IWAHORI SATAKE EQUIVALENCE

Contents

1. Iwahori Satake equivalence

1.1. General outline of argument.

(1) $\mathrm{Shv}^{\heartsuit}_{c,(I_u,\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}_G)$ is highest weight and semisimple.

(2) We identify the map Corollary 2.15 defined by [BGMRR19a] as the inclusion map of adolescent Whittaker categories, [Ras16].

(3) The exactness of such functors is thus a consequence [FR22, Thm. 7.2], of which utilizes the results of [BBM21].

(4) Applying the Casselman-Shalika formula.

We will prove

Theorem 1.1.

$$\mathrm{Shv}_{c,L^+G}(\mathrm{Gr},e) \xrightarrow{\simeq} \mathrm{Shv}_{c,(I_u,\chi^*\mathcal{L}_\psi)}(\mathrm{Gr},e)$$

2. Recollection of Iwahori Whittaker category

2.1. **Definition of Iwahori whittaker category.** In this section we recall the Iwahori-Whittaker category. The stratification is affine, in particular; this makes its highest weight structure clear, see Corollary 2.9.

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Let $\lambda \in X_*$.

$$X_\lambda := I \cdot \varpi^\lambda L^+G, \quad i_\lambda : X_\lambda \hookrightarrow \mathrm{Gr}$$

The standard and costandard objects

$$\Delta_\lambda^{\mathrm{IW}}(e) := \pi_0(i_\lambda)_! e_{X_\lambda}[\dim X_\lambda], \quad \nabla_\lambda^{\mathrm{IW}} := \pi_0(i_\lambda)_* e_{X_\lambda}[\dim X_\lambda]$$

Lemma 2.1. *Let $\pi : \mathbb{A}^1_k \to \mathbb{A}^1_k$ be the map $x \mapsto x^p - x$, the Galois covering with Galois group $\mathbb{F}_p$. Since l is invertible in coefficient e,*

$$\pi_* e_{\mathbb{A}^1_k} \simeq \bigoplus_{\psi : \mathbb{F}_p \to e^\times} \mathcal{L}_\psi$$

We fix a nontrivial morphism $\psi : \mathbb{F}_p \to e^\times$, hence $\mathcal{L}_\psi$. The local system satisfies

$$R\Gamma_c(\mathbb{A}^1_k, \mathcal{L}_\psi) \simeq R\Gamma(\mathbb{A}^1_k, \mathcal{L}_\psi) \simeq 0$$

Remark 2.2. The $\mathcal{L}_{\psi,1}$ character sheaf we have defined previously coincides with this.

Definition 2.3. Let χ denote the composite

$$I_{u,1} \simeq^{\mathrm{ad}\,\check{\rho}(\varpi)} I_u \to N \to N/[N,N] \to \prod_{\alpha \in \Delta^+} \mathbb{G}_a \xrightarrow{\Sigma} \mathbb{G}_a$$

Definition 2.4. Let

$$\mathrm{Shv}_{c,(I_u, \chi^* \mathcal{L}_\psi)}(\mathrm{Gr}, e)$$

be the ∞-category $(I_u, \chi^*\mathcal{L}_\psi)$ equivariant sheaves. We call this the Iwahori-Whittaker category.

Lemma 2.5. *This is a full subcategory* $\mathrm{Shv}_c(\mathrm{Gr}, e)$ *which induces a t-structure on the Iwahori-Whittaker category.*

Proof. Should be similar to [ALWY24, Ch.6]. □

Lemma 2.6. *The orbit X'_λ supports an $(I_u, \chi^*\mathcal{L}^k_{\psi,1})$ equivariant local system if and only if $\lambda \in X_*(T)_{++}$.*

Proof. We follow the proof of [BGMRR19a, Lemma 3.3.]. □

Let us recall the definition of highest weight category,

Definition 2.7. [Ric16] Let $\mathcal{A}$ be a k-linear ordinary category. $\mathcal{A}$ is highest weight if the following conditions holds. Let $\mathcal{S} := \pi_0 \mathrm{Irr}\mathcal{A}$ be the set of isomorphism class of irreducible objects in $\mathcal{A}$, which is equipped with a partial order $\leq$.

(1) For any $s \in \mathcal{S}$, $\{t \in \mathcal{S} : t \leq s\}$ is finite.

(2) For each $s \in \mathcal{S}$, we have $\mathrm{Hom}_{\mathcal{A}}(L_s, L_s) = k$.

