RESEARCH STATEMENT

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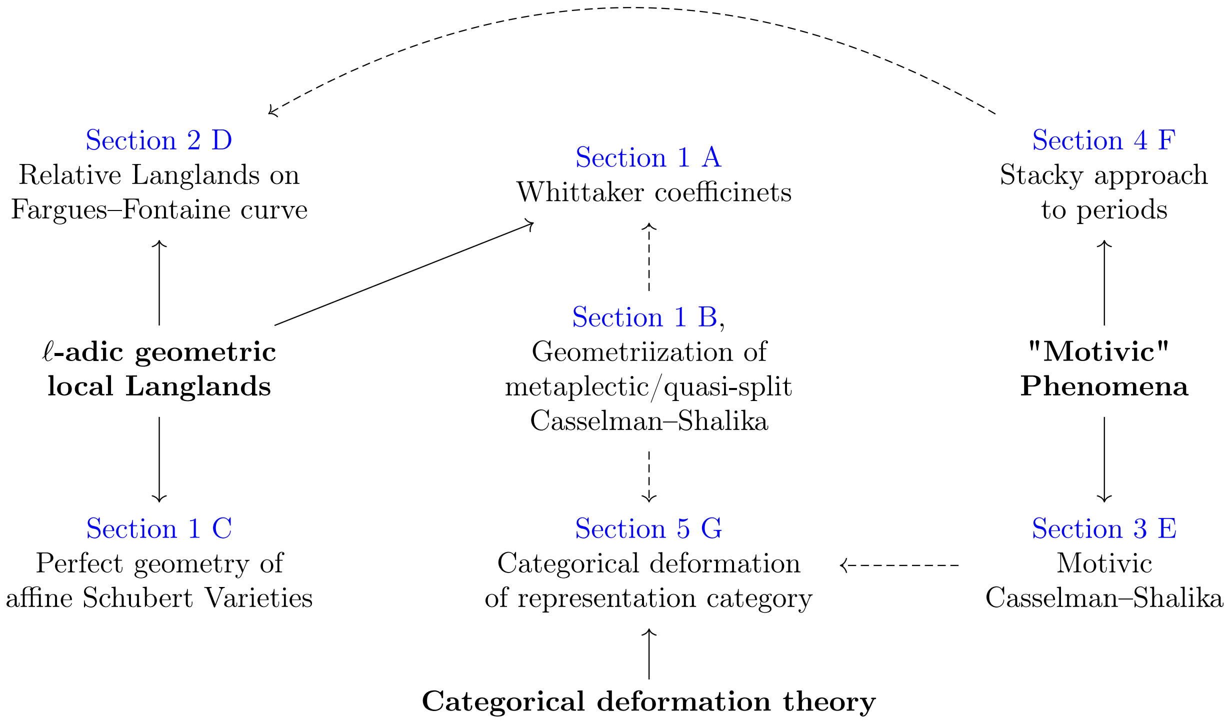
Introduction

My areas of interest in mathematics are:

1. The **Langlands correspondence**, which is now a huge web of conjectures, from special values of L functions to conformal field theory. My particular interest is in various incarnations of **Casselman–Shalika formula**, B, the **ℓ-adic geometric local Langlands correspondence**, A and a local analog of **relative Langlands**, D.
2. **Motivic phenomena**, where I hope to explore a motivic version of Casselman–Shalika formula, E, and a stacky approach to periods, F.
3. **Stable homotopy theory**, where the central problem is to understand he **sphere spectrum**. I hope to understand through **categorical deformation theory**, G.

These research are interconnected, in the following diagram.



Date: September 22, 2024.

The priority of research is listed in the following order[1],

B=D=G>A=E>F> C.

My current research thus focuses on three main areas: (i) the Relative Langlands program on the Fargues–Fontaine curve, (ii) the geometric Casselman–Shalika formula for metaplectic and quasi-split groups, and (iii) categorical deformations in representation theory.

