ON THE COHOMOLOGY OF $\mathsf{GL}(N)$ AND ADJOINT SELMER GROUPS

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ABSTRACT. We prove -under certain conditions (local-global compatibility and vanishing of integral cohomology), a generalization of a theorem of Galatius and Venkatesh. We consider the case of $\mathsf{GL}(N)$ over a CM field and we relate the localization of penultimate non vanishing cuspidal cohomology group for a locally symmetric space to the Selmer group of the Tate dual of the adjoint representation. More precisely we construct a Hecke-equivariant injection from the divisible group associated to the first fundamental group of a derived deformation ring to the Selmer group of the twisted dual adjoint motive with divisible coefficients and we identify its cokernel as its first Tate-Shafarevich group. Actually, we also construct similar maps for higher homotopy groups with values in exterior powers of Selmer groups, although with less precise control on their kernel and cokernel. We generalize this to Hida families as well.

1. INTRODUCTION

Let G be a connected reductive group over $\mathbb{Q}$. Let Z be its center and $A \subset Z$ be the maximal split torus of Z. For any $\mathbb{Q}$-group H, we decompose its group of adelic points as $H(\mathrm{A}) = H_f \times H_\infty$ into its finite and archimedean parts. Let K_∞ be a maximal connected compact subgroup of G_∞. Let $X_G = G_\infty/C_\infty$ where $C_\infty = A_\infty K_\infty$ be the riemannian space associated to G.

Let $d = \dim X_G = \dim G_\infty - \dim C_\infty$; let $\ell_0 = \dim T_{G_\infty} - \dim T_{C_\infty}$ where T_{G_∞}, resp. T_{C_∞} denote a maximal torus of G_∞ resp. C_∞. If G admits a Shimura variety, $\ell_0 = 0$ and d is even. Let $q_0 = (d - \ell_0)/2$. For instance, for F a number field and $G = \mathrm{Res}_{F/\mathbb{Q}}\,\mathsf{GL}(N)$ with $[F : \mathbb{Q}] = r_1 + 2r_2$, we have

$$\ell_0 = (N - [\tfrac{N}{2}])r_1 + Nr_2 - 1, \quad q_0 = [\tfrac{N^2}{4}]r_1 + \frac{N(N-1)}{2}r_2, \quad d = 2q_0 + \ell_0 = \frac{N(N+1)}{2}r_1 + N^2 r_2 - 1$$

Let U be a compact open subgroup of G_f and

$$Y_G(U) = G(\mathbb{Q})\backslash(X_G \times G_f/U)$$

be the locally symmetric space of level U associated to G. Let λ be a dominant weight for G and V_λ the irreducible algebraic representation of G of highest weight λ. Let $\mathrm{H}^\bullet(Y_G(U), V_\lambda(\mathbb{C}))$ be the graded space of cohomology of the corresponding local system over $Y_G(U)$. Let $H_{temp}^\bullet(Y_G(U), V_\lambda(\mathbb{C}))$ be its tempered part (corresponding to tempered automorphic algebraic representations). Let $\mathrm{H}_\bullet^{temp} = \mathrm{H}_{temp}^{d-\bullet}(Y_G(U), V_\lambda(\mathbb{C}))$. Let $h_\mathbb{Z}$ be the spherical $\mathbb{Z}$-Hecke algebra outside $Ram(U)$ acting faithfully on $\mathrm{H}_\bullet^{temp}$.

Recall that Prasanna-Venkatesh [PV16, Sect.3] defined a rank ℓ_0 abelian Lie subalgebra $\mathfrak{a}_G \subset Lie(G_\mathbb{C})$ and an action of the graded ring $\bigwedge^\bullet \mathfrak{a}_G$ on $\mathrm{H}_\bullet^{temp}$; they conjectured that there is a Hecke equivariant isomorphism

$$\mathrm{H}_\bullet^{temp} \cong \mathrm{H}_{q_0}^{temp} \otimes_\mathbb{C} \bigwedge^\bullet \mathfrak{a}_G$$

or

$$\mathrm{H}_\bullet^{temp} \cong \mathrm{H}_{q_0}^{temp} \otimes_{h_\mathbb{Z}\otimes\mathbb{C}} \bigwedge^\bullet_{h_\mathbb{Z}\otimes\mathbb{C}} h_\mathbb{Z} \otimes \mathfrak{a}_G$$

Let p be a prime. We fix embeddings $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$ and $\overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$. Let K be a sufficiently big p-adic field, $\mathcal{O}$ its valuation ring, ϖ a uniformizing parameter, and $k = \mathcal{O}/(\varpi)$ its residue field.

Let $\mathrm{H}^\bullet(A) = \mathrm{H}^\bullet(Y_G(U), V_\lambda(A))$ for $A = \mathcal{O}$ or $k = \mathcal{O}/(\varpi)$. Note that $\mathrm{H}^\bullet(\mathcal{O})$ may have torsion. Let h be the $\mathcal{O}$-Hecke algebra outside $Ram(U)$ acting faithfully on this module. Let S_p be the set of places of F above p. Let π be a cuspidal representation occuring in $H^{q_0}(\mathcal{O})$, let $\lambda_\pi \colon h \to \mathcal{O}$ be the corresponding Hecke eigensystem and $\overline{\lambda} \colon h \to k$ its reduction modulo ϖ. We put $\mathfrak{m} = \mathrm{Ker}(\overline{\lambda})$ the corresponding maximal ideal.

In what follos, we will assume that π has an associated Galois representation $\rho_\pi \colon \mathrm{Gal}(\overline{F}/F) \to {}^L G(\mathcal{O})$ (unramified outside $Ram(U) \cup S_p$) satisfying:

(RLI) The residual representation ρ_π has residual large image.

Recall that this means that its image contains $\phi(H(k'))$ where $k' \subset k$ is a subfield of k and $\phi \colon H \to \widehat{G}'$ is the universal covering of the derived subgroup $\widehat{G}'$ of $\widehat{G}$). When this condition is satisfied, we also say that $\mathfrak{m}$ is strongly non Eisenstein.

Let $H_\mathfrak{m}^\bullet$ (resp. $H_\mathfrak{m}^\bullet(k)$) be the localization of $\mathrm{H}^\bullet(\mathcal{O})$ (resp. $\mathrm{H}^\bullet(k)$) at $\mathfrak{m}$. Recall that Caraiani and Scholze [CaSch15] have proven that if Y_G is a unitary Shimura variety, then $\mathrm{H}_\mathfrak{m}^i = 0$ for $i \neq q_0 = \frac{d}{2}$ which implies that $\mathrm{H}_\mathfrak{m}^{q_0}$ is torsion-free. In general, we put $q_m = q_0$ and $q_s = q_0 + \ell_0$ and consider the assumption

$(Van_\mathfrak{m})$ $\mathrm{H}_\mathfrak{m}^i(k) = 0$ for $i \notin [q_m, q_s]$

When (RLI) holds, it is generally believed that that $(Van_\mathfrak{m})$ is satisfied, but it seems very hard to prove in general. Nevertheless, it is known if $q_0 = 1$. If $q_0 \geq 2$, $\mathrm{H}_\mathfrak{m}^1 = 0$ can be proven for many groups [Cai21, Section 1.2.4]. In particular, if $q_0 \leq 2$, $(Van_\mathfrak{m})$ holds for many groups G. This is the case for:

1) $\mathsf{GL}(2)_{/F}$ for F imaginary quadratic (called in the sequel the Bianchi case, for which $q_0 = \ell_0 = 1, d = 3$) or F CM of degree 4 (called in the sequel the higher Bianchi case, where $q_0 = 2$, $\ell_0 = 3$, $d = 7$),
2) $\mathsf{GL}(3)_{/\mathbb{Q}}$, $(q_m = 2, \ell_0 = 1, d = 5)$,
3) $O(p, q)$, $p + q \leq 5$ (see [Cai21, Section 7])

From now on, we furthermore assume that $G = \mathrm{Res}_{F/\mathbb{Q}} \mathsf{GL}_N$ and F is CM with $[F \colon \mathbb{Q}] = 2d_0$. Therefore, $\ell_0 = Nd_0 - 1$ and $q_0 = \frac{N(N-1)}{2} d_0$. Let $\mathrm{T} = h_\mathfrak{m}$ be the localized Hecke algebra acting faithfully on $\mathrm{H}_\mathfrak{m}^\bullet$. These modules are finite $\mathcal{O}$ but may contain torsion. We assume

$(\mathrm{Gal}_\mathfrak{m})$ There exists a continuous Galois representation $\rho_\mathfrak{m} \colon \mathrm{Gal}(\overline{F}/F) \to \mathsf{GL}_N(\mathrm{T})$ unramified outside $Ram(U) \cup S_p$ and such that for any $v \notin Ram(U) \cup S_p$, $\mathrm{char}(\rho_\mathfrak{m}(\mathrm{Frob}_v)) = Hecke_v(X)$ where the Hecke polynomial at v is given by

$$Hecke_v(X) = X^N - T_{v,1} X^{N-1} + \ldots + (-1)^i T_{v,i} \mathbb{N}v^{\frac{i(i-1)}{2}} X^{N-i} + \ldots + (-1)^N T_{v,N} \mathbb{N}v^{\frac{N(N-1)}{2}}.$$

This conjecture is proven in [Sch15] if one replaces T by T/I where I is a nilpotent ideal of exponent bounded in terms of N and $[F \colon \mathbb{Q}]$. The exponent of I is bounded by 4 in [NT16]. By [CGH+19], the ideal can be taken to be 0 if p splits totally in F.

Let $\mathfrak{n} \subset \mathcal{O}_F$ be a squarefree ideal coprime to p. We denote by S the set of places of F dividing $\mathfrak{n}$. Let $U_0(\mathfrak{n}) \subset GL(N, \widehat{\mathcal{O}}_F)$ be the Iwahori subgroup of level $\mathfrak{n}$, and

$$Y = Y_0(\mathfrak{n}) = \mathsf{GL}_N(F) \backslash GL_N(\mathrm{A}_F)/U_0(\mathfrak{n})C_\infty$$

be the Shimura space of level $\Gamma_0(\mathfrak{n})$ (here $C_\infty = \mathbb{R}^\times U_N(F_\infty^+)$). We assume

(MIN) ρ_π is $\mathfrak{n}$-minimal.

Recall that this means that for any place $v | \mathfrak{n}$, the image $\overline{\rho}_\pi(I_v)$ of the inertia subgroup I_v contains a regular unipotent element. More precisely, let J_N be the standard Jordan block of size N and $t_v \colon I_v \to \mathbb{Z}_p(1)$ be

the p-adic tame homomorphism of F_v given by $\tau(p_v^{1/p^n}) = \zeta_{p^n}^{t_v(\tau)} \cdot p_v^{1/p^n}$ for all n's. Then the condition of $\mathfrak{n}$-minimality at v is that there exists $g_v \in \mathsf{GL}_N(\mathcal{O})$ such that for any $\tau \in I_v$,

$$g_v \cdot \rho_\pi(\tau) \cdot g_v^{-1} = \exp(t_v(\tau)J_N).$$

Let $\lambda = (\lambda_{\tau,i})$ where $\tau\colon F \to \overline{\mathbb{Q}}$ and $i = 1, \ldots, N$. Let $\pi = \pi_f \otimes \pi_\infty$ be a cuspidal automorphic representation such that $dim\, \pi_f^{U_0(\mathfrak{n})} = 1$ and of cohomological weight λ. In other words, there are Hecke equivariant embeddings

$$\pi_f^{U_0(\mathfrak{n})} \subset \mathrm{H}^i(Y, V_\lambda(\mathbb{C})), \quad i = 1, 2$$

Let $S_p = S_{F,p}$ be the set of places of F above p. By [HLLT16], we know that the Galois representation $\rho_\pi\colon \mathrm{Gal}(\overline{F}/F) \to \mathsf{GL}_N(\mathcal{O})$ associated to π exists and by [Ca14], $\rho_\pi|_{G_{F_w}}$ is Fontaine-Laffaille at $w \in S_p$. We will assume either of the two following local conditions at p :

(FL) p is unramified in F and $\lambda_{\tau,1} - \lambda_{\tau,N} < p - N$ for any $\tau \in I_F$.

(ORD_π) F contains an imaginary quadratic field in which p splits, π is unramified and ordinary at all places $v \in S_p$.

To recall what the latter means, we follow the notations of [ACC+18, Section 5.1, before Th.5.5.1] (see also [Ge19, Lemma 2.7.6]). We assume $S \cap S_{F,p} = \emptyset$. Then the ordinarity of π at $v \in S_p$ means that $u_{\lambda,\varpi_v}^{(i)} \in \mathcal{O}^\times$ for $i = 1, \ldots, N$ where the $u_{\lambda,\varpi_v}^{(i)}$'s are defined by

$$Hecke_v(X) = \prod_{i=1}^{N}(X - \mathbb{N}v^{i-1}\frac{u_{\lambda,\varpi_v}^{(i)}}{u_{\lambda,\varpi_v}^{(i-1)}} \cdot \prod_{\tau\colon F_v \to \overline{\mathbb{Q}}_p} \tau(\varpi_v)^{\lambda_{\tau,N-i+1}})$$

with $u_{\lambda,\varpi_v}^{(0)} = 1$ by convention.

For $i = 1, \ldots, N$, let $\chi_{\pi,v,i}\colon G_{F_v} \to \mathcal{O}^\times$ be the character given by

$$\chi_{\pi,v,i} \circ \mathrm{Art}_{F_v}(u) = \epsilon^{-(i-1)}(\mathrm{Art}_{F_v}(u))\prod_{\tau} \tau(u)^{\lambda_{\tau,N+1-i}}$$

for all $u \in \mathcal{O}_{F_v}^\times$, and by

$$\chi_{\pi,v,i} \circ \mathrm{Art}_{F_v}(\varpi_v) = \mathbb{N}v^{i-1}\frac{u_{\lambda,\varpi_v}^{(i)}}{u_{\lambda,\varpi_v}^{(i-1)}} \cdot \prod_{\tau\colon F_v \to \overline{\mathbb{Q}}_p} \tau(\varpi_v)^{\lambda_{\tau,N-i+1}}.$$

We put

$$\underline{\chi}_{\pi,v} = \mathrm{diag}(\chi_{\pi,v,i})_{i=1,\ldots,N}$$

Then the translation of the condition (ORD_π) on the Galois side is as follows. There exists $g_v \in \mathsf{GL}_N(\mathcal{O})$ such that $g_v \cdot \rho_\pi|_{G_{F_v}} \cdot g_v^{-1}$ is upper triangular. Let $B = TN$ be the Levi decomposition of the Borel B of upper triangular matrices with T the subgroup of diagonal matrices. And if we denote by $\underline{\chi}_v\colon G_{F_v} \to T(\mathcal{O})$ the homomorphism $g_v \cdot \rho_\pi|_{G_{F_v}} \cdot g_v^{-1}$ modulo $N(\mathcal{O})$. Then

$$\underline{\chi}_v = \underline{\chi}_{\pi,v}$$

In that situation, we consider the so-called p-distinguished condition

$(DIST)$ We assume that $\overline{\rho}_\pi$ is distinguished at each $v \in S_p$ (or in brief, p-distinguished).

With the notations above, it means that for each $v \in S_p$, the residual characters $\overline{\chi}_{v,i}$ modulo ϖ for $i = 1, \ldots, N$ are mutually distinct. In other words, the assumption of p-distinguishability expresses the fact

that for all $v \in S_p$, the homomorphism $\underline{\overline{\chi}}_v = (\overline{\chi}_{v,i})_{i=1,\dots,N}$ separates the roots of (GL_N, B, T). We will also need sometimes the assumption of strong p-distinguishability:

(STDIST) For all $i < j$'s, the quotients $\overline{\chi}_{\widetilde{v},i} \cdot \overline{\chi}_{\widetilde{v},j}^{-1}$ are neither trivial nor equal to $\overline{\epsilon}^{-1}$.

We now recall briefly the deformation theory of Galois representations that plays a central role in this paper. Let $\mathcal{F}_{\overline{\rho}} \colon {}_{\mathcal{O}}\mathrm{Art}_k \to Sets$ resp. $\mathcal{F}_v$, be the problem of deformations of $\overline{\rho} = \overline{\rho}_\pi$, resp. of $\overline{\rho}_v = \overline{\rho}_\pi|_{G_{F_v}}$. For $v \in S_p$, we consider the subfunctors $\mathcal{F}_v^? \subset \mathcal{F}_v$ where $? = \mathrm{ord}, FL$; assuming (ORD_π), $\mathcal{F}_v^{\mathrm{ord}}$ is the functor of $\underline{\chi}_{\pi,v}$-ordinary deformations ρ's, that is, those for which $\underline{\chi}_\rho = \underline{\chi}_{\pi,v}$ (see Section 2.1.2); similarly, assuming (FL), $\mathcal{F}_v^{FL}$ is the functor of. Fontaine-Laffaille local deformations (see Section 2.1.3). Similarly, for $v \in S$, we consider the subfunctor $\mathcal{F}_v^{min} \subset \mathcal{F}_v$ of minimal deformations (see 2.1.1). Let

$$\mathcal{F}_{loc} = \prod_{v \in S \cup S_p} \mathcal{F}_v \quad \text{and} \quad \mathcal{F}_{loc}^{min,?} = \prod_{v \in S} \mathcal{F}_v^{min} \times \prod_{v \in S_p} \mathcal{F}_v^?$$

and consider the global deformation problem defined as the fiber product

$$\mathcal{D}^? = \mathcal{F}_{\overline{\rho}}^{min,?} = \mathcal{F}_{\overline{\rho}} \times_{\mathcal{F}_{loc}} \mathcal{F}_{loc}^{min,?}.$$

It is pro-prepresentable by a pair $(R^?, \rho^?)$ where $? = \mathrm{ord}, FL$ (see for instance [CHT08, Prop.2.2.9 and Lemma 2.4.1] for $? = FL$ and [CHT08, Section 2.4.4] or [Ti96, Chapt.6] for $? = \mathrm{ord}$). As in [CaGe18], we introduce allowable Taylor-Wiles sets Q disjoint of the set of places dividing $\mathfrak{n}p$ and consider the functors $\mathcal{D}_Q^?$ similar to $\mathcal{D}^?$, where we allow arbitrary ramification at places $v \in Q$. Similarly, let $R_Q^?$ be the universal deformation ring of $\mathcal{D}_Q^?$ and $(\mathrm{T}_Q, \mathfrak{m}_Q)$ be the analogue of $(\mathrm{T}, \mathfrak{m})$ with Taylor-Wiles auxiliary level at Q (see [CaGe18, Section 9.1]). The rings $R_Q^?$ and T_Q both admit a natural structure of $\mathcal{O}[\Delta_Q]$-algebra where Δ_Q is the p-Sylow of $\prod_{v \in Q} T(k_v)$. By $(\mathrm{Gal}_\mathfrak{m})$, there is a Galois representation $\rho_{\mathfrak{m}_Q} \colon \mathrm{Gal}(\overline{F}/F) \to \mathsf{GL}_N(\mathrm{T}_Q)$ associated to $(\mathrm{T}_Q, \mathfrak{m}_Q)$. We will assume the following

Conjecture 1. *(LLC) For any Taylor-Wiles set Q, the conjugacy class $[\rho_{\mathfrak{m}_Q}]$ is in $\mathcal{D}_Q^?(\mathrm{T}_Q)$.*

The local-global compatibility conjecture (LLC) for $\rho_{\mathfrak{m}_Q}$ has been proven (at least if F is not imaginary quadratic but contains an imaginary quadratic field in which p splits) modulo a nilpotent ideal (see [ACC+18]) of exponent bounded by $[F : \mathbb{Q}]$ and N. Assuming (LLC), for each Taylor-Wiles set Q, there is a canonical surjective $\mathcal{O}[\Delta_Q]$-homomorphism $\phi_Q \colon R_Q^? \to \mathrm{T}_Q$. Let $\phi = \phi_\emptyset$. By Taylor-Wiles-Kisin patching, the $\mathcal{O}[\Delta_Q]$-homomorphism ϕ_Q give rise to morphism $S_\infty \to R_\infty^? \to \mathrm{T}_\infty \subset \mathsf{End}_{S_\infty}(C_\infty^\bullet)$ (see [CaGe18]). The graded module $\mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})$ has a natural structure of graded algebra over $\mathrm{Tor}_0^{S_\infty}(R_\infty, \mathcal{O}) = R_\infty \otimes_{S_\infty} \mathcal{O} = R^?$. Let $\mathrm{H}_\bullet^\mathfrak{m} = H_\mathfrak{m}^{d-\bullet}$, then we have:

Theorem 1. *([CaGe18]) Assume $\zeta_p \notin F$, $p > N$, $(\mathrm{Gal}_\mathfrak{m})$, (LLC) (RLI), (MIN), (FL) or $(ORD_\pi) + (DIST)$. Then, the canonical map $R_\infty^? \to \mathrm{T}_\infty$ is an isomorphism, and in particular $R_\infty^? \otimes_{S_\infty} \mathcal{O} = R^? \cong \mathrm{T}$. Moreover, the module $\mathrm{H}_{q_m}^\mathfrak{m}$ is finite and free over T, and there is an isomorphism of graded modules:*

$$\mathrm{H}_\bullet^\mathfrak{m} \cong \mathrm{H}_{q_0}^\mathfrak{m} \otimes_\mathrm{T} \mathrm{Tor}_\bullet^{S_\infty}(\mathrm{T}_\infty, \mathcal{O})$$

in other words $\mathrm{H}_\bullet^\mathfrak{m}$ is a free graded $\mathrm{Tor}_\bullet^{S_\infty}(\mathrm{T}_\infty, \mathcal{O})$-module.

On the other hand, in [GV18], the authors introduced derived notions of deformation problems for $\overline{\rho}_\mathfrak{m}$ denoted $\mathcal{D}^s$, $\mathcal{D}^{s,?}$; they are pro-represented by simplicial pro-artinian rings $\mathcal{R}$, resp. $\mathcal{R}^{s,?}$. Generalizing their main result Th.12.1 and Th.14.1, Y. Cai [Cai21, Theorem 0.6] proved

Theorem 2. *([GV18, Cai21]) Assume that $p > N$, $\zeta_p \notin F$, $(\mathrm{Gal}_\mathfrak{m})$, (LLC) and $(Van_\mathfrak{m})$ hold. Under the assumptions (RLI), (MIN), (FL) or $(ORD_\pi) + (DIST)$, we have an isomorphism of graded commutative rings $\pi_\bullet \mathcal{R} \cong \mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})$, in particular we have an isomorphism of graded objects*

$$\mathrm{H}_\bullet^\mathfrak{m} \cong \mathrm{H}_{q_0}^\mathfrak{m} \otimes_\mathrm{T} \pi_\bullet(\mathcal{R})$$

Our first main result concerns the relation between $\pi_1(\mathcal{R})$ and the minimal and ordinary (resp. Fontaine-Laffaille) Selmer group of $\mathrm{Ad}(\rho_\pi)^*(1)$. Recall $\pi_0(\mathcal{R}) = R$ is the classical $(min, ?)$-deformation ring. Let $\mathcal{O}_n = \mathcal{O}/(\varpi^n)$. We consider the simplicial ring homomorphism

$$\phi_n \colon \mathcal{R} \to R \to \mathcal{O}_n$$

given by the universal property for the deformation $\rho_n = \rho_\pi \pmod{(\varpi^n)}$. Let $M_n = \varpi^{-n}\mathcal{O}/\mathcal{O}$; it is a finite $\mathcal{O}_n$-module. Consider the simplicial ring $\Theta_n = \mathcal{O}_n \oplus M_n[1]$ concentrated in degrees 0 and 1 up to homotopy. It is endowed with a simplicial ring homomorphism $\mathrm{pr}_n \colon \Theta_n \to \mathcal{O}_n$ given by the first projection. Let $L_n(\mathcal{R})$ be the set of homotopy equivalence classes of simplicial ring homomorphisms $\Phi \colon \mathcal{R} \to \Theta_n$ such that $\mathrm{pr}_n \circ \Phi = \phi_n$. Following [GV18, Lemma 15.1], we define (see Section 4) compatible homomorphisms

$$\pi(n, \mathcal{R}) \colon L_n(\mathcal{R}) \to \mathsf{Hom}_R(\pi_1(\mathcal{R}), M_n)$$

Let $\mathrm{Sel}(F, \mathrm{Ad}\,\rho_n(1) \otimes_{\mathrm{T}} M_n^\vee)$ be the minimal and $? = ord, FL$ Selmer group of $\mathrm{Ad}\,\rho_n(1) \otimes_{\mathrm{T}} M_n^\vee$. Formal arguments show that it is the Pontryagin dual of $L_n(\mathcal{R})$. Therefore, by passing to Pontryagin dual, we obtain a homomorphism $GV_n = \pi(n, \mathcal{R})^\vee$:

$$GV_n \colon \mathsf{Hom}_{\mathrm{T}}(\pi_1(\mathcal{R}), M_n)^\vee \hookrightarrow Sel(F, (\mathrm{Ad}\,\rho_n)^*(1) \otimes M_n^\vee)$$

By taking the direct limit we obtain a homomorphism

$$GV \colon \pi_1(\mathcal{R}) \otimes_{R,\lambda} \mathcal{O} \otimes_{\mathcal{O}} K/\mathcal{O} \hookrightarrow \mathrm{Sel}(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})$$

Then our first result (Theorem 13 and Proposition 7 in the text) is

Theorem 3. *Assume $p > N$, $\zeta_p \notin F$, $(\mathrm{Gal}_{\mathfrak{m}})$, (LLC), (RLI), (MIN), (FL) or $(ORD_\pi) + (DIST)$. The natural homomorphism*

$$GV \colon \pi_1(\mathcal{R}) \otimes_{\mathrm{T}} K/\mathcal{O} \hookrightarrow \mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})$$

is injective. The left-hand side module is ϖ-divisible of $\mathcal{O}$-corank ℓ_0. If we are in the ordinary case, let us furthermore assume $(STDIST)$, then the cokernel of GV is the Tate-Shafarevitch group $\mathrm{III}^1(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})$ and is finite. Moreover, its $\mathcal{O}$-Fitting ideal is the same as the one of $\mathrm{Sel}(F, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O})$.

The key step to prove this theorem is Proposition 5. A similar result, under stronger assumptions, is proven in the course of the proof of [GV18, Theorem 15.2]. However, under our assumptions, the proof given there doesn't seem to work. Our approach is different and involves only commutative algebra. The last part of the statement is proved in Proposition 7.

Actually, under the same assumptions , we can also define a graded version $GV^\bullet$ of the Galatius-Venkatesh homomorphism GV. Let us introduce some notations before stating our result. For a graded algebra G, we introduce the truncation

$$\tau_{\leq \ell_0} G = \bigoplus_{j \leq \ell_0} G_j$$

with a partial algebra structure defined only for $g_{j_k} \in G_{j_k}$ such that $j = j_1 + j_2 \leq \ell_0$. For a graded algebra G, we also introduce its largest graded-commutative quotient $\widetilde{G}$, that is, its largest (graded) quotient in which $< a, b >_{j_1, j_2} = (-1)^{j_1 j_2} < b, a >_{j_2, j_1}$ for $a \in \widetilde{G}_{j_1}$, $b \in \widetilde{G}_{j_2}$. We endow $\pi_\bullet(\mathcal{R}^{\otimes \bullet})$ with a structure of graded algebra and we prove (see Theorem 14):

Theorem 4. *For every cuspidal representation π' occuring in $H_{\mathfrak{m}}^\bullet$, for any $m \geq 1$, there is an homomorphism of A_m-modules*

$$(HGV_j) \quad GV_m^j \colon \widetilde{\pi}_j(\mathcal{R}^{\otimes j})_{A_m} \to \bigwedge_{A_m}^j \mathrm{Sel}(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)^{\oplus j}$$

It induces a morphism of truncated graded A_m-algebras

$$(HGV_\bullet) \quad GV_m^\bullet \colon \tau_{\leq \ell_0} \widetilde{\pi}_\bullet(\mathcal{R}^{\otimes \bullet}) \otimes A_m \to \tau_{\leq \ell_0} \overset{\bullet}{\underset{A_m}{\bigwedge}} \mathrm{Sel}(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)^{\oplus \bullet}$$

Moreover, for any j, the composition $a_j \circ GV_m^j \circ \mu_j$ of GV_m^j with the tensor shuffle multiplication map $\mu_j \colon \bigwedge_A^j \pi_1(\mathcal{R})_A \to \widetilde{\pi}_j(\mathcal{R}^{\otimes j})_A$ and the homomorphism a_j induced by the addition

$$\mathrm{Sel}(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)^{\oplus j} \to \mathrm{Sel}(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)$$

coincides with $\bigwedge^j GV_m^1$. For $j = \ell_0$, the cokernel of $GV_m^{\ell_0} \circ \mu_{\ell_0}$ is annihilated by $\mathrm{Fitt}_\mathcal{O}(\mathrm{Sel}(F, Ad(\rho_{\pi'})))$.

This theorem implies that for any $1 \leq j \leq \ell_0$, the shuffle multiplication μ_j is injective[1].

We have similar results for Hida families in Section 7. Let R_h be the universal minimal Λ-ordinary deformation ring prorepresenting the minimal Λ-ordinary deformation problem $\mathcal{D}_h$ (see Section 6.2). Let $\mathcal{R}_h$ the simplical proartinian ring prorepresenting the simplicial minimal Λ-ordinary deformation problem $\mathcal{D}_h^s$ (see Section 6.3). The rings $\mathcal{R}_h$ and R_h are naturally Λ-algebra, where Λ is the Hida-Iwasawa algebra (see Section 6.1). We have $\pi_0(\mathcal{R}_h) = R_h$ (Section 6.3), hence we have a surjective Λ-algebra homomorphism $\mathcal{R}_h \to R_h$. Note that $\pi_\bullet(\mathcal{R}_h)$ is a graded algebra over $\pi_0(\mathcal{R}_h) = R_h$. Let $\mathfrak{m}_h$ be the maximal ideal of Λ-ordinary Hida Hecke algebra acting faithfully on $\mathbb{H}_h^\bullet = e\mathrm{H}^\bullet(Y_1(p^\infty), \mathcal{O})$ associated to $\mathfrak{m}$ and let T_h be its corresponding $\mathfrak{m}_h$-localization. Assuming $(\mathrm{Gal}_\mathfrak{m})$ and (LLC), there is Λ-algebra homomorphism $R_h \to \mathrm{T}_h$. We prove following the Calegari-Geraghty method (Th. 15 and 16 in the text) :

Theorem 5. *Assuming $(\mathrm{Gal}_\mathfrak{m})$ and (LLC), (MIN), $(ORD_\pi) + (DIST)$. Assume $(Van_\mathfrak{m})$ holds for a classical specialization π of T_h of level prime to p. Then*
1) $R_h \to \mathrm{T}_h$ is an isomorphism,
2) there is a isomorphism of graded T_h-modules

$$\mathbb{H}_{h, \mathfrak{m}_h}^\bullet = \mathbb{H}_{h, \mathfrak{m}_h}^{q_s} \otimes_{\mathrm{T}_h} \pi_{d - \bullet}(\mathcal{R}_h)$$

Given a Hida family $\mathrm{T}_h \to \mathbf{I}$ of the $\mathfrak{m}_h$-localization of the ordinary Hida Hecke algebra, let $\Lambda_\mathbf{I}$ be the image of Λ in $\mathbf{I}$ and $\mathcal{R}_\mathbf{I} = \mathcal{R}_h \underline{\otimes}_\Lambda \Lambda_\mathbf{I}$. We prove (Theorem 17 in the text):

Theorem 6. *There are natural $\mathbf{I}$-linear injective homomorphisms*

$$GV_h \colon \pi_1(\mathcal{R}_h) \otimes_{R_h} \widehat{\mathbf{I}} \hookrightarrow \mathrm{H}_\mathcal{L}^1(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_\mathbf{I} \widehat{\mathbf{I}}).$$

and

$$GV_\mathbf{I} \colon \pi_1(\mathcal{R}_\mathbf{I}) \otimes_{R_\mathbf{I}} \widehat{\mathbf{I}} \hookrightarrow \mathrm{H}_\mathcal{L}^1(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_\mathbf{I} \widehat{\mathbf{I}}).$$

and we have $GV_\mathbf{I} \circ \pi_\mathbf{I} = GV_h$ for $\pi_\mathbf{I}$ the homomorphism induced by the map $\mathcal{R}_h \to \mathcal{R}_\mathbf{I}$.

The new feature in this Hida-theoretic context is that the Hida-theoretic cohomology module $\mathbb{H}_h^\bullet = e\mathrm{H}^\bullet(Y_1(p^\infty), \mathcal{O})$ is conjectured to be concentrated in the maximal degree q_s. This is one form of the non-abelian Leopoldt conjecture due to Hida and one of the authors. They are discussed in Section 6.4 in the text. Recall that a simplicial ring is called discrete if it is weakly equivalent to a constant ring. Then, this non-abelian Leopoldt conjecture is equivalent to the following:

Conjecture 2. *The simplicial ring $\mathcal{R}_h$ is discrete and for any arithmetic weight λ', there is an isomorphism of commutative graded rings*

$$\mathrm{Tor}_\bullet^\Lambda(R_h, \Lambda/P_{\lambda'}) = \pi_\bullet(\mathcal{R}_{\lambda'}).$$

---

[1] However it would be more interesting to study the injectivity of the homomorphisms GV_m^j.

Because of this conjecture, the first statement of the theorem above is probably empty. However, the second appears to be interesting and yields the inconditional Theorem below (Theorem 18 in the text). Let $\mathcal{L}$ be the minimal Λ-ordinary local deformation data. Let $\mathrm{Sel}_\mathbf{I} = \mathrm{H}^1_\mathcal{L}(F, \mathrm{Ad}\, \rho_\mathbf{I} \otimes \widehat{\mathbf{I}})$ and $M_\mathbf{I}$ its Pontryagin dual. Let $\Phi_\mathbf{I}$ be the Pontryagin dual of $\pi_1(\mathcal{R}_\mathbf{I}) \otimes_{R_\mathbf{I}} \widehat{\mathbf{I}}$. Let finally $N_\mathbf{I} = \mathrm{Ker}(M_\mathbf{I} \to \Phi_\mathbf{I})$ be the Pontryagin dual of Coker $GV_\mathbf{I}$.

Theorem 7. *Assume moreover (STDIST). Then*
  (1) The $\mathbf{I}$-modules $M_\mathbf{I}$ and $\Phi_\mathbf{I}$ have rank ℓ_0 and $\Phi_\mathbf{I}$ is free.
  (2) The Poitou-Tate-Pontryagin duality induces an isomorphism

$$N_\mathbf{I} \cong \varprojlim_n \mathrm{Sel}_\mathbf{I}[\mathfrak{m}_\mathbf{I}^n]$$

Let $\widetilde{\mathbf{I}}$ be the integral closure of $\mathbf{I}$ and let $X = \mathrm{Supp}(\widetilde{\mathbf{I}}/\mathbf{I})$. For $P \notin X$, we have $\mathbf{I}/P = \widetilde{\mathbf{I}}/P\widetilde{\mathbf{I}}$. For such P, the reduction modulo P of the exact sequence

$$0 \to N_\mathbf{I} \to M_\mathbf{I} \to \Phi_\mathbf{I} \to 0$$

is exactly the Pontryagin dual of the exact sequence given by

$$0 \to \pi_1(\mathcal{R}_P) \overset{GV_P}{\to} \mathrm{Sel}_P \to \mathrm{Coker}\, GV_P \to 0$$

where the natural notations are explained before Corollary 4.