(3) For an $s \in \mathcal{S}$, and ideal $\mathcal{S}' \subset \mathcal{I}$ such that $s \in \mathcal{S}$ is maximal, $\Delta_s \to L_s$ is a projective cover $\cdots$

Lemma 2.8. *Assume k is a field of characteristic 0. Then the i-th cohomology stalks of $\mathrm{IC}_\lambda^{\mathcal{IW}}(k)$ vanish unless $i \equiv 0 \mod \dim(X'_\lambda)$.*

Proof. Observe first that obviously $\overline{X'_\lambda} \subset \mathrm{Gr}_{G,\leq\lambda}$. Choose a preimage w of the Iwahori-Weyl group corresponding to this Schubert cell, we get a smooth morphism $p\colon \mathrm{Fl}_{G,\leq w} \to \mathrm{Gr}_{G,\leq\lambda}$. Choose a reduced expression $\dot{w} = s_1 \dots s_r \omega$. We have the Demazure-Bott-Samuelson resolution $\pi\colon \mathrm{Dem}_{G,\dot{w}} \to \mathrm{Fl}_{G,\leq w}$ whose geometric fibers admit stratifications into affine spaces, see also [Zhu17, Section 1.4.2]. Note that the parity property on stalks may be checked after pulling back to $p^{-1}(\overline{X'_\lambda})$. Then by the decomposition theorem this occurs as a direct summand of $R\pi_* \mathrm{IC}$, where $\mathrm{IC} = j_* \pi^* p^* \mathrm{IC}_\lambda^{\mathcal{IW}}(k)|_{X'_\lambda}[\dim \mathrm{Dem}_{G,\dot{w}}]$, where $j\colon \pi^{-1}p^{-1}(X'_\lambda) \to \pi^{-1}p^{-1}(\overline{X'_\lambda})$ is the open inclusion. This gives the claim. $\square$

Corollary 2.9. *The category $\mathrm{Shv}^{\heartsuit}_{c,(I_u,\mathcal{L}_\psi)}(\mathrm{Gr}, e)$ is a highest weight category with weight poset $(X_{*,++}, \leq)$.*

Proof. Since each stratum is affine, this follows as discussed in [BGS96]. $\square$

2.2. **Whittaker filtration.** Here we briefly recall [Ras16, Ch.2], where one constructs the *rth adolescent Whittaker category.*

Definition 2.10. Let

$$\begin{tikzcd}
\mathcal{P}_{u,r} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & I_r \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^+G \arrow[d] \\
L^r N \arrow[r] & L^r B \arrow[r] & L^r G
\end{tikzcd}$$

and define

$$I_{u,r} := \operatorname{ad} -(r\check{\rho}(\varpi))(\mathcal{P}_{u,r})$$

Example 2.11. In case $G = \mathrm{GL}_2$, $\check{\rho} = \frac{1}{2}(1,-1)$. Thus,

$$I_{u,n} = \begin{pmatrix} 1+\varpi\mathcal{O} & \varpi^{-n}\mathcal{O} \\ \varpi^{2n}\mathcal{O} & 1+\varpi\mathcal{O} \end{pmatrix}$$

We will only be interested in the case $r = 0$, giving $I_{u,0} \simeq L^+G$, and when $r = 1$, where it is the conjugate of the unipotent radical of Iwahori

$$\begin{tikzcd} & I_u \arrow[dl, "\operatorname{ad}\check{\rho}(\varpi),\simeq"'] \arrow[dr] & \\ I_{u,1} & & I \end{tikzcd}$$

In [AB09], $(I_{u,1}, \chi^*\mathcal{L}_\psi)$ -equivariant sheaves are called the *baby Whittaker category.*

Remark 2.12. The natural h fits in the following diagram commute

$$\begin{tikzcd} I_u \cap LN \arrow[r, "\operatorname{ad}\check{\rho}(\varpi)"] \arrow[d] & I_{u,1} \cap LN \arrow[d] \\ N \arrow[d] & L\mathbb{G}_a/L^+\mathbb{G}_a \\ \mathbb{G}_a \arrow[r] & L^{\geq -1}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[u, hook] \end{tikzcd}$$

Proposition 2.13. *We have the following adjunction: $U \hookrightarrow V$, be an inclusion of subgroups,*

$$\begin{tikzcd} \mathrm{Shv}_{c,(V,\mathcal{L})}(X,e) \arrow[r, "fgt"] & \mathrm{Shv}_{c,(U,\mathcal{L}|_U)}(X,e) \arrow[l, bend right, "\mathrm{Av}_! \dim[V/U]"'] \arrow[l, bend left, "\mathrm{Av}_* \, dim[V/U]"] \end{tikzcd}$$

