Chapter 3

Ext groups

Refs.

1. Atiyah-Macdonald, Commutative algebra
2. Rotman, Homological algebra

3.1 Extensions

Given two R-modules A and C, an extension of C by A is a short exact sequence

$$0 \to A \to B \to C \to 0$$

(NB: This terminology is opposite of what Rotman uses, but it is better aligned with the notation to be introduced.) Let us say that another extension

$$0 \to A \to B' \to C \to 0$$

is equivalent to the first if they can be put into a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "id"] & B \arrow[r] \arrow[d, "\phi"] & C \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B' \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

Lemma 3.2. *The map ϕ above is an isomorphism. Equivalence of extensions is an equivalence relation.*

Proof. The first statement, which is a special case of the 5-lemma, is an easy diagram chase. We will omit the proof. Since this implies that ϕ^{-1} exists, we see that this relation is symmetric. It is obviously reflexive, and transitive (use the composite of ϕ and the corresponding map in the third extension). $\square$

Let $ext(C,A)$ denote the set of equivalence classes of extensions. Our goal is to compute this. First observe that this set has a distinguished element

$$0 \to A \to A \oplus C \to C \to 0$$

which we call the trivial extension, and denote this by 0. We say that an extension

$$0 \to A \xrightarrow{j} B \xrightarrow{p} C \to 0$$

splits if there is a homomorphism $i : C \to B$ such that $p \circ i = id$.

Lemma 3.3. *An extension splits iff it is equivalent to the trivial extension.*

Proof. Given a split extension as above, define $\phi : A \oplus C \to B$ by $\phi(a,c) = j(a) + s(c)$. Conversely, if we have such a morphism the $s(c) = \phi(0,c)$ gives a splitting. $\square$

We can now compute it in one case.

Proposition 3.4. *C is projective if and only if $ext(C,A) = \{0\}$ for every A.*

Proof. If C is projective, we proved early that any surjective morphism to C splits. Therefore $ext(C,A) = 0$.

Conversely, suppose $ext(C,A)$ for every A. Given

$$\begin{tikzcd} & & & P \arrow[d, "f"] & \\ 0 \arrow[r] & K \arrow[r] & M \arrow[r, "\pi"] & N \arrow[r] & 0 \end{tikzcd}$$

let $L = \{(m,p) \in (M,P) \mid f(m) = \pi(p)\}$ be the pullback. Then we have an extension

$$0 \to K \to L \to P \to 0$$

This has a splitting $s : P \to L$ by assumption. Composing this with the projection $L \to M$, yields a map $P \to M$ lifting f. $\square$

In order to try to compute $ext(C,A)$ in general, we can try to reduce C to a projective module. We choose a surjection $\pi : P \to C$, with P projective. We could take P to be a free module on a set of generators for P, for example, Then form the sequence

$$0 \to K \xrightarrow{i} P \xrightarrow{\pi} C \to 0$$

Define

$$_{\pi}Ext(C,A) = \text{coker}(Hom(P,A) \to Hom(K,A))$$

We will prove the following later in more form.

Proposition 3.5. *The isomorphism class of $_{\pi}Ext(C,A)$ is independent of f.*

Henceforth, we write $Ext(C, A)$ for ${}_\pi Ext(C, A)$.

Theorem 3.6. *There is a bijection $ext(C, A) \cong Ext(C, A)$ preserving* 0.

Proof. Given $f : K \to A$, let

$$Q_f = P \oplus A / \{(i(k), f(k)) \mid k \in K\}$$

be the pushout. The fits into an extension

$$0 \to A \to Q_f \to C \to 0$$

If $f = F|_K$, with $F \in Hom(P, A)$, then $\phi(p, a) = (F(p), a)$ is an equivalence to the trivial extension. Similarly, one can check that if $g : K \to A$ is another map such that $g - f$ lies in the image of $Hom(P, A)$, then

$$0 \to A \to Q_g \to C \to 0$$

is equivalent to the previous extension. Therefore we have constructed a map from $Ext(C, A) \to ext(C, A)$ preserving 0.

Given an extension of C by A,

$$\begin{tikzcd} 0 \arrow[r] & K \arrow[r] \arrow[d, dashed, "f"] & P \arrow[r] \arrow[d, dashed, "g"] & C \arrow[r] \arrow[d, "id"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C \arrow[r] & 0 \end{tikzcd}$$

we can find g and therefore f using the projectivity of P. This can be checked to give the inverse $ext(C, A) \to Ext(C, A)$.

