Invent. math. 87, 153–215 (1987)

Inventiones mathematicae

Proof of the Deligne-Langlands conjecture for Hecke algebras

David Kazhdan[1] and George Lusztig[2]

[1] Department of Mathematics, Harvard University, Cambridge, MA 02138, USA
[2] Department of Mathematics, M.I.T., Cambridge, MA 02139, USA

Introduction

We shall be concerned with a special case of the general Langlands conjecture [La] on irreducible representations of reductive p-adic groups, namely the case where the group is split with connected centre and the representations have non-zero vectors fixed by an Iwahori subgroup.

According to the conjecture, the representations should be parametrized essentially by representations of the Galois group into the complex dual group. A later refinement of the conjecture, due independently to Deligne and Langlands, has added a unipotent element in the picture. According to them, the representations we are concerned with should be parametrized essentially by conjugacy classes of pairs (u,s) where u (resp. s) is a unipotent (resp. semi-simple) element in the complex dual group such that $sus^{-1}=u^q$, (q = number of elements in the residue field). This was proved by Bernstein, Zelevinskii [BZ], [Z_1] for GL_n. However, in the general case, the conjecture was still not completely precise in this form and in [L_1, 1.5] a third ingredient was added to (u,s) namely an irreducible representation ρ of the (finite) group of components of the simultaneous centralizer of u and s, such that ρ appears in the representation of this finite group on the homology of $\mathscr{B}_u^s$, the variety of Borel subgroups containing u and s. This was motivated in party by the second author's formulation (in 1980) of an analogue (for the unramified principal series of a p-adic group) of our conjecture on multiplicities in Verma modules. The conjecture involved intersection cohomology of a certain variety attached to (u,s) and ρ appeared as a local system. (This conjecture has been found independently (for type A) by Zelevinskii [Z_2]).

As it is well known, the representations we are concerned with are naturally in 1-1 correspondence with the finite dimensional irreducible representations of the Hecke algebra H with respect to the Iwahori subgroup; this paper will be concerned exclusively with H and the p-adic group itself will not be present.

In this paper we shall prove that the irreducible representations of H are indeed in 1-1 correspondence with the conjugacy classes of triples (u, s, ρ) as above.

In [L_4] it has been conjectured that for each (u, s) as above there is a natural H-module structure on the equivariant K-theory of the variety $\mathscr{B}_u$ of all Borel subgroups containing u, with respect to a certain group defined in term of s, which acts on $\mathscr{B}_u$; this has been confirmed by [KL_2] and [G].

In this paper we shall construct for each (u, s, ρ) an H-module by a procedure dual to that of [KL_2] (in the sense that we use equivariant K-homology instead of equivariant K-cohomology) and we prove that this H-module has a unique simple quotient; we also prove that this gives the required 1-1 correspondence.

An important role in our proof is played by the analysis of the equivariant K-homology of the space of triples (u, B, B') where u is a variable unipotent element and B, B' are Borel subgroups of G containing u; we show that this give a model for the regular representation of H, and that it has a natural filtration given by unipotent classes. This is entirely analogous to our proof of completeness of the Springer representations of Weyl groups [KL_1].

The reason we were forced to use equivariant K-homology instead of equivariant K-cohomology is the following one: the variety of triples (u, B, B') as above with u restricted to lie in a fixed unipotent class may have $K^0 \neq 0$, $K^1 \neq 0$, while its K_1 vanishes (at least after tensoring with $\mathbb{C}$). For this reason, our proof could not be carried out in K^*.

We shall now describe the contents of this paper in more detail. § 1 contains essentially a list of properties of equivariant K-homology which are needed in the paper. It also contains a sketch of a definition of equivariant K-homology. This chapter contains very few proofs. In § 2, we fix notations and recollect some results on reductive groups. We also give the definition of the Hecke algebra in terms of generators and relations, following Bernstein; these are a modified form of the Iwahori-Matsumoto relations.

In § 3 we analyse the equivariant K-homology of the variety of triples and we show that it gives a model for the regular representation of H. We also define actions of the generators of H on this K-homology.

In § 4 we show that the variety $\mathscr{B}_u^s$ has no odd rational homology; the proof is a reduction to the case when $s=1$, which is a known (but deep) result. This is used in § 5 to define a filtration of the regular representation of H in terms of unipotent classes and to analyse its successive quotients. In this chapter we also define the "standard H-module" attached to (u, s, ρ) and we prove some of its properties.

In § 6, we prove that under a certain hypothesis, a standard H-module is induced from an analogous module over a smaller Hecke algebra. This is used in § 7 in the proof of the main Theorem (7.12); the proof of this theorem is in part a geometric counterpart of arguments used in the proof of Langlands quotient theorem, (see [BW, IV]).

Finally, § 8 contains the classification of tempered and square-integrable representations of H.

A different construction (to a large extent conjectural) of the standard H-modules has been given in [L_2].

This paper has been written so that it is entirely independent of [KL_2]; we have not tried to establish here the connection (=duality) of the construction here with that in [KL_2]. This can be done without difficulty.

The results of this paper were the object of two lectures given by one of us at Princeton University and Oberwolfach in march and april 1985.

After this paper has been written, we have received (June 1985) Ginsburg's preprint "Deligne-Langlands conjecture and representations of affine Hecke algebras".

One of his main results (Theorem 2.4 in the preprint) is a slightly stronger version of our main Theorem 7.12. However, Ginsburg's proof has some serious errors; in particular his Theorem 2.4 is false as stated.

Acknowledgements. We wish to thank Ginsburg for some interesting correspondence.

We also whish to thank Kato, Segal, Hodgkin, Thomason and especially Bernstein for some useful conversations. We thank the referee for some helpful comments. Both authors were supported in part by the National Science Foundation. We would like to thank I.H.E.S. for its hospitality during the final stages of writing this paper.

Contents

1. Equivariant K-homology 155
2. Preliminaries on reductive groups 165
3. The variety of triples 171
4. A vanishing theorem 182
5. A filtration of $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ 184
6. The induction theorem 195
7. The classification theorem 204
8. Square integrable $\mathbf{H}$-modules 212

1. Equivariant K-homology

1.1. In [Se], [AS], Atiyah and Segal considered for any locally compact space X with a continuous action of a compact group M_0, equivariant K-groups with compact supports $K_{M_0}^i(X)$, $i=0,1$; these are modules over R_{M_0}, the ring of representations of M_0.

In [A], Atiyah suggested a construction of a "dual theory" $K_i^{M_0}(X)$, $i=0,1$, called K-homology (at least for $M_0=\{e\}$ and X compact) which should play the same role for $K_{M_0}^i(X)$ as homology with arbitrary supports plays for cohomology with compact supports.

Atiyah's suggestion has been carried out by Kasparov [K] who defined $K_i^{M_0}(X)$ in the framework of functional analysis.

In Sect. 1.3, we shall present a list of properties of the groups $K_i^{M_0}(X)$. We shall restrict ourselves to the case where X is an algebraic variety over $\mathbb{C}$ and M_0 is a maximal compact subgroup of a linear algebraic group M acting algebraically on X. (In this case we say that X is an M-variety; all algebraic varieties and algebraic groups will be assumed in this paper to be over $\mathbb{C}$). Only a few of the properties we need can be found in [K]. Recently, Thomason [T₁,T₂] gave a purely algebraic construction of $K_i^{M_0}(X)\otimes\mathbb{Z}_\ell$, ($l$=prime number), for X, M_0, M as above and has verified for it several of the properties in 1.3. One can also give a more elementary definition of $K_i^{M_0}(X)$ in terms of "Alexander duality"; this definition is implicit in [A], see also [BFM]. We will give a sketch in 1.4.

The properties will be stated in terms of $\mathbf{K}_i^{M_0}(X)\overset{\text{def}}{=}K_i^{M_0}(X)\otimes\mathbb{C}$.

Since $K_i^{M_0}(X)$, $\mathbf{K}_i^{M_0}(X)$ are independent (up to unique isomorphism) of the choice of maximal compact subgroup M_0 for M, we shall also write $\mathbf{K}_i^M(X)$ instead of $\mathbf{K}_i^{M_0}(X)$. Similarly, we shall write $\mathbf{K}_M^i(X)$ instead of $K_{M_0}^i(X)\otimes\mathbb{C}$. When $M=\{e\}$, we shall write $\mathbf{K}^i(X)$, $\mathbf{K}_i(X)$ instead of $\mathbf{K}_M^i(X)$, $\mathbf{K}_i^M(X)$.

1.2. Let R_M be the Grothendieck group of finite dimensional rational representations of the algebraic group M; it is naturally isomorphic (by restriction) to R_{M_0}, the Grothendieck group of finite dimensional complex continuous representations of a maximal compact subgroup M_0 of M.

We set $\mathbf{R}_M=R_M\otimes\mathbb{C}$. This is a commutative $\mathbb{C}$-algebra of finite type. Taking characters of representations defines an isomorphism of $\mathbf{R}_M$ with the $\mathbb{C}$-algebra of all regular functions $M\to\mathbb{C}$ which are constant on each coset by the unipotent radical of M and also on each conjugacy class in M.

In particular, the maximal ideals of $\mathbf{R}_M$ are in 1-1 correspondence with the semisimple elements (up to conjugacy) in M; if $s\in M$ is a semisimple element, the corresponding maximal ideal I_s of $\mathbf{R}_M$ is the kernel of the algebra homomorphism $\mathbf{R}_M\to\mathbb{C}$, $E\to\operatorname{Tr}(s,E)$.

(a) We shall denote by $\hat{\mathbf{R}}_{M,s}$ the completion of $\mathbf{R}_M$ with respect to the I_s-adic topology.

If i: $M'\to M$ is a homomorphism of algebraic groups, we have a natural algebra homomorphism $\mathbf{R}_M\to\mathbf{R}_{M'}$ obtained by composing representations of M with i.

If i: $M'\to M$ is the embedding of a closed subgroup of M, then the induced homomorphism $\mathbf{R}_M\to\mathbf{R}_{M'}$ corresponds to a finite morphism between the corresponding spaces of maximal ideals.

(b) Hence, if $\mathscr{M}$ is a finitely generated $\mathbf{R}_{M'}$-module and s is semisimple element of M' which is central in M, the natural map $\hat{\mathbf{R}}_{M,s}\underset{\mathbf{R}_M}{\otimes}\mathscr{M}\to\hat{\mathbf{R}}_{M',s}\underset{\mathbf{R}_{M'}}{\otimes}\mathscr{M}$ is an isomorphism.

(c) Given a semisimple element $s\in M$ and an $\mathbf{R}_M$-module $\mathscr{M}$ (resp. $\hat{\mathbf{R}}_{M,s}$-module $\mathscr{M}'$) we shall denote $\mathbb{C}_s\underset{\mathbf{R}_M}{\otimes}\mathscr{M}$ (resp. $\mathbb{C}_s\underset{\hat{\mathbf{R}}_{M,s}}{\otimes}\mathscr{M}'$) the tensor product $\mathbf{R}_M/I_s\underset{\mathbf{R}_M}{\otimes}\mathscr{M}$ (resp. $(\hat{\mathbf{R}}_{M,s}/I_s\hat{\mathbf{R}}_{M,s})\underset{\hat{\mathbf{R}}_{M,s}}{\otimes}\mathscr{M}'$).

1.3. We shall present a list of properties of $\mathbf{K}_i^M(X)$ for X an M-variety.

In the paper we will most of the time need only the properties in the list. (An exception is 4.3, where we use an additional property following from the definition sketched in 1.4.)

(a) For any M-variety X, $\mathbf{K}_i^M(X)$ is an $\mathbf{R}_M$-module of finite type, ($i=0$ or 1).

(b) *Direct image.* If $f\colon X \to X'$ is an M-equivariant proper morphism of M-varieties, then there is a natural $\mathbf{R}_M$-homomorphism $f_*\colon \mathbf{K}_i^M(X) \to \mathbf{K}_i^M(X')$. It is transitive in an obvious sense.

(c) *Inverse image.* If $f\colon X \to X'$ is an M-equivariant smooth morphism of M-varieties, then there is a natural $\mathbf{R}_M$-homomorphism $f^*\colon \mathbf{K}_i^M(X') \to \mathbf{K}_i^M(X)$. It is transitive in an obvious sense.

(d) Let $f_1\colon X_1 \to X_2$, $f_1'\colon X_2' \to X_3$ be as in (b) and let $f_2\colon X_2 \to X_3$, $f_2'\colon X_1 \to X_2'$ be as in (c). Assume that these maps form a cartesian diagram. Then

$$(f_1)_*(f_2')^* = f_2^*(f_1')_*\colon \mathbf{K}_i^M(X_2') \to \mathbf{K}_i^M(X_2).$$

(e) Let $f\colon X \to Y$ be a morphism of algebraic varieties, let H be a linear algebraic group acting on X in such a way that f is a principal H-bundle (locally trivial in the Zariski topology). Assume that M acts on X, Y and on the algebraic group H in such a way that f and the action $H \times X \to X$ are M-equivariant.

If H is unipotent, then $f^*\colon \mathbf{K}_i^M(Y) \to \mathbf{K}_i^M(X)$ is an isomorphism.

If $H = \mathrm{Aut}(A^1)$, (with trivial action of M), $\bar{X} = X \underset{H}{\times} A^1$ and $\bar{f}\colon \bar{X} \to Y$ is induced by f, then $\bar{f}^*\colon \mathbf{K}_i^M(Y) \to \mathbf{K}_i^M(\bar{X})$ is an isomorphism, is (A^1 = affine line).

(f) *Tensor product with a complex.* Let X be an M-variety and let $\mathscr{E} = (\dots 0 \to E_n \to \dots \to E_0 \to 0 \dots)$ be an M-equivariant complex of algebraic vector bundles on X. (Each E_i is an M-equivariant vector bundle and each map of vector bundles $E_i \to E_{i-1}$ is compatible with the M-action). Let X_0 be a closed M-stable subvariety of X such that $\mathscr{E}$ is acyclic on $X - X_0$. Then there is a natural $\mathbf{R}_M$-homomorphism

$$\mathscr{E} \otimes\colon \mathbf{K}_0^M(X) \to \mathbf{K}_0^M(X_0).$$

This is compatible with the operations in (b), (c) as follows. Let $f\colon X' \to X$ be an M-equivariant morphism of M-varieties, let $\mathscr{E}'$ be the pull back of $\mathscr{E}$ under f and let $X_0' = f^{-1}(X_0)$.

Then we have $\mathscr{E}' \otimes\colon \mathbf{K}_0^M(X') \to \mathbf{K}_0^M(X_0')$. Let $f_0\colon X_0' \to X_0$ be the restriction of f.

(f1) If f is proper, then $(\mathscr{E} \otimes) f_* = (f_0)_*(\mathscr{E}' \otimes)\colon \mathbf{K}_0^M(X') \to \mathbf{K}_0^M(X_0)$.

(f2) If f is smooth, then $f_0^*(\mathscr{E} \otimes) = (\mathscr{E}' \otimes) f^*\colon \mathbf{K}_0^M(X) \to \mathbf{K}_0^M(X_0')$.

(f3) If $\mathscr{E} = (\dots 0 \to E_0 \to 0 \dots)$, we can take $X_0 = X$ and we shall write $E_0 \otimes \xi$ instead of $\mathscr{E} \otimes \xi$ for all $\xi \in \mathbf{K}_0^M(X)$. Thus, $E_0 \otimes \xi \in \mathbf{K}_0^M(X)$.

(f4) If V is a rational finite dimensional M-module and $\mathbf{V}$ is the corresponding M-equivariant vector bundle on X (a trivial bundle if we ignore the M-action) then $\mathbf{V} \otimes \xi = V\xi$ ($\xi \in \mathbf{K}_0^M(X)$) where $V\xi$ is given by the $\mathbf{R}_M$-module structure of $\mathbf{K}_0^M(X)$, (see (a)).

(f5) If $\mathscr{E} = (\dots 0 \to E_n \to \dots \to E_1 \to 0 \dots)$ is as above and $X_0 = X$, then $\mathscr{E} \otimes \xi = \sum_i (-1)^i E_i \otimes \xi$, ($\xi \in \mathbf{K}_0^M(X)$).

Let $\pi\colon E\to X$ be an M-equivariant algebraic vector bundle over X and let $\mathscr{E}=(\dots 0\to \Lambda^n\tilde{\pi}E^*\to\dots\to\Lambda^0\tilde{\pi}E^*\to 0\dots)$ be the usual Koszul complex of vector bundles over E. (Here, $\tilde{\pi}E^*$ denotes the pull-back of the dual of E under π). This is acyclic outside the image of the zero section $j\colon X\hookrightarrow E$ of π (which is identified with X). Hence we have $\mathscr{E}\otimes\colon \mathbf{K}_0^M(E)\to\mathbf{K}_0^M(X)$.

Then:

(f6) $\mathscr{E}\otimes$ is the inverse of $\pi^*\colon \mathbf{K}_0^M(X)\xrightarrow{\approx}\mathbf{K}_0^M(E)$.

We have

(f7) $$(\pi^*)^{-1}j_*\xi=\sum_i(-1)^i\Lambda^iE^*\otimes\xi,\qquad (\xi\in\mathbf{K}_0^M(X)).$$

(g) Let X be an M-variety, let F be a closed M-stable subvariety of X and let $j\colon F\hookrightarrow X$, $j'\colon X-F\hookrightarrow X$ be the inclusions. Then there is a natural exact hexagon of $\mathbf{R}_M$-modules:

$$\begin{tikzcd}
 & \mathbf{K}_0^M(X) \arrow[dr, "j'^*"] & \\
\mathbf{K}_0^M(F) \arrow[ur, "j_*"] & & \mathbf{K}_0^M(X-F) \arrow[d] \\
\mathbf{K}_1^M(X-F) \arrow[u] & & \mathbf{K}_1^M(F) \arrow[dl, "j'_*"] \\
 & \mathbf{K}_1^M(X) \arrow[ul, "j^*"] &
\end{tikzcd}$$

(h) Let X be an M-variety and let $M\hookrightarrow\tilde{M}$ be an imbedding of M as a closed subgroup of another algebraic group $\tilde{M}$.

Let $\tilde{X}=M\backslash(\tilde{M}\times X)$ where M acts by $m\colon(\tilde{m},x)\to(\tilde{m}m^{-1},mx)$.

Then $\tilde{X}$ is an $\tilde{M}$-variety, $\tilde{m}\colon(\tilde{m}',x)\mapsto(\tilde{m}\tilde{m}',x)$ and there is a natural isomorphism of $\mathbf{R}_{\tilde{M}}$-modules

(h1) $$\mathbf{K}_i^M(X)\cong\mathbf{K}_i^{\tilde{M}}(\tilde{X})$$

compatible in an obvious sense with the operations (b) and (c). Here, the $\mathbf{R}_M$-module $\mathbf{K}_i^M(X)$ is regarded as an $\mathbf{R}_{\tilde{M}}$-module via the natural homomorphism $\mathbf{R}_{\tilde{M}}\to\mathbf{R}_M$.

The isomorphism (h1) is compatible with the operation $\mathscr{E}\otimes$ (see (f)) as follows. Let $\mathscr{E}$, $X_0\subset X$ be as in the beginning of (f). There is a canonical $\tilde{M}$-equivariant complex of vector bundles $\tilde{\mathscr{E}}$ on $\tilde{X}$ whose pull-back under $\tilde{M}\times X\to\tilde{X}$ is the same as the pull-back of $\mathscr{E}$ under $\mathrm{pr}_2\colon\tilde{M}\times X\to X$. Consider the operation $\tilde{\mathscr{E}}\otimes\colon\mathbf{K}_0^{\tilde{M}}(\tilde{X})\to\mathbf{K}_0^{\tilde{M}}(\tilde{X}_0)$ of (f) where $\tilde{X}_0=M\backslash(\tilde{M}\times X_0)\subset\tilde{X}$. We have a commutative diagram

(h2) $$\begin{tikzcd}
\mathbf{K}_0^M(X) \arrow[r, "\approx"', "(\mathrm{h1})"] \arrow[d, "\mathscr{E}\otimes"'] & \mathbf{K}_0^{\tilde{M}}(\tilde{X}) \arrow[d, "\tilde{\mathscr{E}}\otimes"] \\
\mathbf{K}_0^M(X_0) \arrow[r, "\approx", "(\mathrm{h1})"'] & \mathbf{K}_0^{\tilde{M}}(\tilde{X}_0)
\end{tikzcd}$$

(i) Let X be an M-variety and let $h\colon M'\to M$ be a homomorphism of algebraic groups. Then X can be regarded as an M'-variety (via h) and there is a natural homomorphism of $\mathbf{R}_{M'}$-modules

(i1) $$\bar{h}\colon \mathbf{K}_i^M(X) \to \mathbf{K}_i^{M'}(X).$$

(Here, the $\mathbf{R}_M$-module $\mathbf{K}_i^M(X)$ is regarded as an $\mathbf{R}_{M'}$-module, via h). It is transitive in an obvious sense and $\bar{h}=$identity if $h\colon M \to M$ is the identity. It is compatible with the operations in (b), (c) and with the operator $\mathscr{E}\otimes$ in (f) in an obvious sense.

(j) Let X be an M-variety, let $\alpha\colon M \to M$ be an automorphism of M as an algebraic group and let $\beta\colon X \to X$ be an automorphism of X such that $\beta(mx) = \alpha(m)\beta(x)$ for all $m \in M$, $x \in X$. We can regard X as an M-variety in another way, with a new action of M, $m\colon x \mapsto \alpha(m)x$; we shall denote X_α the variety X with this new action of M. Then β may be regarded as an M-equivariant map $X \to X_\alpha$. It induces $\beta_*\colon \mathbf{K}_i^M(X) \to \mathbf{K}_i^M(X_\alpha)$, see (b).

This is an isomorphism of $\mathbf{R}_M$-modules. On the other hand, applying (i1) to $\alpha\colon M \to M$ we get a $\mathbb{C}$-linear isomorphism $\bar{\alpha}\colon \mathbf{K}_i^M(X) \to \mathbf{K}_i^M(X_\alpha)$. This is not $\mathbf{R}_M$-linear in general; it satisfies $\bar{\alpha}(V\xi) = \tilde{\alpha}(V)\bar{\alpha}(\xi)$, $(V \in \mathbf{R}_M, \xi \in \mathbf{K}_i^M(X_\alpha))$ where $\tilde{\alpha}\colon \mathbf{R}_M \to \mathbf{R}_M$ is the algebra automorphism defined in terms of class functions f on M (see 1.2) by $(\tilde{\alpha}(f))(m) = f(\alpha(m))$, $m \in M$. The composition $(\bar{\alpha})^{-1}\ \beta_*\colon \mathbf{K}_i^M(X) \to \mathbf{K}_i^M(X)$ is then a $\mathbb{C}$-linear isomorphism satisfying $((\bar{\alpha})^{-1}\beta_*)(V\xi) = \tilde{\alpha}^{-1}(V)((\bar{\alpha})^{-1}\beta_*)(\xi)$, $(V \in \mathbf{R}_M,\ \xi \in \mathbf{K}_i^M(X))$. Now let $\tilde{M}$ be an algebraic group containing M as a closed normal subgroup and assume that $\tilde{M}$ acts algebraically on X, extending the given action of M.

Any element $\tilde{m} \in \tilde{M}$ defines an automorphism $\alpha = \alpha_{\tilde{m}}\colon M \to M (m \to \tilde{m} m \tilde{m}^{-1})$ and an automorphism $\beta = \beta_{\tilde{m}}\colon X \to X (x \to \tilde{m} x)$ to which the previous discussion applies. The resulting map $(\bar{\alpha})^{-1}\beta_*$ will be denoted $A_{\tilde{m}}\colon \mathbf{K}_i^M(X) \to \mathbf{K}_i^M(X)$. It is an isomorphism of $\mathbb{C}$-vector spaces satisfying

$$A_{\tilde{m}}(V\xi) = {}^{\tilde{m}}V A_{\tilde{m}}(\xi), \qquad (V \in \mathbf{R}_M, \xi \in \mathbf{K}_i^M(X))$$

where $V \to {}^{\tilde{m}}V$ is the automorphism of $\mathbf{R}_M$ defined in terms of class functions f on M by

(j1) $${}^{\tilde{m}}f(m) = f(\tilde{m}^{-1} m \tilde{m})$$

We shall need the following properties of $A_{\tilde{m}}$.

(j2) $\tilde{m} \mapsto A_{\tilde{m}}$ defines an action of $\tilde{M}$ on $\mathbf{K}_i^M(X)$ (by $\mathbb{C}$-linear automorphisms) whose kernel contains both M and the identity component $\tilde{M}^0$ of $\tilde{M}$

(j3) Let $h\colon M \hookrightarrow \tilde{M}$ be the inclusion. Then the image of $\bar{h}\colon \mathbf{K}_i^{\tilde{M}}(X) \to \mathbf{K}_i^M(X)$ (see (i1)) is contained in $\mathbf{K}_i^M(X)^{\tilde{M}}$, the $\tilde{M}$-invariant part under the action (j2).

(j4) If X' is another $\tilde{M}$-variety and $f\colon X' \to X$ is a proper $\tilde{M}$-equivariant morphism, then for each $\tilde{m} \in \tilde{M}$ we have $f_* A_{\tilde{m}} = A_{\tilde{m}} f_*$.

(k) *Localization.* Let X be an M-variety, let s be a semisimple element in the centre of M and let $j\colon X^s \to X$ be the inclusion of the fixed point set of $s\colon X \to X$. Then the $\mathbf{R}_M$-homomorphism $j_*\colon \mathbf{K}_i^M(X^s) \to \mathbf{K}_i^M(X)$ (see (b)) induces isomorphism on the localization with respect to the maximal ideal I_s of $\mathbf{R}_M$ (see 1.2).

(l) *Completion.* Let X be an M-variety and let s be a semisimple element in the centre of M. Assume that s acts trivially on X and that $\mathbf{K}_1(X)=0$. Let M^1 be the subgroup of M generated by s and by the identity component of M. Then M^1 is normal, of finite index in M; let $h: M^1 \hookrightarrow M$ be the inclusion. Write $\hat{\mathbf{R}}_{M^1}$ (resp. $\hat{\mathbf{R}}_M$) instead of $\hat{\mathbf{R}}_{M^1,s}$ (resp. $\hat{\mathbf{R}}_{M,s}$) (see 1.2), and let $\hat{I}_s^1$ (resp. $\hat{I}_s$) be the unique maximal ideal of $\hat{\mathbf{R}}_{M^1}$ (resp. $\hat{\mathbf{R}}_M$).

Let $\hat{\mathbf{K}}_i^{M^1}(X)=\hat{\mathbf{R}}_{M^1} \underset{\mathbf{R}_{M^1}}{\otimes} \mathbf{K}_i^{M^1}(X)$, $\hat{\mathbf{K}}_i^M(X)=\hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_i^M(X)$. Then:

(l1) $\hat{\mathbf{K}}_1^M(X)=0, \quad \hat{\mathbf{K}}_1^{M^1}(X)=0$

(l2) $\hat{\mathbf{K}}_0^{M^1}(X)$ is a free $\hat{\mathbf{R}}_{M^1}$-module.

(l3) The homomorphism $\bar{h}$: $\mathbf{K}_i^M(X) \to \mathbf{K}_i^{M^1}(X)^M$ (see (j3)) induces an isomorphism $\hat{\mathbf{K}}_i^M(X) \xrightarrow{\sim} (\hat{\mathbf{K}}_i^{M^1}(X))^M$.

(Note that M acts on $\hat{\mathbf{K}}_i^{M^1}(X)$ by tensor product of the action on $\hat{\mathbf{R}}_{M^1}$ induced by the action (j1) on $\mathbf{R}_{M^1}$, with the action (j2) on $\mathbf{K}_i^{M^1}(X)$; this is well defined since s is central in M).

(l4) Let $\langle s\rangle$ be the smallest algebraic subgroup of M^1 containing s, and let h_1: $\langle s\rangle \hookrightarrow M^1$ be the inclusion. Note that $\langle s\rangle$ acts trivially on X. Let J_s (resp. I_s^1) be the maximal ideal of $\mathbf{R}_{\langle s\rangle}$ (resp. $\mathbf{R}_{M^1}$) corresponding to s (see 1.2). The homomorphism $\bar{h}_1$: $\mathbf{K}_0^{M^1}(X) \to \mathbf{K}_0^{\langle s\rangle}(X)$ (see (i1)) defines an isomorphism

$$\hat{\mathbf{K}}_0^{M^1}(X)/\hat{I}_s^1 \hat{\mathbf{K}}_0^{M^1}(X)=\mathbf{K}_0^{M^1}(X)/I_s^1 \mathbf{K}_0^{M^1}(X) \xrightarrow{\approx} \mathbf{K}_0^{\langle s\rangle}(X)/J_s \mathbf{K}_0^{\langle s\rangle}(X).$$

(m) Assume that X is an M-variety with M acting trivially on X. Let h: $M \to \{e\}$ be the canonical homomorphism and let $\bar{h}$: $\mathbf{K}_i(X) \to \mathbf{K}_i^M(X)$ be as in (i1). Then the induced homomorphism $\mathbf{R}_M \underset{\mathbb{C}}{\otimes} \mathbf{K}_i(X) \to \mathbf{K}_i^M(X)$ is an $\mathbf{R}_M$-isomorphism.

Hence the last group in (l4) is

(m1) $$\mathbf{K}_0^{\langle s\rangle}(X)/J_s \mathbf{K}_0^{\langle s\rangle}(X)=(\mathbf{R}_{\langle s\rangle}/J_s) \underset{\mathbb{C}}{\otimes} \mathbf{K}_0(X)=\mathbf{K}_0(X)$$

where we identify $\mathbf{R}_{\langle s\rangle}/J_s=\mathbb{C}$ in the obvious way.

(m2) If X is compact, then $\mathbf{K}_i(X)$ is naturally isomorphic to the direct sum of the homology groups with complex coefficients, of degree congruent to i (mod 2) of X. If X is an M_1-variety then the action of M_1 on $\mathbf{K}_i(X)$ (see (j2)) corresponds to the usual action of M_1 an homology.

(m3) If X is compact and L is a line bundle over X, then the $\mathbb{C}$-linear endomorphism $\xi \mapsto L \otimes \xi$ of $\mathbf{K}_i(X)$ (see (f3)) is unipotent.

(n) *External tensor product.* Let X, X' be two M-varieties; then $X \times X'$ is an M-variety with the diagonal action of M. There is a natural $\mathbf{R}_M$-homomorphism

(n1) $$\boxtimes: \mathbf{K}_0^M(X) \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_i^M(X') \to \mathbf{K}_i^M(X \times X').$$

It is compatible in an obvious sense with the operations (b), (c) in both variables X, X'. It is compatible with the operation $\mathscr{E}\otimes$ of (f) in the following sense. Let $\mathscr{E}$, $X_0 \subset X$ be as in the beginning of (f); they give rise to $\mathscr{E}\otimes$: $\mathbf{K}_0^M(X) \to \mathbf{K}_0^M(X_0)$. Let $\tilde{\mathscr{E}}$ be the pull-back of $\mathscr{E}$ under pr_1: $X \times X' \to X$. It then gives rise similarly to $\tilde{\mathscr{E}}\otimes$: $\mathbf{K}_0^M(X \times X') \to \mathbf{K}_0^M(X_0 \times X')$. If $\xi \in \mathbf{K}_0^M(X)$, $\xi' \in \mathbf{K}_0^M(X')$, then $\tilde{\mathscr{E}} \otimes (\xi \boxtimes \xi') = (\mathscr{E} \otimes \xi) \boxtimes \xi' \in \mathbf{K}_0^M(X_0 \times X')$. There is an analogous result with respect to the X'-variable.

(n2) If M acts trivially on X and X' and $\mathbf{K}_1^M(X')=0$ then, the map (n1) is an isomorphism.

(n3) Assume that M is reductive connected and it has simply connected derived group. Assume further that $\mathbf{K}_1^M(X)=0$ and $\mathbf{K}_0^M(X)$ is a projective $\mathbf{R}_M$-module. Then the map (n1) is an isomorphism.

(o) Assume that X is a compact, smooth M-variety. There is a natural isomorphism:

$$\text{(o1)} \qquad \mathbf{K}_i^M(X) \cong \mathbf{K}_M^i(X).$$

Let X' be another compact, smooth M-variety and let f: $X \to X'$ be an M-equivariant morphism. Then f_*: $\mathbf{K}_i^M(X) \to \mathbf{K}_i^M(X')$ (see (b)) corresponds under (o1) (for X and X') to the direct image map $\mathbf{K}_M^i(X) \to \mathbf{K}_M^i(X')$, defined, for example, in [KL_2, 1.6(b)]. If moreover, f is smooth, then f^*: $\mathbf{K}_i^M(X') \to \mathbf{K}_i^M(X)$ corresponds under (o1), (for X and X'), to the usual pull-back map $\mathbf{K}_M^i(X') \to \mathbf{K}_M^i(X)$.

If E is an M-equivariant vector bundle on X, then the endomorphism $E\otimes$: $\mathbf{K}_0^M(X) \to \mathbf{K}_0^M(X)$ of (f3) corresponds under (o1) to the endomorphism of $\mathbf{K}_M^0(X)$ defined by the usual tensor product of E with a vector bundle over X.

Assume now that f: $X \to X'$ above is a locally trivial fibration with fibers isomorphic to $\mathbb{P}^1$. Assume that X' is connected. Let T' be the cotangent bundle along the fibers of f. Let L be an M-equivariant algebraic line bundle on X. We regard L and T' as elements of $\mathbf{K}_0^M(X)$, by (o1). Let d be the Euler characteristic of the restriction of L to any fiber of f (regarded as a coherent sheaf). Then

$$\text{(o2)} \qquad f^* f_*(L) = \begin{cases} \sum\limits_{0 \leqq i \leqq d-1} L \otimes T'^{\otimes i}, & \text{if } d \geqq 1. \\ -\sum\limits_{d \leqq i \leqq -1} L \otimes T'^{\otimes i}, & \text{if } d \leqq -1. \\ 0, & \text{if } d = 0. \end{cases}$$

(p) Let X be an M-variety, let M' be an algebraic subgroup of M and let h: $M' \hookrightarrow M$ be the inclusion. Let $\bar{h}$: $\mathbf{K}_i^M(X) \to \mathbf{K}_i^{M'}(X)$ be as in (i1). Ther is a unique $\mathbf{R}_M$-homomorphism

$$\text{(p1)} \qquad \bar{\bar{h}}: \mathbf{R}_{M'} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_i^M(X) \to \mathbf{K}_i^{M'}(X)$$

such that $\bar{\bar{h}}(V \otimes \xi) = V\bar{h}(\xi)$ for all $V \in \mathbf{R}_{M'}$, $\xi \in \mathbf{K}_i^M(X)$.

Let us identify

$$\mathbf{K}_0^M(M/M') \underset{(\mathrm{h1})}{=} \mathbf{K}_0^{M'}(\text{point}) \underset{(\mathrm{o1})}{=} \mathbf{K}_{M'}^0(\text{point}) = \mathbf{R}_{M'}$$

and

$$\mathbf{K}_i^M((M/M')\times X) \underset{(\alpha_*^{-1})}{=} \mathbf{K}_i^M(M'\backslash(M\times X)) \underset{(\mathrm{h}1)}{=} \mathbf{K}_i^{M'}(X)$$

where α: $M'\backslash(M\times X) \xrightarrow{\sim} (M/M')\times X$ is defined by $\alpha(m,x)=(m,mx)$ and M' acts on $M\times X$ by m': $(m,x)\to(mm'^{-1}, m'x)$. Under these identifications:

(p2) The map $\boxtimes$: $\mathbf{K}_0^M(M/M')\otimes_{\mathbf{R}_M}\mathbf{K}_i^M(X)\to\mathbf{K}_i^M((M/M')\times X)$ becomes a map $\boxtimes$: $\mathbf{R}_{M'}\otimes_{\mathbf{R}_M}\mathbf{K}_i^M(X)\to\mathbf{K}_i^{M'}(X)$ which coincides with $\bar{h}$ above.

For future reference, note that

(p3) $\mathbf{K}_0^M(M/M')=\mathbf{R}_{M'}$ (see above) and similarly $\mathbf{K}_1^M(M/M')=0$.

1.4. We shall sketch an elementary definition of $K_i^{M_0}(X)$ in the case where X is an M-variety and M_0 is a maximal compact subgroup of M. This definition is closely related to [BFM] (where the case $M=\{e\}$ is considered) and is also implicit in [A]. Assume first that X is compact. Consider a real analytic M_0-equivariant imbedding α: $X\hookrightarrow Y$ of X into a complex manifold on which M_0 acts continuously by holomorphic transformations. There exists a fundamental system $\mathscr{S}$ of M_0-stable open neighbourhoods of X in Y with the following property: if $U\subset U'$ are in $\mathscr{S}$ and j denotes the inclusion of U in U', then j_*: $K_{M_0}^i(U)\to K_{M_0}^i(U')$ is an isomorphism (with j_* defined as in [Se, p. 136]).

(a) We define $K_i^{M_0}(X)$ to be $K_{M_0}^i(U)$ for any $U\in\mathscr{S}$.

This is clearly independent of the choice of U and one shows that it is also independent of the choice of the imbedding α.

