Journal of Algebra 372 (2012) 376–396

Contents lists available at SciVerse ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

On the finiteness of Gorenstein homological dimensions ☆

Ioannis Emmanouil

Department of Mathematics, University of Athens, Athens 15784, Greece

ARTICLE INFO

Article history:
Received 26 March 2012
Available online 11 October 2012
Communicated by Luchezar L. Avramov

Keywords:
Gorenstein projective, injective and flat modules
Relative derived functors
Gorenstein global dimensions

ABSTRACT

In this paper, we study certain properties of modules of finite Gorenstein projective, injective and flat dimensions. We examine conditions which imply that all Gorenstein projective modules are Gorenstein flat and establish the balance of the Gorenstein Tor-functor for modules of finite Gorenstein projective dimension. We also examine the class of rings that have finite Gorenstein global and weak dimensions and compute these dimensions, in terms of certain cohomological invariants of the ring. Finally, we provide some examples of rings of finite Gorenstein global and weak dimensions.

0. Introduction

According to a classical result, established by Auslander, Buchsbaum and Serre in [4] and [33], a commutative Noetherian local ring R is regular if and only if the projective dimension of its residue field is finite. Moreover, in that case, all R-modules have finite projective dimension, i.e. R has finite global dimension. Auslander and Bridger obtained in [2] and [3] a version of that result, characterizing the Gorenstein rings: They defined the G-dimension $\text{G-dim}_R M$ of any finitely generated module M over a commutative Noetherian local ring R and showed that R is Gorenstein if and only if the G-dimension of its residue field is finite. Moreover, in that case, all finitely generated R-modules have finite G-dimension. The G-dimension generalizes the projective dimension, in the sense that if M is a finitely generated R-module of finite projective dimension, then $\text{G-dim}_R M = \text{pd}_R M$.

The above definition was extended to all modules over any ring R by Enochs and Jenda in [17]: A left R-module M is said to be Gorenstein projective if M is a syzygy of a complete projective resolution, i.e. if there exists an acyclic complex of projective left R-modules

☆ Research supported by a GSRT/Greece excellence grant, cofunded by the ESF/EU and National Resources.
E-mail address: emmanoui@math.uoa.gr.

0021-8693/$ – see front matter
http://dx.doi.org/10.1016/j.jalgebra.2012.09.018

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots, \tag{1}$$

which remains acyclic when applying the functor $\mathrm{Hom}_R(\_\_, P)$ for any projective left R-module P, such that $M = \mathrm{im}\,\partial_0$. Then, modules of finite Gorenstein projective dimension are defined in the standard way, by using resolutions by Gorenstein projective modules. As shown in [12, Theorem 4.2.6], the Gorenstein projective dimension of a finitely generated module over a commutative Noetherian ring agrees with its G-dimension. The corresponding classes of Gorenstein injective and Gorenstein flat modules were defined in [17] and [18]. As shown in [5], the resulting relative cohomology theory is closely related to the ordinary cohomology of modules and the complete cohomology, in the case where the latter is defined by means of complete resolutions.

Gorenstein homological algebra has developed rapidly during the past several years. The theory of modules of finite Gorenstein dimensions has found some interesting applications in representation theory; these include the structure of the stable category of Cohen–Macaulay modules, the Auslander–Reiten theory, the existence of Serre duality at the level of perfect complexes and the theory of singularities. As an indicative application in cohomological group theory, we note that (as shown in [6, Theorem 2.5]) the Gorenstein cohomological dimension of a group G is equal to the generalized cohomological dimension of G, which was defined by Ikenaga in [24]. In this way, the formal properties of the Gorenstein projective dimension may prove to be useful in order to study certain group-theoretical problems, such as that of characterizing algebraically the groups that admit a finite dimensional model for the classifying space for proper actions (cf. [6]).

The properties of Gorenstein projective, injective and flat modules diverge from those of their homological algebra counterparts. For example, there is no known robust characterization of Gorenstein projective modules, analogous to the assertion that an R-module M is projective if and only if the functor $\mathrm{Ext}^1_R(M, \_\_)$ vanishes. Even though the flat modules are precisely the filtered colimits of the finitely generated projective modules (cf. [29]), the corresponding assertion is not true in Gorenstein homological algebra: It is shown in [8] and [23] that there exist rings over which not all Gorenstein flat modules can be expressed as filtered colimits of finitely generated Gorenstein projective modules.

It is not at all clear from the definitions that Gorenstein projective modules are Gorenstein flat. As shown by Holm in [21], this is the case if the ring R is right coherent and has finite left finitistic dimension. As a contribution to the study of this problem, we shall obtain a variety of conditions that are equivalent to the assertion that Gorenstein projective modules are Gorenstein flat. Given an acyclic complex of projective left R-modules as in (1), the problem consists in relating the acyclicity of the complexes that are obtained by applying the functors $I \otimes_R \_\_$, where I is an injective right R-module, to the acyclicity of the complexes that are obtained by applying the functors $\mathrm{Hom}_R(\_\_, P)$, where P is a projective left R-module. There is a certain natural transformation, which is ideally suited for the study of this problem. This natural transformation was introduced by Cartan and Eilenberg in [11, Chapter VI, §5], in order to study certain duality problems. More recently, it was used in [15], in order to examine the relation between certain invariants in cohomological group theory. The injectivity of that natural transformation defines a class of modules, which were first studied by Raynaud and Gruson in [32, §II.2.3], the so-called strict Mittag–Leffler modules. We describe the relevance of the class of strict Mittag–Leffler modules in the study of the relation between Gorenstein projective and Gorenstein flat modules in Section 2.

It turns out that all Gorenstein projective left R-modules are Gorenstein flat if and only if the Pontryagin dual of any Gorenstein projective left R-module is a Gorenstein injective right R-module. Even though we do not know whether this is the case for any ring R, we show that the Pontryagin duals of Gorenstein projective modules always possess a Gorenstein injective-like behavior, which has an interesting application in the study of the balance of the Gorenstein Tor-functor. The balance of the relative derived functors of the Hom and tensor product functors with respect to the classes of Gorenstein projective and Gorenstein injective modules has been studied by Holm, who established in [22] the balance of the Gorenstein Ext-functor for modules of finite Gorenstein projective and Gorenstein injective dimension. In the case of the Gorenstein Tor-functor, Holm proved balance for modules of finite Gorenstein projective dimension if the ring R is left and right coherent and has finite left and right finitistic dimension. As we shall explain in Section 3, one may adapt Holm's arguments and use the Gorenstein injective-like behavior of the Pontryagin duals of Gorenstein projective modules, in order to show that the Gorenstein Tor-functor is balanced for modules of finite Gorenstein projective dimension over any ring R.

The theory of Gorenstein projective and Gorenstein injective dimensions is closely related to the study of the invariants $\operatorname{silp} R$ and $\operatorname{spli} R$, that were defined by Gedrich and Gruenberg in [19], in connection with the existence of complete cohomological functors in the category of left R-modules. Here, $\operatorname{silp} R$ is defined as the supremum of the injective lengths of projective left R-modules, whereas $\operatorname{spli} R$ is the supremum of the projective lengths of injective left R-modules. It turns out that the finiteness of these invariants is equivalent to the assertion that the ring R has finite left Gorenstein global dimension, i.e. to the assertion that all left R-modules have finite Gorenstein projective (or, equivalently, finite Gorenstein injective) dimension. Moreover, if both invariants $\operatorname{spli} R$ and $\operatorname{silp} R$ are finite, then these are equal to each other and their common value is the left Gorenstein global dimension of R, i.e. the supremum of the Gorenstein projective (or, equivalently, Gorenstein injective) dimensions of all left R-modules.

As in classical homological algebra, we may examine the finiteness of the Gorenstein weak dimension of R (i.e. the assertion that all left and all right R-modules have finite Gorenstein flat dimension). This finiteness condition is related to the invariant $\operatorname{sfli} R$, which is defined as the supremum of the flat lengths of injective left R-modules. It turns out that the finiteness of the invariants $\operatorname{sfli} R$ and $\operatorname{sfli} R^{op}$ is equivalent to the assertion that the ring R has finite Gorenstein weak dimension. Moreover, if both invariants $\operatorname{sfli} R$ and $\operatorname{sfli} R^{op}$ are finite, then these are equal to each other and their common value is the Gorenstein weak dimension of R, i.e. the supremum of the Gorenstein flat dimensions of all left (or, equivalently, of all right) R-modules. If R is a ring of finite Gorenstein weak dimension, then all Gorenstein projective (left or right) R-modules are Gorenstein flat.

In the final section of the paper, we present several examples of rings of finite Gorenstein global and weak dimensions. The class of these rings contains:

(i) the left and right $\aleph_0$-Noetherian rings R, for which the invariants $\operatorname{silp} R$ and $\operatorname{silp} R^{op}$ are finite; these examples include, in particular, the Iwanaga–Gorenstein rings, i.e. the left and right Noetherian rings of finite left and right self-injective dimension,
(ii) the countable rings of finite left and right self-injective dimension, and
(iii) the integral group rings of those groups that admit a finite dimensional model for the classifying space for proper actions.

Notations and terminology. For any two abelian groups M, N we denote by $\operatorname{Hom}(M,N)$ the group $\operatorname{Hom}_{\mathbb{Z}}(M,N)$ of all additive maps from M to N. If R is a ring, then we denote by R^{op} its opposite ring. We do not distinguish between right R-modules and left R^{op}-modules. In particular, if $\lambda(R)$ is an invariant, which is defined for any ring R in terms of a certain class of left R-modules, then we denote by $\lambda(R^{op})$ the corresponding invariant, which is defined for R in terms of the appropriate class of right R-modules. We denote by D the Pontryagin duality functors from the category of left (resp. right) R-modules to the category of right (resp. left) R-modules, which are both defined by $M \mapsto \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z})$.

1. Preliminary notions

In this section, we collect certain basic notions and preliminary results that will be used in the sequel. These involve:

(i) the strict Mittag–Leffler modules and a related natural transformation,
(ii) the cohomological invariants silp, spli and sfli and their relation to the finitistic dimensions of a ring, and
(iii) certain basic properties of the Gorenstein projective, injective and flat modules.

