Stable conjugacy and endoscopic groups

Talk given in Kyoto the 1st of September 2004

In this talk, I will recall the motivation of Langlands to introduce the notions of stable conjugacy and endoscopic groups. In particular, we will review the how do these notion show up in the stabilization of the trace formula and and the counting of points of Shimura varieties. The references are [Langlands] and [Kottwitz].

1 Stable conjugacy

Let F be a field, $\Gamma = \mathrm{Gal}(\overline{F}/F)$. Let G denote a connected reductive group defined over F, let $\mathfrak{g}$ be its Lie algebra. Let T be a maximal torus of G defined over F, $\mathfrak{t} = \mathrm{Lie}(T)$. Let $W = \mathrm{Nor}(T,G)/T$ be the Weyl group that we consider as a finite group scheme over F acting on T and $\mathfrak{t}$. Let

$$\mathrm{Car} = T/W \qquad \mathrm{car} = \mathfrak{t}/W$$

be the quotients of T and $\mathfrak{t}$ by W in the sense of quotient of an affine variety by a finite group acting on it. Let T^{reg} and $\mathfrak{t}^{reg}$ be the biggest open subvarieties of T and $\mathfrak{t}$ on which W act freely and let denote Car^{reg} and car^{reg} their quotients.

Theorem 1.1 (Chevalley) *There exists a canonical* $\mathrm{Ad}(G)$*-invariant map* $\chi : G \to \mathrm{Car}$ *such that for all geometric point* $a \in \mathrm{Car}^{reg}(\overline{F})$*, the fiber* $\chi^{-1}(a)$ *consists in a unique* $\mathrm{Ad}(G)$*-orbit in* G*.*

Idem, we have a map also denoted $\chi : \mathfrak{g} \to \mathrm{car}$*, with the same properties.*

Let $a \in \mathrm{Car}^{reg}(F)$ be a F-point of Car^{reg}. The set $\chi^{-1}(a)(F)$ is not in general a $G(F)$-orbit. Worse, it can be eventually empty.

Theorem 1.2 (Steinberg) *Let* G *be a quasi-split group over* F*. For all* $a \in \mathrm{car}^{reg}(F)$*, the fiber* $\chi^{-1}(a)$ *in* $\mathfrak{g}$ *has at least one* F*-point. The same is true for* G *under the assumption that the derived group* G^{der} *is simply connected.*

Without the assumption G^{der} simply connected, Kottwitz constructed an obstruction $c(a)$ whose vanishing is equivalent to the existence of a F-point in $\chi^{-1}(a)$. But we are going to be mainly interested to the case of Lie algebra and reductive groups whose derived groups is simply connected. The following simplified definition of stable conjugacy is the correct one only under this assumption.

Definition 1.3 *Two elements $\gamma,\gamma' \in G^{reg}(F)$ are said to be stably conjugate if $\chi(\gamma) = \chi(\gamma')$ in $\mathrm{Car}^{reg}(F)$. In other words, $\gamma,\gamma' \in G^{reg}(F)$ are stably conjugate if there exist $g \in G(\overline{F})$ such that $\gamma' = g\gamma g^{-1}$.*

Stable conjugacy classes are elements $a \in \mathrm{car}^{reg}(F)$ such that $\chi^{-1}(a)(F)$ is non empty.

Let $\gamma,\gamma' \in G(F)$ having the same image $a \in G^{reg}(F)$. Let $g \in G(\overline{F})$ such that $\gamma' = g\gamma g^{-1}$. Since γ,γ' are fixed by Γ, for all $\sigma \in \Gamma$, $\gamma' = \sigma(g)\gamma\sigma(g)^{-1}$. It follows that the map

$$\sigma \mapsto g^{-1}\sigma(g)$$

defines a 1-cocycle of Γ in the centralize G_γ of γ in G whose image in G is obviously a coboundary. We obtains an element

$$\mathrm{inv}(\gamma,\gamma') \in \ker[\mathrm{H}^1(F,G_\gamma) \to \mathrm{H}^1(F,G)].$$

One can check that $\mathrm{inv}(\gamma,\gamma')$ depends only on the $G(F)$-conjugacy classes of γ and γ'.

Proposition 1.4 *Let $a \in G^{reg}(F)$ and let $\gamma \in \chi^{-1}(a)(F)$. Then the map*

$$\gamma' \mapsto \mathrm{inv}(\gamma,\gamma') \in \ker[\mathrm{H}^1(F,G_\gamma) \to \mathrm{H}^1(F,G)]$$

sets up a bijection between the set of $G(F)$-conjugacy class within the stable conjugacy class $\chi^{-1}(a)(F)$ and $\ker[\mathrm{H}^1(F,G_\gamma) \to \mathrm{H}^1(F,G)]$.

