Three lectures on Hodge structures

Damien Mégy[*]

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These are preliminary notes for three lectures given at the spring school "classical and p-adic Hodge theories" in Rennes, France. Latest version available at `http://iecl.univ-lorraine.fr/~Damien.Megy/Megy_Hodge.pdf`, I plan to upload a new version in a few days (hopefully). Comments are most welcome.

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[*]Institut Élie Cartan, université de Lorraine, France. Email: damien.megy@univ-lorraine.fr

Consider your favorite functorial invariant from a category of geometric objects to some abelian or linear category. For example cohomological invariants. First question : inverse problem ? For example:
Question 1: is there a smooth projective variety $\mathrm{X}/\mathbb{C}$ with $\mathrm{H}^1(\mathrm{X}, \mathbb{Z}) \simeq \mathbb{Z}$?

More generally, we can ask if these functors are full, faithfull, essentially surjective... in some cases to get a reasonable answer one must schrink, expand or enrich the categories. In some other cases there is no known (or reasonable) answer.

What are Hodge structures ?

Hodge structures are a way to package several cohomological invariants and comparison isomorphisms between them in one object. They form an abelian category endowed with a forgetful functor to abelian groups, and singular cohomology factors throught an enriched cohomology functor with values in the category of Hodge structures. The question of the fullness of this functor is related to the famous Hodge conjecture (which we won't discuss at all).

What are they useful for ?

- Better invariant : Hodge structures can distinguish between two non-isomorphic elliptic curves, for instance. Singular cohomology cannot because the curves are homeomorphic.
- Hodge theory can answer the above question : No, there cannot be any smooth projective variety $\mathrm{X}/\mathbb{C}$ with $\mathrm{H}^1(\mathrm{X}, \mathbb{Z}) = \mathbb{Z}$. Also can answer similar questions.
- Very strong conditions on the cohomology of smooth projective varieties, and in particular: the Lefschetz Package : a set of beautiful properties satisfied by the cohomology of smooth projective varieties over $\mathbb{C}$. This will be discussed in the second lecture. The relative version of the Hodge package (in both smooth and singular settings) will be explained by Migliorini.

In the first lecture we discuss what it means for a variety to admit a (strong) Hodge decomposition on its cohomology, and we introduce the category of Hodge structures. In lecture 2 we describe the Hodge-Lefschetz package for compact-Kähler manifolds. The third lecture is about spectral sequences and in particular the Frölicher spectral sequence, holomorphic de Rham cohomology, and algebraic de Rham cohomology. All the material is classical and can be found in books such as [Dem], [Huy] and particularly [Voi]. The original papers of Deligne often provide a more conceptual formalism which applies in both the complex and arithmetic settings.

1 The Hodge Decomposition

1.1 Betti Cohomology

1.1.1 Singular cohomology

From the topological point of view, a cohomology theory with coefficients in a ring A is a contravariant functor H from a (nice) category of topological spaces to the category of graded A-modules (or graded-commutative algebras) which statisfies a number of axioms, such as existence of long exact sequences etc. Then, if the category of topological spaces is sufficiently nice, it is possible to prove that such a cohomology theory is essentially unique [May, chap. 13 and 18].

Here, we consider complex manifolds, or more generally topological or differentiable manifolds. These are (very) nice spaces, (for instance, they are CW-complexes) and we can choose the cohomology theory to be singular (or cellular) cohomology. In the rest of the lectures, will call this the *Betti cohomology* of a complex variety.

Betti cohomology can have coefficients in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$ and other rings or modules. The universal coefficients theorem says that there is an isomorphism

$$\mathrm{H}^k(\mathrm{X}, \mathbb{Q}) \simeq \mathrm{H}^k(\mathrm{X}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$$

and similarly for $\mathbb{R}$ and $\mathbb{C}$.

We recall a some relevant features of singular cohomology in the geometric context. Betti cohomology with coefficients in $\mathbb{Q}$ is a contravariant functor

$$\mathrm{H} : \mathcal{V}^{opp} \to \mathrm{G}r\mathrm{V}ec_{\mathbb{Q}}$$

from the category $\mathcal{V}$ of compact orientable smooth manifolds. to the category of finite-dimensional graded $\mathbb{Q}$-vector spaces, such that:

1. for all $\mathrm{X}, \mathrm{Y} \in \mathcal{V}$, there is a canonical isomorphism $\mathrm{H}(\mathrm{X} \cup \mathrm{Y}) \xrightarrow{\sim} \mathrm{H}(\mathrm{X}) \oplus \mathrm{H}(\mathrm{Y})$;
2. (Künneth formula) $\mathrm{H}(\mathrm{X}) \otimes \mathrm{H}(\mathrm{Y}) \xrightarrow{\sim} \mathrm{H}(\mathrm{X} \times \mathrm{Y})$; this means that H is a $\otimes$-functor;
3. $\mathrm{H}^k(\mathrm{X}) = 0$ if $k \notin [0, 2\dim \mathrm{X}]$;
4. for each $\mathrm{X} \in \mathcal{V}$ of dimension d, the choice of an orientation yields an isomorphism[1] $\mathrm{H}^{2d} \xrightarrow{\sim} \mathbb{Q}$, inducing a perfect pairing
$$\mathrm{H}^k(\mathrm{X}) \otimes \mathrm{H}^{2d-k}(\mathrm{X}) \to \mathrm{H}^{2d}(\mathrm{X}) \to \mathbb{Q}.$$

This looks like part of the general definition of a Weil cohomology theory. Here, we are lacking the Tate object, the cycle class map and compatibility with cup-product and more. Hopefully we will come back to this in the third lecture.

[1] Note that there is no way to canonically identify the top cohomology to $\mathbb{Q}$; in the arithmetic setting, such an identification is part of the data, it is called the trace map.

1.1.1 Definition. *For $0 \leq k \leq \dim_{\mathbb{R}} \mathrm{X}$, the Betti numbers b_k are the dimensions of the $\mathbb{Q}$-vector spaces $\mathrm{H}^k(\mathrm{X}, \mathbb{Q})$. The Euler characteristic is $\chi(\mathrm{X}) = \sum_{k \in \mathbb{N}} (-1)^k b_k(\mathrm{X})$.*

1.1.2 Remark. *Poincaré duality can fail if X is not a smooth. Take the projective cone over an complex elliptic curve for example: the Betti numbers are not symmetric (exercise). Thus Betti cohomology is probably not a good invariant when working with singular varieties. See Migliorini's course.*

1.1.3 Example. *All complex elliptic curves are diffeomorphic to $\mathrm{S}^1 \times \mathrm{S}^1$ and thereby have the same Betti cohomology, even though they are in general not algebraically isomorphic. The Betti numbers are $b_0 = 1, \quad b_1 = 2, \quad b_2 = 1$.*

1.1.4 Example. *(Real and complex Tori) A real torus is a real compact manifold diffeomorphic to a product of circles : $\mathrm{M} \simeq (\mathrm{S}^1)^d$. Its Betti numbers are given by Künneth formula : $b_k = \binom{d}{k}$. A complex torus is a complex compact manifold obtained by taking the quotient of a complex vector space by a discrete subgroup $\Lambda \subset \mathbb{C}^n$ of maximal rank : $\mathrm{T} \simeq \mathbb{C}^n/\Lambda$. The underlying real manifold is a real torus.*

1.1.5 Example. *(Hopf surface) The Hopf surface is the quotient of $\mathbb{C}^2 \setminus (0,0)$ by the equivalence relation $(z_1, z_2) \simeq \frac{1}{2}(z_1, z_2)$. It is diffeomorphic to $\mathrm{S}^3 \times \mathrm{S}^1$, and its Betti numbers are $1, 1, 0, 1, 1$. This looks very bad for several reasons. The Hopf surface will be our favorite counter-example.*

1.1.6 Example. *Compute the Euler characteristic of smooth degree d complex hypersurface in $\mathbb{P}^n(\mathbb{C})$. All such hypersurfaces are diffeomorphic so take the Fermat hypersurface $\mathrm{X}_0^d + ... + \mathrm{X}_n^d = 0$ for example.*

1.1.7 Remark. *Betti cohomology can be computed as the cohomology of several complexes of abelian groups : cellular complexes, singular complexes, etc. Note that any complex $\mathrm{K}^\bullet$ of free abelian groups decomposes as the direct sum of objects supported in only two degrees. For a complex of finite-dimensional $\mathbb{Q}$-vector spaces, there is a splitting into indecomposable summands (or* blocks*):*

$$... \to 0 \to \mathbb{Q} \to 0 \to ... \text{ and } .. \to 0 \to \mathbb{Q} \xrightarrow{\sim} \mathbb{Q} \to 0 \to ...$$

Only the first ones contribute to the cohomology of the complex. For example, the complex

$$\mathbb{Q}^2 \xrightarrow{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}} \mathbb{Q}^2 \xrightarrow{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}} \mathbb{Q}^2$$

decomposes as the direct sum of the following blocks:

$$\mathbb{Q} \xrightarrow{\sim} \mathbb{Q} \to 0, \quad 0 \to \mathbb{Q} \xrightarrow{\sim} \mathbb{Q}, \quad \mathbb{Q} \to 0 \to 0, \quad 0 \to 0 \to \mathbb{Q}.$$

Later we will see a similar decomposition result for double complexes.

1.1.2 Comparison with De Rham cohomology

Let X be a smooth manifold, and $(\mathrm{A}^\bullet(\mathrm{X}), d)$ its De Rham complex, given by the exterior derivative acting on global $\mathcal{C}^\infty$ forms on X. The (real) De Rham cohomology groups are :

$$\mathrm{H}^k_{\mathrm{DR}}(\mathrm{X}, \mathbb{R}) = \mathrm{H}^k(\mathrm{A}^\bullet(\mathrm{X}), d) = \frac{\operatorname{Ker} d : \mathrm{A}^k(\mathrm{X}) \to \mathrm{A}^{k+1}(\mathrm{X})}{d(\mathrm{A}^{k-1}(\mathrm{X}))}.$$

De Rham cohomology with complex coefficients is defined as the cohomology of the complex $\mathrm{A}^\bullet(\mathrm{X}) \otimes \mathbb{C}$. It is isomorphic to $\mathrm{H}^k_{\mathrm{DR}}(\mathrm{X}, \mathbb{R}) \otimes \mathbb{C}$.

Let $a \in \mathrm{H}^k_{\mathrm{DR}}(\mathrm{X}, \mathbb{R})$ be the class of a closed k-form α. For every homology class c of degree k, we can choose a smooth representative γ of c. Stokes' theorem gives a well-defined functional $\Psi_a : c \mapsto \int_\gamma \alpha$.

1.1.8 Theorem. *(De Rham) For all $k \in \mathbb{N}$, the map*

$$\Psi : \mathrm{H}^k_{\mathrm{DR}}(\mathrm{X}, \mathbb{R}) \to \mathrm{H}^k_{sing}(\mathrm{X}, \mathbb{R}), \quad a \mapsto \Psi_a$$

is an isomorphism.

Note that $\mathrm{A}^k(\mathrm{X})$ is the space of global sections of the sheaf $\mathcal{A}^k_\mathrm{X}$ of smooth forms. The complex of sheaves

$$\mathbb{R} \to \mathcal{A}^0_\mathrm{X} \to \mathcal{A}^1_\mathrm{X} \to ...$$

is exact by the Poincaré lemma and provides a resolution of the constant sheaf $\mathbb{R}$. On the other hand, the sheaves $\mathcal{A}^k_\mathrm{X}$ are acyclic. Therefore, this resolution computes the sheaf cohomology of $\mathbb{R}$ and we get an isomorphism $\mathrm{H}_{\mathrm{DR}}(\mathrm{X}, \mathbb{R}) \simeq \mathrm{H}^k(\mathrm{X}, \underline{\mathbb{R}})$.

1.1.9 Example. *Let* S *be the Hopf surface. Denote by $r = \sqrt{z \cdot \overline{z}} = \sqrt{z_1\overline{z_1} + z_2\overline{z_2}}$ the radius function on $\mathbb{C}^2$. The closed* 1*-form $\frac{d(r^2)}{r^2} = \frac{d(z \cdot \overline{z})}{z \cdot \overline{z}}$ represents a non-trivial cohomology class in* $\mathrm{H}^1(\mathrm{S}, \mathbb{R})$.

1.1.10 Example. *(choice of a basis) Consider $\mathrm{X} = \mathbb{P}^1(\mathbb{C})$. Topologically, it is a* 2*-sphere and we have*

$$\mathrm{H}^0(\mathrm{X}, \mathbb{C}) = \mathbb{C}, \quad \mathrm{H}^1(\mathrm{X}, \mathbb{C}) = 0, \quad \mathrm{H}^2(\mathrm{X}, \mathbb{C}) \simeq \mathbb{C}.$$

What generator should we choose for $\mathrm{H}^2(\mathrm{X}, \mathbb{C})$? The form $\omega = \frac{i}{2\pi} \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$ (where z is an affine coordinate) has the advantage that $\int_\mathrm{X} \omega = 1$ (exercise), so the cohomology class of ω is actually a generator of $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$, which is nice from a topological point of view. But from an algebraic point of view, the $\frac{i}{2\pi}$ factor looks weird and unnecessary (the algebraic variety seems *to have nothing to do with π or i), and we would like to choose $\omega_{alg} = \frac{dz \wedge d\bar{z}}{(1+|z|^2)^2}$ as a natural generator... but then of course ω_{alg} doesn't belong to $\mathrm{H}^2(\mathrm{X}, \mathbb{Z})$. In fact, there is no way to get rid of $2\pi i$ in this situation, and the interplay between the algebraic and topological structure is precisely the interesting thing here. This kind of data is encoded in Hodge structures and period matrices.*

1.1.3 Integral structure

De Rham Cohomology doesn't compute rational or integral cohomology. Nevertheless we can recover rational or non-torsion integral classes as classes c such that the integral over any chain is integral or rational.

Examples of integral classes are given by cohomology classes of cycles. Consider for instance a codimension c closed submanifold N with orientable normal bundle of a n-dimensional manifold.