1.1. Strict Mittag–Leffler modules

Let R be a ring and consider a left R-module N and an abelian group D. Then, the abelian group $\operatorname{Hom}(N,D)$ can be endowed with the structure of a right R-module, by means of the left R-module structure of N. Hence, if M is another left R-module, then we may consider the tensor product $\mathrm{Hom}(N, D) \otimes_R M$ and define the map

$$\Phi : \mathrm{Hom}(N, D) \otimes_R M \longrightarrow \mathrm{Hom}\big(\mathrm{Hom}_R(M, N), D\big), \tag{2}$$

by letting $\Phi(f \otimes m)(g) = f(g(m))$ for all $f \in \mathrm{Hom}(N, D)$, $m \in M$ and $g \in \mathrm{Hom}_R(M, N)$. The map Φ is well-defined, additive and natural in N, D and M. The natural transformation Φ appears in [11, Chapter VI, §5] and has been also used by Raynaud and Gruson in [32, §II.2.3]. We denote the map (2) above by Φ_M. It is easily seen that Φ_M is an isomorphism if the abelian group D is divisible and the left R-module M is finitely presented.

In order to examine the injectivity of Φ_M, we recall some terminology concerning inverse systems of abelian groups. Let $(A_i)_i$ be such an inverse system indexed by a directed set I and denote by $\sigma_{ij} : A_j \longrightarrow A_i$, $i \leqslant j$, its structural maps. We also consider the inverse limit $A = \varprojlim_i A_i$ and the canonical maps $s_i : A \longrightarrow A_i$, $i \in I$. The inverse system $(A_i)_i$ is said to satisfy the Mittag–Leffler condition if for all $i \in I$ there exists a suitable index $j = j(i) \in I$ with $j \geqslant i$, such that

$$\mathrm{im}(A_j \xrightarrow{\sigma_{ij}} A_i) = \mathrm{im}(A_k \xrightarrow{\sigma_{ik}} A_i)$$

for all $k \in I$ with $k \geqslant j$. The Mittag–Leffler condition was introduced by Grothendieck in [20, §13.1.2]. Assuming that the inverse system $(A_i)_i$ satisfies the Mittag–Leffler condition, we refer to the subgroup $A'_i = \mathrm{im}(A_j \xrightarrow{\sigma_{ij}} A_i)$, where $j = j(i)$ as above, as the stable image. We say that the inverse system $(A_i)_i$ satisfies the strict Mittag–Leffler condition if for all $i \in I$ there exists a suitable index $j = j(i) \in I$ with $j \geqslant i$, such that

$$\mathrm{im}(A_j \xrightarrow{\sigma_{ij}} A_i) = \mathrm{im}(A \xrightarrow{s_i} A_i).$$

It is easily seen that $(A_i)_i$ satisfies the strict Mittag–Leffler condition if and only if it satisfies the Mittag–Leffler condition and the stable image A'_i coincides with the image of the canonical map $s_i : A \longrightarrow A_i$ for all $i \in I$.

The following result, which characterizes those pairs of left R-modules M and N as above, for which the additive map Φ_M is injective for any divisible abelian group D, is proved in [1, Theorem 8.11]; see also [16, Theorem 1.3].

Theorem 1.1. *The following conditions are equivalent for a pair of left R-modules M, N:*

(i) *The natural map Φ_M defined above is injective for any divisible abelian group D.*
(ii) *There exists a representation of M as the direct limit of a direct system $(M_i)_i$ of finitely presented left R-modules, such that the associated inverse system of abelian groups $(\mathrm{Hom}_R(M_i, N))_i$ satisfies the strict Mittag–Leffler condition.*

If these conditions are satisfied, we say that M is a strict Mittag–Leffler module over N.

Having fixed the left R-module N, we denote by $\mathtt{SML}(N)$ the class of those left R-modules M, which are strict Mittag–Leffler over N. The class $\mathtt{SML}(N)$ is closed under direct sums and direct summands. Since the left regular module R is obviously strict Mittag–Leffler over N, it follows that any projective left R-module is contained in $\mathtt{SML}(N)$.

The natural transformation Φ may be derived as follows: We fix a left R-module N and a divisible abelian group D and consider for any left R-module M a projective resolution $P_* \xrightarrow{\varepsilon} M \longrightarrow 0$. Then, Φ induces a chain map

$$\Phi_{P_*} : \mathrm{Hom}(N, D) \otimes_R P_* \longrightarrow \mathrm{Hom}\big(\mathrm{Hom}_R(P_*, N), D\big).$$

By applying homology, we obtain additive maps

$$\Phi_M^{(n)} : \mathrm{Tor}_n^R\big(\mathrm{Hom}(N, D), M\big) \longrightarrow \mathrm{Hom}\big(\mathrm{Ext}_R^n(M, N), D\big),$$

$n \geqslant 0$, which do not depend upon the particular choice of the projective resolution of M. The additive map $\Phi_M^{(0)}$ can be naturally identified with the map Φ_M examined before.

Given a projective resolution of M as above, we denote by $\Omega^n M$ the corresponding n-th syzygy module. Even though $\Omega^n M$ depends (as an R-module) upon the choice of the particular resolution, Schanuel's lemma implies that whether $\Omega^n M$ is strict Mittag–Leffler over N or not is a property that doesn't depend upon the choice of the particular resolution; indeed, as we noted above, the class $\mathtt{SML}(N)$ contains all projective modules and is closed under direct sums and direct summands. If $\Omega^n M \in \mathtt{SML}(N)$, then we say that the n-th syzygy module of M is strict Mittag–Leffler over N. Since $\Omega^0 M = M$, the next result is a higher order version of Theorem 1.1.

Proposition 1.2. *(Cf. [16, Proposition 1.5].) The following conditions are equivalent for a pair of left R-modules M, N and a non-negative integer n:*

(i) *The natural map $\Phi_M^{(n)}$ defined above is injective for any divisible abelian group D.*
(ii) *The n-th syzygy module of M is strict Mittag–Leffler over N.*

In the sequel we will be interested in the special case where the left R-module N is the left regular module R. In that direction, we recall that the right R-module $\mathrm{Hom}(R, D)$ is injective for any divisible abelian group D.

1.2. The invariants silp, spli and sfli

Let R be a ring. In connection with the existence of complete cohomological functors in the category of left R-modules, Gedrich and Gruenberg have defined in [19] the invariant $\mathrm{silp}\, R$ as the supremum of the injective lengths of projective left R-modules and the invariant $\mathrm{spli}\, R$ as the supremum of the projective lengths of injective left R-modules. It is easily seen that $\mathrm{silp}\, R < \infty$ if and only if any projective left R-module has finite injective dimension. In the same way, $\mathrm{spli}\, R < \infty$ if and only if any injective left R-module has finite projective dimension. These invariants are related to the left finitistic dimension $\mathrm{fin.dim}\, R$ of R, which is itself defined as the supremum of the projective dimensions of those left R-modules that have finite projective dimension. The following well-known result will play an important role in the definition of the Gorenstein global dimension of R (cf. Theorem 4.1 below). We include a proof for the reader's convenience; the proof is based on arguments that may be found in the work of Gedrich and Gruenberg [19] and Cornick and Kropholler [14].

Proposition 1.3. *Let R be a ring. Then*:

(i) $\mathrm{fin.dim}\, R \leqslant \mathrm{silp}\, R$.
(ii) *If* $\mathrm{spli}\, R < \infty$ *then* $\mathrm{spli}\, R \leqslant \mathrm{fin.dim}\, R$.
(iii) *If* $\mathrm{silp}\, R < \infty$ *then* $\mathrm{silp}\, R \leqslant \mathrm{spli}\, R$.
(iv) *If* $\mathrm{spli}\, R < \infty$ *and* $\mathrm{silp}\, R < \infty$, *then* $\mathrm{spli}\, R = \mathrm{fin.dim}\, R = \mathrm{silp}\, R$.

Proof. (i) Let M be a left R-module with $\mathrm{pd}_R\, M = n < \infty$. Then, the n-th syzygy module in a projective resolution of M is a projective left R-module P and $\mathrm{Ext}_R^n(M, P) \neq 0$. In particular, $\mathrm{id}_R\, P \geqslant n$ and hence $\mathrm{silp}\, R \geqslant n$.

(ii) Assume that $\mathrm{spli}\, R = n < \infty$. Then, there exists an injective left R-module I with $\mathrm{pd}_R\, I = n$ and hence $\mathrm{fin.dim}\, R \geqslant n$.

(iii) Assume that $\mathrm{silp}\, R = n < \infty$. Then, there exists a projective left R-module P with $\mathrm{id}_R\, P = n$. Then, the n-th cosyzygy module in an injective resolution of P is an injective left R-module I and $\mathrm{Ext}_R^n(I, P) \neq 0$. It follows that $\mathrm{pd}_R\, I \geqslant n$ and hence $\mathrm{spli}\, R \geqslant n$.

(iv) This is an immediate consequence of (i), (ii) and (iii) above. $\square$

If R is a ring, then the invariant $\operatorname{sfli} R$ is defined as the supremum of the flat lengths of injective left R-modules. It is easily seen that $\operatorname{sfli} R < \infty$ if and only if any injective left R-module has finite flat dimension. Since projective modules are flat, it is clear that we have an inequality $\operatorname{sfli} R \leqslant \operatorname{spli} R$. The invariant $\operatorname{sfli} R$ is related to the left finitistic flat dimension $\operatorname{fin.f.dim} R$ of the ring R, which is itself defined as the supremum of the flat dimensions of those left R-modules that have finite flat dimension. The following result is the Tor-version of Proposition 1.3 and will play an important role in the definition of the Gorenstein weak dimension of R (cf. Theorem 5.3 below). We present a proof that uses the duality between flat and injective modules.

Proposition 1.4. *(Cf. [16, Proposition 2.4 and Corollary 2.5].) Let R be a ring. Then*:

(i) $\operatorname{fin.f.dim} R \leqslant \operatorname{sfli} R^{op}$.
(ii) *If* $\operatorname{sfli} R < \infty$, *then* $\operatorname{sfli} R \leqslant \operatorname{fin.f.dim} R$.
(iii) *If* $\operatorname{sfli} R < \infty$ *and* $\operatorname{sfli} R^{op} < \infty$, *then* $\operatorname{sfli} R = \operatorname{fin.f.dim} R = \operatorname{sfli} R^{op} = \operatorname{fin.f.dim} R^{op}$.

Proof. (i) Let M be a left R-module with $\operatorname{fd}_R M = n < \infty$. We may consider the right R-module DM and note that $\operatorname{id}_{R^{op}} DM = n$. Then, the n-th cosyzygy module in an injective resolution of DM is an injective right R-module I and $\operatorname{Ext}^n_{R^{op}}(I, DM) \neq 0$. In view of the standard isomorphism between $\operatorname{Ext}^n_{R^{op}}(I, DM)$ and $\operatorname{Hom}(\operatorname{Tor}^R_n(I, M), \mathbb{Q}/\mathbb{Z})$, we conclude that $\operatorname{Tor}^R_n(I, M) \neq 0$. In particular, $\operatorname{fd}_{R^{op}} I \geqslant n$ and hence $\operatorname{sfli} R^{op} \geqslant n$.

