Introduction to the Gan-Gross-Prasad and Ichino-Ikeda conjectures II

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Global base-change for unitary groups

- K/F quad ext of $\#$ fields, $\mathrm{Gal}(K/F) = \{1, c\}$, (V, h) : n-diml Herm sp over K, then ${}^L U(V) = \mathrm{GL}_n(\mathbb{C}) \rtimes Gal(K/F)$, $cgc^{-1} = J^t g^{-1} J^{-1}$.
- We have $U(V)_K \simeq \mathrm{GL}_{n,K}$ and ${}^L\mathrm{GL}_{n,K} = \mathrm{GL}_n(\mathbb{C})^2 \rtimes Gal(K/F)$ (seen as a gp $/F$) where $c(g_1, g_2)c^{-1} = (g_2, g_1)$.
- Base-change homomorphism : $BC : {}^L U(V) \to {}^L\mathrm{GL}_{n,K}$, $(g, \sigma) \mapsto (g, J^t g^{-1} J^{-1}, \sigma)$.
- Mok, Kaletha-Minguez-Shin-White : for $\pi \hookrightarrow \mathcal{A}_{cusp}([U(V)])$ (where $[U(V)] := U(V)(F)\backslash U(V)(\mathbb{A})$) there exists an automorphic irreducible representation π_K of $\mathrm{GL}_n(\mathbb{A}_K)$ st $\mathcal{L}(\pi_{K,v}) = BC(\mathcal{L}(\pi_v))$ for a.a. places v of F.
- What is the form of π_K ?

Asai L-functions

- Define $\mathrm{As}^{\pm} : {}^L\mathrm{GL}_{n,K} = \mathrm{GL}_n(\mathbb{C})^2 \rtimes Gal(K/F) \to \mathrm{GL}(\mathbb{C}^n \otimes \mathbb{C}^n)$ by
$$\mathrm{As}^{\pm}(g_1, g_2) = g_1 \otimes g_2 \text{ and } \mathrm{As}^{\pm}(c) = \pm\iota$$
where $\iota(v \otimes w) = w \otimes v$.
- The image of $BC : {}^LU(V) \to {}^L\mathrm{GL}_{n,K}$ is the stabilizer of a 'generic' vector for $\mathrm{As}^{(-1)^{n+1}}$.
- Shahidi/Flicker-Rallis : for $\Pi \hookrightarrow \mathcal{A}_{cusp}([\mathrm{GL}_{n,K}])$, the L-functions $L(s, \Pi, \mathrm{As}^+)$ and $L(s, \Pi, \mathrm{As}^-)$ have meromorphic continuations with a possible pole at $s = 1$ iff $\Pi^\vee \simeq \Pi^\sigma$ (i.e. Π is *conjugate self-dual*) in which case exactly one of them has a simple pole at $s = 1$ and the other does not vanish.
- When Π is conjugate self-dual we say it is of the sign $\varepsilon \in \{\pm\}$ if $L(1, \Pi, \mathrm{As}^{\varepsilon}) = \infty$.

Image of base-change

- Isobaric sum : if $\Pi_i \hookrightarrow \mathcal{A}_{cusp}([\mathrm{GL}_{n_i}])$, $1 \leqslant i \leqslant k$, are unitary then

$$\Pi_1 \boxplus \ldots \boxplus \Pi_k = \mathrm{Ind}_{\mathrm{GL}_{n_1} \times \ldots \times \mathrm{GL}_{n_k}}^{\mathrm{GL}_n} (\Pi_1 \otimes \ldots \otimes \Pi_k)$$

  is an irred autom repn of GL_n where $n = n_1 + \ldots + n_k$.
- Mok, Kaletha-Minguez-Shin-White : let $\pi \hookrightarrow \mathcal{A}_{cusp}([U(V)])$, if π_K is generic then it is of the form

$$\pi_K = \Pi_1 \boxplus \ldots \boxplus \Pi_k$$

  where $\Pi_i \hookrightarrow \mathcal{A}_{cusp}([\mathrm{GL}_{n_i,K}])$ are distinct, conjugate self-dual and of sign $(-1)^{n+1}$ (i.e. $L(1, \Pi_i, \mathrm{As}^{(-1)^{n+1}}) = \infty$).
- This decomposition of π_K is unique (up to reordering) and, conversely, any isobaric sum of this form is the base-change of some $\pi \hookrightarrow \mathcal{A}_{cusp}([U(V)])$ for *some* unitary group $U(V)$.
- All of this is to be expected from the Langlands-Tannakian formalism. We set

$$S_\pi := (\mathbb{Z}/2\mathbb{Z})^k$$

  as a substitute for the centralizer of the (non-existing) "global Langlands parameter of π".

