Analytic Stacks

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This text consists of notes on the lecture Analytic Stacks, jointly taught at the University of Bonn and the Institut des hautes études scientifiques by Prof. Peter Scholze and Prof. Dustin Clausen in the winter term (Wintersemester) 2023/24. The lecture firstly introduces light condensed sets/groups/rings/..., secondly introduces analytic rings, thirdly covers Analytic Stacks and lastly goes over a few examples. The lecture series can also be found online on YouTube and Carmin.tv.

Please report typos, mistakes and bugs to my Mail address: jonas@helpsyou.de.

Proofs and remarks denoted with * are changes from the author. So any *Proof**, *Remark**, . . . are only to be used at own risk. Many of such changes are ideas of others which are credited through footnotes.

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1 Lecture 1 (1/24)

1.1 Introduction

Classically there are several different theories of analytic geometry:

1. **Complex analytic spaces:** In the smooth case complex manifolds.
2. **Locally analytic manifolds** $(K, \|\cdot\|)$ **complete valued field:** The idea is to "glue open subsets of K^d along locally analytic maps". When K is non-archimedean (e.g. $K = \mathbb{Q}_p$) it is not geometrically rich because the topology is totally disconnected.
3. **Rigid analytic geometry (over a non-archimedean field):** Here the idea is to focus on local rings of functions instead of local topology. The local rings of functions are (quotients of) functions convergent on a closed polydisc (Tate Algebra). This was generalised by Huber to the theory of **adic spaces**. These still have local rings of functions A, but also include extra data of a certain subring $A^+ \subseteq A$.
4. **Berkovich Theory:** Local models are given by Banach rings $(R, \|\cdot\|) \rightsquigarrow M(R, \|\cdot\|)$, where $M(R, \|\cdot\|)$ is a Berkovich space of multiplicative semi-norms, which are compact Hausdorff spaces.

1.2 Why produce a new theory?

(1)-(4) and their relationships are well understood but they can't all be treated at the same time. Of all the above, the only rich theory allowing archimedean and non-archimedean geometry is Berkovich's, but it is not well worked out. Glueing is only worked out in some "finite type" setting.

Even individually these theories are less flexible than e.g. the theory of schemes. One of the main problems is issues of descent: One main construction in the theory of schemes is $\mathrm{QCoh}(X)$ which can be thought of as modules over a ring glued together to form a sheaf over X. We don't have such things in classical analytic geometry. Even though we could theoretically still have modules locally over a ring, this concept wouldn't be "global" as that wouldn't glue. This would only work with finitely generated modules which would lead to the theory of coherent sheaves.

There is a potentially practical reason, coming from the Langlands program: Fargues-Scholze "geometrized" local Langlands. This was done by replacing $\mathbb{Q}_p$ by a more exotic object, namely the "Fargue-Fontaine curve". This was produced in the language of adic spaces over $\mathbb{Q}_p$, which is not of finite type.

Speculatively, we could also hope for a geometrization of global Langlands by replacing $\mathbb{Z}$ by some exotic analytic space over $\mathbb{Z}$. Whatever such thing might be (if something like this exists), it would need archimedean and non-archimedean aspects and would not be of finite type as well.

The goal of the course is to introduce a new theory of analytic geometry and show its relations to the old theory.

1.3 Condensed Mathematics

The local models classically are in fact topological rings and we need to remember the topology. But topological rings and modules are not suitable to a general theory[1]. That's why we go to the basics and define a replacement for the category of topological spaces. The basic idea: instead of encoding a topological space X traditionally, we single out a collection of "nice test spaces" S and we just record the data of "continuous maps $S \to X$". Formally, the test spaces will be **profinite sets**[2] but this is a very large category.

Definition 1.1. A **light profinite set** is a countable inverse limit of finite sets. A **light condensed set**[3] is a sheaf of sets on the category of light profinite sets on the Grothendieck topology, where covers are finite families of surjective maps.

More explicitly, a light condensed set is a functor

$$\mathrm{LightProf}^{\mathrm{op}} \to \mathrm{Set}$$

with these properties:

1. $X(\varnothing) = \varnothing$
2. $X(S \sqcup T) \xrightarrow{\sim} X(S) \times X(T)$
3. $\exists T \twoheadrightarrow S \Rightarrow X(S) = \mathrm{eq}(X(T) \rightrightarrows X(T \times_S T))$

Example 1.2. Every Topological space X is by Yoneda a Condensed set via $X(S) = \mathrm{Cont}(S, X)$.

Remarks on why we make this precise definition:

1. For any light condensed set X, if we plug in $S = *$ we will get back the "underlying set". Note that if X is a topological space we will just get back the set X (we will still have to put in some work to get back the topology in special cases). If we plug in $S = \mathbb{N} \cup \{\infty\}$ we get something like " the space of convergent series in X with its limit".

[1]One might want to read the motivation in [CS19]. One of the major flaws of topological abelian groups/rings/modules/... is that it's not an Abelian Category. Take e.g. $0 \to \mathbb{R}_{\text{discrete}} \to \mathbb{R} \to 0$ which is exact but $\mathbb{R}_{\text{discrete}} \to \mathbb{R}$ is not an isomorphism. The workaround is to introduce some "condensed" notation whose categories will be Abelian in nice cases.

[2]These can be defined as the inverse limit of finite sets with discrete topology. By Stone duality this is equivalent to compact totally disconnected topological spaces.

[3]For those who read [CS19]: With that category, we don't need to look at extremally disconnected condensed sets anymore. We will see in later lectures what this technically means.

2. Allowing <u>all</u> surjections to count as covers gives a nice simplification of the structure of light condensed sets. It gives us some nice homological Algebra properties, if we pass it to light condensed abelian groups, since we have a lot of flexibility working locally with these covers.

3. Requiring the Grothendieck topology to be finitary gives good categorical compactness properties for light profinite sets. We get these properties even for all metrizable compact Hausdorff spaces, e.g.

$$\{0,1\}^{\mathbb{N}} \twoheadrightarrow [0,1]$$

The compactness from Hausdorff spaces, which we know from general topology, now kind of translates into some categorical notation of compactness.

1.4 Analytic Rings

In the following "condensed sets/rings/..." mean "light condensed sets/rings/..." and rings are always commutative with unit. The definition of a condensed set gives rise to the definition of condensed rings and condensed modules over rings (sheaves of rings and sheaves of modules). Why can't we use condensed rings (alone) as local models for our analytic geometry?

<u>Basic reason:</u> The category of condensed rings has pushouts given by relative tensor product just as for classical rings. Let A, B, k be condensed rings:

$$(A \otimes_k B)(*) = (A(*) \otimes_{k(*)} B(*))$$

$\Rightarrow$ $A \otimes_k B$ just gives a condensed structure on the abstract $- \otimes -$, so in particular it's not giving a completed tensor product, as the completion procedure does change the underlying set.

To fix this, we put additional structure on a condensed ring: We record some class of condensed modules, to be considered as "complete". That would give the notion of Analytic ring.

But what kind of rings do we want? Experience in algebraic geometry shows that the generally correct notion of a fibre product is actually the derived fibre product, which on affine level corresponds to derived relative tensor products of rings. So our rings should be derived, but which kind? There are two basic options:

1. **E_∞-algebras**
2. **Animated commutative rings (presented by simplicial commutative rings)**

For the course we will not allow negative homotopy and use the second option, as it is more closely tied to algebraic geometry. Formally, a **condensed animated ring** is a hypersheaf of animated rings on the site of profinite sets. From now on, another convention for today's lecture is, that by "ring" we mean "animated ring". If we want to stress that the ring lives in degree 0, it will be denoted as classical/static ring. Most of the time, one can just pretend, that everything is an ordinary ring but that's not the general case. The basic invariant of such R is its derived category $D(R)$. If R is static, this is the usual unbounded derived category of the abelian category of condensed R-modules. In particular it has a t-structure as a notion of connected objects and anti-connected objects. In general you still have a t-structure but it's not the derived category of R-modules.

Definition 1.3. An analytic ring is a pair $R = (R^\triangleright, D(R))$ where $R^\triangleright$ is a condensed ring and $D(R) \subseteq D(R^\triangleright)$ such that

1. $D(R)$ is closed under all limits and colimits.
2. If $N \in D(R), M \in D(R^\triangleright)$, then $\underline{\mathrm{RHom}}_{R^\triangleright}(M, N) \in D(R)$.
3. If $\widehat{(-)}_R$ denotes the left adjoint to the inclusion $D(R) \subseteq D(R^\triangleright)$, then $(-)_{\hat{R}}$ sends $D(R^\triangleright)_{\geq 0} \to D(R)_{\geq 0}$.
4. $R^\triangleright \in D(R)$.

A map of analytic rings $R \to S$ is a map of condensed rings $R^\triangleright \to S^\triangleright$ s.t. $D(S^\triangleright) \to D(R^\triangleright)$ restricts to $D(S) \to D(R)$.

Remark 1.4. There is always a t-structure on $D(R)$:

$$D(R)_{\geq 0} = D(R) \cap D(R^\triangleright)_{\geq 0}$$

$$D(R)_{\leq 0} = D(R) \cap D(R^\triangleright)_{\leq 0}$$

$$\Rightarrow \text{We get an abelian category } D(R)^\heartsuit = D(R) \cap D(R^\triangleright)^\heartsuit$$

For a profinite set S, consider the free R-module $R[S] := \widehat{(R^\triangleright[S])}_R$. These generate $D(R)_{\geq 0}$ under colimits.

Intuition: $R^\triangleright[S]$ = "space of $R^\triangleright$-linear combinations of dirac measures on S"

$R[S]$ = "some completion, i.e. a bigger space of measures"

Rule: $M \in D(R^\triangleright)^\heartsuit$ lies in $D(R)^\heartsuit$ $\iff$

$$\begin{tikzcd} \forall f : R^\triangleright[S] \arrow[rr] \arrow[dr] & & M \\ \exists! \text{ extension along} & R[S] \arrow[ur, dotted, "\exists!"'] & \end{tikzcd}$$

In other words, if $f : S \to M$ and $\mu \in R[S]$ then we get a well-defined $\int_S f \, d\mu \in M$.

Colimits in analytic rings: Filtered colimits (more generally sifted colimits):

1. $(\varinjlim_i R_i)^\triangleright = \varinjlim_i (R_i^\triangleright)$
2. $(\varinjlim_i R_i)[S] = \varinjlim_i (R_i[S])$

Pushouts in analytic rings: For maps of analytic rings $A \leftarrow k \rightarrow B$, we have that $D(A \otimes_k B) \subseteq D(A^{\triangleright} \otimes_{k^{\triangleright}} B^{\triangleright})$ is the full subcategory, s.t. the underlying $A^{\triangleright}$-module lies in $D(A)$ and the underlying $B^{\triangleright}$-module lies in $D(B)$.

Warning 1.5. $(A^{\triangleright} \otimes_{k^{\triangleright}} B^{\triangleright}, D(A \otimes_k B))$ is *not* an analytic ring. It satisfies (1)-(3), but not (4). To fix this, apply left adjoint to $D(A \otimes_k B) \subset D(A^{\triangleright} \otimes_{k^{\triangleright}} B^{\triangleright})$.

Solid analytic rings ($\rightsquigarrow$ Adic spaces)
For an analytic ring R it is natural to consider $M_R(\mathbb{N}) = R[\mathbb{N} \cup \infty]/R[\infty]$ which classifies nullsequences in R-modules. It turns out that it is not hard to show, that addition on $\mathbb{N}$ induces a ring structure on $M_R(\mathbb{N})$. As a ring it sits between[4]

$$R[T] \to M_R(\mathbb{N}) \to R[\![T]\!].$$

Definition 1.6. R is **solid** if $M_R(\mathbb{N})/(T-1) = 0$

An interpretation of this, is if you have a measure $\mu \in M$ s.t. $(T-1)\mu = 1 \in M_R(\mathbb{N})$ then μ corresponds to something like "$\sum_n T^n$". One would have to work it out but this means something like "every null-sequence is summable" which is a classic non-archimedean condition.

Theorem 1.7. *There exists a solid analytic ring $Z^{\square} = (\mathbb{Z}, D(\mathbb{Z}^{\square}))$ s.t. an analytic ring is solid $\iff$ $\exists$ (necessarily unique) map $\mathbb{Z}^{\square} \to R$. Moreover for $S = \varprojlim_n (S_n)$ and S infinite:*

1. *$\mathbb{Z}^{\square}(S) = \varprojlim_n \mathbb{Z}[S_n]$ ($\cong \prod_I \mathbb{Z}$ with I countable)*
2. (a) *$\prod_I \mathbb{Z}$ are the compact projective generators of $D(\mathbb{Z}^{\square})^{\heartsuit}$*
   (b) *$\prod_I \mathbb{Z}$ is flat w.r.t. the $(- \otimes_{\mathbb{Z}^{\square}} -)$*
   (c) *$\prod_I \mathbb{Z} \otimes_{\mathbb{Z}^{\square}} \prod_J \mathbb{Z} = \prod_{I \times J} \mathbb{Z}$ (One can find this in Corollary 5.12.)*
3. *The collection of finitely presented objects in $D(\mathbb{Z})^{\heartsuit}$ is abelian, closed under extensions and every finitely presented M has a free resolution by $\prod_I \mathbb{Z}$'s of length[5] ≤ 2.*

[4] Here one might want to think of $M_R(\mathbb{N})$ as $T^{\mathbb{N}}$.
[5] One can interpret this as "$\mathbb{Z}^{\square}$ behaves like a regular ring of dim 2".

2 Lecture 2 (2/24)

Want: "global"/Spec($\mathbb{Z}$) perfectoid space that is modelled by rings with non-trivial topology. This should be "analytic" and include the archimedean and non-archimedean case.

Vague Idea (possibly misguided):

$$\mathbb{C}_p\langle T^{\frac{\pm 1}{p^\infty}}\rangle = \mathbb{C}_p\langle T^{\mathbb{Z}[\frac{1}{p}]}\rangle$$

Globally? $k\langle T^{\mathbb{Q}}\rangle \subset k\langle T^{\mathbb{R}}\rangle$ we first try to understand what the latter object should be. We want to remember the topology of $\mathbb{R}$ but must then "mix" this with the p-adic topology in k. If $k\langle T^{\mathbb{R}}\rangle$ is the algebra what is the geometric space? Such things will be allowed in the new proposed foundation.

Since 2018: Topological sets/groups/rings/... have been replaced by condensed sets/ groups/ rings/ ... . In this setting $k\langle T^{\mathbb{R}}\rangle$ does make sense as a condensed ring.

Attitude: Develop most natural foundations for analytic geometry based on condensed rings and try it out for known formalisms.

2.1 Light Condensed Sets

Starting point: Category Pro(Fin) of profinite sets.

Proposition 2.1 (Stone Duality)**.** *The following categories are equivalent:*

1. Pro(Fin)[6]:
   (a) *Objects:* $\varprojlim_{i\in I} S_i$ *for* S_i *finite, I cofiltered poset* ($I \neq \varnothing, \forall i,j \in I, \exists k \leq i,j$)
   (b) *Morphisms:* $\mathrm{Hom}(\varprojlim_{i\in I} S_i, \varprojlim_{j\in J} T_j) = \varprojlim_{i\in I} \varinjlim_{j\in J} \mathrm{Hom}(S_i, T_j)$
2. *Totally disconnected compact Hausdorff spaces* $\subset$ *Top*
3. *(Boolean Algebras)*$^{\mathrm{op}}$[7]

The functors are:

$$\begin{array}{ccccc}
1) & \longrightarrow & 2) & \longrightarrow & 3) \\
\varprojlim_{i\in I} S_i & \longmapsto & S = \varprojlim_{i\in I} S_i & \longmapsto & \mathrm{Cont}(S,\mathbb{F}_2) = \underset{i}{\underrightarrow{\mathrm{colim}}}\, \mathbb{F}_2^{S_i}
\end{array}$$

$$\varprojlim_i \mathrm{Hom}(A_i, \mathbb{F}_2) \overset{=}{\longleftarrow\!\shortmid} \mathrm{Hom}(A,\mathbb{F}_2) = \mathrm{Spec}(A) \longleftarrow\!\shortmid A$$

[6]We will mostly use this category.
[7]Commutative rings R s.t. $\forall x \in R : x^2 = x$.

Two measures of how "big" S is:

Definition 2.2. Let $S = \varprojlim_{i \in I} S_i$ be a profinite set:

1. The **size** of S is $\chi := |S|$.
2. The **weight** of S is $\lambda := |\mathrm{Cont}(S, \mathbb{F}_2)|$.
3. S is **light** if $\lambda \leq \omega := |\mathbb{N}|$, i.e. countable limit of finite sets.

Remark 2.3. If λ infinite, it is also equal to $|I|$ for the smallest possible I.

Example 2.4.

1. finite sets
2. $\mathbb{N} \cup \{\infty\} = \varprojlim_n \{1, 2, 3, \ldots, n, \infty\}$, $\chi = \omega, \lambda = \omega$
3. Cantor Set $= \{0,1\}^{\mathbb{N}} = \varprojlim_n \{0,1\}^n$, $\chi = 2^\omega, \lambda = \omega$
4. Non-example: $S = \beta\mathbb{N}$ [8] $\rightsquigarrow$ $\{\text{subsets of } \mathbb{N}\} = \mathrm{Cont}(S, \mathbb{F}_2) = \mathrm{Cont}(\beta\mathbb{N}, \mathbb{F}_2) = \mathrm{Cont}(\mathbb{N}, \mathbb{F}_2)$, $\chi = 2^{2^\omega}, \lambda = 2^\omega$

Proposition 2.5. *$\lambda \leq 2^\chi$ and $\chi \leq 2^\lambda$. Also, if $\chi = \omega \Rightarrow \lambda = \omega$.*

Proof. $\lambda = |\mathrm{Cont}(S, \mathbb{F}_2)| \leq |\mathrm{Map}(S, \mathbb{F}_2)| = 2^\chi$. Denote by A the representation as a boolean Algebra: $\chi = |\mathrm{Hom}(A, \mathbb{F}_2)| \leq |\mathrm{Map}(A, \mathbb{F}_2)| = 2^\lambda$. Write $S = \{S_0, S_1, \ldots\}$. For each n inductively choose a quotient $S \twoheadrightarrow S_n (\twoheadrightarrow S_{n-1})$ such that $\{S_0, \ldots, S_n\} \hookrightarrow S_n$. □

Proposition 2.6. *The following categories are equivalent to light profinite sets.*

1. $\mathrm{Pro}_{\mathbb{N}}$*(Fin):*
   (a) *Objects:* $\varprojlim_{n \in \mathbb{N}} S_n$
   (b) *Morphisms:*

$$\begin{aligned} \mathrm{Hom}(\varprojlim_{n \in \mathbb{N}} S_n, \varprojlim_{m \in \mathbb{N}} T_m) &= \varprojlim_{n \in \mathbb{N}} \varinjlim_{m \in \mathbb{N}} \mathrm{Hom}(S_n, T_m) \\ &= \underset{\substack{f: \mathbb{N} \to \mathbb{N} \\ \text{strictly increasing}}}{\varinjlim} \varprojlim_{n \in \mathbb{N}} \mathrm{Hom}(S_n, T_m) \end{aligned}$$

2. *Metrizable totally disconnected compact Hausdorff spaces $\subset$ Top*

3. *(Countable Boolean Algebras)*$^{\mathrm{op}}$

Proposition 2.7. *The Category of light profinite sets has all countable limits. Sequential limits of surjections are surjective.*

This is proved in Proposition 3.5.

[8] Stone-Čech compactification

Proposition 2.8. *Let S be a light profinite set then $\exists$ a surjection $\{0,1\}^{\mathbb{N}} \twoheadrightarrow S$.*

Proof. Take this as a simple exercise: Use the universal property of the limit and find for each S_n a sufficiently large k such that $\{0,1\}^k$ surjects onto S_n. □

Now two properties that are special to *light* profinite sets:

Proposition 2.9. *Let S be a light profinite set, $U \subseteq S$ open. Then U is a countable disjoint union of light profinite sets*[9].

Warning 2.10. In general $\exists\, U \subset S$ open profinite with $\mathrm{H}^i(U, \mathbb{Z}) \neq 0$ for $i > 0$. That would be a problem, as disjoint union takes Cohomology to products and on profinite sets these are totally split so one has no non-trivial coverings, hence cohomology vanishes.

Proof. Let $S = \varprojlim_n S_n$ with $Z = S \setminus U = \varprojlim_n Z_n \subset S$ [10], where $Z_n = \mathrm{im}(Z \to S_n) \subset S_n$. Let π_n be the maps $S \to S_n$. Then $U = \bigcup \pi_n^{-1}(S_n \setminus Z_n)$. Each $\pi_n^{-1}(S_n \setminus Z_n) \underset{\text{clopen}}{\subset} S$.

$$\Rightarrow U = \bigsqcup_n (\pi_{n+1}^{-1}(S_{n+1} \setminus Z_{n+1}) \setminus \pi_n^{-1}(S_n \setminus Z_n))$$

□

Proposition 2.11. *Let S be a light profinite set. Then S is an injective object in* Pro(Fin), *i.e.*

$$\forall\, Z \subset X : \begin{tikzcd} Z \arrow[r] \arrow[d, hook] & S \\ X \arrow[ur, dotted, "\exists"] & \end{tikzcd} .$$

Proof. First of all we need to note, that an element in $\mathrm{Cont}(X, \mathbb{F}_2)$ precisely is a choice of a clopen subset in X.
Case $S = \{0,1\}$: Here, $\mathrm{Cont}(X, \mathbb{F}_2) \twoheadrightarrow \mathrm{Cont}(Z, \mathbb{F}_2)$ (Or any clopen subset of Z extends to a clopen subset of X). In general, write $S = \varprojlim_n S_n$, all $S_{n+1} \twoheadrightarrow S_n$. Induct on n, and extend to S_n: If one already extended the map to S_n then extending further to S_{n+1} the whole situation decomposes into disjoint unions over all the fibers over S_n. Hence w.l.o.g. $S_n = *$ and so S_{n+1} is finite. Every finite set reduces to the first case we looked at. □

Definition 2.12. A **light condensed set** is a sheaf of sets on the category of light profinite sets for the Grothendieck topology generated by:

[9] We already have control over the open sets just via light profinite sets again. This gives us nice properties for when we define our Grothendieck topology.

[10] One way to think about open subsets purely in the language of the pro-category profinite sets, is to think about the closed things in Sets. So closed subsets should themselves be profinite sets and the closed subsets are precisely the injective maps of profinite sets. So here the closed complement $Z \subset X$ should be a profinite set as well.

1. Finite disjoint unions
2. All surjective maps

Equivalently: A functor

$$\begin{aligned} X : \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})^{\mathrm{op}} &\longrightarrow \mathrm{Sets} \\ S &\longmapsto X(S) \end{aligned}$$

with the following properties:

1. $X(\varnothing) = *$
2. $X(S_1 \sqcup S_2) \xrightarrow{\sim} X(S_1) \times X(S_2)$
3. $\forall f : T \twoheadrightarrow S$ it holds that $f^* : X(S) \xrightarrow{\sim} \mathrm{eq}\big(X(T) \overset{p_1^*}{\underset{p_2^*}{\rightrightarrows}} X(T \times_S T)\big)$

Key example: Let A be a topological space. We can produce a light condensed set $\underline{A} : S \mapsto \mathrm{Cont}(S, A)$. In particular $\underline{A}(*) = A$ as a set. So for any light condensed set X we think of $X(*)$ as the "underlying set".

$\underline{A}(\mathbb{N} \cup \infty)$ = convergent sequences in A (with choice of limit point).

$\underline{A}$(Cantor Set) $\circlearrowright$ End(Cantor Set)

Remark 2.13. A light condensed set X is completely determined by X(Cantor Set) and its action on End(Cantor Set). Thus a light condensed set is the data of an abstract set and an abstract monoid.

Proposition 2.14. *The functor*

$$\begin{aligned} \mathrm{Top} &\longrightarrow \mathrm{CondSet}^{\mathrm{light}} \\ A &\longmapsto \underline{A} \end{aligned}$$

has a left adjoint

$$\begin{aligned} \mathrm{CondSet}^{\mathrm{light}} &\longrightarrow \mathrm{Top} \\ X &\longmapsto X(*)_{\mathrm{top}}. \end{aligned}$$

Here $X(*)_{\mathrm{top}} = X(*)$ with the quotient topology[11] *from*[12]

$$\bigsqcup_{S, \alpha \in X(S)} S \longrightarrow X(*) \;\cong\; \bigsqcup_{\alpha \in X(\text{Cantor Set})} \text{Cantor Set} \longrightarrow X(*).$$

It holds that $X(*)_{\mathrm{top}}$ is a "metrizable compactly generated" topological space. If A is any metrizable compactly generated topological space then $A \xleftarrow{\sim} \underline{A}(*)_{\mathrm{top}}$. This is shown in 3.1.

[11] Notice how we don't need to take all profinite sets anymore in comparison to general condensed sets.

[12] For the isomorphism: By Yoneda embedding, $\mathrm{Hom}(\underline{S}, X) = X(S)$. We use that here, as we take the disjoint union of maps of the elements in $X(S)$ applied to $*$.

Corollary 2.15.

$$\{\textit{metrizable compactly generated topological spaces}\} \hookrightarrow \mathrm{CondSet}^{\mathrm{light}}.$$

Remark 2.16. Most spaces arising in nature such as Fréchet spaces or Banach spaces are metrizable compactly generated topological spaces.

Remark 2.17.
1. Johnstone's "Topological Topos" has a similar idea he should rather find topos which is somehow very closely related to topological spaces, based on just $\mathbb{N} \cup \{\infty\}$ and uses the canonical topology (which is not finitary, which leads to bad algebraic properties of topos).
2. Escardó-Xu also have an approach which is closely related to ours here. They take light profinite sets but only take finite disjoint unions as covers. (They use the "set and monoid" description which is reasonable in the finite disjoint union context.)

2.2 Light Condensed Abelian Groups

Recall: For sheaves on any site, sheaves of abelian groups always form a Grothendieck abelian category (filtered colimits are exact + all limits and colimits exist). Hence $\mathrm{CondAb}^{\mathrm{light}}$ is a Grothendieck abelian category.

Example 2.18. Consider $\mathbb{Q} \hookrightarrow \mathbb{R}$, or $\mathbb{R}_{\mathrm{discrete}} \longrightarrow \mathbb{R}$. What is the cokernel here?[13]

$$(\underline{\mathbb{R}}/\underline{\mathbb{Q}})(*) = \mathbb{R}/\mathbb{Q}$$
$$(\underline{\mathbb{R}}/\underline{\mathbb{Q}})(S) = \mathrm{Cont}(S,\mathbb{R})/\mathrm{Cont}(S,\mathbb{Q})$$

More drastic:

$$(\underline{\mathbb{R}}/\underline{\mathbb{R}}_{\mathrm{discrete}})(*) = \mathbb{R}/\mathbb{R} = 0$$
$$(\underline{\mathbb{R}}/\underline{\mathbb{R}}_{\mathrm{discrete}})(S) = \mathrm{Cont}(S,\mathbb{R})/\mathrm{Cont}(S,\mathbb{R}_{\mathrm{discrete}}) \neq 0,$$

since $\mathrm{Cont}(S,\mathbb{R}_{\mathrm{discrete}})$ are again locally constant maps.