(ii) If $\operatorname{sfli} R = n < \infty$, then there exists an injective left R-module I with $\operatorname{fd}_R I = n$ and hence $\operatorname{fin.f.dim} R \geqslant n$.

(iii) This is an immediate consequence of (i) and (ii) above. $\square$

1.3. Gorenstein projective, injective and flat modules

Let R be a ring. An acyclic complex of projective left R-modules

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots,$$

where $n \in \mathbb{Z}$, is a complete projective resolution if the complex of abelian groups

$$\cdots \longrightarrow \operatorname{Hom}_R(P_{n-1}, P) \xrightarrow{\partial_n^*} \operatorname{Hom}_R(P_n, P) \xrightarrow{\partial_{n+1}^*} \operatorname{Hom}_R(P_{n+1}, P) \longrightarrow \cdots$$

is acyclic for any projective left R-module P. A left R-module M is called Gorenstein projective if it is a syzygy of a complete projective resolution, i.e. if there exists a complete projective resolution (P_*, ∂_*) as above, such that $M = \operatorname{im} \partial_n$ for some $n \in \mathbb{Z}$. It is clear that all projective left R-modules are Gorenstein projective. In fact, as shown in [21, Proposition 2.27], a left R-module is projective if and only if it is Gorenstein projective and has finite projective dimension. If M is Gorenstein projective, then the groups $\operatorname{Ext}^i_R(M, P)$ vanish when $i \geqslant 1$ for all projective left R-modules P. We shall denote by $\mathtt{GP}(R)$ the class of Gorenstein projective left R-modules.

Dually, an acyclic complex of injective left R-modules

$$\cdots \longrightarrow I^{n-1} \xrightarrow{\partial^{n-1}} I^n \xrightarrow{\partial^n} I^{n+1} \longrightarrow \cdots,$$

where $n \in \mathbb{Z}$, is a complete injective resolution if the complex of abelian groups

$$\cdots \longrightarrow \operatorname{Hom}_R\big(I, I^{n-1}\big) \xrightarrow{\partial_*^{n-1}} \operatorname{Hom}_R\big(I, I^n\big) \xrightarrow{\partial_*^n} \operatorname{Hom}_R\big(I, I^{n+1}\big) \longrightarrow \cdots$$

is acyclic for any injective left R-module I. A left R-module M is called Gorenstein injective if it is a syzygy of a complete injective resolution, i.e. if there exists a complete injective resolution (I^*, ∂^*) as above, such that $M = \ker \partial^n$ for some $n \in \mathbb{Z}$. It is clear that all injective left R-modules are Gorenstein injective. In fact, it follows from [21, Theorem 2.22] that a left R-module is injective if and only if it is Gorenstein injective and has finite injective dimension. If M is Gorenstein injective, then the groups $\mathrm{Ext}^i_R(I, M)$ vanish when $i \geqslant 1$ for all injective left R-modules I. We shall denote by $\mathtt{GI}(R)$ the class of Gorenstein injective left R-modules.

Finally, an acyclic complex of flat left R-modules

$$\cdots \longrightarrow F_{n+1} \xrightarrow{\partial_{n+1}} F_n \xrightarrow{\partial_n} F_{n-1} \longrightarrow \cdots,$$

where $n \in \mathbb{Z}$, is a complete flat resolution if the complex of abelian groups

$$\cdots \longrightarrow I \otimes_R F_{n+1} \xrightarrow{1\otimes\partial_{n+1}} I \otimes_R F_n \xrightarrow{1\otimes\partial_n} I \otimes_R F_{n-1} \longrightarrow \cdots$$

is acyclic for any injective right R-module I. A left R-module M is called Gorenstein flat if it is a syzygy of a complete flat resolution, i.e. if there exists a complete flat resolution (F_*, ∂_*) as above, such that $M = \mathrm{im}\, \partial_n$ for some $n \in \mathbb{Z}$. It is clear that all flat left R-modules are Gorenstein flat. If M is Gorenstein flat, then the groups $\mathrm{Tor}^R_i(I, M)$ vanish when $i \geqslant 1$ for all injective right R-modules I. We shall denote by $\mathtt{GF}(R)$ the class of Gorenstein flat left R-modules.

Remark 1.5. (Cf. [9, §2].) Let R be a ring. Then, a left R-module is flat if and only if it is Gorenstein flat and has finite flat dimension. Indeed, let M be a Gorenstein flat left R-module of finite flat dimension. If $\mathrm{fd}_R\, M = n$, then (as noted in the proof of Proposition 1.4) we can find an injective right R-module I, such that $\mathrm{Tor}^R_n(I, M) \neq 0$. Since M is Gorenstein flat, the functors $\mathrm{Tor}^R_i(\_\_, M)$ vanish when $i \geqslant 1$ on all injective right R-modules. It follows that $n = 0$ and hence M is flat, as needed.

2. Gorenstein projective versus Gorenstein flat modules

In this section, we examine the relation between Gorenstein projective and Gorenstein flat modules over a ring R and obtain several conditions which are equivalent to the assertion that $\mathtt{GP}(R)$ is a subclass of $\mathtt{GF}(R)$. We begin with the following lemma, which justifies the use of the natural transformation Φ (and hence illustrates the relevance of the strict Mittag–Leffler modules) in the study of this relation.

Lemma 2.1. *Let R be a ring and consider an exact sequence*

$$P' \xrightarrow{f} P \xrightarrow{g} P''$$

of projective left R-modules. We assume that:

(i) *the sequence of abelian groups*

$$\mathrm{Hom}_R\big(P'', R\big) \xrightarrow{g^*} \mathrm{Hom}_R(P, R) \xrightarrow{f^*} \mathrm{Hom}_R\big(P', R\big)$$

is exact, and

(ii) *the left R-module* $\mathrm{im}\, g$ *is strict Mittag–Leffler over R, i.e.* $\mathrm{im}\, g \in \mathtt{SML}(R)$.

Then, the sequence of abelian groups

$$I \otimes_R P' \xrightarrow{1\otimes f} I \otimes_R P \xrightarrow{1\otimes g} I \otimes_R P''$$

is exact for any injective right R-module I.

Proof. If $M = \operatorname{coker} g$, then the exact sequence

$$P' \xrightarrow{f} P \xrightarrow{g} P'' \longrightarrow M \longrightarrow 0$$

is the beginning of a projective resolution of M. In view of (ii), the first syzygy module $\Omega^1 M = \operatorname{im} g$ is strict Mittag–Leffler over R and hence Proposition 1.2 implies that the additive map

$$\Phi_M^{(1)} : \operatorname{Tor}_1^R\big(\operatorname{Hom}(R, D), M\big) \longrightarrow \operatorname{Hom}\big(\operatorname{Ext}_R^1(M, R), D\big)$$

is injective for any divisible abelian group D. Since the abelian group $\operatorname{Ext}_R^1(M, R)$ is trivial (in view of (i)), we conclude that $\operatorname{Tor}_1^R(\operatorname{Hom}(R, D), M) = 0$ for any divisible abelian group D. On the other hand, any right R-module being a submodule of a cofree right R-module, it follows that any injective right R-module is a direct summand of a right R-module of the form $\operatorname{Hom}(R, D)$, for a suitable divisible abelian group D. Therefore, we conclude that $\operatorname{Tor}_1^R(I, M) = 0$ for any injective right R-module I. Of course, we may rephrase the vanishing of the latter group by saying that the sequence of abelian groups

$$I \otimes_R P' \xrightarrow{1 \otimes f} I \otimes_R P \xrightarrow{1 \otimes g} I \otimes_R P''$$

is exact for any injective right R-module I, as needed. □

We are now ready to state and prove the following characterization of the condition that Gorenstein projective left R-modules be Gorenstein flat.

Theorem 2.2. *Let R be a ring. Then, the following conditions are equivalent:*

(i) *Any Gorenstein projective left R-module is Gorenstein flat, i.e.* $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$.
(ii) *Any Gorenstein projective left R-module is strict Mittag–Leffler over R, i.e.* $\mathtt{GP}(R) \subseteq \mathtt{SML}(R)$.
(iii) *Any complete projective resolution is a complete flat resolution.*
(iv) *For any Gorenstein projective left R-module M the right R-module DM is Gorenstein injective.*

If these conditions are satisfied, then we have $\operatorname{Gfd}_R M \leqslant \operatorname{Gpd}_R M$ *for any left R-module M.*

Proof. (i) → (ii): Let M be a Gorenstein projective left R-module. Then, M is a syzygy of a complete projective resolution and hence $M = \Omega^1 N$, where N is a Gorenstein projective left R-module. In view of our assumption, N is Gorenstein flat and hence $\operatorname{Tor}_1^R(I, N) = 0$ for all injective right R-modules I. In particular, $\operatorname{Tor}_1^R(\operatorname{Hom}(R, D), N) = 0$ for any divisible abelian group D. Therefore, Proposition 1.2 implies that $M = \Omega^1 N$ is strict Mittag–Leffler over R, as needed.

(ii) → (iii): Let (P_*, ∂_*) be a complete projective resolution

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots.$$

We note that all syzygies of the complex (P_*, ∂_*) are Gorenstein projective and hence, in view of our hypothesis, all of these modules are strict Mittag–Leffler over R. Therefore, the complex of abelian groups $\operatorname{Hom}_R(P_*, R)$ being acyclic, we may invoke Lemma 2.1, in order to conclude that the complex of abelian groups $(I \otimes_R P_*, 1 \otimes \partial_*)$ is acyclic for any injective right R-module I. It follows that the complex (P_*, ∂_*) is a complete flat resolution, as needed.

(iii) → (i): This is an immediate consequence of the definitions.

(i) → (iv): If the left R-module M is Gorenstein projective, then it is Gorenstein flat (in view of our assumption (i)). Therefore, [21, Theorem 3.6] implies that the right R-module $DM = \operatorname{Hom}(M, \mathbb{Q}/\mathbb{Z})$ is Gorenstein injective.