Local periods for unitary groups

- Setting : (V,h) an n-diml Hermitian sp over K, $V' = V \oplus^{\perp} Ke$ with $h(e,e)=1$, we set $H := U(V) \hookrightarrow G := U(V) \times U(V')$ (diagonal embedding) ;
- Let $\pi = \pi_n \boxtimes \pi_{n+1} \hookrightarrow \mathcal{A}_{cusp}([G])$, the global GGP period is defined by

$$\mathcal{P}_H : \phi \in \pi \mapsto \int_{[H]} \phi(h) d_{Tam} h$$

  where the measure on $[H]$ is the Tamagawa measure.
- Fix factorizations $d_{Tam}h = \prod_v dh_v$ and $\langle .,. \rangle_{Pet} = \prod_v \langle .,. \rangle_v$ where $\langle \phi, \phi \rangle_{Pet} = \int_{[G]} |\phi(g)|^2 d_{Tam} g$.
- Let v a place of F. Assume first π_v *tempered* (a condition on the decay of matrix coefficients of π_v), then we define the *unnormalized* local period

$$\mathcal{P}^{\natural}_{H_v} : (\phi_v, \varphi_v) \in \pi_v \times \pi_v \mapsto \int_{H_v} \langle \pi_v(h_v)\phi_v, \varphi_v \rangle_v dh_v.$$

- Unramified computation (N. Harris) : when G_v, π_v are unr and $\phi_v \in \pi_v^{G(\mathcal{O}_v)}$,

$$\mathcal{P}^{\natural}_{H_v}(\phi_v, \phi_v) = \Delta_v \frac{L(\frac{1}{2}, \pi_{v,K})}{L(1, \pi_v, \mathrm{Ad})} \mathrm{vol}(H(\mathcal{O}_v)) \langle \phi_v, \phi_v \rangle_v$$

  where $L(s, \pi_{v,K}) = L(s, (\pi_{n,K})_v \times (\pi_{n+1,K})_v)$ (local RS L-function) and $\Delta_v = \prod_{k=1}^{n+1} L(k, \eta^k_{K_v/F_v})$ with $\eta_{K_v/F_v} : F_v^\times \to F_v^\times / N(K_v^\times) \subset \{\pm 1\}$.

- For every v, when π_v is tempered, we define a (normalized) local period by

$$\mathcal{P}_{H_v}(\phi_v, \varphi_v) := \Delta_v^{-1} \frac{L(\frac{1}{2}, \pi_{v,K})^{-1}}{L(1, \pi_v, \mathrm{Ad})^{-1}} \mathcal{P}^{\natural}_{H_v}(\phi_v, \varphi_v), \ \phi_v, \varphi_v \in \pi_v$$

  so that if $\phi = \bigotimes'_v \phi_v \in \pi$ then $\mathcal{P}_{H_v}(\phi_v, \phi_v) = 1$ for a.a. v.
- If π_K is generic we expect π_v to be tempered for all v (G^{ed} Ramanujan conj) but it is not known in general. However, we can show that the local periods extend "analytically" to the local components of π.

Conjecture (Ichino-Ikeda, N.Harris)

Assume that π_K is generic. Then, for every $\phi = \otimes'_v \phi_v \in \pi$ we have

$$|\mathcal{P}_H(\phi)|^2 = \frac{\Delta}{|S_\pi|} \frac{L(\frac{1}{2}, \pi_K)}{L(1, \pi, \mathrm{Ad})} \prod_v \mathcal{P}_{H_v}(\phi_v, \phi_v)$$

where as before $L(s, \pi_K) = L(s, \pi_{n,K} \times \pi_{n+1,K})$, $\Delta = \prod_{k=1}^{n+1} L(k, \eta^k_{K/F})$ and S_π is "the centralizer of the Langlands parameter of π".