Notations. Theorems stated have full proof written by either me or joint with collaborators, *unless* it is annotated with: ***expected*** – no proofs have been written down but is believed to hold, or ***in progress*** – sketch outlines of proof are written, available upon request. We use freely the language of higher categories and higher algebra, [Lur09], [Lur18].

1. Mixed Characteristic Geometry and the Local Langlands Program

Let G be a connected reductive group over a nonarchimedean local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld, see [AG15], [Gai14]. We let $\mathrm{LS}_{\hat{G}}$, be the moduli stack of L-parameters of $\hat{G}$, Bun_G the moduli stack of G-bundles on the Fargues–Fontaine curve. They constructed two actions

$$\begin{tikzcd} & \mathrm{IndPerf}(\mathrm{LS}_{\hat{G}}) & \\ \mathrm{IndCoh}^{qc}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}}) \arrow[ur, phantom, "\circlearrowleft"] & & D_{\mathrm{lis}}(\mathrm{Bun}_G) \arrow[ul, phantom, "\circlearrowright"'] \end{tikzcd}$$

The left hand is induced from the natural tensor structure, while the right hand is the so-called *spectral action*, a terminology coined in *Betti Langlands program* [NY19]. For a fixed choice of Whittaker datum, [FS24, I.10.2] conjectured a $\mathrm{IndPerf}(\mathrm{LS}_{\hat{G}})$-linear equivalence

$$\mathbb{L}_G : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \simeq \mathrm{IndCoh}^{\mathrm{qc}}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}})$$

One fundamental aspect is to understand the *Whittaker coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, this corresponds to finding the Fourier coefficients of automorphic functions.

Example 1.1. Let $G = \mathrm{PGL}_2$ be the projective linear group over $\mathbb{Q}$. A modular function, f, has an adelic formulation, $\widetilde{f}$ on $G(\mathbb{A}_{\mathbb{Q}})$. Up to a *normalizing factor*, the Fourier coefficients coincide with the Whittaker coefficients (Jacquet period),

$$a_m \sim \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_{\mathbb{Q}})} \widetilde{f}(n\alpha_m)\psi(-n)\, dn \qquad \text{for } m \geq 1$$

where $\alpha_m \in T(\mathbb{A}^{\mathrm{fin}}_{\mathbb{Q}})$ is m considered as a finite idèle and ψ is a standard character on $N(\mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_{\mathbb{Q}}$, where N is the radical of standard Borel. For more details, see [Gel75, Ch.3].

The first fundamental result is the *global Casselman-Shalika formula*, as proven in [FGV01], which we hope to replicate in the mixed characteristic setting.

Towards this goal, joint with Ashwin Iyengar (Johns Hopkins University) and Konrad Zou (Bonn University) [ILZ24], we proved a *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian Gr_G, analogous to the geometrization carried out in [NP01] for equi-characteristic local fields.

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[1] The alphabet links to the goal rather than the section.

Theorem 1.2. *[ILZ24] The Casselman–Shalika formula holds over the Witt vector Grassmanian.*

Our geometrization suggests the existence of a mixed-characteristic Whittaker category with the action of the spherical Hecke category, fitting into the program of Fargues–Scholze geometrization of the local l-adic Langlands program, [FS24].

Research Goal A. Whittaker coefficient functor in the Fargues–Fontaine setting. *In current joint work with A. Iyengar and K. Zou, we established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), necessary to understand the coefficient functor. This yields, with our results from [ILZ24], an Iwahori–Satake equivalence, [BGMRR19] without using nearby cycles.*

Theorem 1.3 (I.-Lin-Z., in progress)**.** *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [2]

Theorem 1.4 (Expected)**.** *Global–Casselman Shalika formula over the Fargues Fontaine curve (cf. [FGV01], [Gai21]) holds.*

Our end goal is to contribute towards an understanding to the mixed characteristic coefficient functor and the global Casselman–Shalika, Theorem 1.4. As of this writing, D. Hansen, L. Hamman, and L. Mann have made significant progress. My focus has thus oriented towards Research Goal B, exploring the geometrization of Langlands program for covering groups [GGW18].