Assume the normal bundle of N is oriented. Then we have Thom's isomorphism

$$\mathrm{T} : \mathrm{H}^c(\mathrm{M}, \mathrm{M} \setminus \mathrm{N}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{N}, \mathbb{Z}).$$

Now, define the cohomology class of N as follows : take the image of $1 \in \mathrm{H}^0(\mathrm{N}, \mathbb{Z})$ under the inverse of Thom's isomrphism, and then the image under the natural map $\mathrm{H}^k(\mathrm{M}, \mathrm{M} \setminus \mathrm{N}, \mathbb{Z}) \to \mathrm{H}^k(\mathrm{M}, \mathbb{Z})$ in the long exacte sequence of relative cohomology.

1.1.11 Remark. *If the submanifold* N *is compact and oriented, then the cohomology class* $[\mathrm{N}]$ *of* N *can be described as follows. Consider the homoloy class of* N *in* $\mathrm{H}_{n-c}(\mathrm{M}, \mathbb{Z})$. *The cohomology class of* N *in* $\mathrm{H}^c(\mathrm{M}, \mathbb{Z})$ *is the image of this homology class under the Poincaré duality isomorphism* $\mathrm{H}_{n-c}(\mathrm{M}) \simeq \mathrm{H}^c(\mathrm{M}, \mathbb{Z})$.

1.1.12 Remark. *The cohomology class of a submanifold can be trivial: consider for example* $\mathrm{S}^1 \subset \mathrm{S}^2$. *This can occur for compact complex manifolds, too. We will see in the next lecture that if* M *is a compact Kähler manifold and* N *a complex submanifold, the cohomology class* $[\mathrm{N}]$ *is never trivial. This is a very important feature of compact Kähler manifolds – and in particular of smooth projective varieties over* $\mathbb{C}$.

1.2 Complex structure and Dolbeault double complex

1.2.1 Dolbeault cohomology and other invariants

Let X be a n-dimensional compact complex variety. The (almost) complex structure on X allows to define a decomposition by type on de De Rham complex of (complex-valued) differential forms.

Locally, this decomposition can be described as follows. let T an n-dimensional complex vector space (T should be thought of as the tangent space of the complex manifold), and $\mathrm{F} = \mathrm{Hom}_{\mathbb{R}}(\mathrm{V}, \mathbb{C})$. The complex vector space F has two natural complex subspaces T' and T'' of $\mathbb{C}$-linear and $\mathbb{C}$-antilinear forms, and we have $\mathrm{F} = \mathrm{T}' \oplus \mathrm{T}''$. Note that T' is isomorphic to the dual $\mathrm{T}^\vee$ of T and $\mathrm{T}'' = \overline{\mathrm{T}'}$. Let $k \in \mathbb{N}$, and consider $\bigwedge^k \mathrm{F}$. It splits as $\bigwedge^k \mathrm{F} = \bigoplus_{p+q=k} \bigwedge^{p,q} \mathrm{F}$, where $\bigwedge^{p,q} \mathrm{F} := \bigwedge^p \mathrm{T}' \otimes \bigwedge^q \mathrm{T}''$.

We denote by $\mathcal{A}_{\mathrm{X}}^{p,q}$ the $\mathcal{C}_{\mathrm{X}}^\infty$-module of $\mathbb{C}$-valued smooth forms of type (p, q), and by $\mathrm{A}^{p,q}(\mathrm{X})$ the space of global sections. Then, there are decompositions $\mathcal{A}_{\mathrm{X}}^k = \bigoplus\limits_{p+q=k} \mathcal{A}_{\mathrm{X}}^{p,q}$ and $\mathrm{A}^k(\mathrm{X}) = \bigoplus\limits_{p+q=k} \mathrm{A}^{p,q}(\mathrm{X})$, and we define two operators ∂ and $\bar{\partial}$ obtained by composing d with the projections.

1.2.1 Example. *Let* S *be the Hopf surface. The closed* 1*-form* $\frac{d(r^2)}{r^2} = \frac{d(z \cdot \bar{z})}{z \cdot \bar{z}}$ *splits as* $\alpha + \overline{\alpha}$, *with* $\alpha = \frac{\bar{z} \cdot dz}{z \cdot \bar{z}}$ *of type* $(1, 0)$. *Note that* α *and* $\overline{\alpha}$ *are not closed.*

The manifold X is a complex manifold, which means that the almost complex structure is integrable. As a result, the operator $d : \mathrm{A}^k(\mathrm{X}) \to \mathrm{A}^{k+1}(\mathrm{X})$ splits as $\partial + \bar{\partial}$. The identity $d \circ d = 0$ implies that the De Rham complex $(\mathrm{A}^\bullet, d)$ is the total complex associated to the double complex $(\mathrm{A}^{\bullet,\bullet}, \partial, \bar{\partial})$. From this we can build a number of cohomological invariants, for example vertical and horizontal cohomology:

$$\mathrm{H}^{p,q}_{\bar{\partial}}(\mathrm{X}) = \frac{\mathrm{Ker}\,\bar{\partial} \cap \mathrm{A}^{p,q}}{\bar{\partial}\mathrm{A}^{p,q-1}} \quad \mathrm{H}^{p,q}_{\partial}(\mathrm{X}) = \frac{\mathrm{Ker}\,\partial \cap \mathrm{A}^{p,q}}{\partial\mathrm{A}^{p-1,q}}$$

And Bott-Chern and Aeppli cohomology :

$$\mathrm{H}^{p,q}_{\mathrm{BC}}(\mathrm{X}) = \frac{\mathrm{Ker}\,d \cap \mathrm{A}^{p,q}}{\partial\bar{\partial}\mathrm{A}^{p-1,q-1}} \quad \mathrm{H}^{p,q}_{\mathrm{A}}(\mathrm{X}) = \frac{\mathrm{Ker}\,\partial\bar{\partial} \cap \mathrm{A}^{p,q}}{\partial\mathrm{A}^{p-1,q} + \bar{\partial}\mathrm{A}^{p,q-1}}$$

More generally, we can consider

$$\mathrm{H}^{*}_{\bar{\partial}}(\mathrm{X}) = \frac{\mathrm{Ker}\,\bar{\partial}}{\mathrm{Im}\,\bar{\partial}}, \quad \mathrm{H}^{*}_{\partial}(\mathrm{X}) = \frac{\mathrm{Ker}\,\partial}{\mathrm{Im}\,\partial}, \quad \mathrm{H}^{*}_{\mathrm{BC}}(\mathrm{X}) = \frac{\mathrm{Ker}\,\partial \cap \mathrm{Ker}\,\bar{\partial}}{\mathrm{Im}\,\partial\bar{\partial}}, \quad \mathrm{H}^{p,q}_{\mathrm{A}}(\mathrm{X}) = \frac{\mathrm{Ker}\,\partial\bar{\partial}}{\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}}$$

These groups fit into the commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^{*}_{\mathrm{BC}} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathrm{H}^{*}_{\partial} \arrow[dr] & \mathrm{H}^{*}_{\mathrm{DR}}(\mathrm{X}) \arrow[d] & \mathrm{H}^{*}_{\bar{\partial}} \arrow[dl] \\
 & \mathrm{H}^{*}_{\mathrm{A}} &
\end{tikzcd} \tag{1.2.1}$$

and there is a similar diagram for (p, q)-components. All this can of course be defined for any double complex.

The vertical cohomology group $\mathrm{H}^{p,q}_{\bar{\partial}}(\mathrm{X})$ is called the (p, q) Dolbeault cohomology group. In general, we cannot compute it by topological methods, such as singular chains, Morse theory etc. Note that if X is not compact, $\mathrm{H}^{p,q}_{\bar{\partial}}(\mathrm{X})$ has no reason to be finite-dimensional. If it is, then we set $h^{p,q} = \dim \mathrm{H}^{p,q}_{\bar{\partial}}(\mathrm{X})$. The numbers $h^{p,q}$ are called the *Hodge numbers* of the manifold X.

1.2.2 Exercise. *Let* S *be the Hopf surface. Prove that* $h^{0,1}(\mathrm{S}) = 1$ *and* $h^{1,0}(\mathrm{S}) = 0$. *(Use the Noether-Lefschetz formula [**?**] for example).*

1.2.3 Remark. *There is no natural morphism betwen Dolbeault cohmology and De Rham cohomology, except for a few cases. For example there are arrows* $\mathrm{H}^{k}(\mathrm{X}, \mathbb{C}) \to \mathrm{H}^{0,k}_{\bar{\partial}}(\mathrm{X})$ *and* $\mathrm{H}^{k}(\mathrm{X}, \mathbb{C}) \to \mathrm{H}^{k,0}_{\partial}(\mathrm{X})$, *called edge-homomorphisms. In general, if* $k > 1$, *they are neither injective nor surjective. These morphisms come from spectral sequences.*

1.2.4 Remark. *(Construction problem)*
*Is it easy to construct manifolds with given Hodge numbers ? As we will see in the next lectures, Hodge theory gives a lot of restrictions on the possible Betti and Hodge numbers. The question is partially answered in [**?**].*

1.3 Strong Hodge decomposition and the $\partial\bar{\partial}$ lemma

1.3.1 What is a (strong) Hodge decomposition ?

Let X be a compact complex manifold of dimension n We have seen that the spaces of k-forms decompose as the direct sum $\mathrm{A}^k(\mathrm{X},\mathbb{C}) = \bigoplus_{p+q=k} \mathrm{A}^{p,q}(\mathrm{X},\mathbb{C})$. The question is wether this decomposition carries over to cohomology in a natural way. This means a decomposition $\mathrm{H}^k(\mathrm{X},\mathbb{C}) = \bigoplus \mathrm{H}^{p,q}(\mathrm{X})$ as a direct sum of subspaces called the *Hodge subspaces*, which are yet to be defined.

Let α be a closed k-form on X, representing a cohomology class. Then we have $\alpha = \sum_{p,q} \alpha^{p,q}$, but the (p,q) components have no reason to be closed. A (p,q) decomposition in the cohomology group H^k would provide an equality $\alpha = \sum_{p,q} \alpha^{p,q}$ up to an exact k-form, with closed representatives $\alpha^{p,q}$ of the (p,q)-components. We would also like the decomposition to be unique, which means that in the previous decomposition, if α is exact, then all components $\alpha^{p,q}$ are also exact.

In other words, introducing

$$\mathrm{H}^{p,q}(\mathrm{X}) = \frac{\alpha \in \mathrm{A}^{p,q}(\mathrm{X}),\ d\alpha = 0}{\mathrm{I}m(d)},$$

we ask whether the sum of the inclusions $\mathrm{H}^{p,q}(\mathrm{X}) \hookrightarrow \mathrm{H}^k(\mathrm{X},\mathbb{C})$ is an isomorphism onto $\mathrm{H}^k(\mathrm{X})$. We could probably also add some additional conditions on the Dolbeault groups, and comparison conditions with Bott-Chern cohomology.

In general, there is no reason whatsoever for all these conditions to be satisfied.

Note that a that $\mathrm{H}^{p,q}(\mathrm{X})$ and $\mathrm{H}^{q,p}(\mathrm{X})$ have the same dimension because they are complex conjugate to each other. Therefore, a strong Hodge decomposition implies the following property:

$$b_{2l+1}(\mathrm{X}) = \sum_{p+q=2l+1} \dim \mathrm{H}^{p,q} = 2 \sum_{0 \le p \le l} \dim \mathrm{H}^{p,q}(\mathrm{X}) \in 2\mathbb{N}.$$

It also implies that every cohomology class can be represented by a ∂-closed and $\bar{\partial}$-closed smooth form, because each (p,q) component is closed so by purity it is ∂-closed and $\bar{\partial}$-closed.

1.3.1 Example. *Here are a few ways to see that the Hopf surface* S *cannot admit a strong Hodge decomposition.*

1. *We have seen that* $b_1(\mathrm{S}) = 1 \notin 2\mathbb{N}$.
2. *The class of the* 1*-form* $\frac{d(r^2)}{r^2}$ *cannot be represented by a* ∂ *and* $\bar{\partial}$*-closed form.*
3. $\mathrm{H}^{0,1} \oplus \mathrm{H}^{1,0} \to \mathrm{H}^1(\mathrm{X})$ *is not surjective. In fact,* $\mathrm{H}^{1,0}$ *is trivial, because there are no holomorphic* 1*-forms on* S. *This implies that* $\mathrm{H}^{0,1}$ *is also trivial.*
4. $\mathrm{H}^{1,2} \oplus \mathrm{H}^{2,1} \to \mathrm{H}^3(\mathrm{X})$ *is not injective: both Hodge subspaces are equal to* $\mathrm{H}^3(\mathrm{X})$.
5. $\mathrm{H}^{0,1}_{\bar{\partial}}(\mathrm{X})$ *is one-dimensional, spanned by the Dolbeault class of* $\overline{alpha} = \frac{z \cdot d\bar{z}}{z \cdot \bar{z}}$ *which is* $\bar{\partial}$*-closed, whereas* $\mathrm{H}^{1,0}_{\bar{\partial}}(\mathrm{X})$ *is the space of holomorphic* 1*-forms and is trivial.*

1.3.2 Cohomology and decomposition of double complexes

In these notes a double complex $\mathrm{K}^{\bullet,\bullet,\partial,\bar{\partial}}$ is a bigraded object in an abelian category with anti-commuting arrows of $\partial, \bar{\partial}$ of degree $(1,0)$ and $(0,1)$. We choose this convention so that the De Rham double complex of a complex manifold is a double complex. Changing the signs of the differentials gives a complex of complexes.