- Informally (i.e. decomposing L-fns as Euler products outside the range of convergence), the formula can be written as just

$$|\mathcal{P}_H(\phi)|^2 = |S_\pi|^{-1} \prod_v{}' \mathcal{P}^{\natural}_{H_v}(\phi_v, \phi_v).$$

- Both sides of the Ichino-Ikeda formula define $H(\mathbb{A})$-invt sesquilinear forms on π. By general multiplicity one results (Aizenbud-Gourevitch-Rallis-Schiffmann, Sun-Zhu) the space of $H(\mathbb{A})$-invt sesquilinear forms is at most 1-diml hence both sides are proportional to each other.
- More precisely, the I-I conj says that the proportionality constant between the globally defined $H(\mathbb{A})$-invt form $(\phi,\varphi) \mapsto \mathcal{P}_H(\phi)\overline{\mathcal{P}_H(\varphi)}$ and the locally defined one $(\phi,\varphi) \mapsto \prod_v' \mathcal{P}^\natural_{H_v}(\phi_v,\varphi_v)$ (regularized as before) depends only mildly on π (it is given by $|S_\pi|^{-1}$).

Relation to the GGP conjecture and non-vanishing of local periods

- We now recall the global GGP conj : for V_0 another Herm space of dim n, we set $H^{V_0} = U(V_0) \hookrightarrow G^{V_0} = U(V_0) \times U(V_0')$ where $V_0' = V_0 \oplus^{\perp} K.e$.

Conjecture (Gan-Gross-Prasad)

Assume that π_K is generic. The following are equivalent :

1. $L(\frac{1}{2}, \pi_K) \neq 0$ *;*
2. *There exist a n-diml Herm space V_0 and $\sigma \hookrightarrow \mathcal{A}_{cusp}(G^{V_0})$ in the same L-packet as π st $\mathcal{P}_{H^{V_0}} \mid_{\sigma} \not\equiv 0$.*

Moreover the pair (V_0, σ) if it exists is unique.

- Remark : By strong multiplicity one for GL_n the condition that σ and π lie in the same L-packet reads $\sigma_K = \pi_K$.
- The Ichino-Ikeda formula implies

$$\mathcal{P}_H \mid_{\pi} \not\equiv 0 \Leftrightarrow L(1/2, \pi_K) \neq 0 \text{ and } \mathcal{P}_{H_v} \mid_{\pi_v} \not\equiv 0 \ \forall v.$$

Thus to make the link with the GGP conj we need to characterize the non-vanishing of local (normalized) periods $\mathcal{P}_{H_v} \mid_{\pi_v}$.

The local GGP conjecture

- We have the following equivalence (Sakellaridis-Venkatesh) :

$$\mathcal{P}_{H_v} \mid_{\pi_v} \not\equiv 0 \Leftrightarrow \mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \neq 0$$

  (at least when π_v is tempered or v non-Arch).
- By multiplicity one results (Aizenbud-Gourevitch-Rallis-Schiffmann, Sun-Zhu) we have

$$m(\pi_v) := \dim \mathrm{Hom}_{H_v}(\pi_v, \mathbb{C}) \leqslant 1$$

  and the local GGP conjecture roughly seeks to characterize when the dim is 1.
- It can be seen as a generalization of the Saito-Tunnell theorem as the characterization will be in terms of (local) ε-factors.
- For notational simplicity we will suppress the index v : the quad ext K/F is local, V is a n-diml Hermitian space over K, $V' = V \oplus^{\perp} K.e$ with $h(e,e) = 1$, $H = U(V) \hookrightarrow G = U(V) \times U(V')$ and $\pi = \pi_n \boxtimes \pi_{n+1}$ is an irred repn of G with 'multiplicity' $m(\pi) := \dim \mathrm{Hom}_H(\pi, \mathbb{C}) \leqslant 1$.

- Once again, it is better to vary the gps : for V_0 another n-diml Herm space over K, set $V_0' = V_0 \oplus^{\perp} K.e$, $H^{V_0} = U(V_0) \hookrightarrow G^{V_0} = U(V_0) \times U(V_0')$ and for σ irred repn of G^{V_0} we define similarly a 'multiplicity'

$$m(\sigma) := \dim \mathrm{Hom}_{H^{V_0}}(\sigma, \mathbb{C}).$$

- There is also a local notion of 'L-packet' originating from the Local Langlands Correspondence for unitary gps. Roughly : repns in the same L-packet share the same local L-functions and ε-factors.
- Let Π_n and Π_{n+1} be the L-packets of π_n and π_{n+1} : these contain repns of $U(V_0)$ and $U(V_0')$ for various V_0. Let $\Pi := \Pi_n \boxtimes \Pi_{n+1}$ be the L-packet of π.

