The Local Langlands Correspondence for GL(n) over p-adic Fields

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Abstract

This work is intended as an introduction to the statement and the construction of the local Langlands correspondence for $\mathrm{GL}(n)$ over p-adic fields. The emphasis lies on the statement and the explanation of the correspondence.

CONTENTS

Introduction

Let K be a local field, i.e. K is either the field of real or complex numbers (in which case we call K archimedean) or it is a finite extension of $\mathbb{Q}_p$ (in which case we call K p-adic) or it is isomorphic to $\mathbb{F}_q((t))$ for a finite field $\mathbb{F}_q$ (in which case we call K a local function field). The local Langlands conjecture for GL_n gives a bijection of the set of equivalence classes of admissible representations of $\mathrm{GL}_n(K)$ with the set of equivalence classes of n-dimensional Frobenius semisimple representations of the Weil-Deligne group of K. This bijection should be compatible with L- and ε-factors. For the precise definitions see chap. 2 and chap. 3.

If K is archimedean, the local Langlands conjecture is known for a long time and follows from the classification of (infinitesimal) equivalence classes of admissible representations of $\mathrm{GL}_n(K)$ (for $K = \mathbb{C}$ this is due to Želobenko and Naĭmark and for $K = \mathbb{R}$ this was done by Langlands). The archimedean case is particularly simple because all representations of $\mathrm{GL}_n(K)$ can be built up from representations of $\mathrm{GL}_1(\mathbb{R})$, $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_1(\mathbb{C})$. See the survey article of Knapp [Kn] for more details about the local Langlands conjecture in the archimedean case.

If K is non-archimedean and $n = 1$, the local Langlands conjecture is equivalent to local abelian class field theory and hence is known for a long time (due originally to Hasse [Has]). Of course, class field theory predates the general Langlands conjecture. For $n = 2$ the local (and even the global Langlands conjecture) are also known for a couple of years (in the function field case this is due to Drinfeld [Dr1][Dr2], and in the p-adic case due to Kutzko [Kut] and Tunnel [Tu]). Later on Henniart [He1] gave also a proof for the p-adic case for $n = 3$.

If K is a local function field, the local Langlands conjecture for arbitrary n has been proved by Laumon, Rapoport and Stuhler [LRS] generalizing Drinfeld's methods. They use certain moduli spaces of "$\mathcal{D}$-elliptic sheaves" or "shtukas" associated to a global function field.

Finally, if K is a p-adic field, the local Langlands conjecture for all n has been proved by Harris and Taylor [HT] using Shimura varieties, i.e. certain moduli spaces of abelian varieties. A few months later Henniart gave a much

simpler and more elegant proof [He4]. On the other hand, the advantage of the methods of Harris and Taylor is the geometric construction of the local Langlands correspondence and that it establishes many instances of compatibility between the global and the local correspondence.

Hence in all cases the local Langlands conjecture for $GL(n)$ is now a theorem! We remark that in all cases the proof of the local Langlands conjecture for $n > 1$ uses global methods although it is a purely local statement. In fact, even for $n = 1$ (i.e. the case of local class field theory) the first proof was global in nature.

This work is meant as an introduction to the local Langlands correspondence in the p-adic case. In fact, approximately half of it explains the precise statement of the local Langlands conjecture as formulated by Henniart. The other half gives the construction of the correspondence by Harris and Taylor. I did not make any attempt to explain the connections between the local theory and the global theory of automorphic forms. In particular, nothing is said about the proof that the constructed map satisfies all the conditions postulated by the local Langlands correspondence, and this is surely a severe shortcoming. Hence let me at least here briefly sketch the idea roughly:

Let F be a number field which is a totally imaginary extension of a totally real field such that there exists a place w in F with $F_w = K$. The main idea is to look at the cohomology of a certain projective system $X = (X_m)_m$ of projective $(n-1)$-dimensional F-schemes (the X_m are "Shimura varieties of PEL-type", i.e. certain moduli spaces of abelian varieties with polarizations and a level structure depending on m). This system is associated to a reductive group G over $\mathbb{Q}$ such that $G \otimes_{\mathbb{Q}} \mathbb{Q}_p$ is equal to

$$\mathbb{Q}_p^{\times} \times GL_n(K) \times \text{anisotropic mod center factors.}$$

More precisely, these anisotropic factors are algebraic groups associated to skew fields. They affect the local structure of the X_m only in a minor way, so let us ignore them for the rest of this overview. By the general theory of Shimura varieties, to every absolutely irreducible representation ξ of G over $\mathbb{Q}$ there is associated a smooth $\bar{\mathbb{Q}}_{\ell}$-sheaf $\mathcal{L}_{\xi}$ on X where $\ell \neq p$ is some fixed prime. The cohomology $H^i(X, \mathcal{L}_{\xi})$ is an infinite-dimensional $\bar{\mathbb{Q}}_{\ell}$-vector space with an action of $G(\mathbf{A}_f) \times \text{Gal}(\bar{F}/F)$ where $\mathbf{A}_f$ denotes the ring of

finite adeles of $\mathbb{Q}$. We can choose ξ in such a way that $H^i(X, \mathcal{L}_\xi) = 0$ for $i \neq n - 1$.

We have a map from the set of equivalence classes of irreducible admissible representations Π of $G(\mathbf{A}_f)$ to the set of finite-dimensional representations of $W_K \subset \mathrm{Gal}(\bar{F}/F)$ where W_K denotes the Weil group of K by sending Π to

$$R_\xi(\Pi) = \mathrm{Hom}_{G(\mathbf{A}_f)}(\Pi, H^{n-1}(X, \mathcal{L}_\xi)).$$

For every such Π the decomposition

$$G(\mathbf{A}_f) = \mathbb{Q}_p^\times \times GL_n(K) \times \text{remaining components}$$

gives a decomposition

$$\Pi = \Pi_0 \otimes \Pi_w \otimes \Pi^w.$$

If π is a supercuspidal representation of $GL_n(K)$ then we can find a Π as above such that $\Pi_w \cong \pi\chi$ where χ is an unramified character of $K^\times$, such that Π_0 is unramified and such that $R_\xi(\Pi) \neq 0$.

Now we can choose a model $\tilde{X}_m$ of $X_m \otimes_F K$ over O_K and consider the completions $R_{n,m}$ of the local rings of a certain stratum of the special fibre of $\tilde{X}_m$. These completions carry canonical sheaves ψ_m^i (namely the sheaf of vanishing cycles) and their limits ψ^i are endowed with a canonical action $\mathrm{GL}_n(K) \times D_{1/n}^\times \times W_K$ where $D_{1/n}^\times$ is the skew field with invariant $1/n$ and center K. If ρ is any irreducible representation of $D_{1/n}^\times$, $\psi^i(\rho) = \mathrm{Hom}(\rho, \psi^i)$ is a representation of $\mathrm{GL}_n(K) \times W_K$. Via Jacquet-Langlands theory we can associate to every supercuspidal representation π of $\mathrm{GL}_n(K)$ an irreducible representation $\rho = jl(\pi^\vee)$ of $D_{1/n}^\times$. Now there exists an n-dimensional representation $r(\pi)$ of W_K which satisfies

$$[\pi \otimes r(\pi)] = \sum_{i=0}^{n-1} (-1)^{n-1-i}[\psi^i(jl(\pi^\vee))]$$

and

$$n \cdot [R_\xi(\Pi) \otimes \chi(\Pi_0 \circ \mathrm{Nm}_{K/\mathbb{Q}_p})] \in \mathbb{Z}[r(\pi)]$$

where [ ] denotes the associated class in the Grothendieck group. To show this one gives a description of $H^{n-1}(X, \mathcal{L}_\xi)^{\mathbb{Z}_p^\times}$ in which the $[\psi^i(\rho)]$ occur.

This way one gets sufficient information to see that the map

$$\pi \mapsto r(\pi^\vee \otimes | \ |^{\frac{1-n}{2}})$$

defines the local Langlands correspondence.

I now briefly describe the contents of the various sections. The first chapter starts with an introductory section on local abelian class field theory which is reformulated to give the local Langlands correspondence for GL_1. The next section contains the formulation of the general correspondence. The following two chapters intend to explain all terms and notations used in the formulation of the local Langlands correspondence. We start with some basic definitions in the theory of representations of reductive p-adic groups and give the Langlands classification of irreducible smooth representations of $\mathrm{GL}_n(K)$. In some cases I did not find references for the statements (although everything is certainly well known) and I included a short proof. I apologize if some of those proofs are maybe somewhat laborious. After a short interlude about generic and square-integrable representations we come to the definition of $L-$ and ε-factors of pairs of representations. In the following chapter we explain the Galois theoretic side of the correspondence.

The fourth chapter starts with the proof of the correspondence in the unramified case. Although this is not needed in the sequel, it might be an illustrating example. After that we return to the general case and give a number of sketchy arguments to reduce the statement of the existence of a unique bijection satisfying certain properties to the statement of the existence of a map satisfying these properties. The third section contains a small "dictionary" which translates certain properties of irreducible admissible representations of $\mathrm{GL}_n(K)$ into properties of the associated Weil-Deligne representation. In the fourth section the construction of the correspondence is given. It uses Jacquet-Langlands theory, and the cohomology of the sheaf of vanishing cycles on a certain inductive system of formal schemes. These notions are explained in the last chapter.

Nothing of this treatise is new. For each of the topics there is a number of excellent references and survey articles. In many instances I just copied them (up to reordering). In addition to original articles my main sources, which can (and should) be consulted for more details, were [CF], [AT], [Neu],

[Ta2] (for the number theoretic background), [Ca], [BZ1], [Cas1], [Ro] (for the background on representation theory of p-adic groups), and [Kud] (for a survey on "non-archimedean local Langlands"). Note that this is of course a personal choice. I also benefited from the opportunity to listen to the series of lectures of M. Harris and G. Henniart on the local Langlands correspondence during the automorphic semester at the IHP in Paris in spring 2000. I am grateful to the European network in Arithmetic Geometry and to M. Harris for enabling me to participate in this semester.

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NOTATIONS

Throughout we fix the following notations and conventions

- p denotes a fixed prime number.
- K denotes a p-adic field, i.e. a finite extension of $\mathbb{Q}_p$.
- v_K denotes the discrete valuation of K normalized such that it sends uniformizing elements to 1.
- O_K denotes the ring of integers of K. Further $\mathfrak{p}_K$ is the maximal ideal of O_K and π_K a chosen generator of $\mathfrak{p}_K$.
- κ denotes the residue field of O_K, and q the number of elements in κ.
- $|\ |_K$ denotes the absolute value of K which takes π_K to q^{-1}.
- ψ denotes a fixed non-trivial additive character of K (i.e. a continuous homomorphism $K \longrightarrow \{\, z \in \mathbb{C} \mid |z| = 1 \,\}$).
- n denotes a positive integer.
- We fix an algebraic closure $\bar{K}$ of K and denote by $\bar{\kappa}$ the residue field of the ring of integers of $\bar{K}$. This is an algebraic closure of κ.
- K^{nr} denotes the maximal unramified extension of K in $\bar{K}$. It is also equal to the union of all finite unramified extensions of K in $\bar{K}$. Its residue field is equal to $\bar{\kappa}$ and the canonical homomorphism $\mathrm{Gal}(K^{\mathrm{nr}}/K) \longrightarrow \mathrm{Gal}(\bar{\kappa}/\kappa)$ is an isomorphism of topological groups.
- $\Phi_K \in \mathrm{Gal}(\bar{\kappa}/\kappa)$ denotes the geometric Frobenius $x \mapsto x^{1/q}$ and σ_K its inverse, the arithmetic Frobenius $x \mapsto x^q$. We also denote by Φ_K and σ_K the various maps induced by Φ_K resp. σ_K (e.g. on $\mathrm{Gal}(K^{\mathrm{nr}}/K)$).
- If G is any Hausdorff topological group we denote by G^{ab} its maximal abelian Hausdorff quotient, i.e. G^{ab} is the quotient of G by the closure of its commutator subgroup.
- If $\mathcal{A}$ is an abelian category, we denote by $\mathrm{Groth}(\mathcal{A})$ its Grothendieck group. It is the quotient of the free abelian group with basis the isomorphism classes of objects in $\mathcal{A}$ modulo the relation $[V'] + [V''] = [V]$ for objects V, V' and V'' in $\mathcal{A}$ which sit in an exact sequence $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$. For any abelian group X and any function λ which associates to isomorphism classes of objects in $\mathcal{A}$ an element in X and which is additive (i.e. $\lambda(V) = \lambda(V') + \lambda(V'')$ if there exists an exact sequence $0 \longrightarrow V' \longrightarrow V \longrightarrow V'' \longrightarrow 0$) we denote the induced homomorphism of abelian groups $\mathrm{Groth}(\mathcal{A}) \longrightarrow X$ again by λ.

1 The local Langlands correspondence

1.1 THE LOCAL LANGLANDS CORRESPONDENCE FOR $\mathrm{GL}(1)$

(1.1.1) In this introductory section we state the local Langlands correspondence for GL_1 which amounts to one of the main theorems of abelian local class field theory. For the sake of brevity we use Galois cohomology without explanation. Galois cohomology will not be needed in the sequel.

(1.1.2) For any finite extension L of K of degree m and for $\alpha \in K^\times$ we denote by

$$(\alpha, L/K) \in \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

the norm residue symbol of local class field theory. Using Galois cohomology it can be defined as follows (see e.g. [Se1] 2, for an alternative more elementary description see [Neu] chap. IV, V):

The group $H^2(\mathrm{Gal}(L/K), L^\times)$ is cyclic of order m and up to a sign canonically isomorphic to $\frac{1}{m}\mathbb{Z}/\mathbb{Z}$. We use now the sign convention of [Se1]. Let $v_{L/K}$ be the generator of $H^2(\mathrm{Gal}(L/K), L^\times)$ corresponding to $-\frac{1}{m}$. By a theorem of Tate (e.g. [AW] Theorem 12) we know that the map $\hat{H}^q(\mathrm{Gal}(L/K), \mathbb{Z}) \longrightarrow \hat{H}^{q+2}(\mathrm{Gal}(L/K), L^\times)$ which is given by cup-product with $v_{L/K}$ is an isomorphism. Now we have

$$\hat{H}^{-2}(\mathrm{Gal}(L/K), \mathbb{Z}) = H_1(\mathrm{Gal}(L/K), \mathbb{Z}) = \mathrm{Gal}(L/K)^{\mathrm{ab}}$$

and

$$\hat{H}^0(\mathrm{Gal}(L/K), L^\times) = K^*/N_{L/K}(L^\times)$$

where $N_{L/K}$ denotes the norm of the extension L of K. Hence we get an isomorphism

$$\varphi_{L/K} \colon \mathrm{Gal}(L/K)^{\mathrm{ab}} \xrightarrow{\sim} K^*/N_{L/K}(L^\times).$$

We set

$$(\alpha, L/K) = \varphi_{L/K}^{-1}([\alpha])$$

where $[\alpha] \in K^\times/N_{L/K}(L^\times)$ is the class of $\alpha \in K^\times$.

(1.1.3) If L is a finite unramified extension of K of degree m we also have the following description of the norm residue symbol (cf. [Se1] 2.5): Let $\Phi_K \in \mathrm{Gal}(L/K)$ be the geometric Frobenius (i.e. it induces on residue fields the map $\sigma_K^{-1} : x \mapsto x^{-q}$). Then we have for $\alpha \in K^\times$

$$(\alpha, L/K) = \Phi_K^{v_K(\alpha)}.$$

(1.1.4) In the sequel we will only need the isomorphisms $\varphi_{L/K}$. Nevertheless let us give the main theorem of abelian local class field theory:

Theorem: *The map*

$$L \mapsto \phi(L) := N_{L/K}(L^\times) = \mathrm{Ker}(\ , L/K)$$

defines a bijection between finite abelian extensions L of K and closed subgroups of $K^\times$ of finite index. If L and L' are finite abelian extensions of K, we have $L \subset L'$ if and only if $\phi(L) \supset \phi(L')$. In this case L is characterized as the fixed field of $(\phi(L), L'/K)$.

   Proof : See e.g. [Neu] chap. V how to deduce this theorem from the isomorphism $\mathrm{Gal}(L/K) \cong K^\times / N_{L/K}(L^\times)$ using Lubin Tate theory. We note that this is a purely local proof.

(1.1.5) If we go to the limit over all finite extensions L of K, the norm residue symbol defines an isomorphism

$$\varprojlim_L \mathrm{Gal}(L/K)^{\mathrm{ab}} = \mathrm{Gal}(\bar{K}/K)^{\mathrm{ab}} \xrightarrow{\sim} \varprojlim_L K^\times / N_{L/K}(L^\times).$$

The canonical homomorphism $K^\times \longrightarrow \varprojlim_L K^\times / N_{L/K}(L^\times)$ is injective with dense image and hence we get an injective continuous homomorphism with dense image, called the *Artin reciprocity homomorphism*

$$\mathrm{Art}_K \colon K^\times \longrightarrow \mathrm{Gal}(\bar{K}/K)^{\mathrm{ab}}.$$

(1.1.6) Let $O_{\bar{K}}$ be the ring of integers of the algebraic closure $\bar{K}$ of K. Every element of $\mathrm{Gal}(\bar{K}/K)$ defines an automorphism of $O_{\bar{K}}$ which reduces

to an automorphism of the residue field $\bar{\kappa}$ of $O_{\bar{K}}$. We get a surjective map $\pi \colon \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gal}(\bar{\kappa}/\kappa)$ whose kernel is by definition the inertia group I_K of K. The group $\mathrm{Gal}(\bar{\kappa}/\kappa)$ is topologically generated by the arithmetic Frobenius automorphism σ_K which sends $x \in \bar{\kappa}$ to x^q. It contains the free abelian group $\langle \sigma_K \rangle$ generated by σ_K as a subgroup.