Research Goal B. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [BBBG20; BBBG24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

Lastly, I would like to highlight that in our recent work [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\leq\mu}$, for minuscule and quasi-minuscule μ. In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal C. Geometry of general perfect Schubert variety *The first step is to prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhats-Tits building of G, we have an associated group scheme $\mathcal{G}$ over $\mathcal{O}$.*

Theorem 1.5 (Expected)**.** *The smooth locus of $\mathrm{Gr}_{\mathcal{G},\leq\mathfrak{s}}$ is $\mathrm{Gr}_{\mathcal{G},\mathfrak{s}}$ in perfect geometry, in the sense of [Zhu17].*

2. RELATIVE LANGLANDS ON THE FARGUES FONTAINE CURVE

Joint with Yuta Takaya (Tokyo University). We explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and L-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with $(\hat{G}, \hat{X})$ of hyper-spherical varieties. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair

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[2] In the set up of [BGMRR19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

$\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_! \Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \pi_* \omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. Bun_G has a Hardar-Narasimhan straification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G)$. In our case, $G = \mathbb{G}_m$, $\mathrm{Bun}_{\mathbb{G}_m}$ is stratified by $\mathrm{Bun}_{\mathbb{G}_m}^n$ for $n \in \mathbb{Z} = B(T)$. The following theorem explicitly computes the period sheaf.

Theorem 2.1 (Lin-T.)**.** *(1) Let $\mathcal{BC}(n)$ be the Banach-Colmez space of the line bundle $\mathcal{O}(n)$. The relative stack $\mathrm{Bun}_G^{X,n} := \pi^{-1}(\mathrm{Bun}_G^n)$ is a $\mathcal{BC}(n)$-torsor over Bun_G^n.*

(2) The restriction $\mathcal{P}_X^n := \mathcal{P}_X|_{\mathrm{Bun}_G^n}$ is described as follows.

  (a) If $n < 0$, $\mathcal{P}_X^n \simeq \Lambda$ as the trivial character of $\mathbb{Q}_p^\times$.

  (b) If $n > 0$, $\mathcal{P}_X^n \simeq \Lambda[-2n]$.

  (c) If $n = 0$, $\mathcal{P}_X^n \simeq C_c^\infty(\mathbb{Q}_p, \Lambda)$.

On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles (cf. [FW24, Ch. 6]).

Theorem 2.2. *[Lin-T., expected] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[3] $\mathcal{L}_{\hat{X}}$.*

Research Goal D. Relative Langlands on the Fargues Fontaine curve. *Complete Theorem 2.2 as a first step and then the Hecke case, which classically corresponds to Hecke's integral representation of standard L-function for GL_2. Lastly, one can ask whether on the $\mathcal{B}$-side, the same constructions of [BSV, Ch. 11] works for the p-adic (Emerton-Gee) L-parameter stacks, which potentially give new interpretations to p-adic L-functions.*

3. MOTIVIC PHENOMENA

In this section S denotes a base scheme satisfying the Beilinson-Soulé vanishing conjecture, such as $\mathrm{Spec}\,\mathbb{Z}$, finite field. R denotes a regular coherent coefficient ring. I discuss one side project I have done, and one area of research, E, I hope to explore (which extends my joint work [ILZ24]).

In Grothendieck's quest towards the Weil conjectures, he reduced to problems of motives. Beilinson proposed an extension of the notion of pure motives to mixed motives, which was realized through the works of Voevodsky, Levine for a field, and extended by Ayoub and Cisikinki-Déglise, [CD19]. For a finite type S-scheme $X \in \mathrm{Sch}_S^{\mathrm{ft}}$, we can construct the derived categories of motives over X, $\mathrm{DM}(X, R)$. If X is smooth then its Ext-groups,

$$\pi_{-m} \mathrm{Map}_{\mathrm{DM}(X,R)}(1_X, 1_X(n)) \simeq \mathrm{CH}^n(X, 2n-m)_R$$

are Bloch's higher Chow groups. Motivic categories are difficult to explicitly work with due to the lack of motivic t-structures. The study of even the Chow groups, is as subtle of a question. One method is due to the work of Beauville [Bea83], using Fourier transform. In my joint work, [HHLMMM24] we extended the work of Beauville,

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[3] One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on X which we do not discuss.

