1 The remaining axioms of set theory and the power set construction

Week 2: will miss one class due to Labor day.
Reading: [2, Ch.3.1-4], [1, 2].

Learning Objectives

In last lectures, we

- Defined $\mathbb{N}$ axiomatically using the Peano axioms.
- Used induction to prove properties of operations as $+$ and $\times$ on $\mathbb{N}$.

In the next two lectures

- Discuss the remaining axioms of set theory. We begin by discussing new notions: *subsets*, 1.1, We end with the construction of the power set.
- Discuss *equivalence relation*, **??**, and *ordered pairs*, **??**. which constructs the integers and the rationals

1.1 Subcollections

Definition 1.1. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1, 2\} \subset \{1, 2, 3\}$.

1.2 Remaining axioms of set theory

Week 2
In this section we continue from previous lecture and discuss remaining axioms from what is known as the *Zermelo-Fraenkel (ZF) axioms of set theory*, due to Ernest Zermelo and Abraham Fraenkel.

Axiom 1.2. Singleton set axiom. If a is an object. There is a set $\{a\}$ consists of just one element.

Axiom 1.3. Axiom of pairwise union. Given any two sets A, B there exists a set $A \cup B$ whose elements which belong to either A or B or both.

Often we would require a stronger version.

Axiom 1.4. Axiom of union. Let A be a set of sets. Then there exists a set

$$\bigcup A$$

whose objects are precisely the elements of the set.

Example

Let

- $A = \{\{1, 2\}, \{1\}\}$
- $A = \{\{1, 2, 3\}, \{9\}\}$

Discussion

Using the axioms, can we get from $\{1, 3, 4\}$ to $\{2, 4, 5\}$?

We will now state the power set axiom for completeness but revisit again.

Axiom 1.5. Axiom of power set. Let X, Y be sets. Then there exists a set Y^X consists of all functions $f : X \to Y$.

We will review definition of function later, 1.7.

Axiom 1.6. Axiom of replacement. For all $x \in A$, and y any object, suppose there is a statement $P(x, y)$ pertaining to x and y. There is a set

$$\{y : P(x, y) \text{ is true for some } x \in A\}$$

This can intuitively be thought of as the set

$$\{y : y = f(x) \text{ some } x \in A\}$$

1.3 Function

Discussion

How would you intuitively define a function?

Definition 1.7. Let X, Y be two sets. Let

$$P(x, y)$$

be a *property* pertaining to $x \in X$ and $y \in Y$, such that for all $x \in X$, there *exists a unique* $y \in Y$ such that $P(x, y)$ is true. A *function associated to* P is an object

$$f_P : X \to Y$$

such that for each $x \in X$ assigns an output $f(x) \in Y$, to be the unique object such that $P(x, f(x))$ is true.

- X is called the *domain*
- Y is called the *codomain*.

Definition 1.8. The *image*...?

Discussion

When is what kind of properties P does not satisfy the condition of being function?

- "$y^2 = x$".
- "$y = x^2$".

Homework for week 2

Due: Week 3, Wednesday. All questions in 1.4, Boolean algebra is compulsory. Select 3 other questions to be graded.

Reading: We refer to the axioms of set theory we have discussed thus far collectively as the ZF axioms. *The only axiom we did not discuss is the axiom of replacement,* [2, 3.5]. *This will be left as required reading for certain problems.*

Problems

1. Let A, B, C be sets.
   (a) Prove set inclusion, def. 1.1, is reflexive and transitive. $A \subseteq B, B \subseteq C$ then $A \subseteq C$.
   (b) Prove that the union operation $\cup$ on sets 1.3, is associative and commutative:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cup B = B \cup A$$

2. (**) Let I be a set and that for all $\alpha \in I$, I have a set A_α. [1] Read about the axiom of replacement; see [2, Axiom 3.5] or 1.6.
   (a) Prove that under the ZF axioms, one can form the union of the collection:

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha \, : \, \alpha \in I\}$$

   In particular, explain why the following two objects

   i.
$$\{A_\alpha \, : \, \alpha \in I\}$$
   ii.
$$\bigcup \{A_\alpha \, : \, \alpha \in I\}$$

   are sets.

