LEFSCHETZ NUMBERS OF HECKE CORRESPONDENCES

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1. INTRODUCTION

In this announcement we give a formula for the contribution of a fixed point to the Lefschetz number of the action of a Hecke correspondence on the cohomology and the L^2-cohomology of a locally symmetric space. Details will appear elsewhere. For simplicity of exposition we will begin with a semisimple algebraic group G, although the results apply also in the reductive case. The techniques rely heavily on the weighted cohomology theory of [GHM]. In a later paper [GKM] we will add these contributions over all the fixed points (by evaluating an orbital integral) and compare the result with the averaged discrete series characters which appear in the trace formula of [A]. See also [R] and [KR] for a similar fixed point formula in the $\mathbb{Q}$-rank 1 case, which is modelled on the article [La]. Another Lefschetz fixed point theorem has been recently given by M. Stern [S]. It would be interesting to correlate his formula with ours.

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2. THE LOCALLY SYMMETRIC SPACE

Let G be the real points of a connected semisimple algebraic group defined over $\mathbb{Q}$, and let D denote the associated symmetric space. We assume that G acts effectively on D. Fix a basepoint $x_0 \in D$ and let $K = K(x_0)$ be the maximal compact subgroup of G which fixes x_0. Let $\Gamma \subset G(\mathbb{Q})$ be a neat arithmetic subgroup. Then $X = \Gamma\backslash G/K(x_0)$ is a locally symmetric space and D is its universal covering space.

3. PARABOLIC SUBGROUPS

Let P be the real points of a rationally defined parabolic subgroup of G. Let N_P denote the unipotent radical of P and let $L_P = P/N_P$ be the Levy factor. Then $L_P = {}^0L_P A_P^0$ where A_P^0 is the connected component of the identity in the maximal $\mathbb{Q}$-split torus in the center of L_P, and

$$^0L_P = \bigcap_{\chi} \ker(\chi^2)$$

is the intersection of the kernels of the squares of all the rationally defined characters χ on L_P. Associated to the basepoint $x_0 \in D$, there is a Cartan involution $\theta(x_0)$ of G whose fixed point set is $K(x_0)$, and there is a unique θ-stable lift, $L_P(x_0) = {}^0L_P(x_0)A_P^0(x_0) \subset P$ of the Levy factor [BS] §1.9.

4. THE COMPACTIFICATION

Borel and Serre [BS] construct a partial compactification $\tilde{D}$ of D by attaching rational boundary components $e(P) \cong D/A_P^0$ to D, which are in one to one correspondence with rationally defined parabolic subgroups $P \subset G$, and which are defined as follows: since P acts transitively on D we have $D = P/K_P(x_0)$ where $K_P(x_0) = K(x_0) \cap P$ is a maximal compact subgroup of P. The right action of $A_P^0(x_0)$ on P commutes with $K_P(x_0)$ and induces the ***geodesic action***[BS] of A_P^0 on D. Then $e(P) = P/K_P(x_0)A_P^0(x_0)$ is the boundary component in $\tilde{D}$. The group $G(\mathbb{Q})$ of rational points in G acts as the group of automorphisms of the partial compactification $\tilde{D}$, and the quotient $\tilde{X} = \Gamma\backslash\tilde{D}$ is the Borel-Serre compactification of X. Its boundary components are in one to one correspondence with Γ conjugacy classes of rational parabolic subgroups P. We will be concerned with the reductive Borel-Serre compactification $\bar{X}$ of X, which is defined as follows: Form a quotient $\bar{D}$ of $\tilde{D}$ by replacing each boundary component $e(P)$ with the quotient $D_P = e(P)/N_P$ under the action of the unipotent radical N_P of the parabolic subgroup P. The group Γ also acts on $\bar{D}$ and the quotient $\bar{X} = \Gamma\backslash\bar{D}$ is the reductive

Borel-Serre compactification of X. The quotient mapping $\bar{D} \to \tilde{D}$ is Γ-equivariant and induces a surjective (stratified) mapping $\pi_X : \tilde{X} \to \bar{X}$.

Proposition. *The compactification $\bar{X}$ is a disjoint union $\bigcup_P X_P$ of strata*

$$X_P = (\Gamma \cap P)\backslash D_P$$

which are indexed by a set of representatives of Γ-conjugacy classes of rational parabolic subgroups of G. The largest stratum is $X = X_G$.

