Lecture C.2

Classical formal moduli problem: functor $\mathrm{Art}_k^{o} \to \mathrm{Set}$

$$\mathrm{Art}_k = \{\text{artin local rings w/ res field } k\}$$

Def (1) $A \in SCR$ Artin local if

(i) $\pi_0(A)$ Artin local

(ii) $\pi_*(A)$ f.gen $\pi_0(A)$-module

(i.e. $\pi_i(A)$ f.g $\forall i$, $\pi_i(A) \simeq 0$ $\forall i \gg 0$)

Example $\delta : CR \hookrightarrow SCR$ "constant simplicial object"

carries Artin local rings to Artin local simplicial rings

left adjoint $\pi_0 : SCR \to CR$

(2) $A \in SCR$ Artin local $\rightsquigarrow$ residue field $:=$ residue field of $\pi_0(A)$, characterized uniquely by $A \to k$ induces a surjection

$$\pi_0(A) \to \pi_0(k) \simeq k.$$

(3) $\mathrm{Art}_k \subseteq SCR_{/k}$ full subcat spanned by $\varepsilon: A \to k$ with A Artin local and ε exhibiting k as res field.

$\delta : \mathrm{Art}_k^0 \hookrightarrow \mathrm{Art}_k$

<u>Example</u> M simplicial k-module $\rightsquigarrow$ $k \oplus M \in SCR/k$
(sq zero ext. formal lenthen)

if $\pi_*(M)$ f.d. k-vector space then $k \oplus M \in \mathrm{Art}_k$.

<u>Important special case</u>: let $n \geq 0$, $S^n := \Delta^n / \partial\Delta^n \in s\mathrm{Set}$

$$\begin{tikzcd} \partial\Delta^n \arrow[r] \arrow[d] & * \arrow[d] \\ \Delta^n \arrow[r] & \Delta^n/\partial\Delta^n \end{tikzcd}$$

$k[S^n]$ free simplicial k-module

$k[n] := k[S^n]/k[*]$

$$\pi_i k[n] \simeq \begin{cases} k & i = n \\ 0 & o/w \end{cases}$$

$\rightsquigarrow$ $k \oplus k[n] \in \mathrm{Art}_k$

$(n = 0 \rightsquigarrow k[\varepsilon]/\varepsilon^2)$.

<u>Def</u> A <u>derived formal moduli problem</u> $/k$ is a (non-std defn)

homotopy invariant functor $\mathrm{Art}_k \longrightarrow s\mathrm{Set}$

"<u>Example</u>" $R \in SCR_{/k}$ cofibrant

$$\rightsquigarrow \quad F_R(A) := SCR_{/k}(R, A)$$

More generally $R = \{R_j\}_{j \in J}$ pro-object in $SCR_{/k}$, $J \to SCR_{/k}$, (cofiltered)
with each R_j cofibrant. Then

$$(*) \qquad F_R(A) := \operatorname*{colim}_{j \in J} SCR_{/k}(R_j, A) .$$

<u>Def</u> A derived fmp/k is <u>prorepresentable</u> if it's weakly equivalent to one of the form $(*)$ where each $R_j \in Art_k$.

<u>Thm</u> [Lurie] Let k be a field with $\Omega^1_{k/\mathbb{Z}} \simeq 0$.
Then a derived fmp/k $F: Art_k \to sSet$ is pro-rep iff:

formally cohesive {
(1) $F(k)$ weakly contractible
(2) given a <u>homotopy cartesian</u> diagram
$$\begin{array}{ccc} A & \to & B \\ \downarrow & & \downarrow \\ C & \to & D \end{array}$$
in Art_k

with $\pi_0(B) \to \pi_0(D)$ and $\pi_0(C) \to \pi_0(D)$ surjective

the induced diagram
$$\begin{tikzcd} F(A) \arrow[r] \arrow[d] & F(B) \arrow[d] \\ F(C) \arrow[r] & F(D) \end{tikzcd}$$
<u>homotopy cartesian</u>.

(3) $F(\mathrm{hok}[0])$ is homotopy discrete ($\pi_i \simeq 0\ \forall i \geq 0$).

<u>Rmk</u>
$$\begin{tikzcd} & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$
diagram of Kan complexes

$\rightsquigarrow \quad Y \times^h_X Z := (Y \times Z) \times_{(X \times X)} X^{\Delta^1}$

$\rightsquigarrow \quad S \to Y \times^h_X Z \iff S \to Y,\ S \to Z$

+ htpy $S \rightrightarrows X$

In part comm diag
$$\sigma : \begin{tikzcd} S \arrow[r] \arrow[d] & Y \arrow[d] \\ Z \arrow[r] & X \end{tikzcd} \quad \rightsquigarrow \quad S \xrightarrow{f(\sigma)} Y \times^h_X Z$$

<u>Def</u> σ htpy cartesian if $f(\sigma)$ is a weak equivalence.