The fixed field of I_K in $\bar{K}$ is K^{nr}, the union of all unramified extensions of K in $\bar{K}$. By definition we have an isomorphism of topological groups

$$\mathrm{Gal}(K^{\mathrm{nr}}/K) \overset{\sim}{\longrightarrow} \mathrm{Gal}(\bar{\kappa}/\kappa).$$

(1.1.7) The reciprocity homomorphism is already characterized as follows (cf. [Se1] 2.8): Let $f \colon K^{\times} \longrightarrow \mathrm{Gal}(\bar{K}/K)^{\mathrm{ab}}$ be a homomorphism such that:

(a) The composition

$$K^{\times} \overset{f}{\longrightarrow} \mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{Gal}(\bar{\kappa}/\kappa)$$

is the map $\alpha \mapsto \Phi_K^{v_K(\alpha)}$.

(b) For $\alpha \in K^{\times}$ and for any finite abelian extension L of K such that $\alpha \in N_{L/K}(L^{\times})$, $f(\alpha)$ is trivial on L.

Then f is equal to the reciprocity homomorphism Art_K.

(1.1.8) We keep the notations of (1.1.6). The *Weil group of K* is the inverse image of $\langle \sigma_K \rangle$ under π. It is denoted by W_K and it sits in an exact sequence

$$0 \longrightarrow I_K \longrightarrow W_K \longrightarrow \langle \sigma_K \rangle \longrightarrow 0.$$

We endow it with the unique topology of a locally compact group such that the projection $W_K \longrightarrow \langle \sigma_K \rangle \cong \mathbb{Z}$ is continuous if $\mathbb{Z}$ is endowed with the discrete topology and such that the induced topology on I_K equals the the profinite topology induced by the topology of $\mathrm{Gal}(\bar{K}/K)$. Note that this topology is different from the one which is induced by $\mathrm{Gal}(\bar{K}/K)$ via the inclusion $W_K \subset \mathrm{Gal}(\bar{K}/K)$. But the inclusion is still continuous, and it has dense image.

(1.1.9) There is the following alternative definition of the Weil group: As classes in H^2 correspond to extensions of groups, we get for every finite

extension L of K an exact sequence

$$1 \longrightarrow L^{\times} \longrightarrow W(L/K) \longrightarrow \mathrm{Gal}(L/K) \longrightarrow 1$$

corresponding to the class $v_{L/K}$. For $L \subset L'$ we get a diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & L^{\times} & \longrightarrow & W(L/K) & \longrightarrow & \mathrm{Gal}(L/K) & \longrightarrow & 1 \\
 & & {\scriptstyle N_{L'/L}}\uparrow & & & & \uparrow & & \\
1 & \longrightarrow & L'^{\times} & \longrightarrow & W(L'/K) & \longrightarrow & \mathrm{Gal}(L'/K) & \longrightarrow & 1
\end{array}
$$

which can be commutatively completed by an arrow $W(L'/K) \longrightarrow W(L/K)$ such that we get a projective system $(W(L/K))_L$ where L runs through the set of finite extensions of K in $\bar{K}$. Its projective limit is the Weil group of K and the projective limit of the homomorphisms $W(L/K) \longrightarrow \mathrm{Gal}(L/K)$ is the canonical injective homomorphism $W_K \longrightarrow \mathrm{Gal}(\bar{K}/K)$ with dense image.

(1.1.10) Denote by W_K^{ab} the maximal abelian Hausdorff quotient of W_K, i.e. the quotient of W_K by the closure of its commutator subgroup. As the map $W_K \longrightarrow \mathrm{Gal}(\bar{K}/K)$ is injective with dense image, we get an induced injective map

$$W_K^{\mathrm{ab}} \hookrightarrow \mathrm{Gal}(\bar{K}/K)^{\mathrm{ab}}.$$

It follows from (1.1.9) and from the definition of

$$\mathrm{Art}_K \colon K^{\times} \longrightarrow \mathrm{Gal}(\bar{K}/K)^{\mathrm{ab}}$$

that the image of Art_K is W_K^{ab}.

We get an isomorphism of topological groups

$$\mathrm{Art}_K \colon K^{\times} \xrightarrow{\;\sim\;} W_K^{\mathrm{ab}}.$$

This isomorphism maps $O_K^{\times}$ onto the abelianization I_K^{ab} of the inertia group and a uniformizing element to a geometric Frobenius elements, i.e. if π_K is a uniformizer, the image of $\mathrm{Art}_K(\pi_K)$ in $\mathrm{Gal}(\bar{\kappa}/\kappa)$ is Φ_K.

(1.1.11) We can reformulate (1.1.10) as follows: Denote by $\mathcal{A}_1(K)$ the set of isomorphism classes of irreducible complex representations (π, V) of $K^{\times} = \mathrm{GL}_1(K)$ such that the stabilizer of every vector in V is an open subgroup

of K. It follows from the general theory of admissible representations that every (π, V) in $\mathcal{A}_1(K)$ is one-dimensional (see paragraph 2.1 below). Hence $\mathcal{A}_1(K)$ is equal to the set of continuous homomorphisms $K^\times \longrightarrow \mathbb{C}^\times$ where we endow $\mathbb{C}$ with the discrete topology.

On the other hand denote by $\mathcal{G}_1(K)$ the set of continuous homomorphisms $W_K \longrightarrow \mathbb{C}^\times = \mathrm{GL}_1(\mathbb{C})$ where we endow $\mathbb{C}^\times$ with its usual topology. Now a homomorphism $W_K \longrightarrow \mathbb{C}^\times$ is continuous if and only if its restriction to the inertia group I_K is continuous. But I_K is compact and totally disconnected hence its image will be a compact and totally disconnected subgroup of $\mathbb{C}^\times$ hence it will be finite. It follows that a homomorphism $W_K \longrightarrow \mathbb{C}^\times$ is continuous for the usual topology of $\mathbb{C}^\times$ if and only if it is continuous with respect to the discrete topology of $\mathbb{C}^\times$.

Therefore (1.1.10) is equivalent to:

Theorem (Local Langlands for GL_1): *There is a natural bijection between the sets $\mathcal{A}_1(K)$ and $\mathcal{G}_1(K)$.*

The rest of these lectures will deal with a generalization of this theorem to GL_n.

1.2 Formulation of the local Langlands correspondence

(1.2.1) Denote by $\mathcal{A}_n(K)$ the set of equivalence classes of irreducible admissible representations of $\mathrm{GL}_n(K)$. On the other hand denote by $\mathcal{G}_n(K)$ the set of equivalence classes of Frobenius semisimple n-dimensional complex Weil-Deligne representations of the Weil group W_K (see chap. 2 and chap. 3 for a definition of these notions).

(1.2.2) THEOREM (Local Langlands conjecture for GL_n over p-adic fields): *There is a unique collection of bijections*

$$\mathrm{rec}_{K,n} = \mathrm{rec}_n \colon \mathcal{A}_n(K) \longrightarrow \mathcal{G}_n(K)$$

satisfying the following properties:

(1) *For $\pi \in \mathcal{A}_1(K)$ we have*

$$\mathrm{rec}_1(\pi) = \pi \circ \mathrm{Art}_K^{-1}.$$

(2) *For $\pi_1 \in \mathcal{A}_{n_1}(K)$ and $\pi_2 \in \mathcal{A}_{n_2}(K)$ we have*

$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s),$$
$$\varepsilon(\pi_1 \times \pi_2, s, \psi) = \varepsilon(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s, \psi).$$

(3) *For $\pi \in \mathcal{A}_n(K)$ and $\chi \in \mathcal{A}_1(K)$ we have*

$$\mathrm{rec}_n(\pi\chi) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi).$$

(4) *For $\pi \in \mathcal{A}_n(K)$ with central character ω_π we have*

$$\det \circ \mathrm{rec}_n(\pi) = \mathrm{rec}_1(\omega_\pi).$$

(5) *For $\pi \in \mathcal{A}_n(K)$ we have $\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^\vee$ where $(\ )^\vee$ denotes the contragredient.*

This collection does not depend on the choice of the additive character ψ.

(1.2.3) As the Langlands correspondence gives a bijection between representations of $\mathrm{GL}_n(K)$ and Weil-Deligne representations of W_K certain properties of and constructions with representations on the one side correspond to properties and constructions on the other side. Much of this is still an open problem. A few "entries in this dictionary" are given by the following theorem. We will prove it in chapter 4.

Theorem: *Let π be an irreducible admissible representation of $\mathrm{GL}_n(K)$ and denote by $\rho = (r, N)$ the n-dimensional Weil-Deligne representation associated to π via the local Langlands correspondence.*

(1) *The representation π is supercuspidal if and only if ρ is irreducible.*

(2) *We have equivalent statements*

    (i)   *π is essentially square-integrable.*

(ii) ρ *is indecomposable.*

(iii) *The image of the Weil-Deligne group $W_F'(\mathbb{C})$ under ρ is not contained in any proper Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$.*

(3) *The representation π is generic if and only if $L(s, \mathrm{Ad} \circ \rho)$ has no pole at $s = 1$ (here $\mathrm{Ad} \colon \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}(M_n(\mathbb{C}))$ denotes the adjoint representation).*

(1.2.4) We are going to explain all occuring notations in the following two chapters.

2 Explanation of the GL(n)-side

2.1 Generalities on admissible representations

(2.1.1) Throughout this chapter let $\mathbf{G}$ be a connected reductive group over K and set $G = \mathbf{G}(K)$. Then G is a locally compact Hausdorff group such that the compact open subgroups form a basis for the neighborhoods of the identity (this is equivalent to the fact that G has compact open subgroups and they are all profinite). In particular, G is totally disconnected.

To understand (1.2.2) and (1.2.3) we will only need the cases where $\mathbf{G}$ is either a product of GL_n's or the reductive group associated to some central skew field D over K. Nevertheless, in the first sections we will consider the general case of reductive groups to avoid case by case considerations. In fact, almost everywhere we could even work with an arbitrary locally compact totally disconnected group (see e.g. [Vi] for an exposition).

(2.1.2) In the case $\mathbf{G} = \mathrm{GL}_n$ and hence $G = \mathrm{GL}_n(K)$, a fundamental system of open neighborhoods of the identity is given by the open compact subgroups $C_m = 1 + \pi_K^m M_n(O_K)$ for $m \geq 1$. They are all contained in $C_0 = \mathrm{GL}_n(O_K)$, and it is not difficult to see that C_0 is a maximal open compact subgroup and that any other maximal open compact subgroup is conjugated to C_0 (see e.g. [Moe] 2).

(2.1.3) Definition: A representation $\pi \colon G \longrightarrow GL(V)$ on a vector space V over the complex numbers is called *admissible* if it satisfies the following two conditions:

(a) (V, π) is *smooth*, i.e. the stabilizer of each vector $v \in V$ is open in G.

(b) For every open subgroup $H \subset G$ the space V^H of H-invariants in V is finite dimensional.

We denote the set of equivalence classes of irreducible admissible representations of G by $\mathcal{A}(G)$. For $G = \mathrm{GL}_n(K)$ we define

$$\mathcal{A}_n(K) = \mathcal{A}(GL_n(K)).$$

Note that the notions of "smoothness" and "admissibility" are purely algebraic and would make sense if we replace $\mathbb{C}$ by an arbitrary field. In fact for the rest of this survey article we could replace $\mathbb{C}$ by an arbitrary non-countable algebraically closed field of characteristic zero. We could avoid the "non-countability assumption" if we worked consequently only with admissible representations. Further, most elements of the general theory even work over algebraically closed fields of characteristic ℓ with $\ell \neq p$ ([Vi]).

(2.1.4) As every open subgroup of G contains a compact open subgroup, a representation (π, V) is smooth if and only if

$$V = \bigcup_C V^C$$

where C runs through the set of open and compact subgroups of G, and it is admissible if in addition all the V^C are finite-dimensional.

(2.1.5) Example: A smooth one-dimensional representation of $K^\times$ is a *quasi-character of $K^\times$* or by abuse of language a *multiplicative quasi-character of K*, i.e. a homomorphism of abelian groups $K^\times \longrightarrow \mathbb{C}^\times$ which is continuous for the discrete or equivalently for the usual topology of $\mathbb{C}^\times$ (cf. (1.1.11)).

(2.1.6) Let $\mathcal{H}(G)$ be the *Hecke algebra of G*. Its underlying vector space is the space of locally constant, compactly supported measures ϕ on G with complex coefficients. It becomes an associative $\mathbb{C}$-algebra (in general without unit) by the convolution product of measures. If we choose a Haar measure dg on G we can identify $\mathcal{H}(G)$ with the algebra of all locally constant complex-valued functions with compact support on G where the product is given by

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1}) f_2(g) \, dg.$$

(2.1.7) If C is any compact open subgroup of G, we denote by $\mathcal{H}(G//C)$ the subalgebra of $\mathcal{H}(G)$ consisting of those $\phi \in \mathcal{H}(G)$ which are left- and

right-invariant under C. If we choose a Haar measure of G, we can identify $\mathcal{H}(G//C)$ with the set of maps $C\backslash G/C \longrightarrow \mathbb{C}$ with finite support. The algebra $\mathcal{H}(G//C)$ has a unit, given by

$$e_C := \text{vol}(C)^{-1} 1_C$$

where 1_C denotes the characteristic function of C.

If $C' \subset C$ is an open compact subgroup, $\mathcal{H}(G//C)$ is a $\mathbb{C}$-subalgebra of $\mathcal{H}(G//C')$ but with a different unit element if $C \neq C'$. We have

$$\mathcal{H}(G) = \bigcup_C \mathcal{H}(G//C).$$

(2.1.8) If (π, V) is a smooth representation of G, the space V becomes an $\mathcal{H}(G)$-module by the formula

$$\pi(\phi)v = \int_G \pi(g)d\phi$$

for $\phi \in \mathcal{H}(G)$. This makes sense as the integral is essentially a finite sum by (2.1.7).

As $V = \bigcup_C V^C$ where C runs through the open compact subgroups of G, every vector $v \in V$ satisfies $v = \pi(e_C)v$ for some C. In particular, V is a non-degenerate $\mathcal{H}(G)$-module, i.e. $\mathcal{H}(G) \cdot V = V$.

We get a functor from the category of smooth representations of G to the category of non-degenerate $\mathcal{H}(G)$-modules. This functor is an equivalence of categories [Ca] 1.4.

(2.1.9) Let C be an open compact subgroup of G and let (π, V) be a smooth representation of G. Then the space of C-invariants V^C is stable under $\mathcal{H}(G//C)$. If V is an irreducible G-module, V^C is zero or an irreducible $\mathcal{H}(G//C)$-module. More precisely we have

Proposition: *The functor $V \mapsto V^C$ is an equivalence of the category of admissible representations of finite length such that every irreducible subquotient has a non-zero vector fixed by C with the category of finite-dimensional $\mathcal{H}(G//C)$-modules.*

Proof: This follows easily from [Cas1] 2.2.2, 2.2.3 and 2.2.4.

(2.1.10) Corollary: *Let (π, V) be an admissible representation of G and let C be an open compact subgroup of G such that for every irreducible subquotient V' of V we have $(V')^C \neq 0$. Then the following assertions are equivalent:*

(1) *The G-representation π is irreducible.*
(2) *The $\mathcal{H}(G//C)$-module V^C is irreducible.*
(3) *The associated homomorphisms of $\mathbf{C}$-algebras $\mathcal{H}(G//C) \longrightarrow \mathrm{End}_{\mathbf{C}}(V^C)$*
is surjective.

Proof : The equivalence of (1) and (2) is immediate from (2.1.9). The equivalence of (2) and (3) is a standard fact of finite-dimensional modules of an algebra (see e.g. [BouA] chap. VIII, §13, 4, Prop. 5).

(2.1.11) Corollary: *Let (π, V) be an irreducible admissible representation of G and let $C \subset G$ be an open compact subgroup such that $\mathcal{H}(G//C)$ is commutative. Then $\dim_{\mathbf{C}}(V^C) \leq 1$.*

(2.1.12) For $G = GL_n(K)$ the hypothesis that $\mathcal{H}(G//C)$ is commutative is fulfilled for $C = \mathrm{GL}_n(O_K)$. In this case we have

$$\mathcal{H}(G//C) = \mathbf{C}[T_1^{\pm 1}, \ldots, T_n^{\pm 1}]^{S_n}$$

where the symmetric group S_n acts by permuting the variables T_i.

More generally, let $\mathbf{G}$ be unramified, which means that there exists a reductive model of $\mathbf{G}$ over O_K, i.e. a flat affine group scheme over O_K such that its special fibre is reductive and such that its generic fibre is equal to $\mathbf{G}$. This is equivalent to the condition that $\mathbf{G}$ is quasi-split and split over an unramified extension [Ti] 1.10. If C is a hyperspecial subgroup of G (i.e. it is of the form $\tilde{\mathbf{G}}(O_K)$ for some reductive model $\tilde{\mathbf{G}}$), the Hecke algebra $\mathcal{H}(G//C)$ can be identified via the Satake isomorphism with the algebra of invariants under the rational Weyl group of $\mathbf{G}$ of the group algebra of the cocharacter group of a maximal split torus of $\mathbf{G}$ [Ca] 4.1. In particular, it is commutative.

(2.1.13) Under the equivalence of the categories of smooth G-representations and non-degenerate $\mathcal{H}(G)$-modules the admissible representations (π, V)

correspond to those non-degenerate $\mathcal{H}(G)$-modules such that for any $\phi \in \mathcal{H}(G)$ the operator $\pi(\phi)$ has finite rank.

In particular, we may speak of the trace of $\pi(\phi)$ if π is admissible. We get a distribution $\phi \mapsto \mathrm{Tr}(\pi(\phi))$ which is denoted by χ_π and called the *distribution character of π*. It is invariant under conjugation.

(2.1.14) We keep the notations of (2.1.13). If $\{\pi_1, \dots, \pi_n\}$ is a set of pairwise non-isomorphic irreducible admissible representations of G, then the set of functionals

$$\{\chi_{\pi_1}, \dots, \chi_{\pi_2}\}$$

is linearly independent (cf. [JL] Lemma 7.1). In particular, two irreducible admissible representations with the same distribution character are isomorphic.