In learning the material used here, we benefited from a seminar on Galatius-Venkatesh paper held in 2019 at Paris 13, and of discussions with Y. Harpaz and A. Vezzani. Special thanks are due to Yichang Cai whose thesis provided much of the framework for our work. He also taught us several technical aspects of Model Categories and Derived Deformation theory, although misunderstandings and mistakes left in the text are ours.

2. Calegari-Geraghty Theory for GL_N

2.1. **Local deformation conditions.** For each finite place v of F, let $\overline{F}_v$ be an algebraic closure of F_v and G_{F_v} be the local Galois group of $\overline{F}_v$ over F_v, identified to a decomposition group at v in G_F.

By definition, for an artinian local $\mathcal{O}$-algebra A with residue field k, let $\mathcal{F}_{\overline{\rho}}(A)$ (resp. $\mathcal{F}_v(A)$) be the set of conjugacy classes of liftings $\rho\colon G_F \to \mathsf{GL}_N(A)$ of $\overline{\rho}_\pi$ (resp. $\rho_v\colon G_{F_v} \to \mathsf{GL}_N(A)$ of $\overline{\rho}_v$). Let $\mathcal{F}_v^\square$ be the functor of liftings of $\overline{\rho}_v$. Recall that it carries an action of the formal group scheme $\widehat{\mathrm{PGL}_N}$ by conjugation and that $\mathcal{F}_v^\square / \widehat{\mathrm{PGL}_N} \cong \mathcal{F}_v$.

We define below subfunctors $\mathcal{F}_v^{?,\square}$, resp. $\mathcal{F}_v^?$ of local liftings, resp. deformations of $\bar{\rho}_v$, for $? = min, ord, FL$.

Definition 1. *A functor $\mathcal{F}\colon {}_\mathcal{O}\mathrm{Art}_k \to \mathsf{SETS}$ is called liftable if for any surjection $\alpha\colon A_1 \to A_0$, the morphism $\mathcal{F}(A_1) \to \mathcal{F}(A_0)$ is surjective.*

Note that if $\mathcal{F}_v^{?,\square}$ is formally smooth, $\mathcal{F}_v^?$ is liftable. We prove this below for $? = min, \mathrm{ord}, FL$.

2.1.1. *minimal case.* For $v \in S$, let $t_v\colon I_v \to \mathbb{Z}_p^\times$ be the p-adic tame character and let $q = \mathcal{O}_{F_v}/(\varpi_v)$; for $\tau \in I_v^{tame}$, and for ϕ_v an arithmetic Frobenius, we have $\phi_v \circ \tau \circ \phi_v^{-1} = \tau^q$. We consider the problem $\mathcal{F}_v^{min,\square} \subset \mathcal{F}_v^\square$, resp. $\mathcal{F}_v^{min} \subset \mathcal{F}_v$ of v-minimal liftings, resp. deformations:

$$\mathcal{F}_v^{min,\square}(A) = \{\rho_v \in \mathcal{F}_v^\square(A); \text{for } \tau \in I_v, \text{there exists } g_v \in \mathsf{GL}_N(A) \text{ such that for any } \tau \in I_v,$$
$$g_v \cdot \rho(\tau) \cdot g_v^{-1} = \exp(t_v(\tau)J_N)\}$$

and

$$\mathcal{F}_v^{min}(A) = \mathcal{F}_v^{min,\square}(A)/\widehat{\mathrm{PGL}_N}(A).$$

Let $\Phi_v = \mathrm{diag}(q^{N-1}, \ldots, 1)$.

Lemma 1. *The functor $\mathcal{F}_v^{min,\square}$ is prorepresented by a formally smooth $\mathcal{O}$-algebra $R_v^{min,\square}$, isomorphic to a power series ring in N^2 variables. The functor $\mathcal{F}_v^{min}$ is liftable.*

Comments:

1) Note that the proof uses the fact that the matrix J_N is regular nilpotent.
2) This result is also proven in [CHT08, Lemma 2.4.19 and Cor.2.4.20] and [Bo19, Introduction].
3) The functor $\mathcal{F}_v^{min}$ is pro-representable if and only if p does not divide $q_v^i - 1$ $(i = 1, \ldots, N-1)$.

Proof. Let C_0 be the centralizer of J_N in the standard Borel $B \subset G$. The formal scheme $\mathcal{C}$ of centralizers of liftings of $\overline{J}_N$ is isomorphic to the principal homogeneous formal scheme $\widehat{\mathsf{GL}_N / C_0}$, hence is formally smooth of dimension $N^2 - N$. There is a fibration of functors $\mathcal{F}_v^{min,\square} \to \mathcal{C}$ given by $\rho_v \to g_v^{-1} C_0 g_v$. Therefore, to study the formal smoothness of $\mathcal{F}_v^{min,\square}$, one can fix $C \subset B$ and study its fiber $\mathcal{F}_{v,[C]}^{min,\square}$ in $\mathcal{F}_v^{min,\square}$. In the notations below, we take $C = C_0$. We therefore consider ρ_v such that for any $\tau \in I_v$, $\rho_v(\tau) = \exp(t_v(\tau)J_N)$. Let $\Phi_v = \mathrm{diag}(q_v^{N-1}, \ldots, 1)$. We have $\rho_v(\phi_v) = \Psi_v$ with $\Psi_v = \Phi_v g$ with $g \in C_0$. Let $\widehat{C}_0$ be the formal group scheme associated to C_0. By the considerations above, we see easily that the map $g \mapsto ([\rho_g]$ where $\rho_g(\phi_v) = \Phi_v \cdot g$ and $\rho_g(\tau) = \exp(t_v(\tau)J_N)$ for $\tau \in I_v$, is an isomorphism of functors

$$\widehat{C} \cong \mathcal{F}_{v,[C]}^{min,\square}.$$

This isomorphism is compatible to conjugation by $\widehat{G}$. It implies that $\mathcal{F}_v^{min,\square}$ is a torsor under the smooth formal group scheme $\widehat{G}$, hence is formally smooth of dimension N^2. This proves the formal smoothness of $R_v^{min,\square} \cong \mathcal{O}[[X_1, \ldots, X_{N^2}]]$.

Any deformation $[\rho_v] \in \mathcal{F}_v^{min}(A)$ has a representative ρ_v such that for any $\tau \in I_v$, $\rho_v(\tau) = \exp(t_v(\tau)J_N)$. For such a representative, $\rho_v(\phi_v) = \Psi_v$ can be written $\Psi_v = \Phi_v g$ with $g \in C_0$. If ρ_v' is another such representative, we see that there exists $h \in \widehat{C}(A)$ such that

$$\rho_v'(\phi_v) = h\Psi_v h^{-1} = \Phi_v g'$$

Therefore, $g' = \Phi_v^{-1} h \Phi_v h^{-1} \cdot .g$ for an $h \in \widehat{C}(A)$ and conversely. Conjugation by Φ_v is an automorphism of $\widehat{C}(A)$. Thus we find that

$$\mathcal{F}_v^{min} \cong \widehat{C}/(\Phi_v - 1)\widehat{C}$$

which is isomorphic to the functor

$$A \mapsto A/(q-1)A \times A/(q^2-1)A \times \ldots A/(q^{N-1}-1)A$$

This functor is not pro-representable unless p does not divide the $q^i - 1$ (for all $i = 1, \ldots, N-1$). However, it is liftable. $\square$

2.1.2. *The ordinary case.* For $v \in S_p$, we consider the problem $\mathcal{F}_v^{ord} \subset \mathcal{F}_v$ of ordinary deformations of $\overline{\rho}_v$ without fixing the Hodge-Tate weights. This means that $[\rho_v] \in \mathcal{F}_v(A)$ if and only if for a representative ρ_v, there exists $g_v \in \mathsf{GL}_N(A)$ such that $g_v \cdot \rho|_{G_{F_v}} \cdot g_v^{-1}$ takes values in $B(A)$ and that its reduction $\underline{\chi}_{\rho,v} \colon G_{F_v} \to T(A)$ modulo $N(A)$ is a lifting of $\overline{\underline{\chi}}_v \colon G_{F_v} \to T(k)$.

Let $\mu = (\mu_v)_{v \in S_p}$ where $\mu_v \colon \mathcal{O}_{F_v}^\times \to T(\mathcal{O})$ is a continuous character lifting the character $\overline{\underline{\chi}}_v \colon \mathcal{O}_{F_v}^\times \to T(k)$ given by $\bar{\rho}|_{I_v}$. We define the subfunctor $\mathcal{F}_v^{ord,\mu} \subset \mathcal{F}_v^{ord}$ of ordinary deformations of weight μ to be given by $[\rho_v]$'s for which $\underline{\chi}_{\rho,v} \circ \mathrm{Art}_{F_v}|_{\mathcal{O}_{F_v}^\times} = \mu_v$ where μ_v is considered as taking values in $T(A)$ via the canonical morphism $T(\mathcal{O}) \to T(A)$.

In this paper, besides the sections dealing with Hida theory, we only consider the case

$$\mu_v = \underline{\chi}_{\pi,v} \circ \mathrm{Art}_{F_v}|_{\mathcal{O}_{F_v}^\times}.$$

The homomorphism μ is then algebraic, given by the Hodge-Tate weights of ρ_{π_v}.

In the Lemma below, we also consider the subfunctor $\mathcal{F}_v^{\det,\mathrm{ord}} \subset \mathcal{F}_v^{\mathrm{ord}}$ where the determinant of the deformations is fixed equal to $\det \rho_v$.

Lemma 2. *For $v \in S_p$, under the condition (STDIST) (and $p > N$) the functors $\mathcal{F}_v^{\det,\mathrm{ord}}$, $\mathcal{F}_v^{\mathrm{ord}}$, $\mathcal{F}_v^{\mathrm{ord},\mu}$ are pro-representable by a complete noetherian local ring $R_v^{\det,\mathrm{ord}}$, resp.R_v^{ord}, $R_v^{\mathrm{ord},\mu}$. These problems are unobstructed, hence formally smooth. In particular, one has $R_v^{\mathrm{ord}} = R_v^{\det,\mathrm{ord}}[[T_1,\ldots,T_f]]$ (where $f = [F_v : \mathbb{Q}_p]$ is the degree of F_v).*

Proof. Let $\mathfrak{b}$ be the Lie algebra of the standard Borel B and $\mathfrak{b}'$ the subalgebra of trace zero elements. To prove that $\mathcal{F}_v^{\det,\mathrm{ord}}$, resp. $\mathcal{F}_v^{\mathrm{ord},\mu}$, is proprepresentable and unobstructed, it is enough to show that $H^i(\Gamma_v, \mathfrak{b}') = 0$ for $i \neq 1$. By Tate duality, this follows from the strong distinguishability condition. Since $p > N$, any $[\rho] \in \mathcal{F}_v^{\mathrm{ord}}(A)$ has a twist rho_χ by a p-power order character χ whose determinant is equal to $\det \rho_{\pi,v}$. This character χ factors through the universal character $G_{F_v} \to \mathcal{O}[[T_1,\ldots,T_f]]^\times$. Hence $\mathcal{F}_v^{\mathrm{ord}}$ is proprepresentable by $R_v^{\mathrm{ord}} = R_v^{\det,\mathrm{ord}}[[T_1,\ldots,T_f]]$.

$\square$

Comment: The dimension of R_v^{ord} (given in [CHT08, Lemma 2.4.8]) will be recalculated later when applying the Poitou-Tate formula.

Corollary 1. *The functor $\mathcal{F}_v^{\mathrm{ord},\square}$ is pro-representable by a formally smooth ring $R_v^{\mathrm{ord},\square}$.*

Proof. The morphism $\mathcal{F}_v^{\mathrm{ord},\square} \to \mathcal{F}_v^{\mathrm{ord}}$ is a $\widehat{G^{ad}}$-torsor, hence is smooth and the base is smooth. $\square$

In all sections below (except those dealing with Hida theory), we fix $\mu_v = \underline{\chi}_{\pi,v} \circ \mathrm{Art}_{F_v}|_{\mathcal{O}_{F_v}^\times}$ for each $v \in Sp$. We call the deformation problem $\mathcal{F}_v^{\mathrm{ord},\mu}$ the $\underline{\chi}_\pi$-ordinary deformation problem, or for brevity the ordinary deformation problem (with Hodge-Tate weights fixed by π). Hence we write only $\mathcal{F}_v^{\mathrm{ord}}$, although μ is fixed by $\mu = \underline{\chi}_\pi$.

2.1.3. *The Fontaine-Laffaille case.* Finally, for $v \in S_p$, we consider the problem $\mathcal{F}_v^{FL,\square}$, resp. $\mathcal{F}_v^{FL}$, of Fontaine-Laffaille liftings ρ_v, resp. deformations $[\rho_v]$ of $\overline{\rho}_v$. This means that there exists a ϕ-filtered A-module M free of rank N over A, such that ρ_v is isomorphic to $V_{crys}(M)$.

Lemma 3. *The functor $\mathcal{F}_v^{FL,\square}$ is pro-representable by a formally smooth ring $R_v^{FL,\square}$ isomorphic to a power series $\mathcal{O}$-algebra in $N^2 + [F_v : F]\frac{N(N-1)}{2}$ variables.*

Proof. This is [CHT08, Coroll.2.4.3]. $\square$

2.2. **The Theorem of Calegari-Geraghty.** For $? = \mathrm{ord}, FL$, let

$$\mathcal{F}_{loc} = \prod_{v \in S \cup S_p} \mathcal{F}_v \quad \text{and} \quad \mathcal{F}_{loc}^{min,?} = \prod_{v \in S} \mathcal{F}_v^{min} \times \prod_{v \in S_p} \mathcal{F}_v^?$$

We define the global deformation problem as the fiber product

$$\mathcal{D}^? = \mathcal{F}_{\overline{\rho}}^{min,?} = \mathcal{F}_{\overline{\rho}} \times_{\mathcal{F}_{loc}} \mathcal{F}_{loc}^{min,?}.$$

By Schlessinger's criterion, assuming p-distinguishedness, resp. p-smallness, the functor $\mathcal{D}^{\mathrm{ord}}$, resp. $\mathcal{D}^{FL}$, is proprepresentable by a pair $(R^?, \rho^?)$ where $? = \mathrm{ord}, FL$ (see for instance [CHT08, Prop.2.2.9 and Lemma 2.4.1] for $? = FL$ and [CHT08, Section 2.4.4] or [Ti96, Chapt.6] for $? = \mathrm{ord}$). For any proartinian $\mathcal{O}$-algebra A and any object $[\rho] \in \mathcal{D}^?(A)$ there is a unique local $\mathcal{O}$-algebra homomorphism $\phi_\rho \colon R \to A$ such that $\phi_\rho \circ \rho^? \equiv \rho$.

Assuming (LLC), we have $[\rho_\pi] \in \mathcal{D}^?(\mathcal{O})$ and $[\rho_\mathfrak{m}] \in \mathcal{D}^?(T)$ and we therfore have the following commutative diagram of local surjective $\mathcal{O}$-algebra homomorphisms.

$$
\begin{array}{ccc}
R & \xrightarrow{\phi_{\rho_\mathfrak{m}}} & \mathrm{T} \\
& \phi_{\rho_\pi} \searrow & \downarrow \phi_\pi \\
& & \mathcal{O}
\end{array}
$$

Recall that the local-global compatihility condition (LLC) for $\rho_\mathfrak{m}$ has been proven (see [ACC+18]), at least if F is not imaginary quadratic.

Let now S_∞ be the Taylor-Wiles power series ring and let $R_\infty \to \mathrm{T}_\infty$ be the surjective homomorphism of S_∞-algebras introduced by Calegari-Geraghty using Taylor-Wiles systems (see [CaGe18]), and recalled in the introduction.

Theorem 8. *([CaGe18]) Assume $\zeta_p \notin F$, $p > N$, $(\mathrm{Gal}_\mathfrak{m})$, (LLC) (RLI), (MIN), (FL) or $(ORD_\pi) + (DIST)$. Then, the canonical map $R_\infty^? \to \mathrm{T}_\infty$ is an isomorphism, and in particular $R_\infty^? \otimes_{S_\infty} \mathcal{O} = R^? \cong \mathrm{T}$. Moreover, the module $\mathrm{H}_{q_m}^\mathfrak{m}$ is finite and free over T, and there is an isomorphism of graded modules:*

$$
\mathrm{H}_\bullet^\mathfrak{m} \cong \mathrm{H}_{q_0}^\mathfrak{m} \otimes_\mathrm{T} \mathrm{Tor}_\bullet^{S_\infty}(\mathrm{T}_\infty, \mathcal{O})
$$

in other words $\mathrm{H}_\bullet^\mathfrak{m}$ is a free graded $\mathrm{Tor}_\bullet^{S_\infty}(\mathrm{T}_\infty, \mathcal{O})$-module.

Remarks:

1) Poincaré duality shows that $\mathrm{H}_{q_s}^\mathfrak{m}$ is isomorphic to $\mathsf{Hom}(T, \mathcal{O})^m$ as T-module (this reflects also that it is torsion free as $\mathcal{O}$-module).

2) for $G = \mathsf{GL}(2)$ over a number field F, the rank m is 2^{r_1} (this follows from calculations of $(\mathfrak{g}, C_\infty)$-cohomology, see [BW00] or [H94a]); hence $m = 2^{[F:\mathbb{Q}]}$ if F is totally real, or 1 if F is CM.

In the sequel we will focus on the case $G = \mathsf{GL}(N)_{/F}$, with F CM, quadratic over the totally real field F^+, assuming $(Van_\mathfrak{m})$ and (LLC). More general cases are treated in [Cai21].

3. Simplicial Deformation problems and tangent complexes

We follow [GV18, Sections 5 and 7] and [Cai21]. Given a category $\mathcal{C}$, we denote by $\mathcal{C}^s$ the category of simplicial objects of $\mathcal{C}$. Note any object C of $\mathcal{C}$ defines a simplicial object $\mathcal{F}_C \in \mathcal{C}^s$ by putting $\mathcal{F}_C([n]) = C$ and $\mathcal{F}_C(\phi) = \mathrm{Id}_C$ for any $\phi \in \mathsf{Hom}_\Delta([n], [m])$. Such an object is called discrete. We will be mostly concerned with $\mathcal{C} = \mathrm{Mod}_A$, SETS and the category ${}_A\mathrm{CR}_B$ of pairs (X, π) where X is an A-algebra and $\pi\colon X \to B$ is an A-algebra homomorphism. The resulting simplicial categories are model categories (see [Cai21, 3.1.2]). Note it is not the case for the subcategory ${}_A\mathrm{Art}_B \subset {}_A\mathrm{CR}_B$ of simplicial artinian rings. In the sequel, we mostly consider $A = \mathcal{O}$ and $B = \mathcal{O}/(\varpi) = k$.

A functor $\mathcal{F}\colon {}_\mathcal{O}\mathrm{CR}_k^s \to \mathsf{SETS}^s$ is called homotopy invariant if it sends weakly equivalent objects to weakly equivalent objects. Let $* \in \mathsf{SETS}^s$. A simplicial deformation problem of $*$ is a homotopy invariant functor

$$
\mathcal{F}\colon {}_\mathcal{O}\mathrm{Art}_k^s \to \mathsf{SETS}^s
$$

such that $\mathcal{F}(k) = \{*\}$. A diagram

$$
\begin{array}{ccc}
A_1 \times_{A_0}^h A_2 & \xrightarrow{\pi_2} & A_2 \\
\pi_1 \downarrow & & f_2 \downarrow \\
A_1 & \xrightarrow{f_1} & A_0
\end{array}
$$

in ${}_\mathcal{O}\mathrm{Art}_k^s$ is a homotopy pullback if $f_1 \circ \pi_1$ is homotopic to $f_2 \circ \pi_2$. It is unique up to weak equivalence. A simplicial deformation problem $\mathcal{F}$ of an object $*$ is formally cohesive [GV18, Definition 3.8] if it is homotopy invariant and if it preserves homotopy pullback. Let us give an example of such a functor.

Definition 2. *Given two simplicial rings $\mathcal{R}, \mathcal{R}'$, the simplicial set $\mathsf{sHom}_{\mathrm{CR}^s}(\mathcal{R}, \mathcal{R}')$ is defined as $[p] \mapsto$ $\mathsf{Hom}_{\mathrm{CR}^s}(\mathcal{R}, (\mathcal{R}')^{\Delta[p]})$. Here, for each p, $(\mathcal{R}')^{\Delta[p]}$ denotes the simplicial ring given by $[q] \mapsto \mathsf{Hom}_{\mathsf{SETS}^s}(\Delta[p] \times \Delta[q], \mathcal{R}')$; as a simplicial ring, it is weakly equivalent to $\mathcal{R}'$ (see cite[Lemma 2.13]GV18). The structure of simplicial set on $[p] \mapsto \mathsf{Hom}_{\mathrm{CR}^s}(\mathcal{R}, (\mathcal{R}')^{\Delta[p]})$ is given by the natural face and degeneracy maps coming from those between the $\Delta[p]$'s.*

Recall that $\mathcal{R} \in {}_\mathcal{O}\mathrm{CR}^s_k$ is cofibrant if for any $n \geq 0$, $\mathcal{R}_n$ is a free $\mathcal{O}$-algebra. Note this is only a sufficient condition. For precise definition and properties see [Cai21, Section 2.1]. One can show that for any simplicial ring $\mathcal{R}$, there exists a fibration which is a weak equivalence $c(\mathcal{R}) \to \mathcal{R}$ where $c(\mathcal{R})$ is a cofibrant simplicial ring. One can either assume that the $c(\mathcal{R})_n$'s are noetherian for all n's, or that $\mathcal{R} \mapsto c(\mathcal{R})$ is a functorial construction, but in general not both. Any such $c(\mathcal{R})$ is called a cofibrant replacement of $\mathcal{R}$.

Let $\mathcal{R} \in {}_\mathcal{O}\mathrm{CR}^s_k$ be cofibrant; then the functor ${}_\mathcal{O}\mathrm{Art}^s_k \to \mathsf{SETS}^s$ given by

$$A \mapsto \mathsf{sHom}_{\mathcal{O}\mathrm{CR}^s_k}(\mathcal{R}, A) : {}_\mathcal{O}\mathrm{Art}^s_k \to \mathsf{SETS}^s$$

is formally cohesive. Any pro-representable (see [GV18, 2.14 and 2.19] and Section 3.2 below) functor $\mathcal{F}$ should be formally cohesive. J. Lurie gave a criterion for (pro)representability of a simplicial deformation functor $\mathcal{F}$ of, say, a point $* \in \mathsf{SETS}^s$ (see [GV18, 4.6]). The first (necessary) condition is formal cohesiveness. The second involves its tangent complex $\mathsf{t}\mathcal{F}$ which we recall below.

3.1. **Tangent complex.** Let A be a commutative ring and Z be a commutative A-algebra. Then for any Z-module M and $X \in {}_A\mathrm{CR}_Z$, we have natural isomorphisms

$$\mathsf{Hom}_{{}_A\mathrm{CR}_Z}(X, Z \oplus M) \cong \mathrm{Der}_A(X, M) \cong \mathsf{Hom}_Z(\Omega_{X/A} \otimes_X Z, M).$$

So the functor $X \mapsto \Omega_{X/A} \otimes_X Z$ is left ajoint to the functor $M \mapsto Z \oplus M$. The functors $M \mapsto Z \oplus M$, $X \to \Omega_{X/A} \otimes_X Z$ both have extensions to functors between simplicial categories, and we have natural isomorphism

$$\mathsf{Hom}_{{}_A\mathrm{CR}^s_Z}(X, Z \oplus M) \cong \mathsf{Hom}_{\mathrm{Mod}^s_Z}(\Omega_{X/A} \otimes_X Z, M).$$

The functor $M \mapsto Z \oplus M$ preserves fibrations and weak equivalences. So the total left derived functor of its left adjoint functor $X \mapsto \Omega_{X/A} \otimes_X Z$ exists [GJ10, II.7.3]. It is denoted $X \mapsto L_{X/Z}$. In order to give a concrete description of $L_{X/Z}$ and its fundamental property, one often uses the Dold-Kan equivalence. Let $\mathrm{Ch}(Z)$, resp. $\mathrm{Ch}_{\geq 0}(Z)$, be the category of chain complexes of Z-modules, resp. the subcategory of effective complexes $(\dots \to V_n \to \dots \to V_1 \to V_0)$. Recall that the Dold-Kan equivalence is a functor [GV18, Section 4.3.1] and [Cai21, 3.1.4]

$$DK\colon \mathrm{Ch}_{\geq 0}(Z) \to \mathrm{Mod}^s_Z$$

inducing an equivalence of categories that sends weak equivalences to quasi-isomorphisms. It therefore provides an identification[2] of the categories of simplicial Z-modules Mod^s_Z with the category $\mathrm{Ch}_{\geq 0}(Z)$ of non-negative chain complexes $(\dots V_n \to \dots\dots V_1 \to V_0)$ of Z-modules. Now, let $X \in {}_A\mathrm{CR}_Z$; to construct $L_{X/Z}$, we choose a cofibrant replacement $c(X) \to X$ and form the simplicial module $\Omega_{c(X)/A} \otimes_A Z$. By the Dold-Kan equivalence, it comes from a non-negative chain complex, well defined up to quasi-isomorphism, denoted $L_{X/Z}$.

Moreover, given $M \in \mathrm{Ch}_{\geq 0}(Z)$, we can form the simplicial ring $Z \oplus DK(M)$ where the ideal $DK(M)$ is of square zero. We have a weak equivalence of functors of $M \in \mathrm{Ch}_{\geq 0}(Z)$:

$$\mathsf{sHom}_{{}_A\mathrm{CR}^s_Z}(X, Z \oplus DK(M)) \cong \mathsf{sHom}_{\mathrm{Mod}^s_Z}(L_{X/Z}, M).$$

For $A = \mathcal{O}$ and $Z = k$, we will use the following definition.

Definition 3. *The k-cotangent complex functor $\mathcal{R} \mapsto L_{\mathcal{R}/k}$ is the total left derived functor of the functor $\mathcal{R} \mapsto \Omega_{\mathcal{R}/\mathcal{O}} \otimes_\mathcal{R} k$. More precisely, for $\mathcal{R} \in {}_\mathcal{O}\mathrm{CR}^s_k$, we have a simplicial k-module $L_{\mathcal{R}/k} = \Omega_{c(\mathcal{R})/\mathcal{O}} \otimes_{c(\mathcal{R})} k$, where $c(\mathcal{R})$ is a cofibrant replacement of $\mathcal{R}$ in the model category ${}_\mathcal{O}\mathrm{CR}^s_k$, and $L_{\mathcal{R}/k}$ is well-defined in the homotopy category.*

---

[2]When no confusion is possible, we will doing so without specifying it.

The simplicial k-module $\mathsf{sHom}_{\mathrm{Mod}_k^s}(L_{\mathcal{R}/k}, k)$ may be viewed as an element[3] of $\mathrm{Ch}_{\geq 0}(k)$, we denote it by $\mathfrak{t}\mathcal{R}$ and call it the *tangent complex of $\mathcal{R}$ over k*. It is well defined in the derived category of k-vector spaces. We label its terms either as $\mathfrak{t}^{(i)}\mathcal{R}$ ($i \in \mathbb{Z}_{\geq 0}$), or as $\mathfrak{t}_{(-i)}\mathcal{R} = \mathfrak{t}^{(i)}\mathcal{R}$. With these notations, we have, via Dold-Kan identification of simplicial modules and chain complexes: $\pi_{-i}\mathfrak{t}\mathcal{R} = H_{-i}(\mathfrak{t}\mathcal{R}) = H^i(\mathfrak{t}\mathcal{R})$.

More generally, if $\mathcal{R}_1 \to \mathcal{R}_2$ is a morphism of objects $\mathcal{R}_1, \mathcal{R}_2 \in {}_{\mathcal{O}}\mathrm{CR}_k^s$, we fix compatible cofibrant replacements $c(\mathcal{R}_1) \to c(\mathcal{R}_2)$ in the model category ${}_{\mathcal{O}}\mathrm{CR}_k^s$. The relative k-cotangent complex $L_{\mathcal{R}_2/\mathcal{R}_1} = \Omega_{c(\mathcal{R}_2)/c(\mathcal{R}_1)} \otimes_{c(\mathcal{R}_2)} k$ is independent of the choice of the cofibrant replacements, hence is a well-defined object in the homotopy category. The simplicial k-module $\mathsf{sHom}_{\mathrm{Mod}_k^s}(L_{\mathcal{R}_2/\mathcal{R}_1}, k)$ may be viewed as an element of $\mathrm{Ch}_{\leq 0}(k)$, we denote it by $\mathfrak{t}(\mathcal{R}_2, \mathcal{R}_1)$ and call it the *relative tangent complex of $\mathcal{R}_2$ with respect to $\mathcal{R}_1$ over k* (we won't specify this below). It is well defined in the derived category of k-vector spaces.

Lemma 4. *If $\mathcal{R}_1 \to \mathcal{R}_2$ is a morphism of objects $\mathcal{R}_1, \mathcal{R}_2 \in {}_{\mathcal{O}}\mathrm{CR}_k^s$, we have a distinguished triangle*

$$\mathfrak{t}(\mathcal{R}_2, \mathcal{R}_1) \to \mathfrak{t}\mathcal{R}_2 \to \mathfrak{t}\mathcal{R}_1 \overset{+1}{\to}$$

hence a long exact sequence

$$\dots H^i(\mathfrak{t}(\mathcal{R}_2, \mathcal{R}_1)) \to H^i(\mathfrak{t}\mathcal{R}_2) \to H^i(\mathfrak{t}\mathcal{R}_1) \to H^{i+1}(\mathfrak{t}(\mathcal{R}_2, \mathcal{R}_1)) \to \dots$$

Proof. Apply the exact functor $\mathsf{sHom}_{\mathrm{Mod}_k^s}(-, k)$ to the distinguished triangle

$$L_{\mathcal{R}_1/k} \underline{\otimes}_{\mathcal{R}_1} k \to L_{\mathcal{R}_2/k} \to L_{\mathcal{R}_2/\mathcal{R}_1} \to (L_{\mathcal{R}_1/k} \underline{\otimes}_{\mathcal{R}_1} k)[1]$$

similar to [COT, Section 90.7]. To obtain this distinguished triangle, first, take a cofibrant replacement $c(\mathcal{R}_1) \to c(\mathcal{R}_2)$ of $\mathcal{R}_1 \to \mathcal{R}_2$, then form the exact sequence of simplicial $c(\mathcal{R}_2)$-modules

$$0 \to \Omega_{c(\mathcal{R}_1)/\mathcal{O}} \underline{\otimes}_{c(\mathcal{R}_1)} c(\mathcal{R}_2) \to \Omega_{c(\mathcal{R}_2)/\mathcal{O}} \to \Omega_{c(\mathcal{R}_2)/c(\mathcal{R}_1)} \to 0$$

where the first tensor product is taken for simplicial $c(\mathcal{R}_1)$-modules; then, apply $- \otimes_{c(\mathcal{R}_2)} k$.   □

Let $k[n]$ be the object of $\mathrm{Ch}_{\geq 0}(k)$ defined by k in degree n and 0 elsewhere. Then, we have $\mathsf{sHom}_{\mathcal{O}\mathrm{CR}_k^s}(\mathcal{R}, k \oplus DK(k[n])) \cong \mathsf{sHom}_{\mathrm{Mod}_k^s}(L_{\mathcal{R}/k}, DK(k[n]))$ for $\mathcal{R}$ cofibrant, and hence for $i \geq 0$, we have

$$\pi_i \mathsf{sHom}_{\mathcal{O}\mathrm{CR}_k^s}(\mathcal{R}, k \oplus DK(k[n])) \cong H^{i+n}(\mathfrak{t}\mathcal{R}).$$

Proposition 1. *Let $\mathcal{F} : {}_{\mathcal{O}}\mathrm{Art}_k^s \to \mathsf{SETS}^s$ be a formally cohesive functor. Then there exists an object $\mathfrak{t}\mathcal{F} \in \mathrm{Ch}(k)$, unique up to weak equivalence, such that for any non-positive finite dimensional chain complex $V \in \mathrm{Ch}_{\geq 0}(k)$, we have*

$$\mathsf{sHom}_{\mathrm{Ch}(k)}(V, \mathfrak{t}\mathcal{F}) \cong \mathcal{F}(k \oplus DK(V^\vee)).$$

Proof. See [Cai21, Corollary 3.43].   □

The chain complex $\mathfrak{t}\mathcal{F}$ is called the *tangent complex of $\mathcal{F}$*. We denote the terms of this complex as $\mathfrak{t}_i\mathcal{F}$ (with degree -1 differential) for $i \in \mathbb{Z}$, or $\mathfrak{t}^{(i)}\mathcal{F} = \mathfrak{t}_{(-i)}\mathcal{F}$ (with degree $+1$ differential). We also write $\mathfrak{t}^i\mathcal{F} = \mathrm{H}^i(\mathfrak{t}\mathcal{F}) = \mathrm{H}_{-i}(\mathfrak{t}\mathcal{F})$. If we let $V = k[-n]$ and take the i-th homotopy group, then we get $\mathrm{H}^{i+n}\mathfrak{t}\mathcal{F} = \pi_{-i}\mathcal{F}(k \oplus DK(k[n]))$ for any $i, n \geq 0$. We write $\mathfrak{t}^i\mathcal{F}$ to abbreviate $\mathrm{H}^i(\mathfrak{t}\mathcal{F}) = \pi_i \mathfrak{t}\mathcal{F}$ (the last equality is via Dold-Kan, viewing $\mathfrak{t}\mathcal{F}$ as a simplicial k-vector space). To end this section, we recall the important conservativity property of the tangent complex functor.

Lemma 5. *Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be formally cohesive functors from ${}_{\mathcal{O}}\mathrm{Art}_k^s$ to sSETS. Then a natural transformation $T : \mathcal{F}_1 \to \mathcal{F}_2$ is a weak equivalence if and only if T induces isomorphisms $\mathfrak{t}^i\mathcal{F}_1 \cong \mathfrak{t}^i\mathcal{F}_2$ for all i.*

Proof. See [Cai21, Corollary 3.46].   □

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[3]Given a non-positive finite dimensional chain complex $V \in \mathrm{Ch}_{\leq 0}(k)$, its k-dual $V^\vee$ is a finite dimensional non-negative chain complex $V^\vee \in \mathrm{Ch}_{\geq 0}(Z)$.

3.2. **Representability.** Given a projective system $\mathcal{R} = (\mathcal{R}_\alpha)_\alpha$ with $\mathcal{R}_\alpha \in {}_\mathcal{O}\mathrm{Art}_k^s$, we put for any $A \in {}_\mathcal{O}\mathrm{Art}_k^s$

$$\mathsf{sHom}(\mathcal{R}, A) = \mathrm{holim}_\alpha \, \mathsf{sHom}(\mathcal{R}_\alpha, A)$$

Definition 4. *A functor* $\mathcal{F}\colon {}_\mathcal{O}\mathrm{Art}_k^s \to \mathsf{sSETS}$ *is proprepresentable if there exists a projective system of cofibrant simplicial artinian rings* $\mathcal{R} = (\mathcal{R}_\alpha)_\alpha$ *with* $\mathcal{R}_\alpha \in {}_\mathcal{O}\mathrm{Art}_k^s$ *and a functorial weak equivalence:*

$$\mathsf{sHom}(\mathcal{R}, A) \to \mathcal{F}(A)$$

for all $A \in {}_\mathcal{O}\mathrm{Art}_k^s$.