(iv) → (iii): Let (P_*, ∂_*) be a complete projective resolution in the category of left R-modules

$$\cdots \longrightarrow P_{n+1} \xrightarrow{\partial_{n+1}} P_n \xrightarrow{\partial_n} P_{n-1} \longrightarrow \cdots$$

and denote by K_{n-1} the cokernel of the map $\partial_n : P_n \longrightarrow P_{n-1}$ for all $n \in \mathbb{Z}$. We now apply the duality functor D and consider the acyclic complex of injective right R-modules

$$\cdots \longrightarrow DP_{n-1} \xrightarrow{D\partial_n} DP_n \xrightarrow{D\partial_{n+1}} DP_{n+1} \longrightarrow \cdots.$$

We note that the kernel of the map $D\partial_n : DP_{n-1} \longrightarrow DP_n$ can be identified with DK_{n-1} for all $n \in \mathbb{Z}$. Since the right R-module DK_{n-1} is Gorenstein injective (in view of our assumption (iv)), it follows that for any injective right R-module I we have

$$H^n\big(\mathrm{Hom}_{R^{op}}(I, DP_*)\big) = \mathrm{Ext}^1_{R^{op}}(I, DK_{n-1}) = 0$$

for all $n \in \mathbb{Z}$. In view of the standard isomorphism between the complexes $\mathrm{Hom}_{R^{op}}(I, DP_*)$ and $\mathrm{Hom}(I \otimes_R P_*, \mathbb{Q}/\mathbb{Z})$ and the injectivity of the abelian group $\mathbb{Q}/\mathbb{Z}$, we conclude that

$$\mathrm{Hom}\big(H_n(I \otimes_R P_*), \mathbb{Q}/\mathbb{Z}\big) = 0$$

for all $n \in \mathbb{Z}$. Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is faithfully injective, it follows that $H_n(I \otimes_R P_*) = 0$ for all $n \in \mathbb{Z}$, i.e. the complex $(I \otimes_R P_*, 1 \otimes \partial_*)$ is acyclic. Since this is the case for any injective right R-module I, the complex (P_*, ∂_*) is a complete flat resolution, as needed.

It only remains to prove that the equivalent conditions (i)–(iv) imply that $\mathrm{Gfd}_R M \leqslant \mathrm{Gpd}_R M$ for any left R-module M. To that end, we assume that these conditions are satisfied and consider a left R-module M with $\mathrm{Gpd}_R M = n < \infty$. (The inequality to be proved is obvious if $\mathrm{Gpd}_R M = \infty$.) Then, there exists a resolution of M by Gorenstein projective left R-modules of length n. In view of condition (i), this resolution consists of Gorenstein flat left R-modules and hence $\mathrm{Gfd}_R M \leqslant n$. □

Remarks 2.3. (i) Holm has shown in [21, Proposition 3.4] that $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$ if the ring R is right coherent and has finite left finitistic dimension.

(ii) Let R be a ring and consider an acyclic complex of flat left R-modules (F_*, ∂_*) and a right R-module M with $\mathrm{fd}_{R^{op}} M < \infty$. Then, using induction on $\mathrm{fd}_{R^{op}} M$, one can show that the complex of abelian groups $(M \otimes_R F_*, 1 \otimes \partial_*)$ is acyclic. In particular, if R is a ring such that $\mathrm{sfli}\, R^{op} < \infty$, then any complete projective resolution (and, more generally, any acyclic complex of flat left R-modules) is a complete flat resolution and hence $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$; this has been noted in [9, Proposition 3.2].

(iii) Let M be a Gorenstein projective left R-module and consider its Pontryagin dual DM. Then, for any non-negative integer n there exists an exact sequence of right R-modules

$$0 \longrightarrow DN \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow DM \longrightarrow 0,$$

where N is a Gorenstein projective left R-module and I_i is injective for all $i = 1, \ldots, n$. In order to verify this assertion, we argue as follows: Using a complete projective resolution that has M as a syzygy, we may find an exact sequence of left R-modules

$$0 \longrightarrow M \longrightarrow P_1 \longrightarrow \cdots \longrightarrow P_n \longrightarrow N \longrightarrow 0,$$

where P_i is projective for all $i = 1, \ldots, n$ and N is Gorenstein projective. Applying now the duality functor D, we obtain an exact sequence with the required properties; we note that the right R-module DP is injective for any projective left R-module P.

3. Balance of the Gorenstein Tor-functor

The relative cohomology of modules of finite Gorenstein projective dimension has been studied by Avramov and Martsinkovsky in [5] (in the case of finitely generated modules over a left and right Noetherian ring). This topic has been taken up by Holm in [22], who studied therein the balance of the relative derived functors of the Hom and tensor product functors with respect to the classes of Gorenstein projective and Gorenstein injective modules. Holm has established the balance of the Gorenstein Ext-functor for modules of finite Gorenstein projective and Gorenstein injective dimension in [22, Theorem 3.6]. In the case of the Gorenstein Tor-functor, he proved balance for modules of finite Gorenstein projective dimension if the ring R is left and right coherent and has finite left and right finitistic dimension (cf. [22, Theorem 4.8]). In this section, we shall establish the balance of the Gorenstein Tor-functor for modules of finite Gorenstein projective dimension over any ring R.

The technique of the proof consists in adapting Holm's arguments for Gorenstein injective modules to the Pontryagin duals of Gorenstein projective modules. The following lemma describes the Gorenstein injective-like behavior of the Pontryagin duals of Gorenstein projective modules that will be used in the proof.

Lemma 3.1. *Let R be a ring and consider a Gorenstein projective right R-module M and a left R-module K of finite projective dimension. Then*:

(i) $\mathrm{Ext}^1_R(K, DM) = 0$, *and*
(ii) *any injective R-linear map $\iota : K \longrightarrow G$, where G is a Gorenstein projective left R-module, induces a surjective additive map $\iota^* : \mathrm{Hom}_R(G, DM) \longrightarrow \mathrm{Hom}_R(K, DM)$.*

Proof. (i) Let $\mathrm{pd}_R\, K = n$ and consider an exact sequence of left R-modules

$$0 \longrightarrow L \longrightarrow I_1 \longrightarrow \cdots \longrightarrow I_n \longrightarrow DM \longrightarrow 0,$$

where I_i is injective for all $i = 1, \ldots, n$. Such an exact sequence always exists (in fact, as we explained in Remark 2.3(iii), the left R-module L may be chosen to be the Pontryagin dual of a Gorenstein projective right R-module). Then, we have $\mathrm{Ext}^1_R(K, DM) = \mathrm{Ext}^{n+1}_R(K, L) = 0$, as needed.

(ii) Using once more Remark 2.3(iii), we can find a short exact sequence of left R-modules

$$0 \longrightarrow DN \longrightarrow I \xrightarrow{p} DM \longrightarrow 0,$$

where N is a Gorenstein projective right R-module and I is an injective left R-module. Then, we may consider the following commutative diagram of abelian groups with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(G, DN) \arrow[r] \arrow[d] & \mathrm{Hom}_R(G, I) \arrow[r] \arrow[d, "\iota_I^*"] & \mathrm{Hom}_R(G, DM) \arrow[d, "\iota_{DM}^*"] & \\
0 \arrow[r] & \mathrm{Hom}_R(K, DN) \arrow[r] & \mathrm{Hom}_R(K, I) \arrow[r, "p_*"] & \mathrm{Hom}_R(K, DM) \arrow[r] & \mathrm{Ext}^1_R(K, DN).
\end{tikzcd}$$

Since the right R-module N is Gorenstein projective, we may invoke (i) above and conclude that the group $\mathrm{Ext}^1_R(K, DN)$ is trivial. Therefore, the exactness of the bottom row implies that the map p_* therein is surjective. Since the left R-module I is injective, the additive map ι_I^* is surjective as well. Then, in view of the commutativity of the diagram, it follows that the additive map ι_{DM}^* is surjective, as needed. $\square$

An acyclic complex of left R-modules

$$\cdots \longrightarrow N_{i+1} \longrightarrow N_i \longrightarrow N_{i-1} \longrightarrow \cdots \tag{3}$$

is called proper if the induced complex of abelian groups

$$\cdots \longrightarrow \operatorname{Hom}_R(G, N_{i+1}) \longrightarrow \operatorname{Hom}_R(G, N_i) \longrightarrow \operatorname{Hom}_R(G, N_{i-1}) \longrightarrow \cdots$$

is acyclic for any Gorenstein projective left R-module G. If we denote by K_i the image of the map $N_i \longrightarrow N_{i-1}$, then the acyclic complex (3) is proper if and only if the short exact sequence

$$0 \longrightarrow K_{i+1} \longrightarrow N_i \longrightarrow K_i \longrightarrow 0$$

is proper for all i. A proper Gorenstein projective resolution of a left R-module N is a proper acyclic complex of left R-modules

$$\cdots \longrightarrow G_2 \longrightarrow G_1 \longrightarrow G_0 \xrightarrow{\varepsilon} N \longrightarrow 0, \tag{4}$$

such that G_i is Gorenstein projective for all $i \geqslant 0$. Using standard arguments, it is easily seen that any two proper Gorenstein projective resolutions of N are homotopy equivalent. Holm has shown in [21, Theorem 2.10] that any left R-module of finite Gorenstein projective dimension admits a proper Gorenstein projective resolution (of finite length).

Proposition 3.2. *Let R be a ring. We consider a left R-module N with $\operatorname{Gpd}_R N < \infty$ and a proper Gorenstein projective resolution $G_* \xrightarrow{\varepsilon} N \longrightarrow 0$ of it as in (4) above. Then, for any Gorenstein projective right R-module M the induced complex of abelian groups*

$$0 \longrightarrow \operatorname{Hom}_R(N, DM) \xrightarrow{\varepsilon^*} \operatorname{Hom}_R(G_0, DM) \longrightarrow \operatorname{Hom}_R(G_1, DM) \longrightarrow \operatorname{Hom}_R(G_2, DM) \longrightarrow \cdots$$

is acyclic.

Proof. Let M be a Gorenstein projective right R-module and denote by K_i the image of the map $G_i \longrightarrow G_{i-1}$ for all $i \geqslant 1$. We also define $K_0 = N$. Then, an inductive argument that uses [21, Theorem 2.24] shows that $\operatorname{Gpd}_R K_i < \infty$ for all $i \geqslant 0$. In order to verify the acyclicity of the complex in the statement, it suffices to show that the proper short exact sequence of left R-modules

$$0 \longrightarrow K_{i+1} \longrightarrow G_i \longrightarrow K_i \longrightarrow 0 \tag{5}$$

induces an exact sequence of abelian groups

$$0 \longrightarrow \operatorname{Hom}_R(K_i, DM) \longrightarrow \operatorname{Hom}_R(G_i, DM) \longrightarrow \operatorname{Hom}_R(K_{i+1}, DM) \longrightarrow 0 \tag{6}$$

for all $i \geqslant 0$. To that end, we fix the integer i and note that the finiteness of $\operatorname{Gpd}_R K_i$ implies, in view of [21, Theorem 2.10], the existence of a short exact sequence of left R-modules

$$0 \longrightarrow K \longrightarrow G \longrightarrow K_i \longrightarrow 0,$$

where $\operatorname{pd}_R K$ is finite and G is Gorenstein projective; such a short exact sequence is necessarily proper. The exact sequence above is easily seen to be homotopy equivalent to (5) and hence the exactness of (6) is equivalent to the exactness of

$$0 \longrightarrow \mathrm{Hom}_R(K_i, DM) \longrightarrow \mathrm{Hom}_R(G, DM) \longrightarrow \mathrm{Hom}_R(K, DM) \longrightarrow 0.$$

The exactness of the latter sequence follows from Lemma 3.1(ii) and this completes the proof. □

Assuming that the left R-module N admits a proper Gorenstein projective resolution as in (4) above, we define for any right R-module M the relative Tor-group $[\mathrm{Tor}_i^{\mathrm{GP}(R)}(M, \_)]N$ as the i-th homology group of the complex

$$\cdots \longrightarrow M \otimes_R G_2 \longrightarrow M \otimes_R G_1 \longrightarrow M \otimes_R G_0 \longrightarrow 0.$$

This definition does not depend upon the choice of the proper Gorenstein projective resolution of N and is natural in both M and N.