(2.1.15) Let (π, V) be an admissible representation of $G = \mathbf{G}(K)$. By a theorem of Harish-Chandra [HC] the distribution χ_π is represented by a locally integrable function on G which is again denoted by χ_π, i.e. for every $\phi \in \mathcal{H}(G)$ we have

$$\mathrm{Tr}\,\pi(\phi) = \int_G \chi_\pi(g)\,d\phi.$$

The function χ_π is locally constant on the set of regular semisimple (see 5.1.3 for a definition in case $G = \mathrm{GL}_n(K)$) elements in G (loc. cit.), and it is invariant under conjugation. Therefore it defines a function

$$\chi_\pi \colon \{G\}^{\mathrm{reg}} \longrightarrow \mathbb{C}$$

on the set $\{G\}^{\mathrm{reg}}$ of conjugacy classes of regular semisimple elements in G.

(2.1.16) Proposition (Lemma of Schur): *Let (π, V) be an irreducible smooth representation of G. Every G-endomorphism of V is a scalar.*

Proof : We only consider the case that π is admissible (we cannot use (2.1.17), because its proof uses Schur's lemma hence we should not invoke (2.1.17) if we do not want to run into a circular argument; a direct proof of the general case can be found in [Ca] 1.4, it uses the fact that $\mathbb{C}$ is not

countable). For sufficiently small open compact subgroups C of G we have $V^C \neq 0$. Hence we have only to show that every $\mathcal{H}(G//C)$-endomorphism f of a finite dimensional irreducible $\mathcal{H}(G//C)$-module W over $\mathbb{C}$ is a scalar. As $\mathbb{C}$ is algebraically closed, f has an eigenvalue c, and $\mathrm{Ker}(f - c\,\mathrm{id}_W)$ is a $\mathcal{H}(G//C)$-submodule different from W. Therefore $f = c\,\mathrm{id}_W$.

(2.1.17) Proposition: *Let (π, V) be an irreducible and smooth complex representation of G.*

    (1) *The representation π is admissible.*

    (2) *If G is commutative, it is one-dimensional.*

Proof: The first assertion is difficult and can be found for $G = \mathrm{GL}_n(K)$ in [BZ1] 3.25. It follows from the fact that every smooth irreducible representation can be embedded in a representation which is induced from a smaller group and which is admissible (more precisely it is supercuspidal, see below). Given (1) the proof of (2) is easy: By (1) we can assume that π is admissible. For any compact open subgroup C of G the space V^C is finite-dimensional and a G-submodule. Hence $V = V^C$ for any C with $V^C \neq (0)$ and in particular V is finite-dimensional. But it is well known that every irreducible finite-dimensional representation of a commutative group H on a vector space over an algebraically closed field is one-dimensional (apply e.g. [BouA] chap. VIII, §13, Prop. 5 to the group algebra of H).

(2.1.18) Proposition: *Every irreducible smooth representation of $\mathrm{GL}_n(K)$ is either one-dimensional or infinite-dimensional. If it is one-dimensional, it is of the form $\chi \circ \det$ where χ is a quasi-character of $K^\times$, i.e. a continuous homomorphism $K^\times \longrightarrow \mathbb{C}^\times$.*

We leave the proof as an exercise (show e.g. that the kernel of a finite-dimensional representation $\pi \colon \mathrm{GL}_n(K) \longrightarrow \mathrm{GL}_n(\mathbb{C})$ is open, deduce that π is trivial on the subgroup of unipotent upper triangular matrices U, hence π is trivial on the subgroup of $\mathrm{GL}_n(K)$ which is generated by all conjugates of U and this is nothing but $\mathrm{SL}_n(K)$).

(2.1.19) Let $\mathbf{Z}$ be the center of $\mathbf{G}$. As K is infinite, $\mathbf{Z}(K)$ is the center Z of G. In the case $G = GL_n(K)$ we have $Z = K^\times$. For $(\pi, V) \in \mathcal{A}(G)$ we

denote by $\omega_\pi \colon Z \longrightarrow \mathbb{C}^\times$ its *central character*, defined by

$$\omega_\pi(z) \operatorname{id}_V = \pi(z)$$

for $z \in Z$. It exists by the lemma of Schur.

For $G = GL_n(K)$, ω_π is a quasi-character of $K^\times$.

(2.1.20) Proposition Assume that G/Z is a compact group. Then every irreducible admissible representation (π, V) of G is finite-dimensional.

Proof : By hypothesis we can find a compact open subgroup G^0 of G such that $G^0 Z$ has finite index in G (take for example the group G^0 defined in 2.4.1 below). The restriction of an irreducible representation π of G to $G^0 Z$ decomposes into finitely many irreducible admissible representations. By the lemma of Schur, Z acts on each of these representation as a scalar, hence they are also irreducible representations of the compact group G^0 and therefore they are finite-dimensional.

(2.1.21) Let (π, V) be a smooth representation of G and let χ be a quasi-character of G. The *twisted representation* $\pi\chi$ is defined as

$$g \mapsto \pi(g)\chi(g).$$

The G-submodules of (π, V) are the same as the G-submodules of $(\pi\chi, V)$. In particular π is irreducible if and only if $\pi\chi$ is irreducible. Further, if C is a compact open subgroup of G, $\chi(C) \subset \mathbb{C}^\times$ is finite, and therefore χ is trivial on a subgroup $C' \subset C$ of finite index. This shows that π is admissible if and only if $\pi\chi$ is admissible.

If $G = \mathrm{GL}_n(K)$ every quasi-character χ is of the form $\chi' \circ \det$ where χ' is a multiplicative quasi-character of K (2.1.18), and we write $\pi\chi'$ instead of $\pi\chi$.

(2.1.22) Let $\pi \colon G \longrightarrow \mathrm{GL}(V)$ be a smooth representation of G. Denote by V^* the $\mathbb{C}$-linear dual of V. It is a G-module via $(g\lambda)(v) = \lambda(g^{-1}v)$ which is not smooth if $\dim(V) = \infty$. Define

$$V^\vee = \{\, \lambda \in V^* \mid \operatorname{Stab}_G(\lambda) \text{ is open} \,\}.$$

This is a G-submodule $\pi^\vee$ which is smooth by definition. It is called the *contragredient* of the G-module V. Further we have:

(1) π is admissible if and only if $\pi^\vee$ is admissible and in this case the biduality homomorphism induces an isomorphism $V \longrightarrow (V^\vee)^\vee$ of G-modules.

(2) π is irreducible if and only if $\pi^\vee$ is irreducible.

(3) In the case of $G = GL_n(K)$ we can describe the contragredient also in the following way: If π is smooth and irreducible, $\pi^\vee$ is isomorphic to the representation $g \mapsto \pi({}^t g^{-1})$ for $g \in GL_n(K)$.

Assertions (1) and (2) are easy (use that $(V^\vee)^C = (V^C)^*$ for every compact open subgroup C). The last assertion is a theorem of Gelfand and Kazhdan ([BZ1] 7.3).

2.2 Induction and the Bernstein-Zelevinsky classification for $GL(n)$

(2.2.1) Fix an ordered partition $\underline{n} = (n_1, n_2, \ldots, n_r)$ of n. Denote by $G_{\underline{n}}$ the algebraic group $GL_{n_1} \times \cdots \times GL_{n_r}$ considered as a Levi subgroup of $G_{(n)} = GL_n$. Denote by $P_{\underline{n}} \subset GL_n$ the parabolic subgroup of matrices of the form

$$\begin{pmatrix} A_1 & & & \\ & A_2 & & * \\ & & \ddots & \\ & 0 & & \ddots \\ & & & & A_r \end{pmatrix}$$

for $A_i \in GL_{n_i}$ and by $U_{\underline{n}}$ its unipotent radical. If (π_i, V_i) is an admissible representation of $GL_{n_i}(K)$, $\pi_1 \otimes \ldots \otimes \pi_r$ is an admissible reprentation of $G_{\underline{n}}(K)$ on $W = V_1 \otimes \cdots \otimes V_r$. By extending this representation to $P_{\underline{n}}$ and by normalized induction we get a representation $\pi_1 \times \cdots \times \pi_r$ of $GL_n(K)$ whose underlying complex vector space V is explicitly defined by

$$V = \big\{ f \colon GL_n(K) \longrightarrow W \ \big| \ f \text{ smooth}, \ f(umg) = \delta_{\underline{n}}^{1/2}(m)(\pi_1 \otimes \cdots \otimes \pi_r)(m)f(g) \\ \text{for } u \in U_{\underline{n}}(K), \ m \in GL_{\underline{n}}(K) \text{ and } g \in GL_n(K) \big\} .$$

Here we call a map $f\colon GL_n(K) \longrightarrow W$ smooth if its stabilizer

$$\{\, g \in GL_n(K) \mid f(gh) = f(h) \text{ for all } h \in GL_n(K) \,\}$$

is open in $GL_n(K)$ (or equivalently if f is fixed by some open compact of $\mathrm{GL}_n(K)$ acting by right translation), and $\delta_{\underline{n}}^{1/2}$ denotes the positive square root of the modulus character

$$\delta_{\underline{n}}(m) = |\det(\mathrm{Ad}_{U_{\underline{n}}}(m))|.$$

The group $GL_n(K)$ acts on V by right translation.

(2.2.2) Definition: An irreducible smooth representation of $GL_n(K)$ is called *supercuspidal* if there exists no proper partition $\underline{n}$ such that π is a subquotient of a representation of the form $\pi_1 \times \cdots \times \pi_r$ where π_i is an admissible representation of $GL_{n_i}(K)$. We denote by $\mathcal{A}_n^0(K) \subset \mathcal{A}_n(K)$ the subset of equivalence classes of supercuspidal representations of $GL_n(K)$.

(2.2.3) Let π_i be a smooth representation of $\mathrm{GL}_{n_i}(K)$ for $i = 1, \ldots, r$. Then $\pi = \pi_1 \times \cdots \times \pi_r$ is a smooth representation of $\mathrm{GL}_n(K)$ with $n = n_1 + \cdots + n_r$. Further it follows from the compactness of $\mathrm{GL}_n(K)/P_{\underline{n}}$ that if the π_i are admissible, π is also admissible ([BZ1] 2.26). Further, by [BZ2] we have the following

    Theorem: *If the π_i are of finite length (and hence admissible by (2.1.17)) for all $i = 1, \ldots, r$ (e.g. if all π_i are irreducible), $\pi_1 \times \cdots \times \pi_r$ is also admissible and of finite length. Conversely, if π is an irreducible admissible representation of $\mathrm{GL}_n(K)$, there exists a unique partition $n = n_1 + \cdots + n_r$ of n and unique (up to isomorphism and ordering) supercuspidal representations π_i of $GL_{n_i}(K)$ such that π is a subquotient of $\pi_1 \times \cdots \times \pi_r$.*

(2.2.4) If π is an irreducible admissible representation of $\mathrm{GL}_n(K)$ we denote the unique unordered tuple $(\pi_1, \cdots, \pi_r)$ of supercuspidal representations such that π is a subquotient of $\pi_1 \times \cdots \times \pi_r$ the *supercuspidal support*.

(2.2.5) Definition: Let $\pi\colon GL_n(K) \longrightarrow GL(V)$ be a smooth representation. For $v \in V$ and $\lambda \in V^\vee$ the map

$$c_{\pi,v,\lambda} = c_{v,\lambda}\colon G \longrightarrow \mathbb{C}, \qquad g \mapsto \lambda(\pi(g)v)$$

is called the (v,λ)-*matrix coefficient* of π.

(2.2.6) Let (π, V) be an admissible representation.
    (1) For $v \in V = (V^\vee)^\vee$ and $\lambda \in V^\vee$ we have

$$c_{\pi,v,\lambda}(g) = c_{\pi^\vee,\lambda,v}(g^{-1}).$$

    (2) If χ is a quasi-character of $K^\times$ we have

$$c_{\pi\chi,v,\lambda}(g) = \chi(\det(g))c_{\pi,v,\lambda}.$$

(2.2.7) Theorem: *Let π be a smooth irreducible representation of $\mathrm{GL}_n(K)$. Then the following statements are equivalent:*
    (1) *π is supercuspidal.*
    (2) *All the matrix coefficents of π have compact support modulo center.*
    (3) *$\pi^\vee$ is supercuspidal.*
    (4) *For any quasi-character χ of $K^\times$, $\pi\chi$ is supercuspidal.*

    Proof : The equivalence of (1) and (2) is a theorem of Harish-Chandra [BZ1] 3.21. The equivalence of (2), (3) and (4) follows then from (2.2.6).

(2.2.8) For any complex number s and for any admissible representation we define $\pi(s)$ as the twist of π with the character $|\ |^s$, i.e. the representation $g \mapsto |\det(g)|^s\pi(g)$.
    If π is supercuspidal, $\pi(s)$ is also supercuspidal. Define a partial order on $\mathcal{A}_n^0(K)$ by $\pi \leq \pi'$ iff there exists an integer $n \geq 0$ such that $\pi' = \pi(n)$. Hence every finite interval Δ is of the form

$$\Delta(\pi, m) = [\pi, \pi(1), \ldots, \pi(m-1)].$$

The integer m is called the *length* of the interval and nm is called its *degree*. We write $\pi(\Delta)$ for the representation $\pi \times \cdots \times \pi(m-1)$ of $GL_{nm}(K)$.

Two finite intervals Δ_1 and Δ_2 are said to be *linked* if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$, and $\Delta_1 \cup \Delta_2$ is an interval. We say that Δ_1 *precedes* Δ_2 if Δ_1 and Δ_2 are linked and if the minimal element of Δ_1 is smaller than the minimal element of Δ_2.

(2.2.9) Theorem (Bernstein-Zelevinsky classification ([Ze], cf. also [Ro])):

(1) *For any finite interval $\Delta \subset \mathcal{A}_n^0(K)$ of length m the representation $\pi(\Delta)$ has length 2^{m-1}. It has a unique irreducible quotient $Q(\Delta)$ and a unique irreducible subrepresentation $Z(\Delta)$.*

(2) *Let $\Delta_1 \subset \mathcal{A}_{n_1}^0(K), \ldots, \Delta_r \subset \mathcal{A}_{n_r}^0(K)$ be finite intervals such that for $i < j$, Δ_i does not precede Δ_j (this is an empty condition if $n_i \neq n_j$). Then the representation $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ admits a unique irreducible quotient $Q(\Delta_1, \ldots, \Delta_r)$, and the representation $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$ admits a unique irreducible subrepresentation $Z(\Delta_1, \ldots, \Delta_r)$.*

(3) *Let π be a smooth irreducible representation of $GL_n(K)$. Then it is isomorphic to a representation of the form $Q(\Delta_1, \ldots, \Delta_r)$ (resp. $Z(\Delta_1', \ldots, \Delta_{r'}')$) for a unique (up to permutation) collection of intervals $\Delta_1, \ldots, \Delta_r$ (resp. $\Delta_1', \ldots, \Delta_{r'}'$) such that Δ_i (resp. Δ_i') does not precede Δ_j (resp. Δ_j') for $i < j$.*

(4) *Under the hypothesis of (2), the representation $Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ is irreducible if and only if no two of the intervals Δ_i and Δ_j are linked.*

(2.2.10) For $\pi \in \mathcal{A}_n^0(K)$ the set of π' in $\mathcal{A}_n^0(K)$ which are comparable with π with respect to the order defined above is isomorphic (as an ordered set) to $\mathbb{Z}$, in particular it is totally ordered. It follows that given a tuple of intervals $\Delta_i = [\pi_i, \ldots, \pi_i(m_i - 1)]$, $i = 1, \ldots, r$ we can always permute them such that Δ_i does not precede Δ_j for $i < j$.

Denote by $\mathcal{S}_n(K)$ the set of unordered tuples $(\Delta_1, \ldots, \Delta_r)$ where Δ_i is an interval of degree n_i such that $\sum n_i = n$. Then (2) and (3) of (2.2.9) are equivalent to the assertion that the maps

$$Q: \mathcal{S}_n(K) \longrightarrow \mathcal{A}_n(K), \qquad (\Delta_1, \ldots, \Delta_r) \mapsto Q(\Delta_1, \ldots, \Delta_r),$$

$$Z: \mathcal{S}_n(K) \longrightarrow \mathcal{A}_n(K), \qquad (\Delta_1, \ldots, \Delta_r) \mapsto Z(\Delta_1, \ldots, \Delta_r),$$

are bijections.

The unordered tuple of supercuspidal representations $\pi_i(j)$ for $i = 1, \ldots, r$ and $j = 0, \ldots, m_i - 1$ is called the *supercuspidal support*. It is the unique unordered tuple of supercuspidal representations $\rho_1, \ldots, \rho_s$ such that $\pi = Q(\Delta_1, \ldots, \Delta_s)$ and $\pi' = Z(\Delta_1, \ldots, \Delta_r)$ is a subquotient of $\rho_1 \times \cdots \times \rho_s$ ([Ze]).

(2.2.11) If $\mathcal{R}_n(K)$ is the Grothendieck group of the category of admissible representations of $\mathrm{GL}_n(K)$ of finite length and $\mathcal{R}(K) = \bigoplus_{n \geq 0} \mathcal{R}_n(K)$, then

$$([\pi_1], [\pi_2]) \mapsto [\pi_1 \times \pi_2]$$

defines a map

$$\mathcal{R}(K) \times \mathcal{R}(K) \longrightarrow \mathcal{R}(K)$$

which makes $\mathcal{R}(K)$ into a graded commutative ring ([Ze] 1.9) which is isomorphic to the ring of polynomials in the indeterminates Δ for $\Delta \in \mathcal{S}(K) = \bigcup_{n \geq 1} \mathcal{S}_n(K)$ (loc. cit. 7.5).

