Annals of Mathematics, **136** (1992), 51–122

Excision in algebraic K-theory

By ANDREI A. SUSLIN and MARIUSZ WODZICKI*

Contents

Introduction

The algebraic K-theory groups $K_i(R)$, $i \geq 1$, of a ring with unit R are defined as the homotopy groups of $\mathrm{BGL}(R)^+$, the Quillen plus construction [29] applied to the space $\mathrm{BGL}(R)$:

$$K_i(R) = \pi_i(\mathrm{BGL}(R)^+), \qquad i \geq 1. \tag{1}$$

One has $\pi_1(\mathrm{BGL}(R)^+) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)]$ and $\mathrm{H}_*(\mathrm{BGL}(R)) = \mathrm{H}_*(\mathrm{BGL}(R)^+)$, and the space $\mathrm{BGL}(R)^+$ is an infinite loop space (cf. [38]). If $A \subset R$ is a two-sided ideal, let $F(R, A)$ denote the homotopy fiber of the map $\mathrm{BGL}(R)^+ \to \mathrm{B}\overline{\mathrm{GL}}(R/A)^+$, where $\overline{\mathrm{GL}}(R/A) := \mathrm{Im}(\mathrm{GL}(R) \to \mathrm{GL}(R/A))$. The relative K-groups $K_i(R, A)$, $i \geq 1$, are defined as the homotopy groups of $F(R, A)$ so that one gets, with respect to a pair (R, A), a functorial long exact

*The second author was partially supported by NSF Grant DMS-8910928.

sequence

$$\begin{aligned}(2)\qquad \cdots \to K_i(R,A) \to K_i(R) \to K_i(R/A) \to K_{i-1}(R,A) \to \dots \\ \to K_1(R,A) \to K_1(R) \to K_1(R/A).\end{aligned}$$

Algebraic K-theory is extended to the category of rings without unit by setting $K_i(A) = K_i(\tilde{A}, A)$, $i \geq 1$, where $\tilde{A} = \mathbb{Z} \ltimes A$ denotes the ring obtained from A by adjoining the unit. It is easy to see that this agrees with the previous definition if A is unital.

A ring A is said to satisfy excision in algebraic K-theory if for every ring with unit R, which contains A as a two-sided ideal, the canonical map $K_*(A) \to K_*(R,A)$ is an isomorphism. The excision properties in rational K-theory, or in K-theory with finite coefficients, are defined similarly, by the replacement everywhere of the integral K-groups by the rational K-groups, or by K-groups with finite coefficients.

If A satisfies excision and R is an arbitrary ring containing A as a two-sided ideal, the exact sequence (2) takes the following form:

$$\begin{aligned}(3)\qquad \cdots \to K_i(A) \to K_i(R) \to K_i(R/A) \to K_{i-1}(A) \to \dots \\ \to K_1(A) \to K_1(R) \to K_1(R/A).\end{aligned}$$

We do not need to require in (3) that R have a unit.

Recall that a $\mathbb{Q}$-algebra A is said to be *homologically unital* (abbreviated to "H-unital") if the following chain complex is acyclic:

$$A \xleftarrow{b'} A \otimes_{\mathbb{Q}} A \xleftarrow{b'} A \otimes_{\mathbb{Q}} A \otimes_{\mathbb{Q}} A \xleftarrow{b'} \dots,$$

$b'(a_1 \otimes \cdots \otimes a_q) = \sum_{i=1}^{q-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q$; (cf. [43],[40]).

The second author proved in [40] that if a ring A satisfies excision in rational K-theory, then the $\mathbb{Q}$-algebra $A_{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}$ satisfies the analogous excision property in cyclic homology, and that the latter is equivalent to the H-unitality of $A_{\mathbb{Q}}$. He also conjectured the converse, i.e., that the H-unitality of $A_{\mathbb{Q}}$ implies that A satisfies excision in rational algebraic K-theory. Our main objective in this article is to prove this conjecture.

THEOREM A. *For any ring A the following properties are equivalent:*
(a) *A satisfies excision in rational algebraic* K-*theory;*
(b) *$A_{\mathbb{Q}}$ is H-unital.*

A ring A satisfies excision in algebraic K-theory if and only if it satisfies excision in both rational algebraic K-theory and algebraic K-theory with $\mathbb{Z}/p\mathbb{Z}$ coefficients for all primes p. One easily sees that the last excision property holds automatically when A is a $\mathbb{Q}$-algebra (cf. [39] and below). Thus Theorem A implies the following result:

THEOREM B. *For a $\mathbb{Q}$-algebra A the following conditions are equivalent:*
(a) *A satisfies excision in algebraic* K*-theory;*
(b) *A is H-unital.*

Another class of rings for which we prove excision in (integral) algebraic K-theory occurs in our next theorem.

THEOREM C. *Suppose that for any finite collection of elements $a_1, \dots, a_m$ of a ring A there exist $b_1, \dots, b_m, c, d \in A$ such that $a_i = b_i cd$, $1 \le i \le m$, and the left annihilators of elements cd and c coincide* (cf. [41], Property Φ_0). *Then A satisfies excision in algebraic* K*-theory.*

Remarks. (1) If A satisfies excision, so does the opposite ring A^{op}. This observation means that Theorem C is equally valid for rings with the opposite factorization property (i.e., $a_i = dcb_i$).

(2) All rings for which we are able to prove excision in algebraic K-theory are H-unital as $\mathbb{Z}$-algebras (for the meaning of this term see ([40], §3) or §7 of the present article). The exact relation between the H-unitality of rings (as $\mathbb{Z}$-algebras) and the excision in algebraic K-theory is not completely clear yet. It is reasonable to expect that H-unitality implies excision (at least for rings whose additive groups are torsion-free). For partial results concerning K_1 and K_2 see ([36], Thm. 14.2), [18] and [42].

(3) In all cases, when we are able to prove excision in algebraic K-theory we prove, in fact, a stronger assertion, namely that $\mathrm{BGL}(A)^+$ is an infinite loop space and the canonical map $\mathrm{BGL}(A)^+ \to F(R, A)$ is a homotopy equivalence for any ring R containing A as a two-sided ideal. In particular, in all those cases, the groups $K_i(A)$, $i \ge 1$, satisfy equality (1).

Our excision theorems find an immediate application in several problems involving rings of functional-analytic type, since the latter, as was demonstrated in [41], are often H-unital. In the present article we show, e.g., that *all* C^*-algebras satisfy excision in algebraic K-theory and then we use this fact to give a proof of a long-standing conjecture whose origin goes back to M. Karoubi's article [21], which predicts the equality of the algebraic and the topological K-theories on the category of stable C^*-algebras. We shall refer to it as Karoubi's Conjecture.

The proof of Theorems A and B is spread over Sections 1, 2, 4, 5 and 6. Theorem C is proved in Section 3 by the use of the material of Section 1. In Section 7 we further develop the theory of H-unital algebras and prove the following important result conjectured earlier by the second author:

(4) The category of H-unital k-algebras is closed with respect to the tensor product.

(See Theorem 7.10 below; a weaker form of (4) has been proved in [40], Cor. 9.6.) This result is used later in Section 8 to prove the Bar and cyclic acyclicity of matrix algebras of certain preadditive categories (Theorem 8.8). A word of warning to the reader: in the definition of a category we drop the axiom requiring that every object be equipped with the identity morphism. In our view a number of important examples warrant this shift in terminology. The acyclicity results of Section 8 allow us to establish in Section 9 the acyclicity of the "triangular" space $\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A)$ and of its two "affine" variants. These spaces play an essential role in the proof of our excision theorems A, B and C. Finally in Section 10 we prove Karoubi's Conjecture.

Let us mention that the Lie analogue of Volodin's construction, which is the subject of Section 4, plays an important role in Goodwillie's work [13],[12]. It should also be clear that our presentation of Malcev's theory in Section 6 overlaps with [30], Appendix A, and [13].

A few comments about the K_i-groups for $i < 1$. Recall that, in the unital case, $K_0(R)$ is the Grothendieck K-functor of the category of finitely generated, projective right R-modules and that the negative K-groups can be defined by the following recursive formula of Bass ([1],p. 677):

$$K_{-i}(R) := \operatorname{Coker}\bigl(K_{1-i}(R) \to K_{1-i}\bigl(R[t,t^{-1}]\bigr)\bigr) \qquad (i > 0).$$

For general rings one sets $K_{-i}(A) := \operatorname{Ker}(K_{-i}(\tilde{A}) \to K_{-i}(\mathbb{Z}))$. If A is an arbitrary two-sided ideal in a ring R, then a theorem of Bass ([1], Thm. XII.8.3; cf. also [22], pp. 295–298) asserts the existence of a functorial long exact sequence

$$\begin{aligned}(5)\qquad & K_1(R) \to K_1(R/A) \to K_0(A) \to K_0(R) \to \dots \\ & \quad \to K_{-i}(R) \to K_{-i}(R/A) \to K_{-i-1}(A) \to K_{-i-1}(R) \to \dots\ .\end{aligned}$$

We can paraphrase this by saying that *every* ring A satisfies excision in nonpositive algebraic K-theory. This is why in this article we deal almost exclusively with K_i-groups for $i \geq 1$. The sequence (5) can be pasted together with (2), or with (3) if A satisfies excision, to produce a single, doubly infinite, long exact sequence of K-groups.

We close this Introduction with a list of terminological and notational conventions used in the present article:

(1) Rings and algebras are not assumed to possess units unless stated otherwise; the word "algebra" without any adjective stands for "associative algebra over a fixed unital ground ring k";

(2) If A is a ring and m is a positive integer, then $A^m \subset A$ denotes the subring additively generated by products $a_1 \cdots a_m$, where $a_i \in A$, $1 \leq i \leq m$;

(3) $A_{\mathbb{Q}} := A \otimes_{\mathbb{Z}} \mathbb{Q}$;

(4) $\tilde{B} := k \ltimes B$ denotes the algebra obtained by adjoining the unit to a k-algebra B (if A is a ring, we set $\tilde{A} := \mathbb{Z} \ltimes A$);

(5) $\mathrm{GL}_n(A) = \mathrm{Ker}(\mathrm{GL}_n(\tilde{A}) \to \mathrm{GL}_n(\mathbb{Z}))$ and $\mathrm{GL}(A) := \bigcup_{n=1}^{\infty} \mathrm{GL}_n(A)$;

(6) If G is a group, $\mathrm{B}G$ denotes the realization of the standard simplicial model of the classifying space of G (cf. §2); $\mathrm{B}G$ is always a $K(G,1)$ space;

(7) $\mathrm{Ad}\, g : G \to G$ denotes the automorphism $x \mapsto gxg^{-1}$;

(8) If (C_*, ∂_*) is a chain complex, then the complex $C_*[j]$, where $j \in \mathbb{Z}$, is defined as $(C_*[j])_q = C_{q-j}$ and $(\partial[j])_q = (-1)^j \partial_{q-j}$;

(9) $\mathbb{Z}_+ = \{1, 2, 3, \dots\}$.

The results of the present paper were announced in [34].

1. Excision and the homology of affine groups

1.1. *Affine groups.* Let A be a ring and n be a positive integer. We will associate with a subgroup $G \subset \mathrm{GL}_n(A)$ two affine groups

$$\tilde{G} = G \ltimes M_{n1}(A) = \left\{ \begin{pmatrix} \alpha & v \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(A) \,\middle|\, \alpha \in G \text{ and } v \in M_{n1}(A) \right\}$$

and

$$\tilde{\tilde{G}} = M_{1n}(A) \rtimes G = \left\{ \begin{pmatrix} \alpha & 0 \\ w & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(A) \,\middle|\, \alpha \in G \text{ and } w \in M_{1n}(A) \right\}.$$

The natural stabilization maps $\mathrm{GL}_n(A) \hookrightarrow \mathrm{GL}_{n+1}(A)$, $M_{n1}(A) \hookrightarrow M_{n+1,1}(A)$ and $M_{1n}(A) \hookrightarrow M_{1,n+1}(A)$ induce embeddings of the affine groups $\widetilde{\mathrm{GL}}_n(A) \hookrightarrow \widetilde{\mathrm{GL}}_{n+1}(A)$ and $\widetilde{\widetilde{\mathrm{GL}}}_n(A) \hookrightarrow \widetilde{\widetilde{\mathrm{GL}}}_{n+1}(A)$; we set

$$\widetilde{\mathrm{GL}}(A) = \varinjlim \widetilde{\mathrm{GL}}_n(A) = \mathrm{GL}(A) \ltimes M_{\infty 1}(A)$$

and

$$\widetilde{\widetilde{\mathrm{GL}}}(A) = \varinjlim \widetilde{\widetilde{\mathrm{GL}}}_n(A) = M_{1\infty}(A) \rtimes \mathrm{GL}(A).$$

1.2. Let A^{op} denote the ring opposite to A and, for any $a \in A$, let a^{op} denote the same element considered as an element of A^{op}. We extend this to arbitrary matrices by setting

$$(\alpha^{\mathrm{op}})_{ij} = (\alpha_{ji})^{\mathrm{op}}$$

($\alpha^{\mathrm{op}} \in M_{lk}(A^{\mathrm{op}})$ if $\alpha \in M_{kl}(A)$). Since $(\alpha\beta)^{\mathrm{op}} = \beta^{\mathrm{op}} \cdot \alpha^{\mathrm{op}}$, the correspondences

$$\alpha \mapsto (\alpha^{\mathrm{op}})^{-1}, \ (\alpha, v) \mapsto (v^{\mathrm{op}}, \alpha^{\mathrm{op}})^{-1} \text{ and } (w, \alpha) \mapsto (\alpha^{\mathrm{op}}, w^{\mathrm{op}})^{-1}$$

define canonical isomorphisms

$$\mathrm{GL}_n(A) \xrightarrow{\sim} \mathrm{GL}_n(A^{\mathrm{op}}),\ \widetilde{\mathrm{GL}}_n(A) \xrightarrow{\sim} \widetilde{\widetilde{\mathrm{GL}}}_n(A^{\mathrm{op}}) \text{ and } \widetilde{\widetilde{\mathrm{GL}}}_n(A) \xrightarrow{\sim} \widetilde{\mathrm{GL}}_n(A^{\mathrm{op}}).$$

This observation will allow us to reduce all questions concerning $\widetilde{\widetilde{\mathrm{GL}}}(A)$ to the corresponding questions about $\widetilde{\mathrm{GL}}(A^{\mathrm{op}})$. And so we shall be working predominantly with the affine group $\widetilde{\mathrm{GL}}(A)$.

1.3. Let us consider the diagram of natural embeddings

$$\begin{tikzcd} & \widetilde{\mathrm{GL}}(A) \\ \mathrm{GL}(A) \arrow[ur] \arrow[dr] & \\ & \widetilde{\widetilde{\mathrm{GL}}}(A). \end{tikzcd} \tag{6}$$

We will show in this section that A satisfies excision in algebraic K-theory if both embeddings in (6) induce isomorphisms in group homology.

LEMMA 1.4. *The self-embedding* $\mathrm{GL}(A) \hookrightarrow \mathrm{GL}(A)$ *given by*

$$j : \alpha \mapsto \begin{pmatrix} 1 & 0 \\ 0 & \alpha \end{pmatrix}$$

induces an injective map in homology with coefficients in an arbitrary abelian group Λ *viewed as a trivial* $\mathrm{GL}(A)$*-module.*

Proof. For every n, one has the following commutative diagram

$$\begin{tikzcd} & \mathrm{GL}(A) \arrow[dd, "\mathrm{Ad}\,\varepsilon_n"] \\ \mathrm{GL}_n(A) \arrow[ur, "i_n"] \arrow[dr, "j \circ i_n"'] & \\ & \mathrm{GL}(A), \end{tikzcd}$$

where i_n denotes the standard stabilization map and $\mathrm{Ad}\,\varepsilon_n$ is an automorphism of $\mathrm{GL}(A)$ given by conjugation with the permutation matrix

$$\varepsilon_n = \begin{pmatrix} 0 & 1 & & & \\ 1_n & 0 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & \ddots \end{pmatrix}$$

(1_n denotes the $n \times n$ identity matrix). Thus

$$\mathrm{Ker}\big[\mathrm{H}_*(\mathrm{GL}_n(A);\Lambda) \xrightarrow{j \circ i_n} \mathrm{H}_*(\mathrm{GL}(A);\Lambda)\big] = \mathrm{Ker}\big[\mathrm{H}_*(\mathrm{GL}_n(A);\Lambda) \xrightarrow{i_n} \mathrm{H}_*(\mathrm{GL}(A);\Lambda)\big],$$

which implies that $j : \mathrm{H}_*(\mathrm{GL}(A);\Lambda) \to \mathrm{H}_*(\mathrm{GL}(A);\Lambda)$ is injective. $\square$

In Proposition 1.5 and Corollaries 1.6, 1.7 and 1.8 below, a ring A is supposed to possess the following property:

(AH_Λ) The embeddings (6) induce isomorphisms in homology with coefficients Λ.

Here Λ denotes an arbitrary fixed abelian group viewed as a trivial $\mathrm{GL}(A)$-module.

PROPOSITION 1.5. *The natural action by conjugation of* $\mathrm{GL}(\mathbb{Z})$ *on* $\mathrm{H}_*(\mathrm{GL}(A);\Lambda)$ *is trivial.*

Proof. Let $E_{i\infty}(1) = (1, e_i) \in \widetilde{\mathrm{GL}}(\mathbb{Z})$, where $e_i \in M_{\infty 1}(\mathbb{Z})$ denotes the i^{th} basis column-vector. Let us consider the following commutative diagram of adjoint actions

$$\begin{array}{ccc}
\mathrm{GL}(A) & \xrightarrow{\operatorname{Ad} E_{i+1,1}(1)} & \mathrm{GL}(A) \\
\tilde{\jmath}\Big\uparrow & & \Big\uparrow\tilde{\jmath} \\
\widetilde{\mathrm{GL}}(A) & \xrightarrow{\operatorname{Ad} E_{i\infty}(1)} & \widetilde{\mathrm{GL}}(A) \\
p\Big\downarrow & & \Big\downarrow p \\
\mathrm{GL}(A) & = & \mathrm{GL}(A),
\end{array}$$

where $\tilde{\jmath}$ is the embedding

$$\widetilde{\mathrm{GL}}(A) \hookrightarrow \mathrm{GL}(A), \quad (\alpha, v) \mapsto \begin{pmatrix} 1 & 0 \\ v & \alpha \end{pmatrix},$$

p is the canonical projection and $E_{i+1,1}(1) \in \mathrm{GL}(\mathbb{Z})$ is the corresponding elementary matrix.

Since p induces an isomorphism in homology with Λ-coefficients, $\operatorname{Ad} E_{i\infty}(1)$ acts trivially on $\mathrm{H}_*(\widetilde{\mathrm{GL}}(A);\Lambda)$ and $\operatorname{Ad} E_{i+1,1}(1)$ acts trivially on $\mathrm{H}_*\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{GL}(A)\end{smallmatrix}\right);\Lambda\big)$, which, by Lemma 1.4, injects into $\mathrm{H}_*(\mathrm{GL}(A);\Lambda)$. A similar argument with $\widetilde{\mathrm{GL}}$ replaced by $\widetilde{\widetilde{\mathrm{GL}}}$ shows that the adjoint action of $E_{1,i+1}(1)$, $i = 1, 2, \ldots$, on $\mathrm{H}_*\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{GL}(A)\end{smallmatrix}\right);\Lambda\big)$ is trivial too. Since $E_{j1}(1)$, $E_{1k}(1)$ $(j,k = 2,3,\ldots)$ and $-1 \in \mathrm{GL}_1(\mathbb{Z})$, which acts trivially on $\mathrm{H}_*\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{GL}(A)\end{smallmatrix}\right);\Lambda\big)$ by obvious reasons, generate together $\mathrm{GL}(\mathbb{Z})$, we obtain that $\mathrm{H}_*\big(\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{GL}(A)\end{smallmatrix}\right);\Lambda\big) \subset \mathrm{H}_*(\mathrm{GL}(A);\Lambda)$ is fixed under the adjoint action of $\mathrm{GL}(\mathbb{Z})$, hence also under the action of $\left(\begin{smallmatrix} 1 & 0 \\ 0 & \mathrm{GL}(\mathbb{Z})\end{smallmatrix}\right) \subset \mathrm{GL}(\mathbb{Z})$. This proves the proposition. □

COROLLARY 1.6. *For every unital ring* R *containing* A *as a two-sided ideal, the adjoint action of* $\mathrm{GL}(R)$ *on* $\mathrm{H}_*(\mathrm{GL}(A);\Lambda)$ *is trivial.*

Proof. It suffices to show that the adjoint actions of $\mathrm{GL}_n(R)$ on

$$\mathrm{Im}\big(\mathrm{H}_*(\mathrm{GL}_n(A);\Lambda) \xrightarrow{i_n} \mathrm{H}_*(\mathrm{GL}(A);\Lambda)\big) \tag{7}$$

are trivial for all n.

For any $g \in \mathrm{GL}_n(R)$, one has the following factorization in $\mathrm{GL}_{2n}(R)$:

$$\begin{pmatrix} g & 0 \\ 0 & 1_n \end{pmatrix} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} 1_n & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}. \tag{8}$$

All three factors in (8) act on (7) trivially—the middle one by obvious reasons, the left and right ones by Proposition 1.5. □

In the situation of Corollary 1.6 there is an evident map of fibrations

$$\begin{array}{ccccc} \mathrm{BGL}(A) & \longrightarrow & \mathrm{BGL}(R) & \longrightarrow & \mathrm{B}\overline{\mathrm{GL}}(R/A) \\ \varphi\downarrow & & \downarrow & & \downarrow \\ F(R,A) & \longrightarrow & \mathrm{BGL}(R)^+ & \longrightarrow & \mathrm{B}\overline{\mathrm{GL}}(R/A)^+, \end{array} \tag{9}$$

where $\overline{\mathrm{GL}}(R/A) = \mathrm{Im}(\mathrm{GL}(R) \to \mathrm{GL}(R/A))$ and $F(R,A)$ is the homotopy fiber of the map $\mathrm{BGL}(R)^+ \to \mathrm{B}\overline{\mathrm{GL}}(R/A)^+$.

COROLLARY 1.7. *The map* $\varphi : \mathrm{BGL}(A) \to F(R,A)$ *induces an isomorphism in homology with* Λ*-coefficients.*

Proof. The local systems of the homology of the fibers in diagram (9) are trivial. This follows from Corollary 1.6 in the case of the upper fibration; in the case of the bottom one, it is a standard fact about fibrations of infinite loop spaces (cf. [38]). The comparison theorem for spectral sequences [45] completes the proof. □

In the next corollary, Λ denotes one of the following abelian groups: $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{Z}/m\mathbb{Z}$, $m = 2, 3, \dots$.

COROLLARY 1.8. *A satisfies excision in* K*-theory with* Λ*-coefficients.*

Proof. Let us consider the homotopy-commutative triangle

$$\begin{array}{ccc} & \mathrm{BGL}(A) & \\ \swarrow & & \searrow \\ F(\tilde{A},A) & \xrightarrow{\quad e_R \quad} & F(R,A), \end{array}$$

where $\tilde{A} = \mathbb{Z} \ltimes A$. By Corollary 1.7, the map $F(\tilde{A},A) \to F(R,A)$ induces an isomorphism in homology with Λ-coefficients. Since both spaces are simple, a variant of Whitehead's theorem implies that e_R induces an isomorphism on homotopy groups with Λ-coefficients (for the $\mathbb{Z}/m\mathbb{Z}$-variant of Whitehead's theorem, cf. [27], p. 14). Hence $e_R : K_*(A;\Lambda) \xrightarrow{\sim} K_*(R,A;\Lambda)$. □

LEMMA 1.9. *Every $\mathbb{Q}$-algebra possesses Property $\mathrm{AH}_{\mathbb{Z}/m\mathbb{Z}}$ and therefore satisfies excision in algebraic* K*-theory with $\mathbb{Z}/m\mathbb{Z}$-coefficients, $m = 2, 3, \dots$.*[1]

Proof. For a $\mathbb{Q}$-algebra A, one has

$$H_q(M_{\infty 1}(A); \mathbb{Z}/m\mathbb{Z}) = H_q(M_{1\infty}(A); \mathbb{Z}/m\mathbb{Z}) = 0, \quad q > 0.$$

The Hochschild–Serre spectral sequence of the extension

$$1 \to M_{\infty 1}(A) \to \widetilde{\mathrm{GL}}(A) \xrightarrow{p} \mathrm{GL}(A) \to 1 \tag{10}$$

then shows that $p_* : \mathrm{H}_*(\widetilde{\mathrm{GL}}(A); \mathbb{Z}/m\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}_*(\mathrm{GL}(A); \mathbb{Z}/m\mathbb{Z})$. Similarly the same holds true for the other extension

$$1 \to M_{1\infty}(A) \to \widetilde{\widetilde{\mathrm{GL}}}(A) \to \mathrm{GL}(A) \to 1. \qquad \square$$

In the next lemma, Λ denotes again an arbitrary abelian group.

LEMMA 1.10. *For any ring A the following conditions are equivalent:*
(a) $H_1(\mathrm{GL}(A); \Lambda) = H_1(\widetilde{\mathrm{GL}}(A); \Lambda)$,
(b) $A/A^2 \otimes_{\mathbb{Z}} \Lambda = 0$.

Proof. The Hochschild–Serre spectral sequence of (10) gives the short exact sequence

$$0 \to H_1(\mathrm{GL}(A); \Lambda) \to H_1(\widetilde{\mathrm{GL}}(A); \Lambda) \to [M_{\infty,1}(A) \otimes_{\mathbb{Z}} \Lambda]_{\mathrm{GL}(A)} \to 0,$$

and an instantaneous calculation shows that

$$[M_{\infty 1}(A) \otimes_{\mathbb{Z}} \Lambda]_{\mathrm{GL}(A)} \simeq M_{\infty 1}(A/A^2 \otimes_{\mathbb{Z}} \Lambda). \qquad \square$$

In the remaining part of this section we will examine more closely Property $\mathrm{AH}_{\mathbb{Z}}$. For any A and $n \geq 3$, we shall denote by $E_n(A)$ the elementary subgroup of $\mathrm{GL}_n(A)$, i.e, the subgroup generated by elementary matrices $E_{ij}(a)$, $1 \leq i \neq j \leq n$, $a \in A$. Similarly $E(A)$ will denote the elementary subgroup of $\mathrm{GL}(A)$.

LEMMA 1.11 (L.N. Vaserstein [37], Proof of Lemma 1.1). *Suppose that $\alpha \in M_{nk}(A)$ and $\beta \in M_{kn}(A)$ are given such that $1 + \alpha\beta \in \mathrm{GL}_n(A)$. Then $1 + \beta\alpha \in \mathrm{GL}_k(A)$ and*

$$\begin{pmatrix} 1+\alpha\beta & 0 \\ 0 & (1+\beta\alpha)^{-1} \end{pmatrix} \in E_{n+k}(A). \qquad \square$$

COROLLARY 1.12. *For any $g, h \in \mathrm{GL}_n(A^2)$,*
(a) $\left(\begin{smallmatrix} g & 0 \\ 0 & g^{-1} \end{smallmatrix}\right) \in E(A)$, *and*
(b) $[g, h] = ghg^{-1}h^{-1} \in E(A)$.

---

[1] Excision in algebraic K-theory with $\mathbb{Z}/m\mathbb{Z}$-coefficients for $\mathbb{Z}[1/m]$-algebras has been proved by a different method in [39].

Proof. In view of the assumption, a matrix g can be written in the form $1+\alpha\beta$ for certain $\alpha \in M_{nk}(A)$ and $\beta \in M_{kn}(A)$. Now Lemma 1.11 shows that

$$\begin{pmatrix} g & & \\ & g^{-1} & \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1+\alpha\beta & & \\ & 1 & \\ & & (1+\beta\alpha)^{-1} \end{pmatrix} \begin{pmatrix} 1 & & \\ & 1+\alpha\beta & \\ & & (1+\beta\alpha)^{-1} \end{pmatrix}^{-1} \in E_{2n+k}(A),$$

which proves part (a). Part (b) follows from the latter in the standard way:

$$\begin{pmatrix} [g,h] & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & h^{-1} \end{pmatrix} \begin{pmatrix} hg & 0 \\ 0 & (hg)^{-1} \end{pmatrix}^{-1} \in E(A). \qquad \square$$

COROLLARY 1.13. *If $A = A^2$, then*

$$E(A) = [E(A), E(A)] = [\mathrm{GL}(A), \mathrm{GL}(A)].$$

In other words, $E(A)$ is a perfect normal subgroup of $\mathrm{GL}(A)$ *with an abelian quotient* $\mathrm{GL}(A)/E(A)$. $\square$

COROLLARY 1.14. *The adjoint actions of* $\mathrm{GL}(A^2)$ *on* $\mathrm{H}_*(E(A);\mathbb{Z})$ *and of* $\widetilde{\mathrm{GL}}(A^2)$ *on* $\mathrm{H}_*(\tilde{E}(A);\mathbb{Z})$ *are trivial.*

Proof. It suffices to show that the actions of $\mathrm{GL}_n(A^2)$ on both

$$\begin{aligned} &\mathrm{Im}\big[\mathrm{H}_*(E_n(A);\mathbb{Z}) \to \mathrm{H}_*(E(A);\mathbb{Z})\big] \quad \text{and} \\ &\mathrm{Im}\big[\mathrm{H}_*(\tilde{E}_n(A);\mathbb{Z}) \to \mathrm{H}_*(\tilde{E}(A);\mathbb{Z})\big] \end{aligned} \tag{11}$$

are trivial.

