The Luna-Vust Theory of Spherical Embeddings

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The work of Krämer, Luna, Vust, Brion and others [KRÄ,LV,BLV,BP] established the importance of a very distinguished class of homogeneous varieties G/H, those which are now called spherical. Such varieties are homogeneous for a connected reductive group G and are characterized by many equivalent properties, the most important being (see [BLV]):

— Any Borel subgroup B of G has an open orbit in G/H.

— Every equivariant completion of G/H contains only finitely many orbits.

— For every irreducible G-module V and any character χ of H

$$\dim \{v \in V \mid hv = \chi(h)v \text{ for all } h \in H\} \leq 1.$$

Because of the last property spherical varieties are sometimes called "multiplicity free". This class of varieties includes some of the most important spaces, e. g.

— Tori, i. e. G is a torus and H is trivial;

— Symmetric varieties, i. e. H is the set of fixed points of an automorphism of G of order two.

— Horospherical varieties, i. e. H contains a maximal unipotent subgroup of G (for example flag varieties).

The theory of these spaces can be unified with the concept of spherical varieties. For example, all three types of spaces have a nice compactification theory, respectively due to Demazure [DEM], Kempf et al. [KKMS], Miyake-Oda [MO] … (toric varieties[2]), Satake [SAT], De Concini-Procesi [CP] … (symmetric varieties) and Pauer [PAU1] (horospherical varieties). The last case appeared already as an application of the general theory of Luna and Vust on embeddings of arbitrary homogeneous varieties. Unfortunately, in [LV] the classification of embeddings of spherical varieties is buried in a large amount of theory, which makes the paper very hard to read. Therefore the theory of Luna and Vust did not receive the attention which it deserves.

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[2] On this topic see Tadao Oda's article in this volume

In this paper I present the classification of spherical embeddings along the lines of [LV], by cutting down the theory to the absolute necessary. I think this is justified by the importance of spherical varieties and the fact that the Luna-Vust theory becomes much simpler in this case. Nevertheless most arguments apply also to the general case. Therefore this paper can serve as an introduction to [LV].

We follow [LV] very freely. Some theorems don't have an exact counterpart in [LV] or look quite different. For example Corollary 1.7 corresponds to the theory of "quasi-G-stable subspaces" ([LV] 7.3) and the first section of [LV] is replaced in the proof of Theorem 3.1 by an ad-hoc argument.

The last sections contain an account of further developments since the appearance of [LV] which is by no means complete. They are concerned with the structure of the valuation cone, the automorphism group, the rank and dimension of an orbit and an affinity criterion. The results are mainly due to Brion and Pauer.

There are several things to mention which are new: First of all the classification is characteristic-free. It is remarkable that the final result is exactly the same in positive characteristic. Also the proofs are not very much more complicated, once one takes geometric reductivity for granted. For example, the characteristic of the base field occurs explicitly only in the first section. Differences would occur only in the finer points, e. g. the nature of singularities which are not considered in this paper.

Another novelty is the observation that the quite complicated condition (d) of [LV] 8.10 Thm. can be replaced by the much easier condition **SCC**. Also the determination of the connected subgroups containing H and the affinity criterion for spherical embeddings seem to be new.

Missing in this account are all classification results [Akh,Bri3,Bri7,Kac,Krä, Mik,VK], cohomology computations [Bri5,Bri8,CP ...], local structure theorems [Bri8,Bri9,BL,BLV,Pau2], results on the B-orbit structure [Bri1,BL,Vin2] and relations to symplectic geometry [Bri2,GS,HW,Kno1,Mik]. For a survey concerned with some of these topics see [Bri6]. There is also an elementary introduction to spherical embeddings by Pauer [Pau5] and an unpublished one by Luna [Lun].

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Preliminaries: Every variety is defined over an algebraically closed field k. Denote its characteristic exponent by p, i. e. $p = 1$ for $\operatorname{char} k = 0$ and $p = \operatorname{char} k$ otherwise. The group of characters of any group S is denoted by $\mathcal{X}(S)$. By

$$V^{(S)} = \{v \in V \mid v \neq 0 \text{ and } sv = \chi(s)v \text{ for some character } \chi \in \mathcal{X}(S)\}$$

we denote the set of S-eigenvectors of an S-module V. For $v \in V^{(S)}$ let χ_v be the corresponding character. The restriction of a function f to a subset Y is denoted by $f_{|Y}$.

The group G is a connected reductive group and $B \subseteq G$ is a Borel subgroup. All G-modules will be rational. The homogeneous variety G/H is always spherical with open B-orbit Bx_0 unless otherwise stated. The corresponding orbit map is $\pi : G \to G/H$, which we assume to be separable[1]. An *embedding* of G/H is a normal G-variety X together with a G-equivariant open embedding $G/H \hookrightarrow X$. We will always regard G/H as a subset of X. The aim of this paper is to describe X in terms of data involving only G/H.

Observe that there is a list of notations attached at the end of the paper.

Remark: In other papers the term "spherical" is usually not confined to homogeneous varieties, i. e. so is called any G-variety which is normal and has an open B-orbit.

1. The basic tools

In this section I will gather all technical tools needed to perform the classification. We start with a basic theorem on extension of B-eigenfunctions which is an easy consequence of linear (resp. geometric) reductivity [Hab].

Theorem 1.1. ([Gro1] Thms. 1.3 and 2.1) *Let X be an affine G-variety, $Y \subseteq X$ a closed and G-stable subset, and $f \in k[Y]^{(B)}$. Then there exists an $N \in \mathbb{N}$ and an $f' \in k[X]^{(B)}$ with $f'_{|Y} = f^{p^N}$.*

The best known example of spherical embeddings are torus embeddings. The classification of those depends crucially on the following theorem of Sumihiro:

Theorem 1.2. *Let G be a torus acting on a normal variety X. Then every point of X is contained in a G-stable affine open subset.*

The proper generalization of this theorem to arbitrary reductive groups is the following

Theorem 1.3. *Let G be a connected reductive group acting on a normal variety X and let $Y \subseteq X$ be an orbit. There exists a B-stable affine open subset $X_0 \subseteq X$ such that*

a) $X_0 \cap Y \neq \varnothing$.

b) For any $f \in k[X_0 \cap Y]^{(B)}$ there exist $N \in \mathbb{N}$ and $f' \in k[X_0]^{(B)}$ with $f'_{|X_0 \cap Y} = f^{p^N}$.

Proof: a) By a theorem of Sumihiro ([Sum] Thm. 1 or [KKLV]) there is a G-stable open neighborhood of Y which is G-isomorphic to a subvariety of a projective space $\mathbf{P}(V)$ for some finite dimensional G-module V. Thus we may assume $X \subseteq \mathbf{P}(V)$. Let $\overline{X}$ be its closure and $Z := \overline{X} \setminus X$ its boundary. Furthermore, denote the affine cone in V over $\overline{X}$ (resp. over the closure of Y, resp. over Z) by X' (resp. Y', resp. Z').

[1] Only for convenience in some arguments. All results are valid for inseparable orbit maps as well. The details are left to the reader.

Choose a homogeneous B-eigenfunction f_1 on $Y' \cup Z'$ which vanishes on Z' but not on Y'. By Theorem 1.1 we can extend a suitable power of f_1 to a homogeneous B-eigenfunction f_0 on X'. The set $X_0 = \{x \in \overline{X} \mid f_0(x) \neq 0\}$ has the required properties.

b) Choose $d > 0$ with $g := f_0^d f \in k[Y']^{(B)}$. By Theorem 1.1 there is $g' \in k[X']^{(B)}$ which extends some p^N-th power of g. Set $f' := g'/f_0^{dp^N}$. □

In this paper, a *valuation* of a normal variety X is a map

$$v : k(X)^* = k(X) \setminus \{0\} \longrightarrow \mathbb{Q}$$

with the properties

$v(f_1 + f_2) \geq \min\{v(f_1), v(f_2)\}$ whenever $f_1, f_2, f_1 + f_2 \in k(X)^*$,
$v(f_1 f_2) = v(f_1) + v(f_2)$ for all $f_1, f_2 \in k(X)^*$ and
$v(k^*) = 0$.

This includes the valuation which is identically zero on $k(X)^*$. Any prime divisor $D \subseteq X$ induces a valuation v_D. The center of a valuation v is a closed subvariety $Z = Z_v$ such that we have for the local rings $\mathcal{O}_{X,Z} \subseteq \mathcal{O}_v$ and for their maximal ideals $\mathfrak{m}_{X,Z} \subseteq \mathfrak{m}_v$. The center of a valuation is unique if it exists. If X is affine, then v has a center if and only if v is not negative on $k[X]$. In this case Z_v is defined by the ideal $k[X] \cap \mathfrak{m}_v$.

A valuation v of a G-variety X is invariant if $v(f^g) = v(f)$ for all $g \in G$. The set of invariant valuations is denoted by $\mathcal{V}(X)$. We will write $\mathcal{V}$ for $\mathcal{V}(G/H)$. If v is invariant, then its center Z_v is G-stable. Conversely, every G-stable closed subvariety $Z \subseteq X$ is the center of a $v \in \mathcal{V}(X)$: Consider the normalization of the blowing up of X with center Z and let D be a component of the exceptional divisor. Then $Z = Z_{v_D}$.

Whenever a group action on G is considered then G and B always act by left and H by right translations. By $\mathcal{V}(G)$ we understand the G-invariant valuations of $k(G)$.