The different descriptions of $\mathcal{A}_n(K)$ via the maps Q and Z define a map

$$t \colon \mathcal{R}(K) \longrightarrow \mathcal{R}(K), \qquad Q(\Delta) \mapsto Z(\Delta).$$

We have:

Proposition: *(1) The map t is an involution of the graded ring $\mathcal{R}$.*
(2) It sends irreducible representations to irreducible representations.
(3) For $\Delta = [\pi, \pi(1), \ldots, \pi(m-1)]$ we have

$$t(Q(\Delta)) = Q(\pi, \pi(1), \ldots, \pi(m-1))$$

where on the right-hand side we consider $\pi(i)$ as intervals of length 1.
(4) We have
$$t(Q(\Delta_1, \ldots, \Delta_r)) = Z(\Delta_1, \ldots, \Delta_r),$$
$$t(Z(\Delta_1, \ldots, \Delta_r)) = Q(\Delta_1, \ldots, \Delta_r).$$

Proof: Assertions (1) and (3) follow from [Ze] 9.15. The second assertion had been anounced by J.N. Bernstein but no proof has been published. It has been proved quite recently in [Pr] or [Au1] (see also [Au2]). Assertion (4) is proved by Rodier in [Ro] théorème 7 under the assumption of (2).

(2.2.12) For each interval $\Delta = [\pi, \ldots, \pi(m-1)]$ we set

$$\Delta^\vee = [\pi(m-1)^\vee, \ldots, \pi^\vee] = [\pi^\vee(1-m), \ldots, \pi^\vee(-1), \pi^\vee].$$

It follows from [Ze] 3.3 and 9.4 (cf. also [Tad] 1.15 and 5.6) that we have

$$Q(\Delta_1, \ldots, \Delta_r)^\vee = Q(\Delta_1^\vee, \ldots, \Delta_r^\vee),$$
$$Z(\Delta_1, \ldots, \Delta_r)^\vee = Z(\Delta_1^\vee, \ldots, \Delta_r^\vee).$$

In particular we see that the involution on $\mathcal{R}$ induced by $[\pi] \mapsto [\pi^\vee]$ commutes with the involution t in (2.2.11).

(2.2.13) Example: Let $\Delta \subset \mathcal{A}_1(K)$ be the interval

$$\Delta = (|\ |^{(1-n)/2}, |\ |^{(3-n)/2}, \ldots, |\ |^{(n-1)/2}).$$

The associated representation of the diagonal torus $T \subset GL_n(K)$ is equal to $\delta_B^{-1/2}$ where $\delta_B(t) = |\det \mathrm{Ad}_U(t)|_K$ is the modulus character of the adjoint action of T on the group of unipotent upper triangular matrices U and where B is the subgroup of upper triangular matrices in $GL_n(K)$. Hence we see that

$$\pi(\Delta) = |\ |^{(1-n)/2} \times |\ |^{(3-n)/2} \times \ldots \times |\ |^{(n-1)/2}$$

consists just of the space of smooth functions on $B \backslash G$ with the action of G induced by the natural action of G on the flag variety $B \backslash G$. Hence $Z(\Delta)$ is the trivial representation $\mathbf{1}$ of constant functions on G/B. The representation $Q(\Delta) = t(Z(\mathbf{1}))$ is called the *Steinberg representation* and denoted by $\mathrm{St}(n)$. It is selfdual, i.e. $\mathrm{St}(n)^\vee = \mathrm{St}(n)$ (in fact, it is also unitary and even square integrable, see the next section). For $n = 2$ the length of $\pi(\Delta)$ is 2, hence we have $\mathrm{St}(n) = \pi(\Delta)/\mathbf{1}$.

2.3   Square integrable and tempered representations

(2.3.1) We return to the general setting where $\mathbf{G}$ is an arbitrary connected reductive group over K. Every character $\alpha \colon \mathbf{G} \longrightarrow \mathbb{G}_m$ defines on K-valued

points a homomorphism $\alpha\colon G \longrightarrow K^\times$. By composition with the absolute value $|\ |_K$ we obtain a homomorphism $|\alpha|_K\colon G \longrightarrow \mathbb{R}^{>0}$ and we set

$$G^0 = \bigcap_\alpha \mathrm{Ker}(|\alpha|_K).$$

If $\mathbf{G} = GL_n$ then every α is a power of the determinant, hence we have

$$\mathrm{GL}_n(K)^0 = \{\, g \in GL_n(K) \mid |\det(g)|_K = 1 \,\}.$$

Let r be a positive real number. We call an admissible representation (π, V) of G *essentially L^r* if for all $v \in V$ and $\lambda \in V^\vee$ the matrix coefficient $c_{v,\lambda}$ is L^r on G^0, i.e. the integral

$$\int_{G^0} |c_{\lambda,v}|^r \, dg$$

exists (where dg denotes some Haar measure of G^0).

An admissible representation is called L^r if it is essentially L^r and if it has a central character (2.1.19) which is unitary.

Let Z be the center of G. Then the composition $G^0 \longrightarrow G \longrightarrow G/Z$ has compact kernel and finite cokernel. Hence, if (π, V) has a unitary central character ω_π, the integral

$$\int_{Z\backslash G} |c_{v,\lambda}|^r \, dg$$

makes sense, and (π, V) is L^r if and only if this integral is finite.

(2.3.2) Proposition: Let π be an admissible representation of G which is L^r. Then it is $L^{r'}$ for all $r' \geq r$.

Proof: This follows from [Si3] 2.5.

(2.3.3) Definition: An admissible representation of G is called *essentially square integrable* (resp. *essentially tempered*) if it is essentially L^2 (resp. essentially $L^{2+\varepsilon}$ for all $\varepsilon > 0$). We have similar definitions by omitting "essentially".

By (2.3.2), any (essentially) square integrable representation is (essentially) tempered.

(2.3.4) The notion of "tempered" is explained by the following proposition (which follows from [Si1] §4.5 and [Si3] 2.6):

Proposition: *Let π be an irreducible admissible representation of G such that its central character is unitary. Then the following assertions are equivalent:*

(1) *π is tempered.*

(2) *Each matrix coefficient defines a tempered distribution on G (with the usual notion of a tempered distribution: It extends from a linear form on the locally constant functions with compact support on G to a linear form on the Schwartz space of G (the "rapidly decreasing functions on G"), see [Si1] for the precise definition in the p-adic setting).*

(3) *The distribution character of π is tempered.*

(2.3.5) Example: By (2.2.7) any supercuspidal representation is essentially L^r for all $r > 0$. In particular it is essentially square integrable.

(2.3.6) If (π, V) is any smooth representation of G which has a central character, then there exists a unique positive real valued quasi-character χ of G such that $\pi\chi$ has a unitary central character (for $G = GL_n(K)$ this is clear as every quasi-character factors through the determinant (2.1.18), for arbitrary reductive groups this is [Cas1] 5.2.5). Hence for $G = GL_n(K)$ the notion of "essential square-integrability" is equivalent to the notion of "quasi-square-integrability" in the sense of [Ze]. In particular it follows from [Ze] 9.3:

Theorem: *An irreducible admissible representation π of $\mathrm{GL}_n(K)$ is essentially square-integrable if and only if it is of the form $Q(\Delta)$ with the notations of (2.2.9). It is square integrable if and only if Δ is of the form $[\rho, \rho(1), \ldots, \rho(m-1)]$ where the central character of $\rho((m-1)/2)$ is unitary.*

(2.3.7) We also have the following characterization of tempered representations in the Bernstein-Zelevinsky classification (see [Kud] 2.2):

Proposition: *An irreducible admissible representation $Q(\Delta_1, \ldots, \Delta_r)$ of $\mathrm{GL}_n(K)$ is tempered if and only if the $Q(\Delta_i)$ are square integrable.*

(2.3.8) If $\pi = Q(\Delta_1, \ldots, \Delta_r)$ is a tempered representation no two of the intervals $\Delta_i = [\rho_i, \ldots, \rho_i(m_i - 1)]$ are linked as $\mathrm{cent}(\Delta_i) = \rho_i((m_i - 1)/2)$ has unitary central character and all elements in Δ_i different from $\mathrm{cent}(\Delta_i)$ have a non-unitary central character. Therefore we have

$$\pi = Q(\Delta_1) \times \cdots \times Q(\Delta_r).$$

(2.3.9) Let $\pi = Q(\Delta_1, \ldots, \Delta_r)$ be an arbitrary irreducible admissible representation. For each Δ_i there exists a unique real number x_i such that $Q(\Delta_i)(-x_i)$ is square integrable. We can order the Δ_i's such that

$$y_1 := x_1 = \cdots = x_{m_1} > y_2 := x_{m_1+1} = \cdots = x_{m_2}$$

$$> \cdots > y_s := x_{m_{s-1}+1} = \cdots = x_r.$$

In this order Δ_i does not precede Δ_j for $i < j$ and all Δ_i's which correspond to the same y_j are not linked. For $j = 1, \ldots, s$ set

$$\pi_j = Q(\Delta_{m_{j-1}+1})(-y_j) \times \cdots \times Q(\Delta_{m_j})(-y_j)$$

with $m_0 = 0$ and $m_s = r$. Then all π_j are irreducible tempered representation, and π is the unique irreducible quotient of $\pi_1(y_1) \times \cdots \times \pi_s(y_s)$. This is nothing but the Langlands classification which can be generalized to arbitrary reductive groups (see [Si1] or [BW]).

2.4 Generic representations

(2.4.1) Fix a non-trivial additive quasi-character $\psi \colon F \longrightarrow \mathbb{C}^\times$ and let $n(\psi)$ be the *exponent of ψ*, i.e. the largest integer n such that $\psi(\pi_K^{-n} O_K) = 1$.

(2.4.2) Let $U_n(K) \subset \mathrm{GL}_n(K)$ be the subgroup of unipotent upper triangular matrices and define a one-dimensional representation θ_ψ of $U_n(K)$ by

$$\theta_\psi((u_{ij})) = \psi(u_{12} + \cdots + u_{n-1,n}).$$

If π is any representation of $GL_n(K)$ we can consider the space of homomorphisms of $U_n(K)$-modules

$$\mathrm{Hom}_{U_n(K)}(\pi|_{U_n(K)}, \theta_\psi).$$

If π is smooth and irreducible we call π *generic* if this space is non-zero.

(2.4.3) In the next few sections we collect some facts about generic representations of $GL_n(K)$ which can be found in [BZ1], [BZ2] and [Ze]. Note that in loc. cit. the term "non-degenerate" is used instead of "generic". First of all we have:

    Proposition: *(1) The representation π is generic if and only if $\pi^\vee$ is generic.*

    (2) For all multiplicative quasicharacters $\chi \colon K^\times \longrightarrow \mathbb{C}^\times$, π is generic if and only if $\chi\pi$ is generic.

    (3) The property of π being generic does not depend of the choice of the non-trivial additive character ψ.

(2.4.4) Via the Bernstein-Zelevinsky classification we have the following characterization of generic representations ([Ze] 9.7):

    Theorem: *An irreducible admissible representation $\pi = Q(\Delta_1, \ldots, \Delta_r)$ is generic if and only if no two segments Δ_i are linked. In particular we have*

$$\pi \cong Q(\Delta_1) \times \cdots \times Q(\Delta_r).$$

(2.4.5) Corollary: *Every essentially tempered (and in particular every supercuspidal) representation is generic.*

(2.4.6) If (π, V) is generic, it has a Whittaker model: Choose a

$$0 \neq \lambda \in \mathrm{Hom}_{U(K)}(\pi|_{U(K)}, \theta_\psi)$$

and define a map

$$V \longrightarrow \{\, f \colon GL_n(K) \longrightarrow \mathbb{C} \mid f(ug) = \theta(u)f(g) \text{ for all } g \in GL_n(K),\, u \in U(K) \,\},$$
$$v \mapsto (g \mapsto \lambda(\pi(g)v))$$

.

This is an injective homomorphism of $GL_n(K)$-modules if $GL_n(K)$ acts on the right-hand side by right translation, and we call its image the *Whittaker model of π with respect to ψ* and denote it by $\mathcal{W}(\pi, \psi)$.

(2.4.7) The concept of a generic representation plays a fundamental role in the theory of automorphic forms: If π is an irreducible admissible representation of the adele valued group $GL_n(\mathbf{A}_L)$ for a number field L, it can be decomposed in a restricted tensor product

$$\pi = \bigotimes_v \pi_v$$

where v runs through the places of L and where π_v is an admissible irreducible representation of $GL_n(L_v)$ (see Flath [Fl] for the details). If π is cuspidal, all the π_v are generic by Shalika [Sh].

2.5 Definition of L- and epsilon-factors

(2.5.1) Let π and π' be smooth irreducible representations of $GL_n(K)$ and of $GL_{n'}(K)$ respectively. We are going to define L- and ε-factors of the pair (π, π'). We first do this for supercuspidal (or more generally for generic) representation and then use the Bernstein-Zelevinsky classification to make the general definition.

Assume now that our fixed non-trivial additive character ψ (2.4.1) is unitary, i.e. $\psi^{-1} = \bar{\psi}$. Let π and π' be generic representations of $GL_n(K)$ and $GL_{n'}(K)$ respectively. To define L- and ε-factors $L(\pi \times \pi', s)$ and $\varepsilon(\pi \times \pi', s, \psi)$ we follow [JPPS1].

Consider first the case $n = n'$. Denote by $\mathcal{S}(K^n)$ the set of locally constant functions $\phi \colon K^n \longrightarrow \mathbb{C}$ with compact support. For elements $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar{\psi})$ in the Whittaker models and for any $\phi \in \mathcal{S}(K^n)$ define

$$Z(W, W', \phi, s) = \int_{U_n(K)\backslash\mathrm{GL}_n(K)} W(g)W'(g)\phi((0,\ldots,0,1)g)|\det(g)|^s \, dg$$

where dg is a $GL_n(K)$-invariant measure on $U_n(K)\backslash\mathrm{GL}_n(K)$. This is absolutely convergent if $\mathrm{Re}(s)$ is sufficiently large and it is a rational function of

q^{-s}. The set

$$\{ Z(W, W', \phi, s) \mid W \in \mathcal{W}(\pi, \psi),\ W' \in \mathcal{W}(\pi', \bar\psi)\ \text{and}\ \phi \in \mathcal{S}(K^n) \}$$

generates a fractional ideal in $\mathbf{C}[q^s, q^{-s}]$ with a unique generator $L(\pi \times \pi', s)$ of the form $P(q^{-s})^{-1}$ where $P \in \mathbf{C}[X]$ is a polynomial such that $P(0) = 1$.

Further $\varepsilon(\pi \times \pi', s, \psi)$ is defined by the equality

$$\frac{Z(\tilde W, \tilde W', 1 - s, \hat\phi)}{L(\pi^\vee \times \pi'^\vee, 1 - s)} = \omega_{\pi'}(-1)^n \varepsilon(\pi \times \pi', s, \psi) \frac{Z(W, W', s, \phi)}{L(\pi \times \pi', s)}.$$

Here we define $\tilde W$ by $\tilde W(g) = W(w_n{}^t g^{-1})$ where $w_n \in GL_n(K)$ is the permutation matrix corresponding to the longest Weyl group element (i.e. to the permutation which sends i to $n + 1 - i$). Because of (2.1.22) this is an element of $\mathcal{W}(\pi^\vee, \bar\psi)$. In the same way we define $\tilde W' \in \mathcal{W}(\pi'^\vee, \psi)$. Finally $\hat\phi$ denotes the Fourier transform of ϕ with respect to ψ given by

$$\hat\phi(x) = \int_{K^n} \phi(y) \psi({}^t y\, x)\, dy$$

for $x \in K^n$.

Now consider the case $n' < n$. For $W \in \mathcal{W}(\pi, \psi)$, $W' \in \mathcal{W}(\pi', \bar\psi)$ and for $j = 0, 1, \ldots, n - n' - 1$ define

$$Z(W, W', j, s) = \int_{U_{n'}(K) \backslash \mathrm{GL}_{n'}(K)} \int_{M_{j \times n'}(K)} W\left(\begin{pmatrix} g & 0 & 0 \\ x & I_j & 0 \\ 0 & 0 & I_{n-n'-j} \end{pmatrix} \right) W'(g)$$
$$\cdot |\det(g)|^{s - (n - n')/2}\, dx\, dg$$

where dg is a $GL_{n'}(K)$-invariant measure on $U_{n'}(K) \backslash \mathrm{GL}_{n'}(K)$ and dx is a Haar measure on the space of $(j \times n')$-matrices over K. Again this is absolutely convergent if $\mathrm{Re}(s)$ is sufficiently large, it is a rational function of q^{-s} and these functions generate a fractional ideal with a unique generator $L(\pi \times \pi', s)$ of the form $P(q^{-s})^{-1}$ where $P \in \mathbf{C}[X]$ is a polynomial such that $P(0) = 1$. In this case $\varepsilon(\pi \times \pi', s, \psi)$ is defined by

$$\frac{Z(w_{n,n'} \tilde W, \tilde W', n - n' - 1 - j, 1 - s)}{L(\pi^\vee \times \pi'^\vee, 1 - s)} = \omega_{\pi'}(-1)^{n-1} \varepsilon(\pi \times \pi', \psi, s) \frac{Z(W, W', j, s)}{L(\pi \times \pi', s)}$$

where $w_{n,n'}$ is the matrix $\begin{pmatrix} I_{n'} & 0 \\ 0 & w_{n-n'} \end{pmatrix} \in GL_n(K)$.

Finally for $n' > n$ we define

$$L(\pi \times \pi', s) = L(\pi' \times \pi, s), \qquad \varepsilon(\pi \times \pi', \psi, s) = \varepsilon(\pi' \times \pi, \psi, s).$$

In all cases $L(\pi \times \pi', s)$ does not depend on the choice of ψ, and $\varepsilon(\pi \times \pi', s, \psi)$ is of the form cq^{-fs} for a non-zero complex number c and an integer f which depend only on π, π' and ψ.