For any $g \in \mathrm{GL}_n(A)$, we have the following factorization in $\mathrm{GL}_{2n}(A)$:

$$\begin{pmatrix} g & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & g \end{pmatrix}. \tag{12}$$

The adjoint action on both groups in (11) by the second factor in (12) is trivial by obvious reasons. If $g \in \mathrm{GL}_n(A^2)$, the first factor in equality (12) belongs to $E(A)$ (see Corollary 1.12 above), and hence its action on (11) is trivial too. $\square$

PROPOSITION 1.15. *Assume that a ring A possesses Property* $\mathrm{AH}_{\mathbb{Z}}$. *Then for every unital ring R containing A as a two-sided ideal, the canonical up to a homotopy map*

$$\mathrm{BGL}(A)^+ \longrightarrow F(R,A) \tag{13}$$

is a homotopy equivalence.

Notice that $\mathrm{BGL}(A)^+$ *is well defined, since for $A = A^2$, the group* $\mathrm{GL}(A)$ *is quasiperfect* (*Corollary* 1.13).

Proof. Let us consider the canonical (up to a homotopy) map $\varphi : \mathrm{BGL}(A) \to F(R, A)$. The induced homomorphism $\pi_1(\mathrm{BGL}(A)) \to \pi_1(F(R,A))$ kills $E(A) = [\mathrm{GL}(A), \mathrm{GL}(A)]$, since $\pi_1(F(R,A))$ is abelian. Thus φ factors through $\mathrm{BGL}(A)^+$, and this is how we get the map (13). Denote it φ^+. By Corollary 1.7, φ^+ induces an isomorphism in integral homology and therefore is a homotopy equivalence, since both $\mathrm{BGL}(A)^+$ and $F(R,A)$ are weakly simple (i.e., their respective fundamental groups act trivially on the integral homology of the universal cover, cf. [38], Lemma 1.1). The weak simplicity of $\mathrm{BGL}(A)^+$ results from Corollary 1.14, combined with the observation that $BE(A)^+$ is a universal cover of $\mathrm{BGL}(A)^+$, cf., e.g., [24], Prop. 1.1.7. Regarding $F(R,A)$, we find that the latter, being a connected H-space, is simple and hence weakly simple. □

Since $F(\tilde{A}, A)$ is an infinite loop space (cf. [38]), Proposition 1.15 shows, for A possessing Property $\mathrm{AH}_{\mathbb{Z}}$, that $\mathrm{BGL}(A)^+$ is an infinite loop space.

1.16. Let $\mathrm{Emb}(\mathbb{Z}_+)$ denote the monoid of self-embeddings of the set of positive integers. Each $f \in \mathrm{Emb}(\mathbb{Z}_+)$ induces a functorial homomorphism $f_* : \mathrm{GL}(A) \to \mathrm{GL}(A)$ and thus also a map $(Bf_*)^+ : \mathrm{BGL}(A)^+ \to \mathrm{BGL}(A)^+$, if $A = A^2$, cf. Corollary 1.13.

PROPOSITION 1.17. *Assume that a ring A possesses Property* $\mathrm{AH}_{\mathbb{Z}}$. *Then for every $f \in \mathrm{Emb}(\mathbb{Z}_+)$, the map $(Bf_*)^+ : \mathrm{BGL}(A)^+ \to \mathrm{BGL}(A)^+$ is homotopic to the identity map.*

Proof. For a given $f \in \mathrm{Emb}(\mathbb{Z}_+)$ and a positive integer n, one can find a permutation matrix $\sigma \in \mathrm{GL}(\mathbb{Z})$ such that the following diagram commutes:

$$\begin{tikzcd}
 & \mathrm{GL}(A) \arrow[dr, "f_*"] & \\
\mathrm{GL}_n(A) \arrow[ur] \arrow[dr] & & \mathrm{GL}(A) \cdot \\
 & \mathrm{GL}(A) \arrow[ur, "\mathrm{Ad}\,\sigma"'] &
\end{tikzcd}$$

Combined with Proposition 1.5 this means that f_* acts trivially on $\mathrm{H}_*(\mathrm{GL}(A); \mathbb{Z})$ and, since $\mathrm{BGL}(A)^+$ is weakly simple (cf. the proof of Proposition 1.15), that $(Bf_*)^+$ is a self-homotopy equivalence of $\mathrm{BGL}(A)^+$. Thus the correspondence $f \mapsto (Bf_*)^+$ defines a homomorphism of $\mathrm{Emb}(\mathbb{Z}_+)$ into the group of (homotopy classes) of self-homotopy equivalences of $\mathrm{BGL}(A)^+$:

$$\mathrm{Emb}(\mathbb{Z}_+) \to \mathrm{Aut}\,\mathrm{BGL}(A) \quad .$$

The monoid $\mathrm{Emb}(\mathbb{Z}_+)$, however, does not admit any nontrivial homomorphisms into groups ([24], Lemme 1.2.8); thus every $(Bf_*)^+$ has to be homotopic to the identity map $\mathrm{BGL}(A)^+ \to \mathrm{BGL}(A)^+$. □

COROLLARY 1.18. *Assume that A possesses Property* $\mathrm{AH}_{\mathbb{Z}}$. *Then for every unital ring R containing A as a two-sided ideal, the homotopy action of* $\mathrm{GL}(R)$ *on* $\mathrm{BGL}(A)^+$, *which is induced by the adjoint action on* $\mathrm{GL}(A)$, *is trivial. In particular,* $\mathrm{GL}(R)$ *acts trivially on the algebraic* K*-theory groups* $K_*(A)$.

Proof. For a given $g \in \mathrm{GL}_n(R)$ the commutativity of the diagram of homomorphisms

$$\begin{tikzcd} & \mathrm{GL}(A) & \\ \mathrm{GL}(A) \arrow[ur, "f_*"] \arrow[rr, "\mathrm{Ad}_g"'] & & \mathrm{GL}(A), \arrow[ul, "f_*"'] \end{tikzcd}$$

where f denotes the self-embedding induced by the shift $i \mapsto i+n$, implies that $B(\mathrm{Ad}\, g)^+ \circ (Bf_*)^+ = (Bf_*)^+$. □

Proving two further corollaries of Proposition 1.17 is left to the reader.

COROLLARY 1.19. *If A possesses Property* $\mathrm{AH}_{\mathbb{Z}}$, *the space* $\mathrm{BGL}(A)^+$ *is a connected homotopy-commutative and homotopy-associative H-space with respect to the operation given by the direct sum of matrices* (*compare with* [38] *or* [24], *Théorème* 1.2.6). □

COROLLARY 1.20. *The algebraic* K*-theory of a ring having Property* $\mathrm{AH}_{\mathbb{Z}}$ *is Morita invariant: the natural embedding*

$$A \hookrightarrow M_n(A), \quad a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces, for every n, a homotopy equivalence $\mathrm{BGL}(A)^+ \xrightarrow{\sim} \mathrm{BGL}(M_n(A))^+$. *In particular, it induces an isomorphism* $K_*(A) \xrightarrow{\sim} K_*(M_n(A))$. □

PROPOSITION 1.21 ("The long exact sequence in algebraic K-theory"). *Any ring extension*

$$A \overset{i}{\rightarrowtail} R \overset{f}{\twoheadrightarrow} S,$$

such that A possesses Property $\mathrm{AH}_{\mathbb{Z}}$, *induces a functorial and infinite-in-both-directions long exact sequence of algebraic* K*-groups*

$$(14) \qquad \cdots \to K_{q+1}(S) \xrightarrow{\partial_{q+1}} K_q(A) \xrightarrow{i_q} K_q(R) \xrightarrow{f_q} K_q(S) \xrightarrow{\partial_q} \ldots \qquad (q \in \mathbb{Z}).$$

The proof will be preceded by two auxiliary lemmas.

LEMMA 1.22. *For any two-sided ideal A in a unital ring R,* $\pi_q(\mathrm{B}\overline{\mathrm{GL}}(R/A)^+) = K_q(R/A)$, $q \geq 2$, *and* $\pi_1(\mathrm{B}\overline{\mathrm{GL}}(R/A)^+)$ *is included in a* *short exact sequence*

$$0 \to \pi_1(\mathrm{B}\overline{\mathrm{GL}}(R/A)^+) \to K_1(R/A) \to \mathrm{GL}(R/A)/\overline{\mathrm{GL}}(R/A) \to 0.$$

Proof. The subgroup $\overline{\mathrm{GL}}(R/A) \subset \mathrm{GL}(R/A)$ is normal and the quotient group $G := \mathrm{GL}(R/A)/\overline{\mathrm{GL}}(R/A)$ is abelian. It follows from elementary properties of the plus construction that

$$\mathrm{B}\overline{\mathrm{GL}}(R/A)^+ \to \mathrm{BGL}(R/A)^+ \to BG$$

is a homotopy fibration (cf., e.g., [2], Thm. (6.4)(a)). □

1.23. Let A be a two-sided ideal in a ring R. We form the unital ring $\tilde{R} = \mathbb{Z} \ltimes R$ by adjoining a unit to R and consider the following homotopy-commutative diagram of homotopy fibrations:

$$\begin{tikzcd}
\Phi(R,A) \arrow[r] \arrow[d] & F(\tilde{R},R) \arrow[r, "\beta"] \arrow[d] & \bar{F}(\tilde{R}/A) \arrow[d] \\
F(\tilde{R},A) \arrow[r] & \mathrm{BGL}(\tilde{R})^+ \arrow[r] \arrow[d] & \mathrm{B}\overline{\mathrm{GL}}(\tilde{R}/A)^+ \arrow[d, "\alpha"] \\
 & \mathrm{BGL}(\mathbb{Z})^+ \arrow[r, equal] & \mathrm{BGL}(\mathbb{Z})^+.
\end{tikzcd} \tag{15}$$

Here $\bar{F}(\tilde{R}/A)$ denotes the homotopy fiber of α and $\Phi(R,A)$ is the homotopy fiber of β.

LEMMA 1.24. (a) *The map $\Phi(R/A) \to F(\tilde{R},A)$ is a homotopy equivalence.*

(b) $\pi_q(\bar{F}(\tilde{R}/A)) = K_q(R/A)$, $q \geq 2$, *and* $\pi_1(\bar{F}(\tilde{R}/A))$ *appears in a short exact sequence*

$$0 \to \pi_1(\bar{F}(\tilde{R}/A)) \to K_1(R/A) \to \mathrm{GL}(R/A)/\overline{\mathrm{GL}}(R/A) \to 0.$$

Proof. (a) It follows from (15) that both $\Phi(R,A)$ and $F(\tilde{R},A)$ are connected H-spaces and that $\pi_*\Phi(R,A) \to \pi_*F(R,A)$ is an isomorphism.

(b) Consider the map of homotopy fibrations

$$\begin{tikzcd}
\bar{F}(\tilde{R}/A) \arrow[r] \arrow[d] & F(\tilde{R}/A, R/A) \arrow[d] \\
\mathrm{B}\overline{\mathrm{GL}}(\tilde{R}/A)^+ \arrow[r] \arrow[d] & \mathrm{BGL}(\tilde{R}/A)^+ \arrow[d] \\
\mathrm{BGL}(\mathbb{Z})^+ \arrow[r, equal] & \mathrm{BGL}(\mathbb{Z})^+
\end{tikzcd}$$

and then use Lemma 1.22. □

If one defines the relative K-groups $K_q(R, A)$, $q \geq 1$, by

$$K_q(R, A) := \pi_q\big(\Phi(R, A)\big),$$

then Lemma 1.24(a) shows that $K_q(R, A) = K_q(\tilde{R}, A)$. This is also reflected in the fact that $F(R, A)$ and $F(\tilde{R}, A)$ are homotopy equivalent when R is unital.

The following corollary extends Proposition 1.15 to the case where a ring A, satisfying $\mathrm{AH}_{\mathbb{Z}}$, is contained as a two-sided ideal in an arbitrary ring R.

COROLLARY 1.25. *Assume that A possesses Property $\mathrm{AH}_{\mathbb{Z}}$. Then for every ring R containing A as a two-sided ideal, there is a canonical homotopy equivalence*

$$\mathrm{BGL}(A)^+ \xrightarrow{\sim} \Phi(R, A). \tag{16}$$

Proof. The map (16) is induced by the map of homotopy fibrations

$$\begin{tikzcd} \mathrm{BGL}(A) \arrow[r] \arrow[d] & \Phi(R, A) \arrow[d] \\ \mathrm{BGL}(R) \arrow[r] \arrow[d] & F(\tilde{R}, R) \arrow[d] \\ \mathrm{B\overline{GL}}(R/A) \arrow[r] & \bar{F}(\tilde{R}/A). \end{tikzcd} \tag{17}$$

The assertion follows from the homotopy-commutative diagram

$$\begin{tikzcd} \mathrm{BGL}(A)^+ \arrow[rr] \arrow[dr, "\wr"', "1.15"] & & \Phi(R, A) \arrow[dl, "\wr", "1.24(a)"'] \\ & F(\tilde{R}, A) & \end{tikzcd}$$

□

COROLLARY 1.26. *Assume that A is a two-sided ideal in a ring R and that A, R and R/A all have Property $\mathrm{AH}_{\mathbb{Z}}$. Then* (17) *induces a homotopy-commutative diagram of homotopy equivalences*

$$\begin{tikzcd} \mathrm{BGL}(A)^+ \arrow[r, "\sim"] \arrow[d] & \Phi(R, A) \arrow[d] \\ \mathrm{BGL}(R)^+ \arrow[r, "\sim"] \arrow[d] & F(\tilde{R}, R) \arrow[d] \\ \mathrm{B\overline{GL}}(R/A)^+ \arrow[r, "\sim"] & \bar{F}(\tilde{R}/A). \end{tikzcd} \tag{18}$$

In particular, the left column in (18) *is a homotopy fibration.* □

COROLLARY 1.27. *An arbitrary ring extension* $A \overset{i}{\rightarrowtail} R \overset{f}{\twoheadrightarrow} S$ *induces a functorial long exact sequence*

$$\begin{aligned}(19)\qquad \cdots \to K_{q+1}(S) \xrightarrow{\partial_{q+1}} K_q(R,A) \xrightarrow{i_q} K_q(R) \xrightarrow{f_q} K_q(S) \xrightarrow{\partial_q} \dots \\ \to K_2(S) \xrightarrow{\partial_2} K_1(R,A) \xrightarrow{i_1} K_1(R) \xrightarrow{f_1} K_1(S). \qquad \square\end{aligned}$$

Proof of Proposition 1.21. By combining (19) with Proposition 1.15, we get the functorial long exact sequence

$$\begin{aligned}(20)\qquad \cdots \to K_{q+1}(S) \xrightarrow{\partial_{q+1}} K_q(A) \xrightarrow{i_q} K_q(R) \xrightarrow{f_q} K_q(S) \xrightarrow{\partial_q} \dots \\ \to K_2(S) \xrightarrow{\partial_2} K_1(A) \xrightarrow{i_1} K_1(R) \xrightarrow{f_1} K_1(S).\end{aligned}$$

Pasting (20) with sequence (5) gives sequence (14). □

2. Volodin's spaces

2.1. Let X be a set. We denote by $P(X)$ the simplicial set whose p-simplices consist of ordered $(p+1)$-tuples $(x_0, \dots, x_p)$, $x_i \in X$, whose i^{th} face map omits x_i and whose j^{th} degeneracy map repeats x_j twice. The simplicial set $P(X)$ is contractible.

2.2. Let G be a group. It acts freely on $P(G)$ via

$$g \cdot (x_0, \dots, x_p) = (gx_0, \dots, gx_p) \qquad (g \in G);$$

the quotient simplicial set $G\backslash P(G)$ is a standard simplicial model for the classifying space of G. In what follows we shall denote $G\backslash P(G)$ by $\mathrm{B}G$. As usual $[g_1, \dots, g_p]$ will denote the p-simplex $G \cdot (1, g_1, g_1g_2, \dots, g_1g_2\cdots g_p) \in (\mathrm{B}G)_p$.

For any family $\{G_i\}_{i\in I}$ of subgroups of G we shall denote by $V(G, \{G_i\}_{i\in I})$ the simplicial subset of $P(G)$ formed by simplices $(g_0, \dots, g_p)$ satisfying the condition that

$$\text{there exists } i \in I \text{ such that for all } 0 \le j,k \le p, \quad g_j^{-1}g_k \in G_i.$$

The subset $V(G, \{G_i\}_{i\in I}) \subset P(G)$ is invariant under the action of G and the corresponding quotient simplicial set coincides with $\bigcup_{i\in I} \mathrm{B}G_i \subset \mathrm{B}G$. Since the group action on $V(G, \{G_i\}_{i\in I})$ is free, we have the following lemma.

LEMMA 2.3. *For any abelian group* Λ, *there is a natural spectral sequence*

$$E^2_{pq} = H_p\big(G;\, H_q\big(V\big(G, \{G_i\}_{i\in I}\big); \Lambda\big)\big) \Rightarrow H_{p+q}\Big(\bigcup_{i\in I} \mathrm{B}G_i; \Lambda\Big). \qquad \square$$

2.4. *Functoriality of Volodin's space.* Let G' be another group and $\{G'_j\}_{j\in J}$ be a family of subgroups in it. If $\varphi : G \to G'$ is a homomorphism such that each $\varphi(G_i)$ is contained in some G'_j, then φ induces simplicial maps $V(G, \{G_i\}_{i\in I}) \to V(G', \{G'_j\}_{j\in J})$ and $\bigcup_{i\in I} \mathrm{B}G_i \to \bigcup_{j\in J} \mathrm{B}G'_j$ as well as a morphism of the corresponding spectral sequences.

By its very definition, Volodin's space $V(G, \{G_i\}_{i\in I})$ decomposes as

$$V(G, \{G_i\}_{i\in I}) = \bigcup_{i\in I} \bigcup_{g\in G/G_i} g \cdot P(G_i),$$

and the multiple intersection

$$g_0 \cdot P(G_{i_0}) \cap \cdots \cap g_p \cdot P(G_{i_p}) \tag{21}$$

is nonempty if and only if $g_0 G_{i_0} \cap \cdots \cap g_p G_{i_p} \neq \emptyset$, in which case it is equal to $x \cdot P(G_{i_0} \cap \cdots \cap G_{i_p})$, where $x \in G/G_{i_0} \cap \cdots \cap G_{i_p}$ satisfies:

$$x(G_{i_0} \cap \cdots \cap G_{i_p}) = g_0 \cdot G_{i_0} \cap \cdots \cap g_p G_{i_p}.$$

In particular, all nonempty intersections (21) are contractible, and $V(G, \{G_i\}_{i\in I})$ is naturally homotopy equivalent to the nerve $N(G, \{G_i\}_{i\in I})$ of its own covering by subsets (21). The p-simplices of $N(G, \{G_i\}_{i\in I})$ are $(p+2)$-tuples $(i_0, \dots, i_p; x)$, $i_k \in I$, $x \in G/G_{i_0} \cap \cdots \cap G_{i_p}$; the face and degeneracy operators are given by

$$\partial_k(i_0, \dots, i_p; x) = (i_0, \dots, \hat{i}_k, \dots, i_p; r_k(x))$$

and

$$s_k(i_0, \dots, i_p; x) = (i_0, \dots, i_k, i_k, \dots, i_p; x)$$

($r_k : G/G_{i_0} \cap \cdots \cap G_{i_p} \to G/G_{i_0} \cap \cdots \cap \hat{G}_{i_k} \cap \cdots \cap G_{i_p}$ is the canonical quotient map). The above discussion makes the next lemma clear.

LEMMA 2.5. *Assume that there are given:*

(a) *a group G and a family of its subgroups $\{G_i\}_{i\in I}$,*

(b) *a group G' and a family of its subgroups $\{G'_i\}_{i\in I}$,*

(c) *a homomorphism $\varphi : G \to G'$ such that $\varphi(G_i) \subset G'_i$, for each $i \in I$, and such that the induced maps*

$$G/G_{i_0} \cap \cdots \cap G_{i_p} \to G'/G'_{i_0} \cap \cdots \cap G'_{i_p}$$

are bijective for all possible choices of indices $i_0, \dots, i_p \in I$.

Then the canonical map of Volodin's spaces

$$V(G, \{G_i\}_{i\in I}) \to V\big(G', \{G'_i\}_{i\in I}\big)$$

is a homotopy equivalence. □

2.6. Let A be a ring. For a partial ordering σ of the set $\{1,\dots,n\}$, we denote by $T_n^\sigma(A)$ the subgroup of σ-triangular matrices in $\mathrm{GL}_n(A)$:

$$T_n^\sigma(A)=\{(a_{ij})\in \mathrm{GL}_n(A)\mid a_{ij}=0 \text{ if } i\overset{\sigma}{\not<}j \text{ and } a_{ii}=1;\ 1\le i\ne j\le n\}.$$

For $n\ge 3$ the group $T_n^\sigma(A)$ has a presentation as the quotient of the free group generated by elementary matrices $E_{ij}(a)$, where $i\overset{\sigma}{<}j$ and $a\in A$, by the following relations:

$$E_{ij}(a)E_{ij}(a')=E_{ij}(a+a') \qquad (i\overset{\sigma}{<}j;\ a,a'\in A), \tag{22}$$

and

$$[E_{ij}(a),E_{jk}(a')]=E_{ik}(aa') \qquad (i\overset{\sigma}{<}j\overset{\sigma}{<}k;\ a,a'\in A) \tag{23}$$

(see [33]). We shall refer to $\{T_n^\sigma(A)\}_{\sigma\in\Pi_n}$, where $\Pi_n:=\{$partial orderings of $\{1,\dots,n\}\}$, as the family of triangular subgroups of $\mathrm{GL}_n(A)$.

Let us set

$$\begin{aligned} V_n(A)&=V\big(E_n(A),\{T_n^\sigma(A)\}_{\sigma\in\Pi_n}\big),\\ \tilde V_n(A)&=V\big(\tilde E_n(A),\{\tilde T_n^\sigma(A)\}_{\sigma\in\Pi_n}\big),\\ V(A)&=\varinjlim V_n(A)=V\big(E(A);\{T_n^\sigma(A)\}_{\sigma\in\Pi_n,\,n\in\mathbb{Z}_+}\big),\end{aligned}$$

and

$$\tilde V(A)=\varinjlim \tilde V_n(A)=V\big(\tilde E(A);\{\tilde T_n^\sigma(A)\}_{\sigma\in\Pi_n,\,n\in\mathbb{Z}_+}\big).$$

Lemma 2.5 gives us the following corollary:

COROLLARY 2.7. *The inclusion $E_n(A)\subset\tilde E_n(A)$ and the projection $\tilde E_n(A)\to E_n(A)$ induce two mutually inverse homotopy equivalences*

$$V_n(A)\rightleftarrows\tilde V_n(A).$$

Similarly the inclusion $E(A)\subset\tilde E(A)$ and the projection $\tilde E(A)\to E(A)$ induce mutually inverse homotopy equivalences

$$V(A)\rightleftarrows\tilde V(A).$$ □

LEMMA 2.8. *The natural left action of $E_{n+1}(A^2)$ on*

$$\mathrm{Im}\big[\mathrm{H}_*(V_n(A);\Lambda)\to\mathrm{H}_*(V_{n+1}(A);\Lambda)\big] \tag{24}$$

is trivial for all abelian groups of coefficients Λ.

Proof. The group $E_{n+1}(A^2)$ is contained in the subgroup of $E_{n+1}(A)$ generated by matrices $E_{i,n+1}(a)$ and $E_{n+1,i}(a)$; $1\le i\le n$, $a\in A$. We will show that both $E_{i,n+1}(a)$ and $E_{n+1,i}(a)$ act trivially on the group (24). First, we factor the inclusion of Volodin's spaces $V_n(A) \subset V_{n+1}(A)$ into two consecutive embeddings

$$V_n(A) \hookrightarrow \tilde{V}_n(A) \hookrightarrow V_{n+1}(A),$$

which correspond to the embeddings of groups

$$\begin{gathered} E_n(A) \hookrightarrow \tilde{E}_n(A) \hookrightarrow E_{n+1}(A) \\ \alpha \mapsto (\alpha, 0), \qquad (\beta, v) \mapsto \begin{pmatrix} \beta & v \\ 0 & 1 \end{pmatrix}. \end{gathered} \tag{25}$$

Under (25) the matrix $E_{i,n+1}(a) \in E_{n+1}(A)$ corresponds to the element $(1, ae_i) \in \tilde{E}_n(A)$. The latter acts trivially on $\mathrm{H}_*(\tilde{V}_n(A); \Lambda)$ in view of the commutativity of the diagram

$$\begin{tikzcd} \tilde{V}_n(A) \arrow[r, "(1,ae_i)"] \arrow[d, "p"'] & \tilde{V}_n(A) \arrow[d, "p"] \\ V_n(A) \arrow[r, equal] & V_n(A), \end{tikzcd}$$

in which the vertical projection arrows are homotopy equivalences by Corollary 2.7.

We prove similarly the triviality of the action of $E_{n+1,i}(a)$, with $\tilde{E}_n(A)$ and $\tilde{V}_n(A)$ replaced by $\tilde{E}_n(A^{\mathrm{op}})$ and $\tilde{V}_n(A^{\mathrm{op}})$ (cf. 1.2 above). □

COROLLARY 2.9. *The actions of* $E(A^2)$ *on* $\mathrm{H}_*(V(A); \Lambda)$ *and of* $\tilde{E}(A^2)$ *on* $\mathrm{H}_*(\tilde{V}(A); \Lambda)$ *are trivial for all abelian groups of coefficients* Λ. □

THEOREM 2.10. *Let A be a ring satisfying $A = A^2$ and let Λ be a fixed abelian group. Then the following conditions are equivalent:*

(a) $\mathrm{H}_*(\mathrm{GL}(A); \Lambda) = \mathrm{H}_*(\widetilde{\mathrm{GL}}(A); \Lambda)$,

(b) $\mathrm{H}_*(E(A); \Lambda) = \mathrm{H}_*(\tilde{E}(A); \Lambda)$,

(c) $\mathrm{H}_*\big(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A); \Lambda\big) = \mathrm{H}_*\big(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A); \Lambda\big)$.

Proof. (a) $\Leftrightarrow$ (b). Let us consider the map of Hochschild–Serre spectral sequences associated with the morphism of group extensions

$$\begin{tikzcd} 1 \arrow[r] & E(A) \arrow[r] \arrow[d] & \mathrm{GL}(A) \arrow[r] \arrow[d] & \mathrm{GL}(A)/E(A) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & \tilde{E}(A) \arrow[r] & \widetilde{\mathrm{GL}}(A) \arrow[r] & \widetilde{\mathrm{GL}}(A)/\tilde{E}(A) \arrow[r] & 1. \end{tikzcd}$$

By Corollary 1.14 the action of $\mathrm{GL}(A)/E(A) = \widetilde{\mathrm{GL}}(A)/\tilde{E}(A)$ on both $\mathrm{H}_*(E(A); \Lambda)$ and $\mathrm{H}_*(\tilde{E}(A); \Lambda)$ is trivial so that the standard spectral-sequence comparison theorem [45] establishes the equivalence of (a) and (b).

