Nonabelian Poincare Duality (Lecture 8)

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Let M be a compact oriented manifold of dimension n. Then Poincare duality asserts the existence of an isomorphism

$$\mathrm{H}^*(M; A) \simeq \mathrm{H}_{n-*}(M; A)$$

for any abelian group A. Here the essential hypothesis is that M is a manifold: that is, that every point of M admits a neighborhood which is homeomorphic to Euclidean space $\mathbb{R}^n$. It is therefore natural to begin by asking what Poincare duality tells us in the special case $M = \mathbb{R}^n$. Since Euclidean space is not compact, we should consider instead a more general formulation of Poincare duality which applies to noncompact oriented manifolds:

$$\mathrm{H}_c^*(M; A) \simeq \mathrm{H}_{n-*}(M; A).$$

In the case $M = \mathbb{R}^n$, the right hand side of this isomorphism is easy to understand (since M is contractible), and Poincare duality comes down to a calculation

$$\mathrm{H}_c^*(\mathbb{R}^n; A) \simeq \begin{cases} A & \text{if } * = n \\ 0 & \text{otherwise.} \end{cases}$$

From this isomorphism, one can deduce Poincare duality for more general manifolds by means of a local-to-global principle. To formulate it, let $\mathcal{U}(M)$ denote the collection of all open subsets of M, partially ordered by inclusion. For each object $U \in \mathcal{U}(M)$, let $C_*(U; A)$ denote the singular chain complex of U, and let $C_c^*(U; A)$ denote the compactly supported cochain complex of U. Then the constructions

$$U \mapsto C_*(U; A) \qquad U \mapsto C_c^*(U; A)$$

can be regarded as functors from $\mathcal{U}(M)$ to the category Chain of chain complexes of abelian groups, or to the associated ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ described in the previous lecture.

Let $\mathcal{U}_0(M)$ denote the partially ordered subset of $\mathcal{U}(M)$ consisting of those open sets $U \subseteq M$ which are homeomorphic to $\mathbb{R}^n$. In this case, the chain complexes $C_*(U; A)$ and $C_c^*(U; A)$ have their homologies concentrated in a single degree. It follows that they are naturally quasi-isomorphic to chain complexes concentrated in a single degree. The local version of Poincare duality therefore supplies equivalences $C_c^*(U; A) \simeq C_{n-*}(U; A)$ in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. If M is oriented, we can even choose these equivalences to depend functorially on U: that is, the constructions $U \mapsto C_c^*(U; A)$ and $U \mapsto C_{n-*}(U; A)$ determine *equivalent* functors from $\mathcal{U}_0(M)$ into the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$. To obtain information about more general open subsets of M, it is convenient to introduce the following general definition:

Definition 1. Let $\mathcal{C}$ be an ∞-category which admits colimits, and let $\mathcal{F} : \mathcal{U}(M) \to \mathcal{C}$ be a functor. We will say that $\mathcal{F}$ is a $\mathcal{C}$-*valued cosheaf on* M if the following condition is satisfied:

$(*)$ For every open set $U \subseteq M$ and every open cover $\{U_\alpha\}$ of U, the canonical map

$$\varinjlim_{V} \mathcal{F}(V) \to \mathcal{F}(U)$$

is an equivalence in $\mathcal{C}$, where the colimit is taken over all open sets $V \subseteq M$ which are contained in some U_α.

Remark 2. Definition 1 has many equivalent formulations. For example, a functor $\mathcal{F} : \mathcal{U}(M) \to \mathcal{C}$ is a $\mathcal{C}$-valued cosheaf on M if and only if it satisfies the following three conditions:

- For every increasing sequence of open sets $U_0 \subseteq U_1 \subseteq U_2 \subseteq \cdots$, the canonical map $\varinjlim \mathcal{F}(U_i) \to \mathcal{F}(\bigcup U_i)$ is an equivalence in $\mathcal{C}$.
- The object $\mathcal{F}(\emptyset)$ is initial in $\mathcal{C}$.
- For every pair of open sets $U, V \subseteq M$, the diagram

$$\begin{tikzcd} \mathcal{F}(U \cap V) \arrow[r] \arrow[d] & \mathcal{F}(U) \arrow[d] \\ \mathcal{F}(V) \arrow[r] & \mathcal{F}(U \cup V) \end{tikzcd}$$

  is a pushout square in $\mathcal{C}$.