Conjecture (Gan-Gross-Prasad, 1st version)

Assume that Π is 'generic'. Then there exists a unique $\sigma = \sigma_n \boxtimes \sigma_{n+1} \in \Pi$ which is a representation of G^{V_0} for some Herm space V_0 such that

$$m(\sigma) = 1.$$

- There is a more refined version of the conjecture describing the 'distinguished' repn $\sigma \in \Pi$ st $m(\sigma) = 1$. To state it, we need a way to parametrize repns in Π.

- L-packets are parametrized by L-parameters $\phi : \mathcal{L}_F \to {}^L U(V)$ and, following Vogan, repns in the L-packet of ϕ should be param. by characters of $S_\phi := \pi_0(\mathrm{Cent}(\phi))$.
- Again, a substitute for the L-parameter is the (local) base-change $\pi_K = \pi_{n,K} \boxtimes \pi_{n+1,K}$ (an irred repn of $\mathrm{GL}_n(K) \times \mathrm{GL}_{n+1}(K)$). We have decompositions in isobaric sums (parabolic induction) with multiplicities

$$\pi_{n,K} = \boxplus_i m_i \Pi_{n,i},\ \pi_{n+1,K} = \boxplus_j n_j \Pi_{n+1,j}$$

  where the $\Pi_{n,i}$, $\Pi_{n+1,j}$ are (essentially) square-integrable repns of general linear groups. Moreover, $\pi_{n,K}$ is conjugate self-dual and if $\Pi_{n,i}$ is conjugate self-dual of sign $(-1)^n$ (i.e. $L(0, \Pi_{n,i}, \mathrm{As}^{(-1)^n}) = \infty$) then m_i is even. The same holds for $\pi_{n+1,K}$.
- Let I be the set of indexes i st $\Pi_{n,i}$ is conjugate self-dual and m_i is odd (hence $\Pi_{n,i}$ is of sign $(-1)^{n+1}$) and J the set of indexes j st $\Pi_{n+1,j}$ is conjugate self-dual and n_j is odd. The component gp of the centralizer of the Langlands parameter ϕ_π of π can be described as

$$S_\pi = (\mathbb{Z}/2\mathbb{Z})^I \times (\mathbb{Z}/2\mathbb{Z})^J.$$

- Mok, Kaletha-Minguez-Shin-White : there is a bijection (depending on some aux choice)

$$\Pi \simeq \widehat{S_\pi}, \quad \sigma \mapsto \chi_\sigma.$$

- Recall that $\pi_{n,K} = \boxplus_i m_i \Pi_{n,i}$, $\pi_{n+1,K} = \boxplus_j n_j \Pi_{n+1,j}$ and $S_\pi = (\mathbb{Z}/2\mathbb{Z})^I \times (\mathbb{Z}/2\mathbb{Z})^J$ where I (resp. J) is the set of indexes i (resp. j) st m_i is odd (resp. n_j is odd).
- Gan-Gross-Prasad character :

$$\chi_{GGP} : S_\pi \to \{\pm 1\},$$

$$e_i \in (\mathbb{Z}/2\mathbb{Z})^I \mapsto \varepsilon(\Pi_{n,i} \times \pi_{n+1,K}, \psi), \;\; e_j \in (\mathbb{Z}/2\mathbb{Z})^J \mapsto \varepsilon(\pi_{n,K} \times \Pi_{n+1,j}, \psi)$$

where $\psi : K/F \to \mathbb{S}^1$ is nontrivial and is also used to normalize the bijection $\Pi \simeq \widehat{S_\pi}, \sigma \mapsto \chi_\sigma$.

Conjecture (Gan-Gross-Prasad, refined version)

Assume that Π is 'generic'. Then there exists a unique $\sigma \in \Pi$ which is a representation of G^{V_0} for some Herm space V_0 such that

$$m(\sigma) = 1$$

and moreover we have $\chi_\sigma = \chi_{GGP}$.

Remark : We can read the discriminant of V_0 on χ_σ as

$$\eta_{K/F}((-1)^{\lceil n/2 \rceil} \mathrm{disc}(V_0)) = \chi_\sigma(-1,1) = \varepsilon(\pi_{n,K} \times \pi_{n+1,K})$$

where $-1 \in (\mathbb{Z}/2\mathbb{Z})^I$ is the elt with all coordinates 1.

