Proceedings of Symposia in Pure Mathematics
Vol. 33 (1979), part 1, pp. 213–251

FORMS OF GL(2) FROM THE ANALYTIC POINT OF VIEW

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Introduction. Suppose G is a reductive group defined over an A-field F, ρ_{cusp} is the representation of $G(\boldsymbol{A})$ in the space of cusp forms, and φ is a well-behaved function on $G(\boldsymbol{A})$. Then φ defines a trace-class operator $\rho_{\text{cusp}}(\varphi)$, and the goal of the trace formula is to give an explicit formula for this trace. In most cases, one takes $\varphi = \prod_v \varphi_v$ and wants to compute tr $\rho_{\text{cusp}}(\varphi)$ in terms of *local invariant distributions*. Although it is desirable to have *only* invariant distributions, this is not necessary for all applications. In any case, contrary to common belief, the goal is *not* to express the trace in terms of characters of irreducible representations. Indeed the distribution $\varphi(e)$ always appears, and one is quite content to leave it as is.

In the anisotropic case, ρ_{cusp} is actually an induced representation. In this case a general technique—valid for all induced representations and all cocompact groups—gives the trace of $\rho_{\text{cusp}}(\varphi)$ in terms of local orbital integrals. In general, ρ_{cusp} is a subrepresentation of an induced one ρ.

Our purpose here is to describe the analytic theory of the trace formula for forms of GL(2). Since the trace of $\rho_{\text{cusp}}(\varphi)$ is always computed as the integral of the kernel which defines it, we need an explicit formula for this kernel. For GL(2) this turns out to be the kernel of $\rho(\varphi)$ minus the sum and integral of the product of two Eisenstein series. But since only the difference of these kernels is integrable—not one or the other—we need to use a truncation process. For this we closely follow J. Arthur's exposition ([**Ar 1**] and [**Ar 2**]). Any improvement in clarity over earlier references (such as [**DL**], [**JL**], or [**Ge**]) should be credited to him.

A large part of these notes—§§3 through 5—is devoted to the analytic continuation of Eisenstein series. In particular, nearly complete proofs are given for the analytic continuation, functional equation, and location of poles of the *constant term* of the Eisenstein series. This analysis enters precisely because we have to subtract off the continuous spectrum of $\rho(\varphi)$ before we can compute its trace.

§1 is included to show how easy things are when there is no continuous spectrum. §8 is included to give some idea of the power and range of applicability of the trace formula; here discussions with D. Flath on his thesis [**Fl**] were invaluable.

1. Division algebras. First we derive a statement of the trace formula for a division

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AMS (MOS) subject classifications (1970). Primary 22E55; Secondary 10D99.
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algebra whose rank is the square of a prime. Then we give a simple, but in a sense typical, application.

A. *Some generalities.* Let F denote a number field and H a division algebra of degree p^2 over H. The multiplicative group G of H may be regarded as an F-group whose center Z is isomorphic to F^x. Set $\tilde{G} = G/Z$. If ω is a character of $F^x \backslash A^x$ denote by $L^2(\omega, G)$ the space of (classes of) functions f on $G(A)$ such that

$$f(\gamma z g) = \omega(z) f(g), \qquad \gamma \in G(F),\ z \in Z(A) \tag{1.1}$$

and

$$\int_{\tilde{G}(F)\backslash\tilde{G}(A)} |f(g)|^2 \, dg < +\infty. \tag{1.2}$$

Let ρ_ω denote the natural representation of $G(A)$ in $L^2(\omega, G)$ given by right translation.

Suppose now that φ is a function on $G(A)$ satisfying

$$\varphi(zg) = \omega^{-1}(z)\varphi(g) \tag{1.3}$$

for z in $Z(A)$. Suppose also that φ is C^∞ and of compact support mod $Z(A)$. More precisely, let c_i, $1 \leqq i \leqq p^2$, be a basis of H over F. Then, for almost all v, the module generated over the ring of integers R_v of F_v is a maximal order O_v. In particular, $K_v = O_v^*$ is a maximal compact subgroup of H_v^x for almost all v. We assume that $\varphi(g) = \prod \varphi_v(g_v)$ where, for each v, φ_v is a C^∞-function of compact support satisfying the analogue of (1.3); moreover, for almost all v,

$$\begin{aligned} \varphi_v(g_v) &= \omega_v^{-1}(z_v) \quad \text{if } g_v = k_v z_v,\ k_v \in K_v,\ z_v \in Z_v, \\ &= \quad 0 \qquad \text{otherwise.} \end{aligned}$$

Then the operator

$$\rho_\omega(\varphi) = \int_{G(A)} \varphi(g)\rho_\omega(g)\, dg \tag{1.4}$$

is an integral operator in $L^2(\tilde{G}(F)\backslash\tilde{G}(A))$ with kernel

$$K(x, y) = \sum_{\gamma \in \tilde{G}(F)} \varphi(x^{-1}\gamma y). \tag{1.5}$$

When x and y lie in fixed compact sets the sum in (1.5) is actually finite. Therefore $K(x, y)$ is continuous.

THEOREM (1.6). *The operator $\rho_\omega(\varphi)$ is of trace class.*

PROOF. Since $\tilde{G}(F)\backslash\tilde{G}(A)$ is compact, and $K(x, y)$ is continuous, $\rho_\omega(\varphi)$ is at least of Hilbert-Schmidt class. But φ can be written as $\sum \varphi_i^* f_i$ with f_i, φ_i satisfying the same conditions as φ (except at infinity where f_i may only be of class C^m with m large; cf. [**DL**, p. 199]). Thus $\rho_\omega(\varphi)$ is a sum of *products* of Hilbert-Schmidt operators, and hence is of trace class.

COROLLARY (1.7). *The representation ρ_ω decomposes as a (discrete) direct sum of irreducible representations each occurring with finite multiplicity.*

PROOF. According to (1.6) the operator $\rho_\omega(\varphi)$ is compact for well-behaved φ.

Thus the corollary follows from a basic result of functional analysis (cf. **[GGPS]** or **[La 1]**.

If we denote by $m(\pi)$ the multiplicity (possibly zero) with which the irreducible representation π of $G(A)$ occurs in ρ_ω then we have

$$\operatorname{tr} \rho_\omega(\varphi) = \sum m(\pi) \operatorname{tr} \pi(\varphi). \tag{1.8}$$

Note that each component of ρ_ω perforce admits ω as central character, i.e., $\pi(z) = \omega(z)I$, $z \in Z(A)$. Thus the sum in (1.8) need only be extended over representations of $G(A)$ with central character ω. On the other hand, we can also compute the trace of $\rho_\omega(\varphi)$ in terms of the kernel K, viz.,

$$\operatorname{tr} \rho_\omega(\varphi) = \int_{\tilde{G}(F)\backslash\tilde{G}(A)} K(x, x)\, dx. \tag{1.9}$$

B. *The trace formula.* From now on assume that p is a prime. Then any element of $G(F) - Z(F)$ is regular. On the other hand, if $\gamma = e$ (in $\tilde{G}(F)$) then $\varphi(x^{-1}\gamma x) = \varphi(e)$. Thus we may rewrite the right-hand side of (1.9) as

$$\int K(x, x)\, dx = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \int \sum_{\gamma \neq e} \varphi(x^{-1}\gamma x)\, dx. \tag{1.10}$$

But every element ξ of $G(F) - Z(F)$ generates an extension L of F of degree p in H. Moreover, if ξ' is conjugate to ξ in $G(F)$, then the extension L' generated by ξ' is F-isomorphic to L, and there is an F-isomorphism of L' onto L taking ξ' to ξ. Thus, if we let X be a set of representatives for the isomorphism classes of extensions of degree p of F which imbed in H, any element $\gamma \neq e$ of $\tilde{G}(F)$ can be expressed in the form

$$\gamma = \eta^{-1}\xi\eta \tag{$*$}$$

where ξ is in $L^\times/F^\times - \{e\}$ for some L in X and η belongs to a set of representatives for

$$(F^\times\backslash L^\times)\backslash\tilde{G}(F) \simeq L^\times\backslash G(F).$$

Note that the extension $L \in X$ is uniquely determined by γ and, if g_L is the number of F-automorphisms of L, γ admits g_L decompositions like $(*)$. Thus

$$K(x, x) = \varphi(e) + \sum_{L \in X} (g_L)^{-1} \sum_\xi \sum_\eta \varphi(x^{-1}\eta^{-1}\xi\eta x), \qquad \xi \in L^\times/F^\times - \{e\},\ \eta \in L^\times\backslash G(F)$$

and

$$\int K(x, x)\, dx = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \sum_{L \in X} (g_L)^{-1} \sum_\xi \int_{L^\times Z(A)\backslash G(A)} \varphi(x^{-1}\xi x)\, dx.$$

In the last integral, the integrand depends only on the class of x mod $L^\times(A)$. Thus we also have

$$\begin{aligned}\int K(x, x)\, dx &= \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \sum_{L \in X} (g_L)^{-1} \operatorname{vol}(F^\times(A)L^\times\backslash L^\times(A)) \\ &\qquad \sum_{\xi \in (L^\times - F^\times)/F^\times} \int_{L^\times(A)\backslash\tilde{G}(A)} \varphi(x^{-1}\xi x)\, dx = \operatorname{tr} \rho_\omega(\varphi).\end{aligned} \tag{1.11}$$

In this formula the measures on $\tilde{G}(A)$ and $F^\times(A)\backslash L^\times(A)$ are chosen arbitrarily and $L^\times(A)\backslash\tilde{G}(A)$ is given the quotient measure.

Suppose now that the measure on $L^\times(A)\backslash\tilde{G}(A)$ is a product measure; given ξ, for almost all v, ξ is in K_v and $\varphi_v(x_v^{-1}\xi x_v) = 0$ unless $x_v \in K_v L_v^\times$. Thus we have

$$\int \varphi(x^{-1}\xi x)\,dx = \prod_v \int_{L_v^\times\backslash G_v} \varphi_v(x_v^{-1}\xi x_v)\,dx_v$$

almost all factors being equal to $\mathrm{vol}(L_v^\times\backslash L_v^\times K_v)$.

Thus we have expressed $\mathrm{tr}\,\rho_\omega(\varphi)$ in terms of the local invariant distributions $\varphi_v \mapsto \varphi_v(e)$ and $\varphi_v \mapsto \int_{L_v^\times\backslash G_v}\varphi_v(x_v^{-1}\xi x_v)\,dx_v$.

C. *An application.* Let H' denote another division algebra of degree p^2 and suppose H' and H both fail to split at the same set of places S. For $v \notin S$ there is an isomorphism $H_v \to H_v'$ which is uniquely determined up to inner automorphism. For almost all v, we may suppose that this isomorphism takes O_v to a similarly defined maximal order O_v' (and hence $K_v = O_v^\times$ to $K_v' = (O_v')^\times$). The resulting isomorphisms $G_v \to G_v'$ then give rise to an isomorphism of the restricted product groups $G^S = \prod_{v\notin S} G_v$ and $G'^S = \prod_{v\notin S} G_v'$ which is again determined up to inner automorphism.

Let V denote the space of functions on $L^2(\omega, G)$ which are invariant under $G_S = \prod_{v\in S} G_v$. Since G^S and G_S commute, V is invariant for the action of G^S. Thus we have a representation σ of G^S on V and (similarly) a representation σ' of G'^S on V'. Via the isomorphism $G^S \to G'^S$ we may transport σ' to a representation of G^S which we again denote by σ'. (Its class does not depend on the choice of isomorphism.)

THEOREM (1.12). *Suppose $\omega_v = 1$ for all $v \in S$. Then the representations σ and σ' are equivalent.*

PROOF. Let $\theta = \prod_{v\notin S}\theta_v$ be any function on G^S satisfying conditions analogous to those satisfied by φ in §1.A. To prove our theorem it will suffice to show that

$$\mathrm{tr}\,\sigma(\theta) = \mathrm{tr}\,\sigma'(\theta). \tag{1.13}$$

Indeed a basic result in harmonic analysis asserts that σ is a subrepresentation of σ' if and only if

$$\mathrm{tr}\,\sigma(f)\sigma(f)^* \leqq \mathrm{tr}\,\sigma'(f)\sigma'(f)^* \tag{1.14}$$

for sufficiently many "nice" f. Cf. Lemma 16.1.1 of [JL]; it will suffice to apply (1.13) with $\theta = \theta_1 * \theta_1^*$ and $\theta_1^*(g) = \bar{\theta}_1(g^{-1})$.

To prove (1.13), extend θ to a function φ on $G(A)$ by setting $\varphi(g) = \prod_{v\notin S}\theta(g_v)$ and extend θ similarly to φ' by identifying G'^S with G^S. Now take the volume of $\tilde{G}_v$ (resp. $\tilde{G}_v'$) to be one for each $v \in S$ and the measure on $\tilde{G}(A)$ (resp. $\tilde{G}'(A)$) to be a product measure. Furthermore, assume that the isomorphism $G_v \to G_v'$ takes the Haar measure of G_v to the Haar measure of G_v'. Then we find that

$$\mathrm{tr}\,\rho_\omega(\varphi) = \mathrm{tr}\,\sigma(\theta) \quad (\text{resp. } \mathrm{tr}\,\rho_\omega'(\varphi') = \mathrm{tr}\,\sigma'(\theta)). \tag{1.15}$$

Thus (1.13) holds if and only if

$$\mathrm{tr}\,\rho_\omega(\varphi) = \mathrm{tr}\,\rho_\omega'(\varphi'). \tag{1.16}$$

To prove (1.16) we apply the trace formula (1.11) as follows.

Assume that the measure on $F^x(A)\backslash L^x(A)$ is a product measure so that

$$\int_{L^x(A)\backslash\tilde{G}(A)} \varphi(x^{-1}\xi x)\,dx = \left(\prod_{v\in S}\int_{L_v^x\backslash\tilde{G}_v} \theta_v(x^{-1}\xi x)\,dx_v\right)\prod_{v\in S}\operatorname{vol}(L_v^x\backslash\tilde{G}_v).$$

A similar identity holds for $\tilde{G}'$, and the extensions of F which imbed in H or H' are the same. Thus (1.11) implies that

$$\text{(1.17)} \qquad \operatorname{tr}\rho_\omega(\varphi) - \operatorname{tr}\rho_\omega(\varphi') = (c - c')\theta(e).$$

Here $c = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))$ and c' is defined similarly. To prove the theorem it remains to show $c = c'$.

Suppose $c > c'$. If $\theta = \theta_1 * \theta_1^*$ with $\theta_1^*(g) = \bar{\theta}_1(g^{-1})$, then the right side of (1.17) is strictly positive. So by (1.15)—and the result described by (1.14)—σ' must be a subrepresentation of σ, and the quotient representation $\pi = \sigma/\sigma'$ must satisfy the identity

$$\operatorname{tr}(\pi(\theta_1)\,\pi(\theta_1)^*) = (c - c')\,\|\,\theta_1\,\|_2$$

for all nice θ_1.

This implies that the regular representation of G^S is quasi-equivalent to π and hence that the regular representation of G^S decomposes discretely. Since the same is then also true of the regular representation of each G_v, $v \notin S$, we obtain an obvious contradiction. The assumption $c < c'$ yields a similar contradiction and the theorem is proved.

REMARK (1.18). Suitably modified, the proof above shows that $\tilde{G}$ and $\tilde{G}'$ have the same Tamagawa number. The restriction on ω is not really necessary.

2. Cusp forms on GL(2). Henceforth G will denote the group GL(2). The center Z of G is

$$Z = \left\{\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right\}.$$

Again $Z \cong F^x$ and we set $\tilde{G} = G/Z$. We also introduce the subgroups

$$P = \left\{\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}\right\}, \quad A = \left\{\begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}\right\}, \quad N = \left\{\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix}\right\}.$$

If φ is a function on $G(F)\backslash G(A)$ we say that φ is cuspidal if

$$\int_{N(F)\backslash N(A)} \varphi(ng)\,dn \equiv 0, \qquad g \in G(A).$$

As in §1, we introduce a space $L^2(\omega, G)$ and a representation ρ_ω. We denote by $L_0^2(\omega, G)$ the subspace of cuspidal elements in $L^2(\omega, G)$.

THEOREM (2.1). *Let $\rho_{\omega,0}$ denote the restriction of ρ_ω to the invariant subspace $L_0^2(\omega, G)$. Let φ denote a C^∞-function which satisfies* (1.3) *and is of compact support* mod $Z(A)$. *Then the operator $\rho_{\omega,0}(\varphi)$ is Hilbert-Schmidt.*

SKETCH OF PROOF. For each $c > 0$, recall that a Siegel domain $\mathcal{S}$ in $G(A)$ is a set of points of the form

$$g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} tab & 0 \\ 0 & a \end{pmatrix}k$$

where x is in a compact subset of A, t is an idele whose finite components are trivial and whose infinite components equal some fixed number $u > c > 0$, a is arbitrary in $A^\times$, b lies in a compact subset of $A^\times$, and k is in the standard maximal compact subgroup K of $G(A)$. If $\mathscr{G}$ is sufficiently large then $G(A) = G(F)\mathscr{G}$. For a proof of this fact see **[Go 3]**.

By abuse of language, if $\mathscr{G}$ is a Siegel domain in $G(A)$, we call its image in $N(F)\backslash G(A)$ a Siegel domain in $N(F)\backslash G(A)$. If $\mathscr{G}$ is such a domain, we denote by $L^2(\omega, \mathscr{G})$ the space of functions on $\mathscr{G}$ such that $f(zg) = \omega(z)f(g)$ and $\int_{N(F)Z(A)\backslash\mathscr{G}} |f(g)|^2\, dg < +\infty$. Clearly $L^2(\omega, G)$ can be identified with a closed subspace V of $L^2(\omega, \mathscr{G})$ *if* $\mathscr{G}$ is large enough. In this case there is a bounded operator with bounded inverse A from $L^2(\omega, G)$ onto V.