If $\mathcal{F}$ is a cosheaf, then its global behavior is determined by its local behavior. More precisely, one can prove the following:

Proposition 3. *Let $\mathcal{C}$ be an ∞-category which admits colimits and let $\mathcal{F} : \mathcal{U}(M) \to \mathcal{C}$ be a $\mathcal{C}$-valued cosheaf, and let $\mathcal{U}_0(M)$ denote the subset of $\mathcal{U}(M)$ whose elements are open sets $U \subseteq M$ such that U is homeomorphic to $\mathbb{R}^n$. Then the canonical map*

$$\varinjlim_{U \in \mathcal{U}_0(M)} \mathcal{F}(U) \to \mathcal{F}(M)$$

is an equivalence in $\mathcal{C}$.

The proof is mostly formal, using only the fact that $\mathcal{U}_0(M)$ forms a basis for the topology of M (since M is a manifold) and the finite-dimensionality of M.

Proposition 4. *For every abelian group A, the constructions $U \mapsto C_*(U; A)$ and $U \mapsto C^*_c(U; A)$ determine $\mathrm{Mod}_{\mathbf{Z}}$-valued cosheaves on M.*

Proposition 4 is one formulation of the idea that compactly supported cohomology and homology satisfy *excision*. For example, if U and V are open subsets of M, then the existence of a pushout diagram

$$\begin{tikzcd} C_*(U \cap V; A) \arrow[r] \arrow[d] & C_*(U; A) \arrow[d] \\ C_*(V; A) \arrow[r] & C_*(U \cup V; A) \end{tikzcd}$$

in the ∞-category $\mathrm{Mod}_{\mathbf{Z}}$ implies (and is morally equivalent to) the existence of a long exact Mayer-Vietoris sequence

$$\cdots \to \mathrm{H}_*(U \cap V; A) \to \mathrm{H}_*(U; A) \oplus \mathrm{H}_*(V; A) \to \mathrm{H}_*(U \cup V; A) \to \mathrm{H}_{*-1}(U \cap V; A) \to \cdots$$

Using Propositions 3 and 4, we deduce the existence of a canonical equivalence

$$C^*_c(M; A) \xleftarrow{\sim} \varinjlim_{U \in \mathcal{U}_0(M)} C^*_c(U; A) \simeq \varinjlim_{U \in \mathcal{U}_0(M)} C_{n-*}(U; A) \xrightarrow{\sim} C_{n-*}(M; A).$$

Passing to homology, we obtain a proof of Poincare duality.

Our goal in this lecture is to formulate an analogue of Poincare duality to the case of nonabelian coefficients. To formulate this, we first recall that cohomology is a *representable* functor. More precisely, for every abelian group A and every integer $n \geq 0$, one can construct a topological space $K(A, n)$ and a cohomology class $\eta \in \mathrm{H}^n(K(A, n); A)$ with the following universal property: for any sufficiently nice space M, the pullback of η induces a bijection

$$[M, K(A, n)] \simeq \mathrm{H}^n(M; A),$$

where $[M, K(A, n)]$ denotes the set of homotopy classes of maps from M into $K(A, n)$. The space $K(A, n)$ is called an *Eilenberg-MacLane space*. It is characterized (up to weak homotopy equivalence) by the existence of isomorphisms

$$\pi_i K(A, n) \simeq \begin{cases} A & \text{if } i = n \\ 0 & \text{otherwise.} \end{cases}$$

When $n = 1$, one can define an Eilenberg-MacLane space $K(G, n)$ even when the group G is nonabelian. In this case, $K(G, 1)$ is called a *classifying space* of G, and denoted by BG. It can be constructed as the quotient of a contractible space by a free action of G. This motivates one possible definition of nonabelian cohomology:

Definition 5. Let G be a discrete group and let M be a manifold (or any other reasonably nice topological space). We let $\mathrm{H}^1(M; G)$ denote the set of homotopy classes of maps from M into $K(G, 1) = BG$.

