MIXED-CHARACTERISTIC GEOMETRIC CASSELMAN–SHALIKA

Contents

1. Motivation: L-functions

Let G denote a split reductive group over a number field F. Let π denote an automorphic representation of $G(\mathbb{A}_F)$. For example, for GL_2, you can keep in mind the adèlization of a modular form or Maass form of some level. By Flath's theorem, there exists a decomposition

$$\pi = \bigotimes_v \pi_v$$

over the places of F, where each π_v is a smooth admissible representation of $G(F_v)$, and almost all of them are unramified.

One of the central questions in the Langlands program is to understand the various L-function of π, defined as

$$L(s, \pi, \rho) = \prod_v{}' L(s, \pi_v, \rho) = \prod_v{}' \det(\mathrm{id}_n - \rho(g(\pi_v))q_v^{-s})$$

where $\rho : \widehat{G} \to \mathrm{GL}_n(\mathbb{C})$ varies over irreducible algebraic representations of the Langlands dual group $\widehat{G}$, and where $g(\pi_v)$ is the semisimple conjugacy class in $\widehat{G}$ via the theory of unramified representations and the Satake isomorphism.

In general, one wants to show that $L(s, \pi, \rho)$ converges on $\mathrm{Re} >> 0$, and admits a meromorphic continuation to $\mathbb{C}$ and satisfies a functional equation. To do this, the basic strategy is to try to express the L-function as a certain *period integral*, and then prove analytic continuation using integrals, which are part of a more intrinsically analytic theory. One can do this locally, i.e. by expressing each $L(s, \pi_v, \rho)$ as a local period integral.

Example 1.0.1. The Riemann zeta function is the simplest example of this phenomenon. In this case, $G = \mathrm{GL}_1$ over $F = \mathbb{Q}$ and π is the trivial representation. The computation of the local zeta integral becomes

$$\int_{\mathbb{Q}_p^\times} \chi_{\mathrm{triv}}(x)\mathbf{1}_{\mathbb{Z}_p}(x)|x|_p^s \,\mathrm{d}x = \frac{1}{1-p^s} = L(s, \pi, \rho)$$

which is exactly the factor appearing in the Euler factorization.

More generally, you want to express the local L-factor as the integral involving a certain special function defined for unramified representations, and in some (generic) cases this can be done using a so-called "Whittaker function". In certain cases, one does this by for a certain class of π which are called "generic", using the theory of *Whittaker models*. For example, this theory can be used to prove analytic continuation and functional equation when $G = \mathrm{GL}_2$ and π is cuspidal automorphic. It is also used in the Rankin–Selberg method in order to understand $L(s, \pi_1 \times \pi_2)$ for π_1 and π_2 aut. reps of GL_n and GL_m.

2. Whittaker models

Fix a nonarchimedean local field F with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k of size q. Fix G a split reductive group over F, and choose a Borel subgroup $G = TN$ with T a maximal torus and N the unipotent radical.

We are interested in the representation theory of $G(F)$, and in particular we care about representations (π, V) which are smooth and admissible; recall this means that every $v \in V$ is fixed by an open subgroup of $G(F)$, and that . Fix π an unramified representation: recall this means that

$$\pi^{G(\mathcal{O})} \neq 0.$$

Fix $\psi : N(F) \to \overline{\mathbb{Q}}_\ell$ is a character nontrivial on each root subgroup of $N(F)$. Then uniqueness of Whittaker models says that

$$\dim \operatorname{Hom}_G(\pi, \operatorname{Ind}_{N(F)}^{G(F)} \psi) \leq 1$$

and when it is equal to one we say that π has a Whittaker model, which refers to the image of a basis vector. Thus we exhibit π as a space of functions on G, subject to a simple transformation property with respect to $N(F)$.

Not every representation admits a Whittaker model, but the ones that do are called *generic*. For example, if χ is an unramified character of $T(F)$ such that the normalized parabolic induction $\operatorname{Ind}_B^G \chi$ is irreducible, then it's generic.