This finishes the definition of the L- and the ε-factor for the generic case (and in particular for the supercuspidal case). Note that if π and π' are supercuspidal, we have

$$(2.5.1.1) \qquad\qquad L(\pi \times \pi', s) = \prod_{\chi} L(\chi, s)$$

where χ runs over the unramified quasi-characters of $K^\times$ such that $\chi \pi'^\vee \cong \pi$ and where $L(\chi, s)$ is the L-function of a character as defined in Tate's thesis (see also below). In particular we have $L(\pi \times \pi', s) = 1$ for $\pi \in \mathcal{A}_n^0(K)$ and $\pi' \in \mathcal{A}_{n'}^0(K)$ with $n \neq n'$.

It seems that there is no such easy way to define $\varepsilon(\pi \times \pi', \psi, s)$ for supercuspidal π and π'. However, Bushnell and Henniart [BH] prove that $\varepsilon(\pi \times \pi^\vee, \psi, 1/2) = \omega_\pi(-1)^{n-1}$ for every irreducible admissible representation π of $\mathrm{GL}_n(K)$.

(2.5.2) From the definition of the L- and ε-factor in the supercuspidal case we deduce the definition of the L- and ε-factor for pairs of arbitrary smooth irreducible representations π and π' by the following inductive relations using (2.2.9)(cf. [Kud]):

(1) We have $L(\pi \times \pi', s) = L(\pi' \times \pi, s)$ and $\varepsilon(\pi \times \pi', \psi, s) = \varepsilon(\pi' \times \pi, \psi, s)$.

(2) If π is of the form $Q(\Delta_1, \ldots, \Delta_r)$ (2.2.9) and if π' is arbitrary, then

$$L(\pi \times \pi', s) = \prod_{i=i}^{r} L(Q(\Delta_i) \times \pi', s)$$

$$\varepsilon(\pi \times \pi', \psi, s) = \prod_{i=i}^{r} \varepsilon(Q(\Delta_i) \times \pi', \psi, s).$$

(3) If π is of the form $Q(\Delta)$, $\Delta = [\sigma, \sigma(r-1)]$ and $\pi' = Q(\Delta')$, $\Delta' = [\sigma', \sigma'(r'-1)]$ with $r' \geq r$, then

$$L(\pi \times \pi', s) = \prod_{i=1}^{r} L(\sigma \times \sigma', s + r + r' - 1)$$

$$\varepsilon(\pi \times \pi', \psi, s) = \prod_{i=1}^{r} \left(\left(\prod_{j=0}^{r+r'-2i} \varepsilon(\sigma \times \sigma', \psi, s + i + j - 1) \right) \right.$$

$$\left. \times \left(\prod_{j=0}^{r+r'-2i-1} \frac{L(\sigma^\vee \times \sigma'^\vee, 1 - s - i - j)}{L(\sigma \times \sigma', s + i + j - 1)} \right) \right).$$

(2.5.3) Let $\mathbf{1} \colon K^\times \longrightarrow \mathbb{C}^\times$ be the trivial multiplicative character. For any smooth irreducible representation π of $GL_n(K)$ we define

$$L(\pi, s) = L(\pi \times \mathbf{1}, s),$$

$$\varepsilon(\pi, \psi, s) = \varepsilon(\pi \times \mathbf{1}, \psi, s).$$

For $n = 1$, $L(\pi, s)$ and $\varepsilon(\pi, \psi, s)$ are the local L- and ε-factors defined in Tate's thesis. For $n > 1$ and π supercuspidal, we have $L(\pi, s) = 1$, while $\varepsilon(\pi, \psi, s)$ is given by a generalized Gauss sum [Bu].

(2.5.4) Let (π, V) be a smooth and irreducible representation of $\mathrm{GL}_n(K)$. For any non-negative integer t define

$$K_n(t) = \{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_n(O_K) | c \in M_{1 \times n-1}(\pi_K^t O_K), d \equiv 1 \pmod{\pi_K^t O_K} \}.$$

In particular, we have $K_n(0) = GL_n(O_K)$. The smallest non-negative integer t such that $V^{K_n(t)} \neq (0)$ is called the *conductor* of π and denoted by $f(\pi)$. By [JPPS1] (cf. also [BHK]) it is also given by the equality

$$\varepsilon(\pi, \psi, s) = \varepsilon(\pi, \psi, 0) q^{-s(f(\pi) + nn(\psi))}$$

where $n(\psi)$ denotes the exponent of ψ (2.4.1).

3 Explanation of the Galois side

3.1 WEIL-DELIGNE REPRESENTATIONS

(3.1.1) Let W_K be the Weil group of K (1.1.8) and let $\varphi_K \colon W_K \longrightarrow$ $\mathrm{Gal}(\bar{K}/K)$ be the canonical homomorphism.

A *representation of W_K* (resp. *of* $\mathrm{Gal}(\bar{K}/K)$) is a continuous homomorphism $W_K \longrightarrow \mathrm{GL}(V)$ (resp. $\mathrm{Gal}(\bar{K}/K) \longrightarrow \mathrm{GL}(V)$) where V is a finite-dimensional complex vector space. Denote by $\mathrm{Rep}(W_K)$ (resp. $\mathrm{Rep}(\mathrm{Gal}(\bar{K}/K))$) the category of representations of the respective group.

Note that a homomorphism of a locally profinite group (e.g. W_K or $\mathrm{Gal}(\bar{K}/K)$) into $\mathrm{GL}_n(\mathbb{C})$ is continuous for the usual topology of $\mathrm{GL}_n(\mathbb{C})$ if and only if it is continuous for the discrete topology.

(3.1.2) For $w \in W_K$ we set

$$|w| = |w|_K = |\mathrm{Art}_K^{-1}(w)|_K.$$

Then the map $W_K \longrightarrow \mathbb{C}^\times$, $w \mapsto |w|^s$ is a one-dimensional representation (i.e. a quasi-character) of W_K for every complex number s. All one-dimensional representations of W_K which are trivial on I_K (i.e. which are unramified) are of this form ([Ta1] 2.3.1).

(3.1.3) As φ_K is injective with dense image, we can identify $\mathrm{Rep}(\mathrm{Gal}(\bar{K}/K))$ with a full subcategory of $\mathrm{Rep}(W_K)$. A representation in this subcategory is called of *Galois-type*. By [Ta2] 1.4.5 a representation r of W_K is of Galois-type if and only if its image $r(W_K)$ is finite.

Conversely, by [De2] §4.10 and (3.1.2) every irreducible representation r of W_K is of the form $r = r' \otimes |\ |^s$ for some complex number s and for some representation r' of Galois type.

(3.1.4) A representation of Galois-type of W_K is irreducible if and only if it is irreducible as a representation of $\mathrm{Gal}(\bar{K}/K)$. Further, if σ is any irreducible representation of W_K, it is of Galois type if and only if the image of its determinant $\det \circ \sigma$ is a subgroup of finite order of $\mathbb{C}^\times$.

(3.1.5) Let L be a finite extension of K in $\bar{K}$. Then we have a canonical injective homomorphism $W_L \longrightarrow W_K$ with finite cokernel. Hence restriction and induction of representations give functors

$$\mathrm{res}_{L/K}\colon \mathrm{Rep}(W_K) \longrightarrow \mathrm{Rep}(W_L)$$
$$\mathrm{ind}_{L/K}\colon \mathrm{Rep}(W_L) \longrightarrow \mathrm{Rep}(W_K)$$

satisfying the usual Frobenius reciprocity.

More precisely, any representation of W_K becomes a representation of W_L by restriction $r \mapsto r|_{W_K}$. This defines the map $\mathrm{res}_{L/K}$. Conversely, let $r\colon W_L \longrightarrow \mathrm{GL}(V)$ be a representation of W_L. Then we define $\mathrm{ind}_{L/K}(r)$ as the representation of W_K whose underlying vector space consists of the continuous maps $f\colon W_K \longrightarrow V$ such that $f(xw) = r(x)f(w)$ for all $x \in W_L$ and $w \in W_K$.

Note that in the context of the cohomology of abstract groups this functor "induction" as defined above is often called "coinduction".

(3.1.6) Definition: A *Weil-Deligne representation of W_K* is a pair (r, N) where r is a representation of W_K and where N is a $\mathbb{C}$-linear endomorphism of V such that

(3.1.6.1) $$r(\gamma)\, N = |\mathrm{Art}_K^{-1}(\gamma)|_K\, N\, r(\gamma)$$

for $\gamma \in W_K$.

It is called *Frobenius semisimple* if r is semisimple.

(3.1.7) Remark: Let (r, N) be a Weil-Deligne representation of W_K.

(1) Let $\gamma \in W_K$ be an element corresponding to a uniformizer π_K via Art_K. Applying (3.1.6.1) we see that N is conjugate to qN, hence every eigenvalue of N must be zero which shows that N is automatically nilpotent.

(2) The kernel of N is stable under W_K, hence if (r, N) is irreducible, N is equal to zero. Therefore the irreducible Weil-Deligne representations of W_K are simply the irreducible continuous representations of W_K.

(3.1.8) Let $\rho_1 = (r_1, N_1)$ and $\rho_2 = (r_2, N_2)$ be two Weil-Deligne representations on complex vector spaces V_1 and V_2 respectively.

Their tensor product $\rho_1 \otimes \rho_2 = (r, N)$ is the Weil-Deligne representation on the space $V_1 \otimes V_2$ given by

$$r(w)(v_1 \otimes v_2) = r_1(w)v_1 \otimes r_2(w)v_2, \qquad N(v_1 \otimes v_2) = N_1 v_1 \otimes v_2 + v_1 \otimes N_2 v_2$$

for $w \in W_K$ and $v_i \in V_i$, $i = 1, 2$.

Further, $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ becomes the vector space of a Weil-Deligne representation $\underline{\mathrm{Hom}}(\rho_1, \rho_2) = (r, N)$ by

$$(r(w)\varphi)(v_1) = r_2(w)(\varphi(r_1(w)^{-1}v_1)), \qquad (N\varphi)(v_1) = N_2(\varphi(v_1)) - \varphi(N_1(v_1))$$

for $\varphi \in \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$, $w \in W_K$ and $v_1 \in V_1$.

In particular we get the contragredient $\rho^{\vee}$ of a Weil-Deligne representation as the representation $\underline{\mathrm{Hom}}(\rho, \mathbf{1})$ where $\mathbf{1}$ is the trivial one-dimensional representation.

(3.1.9) Consider W_K as a group scheme over $\mathbb{Q}$ (not of finite type) which is the limit of the constant group schemes associated to the discrete groups W_K/J where J runs through the open normal subgroups of I_K. Denote by W'_K the semi-direct product

$$W'_K = W_K \ltimes \mathbb{G}_a$$

where W_K acts on $\mathbb{G}_a$ by the rule $wxw^{-1} = |w|_K x$. This is a group scheme (neither affine nor of finite type) over $\mathbb{Q}$ whose R-valued points for some $\mathbb{Q}$-algebra R without non-trivial idempotents are given by $W_K \ltimes R$, and the law of composition is given by

$$(w_1, x_1)(w_2, x_2) = (w_1 w_2, |w_2|_K^{-1} x_1 + x_2).$$

A Weil-Deligne representation of W_K is the same as a complex finite-dimensional representation of the group scheme W'_K whose underlying W_K-representation is semisimple (to see this use the fact that representations of the additive group on a finite-dimensional vector space over a field in characteristic zero correspond to nilpotent endomorphisms).

The group scheme W'_K (or also its $\mathbb{C}$-valued points $W_K \ltimes \mathbb{C}$) is called the *Weil-Deligne group*.

(3.1.10) It follows from the Jacobson-Morozov theorem that we can also interpret a Weil-Deligne representation as a continuous complex semisimple representation of the group $W_K \times \mathrm{SL}_2(\mathbb{C})$. If η is such a representation, we associate a Weil-Deligne representation (r, N) by the formulas

$$r(w) = \eta(w, \begin{pmatrix} |w|^{\frac{1}{2}} & 0 \\ 0 & |w|^{-\frac{1}{2}} \end{pmatrix})$$

and

$$\exp(N) = \eta(1, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}).$$

A theorem of Kostant assures that two representations of $W_F \times \mathrm{SL}_2(\mathbb{C})$ are isomorphic if and only if the corresponding Weil-Deligne representations are isomorphic (see [Ko] for these facts).

3.2 Definition of L- and epsilon-factors

(3.2.1) Let $\rho = ((r, V), N)$ be a Frobenius semisimple Weil-Deligne representation. Denote by V_N the kernel of N and by $V_N^{I_K}$ the space of invariants in V_N for the action of the inertia group I_K.

The L-factor of ρ is given by

$$L(\rho, s) = \det(1 - q^{-s}\Phi|_{V_N^{I_K}})^{-1}$$

where $\Phi \in W_K$ is a geometric Frobenius. If ρ and ρ' are irreducible Weil-Deligne representations of dimension n, n' respectively, we have

(3.2.1.1) $$L(\rho \otimes \rho', s) = \prod_\chi L(\chi, s)$$

where χ runs through the unramified quasi-characters of $K^\times \cong W_K^{\mathrm{ab}}$ such that $\chi \otimes \rho^\vee = \rho'$ (compare (2.5.1.1)). In particular $L(\rho \otimes \rho', s) = 1$ for $n \neq n'$.

Fix a non-trivial additive character ψ of K and let $n(\psi)$ be the largest integer n such that $\psi(\pi_K^{-n} O_K) = 1$. Further let dx be an additive Haar measure of K.

To define $\varepsilon(\rho, \psi, s)$ we first define the ε-factor of the Weil group representation (r, V). Assume first that V is one-dimensional, i.e. $r = \chi$ is a quasi-character

$$\chi: W_K^{\mathrm{ab}} \longrightarrow \mathbf{C}^\times.$$

Let χ be unramified (i.e. $\chi(I_K) = (1)$ or equivalently $\chi = |\ |^s$ for some complex number s). Then we set

$$\varepsilon(\chi, \psi, dx) = \chi(w) q^{n(\psi)} \operatorname{vol}_{dx}(O_K) = q^{n(\psi)(1-s)} \operatorname{vol}_{dx}(O_K)$$

where $w \in W_K$ is an element whose valuation is $n(\psi)$.

If χ is ramified, let $f(\chi)$ be the conductor of χ, i.e. the smallest integer f such that $\chi(\operatorname{Art}_K(1 + \pi_K^f O_K)) = 1$, and let $c \in K^\times$ be an element with valuation $n(\psi) + f(\chi)$. Then we set

$$\varepsilon(\chi, \psi, dx) = \int_{c^{-1}O_K{}^\times} \chi^{-1}(\operatorname{Art}_K(x))\psi(x)dx.$$

The ε-factors attached to (r, V) with $\dim(V) > 1$ are characterized by the following theorem of Langlands and Deligne [De2]:

Theorem: *There is a unique function ε which associates with each choice of a local field K, a non-trivial additive character ψ of K, an additive Haar measure dx on K and a representation r of W_K a number $\varepsilon(r, \psi, dx) \in \mathbf{C}^\times$ such that*

(1) *If $r = \chi$ is one-dimensional $\varepsilon(\chi, \psi, dx)$ is defined as above.*

(2) *$\varepsilon(\ , \psi, dx)$ is multiplicative in exact sequences of representations of W_K (hence we get an induced homomorphism*

$$\varepsilon(\ , \psi, dx): \operatorname{Groth}(\operatorname{Rep}(W_K)) \longrightarrow \mathbf{C}^\times).$$

(3) *For every tower of finite extensions $L'/L/K$ and for every choice of additive Haar measures μ_L on L and $\mu_{L'}$ on L' we have*

$$\varepsilon(\operatorname{ind}_{L'/L}[r'], \psi \circ \operatorname{Tr}_{L/K}, \mu_L) = \varepsilon([r'], \psi \circ \operatorname{Tr}_{L'/K}, \mu_{L'})$$

for $[r'] \in \operatorname{Groth}(\operatorname{Rep}(W_{L'}))$ with $\dim([r']) = 0$.

Note that we have $\varepsilon(\chi, \psi, \alpha dx) = \alpha\varepsilon(\chi, \psi, dx)$ for $\alpha > 0$ and hence via inductivity $\varepsilon(r, \psi, \alpha dx) = \alpha^{\dim(r)}\varepsilon(\chi, \psi, dx)$. In particular if $[r] \in$

$\mathrm{Groth}(\mathrm{Rep}(W_K))$ is of dimension 0, $\varepsilon([r], \psi, dx)$ is independent of the choice of dx.

Now we can define the ε-factor of the Weil-Deligne representation $\rho = (r, N)$ as

$$\varepsilon(\rho, \psi, s) = \varepsilon(|\ |^s r, \psi, dx) \det(-\Phi|_{V^{I_K}/V_N^{I_K}})$$

where dx is the Haar measure on K which is self-dual with respect to the Fourier transform $f \mapsto \hat{f}$ defined by ψ:

$$\hat{f}(y) = \int f(x)\psi(xy)\, dx.$$

In other words ([Ta1] 2.2.2) it is the Haar measure for which O_K gets the volume $q^{-d/2}$ where d is the valuation of the absolute different of K (if the ramification index e of $K/\mathbb{Q}_p$ is not divided by p, we have $d = e - 1$, in general d can be calculated via higher ramification groups [Se2]).

Note that $\varepsilon(\rho, \psi, s)$ is not additive in exact sequences of Weil-Deligne representations as taking coinvariants is not an exact functor.

(3.2.2) Let ρ be an irreducible Weil-Deligne representation of dimension n, then we can define the *conductor $f(\rho)$ of ρ* by the equality

$$\varepsilon(\rho, \psi, s) = \varepsilon(\rho, \psi, 0) q^{-s(f(\rho) + nn(\psi))}$$

where $n(\psi)$ denotes the exponent of ψ (2.4.1). This is a nonnegative integer which can be explicitly expressed in terms of higher ramification groups (e.g. [Se2] VI,§2, Ex. 2).