□

Corollary 3.7. *$ext(C, A)$ has the structure of an abelian group.*

See Rotman section 7.2.1 for an explicit description of the group structure in terms of extensions.

Example 3.8. *Let $R = \mathbb{Z}$. Consider the exact sequence*

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

where $n \neq 0$. Then "Hom-ing" into A yields

$$A \xrightarrow{n} A \to Ext(\mathbb{Z}/n\mathbb{Z}, A) \to 0$$

Therefore $Ext(\mathbb{Z}/n\mathbb{Z}, A) \cong A/nA$. The calculation can be upgraded to calculate $Ext(B, A)$ for any finitely generated abelian group, Writing $B = \bigoplus \mathbb{Z}/n_{\mathbb{Z}} \oplus \mathbb{Z}^N$, $Ext(\mathbb{Z}/n\mathbb{Z}, A) \cong \bigoplus A/n_i A$.

So far we have been borrowing ideas from topology. Now we are in a position to repay the debt. We defined the cohomology of a simplicial complex earlier, and said that it is roughly dual to homology. Here is a the precise statement.

Theorem 3.9 (Universal coefficient theorem)**.** *Given a simplicial complex S, there is an isomorphism*

$$H^n(S) \cong Hom(H_n(S), \mathbb{Z}) \oplus Ext(H_{n-1}(S), \mathbb{Z})$$

The argument is slightly simpler for finite simplicial complexes. So let us assume this. Then the result will be a consequence of the following result from pure homological algebra.

Theorem 3.10. *If $F_\bullet$ is a complex of finitely generated free abelian groups, there is an isomorphism*

$$H^n(Hom(F_\bullet, \mathbb{Z})) \cong Hom(H_n(F_\bullet), \mathbb{Z}) \oplus Ext(H_{n-1}(F_\bullet), \mathbb{Z})$$

Proof. Let $B_n \subseteq Z_n \subseteq F_n$ be the subgroups of boundaries and cycles. These are free abelian by basic algebra. Therefore the exact sequences

$$0 \to Z_n \to F_n \to B_{n-1} \to 0$$

is split. It follows that

$$0 \to Hom(B_{n-1}, \mathbb{Z}) \to Hom(F_n, \mathbb{Z}) \to Hom(Z_n, \mathbb{Z}) \to 0$$

is also split exact. This can be viewed as an exact sequence of cochain complexes where the complexes on the left and right have zero differential. Having zero differential implies that $Hom(B_{n-1}, \mathbb{Z})$ and $Hom(Z_n, \mathbb{Z})$ are the cohomology groups. The long exact sequence for cohomology is

$$Hom(Z_{n-1}, \mathbb{Z}) \to Hom(B_{n-1}, \mathbb{Z}) \to H^n(Hom(F_n, \mathbb{Z})) \to Hom(Z_n, \mathbb{Z}) \to Hom(B_n, \mathbb{Z})$$

Using the exact sequences

$$0 \to Z_n \to B_n \to H_n(F_\bullet) \to 0$$

we can write the previous sequence as

$$0 \to Ext(H_{n-1}, \mathbb{Z}) \to H^n(Hom(F_n, \mathbb{Z})) \to Hom(H_n, \mathbb{Z}) \to 0$$

Finally, note that Ext is a torsion group and Hom is torsion free, so this must split canonically. □

3.11 Projective resolutions

Let M be an R-module. Choose a projective module P_0 and a surjection $P_0 \to M$. Let K_0 be the kernel. Choose a surjection from another projective module $P_1 \to K_0$. Let K_1 be the kernel of this, and repeat. Composing $P_i \to K_{i-1}$ with $K_{i-1} \to P_{i-1}$ yields an exact sequence

$$\ldots P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective. This is called a projective resolution of M. We have proved that such things exist.

Lemma 3.12. *Every module possesses a projective resolution.*

Such resolutions are not unique, because choices are involved. However, they are unique in a weaker sense that any two projective resolutions are homotopy equivalent.

Theorem 3.13. *If $Q_\bullet \to M \to 0$ is an exact sequence, so perhaps another projective resolution. Then there exists a morphism $f : P_\bullet \to Q_\bullet$ such that*

$$\begin{tikzcd} P_0 \arrow[d, "f"] \arrow[dr] & \\ Q_0 \arrow[r] & M \end{tikzcd}$$

commutes. This is unique up to homotopy, i.e. any other morphism is homotopic to f.