<u>Key properties</u>

(1) invariant under weak equivalence

(2) For $Z = \Delta^0$ $\rightsquigarrow$ homotopy fiber F of $Y \to X$

$\rightsquigarrow$ LES

$$\cdots \to \pi_1(X) \to \pi_0(F) \to \pi_0(Y) \to \pi_0(X)$$

(3) Htpy pullback of a diagram of simpl. ab grps or simpl. rings retains the same structure

For simpl. ab grps

$$Y \times^h_X Z \to Y \oplus Z \to X$$

basically a dist. triangle on other side of Dold-Kan

LES

$$\cdots \to \pi_1(X) \to \pi_0(Y \times^h_X Z) \to \pi_0(Y) \oplus \pi_0(Z) \to \pi_0(X)$$

dist triangle

$$k[n-1] \to 0 \to k[n] \to k[n]$$

<u>For example:</u> $0 \times^h_{k[n]} 0 \simeq k[n-1]$

$\rightsquigarrow$ $k \times^h_{k \oplus k[n]} k \simeq k \oplus k[n-1]$

Prop Let $A \in \mathrm{Art}_k$. Then there is a sequence

$$A \xrightarrow{\simeq} A_m \to A_{m-1} \to \cdots \to A_0 \xrightarrow{\simeq} k$$

and $A_i \simeq A_{i-1} \times^h_{k \oplus k[n_i]} k$ for some $n_i \geq 1$.

$$\left(\begin{tikzcd} \cdot \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & k \arrow[d] \\ A_{i-1} \arrow[r] & k \oplus k[n_i] \end{tikzcd} \right)$$

Rmk Additionally the htpy pullback forms an extension of A_{i-1} by $k[n_i - 1]$

E.g.

$$\begin{tikzcd} \mathbb{Z}/p^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathbb{F}_p \arrow[d, "\text{trivial}"] \\ \mathbb{F}_p \arrow[r] & \mathbb{F}_p \oplus \mathbb{F}_p[1] \end{tikzcd}$$

$\mathbb{F}_p \to \mathbb{F}_p[1]$ classifies $\mathbb{Z}/p^2 \in \mathrm{Ext}^1(\mathbb{F}_p, \mathbb{F}_p)$

<u>Remark</u> Let $F : \mathrm{Art}_k \to \mathrm{sSet}$ formally cohesive.

Let $A \to A'$ be a square-zero ext of ordinary Artin local rings, by k.

$\rightsquigarrow$ htpy fiber square

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A' \arrow[r] & k \oplus k[1] \end{tikzcd}$$

$$\rightsquigarrow \quad F(A) \to F(A') \to F(k \oplus k[1])$$

htpy fiber sequence

$$\rightsquigarrow \quad \pi_0 F(A) \to \pi_0 F(A') \to \pi_0 F(k \oplus k[1])$$

$$\begin{array}{ccc} \rotatebox{90}{$\in$} & & \downarrow \\ x & \longmapsto & o(x) \end{array}$$

"obstruction to lifting x from A' to A"

$$\begin{tikzcd}
\mathbb{Z}/p^2 \arrow[r] \arrow[dr, hook, "\mathrm{cof}"'] & \mathbb{F}_p \\
& A \arrow[u, "\sim"']
\end{tikzcd}$$

A polynomial resolution over $\mathbb{Z}/p^2$

$A \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p$ model $\mathbb{F}_p \otimes^{L}_{\mathbb{Z}/p^2} \mathbb{F}_p$

$$\pi_*\left(A \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p\right) \simeq \mathrm{Tor}_*^{\mathbb{Z}/p^2}(\mathbb{F}_p, \mathbb{F}_p) \simeq \begin{cases} \mathbb{F}_p & * = 0, 1 \\ ? & \text{o/w} \end{cases}$$

$$\begin{tikzcd}
\mathbb{Z}/p^2 \arrow[r] \arrow[d] & \mathbb{F}_p \arrow[d] \arrow[drr, bend left] & & \\
A \arrow[r] & A \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p \arrow[r] & \tau_{\leq 1}\left(A \otimes_{\mathbb{Z}/p^2} \mathbb{F}_p\right) \simeq \mathbb{F}_p \oplus \mathbb{F}_p[1]
\end{tikzcd}$$

$A \simeq \mathbb{F}_p$

outer quadrilateral is htpy cartesian