Theorem 3.1 (Lin et al.)**.** *Let $X \to Y$ be an abelian scheme, whose base Y is smooth and quasi projective over a field. There is an explicit N, such that one obtains a Beauville decomposition*

$$CH^i(X)_{\mathbb{Z}[1/N]} \simeq \bigoplus_s CH^i_{(s)}(X)_{\mathbb{Z}[1/N]}$$

where $CH^i_{(s)}(X)_R := \left\{x \in CH^i(X)_R \,:\, [n]^*_X x = n^{2i-s}x \quad \forall n \in \mathbb{Z}\right\}$ and $[n]$ is the multiplication of an abelian scheme.

This extends to a $\mathfrak{sl}_2$ action, which we discussed in *op.cit*; and if S were an algebraically closed field, this implies various structural results. The key ingredient was using G. Papass' version of integral Grothendieck–Riemann–Roch.

Returning to motivic t-structures, it was shown by Levine that t-structures exist on a nice subcategory $\mathrm{DTM}(X) \hookrightarrow \mathrm{DM}(X)$ of *mixed Tate movies* for nice schemes X. This was extended to schemes with cellular Whitney-Tate stratification by Soergel and Wendt, and to prestacks in [RS20], which led to a series of work in applying the theory of motives to the geometric Langlands.

Research Goal E. Motivic Whittaker categories. *Define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we can prove the same statement in [NP01] in the category of mixed Tate motives, in which all the relevant objects are well defined. The difficulty, however, is that the original argument requires characteristic* 0 *coefficients, whilst in positive characteristic, there are problems of extensions.*

4. STACKY APPROACHES AND PERIODS

From a modern perspective, there is another form of *motive*. Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R)$$
$$X \mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?})$$

For instance, the *de Rham stack* X^{dR} over $\mathbb{Q}$, has points given by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A (cf. [GR14]). This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques. I explored this concept in basic constructions of F. Brown's work on *motivic periods*, [Bro14]. Periods are complex numbers that are integrals of rational differential forms:

$$\log(2) = \int_{1\le z\le 2} \frac{dz}{z}, \quad \zeta(2) = \int_{0\le t_1\le t_2\le 1} \frac{dt_1}{1-t_1}\frac{dt_2}{t_2}$$

More generally, if X were a smooth variety over $\mathbb{Q}$, the matrix coefficient from Grothendieck's comparison theorem

$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^*_{\mathrm{Betti}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

with respect to the $\mathbb{Q}$ structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to X*. These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"[4] which should govern the arithmetic structure of periods.

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the projective space minus three points over $\mathbb{Q}$. The period associated is studied through Chen's de Rham comparison theorem, [Che76], which relates to iterated integrals [Che73] and multiple zeta values [Bro14]. To give a "stacky" perspective of this period one uses

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[4] For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

unipotent types of stacks, which is defined in Toën, [Toë06], and recently in [MR23]. This is an endofunctor $\mathbf{U}$ on stacks, sending a stack X to its unipotent homotopy type. My first result is:

Theorem 4.1. *(Lin) Unipotent de Rham fundamental group,* $\pi_1^{u,dR}(X,x)$ *is isomorphic to* $\pi_1(\mathbf{U}(X^{dR}))$.

Research Goal F. A stacky approach to motivic periods. *This research aims to use geometric techniques in the study of periods. As a proof of concept, we will recover de Rham comparison theorem through the Riemann-Hilbert correspondence between the analytic Betti stack and the de Rham stack.* $\mathbb{P}^1 \backslash \{0,1,\infty\}$, *is not proper, which requires us to incorporate log structures. I expect to prove:*

Theorem 4.2 (Expected)**.** *There exists* X^{Betti} *which is the (analytic) Betti stack of* X*, such that the unipotent (analytic) Betti group* $\pi_1^{u,Betti}(X(\mathbb{C}),x)$ *is isomorphic to* $\pi_1(\boldsymbol{U}(X^{Betti}))$.