This description is not entirely satisfying because it depends on a base point $\gamma \in \chi^{-1}(a)$. Nevertheless, the centralizer G_γ does not depends on this choice for regular a.

Lemma 1.5 *Let $a \in \mathrm{Car}^{reg}(\overline{F})$. The centralizers $G_\gamma,G_{\gamma'}$ of two arbitrary elements $\gamma,\gamma' \in \chi^{-1}(a)(\overline{F})$ are* canonically *isomorphic.*

Proof. There exists $g \in G(\overline{F})$ such that $\gamma' = g\gamma g^{-1}$. Thus $\mathrm{Ad}(g)$ induces an isomorphism $\mathrm{Ad}(g) : G_\gamma \to G_{\gamma'}$. We need to check that this isomorphism does not depend on the choice of g. Let g' be another element such that $\gamma' = g\gamma g'$ thus $g'^{-1}g \in G_\gamma$. The isomorphisms

$$\mathrm{Ad}(g'),\mathrm{Ad}(g) : G_\gamma \to G_{\gamma'}$$

are the same since G_γ is a *commutative* group for regular a. $\square$

For $a \in G^{reg}(F)$, we can define a F-group J_a as the centralizer G_γ for some choice of $\gamma \in \chi^{-1}(a)$, and this does not depend on the choice of γ. In the case that we are going to look at, the set $\ker[\mathrm{H}^1(F,G_\gamma) \to \mathrm{H}^1(F,G)]$ will be an abelian group. The proposition can be rephrased as follows: the set of $G(F)$-conjugacy class within the stable conjugacy class associated to a is a torsor under the group $\ker[\mathrm{H}^1(F,G_\gamma) \to \mathrm{H}^1(F,G)]$. Once this formulation is made rigorous, we can recover the Kottwitz obstruction for the existence of a F-point in $\chi^{-1}(a)$. We will go back to this point in the second lecture.

Why one should be interested in stable conjugacy classes instead of conjugacy classes. The answer is related to Langlands functoriality principle. For instant, let G be a connected reductive group over a global field F and let G^* be its quasi-split form. One can check that the varieties car associated to G and G^* are canonically isomorphic and in particularly they have the same F-points. Let us suppose G^{der} simply connected. Let $\gamma \in G(F)$ and $a \in \mathrm{car}(F)$ its image. Let $\chi^* : G^* \to \mathrm{car}$ be the Chevalley map for G^*. Following Steinberg's theorem, ${\chi^*}^{-1}(a)(F)$ is non empty. This defines a canonical map from the set of $G(F)$-conjugacy class of $G(F)$ to the set of stable conjugacy class of G^*. Similarly, let E be a cyclic extension of F and let σ be a generator of $\mathrm{Gal}(E/F)$. Then one can define a map from the set of σ-conjugacy classes of $G(E)$ to the set of stable conjugacy classes of $G(F)$.

2 Regular elliptic part

Let F be a global field and let G be a connected reductive group over F whose the derived group G^{der} is simply connected. This simplifying assumption implies in particular that for every regular semi-simple element γ of G, G_γ is a torus. Such an element γ is said to be elliptic if G_γ/Z_G is a F-anisotropic torus.

The regular elliptic part of the trace formula consists in a distribution

$$f \mapsto T_e(f) = \sum_{\gamma \in E} \tau(G_\gamma) O_\gamma(f)$$

where E is a set of representatives of regular elliptic conjugacy classes of $G(F)$, $\tau(G_\gamma)$ is the Tamagawa number of the torus G_γ and

$$O_\gamma(f) = \int_{G_\gamma(\mathbb{A})\backslash G(\mathbb{A})} f(g^{-1}\gamma g) \frac{dg}{dg_\gamma}.$$

By definition the distribution O_γ is a tensor product over all the places of F, of local distribution given by the local orbital integrals. We can say that the distribution O_γ is "local".