, which is a adjunct triplet. We also have

$$\mathrm{Av}_! \mathcal{F} = a_!(\mathcal{L} \boxtimes \mathcal{F})[\dim], \quad \mathrm{Av}_* \mathcal{F} = a_*(\mathcal{L} \boxtimes \mathcal{F})[\dim]$$

$\mathcal{L}$ is a character on V, and $\mathcal{L}^\vee$ is the dual character on V, such that $\mathcal{L} \otimes \mathcal{L}^\vee \simeq e_V$. [Milton: this statement is incomplete, but will be modified, [AR15, A.2]]

Proof. Consider the following diagram

$$\begin{tikzcd} & V \times X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ V \arrow[dr] & & X \arrow[dl] \\ & \mathrm{pt} & \end{tikzcd}$$

By the projection formula, and as $\mathcal{L}$ is a character sheaf on V, we have that

$$p_{2*}p_1^*\mathcal{L} \simeq e_X$$

Then we have the following adjuntcions:

$$\begin{tikzcd} \mathrm{Shv}_c(X,e) \arrow[r, shift right, "p^*"'] \arrow[d, shift right, "\mathrm{fgt}"'] & \mathrm{Shv}_{c,V}(V \times X, e) \arrow[l, shift right, "a_!"'] \arrow[d, shift left, "p_1^*\mathcal{L}^\vee \otimes (-)"] \\ \mathrm{Shv}_{c,(V,\mathcal{L})}(X,e) \arrow[r, shift right] \arrow[u, shift right, "\mathrm{Av}_*[\dim]"'] & \mathrm{Shv}_{c,(V,\mathcal{L})}(V \times X, e) \arrow[l, shift right] \arrow[u, shift left, "p_1^*\mathcal{L} \otimes (-)"] \end{tikzcd}$$

The top and bottom adjunctions are equivalences as V is affine. $\square$

We will now consider the following composition

$$\begin{tikzcd} & \mathrm{Shv}_{c,(G(\mathcal{O}) \cap I_{u,1}, \chi^*\mathcal{L}_\psi)}(\mathrm{Gr}_G, e) \arrow[dr, "\mathrm{Av}_![d]"] & \\ \mathrm{Shv}_{c,G(\mathcal{O})}(\mathrm{Gr}_G, e) \arrow[ur] \arrow[rr, "\mathrm{Av}_!^\psi[d]"'] & & \mathrm{Shv}_{c,(I_{u,1},\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}_G, e) \arrow[d, phantom, "\simeq"] \\ & & \mathrm{Shv}_{c,(I_u,\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}, e) \end{tikzcd}$$

$\mathrm{Av}_!$ is the left adjoint of the forgetful functor, as defined in Proposition 2.13. Here $d = 2\langle \check{\rho}, \rho \rangle$. The appearance of this d would be explained in Proposition 3.8.

Remark 2.14. Note that we will consider the following more general situation: whenever we have two subgroups with characters $\{K_i, \psi_i\}_{i=1}^2$

of L^+G such that $\psi_1\Big|_{K_1\cap K_2} = \psi_2\Big|_{K_1\cap K_2}$. We define the composite

$$\begin{tikzcd} & \mathrm{Sh}_{c,(K_1\cap K_2,\mathcal{L}_\psi)}(\mathrm{Gr}_G, e) \arrow[dr] & \\ \mathrm{Shv}_{c,(K_1,\psi)}(\mathrm{Gr}, e) \arrow[ur] & & \mathrm{Shv}_{c,(K_2,\psi_2)}(\mathrm{Gr}, e) \end{tikzcd}$$

as $\mathrm{Av}_!^{\psi_2}$, provided all the functors in diagram are well-defined.

Corollary 2.15. *The composite*

$Av_!^{\psi}[d] : \mathrm{Shv}_{c,G(\mathcal{O})}(\mathrm{Gr}_G, e) \to \mathrm{Shv}_{c,(I_u,1,\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}_G, e) \simeq \mathrm{Shv}_{c,(I_u,\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}_G, e)$

coincides with the construction of [BGMRR19b].

$$\mathcal{A} \mapsto \Delta_\varsigma^{IW} \star \mathcal{A}$$

[Milton: Not clear if this is the correct statement: how does ς come in?]