If f is in $L^2(\omega, G)$ and φ is as in Theorem (2.1), then

$$\rho_\omega(\varphi)f(x) = \int_{N(F)\backslash \bar{G}(A)} H(x, y)f(y)\, dy$$

with

$$H(x, y) = \sum_{\xi\subset N(F)} \varphi(x^{-1}\,\xi y).$$

However, if f is actually cuspidal, we also have

$$\rho_\omega(\varphi)f(x) = \int \tilde{H}(x, y)f(y)\, dy \tag{2.2}$$

with

$$\tilde{H}(x, y) = \sum_{\xi\subset N(F)} \varphi(x^{-1}\,\xi y) - \int_{N(A)} \varphi(x^{-1}ny)\, dn. \tag{2.3}$$

Indeed f cuspidal implies

$$\int_{N(F)\backslash\bar{G}(A)} f(y) \int_{N(A)} \varphi(x^{-1}\, ny)\, dn\, dy = 0.$$

To prove the theorem we need to estimate $\tilde{H}(x, y)$.

If x is in a fixed Siegel domain $\mathscr{G}$ (of $N(F)\backslash G(A)$) it is easy to see that $\tilde{H}(x, y) = 0$ unless y is in another such domain $\mathscr{G}'$. Then using the fact that the term subtracted in (2.3) is precisely the Fourier transform of $\varphi(x^{-1}(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix})y)$ at 0 we see that

$$\iint_{(N(F)Z(A)\backslash\mathscr{G})\,(N(F)Z(A)\backslash\mathscr{G}')} |\tilde{H}(x, y)|^2\, dx\, dy < +\infty,$$

i.e., $\tilde{H}$ defines a Hilbert-Schmidt operator B from $L^2(\omega, \mathscr{G}')$ to $L^2(\omega, \mathscr{G})$. For a detailed discussion of this type of argument see **[Go 1]** or **[La 1]**.

Now enlarge $\mathscr{G}'$ (if necessary) so that $G(A) = G(F)\mathscr{G}'$. Then $\rho_{\omega,0}(\varphi)$ can be written as the composition of the injection $L_0^2(\omega, G) \to L^2(\omega, \mathscr{G}')$, the operator B, the projection of $L^2(\omega, \mathscr{G})$ onto V, and the operator A^{-1}. This proves $\rho_{\omega,0}(\varphi)$ itself is Hilbert-Schmidt.

COROLLARY (2.4). *$\rho_{\omega,0}(\varphi)$ is of trace class.*

PROOF. The conclusion of (2.1) is valid if φ is highly (as opposed to infinitely) differentiable at infinity. Thus one argues as in §1.A (cf. [**DL**, p.199]).

COROLLARY (2.5). *The representation $\rho_{\omega,0}$ decomposes discretely with finite multiplicities.*

3. *P*-series. Suppose f is a function on $G(A)$ satisfying

$$f(n\gamma zg) = \omega(z)f(g) \tag{3.1}$$

with $n \in N(A)$, $\gamma \in P(F)$, and $z \in Z(A)$. Then we call the series

$$F(G) = \sum_{P(F)\backslash G(F)} f(\gamma g) \tag{3.2}$$

a *P-series.* To study the convergence of such a series we need a well-known lemma. Let $H(g)$ be the function on $G(A)$ defined by

$$H(g) = \left|\frac{a}{b}\right| \quad \text{if } g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}k.$$

LEMMA (3.3) (CF. [**JL**, P.197]). *Fix a Siegel domain $\mathfrak{S}$ and a positive number c'. Then the set of γ in $G(F)$ such that $H(\gamma g) \geqq c'$ for some $g \in \mathfrak{S}$ is finite modulo $P(F)$.*

From this lemma we see that if the support of f is contained in a set $\{(g: H(g) \geqq c'\}$—in particular if it is compact mod $N(A)Z(A)P(F)$—then the series (3.2) is finite. Moreover, in the latter case, the function F has compact support mod $G(F)Z(A)$.

Our goal in this section is to analyze the orthocomplement of $L_0^2(\omega, G)$ in terms of these P-series.

A. *The scalar product of two P-series.* First we compute the scalar product of a P-series with a function ψ satisfying (1.1):

$$\begin{aligned}
(\psi, F) &= \int_{\bar{G}(F)\backslash\bar{G}(A)} \psi(g)\bar{F}(g)\,dg \\
&= \int_{\bar{G}(F)\backslash\bar{G}(A)} \psi(g) \sum_{P(F)\backslash G(F)} \bar{f}(\gamma g)\,dg \\
&= \int \Sigma\, \psi(\gamma g)\bar{f}(\gamma g)\,dg = \int_{P(F)Z(A)\backslash G(A)} \psi(g)\bar{f}(g)\,dg \\
&= \int_{P(F)N(A)Z(A)\backslash G(A)} dg \int_{N(F)\backslash N(A)} \psi(ng)\bar{f}(g)\,dg,
\end{aligned}$$

i.e.,

$$(\psi, F) = \int_{P(F)N(A)Z(A)\backslash G(A)} \psi_N(g)\bar{f}(g)\,dg \tag{3.4}$$

where ψ_N is the *constant term* of ψ:

$$\psi_N(g) = \int_{N(F)\backslash N(A)} \psi(ng)\,dn. \tag{3.5}$$

From (3.4) it follows that if ψ is cuspidal then $(\psi, F) = 0$ for any P-series F.

Conversely, if $\phi \in L^2(\omega, G)$ is orthogonal to all P-series (with f of compact support mod $N(A)P(F)Z(A)$), then $\phi_N = 0$ and ϕ is cuspidal. Thus the P-series (with f of compact support mod ...) span a dense subspace of the orthocomplement of $L_0^2(\omega, G)$.

Now we compute the scalar product of two P-series. According to (3.4) we need first to compute the constant term of F. But by Bruhat's decomposition for GL(2), $F(g) = f(g) + \sum_{\gamma \in N(F)} f(w\gamma g)$ with $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. Thus

$$F_N(g) = \int_{N(F)\backslash N(A)} f(ng)\, dn + \int_{N(F)\backslash N(A)} dn \sum f(w\gamma ng).$$

Normalizing the Haar measure on $N(A)$ by the condition $\mathrm{vol}(N(F)\backslash N(A)) = 1$ we see that

$$F_N(g) = f(g) + f'(g) \tag{3.6}$$

where

$$f'(g) = \int_{N(A)} f(wng)\, dn. \tag{3.7}$$

Thus

$$(F_1, F_2) = \int f_1 \cdot \bar{f}_2(g)\, dg + \int f_1' \cdot \bar{f}_2(g)\, dg \tag{3.8}$$

where both integrals extend over the space $P(F)N(A)Z(A)\backslash G(A)$.

To further analyze formula (3.8) we need to carefully investigate the map $f \mapsto f'$. Note that f' still satisfies (3.1). Note also that all the computations above are merely formal unless certain assumptions on f_1 and f_2 are made to insure convergence. For instance, if both f_1 and f_2 have compact support (mod ...) then these computations *are* valid. Under these same conditions, however, the support of f_1' need no longer (in general) be compact.

Another formal relation which is easy to prove is this:

$$\int f_1'(g)\bar{f}_2(g)\, dg = \int f_1(g)\bar{f}_2'(g)\, dg. \tag{3.9}$$

Indeed the left- (resp. right-) hand side of (3.9) may be written as

$$\int_{P(F)Z(A)\backslash G(A)} f_1(wg)\bar{f}_2(g)\, dg$$

(resp. $\int_{P(F)Z(A)\backslash G(A)} f_1(g)\bar{f}_2(wg)\, dg$) and these integrals are equal since w normalizes $P(F)Z(A)$.

B. *Induced representations and intertwining operators.* To investigate the map $f \mapsto f'$ (and the scalar product (3.8)) we need to perform a Mellin transform on the group $Z(A)A(F)\backslash A(A)$. Right now it will suffice to deal with a subgroup of this group. Thus we let F_∞^+ denote the group of ideles whose finite components all equal 1 and whose infinite components all equal some positive number u (independent of the infinite place). By $F^0(A)$ we denote the ideles of norm 1 and by A^0 (resp. A_∞^+) the group of diagonal matrices with entries in $F^0(A)$ (resp. F_∞^+). For convenience *we assume that ω is trivial on F_∞^+.* We also normalize Haar measures as follows.

The Haar measure on F_∞^+ is obtained by transporting the usual measure on $\boldsymbol{R}_+^\times$ through the map $t \mapsto |t|$. Thus

$$\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} ds \int_{F_\infty^+} f(t)|t|^{-s}\, d^\times t = f(e).$$

Recall that the measure on $N(A)$ is chosen so that $\mathrm{vol}(N(F)\backslash N(A)) = 1$. On $A^\times$ we select any Haar measure and give $F^\times\backslash A^\times$ the quotient measure. Then we use the isomorphism $F^\times\backslash A^\times \cong F_\infty^+ \times F^\times\backslash F^0(A)$ to get a Haar measure on $F^\times\backslash F^0(A)$ (which in general will not have volume 1). We also use the isomorphism $\left(\begin{smallmatrix} a & 0 \\ 0 & 1\end{smallmatrix}\right) \mapsto a$ to get a measure on $Z(A)\backslash A(A)$. Finally we select measures on $\bar{G}(A)$ and K such that

$$\int_{\bar{G}(A)} f(g)\, dg = \int_K \int_{A^\times} \int_{N(A)} f\left[\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k\right] dx |a|^{-1}\, d^\times a\, dk. \tag{3.10}$$

Note that vol (K) is not necessarily 1.

Now for each complex number s we introduce a Hilbert space $\mathbf{H}(s)$ of (classes of) functions φ on $G(A)$ such that

$$\varphi\left[\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v\end{pmatrix} g\right] = \omega(a)\left|\frac{u}{v}\right|^{s+1/2} \varphi(g) \tag{3.11}$$

for $\alpha, \beta \in F^\times$, $a \in A^\times$, $u, v \in F_\infty^+$, and $x \in A$. Such functions are completely determined by their restriction to the set of matrices of the form

$$g = \begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k, \qquad a \in F^0(A),\ k \in K. \tag{3.12}$$

For φ to belong to $\mathbf{H}(s)$ we require that

$$\int_K \int_{F^\times\backslash F^0(A)} |\varphi|^2\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix} k\right] da\ dk < +\infty. \tag{3.13}$$

The restriction of φ to matrices of the form (3.12) satisfies

$$\varphi\left[\begin{pmatrix} \alpha a b & 0 \\ 0 & 1\end{pmatrix} k\right] = \varphi\left[\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} b & 0 \\ 0 & 1\end{pmatrix} k\right] \tag{3.14}$$

each time $\alpha \in F^\times$ and $\left(\begin{smallmatrix} b & 0 \\ 0 & 1\end{smallmatrix}\right) \in A(A) \cap K$. Moreover, the group $G(A)$ operates on $\mathbf{H}(s)$ by right translation and the resulting representation is denoted π_s. This representation is continuous but not always unitary. It is, however, unitary when s is purely imaginary.

We may think of the collection $\mathbf{H}(s)$ as a holomorphic fibre bundle of base $\boldsymbol{C}$. The sections over an open set U of $\boldsymbol{C}$ are the functions $\varphi(g, s)$ on $G(A) \times U$ satisfying

$$\varphi\left[\begin{pmatrix} \alpha a u & x \\ 0 & \beta a v\end{pmatrix} g, s\right] = \omega(a)\left|\frac{u}{v}\right|^{s+1/2} \varphi[g, s].$$

Of course this bundle is trivial since every φ in $\mathbf{H}(s)$ is uniquely determined by its restriction to the set of matrices of the form (3.12). Thus we may think of $\mathbf{H}(s)$ as the Hilbert space of functions (*independent of* s) satisfying (3.13) and (3.14).

Put another way, we may set $\mathbf{H} = \mathbf{H}(0)$. Then every element φ of $\mathbf{H}$ defines a section of our bundle over $\boldsymbol{C}$, namely $(g, s) \mapsto \varphi(g)H(g)^s$.

With this understanding let f be a function satisfying (3.1), say of compact support mod $N(A)Z(A)P(F)$. Then define the *Mellin transform* of f by

$$\hat{f}(g, s) = \int_{F_\infty^+} f\left[\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right] |t|^{-s-1/2} d^\times t.$$

Since the integrand has compact support, $\hat{f}$ clearly defines a section of our fibre bundle. The Haar measure is chosen so that

$$f(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \hat{f}(g, s)\, ds$$

with x any real number.

Now it is easy to see that the first term of the inner product formula (3.8) is

$$\begin{aligned} &\int_K \int_{(F^\times \backslash F^0(A))} \int_{F_\infty^+} f_1 \cdot \bar{f}_2\left[\begin{pmatrix} at & 0 \\ 0 & 1 \end{pmatrix} k\right] |t|^{-1}\, d^\times t\, da\, dk \\ &= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \int_{F^\times \backslash F^0} \int_K \hat{f}_1\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k, s\right] \bar{\hat{f}}_2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k, -\bar{s}\right] da\, dk\, ds \\ &= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} (\hat{f}_1(s), \hat{f}_2(-\bar{s}))\, ds. \end{aligned}$$

Here $\hat{f}_1(s)$ denotes the value of the function $\hat{f}_1(g, s)$ at s and the scalar product $(\hat{f}_1(s), \hat{f}_2(-\bar{s}))$ is taken by identifying all fibres with $\mathbf{H}$. Alternatively, observe that if φ_1 is in $\mathbf{H}(s)$ and φ_2 is in $\mathbf{H}(-\bar{s})$ then $\varphi_1\bar{\varphi}_2$ transforms on the left according to the modular function of the group $N(A)Z(A)A_\infty^+A(F)$. Thus if we set

$$(\varphi_1, \varphi_2) = \int_{F^\times \backslash F^0(A)} \int_K \varphi_1 \cdot \bar{\varphi}_2\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right] da\, dk \tag{3.15}$$

we obtain a nondegenerate sesquilinear pairing between $\mathbf{H}(s)$ and $\mathbf{H}(-\bar{s})$ such that

$$(\pi_s(g)\varphi_1, \pi_{-\bar{s}}(g)\varphi_2) = (\varphi_1, \varphi_2). \tag{3.16}$$

This is the pairing which appears in our formula. (For more details see [**Go 2**, pp. 1.24–1.26].)

Similarly we find that the second term of (3.8) is

$$(2\pi i)^{-1} \int_{x-i\infty}^{x+i\infty} (\hat{f}_1'(-s), \hat{f}_2(\bar{s}))\, ds.$$

Here we must integrate on a line where $\hat{f}_1'(-s)$ is given by a convergent integral, i.e., we must have $x > \frac{1}{2}$. It remains then to compute $\hat{f}'(-s)$ in terms of $\hat{f}(s)$.

For $\mathrm{Re}(s) > \frac{1}{2}$ we find that

$$\begin{aligned} \hat{f}'(-s) &= \int f'\left[\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} g\right] |t|^{s-1/2}\, d^\times t \\ &= \int |t|^{s-1/2}\, d^\times t \int f\left[w \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & t^{-1}x \\ 0 & 1 \end{pmatrix} g\right] dx \end{aligned}$$

$$= \int |t|^{s+1/2}\, d^\times t \int f\left[\begin{pmatrix}1 & 0\\ 0 & t\end{pmatrix} w \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx$$

$$= \int |t|^{-s-1/2}\, d^\times t \int f\left[\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} w \begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g\right] dx$$

$$= \int \hat{f}\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix} g, s\right] dx.$$

(Recall that ω was assumed to be trivial on F_∞^+; therefore

$$f\left[\begin{pmatrix}1 & 0\\ 0 & t^{-1}\end{pmatrix} g\right] = f\left[\begin{pmatrix}t & 0\\ 0 & 1\end{pmatrix} g\right]$$

for $t \in F_\infty^+$.) If we define an operator $M(s)$ from $\mathbf{H}(s)$ to $\mathbf{H}(-s)$ by

$$M(s)\varphi(g) = \int_{N(A)} \varphi[wng]\, dn \tag{3.17}$$

when $\mathrm{Re}(s) > \frac{1}{2}$, we conclude that

$$\hat{f}'(-s) = M(s)\hat{f}(s). \tag{3.18}$$

Summing up, we find that the scalar product of two P-series F_1, F_2 (with f_1, f_2 of compact support) is given by the formula

$$\begin{aligned}(F_1, F_2) = {}& \frac{1}{2\pi i}\int (\hat{f}_1(s), \hat{f}_2(-\bar{s}))\, ds \\ & + \frac{1}{2\pi i}\int (M(s)\hat{f}_1(s), \hat{f}_2(\bar{s}))\, ds.\end{aligned} \tag{3.19}$$

The integrals are taken over any vertical line with $\mathrm{Re}(s) > \frac{1}{2}$ and the pairings we have written are on $\mathbf{H}(s) \times \mathbf{H}(-\bar{s})$ and $\mathbf{H}(-s) \times \mathbf{H}(\bar{s})$ (or simply $\mathbf{H} \times \mathbf{H}$). It is worth noting that

$$(M(s)\varphi_1, \varphi_2) = (\varphi_1, M(\bar{s})\varphi_2) \tag{3.20}$$

for φ_1 in $\mathbf{H}(s)$ and φ_2 in $\mathbf{H}(\bar{s})$. (This is analogous to the identity (3.9) and proved just the same way.) If we think of $M(s)$ as an operator from $\mathbf{H}$ to $\mathbf{H}$ then (3.20) simply asserts that

$$M(s)^* = M(\bar{s}). \tag{3.21}$$

Note finally that if f is replaced by $h \mapsto f(hg)$ then F (resp. $\hat{f}(s)$) is replaced by its right translate, i.e., by $\rho_\omega(g)F$ (resp. $\pi_s(g)\hat{f}(s)$). On the other hand, it is also clear that

$$M(s)\pi_s(g) = \pi_{-s}(g)M(s). \tag{3.22}$$

These facts play a key role in the next section.

