TILTING EXERCISES

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This is a geometry-oriented review of the basic formalism of tilting objects (originally due to Ringel, see [Ri][2], §5). In the first section we explain that tilting extensions form a natural framework for the gluing construction from [B1] and [MV]. We show that in case of a stratification with contractible strata, the homotopy category of complexes of tilting perverse sheaves is equivalent to the derived category of sheaves smooth along the stratification. Thus tilting objects play the role similar to projective or injective ones (with advantage of being self-dual and having local origin). In the second section we discuss tilting perverse sheaves smooth along the Schubert stratification of the flag space (or, equivalently, tilting objects in the Bernstein-Gelfand-Gelfand category $\mathcal{O}$). In this case a Radon transform interchanges tilting, projective, and injective modules. As a corollary, we give a short proof of Soergel's Struktursatz [S1], and describe the Serre functor for $D^b(\mathcal{O})$ (as conjectured by M. Kapranov).

We refer to [M] for a much more thorough exposition of many other aspects of the theory.

This article is a modest present to Borya Feigin – with love, and sadness to see him so rarely these days.

..Mr. Fagin took the opportunity of reading Oliver a long lecture on the crying sin of ingratitude: of which he clearly demonstrated he had been guilty, to no ordinary extent, in wilfully absenting himself from the society of his anxious friends..

§1 Generalities.

We consider algebraic varieties over an algebraically closed field k. Below "perverse sheaf" means either plain perverse $\mathbb{Q}_l$-sheaf, $l \neq char(k)$, or perverse sheaf with respect to classical topology with coefficients in any field of characteristic 0 (if $k = \mathbb{C}$), or holonomic $\mathcal{D}$-module (in case $char(k) = 0$). For a variety X we denote by $\mathcal{M}(X)$ the abelian category of perverse sheaves on X, and by $D(X)$ its bounded derived category (which is the same as the usual "topological" derived category of constructible $\mathbb{Q}_\ell$-complexes or complexes of $\mathcal{D}$-modules with holonomic cohomology, see [B2]).

1.1 Let X be an algebraic variety, $i : Y \hookrightarrow X$ a closed subvariety, $j : U := X \smallsetminus Y \hookrightarrow X$ the complementary embedding. Let M be a perverse sheaf on X; then we have[3] $i^! M \in D(Y)^{\geq 0}$, $i^* M \in D(Y)^{\leq 0}$. We say that M is a *tilting* perverse sheaf with respect to Y (or a *Y-tilting* perverse sheaf) if both $i^! M$, $i^* M$ are perverse sheaves. The standard exact triangles together with left (respectively, right) exactness of j_*, $j_!$ show that M is tilting if and only if both $j_* j^* M$ and $j_! j^* M$

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[2] Notice that our terminology differs from the one in *loc. cit.* and some other sources; the term "tilting" there is used for a weaker property.

[3] Here the t-structure is of the middle perversity.

are perverse sheaves and the canonical morphisms $M \to j_* j^* M$, $j_! j^* M \to M$ are, respectively, surjective and injective. The category of Y-tilting perverse sheaves is closed under extensions and Verdier duality.

Proposition. Let M_U be a perverse sheaf on U such that both $j_*(M_U)$ and $j_!(M_U)$ are perverse sheaves on X. Then there exists a Y-tilting perverse sheaf M on X such that $M|_U = M_U$. We call such M a Y-*tilting extension* of M_U to X.

Proof. (a) Set $A := \mathrm{Ker}(j_!(M_U) \to j_*(M_U))$, $B := \mathrm{Coker}(j_!(M_U) \to j_*(M_U))$. These are perverse sheaves supported on Y. Let $c \in \mathrm{Ext}^2(B, A)$ be the Yoneda class of exact sequence $0 \to A \to j_!(M_U) \to j_*(M_U) \to B \to 0$.

(b) If c vanishes, then there exists a perverse sheaf M together with a 3 step filtration $M_0 \subset M_1 \subset M$ and identifications $M_0 = A$, $M_1 = j_!(M_U)$, $M/M_0 = j_*(M_U)$, $M/M_1 = B$ compatible in the obvious manner with the above exact sequence. Then from the exact sequences

$$0 \to A \to M \to j_*(M_U) \to 0,$$

$$0 \to j_!(M_U) \to M \to B \to 0$$

we see that $A \xrightarrow{\sim} i^!(M)$, $i^*(M) \xrightarrow{\sim} B$; thus M is Y-tilting (we call it a *minimal tilting extension* of M_U for obvious reasons), and we are done.