Analogously, one may fix a right R-module M and consider a proper Gorenstein projective resolution

$$\cdots \longrightarrow G'_2 \longrightarrow G'_1 \longrightarrow G'_0 \xrightarrow{\varepsilon'} M \longrightarrow 0. \tag{7}$$

If M admits such a resolution, then we define for any left R-module N the relative Tor-group $[\mathrm{Tor}_i^{\mathrm{GP}(R^{op})}(\_, N)]M$ as the i-th homology group of the complex

$$\cdots \longrightarrow G'_2 \otimes_R N \longrightarrow G'_1 \otimes_R N \longrightarrow G'_0 \otimes_R N \longrightarrow 0.$$

This definition does not depend upon the choice of the proper Gorenstein projective resolution of M and is natural in both M and N.

In particular, if M is a right R-module that admits a proper Gorenstein projective resolution $G'_* \xrightarrow{\varepsilon'} M \longrightarrow 0$ as in (7) above and N is a left R-module that admits a proper Gorenstein projective resolution $G_* \xrightarrow{\varepsilon} N \longrightarrow 0$ as in (4) above, then we may define both abelian groups $[\mathrm{Tor}_i^{\mathrm{GP}(R)}(M, \_)]N$ and $[\mathrm{Tor}_i^{\mathrm{GP}(R^{op})}(\_, N)]M$. In order to show that the latter groups are isomorphic, one may compare them (as in the case of the absolute Tor-functor) with the homology groups of the total complex $\mathrm{Tot}(G'_* \otimes_R G_*)$ of the double complex $G'_* \otimes_R G_*$.

Corollary 3.3. *Let R be a ring and consider a right R-module M of finite Gorenstein projective dimension, a left R-module N of finite Gorenstein projective dimension and two proper Gorenstein projective resolutions $G'_* \xrightarrow{\varepsilon'} M \longrightarrow 0$ and $G_* \xrightarrow{\varepsilon} N \longrightarrow 0$ as above. Then*:

(i) *The augmentation ε induces a quasi-isomorphism $1 \otimes \varepsilon : \mathrm{Tot}(G'_* \otimes_R G_*) \longrightarrow G'_* \otimes_R N$.*
(ii) *The augmentation ε' induces a quasi-isomorphism $\varepsilon' \otimes 1 : \mathrm{Tot}(G'_* \otimes_R G_*) \longrightarrow M \otimes_R G_*$.*

Proof. By symmetry, it suffices to show (i). To that end, we may use a spectral sequence argument and reduce the proof to showing that the complex of abelian groups

$$\cdots \longrightarrow G' \otimes_R G_2 \longrightarrow G' \otimes_R G_1 \longrightarrow G' \otimes_R G_0 \xrightarrow{1\otimes\varepsilon} G' \otimes_R N \longrightarrow 0$$

is acyclic for any Gorenstein projective right R-module G'. Since the abelian group $\mathbb{Q}/\mathbb{Z}$ is faithfully injective, we may use the natural isomorphism $\mathrm{Hom}_R(\_, DG') \simeq \mathrm{Hom}(G' \otimes_R \_, \mathbb{Q}/\mathbb{Z})$ and reduce further the proof to showing that the complex of abelian groups

$$0 \longrightarrow \operatorname{Hom}_R\big(N, DG'\big) \xrightarrow{\varepsilon^*} \operatorname{Hom}_R\big(G_0, DG'\big) \longrightarrow \operatorname{Hom}_R\big(G_1, DG'\big) \longrightarrow \operatorname{Hom}_R\big(G_2, DG'\big) \longrightarrow \cdots$$

is acyclic. But this follows from Proposition 3.2. □

As we have noted above, Holm has shown that any (left or right) R-module of finite Gorenstein projective dimension admits a proper Gorenstein projective resolution (cf. [21, Theorem 2.10]). Hence, the following result is an immediate consequence of Corollary 3.3.

Theorem 3.4. *Let R be a ring and consider a right R-module M and a left R-module N, which both have finite Gorenstein projective dimension. Then, the abelian groups $[\operatorname{Tor}_i^{\mathrm{GP}(R)}(M, \_\_)]N$ and $[\operatorname{Tor}_i^{\mathrm{GP}(R^{op})}(\_\_, N)]M$ are naturally isomorphic for all $i \geqslant 0$.*

4. Rings with finite Gorenstein global dimension

Let R be a ring and consider a left R-module M. We say that M admits a complete projective resolution of coincidence index n if there exists a complete projective resolution (P_*, ∂_*), which coincides with a projective resolution of M in degrees $\geqslant n$. Hence, M admits a complete projective resolution of coincidence index 0 if and only if M is a syzygy of a complete projective resolution, i.e. if and only if M is Gorenstein projective. We say that M admits a complete projective resolution if it admits a complete projective resolution of coincidence index n for some n. Using [21, Proposition 2.7], it is easily seen that the following two conditions are equivalent:

(i) M admits a complete projective resolution, and
(ii) $\operatorname{Gpd}_R M < \infty$.

In fact, if these conditions are satisfied then the Gorenstein projective dimension $\operatorname{Gpd}_R M$ of M is the smallest integer n, such that M admits a complete projective resolution of coincidence index n.

Dually, we say that a left R-module N admits a complete injective resolution of coincidence index n if there exists a complete injective resolution (I^*, ∂^*), which coincides with an injective resolution of N in degrees $\geqslant n$. Hence, N admits a complete injective resolution of coincidence index 0 if and only if N is a syzygy of a complete injective resolution, i.e. if and only if N is Gorenstein injective. We say that M admits a complete injective resolution if it admits a complete injective resolution of coincidence index n for some n. As before, one can easily show that the following two conditions are equivalent:

(i) N admits a complete injective resolution, and
(ii) $\operatorname{Gid}_R N < \infty$.

Moreover, if these conditions are satisfied then the Gorenstein injective dimension $\operatorname{Gid}_R N$ of N is the smallest integer n, such that N admits a complete injective resolution of coincidence index n.

The existence of complete projective and injective resolutions of left R-modules in the above sense has been studied by Gedrich and Gruenberg [19], Cornick and Kropholler [13] and Nucinkis [31], in connection with the existence of complete cohomological functors in the category of left R-modules: Even though they were mainly interested in the case where $R = \mathbb{Z}G$ is the integral group ring of a group G, they obtained the following result, characterizing those rings R over which all left R-modules admit complete projective and injective resolutions, in terms of the finiteness of the invariants $\operatorname{spli} R$ and $\operatorname{silp} R$. An alternative proof of some parts of this result can be found in [10]; as noted therein, the symmetric role played by the Gorenstein projective and the Gorenstein injective modules in this result enables one to define the notion of the Gorenstein global dimension of R, in analogy with the classical notion of global dimension defined in [11, Chapter VI, §2].

Theorem 4.1. *The following conditions are equivalent for a ring R:*

(i) *any left R-module M admits a complete projective resolution,*
(ii) $\mathrm{Gpd}_R\, M < \infty$ *for any left R-module M,*
(iii) *the invariants* $\mathrm{spli}\, R$ *and* $\mathrm{silp}\, R$ *are finite,*
(iv) $\mathrm{spli}\, R = \mathrm{fin.dim}\, R = \mathrm{silp}\, R < \infty$,
(v) *any left R-module N admits a complete injective resolution, and*
(vi) $\mathrm{Gid}_R\, N < \infty$ *for any left R-module N.*

If these conditions are satisfied, then

$$\sup\{\mathrm{Gpd}_R\, M\colon\ M \textit{ a left } R\textit{-module}\} = \sup\{\mathrm{Gid}_R\, N\colon\ N \textit{ a left } R\textit{-module}\} = \mathrm{spli}\, R = \mathrm{silp}\, R.$$

In that case, we say that R has finite left Gorenstein global dimension and define the common value of these four numbers to be its left Gorenstein global dimension $\mathrm{Ggl.dim}\, R$.

Proof. As we mentioned above, it is clear that (i) $\leftrightarrow$ (ii) and (v) $\leftrightarrow$ (vi). The equivalence between (iii) and (iv) has been proved in Proposition 1.3(iv). We also note that the equivalence between (i) and (iii)–(iv) is proved in [13, Theorem 3.10], whereas the equivalence between (v) and (iii)–(iv) is proved in [31, Theorem 7.9].

It only remains to establish the displayed equality, assuming that the six equivalent conditions in the statement of the theorem are satisfied. We shall prove that

$$\sup\{\mathrm{Gid}_R\, N\colon\ N \text{ a left } R\text{-module}\} = \mathrm{spli}\, R = \mathrm{silp}\, R \tag{8}$$

and let the reader supply the dual arguments, that are necessary in order to show that

$$\sup\{\mathrm{Gpd}_R\, M\colon\ M \text{ a left } R\text{-module}\} = \mathrm{spli}\, R = \mathrm{silp}\, R.$$

The validity of (8) will follow if we show the equivalence of the following two assertions for any integer n:

(α) $\mathrm{Gid}_R\, N \leqslant n$ for any left R-module N, and
(β) if $s \in \mathbb{N}$ is the common value of $\mathrm{spli}\, R$ and $\mathrm{silp}\, R$, then $s \leqslant n$.

In order to prove that $(\alpha) \to (\beta)$, we assume that $\mathrm{Gid}_R\, N \leqslant n$ for any left R-module N. Then, using [21, Theorem 2.22], me may conclude that for any injective left R-module I we have $\mathrm{pd}_R\, I \leqslant n$. Therefore, it follows that $s = \mathrm{spli}\, R \leqslant n$.

We shall now prove that $(\beta) \to (\alpha)$. Since $\mathrm{spli}\, R < \infty$, any injective left R-module has finite projective dimension. It follows easily from this that any acyclic complex of injective left R-modules is a complete injective resolution. We may then invoke the inequality $\mathrm{silp}\, R = s \leqslant n$ and use the explicit construction described in [19, §4], in order to show that any left R-module N admits a complete injective resolution of coincidence index n. It follows that $\mathrm{Gid}_R\, N \leqslant n$ for any left R-module N, as needed. □

Remarks 4.2. (i) The finiteness of the right Gorenstein global dimension of a ring R may be defined analogously, by considering right R-modules; the right Gorenstein global dimension of R will be denoted by $\mathrm{Ggl.dim}\, R^{op}$.