Theorem 2.19. $\mathrm{CondAb}^{\mathrm{light}}$ *is a Grothendieck Abelian category*

1. *Countable products are exact (and satisfy (AB6))*
2. $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ *is internally projective*[14]

Remark 2.20. The forgetful functor $\mathrm{CondAb}^{\mathrm{light}} \to \mathrm{CondSet}^{\mathrm{light}}$ has a left adjoint

$$\begin{aligned} \mathrm{CondSet}^{\mathrm{light}} &\longrightarrow \mathrm{CondAb}^{\mathrm{light}} \\ X &\longmapsto \mathbb{Z}[X] \end{aligned}$$

[13] $\mathrm{Cont}(S,\mathbb{Q})$ are locally constant maps.

[14] This is a property that is extremely specific for light condensed abelian groups. In Condensed abelian groups there are many projective objects, but there are no internally projective objects except trivial cases. $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ wouldn't even be projective in Condensed Abelian groups.

3 Lecture 3 (3/24)

Recall:

Light profinite Sets:

$$\begin{aligned}\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}) &\cong \left\{\begin{array}{l}\text{metrizable, totally disconnected,}\\ \text{compact, Hausdorff spaces}\end{array}\right\}\\ &\cong \{\text{Countable Boolean algebras}\}^{\mathrm{op}}\end{aligned}$$

They have a Grothendieck topology with covers defined as:

1. finite disjoint unions
2. all surjective maps

$$\Rightarrow \text{sequential limits of covers are still covers!}$$

$$\begin{tikzcd} & \mathrm{Top} \arrow[dr, "X \mapsto \underline{X}"] & \\ \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}) \arrow[ur, hook] \arrow[rr, hook, "\chi: S \mapsto (T \mapsto \mathrm{Cont}(T,S))"'] & & \mathrm{CondSet}^{\mathrm{light}} = \mathrm{Shv}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})) \end{tikzcd}$$

The image of χ generates $\mathrm{CondSet}^{\mathrm{light}}$ under colimits. The functor

$$\begin{aligned}\mathrm{Top} &\to \mathrm{CondSet}^{\mathrm{light}}\\ X &\mapsto \underline{X}\end{aligned}$$

has a left adjoint $X \mapsto X(*)_{\mathrm{top}}$, where the topology is given by the quotient topology of

$$\bigsqcup_{\alpha \in X(\text{Cantor Set})} (\text{Cantor Set}) \twoheadrightarrow X(*)_{\mathrm{top}}.$$

Also we have in topological spaces the quotient map

$$\bigsqcup_{\gamma \in \underline{\text{Cantor Set}}(\mathbb{N}\cup\{\infty\})} (\mathbb{N}\cup\{\infty\}) \twoheadrightarrow \text{Cantor Set}$$

Hence we can understand

$$\text{Cantor Set} \cong \varinjlim_{\substack{\text{countable, closed}\\ \text{subsets } Z}} (Z) \in \mathrm{Top}.$$

This gives us something like "metrizable compactly generated" = "sequential" and we get

$$\{\text{sequential topological spaces}\} \hookrightarrow \mathrm{CondSet}^{\mathrm{light}}.$$

Remark* 3.1. Hence we can also get the topology of $X(*)_{\mathrm{top}}$ via the quotient topology

$$\coprod_{X(\text{Cantor Set})}\ \coprod_{\underline{\text{Cantor Set}}(\mathbb{N}\cup\{\infty\})} (\mathbb{N}\cup\{\infty\}) \twoheadrightarrow X(*)_{\mathrm{top}}.$$

If X is a qcqs light profinite set embedded via the Yoneda embedding, this coincides with the topology on X as a metrizable totally disconnected compact Hausdorff space[15].

Proof.* [16] Let us denote X with its original topology as X_{lprof}. Since on set level $X_{\mathrm{lprof}} = X(*)_{\mathrm{top}}$ it suffices to show, that $\mathrm{Hom}(X_{\mathrm{lprof}}, Y) = \mathrm{Hom}(X(*)_{\mathrm{top}}, Y)$ for any topological space Y. Now, since light profinite sets are metrizable spaces they are also sequential. Suppose we have a map $X_{\mathrm{lprof}} \to Y$ and consider all convergent sequences of the image of an element $\gamma \in \underline{X}(\text{Cantor Set}) \subseteq X_{\mathrm{lprof}}$. Then we get the diagram where solid arrows are continuous maps and the dotted arrows are maps of sets:

$$\begin{tikzcd}
X(*)_{\mathrm{top}} \arrow[r, dotted] \arrow[dr, dotted, leftrightarrow, "\mathrm{id}_{\mathrm{Set}}"] & Y \\
\coprod(\mathbb{N}\cup\{\infty\}) \arrow[u] \arrow[r] & X_{\mathrm{lprof}} \arrow[u]
\end{tikzcd}$$

Now as $\coprod(\mathbb{N}\cup\{\infty\}) \to X_{\mathrm{lprof}} \to Y$ is continuous and commutes with the factorization through $X(*)_{\mathrm{top}}$, the upper dotted arrow is continuous from the defintion of the quotient topology. For the other way round we get, as $\coprod(\mathbb{N}\cup\{\infty\}) \to Y$ is continuous via the factorization through $X(*)_{\mathrm{top}}$, that for every convergent sequence in the image of γ we get a converging sequence in $\gamma(\text{Cantor Set}) \to Y$. Since X_{lprof} is sequential, this is continuous. As we get local continuity that way, we are done. ☐

3.1 Why allow the Cantor Set?

In any topos, there is an intrinsic notion of being "compact" and of being "Hausdorff".

Definition 3.2.

1. An object X is **quasicompact (qc)** if any cover admits a finite subcover, i.e. for $\coprod_{i\in I} X_i \twoheadrightarrow X$, we get a finite $J \subseteq I$ s.t. $\coprod_{i\in J} X_i \twoheadrightarrow X$.
   In our case $\iff$ $\exists$ surjection (Cantor Set) $\twoheadrightarrow X$ or $X = \varnothing$

2. An object is **quasiseparated (qs)** if for all quasicompact $Y, Z \to X$, the fibre product $Y \times_X Z$ is quasicompact.

Here: For all $f, g : (\text{Cantor Set}) \twoheadrightarrow X$ we have $\{0,1\}^{\mathbb{N}} \times_X \{0,1\}^{\mathbb{N}}$ is quasicompact. If we only allowed $\mathbb{N}\cup\{\infty\}$ in test category then the quasicompact objects would all be countable.

[15] We see that qcqs light profinite sets are metrizable totally disconnected compact Hausdorff spaces in 3.3.

[16] Thanks to Semen for this idea!

Proposition 3.3.

1. *{qcqs (light) condensed sets} ≅ {(metrizable) compact Hausdorff spaces}.*
   For this equivalence to be true we need
   (a) *finitary Grothendieck topology (otherwise our basic objects wouldn't be quasicompact)*
   (b) *allow Cantor Set*

2. *The following holds:*[17]

$$\left\{\begin{array}{l}\textit{qs (light)}\\ \textit{condensed sets}\end{array}\right\} \cong \mathrm{Ind}_{\mathrm{inj}}\left(\left\{\begin{array}{l}\textit{(metrizable,) compact}\\ \textit{Hausdorff spaces}\end{array}\right\}\right)$$
$$\supseteq \left\{\begin{array}{l}\textit{(metrizable,) compactly generated}\\ \textit{weak Hausdorff spaces}\end{array}\right\} =: \mathrm{mCGWH}$$

In mCGWH one can write Cantor Set = $\varinjlim$(countable closed subsets) which we can't in (qs) light condensed sets.
As $[0,1]$ is compact Hausdorff, we have implicitly in 1.:

$$\begin{aligned}\{0,1\}^{\mathbb{N}} &\twoheadrightarrow [0,1]\\ (a_0, a_1, \dots) &\mapsto 0.a_0a_1a_2\dots \quad \text{in binary expansion}\end{aligned}$$

3.2 Light Condensed Abelian Groups

$$\begin{aligned}\mathrm{CondAb}^{\mathrm{light}} &= \text{Abelian group objects in } \mathrm{CondSet}^{\mathrm{light}}\\ &= \mathrm{Shv}(\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin}), \mathrm{Ab})\end{aligned}$$

General theory of sheaves:

1. Grothendieck abelian category
2. has $\otimes$[18]: unit object is $\underline{\mathbb{Z}} : S \mapsto \mathrm{Cont}(S, \mathbb{Z})$
3. The forgetful functor $\mathrm{CondAb}^{\mathrm{light}} \to \mathrm{CondSet}^{\mathrm{light}}$ has a left adjoint "free condensed abelian group" $X \mapsto \mathbb{Z}[X]$ where $\mathbb{Z}[X]$ is the sheafification of $S \mapsto \mathbb{Z}[X(S)]$.

Idea: $\mathbb{Z}[X]$ has "some topology" on $\mathbb{Z}[X(*)]$.

Example 3.4. We will have a look at $\mathbb{Z}[\underline{\mathbb{R}}]$. Firstly we will see what happens at the classical object, i.e. [19]

$$\mathbb{Z}[\underline{\mathbb{R}}](*) = \mathbb{Z}[\mathbb{R}] = \left\{\sum_{x\in\mathbb{R}} n_x[x], n_x \in \mathbb{Z}, \text{almost all } 0\right\}$$

[17] The ind-category is the category of formal direct limits, similar to how a pro-category is the category of formal inverse limits.

[18] $M \otimes N$ is the sheafification of $S \mapsto M(S) \otimes N(S)$.

[19] Note that we don't need to worry about the sheafification, as evaluating at a point is just taking a stalk.

$$= \varinjlim_{I=[-c,c]\subset\mathbb{R}} \mathbb{Z}[I].$$

Here we have $\mathbb{Z}[I] = \bigcup_{n\in\mathbb{N}} \mathbb{Z}[I]_{\le n}$[20] where $\mathbb{Z}[I]_{\le n} := \{\sum_{x\in I} n_x[x] \mid \sum |n_x| \le n\}$. To get results in the more global setting we can use the same construction via this limit.

By general abstract nonsense $\mathbb{Z}[G]$ is naturally a condensed ring.

Theorem 3.5. *In* $\mathrm{CondAb}^{\mathrm{light}}$ *the following hold:*

1. *Countable products are exact.*
2. *Sequential limits of surjective maps are still surjective.*
3. $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ *is internally projective.*[21]

Proof of 1. and 2.

1. reduces to 2.: We need $\forall f_n : M_n \twoheadrightarrow N_n$, that also $\prod f_n : \prod M_n \twoheadrightarrow \prod N_n$ and then we are done, since products of injective maps stay injective. We have $\forall m$,
$$\prod_{n\le m} M_n \times \prod_{n+1\ge m} N_n \twoheadrightarrow \prod_n N_n.$$
Now if we take the limit over this, we get
$$\varprojlim_m \left(\prod_{n\le m} M_n \times \prod_{n+1\ge m} N_n\right) = \prod_m M_n.$$
2. A map of sheaves $F' \to F$ is surjective, iff for every section $s \in F(U)$, there is a covering $\{U_i\}$, s.t. there exists a preimage of $s|_{U_i} \in F(U_i)$ in $F'(U_i)$. Since by Yoneda embedding $\mathrm{Hom}(S, X) \cong X(S)$ for a map $X' \to X$ in $\mathrm{CondAb}^{\mathrm{light}}$ and for all $S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$ a map $X' \to X$ is surjective iff for every diagram of solid arrows, there are dotted arrows that make the following diagram commute:
$$\begin{tikzcd} X' \arrow[r] & X \\ S' \arrow[u, dotted] \arrow[r, dotted, two heads] & S \arrow[u] \end{tikzcd}$$
In our case we have a sequential limit of surjective maps, i.e. a limit of the diagram
$$\dots \twoheadrightarrow M_2 \twoheadrightarrow M_1 \twoheadrightarrow M_0 \ .$$
Hence we get the diagram
$$M_\infty = \varprojlim_m (\dots \twoheadrightarrow M_2 \twoheadrightarrow M_1 \twoheadrightarrow M_0) \xrightarrow{f} M_0.$$

[20] These are compact Hausdorff.

[21] We say that P projective iff $\mathrm{Ext}^i(P, -) = 0$ for $i > 0$. We say that P internally projective iff $\underline{\mathrm{Ext}}^i(P, -) = 0$ for $i > 0$. Here internal Ext is given by $\underline{\mathrm{Ext}}^i(M, N) =$ sheafification of $S \mapsto \mathrm{Ext}^i(M \otimes \mathbb{Z}[S], N)$.

We want to show that f is surjective, hence we want to get a surjective map $S_\infty \twoheadrightarrow S$ in $\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$ and a map $S_\infty \to M_\infty$ s.t.

$$
\begin{tikzcd}
M_\infty = \varprojlim_m (\dots \arrow[r, two heads] & M_2 \arrow[r, two heads] & M_1 \arrow[r, two heads] & M_0) \arrow[r, "f"] & M_0 \\
S_\infty \arrow[u, dotted, "\exists ?"] \arrow[rrrr, two heads] & & & & S_0 \arrow[u]
\end{tikzcd}
$$

commutes. Since inside the maps are surjections, we get a diagram of the following form:

$$
\begin{tikzcd}
M_\infty = \varprojlim_m (& \dots \arrow[r, two heads] & M_2 \arrow[r, two heads] & M_1 \arrow[r, two heads] & M_0) \arrow[r, "f"] & M_0 \\
 & \dots \arrow[u] \arrow[r, two heads] & S_2 \arrow[u] \arrow[r, two heads] & S_1 \arrow[u] \arrow[r, two heads] & S_0 \arrow[u] \arrow[dr, two heads] & \\
S_\infty \arrow[uu, dotted, "\exists ?"] \arrow[rrrrr, two heads] & & & & & S_0 \arrow[uu]
\end{tikzcd} .
$$

Now, as countable limits of surjections are still surjective in $\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$, we can take S_∞ to be the limit of the sequence $S_\infty = \varprojlim_m (\dots \twoheadrightarrow S_2 \twoheadrightarrow S_1 \twoheadrightarrow S_0)$.

Hence we get our induced map from the limit and the following commuting diagram:

$$
\begin{tikzcd}
M_\infty \arrow[r, equal] & \varprojlim_m (\dots \arrow[r, two heads] & M_2 \arrow[r, two heads] & M_1 \arrow[r, two heads] & M_0) \arrow[r, "f"] & M_0 \\
 & \varprojlim_m (\dots \arrow[u] \arrow[r, two heads] & S_2 \arrow[u] \arrow[r, two heads] & S_1 \arrow[u] \arrow[r, two heads] & S_0) \arrow[u] \arrow[dr, two heads] & \\
S_\infty \arrow[uu] \arrow[ur, equal] \arrow[rrrrr, two heads] & & & & & S_0 \arrow[uu]
\end{tikzcd}
$$

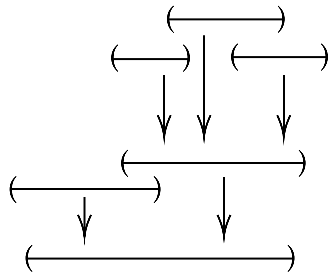
So we are done here.

□

Critical for 2 is that countable limits of covers are covers. This forces one to use totally disconnected spaces as building blocks. The intuition is the picture below, where each line is an interval.

In the limit we get something totally disconnected.

Proof of 3. For this proof we will take two steps: First we show that $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ is



projective. This is only true in $\mathrm{CondAb}^{\mathrm{light}}$ and not true in $\mathrm{CondSet}^{\mathrm{light}}$:

$$\begin{tikzcd} & (2\mathbb{N} \cup \{\infty\}) \sqcup ((2\mathbb{N}+1) \cup \{\infty\}) \arrow[d, two heads] \\ \mathbb{N} \cup \{\infty\} \arrow[r] \arrow[ur, dotted, "\nexists"] & \mathbb{N} \cup \{\infty\} \end{tikzcd}$$

Afterwards, we will show internally projectivity.

1. Projectivity Let $M = \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty]$ which classifies null sequences. Note that if we can show that M is projective we are done, as then $\mathbb{Z}[\mathbb{N} \cup \{\infty\}] = \mathbb{Z}[\infty] \oplus M$, where $\mathbb{Z}[\infty]$ is already projective. Suppose we have

$$\begin{tikzcd} & N' \arrow[d, two heads] \\ M \arrow[r] \arrow[ur, dotted, "\exists?"] & N \end{tikzcd}$$

Now, as $\mathbb{Z}[-]$ is left adjoint to the forgetful functor from $\mathrm{CondAb}^{\mathrm{light}}$ to $\mathrm{CondSet}^{\mathrm{light}}$ and , we have $\mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\mathbb{N} \cup \{\infty\}], M) \cong \mathrm{Hom}_{\mathrm{CondSet}^{\mathrm{light}}}(\mathbb{N} \cup \{\infty\}, M)$. This works as $M \in \mathrm{CondAb}^{\mathrm{light}}$. Thus it suffices to lift the following diagram[22] in $\mathrm{CondSet}^{\mathrm{light}}$:

$$\begin{tikzcd} S \arrow[rr, "\exists g"] \arrow[d, two heads] & & N' \arrow[d, two heads] \\ \mathbb{N} \cup \{\infty\} \arrow[r] & M \arrow[r, "h"] & N \\ \infty \arrow[rr, mapsto] & & 0 \end{tikzcd}$$

As $\mathbb{N} \cup \{\infty\}$ is covered by S at each point, we can choose just any lift (i.e. a point as preimage for each point but ∞) as another cover s.t. w.l.o.g. $S \times_{\mathbb{N} \cup \{\infty\}} \mathbb{N} \xrightarrow{\sim} \mathbb{N}$. We have that $S_\infty := S \times_{\mathbb{N} \cup \{\infty\}} \{\infty\} \overset{i}{\subseteq} S$ is closed. Since S_∞ is light, it is injective in $\mathrm{Prof}(\mathrm{Fin}) \Rightarrow \exists$ retraction $r : S \to S_\infty$. Now consider:

$$\begin{tikzcd} S \arrow[r, "g - g \circ i \circ r"] & N' \\ S_\infty \arrow[u, hook] \arrow[r] & \{0\} \arrow[u, hook] \end{tikzcd}$$

[22] This diagram exists because $\mathbb{N} \cup \{\infty\}$ is a light profinite set and as discussed before, since $N' \twoheadrightarrow N$ is surjective, there must be a $S \in \mathrm{CondSet}^{\mathrm{light}}$ such that this diagram commutes.

But as

$$\begin{tikzcd} S_\infty \arrow[r, hook] \arrow[d] & S \arrow[d, two heads] \\ \{\infty\} \arrow[r] & \mathbb{N}\cup\{\infty\} \end{tikzcd}$$

is a pushout in light condensed sets, we get the diagram:

$$\begin{tikzcd} S \arrow[rr, "g-g\circ i\circ r"] \arrow[dr] & & N' \\ & \mathbb{N}\cup\{\infty\} \arrow[ur, dotted, "\exists f"] & \\ & \{\infty\} \arrow[u, hook] \arrow[dr] & \\ S_\infty \arrow[uuu] \arrow[ur] \arrow[rr] & & \{0\} \arrow[uuu] \end{tikzcd}$$

It's now left to show that this f actually lifts our original diagram:

$$\begin{tikzcd} M = \mathbb{Z}[\mathbb{N}\cup\{\infty\}]/\mathbb{Z}[\infty] \arrow[r, "f"] \arrow[dr, "\overset{?}{=}h"'] & N' \arrow[d, two heads] \\ & N \end{tikzcd}.$$

As $\mathbb{Z}[S] \twoheadrightarrow M$ surjects[23], it suffices to show that the composite is the same as h. But as the map $\mathbb{Z}[S] \to N'$ is induced by $g - g\circ i\circ r$ and $g\circ i\circ r$ projects to 0 on N, we are done by the original diagram.

2. Internally projective* [24] We want that $\underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N}\cup\{\infty\}], -)$ is exact, i.e. if we have an epimorphism $N' \twoheadrightarrow N$, then $\underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N}\cup\{\infty\}], N') \twoheadrightarrow \underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N}\cup\{\infty\}], N)$.

$$\begin{aligned} &\underline{\mathrm{Hom}}\left(\mathbb{Z}[\mathbb{N}\cup\{\infty\}], N\right)(S) \\ &\cong \mathrm{Hom}_{\mathrm{CondSet}^{\mathrm{light}}}\left(S, \underline{\mathrm{Hom}}\left(\mathbb{Z}[\mathbb{N}\cup\{\infty\}], N\right)\right) \\ &\cong \mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}\left(\mathbb{Z}[S], \underline{\mathrm{Hom}}\left(\mathbb{Z}[\mathbb{N}\cup\{\infty\}], N\right)\right) \\ &\cong \mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}\left(\mathbb{Z}[S]\otimes\mathbb{Z}[\mathbb{N}\cup\{\infty\}], N\right) \\ &\cong \mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}\left(\mathbb{Z}[S\times(\mathbb{N}\cup\{\infty\})], N\right) \\ &\cong \mathrm{Hom}_{\mathrm{CondSet}^{\mathrm{light}}}\left(S\times(\mathbb{N}\cup\{\infty\}), N\right) \end{aligned}$$

Hence there is a $T \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$ and a commutative diagram:

$$\begin{tikzcd} T \arrow[r] \arrow[d, two heads, "f"'] & N' \arrow[d, two heads] \\ S\times(\mathbb{N}\cup\{\infty\}) \arrow[r] & N \end{tikzcd}$$

[23] Here we have used the forgetful functor, $\mathbb{Z}[-]$- adjunction again.
[24] Thanks to Ferdinand for this proof!

Now our aim is, to get some S', s.t. $S' \twoheadrightarrow S$ is a cover and s.t. the following diagram commutes:

$$\begin{tikzcd} S' \times (\mathbb{N} \cup \{\infty\}) \arrow[r] \arrow[d, two heads, "f"'] & N' \arrow[d, two heads] \\ S \times (\mathbb{N} \cup \{\infty\}) \arrow[r] & N \end{tikzcd}$$

Following from that, there is a surjection

$$\underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N} \cup \{\infty\}], N')(S') \twoheadrightarrow \underline{\mathrm{Hom}}(\mathbb{Z}[\mathbb{N} \cup \{\infty\}], N')(S),$$

which suffices, as it is surjective on a cover. Note that it is important that the morphism $S' \times \mathbb{N} \cup \{\infty\} \twoheadrightarrow S \times \mathbb{N} \cup \{\infty\}$ is induced by a surjection $S' \twoheadrightarrow S$.
For that, define $T_i := f^{-1}(S \times \{i\}) \subseteq S \times (\mathbb{N} \cup \{\infty\})$ for $i \in \mathbb{N} \cup \{\infty\}$. Consider the fibre product:

$$\begin{tikzcd} & T_1 \times T_2 \times T_\infty \arrow[dl, two heads] \arrow[dr, two heads] & \\ T_1 \arrow[dr, two heads] & & T_2 \arrow[dl, two heads] \\ & S & \end{tikzcd}$$

The limit over indices, i.e. $A = T_1 \times_S T_2 \times_S \ldots \times_S T_\infty$ is still a cover, as sequential limits of surjections stay surjective. Choosing a surjection $\{0,1\}^{\mathbb{N}} \twoheadrightarrow A$, it inductively covers each T_i by $\{0,1\}^{\mathbb{N}}$ via $\{0,1\}^{\mathbb{N}} \twoheadrightarrow A \twoheadrightarrow T_i$. As we never touch T_∞ using this process, one gets the cover $(\{0,1\}^{\mathbb{N}} \times \mathbb{N}) \cup (T_\infty \times \{\infty\})$. Now as it is covered by $(\{0,1\}^{\mathbb{N}} \times \mathbb{N}) \cup (T_\infty \times \{\infty\})$ (at least set theoretical, one would still need to check continuity) and the cover $(\{0,1\}^{\mathbb{N}} \times \mathbb{N}) \cup (T_\infty \times \{\infty\}) \twoheadrightarrow S \times \mathbb{N} \cup \{\infty\}$ is constructed, s.t. it is induced by a map $\{0,1\}^{\mathbb{N}} \twoheadrightarrow S$ we are done. □

Remark 3.6.

1. In CondAb there are all products exact and it has projective generators $\mathbb{Z}[S]$[25] , but none of them are internally projective as $\mathbb{Z}[\beta I \times \beta J]$ is never projective (I, J infinite). One thing that is better in CondAb is that we do have projective generators, other than in $\mathrm{CondAb}^{\mathrm{light}}$. Explicit reason that $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ is <u>not</u> internally projective in CondAb:

$$\begin{tikzcd} \mathbb{Z}[\beta\mathbb{N}] \arrow[r, bend left, two heads] & \mathbb{Z}[\mathbb{N} \cup \{\infty\}] \arrow[l, bend left, dotted, "\nexists"] \end{tikzcd}$$

This is also related to certain questions of Banach spaces: The only known injective Banach spaces[26] are $\mathrm{Cont}(S, \mathbb{R})$ with S extremally disconnected. So

[25] With $S = \beta I$, I discrete, we mean the Stone-Čech compactification, i.e. S is extremally disconnected.
[26] There is some kind of duality, that projective objects in CondAb becomes injective in Banach spaces.

basically Stone-Čech compactifications are retracts of it. It's known that there are Banach spaces which are retracts of continuous functions on a Stone-Čech compactification. $C_0(\mathbb{N})$, the Banach space of null-sequences which corresponds to our $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty]$, is not injective but it is "separably injective", which corresponds to $\mathbb{Z}[\mathbb{N} \cup \{\infty\}]$ being projective in $\mathrm{CondAb}^{\mathrm{light}}$ but not in CondAb.

2. (AB6) holds for countable products in $\mathrm{CondAb}^{\mathrm{light}}$.

3.3 Cohomology

If X is any light condensed set, M an abelian group:

Definition 3.7. $H^i(X, M) := \mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[X], M)$

Here we mean by M the light condensed abelian group $\underline{M} = \mathrm{Const}(-, M)$ where M has discrete topology. In the following lectures we will use the notation M and $\underline{M}$ interchangeably, as it is clear in context what is meant.

Theorem 3.8. *If X is a CW complex, $H^i(\underline{X}, M) \cong H^i_{\mathrm{sing}}(X, M)$.*

4 Lecture 4 (4/24)

This and the next lecture follow the first course on condensed mathematics that was given in the summer term 2019 to some extent. For further reference see [CS19].

4.1 Ext Computations in Light Condensed Abelian Groups

Theorem 4.1. *Let X be a CW-complex, M an abelian group, then*

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{X}], M) \cong H^i_{\mathrm{sing}}(X, M)$$

Proof sketch. $X = \bigcup_i X_i$ with each X_i compact metrizable Hausdorff. As both sides are filtered colimits to derived limits, we can now reduce to the X compact Hausdorff case. □

More general statement:

Theorem 4.2. *Let X metrizable compact Hausdorff, M abelian group. Then*

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{X}], M) \cong H^i_{\mathrm{sheaf}}(X, M).$$

Here, we denote by $H^i_{\mathrm{sheaf}}(X, M)$ sheaf cohomology.

