be treated symmetrically: indeed, there is an automorphism of $\mathfrak{D}^!(H) \otimes \mathcal{C}$ that switches them. $\square$

Corollary 3.3.8. *If H is finite dimensional with* $\dim H = d$*, then there is a natural transformation*

$$\mathsf{Av}_!^H \to \mathsf{Av}_*^H[2d]. \tag{3.3.6}$$

Proof. We showed that $\mathsf{Av}_!^H \simeq p_! \circ \mathsf{coact}$. Under the assumption, $\mathsf{Av}_*^H \simeq p_*^{ren} \circ \mathsf{coact} \simeq p_* \circ \mathsf{coact}[-2d]$. The sought-after map is then induced by the canonical arrow $p_! \to p_*$, where $p : H \to \mathrm{pt}$. $\square$

3.4 Smooth generation

We now discuss a special features of module categories for $(\mathfrak{D}^*(H), \star)$, called *smooth generation.* Namely, if $\mathcal{C}$ is acted upon by H, then any object of c is generated by a sequence of objects each of which is invariant for a high congruence subgroup. More precisely, $\mathcal{C}$ is the colimit of its subcategories $\mathcal{C}^{H^r}$, under the relative forgetful maps.

Notice that we can choose a sequence of *normal* subgroups of H, shrinking to the identity element. Let $\{\widetilde{H}^r\}_r$ be such a sequence, where the tilde indicates that $\widetilde{H}^r$ is in general different from H^r (but commensurable to it). The quotients are now themselves groups: $\widetilde{Q}^r := H/\widetilde{H}^r$. In this subsection we drop the tilde from the notation: we warn the reader that this notation clashes with the one in the previous section.

Lemma 3.4.1. *For any $c \in \mathcal{C}$, there is an equivalence*

$$c \simeq \operatorname*{colim}_{r\geq 1} \Big(\mathsf{oblv}^{H^r} \circ \mathsf{Av}_*^{H^r}(c)\Big).$$

Proof. Consider the delta $\mathfrak{D}^*$-module $\delta_{1,H}$ at $1 \in H$, the monoidal unit of the convolution on $\mathfrak{D}^*(H)$. By (2.1.3), $\delta_{1,H}$ can be written as the colimit

$$\delta_{1,H} \simeq \operatorname*{colim}_{r\geq 1} (\pi_{\infty\to r})^*(\delta_{1,Q^r}),$$

so that

$$c \simeq \delta_{1,H} \star c \simeq \operatorname*{colim}_{r\geq 1} \Big((\pi_{\infty\to r})^*(\delta_{1,Q^r}) \star c \Big).$$

In view of Lemma 3.3.4, it suffices to prove that $(\pi_{\infty\to r})^*(\delta_{1,Q^r}) \simeq k_{H^r}$, for any $r \geq 1$.

To show this, we reconvert the expression $(\pi_{\infty\to r})^*(\delta_{1,Q^r})$ into an object of $\lim_r \mathfrak{D}(Q^r)$ by means of the functor (2.1.4). By almost contractibility we obtain

$$(\pi_{\infty\to r})^*(\delta_{1,Q^r}) \xrightarrow{\mathsf{LC}^{-1}} \big\{ (\pi_{s\to r})^*(\delta_{1,Q^r}) \big\}_{s\geq r},$$

and, by base-change,

$$(\pi_{\infty\to r})^*(\delta_{1,Q^r}) \simeq \{ k_{\mathrm{fib}(Q^s\to Q^r)} \}_{s\geq r}. \tag{3.4.1}$$

Here, $\mathrm{fib}(Q^s \to Q^r)$ is the fiber of the projection $Q^s \to Q^r$ over $1 \in Q^r$, which is of course H^r/H^s. Thus, (3.4.1) coincides with k_{H^r}, by the very construction of the latter. □

Corollary 3.4.2. *If $\mathcal{C}$ is acted on by H, then $\mathcal{C}$ is automatically smoothly generated, i.e.*

$$\mathcal{C} \simeq \operatorname*{colim}_{r\geq 1} \mathcal{C}^{H^r}, \tag{3.4.2}$$

the colimit being taken along the relative forgetful functors $\mathsf{oblv}^{rel} : \mathcal{C}^{H^r} \to \mathcal{C}^{H^{r+1}}$.

Proof. The collection of functors $\mathsf{oblv}^{H^r} : \mathcal{C}^{H^r} \to \mathcal{C}$ yields a functor $\alpha : \operatorname*{colim}_{r\geq 1} \mathcal{C}^{H^r} \to \mathcal{C}$, which is essentially surjective by the above lemma. It remains to prove that α is fully faithful. Let us write

$$\operatorname*{colim}_{r\geq 1,\mathsf{oblv}} \mathcal{C}^{H^r} \simeq \lim_{r\geq 1,\mathsf{Av}_*^{rel}} \mathcal{C}^{H^r},$$

and denote by ins_r and ev_r the structure functors, as usual. Then, $\alpha \circ \mathsf{ins}_r \simeq \mathsf{oblv}^{H^r}$ by definition, so that $\beta := \alpha^R$ satisfies the relation $\mathsf{ev}_r \circ \beta = \mathsf{Av}_*^{H^r}$. In other words, β consists of the inverse family of functors $\mathsf{Av}_*^{H^r} : \mathcal{C} \to \mathcal{C}^{H^r}$. To conclude, it suffices to prove that $\beta \circ \alpha \simeq \mathrm{id}$. For $c \in \mathcal{C}^{H^r}$, we have

$$\beta \circ \alpha(\mathsf{ins}_r(c)) = \{\mathsf{Av}_*^{H^s}(c)\}_{s\geq r} \simeq \{\mathsf{oblv}^{r\to s}(c)\}_{s\geq r} \simeq \operatorname*{colim}_{s\geq r} \mathsf{ins}_s(\mathsf{oblv}^{r\to s} c) = \mathsf{ins}_r(c).$$

□

3.5 Harish-Chandra bimodules and the adjoint representation

In this section we deal with groups of finite type and discuss the relation between weak and strong invariants in detail.

Let G be an affine algebraic group of finite type (as always, defined over $\mathbb{C}$). We freely use the notion of "de Rham" functor, as developed in [GR0]. Specifically, there is a group prestack G_{dR} such that $\mathfrak{D}(G) := \mathrm{QCoh}(G_{\mathrm{dR}})$. This is the realization of $\mathfrak{D}$-modules as *left crystals*.

From this point of view, it is clear that $\mathrm{QCoh}(G)$ acts on $\mathfrak{D}(G)$ via the monoidal functor $\mathbf{ind} : \mathrm{QCoh}(G) \to \mathfrak{D}(G)$, left adjoint to the forgetful functor $\mathsf{oblv}_\ell : \mathfrak{D}(G) \to \mathrm{QCoh}(G)$.

Recall that the action of G on a scheme X of finite type induces a weak (resp., strong) action of G on $\mathrm{QCoh}(X)$ (resp., $\mathfrak{D}(X)$). Slighly more general is the following important example.

Let $\widehat{G}$ denote the formal group of G at $1 \in G$. By definition,

$$\widehat{G} := \{1\} \times_{G_{\mathrm{dR}}} G.$$

Note that $G_{\mathrm{dR}} \simeq G/\widehat{G}$, so that $B\widehat{G} \simeq G_{\mathrm{dR}}/G$. Thus, as a prestack, $B\widehat{G}$ carries an action of G_{dR}, induced by the multiplication of G_{dR}:

$$G_{\mathrm{dR}} \times B\widehat{G} \simeq G_{\mathrm{dR}} \times G_{\mathrm{dR}}/G \xrightarrow{m} G_{\mathrm{dR}}/G \simeq B\widehat{G}. \tag{3.5.1}$$

Remark 3.5.1. We claim that this is a familiar structure. Let $\mathfrak{g}$-mod is the category of representations of the Lie algebra of G. There is an equivalence $\mathrm{QCoh}(B\widehat{G}) \simeq \mathfrak{g}$-mod, under which the strong G-action just described corresponds to the adjoint action of G on $\mathfrak{g}$. This follows from the equivalence between Lie algebras and formal groups, which holds over a field of characteristic zero. The correspondence associates to a Lie algebra $\mathfrak{g}$ the formal group $\mathrm{Spf}(U(\mathfrak{g})^\vee)$ and it is well-known that $U(\mathfrak{g})$ and $\mathcal{O}(\widehat{G})$ are dual Hopf algebras.

We will not use any Lie algebra theory in the present paper, and by the symbol "$\mathfrak{g}$-mod" we understand the category $\mathrm{QCoh}(B\widehat{G})$. Formula (3.5.1) implies that G acts strongly on $\mathfrak{g}$-mod. It will be shown later that

$$(\mathfrak{g}\text{-mod})^{G,s} \simeq \mathrm{Rep}(G),$$

while $(\mathfrak{g}\text{-mod})^{G,w}$ is the *Harish-Chandra* category to be studied in detail in the next paragraphs.

We digress briefly to discuss actions by normal subgroups. Let $i : K \hookrightarrow G$ be a normal subgroup. The category $\mathfrak{D}(K)$ acts on $\mathfrak{D}(G)$, as the functor $i_* : \mathfrak{D}(K) \to \mathfrak{D}(G)$ is monoidal. We prove that G-invariants can be taken in two steps: K-invariants first and then G/K-invariants.

Lemma 3.5.2. *Let $\mathcal{C}$ be a category with a strong action of G. The quotient group G/K acts on $\mathcal{C}^K$ and $\mathcal{C}_K$. Furthermore,*

$$\mathcal{C}^G \simeq \left(\mathcal{C}^K\right)^{G/K} \quad \textit{and} \quad \mathcal{C}_G \simeq \left(\mathcal{C}_K\right)_{G/K}.$$

The statement holds verbatim for weak actions and weak (co)invariants.

Proof. For coinvariants, we have

$$\mathcal{C}_K := \mathrm{Vect} \underset{\mathfrak{D}(K)}{\otimes} \mathcal{C} \simeq \left(\mathrm{Vect} \underset{\mathfrak{D}(K)}{\otimes} \mathfrak{D}(G)\right) \underset{\mathfrak{D}(G)}{\otimes} \mathcal{C}.$$

The expression in parenthesis is by definition the category of K-coinvariants of $\mathfrak{D}(G)$, which is $\mathfrak{D}(G/K)$ by descent. This makes it clear that G/K acts on $\mathcal{C}_K$. Moreover,

$$\left(\mathcal{C}_K\right)_{G/K} \simeq \mathrm{Vect} \underset{\mathfrak{D}(G/K)}{\otimes} \mathcal{C}_K \simeq \mathrm{Vect} \underset{\mathfrak{D}(G/K)}{\otimes} \mathfrak{D}(G/K) \underset{\mathfrak{D}(G)}{\otimes} \mathcal{C} \simeq \mathrm{Vect} \underset{\mathfrak{D}(G)}{\otimes} \mathcal{C} =: \mathcal{C}_G.$$

The argument for invariants is completely analogous. □

As the quotient $G/\widehat{G} \simeq G_{\mathrm{dR}}$ is a group prestack, the above reasoning can be applied to the group morphism $\widehat{G} \to G$, yielding the following result:

Lemma 3.5.3. *If $\mathcal{C}$ admits a weak action of G, then $\mathcal{C}^{\widehat{G}}$ carries a strong action of G and*

$$\mathcal{C}^{G,w} \simeq (\mathcal{C}^{\widehat{G},w})^{G,s} \quad \textit{and} \quad \mathcal{C}_{G,w} \simeq (\mathcal{C}_{\widehat{G},w})_{G,s}.$$

We now explain how to recover strong invariants from weak invariants. (Ultimately, we shall use this to prove that the $*$-averaging functor induces an equivalence $\mathcal{C}_H \simeq \mathcal{C}^H$, whenever H is a pro-unipotent pro-group.)

We need a preliminary lemma. Recall that $\mathfrak{g}$-mod $\simeq \mathrm{QCoh}(B\widehat{G})$ admits a strong action of G.

Lemma 3.5.4. *If $\mathcal{C}$ is a category with a strong action of G, then*

$$\mathcal{C}^{G,w} \simeq \mathrm{Hom}_{\mathfrak{D}(G)}(\mathfrak{g}\text{-mod}, \mathcal{C}) \quad \textit{and} \quad \mathcal{C}_{G,w} \simeq \mathfrak{g}\text{-mod} \underset{\mathfrak{D}(G)}{\otimes} \mathcal{C}.$$

Proof. The defining formula $\mathcal{C}^{G,w} := \mathrm{Hom}_{\mathrm{QCoh}(G)}(\mathrm{Vect}, \mathcal{C})$ becomes, by adjunction,

$$\mathcal{C}^{G,w} \simeq \mathrm{Hom}_{\mathfrak{D}(G)}\left(\mathrm{Vect} \otimes_{\mathrm{QCoh}(G)} \mathfrak{D}(G), \mathcal{C}\right).$$

Using the same logic, the formula for coinvariants becomes

$$\mathcal{C}_{G,w} := \mathrm{Vect} \underset{\mathrm{QCoh}(G)}{\otimes} \mathcal{C} \simeq \mathrm{Vect} \underset{\mathrm{QCoh}(G)}{\otimes} \mathfrak{D}(G) \underset{\mathfrak{D}(G)}{\otimes} \mathcal{C}.$$

In both cases, it remains to show that

$$\mathrm{Vect} \underset{\mathrm{QCoh}(G)}{\otimes} \mathfrak{D}(G) \simeq \mathfrak{g}\text{-mod}.$$

Since

$$\mathrm{Vect} \underset{\mathrm{QCoh}(G)}{\otimes} \mathfrak{D}(G) = \mathfrak{D}(G)^{G,w} \simeq \mathrm{QCoh}(G_{\mathrm{dR}}/G) \simeq \mathrm{QCoh}(B\widehat{G}),$$

this is clear. □

Consequently, if $\mathcal{C}$ admits a strong action of G, its weak invariant and coinvariant categories $\mathcal{C}^{G,w}$ and $\mathcal{C}_{G,w}$ both retain an action of the monoidal category

$$\mathsf{HC} := \mathrm{Hom}_{\mathfrak{D}(G)}(\mathfrak{g}\text{-mod}, \mathfrak{g}\text{-mod}),$$

where the monoidal structure is given by composition. This is called the category of *Harish-Chandra bimodules.* By the above,

$$\mathsf{HC} \simeq \mathrm{Hom}_{\mathrm{QCoh}(G)}(\mathrm{Vect}, \mathfrak{g}\text{-mod}) \simeq (\mathfrak{g}\text{-mod})^{G,w}.$$

Here is yet another description:

Lemma 3.5.5. *There is a natural monoidal equivalence*

$$\Big(\mathrm{QCoh}(G\backslash G_{\mathrm{dR}}/G), \star\Big) \simeq (\mathsf{HC}, \circ).$$

The next result shows that HC is a rigid monoidal category ([G0]). In particular, HC is self-dual and a module category over it is dualizable in HC-**mod** if and only if it is dualizable as a plain category.

Proposition 3.5.6. *The monoidal category* HC *is compactly generated by objects that are left and right dualizable.*

Proof. Let $\mathfrak{q} : G \to G_{\mathrm{dR}}$ be the quotient map. It induces a map, denoted again by $\mathfrak{q}$,

$$\mathfrak{q} : BG \simeq G\backslash G/G \to G\backslash G_{\mathrm{dR}}/G.$$

Using this description, it is easy to see that $\mathfrak{q}_* : \mathrm{Rep}(G) \to \mathsf{HC}$ is monoidal. As any compact object in $\mathrm{Rep}(G)$ is (both left and right) dualizable, it suffices to prove that the right adjoint to $\mathfrak{q}_*$ is continuous and conservative. □

We shall prove the following result:

Theorem 3.5.7. *For any affine algebraic group G over a field of characteristic 0, the functor of weak invariants $inv^w : \mathcal{C} \rightsquigarrow \mathcal{C}^{G,w}$ induces an equivalence*

$$\mathsf{HC}\text{-}\mathbf{mod} \xleftarrow{\simeq} (\mathfrak{D}(G), \star)\text{-}\mathbf{mod}.$$

Before explaining the proof, let us notice that the weak (co)invariant category of any $(\mathrm{QCoh}(G), \star)$-module admits an action of $\mathrm{Vect} \otimes_{\mathrm{QCoh}(G)} \mathrm{Vect} \simeq \mathrm{Rep}(G)$. Thus, there is an adjunction

$$rec : (\mathrm{Rep}(G), \otimes)\text{-}\mathbf{mod} \rightleftarrows (\mathrm{QCoh}(G), \star)\text{-}\mathbf{mod} : inv^w, \tag{3.5.2}$$

where inv^w is the functor of weak G-invariants and rec, the "reconstruction" functor, sends $\mathcal{E}$ to $\mathrm{Vect} \otimes_{\mathrm{Rep}(G)} \mathcal{E}$. It is a theorem, due to Gaitsgory (announced in [AG] and proven in [G7]), that these adjoint functors are mutually inverse equivalences of categories. Our proof of Theorem 3.5.7 is a formal consequence of this.

Proof of Theorem 3.5.7. Recall the adjunction

$$\mathbf{ind} := \mathfrak{D}(G) \otimes_{\mathrm{QCoh}(G)} - : \mathrm{QCoh}(G)\text{-}\mathbf{mod} \rightleftarrows \mathfrak{D}(G)\text{-}\mathbf{mod} : \mathsf{oblv}^{s \to w}. \tag{3.5.3}$$

Combined with (3.5.2), it yields an adjunction

$$\mathsf{F} : \mathrm{Rep}(G)\text{-}\mathbf{mod} \rightleftarrows \mathfrak{D}(G)\text{-}\mathbf{mod} : inv^w, \tag{3.5.4}$$

where

$$\mathsf{F}(\mathcal{E}) = \mathbf{ind} \circ rec \simeq \mathfrak{D}(G) \underset{\mathrm{QCoh}(G)}{\otimes} \mathrm{Vect} \underset{\mathrm{Rep}(G)}{\otimes} \mathcal{E} \simeq \mathfrak{g}\text{-mod} \underset{\mathrm{Rep}(G)}{\otimes} \mathcal{E}.$$

Note that $\mathsf{oblv}^{s \to w}$ is continuous and conservative: in fact, the absolute forgetful functor $\mathsf{oblv} : \mathfrak{D}(G)\text{-}\mathbf{mod} \to \mathrm{DGCat}$ is conservative. Furthermore, the functor inv^w in (3.5.2), being an equivalence, is continuous and conservative as well. Thus, we apply the Barr-Beck theorem to the adjunction (3.5.4): inv^w provides an equivalence between $\mathfrak{D}(G)\text{-}\mathbf{mod}$ and the category of modules for the monad

$$\mathcal{E} \mapsto (\mathfrak{g}\text{-mod} \otimes_{\mathrm{Rep}(G)} \mathcal{E})^{G,w}$$

in $\mathrm{Rep}(G)$-$\mathbf{mod}$. It remains to compute this monad:

$$(\mathfrak{g}\text{-mod} \underset{\mathrm{Rep}(G)}{\otimes} \mathcal{E})^{G,w} = \mathrm{Vect}^\vee \underset{\mathrm{QCoh}(G)}{\otimes} \mathfrak{g}\text{-mod} \underset{\mathrm{Rep}(G)}{\otimes} \mathcal{E} \simeq \widehat{\mathsf{HC}} \underset{\mathrm{Rep}(G)}{\otimes} \mathcal{E}.$$

We have denoted by $\widehat{\mathsf{HC}}$ the category HC, considered as a $(\mathrm{Rep}(G))$-bimodule. In summary, inv^w gives an equivalence between $\mathfrak{D}(G)\text{-}\mathbf{mod}$ and $\widehat{\mathsf{HC}}\text{-mod}(\mathrm{Rep}(G)\text{-}\mathbf{mod})$. Another straightforward application of Barr-Beck yields a canonical equivalence $\widehat{\mathsf{HC}}\text{-mod}(\mathrm{Rep}(G)\text{-}\mathbf{mod}) \simeq \mathsf{HC}\text{-}\mathbf{mod}$, concluding the proof. $\square$

As an immediate corollary, here is the relation between strong and weak (co)invariants:

Corollary 3.5.8. *If $\mathcal{C}$ is equipped with a strong action of G, we can compute its strong invariants as follows:*

$$\mathcal{C}^{G,s} \simeq \mathrm{Hom}_{\mathsf{HC}}(\mathrm{Rep}(G), \mathcal{C}^{G,w}) \quad \textit{and} \quad \mathcal{C}_{G,s} \simeq \mathrm{Rep}(G) \underset{\mathsf{HC}}{\otimes} \mathcal{C}_{G,w}.$$

3.6 Equivalence between invariants and coinvariants for group schemes

We are now ready to prove the following important result.