(c) If $c \neq 0$ then we have to correct our exact sequence. To do this notice that by [B2] the Yoneda Ext's are the same as Ext's in the usual derived category of sheaves on X. The latter can be computed inside the derived category of sheaves on Y, and then as Yoneda Ext of perverse sheaves on Y. Thus one can find an exact sequence $0 \to A \to C \to D \to B \to 0$ of perverse sheaves supported on Y of the class $-c$. Let $0 \to A \to C' \to D' \to B \to 0$ be the Baer sum of the two Yoneda extensions. Its class vanishes, thus there exists a perverse sheaf M together with a 3 step filtration $M_0 \subset M_1 \subset M$ such that $M_0 = A$, $M_1 = C'$, $M/M_0 = D'$, $M/M_1 = B$ compatible in the obvious manner with the above exact sequence. Since $j_!(M_U) \hookrightarrow C' \hookrightarrow M$ we see that $i^*(M) = \mathrm{Coker}(j_!(M_U) \to M)$ is a perverse sheaf, and since $M \twoheadrightarrow D' \twoheadrightarrow j_*(M_U)$ we see that $i^!(M) = \mathrm{Ker}(M \twoheadrightarrow j_*(M_U))$ is perverse; thus M is tilting, and we are done. $\square$

Remarks. (i) The conditions of Proposition are always satisfied if j is an affine embedding.

(ii) If Y is a divisor given by equation $f = 0$ then the "maximal extension" $\Xi_f(M_U)$ from [B1] is a functorial tilting extension.

1.2 Let us show that the gluing construction from [B1] and [MV] fits naturally into the setting of tilting extensions.

Let M_U be a perverse sheaf on U, and let M^{tilt} be a Y-tilting extension of M_U. Set $\Psi := i^! M^{tilt}$, $\Psi' := i^* M^{tilt}$; let $\tau : \Psi \to \Psi'$ be the composition of the canonical morphisms $\Psi \hookrightarrow M^{tilt} \twoheadrightarrow \Psi'$.

Proposition. The category $\mathcal{E}$ of extensions of M_U to X is canonically equivalent to the category $\mathcal{C}$ of diagrams $\Psi \xrightarrow{\alpha} \Phi \xrightarrow{\beta} \Psi'$ where Φ is a perverse sheaf on Y and morphisms α, β are such that $\beta\alpha = \tau$.

Proof.[4] (a) The functor $\mathcal{E} \to \mathcal{C}$ sends $M \in \mathcal{E}$ to $(\Phi, \alpha, \beta) \in \mathcal{C}$ defined as follows. Consider a short complex

$$(1.2.1) \qquad F = F(M) := (j_!(M_U) \to M \oplus M^{tilt} \to j_*(M_U))$$

where the differentials are defined by the property that their restriction to U are, respectively, the diagonal embedding and the anti-diagonal projection. Set $\Phi = \Phi(M) := H^0 F$. Notice that the tilting property of M^{tilt} assures $H^{\neq 0} F = 0$. We have the obvious morphisms

$$(1.2.2) \qquad \mathcal{C}one(M^{tilt} \to j_*(M_U))[-1] \to F \to \mathcal{C}one(j_!(M_U) \to M^{tilt}).$$

Passing to cohomology, we get $\Psi \xrightarrow{\alpha} \Phi \xrightarrow{\beta} \Psi'$. It is clear that $(\Phi, \alpha, \beta) \in \mathcal{C}$.

(b) The inverse functor $\mathcal{C} \to \mathcal{E}$ sends $\Phi = (\Phi, \alpha, \beta) \in \mathcal{C}$ to $M = M(\Phi) \in \mathcal{E}$ defined as follows. Consider a short complex

$$(1.2.3) \qquad G = G(\Phi) := (\Psi \to \Phi \oplus M^{tilt} \to \Psi')$$

where the differentials are, respectively, the sum of α and the canonical embedding $\Psi \to M^{tilt}$ and the difference of β and the canonical projection $M^{tilt} \to \Psi'$. Set $M := H^0 G$. Notice that $H^{\neq 0} G = 0$.

It remains to show that functors from (a) and (b) are mutually inverse. To identify $M(\Phi(M))$ with M let us replace $\Psi \to \Phi \to \Psi'$ in the definition of complex $G(\Phi)$ by (1.2.2). We get a complex whose cohomology equals $M(\Phi(M))$. On the other hand, by construction, this complex carries a 3 step filtration with successive quotients equal to the cone of the identity morphism of $\mathcal{C}one(M^{tilt} \to j_*(M_U))[-1]$, M, and the cone of the identity morphism of $\mathcal{C}one(j_!(M_U) \to M^{tilt})[-1]$. Thus its cohomology equals M. The construction of the isomorphism $\Phi(M(\Phi)) \xrightarrow{\sim} \Phi$ is similar and left to the reader. $\qquad\square$

Remark. It follows from the part (b) of the proof that $i^* M = \mathcal{C}one(\alpha : \Psi \to \Phi)$, $i^! M = \mathcal{C}one(\beta : \Phi \to \Psi')[-1]$. Thus M is tilting if and only if α is injective and β is surjective.