If X is non-compact, one can define in a similar way $K_i^{M_0}(X^+)$ where X^+ is the 1-point compactification of X and then $K_i^{M_0}(X)$ is defined as the cokernel of a natural imbedding $K_i^{M_0}(\text{point})\to K_i^{M_0}(X^+)$.

1.5. We shall now make some remarks on some of the properties (a)–(p) in 1.3 Consider 1.3(d). We want to give a definition of the operation $\mathscr{E}\otimes$ assuming that X is a compact M-variety, in terms of the definition 1.4(a). Let $\mathscr{E}, X_0\subset X$ be as in the beginning of 1.3(f). Let α: $X\hookrightarrow Y$ be as in 1.4. By replacing Y by a suitable open subset containing X, we may assume that $\mathscr{E}$ is the restriction to X of a complex $\tilde{\mathscr{E}}$ of topological M_0-equivariant vector bundles on Y. Let Z be the support of $\tilde{\mathscr{E}}$ (the set of points in Y where $\tilde{\mathscr{E}}$ is not acyclic). Then $Z\cap X\subset X_0$. Let $\mathscr{S}$ be a fundamental system of M_0-stable open neighbourhoods of X in Y as in 1.4 and let $\mathscr{S}_0$ be an analogous fundamental system of M_0-stable open neighbourhoods of X_0 in Y. Let $U_0\in\mathscr{S}_0$. Then $Z\cap(Y-U_0)$ is a closed set in Y; it doesn't meet X_0, hence it doesn't meet X. Hence $Y-(Z\cap(Y-U_0))$ is an open subset of Y containing X. Hence it contains some $U_1\in\mathscr{S}$. If K is a compact set in U_1, then $K\cap Z$ is compact and is contained in U_0. Hence if $\mathscr{F}$ is a complex of M_0-equivariant complex vector bundles on U_1 with compact support K, then $(\tilde{\mathscr{E}}\otimes\mathscr{F})|_{U_0}$ is an M_0-equivariant complex of vector bundles on U_0 with compact support $(=K\cap Z)$. Then $\mathscr{F}\to(\tilde{\mathscr{E}}\otimes\mathscr{F})|_{U_0}$ is a homomorphism $\mathbf{K}_{M_0}^0(U_1)\to\mathbf{K}_{M_0}^0(U_0)$. This is the homomorphism $\tilde{\mathscr{E}}\otimes$: $\mathbf{K}_0^{M_0}(X)\to\mathbf{K}_0^{M_0}(X_0)$, using the identifications $\mathbf{K}_0^{M_0}(X)=\mathbf{K}_{M_0}^0(U_1)$, $\mathbf{K}_0^{M_0}(X_0)=\mathbf{K}_{M_0}^0(U_0)$, see 1.4(a). Consider 1.3(k). The analogous statement in $K_{M_0}^i$-theory is proved in [Se, 4.1]. Consider 1.3(l). These statements can be deduced from the results in [AS]. Consider 1.3(n3). This can be deduced from the analogous result in $K^i_{M_0}$-theory; that result is proved in [Sn], using earlier results of Hodgkin, modulo the following result which is stated in [Sn] as a conjecture.

1.6. **Proposition.** *Assume that M is a reductive connected algebraic group with simply connected derived group and let B be a Borel subgroup of M. Then external tensor product in K^0_M-theory defines an isomorphism*

$$\boxtimes:\ K^0_M(M/B)\bigotimes_{R_M} K^0_M(M/B)\xrightarrow{\sim} K^0_M((M/B)\times(M/B)).$$

Proof. Let T be a maximal torus in B and let W be the corresponding Weyl group. Now B acts naturally on the tangent space at B to M/B. The characters of B (or T) appearing in this representation are called positive roots; they are elements of R_T. It is well known that:

(a) The natural homomorphism $R_M\to R_T$ is injective, with image equal to the W-invariants R_T^W.

In [St], Steinberg has defined some elements $e_v\in R_T$ one for each $v\in W$, such that

(b) The $e_v (v\in W)$ form a basis of R_T as an R_T^W-module and

(c) $$\det(u(e_v))_{(u,v)\in W\times W}=\Delta^{|W|/2}$$

where $\Delta=\Pi(\alpha^{1/2}-\alpha^{-1/2})\in R_T$, (product over all positive roots α). (When $|W|=1$, both sides of (c) are 1).

Let $(\,,\,)$: $K^0_M(M/B)\times K^0_M(M/B)\to R_M$ be the pairing defined by $(E,E')=\pi_*(E\otimes E')$ where π: $M/B\to$point is the natural map and π_* is direct image in K^0_M-theory. Using the natural identification $K^0_M(M/B)=R_B=R_T$, this pairing becomes the pairing $(\,,\,)$: $R_T\times R_T\to R_T^W$ given by $(\alpha,\beta)=\Delta^{-1}\sum_{w\in W}\varepsilon_w w(\alpha\beta\rho)$ where ε_w is the sign of w and ρ^2 is the product of all positive roots. (Here we have used Weyl's character formula). We now show that

(d) $$\det((e_v,e_{v'}))_{(v,v')\in W\times W}=1;$$

(this is a determinant of a matrix with entries in R_T^W).

Indeed, we have $\Delta(e_v,e_{v'})=\sum_w\varepsilon_w w(e_v)\,w(e_{v'})\,w(\rho)$. This is an entry of a product of three $W\times W$-matrices with entries in R_T: the first matrix has entries $w(e_v)$, the second matrix (transposed) has entries $w(e_{v'})$ and the third matrix is diagonal with entries $\varepsilon_w w(\rho)$. It follows that

$$\begin{aligned}\det(\Delta(e_v,e_{v'}))&=\det(w(e_v))^2\prod_w(\varepsilon_w w(\rho))\\&=\Delta^{|W|},\quad\text{by (c),}\end{aligned}$$

and (d) follows. From (d) we see that there is an unique basis $\hat{e}_v (v\in W)$ of R_T as an R_T^W-module such that

(e) $$(e_v,\hat{e}_{v'})=\delta_{v,v'}.$$

We now define $F\colon K_M^0((M/B)\times(M/B))\to K_M^0(M/B)\underset{R_M}{\otimes}K_M^0(M/B)$ by

$$F(\xi)=\sum_{v\in W}\hat{e}_v\boxtimes(\pi_2)_*(\pi_1^*(e_v)\boxtimes\xi)$$

where $\pi_1,\pi_2\colon(M/B)\times(M/B)\to M/B$ are the two projections and π_1^* (resp. $(\pi_2)_*$) are inverse (resp. direct) image in K_M^0-theory; here we regard $e_v,\hat{e}_v$ as elements in $K_M^0(M/B)=R_T$. We now show that

(f) $$F(\eta_1\boxtimes\eta_2)=\eta_1\boxtimes\eta_2,\qquad\text{for any }\eta_1,\eta_2\in K_M^0(M/B).$$

Indeed, the left hand side of (f) equals

$$\sum_v\hat{e}_v\boxtimes(\pi_2)_*(\pi_1^*(e_v\otimes\eta_1)\otimes\pi_2^*\eta_2)=\sum_v\hat{e}_v\boxtimes(\pi^*(\pi_*(e_v\otimes\eta_1))\otimes\eta_2)$$
$$=\sum_v\hat{e}_v\boxtimes(e_v,\eta_1)\eta_2=(\sum_v(e_v,\eta_1)\hat{e}_v)\boxtimes\eta_2=\eta_1\boxtimes\eta_2$$

and (f) is proved. From (f), we see that the map $\boxtimes$ in the proposition is injective and its image is a direct summand as an R_M-module. It is therefore enough to show that $\boxtimes$ is a map between two projective R_M-modules of the same rank $(=|W|^2)$. The R_M-module $K_M^0(M/B)\underset{R_M}{\otimes}K_M^0(M/B)$ is free of rank $|W|^2$, by (b). The variety $(M/B)\times(M/B)$ can be partitioned into locally closed pieces (the orbits of M acting diagonally), one for each $w\in W$. From the exact sequences associated to this partition, we see that $K_M^0((M/B)\times(M/B))$ will be a projective R_M-module of rank $|W|^2$, provided that each piece in the partition has $K_M^1=0$ and K_M^0 a projective R_M-module of rank $|W|$. But each piece is an affine space bundle over M/B and we are reduced to the known statements that $K_M^1(M/B)=0$ and $K_M^0(M/B)$ is a projective R_M-module of rank $|W|$. (See (b)). This completes the proof.

1.7. With the notations in 1.6, let $i\colon M/B\to(M/B)\times(M/B)$ be the diagonal imbedding. Identify $i_*(\mathbb{C})\in K_M^0((M/B)\times(M/B))$ with an element in $R_T\underset{R_M}{\otimes}R_T$, using Proposition 1.6. Let $\varphi\in\mathrm{End}_{R_M}(R_T)$ and let ${}^t\varphi$ be its transpose with respect to the inner product $(\,,)\colon R_T\times R_T\to R_M$ in 1.6. The following is easily checked from the definitions.

(a) $$i_*(\mathbb{C})=\sum_{v\in W}e_v\otimes\hat{e}_v\quad\text{and}\quad(\varphi\otimes1)(i_*(\mathbb{C}))=(1\otimes{}^t\varphi)(i_*(\mathbb{C})).$$

1.8. Let X be an M-variety. Assume that M is reductive connected with simply connected derived group. Let $s\in M$ be a semisimple element, let $Z(s)$ be its centralizer in M and let $h\colon Z(s)\hookrightarrow M$ be the inclusion. Then

(a) the $\mathbf{R}_M$-homomorphism $\bar{h}\colon\mathbf{R}_{Z(s)}\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_i^M(X)\to\mathbf{K}_i^{Z(s)}(X)$, see 1.3(p1), is an isomorphism.

This follows from 1.3(p2) and from the Kunneth formula 1.3(n3) which is applicable since $\mathbf{K}_0^M(M/Z(s))=\mathbf{R}_{Z(s)}$, $\mathbf{K}_1^M(M/Z(s))=0$ (see 1.3(p3)) and $\mathbf{R}_{Z(s)}$ is a free $\mathbf{R}_M$-module by [St].

It is easy to see that the natural map $\mathbf{R}_M \to \mathbf{R}_{Z(s)}$ (corresponding to $Z(s) \hookrightarrow M$) induces an isomorphism

(b) $$\hat{\mathbf{R}}_{M,s} \xrightarrow{\sim} \hat{\mathbf{R}}_{Z(s),s}$$

From (a) and (b) we deduce a natural isomorphism

(c) $$\hat{\mathbf{R}}_{M,s} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_i^M(X) \xrightarrow{\sim} \hat{\mathbf{R}}_{Z(s),s} \underset{\mathbf{R}_{Z(s)}}{\otimes} \mathbf{K}_i^{Z(s)}(X).$$

2. Preliminaries on reductive groups

2.1. In this chapter we shall fix notations and recollect some results on algebraic groups which will be needed in this paper.

Unless otherwise specified, in this paper, G will always denote a connected reductive algebraic group (over $\mathbb{C}$) with simply connected derived group.

We denote by $\mathscr{B}$ the variety of all Borel subgroups of G. The orbits of G on $\mathscr{B} \times \mathscr{B}$ (for the conjugation action on both factors) are naturally indexed by the elements in the Weyl group W; we shall write $B \overset{w}{—} B'$ whenever the orbit of $(B, B') \in \mathscr{B} \times \mathscr{B}$ corresponds to $w \in W$. In general, the dimension of the orbit corresponding to w is $\dim \mathscr{B} + \ell(w)$ where ℓ is the standard length function on W. Let S be the set of simple reflections in W (elements of length 1). If $B \overset{r}{—} B'$ $(r \in S)$ and $B' \overset{w}{—} B''$ then either $B \overset{w}{—} B''$ or $B \overset{rw}{—} B''$; if moreover $\ell(rw) = \ell(w) + 1$ then $B \overset{rw}{—} B''$. We have $B \overset{e}{—} B'$ if and only if $B = B'$.

Let $\leqq$ be the standard partial order on W: we have $w \leqq w'$ if and only if the G-orbit on $\mathscr{B} \times \mathscr{B}$ corresponding to w is contained in the closure of the G-orbit corresponding to w'.

We shall write $B \overset{\geqq w}{—} B'$ instead of: $B \overset{w'}{—} B'$ for some $w' \geqq w$.

We shall write $B \overset{\leqq w}{—} B'$ instead of: $B \overset{w'}{—} B'$ for some $w' \leqq w$.

2.2. Let $r \in S$. A parabolic subgroup P of G is said to be of type r if it is not a Borel subgroup and if for any two Borel subgroups $B \neq B'$ in P we have $B \overset{r}{—} B'$. Let $\mathscr{P}_r$ be the variety of all parabolic subgroups of type r of G. We have a natural map $\pi_r \colon \mathscr{B} \to \mathscr{P}_r$ defined by $\pi_r(B) = P$ whenever $B \in \mathscr{B}$, $P \in \mathscr{P}_r$, $B \subset P$. The fibers of π_r are isomorphic to $\mathbb{P}^1$ and are called "r-lines".

Let P be any parabolic subgroup of G. We associate to P a subset $I \subset S$ (called the type of P) as follows: if $r \in S$, we have $r \in I$ if and only if P contains some parabolic subgroup of type r. It is well known that two parabolic subgroups of G of the same type are conjugate in G.

2.3. We shall denote the Lie algebra of G by $\mathbf{g}$.

Let u be a unipotent element in G. There is an unique homomorphism of algebraic groups $\mathbb{C} \to G$ taking 1 to u; the tangent map takes $1 \in \mathbb{C} = \mathrm{Lie}\ \mathbb{C}$ to a nilpotent element $\log(u) \in \mathbf{g}$. This gives an isomorphism "log" of the variety of unipotent elements in G and the variety of nilpotent elements in $\mathbf{g}$, Its inverse will be denoted exp. For u as above and $\lambda\in\mathbb{C}$, the power u^λ is defined by $\log(u^\lambda)=\lambda\log(u)$.

We shall sometimes write $g\,v$ instead of $\mathrm{Ad}(g)\,v$, for $g\in G$, $v\in\mathfrak{g}$.

2.4. For any element $g\in G$, we denote by $Z(g)$ the centralizer of g in G. If $u\in G$ is unipotent we define

(a) $$\mathscr{B}_u=\{B\in\mathscr{B}\mid u\in B\},$$

(b) $$M(u)=\{(g,q)\in G\times\mathbb{C}^*\mid g\,u\,g^{-1}=u^q\}$$

Then $M(u)$ is a closed subgroup of $G\times\mathbb{C}^*$.

By the Jacobson-Morozov theorem, given u as above there exists a homomorphism of algebraic groups $\varphi\colon SL_2(\mathbb{C})\to G$ such that $\varphi\begin{pmatrix}1&1\\0&1\end{pmatrix}=u$. Moreover, according to Kostant [Ko, 3.6],

(c) φ is uniquely determined by u up to conjugation by an element in $Z(u)$. Using this, one can show in the same way as in [BV, 2.4] that

(d) $M_\varphi=\{(g,q)\in G\times\mathbb{C}^*\mid g\,\varphi(A)\,g^{-1}=\varphi(D(q^{1/2})\,A D(q^{-1/2}))$ for all $A\in SL_2(\mathbb{C})\}$

is a maximal reductive subgroup of $M(u)$. (Here, $q^{1/2}$ denotes a square root of q, and

(e) $$D(\lambda)=\begin{pmatrix}\lambda&0\\0&\lambda^{-1}\end{pmatrix}$$

for any $\lambda\in\mathbb{C}^*$).

We shall need the following result of Mostow [M].

(f) Any linear algebraic group (over $\mathbb{C}$) is the semidirect product of its unipotent radical with any of its maximal reductive subgroups; moreover any two maximal reductive subgroups are conjugate under an element in the unipotent radical.

This implies that any maximal reductive subgroup of $M(u)$ is of the form M_φ for some φ as above. In particular:

(g) if $(s,q)\in M(u)$ is a semisimple element (hence contained in a reductive subgroup of $M(u)$) then there exists φ as above such that $(s,q)\in M_\varphi$.

Moreover

(h) φ in (g) is uniquely determined by (u,s,q) up to conjugacy by an element $x\in Z(s)\cap Z(u)$.

Indeed, if φ' is another choice for φ in (g) then, by (c), φ' is obtained from φ by conjugation by an element $g\in Z(u)$. Then $(s,q)\in M_\varphi$, $(s,q)\in M_{\varphi'}$ hence $(g^{-1}s\,g,q)\in M_\varphi$. Applying the first assertion of (f) to $M(u)$ we see that the element $(g,1)\in M(u)$ can be written as a product of an element $(x,1)\in\mathscr{V}$ (= the unipotent radical of $M(u)$) with an element $(g',1)\in M_\varphi$. We can therefore assume that $g=x$. We have $(s,q)\in M_\varphi$, $(x^{-1}s\,x,q)\in M_\varphi$, $x\in\mathscr{V}$, $s\,x\,s^{-1}\in\mathscr{V}$ and $(x,1)\cdot(x^{-1}s\,x,q)=(s\,x\,s^{-1},\,1)(s,q)\in\mathscr{V}\cdot M_\varphi$. Using the first assertion in (f) for $M(u)$, we deduce that $x^{-1}s\,x=s$. Hence $x\in Z(s)$ and since we also have $x\in Z(u)$, (h) follows.

2.5. Given a unipotent element $u\in G$ and a semisimple element $(s,q)\in M(u)$ we define three algebraic subgroups of $M(u)$:

(a) $$M(u,s)=\{(g,a)\in M(u)\mid g\in Z(s)\}$$

(b) $M^1(u,s)=$subgroup of $M(u,s)$ generated by (s,q) and by the identity component $M^0(u,s)$ of $M(u,s)$.

(Note that $M(u,s)$ and $M^1(u,s)$ are defined without specifying q).

(c) $M(s,q)=$smallest algebraic subgroup of $G\times\mathbb{C}^*$ containing (s,q).

We have clearly: $M(s,q)\subset M^1(u,s)\subset M(u,s)$. We denote

(d) $\bar{M}(u,s)=M(u,s)/M^0(u,s)=$group of connected components of $M(u,s)$.

We also define

(e) $$\mathscr{B}^s=\{B\in\mathscr{B}\mid s\in B\}$$

and

(f) $$\mathscr{B}_u^s=\mathscr{B}_u\cap\mathscr{B}^s=\{B\in\mathscr{B}\mid u\in B, s\in B\}$$

According to [L_1]:

(g) The variety $\mathscr{B}_u^s$ is non-empty.

2.6. Let φ: $SL_2(\mathbb{C})\to G$, $\varphi\begin{pmatrix}1&1\\0&1\end{pmatrix}=u$ be as in 2.4. For each integer i we define

(a) $$\mathbf{g}_i=\{x\in\mathbf{g}\mid\varphi(D(\lambda))\,x=\lambda^i x, \forall\,\lambda\in\mathbb{C}^*\}$$

where $D(\lambda)$ is as in 2.4(e). We have $\mathbf{g}=\bigoplus_i\mathbf{g}_i$, and $\mathbf{g}_0$ is a Lie algebra.

Let G_0 (resp. P_0) be the connected algebraic subgroup of G whose Lie algebra is $\mathbf{g}_0$ (resp. $\bigoplus_{i\geqq 0}\mathbf{g}_i$). Then G_0 acts naturally on each $\mathbf{g}_i$.

We shall need the following result of Malcev (see [Ko, 4]).

(b) G_0 has an unique open orbit on $\mathbf{g}_2$; this orbit contains $\log(u)$.

We shall also need the following known result:

(c) $$M(u)\subset P_0\times\mathbb{C}^*.$$

The maximal reductive subgroup M_φ (see 2.4(d)) is clearly contained in $P_0\times\mathbb{C}^*$ hence it is enough to show that the unipotent radical of $M(u)$ is in $P_0\times\mathbb{C}^*$. This follows from the infinitesimal analogue of (c):

(d) $\operatorname{Ker}(\operatorname{ad}\log(u)\colon\mathbf{g}\to\mathbf{g})\subset\bigoplus_{i\geqq 0}\mathbf{g}_i$ which follows from the representation theory of $SL_2(\mathbb{C})$.

2.7. Let $\mathbf{X}$ be the set of isomorphism classes of algebraic G-equivariant line bundles on $\mathscr{B}$ where G acts on $\mathscr{B}$ by conjugation.

It is a finitely generated free abelian group under tensor product; we shall sometimes write L_1L_2 instead of $L_1\otimes L_2$ and L_1^{-1} instead of the dual of L_1, for $L_1,L_2\in\mathbf{X}$.

For each $r \in S$, let $L_r \in \mathbf{X}$ be the tangent bundle along the fibers of π_r: $\mathscr{B} \to \mathscr{P}_r$; let $\check{\alpha}_r$: $\mathbf{X} \to \mathbb{Z}$ be the homomorphism defined by $\check{\alpha}_r(L) = m$ where $L \in \mathbf{X}$ and the restriction of L to any fiber of $\pi_r (\approx \mathbb{P}^1)$ has Euler characteristic $m+1$ (as a coherent sheaf). We have $\check{\alpha}_r(L_r) = 2$.

Define an automorphism r: $\mathbf{X} \to \mathbf{X}$ by $L \to {}^r L = L \cdot L_r^{-m}$, $m = \check{\alpha}_r(L)$. This extends to a representation w: $L \to {}^w L$ of W on $\mathbf{X}$.

The elements of $\mathbf{X}$ which are in the W-orbit of some $L_r (r \in S)$ are called roots. The roots which are products of $L_r (r \in S)$ with $\geqq 0$ (resp. $\leqq 0$) exponents are called positive (resp. negative) roots.

2.8. Let $B \in \mathscr{B}$. For any $L \in \mathbf{X}$, B acts on the fiber L_B of L at B (according to the G-equivariant structure of L) by

(a) $$b \cdot l = \Psi_L^B(b) \cdot l, \qquad (b \in B,\ l \in L_B)$$

where Ψ_L^B: $B \to \mathbb{C}^*$ is a character.

Obviously,

(b) $$\Psi_L^{gBg^{-1}}(g b g^{-1}) = \Psi_L^B(b), \qquad \text{for all } b \in B,\ g \in G.$$

For fixed B, the map $L \to \Psi_L^B$ is an isomorphism of $\mathbf{X}$ onto the group of characters of B; under this map the notion of roots in 2.7, becomes the usual notion of roots (as characters of B); the L_r become simple roots.

(c) The character $\Psi_{L_r}^B$: $B \to \mathbb{C}^*$ is the character by which B acts on $\mathbf{p}/\mathbf{b}$ where $\mathbf{b}$ is the Lie algebra of B and $\mathbf{p}$ is the Lie algebra of the unique parabolic subgroup of type r containing B.

2.9. We shall denote by $\mathbf{r}(G)$ the set of isomorphism classes of irreducible rational G-modules; we shall assume that we are given in each such class a particular representative.

For any $E \in \mathbf{r}(G)$ we define an element $L_E \in \mathbf{X}$ by the requirement that for any Borel subgroup $B \subset G$, the character by which B acts on the unique B-stable line in E is equal to $\Psi_{L_E}^B$, see 2.8(a).

2.10. We shall need the following *lemma of Langlands.*

Let $Y = \mathrm{Hom}(\mathbf{X}, \mathbb{R})$. Then

(a) For any $f \in Y$ there is a subset $I \subset S$ and a decomposition $f = {}^0 f + {}^1 f$ such that:

(a1) $${}^0 f(L_r) > 0 \quad \text{if } r \in S - I, \qquad {}^0 f(L_r) = 0 \quad \text{if } r \in I$$

(a2) For any $E \in \mathbf{r}(G)$, we have ${}^1 f(L_E) \geqq 0$; if moreover E has a line stable under a parabolic subgroup of type I, then ${}^1 f(L_E) = 0$.

Moreover, I, ${}^0 f$, ${}^1 f$ are uniquely determined by f.

(b) If $f, f' \in Y$ and $f(L_E) \leqq f'(L_E)$ for any $E \in \mathbf{r}(G)$, then ${}^0 f(L_E) \leqq {}^0 f'(L_E)$ for any $E \in \mathbf{r}(G)$.

See [BW, IV, 6.11-6.13] for a proof.

2.11. We shall also need the following known statement (assuming that G is semisimple). If $E \in \mathbf{r}(G)$, then $L_E^b = L_{r_1}^{a_1} L_{r_2}^{a_2} \dots L_{r_n}^{a_n}$ where $S = \{r_1, \dots, r_n\}$, b is an integer >0, and all a_j are integers $\leqq 0$. If moreover, E has a line whose stabilizer in G is a parabolic subgroup of type $S - \{r_i\}$ then $a_i < 0$.

2.12. Bernstein has defined for any root system a certain abstract algebra over

(a) $$\mathscr{A} = \mathbb{C}[\mathbf{q}, \mathbf{q}^{-1}] \quad (\mathbf{q} \text{ an indeterminate})$$

in terms of generators and relations; for type A, he did that together with Zelevinskii.

We shall apply Bernstein's definition to the root system defined in 2.7 in $\mathbf{X}$. We thus consider the abstract algebra $\mathbf{H}$ over $\mathscr{A}$, with unit element 1, whose generators are symbols

$T_r (r \in S)$ and $\theta_L (L \in \mathbf{X})$ subject to the relations (b)–(g) below.

(b) $$T_r^2 = \mathbf{q} \cdot 1 + (\mathbf{q} - 1) T_r, \quad (\forall r \in S)$$

(c) $T_r T_{r'} T_r \dots = T_{r'} T_r T_{r'} \dots$ (μ factors in both products), $\forall r \neq r'$ in S with rr' of order μ in W.

(d) $$T_r \theta_{r_L} = \theta_L T_r - (\mathbf{q} - 1)\theta_L, \quad \forall r \in S, \forall L \in \mathbf{X} \text{ with } \check{\alpha}_r(L) = 1$$

(e) $$T_r \theta_L = \theta_L T_r, \quad \forall r \in S, \forall L \in \mathbf{X} \text{ with } \check{\alpha}_r(L) = 0.$$

(f) $$\theta_L \theta_{L'} = \theta_{LL'}, \quad \forall L, L' \in \mathbf{X}$$

(g) $$\theta_L = 1, \quad \text{if } L = \mathbb{C}.$$

The relation (d) can be replaced by the following relation (cf. [L_1, 4.4(b)]).

(d′) $$T_r \theta_{r_L} T_r = \mathbf{q}\, \theta_L, \quad \forall r \in S, \ \forall L \in \mathbf{X} \text{ with } \check{\alpha}_r(L) = 1.$$

The algebra $\mathbf{H}$ has a natural homomorphism (see [L_1, 4.3]) into the Iwahori-Matsumoto (or Hecke) algebra of the extended affine Weyl group $\mathbf{X} \cdot W$; this is easily seen to be an isomorphism of algebras.

2.13. We shall now state some known properties of $\mathbf{H}$.

For each $w \in W$, there is a unique element $T_w \in \mathbf{H}$ such that $T_w = T_{r_1} T_{r_2} \dots T_{r_p}$ for any reduced expression $w = r_1 r_2 \dots r_p$, ($r_i \in S$). This follows from 2.12(c).

(a) The elements $T_w \theta_L$ ($w \in W$, $L \in \mathbf{X}$) form an $\mathscr{A}$-basis for $\mathbf{H}$.

(b) Similarly, the elements $\theta_L T_w$ ($w \in W$, $L \in \mathbf{X}$) form an $\mathscr{A}$-basis for $\mathbf{H}$.

(c) There is a unique $\mathscr{A}$-linear involutive anti-automorphism $h \to \tilde{h}$ of the algebra $\mathbf{H}$ such that $\tilde{T}_r = T_r$, ($r \in S$) and $\tilde{\theta}_L = \theta_L (L \in \mathbf{X})$.

(d) There is a unique $\mathscr{A}$-linear involutive automorphism $h \to h^*$ of the algebra $\mathbf{H}$ such that $T_r^* = -\mathbf{q} T_r^{-1} = (\mathbf{q} - 1) - T_r$ ($r \in S$), $\theta_L^* = \theta_{L^{-1}} (L \in \mathbf{X})$.

Note that (c) and (d) follow easily from the defining relations 2.12 of $\mathbf{H}$; it is easier to use 2.12(d′) instead of 2.12(d).

The correspondence which attaches to $x=\sum_i a_i L_i \in \mathscr{A}[\mathbf{X}]$, $(a_i \in \mathscr{A},\ L_i \in \mathbf{X})$ the element $\theta_x \stackrel{\mathrm{def}}{=} \sum_i a_i \theta_{L_i} \in \mathbf{H}$ is an isomorphism of the group algebra $\mathscr{A}[\mathbf{X}]$ onto the commutative subalgebra of $\mathbf{H}$ spanned over $\mathscr{A}$ by the elements θ_L, $(L \in \mathbf{X})$. We shall identify these two commutative algebras.

We shall need the following theorem of Bernstein. (See [L₁] for a proof).

(e) The center of the algebra $\mathbf{H}$ is equal to the algebra of W-invariants $\mathscr{A}[\mathbf{X}]^W$ (with the action of W on $\mathscr{A}[\mathbf{X}]$ defined by the action of W on $\mathbf{X}$ in 2.7).

2.14. The representation ring $\mathbf{R}_{G\times\mathbb{C}^*}$ may be identified with $\mathscr{A}[\mathbf{X}]^W$ as follows.

(a) Let $\mathbf{q}$: $G\times\mathbb{C}^* \to \mathbb{C}^*$ be the second projection. We regard $\mathbf{g}$ as a 1-dimensional representation of $G\times\mathbb{C}^*$, hence as an element of $\mathbf{R}_{G\times\mathbb{C}^*}$. The most general irreducible rational representation of $G\times\mathbb{C}^*$ is $E\otimes\mathbf{q}^i$ where $E\in\mathbf{r}(G)$ is regarded as a representation of $G\times\mathbb{C}^*$ trivial on $\mathbb{C}^*$ and i is any integer. Consider the G-equivariant vector bundle $E\times\mathscr{B}\to\mathscr{B}$ with G-action defined by the G-module structure on E and by conjugation on $\mathscr{B}$. This vector bundle admits a filtration by G-equivariant sub-bundles with 1-dimensional subquotients. The formal sum of these subquotients (elements of $\mathbf{X}$) times $\mathbf{q}^i$ is an element of $\mathscr{A}[\mathbf{X}]^W$. Associating this element of $\mathscr{A}[\mathbf{X}]^W$ to $E\otimes\mathbf{q}^i$ defines a $\mathbb{C}$-algebra homomorphism $\mathbf{R}_{G\times\mathbb{C}^*}\to\mathscr{A}[\mathbf{X}]^W$ which is in fact an isomorphism. (See also 1.6(a)). Combining this with 2.13(e) we get an identification

(b) $$(\text{center of } \mathbf{H}) = \mathbf{R}_{G\times\mathbb{C}^*} = \mathscr{A}[\mathbf{X}]^W.$$

Thus, $\mathbf{H}$ appears as an algebra over $\mathbf{R}_{G\times\mathbb{C}^*}$. Note that in this form, $\mathbf{q}$ is not regarded as an indeterminate but as the concrete element of $\mathbf{R}_{G\times\mathbb{C}^*}$ defined by (a).

Later in this paper, we shall also write $\mathbf{q}$ for the restriction of $\mathbf{q}$ to various algebraic subgroups of $G\times\mathbb{C}^*$.

2.15. Let

(a) $$\mathscr{A}[\mathbf{X}] \to \mathbf{K}_0^{G\times\mathbb{C}^*}(\mathscr{B})$$

(where $G\times\mathbb{C}^*$ acts on $\mathscr{B}$ by conjugation on the first factor and with $\mathbb{C}^*$ acting trivially) be the $\mathbb{C}$-linear map which sends $\mathbf{q}^i L \in \mathscr{A}[\mathbf{X}]$ $(L\in\mathbf{X})$ to the $G\times\mathbb{C}^*$-equivariant line bundle $\mathbf{q}^i L \in \mathbf{K}^0_{G\times\mathbb{C}^*}(\mathscr{B}) = \mathbf{K}_0^{G\times\mathbb{C}^*}(\mathscr{B})$ (see 1.3(o1)); here L is regarded as a $G\times\mathbb{C}^*$-equivariant line bundle on $\mathscr{B}$ with trivial $\mathbb{C}^*$-action and $\mathbf{q}$ is as in 2.14(a). Then

(b) The map (a) is an isomorphism, and the natural $\mathscr{A}[\mathbf{X}]^W$-module structure on $\mathscr{A}[\mathbf{X}]$ and the natural $\mathbf{R}_{G\times\mathbb{C}^*}$-module structure on $\mathbf{K}_0^{G\times\mathbb{C}^*}(\mathscr{B})$ correspond to each other under (a) and under 2.14(b).

(c) $$\mathbf{K}_1^{G\times\mathbb{C}^*}(\mathscr{B}) = 0$$

(These results follow from 1.3(p3)).

3. The variety of triples

3.1. Let Λ be the variety of all pairs (u,B) where $B\in\mathscr{B}$ and u is a unipotent element of B. The group $G\times\mathbb{C}^*$ acts on Λ by

(a) $$(g,q)\colon\ (u,B)\mapsto(g\,u^{q^{-1}}g^{-1},gBg^{-1}).$$

Let $r\in S$, and let $\hat{\Lambda}^r$ be the variety of all pairs (u,P) where $P\in\mathscr{P}_r$ and u is a unipotent element of P. The group $G\times\mathbb{C}^*$ acts on $\hat{\Lambda}^r$ by a formula like (a) in which B is replaced by P. Let Λ^r be the variety of all pairs (u,B) where $B\in\mathscr{B}$ and u is a unipotent element such that $B\overset{\leq r}{-\!\!-}uBu^{-1}$. Then $G\times\mathbb{C}^*$ acts on Λ^r by (a). Let $\hat{\pi}^r\colon \Lambda^r\to\hat{\Lambda}^r$ be defined by $(u,B)=(u,\pi_r(B))$. This map is a $\mathbb{P}^1$-bundle and is compatible with the $G\times\mathbb{C}^*$-actions.

Let T^r be the tangent bundle along the fibers of $\hat{\pi}^r$; let T'^r be the corresponding cotangent bundle along fibers.

The fiber of T^r at (u,B) is $\mathbf{p}/\mathbf{b}$, with the notations of 2.8(c). We have $\log(u)\in\mathbf{p}$ and its image in $\mathbf{p}/\mathbf{b}$ gives a section $\mathscr{N}$ of the line bundle T^r. Its zero set is just Λ. The line bundle T^r is naturally $G\times\mathbb{C}^*$-equivariant (for the tangent action); however the section $\mathscr{N}$ is not $G\times\mathbb{C}^*$-invariant (see (a)). Let $V_{\mathbf{q}}$ (resp. $V_{\mathbf{q}^{-1}}$) be the $G\times\mathbb{C}^*$-equivariant vector bundle on Λ^r which is $\mathbb{C}$ with the action of $G\times\mathbb{C}^*$ given by multiplication by the character $\mathbf{q}$ (resp. $\mathbf{q}^{-1}$), see 2.14(a). Then $V_{\mathbf{q}^{-1}}\otimes T^r$ is a $G\times\mathbb{C}^*$-equivariant line bundle. If we ignore the $G\times\mathbb{C}^*$-action, this is just T^r, hence $\mathscr{N}$ can be regarded as a section of $V_{\mathbf{q}^{-1}}\otimes T^r$ which, this time, is $G\times\mathbb{C}^*$-invariant. It defines also a $G\times\mathbb{C}^*$-invariant map ${}^t\mathscr{N}\colon V_{\mathbf{q}}\times T'^r\to\mathbb{C}$.

We use this to form a complex of $G\times\mathbb{C}^*$-equivariant vector bundles

(f) $$\hat{\mathscr{E}}^r=(\ldots 0\to V_{\mathbf{q}}\otimes T'^r\xrightarrow{{}^t\mathscr{N}}\mathbb{C}\to 0\ldots)$$

with $\mathbb{C}$ in degree 0 and $V_{\mathbf{q}}\otimes T'^r$ in degree 1. Then $\hat{\mathscr{E}}^r$ is acyclic precisely at the points where $\mathscr{N}\neq 0$, i.e. on $\Lambda^r-\Lambda$.

3.2. A locally closed subvariety $\Lambda_1\subset\Lambda$ is said to be *r-saturated* ($r\in S$) if

$$\Lambda_1=((\hat{\pi}^r)^{-1}\hat{\pi}^r\Lambda_1)\cap\Lambda.$$

We then set

$$\hat{\Lambda}_1=\hat{\pi}^r\Lambda_1.$$

Let $M\subset G\times\mathbb{C}^*$ be an algebraic subgroup leaving Λ_1 stable for the action 3.1(a). If Λ_1 is r-saturated, we define an $\mathbf{R}_M$-homomorphism

(a) $$\tau^r\colon\ \mathbf{K}_0^M(\Lambda_1)\to\mathbf{K}_0^M(\Lambda_1)$$

as the composition

(b) $$\mathbf{K}_0^M(\Lambda_1)\xrightarrow{j_*}\mathbf{K}_0^M((\hat{\pi}^r)^{-1}\hat{\Lambda}_1)\xrightarrow{\hat{\pi}^r_*}\mathbf{K}_0^M(\hat{\Lambda}_1)\xrightarrow{(\hat{\pi}^r)^*}\mathbf{K}_0^M((\hat{\pi}^r)^{-1}\Lambda_1)\xrightarrow{\hat{\mathscr{E}}^r\otimes}\mathbf{K}_0^M(\Lambda_1).$$

Here, we denote the restriction of $\hat{\pi}^r\colon\Lambda^r\to\hat{\Lambda}^r$ to $(\hat{\pi}^r)^{-1}\hat{\Lambda}_1\to\hat{\Lambda}_1$ again by $\hat{\pi}^r$; $j\colon\Lambda_1\hookrightarrow(\hat{\pi}^r)^{-1}\hat{\Lambda}_1$ is the inclusion, $\hat{\mathscr{E}}^r\otimes$ is as in 1.3(f) with respect to $\hat{\mathscr{E}}^r$ (see 3.1(b)) restricted to $(\hat{\pi}^r)^{-1}\hat{\Lambda}_1$.