Theorem 3.6.1. *For a pro-unipotent group H and a category $\mathcal{C} \in \mathfrak{D}^*(H)$-**mod**, the coinvariant and invariant categories $\mathcal{C}_H$ and $\mathcal{C}^H$ are equivalent. In particular, the operation $\mathcal{C} \rightsquigarrow \mathcal{C}^H$ commutes with colimits and tensor products by categories.*

Proof. Let H be finite dimensional first. Recall the equivalence between $(\mathrm{QCoh}(H), \star)$-**mod** $\simeq$ $\mathrm{Rep}(H)$-**mod**. Since $\mathrm{Rep}(H)$ is a rigid monoidal category, we deduce an equivalence

$$\mathcal{C}^{H,w} = \mathrm{Vect}^\vee \underset{\mathrm{Rep}(H)}{\otimes} \mathcal{C} \simeq \mathrm{Vect} \underset{\mathrm{Rep}(H)}{\otimes} \mathcal{C} = \mathcal{C}_{H,w}$$

between weak invariants and weak coinvariants. From this, we infer the same result for strong invariants. Since HC is a rigid monoidal category and $\mathrm{Rep}(H)$ is self-dual as a plain category, we obtain from Corollary 3.5.8 that

$$\mathcal{C}^H \simeq \mathrm{Rep}(H)^\vee \underset{\mathsf{HC}}{\otimes} \mathcal{C}^{H,w} \simeq \mathrm{Rep}(H) \underset{\mathsf{HC}}{\otimes} \mathcal{C}^{H,w} \simeq \mathrm{Rep}(H) \underset{\mathsf{HC}}{\otimes} \mathcal{C}_{H,w} = \mathcal{C}_H. \tag{3.6.1}$$

This proves the assertion for H finite dimensional.

To extend the result to H being a pro-scheme, we resort to the smooth generation of $\mathcal{C}$: let $\{H^r\}_r$ be a sequence of normal subgroups shrinking to the identity element; by (3.4.2), we have

$$\mathcal{C}_H \simeq \underset{r\geq 1}{\mathrm{colim}}\, (\mathcal{C}^{H^r})_H.$$

As the action of H on $\mathcal{C}^{H^r}$ factors through $H \to H/H^r$, we further obtain

$$\mathcal{C}_H \simeq \underset{r\geq 1}{\mathrm{colim}}\, (\mathcal{C}^{H^r})_{H/H^r} \simeq \underset{r\geq 1}{\mathrm{colim}}\, (\mathcal{C}^{H^r})^{H/H^r}.$$

where the second equivalence follows from finite dimensionality of H/H^r, invoking (3.6.1). Lastly, $(\mathcal{C}^{H^r})^{H/H^r} \simeq \mathcal{C}^H$, which concludes the proof. □

Next, we single out a naturally defined functor $\mathsf{S} : \mathcal{C}_H \to \mathcal{C}^H$ realizing the above equivalence: this is induced by

$$\mathrm{Vect} \otimes \mathcal{C} \to \mathcal{C}, \quad V \otimes c \mapsto V \otimes (k_H \star c).$$

That S descends to a functor $\mathrm{Vect} \underset{\mathfrak{D}^*(H)}{\otimes} \mathcal{C} \to \mathcal{C}$ follows from the canonical equivalence

$$k_H \star (M \star c) \simeq \Gamma_{\mathrm{dR}}(H, M) \otimes k_H \star c, \qquad \text{for any } M \in \mathfrak{D}^*(H),\ c \in \mathcal{C}.$$

proven above. Moreover, S lands in $\mathcal{C}^H$, as $k_H \star - \simeq \mathsf{oblv}^H \circ \mathsf{Av}^H_*$.

Proposition 3.6.2. *The above functor* S *is an equivalence.*

Proof. Any $\mathfrak{D}^*(H)$-module is, via the Bar resolution, a colimit of categories of the form $\mathfrak{D}^*(H) \otimes \mathcal{E}$, where $\mathcal{E}$ is endowed with the trivial action of H. Thus, in view of Theorem 3.6.1, it suffices to prove that S is an equivalence when $\mathcal{C} = \mathfrak{D}^*(H)$ is the regular representation.

In this case, $\mathcal{C}_H$ and $\mathcal{C}^H$ are both equivalent to Vect. More precisely, for coinvariants, the canonical equivalence $\mathrm{Vect} \simeq \mathrm{Vect} \otimes_{\mathfrak{D}^*(H)} \mathcal{C} =: \mathcal{C}_H$ is obtained from the functor

$$\alpha : \mathrm{Vect} \xrightarrow{\simeq} \mathrm{Vect} \underset{\mathfrak{D}^*(H)}{\otimes} \mathcal{C}, \quad \alpha : V \mapsto V \otimes \delta_1.$$

For invariants, the equivalence $\mathrm{Vect} \simeq \mathrm{Hom}_{\mathfrak{D}^*(H)}(\mathrm{Vect}, \mathcal{C}) =: \mathcal{C}^H$ is induced by the map

$$\beta : \mathrm{Vect} \xrightarrow{\simeq} \mathrm{Hom}_{\mathfrak{D}^*(H)}(\mathrm{Vect}, \mathcal{C}), \quad \beta : V \mapsto (\mathbb{C} \mapsto V \otimes k_H).$$

Under α and β, the functor S goes over to the identity $\mathrm{Vect} \to \mathrm{Vect}$. $\square$

Chapter 4

Whittaker actions

We focus now on categories with an action of $\mathbf{N}$ and study their (co)invariants in terms of the invariants for the sequence of $\mathbf{N}_k$. After this, we introduce an additive character χ of $\mathbf{N}$ and study $\mathbf{N}$-(co)invariants of $\mathcal{C}$ against χ. The resulting categories are by definition the Whittaker categories of $\mathcal{C}$.

4.1 Invariants and coinvariants with respect to $N(\!(t)\!)$

Let $i_k : \mathbf{N}_k \to \mathbf{N}_{k+1}$ the inclusion.

Proposition 4.1.1. *There are natural equivalences*

$$\mathcal{C}^{\mathbf{N}} := \lim_{\mathsf{oblv}^{rel}} \mathcal{C}^{\mathbf{N}_k} \quad \textit{and} \quad \mathcal{C}_{\mathbf{N}} := \operatorname*{colim}_{\mathsf{Av}_*^{rel}} \mathcal{C}^{\mathbf{N}_k}.$$

Proof. Let us treat coinvariants first. As the functor $(i_k)_* : (\mathfrak{D}^*(\mathbf{N}_k), \star) \to (\mathfrak{D}^*(\mathbf{N}_{k+1}), \star)$ is monoidal, the equivalence $\mathfrak{D}^*(\mathbf{N}) = \operatorname{colim}_{i_*} \mathfrak{D}^*(\mathbf{N}_k)$ is an equivalence of *monoidal* categories. Hence, we can commute the colimit under the tensor product:

$$\mathcal{C}_{\mathbf{N}} := \operatorname{Vect} \underset{\mathfrak{D}(\mathbf{N})}{\otimes} \mathcal{C} \simeq \operatorname{Vect} \underset{\operatorname*{colim}_{k,i_*} \mathfrak{D}(\mathbf{N}_k)}{\otimes} \mathcal{C} \simeq \operatorname*{colim}_{k,i_*} \Big(\operatorname{Vect} \underset{\mathfrak{D}(\mathbf{N}_k)}{\otimes} \mathcal{C} \Big) \simeq \operatorname*{colim}_{k,i_*} \mathcal{C}_{\mathbf{N}_k}.$$

Next, identifying $\mathcal{C}_{\mathbf{N}_k}$ with $\mathcal{C}^{\mathbf{N}_k}$ via Proposition 3.6.2, the map induced by i_* goes over to $\mathsf{Av}_*^{rel} : \mathcal{C}^{\mathbf{N}_k} \to \mathcal{C}^{\mathbf{N}_{k+1}}$, the right adjoint to the inclusion $\mathsf{oblv}^{rel} : \mathcal{C}^{\mathbf{N}_{k+1}} \to \mathcal{C}^{\mathbf{N}_k}$. Indeed, this follows from the commutativity of the diagram:

$$\begin{tikzcd}
\mathfrak{D}^*(\mathbf{N}_k) \otimes \mathcal{C} \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "i_*"] & \mathcal{C} \arrow[r, "\mathsf{Av}_*^{\mathbf{N}_k}"] \arrow[d, "\mathrm{id}"] & \mathcal{C}^{\mathbf{N}_k} \arrow[d, "\mathsf{Av}_*^{rel}"] \\
\mathfrak{D}^*(\mathbf{N}_{k+1}) \otimes \mathcal{C} \arrow[r, shift left] \arrow[r, shift right] & \mathcal{C} \arrow[r, "\mathsf{Av}_*^{\mathbf{N}_{k+1}}"] & \mathcal{C}^{\mathbf{N}_{k+1}}.
\end{tikzcd} \tag{4.1.1}$$

The computation of $\mathbf{N}$-invariants is easier: $\mathcal{C}^{\mathbf{N}}$ is the limit of $\mathcal{C}^{\mathbf{N}_k}$, along the transition maps $\mathcal{C}^{\mathbf{N}_{k+1}} \to \mathcal{C}^{\mathbf{N}_k}$ induced by $i^! : \mathfrak{D}^!(\mathbf{N}_{k+1}) \to \mathfrak{D}^!(\mathbf{N}_k)$. The relevant diagram

$$\begin{tikzcd}
\mathfrak{D}^!(\mathbf{N}_k) \otimes \mathcal{C} & \mathcal{C} \arrow[l, Leftarrow] & \mathcal{C}^{\mathbf{N}_k} \arrow[l, "\mathsf{oblv}^{\mathbf{N}_k}"'] \\
\mathfrak{D}^!(\mathbf{N}_{k+1}) \otimes \mathcal{C} \arrow[u, "i^!"'] & \mathcal{C} \arrow[l, Leftarrow] \arrow[u, "\mathrm{id}"] & \mathcal{C}^{\mathbf{N}_{k+1}} \arrow[l, "\mathsf{oblv}^{\mathbf{N}_{k+1}}"] \arrow[u, "\mathsf{oblv}^{rel}"']
\end{tikzcd} \tag{4.1.2}$$

is commutative (the assertion for the left square follows by duality from commutativity of the left square of (4.1.1)). This identifies $\mathcal{C}^{\mathbf{N}_{k+1}} \to \mathcal{C}^{\mathbf{N}_k}$ as oblv^{rel}. □

4.2 Whittaker invariants and coinvariants

To introduce the main mathematical objects of this paper, the Whittaker invariant and coinvariant categories of an object of $\mathbf{N}\text{-}\mathbf{rep}$, we need to first discuss $\mathbf{N}$-actions on the trivial category Vect.

Strong actions on Vect

As mentioned before, a strong action of $\mathbf{N}$ on Vect is given by a monoidal functor $\mathfrak{D}^*(\mathbf{N}) \to \mathrm{Vect}$. Equivalently, by a comonoidal functor $\mathrm{Vect} \to \mathfrak{D}^!(\mathbf{N})$. Such functors correspond precisely to *character* $\mathfrak{D}^!$*-modules*, i.e. $\mathfrak{D}^!$-modules $\mathcal{F}$ with an isomorphism

$$m^!(\mathcal{F}) \simeq \mathcal{F} \boxtimes \mathcal{F},$$

satisfying the natural compatibility conditions. For such $\mathcal{F}$, the action map $\mathfrak{D}^*(\mathbf{N}) \otimes \mathrm{Vect} \to \mathrm{Vect}$ is given by

$$M \otimes V \mapsto M \star_{\mathcal{F}} V := \varepsilon_{\mathbf{N}}(M \otimes \mathcal{F}) \otimes V,$$

where, as before, $\varepsilon_{\mathbf{N}}$ is the duality pairing between $\mathfrak{D}^*(\mathbf{N})$ and $\mathfrak{D}^!(\mathbf{N}) := \mathfrak{D}^*(\mathbf{N})^\vee$.

Consider the exponential (right) $\mathfrak{D}$-module exp on $\mathbb{G}_a = \mathbb{A}^1 = \mathrm{Spec}(k[z])$:

$$exp = \frac{\mathfrak{D}_{\mathbb{A}^1}}{(\partial_z - 1)\mathfrak{D}_{\mathbb{A}^1}}. \tag{4.2.1}$$

We point out that exp is a substitute of the Artin-Schreier sheaf in characteristic zero. It is a character $\mathfrak{D}$-module,

$$m^! exp \simeq exp \boxtimes exp, \tag{4.2.2}$$

and the prototype of all the character $\mathfrak{D}$-modules we shall consider. In fact,

Lemma 4.2.1. *Let $\mathcal{G}$ be an ind-pro-scheme (in our case $\mathcal{G} = \mathbf{N}$) equipped with an additive character: $\chi : \mathcal{G} \to \mathbb{G}_a$. Then $\chi^!(exp) \in \mathfrak{D}^!(\mathcal{G})$ is a character $\mathfrak{D}^!$-module.*

Proof. This follows from the equality $m \circ (\chi \times \chi) = \chi \circ m$. $\square$

We write Vect_χ to emphasize that Vect is being considered as a category with the $\mathfrak{D}^!(\mathbf{N})$-coaction corresponding to $\chi^! exp$.

If $\mathbf{N}$ acts on $\mathcal{C}$, we have an action of $\mathbf{N}$ on $\mathcal{C} \otimes \mathrm{Vect}_\chi$: by (3.2.1), this is simply determined in terms of the coaction of $\mathfrak{D}^!(\mathbf{N})$ on $\mathcal{C}$ by the assignment

$$\mathcal{C} \to \mathfrak{D}^!(\mathbf{N}) \otimes \mathcal{C}, \quad c \mapsto \chi^!(exp) \otimes \mathsf{coact}(c),$$

where we have identified $\mathcal{C} \simeq \mathcal{C} \otimes \mathrm{Vect}_\chi$. Dually, the action of $\mathfrak{D}^*(\mathbf{N})$ on $\mathcal{C} \otimes \mathrm{Vect}_\chi$ consists of the composition of the "old" action of $\mathfrak{D}^*(\mathbf{N})$ on $\mathcal{C}$ with the monoidal automorphism

$$\mathfrak{D}^*(\mathbf{N}) \to \mathfrak{D}^*(\mathbf{N}), \;\; M \mapsto \chi^!(exp) \overset{!*}{\otimes} M,$$

where $\overset{!*}{\otimes}$ denotes the action of $\mathfrak{D}^!(\mathbf{N})$ on $\mathfrak{D}^*(\mathbf{N})$.

We define the *Whittaker invariant* and *Whittaker coinvariant* categories of $\mathcal{C}$ respectively as

$$\mathcal{C}^{\mathbf{N},\chi} := \mathrm{Hom}_{\mathfrak{D}^*(\mathbf{N})}(\mathrm{Vect}, \mathrm{Vect}_\chi \otimes \mathcal{C}) \quad \text{and} \quad \mathcal{C}_{\mathbf{N},\chi} := \mathrm{Vect} \underset{\mathfrak{D}^*(\mathbf{N})}{\otimes} (\mathcal{C} \otimes \mathrm{Vect}_\chi).$$

Lemma 4.2.2. *Alternatively,*

$$\mathcal{C}^{\mathbf{N},\chi} := \mathrm{Hom}_{\mathfrak{D}^*(\mathbf{N})}(\mathrm{Vect}_{-\chi}, \mathcal{C}) \quad \textit{and} \quad \mathcal{C}_{\mathbf{N},\chi} := \mathrm{Vect}_{-\chi} \underset{\mathfrak{D}^*(\mathbf{N})}{\otimes} \mathcal{C}.$$

Proof. The automorphism of $\mathfrak{D}^!(\mathbf{N})$ sending $M \mapsto M \otimes (-\chi)^!(exp)$ converts the original definitions into the ones of the lemma. $\square$

We also define

$$\mathcal{C}^{\mathbf{N}_k,\chi} := \mathcal{C}^{\mathbf{N}_k,\chi} := (\mathcal{C} \otimes \mathrm{Vect}_\chi)^{\mathbf{N}_k} \quad \text{and} \quad \mathcal{C}_{\mathbf{N}_k,\chi} := (\mathcal{C} \otimes \mathrm{Vect}_\chi)_{\mathbf{N}_k}.$$

In view of the above Proposition 4.1.1, we have:

$$\mathcal{C}^{\mathbf{N},\chi} := \lim_{\mathsf{oblv}} \mathcal{C}^{\mathbf{N}_k,\chi} \quad \text{and} \quad \mathcal{C}_{\mathbf{N},\chi} := \operatorname*{colim}_{\mathsf{Av}_*} \mathcal{C}^{\mathbf{N}_k,\chi}.$$

We set

$$\chi_{\mathbf{N}_k} := \eta_k(\chi_k^!(exp)) \in \mathfrak{D}^*(\mathbf{N}_k),$$

where χ_k is the restriction of χ to $\mathbf{N}_k$ and $\eta_k : \mathfrak{D}^!(\mathbf{N}_k) \to \mathfrak{D}^*(\mathbf{N}_k)$ is the usual self-duality. The right adjoint to the inclusion $\mathsf{oblv}^{\mathbf{N}_k} : \mathcal{C}^{\mathbf{N}_k,\chi} \hookrightarrow \mathcal{C}$ is denoted $\mathsf{Av}_*^{\mathbf{N}_k,\chi}$.