Proof. Note that $\mathrm{ad}\,\check{\rho}(\varpi)$ is equivalent to $\mathrm{ad}\,\varsigma$. Indeed, ς is chosen such that $\alpha \in \Delta$, $\langle \varsigma, \alpha\rangle = 1$. On the other hand, if $\alpha \in \Delta$, then $s_{\check{\alpha}}(\check{\rho}) = \check{\rho} - \check{\alpha}$. By definition, $s_{\check{\alpha}}(\check{\rho}) = \check{\rho} - \langle\check{\rho}, \alpha\rangle\check{\alpha}$, thus $\langle\check{\rho}, \alpha\rangle = 1$. $\square$

3. Exactness of spherical action

Definition 3.1. We say that a sheaf $\mathrm{Shv}_c(\mathrm{Gr}_G, k)$ is *partially integrable* if it admits a filtration such that each filtered piece admits the structure of a $\mathcal{P}$-equivariant sheaf, where $\mathcal{P}$ is the preimage of a parabolic P^- strictly bigger than B^- under the reduction map $L^+G \to G$

Lemma 3.2. *Let* $\mathrm{Av}_{!,\mathring{I},\chi^*\mathcal{L}_{\psi,1}}$ *denote the left adjoint to the inclusion* $\mathrm{Shv}_{c,(\mathring{I},\chi^*\mathcal{L}^k_{\psi,1})}(\mathrm{Gr}_G, k) \subset \mathrm{Shv}_c(\mathrm{Gr}_G, k)$. *Then the image under* $\mathrm{Av}_{!,\mathring{I},\chi^*\mathcal{L}_{\psi,1}}$ *of any partially integrable object vanishes.*

Proof. Let A be partially integrable, we want to check that $\mathrm{Av}_{\mathring{I},\chi^*\mathcal{L}_{\psi,1}}(A) = 0$. By definition, we may assume that A is $\mathcal{P}$-equivariant where $\mathcal{P}$ is the preimage of a parabolic P^- strictly bigger than B^- under the reduction map $L^+G \to G$. Let G^1 denote the first congruence subgroup of G. In this case $G^1\backslash \mathrm{Gr}_G$ admits an action of G and we can form the category $\mathrm{Shv}_{c,N,\chi'^*\mathcal{L}_{\psi,1}}(G^1\backslash \mathrm{Gr}_G)$ as those sheaves $\mathcal{F}$ on $G^1\backslash \mathrm{Gr}_G$ such that $a^*\mathcal{F} \cong \chi'^*\mathcal{L}_{\psi,1} \boxtimes \mathcal{F}$ where $a\colon N \times G^1\backslash \mathrm{Gr}_G \to \mathrm{Gr}_G$ is the action map and χ' is the composite $N \to N/[N,N] \cong \prod_{\alpha\in\Delta^+} \mathbb{G}_a \xrightarrow{\Sigma} \mathbb{G}_a$. We have a similar averaging functor $\mathrm{Av}_{!,N,\chi'^*\mathcal{L}_{\psi,1}}\colon \mathrm{Shv}_c(G^1\backslash \mathrm{Gr}_G) \to$

$\mathrm{Shv}_{c,N,\chi'^*\mathcal{L}_{\psi,1}}(G^1\backslash \mathrm{Gr}_G)$. Note that by assumption A comes from pullback from a P^--equivariant sheaf A' on $G^1\backslash \mathrm{Gr}_G$ and it suffices to check that $\mathrm{Av}_{!,N,\chi'^*(\mathcal{L}_{\psi,1})}(A') = 0$. This follows from the fact that the !-pushforward of $\chi'^*\mathcal{L}_{\psi,1}$ under $N \to G \to G/P^-$ vanishes. Todo: to see this, consider the following diagram

$$\begin{tikzcd}
 & & N \times G^1\backslash \mathrm{Gr}_G \arrow[dll] \arrow[d] \arrow[drr] & & \\
G^1\backslash \mathrm{Gr}_G \arrow[d] & & N \times P^-\backslash(G^1\backslash \mathrm{Gr}_G) \arrow[dll] \arrow[drr] & & G^1\backslash \mathrm{Gr}_G \arrow[d] \\
P^-\backslash(G^1\backslash \mathrm{Gr}_G) & & & & P^-\backslash(G^1\backslash \mathrm{Gr}_G)
\end{tikzcd}$$

□

Definition 3.3. The category of I^--monodromic sheaves is the essential image of the functor $\mathrm{Shv}(I\backslash \mathrm{Gr}_G) \to \mathrm{Shv}(Gr_G)$.