3. The Casselman–Shalika Formula

Since we seem to care about unramified representations, and perhaps we'll stick to the ones that are generic, how should we understand them? Recall that irreducible unramified representations correspond to simple modules for

$$\mathcal{H}_F = \mathcal{C}_c^\infty(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}), \overline{\mathbb{Q}}_\ell),$$

the (commutative) spherical Hecke algebra for G. The Satake isomorphism gives an explicit description of this algebra, via an explicit isomorphism

$$\mathcal{H}_F \xrightarrow{\sim} K_0(\mathsf{Rep}_{\overline{\mathbb{Q}}_\ell}(\widehat{G})) \cong \overline{\mathbb{Q}}_\ell[X_*(T)_+]$$

to the representation ring of G, where $X_*(T)_+$ is the set of dominant coweights. Write A_λ for the basis vector corresponding to $\lambda \in X_*(T)_+$.

Generic representations embed into the smooth representation $\operatorname{Ind}_{N(F)}^{G(F)} \psi$. We care about unramified representations, so it makes sense to try to understand $(\operatorname{Ind}_{N(F)}^{G(F)})^K$. In fact, for what we want it turns out we can just look at the compact induction. It therefore makes sense to try to understand the invariants

$$W = (\operatorname{c\,Ind}_{N(F)}^{G(F)} \psi)^K = \mathcal{C}_c((N(F), \psi)\backslash G(F)/G(\mathcal{O}), \overline{\mathbb{Q}}_\ell)$$

How do you understand this space? Well, there's a decomposition

$$G(F) = \bigsqcup_{\lambda \in X_*(T)} N(F)\lambda(\varpi)G(\mathcal{O}).$$

So we can study individual cosets, but one can show that a coset only supports a nonzero function if $\lambda \in X_*(T)_+$ is dominant.

So you have an algebra acting on a space of functions, both of which have the same basis. So you might hope that the action plays well with respect to the bases. Let $\phi_\lambda \in W$ denote the unique nonzero function such that $\phi_0(\lambda(\varpi)) = 1$.

Theorem 3.0.1 (Shintani, Kato, Casselman–Shalika, Frenkel–Gaitsgory–Kazhdan–Vilonen)**.**

$$\phi_0 \star A_\lambda = q^{(\lambda, \rho)} \phi_\lambda.$$

so W is free of rank 1 over $\mathcal{H}$.

So the action plays very well with respect to the basis. This formula can be used to give an explicit formula for a certain special unramified Whittaker function, namely the one satisfying

(1) $W \cdot A_\lambda = \mathrm{tr}(g(\pi), V_\lambda) \cdot W$.

so that

$$W = \sum_{\lambda dom} \mathrm{tr}(g(\pi), V_\lambda) \cdot \phi_\lambda$$

4. Geometrization

How should we geometrize this? Note the formula amounts to saying that

$$\int_{N(F)} A_\lambda(x^{-1}\nu(\lambda))\psi(x)dx = \begin{cases} 0 & \text{if } \nu \neq \lambda \\ q^{-(\rho,\nu)} & \text{if } \nu = \lambda. \end{cases}$$

In other words,

$$\int_{N(F)\nu(\varpi)} A_\lambda(x)\psi(x^{-1}t^{-\nu})dx = \begin{cases} 0 & \text{if } \nu \neq \lambda \\ q^{-(\rho,\nu)} & \text{if } \nu = \lambda. \end{cases}$$

In view of the Grothendieck–Lefschetz trace formula, if we can exhibit $N(F)\varpi^\nu$ as the points of some geometric object over $\mathbb{F}_q$, then we can probably geometrize this.

So how does this work? Well, one way to geometrize the Hecke algebra is to use the affine Grassmannian. If we now assume that $F = \mathbb{F}_q((\varpi))$, then there exists an ind-scheme Gr_G over $\mathbb{F}_q$, whose $\mathbb{F}_q$-points are exactly $G(F)/G(\mathcal{O})$. Inside of this lives an $\mathbb{F}_q$-points $\nu(\varpi)G(\mathcal{O})$, and the $N(F)$-orbit of this point can also be given an ind-subscheme structure. Moreover, $\mathrm{Gr}_G = \operatorname{colim} \mathrm{Gr}_{\leq\lambda}$ can be written as the colimit over a explicit set of projective varieties over $\mathbb{F}_q$.

The function A_λ is geometrized by a certain $G(\mathcal{O})$-equivariant perverse sheaf $\mathcal{A}_\lambda$ defined over $\mathrm{Gr}_{G,\overline{\mathbb{F}}_q}$ whose Frobenius-traces recover $\mathcal{A}_\lambda$. This sheaf is supported on $\mathrm{Gr}_{\leq\lambda}$.