1.3.2 Example. *The following double complexes play a particular role in the rest of the lecture. A double complex is a* dot *if all terms are zero except for one* $\mathrm{K}^{p_0,q_0} \neq 0$*. All differentials are zero.*

$$\begin{tikzcd} 0 & 0 & 0 \\ 0 \arrow[r] & \mathrm{K}^{p,q} \arrow[r] \arrow[u] & 0 \\ 0 & 0 \arrow[u] & 0 \end{tikzcd}$$

A square *is a double complex of the form*

$$\begin{tikzcd} \mathrm{K}^{p,q+1} \arrow[r, "\sim"] & \mathrm{K}^{p+1,q+1} \\ \mathrm{K}^{p,q} \arrow[r, "\sim"'] \arrow[u, "\sim"] & \mathrm{K}^{p+1,q} \arrow[u, "\sim"] \end{tikzcd}$$

A zigzag *is a bounded double complex of the form*

$$\begin{tikzcd} \mathrm{K}^{p,q} \arrow[r, "\sim"] & \mathrm{K}^{p+1,q} & & \\ & \mathrm{K}^{p+1,q-1} \arrow[u, "\sim"] \arrow[r, "\sim"] & \mathrm{K}^{p+2,q-1} & \\ & & \mathrm{K}^{p+2,q-2} \arrow[u, "\sim"] \arrow[r, "\sim"] & \cdots \end{tikzcd}$$

of any length, where the length is the number of non-trivial terms (for instance a dot is a zigzag of length one). A zigzag can also start with a vertical arrow:

$$\begin{tikzcd} \mathrm{K}^{p+1,q} & \\ \mathrm{K}^{p+1,q-1} \arrow[u, "\sim"] \arrow[r, "\sim"] & \mathrm{K}^{p+2,q-1} \\ & \cdots \arrow[u, "\sim"] \end{tikzcd}$$

1.3.3 Theorem. *(Does anyone know a precise/convenient reference ? Exercise.)*
Every double complex decomposes as a direct sum of squares, dots, and zigzags.

If $\partial\bar{\partial}$ is always zero, then this should follow from a theorem of Dieudonné. For each element x in some $\mathrm{K}^{p,q}$ such that $\partial\bar{\partial}x \neq 0$, there is a split square.

It seems that the theorem can be recovered from the representation theory of special biserial algebras.

1.3.3 The $\partial\bar{\partial}$-lemma

1.3.4 Proposition. *Let $(\mathrm{K}^{\bullet,\bullet}, \partial, \bar{\partial})$ be a bounded double complex in an abelian category, and denote by $(\mathrm{K}^{\bullet}, d = \partial + \bar{\partial})$ the associated total complex. For each $n \in \mathbb{Z}$, the following conditions are equivalent:*

$(a)_n$ $\mathrm{Ker}\,\partial \cap \mathrm{Ker}\,\bar{\partial} \cap \mathrm{Im}\; d = \mathrm{Im}\; \partial\bar{\partial}$;

$(b)_n$ $\mathrm{Ker}\,\partial \cap \mathrm{Ker}\,\bar{\partial} \cap (\mathrm{Im}\; \partial + \mathrm{Im}\; \bar{\partial}) = \mathrm{Im}\; \partial\bar{\partial}$;

Proof. [DGMS, 5.15] □

1.3.5 Definition. *We say that the $\partial\bar{\partial}$ lemma holds for $\mathrm{K}^{\bullet,\bullet}$ if the above equivalent conditions hold for every $n \in \mathbb{Z}$. We say that it holds for a complex manifold if it holds for its Dolbeault double complex.*

If the lemma holds, the natural maps between Dolbeault, De Rham, Bott-Chern and Aeppli cohomology in the commutative diagram 1.2.1 are isomorphisms.

1.3.6 Example. *The $\partial\bar{\partial}$-lemma fails for any zigzag.*

1.3.7 Exercise. *Find a zig-zag in the Dolbeault double complex of the surface. Hint : start with the 1-form α.*

1.3.8 Theorem. *([DGMS, 5.17])*
Let $(\mathrm{K}^{\bullet,\bullet}, \partial, \bar{\partial})$ be a bounded double complex in a semisimple abelian category, and denote by $(\mathrm{K}^{\bullet}, d = \partial + \bar{\partial})$ the associated total complex. The following conditions are equivalent:

1. *The $\partial\bar{\partial}$ lemma holds for K;*
2. *$\mathrm{K}^{\bullet,\bullet}$ decomposes as a direct sum of squares and dots.*
3. (a) *The differential d of the total complex $\mathrm{K}^{\bullet}$ is strict for the two filtrations on $\mathrm{K}^{\bullet}$ (i.e. $\mathrm{F}^p\mathrm{K}^{\bullet} \cap d\mathrm{K} = d\mathrm{F}^p\mathrm{K}^{\bullet}$) <u>and</u>:*
   (b) *For all n, the two induced filtrations F_1 and F_2 on $\mathrm{H}^n(\mathrm{K}^{\bullet}), d$ are n-opposite, i.e. $\mathrm{H}^n = \mathrm{F}_1^p\mathrm{H}^n \oplus \mathrm{F}_2^{q+1}\mathrm{H}^n$ for all $p + q = n$.*

Proof. (2)⇒ (1) and (2)⇒ (3) : it is enough to prove the results for squares and dots, which is easy.

(1)⇒ (2) : there can be no zigzags in the decomposition into elementary blocks, because the $\partial\bar{\partial}$-lemma doesn't hold for them.

(2)⇔(3) : again, decompose into elementary blocks. There are two types of zigzags of even-length and each kind the prevents one strictness property. Odd-length zigzags are the reason why the n-oppositeness can fail.

Note that the proof given in [DGMS] doesn't make use of the decomposition into elementary blocks. The authors prove (1)⇒ (2) by explicitly splitting each $\mathrm{K}^{p,q}$ by hand, and then prove (3)⇒ (1) by a direct argument. The statement (3.a) can be rephrased in terms of spectral sequences, see third lecture.

□

1.3.9 Corollary. *The cohomology of a complex manifold has a strong Hodge decomposition if and only if the $\partial\bar{\partial}$-lemma holds on* X.

Compact Kähler manifolds (in particular, smooth projective varieties over $\mathbb{C}$) satisfy the $\partial\bar{\partial}$-lemma (second lecture).

The condition 3.b of the theorem leads to one of the definitions of Hodge structures.

1.4 The category of Hodge structures

In this section, A is a noetherian subalgebra of $\mathbb{R}$ such that $\mathrm{A} \otimes \mathbb{Q}$ is a field, for example $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{R}$.[2]

1.4.1 Objects and morphisms

Let $n \in \mathbb{Z}$. A pure A-Hodge structure of weight n is a triple $(\mathrm{V_A}, (\mathrm{V_\mathbb{C}}, \mathrm{F}^\bullet), \alpha)$ where $\mathrm{V_A}$ is an A-module of finite type, $\mathrm{V_\mathbb{C}}$ is a finite-dimensional $\mathbb{C}$-vector space with a finite decreasing filtration $\mathrm{F}^\bullet$, and α is an isomorphism $\mathrm{V_A} \otimes \mathbb{C} \to \mathrm{V_\mathbb{C}}$ such that the filtration $\mathrm{F}^\bullet$ and its complex conjugate (with respect to the real structure induced by α) are n-opposite, that is:

$$\mathrm{V_\mathbb{C}} = \mathrm{F}^p\mathrm{V_\mathbb{C}} \oplus \overline{\mathrm{F}^{n-p+1}\mathrm{V_\mathbb{C}}}.$$

1.4.1 Remark. *This definition can be simplified by setting* $\mathrm{V_\mathbb{C}} := \mathrm{V_A} \otimes \mathbb{C}$ *and forgetting about the comparison isomorphism* α. *But in practice, the filtration* $\mathrm{F}^\bullet$ *is often not defined on* $\mathrm{V_A} \otimes \mathbb{C}$ *itself, but on an isomorphic vector space. Keeping track of the isomorphism can be useful.*

Let V and W be pure A-Hodge structures of weight n. A morphism f of Hodge structures between V and W is a pair $(f_\mathrm{A}, f_\mathbb{C})$ where $f_\mathrm{A} : \mathrm{V_A} \to \mathrm{W_A}$ is a morphism of A-modules, $f_\mathbb{C} : \mathrm{V_\mathbb{C}} \to \mathrm{W_\mathbb{C}}$ is a filtered morphism of filtered $\mathbb{C}$-vector spaces, and $\alpha_\mathrm{W} \circ f_\mathrm{A} = f_\mathbb{C} \circ \alpha_\mathrm{V}$.

If V is a pure A-Hodge structure of weight n, we denote by $\mathrm{V}^{p,q}$ the subspaces $\mathrm{F}^p\mathrm{V_\mathbb{C}} \cap \overline{\mathrm{F}^{n-p}\mathrm{V_\mathbb{C}}}$ of $\mathrm{V_\mathbb{C}}$. These subspaces verify the following properties:

1. Hodge decomposition: $\mathrm{V_\mathbb{C}} = \oplus_{p+q=n}\mathrm{V}^{p,q}$.
2. Hodge symmetry : $\mathrm{V}^{q,p} = \overline{\mathrm{V}^{p,q}}$.

Conversely, given complex subspaces of $\mathrm{V_\mathbb{C}}$ satisfying Hodge decomposition and symmetry, we recover the filtration F as follows : $\mathrm{F}^p\mathrm{V_\mathbb{C}} := \oplus_{p\le r\le n}\mathrm{V}^{r,n-r}$. This filtration and its complex conjugate are n-opposite. The *type* of a Hodge structure is the minimal subset $\mathrm{T} \subset \mathbb{Z}^2$ such that $(p,q) \notin \mathrm{T} \Rightarrow \mathrm{V}^{p,q} = 0$. A Hodge structure is *effective* if $\mathrm{T} \subset \mathbb{N}^2$.

1.4.2 Example.

- *The trivial Hodge structure* $\mathrm{A}(0)$ *is the pure Hodge structure of weight* 0 *given by* $\mathrm{A}(0)_\mathrm{A} = \mathrm{A}$, $\mathrm{A}(0)_\mathbb{C} = \mathbb{C}$, $\mathrm{F}^0 = \mathbb{C}$, $\mathrm{F}^1 = \{0\}$ *and* $\alpha(1) = 1$. *The only non-trivial Hodge subspace is* $\mathrm{A}(0)^{0,0} = \mathbb{C}$.
- *The Tate* A*-Hodge structure is the pure* A*-Hodge structure* $\mathrm{A}(1)$ *of weight* -2 *with* $\mathrm{A}(1)_\mathrm{A} = \mathrm{A}$, $\mathrm{A}(1)_\mathbb{C} = \mathbb{C}$, $\mathrm{F}^{-1} = \mathbb{C}$, $\mathrm{F}^0 = \{0\}$, *and* $\alpha(1) = 2i\pi$. *The only non-trivial Hodge subspace is* $\mathrm{A}(0)^{-1,-1} = \mathbb{C}$.

[2]There are notions of K-Hodge structures for certain subfields K of $\mathbb{C}$ not contained in $\mathbb{R}$. This is implicit or explicit in the work of Deligne–Mostow, Carlson–Toledo and more. Also there is the rather weak notion of $\mathbb{C}$-Hodge structure. Together with polarizations, $\mathbb{C}$-(V)HS are a very useful, see the work of Simpson.

- *More generally, for $n \in \mathbb{Z}$, the Hodge structure* $\mathrm{A}(n)$ *is the weight* $-2n$ *Hodge structure with* $\mathrm{A}(n)_{\mathrm{A}} = \mathrm{A}$, $\mathrm{A}(n)_{\mathbb{C}} = \mathbb{C}$, $\mathrm{F}^{-n} = \mathbb{C}$, $\mathrm{F}^{-n+1} = \{0\}$ *and* $\alpha(1) = (2i\pi)^n$. *The only non-trivial Hodge subspace is* $\mathrm{A}(0)^{-n,-n} = \mathbb{C}$.

1.4.3 Example. *(*$\mathrm{H}^2(\mathbb{P}^1)$ *is the Lefschetz structure)*
Let $\mathrm{H}_{\mathbb{Z}} := \mathrm{H}^2(\mathbb{P}^1, \mathbb{Z})$, $\mathrm{H}_{\mathbb{C}} := \mathrm{H}^2(\mathbb{P}^1, \mathbb{C})$, $\mathrm{F}^1\mathrm{H}_{\mathbb{C}} = \mathrm{H}_{\mathbb{C}}, \mathrm{F}^2\mathrm{H}_{\mathbb{C}} = 0$, *and let* α *be the inverse of the comparison isomorphism between De Rham and singular cohomology :*

$$\mathrm{H}^2(\mathbb{P}^1, \mathbb{C}) \to \mathrm{H}^2(\mathbb{P}^1, \mathbb{Z}) \otimes \mathbb{C} = \mathrm{Hom}(\mathrm{H}_2(\mathbb{P}^1, \mathbb{Z}), \mathbb{C}), \quad [u] \mapsto \left\{\sigma \mapsto \int_\sigma u\right\}.$$

Then, the Hodge structure $(\mathrm{H}_{\mathbb{Z}}, (\mathrm{H}_{\mathbb{C}}, \mathrm{F}^\bullet), \alpha)$ *is isomorphic to the so-called Lefschetz Hodge structure* $\mathbb{Z}(-1)$. *It has weight* 2, *and pure type* $(1,1)$.

1.4.4 Example. *More generally, we will see later that if* X *is a compact Kähler manifold, in particular a smooth projective variety over* $\mathbb{C}$, *then* $\mathrm{H}^k(\mathrm{X}, \mathbb{Z})$ *underlies a pure* $\mathbb{Z}$*-Hodge structure of weight* k.

Pure A-Hodge structures of weight n form a category $\mathrm{A}\mathcal{HS}(n)$.

1.4.5 Theorem. *The category* $\mathrm{A}\mathcal{HS}(n)$ *is abelian.*

Proof. This is not obvious as the category of filtered objects in an abelian category is not abelian. It follows from the important fact that if $f = (f_{\mathrm{A}}, f_{\mathbb{C}})$ is a morphism of pure Hodge structures, then the filtered morphism $f_{\mathbb{C}}$ is strict for the filtrations. See [Del71, 1.2.10]. □

1.4.6 Exercise. *Prove that the category of filtered finite-dimensional vector spaces is not abelian. (find an example such that* $\mathrm{CoIm}(f)$ *is not isomorphic to* $\mathrm{Im}\, f$*)*

The category of A-Hodge structures, denoted $\mathrm{A}\mathcal{HS}$, is the direct sum for $n \in \mathbb{Z}$ of the categories $\mathrm{A}\mathcal{HS}(n)$. This means that the objects of $\mathrm{A}\mathcal{HS}$ are finite direct sums of pure A-HS of some weight, and morphisms are direct sums of morphisms of pure A-Hodge structures.