(c) In the case where $\Lambda_1=(\hat{\pi}^r)^{-1}\hat{\Lambda}_1\subset\Lambda$, the section $\mathcal{N}$ above is identically zero and (a) becomes $\tau^r(\xi)=(\hat{\pi}^r)^*(\hat{\pi}^r)_*(\xi)-\mathbf{q}\,T''\otimes(\hat{\pi}^r)^*(\hat{\pi}^r)_*(\xi)$ (cf. 1.3(f5)).

Consider, for example, a unipotent element $u\in G$. Identify $\mathscr{B}_u$ with the closed subvariety of Λ consisting of all pairs (u,B), $B\in\mathscr{B}_u$. This variety is stable under $M(u)\subset G\times\mathbb{C}^*$, (see 2.3(b)) and is r-saturated for any $r\in S$, hence the operations

(d) $$\tau^r\colon \mathbf{K}_0^M(\mathscr{B}_u)\to\mathbf{K}_0^M(\mathscr{B}_u),\qquad (r\in S)$$

are well defined for any algebraic subgroup $M\subset M(u)$.

3.3. Let Z be the variety of all triples (u,B,B') where $(B,B')\in\mathscr{B}\times\mathscr{B}$ and u is a unipotent element in $B\cap B'$. The group $G\times\mathbb{C}^*$ acts on Z by

(a) $$(g,q)\colon (u,B,B')\to(g\,u^{q^{-1}}g^{-1},gBg^{-1},gB'g^{-1}).$$

Let $r\in S$ be a simple reflection. Let ${}^rZ$ (resp. Z^r) be the variety of all triples (u,B,B') where $(B,B')\in\mathscr{B}\times\mathscr{B}$ and u is a unipotent in B' (resp. B) such that $B\xrightarrow{\leq r}uBu^{-1}$ (resp. $B'\xrightarrow{\leq r}uB'u^{-1}$). Then $G\times\mathbb{C}^*$ acts on ${}^rZ$ (resp. Z^r) by (a). Let ${}^r\hat{Z}$ (resp. $\hat{Z}^r$) be the variety of all triples (u,P,B') (resp. (u,B,P')) where $P\in\mathscr{P}_r$, $B'\in\mathscr{B}$, (resp. $B\in\mathscr{B}$, $P'\in\mathscr{P}_r$) and u is a unipotent element in $P\cap B'$ (resp. $B\cap P'$). The group $G\times\mathbb{C}^*$ acts on ${}^r\hat{Z}$, $\hat{Z}^r$ by a formula like (a) in which B or B' is replaced by P or P'. Let ${}^r\pi\colon {}^rZ\to{}^r\hat{Z}$ be defined by $(u,B,B')\to(u,\pi_r(B),B')$ and let $\pi^r\colon Z^r\to\hat{Z}^r$ be defined by $(u,B,B')\to(u,B,\pi_r(B'))$. These maps are $\mathbb{P}^1$-bundles compatible with the $G\times\mathbb{C}^*$-actions.

Let ${}^r\mathscr{E}$ (resp. $\mathscr{E}^r$) be the complex of $G\times\mathbb{C}^*$-equivariant vector bundles on ${}^rZ$ (resp. Z^r) defined by pulling back $\hat{\mathscr{E}}^r$ (see 3.1(b) under the map ${}^rZ\to\Lambda^r$, $(u,B,B')\to(u,B)$ (resp. $Z^r\to\Lambda^r$, $(u,B,B')\to(u,B')$). Then ${}^r\mathscr{E}$ (resp. $\mathscr{E}^r$) is acyclic at all points of ${}^rZ-Z$ (resp. Z^r-Z).

(b) A locally closed subvariety Π of Z is said to *left-r-saturated* ($r\in S$) if

$$\Pi=(({}^r\pi)^{-1}({}^r\pi)\,\Pi)\cap Z.$$

We then set $\hat{\Pi}={}^r\pi\Pi$. We define similarly the concept of *right-r-saturated* locally closed subvariety of Z by replacing ${}^r\pi$ by π^r in the previous definition.

Let Π be a left-r-saturated subvariety of Z and let $M\subset G\times\mathbb{C}^*$ be an algebraic subgroup leaving Π stable for the action (a). We define an $\mathbf{R}_M$-homomorphism

(c) $${}^r\tau\colon \mathbf{K}_0^M(\Pi)\to\mathbf{K}_0^M(\Pi)$$

as the composition

(d) $$\mathbf{K}_0^M(\Pi)\xrightarrow{j_*}\mathbf{K}_0^M(\tilde{\Pi})\xrightarrow{({}^r\pi)_*}\mathbf{K}_0^M(\hat{\Pi})\xrightarrow{({}^r\pi)^*}\mathbf{K}_0^M(\tilde{\Pi})\xrightarrow{{}^r\mathscr{E}\otimes}\mathbf{K}_0^M(\Pi)$$

Here we set $\tilde{\Pi}=({}^r\pi^{-1})\,\hat{\Pi}$; we denote the restriction of ${}^r\pi\colon{}^rZ\to{}^r\hat{Z}$ to $\tilde{\Pi}\to\hat{\Pi}$ again by ${}^r\pi$; $j\colon \Pi\hookrightarrow\tilde{\Pi}$ is the inclusion, ${}^r\mathscr{E}\otimes$ is as in 1.3(f) with respect to ${}^r\mathscr{E}$ restricted to $\tilde{\Pi}$.

Similarly, if Π is an M-stable right-r-saturated subvariety of Z, ($M \subset G \times \mathbb{C}^*$), we define an $\mathbf{R}_M$-homomorphism

(e) $$\tau^r\colon \mathbf{K}_0^M(\Pi) \to \mathbf{K}_0^M(\Pi)$$

by replacing in the previous definition ${}^r\pi$, ${}^r\mathscr{E}$ by π^r, $\mathscr{E}^r$.

We now give an alternative definition of the operation ${}^r\tau$ in (c). With the notations of (d), let $\Pi' = \tilde{\Pi} \underset{\hat{\Pi}}{\times} \Pi$, $\Pi'' = \Pi \underset{\hat{\Pi}}{\times} \Pi$. Let $pr_1'\colon \Pi' \to \tilde{\Pi}$, $pr_2'\colon \Pi' \to \Pi$, $pr_1''\colon \Pi'' \to \Pi$ be the projections. Let ${}^r\tilde{\mathscr{E}}$ be the complex of M-equivariant vector bundles on Π' obtained by pulling back ${}^r\mathscr{E}|\tilde{\Pi}$ under pr_1'. (M acts naturally on Π', Π''). It is acyclic on $\Pi' - \Pi''$. Then τ^r can be also defined as the composition

(f) $$\mathbf{K}_0^M(\Pi) \xrightarrow{pr_2'^*} \mathbf{K}_0^M(\Pi') \xrightarrow{{}^r\tilde{\mathscr{E}}\otimes} \mathbf{K}_0^M(\Pi'') \xrightarrow{(pr_1'')_*} \mathbf{K}_0^M(\Pi).$$

The equivalence of this definition with the earlier definition (d) follows from the diagram

$$\begin{tikzcd}
\mathbf{K}_0^M(\Pi) \arrow[r, "pr_2'^*"] \arrow[d, "j_*"'] & \mathbf{K}_0^M(\Pi') \arrow[r, "{}^r\tilde{\mathscr{E}}\otimes"] \arrow[d, "(pr_1')_*"] & \mathbf{K}_0^M(\Pi'') \arrow[d, "(pr_1'')_*"] \\
\mathbf{K}_0^M(\tilde{\Pi}) \arrow[r, "({}^r\pi)^*({}^r\pi)_*"] & \mathbf{K}_0^M(\tilde{\Pi}) \arrow[r, "{}^r\mathscr{E}\otimes"] & \mathbf{K}_0^M(\Pi)
\end{tikzcd}$$

This diagram is commutative by 1.3(d) and 1.3(f1).

Now let $\Pi \subset \Pi'$ be two locally closed, M-stable left-r-saturated subvarieties of Z and let $i\colon \Pi \hookrightarrow \Pi'$ be the inclusion. Then we have commutative diagrams

(g) $$\begin{tikzcd}
\mathbf{K}_0^M(\Pi) \arrow[r, "{}^r\tau"] \arrow[d, "j_*"'] & \mathbf{K}_0^M(\Pi) \arrow[d, "j_*"] \\
\mathbf{K}_0^M(\Pi') \arrow[r, "{}^r\tau"] & \mathbf{K}_0^M(\Pi')
\end{tikzcd}$$ if Π is closed in Π'

and

(h) $$\begin{tikzcd}
\mathbf{K}_0^M(\Pi) \arrow[r, "{}^r\tau"] & \mathbf{K}_0^M(\Pi) \\
\mathbf{K}_0^M(\Pi') \arrow[r, "{}^r\tau"] \arrow[u, "j^*"] & \mathbf{K}_0^M(\Pi') \arrow[u, "j^*"']
\end{tikzcd}$$ if Π is open in Π'.

This follows from 1.3(b), (c), (d), (f).

Similar results apply to τ^r.

As an example, the variety $\Pi = Z$ is certainly left and right-r-saturated for any $r \in S$ hence the operations

(i) $${}^r\tau, \tau^r\colon \mathbf{K}_0^{G\times\mathbb{C}}(Z) \to \mathbf{K}_0^{G\times\mathbb{C}}(Z)$$

are well defined. As another example, fix a unipotent element $u\in G$ and identify $\mathscr{B}_u\times\mathscr{B}_u$ with the closed subvariety of Z consisting of all (u,B,B'), $B,B'\in\mathscr{B}_u$. This subvariety is stable under $M(u)\subset G\times\mathbb{C}^*$ (see 2.3(b)) and is again left and right-r-saturated for any $r\in S$; hence the operations

(j)
$$ {}^r\tau,\tau^r\colon\ \mathbf{K}_0^M(\mathscr{B}_u\times\mathscr{B}_u)\to\mathbf{K}_0^M(\mathscr{B}_u\times\mathscr{B}_u)$$

are well defined for any algebraic subgroup $M\subset M(u)$.

Moreover, we have a commutative diagram

(k)
$$\begin{tikzcd}
\mathbf{K}_0^M(\mathscr{B}_u)\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u) \arrow[r, "\tau^r\otimes 1"] \arrow[d, "\boxtimes"'] & \mathbf{K}_0^M(\mathscr{B}_u)\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u) \arrow[d, "\boxtimes"] \\
\mathbf{K}_0^M(\mathscr{B}_u\times\mathscr{B}_u) \arrow[r, "{}^r\tau"] & \mathbf{K}_0^M(\mathscr{B}_u\times\mathscr{B}_u)
\end{tikzcd}$$

and a similar commutative diagram in which $\tau^r\otimes 1$, ${}^r\tau$ are replaced by $1\otimes\tau^r,\tau^r$, (see 1.3(n)); here the upper τ^r is given by 3.2(a) and the lower ${}^r\tau$ is given by (j).

3.4. Consider the diagram

(a)
$$\begin{tikzcd}
\Lambda \arrow[r, "\beta"] \arrow[d, shift left, "\beta'"] & Z \\
\mathscr{B} \arrow[u, shift left, "\delta"] \arrow[r, "\alpha"] & \mathscr{B}\times\mathscr{B} \arrow[u, "j"']
\end{tikzcd}$$

where $\beta'(u,B)=B$, $\beta(u,B)=(u,B,B)$, $\alpha(B)=(B,B)$, $j(B,B)=(1,B,B)$, $\delta(B)=(1,B)$. Note that β' may be regarded as the cotangent bundle of $\mathscr{B}$ and δ as its zero section. These varieties have natural $G\times\mathbb{C}^*$-actions inherited from Z.

(b) Let $\mathbb{1}\in\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ be the image of the trivial bundle $\mathbb{C}$ on $\mathscr{B}$ (=neutral element of $\mathbf{X}$) under the composition

$$\mathbf{K}_0^{G\times\mathbb{C}^*}(\mathscr{B})\xrightarrow{\beta'^*}\mathbf{K}_0^{G\times\mathbb{C}^*}(\Lambda)\xrightarrow{\beta_*}\mathbf{K}_0^{G\times\mathbb{C}^*}Z).$$

(We regard $\mathbb{C}$ as an element of $\mathbf{K}^0_{G\times\mathbb{C}^*}(\mathscr{B})=\mathbf{K}_0^{G\times\mathbb{C}^*}(\mathscr{B})$, see 1.3(o1)).

(c) For each $L\in\mathbf{X}$ we denote by ${}^{\cdot}L$ (resp. $L^{\cdot}$) the pull back of L under the $G\times\mathbb{C}^*$-equivariant map $Z\to\mathscr{B}$, $(u,B,B')\to B$ (resp. $(u,B,B')\to B'$). We regard L as a $G\times\mathbb{C}^*$-equivariant line bundle on $\mathscr{B}$ with trivial action of $\mathbb{C}^*$. Hence ${}^{\cdot}L$, $L^{\cdot}$ are naturally $G\times\mathbb{C}^*$-equivariant line bundles on Z.

We can now state the main result of this chapter.

3.5. **Theorem.** *We regard* $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ *as an* $\mathbf{R}_{G\times\mathbb{C}^*}=\mathscr{A}[\mathbf{X}]^W$*-module* (see 2.14(b)).

(a) *There is a unique left (resp. right)* $\mathbf{H}$*-module structure on* $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ $(h,\xi)\to h\xi$ (resp. $(\xi,h)\to\xi h$), $h\in\mathbf{H}$, $\xi\in\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ *extending the* $\mathscr{A}[\mathbf{X}]^W$*-module* *structure* (*see* 2.13(*e*)) *such that for all* $\xi \in \mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$, $r\in S$, $L\in\mathbf{X}$ *we have*

$$T_r\xi=(\mathbf{q}-{}^r\tau)(\xi)\quad (\textit{resp. } \xi T_r=(\mathbf{q}-\tau^r)(\xi)), \textit{ see } 3.3(\mathrm{i}), \textit{ and}$$
$$\theta_L\xi={}^{\boldsymbol{\cdot}}L\otimes\xi\qquad (\textit{resp. } \xi\theta_L=L^{\boldsymbol{\cdot}}\otimes\xi), \text{ see } 3.4(\mathrm{c}) \text{ and } 1.3(\mathrm{f}3).$$

(b) *The two maps* $\mathbf{H}\to\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$, $h\to h\mathbb{1}$ *and* $h\to\mathbb{1}h$ (*see* 3.4(b)) *coincide.*

(c) *The two maps in* (b) *are isomorphisms.*

The rest of this chapter will be concerned with the proof of this theorem; parts (a) and (b) will be proved in 3.6–3.14 and part (c) in 3.15–3.21.

3.6. Applying the general Definition 3.3(c), (e) (or 3.3(j)) to the variety $\mathscr{B}\times\mathscr{B}$ regarded as a subvariety of Z via j in 3.4(a), we get natural $\mathbf{R}_{G\times\mathbf{C}^*}$-homomorphisms

$${}^r\tau,\tau^r\colon\ \mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})\to\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B}).$$

These are given by formulas analogous to 3.2(c), since the corresponding restrictions of the complexes ${}^r\mathscr{E}$, $\mathscr{E}^r$ have zero maps in this case:

$$\text{(a)}\qquad {}^r\tau(\xi)=(1-\mathbf{q}L_r^{-1}\boxtimes 1)({}_r\varphi)^*({}_r\varphi)_*(\xi)$$
$$\tau^r(\xi)=(1-\mathbf{q}\,1\boxtimes L_r^{-1})(\varphi_r)^*(\varphi_r)_*(\xi)$$

where $\xi\in\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})$, ${}_r\varphi\colon\mathscr{B}\times\mathscr{B}\to\mathscr{P}_r\times\mathscr{B}$ is $\pi_r\times 1$ and $\varphi_r\colon\mathscr{B}\times\mathscr{B}\to\mathscr{B}\times\mathscr{P}_r$ is $1\times\pi_r$. These maps are compatible with ${}^r\tau,\tau^r\in\operatorname{End}\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$ under j_* (j as in 3.4(a)).

Similarly, for each $L\in X$, we define $\theta_L\xi=(L\boxtimes 1)\otimes\xi$, $\xi\theta_L=(1\boxtimes L)\otimes\xi$ $\xi\in\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})$; these maps are compatible with the endomorphisms $\xi\to{}^{\boldsymbol{\cdot}}L\otimes\xi$, $\xi\to L^{\boldsymbol{\cdot}}\otimes\xi$ of $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$ under j_*. Let us admit the following result.

3.7. **Proposition.** (a) *There is a unique left* (*resp. right*) $\mathbf{H}$-*module structure on* $\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})$, $(h,\xi)\to h\xi$ (*resp.* $(\xi,h)\to\xi h$), $h\in\mathbf{H}$, *extending the* $\mathscr{A}[\mathbf{X}]^W$-*module structure* (*see* 2.14(b), 2.13(e)) *such that*

$$T_r\xi=(\mathbf{q}-{}^r\tau)(\xi)\quad(\textit{resp. } \xi T_r=(\mathbf{q}-{}^r\tau)\,\xi), \textit{ see } 3.6(\mathrm{a}),$$
$$\theta_L\xi=(L\boxtimes 1)\otimes\xi,\quad(\textit{resp. } \xi\theta_L=(1\boxtimes L)\otimes\xi).$$

(b) *Let* $A\in\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})$ *be defined as* $\alpha_*(\mathbf{C})$ (*see* 3.4).

Let $\phi^+=\Pi(1-\mathbf{q}L)\in\mathscr{A}[\mathbf{X}]$, *product over all positive roots* $L\in\mathbf{X}$, *and let* θ_{ϕ^+} *be the corresponding element of* $\mathbf{H}$ (see 2.13). *Then, for any* $h\in\mathbf{H}$, *we have* $\theta_{\phi^+}hA=Ah\theta_{\phi^+}$.

3.8. We now show how 3.7 implies 3.5(a) and (b). Fix a complex number $q\neq 1$ and consider the commutative diagram

$$\text{(a)}\qquad \begin{array}{ccc} \mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B}) & \xrightarrow{j_*} & \mathbf{K}_0^{G\times\mathbf{C}^*}(Z) \\ \downarrow & & \downarrow{\scriptstyle f_2} \\ K_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})_{\mathrm{loc}} & \xrightarrow[f_1]{\approx} & \mathbf{K}_0^{G\times\mathbf{C}^*}(Z)_{\mathrm{loc}} \end{array}$$

where j is as in 3.4, "loc" denotes localization with respect to the maximal ideal of $\mathbf{R}_{G\times\mathbf{C}^*}$ corresponding to $(1,q)\in G\times\mathbf{C}^*$ and the vertical maps are the obvious maps. The lower horizontal map f_1 is induced by j_*. It is an isomorphism by 1.3(k). Indeed, the fixed point set of $(1,q)$ on Z is precisely $j(\mathscr{B}\times\mathscr{B})$. (If u is a unipotent element and $u^q=u$, $q\neq 1$, then $u=1$). The right vertical map f_2 in (a) is injective. To show this it is enough to show that $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$ is a projective $\mathbf{R}_{G\times\mathbf{C}^*}$-module. Let us define for each $w\in W$:

(b) $$Z_w=\{(u,B,B')\in Z\mid B\overset{w}{—}B'\}.$$

The Z_w form a partition of Z into locally closed $G\times\mathbf{C}^*$-stable subvarieties. The exact sequences 1.3(g) associated with this partition show that $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$ is projective over $\mathbf{R}_{G\times\mathbf{C}^*}$ provided that for each w:

(c) $\mathbf{K}_1^{G\times\mathbf{C}^*}(Z_w)=0$ and $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z_w)$ is a free $\mathbf{R}_{G\times\mathbf{C}^*}$-module of rank $|W|$.

But $Z_w\to\mathscr{B}$, $(u,B,B')\to B$ is a locally trivial fibration with all fibers isomorphic to $\mathbf{C}^v$, ($v=\dim\mathscr{B}$) and (c) follows from 2.15 and 1.6(b).

Thus f_1 and f_2 in (a) have the stated properties. In 3.5(a) we must prove certain identities for the actions of the generators of $\mathbf{H}$ on $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$. These identities hold for the analogous actions on $\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})$ (by 3.7) hence they hold on $\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B}\times\mathscr{B})_{\text{loc}}$, hence they hold on $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)_{\text{loc}}$, (by the compatibility of these actions and since f_1 is an isomorphism), hence they hold on $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$, since f_2 is injective. The same argument shows that in the resulting $\mathbf{H}$-module structures the centre of $\mathbf{H}$ acts in the same way as $\mathscr{A}[\mathbf{X}]^W$ acts by the $\mathbf{R}_{G\times\mathbf{C}^*}$-module structure. This proves 3.5(a) (assuming 3.7). We now prove 3.5(b) (assuming 3.7). First note that, by 1.3(f7) we have $\delta_*(\mathbf{C})=\beta'^*(\phi^+)$, (notations of 3.4), where ϕ^+, see 3.7(b), is regarded as an element of $\mathbf{K}_0^{G\times\mathbf{C}^*}(\mathscr{B})=\mathscr{A}[\mathbf{X}]$, (see 2.15).

With the notations of 3.4 and 3.7(b) we have:

$$j_*A=j_*\alpha_*(\mathbf{C})=\beta_*\delta_*(\mathbf{C})=\beta_*\beta'^*(\phi^+)=\theta_{\phi^+}\beta_*\beta'^*(\mathbf{C})=\theta_{\phi^+}\mathbb{1}=\mathbb{1}\,\theta_{\phi^+}.$$

(Note that from the definitions it follows immediately that $\theta_L\mathbb{1}=\mathbb{1}\,\theta_L$ for any $L\in\mathbf{X}$). Now let $h_1\in\mathbf{H}$. By 3.7(b) we have $\theta_{\phi^+}h_1A=Ah_1\theta_{\phi^+}$. Apply j_* to the last equality and use the compatibility of the $\mathbf{H}$-actions with j_*. We obtain $\theta_{\phi^+}h_1j_*A=j_*(A)h_1\theta_{\phi^+}$. Substituting $j_*A=\mathbb{1}\,\theta_{\phi^+}=\theta_{\phi^+}\mathbb{1}$, we obtain

(d) $$\theta_{\phi^+}h_1\theta_{\phi^+}\mathbb{1}=\mathbb{1}\,\theta_{\phi^+}h_1\theta_{\phi^+}.$$

We write the identity (d) for h_1 of the form $\theta_{\phi^-}h_2\theta_{\phi^-}$ where $h_2\in\mathbf{H}$ and ϕ^-, θ_{ϕ^-} are defined just like ϕ^+, θ_{ϕ^+} (see 3.7(b) but using negative roots instead of positive roots. Note that $\phi^+\phi^-\in\mathscr{A}[\mathbf{X}]^W$ hence $z=\theta_{\phi^+}\theta_{\phi^-}$ is in the centre of $\mathbf{H}$ (see 2.13(e)). It acts as multiplication by the corresponding element in $\mathbf{R}_{G\times\mathbf{C}^*}$. Thus in our case (d) becomes $zh_2z\mathbb{1}=\mathbb{1}\,zh_2z$ and z commutes with everything hence $z^2(h_2\mathbb{1})=z^2(\mathbb{1}\,h_2)$. As we have seen earlier, $\mathbf{K}_0^{G\times\mathbf{C}^*}(Z)$ is a projective $\mathbf{R}_{G\times\mathbf{C}^*}$-module, hence we can cancel z^2 from the last equality. Hence $h_2\mathbb{1}=\mathbb{1}\,h_2$ as desired. Thus, 3.5(a) and (b) are proved (assuming 3.7). For the proof of 3.7 we shall need a number of lemmas.

3.9. **Lemma.** $[L_4]$ *There is a unique left* **H**-*module structure (denoted* $h \bullet \xi$) *on* $\mathscr{A}[\mathbf{X}]$ *extending the obvious* $\mathscr{A}$-*module structure and such that*

(a) $$T_r \bullet L = \frac{L - {}^r L}{L_r - 1} - \mathbf{q}\frac{L - {}^r L L_r}{L_r - 1}, \quad (r \in S,\ L \in \mathbf{X})$$

(b) $$\theta_{L_1} \bullet L = L_1^{-1} L.$$

(The right hand side of (a) is at first sight an element of the quotient field of $\mathscr{A}[\mathbf{X}]$ but one checks easily that it is actually in $\mathscr{A}[\mathbf{X}]$).

Proof. S. Kato has found a more natural proof than that in $[L_4]$; we shall reproduce it here with his permission.

Let $\Sigma = \sum_{w \in W} T_w \in \mathbf{H}$ and let **I** be the left ideal of **H** generated by Σ. The $\mathscr{A}$-linear map $\mathscr{A}[\mathbf{X}] \to \mathbf{I}$ defined by $L \to \theta_{L^{-1}} \Sigma$ is an isomorphism by 2.13(b). One checks easily that under this isomorphism the endomorphism T_r (resp. θ_{L_1}) defined in (a) (resp. (b)) becomes left multiplication by $T_r \in \mathbf{H}$ (resp. $\theta_{L_1} \in \mathbf{H}$) for the left **H**-module structure of the left ideal **I**. The lemma follows.

3.10. **Lemma.** *There is a unique left* **H**-*module structure (denoted* $h \circ \xi$) *on* $\mathscr{A}[\mathbf{X}]$ *extending the obvious* $\mathscr{A}$-*module structure and such that*

(a) $$T_r \circ L = \frac{{}^r L - L L_r}{L_r - 1} + \mathbf{q}\frac{L L_r - {}^r L L_r^{-1}}{L_r - 1}, \quad (r \in S,\ L \in \mathbf{X})$$

(b) $$\theta_{L_1} \circ L = L_1 L, \quad (L_1, L \in \mathbf{X}).$$

Proof. A simple computation shows that

(c) $$T_r \circ L = (\phi^+)^{-1}(T_r^* \bullet (\phi^+ L))$$

(d) $$\theta_{L_1} \circ L = (\phi^+)^{-1}(\theta_{L_1}^* \bullet (\phi^+ L))$$

where $\bullet$ is as in 3.9, * is as in 2.13(d) and ϕ^+ is as in 3.7(b).

Now the lemma follows from 2.13(d) and 3.9.

3.11. We can now prove 3.7(a). We identify $\mathbf{K}_0^{G \times \mathbf{C}^*}(\mathscr{B}) = \mathscr{A}[\mathbf{X}]$ as in 2.15 and $\mathbf{K}_0^{G \times \mathbf{C}^*}(\mathscr{B} \times \mathscr{B}) = \mathscr{A}[\mathbf{X}] \underset{\mathscr{A}[\mathbf{X}]^W}{\otimes} \mathscr{A}[\mathbf{X}]$, using 1.6.

From 3.6(a) it follows easily that with this identification, the endomorphism $\xi \to T_r \xi$ (resp. $\xi \to \xi T_r$) in 3.7(a) is just

$$L' \boxtimes L'' \to (T_r \circ L') \boxtimes L'' \quad (\text{resp. } L' \boxtimes L'' \to L' \boxtimes (T_r \circ L''))$$

(notation of 3.10); this is easily verified using 1.3(o2).

Similarly the endomorphism $\xi \to \theta_L \xi$ (resp. $\xi \to \xi \theta_L$) in 3.7(a) is just $L' \boxtimes L'' \to (\theta_L \circ L') \boxtimes L''$ (resp. $L' \boxtimes L'' \to L' \boxtimes (\theta_L \circ L'')$), (notation of 3.10).

Therefore 3.7(a) follows from 3.10 (for the left action) and from 3.10 and 2.13(c) (for the right action); the fact that in the resulting **H**-module structure the action of the centre $= \mathscr{A}[\mathbf{X}]^W = \mathbf{R}_{G \times \mathbf{C}^*}$ coincides with the natural $\mathbf{R}_{G \times \mathbf{C}^*}$-module structure on $\mathbf{K}_0^{G \times \mathbf{C}^*}(\mathscr{B} \times \mathscr{B})$ follows immediately from the definitions.

We see also that the **H**-module structures in 3.7(a) satisfy

(a) $$h(L' \boxtimes L'') = (h \circ L') \boxtimes L''$$
$$(L' \boxtimes L'')h = L' \boxtimes (\tilde{h} \circ L'')$$
$$(h \in \mathbf{H},\ L', L'' \in \mathbf{X},\ h \to \tilde{h} \text{ as in 2.13(c)}).$$

3.12. For each $L \in \mathbf{X}$ we set $\mathrm{Alt}(L) = \sum_{w \in W} \varepsilon_w {}^wL, \varepsilon_w = \text{sign of } w$.

Let $L_\rho \in \mathbf{X}$ be the element such that L_ρ^2 is the product of all positive roots. Consider the $\mathscr{A}[\mathbf{X}]^W$-bilinear symmetric pairing $(\,,\,)$: $\mathscr{A}[\mathbf{X}] \times \mathscr{A}[\mathbf{X}] \to \mathscr{A}[\mathbf{X}]^W$ defined by

$(L, L') = \mathrm{Alt}(L\,P\,L' \cdot L_\rho) \cdot \mathrm{Alt}(L_\rho)^{-1}$ (compare 1.6).

The following identity is easily checked from the definitions

(a) $$(T_r^* \bullet L, L') = (L, T_r \circ L'), \quad \text{for all } r \in S,\ L \in \mathbf{X},\ L' \in \mathbf{X}$$

3.13. **Lemma.** *For all $h \in \mathbf{H}$, $\eta, \eta' \in \mathscr{A}[\mathbf{X}]$, we have*

$$((\theta_{\phi^+} h) \circ \eta, \eta') = (\eta, (\theta_{\phi^+} \tilde{h}) \circ \eta')$$

Proof. It is enough to check this when h is one of the generators T_r or θ_L of **H**. The case where $h = \theta_L$ is trivial. The case where $h = T_r$ follows from 3.10(c) and 3.12(a).

3.14. We can now prove 3.7(b). By 3.13, for any $h \in \mathbf{H}$, the $\mathscr{A}[\mathbf{X}]^W$ endomorphism φ: $\eta \mapsto (\theta_{\phi^+} h) \circ \eta$ of $\mathscr{A}[\mathbf{X}]$ has as transpose with respect to $(\,,\,)$ the endomorphism ${}^t\varphi: \eta \mapsto h\theta_{\phi^+} \circ \eta$ of $\mathscr{A}[\mathbf{X}]$. By 1.7(a), we then have $(\varphi \otimes 1)A = (1 \otimes {}^t\varphi)A$ where A of 3.7(b) is regarded as an element of $\mathscr{A}[\mathbf{X}] \underset{\mathscr{A}[\mathbf{X}]^W}{\otimes} \mathscr{A}[\mathbf{X}]$. In view of 3.11(a), the last equality can be expressed as follows: $\theta_{\phi^+} hA = Ah\theta_{\phi^+}$. This proves 3.7(b). Thus 3.7, and hence also 3.5(a) and (b) are proved.

3.15. The rest of this chapter is concerned with the proof of 3.5(c). To simplify notations, *in the rest of this chapter we shall write* $\mathbf{K}(\,)$ *instead of* $\mathbf{K}_0^{G \times \mathbf{C}^*}(\,)$ and $\mathbf{K}_1(\,)$ instead of $\mathbf{K}_1^{G \times \mathbf{C}^*}(\,)$.

We define $Z_{\leqq w} = \bigcup_{y \leqq w} Z_y$ (see 3.8(b)). We define similarly $Z_{<w}, Z_{\geqq w}, Z_{>w}$.

A subset I of W is said to be closed (resp. open) if $w_1 \in I$, $w_2 \leqq w_1 \Rightarrow w_2 \in I$ (resp. if $w_1 \in I$, $w_2 \geqq w_1 \Rightarrow w_2 \in I$). A subset I of W is said to be locally closed if it is of form $I' \cap I''$ where $I' \subset W$ is closed and $I'' \subset W$ is open. For a locally closed subset I of W we define $Z_I = \bigcup_{y \in I} Z_y$. Then Z_y is a locally closed subvariety of Z; moreover, Z_I is a closed (resp. open) subvariety if I is a closed (resp. open) subset of W. For example, $Z_{\leqq w}, Z_{<w}$ above are closed and $Z_{\geqq w}, Z_{>w}$ are open.

3.16. **Lemma.** *If I is a locally closed subset of W, then $\mathbf{K}_1(Z_I) = 0$ and $\mathbf{K}(Z_I)$ is a projective $\mathbf{R}_{G \times \mathbf{C}^*}$-module of rank $|I| \cdot |W|$.*

Proof. Using the partition of Z_I into the pieces Z_y $(y \in I)$ and the associated exact sequences 1.3(g) we are immediately reduced to the case where I has a single element, in which case we can use 3.8(c).

3.17. We now state some results which are immediate consequences of 3.16 and the exact sequences 1.3(g).

(a) The imbeddings $i_{\leqq w}\colon Z_{\leqq w} \hookrightarrow Z$ and $j_w\colon Z_w \hookrightarrow Z_{\geqq w}$ induce injective maps

$$(i_{\leqq w})_*\colon \mathbf{K}(Z_{\leqq w}) \to \mathbf{K}(Z), \qquad (j_w)_*\colon \mathbf{K}(Z_w) \to \mathbf{K}(Z_{\geqq w}).$$

(b) The imbeddings $i_{\geqq w}\colon Z_{\geqq w} \to Z$ and $j'_w\colon Z_w \hookrightarrow Z_{\leqq w}$ induce surjective maps

$$(i_{\geqq w})^*\colon \mathbf{K}(Z) \to \mathbf{K}(Z_{\geqq w}), \qquad (j'_w)^*\colon \mathbf{K}(Z_{\leqq w}) \to \mathbf{K}(Z_w).$$

(c) $$(i_{\geqq w})^*(i_{\leqq w})_* = (j_w)_* j'^*_w\colon \mathbf{K}(Z_{\leqq w}) \to \mathbf{K}(Z_{\geqq w}) \quad \text{(cf. 1.3(d))}.$$

(d) If $I \subset W$ is closed, then the image of the injective map $(i_I)_*\colon \mathbf{K}(Z_I) \to \mathbf{K}(Z)$ induced by $i_I\colon Z_I \hookrightarrow Z$ coincides with the sum of the images of $(i_{\leqq w})_*$ (in (a)) for all $w \in I$.

3.18. We fix $r \in S$ and $w \in W$ such that $rw < w$.

It is clear that $Z_{\leqq w}$ and $Z_{\geqq rw}$ are left-r-saturated subvarieties of Z (see 3.3(b)) hence we have natural operations

$${}^r\tau\colon \mathbf{K}(Z_{\leqq w}) \to \mathbf{K}(Z_{\leqq w}), \qquad {}^r\tau\colon \mathbf{K}(Z_{\geqq rw}) \to \mathbf{K}(Z_{\geqq rw}),$$

and these are compatible with ${}^r\tau\colon \mathbf{K}(Z) \to \mathbf{K}(Z)$ via $(i_{\leqq w})_*$ $(i_{\geqq rw})^*$.

3.19. **Lemma.** *Let $r \in S$, $w \in W$ be such that $rw < w$. Then there is a unique $\mathbf{R}_{G \times \mathbf{C}^*}$-isomorphism $\rho_1\colon \mathbf{K}(Z_{rw}) \xrightarrow{\sim} \mathbf{K}(Z_w)$ such that the diagram*

(a)
$$\begin{tikzcd}
& \mathbf{K}(Z_{rw}) \arrow[rr, "\rho_1", "\approx"'] \arrow[dl, "(j_{rw})_*"'] & & \mathbf{K}(Z_w) \arrow[dr, "(j_w)_*"] & \\
\mathbf{K}(Z_{\geqq rw}) \arrow[rr, "{}^r\tau"] & & \mathbf{K}(Z_{\geqq rw}) \arrow[rr, "d^*"] & & \mathbf{K}(Z_{\geqq w}) \\
& \mathbf{K}(Z) \arrow[ul, "i^*_{\geqq rw}"] \arrow[rr, "{}^r\tau"] & & \mathbf{K}(Z) \arrow[ul] \arrow[ur, "i^*_{\geqq w}"'] &
\end{tikzcd}$$

is commutative. (Here, $d\colon Z_{\geqq w} \hookrightarrow Z_{\geqq rw}$ *is the inclusion*).