The right side of that equation means the following: Consider a topological space X, then $Sh(X, \mathrm{Ab})$ is the category of sheaves on $\mathrm{Op}(X)$ with values in abelian groups. Then the functor

$$\Gamma(X, -) : Sh(X, \mathrm{Ab}) \to \mathrm{Ab}.$$

has a right derived functor

$$H^i_{\mathrm{sheaf}}(X, -) : Sh(X, \mathrm{Ab}) \to \mathrm{Ab}.$$

In the theorem, we apply this functor to the constant sheaf M.

Known: If X is a CW complex,

$$H^i_{\mathrm{sheaf}}(X, M) \cong H^i_{\mathrm{sing}}(X, M).$$

But not in general: If X is a totally disconnected compact Hausdorff, then $\Gamma(X, -)$ is exact so

$$H^i_{\mathrm{sheaf}}(X, M) = \begin{cases} \text{locally constant maps } X \to M & i = 0 \\ 0 & i > 0 \end{cases}$$

However

$$H^i_{\mathrm{sing}}(X, M) = \begin{cases} \underline{\text{all}} \text{ maps } X \to M & i = 0 \\ 0 & i > 0 \end{cases}$$

Remark 4.3. We are **not** considering spaces up to homotopy equivalence.

Further Upgrade: Fix a topological space X. We have two sites:

$$\mathrm{CondSet}^{\mathrm{light}}/\underline{X} = (\mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})/\underline{X})^{\sim} \xrightarrow{\lambda} Op(X)^{\sim} = Sh(X)$$

where $-^{\sim}$ is the functor that passes to the topos of sheaves. For $U \subset X$

$$\lambda^* U = \underline{U} \text{ (together with morphism } \to \underline{X}).$$

Correspondingly we also have a pullback functor of derived categories

$$\lambda^* : D\left(Sh(X, \mathrm{Ab})\right) \to D\left(\mathrm{Ab}(\mathrm{CondSet}^{\mathrm{light}}/\underline{X})\right)$$

with $\mathrm{Ab}(\mathrm{CondSet}^{\mathrm{light}}/\underline{X})$ abelian sheaves.

Theorem 4.4. *Assume that X is a metrizable compact Hausdorff space. On D^+, λ^* is fully faithful. In particular $\forall\, F \in Sh(X, \mathrm{Ab})$*

$$H^i(\mathrm{CondSet}^{\mathrm{light}}/\underline{X}, \lambda^* F) \cong H^i_{\mathrm{sheaf}}(X, F).$$

If we apply this to F = constant sheaf on M, we get previous statement.

Proof sketch. We need: on $A \in D^+\left(Sh(X, \mathrm{Ab})\right)$:

$$A \to R\lambda_* \lambda^* A$$

is an isomorphism. This can be checked on stalks of points $x \in X$

$$A \xrightarrow{\sim ?} R\lambda_* \lambda^* A$$

Key Point: Base Change Property: Taking stalks at x commutes[27] with $R\lambda_*$. □

Here: Use general "cohomology commutes with filtered colimits" results for general "coherent topoi". "Coherent topoi" here means qcqs.

Key "Geometric" Input: $\underline{X}$ is qcqs.

Example 4.5. Here we are trying to calculate more explicitly what happens in 4.4: For a metrizable compact Hausdorff space X we want to calculate $\mathrm{Ext}^i(\mathbb{Z}[\underline{X}], \mathbb{Z})$. Therefore we try to find a projective, or at least acyclic resolution.
Step 1: Show that if $X = S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$ is totally disconnected.

$$\mathrm{Ext}^i(\mathbb{Z}[S], \mathbb{Z}) = \begin{cases} \mathrm{Cont}(S, \mathbb{Z}) & i = 0 \\ 0 & i > 0 \end{cases}$$

[27]For this fact, see second part of the proof of Theorem 3.2 in [CS19]. Here also the property that X is a metrizable compact Hausdorff space plays a role.

This comes down to the following: $\forall$ hypercovers

$$S_\bullet \quad \cdots S_2 \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} S_1 \rightrightarrows S_0 \to S$$

where $S_\bullet$ is a simplicial light profinite set. Concretely that means that we get covers:

$$\begin{array}{lll} & S_0 \twoheadrightarrow S \\ S_1 \twoheadrightarrow S_0 \times_S S_0 & \\ \cdots & \end{array}$$

As the hypercover is a simplical profinite set, one can termwise take the free light condensed abelian groups to obtain a simplicial object in condensed abelian groups. By the usual construction of the Moore complex we obtain a resolution[28]

$$\ldots \longrightarrow \mathbb{Z}(S_2) \longrightarrow \mathbb{Z}(S_1) \longrightarrow \mathbb{Z}(S_0) \longrightarrow \mathbb{Z}[S] \longrightarrow 0$$

which is always exact. Now its just left to show that the dual of this resolution, i.e.

$$0 \longrightarrow \mathrm{Cont}(S, \mathbb{Z}) \longrightarrow \mathrm{Cont}(S_0, \mathbb{Z}) \longrightarrow \mathrm{Cont}(S_1, \mathbb{Z}) \longrightarrow \ldots$$

is exact. We can either treat the object as sheaves on S and reduce it to stalks, or we can write the hypercover as cofiltered limit of hypercovers of finite sets by finite sets, to reduce to case of finite sets, where this is clear.

Step 2: Treat general metrizable compact Hausdorff spaces X. We want to resolve $\mathbb{Z}[\underline{X}]$ by $\mathbb{Z}[S]$'s, $S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$.

$$S_\bullet \ \ldots \ \underbrace{S_0 \times_{\underline{X}} S_0 \times_{\underline{X}} S_0}_{S_2} \substack{\rightarrow\\ \leftarrow\\ \rightarrow\\ \leftarrow\\ \rightarrow} \underbrace{S_0 \times_{\underline{X}} S_0}_{S_1} \rightrightarrows S_0 \twoheadrightarrow \underline{X}$$

The hypercover $S_\bullet \to X$ gives us the complex

$$\ldots \longrightarrow \mathbb{Z}(S_2) \longrightarrow \mathbb{Z}(S_1) \longrightarrow \mathbb{Z}(S_0) \longrightarrow \mathbb{Z}[\underline{X}] \longrightarrow 0$$

which is a resolution by $\mathrm{Ext}^i(-, \mathbb{Z})$-acyclics.
Now $\mathrm{Ext}^i(\mathbb{Z}[\underline{X}, \mathbb{Z}])$ is computed by

$$0 \longrightarrow \mathrm{Cont}(S_0, \mathbb{Z}) \longrightarrow \mathrm{Cont}(S_1, \mathbb{Z}) \longrightarrow \mathrm{Cont}(S_2, \mathbb{Z}) \longrightarrow \ldots$$

The argument for that unravels to treating all terms as global sections of sheaves on X and checking that it resolves the constant sheaf $\mathbb{Z}$ on X.

[28] This is a highly nontrivial result! We will not go further into detail but one might as well dive deeper into it via [AS21].

4.2 Locally Compact Abelian Groups

Let $\mathrm{LCA_m}$ be the category of metrizable locally compact abelian groups. For example it contains discrete abelian groups, $\mathbb{R}$, $\mathbb{C}/\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{A} = \hat{\mathbb{Z}} \otimes \mathbb{Q} \times \mathbb{R}$. (Small Remark: For $\mathbb{A}$, $\mathbb{Q}$ being discrete, the sequence $0 \to \mathbb{Q} \to \mathbb{A} \to \mathbb{A}/\mathbb{Q} \to 0$ is exact.)
Each object in $\mathrm{LCA_m}$ admits a filtration with 3 pieces: discrete, finite dimensional $\mathbb{R}$ vector space, and compact metrizable abelian group. Computing Yoneda-Ext^i's, $\mathrm{Ext}^i = 0$ for $i \geq 2$.

Theorem 4.6. *For $A, B \in \mathrm{LCA_m}$ the following holds:*

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\underline{A}, \underline{B}) = \begin{cases} \mathrm{Hom}_{\mathrm{LCA_m}}(A, B) & i = 0 \\ \mathrm{Ext}^i_{\mathrm{LCA_m}}(A, B) & i = 1 \\ 0 & i \geq 0 \end{cases}$$

Example 4.7.

$$\mathrm{Ext}^i(\underline{A}, \underline{\mathbb{R}/\mathbb{Z}}) = \begin{cases} A^\vee & i = 0 \\ 0 & i > 0 \end{cases}$$

Example 4.8.

$$\underline{\mathrm{Ext}}^i(\underline{\mathbb{R}}, \underline{\mathbb{Z}}) = 0$$

How does one actually compute these things? We need to find something close to a projective resolution of $\underline{A}$.
Key: Breen Deligne resolution[29]:

Theorem 4.9 (Breen-Deligne)**.** *There is a resolution of the form:*

$$\begin{array}{ccccccccccccc} & & & & & & [(a,b)] & \longmapsto & [a]+[b]-[a+b] & & & & \\ \dots & \longrightarrow & \mathbb{Z}[M^{n_i}] & \longrightarrow & \dots & \longrightarrow & \mathbb{Z}[M^2] & \longrightarrow & \mathbb{Z}[M] & \longrightarrow & M & \longrightarrow & 0 \\ & & & & & & & & [m] & \longmapsto & m & & \end{array}$$

This is functorial in abelian groups M.

Warning 4.10. No explicit construction of the higher differentials is known. Thus this resolution is not explicit. But all the differentials are given by universal formulas.

By functoriality, this also works for abelian sheaves on any site:

$$\dots \longrightarrow \mathbb{Z}[\underline{A}^{n_i}] \longrightarrow \dots \longrightarrow \mathbb{Z}[\underline{A}^2] \longrightarrow \mathbb{Z}[\underline{A}] \longrightarrow \underline{A} \longrightarrow 0$$

[29]This is proven in an unpublished letter from Deligne to Breen.

which reduces computation of $\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\underline{A},-)$ to $\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{A}^{n_i}],-)$ which is what we already did in 4.1.

Also need: For all X compact Hausdorff $\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}(\mathbb{Z}[\underline{X}],\mathbb{R}) = \begin{cases} \mathrm{Cont}(X,\mathbb{R}) & i=0 \\ 0 & i>0 \end{cases}$

This also works with $\mathbb{R}$ replaced by any Banach space $\underline{V}$. But it requires local convexity of V (This uses partitions of unity arguments).

Example 4.11. For $\mathrm{Ext}^i(\underline{\mathbb{R}},\underline{\mathbb{Z}})$, we get:

$$\dots \longrightarrow \mathbb{Z}[\underline{\mathbb{R}}^{n_i}] \longrightarrow \dots \longrightarrow \mathbb{Z}[\underline{\mathbb{R}}^2] \longrightarrow \mathbb{Z}[\underline{\mathbb{R}}] \longrightarrow \underline{\mathbb{R}} \longrightarrow 0$$

We know $\mathrm{Ext}^i(\mathbb{Z}[\underline{R}^{n_i}],\mathbb{Z}) = H^i_{\mathrm{sing}}(\mathbb{R}^{n_i},\mathbb{Z}) = \begin{cases} \mathbb{Z} & i=0 \\ 0 & i>0 \end{cases}$

Variant (Maclane's Q-construction, Commelin) is an explicit complex

$$Q(M):\quad \dots \to \mathbb{Z}[M^8] \longrightarrow \mathbb{Z}[M^4] \longrightarrow \mathbb{Z}[M^2] \longrightarrow \mathbb{Z}[M] \longrightarrow 0$$

$$[a,b] \longmapsto [a]+[b]-[a+b]$$

$$[a,b,c,d] \mapsto \begin{array}{l} [(a,b)]+[(c,d)] \\ -[(a+c),(b+d)] \\ -[a,c]-[b,d] \\ +[(a+b),(c+d)] \end{array}$$

Theorem 4.12.

$$Q(M) \simeq Q(\mathbb{Z}) \otimes^L_{\mathbb{Z}} M \Rightarrow (\mathrm{Ext}^i(\underline{A},\underline{B}) = 0\,\forall i \geq 0 \iff \mathrm{Ext}^i(Q(A),B) = 0\,\forall i \geq 0).$$

Using some stable homotopy theory: $Q(\mathbb{Z}) \simeq \bigoplus_{i\geq 0} (\mathbb{Z} \otimes_{\mathbb{S}} \mathbb{Z})^{\oplus 2i}[i]$.

Corollary 4.13. *For all discrete abelian groups M, the following hold:*

$$\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}\left(\prod_{\mathbb{N}} \underline{\mathbb{Z}}, M\right) = \begin{cases} \bigoplus_{\mathbb{N}} M & i=0 \\ 0 & i>0 \end{cases}$$

The object $\prod_{\mathbb{N}} \underline{\mathbb{Z}}$ will be a compact pojective generator of Solid $\subset$ $\mathrm{CondAb}^{\mathrm{light}}$ *where* Solid *embeds as a full subcategory.*

Warning 4.14. $\prod_{\mathbb{N}} \mathbb{Z} = \bigcup_{f:\mathbb{N}\to\mathbb{N}} \prod_{n\in\mathbb{N}} [-f(n),f(n)]$ which becomes a huge limit that is hard to control.

Proof.

$$0 \longrightarrow \prod_{\mathbb{N}} \underline{\mathbb{Z}} \longrightarrow \prod_{\mathbb{N}} \underline{\mathbb{R}} \longrightarrow \prod_{\mathbb{N}} \underline{\mathbb{R}/\mathbb{Z}} \longrightarrow 0$$

So

$$\mathrm{Ext}^i\left(\prod_{\mathbb{N}} \underline{\mathbb{R}/\mathbb{Z}}, M\right) = \begin{cases} 0 & i \neq 1 \\ \bigoplus_{\mathbb{N}} M & i = 1 \end{cases}$$

We need

$$\mathrm{Ext}^i\left(\prod_{\mathbb{N}} \underline{\mathbb{R}}, M\right) = 0 \;\forall\, i \geq 0.$$

It holds that

$$R\mathrm{Hom}_{\underline{\mathbb{R}}}\left(\prod_{\mathbb{N}} \underline{\mathbb{R}}, \underline{R\mathrm{Hom}}(\underline{\mathbb{R}}, M)\right) = 0$$

□

The following theorem is of pure set theoretic nature:

Theorem 4.15. *For all sequential limits* $\ldots \twoheadrightarrow M_1 \twoheadrightarrow M_0$ *of countable discrete abelian groups, and all discrete abelian groups* N*, it is equivalent that*

$$\mathrm{Ext}^i(\varprojlim_n M_n, N) = \varinjlim_n \mathrm{Ext}^i(M_n, N) \;\; (= 0 \textit{ for } i \geq 2) \tag{1}$$

$$\iff \mathrm{Ext}^i\left(\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} \mathbb{Z}, \bigoplus_{I} \mathbb{Z}\right) = 0 \textit{ for } i > 0.$$

It is easy to see that under the continuum hypothesis (1) fails as $\mathrm{Ext}^1 \neq 0$.

Remark 4.16. It holds, that $\mathrm{Ext}^i\left(\prod_{\mathbb{N}} \bigoplus_{\mathbb{N}} \mathbb{Z}, \bigoplus_I \mathbb{Z}\right) = \underset{f:\mathbb{N}\to\mathbb{N}}{\varinjlim} \prod_{n\in\mathbb{N}} \prod_{m \leq f(n)} \mathbb{Z}$.

Theorem 4.17 (Bergfalk, Lambie-Hanson, Hrušák, Bannister)**.**

1. *From (1) it follows that* $2^{\aleph_0} > \aleph_\omega$
2. *It is consistent that (1) holds and* $2^{\aleph_0} = \aleph_{\omega+1}$*. In fact it holds in forcing extension adjoining* J_ω *many Cohen reals.*

While we will not use (1) in the course, it is nice to know it can be enforced as it sometimes simplifies computations.

5 Lecture 5 (5/24)

5.1 Solid Abelian Groups

The goal is to isolate a class of "complete" objects in $\mathrm{CondAb}^{\mathrm{light}}$. For this lecture, whenever we write Hom or Ext^i we mean $\mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}$ or $\mathrm{Ext}^i_{\mathrm{CondAb}^{\mathrm{light}}}$

$$\mathbb{Z}[\![T]\!] \otimes_{\mathrm{Ab}} \mathbb{Z}[\![T]\!] \cong \left(\underline{\mathbb{Z}}[\![T]\!] \otimes_{\mathrm{CondAb}^{\mathrm{light}}} \underline{\mathbb{Z}}[\![T]\!]\right)(*) \neq \mathbb{Z}[\![T,U]\!]$$

Idea: Form completed tensor products instead.
Want: "completeness" defines an abelian category:

$$0 \longrightarrow \underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}_p \longrightarrow \underline{\mathbb{Z}}_p/\underline{\mathbb{Z}} \longrightarrow 0$$

Turns out it is difficult to find a notion for which $\underline{\mathbb{R}}$ is complete, but there is a theory that works well in the non-archimedean context.
Idea: $M \in \mathrm{CondAb}^{\mathrm{light}}$ should be "non-archimedean complete" if any null sequence in M is summable.
Formalization: Consider $P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty] \in \mathrm{CondAb}^{\mathrm{light}}$ the "free condensed abelian group on nullsequences". Recall P is internally projective.

$$\begin{aligned}\underline{\mathrm{Hom}}(P,-):\ \mathrm{CondAb}^{\mathrm{light}} &\longrightarrow \mathrm{CondAb}^{\mathrm{light}}\\ M &\longmapsto (S \mapsto \mathrm{Hom}(P \otimes \mathbb{Z}[S], M))\end{aligned}$$

is exact and preserves all limits and colimits. Consider

$$\begin{aligned}f = id - \mathrm{shift}:\ P &\longrightarrow P\\ [n] &\longmapsto [n] - [n+1]\end{aligned}$$

Definition 5.1. $M \in \mathrm{CondAb}^{\mathrm{light}}$ is **solid** if

$$\begin{aligned}f^*: \underline{\mathrm{Hom}}(P,M) &\xrightarrow{\sim} \underline{\mathrm{Hom}}(P,M)\\ (m_0, m_1, \dots) &\longmapsto (m_0 - m_1, m_1 - m_2, \dots)\\ \text{"}\left(\sum_{i=0}^{\infty} m_i, \sum_{i=1}^{\infty} m_i, \dots\right)\text{"} &\longleftarrow (m_0, m_1, \dots)\end{aligned}$$

We see from that definition, that "null sequences" are summable, which is a "non-archimedean property".

Proposition 5.2. Solid $\subset \mathrm{CondAb}^{\mathrm{light}}$ *is an abelian subcategory, which is stable under kernels, cokernels, extensions, all (co)limits,* $\underline{\mathrm{Hom}}$, $\underline{\mathrm{Ext}}^i$ *and contains* $\underline{\mathbb{Z}}$*. It is also not equal to* $\mathrm{CondAb}^{\mathrm{light}}$*, e.g.* $\underline{\mathbb{R}} \notin$ Solid.

Remark 5.3. This Proposition says something like "Solid has an analytic ring structure on $\underline{\mathbb{Z}}$".

Proof. All, but $\underline{\mathrm{Hom}}$, $\underline{\mathrm{Ext}}^i$ and $\underline{\mathbb{Z}}$ for free because $\underline{\mathrm{Hom}}(P,-)$ is exact, i.e. because P is internally projective and compact.

Stability for internal $\underline{\mathbf{Hom}}$:
M solid, $N \in \mathrm{CondAb}^{\mathrm{light}}$, we want $\underline{\mathrm{Hom}}(N,M)$ solid, i.e. f^* is an isomorphism:

$$\underset{\text{iso?}}{f^* \circlearrowright} \, \underline{\mathrm{Hom}}\left(P, \underline{\mathrm{Hom}}(N,M)\right) \cong \underline{\mathrm{Hom}}(P \otimes N, M) \cong \underline{\mathrm{Hom}}\left(N, \underline{\mathrm{Hom}}(P,M)\right).$$

As f^* is now only acting on $\mathrm{Hom}(P,M)$ we are done, as M is solid.

Stability for internal $\underline{\mathbf{Ext}}^i$: As $\underline{\mathrm{Hom}}(P,-)$ is exact:

$$\underline{\mathrm{Hom}}\left(P, \underline{\mathrm{Ext}}^i(N,M)\right) = \underline{\mathrm{Ext}}^i(P \otimes N, M) \cong \underline{\mathrm{Ext}}^i(N, \underline{\mathrm{Hom}}(P,M)).$$

Now we are done with the same argument as above.

$\underline{\mathbb{Z}}$ is solid:

$$\underline{\mathrm{Hom}}(P, \underline{\mathbb{Z}}) = \bigoplus\nolimits_{\mathbb{N}} \mathbb{Z} \circlearrowleft id - \text{shift is iso}$$

$\square$

Corollary 5.4. *There is a left adjoint to* $\mathrm{Solid} \hookrightarrow \mathrm{CondAb}^{\mathrm{light}}$,

$$\begin{aligned} \mathrm{CondAb}^{\mathrm{light}} &\longrightarrow \mathrm{Solid} \\ M &\longmapsto M^{\square} \end{aligned}$$

called solidification. $\forall\, N \in \mathrm{Solid} : \mathrm{Hom}_{\mathrm{CondAb}^{\mathrm{light}}}(M,N) = \mathrm{Hom}_{\mathrm{Solid}}\left(M^{\square}, N\right)$.
Moreover Solid acquires a unique colimit preserving symmetric monoidal product $\otimes^{\square}$ *making* $M \longmapsto M^{\square}$ *a symmetric monoidal functor.*

Sketch of Proof.

Existence of $M \longmapsto M^{\square}$ is by an adjoint functor theorem.

Symmetric monoidal $\otimes^{\square}$: Define $M \otimes^{\square} N := (M \otimes N)^{\square}$. We want:

$$\forall\, N, M \in \mathrm{CondAb}^{\mathrm{light}} : (M \otimes N)^{\square} \xrightarrow{\sim} \left(M^{\square} \otimes N^{\square}\right)^{\square}$$

<u>Check:</u> $\forall S \in \mathrm{Solid}$, $\mathrm{Hom}(-,S)$ agree. We proceed:

$$\begin{array}{ccc}
\mathrm{Hom}(M \otimes N, S) & \cong & \mathrm{Hom}\left(M^{\square} \otimes N^{\square}, S\right) \\
\cong & & \cong \\
\mathrm{Hom}\left(N, \underline{\mathrm{Hom}}(M,S)\right) & & \mathrm{Hom}\left(M^{\square}, \underline{\mathrm{Hom}}(N^{\square},S)\right) \\
\cong & & \cong \\
\mathrm{Hom}\left(N^{\square}, \underline{\mathrm{Hom}}(M,S)\right) & \cong & \underline{\mathrm{Hom}}\left(M, \underline{\mathrm{Hom}}(N^{\square},S)\right)
\end{array}$$

where $\underline{\mathrm{Hom}}(M, S)$ and $\underline{\mathrm{Hom}}(N^{\square}, S)$ are both solid. Hence all these equations hold. $\square$

Lemma 5.5. $\underline{\mathbb{R}}^{\square} = 0$.

Proof. $\underline{\mathbb{R}}^{\square}$ is a ring, so it is enough to show $1 = 0 \in \underline{\mathbb{R}}^{\square}$. Consider the null sequence

$$1, \frac{1}{2}, \frac{1}{2}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{8}, \cdots$$

which is a map $g : P \to \underline{\mathbb{R}}$. Consider the diagram:

$$\begin{tikzcd} & P \arrow[r, "g"] & \underline{\mathbb{R}} \arrow[d] \\ \underline{\mathbb{Z}} \arrow[r, "{[0]}"] & P \arrow[u, "f"] \arrow[r, dotted, "\exists!"'] & \underline{\mathbb{R}}^{\square} \end{tikzcd}$$

where $1 \in \mathbb{Z}$ has the image $1 =: x \in \underline{\mathbb{R}}^{\square}$, which corresponds on an intuitive level to "$1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \dots$", as we map $[0]$ to some $m_0 \in \underline{\mathbb{R}}^{\square}$ and now by the backwards map in 5.1, it should have this form going over $f \circ g$.
Claim: $x = 1 + x$ (Hence it follows that $1 = 0$.)

$$\begin{aligned} x &= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 1 + \left(\underbrace{\tfrac{1}{2} + \tfrac{1}{2}}_{=1} + \underbrace{\tfrac{1}{4} + \tfrac{1}{4}}_{=\frac{1}{2}} + \underbrace{\tfrac{1}{4} + \tfrac{1}{4}}_{=\frac{1}{2}} + \cdots \right) \\ &= 1 + \left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \cdots \right) = 1 + x \end{aligned}$$

$\square$

Corollary 5.6. *If $M \in \mathrm{CondAb}^{\mathrm{light}}$ admits an $\underline{\mathbb{R}}$-module structure then $M^{\square} = 0$ (even $\mathrm{Ext}^i(M, \mathrm{Solid}) = 0$).*

Proof.

$$\cdots \longrightarrow M \otimes \underline{\mathbb{R}} \otimes \underline{\mathbb{R}} \longrightarrow M \otimes \underline{\mathbb{R}} \longrightarrow M \longrightarrow 0$$

is exact. As solidifying is right exact, we get:

$$0 = M^{\square} \otimes^{\square} \underline{\mathbb{R}}^{\square} = (M \otimes \underline{\mathbb{R}})^{\square} \twoheadrightarrow M^{\square}$$

since $\underline{\mathbb{R}}^{\square} = 0$.
Observation: $\underline{\mathrm{Ext}}^i(M, \mathrm{Solid})$ will be $\underline{\mathbb{R}}^{\square}$-modules as it clearly has $\underline{\mathbb{R}}$–module structure and internal $\underline{\mathrm{Ext}}^i$ into something solid is solid itself. $\square$

Goal: Compute $P^{\square}$

Lemma 5.7. *Let $\prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} := \bigcup_{n \in \mathbb{N}} \prod_{\mathbb{N}} (\mathbb{Z} \cap [-n, n])$. There is the map*

$$\begin{aligned} P &\longrightarrow \prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z} \\ [n] &\longmapsto (0, \dots, 0, 1, 0, \dots) \end{aligned}$$

where the $1-$*entry is the* $n-$*th coordinate. Then*

$$P^{\square} \xrightarrow{\sim} \left(\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z} \right)^{\square}.$$

Even better: $\forall\, M$ *solid,* $\mathrm{Ext}^i (P, M) \xleftarrow{\sim} \mathrm{Ext}^i \left(\prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z}, M \right)$.