1.4.7 Theorem. *The category* $\mathrm{A}\mathcal{HS}$ *is abelian.*

Let V_{A} be a Hodge structure. There is an action ρ of $\mathbb{C}^*$ on $\mathrm{V}_{\mathbb{C}}$ given by $z \cdot v = z^{-p}\bar{z}^{-q}$, for all $v \in \mathrm{V}^{p,q}$. The operator $\rho(i)$ is the endomorphism acting on $\mathrm{V}^{p,q}$ by multiplication by i^{q-p}. It maps $\mathrm{V}_{\mathbb{R}}$ into itself, which gives an endomorphism C of $\mathrm{V}_{\mathbb{R}}$, called the Weil operator. A morphism of real HS is a morphism which commutes with the Weil operators.

The advantage of working with $\mathrm{A}\mathcal{HS}$ instead of $\mathrm{A}\mathcal{HS}(n)$ is that the former has a tensor structure.

Tensor structure If V and V′ are pure A-HS of weight n and n', then the tensor product $\mathrm{V} \otimes \mathrm{V}'$ is a pure Hodge structure of weight $n + n'$, and $\mathrm{Hom}(\mathrm{V}, \mathrm{V}')$ is a pure Hodge structure of weight $n' - n$. In particular, the dual $\mathrm{V}^\vee$ of V is a pure Hodge structure of weight $-n$. We also have the Hodge structures $Sym^k\mathrm{V}$ and $\bigwedge^k \mathrm{V}$, both of weight kn. The category $\mathrm{A}\mathcal{HS}$ is a tensor category.

1.4.8 Exercise. *Prove that if* $n \geq 1$, *then* $\mathrm{A}(n) = \mathrm{A}(1)^{\otimes n}$. *If* $n \leq -1$, *then* $\mathrm{A}(n) = (\mathrm{A}(1)^\vee)^{\otimes -n}$.

If V is A-Hodge structure, we write $\mathrm{V}(n)$ instead of $\mathrm{V} \otimes \mathrm{A}(n)$. Twisting a Hodge structure by some $\mathrm{A}(n)$ is the standard way to modify the weight.

Differences between $\mathbb{Q}$ and $\mathbb{R}$ Hodge structures The notion of $\mathbb{R}$-Hodge structure is much weaker than that of $\mathbb{Q}$ or $\mathbb{Z}$ Hodge structure. For example, any $\mathbb{R}$-Hodge structure of dimension two and weight one is isomorphic to some Hodge structure V_τ with Hodge filtration $\mathbb{C}^2 = \mathrm{F}^0 \supset \mathrm{F}^1 = span\{(1,\tau)\}$, $\mathrm{Im}\ (\tau) > 0$. Any two such Hodge structures V_τ and V_η are isomorphic over A if the two complex numbers τ and η are in the same $\mathrm{GL}_2(\mathrm{A})$-orbit. This is always the case over $\mathbb{R}$, but not over $\mathbb{Q}$ for example.

Invariants of Hodge structures What tools can we use to distinguish between two Hodge structures ? Automorphisms can help : even with the same type, some HS have more automorphisms than others. For example, for weight one and dimension two Hodge structures, V_i has four automorphisms whereas a generic V_τ has none.

In some situations, the automorphisms are the same, but one Hodge structure has more endomorphisms than the other. Consider for example the weight one Hodge structure $\mathrm{V}_{i\sqrt{2}}$: it has no automorphisms, but can be distinguished from a very general Hodge structure V_τ because it has more endomorphisms than just homothetic transformations.

This last example leads to the study of "CM" (complex multiplication) Hodge structures and Mumford-Tate groups.

1.4.2 Geometric origin

One of the first geometric examples of Hodge structures of weight one is the H^1 of a compact Rieman surface. We have a map

$$\{\text{Riemann surfaces of genus } g\} \xrightarrow{\mathrm{H}^1} \{\text{effective weight one HS of rank } 2g\}\,.$$

It is natural to ask if all weight 1 HS come from Riemann surfaces. This is the case if $\mathrm{H}^{0,1} = 1$.

In general, the moduli space of curves of genus g has dimension $3g-3$, and that of Hodge structures of type $(1,0)+(0,1)$ of dimension $2g$ has dimension $g(g+1)/2$. Thus for $g > 2$, there are Hodge structures not coming from curves[3].

In fact at this point it is not clear whether or not all Hodge structures (of any weight) come from (algebraic) geometry[4].

In weight one (or minus one), we have the following positive result.

1.4.9 Proposition. *The functor*

$$\{\textit{Complex tori}\} \longrightarrow \{\mathbb{Z} - \textit{Hodge structures of weight -1 and type } (-1,0)+(0,-1)\}$$

given by $\mathrm{X} \mapsto \mathrm{H}_1(,\mathbb{Z})$*, is an equivalence of categories.*

A quasi-inverse is obtained as follows. Let V be such a Hodge structure. The Weil operator C on $\mathrm{V}_\mathbb{R}$ is a complex structure. The complex variety $\mathrm{V}_\mathbb{R}/\mathrm{V}_\mathbb{Z}$ is a complex torus.

Complex tori up to isogeny correspond to $\mathbb{Q}$-Hodge structures.

1.4.10 Corollary. *The abelian category $\mathbb{Q} - \mathcal{HS}$ is not semisimple.*

We have seen that weight -1 rationnal Hodge structures correspond to complex tori (up to isogeny). It is easy to construct a two-dimensional torus with a non-split subtorus, even up to isogeny. Take a so-called Shafarevich extension of two elliptic curves (ref?).

[3]The image of the map is still not known in detail. The injectivity of the map is exactly the Torelli theorem for curves. For precise statements and definitions of the objects, see Chris Peters' lectures.

[4]Jan Nagel explained in his lectures why it is not the case in general, because of Griffiths transversality

1.4.3 Exercises

$\mathbb{C}^*$-action.

Degree, slope of a filtration. Hodge structures as stable filtrations.

Intermediate jacobians.

Twistors. Hodge polynomial.

2 The Lefschetz Package for Compact Kähler Manifolds

Introduction : what is the (Hodge-)Lefschetz package ? Hodge structure, Lefschetz structure, Hodge-Riemann bilinear relations.

2.1 Strong Hodge decomposition via harmonic theory

2.1.1 Harmonic theory on compact complex manifolds

Let first M denote a smooth compact oriented n-dimensional riemannian manifold with metric $g = \langle -, - \langle$. This metric induces metrics on all exterior powers of the tangent bundle. For ω a smooth k-form on M, define

$$||\omega||_{\mathrm{M}}^2 := \int_{\mathrm{M}} ||\omega||_p^2 d\mathrm{V}ol_g.$$

This is a positive-definite inner product on the space $\mathrm{A}^k(\mathrm{M})$. The idea is to use this metric to find representatives of De Rham cohomology classes.

2.1.1 Definition. *A smooth k-form on* M *is harmonic if it is closed and*

$$||\omega||_{\mathrm{M}} \leq ||\omega + d\beta||_{\mathrm{M}}$$

for all smooth $k-1$-forms β. The space of harmonic k-forms is denoted by $\mathcal{H}^k(\mathrm{M})$.

This means that a harmonic form has minimum norm inside its cohomology class. Note that for a small real number t, we have

$$||\omega + td\beta||_{\mathrm{M}}^2 = ||\omega||_{\mathrm{M}}^2 + 2t(\omega, d\beta) + \mathrm{O}(t^2).$$

We see that if ω harmonic then we must have $(\omega, d\beta) = 0$ for all $\beta \in \mathrm{A}^{k-1}$. The operator d has an adjoint $d^\star$ for $(-,-)_{\mathrm{M}}$. (In an open subset of $\mathbb{R}^n$, it is given by $d^\star u = -\sum_{\mathrm{I},j} \frac{\partial u_{\mathrm{I}}}{\partial x_j} \frac{\partial}{\partial x_j} \neg dx_{\mathrm{I}}$.) Then, $(\omega, d\beta)_{\mathrm{M}} = 0$ implies $(d^\star\omega, \beta)_{\mathrm{M}} = 0$, and this holds for all $\beta \in \mathrm{A}^{k-1}$ if *omega* is harmonic. Thus, a harmonic form is $d^\star$-closed. Conversely, if $d\omega = d^\star\omega = 0$, then $||\omega + d\beta||_{\mathrm{M}}^2 = ||\omega||_{\mathrm{M}}^2 + ||d\beta||_{\mathrm{M}} \geq ||\omega||_{\mathrm{M}}^2$ and ω is harmonic.

2.1.2 Definition. *The Laplace operator $\Delta : \mathrm{A}^k(\mathrm{M}) \to \mathrm{A}^k(\mathrm{M})$ is defined by*

$$\Delta = dd^\star + d^\star d.$$

Note that this operator depends on the metric.

Any harmonic form satisfies the Laplace equation $\Delta\omega = 0$, and conversely, if $\Delta\omega = 0$, then

$$0 = (\Delta\omega, \omega)_{\mathrm{M}} = (dd^\star\omega, \omega)_{\mathrm{M}} + (d^\star d\omega, \omega)_{\mathrm{M}} = ||d^\star\omega||_{\mathrm{M}}^2 + ||d\omega||_{\mathrm{M}}^2,$$

so ω is harmonic.

2.1.3 Theorem. *(Hodge theorem for Δ)*
Let M *be a compact oriented riemannian manifold. For all* k*, the space of smooth* k*-forms decomposes as*

$$\mathrm{A}^k(\mathrm{M}) = \mathcal{H}^k(\mathrm{M}) \oplus \mathrm{Im}\ d \oplus \mathrm{Im}\ d^\star.$$

This sum is orthogonal for the metric on $\mathrm{A}^k(\mathrm{M})$*, the subspace* $\mathcal{H}^k(\mathrm{M})$ *is finite-dimensional and maps isomorphically to* $\mathrm{H}^k(\mathrm{M}, \mathbb{R})$*.*

This shows that a cohomology class contains a unique harmonic representative. If we change the riemannian metric on M, this representative may not be the same.

Harmonic theory on complex manifolds Let X be a compact complex manifold. A hermitian metric h on X is a positive-definite hermitian form on the tangent bundle of X.

The operators ∂ and $\bar\partial$ have adjoints $\partial^\star$ and $\bar\partial^\star$ of type $(-1,0)$ and $(0,-1)$, and we have $d^\star = \partial^\star + \bar\partial^\star$. Now set $\Delta' = \partial\partial^\star + \partial^\star\partial$, and $\Delta'' = \bar\partial\bar\partial^\star + \bar\partial^\star\bar\partial$, and denote by $\mathcal{H}^{p,q}_{\Delta'} \subset \mathrm{A}^{p,q}$ and $\mathcal{H}^{p,q}_{\Delta''} \subset \mathrm{A}^{p,q}$ the corresponding subspaces of Δ'-harmonic and Δ''-harmonic forms. Then we also have decompositions for these operators:

2.1.4 Theorem. *(Hodge theorem for* Δ' *and* Δ''*)*
Let X *be a compact complex manifold. For all* $p, q \geq 0$*, the harmonic subpaces for* Δ' *an* Δ'' *are finite-dimensional and we have orthogonal decompositions*

$$\mathrm{A}^{p,q} = \mathcal{H}^{p,q}_{\Delta'} \oplus \mathrm{Im}\ \partial \oplus \mathrm{Im}\ \partial^\star,$$

$$\mathrm{A}^{p,q} = \mathcal{H}^{p,q}_{\Delta''} \oplus \mathrm{Im}\ \bar\partial \oplus \mathrm{Im}\ \bar\partial^\star.$$

This proves that the Dolbeault cohomology groups of a compact manifold are finite-dimensional, and that any Dolbeault class contains a unique Δ''-harmonic representative.

2.1.2 Proportionality and $\partial\bar\partial$ lemma

Consider the expansion:

$$\Delta = dd^\star + d^\star d = (\partial + \bar\partial)(\partial^\star + \bar\partial^*) + (\partial^\star + \bar\partial^*)(\partial + \bar\partial) = \Delta' + \Delta'' + (\bar\partial\partial^\star + \partial^\star\bar\partial) + (\partial\bar\partial^\star + \bar\partial^\star\partial).$$

The cross-terms $\bar\partial\partial^\star + \partial^\star\bar\partial$ and $\partial\bar\partial^\star + \bar\partial^\star\partial$ have bidegree $(-1,1)$ and $(1,-1)$ and do not vanish in general. Thus the Laplace operator is not bihomogeneous of bidegree $(0,0)$.

2.1.5 Proposition. *Let* X *be a compact complex manifold, and fix a hermitian metric. If* $\Delta = 2\Delta' = 2\Delta''$*, then the* $\partial\bar\partial$ *lemma holds on* X*.*

Proof. The hypothesis implies that the cross-terms vanish and that Δ is bihomogeneous of degree $(0,0)$. Note that every cohomology class can be represented by a ∂ and $\bar\partial$ closed form : choose an harmonic representative, then its (p,q) -components are also harmonic and therefore ∂ and $\bar\partial$ closed.

Now let us prove the pure $\partial\bar\partial$ lemma. Let ω be a closed (p,q) form. If it is d or ∂ or $\bar\partial$ exact, then it is orthogonal to the harmonic subspace $\mathcal{H}^{p,q}$. Thus it sufficient to prove that if ω is orthogonal to $\mathcal{H}^{p,q}$, then it is $\partial\bar\partial$ exact. Apply the Hodge theorem for Δ'' to ω : we get $\omega = \bar\partial\alpha + \bar\partial^\star\beta$. The closed form ω is of pure type so it is $\bar\partial$-closed. It follows that $\bar\partial\bar\partial^\star\beta = 0$, so $0 = (\bar\partial\bar\partial^\star\beta, \beta) = ||\bar\partial^\star\beta||^2$, i.e. $\bar\partial^\star\beta = 0$. As a result, $\omega = \bar\partial\alpha$. The Hodge theorem for Δ' to α gives a decomposition $\alpha = h + \partial\gamma + \partial^\star\varepsilon$. The harmonic component is $\bar\partial$-closed. Thus, we have

$$\omega = \bar\partial\alpha = \bar\partial\partial\gamma + \bar\partial\partial^\star\varepsilon.$$

It is enough to prove that the second component is zero. Recall that proportionality implies $\bar\partial\partial^\star = -\partial^\star\bar\partial$. Consequently, $\omega = -\partial\bar\partial\gamma - \partial^\star\bar\partial\varepsilon$. Now, ω is also ∂-closed, this forces $\partial\partial^\star\bar\partial\varepsilon = 0$ and the same argument as before allows to conclude that $\partial^\star\bar\partial\varepsilon = 0$. □

In the next section we will see that proportionality holds on a compact Kähler manifold.