Lemma 3.4. *Any I^--monodromic sheaf (bounded complex thereof) that is supported on $\mathrm{Gr}_{G,\leq\varsigma} - \mathrm{Gr}_{G,\varsigma}$ is partially integrable.*

Proof. Note that any orbit in $\mathrm{Gr}_{G,\leq\varsigma} - \mathrm{Gr}_{G,\varsigma}$ corresponds to irregular λ. It suffices to show that irreducible I^--equivariant étale sheaves supported on a $\mathrm{Gr}_{G,\lambda}$ as above is in fact partially integrable. Such sheaves are pulled back from irreducible B^--equivariant sheaves on $G^1\backslash \mathrm{Gr}_{G,\lambda}$. This is a G-homogenous space isomorphic to G/P^- for some parabolic P^- strictly bigger than B^-, since λ was irregular. Any such sheaf is supported on an closure of an B^--orbit of G/P^-, however any such orbit is stable under a parabolic strictly bigger than B^-, which shows the claim. □

Corollary 3.5. *For any sheaf $\mathcal{F}$ the cofiber cofib*$(\mathrm{Av}_{!,\mathring{I},\chi^*\mathcal{L}_{\psi,1}}(\mathcal{F}) \to \mathrm{Av}_{*,\mathring{I},\chi^*\mathcal{L}_{\psi,1}}(\mathcal{F})$ *vanishes after applying* $\mathrm{Av}_{!,\mathring{I},\chi^*\mathcal{L}_{\psi,1}}$.

Proof. This is immediate from Lemma 3.4 and Lemma 3.2 □

Lemma 3.6. *Let $K_1 := I_{u,1} \cap I_1^-$, note that we have the exact sequence*

$$K_1 \to I_1^- \to B^-$$

$$K_1 \to I_{u,1} \to N$$

The following diagram commutes

$$\begin{tikzcd} \mathrm{Shv}_{c,L^+G}(\mathrm{Gr}_G, e) \arrow[r, "fgt"] \arrow[drr] & \mathrm{Shv}_{c,I}(\mathrm{Gr}_G, e) \arrow[r, "Av_*"] & \mathrm{Shv}_{c,I_1^-}(\mathrm{Gr}_G, e) \simeq \mathrm{Shv}_c(B^-\backslash(K_1\backslash \mathrm{Gr})) \arrow[d, "Av_!^\psi"] \\ & & \mathrm{Shv}(N\backslash(K_1\backslash \mathrm{Gr})) \simeq \mathrm{Shv}_{c,(I_{u,1},\chi)}(\mathrm{Gr}, e) \end{tikzcd}$$

Proof. This is [FR22, p. 4.3.0.1]. □

Example 3.7.

$$I_1^- = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \varpi\mathcal{O} & \mathcal{O} \end{pmatrix}$$

$$K_1 := I_{u,1} \cap I_1^-$$

Proposition 3.8. Φ *is t-exact.*

Proof. By Lemma 3.6, we reduce the problem of checking cohomological amplitude of each of the following composition

$$\mathrm{Shv}_{c,I}(\mathrm{Gr}, e) \xrightarrow{\mathrm{Av}_*} \mathrm{Shv}_{c,I_1^-}(\mathrm{Gr}, e) \xrightarrow{\mathrm{Av}_!^\psi} \mathrm{Shv}_{c,(I_{u,1},\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}, e)$$

We show that the diagram is equivalent to

(1)

$$\mathrm{Shv}_{c,I}(\mathrm{Gr}, e) \xrightarrow{\mathrm{Av}_*} \mathrm{Shv}_{c,I_1^-}(\mathrm{Gr}, e) \xrightarrow{\mathrm{Av}_*^\psi[2\dim N]} \mathrm{Shv}_{c,(I_{u,1},\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}, e)$$

(2) Let $d' = \left(-2\dim I_1^- \cdot I/I\right)$.

$$\mathrm{Shv}_{c,I}(\mathrm{Gr}, e) \xrightarrow{\mathrm{Av}_![d']} \mathrm{Shv}_{c,I_1^-}(\mathrm{Gr}, e) \xrightarrow{\mathrm{Av}_!^\psi} \mathrm{Shv}_{c,(I_{u,1},\chi^*\mathcal{L}_\psi)}(\mathrm{Gr}, e)$$

1) follows from the result of [BBM21], in particular Theorem A.1.
2) follows from Lemma 3.13, and vanishing of partially integrable objects, Lemma 3.2.