Proof. Consider the diagram

$$\begin{tikzcd} P \otimes \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} \arrow[r, two heads, "g"] & \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} \\ P \otimes \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} \arrow[u, "f \otimes id"] \arrow[r, dotted] & P \arrow[u, hook] \end{tikzcd}$$

Here, g corresponds to a null sequence of maps

$$\prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} \circlearrowleft \ ,$$

which are "projections to coordinates $\geq n$”. (This is again essentially 5.1.)
The composition $P \otimes \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} \longrightarrow \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z}$ corresponds to the null sequence of maps

$$\prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z} \circlearrowleft \ ,$$

which are "projections to the $n-$th coordinate". All of these factor over P, hence we get the dotted arrow.
Now solidify:

$$\begin{tikzcd} \left(P \otimes \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z}\right)^{\square} \arrow[r, two heads] & \left(\prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z}\right)^{\square} \\ \left(P \otimes \prod^{\mathrm{bdd}}_{\mathbb{N}} \mathbb{Z}\right)^{\square} \arrow[u, "(f \otimes id)^{\square}"] \arrow[r] & P^{\square} \arrow[u, hook, "h^{\square}"'] \end{tikzcd}$$

Now $(f \otimes id)^{\square}$ is an isomorphism, since f solidifies to an isomorphism by definition of solid and solidification is symmetric monoidal, hence as both f and id solidfy to isomorphisms, the whole map is an isomorphism. This means that $h^{\square}$ is an split surjection! Also, resulting idempotent of $P^{\square}$ is identity. Now we only need to show by a diagram chase, that going one "round" around the diagram is the identity on $P^{\square}$, which we will not do here. We have essentially the same argument for $\mathrm{Ext}^i(-, \mathrm{Solid})$. $\square$

Lemma 5.8. $\left(\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z}\right)^{\square} \xrightarrow{\sim} \left(\prod_{\mathbb{N}} \mathbb{Z}\right)^{\square} = \prod_{\mathbb{N}} \mathbb{Z}$ *and*

$$\mathrm{Ext}^i\left(\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z},\ \mathit{solid}\right) \xleftarrow{\sim} \mathrm{Ext}^i\left(\prod_{\mathbb{N}} \mathbb{Z},\ \mathit{solid}\right)$$

Proof. Consider the sequence $0 \longrightarrow \prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z} \longrightarrow \prod_{\mathbb{N}} \mathbb{Z} \longrightarrow \prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z} \longrightarrow 0$.
We have to see: $\forall i \geq 0$, M solid:

$$\mathrm{Ext}^i\left(\prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z}, M\right) = 0$$

<u>Claim:</u> $\prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z}$ is a $\underline{\mathbb{R}}$-module!
A different way to write this is $\prod_{\mathbb{N}} \mathbb{Z}/\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z} = \prod_{\mathbb{N}} \underline{\mathbb{R}}/\prod_{\mathbb{N}}^{\mathrm{bdd}} \underline{\mathbb{R}}$, as the ratio between $\prod_{\mathbb{N}} \mathbb{Z}$ and $\prod_{\mathbb{N}} \underline{\mathbb{R}}$ is $\prod_{\mathbb{N}} \underline{\mathbb{R}}/\mathbb{Z}$ and the ratio between $\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z}$ and $\prod_{\mathbb{N}}^{\mathrm{bdd}} \underline{\mathbb{R}}$ is the same, as $\prod_{\mathbb{N}}^{\mathrm{bdd}} \underline{\mathbb{R}}$ surjects onto $\prod_{\mathbb{N}} \underline{\mathbb{R}}/\mathbb{Z}$ with kernel $\prod_{\mathbb{N}}^{\mathrm{bdd}} \mathbb{Z}$. Now we are done by 5.6. ◻

Corollary 5.9. $P^{\square} \xrightarrow{\sim} \prod_{\mathbb{N}} \mathbb{Z}$ *and for all solid* M, $\mathrm{Ext}^i(P, M) \xleftarrow{\sim} \mathrm{Ext}^i\left(\prod_{\mathbb{N}} \mathbb{Z}, M\right)$ *for* $i > 0$

$$\mathrm{Ext}^i\left(\prod_{\mathbb{N}} \mathbb{Z}, M\right) = \begin{cases} \bigoplus_{\mathbb{N}} M & i = 0 \\ 0 & i > 0 \end{cases}$$

$\Rightarrow \prod_{\mathbb{N}} \mathbb{Z}$ *is compact and projective.*

Proposition 5.10. *Let S be any infinite light profinite set. Then $\exists$ map*

$$P \longrightarrow \mathbb{Z}[S]$$

inducing an isomorphism

$$P^{\square} \xrightarrow{\sim} \mathbb{Z}[S]^{\square}$$

and an also an isomorphism

$$\mathrm{Ext}^i(P, \mathrm{Solid}) \xleftarrow{\sim} \mathrm{Ext}^i(\mathbb{Z}[S], \mathrm{Solid})$$

$\Longrightarrow \mathbb{Z}[S]^{\square} \cong \prod_{\mathbb{N}} \mathbb{Z}$
Canonically: $\mathbb{Z}[S]^{\square} \xrightarrow{\sim} \varprojlim \mathbb{Z}[S_n]$, *where* $S = \varprojlim S_n$.

Proof Sketch. $S \twoheadrightarrow \ldots \twoheadrightarrow \ldots \twoheadrightarrow S_2 \twoheadrightarrow S_1 \twoheadrightarrow S_0$
Inductively choose:

- section $i_0 : S_0 \hookrightarrow S$.
- on all elements of S_1 that are not in image of $i_0(S_0)$ choose a section i_1^* on $S_1 \setminus S_0 \longrightarrow S$.
  Now we get a map $i_1 := i_0 \sqcup i_1^* : S_1 \longrightarrow S$.

Enumerate $\mathbb{N} = S_0 \sqcup (S_1 \setminus S_2) \sqcup (S_2 \setminus S_1) \sqcup \ldots$. We get a map

$$g : P \longrightarrow \mathbb{Z}[S]$$

which is on S_0 given by i_0 and on $S_n \setminus S_{n-1}$ given by the difference of maps induced by i_n and $S_n \setminus S_{n-1} \longrightarrow S_{n-1} \xrightarrow{i_{n-1}} S$.

$$\begin{tikzcd}
P \otimes \mathbb{Z}[S] \arrow[r, "h"] & \mathbb{Z}[S] \arrow[l, bend right, dotted, "k"'] \\
P \otimes \mathbb{Z}[S] \arrow[u] \arrow[ur] \arrow[r, dotted] & P \arrow[u]
\end{tikzcd}$$

Where h is the null sequence of maps

$$\mathbb{Z}[S] \circlearrowleft$$

$id, id - i_0\pi_0, id - i_1\pi_1, \ldots$ where π_n is the map $S \to S_n$. We get the section k via

$$\mathbb{Z} \otimes \mathbb{Z}[S] \longrightarrow P \otimes \mathbb{Z}[S] \longrightarrow \mathbb{Z}[S]$$

which is given by $\mathbb{Z} \xrightarrow{[0]} P$. Via construction of h this is the id. Then the diagonal map is given by the null sequence $i_0\pi_0,\ i_1\pi_1 - i_0\pi_0,\ \ldots,\ i_n\pi_n - i_{n-1}\pi_{n-1},\ \ldots$. This is again by 5.1. Now one can check that the diagonal map factors over P as we have constructed our i_n "nicely", giving us the dotted map. Then solidify and use the same arguments as before. $\square$

Remark 5.11. One could also use the strategy from [CS19]. A short summary:

$$\mathbb{Z}[S]^{\blacksquare} = \varprojlim_i \mathrm{Hom}\left(C(S_i, \mathbb{Z}), \mathbb{Z}\right) =$$
$$\mathrm{Hom}(C(S, \mathbb{Z}), \mathbb{Z}) = \mathrm{Hom}\left(\bigoplus \mathbb{Z}, \mathbb{Z}\right) = \prod \mathbb{Z}$$

Corollary 5.12.

$$\begin{tikzcd}
\prod_{\mathbb{N}} \mathbb{Z} \otimes^{\blacksquare} \prod_{\mathbb{N}} \mathbb{Z} \arrow[r, phantom, "\cong"] & \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z} \\
(P \otimes P)^{\blacksquare} \arrow[u, "\cong"] \arrow[r, phantom, "\cong"] & P^{\blacksquare} \arrow[u, "\cong"]
\end{tikzcd}$$

And $\mathbb{Z}[\![T]\!] \otimes^{\blacksquare} \mathbb{Z}[\![T]\!] \cong \mathbb{Z}[\![T, U]\!]$.

*Proof**. [30] For infinite profinite S, T, we have

$$\prod_{\mathbb{N}} \mathbb{Z} \otimes^{\blacksquare} \prod_{\mathbb{N}} \mathbb{Z} = \mathbb{Z}[S]^{\blacksquare} \otimes^{\blacksquare} \mathbb{Z}[T]^{\blacksquare} = \left(\varprojlim \mathbb{Z}[S_i] \otimes \varprojlim \mathbb{Z}[T_j]\right)^{\blacksquare} = \mathbb{Z}[S \times T]^{\blacksquare} = \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}.$$

The intuition for that would be, that for $(P \otimes P)^{\blacksquare} \cong P^{\blacksquare}$ we have two convergent series to 0 on $\mathbb{N} \times \mathbb{N}$ and this is the same as one convergent series, where one takes alternating entries of the two convergent series. $\square$

[30] Thanks to Julius for this proof!

Theorem 5.13. Solid $\subset \mathrm{CondAb}^{\mathrm{light}}$ *is abelian, stable under (co)limits and has a single compact projective generator* $\prod_{\mathbb{N}} \mathbb{Z}$ *(and* $\prod_{\mathbb{N}} \mathbb{Z} \otimes^{\square} \prod_{\mathbb{N}} \mathbb{Z} \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}$*).*
Also $M \in \mathrm{CondAb}^{\mathrm{light}}$ *is solid iff* $\forall S = \varprojlim S_n \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$*, all* $g : S \to M, \exists!$ *extension of* g *to* $S \to \mathbb{Z}[S]^{\square} = \varprojlim \mathbb{Z}[S_n] \dashrightarrow M$.

Proof. Only " $\Leftarrow$ " of last assertion is left to show.
Consider the diagram, where the first row is a presentation[31] of g in $\mathrm{CondAb}^{\mathrm{light}}$:

$$\begin{tikzcd}
\bigoplus_j \mathbb{Z}[S_j] \arrow[r] \arrow[d] & \bigoplus_i \mathbb{Z}[S_i] \arrow[r] \arrow[d] & M \arrow[r] & 0 \\
\bigoplus_j \mathbb{Z}[S_j]^{\square} \arrow[r] \arrow[urr, bend right=60, "0"'] & \bigoplus_i [\mathbb{Z}[S_i]]^{\square} \arrow[ur, dotted, "\exists!"'] & &
\end{tikzcd}$$

Hence M is a cokernel of a map of solids, hence solid itself. $\square$

Philosophical comment:

$$\mathbb{Z}[S]^{\square} = \varprojlim \mathbb{Z}[S_n] = \varprojlim \underline{\mathrm{Hom}}(\mathrm{Cont}(S_n, \mathbb{Z}), \mathbb{Z}) = \underline{\mathrm{Hom}}\,(\mathrm{Cont}(S, \mathbb{Z}), \mathbb{Z})\,,$$

where one can understand the right side as "$\mathbb{Z}$−valued measures on S". If we consider the diagram

$$\begin{tikzcd}
 & \mu \in \mathbb{Z}[S]^{\square} \arrow[dr, dotted, "\exists!"] & \\
g : S \arrow[ur] \arrow[rr] & & M
\end{tikzcd}$$

one can interpret the unique map as "$\int g d\mu$" $\in M$.

Remark 5.14. Note that in 5.13 we used the usual Hom, whereas in the definition of solid we had to use the internal $\underline{\mathrm{Hom}}$.

[31] Here we use the adjunction of the forgetful functor and the free condensed abelian group functor to see, that it suffices to understand maps in this setting.

6 Lecture 6 (6/24)

Recall:

$P := \mathbb{Z}[\mathbb{N} \cup \{\infty\}]/\mathbb{Z}[\infty]$.

$$\begin{aligned} id - \text{shift}: \ P &\longrightarrow P \\ [n] &\longmapsto [n+1] - [n] \end{aligned}$$

Definition 6.1. $M \in \text{CondAb}^{\text{light}}$ is **solid** if

$$\begin{aligned} f^*: \underline{\text{Hom}}(P, M) &\xrightarrow{\sim} \underline{\text{Hom}}(P, M) \\ (m_0, m_1, \dots) &\longmapsto (m_0 - m_1, m_1 - m_2, \dots) \end{aligned}$$

Theorem 6.2. Solid $\subset \text{CondAb}^{\text{light}}$ *is abelian, stable under (co)limits and acquires* $\otimes^{\square}$ *making the left adjoint*

$$\begin{aligned} \text{CondAb}^{\text{light}} &\longrightarrow \text{Solid} \\ M &\longmapsto M^{\square} \end{aligned}$$

symmetric monoidal. It is generated by compact projective $\prod_{\mathbb{N}} \mathbb{Z}$*, where*

$$\prod_{\mathbb{N}} \mathbb{Z} \otimes^{\square} \prod_{\mathbb{N}} \mathbb{Z} \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}.$$

6.1 Derived Categories

Definition 6.3. $A \in D\left(\text{CondAb}^{\text{light}}\right)$ is **solid** if

$$\begin{aligned} &f^*: \underline{R\text{Hom}}(P, A) \xrightarrow{\sim} \underline{R\text{Hom}}(P, A) \\ \iff &\forall i \in \mathbb{Z}\, f^*: \underline{\text{Hom}}\left(P, H^i(A)\right) \xrightarrow{\sim} \underline{\text{Hom}}\left(P, H^i(A)\right) \\ \iff &\text{all } H^i(A) \text{ are solid.} \end{aligned}$$

Corollary 6.4. *Class of solid* A *is a triangulated subcategory, stable under infinite* $\oplus, \prod$ *and* $\underline{R\text{Hom}}$.

Proposition 6.5. $D(\text{Solid}) \longrightarrow D(\text{CondAb}^{\text{light}})$ *which is induced by* Solid $\hookrightarrow \text{CondAb}^{\text{light}}$ *is fully faithful, its essential image consists of the solid* A*'s. The inclusion will have a left adjoint "Derived Solidification"*[32] $A \longmapsto A^{L\square}$ *and a unique symmetric monoidal* $\otimes^{L\square}$ *on* $D(\text{Solid})$ *making* $A \longmapsto A^{L\square}$ *symmteric monoidal. We have*

$$\mathbb{Z}[S]^{L\square} \xrightarrow{\sim} \varprojlim \mathbb{Z}[S_n] = \mathbb{Z}[S]^{\square}$$

[32] Lurie has a strong adjoint functor theorem for presentable infinity categories, requiring all limits and colimits and generation by a set. Then continuity implies the existence of a left adjoint.

$P^{L\square} \xrightarrow{\sim} \prod_{\mathbb{N}} \mathbb{Z}$

$\prod_{\mathbb{N}} \mathbb{Z} \otimes^{L\square} \prod_{\mathbb{N}} \mathbb{Z} \xrightarrow{\sim} \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}$

Proposition 6.6. *Let X be a CW-Complex, then*

$$H_i^{\text{sing}}(X, \mathbb{Z}) = H_i(\mathbb{Z}[\underline{X}]^{L\square}).$$

Even better: $C_\bullet^{\text{sing}}(X, \mathbb{Z}) \simeq \mathbb{Z}[\underline{X}]^{L\square}$.

Example 6.7. $\mathbb{Z}[\underline{[0,1]}]^{L\square} \cong \mathbb{Z}$ and $\mathbb{Z}[\underline{S}^1]^{L\square} \cong \mathbb{Z} \oplus \mathbb{Z}[1]$.

Proof. Formal reduction to the case of finite CW-complexes. So assume X compact and take a covering $S \twoheadrightarrow X$. We get a resolution of condensed abelian groups

$$\ldots \longrightarrow \mathbb{Z}[\underline{S \times_X S}] \longrightarrow \mathbb{Z}[\underline{S}] \longrightarrow \mathbb{Z}[\underline{X}] \longrightarrow 0$$

So $\mathbb{Z}[\underline{X}]^{L\square}$ is computed by

$$\ldots \longrightarrow \mathbb{Z}[\underline{S \times_X S}]^\square \longrightarrow \mathbb{Z}[\underline{S}]^\square = \underline{\mathrm{Hom}}\,(\mathrm{Cont}(S, \mathbb{Z}), \mathbb{Z}) \longrightarrow 0$$

Now as

$$\mathbb{Z}[S]^{L\square} \simeq \underline{R\mathrm{Hom}}\left(\underline{R\mathrm{Hom}}(\mathbb{Z}[S], \mathbb{Z}), \mathbb{Z}\right)$$

it follows that

$$\mathbb{Z}[X]^{L\square} \xrightarrow{\sim} \underbrace{\underline{R\mathrm{Hom}}(\underbrace{\underline{R\mathrm{Hom}}(\mathbb{Z}[\underline{X}], \mathbb{Z})}_{C^\bullet_{\text{sing}}(X,\mathbb{Z})}, \mathbb{Z})}_{C_\bullet^{\text{sing}}(X,\mathbb{Z})}$$

as all $H^i(X, \mathbb{Z})$ are finitely generated. □

Remark 6.8. To some extent passing from light abelian groups to solid abelian groups is like passing to Homotopy types. At least it identifies two homotopy equivalent CW complexes but on totally disconnected spaces it's much finer information.

6.2 Understanding structure of Solid:

Definition 6.9. We call objects in Solid

1. finitely generated, if they are quotients of $\prod_{\mathbb{N}} \mathbb{Z}$.
2. finitely presented, if they are cokernels of a map $\prod_{\mathbb{N}} \mathbb{Z} \to \prod_{\mathbb{N}} \mathbb{Z}$. We denote the category of finitely presented objects as $\mathrm{Solid}^{f.p.}$.

Theorem 6.10. *The finitely presented objects of Solid from an abelian category, i.e. stable under (co)kernels and extensions and we can write Solid as:*

$$\mathrm{Solid} \cong \mathrm{Ind}\left(\mathrm{Solid}^{\textit{fin. pres.}}\right)$$

Any finitely presented $M \in \mathrm{Solid}$ has a resolution

$$0 \longrightarrow \prod_{\mathbb{N}} \mathbb{Z} \longrightarrow \prod_{\mathbb{N}} \mathbb{Z} \longrightarrow M \longrightarrow 0.$$

Key Lemma 6.11. *Any finitely generated submodule M of $\prod_{\mathbb{N}} \mathbb{Z}$ is isomorphic to a product of copies of $\mathbb{Z}$.*

Proof. We know that there is some surjective $a : \prod_{\mathbb{N}} \mathbb{Z} \longrightarrow M$ which embeds back into $\prod_{\mathbb{N}} \mathbb{Z}$

$$\begin{tikzcd} \prod_{\mathbb{N}} \mathbb{Z} \arrow[r, two heads, "a"] \arrow[rr, bend right, "g"'] & M \arrow[r, hook, "b"] & \prod_{\mathbb{N}} \mathbb{Z} \end{tikzcd}$$

g is dual to a map

$$\bigoplus_{\mathbb{N}} \mathbb{Z} \xleftarrow{h} \bigoplus_{\mathbb{N}} \mathbb{Z}$$

as $\mathrm{Hom}\left(\prod_{\mathbb{N}} \mathbb{Z}, \mathbb{Z}\right) = \bigoplus_{\mathbb{N}} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) = \bigoplus_{\mathbb{N}} \mathbb{Z}$.

$\Rightarrow$ $\mathrm{im}(h)$ is free as any subgroup of an abelian free group is free.

$\Rightarrow$ $\ker(h)$ splits as a summand, as the short exact sequence with $\mathrm{im}(h)$ splits.

Hence we can remove $\ker(h)$ from $\prod_{\mathbb{N}} \mathbb{Z}$ and wlog. h is injective.

$$0 \longrightarrow \bigoplus_{\mathbb{N}} \mathbb{Z} \xhookrightarrow{h} \bigoplus_{\mathbb{N}} \mathbb{Z} \longrightarrow Q \longrightarrow 0. \tag{2}$$

So there is a certain quotient of Q:

$$Q \twoheadrightarrow \overline{Q} \hookrightarrow \prod_{\mathrm{Hom}(Q,\mathbb{Z})} \mathbb{Z}$$

for the map $Q \longrightarrow \prod_{\mathrm{Hom}(Q,\mathbb{Z})} \mathbb{Z}$ that sends $q \in Q$ to the tupel that evaluates q in each component.

Fact: Let N be a countable abelian group that embeds into a product of copies of $\mathbb{Z}$. Then N is free.

Now from the **Fact** it follows that $\overline{Q}$ is free hence $Q \twoheadrightarrow \overline{Q}$ splits and we get a section. If we now look at the kernel

$$0 \longrightarrow Q' \hookrightarrow Q \twoheadrightarrow \overline{Q}$$

we see that $\mathrm{Hom}(Q',\mathbb{Z})=0$. (as maps to $\mathbb{Z}$ would lift to Q from splitting and this would not vanish in $\overline{Q}$.) Considering now again the sequence (2), as $\overline{Q}$ is free we have the section and hence the dotted map l:

$$\begin{tikzcd} & & & \overline{Q} \arrow[dll, dotted, bend right, "l"'] \arrow[d, dotted, bend right] & \\ 0 \arrow[r] & \bigoplus_{\mathbb{N}}\mathbb{Z} \arrow[r, "h"] & \bigoplus_{\mathbb{N}}\mathbb{Z} \arrow[r, two heads] & Q \arrow[r] \arrow[u, two heads] & 0 \end{tikzcd}$$

Now we can replace Q by Q' as we can remove the summand $\overline{Q}$ from Q and $\prod_{\mathbb{N}}\mathbb{Z}$ so now wlog. $\mathrm{Hom}(Q,\mathbb{Z})=0$.
As[33] $\mathrm{Hom}\left(\mathrm{Hom}(\prod_{\mathbb{N}}\mathbb{Z},\mathbb{Z})\right)=\prod_{\mathbb{N}}\mathbb{Z}$ we get by dualizing h :

$$0 \longleftarrow \underline{\mathrm{Ext}}^1(Q,\mathbb{Z}) \longleftarrow \prod_{\mathbb{N}}\mathbb{Z} \xleftarrow{g=h^*} \prod_{\mathbb{N}}\mathbb{Z} \hookleftarrow \underline{\mathrm{Hom}}(Q,\mathbb{Z}) \longleftarrow 0$$

But $\underline{\mathrm{Hom}}(Q,\mathbb{Z})=0$ as $S-$valued points are

$$\mathrm{Hom}(Q\otimes\mathbb{Z}[S],\mathbb{Z})=\mathrm{Hom}(Q,\underbrace{\mathrm{Cont}(S,\mathbb{Z})}_{\text{this is free}^{34}})=0.$$

Thus, g is injective and $\mathrm{im}(g)\cong\prod_{\mathbb{N}}\mathbb{Z}$. □

Corollary 6.12. *Any $M\in\mathrm{Solid}^{fin.\ pres.}$ is a product of copies of $\mathbb{Z}$ and a group of form $\underline{\mathrm{Ext}}^i(Q,\mathbb{Z})$ for some countable abelian group Q, with $\mathrm{Hom}(Q,\mathbb{Z})=0$.*

Corollary 6.13. *$\prod_{\mathbb{N}}\mathbb{Z}$ is flat for $\otimes^{\square}$.*

Proof. We need for all $M\in\mathrm{Solid}$, $M\otimes^{L\square}\prod_{\mathbb{N}}\mathbb{Z}$ sits in degree 0. As derived tensor commutes with filtered colimits and we could write M as a filtered colimit of its finitely presented parts, wlog. M is finitely presented, so

$$0\longrightarrow\prod_{\mathbb{N}}\mathbb{Z}\xrightarrow{g}\prod_{\mathbb{N}}\mathbb{Z}\longrightarrow M\longrightarrow 0$$

Now applying the functor $-\otimes^{\square}\prod_{\mathbb{N}}\mathbb{Z}$ g stays injective. In fact $M\otimes^{L\square}\prod_{\mathbb{N}}\mathbb{Z}=\prod_{\mathbb{N}}M$. □

6.3 Some $\otimes^{\square}$-Computations

Let $M\in\mathrm{Ab}$ and $M_{\hat{p}}=\varprojlim M/p^n$. In such situations, we often use $\hat{\otimes}$ = p-adic completion of the local $\otimes$. In full generality it is better to derive everything:

$$M_{\hat{p}}:=R\varprojlim\underbrace{M/^{\mathbb{L}}p^n}_{=[M\xrightarrow{p^n}M]}$$

, where the right M sits in degree 0 and the left M in degree 1.

[33]This holds at least for Condensed Abelian Groups as a Corollary of Proposition 5.7 from [CS19].
[34]This is Theorem 5.4 in [CS19], or more formally [AS23].

Fun little exercise*. Prove that this notation is consistent, i.e. show that the complex with $\varprojlim M/p^n$ as its degree 0 part that vanishes on all other degrees gives the same object as the complex $R\varprojlim M/^{\mathbb{L}}p^n$.

Proposition 6.14. *If $N, M \in D_{\geq 0}(\mathrm{Solid})$ are derived p-complete, then $M \otimes^{L\square} N$ is derived p-complete.*

Corollary 6.15. $\left(\bigoplus_{\mathbb{N}} \mathbb{Z}\right)_{\hat{p}} \otimes^{L\square} \left(\bigoplus_{\mathbb{N}} \mathbb{Z}\right)_{\hat{p}} \cong \left(\bigoplus_{\mathbb{N}\times\mathbb{N}} \mathbb{Z}\right)_{\hat{p}}$.

Remark 6.16. Nothing special about completion to p. For any ring A, $x \in A$ and $M, N \in D_{\geq 0}(\mathrm{Solid}_A)$ which are derived x-complete $\Rightarrow N \otimes^{L\square} M$ is also derived x−complete.[35]

Proof Sketch of Proposition. First we will show, that $\mathbb{Z}_p \otimes^{L\square} \mathbb{Z}_p = \mathbb{Z}_p$. This follows, as we know from 5.12 that

$$\mathbb{Z}[\![U]\!] \otimes^{L\square} \mathbb{Z}[\![V]\!] = \mathbb{Z}[\![U, V]\!].$$

If we now divide out the ideal $(U - p, V - p)$, we get the desired result. Following from that, we see that $D\left(\mathrm{Mod}_{\mathbb{Z}_p}(\mathrm{Solid})\right) \subset D(\mathrm{Solid})$ with its symmetric monoidal $\otimes^{L\square}$.
Consider $M, N \in D\left(\mathrm{Mod}_{\mathbb{Z}_p}(\mathrm{Solid})\right)$.
We will show the case, where $M = N = \hat{\bigoplus}_{\mathbb{N}}\mathbb{Z}_p = \varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N}\\ \text{going to } \infty}} \prod_{\mathbb{N}} p^{f(n)}\mathbb{Z}_p$:
How do we get the right equality? First we get a map from the right to the left: The map goes into the completed direct sum, as modulo some power of p almost all parts on the right will go to 0. This map is also an injection. We need to show surjection: Pick any $S \in \mathrm{Pro}_{\mathbb{N}}(\mathrm{Fin})$ and see, that if it maps to $\hat{\bigoplus}_{\mathbb{N}}\mathbb{Z}_p$, that it also factors through $\varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N}\\ \text{going to } \infty}} \prod_{\mathbb{N}} p^{f(n)}\mathbb{Z}_p$. Suppose we got a map

$$S \xrightarrow{g} \hat{\bigoplus}_{\mathbb{N}}\mathbb{Z}_p = \varprojlim_n \left(\bigoplus_{\mathbb{N}} \mathbb{Z}/p^n\right)$$

The composition of the g and the equality gives a collection of maps $(g_n)_{n\in\mathbb{N}}$, where each g_n factors over $\bigoplus_{m\leq a(n)} \mathbb{Z}/p^n$, as S is compact and maps to a direct sum, hence it factors through a finite direct sum. We can visualize this with Figure 1, where the dots denote nonzero points and the most right points denote $a(n)$: Now we get f as the line of that diagram![36] Hence we construct our surjection this way!
Now

$$M \otimes^{L\square} N = \left(\varinjlim_{\substack{f:\mathbb{N}\to\mathbb{N}\\ \text{going to } \infty}} \prod_{\mathbb{N}} p^{f(n)}\mathbb{Z}_p\right) \otimes^{L\square} \left(\varinjlim_{\substack{g:\mathbb{N}\to\mathbb{N}\\ \text{going to } \infty}} \prod_{\mathbb{N}} p^{g(n)}\mathbb{Z}_p\right)$$

[35] The author thinks this is supposed to mean, that we will later get other notions of completion as well, i.e. I adic completions, where everything will work the same!