(b) $\Leftrightarrow$ (c). Apply the same comparison theorem to the map of spectral sequences

$$\begin{array}{ccc} E^2_{pq} = H_p\big(E(A); H_q(V(A);\Lambda)\big) & & {}'E^2_{pq} = H_p\big(\tilde{E}(A); H_q(\tilde{V}(A);\Lambda)\big) \\ \Downarrow & \longrightarrow & \Downarrow \\ H_{p+q}\left(\bigcup_{n,\sigma} \mathrm{B}T^\sigma_n(A);\Lambda\right) & & H_{p+q}\left(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}^\sigma_n(A);\Lambda\right) \end{array}$$

induced by the inclusions $E(A) \subset \tilde{E}(A)$ and $V(A) \subset \tilde{V}(A)$. One is allowed to do that in view of Corollaries 2.7 and 2.9. $\square$

3. Excision for rings with the Triple Factorization Property

Throughout this section a ring A is assumed to possess the following Triple Factorization Property (cf. [41], Property Φ):

For any finite collection $a_1, \ldots, a_m \in A$, there exist such $b_1, \ldots, b_m, c, d \in A$ that

$$(\mathrm{TF})_{\mathrm{right}} \qquad a_i = b_i cd \qquad (1 \le i \le m)$$

and the left annihilators in A of the elements c and cd are equal.

Recall that the left annihilator $l(a)$ of an element $a \in A$ is defined as $l(a) = \{x \in A \mid xa = 0\}$. Every ring that has Property $(\mathrm{TF})_{\mathrm{right}}$ is *left universally flat*, in the terminology of [41] (cf. also Proposition 8.5 below).

3.1. In this section we will slightly change the notation and consider matrices labelled by ordered pairs of elements of a certain finite set X. Thus we have the corresponding matrix ring $M_X(A)$ and the general linear group $\mathrm{GL}_X(A)$. If X is partially ordered, we have also the triangular subgroup $T_X(A) \subset \mathrm{GL}_X(A)$, which, as usual, is generated by the corresponding elementary matrices $E_{xy}(a)$; $x < y$, $a \in A$. Any injective order-preserving map $\varphi : X \to Y$ between two partially ordered sets induces a monomorphism $T_\varphi : T_X(A) \to T_Y(A)$.

Definition 3.2. Let $X' \subset X$ denote the set of nonminimal elements of a partially ordered set X. An injective and order-preserving map $\varphi : X \to Y$ of partially ordered sets is said to be *n-sparse* if there exist such mappings $\varphi_i : X' \to Y \backslash \varphi(X)$, $1 \le i \le n$, that

(a) $\varphi(x) > \varphi_1(x) > \cdots > \varphi_n(x)$ for all $x \in X'$,

(b) $\varphi_n(x) > \varphi(y)$ for all $x \in X'$ and $y \in X$ such that $x > y$,

(c) $\varphi_i(X') \cap \varphi_j(X') = \emptyset$ for all $1 \le i \ne j \le n$.

If $X' = \phi$, then $T_X(A) = 1$, and every injective and order-preserving map $T_X(A) \to T_Y(A)$ is n-sparse for any $n \ge 1$.

The following result is a straightforward generalization of the main lemma (Lemma 2.2) of [33].

PROPOSITION 3.3. *For any field F, the homomorphism*

$$H_q(T_X(A);F) \to H_q(T_Y(A);F)$$

induced by an n-sparse, injective and order-preserving map $\varphi : X \to Y$ vanishes in dimensions $1 \le q \le n$.

Unless stated otherwise all homology groups below are assumed to have coefficients in a fixed field F. The maps $H_q(G_1) \to H_q(G_2)$ induced by a group homomorphism $f : G_1 \to G_2$ will be denoted by $H_q(f)$, $q = 1, 2, \dots$.

Proof. Property $(\mathrm{TF})_{\mathrm{right}}$ above implies that

$$H_q(T_X(A)) = \bigcup_{c,d} \mathrm{Im}\big[H_q(T_X(Acd)) \to H_q(T_X(A))\big], \tag{26}$$

where the union in equation (26) is taken over all pairs $c, d \in A$ such that the left annihilators $l(cd)$ and $l(c)$ are equal. Thus one has only to verify that the composite maps

$$H_q(T_X(Acd)) \to H_q(T_X(A)) \to H_q(T_Y(A)) \qquad (1 \le q \le n)$$

are zero for all such pairs (c, d). This will be demonstrated by a simultaneous induction on $n \ge 1$ and on the cardinality of X.

Let $\mathrm{Min}\, X := X \backslash X'$ denote the subset of all minimal elements in X. We can clearly assume that $Y = \varphi(X) \cup \bigcup_i \varphi_i(X')$ and hence that $\mathrm{Min}\, Y = \varphi(\mathrm{Min}\, X)$. We set $Y_1 = Y \backslash \varphi_1(X')$ and $Y_2 = \varphi_1(X') \cup \varphi(\mathrm{Min}\, X)$. Then $Y_1 \cap Y_2 = \mathrm{Min}\, Y$. Therefore the subgroups $T_{Y_1}(A)$ and $T_{Y_2}(A)$ centralize each other. Our next step is to introduce three additional pieces of data:

(I) the embedding $\psi : X \hookrightarrow Y_2$,

$$\psi(x) = \begin{cases} \varphi(x), & x \in \mathrm{Min}\, X, \\ \varphi_1(x), & x \in X'; \end{cases}$$

(II) the homomorphism $\alpha : T_X(Acd) \to T_X(A)$,

$$\alpha(E_{xy}(bcd)) = \begin{cases} E_{xy}(bc), & x \in \mathrm{Min}\, X, \\ E_{xy}(dbc), & x \in X'; \end{cases}$$

(III) the distinguished matrix $u = \prod_{x \in X'} E_{\varphi_1(x), \varphi(x)}(d) \in T_Y(A)$.

Note that α is well defined in view of (21) and (22). It is an analogue of the conjugation by the diagonal matrix whose diagonal entries a_{xx} equal 1 or d depending on whether $x \in \mathrm{Min}\, X$ or $x \in X'$.

Since the subgroups $T_{Y_1}(A)$ and $T_{Y_2}(A)$ centralize each other, for any pair of homomorphisms $f : T_X(A) \to T_{Y_1}(A)$ and $g : T_X(A) \to T_{Y_2}(A)$ one can form the product homomorphism $f \cdot g : T_X(A) \to T_Y(A)$ (not to be confused with the composition). A direct computation leads to the following identity

$$(T_\psi \circ \alpha) \cdot (T_{\varphi'} \circ p) = (\mathrm{Ad}\, u) \circ [T_\varphi \cdot (T_\psi \circ \alpha)]$$

between two homomorphisms $T_X(Acd) \to T_Y(A)$, where $\varphi' := \varphi|_{X'}$ and $p : T_X(A) \to T_{X'}(A)$ is the canonical quotient homomorphism.

Since the conjugation by an element of a group acts trivially on its homology, we get

$$\mathrm{H}_*((T_\psi \circ \alpha) \cdot (T_{\varphi'} \circ p)) = \mathrm{H}_*(T_\varphi \cdot (T_\psi \circ \alpha)). \tag{27}$$

The homomorphism $T_\varphi \cdot (T_\psi \circ \alpha)$ is the composite

$$T_X(Acd) \xrightarrow{\Delta} T_X(Ad) \times T_X(Acd) \xrightarrow{T_\varphi \times (T_\psi \circ \alpha)} T_{Y_1}(A) \times T_{Y_2}(A) \xrightarrow{m} T_Y(A),$$

where m is the multiplication homomorphism. The embedding $\varphi : X \hookrightarrow Y_1$ is $(n-1)$-sparse and hence, by the inductive assumption, induces zero maps $H_q(T_X(Acd)) \to H_q(T_{Y_1}(A))$ for $1 \le q \le n-1$. Combined with the Künneth formula this gives the identity:

$$H_n(T_\varphi \cdot (T_\psi \circ \alpha)) = H_n(T_\varphi) + H_n(T_\psi \circ \alpha). \tag{28}$$

In the same way we get

$$H_n((T_\psi \circ \alpha) \cdot (T_{\varphi'} \circ p)) = H_n(T_\psi \circ \alpha) + H_n(T_{\varphi'} \circ p); \tag{29}$$

by combining equations (28) and (29) with (27), we obtain

$$H_n(T_\varphi) = H_n(T_{\varphi'} \circ p) = H_n(T_{\varphi'}) \circ H_n(p). \tag{30}$$

Since $\mathrm{card}\, X' < \mathrm{card}\, X$, the induction hypothesis implies that the right-hand side of (30) vanishes. □

Let σ be a partial ordering on a finite set X. We shall denote by $\sigma \times m$ the lexicographic (partial) ordering of $X \times \{1, \ldots, m\}$ while equipping the set $\mathbf{m} := \{1, \ldots, m\}$ with the ordering inverse to the natural one, i.e., $1 > \cdots > m$. By $\varphi_{(m)}$ we shall denote the embedding

$$X \hookrightarrow X \times \mathbf{m}, \qquad x \mapsto (x, 1). \tag{31}$$

COROLLARY 3.4. *Let there be given partial orderings $\sigma_1, \ldots, \sigma_k$ of a finite set X and a positive integer p. Then for a large enough m, the homomorphisms*

$$H_q\left(\bigcup_{i=1}^{k} \mathrm{B}T_X^{\sigma_i}(A)\right) \to H_q\left(\bigcup_{i=1}^{k} \mathrm{B}T_{X\times\mathbf{m}}^{\sigma_i\times m}(A)\right), \qquad 1 \le q \le p,$$

are zero. □

The proof, which proceeds by induction on k, is a simple application of Mayer–Vietoris long exact sequences (for details the reader is referred to [33], Proof of (2.3), pp. 1564–1565, or to the proof of Lemma 9.3 below).

COROLLARY 3.5. *For any $n \in \mathbb{Z}_+$, the canonical inclusion*

$$\bigcup_{\sigma\in\Pi_n} \mathrm{B}T_n^\sigma(A) \hookrightarrow \bigcup_{j\geq 2,\sigma\in\Pi_j} \mathrm{B}T_j^\sigma(A)$$

induces the zero map on positive-dimensional homology groups with coefficients in arbitrary fields. Consequently,

$$H_q\left(\bigcup_{j,\sigma}\mathrm{B}T_j^\sigma(A);\mathbb{Z}\right) = 0 \qquad (q>0)$$

i.e., the space $\bigcup_{j,\sigma}\mathrm{B}T_j^\sigma(A)$ is acyclic. □

3.6. *The affine case.* For any finite partially ordered set X we can form the corresponding affine triangular groups

$$\tilde{T}_X(A) := T_X(A) \ltimes A^X \quad\text{and}\quad \tilde{\tilde{T}}_X(A) := A^X \rtimes T_X(A).$$

The former coincides with $T_{X\cup\{\infty\}}(A)$, where $\infty > x$ for all $x \in X$; the latter coincides with $T_{\{-\infty\}\cup X}(A)$, where $-\infty < x$ for all $x \in X$. We will consider the two cases separately. Note that the embedding $\tilde{T}_X(A) \hookrightarrow \tilde{T}_Y(A)$ induced by an injective order-preserving map $\varphi : X \to Y$ coincides with the embedding $T_{X\cup\{\infty\}}(A) \to T_{Y\cup\{\infty\}}(A)$ induced by the map $X\cup\{\infty\} \to Y\cup\{\infty\}$, which sends x to $\varphi(x)$ and ∞ to ∞; similarly for the embedding $\tilde{\tilde{T}}_X(A) \hookrightarrow \tilde{\tilde{T}}_Y(A)$.

COROLLARY 3.7. *Let there be given partial orderings $\sigma_1,\dots,\sigma_k$ of a finite set X and a positive integer p. Then there exist:*

(a) *a finite set Y and an embedding $\varphi : X \hookrightarrow Y$, and*

(b) *partial orderings $\tau_1,\dots,\tau_k$ of Y, which satisfy*

$$x_1 \overset{\sigma_i}{<} x_2 \Rightarrow \varphi(x_1) \overset{\tau_i}{<} \varphi(x_2) \qquad (1 \leq i \leq k),$$

such that the maps induced by φ between the homology groups with coefficients in an arbitrary field

$$H_q\left(\bigcup_{i=1}^k \mathrm{B}\tilde{T}_X^{\sigma_i}(A)\right) \to H_q\left(\bigcup_{i=1}^k \mathrm{B}\tilde{T}_Y^{\tau_i}(A)\right)$$

vanish for all $q = 1,\dots,p$.

Proof. Denote by σ_i^+, $1 \leq i \leq k$, the partial ordering of $X \cup \{\infty\}$, which corresponds to σ_i. Then choose an integer m so that the assertion of Corollary 3.4 holds with X replaced by $X \cup \{\infty\}$ and σ_i replaced by σ_i^+, $1 \leq i \leq n$. Finally set

$$Y = (X \cup \infty) \times \mathbf{m} \backslash \{(\infty, 1)\} \quad \text{and} \quad \tau_i = \sigma_i \times m|_Y$$

and note that

$$(x, l) \overset{\sigma_i \times m}{\leq} (\infty, 1) \qquad (x \in X,\ 1 \leq l \leq m)$$

for every $i = 1, \ldots, k$. □

Corollary 3.8. *The space* $\bigcup_{j,\sigma} \mathrm{B}\tilde{T}_j^\sigma(A)$ *is acyclic.* □

Regarding the groups $\tilde{\tilde{T}}_X(A)$, we have the following result:

Corollary 3.9. *Let* $\varphi_{(n)} : X \hookrightarrow X \times \mathbf{n}$ *be as in* (31). *Then the induced homomorphisms between the homology groups with coefficients in an arbitrary field*

$$H_q\big(\tilde{\tilde{T}}_X(A)\big) \to H_q\big(\tilde{\tilde{T}}_{X \times \mathbf{n}}(A)\big)$$

vanish for all $1 \leq q \leq n$.

Proof. Recall that $\tilde{\tilde{T}}_X(A) = T_{\{-\infty\} \cup X}(A)$ and notice that the embedding

$$\{-\infty\} \cup X \hookrightarrow \{-\infty\} \cup (X \times \mathbf{n})$$

satisfies the hypothesis of Proposition 3.3. □

Exactly as before we deduce from Corollary 3.9 the following result:

Corollary 3.10. *The space* $\bigcup_{j,\sigma} \mathrm{B}\tilde{\tilde{T}}_j^\sigma(A)$ *is acyclic.* □

By combining Corollaries 3.5, 3.8 and 3.10 with subsection 1.2 and Theorem 2.10, we obtain the next result:

Theorem 3.11. *For any ring A possessing Property* $(\mathrm{TF})_{\text{right}}$, *the inclusions* $\mathrm{GL}(A) \hookrightarrow \widetilde{\mathrm{GL}}(A)$ *and* $\mathrm{GL}(A) \hookrightarrow \widetilde{\widetilde{\mathrm{GL}}}(A)$ *induce isomorphisms in integral homology; i.e., A satisfies condition* $(\mathrm{AH}_{\mathbb{Z}})$ *of Section* 1. □

In conjunction with the results of Section 1, Theorem 3.11 implies another corollary:

Corollary 3.12. *Every ring possessing Property* $(\mathrm{TF})_{\text{right}}$ *satisfies excision in algebraic* K*-theory.* □

Remarks 3.13. (a) By considering the opposite ring A^{op}, one can immediately extend Theorem 3.11 and Corollary 3.12 above to the class of rings satisfying, instead, the left variant of the Triple Factorization Property $(\mathrm{TF})_{\mathrm{right}}$:

For any finite collection $a_1, \ldots, a_m \in A$, there exist such $b_1, \ldots, b_m, c, d \in A$ that

$(\mathrm{TF})_{\mathrm{left}}$ $$a_i = dcb_i \qquad (1 \leq i \leq m)$$

and the right annihilators in A of the elements c and dc are equal.

(b) Global and functional analysis furnishes numerous examples of rings having property $(\mathrm{TF})_{\mathrm{right}}$ or $(\mathrm{TF})_{\mathrm{left}}$, or both. We mention but a few. It will be shown in Section 10 that every Banach algebra with a bounded right approximate unit satisfies $(\mathrm{TF})_{\mathrm{right}}$. The same is true more generally for locally multiplicatively convex Fréchet algebras with uniformly bounded approximate units (see [41]). The ring of smoothing operators $L^{-\infty}(M, E)$, acting on sections of a vector bundle E on an arbitrary closed C^∞-manifold M, possesses both of these properties.

4. Lie analogue of Volodin's construction

In this section Lie algebras are assumed to be Lie algebras over a fixed field F. The symbol $\otimes$ without a subscript denotes the tensor product over F.

4.1. *The standard Lie algebra resolution.* Let $\mathfrak{g}$ be a Lie algebra and $U = U_{\mathfrak{g}}$ be its universal enveloping algebra. By $P_*(\mathfrak{g})$ we denote the standard U-free resolution of the trivial $\mathfrak{g}$-module F:

$$P_q(\mathfrak{g}) = U \otimes \Lambda^q \mathfrak{g},$$

$$\begin{aligned}\partial_q(u \otimes g_1 \wedge \cdots \wedge g_q) = &\sum_{i=1}^{q} (-1)^i u g_i \otimes g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \wedge g_q \\ &+ \sum_{1 \leq i < j \leq q} (-1)^{i+j-1} u \otimes [g_i, g_j] \wedge g_1 \wedge \cdots \wedge \hat{g}_i \wedge \cdots \\ &\qquad\qquad \wedge \hat{g}_j \wedge \cdots \wedge g_q.\end{aligned}$$

The homology of $\mathfrak{g}$ with coefficients in a right $\mathfrak{g}$-module $\mathfrak{m}$ is calculated by the complex $C_*(\mathfrak{g}; \mathfrak{m}) = \mathfrak{m} \otimes_U P_*(\mathfrak{g})$:

$$\mathrm{H}_*(\mathfrak{g}; \mathfrak{m}) = H(C_*(\mathfrak{g}; \mathfrak{m})).$$

If $\mathfrak{m} = F$, the corresponding complex will be denoted by $C_*(\mathfrak{g})$ and its homology by $\mathrm{H}_*(\mathfrak{g})$.

4.2. *Volodin's Lie complex.* Let $\{\mathfrak{g}_i\}_{i\in I}$ be a family of Lie subalgebras of $\mathfrak{g}$. Each $P_*(\mathfrak{g}_i)$ is a $U_{\mathfrak{g}_i}$-subcomplex of $P_*(\mathfrak{g})$, and $U_{\mathfrak{g}} \otimes_{U_{\mathfrak{g}_i}} P_*(\mathfrak{g}_i)$ is the smallest $U_{\mathfrak{g}}$-subcomplex of $P_*(\mathfrak{g})$ that contains $P_*(\mathfrak{g}_i)$. We define the Volodin Lie complex $v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})$ as the following $U_{\mathfrak{g}}$-free subcomplex of $P_*(\mathfrak{g})$:

$$v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I}) = \sum_{i\in I} U_{\mathfrak{g}} \otimes_{U_{\mathfrak{g}_i}} P_*(\mathfrak{g}_i).$$

One can similarly view $\sum_{i\in I} C_*(\mathfrak{g}_i)$ as a subcomplex of $C_*(\mathfrak{g})$.

LEMMA 4.3. *There exists a natural spectral sequence*

$$E^2_{pq} = H_p\big(\mathfrak{g}; H_q\big(v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})\big)\big) \Rightarrow H_{p+q}\Big(\sum_{i\in I} C_*(\mathfrak{g}_i)\Big). \tag{32}$$

Proof. Since the homogeneous components of $v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})$ are free $U_{\mathfrak{g}}$-modules, one has

$$\mathrm{H}_*\big(\mathfrak{g}; v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})\big) \xrightarrow{\sim} \mathrm{H}_*\big(F \otimes_{U_{\mathfrak{g}}} v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})\big) = \mathrm{H}_*\Big(\sum_{i\in I} C_*(\mathfrak{g}_i)\Big).$$

Then (32) is the corresponding hyperhomology spectral sequence. □

4.4. *Functoriality of Volodin's construction.* Let $\mathfrak{g}'$ be another Lie algebra and $\{\mathfrak{g}'_j\}_{j\in J}$ be a family of its Lie subalgebras. Any homomorphism $f : \mathfrak{g} \to \mathfrak{g}'$ satisfying

$$\text{for every } i \in I \text{ there exists } j \in J \text{ such that } f(\mathfrak{g}_i) \subset \mathfrak{g}'_j,$$

induces chain maps

$$v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I}) \to v_*(\mathfrak{g}', \{\mathfrak{g}'_j\}_{j\in J})$$

and

$$\sum_{i\in I} C_*(\mathfrak{g}_i) \to \sum_{j\in J} C_*(\mathfrak{g}'_j)$$

and the morphism of spectral sequences (32).

4.5. Let M be a vector space over a field F and $\{M_i\}_{i\in I}$ be a family of its linear subspaces. One can associate with $\{M_i\}_{i\in I}$ the simplicial vector space $\mathbf{M}$ whose space of n-simplices is given by

$$\mathbf{M}_n = \bigoplus_{(i_0,\dots,i_n)} M_{i_0\dots i_n} \qquad (M_{i_0\dots i_n} := M_{i_0} \cap \cdots \cap M_{i_n}).$$

The face operator ∂_k restricted to the summand $M_{i_0\dots i_n}$ injects the latter into $M_{i_0\dots \hat{i}_k \dots i_n}$, while the degeneracy operator s_k on $M_{i_0\dots i_n}$ is the identity map $M_{i_0\dots i_n} \xrightarrow{=} M_{i_0\dots i_k i_k \dots i_n}$. The simplicial vector space $\mathbf{M}$ possesses a natural augmentation $\mathbf{M}_0 = \bigoplus_{i\in I} M_i \to \sum_{i\in I} M_i$. We will say that a family $\{M_i\}_{i\in I}$ is *well configured* if there exists a basis $\{e_s\}_{s\in S}$ of M such that each M_i is spanned by some part of it: $\{e_s\}_{s\in S_i}$, $S_i \subset S$.

LEMMA 4.6. *If a family $\{M_i\}_{i\in I}$ is well configured, the augmented simplicial vector space $\mathbf{M} \to \sum_{i\in I} M_i$ is acyclic.*

Proof. The property that a family $\{M_i\}_{i\in I}$ be well configured is equivalent to saying that $\mathbf{M} \simeq F\mathbf{S}$, where $\mathbf{S}$ is a simplicial set associated with some covering $\{S_i\}_{i\in I}$ of S:

$$\mathbf{S}_n = \{(i_0, \dots, i_n; s) \mid i_k \in I,\ s \in S_{i_0} \cap \cdots \cap S_{i_n}\}$$

with obvious face and degeneracy operators. The augmentation $\mathbf{M} \to \sum_{i\in I} M_i$ then corresponds to the augmentation $\mathbf{S} \to \bigcup_{i\in I} S_i$ given by the projection $\mathbf{S}_0 = \coprod_{i\in I} S_i \to \bigcup_{i\in I} S_i$. For every $s \in \bigcup_{i\in I} S_i$, the connected component of $\mathbf{S}$, which is "hanging over" s, coincides with the contractible simplicial set $P(I_s)$, where $I_s := \{i \in I \mid S_i \ni s\}$ (see Section 2 for the definition of P). Thus the augmented simplicial vector space $F\mathbf{S} \to F\left(\bigcup_{i\in I} S_i\right)$ is acyclic. $\square$

4.7. Given a well-configured family $\{\mathfrak{g}_i\}_{i\in I}$ of Lie subalgebras in a Lie algebra $\mathfrak{g}$, we have, for every $n \geq 0$, a well-configured family of vector subspaces $\{U_\mathfrak{g} \otimes \Lambda^n \mathfrak{g}_i\}_{i\in I}$ in $U_\mathfrak{g} \otimes \Lambda^n \mathfrak{g}$. Lemma 4.6 shows then that the complex $v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})$ has the following canonical simplicial resolution:

$$\bigoplus_i U \otimes_{U_{\mathfrak{g}_i}} P_*(\mathfrak{g}_i) \leftleftarrows \bigoplus_{i_0, i_1} U \otimes_{U_{\mathfrak{g}_{i_0 i_1}}} P_*(\mathfrak{g}_{i_0 i_1}) \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \cdots$$

$(U \equiv U_\mathfrak{g};\ \mathfrak{g}_{i_0 \dots i_n} \equiv \mathfrak{g}_{i_0} \cap \cdots \cap \mathfrak{g}_{i_n})$. Since

$$H_q\big(U \otimes_{U_{\mathfrak{g}_{i_0 \dots i_n}}} P_*(\mathfrak{g}_{i_0 \dots i_n})\big) = \begin{cases} U \otimes_{U_{\mathfrak{g}_{i_0 \dots i_n}}} F, & q = 0, \\ 0, & q > 0, \end{cases}$$

we conclude that $v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I})$ is canonically quasiisomorphic to the simplicial U-module:

$$\bigoplus_i U \otimes_{U_{\mathfrak{g}_i}} F \leftleftarrows \bigoplus_{i_0, i_1} U \otimes_{U_{\mathfrak{g}_{i_0 i_1}}} F \overset{\longleftarrow}{\underset{\longleftarrow}{\longleftarrow}} \cdots .$$

PROPOSITION 4.8. *Assume there are given a family $\{\mathfrak{g}_i\}_{i\in I}$ of subalgebras in a Lie algebra $\mathfrak{g}$, a family $\{\mathfrak{g}'_i\}_{i\in I}$ of subalgebras in a Lie algebra $\mathfrak{g}'$ and a homomorphism $f : \mathfrak{g} \to \mathfrak{g}'$ such that $f(\mathfrak{g}_i) \subset \mathfrak{g}'_i$ for all $i \in I$. If both families are well configured and the induced homomorphisms*

$$\mathfrak{g}/\mathfrak{g}_{i_0 \dots i_n} \to \mathfrak{g}'/\mathfrak{g}'_{i_0 \dots i_n}$$

are isomorphisms for all $(i_0, \dots, i_n)$, *then the map between the corresponding Volodin complexes*

$$v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I}) \to v_*(\mathfrak{g}', \{\mathfrak{g}'_i\}_{i\in I})$$

is a quasiisomorphism.

Proof. In view of subsection 4.7, it suffices to show that the maps

$$U_{\mathfrak{g}} \otimes_{U_{\mathfrak{g}_{i_0\dots i_n}}} F \to U_{\mathfrak{g}'} \otimes_{U_{\mathfrak{g}'_{i_0\dots i_n}}} F$$

are isomorphisms for all $(i_0, \dots, i_n)$. This follows immediately from the Poincaré–Birkhoff–Witt theorem. Indeed, if $\{\bar{e}_s\}_{s\in S\setminus S_{i_0\dots i_n}}$ is an *ordered* basis of $\mathfrak{g}/\mathfrak{g}_{i_0\dots i_n}$, then the monomials

$$e_{s_1}\dots e_{s_k} \qquad (s_1 \le \dots \le s_k;\ s_j \in S\setminus S_{i_0\dots i_n})$$

provide a basis for $U_{\mathfrak{g}} \otimes_{U_{\mathfrak{g}_{i_0\dots i_n}}} F$. ☐

COROLLARY 4.9. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mathfrak{m}$ *be a* $\mathfrak{g}$*-module. Then for every well-configured family of subalgebras* $\{\mathfrak{g}_i\}_{i\in I}$ *in* $\mathfrak{g}$, *the canonical maps*

$$v_*(\mathfrak{g}, \{\mathfrak{g}_i\}_{i\in I}) \rightleftarrows v_*(\mathfrak{g} \ltimes \mathfrak{m},\ \{\mathfrak{g}_i \ltimes \mathfrak{m}\}_{i\in I}),$$

which are induced by the pair of obvious homomorphisms

$$\mathfrak{g} \rightleftarrows \mathfrak{g} \ltimes \mathfrak{m},$$

are mutually inverse quasiisomorphisms. ☐

4.10. Let A be an associative F-algebra. For any positive integer n we shall consider the following Lie algebras of $n \times n$ matrices with coefficients in A:

$$\mathfrak{gl}_n(A) = \{(a_{ij}) \mid a_{ij} \in A;\ 1 \le i,j \le n\},$$
$$\mathfrak{sl}_n(A) = \left\{(a_{ij}) \in \mathfrak{gl}_n(A) \mid \sum_{i=1}^{n} a_{ii} \in [A,A]\right\},$$

(33) $\mathfrak{e}_n(A) =$ the smallest Lie subalgebra of $\mathrm{sl}_n(A)$ generated by the elementary matrices $e_{ij}(a) \equiv a \cdot e_{ij}$ $(a \in A;\ 1 \le i \ne j \le n)$, where $(e_{ij})_{kl} = 1$ if $k = i$ and $l = j$, and 0 otherwise.