Lemma 1.4. ([Sum] Lemma 10 & 11, [LV] 3.2 Lemme) *Let v be an arbitrary valuation of $k(G)$. Then there exists a unique $\bar{v} \in \mathcal{V}(G)$ with*

$$\bar{v}(f) = v(f^g)$$

for any $f \in k(G)$ and all g in a nonempty open subset U_f of G.

Proof: It suffices to show that $v(f^g)$ is constant for g in a nonempty open subset of G, because then we define $\bar{v}(f)$ to be this constant value and everything else follows. We may assume $f \in k[G]$. The set $V(q) := \{h \in k[G] \mid v(h) \geq q\}$ is a linear subspace of $k[G]$ for any $q \in \mathbb{Q}$. Then the orbit Gf spans a finite dimensional submodule M of $k[G]$ and therefore there exists a smallest $q_0 \in \mathbb{Q}$ with $M \not\subseteq V(q_0)$. Thus we can take $U_f = \{g \in G \mid f^g \notin M \cap V(q_0)\}$. □

Observe that any $v \in \mathcal{V}(G)$ induces a $v' \in \mathcal{V}(G/H)$ by restriction to the subfield $k(G)^H = k(G/H)$. Conversely,

Corollary 1.5. *Any $v \in \mathcal{V}(G/H)$ can be lifted to a $\bar{v} \in \mathcal{V}(G)$.*

Proof: It is well known that v can be lifted to a possibly non-invariant valuation v' of $k(G)$. The lemma above shows how to make v' invariant. $\square$

Now we are prepared to establish the main technical tool for the classification. For a subspace $M \subseteq k[G]$ let M^n be the subspace spanned by all n-fold products $f_1 \cdots f_n$ with $f_i \in M$.

Theorem 1.6. *Let $v \in \mathcal{V}$, $f \in k(G/H)$ and $h \in k(G)^{(B \times H)}$. Assume that $fh \in k[G]$ and let $M \subseteq k[G]$ be the submodule generated by fh. Then*
  a) $M^n h^{-n} \subseteq k(G/H)$ for all $n \in \mathbb{N}$.
  b) $v(f) = \min\{\frac{1}{n} v(f'/h^n) \mid n = p^N, N \in \mathbb{N}, f' \in (M^n)^{(B)}\}$.

Proof: a) Because the G-action commutes with the H-action, all elements of M^n are H-eigenvectors with character χ_{h^n}. Thus $M^n h^{-n} \subseteq k(G)^H = k(G/H)$.

b) By Corollary 1.5, we can lift v to $\bar{v} \in \mathcal{V}(G)$. Let $f_1 = fh$ and $q_1 := \bar{v}(f_1)$. Observe that $V(q) := \{f \in k[G] \mid \bar{v}(f) \geq q\}$ with $q \in \mathbb{Q}$, is a G-submodule of $k[G]$. This shows $M^n \subseteq V(nq_1)$ and therefore $v(f)$ is less than or equal to the right hand side of b).

To show the converse, consider the graded domain $R := \bigoplus_{n=0}^{\infty} M^n$. We denote the n-th homogeneous component of $r \in R$ by $r_n \in M^n$. Consider the following function v' on $R \setminus \{0\}$:

$$v'(r) := \min\{\bar{v}(r_n) - q_1 n \mid n \in \mathbb{N}, r_n \neq 0\}.$$

It is easily verified that v' is a G-invariant valuation with $v'(r) \geq 0$ for all $r \in R$. Then $\mathfrak{p} := \{r \in R \mid v'(r) > 0\}$ is a homogeneous prime ideal. Because $v'(f_1) = 0$ we can choose $0 \neq r \in (M/M \cap \mathfrak{p})^{(B)}$. By Theorem 1.1, applied to $\operatorname{Spec} R$ and the subvariety defined by $\mathfrak{p}$, there exists an $n = p^N$ such that r^n lifts to an $f' \in (M^n)^{(B)}$ satisfying

$$\begin{aligned} v(f'/h^n) = \bar{v}(f') - n\bar{v}(h) = v'(f') + q_1 n - n\bar{v}(h) = \\ = 0 + \bar{v}(f_1)n - n\bar{v}(h) = nv(f_1/h) = nv(f). \end{aligned}$$

$\square$

This theorem implies an important existence statement for B-eigenfunctions. We denote the set of all B-stable prime divisors in G/H by $\mathcal{D} = \mathcal{D}(G/H)$. Because G/H contains an open B-orbit Bx_0, there are only finitely many of them.

Corollary 1.7. *Let $f \in k[Bx_0]$ and $v_0 \in \mathcal{V}$. Then there exist $n = p^N$ and $f' \in k(G/H)^{(B)}$ such that*

$$\begin{aligned} &v_0(f') = v_0(f^n), \\ &v(f') \geq v(f^n) \qquad \text{for all} \quad v \in \mathcal{V}, \\ &v_D(f') \geq v_D(f^n) \qquad \text{for all} \quad D \in \mathcal{D}. \end{aligned}$$

Proof: There is a finite cover $\tilde{G}$ of G such that $k[\tilde{G}]$ is factorial (see for example [KKLV] 4.6). Without loss of generality we may replace G by $\tilde{G}$. Then there exists $h \in k(G)^{(B\times H)}$ whose divisor (h) is precisely the B-stable part of $(1/f\circ\pi)$. Applying Theorem 1.6b to fh we get $n = p^N$ and $f_1 \in k[G]^{(B)}$ such that $v_0(f_1/h^n) = v_0(f^n)$ and $v(f_1/h^n) \geq v(f^n)$ for all $v \in \mathcal{V}$. Finally, for any $D \in \mathcal{D}$ and any component D' of $\pi^{-1}(D)$ we have

$$v_D(f_1/h^n) = v_{D'}(f_1/h^n) \geq v_{D'}(h^{-n}) = v_D(f^n).$$

Thus $f' = f_1/h^n$ has the required properties. $\square$

Let us take a closer look to the set $k(G/H)^{(B)}$. It is an abelian subgroup of the multiplicative group of the field $k(G/H)$ and the map $f \mapsto \chi_f$ is a homomorphism into the character group of B. Denote its image by $\Lambda = \Lambda_{G/H}$. The kernel consists precisely of the B-invariant rational functions. Since B has an open orbit in G/H, these are all constant. Thus we get a short exact sequence

$$1 \to k^* \to k(G/H)^{(B)} \to \Lambda \to 0.$$

The group $\Lambda \subseteq \mathcal{X}(B)$ is a finitely generated free abelian group. Its rank is called the *rank* of G/H.

Any valuation v induces a homomorphism $k(G/H)^{(B)} \longrightarrow \mathbb{Q}$ by $f \mapsto v(f)$. Because it is trivial on the constant functions it factorizes through Λ and induces an element

$$\varrho_v \in \mathcal{Q} = \mathcal{Q}(G/H) := \mathrm{Hom}(\Lambda, \mathbb{Q}).$$

Corollary 1.8. ([LV] 7.4 Prop.) *The map $\mathcal{V} \to \mathcal{Q} : v \mapsto \varrho_v$ is injective.*

Proof: Let $v_0, v_1 \in \mathcal{V}$ and $f \in k[Bx_0]$ with $v_0(f) < v_1(f)$. By Corollary 1.7 there exists $f' \in k(G/H)^{(B)}$ with $v_0(f') = v_0(f^n) < v_1(f^n) \leq v_1(f')$. $\square$

From now on we will identify $\mathcal{V}$ with its image in $\mathcal{Q}$.

2. Simple spherical embeddings and their characterization

Let X be an embedding of G/H and $Y \subseteq X$ an orbit. By Theorem 1.3 there exists an affine open B-stable subset of X which meets Y. We show that there is even a canonical one. Denote the (finite) set of B-stable prime divisors of X by $\mathcal{D}(X)$. Let $\mathcal{D}_Y(X)$ be the set of all $D \in \mathcal{D}(X)$ with $Y \subseteq D$ and

$$X_0 := X \setminus \bigcup_{D \in \mathcal{D}(X) \setminus \mathcal{D}_Y(X)} D.$$

Theorem 2.1. *X_0 has the following properties:*

a) X_0 is B-stable, affine and open.

b) Y is the only closed orbit of GX_0.

c) $X_0 \cap Y$ is a B-orbit.

Proof: Let $X_1 \subseteq X$ be affine, open and B-stable as in Theorem 1.3. Choose any valuation $v_0 \in \mathcal{V} = \mathcal{V}(X)$ whose center is the closure of Y. Then there exists an $f_0 \in k[X_1]$ with $v_0(f_0) = 0$ and which vanishes on all B-stable divisors of X_1 not containing Y. By Corollary 1.7, we may assume f_0 to be a B-eigenfunction. Thus $X_0 = \{x \in X_1 \mid f_0(x) \neq 0\}$ is affine.

Let $Z \subseteq X$ be an orbit which meets X_1 and assume that Y is not in closure of Z. By Corollary 1.7 we could have chosen f_0 to vanish additionally on Z. Thus $Z \cap X_0 = \varnothing$ and b) follows.

For any $f_1 \in k[X_0 \cap Y]^{(B)}$ there is an $n \in \mathbb{N}$ with $f_0^n f_1 \in k[X_1 \cap Y]$. By Theorem 1.3 some power $(f_0^n f_1)^q$ can be extended to $f' \in k[X_1]^{(B)}$. Then $f'_{|X_0}$ and thus f_1 are invertible. This implies c). □

Corollary 2.2. ([VK], [LV] 7.5 Prop.) *The number of G-orbits in X is finite and each orbit is spherical.*

Proof: There are only finitely many possibilities for X_0 and b) implies that Y is uniquely determined by X_0. By c) all orbits are spherical. □

Remark: For the converse of the first part see [Akh2]. Brion [Bri1] and Vinberg [Vin2] have shown that even B has only finitely many orbits.