4. Analytic continuation of $M(s)$. Our goal is to use the inner product formula (3.19) to construct an intertwining operator between a subrepresentation of ρ_ω and a continuous sum of the representations π_s *with s purely imaginary*. To this end we need to analytically continue the operator $M(s)$. Indeed the integration in (3.19) is over a line $\mathrm{Re}(s) = x$ with $x > \frac{1}{2}$. So if we want to rewrite this formula using purely imaginary s we need to know something about the poles of $M(s)$.

A. *Prelimminary rearks.* Now it will be more convenient to replace π_s and $M(s)$ with the operators we get by performing a Mellin transform on the whole group $F^\times\backslash A^\times$ rather than just F_∞^+. Let $\eta = (\mu, \nu)$ denote a pair of quasi-characters of $F^\times\backslash A^\times$ with $\mu\nu = \omega$. Let $\mathbf{H}(\eta)$ be the space of functions on $G(A)$ such that

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \mu(a)\nu(b)\left|\frac{a}{b}\right|^{1/2}\varphi(g) \tag{4.1}$$

and

$$\int_K |\varphi|^2(k)\, dk < +\infty. \tag{4.2}$$

Once more we can think of the collection $\mathbf{H}(\eta)$ as a fibre-bundle over the space of pairs η. These pairs make up a complex manifold of dimension 1 with infinitely many connected components and our fibre-bundle is trivial over any such component because the character $\left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right) \mapsto \mu(a)\nu(b)$ of $A(A) \cap K$ is fixed there. In particular, $\mathbf{H}(\eta)$ may be regarded as the subspace of functions in $L^2(K)$ such that

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} k\right] = \mu(a)\nu(b)\varphi(k)$$

when $\left(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix}\right) \in K$. In any case, we denote by π_η the natural representation of $G(A)$ on $\mathbf{H}(\eta)$. The fiber $\mathbf{H}(s)$ decomposes as the sum of the fibres $\mathbf{H}(\eta)$, with $\eta = (\mu, \nu)$, and $\mu \circ \nu^{-1}(a) = |a|^s$ for $a \in F_\infty^+$; the representation π_s decomposes correspondingly as the sum of the "principal series representations π_η"; see [Ge, p. 67].

As far as pairings are concerned, we have a natural one between $\mathbf{H}(\eta)$ and $\mathbf{H}(\bar{\eta}^{-1})$ defined by

$$(\varphi_1, \varphi_2) = \int_K \varphi_1(k)\bar{\varphi}_2(k)\, dk.$$

As before, $(\pi_\eta(g)\varphi_1, \varphi_2) = (\varphi_1, \pi_{\bar{\eta}^{-1}}(g^{-1})\varphi_2)$. Moreover, if we set $\tilde{\eta} = (\nu, \mu)$, then formula (3.17) defines an operator $M(\eta)$ from $\mathbf{H}(\eta)$ to $\mathbf{H}(\tilde{\eta})$ satisfying

$$M(\eta)\pi_\eta(g) = \pi_{\tilde{\eta}}(g)M(\eta) \tag{4.3}$$

and

$$(M(\eta)\varphi_1, \varphi_2) = (\varphi_1, M(\bar{\tilde{\eta}}^{-1})\varphi_2). \tag{4.4}$$

Here φ_1 is in $\mathbf{H}(\eta)$ and φ_2 is in $\mathbf{H}(\bar{\tilde{\eta}}^{-1})$; cf. formulas (3.20) and (3.22). In particular, if we identitfy $\mathbf{H}(\eta)$ with $\mathbf{H}(\bar{\eta}^{-1})$ and $\mathbf{H}(\tilde{\eta})$ with $\mathbf{H}(\bar{\tilde{\eta}}^{-1})$ (keeping in mind that η and $\bar{\eta}^{-1}$ belong to the same connected component) then (4.4) reads

$$M(\eta)^* = M(\bar{\tilde{\eta}}^{-1}). \tag{4.5}$$

Note that (3.21) is essentially a direct sum of identities like (4.5).

To continue analytically $M(\eta)$ we are going to decompose it as a tensor product of local intertwining operators and thereby (essentially) reduce the problem to a local one.

B. *Local intertwining operators.* Let v be a place of F and F_v the corresponding local field. If $\eta_v = (\mu_v, \nu_v)$ is a pair of quasi-characters of $F_v^\times$ such that $\mu_v\nu_v = \omega_v$ we can form the space $\mathbf{H}(\eta_v)$ analogous to $\mathbf{H}(\eta)$, the representation π_{η_v}, and the operator $M(\eta_v)\colon \mathbf{H}(\eta_v) \to \mathbf{H}(\tilde{\eta}_v)$ defined by

$$(M(\eta_v)\varphi)(g) = \int_{F_v} \varphi\left[w\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}g\right]dx. \tag{4.6}$$

Since the group G_v operates by right matrix multiplication on $F_v \times F_v$, and the stabilizer of the line $F_v(0, 1)$ is the group P_v, we may define global sections of the bundle $\mathbf{H}(\eta_v)$ by the formula

$$\varphi(g, \eta_v) = \frac{\mu_v\alpha_v^{1/2}(\det g)}{L(1, \mu_v\nu_v^{-1})}\int_{F_v^\times} \Phi[(0, t)g]\mu_v\nu_v^{-1}(t)\,|t|\, d^\times t. \tag{4.7}$$

Here $L(s, \mu_v\nu_v^{-1})$ is the usual Euler factor attached to the character $\mu_v\nu_v^{-1}$, $\alpha_v(x) = |x|_v$, and Φ is a Schwartz-Bruhat function on $F_v \times F_v$. Recall that if χ is a quasi-character of $F_v^\times$, and f is Schwartz-Bruhat *on* F_v, the integral $\int_{F_v^\times} f(t)\,|t|^s\chi(t)\,d^\times t = Z(f,\chi,s)$ converges for $\mathrm{Re}(s) \gg 0$ (> 0 if χ is unitary) and the ratio $Z(f, \chi, s)/L(s, \chi)$ extends to an entire function of s. (This is Tate's theory of the local zeta-function.) Thus the formula (4.7) actually makes sense for all η and indeed defines a section of our fibre-bundle.

Now we want to apply the operator $M(\eta_v)$ defined by (4.6) to an element of $H(\eta_v)$ given by the section (4.7). After a change of variables we get

$$\omega_v(-1)\frac{\mu_v\alpha_v^{1/2}(\det g)}{L(1, \mu_v\nu_v^{-1})}\int_{F_v}\int_{F_v^\times} \Phi[(t, x)g]\mu_v\nu_v^{-1}(t)\, d^\times t\, dx. \tag{4.8}$$

Next recall the functional equation for the quotient $Z(f, \chi, s)/L(s, \chi)$:

$$\frac{Z(\hat{f}, \chi^{-1}, 1-s)}{L(1-s, \chi^{-1})} = \varepsilon(s, \chi, \psi_v)\,\frac{Z(f, \chi, s)}{L(s, \chi)}.$$

Here $\hat{f}$ denotes the usual Fourier transform taken with respect to the fixed additive character ψ_v, and $\varepsilon(s, \chi, \psi_v)$ is an exponential function which also depends on ψ_v. If we define the Fourier transform of Φ to be

$$\hat{\Phi}(x, y) = \iint \Phi(u, v)\,\psi_v(yu - xv)\,du\,dv$$

we see that (4.8) can be written as the product of

$$\frac{L(0, \mu_v\nu_v^{-1})}{L(1, \mu_v\nu_v^{-1})\,\varepsilon(0, \mu_v\nu_v^{-1}, \psi_v)} \tag{4.9}$$

and

$$\frac{\omega_v(-1)}{L(1, \mu_v^{-1}\nu_v)}\int \hat{\Phi}[(0, t)g]\mu_v^{-1}\nu_v(t)\,|t|\, d^\times t(\nu_v\alpha_v^{1/2}(\det g)). \tag{4.10}$$

Note that we pass from (4.7) to (4.10) by the substitutions $\eta \to \tilde{\eta}$, $\Phi \to \hat{\Phi}$, and multiplication by $\omega_v(-1)$.

Next we write $\eta_v = (\mu_v, \nu_v)$, $\mu_v = \chi_1\alpha_v^{s/2}$, $\nu_v = \chi_2\alpha_v^{-s/2}$, where χ_1 and χ_2 are characters. For $\mathrm{Re}\, s > 0$, we can write $M(\eta_v)$ as the product of the scalar (4.9) with an operator $R(\eta_v)$ which (for each η_v) takes the element (4.7) of $\mathbf{H}(\eta_v)$ to the element (4.10) of $\mathrm{H}(\tilde{\eta}_v)$. But for $\mathrm{Re}(s) > -\frac{1}{2}$, every element of $\mathrm{H}(\eta_v)$ can be represented by an integral of the form (4.7) ([**JL**, pp. 97-98]). An easy argument then shows that $R(\eta_v)$ extends to a holomorphic operator valued function of η_v in the domain $\mathrm{Re}(s) > -\frac{1}{2}$ which again takes (4.7) to (4.10). Moreover, from the obvious relations $(\tilde{\eta})^{\sim} = \eta$, $(\Phi^{\wedge})^{\wedge} = \Phi$, and $\omega_v^2(-1) = 1$, we find that

$$R(\tilde{\eta}_v)R(\eta_v) = \mathrm{Id}. \tag{4.11}$$

Since (4.9) is clearly meromorphic, this also gives the analytic continuation of the operator $M(\eta_v)$ in the domain $\mathrm{Re}(s) > -\frac{1}{2}$. Although we do not need to, we note that $R(\eta_v)$ (and hence $M(\eta_v)$) extends to a meromorphic function of all η_v (or $s \in C$).

It is important to note that the operator $R(\eta_v)$ is "normalized" in the following sense. If η_v is unramified then $\mathrm{H}(\eta_v)$ (resp. $\mathrm{H}(\tilde{\eta}_v)$) contains a unique function φ_v (resp. $\tilde{\varphi}_v$) which is invariant under K_v and equal to one on K_v. Then, provided ψ_v has order zero,

$$R(\eta_v)\varphi_v = \tilde{\varphi}_v. \tag{4.12}$$

Moreover, $R(\eta_v)$ is unitary whenever η_v is unitary. Indeed if $\tilde{\eta}_v = \bar{\eta}_v^{-1}$, the operators $M(\eta_v)$: $\mathrm{H}(\eta_v) \to \mathrm{H}(\tilde{\eta}_v)$, $\omega_v(-1)M(\tilde{\eta}_v)$: $\mathrm{H}(\tilde{\eta}_v) \to \mathrm{H}(\eta_v)$ are adjoint to one another. Therefore, since the scalar (4.9) changes to its imaginary conjugate times $\omega_v(-1)$ when η_v is replaced by $\tilde{\eta}_v$, we conclude from (4.11) that $R(\eta_v)^* = R(\eta_v)^{-1}$.

REMARK (4.13) (ON THE RANGE OF $R(\eta_v)$). If $\mu_v\nu_v^{-1} = \alpha_v$ then $\mu_v = \chi_v\alpha_v^{1/2}$ and $\nu_v = \chi_v\alpha_v^{-1/2}$ with $\chi_v^2 = \omega_v$. In this case the space $\mathrm{H}(\eta_v)$ consists of functions φ satisfying

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \chi_v(ab)\left|\frac{a}{b}\right|\varphi(g),$$

and the kernel of $M(\eta_v)$ (or $R(\eta_v)$) has codimension one. The space $\mathrm{H}(\tilde{\eta}_v)$ consists of functions φ satisfying

$$\varphi\left[\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} g\right] = \chi_v(ab)\varphi(g);$$

so the range of $M(\eta_v)$ (or $R(\eta_v)$) must be the one-dimensional space spanned by the function $g \mapsto \chi_v(\det g)$. Moreover, for the sesquilinear pairing between $\mathrm{H}(\eta)$ and $\mathrm{H}(\tilde{\eta})$,

$$(R(\eta)\varphi_1, \varphi_2) = c'(\varphi_1, \chi \circ \det g)\,\overline{(\varphi_2, \chi \circ \det g)}, \tag{4.14}$$

where c' is a known constant.

C. *Global theory.* Our task is to piece together the local intertwining operators $M(\eta_v)$ in order to analytically continue $M(s)$. First we define an operator $R(\eta)$ as the "infinite tensor product" of the local operators $R(\eta_v)$. If φ in $\mathrm{H}(\eta)$ has the form $\prod \varphi_v$, with φ_v invariant under K_v and equal to one on K_v for almost all v, then $R(\eta)\varphi = \prod_v R(\eta_v)\varphi_v$. Because each $R(\eta_v)$ is normalized (cf. (4.12)), $R(\eta)\varphi$ is indeed well-defined in $\mathrm{H}(\tilde{\eta})$. Moreover, we can write $M(\eta)$ as

$$M(\eta) = \frac{L(0, \mu\nu^{-1})}{L(1, \mu\nu^{-1})\,\varepsilon(0, \mu\nu^{-1})} R(\eta). \tag{4.15}$$

This gives the analytic continuation of $M(\eta)$ since the scalar factor in (4.15) has a known meromorphic behavior and the operator $R(\eta)$ is a meromorphic function of η.

To get the functional equation of $M(\eta)$ we use the functional equation of the L-function $L(s, \mu\nu^{-1})$. This allows us to write the scalar in (4.15) as

$$\frac{L(1, \mu^{-1}\nu)}{L(1, \mu\nu^{-1})}. \tag{4.16}$$

Therefore $M(\eta)$ satisfies the functional equation

$$M(\bar{\eta})M(\eta) = \mathrm{Id}. \tag{4.17}$$

(Cf. (4.11).) By (4.5) we also have

$$M(\eta)^* = M(\eta^{-1}) \quad \text{if } \bar{\eta} = \eta^{-1}. \tag{4.18}$$

Thus $M(\eta)$ is *unitary* when η is unitary. Recall that $R(\eta)$ also satisfies (4.17) and (4.18).

We sum up the analytic behavior of $M(\eta)$ as follows. Write $|\mu\nu^{-1}| = \alpha^t$ with t real. Then $M(\eta)$ is meromorphic in the half-plane $t \geqq 0$ and its only poles there are simple ones which occur for $\mu = \chi\alpha^{1/2}, \nu = \chi\alpha^{-1/2}, \chi^2 = \omega$. The residues are scalar multiples of the operator $R(\eta)$ and the ranges are the one-dimensional spaces spanned by the functions $g \mapsto \chi(\det g)$. Going back to $M(s)$ we get (from (4.14)):

THEOREM (4.19). *As a function of s the operator $M(s)$ is meromorphic in the whole complex plane and satisfies the functional equation*

$$M(-s)M(s) = \mathrm{Id}. \tag{4.20}$$

Its only pole in the half-plane $\mathrm{Re}(s) \geqq 0$ *is at* $s = \frac{1}{2}$ *and the residue there is such that*

$$(\mathrm{Re}\, s_{1/2}\, M(\tfrac{1}{2})\hat{f}_1(\tfrac{1}{2}), \hat{f}_2(\tfrac{1}{2})) = c \sum_{\chi^2=\omega} (\hat{f}_1(\tfrac{1}{2}), \chi \circ \det)\, \overline{(\hat{f}_2(\tfrac{1}{2}), \chi \circ \det)}$$

where c is a known constant.

The last assertion follows from (4.14).

D. *Analysis of the continuous spectrum.* Suppose F_1, F_2 are two P-series belonging to f_1, f_2. We know that F_1 and F_2 lie in the orthocomplement of $L_0^2(\omega, G)$ and their scalar product is given by the formula (3.19). If we use the residue theorem, Theorem (4.19), and some simple estimates, we can shift the integration in (3.19) to the imaginary axis and write

$$\begin{aligned}(F_1, F_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} &\{(\hat{f}_1(iy), \hat{f}_2(iy)) + (M(iy)\hat{f}_1(iy), \hat{f}_2(-iy))\}\, dy \\ &+ c \sum_{\chi^2=\omega} (\hat{f}_1(\tfrac{1}{2}), \chi \circ \det)\, \overline{(\hat{f}_2(\tfrac{1}{2}), \chi \circ \det)}.\end{aligned} \tag{4.21}$$

Here the sum is extended over all characters χ of $F^\times \backslash A^\times$ such that $\chi^2 = \omega$ and the scalar product is the pairing between $\mathbf{H}(\frac{1}{2})$ and $\mathbf{H}(-\frac{1}{2})$. (Note that since ω is assumed to be trivial on F_∞^+, so is χ if $\chi^2 = \omega$; thus $\chi \circ \det$ indeed belongs to $\mathbf{H}(-\frac{1}{2})$.) But (cf. (3.15))

$$(\hat{f}_1(\tfrac{1}{2}), \chi \circ \det) = \int_{P(F)N(A)Z(A)\backslash G(A)} f_1(g)\,\bar{\chi}(\det g)\,dg$$
$$= \int_{G(F)Z(A)\backslash G(A)} F_1(g)\,\bar{\chi}(\det g)\,dg$$

since F_1 is the P-series attached to f_1. Therefore the *second* term in (4.21) can also be written as

$$c \sum_{\chi^2=\omega} (F_1, \chi \circ \det)\,\overline{(F_2, \chi \circ \det)}, \tag{4.22}$$

the scalar product now being taken in $L^2(\omega, G)$.

