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Resisting reductionism in mathematics pedagogy

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Although breaking down a mathematical problem into smaller parts can often be an effective solution strategy, when the same reductionist approach is applied to mathematics pedagogy the effects are far from beneficial for students. Mathematics pedagogy in UK schools is gaining an increasingly reductionist flavour, as seen in an excessive focus on bite-sized learning objectives and a tendency for mathematics teachers to path-smooth their students' learning. I argue that a reductionist mathematics pedagogy severely restricts students' opportunities to engage in authentic mathematical thinking and deprives them of the enjoyment of solving richer, more worthwhile problems, which would forge connections across diverse areas of the subject. I attribute the rise of a reductionist mathematics pedagogy partly to an assessment-dominated curriculum and partly to a systemic deprofessionalisation of teachers through a performative accountability culture in which they are constantly required to prove to non-specialist managers that they are effective. I argue that pedagogical reductionism in mathematics must be resisted in favour of a more holistic approach, in which students are able to bring a variety of mathematical knowledge and skills to bear on rich, challenging and non-routine mathematical tasks. Some principles for achieving this are outlined and some examples are given.

Keywords: complexity theory; holism; mathematics curriculum; mathematics education; pedagogy; reductionism; task design; teacher accountability; teacher professionalism

How do you eat an elephant?
One slice at a time! (Anon)

1. Introduction

In this paper, I take reductionism to be the general belief that the best way to analyse complicated phenomena is to look for simpler underlying features. Descartes saw the world as being like a clockwork machine, which

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could be best understood by dismantling it and examining the individual components (Baker & Morris, 2002), and this mechanistic reductionism paved the way for the modern scientific approach, so that now, according to Wilson (1999, p. 59), ‘reductionism is the primary and essential activity of science’. Likewise, in mathematics, when facing complexity, it may be second nature for the experienced problem-solver to seek to *reduce* the problem (Pólya, 1990). This may be to a previously solved ‘known’ problem type (i.e., transforming the problem into a more familiar context) or merely to a smaller problem that it is hoped may be initially more tractable. As Ainley (1995) puts it:

> One way of responding to a task you see as being difficult is to break it down into smaller bits that you feel more confident that you can tackle successfully. This technique is often used by mathematicians to good effect. (Ainley, 1995, p. 10)

Such an approach is implicit in the common description of mathematical problems as ‘multi-step’ (Clement & Konold, 1989); the mathematician often prefers to turn a large problem into a linear list of smaller stages, and then process them one by one.

However, while reductionism has much to offer the processes of mathematical problem solving, it is much less clear that a reductionist approach to mathematics pedagogy is beneficial. In a reductionist pedagogical paradigm, the subject is broken down into numerous tiny skills and pieces of knowledge, which are then taught separately and sequentially. The unstated assumption is that mastering these elements is equivalent to (but more manageable than) learning the original structure. Yet it is widely lamented that, when taught in this way, students often fail to see the purpose of these piecemeal bits of learning, quickly lose the various fragments and struggle to select appropriate ones and combine them when called on to solve more substantial problems (Holt, 1990). By breaking down the mathematics for the student, the teacher attempts to make it easier to learn. However, I argue in this paper that in so doing important transformations take place which drastically curtail the potential benefits that students can derive from their study. When reduction takes place *for* the student, rather than *by* the student, it may be experienced as dangerously disempowering.

In this paper, I consider the problems with reductionism, particularly when applied to mathematics pedagogy, highlighting that the genuine power of the reductive approach when working on *mathematics* risks blinding mathematics educators to its dangers when controlling their *pedagogy*. I explore possible factors contributing to pedagogical reductionism and argue for a more *holistic* pedagogical perspective which allows students to engage productively with complexity in more authentically mathematical experiences. I conclude that reductive approaches in the teaching of mathematics are unduly constraining for students and I offer some holistic alternatives.

2. Reductionism and mathematics

Reductionist ideas can be traced back to the Greek philosopher Thales, who believed that everything was made up of water, and to his pre-Socratic successors, through to Democritus, who thought that the notion that all matter was composed of atoms would enable everything in nature, even the soul, to be explained (Russell, 2006). However, this was soon followed in Aristotle's *Metaphysics* by the famous statement that 'The whole is more than the sum of its parts', evidence that a dichotomy between reductionism and antireductionism goes back a long way (Vega, Hernández, & Rivaud, 2003). Aristotle's approach to the problem of universals was to seek them in what he called the *essence* of things. A simple example of mathematical reduction would be reducing a fraction such as $\frac{8}{10}$ to its lowest terms, $\frac{4}{5}$. This reduced fraction has the same value but is expressed using smaller integers, and might be regarded as seeking the essence of what the number *is*. The assumption is that nothing of value is lost in the process; on the contrary, a purer, simpler, 'better' form of what was initially present is obtained. To take another example, when confronted with an unfamiliar integer, a mathematician might choose to prime factorise it, just as a chemist might put an unknown compound into a mass spectrometer to break it down into its separate fragments. There may be a feeling that until you have done this you have not really probed what it is that you are dealing with.