The embedding X is called *simple* if it contains exactly one closed G-orbit. The corollary implies that any embedding X is covered by finitely many simple open subembeddings, namely those of the form GX_0. This divides the description of all embeddings into two parts: The classification of simple embeddings and the description of how to patch them together.

We want to describe the elements of $\mathcal{D}(X)$ purely in terms of G/H. Note that any $D \in \mathcal{D}(X)$ either meets the open set G/H or is a component of the complement and therefore G-stable. Thus each orbit $Y \subseteq X$ determines two sets:

$$\mathcal{B}_Y(X) := \{v_D \in \mathcal{V} \mid D \in \mathcal{D}_Y(X) \text{ is } G\text{-stable}\}$$

$$\mathcal{F}_Y(X) := \{D \cap G/H \in \mathcal{D} \mid D \in \mathcal{D}_Y(X) \text{ is not } G\text{-stable}\}$$

If X is a simple embedding and Y its unique closed orbit, we will drop the subscript ${}_Y$. For any subset $\mathcal{F}$ of $\mathcal{D}$, let $\complement\mathcal{F}$ be its complement.

Theorem 2.3. ([LV] 8.3 Prop.) *A simple G/H-embedding X is uniquely determined by the pair $(\mathcal{B}(X), \mathcal{F}(X))$.*

Proof: Let X' be a second simple embedding with the same data and let $X_0' \subseteq X'$ be analogous to X_0. Let

$$X_1 := G/H \setminus \bigcup_{D \in \complement\mathcal{F}(X)} D.$$

Then we have

$$k[X_0] = \{f \in k[X_1] \mid v(f) \geq 0 \text{ for all } v \in \mathcal{B}(X)\} = k[X_0'].$$

Therefore the canonical birational map $X \overset{\sim}{\dashrightarrow} X'$ induces an isomorphism $X_0 \overset{\sim}{\to} X_0'$ and thus $X = GX_0 \overset{\sim}{\to} GX_0' = X'$. $\square$

We want to describe all pairs $(\mathcal{B}(X), \mathcal{F}(X))$ arising in this way, but first we organize the data slightly differently. For this we need some additional notation.

A subset $\mathcal{C}$ of $\mathcal{Q}$ is called a *cone* if it is closed under addition and multiplication by $\mathbb{Q}^+ := \{q \in \mathbb{Q} \mid q \geq 0\}$. Its *dual cone* is

$$\mathcal{C}^\vee := \{\alpha \in \mathcal{Q}^\vee \mid \alpha(v) \geq 0 \text{ for all } v \in \mathcal{C}\}.$$

The cone $\mathcal{C}$ is called *strictly convex* if it does not contain a nontrivial linear subspace or equivalently if $\mathcal{C} \cap (-\mathcal{C}) = 0$. It is *finitely generated* if there are finitely many elements $v_1, \ldots, v_s \in \mathcal{Q}$ with $\mathcal{C} = \mathbb{Q}^+ v_1 + \cdots + \mathbb{Q}^+ v_s$. A *face* of a cone $\mathcal{C}$ is a subset of the form $\mathcal{C} \cap \{v \in \mathcal{Q} \mid \alpha(v) = 0\}$ for some $\alpha \in \mathcal{C}^\vee$. The *dimension* of a cone is the dimension of its linear span. A face of dimension one is called an *extremal ray*. Finally the *relative interior* of $\mathcal{C}$ is $\mathcal{C}$ with all proper faces removed. It is denoted by $\mathcal{C}^\circ$. We will consider Λ as a subset of the dual space $\mathcal{Q}^\vee$.

There is a natural map

$$\varrho : \mathcal{D} \longrightarrow \mathcal{Q} : D \mapsto \varrho_{v_D}$$

which in general is not injective. Let $\mathcal{C}(X) = \mathcal{C}_Y(X) \subseteq \mathcal{Q}$ be the cone generated by $\varrho(\mathcal{F}_Y(X))$ and $\mathcal{B}_Y(X)$.

Assume again that X is a simple embedding with closed orbit Y. The next lemma shows that $\mathcal{B}(X)$ can be recovered from $\mathcal{C}(X) := \mathcal{C}_Y(X)$ and $\mathcal{F}(X)$. Observe that $v \in \mathcal{B}(X)$ is uniquely determined in its ray $\mathbb{Q}^+ v$, by the fact that its group of values is precisely $\mathbb{Z} \subset \mathbb{Q}$.

Lemma 2.4. *The sets $\mathbb{Q}^+ v$ with $v \in \mathcal{B}(X)$ are exactly the extremal rays of $\mathcal{C}(X)$ which do not contain an element of $\varrho(\mathcal{F}(X))$.*

Proof: Let $D_0 \subseteq X$ be the G-invariant divisor corresponding to $v \in \mathcal{B}(X)$. By Corollary 1.7 there is an $f \in k(G/H)^{(B)}$ vanishing on all $D \in \mathcal{D}(X)$ except D_0. This implies the lemma. □

The center of an invariant valuation can be determined by $\mathcal{C}(X)$:

Theorem 2.5. *Let X be a simple embedding with closed orbit Y and $v \in \mathcal{V}$.*
a) $k[X_0]^{(B)} = \{f \in k(G/H)^{(B)} \mid \chi_f \in \mathcal{C}_Y(X)^\vee\}$.
b) The center of v exists if and only if $v \in \mathcal{C}(X)$.
c) The center of v is Y if and only if $v \in \mathcal{C}(X)^\circ$.

Proof: a) follows from the definition of $\mathcal{C}(X)$. Because $X = GX_0$, the center of v exists iff v is not negative on $k[X_0]$. Thus b) follows from Corollary 1.7 and a). Finally, the center of v is Y if and only if any $f \in k[X_0]^{(B)}$ with $v(f) = 0$ is invertible. This is equivalent to $v \in \mathcal{C}(X)^\circ$. Thus c) holds. □

We denote the pair $(\mathcal{C}_Y(X), \mathcal{F}_Y(X))$ by $\mathcal{C}_Y^c(X)$.

3. The classification of spherical embeddings

Definition: A *colored cone* is a pair $(\mathcal{C}, \mathcal{F})$ with $\mathcal{C} \subseteq \mathcal{Q}$ and $\mathcal{F} \subseteq \mathcal{D}$ having the following properties:
CC1: $\mathcal{C}$ is a cone generated by $\varrho(\mathcal{F})$ and finitely many elements of $\mathcal{V}$.
CC2: $\mathcal{C}^\circ \cap \mathcal{V} \neq \varnothing$.
A colored cone is called *strictly convex* if the following holds:
SCC: $\mathcal{C}$ is strictly convex and $0 \notin \varrho(\mathcal{F})$.

Theorem 3.1. ([LV] 8.10 Prop.) *The map $X \mapsto \mathcal{C}^c(X)$ is a bijection between isomorphism classes of simple embeddings and strictly convex colored cones.*

Proof: First we show that $\mathcal{C}^c(X)$ is a strictly convex colored cone. **CC1** is clear by construction and **CC2** holds by Theorem 2.5. There exists an $f \in k[X_0]^{(B)}$ vanishing on all B-stable divisors in X_0. This shows **SCC**.

By Theorem 2.3 and Lemma 2.4 the map is injective. To show surjectivity let $(\mathcal{C}, \mathcal{F})$ be a strictly convex colored cone. Choose $g_1, \ldots, g_s \in k(G/H)^{(B)}$ such that the corresponding characters χ_{g_i} generate $\mathcal{C}^\vee \cap \Lambda$ as a monoid. This is possible by **CC1**. Let $D_0 := \bigcup_{D \in \complement\mathcal{F}} D$ and choose $f_0 \in k[G]^{(B \times H)}$ with $\{f_0 = 0\} = \pi^{-1}(D_0) \subseteq G$ and $f_i := f_0 g_i \in k[G]$. Let V be the G-submodule of $k[G]$ generated by $f_0, \ldots, f_s$. Because all elements of V are H-eigenvectors with the same character, we get a morphism $\varphi : G/H \to \mathbf{P}(V^\vee)$. Let $\overline{X}$ be the closure of the image of φ, $X_0' := \{x \in \overline{X} \mid f_0(x) \neq 0\}$ and $X' := GX_0'$.

Let $\mathcal{M} := \{f \in k(G/H)^{(B)} \mid \chi_f \in \mathcal{C}^\vee\}$. By construction we have $\mathcal{M} \subseteq k[X_0']^{(B)}$. Conversely, let $f \in k[X_0']^{(B)}$ and choose v in an extremal ray of $\mathcal{C}$. By **CC1** it is either a multiple of a $\varrho(D)$ with $D \in \mathcal{F}$ or an element of $\mathcal{V}$. In the first case, $v(\chi_f) \geq 0$ holds by construction. Consider the second case. The algebra $k[X'_0]$ is generated by f'/f_0 with $f' \in V$. Therefore Theorem 1.6b implies that v is not negative on $k[X'_0]$, in particular $v(f) \geq 0$. This shows $k[X'_0]^{(B)} = \mathcal{M}$.