1.3 Suppose that our variety X carries a stratification $\{X_\nu\}$; let $i_\nu : X_\nu \hookrightarrow X$ be the locally closed embeddings of the strata. We say that a perverse sheaf M is *tilting* with respect to our stratification if for every ν both complexes $i_\nu^! M, i_\nu^* M$ are perverse sheaves on X_ν.

Assume that each i_ν is an affine embedding.

Proposition. A perverse sheaf M is tilting with respect to our stratification if and only if it satisfies the following two conditions:

1. M can be represented as a successive extension of perverse sheaves of type $i_{\nu*} N_\nu$ where N_ν is a perverse sheaf on X_ν.

2. Same with $i_{\nu*}$ replaced by $i_{\nu!}$.

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[4]This is an immediate generalization of the proof of [B1] 3.1 that dealt with the particular case $M^{tilt} = \Xi_f(M_U)$.

Proof. Our conditions obviously imply that M is tilting (notice that $i_\mu^! i_{\nu*} N_\nu$ equals N_ν if $\mu = \nu$ and 0 otherwise). Conversely, suppose that M is tilting. Choose a closed filtration $X \supset X_1 \supset .. \supset X_n \supset X_{n+1} = \emptyset$ such that $X_i \smallsetminus X_{i+1}$ is a single stratum. Set $j : U := X \smallsetminus X_n \hookrightarrow X$. Using induction by n we can assume that $j^* M$ is a successive extension of perverse sheaves $j^* i_{\nu*} N_\nu$. Thus $j_* j^* M$ is a successive extension of $j_* j^* i_{\nu*} N_\nu = i_{\nu*} N_\nu$, and the tilting property assures that M is an extension of $j_* j^* M$ by a perverse sheaf $i_{n*} i_n^! M$. So condition 1 holds. Condition 2 is checked in the dual manner. $\qquad\square$

1.4 We are in situation of 1.3; assume in addition that every X_ν is smooth and connected. Let $D = D(X, \{X_\nu\}) \subset D(X)$ be the full subcategory of complexes constant along $\{X_\nu\}$, i.e., those $F \in D(X)$ that for every ν the complex $i_\nu^* F$ has constant cohomology sheaves. To assure that D is a reasonable object to deal with, we assume the following two properties:

- The cohomology groups with constant coefficients $H^1(X_\nu)$ vanish. Then D is a triangulated subcategory of $D(X)$. Notice that D is generated by objects $i_{\nu!} M_\nu$ where M_ν are constant (perverse) sheaves on X_ν.

- One has $i_{\nu*} M_\nu \in D$, i.e. D is preserved by the Verdier duality. Then D is a t-category with core $\mathcal{M} = \mathcal{M}(X, \{X_\nu\}) := \mathcal{M} \cap D$; its irreducible objects are middle extensions of constant perverse sheaves of rank 1 on strata.

Suppose, in addition, that $H^2(X_\nu) = 0$ for every ν.

Remark.[5] Under the above assumptions the category $\mathcal{M}$ is what different authors call an abstract Kazhdan-Lusztig category, or a highest weight category, or a quasi-hereditary category (see e.g. [BGS], §3.2 and reference therein). Statements parallel to the next two Propositions are true (and apparently well-known to the experts) for a general category of this sort.

Let $\mathcal{T} = \mathcal{T}(X, \{X_\nu\}) \subset \mathcal{M}$ be the full subcategory of tilting sheaves with respect to our stratification.

Proposition. The support of an indecomposable object $M \in \mathcal{T}$ is irreducible, i.e., it is the closure of some stratum X_ν, and $i_\nu^! M$ is a constant (perverse) sheaf of rank 1 on X_ν. The map $M \to Supp\, M$ is a bijection between the set of isomorphism classes of indecomposable objects in $\mathcal{T}$ and the set of strata.

Proof. Use induction by the number n of strata. We follow notation of the proof of Proposition 1.3. By induction our statement is true for the category $\mathcal{T}_U$ of tilting sheaves on U equipped with the induced stratification. For every object $M_U \in \mathcal{T}_U$ the complexes $j_*(M_U)$, $j_!(M_U)$ are perverse sheaves (use 1.3). The class c from part (a) of the proof of Proposition 1.1 vanishes since $H^2(X_n) = 0$, so M_U admits a minimal tilting extension $M \in \mathcal{T}$ (see ibid., part (b)). Remark in 1.2 implies that for indecomposable M_U the above M is indecomposable, and every indecomposable tilting extension of M_U is isomorphic to M. It also implies that every tilting extension of a decomposable M_U is decomposable. We are done. $\quad\square$

1.5 We are in situation 1.4, and assume, in addition, that $H^{>0}(X_\nu) = 0$.

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[5]We thank the referee to whom this remark is due.