Note that $\mathfrak{t}^i \mathcal{R} = H_{-i}(\mathfrak{t}\mathcal{R})$ vanishes for $i < 0$. So if a formally cohesive functor $\mathcal{F} : {}_\mathcal{O}\mathrm{Art}_k^s \to \mathsf{SETS}^s$ is pro-representable, then $\mathfrak{t}^i \mathcal{F}$ should vanishes for $i < 0$. The converse statement is proved by Lurie.

Theorem 9 (Lurie's derived Schlessinger criterion)**.** *Let* $\mathcal{F}$ *be a formally cohesive functor. Then* $\mathcal{F}$ *is pro-representable if and only if* $\mathfrak{t}^i \mathcal{F}$ *vanishes for each* $i < 0$*. In addition, if all the k-vector spaces* $\mathfrak{t}^i \mathcal{F}$ *are finite, the pro-artinian ring* $\mathcal{R}$ *has a countable indexing category.*

3.3. **Simplicial Galois deformation functors.** In this section we consider a smooth group scheme G defined over $\mathcal{O}$. For our application, $G = \mathsf{GL}_N$. Given an $\mathcal{O}$-algebra A, let $\mathbf{B}G(A)$ be the simplicial set associated to the group $G(A)$ by the bar construction. By definition, it is given by $[p] \mapsto N_p G(A) = G(A)^p$, with standard face and degeneracy maps [Wei94, Example 8.1.7]. Note that for any fixed p, $N_p G(A) = \mathsf{Hom}_{rings}(\mathcal{O}_G^{\otimes p}, A)$. We can therefore say that the functor $N_p G\colon A \mapsto N_p G(A)$ is represented by the ring $\mathcal{O}_{N_p G} = \mathcal{O}_G^{\otimes p}$.

If now A is a simplicial $\mathcal{O}$-algebra, we can define a bisimplicial set $\mathrm{Bi}_G^{naive}(A)$, i.e. a contravariant functor

$$\mathrm{Bi}_G^{naive}(A)\colon \Delta \times \Delta \to \mathsf{SETS}, \quad ([p], [q]) \mapsto \mathsf{Hom}_{\mathrm{CR}}(\mathcal{O}_{N_p G}, A_q) = G(A_q)^p$$

where the p-degeneracy and p-face maps are given by the codegeneracy and coface maps of $\mathcal{O}_{N_\bullet G}$ and the q-degeneracy and q-face maps are given by the degeneracy and face maps of A. Recall that given two simplicial rings R, R', we defined in Definition 2 the simplicial set $\mathsf{sHom}(R, R')$ as $[p] \mapsto \mathsf{Hom}_{\mathrm{CR}^s}(R, (R')^{\Delta[p]})$ with face and degeneracy maps coming from those between the $\Delta[p]$'s. We can consider for each p the bisimplicial set $\mathrm{Bi}_G(A)$ given by

$$([p], [q]) \mapsto \mathsf{Hom}_{\mathrm{CR}^s}(c(\mathcal{O}_{N_p G}), A^{\Delta[q]})$$

where $c(\mathcal{O}_{N_p G})$ denotes a cofibrant replacement of $\mathcal{O}_{N_p G}$, with degeneracy and face maps as for the bisimplicial set $\mathrm{Bi}_G^{naive}(A)$. Note that the natural morphism of simplicial rings $A_q \to A^{\Delta[q]}$ induces an injective morphism of bisimplicial sets

$$\mathrm{Bi}_G^{naive}(A) \hookrightarrow \mathrm{Bi}_G(A).$$

Consider the simplicial set $\mathbf{B}'G(A) = \mathrm{hocolim}_q \mathrm{Bi}_G(A)_q$. It may not be fibrant. So, we fix a fibrant replacement

$$\mathbf{B}G(A) = F(\mathbf{B}'G(A)).$$

For A homotopically discrete, this obviously generalizes the usual definition of $\mathbf{B}G(\pi_0(A))$.

Remark: Recall that in [GV18, Definition 5.1], the authors defined a simplicial set $\mathbf{B}^{GV}G(A)$ as the fibrant replacement Ex^∞ of the simplicial set

$$[p] \mapsto ([p], [p]) \mapsto \mathrm{Bi}_G(A)([p], [p]) = \mathsf{Hom}_{\mathrm{CR}^s}(c(\mathcal{O}_{N_p G}), A^{\Delta[p]}).$$

The two definitions are weakly equivalent but there is a slight difference: in our definition, we have a canonical fibration $\mathbf{B}G(A) \to \mathbf{B}G(k)$, while for $\mathbf{B}G^{GV}(A)$, one has a canonical fibration $\mathbf{B}G(A) \to Ex^\infty \mathbf{B}G(k)$.

The key properties of this construction are that:
- The functor $\mathbf{B}G$ is invariant by homotopy : if $A \to B$ is a weak equivalence, the resulting morphism of simplicial sets $\mathbf{B}G(A) \to \mathbf{B}G(B)$ also is, by smoothness of $\mathcal{O}_G$ (see proof of [GV18, Corollary 5.3]).
- It preserves homotopy pullbacks.
- The $\mathbf{B}G$ construction is functorial in G.

Let us write the $S \cup S_p$-ramified Galois group $\Gamma = G_{F,S \cup S_p}$ as inverse limit of finite Galois groups $\Gamma_\alpha = \mathrm{Gal}(F_\alpha/F)$, where F_α/F is a finite $S \cup S_p$-ramified extension. Let $\mathbf{B}\Gamma$ be the prosimplicial set given by the projective system $(\mathbf{B}\Gamma_\alpha)_\alpha$ of simplicial sets $\mathbf{B}\Gamma_\alpha$. Note that the residual Galois representation

$$\overline{\rho} \colon \Gamma \to G(k)$$

gives rise to an element $[\overline{\rho}]$ of $\mathsf{sHom}(\mathbf{B}\Gamma_\alpha, \mathbf{B}G(k))$ for some α. A natural idea to define the analogue of the classical deformation functor

$$\mathcal{F}_{\overline{\rho}} \colon {}_\mathcal{O}\mathrm{Art}_k \to \mathsf{SETS}, A \mapsto \mathsf{Hom}_{\overline{\rho}}(\Gamma, G(A))/adG(A)$$

is to define for each $A \in Ob({}_\mathcal{O}\mathrm{Art}_k^s)$, and any α, the simplicial set

$$\mathcal{F}_{\alpha,\overline{\rho}}^s(A) = \mathsf{sHom}_{[\overline{\rho}]}(\mathbf{B}\Gamma_\alpha, \mathbf{B}G(A))$$

which is the homotopy fiber in

$$\mathcal{F}_\alpha^s(A) = \mathsf{sHom}(\mathbf{B}\Gamma_\alpha, \mathbf{B}G(A))$$

of $[\overline{\rho}] \in \mathsf{sHom}(\mathbf{B}\Gamma_\alpha, \mathbf{B}G(k))$. Then, one would pass to the inductive limit over α to obtain $\mathcal{F}_{\overline{\rho}}^s$ as the homotopy fiber at $[\overline{\rho}]$ of

$$\mathcal{F}^s = \varinjlim_\alpha \mathcal{F}_\alpha^s.$$

It is homotopy invariant and preserves homotopy pullbacks. However, as explained in [GV18, Section 5.4], when the center Z of G is non trivial, the functor $\mathcal{F}_{\overline{\rho}}^s$ cannot be representable because Z gives rise to non trivial automorphisms. It can be modified in two different ways to be made pro-representable: one is to fix the determinant

$$\mathbf{B}\det \colon \mathbf{B}G(A) \to \mathbf{B}\mathbb{G}_m(A)$$

of the simplicial deformations, the other is to "quotient the action of G by the center Z" (with or without fixing the determinant) as in [GV18, Section 5.4]. The two modifications will be denoted $\mathcal{F}_{\det,\overline{\rho}}^s$ and $\mathcal{F}_{Z,\overline{\rho}}^s$. Let us recall their constructions.

3.4. **Modification by the center.** We first define $\mathcal{F}_{\det,\overline{\rho}}^s$.

Let $\rho_\pi \colon \Gamma \to G(\mathcal{O})$ be the Galois representation that we fixed. For any $A \in Ob({}_\mathcal{O}\mathrm{Art}_k^s)$, we have a morphism $\mathbf{B}G(\mathcal{O}) \to \mathbf{B}G(A)$. On the other hand, the determinant morphism induces morphisms (both denoted by det) of simplicial sets

$$\mathbf{B}G(A) \to \mathbf{B}\mathbb{G}_m(A) \quad \text{and} \mathcal{F}_G^s(A) = \mathsf{sHom}(\mathbf{B}\Gamma, \mathbf{B}G(A)) \to \mathcal{F}_{\mathbb{G}_m}^s(A) = \mathsf{sHom}(\mathbf{B}\Gamma, \mathbf{B}\mathbb{G}_m(A))$$

We recall now the definition of $\mathcal{F}_{Z,\overline{\rho}}^s$ ([GV18, Section 5.4]). Consider the short exact sequence

$$1 \to Z(A) \to G(A) \to PG(A) \to 1$$

It gives rise to a fibration sequence (see the section 3.1.3 of [Cai21]).

$$\mathbf{B}G(A) \to \mathbf{B}PG(A) \to \mathbf{B}^2 Z(A).$$

Let $* = \pi_0 \mathbf{B}\Gamma$. Then the functor $\mathcal{F}_Z^s$ is defined by the homotopy pullback diagram

$$\begin{array}{ccc}
\mathcal{F}_Z^s(A) & \to & \mathsf{Hom}(*, \mathbf{B}^2 Z(A) \\
\downarrow & & \downarrow \\
\mathcal{F}^s(A) & \to & \mathsf{Hom}(\mathbf{B}\Gamma, \mathbf{B}^2 Z(A))
\end{array}$$

Its crucial property is that there is a functorial fibration sequence:

$$\mathsf{sHom}(*, \mathbf{B}Z(A)) \to \mathsf{sHom}(\mathbf{B}\Gamma, \mathbf{B}G(A)) \to \mathcal{F}_Z^s(A)$$

We then define $\mathcal{F}_{Z,\overline{\rho}}^s$ as the homotopy fiber of $[\overline{\rho}]$. It follows from the remarks above that

Lemma 6. *The functor $\mathcal{F}_{Z,\overline{\rho}}^s$ is homotopy invariant and preserves homotopy pullbacks, hence is formally cohesive.*

Let $\mathcal{F}_{Z,\overline{\rho}}^{s,S\cup S_p} \colon {}_{\mathcal{O}}\mathrm{Art}_k^s \to \mathsf{sSETS}$ be the functor given by modifying as above the functor

$$\mathcal{F}_{\overline{\rho}}^s(A) = \mathfrak{s}_{[\overline{\rho}]}(\mathbf{B}\Gamma, \mathbf{B}G(A)) = \mathrm{holim}_\alpha\, \mathsf{sHom}_{[\overline{\rho}]}(\mathbf{B}\Gamma_\alpha, \mathbf{B}G(A))$$

The functor $\mathcal{F}_{Z,\overline{\rho}}^{s,S\cup S_p}$ is called the functor of simplicial deformations of $\overline{\rho}$ unramified outside $S \cup S_p$. To study the representability of this deformation functor, we now need to determine its tangent complex. The tangent complex of $\mathcal{F}_\alpha^s$ can be computed as a Cech complex $C^\bullet(\mathbf{B}\Gamma_\alpha, \mathbf{t}\mathbf{B}G)$ ([GV18, Section 4.7] and [GV18, Example 4.38]). Let $\mathfrak{g}_k = \mathrm{Lie}_k\, G$ be the Lie algebra of G. The tangent complex $\mathbf{t}\mathbf{B}G$ is concentrated in degree 1 and one has $\mathbf{t}\mathbf{B}G = \mathfrak{g}_k[1]$ ([GV18, Lemma 5.5]). Therefore

$$\mathbf{t}\mathcal{F}_\alpha^s = C^{\bullet+1}(\mathbf{B}\Gamma_\alpha, \mathfrak{g}_k)$$

Let $\mathfrak{z}_k$ be the center of $\mathfrak{g}_k$. Then, the functorial fibration 3.4 provides a distinguished triangle

$$C^{\bullet+1}(*, \mathfrak{z}_k) \to C^{*+1}(\mathbf{B}\Gamma, \mathfrak{g}_k) \to \mathbf{t}\mathcal{F}_{Z,\overline{\rho}}^s.$$

From these facts it follows that the Cech complex $C^\bullet(\mathbf{B}\Gamma, \mathbf{t}\mathbf{B}G)$ can be computed as the Galois cohomology standard complex (see [Cai21]). Then by [GV18, Lemma 4.30, (iv)] (Mayer-Vietoris long exact sequence) we have the following key proposition:

Proposition 2. *Suppose that* $\mathrm{H}^0(\Gamma, \mathfrak{g}_k) = \mathfrak{z}_k$. *Then, the homology groups of the tangent complex* $\mathbf{t}\mathcal{F}_{Z,\overline{\rho}}^s$ *satisfy* $\mathbf{t}^i\mathcal{F}_{Z,\overline{\rho}}^s = 0$ *when* $i \leq 0$, *and* $\mathbf{t}^i\mathcal{F}_{Z,\overline{\rho}} = \mathrm{H}^{i+1}(F_S/F, \mathrm{Ad}\,\overline{\rho})$ *when* $i \geq 0$.

and its corollary:

Corollary 2. *Suppose that the adjoint action of* $\overline{\rho}$ *fixes precisely the center of the Lie algebra of* $G(k)$. *The functor* $\mathcal{F}_{Z,\overline{\rho}}^s$ *is pro-representable by a simplicial pro-artinian ring* $\mathcal{R}$. *This means that there exists a functorial weak equivalence*

$$\mathsf{sHom}(\mathcal{R}, A) \to \mathcal{F}_{Z,\overline{\rho}}^s(A)$$

Note that, given a profinite group Γ, the question of prorepresentability of the functor

$$A \mapsto \mathcal{F}_{\overline{\rho}}^s(A) = \mathsf{sHom}_{[\overline{\rho}]}(\mathbf{B}\Gamma, \mathbf{B}G(A))$$

by a proartinian simplicial ring $\mathcal{R}$ can only be posed in terms of the existence of a functorial weak equivalence

$$\mathsf{sHom}_{pro\,\mathrm{Art}_k^s}(R, A) \to \mathcal{F}_{\overline{\rho}}^s(A)$$

and not of a functorial isomorphism, because even is A is discrete, there is a bijection between the set of conjugacy classes of homomorphisms $\Gamma \to G(A)$ and the set of homotopy classes of the simplicial set of morphisms of prosimplicial sets $\mathbf{B}\Gamma \to \mathbf{B}G(A)$, not with the simplicial set of morphisms of prosimplicial sets itself.

Lemma 7. *The functor*

$$\pi_0\mathcal{F}_{Z,\overline{\rho}}^s \colon {}_{\mathcal{O}}\mathrm{Art}_k^s \to \mathsf{SETS} \quad A \mapsto \pi_0\mathcal{F}_{Z,\overline{\rho}}^s(A)$$

coincides with Mazur's functor of $S \cup S_p$-*ramified deformations of* $\overline{\rho}$.

Proof. This is [GV18, Lemma 7.1]. It follows from the fact that for a discrete artinian ring A, $\mathbf{B}G(A)$ is weakly equivalent to the classical classifying space of $G(\pi_0(A))$ defined in [Wei94, Example 8.1.7]. Therefore $\pi_0(\mathsf{sHom}(\mathbf{B}\Gamma, \mathbf{B}G(A)))$ is $\mathsf{Hom}(\mathbf{B}\Gamma, \mathbf{B}G(\pi_0(A)))$, which is naturally $\mathsf{Hom}_{cont}(\Gamma, G(A))/G(A)$. By the definition of $\mathcal{F}_{Z,\overline{\rho}}^s$, this implies that $\pi_0\mathcal{F}_{Z,\overline{\rho}}^s(A)$ is naturally $\mathsf{Hom}_{cont}(\Gamma, G(A))/PG(A) = \mathsf{Hom}_{cont}(\Gamma, G(A))/G(A)$, as desired. $\qquad\square$

3.5. **Simplicial Galois deformation functors with local conditions.** Let us define simplicial local deformation functors $\mathcal{F}_v^{s,?}$. For $v \in S \cup S_p$, let us write $\Gamma_v = G_{F_v}$ as inverse limit of finite Galois groups $\mathrm{Gal}(F_v'/F_v)$. Recall that for $? = min, \mathrm{ord}, \mathrm{ord} -\mu, FL$, under suitable assumptions we have seen (Lemma 1, Lemma 2, Lemma 3, that the ring $R_v^{?,\square}$ representing $\mathcal{F}_v^{?,\square}$ is formally smooth. The functor $\mathcal{F}_v^s$ of arbitrary simplicial deformations of $\overline{\rho}|_{\Gamma_v}$ is given by

$$\mathcal{F}_v^s(A) = \mathsf{sHom}_{\overline{\rho}_v}(\mathbf{B}\Gamma_v, \mathbf{B}G(A)) = \mathrm{holim}_{F_v'} \mathsf{sHom}_{\overline{\rho}_v}(\mathbf{B}\,\mathrm{Gal}(F_v'/F_v), \mathbf{B}G(A))$$

We denote by $\mathcal{F}_{v,Z}^s$ its modification by the center as in [GV18, Section 5.4] (or Section 2.4 above). Note that by [GV18, Lemma 5.2], we have $\mathfrak{t}^{-1}\mathcal{F}_{v,Z}^s = \mathrm{Coker}(\mathrm{H}^0(\{1\}, \mathfrak{z}) \to \mathrm{H}^0(\Gamma_v, \mathfrak{g}_k))$ which doesn't vanish (except in certain FL cases). Therefore, by the converse of Lurie's criterion (Theorem 9 above), $\mathcal{F}_{v,Z}^s$ is not pro-representable. However, by [GV18, Lemma 5.2] we have

Lemma 8. *For any $i \geq 0$, $\mathfrak{t}^i \mathcal{F}_{v,Z}^s = \mathfrak{t}^i \mathcal{F}_v^s \cong \mathrm{H}^{i+1}(\Gamma_v, \mathfrak{g}_k))$*

For $? = min, \mathrm{ord}, FL$, following [GV18, Section 9.2], we shall define local deformation functors $\mathcal{F}_v^{s,?}$ with morphisms

$$\mathcal{F}_v^{s,?} \to \mathcal{F}_v^s$$

which induce embeddings of the tangent complexes. They are defined as "derived quotients" of functors $\mathcal{F}_v^{s,?,\square}$ by the adjoint action of G. Recall that for a simplicial set X with action of a group G, the simplicial quotient $G\backslash X$ is defined as the simplicial set $N(*, G, X)$ given by the diagonal of the bisimplicial set $([p], [q]) \mapsto * \times G^p \times X_q$ where for q fixed, the p-faces and degeneracy maps are given by the bar construction $[p] \mapsto N_p(*, G, X_q) = * \times G^p \times X_q$ while the q-ones are given by the simplicial set X. We now form, so to speak, a formally cohesive replacement of the simplicial analogue of the quotient functor

$$A \mapsto \widehat{G}(A)\backslash \mathcal{F}_{v,*}^{\square,?}(A).$$

Namely, for each simplicial $\mathcal{O}$-algebra A in $_\mathcal{O}\mathrm{Art}_k^s$, we define the simplicial set $\mathcal{F}_v^?(A)$ as

$$[p] \mapsto \mathsf{Hom}_{\mathcal{O}\mathrm{Art}_k^s}(c(R_v^{?,\square} \otimes \mathcal{O}_{N_pG}), A^{\Delta[p]})$$

By the use of the cofibrant replacement of $R_v^{?,\square} \otimes \mathcal{O}_{N_pG}$, we know that $\mathcal{F}_v^?$ is formally cohesive (by [Cai21, Example 2.2.9]). The existence of the natural transformation of functors

$$\mathcal{F}_v^{s,?} \to \mathcal{F}_v^s$$

is established in great generality (which includes $? = min, \mathrm{ord}$) in [Cai21, Proposition 5.15].

Lemma 9. *We have a fibration*

$$\mathcal{F}_v^{s,\square,?} \to \mathcal{F}_v^{s,?} \to \mathsf{hofib}_*(\mathbf{B}G(-) \to BG(k)).$$

Proof. By definition (see [GV18, Definition 5.4 (i)]), $\mathcal{F}_v^{s,?,\square}(A)$ is the homotopy fiber of the composed morphism

$$\mathcal{F}_v^{s,?,\square}(A) \to \mathsf{sHom}_{[\overline{\rho}_v]}(B\Gamma_v, \mathbf{B}G(A)) \to \mathsf{hofib}(\mathbf{B}G(A) \to \mathbf{B}G(k))$$

where the second map is the evaluation on the base point $e = (e_n)_n \in B\Gamma_v$ (with $e_n = (e, \ldots, e) \in \Gamma_v^n$). By definition, $\mathcal{F}_v^{s,?}(A)$ is the diagonal of the bisimplicial set

$$[p] \mapsto \mathsf{sHom}_{\mathcal{O}\mathrm{Art}_k^s}(c(\mathcal{O}_{N_pG} \otimes R_v^{?,\square}), A) \simeq \prod_{i=1}^p \mathsf{sHom}_{\mathcal{O}\mathrm{Art}_k^s}(c(\mathcal{O}_G), A) \times \mathcal{F}_v^{s,\square,?}(A).$$

Similarly, $\mathsf{hofib}_*(\mathbf{B}G(A) \to BG(k))$ (homotopy fiber of the standard marked point in $BG(k)$) is the diagonal of the bisimplicial set

$$[p] \mapsto \mathsf{sHom}_{\mathcal{O}\mathrm{Art}_k^s}(c(\mathcal{O}_{N_pG}), A) \simeq \prod_{i=1}^p \mathsf{sHom}_{\mathcal{O}\mathrm{Art}_k^s}(c(\mathcal{O}_G), A).$$

Therefore, the morphism $s\mathcal{F}_v^{s,\square,?}(A) \to \mathsf{sHom}_{[\overline{\rho}_v]}(B\Gamma_v, \mathbf{B}G(A))$ factors through $s\mathcal{F}_v^{s,\square,?}(A) \to \mathcal{F}_v^{s,?}(A)$. This gives the desired fibration. $\qquad\square$

Remark 1. *Note that in the simplicial context, the classical fibration $\widehat{G}(A) \to \mathcal{F}_v^{\square,?}(A) \to \mathcal{F}_v^?(A)$ is replaced by*

$$\mathcal{F}_v^{s,\square,?} \to \mathcal{F}_v^{s,?} \to \mathsf{hofib}_*(\mathbf{B}G(-) \to BG(k))$$

where, for a classical ring A, $\mathsf{hofib}_*(\mathbf{B}G(A) \to BG(k))$ is to be thought of as $B\widehat{G}(A)[1]$.

Let us put

$$\mathrm{H}^1_{min}(\Gamma_v, \mathfrak{g}_k) := \mathrm{H}^1_{unr}(\Gamma_v, \mathfrak{g}_k), \quad \mathrm{H}^1_{\mathrm{ord}}(\Gamma_v, \mathfrak{g}_k) := \mathrm{Im}(L_v' \to \mathrm{H}^1(\Gamma_v, \mathfrak{g}_k)),$$

where $L_v' = \mathrm{Ker}(\mathrm{H}^1(\Gamma_v, \mathfrak{b}) \to \mathrm{H}^1(I_v, \mathfrak{b}/\mathfrak{n}))$, and

$$\mathrm{H}^1_{FL}(\Gamma_v, \mathfrak{g}_k) = \mathrm{H}^1_f(\Gamma_v, \mathfrak{g}_k).$$

By the long exact sequence above, we conclude:

Lemma 10. *For $? = min, \mathrm{ord}, \mathrm{ord} - \mu, FL$, we have*

(1) $\mathfrak{t}^{-1}\mathcal{F}_v^{s,?} \cong \mathrm{H}^0(\Gamma_v, \mathfrak{g}_k)$,

(2) $\mathfrak{t}^0\mathcal{F}_v^{s,?} \cong \mathrm{H}^1_?(\Gamma_v, \mathfrak{g}_k)$,

(3) $\mathfrak{t}^i\mathcal{F}_v^{s,?} = 0$ *for $i \geq 1$.*

Proof. Recall that for any deformation functor $\mathcal{F}$, $\mathfrak{t}^{n-i}\mathcal{F} \cong \pi_i\mathcal{F}(k \oplus k[n])$. By Lemma 4.30 (4) of [GV18], or [GJ10, Lemma 7.3], the fibration of Lemma 9 gives rise to a long exact sequence

$$0 \to \mathfrak{t}^{-1}s\mathcal{F}_v^{s,?} \to \mathfrak{t}^{-1}\mathsf{hofib}_*(\mathbf{B}G(-) \to BG(k))$$

$$\to \mathfrak{t}^0\mathcal{F}_v^{s,\square,?} \to \mathfrak{t}^0s\mathcal{F}_v^{s,?} \to 0 \to \mathfrak{t}^1\mathcal{F}^{s,\square,?} \to \mathfrak{t}^1s\mathcal{F}_v^{s,?} \to 0$$

(we have used that $\mathsf{hofib}_*(BG(k[\epsilon]) \to BG(k))$ is a $K(\pi, 1)$). Note that $\mathfrak{t}^{-1}\mathsf{hofib}_*(\mathbf{B}G(-) \to BG(k)) \cong \pi_1\mathsf{hofib}_*(BG(k[\epsilon]) \to BG(k))$, and $\pi_1\mathsf{hofib}_*(BG(k[\epsilon]) \to BG(k))$ is $\mathrm{Ker}(G(k[\epsilon]) \to G(k)) = \mathfrak{g}_k$. Therefore, the tangent spaces $\mathfrak{t}^i\mathcal{F}_v^{s,?}$ can be calculated as follows. Since $\mathcal{F}_v^{s,\square,?}$ is prorepresented by $c(R_v^{\square,?})$ which is weakly equivalent to the discrete ring $R_v^{\square,?}$, its tangent complex is given by

$$\mathfrak{t}\mathcal{F}_v^{s,\square,?} = \mathfrak{t}R_v^{\square,?}$$

Moreover, since $R_v^{\square,?}$ is formally smooth, we know that $\mathfrak{t}R_v^{\square,?}$ is quasi-isomorphic to the complex $\mathfrak{t}^0 R_v^{\square,?}[0]$ concentrated in degree 0. It is well-known that $\mathfrak{t}^0 R_v^{\square,?} = Z^1_?(\Gamma_v, \mathfrak{g}_k)$. Hence we can rewrite the long exact sequence above as

$$0 \to \mathfrak{t}^{-1}\mathcal{F}_v^{s,?} \to \mathfrak{g}_k \to Z^1_?(\Gamma_v, \mathfrak{g}_k) \to \mathfrak{t}^0\mathcal{F}_v^{s,?} \to 0 \to 0 \to \mathfrak{t}^1\mathcal{F}_v^{s,?} \to 0$$

This yields the Lemma.

$\qquad\square$

As in the global case, we define also the deformation functors $\mathcal{F}_{v,Z}^s$ and $\mathcal{F}_{v,Z}^{s,?}$ by modifying the center Z. They are inserted in a fibration

$$\mathsf{sHom}(*, \mathbf{B}Z) \to \mathcal{F}_v^{s,?} \to \mathcal{F}_{v,Z}^{s,?}$$

similar to (3.4).

For $v \in S_p$ and $? = \mathrm{ord}$, we can give a more concrete alternative definition of the local nearly ordinary deformation functor. We define $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}}(A)$ as the set of simplicial deformations $\phi \in \mathsf{sHom}(\mathbf{B}G_{F_v}, \mathbf{B}G(A))$ of $\overline{\rho}|_{G_{F_v}}$, which factor through the morphism of simplicial sets $\mathbf{B}B(A) \to \mathbf{B}G(A)$ associated to the inclusion of the standard Borel $B \subset G$. for $? = \mathrm{ord} - \mu$, let $\mathrm{Def}_{v,T}^s = \mathsf{sHom}_{\mathsf{sSETS}_{/BT(k)}}(BI_v, \mathbf{B}T(-))$ be the derived

deformation functor of $\bar{\chi}_v|_{I_v}\colon I_v \to T(k)$. Then there is a natural transformation $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}} \to \mathrm{Def}_{v,T}^s$. Note that μ_v defines a natural transformation $* \to \mathrm{Def}_{v,T}^s$. We define

$$\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu} = \mathsf{hofib}_{\mu_v}(\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}} \to \mathrm{Def}_{v,T}^s).$$

In addition, the natural transformation $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu} \to \mathcal{F}_v^s$ induces $\widetilde{\mathcal{F}}_{v,Z}^{s,\mathrm{ord},\mu} \to \mathcal{F}_{v,Z}^s$.

Lemma 11. *The functors $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}}$ and $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu}$ are formally cohesive. Assuming $(STDIST)$, the tangent complex $\mathfrak{t}\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}} \cong \mathfrak{t}\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu}$, is quasi-isomorphic to $\mathfrak{t}\mathcal{F}_v^{s,\mathrm{ord}} \cong \mathfrak{t}\mathcal{F}_v^{s,\mathrm{ord},\mu}$:*

- $\mathfrak{t}^{-1}\widetilde{\mathcal{F}}_{v,*}^{s,\mathrm{ord}} \cong \mathrm{H}^0(\Gamma_v, \mathfrak{g}_k)$,
- $\mathfrak{t}^0\widetilde{\mathcal{F}}_{v,*}^{s,\mathrm{ord}} \cong \mathrm{H}_?^1(\Gamma_v, \mathfrak{g}_k')$,
- $\mathfrak{t}^i\widetilde{\mathcal{F}}_{v,*}^{s,\mathrm{ord}} = 0$ *for* $i \geq 1$.

Moreover, we have

$$\pi_0\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}} = \mathcal{F}_v^{\mathrm{ord}}.$$

Proof. Note that $\mathrm{Def}_{v,T}^s$ is formally cohesive. We see that $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}}$ is formally cohesive: the functor

$$A \mapsto \mathsf{sHom}_{[\overline{\rho}_v]}(\mathbf{B}G_{F_v}, \mathbf{B}B(A))$$

is homotopy invariant and preserves homotopy pullbacks. From this, it follows by definition that $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu}$ is also formally cohesive.

The tangent complex $\mathfrak{t}^i\widetilde{\mathcal{F}}_v^{\mathrm{ord}}$ is calculated in [GV18, Lemma 5.6] (replacing G by B). More precisely, let $\mathfrak{b} = Lie(B)$. Then we have a quasi-isomorphism of complexes

$$\mathfrak{t}\widetilde{\mathcal{F}}_v^{\mathrm{ord}} \sim C^{*+1}(B\Gamma_v, \mathfrak{b})$$

The assumption (reg^*) implies $\mathrm{H}^0(\Gamma_v, \mathfrak{g}_k/\mathfrak{b}_k) = 0$. Thus, $\mathfrak{t}\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}}$ is quasi-isomorphic to $\mathfrak{t}\mathcal{F}_v^{s,\mathrm{ord}}$. The same holds for $\mathfrak{t}\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu}$. If $A \in Ob(_{\mathcal{O}}\mathrm{Art}_k^s)$ is a discrete artinian ring, $\mathcal{F}_v^{s,\mathrm{ord}}(A)$ is the set of B-conjugacy classes of liftings of $\overline{\rho}_v$; because of (reg), the set of B-conjugacy classes coincide with the set of G-conjugacy classes of representations $\rho_v\colon G_{F_v} \to G(A)$, such that after conjugation by some $g_v \in G(A)$, ρ_v takes values in $B(A)$ and such that the composition of $^{g_v}\rho_v$ and the reduction $B(A) \to T(A) = B(A)/U_B(A)$ is a lifting of $\overline{\chi}_v\colon G_{F_v} \to T(k)$. (see [Ti96, Claim in Proof of Proposition 6.2]). Hence we conclude $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord}}(A) = \mathcal{F}_v^{\mathrm{ord}}(A)$. Similarly for $\widetilde{\mathcal{F}}_v^{s,\mathrm{ord},\mu}(A) = \mathcal{F}_v^{\mathrm{ord},\mu}(A)$. $\square$

We define $\mathcal{F}_{loc}^s$, resp. $\mathcal{F}_{loc}^{s,min,?}$ (and their center modified counterparts) as

$$\mathcal{F}_{loc}^s = \prod_{v \in S \cup S_p} \mathcal{F}_v^s, \quad \mathcal{F}_{loc,Z}^s = \prod_{v \in S \cup S_p} \mathcal{F}_{v,Z}^s$$

resp.

$$\mathcal{F}_{loc}^{s,min,?} = \prod_{v \in S} \mathcal{F}_{v,Z}^{s,min} \times \prod_{v \in S_p} \mathcal{F}_v^{s,?}, \quad \mathcal{F}_{loc,Z}^{s,min,?} = \prod_{v \in S} \mathcal{F}_{v,Z}^{s,min} \times \prod_{v \in S_p} \mathcal{F}_{v,Z}^{s,?}$$

as in [GV18, Definition 9.1]. By definition, the morphisms

$$\mathcal{F}_{\overline{\rho}}^s \to \mathcal{F}_v^s$$

induce morphisms

$$\mathcal{F}_{\overline{\rho},Z}^s \to \mathcal{F}_{v,Z}^s$$

hence a morphism

$$\mathcal{F}_{Z,\overline{\rho}}^s \to \mathcal{F}_{loc,Z}^s$$

We finally put

$$\mathcal{F}^{s,?} = \mathcal{F}_{\overline{\rho}}^{s,gl,?} = \mathcal{F}_{\overline{\rho}}^s \times_{\mathcal{F}_{loc}^s}^h \mathcal{F}_{loc}^{s,min,?}, \quad \mathcal{F}_Z^{s,?} = \mathcal{F}_{Z,\overline{\rho}}^{s,gl,?} = \mathcal{F}_{Z,\overline{\rho}}^s \times_{\mathcal{F}_{loc,Z}^s}^h \mathcal{F}_{loc}^{s,min,?}$$

By Lemma 3 and 4 we have $\pi_0 \mathcal{F}_Z^{s,?} = \mathcal{F}_Z^?$. For $? = \mathrm{ord}, FL$, we put

$$\mathrm{H}^1_{min,?}(\Gamma, \mathrm{Ad}\,\bar{\rho}) = \mathrm{Ker}(\mathrm{H}^1(\Gamma, \mathrm{Ad}\,\bar{\rho}) \to \bigoplus_{v \in S} \frac{\mathrm{H}^1(\Gamma_v, \mathrm{Ad}\,\bar{\rho})}{\mathrm{H}^1_{unr}(\Gamma_v, \mathrm{Ad}\,\bar{\rho})} \oplus \bigoplus_{v \in S_p} \frac{\mathrm{H}^1(\Gamma_v, \mathrm{Ad}\,\bar{\rho})}{L_v^?}$$

where $\mathrm{Ad}\,\bar{\rho} = \mathfrak{g}_k$ with the adjoint Galois action and if $? = \mathrm{ord}$, $L_v^? = \mathrm{Im}(L_v' \to \mathrm{H}^1(\Gamma_v, \mathfrak{g}_k))$ for $L_v' = \mathrm{Ker}(\mathrm{H}^1(\Gamma_v, \mathfrak{b}_k)) \to \mathrm{H}^1(I_v, \mathfrak{b}_k/\mathfrak{n}_k))$, and if $? = FL$, $L_v^? = \mathrm{H}^1_f(\Gamma_v, \mathfrak{g}_k)$ (that is, the image of $\mathrm{H}^1_f(\Gamma_v, \mathfrak{g}_{\mathcal{O}}) \to \mathrm{H}^1(\Gamma_v, \mathfrak{g}_k)$ where $\mathrm{H}^1_f(\Gamma_v, \mathfrak{g}_{\mathcal{O}})$ is defined by Bloch and Kato, [BK90, Definition (3.7.3)].