   (b) Give a one line explanation briefly describing why axiom of union 1.4 is insufficient to construct the set $\bigcup_{\alpha \in I} A_\alpha$.

[1] For example, if $I = \{a, b, c\}$, then I have three sets

$$A_a, A_b, A_c$$

3. The *axiom of regularity* states

   Axiom 1.9. [2, 3.9] If A is a nonempty set, then there is at least one element $x \in A$ which is neither a set, or (if it is a set) disjoint from A.

   Prove (with singleton set axiom) that for all sets $A, A \notin A$.

4. (***) Let A, B, C, D be sets. This exercise shows that we can actually construct *ordered pairs* using the ZF axioms.[2] Prove

   - We can construct the following set [3]

     $$\langle A, B\rangle := \{A, \{A, B\}\}$$

     from the axioms of set theory.
   - $\langle A, B\rangle = \langle C, D\rangle$ if and only if $A = B, C = D$. For this part you will require the *axiom of regularity.* in problem 3. You are free to use the results there.

5. This is a variation of problem 4 [4]. Suppose for two sets A, B we define

   $$[A, B] = \{\{A\}, \{A, B\}\}$$

   In this case, the problem is a lot easier. Prove $[A, B] = [C, D]$ if and only if $A = B, C = D$.

6. (***) Show that the collection

   $$\{Y : Y \text{ is a subset } X\}$$

   is a set using the ZF axioms. We denote this as the power set 2^X, where 2 is regarded as the two elements set $\{0, 1\}$. You will need to use the axiom of replacement.
   Here are two important remarks on possible false solutions:

   (a) (Ryan's) if your property for axiom of replacement $P(x, y) =$ "y is a subset of x" then this is *not correct*. The condition for replacement is that *there is at most one* y, [2, 3.6].

   (b) (Kauí's) You cannot use axiom of comprehension, this is similar to Russell's paradox!

   As a hint: $\{0, 1\}^X$ is a set, by 1.5. For $Y \subset X$, $f \in \{0, 1\}^X$, let $P(Y, f)$ be the property that

   $$Y = f^{-1}(1) := \{x \in X : f(x) = 1\}$$

[2] Another definition is discussed in or [2, 3.5.1], where they assume this as an axiom.
[3] RIP. So another model of this is $\langle A, B\rangle := \{\{A\}, \{A, B\}\}$
[4] which is what I should have written

1.4 Boolean algebras

This section is compulsory. Boolean algebras form the foundation of probability theory.

Definition 1.10. Let Ω be a set. A *Boolean algebra* in Ω is a set $\mathcal{E}$ of subsets of Ω (equivalently, $\mathcal{E} \subseteq 2^{\Omega}$) satisfying

1. $\emptyset \in \mathcal{E}$
2. closed under unions and intersections.

$$E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$$

$$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$$

3. closed under complements.

A *σ-algebra in* Ω is a Boolean algebra in Ω such that it satisfies

4. Countable[5] closure. If $A_i \in \mathcal{E}$ for $i \in \mathbb{N}$, then $\bigcup A_i \in \mathcal{E}$.

Problems

1. Prove that $\mathcal{E} := \{\emptyset, \Omega\}$ is a σ-algebra.
2. Prove that $2^{\Omega} := \{E : E \subset \Omega\}$ is a σ-algebra.
3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Example

(Hard) The *Borel σ-algebra on* $\mathbb{R}$ is the smallest σ-algebra containing the collection of open subsets of $\mathbb{R}$. $E \subset \mathbb{R}$ is *open* if for all $x \in E$, there exist an open interval I containing x and such that $I \subset \mathbb{R}$.

Hints for problems

3. There are 3 cases. What happens $A = \emptyset$ or $A = \Omega$? Now consider the case $A \neq \emptyset$ and $A \neq \Omega$.

[5] A set X is countable if it is in bijection with $\mathbb{N}$. We will explore this word in further detail in the future.

References

[1] Jonathan Pila, *B1.2 set theory*.

[2] Terence Tao, *Analysis I, 4th edition*, 2022.