Proposition. One has canonical equivalences of triangulated categories

(1.5.1) $$K^b(\mathcal{T}) \xrightarrow{\sim} D^b\mathcal{M} \xrightarrow{\sim} D.$$

Here $K^b\mathcal{T}$ is the homotopy category of bounded complexes in $\mathcal{T}$.

Proof. The functors $K^b(\mathcal{T}) \to D^b\mathcal{M} \to D$ in (1.5.1) are the obvious ones.

(i) Let us show that the composition $K^b\mathcal{T} \to D$ is an equivalence of categories. By Proposition 1.4 the image of $K^b\mathcal{T}$ generates D, so it suffices to prove that for every $M, N \in \mathcal{T}$ one has $Ext_D^{>0}(M,N) = 0$. By Proposition 1.3, one needs to check that $Ext_D^{>0}(M,N) = 0$ for $M = i_{\mu!}M_\mu$, $N = i_{\nu*}N_\nu$, where M_μ, N_ν are constant perverse sheaves on strata X_μ, X_ν respectively. This follows by adjunction if $\mu \neq \nu$, and by the vanishing of the higher cohomology of strata if $\mu = \nu$.

(ii) Let us show that $D^b\mathcal{M} \to D$ is an equivalence of categories. This is a t-exact functor which identifies the cores, so it suffices to check that the morphism of the δ-bifunctors $Ext_{D^b\mathcal{M}}^{\cdot} \to Ext_D^{\cdot}$ on $\mathcal{M}^\circ \times \mathcal{M}$ is an isomorphism, or, equivalently, that $Ext_D^{\cdot}$ is effaceable. By (i) our functor $D^b\mathcal{M} \to D$ admits a right inverse, so $Ext_D^{\cdot}$ is a quotient functor of $Ext_{D^b\mathcal{M}}^{\cdot}$, hence it is effaceable, q.e.d. □

Remark. An alternative proof of the second equivalence in (1.5.1) can be found in [BGS], Corollary 3.3.2 on page 500.

§2 The case of Schubert stratification.

2.1. Let G be a semisimple algebraic group. Let $X = G/B$ be the flag variety stratified by the Schubert cells X_w, $w \in W$, where W is the Weyl group. Our stratified space satisfies conditions of 1.5. Set $D := D(X, \{X_w\})$, and let $\mathcal{O} \subset D$ be the category of perverse sheaves.

For $w \in W$ let $L_w, T_w \in \mathcal{O}$ be, respectively, irreducible and indecomposable tilting objects supported on the closure of X_w; let I_w and P_w be, respectively, an injective hull and projective cover of L_w. Let $\mathcal{T}$, $\mathcal{P}$, $\mathcal{I}$ be the categories of, respectively, tilting, projective, and injective objects. We also let $\Delta_w = i_{w!}(M_w)$, $\nabla_w = i_{w*}(M_w)$ where M_w is the constant perverse sheaf of rank 1 on X_w.

Let $\mathcal{O}_{>0} \subset \mathcal{O}$ be the Serre subcategory generated by L_w, $w \neq e$ (where $e \in W$ is the identity); $\mathcal{O}_0 = \mathcal{O}/\mathcal{O}_{>0}$, and $\pi : \mathcal{O} \to \mathcal{O}_0$ be the projection functor (or its extension to the derived categories). We can identify $\mathcal{O}_0$ with the category of modules over $End(P_e)$; the functor π is then identified with $X \to Hom(P_e, X)$.

Proposition. The functor $\pi|_\mathcal{T}$ is fully faithful.

We will need the following standard fact:

Lemma. The socle of Δ_w and the cosocle of ∇_w are isomorphic to L_e.

Proof of Lemma. Let us prove the statement about Δ_w; the one about ∇_w then follows by Verdier duality. We argue by induction in the length $\ell(w)$. If $w = e$ there is nothing to prove, and if $\ell(w) = 1$ then the statement follows from the existence of a non-split exact sequence

(2.1.1) $$0 \to \Delta_e \to \Delta_w \to L_w \to 0$$

of perverse sheaves on $\mathbb{P}^1$.