Proof. The uniqueness of ρ_1 is clear since $(j_w)_*$ is injective (3.17(a)). To prove the existence we introduce some notation. Let

$$\tilde{Z}_{\geqq rw} = \text{set of all } (u, B, B_1, B') \text{ with } (u, B_1, B') \in Z_{\geqq rw} \text{ and } B \overset{\leqq r}{\text{---}} B_1$$

$${}_u\tilde{Z}_{\geqq rw} = \{(u, B, B_1, B') \in \tilde{Z}_{\geqq rw} \mid u \in B\}$$

$$\tilde{Z}_{rw} = \text{set of all } (u, B, B_1, B') \text{ with } (u, B_1, B') \in Z_{rw} \text{ and } B \overset{\leqq r}{\text{---}} B_1$$

$$
\begin{aligned}
{}_u\tilde{Z}_{rw} &= \{(u, B, B_1, B') \in \tilde{Z}_{rw} \mid u \in B\} \\
Z'_{rw} &= \{(u, B, B_1, B') \in \tilde{Z}_{rw} \mid B \overset{r}{—} B_1\} \\
{}_uZ'_{rw} &= \{(u, B, B_1, B') \in Z'_{rw} \mid u \in B\}.
\end{aligned}
$$

Let $\tilde{\mathscr{E}}$ be the complex of $G \times \mathbb{C}^*$-equivariant vector bundles on $\tilde{Z}_{\geqq rw}$ obtained by taking pull back of $\hat{\mathscr{E}}^r$ see 3.1(b) under the map $\tilde{Z}_{\geqq rw} \to \Lambda^r$, $(u, B, B_1, B') \mapsto (u, B)$. Then $\tilde{\mathscr{E}}$ is acyclic on $\tilde{Z}_{\geqq rw} - {}_u\tilde{Z}_{\geqq rw}$. The restriction of $\tilde{\mathscr{E}}$ to the subvariety $\tilde{Z}_{rw}$ (resp. Z'_{rw}) is denoted $\tilde{\mathscr{E}}_1$ (resp. $\tilde{\mathscr{E}}_2$). By 1.3(f) we have

$$
\begin{aligned}
\tilde{\mathscr{E}} \otimes &: \mathbf{K}(\tilde{Z}_{\geqq rw}) \to \mathbf{K}({}_u\tilde{Z}_{\geqq rw}) \\
\tilde{\mathscr{E}}_1 \otimes &: \mathbf{K}(\tilde{Z}_{rw}) \to \mathbf{K}({}_u\tilde{Z}_{rw}) \\
\tilde{\mathscr{E}}_2 \otimes &: \mathbf{K}(Z'_{rw}) \to \mathbf{K}({}_uZ'_{rw}).
\end{aligned}
$$

These fit into the following commutative diagram.

$$
\text{(b)}\quad
\begin{tikzcd}
\mathbf{K}(Z_{\geqq rw}) \arrow[r, "f^*"] & \mathbf{K}(\tilde{Z}_{\geqq rw}) \arrow[r, "\tilde{\mathscr{E}}\otimes"] & \mathbf{K}({}_u\tilde{Z}_{\geqq rw}) \arrow[r, "g_*"] & \mathbf{K}(Z_{\geqq rw}) \arrow[r, "d^*"] & \mathbf{K}(Z_{\geqq w}) \\
\mathbf{K}(Z_{rw}) \arrow[u] \arrow[r, "f_1^*"] \arrow[dr, "f_2^*"'] & \mathbf{K}(\tilde{Z}_{rw}) \arrow[u] \arrow[d] \arrow[r, "\tilde{\mathscr{E}}_1\otimes"] & \mathbf{K}({}_u\tilde{Z}_{rw}) \arrow[u] \arrow[d] \arrow[ur, "(g_1)_*"] \arrow[r, "d_1^*"] & \mathbf{K}(Z_w) \arrow[ur] & \\
 & \mathbf{K}(Z'_{rw}) \arrow[r, "\tilde{\mathscr{E}}_2\otimes"] & \mathbf{K}({}_uZ'_{rw}) \arrow[ur, "(g_2)_*"'] & &
\end{tikzcd}
$$

Here, f_1, f_2 are restrictions of the $\mathbb{P}^1$ bundle f: $\tilde{Z}_{\geqq rw} \to Z_{\geqq rw}((u, B, B_1, B') \mapsto (u, B_1, B'))$ to $\tilde{Z}_{rw}, Z'_{rw}$; g: ${}_u\tilde{Z}_{\geqq rw} \to Z_{\geqq rw}$ is defined by $g(u, B, B_1, B') = (u, B, B')$ and g_1, g_2 are its restrictions to ${}_u\tilde{Z}_{rw}$, ${}_uZ'_{rw}$. The map d_1: $Z_w \hookrightarrow {}_u\tilde{Z}_{rw}$ is defined by $d_1(u, B, B') = (u, B, B_1, B')$ where B_1 is such that $B \overset{r}{—} B_1 \overset{rw}{—} B'$. The remaining un-named upward arrows are direct image maps induced by obvious closed imbeddings; the un-named downward arrows are inverse image maps induced by obvious open imbeddings.

The composition of the arrows on the highest horizontal line in (b) is the same (by the equivalence of the Definitions 3.3(d), 3.3(f)) as the composition $d^{*\,r}\tau$ in (a). We define ρ_1 to be the composition of the arrows in the second to highest horizontal line in (b). It is clear that with this choice of ρ_1 the diagram (a) is commutative. It remains to show that ρ_1, as we have just defined it, is an isomorphism. From (b) we see that it is enough to show that f_2^*, $\tilde{\mathscr{E}}_2 \otimes$, $(g_2)_*$ are isomorphisms. For $(g_2)_*$ this is clear since g_2 is an isomorphism of varieties. The map f_2 is a locally trivial fibration with fibres isomorphic to $\mathbb{C}$; hence f_2^* is an isomorphism by 1.3(e). Let Z''_{rw} be the set of all (u, B, B_1, B', T) where $(u, B, B_1, B') \in Z'_{rw}$ and T is a maximal torus in $B_1 \cap B'$. Let ${}_uZ''_{rw}$ be the subvariety of Z''_{rw} defined by the condition $u \in B$. Let π: $Z''_{rw} \to Z'_{rw}$ be the affine space bundle $(u, B, B_1, B', T) \to (u, B, B_1, B')$ and let π_1 be its restriction ${}_uZ''_{rw} \to {}_uZ'_{rw}$ (again an affine space bundle). Let $\tilde{\mathscr{E}}_3$ be the complex on Z''_{rw}

obtained by pulling back $\tilde{\mathscr{E}}_2$ under π. It gives rise by 1.3(f) to a natural map $\tilde{\mathscr{E}}_3\otimes: \mathbf{K}(Z''_{rw})\to\mathbf{K}({}_uZ''_{rw})$ and we have a commutative diagram

$$\begin{array}{ccc} \mathbf{K}(Z'_{rw}) & \xrightarrow{\tilde{\mathscr{E}}_2\otimes} & \mathbf{K}({}_uZ'_{rw}) \\ \wr\big\downarrow\pi^* & & \wr\big\downarrow\pi^* \\ \mathbf{K}(Z''_{rw}) & \xrightarrow{\tilde{\mathscr{E}}_3\otimes} & \mathbf{K}({}_uZ''_{rw}) \end{array}$$

To show that $\tilde{\mathscr{E}}_2\otimes$ is an isomorphism it is then enough to show that $\tilde{\mathscr{E}}_3\otimes$ is an isomorphism. Now Z''_{rw} is in a natural way a 1-dimensional vector bundle over ${}_uZ''_{rw}$ and the natural inclusion ${}_uZ''_{rw}\hookrightarrow Z''_{rw}$ is the zero section of this vector bundle; we are in the situation considered in 1.3(f6) hence $\tilde{\mathscr{E}}_3\otimes$ is an isomorphism. This completes the proof of the lemma.

3.20. **Lemma.** *Let $r\in S$, $w\in W$ be such that $rw<w$.*

(a) *The image of $(i_{\leqq w})_*$: $\mathbf{K}(Z_{\leqq w})\hookrightarrow\mathbf{K}(Z)$ is stable under ${}^r\tau$.*

(b) *For any $x\in\mathbf{K}(Z_{\leqq w})$ there exists $x'\in\mathbf{K}(Z_{\leqq rw})$ such that $\bar{x}-{}^r\tau\,\bar{x}'$ is in the image of $(i_{<w})_*$: $\mathbf{K}(Z_{<w})\to\mathbf{K}(Z)$. (Here $i_{<w}$: $Z_{<w}\hookrightarrow Z$ is the inclusion and $\bar{x},\bar{x}'$ denote the images of x,x' under $(i_{\leqq w})_*$, $(i_{\leqq rw})_*$ respectively).*

Proof. (a) follows from 3.18. We now prove (b). Let $x\in\mathbf{K}(Z_{\leqq w})$. By 3.17(c) we have $(i_{\geqq w})^*\,\bar{x}=(j_w)_*\,x_1$ for some $x_1\in\mathbf{K}(Z_w)$. Let $x_2=\rho_1^{-1}(x_1)\in\mathbf{K}(Z_{rw})$ (see 3.19). Using again 3.17(b), (c) for rw, we see that $(j_{rw})_*\,x_2=i^*_{\geqq rw}\,\bar{x}''$ for some $x''\in\mathbf{K}(Z_{\leqq rw})$. From the diagram 3.19(a) we have

$$\begin{aligned} i^*_{\geqq w}\,{}^r\tau\,\bar{x}'' &= d^*\,{}^r\tau\, i^*_{\geqq rw}\,\bar{x}''=d^{*r}\,\tau(j_{rw})_*\,x_2\\ &=(j_w)_*\quad \rho_1 x_2=(j_w)_*\,x_1=(i_{\geqq w})^*\,\bar{x}. \end{aligned}$$

Thus $i^*_{\geqq w}({}^r\tau\,\bar{x}''-\bar{x})=0$. Since $\bar{x},\bar{x}''$ are in the image of $(i_{\leqq w})_*$, we see from (a) that ${}^r\tau\,\bar{x}-\bar{x}=(i_{\leqq w})_*\,y$ for some $y\in\mathbf{K}(Z_{\leqq w})$. We have $i^*_{\geqq w}(i_{\leqq w})_*\,y=0$ hence by 3.17(c), $(j_w)_*(j'_w)^*\,y=0$ and by 3.17(a) we have $(j'_w)^*\,y=0$. From the exactness of $\mathbf{K}(Z_{<w})\to\mathbf{K}(Z_{\leqq w})\xrightarrow{(j'_w)^*}\mathbf{K}(Z_w)$, (1.3(g)), we see that y is the image of some $z\in\mathbf{K}(Z_{<w})$ under $\mathbf{K}(Z_{<w})\to\mathbf{K}(Z_{\leqq w})$. We have $(i_{\leqq w})_*\,y=(i_{<w})_*\,z$ and the lemma is proved.

We can now prove the following result which is a strengthening of 3.5(c).

3.21. **Lemma.** *For any $w\in W$, let $\mathbf{H}_{\leqq w}$ be the $\mathscr{A}$-submodule of $\mathbf{H}$ spanned by all $T_{w'}\,\theta_L(w'\leqq w, L\in\mathbf{X})$ or, equivalently, by all $\theta_L\,T_{w'}(w'\leqq w, L\in\mathbf{X})$. Let $K_{\leqq w}$ be the image of $(i_{\leqq w})_*$: $\mathbf{K}(Z_{\leqq w})\to\mathbf{K}(Z)$. Then $h\to h\mathbb{1}$ defines an isomorphism of $\mathbf{H}_{\leqq w}$ onto $K_{\leqq w}$.*

Proof. In the case where $w=e$, the result follows from 2.15(b). Assume now that $w\neq e$ and that the result is known for all $w', w'<w$. Let $r\in S$ be such that $rw<w$. We first show that $\mathbf{H}_{\leqq w}\mathbb{1}\subset K_{\leqq w}$. Since $K_{\leqq w}$ is clearly stable by multiplication by $\theta_L(L\in\mathbf{X})$ it is enough to show that $T_y\mathbb{1}\in K_{\leqq w}$, $(\forall\, y\leqq w)$. If

$y<w$, then by the induction hypothesis, $T_y \mathbb{1} \in K_{\leqq y} \subset K_{\leqq w}$. Assume now that $y=w$. By the induction hypothesis, $T_{rw} \mathbb{1} \in K_{\leqq w}$. By 3.20(a), we have $T_r K_{\leqq w} \subset K_{\leqq w}$ hence $T_w \mathbb{1} = T_r T_{rw} \mathbb{1} \subset K_{\leqq w}$. Hence our map $\mathbf{H}_{\leqq w} \to K_{\leqq w}$ is well defined.

Let $x \in K_{\leqq w}$. By 3.20(b), there exists $x' \in K_{\leqq rw}$ such that $x - (\mathbf{q} - T_r) x'$ is in the image of $(i_{<w})_*$, hence, by 3.17(d) it is of the form $\sum_{i=1}^{m} x_i''$ where $x_i'' \in K_{\leqq w_i}$, $w_i < w$, $(i=1,2,\dots,m)$. By the induction hypothesis we have $x' \in \mathbf{H}_{\leqq rw} \mathbb{1}$, $x_i'' \in \mathbf{H}_{\leqq w_i} \mathbb{1} \subset \mathbf{H}_{\leqq w} \mathbb{1}$. Note that $T_r \mathbf{H}_{\leqq rw} \subset \mathbf{H}_{\leqq w}$. Hence $x \in \mathbf{H}_{\leqq w} \mathbb{1}$. Thus, our map $\mathbf{H}_{\leqq w} \to K_{\leqq w}$ is surjective.

It is an $\mathbf{R}_{G \times \mathbb{C}^*}$-linear map between projective $\mathbf{R}_{G \times \mathbb{C}^*}$-modules of the same rank $|I| \cdot |W|$, where $I = \#\{w' | w' \leqq w\}$. (For $\mathbf{H}_{\leqq w}$ this is clear from 2.13(b), 2.14(b) and 1.6(b); for $K_{\leqq w} \cong \mathbf{K}(Z_{\leqq w})$ this follows from 3.16). This implies that $\mathbf{H}_{\leqq w} \to K_{\leqq w}$ must be an isomorphism. The lemma is proved.

Since 3.5(c) is the special case of the lemma when w is the longest element of W, we see that 3.5(c) is proved.

4. A vanishing theorem

The main result of this chapter is the following

4.1. **Theorem.** *Let $u \in G$ be unipotent and let $(s,q) \in G \times \mathbb{C}^*$ be a semisimple element such that $s u s^{-1} = u^q$. Then $\mathbf{K}_1(\mathscr{B}_u^s) = 0$, or equivalently* (1.3(m2)), *the odd homology groups with complex coefficients of $\mathscr{B}_u^s$ vanish.*

4.2. In the special case where $(s,q) = (1,1)$, so that $\mathscr{B}_u^s = \mathscr{B}_u$, this result is contained in the works [Sh] and [BS] on the Green functions of reductive groups over a finite field; another proof, using character sheaves is contained in [L_3]. Of course, here the hypothesis that G has simply connected derived group is not needed.

Our proof will consist in a reduction of the general case to this special case. In the case where q is a real number >0, the theorem has been proved independently by V. Ginsburg (private communication) using again the special case above.

4.3. Let X be a smooth quasi-projective variety and let X_0 be a compact subvariety of X. An algebraic action $(\lambda, x) \to \lambda x$ of $\mathbb{C}^*$ on X is said to be a contraction to X_0 if X_0 is $\mathbb{C}^*$-stable and if the following condition is satisfied: for any open subset $U \subset X$ containing X_0 and any compact subset $Y \subset X$, there exists a real number $t > 0$ such that $\lambda . Y \subset U$ for all $\lambda \in \mathbb{C}^*$ such that $|\lambda| \geqq t$.

If this condition is satisfied, then

(a) $$\mathbf{K}_i(X_0) \cong \mathbf{K}^i(X).$$

(This follows from 1.4(a) and the definition of a contraction).

The following result is obvious.

(b) If $\mathbb{C}^*$ acts on X as a contraction to X_0 (as above) and if s: $X \to X$ is a semisimple automorphism of X commuting with the C^*-action, then the induced action of $\mathbb{C}^*$ on the fixed point set X^s is a contraction of X^s to X_0^s.

We shall need the following result.

4.4. **Lemma.** *Assume that* $\mathbb{C}^*$ *acts on* X *as a contraction to* X_0 (as in 4.3). *Then the conditions* $\mathbf{K}_1(X_0)=0$ *and* $\mathbf{K}_1(X_0^{\mathbb{C}^*})=0$ *are equivalent.*

Here $X_0^{\mathbb{C}^*}$ denotes the fixed point set of the $\mathbb{C}^*$ action on X.

The proof will be given in 4.6.

4.5. We now prove Theorem 4.1, assuming 4.4. Assume first that $q=1$. Then s and u commute hence each connected component of $\mathscr{B}_u^s$ is isomorphic to the variety of Borel subgroups of $Z(s)$ containing u, hence it has trivial $\mathbf{K}_1$ by 4.2 applied to $Z(s)$.

We now consider the general case. We can find a homomorphism of algebraic groups φ: $SL_2(\mathbb{C}) \to G$ such that $\varphi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}=u$ and $(s,q)\in M_\varphi$ (see 2.3(g)). Following Slodowy [Sl] we consider the subvariety of Λ (see 3.1) given by

(a) $$X_\varphi=\left\{(v,B)\in\Lambda \mid \log v-\log u \text{ commutes with } \log\varphi\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right\}.$$

This subvariety is stable under the action 3.1(a) of M_φ on Λ.

We identify $\mathscr{B}_u$ with the closed subvariety of X_φ defined by the equation $v=u$. According to [Sl], X_φ is a smooth quasi projective variety and the composition of the M_φ-action with the homomorphism $\mathbb{C}^* \to M_\varphi$ $\lambda \to (\varphi(D(\lambda)), \lambda^2)$ (see 2.3(e)) defines a $\mathbb{C}^*$ action on X_φ which is a contraction to $\mathscr{B}_u$.

Let $X_\varphi^{s,q}$ be the fixed point set of $(s,q)\in M_\varphi$ on X_φ. Since the image of $\mathbb{C}^* \to M_\varphi$ above is in the centre of M_φ, the $\mathbb{C}^*$-action on X_φ restricts to a $\mathbb{C}^*$-action on $X_\varphi^{s,q}$ which is a contraction of $X_\varphi^{s,q}$ to $\mathscr{B}_u^s$, (see 4.3(b)).

Applying 4.4 to this last contraction we get

(b) $$\mathbf{K}_1(\mathscr{B}_u^s)=0 \Leftrightarrow \mathbf{K}_1((\mathscr{B}_u^s)^{\mathbb{C}^*})=0.$$

Now let $s_1=s\cdot\varphi\begin{pmatrix}\lambda^{-1} & 0\\ 0 & \lambda\end{pmatrix}$ where $\lambda^2=q$. Then $(s_1,1)\in M_\varphi$ and it is clear that

(c) $$(\mathscr{B}_u^s)^{\mathbb{C}^*}=(\mathscr{B}_u^{s_1})^{\mathbb{C}^*}.$$

Applying now (b) for $(s_1,1)$ instead of (s,q) we get

(d) $$\mathbf{K}_1(\mathscr{B}_u^{s_1})=0 \Leftrightarrow \mathbf{K}_1((\mathscr{B}_u^{s_1})^{\mathbb{C}^*})=0.$$

As we have seen in the beginning of the proof we have $\mathbf{K}_1(\mathscr{B}_u^{s_1})=0$ since s_1 commutes with u. Using this and (b), (c), (d) we see that $\mathbf{K}_1(\mathscr{B}_u^s)=0$. This proves the theorem, assuming 4.4.

4.6. We now prove Lemma 4.4.

Assume first that $\mathbf{K}_1(X_0^{\mathbb{C}^*})=0$, or equivalently, that $\mathbf{K}^1(X_0^{\mathbb{C}^*})=0$. According to [H, Thm. 7.1], there exists a smooth projective variety $\bar{X}$ containing X as an open dense subvariety and a $\mathbb{C}^*$-action on $\bar{X}$ extending the given $\mathbb{C}^*$-action.

By a general result of [BB], there exists a canonical decomposition of $\bar{X}$ into finitely many locally closed subvarieties X^i ($i \in I$) with the following properties. Each X^i is smooth, $\mathbb{C}^*$-stable and the $\mathbb{C}^*$ action on X^i is a contraction of X^i to $(X^i)^{\mathbb{C}^*} = X^i \cap \bar{X}^{\mathbb{C}^*}$, which is a connected component of $\bar{X}^{\mathbb{C}^*}$; moreover, X^i is naturally a vector bundle over $(X^i)^{\mathbb{C}^*}$.

Since $\mathbb{C}^*$ acts as a contraction on X, we have $X^{\mathbb{C}^*} = X_0^{\mathbb{C}^*}$. It follows that for any $i \in I$, $(X^i)^{\mathbb{C}^*}$ is contained either in $X_0^{\mathbb{C}^*}$ or in $\bar{X} - X$.

Let $x \in X$. Then $x \in X^i$ for some i. Assume that $(X^i)^{\mathbb{C}^*} \subset \bar{X} - X$. We can find disjoint open subsets U (containing $(X^i)^{\mathbb{C}^*}$) and U' (containing X_0). Since $\mathbb{C}^*$ contracts X to X_0 and X^i to $(X^i)^{\mathbb{C}^*}$ we have $\lambda x \in U \cap U'$ for $|\lambda|$ large; this is a contraction.

Hence $(X^i)^{\mathbb{C}^*} \subset X_0^{\mathbb{C}^*}$. Conversely, assume that $(X^i)^{\mathbb{C}^*} \subset X_0^{\mathbb{C}^*}$ and $x \in X^i$. If $x \in \bar{X} - X$ then λx stays inside $\bar{X} - X$ for all λ hence it cannot get near $(X^i)^{\mathbb{C}^*}$; this contradics the fact that $\mathbb{C}^*$ contracts X^i to $(X^i)^{\mathbb{C}^*}$. Hence $x \in X$.

We see therefore that X is equal to the union of all X^i such that $(X^i)^{\mathbb{C}^*} \subset X_0^{\mathbb{C}^*}$. For each such X^i we have $\mathbf{K}^1((X^i)^{\mathbb{C}^*}) = 0$, since $\mathbf{K}^1(X_0^{\mathbb{C}^*}) = 0$ and $(X^i)^{\mathbb{C}^*}$ is a connected component of $X_0^{\mathbb{C}^*}$. Since X^i is a vector bundle over $(X^i)^{\mathbb{C}^*}$, it follows that $\mathbf{K}^1(X^i) = 0$. Using now the exact sequences 1.3(g) arising from the partition of X into the pieces X^i (or rather their analogues for $\mathbf{K}^*$) we deduce that $\mathbf{K}^1(X) = 0$. Using now 4.3(a), we see that $\mathbf{K}_1(X_0) = 0$, as desired.

Conversely, we now assume that $\mathbf{K}_1(X_0) = 0$ and we prove that $\mathbf{K}_1(X_0^{\mathbb{C}^*}) = 0$. (This part of the argument doesn't use the hypothesis that $\mathbb{C}^*$ acts as a contraction on X).

Using our assumption and 1.3(11) we deduce that $\hat{\mathbf{R}}_{\mathbb{C}^*,1} \underset{\mathbf{R}_{\mathbb{C}^*}}{\otimes} \mathbf{K}_1^{\mathbb{C}}(X_0) = 0$.

This implies that $\mathbf{K}_1^{\mathbb{C}^*}(X_0)$ is a torsion $\mathbf{R}_{\mathbb{C}^*}$-module. Hence its localization $\mathbf{R}_{\mathbb{C}^*,z}$ with respect to the maximal ideal of $\mathbf{R}_{\mathbb{C}^*}$ corresponding to $z \in \mathbb{C}^*$ is zero for a sufficiently general z. For such z, we have $X_0^{\mathbb{C}^*}$ = fixed point set of z: $X \to X$ and using 1.3(k), we see that $\mathbf{R}_{\mathbb{C}^*,z} \underset{\mathbf{R}_{\mathbb{C}^*}}{\otimes} \mathbf{K}_1^{\mathbb{C}^*}(X_0^{\mathbb{C}^*}) = 0$. This implies by 1.3(m) that $\mathbf{R}_{\mathbb{C}^*,z} \underset{\mathbb{C}}{\otimes} \mathbf{K}_1(X_0^{\mathbb{C}^*}) = 0$ hence $\mathbf{K}_1(X_0^{\mathbb{C}^*}) = 0$, as desired. This completes the proof of 4.4 and of 4.1.

5. A filtration of $\mathbf{K}_0^{G \times \mathbb{C}^*}(Z)$

5.1. For any locally closed subvariety Y of the variety of unipotent elements of G, which is stable under conjugacy by G, we set

(a) $$Z_Y = \{(u, B, B') \in Z \mid u \in Y\}$$

(This notation should not be confused with the notation Z_I in 3.15, where I was a subset of W).

Then Z_Y is stable under the action 3.3(a) of $G \times \mathbb{C}^*$ and is left and right-r-saturated for any $r \in S$, see 3.3(b). Hence the operators ${}^r\tau, \tau^r$: $\mathbf{K}_0^{G \times \mathbb{C}^*}(Z_Y) \to \mathbf{K}_0^{G \times \mathbb{C}^*}(Z_Y)$ are well defined, (3.3(c), (e)). One of the main results in this chapter is the following.

5.2. **Theorem.** *For any Y as above, we have* $\mathbf{K}_1^{G\times\mathbb{C}^*}(Z_Y)=0$.
The proof will be given in 5.7.

5.3. We shall now state some results which are immediate consequences of 5.2 and of the exact sequences 1.3(g).

(a) If Y in 5.2 is closed in G, then the inclusion $i_Y\colon Z_Y \hookrightarrow Z$ induces an injective map $(i_Y)_*\colon \mathbf{K}_0^{G\times\mathbb{C}^*}(Z_Y)\hookrightarrow \mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$.

(b) The same formulas as in 3.5(a) define a left and a right $\mathbf{H}$-module structure on $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_Y)$ which are compatible by $(i_Y)_*$ with the left and right $\mathbf{H}$-module structures of $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ given in 3.5.

Indeed, checking the relations of $\mathbf{H}$ on $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_Y)$ is reduced since $(i_Y)_*$ is injective, to checking the same relations on $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$, where they are known, by 3.5.

(c) For any Y as in 5.2, the inclusion $j_Y\colon Z_Y \hookrightarrow Z_{\bar{Y}}$ induces a surjective map j_Y^*: $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{Y}})\to \mathbf{K}_0^{G\times\mathbb{C}^*}(Z_Y)$.

(d) The same formulas as in 3.5(a) define a left and right $\mathbf{H}$-module structure on $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_Y)$, which are compatible by j_Y^* with the $\mathbf{H}$-module structures on $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{Y}})$ given in (b) above.

Indeed, this clearly follows from (c).

(e) If Y in 5.2 is closed, then the image of $(i_Y)_*\colon \mathbf{K}_0^{G\times\mathbb{C}^*}(Z_Y)\to \mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ coincides with the sum of the images of $(i_{\bar{\mathscr{C}}})_*\colon \mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{\mathscr{C}}})\to \mathbf{K}_0^{G\times\mathbb{C}^*}(Z)$ where $\mathscr{C}$ runs over all unipotent classes in G contained in Y, and $\bar{\mathscr{C}}$ is the closure of $\mathscr{C}$.

(f) Let $\mathbf{E}$ be a simple left $\mathbf{H}$-module. Then there exists a unipotent class $\mathscr{C}$ in G and a surjective homomorphism of left $\mathbf{H}$-modules $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})\to \mathbf{E}$.

(Here $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})$ is regarded as a left $\mathbf{H}$-module as in (d)).

Indeed, we can obviously find a non-zero homomorphism of left $\mathbf{H}$-modules $\mathbf{H}\to\mathbf{E}$ hence by 3.5, we can find a non-zero homomorphism of left $\mathbf{H}$-modules $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)\to\mathbf{E}$. Hence we can find a unipotent class $\mathscr{C}$ in G of minimal possible dimension for which there exists a non-zero homomorphism of left $\mathbf{H}$-modules $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{\mathscr{C}}})\to\mathbf{E}$, since Z itself is $Z_{\bar{\mathscr{C}}}$ for some unipotent class $\mathscr{C}$. From (e) and the minimality of $\mathscr{C}$ it follows that $\mathbf{E}$ cannot be a quotient of $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{\mathscr{C}}-\mathscr{C}})$. Hence the last homomorphism must be zero on the image of $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{\mathscr{C}}-\mathscr{C}})\hookrightarrow \mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\bar{\mathscr{C}}})$ and so it factors (cf. (c)) through a non-zero (hence surjective) homomorphism of left $\mathbf{H}$-modules $\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})\to\mathbf{E}$, as desired.

5.4. We consider a unipotent class $\mathscr{C}$ in G and an element $u\in\mathscr{C}$. We also consider a semisimple class $\mathscr{D}$ in $G\times\mathbb{C}^*$ and an element $(s,q)\in\mathscr{D}$.

The intersection $\mathscr{D}\cap M(u)$ is a union of finitely many conjugacy classes of $M(u)$ which are closed in $M(u)$. (This is a general fact: a semisimple class of an algebraic group is always closed and it meets a closed subgroup in a union of finitely many semisimple classes of that subgroup).

Consider the variety $\mathscr{R}=\{(s',q',u')\in\mathscr{D}\times\mathscr{C}\,|\,s'u's'^{-1}=u'^{q'}\}$.

We have a projection map $pr_2\colon \mathscr{R}\to\mathscr{C}$, $(s',q',u')\to u'$. This is compatible with the actions of $G\times\mathbb{C}^*$ which acts on $\mathscr{R}$ by (g,a): (s',q',u')

$(gs'g^{-1}, q', gu'^{a^{-1}}g^{-1})$ and on $\mathscr{C}$ by (g,a): $u' \to g u'^{a^{-1}} g^{-1}$. Now $G \times \mathbb{C}^*$ acts transitively on $\mathscr{C}$ and the stabilizer of u is $M(u)$; moreover the fibre $pr_2^{-1}(u)$ is $(\mathscr{D} \cap M(u)) \times \{u\}$. Since $M(u)$ acts on this fibre with finitely many orbits which are all closed, it follows that $G \times \mathbb{C}^*$ acts on $\mathscr{R}$ with finitely many orbits which are all closed.

Moreover there is 1-1 correspondence between the $M(u)$-orbits on $pr_2^{-1}(u)$ and the $G \times \mathbb{C}^*$-orbits on $\mathscr{R}$: to an $M(u)$-orbit corresponds the unique $G \times \mathbb{C}^*$-orbit containing it.

We now apply this argument in the opposite direction to the first projection pr_1: $\mathscr{R} \to \mathscr{D}$, $(s', q', u') \mapsto (s', q')$. Again this map is compatible with the actions of $G \times \mathbb{C}^*$, which acts on $\mathscr{D}$ by conjugation hence transitively.

The stabilizer of (s, q) in $G \times \mathbb{C}^*$ is $Z(s) \times \mathbb{C}^*$. It follows that the orbits of $Z(s) \times \mathbb{C}^*$ on the fibre $pr_1^{-1}(s,q)$ are finitely many and all closed; in fact they are as before in 1-1 correspondence with the $G \times \mathbb{C}^*$-orbits on $\mathscr{R}$.

Now $pr_1^{-1}(s,q)$ may be identified with the set

(a) $$\mathscr{T} = \{u' \in \mathscr{C} \mid s u' s^{-1} = u'^q\}$$

and the action of $Z(s) \times \mathbb{C}^*$ becomes

(b) $$(g,a)\colon\ u' \to g u'^{a^{-1}} g^{-1}$$

Hence we have proved the following.

(c) There are only finitely many $Z(s) \times \mathbb{C}^*$-orbits on $\mathscr{T}$ for the action (b) and they are all closed in $\mathscr{T}$. Moreover, they are in natural 1-1 correspondence with the $M(u)$-orbits on $\mathscr{D} \cap M(u)$ (acting by conjugation): the $Z(s) \times \mathbb{C}^*$-orbit of $u' \in \mathscr{T}$ corresponds to the $M(u)$-orbit of $(s', q') \in \mathscr{D} \cap M(u)$ if and only if there exists $(g, a) \in G \times \mathbb{C}^*$ such that $u' = g u^{a^{-1}} g^{-1}$, $s = g s' g^{-1}$, $q = q'$.

We note also the following fact.

(d) The $Z(s) \times \mathbb{C}^*$-orbits on $\mathscr{T}$ coincide with the $Z(s)$-orbits on $\mathscr{T}$, with $Z(s)$ acting by conjugation.

It is sufficient to show that if $u' \in \mathscr{T}$ and $a \in \mathbb{C}^*$ then $u'^a = \sigma u' \sigma^{-1}$ for some $\sigma' \in Z(s)$. Let φ: $SL_2(\mathbb{C}) \to G$ be a homomorphism of algebraic groups such that $\varphi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = u'$, $(s, q) \in M_\varphi$, see 2.4(g). We can then take $\sigma = \varphi(D(a^{1/2}))$ see 2.4(e), where $a^{1/2}$ is a square root of a.

5.5. We fix a unipotent class $\mathscr{C}$ in G, an element $u \in \mathscr{C}$ and a semisimple class $\mathscr{D}$ in $G \times \mathbb{C}^*$. Let $(s_1, q), \ldots, (s_m, q)$ be a set of representatives for the action of $M(u)$ by conjugation on $\mathscr{D} \cap M(u)$, see 5.4. (Note that m can be zero).

Consider the isomorphism 1.3(h1) for $X = \mathscr{B}_u \times \mathscr{B}_u$, $M = M(u)$ acting diagonally on X, $\dot{M} = G \times \mathbb{C}^*$, $\tilde{X} = Z_{\mathscr{C}} = M(u) \backslash ((G \times \mathbb{C}^*) \times \mathscr{B}_u \times \mathscr{B}_u)$; we tensor that isomorphism with $\hat{\mathbf{R}}_{G \times \mathbb{C}^*}$ over $\mathbf{R}_{G \times \mathbb{C}^*}$. (Here, $\hat{\mathbf{R}}_{G \times \mathbb{C}^*}$ denotes the completion of $\mathbf{R}_{G \times \mathbb{C}^*}$ with respect to the powers of the maximal ideal corresponding to $\mathscr{D}$). We thus get

(a) $$\hat{\mathbf{R}}_{G \times \mathbb{C}^*} \underset{\mathbf{R}_{G \times \mathbb{C}^*}}{\otimes} \mathbf{K}_i^{G \times \mathbb{C}^*}(Z_{\mathscr{C}}) \cong \hat{\mathbf{R}}_{G \times \mathbb{C}^*} \underset{\mathbf{R}_{G \times \mathbb{C}^*}}{\otimes} \mathbf{K}_i^{M(u)}(\mathscr{B}_u \times \mathscr{B}_u)$$

For each j, $(1 \leqq j \leqq m)$, we have a natural homomorphism, see 1.3(j3):

$$\mathbf{K}_i^{M(u)}(\mathscr{B}_u \times \mathscr{B}_u) \to \mathbf{K}_i^{M_j^1}(\mathscr{B}_u \times \mathscr{B}_u)^{M_j}$$

where $M_j^1 = M^1(u, s_j)$, $M_j = M(u, s_j)$, (see 2.5).

This induces a homomorphism

(b) $$\hat{\mathbf{R}}_{G \times \mathbf{C}^*} \underset{\mathbf{R}_{G \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{M(u)}(\mathscr{B}_u \times \mathscr{B}_u) \to (\hat{\mathbf{R}}_{M_j^1} \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_i^{M_j^1}(\mathscr{B}_u \times \mathscr{B}_u))^{M_j}$$

Note that M_j acts on $\mathbf{K}_i^{M_j^1}(\mathscr{B}_u \times \mathscr{B}_u)$ and on $\mathbf{R}_{M_j^1}$ as explained in 1.3(j) and this induces an action of M_j on $\mathbf{R}_{M_j^1} \overset{\text{def}}{=} \hat{\mathbf{R}}_{M_j^1, s_j, q}$, since (s_j, q) is central in M_j.

5.6. **Theorem.** *The composition of* 5.5(a) *with* 5.5(b) *for all* j, $(1 \leqq j \leqq m)$, *gives an isomorphism*

(a) $$\hat{\mathbf{R}}_{G \times \mathbf{C}^*} \underset{\mathbf{R}_{G \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{G \times \mathbf{C}^*}(Z_{\mathscr{C}}) \xrightarrow{\sim} \bigoplus_{1 \leqq j \leqq m} (\hat{\mathbf{R}}_{M_j^1} \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_i^{M_j^1}(\mathscr{B}_u \times \mathscr{B}_u))^{M_j}$$

Proof. We shall define the map in the lemma in a different way. First we fix $(s, q) \in \mathscr{D}$. We use the isomorphism (1.8(c)):

(b) $$\hat{\mathbf{R}}_{G \times \mathbf{C}^*} \underset{\mathbf{R}_{G \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{G \times \mathbf{C}^*}(Z_{\mathscr{C}}) \cong \hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{Z(s) \times \mathbf{C}^*}(Z_{\mathscr{C}})$$

where $\hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \overset{\text{def}}{=} \hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*, s, q}$.