2.1.3 Proportionality (and Hodge decomposition) for compact Kähler manifolds

Let (X, h) a complex manifold with a hermitian metric. The associated fundamental form is $\omega = -\mathrm{Im}\ (h)$. This is a positive $(1,1)$-form. The metric h is said to be Kähler if ω is closed. In this case, ω is called the Kähler form. A complex manifolds is Kähler if it admits at least one Kähler metric.

Both $\mathbb{C}^n$, the complex ball $\mathbb{B}^n$ and the projective space $\mathbb{P}^n$ are Kähler, with the metrics

$$\omega_{\mathbb{C}^n} = i\sum dz_k \wedge d\bar z_k, \quad \omega_{\mathbb{B}^n} = i\partial\bar\partial\log(1-||z||^2), \quad \omega_{\mathrm{FS}} = \frac{i}{2\pi}\partial\bar\partial\log(1+||z||^2).$$

The metric ω_{FS} is called the Fubiny-Study metric, it is not canonical and depends on the choice of a hermitian scalar product $||-||^2$ on $\mathbb{C}^n$.

Any complex curve is Kähler, because any $(1,1)$-form on a complex curve is closed. In higher dimension, there are non-Kähler manifolds, for example the Hopf surface.

Every complex torus $\mathrm{T} = \mathbb{C}^n/\Lambda$ is Kähler. Take any scalar product on the $\mathbb{C}^n = \mathbb{R}^{2n}$ compatible by the complex structure of $\mathbb{C}^n$. This defines a constant hence Λ-invariant Kähler metric which descends on T.

Every projective manifold is Kähler. Such a manifold can be embedded in a projective space. Restricting the Fubiny-Study metric yields a Kähler metric.

Lefschetz operator Let (X, ω) be a Kähler manifold, and let L be the operator defined on smooth forms by

$$\mathrm{L} : \alpha \mapsto \omega \wedge \alpha.$$

The closedness condition $d\omega = 0$ is equivalent to $\partial\omega = 0$ and $\bar\partial\omega = 0$. In terms of the Lefschetz operator L, it can be rewritten as

$$[\mathrm{L}, d] = [\mathrm{L}, \partial] = [\mathrm{L}, \bar\partial] = 0.$$

The operator L admits a formal adjoint Λ. For all smooth forms u, v, we have $(\mathrm{L}u, v)_{\mathrm{X}} = (u, \Lambda v)_{\mathrm{X}}$.

One of the key results in the theory of Kähler manifolds is the following

2.1.6 Proposition. *(Kähler identity) Let* X *be a Kähler manifold. Then*

$$[\Lambda, \partial] = i\bar\partial^\star.$$

Several other identities can be obtained by taking the complex conjugate or adjoint of this expression.

Proof. In $\mathbb{C}^n$, this is a computation. The point is precisely that on a Kähler manifold, statements involving the metric up to first derivatives can be checked for the flat metric. See for example [Dem]. There is another proof which doesn't use local coordinates but rather the local Lefschetz decomposition on forms, see [Huy] or [Sab]. $\square$

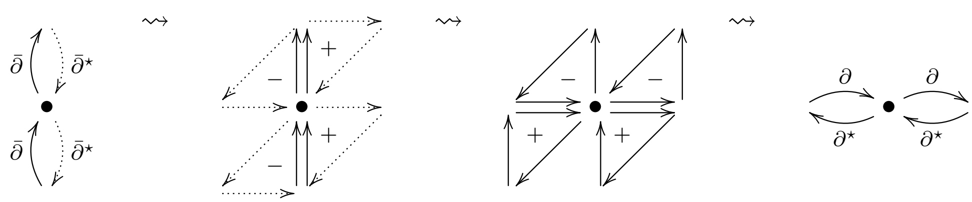
The Kähler identity above can be illustrated by the following diagram : one can replace

$$\begin{tikzcd} \mathrm{A}^{p,q} \arrow[d, "i\bar\partial^\star"] \\ \mathrm{A}^{p,q-1} \end{tikzcd} \quad \text{by} \quad \begin{tikzcd} & \mathrm{A}^{p,q} \arrow[r, "\partial"] \arrow[dl, "\Lambda"'] & \mathrm{A}^{p+1,q} \arrow[dl, "\Lambda"] \\ \mathrm{A}^{p-1,q-1} \arrow[r, "-\partial"'] & \mathrm{A}^{p,q-1} & \end{tikzcd} .$$

Now we see that this identity implies proportionality for the laplacians Δ, Δ' and Δ'' on X. For instance,

$$\partial\bar\partial^\star + \bar\partial^\star\partial = -i(\partial[\Lambda,\partial] + [\Lambda,\partial]\partial) = -i(\partial\Lambda\partial - \partial\partial\Lambda + \Lambda\partial\partial - \partial\Lambda\partial) = 0,$$

and the other cross-term $\bar\partial\partial^\star + \partial^\star\bar\partial$ vanishes too. The equality $\Delta'' = \Delta'$ also follows from a (longer) computation which can be summarized by the following drawing:



The first step uses the Kähler identity (dotted arrows), the second step is just $\partial\bar\partial = -\bar\partial\partial$, and the last step uses the complex conjugate of the Kähler identity: $[\Lambda, \bar\partial] = -i\partial^\star$.

2.1.4 Exercises

Non-Kähler aspects, considerations on volume, Wirtinger, more examples. Holonomy ?

2.2 The Hard Lefschetz property

2.2.1 Linear algebra

See [Del80, 1.6] and [Del68].

Let V be a vector space, and N a nilpotent endomorphism of V of nilpotency index k, i.e. k is the first positive integer such that $\mathrm{N}^{k+1} = 0$.

2.2.1 Proposition. *There exists a unique increasing filtration* $\mathrm{W} = \mathrm{W}(\mathrm{N})$ *of* V*:*

$$0 \subset \mathrm{W}_{-k} \subset ... \subset \mathrm{W}_{k-1} \subset \mathrm{W}_k = \mathrm{V},$$

called the weight filtration of N*, such that:*

1. $\mathrm{N}(\mathrm{W}_k) \subset \mathrm{W}_{k-2}$*;*
2. *For all* $k \geq 0$*,* $\mathrm{N}^k : \mathrm{Gr}_k^{\mathrm{W}}\mathrm{V} \to \mathrm{Gr}_{-k}^{\mathrm{W}}\mathrm{V}$ *is an isomorphism.*

Proof. The subspaces W_k are defined in terms of $\mathrm{Ker}(\mathrm{N}^i)$ and $\mathrm{Im}\,(\mathrm{N}^j)$. $\square$

Exercise : prove the proposition if N has a single Jordan block.

2.2.2 Example. *If the nilpotency index is* 1*, that is,* $\mathrm{N}^2 = 0$*, the filtration is*

$$0 \subset \mathrm{Im}\ \mathrm{N} \subset \mathrm{Ker}\,\mathrm{N} \subset \mathrm{V}.$$

For nilpotency index 2*, the filtration starts with* $0 \subset \mathrm{Im}\ \mathrm{N}^2$ *and ends with* $\mathrm{Ker}\,\mathrm{N}^2 \subset \mathrm{V}$*. COmpute the rest of the filtration. In fact this is the mechanism of the proof in the general case, see Deligne.*

2.2.3 Definition. *Let* $\mathrm{V} = \bigoplus_{l\in\mathbb{Z}} \mathrm{V}^l$ *be a finite-dimensional graded vector space and* N *and endomorphism. We say that the pair* (V,N) *satisfies the* Hard Lefschetz (HL) property *if for all positive integers* l*,* N *send* V^l *to* V^{l+2} *and if*

$$\mathrm{N}^l : \mathrm{V}^{-l} \longrightarrow \mathrm{V}^l$$

is an isomorphism

If (V,N) satisfies (HL), then N is nilpotent. The proposition shows that for any vector space V with a nilpotent endomorphism N, denoting by $\mathrm{W}_\bullet$ the weight filtration of N and $\mathrm{W}^\bullet$ the opposite filtration, then the pair $(\mathrm{G}r_{\mathrm{W}^\bullet}\mathrm{V},\mathrm{N})$ satisfies (HL).

2.2.4 Definition. *(Primitive part) Let* (V,N) *be a pair satisfying (HL). The primitive part of* V^{-l}*, denoted by* ${}_0\mathrm{V}^{-l}$*, is*

$${}_0\mathrm{V}^{-l} = \mathrm{Ker}\left(\mathrm{N}^{l+1} : \mathrm{V}^{-l} \to \mathrm{V}^l\right).$$

2.2.5 Proposition. *(Lefschetz decomposition) Let* (V,N) *be a pair satisfying (HL). For all* $l \geq 0$*, the maps*

$$\bigoplus_{k\geq 0} \mathrm{N}^k : \bigoplus_{k\geq 0} {}_0\mathrm{V}^{-l-2k} \to \mathrm{V}^{-l} \quad \textit{and} \quad \bigoplus_{k\geq 0} \mathrm{N}^{k+l} : \bigoplus_{k\geq 0} {}_0\mathrm{V}^{-l-2k} \to \mathrm{V}^l$$

are isomorphisms.

The Lefschetz decomposition implies in turn the Hard Lefschetz property.

The basic example comes from the local structure of the sheaf of forms on a complex manifold :

2.2.6 Example. *(Local Lefschetz structure in the geometric setting) Let* T *an* n*-dimensional complex vector space and* $\mathrm{F} = \mathrm{Hom}_{\mathbb{R}}(\mathrm{V},\mathbb{C})$*.*

In what follows, we use the notations introduced in **??** *Let* h *be an hermitian scalar product on* T*, and* $\omega = -\mathrm{Im}\ (h)$*. Then* $\omega \in \bigwedge^{1,1}\mathrm{F}$ *is positive in the sense that* $\omega(x, ix) > 0$ *for all* x*. Now let* N *be the operator* $\omega \wedge -$*. With the shifted grading* $\mathrm{V}^l = \bigwedge^{n+l}\mathrm{F}$*, the pair* (V,N) *satisfies (HL). More precisely, let* Λ *be the adjoint of* L *with respect to the hermitian metric on* V *induced by* h*, and define* $\mathrm{B} = (n-l)\mathrm{Id}$ *on* V^l*. Then, we have*

$$[\mathrm{B},\mathrm{L}] = 2\mathrm{L}, \quad [\mathrm{B},\Lambda] = -2\Lambda, \quad [\mathrm{L},\Lambda] = \mathrm{B}.$$

This means that V *is a* $sl_2(\mathbb{C})$*-module, and we recover the Lefschetz decomposition from the decomposition into irreductible submodules.*

2.2.2 Hard Lefschetz property and geometry

2.2.7 Definition. *Let* X *be an n-dimensional complex manifold,* $u \in \mathrm{H}^2(\mathrm{X}, \mathbb{C})$*, and*

$$\mathrm{L} : \mathrm{H}^*(\mathrm{X}, \mathbb{C}) \to \mathrm{H}^*(\mathrm{X}, \mathbb{C}), \quad \alpha \mapsto u \cup \alpha.$$

The pair (X, u) *is said to satisfy the Hard Lefschetz property if* $(\mathrm{H}^*(\mathrm{X}, \mathbb{C}), \mathrm{L})$ *does. The manifold* X *satisfies HL if there is at least one class* u *such that* (X, u) *satisfies HL.*

If a X satisfies the HL property, then the odd and even Betti numbers b_k increase for $k \leq n$ (and decrease for $k \geq n$), and $b_{2l} \neq 0$ for all $0 \leq l \leq n$.

2.2.8 Example. *Consider a smooth projective surface* S*, and a hyperplane class* $u \in \mathrm{H}^2(\mathrm{S}, \mathbb{C})$*. Then* u^2 *generates* $\mathrm{H}^4(\mathrm{S}, \mathbb{C})$*. The non-trivial part of the Lefschetz property says that* $\mathrm{H}^1(\mathrm{S}, \mathbb{C}) \xrightarrow{\cup u} \mathrm{H}^3(\mathrm{S}, \mathbb{C})$ *should be an isomorphism. This is not totally trivial. See exercise 2.5.2 for a possible approach.*

2.2.9 Theorem. *(Hard Lefschetz theorem for Kähler manifolds)*
Let X *be a compact Kähler manifold and* ω *a Kähler form. Then, the pair* $(\mathrm{X}, [\omega])$ *satifies the Hard Lefschetz property.*

The idea of the proof is the following. Write L for the operator $\omega \wedge$ on forms. The first step is the pointwise calculation stated in 2.2.6. This means that the complex vector bundle $\bigwedge^* \mathrm{T}_{\mathrm{X}}^{\vee}$ has a Lefschetz decomposition and that $\mathrm{L}^k : \bigwedge^{n-k} \mathrm{T}_{\mathrm{X}}^{\vee} \to \bigwedge^{n+k} \mathrm{T}_{\mathrm{X}}^{\vee}$ is an isomorphism. In particular, L^k is injective at the level of smooth $(n-k)$-forms.

The second step is that the morphism L preserves harmonic forms. This is a consequence of the equality

$$[\mathrm{L}, \Delta] = 0.$$

By proportionality, it suffices to prove $[\mathrm{L}, \Delta'] = 0$ and this follows from the Kähler identity.

One concludes that $\mathrm{L}^k : \mathcal{H}^{n-k} \to \mathcal{H}^{n+k}$ is injective, so it is bijective because both spaces have same dimension.

Note that if X is compact Kähler and the (X, u) satisfies the Lefschetz property, the class u doesn't have to be a Kähler class. (For example, if u works, so does $-u$.)