Now note that (4.20) and (3.21) together imply that, for $y \in \boldsymbol{R}$, $M(iy)^*M(iy) = \mathrm{Id}$. In particular, $M(s)$ is unitary on the imaginary axis and, if we set

$$a(iy) = \tfrac{1}{2}\{\hat{f}(iy) + M(-iy)\hat{f}(-iy)\}, \tag{4.23}$$

then

$$M(-iy)a(-iy) = a(iy), \tag{4.24}$$

and the first term in (4.21) can be written as

$$\frac{1}{\pi}\int_{-\infty}^{\infty} (a_1(iy), a_2(iy))\,dy. \tag{4.25}$$

This is significant for the decomposition of ρ_ω because if we replace f_1 by $h \mapsto f_1(hg)$ then F_1 is replaced by $\rho_\omega(g)F_1$, $y \mapsto \hat{f}(iy)$ by $y \mapsto \pi_s(g)\hat{f}(iy)$, and $y \mapsto a(iy)$ by $y \mapsto \pi_s(h)a(iy)$.

To sum up, let $\mathscr{L}$ denote the Hilbert space of square-integrable functions a on $i\boldsymbol{R}$ with values in H (i.e., square-integrable sections of our bundle over $i\boldsymbol{R}$) *satisfying* (4.24). Equip $\mathscr{L}$ with the inner product (4.25) and let π denote the representation of $G(\boldsymbol{A})$ on $\mathscr{L}$ given by $\pi(g)a(iy) = \pi_s(g)a(iy)$. Because of (4.24), we may also regard $\mathscr{L}$ as the space of square-integrable functions $a(y)$ from $\boldsymbol{R}_+$ to $\mathbf{H}$, the scalar product being given by

$$(a_1, a_2) = \frac{2}{\pi}\int_0^{+\infty} (a_1(y), a_2(y))\,dy.$$

Thus π is a continuous sum of the representations π_{iy}.

For each χ such that $\chi^2 = \omega$ we let $\mathscr{L}_\chi$ denote the space of the one-dimensional representation $g \mapsto \chi(\det g)$. Then, since (F_1, F_2) is the sum of (4.22) and (4.25), it follows that there is an isometric map (with dense domain) from $(L_0^2)^\perp$ to $\mathscr{L} \oplus (\bigoplus_\chi \mathscr{L}_\chi)$. This map (given by $F \mapsto (a(iy), \hat{f}(\tfrac{1}{2}))$) extends by continuity to a map T from $(L_0^2)^\perp$ to a dense subspace of $\mathscr{L} \oplus (\bigoplus_\chi \mathscr{L}_\chi)$ and, by the remarks above, it is also an intertwining operator.

To conclude, we see that $L^2(\omega, G)$ decomposes as a direct sum

$$L_0^2(\omega, G) \oplus L^2_{\mathrm{cont}}(\omega, G) \oplus L^2_{\mathrm{sp}}(\omega, G)$$

where L^2_{sp} is the space spanned by the functions $\chi(\det g)$, i.e., L^2_{sp} is the space $\bigoplus \mathscr{L}_\chi$, and L^2_{cont} is isomorphic to $\mathscr{L}$ via the intertwining operator S: $L^2_{\mathrm{cont}}(\omega, G) \to \mathscr{L}$.

Moreover, (4.22) is the scalar product of the orthogonal projections of F_1 and F_2 on L^2_{sp}. Therefore $c = [\mathrm{vol}(\tilde{G}(F)\backslash\tilde{G}(\boldsymbol{A}))]^{-1}$ and we have computed this volume.

5. Eisenstein series and the truncation process. Let P_{cusp} denote the orthogonal projection onto $L_0^2(\omega, G)$. Since we need an explicit formula for the kernel of the Hilbert-Schmidt operator $P_{\text{cusp}}\rho_\omega(\varphi)P_{\text{cusp}}$ we need an explicit description of the operator S of §4. This description is given in terms of Eisenstein series.

A. *Eisenstein series.* If φ is a section of the fibre-bundle $\mathbf{H}(s)$ we set

$$E(\varphi(s), g) = \sum_{\gamma \in P(F)\backslash G(F)} \varphi(\gamma g, s). \tag{5.1}$$

This series converges only for $\operatorname{Re}(s) > \frac{1}{2}$ and in general has to be defined by analytic continuation.

Recall that if f satisfies (3.1) and has compact support mod $N(A)Z(A)P(F)$ then $f(g) = (2\pi i)^{-1} \int_{x-i\infty}^{x+i\infty} \hat{f}(g, s)\, ds$. Interchanging summation and integration we see that the P-series F defined by f is given by

$$F(g) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}(s), g)\, ds \tag{5.2}$$

provided $x > \frac{1}{2}$. Thus the functions of a dense subspace of $(L_0^2)^\perp$ are "continuous sums of Eisenstein series". However, as before, to obtain a useful formula we must analytically continue E and shift the integration in (5.2) to the imaginary axis.

Since E is a P-series, its constant term is easily computed (cf. (3.6) and (3.7)):

$$E_N(\varphi(s), g) = \varphi(g, s) + \int \varphi[wng, s]\, dn$$

or

$$E_N(\varphi(s), g) = \varphi(s)(g) + [M(s)\varphi(s)](g). \tag{5.3}$$

In both equations, $\operatorname{Re}(s) > \frac{1}{2}$.

To analytically continue E we shall primarily deal with sections obtained from identifying $\mathbf{H}(s)$ with $\mathbf{H}$. In other words, if $h \in \mathbf{H} = \mathbf{H}(0)$, we define a section $h(s)$ by the formula

$$h(g, s) = h(g)H(g)^s. \tag{5.4}$$

The corresponding Eisenstein series will be denoted $E(h(s), g)$.

B. *Properties of the truncation operator.* We shall use the truncation operator to obtain the analytic continuation of E. If $c > 1$ we let χ_c denote the characteristic function of $[c, +\infty)$. For any function φ on $G(F)\backslash G(A)$ set

$$\Lambda^c\varphi(g) = \varphi(g) - \sum_{P(F)\backslash G(F)} \varphi_N(\gamma g)\chi_c(H(\gamma g)). \tag{5.5}$$

The second term here is a P-series attached to a function with support in the set $\{g\colon H(g) > c\}$. Thus if g is in a Siegel domain *the series has only finitely many terms* (cf. Lemma (3.3)).

If φ is a cuspidal function, i.e., $\varphi_N \equiv 0$, then clearly $\Lambda^c\varphi = \varphi$. In general, we need to appeal to the following lemma:

LEMMA (5.6) (CF. [**DL**, P. 197]). *If there is a g in $G(A)$ such that $H(g) > 1$ and $H(\gamma g) > 1$ then γ belongs to $P(F)$.*

This lemma shows that, for a given g, the series in (5.5) has at most one term for $c > 1$. In particular,

$$(5.7)\qquad \begin{aligned} \Lambda^c\varphi(g) &= \varphi(g) - \varphi_N(g)\chi_c(H(g)) && \text{if } H(g) > 1,\ c > 1, \\ &= \varphi(g) - \varphi_N(g) && \text{if } H(g) > c > 1. \end{aligned}$$

Now let $\mathscr{S}$ be a Siegel domain and Ω the set of g in $\mathscr{S}$ such that $H(g) \leqq c$. Then Ω is compact (mod $Z(A)$) and, if $c > 1$, $\Lambda^c\varphi$ is given on $\mathscr{S} - \Omega$ by the second formula in (5.7). Thus $\Lambda^c\varphi(g)$ is bounded on $\mathscr{S} - \Omega$ *under very mild assumptions* on φ. For instance, this is the case if φ is the convolution of a "slowly increasing" or square-integrable function on $G(F)\backslash G(A)$ with a C^∞-function of compact support. Actually (5.7) will then be "rapidly decreasing".

On the other hand, if Ω is any compact set, then

$$(5.8)\qquad \Lambda^c\varphi(g) = \varphi(g) \quad \text{for } g \in \Omega \textit{ and } c \textit{ large.}$$

Indeed suppose $g \in \Omega$ and $\Lambda^c\varphi(g) \neq \varphi(g)$ with $c > 1$. Then there is a finite nonempty set of elements γ of $P(F)\backslash G(F)$ such that $H(\gamma g) > c > 1$ (cf. Lemma (3.3) again; this finite set depends on Ω but not on c). Since the resulting element γg must belong to a compact set mod $P(F)N(A)Z(A)$, and since H is continuous, we must have $H(\gamma g) < c_0$. But if $c > c_0$ we get a contradiction. Therefore (5.8) must hold. In other words, $\Lambda^c\varphi \to \varphi$ uniformly on compact sets as $c \to +\infty$.

We also want to point out that Λ^c is a continuous *hermitian* operator on $L^2(\omega, G)$:

$$(5.9)\qquad (\Lambda^c\varphi_1, \varphi_2) = (\varphi_1, \Lambda^c\varphi_2).$$

Indeed the left-hand side is (φ_1, φ_2) minus the scalar product of a P-series with φ_2. In particular, by (3.4) the left-hand side equals

$$(\varphi_1, \varphi_2) - \int_{P(F)N(A)Z(A)\backslash G(A)} \varphi_{1,N}(g)\chi_c(H(g))\,\overline{\varphi_{2,N}(g)\,dg}.$$

Similarly the right-hand side is $(\varphi_1, \varphi_2) - \int\varphi_{1,N}(g)\bar{\varphi}_{2,N}(g)\bar{\chi}_c(H(g))\,dg$; since χ_c is real the desired equality follows.

We also have, for $c > 1$,

$$(5.10)\qquad ((1 - \Lambda^c)\varphi_1, \Lambda^2\varphi_2) = 0,$$

i.e., *Λ^c is an orthogonal projection in $L^2(\omega, G)$*. Indeed $(1 - \Lambda^c)\varphi_1$ is also a P-series. So the left-hand side of (5.10) is

$$\int_{N(A)P(F)Z(A)\backslash G(A)} \varphi_{1,N}(g)\chi_c(H(g))\left(\int_{N(F)\backslash N(A)} \Lambda^c\varphi_2(ng)\,dn\right)dn.$$

But $\chi_c(H(g)) = 0$ unless $H(g) > c > 1$, in which case the inner integral is (by (5.7))

$$\int_{N(F)\backslash N(A)} (\varphi_2(ng) - \varphi_{2,N}(ng))\,dn = 0$$

and (5.10) follows. (In fact (5.10) is sometimes true even when φ_i does not belong to $L^2(\omega, G)$.)

C. *Analytic continuation of Eisenstein series.* Using the truncation operator (5.5) we can write our Eisenstein series as

$$E(h(s), g) = \Lambda^c E(h(s), g) + \sum_{P(F)\backslash G(F)} E_N(h(s), \gamma g)\chi_c(H(\gamma g)). \tag{5.11}$$

The second term on the right side is the P-series attached to a function with support in the set $\{g: H(g) \geqq c\}$. Thus, as noted before, for g in a Siegel domain the second term has only finitely many terms. In particular, it represents a meromorphic function of s whose singularities are at most those of $M(s)$ (cf. (5.3)).

On the other hand, the first term on the right side of (5.11) is initially defined only for $\mathrm{Re}(s) > \frac{1}{2}$. However, since it is square-integrable (cf. (5.7) and the remarks immediately following it), it will suffice to continue it analytically as a $L^2(\omega, G)$-valued function. Thus we need to examine the inner product

$$(\Lambda^c E(h_1(s_1)), \Lambda^c E(h_2(\bar{s}_2))). \tag{5.12}$$

By (5.10) (which is true in this case) the inner product (5.12) is just $(E(h_1(s_1)), \Lambda^c E(h_2(\bar{s}_2)))$, which, since $\Lambda^c E$ is a difference of P-series, we compute (using (3.4)) to be

$$\int_{N(A)P(F)Z(A)\backslash G(A)} E_N(h_1(s_1), g)\{\bar{h}_2(g, \bar{s}_2) - \bar{E}_N(h_2(\bar{s}_2), g)\chi_c(H(g))\}\, dg.$$

Now recall that E_N is given by (5.3) with $\varphi(g, s) = h(g, s)$ and

$$h\left[\begin{pmatrix} ta & x \\ 0 & 1 \end{pmatrix} k, s\right] = |t|^{s+1/2}\, h\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} k\right]$$

for $t \in F_\infty^+$, $a \in F^0(A)$, and $k \in K$. So by Iwasawa's decomposition, we compute

$$\begin{aligned} &(\Lambda^c E(h_1, (s_1)), \Lambda^c E(h_2(\bar{s}_2))) \\ &\quad = \{(h_1, h_2)c^{s_1+s_2} - (M(s_1)h_1, M(\bar{s}_2)h_2)c^{-(s_1+s_2)}\}\frac{1}{s_1 + s_2} \\ &\qquad + \{(h_1, M(\bar{s}_2)h_2)c^{s_1-s_2} - (M(s_1)h_1, h_2)^{-(s_1-s_2)}\}\frac{1}{s_1 - s_2}. \end{aligned} \tag{5.13}$$

Here $\mathrm{Re}(s_1) > \mathrm{Re}(s_2) > \frac{1}{2}$.

Note that the right side of (5.13) is a meromorphic function of (s_1, s_2) which seems to have a singularity on the line $s_1 + s_2 = 0$. However, by (3.21) and (4.20), the expression in the first bracket vanishes precisely along this line. Similarly the expression in the second bracket vanishes when $s_1 - s_2 = 0$. Thus we conclude that (5.13) is meromorphic in s_1, s_2 with singularities at most those of $M(s_1)$ and $M(s_2)$. In particular, for $s = s_1 = -s_2 \neq 0$,

$$\begin{aligned} (\Lambda^c E(h_1(s)), \Lambda^c E(h_2(-\bar{s}))) &= 2(h_1, h_2)\log c + (M(-s)M'(s)h_1, h_2) \\ &\quad + \{(h_1, M(-\bar{s})h_2)c^{2c} - (M(s)h_1, h_2)c^{-2s}\}\frac{1}{2s}. \end{aligned} \tag{5.14}$$

Now suppose that for $h_1 = h_2 = h$, the right-hand side of (5.13) is analytic in the polydisc $|s_1 - s_0| < R$, $|s_2 - s_0| < R$, while $E(h(s))$ is holomorphic in some small disc centered at s_0. Then the double Taylor series of the left-hand side converges in this polydisc. Since

$$\left\| \frac{\partial^n}{\partial s^n} \Lambda^c E(h(s)) \right\|^2 = \left(\frac{\partial^n}{\partial s^n} \Lambda^c E(h(s)), \frac{\partial^n}{\partial s^n} \Lambda^c E(h(s)) \right)$$
$$= \frac{\partial^{2n}}{\partial s_1^n \partial \bar{s}_2^n} (\Lambda^c E(h(s_1)), \Lambda^c E(h(\bar{s}_2))) \Big\|_{s_1=s_2=s}$$

it is easy to conclude that the Taylor series of $\Lambda^c E(h(s))$ converges in the disc $|s-s_0| < R$, i.e., $\Lambda^c E(h(s))$ is analytic in this disc. Thus we conclude $\Lambda^c E(h(s))$ (and hence $E(h(s), g)$) is meromorphic in $\mathrm{Re}(s) \geqq 0$ with singularities at most those of $M(s)$. Note that in (5.13) and (5.14) we needed to identify $\mathbf{H}(s)$ with $\mathbf{H}$ to define the scalar products and the derivative $M'(s)$ of $M(s)$.

Summing up, we know that if $\varphi(s)$ is any meromorphic section of our bundle $\mathbf{H}(s)$ then $E(\varphi(s), g)$ is defined as a meromorphic function of s (at least for $\mathrm{Re}(s) \geqq 0$). Moreover,

$$E(M(s)\varphi(s), g) = E(\varphi(s), g). \tag{5.15}$$

Indeed since $\varphi(s)$ belongs to $\mathbf{H}(s)$ and $M(s)\varphi(s)$ belongs to $\mathbf{H}(-s)$, the function $g \to E_N(M(s)\varphi(s), g)$ is equal to

$$M(s)\varphi(s) + M(-s)M(s)\varphi(s) = \varphi(s) + M(s)\varphi(s) = E_N(\varphi(s), g).$$

Thus $E_N(M(s)\varphi(s), g) = E_N(\varphi(s), g)$, i.e., the difference between the two sides of (5.15) is a cuspidal function. Therefore, since any P-series (or its analytic continuation) is orthogonal to all cuspidal functions, this difference vanishes as claimed.

In general, say for a group whose derived group has F-rank one, the same facts can be proved. The only difference is that the operator $M(s)$ may have a finite number of poles and these poles are not necessarily known. Nevertheless, the operator $M(s)$ still similarly controls the analytic behavior of the Eisenstein series (cf. **[Ar 1]** and **[La 1]**).

D. *The kernel of* $\rho_\omega(\varphi)$ *in* L^2_{cont}. Suppose $F_1(g)$ is a P-series attached to f_1 (of compact support mod $Z(A)N(A)P(F)$). Our immediate goal is to prove that, for all $h \in \mathbf{H}$,

$$(h, SF_1(iy)) = \frac{1}{2} \int_{G(F)Z(A)\backslash G(A)} E(h(iy), g)\bar{F}_1(g)\, dg \tag{5.16}$$

for almost every y. Here $E(h(iy), g)$ is defined by analytic continuation and $SF_1(iy) = a_1(iy)$ as in (4.23).