Let $Z_{\mathscr{C}}^{s,q} = \{(u', B, B') \in Z_{\mathscr{C}} \mid s u' s^{-1} = u'^q, B, B' \in \mathscr{B}_{u'}^s\}$; this is the fixed point set of (s, q) acting on $Z_{\mathscr{C}}$ by 3.3(a). By 1.3(k), the imbedding $Z_{\mathscr{C}}^{s,q} \hookrightarrow Z_{\mathscr{C}}$ induces an isomorphism

(c) $$\hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{Z(s) \times \mathbf{C}^*}(Z_{\mathscr{C}}^{s,q}) \xrightarrow{\approx} \hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{Z(s) \times \mathbf{C}^*}(Z_{\mathscr{C}})$$

By 5.4, $Z_{\mathscr{C}}^{s,q}$ decomposes into finitely many pieces $Z_{\mathscr{C},j}^{s,q}$ according to the $Z(s) \times \mathbf{C}^*$-orbits $\mathscr{T}_j$ on $\mathscr{T}$ see 5.4(a), (b); these pieces are indexed by j, $1 \leqq j \leqq m$ and are naturally in $1-1$ correspondence with $(s_1, q), \ldots, (s_m, q)$, (see 5.5). We have $Z_{\mathscr{C},j}^{s,q} = \{(u', B, B') \in Z_{\mathscr{C}}^{s,q} \mid u' \in \mathscr{T}_j\}$.

By 5.4(c) each $\mathscr{T}_j$ is both open and closed in $\mathscr{T}$; hence each $Z_{\mathscr{C},j}^{s,q}$ is both open and closed in $Z_{\mathscr{C}}^{s,q}$. Thus

(d) $$\hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{Z(s) \times \mathbf{C}^*}(Z_{\mathscr{C}}^{s,q}) \cong \bigoplus_{1 \leqq j \leqq m} \hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{Z(s) \times \mathbf{C}^*}(Z_{\mathscr{C},j}^{s,q})$$

Let us choose an element $u_j \in \mathscr{T}_j$ for each j, $(1 \leqq j \leqq m)$.

We now consider the isomorphism 1.3(h1) for $X = \mathscr{B}_{u_j}^s \times \mathscr{B}_{u_j}^s$, $M = M(u_j, s)$, $\tilde{M} = Z(s) \times \mathbf{C}^*$, $\tilde{X} = Z_{\mathscr{C},j}^{s,q} = M(u_j, s) \backslash (Z(s) \times \mathbf{C}^* \times \mathscr{B}_{u_j}^s \times \mathscr{B}_{u_j}^s)$; we tensor it with $\hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*}$ over $\mathbf{R}_{Z(s) \times \mathbf{C}^*}$ and we obtain

(e) $$\begin{aligned} &\bigoplus_{1 \leqq j \leqq m} \hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{Z(s) \times \mathbf{C}^*}(Z_{\mathscr{C},j}^{s,q}) \\ &\cong \bigoplus_{1 \leqq j \leqq m} \hat{\mathbf{R}}_{Z(s) \times \mathbf{C}^*} \underset{\mathbf{R}_{Z(s) \times \mathbf{C}^*}}{\otimes} \mathbf{K}_i^{jM}(\mathscr{B}_{u_j}^s \times \mathscr{B}_{u_j}^s) \\ &\cong \bigoplus_{1 \leqq j \leqq m} \hat{\mathbf{R}}_{jM} \underset{\mathbf{R}_{jM}}{\otimes} \mathbf{K}_i^{jM}(\mathscr{B}_{u_j}^s \times \mathscr{B}_{u_j}^s) \end{aligned}$$

where ${}_jM=M(u_j, s)$ and the second isomorphism is given by 1.2(b); we have set $\hat{\mathbf{R}}_{{}_jM}=\hat{\mathbf{R}}_{{}_jM,s,q}$.

Using 1.3(13), which is applicable in view of 4.1 for $G\times G$, we get (f) an isomorphism of the last group in (e) with $(\hat{\mathbf{R}}_{{}_jM^1}\underset{\mathbf{R}_{{}_jM^1}}{\otimes} K_i^{{}_jM^1}(\mathscr{B}^s_{u_j}\times\mathscr{B}^s_{u_j}))^{{}_jM}$; here, ${}_jM^1=M^1(u_j, s)$ and $\hat{\mathbf{R}}_{{}_jM^1}=\hat{\mathbf{R}}_{{}_jM^1,s,q}$.

Next, we use 1.3(k) for the imbedding $\mathscr{B}^s_{u_j}\times\mathscr{B}^s_{u_j}\hookrightarrow\mathscr{B}_{u_j}\times\mathscr{B}_{u_j}$:

(g) $$\bigoplus_{1\leqq j\leqq m}(\hat{\mathbf{R}}_{{}_jM^1}\underset{\mathbf{R}_{{}_jM^1}}{\otimes}\mathbf{K}_i^{{}_jM^1}(\mathscr{B}^s_{u_j}\times\mathscr{B}^s_{u_j}))\xrightarrow{\approx}\bigoplus_{1\leqq j\leqq m}(\hat{\mathbf{R}}_{{}_jM^1}\underset{\mathbf{R}_{{}_jM^1}}{\otimes}\mathbf{K}_i^{{}_jM^1}(\mathscr{B}_{u_j}\times\mathscr{B}_{u_j})).$$

(h) We now consider the isomorphism obtained by taking ${}_jM$ invariants (see 1.3(j2)) in both sides of (g).

By the choice of indexation for s_j, u_j, there exists $(g_j, a_j)\in G\times\mathbb{C}^*$ such that $g_jsg_j^{-1}=s_j$, $g_ju^{a_j^{-1}}g_j^{-1}=u$, (see 5.4(c)).

This (g_j, a_j) will conjugate ${}_jM$ to M_j, ${}_jM^1$ to M_j^1, $\mathscr{B}_{u_j}$ to $\mathscr{B}_u$ and defines an isomorphism

(i) $$(\hat{\mathbf{R}}_{{}_jM^1}\underset{\mathbf{R}_{{}_jM^1}}{\otimes}\mathbf{K}_i^{{}_jM^1}(\mathscr{B}_{u_j}\times\mathscr{B}_{u_j}))^{{}_jM}\cong(\hat{\mathbf{R}}_{M_j^1}\underset{\mathbf{R}_{M_j^1}}{\otimes}\mathbf{K}_i^{M_j^1}(\mathscr{B}_u\times\mathscr{B}_u))^{M_j}.$$

(This isomorphism is independent of the choice of g_j, since another choice is of form gg_1, $g_1\in{}_jM$ and in the left hand side of (i) we have ${}_jM$-invariant).

(j) We now take the direct sum of the isomorphism in (i) over all j, $1\leqq j\leqq m$. Composing the isomorphisms (b), (c), (d), (e), (f), (g), (h), (j) (or their inverses) we get a homomorphism

$$\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_1^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})\to\bigoplus_{1\leqq j\leqq m}(\hat{\mathbf{R}}_{M_j^1}\underset{\mathbf{R}_{M_j^1}}{\otimes}\mathbf{K}_i^{M_j^1}(\mathscr{B}_u\times\mathscr{B}_u))^{M_j}$$

which on the one hand is an isomorphism, by (h)–(j), and on the other hand (as the reader may verify for himself) it coincides with the map (a).

This completes the proof of the theorem.

5.7. We can now prove Theorem 5.2. Using 1.3(11), which is applicable in view of 4.1, we see that the last group in 5.6(e) is zero for $i=1$. From the proof of 5.6 it then follows that the left hand side of 5.6(a) is zero for $i=1$. Since in 5.6(a), $\hat{\mathbf{R}}_{G\times\mathbb{C}^*}$ is the completion of $\mathbf{R}_{G\times\mathbb{C}^*}$ at an arbitrary maximal ideal, it follows that $\mathbf{K}_1^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})=0$, and Theorem 5.2 is proved.

5.8. **Proposition.** *Let $u\in G$ be a unipotent element and let $(s,q)\in M(u)$ be semi-simple. Let M be any algebraic subgroup of $M(u,s)$ such that M is generated by (s,q) and by its identity component M^0. (In particular, we have $M(s,q)\subset M\subset M^1(u,s)$). Let $\hat{\mathbf{R}}_M\overset{\text{def}}{=}\hat{\mathbf{R}}_{M,s,q}$. Then*

(a) $\hat{\mathbf{R}}_M\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_i^M(\mathscr{B}_u)$ *is zero for $i=1$ and is a free $\hat{\mathbf{R}}_M$-module for $i=0$.*

(b) *The natural map* 1.3(p1): $\hat{\mathbf{R}}_M\underset{\mathbf{R}_{M^1(u,s)}}{\otimes}\mathbf{K}_0^{M^1(u,s)}(\mathscr{B}_u)\to\hat{\mathbf{R}}_M\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u)$ *is an isomorphism.*

(c) *The external tensor product* 1.3(n1) *induces an isomorphism*

$$\boxtimes:\ \hat{\mathbf{R}}_M\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u)\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u)\xrightarrow{\approx}\hat{\mathbf{R}}_M\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u\times\mathscr{B}_u)$$

Proof. By 1.3(k) we have $\hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_i^M(\mathscr{B}_u^s) \xrightarrow{\approx} \hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_i^M(\mathscr{B}_u)$ and we see that (a) follows from 1.3(11), (12) applied to $\mathscr{B}_u^s$; this is applicable, in view of 4.1. From 1.3(k) we see also that the map in (b) may be identified with the map

(d) $$\hat{\mathbf{R}}_M \underset{\mathbf{R}_{M^1(u,s)}}{\otimes} \mathbf{K}_0^{M^1(u,s)}(\mathscr{B}_u^s) \to \hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u^s).$$

This is a homomorphism between free $\hat{\mathbf{R}}_M$ modules. To show that it is isomorphism it is enough (by Nakayama's lemma) to show that it becomes isomorphism after tensoring it over $\hat{\mathbf{R}}_M$ with the residue field of $\hat{\mathbf{R}}_M$. After this tensoring the map (d) becomes $\mathbf{K}_0(\mathscr{B}_u^s) \xrightarrow{\mathrm{id}} \mathbf{K}_0(\mathscr{B}_u^s)$ (cf. 1.3(14), (m1) for the M-variety $\mathscr{B}_u^s$ and for the $M^1(u,s)$-variety $\mathscr{B}_u^s$; this is applicable in view of 4.1). This proves (b).

The proof of (c) is similar. First, using 1.3(k) we are reduced to showing that

(e) $$\boxtimes: \hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u^s) \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u^s) \to \hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u^s \times \mathscr{B}_u^s)$$

is an isomorphism. By (a) for G and for $G \times G$, both sides of the isomorphism in (c), hence both sides of (e) are free $\hat{\mathbf{R}}_M$-modules. To show that (e) is an isomorphism it is again enough to show that the map deduced from (e) by tensoring over $\hat{\mathbf{R}}_M$ with the residue field of $\hat{\mathbf{R}}_M$ is an isomorphism. Using again 1.3(14), (m1) and 4.1 for G and $G \times G$, we are reduced to showing that $\boxtimes$: $\mathbf{K}_0(\mathscr{B}_u^s) \underset{\mathbf{C}}{\otimes} \mathbf{K}_0(\mathscr{B}_u^s) \to \mathbf{K}_0(\mathscr{B}_u^s \times \mathscr{B}_u^s)$ is an isomorphism. This follows from 1.3(n2). This completes the proof.

5.9. **Corollary.** *With the notations of* 5.5, *there exists an isomorphism of* $\hat{\mathbf{R}}_{G \times \mathbf{C}^*}$-*modules*

(a) $$\hat{\mathbf{R}}_{G \times \mathbf{C}^*} \underset{\mathbf{R}_{G \times \mathbf{C}^*}}{\otimes} \mathbf{K}_0^{G \times \mathbf{C}^*}(Z_{\mathscr{C}}) \xrightarrow{\sim} \bigoplus_{1 \leq j \leq m} (\hat{\mathbf{R}}_{M_j} \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_0^{M_j^1}(\mathscr{B}_u) \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_0^{M_j^1}(\mathscr{B}_u))^{M_j}$$

with the following property: the operators

$$E \otimes \xi \to E \otimes (T_r \xi),\ E \otimes \xi \to E \otimes (\xi T_r),\ E \otimes \xi \to E \otimes (\theta_L \xi),\ E \otimes \xi \to E \otimes (\xi \theta_L)$$

($r \in S, L \in \mathbf{X}$) *on the left hand side of* (a) *correspond respectively to the operators*

$$E_1 \otimes \xi_1 \otimes \xi_2 \to E_1 \otimes ((\mathbf{q} - \tau^r)\xi_1) \otimes \xi_2,\ E_1 \otimes \xi_1 \otimes \xi_2 \to E_1 \otimes \xi_1 \otimes (\mathbf{q} - \tau^r)\xi_2,$$
$$E_1 \otimes \xi_1 \otimes \xi_2 \to E_1 \otimes (L \otimes \xi_1) \otimes \xi_2, \quad E_1 \otimes \xi_1 \otimes \xi_2 \to E_1 \otimes \xi_1 \otimes (L \otimes \xi_2)$$

(*see* 3.2(d)) *on the right hand side of* (a). (*The last four operators are defined before taking M_j-invariants but they are compatible with the M_j-action hence define operators on the M_j-invariants*).

Proof. We define (a) as the composition of 5.6(a) with the direct sum over j of the isomorphisms

$$(\hat{\mathbf{R}}_{M_j^1} \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_0^{M_j^1}(\mathscr{B}_u \times \mathscr{B}_u))^{M_j^1} \xrightarrow{\approx} (\hat{\mathbf{R}}_{M_j^1} \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_0^{M_j^1}(\mathscr{B}_u) \underset{\mathbf{R}_{M_j^1}}{\otimes} \mathbf{K}_0^{M_j^1}(\mathscr{B}_u))^{M_j}$$

obtained by taking the inverses of the isomorphisms 5.8(c) for $M=M_j^1$ and (s_j, q), and then taking M_j-invariant parts.

The fact that (a) has the required properties follows from the definitions and from the naturality properties of the operations τ^r, ${}^r\tau$ (see especially 3.3(k)). Note that it is important to define the isomorphism 5.6(a) as in the statement of that theorem and not as in its proof; the maps in that proof are not compatible with operations ${}^r\tau$, τ^r, they are only a device to show that the map 5.6(a) is an isomorphism.

5.10. Let $u\in G$ be unipotent and let M be a diagonalizable algebraic subgroup of $M(u)$. Let $\mathbf{c}$ be a connected component of M such that the image of $\mathbf{c}$ in the group of components M/M^0 is a generator of that group. Identify $\mathbf{R}_M$ with the coordinate ring of M as in 1.2 and consider $\mathbf{R}_{M,\mathbf{c}}$, the coordinate ring of $\mathbf{c}$, as a quotient algebra of $\mathbf{R}_M$ (via restriction of functions). For any $\mathbf{R}_M$-module $\mathscr{M}$, write $\mathscr{M}_\mathbf{c}$ for $\mathbf{R}_{M,\mathbf{c}} \bigotimes_{\mathbf{R}_M} \mathscr{M}$.

5.11. **Proposition.** *With the notations of* 5.10, *the following hold.*

(a) $\mathbf{K}_0^M(\mathscr{B}_u)_\mathbf{c}$ *is a projective* $\mathbf{R}_{M,\mathbf{c}}$*-module and* $\mathbf{K}_1^M(\mathscr{B}_u)_\mathbf{c}=0$.

(b) *There is a unique left* $\mathbf{R}_{M,\mathbf{c}} \bigotimes_{\mathbf{R}_{G\times\mathbb{C}^*}} \mathbf{H}$*-module structure on* $\mathbf{K}_0^M(\mathscr{B}_u)_\mathbf{c}$ *extending the obvious* $\mathbf{R}_{M,\mathbf{c}}$*-module structure and such that*

$$T_r(E_1\otimes\xi)=E_1\otimes(\mathbf{q}-\tau^r)\xi$$
$$\theta_L(E_1\otimes\xi)=E_1\otimes(L\otimes\xi)$$

for all $E_1\in\mathbf{R}_{M,\mathbf{c}}$, $\xi\in\mathbf{K}_0^M(\mathscr{B}_u)$, $r\in S$, $L\in\mathbf{X}$, (τ^r *as in* 3.2(d)).

(c) *For any* $(s,q)\in\mathbf{c}$, *the inclusions* $M(s,q)\subset M\subset M^1(u,s)$ *induce* (1.3(p1)) *isomorphisms:*

$$\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_{M^1(u,s)}}\mathbf{K}_0^{M^1(u,s)}(\mathscr{B}_u)\xrightarrow{\sim}\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_M}\mathbf{K}_0^M(\mathscr{B}_u)\xrightarrow{\sim}\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_{M(s,q)}}\mathbf{K}_0^{M(s,q)}(\mathscr{B}_u),$$

where $\mathbb{C}_{s,q}$ *is as in* 1.2(c).

Proof. Note that M and $M(s,q)$ satisfy the hypothesis of 5.8 for $(s,q)\in\mathbf{c}$. Hence (c) follows from 5.8(b) applied to M and $M(s,q)$. Similarly, (a) follows by applying 5.8(a) at each point (s,q) in $\mathbf{c}$. To check that the endomorphisms in (b) satisfy the relations of $\mathbf{H}$ it is enough to check (in view of (a)) that these relations hold for the analogous endomorphisms of $\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_M}\mathbf{K}_0^M(\mathscr{B}_u)$, for any $(s,q)\in\mathbf{c}$; by (c), we may assume that $M=M(s,q)$. In the rest of the proof, we shall assume that $M=M(s,q)$, (s,q) fixed. Let $\hat{\mathbf{R}}_M=\hat{\mathbf{R}}_{M,s,q}$, $\hat{\mathbf{R}}_{M^1(u,s)}=\hat{\mathbf{R}}_{M^1(u,s),s,q}$.

From 5.9 and 5.3(d) it follows that the operators

$$\text{(d)}\qquad \begin{cases} T_r\colon\ E_1\otimes\xi_1\otimes\xi_2\to E_1\otimes((\mathbf{q}-\tau^r)\xi_1)\otimes\xi_2 & (r\in S)\\ \theta_L\colon\ E_1\otimes\xi_1\otimes\xi_2\to E_1\otimes(L\otimes\xi_1)\otimes\xi_2 & (L\in\mathbf{X})\end{cases}$$

on $\mathscr{K}=\hat{\mathbf{R}}_{M^1(u,s)}\bigotimes_{\mathbf{R}_{M^1(u,s)}}\mathbf{K}_0^{M^1(u,s)}(\mathscr{B}_u)\bigotimes_{\mathbf{R}_{M^1(u,s)}}\mathbf{K}_0^{M^1(u,s)}(\mathscr{B}_u)$ satisfy the relations of $\mathbf{H}$ at least on the $M(u,s)$ invariant part of $\mathscr{K}$. (These operators certainly commute with the action of $M(u,s)$; however multiplication by $\hat{\mathbf{R}}_{M^1(u,s)}$ doesn't in general commute with the $M(u,s)$ action). We have by 5.8(b):

(e) $$\hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u) \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u) \cong \hat{\mathbf{R}}_M \underset{\hat{\mathbf{R}}_{M^1(u,s)}}{\otimes} \mathscr{K}$$

(compatible with the $M(u,s)$ action). It follows that the operators on the left hand side of (e), defined as in (d), satisfy the relations of $\mathbf{H}$, at least on the $M(u,s)$ invariant part. This time, not only these operators but also the action of $\hat{\mathbf{R}}_M$ commutes with the action of $M(u,s)$ since M is contained in the centre of $M(u,s)$. We set $\hat{\mathbf{K}}_0^M(\mathscr{B}_u) = \hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u)$. Consider the $\hat{\mathbf{R}}_M$-homomorphism

$$\psi\colon \hat{\mathbf{K}}_0^M(\mathscr{B}_u) \underset{\hat{\mathbf{R}}_M}{\otimes} \hat{\mathbf{K}}_0^M(\mathscr{B}_u) \underset{\hat{\mathbf{R}}_M}{\otimes} \mathrm{Hom}_{\hat{\mathbf{R}}_M}(\hat{\mathbf{K}}_0^M(\mathscr{B}_u), \hat{\mathbf{R}}_M) \to \hat{\mathbf{K}}_0^M(\mathscr{B}_u)$$

defined by $\alpha \otimes \beta \otimes \gamma \to \gamma(\beta)\alpha$. This is compatible with the operators on $\hat{\mathbf{K}}_0^M(\mathscr{B}_u)$ defined as in (b) and with the analogous operators acting on the first component of the triple tensor product; these last operators satisfy the relations of $\mathbf{H}$ on the subspace

(f) $$(\hat{\mathbf{K}}_0^M(\mathscr{B}_u) \underset{\hat{\mathbf{R}}_M}{\otimes} \hat{\mathbf{K}}_0^M(\mathscr{B}_u))^{M(u,s)} \underset{\hat{\mathbf{R}}_M}{\otimes} \mathrm{Hom}_{\hat{\mathbf{R}}_M}(\hat{\mathbf{K}}_0^M(\mathscr{B}_u), \hat{\mathbf{R}}_M).$$

(Note that this subspace is well defined since $M(u,s)$ acts $\hat{\mathbf{R}}_M$-linearly). If we can show that the restriction of Ψ to the subspace (f) is surjective then it will follow that the relations of $\mathbf{H}$ are also satisfied on $\hat{\mathbf{K}}_0^M(\mathscr{B}_u)$ and hence on $\mathbb{C}_{s,q} \underset{\hat{\mathbf{R}}_M}{\otimes} \hat{\mathbf{K}}_0^M(\mathscr{B}_u) = \mathbb{C}_{s,q} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u)$ and this will end the proof.

The last statement on the restriction of Ψ is a consequence of the following statement.

(g) Let ρ be an irreductible (complex) representation of $\bar{M}(u,s)$, see 2.5(d), and let ρ^* be its dual. If the ρ-isotypic component of the $\bar{M}(u,s)$ module $\hat{\mathbf{K}}_0^M(\mathscr{B}_u)$ is non zero, then so is the ρ^*-isotypic component. (Note that $M(u,s)$ acts on $\hat{\mathbf{K}}_0^M(\mathscr{B}_u)$ through its finite quotient $\bar{M}(u,s)$, see 1.3(j2)).

We prove (g) as follows. Using 1.3(k) we are reduced to the analogous statement for the $\bar{M}(s,u)$-module $\hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u^s)$. Since M acts trivially on $\mathscr{B}_u^s$, we have by 1.3(m): $\hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u^s) \cong \hat{\mathbf{R}}_M \underset{\mathbb{C}}{\otimes} \mathbf{K}_0(\mathscr{B}_u^s)$. (Recall that $\bar{M}(u,s)$ acts trivially on $\hat{\mathbf{R}}_M$). This $\bar{M}(u,s)$-module comes by extension of scalars from a representation defined over $\mathbb{Q}$, see 1.3(m2), in particular, it is isomorphic to its dual. Thus, (g) holds. The proposition is proved.

5.12. Let u be a unipotent element of G and let (s,q) be a semisimple element of $M(u)$. We shall denote $M = M(s,q)$. From 5.11 we see that $\mathbb{C}_{s,q} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u)$ is in a natural way a left $\mathbf{H}$-module. The centre $\mathbf{R}_{G \times \mathbb{C}^*}$ of $\mathbf{H}$ acts via the homomorphism $\mathbf{R}_{G \times \mathbb{C}^*} \to \mathbf{R}_M \to \mathbb{C}_{s,q} = \mathbb{C}$; in particular, the maximal ideal $I_{s,q}$ of $\mathbf{R}_{G \times \mathbb{C}^*}$ corresponding to (s,q) acts as zero. Moreover $\bar{M}(u,s)$ acts naturally on $\mathbb{C}_{s,q} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u)$ (trivially on $\mathbf{R}_M$ and $\mathbb{C}_{s,q}$), see 1.3(j2) and this action commutes with the action of $\mathbf{H}$.

(a) Let $\mathbf{r}(\bar{M}(u,s))$ be the set of irreducible complex representations (up to isomorphism) of $\bar{M}(u,s)$.

For each $\rho \in \mathbf{r}(\bar{M}(u,s))$ we define

(b) $$\mathscr{M}_{u,s,q,\rho} = (\rho^* \underset{\mathbf{C}}{\otimes} (\mathbf{C}_{s,q} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u)))^{\bar{M}(u,s)}$$

where ρ^* is the representation dual to ρ, and $\bar{M}(u,s)$ acts diagonally. We then have naturally

(c) $$\mathbf{C}_{s,q} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u) \cong \bigoplus_{\rho} (\rho \underset{\mathbf{C}}{\otimes} \mathscr{M}_{u,s,q,\rho}).$$

$\mathscr{M}_{u,s,q,\rho}$ will be called a *standard* **H**-*module*. It is a left **H**-module of finite dimension over $\mathbf{C}$, annihilated by $I_{s,q}$.

It is clear that $\mathscr{M}_{u,s,q,\rho}$ depends only on the G-conjugacy class of (u,s,ρ). We set

(d) $$\begin{aligned}\mathbf{r}_0(\bar{M}(u,s)) &= \{\rho \in \mathbf{r}(\bar{M}(u,s)) \mid \mathscr{M}_{u,s,q,\rho} \neq 0\} \\ &= \{\rho \in \mathbf{r}(\bar{M}(u,s)) \mid \rho \text{ appears in the natural representation} \\ &\qquad \text{of } \bar{M}(u,s) \text{ on } \mathbf{K}_0(\mathscr{B}_u^s)\}.\end{aligned}$$

(The equivalence of these two definitions is shown as in the proof of 5.11(g)).

5.13. **Proposition.** *Let* **E** *be any simple (left)* **H**-*module.*

Then there exists a standard **H**-*module* $\mathscr{M}_{u,s,q,\rho}$ *(see* 5.12) *and a non-zero homomorphism of* **H**-*modules* $\mathscr{M}_{u,s,q,\rho} \to \mathbf{E}$.

Proof. By 5.3(f), there exists a unipotent class $\mathscr{C}$ in G and a non-zero **H**-homomorphism $\Psi_1: \mathbf{K}_0^{G \times \mathbf{C}^*}(Z_{\mathscr{C}}) \to \mathbf{E}$. A known argument of Dixmier (applicable since **H** has countable dimension over $\mathbf{C}$) shows that the centre of **H** $(=\mathbf{R}_{G \times \mathbf{C}^*})$ acts on **E** by scalar operators. Hence there exists a maximal ideal I in $\mathbf{R}_{G \times \mathbf{C}^*}$ (corresponding to a semisimple element $(s,q) \in G \times \mathbf{C}^*$), which acts as zero on **E**. It follows that Ψ_1 factors through a non-zero **H**-homomorphism

$$\hat{\mathbf{R}}_{G \times \mathbf{C}^*} \underset{\mathbf{R}_{G \times \mathbf{C}^*}}{\otimes} \mathbf{K}_0^{G \times \mathbf{C}^*}(Z_{\mathscr{C}}) \to \mathbf{E}$$

which is trivial on $\hat{I} = I \cdot \hat{\mathbf{R}}_{G \times \mathbf{C}^*}$. (Here, $\hat{\mathbf{R}}_{G \times \mathbf{C}^*} = \hat{\mathbf{R}}_{G \times \mathbf{C}^*, s, q}$). Using now 5.9, we see that there exists $u \in \mathscr{C}$ such that $(s,q) \in M(u)$ and an **H**-linear non-zero map Ψ_2: $\mathscr{V}^{\bar{M}(u,s)} \to \mathbf{E}$, which is zero on $\hat{I} \cdot \mathscr{V}^{\bar{M}(u,s)}$, where

$$\mathscr{V} = \hat{\mathbf{R}}_{M^1} \underset{\mathbf{R}_{M^1}}{\otimes} \mathbf{K}_0^{M^1}(\mathscr{B}_u) \underset{\mathbf{R}_{M^1}}{\otimes} \mathbf{K}_0^{M^1}(\mathscr{B}_u), \qquad M^1 = M^1(u,s)$$

and $\hat{\mathbf{R}}_{M^1} = \hat{\mathbf{R}}_{M^1, s, q}$. Here $\mathscr{V}^{\bar{M}(u,s)}$ is regarded as a left **H**-module in the same way as the right hand side of 5.9(a). (The operators T_r, θ_L, q act on the whole of $\mathscr{V}$ but we do not known if they satisfy the relations of **H** on the whole of $\mathscr{V}$). Composing Ψ_2 with the canonical $\bar{M}(u,s)$-invariant projection $\mathscr{V} \to \mathscr{V}^{\bar{M}(u,s)}$ (which is compatible with the operators $T_r, \theta_L, \mathbf{q}$) we obtain a non-zero $\mathbf{C}$-linear map Ψ_3: $\mathscr{V} \to \mathbf{E}$, compatible with the operators $T_r, \theta_L, \mathbf{q}$, and such that

$\Psi_3(\hat{I}\mathscr{V})=0$. There exists $\alpha \in \mathbf{K}^{M^1}(\mathscr{B}_u)$ such that the map

$$\Psi_4\colon \mathscr{V}_1=\hat{\mathbf{R}}_{M^1} \underset{\mathbf{R}_{M^1}}{\otimes} \mathbf{K}_0^{M^1}(\mathscr{B}_u) \to \mathbf{E}$$

defined by $\Psi_4(\beta)=\Psi_3(\beta\otimes\alpha)$, is non-zero; it is clearly compatible with the operators $T_r, \theta_L, \mathbf{q}$. Note that T_r, θ_L act on $\mathbf{K}_0^{M^1}(\mathscr{B}_u)$ by a formula like that in 5.11(b), but we do not know if they satisfy the relations of $\mathbf{H}$). Moreover, we have $\Psi_4(\hat{I}\mathscr{V}_1)=0$. Let J be the maximal ideal of $\hat{\mathbf{R}}_{M^1}$. Then by the Nullstellensatz, there exists $t\geqq 1$ such that $J^t\subset \hat{I}\hat{\mathbf{R}}_{M^1}$. Hence $\Psi_4(J^t\mathscr{V}_1)=0$. Let $\mathscr{V}_2=\mathscr{V}_1/J^t\mathscr{V}_1$. Then $\mathscr{V}_2$ inherits an $\hat{\mathbf{R}}_{M^1}/J^t\hat{\mathbf{R}}_{M^1}$-module structure from $\mathscr{V}_1$; it also inherits the operators $T_r, \theta_L, \mathbf{q}$ from $\mathscr{V}_1$, (since those operators are $\hat{\mathbf{R}}_{M^1}$-linear on $\mathscr{V}_1$). Moreover, there exists a non-zero $\mathbb{C}$-linear map $\Psi_5\colon \mathscr{V}_2\to\mathbf{E}$ compatible with the operators $T_r, \theta_L, \mathbf{q}$. We shall admit for a moment the following result to which we will return.

(a) There exists a non-zero $\mathbb{C}$-linear map $\mathscr{V}_2/J\mathscr{V}_2=\mathscr{V}_1/J\mathscr{V}_1\to\mathbf{E}$ which commutes with the operators $T_r, \theta_L, \mathbf{q}$.

Now let $M=M(s,q)$, $\hat{\mathbf{R}}_M=\hat{\mathbf{R}}_{M,s,q}$ and let J' be the maximal ideal of $\hat{\mathbf{R}}_M$. Let $\mathscr{V}_3=\hat{\mathbf{R}}_M \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\mathscr{B}_u)$. From 5.11(c) it follows that $\mathscr{V}_2/J\mathscr{V}_2 \xrightarrow{\sim} \mathscr{V}_3/J'\mathscr{V}_3$ and this is clearly compatible with the action of the operators $T_r, \theta_L, \mathbf{q}$. Moreover from 5.11(b), we know that these operators satisfy the relations of $\mathbf{H}$ on $\mathscr{V}_3/J'\mathscr{V}_3$. Hence from (a) we deduce that there exists a non-zero $\mathbf{H}$-linear map $\Psi_6\colon \mathscr{V}_3/J'\mathscr{V}_3\to\mathbf{E}$. From the definition of standard $\mathbf{H}$-modules (5.12) it follows that $\mathscr{V}_3/J'\mathscr{V}_3$ is a direct sum of standard $\mathbf{H}$-modules $\mathscr{M}_{u,s,q,\rho}$. The restriction of Ψ_6 to at least one of these summands must be non-zero, and the proposition follows, except that we should still verify (a). This is a consequence of the following result.

5.14. **Lemma.** *Let Y be a finite dimensional $\mathbb{C}$-vector space which is a module over a commutative local $\mathbb{C}$-algebra R with* 1 *and with maximal ideal J. Let H_1 be an associative $\mathbb{C}$-algebra with* 1, *with a given $\mathbb{C}$-algebra homomorphism $H_1\to\operatorname{End}_R Y$. Let $\mathbf{E}$ be a simple H_1-module, finite dimensional over $\mathbb{C}$. Assume that there exists a non-zero H_1-linear map $\varphi\colon Y\to\mathbf{E}$. Then there exists a non-zero H_1-linear map $Y/JY\to\mathbf{E}$.*

Proof. We can assume that H_1 is finite dimensional over $\mathbb{C}$. Let $\bar{H}_1=H_1/\operatorname{Rad}(H_1)$, a semisimple algebra. Now $\operatorname{Rad}H_1$ acts as zero on $\mathbf{E}$ hence $\varphi(\operatorname{Rad}(H_1)Y)=0$. Hence φ factors through a non-zero $\bar{H}_1$-linear map $\bar{Y}\to\mathbf{E}$ where $\bar{Y}=Y/\operatorname{Rad}(H_1)Y$. We can decompose $\bar{Y}$ into a direct sum of $\bar{H}_1$-isotypic subspaces. The $\mathbf{E}$-isotypic subspace $\bar{Y}_{\mathbf{E}}$ is non-zero. It is clearly an R-module. By Nakayama's lemma, we have $\bar{J}\cdot\bar{Y}_{\mathbf{E}}\neq 0$. Hence there exists a non-zero H_1-linear map $\bar{Y}_{\mathbf{E}}/J\bar{Y}_{\mathbf{E}}\to\mathbf{E}$ and hence also a non-zero H_1-linear map $\bar{Y}/J\bar{Y}\to\mathbf{E}$. Composing this with the canonical surjection $Y/JY\to\bar{Y}/J\bar{Y}$ we find the required map $Y/JY\to\mathbf{E}$.

This completes the proof of the lemma, hence that of 5.14.

5.15. **Theorem.** (a) *Let (s,q) be a semisimple element of $G\times\mathbb{C}^*$ such that the following condition is satisfied: the variety $\mathscr{T}=\{\bar{u}\in G\,|\,\bar{u}$ unipotent, $s\bar{u}s^{-1}=\bar{u}^q\}$ is* *irreducible. (This condition is automatically satisfied if $q\in\mathbb{C}^*$ is of infinite order). Let $\mathscr{T}_0$ be the unique open orbit of $Z(s)\times\mathbb{C}^*$ acting on $\mathscr{T}$ by (g,a): $\bar{u}\to g\bar{u}^{a^{-1}}g^{-1}$. (The existence of $\mathscr{T}_0$ follows from* 5.4(c)). *Let $u\in\mathscr{T}_0$ and let $\rho\in\mathbf{r}_0(\bar{M}(u,s))$ (see* 5.12(d)). *Then the standard* $\mathbf{H}$*-module $\mathscr{M}_{u,s,q,\rho}$ is simple.*

(b) *Let (u',s',q',ρ') be another set of data satisfying the same assumptions as (u,s,q,ρ) in* (a). *Assume that $\mathscr{M}_{u',s',q',\rho'}$ is isomorphic to $\mathscr{M}_{u,s,q,\rho}$ as an* $\mathbf{H}$*-module. Then $q=q'$ and (u,s,ρ), (u',s',ρ') are G-conjugate.*

Proof. We shall use the notations $M=M(s,q)$, $M^1=M^1(u,s)$, $\hat{\mathbf{R}}_M, \hat{\mathbf{R}}_{M^1}, \hat{\mathbf{R}}_{G\times\mathbb{C}^*}$ $I,\hat{I},J,J'$, $\mathscr{V}$ with the same meaning as in the proof of 5.13. We shall also denote $\mathscr{V}'=\hat{\mathbf{R}}_M\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u)\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u)$.

Let $\mathscr{C}$ be the G-conjugacy class of u and let $\tilde{\mathscr{C}}$ be the union of all unipotent classes of G which contain $\mathscr{C}$ in their closure. Then $Z_{\mathscr{C}}$ is closed in $Z_{\tilde{\mathscr{C}}}$ and $Z_{\tilde{\mathscr{C}}}$ is open in Z, (see 5.1). From our assumption it follows that $\mathscr{T}_0\subset\mathscr{C}\cap\mathscr{T}$ $=\tilde{\mathscr{C}}\cap\mathscr{T}$. Hence the fixed point set $Z^{s,q}_{\tilde{\mathscr{C}}-\mathscr{C}}$ of (s,q) on $Z_{\tilde{\mathscr{C}}-\mathscr{C}}$ is empty.