For any $D \in \complement\mathcal{F}$ we have $\varphi(D) \subseteq \{f_0 = 0\}$. By **SCC**, there is an $h \in \mathcal{M}$ vanishing on all $D \in \mathcal{F}$ and therefore $\varphi(D) \subseteq \{h = 0\}$. Thus no $D \in \mathcal{D}$ maps dominantly to X', which implies $\varphi^{-1}(\varphi(Bx_0)) = Bx_0$. Therefore all fibers of φ are affine.

Choose any complete embedding X'' of G/H and let $\widetilde{X}$ be the closure of G/H in $X' \times X''$, embedded diagonally. The projection $\widetilde{X} \to X'$ is proper. Assume there is a G-subvariety $Z \subseteq \widetilde{X} \setminus (G/H)$ mapping dominantly to X' and let $v \in \mathcal{V}$ have Z as center. Then v vanishes on $k(X')$ and in particular on $\mathcal{M}$. **SCC** implies that $k(G/H)^{(B)}$ is generated as a group by $\mathcal{M}$. Therefore v vanishes on $k(G/H)^{(B)}$ in contradiction to Corollary 1.8. Hence the fibers of φ are both affine and complete, and therefore they are finite. Thus φ factors uniquely in an embedding $G/H \hookrightarrow X$ and a finite morphism $\psi : X \to X'$. For $X_0 := \psi^{-1}(X'_0)$ we still have $k[X_0]^{(B)} = \mathcal{M}$.

By **CC2** there exists a $v_0 \in \mathcal{C}^\circ \cap \mathcal{V}$. In particular v_0 is not negative on $k[X_0]^{(B)}$. By Corollary 1.7, the same is true for $k[X_0]$. Thus v_0 has a center Y in X.

Let $Z \subseteq X$ be a closed G-stable subvariety and choose a valuation $v_1 \in \mathcal{V}$ whose center is Z. Because v_1 is not negative on $k[X_0]$, we have $v_1 \in \mathcal{C}$. Assume $Y \not\subseteq Z$. By Corollary 1.7, there exists an $f \in k[X_0]^{(B)}$ with $v_0(f) = 0$ and $v_1(f) > 0$. But $v_0 \in \mathcal{C}^\circ$ and $v_0(f) = 0$ imply $v(f) = 0$ for all $v \in \mathcal{C}$. Thus $Y \subseteq Z$ and X is a simple embedding with closed orbit Y.

The same argument shows that Y is contained in all B-stable divisors of X_0. This implies $\mathcal{C}^\vee \cap \Lambda = \mathcal{C}(X)^\vee \cap \Lambda$ and therefore $\mathcal{C}(X) = \mathcal{C}$. It also shows $\mathcal{F} \subseteq \mathcal{F}(X)$. Finally the opposite inclusion holds because $\varphi(D) \subseteq \{f_0 = 0\}$ for any $D \in \complement\mathcal{F}$. □

A pair $(\mathcal{C}_0, \mathcal{F}_0)$ is called a *face* of the colored cone $(\mathcal{C}, \mathcal{F})$ if $\mathcal{C}_0$ is a face of $\mathcal{C}$, $\mathcal{C}_0^\circ \cap \mathcal{V} \neq \varnothing$ and $\mathcal{F}_0 = \mathcal{F} \cap \varrho^{-1}(\mathcal{C}_0)$.

Lemma 3.2. *Let X be an embedding and $Y \subseteq X$ an orbit. Then $Z \mapsto \mathcal{C}^c_Z(X)$ is a bijection between orbits whose closure contain Y and faces of $\mathcal{C}^c_Y(X)$.*

Proof: We may assume X to be simple with closed orbit Y. Let $(\mathcal{C}, \mathcal{F})$ be a face of $\mathcal{C}^c(X)$. By definition there is a $v_0 \in \mathcal{C}^\circ \cap \mathcal{V}$, and by Theorem 2.5b v_0 has a center $\overline{Z}$ in X with open orbit Z. Let $D \in \mathcal{D}_Y(X)$. By Corollary 1.7, we have $Z \not\subseteq D$ if and only if there is an $f \in k[X_0]^{(B)}$ with $v_0(f) = 0$ and $v_D(f) > 0$. This implies $\mathcal{C}^c_Z(X) = (\mathcal{C}, \mathcal{F})$.

Conversely, let $Z \subseteq X$ be an orbit. By Corollary 1.7, there is an $f \in k[X_0]^{(B)}$ with $f_{|Z} \neq 0$ which vanishes on all $D \in \mathcal{D}_Y(X)$ with $Z \not\subseteq D$. This implies that $\mathcal{C}^c_Z(X)$ is the face defined by χ_f. □

Definition: A *colored fan* is a nonempty finite set $\mathfrak{F}$ of colored cones with the following properties:

CF1: Every face of $\mathcal{C}^c \in \mathfrak{F}$ belongs to $\mathfrak{F}$.
CF2: For every $v \in \mathcal{V}$ there is at most one $(\mathcal{C}, \mathcal{F}) \in \mathfrak{F}$ with $v \in \mathcal{C}^\circ$.
A colored fan $\mathfrak{F}$ is called *strictly convex* if $(0, \varnothing) \in \mathfrak{F}$, or equivalently, if all elements of $\mathfrak{F}$ are strictly convex.

For an embedding X denote

$$\mathfrak{F}(X) := \{\mathcal{C}^c_Y(X) \mid Y \subseteq X \text{ is a } G\text{-orbit}\}.$$

Inclusion of closures gives an order relation on the set of orbits. Theorem 3.1 and Lemma 3.2 imply that $Y \mapsto \mathcal{C}^c_Y(X)$ is an order-reversing bijection between the set of orbits of X and $\mathfrak{F}(X)$. The open orbit corresponds to the cone $(0, \varnothing)$. The main theorem of this paper is:

Theorem 3.3. *The map $X \mapsto \mathfrak{F}(X)$ induces a bijection between isomorphism classes of embeddings and strictly convex colored fans.*

Proof: Let X be an embedding. Then **CF1** holds by Lemma 3.2. Let $Y_1, Y_2 \subseteq X$ be orbits and $v_0 \in \mathcal{C}_{Y_1}(X)^\circ \cap \mathcal{C}_{Y_2}(X)^\circ \cap \mathcal{V}$. Then $\overline{Y_1}$ and $\overline{Y_2}$ are both centers for v_0 (Theorem 2.5c) and therefore equal. This implies **CF2**.

The injectivity of the map follows from Theorem 3.1. Given a strictly convex colored fan $\mathfrak{F}$, for any $\mathcal{C}^c \in \mathfrak{F}$ there is a simple embedding $X(\mathcal{C}^c)$ which we paste along maximal isomorphic open subsets. We thus obtain a prevariety X. Let $\mathcal{C}^c_1, \mathcal{C}^c_2 \in \mathfrak{F}$ with embeddings X_1, X_2, and let X_3 be the normalization of the closure of G/H, embedded diagonally in $X_1 \times X_2$. The condition for X to be separated is precisely that $\varphi_1 : X_3 \to X_1$ and $\varphi_2 : X_3 \to X_2$ are open embeddings. Let $Y_3 \subset X_3$ be an orbit, $Y_1 = \varphi_1(Y_3)$, $Y_2 = \varphi_2(Y_3)$ and $v_0 \in \mathcal{V}$, with center Y_3. We may assume that Y_1 and Y_2 are the only closed orbits. Then we have $v_0 \in \mathcal{C}_{Y_1}(X)^\circ \cap \mathcal{C}_{Y_2}(X)^\circ \cap \mathcal{V}$ and **CF2** implies $X_1 = X_2 = X_3$. $\square$

4. Morphisms between spherical embeddings

Let G/H' be a second spherical variety and $\varphi : G/H \to G/H'$ be a dominant G-equivariant morphism. This induces an injection $\varphi^* : \Lambda_{G/H'} \hookrightarrow \Lambda_{G/H}$ and thus

$$\varphi_* : \mathcal{Q}(G/H) \twoheadrightarrow \mathcal{Q}(G/H').$$

It follows easily from Corollary 1.5 that

$$\varphi_*(\mathcal{V}(G/H)) = \mathcal{V}(G/H').$$

On the other hand, let $\mathcal{F}_\varphi$ be the set of those $D \in \mathcal{D}(G/H)$ for which D maps dominantly to G/H'. Thus we get a map

$$\varphi_* : \complement\mathcal{F}_\varphi \to \mathcal{D}(G/H').$$

Definition: a) Let $(\mathcal{C},\mathcal{F})$, $(\mathcal{C}',\mathcal{F}')$ be colored cones for G/H, G/H' respectively. Then we say that $(\mathcal{C},\mathcal{F})$ *maps to* $(\mathcal{C}',\mathcal{F}')$ if the following conditions hold:
CM1: $\varphi_*(\mathcal{C}) \subseteq \mathcal{C}'$.
CM2: $\varphi_*(\mathcal{F} \setminus \mathcal{F}_\varphi) \subseteq \mathcal{F}'$.
b) Let $\mathfrak{F}$, $\mathfrak{F}'$ be colored fans for G/H, G/H' respectively. Then we say that $\mathfrak{F}$ *maps to* $\mathfrak{F}'$ if every element of $\mathfrak{F}$ maps to some element of $\mathfrak{F}'$.

Theorem 4.1. *Let X, X' be embeddings of G/H, G/H' respectively. Then φ extends to a morphism $X \to X'$ if and only if $\mathfrak{F}(X)$ maps to $\mathfrak{F}(X')$.*

Proof: We may assume that X and X' are both simple, with closed orbits Y and Y'. Let X_0, resp. X_0', be their canonical B-stable subsets. Assume $X \to X'$ exists. Then $D \in \mathcal{F}$ implies $Y' \subseteq \overline{\varphi(D)}$, hence **CM2** holds. Let $f \in k[X_0']^{(B)}$. Then $f \circ \varphi \in k[X_0]^{(B)}$, which implies **CM1**.