Note first that if F_2 is the P-series attached to f_2 we get from (5.2) that

$$(F_1, F_2) = \int_{\bar{G}(F)\backslash \bar{G}(A)} \left\{ \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} E(\hat{f}_1(s), g)\, ds \right\} \bar{F}_2(g)\, dg$$
$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} ds \int_{\bar{G}(F)\backslash \bar{G}(A)} E(\hat{f}_1(s), g)\bar{F}_2(g)\, dg$$

for $\mathrm{Re}(s) = x > \frac{1}{2}$. Shifting the integration to the imaginary axis we then get

$$(F_1, F_2) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dy \int E(\hat{f}_1(iy), g)\bar{F}_2(g)\, dg + c \sum_{\chi^2=\omega} (F_1, \chi)(\chi, F_2). \tag{5.17}$$

But (5.15) says that the integral in (5.17) is unchanged if we replace $\hat{f}_1(iy)$ by $M(iy)\hat{f}_1(iy)$ and then change y to $-y$. Thus, with $a_1(iy)$ as in (4.23), formula (5.17) reads

$$(F_1, F_2) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dy \int E(a_1(iy), g)\bar{F}_2(g)\, dg$$
$$+ c \sum_{\chi^2=\omega} (F_1, \chi)(\chi, F_2).$$

On the other hand, computing the inner product as in (4.21), we also have

$$(F_1, F_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} (a_1(iy), a_2(iy))\, dy + c \sum_{\chi^2=\omega} (F_1, \chi)(\chi, F_2).$$

Thus we conclude that

$$\int_{-\infty}^{\infty} (a(iy), SF(iy))\, dy = \frac{1}{2}\int_{-\infty}^{\infty} \left\{\int_{\tilde{G}(F)\backslash\tilde{G}(A)} E(a(iy), g)\, \bar{F}(g)\, dg\right\} dy \tag{5.18}$$

for all a in $\mathscr{L}$. This formula is then also true for any square-integrable function $a(y)$ with values in $\mathbf{H}$ since both sides remain unchanged when $a(iy)$ is replaced by $\frac{1}{2}(a(iy) + M(-iy)a(-iy))$. Thus we may take $a(iy) = c(y)h(iy)$ with c a scalar function and h a "constant section" to conclude from (5.18) that (5.16) indeed holds.

Now extend the map $S\colon L^2_{\mathrm{cont}}(\omega, G) \to \mathscr{L}$ to $S\colon L^2(\omega, G) \to \mathscr{L}$ by setting it equal to 0 on L^2_0 and $L^2_{\mathrm{sp}}(\omega, G)$. Then S^*S is the orthogonal projection of $L^2(\omega, G)$ onto L^2_{cont}. Since S is an intertwining operator, $S^*S\rho_\omega(\varphi)S^*S = S^*\pi(\varphi)S$ for any C^∞ function φ of compact support.

Therefore, if F_1 and F_2 are in $L^2(\omega, G)$ then

$$\begin{aligned}(S^*S\rho_\omega(\varphi)S^*SF_1, F_2) &= (S^*\pi(\varphi)SF_1, F_2) = (\pi(\varphi)SF_1, SF_2)\\ &= \frac{1}{\pi}\int_{-\infty}^{\infty} (\pi_{iy}(\varphi)SF_1(iy), SF_2(iy))\, dy.\end{aligned} \tag{5.19}$$

Before applying the identity (5.16) we make the following observation. Although (5.16) was proved only for F_1 in the orthocomplement of $L^2_0(\omega, G)$ *it is actually true for all* F_1 *in* $L^2(\omega, G)$. Indeed if F is in $L^2_0(\omega, G)$ then $SF \equiv 0$ by definition. On the other hand, since every Eisenstein series is orthogonal to any cuspidal function (in the domain of convergence at first but for all s by analytic continuation), the right side of (5.16) is also identically zero.

If $\{\Phi_\alpha\}$ is an orthonormal basis for $\mathbf{H}$ we can finally apply (5.16) to (5.19) to get

$$\begin{aligned}(S^*S\rho_\omega(\varphi)S^*SF_1, F_2) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \sum_\alpha (\pi_{iy}(\varphi)SF_1(iy), \Phi_\alpha)(\Phi_\alpha, SF_2(iy))\, dy\\ &= \frac{1}{4\pi}\int_{-\infty}^{\infty} dy \sum_\alpha \int F_1(g)\bar{E}(\pi^*_{iy}(\varphi)\Phi_\alpha(iy), g)\, dg\\ &\quad\cdot \int \bar{F}_2(h)\, E(\Phi_\alpha(iy), h)\, dh\end{aligned}$$

with g, h in $\tilde{G}(F)\backslash\tilde{G}(A)$. Interchanging the integrations and summations then yields

$$\begin{aligned}&(S^*S\rho_\omega(\varphi)S^*SF_1, F_2)\\ &= \iint F_1(g)\bar{F}_2(h)\, dg\, dh \left\{\sum_\alpha \frac{1}{4\pi}\int_{-\infty}^{\infty} E(\Phi_\alpha(iy), h)\,\bar{E}(\pi^*_{iy}(\varphi)\Phi_2(iy), g)\, dy\right\}.\end{aligned} \tag{5.20}$$

So if we let P_{cont} denote the projection S^*S onto L^2_{cont} we find that the kernel of

$P_{\text{cont}}\,\rho_\omega(\varphi)P_{\text{cont}}$ is precisely the expression in brackets in (5.20). Alternately, since $\pi^*_{iy}(\varphi)\Phi_\alpha = \sum(\pi^*_{iy}(\varphi)\Phi_\alpha, \Phi_\beta)\,\Phi_\beta$, we also have

$$(5.21)\qquad K_{\text{cont}}(h, g) = \frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta, \Phi_\alpha)\,E(\Phi_\alpha(iy), h)\,\bar{E}(\Phi_\beta(iy), g)\,dg.$$

On the other hand, if P_{sp} is the projection onto L^2_{sp}, then the operator $P_{\text{sp}}\rho_\omega(\varphi)P_{\text{sp}}$ is defined by the kernel

$$K_{\text{sp}}(h, g) = [\operatorname{vol}(\bar{G}(F)\backslash\bar{G}(A))]^{-1}\sum_{\chi^2=\omega}\chi(h)\bar{\chi}(g)\int_{\bar{G}(A)}\varphi(g)\bar{\chi}(\det g)\,dg.$$

The operator $\rho_\omega(\varphi)$ of course is still defined by the kernel $K(h, g) = \sum_{\bar{G}(F)}\varphi(h^{-1}\gamma g)$. Thus we conclude that

$$(P_{\text{cusp}}\,\rho_\omega(\varphi)P_{\text{cusp}}F)(h) = \int K_{\text{cusp}}(h, g)\bar{F}(g)\,dg,$$

where $K_{\text{cusp}}(h, g) = K(h, g) - K_{\text{cont}}(h, g) - K_{\text{sp}}(h, g)$.

6. The trace formula. Suppose φ is a C^∞-function on $G(A)$ which has compact support mod $Z(A)$ and satisfies (1.3) and the conditions immediately following it. Then we have observed that $P_{\text{cusp}}\,\rho_\omega(\varphi)P_{\text{cusp}}$ is of Hilbert-Schmidt class and also of trace class. In fact the technique we used can also be used to show that the kernel K_{cusp} is square-integrable, continuous, and integrable over the diagonal (cf. [**DL**]). Moreover,

$$(6.1)\qquad \operatorname{tr}(P_{\text{cusp}}\rho_\omega(\varphi)P_{\text{cusp}}) = \int_{\bar{G}(F)\backslash\bar{G}(A)} K_{\text{cusp}}(x, x)\,dx.$$

What we are going to do now is give an explicit formula for the right-hand side of (6.1). Note that for f in $L^2_0(\omega, G)^\perp$, $\int K_{\text{cusp}}(x, y)f(y)\,dy = 0$. Thus for each x, the function $y \mapsto \bar{K}(x, y)$ is orthogonal to $L^2_0(\omega, G)^\perp$ and hence in $L^2_0(\bar{\omega}, G)$. In other words, it is a cuspidal function of y.

If we denote by Λ^c_2 the truncation operator *with respect to the second variable* then

$$(6.2)\qquad K_{\text{cusp}}(x, y) = \Lambda^c_2K_{\text{cusp}}(x, y) = \Lambda^c_2K(x, y) - \Lambda^c_2K_{\text{cont}}(x, y) - \Lambda^c_2K_{\text{sp}}(x, y).$$

But one can show that each term in (6.2) is integrable over the diagonal. Thus

$$(6.3)\quad \operatorname{tr}(P_{\text{cusp}}\rho_\omega(\varphi)P_{\text{cusp}}) = \int\Lambda^c_2K(x, x)\,dx - \int\Lambda^c_2K_{\text{cont}}(x, x)\,dx - \int\Lambda^c_2K_{\text{sp}}(x, x)\,dx.$$

We shall content ourselves with computing each of these integrals.

Note that since the left-hand side of (6.3) does not depend on c, we can let c tend to $+\infty$ and—when computing—*ignore all terms which tend to zero.*

A. *Contribution from the kernel* Λ^c_2K. When $x = y$, $\Lambda^c_2K(x, y)$ is by definition

$$(6.4)\qquad \begin{aligned}\Lambda^c_2K(x, x) = {}&\sum_{\gamma\in\bar{G}(F)}\varphi(x^{-1}\gamma x)\\ &-\sum_{\xi\in P(F)\backslash G(F)}\int_{N(F)\backslash N(A)}dn\sum_{\gamma\in\bar{G}(F)}\varphi(x^{-1}\gamma n\xi x)\chi_c(H(\xi x)).\end{aligned}$$

Let us recall the following lemma:

LEMMA (6.5) (CF. [**Ge**, P. 201]). *If Ω is a compact set in $Z(A)\backslash G(A)$ then there exists* *a number d_Ω with the property that if γ in $G(F)$ and n in $N(A)$ are such that $x^{-1}\gamma n x \in \Omega$ for some x in $G(A)$ with $H(x) > d_\Omega$ then $\gamma \in P(F)$.*

We shall apply this to the support Ω of φ. After changing γ to $\xi^{-1}\gamma$ we may write the second term in (6.4) as

$$\sum_{\xi \in P(F)\backslash G(F)} \int_{N(F)\backslash N(A)} dn \sum_{\gamma \in \tilde{G}(F)} \varphi(x^{-1}\xi^{-1}\gamma n \xi x)\chi_c(H(\xi x)). \tag{6.6}$$

Here we need only sum over those γ for which there is x, ξ, n such that $g = \xi x$ and $g^{-1}\gamma n g$ is in Ω. Thus if c is sufficiently large (φ being given) it follows from Lemma (6.5) that γ must belong to $P(F)$, i.e., if c is large enough, we need only sum over those γ which lie in the image of $P(F)$ in $\tilde{G}(F)$. But for such γ we can write

$$\gamma = \mu\nu \quad \text{with } \mu = \begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} \text{and } \nu \in N(F). \tag{6.7}$$

Thus in (6.6) we can combine the integral over $N(F)\backslash N(A)$ with a sum over $N(F)$ and rewrite (6.4) as

$$\sum_{\gamma \in \tilde{G}(F)} \varphi(x^{-1}\gamma x) - \sum_{\xi \in P(F)\backslash G(F)} \int_{N(A)} \sum_{\mu} \varphi(x^{-1}\xi^{-1}\mu n \xi x)\chi_c(H(\xi x))\, dn \tag{6.8}$$

with μ as in (6.7).

Now we break up the sums in (6.8) and write

$$\Lambda_2^c K(x, x) =$$

$$\varphi(e) \tag{6.9}$$

$$+ \sum \varphi(x^{-1}\gamma x) \qquad (\gamma\ F\text{-elliptic}) \tag{6.10}$$

$$+ \sum_{\gamma} \varphi(x^{-1}\gamma x) - \sum_{\xi \in P(F)\backslash G(F)} \int_{N(A)} \varphi(x^{-1}\xi^{-1} n \xi x)\chi_c(H(\xi x))\, dn \qquad (\gamma \text{ nilpotent regular}) \tag{6.11}$$

$$+ \sum_{\gamma} \varphi(x^{-1}\gamma x) - \sum_{\xi \in P(F)\backslash G(F)} \int_{N(A)} \sum_{\alpha \neq 1} \varphi\left(x^{-1}\xi^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1 \end{pmatrix} n \xi x\right)\chi_c(H(\xi x))\, dn \qquad (\gamma\ F\text{-hyperbolic regular}). \tag{6.12}$$

Here γ *F-elliptic* means γ is not $G(F)$-conjugate to anything in $P(F)$; γ *nilpotent* regular means γ is conjugate to a nontrivial element of $N(F)$ and γ *F-hyperbolic* regular means γ is conjugate to some $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1 \end{smallmatrix}\right)$ with $\alpha \neq 1$ in $F^\times$. One can see that each of the terms (6.9)—(6.12) is integrable over $\tilde{G}(F)\backslash\tilde{G}(A)$; our task is to evaluate the resulting integrals.

The integral of (6.9) is clearly

$$\operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e). \tag{6.13}$$

As for (6.10), it has compact support mod $\tilde{G}(F)$. Indeed if $\varphi(x^{-1}\gamma x) \neq 0$ then $x^{-1}\gamma x$ belongs to $\Omega = \text{support}(\varphi)$. So since γ is elliptic, Lemma (6.5) implies $H(x) < d_\Omega$. The integral of (6.10) over $\tilde{G}(F)\backslash\tilde{G}(A)$ is

$$\int_{\bar{G}(F)\backslash\bar{G}(A)} \sum_{\gamma} \varphi(x^{-1}\gamma x)\, dx \qquad (\gamma \text{ elliptic}) \tag{6.14}$$

and this can be transformed further just as in the division algebra case.

In (6.11) we can write γ in the form $\gamma = \xi^{-1}\left(\begin{smallmatrix}1 & \eta\\ 0 & 1\end{smallmatrix}\right)\xi$ with $\xi \in P(F)\backslash G(F)$, $\eta \neq 0$. Thus (6.11) takes the form

$$\sum_{\xi\in P(F)\backslash G(F)} \left\{ \sum_{\eta\neq 0} \varphi\left[x^{-1}\xi^{-1}\begin{pmatrix}1 & \eta\\ 0 & 1\end{pmatrix}\xi x\right] - \int_{N(A)} \varphi(x^{-1}\xi^{-1}n\xi x)\,\chi_c(H(\xi x))\, dn\right\}$$

which we now have to integrate over $\bar{G}(F)\backslash\bar{G}(A)$. This is the same as the integral over $P(F)Z(A)\backslash G(A)$ of

$$\sum_{\eta\neq 0} \varphi\left[x^{-1}\begin{pmatrix}1 & \eta\\ 0 & 1\end{pmatrix}x\right] - \int_A \varphi\left[x^{-1}\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}x\right]\chi_c(H(x))\, du. \tag{6.15}$$

To evaluate this integral we shall use Poisson's summation formula.

Set

$$F(x) = \int_K \varphi\left[k^{-1}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right] dk \tag{6.16}$$

so that $F(x)$ is a Schwartz-Bruhat function on A. Using Iwasawa's decomposition we get that the integral of (6.15) over $P(F)Z(A)\backslash G(A)$ is

$$\int_{F^\times\backslash A^\times} \left\{ \sum_{\eta\neq 0} F(a\eta) - \hat{F}(0)\chi_c(|a|^{-1})|a|^{-1}\right\} |a|\, d^\times a. \tag{6.17}$$

Using Poisson's summation formula we find this is

$$\int_{|a|\geq 1}\left(\sum_{\eta\neq 0} F(a\eta)\right)|a|\, d^\times a + \int_{|a|\leq 1} \sum_{\eta\neq 0}\hat{F}(a^{-1}\eta)\, d^\times a + \hat{F}(0)$$
$$\cdot \int_{|a|\leq 1} (1 - \chi_c(|a|^{-1}))\, d^\times a - F(0)\int_{|a|\leq 1} |a|\, d^\times a.$$

This shows that the integral converges. The term involving $\hat{F}(0)$ depends on c and equals $(\log c)\hat{F}(0)\mathrm{vol}(F^\times\backslash F^0(A))$.