5. MORPHISMS AND HECKE CORRESPONDENCES

Let $\Gamma' \subset G(\mathbb{Q})$ be another neat arithmetic subgroup and set $C = \Gamma'\backslash D$. Suppose there exists $g \in G(\mathbb{Q})$ such that Γ' is a subgroup of finite index in $g^{-1}\Gamma g$. Then multiplication from the left by g determines a local isometry $c : C = \Gamma'\backslash D \to X = \Gamma\backslash D$ which is an umramified covering of degree $[g^{-1}\Gamma g : \Gamma']$. This covering map depends only on the double coset $\Gamma g \Gamma'$ of g and it extends to finite maps $\tilde{c} : \tilde{C} \to \tilde{X}$ and $\bar{c} : \bar{C} \to \bar{X}$ on the Borel-Serre and the reductive Borel-Serre compactifications, so that the following diagram commutes:

$$\begin{array}{ccccc} & & \tilde{c} & & \\ & \tilde{C} & \rightarrow & \tilde{X} & \\ \pi_C & \downarrow & & \downarrow & \pi_X . \\ & \bar{C} & \rightarrow & \bar{X} & \\ & & \bar{c} & & \end{array}$$

Thus we define

Definition. The set of morphisms from C to X is

$$\mathrm{Mor}(C, X) = \{\Gamma g \Gamma' \in \Gamma\backslash G(\mathbb{Q})/\Gamma' \mid \Gamma' \text{ is a subgroup of finite index in } g^{-1}\Gamma g\}.$$

A ***correspondence*** on the locally symmetric space $X = \Gamma\backslash D$ is a neat arithmetic subgroup $\Gamma' \subset G(\mathbb{Q})$ together with two morphisms $c_1, c_2 \in \mathrm{Mor}(C, X)$ (where $C = \Gamma'\backslash D$). Two correspondences $(c_1, c_2) : C \rightrightarrows X$ and $(c_1', c_2') : C' \rightrightarrows X$ are said to be ***isomorphic*** if there is an invertible morphism $\alpha : C \to C'$ such that $c_i' \circ \alpha = c_i$ (for $i = 1, 2$).

Each $g \in G(\mathbb{Q})$ gives rise to a particular correspondence, $(c_1, c_2) : C \rightrightarrows X$ as follows: set $\Gamma' = \Gamma \cap g^{-1}\Gamma g$, $C = \Gamma'\backslash D$, and for each $y \in D$ define $c_1(\Gamma' y) = \Gamma y$, and $c_2(\Gamma' y) = \Gamma g y$.

Definition. The correspondence $T(g)$ arising in this way from $g \in G(\mathbb{Q})$ is called a ***Hecke correspondence***.

Proposition. *The isomorphism class of the Hecke correspondence $T(g)$ depends only on the double coset $\Gamma g \Gamma \in \Gamma \backslash G(\mathbb{Q})/\Gamma$. Every correspondence is isomorphic to a finite covering of a Hecke correspondence.*

6. THE FIXED POINTS

Suppose $(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$ is a Hecke correspondence. A point $p \in \bar{C}$ is ***fixed*** if $c_1(p) = c_2(p)$. We denote by $\mathcal{F}$ the set of fixed points of the correspondence. Fix a stratum $C_Q \subset \bar{C}$ (which corresponds to a Γ'-conjugacy class of rational parabolic subgroups $Q \subset G$). The intersection $\mathcal{F} \cap C_Q$ decomposes into connected components, and we now focus on a single such connected component F. We will associate to the component F two integers r and s, and an important characteristic group element $\gamma \in G(\mathbb{Q})$. Our formula for the contribution to the Lefschetz number arising from the fixed component F will depend only on r, s, and γ. Choose a fixed point $p \in F \subset C_Q \subset \bar{C}$.