From 1.3(g) and 5.2 we have exact sequences

(c) $$\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)\to\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\tilde{\mathscr{C}}})\to 0$$

(d) $$0\to\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})\to\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\tilde{\mathscr{C}}})\to\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\tilde{\mathscr{C}}-\mathscr{C}}).$$

Tensoring (d) with $\hat{\mathbf{R}}_{G\times\mathbb{C}^*}$ over $\mathbf{R}_{G\times\mathbb{C}^*}$ gives again an exact sequence; moreover, from 1.3(k) it follows that

$$\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\tilde{\mathscr{C}}-\mathscr{C}})=\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z^{s,q}_{\tilde{\mathscr{C}}-\mathscr{C}})=0,$$

since $Z^{s,q}_{\tilde{\mathscr{C}}-\mathscr{C}}=\emptyset$. Hence (d) becomes

$$\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})\xrightarrow{\sim}\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}(Z_{\tilde{\mathscr{C}}}).$$

Combining this with (c) we get a surjective map

(e) $$\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z)\to\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z_{\mathscr{C}})$$

Consider the diagram

$$\begin{tikzcd}
\hat{\mathbf{H}}\overset{\mathrm{def}}{=}\hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{H} \arrow[r, equal, "(3.5)"] & \hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z) \arrow[r, "(e)"] & \hat{\mathbf{R}}_{G\times\mathbb{C}^*}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{K}_0^{G\times\mathbb{C}^*}(Z) \arrow[dll, "\alpha"] & \\
\mathscr{V}^{\bar{M}(u,s)} \arrow[r, "\beta"] & (\mathscr{V}/J\mathscr{V})^{\bar{M}(u,s)} \arrow[r, equal, "5.11(c)"] & (\mathscr{V}'/J'\mathscr{V}')^{\bar{M}(u,s)} \arrow[r, equal, "5.12(c)"] & \bigoplus\limits_{\rho}(\mathscr{M}_{u,s,q,\rho}\underset{\mathbb{C}}{\otimes}\mathscr{M}_{u,s,q,\rho^*})
\end{tikzcd}$$

where ρ runs over $\mathbf{r}_0(\bar{M}(u,s))$, α is the map 5.9(a) followed by the projection onto the direct summand $\mathscr{V}^{\bar{M}(u,s)}$, β is the obvious map; note that although $\bar{M}(u,s)$ acts non-trivially on $\hat{\mathbf{R}}_{M^1}$, it maps the maximal ideal J into itself. All these maps are $\hat{\mathbf{R}}_{G\times\mathbb{C}^*}$-linear and are compatible with the left and right operators T_r, θ_L which are known to satisfy the relations of $\mathbf{H}$ at the two ends, but not in the middle. Composing these maps we get a surjective $\hat{\mathbf{R}}_{G\times\mathbb{C}^*}$-linear map which is zero on $\hat{I}\hat{\mathbf{H}}$ hence it factors through $\hat{\mathbf{H}}/\hat{I}\hat{\mathbf{H}}=\mathbf{H}/I\mathbf{H}$. Hence it gives rise to a surjective $\mathbb{C}$-linear map $\Phi\colon \mathbf{H}\to\bigoplus_\rho(\mathcal{M}_\rho\otimes\mathcal{M}_{\rho^*})$, $(\mathcal{M}_\rho=\mathcal{M}_{u,s,q,\rho})$, such that left (resp. right) multiplication by T_r, θ_L on $\mathbf{H}$ corresponds to left multiplication by T_r, θ_L on the $\mathcal{M}_\rho$-factor (resp. $\mathcal{M}_{\rho^*}$-factor).

Let $\mathcal{M}_{\rho^*}^*\overset{\text{def}}{=}\operatorname{Hom}_{\mathbb{C}}(\mathcal{M}_{\rho^*},\mathbb{C})$; we make this into a left $\mathbf{H}$-module by setting $(hf)(m)=f(\tilde{h}m)$, $(h\in\mathbf{H}, m\in\mathcal{M}_{\rho^*}, f\in\mathcal{M}_{\rho^*}^*)$, $\tilde{h}$ as in 2.13(c)). Then Φ can be identified with a *surjective* $\mathbb{C}$-linear map

$$\tilde{\Phi}\colon \mathbf{H}\to\bigoplus_\rho \operatorname{Hom}_{\mathbb{C}}(\mathcal{M}_{\rho^*}^*,\mathcal{M}_\rho)$$

such that $\tilde{\Phi}_\rho(h_1hh_2)=h_1\circ\tilde{\Phi}_\rho(h)\circ h_2$, $(h_1,h,h_2\in\mathbf{H})$; here $\tilde{\Phi}_\rho=pr_\rho\circ\tilde{\Phi}$ and pr_ρ is projection onto the ρ-summand; ρ runs over $\mathbf{r}_0(\bar{M}(u,s))$. (In the composition $h_1\circ\tilde{\Phi}_\rho(h)\circ h_2$, we regard $h_1\in\operatorname{End}(\mathcal{M}_\rho)$, $h_2\in\operatorname{End}(\mathcal{M}_{\rho^*}^*)$, using the left $\mathbf{H}$-module structures on these spaces). We now fix an index ρ. It follows that $\tilde{\Phi}_\rho(h)=h\circ\tilde{\Phi}_\rho(1)=\tilde{\Phi}_\rho(1)\circ h$, for all $h\in\mathbf{H}$. In particular, $\tilde{\Phi}_\rho(1)\colon\mathcal{M}_{\rho^*}^*\to\mathcal{M}_\rho$ is an $\mathbf{H}$-linear map. Suppose that $\operatorname{Ker}\tilde{\Phi}_\rho(1)\neq 0$. Then $\tilde{\Phi}_\rho(h)=h\circ\tilde{\Phi}_\rho(1)$ is always zero on $\operatorname{Ker}\tilde{\Phi}_\rho(1)$, which contradicts the fact that all $\mathbb{C}$-linear maps $\mathcal{M}_{\rho^*}^*\to\mathcal{M}_\rho$ are of form $\tilde{\Phi}_\rho(h)$. Thus, $\operatorname{Ker}\tilde{\Phi}_\rho(1)=0$. Similarly, we see that $\operatorname{Coker}\tilde{\Phi}_\rho(1)=0$. Hence $\tilde{\Phi}_\rho(1)$ is an isomorphism of $\mathbf{H}$-modules. Assume that $0\subsetneqq V\subsetneqq\mathcal{M}_{\rho^*}^*$ is an $\mathbf{H}$-submodule. Let $V'=\tilde{\Phi}_\rho(1)(V)$. Then $0\subsetneqq V'\subsetneqq\mathcal{M}_\rho$, $\tilde{\Phi}_\rho(h)V=\tilde{\Phi}_\rho(1)hV\subset V'$. This again contradicts the fact that any $\mathbb{C}$-linear map $\mathcal{M}_{\rho^*}^*\to\mathcal{M}_\rho$ is of form $\tilde{\Phi}_\rho(h)$. It follows that both $\mathcal{M}_{\rho^*}^*$ and $\mathcal{M}_\rho$ are simple $\mathbf{H}$-modules. It is obvious that the kernel of $\tilde{\Phi}_\rho$ is exactly the annihilator of the $\mathbf{H}$-module $\mathcal{M}_\rho$. Let ρ' be an index $\neq\rho$ and assume that $\mathcal{M}_\rho$, $\mathcal{M}_{\rho'}$ are isomorphic as $\mathbf{H}$-modules.

Then these $\mathbf{H}$-modules have the same annihilator in $\mathbf{H}$, hence $\operatorname{Ker}\tilde{\Phi}_\rho=\operatorname{Ker}\tilde{\Phi}_{\rho'}$. Since $\tilde{\Phi}$ is surjective and $\rho\neq\rho'$, there exists $h\in\mathbf{H}$ such that $\tilde{\Phi}_\rho(h)=0$, $\tilde{\Phi}_{\rho'}(h)\neq 0$. This contradicts $\operatorname{Ker}\tilde{\Phi}_\rho=\operatorname{Ker}\tilde{\Phi}_{\rho'}$. Hence $\mathcal{M}_\rho\approx\mathcal{M}_{\rho'}$ implies $\rho=\rho'$.

To prove (a) it remains to show that for q of infinite order in $\mathbb{C}^*$, the variety $\mathcal{T}$ is irreducible. It is enough to prove that the nilpotent elements in the q-eigenspace of $\operatorname{Ad}(s)\colon\mathbf{g}\to\mathbf{g}$ form an irreducible variety. In fact, they form an affine space, since all elements in that q-eigenspace are nilpotent for q of infinite order.

We now prove (b). The centre of $\mathbf{H}$ acts on $\mathcal{M}_{u,s,q,\rho}$, $\mathcal{M}_{u',s',q',\rho'}$ in the same way. Hence $I_{s,q}=I_{s',q'}$, see 5.12. This implies that (s,q), (s',q') are $G\times\mathbb{C}^*$-conjugate. Hence $q=q'$ and we may assume that $s=s'$. Then both u,u' are in $\mathcal{T}_0$ which is a single $Z(s)\times\mathbb{C}^*$-orbit. Hence they are also in the same $Z(s)$-orbit, see 5.4(d). Hence we can also assume that $u=u'$. An earlier argument in the proof shows that $\rho=\rho'$. This completes the proof.

5.16. **Corollary** *(of the proof). The* $\mathbf{H}$*-modules* $\mathcal{M}_\rho$ *and* $\mathcal{M}_{\rho^*}^*$ *(in the proof of* 5.15) *are isomorphic.*

6. The induction theorem

6.1. In this chapter we assume given a unipotent element $u\in G$ and a parabolic subgroup $P\subset G$ with Levi subgroup L containing u. We assume given a diagonalizable algebraic subgroup M of $M(u)\cap(L\times\mathbb{C}^*)$ containing $\mathscr{Z}_L^0\times\{e\}$, where $\mathscr{Z}_L^0$ is the connected centre of L. We also assume given a connected component $\mathbf{c}$ of M such that the image of $\mathbf{c}$ in M/M^0 is a generator of that group.

Let U be the unipotent radical of P and let $\mathbf{u}, \mathbf{l}, \mathbf{p}$ be the Lie algebras of U, L, P.

(a) Let $\mathbf{c}_0$ be the set of all $(g,q)\in\mathbf{c}$ such that $\det(1-gq, (\mathbf{g}/\mathbf{p})_u)\neq 0$. Here, $(\mathbf{g}/\mathbf{p})_u = \mathrm{Ker}(1-u\colon \mathbf{g}/\mathbf{p}\to\mathbf{g}/\mathbf{p})$. Then $\mathbf{c}_0$ is an open subvariety of $\mathbf{c}$. *We shall assume that* $\mathbf{c}_0$ *is non-empty, hence dense in* $\mathbf{c}$.

As in 5.10, we shall identify $\mathbf{R}_M$ with the coordinate ring of M. If $\mathscr{U}$ is an open dense subvariety of $\mathbf{c}$, we shall write $\mathbf{R}_{M,\mathscr{U}}$ for the coordinate ring of $\mathscr{U}$; we have a natural homomorphism $\mathbf{R}_M\to\mathbf{R}_{M,\mathscr{U}}$ given by restriction of functions. For any $\mathbf{R}_M$-module $\mathscr{M}$, we shall write $\mathscr{M}_{\mathscr{U}}$ instead of $\mathbf{R}_{M,\mathscr{U}}\underset{\mathbf{R}_M}{\otimes}\mathscr{M}$.

(b) Let Δ be the invertible element of $\mathbf{R}_{M,\mathbf{c}_0}$ defined by the function $(g,q)\to\det(1-qg,(\mathbf{g}/\mathbf{p})_u)$. Let $\hat{\mathscr{B}}$ be the variety of all Borel subgroups of P/U and let $\hat{\mathscr{B}}_{\hat{u}}$ be the subvariety of $\hat{\mathscr{B}}$ consisting of all Borel subgroups containing $\hat{u}$, the image of u under $\pi\colon P\to P/U$. We have a natural isomorphism

(c) $\alpha\colon \hat{\mathscr{B}}_{\hat{u}}\xrightarrow{\approx}\mathscr{B}_u^P\overset{\text{def}}{=}\mathscr{B}_u\cap\mathscr{B}^P$, $(\hat{B}\to\pi^{-1}\hat{B})$, where $\mathscr{B}^P=\{B\in\mathscr{B}\mid B\subset P\}$.

The group M which acts naturally on $\mathscr{B}_u$ leaves $\mathscr{B}_u^P$ stable. It can be also regarded as a subgroup of $(P/U)\times\mathbb{C}^*$ hence it acts naturally on $\hat{\mathscr{B}}_{\hat{u}}$. By 5.11 (for P/U instead of G), $\mathbf{K}_0^M(\hat{\mathscr{B}}_{\hat{u}})_{\mathbf{c}}$ is a projective $\mathbf{R}_{M,\mathbf{c}}$-module and it has a natural left module structure over the Hecke algebra $\mathbf{H}_{P/U}$ (defined as $\mathbf{H}$ in terms of P/U instead of G). We shall identify $\mathbf{H}_{P/U}$ with the subalgebra of $\mathbf{H}$ spanned as an $\mathscr{A}$-module by the products $T_w\theta_L$ ($w\in W^*$, $L\in X$). (Here, W^* is the subgroup of W generated by the following subset S^* of S: if $r\in S$, we have $r\in S^*$ if and only if P contains some parabolic subgroup of type r.) We identify the groups $\mathbf{X}$ for G and P/U as follows: each $L\in\mathbf{X}$ (for G) may be restricted to $\mathscr{B}^P$ and regarded as a line bundle on $\hat{\mathscr{B}}$ hence as an element $\mathbf{X}$ (for P/U)). Similarly, by 5.11, $\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}}$ is a projective $\mathbf{R}_{M,\mathbf{c}}$-module with a natural left $\mathbf{H}$-module structure. Let $j\colon \mathscr{B}_u^P\to\mathscr{B}_u$ be the inclusion and let $\bar{j}$ be the composition $\hat{\mathscr{B}}_{\hat{u}}\underset{\approx}{\xrightarrow{\alpha}}\mathscr{B}_u^P\overset{j}{\hookrightarrow}\mathscr{B}_u$. It induces a map

(d) $$\bar{j}\colon \mathbf{K}_0^M(\hat{\mathscr{B}}_{\hat{u}})_{\mathbf{c}}\to\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}}$$

which is clearly $\mathbf{R}_{M,\mathbf{c}}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{H}_{P/U}$-linear. (Note that $\mathscr{B}_u^P$ is r-saturated for each $r\in S^*$). Hence $\bar{j}_*$ induces an $\mathbf{H}$-linear map

(e) $$\mathbf{H}\underset{\mathbf{H}_{P/U}}{\otimes}\mathbf{K}_0^M(\hat{\mathscr{B}}_{\hat{u}})_{\mathbf{c}}\to\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}}.$$

(Here we regard $\mathbf{H}$ in a natural way as a right $\mathbf{H}_{P/U}$-module, via $\mathbf{H}_{P/U}\hookrightarrow\mathbf{H}$). With these notations, we can state the main result of this chapter.

6.2. **Theorem.** *Recall that* $\mathbf{c}_0\neq\emptyset$. *The map*

(a) $$\mathbf{H}\underset{\mathbf{H}_{P/U}}{\otimes}\mathbf{K}_0^M(\hat{\mathscr{B}}_{\hat{u}})_{\mathbf{c}_0}\to\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}_0}$$

obtained by restricting 6.1(e) *to* $\mathbf{c}_0$ *is an isomorphism of* $\mathbf{R}_{M,\mathbf{c}_0}\underset{\mathbf{R}_{G\times\mathbb{C}^*}}{\otimes}\mathbf{H}$*-modules.*

6.3. We shall denote

(a) $\quad {}^P\mathscr{B}=\{B\in\mathscr{B}\mid B \text{ opposed to } P\}, \quad {}^P\mathscr{B}_u=\mathscr{B}_u\cap{}^P\mathscr{B}.$

We have a natural map Ψ_0: ${}^P\mathscr{B}\to\mathscr{B}^P$, $B\to(B\cap P)U$. It is clear that Ψ_0 is a principal U-bundle. (U acts by conjugation on ${}^P\mathscr{B}$). Let Ψ: ${}^P\mathscr{B}_u\to\mathscr{B}_u^P$ be the restriction of Ψ_0. Then

(b) Ψ is a principal $U\cap Z(u)$-bundle.

Indeed, $U\cap Z(u)$ acts freely by conjugation on ${}^P\mathscr{B}_u$ and it maps each fibre of Ψ into itself. Let $B, B'\in{}^P\mathscr{B}_u$ be such that $\Psi(B)=\Psi(B')$. Then $B=vB'v^{-1}$ for a unique $v\in U$. Let Q be the unique parabolic subgroup containing B such that $P\cap Q$ is a Levi subgroup of both P and Q; let U' be the unipotent radical of Q. Since $vuv^{-1}\in B\subset Q$, we have $vuv^{-1}=u'l'$, ($u'\in U'$, $l'\in P\cap Q$). Since $u\in B\cap P$ we have $u\in P\cap Q$. Since $u\in P$ and $v\in U$, we have $u^{-1}vu\in U$. Hence $u'l'v=u(u^{-1}vu)$ with $u'\in U'$, $l'\in P\cap Q$, $v\in U$, $u\in P\cap Q$, $u^{-1}vu\in U$. An element of G can be written in at most one way as a product $xyz(x\in U'$, $y\in P\cap Q$, $z\in U)$. It follows that $u'=e$, $l'=u$, $v=u^{-1}vu$. In particular we have $v\in U\cap Z(u)$, as desired, and (b) follows.

From (b) and 1.3(e), it follows that

(c) Ψ^*: $\mathbf{K}_0^M(\mathscr{B}_u^P)_{\mathbf{c}_0}\to\mathbf{K}_0^M({}^P\mathscr{B}_u)_{\mathbf{c}_0}$ is an isomorphism.

The inclusion j': ${}^P\mathscr{B}_u\hookrightarrow\mathscr{B}_u$ induces j'^*: $\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}_0}\to\mathbf{K}_0^M({}^P\mathscr{B}_u)_{\mathbf{c}_0}$. Let w_0 (resp. w_0^*) be the longest element of W (resp. W^*). We can now state the following result.

6.4. **Proposition.** (a) *If* $w\in W$, *the composition*

$$\mathbf{K}_0^M(\mathscr{B}_u^P)_{\mathbf{c}_0}\xrightarrow{j_*}\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}_0}\xrightarrow{T_w}\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}_0}\xrightarrow{j'^*}\mathbf{K}_0^M({}^P\mathscr{B}_u)_{\mathbf{c}_0}\xrightarrow{\Psi^{*-1}}\mathbf{K}_0^M(\mathscr{B}_u^P)_{\mathbf{c}_0}$$

is equal to $(-1)^{\ell(w_0w_0^*)}\Delta T_y$ *if* $w=w_0w_0^*y$ *and is* 0 *if* $w\notin w_0w^*$.

(b) $\operatorname{rank}\mathbf{K}_0^M(\mathscr{B}_u)_{\mathbf{c}_0}=|W/W^*|\operatorname{rank}\mathbf{K}_0^M(\mathscr{B}_u^P)_{\mathbf{c}_0}$, (*rank as projective* $\mathbf{R}_{M,\mathbf{c}_0}$-*modules*).

We shall first show how 6.2 can be deduced from 6.4 and after that we shall prove 6.4. In the following lemma, W_* denotes the set of elements of W such that w has minimal length among the elements in the coset wW^*. We fix $(s,q)\in\mathbf{c}_0$. We shall write $\mathscr{M}=\mathbf{C}_{s,q}\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u)$, $\bar{\mathscr{M}}=\mathbf{C}_{s,q}\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^M(\mathscr{B}_u^P)$. The maps j_*, T_w, ... in 6.4 define analogous maps after the tensoring $\mathbf{C}_{s,q}\underset{\mathbf{R}_M}{\otimes}$, which will be denoted again j_*, $T_{w'}$, ...

6.5. **Lemma.** *Let* I *be a closed subset of* W (*see* 3.15) *such that* $IW^*=I$. *Then any element* $z\in\mathscr{M}$ *of the form* $\sum_{w\in I}T_wj_*(z_w)$, $(z_w\in\bar{\mathscr{M}})$ *can be written uniquely in the form* $\sum_{w\in I\cap W_*}T_wj_*(z'_w)$, $(z'_w\in\bar{\mathscr{M}})$.

Proof. The existence of z'_w is clear from the fact that j_* commutes with T_y, $y\in W^*$. We prove the uniqueness by induction on $|I|$. For I empty the result is clear. Assume that I is non empty and let w be an element of maximal length of I. Let $I' = I - wW^*$. Then I' is again closed and $I' \cdot W^* = I'$. Hence we may assume that the conclusion of the lemma holds for I'; assume it doesn't hold for I. Then there exists $z' \in \bar{\mathcal{M}}$, $z' \neq 0$ such that

(a) $$T_{ww_0^*} j_*(z') = \sum_{w' \in I' \cap W_*} T_{w'} j_*(z'_{w'}), (z'_{w'} \in \bar{\mathcal{M}}).$$

For all w' in the sum, we have $w' \not\leq w$; this implies (we leave the verification to the reader) that in $\mathbf{H}$, $T_{w_0 w^{-1}} T_{w'}$ is an $\mathcal{A}$-linear combination of elements $T_{w''}$, $(w'' \notin w_0 W^*)$. Moreover $T_{w_0 w^{-1}} T_{ww_0^*} = T_{w_0 w_0^*}$. Hence, by applying $T_{w_0 w^{-1}}$ to both sides of (a) we get

$$T_{w_0 w_0^*} j_*(z') = \sum_{w'' \notin w_0 W^*} T_{w''} j_*(z''_{w''}), \quad (z''_{w''} \in \bar{\mathcal{M}}).$$

We now apply $\Psi^{*-1} j'^*$ to both sides of the last equality and we use 6.4; we find $\pm \Delta(s,q)$ $z' = 0$. Since $\Delta(s,q) \neq 0$, we have $z' = 0$. This contradiction proves the lemma.

We now show how 6.4 implies 6.2. Let $\hat{\mathcal{M}} = \mathbb{C}_{s,q} \underset{\mathbf{R}_M}{\otimes} \mathbf{K}_0^M(\hat{\mathcal{B}}_{\hat{u}})$. Then α_*: $\hat{\mathcal{M}} \xrightarrow{\approx} \bar{\mathcal{M}}$. Applying the lemma (which was proved using 6.4(a)) to $I = W$, we see that the $\mathbb{C}$-subspace of $\mathcal{M}$ spanned by all $T_w j_* \bar{\mathcal{M}} = T_w \bar{j}_* \hat{\mathcal{M}}$, $(w \in W_*)$, has dimension $|W/W^*| \dim_{\mathbb{C}} \bar{\mathcal{M}}$ which equals $\dim_{\mathbb{C}} \mathcal{M}$ by 6.4(b). It follows that the map 6.2(a) is onto. On the other hand it is a map between projective $\mathbf{R}_{M,\mathbf{c}_0}$-modules of the same rank (using 6.4(b) and the fact that $\mathbf{H}$ is a free $\mathbf{H}_{P/U}$-module of rank $|W/W^*|$). Hence it is an isomorphism.

6.6. It remains to prove 6.4. Let $\mathbf{c}_1$ be the subvariety of $\mathbf{c}$ consisting of all $m \in \mathbf{c}$ such that $\chi(m) \neq \chi'(m)$ for any two characters $\chi \neq \chi'$: $M \to \mathbb{C}^*$ which appear in the M-module $\mathbf{g}$; M acts on $\mathbf{g}$ by adjoint action via the first projection. It is clear that $\mathbf{c}_1$ is an open dense subset of $\mathbf{c}$. If $m \in \mathbf{c}_1$, then any subspace of $\mathbf{g}$ which is stable under m, is stable under the whole M and hence also under $\mathcal{Z}_L^0 \times \{e\}$, which is contained is M. Applying this to the Lie algebra of a Borel subgroup, we see that

(a) If $(s,q) \in \mathbf{c}_1$ and $B \in \mathcal{B}$ satisfies $s \in B$, $\mathcal{Z}_L^0 \subset B$ and $M \subset B \times \mathbb{C}^*$.

Then $\mathbf{c}_0 \cap \mathbf{c}_1$ is an open dense subvariety of $\mathbf{c}_0$. Since $\mathbf{K}_0^M(\mathcal{B}_u)_{\mathbf{c}_0}$, $\mathbf{K}_0^M(\hat{\mathcal{B}}_{\hat{u}})_{\mathbf{c}_0}$ are projective $\mathbf{R}_{M,\mathbf{c}_0}$-modules, it is enough to prove instead of 6.4 the statement obtained from 6.4 by replacing $\mathbf{c}_0$ by $\mathbf{c}_0 \cap \mathbf{c}_1$. In the rest of this chapter, we shall fix $(s,q) \in \mathbf{c}_0 \cap \mathbf{c}_1$; for any M-variety X, we shall write $\mathbf{K}(X)$ instead of the localization of the $\mathbf{R}_M$-module $\mathbf{K}_0^M(X)$ with respect to the maximal ideal $I_{s,q}$ of $\mathbf{R}_M$ corresponding to (s,q). It will be enough to prove the statement of 6.4 after localization at $I_{s,q}$.

6.7. The orbits of L on the set $\{B \in \mathcal{B} \mid B \supset \mathcal{Z}_L^0\}$ are both open and closed; they are indexed by W_* as follows: to $w \in W_*$ corresponds the L-orbit $\mathcal{O}_w = \{B \in \mathcal{B} \mid B \supset \mathcal{Z}_L^0,\ B \xrightarrow{w} B_1$ for some $B_1 \subset P\}$. Here, B_1 is in fact uniquely determined by B and $B \to B_1$ is an L-equivariant isomorphism $\mathcal{O}_w \xrightarrow{\approx} \mathcal{O}_e$. From the assumption that $(s,q) \in \mathbf{c}_0 \cap \mathbf{c}_1$, it follows that $\mathcal{B}_u^s = \coprod_{w \in W_*} \mathcal{B}_{u,w}^s$, where

$\mathscr{B}^s_{u,w} = \mathscr{B}^s_u \cap \mathcal{O}_w$ is both open and closed in $\mathscr{B}^s_u$. (See 6.6(a)). The isomorphism $\mathcal{O}_w \xrightarrow{\approx} \mathcal{O}_e$ considered above restricts to an isomorphism

(a) $$\lambda_w\colon \mathscr{B}^s_{u,w} \xrightarrow{\approx} \mathscr{B}^s_{u,e}.$$

We shall also consider the orbits of P on $\mathscr{B}$; they are also indexed by W_*; to $w \in W_*$ corresponds the P-orbit $\mathscr{B}_w = \{B \in \mathscr{B} \mid B \xrightarrow{w} B_1$ for some $\mathscr{B}_1 \subset P\}$. We shall set $\mathscr{B}_{u,w} = \mathscr{B}_u \cap \mathscr{B}_w$. The $\mathscr{B}_w$ (resp. $\mathscr{B}_{u,w}$) form a partition of $\mathscr{B}$ (resp. $\mathscr{B}_u$) into locally closed subvarieties. For example, $\mathscr{B}_{u,e} = \mathscr{B}^P_u$ and $\mathscr{B}_{u,w_0 w_0^*} = {}^P\mathscr{B}_u$. We shall denote by $\bar{\mathscr{B}}_{u,w}$ the closure of $\mathscr{B}_{u,w}$. We have $\mathscr{B}_{u,w} \cap \mathscr{B}^s_u = \mathscr{B}^s_{u,w}$. By 1.3(k), the inclusion $\mathscr{B}^s_{u,w} \hookrightarrow \mathscr{B}_{u,w}$ induces an isomorphism

(b) $$\mathbf{K}(\mathscr{B}^s_{u,w}) \xrightarrow{\approx} \mathbf{K}(\mathscr{B}_{u,w}).$$

6.8. **Lemma.** *Let $\hat{\Delta}$ be the element of $\mathbf{R}_{M,\mathbf{c}_0 \cap \mathbf{c}_1}$ defined by the function on* $\mathbf{c}_0 \cap \mathbf{c}_1$: $(s', q') \to \det(1 - s', {}_u(\mathbf{g}/\mathbf{p}))$, *where* ${}_u(\mathbf{g}/\mathbf{p}) = \operatorname{coker}(1 - u\colon \mathbf{g}/\mathbf{p} \to \mathbf{g}/\mathbf{p})$. *Then we have a commutative diagram*

(a) $$\begin{array}{ccccc} \mathbf{K}(\mathscr{B}^s_{u,e}) & \xrightarrow{\hat{\Delta}} & \mathbf{K}(\mathscr{B}^s_{u,e}) & \xrightarrow[\approx]{6.7(\mathrm{b})} & \mathbf{K}(\mathscr{B}^P_u) \\ \Big\downarrow{\scriptstyle (\lambda_{w_0 w_0^*})^* \,\wr} & & \Big\downarrow{\scriptstyle \wr} & & \Big\downarrow{\scriptstyle \wr\, \Psi^*} \\ \mathbf{K}(\mathscr{B}^s_{u,w_0 w_0^*}) & \xrightarrow[\approx]{} & \mathbf{K}(\Psi^{-1}(\mathscr{B}^s_{u,e})) & \xrightarrow[\approx]{} & \mathbf{K}({}^P\mathscr{B}_u) \end{array}$$

(The middle vertical arrow is induced by the affine space bundle $\Psi^{-1}(\mathscr{B}^s_{u,e}) \to \mathscr{B}^s_{u,e}$ defined by restricting $\Psi\colon {}^P\mathscr{B}_u \to \mathscr{B}^P_u$. The lower horizontal maps are induced by the inclusions $\mathscr{B}^s_{u,w_0 w_0^*} \hookrightarrow \Psi^{-1}(\mathscr{B}^s_{u,e}) \hookrightarrow {}^P\mathscr{B}_u$; they are isomorphisms by 1.3(k)).

Proof. We can also describe $\hat{\Delta}$ in terms of the function $(s', q') \to \det(1 - s'^{-1}, \mathbf{u}_u)$ where $\mathbf{u}_u = \operatorname{Ker}(1 - u\colon \mathbf{u} \to \mathbf{u})$. Indeed, the Killing form gives a perfect pairing $\mathbf{u}_u \times {}_u(\mathbf{g}/\mathbf{p}) \to \mathbb{C}$.

The right rectangle in (a) is commutative by 1.3(d). We define

$$\mathbf{u}_u \times \mathscr{B}^s_{u,w_0 w_0^*} \xrightarrow{\sim} \Psi^{-1}(\mathscr{B}^s_{u,e}) \quad \text{by} \quad (v, B) \to \exp(v) B \exp(v)^{-1},$$

see 6.3(b). Then the commutativity of the left rectangle in (a) is equivalent to the commutativity of the triangle

$$\begin{array}{ccc} \mathbf{K}(Y_1) & \xrightarrow{pr_2^*} & \mathbf{K}(\mathbf{u}_u \times Y_1) \\ & {\scriptstyle \hat{\Delta}} \nwarrow \quad \nearrow {\scriptstyle i_*} & \\ & \mathbf{K}(Y_1) & \end{array}$$

where $Y_1 = \mathscr{B}^s_{u,w_0 w_0^*}$, $pr_2\colon \mathbf{u}_u \times Y_1 \to Y_1$ is the second projection and $i\colon Y_1 \hookrightarrow \mathbf{u}_u \times Y_1$ is $y \to (0, y)$. The commutativity of this triangle follows from 1.3(f7).

6.9. We now fix two elements $w', w \in W_*$ and $r \in S$ such that $w = rw' > w'$. Let $X_{w'} = \mathscr{B}^s_{u,w'}$, $X_w = \mathscr{B}^s_{u,w}$. Let Y be the variety of all Borel subgroups $B \in \mathscr{B}$ such that $B \overset{\leqq r}{\text{---}} B_1$ for some $B_1 \in X_{w'}$ (or, equivalently $B \overset{\leqq r}{\text{---}} B_2$ for some $B_2 \in X_w$). Then

Y is a union of r-lines (see 2.2) in $\mathscr{B}$. Each r-line in Y meets $X_{w'}$ in a single point and it meets X_w in a single point; it is therefore contained in $\mathscr{B}_u$. (An r-line which intersects $\mathscr{B}_u$ in more than one point must be entirely contained in $\mathscr{B}_u$). It follows that $Y \subset \mathscr{B}_u$. It is clear that Y is r-saturated, hence we have a natural map τ^r: $\mathbf{K}(Y) \to \mathbf{K}(Y)$, (see 3.2(a)). We shall set $T_r = \mathbf{q} - \tau^r$: $\mathbf{K}(Y) \to \mathbf{K}(Y)$.

Let $\hat{Y}$ be the space of all parabolic subgroups of type r which contain some Borel subgroup in Y. Then π_r: $\mathscr{B} \to \mathscr{P}_r$ (see 2.2) restricts to a map $\hat{\pi}$: $Y \to \hat{Y}$. This is a $\mathbb{P}^1$-bundle with two disjoint sections $X_{w'}$, X_w. Let j_1: $X_{w'} \hookrightarrow Y$, i: $Y - X_{w'} \hookrightarrow Y$, j_2: $X_w \hookrightarrow Y - X_{w'}$ be the inclusions. We now define an endomorphism $\Delta_{w,w'}$ of $\mathbf{K}(\mathscr{B}^s_{u,e})$ as follows. Write $\mathscr{B}^s_{u,e} = \coprod_\alpha \mathscr{B}^{s,\alpha}_{u,e}$ (union of connected components). Correspondingly, we have $\mathbf{K}(\mathscr{B}^s_{u,e}) = \bigoplus_\alpha \mathbf{K}(\mathscr{B}^{s,\alpha}_{u,e})$ and $\Delta_{w,w'}$, on the summand $\mathbf{K}(\mathscr{B}^{s,\alpha}_{u,e})$, is multiplication by the element of $\mathbf{R}_{M, \mathbf{c}_0 \cap \mathbf{c}_1}$ defined by the function on $\mathbf{c}_0 \cap \mathbf{c}_1$: $(s', q') \to \dfrac{1 - q' \lambda^{-1}}{1 - \lambda^{-1}}$; here λ is the eigenvalue of $\mathrm{Ad}(s')$ on $(\mathbf{b} + \mathbf{b}')/\mathbf{b}$, where $B_1 \in \mathscr{B}^{s,\alpha}_{u,e}$, $B = \lambda_w^{-1} B_1$, $B' = \lambda_{w'}^{-1} B_1$ and $\mathbf{b}, \mathbf{b}'$ are the Lie algebras of B, B'. (Note that λ is independent of the choices since a regular function on $\mathscr{B}^{s,\alpha}_{u,e}$ must be constant).

We can now state

6.10. **Lemma** (a) *The composition*

$$\mathbf{K}(X_{w'}) \xrightarrow{(j_1)_*} \mathbf{K}(Y) \xrightarrow{T_r} \mathbf{K}(Y) \xrightarrow{i^*} \mathbf{K}(Y - X_{w'}) \underset{\approx}{\xrightarrow{(j_2)_*^{-1}}} \mathbf{K}(X_w)$$

is given by $-(\lambda_w^{-1})_* \Delta_{w,w'} (\lambda_{w'})_*$.

Proof. Let π_1: $Y - X_{w'} \to X_w$ be the map defined by $\pi_1(B') = B$ where $B \in X_w$, $B \overset{\leq r}{—} B'$. Then π_1 is an affine line-bundle. Applying to it 1.3(f7) we have $\pi_1^* a = (j_2)_* \left(\dfrac{a}{1 - T^*}\right)$, where we denote by T^* (resp. T_Y^*) the cotangent bundle along the fibers of $\hat{\pi}$: $Y \to \hat{Y}$ (or π_r: $\mathscr{B} \to \mathscr{P}_r$) restricted to X_w (resp. Y); we regard it as an M-equivariant vector bundle for the cotangent action of M. Note that M acts trivially on X_w, by 6.6(a).

We now apply 3.2(c) to compute τ^r; this is applicable since $Y \subset \mathscr{B}_u$. We have

$$\begin{aligned}
(j_2)_*^{-1} i^* T_r (j_1)_* \xi &= -(j_2)_*^{-1} i^* \tau^r (j_1)_* \xi \\
&= -(j_2)_*^{-1} i^* (1 - q T_Y^*) \hat{\pi}^* \hat{\pi}_* (j_1)_* \xi \\
&= -(1 - q T^*)(j_2)_*^{-1} i^* \hat{\pi}^* \hat{\pi}_* (j_1)_* \xi \\
&= -(1 - q T^*)(j_2)_*^{-1} \pi_1^* (\lambda_w^{-1} \lambda_{w'})_* \xi \\
&= \frac{1 - q T^*}{1 - T^*} (\lambda_w^{-1} \lambda_{w'})_* \xi.
\end{aligned}$$

It remains to note that the fibre of T^* at $B \in X_w$, (with Lie algebra $\mathbf{b}$) is naturally the dual space of $\mathbf{p}'/\mathbf{b}$ where $P' \in \mathscr{P}_r$, $P' \supset B$ and P' has Lie algebra $\mathbf{p}'$.