(3.2.3) For any $m \geq 1$ we define the Weil-Deligne representations $\mathrm{Sp}(m) = ((r, V), N)$ by $V = \mathbb{C}e_0 \oplus \cdots \oplus \mathbb{C}e_{m-1}$ with

$$r(w)e_i = |w|^i e_i$$

and

$$Ne_i = e_{i+1} \quad (0 \leq i < m - 1), \qquad Ne_{m-1} = 0.$$

In this case we have $V_N = V_N^{I_K} = \mathbb{C}e_{m-1}$ and $\Phi e_i = q^{-i}e_i$ for a geometric Frobenius $\Phi \in W_K$. Hence the L-factor is given by

$$L(\rho, s) = \frac{1}{1 - q^{1-s-m}}.$$

Let ψ be an additive character such that $n(\psi) = 0$ and let dx be the Haar measure on K which is self-dual with respect to Fourier transform as above. Then we have $\varepsilon(r, \psi, dx) = q^{-md/2}$ where d is the valuation of the absolute different of K. Hence the ε-factor is given by

$$\varepsilon(\rho, \psi, s) = (-1)^{m-1} q^{\frac{-md - (m-2)(m-1)}{2}}.$$

(3.2.4) A Frobenius semisimple Weil-Deligne representation ρ is indecomposable if and only if it has the form $\rho_0 \otimes \mathrm{Sp}(m)$ for some $m \geq 1$ and with ρ_0 irreducible. Moreover, the isomorphism class of ρ_0 and m are uniquely determined by ρ ([De1] 3.1.3(ii)).

Further (as in every abelian category where all objects have finite length) every Frobenius semisimple Weil-Deligne representation is the direct sum of unique (up to order) indecomposable Frobenius semisimple Weil-Deligne representations.

4 Construction of the correspondence

4.1 THE CORRESPONDENCE IN THE UNRAMIFIED CASE

(4.1.1) Definition: An irreducible admissible representation (π, V) of $\mathrm{GL}_n(K)$ is called *unramified*, if the space of fixed vectors under $C = \mathrm{GL}_n(O_K)$ is non-zero, i.e. if its conductor (2.5.4) is zero.

(4.1.2) Example: A multiplicative quasi-character $\chi: K^\times \longrightarrow \mathbb{C}^\times$ is unramified if and only if $\chi(O_K^\times) = \{1\}$. An unramified quasi-character χ is uniquely determined by its value $\chi(\pi_K)$ which does not depend on the choice of the uniformizing element π_K. It is of the form $|\ |^s$ for a unique $s \in \mathbb{C}/(2\pi i (\log q)^{-1})\mathbb{Z}$.

(4.1.3) Let $(\chi_1, \ldots, \chi_n)$ be a family of unramified quasi-characters which we can view as intervals of length zero in $\mathcal{A}_1^0(K)$. We assume that for $i < j$, χ_i does not precede χ_j, i.e. $\chi_i^{-1}\chi_j \neq |\ |_K$. Then $Q(\chi_1, \ldots, \chi_n)$ is an unramified representation of $\mathrm{GL}_n(K)$. Conversely we have [Cas2]

Theorem: *Every unramified representation π of $\mathrm{GL}_n(K)$ is isomorphic to a representation of the form $Q(\chi_1, \ldots, \chi_n)$ where the χ_i are unramified quasi-characters of $K^\times$.*

(4.1.4) An unramified representation π of $\mathrm{GL}_n(K)$ is supercuspidal if and only if $n = 1$ and π is an unramified quasi-character of $K^\times$.

(4.1.5) Let π be an unramified representation associated to unramified quasi-characters $\chi_1, \ldots, \chi_n$. This tuple of unramified quasi-characters induces a homomorphism

$$T/T_c \longrightarrow \mathbb{C}^\times$$

where $T \cong (K^\times)^n$ denotes the diagonal torus of G and where $T_c \cong (O_K^\times)^n$ denotes the unique maximal compact subgroup of T of diagonal matrices with coefficients in $O_K^\times$.

Thus the set of unramified representations may be identified with the set of orbits under the Weyl group S_n of GL_n in

$$\hat{T} = \mathrm{Hom}(T/T_c, \mathbb{C}^\times) = (\mathbb{C}^\times)^n$$

where the last isomorphism is given by the identification

$$T/T_c = \mathbb{Z}^n, \quad \mathrm{diag}(t_1, \ldots, t_n) \mapsto (v_K(t_1), \ldots, v_K(t_n)).$$

(4.1.6) To shorten notations set $C = \mathrm{GL}_n(O_K)$. The Hecke algebra $\mathcal{H}(\mathrm{GL}_n(K)//C)$ is commutative and canonically isomorphic to the S_n-invariants of the group algebra ([Ca] 4.1)

$$\mathbb{C}[X^*(\hat{T})] = \mathbb{C}[X_*(T)] \cong \mathbb{C}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}].$$

If (π, V) is an unramified representation, V^C is one-dimensional (2.1.11), hence we get a canonical homomorphism

$$\lambda_\pi \colon \mathcal{H}(\mathrm{GL}_n(K)//C) \longrightarrow \mathrm{End}(V^C) = \mathbb{C}.$$

For every $h \in \mathcal{H}(\mathrm{GL}_n(K)//C)$ the map

$$\hat{T}/\Omega_{\mathrm{GL}_n} \longrightarrow \mathbb{C}, \qquad \pi \mapsto \lambda_\pi(h)$$

can be considered as an element in $\mathbb{C}[X^*(\hat{T})]^{S_n}$ and this defines the isomorphism

$$\mathcal{H}(\mathrm{GL}_n(K)//C) \overset{\sim}{\longrightarrow} \mathbb{C}[X^*(\hat{T})]^{S_n}.$$

(4.1.7) Definition: An n-dimensional Weil-Deligne representation $\rho = ((r, V), N)$ is called *unramified* if $N = 0$ and if $r(I_K) = \{1\}$.

(4.1.8) Every unramified n-dimensional Weil-Deligne representation $\rho = ((r, \mathbb{C}^n), N)$ is uniquely determined by the $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of $r(\Phi) =: g_\rho$ for a geometric Frobenius Φ. By definition this element is semisimple and hence we can consider this as an S_n-orbit of the diagonal torus $(\mathbb{C}^\times)^n$ of $\mathrm{GL}_n(\mathbb{C})$. Hence we get a bijection rec_n between unramified representations of $GL_n(K)$ and unramified n-dimensional Weil-Deligne representations.

This is normalized by the following two conditions:

(i)   An unramified quasi-character χ of $\mathbb{K}^\times$ corresponds to an unramified quasi-character $\mathrm{rec}_1(\chi)$ of W_K^{ab} via the map Art_K from local class field theory.

(ii) The representation $Q(\chi_1, \ldots, \chi_n)$ (4.1.3) corresponds to the unramified Weil-Deligne representation

$$\mathrm{rec}_1(\chi_1) \oplus \cdots \oplus \mathrm{rec}_1(\chi_n).$$

By the inductive definition of L- and ε-factors it follows that the bijection rec_n satisfies condition (2) of (1.2.2) (see also (2.5.1.1) and (3.2.1.1)). Further condition (3) for unramified characters and condition (4) are clearly okay, and condition (5) follows from the obvious fact that if unramified elements in $\mathcal{A}_n(K)$ or in $\mathcal{G}_n(K)$ correspond to the S_n-orbit of $\mathrm{diag}(t_1, \ldots, t_n)$ their contragredients correspond to the orbit of $\mathrm{diag}(t_1, \ldots, t_n)^{-1} = \mathrm{diag}(t_1^{-1}, \ldots, t_n^{-1})$.

(4.1.9) From the global point of view, unramified representations are the "normal" ones: If

$$\pi = \bigotimes_v \pi_v$$

is an irreducible admissible representation of the adele valued group $\mathrm{GL}_n(\mathbf{A}_L)$ for a number field L as in (2.4.7), all but finitely many π_v are unramified.

4.2   Some reductions

(4.2.1) In this paragraph we sketch some arguments (mostly due to Henniart) which show that it suffices to show the existence of a family of maps (rec_n) satisfying all the desired properties between the set of isomorphism classes of supercuspidal representations and the set of isomorphism classes of irreducible Weil-Deligne representations. We denote by $\mathcal{A}_n^0(K)$ the subset of $\mathcal{A}_n(K)$ consisting of the supercuspidal representations of $\mathrm{GL}_n(K)$. Further let $\mathcal{G}_n^0(K)$ be the set of irreducible Weil-Deligne representations in $\mathcal{G}_n(K)$.

(4.2.2) Reduction to the supercuspidal case: *In order to prove the local Langlands conjecture (1.2.2), it suffices to show that there exists a unique collection of bijections*

$$\mathrm{rec}_n \colon \mathcal{A}_n^0(K) \longrightarrow \mathcal{G}_n^0(K)$$

satisfying (1.2.2) (1) to (5).

Reasoning : This follows from (2.2.9) and from (3.2.4). More precisely, for any irreducible admissible representation $\pi \cong Q(\Delta_1, \ldots, \Delta_r)$, with $\Delta_i = [\pi_i, \pi_i(m_i - 1)]$ and $\pi_i \in \mathcal{A}_{n_i}^0(K)$ define

$$\mathrm{rec}_{n_1 m_1 + \cdots + n_r m_r}(\pi) = \bigoplus_{i=1}^{r} \mathrm{rec}_{n_i}(\pi_i) \otimes \mathrm{Sp}(m_i).$$

Properties (1.2.2) (1) to (5) follow then (nontrivially) from the inductive description of the L- and the ε-factors.

(4.2.3) Reduction to an existence statement: If there exists a collection of bijections $(\mathrm{rec}_n)_n$ as in (4.2.2), it is unique. This follows from the fact that representations $\pi \in \mathcal{A}_n^0(K)$ are already determined inductively by their ε-factors in pairs and that by (1.2.2)(1) rec_1 is given by class field theory. More precisely, we have the following theorem of Henniart [He3]:

Theorem: *Let $n \geq 2$ be an integer and let π and π' be representations in $\mathcal{A}_n^0(K)$. Assume that we have an equality*

$$\varepsilon(\pi \times \tau, \psi, s) = \varepsilon(\pi' \times \tau, \psi, s)$$

for all integers $r = 1, \ldots, n-1$ and for every $\tau \in \mathcal{A}_r^0(K)$. Then $\pi \cong \pi'$.

(4.2.4) Injectivity: Every collection of maps rec_n as in (4.2.2) is automatically injective: If χ is a quasi-character of $K^\times$, its L-function $L(\chi, s)$ is given by

$$L(\chi, s) = \begin{cases} (1 - \chi(\pi)q^s)^{-1}, & \text{if } \chi \text{ is unramified,} \\ 1, & \text{if } \chi \text{ is ramified.} \end{cases}$$

In particular, it has a pole in $s = 0$ if and only if $\chi = 1$. Hence by (2.5.1.1) and (3.2.1.1) we have for $\pi, \pi' \in \mathcal{A}_n^0(K)$:

$$
\begin{aligned}
\mathrm{rec}_n(\pi) = \mathrm{rec}_n(\pi') &\Leftrightarrow L(\mathrm{rec}_n(\pi)^\vee \otimes \mathrm{rec}_n(\pi'), s) \text{ has a pole in } s = 0 \\
&\Leftrightarrow L(\pi^\vee \times \pi', s) \text{ has a pole in } s = 0 \\
&\Leftrightarrow \pi = \pi'.
\end{aligned}
$$

(4.2.5) Surjectivity: *In order to prove the local Langlands conjecture it suffices to show that there exists a collection of maps*

$$
\mathrm{rec}_n \colon \mathcal{A}_n^0(K) \longrightarrow \mathcal{G}_n^0(K)
$$

satisfying (1.2.2) (1) to (5).

Reasoning : Because of the preservation of ε-factors in pairs it follows from (3.2.2) and (2.5.4) that rec_n preserves conductors. But by the numerical local Langlands theorem of Henniart [He2] the sets of elements in $\mathcal{A}_n^0(K)$ and $\mathcal{G}_n^0(K)$ which have the same given conductor and the same central character are finite and have the same number of elements. Hence the bijectivity of rec_n follows from its injectivity (4.2.4).

4.3   A RUDIMENTARY DICTIONARY OF THE CORRESPONDENCE

(4.3.1)   In this section we give some examples how certain properties of admissible representations can be detected on the corresponding Weil-Deligne representation and vice versa. Throughout (π, V_π) denotes an admissible irreducible representation of $\mathrm{GL}_n(K)$, and $\rho = ((r, V_r), N)$ the n-dimensional Frobenius-semisimple Weil-Deligne representation associated to it via the local Langlands correspondence (1.2.2).

(4.3.2)   First of all, we have of course:

Proposition: *The admissible representation π is supercuspidal if and only if ρ is irreducible (or equivalently iff r is irreducible and $N = 0$).*

(4.3.3) Write $\pi = Q(\Delta_1, \ldots, \Delta_s)$ in the Bernstein-Zelevinsky classification (2.2.9), where $\Delta_i = [\pi_i, \ldots, \pi_i(m_i - 1)]$ is an interval of supercuspidal representations of $\mathrm{GL}_{n_i}(K)$.

By (4.2.2) we have

$$\rho = \bigoplus_{i=1}^{s} (\mathrm{rec}_{n_i}(\pi_i) \otimes \mathrm{Sp}(m_i)).$$

Set $\rho_i = ((r_i, V_{r_i}), 0) = \mathrm{rec}_{n_i}(\pi_i)$. The underlying representation of the Weil group of $\pi_i \otimes \mathrm{Sp}(m_i)$ is then given by

$$r_i \oplus r_i(1) \oplus \cdots \oplus r_i(m_i - 1)$$

where $r(x)$ denotes the representation $w \mapsto r(w)|w|^x$ for any representation r of W_K and any real number x. We have $(r_i(j), 0) = \mathrm{rec}_{n_i}(\pi_i(j))$.

Further, if N_i is the nilpotent endomorphism of $\rho_i \otimes \mathrm{Sp}(m_i)$, its conjugacy class (which we can consider as a non-ordered partition of $n_i m_i$ by the Jordan normal form) is given by the partition

$$n_i m_i = \underbrace{m_i + \cdots + m_i}_{n_i\text{-times}}.$$

Hence we get:

Proposition: *The underlying W_K-representation r of ρ depends only on the supercuspidal support $\tau_1, \ldots, \tau_t$ of π (2.2.10). More precisely, we have an isomorphism of Weil-Deligne representations*

$$(r, 0) \cong \mathrm{rec}(\tau_1) \oplus \ldots \oplus \mathrm{rec}(\tau_t).$$

The conjugacy class of N is given by the degree n_i of π_i and the length m_i of the intervals Δ_i as above. In particular, we have $N = 0$ if and only if all intervals Δ_i are of length 1.

(4.3.4) Example: The Steinberg representation $\mathrm{St}(n)$ (2.2.13) corresponds to the Weil-Deligne representation $|\ |^{(1-n)/2} \mathrm{Sp}(n)$.

(4.3.5) Recall from (4.1.3) that π is unramified if and only if all intervals Δ_i are of length 1 and consist of an unramified quasi-character of $K^\times$. Hence

(4.3.3) shows that π is unramified if and only if ρ is unramified. We used this already in (4.1).

(4.3.6) The "arithmetic information" of W_K is encoded in the inertia subgroup I_K. The quotient W_K/I_K is the free group generated by Φ_K and hence "knows" only the number q of elements in the residue field of K. Therefore Weil-Deligne representations $\rho = (r, N)$ with $r(I_K) = 1$ should be particularly simple. We call such representations I_K-*spherical.* Then r is a semisimple representation of $< \Phi_K > \cong \mathbb{Z}$. Obviously, every finite-dimensional semisimple representation of $\mathbb{Z}$ is the direct sum of one-dimensional representations. Hence r is the direct sum of quasi-characters of W_K which are necessarily unramified.

On the $\mathrm{GL}_n(K)$-side let $\mathcal{I} \subset \mathrm{GL}_n(K)$ be an Iwahori subgroup, i.e. $\mathcal{I}$ is an open compact subgroup of $\mathrm{GL}_n(K)$ which is conjugated to the group of matrices $(a_{ij}) \in \mathrm{GL}_n(O_K)$ with $a_{ij} \in \pi_K O_K$ for $i > j$. We say that π is $\mathcal{I}$-*spherical* if the space of $\mathcal{I}$-fixed vectors is non-zero. By a theorem of Casselman ([Ca] 3.8, valid for arbitrary reductive groups - with the appropriate reformulation) an irreducible admissible representation is $\mathcal{I}$-spherical if and only if its supercuspidal support consists of unramified quasi-characters. Altogether we get:

Proposition: *We have equivalent assertions:*
(1) *The irreducible admissible representation π is $\mathcal{I}$-spherical.*
(2) *The supercuspidal support of π consists of unramified quasi-characters.*
(3) *The corresponding Weil-Deligne representation ρ is I_K-spherical.*

By (2.1.9) the irreducible admissible $\mathcal{I}$-spherical representations are nothing but the finite-dimensional irreducible $\mathcal{H}(\mathrm{GL}_n(K)//\mathcal{I})$-modules. The structure of the $\mathbb{C}$-algebra $\mathcal{H}(\mathrm{GL}_n(K)//\mathcal{I})$ is known in terms of generators and relations ([IM]) and depends only on the isomorphism class of GL_n over some algebraically closed field (i.e. the based root datum of GL_n) and on the number q.