(ii) If the ring R has finite left global dimension, then R has also finite left Gorenstein global dimension and $\mathrm{Ggl.dim}\, R = \mathrm{gl.dim}\, R$. Indeed, as shown in [19, Corollary 1.7], the finiteness of $\mathrm{gl.dim}\, R$ implies that $\mathrm{silp}\, R = \mathrm{spli}\, R = \mathrm{gl.dim}\, R$. It is well known that the left and the right global dimensions of a ring may be different; in fact, Jategaonkar has shown in [26] that any integer may be realized as the difference between these dimensions. Therefore, it follows that the left and the right Gorenstein global dimensions of a ring may be different.

(iii) A ring R is called left $\aleph_0$-Noetherian if all left ideals of it are countably generated. Right $\aleph_0$-Noetherian rings are defined analogously. Jensen has shown in [27] that the left and the right global dimensions of a left and right $\aleph_0$-Noetherian ring are either equal to each other or else they differ by one. A similar result holds for the Gorenstein global dimensions of such a ring: Indeed, let R be a left and right $\aleph_0$-Noetherian ring of finite left and right Gorenstein global dimension. Then, as $\operatorname{Ggl.dim} R = \operatorname{spli} R$ and $\operatorname{Ggl.dim} R^{op} = \operatorname{spli} R^{op}$, we may invoke [16, Theorem 2.9] and conclude that the Gorenstein global dimensions $\operatorname{Ggl.dim} R$ and $\operatorname{Ggl.dim} R^{op}$ are either equal to each other or else they differ by one. In that direction, see also Remark 5.4(iii) below.

We shall now list a few properties that are enjoyed by rings of finite (left or right) Gorenstein global dimension.

Corollary 4.3. *(Cf. [10, Corollary 2.7].) Let R be a ring of finite left Gorenstein global dimension. Then, the following conditions are equivalent for a left R-module M:*

(i) $\operatorname{fd}_R M < \infty$,
(ii) $\operatorname{pd}_R M < \infty$, *and*
(iii) $\operatorname{id}_R M < \infty$.

Moreover, if these conditions are satisfied, then all three numbers $\operatorname{fd}_R M$, $\operatorname{pd}_R M$ and $\operatorname{id}_R M$ are $\leqslant \operatorname{Ggl.dim} R$.

Proof. Our assumption on R implies that $\operatorname{fin.dim} R = \operatorname{Ggl.dim} R < \infty$ and hence any flat left R-module has finite projective dimension (cf. [28, Proposition 6]). This shows that (i) → (ii). The implication (ii) → (iii) follows since any projective left R-module has finite injective dimension ($\operatorname{silp} R = \operatorname{Ggl.dim} R < \infty$), whereas the implication (iii) → (i) follows since any injective left R-module has finite flat dimension ($\operatorname{sfli} R \leqslant \operatorname{spli} R = \operatorname{Ggl.dim} R < \infty$).

The final assertion in the statement follows since for any left R-module M that satisfies (ii) and (iii) we have $\operatorname{fd}_R M \leqslant \operatorname{pd}_R M = \operatorname{Gpd}_R M \leqslant \operatorname{Ggl.dim} R$ and $\operatorname{id}_R M = \operatorname{Gid}_R M \leqslant \operatorname{Ggl.dim} R$ (cf. [21, Theorem 2.22 and Proposition 2.27]). □

Corollary 4.4. *If R is a ring of finite right Gorenstein global dimension, then $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$.*

Proof. Our assumption on R implies that $\operatorname{sfli} R^{op} \leqslant \operatorname{spli} R^{op} = \operatorname{Ggl.dim} R^{op} < \infty$. Then, the result follows invoking Remark 2.3(ii). □

5. Rings with finite Gorenstein weak dimension

Even though the left and the right global dimensions of a ring R may be different, the weak dimension of R is always left–right symmetric. In this section, we examine the extent to which a similar phenomenon occurs in Gorenstein homological algebra as well.

Lemma 5.1. *Let R be a ring and assume that $\operatorname{Gfd}_R N < \infty$ for any left R-module N. Then, $\operatorname{sfli} R^{op} < \infty$.*

Proof. We have to show that any injective right R-module I has finite flat dimension. To that end, we fix an injective right R-module I and consider for any left R-module N the subset $A_N \subseteq \mathbb{N}$, which is defined by letting

$$A_N = \{i \in \mathbb{N}\colon \operatorname{Tor}_j^R(I, N) = 0 \text{ for all } j > i\}.$$

Since $\operatorname{Gfd}_R N < \infty$, the set A_N is non-empty; in fact, $\operatorname{Gfd}_R N \in A_N$. Let $i_N = \inf A_N$ be the smallest element of A_N.

Then, the set $\{i_N\colon\ N \text{ a left } R\text{-module}\}$ is bounded. In order to verify this claim, it suffices to show that for any sequence N_t, $t = 0, 1, 2, \ldots$ of left R-modules there is an integer i, such that $i_{N_t} \leqslant i$ for all t. To that end, we consider the direct sum $N = \bigoplus_{t=0}^{\infty} N_t$ and let $i = i_N$. Since the group $\mathrm{Tor}_j^R(I, N)$ vanishes (if and) only if the groups $\mathrm{Tor}_j^R(I, N_t)$ vanish for all t, we conclude that $i_{N_t} \leqslant i_N = i$ for all $t = 0, 1, 2, \ldots$. Hence, the set $\{i_N\colon\ N \text{ a left } R\text{-module}\}$ is indeed bounded.

We now consider the maximum element i of the set $\{i_N\colon\ N \text{ a left } R\text{-module}\}$. Then, we have $\mathrm{Tor}_{i+1}^R(I, N) = 0$ for all left R-modules N (since $i + 1 > i \geqslant i_N$) and hence we conclude that $\mathrm{fd}_{R^{op}} I \leqslant i < \infty$, as needed. $\square$

Lemma 5.2. *Let R be a ring and assume that* $\mathrm{sfli}\, R = n < \infty$*. Then, for any left R-module N there exists an acyclic complex of flat left R-modules*

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots,$$

which coincides in degrees $\geqslant n$ with a projective resolution

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

of N.

Proof. We follow closely the argument by Gedrich and Gruenberg in [19, §4]: We begin with an injective resolution

$$0 \longrightarrow N \longrightarrow I^0 \longrightarrow I^1 \longrightarrow I^2 \longrightarrow \cdots$$

and then construct a projective resolution of the latter as in [11, Chapter XVII, §1]

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_1 & \longrightarrow & P_1^0 & \longrightarrow & P_1^1 & \longrightarrow & P_1^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P_0 & \longrightarrow & P_0^0 & \longrightarrow & P_0^1 & \longrightarrow & P_0^2 & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & I^2 & \longrightarrow & \cdots
\end{array}$$

In this way, the images of the vertical maps at the n-th level form an exact sequence

$$0 \longrightarrow \Omega^n N \longrightarrow \Omega^n I^0 \longrightarrow \Omega^n I^1 \longrightarrow \Omega^n I^2 \longrightarrow \cdots.$$

In view of our assumption about the value of $\mathrm{sfli}\, R$, the left R-modules $\Omega^n I^0, \Omega^n I^1, \Omega^n I^2, \ldots$ are flat. Therefore, gluing the exact sequence above with the exact sequence

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \Omega^n N \longrightarrow 0,$$

we obtain an exact sequence

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow \Omega^n I^0 \longrightarrow \Omega^n I^1 \longrightarrow \Omega^n I^2 \longrightarrow \cdots,$$

that has all of the required properties. □

Theorem 5.3. *The following conditions are equivalent for a ring R:*

(i) $\mathrm{Gfd}_{R^{op}} M < \infty$ *for any right R-module M and* $\mathrm{Gfd}_R N < \infty$ *for any left R-module N,*
(ii) *the invariants* $\mathrm{sfli}\, R$ *and* $\mathrm{sfli}\, R^{op}$ *are finite,*
(iii) $\mathrm{sfli}\, R = \mathrm{fin.f.dim}\, R = \mathrm{sfli}\, R^{op} = \mathrm{fin.f.dim}\, R^{op} < \infty$.

If these conditions are satisfied, then

$$\sup\{\mathrm{Gfd}_{R^{op}} M\colon\ M \text{ a right } R\text{-module}\} = \sup\{\mathrm{Gfd}_R N\colon\ N \text{ a left } R\text{-module}\} = \mathrm{sfli}\, R = \mathrm{sfli}\, R^{op}.$$

In that case, we say that R has finite Gorenstein weak dimension and define the common value of these four numbers to be its Gorenstein weak dimension $\mathrm{Gw.dim}\, R$.

Proof. The implication (i) → (ii) is an immediate consequence of Lemma 5.1, whereas the equivalence between (ii) and (iii) has been proved in Proposition 1.4(iii). Therefore, it only remains to prove that (ii) → (i). To that end, we note that the hypothesis about the finiteness of $\mathrm{sfli}\, R^{op}$ implies that any injective right R-module has finite flat dimension; it follows that any acyclic complex of flat left R-modules is a complete flat resolution. Therefore, if N is a left R-module and $\mathrm{sfli}\, R = n$, then Lemma 5.2 provides us with a complete flat resolution

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow F_{n-1} \longrightarrow F_{n-2} \longrightarrow \cdots,$$

which coincides in degrees $\geqslant n$ with a projective resolution

$$\cdots \longrightarrow P_{n+1} \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

of N. We now consider the Gorenstein flat left R-module $F = \mathrm{coker}(P_{n+1} \longrightarrow P_n)$ and note that the exact sequence

$$0 \longrightarrow F \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_0 \longrightarrow N \longrightarrow 0$$

is a resolution of N by Gorenstein flat left R-modules; hence, $\mathrm{Gfd}_R N \leqslant n$. Since condition (ii) is left–right symmetric, a similar argument shows that $\mathrm{Gfd}_{R^{op}} M < \infty$ for any right R-module M, completing thereby the proof of the implication (ii) → (i).