Proof. A morphism f is a collection of homomorphisms $f_n : P_n \to Q_n$, which can be inductilvely The first map f_0 exists by projectivity of P_0

$$\begin{tikzcd} & P_0 \arrow[d] \arrow[dl, dashed, "f_0"] \\ Q_0 \arrow[r] & M \end{tikzcd}$$

Suppose $f_n, f_{n-1} \ldots$ have been constructed. Let us write $d_\bullet$ and $d'_\bullet$ for the differentials of $P_\bullet$ and $Q_\bullet$. Then we have that $f_{n-1}d_n = d'_n f_n$. So that $d'_n f_n d_{n+1} = f_{n-1}d_n d_{n+1} = 0$. Therefore $f_n d_{n+1} \subseteq \ker d'_n = \operatorname{im} d'_{n+1}$. So we have a diagram

$$\begin{tikzcd} & P_{n+1} \arrow[d] \arrow[dl, dashed, "f_{n+1}"] \\ Q_{n+1} \arrow[r] & \operatorname{im} d'_{n+1} \end{tikzcd}$$

Projectivity of P_{n+1} shows the existence of f_{n+1} making this commute.

Given a second morphism $g : P_\bullet \to Q_\bullet$, we have to construct a homotopy h between, that is sequence of maps $h_n : P_n \to Q_{n+1}$ satisfying

$$f_n - g_n = d_{n+1}h_n + h_{n-1}d_n$$

This is again constructed by induction, using projectivity of each P_n. See p342 of Rotman for details. □

Remark 3.14. *The same proof actually something stronger, namely that if $P_\bullet \to M$ is a complex, with each P_n projective, then $f : P_\bullet \to Q_\bullet$ exists and is unique up to homotopy.*

Corollary 3.15. *If $Q_\bullet \to M$ is another projective resolution, there exists a homotopy equivalence $f : P_\bullet \to Q_\bullet$. (Recall that this means that there is $g : Q_\bullet \to P_\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities.)*

3.16 Higher Ext groups

Given a pair of modules M and N fix a projective resolution $P_\bullet \to M$. Let $\partial : P_n \to P_{n-1}$ denote the maps. Since $P\bullet$ is exact, it forms a complex i.e. $\partial^2 = 0$. Then

$$C^n = Hom(P_n, N)$$

carries maps

$$d : C^n \to C^{n+1}$$

dual to ∂. We necessarily have $d^2 = 0$, so $C^\bullet$ forms a cochain complex.

Theorem/Def 3.17. *The isomorphism classes of the cohomology groups*

$$Ext^n_R(M, N) = H^n(Hom_R(P_\bullet, N))$$

depend only on M and not on the choice of resolution $P_\bullet$.

Proof. If $Q_\bullet$ is another projective resolution, we have morphisms $f : P_\bullet \to Q_\bullet$ and $g : Q_\bullet \to P_\bullet$ such that $f \circ g$ and $g \circ f$ are homotopic to the identities. These induces morphisms between $Hom(P^\bullet, N)$ and $Hom(Q_\bullet, N)$ whose compositions are again homotopic to the identities. This implies that they have isomorphic cohomology by proposition 2.22.

□

Corollary 3.18. *There are isomorphisms*

$$Ext^0_R(M, N) \cong Hom_R(M, N)$$

and

$$Ext^1_R(M, N) \cong Ext(M, N)$$

where the last group is the one constructed in a previous section.

Proof. Given a projective resolution $P_\bullet \to M$, we can form an exact sequence

$$0 \to K \to P_0 \to M \to 0$$

where $K = \operatorname{im} P_1 \to P_0$. Then

$$0 \to Hom(M, N) \to Hom(P_0, N) \to Hom(K, N)$$

and

$$0 \to Hom(K, N) \to Hom(P_1, N)$$

are exact. This implies that

$$Hom(M, N) = H^0(Hom(P_\bullet, N))$$

The proof of the second isomorphism is similar. □

The previous theorem is not that useful as stated. In fact, we will show that $Ext^n(-,-)$ is a functor in both variables, and that it fits into natural exact sequences. It are these properties that make it a powerful tool.