Definition 5 has many other formulations: the set $\mathrm{H}^1(M; G)$ can be identified with the set of isomorphism classes of G-bundles on M, or (in the case where M is connected) with the set of conjugacy classes of group homomorphisms $\pi_1 M \to G$. However, the formulation given above suggests the possibility of giving a far more general interpretation of nonabelian cohomology:

Definition 6. Let Y be a topological space, and let M be a manifold (or any other sufficiently nice space). Then the *cohomology of M with coefficients in* Y is the set of homotopy classes of maps from M into Y, which we will denote by $[M, Y]$.

We have the following table of analogies:

Abelian Cohomology	Nonabelian Cohomology
Abelian group A	Pointed topological space (Y, y)
$\mathrm{H}^n(M; A)$	$[M, Y] = \pi_0 \operatorname{Map}(M, Y)$
$C^*(M; A)$	$\operatorname{Map}(M, Y)$
$C^*_c(M; A)$	$\operatorname{Map}_c(M, Y)$
$C_*(M; A)$	???

Here $\operatorname{Map}(M, Y)$ denotes the space of continuous maps from M into Y, and $\operatorname{Map}_c(M, Y)$ denotes the subspace consisting of maps which are *compactly supported*: that is, maps $f : M \to Y$ such that the set $\{x \in M : f(x) \neq y\}$ has compact closure.

We can now ask if there is any analogue of Poincare duality in the nonabelian setting. That is, if M is a manifold, does the space $\operatorname{Map}_c(M, Y)$ of compactly supported maps from M into Y admit some sort of "homological" description? By analogy with classical Poincare duality, we can break this question into two parts:

(*a*) What does the mapping space $\operatorname{Map}_c(M, Y)$ look like when $M \simeq \mathbb{R}^n$?

(*b*) Can we recover the mapping space $\operatorname{Map}_c(M, Y)$ from the mapping spaces $\operatorname{Map}_c(U, Y)$, where U ranges over the open disks in M?

Question (a) is easy to address. The space of compactly supported maps from $\mathbb{R}^n$ into a pointed space (Y, y) is homotopy equivalent to the space of maps which are supported in the unit ball of $\mathbb{R}^n$: that is, the n-fold loop space $\Omega^n(Y)$.

To address question (b), we note that the construction $U \mapsto \mathrm{Map}_c(U, Y)$ can be regarded as a *covariant* functor of U: any compactly supported map from U into Y can be extended to a compactly supported map on any open set containing U (by carrying the complement of U to the base point of Y). We can regard this construction as a functor from the partially ordered set $\mathcal{U}(M)$ to an ∞-category $\mathcal{S}$ of spaces (whose 0-simplices are nice topological spaces, 1-simplices are continuous maps, 2-simplices are given by homotopies between continuous maps, and so forth). We might then ask the following:

Question 7. Let (Y, y) be a topological space. Is the construction $U \mapsto \mathrm{Map}_c(U, Y)$ a $\mathcal{S}$-valued cosheaf on M?

For example, Question 7 asks if, for any pair of open sets $U, V \subseteq M$, the diagram of spaces

$$\begin{tikzcd} \mathrm{Map}_c(U \cap V, Y) \arrow[r] \arrow[d] & \mathrm{Map}_c(U, Y) \arrow[d] \\ \mathrm{Map}_c(V, Y) \arrow[r] & \mathrm{Map}_c(U \cup V, Y) \end{tikzcd}$$

is a pushout square in the ∞-category $\mathcal{S}$ (such a diagram of spaces is commonly referred to as a *homotopy pushout* square). This is an unreasonable demand: if it were true, then the diagram

$$\begin{tikzcd} \pi_0 \mathrm{Map}_c(U \cap V, Y) \arrow[r] \arrow[d] & \pi_0 \mathrm{Map}_c(U, Y) \arrow[d] \\ \pi_0 \mathrm{Map}_c(V, Y) \arrow[r] & \pi_0 \mathrm{Map}_c(U \cup V, Y) \end{tikzcd}$$

would be a pushout square in the ordinary category of sets. In other words, any compactly supported map from $U \cup V$ into Y would need to be homotopic (through compactly supported maps) to a map which is supported either in U or in V. This is generally not true.