This essence seeking is embodied in the ubiquitous practice of seeing a mathematical object as an example of a deeper, simpler generalisation (Watson & Mason, 2005), leading to the classification of mathematical problems into types. Consequently, students have tables of formulae for every conceivable eventuality, and such things as 'standard integrals', each of which encompasses countless special cases but can be handled in a single well-characterised way (Merzbach & Boyer, 2011). As a result, great savings can be made, in that relatively few tools are sufficient for doing much varied work. Moving from a number of special cases to an overall generalisation involves capturing the essence of what they are all about. In one sense this is reductive, since there is a simplification resulting from 'throwing away' details, but without seeming to lose the core. On the other hand, it may be viewed in the opposite way, as standing back and seeing 'the bigger picture'. The expression 'seeing the wood for the trees' nicely encapsulates this ambiguity: does our perception of a tree bring to light the entire 'wood' (i.e., the forest) that it is part of, or the 'wood' (i.e., the material) that each individual tree is composed of?

3. Problems with reductionism

Despite its evident successes, a reductionist paradigm is nonetheless severely limited. As Sheldrake (2012, pp. 46–47) starkly puts it in the realm of science, 'Attempts to explain organisms in terms of their chemical constituents are rather like trying to understand a computer by grinding it up and analysing its component elements . . .'. Dennett (1995) refers to ventures such as radical behaviourism, which seek to reduce the complexity of human behaviour to mere sequences of actions, as 'greedy reductionism', which *explain away* rather than explain. As Sheldrake (2012) remarks:

> It is relatively easy to break things up and analyse the parts. The problem is to understand the whole; not just the parts but also their interactions need to be understood. And these interactions are not contained in the parts themselves. (Sheldrake, 2012, p. 146)

For Jones (2000, p. 17), the limitations of reductionism 'are seen in a phrase once notorious in British politics, the late [*sic*] Prime Minister Mrs Thatcher's statement that "There is no such thing as society, there are only individuals."' Dismissing society as merely individuals or human beings as merely social animals is a gross over-simplification. According to Raman (2005, p. 251), 'Reductionism reveals what *are* underneath everything, much more than how they converge into something very different. It reveals the *being* part of the world, but not always the *becoming* aspect.' Reduction to fundamentals may trap a static elemental view but miss a more dynamic and complete picture.

The fallacy that the reduction contains everything of importance has been ridiculed by Medawar (1961) as 'nothing buttery'. In such a view, psychology is 'nothing but' biology, which itself is 'nothing but' chemistry, which is 'nothing but' the physics of sub-atomic particles. Indeed, some go further and say that physics is 'nothing but' mathematics (Tegmark, 2007, 2008). This kind of reduction would seem to be quite different from Aristotle's essence seeking. Here, as the microscope zooms in closer and closer, an increasingly *limited* image appears, with important (perhaps the most important) factors missing. The statement, for instance, that the human brain is 'nothing but' atoms and molecules is no doubt true in the sense that if you took away the atoms and molecules there would be nothing left, but it is equally clear that it is false if it is taken to mean that the human brain has no properties other than those possessed by atoms and molecules. Consequently, complexity scientists have begun to acknowledge more explicitly the importance of *emergent properties* (Bedau, 2002), which arise gradually as complexity grows but cannot be identified with specific small-scale features. These are the properties that slip through reductionism's fingers.

In mathematics, the axioms of Euclidean geometry do not in themselves capture the entirety of the subject. Likewise, as Suppes (1978) comments, the attempt to reduce all of mathematics to set theory has:

> a kind of barren formality about it . . . Moreover, as we have reached for a deeper understanding of the foundations of mathematics we have come to realize that the foundations are not to be built on a bedrock of certainty but that, in many ways, developed parts of mathematics are much better understood than the foundations themselves. (Suppes, 1978, p. 8)

A reduced version does not have identical properties in kind to the less reduced version, simply to a different degree; it can be different in crucial, qualitative ways. As Davis and Sumara (2008, p. 49) put it, 'with certain phenomena, "more" is not simply "more," but "different"'.

Mathematical objects frequently possess emergent properties that are not features of any of their constituents. For instance, an equilateral triangle is composed of three equal sides, but the triangle has rotational symmetry of order 3, whereas none of the sides does. Similarly, Baas and Emmeche (1997, p. 9) ask 'in a knot – where is the knottedness? It is a global property, having no meaning locally. Or in a Moebius band – where is the *twist*?' You can go round and round looking for it, but it cannot be identified with any specific location. In arithmetic, divisibility by 4, for instance, is a property of the number 8, but if 8 is partitioned into 2 and 6, neither of those is divisible by 4 – the divisibility *emerges* when the 2 and the 6 are added (Vega et al., 2003). In this case a different reduction, to 4 and 4, *would* preserve the divisibility, so which properties are retained or lost in a reductive process depends on the details of the particular reduction.