2.2.10 Remark. *The Hard Lefschetz property is purely topological and makes sense even if* X *is not a complex manifold : for any topological space* X *together with a fixed integer* n *and a class in* $\mathrm{H}^2(\mathrm{X})$*, we can ask if the maps* $\cup u^k : \mathrm{H}^{n-k}(\mathrm{X}, \mathbb{Q}) \to \mathrm{H}^{n+k}(\mathrm{X})$ *are bijective. The Hard Lefschetz property is intensively studied in symplectic geometry : in that context, it is equivalent to a symplectic version of the* $\partial\bar{\partial}$*-lemma.*

More generally, the Hard Lefschetz makes sense for algebraic varieties and several cohomology theories. If X *is a smooth projective variety of dimension* n *over a field* k*, let* $\mathrm{H} := \mathrm{H}_t^*(\mathrm{X}_{\bar{k}}, \mathbb{Q}_l)$ *the* l*-adic cohomology of* X *and* $u \in \mathrm{H}^2(\mathrm{X}, \mathbb{Q}_l)$ *the first Chern class of an ample line bundle. Then the pair* (H, u) *satisfies the Hard Lefschetz property. This is very hard ([Del80, Théorème 4.1.1]). There is an analogue of this for crystalline cohomology. See the other lectures for the arithmetic side of this story.*

2.3 Hodge-Riemann bilinear relations

Let (X,ω) a compact complex manifold. Part of the Hard Lefschetz property can be rephrased in the following way. Consider the pairing

$$(a,b)\mapsto\int_{\mathrm{X}} a\wedge b\wedge\omega^{n-k}.$$

This is a $(-1)^k$-symmetric bilinear form and the Hard Lefschetz property implies that it is non-degenerate.

Now consider the associated sesquilinear pairing

$$(a,b)\mapsto i^k\int_{\mathrm{X}} a\wedge\overline{b}\wedge\omega^{n-k}.$$

We would like to study the signature of these pairings.

Let us first look at one example. If X is a Riemann surface, and consider the pairing $h(a,b)=i\int_{\mathrm{X}}a\wedge\overline{b}$ on $\mathrm{H}^1(\mathrm{X})$. If $u\in\mathrm{H}^{1,0}$, it is represented by a holomorphic form which can be written $u=f.dz$ in local coordinates. Then, $iu\wedge\bar{u}=i|f|^2dz\wedge d\bar{z}$ is positive, thus

$$h(u,u)>0.$$

Similarly, if $u\in\mathrm{H}^{0,1}$, we have $h(u,u)<0$.

It is also possible to say something about compact Kähler surfaces. Consider this time the sesquilinear pairing $h(a,b)=\int_{\mathrm{X}}a\wedge\overline{b}$ on the second cohomology group $\mathrm{H}^2(\mathrm{X})$. It is positive-definite on $\mathrm{H}^{2,0}$ and $\mathrm{H}^{0,2}$ (just write a $(2,0)$ form in local coordinates as above), and it can be shown that if α is a primitive and harmonic $(1,1)$-form, then $h(\alpha,\alpha)<0$ (exercise : write the form in local coordinates).

The conclusion is that it seems that these sesquilinear pairings are definite on the primitive summands ${}_0\mathrm{H}^{p,q}(\mathrm{X})$. What should be the sign ? The following linear algebra lemma gives a plausible answer:

2.3.1 Lemma. *(local Hodge-Riemann relations [Huy, 1.2.36]) Let* (T,h) *a hermitian complex vector space as in 2.2.6, and* ω *the associated fundamental form. Recall that* $\mathrm{V}^*:=\bigwedge^*\mathrm{Hom}_{\mathbb{R}}(\mathrm{T},\mathbb{C})$*. The Hodge-Riemann pairing is the bilinear form*

$$\mathrm{Q}:\mathrm{V}^k\times\mathrm{V}^k\to\mathbb{C},\quad (a,b)\mapsto(-1)^{k(k-1)/2}a\wedge b\wedge\omega^{n-k},$$

where V^{2n} *is identified to* $\mathbb{C}$ *via the volume form. The, the decomposition* $\mathrm{V}=\oplus_{p,q}\mathrm{V}^{p,q}$ *is orthogonal for* Q*, and:*

$$i^{p-q}\mathrm{Q}(a,\bar{a})>0$$

for all primitive elements a *of bidegree* (p,q)*.*

Proof. See [Huy], p. 39, or [Sab]. □

This local lemma gives the sign $(-1)^{k(k-1)/2}$ and implies rather directly the global result which we now state.

2.3.2 Proposition. *(Hodge-Riemann bilinear relations) Let* (X,ω) *be a compact Kähler manifold, and let*

$$\mathrm{Q}:\mathrm{H}^k\times\mathrm{H}^k\to\mathbb{C},\quad (a,b)\mapsto(-1)^{k(k-1)/2}a\wedge b\wedge\omega^{n-k}$$

be the Hodge-Riemann pairing and $h(a,b)=i^k\mathrm{Q}(a,\overline{b})$ *the associated sesquilinear pairing.*

1. *The Hodge decomposition on* $\mathrm{H}^k(\mathrm{X}, \mathbb{C})$ *is orthogonal with respect to* h.
2. *For all primitive* (p, q) *cohomology class* u*, we have:*

$$i^{p-q}\mathrm{Q}(u, \bar{u}) > 0.$$

In other words, the restriction of the hermitian form $(-1)^q h$ *to the primitive Hodge subspace* ${}_0\mathrm{H}^{p,q}(\mathrm{X})$ *is positive-definite.*

Proof. The first property holds because if the types of a and b are different, then $a \wedge b \wedge \omega^{n-k}$ is not of type (n, n). Let us prove the second assertion. The primitive class u can be represented by an harmonic form α, primitive at each point. The pointwise computation in the preceeding lemma (**??**) allows to conclude. □

In terms of Hodge structures, the Hodge-Riemann bilinear relations say that the bilinear form Q is a polarization of the primitive cohomology.

2.4 Polarizable Hodge structures

2.4.1 Definition. *A polarization on* V *is a morphism of Hodge structures* $\phi : \mathrm{V} \otimes \mathrm{V} \to \mathrm{A}(-n)$ *such that* $\mathrm{Q}(x, y) = (2i\pi)^n \phi(\mathrm{C}x, y)$ *is a symmetric positive definite bilinear form on* $\mathrm{V}_{\mathbb{R}}$.

Since ϕ is a morphism of Hodge structures, we have

$$\phi(x, y) = \phi(\mathrm{C}x, \mathrm{C}y) = \phi(\mathrm{C}^2 y, x) = (-1)^n \phi(y, x)$$

Hence ϕ is symmetric if n is even, and alternating if m is odd.

A pure Hodge structure is polarizable if it admits a polarization. If V is a polarizable Hodge structure and W is a sub-Hodge structure, then any polarization of V restricts to a polarization of W.

The category of polarizable A-Hodge is denoted by $\mathrm{A}\mathcal{PHS}$. If A is a field, it is semisimple.

2.4.2 Theorem. *The category of polarizable* $\mathbb{Q}$*-Hodge structures is semisimple.*

Under the equivalence between complex tori and weight -1 $\mathbb{Z}$-Hodge structures of type $(-1, 0) + (0, -1)$, abelian varieties correspond to polarizable Hodge structures, polarized abelian correspond to polarized Hodge structures, and abelian varieties up to usigeny correspond to $\mathbb{Q}$-Hodge structures.

2.5 Exercises

2.5.1 Exercise. *Let* V *be a real finite-dimensional vector space, and* Q *a symmetric bilinear form on* V*. Describe the set of effective Hodge structures of weight two on* V *and hodge numbers* $h^{2,0}, h^{1,1}, h^{0,2} = h^{2,0}$*, polarized by the form* Q*, as a subset of* $\mathrm{Grass}(\mathrm{V}, \mathrm{h}^{2,0})$.

2.5.2 Exercise. *(Another approach to the Hard Lefschetz theorem)*

Let S *be a smooth projective surface,* C *a smooth hyperplane section of* S*, and* $u = [\mathrm{C}] \in \mathrm{H}^2(\mathrm{X}, \mathbb{Z})$ *its cohomology class. Prove that*

$$\mathrm{L} = u\cup : \mathrm{H}^1(\mathrm{S}, \mathbb{C}) \to \mathrm{H}^3(\mathrm{S}, \mathbb{C})$$

is an isomorphism by using the following tools:

- *Weak Lefschetz : the inclusion* $i : \mathrm{C} \hookrightarrow \mathrm{S}$ *induces an injective map in cohomology* $\mathrm{H}^1(\mathrm{S}) \xrightarrow{i^*} \mathrm{H}^1(\mathrm{C})$.

- *Hodge-Riemann bilinear relations on the curve* C *(see example 2.3 : this can be proved whithout Hard Lefschetz). This means that* $\mathrm{H}^1(\mathrm{C})$ *is a polarized Hodge structure of weight one.*

- *The Gysin morphism* $i_! : \mathrm{H}^1(\mathrm{C}) \to \mathrm{H}^3(\mathrm{S})$ *(the Poincaré dual of* i^**, thereby surjective), and the commutative diagram*

$$\begin{tikzcd} \mathrm{H}^1(\mathrm{S},\mathbb{C}) \arrow[r, "\mathrm{L}"] \arrow[d, "i^*"'] & \mathrm{H}^3(\mathrm{S},\mathbb{C}) \\ \mathrm{H}^1(\mathrm{C},\mathbb{C}) \arrow[ur, "i_!"'] & \end{tikzcd}$$

First prove that the intersection form on $\mathrm{H}^1(\mathrm{C})$ *restricted to* $i^*\mathrm{H}^1(\mathrm{S})$ *is nondegenerate. Show that this implies the injectivity of* $i_!$ *on the subspace* $i^*\mathrm{H}^1(\mathrm{S})$.

This approach is the one used by Deligne to prove Hard Lefschetz by induction, see [Del80, 4.1]. Instead of using the Hodge-Riemann bilinear relations, Deligne puts the hyperplane section inside a Lefschetz pencil and then applies his global invariant cycle and semisimplicity theorems to the monodromy representation of the Lefschetz pencil.

3 Spectral sequences, holomorphic and algebraic de Rham Cohomology

3.1 Spectral sequences

3.1.1 Motivation and definitions

Let $\mathcal{A}$ be an abelian category.

3.1.1 Definition. *A differential object is a pair* (X, d) *with* $\mathrm{X} \in \mathrm{Ob}(\mathcal{A})$ *and* d *is an endomorphism of* X *with* $d^2 = 0$. *A morphism of differential objects* $f : (\mathrm{X}, d_\mathrm{X}) \to (\mathrm{Y}, d_\mathrm{Y})$ *is a morphism* $f : \mathrm{X} \to \mathrm{Y}$ *such that the following square commutes:*

$$\begin{tikzcd} \mathrm{A} \arrow[r, "f"] \arrow[d, "d_\mathrm{X}"] & \mathrm{Y} \arrow[d, "d_\mathrm{Y}"] \\ \mathrm{X} \arrow[r, "f"'] & \mathrm{Y} \end{tikzcd}$$

This defines a category $\mathrm{D}iff(\mathcal{A})$. It is abelian.

3.1.2 Definition. *Let* $(\mathrm{X}, d) \in \mathrm{D}iff(\mathcal{A})$. *We define the cycles and boundaries to be* $\mathrm{Z}(\mathrm{X}) = \mathrm{Ker}\, d$ *and* $\mathrm{B}(\mathrm{X}) = \mathrm{Im}\, d$. *The cohomology of* X *is* $\mathrm{H}(\mathrm{X}) := \frac{\mathrm{Z}(\mathrm{X})}{\mathrm{B}(\mathrm{X})}$. *This defines a functor* $\mathrm{H} : \mathrm{D}iff(\mathcal{A}) \to \mathcal{A}$.

Computing H(X) can be difficult. Additional structure on X may be useful.

3.1.3 Definition. *A* filtered differential object *is a filtered object* $(\mathrm{X}, \mathrm{F}^\bullet)$ *with a filtered endomorphism* $d : \mathrm{X} \to \mathrm{X}$*, such that* $d^2 = 0$*. A filtration on a differential object induces a filtration on its cohomology* $\mathrm{H}(\mathrm{X})$*.*

How can filtrations be useful when computing cohomology ? First consider the most elementary case : a filtration with only two graded pieces. This means that we have a filtration

$$0 \subset \mathrm{Y} \subset \mathrm{X}.$$

Then, the cohomology of X can be computed with the help of the long exact sequence

$$\ldots \to \mathrm{H}(\mathrm{Y}) \to \mathrm{H}(\mathrm{X}) \to \mathrm{H}(\mathrm{X}/\mathrm{Y}) \to \mathrm{H}(\mathrm{Y}) \to \ldots,$$

which is known to be very useful. Let us now consider a less elementary example, for example a filtration

$$0 \subset \mathrm{Z} \subset \mathrm{Y} \subset \mathrm{X}.$$

Then, there are three short exact sequences :

$$\mathrm{Z} \to \mathrm{Y} \to \mathrm{Y}/\mathrm{Z}, \quad \mathrm{Y} \to \mathrm{X} \to \mathrm{X}/\mathrm{Y}, \quad \mathrm{Z} \to \mathrm{X} \to \mathrm{X}/\mathrm{Z}.$$

These three exact sequences give rise to three long exact sequences in cohomology. This kind of data is what is encoded in a spectral sequence.

More generally, the idea of spectral sequences is to use the filtration on X to compute the cohomology $\mathrm{H}(\mathrm{X})$ or at least its associated graded $\mathrm{G}r\mathrm{H}(\mathrm{X})$. The idea is to approximate the cycles and boundaries Z and B and then compute Z/B as some kind of limit of $\mathrm{Z}_r/\mathrm{B}_r$. What does it mean to approximate or to take the limit ?

3.1.4 Definition. *(Abstract spectral sequence)*
A bigraded spectral sequence in the abelian category $\mathcal{A}$ (starting at r_0) is the following data: for all $p, q \in \mathrm{Z}$ and $r \geq r_0$:

- *Objects* $\mathrm{E}_r^{p,q}$.
- *Morphisms* $d_{p,q} : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$ *such that* $d^{p+r,q-r+1} \circ d^{p,q} = 0$.
- *Isomorphisms* $\alpha^{p,q} : \mathrm{E}_{r+1}^{p,q} \simeq \frac{\mathrm{Ker}\, d_r^{p,q}}{\mathrm{Im}\ d_r^{p-r,q+r-1}}$.