Why does Question 7 have a negative answer, when Proposition 4 is true? The difference stems from the type of presheaves we are considering. Recall that Proposition 4 implies (and is essentially equivalent to) the existence of Mayer-Vietoris sequences

$$\cdots \to \mathrm{H}_c^*(U \cap V; A) \to \mathrm{H}_c^*(U; A) \oplus \mathrm{H}_c^*(V; A) \to \mathrm{H}_c^*(U \cup V; A) \xrightarrow{\delta} \mathrm{H}_c^{*+1}(U \cap V; A) \to \cdots$$

The existence of such a sequence says that any compactly supported cohomology class $u \in \mathrm{H}_c^n(U \cup V; A)$ satisfying the condition $\delta(u) = 0$ can be written as a sum $u = u' + u''$, where u' is supported on U and u'' is supported on V. Here it is crucial that we can add cohomology classes (and the cocycles that represent them): there is no reason to expect that we can arrange that u' or u'' is equal to zero.

In the setting of nonabelian cohomology, there is generally no way to "add" a compactly supported map $u' : U \to Y$ to a compactly supported map $u'' : V \to Y$ to obtain a compactly supported map from $u : U \cup V \to Y$. However, there is an obvious exception: if U and V are disjoint, then there is a canonical homeomorphism $\mathrm{Map}_c(U, Y) \times \mathrm{Map}_c(V, Y) \simeq \mathrm{Map}_c(U \cup V, Y)$, which we can think of as a type of "addition". It turns out that if we take this structure into account, then we can salvage Question 7.

Theorem 8 (Nonabelian Poincare Duality)**.** *Let M be a manifold of dimension n, let $\mathcal{U}_1(M)$ denote the collection of all open subsets of M which are homeomorphic to a disjoint union of finitely many open disks, and let Y be a space which is $(n-1)$-connected. Then the canonical map*

$$\varinjlim_{U \in \mathcal{U}_1(M)} \mathrm{Map}_c(U, Y) \to \mathrm{Map}_c(M, Y)$$

is an equivalence in the ∞-category $\mathcal{S}$. In other words, $\mathrm{Map}_c(M,Y)$ *can be realized as the homotopy colimit of the diagram* $\varinjlim_{U \in \mathcal{U}_1(M)} \mathrm{Map}_c(U,Y)$.

Remark 9. The formation of chain complexes $T \mapsto C_*(T;\mathbf{Z})$ determines a functor of ∞-categories $\mathcal{S} \to \mathrm{Mod}_{\mathbf{Z}}$ which preserves colimits. Consequently, Theorem 8 implies that the chain complex $C_*(\mathrm{Map}_c(M,Y);A)$ can be realized as a colimit

$$\varinjlim_{U_1,\ldots,U_n} C_*(\mathrm{Map}_c(U_1 \cup \cdots \cup U_n, Y);\mathbf{Z}) \simeq \varinjlim_{U_1,\ldots,U_n} \bigotimes C_*(\mathrm{Map}_c(U_i,Y);\mathbf{Z}),$$

where the U_i range over all collections of disjoint open disks in M. This expresses the informal idea that $C_*(\mathrm{Map}_c(M,Y);\mathbf{Z})$ can be obtained as a continuous tensor product of copies of cochain complex $C_*(\Omega^n Y;\mathbf{Z})$, indexed by the points of M (or open disks in M).

Remark 10. Theorem 8 provides a convenient mechanism for analyzing the homotopy type of the mapping space $\mathrm{Map}_c(M,Y)$: the partially ordered set indexing the colimit depends only on manifold M, and the individual factors $\mathrm{Map}_c(U,Y)$ are homotopy equivalent to $\Omega^n Y$, which depends only on Y.

Remark 11. The hypothesis that Y be $(n-1)$-connected is necessary for Theorem 8. For example, if $n > 0$ and Y is disconnected, then the constant map from a compact manifold M to a point $y' \in Y$ belonging to a different connected component than the base point $y \in Y$ cannot be homotopic to a map which is supported in a union of open disks of M.

On the other hand, suppose that Y is $(n-1)$-connected and that M is a compact smooth manifold, so that we can choose a triangulation of M. Then any continuous map $f : M \to Y$ is nullhomotopic on the $(n-1)$-skeleton of Y, and therefore homotopic to a map which is support on the interiors of the n-simplices of Y. This should make Theorem 8 sound at least plausible.

Remark 12. One can use Theorem 8 to deduce Proposition 4 (and therefore the classical Poincare duality theorem).