Theorem 3.19. *If $g : N \to N'$ is a morphism there is an induced morphism $g_* : Ext^n_R(M, N) \to Ext^n_R(M', N)$. This makes $Ext^n_R(M, -)$ a covariant functor from $Mod_R \to Ab$. If*

$$0 \to N \to N' \to N'' \to 0$$

is exact, then there is a long exact sequence

$$\ldots Ext^n_R(M, N) \to Ext^n_R(M, N') \to Ext^n_R(M, N'') \to Ext^{n+1}_R(M, N) \ldots$$

Proof. Fix a projective resolution $P_\bullet \to M$. Then we get a morphism of complexes

$$Hom(P_\bullet, N) \to Hom(P_\bullet, N')$$

The induced map on cohomology yields

$$Ext^n_R(M, N) \to Ext^n_R(M', N)$$

Suppose that

$$0 \to N \to N' \to N'' \to 0$$

is exact. Since P_i is projective, $Hom(P_i, -)$ is an exact functor. Therefore we get a short exact sequence of complexes

$$0 \to Hom(P_\bullet, N) \to Hom(P_\bullet, N') \to Hom(P_\bullet, N'') \to 0$$

This yields a long exact sequence

$$\ldots Ext^n_R(M, N) \to Ext^n_R(M, N') \to Ext^n_R(M, N'') \to Ext^{n+1}_R(M, N) \ldots$$

□

Theorem 3.20. *If $h : M \to M'$ is a morphism, there is an induced morphism $h^* : Ext^n_R(M', N) \to Ext^n_R(M, N)$. This makes $Ext^n_R(-, N)$ into a contravariant functor from $Mod_R \to Ab$. If*

$$0 \to M \to M' \to M'' \to 0$$

is exact, then there is a long exact sequence

$$\ldots Ext^n_R(M'', N) \to Ext^n_R(M', N) \to Ext^n_R(M, N) \to Ext^{n+1}_R(M'', N) \ldots$$

Proof. If $P'_\bullet \to M'$ is a projective resolution, the above remark 3.14 allows us to construct a morphism $\tilde{h} : P_\bullet \to P'_\bullet$ unique up to homotopy. This indices a morphism

$$Hom(P'_\bullet, N) \to Hom(P_\bullet, N)$$

which induces h^*. If $\ell : M' \to M''$ is another morphism. Choose a projective resolution $P''_\bullet \to M''$ and construct the corresponding morphism $\tilde{\ell} : P'_\bullet \to P''_\bullet$.

The uniqueness shows that $\widetilde{\ell \circ h}$ and $\tilde{\ell} \circ \tilde{h}$ are homotopy equivalent. This implies $(\ell \circ h)^* = h^* \circ \ell^*$. Therefore we have a functor.

For the last statement, we claim that we can construct projective resolutions fitting into a diagram

$$\begin{tikzcd}
 & {} \arrow[d] & {} \arrow[d] & {} \arrow[d] \arrow[dl, dashed] & \\
0 \arrow[r] & P_0 \arrow[r] \arrow[d, "i"] & P_0' \arrow[r] \arrow[d, "f"] & P_0'' \arrow[r] \arrow[d] \arrow[dl, dashed, "g"] & 0 \\
0 \arrow[r] & M \arrow[r] & M' \arrow[r] & M'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows. To prove this, choose resolutions $P_\bullet$ and $P_\bullet''$, and set $P_\bullet' = P_\bullet \oplus P_\bullet''$ as a graded module. Since P_0'' is projective, we can construct g above. Set $f : P_0 \oplus P_0'' \to M'$ to $i + g$. The differentials of $P_\bullet'$ are built similarly.

From the claim, we have an exact sequence of complexes

$$0 \to P_\bullet \to P_\bullet' \to P_\bullet'' \to 0$$

which, by construction, is split as a sequence of graded modules. It follows that

$$0 \to Hom(P_\bullet'', N) \to Hom(P_\bullet', N) \to Hom(P_\bullet, N) \to 0$$

is an exact sequence of complexes. Applying theorem 2.13 to this, gives a long exact sequence of Ext groups.

□

Example 3.21. *If $R = \mathbb{Z}$, using the projective resolution,*

$$0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z} \to 0$$

we find that

$$Ext^1(\mathbb{Z}/n\mathbb{Z}, A) = A/nA$$

and

$$Ext^i(\mathbb{Z}/n\mathbb{Z}, A) = 0$$

for $i > 1$.

3.22 Characterization of projectives and injectives

Theorem 3.23. *Let P be an R-module. The following are equivalent.*

(a) P is projective.

(b) $Ext^n_R(P, M) = 0$ for all $n > 0$ and for all modules M.

(c) $Ext^1_R(P,M)=0$ *for all modules* M.