For $v \in S$, let $\widetilde{L}_v^{min}$ be the inverse image of $\mathrm{H}^1_{unr}(\Gamma_v, \mathfrak{g}_k)$ by $Z^1(\Gamma_v, \mathfrak{g}_k) \to \mathrm{H}^1(\Gamma_v, \mathfrak{g}_k)$, and for $v \in S_p$, let $\widetilde{L}_v^?$ be the inverse image of $L_v^?$ by $Z^1(\Gamma_v, \mathfrak{g}_k) \to \mathrm{H}^1(\Gamma_v, \mathfrak{g}_k)$.

Definition 5. *Given a morphism of cochain complexes $f\colon A^\bullet \to B^\bullet$, its mapping cone $MC(f)^\bullet$ is defined by $MC(f)^n = B^n \oplus A^{n+1}$ with $d(b,a) = (db + fa, -da)$. This give rise to distinguished triangles*

$$A^\bullet \to B^\bullet \to MC(f)^\bullet \to A^\bullet[1]$$

and

$$MC(f)^\bullet[-1] \to A^\bullet \to B^\bullet \to MC(f)^\bullet$$

We define the complex

$$C^\bullet_{min,?}(\Gamma, \mathfrak{g}_k) = MC(f)[-1]$$

for the morphism

$$f\colon C^\bullet(\Gamma, \mathfrak{g}_k) \to \bigoplus_{v \in S} C^\bullet(\Gamma_v, \mathfrak{g}_k))/\widetilde{L}_v^{\bullet,min} \oplus \bigoplus_{v \in S_p} C^\bullet_?(\Gamma_v, \mathfrak{g}_k))/\widetilde{L}_v^{\bullet,?}$$

where for $? = min, \mathrm{ord}, FL$, $\widetilde{L}_v^{0,?} = C^0(\Gamma_v, \mathfrak{g}_k)$, $\widetilde{L}_v^{1,?} = \widetilde{L}_v^?$ and $\widetilde{L}_v^{2,min} = 0$.

We write its cohomology as $\mathrm{H}^*_{min,?}(\Gamma, \mathfrak{g}_k) = \mathrm{H}^*(C^\bullet_{min,?}(\Gamma, \mathfrak{g}_k))$. In accordance with Wiles' notations, we also write sometimes this cohomology as $\mathrm{H}^*_{\mathcal{L}}(\Gamma, \mathfrak{g}_k)$ where $\mathcal{L} = (L_v)_{v \in S \cup S_p}$. We have the following exact sequence:

$$0 \to \mathrm{H}^1_{min,?}(\Gamma, \mathrm{Ad}\,\bar{\rho}) \to \mathrm{H}^1(\Gamma, \mathrm{Ad}\,\bar{\rho}) \to \bigoplus_{v \in S} \tfrac{\mathrm{H}^1(\Gamma_v, \mathrm{Ad}\,\bar{\rho})}{\mathrm{H}^1_{unr}(\Gamma_v, \mathrm{Ad}\,\bar{\rho})} \oplus \bigoplus_{v \in S_p} \tfrac{\mathrm{H}^1(\Gamma_v, \mathrm{Ad}\,\bar{\rho})}{L_v^?}$$
$$\to \mathrm{H}^2_{min,?}(\Gamma, \mathrm{Ad}\,\bar{\rho}) \to \mathrm{H}^2(\Gamma, \mathrm{Ad}\,\bar{\rho}) \to \bigoplus_{v \in S \cup S_p} \mathrm{H}^2(\Gamma_v, \mathrm{Ad}\,\bar{\rho}) \ldots$$

Recall that we are interested in the deformation functor $\mathcal{F}_Z^{s,?} = \mathcal{F}_{Z,\bar{\rho}}^{s,gl,?}$.

Theorem 10. *We have $\mathfrak{t}^{-1}\mathcal{F}_Z^{s,?} = 0$ and for any $i \geq 0$, the i-th cohomology of the tangent complex $\mathfrak{t}\mathcal{F}_Z^{s,?}$ is naturally identified with the cohomology with local conditions:*

$$\mathfrak{t}^i \mathcal{F}^{s,?} \cong \mathrm{H}^{i+1}_{min,?}(\Gamma, \mathrm{Ad}\,\bar{\rho}).$$

Proof. Since $\mathcal{F}_Z^{s,?} = \mathcal{F}_{Z,\bar{\rho}}^s \times^h_{\mathcal{F}^s_{loc,Z}} \mathcal{F}_{loc,Z}^{min,?,s}$, we can apply [GV18, Lemma 4.30 (iv)] and we get a long exact sequence of finite dimensional k-vector spaces

$$0 \to \mathfrak{t}^{-1}\mathcal{F}_Z^{s,?} \to \mathfrak{t}^{-1}\mathcal{F}_{Z,\bar{\rho}}^s \oplus \mathfrak{t}^{-1}\mathcal{F}_{loc,Z}^{min,?,s} \to \mathfrak{t}^{-1}\mathcal{F}_{loc,Z}^s \to$$

$$\to \mathfrak{t}^0\mathcal{F}_Z^{s,?} \to \mathfrak{t}^0\mathcal{F}_{Z,\bar{\rho}}^s \oplus \mathfrak{t}^0\mathcal{F}_{loc,Z}^{min,?,s} \to \mathfrak{t}^0\mathcal{F}_{loc,Z}^s \to$$

$$\to \mathfrak{t}^1\mathcal{F}_Z^{s,?} \to \mathfrak{t}^1\mathcal{F}_{Z,\bar{\rho}}^s \oplus \mathfrak{t}^1\mathcal{F}_{loc,Z}^{min,?,s} \to \mathfrak{t}^1\mathcal{F}_{loc,Z}^s \to \mathfrak{t}^1\mathcal{F}^{s,?} \ldots$$

We know by Proposition 2 that $\mathfrak{t}^{-1}\mathcal{F}_{Z,\bar{\rho}}^s = 0$ and $\mathfrak{t}^i\mathcal{F}_{Z,\bar{\rho}}^s \cong \mathrm{H}^{i+1}(\Gamma, \mathfrak{g}_k)$ for $i \geq 0$. By Lemma 8, we have $\mathfrak{t}^i\mathcal{F}_v^s \cong \mathrm{H}^{i+1}(\Gamma_v, \mathfrak{g}_k)$ for all $i \geq -1$ and $\mathfrak{t}^{-1}\mathcal{F}_{v,Z}^s \cong \mathrm{Coker}(\mathfrak{z}_k \to \mathrm{H}^0(\Gamma_v, \mathfrak{g}_k))$. By Lemma 10, we have for

$v \in S \cup S_p$, $\mathfrak{t}^{-1}\mathcal{F}_v^{s,?} \cong \mathrm{H}^0(\Gamma_v, \mathfrak{g}_k)$, $\mathfrak{t}^{-1}\mathcal{F}_{v,Z}^{s,?} \cong \mathrm{Coker}(\mathfrak{z} \to \mathrm{H}^0(\Gamma_v, \mathfrak{g}_k))$, $\mathfrak{t}^0\mathcal{F}_v^{s,?} \cong \mathrm{H}_?^1(\Gamma_v, \mathfrak{g}_k)$, $\mathfrak{t}^i\mathcal{F}_v^{s,?} = 0$ for $i \geq 1$.
We have therefore $\mathfrak{t}^{-1}\mathcal{F}^{s,?} = 0$ and

$$0 \to \mathfrak{t}^0\mathcal{F}_Z^{s,?} \to \mathrm{H}^1(\Gamma, \mathfrak{g}_k) \oplus \bigoplus_{v \in S \cup S_p} \mathrm{H}_?^1(\Gamma_v, \mathfrak{g}_k) \to \bigoplus_{v \in S \cup S_p} \mathrm{H}^1(\Gamma_v, \mathfrak{g}_k) \to$$

$$\to \mathfrak{t}^1\mathcal{F}_Z^{s,?} \to \mathrm{H}^2(\Gamma, \mathfrak{g}_k) \to \bigoplus_{v \in S \cup S_p} \mathrm{H}^2(\Gamma_v, \mathfrak{g}_k).$$

By comparing to the long exact sequence of cohomology, we conclude that for any $i \geq 0$,

$$\mathfrak{t}^i\mathcal{F}_Z^{s,?} \cong \mathrm{H}_{min,?}^{i+1}(\Gamma, \mathrm{Ad}\,\bar{\rho}).$$

$\square$

Proposition 3. *Under the assumption $(STDIST)$, the global deformation problem $\mathcal{F}_Z^{s,\mathrm{ord}}$ is pro-representable. Under the assumption (FL), the deformation problem $\mathcal{F}_Z^{s,FL}$ is pro-representable.*

Note that $\mathcal{F}_{Z,\bar{\rho}}^s$ is pro-representable, but not $\mathcal{F}_{loc,Z}^{min,?,s}$.

Proof. We apply Lurie's criterion Theorem 9. We know that $\mathcal{F}_{Z,\bar{\rho}}^s$, $\mathcal{F}_{loc,Z}^s$ and $\mathcal{F}_{loc,Z}^{min,?,s}$ are formally cohesive, hence so is their homotopy fiber product $\mathcal{F}_Z^{s,?}$. By Theorem 10, $\mathfrak{t}^{-1}\mathcal{F}_Z^{s,?} = 0$ and for any $i \geq 0$, $\mathfrak{t}^i\mathcal{F}_Z^{s,?}$ is finite dimensional over k (and vanishes for $i > 1$). $\square$

3.6. **Theorems of Galatius-Venkatesh and Cai.** In this section we state generalizations of the main theorems of Galatius and Venkatesh [GV18] by Y. Cai [Cai21]. We recall the notations and assumptions. Let π be a cohomological cuspidal representation on $\mathsf{GL}_N(F)$ (F a CM field), with squarefree conductor $\mathfrak{n}$, and level group $U = U_0(\mathfrak{n})$. Let S be the set of places dividing $\mathfrak{n}$. It occurs in $\mathrm{H}^\bullet(X_U, V_\lambda(\mathbb{C}))$ for a weight $\lambda = (\lambda_{\tau,i})_{\tau \in I_F, 1 \leq i \leq N} \in (X^*(\mathbf{T}_K)^+)^{\mathsf{Hom}(F,\mathbb{C})}$, *i.e.* $\lambda_{\tau,1} \geq \ldots \geq \lambda_{\tau,N}$ for any $\tau \in I_F$. Let $p > 2$ be a rational prime relatively prime to $\mathfrak{n}$, unramified in F; let S_p be the set of places of F above p. We assume either

$$(FL) \quad \lambda_{\tau,1} - \lambda_{\tau,N} < p - N, \forall \tau \in I_F,$$

or

$$(\mathrm{ord}) \quad \pi_v \text{ is ordinary for all } v \in S_p.$$

Let $\Gamma = \mathrm{Gal}(F_{S \cup S_p}/F)$. Let K_0 be a sufficiently big number field containing the Hecke eigenvalues of π. Fix a p-adic place v_0 and let K be its p-adic completion, $\mathcal{O}$ its valuation ring, ϖ a uniformizing parameter; k its residue field. Let $\rho_\pi\colon \Gamma \to \mathsf{GL}_N(\mathcal{O})$ be the Galois representation associated to π and $\bar{\rho}\colon \Gamma \to \mathsf{GL}_N(k)$ its reduction modulo ϖ. Assume

(MIN) for any places $v \in S$, the image $\overline{\rho}_\pi(I_v)$ of the inertia subgroup I_v contains a regular unipotent element.

Recall that for any place $v \in S_p$, it is known that (FL), resp. (ord), implies that $\rho_\pi|_{\Gamma_v}$ and $\bar{\rho}|_{\Gamma_v}$ are both Fontaine-Laffaille, resp. ordinary. Assume that the image of $\bar{\rho}$ is enormous:

(RLI) $\bar{\rho}(\mathrm{Gal}_{F(\zeta_p)}) \supset k^\times \mathsf{SL}_N(k')$ for some subfield k' of the residue field k.

Recall that a Taylor-Wiles prime v is a place $v \notin S \cup S_p$ of residual characteristic q_v such that $q_v \equiv 1 \pmod{p}$ and $\bar{\rho}(\mathrm{Frob}_v)$ is conjugated to a strongly regular element of $T(k)$ (i.e. an element $t \in T(k)$ whose centralizer in $G(k)$ coincides with $T(k)$). In particular, for a Taylor-Wiles prime v, we may choose a representation $\bar{\rho}_v\colon \Gamma_v \to T(k)$ such that the composition with $T \hookrightarrow G$ is conjugate to $\bar{\rho}|_{\Gamma_v}$. Note that by definition, $\bar{\rho}_v$ factors through the Galois group Γ_v^{unr} of the maximal unramified extension F_v^{unr}/F_v. Let $\mathcal{F}_v^s$, resp. $\mathcal{F}_v^{\mathrm{unr},s}$ be the simplicial deformation functor of $\bar{\rho}|_{\Gamma_v}$. As in [GV18, Section 8.1], in order to study them, one needs to compare them to the (simpler) simplicial deformation functors of $\bar{\rho}_v$ with target in T. Let $\mathcal{F}_v^{s,T}$, resp. $\mathcal{F}_v^{s,T,\mathrm{unr}}$,

be the T-valued derived deformation functor, resp. its unramified analogue, of $\bar{\rho}_v$. Since T is commutative, its center is T itself, so their center-modified analogues make use of T; they are denoted by $\mathcal{F}_{v,T}^{s,T}$, resp. $\mathcal{F}_{v,T}^{s,T,\mathrm{unr}}$. They are pro-representable. Let Q be a finite set of Taylor-Wiles primes; we consider the functors

$$\mathcal{F}_{Q,loc}^s = \prod_{v \in S \cup S_p \cup Q} \mathcal{F}_v^s,$$

$$\mathcal{F}_{Q,loc}^{s,min,?} = \prod_{v \in S} \mathcal{F}_v^{s,min} \times \prod_{v \in S_p} \mathcal{F}_v^{s,?} \times \prod_{v \in Q} \mathcal{F}_v^s,$$

let $\Gamma_Q = \mathrm{Gal}(F_{S \cup S_p \cup Q}/F)$ and $\mathcal{F}_{Q,\bar{\rho}}^s$ be the global derived deformation problem $A \mapsto \mathsf{Hom}_{[\bar{\rho}]}(B\Gamma_Q, \mathbf{B}G(A))$; we then define

$$\mathcal{F}_Q^{s,min,?} = \mathcal{F}_{Q,\bar{\rho}}^{s,gl/loc,?} = \mathcal{F}_{Q,\bar{\rho}}^s \times_{\mathcal{F}_{Q,loc}^s}^h \mathcal{F}_{Q,loc}^{s,min,?}$$

and their center-modified analogues $\mathcal{F}_{Z,Q}^{s,min,?}$ and $\mathcal{F}_{Z,Q}^{s,min,?}$, involving products of center-modified global and local factors. We also introduce $\mathcal{F}_{Q,T}^{s,T} = \prod_{v \in Q} \mathcal{F}_{v,T}^{s,T}$ and $\mathcal{F}_{Q,T}^{s,T,\mathrm{unr}} = \prod_{v \in Q} \mathcal{F}_{v,T}^{s,T,\mathrm{unr}}$.

Definition 6. *An allowable Taylor-Wiles datum of level n is a set of Taylor-Wiles primes $Q = \{v_1, \ldots, v_r\}$, together with a strongly regular element $t_{v_i} \in T(k)$ conjugate to $\bar{\rho}(\mathrm{Frob}_{v_i})$ for each $i \in \{1, \ldots, r\}$, such that:*

(1) *Each $q_{v_i} \equiv 1 \pmod{p^n}$, $i \in \{1, \ldots, r\}$.*
(2) *We have $\mathrm{H}^2_{\mathcal{L}_Q}(\Gamma_Q, \mathrm{Ad}\,\bar{\rho}) = 0$, where $\mathcal{L}_Q = (L_{Q,v})_v$ with*

$$L_{Q,v} = \left\{ \begin{array}{ll} L_v, & v \in S \cup S_p; \\ \mathrm{H}^1(F_v, \mathrm{Ad}\,\bar{\rho}), & v \in Q. \end{array} \right.$$

As noted in [GV18, Remark after Definition 6.2], we have

Remark 2. *Under the assumption that $k^\times \cdot \mathsf{SL}_N(k') \subset \mathrm{Im}\,\bar{\rho}$, for any level $n \geq 1$, there exist infinitely many allowable Taylor-Wiles data $(Q, (t_v)_{v \in Q})$ of degree n.*

Let $\Gamma_Q = \mathrm{Gal}(F_{S \cup S_p \cup Q}/F)$ and for each $v \in Q$, For an allowable Taylor-Wiles datum Q, we have the long exact sequence:

$$0 \to \mathrm{H}^1_{\mathcal{L}_Q}(\Gamma_Q, \mathrm{Ad}\,\bar{\rho}) \to \mathrm{H}^1(\Gamma_Q, \mathrm{Ad}\,\bar{\rho}) \xrightarrow{A} \bigoplus_{l \in S} \mathrm{H}^1(\Gamma_v, \mathrm{Ad}\,\bar{\rho})/L_v$$

$$\to 0 \to \mathrm{H}^2(\Gamma_Q, \mathrm{Ad}\,\bar{\rho}) \xrightarrow{B} \bigoplus_{v \in S \cup Q} \mathrm{H}^2(\Gamma_v, \mathrm{Ad}\,\bar{\rho}).$$

Moreover, the cokernel of B is dual to $\mathrm{H}^0(\mathbb{Z}[\frac{1}{S(Q)}], (\mathrm{Ad}\,\bar{\rho})^*)$ by global duality. We want to approximate $\mathcal{R}$ by larger rings allowing ramification at Taylor-Wiles primes. Let $(Q_n, (t_v)_{v \in Q_n})$ be an allowable Taylor-Wiles datum of level n. Let $\mathcal{F}_n^s = \mathcal{F}_{Q_n, Z}^{s,?}$ and let $\mathcal{R}_n$ be a representing ring for $\mathcal{F}_n^s$. Let $\mathcal{F}_n^{s,\mathrm{loc}} := \mathcal{F}_{Q_n,T}^{s,T}$ with representing ring $\mathcal{S}_n$. Let $\mathcal{F}_n^{s,\mathrm{loc},\mathrm{ur}} := \mathcal{F}_{Q_n,T}^{s,T,\mathrm{unr}}$ with representing ring $\mathcal{S}_n^{\mathrm{ur}}$. As noticed in [Cai21,], $\mathcal{F}_n^{s,\mathrm{loc}}$ is objectwise weakly equivalent to the derived framed deformation functor $\prod_{v \in Q_m} \mathcal{F}_v^{s,T,\square}$, and similarly for $\mathcal{F}_{Q_n,T}^{s,T,\mathrm{unr}}$ and $\prod_{v \in Q_n} \mathcal{F}_v^{s,T,\square}$. Therefore, the simplicial deformation rings $\mathcal{S}_n$ and $\mathcal{S}_n^{\mathrm{unr}}$ can be described explicitly as in [GV18, Remark 8.7].

Lemma 12. *We have a homotopy pullback square*

$$
\begin{array}{ccc}
\mathcal{F}_Z^{s,?} & \longrightarrow & \mathcal{F}_n^{s,\mathrm{loc},\mathrm{ur}} \\
\downarrow & & \downarrow \\
\mathcal{F}_n^s & \longrightarrow & \mathcal{F}_n^{s,\mathrm{loc}}.
\end{array}
$$

We have an objectwise weak equivalence

$$\mathcal{F}_n^{s,?} \times_{\mathcal{F}_n^{s,\mathrm{loc}}}^h \mathcal{F}_n^{s,\mathrm{loc,ur}} \sim \mathcal{F}_Z^{s,?}.$$

Proof. This is [Cai21, Lemma 6.6 and Corollary 6.7]. $\qquad\square$

Now we pass to the level of rings. Recall the derived tensor product $\underline{\otimes}$ of [GV18, Definition 3.3], we remark that $\mathcal{R}_1 \underline{\otimes}_{\mathcal{R}_3} \mathcal{R}_2$ represents the homotopy pullback of represented functors after cofibrant replacements, and since all pro-rings for us are indexed by natural numbers, the derived tensor product can be explicitly constructed levelwise (see [GV18, Definition 3.3] for details). The above corollary gives a weak equivalence

$$\mathcal{R} \sim \mathcal{R}_n \underline{\otimes}_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{unr}}.$$

We say a pro-object $\mathcal{R}$ of Art^s is homotopy discrete if the map $\mathcal{R} \to \pi_0 \mathcal{R}$ induces an equivalence on represented functors after applying level-wise cofibrant replacement (see [GV18, Definition 7.4]). Let $F_{Q_n}^\times = \prod_{v \in Q_n} F_v^\times$, $\mathcal{O}_{Q_n}^\times = \prod_{v \in Q_n} \mathcal{O}_v^\times$. By choosing uniformizing parameters at v's in Q_n, we have an isomorphism

$$F_{Q_n}^{(p)}/\mathcal{O}_{Q_n}^{(p)} \cong \mathbb{Z}^{Q_n}.$$

hence a decomposition

$$F_{Q_n}^\times = \mathcal{O}_{Q_n}^\times \times \mathbb{Z}^{Q_n}.$$

Let $\Delta_{Q_n}' = \mathcal{O}_{Q_n}^{(p)}$ be the pro-p completion of $\mathcal{O}_{Q_n}^\times$. It is a finite group isomorphic to the p-Sylow of $\prod_{v \in Q} k_v^\times$. Let $\Delta_{Q_n} = \Delta_{Q_n}'/(p^n)$ and

$$F_{Q_n}^{(p)} = \Delta_{Q_n} \times \mathbb{Z}^{Q_n}.$$

By definition of the level, this group is free of rank r over $\mathbb{Z}/p^n\mathbb{Z}$. Let $S_n = \mathcal{O}[\Delta_{Q_n}]$; it is a complete intersection ring. Actually let $S_\infty = \mathcal{O}[[Y_1, \ldots, Y_{Nr}]]$. The ideal $J_n = ((1 + Y_1)^n - 1, \ldots, (1 + Y_{Nr})^n - 1)$, and an isomorphism

$$i_n \colon S_\infty/J_n \cong S_n$$

Let

$$\Sigma_n = \mathcal{O}[[X^*(T) \otimes F_{Q_n}^{(p)}]] = S_n[[F_{Q_n}^{(p)}/\mathcal{O}_{Q_n}^{(p)}]] \overset{i}{\cong} S_n[[X_1, \ldots, X_{Nr}]],$$
$$\Sigma_\infty = S_\infty[[X_1, \ldots, X_{Nr}]]$$

The isomorphism i_n induces an isomorphism $\Sigma_\infty/J_n\Sigma_\infty \cong \Sigma_n$. Let also

$$\Sigma_n^{\mathrm{unr}} = \mathcal{O}[[X^*(T) \otimes F_{Q_n}^{(p)}/\mathcal{O}_{Q_n}^{(p)}]] \overset{i_{\mathrm{unr}}}{\cong} \mathcal{O}[[X_1, \ldots, X_{Nr}]].$$

Note that the isomorphisms i and i_{unr} are compatible to the augmentation homomorphism $S_n = \mathcal{O}[\Delta_{Q_n}] \to \mathcal{O}$.

Lemma 13. *The simplicial rings $\mathcal{S}_n$ and $\mathcal{S}_n^{\mathrm{unr}}$ are homotopy discrete. Actually $\mathcal{S}_n$, resp. $\mathcal{S}_n^{\mathrm{unr}}$, is a cofibrant replacement of the complete intersection classical ring Σ_n, resp. the formally smooth classical ring Σ_n^{unr}.*

Proof. See [GV18,] and [Cai21,]. $\qquad\square$

Since we can choose $\mathcal{S}_n$ and $\mathcal{S}_n^{\mathrm{unr}}$ up to weak equivalence, we suppose in the following that $\mathcal{S}_n = c(\Sigma_n)$ and $\mathcal{S}_n^{\mathrm{unr}} = c(\Sigma_n^{\mathrm{unr}})$. Let $\bar{S}_m = S_\infty/(p^m, J_m)$ and $\bar{\Sigma}_m = \Sigma_\infty/(p^m, J_m\Sigma_\infty)$. Note that $\bar{S}_m$ is finite and if M_m is a Σ_m-module, $\bar{M}_m = M_m \otimes_{\Sigma_m} \bar{\Sigma}_m = M_m \otimes_{S_m} \bar{S}_m$ and $\bar{M}_m \otimes_{\bar{\Sigma}_m} \Sigma_m^{\mathrm{unr}} = \bar{M}_m \otimes_{\bar{S}_m} \mathcal{O}/(p^m)$. This can be rewritten in terms of simplicial rings as follows. By quotienting by the ideal (p^m, J_m) the morphism $S_\infty \to \mathcal{S}_m$, we get the commutative diagram

$$\begin{array}{ccc}
\bar{S}_m & \longrightarrow & \mathcal{S}_m/(p^m, J_m) \\
\downarrow & & \downarrow \\
\mathcal{O}/p^m & \longrightarrow & \mathcal{S}_m^{\mathrm{unr}}/p^m
\end{array}$$

which induces a homotopy pullback square of represented functors. In consequence, for any $\mathcal{S}_m/(p^m, J_m)$-module $\mathcal{M}_m$, we have a weak equivalence $\mathcal{M}_m \underline{\otimes}_{\bar{S}_m} \mathcal{O}/p^m \sim \mathcal{M}_m \underline{\otimes}_{\mathcal{S}_m/(p^m, J_m)} \mathcal{S}_m^{\mathrm{unr}}/p^m$.

Let R_m, resp. $\bar{R}_m$, be the quotient of the classical universal deformation ring which represents the $(min, ?)$-deformations with Q_m-ramification, resp. the $(min, ?)$-deformations modulo p^m whose Q_m-ramification has of inertial level $\leq m$ (*i.e.* such that the restriction to inertia factors modulo p^m for all $v \in Q_m$). By definition, the ring $\bar{R}_m$ is a natural quotient of $R_m/(p^m, J_m)$ and is an $\mathcal{S}_m/(p^m, J_m)$-algebra. The map from $\mathcal{R}_m \leftarrow \mathcal{S}_m \to \mathcal{S}_m^{\mathrm{unr}}$ to $\bar{R}_m \leftarrow \mathcal{S}_m/(p^m, J_m) \to \mathcal{S}_m^{\mathrm{unr}}/p^m$ induces

$$(1) \qquad \mathcal{R} \sim \mathcal{R}_m \underline{\otimes}_{\mathcal{S}_m} \mathcal{S}_m^{\mathrm{unr}} \to \bar{R}_m \underline{\otimes}_{\mathcal{S}_m/(p^m, J_m)} \mathcal{S}_m^{\mathrm{unr}}/p^m \sim \bar{R}_m \underline{\otimes}_{\bar{S}_m} \mathcal{O}/p^m.$$

A Taylor-Wiles patching argument provides projective systems out of the $\bar{R}_m \underline{\otimes}_{\bar{S}_m} \mathcal{O}/p^m$'s. Let R_∞ be the projective limit for one of these projective systems (it depends on the choice of such a projective system). The main result of [GV18] (Theorem 14.1) compares $\pi_* \mathcal{R}$ to $\mathrm{Tor}_*^{S_\infty}(R_\infty, \mathcal{O})$ under certain conditions.

For $n \geq m$, there is a natural $\bar{R}_n \underline{\otimes}_{\bar{S}_n} \mathcal{O}/p^n \to \bar{R}_m \underline{\otimes}_{\bar{S}_m} \mathcal{O}/p^m$. Let $f_{n,m}$ be the composition

$$\mathcal{R} \to \bar{R}_n \underline{\otimes}_{\bar{S}_n} \mathcal{O}/p^n \to \bar{R}_m \underline{\otimes}_{\bar{S}_m} \mathcal{O}/p^m.$$

In order to apply [GV18, Theorem 12.1], recall the key result [Cai21, Proposition 6.11] generalizing [GV18, Theorem 11.1]:

Proposition 4. *Under the assumptions (RLI), (MIN), (FL) or (ORD_π), let $n \geq m$, the map $f_{n,m}$ induces an isomorphism on $\mathfrak{t}^0$ and a surjection on $\mathfrak{t}^1$.*

Based on the fact that the homology of $\bar{R}_m \underline{\otimes}_{\bar{S}_m} \mathcal{O}/p^m$ is $\mathrm{Tor}_\bullet^{\bar{S}_m}(\bar{R}_m, \mathcal{O}/p^m)$, this Proposition yields the following

Theorem 11. *We have an isomorphism of graded commutative rings $\pi_\bullet \mathcal{R} \cong \mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})$.*

Proof. Recall that $\pi_\bullet(\mathcal{R})$ is a $\pi_0(\mathcal{R})$-algebra (see [Cai21, Lemma 3.45]) and that $\pi_0(\mathcal{R}) = R_0$. The theorem is proven in [Cai21, Proposition 6.13] by applying Proposition 4 to [GV18, Theorem 12.1]. Note that Theorem 12.1 requires a numerical coincidence (12.1), which is satisfied when the (framed) local deformation rings are formally smooth (here it is the case by Section 2.1). $\square$

This Theorem implies the Main Theorem ([GV18, Th.14.1], generalized in [Cai21, Theorem 7.3]) below. We need to add few more notations and assumptions. (this implies Taylor-Wiles data exist, see [ACC+18, 6.2.28]). Let $C^\bullet(X_U, V_\lambda(\mathcal{O}))$ be the integral cochain complex; its cohomology is $\mathrm{H}^\bullet(X_U, V_\lambda(\mathcal{O}))$. There is a natural homomorphism

$$\mathcal{H}^{S, \mathrm{univ}} \to \mathsf{End}_{D(\mathcal{O})}(C^\bullet(X_U, V_\lambda(\mathcal{O})))$$

from the abstract spherical Hecke algebra outside S to the endomorphisms of $C^\bullet$ in the derived category. Let $\mathrm{T}(X_U, V_\lambda(\mathcal{O}))$ be the image of this map. Then $\mathrm{T}(X_U, V_\lambda(\mathcal{O}))$ is finite over $\mathcal{O}$. see [NT16]. Let $\mathfrak{m}$ be a non-Eisenstein maximal ideal of $\mathrm{T}(X_U, V_\lambda(\mathcal{O}))$ associated to π modulo v_0. For simplicity, we write $C_{\lambda, \mathfrak{m}}^\bullet$ for $C^\bullet(X_U, V_\lambda(\mathcal{O}))_\mathfrak{m}$ and write $\mathrm{T}_{\lambda, \mathfrak{m}}$ for $\mathrm{T}(X_U, V_\lambda(\mathcal{O}))_\mathfrak{m}$. We assume

$(Gal_\mathfrak{m})$ For $? = FL, \mathrm{ord}$, there exists a lifting $\rho_\mathfrak{m} \colon G_{F, S \cup S_p} \to \mathsf{GL}_N(\mathrm{T}_{\lambda, \mathfrak{m}})$ of $\bar{\rho}$ such that $[\rho_\mathfrak{m}] \in \mathcal{F}^?(\mathrm{T}_\mathfrak{m})$. In particular, there is a natural map

$$\mathcal{R} \to \mathrm{T}_{\lambda, \mathfrak{m}} \hookrightarrow \mathsf{End}_{D(\mathcal{O})}(C_{\lambda, \mathfrak{m}}^\bullet).$$

Let $\mathrm{H}_\mathfrak{m}^\bullet = \mathrm{H}^\bullet(X_U, V_\lambda(\mathcal{O}))_\mathfrak{m}$. Let $\mathrm{H}_i^\mathfrak{m} = \mathrm{H}_\mathfrak{m}^{d-i}$ for all $i \geq 0$ and let $\mathrm{H}_\bullet^\mathfrak{m} = \bigoplus_{i \geq 0} \mathrm{H}_i^\mathfrak{m}$ the corresponding graded module. We assume

$(Van_\mathfrak{m})$ $\mathrm{H}_\mathfrak{m}^i(k) = 0$ if $i \notin [q_0, q_0 + \ell_0]$.

Theorem 12. *Assume* (RLI), (MIN), (FL) *or* $(ORD_\pi) + (DIST)$, $(\mathrm{Gal_m})$, (Van_m) *as above. Then,*

(i) we have an isomorphism of graded commutative rings $\pi_*\mathcal{R} \cong \mathrm{Tor}_*^{S_\infty}(R_\infty, \mathcal{O})$,

(ii) $\pi_0(\mathcal{R}) = R_\infty \otimes_{S_\infty} \mathcal{O} = R \cong \mathrm{T}_{\lambda,\mathrm{m}}$, $\mathrm{H}_{q_0}^\mathrm{m} = \mathrm{H}_\mathrm{m}^{q_0+\ell_0}$ *is free of rank* 1 *over* $\mathrm{T}_{\lambda,\mathrm{m}}$,

(iii) There is an isomorphism of graded $\pi_\bullet\mathcal{R}$*-modules*

$$\mathrm{H}_\bullet^\mathrm{m} \cong \mathrm{H}_{q_0}^\mathrm{m} \otimes_{\mathrm{T}_{\lambda,\mathrm{m}}} \pi_\bullet(\mathcal{R}).$$

Remark 3. *As in* [GV18, Theorem 14.1]*, this statement relies on* [CaGe18, Theorem 12.1]*. See examples where these assumptions are satisfied in* [Cai21, Section 8]*.