(4.3.7) Finally, we translate several notions which have been defined for

admissible representations of $\mathrm{GL}_n(K)$ into properties of Weil-Deligne representations:

Proposition: *Let π be an irreducible admissible representation of $\mathrm{GL}_n(K)$ and let $\rho = (r, N)$ be the corresponding Weil-Deligne representation.*

(1) *We have equivalent statements*

    (i)   *π is essentially square-integrable.*

    (ii)   *ρ is indecomposable.*

    (iii) *The image of the Weil-Deligne group $W_F'(\mathbb{C})$ under ρ is not contained in any proper Levi subgroup of $\mathrm{GL}_n(\mathbb{C})$.*

(2) *We have equivalent statements*

    (i)   *π is tempered.*

    (ii)   *Let η be a representation of $W_K \times \mathrm{SL}_2(\mathbb{C})$ associated to ρ (unique up to isomorphism) (3.1.10). Then $\eta(W_F)$ is bounded.*

    (iii) *Let η be as in (ii) and let $\Phi \in W_K$ a geometric Frobenius. Then $\eta(\Phi)$ has only eigenvalues of absolute value 1.*

(3) *The representation π is generic if and only if $L(s, \mathrm{Ad} \circ \rho)$ has no pole at $s = 1$ (here $\mathrm{Ad} \colon \mathrm{GL}_n(\mathbb{C}) \longrightarrow \mathrm{GL}(M_n(\mathbb{C}))$ denotes the adjoint representation).*

Proof : (1): The equivalence of (i) and (ii) follows from (2.3.6) and (3.2.4), the equivalence of (ii) and (iii) is clear as any factorization through a Levi subgroup $\mathrm{GL}_{n_1}(\mathbb{C}) \times \mathrm{GL}_{n_2}(\mathbb{C}) \subset \mathrm{GL}_n(\mathbb{C})$ would induce a decomposition of ρ.

(3): This is [Kud] 5.2.2.

(2): The equivalence of (ii) and (iii) follows from the facts that the image of the inertia group I_K under η is finite, as I_K is compact and totally disconnected, and that a subgroup H of semisimple elements in $\mathrm{GL}_n(\mathbb{C})$ is bounded if and only if every element of H has only eigenvalues of absolute value 1 (use the spectral norm).

Now ρ is indecomposable if and only if η is indecomposable. To prove the equivalence of (i) and (iii) we can therefore assume by (2.3.7) and (4.3.3) that ρ is indecomposable, i.e. that $\pi = Q(\Delta)$ is essentially square integrable. Let $x \in \mathbb{R}$ be the unique real number such that $Q(\Delta)(x)$ is square integrable (2.3.6). Then the description of $\mathrm{rec}(Q(\Delta))$ in (4.3.3) shows that

$$|\det(\eta(w))| = |\det(r(w))| = |w|^{nx}.$$

Now π is square integrable if and only if its central character is unitary. But by property (4) of the local Langlands classification the central character of π is given by $\det \circ r$. Hence (iii) is equivalent to (i) by the following lemma whose proof we leave as an exercise:

Lemma: *Let π be a supercuspidal representation of $\mathrm{GL}_n(K)$ and denote by ω_π its central character. For an integer $m \geq 1$ let δ be the interval $[\pi((1-m)/2), \pi((m-1)/2)]$. Let η be a representation of $W_K \times \mathrm{SL}_2(\mathbb{C})$ associated to $(r, N) = \mathrm{rec}(Q(\Delta))$ (3.1.10). Then for $w \in W_K$ all absolute values of eigenvalues of $\eta(w)$ are equal to $|\omega_\pi(\mathrm{Art}_K^{-1}(w))|^{1/n}$. In particular all eigenvalues of $\eta(w)$ have the same absolute value.*

Hint : First show the result for $m = 1$ where there is no difference between η and r. Then the general result can be checked by making explicit the Jacobson-Morozov theorem in the case of $\mathrm{GL}_{nm}(K)$.

4.4 THE CONSTRUCTION OF THE CORRESPONDENCE AFTER HARRIS AND TAYLOR

(4.4.1) Fix a prime $\ell \neq p$ and an isomorphism of an algebraic closure $\bar{\mathbb{Q}}_\ell$ of $\mathbb{Q}_\ell$ with $\mathbb{C}$. Denote by κ the residue field of O_K and by $\bar{\kappa}$ an algebraic closure of κ. For $m \geq 0$ and $n \geq 1$ let $\Sigma_{K,n,m}$ be the unique (up to isomorphism) one-dimensional special formal O_K-module of O_K-height n with Drinfeld level $\mathfrak{p}_K^m$-structure over $\bar{k}$. Its deformation functor on local Artinian O_K-algebras with residue field $\bar{\kappa}$ is prorepresented by a complete noetherian local O_K-algebra $R_{K,n,m}$ with residue field $\bar{\kappa}$. Drinfeld showed that $R_{K,n,m}$ is regular and that the canonical maps $R_{K,n,m} \longrightarrow R_{K,n,m+1}$ are finite and flat. The inductive limit (over m) of the formal vanishing cycle sheaves of $\mathrm{Spf}(R_{K,n,m})$ with coefficients in $\bar{\mathbb{Q}}_\ell$ gives a collection $(\Psi_{K,\ell,h}^i)$ of infinite-dimensional $\bar{\mathbb{Q}}_\ell$-vector spaces with an admissible action of the subgroup of $GL_h(K) \times D_{K,1/n}^\times \times W_K$ consisting of elements (γ, δ, σ) such that

$$|\mathrm{Nrd}\delta||\det\gamma|^{-1}|\mathrm{Art}_K^{-1}\sigma| = 1.$$

For any irreducible representation ρ of $D_{K,1/n}^\times$ set

$$\Psi_{K,\ell,n}^i(\rho) = \mathrm{Hom}_{D_{K,1/n}^\times}(\rho, \psi_{K,\ell,n}^i).$$

This is an admissible $(GL_n(K) \times W_K)$-module. Denote by $[\Psi_{K,\ell,n}(\rho)]$ the virtual representation

$$(-1)^{n-1} \sum_{i=0}^{n-1} (-1)^i [\Psi^i_{K,\ell,n}(\rho)].$$

Then the first step is to prove:

Construction theorem: *Let π be an irreducible supercuspidal representation of $\mathrm{GL}_n(K)$. Then there is a (true) representation*

$$r_\ell(\pi): W_K \longrightarrow GL_n(\bar{\mathbb{Q}}_\ell) = \mathrm{GL}_n(\mathbb{C})$$

such that in the Grothendieck group

$$[\Psi_{K,\ell,n}(\mathrm{JL}(\pi)^\vee)] = [\pi \otimes r_\ell(\pi)]$$

where JL denotes the Jacquet-Langlands bijection between irreducible representations of $D^\times_{K,1/n}$ and essentially square integrable irreducible admissible representations of $GL_n(K)$.

Using this theorem we can define $\mathrm{rec}_n = \mathrm{rec}_{K,n}: \mathcal{A}^0_n(K) \longrightarrow \mathcal{G}_n(K)$ by the formula

$$\mathrm{rec}_n(\pi) = r_\ell(\pi^\vee \otimes (|\ |_K \circ \det)^{(1-n)/2}).$$

That this map satisfies (1.2.2) (1) - (5) follows from compatibility of r_ℓ with many instances of the global Langlands correspondence. The proof of these compatibilities and also the proof of the construction theorem follow from an analysis of the bad reduction of certain Shimura varieties. I am not going into any details here and refer to [HT].

(4.4.2) In the rest of this treatise we explain the ingredients of the construction of the collection of maps (rec_n).

5 Explanation of the correspondence

5.1 JACQUET-LANGLANDS THEORY

(5.1.1) We collect some facts about skew fields with center K (see e.g. [PR] as a reference).

Let $\mathrm{Br}(K)$ be the Brauer group of K. As a set it can be identified with the set of isomorphism classes of finite-dimensional division algebras over K with center K. For $D, D' \in \mathrm{Br}(K)$, $D \otimes D'$ is again a central simple algebra over K, hence it is isomorphic to a matrix algebra $M_r(D'')$ for some $D'' \in \mathrm{Br}(K)$. If we set $D \cdot D' := D''$, this defines the structure of an abelian group on $\mathrm{Br}(K)$.

This group is isomorphic to $\mathbb{Q}/\mathbb{Z}$ where the homomorphism $\mathbb{Q}/\mathbb{Z} \longrightarrow \mathrm{Br}(K)$ is given as follows: For a rational number λ with $0 \le \lambda < 1$ we write $\lambda = s/r$ for integers r, s which are prime to each other and with $r > 0$ (and we make the convention $0 = 0/1$). Then the associated skew field D_λ is given by $D_\lambda = K_r[\Pi]$ where K_r is the (unique up to isomorphism) unramified extension of K of degree r and where Π is an indeterminate satisfying the relations $\Pi^r = \pi_k^s$ and $\Pi a = \sigma_K(a)\Pi$ for $a \in K_r$.

We call r the index of D_λ. It is the order of D_λ as an element in the Brauer group and we have

$$\dim_K(D_\lambda) = r^2.$$

If B is any simple finite-dimensional K-algebra with center K, it is isomorphic to $M_r(D)$ for some skew field D with center K. Further, $B \otimes_K L$ is a simple L-algebra with center L for any extension L of K. In particular $B \otimes_K \bar{K}$ is isomorphic to an algebra of matrices over $\bar{K}$ as there do not exist any finite-dimensional division algebras over algebraically closed fields $\bar{K}$ except $\bar{K}$ itself.

Conversely, if D is a skew field with center K which is finite-dimensional over K we can associate the invariant $\mathrm{inv}(D) \in \mathbb{Q}/\mathbb{Z}$: As $D \otimes_K \bar{K}$ is isomorphic to some matrix algebra $M_r(\bar{K})$, we have $\dim_K(D) = r^2$. The valuation v_K on K extends uniquely to D by the formula

$$v_D(\delta) = \frac{1}{r} v_K(\mathrm{Nrd}_{D/K}(\delta))$$

for $\delta \in D$. Moreover D is complete in the topology given by this valuation. It follows from the definition of v_D that the ramification index e of D over K is smaller than r. Set

$$O_D = \{\, \delta \in D \mid v_D(\delta) \geq 0 \,\}, \qquad \mathfrak{P}_D = \{\, \delta \in D \mid v_D(\delta) > 0 \,\}.$$

Clearly $\mathfrak{P}_D$ is a maximal right and left ideal of O_D and the quotient $\kappa_D = O_D/\mathfrak{P}_D$ is a skew field which is a finite extension of κ, hence it is finite and has to be commutative. Let $L \subset D$ be the unramified extension corresponding to the extension κ_D of κ. As no skew field with center K of dimension r^2 can contain a field of K-degree bigger than r we have for the inertia index f of D over K

$$f = [\kappa_D : \kappa] = [L : K] \leq r.$$

Hence the formula $r^2 = ef$ shows that $e = f = r$. Further we have seen that D contains a maximal unramified subfield. The extension L/K is Galois with cyclic Galois-group generated by the Frobenius automorphism σ_K. By the Skolem-Noether theorem (e.g. [BouA] VIII, §10.1), there exists an element $\delta \in D^\times$ such that $\sigma_K(x) = \delta x \delta^{-1}$ for all $x \in L$. Then

$$\mathrm{inv}(D) = v_D(\delta) \in \frac{1}{r}\mathbb{Z}/\mathbb{Z} \subset \mathbb{Q}/\mathbb{Z}$$

is the invariant of D.

(5.1.2) For every $D \in \mathrm{Br}(K)$ we can consider its units as an algebraic group over K. More precisely, we define for every K-algebra R

$$D^\times(R) = (D \otimes_K R)^\times.$$

This is an inner form of $GL_{n,K}$ if n is the index of D.

(5.1.3) Let $D \in \mathrm{Br}(K)$ be a division algebra with center K of index n. Let $\{d\}$ be a $D^\times$-conjugacy class of elements in $D^\times$. The image of $\{d\}$ in $D \otimes_K \bar{K} \cong M_n(\bar{K})$ is a $GL_n(\bar{K})$-conjugacy class $\{d\}'$ of elements in $GL_n(\bar{K})$ which does not depend on the choice of the isomorphism $D \otimes_K \bar{K} \cong M_n(\bar{K})$ as any automorphism of $M_n(\bar{K})$ is an inner automorphism. Further, $\{d\}'$ is fixed by the natural action of $\mathrm{Gal}(\bar{K}/K)$ on conjugacy classes of

$GL_n(\bar{K})$. Hence its similarity invariants in the sense of [BouA] chap. 7, §5 are polynomials in $K[X]$ and it follows that there is a unique $GL_n(K)$-conjugacy class of elements $\alpha(\{d\})$ in $GL_n(K)$ whose image in $GL_n(\bar{K})$ is $\{d\}'$. Altogether we get a canonical injective map α from the set of $D^\times$-conjugacy classes $\{D^\times\}$ in $D^\times$ into the set of $GL_n(K)$-conjugacy classes $\{GL_n(K)\}$ in $GL_n(K)$.

The image of α consists of the set of conjugacy classes of elliptic elements in $GL_n(K)$. Recall that an element $g \in GL_n(K)$ is called *elliptic* if it is contained in a maximal torus $T(K)$ of $GL_n(K)$ such that $T(K)/K^\times$ is compact. Equivalently, g is elliptic if and only if $K[g]$ is a field.

We call a conjugacy class $\{g\}$ in $GL_n(K)$ *semisimple* if it consists of elements which are diagonalizable over $\bar{K}$ or, equivalently, if $K[g]$ is a product of field extensions for $g \in \{g\}$. A conjugacy class $\{g\}$ is called *regular semisimple* if it is semisimple and if all eigenvalues of elements in $\{g\}$ in $\bar{K}$ are pairwise different. Note that every elliptic element is semisimple. We make the same definitions for conjugacy classes in $D^\times$, or equivalently we call a conjugacy class of $D^\times$ *semisimple* (resp. *regular semisimple*) if its image under $\alpha\colon \{D^\times\} \longrightarrow \{G\}$ is semisimple (resp. regular semisimple).

(5.1.4)   Denote by $\mathcal{A}^2(G)$ the set of isomorphism classes of irreducible admissible essentially square integrable representations of G. We now have the following theorem which is due to Jacquet and Langlands in the case $n = 2$ and due to Rogawski and Deligne, Kazhdan and Vigneras in general ([Rog] and [DKV]):

Theorem: *Let D be a skew field with center K and with index n. There exists a bijection, called Jacquet-Langlands correspondence,*

$$\mathrm{JL}\colon \mathcal{A}^2(D^\times) \leftrightarrow \mathcal{A}^2(\mathrm{GL}_n(K))$$

which is characterized on characters by

$$(5.1.4.1) \qquad\qquad \chi_\pi = (-1)^{n-1}\chi_{\mathrm{JL}(\pi)}.$$

Further JL satisfies the following conditions:
  (1) We have equality of central characters

$$\omega_\pi = \omega_{\mathrm{JL}(\pi)}.$$

(2) *We have an equality of L-functions and of ε-functions up to a sign*

$$L(\pi, s) = L(\mathrm{JL}(\pi), s), \qquad \varepsilon(\pi, \psi, s) = \varepsilon(\mathrm{JL}(\pi), \psi), s)$$

(for the definition of L- and ε-function of irreducible admissible representations of $D^\times$ see e.g. [GJ]).

(3) *The Jacquet-Langlands correspondence is compatible with twist by characters: If χ is a multiplicative quasi-character of K, we have*

$$\mathrm{JL}(\pi(\chi \circ \mathrm{Nrd})) = \mathrm{JL}(\pi)(\chi \circ \det).$$

(4) *It is compatible with contragredient:*

$$\mathrm{JL}(\pi^\vee) = \mathrm{JL}(\pi)^\vee.$$

(5.1.5) Remark: Note that for $G = D^\times$ every admissible representation is essentially square integrable as $D^\times / K^\times$ is compact.

5.2 SPECIAL p-DIVISIBLE O-MODULES

(5.2.1) Let R be a ring. A *p-divisible group* over R is an inductive system $G = (G_n, i_n)_{n \geq 1}$ of finite locally free commutative group schemes G_n over $\mathrm{Spec}(R)$ and group scheme homomorphisms $i_n \colon G_n \to G_{n+1}$ such that for all integers n there is an exact sequence

$$0 \longrightarrow G_1 \xrightarrow{i_{n-1} \circ \cdots \circ i_1} G_n \xrightarrow{p} G_{n-1} \longrightarrow 0$$

of group schemes over $\mathrm{Spec}(R)$. We have the obvious notion of a homomorphism of p-divisible groups. This way we get a $\mathbb{Z}_p$-linear category.

As the G_n are finite locally free, their underlying schemes are by definition of the form $\mathrm{Spec}(A_n)$ where A_n is an R-algebra which is a finitely generated locally free R-module. In particular, it makes sense to speak of the rank of A_n which we also call the rank of G_n. From the exact sequence above it follows that G_n is of rank p^{nh} for some non-negative locally constant function $h \colon \mathrm{Spec}(R) \longrightarrow \mathbb{Z}$ which is called the *height of G*.

(5.2.2) Let $G = (G_n)$ be a p-divisible group over some ring R and let R' be an R-algebra. Then the inductive system of $G_n \otimes_R R'$ defines a p-divisible group over R' which we denote by $G_{R'}$.

(5.2.3) Let $G = (G_n)$ be a p-divisible group over a ring R. If there exists some integer $N \geq 1$ such that $p^N R = 0$, the Lie algebra $\mathrm{Lie}(G_n)$ is a locally free R-module for $n \geq N$ whose rank is independent of $n \geq N$. We call this rank the *dimension of G*. More generally, if R is p-adically complete we define the dimension of G as the dimension of the p-divisible group $G_{R/pR}$ over R/pR.

(5.2.4) Let R be an O_K-algebra. A *special p-divisible O_K-module over R* is a pair (G, ι) where G is a p-divisible group over R and where $\iota : O_K \longrightarrow \mathrm{End}(G)$ is a homomorphism of $\mathbb{Z}_p$-algebras such that for all $n \geq 1$ the O_K-action induced by ι on $\mathrm{Lie}(G_n)$ is the same as the O_K-action which is induced from the R-module structure of $\mathrm{Lie}(G)$ via the O_K-module structure of R. In other words the induced homomorphism $O_K \otimes_{\mathbb{Z}_p} O_K \longrightarrow \mathrm{End}(\mathrm{Lie}(G_n))$ factorizes through the multiplication $O_K \otimes_{\mathbb{Z}_p} O_K \longrightarrow O_K$.