Remark 13. Theorem 8 asserts the existence of a homotopy equivalence

$$\varinjlim_{U \in \mathcal{U}_1(M)} \mathrm{Map}_c(U,Y) \to \mathrm{Map}_c(M,Y),$$

whose codomain can be viewed as a kind measuring the (compactly supported) nonabelian cohomology of the manifold M with coefficients in Y. To argue that this is an analogue of Poincare duality, we need to interpret the left hand side as some kind of homology. We will return to this point in a future lecture.

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whose codomain can be viewed as a kind measuring the (compactly supported) nonabelian cohomology of the manifold M with coefficients in Y. As in the case of classical Poincare duality, the left hand side can be viewed as a kind of homology. However, it is not the homology of M itself, but of the Ran space $\mathrm{Ran}(M)$.

Definition 14. Let M be a topological space. We let $\mathrm{Ran}(M)$ denote the collection of all nonempty finite subsets of M. For every collection of disjoint open sets $U_1, \ldots, U_M \subseteq M$, let $\mathrm{Ran}(U_1, \ldots, U_m)$ denote the subset of $\mathrm{Ran}(M)$ consisting of those nonempty finite sets $S \subseteq M$ satisfying

$$S \subseteq U_1 \cup \cdots \cup U_m \qquad S \cap U_1 \neq \emptyset \qquad \cdots \qquad S \cap U_m \neq \emptyset.$$

The collection of sets $\mathrm{Ran}(U_1, \ldots, U_m)$ form a basis for a topology on $\mathrm{Ran}(M)$. We will refer to $\mathrm{Ran}(M)$ as the *Ran space* of M.

Remark 15. Suppose that the topology on M is defined by a metric d. Then the topology on $\mathrm{Ran}(M)$ is also defined by a metric, where the distance from a nonempty finite set $S \subseteq M$ to another nonempty finite set $T \subseteq M$ is given by

$$\max\{\max_{s \in S} \min_{t \in T} d(s,t), \max_{t \in T} \min_{s \in S} d(s,t)\}.$$

Theorem 16 (Nonabelian Poincare Duality Reformulated)**.** *Let M be a topological manifold of dimension n and let Y be a pointed space which is $(n-1)$-connected. Then there exists an $\mathcal{S}$-valued cosheaf $\mathcal{F}$ on the topological space* $\mathrm{Ran}(M)$ *with the following property: for every collection of disjoint connected open sets $U_1, \ldots, U_k \subseteq M$, we have*

$$\mathcal{F}(\mathrm{Ran}(U_1, \ldots, U_k)) \simeq \mathrm{Map}_c(U_1, Y) \times \cdots \times \mathrm{Map}_c(U_k, Y).$$

Theorem 16 is essentially a reformulation of Theorem 8. If M is connected, it implies that we can recover $\mathrm{Map}_c(M, Y) \simeq \mathcal{F}(M)$ as a homotopy colimit

$$\varinjlim_{U_1, \ldots, U_k} \mathrm{Map}_c(U_1, Y) \times \cdots \times \mathrm{Map}_c(U_k, Y),$$

where the colimit is taken over all collections of disjoint open disks in M (this follows from the fact that sets of the form $\mathrm{Ran}(U_1, \ldots, U_k)$ form a basis for the topology of $\mathrm{Ran}(M)$). This is essentially the same as the colimit which appears in the statement of Theorem 8 (though there are a few subtleties; see [1] for a more detailed discussion).

Remark 17. The cosheaf $\mathcal{F}$ appearing in the statement of Theorem 16 is not locally constant. Unwinding the definitions, one can identify the costalk of $\mathcal{F}$ at a point $S = \{x_1, \ldots, x_m\} \in \mathrm{Ran}(M)$ with the product $\prod_i \mathrm{Map}_c(U_i, M)$, where $\{U_i\}_{1 \leq i \leq m}$ is a collection of disjoint open disks around the points $\{x_i\}_{1 \leq i \leq m}$. In particular, the costalk of $\mathcal{F}$ at S is noncanonically equivalent to $\Omega^n(Y)^m$: a homotopy type which depends only on the target space Y, and not on the manifold M.

In the next lecture, we will formulate a nonabelian version of Theorem 16.

References

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