3.7. **Divisible part of the Dual Adjoint Selmer group.** Let P be a finite free $\mathcal{O}$-module with action of $\Gamma = G_{F,S \cup S_p}$. For each $v \in S_p$, let $Fil_v^+ P \subset Fil_v P \subset P$ be direct factors submodules stable by Γ_v. We define the minimal $(Fil_v^+ P)_v$-ordinary Selmer group as

$$\mathrm{Sel}_\mathrm{ord}(P) = \mathrm{H}_{min,\mathrm{ord}}^1(F, P \otimes \mathbb{Q}/\mathbb{Z}) = \mathrm{Ker}(\mathrm{H}^1(F, P \otimes \mathbb{Q}/\mathbb{Z}) \to \bigoplus_v \frac{\mathrm{H}^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z})}{\mathrm{H}_\mathrm{ord}^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z})}))$$

where

$$\mathrm{H}_\mathrm{ord}^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z}) = \mathrm{Im}(L_v' \to \mathrm{H}^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z}))$$

$$L_v' = \mathrm{Ker}(\mathrm{H}^1(F_v, Fil_v P \otimes \mathbb{Q}/\mathbb{Z}) \to \mathrm{H}^1(F_v, Fil_v P \otimes \mathbb{Q}/\mathbb{Z}))/\mathrm{H}^1(I_v, Fil_v^+ P \otimes \mathbb{Q}/\mathbb{Z}))$$

Similarly,

$$\mathrm{Sel}_f(P) = \mathrm{H}_f^1(F, P \otimes \mathbb{Q}/\mathbb{Z}) = \mathrm{Ker}(\mathrm{H}^1(F, P \otimes \mathbb{Q}/\mathbb{Z}) \to \bigoplus_v \frac{\mathrm{H}^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z})}{\mathrm{H}_f^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z})}))$$

where

$$\mathrm{H}_f^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z}) = \mathrm{Im}(\mathrm{H}_f^1(F_v, P \otimes \mathbb{Q}) \to \mathrm{H}^1(F_v, P \otimes \mathbb{Q}/\mathbb{Z})).$$

In the sequel we shall take $P = \mathrm{Ad}(\rho_\pi) = \mathfrak{g}_\mathcal{O}$, $Fil_v P = \mathfrak{b}_\mathcal{O}$, $Fil_v^+ P = \mathfrak{n}_\mathcal{O}$, or $P = \mathrm{Ad}(\rho_\pi)^\vee(1) = \mathfrak{g}_\mathcal{O}^\vee(1)$, $Fil_v = \mathfrak{n}_\mathcal{O}^\perp(1)$, $Fil_v^+ P = \mathfrak{b}_\mathcal{O}^\perp(1)$. For $? = \mathrm{ord}, f$, Theorem (12) implies that $\mathrm{Sel}_?(\mathrm{Ad}(\rho_\pi))$ is finite. Let

$$\mathrm{III}_?(P) = \mathrm{H}_?^1(F, P \otimes \mathbb{Q}/\mathbb{Z})/\mathrm{H}_?^1(F, P \otimes \mathbb{Q}/\mathbb{Z})_{\varpi-div}$$

It is the torsion quotient of $\mathrm{H}_?^1(F, P \otimes \mathbb{Q}/\mathbb{Z})$. For $P = \mathfrak{g}_\mathcal{O}^\vee(1)$, We will recall below, using Poitou-Tate duality, that it is Pontryagin dual to $\mathrm{Sel}_?(\mathrm{Ad}(\rho_\pi))$. Recall that Beilinson conjecture predicts that

$$(B) \quad cork_\mathcal{O} \underset{?}{\mathrm{Sel}}(\mathfrak{g}_\mathcal{O}^\vee(1)) = \ell_0.$$

In the next section, we'll see that this conjecture follows from Theorem 12. In the next section, assuming the assumption of Theorem 12, we will provide an automorphic description of a ϖ-divisible subgroup of a corank ℓ_0 of $\mathrm{Sel}_?(\mathfrak{g}_\mathcal{O}^\vee(1))$. By Proposition 7 below, $\mathrm{Sel}_?(\mathfrak{g}_\mathcal{O}^\vee(1))$, assuming Theorem 12, Conjecture (B) holds. It implies that the cokernel of this embedding is therefore $\mathrm{III}_?(\mathfrak{g}_\mathcal{O}^\vee(1))$, which is Pontryagin dual to $\mathrm{Sel}_?(\mathfrak{g}_\mathcal{O})$.

4. THE GALATIUS-VENKATESH HOMOMORPHISM

We redefine the map of Lemma 15.1 of [GV18] as follows. Let $\mathcal{O}_n = \mathcal{O}/(\varpi^n)$. We consider the simplicial ring homomorphism

$$\phi_n \colon \mathcal{R} \to R \to \mathcal{O}_n$$

given by the universal property for the deformation $\rho_n = \rho_\pi \pmod{(\varpi^n)}$. Let M_n be a finite $\mathcal{O}_n$-module. Consider the simplicial ring $\Theta_n = \mathcal{O}_n \oplus \mathrm{DK}(M_n[1])$ where $M_n[1]$ is the chain complex concentrated in degree 1 up to homotopy. For an explicit description of the simplicial module $DK(M_n[1])$, see [Cai21, Section 3.4]. The simplicial ring Θ_n is endowed with a simplicial ring homomorphism $\mathrm{pr}_n \colon \Theta_n \to \mathcal{O}_n$ given by the

first projection. Let $L_n(\mathcal{R})$ be the set of homotopy equivalence classes of simplicial ring homomorphisms $\Phi \colon \mathcal{R} \to \Theta_n$ such that $\mathrm{pr}_n \circ \Phi = \phi_n$. There is a canonical bijection

$$L_n(\mathcal{R}) \cong \mathrm{H}^2_f(F, \mathrm{Ad}(\rho_n) \otimes M_n)$$

Moreover, as in [GV18, Lemma 15.1], there is a map

$$\pi(n, \mathcal{R}) \colon L_n(\mathcal{R}) \to \mathsf{Hom}_R(\pi_1(\mathcal{R}), M_n)$$

sending (the homotopy class of) Φ to the homomorphism $\pi(n, \mathcal{R})(\Phi)$ which sends the homotopy class $[\gamma]$ of a loop γ to $\Phi \circ \gamma \in \mathsf{Hom}_{\mathsf{sSETS}}(\Delta[1], M_n[1]) = M_n$. Recall a loop γ is a morphism of sSETS

$$\gamma \colon \Delta[1] \to \Theta_n$$

from the simplicial interval $\Delta[1]$ to the simplicial set Θ_n which sends the boundary $\partial\Delta[1]$ to 0. Note that $\pi(n, \mathcal{R})(\Phi)$ is R-linear by definition of the structure of $\pi_0(\mathcal{R})$-module on $\pi_1(\mathcal{R})$.

Proposition 5. *For any $n \geq 1$, the map $\pi(n, \mathcal{R})$ is surjective.*

We'll give two proofs of this Proposition. Note first that by the isomorphisms (14.4), we have a diagram

$$
\begin{array}{ccccc}
\pi(n, \mathcal{R}) & : & L_n(\mathcal{R}) & \to & \mathsf{Hom}_{\pi_0(\mathcal{R})}(\pi_1(\mathcal{R}), M_n) \\
& & \uparrow & & \uparrow \\
\pi(R_n \overset{L}{\otimes}_{S_n} \mathcal{O}_n) & : & L_n(R_n \overset{L}{\otimes}_{S_n} \mathcal{O}_n) & \to & \mathsf{Hom}_{\pi_0(\mathcal{R})}(\pi_1(R_n \overset{L}{\otimes}_{S_n} \mathcal{O}_n), M_n) \\
& & \downarrow & & \downarrow \\
\pi(R_\infty \overset{L}{\otimes}_{S_\infty} \mathcal{O}) & : & L_n(R_\infty. \overset{L}{\otimes}_{S_\infty} \mathcal{O}) & \to & \mathsf{Hom}_{\pi_0(\mathcal{R})}(\pi_1(R_\infty \overset{L}{\otimes}_{S_\infty} \mathcal{O}), M_n)
\end{array}
$$

Note that $\pi_1 \mathcal{R} = \pi_1(R_\infty \overset{L}{\otimes}_{S_\infty} \mathcal{O})$ by [GV18, Theorem 14.1]. So it is enough to prove that the map on the last line is surjective. In the sequel, since n is fixed, we drop the indices n and write simply $A = A_n$, $\Theta = \Theta_n$, $M = M_n$.

Proof. First Proof: Recall that by finiteness of the Calegari-Geraghty complex of finite free S_∞-modules constructed in [CaGe18], R_∞ is a finite S_∞ algebra. Let be a cofibrant replacement $P_\bullet \to R_\infty$ by noetherian polynomial rings $\mathbf{P}_n$ over S_∞. Then, one can take as $R_\infty \overset{L}{\otimes}_{S_\infty} \mathcal{O}$ the complex associated to the simplicial ring $\mathbf{R}_\bullet = \mathbf{P}_\bullet \otimes_{S_\infty} \mathcal{O}$, which consists in (noetherian) polynomial $\mathcal{O}$-algebras and is a cofibrant fibration of $R_\infty \otimes_{S_\infty} \mathcal{O}$ (but is not a weak equivalence). By the adjunction formula of Section 3.1, we have

$$\mathsf{sHom}_{\mathcal{O}\mathrm{CR}^s_A}(\mathbf{R}_\bullet, A \oplus \mathrm{DK}(M[1])) \cong \mathsf{sHom}_{\mathrm{Mod}^s_A}(L_{\mathbf{R}_\bullet/\mathcal{O}} \otimes_{\mathbf{R}_\bullet} A, M[1]).$$

Note that by construction, since $\mathbf{R}_\bullet$ is cofibrant, we have

$$L_{\mathbf{R}_\bullet/\mathcal{O}} = \Omega_{\mathbf{R}_\bullet/\mathcal{O}}.$$

Therefore, the datum of a homomorphism of chain complexes $\Phi \colon L_{\mathbf{R}_\bullet/\mathcal{O}} \otimes_{\mathbf{R}_\bullet} A \to M[1]$ is equivalent to that of a homomorphism of modules $\psi_1 \colon \Omega^1_{\mathbf{R}_1/\mathcal{O}} \otimes_{\mathbf{R}_1} A \to M$ placed in degree 1 which is itself equivalent to the datum of a classical $\mathcal{O}$-algebra homomorphism $\Phi_1 \colon \mathbf{R}_1 \to A \oplus M$. Thus we need to lift a given $\varphi_1 \colon \pi_1(\mathbf{R}_\bullet) \to M$ to a homomorphism Φ_1.

Following [COT, Section 09D4 Example 5.9], we give an explicit description of $\mathbf{P}_i$ $(i = 0, 1, 2)$ and their face and degeneracy maps. We denote $\mathbf{P}_0 = S_\infty[u_1, \ldots, u_k]$ which we abbreviate as $\mathbf{P}_0 = S_\infty[u_j]$, and similarly for the other rings: $\mathbf{P}_1 = S_\infty[u_j, x_t]$, $\mathbf{P}_2 = S_\infty[u_j, x_t, v_r, y_t, z_t, w_{t,t'}]$, where $u_i, x_t, v_r, y_t, z_t, w_{t,t'}$ are independent variables. One fixes $\mathbf{P}_0 \to R_\infty$ a surjective S_∞-algebra homomorphism, and a system (f_t) of generators of $\mathrm{Ker}(\mathbf{P}_0 \to R_\infty)$. Here $r = (r_t)$ runs over the (finitely generated) module of systems of relations between the f_t's in $\mathbf{P}_0 : \sum_t r_t \cdot f_t = 0$.

Let $d_0, d_1 \colon \mathbf{P_1} \to \mathbf{P_0}$ be the two face maps given by $u_i \mapsto u_i$ and $x_t \mapsto 0$, resp. $x_t \mapsto f_t$. Let $\delta_0, \delta_1, \delta_2 \colon \mathbf{P_2} \to \mathbf{P_1}$ the three face maps. See their definitions in [COT, Section 09D4 Example 5.9]. We simply recall that $\delta_0(v_r) = \delta_1(v_r) = 0$ and $\delta_2(v_r) = \sum_t r_t x_t$.

By tensoring by $\mathcal{O}$ over $\mathbf{S_\bullet}$, these maps yield the face and degeneracy maps of $\mathbf{R_i}$, $i = 0, 1, 2$. We still denote by $d_0, d_1 \colon \mathbf{R_1} \to \mathbf{R_0}$ and $\delta_0, \delta_1, \delta_2 \colon \mathbf{R_2} \to \mathbf{R_1}$ the resulting face maps. We have $\pi_1(\mathbf{R_\bullet}) = \mathrm{Ker}(d_0 - d_1)/\mathrm{Im}(\delta_0 - \delta_1 + \delta_2)$. For any homomorphism $\varphi_1 \colon \mathrm{Ker}(d_0 - d_1) \to M$, such that $\varphi_1 \circ (\delta_0 - \delta_1 + \delta_2) = 0$, we consider $\Phi_0 \colon \mathbf{P_0} \otimes_{\mathbf{S_0}} \mathcal{O} \to A$ as the composition of $\mathbf{P_0} \otimes_{S_\infty} \mathcal{O} \to R_\infty \otimes_{S_\infty} \mathcal{O} = R$ and $R \to A$. We need to define $\Phi_1 \colon \mathbf{P_1} \otimes_{\mathbf{S_1}} \mathcal{O} \to A \oplus M$, compatible to Φ_0 for the d_i $(i = 0, 1)$ and s_0, in terms of φ_1.

One can rewrite this more explicitly, $\mathbf{R_0} = \mathbf{P_0} \otimes_{S_\infty} \mathcal{O} = \mathcal{O}[[u_j]]$ and $\mathbf{R_1} = \mathbf{P_1} \otimes_{S_\infty} \mathcal{O} = \mathcal{O}[[u_j, x_t]]$. Then $\mathrm{Ker}(d_0 - d_1)$ is the $\mathcal{O}[[u_j]]$-module of $P(u_j, x_t)$ such that

$$P(u_j, 0) = P(u_j, f_t).$$

Moreover, $R_\infty \otimes_{S_\infty} \mathcal{O}$ is a quotient of $\mathbf{R_0}$. We can extend φ_1 to $\mathcal{O}[[u_j, x_t]]$ as follows. We first write $P = P_1 + K_1$ where $P_1 = P(u_j, 0) + \sum_t Q_t(u_j) \cdot x_t$ and $K_1 \in \mathrm{Ker}(d_0 - d_1)$; indeed, if we write $P(u_j, f_t) - P(u_j, 0) = \sum_t Q_t(u_j) \cdot f_t$, then if we put $P_1(u_j, x_t) = P(u_j, 0) + \sum_t Q_t(u_j) \cdot x_t$, we see that $K_1 = P - P_1 \in \mathrm{Ker}(d_0 - d_1)$. Then we put

$$\Phi_1(P) = \Phi_0(d_0(P_1)) + \varphi_1(K_1)$$

Note that $\Phi_0(d_0(P_1)) = \Phi_0(d_0(P))$. Let us check that Φ_1 is well defined. If we write

$$P(u_j, f_t) - P(u_j, 0) = \sum_k Q_t(u_j) \cdot f_t = \sum_k Q'_t(u_j) \cdot f_t$$

for another system (Q'_t) of elements of $\mathcal{O}[[u_j]]$; let $K'_1 = P - P'_1$ where $P'_1 = P(u_j, 0) + \sum_t Q'_t x_t$. We need to show that $\varphi_1(K_1) = \varphi_1(K'_1)$. The system $r = (r_t)$ defined by $r_t = Q_t - Q'_t$ is a relation between the f_t's: $\sum_t r_t f_t = 0$. By [COT, 09D4], we see that

$$\varphi_1(K_1) - \varphi_1(K'_1) = \varphi_1(\sum_t r_t x_t) = \varphi_1(\delta_0 - \delta_1 + \delta_2)(v_r)) = 0$$

hence Φ_1 is well defined. This yields the desired lifting

$$\Phi \colon R_\bullet \to A \oplus \mathrm{DK}(M[1]))$$

of $\varphi_1 \colon \pi_1(R_\bullet) \to M$.

To conclude the proof, we have compatible morphisms $\mathcal{R} \to R_\bullet/\mathfrak{a}_n$ by [GV18, (14.4)] which induce an isomorphism $\pi_1(\mathcal{R}) \cong \pi_1(R_\bullet)$. Therefore, the surjectivity of $\pi(n, \mathcal{R})$ follows from that of $\pi(n, R_\bullet)$. $\square$

Proof. Second Proof: For a B-algebra C, we denote by $B[C]^{(j)}$ the ring defined by induction as follows : $B[C]^{(0)}$ is the polynomial B-algebra with variables the elements of C, and for $j \geq 1$, $B[C]^{(j)}$ is the polynomial B-algebra with variables the elements of $B[C]^{(j-1)}$. As explained in [SIMP, Section 14.34.5], these sets form a simplicial set. In particular the d_k's, $k = 0, \ldots, j$ from $B[C]^{(j)}$ to $B[C]^{(j-1)}$ are given as follows. Let

$$P_j = P_j([P_{j-1}(\ldots ([P_0([\underline{c}])]) \ldots]) \in B[C]^{(j)}.$$

Then, $d_k P_j$ is obtained by removing the k-th bracket (the zeroth being the innermost one).

Let $B = S_\infty$ and $C = R_\infty$ We choose the functorial cofibrant replacement $P_\bullet$ of R_∞ by rings $\mathbf{P}_j = S_\infty[R_\infty]^{(j)}$ Then, one can take as $R_\infty \overset{L}{\otimes}_{S_\infty} \mathcal{O}$ the complex associated to the simplicial ring $\mathbf{R_\bullet} = \mathbf{P_\bullet} \otimes_{S_\infty} \mathcal{O}$; tt follows from [Gil13, Corollary 7.10.5] that $\mathbf{R_\bullet}$ is a cofibrant fibration of $R_\infty \otimes_{S_\infty} \mathcal{O}$ (but is not a weak equivalence). It can also be proven by verifying first that the morphism of simplicial rings $\mathcal{O} \to \mathbf{R_\bullet}$ is free in the sense of [Gil13, Definition 7.6.2] (that is, by verifying Conditions (1)-(3) which follow this definition), then, by quoting [Gil13, Theorem 7.6.13] stating that any free simplicial morphism is a cofibration.

Let $P_1 = P_1([P_0([\underline{c}])]) \in \mathbf{R}_1$ where P_1, resp. P_0, has coefficients in $\mathcal{O}$, resp. in B. Let $P_0^{(1,0)}, \ldots, P_0^{(m,0)}$ be the finite list of polynomials P_0 which actually occur as variables in P_1. We write $P_1 = P_1([P_0(\underline{c})]) + \sum_{s=1}^{m}([P_0^{(s,0)}] - [P^{(s,0)}(\underline{c})])E_s$, where $E_s \in \mathbf{R}_1$. Let $P_0^{(1,1)}, \ldots, P_0^{(n,1)}$ be the list of polynomials P_0 which occur as variables in E_s. Then we redecompose E_s as $E_s = E_s(P_0([\underline{c}])) + \sum_{t=1}^{n}([P_0^{(t,1)}] - P^{(t,1)}([\underline{c}]))F_{s,t}$. If we put

$$Q_1 = P_1([P_0(\underline{c})]) + \sum_{s=1}^{m}([P_0^{(s,0)}] - [P^{(s,0)}(\underline{c})]) \cdot E_s(P_0([\underline{c}]))$$

and

$$K_1 = \sum_{s,t}([P_0^{(s,0)}] - [P^{(s,0)}(\underline{c})]) \cdot ([P_0^{(t,1)}] - [P^{(t,1)}(\underline{c})]) \cdot F_{s,t}$$

we see that $P_1 = Q_1 + K_1$, with $Q_1 \in \mathbf{R}_0$ and $K_1 \in ZN_1(\mathbf{R}_\bullet)$ since we have clearly $d_k P_1 = d_k Q_1$ for $k = 0, 1$. Moreover, if $P_1 = Q_1 + K_1 = Q'_1 + K'_1$, we have $Q_1 = d_0 \widetilde{Q}_1$ and $Q'_1 = d_0 \widetilde{Q'}_1$ (where $\widetilde{Q}_1$, resp. $\widetilde{Q'}_1$, is defined by inserting an inner bracket: if $Q_1 = Q_1([c])$, put $\widetilde{Q}_1 = Q_1([[[c]]])$) and similarly for $\widetilde{Q'}_1$), hence $Q_1 - Q'_1 \in BN_1(\mathbf{R}_\bullet)$. $\qquad\square$

Let $\lambda \colon R \to \mathcal{O}$ be a Hecke eigensystem associated to an automorphic representation π' occuring in T. Let us consider $M_n = \varpi^{-n}\mathcal{O}/\mathcal{O})$ as a R-module through λ; we take the Pontryagin dual $\pi(n, \mathcal{R})^\vee$ and apply Poitou-Tate duality. We obtain a T-linear injection :

$$GV_n \colon \mathsf{Hom}_R(\pi_1(\mathcal{R}), M_n)^\vee \hookrightarrow \mathrm{H}^1_f(F, \mathrm{Ad}\ \rho_n(1) \otimes M_n^\vee)$$

We have $\mathsf{Hom}_R(\pi_1(\mathcal{R}), M_n) = \mathsf{Hom}_\mathcal{O}(\pi_1(\mathcal{R}) \otimes_{R,\lambda} \mathcal{O}/\varpi^n, \varpi^{-n}\mathcal{O}/\mathcal{O}_n)$. The Pontryagin dual of this module is equal to $\pi_1(\mathcal{R}) \otimes_{R,\lambda} \mathcal{O}/\varpi^n$.

The right hand side is $\mathrm{H}^1_f(F, \mathrm{Ad}\ \rho_\lambda(1) \otimes \mathcal{O}/\varpi^n) = \mathrm{H}^1_f(F, \mathrm{Ad}\ \rho_\lambda(1)/(\varpi^n)) = \mathrm{Sel}(\mathrm{Ad}(\rho_\lambda)(1) \otimes \varpi^{-n}\mathcal{O}/\mathcal{O})$. Taking inductive limit on both sides we obtain a linear injection

$$GV \colon \pi_1(\mathcal{R}) \otimes_{R,\lambda} \mathcal{O} \otimes_\mathcal{O} K/\mathcal{O} \hookrightarrow \mathrm{Sel}(\mathrm{Ad}(\rho_\lambda)(1) \otimes K/\mathcal{O})$$

Theorem 13. *Assume $p > N$, $\zeta_p \notin F$, $(\mathrm{Gal}_\mathfrak{m})$, (LLC), (RLI), (MIN), (FL) or $(ORD_\pi) + (DIST)$. The natural homomorphism*

$$GV \colon \pi_1(\mathcal{R}) \otimes_{R,\lambda} \mathcal{O} \otimes_\mathcal{O} K/\mathcal{O} \hookrightarrow \mathrm{Sel}(\mathrm{Ad}(\rho_\lambda)(1) \otimes K/\mathcal{O})$$

is injective. The left hand side module is ϖ-divisible of $\mathcal{O}$-corank ℓ_0.

Proof. The left-hand side module is the tensor product of a finitely generated $\mathcal{O}$-module by $K/\mathcal{O}$, hence it is ϖ-divisible. Moreover, we have $R = T$. Let us show that $\pi_1(\mathcal{R}) \otimes_{R,\lambda} \mathcal{O} \otimes_\mathcal{O} K/\mathcal{O}$ has corank ℓ_0: Indeed, by Borel-Wallach, we have

$$\dim \mathrm{H}^{q_s-1}_{temp} = \ell_0 \cdot \dim \mathrm{H}^{q_s}_{temp}$$

and and, for F a CM field,

$$\dim \mathrm{H}^{q_s}_{cusp} = \dim A_{cusp}(\lambda)$$

where $A_{cusp}(\lambda)$ denotes the space of cuspidal automorphic forms of level K and cohomological weight λ. This implies that $H^{q_s}_\mathfrak{m}$ is free of rank one over T, hence $\mathrm{H}^{q_s-1}_\mathfrak{m}$ is free of rank ℓ_0 over T. This concludes the proof. $\qquad\square$

As in [TU19, Lemma 10], one shows by using Poitou-Tate duality that the right-hand side $\mathcal{O}$-module is of cofinite rank ℓ_0. In other words, the left-hand side is the maximal ϖ-divisible submodule of the right-hand side. Therefore by [BK90, Definition 5.13], we see that $Coker\ GV = \text{Ш}(\mathrm{Ad}(\rho_\mathfrak{m})(1))$. By Poitou-Tate duality, this group is finite and is isomorphic to the Pontryagin dual of $\mathrm{Sel}(\mathrm{Ad}\ \rho_\lambda \otimes K/\mathcal{O})$. By deformation theory, this Pontryagin dual is isomorphic to $C_1(R, \lambda) = \Omega_{R/\mathcal{O}} \otimes_{T,\lambda} \mathcal{O}$. By the $R = T$ theorem of Calegari-Geraghty, this group is isomorphic to $C_1(R, \lambda) = \Omega_{T/\mathcal{O}} \otimes_{T,\lambda} \mathcal{O}$, which is finite.

5. A GRADED VERSION OF THE GALATIUS-VENKATESH HOMOMORPHISM

Under the assumptions $p > N$, (RLI), (MIN), (FL) or $(ORD_\pi) + (DIST)$, we can also define a graded version $GV^\bullet$ of the Galatius-Venkatesh homomorphism GV. For every cuspidal representation π' occuring in $H_{\mathfrak{m}}^\bullet$, let $\phi = \lambda_{\pi'}^{\mathrm{gal}} \colon R^? \to \mathcal{O}$ be the algebra homomorphism associated to $\rho_{\pi'}$. Recall that the derived universal ring $\mathcal{R}$ is given by a projective system of cofibrant simplicial artinian rings $(\mathcal{R}_\alpha)$. For any $j \geq 1$, we denote by $\mathcal{R}^{\otimes j}$ the pro-artinian simplicial ring defined by the projective system of the strict tensor product simplicial artinian $\mathcal{O}$-algebras $(\mathcal{R}_\alpha^{\otimes j})$. Recall that the strict tensor product of two simplicial $\mathcal{O}$-modules $A = (\mathcal{A}_n)$ and $\mathcal{B} = (\mathcal{B}_n)$ is $\mathcal{A} \otimes \mathcal{B}$ such that for any $n \geq 0$, $(\mathcal{A} \otimes \mathcal{B})_n = \mathcal{A}_n \otimes_{\mathcal{O}} \mathcal{B}_n$. If $\mathcal{A}$ and $\mathcal{B}$ are simplicial $\mathcal{O}$-algebras, the result is a simplicial $\mathcal{O}$-algebra.

Let $m \geq 1$ and $A = \mathcal{O}/(\varpi^m)$ and $\pi_k(\mathcal{R})_A = \pi_k(\mathcal{R}) \otimes_{\pi_0(\mathcal{R}),\phi} A$. Note that for any $j \geq 0$, we have a natural ring homomorphism $\pi_0(\mathcal{R})^{\otimes j} \to \pi_0(\mathcal{R}^{\otimes j})$, hence a natural structure of $\pi_0(\mathcal{R})^{\otimes j}$-module on $\pi_j(\mathcal{R}^{\otimes j})$. Let us consider $\phi^{\otimes j} \colon \pi_0(\mathcal{R})^{\otimes j} \to A$, via the identification $A^{\otimes j} \cong A$. We can form $\pi_j(\mathcal{R}^{\otimes j})_A = \pi_j(\mathcal{R}^{\otimes j}) \otimes_{\pi_0(\mathcal{R})^{\otimes j},\phi^{\otimes j}} A$.

In this subsection, we assume $p > \ell_0 = N d_0 - 1$.

Let $\mathcal{T}$ be a simplicial $\mathcal{O}$-algebra. For $j_1, j_2 \geq 0$ and $j = j_1 + j_2$, consider the subset $P_{j_1,j_2} \subset \mathbb{S}_j$ of permutations (σ, τ) where $\sigma(i) = \sigma_i$, $i = 1, \ldots, j_1$ with $\sigma_1 < \ldots \sigma_{j_1}$, and $\tau(i + j_1) = \tau_i$, $i = 1, \ldots, j_2$, with $\tau_1 < \ldots < \tau_{j_2}$. Recall that the graded $\mathcal{O}$-module $\pi_\bullet(\mathcal{T})$ is endowed with a structure of graded $\mathcal{O}$-algebra by the shuffle product

$$[-,-]_{j_1,j_2} \colon \pi_{j_1}(\mathcal{T}) \times \pi_{j_2}(\mathcal{T}) \to \pi_j(\mathcal{T})$$

induced by the Eilenberg-Zilber map: for $t_i \in \mathcal{T}_{j_i}$ by

$$\sum_{(\sigma,\tau) \in P_{j_1,j_2}} sgn(\sigma,\tau) \cdot A(\sigma)(t_1) \cdot A(\tau)(t_2)$$

see [Gil13, Section 8.3] or [EZ].

Consider the graded $\mathcal{O}$-algebra

$$\pi_\bullet(\mathcal{T}^{\otimes \bullet}) = \bigoplus_j \pi_j(\mathcal{T}^{\otimes j})$$

for the multiplication given, for $j_1, j_2 \geq 0$ and $j_1 + j_2 = j$, by

$$< -, - >_{j_1,j_2} \colon \pi_{j_1}(\mathcal{T}^{\otimes j_1}) \times \pi_{j_2}(\mathcal{T}^{\otimes j_2}) \to \pi_j(\mathcal{T}^{\otimes j})$$

defined as the composition of the normalized shuffle product for the simplicial $\mathcal{O}$-algebra $\mathcal{T}^{\otimes j}$:

$$[-,-]_{j_1,j_2} \colon \pi_{j_1}(\mathcal{T}^{\otimes j}) \times \pi_{j_2}(\mathcal{T}^{\otimes j}) \to \pi_j(\mathcal{T}^{\otimes j})$$

with the morphism

$$\pi_{j_1}(\mathcal{R}^{\otimes j_1}) \times \pi_{j_2}(\mathcal{T}^{\otimes j_2}) \to \pi_{j_1}(\mathcal{T}^{\otimes j}) \times \pi_{j_2}(\mathcal{T}^{\otimes j})$$

induced by $\iota_1 \otimes \iota_2$ where $\iota_1 \colon \mathcal{T}^{\otimes j_1} \to \mathcal{T}^{\otimes j_1 + j_2}, x \mapsto x \otimes 1$ and $\iota_2 \colon \mathcal{R}^{\otimes j_2} \to \mathcal{R}^{\otimes j_1 + j_2}, x \mapsto 1 \otimes x$. Note that $\pi_\bullet(\mathcal{T}^{\otimes \bullet})$ endowed with $< -, - >$ is a graded associative algebra but is not graded-commutative in general. It comes with a graded-algebra homomorphism

$$c_\bullet \colon \pi_\bullet(\mathcal{T}^{\otimes \bullet}) \to \pi_\bullet(\mathcal{T})$$

given degreewise by the multiplication $m_j \colon \mathcal{T}^{\otimes j} \to \mathcal{T}$. It is compatible with the (tensor) shuffle product: for $j = j_1 + j_2$, we have

$$[-,-]_{j_1,j_2} \circ (c_{j_1} \otimes c_{j_2}) = c_j \circ < -, - >_{j_1,j_2} .$$

Let $\widetilde{\pi}_\bullet(\mathcal{T}^{\otimes \bullet})$ be the largest graded-commutative quotient of $\pi_\bullet(\mathcal{T}^{\otimes \bullet})$. It is the graded $\mathcal{O}$-algebra sum of the quotients $\widetilde{\pi}_j(\mathcal{T}^{\otimes j})$ of $\widetilde{\pi}_j(\mathcal{T}^{\otimes j})$, defined by induction as the sub-$\mathcal{O}$-module generated by the products $< t_1, t_2 > -(-1)^{j_1 j_2} < t_2, t_1 >_{j_2,j_1}$, $t_i \in \widetilde{\pi}_{j_i}(\mathcal{T}^{\otimes j_i})$. The homomorphism $c_\bullet$ factors through $\widetilde{\pi}_\bullet(\mathcal{T}^{\otimes \bullet})$ as

$$\widetilde{c}_\bullet \colon \widetilde{\pi}_\bullet(\mathcal{T}^{\otimes \bullet}) \to \pi_\bullet(\mathcal{T})$$

Note that the iterated tensor shuffle product

$$\pi_1(\mathcal{T})^{\otimes \bullet} \to \pi_\bullet(\mathcal{T}^{\otimes \bullet})$$

induces a homomorphism of graded-commutative algebras

$$s_\bullet \colon \bigwedge^\bullet \pi_1(\mathcal{T}) \to \widetilde{\pi}_\bullet(\mathcal{T}^{\otimes \bullet})$$

The composition $c_\bullet \circ s_\bullet$ coincides with the iterated classical shuffle product

$$\bigwedge^\bullet \pi_1(\mathcal{T}) \to \pi_\bullet(\mathcal{T}).$$

For certain simplicial rings $\mathcal{T}$, as seen below, $\widetilde{c}_\bullet$ is an isomorphism. However, this may not be always the case.

Remark 4. *1) Given a simplicial $\mathcal{O}$-algebra $\mathcal{T}$, the relation between $\pi_\bullet(\mathcal{T})$ and $\pi_\bullet(\mathcal{T}^{\otimes \bullet})$ is similar to the relation, for a finite free $\mathcal{O}$-module T, between $\bigwedge^\bullet T$ and $\bigwedge^\bullet (T^{\oplus \bullet})$, using inclusions $\iota_k \colon T^{\oplus j_k} \to T^{\oplus j}$ for $j = j_1 + j_2$ and wedge products*

$$\bigwedge^{j_1} T^{\oplus j} \times \bigwedge^{j_2} T^{\oplus j} \to \bigwedge^{j} T^{\oplus j}.$$

The graded algebra homomorphism

$$c_\bullet \bigwedge^\bullet T^{\oplus \bullet} \to \bigwedge^\bullet T$$

analogue to

$$c_\bullet \colon \pi_\bullet(\mathcal{T}^{\otimes \bullet}) \to \pi_\bullet(\mathcal{T})$$

is induced by the addition morphisms $a_j \colon T^{\oplus j} \to T$ (instead of the multiplications m_j).

2) For $\mathcal{T} = \mathcal{R}$ the universal derived deformation ring, we know that the graded-commutative $\mathcal{O}$-algebra $\pi_\bullet(\mathcal{R})$ is finite. Here, the (non-commutative) graded algebra $\pi_\bullet(\mathcal{R}^{\otimes \bullet})$ is not finite in general. Consider the graded-commutative algebra homomorphism

$$\widetilde{c}_\bullet \colon \widetilde{\pi}_\bullet(\mathcal{R}^{\otimes \bullet}) \to \pi_\bullet(\mathcal{R})$$

One can ask whether it is an isomorphism.

3) As graded $\mathcal{O}$-modules $\pi_\bullet(\mathcal{R}^{\otimes \bullet})$ and $\pi_\bullet(\mathcal{R})$ can be related by induction, using the Tor-spectral sequences ([Qu67, Theorem 6,Sect.6,II]):

$$E^2 = \mathrm{Tor}_\bullet^\mathcal{O}(\pi_\bullet(\mathcal{R}^{\otimes j - 1}), \pi_\bullet(\mathcal{R})) \Rightarrow \pi_\bullet(\mathcal{R}^{\otimes j})$$

4) The graded $\mathcal{O}$-algebra $\pi_\bullet(\mathcal{R}^{\otimes \bullet})$ is also a graded algebra over the graded $\mathcal{O}$-algebra $\pi_0(\mathcal{R})^{\otimes \bullet}$. Let $A = \mathcal{O}/(\varpi^m)$. The system of homomorphisms $\phi^{\otimes j} \colon \pi_0(\mathcal{R})^{\otimes j} \to A$, $j = 1, \ldots$, defined above give rise to an $\mathcal{O}$-algebra homomorphism

$$\phi^{\otimes \bullet} \colon \pi_0(\mathcal{R})^{\otimes \bullet} \to A,$$

and the tensor products $\pi_j(\mathcal{R}^{\otimes j})_A = \pi_j(\mathcal{R}^{\otimes j}) \otimes_{\pi_0(\mathcal{R})^{\otimes j}, \phi^{\otimes j}} A$. give rise to a tensor product over the graded tensor algebra $\pi_0(\mathcal{R})^{\otimes \bullet}$

$$\pi_(\mathcal{R}^{\otimes \bullet})_A = \pi_\bullet(\mathcal{R}^{\otimes \bullet}) \otimes_{\pi_0(\mathcal{R})^{\otimes \bullet}, \phi^{\otimes \bullet}} A.$$

Let V be a finite free A-module; consider the simplicial A-algebra $\mathcal{S}_V = A \oplus \mathrm{DK}(V[1])$ and the graded algebra $\pi_\bullet(\mathcal{S}_V^{\otimes \bullet})$ endowed with $< -, - >$.