An important area of modern mathematics for illuminating some of the possible problems of reductionism is fractal geometry. A key feature of a fractal is that it is self-similar, appearing equally detailed and complex at every scale – zooming in does not afford a simplification. As Davis and Sumara (2008) comment:

> Perhaps the most significant contribution of fractal geometry has been conceptual, as a source of new images and metaphors. Complex phenomena, it seems, are much more fractal-like than Euclidean. They are incompressible, recursively elaborated, often surprising. Further, fractal geometry presents a challenge to the pervasive assumption of linearity that has long been inscribed in analytic science. (Davis & Sumara, 2008, p. 45)

If the structure of mathematics itself has fractal qualities, then we cannot assume that a reduced version preserves the most important features and can be unproblematically scaled back up. Thoughtlessly reductive approaches are like peeling off the layers of an onion in an attempt to expose the core – by the time you have finished, there is nothing left

(Fuenmayor, 1991). Much of importance may be lost when mathematics is dismantled into its so-called fundamental parts.

4. Reasons for pedagogical reductionism

There are clearly considerable practical difficulties in converting the rich complexities of a discipline such as mathematics into a curriculum which can be accommodated within the artificial school experience of learning, where days are fragmented into discrete lessons of up to an hour or so. Yet mathematics teaching can become *excessively* fragmented beyond this. Ollerton (1994) condemns fragmented teaching where:

> for one or two lessons children are responding to a set of short questions in an exercise, such as ‘solve the following equations’, and then the following day or week they are working on another skill such as adding fractions or working out areas of triangles. (Ollerton, 1994, p. 63)

He likens the meaninglessness of such an experience to that described in Henry Reed’s poem ‘Naming of Parts’ (Ricks, 1999, p. 619). These comments echo the concerns of the Cockcroft report, which lamented that:

> Mathematics lessons in secondary schools are very often not about anything. You collect like terms, or learn the laws of indices, with no perception of why anyone needs to do such things. There is excessive preoccupation with a sequence of skills and quite inadequate opportunity to see the skills emerging from the solution of problems. (DES, 1982, para 462, p. 141)

The report urged teachers to ‘relate the content of the mathematics course to pupils’ experience of everyday life’ (DES, 1982, p. 142). The treatment of isolated skills risks locking students into a technical view of the subject, divorced from its applications. *Explanatory reductionism*, the idea that mathematics is best learned by beginning with ‘building blocks’ and assembling them into secondary structures, constitutes a misunderstanding of constructivism (Simon, 1995).

There is little doubt that UK mathematics teaching is still dominated by traditional *triple-X lessons*: ***ex***planation, ***ex***amples, ***ex***ercises (Swain & Swan, 2007), in which mathematics has been broken down into piecemeal facts and skills (Gates, 2006). The almost universality of the term ‘exercise books’ in UK mathematics classrooms, for instance, seems socially and psychologically programmed to constrain the kinds of activity that will take place, emphasising the routine repetition of procedures to develop facility over and above anything richer. ACME (2012) stresses the need for greater depth and richness in the UK mathematics curriculum, pointing to the lack of challenge currently offered to many of the highest achievers. According to the most recent Ofsted mathematics report (2012, p. 9), schools are aware of 'the need to improve pupils' problem-solving and investigative skills, but such activities were rarely integral to learning'. They report a continued emphasis on students completing short, closed exercises – the ultimate reductive approach.

I suggest below that there are two important and related reasons for the prevalence of pedagogical reductionism in mathematics education: the backwash effects of high-stakes assessments and a systemic de-professionalisation of teachers through a performative accountability culture.

4.1. Backwash effects of assessment

It is frequently stated that 'what you test is what you get, and . . . how you test is how it gets taught' (Taleporos, 2005). In the UK educational culture of high-stakes mathematics examinations consisting of short, closed questions, it would be surprising if teachers did not feel under pressure to form their teaching accordingly. For Watson (1999):

> Assessment both contributes to, and is partly formed by, the classroom culture as a whole. The mechanisms of assessment reflect what is valued by teachers and others, explicate such values, bestow status and also shape classroom activities so that valued behaviour is generated. (Watson, 1999, p. 106)

An assessment system that focuses on bite-sized pieces of mathematics, because they are quick and easy to test and score, is bound to encourage such reductive practices in the classroom. In their most recent report, Ofsted (2012, p. 18) generally observed few 'lessons that were helping pupils to gain a better understanding of mathematics', as opposed to those with 'a strong focus in teaching to the next examination'.

Tarrant (2000, p. 78) summarises epistemological, ethical and political objections to the 'competency model', criticising 'a crude behaviourism that equates knowing with a performance' and stressing the crucial nature of 'underpinning knowledge'. For him, it is naive to assume knowledge on the evidence of behaviour: 'For example, the student may have connected the correct wires [in a practical electrical problem] on the basis of a lucky guess, or because he or she was told which wires to connect by a course member in the corridor'. Tarrant (2000, p. 79) argues that, 'the tendency to bifurcate knowledge of how to do something and knowledge that something is the case arises when too many examples of very basic practical skills are selected for analysis'. Knowledge and skills must be seen as a unity, otherwise understanding is only instrumental (Skemp, 1976) and is developed only to the point where the particular action can be carried out correctly. In this way, mathematical knowledge becomes the means to a performative end and not the focus of the learning process. One example of the sort of difficulties that arise would be the attempt to itemise the various forms of knowledge required for teaching mathematics, so as to test teaching ability (Hill, Schilling, & Ball, 2004).