Lemma 4.2.3. *The functor* $\mathsf{oblv}^{\mathbf{N}_k} \circ \mathsf{Av}_*^{\mathbf{N}_k,\chi}$ *in given by convolution with* $\chi_{\mathbf{N}_k}$.

Proof. Let coact_χ be the coaction $\mathcal{C} \otimes \mathrm{Vect}_\chi \to \mathfrak{D}^!(\mathbf{N}_k) \otimes \mathcal{C} \otimes \mathrm{Vect}_\chi$: under the identification $\mathcal{C} \otimes \mathrm{Vect}_\chi \simeq \mathcal{C}$, it is simply given by $\chi_k^! exp \otimes \mathsf{coact}$.

Now, convolution with $\chi_{\mathbf{N}_k}$ is given by $p_* \circ \eta_k(\chi_k^! exp \otimes \mathsf{coact}(-)) \simeq p_* \circ \eta_k \circ \mathsf{coact}_\chi$; on the other hand, base-change yields

$$\mathsf{oblv}^{\mathbf{N}_k} \circ \mathsf{Av}_*^{\mathbf{N}_k,\chi} \simeq (p^!)^R \circ \mathsf{coact}_\chi,$$

concluding the proof in view of (3.3.4). $\square$

The two conjectures that follow have been proposed by Gaitsgory. First,

Conjecture 4.2.4. *For any* $\mathcal{C}$ *with an action of* $\mathbf{G}$*, there is an equivalence* $\mathcal{C}^{\mathbf{N},\chi} \simeq \mathcal{C}_{\mathbf{N},\chi}$.

This conjecture can be refined. Indeed, there is always a natural functor (or rather, a $\mathbb{Z}$-family of such) mapping $\mathcal{C}_{\mathbf{N},\chi} \to \mathcal{C}^{\mathbf{N},\chi}$. Such functor depends on the choice of a sequence of integers $\{d_i\}_i$ such that $d_\ell - d_k = \dim(\mathbf{N}_\ell/\mathbf{N}_k)$ for any $\ell \geq k$. We need the following fact:

Lemma 4.2.5. *For any* $\ell \geq k$*, there is a canonical morphism of sheaves*

$$\chi_{\mathbf{N}_k} \to \chi_{\mathbf{N}_\ell}[2(d_\ell - d_k)]. \tag{4.2.3}$$

Furthermore, this system of maps is transitive in a natural way.

Proof. Let $i : \mathbf{N}_k \to \mathbf{N}_\ell$ be the closed embedding given by the inclusion. It suffices to find a natural map

$$i_* (\chi_{\mathbf{N}_k}) \to \chi_{\mathbf{N}_\ell}[2(d_\ell - d_k)]$$

of $\mathfrak{D}^*$-modules on $\mathbf{N}_\ell$. Under η_ℓ^{-1}, such map corresponds to

$$i_*^{ren} \chi_k^!(exp) \to \chi_\ell^!(exp)[2(d_\ell - d_k)],$$

where $i_*^{ren} : \mathfrak{D}^!(\mathbf{N}_k) \to \mathfrak{D}^!(\mathbf{N}_\ell)$ is the functor $\eta_\ell^{-1} \circ i_* \circ \eta_k$. Since the LHS is isomorphic to $i_*^{ren} i^! \chi_\ell^!(exp)$, it suffices to exhibit a natural map

$$i_*^{ren} i^!(M) \to M[2(d_\ell - d_k)]$$

for any arbitrary $M \in \mathfrak{D}^!(\mathbf{N}_\ell)$.

Let us represent $\mathbf{N}_\ell$ as a pro-scheme by $\mathbf{N}_\ell = \lim_r Q^r$, where each $Q^r = \mathbf{N}_\ell/\mathbf{N}^r$. The induced presentation $\mathbf{N}_k = \lim_r P^r$ is obtained by setting $P^r = \mathbf{N}_k/\mathbf{N}^r$, so that the inclusion i corresponds to the inverse system of closed embeddings $i^r : P^r \hookrightarrow Q^r$. We also set $q_r = \dim Q^r$ and $p_r = \dim P^r$.

Let $M \in \mathfrak{D}^!(\mathbf{N}_\ell)$ correspond, in the realization of $\mathfrak{D}^!(\mathbf{N}_\ell)$ as a limit, to the family of sheaves $M^r \in \mathfrak{D}(Q^r)$. We wish to show that $i^!(M)$ can be computed value-wise. By definition, the functor $i^!$ is computed as the colimit of the directed system of functors $(i^r)^!$, according to the leftmost diagram below. In the present case, however, such diagram is *right adjointable*, which means that the rightmost diagram commutes as well.

$$\begin{tikzcd}
\mathfrak{D}(P^{r+1}) & \mathfrak{D}(P^r) \arrow[l, "\pi^!"'] & \mathfrak{D}(P^{r+1}) \arrow[r, "{\pi_*[-2(p_{r+1}-p_r)]}"] & \mathfrak{D}(P^r) \\
\mathfrak{D}(Q^{r+1}) \arrow[u, "(i^{r+1})^!"] & \mathfrak{D}(Q^r) \arrow[l, "\pi^!"'] \arrow[u, "(i^r)^!"'] & \mathfrak{D}(Q^{r+1}) \arrow[u, "(i^{r+1})^!"] \arrow[r, "{\pi_*[-2(q_{r+1}-q_r)]}"] & \mathfrak{D}(Q^r) \arrow[u, "(i^r)^!"']
\end{tikzcd} \tag{4.2.4}$$

This is a consequence of base-change, which holds thanks to $P_{r+1} \simeq P_r \times_{Q^r} Q^{r+1}$, combined with the equality $p_{r+1} - p_r = q_{r+1} - q_r$. This guarantees that $i^!(M) \simeq \{(i^r)^!(M^r)\}_r$, as wanted.

We then have

$$i_*^{ren} i^!(M) \simeq \Big\{ i_*^r \circ (i^r)^!(M^r)[2q_r - 2p_r] \Big\}_r \simeq \Big\{ M^r \otimes i_*^r(\omega_{P^r}) \Big\}_r [2(d_\ell - d_k)].$$

Indeed $q_r - p_r = d_\ell - d_k$ for any r. For the above formula, the sought-after map is induced by the canonical "trace" $i_*^r(\omega_{P^r}) \to \omega_{Q^r}$.

The construction also shows that the composition

$$\chi_{\mathbf{N}_k} \to \chi_{\mathbf{N}_\ell}[2(d_\ell - d_k)] \to \chi_{\mathbf{N}_m}[2(d_m - d_\ell)][2(d_\ell - d_k)] \tag{4.2.5}$$

is canonically isomorphic to $\chi_{\mathbf{N}_k} \to \chi_{\mathbf{N}_m}[2(d_m - d_k)]$, for any $m \geq \ell \geq k$. $\square$

Equation (4.2.3) and Lemma 4.2.3 yield a natural transformation

$$\mathsf{Av}_*^{\mathbf{N}_k,\chi}[2d_k] \longrightarrow \mathsf{Av}_*^{\mathbf{N}_\ell,\chi}[2d_\ell].$$

Transitivity of such natural transformations, (4.2.5), allows to form the functor:

$$\mathsf{T} : \mathcal{C} \to \mathcal{C}, \quad \mathsf{T}(c) = \operatorname*{colim}_{k \in \mathbb{N}} \big(\mathsf{Av}_*^{\mathbf{N}_k,\chi}(c)[2d_k] \big). \tag{4.2.6}$$

Notice that the image of T is contained in $\mathcal{C}^{\mathbf{N},\chi}$, as the tail of the sequence is in $\mathcal{C}^{\mathbf{N}_k,\chi}$ for any $k \in \mathbb{N}$.

Lemma 4.2.6. *The above functor* $\mathsf{T} : \mathcal{C} \to \mathcal{C}^{\mathbf{N},\chi}$ *descends to a functor (denoted by the same name)* $\mathsf{T} : \mathcal{C}_{\mathbf{N},\chi} \to \mathcal{C}^{\mathbf{N},\chi}$.

Proof. For any $M \in \mathfrak{D}^*(\mathbf{N})$, we need to provide a natural equivalence between $\mathsf{T}(M \star c)$ and $\varepsilon_{\mathbf{N}}(M, (-\chi)^! exp) \otimes \mathsf{T}(c)$. It suffices to exhibit a natural equivalence

$$M \star \chi_{\mathbf{N}_k} \simeq \varepsilon_{\mathbf{N}_k}\big(M, (-\chi_{\mathbf{N}_k})^!(exp)\big) \otimes \chi_{\mathbf{N}_k},$$

for $M \in \mathfrak{D}^*(\mathbf{N}_k)$ and any $k \in \mathbb{N}$. Let $\sigma : \mathbf{N}^2 \to \mathbf{N}^2$ be the automorphism $(x, y) \mapsto (x, xy)$, whose inverse is $\tau := (\mathrm{id} \times m) \circ (\widetilde{\Delta} \times \mathrm{id})$, where $\widetilde{\Delta} : x \mapsto (x, x^{-1})$. It follows that σ_* and $(\tau^!)^{ren} := \eta \circ \tau^! \circ \eta^{-1}$, so that $m_* \simeq (p_2)_* \circ (\tau^!)^{ren}$. Hence,

$$M \star \chi_{\mathbf{N}_k} \simeq m_*\big(M \boxtimes \eta(\chi^! exp)\big) \simeq (p_2)_* \circ \eta \circ (\widetilde{\Delta} \times \mathrm{id})^! (\mathrm{id} \times m)^! (\eta^{-1} M \boxtimes \chi^! exp)$$

$$\simeq (p_2)_* \circ \eta \circ \Big(\big(\eta^{-1} M \otimes (-\chi)^! exp\big) \boxtimes \chi^! exp\Big) \simeq \Gamma(M \overset{*!}{\otimes} (-\chi)^! exp) \otimes \eta(\chi^! exp).$$

The conclusion is now evident, in view of the definition of the pairing ε (see (2.1.8)). $\square$

Conjecture 4.2.7. *Let $\mathcal{C}$ be a category with an action of $\mathbf{G}$. For any choice of d_k as above, $\mathsf{T} : \mathcal{C}_{\mathbf{N},\chi} \to \mathcal{C}^{\mathbf{N},\chi}$ is an equivalence of categories.*

We prove this conjecture for $G = GL_n$ in Section 7, but first we need to study actions of loop vector spaces. This is the subject of the next section.

Chapter 5

Fourier transform and actions by loop vector groups

If $G = GL_2$, then $N \simeq \mathbb{A}^1$ is abelian, so that all the notions discussed above (group actions, invariants, coinvariants, averaging functors...) can be understood via Fourier transform. Slightly more generally, we consider the case of a vector group $\mathbb{A}^n$ and its loop group $\mathbf{A} := \mathbb{A}^n(\!(t)\!)$, which is of course the main example of an ind-pro-vector space.

5.1 The Fourier transform on a finite dimensional vector space

We start by reviewing the well-known Fourier-Deligne transform in the finite dimensional case, following, for the most part, [La].

Let $V \simeq \mathbb{A}^n$ be a finite dimensional vector space, with dual $V^\vee$. We indicate by m the addition in V, $V^\vee$ or $\mathbb{G}_a$ (depending on the context) and by $Q : V \times V^\vee \to \mathbb{G}_a$ the duality pairing. Let p_1 and p_2 be the projections from $V \times V^\vee$ to V and $V^\vee$, respectively.

Recall the $\mathfrak{D}$-module exp on $\mathbb{G}_a$, as in formula (4.2.1). The Fourier transform kernel is

$$exp^Q := Q^!(exp) \in \mathfrak{D}(V \times V^\vee).$$

Here are some (well-known) key features of this “integral kernel”.

Lemma 5.1.1. $(p_1)_*(exp^Q) \simeq \delta_{0,V}[2d_V]$.

Proof. It is enough to show that the !-fibers of $(p_1)_*(exp^Q)$ at $v \in V$ are zero for $v \neq 0$ and that $(p_1)_*(exp^Q)\big|_0 \simeq \mathbb{C}[2d_V]$.

In either case, the !-fiber of $(p_1)_*(exp^Q)$ at $v \in V$ is isomorphic to the cohomology $\Gamma(V^\vee, exp^v)$, where $exp^v := v^!(exp)$ and v is thought of as a character of $V^\vee$. This cohomology is known to be zero if $v \neq 0$. For $v = 0$, we compute

$$\Gamma(V^\vee, exp^0) \simeq \Gamma(V^\vee, \omega_{V^\vee}) \otimes exp|_0 \simeq \mathbb{C}[2d_V],$$

as claimed. $\square$

Lemma 5.1.2. *Let p_{13}, $p_{23} : V \times V \times V^\vee \to V \times V^\vee$ be the obvious projections. There is a canonical isomorphism*

$$p_{13}^!(exp^Q) \otimes p_{23}^!(exp^Q) \simeq (m \times id_{V^\vee})^!(exp^Q). \tag{5.1.1}$$

Proof. This follows from the commutative diagram

$$\begin{tikzcd} V \times V \times V^\vee \arrow[r, "m\times \mathrm{id}_{V^\vee}"] \arrow[d, "Q_{13}\times Q_{23}"'] & V \times V^\vee \arrow[d, "Q"] \\ \mathbb{G}_a \times \mathbb{G}_a \arrow[r, "m"] & \mathbb{G}_a \end{tikzcd} \tag{5.1.2}$$

together with (4.2.2). $\square$

The *Fourier transform* $\mathsf{FT} = \mathsf{FT}_V$ is the functor

$$\mathsf{FT} : \mathfrak{D}(V) \to \mathfrak{D}(V^\vee), \;\; M \mapsto (p_2)_*(p_1^!(M) \otimes exp^Q).$$

We also define the *inverse Fourier transform* $\mathsf{IFT} = \mathsf{IFT}_V$ as

$$\mathsf{IFT} : \mathfrak{D}(V^\vee) \to \mathfrak{D}(V), \;\; M \mapsto (p_1)_*(p_2^!(M) \otimes exp^{-Q})[-2d_V].$$

This name is justified by the following result:

Proposition 5.1.3. *For any finite dimensional vector space V, the functors FT_V and IFT_V are mutually inverse equivalences of categories.*

Proof. By symmetry, it suffices to prove a natural isomorphism $\mathsf{IFT} \circ \mathsf{FT} \simeq \mathrm{id}_V$. Base-change yields

$$\mathsf{IFT} \circ \mathsf{FT}(M) \simeq (p_1 \circ p_{13})_* \Big(p_{13}^!(exp^{-Q}) \otimes p_{23}^!(exp^Q) \otimes p_{23}^! p_1^!(M) \Big)[-2d_V].$$

A computation similar to the proof of (5.1.1) further gives

$$\mathsf{IFT} \circ \mathsf{FT}(M) \simeq (\widehat{p}_1)_* \Big((\xi \times \mathrm{id}_{V^\vee})^!(exp^Q) \otimes \widehat{p}_2^!(M) \Big)[-2d_V],$$

where $\xi : V \times V \to V$ sends $(v, v') \mapsto (v' - v)$ and $\widehat{p}_i : V \times V \times V^\vee \to V$ are the projections. Rewriting $\widehat{p}_1 = p_1 \circ p_{12}$ and $\widehat{p}_2 = p_2 \circ p_{12}$, we obtain

$$\begin{aligned} \mathsf{IFT} \circ \mathsf{FT}(M) &\simeq (p_1)_*\Big((p_{12})_*(\xi \times \mathrm{id}_{V^\vee})^!(exp^Q) \otimes p_2^!(M)\Big)[-2d_V] \\ &\simeq (p_1)_*\Big(\xi^!\,(p_1)_*(exp^Q) \otimes p_2^!(M)\Big)[-2d_V]. \end{aligned} \tag{5.1.3}$$

By Lemma 5.1.1, we can simplify $\xi^!\,(p_1)_*(exp^Q) \simeq \Delta_*(\omega_V)[2d_V]$, and the result follows by the projection formula. □

Emphasizing the dependence on Q, it is obvious that

$$\mathsf{FT}_V^Q \simeq \mathsf{IFT}_{V^\vee}^{-Q}[2d_V].$$

Remark 5.1.4. The proof of Lemma 5.1.1 shows that $\mathsf{FT}_V(\delta_v) \simeq v^!(exp)$. Here are two handy consequences:

$$\mathsf{FT}_{\mathbb{G}_a}(\delta_1) \simeq exp, \qquad \mathsf{FT}_{\mathbb{G}_a}(exp) \simeq \delta_{-1}[2]. \tag{5.1.4}$$

We now recall how the formalism of correspondences allows to handle base-change in the setting of DG categories. Let Sch^{ft} the ordinary category of schemes of finite type. We form Sch^{ft}_{corr}, the 1-category whose objects are the same as Sch^{ft} and whose morphisms are given by correspondences:

$$\mathrm{Hom}_{\mathrm{Sch}^{ft}_{corr}}(S, T) = \Big\{ S \xleftarrow{\alpha} H \xrightarrow{\beta} T \ : H \in \mathrm{Sch}^{ft} \Big\}.$$

Such correspondences compose under fiber product. Clearly, Sch^{ft}_{corr} is symmetric monoidal via Cartesian product.

Theorem 5.1.5 ([GR2]). *The assignment $S \rightsquigarrow \mathfrak{D}(S)$ upgrades to a* symmetric monoidal *functor*

$$\mathfrak{D} : \mathrm{Sch}^{ft}_{corr} \to \mathrm{DGCat}$$

which sends $S \xleftarrow{\alpha} H \xrightarrow{\beta} T$ to the functor $\mathfrak{D}(S) \xrightarrow{\beta_* \circ \alpha^!} \mathfrak{D}(T)$.

In particular, by restricting the domain of the above functor to the 1-category of finite dimensional vector spaces (in schemes) and linear maps under correspondences, we obtain the theory of $\mathfrak{D}$-modules on vector spaces. To discuss the Fourier transform, we shall need a mild generalization of this theory.

Let $\mathsf{Corr}' := \mathrm{Vect}^{fd, \mathbb{G}_a}_{corr}$ be the following 1-category: its objects are finite dimensional vector spaces; given two such V, W, the set of morphisms $V \dashrightarrow W$ consists of all diagrams of the form

$$\begin{tikzcd} & \mathbb{G}_a & \\ & H \arrow[u, "f"] \arrow[dl, "\alpha"'] \arrow[dr, "\beta"] & \\ V & & W, \end{tikzcd} \tag{5.1.5}$$

where H is also a finite dimensional vector space and all maps are linear. For short, we write $(V \xleftarrow{\alpha} H \xrightarrow{\beta} W; f)$ to indicate the morphism $V \dashrightarrow W$ in (5.1.5). The composition of $(V \xleftarrow{\alpha} H \xrightarrow{\beta} W; f)$ with $(W \xleftarrow{\gamma} K \xrightarrow{\delta} Z; g)$ is the correspondence

$$\left(V \leftarrow H \underset{W}{\times} K \to Z; h \right),$$

with $h : H \times_W K \to H \times K \xrightarrow{f+g} \mathbb{G}_a$. It is straightforward to check that $\mathrm{Vect}_{corr}^{fd,\mathbb{G}_a}$ is indeed a 1-category. The identity morphism $V \dashrightarrow V$ is given by $(V \xleftarrow{\mathrm{id}} V \xrightarrow{\mathrm{id}} V; 0)$.