We shall now establish the displayed equality, assuming that the three equivalent conditions in the statement of the theorem are satisfied. By left–right symmetry, we only need to show that

$$\sup\{\mathrm{Gfd}_R N\colon\ N \text{ a left } R\text{-module}\} = \mathrm{sfli}\, R = \mathrm{sfli}\, R^{op}.$$

Let $m = \sup\{\mathrm{Gfd}_R N\colon\ N \text{ a left } R\text{-module}\}$ and define n to be the common value of $\mathrm{sfli}\, R$ and $\mathrm{sfli}\, R^{op}$. The argument used above in order to prove that (ii) → (i) shows that $\mathrm{Gfd}_R N \leqslant n$ for any left R-module N and hence $m \leqslant n$. On the other hand, we have $\mathrm{Gfd}_R N \leqslant m$ for any left R-module N. Hence, if I is an injective right R-module, then the group $\mathrm{Tor}^R_{m+1}(I, N)$ is trivial for any left R-module N. It follows that $\mathrm{fd}_{R^{op}} I \leqslant m$ and hence $n = \mathrm{sfli}\, R^{op} \leqslant m$. □

Remarks 5.4. (i) A weaker form of Theorem 5.3 has been proved in [30, Theorem 2.4].

(ii) Let R be a ring of finite left and right Gorenstein global dimension. Then, we have $\operatorname{sfli} R \leqslant \operatorname{spli} R = \operatorname{Ggl.dim} R$ and $\operatorname{sfli} R^{op} \leqslant \operatorname{spli} R^{op} = \operatorname{Ggl.dim} R^{op}$. Therefore, R has finite Gorenstein weak dimension; in fact, we have

$$\operatorname{Gw.dim} R \leqslant \min\{\operatorname{Ggl.dim} R, \operatorname{Ggl.dim} R^{op}\}.$$

The latter inequality strengthens that obtained in [10, Corollary 1.2(1)].

(iii) Let R be a left and right $\aleph_0$-Noetherian ring of finite left and right Gorenstein global dimension and consider its Gorenstein weak dimension $\operatorname{Gw.dim} R = n$. In this case, we may obtain a sharper version of the inequality in (ii) above: Indeed, since $n = \operatorname{Gw.dim} R = \operatorname{sfli} R = \operatorname{sfli} R^{op}$, $\operatorname{Ggl.dim} R = \operatorname{spli} R$ and $\operatorname{Ggl.dim} R^{op} = \operatorname{spli} R^{op}$, we may invoke [16, Theorem 2.9] and conclude that each of the numbers $\operatorname{Ggl.dim} R$ and $\operatorname{Ggl.dim} R^{op}$ is either equal to n or to $n+1$.

(iv) If the ring R has finite weak dimension, then R has also finite Gorenstein weak dimension; in fact, we then have $\operatorname{Gw.dim} R = \operatorname{w.dim} R$. Indeed, it is clear that both $\operatorname{sfli} R$ and $\operatorname{sfli} R^{op}$ are bounded by the weak dimension $\operatorname{w.dim} R$ of R and hence $\operatorname{sfli} R = \operatorname{sfli} R^{op} \leqslant \operatorname{w.dim} R$. On the other hand, if $\operatorname{w.dim} R = n$, then $\operatorname{Tor}_n^R(M,N) \neq 0$ for a suitable pair consisting of a right R-module M and a left R-module N. If I is an injective right R-module containing M, then it is easily seen that we also have $\operatorname{Tor}_n^R(I,N) \neq 0$. It follows that $\operatorname{fd}_{R^{op}} I \geqslant n$ and hence $\operatorname{sfli} R^{op} \geqslant n$.

Corollary 5.5. *If R is a ring of finite Gorenstein weak dimension, then $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$ and $\mathtt{GP}(R^{op}) \subseteq \mathtt{GF}(R^{op})$.*

Proof. In view of our assumption on R, both invariants $\operatorname{sfli} R$ and $\operatorname{sfli} R^{op}$ are finite; hence, the result follows invoking Remark 2.3(ii). □

6. Some examples

In this final section, we present a few examples of rings of finite Gorenstein global or weak dimension.

6.1. $\aleph_0$-Noetherian rings

We recall that a Gorenstein ring is a commutative Noetherian ring of finite self-injective dimension. In the non-commutative setting, one defines a ring R to be an Iwanaga–Gorenstein ring if R is left and right Noetherian and has finite left and right self-injective dimensions. We present below a class consisting of certain rings of finite left and right Gorenstein global dimensions, which contains the class of Iwanaga–Gorenstein rings.

To begin with, we consider a ring R and note that the inequality $\operatorname{id}_R R \leqslant \operatorname{silp} R$ is actually an equality, in the case where R is left Noetherian. Indeed, assuming that $\operatorname{id}_R R = n < \infty$, we may construct an injective resolution of the left regular module R of length n and then obtain an injective resolution of any free left R-module of length n, by taking direct sums; the point is that, in this case, a direct sum of injective left R-modules is injective as well. In view of this observation, the class of rings described below includes the Iwanaga–Gorenstein rings.

Proposition 6.1. *Let R be a left and right $\aleph_0$-Noetherian ring and assume that both invariants $\operatorname{silp} R$ and $\operatorname{silp} R^{op}$ are finite. Then, R has finite left and right Gorenstein global dimensions; in fact, we have $\operatorname{Ggl.dim} R = \operatorname{silp} R$ and $\operatorname{Ggl.dim} R^{op} = \operatorname{silp} R^{op}$.*

Proof. As shown in [15, Theorem 3.6], our assumptions on the ring R imply that $\operatorname{silp} R = \operatorname{spli} R < \infty$ and $\operatorname{silp} R^{op} = \operatorname{spli} R^{op} < \infty$. The result is then an immediate consequence of Theorem 4.1. □

Remarks 6.2. (i) If R is an Iwanaga–Gorenstein ring, then $\mathrm{id}_R R = \mathrm{id}_{R^{op}} R$; this is shown in [36, §5, Lemma A]. Moreover, the left and the right Gorenstein global dimensions of R are then equal to the common value of $\mathrm{id}_R R$ and $\mathrm{id}_{R^{op}} R$; this follows since $\mathrm{Ggl.dim}\, R = \mathrm{silp}\, R = \mathrm{id}_R R$ and $\mathrm{Ggl.dim}\, R^{op} = \mathrm{silp}\, R^{op} = \mathrm{id}_{R^{op}} R$. The class of Iwanaga–Gorenstein rings consists precisely of those rings of finite left and right Gorenstein global dimensions that are left and right Noetherian.

(ii) If R is a left coherent ring, then $\mathrm{sfli}\, R^{op} \leqslant \mathrm{id}_R R$. In order to verify this assertion, we may assume that $\mathrm{id}_R R = n < \infty$. (If $\mathrm{id}_R R = \infty$, then the inequality to be proved is obvious.) Then, the left regular module R has an injective resolution of length n. Since R is left coherent, the right R-module $\mathrm{Hom}(I, D)$ is flat for any injective left R-module I and any divisible abelian group D (cf. [35, Lemma 3.1.4]). It follows that the right R-module $\mathrm{Hom}(R, D)$ has a flat resolution of length n (and hence its flat dimension is $\leqslant n$) for any divisible abelian group D. Since any injective right R-module is a direct summand of $\mathrm{Hom}(R, D)$, for some divisible abelian group D, we conclude that $\mathrm{sfli}\, R^{op} \leqslant n$, as needed.

(iii) Let R be a ring, which is left and right coherent and has finite left and right self-injective dimensions.[1] Then, R has finite Gorenstein weak dimension and

$$\mathrm{Gw.dim}\, R \leqslant \min\{\mathrm{id}_R R, \mathrm{id}_{R^{op}} R\}.$$

Indeed, the left coherence of R implies that $\mathrm{sfli}\, R^{op} \leqslant \mathrm{id}_R R < \infty$ (cf. (ii) above) and we also have the symmetric inequality $\mathrm{sfli}\, R \leqslant \mathrm{id}_{R^{op}} R < \infty$, as a consequence of the right coherence of R. Then, the invariants $\mathrm{sfli}\, R$ and $\mathrm{sfli}\, R^{op}$ are equal to each other (as shown in Proposition 1.4(iii)) and their common value is $\leqslant \min\{\mathrm{id}_R R, \mathrm{id}_{R^{op}} R\}$.

As shown in [16, Corollary 3.2], if R is a left $\aleph_0$-Noetherian ring, then $\mathrm{silp}\, R = \mathrm{id}_R R^{(\mathbb{N})}$, where $R^{(\mathbb{N})}$ is the free left R-module with an infinite countable basis. Even though the inequality $\mathrm{id}_R R \leqslant \mathrm{silp}\, R$ may very well be strict, it is shown in [7, Theorem 3.3] that $\mathrm{id}_R R = \mathrm{silp}\, R$ if the ring R is countable. Since countable rings are, of course, left and right $\aleph_0$-Noetherian, the following result is really a special case of Proposition 6.1.

Corollary 6.3. *A countable ring R of finite left and right self-injective dimensions has finite left and right Gorenstein global dimensions; in fact, we have* $\mathrm{Ggl.dim}\, R = \mathrm{id}_R R$ *and* $\mathrm{Ggl.dim}\, R^{op} = \mathrm{id}_{R^{op}} R$.

Some information about the relation between Gorenstein projective and Gorenstein flat modules may be also obtained from one-sided assumptions on the ring R, as described below.

Proposition 6.4.

(i) *If R is a left $\aleph_0$-Noetherian ring and* $\mathrm{silp}\, R < \infty$, *then* $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$.
(ii) *If R is a countable ring of finite left self-injective dimension, then* $\mathtt{GP}(R) \subseteq \mathtt{GF}(R)$.

Proof. (i) In view of [15, Proposition 3.2], our assumptions on the ring R imply that we also have $\mathrm{sfli}\, R^{op} < \infty$. Hence, the result follows invoking Remark 2.3(ii).

(ii) Since the ring R is countable, we have $\mathrm{silp}\, R = \mathrm{id}_R R$ (cf. [7, Theorem 3.3]). Hence, the result is a special case of (i). □

6.2. Group rings

We now consider the special case where $R = \mathbb{Z}G$ is the integral group ring of a group G. In this case, the ring R is isomorphic with its opposite ring R^{op} and hence there is no need to distinguish

[1] These conditions on a ring are weaker than those defining the Iwanaga–Gorenstein rings.

between left and right modules. Moreover, as shown in [15, Corollary 4.5], we always have an equality $\operatorname{silp}\mathbb{Z}G = \operatorname{spli}\mathbb{Z}G$. Therefore, the following result is an immediate consequence of Theorem 4.1, Theorem 5.3 and Remark 2.3(ii):

Proposition 6.5. *Let G be a group. Then*:

(i) *The group ring $\mathbb{Z}G$ has finite Gorenstein global dimension if and only if* $\operatorname{spli}\mathbb{Z}G < \infty$.
(ii) *The group ring $\mathbb{Z}G$ has finite Gorenstein weak dimension if and only if* $\operatorname{sfli}\mathbb{Z}G < \infty$.
(iii) *If the assumptions in either* (i) *or* (ii) *above are satisfied, then* $\mathtt{GP}(\mathbb{Z}G) \subseteq \mathtt{GF}(\mathbb{Z}G)$.