4.2. De-professionalisation of teachers

Intertwined with an increasingly assessment-heavy curriculum is a systemic political de-professionalisation of teachers. Beck (2008) describes the 'coercive accountability' which is fundamental to this de-professionalisation:

> [T]he now dominant common-sense, after three decades of intensifying government prescription, audit and managerialism, is one which takes the prioritising of instrumental purposes in education for granted, and which has no time for conceptions of accountability except those that require teachers to 'measure up' to externally imposed performativity demands. (Beck, 2008, p. 10)

In such an 'audit culture', teachers are constantly required, just in order to survive, to prove to their schools that they are 'effective'. Often this is to non-specialist school managers who may have little knowledge of mathematics-specific pedagogy beyond that inferred from experiences in their own schooldays. Thus, whenever the teacher's lesson is observed – perhaps at no or very little notice – there must be obvious markers of students' 'progress' during the lesson. The senior teacher who 'pops in' might have time only for a 15-minute visit, in which they will make judgements that can have serious consequences for the teacher. So an understandable defensive strategy for the teacher against these intrusions is to break up the lesson into episodes of no longer than 15 minutes, during each of which some superficial public student assessment takes place, which no observer can fail to miss, and which highlights what students have achieved during this period. It is hard to see how such a highly oppressive constraint can fail to disconnect the students' learning of mathematics into fragmented 'moments' in which some technical fact or skill is acquired. Unless trust can be rediscovered, many of the recommendations of mathematics education research will simply be ruled out by teachers inhabiting this system as just too risky.

5. Consequences of pedagogical reductionism

In this section, I draw attention to two consequences of pedagogical reductionism in mathematics education: the prioritising of short-term easily measured learning objectives in the classroom and teachers' desire to path-smooth their students' learning.

5.1. Micro-learning objectives agenda

One very common feature of lessons in UK schools is a strong emphasis on highly specific learning objectives for each lesson, which students frequently spend the beginning of the lesson copying into their books. Noyes (2007, p. 125) comments that, 'Writing the Learning Objective on a little whiteboard, normally about 1 metre to the side of the larger whiteboard, is de rigueur – like a talisman that somehow ensures quality learning.' This narrow vision of what constitutes mathematical learning leads to a focus on small units of 'competence'; in the extreme this is reduced to binary yes/no measures of performance and proficiency, diminishing the student to the status of a technician (Talbot, 2004). Thus mathematics is diminished to a minimalist set of procedural skills to be mastered sequentially. Where students have difficulties in achieving a particular micro-objective, these are diagnosed as pathologies and extra practice may be prescribed as a 'magic bullet' to tackle that one specific deficient area (Prestage & Perks, 2006).

The frequent institutional requirement to specify objectives beforehand, and communicate them to students, can discourage teachers from using rich, open tasks, where the outcome for any particular learner will be less certain. Moreover, when objectives are tightly specified, the possibility arises of achieving them in ways that were not anticipated by the teacher: the widespread phenomenon of 'hitting the target but missing the point' in education has been commented on extensively (Foster, 2006). Mason (2000, p. 97) describes the dilemma at the heart of the 'didactic contract' as: 'the more clearly the teacher indicates the behaviour sought, the easier it is for students to display that behaviour without generating it from understanding'. In such circumstances, all sense of *why* students are being asked to complete particular tasks may be lost. Gray (2002) even goes so far as to suggest that self-awareness of goals can be counterproductive:

> Self-awareness is as much a disability as a power. The most accomplished pianist is not the one who is most aware of her movements when she plays. … Very often we are at our most skilful when we are least self-aware. … In Japan, archers are taught that they will hit the target only when they no longer think of it – or themselves. The meditative states that have long been cultivated in Eastern traditions are often described as techniques for heightening consciousness. In fact they are ways of bypassing it … Subliminal perception – perception that occurs without conscious awareness – is not an anomaly but the norm. (Gray, 2002, pp. 62–63)

Skilful action does not always benefit from being conscious of its aims (Atkinson & Claxton, 2003).

It is not necessary, or perhaps even possible, for students to know in detail what they are going to learn before they start; such attempted predictions may simply be a distraction. In a setting where significant and complex learning is taking place, trying to specify the details of what will happen in advance will unduly constrain the learning process for both student and teacher. Rodd (1995, p. 238) remarks that 'the school environment tends to reinforce an "atomistic" view as teachers are required to "deliver items" of the National Curriculum'. Prestage and Perks (2006, p. 69) have reported one consequence of this as being an excessive amount of 'practising the finished product'. A bitty, fragmented curriculum is more convenient to administer and test and easier for non-specialists to 'deliver'. It may also correspond to 'safe' common-sense notions of 'keep it simple', doing one thing at a time (Adamson, 2006).