Definition. Let r and s denote the local degrees of the finite maps c_1 and c_2 respectively, at the fixed point p. In other words, for a sufficiently small neighborhood $\mathrm{U} \subset \bar{C}$ and for points q sufficiently close to p, we have

$$r = |\ \mathrm{U} \cap c_1^{-1}(q)\ | \quad \text{and} \quad s = |\ \mathrm{U} \cap c_2^{-1}(q)\ |.$$

Corresponding to this choice of fixed point p, we have a point

$$x \in F' = c_1(F) = c_2(F) \subset X_P \subset \bar{X}$$

(where X_P denotes the stratum of $\bar{X}$ containing F'). Choose lifts D_Q and $D_P \subset \bar{D}$ of the strata C_Q and X_P respectively. (This amounts to choosing a particular rational parabolic Q in the Γ'-conjugacy class of parabolics corresponding to C_Q and a rational parabolic P in the Γ-conjugacy class of parabolics corresponding to X_P.) Choose a lift $\bar{p} \in D_Q$ of the point p, and a lift $\bar{x} \in D_P$ of the point x.

Lemma. *There exist automorphisms d_1 and d_2 of $\bar{D}$ which cover the maps c_1 and c_2 such that $d_1(\bar{p}) = d_2(\bar{p}) = \bar{x}$, so $d_1(D_Q) = d_2(D_Q) = D_P$.*

Diagram.

$$\begin{array}{ccccccccccc}
 & & & & & (d_1, d_2) & & & & & \\
\bar{p} & \in & D_Q & \subset & \bar{D} & \rightrightarrows & \bar{D} & \supset & D_P & \ni & \bar{x} \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
p & \in & C_Q & \subset & \bar{C} & \rightrightarrows & \bar{X} & \supset & X_P & \ni & x \\
 & & & & & (c_1, c_2) & & & & &
\end{array}$$

Definition. Let $\gamma \in G(\mathbb{Q})$ be the automorphism given by $d_2 d_1^{-1} : \bar{D} \to \bar{D}$.

Proposition. *The element γ represents the same Hecke correspondence as g, i.e. $\gamma \in \Gamma g \Gamma$. The element γ is an element of the parabolic subgroup $P(\mathbb{Q})$. It is well defined in $P(\mathbb{Q})$ up to conjugation by elements of $\Gamma_P = \Gamma \cap P$, and is otherwise independent of the choices of the point $p \in F$, the lifts $\bar{p}$ of p and $\bar{x}$ of x, and of the representative g of the Hecke correspondence within the double coset $\Gamma g \Gamma$. The projection γ_L of γ to the Levy factor $L = P/N_P$ is semisimple.*

7. THE NILMANIFOLD CORRESPONDENCE

Let $x = c_1(p) = c_2(p) \in X_P$ be the image of the fixed point, as above. If $\pi_X : \tilde{X} \to \bar{X}$ is the projection of the Borel-Serre compactification to the reductive Borel-Serre compactification, then $S = \pi_X^{-1}(x)$ is a nilmanifold. The Hecke correspondence on the Borel-Serre compactification restricts to a correspondence $(s_1, s_2) : S' \rightrightarrows S$ on this nilmanifold, where

$$S' = \pi_C^{-1}(p) \subset \tilde{C}$$

is the associated nilmanifold in the total space of the correspondence. The induced homomorphism on the cohomology of the nilmanifold S is a key ingredient in the formula for the contribution to the Lefschetz number.

This nilmanifold correspondence may be explicitly described as follows: Let

$$\begin{aligned} \Gamma_N &= \Gamma \cap N_P, \\ \Gamma'_N &= \Gamma_N \cap \gamma^{-1} \Gamma_N \gamma = \Gamma' \cap N_P, \end{aligned}$$

and define nilmanifolds $T = \Gamma_N \backslash N_P$ and $T' = \Gamma'_N \backslash N_P$. We obtain a correspondence

$$(t_1, t_2) : T' \rightrightarrows T$$

by $t_1(\Gamma'_N x) = \Gamma_N x$ and $t_2(\Gamma'_N x) = \Gamma_N \gamma x$. The map t_2 is well defined since N_P is normal in P and $\gamma \Gamma'_N \subset \Gamma_N \gamma$.

