3. Lecture III: Hyperplane arrangements & zonotopes; Inequalities: a first look

3.1. Zonotopes.

Recall that a zonotope Z can be described as the Minkowski sum of line segments:

$$Z = [\mathbf{0}, \mathbf{v_1}] + \cdots + [\mathbf{0}, \mathbf{v_k}].$$

Associated with a zonotope is a hyperplane arrangement given by the set of k hyperplanes $H_1, \ldots, H_k$, where the hyperplane H_i is the subspace orthogonal to the normal vector $\mathbf{v_i}$, for $i = 1, \ldots, k$.

Given a hyperplane arrangement $\mathscr{H} = \{H_1, \ldots, H_k\}$ in $\mathbb{R}^n$, there are two associated lattices. The *intersection lattice* L consists of all the intersections of the hyperplanes in $\mathscr{H}$ ordered with respect to reverse inclusion. Thus the minimal element is the empty intersection $\mathbb{R}^n$ and the maximal element of L is the intersection of all the hyperplanes, that is, the zero vector. The second lattice is the more complicated *lattice of regions* T. It is formed by intersecting the arrangement $\mathscr{H}$ with an n-dimensional sphere. This gives a decomposition of the n-sphere and the open cells can be ordered by inclusion to form T.

The important function we will work with is the omega map ω, which we now describe.

Definition 3.1.1. *Define a linear function $\omega : \mathbb{Z}\langle \mathbf{a}, \mathbf{b} \rangle \longrightarrow \mathbb{Z}\langle \mathbf{c}, 2\mathbf{d} \rangle$ as follows: For an* **ab***-monomial v we compute $\omega(v)$ by replacing each occurrence of* **ab** *in the monomial v with* $2\mathbf{d}$*, and then replacing the remaining letters with* **c***'s. Extend this definition by linearity to* **ab***-polynomials.*

The function ω takes an **ab**-polynomial of degree n into a **c**-2**d**-polynomial of degree n. As an example $\omega(\mathbf{bbaababba}) = \mathbf{c}^3 \cdot 2\mathbf{d} \cdot 2\mathbf{d} \cdot \mathbf{c}^2$.

We can now link the **ab**-index of the intersection lattice with the **cd**-index of the corresponding zonotope. Here an *essential* hyperplane arrangement $\mathscr{H}$ is one where there is no non-zero vector orthogonal to all of the hyperplanes in $\mathscr{H}$.

Theorem 3.1.2 (Billera–Ehrenborg–Readdy)**.** *Let $\mathscr{H}$ be an essential hyperplane arrangement and let L be the intersection lattice of $\mathscr{H}$. Let Z be the zonotope corresponding to $\mathscr{H}$. Then the* **c**-2**d***-index of the zonotope Z is given by*

$$\Psi(Z) = \omega(\mathbf{a} \cdot \Psi(L)).$$

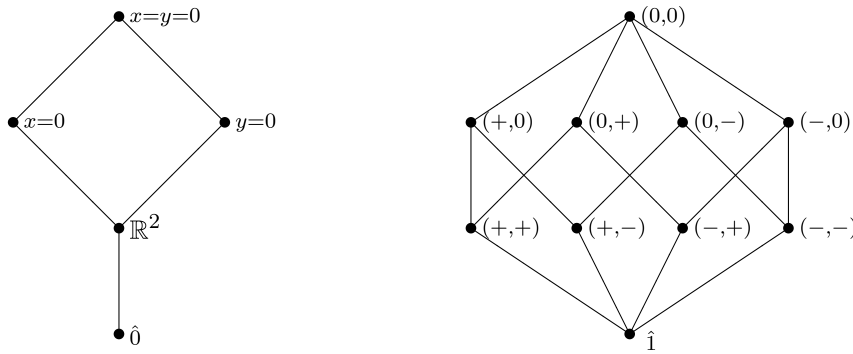

FIGURE 1. The lattice $\widehat{L}$ and the lattice T^*, the dual of the lattice of regions.

For example, the intersection lattice corresponding to the hexagonal prism has **ab**-index $\Psi(L) = \mathbf{aa} + 3 \cdot \mathbf{ba} + 3 \cdot \mathbf{ab} + 2 \cdot \mathbf{bb}$, so

$$\begin{aligned}\Psi(Z) &= \omega(\mathbf{a} \cdot (\mathbf{aa} + 3 \cdot \mathbf{ba} + 3 \cdot \mathbf{ab} + 2 \cdot \mathbf{bb})) \\ &= \omega(\mathbf{aaa} + 3 \cdot \mathbf{aba} + 3 \cdot \mathbf{aab} + 2 \cdot \mathbf{abb}) \\ &= \mathbf{c}^3 + 3 \cdot 2\mathbf{dc} + 3 \cdot \mathbf{c} \cdot 2\mathbf{d} + 2 \cdot 2\mathbf{d} \cdot \mathbf{c}) \\ &= \mathbf{c}^3 + 10 \cdot \mathbf{cd} + 6 \cdot \mathbf{dc}\end{aligned}$$

Theorem 3.1.2 was originally stated in terms of oriented matroids. For further details, see [10].