A topic such as trigonometry can be seen to embrace many diverse aspects of elementary mathematics: angles, approximation, enlargement, equations, measurement, proportion, ratio, etc. But it is much more than simply the sum total of those things. Attempting to tick off each of them in isolation will not give students an experience of trigonometry. As Mason (2010, p. 32) comments, 'a succession of experiences does not add up to an experience of that succession ... Just because I engage in mathematical activity, it does not follow that I am aware of the activity itself as a whole.' Reductionism leads to 'topics' such as 'collecting terms' or learning the 'order of operations', which can be difficult to justify as worthwhile objects in their own right. These become isolated 'tricks', which students are required to perform on demand merely to satisfy a teacher or examiner. Consequently, students do not sufficiently appreciate mathematical connections and coherence (Steinbring, 1991).

In the teaching of physical education, for instance, very careful judgements may be made regarding whether to train a behaviour by 'breaking it down' into sub-elements or whether to persevere working with the whole movement (Coker, 2006). It may sometimes be helpful to 'go backwards in order to go forwards' and perhaps to invite a learner who, for instance, is apparently working well on fairly advanced mathematics to take some time to review procedures concerning a more 'basic' topic such as algebraic fractions in order to support their current learning. However, determining whether identified areas of weakness are better addressed as incidental elements of large-scale tasks or by focusing entirely on them is a matter for the teacher's wise judgement. There would seem to be much potential for similar research and discussion in mathematics education. Currently, there is too great a tendency to assume that if a student can be trained to do the 'component parts', they will automatically, inevitably be successful at 'the whole'. The familiar experience of a young child who spells out 'c...a...t' phonetically but who, despite repeating the separate sounds, cannot merge them fluently into the word 'cat' (Smith, 1992) might be seen as a simple example where competent performance at the micro-level does not guarantee larger-scale proficiency. Too often, developing mathematical fluency on particular technical skills is seen as a prerequisite for tackling more complex problems. Instead, rich problems might provide more motivating and meaningful contexts for developing the desired fluencies (Foster, 2013a).

Ariely (2010) laments some aspects of the 'division of labour' prevalent in modern society:

> Modern IT infrastructure allows us to break projects into very small, discrete parts and assign each person to do only one of the many parts. In so doing, companies run the risk of taking away employees' sense of the big picture, purpose, and sense of completion. (Ariely, 2010, p. 79)

It would seem that much the same thing can happen on an individual basis, when a student loses sight of the bigger *mathematical* picture through being given tiny one-step procedures to perform, perhaps leading to some *self-alienation* from the content (Boaler, William, & Zevenbergen, 2000): students simply do not know why they are doing what they are doing (Duffin & Simpson, 1993).

5.2. *Path-smoothing of learning*

A second consequence of pedagogical reductionism is the tendency for teachers to attempt to micromanage the details of the students' learning; in particular, to make the path 'smooth' (Wigley, 1992). Baas and Emmeche (1997) point to *emergent phenomena* within the subjective experience of doing mathematics:

> [I]t is a common experience of mathematicians to have sudden 'flashes of insights' . . . The sudden 'flash' is experienced as the appearance of the solution as a new structure, that in a sense can be observed at once, but nevertheless has to be worked out deductively in great detail and tested formally before it can be trusted. (Baas & Emmeche, 1997, p. 16)

The discontinuous nature of such accounts stands in stark contrast to the supposedly gradual, accumulative way in which mathematics is commonly thought to be learned. It follows that when students have difficulties with mathematics, these will not necessarily be easily located in one 'step' or idea; they may be more global in nature. When a teacher tries to help students understand some mathematics which they 'don't get', one often hears the teacher ask: 'Which *step* is it that you don't understand?' Yet the problem that the student is having may be less easily pinned down than that.

Having mathematics broken down for you is a very different experience from doing it for yourself. When faced with a student who is stuck, mathematics teachers may hope that by breaking down the problem they may help the student to develop the ability to do this for themselves. But Ainley (1995, p. 10) comments that although this approach 'is deeply rooted in the pedagogic tradition of mathematics education ... [i]t is less clear that it is an effective technique for supporting children's mathematical learning'. For reduction to be a powerful mathematical process, it must increasingly reside under the student's control rather than be imposed from outside. If reducing mathematical problems for students is a helpful scaffold, then it must fade (Collins, Brown, & Holum, 1991).