Assume now that $w = w's$, where s is the simple reflection corresponding to a simple root α, and $\ell(w) > \ell(w')$. Let X^α be the corresponding partial flag variety, and $pr_\alpha : X \to X^\alpha$ be the projection; thus pr_α is a fibration with projective lines as fibers. Set $X_w^\alpha = pr_\alpha(X_w)$; $X_w' = pr_\alpha^{-1}(X_w^\alpha)$, and let $i_w^\alpha : X_w^\alpha \hookrightarrow X^\alpha$, $i_w' : X_w' \hookrightarrow X$ be the embeddings. Then i_w' is an affine morphism because it is a base change of the affine morphism i_w^α. Hence the functor $i_{w'!}'$ is exact. The fibration pr_α is trivial over $pr(X_w)$, so we have $X_w' \cong \mathbb{P}^1 \times X_{w'}$. Applying the functor $i_{w'!}' \circ pr_1^*[\ell(w) - 1]$ to (2.1.1) (where $pr_1 : X_w' \to \mathbb{P}^1$ is the projection) we get an exact sequence in $\mathcal{O}$

$$0 \to i_{w'!}(M_1) \to i_{w!}(M_2) \to i_{w!}'(M_3) \to 0,$$

where M_1, M_2, M_3 are constant perverse sheaves on the corresponding varieties. Let $L_u \subset \Delta_w$ be a simple subobject. Suppose first that the composition $L \to i_{w!}'(M_3)$ is nonzero. It is easy to see that this only can happen if $\ell(u \cdot s_\alpha) < \ell(u)$ so that $L_u = pr_\alpha^*(L')[1]$ for a certain irreducible perverse sheaf L' on X^α. We arrive to a contradiction since $Hom(L_u, i_{w!}(M_2)) = Hom\,(L'[1], pr_{\alpha\bullet}i_{w!}(M_2)) = Hom\,(L'[1], i_{w!}^\alpha(M_4[-1])) = Ext^{-2}\,(L', i_{w!}^\alpha(M_4)) = 0$ (here M_4 is a rank 1 perverse constant sheaf on X_w^α; and we use the notation $f_\bullet := f_* = f_!$ for a proper morphism f). Thus we have $L \subset i_{w'!}(M_1)$, so we get the desired statement by induction. $\square$

Proof of Proposition. Let $\mathcal{A}$ be an abelian category, and $\mathcal{B} \subset \mathcal{A}$ be a Serre subcategory. Define the left and right orthogonals to $\mathcal{B}$ in $\mathcal{A}$ by

$$^\perp\mathcal{B} = \{A \in \mathcal{A} \mid Hom(A, X) = 0 \ \ \forall X \in \mathcal{B}\}$$

$$\mathcal{B}^\perp = \{A \in \mathcal{A} \mid Hom(X, A) = 0 \ \ \forall X \in \mathcal{B}\}.$$

It follows from the definitions that if $A \in {}^\perp\mathcal{B}$ and $B \in \mathcal{B}^\perp$, then $Hom_\mathcal{A}(A, B) \xrightarrow{\sim} Hom_{\mathcal{A}/\mathcal{B}}(A, B)$. The lemma implies that $\Delta_w \in \mathcal{O}_{>0}^\perp$, $\nabla_w \in {}^\perp\mathcal{O}_{>0}$ for all w. Hence $\mathcal{T} \subset {}^\perp\mathcal{O}_{>0} \cap \mathcal{O}_{>0}^\perp$, so our proposition is proved. $\square$

2.2. We recall the intertwining functors (Radon transforms) acting on D. Let $\ell(w) = \dim(X_w)$ be the length function. For $w \in W$ let $X_w^2 \subset X^2$ be the G-orbit corresponding to w (thus $X_w^2 = G(X_e \times X_w)$). Let $pr_i^w : X_w^2 \to X$ for $i = 1, 2$ be the projections. Set $R_w^?(X) = pr_{2?}pr_1^*(X)[\ell(w)]$, where $? =!$ or $*$. We need a standard

Fact. For $? =!$ or $*$ we have:

(a) $R_{w_1}^? \circ R_{w_2}^? \cong R_{w_1 w_2}^?$ provided $\ell(w_1 w_2) = \ell(w_1) + \ell(w_2)$.

(b) $R_w^! \circ R_{w^{-1}}^* \cong id \cong R_{w^{-1}}^* \circ R_w^!$.

(c) $\pi \circ R_w^? \cong \pi$.

Proof. (a) and (b) are well known (see e.g. [BB]). Using (a) we see that it is enough to check (c) for w of length 1; so assume that $w = s_\alpha$ is a simple reflection. We treat the case $? =!$, the other case follows. Let $\overline{X}_{s_\alpha}^2$ be the closure of $X_{s_\alpha}^2$, and let $\overline{pr}_1, \overline{pr}_2 : \overline{X}_{s_\alpha}^2 \to X$ be the projections. Thus $\overline{pr}_1, \overline{pr}_2$ are fibrations with fiber $\mathbb{P}^1$. For $M \in D$ we have a canonical exact triangle

$$\delta_*(M) \to i_! pr_1^* M[1] \to \overline{pr}_1^* M[1]$$

where $\delta : X \to X^2$ is the diagonal embedding, and $i : X^2_{s_\alpha} \hookrightarrow X^2$ is the embedding. Applying $\overline{pr}_{2!}$ to it, we see that it suffices to check that