6.11. For each $w \in W_*$, we define an endomorphism Δ_w of $\mathbf{K}(\mathscr{B}^s_{u,e})$ as follows. Write $\mathscr{B}^s_{u,e} = \coprod_\alpha \mathscr{B}^{s,\alpha}_{u,e}$ (union of connected components). Correspondingly, we have $\mathbf{K}(\mathscr{B}^s_{u,e}) = \bigoplus_\alpha \mathbf{K}(\mathscr{B}^{s,\alpha}_{u,e})$ and Δ_w on $\mathbf{K}(\mathscr{B}^{s,\alpha}_{u,e})$ is multiplication by the element of $\mathbf{R}_{M,\mathbf{c}_0\cap\mathbf{c}_1}$ defined by the function on $\mathbf{c}_0\cap\mathbf{c}_1$:

$$(s',q') \to \det(1-q's', (\tilde{\mathbf{b}}+\mathbf{p})/\mathbf{p}) \cdot \det(1-s', (\tilde{\mathbf{b}}+\mathbf{p})/\mathbf{p})^{-1}$$

where $B \in \mathscr{B}^{s,\alpha}_{u,e}$, $\tilde{B} \in \mathscr{B}^s_{u,w}$ is defined by $\lambda_w(\tilde{B}) = B$ and $\tilde{\mathbf{b}}$ is the Lie algebra of $\tilde{B}$.

6.12. **Lemma.** *Assume that* $w, w' \in W_*$, $r \in S$, $w' = rw < w$. *Then* $\Delta_w = \Delta_{w'} \Delta_{w,w'}$ *where* $\Delta_{w,w'}$ *is as in* 6.9.

Proof. Let $B, B', B_1 \in \mathscr{B}$ be three Borel subgroups containing s such that $B \overset{r}{—} B' \overset{w'}{—} B_1 \subset P$ and let $\mathbf{b}, \mathbf{b}', \mathbf{b}_1$ be their Lie algebras. It is sufficient to prove that

$$\det(1-q's', (\mathbf{b}+\mathbf{p})/\mathbf{p}) = \det(1-q's', (\mathbf{b}'+\mathbf{p})/\mathbf{p}) \cdot (1-q'\lambda^{-1})$$
$$\det(1-s', (\mathbf{b}+\mathbf{p})/\mathbf{p}) \quad = \det(1-s', (\mathbf{b}'+\mathbf{p})/\mathbf{p}) \cdot (1-\lambda^{-1})$$

where λ is the eigenvalue of $\mathrm{Ad}(s')$ on $(\mathbf{b}+\mathbf{b}')/\mathbf{b}$.

This follows from the exact sequence

$$0 \to (\mathbf{b}'+\mathbf{p})/\mathbf{p} \to (\mathbf{b}+\mathbf{p})/\mathbf{p} \to (\mathbf{b}+\mathbf{p})/(\mathbf{b}'+\mathbf{p}) \to 0,$$

the isomorphism

$$\mathbf{b}/(\mathbf{b}\cap\mathbf{b}') \xrightarrow{\approx} (\mathbf{b}+\mathbf{p})/(\mathbf{b}'+\mathbf{p})$$

and the fact that

$$\mathbf{b}/(\mathbf{b}\cap\mathbf{b}'),\ (\mathbf{b}+\mathbf{b}')/\mathbf{b}$$

are dual to each other under the Killing form.

6.13. **Lemma.** *Let* $w \in W_*$.

(a) *There is a unique map* R_w: $\mathbf{K}(\mathscr{B}^s_{u,e}) \to \mathbf{K}(\bar{\mathscr{B}}_{u,w})$ *such that the diagram*

$$\begin{tikzcd} \mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[r, "R_w"] \arrow[d] & \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \arrow[d] \\ \mathbf{K}(\mathscr{B}_u) \arrow[r, "T_w"] & \mathbf{K}(\mathscr{B}_u) \end{tikzcd}$$

(*with vertical arrows induced by the inclusions* $\mathscr{B}^s_{u,e} \hookrightarrow \mathscr{B}_u$, $\bar{\mathscr{B}}_{u,w} \hookrightarrow \mathscr{B}_u$) *is commutative.*

(b) *The composition*

$$\mathbf{K}(\mathscr{B}^s_{u,e}) \xrightarrow{R_w} \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \xrightarrow{i^*} \mathbf{K}(\mathscr{B}_{u,w}) \xrightarrow[6.7(\mathrm{b})]{(j_w)_*^{-1}} \mathbf{K}(\mathscr{B}^s_{u,w}) \xrightarrow{(\lambda_w)_*} \mathbf{K}(\mathscr{B}^s_{u,e})$$

(*where* j_w: $\mathscr{B}^s_{u,w} \hookrightarrow \mathscr{B}_{u,w}$, i: $\mathscr{B}_{u,w} \hookrightarrow \bar{\mathscr{B}}_{u,w}$ *are the inclusions*) *is equal to* $(-1)^{\ell(w)} \Delta_w$, *see* 6.11.

Proof. We argue by induction on $\ell(w)$. For $w=e$, the result is trivial; we have $\mathscr{B}^s_{u,e} = \mathscr{B}_{u,e} = \bar{\mathscr{B}}_{u,e}$ and R_e = identity. Assume now that $w \neq e$. Let $r \in S$ be such that $w' = rw < w$. Then $w' \in W_*$ and we may assume the result known for w'. Since $rw < w$, the variety $\bar{\mathscr{B}}_{u,w}$ is r-saturated. Hence there are natural operators τ^r (see 3.2(a)) and $T_r = \mathbf{q} - \tau^r$ on $\mathbf{K}(\bar{\mathscr{B}}_{u,w})$. We define R_w as the composition

$$\mathbf{K}(\mathscr{B}^s_{u,e}) \xrightarrow{R_{w'}} \mathbf{K}(\bar{\mathscr{B}}_{u,w'}) \longrightarrow \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \xrightarrow{T_r} \mathbf{K}(\bar{\mathscr{B}}_{u,w})$$

where the middle map is induced by the inclusion $\bar{\mathscr{B}}_{u,w'} \hookrightarrow \bar{\mathscr{B}}_{u,w}$.

Then R_w satisfies the requirement of (a) since $T_w = T_r T_{w'}$ on $\mathbf{K}(\mathscr{B}_u)$, and the operators T_r on $\mathbf{K}(\bar{\mathscr{B}}_{u,w})$ and $\mathbf{K}(\mathscr{B}_u)$ are compatible with the homomorphism i'_*: $\mathbf{K}(\bar{\mathscr{B}}_{u,w}) \to \mathbf{K}(\mathscr{B}_u)$ induced by the inclusion i': $\bar{\mathscr{B}}_{u,w} \hookrightarrow \mathscr{B}_u$.

The uniqueness of R_w follows from the fact that i'_* is injective. Indeed, using 1.3(k) this is equivalent to the statement that the corresponding map $\mathbf{K}(\bar{\mathscr{B}}_{u,w} \cap \mathscr{B}^s_u) \to \mathbf{K}(\mathscr{B}^s_u)$ is injective; but this is clear since $\bar{\mathscr{B}}_{u,w} \cap \mathscr{B}^s_u$ is both open and closed in $\mathscr{B}^s_u$, (see 6.7).

To prove (b) we shall consider the closed subvariety Y (see 6.9) of $\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'}$. Since both Y and $\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'}$ are r-saturated, we have natural maps τ^r (and $T_r = \mathbf{q} - \tau^r$) on $\mathbf{K}(Y)$, $\mathbf{K}(\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'})$, compatible with the map $(i_1)_*$: $\mathbf{K}(Y) \hookrightarrow \mathbf{K}(\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'})$ induced by the inclusion i_1: $Y \hookrightarrow \mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'}$. Note that $(i_1)_*$ is an isomorphism; this follows from 1.3(k) since s has the same fixed point set on both Y and $\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'}$.

We have a commutative diagram:

$$\begin{tikzcd}
\mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[dr, "R_w"] & & & & & \\
& \mathbf{K}(\bar{\mathscr{B}}_{u,w'}) \arrow[r] \arrow[d] & \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \arrow[r, "T_r"] \arrow[d] & \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \arrow[rr, "="] \arrow[d] & & \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \arrow[d] \\
& \mathbf{K}(\mathscr{B}_{u,w'}) \arrow[r] & \mathbf{K}(\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'}) \arrow[r, "T_r"] & \mathbf{K}(\mathscr{B}_{u,w} \cup \mathscr{B}_{u,w'}) \arrow[rr] & & \mathbf{K}(\mathscr{B}_{u,w}) \\
& \mathbf{K}(\mathscr{B}^s_{u,w'}) \arrow[r] \arrow[u, "\approx"] \arrow[dr, "(\lambda_{w'})_*", "\approx"'] & \mathbf{K}(Y) \arrow[r, "T_r"] \arrow[u, "\approx", "(i_1)_*"'] & \mathbf{K}(Y) \arrow[r] \arrow[u, "\approx", "(i_1)_*"'] & \mathbf{K}(Y - \mathscr{B}^s_{u,w'}) \arrow[r] & \mathbf{K}(\mathscr{B}^s_{u,w}) \arrow[u, "\approx"'] \arrow[dll, "\approx", "(\lambda_w)_*"'] \\
& & \mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[r, "-\Delta_{w,w'}"] & \mathbf{K}(\mathscr{B}^s_{u,e}) & &
\end{tikzcd}$$

where the last two lines are as in Lemma 6.10, the vertical downward maps are induced by inclusions of open subvarieties and the upward maps are induced by inclusion of closed subvarieties.

Going on the periphery of this diagram in two different ways shows that the composition in (b) for w, is equal to the analogous composition for w', composed with $-\Delta_{w,w'}$. It remains to use Lemma 6.12.

6.14. We now prove Proposition 6.4. Note that by 1.3(k), we have rank $\mathbf{K}(\mathscr{B}^P_u)$ $=$ rank $\mathbf{K}(\mathscr{B}^s_{u,e})$,

$$\begin{aligned}\operatorname{rank} \mathbf{K}(\mathscr{B}_u) &= \operatorname{rank} \mathbf{K}(\mathscr{B}_u^s) = \operatorname{rank} \mathbf{K}\Big(\coprod_{w\in W_*} \mathscr{B}^s_{u,w}\Big) = \sum_{w\in W_*} \operatorname{rank} \mathbf{K}(\mathscr{B}^s_{u,w})\\ &= |W_*| \operatorname{rank} \mathbf{K}(\mathscr{B}^s_{u,e})\end{aligned}$$

(using 6.7). Hence 6.4(b) follows.

To prove 6.4(a), we first note that it is enough to consider the case where $w \in W_*$; the general case reduces to this, since j_* in 6.4(a) commutes with T_y, $(y \in W^*)$. We now consider the case where $w \in W_*$, $w \neq w_0 w_0^*$. We then have $\bar{\mathscr{B}}_{u,w} \subset \mathscr{B}_u - {}^P\mathscr{B}_u$ hence the composition $\mathbf{K}(\bar{\mathscr{B}}_{u,w}) \xrightarrow{i'_*} \mathbf{K}(\mathscr{B}_u) \xrightarrow{j'^*} \mathbf{K}({}^P\mathscr{B}_u)$ is zero by 1.3(g). (Here $i': \bar{\mathscr{B}}_{u,w} \hookrightarrow \mathscr{B}_u$, $j': {}^P\mathscr{B}_u \hookrightarrow \mathscr{B}_u$ are the inclusions). It follows that in the commutative diagram

$$\begin{tikzcd}
 & \mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[r, "R_w"] \arrow[dl, "\approx"'] \arrow[d] & \mathbf{K}(\bar{\mathscr{B}}_{u,w}) \arrow[d, "i'_*"] & & \\
\mathbf{K}(\mathscr{B}^P_u) \arrow[r] & \mathbf{K}(\mathscr{B}_u) \arrow[r, "T_w"] & \mathbf{K}(\mathscr{B}_u) \arrow[r] & \mathbf{K}({}^P\mathscr{B}_u) \arrow[r] & \mathbf{K}(\mathscr{B}^P_u)
\end{tikzcd}$$

with the lower line as in 6.4(a) and the square as in 6.13(a), the composition of the lower horizontal arrows is zero, as asserted in 6.4(a).

Next, we consider the case where $w = w_0 w_0^*$. In this case, $\bar{\mathscr{B}}_{u,w} = \mathscr{B}_u$, $\mathscr{B}_{u,w} = {}^P\mathscr{B}_u$. We have a commutative diagram

$$\begin{tikzcd}
 & \mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[r, "R_w"] \arrow[dl, "\approx"'] \arrow[dd] & \mathbf{K}(\mathscr{B}_u) \arrow[r] \arrow[dd, equal] & \mathbf{K}(\mathscr{B}_{u,w}) \arrow[r, "(j_w)_*^{-1}"] \arrow[dd, equal] & \mathbf{K}(\mathscr{B}^s_{u,w}) \arrow[r] \arrow[ddl, "\approx"'] & \mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[d, "\hat{\Delta}"'] \\
 & & & & & \mathbf{K}(\mathscr{B}^s_{u,e}) \arrow[dl, "\approx"] \\
\mathbf{K}(\mathscr{B}^P_u) \arrow[r] & \mathbf{K}(\mathscr{B}_u) \arrow[r, "T_w"] & \mathbf{K}(\mathscr{B}_u) \arrow[r] & \mathbf{K}({}^P\mathscr{B}_u) \arrow[r, "\Psi^{*-1}"] & \mathbf{K}(\mathscr{B}^P_u) &
\end{tikzcd}$$

with the upper line as in 6.13(b) and the lower line as in 6.4(a); the right part of the diagram is as in Lemma 6.8.

Using 6.13(b) we see that the composition of the upper horizontal arrows is multiplication by the element of $\mathbf{R}_{M, \mathbf{c}_0 \cap \mathbf{c}_1}$ defined by the function on $\mathbf{c}_0 \cap \mathbf{c}_1$:

(a) $$(s', q') \mapsto (-1)^{\ell(w_0 w_0^*)} \det(1 - q's', \mathbf{g}/\mathbf{p}) \det(1 - s', \mathbf{g}/\mathbf{p})^{-1}.$$

The commutativity of the diagram shows that the composition of the lower horizontal arrows is multiplication by the element of $\mathbf{R}_{M, \mathbf{c}_0 \cap \mathbf{c}_1}$ defined by the function on $\mathbf{c}_0 \cap \mathbf{c}_1$ which is the product of (a) with $\hat{\Delta}$. It remains to prove that this product is equal to Δ. (See 6.1(b)). Thus, we are reduced to proving the following lemma.

6.15. **Lemma.** *For any semisimple element $(s, q) \in M(u) \cap L$ we have the identity.*

$$\det(1 - qs, \mathbf{g}/\mathbf{p}), \det(1 - s, {}_u(\mathbf{g}/\mathbf{p})) = \det(1 - s, \mathbf{g}/\mathbf{p}) \det(1 - qs, (\mathbf{g}/\mathbf{p}_u))$$

(Recall that $(\mathbf{g}/\mathbf{p})_u$, ${}_u(\mathbf{g}/\mathbf{p})$ denote the kernel and cokernel of $1 - \operatorname{Ad}(u)$: $\mathbf{g}/\mathbf{p} \to \mathbf{g}/\mathbf{p}$).

Proof. Let f: $\mathbf{g}/\mathbf{p} \to \mathbf{g}/\mathbf{p}$ be the endomorphism induced by $x \to [\log(u), x]$. Then $(\mathbf{g}/\mathbf{p})_u$: $\ker f$, ${}_u(\mathbf{g}/\mathbf{p}) = \operatorname{coker} f$. The identity in the lemma can be also written as follows: $\det(1-s \text{ image } (f)) = \det(1-qs, (\mathbf{g}/\mathbf{p})/\ker f)$. Now $f: (\mathbf{g}/\mathbf{p})/\ker f \to \operatorname{image} f$ is an isomorphism under which the endomorphisms $1-qs$ (on the first space) and $1-s$ (on the second space) correspond to each other. Hence they have the same determinant. (We have used the identity $[\log u, (1-qs)x] = (1-s)[\log u, x]$, $x \in \mathbf{g}$). The lemma is proved.

7. The classification theorem

7.1. Let u be a unipotent element of G and let (s, q) be a semisimple element of $M(u)$. We shall set $N = \log(u) \in \mathbf{g}$. We assume given a homomorphism V: $\mathbb{C}^* \to \mathbb{R}$ (not necessarity continuous) such that $V(q) \geqq 0$.

We shall attach to u, s, q, V a canonical parabolic subgroup P of G as follows.

Choose a homomorphism of algebraic groups $\varphi: SL_2(\mathbb{C}) \to G$ such that $\varphi\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix} = u$, $(s, q) \in M_\varphi$, (see 2.4(g)). Let $\lambda = q^{1/2}$ be a square root of q and let $s_1 = \varphi(D(\lambda^{-1}))$. Then s_1 commutes with $\varphi(SL_2(\mathbb{C}))$ and with s. Hence we have a direct sum decomposition

$$\mathbf{g} = \bigoplus_{\substack{\alpha \in \mathbb{C}^*\\ i \in \mathbb{Z}}} \mathbf{g}_{\alpha, i}, \qquad \mathbf{g}_{\alpha, i} = \{x \in \mathbf{g} \mid s_1 x = \alpha x,\ \varphi(D(\mu))x = \mu^i x, \text{ for all } \mu \in \mathbb{C}^*\}$$

Then $\mathbf{p} = \bigoplus_{V(\alpha) \leqq 0} \mathbf{g}_{\alpha, i}$ is the Lie algebra of a parabolic subgroup P of G, with Levi subgroup L and unipotent radical U such that the Lie algebra of L is $\mathbf{l} = \bigoplus_{V(\alpha) = 0} \mathbf{g}_{\alpha, i}$ and the Lie algebra of U is $\mathbf{u} = \bigoplus_{V(\alpha) < 0} \mathbf{g}_{\alpha, i}$.

Note that L contains both s_1 and $\varphi(SL_2(\mathbb{C}))$.

7.2. **Lemma.** (a) *We have* $M(u, s) \subset P \times \mathbb{C}^*$

(b) *P is independent of the choice of φ and λ above.*

(c) $\det(1-qs, (\mathbf{g}/\mathbf{p})_u) \neq 0$, ($(\mathbf{g}/\mathbf{p})_u$ is as in 6.1(a)).

Proof. Let $x \in \mathbf{g}_{\alpha, i}$, $V(\alpha) \leqq 0$. If $z \in Z(s)$, we have

$$\operatorname{Ad}(s) \operatorname{Ad}(z)x = \operatorname{Ad}(z) \operatorname{Ad}(s)x = \operatorname{Ad}(z)\alpha\lambda^i x = \alpha\lambda^i \operatorname{Ad}(z)x$$

hence

(d) $\operatorname{Ad}(z)x \in \bigoplus \mathbf{g}_{\alpha', i'}$, sum over all α', i' such that $\alpha'\lambda^{i'} = \alpha\lambda^i$.

On the other hand, if $(z, q') \in M(u)$, then using 2.6(c) we see that

(e) $\operatorname{Ad}(z)x \in \bigoplus_{j \geqq i} \mathbf{g}_{\beta, j}$.

If now $(z, q') \in M(u, s)$ then both (d) and (e) hold, hence $\operatorname{Ad}(z) \in \bigoplus \mathbf{g}_{\alpha', i'}$, sum over all α', i' such that $\alpha'\lambda^{i'} = \alpha\lambda^i$, $i' \geqq i$. For any such α', i' we have

$$V(\alpha') = V(\alpha\lambda^{i-i'}) = V(\alpha) + (i-i')V(\lambda) \leqq 0$$

since $V(\alpha) \leqq 0$, $i' \leqq i$ and $V(\lambda) \geqq 0$.

Thus $\mathrm{Ad}(z)x \in \mathbf{p}$. It follows that $\mathrm{Ad}(z)\mathbf{p} \subset \mathbf{p}$, hence $z \in P$ and (a) is proved.

We now prove (b). By 2.4(h) any other choice of φ is of the form $\varphi'(A) = z\varphi(A)z^{-1}$ for some $z \in Z(s) \cap Z(u)$; moreover λ could be only changed into $\lambda' = \varepsilon\lambda$, $(\varepsilon = \pm 1)$. Applying the construction of P in 7.1 starting from φ', λ' instead of φ, λ we shall get $zs_1\varphi(D(\varepsilon))z^{-1}$ instead of s_1 and zPz^{-1} instead of P. (Note that $V(\pm 1) = 0$.) By (a) we have $z \in P$ hence $zPz^{-1} = P$ and (b) is proved.

We now prove (c). It is enough to show that any eigenvalue β of qs on $\ker(\mathrm{ad}(N)\colon \mathbf{g}/\mathbf{p} \to \mathbf{g}/\mathbf{p})$, (or equivalently, on $\ker(\mathrm{ad}(N)\colon \bigoplus_{V(\alpha)>0} \mathbf{g}_{\alpha,i} \to \bigoplus_{V(\alpha)>0} \mathbf{g}_{\alpha,i})$) satisfies $V(\beta) > 0$ (hence $\beta \neq 1$). Now $\ker(\mathrm{ad}(N)\colon \mathbf{g} \to \mathbf{g}) \subset \bigoplus_{i \geqq 0} \mathbf{g}_{\alpha,i}$, (2.6(d)), and hence β is also an eigenvalue of qs on the bigger vector space $\bigoplus_{\substack{V(\alpha)>0\\ i \geqq 0}} \mathbf{g}_{\alpha,i}$. Hence $\beta = \alpha q^{(i/2)+1}$ and we have

$$V(\alpha q^{(i/2)+1}) = V(\alpha) + ((i/2)+1)\,V(q) \geqq V(\alpha) > 0,$$

as required. The following lemma and its proof are closely related to [L_1, 2.8, 2.9].

7.3. **Lemma.** *Let (π, E) be an irreducible rational G-module.*

(a) *If $B \in \mathscr{B}_u^s$ and $B_1 \in \mathscr{B}^{s_1}$, $B_1 \subset P$ then $V(\Psi_{LE}^B(s)) \geqq V(\Psi_{LE}^{B_1}(s_1))$ (notation of 2.8, 2.9).*

Moreover, $V(\Psi_{LE}^B(s_1))$ is independent of the choice of B_1: it is the minimum value t of $V(\nu)$ where ν runs over the eigenvalues of $\pi(s_1)\colon E \to E$.

(b) *If, in addition, E above contains a P-stable line then*

$$\Psi_{LE}^B(s) = \Psi_{LE}^{B_1}(s_1) \qquad \text{for all } B \in \mathscr{B}_u^s,\ B \subset P.$$

(c) *If, in addition, E above contains a line whose stabilizer in G is exactly P then $V(\Psi_{LE}^B(s)) > V(\Psi_{LE}^{B_1}(s_1))$ for all $B \in \mathscr{B}_u^s$, $B \not\subset P$.*

Proof. Let us decompose $E = \bigoplus_\nu E^\nu$ where E^ν is the ν-eigenspace of $\pi(s_1)\colon E \to E$. We have clearly $\mathbf{g}_{\alpha,i} E^\nu \subset E^{\alpha\nu}$ for the $\mathbf{g}$-module structure on E defined by π. Let t be as in (a) and let $E_{\min} = \bigoplus_{V(\nu)=t} E^\nu$. It follows that $\mathbf{g}_{\alpha,i} E_{\min} \subset E_{\min}$ whenever $V(\alpha) \leqq 0$ and $\mathbf{g}_{\alpha,i} E_{\min} = 0$ if $V(\alpha) < 0$. Hence $\mathbf{p}E_{\min} \subset E_{\min}$ and $\mathbf{u}E_{\min} = 0$.

Let $\mathbf{b}_1$ be the Lie algebra of B_1. Then $\mathbf{b}_1 E_{\min} \subset E_{\min}$, since $\mathbf{b}_1 \subset \mathbf{p}$. Hence, by Lie's theorem, $E_{\min}$ (which is $\neq 0$) must have a $\mathbf{b}_1$-stable line. This must be the unique B_1-stable line in E and it follows that $V(\Psi_{LE}^{B_1}(s_1)) = t$, proving the second assertion of (a).

Now let y be a non-zero vector in E such that $\pi(b)y = \Psi_{LE}^B(b)y$ for all $b \in B$. We set $\Psi_{LE}^B(s) = \alpha$; we must show that $V(\alpha) \geqq t$. We write $y = \sum_\nu y_\nu$, $y_\nu \in E^\nu$. Each E^ν is stable under $\pi(s)$ and $\pi(\varphi(SL_2(\mathbb{C})))$, since these commute with $\pi(s_1)$. We have $\pi(s)y = \sum_\nu \pi(s)y_\nu = \alpha \sum_\nu y_\nu$ hence $\pi(s)y_\nu = \alpha y_\nu$ for all ν. Similarly, since $u \in B$, we have $\pi(u)y = y$ and $\pi(u)y_\nu = y_\nu$ for all ν. Let ν be such that $y_\nu \neq 0$. We have

$$\pi(s)y_\nu = \pi(\varphi(D(q^{1/2})))\pi(s_1)y_\nu = \nu\pi(\varphi(D)q^{1/2}))\,y_\nu$$

hence

$$\pi(\varphi(D(q^{1/2})))y_\nu = \alpha\nu^{-1}y_\nu, \quad \text{and} \quad \pi(u)y_\nu = y_\nu.$$

This together with a general property ((d) below) of $SL_2(\mathbb{C})$-modules applied to the $SL_2(\mathbb{C})$-module E^ν implies that $V(\alpha\nu^{-1}) \geqq 0$.

(d) Let $\bar{E}$ be a finite dimensional rational $SL_2(\mathbb{C})$-module and let $\bar{y}$ be a non-zero vector of $\bar{E}$ such that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \bar{y} = \bar{y}$ and $D(\lambda)\bar{y} = \xi\bar{y}$ for some $\lambda, \xi \in \mathbb{C}^*$.

Then $\xi = \lambda^m$ for some integer $m \geqq 0$. Hence if $V(\lambda) \geqq 0$, then $V(\xi) \geqq 0$. (It is enough to verify this for $\bar{E}$ irreducible).

From $V(\alpha\nu^{-1}) \geqq 0$ we now get $V(\alpha) \geqq V(\nu)$. As we have seen earlier, we have $V(\nu) \geqq t$ hence $V(\alpha) \geqq t$ and (a) is proved.

Now let E be as in (b). Let $\omega\colon P \to \mathbb{C}^*$ be the character by which P acts on the unique P-stable line in E. We have $\Psi^B_{L_E}(s) = \omega(s)$, $\Psi^{B_1}_{L_E}(s_1) = \omega(s_1)$. Since $\varphi(SL_2(\mathbb{C}))$ is contained in the derived group of P, we have $\omega(\varphi(D(q^{1/2}))) = 1$ hence $\omega(s) = \omega(s_1)$. It follows that $\Psi^B_{L_E}(s) = \Psi^{B_1}_{L_E}(s_1)$, proving (b). Finally, let E be as in (c). Assume that in (a) we have equality: $V(\Psi^B_{L_E}(s)) = V(\Psi^{B_1}_{L_E}(s_1))$. From the proof of (a), we see that for each ν such that $y_\nu \neq 0$, we have $V(\alpha) = V(\nu) = t$ (with notations in that proof). Hence $y = \sum_{V(\nu)=t} y_\nu \in E_{\min}$. As we have seen earlier, $E_{\min}$ is annihilated by $\mathbf{u}$. From our hypothesis on E, it follows that the space of vectors of E annihilated by $\mathbf{u}$ is one-dimensional and hence it coincides with $E_{\min}$. It also follows that $E_{\min}$ is spanned by y. Since the stabilizer in G of the line spanned by y contains B and the stabilizer of $E_{\min}$ is exactly P, it follows that $B \subset P$. This completes the proof.

7.4. **Corollary.** *The variety $\mathscr{B}^s_u \cap \mathscr{B}^P$ (see 6.1(c)) is both open and closed in $\mathscr{B}^s_u$.*

Proof. For any $E \in \mathbf{r}(G)$, the function $B \to \Psi^B_{L_E}(s)$ defined on $\mathscr{B}^s_u$ is constant on the connected components of $\mathscr{B}^s_u$ (since they are compact varieties). In view of 7.3(b), (c), this implies that any connected component of $\mathscr{B}^s_u$ must be either contained in $\mathscr{B}^s_u \cap \mathscr{B}^P$ or disjoint from it. Hence $\mathscr{B}^s_u \cap \mathscr{B}^P$ is a union of connected components of $\mathscr{B}^s_u$ and the corollary follows.

7.5. Let $\mathbf{E}$ be an $\mathscr{A}[\mathbf{X}]$-module which is finite dimensional over $\mathbb{C}^*$; we shall write the action of $L \in \mathbf{X}$ on $\mathbf{E}$ as $\theta_L\colon \mathbf{E} \to \mathbf{E}$. Since these operators commute, there is a canonical direct sum decomposition $\mathbf{E} = \bigoplus_\chi {}_\chi\mathbf{E}$ indexed by the homomorphisms $\chi\colon \mathbf{X} \to \mathbb{C}^*$, with the following property: for any $\chi \in \mathrm{Hom}(\mathbf{X}, \mathbb{C}^*)$ and any $L \in \mathbf{X}$, the operator θ_L acts on ${}_\chi\mathbf{E}$ as the scalar $\chi(L)$ times a unipotent transformation.

We say that ${}_\chi\mathbf{E}$ is the *χ-weight space* of the $\mathscr{A}[\mathbf{X}]$-module $\mathbf{E}$; if ${}_\chi\mathbf{E} \neq 0$, we say that χ is a *weight* of $\mathbf{E}$.

7.6. We now show how the weight spaces of the $\mathbf{H}$-module $\mathbb{C}_{s,q} \underset{\mathbf{R}_{M'}}{\otimes} \mathbf{K}^{M'}_0(\mathscr{B}_u)$ ($M' = M(s,q)$) can be related to the connected components of $\mathscr{B}^s_u$. (The existence of such a relation has been predicted in [L_1] before the K-theory methods were found). The decomposition $\mathscr{B}^s_u = \coprod_\alpha \mathscr{B}^{s,\alpha}_u$ into connected components gives rise to a direct sum decomposition

$$\mathbf{K}_0^{M'}(\mathscr{B}_u^s)=\bigoplus_\alpha \mathbf{K}_0^{M'}(\mathscr{B}_u^{s,\alpha})$$

which is compatible with the natural action of $\mathscr{A}[\mathbf{X}]$, by $\theta_L(\xi)=L\otimes\xi$. Note that M' acts trivially on $\mathscr{B}_u^s$. From 1.3(m) we see that for $L\in\mathbf{X}$, the endomorphism of $\mathbb{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u^{s,\alpha})$ defined by θ_L is scalar multiplication by $\Psi_L^B(s)$ (for any $B\in\mathscr{B}_u^{s,\alpha}$) times a unipotent automorphism. (The last automorphism may be identified with the automorphism $\xi\to L\otimes\xi$ of $\mathbf{K}_0(\mathscr{B}_u^{s,\alpha})$; we have used 1.3(m3)). Hence each summand in

(a) $$\mathbb{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u^s)=\bigoplus_\alpha \mathbb{C}_{s,q}\underset{\mathbf{R}_M}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u^{s,\alpha})$$

is contained in a weight space of this $\mathscr{A}[\mathbf{X}]$-module.

Applying 1.3(k) to the inclusion i: $\mathscr{B}_u^s\hookrightarrow\mathscr{B}_u$, we get an isomorphism

$$i_*:\mathbb{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u^s)\xrightarrow{\approx}\mathbb{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u).$$

which is compatible with the operators θ_L ($L\in\mathbf{X}$). Hence we can carry over by i_* the decomposition (a) to get a decomposition

(b) $$\mathscr{M}=\bigoplus_\alpha \mathscr{M}^\alpha \quad \text{of } \mathscr{M}=\mathbb{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u)$$

with the following property:

(c) For each $L\in\mathbf{X}$, θ_L acts on $\mathscr{M}^\alpha$ as scalar multiplication by $\Psi_L^B(s)$ (for any $B\in\mathscr{B}_u^{s,\alpha}$) times a unipotent transformation.

We now consider the natural action of $\bar{M}(u,s)$ on $\mathscr{M}$, (see 5.12).

This action is compatible with the natural action of $\bar{M}(u,s)$ on $\mathbb{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u^s)$. Note that $\bar{M}(u,s)$ permutes among themselves the irreducible components of $\mathscr{B}_u^s$. It follows that the decomposition (b) gives rise, for each $\rho\in\mathbf{r}_0(\bar{M}(u,s))$, to a decomposition

(d) $\mathscr{M}_\rho=\bigoplus_{(\alpha)}\mathscr{M}_\rho^{(\alpha)}$, ($\mathscr{M}_\rho=\mathscr{M}_{u,s,q,\rho}$, see 5.12), indexed by the $\bar{M}(u,s)$-orbits (α) on the set of connected components of $\mathscr{B}_u^s$, with the following property:

(e) for each $L\in\mathbf{X}$, θ_L acts on $\mathscr{M}_\rho^{(\alpha)}$ as scalar multiplication by $\Psi_L^B(s)$ (for any $B\in\mathscr{B}_u^{s,\alpha}$ and any $\alpha\in(\alpha)$) times a unipotent transformation.

Here, $\mathscr{M}_\rho^{(\alpha)}=(\rho^*\otimes(\bigoplus_{\alpha\in(\alpha)}\mathscr{M}^\alpha))^{\bar{M}(u,s)}$.

We see that:

(f) $\dim\mathscr{M}_\rho^{(\alpha)}$ is the multiplicity of ρ in the $\bar{M}(u,s)$-module $\bigoplus_{\alpha\in(\alpha)}\mathscr{M}^\alpha$, or equivalently (1.3(m)), in the $\bar{M}(u,s)$-module $\bigoplus_{\alpha\in(\alpha)}\mathbf{K}_0(\mathscr{B}^{s,\alpha})$.

7.7. With notations in 7.6, we consider the decomposition

(a) $$\mathscr{M}_\rho=\mathscr{M}_\rho^{\mathrm{I}}\oplus\mathscr{M}_\rho^{\mathrm{II}}$$

where $\mathscr{M}_\rho^{\mathrm{I}}$ (resp. $\mathscr{M}_\rho^{\mathrm{II}}$) is the sum of all $\mathscr{M}_\rho^{(\alpha)}$ such that $\mathscr{B}_u^{s,\alpha}\subset\mathscr{B}^P$ (resp. $\mathscr{B}_u^{s,\alpha}\subset\mathscr{B}-\mathscr{B}^P$) for some $\alpha\in(\alpha)$; note that for any α, we have $\mathscr{B}_u^{s,\alpha}\subset\mathscr{B}^P$ or $\mathscr{B}_u^{s,\alpha}\subset\mathscr{B}-\mathscr{B}^P$ (see 7.4), and if one of these conditions is satisfied for some α in (α) then it is satisfied by all α in (α) (since $M(u,s)\subset P\times\mathbb{C}^*$, see 7.2(a)).

From the definition and from 7.6(e) it follows that:

(b) $\mathscr{M}_\rho^{\mathrm{I}}$ and $\mathscr{M}_\rho^{\mathrm{II}}$ are sums of weight spaces of $\mathscr{M}_\rho$.

The following result is an immediate consequence of the definitions and of 7.3(b), (c).

(c) If χ^{I} (resp. χ^{II}) is a weight of $\mathscr{M}_\rho^{\mathrm{I}}$ (resp. $\mathscr{M}_\rho^{\mathrm{II}}$) and if $E\in\mathbf{r}(G)$ contains a line whose stabilizer is exactly P, then $V(\chi^{\mathrm{II}}(L_E))>V(\chi^{\mathrm{I}}(L_E))$, ($L_E$ as in (2.9)). Hence any weight space of $\mathscr{M}_\rho$ is contained either in $\mathscr{M}_\rho^{\mathrm{I}}$ or in $\mathscr{M}_\rho^{\mathrm{II}}$.

7.8. We now apply Theorem 6.2 in our situation: u, s, q, P, L are as in 7.1; we take M in 6.2 to be the smallest algebraic subgroup of $G\times\mathbb{C}^*$ containing (s,q) and $\mathscr{Z}_L^0\times\{e\}$ and we take $\mathbf{c}$ in 6.2 to be the connected component of M containing (s,q). It is clear that the image of $\mathbf{c}$ in M/M^0 generates M/M^0; moreover, the set $\mathbf{c}_0$ of 6.1(a) contains (s,q) by 7.2(c). Hence 6.2 is applicable; we now write the isomorphism of 6.2 after tensoring over $\mathbf{R}_{M,\mathbf{c}_0}$ with the residue field of $\mathbf{R}_{M,\mathbf{c}_0}$ at the maximal ideal corresponding to (s,q):

$$\text{(a)}\qquad \mathbf{H}\bigotimes_{\mathbf{H}_{P/U}}(\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_M}\mathbf{K}_0^M(\mathscr{B}_u))\xrightarrow{\approx}\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_M}\mathbf{K}_0^M(\mathscr{B}_u)$$

Applying now 5.11(c) to G and to P/U, we can replace M by $M'=M(s,q)$ in (a) and we get an isomorphism

$$\text{(b)}\qquad \mathbf{H}\bigotimes_{\mathbf{H}_{P/U}}(\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_{M'}}\mathbf{K}_0^{M'}(\mathscr{B}_u))\xrightarrow{\approx}\mathbb{C}_{s,q}\bigotimes_{\mathbf{R}_{M'}}\mathbf{K}_0^{M'}(\mathscr{B}_u).$$

Note that the group $\overline{M}(u,s)$ defined in terms of G is canonically the same, by 7.2(a), as the analogous group $\overline{M}(\hat{u},\hat{s})$ defined in terms of P/U and of the images $\hat{u}$, $\hat{s}$ of u, s in P/U; we shall identify these two groups. Now $\overline{M}(u,s)$ acts naturally on both sides of (b) compatibly with the isomorphism. Taking $(\rho^*\otimes(\;))^{\overline{M}(u,s)}$ in both sides of (b) we therefore get

$$\text{(c)}\qquad i\colon \mathbf{H}\bigotimes_{\mathbf{H}_{P/U}}\hat{\mathscr{M}}_\rho\xrightarrow{\approx}\mathscr{M}_\rho,\qquad \rho\in\mathbf{r}(\overline{M}(u,s))$$

where $\mathscr{M}_\rho=\mathscr{M}_{u,s,q,\rho}$ (see 5.12) and $\hat{\mathscr{M}}_\rho$ is the standard $\mathbf{H}_{P/U}$-module defined with respect to $\hat{u},\hat{s},q,\rho$. In particular, we see that the set $\mathbf{r}_0(\overline{M}(u,s))$, (see 5.12(d)) defined in terms of G is the same as the analogous set defined in terms of P/U. Assume that $\rho\in\mathbf{r}_0(\overline{M}(u,s))$.