Proposition. *An appropriate choice of basepoint $x_0 \in D$ determines isomorphisms $\alpha : T \to S$ and $\alpha' : T' \to S'$ such that $\alpha' s_i = t_i \alpha$ (for $i = 1, 2$). Furthermore the local degrees r and s of the maps c_1 and c_2 are given by*

$$r = \deg(c_1) = \deg(t_1) = [\Gamma_N : \Gamma'_N],$$

$$s = \deg(c_2) = \deg(t_2) = [\gamma^{-1} \Gamma_N \gamma : \Gamma'_N].$$

8. COHOMOLOGY

Let E denote a (rationally defined) finite dimensional complex representation of G. This representation restricts to a representation of Γ which we identify with the fundamental group of X, $\Gamma \cong \pi_1(X, x_0)$. In this way we obtain a local coefficient system (which we also denote by E) on the locally symmetric space X. We will be primarily concerned with two cohomology theories:

(a) The cohomology of the group Γ, which may be identified with

$$H^*(X; E) = H^*(\bar{X}; Ri_* E)$$

(where $i : X \to \bar{X}$ denotes the inclusion of the locally symmetric space into its reductive Borel-Serre compactification)

(b) The L^2 cohomology $H_2^*(X; E)$ of X, which we consider only in the case that D is a Hermitian symmetric space, and which may be identified with the intersection cohomology $IH^*(\hat{X}; E)$ of the Baily-Borel compactification of X [SS],[L], or with the middle weighted cohomology $W^*(\bar{X};\ E)$ [GHM] of the reductive Borel-Serre compactification.

In either case the Hecke correspondence described above induces a self homomorphism on these cohomology groups, which we now denote simply by H^* (so as to treat both cases simultaneously). We consider the alternating sum of the traces of this homomorphism

$$L(\Gamma, g) = \sum_{i=0}^{\dim(X)} (-1)^i \mathrm{Tr}(\bar{C}^* : H^i \to H^i).$$

It follows from the Lefschetz fixed point theorem of Grothendieck, Verdier, and Illusie [GI] that in each of the cases (a) and (b) above, there are locally defined contributions LC(F) which may be associated to each fixed point stratum such that

Theorem (Version 0). *The Lefschetz number $L(\Gamma, g)$ is given by*

$$L(\Gamma, g) = \sum_P \sum_F LC(F),$$

where the outer sum is taken over a set of representatives P of Γ'-conjugacy classes of rational parabolic subgroups of G, and the second sum is taken over all connected components F of the intersection of the fixed point set with the boundary component $C_P \subset \bar{C}$.

9. LOCAL CONTRIBUTIONS

The parabolic subgroup P is contained in a collection $Q_1, Q_2, \dots, Q_\ell$ of maximal parabolic subgroups. These are in one to one correspondence with the simple roots $\Delta = \{\sigma_1, \sigma_2, \dots, \sigma_\ell\}$ of A_P occurring in $\mathfrak{n}_P = Lie(N_P)$. They determine a canonical isomorphism $A_P^0 \to (\mathbb{R}_+)^k$ by $a \mapsto (a^{\sigma_1}, a^{\sigma_2}, \dots a^{\sigma_\ell})$. (This set of roots is denoted $\Delta - I$ in [BS] §4.2, but note as in [Z2] §1.2 that in our case, G acts on X from the left, so $(\mathbb{R}_+)^k$ is compactified by adding points at infinity rather than by adding points at 0 as in [BS].) For any $y \in P$ we denote by y_A its projection to A_P^0 under the composition $P \to L = P/N_P \to A_P^0 = L/{}^0L$. Then $y_A^{\sigma_k} < 1$ iff $y \in P$ moves points in $\bar{D}$ away from the boundary component D_{Q_k}, and in this case we will say that y acts in an "expanding" manner near the boundary component D_{Q_k}. (See also [Z1] §3.19.) Define

$$I(\gamma) = \left\{k | \gamma^{2\sigma_k} \leq 1\right\}$$

to be the set of indices k for which the action of γ near D_{Q_k} is not contracting. The local system E on X restricts to a local system (which we also denote by E) on the nilmanifold S.

Theorem (Version 1a). *In case (a) above, the local contribution LC(F) to the Lefschetz number is given by*

$$LC(F) = \begin{cases} \chi(F) \sum_i (-1)^i (-1)^{|I(\gamma)|} \operatorname{Tr}((s_1)_*(s_2)^* : H^i(S;E) \to H^i(S;E)) & \text{if } I(\gamma) = \phi; \\ 0, & \text{if } I(\gamma) \neq \phi; \end{cases}$$

where $\chi(F)$ denotes the Euler characteristic of F, and $\gamma \in P$ is the characteristic element associated to any fixed point in F as defined above.

The analogous formula in case (b) is more complicated because it involves decomposing the cohomology $H^i(S;E)$ into pieces under the action of Looijenga's local Hecke correspondences [L]. In the statement of Version 2 of our formula for the local contribution, the action of these local Hecke correspondences will be replaced by an action of the torus A_P^0 on the Lie algebra cohomology of $\mathfrak{n} = Lie(N_P)$.