Conversely, assume that $\mathcal{C}^c(X)$ maps to $\mathcal{C}^c(X')$. Let X_1 (resp. X_1') be the intersection of X_0 (resp. X_0') with the open G-orbit. Then **CM2** implies $\varphi(X_1) \subseteq X_1'$. It follows from **CC1** that

$$\varphi^*(k[X_0']^{(B)}) \subseteq k[X_0]^{(B)}.$$

This inclusion together with Corollary 1.7 imply $k[X_0'] \overset{\varphi}{\hookrightarrow} k[X_0]$. Thus $X = GX_0 \to GX_0' = X'$ exists. □

For any colored fan $\mathfrak{F}$ define the *support* of $\mathfrak{F}$ by:

$$\operatorname{Supp}\mathfrak{F} := \mathcal{V}(G/H) \cap \bigcup \{\mathcal{C} \mid (\mathcal{C},\mathcal{F}) \in \mathfrak{F}\}.$$

Theorem 4.2. *Let $\varphi : X \to X'$ be a dominant G-morphism between a G/H- and a G/H'-embedding. Then φ is proper if and only if*

$$\operatorname{Supp}\mathfrak{F}(X) = \varphi_*^{-1}(\operatorname{Supp}\mathfrak{F}(X')).$$

In particular, X is complete if and only if $\operatorname{Supp}\mathfrak{F}(X) = \mathcal{V}(G/H)$.

Proof: $\operatorname{Supp}\mathfrak{F}(X)$ consists precisely of those valuations whose center exists in X. Thus the necessity follows from the valuative criterion of properness.

On the other hand, let $\varphi(v) \in \operatorname{Supp}\mathfrak{F}(X')$ but $v \notin \operatorname{Supp}\mathfrak{F}(X)$. Then $\mathfrak{F}^* := \mathfrak{F}(X) \cup \{(\mathbb{Q}^+ v, \varnothing)\}$ is again a colored fan mapping to $\mathfrak{F}(X')$. Let X^* be the corresponding embedding. Because φ factors through $X \hookrightarrow X^*$ it cannot be proper. □

One can reformulate the results of this section without reference to a second spherical variety G/H'. A *colored subspace* is a colored cone $(\mathcal{C},\mathcal{F})$ such that $\mathcal{C} \subseteq \mathcal{Q}$ is a subspace. Observe that condition **CC2** is automatically fulfilled because $0 \in \mathcal{C} = \mathcal{C}^\circ$. Let

$$\mathcal{C}_\varphi := \{v \in \mathcal{Q} \mid v(\chi_f) = 0 \text{ for all } f \in k(G/H')^{(B)}\}.$$

Definition: An *integral submersion* of G/H is a normal G-variety X together with a dominant G-morphism $\varphi : G/H \to X$ such that all fibers of φ are reduced and irreducible.

Lemma 4.3. *Let $\varphi : G/H \to G/H'$ be integral and let $X_1 = G/H \setminus \bigcup_{D \in \complement\mathcal{F}_\varphi} D$.*

a) $k[B\varphi(x_0)] = \{f \in k[X_1] \mid v(f) \geq 0$ for all $v \in \mathcal{C}_\varphi \cap \mathcal{V}\}$

b) $(\mathcal{C}_\varphi, \mathcal{F}_\varphi)$ is a colored subspace.

Proof: a) Choose an embedding $G/H \hookrightarrow X$ such that φ extends to a proper morphism $\overline{\varphi} : X \to G/H'$. Let $X_0' := B\varphi(x_0) \subseteq G/H'$ and $X_0 = \overline{\varphi}^{-1}(X_0')$. Because $\overline{\varphi}$ is integral and proper, we have $k[X_0'] = k[X_0]$. By definition $X_1 = G/H \cap X_0$ and the closure of $X_0 \setminus (G/H)$ is G-stable. Thus $f \in k[X_0]$ if and only if $f \in k[X_1]$ and $v(f) \geq 0$ for all $v \in \mathcal{V}$ whose center meets X_0. But these are exactly the $v \in \mathcal{C}_\varphi \cap \mathcal{V}$.

b) Let $f \in k(G/H)^{(B)}$ such that χ_f is not negative on $\mathcal{C}_\varphi \cap \mathcal{V}$ and $\varrho(\mathcal{F}_\varphi)$. Then a) implies $f \in k(G/H')^{(B)}$. Thus χ_f vanishes on $\mathcal{C}_\varphi$. This shows **CC1**. □

With this lemma one proves as in Theorem 2.3 and Theorem 3.1:

Theorem 4.4. *The map $\varphi \mapsto (\mathcal{C}_\varphi, \mathcal{F}_\varphi)$ is a bijection between surjective integral submersions and colored subspaces. In this case*

$$0 \longrightarrow \mathcal{C}_\varphi \longrightarrow \mathcal{Q}(G/H) \xrightarrow{\varphi_*} \mathcal{Q}(G/H') \longrightarrow 0$$

is exact. Furthermore, $\mathcal{V}(G/H') = \varphi_*(\mathcal{V}(G/H))$ and $\mathcal{D}(G/H') = \complement\mathcal{F}_\varphi$.

Let $\varphi : G/H \to X$ be an arbitrary integral submersion and let $Y \subseteq X$ be an orbit. We define

$$\mathcal{C}_Y(X)^c = (\mathcal{C}_Y(X), \mathcal{F}_Y(X)) = \left(\varphi_*^{-1}(\mathcal{C}_Y'(X)), \varphi_*^{-1}(\mathcal{F}_Y'(X)) \cup \mathcal{F}_\varphi\right)$$

where the prime $'$ denotes the data with respect to G/H'. Finally, let $\mathfrak{F}(X)$ be the set of all $\mathcal{C}_Y^c(X)$, where Y runs through all orbits of X. Combining all the results we obtain

Theorem 4.5. *The functor $X \mapsto \mathfrak{F}(X)$ is an equivalence between the category of integral submersions and the category of colored fans.*

Theorem 4.4 has also the following interpretation:

Corollary 4.6. *Let $H \subseteq G$ be connected. Then there is a order-preserving bijection between intermediate connected subgroups and colored subspaces.*

Let for example $X = G/B$. Then $\mathcal{Q} = \mathcal{V} = 0$ and the elements of $\mathcal{F}$ correspond to Schubert varieties of codimension one and therefore to simple roots. Thus a colored subspace is simply a set of simple roots, in agreement with the classification of parabolic subgroups containing B.

5. The set of invariant valuations

In this section we consider the structure of $\mathcal{V}$ inside $\mathcal{Q}$. For this, let $g_1, \ldots, g_s \in k(G/H)^{(B)}$, $h \in k[G]^{(B\times H)}$ with $f_i = hg_i \in k[G]$ and let M_i be the G-module generated by f_i. As in the first part of Theorem 1.6 one proves: For any $f' \in (M_1 \cdots M_s)^{(B)}$ holds $f'/h^s \in k(G/H)^{(B)}$ and

$$v(f'/h^s) \geq \sum_i v(g_i)$$

in case $v \in \mathcal{V}$. Let

$$\delta = \delta(g_1, \ldots, g_s, h, f') := \sum_i \chi_{g_i} - \chi_{f'/h^s} \in \Lambda.$$

Then the above inequality is equivalent to $v(\delta) \leq 0$. Let Δ be the set of all δ's where $g_1, \ldots, g_s, h, f'$ run through all possible choices.

Lemma 5.1. ([Pau4] 2.1 Prop.) $\mathcal{V} = \{v \in \mathcal{Q} \mid v(\delta) \leq 0$ *for all* $\delta \in \Delta\}$.

Proof: We have already seen "$\subseteq$". For the converse we repeat the construction from the proof of Theorem 3.1 for $\mathcal{C} = \mathbb{Q}^+ v$ and $\mathcal{F} = \varnothing$. Thus let f_0, g_i, f_i, X', X_0' and $\mathcal{M}$ as in that proof. Then $\mathcal{M} \subseteq k[X_0']^{(B)}$ and we want to prove the converse. Let M_i be the module spanned by f_i. Any function $g' \in k[X_0']^{(B)}$ is of the form f'/f_0^N with

$$f' \in (M_0^{n_0} \cdots M_s^{n_s})^{(B)}$$

and $n_i \in \mathbb{N}$; $N = \sum_i n_i$. By the definition of Δ we have

$$\delta = \sum_i n_i \chi_{g_i} - \chi_{g'} \in \Delta,$$

which implies $v(\chi_{g'}) \geq \sum_i n_i v(\chi_{g_i}) \geq 0$. Thus $\mathcal{M} = k[X_0']^{(B)}$.

The same reasoning as in Theorem 3.1 implies that $G/H \to X'$ has finite fibers. Thus we can construct X, X_0 with $k[X_0]^{(B)} = k[X_0']^{(B)}$. From $k[X_0]^{(B)} \neq k(G/H)^{(B)}$ and $D \cap X_0 = \varnothing$ for all $D \in \mathcal{D}$ we get $X \neq G/H$. Thus there is a closed orbit $Y \subsetneqq X$. Let $v' \in \mathcal{V}$ have Y as center. Then v' is not negative on $\mathcal{M}$ and therefore $v \in \mathbb{Q}^+ v'$. $\square$

Definition: An embedding X is called *toroidal* if no $D \in \mathcal{D}$ contains a G-orbit in its closure, or equivalently if $\mathcal{F}_Y(X) = \varnothing$ for all G-orbits Y.