Next, if σ is a partial ordering of the set $\{1, \dots, n\}$, we shall also consider the triangular Lie algebra $\mathfrak{t}_n^\sigma(A)$ associated with σ:

$$\mathfrak{t}_n^\sigma(A) = \left\{(a_{ij}) \in \mathfrak{gl}_n(A) \mid a_{ij} = 0 \ \text{ if } \ i \overset{\sigma}{\not<} j\right\}.$$

The algebra $\mathfrak{t}_n^\sigma(A)$ is generated by elementary matrices

$$e_{ij}(a) \qquad \big(a \in A;\, i \overset{\sigma}{<} j\big).$$

We shall also encounter the stable versions of Lie algebras (33): $\mathfrak{gl}(A) = \bigcup_{n\geq 1} \mathfrak{gl}_n(A)$, $\mathfrak{sl}(A) = \bigcup_{n\geq 1} \mathfrak{sl}_n(A)$ and $\mathfrak{e}(A) = \bigcup_{n\geq 1} \mathfrak{e}_n(A)$, as well as the Volodin chain complexes:

$$v_n(A) := v_*\big(\mathfrak{e}_n(A), \{\mathfrak{t}_n^\sigma(A)\}_{\sigma\in\Pi_n}\big), \text{ and} \tag{34}$$

$$v(A) := \varinjlim_n v_n(A) = v_*\big(\mathfrak{e}(A), \{\mathfrak{t}_n^\sigma(A)\}_{\sigma\in\Pi_n, n\in\mathbb{Z}_+}\big) \tag{35}$$

(Π_n denotes the set of all partial orderings of the set $\{1, \dots, n\}$).

The algebra $\mathfrak{gl}_n(A)$ acts naturally on the module of n-column vectors $M_{n,1}(A)$. And for any subalgebra $\mathfrak{g} \subset \mathfrak{gl}_n(A)$ we can form the semidirect product

$$\tilde{\mathfrak{g}} := \mathfrak{g} \ltimes M = \left\{ \begin{pmatrix} \alpha & v \\ 0 & 0 \end{pmatrix} \in \mathfrak{gl}_{n+1}(A) \mid \alpha \in \mathfrak{g},\, v \in M_{n,1}(A) \right\}.$$

Finally we have the corresponding affine versions of the Volodin complexes (34) and (35):

$$\tilde{v}_n(A) := v\big(\tilde{\mathfrak{e}}_n(A), \{\tilde{\mathfrak{t}}_n^\sigma\}_{\sigma\in\Pi_n}\big),$$

and

$$\tilde{v}(A) := \varinjlim_n \tilde{v}_n(A) = v_*\big(\tilde{\mathfrak{e}}(A), \{\tilde{\mathfrak{t}}_n^\sigma\}_{\sigma\in\Pi_n, n\in\mathbb{Z}_+}\big).$$

LEMMA 4.11. *The natural inclusion and projection maps*

$$\mathfrak{e}_n(A) \overset{\hookrightarrow}{\leftarrow} \tilde{\mathfrak{e}}_n(A) \quad \textit{and} \quad \mathfrak{e}(A) \overset{\hookrightarrow}{\leftarrow} \tilde{\mathfrak{e}}(A)$$

induce pairs of mutually inverse quasiisomorphisms $v_n(A) \rightleftarrows \tilde{v}_n(A)$ and, respectively, $v(A) \rightleftarrows \tilde{v}(A)$. □

This is a special case of Corollary 4.9.

Our next lemma is an analogue of Lemma 2.8, and it has a similar proof. In the assertions 4.12 through 4.16 below, A is assumed to satisfy $A = A^2$.

LEMMA 4.12. *The action of $\mathfrak{e}_{n+1}(A)$ on*

$$\mathrm{Im}\big[\mathrm{H}_*(v_n(A)) \to \mathrm{H}_*(v_{n+1}(A))\big]$$

is trivial. □

COROLLARY 4.13. *The actions of $\mathfrak{e}(A)$ on $\mathrm{H}_*(v(A))$ and of $\tilde{\mathfrak{e}}(A)$ on $\mathrm{H}_*(\tilde{v}(A))$ are trivial.* □

In the following lemma we assume that $n \geq 3$.

LEMMA 4.14. (a) $\mathfrak{sl}_n(A) = \mathfrak{e}_n(A)$,
(b) $\mathfrak{sl}_n(A) = [\mathfrak{sl}_n(A), \mathfrak{sl}_n(A)] = [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)]$,
(c) $\widetilde{\mathfrak{sl}}_n(A) = [\widetilde{\mathfrak{sl}}_n(A), \widetilde{\mathfrak{sl}}_n(A)] = [\widetilde{\mathfrak{gl}}_n(A), \widetilde{\mathfrak{gl}}_n(A)]$.

Proof. The following inclusions are clear:

$$\begin{array}{ccc} \mathfrak{sl}_n(A) \supseteq \mathfrak{e}_n(A) & \supseteq & [\mathfrak{e}_n(A), \mathfrak{e}_n(A)] \\ \cup\!\vert & & \cap\!\vert \\ [\mathfrak{gl}_n(A), \mathfrak{gl}_n(A)] & \supseteq & [\mathfrak{sl}_n(A), \mathfrak{sl}_n(A)]. \end{array}$$

For the proof of (a) and (b) it is therefore sufficient to demonstrate the equality $\mathfrak{sl}_n(A) = [\mathfrak{e}_n(A), \mathfrak{e}_n(A)]$. The latter is an easy consequence of the following identities:

$$h_i(ab) - h_j(ab) = [e_{ik}(a), e_{ki}(b)] - [e_{jk}(a), e_{kj}(b)],$$

and

$$h_i([a,b]) = [e_{ij}(a), e_{ji}(b)] + [e_{jk}(b), e_{kj}(a)] + [e_{ki}(a), e_{ik}(b)],$$

where $a, b \in A$ and $i \neq j$, $j \neq k$, $k \neq i$ and $h_i(c)$ is the diagonal matrix

$$(h_i(c))_{jk} = \begin{cases} c, & j,k = i, \\ 0, & \text{otherwise.} \end{cases}$$

Part (c) follows easily from (b). □

COROLLARY 4.15. *The actions of* $\mathfrak{gl}(A)$ *on* $\mathrm{H}_*(\mathfrak{sl}(A))$ *and of* $\widetilde{\mathfrak{gl}}(A)$ *on* $\mathrm{H}_*(\widetilde{\mathfrak{sl}}(A))$ *are trivial.*

The proof is similar to that of Corollary 1.14. □

We end this section with the Lie analogue of the comparison theorem Theorem 2.10.

THEOREM 4.16. *The following conditions are equivalent:*
(a) $\mathrm{H}_*(\mathfrak{gl}(A)) = \mathrm{H}_*(\widetilde{\mathfrak{gl}}(A))$,
(b) $\mathrm{H}_*(\mathfrak{e}(A)) = \mathrm{H}_*(\tilde{\mathfrak{e}}(A))$,
(c) $\mathrm{H}_*\big(\sum_{n,\sigma} C_*\big(\mathfrak{t}_n^\sigma(A)\big)\big) = \mathrm{H}_*\big(\sum_{n,\sigma} C_*\big(\tilde{\mathfrak{t}}_n^\sigma(A)\big)\big)$.

The proof is the same as in the group case. □

5. The homology of nilpotent groups and nilpotent Lie algebras

5.1. *Malcev's theory.* A theory due to A.I. Malcev [25] puts nilpotent Lie algebras over $\mathbb{Q}$ into a one-to-one correspondence with uniquely divisible nilpotent groups. This correspondence can be described as follows: For a nilpotent Lie $\mathbb{Q}$-algebra L let $U = U(L)$ be its universal enveloping algebra and $\mathcal{J} = \mathcal{J}(L)$ be the augmentation ideal of U. After completing U in the $\mathcal{J}$-adic topology $\hat{U} = \varprojlim_n U/\mathcal{J}^n$, we have the exponentiation mapping $\exp :$ $\hat{\mathcal{J}} \to 1 + \hat{\mathcal{J}}$ given by the usual power series, which is convergent in the $\mathcal{J}$-adic topology. Set

$$G(L) := \exp(L) \subset (1 + \hat{\mathcal{J}})^{\times}.$$

The Campbell–Hausdorff formula shows that $G(L)$ is a subgroup of the multiplicative group $(1 + \hat{\mathcal{J}})^{\times}$, and one verifies easily that $G(L)$ is nilpotent and uniquely divisible.

LEMMA 5.2. *For a nilpotent associative $\mathbb{Q}$-algebra B,*

$$G(B_{\text{Lie}}) = (1 + B)^{\times}$$

(B_{Lie} *denotes the Lie algebra* $(B; [\,,\,])$, *where* $[a, b] = ab - ba$).

Proof. Let $U = U(B_{\text{Lie}})$ and $\mathcal{J} = \mathcal{J}(B_{\text{Lie}})$, and let $\tilde{B} = \mathbb{Q} \ltimes B$ denote the $\mathbb{Q}$-algebra obtained from B by adjoining the unit. The identity map $B_{\text{Lie}} \to B$ gives rise to the commutative diagram

$$\begin{tikzcd} & U \arrow[dd, "f"] \\ B_{\text{Lie}} \arrow[ur] \arrow[dr, hook] & \\ & \tilde{B} \end{tikzcd}.$$

Since $f(\mathcal{J}) = B$ and B is nilpotent, we have $f(\mathcal{J}^n) = 0$ for n sufficiently large. The map f, being continuous in the $\mathcal{J}$-adic topology, extends to a map $\hat{\mathcal{J}} \to B$ and thus induces a group homomorphism $(1+\hat{\mathcal{J}})^{\times} \to (1+B)^{\times}$ making the following diagram commute:

$$\begin{tikzcd} B_{\text{Lie}} \arrow[r, hook] \arrow[dr, equal] & \hat{\mathcal{J}} \arrow[r, "\exp", "\sim"'] \arrow[d, "f"] & (1 + \hat{\mathcal{J}})^{\times} \arrow[d] \\ & B \arrow[r, "\exp", "\sim"'] & (1 + B)^{\times} \end{tikzcd}.$$

Since the image of B_{Lie} in $(1 + \hat{\mathcal{J}})^{\times}$ is, by definition, $G(B_{\text{Lie}})$, we have $G(B_{\text{Lie}}) \simeq (1 + B)^{\times}$. □

Let A be a $\mathbb{Q}$-algebra, σ be a partial ordering of the set $\{1, \ldots, n\}$, and $T_n^{\sigma}(A)$ and $\mathfrak{t}_n^{\sigma}(A)$ be the corresponding triangular group and, respectively, Lie algebra (see Sections 2 and 4). We then have the following corollary:

COROLLARY 5.3. *For any $\mathbb{Q}$-algebra A,*

$$G(\mathfrak{t}_n^{\sigma}(A)) = T_n^{\sigma}(A). \qquad \square$$

5.4. As is well known, the homology of a nilpotent Lie $\mathbb{Q}$-algebra L and the rational homology of the corresponding nilpotent group $G = G(L)$ coincide [28]:

$$\mathrm{H}_*(\mathrm{B}G;\mathbb{Q}) = \mathrm{H}_*(L;\mathbb{Q}).$$

It will be convenient for our purposes to know that this isomorphism is induced by a functorial morphism of chain complexes:

$$C_*(\mathrm{B}G;\mathbb{Q}) \to C_*(L;\mathbb{Q}).$$

To produce such a morphism we need a functorial contracting homotopy of the standard resolution $P_*(L)$ introduced in Section 4.

5.5. *A functorial contracting homotopy for $P_*(L)$.* Throughout this subsection, L denotes an arbitrary Lie algebra over a field F of characteristic zero. The universal enveloping algebra $U = U(L)$ has a canonical increasing filtration

$$F_pU := \operatorname{lin\,span}\{l_1 \dots l_j \mid l_i \in L;\ 0 \le i \le j \le p\}. \tag{36}$$

(The empty product of elements is set to be equal to $1 \in U$.) It follows from the Poincaré–Birkhoff–Witt theorem that the subsequent quotients $F_pU/F_{p-1}U$ coincide with the symmetric powers $S^p(L)$ of L. The map $\varphi_p : S^p(L) \to F_pU$,

$$\varphi_p(l_1, \dots, l_p) = \frac{1}{p!}\sum_{\sigma\in S_p} l_{\sigma(1)} \dots l_{\sigma(p)},$$

provides a vector-space splitting of the projection $F_pU \to S^p(L)$. Thus $F_pU = S^p(L) \oplus F_{p-1}U$, and we get a canonical vector-space isomorphism

$$\varphi_* : S^*(L) \xrightarrow{\sim} U.$$

We use it to define the following map $s_0 : U \otimes \Lambda^*(L) \to U \otimes \Lambda^{*+1}(L)$,

$$\begin{aligned} s_0\bigl(\varphi_p(l_1, \dots, l_p) \otimes l_{p+1} \wedge \dots \wedge l_{p+q}\bigr) = \frac{1}{(p-1)!(p+q)} &\sum_{\sigma\in S_p} (l_{\sigma(1)} \dots l_{\sigma(p-1)}) \\ &\otimes l_{\sigma(p)} \wedge l_{p+1} \wedge \dots \wedge l_{p+q} \end{aligned}$$

and

$$s_0(1 \otimes l_1 \wedge \dots \wedge l_q) = 0.$$

The map s_0 respects the filtration on $P_*(L)$, which is a natural extension of filtration (36):

$$F_n(U \otimes \Lambda^q(L)) := F_{n-q}U \otimes \Lambda^q(L) \qquad (q \ge 0).$$

A straightforward calculation shows that the mapping $t : P_*(L) \to P_*(L)$, given by $t = 1 - s_0\partial - \partial s_0$, is the identity map on $F_0P_*(L)$ and maps $F_nP_*(L)$

into $F_{n-1}P_*(L)$ if $n > 0$. After setting

$$s := s_0 + s_0 t + s_0 t^2 + \dots, \tag{37}$$

we can easily verify that $s\partial + \partial s = 1 - \varepsilon$, where $\varepsilon : U \to U$ is the augmentation $U \to F$ composed with the subsequent inclusion $F \to F \cdot 1 \subset U$. The sum in formula (37) is finite on each $F_n P_*(L)$ in view of the equality $t^n|_{F_n P_*(L)} = 0$.

5.6. We assume now again that the Lie algebra L is nilpotent. We shall show that the homotopy operator $s : P_*(L) \to P_{*+1}(L)$ defined above is continuous in the $\mathcal{J}$-adic topology.

Consider the lower central series of L:

$$L^{(1)} = L, \quad L^{(2)} = [L, L], \quad \dots, \quad L^{(i)} = [L, L^{(i-1)}], \quad \dots \tag{38}$$

and denote by N the nilpotency class of L (i.e., $L^{(N+1)} = 0$ and $L^{(N)} \neq 0$). The following properties of (38) are rather obvious:

$$[L^{(i)}, L^{(j)}] \subset L^{(i+j)} \tag{39}$$

and

$$L^{(i)} \subset \mathcal{J}^i. \tag{40}$$

Let us define the degree function $\deg : L \to \mathbb{Z}_+ \cup \{\infty\}$ by setting $\deg(0) = \infty$ and $\deg(l) = n$ if $l \in L^{(n)} \backslash L^{(n-1)}$. Then (39) implies that

$$\deg([l_1, l_2]) \geq \deg(l_1) + \deg(l_2) \qquad (l_1, l_2 \in L). \tag{41}$$

Finally we define the decreasing filtration on $S^p(L)$, $p \geq 0$, by

$$S^p(L)^{(n)} = \operatorname{lin\,span}\left\{ \varphi_p(l_1, \dots, l_p) \mid \sum_{i=1}^{p} \deg(l_i) \geq n \right\}. \tag{42}$$

Clearly $S^p(L)^{(p)} = S^p(L)$ and

$$S^p(L)^{(n)} \subset S^p(L) \cap \mathcal{J}^n,$$

where $S^p(L)$ is identified with a subspace of U via φ_p.

The converse inclusion holds as well, as one readily sees from the following lemma:

LEMMA 5.7. *If $l_1, \dots, l_p \in L$ and $\sum_{i=1}^{p} \deg(l_i) \geq n$, then*

$$l_1 \cdots l_p \in \bigoplus_{i=1}^{p} S^i(L)^{(n)}.$$

Proof. We will proceed by induction on p. The assertion is obvious for $p = 1$. For $p > 1$ we have

$$l_1 \cdots l_p = \varphi_p(l_1, \dots, l_p) + \frac{1}{p!} \sum_{\sigma \in S_p} (l_1 \cdots l_p - l_{\sigma(1)} \cdots l_{\sigma(p)}),$$

and each term

$$l_1 \cdots l_p - l_{\sigma(1)} \cdots l_{\sigma(p)} \qquad (\sigma \in S_p)$$

is a sum of expressions of the form

$$l_{\tau(1)} \cdots l_{\tau(k-1)} \cdot [l_{\tau(k)}, l_{\tau(k+1)}] \cdot l_{\tau(k+2)} \cdots l_{\tau(p)}.$$

In particular,

$$l_1 \cdots l_p - \varphi_p(l_1, \dots, l_p)$$

is a sum of products of the form $m_1 \cdots m_{p-1}$ with $\sum_{i=1}^{p-1} \deg(m_i) \geq n$. It remains to invoke the induction hypothesis. □

COROLLARY 5.8. $\mathcal{J}^n = \bigoplus_{i=1}^n S^i(L)^{(n)}$. □

COROLLARY 5.9. *The homotopy operator* $s : P_*(L) \to P_{*+1}(L)$ *is continuous in the* $\mathcal{J}$*-adic topology and hence defines a homotopy operator*

$$s : P_*(L)^\wedge \to P_{*+1}(L)^\wedge$$

(*i.e.,* $s\partial + \partial s = 1 - \varepsilon$ *on* $P_*(L)^\wedge$).

Proof. Let us extend the decreasing filtration (42) to the whole $P_*(L)$:

$$P_q(L)^{(n)} := \operatorname{lin\,span}\left\{ (l_1 \cdots l_k) \otimes l_{k+1} \wedge \cdots \wedge l_{k+q} \mid \sum_{i=1}^{k+q} \deg(l_i) \geq n;\ k \geq 0 \right\}. \tag{43}$$

Clearly $\partial(P_*(L)^{(n)}) \subset P_{*-1}(L)^{(n)}$; Lemma 5.7 shows that $s_0(P_*(L)^{(n)}) \subset P_{*+1}(L)^{(n)}$. Hence $s(P_*(L)^{(n)}) \subset P_{*+1}(L)^{(n)}$.

Finally notice that the filtration (43) induces a topology on $U \otimes \Lambda^*(L)$, which is equivalent to the $\mathcal{J}$-adic topology

$$\mathcal{J}^k \otimes \Lambda^q(L) \subset [U \otimes \Lambda^q(L)]^{(k+q)} \subset \mathcal{J}^{k-(N-1)q} \otimes \Lambda^q(L)$$

(recall that N denotes the nilpotency class of L). □

5.10. Now let L be a finite-dimensional, nilpotent, Lie $\mathbb{Q}$-algebra and $G = G(L)$ be the corresponding nilpotent group. We denote by I the augmentation ideal in the group algebra $\mathbb{Q}[G]$. The inclusion $G \subset \hat{U}^*$ defines an algebra homomorphism $\mathbb{Q}[G] \to \hat{U}$, which is continuous if $\mathbb{Q}[G]$ is equipped with its

I-adic and $\hat{U}$ with the $\mathcal{J}$-adic topology. The induced map $\mathbb{Q}[G]^\wedge \to \hat{U}$ is an isomorphism (see [28]). We will use that map to identify the two algebras. The standard Artin–Rees property argument shows that $\mathbb{Q}[G]^\wedge$ is flat as both a right and a left $\mathbb{Q}[G]$-module, while $\hat{U}$ is flat as a U-module ([28], pp. 414–415). Therefore

$$\mathbb{Q}[G]^\wedge \otimes_{\mathbb{Q}[G]} C_*(P(G),\mathbb{Q})$$

is a free resolution of the augmentation $\mathbb{Q}[G]^\wedge$-module $\mathbb{Q}$ ($P(G)$ being the simplicial set introduced in Section 2) and

$$\hat{U} \otimes_U P_*(L) \equiv P_*(L)^\wedge \tag{44}$$

is a free resolution of the $\hat{U}$-module $\mathbb{Q}$. In addition, (44) comes with the canonical contracting homotopy s (see Corollary 5.9).

We shall now construct a morphism of resolutions

$$\begin{array}{ccc} \mathbb{Q} & \longleftarrow & \mathbb{Q}[G]^\wedge \otimes_{\mathbb{Q}[G]} C_*(P(G);\mathbb{Q}) \\ \| & & \downarrow f \\ \mathbb{Q} & \longleftarrow & \hat{U} \otimes_U P_*(L) \end{array}$$

by asking that, in dimension zero, f be the identification $\mathbb{Q}[G]^\wedge \xrightarrow{\sim} \hat{U}$ and by extending f to higher dimensions by means of the inductive formula

$$f([g_1,\dots,g_q]) = (sf\partial)([g_1,\dots,g_q])$$

(The vectors $[g_1,\dots,g_q] \equiv (1,g_1,g_1g_2,\dots,g_1g_2\cdots g_q)$ are the usual $\mathbb{Q}[G]$-basis vectors in $\mathbb{Q}[G^{q+1}]$.) By tensoring f over $\mathbb{Q}[G]^\wedge = \hat{U}$ with $\mathbb{Q}$, we get a functorial quasiisomorphism

$$\begin{aligned} f : C_*(\mathrm{B}G;\mathbb{Q}) &= \mathbb{Q}\otimes_{\mathbb{Q}[G]} C_*(PG;\mathbb{Q}) = \mathbb{Q}\otimes_{\mathbb{Q}[G]^\wedge} \otimes\big(\mathbb{Q}[G]^\wedge \otimes_{\mathbb{Q}[G]} C_*(PG;\mathbb{Q})\big) \\ &\longrightarrow \mathbb{Q}\otimes_{\hat{U}} \big(\hat{U}\otimes_U P_*(L)\big) = \mathbb{Q}\otimes_U P_*(L) = C_*(L). \end{aligned} \tag{45}$$

Since every nilpotent Lie $\mathbb{Q}$-algebra is a filtered inductive limit of its finite-dimensional subalgebras, and since the chain complex functors, which occur in equation (45), commute with filtered inductive limits, we arrive at the following theorem:

THEOREM 5.11. *On the category of nilpotent Lie $\mathbb{Q}$-algebras there exists a functorial quasiisomorphism*[2]

$$f : C_*(\mathrm{B}G(L);\mathbb{Q}) \to C_*(L). \qquad \square$$

5.12. Let A be an associative $\mathbb{Q}$-algebra and σ be a partial ordering of the set $\{1,\dots,n\}$. Corollary 5.3 combined with Theorem 5.11 gives a functorial

[2] A slightly weaker statement is proved by T.G. Goodwillie in [13], Prop. III.5.

quasiisomorphism

$$f : C_*(\mathrm{B}T_n^\sigma(A);\mathbb{Q}) \to C_*(\mathfrak{t}_n^\sigma(A)), \tag{46}$$

where $T_n^\sigma(A) \subset \mathrm{GL}_n(A)$ and $\mathfrak{t}_n^\sigma(A) \subset \mathfrak{gl}_n(A)$ are the corresponding triangular group and, respectively, Lie algebra. The functoriality of (46) implies that the quasiisomorphisms (46), for different σ, patch together into a single morphism of chain complexes

$$f : \sum_\sigma C_*(\mathrm{B}T_n^\sigma(A);\mathbb{Q}) = C_*\left(\bigcup_\sigma \mathrm{B}T_n^\sigma(A);\mathbb{Q}\right) \to \sum_\sigma C_*(\mathfrak{t}_n^\sigma(A)). \tag{47}$$

(The summation extends over all partial orderings of the set $\{1,\dots,n\}$.) The Mayer–Vietoris spectral sequence, or the repetitive use of the Mayer–Vietoris long exact sequence, shows that (47) is again a quasiisomorphism. Exactly in the same way one also obtains the corresponding quasiisomorphism of affine complexes:

$$C_*\left(\bigcup_\sigma \mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Q}\right) \to \sum_\sigma C_*(\tilde{\mathfrak{t}}_n^\sigma(A)).$$

THEOREM 5.13. *For any $\mathbb{Q}$-algebra A and any n there are canonical isomorphisms*

$$\tilde{\mathrm{H}}_*\left(\bigcup_\sigma \mathrm{B}T_n^\sigma(A);\mathbb{Z}\right) \xrightarrow{\sim} \tilde{\mathrm{H}}_*\left(\sum_\sigma C_*(\mathfrak{t}_n^\sigma(A))\right)$$

and

$$\tilde{\mathrm{H}}_*\left(\bigcup_\sigma \mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Z}\right) \xrightarrow{\sim} \tilde{\mathrm{H}}_*\left(\sum_\sigma C_*(\tilde{\mathfrak{t}}_n^\sigma(A))\right)$$

($\tilde{H}_0 \equiv 0$ *and* $\tilde{H}_q \equiv H_q$ *for* $q > 0$).

Proof. An easy argument involving the Hochschild–Serre spectral sequence shows that $\tilde{\mathrm{H}}_*(\mathrm{B}T_n^\sigma(A);\mathbb{Z}/m\mathbb{Z}) = 0$ for any $m \in \mathbb{Z}_+$ and σ. The Mayer–Vietoris spectral sequence then gives

$$\tilde{\mathrm{H}}_*\left(\bigcup_\sigma \mathrm{B}T_n^\sigma(A);\mathbb{Z}/m\mathbb{Z}\right) = 0 \qquad (m \in \mathbb{Z}_+),$$

which, in conjunction with (46), implies that

$$\tilde{\mathrm{H}}_*\left(\bigcup_\sigma \mathrm{B}T_n^\sigma(A);\mathbb{Z}\right) = \tilde{\mathrm{H}}_*\left(\bigcup_\sigma \mathrm{B}T_n^\sigma(A);\mathbb{Q}\right) \xrightarrow{\sim} \tilde{\mathrm{H}}_*\left(\sum_\sigma C_*(\mathfrak{t}_n^\sigma(A))\right).$$

A similar argument proves the other isomorphism. □

COROLLARY 5.14. *For any $\mathbb{Q}$-algebra A there are canonical isomorphisms*

$$\tilde{\mathrm{H}}_*\left(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A);\mathbb{Z}\right) \xrightarrow{\sim} \tilde{\mathrm{H}}_*\left(\sum_{n,\sigma} C_*(\mathfrak{t}_n^\sigma(A))\right)$$

and

$$\tilde{\mathrm{H}}_*\Big(\bigcup_{n,\sigma}\mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Z}\Big)\ \tilde{\to}\ \tilde{\mathrm{H}}_*\Big(\sum_{n,\sigma}C_*(\tilde{\mathfrak{t}}_n^\sigma(A))\Big).$$

Proof. The quasiisomorphisms (47) defined for different n agree with each other. □

5.15. *Nilpotent modules of coefficients.* Let $\mathcal{G}$ be a group and $\mathcal{M}$ be a $\mathcal{G}$-module. We call $\mathcal{M}$ *nilpotent* if there exists a finite filtration by $\mathcal{G}$-submodules:

$$0=\mathcal{M}_{-1}\subset\mathcal{M}_0\subset\mathcal{M}_1\subset\cdots\subset\mathcal{M}_k=\mathcal{M}$$

such that $\mathcal{G}$ acts trivially on $\mathcal{M}_i/\mathcal{M}_{i-1}$, $0\le i\le k$.

PROPOSITION 5.16. *Let A be a ring and M be any nilpotent $\mathbb{Q}[T_n^\sigma(A_\mathbb{Q})]$-module. Then for any $n\in\mathbb{Z}_+$ and any partial ordering of the set $\{1,\dots,n\}$,*

$$\mathrm{H}_*(T_n^\sigma(A);M)=\mathrm{H}_*(T_n^\sigma(A_\mathbb{Q});M).$$

Proof. We proceed by induction on n. It is clearly sufficient to consider only the case of trivial coefficients $M=\mathbb{Q}$. Let $k\in\{1,\dots,n\}$ be an element maximal for σ. In order to keep the notation simple we will assume that n is maximal and take $k=n$. Then the deletion of the n^{th} row and n^{th} column defines the group epimorphism

$$T_n^\sigma(A)\twoheadrightarrow T_{n-1}^{\sigma'}(A), \tag{48}$$

where $\sigma'=\sigma|_{\{1,\dots,n-1\}}$. The kernel of (48) is abelian and isomorphic to the additive group $M_{l,1}(A)$, where $l=\operatorname{card}\{i\mid i\overset{\sigma}{<}n\}$. As a $T_{n-1}^{\sigma'}(A)$-module, $M_{l,1}(A)$ is nilpotent.