Theorem 3.1.3. *[Billera–Ehrenborg–Readdy] Let $\mathscr{M}$ be an oriented matroid, T the lattice of regions of $\mathscr{M}$ and L the lattice of flats of $\mathscr{M}$. Then the* **c**-2**d**-*index of the lattice of regions T is given by*

$$\Psi(T) = \omega(\mathbf{a} \cdot \Psi(L))^*.$$

The proof of Theorem 3.1.3 involves three ingredients. First, one orients the hyperplane arrangement so that each region has an associated sign vector. There is a map z, called the zero map, from the dual of the lattice of regions to the lattice $L \cup \{\hat{0}\}$ which sends a sign vector with zero coordinates $I = \{i_1, \ldots, i_k\}$ to the intersection $H_{i_1} \cap \cdots \cap H_{i_k}$. See Figure 1. Secondly, a result of Bayer and Sturmfels gives the cardinality of the inverse image of a chain in $L \cup \{\hat{0}\}$ [5].

Theorem 3.1.4 (Bayer–Sturmfels)**.** *For a chain $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in $L \cup \{\hat{0}\}$, the cardinality of its inverse image is given by*

$$|z^{-1}(c)| = \prod_{i=1}^{k-1} \sum_{x_i \le y \le x_{i+1}} (-1)^{\rho(x_i, y)} \cdot \mu(x_i, y).$$

Finally, coalgebraic techniques from [31] allow one to translate this into a straightforward-to-compute expression.

3.2. **Application: R-labelings.**

Let P be a graded poset with $\hat{0}$ and $\hat{1}$. We say $\lambda : E(P) \to \mathbb{Z}$ is an R-*labeling* if for every interval $[x, y]$ of P, there exists a unique saturated chain that is rising, that is, $c : x = x_0 \prec x_1 \prec \cdots \prec x_k = y$ with

$$\lambda(x_0, x_1) \leq \lambda(x_1, x_2) \leq \cdots \leq \lambda(x_{k-1}, x_k).$$

The classical R-labeling on the Boolean algebra is to label an edge $S \prec T$ by the unique element $T - S$. The $n!$ maximal chains in the Boolean algebra B_n then correspond to the $n!$ permutation in $\mathfrak{S}_n$. See [66, Chapter 3].

Theorem 3.2.1 (Björner; Stanley)**.** *Let P be a poset with R-labeling λ. Then*

$$h_S = \text{ \# maximal chains from } \hat{0} \text{ to } \hat{1} \text{ in } P \text{ with descent set } S.$$

As a first application, the theory of R-labelings gives the **ab**-index of the Boolean algebra as

$$\Psi(B_n) = \sum_{\pi \in \mathfrak{S}_n} u_{D(\pi)},$$

where $D(\pi)$ is the descent word of the permutation π. Since this corresponds to the **ab**-index of the hyperplane arrangement consisting of the n coordinate hyperplanes in $\mathbb{R}^n$, we can apply Theorem 3.1.2 to obtain the **cd**-index of the zonotope, that is, the cubical lattice.

Theorem 3.2.2 (Billera–Ehrenborg–Readdy)**.** *The* $\mathbf{c}$-$2\mathbf{d}$*-index of the n-dimensional cube C_n is given by*

$$\Psi(C_n) = \sum_{\pi \in \mathfrak{S}_n} \omega(\mathbf{a} \cdot u_{D(\pi)}).$$

As a second application, Stanley conjectured that the **cd**-index of any convex polytope, and more generally, any Gorenstein* lattice, is coefficient-wise greater than or equal to the **cd**-index of the simplex of the same dimension, i.e., the Boolean algebra of the same rank. We obtain a zonotopal analogue of this conjecture.

Corollary 3.2.3 (Billera–Ehrenborg–Readdy)**.** *Among all zonotopes of dimension n, the n-dimensional cube has the smallest* $\mathbf{c}$-$2\mathbf{d}$*-index.*

3.3. **Kalai convolution and 4-polytope inequalities.**

Knowing inequalities for the **cd**-index implies inequalities for the flag h-vector and the flag f-vector. This follows from expanding the **cd**-index back into the **ab**-index ($\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{d} = \mathbf{ab} + \mathbf{ba}$ are each non-negative linear combinations of monomials in $\mathbf{a}$ and $\mathbf{b}$), then expanding the **ab**-index back into the flag f-vector via equation (1.4) (another non-negative linear combination).