Theorem 5.1.6. *The assignment $V \rightsquigarrow \mathfrak{D}(V)$ upgrades to a symmetric monoidal functor*

$$\mathfrak{D}^{enh} : \mathrm{Vect}_{corr}^{fd,\mathbb{G}_a} \to \mathrm{DGCat}$$

which sends $(V \xleftarrow{\alpha} H \xrightarrow{\beta} W; f)$ to the functor

$$\mathfrak{D}(V) \xrightarrow{\beta_*(f^!(exp)\otimes\alpha^!(-))} \mathfrak{D}(W).$$

Proof. The following argument is due to D. Gaitsgory and holds in greater generality. First of all, for an algebraic group of finite type, Theorem 5.1.5 holds in the G-equivariant setting, namely $\mathfrak{D}$ upgrades to a symmetric monoidal functor

$$\mathfrak{D}_G : (\mathrm{Sch}_G^{\mathrm{ft}})_{corr} \to \mathfrak{D}(G)\text{-}\mathbf{mod},$$

from the 1-category of G-schemes under correspondences to the ∞-category G-**rep**. If fact, more generally, Theorem 5.1.5 holds for simplicial schemes. Let Corr_G denote the 1-category of finite dimensional vector spaces with G-action, under correspondences.

Consider the following functor $\Psi : \mathsf{Corr}' \to \mathsf{Corr}_{\mathbb{G}_a}$: at the level of objects, it sends $V \mapsto V \times \mathbb{G}_a$, where $\mathbb{G}_a$ acts freely on the second factor of $V \times \mathbb{G}_a$. At the level of morphisms, Ψ sends

$$(V \xleftarrow{\alpha} H \xrightarrow{\beta} W; f) \mapsto (V \times \mathbb{G}_a \xleftarrow{\sigma} H \times \mathbb{G}_a \xrightarrow{\beta\times\mathrm{id}} W \times \mathbb{G}_a),$$

where $\sigma(v, x) = (\alpha(v), f(v) + x)$. One can easily check that Ψ is indeed a functor.

Let $\mathsf{inv}^{\mathbb{G}_a,exp} : \mathbb{G}_a\text{-}\mathbf{rep} \to \mathrm{DGCat}$ be the functor of $(\mathbb{G}_a, exp)$-invariants. Recall the canonical equivalence

$$\mathfrak{D}(V) \simeq \mathsf{inv}^{\mathbb{G}_a,exp}\big(\mathfrak{D}(V \times \mathbb{G}_a)\big)$$

induced by the functor $\mathfrak{D}(V) \to \mathfrak{D}(V \times \mathbb{G}_a)$ sending $M \mapsto M \boxtimes exp$. We now claim that, under this equivalence, our functor $\mathfrak{D}^{enh}$ is the composition

$$\mathfrak{D}^{enh} \simeq \mathsf{inv}^{\mathbb{G}_a,exp} \circ \mathfrak{D}_{\mathbb{G}_a} \circ \Psi.$$

Noticing that the above map σ equals the composition $(\mathrm{id} \times m) \circ (\alpha \times f \times \mathrm{id}) \circ (\Delta \times \mathrm{id})$, the proof of the claim is routine. $\square$

Tautologically, FT_V is the value of $\mathfrak{D}^{enh}$ on the following "arrow" (which we also call FT):

$$V \overset{\mathsf{FT}}{\dashrightarrow} V^\vee \quad := \quad \begin{tikzcd} & \mathbb{G}_a & \\ & V \times V^\vee \arrow[u, "Q"] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ V & & V^\vee. \end{tikzcd} \tag{5.1.6}$$

Furthermore,

Proposition 5.1.7. FT *is a monoidal equivalence between* $(\mathfrak{D}(V), \star)$ *and* $(\mathfrak{D}(V^\vee), \otimes)$.

Proof. Recall that the convolution monoidal structure on $\mathfrak{D}(V)$ arises from the algebra structure on V given by

$$V \times V \overset{m}{\dashrightarrow} V := \Big(V \times V \xleftarrow{\mathrm{id}} V \times V \xrightarrow{m} V; 0\Big)$$

and that the pointwise monoidal structure on $\mathfrak{D}(V^\vee)$ from the algebra structure

$$V^\vee \times V^\vee \overset{\Delta}{\dashrightarrow} V^\vee := \Big(V^\vee \times V^\vee \xleftarrow{\Delta} V^\vee \xrightarrow{\mathrm{id}} V^\vee; 0\Big).$$

We just need to prove that $\mathsf{FT} : V \dashrightarrow V^\vee$ intertwines the two algebra structures, or equivalently that the following diagram in $\mathrm{Sch}^{\mathrm{ft},\mathbb{G}_a}_{corr}$ commutes:

$$\begin{tikzcd} V \times V \arrow[r, dashed, "\mathsf{FT}\times\mathsf{FT}"] \arrow[d, dashed, "m"'] & V^\vee \times V^\vee \arrow[d, dashed, "\Delta"] \\ V \arrow[r, dashed, "\mathsf{FT}"] & V^\vee. \end{tikzcd} \tag{5.1.7}$$

We leave it to the reader to check that both paths in the above diagram coincide with

$$\Big(V \times V \xleftarrow{p_{12}} V \times V \times V^\vee \xrightarrow{m \times id_{V^\vee}} V^\vee; f\Big),$$

where $f : V \times V \times V^\vee \to \mathbb{G}_a$ sends $(v, w, \phi) \mapsto Q(\phi, w - v)$. $\square$

This result immediately implies a crucial fact for us. First, recall that an action of the vector group V on a category amounts to an action of the monoidal category $(\mathfrak{D}(V), \star)$: thus we have

$$\mathsf{FT} : V\text{-}\mathbf{rep} := (\mathfrak{D}(V), \star)\text{-}\mathbf{mod} \xrightarrow{\simeq} (\mathfrak{D}(V^\vee), \otimes)\text{-}\mathbf{mod}.$$

In other words,

Corollary 5.1.8. *Fourier transform indentifies categorical representations of V and crystals of categories over $V^\vee$, that is, categories with an action of $(\mathfrak{D}(V^\vee), \otimes)$.*

The action of $(\mathfrak{D}(V^\vee), \otimes)$ on $\mathcal{C} \in V\text{-}\mathbf{rep}$ is given by $P \otimes c \mapsto \mathsf{IFT}(P) \star c$.

Let us study the effect of Fourier transform on pushforwards and pullbacks along linear maps.

Lemma 5.1.9. *Given a linear map of finite dimensional vector spaces $f : W \to V$ and its dual $\phi : V^\vee \to W^\vee$, there exist natural equivalences of functors:*

$$\phi^! \circ \mathsf{FT}_W \simeq \mathsf{FT}_V \circ f_* \tag{5.1.8}$$

$$\mathsf{FT}_{V^\vee} \circ \phi^![-2d_V] \simeq f_* \circ \mathsf{FT}_{W^\vee}[-2d_W]. \tag{5.1.9}$$

Proof. We only prove the first formula, the second follows by applying the inverse Fourier trasform. Consider the arrows

$$W \overset{(id,f)}{\dashrightarrow} V := \left(W \xleftarrow{\mathrm{id}} W \xrightarrow{f} V; 0\right)$$

and

$$W^\vee \overset{(\phi,\mathrm{id})}{\dashrightarrow} V^\vee := \left(W^\vee \xleftarrow{\phi} V^\vee \xrightarrow{id} V^\vee; 0\right).$$

It suffices to prove that the following diagram is commutative:

$$\begin{tikzcd} W \arrow[r, dashed, "\mathsf{FT}_W"] \arrow[d, dashed, "(\mathrm{id},f)"'] & W^\vee \arrow[d, dashed, "(\phi,\mathrm{id})"] \\ V \arrow[r, dashed, "\mathsf{FT}_V"] & V^\vee. \end{tikzcd} \tag{5.1.10}$$

As in the proof above, it is routine to verify that both paths coincide with

$$\left(W \xleftarrow{p_1} W \times V^\vee \xrightarrow{p_2} V^\vee; h\right),$$

where $h = Q \circ (f \times \mathrm{id}_{V^\vee}) = Q \circ (\mathrm{id}_W \times \phi)$. □

Our last computation in the finite dimensional case involves a character $\chi \in V^\vee$. We first need a small result:

Corollary 5.1.10. $\mathsf{FT}_V(\chi^!(exp)) \simeq \delta_{-\chi,V^\vee}[2d_V]$. *In particular, by putting* $\chi = 0$, *we obtain* $\mathsf{FT}_V(k_V) \simeq \delta_{0,V^\vee}$.

Proof. By (5.1.9) and (5.1.4),

$$\mathsf{FT}_V(\chi^!(exp)) \simeq (\chi^\vee)_*\mathsf{FT}_{\mathbb{G}_a}(exp)[2d_V - 2] \simeq (\chi^\vee)_* \left(\delta_{-1,\mathbb{G}_a}[2]\right)[2d_V - 2].$$

Since $\chi^\vee : \mathbb{G}_a \to V^\vee$ is the linear map sending 1 to χ, the conclusion follows. □

Proposition 5.1.11. *Let* $i : W \hookrightarrow V$ *be the embedding of a vector subspace and* $\chi \in V^\vee$ *a character. Let* $\overline{\chi} : W \to \mathbb{G}_a$ *be its restriction to* W *and* $\chi_W := i_* \circ \overline{\chi}^! exp \in \mathfrak{D}(V)$. *Then*

$$\mathsf{FT}_V(\chi_W)[-2d_W] \simeq \omega_{-\chi+W^\perp},$$

where $W^\perp$ *is the annihilator of* W *in* $V^\vee$.

Proof. Using (5.1.8) and the corollary above, we have

$$\mathsf{FT}_V(\chi_W) \simeq \pi^!\big(\mathsf{FT}_W(\overline{\chi}^! exp)\big) \simeq \mathsf{FT}_W(\overline{\chi}^! exp) \simeq \delta_{-\overline{\chi},W^\vee}[2d_W],$$

where $\pi : V^\vee \to W^\vee$ is the projection dual to i. Noting that the fiber of the projection $\pi : V^\vee \to W^\vee$ at $\{-\overline{\chi}\}$ is precisely $-\chi + W^\perp$, we conclude

$$\mathsf{FT}_V(\chi_W) = \omega_{-\chi+W^\perp}[2d_W],$$

as desired. □

5.2 Fourier transform on a loop vector space

We now develop the notion of Fourier transform for the loop group $\mathbf{A} := \mathbb{A}^n(\!(t)\!)$ of the affine scheme $\mathbb{A}^n$, thought of as a vector group. More generally, we define the Fourier transform functor of a vector space of ind-pro type and estabilish its properties, parallel to the ones of the previous section.

Let $\mathbf{V}$ be a vector space of ind-pro-type, with presentation

$$\mathbf{V} \simeq \operatorname*{colim}_{n\in\mathcal{N}} \mathbf{V}_n \simeq \operatorname*{colim}_{n\in\mathcal{N}} \left(\lim_{r\in\mathcal{R}} V_{n,r}\right).$$

Its dual, as a topological vector space, can be written as

$$\mathbf{V}^\vee \simeq \lim_n \left(\operatorname{colim}_r V_{n,r}^\vee\right).$$

Lemma 5.2.1. *Suppose that, for each $n \to n'$ and $r \to r'$, the square*

$$\begin{tikzcd} V_{n,r'} \arrow[r] \arrow[d] & V_{n',r'} \arrow[d] \\ V_{n,r} \arrow[r] & V_{n',r} \end{tikzcd} \tag{5.2.1}$$

is Cartesian. Then the natural map

$$\mathrm{colim}_r \left(\lim_n V_{n,r}\right) \to \lim_n \left(\mathrm{colim}_r V_{n,r}\right)$$

is an equivalence.

Proof. The assignment $(n,r) \mapsto V_{n,r}$ upgrades to a functor $\Xi : \mathcal{N} \times \mathcal{R}^{\mathrm{op}} \to \mathrm{Sch}$. By Cartesianity of all the squares involved, the two expressions in the lemma are both equivalent to the colimit of Ξ, calculated in two ways (either first along columns and then along rows, or viceversa). $\square$

Let us refer to ind-pro vector spaces $\mathbf{V}$ satisfying the condition of Lemma 5.2.1 as *Cartesian ind-pro vector spaces*. For intance, $\mathbf{A} := \mathbb{A}^n(\!(t)\!)$ and $\mathbf{A}^\vee$ are such. From the above discussion, the 1-category of Cartesian ind-pro vector spaces admits duals.

We wish to extend the Fourier Transform equivalence to such vector spaces. This amounts to a combination of a left and a right Kan extension. We start by considering FT_V as a functor of V:

$$\mathsf{FT} : \mathrm{Vect}^{fd} \to \left(\mathrm{DGCat}^{\mathrm{SymmMon},\simeq}\right)^{\Delta^1}. \tag{5.2.2}$$

Namely, at the level of objects, FT sends a finite dimensional vector space to the equivalence of symmetric monoidal categories categories $(\mathfrak{D}(V), \star) \to (\mathfrak{D}(V^\vee), \otimes)$. At the level of morphisms, FT sends the linear map $f : W \to V$ to the natural transformation

$$\begin{tikzcd} \mathfrak{D}(W) \arrow[r, "\mathsf{FT}_W"] \arrow[d, "f_*"'] & \mathfrak{D}(W^\vee) \arrow[d, "\phi^!"] \\ \mathfrak{D}(V) \arrow[r, "\mathsf{FT}_V"] & \mathfrak{D}(V^\vee), \end{tikzcd} \tag{5.2.3}$$

where $\phi = f^\vee : V^\vee \to W^\vee$. That this defines a functor is the content of Lemma 5.1.9. Note that f_* and $\phi^!$ are compatible with the symmetric monoidal structures.

We now define the functor

$$\mathsf{FT} : \mathrm{Vect}^{pro} \to \left(\mathrm{DGCat}^{\mathrm{SymmMon},\simeq}\right)^{\Delta^1}. \tag{5.2.4}$$

by right Kan extension of (5.2.2) along the inclusion $\mathrm{Vect}^{fd} \hookrightarrow \mathrm{Vect}^{pro}$. By construction, the new FT yields and equivalence of symmetric monoidal categories $(\mathfrak{D}^*(V), \star) \to (\mathfrak{D}(V^\vee), \otimes)$, by the very definition of $\mathfrak{D}^*$ for pro-schemes. In fact, FT_V (for a pro-scheme V) is built as a limit of monoidal equivalences along symmetric monoidal functors.

To define FT at the level of ind-pro-schemes, we simply left Kan extend (5.2.4) along $\mathrm{Vect}^{pro} \hookrightarrow \mathrm{Ind}(\mathrm{Vect}^{pro})$:

$$\mathsf{FT} : \mathrm{Ind}(\mathrm{Vect}^{pro}) \to \Big(\mathrm{DGCat}^{\mathrm{SymmMon},\simeq} \Big)^{\Delta^1}. \tag{5.2.5}$$

As before, by construction and Lemma 5.2.1, for any *Cartesian* ind-pro vector space $\mathbf{V}$, we obtain a monoidal equivalence $(\mathfrak{D}^*(\mathbf{V}), \star) \xrightarrow{\simeq} (\mathfrak{D}^!(\mathbf{V}^\vee), \otimes)$.

We now express the above functors in more explicit terms. Let $Q : \mathbf{V} \times \mathbf{V}^\vee \to \mathbb{G}_a$ be a perfect pairing; we shall use the sheaf $Q^!(exp) \in \mathfrak{D}^!(\mathbf{V} \times \mathbf{V}^\vee)$ as the kernel. Pick a self-duality equivalence $\eta_{\mathbf{V}} : \mathfrak{D}^!(\mathbf{V}) \to \mathfrak{D}^*(\mathbf{V})$: while this is not canonical, it determines preferred $\eta_{\mathbf{V}^\vee}$ and $\eta_{\mathbf{V}\times\mathbf{V}^\vee}$. We stipulate that if formula involves two or more η functors, these are chosen coherently.

We claim that $\mathsf{FT}_{\mathbf{V}}$, as defined above, is the functor

$$\mathsf{FT}_{\mathbf{V}} : \mathfrak{D}^*(\mathbf{V}) \to \mathfrak{D}^!(\mathbf{V}^\vee), \quad M \mapsto (p_2)_*^{ren}\Big(exp^Q \otimes p_1^!\big(\eta_{\mathbf{V}}^{-1}(M)\big)\Big),$$

while the inverse Fourier transform $\mathsf{IFT}_{\mathbf{V}}$ is

$$\mathsf{IFT}_{\mathbf{V}} : \mathfrak{D}^!(\mathbf{V}^\vee) \to \mathfrak{D}^*(\mathbf{V}), \quad M \mapsto \eta_{\mathbf{V}} \circ (p_1)_!\Big(exp^{-Q} \otimes p_2^!(M)\Big).$$

Remark 5.2.2. The above claim implies that these functors are canonical, that is, independent of the choice of $\eta_{\mathbf{V}}$. Also, $\mathsf{IFT}_{\mathbf{V}}$, which involves the partially defined functor $(p_1)_!$ will be shown to be well-defined, being the inverse of $\mathsf{FT}_{\mathbf{V}}$. Note that $\mathsf{FT}_{\mathbf{V}}$ and $\mathsf{IFT}_{\mathbf{V}}$ extend their finite dimensional analogs.

The proof of the claim is a tedious calculation, relying on a explicit formula for exp^Q (provided below); the proof is left to the reader (in this paper, we will only use FT_V with its definition of left-right Kan extension). Nevertheless, we record:

Theorem 5.2.3. *For any Cartesian vector space* $\mathbf{V}$ *of ind-pro type, the functors*

$$\mathsf{FT}_{\mathbf{V}} : (\mathfrak{D}^*(\mathbf{V}), \star) \rightleftarrows (\mathfrak{D}^!(\mathbf{V}^\vee), \otimes) : \mathsf{IFT}_{\mathbf{V}}$$

are mutually inverse monoidal equivalences of categories.

For completeness, we analyze how the "kernel" exp^Q on $\mathbf{V} \times \mathbf{V}^\vee$ looks like. Let $\mathbf{Z}_r$ be the pro vector space

$$\mathbf{Z}_r := \lim_{n, \iota^\vee} V^\vee_{n,r},$$

so that

$$\mathbf{V} \times \mathbf{V}^\vee \simeq \operatorname*{colim}_{m, (\iota \times \pi^\vee)} \mathbf{V}_m \times \mathbf{Z}_m$$

is a presentation of $\mathbf{V} \times \mathbf{V}^\vee$ as an ind-scheme. It is straightforward to check that the restriction of Q to $\mathbf{V}_m \times \mathbf{Z}_m$, which we denote by Q_m, factors through the quotient

$$\pi_m \times \iota^\vee_m : \mathbf{V}_m \times \mathbf{Z}_m \twoheadrightarrow V_{m,m} \times V^\vee_{m,m}.$$

Unraveling the definitions, we obtain that exp^Q consists of the compatible family

$$Q^! exp = \Big\{ (\pi_m \times \iota^\vee_m)^! \big(Q^!_{m,m}(exp) \big) \Big\}_m \in \mathfrak{D}^!(\mathbf{V} \times \mathbf{V}^\vee) \simeq \lim_{m, (\iota \times \pi^\vee)^!} \mathfrak{D}^!(\mathbf{V}_m \times \mathbf{Z}_m),$$

where $Q_{m,m}$ is the evaluation on $V_{m,m} \times V^\vee_{m,m}$.