Theorem 4.3 (Expected)**.** *A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1], i.e.* $\pi_1^{u,dR}(X,x) \simeq \pi_1^{u,Betti}(X(\mathbb{C}),x) \otimes \mathbb{C}$

By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [5] *We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].*

5. DEFORMATION THEORY AND THE SPHERE SPECTRUM $\mathbb{S}$

Let $\mathcal{S}$ denote the ∞-category of ∞-groupoids/anima, the sifted cocompletion of Set. The *stabilization* of $\mathcal{S}$ is Sp, the ∞-category of spectra. This is the natural category to study cohomological invariants and fits into the following diagram:

$$\begin{tikzcd}
(\mathrm{Ab}, \otimes_{\mathbb{Z}}) \arrow[r, shift right] \arrow[d, shift left, "\mathrm{Fgt}"] & (\mathrm{Sp}, \otimes_{\mathbb{S}}) \arrow[l, shift right] \arrow[d, shift left, "\Omega^\infty"] \\
(\mathrm{Set}, \times) \arrow[u, shift left, "{\mathbb{Z}[-]}"] \arrow[r, shift right] & \mathcal{S} \arrow[l, shift right] \arrow[u, shift left, "\Sigma_+"]
\end{tikzcd}$$

The category of abelian groups sits as the heart of its natural t-structure, and Ω^∞ takes a *spectrum* to its 0th space. Within Sp, lies the universal cohomology theory, $\mathbb{S}$, the *sphere spectrum* which is also the unit ring of *brave new algebra*.

The theory of connected reductive groups over $\mathbb{C}$, has been a fruitful area. By Chevalley's works, such group has a canonical split $\mathbb{Z}$-form $G_{\mathbb{Z}}$, see [Con15]. One can analogously ask: *is there a* $\mathbb{S}$*-form for algebraic groups*? A first approximation is the existence of an algebraic category $\mathrm{Rep}_{\mathbb{S}}(G_{\mathbb{S}})$, which *deforms* to $\mathrm{Rep}_{\mathbb{Z}}(G_{\mathbb{Z}})$. This is a far reach so far, but we can begin with *formal deformation of categories.*

Let $\mathcal{C}$ be a symmetric monoidal category. There is a natural hierarchy of commutativity fitting in the diagram

$$\cdots \to \mathbb{E}_n(\mathcal{C}) \to \mathbb{E}_{n-1}(\mathcal{C}) \to \cdots \to \mathbb{E}_1(\mathcal{C})$$

where $\mathrm{CAlg}(\mathcal{C}) := \mathbb{E}_\infty(\mathcal{C}) := \varprojlim \mathbb{E}_n(\mathcal{C})$ of symmetric algebra objects can be identified with the limit. One can formalize the notion of $\mathbb{E}_n$-algebra objects via *disk operads*, or *little cubes operads.*

Example 5.1. Let $\mathcal{C} = (\mathcal{S}, \times)$. Let $Y \in \mathcal{S}_*$ is a pointed ∞-groupoid, its *k-fold based loop spaces*, $\Omega_*^k Y$ is a classical example of $\mathbb{E}_k$ algebra object in $(\mathcal{S}, \times)$.

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[5] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

Let $R \in \mathbb{E}_n(\mathrm{Sp})$ be an $\mathbb{E}_n$ ring, and consider LMod_R, the derived category of left R-modules, as an $\mathbb{E}_1$ object in $\mathrm{Pr}^{\mathrm{st}}$, the category of presentable stable categories. This defines $\mathrm{RMod}_{\mathrm{LMod}_R}(\mathrm{Pr}^{\mathrm{st}})$, the category of presentable stable (right) R-linear categories, [Lur18, Appendix D]. Set $\mathrm{Pr}_R^{\mathrm{st,cg}}$, as the full subcategory spanned by those whose underlying category is compactly generated. [6] For G a connected reductive group over a field k, $D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$, the bounded derived category of finite dimensional algebraic representations with k coefficients lies in $\mathrm{Pr}_R^{\mathrm{st,cg}}$.