Lemma 5.2. ([BP] 3.1) *There exists a complete toroidal embedding of* G/H.

Proof: Choose $f_0 \in k[G]^{(B\times H)}$ vanishing on any divisor $\pi^{-1}(D)$ with $D \in \mathcal{D}$ and let $V \subseteq k[G]$ be the G-module generated by f_0. This induces a G-morphism $\psi : G/H \to \mathbf{P}(V^\vee)$. Let X' be the closure of the image. We may regard f_0 as a linear function on $V^\vee$. Let L be its set of zeros in $\mathbf{P}(V^\vee)$. Then the fact that V is generated by f_0

translates to $\bigcap_{g\in G} L^g = \varnothing$ or equivalently to: L contains no G-orbit. By construction $\psi(D) \subseteq L$ for any $D \in \mathcal{D}$. Now choose any complete embedding X''. Then embed G/H in $X' \times X''$ diagonally, close it up, normalize it and the result is a complete toroidal embedding. □

Let $\mathcal{Q}_0 := \mathrm{Hom}(\mathcal{X}(B), \mathbb{Q})$ and

$$C := \{v \in \mathcal{Q}_0 \mid v(\alpha) \leq 0 \text{ for all positive roots } \alpha\}$$

be the antidominant Weyl chamber. Then $\Lambda \subseteq \mathcal{X}(B)$ yields $\mathcal{Q}_0 \twoheadrightarrow \mathcal{Q}$. We denote the image of C by $C_{G/H}$.

Corollary 5.3. ([BP] 3.2) *The set of valuations $\mathcal{V}$ is a finitely generated cone containing $C_{G/H}$.*

Proof: The definition of Δ implies that any $\delta \in \Delta$ is the sum of positive roots. Thus it follows from Lemma 5.1 that $\mathcal{V}$ is a cone containing $C_{G/H}$. By Lemma 5.2, **CC1** and Theorem 4.2 it is finitely generated. □

This corollary implies in particular that $\mathcal{V}$ spans $\mathcal{Q}$ as a vector space. In case G is a torus there are no roots and therefore $\mathcal{V} = \mathcal{Q}$. This shows that $\mathcal{V}$ need not be strictly convex. Toroidal embeddings are classified by ordinary fans in $\mathcal{Q}$ whose support is in $\mathcal{V}$. For $\mathrm{char}\, k = 0$ it follows from the local structure theorem [BLV] that toroidal embeddings are also toroidal in the sense of [KKMS] Chap. II §1.

In characteristic zero a much more precise description of $\mathcal{V}$, due to Brion [Bri4], is available. Let $T \subseteq B$ be a maximal torus, $W = N_G(T)/C_G(T)$ the Weyl group, $N(\Lambda) \subseteq W$ the stabilizer of $\Lambda \subseteq \mathcal{X}(B) = \mathcal{X}(T)$ and $W(\Lambda)$ the group of automorphisms of Λ induced by $N(\Lambda)$. Then $W(\Lambda)$ acts on $\mathcal{Q}$ as well.

Theorem 5.4. ($\mathrm{char}\, k = 0$)[1] *There is a subgroup $W_{G/H} \subseteq W(\Lambda)$ generated by reflections such that $\mathcal{V}$ is a fundamental domain for the action of $W_{G/H}$ on $\mathcal{Q}$.*

This theorem implies in particular that $\mathcal{V}$ is a cosimplicial cone, i. e. there exist linear independent forms $\chi_1, \ldots, \chi_s \in \mathcal{Q}^\vee$ such that $\mathcal{V} = \{v \in \mathcal{Q} \mid \chi_i(v) \geq 0 \text{ for all } i\}$. The proof in [Bri4] is rather involved and rests ultimately on a case-by-case consideration. For a conceptual but equally involved proof see [Kno3], where the embedding theory of G/H is linked to the geometry of the moment map on the cotangent bundle $T^*_{G/H}$. This theory gives also some clue how to calculate $W_{G/H}$. Let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras of G, H respectively.

Theorem 5.5. ([Kno1] Kor. 7.2; $\mathrm{char}\, k = 0$) *There is a canonical degree-preserving isomorphism*

$$S(\mathfrak{g}/\mathfrak{h})^H = S(\mathcal{Q})^{W_{G/H}} \otimes_{\mathbb{Q}} k.$$

[1] I conjecture that this theorem is true in arbitrary characteristic.

Consider for example the case of a symmetric variety. Let $\vartheta \in \operatorname{Aut} G$ with $\vartheta^2 = \mathrm{id}_G$ and $H = G^\vartheta$ be the set of fixed points. Choose a ϑ-stable maximal torus T of G such that $T \cap H$ has minimal dimension. For any Borel subgroup B containing T, $BH \subseteq G$ is open [CP]. In particular Λ may be identified with $\mathcal{X}(A)$, where $A = T/(T \cap H)$. It is well known (see [VIN1]) that $S(\mathfrak{g}/\mathfrak{h})^H$ may be identified with $S(\operatorname{Lie} A)^{W(\Lambda)}$. Thus the above theorem implies $W_{G/H} = W(\Lambda) =$"the little Weyl group of G/H"; in particular $\mathcal{V} = C_{G/H}$. This result has been obtained by Vust [VUS] in a different way.

6. Further properties

In this section we derive some properties of a spherical embedding which can be read off from its colored fan.

Theorem 6.1. ([BP] 5.2) *The G-automorphism group $A = N_G(H)/H$ of G/H is an extension of a diagonalizable group by a finite p-group. In particular, the connected component of the identity A^0 is a central torus. Furthermore*[1], $\dim A = \dim \mathcal{V} \cap (-\mathcal{V})$.

Proof: Let $L \subseteq A$ be a connected, H' its preimage in $N_G(H) \subseteq G$ and $\varphi : G/H \to G/H'$. Then H' corresponds to the colored subspace $(\mathcal{C}_\varphi, \mathcal{F}_\varphi)$. Because any automorphism leaves the open B-orbit stable we have $Bx_0 = \varphi^{-1}(B\varphi(x_0))$. This implies $\mathcal{F} = \varnothing$ and therefore $0 \neq \mathcal{C}_\varphi \subseteq \mathcal{V} \cap (-\mathcal{V})$. For $L = A^0$, this implies $\dim A \leq \dim \mathcal{V} \cap (-\mathcal{V})$.

Choose L one dimensional and any $f \in k(G/H)^{(B)}$ such that χ_f does not vanish on $\mathcal{C}_\varphi$, i. e. $f \notin k(G/H')^{(B)}$. Then f is a nonconstant invertible function on $L = \varphi^{-1}(\varphi(x_0)) \subseteq Bx_0$. This implies that L and therefore A^0, is a torus. From $A \subseteq \operatorname{Aut}_B Bx_0$ we get that A is a subquotient of B. This implies the assertion on A.

Assume $0 \neq v \in \mathcal{V} \cap (-\mathcal{V})$ and let $H' \supseteq H$ be the subgroup corresponding to the colored subspace $(\mathbb{Q}v, \varnothing)$. By choosing a function f as above we obtain an invertible function on H' via $H' \to H'x_0 \subseteq Bx_0$. This function is a character χ times a constant (see for example [KKV] 1.2). The kernel of χ corresponds to a proper subspace of $\mathbb{Q}v$ and therefore coincides with H. This shows that H' normalizes H. Because a connected group of automorphisms of H' acts trivially on $\mathcal{X}(H')$, it follows $N_G(H')^0 \subseteq N_G(H)^0$. By induction we get

$$\dim \mathcal{V} \cap (-\mathcal{V}) = 1 + \dim N_G(H')/H' \leq \dim N_G(H)/H \leq \dim \mathcal{V} \cap (-\mathcal{V}). \quad \square$$

This theorem shows that G/H has a simple completion if and only if its automorphism group is finite. In that case there is even a canonical one, namely the completion $\overline{X}$ with $\mathcal{C}^c(\overline{X}) = (\mathcal{V}, \varnothing)$. It is maximal in the sense that it dominates any other simple completion. There is a conjecture of Brion [BRI4] which states:

If $\operatorname{char} k = 0$ and $\operatorname{Aut}_G G/H = 1$, then $\overline{X}$ is smooth.

[1] This is the only result in this paper for which the separability of the orbit map π is essential.

The fact that $\mathcal{V}$ is cosimplicial is already equivalent to the fact that $\overline{X}$ has at worst abelian quotient singularities ([BLV], $\operatorname{char} k = 0$).

For symmetric varieties, the simple completions have been constructed by Satake [Sat]. Brion's conjecture is true in that case by the results of De Concini and Procesi [CP].

There is also the following corollary due to Pauer [Pau4] which describes the other extreme.

Corollary 6.2. *The spherical variety G/H is horospherical, i. e. H contains a maximal unipotent subgroup U of G, if and only if $\mathcal{V} = \mathcal{Q}$.*

Proof: The normalizer of U is B. Therefore Theorem 6.1 implies $\mathcal{V}(G/U) = \mathcal{Q}(G/U)$. We may assume H to be connected. For $H \supseteq U$ the assertion follows from Theorem 4.4.

Assume now $\mathcal{V} = \mathcal{Q}$ and let P be the subgroup corresponding to $(\mathcal{V}, \varnothing)$. Then H is normal in P with a torus as quotient and $\operatorname{rk} G/P = 0$. Thus P is parabolic and H horospherical. $\square$

Now we determine some properties of the orbits. We may assume X to be simple with closed orbit Y.