Proof. If P is projective, then $P=P$ is a projective resolution. Therefore (b) follows. Clearly (b) implies (c). If (c) holds, then for any exact sequence

$$0\to K\to M\to N\to 0$$

we have

$$Hom_R(P,M)\to Hom_R(P,N)\to Ext^1_R(P,K)=0$$

This implies that P is projective. □

We prove an analogous characterization for injectives. However, due to the asymmetry of the definition, the proof will be completely different.

Theorem 3.24. *Let E be an R-module. The following are equivalent.*

(a) *E is injective.*

(b) $Ext^1_R(M,E)=0$ *for all modules* M.

(c) $Ext^n_R(M,E)=0$ *for all* $n>0$ *and for all modules* M.

Proof. Suppose that E is injective. Injectivity will imply that given an exact sequence

$$0\to E\xrightarrow{i} N\to M\to 0$$

we can find a homomorphism $r:N\to E$ such that $r\circ i=id$. This means that the sequence splits. By an earlier characterization, $Ext^1_R(M,E)$ is the equivalence class of extensions as above. Therefore it must be zero. Conversely, if (b) holds then any extension must split. So E can be seen to be injective.

Clearly (c) implies (b). We just have to prove the converse. We use induction on n and a trick called "dimension shifting". Following Grothendieck, algebraic geometers also refer this type of argument more broadly as "devissage", which translates roughly as "untwisting". Suppose that (c) holds for a fixed $n>0$ for all M. Given M we can find an exact sequence

$$0\to K\to P\to M\to 0$$

with P projective. Then we have an exact sequence

$$Ext^n_R(K,E)\to Ext^{n+1}_R(M,E)\to Ext^{n+1}_R(P,E)$$

The group on the left is zero by induction, while the group on the right is zero by projectivity of P.

□

For the remainder of this section, let us assume that R is commutative. Then $Hom_R(M,N)$ is naturally an R-module via $(rf)(m)=rf(m)=f(rm)$. Therefore

$$Ext^n_R(M,N)=H^n(Hom_R(P_\bullet,N))$$

is also an R-module. Moreover, the previous arguments can be modified to show that this structure is independent of the resolution.

Recall that if $S \subset R$ is a multiplicatively closed set, we can form a new ring $S^{-1}R$ by inverting elements of S. This operation extends to an exact functor $S^{-1} : Mod_R \to Mod_{S^{-1}R}$. See Atiyah-Macdonald for details.

Lemma 3.25. *If P is projective, then $S^{-1}P$ is projective.*

Proof. P is projective if and only if it is a summand of a free module. The last condition is stable under localization. □

Suppose now in addition that R is noetherian. If M is finitely generated over R, then we can find a surjection

$$R^{n_0} \to M \to 0$$

for some n_0. Since the kernel is finitely generated (by noetherianness), we can prolong this to an exact sequence

$$R^{n_1} \to R^{n_0} \to M \to 0$$

and so on to obtain

Lemma 3.26. *If M is finitely generated, then it has a free resolution by finitely generated free modules.*

Lemma 3.27. *If M is finitely generated, then for any multiplicative set*

$$S^{-1}Hom_R(M, N) \cong Hom_{S^{-1}R}(S^{-1}M, S^{-1}N)$$

Proof. If $M = R^n$, then this amounts to the isomorphism

$$S^{-1}(M^n) = (S^{-1}M)^n$$

We can form a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}Hom(M,N) \arrow[r] \arrow[d, "f"] & S^{-1}Hom(R^{n_0},N) \arrow[r] \arrow[d, "\cong"] & S^{-1}Hom(R^{n_1},N) \arrow[d, "\cong"] \\
0 \arrow[r] & Hom(S^{-1}M, S^{-1}N) \arrow[r] & Hom(S^{-1}R^{n_0}, S^{-1}N) \arrow[r] & Hom(S^{-1}R^{n_1}, S^{-1}N)
\end{tikzcd}$$

The last two maps are isomorphisms by what we said above. Therefore f is an isomorphism by a diagram chase.

□

Combining the last two lemmas, we find that

Theorem 3.28. *If M is finitely generated, then*

$$S^{-1}Ext^n_R(M, N) \cong Ext^n_{S^{-1}R}(M, N)$$

Corollary 3.29. *A finitely generated R-module P is projective if and only if it is locally free.*

Proof. Suppose that P is finitely generated and locally free. We have to show that $E = Ext^1_R(P, N) = 0$ for any N. It suffices to prove that localizations of $E_p = 0$ at primes $p \in \operatorname{Spec} R$. By the theorem

$$E_p = Ext^1_{R_p}(P_p, N_p) = 0$$

for any $p \in \operatorname{Spec} R$. $\square$