Henningsen and Stein (1997) found that teachers sometimes reduced the complexity of high-level mathematics tasks when students appeared to struggle and, in the context of working with student teachers, Rodd (1995) describes what she calls the 'holistic-atomistic' tension as a need to be:

> mindful that pressure to 'break things down' so that the pupils only have to understand a small piece of new mathematics at a time, does not result in a fragmentary conception of mathematics for the pupil. (Rodd, 1995, p. 237)

When teachers *path-smooth* the process, they assist the student in solving that particular problem on that particular occasion, but may be hampering the student's more long-term development (Wigley, 1992). Students who find mathematics bewildering may be appreciative of teachers who appear to simplify the subject for them by offering bite-sized pieces in a palatable order. For example, Corbett and Wilson (2002, p. 20) report a teenage student who praised teachers who 'feed it into our head real good; they do it step-by-step and they break it down'. Such encouragement confirms the teacher in their behaviour, and a mutual dependency 'enabling' relationship may be set up, to the detriment of mathematical learning for the student and developing pedagogy for the teacher (D'Errico, Leone, & Poggi, 2010). Holt (1990, p. 199) warns against thinking that, 'guiding children to answers by carefully chosen leading questions is in any important respect different from just telling them the answers in the first place'. Employing 'funnelling questions' (Bauersfeld, 1995) may result in the student 'going through the motions' but not leave them empowered to tackle future problems, even of a very similar type.

In such circumstances, it is easy for a mathematics teacher to underestimate the significance of their intervention. A teacher assisting a student to solve a mathematical problem may believe that in separating out the distinct elements and itemising them they are simply keeping track of what is going on while the student does the *real* work of dealing with each part. However, the student may find themselves unable to proceed without such help, and this may perplex them both. For example, the exchange below was observed in a UK secondary mathematics classroom, following a teacher giving some one-to-one help:

Student	I can do it when you're with me but I can't do it by myself!
Teacher	Of course you can! You don't need me. I wasn't really doing anything – *you* did all the maths!
Student	I only know how to do it if you tell me what to do.

Here, the teacher perceives the challenge solely in terms of the individual mathematical steps, which the student has carried out largely unaided ('*you* did all the maths'), whereas the student's inability to decompose the problem into appropriate units or identify their nature prevents them from making progress alone. Perhaps because the teacher's prior teaching has tended to focus on the isolated mathematical techniques, she now assumes that one of these must be where the student is having difficulty. Or the teacher may suppose that the problem is more psychological than mathematical, and that the student merely wants a reassuring 'crutch' by their side. It does not seem to occur to the teacher that the difficulty could be more global and to do with the student's problem in managing large-scale aspects, such as breaking down the task for themselves. When understanding a worked example, more important than 'knowing the next step' may be how the student knew to *make* that the next step (Atkinson, Derry, Renkl, & Wortham, 2000) – having it 'come to mind' (Mason & Spence, 1999).

The term *zone of proximal development* (ZPD) (Vygotsky, 1978) has sometimes been misused to defend teachers forcing their approaches onto students and pushing them into mathematical actions that they would never do if left to themselves. This is *not* what Bruner (1986) meant by 'consciousness for two'. Levykh (2008) comments that:

> the dynamic process of establishing and maintaining the ZPD is successful only when emotionally laden reciprocal relations between the learner and the instructor allow for participants' comfort and trust, which are manifested in constant negotiation of the subject of inquiry and the way it is presented and acquired. (Levykh, 2008, p. 97)

What Vygotsky seems to have meant by the ZPD is the student's progression from being able to do something *with* help to being able to do it *unaided* (van der Veer & Valsiner, 1991), which has since been further conceptualised as 'scaffolding with fading' (Collins et al., 1991). The principle 'only do for people what they cannot yet do for themselves' (Mason, 2000, p. 101), though difficult to apply, remains a worthy goal. An invalid may appreciate having their food cut up for them, but to do this for a normally able child would risk severely de-skilling them and inculcating a 'learned helplessness' (Peterson, Maier, & Seligman, 1995).

Path-smoothing approaches are seen as particularly important for students who are socially constructed by teachers and school managers as 'low achievers'. In the context of advice for dealing with 'slow students', for instance, Hoffman (1968, p. 89) advises helping 'students to analyze an algorithm or a process of computation and break it down into its basic parts'. But, particularly in special educational needs teaching, it is necessary to eschew quick fixes and stop-gap measures; frequently, short-term resolutions (such as getting the student as painlessly as possible through one particular mathematical problem) are in conflict with longer-term benefits (Watson, 2006b). Watson, De Geest, and Prestage (2003, p. 37) found that successful teachers of students socially constructed as 'low-attaining' 'did not simplify mathematics for students, nor did they fudge difficult issues. Instead they saw their job as helping students learn mathematics, with all its complexities.' It is easy for teachers unwittingly to over-help students from the best of intentions (D'Errico et al., 2010). Some teacher interventions are limiting, constraining, preventing student action, closing down possibilities, removing complexity, reducing challenge and making problems easier. The move towards increasingly highly structured examination questions in post-16 mathematics assessment has been an attempt to improve accessibility and cater for a wider spread of attainment, but is thought by many to have reduced the demands and led to 'dumbing down' (Kounine, Marks, & Truss, 2008). Schoenfeld (1988) describes students who gave up when they could not solve a problem quickly, because they assumed that this meant that they must have misunderstood something and were therefore doomed to fail.