5.3 Invariants and coinvariants via Fourier transform

Let $\mathbf{V}$ be a Cartesian ind-pro vector space as in the above section. Suppose that $\mathbf{V}$ acts on $\mathcal{C}$; as usual, we indicate by $\star$ the action. We wish to express the invariant and coinvariant categories $\mathcal{C}^{\mathbf{V}}$ and $\mathcal{C}_{\mathbf{V}}$ in terms of the action of $(\mathfrak{D}^!(\mathbf{V}^\vee), \otimes)$ on $\mathcal{C}$, which is given by

$$\mathfrak{D}^!(\mathbf{V}^\vee) \otimes \mathcal{C} \to \mathcal{C}, \quad P \otimes c \mapsto \mathsf{IFT}_{\mathbf{V}}(P) \star c. \tag{5.3.1}$$

We begin by studying invariants and coinvariants for the action of a pro-finite dimensional vector group V.

Lemma 5.3.1. *Under Fourier transform,*

$$\mathcal{C}_V \simeq \mathfrak{D}(0) \underset{\mathfrak{D}(V^\vee)}{\otimes} \mathcal{C} \quad \textit{and} \quad \mathcal{C}^V \simeq \mathrm{Hom}_{\mathfrak{D}(V^\vee)}(\mathfrak{D}(0), \mathcal{C})$$

where the action of $\mathfrak{D}(V^\vee)$ *on* $\mathrm{Vect} \simeq \mathfrak{D}(0)$ *is by pullback along the inclusion* $0 \hookrightarrow V^\vee$.

Proof. Using the equivalence between $\mathfrak{D}^*(V)$ and $\mathfrak{D}(V^\vee)$ provided by (5.2.4), the categories $\mathcal{C}_V$ and $\mathcal{C}^V$ go over to

$$\mathcal{C}_V \simeq \mathrm{Vect} \underset{\mathfrak{D}(V^\vee)}{\otimes} \mathcal{C}, \quad \mathcal{C}^V \simeq \mathrm{Hom}_{\mathfrak{D}(V^\vee)}(\mathrm{Vect}, \mathcal{C}),$$

where the action $\mathfrak{D}(V^\vee) \otimes \mathcal{C} \to \mathcal{C}$ is as in (5.3.1) and the action of $(\mathfrak{D}(V^\vee), \otimes)$ on Vect is given by the monoidal functor

$$\mathfrak{D}(V^\vee) \to \mathrm{Vect}, \quad P \mapsto \Gamma_{\mathrm{dR}}(\mathsf{IFT}_V(P)).$$

By the pro-finite version of Lemma 5.1.9 discussed after (5.2.4), we obtain that $\Gamma_{\mathrm{dR}}(\mathsf{IFT}_V(P)) \simeq (i_0)^!(P)$. This proves both formulas. $\square$

Remark 5.3.2. Note that Vect is self-dual as a module for $\mathfrak{D}(V^\vee)$. Using this, the functor oblv^V goes over to push-forward along the embedding $0 \in V^\vee$:

$$(i_0)_* \simeq (i_0)_! : \mathfrak{D}(0) \otimes_{\mathfrak{D}(V^\vee)} \mathcal{C} \hookrightarrow \mathfrak{D}(V^\vee) \otimes_{\mathfrak{D}(V^\vee)} \mathcal{C} \simeq \mathcal{C}.$$

By adjunction, Av_*^V corresponds to the restriction

$$\mathfrak{r} := (i_0)^! : \mathcal{C} \simeq \mathfrak{D}(V^\vee) \otimes_{\mathfrak{D}(V^\vee)} \mathcal{C} \longrightarrow \mathfrak{D}(0) \otimes_{\mathfrak{D}(V^\vee)} \mathcal{C}.$$

Lemma 5.3.3. *Let V be a pro-vector space and $\chi : V \to \mathbb{G}_a$ a character. Under Fourier transform, the action of $\mathfrak{D}(V^\vee)$ on Vect_χ corresponds to the restriction functor $(i_{-\chi})^! : \mathfrak{D}(V^\vee) \to \mathrm{Vect}$.*

Proof. As the action of $\mathfrak{D}^*(V)$ on Vect corresponds to the monoidal functor

$$\alpha : \mathfrak{D}^*(V) \to \mathrm{Vect}, \quad M \mapsto p_*(M \overset{*!}{\otimes} \chi^!(exp)),$$

it suffices to verify that the composition

$$\mathfrak{D}(V^\vee) \xrightarrow{\mathsf{IFT}} \mathfrak{D}^*(V) \xrightarrow{\alpha} \mathrm{Vect}, \quad P \mapsto p_*(\mathsf{IFT}(P) \overset{*!}{\otimes} \chi^!(exp))$$

is equivalent to $(i_{-\chi})^!$. By construction, given $P = \{P^s\}_s \in \mathfrak{D}(V^\vee)$, we get

$$p_*\Big(\mathsf{IFT}(P) \overset{*!}{\otimes} \chi^!(exp)\Big) = p_*\Big(\{\mathsf{IFT}_{V^s}(P^s)\}_s \overset{*!}{\otimes} \chi^!(exp)\Big).$$

Let V_r be a quotient under which χ factors. Equivalently, χ, as a point of the ind-scheme $V^\vee$, comes from the scheme ${V_r}^\vee$. Writing $\chi^!(exp) = \{(\pi_{s\to r})^!(\chi_r)^! exp\}_{s \in \mathcal{R}_{/r}}$, we further obtain

$$\begin{aligned}
p_*\Big(\mathsf{IFT}(P) \overset{*!}{\otimes} \chi^!(exp)\Big) &= p_*\Big\{\mathsf{IFT}_{V^s}(P^s) \otimes (\pi_{s\to r})^!(\chi_r)^! exp\Big\}_s \\
&\simeq \Big\{(p_r)_*\Big((\pi_{s\to r})_*\big(\mathsf{IFT}_{V^s}(P^s)\big) \otimes (\chi_r)^! exp\Big)\Big\}_s \\
&\simeq \Big\{(p_r)_*\big(\mathsf{IFT}_{V^r}(P^r) \otimes \mathsf{IFT}_{V^r}(\delta_{-\chi_r})\big)\Big\}_s \\
&\simeq \Big\{(p_r)_*\big(\mathsf{IFT}_{V^r}(P^r \star \delta_{-\chi_r})\big)\Big\}_s \simeq (P^r \star \delta_{-\chi_r})\big|_0 \simeq P^r\big|_0 .
\end{aligned}$$

The latter is exactly the fiber of P at $-\chi \in V^\vee$. $\square$

Combining the above results, we obtain:

Corollary 5.3.4. *Under Fourier transform,*

$$\mathcal{C}_{V,\chi} \simeq \mathfrak{D}(\chi) \underset{\mathfrak{D}(V^\vee)}{\otimes} \mathcal{C} \quad \textit{and} \quad \mathcal{C}^{V,\chi} \simeq \mathrm{Hom}_{\mathfrak{D}(V^\vee)}(\mathfrak{D}(\chi), \mathcal{C})$$

where the action of $\mathfrak{D}(V^\vee)$ on $\mathfrak{D}(\chi) \simeq \mathrm{Vect}$ is by pullback along $\{\chi\} \hookrightarrow V^\vee$. Under this equivalence, the averaging functor $\mathsf{Av}_*^{V,\chi}$ goes over to the functor of restriction at χ.

Proof. The pair of equivalences is an immediate consequence of Lemma 4.2.2, Lemma 5.3.1 and Lemma 5.3.3. □

Invariants with respect to a subspace

Consider again a category $\mathcal{C}$ with an action of $\mathbf{A} = \mathbb{A}^n((t))$ and let $W \subset \mathbf{A}$ a pro-finite dimensional subspace. We wish to describe the procedure of taking (co)invariants of $\mathcal{C}$ with respect to W via Fourier transform. Let $\mathfrak{p} : \mathbf{A}^\vee \to W^\vee$ be the projection dual to $W \hookrightarrow \mathbf{A}$. Let $W^\perp \subseteq \mathbf{A}^\vee$ denote the annihilator of W. If W is infinite dimensional, then $W^\perp$ is a pro-scheme, too.

Proposition 5.3.5. *Let $\chi \in \mathbf{A}^\vee$ be a character. Under Fourier transform,*

$$\mathcal{C}_{W,\chi} \simeq \mathfrak{D}^!(W^\perp + \{\chi\}) \underset{\mathfrak{D}^!(\mathbf{A}^\vee)}{\otimes} \mathcal{C}, \qquad \mathcal{C}^{W,\chi} \simeq \mathrm{Hom}_{\mathfrak{D}^!(\mathbf{A}^\vee)}\left(\mathfrak{D}^!(W^\perp + \{\chi\}), \mathcal{C}\right),$$

where $\mathfrak{D}^!(\mathbf{A}^\vee)$ acts on $\mathfrak{D}^!(W^\perp + \{\chi\})$ via $!$-pullback along the inclusion $W^\perp + \{\chi\} \subseteq \mathbf{A}^\vee$.

Proof. We can arrange that $\mathbf{A} = \lim_j \mathrm{colim}_i A_{i,j}$ and $W = \lim_j W_j$ are presentations as an ind-pro-scheme and pro-scheme respectively, such that the linear map $W \hookrightarrow \mathbf{A}$ is induced by the compatible family of inclusions $W_j \hookrightarrow \mathrm{colim}_i A_{i,j}$.

By Corollary 5.3.4, we have

$$\mathcal{C}_{W,\chi} \simeq \mathfrak{D}(\chi) \underset{\mathfrak{D}(W^\vee)}{\otimes} \mathcal{C}.$$

However, $\mathcal{C}$ lives over $\mathbf{A}^\vee$ and the action of $\mathfrak{D}(W^\vee)$ on $\mathcal{C}$ factors through $\mathfrak{D}(W^\vee) \xrightarrow{\mathfrak{p}^!} \mathfrak{D}^!(\mathbf{A}^\vee)$, which lets us write

$$\mathcal{C}_{W,\chi} \simeq \mathfrak{D}(\chi) \underset{\mathfrak{D}(W^\vee)}{\otimes} \mathcal{C} \simeq \mathfrak{D}(\chi) \underset{\mathfrak{D}(W^\vee)}{\otimes} \mathfrak{D}^!(\mathbf{A}^\vee) \underset{\mathfrak{D}^!(\mathbf{A}^\vee)}{\otimes} \mathcal{C}.$$

It remains to prove that

$$\mathfrak{D}(\chi) \underset{\mathfrak{D}(W^\vee)}{\otimes} \mathfrak{D}^!(\mathbf{A}^\vee) \simeq \mathfrak{D}(\{\chi\} + W^\perp) :$$

to show this, we note that $\{\chi\} + W^{\perp} \simeq \{\chi\} \times_{W^\vee} \mathbf{A}^\vee$ and invoke the result of Lemma 2.2.4. In more detail, we have the presentation

$$\{\chi\} \times_{W^\vee} \mathbf{A}^\vee \simeq \mathrm{colim}_j \left(\chi \times_{W_j^\vee} \lim_i A_{i,j}^\vee\right)$$

of $\{\chi\} \times_{W^\vee} \mathbf{A}^\vee$ as an indscheme, so that

$$\begin{aligned}
\mathfrak{D}^!(\{\chi\} \times_{W^\vee} \mathbf{A}^\vee) &\simeq \lim_j \mathfrak{D}^!(\{\chi\} \times_{W_j^\vee} \lim_i A_{i,j}^\vee) \\
&\simeq \lim_j \Big(\mathfrak{D}(\chi) \underset{\mathfrak{D}(W_j^\vee)}{\otimes} \mathfrak{D}^!(\lim_i A_{i,j}^\vee)\Big) \\
&\simeq \mathfrak{D}(\chi) \underset{\mathfrak{D}(W^\vee)}{\otimes} \big(\lim_j \mathfrak{D}^!(\lim_i A_{i,j}^\vee)\big) \\
&\simeq \mathfrak{D}(\chi) \underset{\mathfrak{D}(W^\vee)}{\otimes} \mathfrak{D}^!(\mathbf{A}).
\end{aligned}$$

The proof of the formula about invariants is completely analogous and is left to the reader. $\square$

Chapter 6

Categories fibering over quotient stacks

Let X/G be a quotient stack, where X is a scheme of finite type and G a finite dimensional affine algebraic group: a category over X/G (for us, always with connection) is an object of $\mathrm{ShvCat}((X/G)_{\mathrm{dR}})$. We shall discuss the relation between the averaging functor and the functor of restriction to a subscheme of X. However, if X and G are of infinite type, the notion of de Rham functor is not yet well understood. We propose an alternative definition, inspired by the following result.

Proposition 6.0.6. *With the notation above, recall that $\mathfrak{D}(X)$ acquires the structure of algebra object in $\mathfrak{D}(G)$ -* **comod**. *There is an equivalence*

$$\mathrm{ShvCat}((X/G)_{\mathrm{dR}}) \xrightarrow{\simeq} \mathfrak{D}(G) \ltimes \mathfrak{D}(X)\textbf{-mod}.$$

Proof. By definition of ShvCat and the 1-affiness of $G^n \times X$, the LHS is the totalization of the cosimplicial ∞-category

$$\mathfrak{D}(X)\textbf{-mod} \rightrightarrows \mathfrak{D}(G \times X)\textbf{-mod} \rightrightarrows \mathfrak{D}(G \times G \times X)\textbf{-mod} \cdots .$$

A standard application of Barr-Beck-Lurie's theorem shows that $\mathrm{ShvCat}((X/G)_{\mathrm{dR}})$ is equivalent to the ∞-category of comodules in $\mathfrak{D}(X)$**-mod** for the comonad given by tensoring up with $\mathfrak{D}(G \times X)$ over $\mathfrak{D}(X)$. Obviously, this is equivalent to tensoring by $\mathfrak{D}(G)$. Hence,

$$\mathrm{ShvCat}((X/G)_{\mathrm{dR}}) \simeq \mathfrak{D}(G)\textbf{-comod}(\mathfrak{D}(X)\textbf{-mod}) \simeq \mathfrak{D}(X)\textbf{-mod}(\mathfrak{D}(G)\textbf{-comod}).$$

The latter expression is precisely the definition of $\mathfrak{D}(G) \ltimes \mathfrak{D}(X)$**-mod**, given in Section 3. $\square$

6.1 Quotients by pro-unipotent groups

Let us now move to the infinite dimensional setting. We only need the case of pro-unipotent G.

Let K be a pro-unipotent group and X an ind-pro-scheme endowed with an action of K. In this situation, $\mathfrak{D}^!(X)$ is algebra object in the ∞-category $\mathfrak{D}^!(G)$ - **comod** and we can form the crossed product monoidal category $\mathfrak{D}^*(K) \ltimes \mathfrak{D}^!(X)$. We say that $\mathcal{C} \in$ DGCat is *a category fibered over* X/K if it is endowed with the structure of a module category for $\mathfrak{D}^*(K) \ltimes \mathfrak{D}^!(X)$. To reduce ambiguity in the notation, we indicate by $M \diamond c$ the action of $M \in \mathfrak{D}^!(X)$ on $c \in \mathcal{C}$, while the action of $\mathfrak{D}^*(K)$ on $\mathcal{C}$ is the usual $\star$.

The construction $\mathfrak{D}^*(K) \ltimes \mathfrak{D}^!(X)$ entails the following compatibility between the actions of K and $\mathfrak{D}^!(X)$ on $\mathcal{C}$: for each M and c as above, there is a canonical isomorphism

$$\mathsf{coact}_K(M \diamond c) \simeq \mathsf{act}^!_{K,X}(M) \diamond \mathsf{coact}_K(c). \tag{6.1.1}$$

To be precise, the symbol $\diamond$ in the RHS means $\otimes$ on the $\mathfrak{D}^!(K)$-factor and action of $\mathfrak{D}(X)$ on $\mathcal{C}$. Let

$$\delta_x^{ren} := (i_x)_*^{ren}(\mathbb{C}) \simeq \eta_X^{-1}(\delta_x).$$

be the renormalized delta $\mathfrak{D}^!$-module at $x \in X$. If $M = \delta_x^{ren}$, we obtain

$$\mathsf{coact}_K(\delta_x^{ren} \diamond c) \simeq \mathsf{act}^!_{K,X}(\delta_x^{ren}) \diamond \mathsf{coact}_K(c), \tag{6.1.2}$$

a formula that will be particularly useful later.

We pause to give one important example of category over a quotient stack. Let K is a pro-unipotent group scheme action on an ind-pro-vector space $\mathbb{V}$. If $K \ltimes \mathbb{V}$ acts on a category $\mathcal{C}$, then

- $\mathcal{C}$ fibers over $X := V^\vee$, by Fourier transform;
- K acts on X via the dual action;
- K acts on $\mathcal{C}$ via the embedding $K \to K \ltimes V$.

The proposition below makes it precise that these three pieces of data are compatible.

Proposition 6.1.1. *With the above notation, Fourier transform induces an equivalence*

$$id_{\mathfrak{D}^*(K)} \otimes \mathsf{FT}_V : \mathfrak{D}^*(K \ltimes V) \xrightarrow{\simeq} \mathfrak{D}^*(K) \ltimes \mathfrak{D}^!(V^\vee).$$

Proof. It suffices to notice that $\mathfrak{D}^*$ preserves $\ltimes$, so that $\mathfrak{D}^*(K \ltimes V) \simeq \mathfrak{D}^*(K) \ltimes \mathfrak{D}^*(V)$. $\square$

6.2 Interactions between (co)invariants and restrictions

Let us continue with the general treatment. We say that a map $f : Y \to X$ of pro-schemes is *q-closed* if it can be represented as a limit of closed embeddings $f^r : Y^r \hookrightarrow X^r$ in $\mathrm{Sch}^{\mathrm{ft}}$. For instance, the embedding of a point into a pro-scheme of infinite dimension is obviously q-closed, but not closed: in fact, it is not even proper. Analogously, we say that f is *q-proper* if it arises as a limit of proper maps. The letter "q" stands for "quotient-wise".