(2.2.1) $$\pi(\overline{pr}_{2!}\overline{pr}_1^* M) = 0$$

This is clear since $\overline{pr}_{2!}\overline{pr}_1^* M = pr_\alpha^* pr_{\alpha!} M$ (we use notation of the proof of Lemma 2.1). Indeed, the pull-back functor from X_α identifies irreducible perverse sheaves constant along the Schubert stratification on X_α with irreducible objects of $\mathcal{O}$ constant along the fibers of p_α, so L_e cannot occur in $pr_\alpha^* pr_{\alpha!} M$. $\qquad\square$

2.3. The following result, inspired by W. Soergel's article [S2], appears in [BG] (see *loc.cit.* Theorem 6.10(i)); it was also found independently by R. Rouquier (unpublished). We include a proof for the reader's convenience.

Let $w_0 \in W$ be the longest element.

Proposition. We have $R^!_{w_0}(I_w) \cong T_{ww_0}$; $R^!_{w_0}(T_w) = P_{ww_0}$.

Lemma. Assume we are in the situation of 1.5; denote by M_ν a constant perverse sheaf of rank 1 on X_ν. Let $M \in D(X, \{X_\nu\})$ be any perverse sheaf.

If $Ext^a(M, i_{\nu!}(M_\nu)) = 0$ for every $a > 0$ and ν, then M is projective.

If $Ext^a(i_{\nu*}(M_\nu), M) = 0$ for every $a > 0$ and ν, then M is injective.

Proof of Lemma. We prove the first statement, the second one is similar. We will say that an object A of a triangulated category is filtered by objects B_i if there exist objects $A_0 = 0, A_1, \ldots, A_n = A$ and exact triangles $A_{i-1} \to A_i \to B_i$. Then the definition of the perverse t structure implies that any perverse sheaf N is filtered (as an object of the triangulated category $D(X, \{X_\nu\})$) by objects of the form $i_{\nu!}(M_\nu)[d]$, $d \le 0$. Thus the condition implies that $Ext^a(M, N) = 0$ for all $a > 0$. $\qquad\square$

Proof of Proposition. We prove the first isomorphism, the second one is similar. Let us first see that $R^!_{w_0}(T_w)$ is a projective object of $\mathcal{O}$. By Fact above we have

$$R^!_{w_0}(\nabla_w) = R^!_{w_0 w} \circ R^!_{w^{-1}}(R^*_w(\Delta_e)) = R^!_{w_0}(\Delta_e) = \Delta_{w_0 w}.$$

It follows that $R^!_{w_0}(\nabla_w) \in D$ is filtered by the objects ∇_w, in particular, it lies in $\mathcal{O}$. We also have

$$Ext^a(R^!_{w_0}(T_w), \Delta_w) = Ext^a(R^!_{w_0}(T_w), R^!_{w_0}(\nabla_{w_0 w})) = Ext^a(T_w, \nabla_{w_0 w}) = 0$$

for $a > 0$, where the last equality follows from the fact that T_w is filtered by objects Δ_u, and $Ext^{>0}(\Delta_u, \nabla_v) = 0$. Thus, by Lemma, we see that $R^!_{w_0}(T_w)$ is projective. Moreover, $R^!_{w_0}(T_w)$ is indecomposable, and it follows from the above that it is filtered by objects of the form Δ_{w_i} where $w_1 = w_0 w$, and $w_i \succ w_0 w$ for $i > 1$. It follows that $R^!_{w_0}(T_w) \cong P_{w_0 w}$. We have proved the second isomorphism; by Verdier duality it implies that $R^*_{w_0}(T_w) \cong I_{w_0 w}$; and applying $R^!_{w_0}$ to both sides we get the first isomorphism. $\qquad\square$

2.4 Corollary. (Soergel's Struktursatz, [S1], p.433) The functors $\pi|_{\mathcal{I}}$, $\pi|_{\mathcal{P}}$ are fully faithful. $\qquad\square$

2.5. Recall some definitions of Bondal and Kapranov [BK]. Let $\mathcal{D}$ be a k-linear category such that $Hom(X, Y)$ is a finite-dimensional vector space for all $X, Y \in \mathcal{D}$. Suppose that $\mathcal{D}$ admits an endofunctor S equipped with a natural isomorphism $\alpha : Hom(X, S(Y)) \overset{\sim}{\to} Hom(Y, X)^*$. Such (S, α) is evidently unique.[6] It is called *the Serre functor* if S is actually an auto-equivalence of $\mathcal{D}$.[7] If $\mathcal{D}$ is a triangulated category, then S is naturally a triangulated functor.