6. A more holistic approach

The dominance of a reductionist pedagogical paradigm is increasingly being challenged by the rise of complexity thinking, which advocates a more holistic viewpoint (Davis & Simmt, 2003). Davis and Sumara (2008, p. xi) regard this as offering, 'a powerful alternative to the linear, reductionist approaches to inquiry that have dominated the sciences for half a millennium – and educational research for more than a century'. We know that mathematics is not learned in a linear, unidirectional, ladder-like fashion (Denvir & Brown, 1986). The Pirie-Kieren model of growth of understanding, where students move back and forth between onion-like layers of thinking, is much more realistic (Kieren, Pirie, & Gordon Calvert, 1999), and a helpful development of Bruner's notion of spiral learning. So given what we know it seems completely inappropriate to take a reductionist approach to mathematical pedagogy.

If learning is nonlinear, then pedagogy must be also. According to Strogatz (2004, p. 182), the 'synergistic character of nonlinear systems is precisely what makes them so difficult to analyze. They can't be taken apart. The whole system has to be examined all at once, as a coherent entity.' A *systems approach* to pedagogy would reflect developments in social science research, where 'sterile' laboratory conditions do not always lead to the simplest or most useful outcomes, and the complexity of real-life situations is increasingly embraced (Cohen, Manion, & Morrison, 2011). Davis and Sumara (2008, p. 53) propose reframing, 'the architecture of mathematics ... in terms of a nested, scale-free network ... [in order to] prompt attentions away from assumptions of universal basics and linear progress toward notions of highly connected ideas/nodes and neighborhoods of ideas'. Working more holistically in the mathematics classroom means to some extent relinquishing teacher control ('teacher lust') over micromanaging every detail (Boole, 1931; Tyminski, 2010). It also entails a classroom focused on longer timescales. Ollerton (1994, p. 63) describes holistic mathematics teaching as, 'planning modules of work that can be sustained for two, three or four weeks, rather than in deconstructed and fragmented ways'. Watson et al. (2003, p. 23) found that:

> Rather than rushing through topics, [effective] teachers gave extended time for learning . . . Some teachers extended a single topic over several weeks, in order to use many different representations, to ensure progression in the topic . . . Deep progress was ensured by such extended tasks.

Ausubel distinguishes between *rote learning*, where new information is uncritically accumulated in the memory and *meaningful learning,* where new ideas are analytically evaluated and integrated into what the student already knows (Novak, 2002). Viewed in this way, learning cannot be meaningful if ideas are encountered only one at a time. E. M. Forster said that 'Only what is seen sideways sinks deep' (Forster & Gardner, 1985), and it would seem that tackling multiple curriculum areas simultaneously gives students the opportunity to examine familiar and unfamiliar material from various angles, potentially leading to new insights. Bell (1993, p. 7) advocates that mathematics teaching should be designed so that, 'the pupils' main lesson experience should be of genuine and substantial mathematical activities, which bring into play general mathematical strategies such as abstracting, representing, symbolizing, generalizing, proving, and formulating new questions'. In a related way, Sfard (1998) contrasts metaphors of *acquisition* and of *participation*, the latter being a more active and persistent process involving deeper engagement with subject content and with other people in the learning process. All of this is the very antithesis to a reductive approach.

Features of working more holistically could include:

- giving students richer, more complex mathematical problems with a deeper degree of challenge, so that solutions are not straightforward or obvious;
- deliberately using problems which simultaneously call on a range of different areas of the curriculum, encouraging students to 'see sideways' and make connections;

- using ‘open’ tasks, where students can exercise a significant degree of choice about how they define the task and how they approach it – importantly, the teacher does not have one fixed outcome in mind;
- giving students sufficient time to explore different pathways without the pressure to arrive at ‘an answer’ quickly;
- encouraging a view that being stuck or confused and not knowing what to do is normal and can be productive, that ambiguities can be beneficial for a time (Foster, 2011a), and that seeking not to ‘move students on’ too quickly can deepen their opportunities to learn (Dweck, 2000).

Various endeavours in mathematical task design have attempted to meet some or all of these criteria. The Unified Science and Mathematics for Elementary Schools (USMES) project in the USA in the 1970s aimed to integrate science and mathematics education. In their evaluation, Shann, Reali, Bender, Aiello, and Hench (1975, p. 10) found that ‘there were indications that children felt capable of dealing with their environment, and that teachers, through less directive teaching, were encouraging children to solve their own problems’. In the UK in the 1980s, the establishment by the Association of Teachers of Mathematics of a ‘100% coursework’ mathematics qualification (SEG, 1988) embodied many desirable features of student-centred holistic mathematics learning (Ollerton & Watson, 2007; Watson, 2006a). The Centre for Research in Mathematics Education at the University of Nottingham (also known as the Shell Centre) has a long tradition of rich, innovative task design in which more holistic principles have been adopted (Swan, 2006), and the *Bowland Maths* project has led to extended tasks addressing a multitude of curriculum areas, which have proved highly popular (Thompson, 2011). There are also indications that some of the newer ‘Use of Mathematics’ Advanced Subsidiary (AS) specifications are leading to resources that are much richer and better at encouraging deep mathematical thinking.