For $\mathrm{Re}(s) > 1$ we also have

$$\int_{A^\times} F(a)|a|^s\, d^\times a = \int_{F^\times\backslash A^\times} \sum_{\eta\neq 0} F(a\eta)|a|^s\, d^\times a$$
$$= \int_{|a|\geq 1}\left(\sum_{\eta\neq 0} F(a\eta)\right)|a|^s\, d^\times a + \int_{|a|\leq 1}\left(\sum_{\eta\neq 0}\hat{F}(a^{-1}\eta)\right)|a|^{s-1}\, d^\times a$$
$$+ \hat{F}(0)\int_{|a|\leq 1} |a|^{s-1} d^\times a - F(0)\int_{|a|\leq 1} |a|^s\, d^\times a.$$

Here each term is analytic at $s = 1$ except the term involving $\hat{F}(0)$ which equals $(\hat{F}(0)\,\mathrm{vol}(F^\times\backslash F^0A))/s - 1$. Thus we conclude that (6.17) equals

$$\mathrm{f.p.}\left(\int_{A^\times} F(a)|a|\, d^\times a\right) + (\log c)\hat{F}(0)\mathrm{vol}(F^\times\backslash F^0(A))$$

where the symbol f.p. means f.p.$(\ldots)$ = value at 1 of $\{\int_{A^\times} F(a)|a|^s\, d^\times a$ − principal part at 1}. Changing a to a^{-1} in the "f.p. integral" we get (finally) that the integral of (6.11) is

$$\text{(6.18)}\quad \begin{aligned} &\text{f.p.} \int_{A^\times}\int_K \varphi\left[k^{-1}\begin{pmatrix} a^{-1} & 0 \\ 0 & 1\end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1\end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & 1\end{pmatrix}k\right] |a|^{-1}\, d^\times a\, dk \\ &\quad + (\log c)\mathrm{vol}(F^\times\backslash F^0(A)) \int_K\int_A \varphi\left(k^{-1}\begin{pmatrix} 1 & x \\ 0 & 1\end{pmatrix}k\right] dx\, dk. \end{aligned}$$

Now we deal with the term (6.12). In this term we can write γ in the form $\gamma = \xi^{-1}\left(\begin{smallmatrix} \alpha & 0 \\ 0 & 1\end{smallmatrix}\right)\xi$ with ξ in $A(F)\backslash G(F)$, α in $F^\times$, $\alpha \neq 1$. But when ξ varies through a system of representatives for $A(F)\backslash G(F)$ we obtain each γ *twice* since

$$w\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix} w^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \alpha\end{pmatrix} = \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix} \quad \text{in } \tilde{G}.$$

Thus the first term in (6.12) is

$$\begin{aligned} &\frac{1}{2}\sum_{\xi\in A(F)\backslash G(F)}\ \sum_{\alpha\neq 1} \varphi\left[x^{-1}\xi^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\xi x\right] \\ &\quad = \frac{1}{2}\sum_{\xi\in P(F)\backslash G(F)}\ \sum_{\nu\in N(F)}\ \sum_{\alpha\neq 1} \varphi\left[x^{-1}\xi^{-1}\nu^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\nu\xi x\right]. \end{aligned}$$

The integral of all of (6.12) is therefore the integral over $\tilde{G}(F)\backslash\tilde{G}(A)$ of

$$\begin{aligned} &\frac{1}{2}\sum_{\xi\in P(F)\backslash G(F)}\ \sum_{\nu\in N(F)}\ \sum_{\alpha\neq 1} \varphi\left[x^{-1}\xi^{-1}\nu^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\nu\xi x\right] \\ &\quad - \sum_{\xi\in P(F)\backslash G(F)} \int_{N(A)} \sum_{\alpha\neq 1} \varphi\left[x^{-1}\xi^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}n\xi x\right]\chi_c(H(\xi x)). \end{aligned}$$

Making the change of variables on $N(A)$ given by

$$n \to \begin{pmatrix} \alpha^{-1} & 0 \\ 0 & 1\end{pmatrix}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}n \qquad (\alpha \neq 1)$$

we see that the integral of (6.12) over $\tilde{G}(F)\backslash\tilde{G}(A)$ equals the integral over $P(F)Z(A)\backslash G(A)$ of

$$\begin{aligned} &\frac{1}{2}\sum_{\alpha\neq 1}\ \sum_{\nu\in N(F)} \varphi\left[x^{-1}\nu^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}\nu x\right] \\ &\quad - \int_{N(A)} \sum_{\alpha\neq 1} \varphi\left[x^{-1}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}nx\right]\chi_c(H(x))\, dn. \end{aligned}$$

To compute this last integral we have to first integrate over $N(F)\backslash N(A)$ and then over $N(A)A(F)Z(A)\backslash G(A)$. The integration over $N(F)\backslash N(A)$ gives

$$\begin{aligned} &\frac{1}{2}\sum_{\alpha\neq 1} \int_{N(A)} \Sigma\, \varphi\left[x^{-1}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}nx\right] dn \\ &\quad - \int_{N(A)} \sum_{\alpha\neq 1} \varphi\left[x^{-1}n^{-1}\begin{pmatrix} \alpha & 0 \\ 0 & 1\end{pmatrix}nx\right]\chi_c(H(x))\, dn \end{aligned}$$

and the subsequent integral over $N(A)A(F)Z(A)\backslash G(A)$ is the same as the integral of

$$\sum_{\alpha\neq 1}\varphi\left[x^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}x\right]\left(\frac{1}{2}-\chi_c(H(x))\right) \tag{6.19}$$

over $A(F)Z(A)\backslash G(A)$.

Note now that the first factor in (6.19) does not change when x is replaced by wx but the second factor does. In any case, w normalizes $A(F)$. Thus we see that the integral of (6.12) is also the integral over $A(F)Z(A)\backslash G(A)$ of

$$\frac{1}{2}\sum_{\alpha\neq 1}\varphi\left[x^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}x\right](1-\chi_c(H(x))-\chi_c(H(wx))).$$

Using Iwasawa's decomposition we get that this integral is

$$\begin{aligned}&\frac{1}{2}\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]\\ &\qquad\cdot\left(\int_{F^\times\backslash A^\times}(1-\chi_c(|a|)-\chi_c(|a|^{-1}H(wn)))\,d^\times a\right)dk\,du.\end{aligned} \tag{6.20}$$

But $c>1$ and $H(wn)\leqq 1$. Therefore in (6.20) the integrand in the inner integral vanishes unless $c^{-1}H(wn)<|a|<c$ in which case it equals 1. Thus the integral of (6.12) over $\tilde{G}(F)\backslash\tilde{G}(A)$ is

$$\begin{aligned}&(\log c)\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]dk\,dn\\ &-\frac{1}{2}\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[k^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]\log H(wnk)\,dn\,dk.\end{aligned} \tag{6.21}$$

Summing up (6.13), (6.14), (8.16), and (6.21), we get

PROPOSITION (6.22).

$$\int\Lambda_2^c K(x,x)\,dx$$

$$= \mathrm{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e)+\int_{\tilde{G}(F)\backslash\tilde{G}(A)}\sum_{\gamma\text{ elliptic}}\varphi(x^{-1}\gamma x)\,dx$$

$$+\text{ f.p.}\int_{Z(A)N(A)\backslash G(A)}\varphi\left[g^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g\right]dg \tag{6.23}$$

$$+(\log c)\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\in F^\times}\varphi\left[k^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nk\right]dn\,dk \tag{6.24}$$

$$-\frac{1}{2}\mathrm{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq 1}\varphi\left[h^{-1}n^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}nh\right]\log H(wnk)\,dn\,dk. \tag{6.25}$$

REMARK. The term (6.24) depends on c but $\operatorname{tr}\rho_\omega(\varphi)$ does not. Thus we can expect (6.24) to cancel with another term later on.

B. *Contribution from the continuous spectrum.* Recall that

$$\Lambda_2^c K_{\mathrm{cont}}(g,h)=\Lambda_2^c\left\{\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)E(\Phi_\alpha(iy),g)\bar{E}(\Phi_\beta(iy),h)\,dy\right\}.$$

It is not hard to see that this also equals

$$\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)E(\Phi_\alpha(iy),g)\Lambda^c\bar{E}(\Phi_\beta(iy),h)\,dy.$$

So taking it for granted that we can interchange orders of integration, we find

$$\int \Lambda_2^c K_{\text{cont}}(x,x)\,dx$$

$$=\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\,dy\int_{\bar{G}(F)\backslash\bar{G}(A)}E(\Phi_\alpha(iy),x)\Lambda^c\bar{E}(\Phi_\beta(iy),x)\,dx.$$

Note that the inner integral here has already been computed in §5 (cf. (5.14) with $h_1(s)=\Phi_\alpha(s)$, $h_2(s)=\Phi_\beta(s)$). Plugging in (5.14) then gives

$$\int \Lambda_2^c K_{\text{cont}}(x,x)\,dx$$

$$=\frac{(\log c)}{2\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\,(\Phi_\alpha,\Phi_\beta)\,dy \tag{6.26}$$

$$-\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(M(-iy)M'(iy)\Phi_\alpha,\Phi_\beta)(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\,dy \tag{6.27}$$

$$+\frac{1}{4\pi}\sum_{\alpha,\beta}\int_{-\infty}^{\infty}(\pi_{iy}(\varphi)\Phi_\beta,\Phi_\alpha)\left\{(\Phi_\alpha,M(iy)\Phi_\beta)\frac{c^{2iy}}{2iy}-(M(iy)\Phi_\alpha,\Phi_\beta)\frac{c^{-2iy}}{2iy}\right\}dy. \tag{6.28}$$

After exchanging $\sum$ and $\int$ the term (6.26) can also be written $(\log c/2\pi)\cdot\int_{-\infty}^{\infty}\operatorname{tr}\,\pi_{iy}(\varphi)\,dy$. But π_{iy} is an induced representation whose trace is easily computed to be

$$\operatorname{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\in F^\times}\left(\int_{F_\infty^+}\varphi\left(k^{-1}\begin{pmatrix}t\alpha&0\\0&1\end{pmatrix}nk\right)|t|^{iy+1/2}d^\times t\right)dn\,dk.$$

So after using the Fourier inversion formula we find that (6.26) precisely equals (6.24), i.e., it cancels (6.24) just as expected.

Now rewrite (6.27) as

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(\varphi))\,dy. \tag{6.29}$$

As for (6.28), it can be written as

$$\frac{1}{4\pi}\sum_{\beta}\int_{-\infty}^{\infty}\left\{(\pi_{iy}(\varphi)\Phi_\beta,M(iy)\Phi_\beta)\frac{c^{2iy}}{2iy}-(\pi_{iy}(\varphi)\Phi_\beta,M(-iy)\Phi_\beta)\frac{c^{-2iy}}{2iy}\right\}dy$$

or

$$\begin{aligned}&\frac{1}{4\pi}\sum_{\beta}\int_{-\infty}^{\infty}(M(-iy)\pi_{iy}(\varphi)\Phi_\beta,\Phi_\beta)\frac{c^{2iy}-c^{-2iy}}{2iy}\,dy\\&+\frac{1}{4\pi}\sum_{\beta}\int_{-\infty}^{\infty}\{(\pi_{iy}(\varphi)\Phi_\beta,M(iy)\Phi_\beta)-(\pi_{iy}(\varphi)\Phi_\beta,M(-iy)\Phi_\beta)\}\frac{c^{-2iy}}{2iy}\,dy.\end{aligned} \tag{6.30}$$

But the last term here is the Fourier transform of an integrable function (namely $\sum(\pi_{iy}(\varphi)\Phi_\beta,M(iy)\Phi_\beta)\ldots$) at $\log c/\pi$ (cf. Lemma 9.14 of [Ge]). Thus it tends to zero as $c\to+\infty$. To evaluate the first term we need the following lemma:

LEMMA (6.31). *If F is continuous on $\mathbf{R}$, and F, $\hat{F}$ are integrable, then*

$$\lim_{x\to+\infty}\int \frac{e^{2\pi ixy}-e^{-2\pi ixy}}{y}F(y)\,dy = 2\pi iF(0).$$

PROOF. Set $G(x) = \int_{-\infty}^{\infty}((e^{2\pi ixy}-e^{-2\pi ixy}/y))F(y)\,dy$. Then $G(0)=0$, and

$$G'(x) = 2\pi i\int(e^{2\pi ixy}+e^{-2\pi ixy})F(y)\,dy$$
$$= 2\pi i(\hat{F}(x)+\hat{F}(-x)).$$

Therefore

$$G(x) = 2\pi i\int_0^x(\hat{F}(t)+\hat{F}(-t))\,dt$$
$$= 2\pi i\int_{-x}^x \hat{F}(t)\,dt \qquad \text{for } x>0.$$

So as $x\to+\infty$, $G(x)$ tends to $2\pi iF(0)$.

Applying Lemma (6.31) to (6.30) we conclude that (6.28) tends to

$$\tfrac{1}{4}\sum_\beta(M(0)\pi_0(\varphi)\Phi_\beta,\Phi_\beta) = \tfrac{1}{4}\operatorname{tr}M(0)\pi_0(\varphi) \tag{6.32}$$

as $c\to+\infty$.

C. *Trace formula.* We leave it to the reader to check that

$$\int \Lambda_2^c K_{\mathrm{sp}}(x,x)\,dx \to \int K_{\mathrm{sp}}(x,x)\,dx$$
$$= [\operatorname{vol}(\tilde{G}(F)/\tilde{G}(A))]^{-1}\sum_{\chi^2=\omega}\int\varphi(x)\bar{\chi}(x)\,dx$$

as $c\to+\infty$. Therefore, by combining (6.3), Proposition (6.22), (6.26), (6.27), and (6.32) we obtain

THEOREM (6.33).

$$\operatorname{tr}\rho_{\mathrm{cusp}}(\varphi)+\operatorname{tr}\rho_{\mathrm{sp}}(\varphi) = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e)$$
$$+\int_{G(F)\backslash\tilde{G}(A)}\Big(\sum_{\gamma\ \mathit{elliptic}}\varphi(x^{-1}\gamma x)\Big)\,dx$$
$$+\ \text{f.p.}\int_{Z(A)N(A)\backslash G(A)}\varphi\left[g^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right]dg \tag{6.34}$$
$$-\frac{1}{2}\operatorname{vol}(F^\times\backslash F^0(A))\int_K\int_{N(A)}\sum_{\alpha\neq1}\varphi\left[k^{-1}n^{-1}\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}nk\right]\log H[wnk]\,dn\,dk \tag{6.35}$$
$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\operatorname{tr}(M(-iy)M'(iy)\pi_{iy}(\varphi))\,dy \tag{6.36}$$
$$-\tfrac{1}{4}\operatorname{tr}(M(0)\pi_0(\varphi)). \tag{6.37}$$

Here the f.p. term is computed as the value at $s=1$ of

$$\left\{\int \varphi\left[k^{-1}\begin{pmatrix}1 & a\\ 0 & 1\end{pmatrix}k\right]|a|^s\, d^\times a\, dk - \text{principal part at } s = 1\right\}.$$

Recall also that we have identified all fibres $\mathbf{H}(s)$ with $\mathbf{H}$ in order to define $M'(s) = dM/ds$.

7. A second form of the trace formula. Our next goal is to express the right-hand side of (6.33) in terms of *local* distributions, most of them invariant. This involves replacing the Mellin transform on F_∞^+ by a Mellin transform on $F^\times\backslash F^\times(A)$, the fiber bundle $\mathbf{H}(s)$ by the bundle $\mathbf{H}(\eta)$, and the operator $M(s)$ by the operator $M(\eta)$.

To express the right-hand side of (6.33) as a sum *only* of invariant distributions entails applying a form of Poisson summation to some nonsmooth functions. This analysis is carried out in §7 of **[La 2]** but is not needed for the application we have in mind.

A. *Normalization of Haar measures*. We normalize the Haar measure on $A^\times$ as follows. Select (in any way) a nontrivial additive character $\psi = \pi\psi_v$ of $F\backslash A$. For each place v let dx_v be the self-dual Haar measure on F_v with respect to ψ_v. Then the Haar measure we take on $F_v^\times$ is $d^\times x_v = L(1, 1_v)dx_v/|x_v|$. Note that on A the Haar measure $dx = \bigotimes dx_v$ is self-dual. On $A^\times$ the (normalized) Tamagawa measure is

$$d^\times x = \frac{1}{\lambda_{-1}} \otimes d^\times x_v \tag{7.1}$$

where

$$\lambda_{-1} = \lim_{s\to 1} (s - 1)L(s, 1_F). \tag{7.2}$$

The map $x \mapsto |x|$ allows us to identify $F^0(A)\backslash A^\times$ with $\mathbf{R}_+^\times$. The Tamagawa measure μ on $F^0(A)$ is such that the quotient measure $d^\times x/\mu$ (on $F^0(A)\backslash A^\times$ or $\mathbf{R}_+^\times$) is dt/t and—as shown in Tate's thesis—

$$\operatorname{vol}(F^\times\backslash F^0(A)) = 1. \tag{7.3}$$

To define the Tamagawa measure on $\tilde{G}(A)$ we select in any way an invariant differential form ω of degree 3 defined over F. In particular, we may take ω to be the form whose pull-back through the map

$$(a, x, y) \mapsto \begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y & 1\end{pmatrix}$$

is the form $(da/a)\, dx\, dy$. Then for $f = \prod f_v$ on $\tilde{G}(A)$,

$$\begin{aligned}\int f(g)\, dg &= \int f(g)|\omega(g)| = \prod_v \int f_v(g_v)|\omega_v(g)| \\ &= \prod_v \int f_v\left[\begin{pmatrix}a_v & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x_v\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ y_v & 1\end{pmatrix}\right]\frac{da_v}{|a_v|}\, dx_v\, dy_v.\end{aligned} \tag{7.4}$$

On the other hand, the Haar measure on K is normalized by the condition

$$\int f(g)\, dg = \int f\left[\begin{pmatrix}a & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & x\\ 0 & 1\end{pmatrix}k\right]d^\times a\, dx\, dk. \tag{7.5}$$

Similarly the Haar measure on K_v is defined by

$$\int f(g_v)\,dg_v = \int f\left[\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & x_v \\ 0 & 1 \end{pmatrix} k_v\right] d^\times a_v\, dx_v\, dk_v. \tag{7.6}$$

Then a simple computation shows that

$$\mathrm{vol}(K_v) = \frac{1}{L(2, 1_v)\varepsilon(0, 1_v, \psi_v)}. \tag{7.7}$$

Since $dk = \lambda_{-1} \otimes_v dk_v$ we also have

$$\mathrm{vol}(K) = \frac{\lambda_{-1}}{L(2, 1_F)\varepsilon(0, 1_F, \psi_F)}. \tag{7.8}$$

Now we replace the Mellin transform on F_∞^+ (introduced in §3) by a Mellin transform on $A^\times$, namely

$$\hat{f}(\chi)(g) = \int_{A^\times} f\left[\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right] \chi^{-1}(a)|a|^{-1/2}\, d^\times a.$$

Then formula (3.19) should be replaced by

$$\begin{aligned} (F_1, F_2) = {} & \frac{1}{2\pi i} \sum_\chi \int_{x-i\infty}^{x+i\infty} (\hat{f}_1(\chi\alpha^s), \hat{f}_2(\chi\alpha^{-\bar{s}}))\, ds \\ & + \frac{1}{2\pi i} \sum_\chi \int_{x-i\infty}^{x+i\infty} (M(\chi\alpha^s, \chi^{-1}\omega\alpha^{-s})\hat{f}_1(\chi\alpha^s), \hat{f}_2(\chi^{-1}\omega\alpha^{\bar{s}}))\, ds. \end{aligned} \tag{7.9}$$

Here χ runs through the set of characters of $F^\times(A)$ trivial on $F^\times F_\infty^+$; cf. [Ge, p. 167].