Lemma 14. *The graded algebra $\pi_\bullet(\mathcal{S}_V^{\otimes \bullet})$ is canonically isomorphic to the graded A-algebra $\bigotimes^\bullet V$; its largest graded-commutative quotient $\widetilde{\pi}_\bullet(\mathcal{S}_V^{\otimes \bullet})$ is isomorphic to $\bigwedge^\bullet V$. Moreover, $\widetilde{c}_\bullet$ is an isomorphism, as well as $s_\bullet$; they yield canonical identifications*

$$\widetilde{\pi}_\bullet(\mathcal{S}_V^{\otimes \bullet}) = \pi_\bullet(\mathcal{S}_V) = \bigwedge^\bullet \pi_1(\mathcal{S}_V).$$

Remark 5. *For instance, for $V = A$, $\pi_\bullet(\mathcal{S}_A^{\otimes\bullet})$ is isomorphic to the (strictly) commutative graded algebra $A[X]$ of one variable polynomials, hence $\pi_\bullet(\mathcal{S}_A) \cong A[X]/(X^2)$ as graded-commutative algebras, in such a way that the algebra homomorphism*

$$c_\bullet \colon \pi_\bullet(\mathcal{S}_A^{\otimes\bullet}) \to \pi_\bullet(\mathcal{S}_A)$$

identifies to the quotient homomorphism $A[X] \to A[X]/(X^2)$.

Proof. We first note that given two simplicial $\mathcal{O}$-modules $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{A} \otimes \mathcal{B}$ their strict tensor product (degreewise tensor product), the Eilenberg-Zilber map

$$C(\mathcal{A}) \otimes C(\mathcal{B}) \to C(\mathcal{A} \otimes \mathcal{B})$$

induces an isomorphism of complexes

$$N(\mathcal{A}) \otimes N(\mathcal{B}) \to N(\mathcal{A} \otimes \mathcal{B})$$

by the Eilenberg-MacLane theorem (its inverse is the Alexander-Whitney map, see for instance [Gil13, Section 5.8, Page 76] or [GJ10, Theorem 4.2.4, Page 205]). Therefore, by applying the Dold-Kan functor to this isomorphism we see that for two chain complexes $C, D \in Ch_+(\mathcal{O})$, there is a natural weak equivalence

$$\mathrm{DK}(C \otimes D) \cong \mathrm{DK}(C) \otimes DK(D).$$

In particular, if $V[1]^{\otimes k}$ denotes the total tensor kth-power complex of $V[1]$, we have a weak equivalence

$$\mathrm{DK}(V[1])^{\otimes k} \cong \mathrm{DK}(V[1]^{\otimes k}).$$

Moreover, for any $k \geq 0$, the tensor product induces an isomorphism of complexes

$$m_k \colon V[1]^{\otimes k} \cong V^{\otimes k}[k]$$

By composing these maps, we obtain a weak equivalence of simplicial A-modules

$$\mathcal{S}_V^{\otimes j} \cong \bigoplus_{k=0}^{j} \binom{j}{k} \cdot \mathrm{DK}(V^{\otimes k}[k])$$

which we can rewrite as a quasi-isomorphism

$$N(\mathcal{S}_V^{\otimes j}) \cong \bigoplus_{k=0}^{j} \binom{j}{k} \cdot V^{\otimes k}[k].$$

Now $\pi_j(\mathrm{DK}(W[k])) = \mathrm{H}_j(W[k]) = 0$ for any $k < j$ and any finite free A-module W, hence $m_j \circ pr_2^{\otimes j}$ induces an isomorphism $\pi_j(\mathcal{S}_V^{\otimes j}) = \pi_j(\mathrm{DK}(V^{\otimes j}[j]))$ which yields $\pi_j(\mathcal{S}_V^{\otimes j}) \cong \mathrm{H}_j(V^{\otimes j}[j]) = V^{\otimes j}$. Hence we have an isomorphism of graded A-modules

$$\pi_\bullet(\mathcal{S}_V^{\otimes\bullet}) \cong \overset{\bullet}{\bigotimes} V$$

It remains to show it is multiplicative. For this, we use again the Eilenberg-MacLane theorem: Let us consider the Eilenberg-Zilber map

$$EZ_{j_1,j_2} \colon C(\mathcal{S}_V^{\otimes j_1}) \otimes C(\mathcal{S}_V^{\otimes j_2}) \to C(\mathcal{S}_V^{\otimes j_1} \otimes \mathcal{S}_V^{\otimes j_2})$$

followed by the canonical identification

$$C(\mathcal{S}_V^{\otimes j_1} \otimes \mathcal{S}_V^{\otimes j_2}) \cong C(\mathcal{S}_V^{\otimes j}).$$

The homomorphism $< -, - >_{j_1,j_2}$ is given by the restriction to the (j_1, j_2)-component on the left-hand side of EZ_{j_1,j_2}. Using the identifications $V^{\otimes k} = \mathrm{H}_0(V^{\otimes k}[0]) = \mathrm{H}_k(V^{\otimes k}[k]) = \mathrm{H}_k(C(\mathcal{S}_V)^{\otimes k})$, we see that the isomorphism $V^{\otimes j_k} \cong \pi_{j_k}(\mathcal{S}_V^{\otimes j_k})$, $k = 1, 2$, is given by the j_k-th homology $\mathrm{H}_{j_k}(EZ_{j_k})$, of the morphism of chain complexes

$$EZ_{j_k} \colon C(\mathcal{S}_V)^{\otimes j_k} \to C(\mathcal{S}_V^{\otimes j_k}).$$

Let $m \colon V^{\otimes j_1} \otimes_A V^{\otimes j_2} \to V^{\otimes j}$ be the multiplication isomorphism. By the identification $\mathrm{H}_0(W[0]) = W$, we have $EZ_{0,0} = m$, hence

$$\mathrm{H}_{j_1}(V^{\otimes j_1}[j_1]) \otimes \mathrm{H}_{j_2}(V^{\otimes j_2}[j_2]) \to \mathrm{H}_j(V^{\otimes j}[j])$$

is given by m via $\mathrm{H}_k(V^{\otimes k}[k]) = \mathrm{H}_0(V^{\otimes k}[0]) = V^{\otimes k}$. On the other hand, we have by associativity of the shuffles

$$EZ_{j_1,j_2} \circ (EZ_{j_1} \otimes EZ_{j_2}) = EZ_j \circ EZ_{0,0}.$$

This shows that, via the identifications $EZ_{j_k} \colon V^{\otimes j_k} \cong \pi_{j_k}(\mathcal{S}_V^{\otimes j_k})$ and $EZ_j \colon V^{\otimes j} \cong \pi_j(\mathcal{S}_V^{\otimes j})$, the multiplication

$$<-,->_{j_1,j_2} \colon \pi_{j_1}(\mathcal{S}_V^{\otimes j_1}) \times \pi_{j_2}(\mathcal{S}_V^{\otimes j_2}) \to \pi_j(\mathcal{S}_V^{\otimes j})$$

becomes the multiplication $V^{\otimes j_1} \times V^{\otimes j_2} \to V^{\otimes j}$, as desired.

$\square$

Recall that if $p > \ell_0$, for any $j \leq \ell_0$, for any A-module W, the largest submodule $\bigwedge^j W$ and the largest quotient $\bigwedge^j W$ of $W^{\otimes j}$ are naturally isomorphic. For this reason, for a graded algebra G, we introduce the truncation

$$\tau_{\leq \ell_0} G = \bigoplus_{j \leq \ell_0} G_j$$

with a partial algebra structure defined only for $g_{j_k} \in G_{j_k}$ such that $j = j_1 + j_2 \leq \ell_0$.

Theorem 14. *For every cuspidal representation π' occuring in $H_{\mathfrak{m}}^\bullet$, for any $m \geq 1$, there is an homomorphism of A_m-modules*

$$(HGV_j) \quad GV_m^j \colon \widetilde{\pi}_j(\mathcal{R}^{\otimes j})_{A_m} \to \bigwedge_{A_m}^j \mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)^{\oplus j}$$

It induces a morphism of truncated graded A_m-algebras

$$(HGV_\bullet) \quad GV_m^\bullet \colon \tau_{\leq \ell_0} \widetilde{\pi}_\bullet(\mathcal{R}^{\otimes \bullet}) \otimes A_m \to \tau_{\leq \ell_0} \bigwedge_{A_m}^\bullet \mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)^{\oplus \bullet}$$

Moreover, for any j, the composition $a_j \circ GV_m^j \circ \mu_j$ of GV_m^j with the tensor shuffle multiplication map $\mu_j \colon \bigwedge_A^j \pi_1(\mathcal{R})_A \to \widetilde{\pi}_j(\mathcal{R}^{\otimes j})_A$ and the homomorphism a_j induced by the addition

$$\mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)^{\oplus j} \to \mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m)$$

coincides with $\bigwedge^j GV_m^1$.
For $j = \ell_0$, the cokernel of $GV_m^{\ell_0} \circ \mu_{\ell_0}$ is annihilated $\mathrm{Fitt}(\mathrm{Sel}(Ad(\rho_{\pi'})))$.

Proof. Let $m \geq 1$. We define first for any $j \geq 1$ a homomorphism

$$GV_m^j \colon \widetilde{\pi}_j(\mathcal{R}^{\otimes j}) \otimes_{\pi_0(\mathcal{R}), \lambda_{\pi'}^{\mathrm{gal}}} \mathcal{O}/\varpi^m\mathcal{O} \to \bigwedge_{\mathcal{O}/(\varpi^m)}^j \mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes \mathcal{O}/\varpi^m\mathcal{O})^{\oplus j}.$$

Let $A = \mathcal{O}/(\varpi^m)$. Let $\rho = \rho_{\pi'} \pmod{\varpi^m}$. We will proceed by Pontryagin duality, and define an A-linear homomorphism $\Theta_{m,V}^j \colon \pi_0 \mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_V)^{\otimes_A j} \to \mathsf{Hom}(\pi_j(\mathcal{R}^{\otimes j}), V^{\otimes j})$. Let $\Phi = \phi_1 \otimes \ldots \otimes \phi_j \in \pi_0 \mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_V)^{\otimes_A j}$. It defines a homomorphism

$$\Phi \colon \mathcal{R}^{\otimes_A j} \to \mathcal{S}_V^{\otimes_A j}$$

and, by passing to the homotopy groups: $\pi_j(\Phi) \colon \pi_j(\mathcal{R}^{\otimes j}) \to \pi_j(\mathcal{S}_V^{\otimes_A j})$. By Lemma 14, we have canonically $\pi_j(\mathcal{S}_V^{\otimes j}) = V^{\otimes j}$. Thus, we can view $\pi_j(\Phi)$ as a homomorphism

$$\pi_j(\Phi) \colon \pi_j(\mathcal{R}^{\otimes j}) \to V^{\otimes j}.$$

and put $\Theta_{m,V}^j(\Phi) = \pi_j(\Phi)$. By Lemma 14 again, the direct sum of these homomorphisms provides a graded algebra homomorphism

$$\Theta_{m,V}^\bullet \colon \pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_V)^{\otimes\bullet} \to \mathsf{Hom}_{gralg}(\pi_\bullet(\mathcal{R}^{\otimes\bullet}), V^{\otimes\bullet})$$

By composing with the quotient morphism $q_j \colon V^{\otimes\bullet} \to \bigwedge^\bullet V$, any graded algebra homomorphism $\pi_\bullet(\mathcal{R}^{\otimes\bullet}) \to V^{\otimes\bullet}$ induces a homomorphism

$$\widetilde{\pi}_\bullet(\mathcal{R}^{\otimes\bullet}) \to \bigwedge{}^\bullet V.$$

For any A-module W, let $\widetilde{\bigwedge}^\bullet W$ be the largest graded-commutative submodule of $W^{\otimes\bullet}$. By restricting $\Phi \mapsto q_j \circ \pi_j(\Phi)$ to $\widetilde{\bigwedge}^\bullet \pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_V)$, we obtain a homomorphism

$$\widetilde{\bigwedge}{}^\bullet \pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_V) \to \mathsf{Hom}_{gralg}(\widetilde{\pi}_\bullet(\mathcal{R}^{\otimes\bullet}), \bigwedge{}^\bullet V).$$

which we still denote by $\Theta_{m,V}^\bullet$. In particular, for every $j \geq 1$, we have a homomorphism

$$\Theta_{m,V}^j \colon \widetilde{\bigwedge}{}^j \pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_V) \to \mathsf{Hom}_A(\widetilde{\pi}_j(\mathcal{R}^{\otimes j})_A, \bigwedge{}^j V).$$

For each fixed j, let us choose $V = V_j := A^j$ so that $\bigwedge^j V_j = A$. We obtain

$$\Theta_{m,V_j}^j \colon \widetilde{\bigwedge}{}^j \pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_{V_j}) \to \mathsf{Hom}_A(\widetilde{\pi}_j(\mathcal{R}^{\otimes j})_A, A).$$

Since $p > \ell_0$, we know that for any A-module W and for any $j \leq \ell_0$, $\widetilde{\bigwedge}^j W$ is a direct factor of $W^{\otimes j}$ and is canonically isomorphic to the largest exterior product quotient $\bigwedge^j W$. Now we remark that

$$\pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_{V_j}) \cong \mathrm{H}_f^2(F, \mathrm{Ad}(\rho_m) \otimes V_j = \mathrm{H}_f^2(F, \mathrm{Ad}(\rho_m)^{\oplus j},$$

hence, its Pontryagin dual is canonically isomorphic to $\mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j}$. Therefore, by passing to the A-dual, we obtain A-linear homomorphisms

$$GV_m^j \colon \widetilde{\pi}_j(\mathcal{R}^{\otimes j})_A \to \left(\bigwedge^j (\mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j})^\vee \right)^\vee$$

For finite $A = \mathcal{O}/(\varpi^m)$-modules, we see by the structure theorem of these modules that the Kronecker homomorphism

$$\bigwedge^j \mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j} \to \left(\bigwedge^j (\mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j})^\vee \right)^\vee$$

is an isomorphism. Hence we can rewrite GV_m^j as

$$GV_m^j \colon \widetilde{\pi}_j(\mathcal{R}^{\otimes j})_A \to \bigwedge^j \mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j}.$$

We now need to show that $GV_m^\bullet = \bigoplus_j GV_m^j$ is a graded-commutative algebra homomorphism. On one hand, let us compute $GV_m^j(<a_1, a_2>_{j_1,j_2})$; let $\Phi_k \in \widetilde{\bigwedge}_k^j \pi_0\,\mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_{V_{j_k}})$, $k = 1, 2$. By composing with the inclusions $\iota_k \colon V_{j_k} \to V_j$, we can view Φ_k as taking values in $\mathcal{S}_{V_j}$ and form $\Phi = \Phi_1 \wedge \Phi_2$; then $GV_m^j(<a_1, a_2>_{j_1,j_2})(\Phi)$ is given by $q_{j_1} \circ \Phi_1(a_1) \wedge q_{j_2} \circ \Phi_2(a_2) \in \bigwedge^j V_j = A$. On the other hand, $GV_m^{j_k}(a_k)$ sends Φ_k to $q_{j_k} \circ \Phi_k(a_k) \in \bigwedge^{j_k} V_{j_k} = A$. By Lemma 14, we conclude that

$$GV_m^j(<a_1, a_2>_{j_1,j_2})(\Phi) = GV_m^{j_k}(a_k)(\Phi_1) \cdot GV_m^{j_k}(a_k)(\Phi_2).$$

Finally, for $j \leq \ell_0$, let $\mu_j \colon \bigwedge^j \pi_1(\mathcal{R}) \to \widetilde{\pi}_j(\mathcal{R}^{\otimes j})$ and

$$a_j \colon \bigwedge^j \mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j} \to \bigwedge^j \mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))$$

be the projection induced by the addition $a_j \colon \mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))^{\oplus j} \to \mathrm{Sel}(\mathrm{Ad}(\rho_m)^*(1))$. Let us show that the resulting map $a_j \circ GV^j_{m,V_j} \circ \mu_j$ coincides with $\bigwedge^j GV^1_m$. For $a_k \in \pi_1(\mathcal{R})$ and $\phi_k \in \pi_0 \, \mathsf{sHom}_{\phi_\rho}(\mathcal{R}, \mathcal{S}_{V_1^{(k)}})$ (where $V_1^{(k)} = A$) for $k = 1 \ldots, j$, we need to compute $GV^j_m(a_1 \cdot \ldots \cdot a_j)(\phi_1 \wedge \ldots \wedge \phi_j)$. We view $A^j = V_j$ as the sum $V_1^{(1)} \oplus \ldots \oplus V_1^{(j)}$. Therefore $\bigwedge^j V_j = V_1^{(1)} \otimes \ldots \otimes V_1^{(j)}$, and by Lemma 14 again, we find

$$GV^j_m(a_1 \cdot \ldots \cdot a_j)(\phi_1 \wedge \ldots \wedge \phi_j) = GV^1_m(a_1)(\phi_1) \cdot \ldots \cdot GV^1_m(a_j)(\phi_j).$$

This implies that the multiplication homomorphism

$$\mu_j \colon \bigwedge^j \pi_1(\mathcal{R}) \to \widetilde{\pi}_j(\mathcal{R}^{\otimes j})$$

is injective since its composition with $a_j \circ GV^j_m$ is equal to $\bigwedge^j_A GV^1_m$ which is injective (since GV^1_m is injective and that a tensor product of A-linear injections is injective). The formula $GV^1_m = GV[\varpi^m]$ is obvious.

We now prove the last statement. Recall that by our assumptions $(\mathrm{Gal}_\mathfrak{m})$ and (LLC), we have $\lambda^{\mathrm{gal}}_{\pi'} = \lambda_{\pi'}$ and that $\mathrm{Im}\,\mu_j$ has bounded index in $\pi_j(\mathcal{R})_A$ when m grows and that the restriction of GV^j_m to this submodule is an injection into $\bigwedge^j_A \mathrm{Sel}(\mathrm{Ad}\,\rho^*_{\pi'}(1)_A)$. By definition of the Fitting ideal, we therefore see that $\mathrm{Coker}\,GV^{\ell_0}_m \circ \mu_{\ell_0}$ is annihilated by $\mathrm{Fitt}(\mathrm{Sel}(\mathrm{Ad}(\rho_{\pi'})))$.

$\square$

Remark 6. *1) As proved above, for any $1 \leq j \leq \ell_0$, the shuffle multiplication μ_j is injective. However it would be interesting to study the injectivity of the homomorphisms GV^j_m.*

2) By Remark 4, 1) above, one can apply the homomorphisms c_j to both sides of (HGV_j) however it is not clear if $GV^\bullet_m$ factors through $c_\bullet$ into a homomorphism

$$\pi_\bullet(\mathcal{R}) \otimes A_m \to \bigwedge^\bullet_{A_m} \mathrm{Sel}(\mathrm{Ad}(\rho_\pi)^*(1) \otimes A_m).$$

We finish this section with an observation. Let $K = Frac(\mathcal{O})$ and $V = K$ be a one dimensional K-vector space. Consider the simplicial K-algebras $\mathcal{S}_V^{\otimes \ell_0}$ and $\mathcal{R}_K = \mathcal{R} \otimes_{\phi_{\rho_{\pi'}}} K$ for a cuspidal representation π' occuring in $\mathrm{H}^\bullet_\mathfrak{m}$. Then we have:

Proposition 6. *There are canonical isomorphisms of graded K-algebras*

$$\pi_\bullet(\mathcal{R}_K) \cong \bigwedge^\bullet_K \pi_1(\mathcal{R})_K$$

and

$$\pi_\bullet(\mathcal{S}_V^{\otimes \ell_0}) \cong \bigwedge^\bullet_K \pi_1(\mathcal{S}_V^{\otimes \ell_0}).$$

By fixing identifications $\pi_1(\mathcal{R})_K = K^{\ell_0} = \pi_1(\mathcal{S}_V^{\otimes \ell_0})$, we have an identification of graded K-algebras

$$\pi_\bullet(\mathcal{R}_K) = \pi_\bullet(\mathcal{S}_V^{\otimes \ell_0}).$$

There exists a weak equivalence of simplicial modules

$$\mathcal{R}_K \sim \mathcal{S}_V^{\otimes \ell_0}$$

If $\ell_0 = 1$, there is a (non canonical) weak equivalence of simplicial K-algebras

$$\phi \colon \mathcal{R}_K \to \mathcal{S}_V^{\otimes \ell_0}.$$

Remark 7. *There exists simplicial K-algebra homomorphisms $\mathcal{R}_K^{\otimes \ell_0} \to \mathcal{R}_K$ (given by the multiplication μ_{ℓ_0}) and $\mathcal{R}_K^{\otimes \ell_0} \to \mathcal{S}_V^{\otimes \ell_0}$. The second is given by the choice of a basis of the ℓ_0-dimensional K-vector space $\mathrm{Sel}(\mathrm{Ad}(\rho_{\pi'}^*)(1))_K^*$. Indeed, this choice gives rise to homomorphisms*

$$\phi_1, \ldots, \phi_{\ell_0} \colon \mathcal{R}_K \to \mathcal{S}_V$$

above $\phi_{\rho_{\pi'}}$. By tensor product, we have a simplicial ring homomorphism

$$\mathcal{R}_K^{\otimes \ell_0} \to \mathcal{S}_V^{\otimes \ell_0}$$

It is not clear if it can be adjusted to factor through the multiplication homomorphism $\mu_{\ell_0} \colon \mathcal{R}_K^{\otimes \ell_0} \to \mathcal{R}_K$. In other words, it is not clear that this second homomorphism factors through the first into a homomorphism of simplicial K-algebras $\mathcal{R}_K \to \mathcal{S}_V^{\otimes \ell_0}$. We can ask whether there exists a zig-zag of weak equivalence maps of simplicial K-algebras

$$\mathcal{R}_K \leftarrow \ldots \to \mathcal{S}_V^{\otimes \ell_0}.$$

In other words, this poses the question of whether $\mathcal{R}_K$ is weakly equivalent to $\mathcal{S}_K^{\otimes \ell_0}$ as simplicial K-algebra. The answer is yes if $\ell_0 = 1$.

Proof. We first note that

$$\mathcal{S}_V^{\otimes \ell_0} = \bigoplus_{j=0}^{\ell_0} \binom{\ell_0}{j} \cdot \mathrm{DK}(V^{\otimes j}[j])$$

hence

$$\pi_\bullet(\mathcal{S}_V^{\otimes \ell_0}) = \bigoplus_{j=0}^{\ell_0} \binom{\ell_0}{j} \cdot \pi_j(\mathrm{DK}(V^{\otimes j}[j])) = \bigoplus_{j=0}^{\ell_0} K^{\oplus \binom{\ell_0}{j}} = \bigwedge^\bullet K^{\oplus \ell_0}$$

as graded K-module. We also know by [GV18] that

$$\pi_\bullet(\mathcal{R}_K) = \bigwedge^\bullet K^{\oplus \ell_0}$$

as graded K-module, hence the simplicial K-modules $\mathcal{R}_K$ and $\mathcal{S}_V^{\otimes \ell_0}$ have the same homotopy graded module. By Section 2.5 of [Ke04], it implies that there exists a zig-zag of quasi-isomorphisms between them.

Actually, by [Cai21, Theorem 5.31], if we put $\mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})_K = \mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O}) \otimes_{\phi_{\rho_{\pi'}}} K$, we have an isomorphism of graded K-algebras $\pi_\bullet(\mathcal{R}_K) = \mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})_K$ and by [GV18, Formula (15.11)], we have an isomorphism of graded K-algebras

$$\mathrm{Tor}_\bullet^{S_\infty}(R_\infty, \mathcal{O})_K = \bigwedge^\bullet \pi_1(\mathcal{R})_K$$

hence we have an isomorphism of graded K-algebras

$$\pi_\bullet(\mathcal{R}_K) = \bigwedge^\bullet K^{\oplus \ell_0}$$

Similarly, by definition of the Eilenberg-Zilber map, the shuffle product

$$\bigwedge^j \pi_1(\mathcal{S}_V^{\otimes \ell_0}) \to \pi_j(c\mathcal{S}_V^{\otimes \ell_0})$$

is induced by the canonical isomorphism of complexes

$$\bigotimes^{j}(K[1]^{\oplus \ell_0}) \to \bigoplus_{k_1,\dots,k_j} K[1]^{\otimes j}$$

(recall $V = K$); this induces an isomorphism

$$\bigwedge^{j}(K[1]^{\oplus \ell_0}) \to \bigoplus_{k_1<\dots<k_j} K[j].$$

this shows that we have a graded algebra isomorphism

$$\bigwedge^{\bullet} \pi_1(\mathcal{S}_V^{\otimes \ell_0}) \to \pi_\bullet(\mathcal{S}_V^{\otimes \ell_0})$$

Hence we conclude that we have graded algebra isomorphisms

$$\pi_\bullet(\mathcal{S}_V^{\otimes \ell_0}) = \bigwedge^{\bullet}(K^{\oplus \ell_0}) = \pi_\bullet(\mathcal{R}_K).$$

For $\ell_0 = 1$, recall that by Proposition 5, we have a morphism $\phi \colon \mathcal{R}_K \to \mathcal{S}_V$ of simplicial K-algebras above ϕ_ρ which induces an isomorphism $\pi_1(\mathcal{R}_K) \cong K$ (this isomorphism is defined up to a scalar). Since $\pi_k(\mathcal{R}_K) = \pi_k(\mathcal{S}_V) = 0$ for $k > 1$, we see that ϕ is a weak equivalence. $\square$

5.1. **The Bloch-Kato conjecture.** Let K_π be a number field over which the cohomological cuspidal representation π of $\mathsf{GL}_N(F)$ is defined. Assume there exists a rank N motive M_π defined over F with coefficients in K_π associated to π. Let $\mathrm{Ad}\, M_\pi = \mathsf{End}(M_\pi)$; let $(\mathrm{Ad}\, M_\pi)^*(1)$ be the twisted dual of $\mathrm{Ad}\, M_\pi$.

A conjecture of Beilinson predicts that the motivic cohomology group

$$\mathrm{H}^1_{mot}(\mathrm{Ad}\, M_\pi^*, K_\pi(1)) = Ext^1_{\mathcal{MM}}(\mathrm{Ad}(M_\pi), K_\pi(1))$$

is of rank ℓ_0 over K_π. Here $\mathcal{MM}$ is the conjectural category of mixed motives. Part of the conjectural framework is the existence of archimedean and p-adic regulator maps:

$$\mathrm{H}^1_{mot}((\mathrm{Ad}\, M_\pi)^*, K_\pi(1)) \xrightarrow{r_p} \mathrm{H}^1_f(F, (\mathrm{Ad}\, \rho_\pi)^*(1))$$

$$\left. r_\infty \right\downarrow$$

$$\mathrm{H}^1_{\mathcal{D}}(\mathrm{Ad}(M_\pi)^*_{\mathbb{R}}, \mathbb{R}(1))$$

where $\mathrm{H}^1_f(F, \mathrm{Ad}(\rho_\pi)^*(1))$ is the Bloch-Kato Selmer group attached to the Galois representation $\mathrm{Ad}(\rho_\pi)^*(1)$ and $\mathrm{H}^1_{\mathcal{D}}$ stands for Deligne cohomology and, for any motive M, $M_{\mathbb{R}}$ denotes the real Hodge structure of its Betti realization $\mathrm{H}_B(M)$. By a standard computation the Deligne cohomology is equal to the cokernel of Deligne's period map

$$c^+_\infty \colon \mathrm{H}_B(\mathrm{Ad}(M_\pi)^*(1))^+_{\mathbb{R}} \longrightarrow \mathrm{H}_{dR}(\mathrm{Ad}(M_\pi)^*(1))_{\mathbb{R}}/F^0\mathrm{H}_{dR}(\mathrm{Ad}(M_\pi)^*(1))_{\mathbb{R}}$$

which is injective because the weight of $\mathrm{Ad}(M_\pi)^*(1)$ is negative. Note that

$$F^+ = F^-\mathrm{H}_{dR}(\mathrm{Ad}(M_\pi)^*(1))_{\mathbb{R}} = F^0\mathrm{H}_{dR}(\mathrm{Ad}(M_\pi)^*(1))_{\mathbb{R}}.$$

Our motive is not critical. From this description, one sees immediately that the Deligne cohomology, namely Coker c^+_∞, has a rational structure and even a p-integral structure if p prime to N and $p > max(HT(\rho_\pi))$ so that the Hodge-Tate weights of $\mathrm{Ad}(\rho_\pi)$ are in the Fontaine-Lafaille range.

A first conjecture by Beilinson and Bloch-Kato is that:

(Bei) The regulator map r_p (resp. r_∞) induces an isomorphism after the extension of scalar to K (resp, to $\mathbb{R}$).

Assuming (Bei), we can define a regulator $R_\pi \in \mathbb{R}^\times / O_{\pi,(p)}^\times$ as follows. Let $L_\pi \subset \rho_\pi$ be a stable $\mathcal{O}$-lattice. Because the Hodge-Tate weights are in the Fontaine-Lafaille range, this lattice induces a lattice of $D_{dR}(\rho_\pi)$ and therefore define a integral $\mathcal{O}_{(\pi,(p))}$- lattice of $\mathrm{H}_{dR}((\mathrm{Ad}\ \rho_\pi)^*(1))$ and of $\mathrm{H}_B((\mathrm{Ad}\ \rho_\pi)^*(1))$ via the comparison isomorphisms c_∞, c_p and c_{dR}. This yields an $\mathcal{O}_{(\pi,(p))}$ -integral structure on $coker(\beta) = \mathrm{H}_{\mathcal{D}}^1((\mathrm{Ad}(M_\pi)^*, \mathbb{R}(1))$. Let $A(F)$ be the inverse image of $\mathrm{H}_f^1(F, \mathrm{Ad}(L_\pi)^*(1))$ via r_p. The archimedean regulator of $(\mathrm{Ad}(M_\pi)^*(1)$ is then defined as the co-volume of the image of $A(F)$ by r_∞ with respect to the Haar measure induced by the $\mathcal{O}_{(\pi,(p))}$ -integral structure defined above. It gives an element

$$R_\pi \in \mathbb{R}^\times / \mathcal{O}_{(\pi,(p))}^\times$$

We now review the definition of the Tate-Shafarevitch group $\mathrm{III}_*^1(F, (\mathrm{Ad}(M_\pi)^*(1))$ following Bloch-Kato ($* =$ ord, f). It depends on the choice of the stable lattice L_π. It is defined as:

$$\mathrm{Ker}\left(\frac{\mathrm{H}^1(F, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})}{\Phi_\mathcal{O} \otimes K/\mathcal{O}} \to \bigoplus_v \frac{\mathrm{H}^1(F_v, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})}{\mathrm{H}_*^1(F_v, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})} \right)$$

where $\Phi_\mathcal{O} = \mathrm{H}_*^1(F, \mathrm{Ad}(\rho_\pi)^*(1)) \cap \mathrm{H}^1(F, \mathrm{Ad}(L_\pi)^*(1))$.

$A(F \otimes \mathbb{R})/A(F)$ with

$$A(F \otimes \mathbb{R}) := D_\mathbb{R}/(F^0 D_\mathbb{R} + W^+)$$

with $D = \mathrm{H}_{dR}((\mathrm{Ad}(M_\pi)^*(1))$, $W = c_p^{-1}(\mathrm{Ad}(L_\pi)(1)) \cap \mathrm{H}_B((\mathrm{Ad}(M_\pi)^*(1))$ and where we have identified $A(F)$ by its image under the regulator map r_∞ We have the exact sequence

$$0 \to W_\mathbb{R}^+/W^+ \to A(F \otimes \mathbb{R})/A(F) \to \mathrm{H}_{\mathcal{D}}^1(\mathrm{Ad}(M_\pi)^*, \mathbb{R}(1))/A(F) \to 0$$

Proposition 7. *Let $p > N$ and $\zeta_p \notin F$. In the Fontaine-Laffaille case, assume (FL). In the ordinary case, make the hypothesis of strong distinguishability (STDIST) : $\alpha \circ \underline{\chi}_v \neq 1, \omega^{-1}$ for any positive root α. Let $* = o, f$ (ordinary or crystalline Fontaine-Laffaille). Then, the Selmer group $\mathrm{H}_{m*}^1(\Gamma, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})$ has $\mathcal{O}$-corang ℓ_0. Moreover $\mathrm{III}_*^1(\Gamma, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})$ and $\mathrm{H}_{m*}^1(\Gamma, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O}))$ have same $\mathcal{O}$-Fitting ideal.*

Proof. Since the $\mathcal{O}$-algebra $R = \mathrm{T}$ is finite, we know that $\mathrm{H}_{m*}^1(\Gamma, \mathrm{Ad}(\rho_\mathrm{T}) \otimes K/\mathcal{O}))$ and $\mathrm{H}_{m*}^1(\Gamma, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O}))$ are finite. Let us write $W_n := \mathrm{Ad}(\rho_\pi) \otimes \varpi^{-n}\mathcal{O}/\mathcal{O} = \mathfrak{g}_n$ and $q = \#\mathcal{O}/(\varpi)$. Since $\mathrm{H}^0(F, W_n) = \mathcal{O}/(\varpi^n)$ and $\mathrm{H}^0(F, W_n(1)) = \{0\}$ because $\zeta_p \notin F$, we have by the Euler-Poincaré characteristic formula due to Greenberg-Wiles

$$\frac{\#\mathrm{H}_{m*}^1(F, W_n)}{\#\mathrm{H}_{m*}^1(F, W_n^*(1))\#\mathrm{H}^0(F, W_n)} = \frac{1}{\prod_{v|\infty} \#\mathrm{H}^0(F_v, W_n)} \cdot \prod_{v|N} \frac{\#\mathrm{H}_{unr}^1(F_v, W_n)}{\#\mathrm{H}^0(F_v, W_n)} \cdot \prod_{v|p} \frac{\#\mathrm{H}_*^1(F_v, W_n)}{\#\mathrm{H}^0(F_v, W_n)}$$

Here $m*$ means minimal and $*$. We note that we have $\#\mathrm{H}_{unr}^1(F_v, W_n) = \#\mathrm{H}^0(F_v, W_n)$ for all $v|N$. Moreover, for each $v|\infty$, $\mathrm{H}^0(F_v, W_n) = (\mathcal{O}/\varpi^n\mathcal{O})^{N^2}$.

(1) The case $* = $ ord.

For any $v|p$, let us compute $\frac{\#\mathrm{H}_{ord}^1(F_v, W_n)}{\#\mathrm{H}^0(F_v, W_n)}$. Let $\mathfrak{b} = \mathrm{Lie}(B; \mathcal{O})$, $\mathfrak{n} = \mathrm{Lie}(U_B; \mathcal{O})$, $\mathfrak{g} = \mathrm{Lie}(G; \mathcal{O})$ and $\mathfrak{b}_n = \mathfrak{b} \otimes \mathcal{O}/\varpi^n\mathcal{O}$, $\mathfrak{n}_n = \mathfrak{n} \otimes \mathcal{O}/\varpi^n\mathcal{O}$; consider the fiber product

$$L_v' = \mathrm{H}^1(F_v, \mathfrak{b}_n) \times_{\mathrm{H}^1(F_v, \mathfrak{b}_n/\mathfrak{n}_n)} \mathrm{H}_{unr}^1(F_v, \mathfrak{b}_n/\mathfrak{n}_n)$$

Then, we have $\mathrm{H}_{ord}^1(F_v, W_n) = \mathrm{Im}(L_v' \to \mathrm{H}^1(F_v, W_n))$.

