# THESIS SUMMARY

MILTON LIN

## Introduction

My thesis explores multiple aspects of the Geometric Langlands program. Within the mixed characteristic setting, I studied a geometric Casselman–Shalika formula Section 1. In Fargues-Scholze's geometrization of the local Langlands program, I analyzed period and $L$-sheaves Section 2. Finally, I apply insights from these studies to the categorical deformations of representation categories Section 3.

Each section is marked with the project's completion percentage and anticipated completion date. We freely utilize the language of higher categories and higher algebra, as in [Lur09] and [Lur18].

## 1. Mixed Characteristic Geometry and the Casselman–Shalika formula

Let $G$ be a connected reductive group over a nonarchimedian local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld [AG15], [Gai14]. For a fixed choice of Whittaker datum, [FS24, I.10.2] conjectured a $\mathrm{IndPerf}(\mathrm{LS}_{\hat{G}})$-linear equivalence

$$\mathbb{L}_G : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}})$$

which can be thought of as a generalization of the classical local Langlands. One fundamental aspect of the program is to understand the *Whittaker Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, this corresponds to finding the Fourier coefficients of automorphic functions, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which we aim to replicate in the mixed characteristic setting. In joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou [ILZ24], we proved a variation of this problem: the *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian $\mathrm{Gr}_G$, analogous to the equi-characteristic geometrization carried out in [NP01].

**Theorem 1.1.** *[ILZ24] The geometric Casselman–Shalika formula holds over the Witt vector Grassmanian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori–Satake equivalence, [Bez+19] without using nearby cycles.

**Theorem 1.2** (I.-Lin-Z., **in progress 70% , done by March 2025**). *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [1]

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[1] In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a simpler categorical structure.

## 2. Relative langlands on the Fargues Fontaine curve

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and $L$-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: $(G, X)$ with $(\hat{G}, \hat{X})$ of hyper-spherical varieties. Let $\Lambda = \bar{\mathbb{Q}}_l$. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_!\Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \pi_*\omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. $\mathrm{Bun}_G$ has a Hardar-Narasimhan straification by locally closed substacks $\mathrm{Bun}_G^b$ indexed by the Kottwitz set $B(G)$. In our case, $G = \mathbb{G}_m$, $\mathrm{Bun}_{\mathbb{G}_m}$ is stratified by $\mathrm{Bun}_{\mathbb{G}_m}^n$ for $n \in \mathbb{Z} = B(T)$. Interesting phenomena occurs for $n > 0$, and the study of period sheaves reduces to the study of $\mathrm{Bun}_G^X$ restricted to $\mathrm{Bun}_G^n$. This corresponds to the Abel-Jacobi map previously studied by Fargues [Far20] and Hansen [Han21]. On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles, [FW24, Ch. 6] recently developed by Le-Bras et al.

**Conjecture 2.1.** [Lin-T., **in progress 20% , done by May 2025**] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[2] $\mathcal{L}_{\hat{X}}$.

## 3. Applications to categorical deformations

This builds upon my current research on the Whittaker category, from the point of view of deformation theory. We will first document a careful proof (**in progress, 50%, done by March 2025**) of Lurie's theorem, [Lur10, Thm 10.10], which describe *formal deformation of categories*, as gerbes see [Lur10, Ch.8-10] for definitions. We hope this will help explore deformations of representation of Lusztig's small quantum group, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.

[2] One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on $X$ which we do not discuss.

## References


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