B. *The* "f.p." *integral.* In order to compute (6.34) we have to remove from $\int \varphi[k^{-1}(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix})k]\,|a|^s\, d^\times a\, dk$ the principal part at $s = 1$ and then set $s = 1$. So first we write this integral as the product $L(s, 1_F)\theta(s)$ where

$$\theta(s) = \frac{1}{L(s, 1_F)} \int \varphi\left[k^{-1}\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} k\right] |a|^s\, d^\times a\, dk. \tag{7.10}$$

At $s = 1$ the function $\theta(s)$ is holomorphic. On the other hand, $L(s, 1_F) = \lambda_{-1}/(s-1) + \lambda_0 + \cdots$. Therefore the f.p. integral is $\lambda_{-1}\theta'(1) + \lambda_0\theta(1)$. But $d^\times a = (\lambda_{-1})^{-1} \otimes d^\times a_v$ and $dk = \lambda_{-1} \otimes dk_v$. So for $\mathrm{Re}(s) > 1$,

$$\theta(s) = \prod_v \frac{1}{L(s, 1_v)} \int \varphi_v\left[k_v^{-1}\begin{pmatrix} 1 & a_v \\ 0 & 1 \end{pmatrix} k_v\right] |a_v|^s\, d^\times a_v\, dk_v$$

and since almost all factors here are equal to one,

$$\begin{aligned} \theta(1) &= \prod_v \frac{1}{L(1, 1_v)} \int \varphi_v\left[k_v^{-1}\begin{pmatrix} a_v^{-1} & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_v & 0 \\ 0 & 1 \end{pmatrix} k_v\right] |a_v|\, d^\times a_v\, dk_v \\ &= \prod_v \frac{1}{L(1, 1_v)} \int_{Z_v N_v \backslash \bar{G}_v} \varphi_v\left[g_v^{-1}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} g_v\right] dg_v. \end{aligned}$$

I.e., $\theta(1)$ is the product of the orbital integral for $(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix})$ with a convergence factor (which is "missing" in the original f.p. integral).

Similarly, taking the derivative of $\theta(s)$ at $s = 1$, we get

$$\theta'(1) = \sum_u \prod_{v \neq u} \int_{Z_v N_v \backslash G_v} \varphi_v\left[g_v^{-1}\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}g_v\right] dg_v \tag{7.11}$$
$$\times \frac{d}{ds}\bigg|_{s=1} \int \varphi_u\left[k_u^{-1}\begin{pmatrix}1 & a_u\\ 0 & 1\end{pmatrix}k_u\right]|a_u|^s \, d^\times a_u / L(s, 1_u).$$

Here the sum is extended over all places u, but only finitely many have a nonzero contribution.

C. *Computation of* (6.35). Recall that $\mathrm{vol}(F^\times \backslash F^0(A)) = 1$, $dk = \lambda_{-1} \otimes dk_v$, $dn = \otimes\, dn_v$, and $H(g) = \prod_v H_v(g_v)$ where

$$H\left[\begin{pmatrix}a_v & x_v\\ 0 & b_v\end{pmatrix}k_v\right] = \left|\frac{a_v}{b_v}\right|.$$

Thus (6.35) is equal to

$$\frac{-\lambda_{-1}}{2}\sum_u \sum_{\alpha \neq 1} \prod_{v \neq u} \int \varphi_v\left[k_v^{-1}n_v^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n_v k_v\right] dn_v\, dk_v$$
$$\cdot \int \varphi_u\left[k_u^{-1}n_u^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}n_u k_u\right] \log(H_u[wn_u])\, dn_u$$

or

$$\frac{-\lambda_{-1}}{2} \sum_{u;\alpha\neq 1} \prod_{v \neq u} \int \varphi_v\left[g_v^{-1}\begin{pmatrix}\alpha & 0\\ 1 & 0\end{pmatrix}g_v\right] dg_v \int \varphi_u\left[g_u^{-1}\begin{pmatrix}\alpha & 0\\ 0 & 1\end{pmatrix}g_u\right]\mu_u(g_u)\, dg_u \tag{7.12}$$

where

$$\mu_u\left[\begin{pmatrix}a_v & 0\\ 0 & b_v\end{pmatrix}\begin{pmatrix}1 & n\\ 0 & 1\end{pmatrix}k\right] = \log H_u[wn_u], \qquad g_v \in A_v \backslash G_v.$$

Since μ_u vanishes on K_u, the sum above has only finitely many nonzero terms.

D. *Computation of* (6.36) *and* (6.37). To begin with, (6.36) can be written as

$$\frac{1}{4\pi}\sum_\chi \int_{-i\infty}^{i\infty} \mathrm{tr}(M(\eta)^{-1}M'(\eta)\pi_\eta(\varphi))\, dy$$

where $\eta = (\chi\alpha^{iy}, \chi^{-1}\omega\alpha^{-iy})$, π_η is the representation of $G(A)$ on $H(\eta)$, and the sum is over all characters χ of $F^\times \backslash A^\times$ whose restriction to F_∞^+ is trivial The derivative is defined by $M'(\eta)\varphi = (d/ds)M(\eta)\varphi$ if $\eta = (\chi\alpha^s, \chi^{-1}\omega\alpha^{-s})$ and $\varphi|_K$ is independent of s. (To define the derivative we have once again trivialized the fibre-bundle.) Since $M(\eta) = m(\eta) \otimes_v R_v(\eta_v)$ (where $m(\eta)$ is the scalar described by (4.15)) we get (by taking the logarithmic derivative) that

$$M'(\eta)M^{-1}(\eta) = m'(\eta)I + \sum_u R_u^{-1}(\eta_u)R_u'(\eta_u) \underset{v\neq u}{\otimes} I_v.$$

But if $A = \otimes_v A_v$ where A_v is an operator on $H(\eta_v)$ then $\mathrm{tr}(A) = \prod_v \mathrm{tr}(A_v)$. Thus (6.36) is equal to

$$\frac{1}{4\pi}\sum_\chi \int_{-i\infty}^{i\infty} m'(\eta)\mathrm{tr}(\pi_\eta(\varphi))\, dy \tag{7.13}$$
$$+ \sum_u \frac{1}{4\pi}\sum_\chi \int_{-i\infty}^{i\infty} \prod_{v\neq u} \mathrm{tr}(\pi_{\eta_v}(\varphi_v))\mathrm{tr}(R_u(\eta_u)^{-1}R'(\eta_u)\pi_{\eta_u}(\varphi_u))\, dy.$$

Recall that if $\varphi = 1$ on K_u then $R(\eta_u)\varphi = \varphi$ for all η_u. Thus $R'(\eta_u)\varphi = 0$, and the sum above once again has only finitely many nonzero terms.

As for (6.37), we write it as

$$-\frac{1}{4}\sum_{\chi^2=\omega} \operatorname{tr}(M(\chi,\chi)\pi_{(\chi,\chi)}(\varphi)).$$

E. *Summing up.* We have that tr $\rho_{\text{cusp}}(\varphi)$ + tr $\rho_{\text{sp}}(\varphi)$ is equal to the sum of $\operatorname{vol}(\bar{G}(F)\backslash\bar{G}(A))\varphi(e)$,

$$(*) \qquad \sum_{\gamma \text{ elliptic}} \int_{\bar{G}_\gamma(F)\backslash \bar{G}(A)} \varphi(g^{-1}\gamma g)\,dg$$

and a complementary term. As in the case of division algebras, $(*)$ can be expressed in terms of local elliptic *orbital integrals* of the form

$$\int_{G_\gamma(F_v)\backslash G_v} \varphi_v(g_v^{-1}\gamma g)\,dg_v, \quad \gamma \text{ elliptic}.$$

The complementary term can be expressed explicitly in terms of the following local distributions:

$$(7.14) \qquad \int_{Z_vN_v\backslash G_v} \varphi_v\left[g_v^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}g_v\right]dg_v,$$

$$(7.15) \qquad \int_{A_v\backslash G_v} \varphi_v\left[g_v^{-1}\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}g_v\right]dg_v,$$

$$(7.16) \qquad \operatorname{tr} \pi_{\eta_v}(\varphi_v),$$

$$(7.17) \qquad \frac{d}{ds}\bigg|_{s=1} \frac{1}{L(s,1_v)} \int \varphi_v\left[k_v^{-1}\begin{pmatrix}1&a_v\\0&1\end{pmatrix}k_v\right]|a_v|^s\,d^\times a_v\,dk_v,$$

$$(7.18) \qquad \int \varphi_v\left[g_v^{-1}\begin{pmatrix}\alpha&0\\0&1\end{pmatrix}g_v\right]\mu_v(g_v)\,dg_v \qquad (\alpha \neq 1)$$

and

$$(7.19) \qquad \operatorname{tr}(R_v(\eta_v)^{-1}R'_v(\eta_v)\pi_{\eta_v}(\varphi_v)).$$

The distributions (7.14) and (7.15) are orbital integrals. The distributions (7.16) are invariant and can also be computed in terms of orbital integrals (cf. **[DL]** for example). The distributions (7.17)—(7.19), however, are *not* invariant.

F. *A special case.* The distributions (7.14)—(7.16) enjoy the following property. Suppose

$$(7.20) \qquad \int_{A_v\backslash G_v} \varphi_v\left(g^{-1}\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right)d^\times a = 0$$

for all $a \neq 1$, i.e., the orbital integrals of φ_v vanish for all regular hyperbolic elements. Then each of the distributions (7.14)—(7.16) vanishes. Indeed there is nothing to prove for (7.15), and (7.16) is an integral of orbital integrals of this type. As for (7.14) we note that

$$\int \varphi\left[g^{-1}\begin{pmatrix}1&1\\0&1\end{pmatrix}g\right]dg = \lim_{a\to 1}|1-a^{-1}| \int \varphi\left[g^{-1}\begin{pmatrix}a&0\\0&1\end{pmatrix}g\right]dg,$$

i.e., the nilpotent orbital integral is a limit of hyperbolic orbital integrals.

It follows that if (7.20) is satisfied for a certain place u_1 then the sums in (7.11), (7.12), and (7.13) reduce to the term involving $u = u_1$. On the other hand, tr $\pi_\eta(\varphi)$ vanishes. Thus we have:

THEOREM (7.21). *Suppose condition* (7.20) *is satisfied for two places* $u_1 \neq u_2$. *Then* (7.11)—(7.13) *vanish and we have*

$$\operatorname{tr} \rho_{\text{cusp}}(\varphi) + \operatorname{tr} \rho_{\text{sp}}(\varphi) = \operatorname{vol}(\tilde{G}(F)\backslash\tilde{G}(A))\varphi(e) + \sum_{\gamma \text{ elliptic}} \int_{G(F)\backslash G(A)} \varphi(g^{-1}\gamma g)\, dg.$$

The reader will note that this formula closely resembles formula (1.11).

8. Applications to quaternion algebras. As before, G is the group GL(2) regarded as an F-group and $Z \simeq \mathrm{GL}(1)$ is its center. Let D be a quaternion algebra of center F and S the (finite) set of places of F where D does not split. Regard the multiplicative group of D as an algebraic F-group G' and let Z' denote its center. Then for all $v \notin S$ the local groups G_v and G'_v are isomorphic. More precisely, as in §1, the isomorphism $D_v \approx M(2, F_v)$ induces an isomorphism $G_v \approx G'_v$ defined up to inner automorphism. If ε_i, $1 \leqq i \leqq 4$, is an F basis of D then for almost all v we can assume that $\sum R_v \varepsilon_i$ maps to $M(2, R_v)$. Also $K_v = \mathrm{GL}(2, R_v)$ maps to the compact subgroup K'_v of G'_v and the isomorphisms $G_v \simeq G'_v$ give rise to an isomorphism of the restricted products $G^S = \prod_{v \notin S} G_v$, $G'^S = \prod_{v \notin S} G'_v$.

A. *Statement of results.* For the moment, let F be a local field and D a division algebra of center F so that $G'(F) = D^\times$; let ν denote the reduced norm on D. Denote by $\mathscr{E}(G(F))$ the set of classes of irreducible admissible representations of $G(F)$ and by $\mathscr{E}_2(G(F))$ the subset of those which are square-integrable (modulo the center). Define $\mathscr{E}(G'(F)) = \mathscr{E}_2(G'(F))$ similarly.

THEOREM (8.1). *There is a unique bijection* $\pi' \leftrightarrow \pi$ *from* $\mathscr{E}(G'(F))$ *to* $\mathscr{E}_2(G(F))$ *such that the characters* $\theta_{\pi'}$ *and* θ_π *of* π' *and* π *satisfy the relation*

$$(8.2) \qquad \theta_{\pi'}(t') = -\theta_\pi(t)$$

each time t *and* t' *are regular semisimple elements of* $G'(F)$ *and* $G(F)$ *related by the identities* $\operatorname{tr}(t') = \operatorname{tr}(t)$, $\nu(t') = \det(t)$.

This condition implies that the central quasi-characters of π and π' are the same. Note that G' is an inner twisting of G and $\Phi(G) \subset \Phi(G')$ (cf. **[Bo]**). Thus if $F = \boldsymbol{R}$, the correspondence $\pi' \leftrightarrow \pi$ is the one specified by Langlands; in the *non*-archimedean case one can at least construct the map using "Weil's representation" (as in **[JL]**).

Now let F be a number field and D a division algebra with center F. Let $\mathscr{A}(G')$ be the set of (classes of) automorphic representations of $G'(A)$ and $\mathscr{A}_*(G')$ the subset of those which are not one dimensional. Similarly let $\mathscr{A}_0(G)$ be the set of cuspidal representations of $G(A)$.

THEOREM (8.3) (GLOBAL). *If* π' *is in* $\mathscr{A}_*(G')$ *let* π *be the representation of* $G(A)$ *such that, for* $v \in S$, $\pi'_v \leftrightarrow \pi_v$ *as in* (8.1), *and for* $v \notin S$, $\pi_v \approx \pi'_v$ *via the isomorphism* $G_v \approx G'_v$. *Then* π *is in* $\mathscr{A}_0(G)$. *Moreover, the map* $\pi' \mapsto \pi$ *is a bijection between* $\mathscr{A}_*(G')$ *and the set of* π *in* $\mathscr{A}_0(G)$ *such that* $\pi_v \in \mathscr{E}_2(G_v)$ *for all* $v \in S$.

We shall give two proofs of this result, both somewhat different from the one in **[JL]**. First we shall take Theorem (8.1) for granted and indicate how it together with the trace formula implies Theorem (8.3). Then we shall sketch an alternate proof that essentially implies (rather than depends on) Theorem (8.1). Both proofs use orbital integral techniques (cf. **[Sa]**, **[Sh]**, and **[La 2]**).

B. *Matching orbital integrals* (*and consequences*). Let ω and ω' be differential forms invariant and of maximal degree on $\tilde{G}$ and $\tilde{G}'$ respectively. We assume that ω and ω' are related to one another as in **[JL**, pp. 475 and 503]. Then for each $v \notin S$, $\omega_v = \pm\omega_v'$ and $|\omega_v| = |\omega_v'|$.

For $v \in S$ let us say that $f \in C_c^\infty(G_v, \omega_v^{-1})$ and $\varphi \in C_c^\infty(G_v, \omega_v^{-1})$ have *matching orbital integrals* if

$$(8.4) \qquad \int_{Z_v\backslash G_v} g(g)h(\operatorname{tr} g, \det g)\, dg = \int_{Z_v'\backslash G_v'} \varphi(g)h(\operatorname{tr}(g), \nu(g))\, dg'$$

for "any function" h on $F \times F^\times$. In particular, if π_v and π_v' are related as in Theorem (8.1), then

$$(8.5) \qquad \operatorname{tr} \pi_v(f) = -\operatorname{tr} \pi_v'(\varphi).$$

On the other hand,

$$(8.6) \qquad \operatorname{tr} \pi_v(f) = \operatorname{tr} \pi_v'(\varphi)$$

if $\pi_v = \chi \circ \det$ and $\pi_v' = \chi \circ \nu$.

Condition (8.4) can also be expressed in terms of orbital integrals as follows:

(i) The hyperbolic (regular) orbital integrals of f vanish, i.e.,

$$(8.7) \qquad \int_{A_v\backslash G_v} f\left(g^{-1}\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} g\right) dg = 0 \quad \text{for } a \neq 1.$$

(ii) Let $L \subset M(2, F_v)$ and $L' \subset D$ be isomorphic quadratic extensions; then

$$\int_{L_v^\times\backslash G_v} f(g^{-1} tg)\, dg = \int_{L_v'^\times\backslash G_v'} \varphi(g^{-1}\, t'g)\, dg$$

if there is an isomorphsim $L_v \to L_v'$ taking t in $L^\times - F^\times$ to $t' \in L'^\times - F^\times$. (Here we select in any way a Haar measure on $F_v^\times\backslash L_v^\times$ and transport it to $F_v^\times\backslash L_v'^\times$ via the isomorphism $L_v \to L_v'$; the quotient spaces are then given the quotient measures.) A consequence of the last few identities is

$$(8.8) \qquad f(e) = \varphi(e).$$

(This follows, for instance, from the Plancherel formula.) Note too that if π_v is a representation of G_v which is neither square-integrable nor finite dimensional then

$$(8.9) \qquad \operatorname{tr} \pi_v(f) = 0.$$

The following lemma results from the characterization of orbital integrals in §4 of **[La 2]**.