We can insert L_v' in the long exact sequence

$$0 \to \mathrm{H}^0(F_v, \mathfrak{n}_n) \to \mathrm{H}^0(F_v, \mathfrak{b}_n) \to \mathrm{H}^0(F_v, \mathfrak{b}_n/\mathfrak{n}_n) \to \mathrm{H}^1(F_v, \mathfrak{n}_n) \to L_v' \to \mathrm{H}^1(G_{F_v}/I_v, \mathfrak{b}_n'/\mathfrak{n}_n) \to 0$$

Hence, taking the orders of the terms of the sequence, we have :

$$\#L_v' \cdot (\#\mathrm{H}^0(F_v, \mathfrak{b}_n))^{-1} = \#\mathrm{H}^1(G_{F_v}/I_v, \mathfrak{b}_n/\mathfrak{n}_n) \cdot (\#\mathrm{H}^0(F_v), \mathfrak{b}_n/\mathfrak{n}_n))^{-1} \cdot \#\mathrm{H}^1(F_v, \mathfrak{n}_n) \cdot (\#\mathrm{H}^0(F_v, \mathfrak{n}_n))^{-1}$$

We have

$$\#\mathrm{H}^1(G_{F_v}/I_v, \mathfrak{b}_n/\mathfrak{n}_n) \cdot (\#\mathrm{H}^0(F_v, \mathfrak{b}_n/\mathfrak{n}_n))^{-1} = 1$$

$$\#\mathrm{H}^1(G_v, \mathfrak{n}_n) \cdot (\#\mathrm{H}^0(G_v, \mathfrak{n}_n))^{-1} = \#\mathfrak{n}_n^{[F_v\colon \mathbb{Q}_p]} \cdot \#\mathrm{H}^0(G_v, \mathfrak{n}_n^*(1))$$

and $\#\mathrm{H}^0(G_v, \mathfrak{n}_n^*(1)) = 1$ by strong distinguishability. Note also for the same reason that $\mathrm{H}^0(F_v, W_n) = \mathrm{H}^0(F_v, \mathfrak{b}_n)$ (because $\mathrm{H}^0(F_v, W_n/\mathfrak{b}_n) = 0$). Since $L'_v \to L_v$ is injective by strong distinguishability, we conclude that for every $v|p$, $\#\mathrm{H}^1_{\mathrm{ord}}(F_v, W_n) \cdot (\#\mathrm{H}^0(F_v, W_n))^{-1} = q^{(\frac{N(N-1)}{2}) \cdot n[F_v\colon \mathbb{Q}_p]}$.

We have $\sum_{v|p}[F_v\colon \mathbb{Q}_p] = 2d_0$. Note that $d_0 N^2 - 1 - 2d_0(\frac{N(N-1)}{2}) = Nd_0 - 1$. Recall that $\ell_0 = Nd_0 - 1$.
Therefore, we get

$$\tag{2} \#\mathrm{H}^1_{mo}(\Gamma, W_n^*(1)) = q^{n\ell_0} \cdot \#\mathrm{H}^1_{mo}(\Gamma, W_n)$$

where the sequence $(\#\mathrm{H}^1_{mo}(\Gamma, W_n))_n$ is bounded.

(2) The Fontaine-Laffaille case.

For any $v|p$, let us compute $\frac{\#\mathrm{H}^1_f(F_v, W_n)}{\#\mathrm{H}^0(F_v, W_n)}$. Let $\mathbb{G}_v$ be the Fontaine-Laffaille functor defined in [CHT08, Section 2.4.1)]. Let $V_n = \rho_\pi \pmod{\varpi^n}$ and $M_n = \mathbb{G}(V_n)$.

We have

$$\#\mathrm{H}^1_f(F_v, W_n) \cdot (\#\mathrm{H}^0(F_v, W_n))^{-1} = \#\operatorname{Ext}^1_{MF}(M_n, M_n) \cdot (\#\operatorname{\mathsf{Hom}}_{MF}(M_n, M_n))^{-1}.$$

We apply [CHT08, Lemma 2.4.2 and Corollary 2.4.3] for $M = N = M_n$ and we obtain

$$= q^{(\frac{N(N-1)}{2}) \cdot n[F_v\colon \mathbb{Q}_p]}.$$

Note that $d(N^2 - 1) - 2d(\frac{N(N-1)}{2}) = (N-1)d = \ell_0$. Therefore,

$$\tag{3} \#\mathrm{H}^1_{mf}(\Gamma, W_n^*(1)) = q^{n\ell_0} \cdot \#\mathrm{H}^1_{mf}(\Gamma, W_n)$$

In case (1) and (2), we conclude by passing to the inductive limit on n. Let $\varinjlim_n W_n = W_\infty = \operatorname{Ad} \rho_\pi \otimes K/\mathcal{O}$ and $\varinjlim_n W_n^*(1) = W_\infty^*(1)$. Then by (2) and (3), there exists $c \geq 0$ such that for all n sufficiently large, the order of

$$\mathrm{H}^1_{m*}(\Gamma, W_\infty^*(1))[\varpi^n] = \mathrm{H}^1_{m*}(\Gamma, W_n^*(1))$$

is $q^{n\ell_0+c}$. Passing to the Pontryagin dual and using Nakayama's lemma one sees easily that this implies there exists an exact sequence

$$0 \to \Phi_\mathcal{O} \otimes \varpi^{-n}\mathcal{O}/\mathcal{O} \to \mathrm{H}^1_{m*}(\mathbb{Q}, W_n(1)) \to \mathrussic{Ш}^1_*(\Gamma, \operatorname{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})[\varpi^n] \to 0$$

where $\Phi_\mathcal{O}$ is a finitely generated $\mathcal{O}$-module of rank ℓ_0. Note that $\Phi_\mathcal{O}$ can be supposed to be free and that for n sufficiently large,

$$\mathrussic{Ш}^1_*(\Gamma, \operatorname{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})[\varpi^n] = \mathrussic{Ш}^1_*(\Gamma, \operatorname{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O}).$$

From (2) and (3), we indeed deduce that $\#\mathrussic{Ш}^1_*(F, \operatorname{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O}) = \#\mathrm{H}^1_{m*}(\Gamma, W_n)$ for all $n > 0$ sufficiently large. $\qquad\square$

For any finite S-ramified extension F'/F, let S' the set of places of F' above S and $U^S_{F'}$ be the image of $\prod_{v' \notin S'} \mathcal{O}^\times_{F'_{v'}}$ in the group C_F of idèle classes of F. Let $C_S(F') = C_{F'}/U^S_{F'}$, $C_S = \varinjlim_{F' \subset F_S} C_S(F')$. It is known [NSW99, Sect.X.9] that C_S is divisible (assuming $S_p \subset S$). Let $C_S(p)$ be the p-divisible part of C_S. Moreover, there is an isomorphism [NSW99, Proposition 8.1.21 and Proposition 8.3.8]:

$$tr\colon \mathrm{H}^2(\Gamma, C_S(p)) \cong \mathbb{Q}_p/\mathbb{Z}_p$$

Let us write $W_n := \operatorname{Ad}(\rho_\pi) \otimes \varpi^{-n}\mathcal{O}/\mathcal{O}$ as in the previous Lemma. By [NSW99, Theorem 3.4.6], we have a perfect pairing induced by the cup-product and trace map:

$$\mathrm{H}^1(\Gamma, W_n^*(1)) \times \mathrm{H}^1(\Gamma, W_n) \to \mathrm{H}^2(\Gamma, C_S(p) \otimes K/\mathcal{O}) \cong K/\mathcal{O}$$

by using Tate local duality, we obtain by restriction a pairing

$$< \bullet, \bullet >_n \colon \mathrm{H}^1_{m*}(\Gamma, W_n^*(1)) \times \mathrm{H}^1_{m*}(\Gamma, W_n) \to K/\mathcal{O}$$

which is non-degenerate on the right: if $< x, y >_n = 0$ for any x, then $y = 0$. We have seen in the proof of Lemma 7 that

$$\mathrm{H}^1_{m*}(\Gamma, W_\infty^*(1))[\varpi^n] = \mathrm{H}^1_{m*}(\Gamma, W_n^*(1))$$

Therefore, by passing to the direct limit over n, we obtain a pairing

$$< \bullet, \bullet > \colon \mathrm{H}^1_{m*}(\Gamma, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O}) \times \mathrm{H}^1_{m*}(\Gamma, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O}) \to K/\mathcal{O}$$

which is non-degenerate on the right.

Corollary 3. *With the notations of Lemma 7, for $* = FL, \mathrm{ord}$, the pairing $< \bullet, \bullet >$ induces a perfect pairing*

$$\mathrm{III}^1_*(\Gamma, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O}) \times \mathrm{H}^1_{m*}(\Gamma, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O}) \to K/\mathcal{O}$$

between two finite $\mathcal{O}$-modules.

Proof. If $x \in \mathrm{H}^1_{m*}(\Gamma, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O})$ is divisible, we see that $< x, y >=< \varpi^n x_n, y >=< x_n, \varpi^m y >= 0$ for n sufficiently large (because $\mathrm{H}^1_{m*}(\Gamma, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O})$ is finite). Hence, the pairing factors as

$$\mathrm{III}^1_*(\Gamma, \mathrm{Ad}(\rho_\pi)^*(1) \otimes K/\mathcal{O}) \times \mathrm{H}^1_{m*}(\Gamma, \mathrm{Ad}(\rho_\pi) \otimes K/\mathcal{O}) \to K/\mathcal{O}.$$

By Lemma 7, the two groups have the same order and the pairing is non-degenerate on the right. Therefore it is perfect. $\square$

6. Galatius and Venkatesh Theory for Hida families

6.1. Nearly ordinary Hida-Hecke algebra.
Let E be an imaginary quadratic field in which p splits and F^+ be a totally real field; let $d_0 = [F^+ \colon F]$. We consider the CM field $F = F^+ E$. It has a natural CM type (embeddings fixing E). It is for this type of CM fields that the local properties of the Scholze Galois representation are established (at least modulo a nilpotent ideal of explicitly bounded nilpotency exponent). Since p splits in E, say $p\mathcal{O}_E = \mathfrak{p}\mathfrak{p}^c$. Thus, F has also a natural p-adic CM type, namely the set of primes above $\mathfrak{p}$.

Let $B = TU_B$ be the upper triangular Borel subgroup of the F-algebraic group $G = \mathsf{GL}(N)$. Let S_p^+ be the set of primes obove p in F^+. We assume that all the places $v \in S_p^+$ split in F. Let $U = U_0(\mathfrak{n})$ and for each $m \geq 0$

$$U_1(p^m) = \{g \in U, g \pmod{p^m} \in U_B(\mathcal{O}_{F,p}/(p^m))\}$$

Let $Y_1(p^m) = G(F) \backslash X_G \times G(F_f)/U_1(p^m)$; we write $Y = Y_1(1)$. Let $\pi_m \colon Y_1(p^m) \to Y_1(p^{m-1})$ be the transition maps. They are finite. These locally symmetric spaces form a projective system $Y_1(p^\infty) \to Y$ of Galois group $G(\mathcal{O}_{F,p})$. Let $\mathrm{H}^\bullet(Y_1(p^\infty), \mathcal{O})$ be the projective limit of the $\mathcal{O}$-modules $\mathrm{H}^\bullet(Y_1(p^m), \mathcal{O})$ for the transition maps $\mathrm{Tr}_{\pi_m} \colon \mathrm{H}^\bullet(Y_1(p^m), \mathcal{O}) \to \mathrm{H}^\bullet(Y_1(p^{m-1}), \mathcal{O})$. Let $h(Y_1(p^\infty), \mathcal{O})$ be the closed $\mathcal{O}$-algebra generated by the Hecke operators outside $\mathfrak{n}$ acting faithfully on $\mathrm{H}^\bullet(Y_1(p^\infty), \mathcal{O})$. Let U_p be the Hecke operator associated to $\mathrm{diag}(p^{N-1}, p^{N-2}, \ldots, 1)$ and e be the idempotent of the Hecke algebra $h(Y_1(p^\infty), \mathcal{O})$ associated to U_p. In particular, the torus $T(\mathcal{O}_{F,p})$ acts on $\mathrm{H}^\bullet(Y_1(p^\infty), \mathcal{O})$ by right translation. This action factors through the quotient by the closure of central global units: $T(\mathcal{O}_{F,p})/\overline{Z(\mathcal{O}_F)}$. Actually, the class group $Cl_{U,p^\infty} = F_\mathbb{A}^\times/\overline{F}^\times \cdot (U^p \cap Z(F_f)) \cdot F_\infty^\times$ (where $Z \subset T \subset G$ denotes the center) also acts by right translation on $Y_1(p^\infty)$. This gives an action of the Hida-Iwasawa group H defined as the amalgamated product $\mathrm{H} = T(\mathcal{O}_{F,p}) \times_{Z(\mathcal{O}_{F,p})} Cl_{U,p^\infty}$.

We can decompose $\mathrm{H} = \mathrm{H}_t \times \mathbf{W}$, where H_t is finite and $\mathbf{W}$ is a free $\mathbb{Z}_p$-module of rank $2Nd - (d - 1 - \delta) = 2(N-1)d + d + 1 + \delta$ where δ is the Leopoldt defect for (F, p).

Definition 7. *The completed group algebra $\Lambda = \mathcal{O}[[\mathbf{W}]]$ is called the Hida-Iwasawa algebra.*

Let $\mathbb{H}^\bullet = e\mathrm{H}^\bullet(Y_1(p^\infty), \mathcal{O})$. Let $I \subset G(\mathcal{O}_{F,p})$ be the Iwahori subgroup and $N = U_B(\mathcal{O}_{F,p})$; let $\mathcal{C}(I/N, \mathcal{O})$ be the Banach $\mathcal{O}$-module of continuous functions on I/N. Let $\mathbb{D}$ be the $\mathcal{O}$-linear dual of $\mathcal{C}(I/N, \mathcal{O})$. It is a ring for the convolution multiplication: $< f, d * d' > = << f(gh), d_g >, d'_h >$. It is called the ring of integral distributions (or measures) on I/N.

For any p-adic weight $\lambda \in Hom_{cont}(T(\mathcal{O}_{F,p}), \mathcal{O})$, we denote by $\mathbb{D}_\lambda(\mathcal{O})$ the continuous $\mathcal{O}$-dual of $\mathcal{C}_\lambda(I/N, \mathcal{O})$ which is the sub-module of continuous functions $f \in \mathcal{C}(I/N, \mathcal{O})$ such that $f(tg) = \lambda(t)f(g)$ for all $t \in T(\mathcal{O}_{F,p})$ and $g \in I$.

By applying Poincaré duality to the result [H98, (6.14)], we have that
$$e\mathrm{H}^{q_s}(Y_1(p^\infty), \mathcal{O}) = e\mathrm{H}^{q_s}(Y_0(p), \mathbb{D}).$$

Let λ be a dominant weight of T and V_λ the irreducible representation of highest weight λ. Let $\lambda^\vee$ be the highest weight of the dual of V_λ.

Definition 8. *We say that $\lambda\colon T(\mathcal{O}_{F,p}) \to \mathcal{O}^\times$ is arithmetic if it is dominant algebraic and it is trivial on (the p-adic closure of) $Z(\mathcal{O}_F)$. In other words, it defines an $\mathcal{O}$-algebra homomorphism $\phi_\lambda\colon \Lambda \to \mathcal{O}$. The prime ideal $P_\lambda = \mathrm{Ker}\,\phi_\lambda$ of Λ is also called arithmetic.*

For an arithmetic character λ, the inclusion $V_{\lambda^\vee}(\mathcal{O}) \hookrightarrow \mathcal{C}_\lambda(I/N, \mathcal{O}) \subset \mathcal{C}(I/N, \mathcal{O})$ provides by duality a $\mathcal{O}[U]$-linear homomorphism $\mathbb{D} \to \mathbb{D}_\lambda(\mathcal{O}) \to V_\lambda(\mathcal{O})$ hence a Λ-linear homomorphism
$$e\mathrm{H}^{q_s}(Y_0(p), \mathbb{D}) \to e\mathrm{H}^{q_s}(Y_0(p), V_\lambda(\mathcal{O})).$$
By [H95, Theorem 5.2] or [H98, Theorem 6.2], we have an isogeny
$$\mathbb{H}^{q_s}/P\mathbb{H}^{q_s} \to e\mathrm{H}^{q_s}(Y_0(p), V_\lambda(\mathcal{O})).$$

This implies that $\mathbb{H}^{q_s}$ is a finitely generated Λ-module. But for F CM, it is a torsion Λ-module (see [H98, Section 6]).

Let π be a p-ordinary cuspidal cohomological automorphic representation occuring in $e\mathrm{H}^{q_s}(Y_0(p), V_\lambda(\mathcal{O})) \otimes \mathbb{C}$. Assume its Hecke values belong to $\mathcal{O}$. Its Hecke eigensystem $\theta_\pi\colon h_\lambda(Y_0(p), \mathcal{O}) \to \mathcal{O}$ gives rise by composition with $h(Y_1(p^\infty), \mathcal{O}) \to h_\lambda(Y_0(p), \mathcal{O})$ to a homomorphism $\theta\colon eh(Y_1(p^\infty), \mathcal{O}) \to \mathcal{O}$. Let $\mathfrak{m}$ be the maximal ideal of the nearly ordinary Hecke algebra $eh(Y_1(p^\infty), \mathcal{O})$ given by θ modulo ϖ. We assume it is non Eisenstein. Let $\mathrm{T}_h = eh(Y_1(p^\infty), \mathcal{O})_\mathfrak{m}$ and $\mathrm{T}_\lambda = eh_\lambda(Y_0(p), \mathcal{O})_\mathfrak{m}$. We have an algebra homomorphism $\Lambda \to \mathrm{T}_h$.

Definition 9. *A prime ideal $\mathfrak{P}$ of T_h is called of weight λ if its inverse image in Λ is arithmetic P_λ and we write $\mathfrak{P}|P_\lambda$. It is said arithmetic if λ is an arithmetic weight. The prime $\mathfrak{P}$ is called Hida-automorphic if $\mathfrak{P} \in \mathrm{Supp}_{\mathrm{T}_h}(\mathbb{H}^\bullet(Y_0(p), \mathbb{D}_\lambda(\mathcal{O}))$. We denote by $\widetilde{\Sigma}_h$ the set of Hida-automorphic primes and by $\widetilde{\Sigma}_h^a \subset \widetilde{\Sigma}_h$ its subset of arithmetic Hida-automorphic primes. We also introduce subset Σ_h and Σ_h^a of $Spec(\Lambda)$ defined as the respective images of $\widetilde{\Sigma}_h$ and $\widetilde{\Sigma}_h^a$ by the map $Spec(\mathrm{T}_h) \to Spec(\Lambda)$.*

We can interpret these sets in terms the support over Λ or $\mathbf{T}_h$ of the cohomology with coefficient in $\mathbb{D}$.

Lemma 15. *We have*
$$\tilde{\Sigma}_h = \mathrm{Supp}_{\mathrm{T}_h} H^\bullet(Y_0(p), \mathbb{D})) \cap Spec(\mathrm{T}_h)(\overline{\mathbb{Q}}_p)$$
and
$$\Sigma_h = \mathrm{Supp}_\Lambda(\mathbb{H}^\bullet(Y_0(p), \mathbb{D})) \cap Spec(\Lambda)(\overline{\mathbb{Q}}_p)$$

Proof. It is sufficient to prove the second statement. Let $P_\lambda \in Spec(\Lambda)(\mathcal{O})$. We have a Spectral squence
$$Tor_j^\Lambda((H^i(Y_0(p), \mathbb{D})_{P_\lambda}, \Lambda/P_\lambda) \Rightarrow H^{i-j}(Y_0(p), \mathbb{D}_\lambda(\mathcal{O}))) \otimes Frac(\mathcal{O})$$
If $\lambda \in \Sigma_h$, then clearly one of the terms of the spectral sequence does not vanish. It therefore implies that $H^i(Y_0(p), \mathbb{D}))_{P_\lambda} \neq 0$ for some i and thus
$$\Sigma_h \subset \mathrm{Supp}_\Lambda(\mathbb{H}^\bullet(Y_0(p), \mathbb{D})) \cap Spec(\Lambda)(\overline{\mathbb{Q}}_p)$$

Conversely if $P_\lambda \in \mathrm{Supp}_\Lambda(\mathbb{H}^\bullet(Y_0(p), \mathbb{D})) \cap Spec(\Lambda)(\overline{\mathbb{Q}}_p)$, let q_λ the largest integer i such that $H^i(Y_0(p), \mathbb{D}))_{P_\lambda} \neq 0$. Then from the above spectral, we have

$$H^{q_\lambda}Y_0(p), \mathbb{D}))_{P_\lambda} \otimes \Lambda/P_\lambda \cong H^{q_\lambda}(Y_0(p), \mathbb{D}_\lambda) \otimes Frac(\mathcal{O}) \neq 0.$$

So $P_\lambda \in \Sigma_h$.

$\square$

6.2. **Minimal ordinary deformation rings over Λ.** We consider the problem $\mathcal{D}_h$ of minimal (Λ-)ordinary deformations of $\overline{\rho}$.

A deformation $\rho \colon \Gamma \to \widehat{G}(A)$ is called minimal at $v \in S$, if there exists $g_v \in \widehat{G}(A)$ such that for any $\sigma \in I_v$, $g_v \cdot \rho(\sigma) \cdot g_v^{-1} = \exp(t_v(\sigma)J)$ where J is the standard regular nilpotent Jordan matrix and $t_v \colon I_v \to \mathbb{Z}_p(1)$ is the p-adic tame inertia homomorphism.

It is called (Λ-)ordinary at $v \in S_p$ if there exists $g_v \in \widehat{G}(A)$ such that $g_v \cdot \rho \cdot g_v^{-1} \colon G_{F_v} \to \widehat{G}(A)$ takes values in the standard Borel $\widehat{B}$ and its reduction modulo its unipotent radical $\widehat{N} \colon \chi_\rho \colon G_{F_v} \to \widehat{T}(A)$, is a lifting of the regular homomorphism $\overline{\chi}_{\overline{\rho}} \colon G_{F_v} \to \widehat{T}(k)$.

Note that in the definition of Λ-ordinary deformations, contrary to $\underline{\chi}_\pi$-ordinary deformations, the Hodge-Tate weights are left variable.

By [CHT08, Section 2.4.4] or [Ti96, Chapt.6], $\mathcal{D}_h$ is pro-representable. Let R_h be the universal deformation ring for minimal ordinary deformations. We also have a group homomorphism $\mathbb{H} \to R_h^\times$ defined by duality: Let ρ be the universal deformation $\Gamma \to \widehat{G}(R_h)$. For each place $v \in S_p$, we have a homomorphism

$$\chi_{\rho,v} \colon G_v = \mathrm{Gal}(\overline{F}_v/F_v) \to \widehat{T}(R_h) = \widehat{B}(R_h)/U_{\widehat{B}}(R_h)$$

obtained by reduction mod. $U_{\widehat{B}}(R_h)$ of a conjugate of $\rho|_{G_{\widetilde{v}}}$. By class field theory, it gives rise to $\mathcal{O}_{F_v}^\times \to \widehat{T}(R_h)$ which can be interpreted as a homomorphism $T(\mathcal{O}_{F_v}) \to R_h^\times$. Putting all places together we have $T(\mathcal{O}_{F,p}) \to R_h^\times$. Using the determinant $\det \rho$, we also have a homomorphism $Cl_{U,p^\infty} \to R_h^\times$; both coincide on $Z(\mathcal{O}_{F,p})$. On the other hand, by [Sch15] and [CGH+19] and [ACC+18], for each dominant weight λ, there exists a continuous Galois representation

$$\rho_\lambda \colon \Gamma \to \widehat{G}(\mathrm{T}_\lambda)$$

which is minimal and ordinary. Since the residual representation $\overline{\rho}$ is absolutely irreducible, these representation glue into a deformation of $\overline{\rho}$

$$\rho_\mathrm{T} \colon \Gamma \to \widehat{G}(\mathrm{T}_h)$$

which is minimal and p-ordinary. This is due to Carayol for $\widehat{G} = \mathsf{GL}_N$; for the general case, see [BHKT19, Theorem 4.10]. It gives rise to a surjective ring homomorphism $\phi \colon R_h \to \mathrm{T}_h$.

For Galois representations coming from automorphic representations on a unitary group $U(N)$, it is well-known that it is Λ-linear (see [Ge19, Cor.2.5], [HT17, Proposition 2.8]) For Galois representations coming from the cohomology of GL_N by [Sch15] and [CGH+19], the Λ-linearity follows from [ACC+18, Theorem 5.5.1] (at least modulo a nilpotent ideal of nilpotency order explicitely bounded in terms of d and N).

In a manner similar to [CaGe18, Section 5], we introduce a Taylor-Wiles system $\{Q_m\}$ of finite sets of finite places of F in order to define rings S_∞^Λ, $R_{h,\infty}$ and a ring homomorphism $\alpha \colon S_\infty^\Lambda \to R_{h,\infty}$. Let $r = \dim \mathrm{H}^1_\mathcal{L}(\Gamma, (\mathrm{Ad}\,\overline{\rho})^*(1))$. An element $t \in \widehat{T}(k)$ is called strongly regular if for any root $\alpha^\vee$ of $\widehat{T}$, $\alpha^\vee(t) \neq 1$. Let $\mathcal{L} = (L_v)_{v\,finite}$ where

-for $v \in S_p$, $L_v = \mathrm{Im}(L'_v \to \mathrm{H}^1(F_v, \mathfrak{g}))$ where $L'_v = \mathrm{Ker}(\mathrm{H}^1(G_{F_v}, \mathfrak{b}) \to \mathrm{H}^1(I_v), \mathfrak{b}/\mathfrak{n}))$

-for $v \notin S_p$, $L_v = \mathrm{H}^1_{unr}(G_{F_v}, \mathrm{Ad}\,\overline{\rho})$, and

For a finite set Q disjoint of $S \cup S_p$, we write $\mathcal{L}_Q = (L_v^Q)_{v\,finite}$ where L_v^Q is as above for $v \in S_p$ or $v \notin S_p \cup Q$, and $L_v^Q = \mathrm{H}^1(G_{F_v}, \mathfrak{g})$ for $v \in Q$.

Definition 10. *A finite set Q of finite places of F is called a Taylor-Wiles set if '*

- $Q \cap (S \cup S_p) = \emptyset$, $\sharp Q = r$,
- for any $v \in Q$, $Nv \equiv 1 \pmod{p}$
- for any $v \in Q$, $\overline{\rho}(\mathrm{Frob}_v)$ is conjugate to a strongly regular element of $\widehat{T}(k)$
- we have $\mathrm{H}^1_{\mathcal{L}^\perp_{\overline{Q}}}(G_{F,S \cup S_p \cup S_Q}, Ad(\overline{\rho})^*(1)) = 0$.

As in [CHT08, 2.5], the existence of such sets follows from the assumption of big image of $\overline{\rho}$.

Let Q be such a set; for $v \in Q$, let Δ_v, resp. $T(k(v))^p$, be the p-Sylow, resp. the prime to p part, of $T(k(v))$, where $k(v)\mathcal{O}_{F_v}/(\varpi_v)$ is the residue field of $_F$ at v. We have canonically $T(k(v)) = \Delta_v \times T(k(v))^p$.

Let $\Delta_Q = \prod_{v \in Q} \Delta_v$.

The p-rank of Δ_Q is Nr. Let $R_{h,Q}$ be the universal deformation ring of $\overline{\rho}$ representing the covariant functur $\mathcal{D}_{h,Q} \colon {}_{\mathcal{O}}\mathrm{Art}_k \to \mathsf{SETS}$ of deformations ρ which are minimal ordinary and unramified outside $S \cup Q$. As above, it comes with a natural Λ-algebra structure $\Lambda \to R_{h,Q}$. For each $v \in Q$, let us fix an ordering $\underline{\alpha}_v = (\alpha_{v,1}, \ldots, \alpha_{v,N})$ of the distinct roots of $\overline{\rho}(\mathrm{Frob}_v)$ in k. Let $\rho_Q \colon G_{F,S \cup S_p \cup S_Q} \to \widehat{G}(R_{h,Q})$ be the universal Galois representation. Then for each $v \in Q$ one can diagonalize $\rho_Q(\mathrm{Frob}_v)$ in a basis adapted to the ordering $\underline{\alpha}_v$. An immediate generalization of the induction in [TW95, Lemma 7] (mentioned in [ACC+18, Lemma 6.2.19]) shows that for any fixed $v \in Q$, the restriction $\rho_{h,Q}|_{G_{F_v}}$ to a decomposition subgroup at $v \in Q$ takes values after conjugation, in $\widehat{T}(R_{h,Q})$. Moreover, for any $v \in Q$, the restriction of the homomorphism $G_{F_v} \to \widehat{T}(R_{h,Q})$ to the inertia subgroup I_v of G_{F_v} can be viewed as a homomorphism $T(k(v)) \to R^\times_{h,Q}$ of p-power order. We thus obtain a group homomorphism $\Delta_Q \to R^\times_{h,Q}$, hence an $\mathcal{O}$-algebra homomorphism

$$\alpha_Q \colon \Lambda[\Delta_Q] \to R_{h,Q}.$$

Let I_Q be the augmentation ideal of $\Lambda[\Delta_Q]$. the natural homomorphism $R_{h,Q} \to R_{h,\emptyset} = R_h$ induces an isomorphism $R_{h,Q}/I_Q R_{h,Q} \cong R_h$.

Let I_v, resp. I_v^+, be the Iwahori, resp. the pro-p Iwahori subgroup of $\mathsf{GL}_N(\mathcal{O}_{F_v})$.

Let $U_0(Q) = U^Q \times \prod_{v \in Q} I_v$, $\quad U_1(Q) = U^Q \times \prod_{v \in Q} I_v^+$ For any $v \in Q$, we have a canonical isomorphism $i_v \colon I_v/I_v^+ \cong T(k(v))$. Let U_Q be the level subgroup $U_1(Q) \subset U_Q \subset U_0(Q)$ such that via the isomorphism $i_Q = \prod_{v \in Q} i_v$ we have

$$U_Q/U_1(Q) \cong \prod_{v \in Q} T(k(v))^p.$$

Let $Y_{0,1}(Q, p^\infty)$, resp. $Y_1^Q(p^\infty)$, be the provariety associated to the level group $U_0(Q) \cap U_1(p^\infty)$, resp. $U_Q \cap U_1(p^\infty)$. We denote by $< \bullet >_Q$ the isomorphism

$$< \bullet >_Q \colon \Delta_Q \cong \mathrm{Gal}(Y_1^Q(p^\infty)/Y_{0,1}(Q, p^\infty))$$

Let $h^-(Y_1^Q(p^\infty), \mathcal{O})$ be the Hecke algebra outside $S \cup Q$ acting faithfully on $e\mathrm{H}^{q_s}(Y_1^Q(p^\infty), \mathcal{O})$. There is a natural surjective algebra homomorphism

$$h^-(Y_1^Q(p^\infty), \mathcal{O}) \to h(Y_1(p^\infty), \mathcal{O})$$

We still denote by $\mathfrak{m}$ the inverse image of the maximal ideal $\mathfrak{m}$ by this homomorphism.

For $v \in Q$, let $\alpha_{v,i} \in k_v^\times$ $(i = 1, \ldots, N)$ be the (simple) roots of $\mathrm{char}(\overline{\rho}(\mathrm{Frob}_v))$. Let $\widetilde{\alpha_{v,i}} \in \mathrm{T}_h$ the simple root of $\mathrm{char}(\rho_h(\mathrm{Frob}_v))$ lifting $\alpha_{v,i}$. Let $U_{v,i} = [U_Q t_{v,i} U_Q]$ where $t_{v,i} = \mathrm{diag}(\varpi_v \cdot 1_i, 1_{n-i})$. Let

$$h(Y_1^Q(p^\infty), \mathcal{O}) = h^-(Y_1^Q(p^\infty), \mathcal{O})[U_{v,i} (v \in Q, i = 1, \ldots, N-1), < x >_Q, x \in \Delta_Q]$$

We define the maximal ideal $\mathfrak{m}_Q$ of $h(Y_1^Q(p^\infty), \mathcal{O})$ as

$$\mathfrak{m}_Q = \mathfrak{m} + (U_{v,i} - \prod_{j \leq i} N(v)^{-(j-1)} \widetilde{\alpha_{v,j}})_{v \in Q}$$

(see [Ge19, Corollary 2.7.8]). Let

$$T_{h,Q} = eh(Y_1^Q(p^\infty), \mathcal{O})_{\mathfrak{m}_Q}$$

It acts faithfully on $eH^{q_s}(Y_1^Q(p^\infty), \mathcal{O})_{\mathfrak{m}_Q}$. By Scholze and [CGH+19], there exists a Galois representation

$$\rho_{T_{h,Q}} \colon G_{F, S \cup S_p \cup Q} \to \mathsf{GL}_N(T_{h,Q})$$

associated to $T_{h,Q}$. Modulo a nilpotent ideal (or assuming Conjecture A), it is ordinary at places above p and minimal at places dividing $\mathfrak{n}$. Moreover, for any $v \in Q$, let I_v be the inertia subgroup at v; again by (a trivial generalization of) [TW95, Lemma 7], we may assume after conjugation that it takes values in $\widehat{T}(T_{h,Q})$. Let us determine the character $\Delta_v \to T_{h,Q}^\times$.

-On the maximal conjugate self-adjoint quotient of $T_{h,Q}$, it follows from [CHT08, Proposition 3.4.4 (8)] that it is given by $a \mapsto <a>_Q$ for $a \in \Delta_v$. Note that these authors define a Hecke algebra $eh(Y^Q, \mathcal{O})$ using a level group U_Q associated to maximal parahoric subgroups instead of Iwahori subgroups at $v \in Q$, hence their diamond operator is defined on $k(v)^\times$ instead of $T(k(v))$, but the proof is the same.

-For $T_{h,Q}$ itself, it follows from [ACC+18, Proposition 6.5.11] that this is also the case, modulo a nilpotent ideal with nilpotence index bounded in terms of N and $[F : \mathbb{Q}]$. Conjecture A of [CaGe18], generalized to GL_N of a CM field asserts in particular that this nilpotent ideal can be chosen to be 0.

This implies that, assuming Conjecture A, the Galois representation $\rho_{T_{h,Q}}$ satisfies all the local conditions of the problem $\mathcal{D}_{h,Q}$. Therefore, there exists a canonical Λ-algebra homomorphism

$$\phi_Q \colon R_{h,Q} \to T_{h,Q}$$

associated to $\rho_{T_{h,Q}}$.

Definition 11. *A Taylor-Wiles system is a collection $\{Q_m\}$ of mutually disjoint Taylor-Wiles sets such that for any $v \in Q_m$, $Nv \equiv 1 \pmod{p^m}$.*

Let $Y_1^{Q_m}(p^\infty)$ be the provariety associated to the level group $U_{Q_m} \cap U_1(p^\infty)$. Let $T_{h,Q_m} = h(Y^{Q_m}(p^\infty), \mathcal{O})_{\mathfrak{m}_{Q_m}}$.
We thus have a collection of surjective ring homomorphisms $\phi_{Q_m} \colon R_{h,Q_m} \to T_{h,Q_m}$ which reduce to $\phi \colon R_h \to T_h$ modulo I_{Q_m}.

Recall that for $G = \mathrm{Res}_{F/\mathbb{Q}} \mathsf{GL}_N$, $\ell_0 = Nd_0 - 1$. Recall we assume all primes above p in F^+ split in F.