If $\gamma,\gamma' \in E$ are stably conjugate, G_γ and $G_{\gamma'}$ are isomorphic so that $\tau(G_\gamma) = \tau(G'_\gamma)$. Let γ_0 be an element of $G(F)$ and let $E(\gamma_0)$ the set of the elements $\gamma \in E$ that are stably conjugate to γ_0. We can consider the sum

$$\sum_{\gamma \in E(\gamma_0)} O_\gamma(f)$$

of $T_e(f)$. This sum looks "stable" but not "local" anymore in the sense that it can not be factorized in to product of local distributions. Let $|F|$ be the set of valuations of F. For all $v \in |F|$, let γ_v be an element of $G(F_v)$ which is stably conjugate to γ_0. There is an *obstruction*, depending on the collection of the local invariants $\mathrm{inv}(\gamma_0,\gamma_v) \in \ker[\mathrm{H}^1(F_v,G_{\gamma_0}) \to \mathrm{H}^1(F_v,G)]$, whose the vanishing is equivalent to the existence of a rational conjugacy class $\gamma \in E$ such that for all $v \in |F|$, γ is $G(F_v)$-conjugate to γ_v. To make precise the definition of this obstruction, we will need a few preparation in Galois cohomology.

3 Local and global duality

Kottwitz has given a very convenient reformulation of local and global Tate-Nakayama duality. He also extends this duality from tori to arbitrary reductive groups.

Let F be any field. Let T be a F-torus. Let $X^*(T)$ be the group of characters $T \to \mathbb{G}_m$ which is a free finitely generated abelian group which is equipped with a finite action of $\Gamma = \mathrm{Gal}(\overline{F}/F)$. The usual formulation of Tate-Nakayama duality expresses a pairing between the Galois cohomology of T and of $X^*(T)$. Following Kottwitz, we define its complex dual torus to be

$$\hat{T} = \mathrm{Hom}(X^*(T),\mathbb{C}^\times)$$

which is also equipped with a finite action of Γ. The local duality can be expressed as a "computation" of the Galois cohomology of T in term of $\hat{T}$. More generally, one can associate to any connected reductive group G over F its Langlands dual which is complexe reductive group $\hat{G}$ equipped with a finite action of Γ. The dual group $\hat{G}$ is only well defined up to inner automorphism.

Theorem 3.1 (Kottwitz) *Let F_v be a local field. For any torus T over F_v,* *there is a canonical isomorphism*

$$\alpha_T : \mathrm{H}^1(F_v,T) \to \pi_0(\hat{T}^{\Gamma_v})^*.$$

More generally, for any connected reductive group G over F_v, there is a canonical homomorphism

$$\alpha_G : \mathrm{H}^1(F_v,G) \to \pi_0(Z(\hat{G}))^*$$

which is functorial up to homomophisms $H \to G$ with normal image. Moreover, α_G is an isomorphism if F_v is a non-archimedean local field. In that case, $\mathrm{H}^1(F_v,G)$ is an abelian group.

Let G be a connected reductive group over F_v and let T be a maximal torus of G which is defined over F_v. The torus T is not a normal subgroup of G thus there might not be a Γ-equivariant inclusion $\hat{T} \to \hat{G}$. But there is an inclusion $j : \hat{T} \to \hat{G}$ which is Γ-equivariant up to inner automorphisms that means for all $\sigma \in \Gamma_v$, there exists $g_\sigma \in \hat{G}$ such that

$$\sigma(j(t)) = \mathrm{Ad}(g_\sigma)j(\sigma(t))$$

for all $t \in \hat{T}$. The image of $\hat{T}$ is a maximal torus of $\hat{G}$ and contains its center $Z(\hat{G})$. The inclusion $Z(\hat{G}) \to \hat{T}$ is then Γ_v-equivariant.

Corollary 3.2 *An element of*

$$\ker[\mathrm{H}^1(F_v,T) \to \mathrm{H}^1(F_v,G)]$$

consists in a character $\hat{T}^{\Gamma_v} \to \mathbb{C}^\times$ whose both the restrictions to the neutral connected component of $\hat{T}^{\Gamma_v}$ and to $Z(\hat{G})^{\Gamma_v}$ are trivial.