[36] Don't get confused, $f = a$ now!

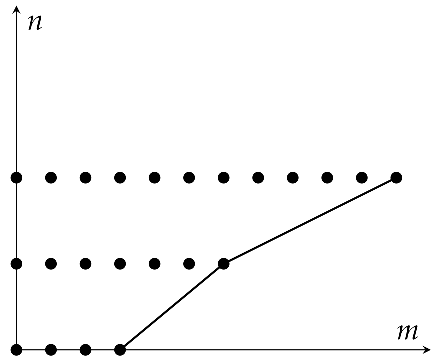


Figure 1

There is an obvious map:

$$M \otimes^{L\square} N \xrightarrow{\sim} \underset{\substack{g,f:\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}}{\varinjlim} \prod_{\mathbb{N}\times\mathbb{N}} p^{f(n)+g(m)}\mathbb{Z}_p$$

$$\xrightarrow{\sim} \underset{\substack{h:\mathbb{N}\times\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}}{\varinjlim} \prod_{\mathbb{N}\times\mathbb{N}} p^{h(n,m)}\mathbb{Z}_p = \hat{\bigoplus}_{\mathbb{N}\times\mathbb{N}}\mathbb{Z}_p$$

As we can see, for non-obvious reasons, $\otimes^{L\square}$ gives the correct answer. □

6.4 Solid Functional Analysis

We will work over $\mathbb{Q}_p$ but most will also work over all non-archimedean fields. We have that

$$D\left(\mathrm{Mod}_{\mathbb{Q}_p}(\mathrm{Solid})\right) \subset D\left(\mathrm{Mod}_{\mathbb{Z}_p}(\mathrm{Solid})\right) \subset D\left(\mathrm{Solid}\right)$$

where $D(\mathrm{Solid})$ has the compact projective generator $\left(\prod_{\mathbb{N}} \mathbb{Z}_p\right)[\frac{1}{p}]$ which is not really a normed vector space with the "canonical norm". We call this space the "p-adic Smith space"

More usual objects are p-adic Banach spaces, e.g. $\left(\bigoplus_{\mathbb{N}} \mathbb{Z}_p\right)_{\hat{p}}[\frac{1}{p}]$

Proposition 6.17.

$$((\textit{light}) \textit{ Smith Spaces}^{\mathrm{op}}) \cong ((\textit{separable}) \textit{ Banach spaces})$$

$$V \longmapsto \underline{\mathrm{Hom}}(V, \mathbb{Q}_p)$$

$$\underline{\mathrm{Hom}}(W, \mathbb{Q}_p) \longleftarrow W$$

Remark 6.18. What about the derived duality?

$$V \longmapsto \underline{R\mathrm{Hom}}(V, \mathbb{Q}_p) = \underline{\mathrm{Hom}}(V, \mathbb{Q}_p)$$

$$\underline{\mathrm{Hom}}(W, \mathbb{Q}_p) \overset{?}{=} \underline{R\mathrm{Hom}}(W, \mathbb{Q}_p) \longleftarrow W$$

$\overset{?}{=}$ depends on the model of set theory. It is definitely true under (1).

Fréchet Spaces = Countable limits of Banach spaces under dense transition maps. We have a standard notion of $\hat{\otimes}$ for them, compatible with such limits.

Proposition 6.19. *Let V, W be Fréchet $\mathbb{Q}_p$-vector spaces, then $\underline{V} \otimes^{L\square} \underline{W} \cong \underline{V \hat{\otimes} W}$, e.g. $\prod_{\mathbb{N}} \mathbb{Q}_p \otimes^{L\square} \prod_{\mathbb{N}} \mathbb{Q}_p \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Q}_p$*

Proof of example. $\prod_{\mathbb{N}} \mathbb{Q}_p = \underset{\substack{f:\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}}{\varinjlim} \left(\prod_{\mathbb{N}} p^{-f(n)}\mathbb{Z}_p\right)[\frac{1}{p}]$.

So

$$\prod_{\mathbb{N}} \mathbb{Q}_p \otimes^{L\square} \prod_{\mathbb{N}} \mathbb{Q}_p \cong \prod_{\mathbb{N}\times\mathbb{N}} \mathbb{Q}_p = \underset{\substack{f,g:\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}}{\varinjlim} \left(\prod_{\mathbb{N}\times\mathbb{N}} p^{-f(n)-g(m)}\mathbb{Z}_p\right)\left[\frac{1}{p}\right]$$

$$= \underset{\substack{h:\mathbb{N}\times\mathbb{N}\to\mathbb{N} \\ \text{going to } \infty}}{\varinjlim} \left(\prod_{\mathbb{N}\times\mathbb{N}} p^{-h(n,m)}\mathbb{Z}_p\right)\left[\frac{1}{p}\right]$$

□

7 Lecture 7 (7/24)

Recall:

$$\begin{tikzcd} \mathrm{Solid}_{\mathbb{Z}} \arrow[r, phantom, "\subseteq"] & \mathrm{Cond}_{\mathbb{Z}}^{\mathrm{light}} \arrow[l, dotted, bend right, "(-)^{\square}"'] \end{tikzcd}$$

where the left side is to be understood as a complete non-archimedean topological groups and the inclusion is as an abelian subcategory and is closed under (co)lim, extensions, ... The dotted arrow above is the so-called solidification and is adjoint to the inclusion.
Recap, that P is also a ring and $\exists$ a ring map

$$\mathbb{Z}[T] \longrightarrow P,$$

where multiplication by T is a shift in P. Solidifying this map we get

$$\mathbb{Z}[T] \longrightarrow \mathbb{Z}[\![T]\!]$$

Lemma 7.1.

$$\mathbb{Z}[\![T]\!] \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}[\![T]\!] \cong \mathbb{Z}[\![T]\!]$$

Proof.

$$\mathbb{Z}[\![T_1]\!] \otimes^{\square}_{\mathbb{Z}} \mathbb{Z}[\![T_2]\!] = \mathbb{Z}[\![T_1, T_2]\!]$$

Now to get the desired result, we just mod out the ideal $(T_1 - T_2)$. □

7.1 Intuition of the Open Unit Disk

The interpretation of $\mathbb{Z}[T]$ now is something like $\mathbb{A}^1$ and the interpretation of $\mathbb{Z}[\![T]\!]$ is something like some subspace of $\mathbb{A}^1$. The naive interpretation of that subspace would be an open neighbourhood of the origin in $\mathbb{A}^1$. But this intuition is getting problematic as we look at base change:
E.g.

$$\mathbb{Q}_p \otimes^{\square}_{\mathbb{Z}} \mathbb{Z}[\![T]\!] = (\mathbb{Z}_p \otimes \mathbb{Z}[\![T]\!])\left[\frac{1}{p}\right] = \mathbb{Z}_p[\![T]\!]\left[\frac{1}{p}\right]$$

The last equation holds, as $\mathbb{Z}_p = \mathbb{Z}[\![U]\!]/(U-p)$. But $\mathbb{Z}_p[\![T]\!]\left[\frac{1}{p}\right] \subsetneq \mathbb{Q}_p[\![T]\!]$, where the inclusion only has coefficients with bounded p-adic norm.
Hence we may interpret $\mathbb{Z}_p[\![T]\!]\left[\frac{1}{p}\right]$ as the ring of bounded functions on the open unit disc in $\mathbb{A}^1_{\mathbb{Q}_p}$.
As an interpretation that means, if we take an element of $\mathbb{Q}_p$, it converges on all $f \in \mathbb{Z}_p[\![T]\!]\left[\frac{1}{p}\right]$, plugged in as T, iff its absolute value is smaller than 1.
This suggests the interpretation that "$\mathbb{Z}[\![T]\!] \hookrightarrow \mathbb{Z}[T]$" corresponds to the open unit disk.

But what about the closed unit disk?
From complex geometry we know, that taking the complement of the closed unit disk in $\mathbb{P}^1_{\mathbb{C}}$ can be portrayed as an open simply connected properly contained set in $\mathbb{C} = D(z)$ (we can take this chart as we excluded the origin). Now by Riemann mapping theorem this is isomorphic to the open unit disk at ∞. Denote this disk as U_∞. So the intuition is that we want to construct the open unit disk at ∞ as the complement. But that is just the same as taking $\mathbb{C} \setminus (U_\infty \setminus \{\infty\})$.
Hence it can be thought of as

$$\mathbb{Z}((T^{-1})) = \mathbb{Z}[\![T^{-1}]\!] \otimes_{\mathbb{Z}[T^{-1}]} \mathbb{Z}[T, T^{-1}] \longleftarrow \mathbb{Z}[T] \tag{3}$$

To get the closed unit disk, "kill" $\mathbb{Z}((T^{-1}))$.

Note:

$$\mathbb{Z}[\![U]\!][T] \xrightarrow{UT-1} \mathbb{Z}[\![U]\!][T] \longrightarrow \mathbb{Z}((T^{-1}))$$

2-term resolution by compact projective generator of $\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$.

So killing $\mathbb{Z}((T^{-1}))$ is the same as requesting the $UT-1$ map to become an isomorphism. This suggests:

Definition 7.2. $M \in \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$ is $\mathbb{Z}[T]$**-solid** iff

$$\begin{aligned} &\underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M) \xrightarrow{\sigma T-1} \underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M) \\ \Longleftrightarrow\ &\underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(P \otimes_{\mathbb{Z}} \mathbb{Z}[T], M) \xrightarrow{\sigma T-1} \underline{\mathrm{Hom}}_{\mathbb{Z}[T]}(P \otimes_{\mathbb{Z}} \mathbb{Z}[T], M) \end{aligned}$$

is an isomorphism, where σ is the shift.

The **Note** from above is suggesting, that this is equivalent to $\underline{R\mathrm{Hom}}(\mathbb{Z}((T^{-1})), M) = 0$.

Lemma* 7.3. [37] *For $D(\mathbb{Z}[T])$-modules M and N where T_M denotes the T-action on M and T_N analogous on N, we have the equality*

$$\underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}(M, N) \simeq \mathrm{fib}\left(\underline{R\mathrm{Hom}}_{\mathbb{Z}}(M, N) \xrightarrow{T_N - T_M} \underline{R\mathrm{Hom}}_{\mathbb{Z}}(M, N)\right)$$

Proof.*

$$\begin{aligned} M &\simeq M \otimes^L_{\mathbb{Z}[T]} \mathbb{Z}[T] \\ &\simeq M \otimes^L_{\mathbb{Z}[T_M]} \mathrm{cofib}\left(\mathbb{Z}[T_M, T_N] \xrightarrow{(T_N - T_M)} \mathbb{Z}[T_M, T_N]\right) \\ &\simeq \mathrm{cofib}\left(M \otimes^L_{\mathbb{Z}[T]} (\mathbb{Z}[T_M] \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N]) \xrightarrow{(T_N - T_M)} M \otimes^L_{\mathbb{Z}[T]} (\mathbb{Z}[T_M] \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N])\right) \end{aligned}$$

[37] Thanks to Ferdinand to explaining this Lemma with proof to us! We will need this as a technical statement for a proof in the next Theorem!

$$\simeq \operatorname{cofib}\left(M \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N] \xrightarrow{T_M - T_N} M \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N]\right)$$

Now

$$\begin{aligned}
&\underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}(M,N)\\
&\simeq \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\operatorname{cofib}\left(M \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N] \xrightarrow{T_M - T_N} M \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N]\right), N\right)\\
&\simeq \operatorname{fib}\left(\underline{R\mathrm{Hom}}_{\mathbb{Z}[T_N]}(M \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N], N) \xrightarrow{T_M - T_N} \underline{R\mathrm{Hom}}_{\mathbb{Z}[T_N]}(M \otimes^L_{\mathbb{Z}} \mathbb{Z}[T_N], N)\right)\\
&\simeq \operatorname{fib}\left(\underline{R\mathrm{Hom}}_{\mathbb{Z}}(M,N) \xrightarrow{T_M - T_N} \underline{R\mathrm{Hom}}_{\mathbb{Z}}(M,N)\right)
\end{aligned}$$

□

Lemma* 7.4. [38] *Let X be a chain complex in $D(\mathrm{Mod}(\mathbb{Z}[T]))$ and $f : X \to X$, then*

$$X \simeq \operatorname{cofib}\left(X[T] \xrightarrow{T-f} X[T]\right)$$

Proof.*

$$\begin{aligned}
&\operatorname{cofib}\left(X[T] \xrightarrow{T-f} X[T]\right)\\
&\simeq \operatorname{cofib}\left(X \otimes^L_{\mathbb{Z}[f]} \mathbb{Z}[f,T] \xrightarrow{T-f} X \otimes^L_{\mathbb{Z}[f]} \mathbb{Z}[f,T]\right)
\end{aligned}$$

From here everything analogous to last lemma. □

Theorem 7.5.

$$\mathrm{Solid}_{\mathbb{Z}[T]} \subset \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}}) \subset \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{CondAb}^{\mathrm{light}})$$

The following holds:

- $\mathrm{Solid}_{\mathbb{Z}[T]}$ *is closed under (co)lim, extensions*
- *for any* $M \in \mathrm{Cond}^{\mathrm{light}}_{\mathbb{Z}[T]} \Rightarrow \underline{\mathrm{Ext}}^i(M,N) \in \mathrm{Solid}_{\mathbb{Z}[T]}$ *for* $N \in \mathrm{Solid}_{\mathbb{Z}[T]}$
- *there is a left adjoint* $(-)^{T\square}$ *to the first inclusion*
- *there is a symmetric monoidal structure on* $(-)^{T\square}$
- *the derived analog holds as well*
- $\left(\left(\prod_{\mathbb{N}} \mathbb{Z}\right)[T]\right)^{T\square} = \prod_{\mathbb{N}} \mathbb{Z}[T]$

Remark 7.6. As noted before, $\mathrm{Solid}_{\mathbb{Z}}$ is analogous to complete abelian non-archimedean topological groups, where we have a basis of neighbourhoods of the origin consisting of abelian subgroups. The interpretation of $\mathrm{Solid}_{\mathbb{Z}[T]}$ should also be complete non-archimedean, i.e. there is a basis of neighbourhood consisting of $\mathbb{Z}[T]$ submodules. A non-archimedean $\mathbb{Z}[T]$-module which is an element of $\mathrm{Solid}_{\mathbb{Z}}$ but not in $\mathrm{Solid}_{\mathbb{Z}[T]}$ would be $\mathbb{Z}((T^{-1}))$. Hence what the theorem tells us, is that "killing" such objects explains the difference of those notions.

[38] Also here thanks to Ferdinand for this lemma and proof!

Proof of Theorem. The proof will consist of a few claims which are going to be sketched step by step. All except last property are exactly the same as for $\mathrm{Solid}_{\mathbb{Z}}$.
Claim: For $M \in \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$:

$$M^{LT\square} = \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)$$

Now this gives us what we want, as we get a map

$$M \simeq \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}(\mathbb{Z}[T], M) \longrightarrow \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)$$

which gives us the derived T-solidification.

*Proof of Claim**.[39] We first show that $\mathbb{Z}[T]$-modules of that form are solid. For this we need to show, that

$$\underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1})), \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)\right) = 0$$

To show this, we will use that $\mathbb{Z}((T^{-1})) \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1})) = \mathbb{Z}((T^{-1}))$ which follows from (3). We can write the above expression as

$$\begin{aligned}&\underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1})) \otimes_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)\\ &\simeq \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1})) \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)\end{aligned}$$

To show that this equality holds, we show that for all K:

$$\mathrm{Hom}_{\mathbb{Z}}\left(K, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(N^{L\square}, M)\right) \xrightarrow[\sim?]{\text{よ}} \mathrm{Hom}_{\mathbb{Z}}\left(K, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(N, M)\right)$$ [40]

By left adjointness of solidification, $\mathrm{Hom}_{\mathbb{Z}}(N, M) = \mathrm{Hom}_{\mathbb{Z}}(N^{L\square}, M)$ holds for solid M modules. Using that, the above map is an isomorphism by Tensor-Hom-adjunction, as

$$\begin{aligned}&\mathrm{Hom}_{\mathbb{Z}}\left(K, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(N^{L\square}, M)\right) \simeq \mathrm{Hom}_{\mathbb{Z}}\left(N^{L\square}, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(K, M)\right)\\ &\simeq \mathrm{Hom}_{\mathbb{Z}}\left(N, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(K, M)\right) \simeq \mathrm{Hom}_{\mathbb{Z}}\left(K, \underline{R\mathrm{Hom}}_{\mathbb{Z}}(N, M)\right)\end{aligned}$$

Now it's left to show, that $\underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1})) \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right) \simeq 0$ which follows from

$$\begin{aligned}&\mathbb{Z}((T^{-1})) \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1]\\ &\simeq \mathrm{cofib}\left(\mathbb{Z}((T^{-1})) \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}[T] \to \mathbb{Z}((T^{-1})) \otimes^{L\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))\right)\\ &\simeq \mathrm{cofib}\left(\mathbb{Z}((T^{-1})) \to \mathbb{Z}((T^{-1}))\right) \simeq 0\end{aligned}$$

[39] Thanks to Ferdinand for explaining this to us!

[40] This suffices as the adjunction on $\mathbb{Z}$ level induces an adjunction between $\mathrm{Mod}_{\mathbb{Z}[T]}\left(D(\mathrm{CondAb}^{\mathrm{light}})\right)$ and $\mathrm{Mod}_{\mathbb{Z}[T]}\left(D(\mathrm{Solid}_{\mathbb{Z}})\right)$.

So now any module of the form of our functor is solid. Note that objects will still be mapped to the same thing if you apply the functor twice:

$$\begin{aligned}
&\underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)\right) \\
&\simeq \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1] \otimes_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right) \\
&\simeq \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M\right)
\end{aligned}$$

From here there a still a few things left to show which we will not cover.

<u>Next Claim</u>:

$$\begin{aligned}
D(\mathrm{Solid}_{\mathbb{Z}}) &\longrightarrow D(\mathrm{Solid}_{\mathbb{Z}[T]}) \\
M &\longmapsto (M \otimes_{\mathbb{Z}} \mathbb{Z}[T])^{LT\square}
\end{aligned}$$

is t-exact, preserves (co)limits and sends $\mathbb{Z} \mapsto \mathbb{Z}[T]$.

Note that if this functor commutes with products, this claim already proves the last point of the Theorem, as we would only need to understand what happens on $\mathbb{Z}$.

Proof of Claim.

$$\begin{aligned}
&(M \otimes \mathbb{Z}[T])^{LT\square} \simeq \underline{R\mathrm{Hom}}_{\mathbb{Z}[T]}\left(\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1], M[T]\right) = \\
&\mathrm{fib}\left(\underline{R\mathrm{Hom}}_{\mathbb{Z}}\left(U\mathbb{Z}[\![U]\!][-1], M[T]\right) \xrightarrow{T-\sigma} \underline{R\mathrm{Hom}}_{\mathbb{Z}}\left(U\mathbb{Z}[\![U]\!][-1], M[T]\right)\right)
\end{aligned}$$

where σ denotes the shift operator induced by $\cdot T$ in $\mathbb{Z}((T^{-1}))/\mathbb{Z}[T][-1]$. The last equation holds because of Lemma 7.3. Now as $U\mathbb{Z}[U][-1]$ is compact projective over $\mathbb{Z}$, we get

$$\mathrm{fib}\left(\underline{R\mathrm{Hom}}_{\mathbb{Z}}\left(U\mathbb{Z}[U][-1], M\right)[T] \xrightarrow{T-\sigma} \underline{R\mathrm{Hom}}_{\mathbb{Z}}\left(U\mathbb{Z}[\![U]\!][-1], M\right)[T]\right),$$

since $\underline{R\mathrm{Hom}}$ with a compact internally projective domain commutes with colimits. By Lemma 7.4 we get

$$\underline{R\mathrm{Hom}}_{\mathbb{Z}}(U\mathbb{Z}[\![U]\!][-1], M).$$

On the way we "lost" the $\mathbb{Z}[T]$–module structure, which can be recovered, by denoting the multiplication with T on the first component, that sends

$$\begin{aligned}
U &\mapsto 0 \\
U^2 &\mapsto U \\
U^3 &\mapsto U^2
\end{aligned}$$

Now as $U\mathbb{Z}[\![U]\!][-1]$ is internally projective, this functor is t-exact and preserves (co)limits in $\mathrm{Solid}_{\mathbb{Z}}$. But (co)limits in $\mathrm{Solid}_{\mathbb{Z}[T]}$ are calculated on the underlying level, as it is part of a module category.

For $\mathbb{Z} \mapsto \mathbb{Z}[T]$ we plug in $\mathbb{Z}$ into $\underline{R\mathrm{Hom}}_{\mathbb{Z}}(U\mathbb{Z}[\![U]\!], -)$. Observe that taking $R\mathrm{Hom}$ from this product of copies of $\mathbb{Z}$ to $\mathbb{Z}$ turns into a direct product of copies of $\mathbb{Z}$. One can check that this is just $\mathbb{Z}[T]$ with its $\mathbb{Z}[T]$-module structure. □

Warning 7.7. For the second claim of the proof, it is important to note, that T-solidification itself is <u>not</u> t-exact, only the composition!

Example 7.8.

$$(\mathbb{Q}_p[T])^{T\square} = (\mathbb{Z}_p[T]^{T\square})\left[\frac{1}{p}\right] = \left(\varprojlim ((\mathbb{Z}/p^n\mathbb{Z})[T])^{T\square}\right)\left[\frac{1}{p}\right] = \left(\varprojlim(\mathbb{Z}/p^n\mathbb{Z})[T]\right)\left[\frac{1}{p}\right]$$
$$= \mathbb{Z}[T]_{\hat{p}}\left[\frac{1}{p}\right]$$

where the last object can be interpreted as the functions on the closed unit disk, as it is the subset of $\mathbb{Q}_p[\![T]\!]$ where the coefficients converge to 0 p-adically.

Some philosophical thoughts: What should be considered as "closed", what should be considered as "open"?

$$\begin{tikzcd}
D\big(\mathrm{Mod}_{\mathbb{Z}((T^{-1}))}(\mathrm{Solid}_{\mathbb{Z}})\big) \arrow[r, "i_*"'] & D\big(\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})\big) \arrow[l, bend right=50, "-\otimes_{\mathbb{Z}[T]}\mathbb{Z}((T^{-1}))=:i^*"'] \arrow[l, bend left=50, "i^!"] \arrow[r, "(-)^{LT\square}", "=:j^*"'] & D\big(\mathrm{Solid}_{\mathbb{Z}[T]}\big) \arrow[l, bend right=50, "M\mapsto[M\to M\otimes_{\mathbb{Z}[T]}\mathbb{Z}((T^{-1}))]=:j_!"'] \arrow[l, bend left=50, "\supseteq=:j_*"]
\end{tikzcd}$$

where the upper arrows are the left adjoints and the lower arrows the right adjoints to the straight arrows. With our interpretation of these objects from the beginning of the lecture, this behaves exactly like:
X top. space, $Z \subset X$ closed and $U = X \setminus Z$ open

$$\begin{tikzcd}
D(Sh(Z;\mathbb{Z})) \arrow[r, "i_*"'] & D(Sh(X;\mathbb{Z})) \arrow[l, bend right=50, "i^*"'] \arrow[l, bend left=50, "i^!"] \arrow[r, "j^*"'] & D(Sh(U;\mathbb{Z})) \arrow[l, bend right=50, "j_!"'] \arrow[l, bend left=50, "j_*"]
\end{tikzcd}$$

Now it's going to get confusing: As these formalisms work exactly analogous, we see, that the closed unit disk should be thought of as something open and the open unit disk as being something closed.

7.2 More general setting for "Geometry"

Vista: Look at solid rings, $R \in \mathrm{Alg}(\mathrm{Solid}_{\mathbb{Z}}, \otimes)$. The main idea is to take the things we know about $\mathbb{Z}[T]$-modules and base change them along any map from $\mathbb{Z}[T]$ into R. These maps into R are something like all possible maps in R. What we will end up with, is that $D(\mathrm{Mod}_R(\mathrm{Solid}_{\mathbb{Z}}))$ localizes along $\mathrm{Spv}(R(*))$[41]. Small reminder on Spv: The basic

[41] This is called the "valuative spectrum".

opens of that topology are **rational opens**: $X\left(\frac{f_1,\dots,f_n}{g}\right)$ for $g \neq 0$ and which consist of valuations $|f_i| \leq |g|$.
We are going to attach to

$$X\left(\frac{f_1,\dots,f_n}{g}\right) \mapsto \left\{M \in D(\mathrm{Mod}_R(\mathrm{Solid}_{\mathbb{Z}}))\right\}^{42}$$

s.t.

1. $\cdot g : M \xrightarrow{\sim} M$
2. $\underline{\mathrm{Hom}}(P,M) \xrightarrow[\simeq]{\frac{f_i}{g}\sigma-1} \underline{\mathrm{Hom}}(P,M)\ \forall i$

We want to interpret that as

$$\begin{tikzcd} g : \mathrm{Spec}(R) \arrow[r] \arrow[dr] & \mathbb{A}^1 \\ & \mathbb{A}^1 \setminus 0 \arrow[u, hook] \end{tikzcd} \qquad \begin{tikzcd} \frac{f_i}{g} : \mathrm{Spec}(R) \arrow[r] \arrow[dr] & \mathbb{A}^1 \\ & \mathbb{D} \arrow[u, hook] \end{tikzcd}$$

In particular, one gets a structure sheaf on $\mathrm{Spv}(R(*))$.

Goal for the rest of the lecture:

1. Make $\mathcal{O}\left(X\left(\frac{f_1,\dots,f_n}{g}\right)\right)$ explicit
2. Compare it to Huber's theory.

Definition 7.9. Let $R \in \mathrm{CAlg}(\mathrm{Solid}_{\mathbb{Z}}, \otimes)$, $f \in R(*)$. This gives us a map $\mathbb{Z}[T] \xrightarrow{T\mapsto f} R$. Now f is called **topologically nilpotent** if the above map factors through $\mathbb{Z}[\![T]\!] = P^{\square}$.[43] f is **power-bounded** if $R \in \mathrm{Solid}_{\mathbb{Z}[T]}$ (this is equivalent to $\underline{\mathrm{Hom}}(P,R) \xrightarrow[\simeq]{f\sigma-1} \underline{\mathrm{Hom}}(P,R)$).

We define $R^{\circ} \subseteq R(*)$ as the set of power-bounded elements and $R^{\circ\circ} \subseteq R(*)$ as the set of topological nilpotent elements.

Lemma 7.10.

1. *$R^{\circ} \subseteq R(*)$ is an integrally closed subring.*
2. *$R^{\circ\circ} \subseteq R^{\circ}$ is a radical ideal.*

Proof.