Status

- In a series of 5 papers Waldspurger and Mœglin-Waldspurger have established the analog local conjecture for p-adic orthogonal gps.
- This proof was adapted to deal with p-adic unitary gps (B.-P., Gan-Ichino) and also to give the 1st version of the conjecture (multiplicity one in L-packets) for *tempered* L-packets of real unitary gps (B.-P.).
- H. He has also proved the full refined conj for *discrete* L-packets of real unitary gps.

Relation to the global conjectures

- We return to the global setting : K/F quad ext of $\#$ fields, V n-diml Herm space over K, $V' = V \oplus^{\perp} K.e$ and $H = U(V) \hookrightarrow G = U(V) \times U(V')$. Let $\pi = \pi_n \boxtimes \pi_{n+1} \hookrightarrow \mathcal{A}_{cusp}([G])$ be st $\pi_K = \pi_{n,K} \boxtimes \pi_{n+1,K}$ is generic.
- We look for a global automorphic repn $\sigma = \bigotimes_v' \sigma_v$ in the same L-packet as π st $m(\sigma_v) = 1$ for all v (so that the local I-I period is nonzero on σ_v).
- Let Π_v be the local L-packet of π_v and $\Pi = \bigotimes_v' \Pi_v$ the global L-packet of Π. By the local conjecture, there is an unique $\sigma = \bigotimes_v' \sigma_v \in \Pi$ st $m(\sigma_v) = 1$ $\forall v$. Is σ automorphic ?
- First we need a group : for every v, σ_v is a repn of $G^{V_{0,v}} = U(V_{0,v}) \times U(V'_{0,v})$ for some local Herm space $V_{0,v}$. Does $(V_{0,v})_v$ comes from a global Herm space ?
- By CFT, this is equivalent to

$$\prod_v \eta_{K_v/F_v}(\mathrm{disc}(V_{0,v})) = 1$$

which by the refined local conjecture reads

$$\varepsilon(\pi_{n,K} \times \pi_{n+1,K}) = \prod_v \varepsilon(\pi_{n,K,v} \times \pi_{n+1,K,v}) = 1.$$

- Assuming this is the case, we get a global Herm space V_0 st σ is a repn of $G^{V_0}(\mathbb{A})$. Do we have $\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}])$? The answer is provided by *Arthur's multiplicity formula*.

Arthur's multiplicity formula

- Recall that the "centralizer of the global L-parameter of π" is
$$S_\pi = (\mathbb{Z}/2\mathbb{Z})^I \times (\mathbb{Z}/2\mathbb{Z})^J$$
where
$$\pi_{n,K} = \boxplus_{i\in I}\Pi_{n,i},\ \pi_{n+1,K} = \boxplus_{j\in J}\Pi_{n+1,j}.$$
- For every v, by the LLC, σ_v determines a character $\chi_{\sigma_v} \in \widehat{S_{\pi_v}}$ and we have natural morphisms $S_\pi \to S_{\pi_v}$.
- Arthur's multiplicity formula : we have $\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}])$ iff the restriction of $\prod_v \chi_{\sigma_v}$ to S_π is trivial in which case it appears with multiplicity one.
- By the refined local GGP conj, we have a formula for χ_{σ_v} in terms of (local) ε-factors. Combined with Arthur's multiplicity formula this gives
$$\sigma \hookrightarrow \mathcal{A}_{cusp}([G^{V_0}]) \Leftrightarrow \varepsilon(\Pi_{n,i} \times \pi_{n+1,K}) = \varepsilon(\pi_{n,K} \times \Pi_{n+1,j}) = 1\ \forall (i,j) \in I \times J.$$
- Note that
$$L(\frac{1}{2}, \pi_K) = L(\frac{1}{2}, \pi_{n,K} \times \pi_{n+1,K}) = \prod_{i\in I} L(\frac{1}{2}, \Pi_{n,i} \times \pi_{n+1,K}) = \prod_{j\in J} L(\frac{1}{2}, \pi_{n,K} \times \Pi_{n+1,j})$$
therefore σ is automorphic unless "$L(1/2, \pi_K)$ vanishes for obvious reasons".
- More concretely : Global Ichino-Ikeda conj+Local GGP conj $\Rightarrow$ Global GGP conj.
- This is only to illustrate the internal structure since (in the cases these are known) the global GGP conj is usually established before the I-I conj.