The characters σ_k restrict to characters on P which are trivial on N_P so they define a rational basis of the character group of A_P. ([BS] §4.2) Choose a set $t_1, t_2, \dots, t_\ell \in A_P^0$ of dual "contracting" elements, in other words $\sigma_k(t_j) = 1$ if $j \neq k$, and $\sigma_k(t_k) > 1$. The elements $t_1, t_2, \dots, t_\ell$ may even be chosen to be sufficiently divisible (see [L] § 3.6), i.e. for an appropriately chosen rationally defined lift $A'_P \subset P$ of A_P, the corresponding elements t'_k satisfy $t'_k \Gamma_N t_k'^{-1} \subset \Gamma_N$. Conjugation by such an element t'_k gives a self map $\Psi_k : S \to S$ and an induced homomorphism on (ordinary) cohomology,

$$\Psi_k^* : H^*(S; E) \to H^*(S; E).$$

Decompose the cohomology $H^i(S; E)$ into simultaneous eigenspaces H^{ij} under the action of the commuting endomorphisms $\Psi_1, \Psi_2, ..., \Psi_\ell$ and denote the corresponding eigenvalues by λ_{ijk}. (Then the sufficiently divisible elements act on H^{ij} through a character λ_{ij} of A_P and $\lambda_{ijk} = t_k^{\lambda_{ij}}$.) We also denote by ζ_k the eigenvalue of Ψ_k^* on the one-dimensional group $H^{\text{top}}(S; \mathbb{C})$. (Then ζ is a character of A_P with $\zeta_k = t_k^{\zeta}$.) Define

$$I(H^{ij}) = \{k | \lambda_{ijk}^2 \geq \zeta_k\}.$$

Theorem (Version 1b). *In case (b) the local contribution to the Lefschetz number from the fixed point stratum F is given by*

$$LC(F) = \chi(F) \sum_{i,j} (-1)^i (-1)^{|I(\gamma)|} \mathrm{Tr}(s_{1*} s_2^* : H^{ij} \to H^{ij}),$$

where the sum is taken over all pairs (i, j) such that $I(\gamma) = I(H^{ij})$.

Remarks. Thus, in the case of the cohomology of the group Γ, the only contributions to the Lefschetz number arise from fixed points which are "contracting", as predicted by [H1] (see also [H2] and [B]). In the case of L^2 cohomology, the contracting-expanding properties of the fixed point determine which piece of the cohomology of S will contribute.

Although version 1 is the least explicit version of our formula, the proof involves several steps:

(1) using the Zucker conjecture [L], [SS] we identify the L^2 cohomology of X with the "middle weighted cohomology" $W^*(\bar{X}, E)$ [GHM] of $\bar{X}$ (rather than with the intersection cohomology of the Baily-Borel compactification of X, as one might have expected).

(2) Using a new topological result [GM] we find that the contribution $LC(F)$ to the Lefschetz number is equal to the trace of the induced homomorphism on the local cohomology (of the weighted cohomology complex) at F, with supports which are compact in the expanding directions and closed in the contracting directions.

(3) Identify this stalk cohomology with supports as the sum of those pieces H^{ij} of the cohomology of S such that $I(\gamma) = I(H^{ij})$ (and with a degree shift by $|I(\gamma)|$).

10. $\mathfrak{n}$-COHOMOLOGY

Now we use the theorem of Nomizu [N] and Van Est [V] to identify the cohomology of S with the Lie algebra cohomology of N_P. Let $\mathfrak{n}$ denote the (real) Lie algebra of N_P. The adjoint action of the parabolic subgroup P on N_P induces an adjoint action of the Levy factor $L = P/N_P$ on the Lie algebra $\mathfrak{n}$ and also on the Lie algebra cohomology $H^i(\mathfrak{n}; E)$.