How do you geometrize ψ?. Well, the $N(F)$-orbit of $\nu(\varpi)$ also admits an ind-scheme structure, and we call this $S_\nu \subset \mathrm{Gr}_G$. Since there's a group homomorphism $\mathbb{F}_q((t)) \to \mathbb{F}_q$ sending $f(x) = \sum c_i x^i \mapsto c_{-1}$, we may as well take ψ to be the map

$$N(F) \xrightarrow{c_{-1}} N(\mathbb{F}_q) \to \mathbb{F}_q \to \overline{\mathbb{Q}}_\ell^\times.$$

This naturally admits a geometrization $h : S_\nu \to \mathbb{A}^1$. But $\mathbb{A}^1$ admits a natural étale $\mathbb{F}_q$-torsor by the Lang isogeny, so one gets an Artin–Schreier sheaf $\mathcal{L}_\psi$ over $\mathbb{A}^1$.

Then:

Theorem 4.0.1 (Geometric Casselman–Shalika, due to Ngo–Polo, Frenkel–Gaitsgory–Vilonen)**.**

$$R\Gamma_c^{\text{ét}}(S_\nu \cap \mathrm{Gr}_{\leq\lambda}, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi) = \begin{cases} 0 & \textit{if } \nu \neq \lambda \\ \overline{\mathbb{Q}}_\ell[2(\rho,\nu)](-(\rho,\nu)) & \textit{if } \nu = \lambda. \end{cases}$$

Remark 4.0.2. Frenkel–Gaitsgory–Vilonen categorify this picture, by defining a "Whittaker category" and defining an action of the Satake category on the Whittaker category, and studying this action, yielding the theorem as a corollary. Ngo–Polo have a more direct proof.

4.1. Mixed Characteristic.

What happens in mixed characteristic?

(1) We can no longer use the usual affine Grassmannian. Instead, we have to use the Witt vector affine Grassmannian. Before, Gr_G was the sheafification of

$$R \mapsto R((t))/R[[t]].$$

In mixed characteristic, it should be the sheafification of

$$R \mapsto W(R)[1/p]/W(R)$$

except that Witt vectors don't behave well unless R is a perfect $\mathbb{F}_q$-algebra.

Xinwen Zhu fixes this problem by simply restricting to perfect R-algebras, or in other words by taking the perfection. He and Bhatt–Scholze then show that this functor is represented by an ind-(perfection of projective variety).

He defines similar subspaces S_ν and $\mathrm{Gr}_{\le\lambda}$, and shows that they behave very similarly.

(2) A more serious issue is that the additive character

$$\mathbb{F}_q((t)) \to \overline{\mathbb{Q}}_\ell^\times$$

can be chosen to factor through $\mathbb{F}_q$. But if $F = \mathbb{Q}_p$, then any nontrivial $\mathbb{Q}_p \to \overline{\mathbb{Q}}_\ell^\times$ cannot factor through $\mathbb{F}_p$ in any way. In fact, they're all equal to

$$x \mapsto e^{2\pi i a x}.$$

for some $a \in \mathbb{Q}_p$. If we take $a = 1$, then at least it factors through $\mathbb{Q}_p/\mathbb{Z}_p$. But these are exactly the $\mathbb{F}_p$-points of $\mathrm{Gr}_{\mathbb{G}_a}$. So we can still define a morphism

$$h : S_\nu \to \mathrm{Gr}_{\mathbb{G}_a}.$$

But $\mathrm{Gr}_{\mathbb{G}_a}$ is still an infinite-dimensional thing, which makes trying to define some character local system a real headache. But the key observation is that luckily

$$h|_{S_\nu \cap \mathrm{Gr}_{\le\lambda}}$$

lands in some finite-dimensional sub-(perfect scheme), whose $\mathbb{F}_q$-points are $p^{-r}\mathbb{Z}_p/\mathbb{Z}_p$ (in fact this is the perfection of an affine space).

So

Theorem 4.1.1 (I.–Lin, in progress)**.** *The geometric Casselman–Shalika formula is true in mixed-characteristic.*

Our proof essentially blends Ngo–Polo with Zhu's work. The overall strategy is similar, but there are a number of technical differences, due to the fact that h is defined differently, and due to the fact that there is a difference in the way a certain smooth resolution of $\mathrm{Gr}_{\le\mu}$ is constructed for quasi–minuscule coweights.