Let us consider the canonical morphism of two Hochschild–Serre spectral sequences associated with the epimorphism (48):

$$\begin{array}{ccc} H_p(T_{n-1}^{\sigma'}(A);H_q(M_{l,1}(A);\mathbb{Q})) & & H_p(T_{n-1}^{\sigma'}(A_\mathbb{Q});H_q(M_{l,1}(A_\mathbb{Q});\mathbb{Q})) \\ \Downarrow & \longrightarrow & \Downarrow \\ H_{p+q}(T_n^\sigma(A);\mathbb{Q}) & & H_{p+q}(T_n^\sigma(A_\mathbb{Q});\mathbb{Q}). \end{array} \tag{49}$$

Since $H_q(M_{l,1}(A);\mathbb{Q})=\Lambda^q_\mathbb{Q}M_{l,1}(A)=H_q(M_{l,1}(A_\mathbb{Q});\mathbb{Q})$ and the action of $T_{n-1}^{\sigma'}(A_\mathbb{Q})$ on it is nilpotent, we deduce from the induction hypothesis that (49) is an isomorphism of spectral sequences. □

COROLLARY 5.17. *For any ring A and any n,*

$$\mathrm{H}_*(\tilde{T}_n^\sigma(A);\mathbb{Q})=\mathrm{H}_*(\tilde{T}_n^\sigma(A_\mathbb{Q});\mathbb{Q}).$$

Proof. The morphism of the Hochschild–Serre spectral sequences

$$\begin{array}{ccc} H_p(T_n^\sigma(A); H_q(M_{n,1}(A);\mathbb{Q})) & & H_p(T_n^\sigma(A_\mathbb{Q}); H_q(M_{n,1}(A_\mathbb{Q});\mathbb{Q})) \\ \Downarrow & \longrightarrow & \Downarrow \\ H_{p+q}(\tilde{T}_n^\sigma(A);\mathbb{Q}) & & H_{p+q}(\tilde{T}_n^\sigma(A_\mathbb{Q});\mathbb{Q}) \end{array}$$

is an isomorphism by Proposition 5.16. □

We deduce the next two corollaries from Proposition 5.16 and Corollary 5.17 by using the Mayer–Vietoris spectral sequence.

COROLLARY 5.18. *For any ring A and any n,*

$$\mathrm{H}_*\Big(\bigcup_\sigma \mathrm{B}T_n^\sigma(A);\mathbb{Q}\Big) = \mathrm{H}_*\Big(\bigcup_\sigma \mathrm{B}T_n^\sigma(A_\mathbb{Q});\mathbb{Q}\Big), \text{ and}$$

$$\mathrm{H}_*\Big(\bigcup_\sigma \mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Q}\Big) = \mathrm{H}_*\Big(\bigcup_\sigma \mathrm{B}\tilde{T}_n^\sigma(A_\mathbb{Q}\Big);\mathbb{Q}. \quad \square$$

COROLLARY 5.19. *For any ring A,*

$$\mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A);\mathbb{Q}\Big) = \mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A_\mathbb{Q});\mathbb{Q}\Big), \text{ and}$$

$$\mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Q}\Big) = \mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A_\mathbb{Q});\mathbb{Q}\Big). \quad \square$$

6. A proof of the main theorems

We begin this section by proving the following proposition:

PROPOSITION 6.1. (a) *Let A be an H-unital $\mathbb{Q}$-algebra. Then*

$$\mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A);\mathbb{Z}\Big) = \mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Z}\Big).$$

(b) *Let A be a ring with the property that the $\mathbb{Q}$-algebra $A_\mathbb{Q}$ be H-unital. Then*

$$\mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A);\mathbb{Q}\Big) = \mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A);\mathbb{Q}\Big).$$

Proof. (a) Consider the following algebra of 2×2 matrices:

$$A_1 := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \middle| \, a, b \in A \right\}.$$

According to [40], Theorem 11.1, if A is H-unital, so is A_1, and the inclusion

$$A \hookrightarrow A_1, \qquad a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix},$$

induces an isomorphism in cyclic homology. From Hanlon's results [14], it follows that, for an H-unital $\mathbb{Q}$-algebra A, the Lie algebra homology $\mathrm{H}_*(\mathfrak{gl}(A))$ is a free graded-commutative algebra freely generated by $HC_*(A)[1]$. Thus the inclusion

$$\mathfrak{gl}(A) \hookrightarrow \mathfrak{gl}(A_1) \tag{50}$$

induces an isomorphism $\mathrm{H}_*(\mathfrak{gl}(A)) \xrightarrow{\sim} \mathrm{H}_*(\mathfrak{gl}(A_1))$. Now

$$\mathfrak{gl}(A_1) = \mathfrak{gl}(A) \ltimes \mathcal{M}(A),$$

where the $\mathfrak{gl}(A)$-module $\mathcal{M}(A)$ is the countable direct sum $\bigoplus_{j=1}^{\infty} M_{\infty 1}(A)$ of copies of $M_{\infty 1}(A)$. This implies that (50) is a composite of two Lie algebra embeddings

$$\mathfrak{gl}(A) \overset{i_1}{\hookrightarrow} \widetilde{\mathfrak{gl}}(A) = \mathfrak{gl}(A) \ltimes M_{\infty 1}(A) \overset{i_2}{\hookrightarrow} \mathfrak{gl}(A_1),$$

where the second one is induced by the inclusion of $M_{\infty 1}(A)$ into $\bigoplus_{j=1}^{\infty} M_{\infty 1}(A)$ via

$$\beta \mapsto (\beta, 0, 0, \dots) \qquad (\beta \in M_{\infty 1}(A)).$$

Since $\widetilde{\mathfrak{gl}}(A)$ is a retract of $\mathfrak{gl}(A_1)$, and since the composite map $i_2 \circ i_1$ induces an isomorphism in homology, we conclude that

$$\mathrm{H}_*(\mathfrak{gl}(A)) = \mathrm{H}_*(\widetilde{\mathfrak{gl}}(A)).$$

The following chain of implications

$$\mathrm{H}_*(\mathfrak{gl}(A)) = \mathrm{H}_*(\widetilde{\mathfrak{gl}}(A))$$

$$\Downarrow \text{ Thm. 4.16}$$

$$\mathrm{H}_*\Big(\sum_{n,\sigma} C_*(\mathfrak{t}_n^\sigma(A))\Big) = \mathrm{H}_*\Big(\sum_{n,\sigma} C_*(\tilde{\mathfrak{t}}_n^\sigma(A))\Big)$$

$$\Downarrow \text{ Cor. 5.14}$$

$$\mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A); \mathbb{Z}\Big) = \mathrm{H}_*\Big(\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A); \mathbb{Z}\Big)$$

completes the proof of part (a).

Part (b) follows from part (a) and Corollary 5.19. □

6.2. *Proof of Theorem* B. As already indicated, the implication

$$\begin{array}{l}\text{a } \mathbb{Q}\text{-algebra } A \text{ satisfies excision}\\ \text{in algebraic K-theory}\end{array} \quad \Rightarrow \quad A \text{ is } H\text{-unital}$$

has been proved in [40]. Here we prove the converse.

Let A be an H-unital $\mathbb{Q}$-algebra. Proposition 6.1(a) in conjunction with Theorem 2.10 implies that

$$\mathrm{H}_*(\mathrm{GL}(A);\mathbb{Z}) = \mathrm{H}_*\big(\widetilde{\mathrm{GL}}(A);\mathbb{Z}\big).$$

The same argument, when applied to the opposite algebra A^{op}, which is H-unital simultaneously with A, also gives

$$\mathrm{H}_*(\mathrm{GL}(A);\mathbb{Z}) = \mathrm{H}_*\Big(\widetilde{\widetilde{\mathrm{GL}}}(A);\mathbb{Z}\Big)$$

(cf. 1.2). In other words, A possesses Property $\mathrm{AH}_{\mathbb{Z}}$ and hence satisfies excision in algebraic K-theory (see Proposition 1.15). $\square$

6.3. The rest of this section is devoted to proving Theorem A. If A is a ring such that $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$ is H-unital and, in addition, $A = A^2$, then the proof goes almost as in the case of Theorem B. More specifically, Proposition 6.1(b) combined with Theorem 2.10 shows that A has Property $\mathrm{AH}_{\mathbb{Q}}$, and then Corollary 1.8 shows that A satisfies excision in rational algebraic K-theory. For general A, the H-unitality of $A_{\mathbb{Q}}$ implies only that A/A^2 is a torsion abelian group and, as a result, we are not able to use Theorem 2.10. We need a more precise result on the comparison of spectral sequences than the standard comparison theorem [45], on which Theorem 2.10 is based. Our proof of the general case of Theorem A relies, instead, on the following theorem:

THEOREM 6.4. *Let A be a ring with the property that the $\mathbb{Q}$-algebra $A_{\mathbb{Q}}$ be H-unital. Then for every subgroup $H \subset \mathrm{GL}(A)$, which contains $E(A^i)$ for some i, the inclusion $H \hookrightarrow \tilde{H}$ induces an isomorphism in rational homology:*

$$\mathrm{H}_*(H;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*(\tilde{H};\mathbb{Q}).$$

Recall that $\tilde{H}$ denotes the affine group $H \ltimes M_{\infty 1}(A)$.

6.5. *Derivation of Theorem* A *from Theorem* 6.4. Let A be a ring with the property that $A_{\mathbb{Q}}$ be H-unital. By Theorem 6.4 we have

$$\mathrm{H}_*(\mathrm{GL}(A);\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*\big(\widetilde{\mathrm{GL}}(A);\mathbb{Q}\big).$$

The same applied to A^{op} gives

$$\mathrm{H}_*(\mathrm{GL}(A);\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}_*\Big(\widetilde{\widetilde{\mathrm{GL}}}(A);\mathbb{Q}\Big),$$

and we conclude that A possesses Property $\mathrm{AH}_{\mathbb{Q}}$. Corollary 1.8 then shows that A satisfies excision in rational algebraic K-theory.

6.6. A series of auxiliary lemmas will precede the proof of Theorem 6.4. We begin by examining more closely the normal structure of the group $G = \mathrm{GL}(A)$. Let

$$G = G^{(1)} \supset G^{(2)} \supset \dots$$

denote the lower central series of G; i.e., $G^{(1)} = G$ and $G^{(i)} = [G, G^{(i-1)}]$, $i > 1$.

LEMMA 6.7. *For any ring A and any $n \geq 1$ there are the following inclusions:*

(a) $E(A^n) \subset G^{(n)} \subset \mathrm{GL}(A^n)$,
(b) $[\mathrm{GL}(A), \mathrm{GL}(A^{2n})] \subset E(A^n)$,
(c) $G^{(2n+1)} \subset E(A^n)$.

Proof. (a) This follows by an easy induction on n.

(b) Let $g \in \mathrm{GL}_k(A)$ and $h \in \mathrm{GL}_k(A^{2n})$. The matrix h can be written in the form $1 + \alpha\beta$ for some $\alpha \in M_{kl}(A^n)$, $\beta \in M_{lk}(A^n)$ and $l \in \mathbb{Z}_+$. Then one has the following identity in $\mathrm{GL}_{k+l}(A)$:

$$\begin{pmatrix} [g,h] & 0 \\ 0 & 1_l \end{pmatrix} = \begin{pmatrix} 1 + (g\alpha)(\beta g^{-1}) & 0 \\ 0 & (1 + (\beta g^{-1})(g\alpha))^{-1} \end{pmatrix} \times \begin{pmatrix} 1 + \alpha\beta & 0 \\ 0 & (1 + \beta\alpha)^{-1} \end{pmatrix}^{-1}. \tag{51}$$

Both matrices on the right-hand side of (51) belong to $E_{k+l}(A^n)$ in view of Lemma 1.11.

Part (c) follows immediately from the combination of (a) and (b). □

LEMMA 6.8. *Let A be a ring with the property that A/A^2 be a torsion abelian group. If a normal subgroup N of $E(A)$ contains $E(A^n)$ for some $n \geq 1$, then the quotient $E(A)/N$ is a torsion nilpotent group.*

Proof. The nilpotency of $E(A)/N$ follows from Lemma 6.7(c). In particular, the quotient group $E(A)/E(A^n)$ is nilpotent. If A/A^2 is torsion, so is A/A^n; therefore, for every $a \in A$, there exists $m \in \mathbb{Z}_+$ such that $ma \in A^n$. In terms of the elementary matrices $E_{ij}(a) \in E(A)$ this translates into the statement that

$$(E_{ij}(a))^m = E_{ij}(ma) \in E(A^n)$$

for some $m \in \mathbb{Z}_+$ depending on a, in general. Thus the group $E(A)/E(A^n)$ is generated by elements of finite order. Since the elements of finite order in a nilpotent group form a subgroup (cf., e.g., [19], Thm. 16.2.7), we conclude that $E(A)/E(A^n)$ consists entirely of torsion. This applies also to $E(A)/N$, which is a homomorphic image of $E(A)/E(A^n)$. □

LEMMA 6.9. *For any torsion solvable group H and any $\mathbb{Q}[H]$-module V,*

$$H_q(H;V)=0 \qquad (q>0). \tag{52}$$

Proof. Any torsion abelian group is the filtered inductive limit of its own finite subgroups, for which equality (52) is clear. Since filtered inductive limits commute with the group homology, we obtain (52) in the case of a torsion abelian H. In the general case, one proceeds by induction on the length of the derived series of H using the Hochschild–Serre spectral sequence

$$E^2_{pq}=H_p\big(H/[H,H];\ H_q([H,H];V)\big)\Rightarrow H_{p+q}(H;V). \qquad \square$$

6.10. As a last step in the preparations for proving Theorem 6.4, we formulate two simple results concerned with the comparison of spectral sequences. For convenience we consider the following category of first-quadrant spectral sequences, whose objects are quadruples (E,B,F,φ) such that:

(a) $E=(E^k_{pq})$, $k=2,3,\dots,\infty$, is a first-quadrant homological spectral sequence of abelian groups,

(b) $B=\bigoplus_{n\geq 0}B_n$ is a graded abelian group,

(c) each B_n, $n\geq 0$, is equipped with an increasing filtration

$$0=F_{-1}B_n\subset F_0B_n\subset\cdots\subset F_nB_n=B_n,$$

(d) φ is an identification $E^\infty_{pq}\xrightarrow{\sim}F_pB_{p+q}/F_{p-1}B_{p+q}$; $p,q\geq 0$.

Morphisms are defined in a natural way and are supposed to be compatible with all the above data. We will use a shorter notation (E,B) instead of (E,B,F,φ).

LEMMA 6.11. *Let $f:(E,B)\to(\tilde{E},\tilde{B})$ be a morphism between two spectral sequences. If the maps $E^2_{pq}\to\tilde{E}^2_{pq}$ are isomorphisms for $p=0,\dots,n-1$, so are the maps $B_k\to\tilde{B}_k$ for $k=0,\dots,n-2$.*

If, in addition, $B_{n-1}\to\tilde{B}_{n-1}$ and $B_n\to\tilde{B}_n$ also are isomorphisms, so is $E^2_{n0}\to\tilde{E}^2_{n0}$. $\square$

LEMMA 6.12. *Let $f:(E,B)\to(\tilde{E},\tilde{B})$ be a morphism of spectral sequences. If the maps $E^2_{pq}\to\tilde{E}^2_{pq}$ are isomorphisms for $q=0,\dots,n$, so are the maps $B_k\to\tilde{B}_k$ for $k=0,\dots,n$.*

If, in addition, $B_{n+1}\to\tilde{B}_{n+1}$ is onto, then $E^2_{0,n+1}\to\tilde{E}^2_{0,n+1}$ is onto as well. $\square$

Proof of Theorem 6.4. Let A and $H\subset\mathrm{GL}(A)$ satisfy the hypothesis of the theorem. We will prove by induction on $n\geq 0$ that the canonical map

$$H_n(H;\mathbb{Q})\to H_n(\tilde{H};\mathbb{Q}) \tag{53}$$

is an isomorphism.

There is nothing to prove if $n = 0$. Let us assume that

$$H_j(H;\mathbb{Q}) \to H_j(\tilde{H};\mathbb{Q})$$

is an isomorphism for all $j < n$ as well as for all A and H satisfying the hypothesis of the theorem.

Case $H = E(A)$. Consider the morphism of spectral sequences from the proof of Theorem 2.10:

$$\begin{array}{ccc} E^2_{pq} = H_p(E(A);V_q) & & \tilde{E}^2_{pq} = H_p\big(\widetilde{E(A)};V_q\big) \\ \Downarrow & \longrightarrow & \Downarrow \\ H_{p+q}\Big(\bigcup\limits_{n,\sigma} \mathrm{B}T^\sigma_n(A);\mathbb{Q}\Big) & & H_{p+q}\Big(\bigcup\limits_{n,\sigma} \mathrm{B}\tilde{T}^\sigma_n(A);\mathbb{Q}\Big), \end{array}$$

where $V_q := H_q(V(A);\mathbb{Q}) = H_q(\tilde{V}(A);\mathbb{Q})$ (in particular, $V_0 = \mathbb{Q}$).

Our objective is to show that $E^2_{pq} \to \tilde{E}^2_{pq}$ are isomorphisms for $p = 0,\dots,n-1$. Denote by N the normal closure of $E(A^2)$ in $E(A)$ and consider the morphisms of the Hochschild–Serre spectral sequences

$$\begin{array}{ccc} {}'E^2_{st}(q) = H_s\big(E(A)/N;H_t(N;V_q)\big) & & {}'\tilde{E}^2_{st}(q) = H_s\big(\widetilde{E(A)}/\tilde{N};H_t(\tilde{N};V_q)\big) \\ \Downarrow & \longrightarrow & \Downarrow \\ H_{s+t}\big(E(A);V_q\big) & & H_{s+t}\big(\widetilde{E(A)};V_q\big), \end{array} \tag{54}$$

one for each $q \geq 0$. Corollary 2.9 shows that N and the affine group $\tilde{N} \subset \widetilde{E(A)}$ act trivially on V_q. This remark combined with Lemmas 6.8 and 6.9 implies that (54) reduces to the sequence of commutative diagrams

$$\begin{array}{ccc} H_0\big(E(A)/N;H_p(N;\mathbb{Q})\otimes V_q\big) & \longrightarrow & H_0\big(\widetilde{E(A)}/\tilde{N};H_p(\tilde{N};\mathbb{Q})\otimes V_q\big) \\ \big\downarrow\wr & & \big\downarrow\wr \\ H_p\big(E(A);V_q\big) & \longrightarrow & H_p\big(\widetilde{E(A)};V_q\big) \end{array} \tag{55}$$

$(p,q \geq 0)$. Note that $E(A)/N = \widetilde{E(A)}/\tilde{N}$ and, by the induction hypothesis, $H_p(N;\mathbb{Q}) = H_p(\tilde{N};\mathbb{Q})$ for $p = 0,\dots,n-1$. Using (55), we conclude that

$$E^2_{pq} = H_p\big(E(A);V_q\big) \to \tilde{E}^2_{pq} = H_p\big(\widetilde{E(A)};V_q\big)$$

is an isomorphism for $p = 0,\dots,n-1$. Lemma 6.11 and Proposition 6.1(b) then imply that

$$H_n\big(E(A);\mathbb{Q}\big) = E^2_{n0} \to \tilde{E}^2_{n0} = H_n\big(\widetilde{E(A)};\mathbb{Q}\big)$$

is also an isomorphism.

Case $H = E(A^i)$, $i \geq 1$. The previous case, which has been proved above, when applied to the ring A^i gives an isomorphism

$$H_n\big(E(A^i);\mathbb{Q}\big) \xrightarrow{\sim} H_n\big(E(A^i) \ltimes M_{\infty 1}(A^i);\mathbb{Q}\big). \tag{56}$$

The subgroup

$$E(A^i) \ltimes M_{\infty 1}(A^i) \subset \widetilde{E(A^i)} \equiv E(A^i) \ltimes M_{\infty 1}(A)$$

is normal in $\widetilde{E(A^i)}$, and the quotient is isomorphic to the torsion abelian group $M_{\infty 1}(A/A^i)$. The Hochschild–Serre spectral sequence gives, in view of Lemma 6.9, the isomorphism

$$H_0\big(M_{\infty 1}(A/A^i);\, H_n\big(E(A^i) \ltimes M_{\infty 1}(A^i);\mathbb{Q}\big)\big) \xrightarrow{\sim} H_n\big(\widetilde{E(A^i)};\mathbb{Q}\big);$$

when combined with (56) this shows that the map

$$H_n\big(E(A^i);\mathbb{Q}\big) \to H_n\big(\widetilde{E(A^i)};\mathbb{Q}\big) \tag{57}$$

is surjective and therefore an isomorphism ($E(A^i)$ is a retract of $\widetilde{E(A^i)}$).

The general case. H is now assumed to be any subgroup of $G = \mathrm{GL}(A)$ that contains some $E(A^i)$. Then $E(A^i)/G^{(2i+1)}$ becomes a subgroup of a nilpotent group, $H/G^{(2i+1)}$. In a nilpotent group every subgroup is subnormal (cf., e.g., [19], Thm. 16.2.2). Therefore there exists a finite tower of groups

$$H = H_0 \supset H_1 \supset \cdots \supset H_k = E(A^i)$$

such that H_j is normal in H_{j-1}, $1 \le j \le k$. The isomorphism (57) and Lemma 6.13 below show that the map (53) is an isomorphism. □

LEMMA 6.13. *Suppose that $H_1 \subset H \subset \mathrm{GL}(A)$ and that $H_1 \subset H$ is a normal subgroup of H. If $H_i(H_1;\mathbb{Q}) = H_i(\tilde{H}_1;\mathbb{Q})$ for $i = 0,\dots,n$, then*

$$H_i(H;\mathbb{Q}) = H_i(\tilde{H};\mathbb{Q})$$

for the same range of i's.

Proof. Consider the morphism of the Hochschild–Serre spectral sequences

$$\begin{array}{ccc} H_p\big(H/H_1; H_q(H_1;\mathbb{Q})\big) & & H_p\big(\tilde{H}/\tilde{H}_1; H_q(\tilde{H}_1;\mathbb{Q})\big) \\ \Downarrow & \longrightarrow & \Downarrow \\ H_{p+q}(H;\mathbb{Q}) & & H_{p+q}(\tilde{H};\mathbb{Q}) \end{array}$$

and apply to it Lemma 6.12. □

7. The tensor product of H-unital algebras

In this section k is an arbitrary commutative ring with unit and $\otimes$ denotes the tensor product over k. We begin by recalling some material from [40, §§ 2 and 9].

7.1. *Bar complexes.* Let A be a k-algebra. We denote by $(B_*(A), b')$ the chain complex

$$\begin{aligned} B_q(A) &= A^{\otimes q} \quad (q \geq 1), \\ b'(a_1 \otimes \cdots \otimes a_q) &= \sum_{i=1}^{q-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q. \end{aligned} \tag{58}$$

For an arbitrary k-module V, the homology of $B_*(A) \otimes V$ will be denoted by $\mathrm{H}B_*(A; V)$, or $\mathrm{H}B_*(A)$ if $V = k$. If $\mathrm{H}B_*(A; V) \equiv 0$ for all V, the algebra A is called H-unital.

Let M be a right A-module. We denote by $(B'_*(A; M), b')$ the chain complex

$$\begin{aligned} B'_q(A; M) &= M \otimes A^{\otimes q} \quad (q \geq 0), \\ b'(m \otimes a_1 \otimes \cdots \otimes a_q) &= m a_1 \otimes a_2 \otimes \cdots \otimes a_q \\ &\quad + \sum_{i=1}^{q-1} (-1)^i m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q. \end{aligned} \tag{59}$$

Its homology will be denoted by $\mathrm{H}B'_*(A; M)$. If $\mathrm{H}B'_*(A; V \otimes M) \equiv 0$ for all k-modules V, the A-module M is called *H-unitary* (in contrast to [40, §9], we do not assume here that A is H-unital). We define H-unitary left A-modules N using the right variant of (59):

$$\begin{aligned} B'_q(A; N) &= A^{\otimes q} \otimes N \quad (q \geq 0), \\ b'(a_1 \otimes \cdots \otimes a_q \otimes n) &= \sum_{i=1}^{q-1} (-1)^{i-1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_q \otimes n \\ &\quad + (-1)^{q-1} a_1 \otimes \cdots \otimes a_{q-1} \otimes a_q n. \end{aligned}$$

7.2. *Purity.* Recall that a short exact sequence of modules over a (not necessarily commutative) ring k is called *pure* (or *k-pure* to indicate the dependence on the ground ring k) if it stays exact after being tensored with an arbitrary k-module V. We say that a monomorphism $f : U_1 \rightarrowtail U_2$ and an epimorphism $g : W_1 \twoheadrightarrow W_2$ are *pure* if the corresponding short exact sequences

$$U_1 \overset{f}{\rightarrowtail} U_2 \twoheadrightarrow \operatorname{Coker} f$$

and

$$\operatorname{Ker} g \rightarrowtail W_1 \overset{g}{\twoheadrightarrow} W_2$$

are pure. These definitions are due to P.M. Cohn [7].

A chain complex of k-modules (C_*, ∂_*) is called *pure acyclic* (cf. [4], Exerc. §5.1b) if the canonical maps $C_{q+1} \to \operatorname{Ker} \partial_q$ are pure surjective for all q (we do not assume C_* to be bounded from below).

The definitions of subsection 7.1 can now be rephrased as follows:

(i) An algebra A is H-unital if the Bar complex (58) is k-pure acyclic (i.e., the underlying complex of k-modules is pure acyclic);

(ii) An A-module M is H-unitary if the Bar complex (59) is k-pure acyclic.

LEMMA 7.3. *In a k-pure exact sequence of right or left A-modules*

$$M_1 \rightarrowtail M_2 \twoheadrightarrow M_3,$$

the H-unitarity of any two modules implies the H-unitarity of the third one. □

LEMMA 7.4. *For an H-unitary right A-module M (respectively, left A-module N) the structural morphism $\mu_M : M \otimes A \to M$, $m \otimes a \mapsto ma$ (respectively, the morphism $\mu_N : A \otimes N \to N$, $a \otimes n \mapsto an$) is k-pure surjective.* □

7.5. Any algebra A defines the augmented semisimplicial[3] k-module

$$\mathbf{B}(A) = \left\{ A \overset{\partial_0}{\longleftarrow} A^{\otimes 2} \underset{\partial_1}{\overset{\partial_0}{\leftleftarrows}} A^{\otimes 3} \begin{smallmatrix}\leftarrow\\ \leftarrow\\ \leftarrow\end{smallmatrix} \dots \right\}$$

($\mathbf{B}(A)_n = A^{\otimes(n+2)}$, $n \geq -1$; $\partial_i(a_0 \otimes \cdots \otimes a_{n+1}) = a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}$, $0 \leq i \leq n$). We put

$$L_n(A) = \bigcap_{i=0}^{n} \operatorname{Ker}[\, \partial_i : \mathbf{B}(A)_n \to \mathbf{B}(A)_{n-1}\,] \quad (n \geq 0)$$

and $L_{-1}(A) = A$. If V is a k-module, we also define

$$L_n(A;V) = \bigcap_{i=0}^{n} \operatorname{Ker}[\, \partial_i \otimes \mathrm{id}_V : \mathbf{B}(A)_n \otimes V \to \mathbf{B}(A)_{n-1} \otimes V\,] \quad (n \geq 0)$$

and $L_{-1}(A;V) = A \otimes V$. Note that $\mathbf{B}(A)_n \otimes V$ is filtered by A-bi-submodules

$$L_{n-q}(A; A^{\otimes q} \otimes V) = \bigcap_{i=0}^{n-q} \operatorname{Ker}[\, \partial_i \otimes \mathrm{id}_V : \mathbf{B}(A)_n \otimes V \to \mathbf{B}(A)_{n-1} \otimes V\,]$$

$$(0 \leq q \leq n).$$

We will denote the right A-module structure maps $L_n(A;V) \otimes A \to L_n(A;V)$ by $\mu_{V,n}$, or by μ_n if $V = k$.

LEMMA 7.6. *If the structural morphisms $L_p(A) \otimes A \xrightarrow{\mu_p} L_p(A)$ are k-pure surjective for $p = -1, 0, \dots, n-1$, then*

[3] In the sense of S. Eilenberg and J.A. Zilber ([11], §1); i.e. $\mathbf{B}(A)$ is equipped with face maps, which satisfy standard identities, but it is not assumed to have degeneracy maps.