Let F be a global field and let G be a conneted reductive group over F whose the derived group is simply connected. Let γ_0 be a regular semi-simple element of $G(F)$. Its centralizer is a torus T. For all places $v \in |F|$ let $\gamma_v \in G(F_v)$ be an element stably conjugate to γ_0. We will consider the invariants $\mathrm{inv}(\gamma_0,\gamma_v)$ as a character $\hat{T}^{\gamma_v} \to \mathbb{C}^\times$ verifying additional properties as above. Using the global Tate-Nakayama duality, Langlands and Kottwitz obtain the following:

Theorem 3.3 *With the above notations and assumptions, there exists $\gamma \in G(F)$ such that for all v, γ is $G(F_v)$-conjugate to γ_v if and only if for almost all v the invariants $\mathrm{inv}(\gamma_0,\gamma_v) : \hat{T}^{\Gamma_v} \to \mathbb{C}^\times$ are trivial and the sum of their restrictions to $\hat{T}^\Gamma$*

$$\mathrm{obs}(\gamma_0;(\gamma_v)) := \sum_{v\in|F|} \mathrm{inv}(\gamma_0,\gamma_v)|_{\hat{T}^\Gamma}$$

vanishes. In that case, the number of $G(F)$-conjugacy classes γ that are $G(\gamma_v)$-conjugate to γ_v is equal to

$$\tau(G)\tau(T)^{-1}|\pi_0(\hat{T}^\Gamma)|.$$

In the case of Lie algebra, we don't need the hypothesis that G_{der} is simply connected.

4 Pre-stabilization

We are now coming back to the trace formula. Let F be a global field and let G be a connected reductive group over F whose the derived group is simply connected. Let E_0 be a set of representatives of stable conjugacy classes of $G(F)$ that are regular semi-simple. We have a map from the set E of representatives of conjugacy classes to E_0 mapping $\gamma \mapsto \gamma_0$ with γ_0 stably conjugate to γ. We can rewrite the sum

$$T_e(f) = \sum_{\gamma \in E} \tau(G_\gamma) O_\gamma(f) = \sum_{\gamma_0 \in E_0} \tau(G_{\gamma_0}) \sum_{\substack{\gamma \in E \\ \gamma \mapsto \gamma_0}} O_\gamma(f).$$

Let suppose that the test function if pure tensor $f = \bigotimes f_v$ where f_v is the characteristic function of the maximal compact subgroup for almost all places v. The above expression can be now rewritten as follows

$$T_e(f) = \sum_{\gamma_0 \in E_0} \tau(G)|\pi_0(\hat{G}_{\gamma_0}^\Gamma)| \sum_{\mathrm{obs}(\gamma_0;\gamma_v)=0} \prod_{v \in |F|} O_{\gamma_v}(f_v).$$

Using now the Fourier transform all the finite abelian group $\pi_0(\hat{G}_{\gamma_0}^\Gamma)$, we obtain

$$T_e(f) = \tau(G) \sum_{\gamma_0 \in E_0} \sum_{\kappa \in \pi_0(\hat{G}_{\gamma_0}^\Gamma)} O_{\gamma_0}^\kappa(f)$$

where the κ-orbital integrals are

$$O_{\gamma_0}^\kappa(f) = \prod_{v \in |F|} \sum_{\gamma_v} \langle \mathrm{inv}(\gamma_0,\gamma_v),\kappa \rangle O_{\gamma_v}(f_v)$$

the inner term consists in the sum over all $G(F_v)$-conjugacy classes in $G(F_v)$ that are stably conjugate to γ_0.

The terms corresponding to $\kappa = 1$ is called the stable part. The others κ-part should be equal to the stable part of endoscopic groups. This is the contents of the principle of transfer and of the fundamental lemma, stated in unprecise terms. To make this precise, we need to introduce the notion of endoscopic groups.

Let us make an observation : in order to talk about the κ-part of the trace formula, one would prefer to reverse the order of the summation from $\sum_{\gamma_0}\sum_{\kappa}$ to $\sum_{\kappa}\sum_{\kappa_0}$. It's not easy to do so here because we sum over the set of κ depending in γ_0. As we will see in the second talk, from the geometric point of view, we really get a sum over κ first.

5 Shimura varieties

It was observed by Langlands and Kottwitz that the formula expressing the number of points of Shimura varieties has a very similar form. A Shimura variety Sh is attaced to a Shimura datum (G,h) where G is a reductive group over $\mathbb{Q}$ and $h : \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{G}_m \to G$. The set of isogeny classes of $\mathbb{F}_q$-points of Sh should correspond bijectively the set of triples

$$(\gamma_0;\gamma,\delta)$$

where γ_0 is a stable conjugacy class of $G(\mathbb{Q})$, $\gamma = (\gamma_v)_{v\neq p}$ is a $G(\mathbb{A}_f^p)$-conjugacy class in $G(\mathbb{A}_f^p)$ within the stable conjugacy class γ_0 and finally, δ is a σ-conjugacy class of $G(F)$ where F is the unramified extension of the reflex field E_p at p whose residue field is $\mathbb{F}_q$ that satisfy the Kottwitz vanishing condition

$$\sum_v \mathrm{inv}(\gamma_0,\gamma_v) + \mathrm{inv}(\gamma_0,\delta) + \mathrm{inv}_\infty(\gamma_0) = 0.$$