The projection formula shows then that ι_*^{ren} is (symmetric) monoidal. Thus, given $\mathcal{C} \in \mathfrak{D}^!(X)$-**mod**, it makes sense to consider the category $\mathcal{C}|_Y := \mathfrak{D}^!(Y) \otimes_{\mathfrak{D}^!(X)} \mathcal{C}$, which comes with a natural functor

$$\boldsymbol{\iota}_*^{ren} : \mathcal{C}|_Y = \mathfrak{D}^!(Y) \underset{\mathfrak{D}^!(X)}{\otimes} \mathcal{C} \to \mathfrak{D}^!(X) \underset{\mathfrak{D}^!(X)}{\otimes} \mathcal{C} \simeq \mathcal{C}.$$

As $\boldsymbol{\iota}_*^{ren}$ is a limit of fully faithful functors (the inverse family $r \mapsto (\iota^r)_*^{ren} \otimes \mathrm{id}_{\mathcal{C}}$), it is itself fully faithful. Hence, we consider $\mathcal{C}|_Y$ as a subcategory of $\mathcal{C}$.

We now provide a criterion for the existence of the !-averaging functor $\mathsf{Av}_!^K$. Let X be a pseudo-contractible pro-scheme and K a pro-unipotent pro-group acting on X. We impose the technical condition that K acts on X *quotient-wise*. By definition, this means that there exist pro-scheme presentations of $X \simeq \lim_r X^r$ and $K = \lim_r K^r$ such that $\mathsf{act} : K \times X \to X$ is the limit of maps $\mathsf{act}^r : K^r \times X^r \to X^r$. We emphasize that K^r might not be a group, so that act^r, in spite of the name, is not an action.

Proposition 6.2.1. *Let X and K as above, and $Y \subset X$ be a q-closed embedding with Y also pseudo-contractible. If $K \times Y \xrightarrow{\mathsf{act}_{K,Y}} X$, the restriction of the action to Y, is q-proper, then $\mathsf{Av}_!^K$ is defined on $\mathcal{C}|_Y \subseteq \mathcal{C}$.*

Proof. Each object of $c \in \mathcal{C}|_Y$ is written as $\iota_*^{ren}(\omega_Y) \diamond c$, so that (6.1.1) gives

$$\mathsf{coact}_K(\iota_*^{ren}(\omega_Y) \diamond c) \simeq \mathsf{act}_{K,X}^!(\iota_*^{ren}(\omega_Y)) \diamond \mathsf{coact}_K(c).$$

Thus, we need to prove that $(p_K)_!$ is defined on the RHS of the above formula. As $\mathsf{coact}_K(c)$ can be expressed as a colimit of simple tensors, it suffices to prove that

$$(p_K)_! \Big(\mathsf{act}_{K,X}^!(\iota_*^{ren}(\omega_Y)) \diamond (Q \otimes d) \Big) =: d' \in \mathcal{C}$$

is well-defined for arbitrary $Q \in \mathfrak{D}^!(K)$ and $d \in \mathcal{C}$. In turn, it is just enough to check that

$$(p_K)_! \Big(\mathsf{act}_{K,X}^!(\iota_*^{ren}(\omega_Y)) \underset{\mathfrak{D}^!(K \times X)}{\otimes} (Q \boxtimes \omega_X) \Big) =: M' \in \mathfrak{D}^!(X)$$

is well-defined for arbitrary $Q \in \mathfrak{D}^!(K)$. (Indeed, if M' were defined, then $d' = M' \diamond d$.)

Existence of M' is shown as follows. Consider the following Cartesian diagram:

$$\begin{tikzcd} K \times Y \arrow[r, "\zeta\colon (k,y)\mapsto(k^{-1},k\cdot y)"] \arrow[d, "p_2"'] & K \times X, \arrow[d, "\mathsf{act}_{K,X}"] \\ Y \arrow[r, hook, "\iota"] & X. \end{tikzcd} \tag{6.2.1}$$

We claim

Lemma 6.2.2. *There is a canonical equivalence*

$$\mathsf{act}^!_{K,X}(\iota^{ren}_*(\omega_Y)) \simeq \zeta^{ren}_*(\omega_{K\times Y}). \tag{6.2.2}$$

The proof of this will be supplied in Sect. 6.2. Combined with the projection formula, (6.2.2) yields

$$(p_K)_!\Big(\mathsf{act}^!_{K,X}(\omega_Y) \otimes (Q \boxtimes \omega_X)\Big) \simeq (p_K)_! \circ \zeta^{ren}_* \circ \zeta^!(Q \boxtimes \omega_X).$$

To conclude, it suffices to check that $(p_K)_! \circ \zeta^{ren}_* \simeq (\mathsf{act}_{K,Y})^{ren}_*$. For $M = \{M^r\}_r \in \mathfrak{D}^!(K\times Y)$, we have

$$\zeta^{ren}_*(M) = \{(\zeta^r)_*(M^r)[2n_r]\}_r \simeq \operatorname*{colim}_{r\in\mathcal{R}^{\mathrm{op}}} (\pi_{\infty\to r})^!\Big((\zeta^r)_*(M^r)[2n_r]\Big).$$

As ζ is q-proper, we have $(\zeta^r)_! \simeq (\zeta^r)_*$ for any r, so that

$$(p_K)_! \circ \zeta^{ren}_*(M) \simeq \operatorname*{colim}_{r\in\mathcal{R}^{\mathrm{op}}} (\pi_{\infty\to r})^!\Big((p_{K^r})_! \circ (\zeta^r)_!(M^r)[2n_r]\Big).$$

It remains to notice that $(p_{K^r} \circ \zeta^r)_! \simeq (\mathsf{act}^r_{K,Y})_! \simeq (\mathsf{act}^r_{K,Y})_*$, by the q-properness of $\mathsf{act}_{K,Y}$. □

Corollary 6.2.3. *In the setting of the above proposition, if K is finite dimensional, the natural transformation $\mathsf{Av}^K_! \to \mathsf{Av}^K_*[2d_K]$ (see Corollary 3.3.8) is an equivalence on the subcategory $\mathcal{C}|_Y$.*

Proof. If K is finite dimensional, the equivalence $\eta_K : \mathfrak{D}^!(K) \to \mathfrak{D}^*(K)$ is the shift functor $[-2d_K]$. So, Lemma 3.3.3 gives

$$\mathsf{Av}^K_*(c) \simeq (p_K)_*\big(\mathsf{coact}_K(c)\big)[-2d_K].$$

On the other hand, for $c \in \mathcal{C}|_Y$, the proof of the above proposition shows that

$$\mathsf{Av}^K_!(c) \simeq (p_K)_*\big(\mathsf{coact}_K(c)\big),$$

whence the conclusion. □

We now supply the proof of Formula 6.2.2, which will also be of use later.

Proof of Lemma 6.2.2. Since the action of K on Y is quotient-wise, the diagram (6.2.1) is the limit of the $\mathcal{R}$-family of Cartesian diagrams

$$\begin{tikzcd} K^r \times Y^r \arrow[r, "\zeta^r : (k,y) \mapsto (k^{-1}, k\cdot y)"] \arrow[d, "p_2"'] & K^r \times X^r, \arrow[d, "\mathsf{act}^r_{K,X}"] \\ Y^r \arrow[r, hook, "\iota^r"] & X^r. \end{tikzcd} \tag{6.2.3}$$

We stress that act^r is not necessarily the action of a group. By pseudo-contractibility, $\iota^{ren}_*(\omega_Y) \simeq \{\iota^r_*(\omega_{Y^r})[n_r]\}_r$; using base-change for the diagram (6.2.3), we compute

$$\begin{aligned} \mathsf{act}^!_{K,X}(\iota^{ren}_*(\omega_Y)) &\simeq \operatorname{colim}_r(\pi_{\infty\to r})^!\Big((\mathsf{act}^r)^! \circ \iota^r_*(\omega_{Y^r})[n_r]\Big) \\ &\simeq \operatorname{colim}_r(\pi_{\infty\to r})^!\Big((\zeta^r)_*(\omega_{K^r\times Y^r})[n_r]\Big). \end{aligned}$$

The latter expression matches $\zeta^{ren}_*(\omega_{K\times Y})$ via the equivalence (2.1.3). $\square$

Here is the special case in which Y is a point.

Corollary 6.2.4. *If $\iota_x : \mathrm{pt} \hookrightarrow X$ is the inclusion of a point in a pseudo-contractible X on which K acts quotient-wise, then*

$$\mathsf{act}^!(\delta^{ren}_x) \simeq (\zeta_x)^{ren}_*(\omega_K), \tag{6.2.4}$$

where $\zeta_x : k \mapsto (k^{-1}, k\cdot x)$.

Example: the Heisenberg group

Let V be a finite dimensional vector space, $V^\vee$ its dual and Q a duality pairing. The *Heisenberg group* of (V, Q) is defined to be

$$\mathsf{Heis}(V) := V \ltimes (V^\vee \times \mathbb{G}_a),$$

where V acts on $V^\vee \times \mathbb{G}_a$ via $v \cdot (\phi, x) = (\phi, Q(\phi, v) + x)$.

Lemma 6.2.5. *Let $\mathcal{C}$ be a module category for $\mathfrak{D}(\mathsf{Heis}(V))$ and χ_a be the character on $V^\vee \times \mathbb{G}_a$ given by $(0,1) \in V \times \mathbb{G}_a$. The functor*

$$\mathsf{Av}^V_! : \mathcal{C}^{V^\vee\times\mathbb{G}_a,\chi_a} \to \mathcal{C}^V$$

is defined and naturally isomorphic to $\mathsf{Av}^V_*[2d_V]$.

Proof. According to Proposition 6.1.1, a category acted on by $\mathsf{Heis}(V)$ becomes a category over the stack $(V \times \mathbb{G}_a)/V$, where the action of V on $V \times \mathbb{G}_a$ is given by $\check{\mathsf{act}}(v,(u,y)) = (u + yv, y)$, as one easily checks. Furthermore, Fourier transform gives the equivalence

$$\mathcal{C}^{V^\vee \times \mathbb{G}_a, \chi_a} \simeq \mathcal{C}|_{(0,1)}\,.$$

Since the map

$$V \times \{(0,1)\} \hookrightarrow V \times (V \times \mathbb{G}_a) \xrightarrow{\check{\mathsf{act}}} V \times \mathbb{G}_a, \;\; v \mapsto (v,1)$$

is proper, we conclude by invoking Proposition 6.2.1. $\square$

Let $\iota : Y \hookrightarrow X$ be a q-closed embedding and $S \subseteq K$ the subgroup preserving Y. Let $\boldsymbol{\iota}^! : \mathcal{C} \to \mathcal{C}|_Y$ be the restriction map induced by $\iota^! : \mathfrak{D}^!(X) \to \mathfrak{D}(Y)$ upon tensoring up with $\mathcal{C}$ over $\mathfrak{D}^!(X)$.

Lemma 6.2.6. *The K-action on $\mathcal{C}$ restricts to an S-action on the subcategory $\mathcal{C}|_Y \subseteq \mathcal{C}$. Also, $\boldsymbol{\iota}^!$ descends to a functor $\mathcal{C}^K \to (\mathcal{C}|_Y)^S$.*

Proof. As for the first claim, it suffices to show that the composition

$$\mathcal{C}|_Y \xrightarrow{\boldsymbol{\iota}_*^{ren}} \mathcal{C} \xrightarrow{\mathsf{coact}_S} \mathfrak{D}^!(S) \otimes \mathcal{C}$$

lands in the subcategory $\mathfrak{D}^!(S) \otimes \mathcal{C}|_Y$. Compatibility yields

$$\mathsf{coact}_S(\iota_*^{ren}(\omega_Y) \diamond c) \simeq \mathsf{act}^!_{S,X}\big(\iota_*^{ren}(\omega_Y)\big) \diamond \mathsf{coact}_S(c). \tag{6.2.5}$$

Since $\mathsf{act}^!_{S,X}\big(\iota_*^{ren}(\omega_Y)\big) \simeq \omega_S \boxtimes \iota_*^{ren}(\omega_Y)$, the assertion follows.

Next, we must show that the functor $\boldsymbol{\iota}^! \circ \mathsf{oblv}^K : \mathcal{C}^K \to \mathcal{C}|_Y$ lands in $(\mathcal{C}|_Y)^S$: in other words, that $c \in \mathcal{C}^K$ implies

$$\mathsf{coact}_S\big(\iota_*^{ren}(\omega_Y) \diamond c\;\big) \simeq \omega_S \boxtimes \big(\iota_*^{ren}(\omega_Y) \diamond c\big).$$

This is clear, as $\mathsf{coact}_S(c) \simeq \omega_S \otimes c$ for such c. $\square$

Lemma 6.2.7. *With the same hypotheses as above, assume that the embedding $\lambda : S \hookrightarrow K$ is finitely presented. Then, there is a canonical isomorphism*

$$\boldsymbol{\iota}^! \circ (k_K \star -) \circ \boldsymbol{\iota}_*^{ren} \simeq (k_S \star -)[-2d_{K/S}]. \tag{6.2.6}$$

Proof. For $c \in \mathcal{C}|_Y$, we have a chain of isomorphisms

$$\begin{aligned} \boldsymbol{\iota}^! \circ \mathsf{Av}_*^K\big(\iota_*^{ren}(\omega_Y) \diamond c\big) &\simeq \boldsymbol{\iota}^! \circ (p^{K\times X\to X})_*^{ren}\Big(\mathsf{act}^!(\iota_*^{ren}(\omega_Y)) \diamond \mathsf{coact}_K(c)\Big) \\ &\simeq (p^{K\times X\to X})_*^{ren} \circ \widetilde{\boldsymbol{\iota}}^!\Big(\mathsf{act}^!(\iota_*^{ren}(\omega_Y)) \diamond \mathsf{coact}_K(c)\Big) \\ &\simeq (p^{K\times X\to X})_*^{ren}\Big(\widetilde{\iota}^!\big(\mathsf{act}^!(\iota_*^{ren}(\omega_Y))\big) \diamond \widetilde{\boldsymbol{\iota}}^!\mathsf{coact}_K(c)\Big), \end{aligned}$$

where $\widetilde{\iota} = \mathrm{id}_K \times \iota$. Thanks to the hypotheses on the action and on λ, one quickly computes that

$$\widetilde{\iota}^!\big(\mathsf{act}^!(\iota_*^{ren}(\omega_Y))\big) \simeq \widetilde{\iota}^!\big((\lambda \times \iota)_*^{ren}(\omega_{S\times Y})\big)[-2d_{K/S}],$$

whence

$$\begin{aligned} \boldsymbol{\iota}^! \circ \mathsf{Av}_*^K\big(\iota_*^{ren}(\omega_Y) \diamond c\big) &\simeq \boldsymbol{\iota}^! \circ (p^{K\times X\to X})_*^{ren}\Big((\lambda \times \iota)_*^{ren}(\omega_{S\times Y}) \diamond \mathsf{coact}_K(c)\Big)[-2d_{K/S}] \\ &\simeq \boldsymbol{\iota}^! \circ (p^{K\times X\to X})_*^{ren} \circ (\lambda \times \boldsymbol{\iota})_*^{ren}\Big((\lambda \times \boldsymbol{\iota})^!\mathsf{coact}_K(c)\Big)[-2d_{K/S}] \\ &\simeq (p^{S\times Y\to Y})_*^{ren}\big(\mathsf{coact}_S(c)\big)[-2d_{K/S}] \\ &\simeq \mathsf{Av}_*^S(c)[-2d_{K/S}]. \end{aligned}$$

The second isomorphism follows from the projection formula. ☐

6.3 Transitive actions

If $Y = \{x\}$ is a $\mathbb{C}$-point, then obviously $S \simeq \mathrm{Stab}_K(x)$ is the stabilizer of x in K. We shall investigate the situation when K acts *transitively* on X. For clarity, we denote $\iota_x : x \hookrightarrow X$ the inclusion.

Lemma 6.3.1. *If K acts quotient-wise and transitively on X, then $\mathsf{Av}_*^K(\delta_x^{ren}) \simeq \omega_X$.*

Proof. Using (6.2.4) and the formula $\mathsf{Av}_*^K \simeq (p^K)_*^{ren} \circ \mathsf{act}^!$, we get

$$\mathsf{Av}_*^K(\delta_x^{ren}) \simeq (p^K)_*^{ren} \circ (\zeta_x)_*^{ren}(\omega_K) \simeq (\mathsf{act}_x)_*^{ren}(\omega_K).$$

Since $\mathsf{act}_x : K \to X$ is a projection with contractible fibers, the push-forward $(\mathsf{act}_x)_*$ sends $k_K \mapsto k_X$. ☐

Proposition 6.3.2. *If K acts* transitively *and quotient-wise on X, the functor $\alpha = \boldsymbol{\iota}_x^! \circ \mathsf{oblv}^K : \mathcal{C}^K \to (\mathcal{C}|_x)^S$ of Lemma 6.2.6 is an equivalence with inverse*

$$\beta := \mathsf{Av}_*^K \circ\, (\boldsymbol{\iota}_x)_*^{ren} \circ \mathsf{oblv}^S.$$

Moreover, the two diagrams below are commutative:

$$
\begin{tikzcd}
(\mathcal{C}|_x)^S \arrow[r, "\mathsf{oblv}^S"] & \mathcal{C}|_x \\
\mathcal{C}^K \arrow[u, "\alpha"', "\simeq"] \arrow[r, "\mathsf{oblv}^K"] & \mathcal{C} \arrow[u, "\boldsymbol{\iota}_x^!"']
\end{tikzcd}
\qquad
\begin{tikzcd}
(\mathcal{C}|_x)^S & \mathcal{C}|_x \arrow[l, "\mathsf{Av}_*^S"'] \arrow[d, "(\boldsymbol{\iota}_x)_*^{ren}"] \\
\mathcal{C}^K \arrow[u, "\alpha"', "\simeq"] & \mathcal{C}. \arrow[l, "\mathsf{Av}_*^K"']
\end{tikzcd}
\tag{6.3.1}
$$

Proof. We will check that $\alpha \circ \beta$ and $\beta \circ \alpha$ are naturally isomorphic to the identity functors. As for the former,

$$\alpha \circ \beta(c) \simeq \boldsymbol{\iota}_x^! \circ (k_K \star -) \circ (\delta_x^{ren} \diamond \mathsf{oblv}^S(c))$$

We now claim that

$$\boldsymbol{\iota}_x^! \circ (k_K \star -) \circ (\boldsymbol{\iota}_x)_*^{ren} \simeq (k_S \star -) \simeq \mathsf{Av}_*^S; \tag{6.3.2}$$

this would imply that $\alpha \circ \beta \simeq \mathsf{Av}_*^S \circ \mathsf{oblv}^S \simeq \mathrm{id}$.