Proposition 8. *Assume (ST). Then, for any $m \geq 1$, R_{h,Q_m} can be generated by $s = rN - \ell_0$ elements.*

Proof. By [DDT95, Theorem 2.18], we know that R_{h,Q_m} can be generated by s elements with

$$s = h^0(F, \mathfrak{g}) - \sum_{v|\infty} \dim_k \mathfrak{g} + \sum_{v \in S_p} (\ell_v - h^0(G_v, \mathfrak{g})) + \sum_{v \in Q} (\dim_k(\ell_v - h^0(G_v, \mathfrak{g})).$$

where $\ell_v = \dim_k L_v$. We have $h^0(F, \mathfrak{g}) = 1$ and
$-\sum_{v|\infty} \dim_k \mathfrak{g} = -N^2 d_0$
for $v \in S_p$, let us compute $\ell_v - h^0(G_v, \mathfrak{g})$; we recall that L_v is the image in $\mathrm{H}^1(G_v, \mathfrak{g})$ of the fiber product

$$L_v' = \mathrm{H}^1(G_v, \mathfrak{b}) \times_{\mathrm{H}^1(G_v, \mathfrak{b}/\mathfrak{n})} \mathrm{H}^1_{unr}(G_v, \mathfrak{b}/\mathfrak{n})$$

We can insert L_v' in the long exact sequence

$$0 \to \mathrm{H}^0(G_v, \mathfrak{n}) \to \mathrm{H}^0(G_v, \mathfrak{b}) \to \mathrm{H}^0(G_v, \mathfrak{b}/\mathfrak{n}) \to \mathrm{H}^1(G_v, \mathfrak{n}) \to L_v' \to \mathrm{H}^1(G_v/I_v, \mathfrak{b}/\mathfrak{n}) \to 0$$

Hence, we have :

$$\dim_k L_v' - h^0(G_v, \mathfrak{b}) = h^1(G_v/I_v, \mathfrak{b}/\mathfrak{n}) - h^0(G_v/I_v, \mathfrak{b}/\mathfrak{n}) + h^1(G_v, \mathfrak{n}) - h^0(G_v, \mathfrak{n})$$

We have

$$h^1(G_v/I_v, \mathfrak{b}/\mathfrak{n}) = h^0(G_v/I_v, \mathfrak{b}),$$
$$h^1(G_v, \mathfrak{n}) - h^0(G_v, \mathfrak{n}) = [F_v \colon \mathbb{Q}_p] \dim_k \mathfrak{n} + h^0(G_v, \mathfrak{n}^*(1))$$

and $h^0(G_v, \mathfrak{n}^*(1)) = 0$ by strong distinguishability. Since $L'_v \to L_v$ is injective by strong distinguishability, we conclude $\ell_v - h^0(G_v, \mathfrak{g}) = [F_v : \mathbb{Q}_p] \cdot N(N-1)/2$. So

$$\sum_{v \in S_p} (\ell_v - h^0(G_v, \mathfrak{g})) = N(N-1)d_0$$

Finally, vor $v \in Q$, $L_v = \mathrm{H}^1(G_v, \mathfrak{g})$. By inflation restriction,

$$0 \to \mathrm{H}^1_{unr}(G_v, \mathfrak{g}) \to \mathrm{H}^1(G_v, \mathfrak{g}) \to \mathsf{Hom}_{G_v/I_v}(\mathbb{Z}_p(1), \mathfrak{g}) \to 0.$$

The kernel and cokernel are N-dimensional. Moreover $\mathrm{H}^0(G_v, \mathfrak{g})$ is diagonal, hence N-dimensional. Therefore $\ell_v - h^0(G_v, \mathfrak{g}) = 2N - N = N$.

We conclude that

$$s = 1 - N^2 d_0 + N(N-1)d_0 + rN = Nr - Nd_0 + 1 = rN - \ell_0$$

as desired. $\qquad\square$

Let $S_\infty^\Lambda = \Lambda[[S_1, \dots, S_{Nr}]]$ and $R_\infty^\Lambda = \Lambda[[X_1, \dots, X_s]]$. For any TW set Q_m as above, there exists (several) surjective Λ-algebra homomorphisms $r_{Q_m} : R_\infty^\Lambda \to R_{h,Q_m}$.

Lemma 16. *There exists a Λ-algebra homomorphism $\alpha_\infty : S_\infty^\Lambda \to R_\infty^\Lambda$, a sequence of Taylor-Wiles sets Q_{m_j}, and Λ-algebra homomorphisms Λ-algebra homomorphisms $r_{Q_{m_j}} : R_\infty^\Lambda \to R_{h,Q_{m_j}}$ such that for any j, the diagram*

$$\begin{array}{ccc} S_\infty^\Lambda & \xrightarrow{\alpha_\infty} & R_\infty^\Lambda \\ \downarrow & & \downarrow r_{Q_{m_j}} \\ \Lambda[\Delta_{Q_{m_j}}] & \xrightarrow{\alpha_{Q_{m_j}}} & R_{h,Q_{m_j}} \end{array}$$

commutes.

The argument of [CaGe18, Part II] and [GV18, Section 13 (Theorem 13.1 and its proof)] goes through for $S_\infty^\Lambda \to R_\infty^\Lambda$ to prove

Theorem 15. *1) $\phi : R_h \to \mathbb{T}_h$ is an isomorphism of Λ-algebras (which may not locally complete intersections over $\mathcal{O}$).*

2) As graded module over the commutative graded ring $\mathrm{Tor}_\bullet^{S_\infty^\Lambda}(R_\infty^\Lambda, \Lambda)$, we have

$$\mathbb{H}_{\mathfrak{m}}^{q_s - \bullet} \cong \mathbb{H}_{\mathfrak{m}}^{q_s} \otimes \mathrm{Tor}_\bullet^{S_\infty^\Lambda}(R_\infty^\Lambda, \Lambda).$$

6.3. **Simplicial deformation ring.** Let $\mathcal{D}_h^s : \mathcal{O}_k \to \mathsf{sSETS}$ be the problem of minimal ordinary simplicial deformations (without specifying the Hodge-Tate weights). By the same argument proving the prorepresentability of $\mathcal{D}_\lambda^s$, one sees that $\mathcal{D}_h^s$ is pro-representable by a simplicial deformation ring $\mathcal{R}_h$ which satisfies $\pi_0(\mathcal{R}_h) = R_h$. Exactly as in Theorem 11, we prove under the same assumptions as Theorem 15:

Theorem 16. *There is a natural isomorphism of graded $\mathbb{T}_h$-algebras*

$$\pi_\bullet(\mathcal{R}_h) \cong \mathrm{Tor}_\bullet^{S_\infty^\Lambda}(R_\infty^\Lambda, \Lambda)$$

Let $\mathcal{R}_{\lambda'}$ be the simplicial deformation ring at bottom level $U_0(p)$ and weight λ', as in Section 3.6. Recall that if $\mathcal{R}_1$ and $\mathcal{R}_2$ are two simplicial $\mathcal{O}$-algebras, one can form a tensor product simplicial $\mathcal{O}$-algebra $\mathcal{R}_1 \underline{\otimes} \mathcal{R}_2$. Note that for the prime ideal $P_{\lambda'}$ of Λ associated to the arithmetic weight λ', we have a weak equivalence of simplicial rings

$$\mathcal{R}_h \underline{\otimes}_\Lambda \Lambda/P_{\lambda'}\mathcal{R}_h \cong \mathcal{R}.$$

We first formulate the conjecture of Concentration in Supremum Degree (CinS):

Conjecture 3. *The following two equivalent statements are true*
  1) $H^\bullet_\mathfrak{m}$ *is concentrated in degree* q_s,
  2) $\mathcal{R}_h$ *is discrete and for any arithmetic weight* λ', *there is an isomorphism of commutative graded rings*

$$\mathrm{Tor}^\Lambda_\bullet(R_h, \Lambda/P_{\lambda'}) = \pi_\bullet(\mathcal{R}_{\lambda'}).$$

Comments:
  1) The equivalence between the two statements follows from Theorem 15.
  2) By Theorem 15, this conjecture also implies that
  (a) $H^\bullet_\mathfrak{m}$ is free of rank one over $R_h = T_h$,
  (b) for any arithmetic weight λ', for any $i = 0, \ldots, \ell_0$,

$$\mathrm{Tor}^\Lambda_i(H^{q_s}_\mathfrak{m}, \Lambda/P_{\lambda'}) = H^{q_s-i}(Y_0(p), V_{\lambda'}(\mathcal{O}))_\mathfrak{m}.$$

  3) All the statements are easy to prove for $N = 2$ and F quadratic because $H^1_\mathfrak{m} = 0$ since it is torsion free over $\Lambda = \mathcal{O}[[T_1, T_2, X_1, X_2]]$ (where X_i's are the twist variables) but is annihilated by $T_1 - T_2$.
  This conjecture is motivated by non abelian Leopoldt conjectures due to Hida and one of the authors (E. Urban).

6.4. **Non-abelian Leopoldt Conjectures.** Recall a first version of the non abelian Leopoldt conjecture which we call the deformation theoretic non abelian Leopoldt conjecture (DNAL), cf. [Ti96, Section 9, Example 1]:

Conjecture 4.

$$\dim R_h = \dim \Lambda - \ell_0$$

Proposition 9. *Assume that Conjecture (DNAL) holds; then Conjecture (CinS) holds.*

Proof. Let $q \in [q_m, q_s]$ be the smallest integer such that $H^q_h \neq 0$. Then, the complex $C^{q_m} \to \ldots \to C^q_h$ is a resolution of $M = C^q_h/B^q_h$ where $B^q_h = \mathrm{Im}(C^{q-1}_h \to C^q_h)$. We have $H^q_h \subset M$ and $\mathsf{projdim}_\Lambda M \leq q - q_h$ hence by Auslander-Buchsbaum formula, we have $\mathrm{depth}_\Lambda M \geq \dim_\Lambda M - (q - q_h)$. Recall Ischebeck's Lemma ([?], (15.E) Lemma 2]: if A, B are two finitely generated Λ-modules, for any $i < depth_\Lambda B - \dim_\Lambda A$,

$$\mathrm{Ext}^i(A, B) = 0.$$

Here, we consider $\mathrm{Ext}^0(H^q_h, M)) \neq 0$ hence $0 \geq \mathrm{depth}_\Lambda M - \dim_\Lambda H^q_h$, so

$$\dim_\Lambda H^q_h \geq \mathrm{depth}_\Lambda M \geq \dim_\Lambda -\ell_0.$$

$\square$

Recall $\Sigma_{coh} \subset \mathrm{Supp}_\Lambda(H^\bullet_\mathfrak{m})$. Note that it is Zariski-dense in $\mathrm{Supp}_\Lambda(H^\bullet_\mathfrak{m})$. We know that for $P_\lambda \in \Sigma_{coh}$, $H^i(Y_0(p), V_\lambda(K))_\mathfrak{m} \neq 0$ if and only f $i \in [q_m, q_s]$ (where $q_m = q_0$ and $q_s = q_0 + \ell_0$). Let us consider the integral Tor-spectral sequence

$$E^{i,j}_2(\lambda) = \mathrm{Tor}^\Lambda_i(H^j_\mathfrak{m}, \Lambda/P_\lambda) \Rightarrow H^{j-i}(Y_0(p), V_\lambda(\mathcal{O}))_\mathfrak{m}$$

$(i \leq 0, j \geq 0)$. Let λ be an arietic weight. Consider the condition

$(INTDEG)_\lambda$ the Tor-spectral sequence degenerates at E_2 and we have

$$H^q(Y_0(p), V_\lambda(\mathcal{O}))_\mathfrak{m} = \bigoplus_{j-i=q} \mathrm{Tor}^\Lambda_i(H^j_\mathfrak{m}, \Lambda/P_\lambda).$$

Note that Conjecture (CinS) implies $(INTDEG)_\lambda$ for all arithmetic weights λ. Recall that for a finitely generated module M over a noetherian ring A, one defines $\mathrm{codim}_A M = min\{ht\mathfrak{p}; \mathfrak{p} \in \mathrm{Supp}(M)\}$. If A is Cohen-Macaulay, one has $\dim A = \dim_A M + \mathrm{codim}_A M$ and $\dim_A M = \mathrm{depth}_A M$. If A is regular, it follows

from the Auslander-Buchsbaum formula that $\mathsf{codim}_A M = \mathsf{projdim}_A M$ (which is finite). Then, assuming Conjecture (CinS), we see that for any $i \in [0, \ell_0]$,

$$\mathrm{Tor}_i^\Lambda(\mathrm{H}_{\mathfrak{m}}^{q_s}, \Lambda/P) \neq 0$$

for any $P \in \Sigma_{coh}$ Since the subset Σ_{coh} is Zariski-dense in $\mathrm{Supp}_\Lambda(\mathrm{H}_{\mathfrak{m}}^\bullet)$, we conclude that

$$\mathsf{projdim}_\Lambda \mathrm{H}_{\mathfrak{m}}^{q_s} = \ell_0$$

This implies that $\mathrm{Supp}(\mathrm{H}_{\mathfrak{m}}^{q_s})$ has Λ-codimension $\leq \ell_0$. Let us recall the cohomological non abelian Leopoldt Conjecture due to Hida and E. Urban.

Conjecture 5. *(CNLA)* $\mathsf{codim}_\Lambda \Sigma_h = \ell_0$

Remark 8. *Hida conjectured that $\mathsf{codim}_\Lambda \mathrm{H}_{\mathfrak{m}}^\bullet \otimes \mathbb{Q}_p = \ell_0$ and Urban conjectured that $\mathsf{codim}_\Lambda \Sigma_h = \ell_0$. But by Lemma 15, we have $\Sigma_h = \mathrm{Supp}_\Lambda(\mathbb{H}_{\mathfrak{m}}^\bullet) \cap Spec(\Lambda)(\overline{\mathbb{Q}}_p)$, therefore these two statements are equivalent.*

Lemma 17. *Assume that Condition $(INTDEG)_\lambda$ holds for some arithmetic weight λ and that Conjecture $(CNAL_h)$ holds. Then Conjecture (CinS) holds.*

Proof. Let q with $q_m \leq q < q_s$ be the minimal integer such that $\mathrm{H}_{\mathfrak{m}}^q \neq 0$. By (INTDEG) , we see that for all i's $i > \ell_0 - (q_s - q)$, we have $\mathrm{Tor}_i^\Lambda(\mathrm{H}^{q_s}, \Lambda/P) \neq 0$ for all $P \in \Sigma_{coh}(\subset \mathrm{Supp}_\Lambda(\mathrm{H}_{\mathfrak{m}}^{q_s}))$. This implies $\mathsf{projdim}\, \mathrm{H}_{\mathfrak{m}}^q \leq \ell_0 - (q_s - q) < \ell_0$. Contradiction.

$\square$

7. The Galatius-Venkatesh homomorphism for Hida families

Let $\mathrm{Spec}\,\mathbf{I}$ be an irreducible component of $\mathrm{Spec}\,\mathrm{T}_h$; let $\mathfrak{m}_\mathbf{I}$ be its maximal ideal. Note that $\mathbf{I}$ is not necessary flat over Λ. We have a Λ-linear algebra homomomorphism $\theta \colon \mathrm{T}_h \to \mathbf{I}$. Let $\Lambda_\mathbf{I}$ be the image of Λ in $\mathbf{I}$. Let $\mathcal{R}_\mathbf{I} = \mathcal{R}_h \underline{\otimes}_\Lambda \Lambda_\mathbf{I}$ and $R_\mathbf{I} = R_h \otimes_\Lambda \Lambda_\mathbf{I}$. We have $\pi_0(\mathcal{R}_\mathbf{I}) = R_\mathbf{I}$ and there is a natural R_h-algebra homomorphism $\pi_\mathbf{I} \colon \pi_1(\mathcal{R}_h) \to \pi_1(\mathcal{R}_\mathbf{I})$. However, we conjecture that $\pi_1(\mathcal{R}_h) = 0$ (hence $\pi_\mathbf{I} = 0$) while $\pi_1(\mathcal{R}_\mathbf{I}) \otimes_{R_\mathbf{I}} \widehat{\mathbf{I}}$ will be shown to interpolate the classical $\pi_1(\mathcal{R}_{\lambda'}) \otimes_{R_{\lambda'}, \theta_{\pi'}} K/\mathcal{O}$ for all arithmetic weights λ' congruent to λ modulo p and all classical Hecke eigensystems $\theta_{\pi'}$ of weight λ'.

Theorem 17. *There are natural $\mathbf{I}$-linear injective homomorphisms*

$$GV_h \colon \pi_1(\mathcal{R}_h) \otimes_{R_h} \widehat{\mathbf{I}} \hookrightarrow \mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_\mathbf{I} \widehat{\mathbf{I}}).$$

and

$$GV_\mathbf{I} \colon \pi_1(\mathcal{R}_\mathbf{I}) \otimes_{R_\mathbf{I}} \widehat{\mathbf{I}} \hookrightarrow \mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_\mathbf{I} \widehat{\mathbf{I}}).$$

and we have $GV_\mathbf{I} \circ \pi_\mathbf{I} = GV_h$.

Proof. Let $n \geq 1$, let $\mathbf{I}_n = \mathbf{I}/\mathfrak{m}_\mathbf{I}^n$. For any finite $\mathbf{I}_n$-module M (which can be viewed as a T_h-module), we consider the simplicial ring $\Theta_n = \mathbf{I}_n \oplus M[1]$. Let $L_n(\mathcal{R}_h)$ be the set of homotopy equivalence classes of simplicial ring homomorphisms $\Phi \colon \mathcal{R}_h \to \Theta_n$ such that $\mathrm{pr}_n \circ \Phi = \phi_n$. Note that any such Φ factors through $\mathcal{R}_h \to \mathcal{R}_\mathbf{I}$ into a homomorphism $\Phi_\mathbf{I} \colon \mathcal{R}_\mathbf{I} \to \Theta_n$. So, $L_n(\mathcal{R}_h)$ can be reinterpreted as the set $L_n(\mathcal{R}_\mathbf{I})$ of homotopy equivalence classes of

$$\mathsf{Hom}_{\mathrm{pr}_n}(\mathcal{R}_\mathbf{I}, \Theta_n).$$

There is a canonical bijection

$$L_n(\mathcal{R}_h) \cong \mathrm{H}_{\mathcal{L}}^2(F, \mathrm{Ad}(\rho_\theta) \otimes_{\mathbf{I}_n} M)$$

where $\mathcal{L}$ is the minimal ordinary condition. Moreover, as in [GV18, Lemma 15.1] and in Formula (4), for each $n \geq 1$, we have a T_h-linear homomorphism

$$\pi(n, \mathcal{R}_*) \colon L_n(\mathcal{R}_*) \to \mathsf{Hom}_{\mathrm{T}_h}(\pi_1(\mathcal{R}_*), M)$$

where $* = h, \mathbf{I}$. As in Proposition 5, this homomorphism is surjective. We take $M = \mathbf{I}_n$. Its Pontryagin dual is $M^\vee = \mathsf{Hom}_{\mathcal{O}}(\mathbf{I}, K/\mathcal{O})[\mathfrak{m}_{\mathbf{I}}^n]$. Let $\widehat{\mathbf{I}} = \mathsf{Hom}_{\mathcal{O}}(\mathbf{I}, K/\mathcal{O})$. Let $\pi_1 \mathcal{R}_{*,\theta} = \pi_1 \mathcal{R}_* \otimes_{\mathrm{T}_h, \theta} \mathbf{I}$. We see that we can rewrite the Pontryagin dual of the right hand side of (7) as

$$\mathsf{Hom}_{\mathcal{O}}(\mathsf{Hom}_{\mathrm{T}_h}(\pi_1(\mathcal{R}_*), M), K/\mathcal{O}) = \mathsf{Hom}_{\mathbf{I}}(\mathsf{Hom}_{\mathbf{I}}(\pi_1(\mathcal{R}_*)_\theta, \mathbf{I}/\mathfrak{m}_{\mathbf{I}}^n), \widehat{\mathbf{I}})$$

Since $\mathsf{Hom}_{\mathbf{I}}(\widehat{\mathbf{I}}, \widehat{\mathbf{I}}) = \mathbf{I}$, we have $\mathsf{Hom}_{\mathbf{I}}(\widehat{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n], \widehat{\mathbf{I}}) = \mathbf{I}/\mathfrak{m}_{\mathbf{I}}^n$ hence

$$\mathsf{Hom}_{\mathbf{I}}(\pi_1(\mathcal{R}_*)_\theta, \mathbf{I}/\mathfrak{m}_{\mathbf{I}}^n) = \mathsf{Hom}_{\mathbf{I}}(\pi_1(\mathcal{R}_*)_\theta, \mathsf{Hom}_{\mathbf{I}}(\widehat{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n], \widehat{\mathbf{I}}))) =$$

$$\mathsf{Hom}_{\mathbf{I}}(\pi_1(\mathcal{R}_*)_\theta \otimes_{\mathbf{I}} \widehat{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n], \widehat{\mathbf{I}})),$$

but for $S = \pi_1(\mathcal{R}_*)_\theta \otimes_{\mathbf{I}} \widehat{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n]$, we have $\mathsf{Hom}_{\mathbf{I}}(\mathsf{Hom}_{\mathbf{I}}(S, \widehat{\mathbf{I}}), \widehat{\mathbf{I}}) = S$, so we conclude

$$\mathsf{Hom}_{\mathcal{O}}(\mathsf{Hom}_{\mathrm{T}_h}(\pi_1(\mathcal{R}_*), M), K/\mathcal{O}) = \pi_1 \mathcal{R}_{*,\theta} \otimes_{\mathbf{I}} \widehat{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n].$$

Similarly, the Pontryagin dual of the left hand side of (7) is

$$\mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}\, \rho_\theta^*(1) \otimes_{\mathcal{O}} M^\vee)$$

which can be rewritten as

$$\mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}\, \rho_\theta^*(1) \otimes_{\mathbf{I}} \widehat{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n])$$

Let $GV_{*,n}$ be the Pontryagin dual of $\pi(n, \mathcal{R}_*)$. Let $GV_* = \varinjlim_n GV_{*,n}$, it provides the desired injection

$$GV_* \colon \pi_1(\mathcal{R}_*) \otimes_{\mathrm{T}_h} \widehat{\mathbf{I}} \hookrightarrow \mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_{\mathbf{I}} \widehat{\mathbf{I}}).$$

$\qquad\square$

7.1. **Interpolation properties.** Let $P \in \mathrm{Spec}\,\mathbf{I}$ be a codimension one prime; assume that $\mathcal{O}_P = \mathbf{I}/P$ is a dvr. Let $K_P = \mathrm{Frac}(\mathcal{O}_P)$. We have $\mathsf{Hom}(\mathcal{O}_P, K/\mathcal{O}) = K_P/\mathcal{O}_P$. The following exact control theorem holds for the minimal ordinary Selmer group. We put

$$\mathrm{Sel}_{\mathbf{I}}^* = \mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_{\mathbf{I}} \widehat{\mathbf{I}}), \quad \mathrm{Sel}_P^* = \mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}^* \rho_P(1) \otimes_{\mathcal{O}_P} K_P/\mathcal{O}_P)$$

Proposition 10. *For any prime ideal $P \subset \mathbf{I}$,*

$$\mathrm{Sel}_{\mathbf{I}}^*[P] = \mathrm{Sel}_P^*.$$

Proof. We apply [SU14, Proposition 3.2.8]: Let T be a $\mathbf{I}[\Gamma]$-module, which is finite free over $\mathbf{I}$. Then, for any ideal $\mathfrak{a} \subset \mathbf{I}$, $\mathrm{Sel}(T \otimes_{\mathbf{I}} \widehat{\mathbf{I}})[\mathfrak{a}] = \mathrm{Sel}(T \otimes_{\mathbf{I}} \widehat{\mathbf{I}/\mathfrak{a}})$ under the assumption that $T \otimes_{\mathbf{I}} \widehat{\mathbf{I}}$ has no non trivial subquotient with trivial Galois action.

Let us check this condition for $T = \mathrm{Ad}\, \rho_{\mathbf{I}}^*(1)$. Let M be a $\mathbf{I}[\Gamma]$-submodule of $T \otimes_{\mathbf{I}} \widehat{\mathbf{I}}$ and N a $\mathbf{I}[\Gamma]$-quotient of M. By applying $\mathsf{Hom}_{\mathbf{I}}(-, \widehat{\mathbf{I}})$, one sees that $\widehat{M}$ is a $\mathbf{I}[\Gamma]$-quotient of $\mathrm{Ad}\, \rho_{\mathbf{I}}$. The reduction of this homomorphism modulo $\mathfrak{m}_{\mathbf{I}}$ is null by absolute irreducibility of $\mathrm{Ad}\, \overline{\rho}$. By Nakayama's lemma, this implies $\widehat{M} = 0$. The same holds for the $\mathbf{I}[\Gamma]$-submodule $\widehat{N} \subset \widehat{M}$. $\qquad\square$

Let $M_{\mathbf{I}}$ be the Pontryagin dual of $\mathrm{Sel}_{\mathbf{I}}^*$. Let $\Phi_{\mathbf{I}}$ be the Pontryagin dual of $\pi_1(\mathcal{R}_{\mathbf{I}}) \otimes_{R_{\mathbf{I}}} \widehat{\mathbf{I}}$. We have

$$\Phi_{\mathbf{I}} = \mathsf{Hom}_{R_{\mathbf{I}}}(\pi_1(\mathcal{R}_{\mathbf{I}}), \mathbf{I}) = \mathsf{Hom}_{\mathbf{I}}(\pi_1(\mathcal{R}_{\mathbf{I}})_\theta, \mathbf{I}).$$

It is a finitely generated torsion free $\mathbf{I}$-module. Let $N_{\mathbf{I}} = \mathrm{Ker}(M_{\mathbf{I}} \to \Phi_{\mathbf{I}})$ be the Pontryagin dual of $\mathrm{Coker}\, GV_{\mathbf{I}}$. We have a short exact sequence

$$0 \to N_{\mathbf{I}} \to M_{\mathbf{I}} \to \Phi_{\mathbf{I}} \to 0$$

Let $\mathrm{Sel}_{\mathbf{I}} = \mathrm{H}_{\mathcal{L}}^1(\Gamma, \mathrm{Ad}\, \rho_{\mathbf{I}} \otimes \widehat{\mathbf{I}})$.

Theorem 18. *(1) The $\mathbf{I}$-modules $M_{\mathbf{I}}$ and $\Phi_{\mathbf{I}}$ have rank ℓ_0 and $\Phi_{\mathbf{I}}$ is free.*

(2) Let $T_{\mathfrak{m}_{\mathbf{I}}}\mathrm{Sel}_{\mathbf{I}} = \varprojlim_n \mathrm{Sel}_{\mathbf{I}}[\mathfrak{m}_{\mathbf{I}}^n]$. The Poitou-Tate-Pontryagin duality induces an isomorphism

$$N_{\mathbf{I}} \cong T_{\mathfrak{m}_{\mathbf{I}}}\mathrm{Sel}_{\mathbf{I}}$$

Proof. (1) Let P be an arithmetic prime of weight λ' congruent to λ modulo p; let $\mathcal{O}_P = \mathbf{I}/P$. Generically, it is a dvr. We compare

$$GV_{\mathbf{I}} \colon \pi_1(\mathcal{R}_{\mathbf{I}}) \otimes_{R_{\mathbf{I}}} \widehat{\mathbf{I}} \hookrightarrow \mathrm{H}^1_{\mathcal{L}}(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_{\mathbf{I}} \widehat{\mathbf{I}})$$

to the "classical" map

$$GV_P \colon \pi_1(\mathcal{R}_{\lambda'}) \otimes_{R_{\lambda'}} K_P/\mathcal{O}_P \hookrightarrow \mathrm{H}^1_{\mathcal{L}}(\Gamma, \mathrm{Ad}^* \rho_P(1) \otimes_{\mathcal{O}_P} K_P/\mathcal{O}_P)$$

We have a commutative diagram

$$
\begin{array}{ccc}
GV_{\mathbf{I}} \quad : & \pi_1(\mathcal{R}_{\mathbf{I}}) \otimes_{R_{\mathbf{I}}} \widehat{\mathbf{I}} & \hookrightarrow & \mathrm{H}^1_{\mathcal{L}}(\Gamma, \mathrm{Ad}^* \rho_\theta(1) \otimes_{\mathbf{I}} \widehat{\mathbf{I}}) \\
& \uparrow & & \uparrow \\
GV_P \quad : & \pi_1(\mathcal{R}_{\lambda'}) \otimes_{R_{\lambda'}} K_P/\mathcal{O}_P & \hookrightarrow & \mathrm{H}^1_{\mathcal{L}}(\Gamma, \mathrm{Ad}^* \rho_P(1) \otimes_{\mathcal{O}_P} K_P/\mathcal{O}_P)
\end{array}
$$

Let M_P be the Pontryagin dual of Sel^*_P. Similarly, let Φ_P denote the Pontryagin dual of $\pi_1(\mathcal{R}_{\lambda'}) \otimes_{R_{\lambda'}} K_P/\mathcal{O}_P$. By Pontryagin duality we obtain a diagram

$$
\begin{array}{ccc}
M_{\mathbf{I}} & \to & \Phi_{\mathbf{I}} \\
\downarrow & & \downarrow \\
M_P & \to & \Phi_P
\end{array}
$$

where the two horizontal arrow are surjective and where the left downarrow induces an isomorphism by exact control of the dual Selmer groups:

$$M_{\mathbf{I}}/P M_{\mathbf{I}} \cong M_P.$$

It implies that the right downarrow

$$\Phi_{\mathbf{I}}/P\Phi_{\mathbf{I}} \to \Phi_P$$

is surjective. Let us prove it is an isomorphism. Note that Φ_P is free of rank ℓ_0 by Theorem 13. We have

$$\Phi_{\mathbf{I}}/P\Phi_{\mathbf{I}} = \mathsf{Hom}_{\mathbf{I}}(\pi_1(\mathcal{R}_{\mathbf{I}})_\theta, \mathbf{I}/P) = \mathsf{Hom}_{\mathbf{I}/P}(\pi_1(\mathcal{R}_{\mathbf{I}})_\theta \otimes_{\mathbf{I}} \mathbf{I}/P, \mathbf{I}/P)$$

Let $\widetilde{\mathbf{I}}$ be the integral closure of $\mathbf{I}$ and let $X = \mathrm{Supp}(\widetilde{\mathbf{I}}/\mathbf{I})$. For $P \notin X$, we have $\mathbf{I}/P = \widetilde{\mathbf{I}}/P\widetilde{\mathbf{I}}$. Therefore, we can assume $\mathbf{I}$ is integrally closed provided we consider only arithmetic primes $P \notin X$. We know that $M_{\mathbf{I}}$ and $\pi_1(\mathcal{R}_{\mathbf{I}})_\theta$ (hence $\Phi_{,\mathbf{I}}$) are finitely generated $\mathbf{I}$-modules. By the structure theorem of modules over a Krull ring, $M_{\mathbf{I}}$, resp. $\Phi_{\mathbf{I}}$, is pseudo-isomorphic to $\mathbf{I}^{r_1} \oplus \Theta_1$, resp. $\mathbf{I}^{r_2} \oplus \Theta_2$ where Θ_i ($i = 1, 2$) is a torsion $\mathbf{I}$-module. By reduction modulo $P \notin X$, we know by Lemma 7 that $r_1 = \ell_0$ and $r_2 \leq \ell_0$. Note that

$$\Phi_{\mathbf{I}}/P\Phi_{\mathbf{I}} = \mathsf{Hom}_{\mathcal{O}_P}(\pi_1(\mathcal{R}_{\mathbf{I}})_\theta \otimes_{\mathbf{I}} \mathcal{O}_P, \mathcal{O}_P)$$

the right member is free of rank r_2 over $\mathcal{O}_P$. Hence, $\Phi_{\mathbf{I}}/P\Phi_{\mathbf{I}} \cong \mathcal{O}_P^{r_2}$; since this $\mathcal{O}_P$-module maps surjectively to $\mathcal{O}_P^{\ell_0}$, we conclude that $r_2 = \ell_0$ and that

$$\Phi_{\mathbf{I}}/P\Phi_{\mathbf{I}} \to \Phi_P$$

is an isomorphism. Thus, by Nakayama's lemma, there exists a surjective $\mathbf{I}$-linear homomorphism

$$j \colon \mathbf{I}^{\ell_0} \to \Phi_{\mathbf{I}}$$

Let $\mathcal{K} = \mathrm{Ker}\, j$. For $P \notin X$, we have $\mathcal{K} \subset P \cdot \mathbf{I}^{\ell_0}$. Note that $\bigcap_{P \notin X} P = 0$ in $\mathbf{I}$, hence $\mathcal{K} = 0$ and j is an isomorphism as desired.

(2) We use notations from the proof of Theorem 17. By Lemma 10, $N_{\mathbf{I}}/\mathfrak{m}_{\mathbf{I}}^n N_{\mathbf{I}}$ is Pontryagin dual of $\mathrm{Coker}\, GV_{\mathbf{I},n}$. By Poitou-Tate duality (Lemma 3), the group $\mathrm{Coker}\, GV_{\mathbf{I},n}$ is Pontryagin dual to

$$\mathrm{Sel}_{\mathbf{I}_n} = \mathrm{H}^1_{\mathcal{L}}(\Gamma, \mathrm{Ad}\, \rho_n \otimes_{\mathbf{I}_n} \widehat{\mathbf{I}}_n)$$

By taking projective limits over n, we find that $N_{\mathbf{I}}$ is Pontryagin dual to $T_{\mathfrak{m}_{\mathbf{I}}} \mathrm{Sel}_{\mathbf{I}}$.

$\square$

Let $\widetilde{\mathbf{I}}$ be the integral closure of $\mathbf{I}$ and let $X = \mathrm{Supp}(\widetilde{\mathbf{I}}/\mathbf{I})$. For $P \notin X$, we have $\mathbf{I}/P = \widetilde{\mathbf{I}}/P\widetilde{\mathbf{I}}$. Let $L_{\mathbf{I}}^{alg}$ be the Fitting ideal of the torsion $\mathbf{I}$-module $N_{\mathbf{I}}$. It plays the role of the algebraic p-adic L function for the family of motives $\mathrm{Ad}(\rho_{\mathbf{I}})$. For any arithmetic prime P of $\mathbf{I}$, let N_P be the Pontryagin dual of $\mathrm{Coker}\, GV_P$. Let $L_{\mathbf{I}}^{alg}(P)$ be the image of $L_{\mathbf{I}}^{alg}$ in $\mathbf{I}/P$.

Corollary 4. *For any $P \notin X$,*

(1) the map $\mathrm{Coker}\, GV_P \to \mathrm{Coker}\, GV_{\mathbf{I}}[P]$ *is an isomorphism*

(2) the specialization $L_{\mathbf{I}}^{alg}(P)$ of $L_{\mathbf{I}}^{alg}$ at P is a generator of $\mathrm{Fitt}_{\mathcal{O}_P} N_P = \mathrm{Fitt}_{\mathcal{O}_P} \mathrm{Coker}\, GV_P$.

Proof. By Pontryagin duality, both statements are equivalent to $N_{\mathbf{I}}/PN_{\mathbf{I}} = N_P$. This statement follows from the formulas $M_{\mathbf{I}}/PM_{\mathbf{I}} = M_P$, $\Phi_{\mathbf{I}}/P\Phi_{\mathbf{I}} = \Phi_P$ and $\mathrm{Tor}_1^{\mathbf{I}}(\Phi_{\mathbf{I}}, \mathbf{I}/P) = 0$ proven in Theorem 18. $\qquad\square$

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