The invariants $\mathrm{inv}(\gamma_0,\gamma_v)$ are those defined in the previous section. The invariant $\mathrm{inv}(\gamma_0,\delta)$ is also defined in the similar way, see [Kottwitz-AnnArbor]. The number of points of Shimura varieties over $\mathbb{F}_q$ should have the form

$$\sum_{(\gamma_0,\gamma,\delta)} c(\gamma_0;\gamma,\delta)\, TO_\delta(\phi_\mu) \prod_{v\neq p} O_\gamma(1_v)$$

and this has been proved in the PEL cases by Kottwitz. Here $c(\gamma_0;\gamma,\delta)$ is a Tamagawa number of some torus associated to $(\gamma_0;\gamma,\delta)$, ϕ_μ is the spherical function at q associated to h, and at the other places 1_v are the unit spherical function.

In order to compute the Hasse-Weil zeta function of Shimura varieties in terms of automorphic L-functions, we need to stabilize this formula. This is is very similar to the stabilization of the trace formula, in particular, the same endoscopic groups will show up in the final formula.

6 Endoscopic groups

Let G be a connected reductive group over a global field F whose the derived group is simply connected. Let γ_0 be an regular semi-simple element of $G(F)$, T its centralizer and κ an element of $\pi_0(\hat{T}^\Gamma)$. Lets choose a lift $\kappa \in \hat{T}^\Gamma$. There might not be a Γ-equivariant embedding $\hat{T} \to \hat{G}$ but there existe an embedding $j : \hat{T} \to \hat{G}$ that is Γ-equivariant up to inner automorphism. Let denote κ the image of κ in $\hat{G}$. In $\hat{G}$, κ is a semi-simple element which is not necessarily Γ-invariant. Since T is an elliptic torus, $\hat{T}^\Gamma/Z(\hat{G})$ is a finite group thus κ is an element of finite order modulo the center of $\hat{G}$.

Let $\hat{H}$ be the neutral connected component of the centralizer $\hat{G}_\kappa$ of κ in $\hat{G}$. The action of Γ in $\hat{G}$ does not necessarily preserve $\hat{H}$ since κ in not Γ-invariant in $\hat{G}$. Nevertheless, there exists a canonical homomorphism

$$\rho_H : \Gamma \to \mathrm{Out}(\hat{H})$$

that defines an action of Γ on $\hat{H}$ up to inner automorphisms. Its definition, due to Langlands, is a rather subtle exercice on root systems equipped with Galois action. The homomorphism ρ_H enjoys the following properties.

Proposition 6.1 *The homomorphism $\rho_H : \Gamma \to \mathrm{Out}(\hat{H})$ induces a finite action of Γ on $\hat{H}$, well-defined up to inner automorphisms. The embeddings that $\hat{T} \to \hat{H}$ and $\hat{H} \to \hat{G}$ are Γ-equivariant un to inner automorphisms. Moreover, ρ_H differs from the action ρ_T on $\hat{T}$ by a 1-cocycle in $\mathrm{H}^1(\Gamma, W_H)$.*

The homomorphism $\rho_H : \Gamma \to \mathrm{Out}(\hat{H})$ determines uniquely up to inner automorphism a quasi-split reductive group H over F. It follows from the property that ρ_H differs from the action ρ_T on $\hat{T}$ by a 1-cocycle in $\mathrm{H}^1(\Gamma, W_H)$, that there exist an embedding $T \to H$ which is defined over F and which is uniquely determined up to stable conjugacy. The image of $\gamma_0 \in T(F)$ is an element $\gamma_H \in H(F)$ well determined up to stable conjugacy.

7 Transfer conjecture and fundamental lemma

These conjectures are initially stated in [Langlands-LesDebuts]. I also recommend the lecture of the recent survey article [Hales-AStatement].

Transfer factor. Langlands and Shelstad have introduced the numbers $\Delta_v(\gamma_H, \gamma_v) \in \mathbb{C}$ for all places $v \in |F|$ and for any $G(F_v)$-conjugacy classes that are stably conjugate to γ_0 verifying the following properties and others

1. $\Delta_v(\gamma_H, \gamma'_v) = \Delta_v(\gamma_H, \gamma_v)\langle \mathrm{inv}(\gamma_v, \gamma'_v), \kappa\rangle$

2. Let γ be a $G(F)$-conjugacy class in $G(F)$ which is stably conjugate to γ_0, then we have the product formula $\prod_{v\in|F|}\Delta_v(\gamma_H,\gamma_v)=1$.