$(\alpha)_n$ *there exist functorial both left and right A-linear maps $j_p : L_{p+1}(A) \to L_p(A) \otimes A$ such that the sequences*

$$0 \leftarrow L_p(A) \xleftarrow{\mu_p} L_p(A) \otimes A \xleftarrow{j_p} L_{p+1}(A) \leftarrow 0$$

are k-pure exact for $p = -1, 0, \dots, n-1$;

$(\beta)_n$ *the chain complex*

$$0 \leftarrow A = L_{-1}(A) \xleftarrow{\mu_{-1}} L_{-1}(A) \otimes A \xleftarrow{j_{-1} \circ \mu_0} \dots \xleftarrow{j_{n-2} \circ \mu_{n-1}} L_{n-1}(A) \otimes A \xleftarrow{j_{n-1}} L_n(A) \leftarrow 0$$

is k-pure acyclic;

$(\gamma)_n$ *the canonical maps $i_{V,p} : L_p(A) \otimes V \to L_p(A; V)$ are isomorphisms for $p = -1, 0, \dots, n$.*

Proof. We will proceed by induction on n. There is nothing to prove if $n = -1$. Assume that Lemma 7.6 holds for $n - 1$. Consider the commutative diagram with exact rows

(60)
$$\begin{tikzcd}
0 & L_{n-1}(A;V) \arrow[l] \arrow[d, hook] & L_{n-1}(A;V) \otimes A \arrow[l, "\mu_{V,n-1}"'] \arrow[d, "\lambda"] & & \\
& A^{\otimes(n+1)} \otimes V & L_{n-1}(A; A \otimes V) \arrow[l, "\partial_n"'] & L_n(A;V) \arrow[l, "\nu"'] & 0, \arrow[l]
\end{tikzcd}$$

in which ν is the canonical injection and λ is another canonical map making the following triangle commutative:

(61)
$$\begin{tikzcd}
& L_{n-1}(A;V) \otimes A \arrow[dr, "\lambda"] & \\
L_{n-1}(A) \otimes (A \otimes V) \arrow[ur, "i_{V,n-1} \otimes 1_A"] \arrow[rr, "i_{A \otimes V, n-1}"'] & & L_{n-1}(A; A \otimes V).
\end{tikzcd}$$

Since the other two arrows in (61) are isomorphisms by $(\gamma)_{n-1}$, we infer that λ is an isomorphism. This allows us to define a canonical map $j_{V,n} := \lambda^{-1} \circ \nu$ from $L_n(A;V)$ to $L_{n-1}(A;V) \otimes A$ so that the sequence

(62) $$0 \leftarrow L_{n-1}(A;V) \xleftarrow{\mu_{V,n-1}} L_{n-1}(A;V) \otimes A \xleftarrow{j_{V,n}} L_n(A;V) \leftarrow 0$$

is k-pure exact. If $V = k$, we will write j_n instead of $j_{V,k}$. Then the exact sequence (62) can be inserted into the commutative diagram

(63)
$$\begin{tikzcd}
0 & L_{n-1}(A;V) \arrow[l] & L_{n-1}(A;V) \otimes A \arrow[l] & L_n(A;V) \arrow[l] & 0 \arrow[l] \\
0 & L_{n-2}(A) \otimes V \arrow[l] \arrow[u, "\wr", "i_{V,n-1}"'] & (L_{n-1}(A) \otimes A) \otimes V \arrow[l, "\mu_{n-1} \otimes 1_V"] \arrow[u, "\wr", "i_{V,n-1} \otimes 1_A"'] & L_n(A) \otimes V \arrow[l, "j_n \otimes 1_V"] \arrow[u, "i_{V,n}"'] & 0 \arrow[l]
\end{tikzcd}$$

whose bottom row is exact by the already proved part $(\alpha)_n$ combined with the assumption of pure surjectivity of $\mu_{n-1} : L_{n-1}(A) \otimes A \to L_{n-1}(A)$; two out of the three vertical arrows in (63) are isomorphisms by the inductive hypothesis $(\gamma)_{n-1}$. This implies that $i_{V,n}$ is an isomorphism. $\square$

7.7. *Pure (relatively) pseudofree resolutions.* We shall call a right A-module M *relatively pseudofree* if it is of the form $M = U \otimes A$ for some k-module U. Relatively pseudofree left A-modules are defined similarly. The functor $k-\mathrm{Mod} \rightsquigarrow \mathrm{Mod}-A$, $U \mapsto U \otimes A$ is not, in general, left adjoint to the forgetful functor $\mathrm{Mod}-A \rightsquigarrow k-\mathrm{Mod}$, hence the prefix "pseudo" in its name.

We shall say that a right A-module M admits a *pure relatively pseudofree* resolution if there exists a k-pure acyclic chain complex of right A-modules

$$0 \leftarrow M \leftarrow M_0 \leftarrow M_1 \leftarrow \dots$$

with M_q relatively pseudofree $(q \geq 0)$. For brevity we will call such resolutions "pure pseudofree," though that term should rather be reserved for the case when all M_q's are *pseudofree*, i.e., direct sums of copies of A. The latter case will not be discussed in this article.

THEOREM 7.8. *For an algebra A, the following conditions are equivalent:*

(a) *A is H-unital;*

(b) *all $L_n(A)$, $n \geq -1$, are H-unitary as right (respectively, left) A-modules;*

(c) *the structural maps $L_n(A) \otimes A \xrightarrow{\mu_n} L_n(A)$, $n \geq -1$, are k-pure surjective;*

(d) *the multiplication map $A \otimes A \xrightarrow{\mu} A$ is k-pure surjective and $L_0(A) \equiv \mathrm{Ker}(A \otimes A \xrightarrow{\mu_0} A)$ admits a pure pseudofree resolution as a right (respectively, left) A-module;*

(e) *the multiplication map $A \otimes A \xrightarrow{\mu} A$ is k-pure surjective and*

$$0 \leftarrow L_0(A) \xleftarrow{\mu_0} L_0(A) \otimes A \xleftarrow{j_0 \circ \mu_1} L_1(A) \otimes A \xleftarrow{j_1 \circ \mu_2} L_2(A) \otimes A \leftarrow \dots$$

is k-pure acyclic (and hence a pure pseudofree resolution of the right A-module $L_0(A)$).

Proof. The proof will be organized around the following diagram of implications:

$$\begin{tikzcd}[column sep=small]
 & (a) \arrow[dl, Rightarrow] & & (e) \arrow[dd, Rightarrow] \\
(b)+(c) \arrow[dr, Rightarrow] & & (c) \arrow[ul, Rightarrow] \arrow[ur, Rightarrow] & \\
 & (b) \arrow[ur, Rightarrow] & & (d) \arrow[ul, Rightarrow]
\end{tikzcd}$$

$(a) \Rightarrow (b) + (c)$. Consider these partial assertions:

$(b)_n$ the right A-modules $L_p(A)$ are H-unitary for $p = -1, \dots, n$;

$(c)_n$ the maps $L_p(A) \otimes A \xrightarrow{\mu_p} L_p(A)$ are k-pure surjective for $p = -1, \dots, n$.

We have the following chain of implications:

$$\{H-\text{unitality of }A\}\equiv(b)_{-1}\overset{7.4}{\Longrightarrow}(c)_{-1}\Rightarrow\dots$$
$$\Rightarrow(b)_n\overset{7.4}{\Longrightarrow}(c)_n\overset{7.6}{\Longrightarrow}(\alpha)_{n+1}\overset{7.3+(b)_n}{\Longrightarrow}(b)_{n+1}\Rightarrow\dots\ .$$

$(b)+(c)\Rightarrow(b)\Rightarrow(c)$. Obvious.

$(c)\Rightarrow(a)$. Let V be a k-module. We filter the Bar complex $B_*(A)\otimes V$ by

$$F_p\,[\,B_n(A)\otimes V\,]=\begin{cases}B_n(A)\otimes V, & p\geq 0,\\ L_{1-p}(A\,;A^{\otimes n+p-1}\otimes V), & 1-n\leq p\leq -1,\\ 0, & p\leq -n.\end{cases}\tag{64}$$

Filtration (64) defines a spectral sequence $E^k_{pq}:=E^k_{pq}(A\,;V)$ of the second quadrant, which by dimensional reasons is strongly convergent to $\mathrm{H}B_{p+q}(A;V)$. Its E^0-term can easily be determined by the use of the identifications provided by Lemma 7.6 (Parts $(\alpha)_n$ and $(\gamma)_n$) and the short exact sequences (62):

$$E^0_{pq}\simeq\begin{cases}B_{2p+q-2}(A;L_{-1-p}(A;V)), & (p\leq 0;q\geq 3-2p),\\ L_{-1-p}(A;V), & (p\leq 0;q=1-2p\text{ or }2-2p),\\ 0, & \text{otherwise.}\end{cases}$$

The differentials $d^0_{pq}:E^0_{pq}\to E^0_{p,q-1}$ are given by

$$d^0_{pq}\simeq\begin{cases}(-1)^{1-p}b', & (p\leq 0;q\geq 4-2p),\\ 1_{L_{-1-p}(A;V)}, & (p\leq 0;q=2-2p),\\ 0, & \text{otherwise}\end{cases}$$

(see the table on the next page).

Consequently the E^1-term is as follows:

$$E^1_{pq}=\begin{cases}\mathrm{H}B_{2p+q-2}(A;L_{-1-p}(A;V)), & (p\leq 0;q\geq 3-2p),\\ 0, & \text{otherwise;}\end{cases}\tag{65}$$

i.e., the diagonals $p+q=1$ or 2 contain no nonzero terms and the diagonals $p+q=n\geq 3$ contain the following $n-2$ terms

$$\mathrm{H}B_{n-2}(A;A\otimes V),\quad \mathrm{H}B_{n-3}(A;L_0(A;V)),\quad \dots,\quad \mathrm{H}B_1(A;L_{n-4}(A;V)),$$

which are mentioned in the decreasing order of $p=0,-1,-2,\dots$.

In particular, $\mathrm{H}B_1(A;V)=\mathrm{H}B_2(A;V)=0$ for *all* k-modules V. Plugging this information into (65) yields $\mathrm{H}B_3(A;V)=\mathrm{H}B_4(A;V)=0$ for *all* V, which plugged again into (65) yields the vanishing of the subsequent groups $\mathrm{H}B_5(A;V)$ and $\mathrm{H}B_6(A;V)$, and so on. This eventually shows that $E^1_{**}\equiv 0$ and that $\mathrm{H}B_*(A;V)=0$.

$$
\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
 & \downarrow 0 & \downarrow b' & \downarrow -b' & \vdots \\
\ddots & L_1(A;V) & B_2(A;L_0(A;V)) & B_4(A;A\otimes V) & 6 \\
 & \downarrow 1 & \downarrow b' & \downarrow -b' & \\
 & L_1(A;V) & B_1(A;L_0(A;V)) & B_3(A;A\otimes V) & 5 \\
 & & \downarrow 0 & \downarrow -b' & \\
 & & L_0(A;V) & B_2(A;A\otimes V) & 4 \\
 & & \downarrow 1 & \downarrow -b' & \\
 & & L_0(A;V) & B_1(A;A\otimes V) & 3 \\
 & & & \downarrow 0 & \\
 & & & A\otimes V & 2 \\
 & & & \downarrow 1 & \\
 & & & A\otimes V & 1 \\
 & & & & \\
\dots & -2 & -1 & 0 &
\end{array}
$$

TABLE. The Term $E^0_{**}(A;V)$

$(d) \Rightarrow (c)$. Strictly speaking we should distinguish between two cases: $(d)_{\text{right}}$ and $(d)_{\text{left}}$, depending on whether $L_0(A)$ is considered as a left or as a right A-module. In fact, we shall prove here the implication

$$(d)_{\text{left}} \Rightarrow (c). \tag{66}$$

The other implication $(d)_{\text{right}} \Rightarrow (c)$ will follow from:

(I) applying (66) and the already proved implication $(c) \Rightarrow (a)$ to the opposite ring A^{op}; then

(II) noting that the H-unitality of A^{op} is equivalent to the H-unitality of A; and finally

(III) applying the chain of implications $(a) \Rightarrow (b) + (c) \Rightarrow (c)$ to the ring A.

In order to prove (66) it suffices to demonstrate the implications

$$(d)_{\text{left}} + (c)_{n-1} \Rightarrow (c)_n$$

for all $n \geq 0$ (the implication $(d)_{\text{left}} \Rightarrow (c)_{-1}$ holds trivially). Let

$$0 \leftarrow L_0(A) \xleftarrow{d_0} A \otimes U_0 \xleftarrow{d_1} A \otimes U_1 \xleftarrow{d_2} \dots$$

be any pure pseudofree resolution of the left A-module $L_0(A)$. We augment it, using the fact that $\mu : A \otimes A \to A$ is k-pure surjective, to the k-pure acyclic complex

$$0 \leftarrow A \xleftarrow{\mu} A \otimes A \xleftarrow{d_0} A \otimes U_0 \xleftarrow{d_1} A \otimes U_1 \xleftarrow{d_2} \dots . \tag{67}$$

By Lemma 7.6$(\beta)_n$ we also have a k-pure acyclic chain complex

$$\begin{aligned} 0 \leftarrow A \xleftarrow{\mu} A \otimes A \xleftarrow{\bar{\mu}_0} L_0 \otimes A \xleftarrow{\bar{\mu}_1} L_1 \otimes A \xleftarrow{\bar{\mu}_2} \dots \\ \xleftarrow{\bar{\mu}_{n-2}} L_{n-2} \otimes A \xleftarrow{\bar{\mu}_{n-1}} L_{n-1} \otimes A \xleftarrow{j_n} L_n \leftarrow 0 \end{aligned} \tag{68}$$

$(L_p \equiv L_p(A),\ \bar{\mu}_p \equiv j_{p-1} \circ \mu_p)$. We use complexes (67) and (68) to assemble the following diagram with $n+2$ rows and infinitely many columns:
(69)

$$\begin{tikzcd}
L_{n-1} \otimes A \arrow[d, "\bar{\mu}_{n-1}"] & L_{n-1} \otimes A \otimes A \arrow[l, "1_{L_{n-1}} \otimes \mu"'] \arrow[d, "-\bar{\mu}_{n-1} \otimes 1_A"] & L_{n-1} \otimes A \otimes U_0 \arrow[l, "1_{L_{n-1}} \otimes d_0"'] \arrow[d, "\bar{\mu}_{n-1} \otimes 1_{U_0}"] & L_{n-1} \otimes A \otimes U \arrow[l, "1_{L_{n-1}} \otimes d_1"'] \arrow[d, "-\bar{\mu}_{n-1} \otimes 1_{U_1}"] & \dots \arrow[l] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
L_0 \otimes A \arrow[d, "\bar{\mu}_0"] & L_0 \otimes A \otimes A \arrow[l, "1_{L_0} \otimes \mu"'] \arrow[d, "-\bar{\mu}_0 \otimes 1_A"] & L_0 \otimes A \otimes U_0 \arrow[l, "1_{L_0} \otimes d_0"'] \arrow[d, "\bar{\mu}_0 \otimes 1_{U_0}"] & L_0 \otimes A \otimes U_1 \arrow[l, "1_{L_0} \otimes d_1"'] \arrow[d, "-\bar{\mu}_0 \otimes 1_{U_1}"] & \dots \arrow[l] \\
A \otimes A \arrow[d, "\mu"] & A \otimes A \otimes A \arrow[l, "1_A \otimes \mu"'] \arrow[d, "-\mu \otimes 1_A"] & A \otimes A \otimes U_0 \arrow[l, "1_A \otimes d_0"'] \arrow[d, "\mu \otimes 1_{U_0}"] & A \otimes A \otimes U_1 \arrow[l, "1_A \otimes d_1"'] \arrow[d, "-\mu \otimes 1_{U_1}"] & \dots \arrow[l] \\
A & A \otimes A \arrow[l, "\mu"'] & A \otimes U_0 \arrow[l, "d_0"'] & A \otimes U_1 \arrow[l, "d_1"'] & \dots \arrow[l]
\end{tikzcd}$$

Its anticommutativity follows from Lemma 7.9 below.

LEMMA 7.9. *Let A be a k-algebra and let V_0, V_1, W_0, W_1 be k-modules. Suppose there are given a right A-module map $f : V_1 \otimes A \to V_0 \otimes A$ and a left A-module map $g : A \otimes W_1 \to A \otimes W_0$. If $A = A^2$, then the square*

$$\begin{tikzcd}
V_1 \otimes A \otimes W_0 \arrow[d, "f \otimes 1_{W_0}"'] & V_1 \otimes A \otimes W_1 \arrow[l, "1_{V_1} \otimes g"'] \arrow[d, "f \otimes 1_{W_1}"'] \\
V_0 \otimes A \otimes W_0 & V_0 \otimes A \otimes W_1 \arrow[l, "1_{V_0} \otimes g"]
\end{tikzcd} \tag{70}$$

commutes.

Proof. Since $A \otimes A \to A$ is surjective, (70) is an image of the evidently commutative diagram $f \boxtimes g$:

$$\begin{array}{ccc}
(V_1 \otimes A) \otimes (A \otimes W_0) & \xleftarrow{1_{V_1 \otimes A} \otimes g} & (V_1 \otimes A) \otimes (A \otimes W_1) \\
{\scriptstyle f \otimes 1_{A \otimes W_0}} \downarrow & & \downarrow {\scriptstyle f \otimes 1_{A \otimes W_1}} \\
(V_0 \otimes A) \otimes (A \otimes W_0) & \xleftarrow[1_{V_0 \otimes A} \otimes g]{} & (V_0 \otimes A) \otimes (A \otimes W_1);
\end{array}$$

thus (70) commutes itself. □

Let us tensor diagram (69) with an arbitrary k-module V. We get a double complex $C_{**} \otimes V$ whose rows are exact. On the other hand, the E^1-term of the first spectral sequence of $C_{**} \otimes V$ consists of the single row

$$E^1_{*,n+1} = \left\{ 0 \leftarrow L_n \otimes V \xleftarrow{d^1} L_n \otimes A \otimes V \xleftarrow{d^1} L_n \otimes U_0 \otimes V \xleftarrow{d^1} \dots \right\}. \tag{71}$$

The differential $d^1_{1,n+1} : L_n \otimes A \otimes V \to L_n \otimes V$ identifies with the map $\mu_n \otimes 1_V$. This follows if one notes that all of the arrows in diagram (60) are right A-linear. The acyclicity of (71) thus implies that $\mu_n : L_n \otimes A \to L_n$ is k-pure surjective.

The implication $(e) \Rightarrow (d)$ is obvious, and $(c) \Rightarrow (e)$ follows from Lemma $7.6(\beta)_n$. This completes the proof of Theorem 7.8. □

We obtain the following important theorem as a corollary:

THEOREM 7.10. *The tensor product $A \otimes B$ of any two H-unital algebras A and B is again H-unital.*

Proof. Take the truncated Bar complexes $B_*(A)_{\geq 2}$ and $B_*(B)_{\geq 2}$. The total complex of their tensor product gives rise to the augmented k-pure acyclic chain complex

$$0 \leftarrow A \otimes B \xleftarrow{\mu_A \otimes \mu_B} M_*,$$

where $M_q := (A \otimes B) \otimes W_q \otimes (A \otimes B)$ and $W_q := \bigoplus_{i=0}^q A^{\otimes i} \otimes B^{\otimes (q-i)}$, $q \geq 0$, and whose boundary maps are both left and right $A \otimes B$-linear. In particular, $\mu_A \otimes \mu_B : (A \otimes B)^{\otimes 2} \to A \otimes B$ is pure surjective, and

$$0 \leftarrow L_0(A \otimes B) \leftarrow M_1 \leftarrow M_2 \leftarrow M_2 \leftarrow \dots$$

is a pure pseudofree resolution of equally the left and the right $A \otimes B$-module structures on $L_0(A \otimes B)$. It remains to apply Theorem 7.8. □

The special case of Theorem 7.10, when one of the two algebras, say, B, has a unit and, in addition, the map $k \to B$, $1_k \mapsto 1_B$, is pure injective, has been proved in [40], Corollary 9.6.

8. The matrix ring of a small category

8.1. *"Nonunital" categories.* Studying triangular matrix groups and Lie algebras has brought us to the generalization of the notion of a category which discards the axiom requiring the existence of identity morphisms while retaining all the other axioms. The term "category" will always be used below in this more general sense.

Now "a semigroup" and "a category with one object" become synonymous. This more general terminology allows also for the existence of objects x with $\mathrm{Hom}(x,x)=\emptyset$ ("objects with no endomorphisms"). The definition of a preadditive category (i.e., a category $\mathcal{C}$, in which all Hom sets are abelian groups and the composition of morphisms is bilinear) extends to this more general context without modifications. A preadditive category with one object is exactly the same as a (nonunital) ring. Another example of a category in our sense appears in the recent work of G. Segal [31].

Notation. If $a\in\mathrm{Hom}_{\mathcal{C}}(x,y)$, we denote x by $s(a)$ (the source of a) and y by $t(a)$ (the target of a).

8.2. *The matrix ring of a small preadditive category.* Let $\mathcal{A}$ be a small preadditive category. We form a ring $M(\mathcal{A})$ consisting of matrices (a_{yx}) labelled by pairs of objects (y,x) of $\mathcal{A}$ and such that:

(I) $a_{yx}\in\mathrm{Hom}_{\mathcal{A}}(x,y)$,

(II) all but finitely many a_{yx}'s vanish.

The matrices are multiplied as usual with $a_{zy}b_{yx}$ being the composition of morphisms. If $a\in\mathrm{Mor}\,\mathcal{A}$, we will denote by a also the corresponding "elementary" matrix (a_{yx}):

$$a_{yx}=\begin{cases} a, & x=s(a),\ y=t(a),\\ 0, & \text{otherwise.}\end{cases}$$

If $\mathcal{C}$ is an arbitrary small category and R is a ring, we form a preadditive category $R\mathcal{C}$, which has the same objects as $\mathcal{C}$ and

$$\mathrm{Hom}_{R\mathcal{C}}(x,y)=\bigoplus_{\mathrm{Hom}_{\mathcal{A}}(x,y)} R.$$

The matrix ring $M(R\mathcal{C})$ will be also denoted $M_{\mathcal{C}}(R)$. This is a generalization of the definition of the ring $M_X(R)$ from Section 3. Until the end of this section all categories are supposed to be small.

8.3. *Categories with triple factorization.* We will consider the following category-theoretic variant of the Triple Factorization Property formulated in

Section 3 for rings:

> For any finite collection of morphisms $a_1, \ldots, a_m \in \operatorname{Mor}\mathcal{C}$ with the common source $x \in \operatorname{Ob}\mathcal{C}$ there exist:
>
> (I) objects $x', x'' \in \operatorname{Ob}\mathcal{C}$,
>
> (II) morphisms $c \in \operatorname{Hom}_{\mathcal{C}}(x', x'')$ and $d \in \operatorname{Hom}_{\mathcal{C}}(x, x')$,
>
> $(\mathcal{TF})_{\text{right}}$ (III) morphisms $b_i \in \operatorname{Hom}_{\mathcal{C}}(x'', t(a_i))$,
>
> such that
>
> $$a_i = b_i \circ c \circ d \qquad (1 \le i \le m)$$
>
> and any pair of parallel arrows equalized by $c \circ d$ is equalized by c alone.

Recall that $\overset{z}{\cdot} \overset{f}{\underset{g}{\leftleftarrows}} \overset{y}{\cdot}$ is equalized by $\overset{y}{\cdot} \xleftarrow{h} \overset{w}{\cdot}$ if $f \circ h = g \circ h$. One can also define the dual property of left factorization $(\mathcal{TF})_{\text{left}}$. A category $\mathcal{C}$ possesses $(\mathcal{TF})_{\text{left}}$ if and only if the opposite category $\mathcal{C}^{\text{op}}$ possesses $(\mathcal{TF})_{\text{right}}$. For trivial reasons, any category with identity morphisms has both of these properties.

LEMMA 8.4. *Assume that $\mathcal{C}$ is a category with Property $(\mathcal{TF})_{\text{right}}$ and that R is a ring with Property* $(\text{TF})_{\text{right}}$ (*see Section* 3). *Then the preadditive category $R\mathcal{C}$ has Property $(\mathcal{TF})_{\text{right}}$.*

Proof. Consider an arbitrary finite pencil

$$\begin{tikzcd}
y_1 \cdot & & \\
y_2 \cdot & & \cdot x \arrow[ull, "a_1"'] \arrow[ll, "a_2"'] \arrow[ddll, "a_m"] \\
\vdots & & \\
y_m \cdot & &
\end{tikzcd}$$

of morphisms in $R\mathcal{C}$, where each a_i decomposes as follows:

$$a_i = \sum_{j=1}^{n(i)} t_{ij}\alpha_{ij}$$

($t_{ij} \in R$, $\alpha_{ij} \in \operatorname{Hom}_{\mathcal{C}}(x, y_i)$ and, for a given $i = 1, \ldots, m$, all α_{ij}'s are different; y_i's are not necessarily different themselves). Properties $(\mathcal{TF})_{\text{right}}$ and $(\text{TF})_{\text{right}}$ provide respective factorizations: $\alpha_{ij} = \beta_{ij}\gamma\delta$ in $\mathcal{C}$, and $t_{ij} = u_{ij}vw$ in R. Set

$$b_i = \sum_{j=1}^{n(i)} u_{ij}\beta_{ij}, \quad c = v\gamma \quad \text{and} \quad d = w\delta.$$

It remains to show that if the composite

$$\underset{z}{\cdot} \xleftarrow{f} \cdot \xleftarrow{c} \cdot \xleftarrow{d} \underset{x}{\cdot}$$

is zero for some $z \in \operatorname{Ob} R\mathcal{C}$ and $f \in \operatorname{Hom}_{R\mathcal{C}}(t(c), z)$, then $f \circ c = 0$. Let us write $f = \sum_{j=1}^{l} r_j\varphi_j$ for some $r_j \in R$ and $\varphi_j \in \operatorname{Hom}_{\mathcal{C}}(t(c), z)$. Then

$$0 = \sum_{j=1}^{l}(r_jvw)(\varphi_j\gamma\delta) = \sum_{\psi \in \operatorname{Hom}_{\mathcal{C}}(t(c),z)} \left(\sum_{j\in I_\psi} r_jvw\right)\psi, \tag{72}$$

where $I_\psi := \{j \mid \varphi_j\gamma\delta = \psi;\ 1 \le j \le l\}$ and the summation in (72) extends over all ψ such that $I_\psi \neq \emptyset$. Since the ψ's are part of a "basis" of the R-module $\operatorname{Hom}_{R\mathcal{C}}(t(c), z)$, we get

$$\sum_{j\in\psi} r_jvw = 0 \tag{73}$$

for all ψ such that $I_\psi \neq \emptyset$. The assumption on R (see $(\mathrm{TF})_{\mathrm{right}}$) and equality (73) imply that

$$\sum_{j\in I_\psi} r_jv = 0. \tag{74}$$

The assumption on $\mathcal{C}$ (see $(\mathcal{TF})_{\mathrm{right}}$) yields

$$\varphi_{j_1}\gamma = \varphi_{j_2}\gamma \tag{75}$$

for all $j_1, j_2 \in I_\psi$. By combining (74) and (75), we finally get

$$0 = \sum_{\psi}\left(\sum_{j\in I_\psi} r_jv\varphi_j\gamma\right) = f \circ c,$$

as desired. □

PROPOSITION 8.5. *Let $\mathcal{A}$ be a preadditive category possessing Property $(\mathcal{TF})_{\mathrm{right}}$. Then the matrix ring $M(\mathcal{A})$ is left universally flat.*

Recall that a ring R is left universally flat (see [41]) if for every unital ring S containing R as a left ideal, R is a flat S-module.

The ring $M(\mathcal{A})$ is a direct sum of its left ideals

$$M(\mathcal{A}) = \bigoplus_{x\in\operatorname{Ob}\mathcal{A}} \operatorname{Col}_x(\mathcal{A}), \tag{76}$$

where $\operatorname{Col}_x(\mathcal{A}) := \{\alpha \in M(\mathcal{A}) \mid \alpha_{vu} = 0 \text{ if } u \neq x\}$. For $\alpha \in M(\mathcal{A})$ we will write $\alpha \equiv \sum_x C_x(\alpha)$, where $C_x(\alpha) \in \operatorname{Col}_x(\mathcal{A})$.

LEMMA 8.6. *Let $\mathcal{A}$ be a preadditive category in which every morphism $a \in \operatorname{Hom}_{\mathcal{A}}(x,y)$ (x, y arbitrary) admits a factorization*

$$\underset{y}{\cdot} \xleftarrow{a'} \underset{z}{\cdot} \xleftarrow{a''} \underset{x}{\cdot}$$

for some $z \in \operatorname{Ob}\mathcal{A}$, $a' \in \operatorname{Hom}_{\mathcal{A}}(z,y)$ and $a'' \in \operatorname{Hom}_{\mathcal{A}}(x,z)$.