Let us return to our situation. The Serre functor on D exists by the results [BK] (compare [BK], Corollary 3.5 with either Theorem 2.11 or Corollary 2.10 in *loc. cit.*). In fact, the bounded derived category $D^b(A)$ has the Serre functor whenever A is an Artinian abelian category of finite homological dimension having enough projectives and finitely many isomorphism classes of irreducible objects.

The following result was conjectured by Kapranov:

Proposition. The Serre functor S for D is isomorphic to $(R_{w_0}^*)^2$ as a triangulated functor.

Proof. It takes two steps:

(i) Our functors send $\mathcal{P}$ to $\mathcal{I}$ and their restrictions to $\mathcal{P}$ are isomorphic.

(ii) Any isomorphism of functors $(R_{w_0}^*)^2|_{\mathcal{P}} \overset{\sim}{\to} S|_{\mathcal{P}}$ extends in a canonical way to an isomorphism of triangulated functors $(R_{w_0}^*)^2 \overset{\sim}{\to} S$.

Proof of (i). Notice that for each $w \in W$ one has $S(P_w) \cong I_w \cong (R_{w_0}^*)^2(P_w)$ as follows, respectively, from [BK] 3.2(3) and 2.3.

We will prove that any isomorphism $S(P_e) \overset{\sim}{\to} (R_{w_0}^*)^2(P_e)$ extends uniquely to an isomorphism of functors $S|_{\mathcal{P}} \overset{\sim}{\to} (R_{w_0}^*)^2|_{\mathcal{P}}$.

According to 2.4, we can replace our functors by their composition with π. By Fact 2.2(c), one has $\pi(R_{w_0}^*)^2 \cong \pi$, so we can reformulate our claim as follows: Any isomorphism $\alpha : \pi S(P_e) \overset{\sim}{\to} \pi(P_e)$ extends uniquely to an isomorphism of functors $\pi S|_{\mathcal{P}} \overset{\sim}{\to} \pi|_{\mathcal{P}}$. Since $\pi = Hom(P_e, \cdot)$, it suffices to check that α commutes with the action of $End(P_e)$. As follows from the definition of the Serre functor, S commutes with any endomorphism of the identity functor Id_D. Now any endomorphism of P_e comes from an endomorphism of Id_D, as follows from 2.4 and commutativity of $End(P_e)$ established in [S1], Lemma 5 on p. 430,[8] and we are done.

Proof of (ii). For a k-linear functor $\phi : \mathcal{P} \to \mathcal{I}$ let $C(\phi) : C^b(\mathcal{P}) \to C^b(\mathcal{I})$ be its DG extension to the category of bounded complexes and $D(\phi)$ the triangulated endofunctor of $D = K^b(\mathcal{P}) = K^b(\mathcal{I})$ defined by $C(\phi)$. We have seen that the restrictions of S and $(R_{w_0}^*)^2$ to $\mathcal{P}$ are isomorphic functors $\mathcal{P} \to \mathcal{I}$. We will show that there are *canonical* identifications of triangulated endofunctors $S \overset{\sim}{\to} D(S|_{\mathcal{P}})$, $(R_{w_0}^*)^2 \overset{\sim}{\to} D((R_{w_0}^*)^2|_{\mathcal{P}})$; this yields (ii).

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[6] And for given S all possible α form a torsor with respect to an obvious action of the group of automorphisms of the identity functor $Id_{\mathcal{D}}$.

[7] Additional requirements on S imposed in the definition of the Serre functor in [BK] are actually redundant, see the proof of Proposition 3.4 in *loc. cit.*

[8] In fact, Soergel's "Endomorphismensatz" (*loc. cit.* p. 428; see also [B]) provides a very explicit description of $End(P_e)$, see Remark (ii) below.

The statement about S is clear. Indeed, since S is the Serre functor, we have a natural isomorphism $Hom(X, S(Y)) \xrightarrow{\sim} Hom(Y, X)^*$ for $X \in \mathcal{O}$, $Y \in \mathcal{P}$. It extends canonically to a natural DG isomorphism $Hom(X, S(Y)) \xrightarrow{\sim} Hom(Y, X)^*$ for $X \in C^b(\mathcal{O})$, $Y \in C^b(\mathcal{P})$, which makes $D(S|_{\mathcal{P}})$ the Serre functor.