The structure of the proof of (6.3.2) is formally equal to the one of (6.2.6); the two differences are that now the action is transitive and that $\lambda : S \hookrightarrow K$ is no longer required to be finitely presented (so that $\dim_{K/S}$ does not make sense). The crucial computation is

$$\widetilde{\iota}^!(\mathsf{act}^!(\delta_x^{ren})) \simeq \lambda_*^{ren}(\omega_S),$$

obtained as follows:

$$
\begin{aligned}
\widetilde{\iota}^!(\mathsf{act}^!(\delta_x^{ren})) &\simeq \operatorname{colim}_r (\pi_{\infty\to r})^! (\widetilde{\iota}^r)^! (\mathsf{act}^r)^! \big(\delta_{x^r, X^r}[2d_{X^r}]\big) \\
&\simeq \operatorname{colim}_r (\pi_{\infty\to r})^! \lambda_*^r(\omega_{S^r})[2d_{X^r}] \\
&\simeq \operatorname{colim}_r (\pi_{\infty\to r})^! \lambda_*^r(\omega_{S^r})[2(d_{K^r} - d_{S^r})] \simeq \lambda_*^{ren}(\omega_S).
\end{aligned}
$$

The second $\simeq$ is base-change.

The opposite composition simplifies as

$$\beta \circ \alpha \simeq \mathsf{Av}_*^K \circ (i_x)_*^{ren} \circ (i_x)^! \circ \mathsf{oblv}^K.$$

For $c \in \mathcal{C}^K$, we thus have

$$
\begin{aligned}
\beta \circ \alpha(c) &\simeq \mathsf{Av}_*^K(\delta_x^{ren} \diamond c) \\
&\simeq p_*^K \circ \eta_K \big(\mathsf{coact}_K(\delta_x^{ren} \diamond c)\big) \\
&\simeq p_*^K \circ \eta_K \big(\mathsf{act}_{K,X}^!(\delta_x^{ren}) \diamond \mathsf{coact}_K(c)\big) \\
&\simeq \big(p_*^K \circ \eta_K \circ \mathsf{act}_{K,X}^!(\delta_x^{ren})\big) \diamond c
\end{aligned}
$$

and the latter expression is equivalent to $\mathsf{Av}_*^K(\delta_x^{ren}) \diamond c$, which in turn is c by Lemma 6.3.1.

Finally, commutativity of the leftmost diagram in the proposition is obvious; commutativity of the second one is precisely (6.3.2). $\square$

The results above generalize easily to include averages against characters. For instance:

Corollary 6.3.3. *If K is endowed with a character θ, restriction induces an equivalence $\alpha : \mathcal{C}^{H,\theta} \simeq (\mathcal{C}|_x)^{S,\theta}$ such that the diagrams like in (6.3.1) commute.*

Proof. Substituting $\mathcal{C}$ with $\mathcal{C} \otimes \mathrm{Vect}_\theta$, this is immediate from the above proposition. □

6.4 Actions by semi-direct products

Let L and K be two pro-unipotent group subschemes of $G((t))$, with K acting *quotient-wise* on L. We form the semi-direct product $K \ltimes L$. First of all, let us record the useful formula

$$k_{K \ltimes L} \simeq k_K \star k_L, \tag{6.4.1}$$

whose proof follows immediately from the finite dimensional case (using that K and L pro-objects in the category of group schemes). More generally, using (4.2.2):

Lemma 6.4.1. *Assume that K and L are endowed with characters μ and ν and that ν is compatible with the action of K. Then, $\psi := \mu + \nu$ is a character on $K \ltimes L$. In this situation,*

$$\psi_{K \ltimes L} \simeq \mu_K \star \nu_L, \tag{6.4.2}$$

where $\mu_K := (i_K)_* \big(\eta_K(\mu^! exp)\big)$ and ν_L, $\psi_{K \ltimes L}$ are defined accordingly.

Assume now that $K \ltimes L$ acts on a category $\mathcal{C}$. This is a special case of the situation considered in Section 3.5: indeed, L is normal in $K \ltimes L$ and $(K \ltimes L)/L \simeq K$. The statements of Lemma 3.5.3 adapt verbatim (up to changing $\mathfrak{D}$ with $\mathfrak{D}^*$) to the case of pro-groups. Thus, there is an equivalence

$$(\mathcal{C}^L)^K \simeq \mathcal{C}^{K \ltimes L}.$$

This equivalence is compatible with the forgetful functors to $\mathcal{C}$:

Lemma 6.4.2. *The composition $(\mathcal{C}^L)^K \xrightarrow{\mathsf{oblv}^K} \mathcal{C}^L \xrightarrow{\mathsf{oblv}^L} \mathcal{C}$ factors as follows:*

$$\begin{tikzcd}
\mathcal{C}^L \arrow[r, "\mathsf{oblv}^L"] & \mathcal{C} \\
(\mathcal{C}^L)^K \arrow[u, "\mathsf{oblv}^K"] \arrow[r, dotted, "\simeq"] & \mathcal{C}^{K \ltimes L}. \arrow[u, "\mathsf{oblv}^{K \ltimes L}"']
\end{tikzcd} \tag{6.4.3}$$

Proof. First, we show that $\mathsf{Av}_*^{K \ltimes L} \circ \mathsf{oblv}^K \circ \mathsf{oblv}^L \simeq \mathsf{oblv}^K \circ \mathsf{oblv}^L$. This implies the existence of the dotted arrow making the diagram commutative. Using (6.4.1), we have

$$\mathsf{oblv}^{K \ltimes L} \circ \mathsf{Av}_*^{K \ltimes L} \circ \mathsf{oblv}^K \circ \mathsf{oblv}^L(-) \simeq k_L \star (k_K \star (\mathsf{oblv}^K \circ \mathsf{oblv}^L(-))) \simeq \mathsf{oblv}^K \circ \mathsf{oblv}^L(-).$$

Let us call the dotted arrow $\lambda : (\mathcal{C}^L)^K \to \mathcal{C}^{K \ltimes L}$. Next, we show that λ is an equivalence. Clearly,

$$\lambda \simeq \mathsf{Av}_!^{K \ltimes L} \circ \mathsf{oblv}^L \circ \mathsf{oblv}^K$$

so that its right adjoint is

$$\lambda^R \simeq \mathsf{Av}_*^K \circ \mathsf{Av}_*^L \circ \mathsf{oblv}^{K \ltimes L}.$$

It is then straightforward to check that the compositions $\lambda \circ \lambda^R$ and $\lambda^R \circ \lambda$ are isomorphic to the identity functors. $\square$

Corollary 6.4.3. *Diagram (6.4.3) is* right adjointable*, that is, the following diagram is still commutative:*

$$\begin{tikzcd} \mathcal{C}^L & \mathcal{C} \arrow[l, "\mathsf{Av}_*^L"'] \\ (\mathcal{C}^L)^K \arrow[u, "\mathsf{oblv}^K"] & \mathcal{C}^{K \ltimes L}. \arrow[l, "\lambda^{-1}"'] \arrow[u, "\mathsf{oblv}^{K \ltimes L}"'] \end{tikzcd} \tag{6.4.4}$$

Proof. The above lemma shows that $\lambda^{-1} \simeq \lambda^R \simeq \mathsf{Av}_*^K \circ \mathsf{Av}_*^L \circ \mathsf{oblv}^{K \ltimes L}$. It is obvious that the two paths in the diagram are both canonically isomorphic to $\mathsf{oblv}^{rel} : \mathcal{C}^{K \ltimes L} \hookrightarrow \mathcal{C}^L$. $\square$

Variant including characters In the situation of Lemma 6.4.1, there is still an equivalence

$$(\mathcal{C}^{L,\nu})^{K,\mu} \simeq \mathcal{C}^{K \ltimes L, \psi}$$

and the analogous diagrams still commute. The proofs are identical.

Assume now that $L = V$, a pro-vector space. Then, K acts on $V^\vee$ as well and the above results, combined with the Fourier transform, yield

Lemma 6.4.4. *With the above set-up, let $\mathcal{C}|_\chi$ denote the category $\mathrm{Hom}_{\mathfrak{D}^!(\mathbf{A}^\vee)}(\mathfrak{D}(0), \mathcal{C})$. The following two diagrams are canonically commutative:*

$$\begin{tikzcd} \mathcal{C}|_\chi \arrow[r, "\mathsf{oblv}"] & \mathcal{C} \\ (\mathcal{C}|_\chi)^{K,\psi} \arrow[u, "\mathsf{oblv}^K"] \arrow[r, "\simeq"] & \mathcal{C}^{K \ltimes V, \chi+\psi}. \arrow[u, "\mathsf{oblv}^{K \ltimes V}"'] \end{tikzcd} \qquad \begin{tikzcd} \mathcal{C}|_\chi & \mathcal{C} \arrow[l, "i_\chi^!"'] \\ (\mathcal{C}|_\chi)^{K,\psi} \arrow[u, "\mathsf{oblv}^K"] & \mathcal{C}^{K \ltimes V}. \arrow[l, "\simeq"'] \arrow[u, "\mathsf{oblv}^{K \ltimes V}"'] \end{tikzcd} \tag{6.4.5}$$

We shall exploit this in two situations of interest:

1. Writing $\mathbf{N} \simeq \mathbf{N}' \ltimes \mathbf{A}^{n-1}$, we see that χ is the sum of two characters: χ_a (the restriction of χ on $\mathbf{A}^{n-1}$) and χ' (the projection of χ to $\mathbf{N}'$). Moveover, χ_a is constant on the $\mathbf{N}'$-orbits of $\mathbf{A}^{n-1}$. Thus, according to the above results (adapted to the case of ind-pro-schemes), the procedure of taking Whittaker invariants consists of two steps: first take $(\mathbf{A}^{n-1}, \chi_a)$-invariants and then take $(\mathbf{N}', \chi')$-invariants. This is the observation that allows the use of induction.

2. We also apply the above results to the semi-direct product $\mathbf{H}_k = \mathbf{G}_k \ltimes \mathbf{A}_k$, endowed with the character ψ, which is the sum of two characters: χ_g on $\mathbf{G}_k$ and χ_a on $\mathbf{A}_k$.

Chapter 7

Actions by the loop group of GL_n

In this section, unless otherwise stated, we will assume that $G = GL_n$. Thus, B is the Borel subgroup of upper triangular matrices and $N \subseteq B$ its unipotent radical: upper triangular matrices with 1's on the diagonal. Further, let T be the torus of diagonal matrices and B_-, N_- be the opposite Borel and its unipotent radical. The character χ on $\mathbf{N}$ simply computes the sum of the residues of the entries in the diagonal $(i, i+1)$.

We will also need analogous notations for subgroups of $G' := GL_{n-1}$, their loop groups and so on. Thus $B', N', \mathbf{G}', \mathbf{N}', \chi'$ have their obvious meanings.

7.1 Statement of the main theorem

Let (d_k) be a sequence of integers with the property that $d_{k+1} - d_k = \dim(\mathbf{N}_{k+1}/\mathbf{N}_k)$ and such that $d_k = 0$ for some $k \geq 1$. Of course, this sequence is just determined by the positive integer k: let T_k be the corresponding functor, as defined in formula (4.2.6). When k is clear from the context, we omit it from the notation. Our goal is to prove:

Theorem 7.1.1. *For any choice of (d_k) as above, $\mathsf{T}_k : \mathcal{C}_{\mathbf{N},\chi} \to \mathcal{C}^{\mathbf{N},\chi}$ is an equivalence of categories.*

The proof will occupy the remainder of the text.

If we were just interested in the existence of *an* equivalence $\mathcal{C}_{\mathbf{N},\chi} \simeq \mathcal{C}^{\mathbf{N},\chi}$, then, in view of Proposition 3.6.2, the following theorem would be enough:

Theorem 7.1.2. *For any $k \geq 1$, there exists a group scheme $\mathbf{H}_k \subset \mathbf{G}$ endowed with a character, such that $\mathcal{C}^{\mathbf{N},\chi} \simeq \mathcal{C}^{\mathbf{H}_k,\psi}$ and $\mathcal{C}_{\mathbf{N},\chi} \simeq \mathcal{C}_{\mathbf{H}_k,\psi}$.*

The group $\mathbf{H}_k$ is presented in the next section. However, to show that T is an equivalence, we need to keep track of the functors more carefully. We will prove a pair of twin statements:

Theorem 7.1.3. *For any $k \geq 1$, the functors*

$$\Psi := \mathsf{Av}_*^{\mathbf{H}_k,\psi} \circ \mathsf{oblv}^{\mathbf{N}} : \mathcal{C}^{\mathbf{N},\chi} \longrightarrow \mathcal{C}^{\mathbf{H}_k,\psi}$$

$$\Upsilon := \mathrm{pr} \circ \mathsf{oblv}^{\mathbf{H}_k} : \mathcal{C}^{\mathbf{H}_k,\psi} \to \mathcal{C}_{\mathbf{N},\chi}$$

are equivalences of categories.

Finally, the proof of Theorem 7.1.1 will be completed by showing:

Proposition 7.1.4. *For k chosen as in the above theorem, we have $\mathsf{T}_k \simeq \Psi^{-1} \circ \Upsilon^{-1}$.*

7.2 Some combinatorics of GL_n

Let us introduce the long-awaited group $\mathbf{H}_k$. For $k \geq 1$ we consider $\mathbf{B}_-^k \cdot N(\mathcal{O}) \subset G(\mathcal{O})$. To show it is a group, notice that it is the preimage of $N[t]/t^n$ under the group epimorphism $\mathfrak{p} : G(\mathcal{O}) \to G[t]/t^k$.

For convenience, we redefine the cofinal sequence of group schemes $\mathbf{N}_k$ used above to approximate $\mathbf{N}$: consider the diagonal element $\gamma := \operatorname{diag}\left(t^{nk}, t^{(n-1)k}, \ldots, t^k\right) \in \mathbf{T}$. We let $\mathbf{N}_k := \gamma^{-1} \cdot N[[t]] \cdot \gamma$.

We first define the group

$$\mathbf{G}_k := \mathrm{Ad}_{\gamma'^{-1}}\left((\mathbf{B}'_-)^k \cdot N'(\mathcal{O})\right) = \mathrm{Ad}_{\gamma'^{-1}}\left((\mathbf{N}'_-)^k\right) \cdot (\mathbf{T}')^k \cdot \mathbf{N}'_k.$$

For example, when $n = 2$ and $n = 3$ and $n = 4$, we have

$$\mathbf{G}_k^{(2)} = (1 + t^k\mathcal{O}), \;\; \mathbf{G}_k^{(3)} = \begin{pmatrix} 1 + t^k\mathcal{O} & t^{-k}\mathcal{O} \\ t^{2k}\mathcal{O} & 1 + t^k\mathcal{O} \end{pmatrix}, \;\; \mathbf{G}_k^{(4)} = \begin{pmatrix} 1 + t^k\mathcal{O} & t^{-k}\mathcal{O} & t^{-2k}\mathcal{O} \\ t^{2k}\mathcal{O} & 1 + t^k\mathcal{O} & t^{-k}\mathcal{O} \\ t^{3k}\mathcal{O} & t^{2k}\mathcal{O} & 1 + t^k\mathcal{O} \end{pmatrix}.$$

For higher n, the structure of $\mathbf{G}_k$ follows the evident pattern. The first important feature of $\mathbf{G}_k$ is the following:

Lemma 7.2.1. *The group $\mathbf{G}_k$ is endowed with a character χ_g that extends the character χ' on $\mathrm{Ad}_{\gamma'^{-1}}(N'(\mathcal{O})) = \mathbf{N}'_k$ and that is trivial on $\mathrm{Ad}_{\gamma'^{-1}}\left((\mathbf{B}'_-)^k\right)$.*

Proof. Each element of $\mathbf{G}_k$ can be written uniquely as $\gamma'^{-1} \cdot y \cdot \gamma'$. We set

$$\chi_g(\gamma'^{-1} \cdot y \cdot \gamma') := \chi'(\gamma'^{-1} \cdot \mathfrak{p}'(y) \cdot \gamma'),$$

where $\mathfrak{p}' : (\mathbf{B}'_-)^k \cdot N'(\mathcal{O}) \to N'[t]/t^k$ is the projection. $\square$

Consider now the vector group

$$\mathbf{A}_k := \mathbf{A}_k^{n-1} := \begin{pmatrix} t^{-(n-1)k}\mathcal{O} \\ t^{-(n-2)k}\mathcal{O} \\ \vdots \\ t^{-k}\mathcal{O} \end{pmatrix}$$

We consider the character χ_a on $\mathbf{A}_k$ that computes the residue of the *last entry*. The second important feature of $\mathbf{G}_k$ is that it acts on $\mathbf{A}_k$. We form the semidirect product

$$\mathbf{H}_k := \mathbf{G}_k \ltimes \mathbf{A}_k = \begin{pmatrix} \mathbf{G}_k & \mathbf{A}_k \\ O & 1 \end{pmatrix}.$$

The latter also admits a character $\psi : \mathbf{H}_k \to \mathbb{G}_a$, which is the sum of the characters χ_g on $\mathbf{G}_k$ and χ_a on $\mathbf{A}_k$.

The annihilator of $\mathbf{A}_k$ in $(\mathbf{A}^{n-1})^\vee \simeq \mathbf{A}^{n-1}$ is easily computed:

$$\mathbf{A}_k^\perp = \begin{pmatrix} t^{(n-1)k}\mathcal{O} \\ t^{(n-2)k}\mathcal{O} \\ \vdots \\ t^{k}\mathcal{O} \end{pmatrix}$$

Let $\mathbf{L}_k := \{e_{n-1}\} + \mathbf{A}_k^\perp \subset (\mathbf{A}^{n-1})^\vee$. For future use, notice the third important feature of $\mathbf{G}_k$, which is actually one of the main motivations for the theory of Section 6.

Lemma 7.2.2. $\mathbf{G}_k$ *acts on* $\mathbf{L}_k$ *(via the dual action) transitively, and the stabilizer of* e_{n-1} *is exactly* $\mathbf{H}'_k$.

Proof. This is straightforward linear algebra. For instance, to show the second claim, it suffices to notice that $\mathbf{H}'_k$ is obtained from $\mathbf{G}_k$ by setting the last row of the latter to be $(0, \dots, 0, 1)$. $\square$

This result allows to apply Lemma 6.3.2 in the example where $X = \mathbf{L}_k$, $K = \mathbf{G}_k$, $x = e_{n-1}$ and $S = \mathbf{H}'_k$:

Corollary 7.2.3. *Let* $\mathcal{C}$ *be a category acted on by* $GL_n(\!(t)\!)$. *In particular* $\mathcal{C}$ *is acted on by* $\mathbf{A}$ *and, by Fourier transform, we view it as a category over* $\mathbf{A}^\vee \simeq \mathbf{A}$. *Then, the restriction functor* $\boldsymbol{\iota}_x^! : \mathcal{C}|_{\mathbf{L}_k} \to \mathcal{C}|_{e_{n-1}}$ *yields an equivalence*

$$\alpha : \left(\mathcal{C}|_{\mathbf{L}_k}\right)^{\mathbf{G}_k, \chi_g} \xrightarrow{\simeq} \left(\mathcal{C}|_{e_{n-1}}\right)^{\mathbf{H}'_k, \psi'}.$$

Proof. After noting that the restriction of χ_g to $\mathbf{H}'_k$ is exactly ψ', this is an immediate consequence of Corollary 6.3.3. $\square$

7.3 Proof of the main theorem: step 1

We have introduced all the necessary tools to prove Theorem 7.1.3.