Then for every ring S containing $M(\mathcal{A})$ as a left ideal, each $\operatorname{Col}_x(\mathcal{A})$ *is itself a left ideal in S.*

Proof. Since

$$\operatorname{Col}_x(\mathcal{A}) = \bigoplus_{y \in \operatorname{Ob}\mathcal{A}} \operatorname{Hom}_{\mathcal{A}}(x,y),$$

it suffices to show that $S \cdot \operatorname{Hom}_{\mathcal{A}}(x,y) \subset \operatorname{Col}_x(\mathcal{A})$ for all y. Let $s \in S$ and $a = a'a''$, where $a' \in \operatorname{Hom}_{\mathcal{A}}(z,y)$ and $a'' \in \operatorname{Hom}_{\mathcal{A}}(x,z)$. The matrix elements of sa must satisfy

$$(sa)_{vu} = \left((sa')a''\right)_{vu} = \sum_{w \in \operatorname{Ob}\mathcal{A}} (sa')_{vw}(a'')_{wu} = 0$$

if $u \neq x$. □

Proof of Proposition 8.5. Let

$$\sum_{i \in I} s_i \alpha_i = 0 \qquad (s_i \in S;\ \alpha_i \in M(\mathcal{A});\ I \text{ a finite set}) \tag{77}$$

be an arbitrary linear relation in $M(\mathcal{A})$. To prove the flatness of $M(\mathcal{A})$ over S one has to show that (77) is a consequence of linear relations in S; i.e., that there exist a finite set J and elements $\sigma_{ij} \in S$ and $\beta_j \in M(\mathcal{A})$, $j \in J$, such that

$$\sum_{i \in I} s_i \sigma_{ij} = 0 \qquad (\text{for all } j \in J) \tag{78}$$

and

$$\alpha_i = \sum_{j \in J} \sigma_{ij}\beta_j \quad (\text{for all } i \in I). \tag{79}$$

(See, e.g., [6], Exerc. VI.6, or [5], Cor. I.2.11.1.)

The direct sum decomposition (76) combined with Lemma 8.6 gives

$$\sum_{i \in I} s_i C_x(\alpha_i) = 0 \qquad (\text{for all } x \in \operatorname{Ob}\mathcal{A}). \tag{80}$$

Denote by J the set of those x for which at least one $C_x(\alpha_i)$ is nonzero. Property $(\mathcal{TF})_{\text{right}}$ provides, for each $x \in J$, factorizations

$$(\alpha_i)_{yx} = b_{i;yx} \circ c_x \circ d_x, \tag{81}$$

where

(I) $y \in Y_x := \{y \in \operatorname{Ob}\mathcal{A} \mid (\alpha_i)_{yx} \neq 0 \text{ for some } \alpha_i,\ i \in I\}$,

(II) $b_{i;yx} \in \operatorname{Hom}_{\mathcal{A}}(x'', x)$ for some $x'' \in \operatorname{Ob}\mathcal{A}$ (x' and x'' depend on x, in general),

(III) the morphisms $c_x \in \operatorname{Hom}_{\mathcal{A}}(x', x'')$ and $d_x \in \operatorname{Hom}_{\mathcal{A}}(x, x')$ are chosen so that any parallel pair of morphisms equalized by $c_x \circ d_x$ is equalized by c_x alone.

Now we set σ_{ix} to be an element of $M(\mathcal{A}) \subset S$ whose matrix elements are given by

$$(\sigma_{ix})_{vu} = \begin{cases} b_{i;yx} \circ c_x, & (u = x',\ v = y), \\ 0, & \text{otherwise.} \end{cases}$$

In particular, σ_{ix} has only one nonzero column:

$$C_u(\sigma_{ix}) = 0 \text{ if } u \neq x'$$

(notice that the choice of factorization (81) makes x' a function of x). Finally we set β_x to be the elementary matrix defined by d_x.

Let us verify identities (78) and (79). From equality (80) we get

$$\left(\sum_{i\in I} s_i\sigma_{ix}\right) \cdot \beta_x = 0 \qquad (x \in J). \tag{82}$$

If we put

$$\tau_x = \sum_{i\in I,\ y\in Y_x} s_i b_{i;yx} \in M(\mathcal{A}),$$

then (82) becomes

$$\tau_x c_x d_x = 0 \qquad (x \in J),$$

and condition (III) above implies that

$$0 = \tau_x c_x \equiv \sum_{i\in I} s_i\sigma_{ix} \qquad (x \in J).$$

The second set of identities (79) is equivalent to the factorizations (81). □

8.7. *Preadditive categories with no cycles.* Assume that we are given an arbitrary finite collection $(x_1, \dots, x_n)$ of objects of a preadditive category $\mathcal{A}$. We will say that $\mathcal{A}$ *has no cycles* if $\operatorname{Hom}_{\mathcal{A}}(x_i, x_{i+1}) = 0$ for some $i = 1, \dots, n$ ($x_{i+1} \equiv x_1$ if $i = n$).

THEOREM 8.8. *Assume that there are given:*

(I) *a preadditive category $\mathcal{A}$, which possesses the right factorization property $(\mathcal{TF})_{\text{right}}$ or its left analogue $(\mathcal{TF})_{\text{left}}$,*

(II) *a commutative ring with unit k and a flat k-algebra structure on $M(\mathcal{A})$,*

(III) *an H-unital k-algebra B.*

Then

(a) *the k-algebra $M(\mathcal{A}) \otimes B$ is H-unital,*

(b) *if, in addition, $\mathcal{A}$ has no cycles, the Hochschild and cyclic homology groups* $\mathrm{HH}_*(M(\mathcal{A}) \otimes B)$ *and, respectively,* $\mathrm{H}C_*(M(\mathcal{A}) \otimes B)$ *vanish in all dimensions.*

Proof. (a) In view of Theorem 7.10 it suffices to prove that the k-algebra $M(\mathcal{A})$ is H-unital. Assume that $\mathcal{A}$ possesses property $(\mathcal{TF})_{\text{right}}$. By Proposition 8.5, $M(\mathcal{A})$ is a flat left ideal in the ring $k \ltimes M(\mathcal{A})$. Since $M(\mathcal{A})$ is also flat as a k-module, we have

$$\mathrm{H}B_q(M(\mathcal{A})) \simeq \operatorname{Tor}_{q-1}^{k\ltimes M(\mathcal{A})}(k, M(\mathcal{A})) = 0 \qquad (q > 1).$$

In addition, the factorization property $(\mathcal{TF})_{\text{right}}$ shows that $M(\mathcal{A}) = M(\mathcal{A})^2$. Hence $\mathrm{H}B_1(M(\mathcal{A})) \equiv M(\mathcal{A})/M(\mathcal{A})^2 = 0$ and the Bar complex (58) of $M(\mathcal{A})$ is acyclic. It is then automatically k-pure acyclic, since $M(\mathcal{A})$ is flat over k.

Passing to the opposite category $\mathcal{A}^{\text{op}}$ reduces the case of the left factorization property $(\mathcal{TF})_{\text{left}}$ to the one considered above.

(b) Let us represent elements of $A := M(\mathcal{A}) \otimes_k B$ as linear combinations of tensors $a = \alpha b \equiv \alpha \otimes b$, where $\alpha \in \operatorname{Hom}_{\mathcal{A}}(x, y)$ for some $x, y \in \operatorname{Ob} \mathcal{A}$, and $b \in B$. We extend the definition of the source and the target of a morphism to such tensors by setting

$$s(a) := s(\alpha) \;\text{ and }\; t(a) := t(\alpha).$$

For any pair $x, y \in \operatorname{Ob} \mathcal{A}$ let $B_*(y, x)$ denote the subcomplex of the Bar complex $B_*(A)$ spanned by tensors $a_1 \otimes \cdots \otimes a_q$ satisfying

$$s(a_i) = t(a_{i+1}) \qquad (1 \le i \le q-1)$$

and $t(a_1) = y$ and $s(a_q) = x$. The Bar complex $B_*(A)$ decomposes in terms of $B_*(y, x)$'s into the following direct sum of chain complexes:

$$B_*(A) = \bigoplus_{l=1}^{\infty} \bigoplus_{(y_1,x_1;\ldots;y_l,x_l)} \operatorname{Tot}\bigl(B_*(y_1, x_1) \otimes \cdots \otimes B_*(y_l, x_l)\bigr), \tag{83}$$

where the second direct sum in (83) extends over all l-tuples of pairs (y_j, x_j) satisfying condition

$$x_j \ne y_{j+1} \qquad (1 \le j \le l-1).$$

The decomposition (83) can be rewritten in the form

$$B_*(A) = B_*(A)_{\text{eq}} \oplus B_*(A)_{\text{dif}},$$

where $B_*(A)_{\text{eq}}$ is the direct sum of terms in (83) with $x_l = y_1$ and $B_*(A)_{\text{dif}}$ is the direct sum of terms with $x_l \ne y_1$.

The pure acyclicity of $B_*(A)$ proven in part (a) tells us that all

$$B_*(y,x) \qquad (x,y \in \operatorname{Ob}\mathcal{A})$$

as well as $B_*(A)_{\mathrm{eq}}$ and $B_*(A)_{\mathrm{dif}}$ are pure acyclic. Using this information, we shall prove that $\mathrm{HH}_*(A)=0$.

The Hochschild homology of an H-unital algebra is calculated by the standard Hochschild complex $(C_*(A,A),b)$:

$$C_q(A) = A^{\otimes(q+1)}$$

and

$$\begin{aligned} b(a_0\otimes\cdots\otimes a_q) = &\sum_{i=0}^{q-1}(-1)^j a_0\otimes\cdots\otimes a_ia_{i+1}\otimes\cdots\otimes a_q \\ &+(-1)^q a_qa_0\otimes a_1\otimes\cdots\otimes a_{q-1}. \end{aligned}$$

We filter $C_*(A,A)$ by

$$\begin{aligned} &F_pC_{p+q}(A,A) \\ &\quad = \operatorname{lin\,span}\{a_0\otimes\cdots\otimes a_{p+q} \mid \text{ there exist } q\le i\le p+q : s(a_i)\ne t(a_{i+1})\}, \end{aligned}$$

where we assume that $a_{p+q+1}\equiv a_0$. The assumption that $\mathcal{A}$ has no cycles implies that $C_*(A,A)=\bigcup_p F_pC_*(A,A)$. More precisely one has

$$0=F_{-1}C_n\subset F_0C_n\subset\cdots\subset F_nC_n=C_n \qquad (n\ge 0).$$

In particular, the associated spectral sequence $E^k_{pq}\Rightarrow \mathrm{HH}_{p+q}(A)$ is a spectral sequence of the first quadrant. Its E^0-term is described by the next lemma.

LEMMA 8.9. *For every $p\ge 0$ there is a canonical isomorphism*

$$(E^0_{p*},d^0_{p*}) \simeq \begin{cases} (B_*(A)_{\mathrm{dif}})[-1], & p=0, \\ \bigoplus_{\substack{u,v,w\in\operatorname{Ob}\mathcal{A}\\ v\ne w}} {}'B_*(u,w)\otimes B_*(v,u)[-1], & p\ge 1, \end{cases}$$

where the chain complex ${}'B_*(u,w)$ is the direct sum of terms in (83) *with $y_1=u$ and $x_l=w$ and where the k-module $B_p(v,u)$ is treated as a trivial complex concentrated in degree zero.* □

The pure acyclicity of $B_*(A)$, and hence of the complexes $B_*(A)_{\mathrm{dif}}$ and ${}'B_*(u,w)(u,w\in\operatorname{Ob}\mathcal{A})$, implies then that $E^1_{**}\equiv 0$ and, in conclusion, that $\mathrm{HH}_*(A)=0$.

The Connes long exact sequence, which relates the cyclic homology to the Hochschild homology, implies that $\mathrm{H}C_*(A)=0$. □

8.10. Let $\mathcal{C}$ be a category satisfying:
(a) $\operatorname{Hom}_{\mathcal{C}}(x,y)$ has at most one element for all $x,y\in\operatorname{Ob}\mathcal{C}$,
(b) $\operatorname{Hom}_{\mathcal{C}}(x,x)=\emptyset$ for all $x\in\operatorname{Ob}\mathcal{C}$; i.e., $\mathcal{C}$ has no endomorphisms.

Every such category is equivalent to partially ordering the set of its objects. Given a partially ordered set P, we define the category $\mathcal{C}_P$:

$$\operatorname{Ob}\mathcal{C}_P = P, \quad \operatorname{Hom}_{\mathcal{C}_P}(x,y) = \begin{cases} \{x<y\}, & \text{if } x<y, \\ \emptyset, & \text{otherwise.} \end{cases}$$

If R is a ring, we will denote the matrix ring $M_{\mathcal{C}_P}(R) \equiv M(R\mathcal{C}_P)$ by $M_P(R)$ and call it the ring of *P-triangular matrices* with coefficients in R.

The following is an analogue of the Triple Factorization Property $(\mathcal{TF})_{\text{right}}$ for partially ordered sets P:

(D)$_{\text{below}}$ For every finite collection $x, y_1, \dots, y_m \in P$ satisfying inequalities

$$x < y_i \qquad (1 \le i \le m)$$

there exists such $x' \in P$ that

$$x < x' < y_i \qquad (1 \le i \le m).$$

Partially ordered sets satisfying condition (D)$_{\text{below}}$ will be called *dense from below.* One obtains the property of *density from above* (D)$_{\text{above}}$ by reversing all inequalities in (D)$_{\text{below}}$.

COROLLARY 8.11. *Let P be a partially ordered set, which is dense from below or from above, and let A be an H-unital k-algebra. Then the k-algebra of P-triangular matrices $M_P(A)$ is H-unital, and*

$$\mathrm{HH}_*(M_P(A)) = \mathrm{H}C_*(M_P(A)) = 0.$$

Proof. The category $\mathcal{C}_P$ associated with a dense-from-below P possesses the right Triple Factorization Property $(\mathcal{TF})_{\text{right}}$. Lemma 8.4 implies that the preadditive category $k\mathcal{C}_P$ has property $(\mathcal{TF})_{\text{right}}$ too. For any commutative ring with unit k, the algebra $M_P(k) \equiv M(k\mathcal{C}_P)$ is k-free. It remains to apply Theorem 8.8 to $\mathcal{A} = k\mathcal{C}_P$ and $B = A$. The case of P dense from above follows if one notes that

$$M_P(A)^{\text{op}} \simeq M_{P^{\text{op}}}(A^{\text{op}}),$$

where P^{op} denotes P with the reverse partial ordering. ☐

9. The acyclicity of triangular complexes

In Section 4 we introduced the "triangular" chain complexes $\sum_{n,\sigma} C_*(\mathfrak{t}_n^\sigma(A))$. Here we prove the following important theorem:

THEOREM 9.1. *For every H-unital algebra A over a field of characteristic zero, the chain complex $\sum_{n,\sigma} C_*(\mathfrak{t}_n^\sigma(A))$ is acyclic in positive degrees.*

We precede the proof by some notational conventions.

Throughout this section $\mathfrak{t}_X^\sigma(A)$ denotes the Lie algebra of generalized triangular matrices associated with a finite partially ordered set (X,σ):

$$\mathfrak{t}_X^\sigma(A) := \left\{(a_{xy})_{x,y\in X} \mid a_{xy}\in A \text{ and } a_{xy}=0 \text{ if } x \overset{\sigma}{\not<} y\right\}.$$

An order-preserving injective map of partially ordered sets $(X,\sigma)\to(Y,\tau)$ induces a natural homomorphism of Lie algebras $\mathfrak{t}_X^\sigma(A)\to\mathfrak{t}_X^\tau(A)$. As before, $\mathbf{m}$ will denote the set $\{1,\dots,m\}$ and $\sigma\times m\times n^\delta$ will denote the lexicographical partial ordering of $X\times\mathbf{m}\times\mathbf{n}$, in which $\mathbf{m}$ is equipped with the ordering

$$1>2>\cdots>m$$

and $\mathbf{n}$ is equipped with the discrete partial ordering (i.e., $i<j$ for no $i,j\in\mathbf{n}$).

We shall make $\mathbb{Z}_+\times\mathbb{Z}_+$ into a directed set by means of the partial ordering

$$(m,n)\preccurlyeq(m',n') \Leftrightarrow m \text{ divides } m' \text{ and } n\le n'.$$

For every $(m,n)\preccurlyeq(m',n')$ we define an order-preserving embedding

$$\begin{aligned}\varphi_{(m',n';m,n)}: (X\times\mathbf{m}\times\mathbf{n},\,\sigma\times m\times n^\delta) &\hookrightarrow (X\times\mathbf{m}'\times\mathbf{n}',\,\sigma\times m'\times(n')^\delta),\\ (x,i,j)&\mapsto\left(x,1+\tfrac{m'}{m}(i-1),j\right).\end{aligned}\tag{84}$$

Clearly (84) defines an inductive system of partially ordered sets, which is indexed by $(\mathbb{Z}_+\times\mathbb{Z}_+,\preccurlyeq)$. If we set $n=1$, we obtain a subsystem whose limit will be denoted by

$$P(X,\sigma)=\varinjlim_m(X\times\mathbf{m},\sigma\times m).$$

The functoriality of $\varphi_{(m',n';m,n)}$ with respect to (X,σ) implies that, for any collection of partial orderings $\sigma_1,\dots,\sigma_k$ of the set X, one has canonical morphisms of chain complexes

$$\sum_{l=1}^k C_*\left(\mathfrak{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma_l\times m\times n^\delta}(A)\right)\to\sum_{l=1}^k C_*\left(\mathfrak{t}_{X\times\mathbf{m}'\times\mathbf{n}'}^{\sigma_l\times m'\times(n')^\delta}(A)\right)\tag{85}$$

whenever $(m',n')\succcurlyeq(m,n)$. The morphisms (85) form an inductive system of chain complexes.

LEMMA 9.2. *Let A be an H-unital algebra over a field of characteristic zero, $q,m,n\in\mathbb{Z}_+$ and*

$$\gamma\in H_q\left(\mathfrak{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma\times m\times n^\delta}(A)\right)$$

be a homology class. Then there exists $(m',n')\succcurlyeq(m,n)$ such that the map induced by $\varphi_{(m',n';m,n)}$

$$H_q\left(\mathfrak{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma\times m\times n}(A)\right)\to H_q\left(\mathfrak{t}_{X\times\mathbf{m}'\times\mathbf{n}'}^{\sigma\times m'\times(n')^\delta}(A)\right)$$

annihilates γ.

Proof. The partially ordered set $P(X,\sigma)$ is isomorphic to the set $X \times (\mathbb{Q} \cap (0,1])$ partially ordered by the relation

$$(x,s) < (y,t) \Leftrightarrow x \overset{\sigma}{<} y \quad \text{or} \quad x = y \quad \text{and} \quad s < t.$$

The canonical inclusion $X \times \mathbf{m} \hookrightarrow P(X,\sigma)$ corresponds to the embedding

$$X \times \mathbf{m} \hookrightarrow X \times (\mathbb{Q} \cap (0,1]), \quad (x,i) \mapsto \left(x, \frac{m-i+1}{m}\right).$$

In particular, the opposite partially ordered set $\Pi = P(X,\sigma)^{\mathrm{op}}$ is always dense from below (cf. subsection 8.10), and Corollary 8.11 implies that the algebra $M_\Pi(A)$ is H-unital and its cyclic homology vanishes. In combination with the results of Hanlon [14], already quoted in Section 6, this gives

$$H_q\big(\mathfrak{gl}_n(M_\Pi(A))\big) = 0 \qquad (n >> 0). \tag{86}$$

Since, for a fixed n,

$$\mathfrak{gl}_n(M_\Pi(A)) = \varinjlim_m \mathfrak{t}^{\sigma \times m \times n^\delta}_{X \times \mathbf{m} \times \mathbf{n}}(A)$$

and the stabilization maps $\mathfrak{gl}_n \hookrightarrow \mathfrak{gl}_{n'}$, $n \leq n'$, agree with the embeddings (84), we obtain the assertion of the lemma. □

LEMMA 9.3. *Assume that there are given:*
(a) *an H-unital algebra A,*
(b) *a finite collection $\sigma_1, \ldots, \sigma_k$ of partial orderings of a finite set X,*
(c) *positive integers q, m and n, and a homology class*

$$\gamma \in H_q\left(\sum_{i=1}^k C_*\left(\mathfrak{t}^{\sigma_i \times m \times n^\delta}_{X \times \mathbf{m} \times \mathbf{n}}(A)\right)\right).$$

Then there exists such $(m', n') \succcurlyeq (m, n)$ that the map induced by (85)

$$H_q\left(\sum_{l=1}^k C_*\left(\mathfrak{t}^{\sigma_l \times m \times n^\delta}_{X \times \mathbf{m} \times \mathbf{n}}(A)\right)\right) \to H_q\left(\sum_{l=1}^k C_*\left(\mathfrak{t}^{\sigma_l \times m' \times (n')^\delta}_{X \times \mathbf{m}' \times \mathbf{n}'}(A)\right)\right)$$

annihilates γ.

Proof. Let $\sigma_1, \ldots, \sigma_k$ be arbitrary partial orderings of a finite set X. The chain maps (85) give rise to the inductive system of Mayer–Vietoris long exact sequences (87). A simple diagram chasing reduces the assertion to the similar assertion involving collections of partial orderings of cardinality $k-1$. Since the assertion is true for $k=1$ (Lemma 9.2 above), an induction on k proves the lemma. □

(87)

$$(m,n) \preccurlyeq (m',n')$$

$$\begin{tikzcd}
\cdots H_q\Big(\sum_{l=1}^{k-1} C_*\big(\mathbf{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma_l\times m\times n^\delta}\big) \oplus H_q\big(\mathbf{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma_k\times m\times n^\delta}\big)\Big) \arrow[r] \arrow[d] & H_q\Big(\sum_{l=1}^{k} C_*\big(\mathbf{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma_l\times m\times n^\delta}\big)\Big) \arrow[r] \arrow[d] & \tilde{H}_{q-1}\Big(\sum_{l-1}^{k-1} C_*\big(\mathbf{t}_{X\times\mathbf{m}\times\mathbf{n}}^{(\sigma_l\cap\sigma_k)\times m\times n^\delta}\big)\Big)\cdots \arrow[d] \\
\cdots H_q\Big(\sum_{l=1}^{k-1} C_*\big(\mathbf{t}_{X\times\mathbf{m}'\times\mathbf{n}'}^{\sigma_l\times m'\times (n')^\delta}\big) \oplus H_q\big(\mathbf{t}_{X\times\mathbf{m}'\times\mathbf{n}'}^{\sigma_k\times m'\times (n')^\delta}\big)\Big) \arrow[r] & H_q\Big(\sum_{l=1}^{k} C_*\big(\mathbf{t}_{X\times\mathbf{m}'\times\mathbf{n}'}^{\sigma_l\times m'\times (n')^\delta}\big)\Big) \arrow[r] & \tilde{H}_{q-1}\Big(\sum_{l-1}^{k-1} C_*\big(\mathbf{t}_{X\times\mathbf{m}'\times\mathbf{n}'}^{(\sigma_l\cap\sigma_k)\times m'\times (n')^\delta}\big)\Big)\cdots
\end{tikzcd}$$

$(\mathbf{t}_y^\tau \equiv \mathbf{t}_y^\tau(A)$; $\tilde{H}_0 = 0$ and $\tilde{H}_j = H_j$ if $j > 0)$.

COROLLARY 9.4. *Let A be an H-unital algebra and $\sigma_1, \dots, \sigma_k$ be partial orderings of a finite set X. Then for any homology class*

$$\gamma \in H_q\left(\sum_{l=1}^{k} C_*\left(t_X^{\sigma_i}(A)\right)\right),$$

there exist such $(m,n) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ that the map

$$H_q\left(\sum_{l=1}^{k} C_*\left(\mathfrak{t}_X^{\sigma_i}(A)\right)\right) \to H_q\left(\sum_{l=1}^{k} C_*\left(\mathfrak{t}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma_i\times m\times n^\delta}(A)\right)\right)$$

annihilates γ. □

Theorem 9.1 immediately follows from Corollary 9.4.

Remark. One can show that the assertions of Lemmas 9.2 and 9.3 hold for $n' = \max(n, 2q+1)$, which does not depend on the choice of m, an algebra A, partial orderings $\sigma_1, \dots, \sigma_k$ or a set X.

We prove next an analogue of Theorem 9.1 for the affine triangular Lie complex $\sum_{n,\sigma} C_*\left(\tilde{\mathfrak{t}}_n^\sigma(A)\right)$; cf. subsection 5.12.

THEOREM 9.5. *For every H-unital algebra A over a field of characteristic zero, the chain complex $\sum_{n,\sigma} C_*\left(\tilde{\mathfrak{t}}_n^\sigma(A)\right)$ is acyclic in positive degrees.*

Proof. The proof will again consist of two lemmas. Notation used is

$$\tilde{\mathfrak{t}}_X^\sigma(A) := \mathfrak{t}_X^\sigma(A) \ltimes A^X,$$

where $A^X = \bigoplus_{x\in X} A$ is considered as a left $\mathfrak{t}_X^\sigma(A)$-module. An order-preserving injective map of partially ordered sets $(X,\sigma) \to (Y,\tau)$ induces a homomorphism of Lie algebras $\tilde{\mathfrak{t}}_X^\sigma(A) \to \tilde{\mathfrak{t}}_Y^\tau(A)$.

For every finite collection of partial orderings $\sigma_1, \dots, \sigma_k$ of X, we have canonical chain maps (cf. (85))

$$\sum_{l=1}^{k} C_*\left(\tilde{\mathfrak{t}}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma_l\times m\times n^\delta}(A)\right) \to \sum_{l=1}^{k} C_*\left(\tilde{\mathfrak{t}}_{X\times\mathbf{m}'\times\mathbf{n}'}^{\sigma_l\times m'\times (n')^\delta}(A)\right) \qquad ((m,n) \preccurlyeq (m',n')),$$

which form an inductive system of chain complexes indexed by $\mathbb{Z}_+ \times \mathbb{Z}_+ = \{(m,n)\}$.

In the two lemmas that follow we assume that A is an H-unital algebra over a field of characteristic zero.

LEMMA 9.6. *For any $q, m, n \in \mathbb{Z}_+$ and a homology class*

$$\gamma \in H_q\left(\tilde{\mathfrak{t}}_{X\times\mathbf{m}\times\mathbf{n}}^{\sigma\times m\times n^\delta}(A)\right)$$

there exist such $(m', n') \succcurlyeq (m, n)$ *that the map*

$$H_q\left(\tilde{\mathfrak{t}}^{\sigma\times m\times n^\delta}_{X\times\mathbf{m}\times\mathbf{n}}(A)\right) \to H_q\left(\tilde{\mathfrak{t}}^{\sigma\times m'\times(n')^\delta}_{X\times\mathbf{m}'\times\mathbf{n}'}(A)\right)$$

annihilates γ.

LEMMA 9.7. *For any* $q, m, n \in \mathbb{Z}_+$, *any finite collection* $\sigma_1, \ldots, \sigma_k$ *of partial orderings of a finite set* X *and a homology class*

$$\gamma \in H_q\left(\sum_{l=1}^{k} C_*\left(\tilde{\mathfrak{t}}^{\sigma_l\times m\times n^\delta}_{X\times\mathbf{m}\times\mathbf{n}}(A)\right)\right),$$

there exist such $(m', n') \succcurlyeq (m, n)$ *that the map*

$$H_q\left(\sum_{l=1}^{k} C_*\left(\tilde{\mathfrak{t}}^{\sigma_l\times m\times n^\delta}_{X\times\mathbf{m}\times\mathbf{n}}(A)\right)\right) \to H_q\left(\sum_{l=1}^{k} C_*\left(\tilde{\mathfrak{t}}^{\sigma_l\times m'\times(n')^\delta}_{X\times\mathbf{m}'\times\mathbf{n}'}(A)\right)\right)$$

annihilates γ.