Consider $(R^*_{w_0})^2$. This is the restriction to D of the endofunctor $(R^*_{w_0})^2$ of the derived category $D(X)$ of constructible complexes on X. The latter functor has "geometric origin" hence it lifts canonically to a triangulated endofunctor $(R^{*F}_{w_0})^2$ of the filtered derived category $DF(X)$ of finitely filtered constructible complexes. Recall (see [BBD] 3.1) that there is a canonical fully faithful embedding $C^b(\mathcal{M}(X)) \hookrightarrow DF(X)$ whose essential image consists of those filtered complexes P that $\mathrm{gr}^i P \in \mathcal{M}(X)[-i]$ for any i (the inverse functor identifies such P with the complex of perverse sheaves $.. \to \mathrm{gr}^i P \to \mathrm{gr}^{i+1} P \to ..$ where the differential is the third side of the triangle $\mathrm{gr}^{i+1} P \to P_i/P_{i+2} \to \mathrm{gr}_i P$). The equivalence $D^b(\mathcal{O}) \xrightarrow{\sim} D \subset D(X)$ from 1.5.1 comes from the composition $C^b(\mathcal{O}) \hookrightarrow C^b(\mathcal{M}(X)) \hookrightarrow DF(X) \to D(X)$ where the third arrow is the forgetting of filtration functor. Now the restriction of $(R^{*F}_{w_0})^2$ to $C^b(\mathcal{P}) \subset C^b(\mathcal{M}(X))$ is $C^b(\mathcal{P}) \to C^b(\mathcal{I}) \subset C^b(\mathcal{M}(X)) \subset DF(X)$ where the arrow is $C((R^*_{w_0})^2|_{\mathcal{P}})$. Since $(R^{*F}_{w_0})^2$ lifts $(R^*_{w_0})^2$, we are done. $\qquad\square$

2.6 Remarks. (i) Let $\tilde{W}$ be the braid group associated to the root system of G, and for $w \in W$ let $\tilde{w} \in \tilde{W}$ be its canonical (minimal length) lifting. According to 2.2, for $? = *, !$ the map $w \to R^?_w$ extends to a weak action of $\tilde{W}$ on D (extending it to a strong action in the sense of [D] requires more work; this is done in [R]). Notice that $\tilde{w_0}^2$ is a central element in $\tilde{W}$. This conforms with the general fact that for any triangulated category $\mathcal{D}$ with a Serre functor S, and any other functor $F : \mathcal{D} \to \mathcal{D}$ which admits a left adjoint LF we have a canonical isomorphism

$$F \circ S \cong S \circ L(LF)$$

(where $L(LF)$ denotes the left adjoint to LF). In particular, if F is invertible we have $LF \cong F^{-1}$, so $L(LF) \cong F$, i.e. F commutes with S.

(ii) In the step (i) of the proof of Proposition we have shown that the set of isomorphism of functors $(R^*_{w_0})^2|_{\mathcal{P}} \xrightarrow{\sim} S|_{\mathcal{P}}$ identifies canonically with the $Z^\times$-torsor of invertible elements in the Z-module $K := Hom((R^*_{w_0})^2(P_e), S(P_e))$ where $Z := End(P_e)$. Now $K \xrightarrow{\sim} Hom(P_e, (R^!_{w_0})^2(P_e))^* \xrightarrow{\sim} Hom(\pi(P_e), \pi((R^!_{w_0})^2(P_e)))^*$ which equals $Hom(\pi(P_e), \pi(P_e))^* \xrightarrow{\sim} Hom(P_e, P_e)^* \xrightarrow{\sim} Z^*$ (the k-linear dual to Z) by 2.2(c). Thus we have a canonical isomorphism of Z-modules $K \xrightarrow{\sim} Z^*$. Accordnig to [S1], [B], there is a canonical isomorphism of algebras $Z \xrightarrow{\sim} H^*(X^\vee)$, where $X^\vee$ is the flag space for the Langlands dual group $G^\vee$.[9] So the trace map $H^*(X^\vee) \to k$ provides a canonical generator of the Z-module Z^*. It yields a canonical isomorphism of functors $(R^*_{w_0})^2|_{\mathcal{P}} \xrightarrow{\sim} S|_{\mathcal{P}}$ hence, by step (ii) of the proof of Proposition, an identification of the triangulated functors $(R^*_{w_0})^2 \xrightarrow{\sim} S$.

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[9]Actually [S1], [B] work with modules over the enveloping algebra, so one has to invoke the localization theorem to derive the computation of $End(P_e)$ from their results. There is an equivalent, purely topological, construction (see [BGS], p. 525) of the morphism $H^\cdot(X^\vee) \to A$. One knows that $H^\cdot(X^\vee)$ is generated by $H^2(X^\vee)$, and the Chern class for the $T^\vee$-torsor $G^\vee/N^\vee$ over $X^\vee$ provides a canonical identification $\mathfrak{t} \xrightarrow{\sim} H^2(X^\vee)$, where $\mathfrak{t}$ is the Cartan algebra of G. So our morphism is determined by a linear map $\mathfrak{t} \to Z$. Our perverse sheaves are monodromic (of unipotent monodromy) with respect to the action of (any) maximal torus $T \subset G$ on X. Now $\mathfrak{t} \to Z$ is the logarithm of the monodromy map.

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