□

Definition 3.9. We say a morphism $f: X \to Y$ is cohomologically contractible if it is cohomologically smooth and we have $f_! f^! e \cong e$.

Lemma 3.10. *The following conditions are equivalent for a cohomologically smooth morphism* $f: X \to Y$*:*

(1) f is cohomologically contractible.

(2) The functor $f^!\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ is fully faithful.

(3) The functor $f^*\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(X)$ is fully faithful.

(4) The natural transformation $\mathrm{id} \to f_* f^*$ is an isomorphism.

Proof. Since f is cohomologically smooth we see that Item 2 is equivalent to Item 3. It is clear that Item 2 implies Item 1, conversely by the projection formula we have $f_! f^! \mathcal{F} \cong f_!(f^! e \otimes f^* \mathcal{F} \cong f_! f^! e \otimes \mathcal{F}$, so that Item 1 implies Item 2. The equivalence between Item 4 and Item 3 is standard. $\square$

Definition 3.11. For a cohomologically smooth morphism $f\colon X \to Y$ we write $f_\natural$ for the left adjoint of f^*. We have a natural isomorphism $f_\natural \cong f_!(f^! e \otimes -)$ and it is easy to check that similarly to $f_!$ the functor $f_\natural$ satisfies the projection formula and base change.

Lemma 3.12. *Let $f\colon X \to Y$ be cohomologically contractible. Then $p_\natural p^*\colon \mathrm{Shv}(Y) \to \mathrm{Shv}(Y)$ is an equivalence of categories with inverse $p_* p^*$.*

Proof. For this we have to check that $p_* p^* p_\natural p^*$ and $p_\natural p^* p_* p^*$ are naturally isomorphic to the identity functors. This follows easily from base change and the projection formula as well as from the natural isomorphism $\mathrm{id} \cong p_* p^*$ we have from Lemma 3.10. $\square$

Lemma 3.13. *Now we study $Av_!^{I \to I_1^-}$.*

(1)

$$\mathrm{Shv}_{c,I_1^-}(\mathrm{Gr}, e) \underset{Av_*^{I_1^- \to I}}{\overset{Av_!^{I \to I_1^-}\langle \dim(I_1^- \cdot I/I)\rangle}{\rightleftarrows}} \mathrm{Shv}_{c,I}(\mathrm{Gr}, e)$$

is are mutally inverse equivalences. Here we write $\langle d\rangle = [2d](d)$ for the usual shift and Tate twist.

Proof. Note that by definition we have $\mathrm{Shv}_{c,I_1^-}(\mathrm{Gr}, e) \simeq \mathrm{Shv}_c(I_1^- \backslash \mathrm{Gr}, e)$ as well as $\mathrm{Shv}_{c,I}(\mathrm{Gr}, e) \simeq \mathrm{Shv}_c(I \backslash \mathrm{Gr}, e)$. Observe that $I_1^- \backslash \mathrm{Gr} \cong I \backslash \mathrm{Gr}$. This induces an equivalence of categories $\mathrm{Shv}_c(I_1^- \backslash \mathrm{Gr}) \simeq \mathrm{Shv}_c(I \backslash \mathrm{Gr})$. Consider the map $p\colon I \backslash \mathrm{Gr} \to (I_1^- \cap I) \backslash \mathrm{Gr}$. This is a fibration with fibers $I/I_1^- \cap I$, which is an affine space. We deduce that p is cohomologically contractible. Using the identifications mentioned above we can compute that $\mathrm{Av}_!^{I \to I_1^-} \left\langle \dim(I_1^- \cdot I/I) \right\rangle \cong p_\natural p^*$ and that $\mathrm{Av}_*^{I_1^- \to I} \cong p_* p^*$. The claim now follows from Lemma 3.12. □

Appendix A. Properties of averaging functor

In this appendix we record various properties of averaging functors.

Theorem A.1. *Let $N \hookrightarrow G$ be unipotent radical of parabolic subgroup $P \hookrightarrow G$. $\psi : N \to \mathbb{G}_a$ a nondegenerate character. Let N^-, P^- be the associated opposite unipotent radical of the opposite parabolic subgroup.*

(1) If $\mathcal{F} \in \mathrm{Shv}^b(N^- \backslash X)$, then

$$Av_{N,\psi,!}\mathcal{F} \simeq Av_{N,\psi,*}\mathcal{F}$$

Proof. The is from [BBM21]. This is a consequence of the cleanness property of the inclusion

$$j : N \times X \hookrightarrow G \times_{P^-} X$$

□

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