LEMMA (8.10). *Given φ in $C_c^\infty(G_v', \omega^{-1})$ there are* (*many*) *f in $C_c^\infty(G_v, \omega^{-1})$ with matching orbital integrals.*

REMARK (8.11). Fix a representation π_v' of G_v' and let π_v be the corresponding representation of G_v given by (8.1). There is easily seen to be a φ in $C_c^\infty(G_v', \omega_v^{-1})$ such that tr $\pi_v'(\varphi) \neq 0$, but tr $\sigma'(\varphi) = 0$ if σ' is any irreducible representation of G_v' (with central quasi-character ω_v) not equivalent to π_v'. Thus if f has matching orbital integrals, tr $\pi_v(f) \neq 0$, but tr $\sigma(f) = 0$ whenever σ is not equivalent to π_v (or finite dimensional).

Now suppose f is a function on $G(A)$ such that $f(zg) = \omega^{-1}(z)f(g)$. Moreover, suppose as before that $f(g) = \prod_v f_v(g_v)$ where $f_v \in C_c^\infty(G_v, \omega_v^{-1})$ and f_v for almost all v is the function defined by

$$\begin{aligned} f_v(g_v) &= \omega_v^{-1}(z) \quad \text{if } g_v = z_v k_v,\ k_v \in K_v,\ z_v \in Z_v, \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Suppose φ is a function on $G'(A)$ satisfying similar conditions, $\varphi_v \sim f_v$ via the isomorphism $G_v \approx G_v'$ for $v \notin S$, and φ_v and f_v have matching orbital integrals for $v \in S$.

PROPOSITION (8.12). *With φ and f as above,*

$$\text{tr } \rho_\omega'(\varphi) = \text{tr } \rho_{\omega,0}(f) + \text{tr } \rho_{\omega,\mathrm{sp}}(f). \tag{8.13}$$

PROOF. By (1.11) the left-hand side is

$$\begin{aligned} &\operatorname{vol}(Z(A)G'(F)\backslash G'(A))\varphi(e) \\ &+ \frac{1}{2}\sum_{L'\in X'} \operatorname{vol}(A^\times L^\times\backslash L^\times(A)) \sum_{\xi\in F^\times\backslash L'^\times;\xi\neq 1} \int_{L'^\times(A)\backslash G'(A)} \varphi[g^{-1}\xi g]\, dg \end{aligned} \tag{8.14}$$

the sum extending over a set X' of representatives for the classes of quadratic extension L' of F embedded in D. Moreover

$$\int_{L'^\times(A)\backslash G'(A)} \varphi[g^{-1}\xi g]\, dg = \prod_v \int_{L_v'^\times\backslash G_v'} \varphi_v[g_v^{-1}\xi g_v]\, dg_v.$$

Here we have selected for each v a Haar measure on $F_v^\times\backslash L_v^\times$ such that (units $F_v^\times\backslash L_v^\times$) $= 1$ for almost all v; to $A^\times\backslash L^\times(A)$ we have given the product measure.

On the GL(2) side, note that S has at least two places v_1, v_2 and for these places f_{v_1}, f_{v_2} have vanishing hyperbolic orbital integrals. Thus by (7.21), the right-hand side of (8.13) is

$$\begin{aligned} &\operatorname{vol}(Z(A)G(F)\backslash G(A))f(e) \\ &+ \frac{1}{2}\sum_{L} \operatorname{vol}(A^\times L^\times\backslash L^\times(A)) \sum_{\xi\in F^\times\backslash L^\times;\xi\neq 1} \int_{L^\times(A)\backslash G(A)} f(g^{-1}\xi g)\, dg \end{aligned} \tag{8.15}$$

where L runs through a set of representatives for the classes of quadratic extensions of F in $M(2, F)$. The measures are selected as above, and

$$\int f(g^{-1}\xi g^{-1})\, dg = \prod_v \int_{L_v^\times\backslash G_v} f(g_v^{-1}\xi g_v)\, dg_v.$$

Note that if $v \in S$ and L_v is split then the corresponding local integral is 0 by (8.7) (i). Thus we need only sum over the set X of L which do *not* split at any $v \in S$, that is, which embed in D. But for every $L \in X$ there is an L' in X' and an isomorphism

$L \to L'$. So if we assume that the Haar measures of $F_v^\times \backslash L_v^\times$ and $F_v^\times \backslash L_v'^\times$ correspond to one another we have

$$\int_{L_v^\times \backslash G_v} f_v(g^{-1}\xi g)\, dg_v = \int_{L_v'^\times \backslash G_v'} \varphi_v(g^{-1}\xi g)\, dg_v$$

for all v. Indeed for $v \in S$ this is clear, and for $v \notin S$ it follows from the fact that we may modify the isomorphism $G_v \to G_v'$ so as to make it compatible with $L_v^\times \to L_v'^\times$.

We conclude from (8.7) and our choice of φ_v, f_v for $v \notin S$ that in (8.14) and (8.15) the series on L and L' are equal. Since we have used the Tamagawa measures on G and G' we also know that

$$\mathrm{vol}(Z(A)G'(F)\backslash G'(A)) = \mathrm{vol}(Z(A)G(F)\backslash G(A)).$$

Therefore, taking (8.8) into account, (8.12) follows.

C. *Proofs of the main result.* Let $\rho'_{\omega,0}$ be the representation of $G'(A)$ in the orthocomplement of the space spanned by the functions $\chi \circ \nu$ with $\chi^2 = \omega$. From (8.6) and our choice of f_v, φ_v for $v \notin S$ we get

$$\int_{Z(A)\backslash G(A)} f(g)\chi(\det g)\, dg = \int_{Z(A)\backslash G'(A)} \varphi(g)\chi(\nu(g))\, dg.$$

Thus (8.12) implies that

$$\mathrm{tr}\, \rho_{\omega,0}(f) = \mathrm{tr}\, \rho'_{\omega,0}(\varphi). \tag{8.16}$$

FIRST PROOF OF THEOREM (8.2). For each place $v \in S$ fix an irreducible representation σ_v' of G_v' and let σ_v be the corresponding representation of G_v. Choose $\varphi_v \in C_c^\infty(G_v', \omega_v^{-1})$ such that $\mathrm{tr}\, \sigma_v'(\varphi_v) = 1$, and $\mathrm{tr}\, \pi_v'(\varphi_v) = 0$ for π_v' inequivalent to σ_v'. Thus identity (8.16) reads

$$\sum n(\pi')\, \mathrm{tr}\, \pi'(\varphi) = \sum n(\pi)\, \mathrm{tr}\, \pi(f)$$

where the sum on the right-hand (resp. left-hand) side is over all irreducible representations π of $G(A)$ (resp. π' of $G'(A)$) such that $\pi_v \simeq \sigma_v$ (resp. $\pi_v' \simeq \pi_v$) for all v in S, and $n(\pi)$ (resp. $n(\pi')$) is the multiplicity of π in $\rho_{\omega,0}$ (resp. $\rho'_{\omega,0}$). Moreover, if π and π' satisfy these conditions, then, because S has even cardinality, we get $1 = \prod_{v \in S} \pi_v'(\varphi_v) = \prod_{v \in S} \pi_v^v(f_v)$. Thus

$$\sum n(\pi') \prod_{v \notin S} \mathrm{tr}\, \pi_v'(\varphi_v) = \sum n(\pi) \prod_{v \notin S} \mathrm{tr}\, \pi_v(f_v)$$

$$(\pi_v' \simeq \sigma_v \text{ for } v \in S,\ \pi_v \simeq \sigma_v \text{ for } v \in S),$$

and Theorem (8.2) follows from fundamental principles of functional analysis (applied to the isomorphic groups G^S and G'^S). Moreover, if π' corresponds to π, then $n(\pi') = n(\pi)$. So since we already know from **[JL]** that $n(\pi)$ is at most 1, we have also proved multiplicity one for π'.

SECOND PROOF. We rewrite (8.16) as

$$\sum \prod_v \mathrm{tr}\, \pi_v(f_v) = \sum \prod_v \mathrm{tr}\, \pi_v'(\varphi_v), \tag{8.17}$$

where $\pi = \otimes \pi_v$ (resp. $\pi' = \otimes \pi_v'$) runs through all irreducible subrepresentations of $\rho_{\omega,0}$ (resp. $\rho'_{\omega,0}$). What we shall do now is manipulate (8.17) until it equates the right side to the trace of just *one* representation. (Since we are summing over subrepresentations rather than classes, two representations on the right side of (8.17) may, a priori, be equivalent.)

LEMMA (8.18). *Fix an arbitrary place w outside S and an irreducible unitary representation τ_w of G_w. If $f = \prod f_v$ and $\varphi = \prod \varphi_v$ are as in* (8.12) *then*

$$\sum_{\pi=\otimes\pi_v} \prod_{v\neq w} \operatorname{tr} \pi_v(f) = \sum_{\pi'=\otimes\pi'_v} \prod_{v\neq w} \operatorname{tr} \pi'_v(\varphi_v)$$

$$(\pi \subset \rho_{\omega,0};\, \pi_w \simeq \tau_w)\,(\pi' \subset \rho'_{\omega,0};\, \pi'_w \simeq \tau_w).$$

PROOF. Set

$$a_{\tau_w} = \sum \prod_{v\neq w} \operatorname{tr} \pi_v(f_v) - \sum \prod_{v\neq w} \operatorname{tr} \pi'_v(\varphi_v)$$

$$(\pi \subset \rho_{\omega,0},\, \pi_w \simeq \tau_w)\,(\pi' \subset \rho'_{\omega,0},\, \pi'_w \simeq \tau_w).$$

It suffices to prove $a_{\tau_w} = 0$. But (8.17) can be rewritten as

$$\sum \operatorname{tr} \pi_w(f_w) \prod_{v\neq w} \operatorname{tr} \pi_v(f_v) = \sum_{\pi'\subset\rho'_{\omega,0}} \operatorname{tr}\ \pi'_\omega(\varphi_w) \prod_{v\neq w} \operatorname{tr} \pi'_v(\varphi_v)$$

or, since $G_w \simeq G'_w$ and $f_w \simeq \varphi_w$, as

$$0 = \sum_{\tau_w \text{ in } \mathscr{E}(G_w)} a_{\tau_w} \operatorname{tr} \tau_w(f_w).$$

Moreover, f_w is completely arbitrary in $C_c^\infty(G_w, \omega_w^{-1})$. Therefore a_{τ_w} must be zero by the generalized "linear independence of characters for GL(2)" (cf. [**LL**]).

LEMMA (8.19). *For every $w \notin S$ fix τ_w (which is unramified for almost all w). Then if f_v and φ_v have matching orbital integrals for $v \in S$,*

$$\sum \prod_{v\in S} \operatorname{tr} \pi_v(f_v) = \sum \prod_{v\in S} \operatorname{tr} \pi'_v(\varphi_v) \tag{8.20}$$

$$(\pi \subset \rho_{\omega,0},\, \pi_w \simeq \tau_w,\, w \notin S)\,(\pi' \subset \rho'_{\omega,0},\, \pi_w \simeq \tau_w,\, w \notin S).$$

PROOF. Apply the argument above to the restricted direct product $G^S = \prod_{w\notin S} G_w$.

PROPOSITION (8.21). *Given $\pi' = \otimes\pi'_v$ in $\rho'_{\omega,0}$ there exists a unique $\pi = \otimes\pi_v$ in $\rho_{\omega,0}$ such that $\pi_v \approx \pi'_v$ for all $v \notin S$. Moreover $\pi_v \in \mathscr{E}_2(G_v)$ for all v in S.*

PROOF. Uniqueness follows from the strong multiplicity one theorem for GL(2). What remains to be shown is that there is at least one such π, and that π_v belong to $\mathscr{E}_2(G_v)$ for all v in S.

Suppose no such π exists. Then applying (8.19) with $\tau_w = \pi'_w$ for all $w \notin S$ we conclude

$$\sum \prod_{v\in S} \operatorname{tr} \sigma'_v(\varphi_v) = 0 \qquad (\sigma' \subset \rho'_{\omega,0},\, \sigma'_w \simeq \pi'_w \text{ for } w \in S). \tag{8.22}$$

But (8.22) is a sum of characters of the group $G'_S = \prod_{v\in S} G'_v$. Thus by the linear independence of characters for G'_S we get a contradiction.

Now suppose $\pi_{v_0} \notin \mathscr{E}_2(G_v)$ for some v_0 in S. The fact that π is cuspidal excludes the possibility that π_{v_0} is finite dimensional. So by (8.9) we have $\pi_{v_0}(f_{v_0}) = 0$, and again by (8.22) we get a contradiction.

COROLLARY (8.23). *The left-hand side of* (8.20) *reduces to exactly one term when* $\tau_w = \pi'_w$ *and* $\otimes\pi'_v$ *is in* $\rho'_{\omega,0}$.

PROPOSITION (8.24). *Given* $\pi = \otimes\pi_v$ *in* $\rho_{\omega,0}$ *such that* $\pi_v \in \mathscr{E}_2(G_v)$ *for all* $v \in S$ *there exists a unique* $\pi' = \otimes\pi'_v$ *in* $\rho'_{\omega,0}$ *such that* $\pi'_v = \pi_v$ *for all* $v \notin S$. *Moreover, for each* $v \in S$, *there is* $\varepsilon_v = \pm 1$ *such that* $\theta'_{\pi_v}(t') = \varepsilon_v\theta_{\pi_v}(t)$ *each time* $t' \in G_v$ *and* $t \in G_v$ *satisfy the conditions of* (8.1).

Later we shall see that ε_v in fact equals 1.

PROOF. For $v \in S$ let X_v denote a set of representatives for the equivalence classes of quadratic extensions L_v of F_v contained in $M(2, F_v)$. As in [**JL**, §15], we can introduce a measure μ_v on $\bigcup F_v^\times\backslash L_v^\times$ ($L_v \in X_v$) and we consider the Hilbert space $\mathscr{H}_v = \mathscr{H}_v(\omega_v, \mu_v)$ of all functions h on that union which satisfy $h(zl) = \omega_v(z)h(l)$, for $z \in F_v^\times$, and $\int |h(x)|^2\, d\mu_v(x) < +\infty$; the functions θ_λ, $\lambda \in \mathscr{E}_2(G_v, \omega_v)$, then determine an orthonormal basis of $\mathscr{H}_v$.

Similarly we can introduce a Hilbert space $\mathscr{H}'_v$ for the group G'_v. But since any quadratic extension of F_v embeds into D_v, we may identify $\mathscr{H}_v$ and $\mathscr{H}'_v$.

Recall $\pi = \otimes_v\pi_v \subset \rho_{\omega,0}$ is such that $\pi_v \in \mathscr{E}_2(G_v)$ for all $v \in S$. So for each $v \in S$ there is $\sigma_v \in \mathscr{E}(G'_v, \omega_v)$ such that (when we regard θ_{π_v} and θ_{σ_v} as elements of $\mathscr{H}_v$) $\langle\theta_{\pi_v}, \theta_{\sigma_v}\rangle = a_v \neq 0$. Since both θ_{π_v} and θ_{σ_v} are unit vectors, it follows that $|a_v| \leqq 1$.

Now, fix φ_v to be θ_{σ_v} when $v \in S$. Then for any $\pi'_v \in \mathscr{E}(G'_v, \omega_v)$,

$$\begin{aligned} \operatorname{tr} \pi'_v(\varphi_v) &= 1 \quad \text{if } \pi'_v \simeq \sigma_v, \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

In particular, if $f_v \in C_c^\infty(G_v, \omega_v^{-1})$ and φ_v have matching orbital integrals, then (using the notations of [**JL**, §15])

$$\begin{aligned} \operatorname{tr} \pi_v(f_v) &= \int_{Z_v\backslash G_v} f_v(g)\theta_{\pi_v}(g)\, dg \\ &= \sum_{L_v} \frac{1}{2} \int_{Z_v\backslash L_v^\times} \delta(b)\theta_{\pi_v}(b) \int_{L_v'^\times\backslash G'_v} \theta_{\sigma_v}(g^{-1}bg)\, dg \\ &= \langle\theta_{\pi_v}, \theta_{\sigma_v}\rangle = a_v. \end{aligned}$$

This means (8.20) with $\tau_v = \pi_v$ for $v \notin S$ reduces to the identity

$$\prod_{v \in S} a_v = \sum 1 \qquad (\pi' \subset \rho'_{\omega,0}, \pi'_v \simeq \sigma_v \text{ for } v \in S, \pi'_v \simeq \pi_v \text{ for } v \notin S).$$

From this we conclude that the right-hand side has only 1 term (once again proving multiplicity one for G'). We also conclude that $|a_v| = 1$. So since a simple argument involving π_v and $\tilde{\pi}_v$ implies a_v must be real, we conclude finally that $a_v = \pm 1$, i.e., $\langle\theta_{\pi_v}, \theta_{\sigma_v}\rangle = \pm 1$. But this implies that $\pi'_v = \sigma_v$ is completely determined by π_v and that $\theta_{\pi_v} = \pm\,\theta_{\pi'_v} = \theta_{\sigma_v}$ (regarded as elements of $\mathscr{H}$). Thus the proposition follows.

It remains to complete the local correspondence asserted by Theorem (8.1) and to show that θ_{π_v} actually equals $-\theta_{\pi'_v}$. For this we need to embed an arbitrary local representation π'_v of G'_v in an automorphic representation π' and exploit the fact that the cardinality of S is even. For details, see the original arguments in [**Fl**].

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