1. $\underline{R^{\circ}\text{ subring:}}$
   (a) $1 \in R^{\circ}$: This follows from the definition of being solid, plug in 1 as f.

[42] This is not a set, but a category.
[43] Note that this is the same as the classical notion of being topological nilpotent: the map sends $T^n \mapsto f^n$. If this factors through $P^{\square}$ this means that the sequence $1, f^1, f^2, \dots$ extends to a nullsequence.

(b) Closed under addition and multiplication: $f, g \in R^\circ$ then we want to show for any $F \in \mathbb{Z}[X, Y]$, then $F(f, g) \in R^\circ$

$$\mathbb{Z}[X, Y] \xrightarrow[Y \mapsto g]{X \mapsto f} R$$

As f and g are already powerbounded, we have that R is a solid $\mathbb{Z}[X]$ and a solid $\mathbb{Z}[Y]$ module. What we need to show, is that for any map

$$\mathbb{Z}[T] \xrightarrow{F} \mathbb{Z}[X, Y]$$

we get an induced solid $\mathbb{Z}[T]$-structure on R.
Resolve R by $\bigoplus \prod_{\mathbb{N}} \mathbb{Z}[X, Y]$'s.

Solidfy $X^\square$: $\bigoplus (\prod \mathbb{Z}[X]) [Y]$
Solidfy $Y^\square$: $\bigoplus (\prod \mathbb{Z}[X, Y])$

After the last step, we see, that we can resolve R by $\bigoplus (\prod \mathbb{Z}[X, Y])$. Each of these are solid modules over $\mathbb{Z}[T]$, as it is a colimit of limits of something that is solid[44].

<u>R° integrally closed:</u>
Suppose we have an equation $f^n + c_{n-1} f^{n-1} + \cdots + c_0 = 0$ with $c_i \in R^\circ$.
Now consider

$$\mathbb{Z}[x_0, \ldots, x_{n-1}] \longrightarrow \mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0)$$

By hypothesis, if we have a map $\mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0) \to R$ the composition gives R a solid module structure over each variable of $\mathbb{Z}[x_0, \ldots, x_{n-1}]$. We want to show, that if we have a map

$$\mathbb{Z}[T] \longrightarrow \mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0),$$

that R is a solid $\mathbb{Z}[T]$-module. Now we do the same trick as before: resolve R as a $\mathbb{Z}[x_0, \ldots, x_{n-1}]$-module, solidify at each variable respectively, tensor it up to be a $\mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0)$-module, by computations[45] one gets, that we have sums of products of $\mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0)$ and as $\mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0)$ is individually solid, and as solid is closed under limits and colimits, one deduces, that it is solid over $\mathbb{Z}[T]$ as well.

2. $R^{\circ\circ} \subseteq R^\circ$:
If we consider the map

$$\mathbb{Z}[\![T]\!] \xrightarrow{T \mapsto f} R$$

[44] $\mathbb{Z}[X, Y]$ is solid over $\mathbb{Z}[T]$ as it is discrete.

[45] Here it comes into play, that $\mathbb{Z}[x_0, \ldots, x_{n-1}, T]/(T^n + x_{n-1}T^{n-1} + \cdots + x_0)$ is finite free over $\mathbb{Z}[x_0, \ldots, x_{n-1}]$, such that we can bring in the tensor into the product.

we get a $\mathbb{Z}[\![T]\!]$ module structure on R. Now if we resolve R as before and then tensor it with $\mathbb{Z}[\![T]\!]$ we get

$$\left(\bigoplus \prod_{\mathbb{N}} \mathbb{Z} \otimes \mathbb{Z}[\![T]\!]\right) = \left(\bigoplus \prod_{\mathbb{N}} \mathbb{Z}[\![T]\!]\right)$$

But this is already solid over $\mathbb{Z}[T]$. The rest is left as a fun exercise.

$\square$

Proposition 7.11. *Let $R \in \mathrm{CAlg}(\mathrm{Solid}_{\mathbb{Z}})$, $g, f_1, \ldots, f_n \in R(*)$. There is a initial solid ring*

$$R \longrightarrow R\left\langle \frac{f_1, \ldots, f_n}{g} \right\rangle^{\mathrm{Solid}}$$

s.t.

1. *g is invertible in $R\left\langle \frac{f_1,\ldots,f_n}{g} \right\rangle^{\mathrm{Solid}}$*
2. *$\frac{f_i}{g}$ is power-bounded in $R\left\langle \frac{f_1,\ldots,f_n}{g} \right\rangle^{\mathrm{Solid}}$ for all i*

The candidate is

$$R[x_1, \ldots, x_n]^{x_1\square,\ldots,x_n\square}/(gx_1 - f_1, gx_2 - f_2, \ldots, gx_n - f_n)\left[\frac{1}{g}\right]$$

The reasoning why it satisfies the proposed properties can be easily calculated and to see, that it is a solid object, one sees that all the operations that happen after solidification are just colimits.

Warning 7.12. This is <u>not</u> exactly the value of the structure sheaf on $X\left(\frac{f_1,\ldots,f_n}{g}\right)$. It is indeed π_0(value of structure sheaf). In almost all practical cases, all $\pi_i = 0$ for $i > 0$.

Warning 7.13. Even if R is nice (complete Huber ring), this $R\left\langle \frac{f_1,\ldots,f_n}{g} \right\rangle^{\mathrm{Solid}}$ may not be quasi separated. But again in almost all practical cases, it is.

Relation to Huber's theory:
Let R be a complete Huber ring and $(f_1, \ldots, f_n) \subseteq R$ open. Here Huber defines $R\left\langle \frac{f_1,\ldots,f_n}{g} \right\rangle^{\mathrm{Huber}}$.
This is the "quasi-separafication" of $\pi_0\left(R\left\langle \frac{f_1,\ldots,f_n}{g} \right\rangle^{\mathrm{Solid}}\right)$.

8 Lecture 8 (8/24)

8.1 Motivation

By now, we have seen several examples of a pair:

light condensed ring A
+ notion of "complete" condensed A-modules

We want to connect these pairs to the notion of so-called *Huber pairs*. The bridge between these will be given by *Analytic Rings*.

Definition 8.1. (Huber)

1. A **Huber ring** is a topological ring A, that admits an open subring $A_0 \subseteq A$ such that there is a finitely generated A_0-ideal $I \subset A_0$, s.t. A_0 has the I-adic topology.
2. A **ring of integral elements** in a Huber ring A, is an open and integrally closed subring $A^+ \subseteq A^\circ = \{\text{powerbounded elements}\}$.
3. A **Huber pair** is a pair of (A, A^+) of a Huber ring A and a ring of integral elements A^+ of A

Example 8.2.

1. Any discrete ring is a Huber ring. ($A_0 = A, I = (0)$)
2. Any topological ring with I-adic topology for some finitely generated ideal J is Huber. ($A_0 = A, I = J$)
3. $\mathbb{Q}_p$ is Huber. ($A_0 = \mathbb{Z}_p, I = (p)$)

Remark 8.3. The completion $\hat{A}$ of any Huber ring A is again a Huber ring. In particular we have for

$$A \supseteq A_0 \supseteq I$$

the explicit completion

$$\hat{A} \supseteq \hat{A}_0 = I - \text{adic completion of } A_0$$

We will always consider complete Huber rings/ Huber pairs.

Definition 8.4. For A Huber ring. Then

$$\left\{f \in A \mid f^n \xrightarrow{n\to\infty} 0\right\} =: A^{\circ\circ} \subseteq A^\circ := \left\{f \in A \mid \{f^n\}_n \text{ is bounded}\right\} \subseteq A$$

In fact, for all $I \subseteq A_0 \subseteq A$ as above, we get

$$\begin{array}{ccc} A^{\circ\circ} & \subseteq & A^{\circ} \\ \cup\!| & & \cup\!| \\ I & \subseteq & A_0 \end{array}$$

Also one can calculate that

$$A^{\circ\circ} = \varinjlim_{\substack{\text{(filtered)}\\ I\subseteq A_0\subseteq A}} I \subseteq A^{\circ} = \varinjlim_{\substack{\text{(filtered)}\\ A_0\subseteq A}} A_0 \subseteq A$$

Example 8.5. $\mathbb{Z}[T]$ has several possible rings of integral elements, which will "lead up" to different theories. For example:

1. $\mathbb{Z} \subset \mathbb{Z}[T] \rightsquigarrow \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})$
2. $\mathbb{Z}[T] \subseteq \mathbb{Z}[T] \rightsquigarrow \mathrm{Solid}_{\mathbb{Z}[T]}$

Notation: Huber uses single letter A to denote a Huber pair $(A^{\triangleright}, A^{+})$. We will follow this notation.

8.2 Analytic Rings

Let $A^{\triangleright}$ be a light condensed ring.

Definition 8.6 (Analytic Ring)**.** An **analytic ring structure** on $A^{\triangleright}$ is a full subcategory

$$\mathrm{Mod}_A \subset \mathrm{Cond}(A^{\triangleright}) = \left\{\substack{\text{(light) condensed } A^{\triangleright}\text{-modules} \\ + \text{ map: } A^{\triangleright}\otimes M \longrightarrow M}\right\}$$

that is an abelian subcategory, stable under all (co)limits, all extensions, all $\underline{\mathrm{Ext}}^i_{A^{\triangleright}}(N, -)$ for $N \in \mathrm{Cond}(A^{\triangleright})$ and contains $A^{\triangleright}$.
A map of analytic rings

$$A = (A^{\triangleright}, \mathrm{Mod}_A) \longrightarrow B = (B^{\triangleright}, \mathrm{Mod}_B)$$

is a map $A^{\triangleright} \longrightarrow B^{\triangleright}$, s.t.

$$\begin{tikzcd} \mathrm{Mod}_B \arrow[d, dashed, "!"'] & \subseteq & \mathrm{Cond}(B^{\triangleright}) \arrow[d, "\text{restr.}"] \\ \mathrm{Mod}_A & \subseteq & \mathrm{Cond}(A^{\triangleright}) \end{tikzcd}$$

Remark* 8.7. It is immediate to see, that this kind of generalizes the concept of solid modules over a light condensed ring.

Theorem 8.8 (Reflection Principle)**.** *If $\mathcal{C}$ is presentable, $\mathcal{D} \subset \mathcal{C}$ full and closed under all limits, and $\exists$ regular cardinal κ such that $\mathcal{D}$ is closed under κ-filtered colimits, then $\mathcal{D}$ is presentable and there exists a left adjoint to the inclusion.*

The ∞-Category version of this theorem was proved by Ragimov-Schlank.

Proposition 8.9. *There is a left adjoint to the inclusion:*

$$\begin{aligned} \mathrm{Cond}(A^{\triangleright}) &\longrightarrow \mathrm{Mod}_A \\ {}''M &\longmapsto M \otimes_{A^{\triangleright}} A'' \end{aligned}$$

The kernel of this functor is a $\otimes$-ideal, and Mod_A acquires a unique symmetric monoidal structure making $(- \otimes_{A^{\triangleright}} A)$ symmetric monoidal.

Proof.

Existence of left adjoint: formal nonsense + Theorem 8.8

$\otimes$-ideal: Assume: $M \in \mathrm{Cond}(A^{\triangleright})$ s.t. $M \otimes_{A^{\triangleright}} A = 0$. Let $N \in \mathrm{Cond}(A^{\triangleright})$ arbitrary.

<u>To show:</u>

$$\begin{aligned} &\left(N \otimes_{A^{\triangleright}} M\right) \otimes_{A^{\triangleright}} A = 0 \\ \iff &\forall L \in \mathrm{Mod}_A : \mathrm{Hom}_{A^{\triangleright}}\left(N \otimes_{A^{\triangleright}} M, L\right) = 0 \end{aligned}$$

but as

$$\begin{aligned} \mathrm{Hom}_{A^{\triangleright}}\left(N \otimes_{A^{\triangleright}} M, L\right) &= \mathrm{Hom}_{A^{\triangleright}}\Big(M, \underbrace{\underline{\mathrm{Hom}}_{A^{\triangleright}}(N, L)}_{\in \mathrm{Mod}_A}\Big) \\ &= \mathrm{Hom}_{\mathrm{Mod}_A}\Big(\underbrace{M \otimes_{A^{\triangleright}} A}_{=0}, \underline{\mathrm{Hom}}_{A^{\triangleright}}(N, L)\Big) = 0 \end{aligned}$$

unique symmetric monoidal structure: $\otimes_A$ has to be given by

$$M \otimes_A N := \left(M \otimes_{A^{\triangleright}} N\right) \otimes_{A^{\triangleright}} A$$

to check: for all $M, N \in \mathrm{Cond}(A^{\triangleright})$

$$\begin{aligned} \left(M \otimes_{A^{\triangleright}} N\right) \otimes_{A^{\triangleright}} A &\underset{?}{\xrightarrow{\sim}} \left(M \otimes_{A^{\triangleright}} A\right) \otimes_A \left(N \otimes_{A^{\triangleright}} A\right) \\ &:= \left(\left(M \otimes_{A^{\triangleright}} A\right) \otimes_{A^{\triangleright}} \left(N \otimes_{A^{\triangleright}} A\right)\right) \otimes_{A^{\triangleright}} A \end{aligned}$$

For that, use the same argument as in the Solid case, see Corollary 5.4.

□

Remark 8.10. Hence for a map of analytic rings $A \longrightarrow B$, we get via the left-adjoint of $\mathrm{Mod}_B \subseteq \mathrm{Cond}(B^{\triangleright})$ and the left adjoint of the restriction $\mathrm{Cond}(B^{\triangleright}) \to \mathrm{Cond}(A^{\triangleright})$ a functor which is symmetric monoidal and also left-adjoint to $\mathrm{Mod}_B \to \mathrm{Mod}_A$:

$$(- \otimes_A B) : \mathrm{Mod}_A \xrightarrow{\otimes-\text{functor}} \mathrm{Mod}_B$$

8.3 Analytic Ring structure on Derived Categories

Definition 8.11. Let A be an analytic ring structure on $A^{\triangleright}$. Then $D(A) \subseteq D(\mathrm{Cond}(A^{\triangleright}))$ is defined as the full subcategory of all $M \in D(\mathrm{Cond}(A^{\triangleright}))$ s.t. $\forall i \in \mathbb{Z}$, $H_i(M) \in \mathrm{Mod}_A$.

Warning 8.12. There is a functor $D(\mathrm{Mod}_A) \to D(A)$ but not always an equivalence.

Proposition 8.13. *$D(A) \subseteq D(\mathrm{Cond}(A^{\triangleright}))$ triangulated subcategory stable under all $\bigoplus$, all $\prod$. The inclusion has a left adjoint*

$$(- \otimes^L_{A^{\triangleright}} A) : D(\mathrm{Cond}(A^{\triangleright})) \longrightarrow D(A).$$

The kernel of this functor is a $\otimes$-ideal and there is a unique monoidal tensor product $(- \otimes^L_A -)$ on $D(A)$ making $(- \otimes^L_{A^{\triangleright}} A)$ symmetric monoidal.

Proof.

triangulated: Let

$$M' \longrightarrow M \longrightarrow M'' \longrightarrow M'[1]$$

be a triangle in $D(\mathrm{Cond}(A^{\triangleright}))$ and $M', M'' \in D(A)$.
<u>To show:</u> $M \in D(A)$

$$\begin{tikzcd}
H_{i+1}(M'') \arrow[r] & H_i(M') \arrow[rr] \arrow[dr] & & H_i(M) \arrow[rr] \arrow[dr] & & H_i(M'') \arrow[r] & H_{i-1}(M') \\
& & Q \arrow[ur] \arrow[dr] & & K \arrow[ur] \arrow[dr] & & \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 &
\end{tikzcd}$$

with $K, Q \in \mathrm{Mod}_A$ and stability under extensions.
Stability under $\bigoplus$ and stability under countable $\prod_{\mathbb{N}}$ reduce to H_i's.
For an arbitrary collection $(M_a)_{a \in L}$ of objects in Mod_A, we need that

$$R^i \prod_{a \in L} M_a \in \mathrm{Mod}_A \, \forall i.$$

Set $M = \bigoplus_{a \in L} M_a$. Then $R^i \prod_{a \in L} M_a$ is a retract of

$$R^i \prod_{a \in L} M = \underline{\mathrm{Ext}}^i_{A^{\triangleright}}\Big(\bigoplus_{a \in L} A^{\triangleright}, M\Big),$$

thus it is in $\mathrm{Mod}_A \, \forall i$.

Existence of left adjoint: Follows from the last point and Theorem 8.8.

⊗-ideal:

$$\begin{aligned} &M \in \ker\left(D(\mathrm{Cond}(A^{\triangleright})) \longrightarrow D(A)\right) \\ &N \in D\left(\mathrm{Cond}(A^{\triangleright})\right) \end{aligned}$$

To show: $\forall L \in D(A)$:

$$R\mathrm{Hom}_{D(\mathrm{Cond}(A^{\triangleright}))}\left(M \otimes^{L}_{A^{\triangleright}} N, L\right) = 0$$

Light condensed sets are "replete". This means countable limits of surjections are surjections.

$$\Rightarrow K \in D\left(\mathrm{Cond}(A^{\triangleright})\right), K \xrightarrow{\sim} R\varprojlim_{n} \tau_{\leq n} K$$

This is called "Postnikov limit".

By Tensor-Hom adjunction, it suffices to show

$$\underline{R\mathrm{Hom}}_{D(\mathrm{Cond}(A^{\triangleright}))}(N, L) \in D(A)$$

By the above fact, we can assume: $L \in D^{+}$. On the other hand we can write $N = \varinjlim_{n} \tau_{\geq -n} N$. As N lies in the first component, pulling out the colimit of the $\underline{R\mathrm{Hom}}$ gives a limit and we can assume that $N \in D^{-}$. But then a spectral sequence reduces one to $L \in \mathrm{Mod}_{A}, N \in \mathrm{Cond}(A^{\triangleright})$.
$\rightsquigarrow$ condition $\underline{\mathrm{Ext}}^{i}_{A^{\triangleright}} \in \mathrm{Mod}_{A}$. Now existence of $(- \otimes^{L}_{A} -)$ is formal.

□

Remark 8.14. For a map of analytic rings $A \to B$ we get via the left adjoint of $D(B) \subseteq D(\mathrm{Cond}(B^{\triangleright}))$ the map:

$$(- \otimes^{L}_{A} B) : D(A) \longrightarrow D(B)$$

Proposition 8.15. *$D(A)$ has a natural t-structure making $\left(D(A) \hookrightarrow D(\mathrm{Cond}(A^{\triangleright}))\right)$ t-exact and $(- \otimes^{L}_{A^{\triangleright}} A)$ preserve $D_{\geq 0}$.*

Proof. $D(A)$ is stable under all $\tau_{\leq n}, \tau_{\geq n}$ by definition. Left adjoints to t-exact functors preserve $D_{\geq 0}$. □

The $\heartsuit$ of the t-structure is:

$$\begin{aligned} D(A)^{\heartsuit} &= \mathrm{Mod}_{A} \\ (- \otimes^{L}_{A^{\triangleright}} A)^{\heartsuit} &= (- \otimes_{A^{\triangleright}} A) \\ (- \otimes^{L}_{A} -)^{\heartsuit} &= (- \otimes_{A} -) \end{aligned}$$

Back to comparison with Huber Rings

$$\begin{tikzcd}
 & \text{Solid Rings} \arrow[dr, hook] & \\
\text{Huber rings} \arrow[ur, hook] \arrow[rr, hook] & & \text{CondRings} \\
A^{\triangleright} \arrow[rr, mapsto] & & \underline{A}^{\triangleright}
\end{tikzcd}$$

Definition 8.16. Let $A^{\triangleright}$ be a solid ring, we can define as last time

$$A^{\circ\circ} \subseteq A^{\circ} \subseteq A^{\triangleright}(*)$$

With

$$A^{\circ\circ} := \left\{ f \in A^{\triangleright}(*) \mid {\scriptstyle \underset{T \longmapsto f}{\mathbb{Z}[T] \longrightarrow A^{\triangleright}}} \text{ factors over } \mathbb{Z}[\![T]\!] \right\}$$

and

$$A^{\circ} := \left\{ f \in A^{\triangleright}(*) \mid A^{\triangleright} \in \mathrm{Solid}_{\mathbb{Z}[T]} \text{ via } {\scriptstyle \underset{T \longmapsto f}{\mathbb{Z}[T] \longrightarrow A^{\triangleright}}} \right\}$$

Remember that $A^{\circ\circ} \subseteq A^{\circ}$ is a radical ideal and A° is an integrally closed subring of $A^{\triangleright}(*)$.

Definition 8.17. An analytic ring $A = \left(A^{\triangleright}, \mathrm{Mod}_A\right)$ is **solid** if it admits a (necessarily unique) map of analytic rings

$$\begin{aligned}
&\mathbb{Z}_{\square} \longrightarrow A \\
\iff& \text{All } M \in \mathrm{Mod}_A \text{ are solid} \\
\iff& 1 - \text{shift} : P \otimes_{\mathbb{Z}} A \xrightarrow{\sim} P \otimes_{\mathbb{Z}} A
\end{aligned}$$

Definition 8.18. Assume A is a solid analytic ring structure on solid ring $A^{\triangleright}$. Then

$$\begin{aligned}
A^{+} &= \left\{ f \in A^{\triangleright}(*) \mid {\scriptstyle \underset{T \longmapsto f}{\mathbb{Z}[T] \longrightarrow A^{\triangleright}}} \text{induces map of analytic rings } \mathbb{Z}[T]_{\square} \longrightarrow A \right\} \\
&= \left\{ f \in A^{\triangleright}(*) \mid 1 - f \cdot \text{shift:} \, P \otimes^{L}_{\mathbb{Z}} A \xrightarrow{\text{isom.}} P \otimes^{L}_{\mathbb{Z}} A \right\}
\end{aligned}$$

Remark* 8.19.

1. For the last equality consider the map $\mathbb{Z}_{\square} \xrightarrow{T \mapsto f} A^{\triangleright}$. Naturally we can now consider elements of Mod_A as $\mathbb{Z}[T]$-modules via this map. As discussed in last lecture, these are in $\mathrm{Solid}_{\mathbb{Z}[T]}$ iff $R\underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M) \xrightarrow[\sim]{\text{shift}\cdot f - 1} R\underline{\mathrm{Hom}}_{\mathbb{Z}}(P, M)$ for all $M \in D(A)$. This is by adjunction the same as $R\underline{\mathrm{Hom}}_{A}(P \otimes_{\mathbb{Z}} A, M) \xrightarrow[\sim]{\text{shift}\cdot f - 1} R\underline{\mathrm{Hom}}_{A}(P \otimes_{\mathbb{Z}} A, M)$. By some computation, this is an iso iff $P \otimes^{L}_{\mathbb{Z}} A \xrightarrow[\sim]{\text{shift}\cdot f - 1} P \otimes^{L}_{\mathbb{Z}} A$.
2. To emphasize the difference between A^{+} and A°: elements in A^{+} induce a solid $\mathbb{Z}[T]$-module structure for <u>all</u> modules in Mod_A whereas elements in A° only induce a solid $\mathbb{Z}[T]$-module structure for $A^{\triangleright}$.

Proposition 8.20. *For a solid analytic ring structure A on $A^{\triangleright}$, we get:*

$$A^{\circ\circ} \subseteq A^{+} \subseteq A^{\circ},$$

where $A^{+} \subseteq A^{\circ}$ is an integrally closed subring.

Proof. For $f \in A^{\circ\circ}$, we get an induced map $\mathbb{Z}[\![T]\!] \to A^{\triangleright}$. As

$$\mathrm{Mod}_{\mathbb{Z}[\![T]\!]}(\mathrm{Mod}_A) \subseteq \mathrm{Mod}_{\mathbb{Z}[\![T]\!]}(\mathrm{Solid}_{\mathbb{Z}}) \subseteq \mathrm{Solid}_{\mathbb{Z}[T]},$$

the induced map by f gives elements in Mod_A an solid $\mathbb{Z}[T]$ structure, hence $f \in A^{+}$ by definition.
For $f \in A^{+}$, we get a map $\mathbb{Z}[T]_{\square} \to A \Rightarrow A^{\triangleright} \in \mathrm{Mod}_A \to \mathrm{Solid}_{\mathbb{Z}[T]}$
Integrally closed subring: same argument as last lecture $\square$

Theorem 8.21. *For a Huber ring $A^{\triangleright}$:*

$$\left\{\begin{matrix}\textit{rings of integral}\\ \textit{elements}\end{matrix}\right\} \longleftarrow \left\{\begin{matrix}\textit{solid analytic ring}\\ \textit{structures on } \underline{A}^{\triangleright}\end{matrix}\right\}$$
$$A^{+} \longleftarrow\!\shortmid A$$

This has a left adjoint:

$$\textit{Huber pair: } (A^{\triangleright}, A^{+}) \longmapsto A = \begin{matrix}(A^{\triangleright}, A^{+})_{\square} \\ \mathrm{Mod}_A = \left\{ M \in \mathrm{Cond}(A^{\triangleright}) \;\middle|\; \begin{matrix}\forall f \in A^{+}: \\ \underline{\mathrm{Hom}}(P,M) \xrightarrow[\sim]{1-f\cdot \mathrm{shift}} \underline{\mathrm{Hom}}(P,M)\end{matrix} \right\}\end{matrix}$$

Example 8.22.

$$(\mathbb{Z},\mathbb{Z})_{\square} = \mathbb{Z}_{\square} = (\mathbb{Z}, \mathrm{Solid}_{\mathbb{Z}})$$
$$(\mathbb{Z}[T],\mathbb{Z})_{\square} = \big(\mathbb{Z}[T], \mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}})\big)$$
$$(\mathbb{Z}[T],\mathbb{Z}[T])_{\square} = \big(\mathbb{Z}[T], \mathrm{Solid}_{\mathbb{Z}[T]}\big)$$

Remark* 8.23. One does not need to worry about this definition regarding $\mathbb{Z}$-solid modules: In definition 5.1 we defined solid $\mathbb{Z}$ modules to be the modules, for that the morphism

$$\underline{\mathrm{Hom}}(P,M) \xrightarrow{1-\sigma} \underline{\mathrm{Hom}}(P,M)$$

is an isomorphism. In fact this isomorphism suffices for the morphism

$$\underline{\mathrm{Hom}}(P,M) \xrightarrow{1-n\sigma} \underline{\mathrm{Hom}}(P,M)$$

to be an isomorphism:

On an intuitive level, for a zero sequence $m_0, m_1, \ldots$, one wants to construct $\sum_{i=j}^{\infty} m_i n^i$, as this would behave as the inverse of $1 - n\sigma$ for each entry. As we already know that for any zero sequence we can construct $\sum_{i=0}^{\infty} m_i$ via the inverse of $1-\sigma$, we can just take the zero sequence

$$(m_0, \underbrace{m_1, \dots, m_1}_{n-\text{times}}, \underbrace{m_2, \dots, m_2}_{n^2-\text{times}})$$

which maps via the inverse map of $1-\sigma$ to $\sum_{i=0}^{\infty} m_i n^i$ in the zero-th entry. To do that on a technical level, one needs to pay attention to the fact, that for every entry, one needs a different zero sequence, which looks similar. E.g. for the next entry one needs the sequence

$$(m_0, m_1, \underbrace{m_2, \dots, m_2}_{n-\text{times}}, \underbrace{m_3, \dots, m_3}_{n^2-\text{times}})$$

and so on. This way one can construct an inverse to $1-n\sigma$.
For negative n one can do just the same, alternating signs of the m_i accordingly.