One example of a more holistic mathematics task, suitable for students aged 11–14, begins with students being invited to imagine receiving a large sum of money, such as UK £1 million (Foster, 2013b). After discussing what they might do with it, the students are told that the money is going to be provided in £1 coins, and they are invited to pose mathematical questions. Typical questions involve asking whether all the coins would fit in their classroom or bedroom, how many trips a student would have to make to get them home (or to the bank) one backpack at a time or what kind of vehicle would be most suitable and how many trips it would have to make. Details of the mass and size of the coins can be provided or determined experimentally by the students. Students sometimes pose additional questions such as how much their own body weight in pound coins or in gold would be worth or how big a gold coin (not necessarily cylindrical) would have to be in order to be worth £1 (Foster, 2005, 2011b, 2013b). This task involves students in making choices, being creative, asking 'What if?' questions about the scenario given, and bringing to bear their knowledge from topics such as circles, volume/density/mass, scale factors, estimation, rounding, units, large numbers and even trigonometry. More confident students can ask and answer more demanding questions drawing on more sophisticated knowledge and skills.

Another example of a holistic mathematical task involves the teacher producing a cup of lolly sticks, each stick with a different student's name written on it, one for each member of the class (Foster, 2013c). The teacher begins by asking the students questions about the use of the lolly sticks, choosing the student to answer each time by removing a lolly stick and reading the name. Eventually the teacher poses the question, 'When I pick out a lolly stick and someone answers a question, do you think I should put the stick back in the cup or not? Why?' (Foster, 2013c). This question provides students with the opportunity to engage with the teacher's possible purposes in using the lolly sticks for asking questions as well as with the nature of probability. Eventually, students are asked to suppose that the sticks *are* replaced each time and are invited to work out how many lessons it will take, on average, before everyone has answered at least one question. Students can explore this experimentally, perhaps with different groups of students working with different total numbers of lolly sticks and then comparing their results. They can also examine the problem theoretically (Foster, 2013c). This task, based on the 'coupon-collecting problem', naturally extends across several lessons. Such rich tasks as the two described here seek to encourage a deeper, more connected understanding of mathematics and give students much greater scope to use their mathematical powers.

7. Conclusion

Reductionist approaches have an important role to play for the mathematician when seeking to simplify a complex problem, yet their prevalence in mathematics pedagogy is highly limiting for the student. Raman (2005, p. 252) suggests that 'Reductionism and holism are like the microscope and the telescope' – different but equally important tools. In a related way, Mason (2003) refers to two different forms, states or structures of attention as 'holding the whole' and 'discerning details'. For fluent mathematical thinking, students need to be able to move comfortably between them. However, the overriding emphasis currently in the UK seems to be on a reduction that is done *for* the student, rather than under their own control, and which consequently is disempowering rather than beneficial.

Currently, many mathematics lessons have very meagre intentions. Feynman (2005) captures the problem well in the context of arithmetic:

> What we have been doing in the past is teaching just one fixed way to do arithmetic problems, instead of teaching flexibility of mind – the various possible ways of writing down a problem, the possible ways of thinking about it, and the possible ways of getting at the problem. (Feynman, 2005, p. 448)

He describes such a flexible 'attitude of mind' as belonging both to users of mathematics and to truly creative pure mathematicians. Because it is not obvious in final proofs, which are polished conclusions, it is easily overlooked. For Feynman (2005, p. 448), 'mathematical thinking ... is a free, intuitive business'.

With 'lossy' data compression in information technology, the amount of data that needs to be kept is reduced by discarding some – often quite a large amount (Sayood, 2000) – so that it is impossible to recover the entirety of the original. If, as I have argued here, it is the case that mathematics too is not divisible without loss, then the common practice of reducing the curriculum into component pieces risks throwing away the very connections that are fundamental to the subject. We know that teachers who stress mathematical connections are more effective (Askew, Brown, Rhodes, Johnson, & Wiliam, 1997). Taking a holistic approach to the learning of mathematics has the potential to build on students' innate mathematical powers and lead to an experience in classrooms that is far more authentically mathematical.

Attempts at holistic mathematics education will always come with the feeling of swimming against the tide of more traditional approaches (Wright, 2012). Operating in more meaningful ways may take longer, be less predictable and can initially be uncomfortable – frightening, even – for students and teachers who are not used to facing complex mathematics and being unsure and stuck for extended periods of time. Skovsmose (2011, p. 48) regards the 'exercise paradigm' as contributing to predictability and providing a comfort zone for both the teacher and the students, yet advocates entering instead a 'risk zone', commenting that 'Dealing with risk also means creating new possibilities'. It is essential that mathematics educators continue to combat reductionist pedagogical pressures so that students can experience real mathematics in all its wonder and complexity.

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