Remark. The Lie algebra cohomology $H^i(\mathfrak{n}, E)$ is the cohomology of the cochain complex $\mathrm{Hom}(\wedge^i\mathfrak{n}, E)$ on which there are two actions of P : The (usual) left action by $x \in P$ is given by $x \bullet f(v) = \phi(x)f(x^{-1}vx)$ where $f \in \mathrm{Hom}(\wedge^i\mathfrak{n}, E)$ and $\phi : G \to Gl(E)$ is the representation of G on the coefficients. Unfortunately, for $x = \gamma$ this disagrees with the geometrically defined action on the cohomology $H^i(S; E)$ of the nilmanifold S which we consider above. Therefore we will refer to the ***geometric*** action of an element $x \in P$ on the Lie algebra cohomology as the action which is given by $x \bullet f(v) = \phi(x^{-1})f(xvx^{-1})$. It coincides with the (usual) action by x^{-1} and defines a right action of P on $H^i(\mathfrak{n}, E)$.

We decompose the induced L-representation on $H^i(\mathfrak{n}, E)$ into irreducible components $\mathfrak{n}_{ij}$ (which are the same under either the right action or the left action). Let γ_{ij} denote the trace of the (geometric) action of γ on the component $\mathfrak{n}_{ij}$. The geometric action of the $t_k \in A_P^0 \subset L$ on $\mathfrak{n}_{ij}$ is by some multiple λ_{ijk} of the identity. The action of t_k on $\bigwedge^{top}(\mathfrak{n})$ is given by multiplication with some number $\zeta_k = t_k^{\zeta}$. (The decomposition of the $\mathfrak{n}$-cohomology into components $\mathfrak{n}_{ij}$ will refine the previous decomposition of $H^i(S; E)$ into A_P-isotypical components, so the use of the same symbols λ_{ijk} and ζ_k in this setting is a slight abuse of notation.)

Definition. Set $I(\mathfrak{n}_{ij}) = \left\{k | \lambda_{ijk}^2 \geq \zeta_k\right\}$.

Theorem (version 2). *The local contribution LC(F) to the Lefschetz number $L(\Gamma, g)$ is given by*

$$LC(F) = \chi(F) r \sum_{(i,j)} (-1)^i (-1)^{|I(\gamma)|} \gamma_{ij},$$

where in case (b), the sum is taken over all pairs (i, j) such that $I(\gamma) = I(\mathfrak{n}_{ij})$, while in case (a) the sum is taken over all pairs (i, j) if $I(\gamma) = \phi$, and is taken over no pairs (i, j) otherwise (i.e. the contribution is 0 if $I(\gamma) \neq \phi$).

11. KOSTANT'S THEOREM

The numbers γ_{ij} may be determined using Kostant's theorem on $\mathfrak{n}$-cohomology ([K] [V]). If γ_L denotes the projection of γ to the Levy factor $L = P/N_P$, then γ_L is semisimple (in fact $\mathrm{ad}(\gamma_L)$ is elliptic). The trace of its (geometric) action on $H^i(\mathfrak{n}; E)$ is equal to the trace of the action of its inverse, γ_L^{-1} through the (usual action of the) complexification $L(\mathbb{C}) \supset L(\mathbb{R}) \ni \gamma_L^{-1}$ on $H^i(\mathfrak{n}(\mathbb{C}); E)$. Choose a maximally split θ-stable Cartan subgroup $H \supset A_P$ of L. Choose a system of positive roots $\Phi^+ = \Phi^+(\mathfrak{g}(\mathbb{C}), H)$ for H in $\mathfrak{g}(\mathbb{C})$ such that $\Phi^+ \supset \Phi(\mathfrak{n}(\mathbb{C}), H)$. Then

$$\Phi^+(L) = \Phi^+ \cap \Phi(\mathfrak{l}(\mathbb{C}), H)$$

is a system of positive roots for $\mathfrak{l}(\mathbb{C})$, and $\Phi^+ = \Phi^+(L) \cup \Phi(\mathfrak{n}(\mathbb{C}), H)$. Corresponding to this union we have $\rho = \rho_G = \rho_L + \rho_N$ where

$$\rho_G = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha \ , \ \rho_L = \frac{1}{2} \sum_{\alpha \in \Phi^+(L)} \alpha, \quad \text{and} \quad \rho_N = \frac{1}{2} \sum_{\alpha \in \Phi(\mathfrak{n}(\mathbb{C}))} \alpha.$$