Proposition 8.24. *Under the maps of Theorem 8.21:*

$$\begin{tikzcd} & \left(A^{\triangleright}, A^+\right) \arrow[r, mapsto] & A = \left(A^{\triangleright}, A^+\right)_{\square} \arrow[dll, mapsto, bend left] \\ A^+ = (A^{\triangleright}, A^+)_{\square}^+ & & \end{tikzcd}$$

Equivalently, we get a fully faithful functor

$$\begin{aligned} \{\textit{Huber pairs}\} &\hookrightarrow \{\textit{(solid) analytic rings}\} \\ \left(A^{\triangleright}, A^+\right) &\longmapsto \left(A^{\triangleright}, A^+\right)_{\square} \end{aligned}$$

*Proof**. [46] The argument for that can be looked up in Proposition 13.16 of [CS20]. Here is a rough sketch:

faithful faithful: Consider two Huber pairs (A, A^+) and (B, B^+). By definition maps of analytic rings $(A, A^+)_{\square} \to (B, B^+)_{\square}$ embed into maps $\underline{A} \to \underline{B}$. As A and B are metrizable these correspond to maps $A \to B$.
We need to show that a map $A \to B$ sends $A^+ \to B^+$ iff it induces a map of analytic rings $f : (A, A^+)_{\square} \to (B, B^+)_{\square}$. Forward direction is clear. For the backward direction we want to show, that for any element $g = f(a)$ for $a \in A^+$ lies in B^+. From composing with the map $(\mathbb{Z}[T], \mathbb{Z}[T]) \to (A, A^+)$, we can assume wlog. $(A, A^+) = (\mathbb{Z}[T], \mathbb{Z}[T])$. From lecture 7 now follows:

$$\begin{aligned} &\mathbb{Z}((T^{-1})) \otimes_{(\mathbb{Z}[T], \mathbb{Z})_{\square}} (B, B^+)_{\square} \\ &\quad = \underbrace{\left(\mathbb{Z}((T^{-1})) \otimes_{(\mathbb{Z}[T], \mathbb{Z})_{\square}} (\mathbb{Z}[T], \mathbb{Z}[T])_{\square} \right)}_{=0} \otimes_{(\mathbb{Z}[T], \mathbb{Z}[T])_{\square}} (B, B^+)_{\square} = 0 \end{aligned}$$

[46] Thanks to Ferdinand's help!

As

$$\varinjlim_{\substack{\text{filtered}\\ \text{fin.gen.}\\ R\subseteq B^+}} \left(\mathbb{Z}((T^{-1}))\otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square}(B,R)_\square\right) = \mathbb{Z}((T^{-1}))\otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square}(B,B^+)_\square$$

and asking for an algebra to be 0 is the same as asking $1 = 0$, there is a finitely generated $R \subseteq B^+$, s.t.

$$\mathbb{Z}((T^{-1}))\otimes_{(\mathbb{Z}[T],\mathbb{Z})_\square}(B,R)_\square = 0$$

Using the resolution

$$0 \to \prod_{\mathbb{N}} \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}[T] \xrightarrow{1-\text{shift}\otimes T} \prod_{\mathbb{N}} \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}[T] \to \mathbb{Z}((T^{-1})) \to 0$$

which is the same as

$$0 \to P^\square \otimes_{\mathbb{Z}}\mathbb{Z}[T] \xrightarrow{1-\text{shift}\otimes T} P^\square \otimes_{\mathbb{Z}}\mathbb{Z}[T] \to \mathbb{Z}((T^{-1})) \to 0$$

As P is flat, by applying base change twice (first with $R[g]$ and then with $\underline{B}$) this induces the isomorphism

$$(P\otimes_{\mathbb{Z}} R)^{R_\square}\otimes_{R_\square}\underline{B} \xrightarrow{1-\text{shift}\otimes g} (P\otimes_{\mathbb{Z}} R)^{R_\square}\otimes_{R_\square}\underline{B}$$

which is the same as

$$\prod_{\mathbb{N}}\mathbb{Z}\otimes_{R_\square}\underline{B} \xrightarrow{1-\text{shift}\otimes g} \prod_{\mathbb{N}}\mathbb{Z}\otimes_{R_\square}\underline{B}. \tag{4}$$

Now choose a ring of definition $B_0 \subseteq B$ with ideal of definition $I \subseteq B_0$. We want to show, that any g, such that (4) is an iso, is integral over $R + I$. As $R + I$ is a subset of B^+ by definition, and B^+ is integrally closed, it follows that $g \in B^+$ and we are done. Now consider[47]

$$\begin{aligned}\prod_{\mathbb{N}} R\otimes_{R_\square} B \twoheadrightarrow \prod_{\mathbb{N}} R\otimes_{R_\square} B/I &= \varinjlim_{\substack{\text{fin.gen.}\\ M\subseteq B/I}} \prod_{\mathbb{N}} R\otimes_{R_\square} M\\ &= \varinjlim \prod_{\mathbb{N}} M \hookrightarrow \prod_{\mathbb{N}} B/I\end{aligned}$$

Now we have an injective map $\phi : \prod_{\mathbb{N}} R\otimes_{R_\square} B/I \hookrightarrow \prod_{\mathbb{N}} B/I$ whose image consists of elements $(b_0, b_1, \dots)$, s.t. the submodule of B/I generated by $\{b_0, \dots\}$ is finitely generated. Now we can see, that $(1, 0, 0, \dots)$ is in the image, as it is already in $\prod_{\mathbb{N}} R$. Taking the preimage of this element under

$$\prod_{\mathbb{N}}\mathbb{Z}\otimes_{R_\square} B/I \xrightarrow{1-\text{shift}\otimes g} \prod_{\mathbb{N}}\mathbb{Z}\otimes_{R_\square} B/I \to \varprojlim_n \prod_{\mathbb{N}} B$$

[47]Note that the colimit works so well, as B/I is discrete.

and then mapping it through ϕ, we see that the submodule of B/I generated by $(g^0, g^1, g^2, \dots)$ is finitely generated. Hence we get a relation

$$g^i + \sum_{n=0}^{i-1} b_i g^i = 0 \mod B/I$$

which shows as desired, that g is integral over $R + I$.

□

9 Lecture 9 (9/24)

Recall:

Suppose $R^{\triangleright} \in \mathrm{CAlg}(\mathrm{Solid}_{\mathbb{Z}})$ and let $R^{\circ} \subseteq R^{\triangleright}(*)$ be the set of power bounded elements. For any $R^{+} \subseteq R^{\circ}$, one gets an analytic ring structure on $R^{\triangleright}$, denoted by $(R^{\triangleright}, R^{+})_{\square}$ with

$$\mathrm{Mod}_{(R^{\triangleright},R^{+})_{\square}} = \left\{ M \in \mathrm{Mod}_{R^{\triangleright}} \;\Big|\; \overset{\forall f \in R^{+}}{\underline{\mathrm{Hom}}(P,M) \xrightarrow[\sim]{1-\sigma f} \underline{\mathrm{Hom}}(P,M)} \right\}$$

Remark 9.1.

- $1 \in R^{+} \Rightarrow M \in \mathrm{Mod}_{R}$ is in $\mathrm{Solid}_{\mathbb{Z}}$.
- $R^{+} \subseteq R^{\circ} \Rightarrow R^{\triangleright} \in \mathrm{Mod}_{R}$.
- No loss in assuming $R^{+} \subseteq R(*)$ is an integrally closed subring with $R^{\circ\circ} \subseteq R^{+}$.

Example 9.2. If $R^{\triangleright}$ condensed Huber ring we get that integrally closed subrings R^{+} of the form $R^{\circ\circ} \subseteq R^{+} \subseteq R^{\circ}$ are the same thing as open integrally closed subrings $R^{+} \subseteq R^{\circ}$ which are the same as the R^{+} in Huber's theory. This notation will from now on be used interchangeably.

9.1 Localization

Remark 9.3. Let R be a solid ring and consider the pair (R, R^{+}). Then, for the pair $(\underline{R(*)}, R^{+})$ the following equation holds:

$$\mathrm{Mod}_{(R,R^{+})_{\square}} = \mathrm{Mod}_{R}\left(\mathrm{Mod}_{(\underline{R(*)},R^{+})_{\square}}\right)$$

This holds, as being a solid module is being calculated in the discrete underlying ring. Hence it suffices to build our theory over discrete cases as we only need to put R-module structure over the discrete case for arbitrary R.

Analogy: Let R be a commutative ring, and the usual derived category of R-modules $D(R)$ localizes over $\mathrm{Spec}(R)$.
$\mathrm{Spec}(R)$ is the set of prime ideals and has the following properties:

1. has a basis of qc. opens closed under finite intersections, which are the so-called "distinguished opens", and which are of the form $U(f) = \{\mathfrak{p} : f \notin \mathfrak{p}\}$.
2. has a structure sheaf $\mathcal{O}(U(f)) = R\left[\frac{1}{f}\right]$.
3. $U(f) \cong \mathrm{Spec}(R[\frac{1}{f}])$ matches up distinguished opens.
4. for U distinguished open $\rightsquigarrow D(\mathcal{O}(U))$
   for $U \supseteq V \rightsquigarrow$ base change $D(\mathcal{O}(U)) \to D(\mathcal{O}(V))$

Theorem 9.4 (Lurie)**.** *This presheaf is a sheaf of ∞-categories.*

Warning 9.5. In the setting that is going to be discussed next, we will only be able to get a sheaf on derived level and not on the level of abelian categories, as localization maps will not be flat in general.

Notation 9.6. As known, we can assign to a Huber pair (A, A^+) the analytic ring $(A, A^+)_\square$. Then the solid derived modules of this analytic ring will be denoted as $D(A, A^+)_\square$.

Properties for spv: Let R be a discrete Huber pair (i.e. R commutative ring and, $R^+ \subseteq R$ int. closed subring). Then the claim is, that $D(R, R^+)_\square$ localizes on $\mathrm{Spv}(R, R^+)$, where

$$\mathrm{Spv}(R, R^+) = \left\{ v : R \to \Gamma \cup \{0\} \quad \middle| \quad \begin{smallmatrix} v(gf)=v(g)v(f) \\ v(f+g)\leq \max\{v(f),v(g)\} \\ v(0)=0, v(1)=1 \\ v(R^+)\leq 1 \end{smallmatrix} \right\} / \sim$$

By Γ we denote an ordered abelian group written multiplicatively and the equivalence relation $\sim$ is given by the usual equivalence of valuations[48]. But this is also just the same thing as

$$\left\{(A, \mathfrak{p}) \mid A \subseteq k(\mathfrak{p}) \text{ valuation domain}, \mathrm{im}(R^+) \subseteq A\right\}$$

Definition 9.7. We now define for a discrete Huber pair R:

1. *Basis of qc opens:* "rational opens", closed under finite intersections, for $g, f_i \in R$:
$$U\left(\frac{f_1, \ldots, f_n}{g}\right) := \{v \mid v(g) \neq 0, v(f_i) \leq v(g) \quad \forall i\}$$
2. *Structure sheaves:*
$$\mathcal{O}\left(U\left(\frac{f_1, \ldots, f_n}{g}\right)\right) = R\left[\frac{1}{g}\right]$$
$$\mathcal{O}^+\left(U\left(\frac{f_1, \ldots, f_n}{g}\right)\right) = \overline{R^+\left[\frac{f_1}{g}, \ldots, \frac{f_n}{g}\right]}$$
where $\overline{(-)}$ denotes the integral closure.
3. *Matching rational opens:*
$$U\left(\frac{f_1, \ldots, f_n}{g}\right) \cong \mathrm{Spv}\left(\mathcal{O}(U), \mathcal{O}^+(U)\right)$$
4. U rational open $\rightsquigarrow D\left(\mathcal{O}(U), \mathcal{O}^+(U)\right)_\square$
$U \supseteq V \rightsquigarrow$ pullback $D\left(\mathcal{O}(U), \mathcal{O}^+(U)\right)_\square \to D\left(\mathcal{O}(V), \mathcal{O}^+(V)\right)_\square$ [49]

Theorem 9.8. *This presheaf is a sheaf of ∞-categories.*

As mentioned earlier, this is not true on abelian level in contrast to the classical case, as pullback is not (but almost) t-exact.

[48]Note that the difference to Spa is, that we don't require the valuations to be continuous. One could describe Spv as Spa where we take the discrete topology on R.

[49]This comes from a map of analytic rings $(\mathcal{O}(U), \mathcal{O}^+(U))_\square \to (\mathcal{O}(V), \mathcal{O}^+(V))_\square$.

9.2 Sketch of proof of Lurie's theorem

From here on we are going to give a proof of the classical theorem (i.e. we are proving Zariski descent for $D(R)$). Here we are working with the site of distinguished opens in $\mathrm{Spec}(R)$ with the open cover topology.

Lemma 9.9. *To show that a presheaf has sheaf conditions on* $\mathrm{Spec}(R)$*, it suffices to check sheaf conditions for covers of the form: for* $f \in \mathcal{O}(U)$

$$\begin{tikzcd} U(f) \arrow[dr] & & U(1-f) \arrow[dl] \\ & U & \end{tikzcd}$$

Proof. Consider $f_1, \dots, f_n \in \mathcal{O}(U)$ s.t. $\exists x_1, \dots, x_n \in \mathcal{O}(U)$, $x_1 f_1 +, \dots, x_n f_n = 1$, which form a general cover $\{U(f_i)\}_{i \in I}$ covering U. But this cover is refined by a cover $\{U(f_i x_i)\}_{i \in I}$ hence we can assume w.l.o.g. $f_1 + \dots + f_n = 1$. Now we induct on n, i.e. we cover U by $U(f_1 + \dots + f_{n-1})$ and $U(f_n)$. Now to cover $U(f_1 + f_{n-1})$ we do the same step.
The induction idea is the following: let $f_1 + f_2 + f_3 = 1$; then consider, that $U(f_1 + f_2) \cup U(f_3)$ satisfies sheaf conditions. We want to show from here, that the cover $U(f_1) \cup U(f_2) \cup U(f_3)$ satisfies sheaf conditions. As it is a split cover on $U(f_3)$, we are done on that side. Now we need to show that sheaf conditions on $U(f_1) \cup U(f_2)$. We can do that algebraically as

$$\begin{aligned} U(f_1 + f_2) &= \mathrm{Spec}\Big(R\Big[\frac{1}{f_1 + f_2}\Big]\Big) \\ &= \mathrm{Spec}\Big(R\Big[\frac{1}{f_1 + f_2}\Big]\Big[\frac{1}{f_1/(f_1 + f_2)}\Big]\Big) \cup \mathrm{Spec}\Big(R\Big[\frac{1}{f_1 + f_2}\Big]\Big[\frac{1}{f_2/(f_1 + f_2)}\Big]\Big) \\ &= \mathrm{Spec}\Big(R\Big[\frac{1}{f_1 + f_2}\Big]\Big[\frac{1}{f_1}\Big]\Big) \cup \mathrm{Spec}\Big(R\Big[\frac{1}{f_1 + f_2}\Big]\Big[\frac{1}{f_2}\Big]\Big) \\ &= \Big(U(f_1 + f_2) \cap U(f_1)\Big) \cup \Big(U(f_1 + f_2) \cap U(f_2)\Big) \end{aligned}$$

□

Now we are done if we check that the sheaf condition holds for

$$\begin{tikzcd} U(f) \arrow[dr] & & U(1-f) \arrow[dl] \\ & U & \end{tikzcd}$$

Hence we need to check whether

$$\begin{tikzcd} & D(R[\frac{1}{f}]) \arrow[dr] & \\ D(R) \arrow[ur] \arrow[dr] & & D(R[\frac{1}{f(1-f)}]) \\ & D(R[\frac{1}{1-f}]) \arrow[ur] & \end{tikzcd}$$

is a pullback of ∞−categories. This is the same as claiming that

$$D(R) \xrightarrow{\sim} D\Big(R\Big[\frac{1}{f}\Big]\Big) \times_{D\left(R\left[\frac{1}{f(1-f)}\right]\right)} D\Big(R\Big[\frac{1}{1-f}\Big]\Big)$$

Proof. Note: each base change functor has a right adjoint
$\Rightarrow D(R) \to D\big(R\big[\frac{1}{f}\big]\big) \times_{D\left(R\left[\frac{1}{f(1-f)}\right]\right)} D\big(R\big[\frac{1}{1-f}\big]\big)$ does too:

$$\begin{aligned} D(R) &\leftarrow D\Big(R\Big[\frac{1}{f}\Big]\Big) \times_{D\left(R\left[\frac{1}{f(1-f)}\right]\right)} D\Big(R\Big[\frac{1}{1-f}\Big]\Big) \\ M \times_{M\left[\frac{1}{f}\right]} N &\leftarrow\!\shortmid (M, N, \alpha) \end{aligned}$$

with $\alpha : M\big[\frac{1}{1-f}\big] \cong N\big[\frac{1}{f}\big]$. We need to check, that the unit and counit are isomorphisms. Exemplary For $M \in D(R)$ we need to show for the unit:

$$\begin{tikzcd} M \arrow[r] \arrow[d] & M[\frac{1}{f}] \arrow[d] \\ M[\frac{1}{1-f}] \arrow[r] & M[\frac{1}{f(1-f)}] \end{tikzcd}$$

is a pullback. This follows from: $N \in D(R)$, s.t. $N\big[\frac{1}{f}\big] = N\big[\frac{1}{1-f}\big] = 0 \Rightarrow N = 0$
Analog for counit. □

Remark 9.10. Formally, what is used is the following:

1. each base change is a localization (right adjoint fully faithful)
2. Those localizations $M \mapsto M\big[\frac{1}{f}\big], M\big[\frac{1}{1-f}\big]$
3. if $M \mapsto 0$ on each element of the cover, then $M = 0$.

9.3 Proof of solid analog

Now we want to proof the solid analog:
We consider the site of rational opens $U \subseteq \mathrm{Spv}(R, R^+)$ with the Grothendieck topology of open covers.

Lemma 9.11. *To show the sheaf condition on this site, we only need to show it for the empty cover and for covers of the following form: for rational open: $U \subseteq \operatorname{Spv}(R, R^+)$ and all $f \in \mathcal{O}(R)$*

$$\begin{tikzcd} U(\frac{f}{1}) \arrow[dr] & & U(\frac{1}{f}) \arrow[dl] \\ & U & \end{tikzcd} \quad \& \quad \begin{tikzcd} U(\frac{1}{f}) \arrow[dr] & & U(\frac{1}{1-f}) \arrow[dl] \\ & U & \end{tikzcd}$$

Proof. Huber showed, that every cover can be refined by a cover of the form

$$\left\{U\left(\frac{f_1, \ldots, \hat{f_i}, \ldots, f_n}{f_i}\right)\right\}_{i \in I}$$

where there is a collection $f_1, \ldots, f_n$ generating the unit ideal. From here we use the same argument as in the classical proof. □

Now the theorem reduces to (R, R^+) discrete Huber pair, $f \in R$, then we need that

$$D(R, R^+)_\square \xrightarrow{\sim} D\left(R\left[\frac{1}{f}\right], \widetilde{R^+\left[\frac{1}{f}\right]}\right)_\square \times_{D\left(R\left[f, \frac{1}{f}\right], \widetilde{R^+\left[f, \frac{1}{f}\right]}\right)_\square} D(R[f], \widetilde{R^+[f]})_\square$$

and

$$D(R, R^+)_\square \xrightarrow{\sim} D\left(R\left[\frac{1}{f}\right], \widetilde{R^+\left[\frac{1}{f}\right]}\right)_\square \times_{D\left(R\left[\frac{1}{(1-f)f}\right], \widetilde{R^+\left[f, \frac{1}{(1-f)f}\right]}\right)_\square} D\left(R\left[\frac{1}{1-f}\right], \widetilde{R^+\left[\frac{1}{1-f}\right]}\right)_\square$$

For $f \in R$:

1. For $U\left(\frac{f}{1}\right)$
$$D(R, R^+) \to D(R, \widetilde{R^+[f]})$$
is just T–solidification for $\mathbb{Z}[T] \xrightarrow{T \mapsto f} R$, hence it is $\underline{RHom}_{\mathbb{Z}[T]}(\mathbb{Z}((T^{-1}))/Z[T][-1], -)$

2. For $U\left(\frac{1}{f}\right)$
$$D(R, R^+) \to D\left(R\left[\frac{1}{f}\right], \widetilde{R^+\left[\frac{1}{f}\right]}\right)$$
is the functor, that first inverts f, then T–solidifies for $\mathbb{Z}[T] \xrightarrow{T \mapsto \frac{1}{f}} R\left[\frac{1}{f}\right]$

Base change is given by

$$\underline{RHom}_{\mathbb{Z}[T]}(\mathbb{Z}[\![T]\!]/\mathbb{Z}[T][-1], -)$$

where $\mathbb{Z}[T] \xrightarrow{T \mapsto f} R$. By lecture 7, this the localization killing the idempotent object $\mathbb{Z}[\![T]\!] \in D(\mathbb{Z}[T], \mathbb{Z})_\square = D(\mathrm{Mod}_{\mathbb{Z}[T]}(\mathrm{Solid}_{\mathbb{Z}}))$. (Automatically inverts f because anything T–torsion is a $\mathbb{Z}[T]$–module).

Now in the solid setting, the conditions of 9.10 are satisfied:

2. The localizations commute, as they are both given by $\underline{R\mathrm{Hom}}$ out of objects and these commute from tensor-hom adjunction.

3. translates for $\left(U = U\left(\frac{1}{f}\right) \cup U\left(\frac{f}{1}\right)\right)$ to $\mathbb{Z}[\![T]\!] \otimes^{\square}_{\mathbb{Z}[T]} \mathbb{Z}((T^{-1})) = \mathbb{Z}[\![T,U]\!]/(1-UT) = 0$.
   For the last equation one can use the geometric series.
   Similary we need that $\mathbb{Z}[\![T]\!] \otimes_{\mathbb{Z}[T]} \mathbb{Z}[\![1-T]\!] = \mathbb{Z}[\![T,U]\!]/(1-(U+T)) = 0$

The interpretation of a covering of the form $U = U = U\left(\frac{1}{f}\right) \cup U\left(\frac{f}{1}\right)$ is the following: The left side is localizing away from open unit disk and the right side is localizing to the closed unit disk.

Corollary 9.12. *R solid ring, $R^+ \subseteq R^\circ \subseteq R(*) \Rightarrow D(R,R^+)$ localizes on $\mathrm{Spv}(R(*),R^+)$ via*

$$U \mapsto \mathrm{Mod}_R\left(D(\mathcal{O}(U),\mathcal{O}^+(U))\right)$$

for U rational open

Remark 9.13. In fact, $D(R,R^+)$ actually lies on the closed subset $\mathrm{Spv}(R(*),R^+,R^{\circ\circ})$.[50] On the other hand Huber studies

$$\begin{aligned}\mathrm{Spa}(R,R^+) &= \{\text{continuous valuations on } R, \text{ which are } \leq 1 \text{ on } R^+\}\\ &= \text{subset of } \mathrm{Spv}(R(*),R^+,R^{\circ\circ}) \text{ s.t. } f \in R^{\circ\circ}\\ &\Rightarrow \forall \gamma \in \Gamma, \exists N \in \mathbb{N} \text{ s.t. } v(f^N) < \gamma\end{aligned}$$

$D(R,R^+)$ does live over $\mathrm{Spa}(R,R^+) \subseteq \mathrm{Spv}(R,R^+)$ but $\exists$ retraction

$$\mathrm{Spv}(R,R^+) \xrightarrow{r} \mathrm{Spa}(R,R^+)$$

realizing $\mathrm{Spa}(R,R^+)$ as a quotient $\Rightarrow$ sheaf of categories on $\mathrm{Spa}(R,R^+)$.

In general one gets more flexibility for localization using this machinery than in Huber's machinery as the kind of extra things one can get are $\mathbb{Z}_p[\![x]\!]\left[\frac{1}{p}\right]$ which arises from the structure sheaf of the general setting but not in the structure sheaf of Huber's theory.

[50]Here with this information we add the condition , that for $f \in R^{\circ\circ} \Rightarrow v(f) < 1$

10 Lecture 10 (10/24)

Warning 10.1 (Conflict of Terminology)**.** All adic spaces give rise to analytic spaces in our sense. But within the theory of adic spaces, there is a subclass of "analytic" ones. We will call them "Tate" in this lecture.

Definition 10.2. A Huber ring A is called **Tate**, if it has a topological nilpotent unit (called "pseudo-uniformizer").

Example 10.3. $\mathbb{Q}_p \ni p$ or any non-archimedean local field or any Huber ring over a local field.

Definition 10.4. An adic space X is **Tate** (usually analytic) if it is covered by $\mathrm{Spa}(A, A^+)$ with A Tate ($\iff$ the (completed) residue fields at all points of X are not discrete (i.e. are non-archimedean fields))

Intuition:

Schemes	$\subseteq$	Formal Schemes	$\subseteq$	Adic Spaces	$\supseteq$	Tate Adic Spaces
$\mathrm{Spec}(\mathbb{F}_p)$		$\mathrm{Spf}(\mathbb{Z}_p)$		$\mathrm{Spa}(\mathbb{Z}_p, \mathbb{Z}_p)$		$\mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$

10.1 Structure of Tate Huber rings A

Consider a Huber ring A with $\pi \in A$ topologically nilpotent unit and $A_0 \subseteq A$ a ring of definition, hence w.l.o.g. $\pi \in A_0$ as this is true for a sufficiently high power. Then A_0 carries the π-adic topology. Then one can turn A into a Banach algebra with unit ball A_0 via

$$
\begin{aligned}
&|f| = 2^{-n}, \text{ where } n = \sup\{n \mid \pi^{-n} f \in A_0\} \\
&|\pi| = \frac{1}{2}
\end{aligned}
$$

Hence we get an equivalence (without the 0 ring on both sides)

$$
\left\{ \begin{array}{l} \text{complete Tate} \\ \text{Huber rings} \end{array} \right\} \xrightarrow{\sim} \left\{ \begin{array}{l} \text{complete ultra-metric Banach rings B} \\ \text{admitting } \pi \in B \text{ with } 0 < |\pi| < 1 \\ \text{and continuous maps } |\pi||\pi^{-1}| = 1 \end{array} \right\}
$$

Definition 10.5. (A, A^+) is **sheafy** if $\mathcal{O}^{\mathrm{Huber}}_{\mathrm{Spa}(A,A^+)}$ is a sheaf.

Theorem 10.6 (Kedlaya-Liu)**.** *Assume A Tate and (A, A^+) sheafy. Then*

$$
\begin{aligned}
\mathrm{FinProj} &\xrightarrow{\sim} \mathrm{VB}\big(\mathrm{Spa}(A, A^+), \mathcal{O}^{\mathrm{Huber}}_{\mathrm{Spa}(A,A^+)}\big) \\
M &\mapsto M \otimes_A \mathcal{O}^{\mathrm{Huber}}_{\mathrm{Spa}(A,A^+)}
\end{aligned}
$$

is an equivalence, i.e. one can glue finite projective modules. Here by VB *we denote finite locally free sheaves of modules.*

Of this theorem, there is also a version for (pseudo-)coherent[51] modules.