It follows immediately from (c) and the definitions that:

(d) the map $\hat{\mathscr{M}}_\rho\to\mathscr{M}_\rho$ defined by $x\to i(1\otimes x)$, see (c), is injective; its image is exactly $\mathscr{M}_\rho^{\mathrm{I}}$, see 7.7(a), and it generates $\mathscr{M}_\rho$ as an $\mathbf{H}$-module.

In particular:

(e) $\mathscr{M}_\rho^{\mathrm{I}}$ is an $\mathbf{H}_{P/U}$ submodule of $\mathscr{M}_\rho$, isomorphic to $\hat{\mathscr{M}}_\rho$.

We shall say that a submodule of a module is *proper* if it is not equal to the whole module.

7.9. **Proposition.** *Let notations be as in 7.8, in particular, let $\rho \in \mathbf{r}_0(\overline{M}(u,s))$. Assume that in the $\mathbf{H}_{P/U_p}$-module $\hat{\mathscr{M}}_\rho$, the sum of all proper submodules is a proper submodule $\hat{\mathscr{M}}_{\rho,\max}$. Then, in the $\mathbf{H}$-module $\mathscr{M}_\rho$, the sum of all proper submodules is a proper submodule $\mathscr{M}_{\rho,\max}$. Moreover, the natural inclusion $\hat{\mathscr{M}}_\rho \to \mathscr{M}_\rho$ (see 7.8(d)) maps $\hat{\mathscr{M}}_{\rho,\max}$ into $\mathscr{M}_{\rho,\max}$ and induces an injective map*

(a) $$\hat{\mathscr{M}}_\rho/\hat{\mathscr{M}}_{\rho,\max} \hookrightarrow \mathscr{M}_\rho/\mathscr{M}_{\rho,\max}.$$

Proof. We identify $\hat{\mathscr{M}}_\rho = \mathscr{M}^{\mathrm{I}}_\rho$ by 7.8(d). Let $F = \mathscr{M}_{\rho,\max} \oplus \mathscr{M}^{\mathrm{II}}_\rho$, see 7.7(a).

Let F_1 be any proper $\mathbf{H}$-submodule of $\mathscr{M}_\rho$.

By 7.7(c), any weight space of F_1 is contained in $\mathscr{M}^{\mathrm{I}}_\rho$ or in $\mathscr{M}^{\mathrm{II}}_\rho$, hence $F_1 = (F_1 \cap \mathscr{M}^{\mathrm{I}}_\rho) \oplus (F_1 \cap \mathscr{M}^{\mathrm{II}}_\rho)$.

Assume that $F_1 \cap \mathscr{M}^{\mathrm{I}}_\rho = \mathscr{M}^{\mathrm{I}}_\rho$. Since $\mathscr{M}^{\mathrm{I}}_\rho$ generates $\mathscr{M}_\rho$ as an $\mathbf{H}$-module (7.8(d)) it would follow that F_1 also generates $\mathscr{M}_\rho$ as an $\mathbf{H}$-module so $F_1 = \mathbf{H}$, a contradiction. Hence $F_1 \cap \mathscr{M}^{\mathrm{I}}_\rho \subsetneqq \mathscr{M}^{\mathrm{I}}_\rho$. Now $F_1 \cap \mathscr{M}^{\mathrm{I}}_\rho$ is an $\mathbf{H}_{P/U}$-submodule of $\mathscr{M}^{\mathrm{I}}_\rho$. By the definition of $\hat{\mathscr{M}}_{\rho,\max}$, we then have $F_1 \cap \mathscr{M}^{\mathrm{I}}_\rho \subset \hat{\mathscr{M}}_{\rho,\max}$. Thus, $F_1 \subset \hat{\mathscr{M}}_{\rho,\max} \oplus \mathscr{M}^{\mathrm{II}}_\rho$. Since the last space is $\neq \mathscr{M}_\rho$, it follows that the sum of all proper $\mathbf{H}$-submodules of $\mathscr{M}_\rho$ is a proper submodule $\mathscr{M}_{\rho,\max}$ of $\mathscr{M}_\rho$, contained in $\hat{\mathscr{M}}_{\rho,\max} \oplus \mathscr{M}^{\mathrm{II}}_\rho$.

If we tensor the exact sequence $0 \to \hat{\mathscr{M}}_{\rho,\max} \to \hat{\mathscr{M}}_\rho \to \hat{\mathscr{M}}_\rho/\hat{\mathscr{M}}_{\rho,\max} \to 0$ with $\mathbf{H}$ over $\mathbf{H}_{P/U}$ we get again an exact sequence, since $\mathbf{H}$ is free over $\mathbf{H}_{P/U}$. Moreover, $\mathbf{H} \underset{\mathbf{H}_{P/U}}{\otimes} (\hat{\mathscr{M}}_\rho/\hat{\mathscr{M}}_{\rho,\max}) \neq 0$ since $\hat{\mathscr{M}}_\rho/\hat{\mathscr{M}}_{\rho,\max} \neq 0$ and $\mathbf{H}$ is free over $\mathbf{H}_{P/U}$. It follows that the image of $\mathbf{H} \underset{\mathbf{H}_{P/U}}{\otimes} \hat{\mathscr{M}}_{\rho,\max} \hookrightarrow \mathbf{H} \underset{\mathbf{H}_{P/U}}{\otimes} \hat{\mathscr{M}}_\rho$ is a proper $\mathbf{H}$-submodule, hence, using 7.8(c) we see that the $\mathbf{H}$-submodule of $\mathscr{M}_\rho$ generated by $\hat{\mathscr{M}}_{\rho,\max}$ is proper. In other words, $\hat{\mathscr{M}}_{\rho,\max} \subset \mathscr{M}_{\rho,\max}$.

Note that $\hat{\mathscr{M}}_\rho \not\subset \mathscr{M}_{\rho,\max}$ (since $\hat{\mathscr{M}}_\rho$ generates $\mathscr{M}_\rho$ as an $\mathbf{H}$-module); it follows that the map (a) is non-zero. It is an $\mathbf{H}_{P/U}$-linear map and $\hat{\mathscr{M}}_\rho/\hat{\mathscr{M}}_{\rho,\max}$ is a simple $\mathbf{H}_{P/U}$-module, hence (a) must be injective. This completes the proof.

7.10. Now assume that in addition to $u, s, q, \varphi, s_1, L, P$ as in 7.1 we are given another set of data $u', s', q' = q, \varphi', s'_1, L', P'$ of the same kind (with P', L' defined in terms of the same function V). Recall that $V(q) \geqq 0$. Assume that P, P' contain a common Borel subgroup.

Let $\mathscr{M}_\rho$, $\hat{\mathscr{M}}_\rho$ be as in 7.8 ($\rho \in \mathbf{r}_0\overline{\mathscr{M}}(u,s)$)) and let $\mathscr{M}'_{\rho'}$, $\hat{\mathscr{M}}'_{\rho'}$ be the analogous objects defined in terms of $u', s', \ldots$ ($\rho' \in \mathbf{r}_0(\overline{\mathscr{M}}(u', s'))$). Thus, $\mathscr{M}_\rho$, $\mathscr{M}'_{\rho'}$ are standard $\mathbf{H}$-modules and $\hat{\mathscr{M}}_\rho$ is a standard $\mathbf{H}_{P/U}$-module and $\hat{\mathscr{M}}'_{\rho'}$ is a standard $\mathbf{H}_{P'/U'}$-module (U' = unipotent radical of P').

Assume that the sum of all proper $\mathbf{H}_{P/U}$- (resp. $\mathbf{H}_{P'/U'}$) submodules of $\hat{\mathscr{M}}_\rho$ (resp. $\hat{\mathscr{M}}'_{\rho'}$) is a proper submodule $\hat{\mathscr{M}}_{\rho,\max}$ (resp. $\hat{\mathscr{M}}'_{\rho',\max}$) and let $\mathscr{M}_{\rho,\max}$ (resp. $\mathscr{M}'_{\rho',\max}$) be the largest proper $\mathbf{H}$-submodule of $\mathscr{M}_\rho$ (resp. $\mathscr{M}'_{\rho'}$) (cf. 7.9).

With these assumptions we can state.

7.11. **Proposition.** *Assume further that the simple $\mathbf{H}$-modules $\mathscr{M}/\mathscr{M}_{\rho,\max}$, $\mathscr{M}'_{\rho'}/\mathscr{M}'_{\rho',\max}$ are isomorphic. Then $P = P'$ and the simple $\mathbf{H}_{P/U}$-modules $\hat{\mathscr{M}}_\rho/\hat{\mathscr{M}}_{\rho,\max}$, $\hat{\mathscr{M}}'_{\rho'}/\hat{\mathscr{M}}'_{\rho',\max}$ are isomorphic.*

Proof. Let $\mathbf{t} \in \mathrm{Hom}(\mathbf{X}, \mathbb{R})$ be defined by

(a) $$\mathbf{t}(L) = V(\Psi^{B_1}_L(s_1))$$

where B_1 is any Borel subgroup containing s_1 and contained in P and $\Psi_L^{B_1}$ is an in 2.8. This is independent of the choice of B_1, at least when $L = L_E$ (see 7.3(a)), and hence in general since the L_E generate $\mathbf{X}$.

We define similarly $\mathbf{t}' \in \operatorname{Hom}(\mathbf{X}, \mathbb{R})$, replacing s_1, P by s_1', P' in the definition of $\mathbf{t}$.

Let χ be any weight of $\mathcal{M}_\rho / \mathcal{M}_{\rho, \max}$ which is also a weight of $\hat{\mathcal{M}}_\rho / \hat{\mathcal{M}}_{\rho, \max}$. (Such χ exists by 7.9(a)).

Similarly let χ' be any weight of $\mathcal{M}'_{\rho'} / \mathcal{M}'_{\rho', \max}$ which is also a weight of $\hat{\mathcal{M}}'_{\rho'} / \hat{\mathcal{M}}'_{\rho', \max}$.

Composing χ, χ' with $V: \mathbb{C}^* \to \mathbb{R}$, we get elements $V \circ \chi$, $V \circ \chi'$ in $\operatorname{Hom}(\mathbf{X}, \mathbb{R})$. We shall show that

(b) $\mathbf{t}(L_r) > 0$ if $r \in S - S^*$, $\mathbf{t}(L_r) = 0$ if $r \in S^*$, (S^* as in 6.1).

(c) $(V \circ \chi - \mathbf{t})(L_E) \geqq 0$ if $E \in \mathbf{r}(G)$.

(d) $(V \circ \chi - \mathbf{t})(L_E) = 0$ if $E \in \mathbf{r}(G)$ has a P-stable line.

Let B_1 be as in (a), let $\mathbf{b}_1$ be its Lie algebra and let $\mathbf{p}_1$ be the Lie algebra of the unique parabolic subgroup of type r containing B_1. Let λ_r be the eigenvalue of $\operatorname{Ad}(s_1)$ on $\mathbf{p}_1/\mathbf{b}_1$. Then $\mathbf{t}(L_r) = V(\lambda_r)$ (see (a) and 2.8(e)). If $r \in S^*$, then $\mathbf{p}_1 \subset \mathbf{p}$ and $\mathbf{p}_1/\mathbf{b}_1 = (\mathbf{p}_1 \cap \mathbf{l})/(\mathbf{b}_1 \cap \mathbf{l})$; it follows that $V(\lambda_r) = 0$, (by the definition of $\mathbf{l}$).

If $r \in S - S^*$, then $\mathbf{p}_1 \not\subset \mathbf{p}$ and from the definition of $\mathbf{p}$ (see 7.1), it follows that $V(\lambda_r) > 0$. This proves (b). The statements (c), (d) follow from 7.3(a), (b) and from 7.6(e).

From (b), (c) and (d), we deduce using Langlands' Lemma 2.10(a), that

(e) $$^\circ(V \circ \chi) = \mathbf{t}$$

(with the notation of 2.10). Similarly, we have

(f) $$^\circ(V \circ \chi_0') = \mathbf{t}'.$$

Next, we note that, since $\mathcal{M}_\rho / \mathcal{M}_{\rho, \max}$ is isomorphic to $\mathcal{M}'_{\rho'} / \mathcal{M}'_{\rho', \max}$ as an $\mathbf{H}$-module, χ must be also a weight of $\mathcal{M}'_{\rho'} / \mathcal{M}'_{\rho', \max}$ and χ' must be also a weight of $\mathcal{M}_\rho / \mathcal{M}_{\rho, \max}$.

Now (c) above holds when χ is replaced by any weight of $\mathcal{M}_\rho / \mathcal{M}_{\rho, \max}$ (cf. 7.3(a)). In particular, we have

(g) $$(V \circ \chi' - \mathbf{t})(L_E) \geqq 0, \quad \text{if } E \in \mathbf{r}(G),$$

and similarly

(h) $$(V \circ \chi - \mathbf{t}')(L_E) \geqq 0, \quad E \in \mathbf{r}(G).$$

Using now 2.10(b), we deduce from (g), (h) that

(i) $(^\circ(V \circ \chi') - {}^\circ\mathbf{t})(L_E) \geqq 0$, $(^\circ(V \circ \chi) - {}^\circ(\mathbf{t}'))(L_E) \geqq 0$ for all $E \in \mathbf{r}(G)$.

We substitute (e) and (f) in (i) and note that $^\circ\mathbf{t} = \mathbf{t}$, $^\circ(\mathbf{t}') = \mathbf{t}'$.

(In general, for any $f \in \operatorname{Hom}(\mathbf{X}, \mathbb{C}^*)$ we have $^\circ(^\circ f) = {}^\circ f$ by the uniqueness in 2.10(a)). We obtain

$$(\mathbf{t}'-\mathbf{t})(L_E)\geqq 0, \quad (\mathbf{t}-\mathbf{t}')(L_E)\geqq 0$$

hence $(\mathbf{t}'-\mathbf{t})(L_E)=0$ for all $E\in\mathfrak{r}(G)$. Since the L_E generate $\mathbf{X}$ it follows that $\mathbf{t}=\mathbf{t}'$.

From (b), we see that S^* is completely determined by $\mathbf{t}$; similarly, the analogous set S'^* associated to P' is completely determined by $\mathbf{t}'$. Thus $S^*=S'^*$, hence P, P' have the same type. Since they contain a common Borel subgroup, they coincide: $P=P'$.

Let $E\in\mathfrak{r}(G)$ be such that E contains a line whose stabilizer in G is exactly P. The decomposition $\mathcal{M}_\rho=\mathcal{M}_\rho^{\mathrm{I}}\oplus\mathcal{M}_\rho^{\mathrm{II}}$ in 7.7(a) induces a decomposition $\mathcal{M}_\rho/\mathcal{M}_{\rho,\max}=(\mathcal{M}_\rho/\mathcal{M}_{\rho,\max})^{\mathrm{I}}\oplus(\mathcal{M}_\rho/\mathcal{M}_{\rho,\max})^{\mathrm{II}}$ in which the first (resp. second) summand is the sum of all χ-weight spaces such that $(V\circ\chi-\mathbf{t})(L_E)=0$ (resp. $(V\circ\chi-\mathbf{t})(L_E)>0$). (See 7.3). We have $(\mathcal{M}_\rho/\mathcal{M}_{\rho,\max})^{\mathrm{I}}=\hat{\mathcal{M}}_\rho/\hat{\mathcal{M}}_{\rho,\max}$. Thus $\hat{\mathcal{M}}_\rho/\hat{\mathcal{M}}_{\rho,\max}$ can be characterized as the sum of all χ-weight spaces of $\mathcal{M}_\rho/\mathcal{M}_{\rho,\max}$ for the χ such that $(V\circ\chi-\mathbf{t})(L_E)=0$. An analogous characterization holds (in terms of $\mathbf{t}'$) for the subspace $\hat{\mathcal{M}}'_{\rho'}/\hat{\mathcal{M}}'_{\rho',\max}$ of $\mathcal{M}'_{\rho'}/\mathcal{M}'_{\rho',\max}$. Since $\mathbf{t}=\mathbf{t}'$, it follows that an isomorphism of $\mathbf{H}$-modules $\mathcal{M}_\rho/\mathcal{M}_{\rho,\max}\xrightarrow{\sim}\mathcal{M}'_{\rho'}/\mathcal{M}'_{\rho',\max}$ must carry the subspace $\hat{\mathcal{M}}_\rho/\hat{\mathcal{M}}_{\rho,\max}$ onto the subspace $\hat{\mathcal{M}}'_\rho/\hat{\mathcal{M}}'_{\rho',\max}$ and will therefore define an $\mathbf{H}_{U/P}$-module isomorphism between these subspaces. This completes the proof.

We now state the main theorem of this paper.

7.12. **Theorem.** *Assume that $q\in\mathbb{C}^*$ is not a root of* 1. (a) *Let u be a unipotent element of G and let s be a semisimple element of G such that $sus^{-1}=u^q$. Let ρ be an irreducible representation of $\bar{\mathcal{M}}(u,s)$ which appears in the natural representation of $\bar{\mathcal{M}}(u,s)$ on $\mathbf{K}_0(\mathcal{B}_u^s)$. Then the standard $\mathbf{H}$-module $\mathcal{M}_{u,s,q,\rho}$ (see* 5.12) *has a unique simple quotient, which depends only on the G-conjugacy class of (u,s,ρ).*

(b) *Any simple $\mathbf{H}$-module $\mathbf{E}$ in which $\mathbf{q}$ acts as multiplication by q is obtained from some triple (u,s,ρ) as in* (a), *and the triple (u,s,ρ) is uniquely determined by $\mathbf{E}$ up to G-conjugacy.*

Proof. Since q is not a root of 1 we can find a homomorphism $V\colon\mathbb{C}^*\to\mathbb{R}$ as in 6.1 such that $V(q)>0$. We can assume that $G\neq\{e\}$ and that the theorem is already proved for groups of smaller dimension.

Let u,s be as in (a). We fix a square root $q^{1/2}$ of q. We attach to u,s,q,V (as in 7.1) s_1,φ and P. To prove (a) we can reduce ourselves using 7.8 and the induction hypothesis to the case where $P=G$. Then all eigenvalues α of $\mathrm{Ad}(s)\colon\mathbf{g}\to\mathbf{g}$ satisfy $V(\alpha)=0$.

With the notation of 7.1, the q-eigenspace of $\mathrm{Ad}(s)\colon\mathbf{g}\to\mathbf{g}$ is $\oplus\mathbf{g}_{\alpha,i}$ where (α,i) is subject to $\alpha q^{i/2}=q$, $V(\alpha)=0$.

For such (α,i) we have $V(q)=V(\alpha q^{i/2})=V(\alpha)+(i/2)V(q)=(i/2)V(q)$, and since $V(q)>0$, it follows that $i=2$. Hence the q-eigenspace of $\mathrm{Ad}(s)\colon\mathbf{g}\to\mathbf{g}$ is $\oplus\mathbf{g}_{\alpha,2}$.

Similarly, the 1-eigenspace of $\mathrm{Ad}(s)\colon\mathbf{g}\to\mathbf{g}$ is $\oplus\mathbf{g}_{\alpha,0}$. Hence $Z(s)$ has Lie algebra equal to $\oplus\mathbf{g}_{\alpha,0}$. From 2.6(b) it follows that $Z(s)$ has a unique open orbit on $\bigoplus_\alpha\mathbf{g}_{\alpha,2}$, and this orbit contains $\log u$. In other words, $Z(s)$ has a unique open orbit on the q-eigenspace of $\mathrm{Ad}(s)\colon\mathbf{g}\to\mathbf{g}$, and that orbit contains $\log u$. From 5.15(a) it now follows that $\mathcal{M}_{u,s,q,\rho}$ is a simple $\mathbf{H}$-module. This completes the proof of (a).

The first assertion of (b) follows from (a) and 5.13.

Assume now that $\mathbf{E}$ as in (b) is obtained from two triples (u,s,ρ), (u',s',ρ') as in (a). We must prove that (u,s,ρ), (u',s',ρ') are G-conjugate. Let P be the parabolic subgroup attached to u,s,q,V as in 7.1 and let P' be the analogous parabolic subgroup for u',s',q,V. By conjugating if necessary (u',s',ρ'), we may assume that P,P' contain a common Borel subgroup. We can apply 7.11 since (a) is already proved. We deduce that $P=P'$. If $P\neq G$, we also see from 7.11 and the induction hypothesis that (u,s,ρ), (u',s',ρ') are conjugate when regarded as objects for P/U, (U as in 7.1). Since (u,s) is contained in some Levi subgroup of P, and similarly (u',s') in some Levi subgroup of P, it follows that (u,s,ρ), (u',s',ρ') are conjugate under P.

Thus we are reduced to the case where $P=P'=G$. In this case, as we have seen in the proof of (a), the hypothesis of 5.15 are verified and it remains to apply 5.15(b). This completes the proof.

8. Square integrable H-modules

8.1. In this chapter we shall fix $q\in\mathbb{C}^*$ and $V\colon \mathbb{C}^*\to\mathbb{R}$ as in 7.1. We shall assume that $V(q)>0$. We shall also assume that G is semisimple. Let $\mathcal{M}$ be an $\mathbf{H}$-module of finite dimension over $\mathbb{C}$, with $\mathbf{q}$ acting as multiplication by q. We say that $\mathcal{M}$ is *V-square integrable* if for all $E\in\mathbf{r}(G)$ other then the unit representation, all eigenvalues v of $\theta_{L_E}\colon \mathcal{M}\to\mathcal{M}$ satisfy $V(v)>0$. We shall say that $\mathcal{M}$ is *V-tempered* if for all $E\in\mathbf{r}(G)$, all eigenvalues v of $\theta_{L_E}\colon \mathcal{M}\to\mathcal{M}$ satisfy $V(v)\geqq 0$. (The usual notion of tempered and square integrable $\mathbf{H}$-module is obtained by taking q to be a prime power and $V(z)=\log|z|$, by a criterion of Casselman).

We shall prove the following result.

8.2. **Theorem.** *In the setup of* 8.1, *let $u\in G$ be a unipotent element and let $s\in G$ be a semisimple element such that $sus^{-1}=u^q$. Let P be the parabolic subgroup attached to s,u,q,V, in* 7.1. *Let $\mathcal{M}_\rho=\mathcal{M}_{u,s,q,\rho}$ $(\rho\in\mathbf{r}_0(\bar{\mathcal{M}}(u,s)))$. The following conditions are equivalent.*

(a) *$\mathcal{M}_\rho$ is V-tempered.*

(b) *The unique simple quotient of $\mathcal{M}_\rho$ is V-tempered.*

(c) *$P=G$.*

If these conditions are satisfied, then $\mathcal{M}_\rho$ is a simple $\mathbf{H}$*-module.*

Proof. Assume first that $P=G$. Let s_1,φ be defined in terms of u,s,q as in 7.1. Let $\mathbf{t}\in\mathrm{Hom}(\mathbf{X},\mathbb{R})$ be defined in terms of s_1 as in 7.10(a). By 7.10(b) we have $\mathbf{t}(L_r)=0$ for all $r\in S$ hence $\mathbf{t}=0$ (since G is semisimple). From 7.10(c) which is valid for all weights χ of $\mathcal{M}_\rho$, we get $V(\chi(L_E))\geqq 0$ for all $E\in\mathbf{r}(G)$, hence $\mathcal{M}_\rho$ is V-tempered.

Thus, we have (c) $\Rightarrow$ (a).

Assume now that $P\neq G$. Let $E\in\mathbf{r}(G)$ be such that E has a line whose stabilizer in G is a maximal parabolic subgroup $(\neq G)$ containing P. Attach $b,a_1,\ldots,a_n$ to E as in 2.11. Then $b\mathbf{t}(L_E)=\sum_j a_j\mathbf{t}(L_{r_j})$; the right hand side of this equality is <0 by 7.11(b) and 2.11. Since $b>0$, it follows that $\mathbf{t}(L_E)<0$. Now let χ be a weight of $\mathcal{M}_\rho/\mathcal{M}_{\rho,\max}$ which is also a weight of $\hat{\mathcal{M}}_\rho/\hat{\mathcal{M}}_{\rho,\max}$ (with notations in the proof of 7.11).

By 7.11(d), we have $(V\circ\chi)L_E=\mathbf{t}(L_E)$. Hence $V(\chi(L_E))<0$. It follows that $\mathcal{M}_\rho/\mathcal{M}_{\rho,\max}$ is not V-tempered. Thus, we have (b) $\Rightarrow$ (c).

The implication (a) $\Rightarrow$ (b) is trivial. The last assertion of the theorem follows from 5.15(a) as in the proof of 7.12(a). The theorem is proved.

8.3. **Theorem.** *In the setup of* 8.1, *let $u\in G$ be a unipotent element and let $s\in G$ be a semisimple element such that $sus^{-1}=u^q$; Let $\mathcal{M}_\rho=\mathcal{M}_{u,s,q,\rho}$, $\rho\in\mathbf{r}_0(\bar{M}(u,s))$.*

We fix a square root $q^{1/2}$ of q.

The following conditions are equivalent.

(a) *There is no proper parabolic subgroup of G with Levi subgroup L, with L containing both u and s.*

(b) *There exists a semisimple element $s_1\in G$ such that $Z(s_1)$ is a semisimple group and a homomorphism of algebraic groups $\varphi\colon SL_2(\mathbb{C})\to Z(s_1)$ such that $\varphi\begin{pmatrix}1&1\\0&1\end{pmatrix}=u$, $s=s_1\varphi(D(q^{1/2}))$; moreover $Z(s_1)\cap Z(u)$ contains no torus $\neq e$.*

(c) *$\mathcal{M}_\rho$ is V-square integrable.*

Proof. The implication (a) $\Rightarrow$ (b) is in [L_1, 2.5]. Assume now that (b) holds. to prove (c) it is enough to verify the following statement: "Let $B\in\mathcal{B}_u^s$ and let $E\in\mathbf{r}(G)$ be other than the unit representation. Then $\psi_{L_E}^B(s)=\varepsilon q^{n/2}$, where ε is a root of 1 and n is an integer $\geqq 1$". This is proved assuming (b) in [L_1, 2.8]. Finally we shall prove that (c) $\Rightarrow$ (a).

Assume that (c) holds but (a) doesn't hold. Since (c) holds we see that $\mathcal{M}_\rho$ is V-tempered hence by 8.2, the condition 8.2(c) holds.

Since (a) doesn't hold, we can find a proper parabolic subgroup P of G with a Levi subgroup L and with unipotent radical U such that both u and s are contained in L. Let M be the smallest algebraic subgroup of $L\times\mathbb{C}^*$ containing (s,q) and $\mathcal{Z}_L^0\times\{e\}$; let $\mathbf{c}$ be the connected component of M containing (s,q). We shall verify that the assumptions of Theorem 6.2 are verified in our case and more precisely, that the set $\mathbf{c}_0$ in 6.1(a) contains (s,q). Let $\varphi\colon SL_2(\mathbb{C})\to L$ be a homomorphism of algebraic groups such that $\varphi\begin{pmatrix}1&1\\0&1\end{pmatrix}=u$ and $(s,q)\in M_\varphi$, see 2.4(g). Let $s_1=s\varphi(D(q^{-1/2}))$. Define a decomposition $\mathbf{g}=\oplus\,\mathbf{g}_{\alpha,i}$ in terms of s_1 and φ by the same formula as in 7.1. Here α runs over the eigenvalues of s_1, they all satisfy $V(\alpha)=0$ since by our assumption, condition 8.2(c) is verified. The index i runs through the integers. Let $\mathbf{p}$ be the Lie algebra of P. To verify the condition $\det(1-qs,(\mathbf{g}/\mathbf{p})_u)\neq 0$, it is enough to show that all eigenvalues of qs on $\ker(\operatorname{ad}\log(u)\colon \mathbf{g}\to\mathbf{g})$ are $\neq 1$. The last space is contained in $\bigoplus_{i\geqq 0}\mathbf{g}_{\alpha,i}$ (by 2.6(d)). The eigenvalue of qs on $\mathbf{g}_{\alpha,i}$ is $q^{1+(i/2)}\alpha$, and we have $V(q^{(i/2)+1}\alpha)=((i/2)+i)\,V(q)+V(\alpha)>0$ since $i\geqq 0$, $V(q)>0$, $V(\alpha)=0$. Thus, Theorem 6.2 is applicable and it implies the existence of an isomorphism like 7.8(b), in our case:

$$\mathbf{H}\underset{\mathbf{H}_{P/U}}{\otimes}\hat{\mathcal{M}}\xrightarrow{\sim}\mathcal{M}$$

where $\hat{\mathscr{M}}=\mathbf{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\hat{\mathscr{B}}_{\hat{u}})$, $\mathscr{M}=\mathbf{C}_{s,q}\underset{\mathbf{R}_{M'}}{\otimes}\mathbf{K}_0^{M'}(\mathscr{B}_u)$, $M'=M(s,q)$ and $\hat{\mathscr{B}}_{\hat{u}}$ is as in 6.1.

From this we see that the natural $\mathbf{H}_{P/U}$-homomorphism $i\colon \hat{\mathscr{M}}\to\mathscr{M}$ is injective, and its image generates $\mathscr{M}$ as an $\mathbf{H}$-module.

Now $\mathscr{M}$ is a direct sum of simple (V-tempered) $\mathbf{H}$-modules (by 8.2): $\mathscr{M}=\mathscr{M}_1\oplus\ldots\oplus\mathscr{M}_k$. Let $pr_j\colon \mathscr{M}\to\mathscr{M}_j$ be the natural projection.

Then, for any j, the composition $\hat{\mathscr{M}}\xrightarrow{i}\mathscr{M}\xrightarrow{pr_j}\mathscr{M}_j$ must be non-zero (otherwise, $i(\hat{\mathscr{M}})$ would be contained in the $\mathbf{H}$-submodule $\mathscr{M}_1\oplus\ldots\oplus\mathscr{M}_{j-1}\oplus\mathscr{M}_{j+1}\oplus\ldots\oplus\mathscr{M}_k$ of $\mathscr{M}$ and it couldn't generate $\mathscr{M}$ as an $\mathbf{H}$-module). In particular, we can find a non-zero $\mathbf{H}_{P/U}$-homomorphism $\hat{\mathscr{M}}\to\mathscr{M}_\rho$.

It follows that there exists $\chi\in\mathrm{Hom}(\mathbf{X},\mathbf{C}^*)$ which is a weight both for $\hat{\mathscr{M}}$ and for $\mathscr{M}_\rho$. Since χ is a weight for $\hat{\mathscr{M}}$, we see from 7.6(c) for P/U that there exists $B\in\mathscr{B}_u^s\cap\mathscr{B}^P$ such that $\chi(L)=\Psi_L^B(s)$ for all $L\in\mathbf{X}$.

Now the proof of Lemma 7.3(b) applies without change in our case and it shows that for any $E\in\mathbf{r}(G)$ with a P-stable line and for any $B_1\in\mathscr{B}^{s_1}\cap\mathscr{B}^P$, we have $\Psi_{L_E}^B(s)=\Psi_{L_E}^{B_1}(s_1)$.

Hence

$$\begin{aligned}V(\Psi_{L_E}^B(s))&=V(\Psi_{L_E}^{B_1}(s_1))\\&=\mathbf{t}(L_E),\quad \text{see 7.10(a).}\\&=0,\quad \text{see the proof of 8.2.}\end{aligned}$$

It follows that

$$V(\chi(L_E))=0 \qquad \text{for all } E\in\mathbf{r}(G) \text{ with a } P\text{-stable line.}$$

Since χ is a weight of $\mathscr{M}_\rho$, and $P\neq G$, this implies that $\mathscr{M}_\rho$ cannot be V-square integrable. This contradiction completes the proof.

References

[A] Atiyah, M.F.: Global theory of elliptic operators. Proc. Int. Conf. Funct. Anal. Rel. Topics, Tokyo 1969

[AS] Atiyah, M.F., Segal, G.B.: Equivariant K-theory and completion. J. Differ. Geom. **3**, 1–18 (1969)

[BV] Barbash, D., Vogan, D.: Unipotent representation of complex semisimple groups. Ann. Math. **121**, 41–110 (1985)

[BFM] Baum, P., Fulton, W., MacPherson, R.: Riemann-Roch and topological K-theory for singular varieties. Acta Math. **143**, 155–192 (1979)

[BZ] Bernstein, J., Zelevinskii, A.V.: Induced representations of reductive p-adic groups, I. Ann. Sci. E.N.S. **10**, 441–472 (1977)

[BS] Beynon, W.M., Spaltenstein, N.: Green functions of finite Chevalley groups of type E_n ($n=6,7,8$). J. Algebra **88**, 584–614 (1984)

[BB] Bialynicky-Birula, A.: Some theorems on actions of algebraic groups. Ann. Math. **98**, 480–497 (1973)

[BW] Borel, A., Wallach, N.: Continuous cohomology, discrete subgroups and representations of reductive groups. Ann. Math. Stud. **94** (1980), Princeton Univ. Press

[G] Ginsburg, V.: Lagrangian construction for representations of Hecke algebras. (Preprint 1984)

[H] Hironaka, H.: Bimeromorphic smoothing of a complex analytic space. Acta Math. Vietnam. **2**, (n° 2), Hanoi 1977

[K] Kasparov, G.G.: Topological invariants of elliptic operators I., K-homology. Izv. Akad. Nauk. SSSR **39**, 796–838 (1975)

[KL_1] Kazhdan, D., Lusztig, G.: A topological approach to Springer's representations. Adv. Math. **38**, 222–228 (1980)

[KL_2] Kazhdan, D., Lusztig, G.: Equivariant K-theory and representations of Hecke algebras II. Invent. Math. **80**, 209–231 (1985)

[Ko] Kostant, B.: The principal 3-dimensional subgroup and the Betti numbers of a complex Lie group. Am. J. Math. **81**, 973–1032 (1959)

[La] Langlands, R.P.: Problems in the theory of automorphic forms. Lect. Modern Anal. Appl., Lect. Notes Math., vol. 170, pp. 18–86. Berlin-Heidelberg-New York: Springer 1970

[L_1] Lusztig, G.: Some examples of square integrable representations of semisimple p-adic groups. Trans. Am. Math. Soc. **277**, 623–653 (1983)

[L_2] Lusztig, G.: Cells in affine Weyl groups. In: Algebraic groups and related topics, Hotta, R. (ed.), Advanced Studies in Pure Mathematics, vol. 6. Kinokunia, Tokyo and North-Holland, Amsterdam, 1985

[L_3] Lusztig, G.: Character sheaves *V*. Adv. Math. **61**, 103–155 (1986)

[L_4] Lusztig, G.: Equivariant K-theory and representations of Hecke algebras. Proc. Am. Math. Soc. **94**, 337–342 (1985)

[M] Mostow, G.D.: Fully reducible subgroups of algebraic groups. Am. J. Math. **78**, 200–221 (1956)

[Se] Segal, G.B.: Equivariant K-theory. Publ. Math. IHES **34**, 129–151 (1968)

[Sh] Shoji, T.: On the Green polynomials of classical groups. Invent. math. **74**, 239–264 (1983)

[Sl] Slodowy, P.: Four lectures on simple groups and singularities. Commun. Math. Inst., Rijksuniv. Utr. **11** (1980)

[Sn] Snaith, V.: On the Künneth spectral sequence in equivariant K-theory. Proc. Camb. Philos. Soc. **72**, 167–177 (1972)

[St] Steinberg, R.: On a theorem of Pittie. Topology **14**, 173–177 (1975)

[T_1] Thomason, R.: Algebraic K-theory of group scheme actions. Proc. Topol. Conf. in honor J. Moore, Princeton 1983

[T_2] Thomason, R.: Comparison of equivariant algebraic and topological K-theory. Preprint

[Z_1] Zelevinskii, A.V.: Induced representations of reductive p-adic groups. II. On irreducible representations of GL_n. Ann. Sci. E.N.S. **13**, 165–210 (1980)

[Z_2] Zelevinskii, A.V.: A p-adic analogue of the Kazhdan-Lusztig conjecture. Funkts. Anal. Prilozh. **15**, 9–21 (1981)

Oblatum 22-VII-1985 & 11-III-1986 & 26-V-1986