Let $W = W(\mathfrak{g}(\mathbb{C}), H)$ denote the Weyl group of H in $\mathfrak{g}(\mathbb{C})$ and for $\mathbf{w} \in W$, set

$$\begin{aligned} \Phi^+(w) &= \Phi^+ \cap w\Phi^- \\ \ell(w) &= |\Phi^+(w)| \\ I(w) &= \{k | t_k^{w(-\lambda-\rho)+\rho} \le t_k^{\rho_N})\}, \end{aligned}$$

where λ denotes the highest weight of the representation E and where $t_k^{\rho_N}$ means $\sqrt{t_k^{2\rho_N}}$ in the event that ρ_N is not a character of A_P. (Note that $w(-\lambda - \rho) + \rho$ is always a character.) We also define

$$W' = \{w \in W | \Phi^+(w) \subset \Phi(\mathfrak{n}(\mathbb{C}), H)\}.$$

For any dominant weight μ of L, let ψ_μ denote the trace of the L-representation with highest weight μ.

Theorem (Version 3). *Suppose the representation E of G is irreducible, with highest weight λ. Then the local contribution LC(F) to the Lefschetz number $L(\Gamma, g)$ is given by*

$$LC(F) = (-1)^{|I(\gamma)|}\chi(F) r \sum_w (-1)^{\ell(w)} \psi_{w(\lambda+\rho)-\rho}(\gamma_L^{-1}),$$

where in case (b) the sum is over all $w \in W'$ such that $I(w) = I(\gamma)$, while in case (a) the sum is over all $w \in W'$ if $I(\gamma) = \phi$, and the last sum is over no elements w if $I(\gamma) \neq \phi$ (i.e. the contribution from F is 0 if $I(\gamma) \neq \phi$).

12. Sp(4)

In this example we suppose Γ is a torsion free arithmetic subgroup of $Sp(4,\mathbb{Z})$ and $\bar{X}$ is the reductive Borel-Serre compactification of the locally symmetric space $\Gamma\backslash \mathrm{Sp}_4(\mathbb{R})/\mathrm{U}(2)$. Then $\dim_{\mathbb{R}}(\bar{X}) = 6$ and there are three kinds of boundary components (see [G]) X_P corresponding to the three kinds of proper parabolic subgroups P:

(I) P is the stabilizer of a rational isotropic line $L \subset \mathbb{R}^4$. Then X_P is a locally symmetric space for the group $\mathrm{Sp}(L^{\perp}/L)$ and has (real) dimension 2.

(II) P is the stabilizer of a rational Lagrangian plane $Q \subset \mathbb{R}^4$. Then X_P is a locally symmetric space for the group $Gl(Q)$ and has (real) dimension 2.

(III) P is a Borel subgroup, i.e. the stabilizer of a rational isotropic flag $0 \subset L \subset Q \subset \mathbb{R}^4$. Then X_P is a point.

Suppose $g \in \mathrm{Sp}_4(\mathbb{Q})$ defines a Hecke correspondence

$$(c_1, c_2) : \bar{C} \rightrightarrows \bar{X}$$

on $\bar{X}$ with a fixed point F at a boundary component $X_P = \{F\}$ of type (III). (This is the most interesting case since $\dim(A_P) = 2$.) We denote by Q_1 and Q_2 the maximal parabolic subgroups (of type I and II respectively) which contain P, and we denote by X_1 and X_2 the corresponding boundary components.

Let $\gamma \in P(\mathbb{Q})$ be the characteristic element which determines the local behavior of the correspondence near the fixed point. For simplicity we will assume that F is an isolated fixed point and that the projection γ_P of γ to the split torus $A_P = P/N_P$ lies in the identity component A_P^0 and is regular. Let $r = [\Gamma_N : \Gamma'_N]$ and $s = [\gamma^{-1}\Gamma_N\gamma : \Gamma'_N]$ denote the local degrees of the finite maps c_1 and c_2 near the fixed point F as above. Consider the (positive) roots of A_P in N_P, $\Phi^+ = \{\sigma_1, \sigma_2, \sigma_1 + 2\sigma_2, \sigma_1 + \sigma_2\}$, where σ_1 is the long simple root. Define $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$.