I will not recall the definition of Langlands-Shelstad transfer factors which is rather elaborated. Let me just say that $\Delta_v(\gamma_H,\gamma_v)$ is a product of a sign and a power of q where the power of q is a discriminant easy to be defined. The difficulty is how to define the sign. In the case of Lie algebra, Kottwitz has discovered a nice property chich characterize this sign: it is 1 if γ_v is the $G(F_v)$-conjugacy class of the Kostant section, see [Kottwitz-TransferFactor]. In the case of classical groups, Waldspurger made useful computation of transfer factor in terms of linear algebra, see [Waldspurger-Asterisque].

Transfer conjecture. For all place $v\in|F|$, for all test function $f_v\in C_c^\infty(G(F_v))$ there exists a function $f_v^H\in C_c^\infty(H(F_v))$ such that

$$SO_{\gamma_H}(f_v^H)=\sum_{\gamma_v}\Delta(\gamma_H,\gamma_v)O_{\gamma_v}(f_v).$$

Fundamental lemma. Let v be a place of F where G is unramified i.e. there exists a groups scheme $\mathcal{G}$ over $\mathcal{O}_v$ whose generic fiber is G_v and whose special is also a connected reductive group. Let $K_v=\mathcal{G}(\mathcal{O}_v)$ which is a hyperspecial maximal compact subgroup of $G(F_v)$ and f_v be the characteristic function of K_v, we say f_v is the "unit function".

If H_v is also an unramified reductive group over the local field F_v, then we take f_v^H to be the characteristic function of a hyperspcial maximal compact. Then we should the equality

$$SO_{\gamma_H}(f_v^H)=\sum_{\gamma_v}\Delta(\gamma_H,\gamma_v)O_{\gamma_v}(f_v)$$

holds. If H_v is ramified then $\sum_{\gamma_v}\Delta(\gamma_H,\gamma_v)O_{\gamma_v}(f_v)=0$.

There is a more general version of the fundamental lemma that concerns all Hecke function instead of the unit function.

Global setting These local conjectures implies the formula we desire in the global setting. Let $f=\bigotimes_v f_v$ where f_v are the function for all places v out of some finite set S. Let take $f^H=\bigotimes_v f_v^H$ where f_v^H is the unit functio of H for $v\notin S$. For the remaining places $v\in S$, we take f_v^H to be one of the function whose the existence is insured by the transfer conjecture. Put these conjectures together, and using the product formula for transfer factor, we get

$$O_{\gamma_0}^\kappa(f)=SO_{\gamma_H}(f^H).$$

Let us observe that the global formula is somehow simpler because we don't need the definition of local transfer factor.

Using the above formula, and by analyzing the combinatoric of the set of summation, Langlands proved the desired stabilization

$$T_e^{reg}(f) = \sum_H \iota(G,H) ST_e^{G-reg}(f^H)$$

expressing the regular elliptic part of the trace formula for G as a linear combination of stable trace formula for endoscopic groups. Kottwitz extends this formula for the whole elliptic part of the trace formula. We refer to their original paper for the definition of the constants $\iota(G,H)$.

Results known to me

- Shelstad proved the transfer conjecture for real groups before Langlands stated the above conjectures over the non-archimedean places.
- Labesse-Langlands achieved the case SL(2).
- Rogawsky solved the case $U(3)$.
- Kottwitz proved the fundamental lemma for stable base change.
- Clozel and Labesse a generalized Kottwitz results to other Hecke functions. Hales proved that this is also true for other fundamental lemma.
- Hales and Weissauer solved the case Sp(4).
- Waldspurger solved the case SL(n).

Hales and Waldspurger have etablished the closed relationship between the transfer conjecture, the fundamental lemma, and a variant of the fundamental lemma for Lie algebra. I understand that "obvious extrapolation from [Hales-SimpleDefinition] leads to a proof that fundamental lemma for Lie algebra implies the fundamental lemma for groups" (Waldspurger). Waldspurger also shown that the fundamental lemma for Lie algebra implies the transfer. Very recently, Waldspurger proved that fundamental lemma in the p-adic local field is equivalent to the fundamental lemma for local field in equal characteristic. The case of local field of equal characteristic is more accessible if one wants to use algebraic geometry.

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