Proposition 6.6. *Let G be a group, which may be expressed as the filtered union of a family $(G_\lambda)_\lambda$ of subgroups of it, such that* $\operatorname{sfli}\mathbb{Z}G_\lambda < \infty$ *for any index λ. Then,* $\mathtt{GP}(\mathbb{Z}G) \subseteq \mathtt{GF}(\mathbb{Z}G)$.

Proof. In view of Theorem 2.2, we have to show that any complete projective resolution of $\mathbb{Z}G$-modules is a complete flat resolution. In fact, we shall prove that any acyclic complex (F_*, ∂_*) of flat $\mathbb{Z}G$-modules is a complete flat resolution. To that end, we fix an index λ and consider the subgroup G_λ of G. The complex (F_*, ∂_*) may be viewed as an acyclic complex of flat $\mathbb{Z}G_\lambda$-modules. Since $\operatorname{sfli}\mathbb{Z}G_\lambda < \infty$, all injective $\mathbb{Z}G_\lambda$-modules have finite flat dimension. It follows that the functor $I \otimes_{\mathbb{Z}G_\lambda} \_\_$ preserves the acyclicity of complexes of flat $\mathbb{Z}G_\lambda$-modules for any injective $\mathbb{Z}G_\lambda$-module I. Hence, the complex (F_*, ∂_*) is a complete flat resolution over $\mathbb{Z}G_\lambda$. This being the case for any index λ, we claim that the complex (F_*, ∂_*) is, in fact, a complete flat resolution over $\mathbb{Z}G$. To prove that claim, let I be an injective $\mathbb{Z}G$-module. Then, I is an injective $\mathbb{Z}G_\lambda$-module and hence the complex of abelian groups $(I \otimes_{\mathbb{Z}G_\lambda} F_*, 1 \otimes \partial_*)$ is acyclic for any index λ. Since the complex $(I \otimes_{\mathbb{Z}G} F_*, 1 \otimes \partial_*)$ is the filtered colimit of the complexes $(I \otimes_{\mathbb{Z}G_\lambda} F_*, 1 \otimes \partial_*)$, it follows that $(I \otimes_{\mathbb{Z}G} F_*, 1 \otimes \partial_*)$ is acyclic as well. Therefore, (F_*, ∂_*) is a complete flat resolution over $\mathbb{Z}G$, as needed. □

Corollary 6.7. *Let G be a group and assume that* $\operatorname{sfli}\mathbb{Z}H$ *is finite for any finitely generated subgroup $H \subseteq G$. Then,* $\mathtt{GP}(\mathbb{Z}G) \subseteq \mathtt{GF}(\mathbb{Z}G)$.

Proof. Since G is the filtered union of its finitely generated subgroups, the result is an immediate consequence of Proposition 6.6. □

Examples 6.8. (i) Let G be the ascending union of a sequence of countable subgroups $(G_n)_n$ of finite cohomological dimension, such that $\lim_n \operatorname{cd} G_n = \infty$. (For example, G may be the free abelian group of infinite countable rank with $G_n = \mathbb{Z}^n$ for all n.) Then, $\operatorname{sfli}\mathbb{Z}G_n \leqslant \operatorname{spli}\mathbb{Z}G_n = \operatorname{silp}\mathbb{Z}G_n \leqslant \operatorname{cd} G_n + 1 < \infty$ (for the second inequality, see [25, Lemma 1.8(a)]) and hence we may conclude, invoking Proposition 6.6, that $\mathtt{GP}(\mathbb{Z}G) \subseteq \mathtt{GF}(\mathbb{Z}G)$. On the other hand, we have

$$\operatorname{cd} G_n \leqslant \operatorname{silp}\mathbb{Z}G_n = \operatorname{spli}\mathbb{Z}G_n \leqslant \operatorname{sfli}\mathbb{Z}G_n + 1 \leqslant \operatorname{sfli}\mathbb{Z}G + 1.$$

The second inequality above follows from [16, Theorem 4.8(i)], since G_n is countable, whereas the third one follows since for any subgroup $H \subseteq G$ we have $\operatorname{sfli}\mathbb{Z}H \leqslant \operatorname{sfli}\mathbb{Z}G$.[2] Hence, we conclude that $\operatorname{sfli}\mathbb{Z}G = \infty$. It follows that the hypothesis about the finiteness of the invariant sfli in Remark 2.3(ii), even though sufficient, is not a necessary condition for the class of Gorenstein projective modules to be contained in the class of Gorenstein flat modules.

(ii) Let G be a group admitting a finite dimensional model for $\underline{E}G$, the classifying space for proper actions. Then, as shown in [34, Proposition 3.1], we have $\operatorname{spli}\mathbb{Z}G < \infty$; in fact, if there is an n-dimensional model for $\underline{E}G$, then $\operatorname{spli}\mathbb{Z}G \leqslant n+1$. It follows that the group ring $\mathbb{Z}G$ has finite Gorenstein global dimension and, in particular, we have $\mathtt{GP}(\mathbb{Z}G) \subseteq \mathtt{GF}(\mathbb{Z}G)$.

[2] This is clear, since any injective $\mathbb{Z}H$-module is a direct summand of the restriction to H of an injective $\mathbb{Z}G$-module, whereas the restriction to H of any flat $\mathbb{Z}G$-module is a flat $\mathbb{Z}H$-module.

References

[1] L. Angeleri-Hügel, D. Herbera, Mittag–Leffler conditions on modules, Indiana Univ. Math. J. 57 (2008) 2459–2517.
[2] M. Auslander, Anneaux de Gorenstein, et torsion en algèbre commutative, in: Séminaire d'Algèbre Commutative dirigé par Pierre Samuel, 1966/67, Secrétariat mathématique, Paris, 1967; Texte rédigé, d'après des exposés de Maurice Auslander, par Marguerite Mangeney, Christian Peskine et Lucien Szpiro, École Normale Supérieure de Jeunes Filles.
[3] M. Auslander, M. Bridger, Stable Module Theory, Mem. Amer. Math. Socs., vol. 94, American Mathematical Society, Providence, RI, 1969.
[4] M. Auslander, D.A. Buchsbaum, Homological dimension in local rings, Trans. Amer. Math. Soc. 85 (1957) 390–405.
[5] L.L. Avramov, A. Martsinkovsky, Absolute, relative and Tate cohomology of modules of finite Gorenstein dimension, Proc. Lond. Math. Soc. 85 (2002) 393–440.
[6] A. Bahlekeh, F. Dembegioti, O. Talelli, Gorenstein dimension and proper actions, Bull. Lond. Math. Soc. 41 (2009) 859–871.
[7] D. Ballas, On certain homological finiteness conditions, preprint.
[8] A. Beligiannis, H. Krause, Thick subcategories and virtually Gorenstein algebras, Illinois J. Math. 52 (2008) 551–562.
[9] D. Bennis, A note on Gorenstein flat dimension, Algebra Colloq. 18 (2011) 155–161.
[10] D. Bennis, N. Mahbou, Global Gorenstein dimensions, Proc. Amer. Math. Soc. 138 (2010) 461–465.
[11] H. Cartan, S. Eilenberg, Homological Algebra, Princeton University Press, 1956.
[12] L.W. Christensen, Gorenstein Dimensions, Lecture Notes in Math., vol. 1747, Springer, Berlin, 2000.
[13] J. Cornick, P.H. Kropholler, On complete resolutions, Topology Appl. 78 (1997) 235–250.
[14] J. Cornick, P.H. Kropholler, Homological finiteness conditions for modules over group algebras, J. Lond. Math. Soc. 58 (1998) 49–62.
[15] I. Emmanouil, On certain cohomological invariants of groups, Adv. Math. 225 (2010) 3446–3462.
[16] I. Emmanouil, O. Talelli, On the flat length of injective modules, J. Lond. Math. Soc. 84 (2011) 408–432.
[17] E.E. Enochs, O.M.G. Jenda, Gorenstein injective and projective modules, Math. Z. 220 (1995) 611–633.
[18] E.E. Enochs, O.M.G. Jenda, B. Torrecillas, Gorenstein flat modules, Nanjing Daxue Xuebao Shuxue Bannian Kan 10 (1993) 1–9.
[19] T.V. Gedrich, K.W. Gruenberg, Complete cohomological functors on groups, Topology Appl. 25 (1987) 203–223.
[20] A. Grothendieck, EGA III, Publ. Math. Inst. Hautes Etudes Sci. 11 (1961).
[21] H. Holm, Gorenstein homological dimensions, J. Pure Appl. Algebra 189 (2004) 167–193.
[22] H. Holm, Gorenstein derived functors, Proc. Amer. Math. Soc. 132 (2004) 1913–1923.
[23] H. Holm, P. Jørgensen, Rings without a Gorenstein analogue of the Govorov–Lazard theorem, Q. J. Math. 62 (2011) 977–988.
[24] B.M. Ikenaga, Homological dimension and Farrell cohomology, J. Algebra 87 (1984) 422–457.
[25] B.M. Ikenaga, Generalized cohomological dimension, J. Pure Appl. Algebra 40 (1986) 125–134.
[26] A.V. Jategaonkar, A counter-example in ring theory and homological algebra, J. Algebra 12 (1969) 418–440.
[27] C.U. Jensen, On homological dimensions of rings with countably generated ideals, Math. Scand. 18 (1966) 97–105.
[28] C.U. Jensen, On the vanishing of $\varprojlim^i$, J. Algebra 15 (1970) 151–166.
[29] D. Lazard, Autour de la platitude, Bull. Soc. Math. France 97 (1969) 81–128.
[30] N. Mahbou, M. Tamekkante, Note on (weak) Gorenstein global dimensions, arXiv:0910.5752v1 [math.AC].
[31] B.E.A. Nucinkis, Complete cohomology for arbitrary rings using injectives, J. Pure Appl. Algebra 131 (1998) 297–318.
[32] M. Raynaud, L. Gruson, Critères de platitude et de projectivité, Invent. Math. 13 (1971) 1–89.
[33] J.P. Serre, Sur la dimension des anneaux et des modules noethériens, in: Proceedings of the International Symposium on Algebraic Number Theory, Tokyo and Nikko, 1955, Science Council of Japan, Tokyo, 1956.
[34] O. Talelli, On groups of type Φ, Arch. Math. 89 (2007) 24–32.
[35] J. Xu, Flat Covers of Modules, Lecture Notes in Math., vol. 1634, Springer, Berlin, 1996.
[36] A. Zaks, Injective dimension of semi-primary rings, J. Algebra 13 (1969) 73–86.