Theorem 9.5 follows easily from these two lemmas. □

Proof of Lemma 9.6. Let $\Pi = P(X, \sigma)^{\mathrm{op}}$ be the partially ordered set from the proof of Lemma 9.2. We recall that the matrix algebra $M_\Pi(A)$ is H-unital and its cyclic homology vanishes. By Theorem 11.1 of [40] then the algebra

$$\begin{pmatrix} M_\Pi(A) & M_\Pi(A) \\ 0 & 0 \end{pmatrix} \subset \mathrm{Mat}_2\big(M_\Pi(A)\big)$$

is also H-unital and its cyclic homology vanishes. Combined with the previously mentioned results of Hanlon [14] this gives

$$H_q\big(\mathfrak{gl}_n\big(M_\Pi(A)\big) \ltimes \mathrm{Mat}_n\big(M_\Pi(A)\big)\big) = 0 \qquad (n >> 0). \tag{88}$$

The existence of the natural split epimorphism of Lie algebras

$$\mathfrak{gl}_n(M_\Pi(A)) \ltimes \mathrm{Mat}_n(M_\Pi(A)) \twoheadrightarrow \varinjlim_m \tilde{\mathfrak{t}}^{\sigma\times m\times n^\delta}_{X\times\mathbf{m}\times\mathbf{n}}(A)$$

in conjunction with equation (88) shows that

$$\varinjlim_m H_q\big(\tilde{\mathfrak{t}}^{\sigma\times m\times n^\delta}_{X\times\mathbf{m}\times\mathbf{n}}(A)\big) = 0 \qquad (n >> 0).$$ □

Proof of Lemma 9.7. Repeat the proof of Lemma 9.3 with Lemma 9.6 used in place of Lemma 9.2. □

Theorems 9.1 and 9.5 combined with the results of Section 5 (Corollaries 5.14 and 5.19) give the following result:

THEOREM 9.8. (a) *For any* $H_{\mathbb{Z}}$*-unital,* $\mathbb{Q}$*-algebra* A*, the spaces* $\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A)$, $\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A)$ *and* $\bigcup_{n,\sigma} \mathrm{B}\tilde{\tilde{T}}_n^\sigma(A)$ *are acyclic; i.e., their integral homology is zero in positive dimensions.*

(b) *For any ring* A *with the property that the* $\mathbb{Q}$*-algebra* $A_{\mathbb{Q}} = A \otimes_{\mathbb{Z}} \mathbb{Q}$ *be* H*-unital, the spaces* $\bigcup_{n,\sigma} \mathrm{B}T_n^\sigma(A)$, $\bigcup_{n,\sigma} \mathrm{B}\tilde{T}_n^\sigma(A)$ *and* $\bigcup_{n,\sigma} \mathrm{B}\tilde{\tilde{T}}_n^\sigma(A)$ *are rationally acyclic; i.e., their rational homology is zero in positive dimensions.* □

10. A proof of Karoubi's Conjecture

In this section we apply our main results to rings of functional-analytic type. Let A be a (real or complex) Banach algebra. Recall that A is said to have a right bounded approximate unit (b.a.u.) if

$$\begin{array}{l} \text{there exists a constant } M > 0 \text{ such that} \\ \text{for every } a \in A \text{ and } \varepsilon > 0, \text{one has} \\ \qquad\qquad \|a - au\| < \varepsilon \\ \text{for some } u \in A \text{ whose norm is less than } M. \end{array} \tag{89}$$

A left b.a.u. is defined similarly. Property (89) is equivalent to the existence of a bounded net $(e_\lambda)_{\lambda\in\Lambda}$ in A (i.e., $\sup_{\lambda\in\Lambda} \|e_\lambda\| < \infty$; Λ is a directed set) such that $ae_\lambda \to a$ for all $a \in A$ (cf. [10], §9). A net $(e_\lambda)_{\lambda\in\Lambda}$ with the above property is usually called a right bounded approximate identity.

THEOREM 10.1. *Every Banach algebra with a right or left* b.a.u. *possesses Property* $\mathrm{AH}_{\mathbb{Z}}$ *(cf. Section* 1*). In particular, it satisfies excision in algebraic* K*-theory.*

The proof of Theorem 10.1 is obtained by the combination of Corollary 3.12 with the following proposition:

PROPOSITION 10.2. *Every Banach algebra with a right* b.a.u. *has the factorization property* $(\mathrm{TF})_{\mathrm{right}}$ *(see Section* 3*). Similarly a Banach algebra with a left* b.a.u. *has the factorization property* $(\mathrm{TF})_{\mathrm{left}}$.

Proof. Assume that we are given elements $a_1, \ldots, a_m \in A$ and that A has a right b.a.u. From the Cohen–Hewitt factorization theorem ([16], Thm. 2.5), applied to the right A-module $A^{\oplus m}$ we infer that there exist $b_1, \ldots, b_m$, $\alpha \in A$, such that $a_i = b_i\alpha$ for all $i = 1, \ldots, m$. By again applying the Cohen–Hewitt theorem, this time to the right A-module A, we obtain a factorization $\alpha = cd$ for some $c, d \in A$. Moreover the left factor c can be chosen so that $c \in \overline{\alpha A}$ (the horizontal bar denotes the closure in A). Therefore the left annihilator $l(c) = \{x \in A \mid xc = 0\}$ contains the left annihilator $l(\alpha)$ of $\alpha = cd$. As the opposite inclusion $l(c) \subset l(cd)$ is always true and trivial, we get the equality

$l(c) = l(cd)$. This proves that A possesses Property $(\mathrm{TF})_{\mathrm{right}}$. The case of a left b.a.u. is handled by the consideration of the opposite Banach algebra A^{op}. □

Recall that the topological K-theory of a Banach algebra A is defined as

$$K_q^{\mathrm{top}}(A) = \begin{cases} K_0(A), & q = 0, \\ \pi_i(\mathrm{BGL}^{\mathrm{top}}(A)), & q > 0, \end{cases}$$

where $\mathrm{GL}^{\mathrm{top}}(A) = \varinjlim \mathrm{GL}_n^{\mathrm{top}}(A)$ and each $\mathrm{GL}_n^{\mathrm{top}}(A)$ is equipped with the topology induced from the Banach space $M_n(A)$. The canonical map $\mathrm{BGL}(A) \to \mathrm{BGL}^{\mathrm{top}}(A)$ induces the map

$$F(\tilde{A}, A) \to \mathrm{BGL}^{\mathrm{top}}(A)$$

(or $\mathrm{BGL}(A)^+ \to \mathrm{BGL}^{\mathrm{top}}(A)$, if A has a unit). After passing to homotopy groups, we obtain the comparison homomorphisms

$$\iota_q : K_q(A) \to K_q^{\mathrm{top}}(A), \qquad q \geq 1. \tag{90}$$

One finds it convenient to extend (90) to $q = 0$ by defining ι_0 as the identity map $K_0(A) \to K_0^{\mathrm{top}}(A)$.

Corollary 10.3. *Any extension of Banach algebras possessing right or left* b.a.u.*'s*

$$A \overset{j}{\rightarrowtail} B \overset{f}{\twoheadrightarrow} C \tag{91}$$

induces the morphism between long exact sequences of algebraic and topological K*-groups*

$$\begin{array}{ccccccccc}
\ldots \to K_{q+1}(C) & \xrightarrow{\partial_{q+1}} & K_q(A) & \xrightarrow{j_q} & K_q(B) & \xrightarrow{f_q} & K_q(C) & \xrightarrow{\partial_q} & \ldots \\
\downarrow \iota_{q+1} & & \downarrow \iota_q & & \downarrow \iota_q & & \downarrow \iota_q & & \\
\ldots \to K_{q+1}^{\mathrm{top}}(C) & \xrightarrow{\partial_{q+1}} & K_q^{\mathrm{top}}(A) & \xrightarrow{j_q} & K_q^{\mathrm{top}}(B) & \xrightarrow{f_q} & K_q^{\mathrm{top}}(C) & \xrightarrow{\partial_1} & \ldots \\
\to K_2(C) & \xrightarrow{\partial_2} & K_1(A) & \xrightarrow{j_1} & K_1(B) & \xrightarrow{f_1} & K_1(C) & & \\
\downarrow \iota_2 & & \downarrow \iota_1 & & \downarrow \iota_1 & & \downarrow \iota_1 & & \\
\to K_2^{\mathrm{top}}(C) & \xrightarrow{\partial_2} & K_1^{\mathrm{top}}(A) & \xrightarrow{j_1} & K_1^{\mathrm{top}}(B) & \xrightarrow{f_1} & K_1^{\mathrm{top}}(C). & &
\end{array} \tag{92}$$

Proof. From Theorem 10.1 and Corollary 1.26 it follows that (91) induces a natural morphism of homotopy fibrations

$$\begin{array}{ccccc}
\mathrm{BGL}(A)^+ & \longrightarrow & \mathrm{BGL}(B)^+ & \longrightarrow & \mathrm{B}\overline{\mathrm{GL}}(B/A)^+ \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BGL}^{\mathrm{top}}(A) & \longrightarrow & \mathrm{BGL}^{\mathrm{top}}(B) & \longrightarrow & \mathrm{B}\overline{\mathrm{GL}}^{\mathrm{top}}(B/A)^+.
\end{array} \tag{93}$$

Note that the quotient group $G = \mathrm{GL}^{\mathrm{top}}(B/A)/\overline{\mathrm{GL}}^{\mathrm{top}}(B/A)$ is discrete and abelian; thus we have a morphism of homotopy fibrations

$$\begin{array}{ccccc} \mathrm{B}\overline{\mathrm{GL}}(B/A)^+ & \longrightarrow & \mathrm{BGL}(C)^+ & \longrightarrow & \mathrm{B}G \\ \downarrow & & \downarrow & & \| \\ \mathrm{B}\overline{\mathrm{GL}}^{\mathrm{top}}(B/A) & \longrightarrow & \mathrm{BGL}^{\mathrm{top}}(C) & \longrightarrow & \mathrm{B}G. \end{array} \tag{94}$$

By combining morphisms (93) and (94) with Lemma 1.24 and diagram (18), we get (92). □

Since every C^*-algebra admits a two-sided b.a.u. (cf. [32], Lemma 1.1), the following corollary also holds:

COROLLARY 10.4. *All C^*-algebras possess Property* $\mathrm{AH}_{\mathbb{Z}}$. *In particular, they satisfy excision in algebraic* K*-theory.*

10.5. Recall the definition of a homotopy-invariant functor from the category of C^*-algebras to the category of abelian groups. A functor F is said to be *homotopy invariant* if $F(\varphi_0) = F(\varphi_1)$ for any pair of homotopic $*$-homomorphisms $\varphi_0, \varphi_1 : A \to B$ between arbitrary C^*-algebras A and B. We say homomorphisms φ_1 and φ_1 are homotopic if there exists a $*$-homomorphism $\varphi : A \to B \tilde{\otimes} C[0,1]$ such that φ composed with the evaluation at $t = 0$ gives φ_0, and when composed with the evaluation at $t = 1$, gives φ_1. Throughout this section $B \tilde{\otimes} C$ will denote the *spatial* tensor product of C^*-algebras B and C, while $\mathcal{K} = \mathcal{K}(\mathcal{H})$ will be the C^*-algebra of compact operators on the standard, infinite-dimensional, separable Hilbert space $\mathcal{H}$.

PROPOSITION 10.6. *The functors*

$$A \mapsto K_q(A \tilde{\otimes} \mathcal{K}) \qquad (q \geq 1) \tag{95}$$

are homotopy invariant.

The same fact is obvious for K_0, since $K_0(C) = \pi_1\big(\mathrm{GL}^{\mathrm{top}}(C)\big)$ for any C^*-algebra C by Bott's periodicity.

Proof. J. Cuntz and N. Higson (see [17], Thm. 3.2.2) extracted from the earlier work of G.G. Kasparov [23] two simple properties of a functor F, which secure its homotopy invariance. These are:

(i) *Stability.* For any C^*-algebra A, the inclusion

$$A \hookrightarrow A \tilde{\otimes} \mathcal{K}, \qquad q \mapsto a \otimes p, \tag{96}$$

where $p : \mathcal{H} \to \mathcal{H}$ denotes the orthogonal projection onto the first basis vector $e_1 \in \mathcal{H}$, induces an isomorphism $F(A) \xrightarrow{\sim} F(A \tilde{\otimes} \mathcal{K})$.

(ii) *Split exactness.* Any split extension in the category of C^*-algebras

$$A \underset{i}{\rightarrowtail} B \underset{\varphi}{\twoheadrightarrow} C$$

induces the short exact sequence

$$0 \to F(A) \xrightarrow{F(i)} F(B) \xrightarrow{F(\varphi)} F(C) \to 0.$$

We claim that the functors (95) possess both of these properties. Indeed the inclusion (96) applied to the algebra $A \tilde{\otimes} \mathcal{K}$ becomes isomorphic to the canonical inclusion $A \tilde{\otimes} \mathcal{K} \to A \tilde{\otimes} M_2(\mathcal{K}) = M_2(A \tilde{\otimes} \mathcal{K})$, $\alpha \mapsto \left(\begin{smallmatrix} \alpha & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The isomorphism is induced by the identification

$$\left(\mathbb{C}e_1 \oplus (\mathbb{C}e_1)^{\perp}\right) \otimes \mathcal{H} \simeq \mathcal{H} \oplus \mathcal{H},$$

where $\otimes$ denotes the tensor product in the category of Hilbert spaces. Thus the stability of the functors (95) follows from the combination of Corollaries 10.4 and 1.20. The split-exactness of (95) follows immediately from Corollary 10.4 and Proposition 1.21. □

Remark 10.7. It is a standard and easy fact that the topological K-theory K_*^{top} is homotopy invariant (cf., e.g., [3], Prop. 4.3.4).

COROLLARY 10.8. *Let $\mathcal{F} \subset C[0,1]$ denote the subalgebra of functions vanishing at $t = 0$. Then for every C^*-algebra A,*

$$K_q(A \tilde{\otimes} \mathcal{F} \tilde{\otimes} \mathcal{K}) = K_q^{\mathrm{top}}(A \tilde{\otimes} \mathcal{F} \tilde{\otimes} \mathcal{K}) = 0 \qquad (q \geq 0).$$

Proof. The identity and zero endomorphisms of $A \tilde{\otimes} \mathcal{F} \tilde{\otimes} \mathcal{K}$ are homotopic. Thus for K_q and $q \geq 1$, the assertion follows from Proposition 10.6; for K_q^{top} and $q \geq 0$, the assertion follows from Remark 10.7; and for K_0, from the trivial equality $K_0 = K_0^{\mathrm{top}}$. □

Corollaries 10.4 and 10.8 together lead to the following theorem:

THEOREM 10.9 (Karoubi's Conjecture). *For every C^*-algebra A, the canonical comparison maps*

$$\iota_q : K_q(A \tilde{\otimes} \mathcal{K}) \to K_q^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K}) \qquad (q \geq 0)$$

are isomorphisms.

Proof. There is nothing to prove for $q = 0$, and for $q = 1$ the assertion is well known and its proof elementary (see [15], Appendix, Prop. A.1). We therefore assume that $q \geq 2$.

Let $C_0(S^n)$ denote the algebra of complex-valued continuous functions on the n-dimensional sphere, which vanish at the "Northern Pole." For any C^*-algebra A we consider the C^*-algebra extension

$$A \tilde{\otimes} C_0(S^1) \tilde{\otimes} \mathcal{K} \overset{j}{\rightarrowtail} A \tilde{\otimes} \mathcal{F} \tilde{\otimes} \mathcal{K} \overset{f}{\twoheadrightarrow} A \tilde{\otimes} \mathcal{K},$$

where f is induced by the evaluation map $\mathcal{F} \to \mathbb{C}$, $g \mapsto g(1)$. Combining Corollaries 10.3, 10.4 and 10.8 then produces a sequence of commutative diagrams

$$\text{(97)} \qquad \begin{array}{ccc} K_q(A \tilde{\otimes} \mathcal{K}) & \xrightarrow[\sim]{\partial_q} & K_{q-1}\big(A \tilde{\otimes} C_0(S^1) \tilde{\otimes} \mathcal{K}\big) \\ \iota_q \downarrow & & \downarrow \iota_{q-1} \\ K_q^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K}) & \xrightarrow[\sim]{\partial_q} & K_{q-1}^{\mathrm{top}}\big(A \tilde{\otimes} C_0(S^1) \tilde{\otimes} \mathcal{K}\big) \end{array} \qquad (q \geq 2)$$

whose horizontal arrows are isomorphisms. By applying (97) to the algebras $A \tilde{\otimes} C_0(S^1)^{\tilde{\otimes} n} = A \tilde{\otimes} C_0(S^n)$, $n = 0, 1, \ldots$, we obtain the commutative diagrams

$$\begin{array}{ccc} K_q(A \tilde{\otimes} \mathcal{K}) & \xrightarrow[\sim]{\partial_2 \circ \partial_3 \circ \ldots \circ \partial_q} & K_1\big(A \tilde{\otimes} C_0(S^q) \tilde{\otimes} \mathcal{K}\big) \\ \iota_q \downarrow & & \iota_1 \downarrow \\ K_q^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K}) & \xrightarrow[\sim]{\partial_2 \circ \partial_3 \circ \ldots \circ \partial_q} & K_1^{\mathrm{top}}\big(A \tilde{\otimes} C_0(S^q) \tilde{\otimes} \mathcal{K}\big), \end{array}$$

which reduce the assertion to the case $q = 1$, where it is known to hold. □

Karoubi's Conjecture was previously settled in the special case $q = 2$ by N. Higson ([17], Thm. 4.2.7) and by M. Karoubi ([20], Cor. 4.10). However the conjecture has not been verified for a single C^*-algebra $A \neq 0$ when $q > 2$.[4]

Since $\mathrm{BGL}^{\mathrm{top}}(C)$ has, for a Banach algebra C, the homotopy type of a CW-complex (cf. [26],p. 277) we obtain the following corollary:

Corollary 10.10. *For every C^*-algebra A, the canonical map*

$$\mathrm{BGL}(A \tilde{\otimes} \mathcal{K})^+ \to \mathrm{BGL}^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K})$$

is a homotopy equivalence. □

The topological K-theory is *stable* (cf. [21], Example 1.7 plus [35], Prop. 8.4); i.e., the inclusion (96), $A \hookrightarrow A \tilde{\otimes} \mathcal{K}$, induces a homotopy equivalence $\mathrm{BGL}^{\mathrm{top}}(A) \xrightarrow{\sim} \mathrm{BGL}^{\mathrm{top}}(A \tilde{\otimes} \mathcal{K})$. Thus Corollary 10.10 yields a homotopy equivalence

$$\mathrm{BGL}(A \tilde{\otimes} \mathcal{K})^+ \simeq \mathrm{BGL}^{\mathrm{top}}(A).$$

10.11. In their work on the multiplicative character of Fredholm modules ([8],[9]) A. Connes and M. Karoubi consider the following two C^*-subalgebras

[4] After this article was submitted, the second author obtained a completely different proof of Theorem 10.9 and also proved several generalizations and improvements of Karoubi's Conjecture.

of the algebra $\mathcal{B}(\mathcal{H}\oplus\mathcal{H})$ of bounded operators on $\mathcal{H}\oplus\mathcal{H}$:

$$\mathcal{M}_0 := \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix} \middle| \, \alpha, \beta \in \mathcal{B}(\mathcal{H}) \text{ and } \alpha - \beta \in \mathcal{K}(\mathcal{H}) \right\}$$

and

$$\mathcal{M}_1 := \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \middle| \, a_{ij} \in \mathcal{B}(\mathcal{H}),\ 1 \leq i,j \leq 2;\ a_{12}, a_{21} \in \mathcal{K}(\mathcal{H}) \right\}.$$

COROLLARY 10.12. (a) *Let* $\mathcal{A} = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ *denote the Calkin algebra. Then the map* $\mathrm{BGL}(\mathcal{A})^+ \to \mathrm{BGL}^{\mathrm{top}}(\mathcal{A})$ *is a homotopy equivalence.*

(b) *The maps* $\mathrm{BGL}(\mathcal{M}_i)^+ \to \mathrm{BGL}^{\mathrm{top}}(\mathcal{M}_i)$, $i = 0, 1$, *are homotopy equivalences.*

Proof. (a) Apply Corollary 10.3 to the extension

$$\mathcal{K} \rightarrowtail \mathcal{B} \twoheadrightarrow \mathcal{A} \qquad (\mathcal{B} \equiv \mathcal{B}(\mathcal{H}))$$

and subsequently use Theorem 10.9 (for $A = \mathbb{C}$) and the fact that $K_*(\mathcal{B}) = K_*^{\mathrm{top}}(\mathcal{B}) = 0$ (cf., e.g., [38], Cor. 2.5).

(b) Apply Corollary 10.3 to the extensions

$$\mathcal{K}\oplus\mathcal{K} \rightarrowtail \mathcal{M}_0 \twoheadrightarrow \mathcal{A} \tag{98}$$

$$M_2(\mathcal{K}) \rightarrowtail \mathcal{M}_1 \twoheadrightarrow \mathcal{A}\oplus\mathcal{A} \tag{99}$$

and then use Theorem 10.9 for $A = \mathbb{C}$ or $M_2(\mathbb{C})$ and the already proven part (a). □

One can easily show using the extensions (98) and (99) that

$$K_q(\mathcal{M}_0) = K_q^{\mathrm{top}}(\mathcal{M}_0) \simeq \begin{cases} \mathbb{Z}, & q = 0, \\ 0, & q = 1, \end{cases}$$

$$K_q(\mathcal{M}_1) = K_q^{\mathrm{top}}(\mathcal{M}_1) \simeq \begin{cases} 0, & q = 0, \\ \mathbb{Z}, & q = 1, \end{cases}$$

(see [8],[9]).

Remark 10.13. Let k be an arbitrary unital subring of $\mathbb{C}$. The second author proved the following additive analogue of Karoubi's Conjecture: For every C^*-algebra, one has

$$\mathrm{H}C_*(A \tilde{\otimes} \mathcal{K}/k) = \mathrm{H}C_*^{\mathrm{cont}}(A \tilde{\otimes} \mathcal{K}) = 0$$

($\mathrm{H}C_*(\ /k)$ denotes the k-algebra cyclic homology and $\mathrm{H}C_*^{\mathrm{cont}}$ denotes the continuous cyclic homology; cf. [44],[41]).

STEKLOV MATHEMATICAL INSTITUTE, ST. PETERSBURG, RUSSIA
UNIVERSITY OF CALIFORNIA, BERKELEY, CALIFORNIA

References

[1] H. Bass, *Algebraic K-Theory*, W.A. Benjamin, New York, Amsterdam, 1968.

[2] A.J. Berrick, *An Approach to Algebraic K-Theory*, Research Notes in Math. **56**, Pitman, 1982.

[3] B. Blackadar, *K-Theory for Operator Algebras*, M.S.R.I. Publ. **5**, Springer–Verlag, 1986.

[4] N. Bourbaki, *Algèbre Homologique*, Masson, Paris, 1980.

[5] ———, *Algèbre Commutative*, Masson, Paris, 1985, Chap. 1 à 4.

[6] H. Cartan and S. Eilenberg, *Homological Algebra*, Princeton University Press, Princeton, 1956.

[7] P.M. Cohn, On the free product of associative rings, Math. Z. **71** (1959), 380-398.

[8] A. Connes and M. Karoubi, Caractère multiplicatif d'un module de Fredholm, C.R. Acad. Sci. Paris **299** (1984), 963-968.

[9] ———, Caractère multiplicatif d'un module de Fredholm, K-Theory **2** (1988), 431-463.

[10] R.S. Doran and J. Wichmann, *Approximate Identities and Factorization in Banach Modules*, Lecture Notes in Math. **768**, Springer–Verlag, 1979.

[11] S. Eilenberg and J.A. Zilber, Semi-simplicial complexes and singular homology, Ann. of Math. **51** (1950), 499–513.

[12] T.G. Goodwillie, Relative algebraic K-theory and cyclic homology, Ann. of Math. **124** (1986), 347–402.

[13] ———, On the general linear group and Hochschild homology, Ann. of Math. **121** (1985), 383–407.

[14] P. Hanlon, On the complete $\mathrm{GL}(n,\mathbb{C})$-decomposition of the stable cohomology of $\mathfrak{gl}_n(A)$, Trans. A.M.S. **308** (1988), 209-225.

[15] P. de la Harpe and G. Skandalis, Sur la simplicité essentielle du groupe des inversibles et du groupe unitaire dans une C^*-algèbre simple, J. Funct. Anal. **62** (1985), 354-378.

[16] E. Hewitt, The ranges of certain convolution operators, Math. Scand. **15** (1964), 147-155.

[17] N. Higson, Algebraic K-theory of stable C^*-algebras, Adv. Math. **67** (1988), 1-140.

[18] W. van der Kallen, A note on excision for K_2, Lecture Notes in Math. **1046**, Springer–Verlag, 1984, pp. 173-177.

[19] M.I. Kargapolov and Y.I. Merzlyakov, *Fundamentals of the Theory of Groups*, 3rd ed., Nauka, Moscow, 1982 (English trans. (2nd Russian ed.) Graduate Texts in Math. **62**, Springer–Verlag, 1979).

[20] M. Karoubi, Homologie des groupes discrets associés à des algèbres d'opérateurs, J. Oper. Theory **15** (1986), 109-162.

[21] ———, K-Théorie algébrique de certaines algèbres d'opérateurs, Lecture Notes in Math. **725**, Springer–Verlag, 1979, pp. 254-290.

[22] M. Karoubi and O. Villamayor, K-Théorie algébrique et K-théorie topologique, I, Math. Scand. **28** (1971), 265-307.

[23] G.G. Kasparov, The operator K-functor and extensions of C^*-algebras, Izv. A.N. S.S.S.R. **44** (1980), 571-636 (in Russian); Math. USSR Izv. **16** (1981), 513-572 (English trans.).

[24] J.-L. Loday, K-Théorie algébrique et representations de groupes, Ann. Scient. Ec. Norm. Sup., 4e Sér. **9** (1976), 309-377.

[25] A.I. Malcev, Nilpotent torsion-free groups, Izv. A.N. S.S.S.R. **13** (1949), 201-212 (in Russian).

[26] J. Milnor, On spaces having the homotopy type of a CW-complex, Trans. A.M.S. **90** (1959), 272-280.

[27] J. Neisendorfer, *Primary Homotopy Theory*, Mem. A.M.S. **232**, Providence, R.I., 1980.

[28] P.F. Pickel, Rational cohomology of nilpotent groups and Lie algebras, Comm. Alg. **6** (1978), 409-419.

[29] D.G. Quillen, Cohomology of groups, in *Proc. Int. Congr. Math.*, Nice, 1970, vol. **2**, Gauthier–Villars, 1971, pp. 47-51.

[30] D.G. Quillen, Rational homotopy theory, Ann. of Math. **90** (1969), 205–295.

[31] G. Segal, The definition of conformal field theory, Oxford, 1987, manuscript.

[32] I.E. Segal, Irreducible representations of operator algebras, Bull. A.M.S. **53** (1947), 73-88.

[33] A.A. Suslin, On the equivalence of K-theories, Comm. Alg. **9** (1981), 1559-1566.

[34] A.A. Suslin and M. Wodzicki, Excision in algebraic K-theory and Karoubi's Conjecture, Proc. Nat. Acad. Sci. USA **87** (1990), 9582-9584.

[35] J.L. Taylor, Banach algebras and topology, in *Algebras in Analysis*, J.H. Williamson, ed., Academic Press, New York, 1975, pp. 118-186.

[36] L.N. Vaserstein, Foundations of algebraic K-theory, Uspekhi mat. nauk **31:4** (1976), 87-149 (in Russian); Russian Math. Surveys **31:4** (1976), 89-156 (English trans.).

[37] ———, On the stabilization of the general linear group over a ring, Mat. Sbornik **79** (1969), 405-424 (in Russian); Math. USSR Sbornik **8** (1969), 383-400 (English trans.).

[38] J.B. Wagoner, Delooping classifying spaces in algebraic K-theory, Topology **11** (1972), 349-370.

[39] C.A. Weibel, Mayer–Vietoris sequences and mod p K-theory, in *Proc. Conf. on Algebraic K-Theory*, Oberwolfach, 1980, Lecture Notes in Math. **966**, Springer–Verlag, 1982, pp. 390-406.

[40] M. Wodzicki, Excision in cyclic homology and in rational algebraic K-theory, Ann. of Math. **129** (1989), 591-639.

[41] ———, Homological properties of rings of functional-analytic type, Proc. Nat. Acad. Sci. USA **87** (1990), 4910-4911.

[42] ———, On the algebraic K-theorem of nonunital rings, in preparation.

[43] ———, The long exact sequence in cyclic homology associated with an extension of algebras, C.R. Acad. Sci. Paris **306** (1988), 399-404.

[44] ———, Vanishing of cyclic homology of stable C^*-algebras, C.R. Acad. Sci. Paris **307** (1988), 329-334.

[45] E.C. Zeeman, A proof of the comparison theorem for spectral sequences, Proc. Camb. Philos. Soc. **53** (1957), 57-62.

(Received September 28, 1990)