We first prove that

$$\Psi := \mathsf{Av}_*^{\mathbf{H}_k,\psi} \circ \mathsf{oblv}^{\mathbf{N}} : \mathcal{C}^{\mathbf{N},\chi} \longrightarrow \mathcal{C}^{\mathbf{H}_k,\psi}$$

is an equivalence. We proceed by induction on n, the claim being tautologically true for $n = 1$. Consider the following (possibly non-commutative) diagram.

$$\begin{tikzcd}
(\mathcal{C}|_{\chi_a})^{\mathbf{N}',\chi'} \arrow[r, hook, "\mathsf{oblv}^{\mathbf{N}'}"] & \mathcal{C}|_{\chi_a} \arrow[r, two heads, "\mathsf{Av}_*^{\mathbf{H}'_k,\psi'}"] & (\mathcal{C}|_{\chi_a})^{\mathbf{H}'_k,\psi'} \\
 & & (\mathcal{C}|_{\chi_a+\mathbf{A}_k^{\perp}})^{\mathbf{G}_k,\chi_g} \arrow[u, "\alpha_3"] \\
\mathcal{C}^{\mathbf{N},\chi} = (\mathcal{C}|_{\mathbf{A}^\vee})^{\mathbf{N},\chi} \arrow[uu, "\mathfrak{r}_1"] \arrow[r, hook, "\mathsf{oblv}^{\mathbf{N}}"] & \mathcal{C} = \mathcal{C}|_{\mathbf{A}^\vee} \arrow[r, two heads, "\mathsf{Av}_*^{\mathbf{H}_k,\psi}"] & \mathcal{C}^{\mathbf{H}_k,\psi} = (\mathcal{C}|_{\mathbf{A}^\vee})^{\mathbf{H}_k,\psi} \arrow[u, "\mathfrak{r}_2"]
\end{tikzcd} \tag{7.3.1}$$

We note that:

- the bottom edge is our functor Ψ;
- the top edge is Ψ', applied to $\mathcal{C}^{\mathbf{A},\chi_a} \simeq \mathcal{C}|_{\chi_a}$, and it is an equivalence by the induction hypothesis;
- the leftmost vertical functor $\mathfrak{r}_1$ is induced by the restriction $\mathfrak{r} : \mathcal{C} \to \mathcal{C}|_{\chi_a}$ and it is an equivalence thanks to Lemma 6.4.2;
- the vertical functor $\mathfrak{r}_2$ is induced by restriction and it is an equivalence, also by Lemma 6.4.2;
- the vertical functor α_3 is induced by restriction and it is an equivalence thanks to Corollary 7.2.3.

Consequently, it remains to show that the above diagram *commutes*. To this goal, we insert an extra vertex and four extra arrows.

$$
\begin{tikzcd}
(\mathcal{C}|_{\chi_a})^{\mathbf{N}',\chi'} \arrow[r, "\mathsf{oblv}^{\mathbf{N}'}"] & \mathcal{C}|_{\chi_a} \arrow[r, "\mathsf{Av}_*^{\mathbf{H}'_k,\psi'}"] \arrow[d, "\boldsymbol{\iota}_3"'] \arrow[dr, phantom, "\mathsf{D}_3"] & (\mathcal{C}|_{\chi_a})^{\mathbf{H}'_k,\psi'} \\
 & \mathcal{C}|_{\chi_a+\mathbf{A}_k^\perp} \arrow[r, "\mathsf{Av}_*^{\mathbf{G}_k,\chi_g}"] \arrow[dr, phantom, "\mathsf{D}_2"] & (\mathcal{C}|_{\chi_a+\mathbf{A}_k^\perp})^{\mathbf{G}_k,\chi_g} \arrow[u, "\alpha_3"] \\
\mathcal{C}^{\mathbf{N},\chi} = (\mathcal{C}|_{\mathbf{A}^\vee})^{\mathbf{N},\chi} \arrow[uu, "\mathfrak{r}_1"] \arrow[ur, "\gamma"] \arrow[r, "\mathsf{oblv}^{\mathbf{N}}"] & \mathcal{C} = \mathcal{C}|_{\mathbf{A}^\vee} \arrow[u, "\mathfrak{r}_2"] \arrow[r, "\mathsf{Av}_*^{\mathbf{H}_k,\psi}"] & \mathcal{C}^{\mathbf{H}_k,\psi} = (\mathcal{C}|_{\mathbf{A}^\vee})^{\mathbf{H}_k,\psi} \arrow[u, "\mathfrak{r}_2"]
\end{tikzcd}
\tag{7.3.2}
$$

Here,

$$\gamma : \mathcal{C}^{\mathbf{N},\chi} \longrightarrow \mathcal{C}|_{\chi_a+\mathbf{A}_k^\perp} \simeq \mathcal{C}^{\mathbf{A}_k,\chi_a} \quad \text{and} \quad \boldsymbol{\iota}_3 : \mathcal{C}^{\mathbf{A},\chi_a} \simeq \mathcal{C}|_{\chi_a} \longrightarrow \mathcal{C}|_{\chi_a+\mathbf{A}_k^\perp} \simeq \mathcal{C}^{\mathbf{A}_k,\chi_a}$$

are just the relative forgetful functors. By pro-unipotence of all the groups involved, commutativity of the triangle and of the trapezoid follow immediately.

Now, diagram D_2 commutes in view of the second assertion of Lemma 6.4.2 after passing to right adjoints. Diagram D_3 commutes in view of Proposition 6.3.2.

This concludes the proof the first equivalence in Theorem 7.1.3. Before proceeding with the analysis of Υ, let us notice the following immediate consequence:

Corollary 7.3.1. *For any $k \geq 1$, the partially defined functor*

$$\Phi := \mathrm{colim}_k \, \mathsf{Av}_!^{\mathbf{N}_k,\chi} \circ \mathsf{oblv}^{\mathbf{H}_k} : \mathcal{C}^{\mathbf{H}_k,\psi} \longrightarrow \mathcal{C}^{\mathbf{N},\chi}$$

is everywhere defined and it is the inverse of Ψ.

Proof. One readily verifies that Φ is (where defined) left adjoint to Ψ. Since Ψ is an equivalence by Theorem 7.1.3, Φ is everywhere defined and inverse to Ψ. $\square$

Second equivalence in Theorem 7.1.3

The proof that Υ is an equivalence is completely analogous, if not easier: it amounts to proving that the following diagram is commutative.

$$
\begin{tikzcd}
(\mathcal{C}|_{\chi_a})^{\mathbf{H}'_k,\psi'} \arrow[r, hook, "\mathsf{oblv}^{\mathbf{H}'_k}"] & \mathcal{C}|_{\chi_a} \arrow[r, two heads, "\mathrm{pr}"] & (\mathcal{C}|_{\chi_a})_{\mathbf{N}',\chi'} \\
(\mathcal{C}|_{\chi_a+\mathbf{A}_k^\perp})^{\mathbf{G}_k,\chi_g} \arrow[u, "\alpha_3"] \arrow[r, hook, "\mathsf{oblv}^{\mathbf{G}_k}"] & \mathcal{C}|_{\chi_a+\mathbf{A}_k^\perp} \arrow[u, "\mathfrak{r}_3"] & \\
\mathcal{C}^{\mathbf{H}_k,\psi} \arrow[u, "\mathfrak{r}_2"] \arrow[r, hook, "\mathsf{oblv}^{\mathbf{H}_k}"] & \mathcal{C} \arrow[u, "\mathfrak{r}_2"] \arrow[r, two heads, "\mathrm{pr}"] & \mathcal{C}_{\mathbf{N},\chi} \arrow[uu, "\mathfrak{r}_1"]
\end{tikzcd}
\tag{7.3.3}
$$

Commutativity of the rightmost square and of the bottom rectangle follow from Lemma 6.4.2, while commutativity of the top rectangle is a consequence of Corollary 6.3.3.

7.4 Proof of the main theorem: step 2

Proposition 7.1.4 is what remains to be proven. I.e., we need to show that the compositions

$$\mathcal{C}^{\mathbf{H}_k,\psi} \xrightarrow{\mathsf{oblv}} \mathcal{C} \xrightarrow{\mathsf{T}} \mathcal{C} \quad \text{and} \quad \mathcal{C}^{\mathbf{H}_k,\psi} \xrightarrow{\mathsf{oblv}} \mathcal{C} \xrightarrow{\mathsf{Av}_!^{\mathbf{N},\chi}} \mathcal{C}$$

are canonically equivalent. This will be proven again by induction on n and, as before, the statement is trivial for $G = GL_1$. Note that

$$\mathsf{Av}_!^{\mathbf{N},\chi} := \operatorname*{colim}_{k\in\mathbf{N}} (\mathsf{Av}_!^{\mathbf{N}_k,\chi})$$

is indeed the left adjoint to $\mathsf{oblv}^{\mathbf{N},\chi} : \mathcal{C}^{\mathbf{N},\chi} \to \mathcal{C}$.

Once more, induction is made possible by the isomorphism of functors:

$$\mathsf{Av}_*^{\mathbf{N}_\ell,\chi} \simeq \mathsf{Av}_*^{\mathbf{N}'_\ell,\chi} \circ \mathsf{Av}_*^{\mathbf{A}_\ell,\chi_a} : \mathcal{C}^{\mathbf{H}_k,\psi} \to \mathcal{C}$$

which is an instance of Lemma 6.4.1.

Since $d_\ell = d_\ell^{\mathbf{A}} + d_\ell^{\mathbf{N}'}$, we also have

$$\mathsf{Av}_*^{\mathbf{N}_\ell,\chi}[2d_\ell] \simeq \mathsf{Av}_*^{\mathbf{N}'_\ell,\chi}[2d_\ell^{\mathbf{N}'}] \circ \mathsf{Av}_*^{\mathbf{A}_\ell,\chi_a}[2d_\ell^{\mathbf{A}}].$$

We shall treat the two averaging functors in right-hand side separately. We begin with $\mathsf{Av}_*^{\mathbf{A}_\ell,\chi_a}$.

The abelian case

We wish to show that

$$\operatorname*{colim}_{k\in\mathbb{N}} (\mathsf{Av}_!^{\mathbf{N}_k,\chi}) \simeq \operatorname*{colim}_{\ell\geq k} (\mathsf{Av}_*^{\mathbf{A}_k,\chi})[2d_\ell^{\mathbf{A}}].$$

We will prove a stronger statement, namely that this equivalence holds at each stage ℓ:

Proposition 7.4.1. *For any $\ell \geq k \geq 1$, there is a natural equivalence*

$$\mathsf{Av}_!^{\mathbf{A}_\ell,\chi_a} \xrightarrow{\simeq} \mathsf{Av}_*^{\mathbf{A}_\ell,\chi_a}[2d_\ell^{\mathbf{A}}]$$

of functors $\mathcal{C}^{\mathbf{H}_k,\psi} \to \mathcal{C}$.

The proof of this requires some preparation and some more notation. Let us onsider the subgroup of $\mathbf{G}_k$ defined by requiring that all the rows except for the last one equal the corresponding rows of the identity matrix. We name this subgroup $\mathbf{K}_k$ and form the semi-direct product

$$\mathsf{Heis}_k := \mathbf{K}_k \ltimes \mathbf{A}_k.$$

The name is chosen because Heis_k looks like an "extended Heisenberg group"; e.g., for $G = GL_4$,

$$\mathsf{Heis}_k = \begin{pmatrix} 1 & 0 & 0 & t^{-3k}\mathcal{O} \\ 0 & 1 & 0 & t^{-2k}\mathcal{O} \\ t^{3k}\mathcal{O} & t^{2k}\mathcal{O} & 1+t^k\mathcal{O} & t^{-k}\mathcal{O} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

By construction, Heis_k is a subgroup of $\mathbf{H}_k$ and $\psi = \chi_a$ on it.

Fix now $\ell \geq k \geq 1$. On $\mathcal{C}^{\mathsf{Heis}_k}$, the natural forgetful map $\mathsf{Av}_*^{\mathsf{Heis}_\ell,\chi} \to \mathsf{Av}_*^{\mathbf{A}_\ell,\chi}$ is an equivalence. Similarly, on $\mathcal{C}^{\mathsf{Heis}_\ell}$, the natural forgetful map $\mathsf{Av}_*^{\mathsf{Heis}_k,\chi} \to \mathsf{Av}_*^{\mathbf{K}_k}$ is an equivalence. Thus, Proposition 7.4.1 follows immediately from the next lemma.

Lemma 7.4.2. *Let $\mathcal{C}$ be endowed with an action of the mirabolic group of $GL_n(\!(t)\!)$. For any $\ell \geq k \geq 1$, the functors*

$$\mathsf{Av}_*^{\mathsf{Heis}_\ell,\chi}[2(d_\ell^{\mathbf{A}} - d_k^{\mathbf{A}})] : \mathcal{C}^{\mathsf{Heis}_k,\chi} \rightleftarrows \mathcal{C}^{\mathsf{Heis}_\ell,\chi} : \mathsf{Av}_*^{\mathsf{Heis}_k,\chi}$$

are mutually inverse equivalences of categories.

For $G = GL_2$, this lemma was originally stated and proved by D. Gaitsgory in a more direct way, i.e., without using the Fourier transform.

Proof. We use the same strategy as in Proposition 6.3.2. Let $D := 2(d_k^{\mathbf{A}} - d_\ell^{\mathbf{A}})$. Let Av_ℓ and Av_k denote the functors $\mathsf{Av}_*^{\mathsf{Heis}_\ell,\chi}$ and $\mathsf{Av}_*^{\mathsf{Heis}_k,\chi}$ respectively. We shall show that the two compositions are naturally isomorphic to the shift functor $\mathrm{id}[D]$. By Fourier transform, the datum of an action of the mirabolic group on $\mathcal{C}$ makes $\mathcal{C}$ a category over the quotient stack $\mathbf{A}^\vee/\mathbf{G}'$. From this point of view, the functor Av_ℓ is equivalent to restricting to $\chi + \mathbf{A}_\ell^\perp$, which we identify with $\mathbf{K}_\ell$. Thus, we view $\mathcal{C}^{\mathsf{Heis}_k,\chi}$ as a category over $\mathbf{K}_k$ and we are going to analyze its restriction to $\mathbf{K}_\ell$. Let $\iota : \mathbf{K}_\ell \hookrightarrow \mathbf{K}_k$ be the inclusion; note that ι is a closed embedding (finitely presented, in particular) and $\mathbf{K}_k$ acts quotient-wise on $\mathbf{K}_\ell$. Then, Av_ℓ consists of acting by $\iota_\bullet(\omega_{\mathbf{K}_\ell}) \simeq \iota_*^{ren}(\omega_{\mathbf{K}_\ell})[D]$.

Hence, the composition $\mathsf{Av}_k \circ \mathsf{Av}_\ell : \mathcal{C}^{\mathsf{Heis}_k,\chi} \to \mathcal{C}^{\mathsf{Heis}_k,\chi}$ becomes:

$$c \mapsto \mathsf{Av}_*^{\mathbf{K}_k}\big(\iota_*^{ren}(\omega_{\mathbf{K}_\ell})[D] \diamond c\big).$$

Thanks to (6.1.1) and the $\mathbf{K}_k$-invariance of c, it is enough to prove that $\mathsf{Av}_*^{\mathbf{K}_k}(\iota_*^{ren}(\omega_{\mathbf{K}_\ell})) \simeq \omega_{\mathbf{K}_k}$. By (6.2.2),

$$\mathsf{Av}_*^{\mathbf{K}_k}(\iota_*^{ren}(\omega_{\mathbf{K}_\ell})) \simeq p_*^{ren} \circ \zeta_*^{ren}(\omega_{\mathbf{K}_\ell \times \mathbf{K}_k}) \simeq m_*^{ren}(\iota_*^{ren}(\omega_{\mathbf{K}_\ell}) \boxtimes \omega_{\mathbf{K}_k}) \simeq \omega_{\mathbf{K}_k}.$$

The opposite composition is written as

$$\mathsf{Av}_\ell \circ \mathsf{Av}_k : \mathcal{C}^{\mathsf{Heis}_\ell,\chi} \to \mathcal{C}^{\mathsf{Heis}_\ell,\chi}, \;\; c \mapsto \boldsymbol{\iota}^! \circ \mathsf{Av}^{\mathbf{K}_k}(\boldsymbol{\iota}_*^{ren} c).$$

By (6.2.6), the latter expression is canonically equivalent to

$$\mathsf{Av}_*^{\mathbf{K}_\ell}(c)[-2\dim_{\mathbf{K}_k/\mathbf{K}_\ell}] \simeq c[2(d_k^{\mathbf{A}} - d_\ell^{\mathbf{A}})],$$

as desired. □

We can now finish the proof that T is an equivalence.

Conclusion of the proof of Proposition 7.1.4. As the diagonal $\{(\ell,\ell) : \ell \geq k\}$ is cofinal in the half-quadrant $\{(h,\ell) : h \geq \ell \geq k\}$, we have

$$\mathsf{T} := \operatorname*{colim}_{\ell \geq k} \mathsf{Av}_*^{\mathbf{N}_\ell,\chi}[2d_\ell] \simeq \operatorname*{colim}_{h\geq \ell \geq k} \Big(\mathsf{Av}_*^{\mathbf{N}'_\ell,\chi}[2d_\ell^{\mathbf{N}'}] \circ \mathsf{Av}_*^{\mathbf{A}_h,\chi_a}[2d_h^{\mathbf{A}}] \Big).$$

Thus, we compute

$$\begin{aligned} \mathsf{T} &\simeq \operatorname*{colim}_{h\geq \ell \geq k} \Big(\mathsf{Av}_*^{\mathbf{N}'_\ell,\chi}[2d_\ell^{\mathbf{N}'}] \circ \mathsf{Av}_*^{\mathbf{A}_h,\chi_a}[2d_h^{\mathbf{A}}] \Big) \\ &\simeq \operatorname*{colim}_{\ell \geq k} \Big(\mathsf{Av}_*^{\mathbf{N}'_\ell,\chi}[2d_\ell^{\mathbf{N}'}] \circ \operatorname*{colim}_{h \geq \ell} \mathsf{Av}_*^{\mathbf{A}_h,\chi_a}[2d_h^{\mathbf{A}}] \Big) \\ &\simeq \operatorname*{colim}_{\ell \geq k} \Big(\mathsf{Av}_*^{\mathbf{N}'_\ell,\chi}[2d_\ell^{\mathbf{N}'}] \Big) \circ \mathsf{Av}_!^{\mathbf{A},\chi_a}, \end{aligned}$$

where the last isomorphism is a consequence of Lemma 7.4.1. By induction hypothesis,

$$\mathsf{T} \simeq \operatorname*{colim}_{\ell \geq k} \Big(\mathsf{Av}_*^{\mathbf{N}'_\ell,\chi}[2d_\ell^{\mathbf{N}'}] \Big) \circ \mathsf{Av}_!^{\mathbf{A},\chi_a} \simeq \mathsf{Av}_!^{\mathbf{N}',\chi} \circ \mathsf{Av}_!^{\mathbf{A},\chi_a} \simeq \mathsf{Av}_!^{\mathbf{N},\chi},$$

as desired. □

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