Theorem 6.3. ([Bri9] 1.2) $\qquad \operatorname{rk} Y = \operatorname{rk} X - \dim \mathcal{C}(X)$.

Proof: Let $L \subseteq \Lambda$ be the set of characters vanishing on $\mathcal{C}(X)$. Then restriction of B-eigenfunctions gives $L \hookrightarrow \Lambda_Y$. By Theorem 1.3b, this inclusion surjective up to p-torsion. $\square$

Next we calculate the dimension of Y. One case is easy: If $\mathcal{F}(X) = \varnothing$ and $\dim \mathcal{C}(X) = 1$ then either Y is a divisor or $X = Y$ is the quotient of G/H by a one dimensional torus. In both cases $\dim Y = \dim G/H - 1$. This can be generalized to:

Lemma 6.4. *Let $(\mathcal{C}, \mathcal{F})$ be a colored cone, $(\mathcal{C}', \mathcal{F}')$ a face with $\mathcal{F}' = \mathcal{F}$ and let X, X' be the corresponding varieties with closed orbits Y, Y' respectively. Then $\dim Y = \dim Y' - (\dim \mathcal{C} - \dim \mathcal{C}')$.*

Proof: By induction we may assume $\dim \mathcal{C} - \dim \mathcal{C}' = 1$. Since we have a natural inclusion $X' \hookrightarrow X$, we can form the normalization $\widetilde{Y}'$ of the closure of Y' in X. Let $\widetilde{Y}$ be the preimage of Y in $\widetilde{Y}'$. This gives the following diagram:

$$\begin{array}{ccccc} \widetilde{Y} & \hookrightarrow & \widetilde{Y}' & & \\ \downarrow & & \downarrow & & \\ Y & \hookrightarrow & \overline{Y'} & \hookrightarrow & X \end{array}$$

Then $\widetilde{Y}'$ is a simple spherical embedding with closed orbit $\widetilde{Y}$. Let $\widetilde{D} \in \mathcal{F}(\widetilde{Y}')$. It follows easily from Theorem 1.3 that the image of $\widetilde{D}$ in X is contained in a $D \in \mathcal{D}(X)$ with $\widetilde{Y}' \not\subseteq D$. Therefore $\mathcal{F} = \mathcal{F}'$ implies $\mathcal{F}(\widetilde{Y}') = \varnothing$. From Theorem 6.3 follows $\operatorname{rk} \widetilde{Y} = \operatorname{rk} \widetilde{Y}' - 1$ and therefore $\dim \mathcal{C}(\widetilde{Y}') = 1$. $\square$

For any $D \in \mathcal{D}$, let P_D be the stabilizer of D. Since $B \subseteq P_D$, this is a parabolic subgroup of G. For any subset $\mathcal{F} \subseteq \mathcal{D}$, let $P_{\mathcal{F}} = \bigcap_{D \in \mathcal{F}} P_D$.

Lemma 6.5. *Assume* $\dim \mathcal{C}(X) = \operatorname{rk} X$. *Then any isotropy group of* Y *is conjugate to the opposite parabolic of* $P_{\complement\mathcal{F}(X)}$.

Proof: Because of its dimension, $\mathcal{C}(X)$ is never a proper face of some other cone. This implies that Y is complete and thus that the isotropy groups are parabolic. Let By_0 be the open orbit in Y and P its stabilizer. This is the parabolic opposite to G_{y_0}. Then Theorem 2.1c implies $P_{\complement\mathcal{F}(X)} \subseteq P$.

Let $D \in \complement\mathcal{F}(X)$. The natural map $P \times^B D \to X$ is proper, and hence PD is closed in X. On the other hand $y_0 \notin PD$, which implies $PD = D$. This shows $P \subseteq P_{\complement\mathcal{F}(X)}$. □

Combining all these results we obtain:

Theorem 6.6. ([Bri9] 1.2) *The dimension of* Y *is given by*

$$\operatorname{rk} G/H - \dim \mathcal{C}_Y(X) + \dim G/P_{\complement\mathcal{F}(X)}.$$

Proof: Observe that any colored cone is the face of a cone of maximal dimension having the same set of colors $\mathcal{F}$. □

Finally I give a criterion for a spherical embedding to be affine:

Theorem 6.7. *A spherical embedding* X *is affine if and only if* X *is simple and there exists a* $\chi \in \Lambda$ *with*

$$\chi_{|\mathcal{V}} \leq 0; \qquad \chi_{|\mathcal{C}(X)} = 0; \qquad \chi_{|\varrho(\complement\mathcal{F}(X))} > 0.$$

Proof: Let $\mathcal{C}_0$ be the cone spanned by $\mathcal{C}(X) \cup \varrho(\mathcal{D})$. First, let X be affine. Then all G-invariants are constant which implies that there is only one closed orbit Y, i. e. X is simple. For any $v \in \mathcal{V} \setminus \mathcal{C}(X)$ there exists by Corollary 1.7 an $f \in k[X]^{(B)}$ vanishing on all $D \in \mathcal{D}(X)$ with $v(f) < 0$. This shows $\mathcal{C}_0 \cap \mathcal{V} = \mathcal{C}(X) \cap \mathcal{V} =: \mathcal{C}_1$. Let L be the subspace spanned by $\mathcal{C}_1$ and α the projection $\mathcal{Q} \twoheadrightarrow \mathcal{Q}/L$. The convexity of $\mathcal{C}_1$ implies that $\alpha(\mathcal{C}_0)$ and $\alpha(\mathcal{V})$ are cones intersecting only in the origin. Also by Corollary 1.7 there is an $f \in k[X]^{(B)}$ vanishing on all $D \in \complement\mathcal{F}(X)$ but not on Y. This shows that $\alpha \circ \varrho(\complement\mathcal{F}(X))$ does not intersect $\alpha(\mathcal{C}(X))$. All that implies the existence of χ.

Conversely, let X and χ satisfy the above conditions and let $A = k[X]$ be the algebra of global functions on X. Then $f \in k(G/H)^{(B)}$ is in A if and only if $\chi_f \in \mathcal{C}_0^\vee$. This shows that A^U is finitely generated where U is the unipotent radical of B. As in [Gro2] Thm. 4 this implies that A is finitely generated as well. Let $X' := \operatorname{Spec} A$. Because X maps to X' we have $\mathcal{C}(X) \subseteq \mathcal{C}(X')$ and $\mathcal{F}(X) \subseteq \mathcal{F}(X')$. Furthermore $k[X]^{(B)} = k[X']^{(B)}$ implies that also $\mathcal{C}(X') \cup \varrho(\mathcal{D})$ spans $\mathcal{C}_0$. In particular $\mathcal{C}(X') \subseteq \mathcal{C}_0$. It follows from the assumptions on χ that $\mathcal{C}(X)$ is the maximal face of $\mathcal{C}_0$ having property **CC2**. Thus $\mathcal{C}(X') \subseteq \mathcal{C}(X)$. These assumptions also imply that $\mathcal{F}(X)$ is the set of all $D \in \mathcal{D}$ with $\varrho(D) \in \mathcal{C}(X)$. Thus $\mathcal{F}(X') \subseteq \mathcal{F}(X)$, and $X = X'$ is affine. □

This theorem implies in particular that X is affine if $\mathcal{F}(X) = \mathcal{D}$, because in this case one can choose $\chi = 0$. On the other hand, we get that G/H is affine if and only if $\mathcal{V}$ and $\varrho(\mathcal{D})$ can be separated by a hyperplane. Finally, G/H is quasiaffine if and only if $\varrho(\mathcal{D})$ does not contain 0 and spans a strictly convex cone.

7. References

[Akh1] Akhiezer, D. N.: Equivariant completion of homogeneous algebraic varieties by homogeneous divisors. *Ann. Global Anal. Geom.* **1** (1983), 49–78

[Akh2] Akhiezer, D. N.: Actions with a finite number of orbits. *Func. Anal. Appl.* **19** (1985), 1–4

[Bri1] Brion, M.: Quelques propriétés des espaces homogènes sphériques. *Manuscr. Math.* **55** (1986), 191–198

[Bri2] Brion, M.: Sur l'image de l'application moment. In: *Séminaire d'Algèbre Paul Dubreil et Marie-Paule Malliavin.* Lect. Notes Math. **1296**, Heidelberg-Berlin-New-York: Springer-Verlag 1987

[Bri3] Brion, M.: Classification des espaces homogènes sphériques. *Compos. Math.* **63** (189–208), 1987

[Bri4] Brion, M.: Vers une généralisation des espaces symétriques. *Preprint, Inst. Fourier* (to appear in *J. Algebra*) **79** (1987)

[Bri5] Brion, M.: Groupe de Picard et nombres caractéristiques des variétés sphériques. *Duke Math. J.* **58** (1989), 397–424

[Bri6] Brion, M.: Spherical varieties: An introduction. In: *Topological methods in algebraic transformation groups.* (H. Kraft, T. Petrie and G. Schwarz eds.) Prog. Math. **80**, Basel-Boston-Berlin: Birkhäuser 1989, 11–26

[Bri7] Brion, M.: On spherical varieties of rank one. *CMS Conf. Proc.* **10** (1989), 31–41

[Bri8] Brion, M.: Une extension du théorème de Borel-Weil. *Math. Ann.* **286** (1990), 655–660

[Bri9] Brion, M.: Sur la géométrie des variétés sphériques. *Preprint* (1990)