Lemma. *The value of ρ on γ_P is given by $\gamma_P^{\rho} = \sqrt{\frac{s}{r}}$.*

Substituting this into version 3 of the theorem we obtain

Theorem. *The contribution to the Lefschetz number from an isolated fixed point F in a minimal boundary component is*

$$LC(F) = \sqrt{rs}(-1)^{|I(\gamma)|}\sum_{w\in W_\gamma}(-1)^{\ell(w)}\gamma_P^{-w(\lambda+\rho)}$$

where λ is the highest weight of the representation giving rise to the local system E on X, and in the case (b) of L^2 cohomology,

$$W_\gamma = \{w \in W | I(\gamma) = I(w)\}$$

while in the case (a) of ordinary cohomology,

$$W_\lambda = \begin{cases} W, & \text{if } I(\gamma) = \phi; \\ \phi, & \text{if } I(\gamma) \neq \phi. \end{cases}$$

The surprising factor of $\sqrt{rs}$ was predicted by R. Kottwitz. The correspondence restricts to correspondences on the neighboring boundary components X_1 and X_2. Denote by r_1, s_1 the local degrees of the induced correspondence on the boundary component X_1, and denote by r_2, s_2 the local degrees of the induced correspondence on the boundary component X_2. (These local degrees also have descriptions in terms of the indices of certain discrete subgroups.) The regularity assumption on γ implies that $r_1 \neq s_1$ and $r_2 \neq s_2$.

Lemma. *The values of the roots on γ_P are given by:*

$$\gamma_P^{\sigma_1} = \frac{s_2}{r_2} \qquad \gamma_P^{\sigma_2} = \frac{s_1}{r_1}$$

with

$$r = r_1^4 r_2^3 \quad \text{and} \quad s = s_1^4 s_2^3.$$

Thus there are four cases (analogous to the "graph" and "anti-graph" cases in [KR] *or to the cases "$d > a$" and "$a > d$" of* [La] §*7.12):*

$$I(\gamma) = \begin{cases} \phi, & \text{if} \quad s_1 \geq r_1 \quad \text{and} \quad s_2 \geq r_2; \\ \{1\}, & \text{if} \quad s_1 \geq r_1 \quad \text{and} \quad s_2 < r_2; \\ \{2\}, & \text{if} \quad s_1 < r_1 \quad \text{and} \quad s_2 \geq r_2; \\ \{1,2\}, & \text{if} \quad s_1 < r_1 \quad \text{and} \quad s_2 < r_2; \end{cases}$$

and

$$I(w) = \begin{cases} \phi, & \text{if} \quad w = S_2 S_1 S_2 \text{ or } S_1 S_2 S_1 \text{ or } I; \\ \{1\}, & \text{if} \quad w = S_1 S_2; \\ \{2\}, & \text{if} \quad w = S_2 S_1; \\ \{1,2\}, & \text{if} \quad w = S_1 \text{ or } S_2 \text{ or } S_1 S_2 S_1 S_2, \end{cases}$$

where S_1 and S_2 are the simple reflections through the hyperplanes $\ker(\sigma_1)$ *and* $\ker(\sigma_2)$ *respectively. It is now possible to evaluate the preceding formula in terms of r_i and s_i . The formula simplifies somewhat if we take $\lambda = 0$. This gives*

Theorem. *Suppose the local system E on X is trivial, so $\lambda = 0$. Then the local contribution to the Lefschetz number from the fixed point F (in the case (b) of L^2 cohomology) is*

$$LC(F) = \begin{cases} r_1^3 r_2^2(r_1r_2 - r_1s_2 - r_2s_1), & \text{if } s_1 \geq r_1 \text{ and } s_2 \geq r_2; \\ -r_1s_1^3r_2^2s_2, & \text{if } s_1 \geq r_1 \text{ and } s_2 < r_2; \\ -r_1^3s_1r_2s_2^2, & \text{if } s_1 < r_1 \text{ and } s_2 \geq r_2; \\ s_1^3s_2^2(s_1s_2 - r_1s_2 - r_2s_1), & \text{if } s_1 < r_1 \text{ and } s_2 < r_2; \end{cases}$$

while in the case (a) of ordinary cohomology the local contribution is

$$LC(F) = \begin{cases} 0, & \text{if } s_1 < r_1 \text{ or } s_2 < r_2; \\ p, & \text{if } s_1 \geq r_1 \text{ and } s_2 \geq r_2, \end{cases}$$

where p is the polynomial

$$p = r_1^4r_2^3 - r_1^4r_2^2s_2 - r_1^3s_1r_2^3 + r_1s_1^3r_2^2s_2 + r_1^3s_1r_2s_2^2 + s_1^4s_2^3 - r_1s_1^3s_2^3 - s_1^4r_2s_2^2.$$

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