Note that if we set $\mathrm{E}_r^n = \bigoplus_{p+q=n} \mathrm{E}_r^{p,q}$ and $d_r = \bigoplus_{p+q=n} d_r^{p,q}$, then $d_r : \mathrm{E}_r^n \to \mathrm{E}_r^{n+1}$, $(\mathrm{E}_r^\bullet, d_r)$ is a complex and its cohomology objects are E_{r+1}^*.

3.1.5 Definition. *The spectral sequence* converges *is for all (p, q), the sequence $(\mathrm{E}_r^{p,q})_r$ is eventually constant. The limit is then denoted by $\mathrm{E}_\infty^{p,q}$. The spectral sequence* collapses *(or* degenerates*) at E_r if all differentials $d_s^{p,q}$ vanish for $s \geq r$.*

A filtration $\mathrm{F}^\bullet$ of a complex $\mathrm{K}^\bullet$ is said to be biregular if for each degree l there are indices p_{min} and p_{max} such that $\mathrm{F}^p\mathrm{K}^l = \mathrm{K}^l$ for $p < p_{min}$ and $\mathrm{F}^p\mathrm{K}^l = 0$ for $p > p_{max}$.

3.1.6 Proposition. *(Spectral sequence of a filtered complex, Leray) Let $\mathrm{K}^\bullet$ be a complex with a biregular filtration $\mathrm{F}^\bullet$. There exists a spectral sequence $(\mathrm{E}_r^{p,q})$, such that*

1. $\mathrm{E}_0^{p,q} = \mathrm{G}r^p\mathrm{K}^{p+q}$.

2. $\mathrm{E}_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}^p\mathrm{K}^\bullet, d)$.

3. *The spectral sequence converges and the limit is* $\mathrm{Gr}\mathrm{H}^*(\mathrm{K}^\bullet)$ *in the sense that for all* (p, q) *the sequence* $\mathrm{E}_r^{p,q}$ *is stationary for large* r *and this limit is* $\mathrm{Gr}^p\mathrm{H}^{p+q}(\mathrm{K}^\bullet, d)$.

Proof. Define $\mathrm{Z}_r^{p,q} = \{x \in \mathrm{F}^p\mathrm{K}^{p+q},\ dx \in \mathrm{F}^{p+r}\mathrm{K}^{p+q+1}\}$ and

$$\mathrm{E}_r^{p,q} := \frac{\mathrm{Z}_r^{p,q}}{\mathrm{Z}_{r-1}^{p+1,q-1} + d\mathrm{Z}_{r-1}^{p-r+1,q+r-2}},$$

and check the remaining properties [Del71, 1.3.1]. See also [Voi]. □

Let $\mathrm{K}^{\bullet,\bullet}$ be a double complex, and denote by $\mathrm{K}^\bullet$ the associated total complex. The $\mathrm{K}^\bullet$ has two natural filtrations $\mathrm{F}_1^\bullet$ and $\mathrm{F}_2^\bullet$ given by:

$$\mathrm{F}_1^p\mathrm{K}^\bullet = \bigoplus_{r \geq p, s} \mathrm{K}^{r,s}$$

$$\mathrm{F}_2^p\mathrm{K}^\bullet = \bigoplus_{r, s \geq p} \mathrm{K}^{r,s}$$

This yields two different spectral sequences, one for each filtration.

We will mostly use the first filtration in our examples and applications. In this case, we have:

$$\mathrm{E}_0^{p,q} = \mathrm{K}^{p,q}, d_0 = \bar{\partial}; \quad \mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{K}^{p,\bullet}, \bar{\partial}), d_1 = \partial.$$

This follows from the description of the spectral sequence of a filtered complex.

3.1.7 Proposition. *(Deligne, [Del71, 1.3.2])*
Let $(\mathrm{K}^\bullet, d)$ *be a complex with a biregular filtration* F. *The following two conditions are equivalent:*

1. *The spectral sequence degenerates at* E_1.

2. *The morphisms* $d : \mathrm{K}^l \to \mathrm{K}^{l+1}$ *are strict for the filtration*

Proof. (particular case : double complexes)

If $\mathrm{K}^\bullet$ is the total complex of a double complex $\mathrm{K}^{\bullet,\bullet}$, let us prove that both statements are equivalent to:

3. There are no even-length zigzags ending with an horizontal arrow in the decomposition of $\mathrm{K}^{\bullet,\bullet}$.

(1) ⇔ (3) follows from the following simple remarks.

- The spectral sequence of a dot is constant, i.e. degenerates at E_0.
- The spectral sequence of a square collapses to zero at E_1.
- The spectral sequence of an odd-length zigzag has only one non-zero term at the E_1 stage : the beginning of the zigzag if it begins with a horizontal arrow, the end of the zigzag otherwise. In any case it degenerates at E_1.

- The spectral sequence of an even-length zigzag starting (and thereby also ending) with a vertical arrow collapses to zero at E_1.
- The spectral sequence of a zigzag of length $2k$ starting (and ending) with a horizontal arrow has exactly one nonzero differential d_k. It degenerates at E_{k+1}, not before.

(2) $\Leftrightarrow$ (3) is left as an exercise.

The proof in the general case is not very difficult but uses an explicit description of the arrows $d_r : \mathrm{E}_r^{p,q} \to \mathrm{E}_r^{p+r,q-r+1}$. See Hodge II.

$\square$

3.1.2 Frölicher spectral sequence

Let X be a compact complex manifold. The Hodge Frölicher (or Frölicher) spectral sequence of X is the spectral sequence associated to the double complex $\mathrm{A}^{\bullet,\bullet}(\mathrm{X}), \partial, \bar{\partial})$, filtered by

$$\mathrm{F}^p\mathrm{A}^{\bullet,\bullet}, \partial, \bar{\partial}) = \mathrm{A}^{\bullet \geq p,\bullet}, \partial, \bar{\partial}) = \bigoplus_{r \geq p, s} \mathrm{A}^{r,s}(\mathrm{X}).$$

It starts with

$$\mathrm{E}_1^{p,q} = \mathrm{H}^{p+q}(\mathrm{Gr}^p\mathrm{A}^\bullet = \mathrm{H}^q(\mathrm{A}^{p,\bullet}, \bar{\partial}) = \mathrm{H}_{\bar{\partial}}^{p,q}(\mathrm{X}).$$

The filtration F^p on the total De Rham complex $\mathrm{A}^\bullet(\mathrm{X})$ induces a filtration on its cohomology $\mathrm{H}^*(\mathrm{X}, \mathbb{C})$. We call it the *Hodge-Frölicher filtration.* The associated graded $\mathrm{Gr}_\mathrm{F}\mathrm{H}^*(\mathrm{X}, \mathbb{C})$ is the limit of the spectral sequence.

3.1.8 Proposition. *(Degeneration)*
The existence of an isomorphism $\mathrm{H}^k(\mathrm{X}, \mathbb{C}) \simeq \bigoplus_{p+q=k} \mathrm{H}_{\bar{\partial}}^{p,q}(\mathrm{X})$ *is equivalent to the degeneration of the Frölicher spectral sequence at* E_1. *In this case, we say that* X *admits a (weak) Hodge decomposition.*

Proof. The limit terms $\mathrm{E}_\infty^{p,q}$ are subquotients of $\mathrm{E}_1^{p,q}$, thus $\dim \mathrm{E}_\infty^{p,q} \leq \dim \mathrm{E}_1^{p,q}$. Existence of an isomorphism $\mathrm{H}^k(\mathrm{X}, \mathbb{C}) \simeq \bigoplus_{p+q=k} \mathrm{H}_{\bar{\partial}}^{p,q}(\mathrm{X})$ thus implies

$$\dim \mathrm{H}^k(\mathrm{X}, \mathbb{C}) = \sum_{p+q=k} \dim \mathrm{E}_\infty^{p,q} \leq \sum_{p+q=k} \dim \mathrm{E}_1^{p,q} = \dim \mathrm{H}^k(\mathrm{X}, \mathbb{C}).$$

This forces the equalities $\dim \mathrm{E}_\infty^{p,q} = \dim \mathrm{E}_1^{p,q}$ for all p, q. As a result, the spectral sequence is E_1-degenerate. $\square$

In terms of forms, E_1-degeneration means the following. Let $a \in \mathrm{H}_{\bar{\partial}}^{p,q}$ a Dolbeault class. Then, there exists a closed k-form $\alpha = \alpha^{p,q} + \alpha^{p+1,q-1} + ... + \alpha^{k-1,1} + \alpha^{k,0}$ in $\mathrm{F}^p\mathrm{A}^k(\mathrm{X})$, such that the Dolbeault class of the (p,q)-component $\alpha^{p,q}$ is equal to a. This is weaker than strong Hodge decomposition (or equivalently the $\partial\bar{\partial}$-lemma), which says that in this setting, there exists a closed form α of pure type (p,q) representing a, and that the associated De Rham class $[\alpha]_{\mathrm{DR}} \in \mathrm{H}^k$ is uniquely determined by the Dolbeault class a.

Every smooth projective or compact Kähler manifold has of course a strong Hodge decomposition. The Hopf surface has a weak Hodge decomposition, but not a strong one. In fact, the

Frölicher spectral sequence of any compact complex surface degenerates at E_1 [**?**]! The non-trivial part consists in proving that the arrow $d_1 = \partial : E_1^{0,1} = H^1(S, \mathcal{O}_S) \to E_1^{1,1} = H^1(S, \Omega_S)$ is zero.

In higher dimension, the Frölicher spectral sequence need not degenerate at E_1. The Iwasawa manifold [Dem, VI, 8.10] is a three-dimensional compact manifold with a non-closed holomorphic 1-form (!)[5], and the arrow $d_1 : E_1^{1,0} \to E_1^{2,0}$ is not zero. Recently, Rollenske constructed examples with arbitrarily non-degenerate Frölicher spectral sequence. Even if the spectral sequence degenerates at E_1, the groups $F^pH^k \cap \overline{F^qH^k}$, where $F^\bullet$ denotes the Hodge-Frölicher filtration on cohomology, are not necessarily equal to the "Hodge subspaces" we have defined before. For this we really the $\partial\bar{\partial}$-lemma on the double complex.

3.1.3 Effect of a modification on the $\partial\bar{\partial}$-lemma

3.1.9 Theorem. *[Del68, 4.3, 5.3][DGMS, 5.22] Let* X *and* Y *be compact complex manifolds, and* $f : X \to Y$ *a holomorphic (birational) surjective morphism. Id the* $\partial\bar{\partial}$ *lemma holds for* X*, then it holds for* Y*.*

Proof. We first prove that the Frölicher spectral sequence for Y degenerates at E_1, and then, that the limit filtration and its conjugate are k-opposite on $H^k(Y)$. By **??**, this is equivalent to the $\partial\bar{\partial}$-lemma.

First step : degeneration

The map $H^q(Y, \Omega^p) \xrightarrow{f^*} H^q(X, \Omega^p)$ is injective because there is a map f_* such that

$$f_* \circ f^* : H^q(Y, \Omega^p) \xrightarrow{f^*} H^q(X, \Omega^p) \xrightarrow{f_*} H^q(Y, \Omega^p)$$

is the identity. In [Del68], the map f is a proper map between algebraic varieties and f_* is the relative trace map Tr_f obtained from $(R)f^!$. The following analytic construction of such a map can be found in [Dem, VI,12.2]. The complex of $(p, \bullet)$ currents is a resolution of the sheaf Ω^p [Dem, IV,6.17] and can be used to compute Dolbeault cohomology. There is a well-defined push-forward (or direct image) at the level of currents [Dem, I, 2.14]. The direct image commutes with d and satisfies the projection formula : $f_*(T \wedge f^*\alpha) = f_*T \wedge \alpha$, for any smooth form α. It induces a direct image $f_* : H^q(X, \Omega^p_X) \to H^q(Y, \Omega^p_Y)$. Now, $f_*f^* = Id$ on $H^q(Y, \Omega^p_Y)$ follows from the fact that f is a biholomorphism outside a set of measure zero.

This proves that the first page of the Frölicher sequence of X contains that of Y as a direct summand. An induction shows that $E_r(Y) = E_{r+1}(Y)$, $f^* : E_r(Y) \to E_r(X)$ is injective, and that $d_r : E_r(Y) \to E_r(Y)$ vanishes for all $r \geq 1$, so the spectral sequence degenerates at E_1.

Step two : k-opposite filtrations

The $\partial\bar{\partial}$ lemma holds for X, thus $H^k(X) = F^pH^k(X) \oplus \overline{F^{k-p+1}H^k(X)}$. The map f^* sends $F^pH^k(Y)$ injectively into $F^pH^k(X)$ and commutes with complex conjugation. As a result, the intersection $F^pH^k(Y) \cap \overline{F^{k-p+1}H^k(Y)} = \{0\}$ is zero (the filtrations F and $\bar{F}$ are k-transverse). This holds for all k, and in particular, for each k it holds for $2n-k$. A Serre duality argument then implies that $F^pH^k(Y) + \overline{F^{k-p+1}H^k(X)} = H^k(Y)$. See [Del68, 5.3]. □

[5] This wouldn't happen on a Kähler manifold. An holomorphic 1-form is $\bar{\partial}$-closed by definition and $\bar{\partial}^\star$-closed for type reasons. So it is Δ''-harmonic. Proportionality would imply that the form is harmonic, thus closed.

Remark : $f^*\mathrm{H}^*(\mathrm{Y}, \mathbb{C})$ is a sub-Hodge structure of $\mathrm{H}^*(\mathrm{X}, \mathbb{C})$. Demailly [Dem, VI, 12.9] gives another proof, using a sheaf-theoretic description of Bott-Chern cohomology. (Same idea : there are injective pull-backs and surjective push-forward between Bott-Chern (p, q)-groups).