[BL] Brion, M.; Luna, D.: Sur la structure locale des variétés sphériques. *Bull. Soc. Math. Fr.* **115** (1987), 211–226

[BLV] Brion, M.; Luna, D.; Vust, Th.: Espaces homogènes sphériques. *Invent. Math.* **84** (1986), 617–632

[BP] Brion, M.; Pauer, F.: Valuations des espaces homogènes sphériques. *Comment. Math. Helv.* **62** (1987), 265–285

[CP] De Concini, C.; Procesi, C.: Complete symmetric varieties I. In: *Invariant theory, Proceedings, Montecatini.* (F. Gherardelli ed.) Lect. Notes Math. **996**, Heidelberg-Berlin-New York: Springer-Verlag 1983, 1–44

[Dem] Demazure, M.: Sous-groupes algébriques de rang maximum du groupe de Cremona. *Ann. Sci. Éc. Norm. Supér.* **3** (1970), 507–588

[Gro1] Grosshans, F.: The invariants of unipotent radicals of parabolic subgroups. *Invent. Math.* **73** (1983), 1–9

[Gro2] Grosshans, F.: Hilbert's fourteenth problem for non-reductive groups. *Math. Z.* **193** (1986), 95–103

[GS] Guillemin,V.; Sternberg, S.: Multiplicity-free spaces. *J. Differ. Geom.* **19** (1984), 31–56

[Hab] Haboush, W.: Reductive groups are geometrically reductive. *Ann. Math.* **102** (1975), 67–83

[HW] Huckleberry, A.; Wurzbacher, T.: Multiplicity-free complex manifolds. *Math. Ann.* **286** (1990), 261–280

[Kac] Kac, V.: Some remarks on nilpotent orbits. *J. Algebra* **64** (1980), 190–213

[KKMS] Kempf, G.; Knudson, F.; Mumford, D.; Saint-Donat, B.: Toroidal embeddings, I. Lect. Notes Math. **339**, Heidelberg-Berlin-New York: Springer-Verlag 1973

[Kno1] Knop, F.: Weylgruppe und Momentabbildung. *Invent. Math.* **99** (1990), 1–23

[Kno2] Knop, F.: Invariante Bewertungen. *Preprint* (1990)

[Kno3] Knop, F.: Weylgruppe und äquivariante Einbettung. *Preprint* (1990)

[KKLV] Knop, F.; Kraft, H.; Luna, D.; Vust, Th.: Local properties of algebraic group actions. In: *Algebraische Transformationsgruppen und Invariantentheorie.* (H. Kraft, P. Slodowy, T. Springer eds.) DMV Semin. **13**, Basel-Boston-Berlin: Birkhäuser 1989, 63–76

[KKV] Knop, F.; Kraft, H.; Vust, Th.: The Picard group of a G-variety. In: *Algebraische Transformationsgruppen und Invariantentheorie.* (H. Kraft, P. Slodowy, T. Springer eds.) DMV Semin. **13**, Basel-Boston-Berlin: Birkhäuser 1989, 77–88

[Krä] Krämer, M.: Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen. *Compos. Math.* **38** (1979), 129–153

[Lun] Luna, D.: Report on spherical varieties. *Manuscript* (1986)

[LV] Luna, D.; Vust, Th.: Plongements d'espaces homogènes. *Comment. Math. Helv.* **58** (1983), 186–245

[Mik] Mikityuk, I. V.: On the integrability of invariant hamiltonian systems with homogeneous configuration spaces. *Math. USSR, Sb.* **57** (1987), 527–546

[MO] Miyake, K.; Oda, T.: Almost homogeneous algebraic varieties under algebraic torus actions. In: *Manifolds—Tokyo, 1973.* (A. Hattori, ed.) Tokyo: Univ. of Tokyo Press 1975, 373–381

[Pau1] Pauer, F.: Normale Einbettungen von G/U. *Math. Ann.* **257** (1981), 371–396

[Pau2] Pauer, F.: Glatte Einbettungen von G/U. *Math. Ann.* **262** (1983), 421–429

[Pau3] Pauer, F.: Plongements normaux de l'espace homogène $SL(3)/SL(2)$. *C. R. 108*e *Congrès nat. Soc. sav.* **3** (1983), 87–104

[Pau4] Pauer, F.: "Caracterisation valuative" d'une classe de sous-groupes d'un groupe algébrique. *C. R. 109*e *Congrès nat. Soc. sav.* **3** (1984), 159–166

[Pau5] Pauer, F.: Normale Einbettungen von sphärischen homogenen Räumen. In: *Algebraische Transformationsgruppen und Invariantentheorie.* (H. Kraft, P. Slodowy, T. Springer eds.) DMV Semin. **13**, Basel-Boston-Berlin: Birkhäuser 1989, 145–156

[Pau6] Pauer, F.: Über gewisse G-stabile Teilmengen in projektiven Räumen. *Manuscr. Math.* **66** (1989), 1–16

[Pan] Panyushev, D. I.: Complexity and rank of homogeneous spaces (in russian). *Dokl. Akad. Nauk. SSSR* **307** (1989), 276–279

[Sat] Satake, I.: On representations and compactifications of symmetric Riemannian spaces. *Ann. Math.* **71** (1960), 77–110

[Sum] Sumihiro, H.: Equivariant completion. *J. Math. Kyoto Univ.* **14** (1974), 1–28

[Vin1] Vinberg, E. B.: The Weyl group of a graded Lie algebra. *Math. USSR, Izv.* **10** (1977), 463–495

[Vin2] Vinberg, E. B.: Complexity of action of reductive groups. *Funct. Anal. Appl.* **20** (1986), 1–11

[VK] Vinberg, E. B.; Kimel'fel'd, B. N.: Homogeneous domains on flag manifolds and spherical subgroups. *Func. Anal. Appl.* **12** (1978), 168–174

[Vus] Vust, Th.: Plongements d'espaces symétriques algébriques: une classification. *Preprint* (to appear in *Ann. Sc. Norm. Super. Pisa*) (1987)

8. Notations

k	Algebraically closed field
p	Characteristic exponent of k
$k[X]$	Algebra of regular functions on a variety X
$k(X)$	Field of rational functions
G	Connected reductive group
B	Borel subgroup of G
G/H	Homogeneous variety such that B has an open orbit in G/H
Bx_0	Open B-orbit in G/H
π	Orbit map $G \to G/H$
X	Normal G-variety together with an open embedding $G/H \hookrightarrow X$
$\mathcal{X}(S)$	Group of characters of a group S
$V^{(S)}$	Set of common eigenvectors of S in an S-module V
χ_v	Character of S corresponding to $v \in V^{(S)}$
$\Lambda = \Lambda_{G/H}$	$\{\chi_f \in \mathcal{X}(B) \mid f \in k(G/H)^{(B)}\}$
$\operatorname{rk} X$	$\operatorname{rk} \Lambda_{G/H}$ the rank of X
$\mathcal{Q} = \mathcal{Q}(G/H)$	$\operatorname{Hom}(\Lambda_{G/H}, \mathbb{Q})$
$\mathcal{V} = \mathcal{V}(G/H)$	Set of G-invariant, $\mathbb{Q}$-valued valuations of $k(G/H)$. By Corollary 1.8 this set is regarded as a subset of $\mathcal{Q}(G/H)$.
$\mathcal{D}(X)$	Set of B-invariant prime divisors in X
$\mathcal{D}$	$\mathcal{D}(G/H)$
$\complement\mathcal{F}$	$\mathcal{D} \setminus \mathcal{F}$
$\mathcal{D}_Y(X)$	Set of B-invariant prime divisors in X containing the orbit Y
$\mathcal{B}_Y(X)$ [1]	$\{v_D \in \mathcal{V}(G/H) \mid D \in \mathcal{D}_Y(X) \text{ is } G\text{-stable}\}$
$\mathcal{F}_Y(X)$ [1]	$\{D \cap G/H \in \mathcal{D}(G/H) \mid D \in \mathcal{D}_Y(X) \text{ is not } G\text{-stable}\}$
ϱ_v	Evaluation map $[f \mapsto v(f)] \in \mathcal{Q}(G/H)$ with $f \in k(G/H)^{(B)}$
ϱ	Map $\mathcal{D}(G/H) \to \mathcal{Q}(G/H) : D \mapsto \varrho_{v_D}$
$\mathcal{C}^\vee$	Dual cone: $\{\alpha \in \mathcal{Q}^\vee \mid \alpha_{\mid\mathcal{C}} \geq 0\}$
$\mathcal{C}^\circ$	Relative interior: $\{v \in \mathcal{C} \mid \alpha \in \mathcal{C}^\vee, \alpha(v) = 0 \Rightarrow \alpha_{\mid\mathcal{C}} = 0\}$
$\mathcal{C}_Y(X)$ [1]	Cone in $\mathcal{Q}(G/H)$ generated by $\mathcal{B}_Y(X)$ and $\varrho(\mathcal{F}_Y(X))$
$\mathcal{C}^c_Y(X)$ [1]	Pair $(\mathcal{C}_Y(X), \mathcal{F}_Y(X))$
$\mathfrak{F}(X)$	$\{\mathcal{C}^c_Y(X) \mid Y \subseteq X \text{ is an orbit}\}$
$C_{G/H}$	Image of the antidominant Weyl chamber in $\mathcal{Q}(G/H)$
$P_\mathcal{F}$	Common stabilizer of all divisors in $\mathcal{F} \subseteq \mathcal{D}(G/H)$

[1] If X is a simple embedding with Y as unique closed orbit we drop the subscript $_Y$.