Definition* 10.7. For Tate Huber pairs R we now define the structure sheaf analogous to Definition 9.7:

$$\mathcal{O}\left(U\left(\frac{f_1,\ldots,f_n}{g}\right)\right)=R\left[\frac{1}{g}\right]^{L\frac{f_1}{g},\ldots,\frac{f_n}{g}\square}$$

If this lives in degree 0, one can also define a sheaf of integral elements which is the same as the one in Definition 9.7:

$$\mathcal{O}^+\left(U\left(\frac{f_1,\ldots,f_n}{g}\right)\right)=\overline{R^+\left[\frac{f_1}{g},\ldots,\frac{f_n}{g}\right]}$$

Remark* 10.8. Note, that this agrees with Definition 9.7, as solidifying in a discrete ring doesn't change anything (elements in $\underline{\mathrm{Hom}}(P,R)$ are sequences which are constant at some point). This definition is also parallel to Huber's theory, as solidifying can be thought of a completion in the classical case, which is what we are going to defend in the following section.

Goal: Give a proof of Theorem 10.6 using $D(-)^\square$.
Outline:

1. Relation between $\mathcal{O}^{\mathrm{Huber}}$ and $\mathcal{O}$.
2. In general $U=U\left(\frac{f_1,\ldots,f_n}{g}\right)\mapsto D^{\mathrm{pcoh}}_{\geq 0}(\mathcal{O}(U))\subseteq D\left((\mathcal{O}(U),\mathcal{O}^+(U))_\square\right)$ is a sheaf of ∞−categories

Reference: Andreychev.

1. Relation between $\mathcal{O}^{\mathrm{Huber}}$ & $\mathcal{O}$

Proposition 10.9. *Let $U=U\left(\frac{f_1,\ldots,f_n}{g}\right)\subseteq \mathrm{Spa}(A,A^+)$ be a rational subset (so $(f_1,\ldots,f_n)\subseteq A$ is open, but once one has a pseudo-uniformizer it follows $(f_1,\ldots,f_n)=A$). Then*

$$\mathcal{O}^{\mathrm{Huber}}(U)=A\langle T_1,\ldots,T_n\rangle/\overline{(gT_1-f_1,\ldots,gT_n-f_n)}$$
$$\mathcal{O}(U)=A\langle T_1,\ldots,T_n\rangle/^{\mathbb{L}}(gT_1-f_1,\ldots,gT_n-f_n)$$

where for a (condensed) ring R and $x_1,\ldots,x_n\in R$,

$$R/^{\mathbb{L}}(x_1,\ldots,x_n)=R\otimes^L_{\mathbb{Z}[x_1,\ldots,x_n]}\mathbb{Z}$$
$$=\left(R\to\bigoplus_{i=1}^n R\to\cdots\to\bigwedge^i\left(\bigoplus_{l=1}^n R\right)\to\cdots\to\bigoplus_{l=1}^n R\xrightarrow{(x_i)_i}R\right)\textit{Koszul complex}$$

[51] We will define pseudo-coherence in 10.16 again.

Sketch of Proof. Start with $A\left[\frac{1}{g}\right] = A[T_1, \dots, T_n]/^{\mathbb{L}}(gT_1 - f_1, \dots, gT_n - f_n)$ and $T_1, \dots, T_n$-solidify. This is an exact operation on derived categories. As $A\left[\frac{f_1,\dots,f_n}{g}\right]$ is calculated by the Koszul complex, which is a finite resolution of finite direct sums of A, one just has to understand what happens when solidifying A. As we can write $A = \varprojlim A/(\pi^n)$ and adjoining plus solidifying T_i commutes with limits, we get

$$A[T_1, \dots, T_n]^{LT_1,\dots,T_n\square} = A\langle T_1, \dots, T_n\rangle$$

□

Definition 10.10. In a Topos T a map $f : X \to Y$ is quasicompact, if for all quasicompact objects $Z \in T$ the fibre product $Y \times_X Z$ is quasicompact.

Remark 10.11. The inclusion $\mathrm{CondSet}^{\mathrm{qs}} \subseteq \mathrm{CondSet}$ has a left adjoint

$$\mathrm{CondSet} \to \mathrm{CondSet}^{\mathrm{qs}}$$
$$X \mapsto X^{\mathrm{qs}}$$

which can be imagined like "universal Hausdorff quotient". This commutes with finite products and hence preserves algebraic structures (like being a group, ring,. . .). Concretely if $X = \tilde{X}/R$ with $\tilde{X}$ quasi-separated & $R \subseteq \tilde{X} \times \tilde{X}$ equivalence relation. Then $\exists$ minimal quasi-compact injection $\overline{R} \subseteq \tilde{X} \times \tilde{X}$ that is also an equivalence relation with $R \subseteq \overline{R}$, then $X^{\mathrm{qs}} = \tilde{X}/\overline{R}$.

Corollary 10.12. $\mathcal{O}^{\mathrm{Huber}}(U) = (\pi_0\mathcal{O}(U))^{\mathrm{qs}}$

Lemma 10.13. *Let X be any sequential topological space. Then*

$$\{\textit{closed subsets of } X\} \xrightarrow{\sim} \left\{\textit{quasi-compact injection into} \underline{X}\right\}$$
$$Z \mapsto \underline{Z} \subseteq \underline{X}$$

Proof of Corollary.

$$\begin{aligned}\pi_0\mathcal{O}(U) &= A\langle T_1, \dots, T_n\rangle/(gT_1 - f_1, \dots, gT_n - f_n)\\ (\pi_0\mathcal{O}(U))^{\mathrm{qs}} &= A\langle T_1, \dots, T_n\rangle/\overline{(gT_1 - f_1, \dots, gT_n - f_n)}\\ &= \mathcal{O}^{\mathrm{Huber}}(U)\end{aligned}$$

□

Lemma 10.14. *Let A be a Tate Huber ring, $f \in A$. Assume*

$$0 \longrightarrow A \longrightarrow A\left\langle\frac{f}{1}\right\rangle \oplus A\left\langle\frac{1}{f}\right\rangle \longrightarrow A\left\langle\frac{f}{1}, \frac{1}{f}\right\rangle \longrightarrow 0 \tag{5}$$

is exact[52] *(e.g. A is sheafy). Then*

$$A\left\langle\frac{f}{1}\right\rangle = A\langle T\rangle/^{\mathbb{L}}(T - f)$$

[52] E.g. $0 \to \mathcal{O}^{\mathrm{Huber}}(X) \to \mathcal{O}^{\mathrm{Huber}}(\{|f| \le 1\}) \oplus \mathcal{O}^{\mathrm{Huber}}(\{|f| \ge 1\}) \to \mathcal{O}^{\mathrm{Huber}}(\{|f| = 1\}) \to 0$

$$A\left\langle \frac{1}{f} \right\rangle = A\langle T\rangle /^{\mathbb{L}} (1 - Tf)$$

Proof. To see:

$$A\langle T\rangle \xrightarrow[\text{(or } 1-Tf)]{T-f} A\langle T\rangle$$

is injective with closed image. Using (5) this question can be analyzed over $A\langle \frac{f}{1}\rangle$ and $A\langle \frac{1}{f}\rangle$. In other words, assume $|f| \leq 1$ or $|f| \geq 1$. But this can be checked by hand. □

Theorem 10.15 (essentially Kedlaya)**.**

$$(A, A^+) \textit{ sheafy} \iff \mathcal{O}^{\text{Huber}} = \mathcal{O}$$
$$\left(\iff \forall U \textit{ rational}, \mathcal{O}(U) \textit{ sits in degree } 0 \textit{ and is quasi-separated}\right)$$

Proof.

"⇐" is clear.

"⇒" suffices to check good properties of $\mathcal{O}(U)$ after possibly further refinement. Also one can always refine by "simple Laurent covers"

$$\{|f| \leq 1\} \cup \{|f| \geq 1\}$$

Now this reduces to Lemma 10.14. □

2. Gluing of pseudocoherent complexes.

Definition 10.16 ([SGA_6])**.** Let R be a ring. Then $C \in D(R)$ is **pseudocoherent** if it can be represented by a complex

$$\cdots \to C_n \to \cdots \to C_1 \to C_0 \to \cdots \to C_{-n} \to \cdots \to 0 \to 0 \to \cdots$$

where each C_i is a finite projective R–module[53].
C is **perfect** if there is such a complex that is bounded.

Theorem 10.17. *Let (A, A^+) be any Huber pair with A Tate. Then the assignments for $V \subseteq \mathrm{Spa}(A, A^+)$ rational*

$$V \longmapsto \begin{cases} D^{\mathrm{pcoh}}(\mathcal{O}(V)(*)) \\ D^{\mathrm{pcoh}}_{\geq n}(\mathcal{O}(V)(*)) \\ \mathrm{Perf}(\mathcal{O}(V)(*)) \\ \mathrm{Perf}^{[a,b]}(\mathcal{O}(V)(*)) \end{cases}$$

are sheaves of ∞–categories. Here $\mathrm{Perf}^{[a,b]}(\mathcal{O}(V)(*))$ denotes non-zero entries in between degrees a and b.

[53]This roughly says that each homology is coherent.

Remark 10.18. The case $\mathrm{Perf}^{[0,0]} = \mathrm{VB}$ recovers the theorem of Kedlaya-Liu.

Outline of Argument: We know[54]

$$V \mapsto D\left((\mathcal{O}(V), \mathcal{O}^+(V))_\square\right) \supseteq D(\mathcal{O}(V)(*)) \supseteq \text{all categories above}$$

is a sheaf of ∞−categories.

This already reduces us to:
If $M \in D\left((A, A^+)_\square\right)$ s.t. over a cover, $M \otimes^L_{(A,A^+)_\square} (\mathcal{O}(U), \mathcal{O}^+(U))_\square$ lies in one of these subcategories, then so does M.

Warning 10.19. The condition $M \in D(A(*)) \subseteq D\left((A, A^+)_\square\right)$ does not globalize.

Example 10.20. Consider the Tate-elliptic curve

$$\mathbb{G}_{m,\mathbb{Q}_p} \xrightarrow{a} \mathbb{G}^{\mathrm{ad}}_{m,\mathbb{Q}_p}/p^{\mathbb{Z}}$$

Assume $\mathrm{Spa}(A, A^+)$ is a large open subset of $\mathbb{G}^{\mathrm{ad}}_{m,\mathbb{Q}_p}/p^{\mathbb{Z}}$. Then

$$a_! \mathcal{O}_{\mathbb{G}^{\mathrm{ad}}_{m,\mathbb{Q}_p}}\Big|_{\mathrm{Spa}(A,A^+)} \in D(A, A^+)_\square$$

and is locally discrete but globally not discrete in $D\left(A(*)\right)$.

10.2 Pseudocoherent and Nuclear modules

Instead we will do the following to replace the old definition:

1. Define $D^{\mathrm{pcoh}}\left((A, A^+)_\square\right) \subseteq D\left((A, A^+)_\square\right)$ complexes represented by

$$\cdots \to C_n \to C_{n-1} \to \cdots \to C_1 \to C_0 \to \cdots \to C_{-n} \to 0 \to 0 \to \cdots$$

with all $C_i = P^\square$

$$\iff \text{bounded to the right and } \forall i \in \mathbb{Z}, \mathrm{Ext}^i(C, -) : \mathrm{Mod}(\mathrm{Ab}) \to \mathrm{Ab} \text{ commutes with filtered colimits}$$

and show that the property of lying in there globalizes. This is formal using that localizations have finite Tor-dimension.

2. Define nuclear modules $D^{\mathrm{nuc}}((A, A^+)_\square) \subseteq D((A, A^+)_\square)$ and show that this property globalizes.

3. Show that $D^{\mathrm{pcoh}}(A(*)) \subseteq D^{\mathrm{pcoh}}((A, A^+)_\square) \cap D^{\mathrm{nuc}}((A, A^+)_\square)$ is an equality.

[54] The first inclusion is by the embedding, which firstly chooses free resolutions as representatives in $D(A(*))$ and then maps $\bigoplus_I A(*) \mapsto \bigoplus_I A$. Elements in the image of that functor are called discrete.

As 1. follows from formal properties, we will now do 2.

2. Define nuclear modules and show that it globalizes:

Definition 10.21. $C \in D((A, A^+)_\square)$ is **nuclear**, if

$$\underline{\mathrm{Hom}}(P, C) \xleftarrow{\sim} C \otimes^L_{(A,A^+)_\square} \underline{\mathrm{Hom}}(P, A)$$

is an isomorphism. ("All maps from P are trace-class")

Remark 10.22.

1. Nuclear modules are generated under colimits and shift by $C_0(\mathbb{N}, A)\,(= A\langle T\rangle)$.
2. In general[55] one can generate them by $\varinjlim(Q_0 \to Q_1 \to \dots)$ where the sequence consists of trace-class morphisms, meaning they come from $Q_i^\vee \otimes Q_{i+1}$.

So over $\mathbb{Q}_p$ nuclear modules are generated by $\mathbb{Q}_p$–Banach-spaces.
Now we would need to show that this property globalizes:

Key: $\underline{\mathrm{Hom}}(P, -)$ commutes with localizations. But this is okay, as localizations are also given by some $\underline{\mathrm{Hom}}(-, -)$.

3. Show that $D^{\mathrm{pcoh}}(A(*)) \subseteq D^{\mathrm{pcoh}}((A, A^+)_\square) \cap D^{\mathrm{nuc}}((A, A^+)_\square)$ is an equality.

Sketch of Proof. (The very formal proof of this assertion can be found in [CS22] and uses some homotopy theory.)
Assume $C \in D^{\mathrm{pcoh}}((A, A^+)_\square) \cap D^{\mathrm{nuc}}((A, A^+)_\square)$ and consider the map $\alpha : P \to C$ of the following form:

$$\begin{tikzcd}
C: & \dots \arrow[r] & P \arrow[r] & P \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \dots \\
& \dots \arrow[r] & 0 \arrow[r] & P \arrow[r] \arrow[u, "id"] & 0 \arrow[r] & 0 \arrow[r] & \dots
\end{tikzcd}$$

As this is surjective on H_0 and C is nuclear, we get the following commuting diagram:

$$\begin{tikzcd}
H_0(\underline{R\mathrm{Hom}}(P, P))(*) \arrow[d, "\alpha"] & H_0(P \otimes^L_{(A,A_+)_\square} \underline{R\mathrm{Hom}}(P, A))(*) =: M \arrow[l] \arrow[d, twoheadrightarrow, "\alpha"] \\
H_0(\underline{R\mathrm{Hom}}(P, C))(*) & H_0(C \otimes^L_{(A,A_+)_\square} \underline{R\mathrm{Hom}}(P, A))(*) =: N \arrow[l, "\simeq"']
\end{tikzcd}$$

So by taking $\alpha \in H_0(\underline{R\mathrm{Hom}}(P, C))(*)$, identifying it in N and finding a preimage in M, we get a trace class $g \in H_0(\underline{R\mathrm{Hom}}(P, P))(*)$, with the following commuting diagram:

$$\begin{tikzcd}
P \arrow[d, dotted, "g"'] \arrow[dr, "\alpha"] & \\
P \arrow[r, "\alpha"] & C
\end{tikzcd}$$

[55] I.e. for more general (non-Tate) analytic rings.

Now this factors uniquely through

$$\begin{tikzcd}
P \arrow[d, "g"'] \arrow[drr, "\alpha"] & & \\
P \arrow[rr, "\alpha"] \arrow[d] & & C \\
\mathrm{Cone}(P \xrightarrow{1-g} P) \arrow[urr, dotted] & &
\end{tikzcd}$$

Then by inducting on the degree, it is enough to show, that $\mathrm{Cone}(P \xrightarrow{1-g} P)$ lies in $\mathrm{Perf}(A(*))$, but this is classical trace norm theory [56]. □

[56]"This is a step where you actually feel like you're doing somewhat a little bit of Analysis."

11 Lecture 11 (11/24)

Recall

For a pair $(A^\triangleright, A^+)$ with a solid ring $A^\triangleright$ and a subring of $A^\circ \subseteq A^\triangleright(*)$ we can associate this to a certain analytic ring

$$\mathrm{Mod}_{(A^\triangleright, A^+)_\square} \subseteq \mathrm{Mod}_{A^\triangleright}(\mathrm{Solid}_\mathbb{Z})$$

by enforcing the elements of A^+ to become solidified variables. We also defined

$$D(A^\triangleright, A^+)_\square \subseteq D\big(\mathrm{Mod}_{A^\triangleright}(\mathrm{Solid}_\mathbb{Z})\big)$$

as the full subcategory of $D\big(\mathrm{Mod}_{A^\triangleright}(\mathrm{Solid}_\mathbb{Z})\big)$, where all homologies of elements $M \in D(A^\triangleright, A^+)_\square$ live in $\mathrm{Mod}_{(A^\triangleright, A^+)}$. For both inclusions we have left adjoints. Furthermore we have a compact projective generator

$$\underbrace{\prod\nolimits_\mathbb{N} \mathbb{Z}}_{\text{flat in } \mathbb{Z}_\square} \otimes_{\mathbb{Z}_\square} A^\triangleright \in \mathrm{Mod}_{A^\triangleright}(\mathrm{Solid}_\mathbb{Z})$$

and a compact generator

$$\Big(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} A^\triangleright\Big)^{A^+ L\square} \in D(A^\triangleright, A^+)_\square \tag{6}$$

Lemma 11.1. *Recall that*

- $\big(M \otimes_\mathbb{Z} \mathbb{Z}[T]\big)^{\mathbb{Z}[T]\square}$ *commutes with limits in M and sends $\mathbb{Z} \mapsto \mathbb{Z}[T]$*
- $M^{T_1, T_2\square} = \big(M^{T_1\square}\big)^{T_2\square} = \big(M^{T_2\square}\big)^{T_1\square}$

Theorem 11.2. *The compact generator (6) lives in degree 0. Hence it is a compact projective generator for $\mathrm{Mod}_{(A^\triangleright, A^+)_\square}$ and $D\big(\mathrm{Mod}_{(A^\triangleright, A^+)_\square}\big) = D\big(A^\triangleright, A^+\big)_\square$.*[57]

Lemma 11.3.
$$M^{LA^+\square} = \varinjlim_{\substack{R \subseteq A^+ \\ R \text{ finite type over } \mathbb{Z}}} M^{LR\square} =: N$$

which is a filtered colimit.

Proof. There are two things to check:

1. N is derived A^+-solid:

   We have that derived A^+-solid is the same as derived R-solid $\forall R \subseteq A^+$ of finite type. So certainly, for fixed R, the R-th term of the filtered colimit is R-solid, but also every further term is derived R-solid, as it has the stronger property of being R'-solid with $R \subseteq R'$. So there is a cofinal system of R-solid modules in the filtered colimit and a filtered colimit of R-solids is R-solid.

[57] This is not contrary to Warning 8.12 as this is only a subclass of analytic rings.

2. $\mathrm{Hom}_{(A,A^+)_\square}(M,L) = \mathrm{Hom}_{(A,A^+)_\square}(N,L)$ for all L derived A^+-solid:

   By the same fact as before a A^+-solid map is the same as a map of R-solids for every $R \subseteq A^+$ of finite type. Then the statement follows by a similar argument. $\square$

Proof of Theorem 11.2. It now suffices to show

$$\left(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} A\right)^{LR\square}$$

lives in degree 0. As

$$\left(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} A\right)^{LR\square} = \left(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} R\right)^{LR\square} \otimes^L_{R_\square} A$$

it suffices to see:

$$(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} R)^{LR\square} \text{ lives in degree 0 and it is flat with respect to } \otimes_{R_\square}.$$

The flatness part is going to be proved in Lemma 11.8.
<u>Claim:</u> For R of finite type[58] over $\mathbb{Z}$:

$$\left(\prod_{\mathbb{N}} \mathbb{Z} \otimes_{\mathbb{Z}_\square} \mathbb{R}\right)^{LR\square} = \prod_{\mathbb{N}} R$$

Proof of Claim. First consider $R = \mathbb{Z}[x_1, \dots, x_n]$. Now by Lemma 11.1:

$$\begin{aligned}\prod_{\mathbb{N}} \mathbb{Z} \otimes_{\mathbb{Z}_\square} \mathbb{Z}[x_1,\dots,x_n]^{Lx_1,\dots,x_n\square} &= \left(\prod_{\mathbb{N}} \mathbb{Z} \otimes_{\mathbb{Z}_\square} \mathbb{Z}[x_1,\dots,x_n]^{Lx_1\square}\right)^{Lx_2,\dots,x_n\square} \\ &= \left(\prod \mathbb{Z}[x_1] \otimes_{\mathbb{Z}[x_1]_\square} \mathbb{Z}[x_2,\dots,x_n]\right)^{Lx_2,\dots,x_n\square} \\ &= \dots \\ &= \prod \mathbb{Z}[x_1,\dots,x_n]\end{aligned}$$

General case: $\mathbb{Z}[x_1,\dots,x_n] \twoheadrightarrow R$

$$\begin{aligned}\left(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} R\right)^{LR\square} &= \left(\prod \mathbb{Z} \otimes_{\mathbb{Z}_\square} \mathbb{Z}[T_1,\dots,T_n]\right)^{L\mathbb{Z}[T_1,\dots,T_n]\square} \otimes^L_{\mathbb{Z}[T_1,\dots,T_n]} R \\ &= \left(\prod \mathbb{Z}[T_1,\dots,T_n]\right) \otimes^L_{\mathbb{Z}[T_1,\dots,T_n]} R\end{aligned}$$

As $\mathbb{Z}[T_1,\dots,T_n]$ is noetherian, R can be resolved by finite free $\mathbb{Z}[T_1,\dots,T_n]$-modules. Hence

$$\prod \mathbb{Z}[x_1,\dots,x_n] \otimes^L_{\mathbb{Z}[x_1,\dots,x_n]} R \xrightarrow{\sim} \prod R$$

$\square$

[58] If it is not of finite type, then it would be equal to $\bigcup_{\substack{R' \subseteq R \\ \text{finite type}}} \prod R'$.

Definition 11.4. For an arbitrary commutative ring R we define $\mathrm{Solid}_R^{f.p.}$ as the category of cokernels of maps[59] $\prod R \to \prod R$.

Corollary 11.5. *Let R be an arbitrary commutative ring. Then*

$$\mathrm{Solid}_R \overset{\mathrm{def}}{=} \mathrm{Solid}_{(R,R)} = \mathrm{Ind}(\mathrm{Solid}_R^{f.p.})$$

and

$$\mathrm{Solid}_R^{f.p.} = \varinjlim_{\substack{R' \subseteq R \\ \text{finite type}}} \mathrm{Solid}_{R'}^{f.p.}$$

11.1 Structure of Solid_R

Lemma 11.6. *If $M \in \mathrm{Solid}_R$ is quasiseparated, then TFAE:*

1. *M is finitely presented*
2. *M is finitely generated*
3. *$M = \varprojlim_n M_n$, M_n finitely generated R-module, $M_{n+1} \twoheadrightarrow M_n$*

Proof. "1. ⇒ 2.": clear.
"2. ⇒ 3.": By definition we have an exact sequence

$$0 \to K \to \prod R \twoheadrightarrow M \to 0.$$

Since M is quasiseparated, $K \to \prod R$ is a quasicompact map. As $\prod R$ is metrizable, by Lemma 10.13 this is the same as a closed subset in $\prod R$ with product topology. Now by defining K_n as the image of

$$K \to \prod R \to \prod_{i \leq n} R,$$

one gets that $K \xrightarrow{\sim} \varprojlim K_n$ as in 3. . Furthermore we get that $M = \varprojlim \prod_{i \leq n} R/K_n$.
"3. ⇒ 1.": It suffices to prove "3. ⇒ 2.", as if M is finitely generated, then we can get again get an exact sequence

$$0 \to K \to \prod R \twoheadrightarrow M \to 0$$

and by the proof before, we see that if M is finitely generated, then the kernel is also of the form in 3. which then is also finitely generated which gives [60] us 1. Now for

[59] Compare this definition to Definition 6.9!

[60] The map $\prod R \twoheadrightarrow K$, where K is the kernel of $\prod R \twoheadrightarrow M$, induces the exact sequence $\prod R \to \prod R \to M \to 0$

sufficiently big d_n one gets maps:

$$\begin{tikzcd}
R^{\oplus d_3} \arrow[r, two heads] & \ker(f_2) \arrow[r] & M_3 \arrow[d, two heads, "f_2"] \\
R^{\oplus d_2} \arrow[r, two heads] & \ker(f_1) \arrow[r] & M_2 \arrow[d, two heads, "f_1"] \\
R^{\oplus d_1} \arrow[rr, two heads] & & M_1
\end{tikzcd}$$

This induces:

$$\begin{tikzcd}
\cdots \arrow[d, two heads] & \cdots \arrow[d, two heads] \\
R^{\oplus d_1+d_2+d_3} \arrow[r, two heads] \arrow[d, two heads] & M_3 \arrow[d, two heads, "f_2"] \\
R^{\oplus d_1+d_2} \arrow[r, two heads] \arrow[d, two heads] & M_2 \arrow[d, two heads, "f_1"] \\
R^{\oplus d_1} \arrow[r, two heads] & M_1
\end{tikzcd}$$

taking the limit of this diagram we get a map $\prod R \twoheadrightarrow M$. $\square$

Theorem 11.7. *Let R be a ring of finite type*[61] *over $\mathbb{Z}$. Then we have "Coherence": $\mathrm{Solid}_R^{f.p.}$ is closed under* $\ker$, coker *and extensions.*

Reminder. A classical commutative ring is called **coherent** if the finitely presented R−modules form an Abelian subcategory. By some short exact sequences one can show, that one only needs to check, that every finitely generated ideal is actually finitely presented, in order to prove that a ring is coherent.

Proof of Theorem. By the same arguments as in the classical case it suffices to check that every finitely generated subobject of $\prod R$ is finitely presented. Note, that every subobject of $\prod R$ is quasiseparated, as $\prod R$ is quasiseparated. So now we only need to show, that every finitely generated quasiseparated module is finitely presented. This follows from Lemma 11.6. $\square$

Lemma 11.8. *If M is finitely presented in* Solid_R, *we get*

$$M \otimes^L_{R^\square} \prod R \xrightarrow{\sim} \prod M$$

In particular $M \otimes^L_{R^\square} \prod R$ lives in degree 0.

Remark 11.9. This implies that $\prod R$ is flat with respect to $\otimes_{R^\square}$, as any object is a filtered colimit of finitely presented objects, hence for any M, $M \otimes^L_{R^\square} \prod R$ lives in degree 0.

[61] Finite type is used in the proof, in order that $\prod R$ is the projective generator.

Proof. As M is finitely presented, we have the surjection $\prod_R \to M \to 0$, where the kernel is finitely generated. As the kernel is also quasiseparated, it will also be finitely presented. This way we get an infinite resolution

$$\cdots \to \prod R \to \prod R \to M \to 0$$

This reduces us to the case $M = \prod R$, i.e. we need to show that $\prod R \otimes^L_{R^\square} \prod R = \prod R$. As $\prod R = \left(\prod \mathbb{Z} \otimes_{\mathbb{Z}^\square} R\right)^{R\square}$ the fact descends to the analogous fact for $\prod \mathbb{Z}$, which was proven in Corollary 5.12. $\square$

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