Before examining inequalities for the **cd**-index, we begin with a critique of the known linear inequalities for 4-dimensional polytopes. See Theorem 2.7.1.

Kalai's convolution is a method to lift known inequalities on flag vectors of m and n-dimensional polytopes to an inequality which holds for $(m+n+1)$-dimensional polytopes [44]. We follow [26].

Definition 3.3.1. *The* Kalai convolution *is*

$$f_S^m * f_T^n = f_{S\cup\{m\}\cup(T+m+1)}^{m+n+1},$$

where $S \subseteq \{0,\ldots,m-1\}$ and $T \subseteq \{0,\ldots,n-1\}$. The superscripts indicate the dimension of the polytope that the flag vector is from, and $T+m+1$ denotes shifting all the elements of the subset of T by $m+1$.

The Kalai product implies that for linear operators M and N defined on m-, respectively n-, dimensional polytopes yields a linear functional on $(m+n+1)$-dimensional polytopes P by

$$(M^m * N^n)(P) = \sum_{\substack{x \\ \dim(x)=m}} M^m([\hat{0},x]) \cdot N^n([x,\hat{1}]).$$

Corollary 3.3.2. *If M and N are two linear functionals that are non-negative on polytopes, then so is their Kalai convolution $M * N$.*

Example 3.3.3. Since every 2-dimensional face has at least 3 vertices, we have

$$0 \leq (f_0^2 - 3f_\emptyset^2) * f_\emptyset^1 = f_{0,2}^4 - 3f_2^4.$$

This is (1) of Theorem 2.7.1. The dual is

$$0 \leq f_\emptyset^1 * (f_0^2 - 3f_\emptyset^2) = f_{1,2}^4 - 3f_1^4 = \frac{1}{2} f_{012}^4 - 3f_1^4 = f_{02}^4 - 3f_1^4,$$

which is (2) of Theorem 2.7.1. This inequaity states that each edge of a 4-dimensional polytope is surrounded by three 2-faces.

We continue discussing the inequalities of Theorem 2.7.1. Inequality (3) of is the toric g-vector inequality $g_2 \geq 0$; see [43, 44].

Inequality (4) comes from the following computation:

$$0 \leq f_\emptyset^0 * (f_0^2 - 3f_\emptyset^2) * f_\emptyset^0 = f_{013}^4 - 3f_{03}^4 = 6f_1^4 - 6f_0^4 - f_{02}^4, \tag{3.1}$$

where the last equality is Exercise 3.7.2.

Finally, inequalities (5) and (6) state that every 4-dimensional polytope has at least five vertices and at least five 3-dimensional faces.

3.4. **Shelling and cd-index inequalities.**

Let us return to inequalities for the **cd**-index. Recall that Stanley proved the nonnegativity of the **cd**-index for polytopes, and more generally, for spherically-shellable regular CW-spheres. See Theorem 1.3.2. Stanley conjectured that for n-dimensional polytopes, more generally, Gorenstein* lattices, the **cd**-index was minimized on the simplex of the same dimension, respectively Boolean algebra of the same rank. Both of these conjectures were shown to be true. See [9, 28].

Theorem 3.4.1 (Billera–Ehrenborg)**.** *The* **cd***-index of a convex n-polytope is coefficient-wise greater than or equal to the* **cd***-index of the n-simplex.*

Theorem 3.4.2 (Ehrenborg–Karu)**.** *The* **cd***-index of a Gorenstein* lattice of rank n is coefficient-wise greater than or equal to the* **cd***-index of the Boolean algebra B_n.*

Define an inner product on $\mathbf{k}\langle \mathbf{c}, \mathbf{d}\rangle$ by

$$\langle u|v\rangle = \delta_{u,v}$$

where u and v are **cd**-monomials and extend by linearity. We can use this notation to encode inequalities easily. For example,

$$\left\langle \mathbf{d} - \mathbf{c}^2 | \Psi(P) \right\rangle \geq 0$$

says the for a 2-dimensional polytope the coefficient of $\mathbf{d}$ minus the coefficient of $\mathbf{c}^2$ is nonnegative. (True, as $(n-2)-1 \geq 0$ for $n \geq 3$.) We can now state Ehrenborg's lifting technique [26, Theorem 3.1].