From now on $k = \mathbb{C}$. Let $\mathrm{Art}_k^{(n)}$ denote the category of $\mathbb{E}_n$ *artinian* ring spectrum, for $n \geq 0$ over k. We refer to [Lur18, Ch. 15] for definition. This extends the classical definition of Artinian local ring, in particular, $R \in \mathrm{Art}_k^{(n)}$ admits an augmentation map $\epsilon : R \to k$. One defines the $\mathbb{E}_{n+2}$-formal moduli problem,

$$\mathrm{Cat}\mathcal{D}\mathrm{ef}^{(n)}(\mathcal{C}) : \mathrm{Art}_k^{(n+2)} \to \hat{\mathcal{S}}$$
$$R \mapsto |\,\{\mathcal{C}\} \times_{\mathrm{Pr}_R^{\mathrm{st,cg}}} \mathrm{Pr}_k^{\mathrm{st,cg}}|$$

where $|\quad|$ is the underlying Kan complex of the ∞-category. An object consists of: a $\mathcal{C}_B$ right stable R-linear category, and an equivalence $\mathcal{C}_B \otimes_{\mathrm{LMod}_B} \mathrm{LMod}_k \simeq \mathcal{C}$. Our $\mathbb{E}_4$-moduli problem is when $n = 2$ and $\mathcal{C} = D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$. The Casselman–Shalika formula [FGV01] implies

$$\mathrm{Whit}(\mathrm{Gr}_{\hat{G}}) \simeq D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$$

an $\mathbb{E}_2$-algebra equivalence of $\mathcal{C}$ with the *Whittaker sheaves* on the *affine Grassmanian* $\mathrm{Gr}_{\hat{G}}$.

The homotopy type of $\mathrm{Gr}_{\hat{G}}$ is $\Omega^2 B\hat{G}(\mathbb{C}) \in \mathbb{E}_2(\mathcal{S})$, whose structure induces the *fusion product* on the Whittaker category. The categorical equivalence suggests $\mathbb{E}_2$ deformations are controlled by twists of line bundles. For $X \in \mathbb{E}_2(\mathcal{S})$, define the moduli of functor of $\mathbb{G}_m$*-gerbes over* X

$$\mathrm{Ge}_{\mathbb{G}_m}(X) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(X, B^2R^\times) \quad R \in \mathrm{Art}_k^{(4)}$$

where $R^\times \subset \Omega^\infty R$ are the invertible elements of the underlying space of R [7] and B^2 is the second deloop. It was stated without proof in [Lur10]

Theorem 5.2 (Lurie, expected)**.** *There is an equivalence of formal* $\mathbb{E}_4$*-moduli problems*

$$\widehat{Ge_{\mathbb{G}_m}}(\mathrm{Gr}_{\hat{G}}) \xrightarrow{\simeq} Cat\mathcal{D}ef^{(2)}(\mathrm{Rep}\,G)$$

where $\widehat{-}$ *is formal completion of the moduli functor at a base point.*

Research Goal G. Categorical deformations for the sphere spectrum. *As the first step, we will prove Lurie's theorem, Theorem 5.2 in detail. There are two direct problems one can ask: i) deformations with respect to other rings k –this would be closely related to a motivic Whittaker category, Section 3. ii) deformations of representation of Lusztig's small quantum group, see op.cit. Remark 10.12. This would be approachable due to recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.*

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[6] The compact generation is only a smallness condition for our version.

[7] is the union of the connected components of invertibles in the $\pi_0 R$ of the 0th space of R and is equivalent to the nth loop space of some space, $R^\times \simeq \Omega^n Z$, $n \geq 4$, hence the deloop $B^2R^\times \in \mathbb{E}_2(\mathcal{S})$

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