3.2 Holomorphic De Rham complex, Hodge-De Rham spectral sequence

Let X be an n-dimensional complex manifold. The holomorphic De Rham complex $\Omega^\bullet_{\mathrm{X}}$ is the complex of sheaves
$$0 \to \mathcal{O}_{\mathrm{X}} \xrightarrow{\partial} \Omega^1_{\mathrm{X}} \xrightarrow{\partial} \dots \xrightarrow{\partial} \Omega^n_{\mathrm{X}} \to 0.$$

By the holomorphic Poincaré lemma, this complex is a resolution of the constant sheaf $\mathbb{C}$ on X. The quasi-isomorphism $\mathbb{C} \hookrightarrow \Omega^\bullet_{\mathrm{X}}$ induces an isomorphism in hypercohomology :
$$\mathrm{H}^k(\mathrm{X}, \mathbb{C}) \xrightarrow{\sim} \mathbb{H}^k(\mathrm{X}, \Omega^\bullet_{\mathrm{X}}).$$

The holomorphic De Rham complex admits the naïve[6] filtration $\mathrm{F}^p\Omega^\bullet = \Omega^{\bullet \geq p}$. It induces the filtration
$$\mathrm{F}^p\mathrm{H}^k(\mathrm{X}, \mathbb{C}) = \mathrm{Im}\ (\mathbb{H}^k(\mathrm{X}, \Omega^{\bullet\geq p}_{\mathrm{X}}) \to \mathbb{H}^k(\mathrm{X}, \Omega^\bullet_{\mathrm{X}})),$$
and the associated graded is the limit of a spectral sequence which starts with $\mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{X}, \mathrm{G}r^p\Omega^\bullet_{\mathrm{X}}) = \mathrm{H}^q(\mathrm{X}, \Omega^p)$.

3.3 Algebraic De Rham cohomology and comparison theorem

Let us now consider algebraic varieties. What cohomology theory should we use ? Basically, we would like to consider cohomological invariants that behave like singular cohomology in the analytic context. A partial list of expected properties has been given in **??**.

There are notions of "well-behaved" cohomology theories, for example the so-called Weil cohomology theories, see []. In addition to what has already been discussed, there are requirements on the multiplicative structure, the cycle class maps, and the way the interact.

A discussion on Weil cohomology theories would lead us too far from the subject of this lectures, but the following remark may be useful. When given a cohomology theory, it can be enlightening to test it on $\mathbb{A}^n_k \setminus \{0\}$ and on curves over k. The natural expectation is that we should get the same result as in the complex analytic setting. For example, the cohomology of $\mathbb{A}^n_k \setminus \{0\}$ should look like the cohomology of topological spheres.

How to construct a good cohomology theory for algebraic varieties over a field k? The naïve analogue of the Betti cohomology for complex varieties would be to consider the sheaf cohomology of the constant sheaf k_{X} on X, where X is endowed with its Zariski topology. But this is not well-behaved. In fact, due to Grothendieck's vanishing theorem, the cohomology $\mathrm{H}^l(\mathrm{X}, \mathcal{F})$ of *any* sheaf $\mathcal{F}$ on X vanishes for $l > \dim \mathrm{X}$. But one of the requirements of a good cohmology theory is precisely that the cohomology on degree $2 \dim \mathrm{X}$ should be isomorphic to the field, just as in the analytic case, where the isomorphism is given by integration of forms. This remark rules out sheaf cohomology.

Now, recall that singular cohomology of complex varieties can be computed by holomorphic differentials. This remarkable fact shows us how to use the algebraic structure, in the agebraic setting.

[6] as opposed to the *standard* truncation of a complex $\sigma_{\geq p}\mathrm{K}^\bullet = \dots \to 0 \to \mathrm{Coker}\, d_p \xrightarrow{d_p} \mathrm{K}^{p+1} \xrightarrow{d_{p+1}} \mathrm{K}^{p+2} \to \dots$ which has much better functorial properties : adjunction etc.

Let X be a non-singular n-dimensional quasi-projective variety over a field K of characteristic zéro, $\Omega^1_{\mathrm{X/K}}$ the sheaf of Khähler differentials, and $\Omega^l_{\mathrm{X/K}} = \bigwedge^l \Omega^1 \mathrm{X/K}$. The algebraic De Rham complex of $\mathrm{X/K}$ is

$$\mathcal{O}_{\mathrm{X}} \xrightarrow{d} \Omega^1_{\mathrm{X/K}} \xrightarrow{d} \ldots \to \Omega^n_{\mathrm{X/K}}.$$

3.3.1 Definition. *The algebraic De Rham cohomology consists of the* K*-vector spaces*

$$\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/K}) := \mathbb{H}^i \left(\mathcal{O}_{\mathrm{X}} \xrightarrow{d} \Omega^1_{\mathrm{X/K}} \xrightarrow{d} \ldots \to \Omega^n_{\mathrm{X/K}} \right).$$

Given a morphism $f : \mathrm{X} \to \mathrm{Y}$, we obtain a map of complexe $f^* \Omega^\bullet_{\mathrm{Y/K}} \to \Omega^\bullet_{\mathrm{X/K}}$, which gives rise to a map

$$f^* : \mathrm{H}^i_{\mathrm{DR}}(\mathrm{Y/K}) \to \mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/K}).$$

The inclusion of complexes $\sigma_p : \Omega^{\bullet \geq p}_{\mathrm{X/K}} \hookrightarrow \Omega^\bullet_{\mathrm{X/K}}$ induces a map in hypercohomology, whose image in $\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/K})$ we denote by $\mathrm{F}^p \mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/K})$. This defines a decreasing filtration on $\mathrm{H}^i_{\mathrm{DR}}(\mathrm{X/K})$ by K-vector spaces.

At this point, it is not clear whether this cohomology theory is well-behaved or not. Before proving the general theorem, let us compute a few elementary examples.

3.3.2 Example. *(Punctured line)*
Let $\mathrm{X} = \mathbb{A}^1_{\mathbb{Q}} \setminus 0 = \mathrm{Spec}\mathbb{Q}[z, z^{-1}]$*. The de Rham cohomology* $\mathbb{H}^k(\mathrm{X}, \Omega^\bullet_{\mathrm{X}})$ *can be computed using the Hodge-to-de Rham spectral sequence* $\mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{X}, \Omega^p) \Rightarrow \mathbb{H}^{p+q}(\mathrm{X}, \Omega^\bullet_{\mathrm{X}})$*.*

In this particular case, X *has dimension one, so* $\mathrm{E}_1^{p,q} = 0$ *for* $p > 1$*, and* X *is affine, so* $\mathrm{E}_1^{p,q} = 0$ *for* $q > 0$*. The first page of the spectral sequence is*

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \\ 0 & \mathrm{E}_1^{0,0} & \xrightarrow{d_1} \mathrm{E}_1^{1,0} & 0 \end{array}$$

This means that the spectral sequence has to degenerate at E_2*, and we have:*

$$\mathrm{H}^0_{d\mathrm{R}}(\mathrm{X}/\mathbb{Q}) \simeq \mathrm{E}_2^{0,0} = \mathrm{Ker}\left(d : \mathrm{H}^0(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}})\right) \simeq \mathbb{Q},$$

$$\mathrm{H}^1_{d\mathrm{R}}(\mathrm{X}/\mathbb{Q}) \simeq \mathrm{E}_2^{0,0} = \mathrm{Coker}\left(d : \mathrm{H}^0(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) \to \mathrm{H}^0(\mathrm{X}, \Omega^1_{\mathrm{X}})\right) = \mathbb{Q}\frac{dz}{z} \simeq \mathbb{Q}.$$

After taking the tensor product with $\mathbb{C}$*, we get the expected result, that is, the* H^0 *and* H^1 *of the punctured complex plane* $\mathbb{C}^*$ *are one-dimensional.*

3.3.3 Example. *(Complete curve)*
Let X *be a smooth projective curve over* $\mathbb{Q}$*. The Hodge-to-de Rham spectral sequence starts with* $\mathrm{E}_1^{p,q} = \mathrm{H}^q(\mathrm{X}, \Omega^p)$*. We have* $\mathrm{E}_1^{p,q} = 0$ *for* $p > 1$ *because* X *is a curve, and* $\mathrm{E}_1^{p,q} = 0$ *for* $q > \dim \mathrm{X} = 1$ *by Grothendieck's vanishing.*

$$\begin{array}{cccc} 0 & 0 & 0 & 0 \\ \\ 0 & \mathrm{E}_1^{0,1} & \xrightarrow{d_1} \mathrm{E}_1^{1,1} & 0 \\ \\ 0 & \mathrm{E}_1^{0,0} & \xrightarrow{d_1} \mathrm{E}_1^{1,0} & 0 \end{array}$$

The spectral sequence must degenerate at E_2*, but this time there can be several graded pieces. In fact, the spectral sequence degenerates at* E_1*. The differential* $E_1^{0,0} \xrightarrow{d_1} E_1^{1,0}$ *is given by* $H^0(X, \mathcal{O}_X) \xrightarrow{d} H^0(X, \Omega^1_X)$ *which vanishes because the curve is projective, so regular functions are constant. Serre duality yields* $E_1^{0,1} = H^0(X, \Omega^1_X) = H^1(X, \mathcal{O}_X)^\vee$ *and* $E_1^{1,1} = H^1(X, \Omega_X) = H^0(X, \mathcal{O}_X)^\vee$*. The map* $E_1^{0,1} \xrightarrow{d_1} E_1^{1,1}$ *is dual to* $E_1^{0,0} \xrightarrow{d_1} E_1^{1,0}$*, therefore it is also zero.*

In the end, we have

$$H^0_{dR}(X/\mathbb{Q}) \simeq E_1^{0,0} = \mathbb{Q},$$
$$H^2_{dR}(X/\mathbb{Q}) \simeq E_1^{1,1} = H^1(X, \Omega_X) \simeq \mathbb{Q},$$

and a short exact sequence :

$$0 \to H^0(X, \Omega^1_X) \to H^1_{dR}(X/\mathbb{Q}) \to H^1(X, \mathcal{O}_X) \to 0.$$

We get the expected Betti numbers $1, 2g, 1$*, with* $g := \dim H^0(X, \Omega^1_X)$*.*

In both cases, the result is what we expected. This is a consequence of the following general comparison theorem.

3.3.4 Theorem. *(Grothendieck, [Gro])*
Let X *be a smooth quasiprojective variety over* $\mathbb{C}$*, and denote by* X^{an} *the associated complex manifold. The canonical map of ringed spaces* $(X^{an}, \mathcal{O}_{X^{an}}) \to (X, \mathcal{O}_X)$ *induces an isomorphism*

$$H^i_{DR}(X/\mathbb{C}) \to H^i(X^{an}, \mathbb{C}).$$

If X *is projective, the image of the filtration* $F^\bullet$ *on* $H^i_{DR}(X/\mathbb{C})$ *under this isomorphism is the Hodge filtration on Betti cohomology.*

Proof. We give the proof only for projective varieties. □

Extension of scalars and comparison of rational structures Let $K \subset L$ be a field extension and $X_L = X \times_K Spec L$, then we have $\Omega^1_{X/L} \simeq \Omega^1_{X/K} \otimes_K L$, $\Omega^i_{X/L} \simeq \Omega^i_{X/K} \otimes_K L$, and thus

$$H^i_{DR}(X/L) \simeq H^i_{DR}(X/K) \otimes_K L.$$

Suppose that the projective variety X is defined over $\mathbb{Q}$. Then composing with the base field extension isomorphism and the universal coefficients theorm, we get an isomorphism

$$H^i_{DR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H^i(X^{an}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C}.$$

It seems a natural question to ask whether or not this isomorphism of $\mathbb{C}$-vector spaces comes from a natural isomorphism

$$H^i_{DR}(X/\mathbb{Q}) \xrightarrow{???} H^i(X^{an}, \mathbb{Q})$$

of $\mathbb{Q}$-vector spaces...

The answer is no. For instance, if $X = \mathbb{A}^1_{\mathbb{Q}} \backslash \{0\}$, the comparison isomorphism $H^1_{DR}(X/\mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \to H^1(X^{an}, \mathbb{C})$ maps the generator $\frac{dt}{t}$ of $H^1_{DR}(X/\mathbb{Q})$ to $2i\pi$. This map is not defined over $\mathbb{Q}$. For an example with projective varieties, consider $X = \mathbb{P}^1_{\mathbb{Q}}$ and H^2. It is more or less the same example.

With respect to basis of the $\mathbb{Q}$-vector spaces $H^i_{DR}(X/\mathbb{Q})$ and $H^i(X^{an}, \mathbb{Q})$, the comparison isomorphism is given by a matrix with complex coefficients. It is called the period matrix.

3.3.5 Definition. *Complex numbers that appear in period matrices are called periods.*

The basic example of a period is $2\pi i$.
We finish by stating an important result :

3.3.6 Theorem. *In characteristic zero, algebraic De Rham cohomology is a Weil cohomology theory.*

This means in particular that there is a good notion of cup-product and cycle class map on algebraic de Rham cohomology.

References

[Del68] P. Deligne, Théorème de Lefschetz et critères de dégénérescence de suites spectrales, Publ. Math. IHÉS, 1968

[Del71] P. Deligne, Théorie de Hodge II, Publ. Math. IHÉS 40 (1971)

[Del80] P. Deligne, La conjecture de Weil, II, Publ. Math. IHÉS (1980)

[DGMS] P. Deligne, P. Griffiths, J. Morgan, D. Sullivan, Real Homotopy Theory of K ihler Manifolds, Invent. Math. 29 (1975) 245–274

[Dem] J-P. Demailly, Complex analytic and differential geometry, available at `http://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf`

[Gro] A. Grothendieck, On the de Rham cohomology of algebraic varieties, Publ. Math. IHÉS 29 (1966) 95–103

[Huy] D. Huybrechts, Complex geometry, an introduction, Universitext, Springer, 2004

[May] J. P. May, A Concise Course in Algebraic Topology, University of Chicago Press 1999, available at `http://www.math.uchicago.edu/~may/CONCISE/ConciseRevised.pdf`

[Sab] C. Sabbah, Théorie de Hodge et théorèmes de Lefschetz difficiles, Notes de cours, Strasbourg 2000, Bordeaux 2001, available at `www.math.polytechnique.fr/~sabbah/hodge-str.pdf`

[Voi] C. Voisin, Théorie de Hodge et géométrie algébrique complexe, cours spécialisés 10, SMF (2002)