Theorem 3.4.3 (Ehrenborg)**.** *Let u and v be two* **cd***-monomials. Suppose u does not end in* **c** *and v does not begin with* **c***. Then the inequality*

$$\langle H|\Psi(P)\rangle \geq 0 \text{ implies } \langle u \cdot H \cdot v|\Psi(P)\rangle \geq 0.$$

where H is a **cd***-polynomial such that the inequality $\langle H|\Psi(P)\rangle \geq 0$ holds for all polytopes P.*

Corollary 3.4.4. *For two* **cd***-monomials u and v the following inequality holds for all polytopes P:*

$$\langle u \cdot \mathbf{d} \cdot v|\Psi(P)\rangle \geq \left\langle u \cdot \mathbf{c}^2 \cdot v|\Psi(P)\right\rangle.$$

This corollary says the coefficient of a **cd**-monomial increases when replacing a $\mathbf{c}^2$ with a $\mathbf{d}$.

3.5. **A word about shellings.**

A pure n-dimensional polytopal complex is *shellable* if there is an ordering of its facets $F_1, \ldots, F_s$, called a *shelling order*, such that (i) ∂F_1 is shellable, (ii) for all $1 \leq k \leq s$, the intersection of $F_k \cap \cup_{i=1}^{k-1} F_i$ is shellable of dimension $n-1$. If a polytopal complex is of dimension 0, then any order of its vertices is declared to be a valid shelling order.

In Section 1.4, it was pointed out that many proofs for results about polytopes were incomplete as they assumed all polytopes (that is, the complex formed by the boundary of the polytope) are shellable. Shellability of polytopes was settled in 1971 by Bruggesser and Mani [20].

Theorem 3.5.1 (Bruggesser–Mani)**.** *Polytopes are shellable.*

Proof. The idea of the proof is to treat the boundary of a polytope as a planet and to send a space rocket off from the planet. Unlike NASA, your rocket travels in a straight line. As you are taking off, you should write down the order of the new facets you are seeing, starting with the first facet you took off from. Eventually you will see all the facets on one side of the polytope. The rocket goes off to infinity, then returns from the other direction along the same straight line. You begin to descend on the other side of the polytope. Now record the facets which begin to disappear as you approach your landing spot. The order of the facets you recorded is a shelling order. □

The shelling order in Bruggesser–Mani is called a *line shelling*.

Given a shelling order for a polytope, observe this builds the polytope one facet at a time a polytope one facet at a time so that at each shelling step the polytopal complex is topologically a ball except the last step when it becomes a sphere.

The notion of spherical shellability is closely related to shellability. The **cd**-index is only defined for regular decompositions of a sphere?? In order for Stanley to proof of the nonnegativity of the **cd**-index, he had to work with spherical objects. At each shelling step of a polytope, he attached an artificial facet to close off the complex $F_1 \cup \cdots \cup F_i$ into a sphere. He was then able to show at each shelling step that the coefficients of the **cd**-index were weakly increasing and hence nonnegative.

Proposition 3.5.2. *[Stanley] Let $F_1, \ldots, F_s$ be a spherical-shelling of a regular cellular sphere Ω. Then*

$$0 \le \Psi(F_1') \le \Psi((F_1 \cup F_2)') \le \cdots \le \Psi((F_1 \cup \cdots \cup F_{n-1})') = \Psi(\Omega), \quad (3.2)$$

where the notation Γ' indicates attaching a cell to the boundary $\partial\Gamma$ of the complex Γ so that is topologically a sphere.

The inequalities in Proposition 3.5.2 were essential in the proof of Theorem 3.4.1, that is, that the n-simplex minimizes the **cd**-index coefficient-wise for all n-polytopes. The proof also required using coalgebra techniques to derive a number of identities, and combining the inequalities into the desired inequality. The proof of Theorem 3.4.3 also used shellings. However, the inequality relations in Proposition 3.5.2 were replaced with a different type of inequality.

3.6. **Notes.**

The labels in an R-labeling do not necessarily have to be the integers, but instead elements from some poset. There are other notions of edge labelings, including EL-labelings (edge-lexicographic labelings), CL-labelings (chain-lexicograpic labelings), and analogues for nonpure complexes. See [18] and the references therein.

In [26] Ehrenborg has determined the best linear inequalities for polytopes of dimension up to dimension 8. There has been some work on finding *quadratic* inequalities for flag vectors of polytopes due to Ling [50]. Bayer's 1987 paper also includes some quadratic inequalities [2].

3.7. **Exercises.**

Exercise 3.7.1. The 3-dimensional permutahedron, depicted on the WAM poster, is a zonotope. Describe the associated hyperplane arrangement, intersection lattice and compute the **cd**-index using Theorem 3.1.2.

Exercise 3.7.2. Finish the computation in (3.1).

Exercise 3.7.3. Use line shellings to prove the Euler–Poincaré formula.

Exercise 3.7.4. Prove Corollary 3.4.4.