The height $\mathrm{ht}(G)$ of a special p-divisible O_K-module (G, ι) is always divisible by $[K : \mathbb{Q}_p]$ and we call $\mathrm{ht}_{O_K}(G) := [K : \mathbb{Q}_p]^{-1}\mathrm{ht}(G)$ the O_K-*height of (G, ι)*.

(5.2.5) If $(G = (G_n), \iota)$ is a special p-divisible O_K-module over an O_K-algebra R and if $R \longrightarrow R'$ is an R-algebra, we get an induced O_K-action ι' on $G_{R'}$ and the pair $(G_{R'}, \iota')$ is a special p-divisible O_K-module over R' which we denote by $(G, \iota)_{R'}$.

(5.2.6) Let k be a perfect extension of the residue class field κ of O_K. Denote by $W(k)$ the ring of Witt vectors of k. Recall that this is the unique (up to unique isomorphism inducing the identity on k) complete discrete valuation ring with residue class field k whose maximal ideal is generated by p. Further $W(k)$ has the property that for any complete local noetherian ring R with residue field k there is a unique local homomorphism $W(k) \longrightarrow R$ inducing the identity on k.

In particular, we can consider $W(\kappa)$. It can be identified with the ring of integers of the maximal unramified extension of $\mathbb{Q}_p$ in K (use the universal property of the ring of Witt vectors). Set

$$W_K(k) = W(k) \otimes_{W(\kappa)} O_K.$$

This is a complete discrete valuation ring of mixed characteristic with residue field k which is a formally unramified O_K-algebra (i.e. the image of $\mathfrak{p}_K$ generates the maximal ideal of $W_K(k)$). There exists a unique continuous automorphism σ_K of $W(k)$ which induces the automorphism $x \mapsto x^q$ on k. We denote the induced automorphism $\sigma_K \otimes \mathrm{id}_{O_K}$ again by σ_K.

(5.2.7) Proposition: *The category of special p-divisible O_K-modules (G, ι) over k and the category of triples (M, F, V) where M is a free $W_K(k)$-module of rank equal to the O_K-height and F (resp. V) is a σ- (resp. σ^{-1}-) linear map such that $FV = VF = \pi_K \, \mathrm{id}_M$ are equivalent. Via this equivalence there is a canonical functorial isomorphism*

$$M/VM \cong \mathrm{Lie}(G).$$

We call $(M, F, V) = M(G, \iota)$ the Dieudonné module of (G, ι).

Proof : To prove this we use covariant Dieudonné theory for p-divisible groups as in [Zi1] for example. Denote by σ the usual Frobenius of the ring of Witt vectors. Covariant Dieudonné theory tells us that there is an equivalence of the category of p-divisible groups over k with the category of triples (M', F', V') where M' is a free $W(k)$-module of rank equal to the height of G and with a σ-linear (resp. a σ^{-1}-linear) endomorphism F' (resp. V') such that $F'V' = V'F' = p \, \mathrm{id}_{M'}$ and such that $M'/V'M' = \mathrm{Lie}(G)$. Let us call this functor M'. Let (G, ι) be a special p-divisible O_K-module. Then the Dieudonné module $M'(G)$ is a $W(k) \otimes_{\mathbb{Z}_p} O_K$-module, the operators F' and V' commute with the O_K-action and the induced homomorphism $O_K \otimes_{\mathbb{Z}_p} k \longrightarrow \mathrm{End}(M'/V'M')$ factors through the multiplication $O_K \otimes_{\mathbb{Z}_p} k \longrightarrow k$. We have to construct from these data a triple (M, F, V) as in the claim of the proposition. To do this write

$$W(k) \otimes_{\mathbb{Z}_p} O_K = W(k) \otimes_{\mathbb{Z}_p} W(\kappa) \otimes_{W(\kappa)} O_K = \prod_{\mathrm{Gal}(\kappa/\mathbb{F}_p)} W_K(k).$$

By choosing the Frobenius $\sigma = \sigma_{\mathbf{Q}_p}$ as a generator of $\text{Gal}(\kappa/\mathbb{F}_p)$ we can identify this group with $\mathbb{Z}/r\mathbb{Z}$ where $p^r = q$. We get an induced decomposition $M' = \oplus_{i \in \mathbb{Z}/r\mathbb{Z}} M_i'$ where the M_i are $W_K(k)$-modules defined by

$$M_i = \{\, m \in M' \mid (a \otimes 1)m = (1 \otimes \sigma^{-i}(a))m \text{ for all } a \in W(\kappa) \subset O_K \,\}.$$

The operator F' (resp. V') is homogeneous of degree -1 (resp. $+1$) with respect to this decomposition. By the condition on the O_K-action on the Lie algebra we know that $M_0'/VM_{r-1}' = M'/V'M'$ and hence that $VM_{i-1}' = M_i'$ for all $i \neq 0$. We set $M = M_0'$ and $V = (V')^r|_{M_0'}$. It follows that we have $M/VM = M'/V'M'$. Further the action of π_K on $M/VM = M'/VM'$ equals the scalar multiplication with the image of π_K under the map $O_K \longrightarrow \kappa \longrightarrow k$ but this image is zero. It follows that VM contains $\pi_K M$ and hence we can define $F = V^{-1}\pi_K$. Thus we constructed the triple (M, F, V) and it is easy to see that this defines an equivalence of the category of triples (M', F', V') as above and the one of triples (M, F, V) as in the claim.

(5.2.8) Let (G, ι) be a special p-divisible O_K-module over a ring R. We call it *étale* if it is an inductive system of finite étale group schemes. This is equivalent to the fact that its Lie algebra is zero. If p is invertible in R, (G, ι) will be always étale.

Now assume that $R = k$ is a perfect field of characteristic p and let (M, F, V) be its Dieudonné module. Then (G, ι) is étale if and only if $M = VM$.

In general there is a unique decomposition $(M, F, V) = (M_{\text{ét}}, F, V) \oplus (M_{\text{inf}}, F, V)$ such that V is bijective on $M_{\text{ét}}$ and such that $V^N M_{\text{inf}} \subset \pi_K M_{\text{inf}}$ for large N (define $M_{\text{ét}}$ (resp. M_{inf}) as the projective limit over n of $\bigcap_m V^m(M/\pi_K^n M)$ (resp. of $\bigcup_m \text{Ker}(V^m|_{M/\pi_K^n M})$)). We call the $W_K(k)$-rank of $M_{\text{ét}}$ the *étale O_K-height of (M, F, V)* or *of (G, ι)*.

We call (G, ι) *formal* or also *infinitesimal* if its étale O_K-height is zero.

(5.2.9) Proposition: *Let k be an algebraically closed field of characteristic p. For all non-negative integers $h \leq n$ there exists up to isomorphism exactly one special p-divisible O_K-module of O_K-height n, étale O_K-height h and of dimension one. Its Dieudonné module (M, F, V) is the free $W_K(k)$-module*

with basis $(d_1, \ldots, d_h, e_1, \ldots, e_{n-h})$ such that V is given by

$$V d_i = d_i, \qquad\qquad i = 1, \ldots, h$$
$$V e_i = e_{i+1}, \qquad\qquad i = 1, \ldots, n - h - 1$$
$$V e_{n-h} = \pi_K e_1.$$

This determines also F by the equality $F = V^{-1}\pi_K$.

The key point to this proposition ist the following lemma due to Dieudonné:

Lemma: *Let M be a free finitely generated $W_K(k)$-module and let V be a σ_K^a-linear bijection where a is some integer different from zero. Then there exists a $W_K(k)$-basis $(e_1, \ldots, e_n)$ of M such that $V e_i = e_i$.*

A proof of this lemma in the case of $K = \mathbb{Q}_p$ can be found in [Zi1] 6.26. The general case is proved word by word in the same way if one replaces everywhere p by π_K.

Proof of the Proposition: Let (G, ι) be a special p-divisible O_K-module as in the proposition and let (M, F, V) be its Dieudonné module. We use the decomposition $(M, F, V) = (M_{\text{ét}}, F, V) \oplus (M_{\text{inf}}, F, V)$ and can apply the lemma to the étale part. Hence we can assume that $h = 0$ (note that $M_{\text{inf}}/V M_{\text{inf}} = M/VM$). By definition of M_{inf}, V acts nilpotent on $M/\pi_K M$. We get a decreasing filtration $M/\pi_K M \supset V(M/\pi_K M) \supset \cdots \supset V^N(M/\pi_K M) = (0)$. The successive quotients have dimension 1 because $\dim_k(M/VM) = 1$. Hence we see that $V^n M \subset \pi_K M$. On the other hand we have

$$\text{length}_{W_K(k)}(M/V^n M) = n \,\text{length}_{W_K(k)}(M/VM)$$
$$= n = \text{length}_{W_K(k)}(M/\pi_K M)$$

which implies $V^n M = \pi_K M$. Hence we can apply the lemma to the operator $\pi_K^{-1} V^n$ and we get a basis of elements f satisfying $V^n f = \pi_K f$. Choose an element f of this basis which does not lie in VM. Then the images of $e_i := V^{i-1} f$ for $i = 1, \ldots, n$ in $M/\pi_K M$ form a basis of the k-vector space $M/\pi_K M$. Hence the e_i form a $W_K(k)$-basis of M, and V acts in the desired form.

(5.2.10) Definition: We denote the unique formal p-divisible O_K-module

of height h and dimension 1 over an algebraically closed field k of characteristic p by $\Sigma_{h,k}$.

(5.2.11) Denote by $D'_{K,1/h}$ the ring of endomorphisms of $\Sigma_{h,k}$ and set $D_{K,1/h} = D'_{K,1/h} \otimes_{\mathbb{Z}} \mathbb{Q}$. Then this is "the" skew field with center K and invariant $1/h \in \mathbb{Q}/\mathbb{Z}$ (5.1.1). This follows from the following more general proposition:

Proposition: *Denote by L the field of fractions of $W_K(k)$ and fix a rational number λ. We write $\lambda = r/s$ with integers r and s which are prime to each other and with $s > 0$ (and with the convention $0 = 0/1$). Denote by $N_\lambda = (N, V)$ the pair consisting of the vector space $N = L^s$ and of the σ_K^{-1}-linear bijective map V which acts on the standard basis via the matrix*

$$\begin{pmatrix} 0 & 0 & \ldots & 0 & \pi_K^r \\ 1 & 0 & & \ldots & 0 \\ 0 & 1 & 0 & \ldots & 0 \\ & & \ldots & & \\ 0 & \ldots & 0 & 1 & 0 \end{pmatrix}.$$

Then $\operatorname{End}(N_\lambda) = \{\, f \in \operatorname{End}_L(N) \mid f \circ V = V \circ f \,\}$ is the skew field D_λ with center K and invariant equal to the image of λ in $\mathbb{Q}/\mathbb{Z}$ (cf. (5.1.1)).

Proof: We identify $\mathbb{F}_{q^s}$ with the subfield of k of elements x with $x^{q^s} = x$. This contains the residue field κ of O_K and we get inclusions

$$O_K \subset O_{K_s} := W(\mathbb{F}_{q^s}) \otimes_{W(\kappa)} O_K \subset W_K(k)$$

and hence

$$K \subset K_s \subset L.$$

These extensions are unramified, $[K_s : K] = s$, and K_s can be described as the fixed field of σ_K^s in L.

To shorten notations we set $A_\lambda = \operatorname{End}(N_\lambda)$. As N_λ does not have any non-trivial V-stable subspaces (cf. [Zi1] 6.27), A_λ is a skew field and its center contains K. For a matrix $(u_{ij}) \in \operatorname{End}(L^s)$ an easy explicit calculation

shows that $(u_{ij}) \in A_\lambda$ if and only if we have the relations

$$
\begin{aligned}
u_{11} &= \sigma_K^{-1}(u_{ss}), \\
u_{i+1,j+1} &= \sigma_K^{-1}(u_{ij}), & 1 \leq i,j \leq s-1, \\
u_{1,j+1} &= \pi_K^r \sigma_K^{-1}(u_{sj}), & 1 \leq j \leq s-1, \\
u_{i+1,j} &= \pi_K^{-r} \sigma_K^{-1}(u_{is}), & 1 \leq i \leq s-1.
\end{aligned}
$$

It follows that $\sigma_K^s(u_{ij}) = u_{ij}$ for all i, j, and hence $u_{ij} \in K_s$. Further, sending a matrix $(u_{ij}) \in A_\lambda$ to its first column defines a K_s-linear isomorphism $A_\lambda \cong K_s^s$, hence $\dim_{K_s}(A_\lambda) = s$.

The K_s-algebra homomorphism

$$
\varphi \colon K_s \otimes_K A_\lambda \longrightarrow M_s(K_s), \qquad \alpha \otimes x \mapsto \alpha x
$$

is a homomorphism of $M_s(K_s)$-left modules and hence it is surjective as the identity matrix is in its image. Therefore φ is bijective. In particular, K_s is the center of $K_s \otimes_K A_\lambda$ and hence the center of A_λ is equal to K.

Now define

$$
\Pi = \begin{pmatrix}
0 & 0 & \ldots & 0 & \pi_K^r \\
1 & 0 & & \ldots & 0 \\
0 & 1 & 0 & \ldots & 0 \\
& & \ldots & & \\
0 & \ldots & 0 & 1 & 0
\end{pmatrix} \in A_\lambda.
$$

Then we have the relations $\Pi^s = \pi_K^r$ and $\Pi d = \sigma_K(d)\Pi$ for $d \in A_\lambda$. We get an embedding $D_\lambda = K_s[\Pi] \hookrightarrow A_\lambda$ by

$$
\Pi \mapsto \Pi
$$

$$
K_s \ni \alpha \mapsto \begin{pmatrix}
\sigma_K^{-1}(\alpha) & & & \\
& \sigma_K^{-2}(\alpha) & & \\
& & \ldots & \\
& & & \alpha
\end{pmatrix} \in A_\lambda \subset M_s(K_s)
$$

which has to be an isomorphism because both sides have the same K-dimension.

(5.2.12) Over a complete local noetherian ring R with perfect residue field k we have the following alternative description of a special formal p-divisible

O_K-module due to Zink [Zi2]. For this we need a more general definition of the Witt ring.

Let R be an arbitrary commutative ring with 1. The Witt ring $W(R)$ is characterized by the following properties:

(a) As a set it is given by $R^{\mathbb{N}_0}$, i.e. elements of $W(R)$ can be written as infinite tuples $(x_0, x_1, \ldots, x_i, \ldots)$.

(b) If we associate to each ring R the ring $W(R)$ and to each homomorphism of rings $\alpha\colon R \longrightarrow R'$ the map

$$W(\alpha)\colon (x_0, x_1, \ldots) \mapsto (\alpha(x_0), \alpha(x_1), \ldots),$$

then we obtain a functor from the category of rings into the category of rings.

(c) For all integers $n \geq 0$ the so called Witt polynomials

$$w_n\colon W(R) \longrightarrow R$$
$$(x_0, x_1, \ldots) \mapsto x_0^{p^n} + p x_1^{p^{n-1}} + \ldots + p^n x_n$$

are ring homomorphisms.

For the existence of such a ring see e.g. [BouAC] chap. IX, §1. If we endow the product $R^{\mathbb{N}_0}$ with the usual ring structure the map

$$\underline{x} \mapsto (w_0(\underline{x}), w_1(\underline{x}), \ldots)$$

defines a homomorphism of rings

$$W_*\colon W(R) \longrightarrow R^{\mathbb{N}_0}.$$

The ring $W(R)$ is endowed with two operators τ and σ which are characterized by the property that they are functorial in R and that they make the following diagrams commutative

$$
\begin{array}{ccc}
W(R) & \xrightarrow{\ \ \tau\ \ } & W(R) \\
{\scriptstyle W_*}\downarrow & & \downarrow{\scriptstyle W_*} \\
R^{\mathbb{N}_0} & \xrightarrow[\underline{x} \mapsto (0, px_0, px_1, \ldots)]{} & R^{\mathbb{N}_0},
\end{array}
$$

$$
\begin{array}{ccc}
W(R) & \xrightarrow{\ \ \sigma\ \ } & W(R) \\
{\scriptstyle W_*}\downarrow & & \downarrow{\scriptstyle W_*} \\
R^{\mathbb{N}_0} & \xrightarrow[\underline{x} \mapsto (x_1, x_2, \ldots)]{} & R^{\mathbb{N}_0}.
\end{array}
$$

The operator τ can be written explicitly by $\tau(x_0, x_1, \ldots) = (0, x_0, x_1, \ldots)$ and it is called *Verschiebung of $W(R)$*. It is an endomorphism of the additive group of $W(R)$. If R is of characteristic p (i.e. $pR = 0$), σ can be described as $(x_0, x_1, \ldots) \mapsto (x_0^p, x_1^p, \ldots)$. For an arbitrary ring, σ is a ring endomorphism and it is called *Frobenius of $W(R)$*.

There are the following relations for σ and τ:

(i) $\sigma \circ \tau = p \cdot \mathrm{id}_{W(R)}$,

(ii) $\tau(x\sigma(y)) = \tau(x)y$ for $x, y \in W(R)$,

(iii) $\tau(x)\tau(y) = p\tau(xy)$ for $x, y \in W(R)$,

(iv) $\tau(\sigma(x)) = \tau(1)x$ for $x \in W(R)$, and we have $\tau(1) = p$ if R is of characteristic p.

We have a surjective homomorphism of rings

$$w_0 \colon W(R) \longrightarrow R, \qquad (x_0, x_1, \ldots) \mapsto x_0,$$

and we denote its kernel $\tau(W(R))$ by I_R. We have $I_R^n = \tau^n(W(R))$ and $W(R)$ is complete with respect to the I_R-adic topology.

If R is a local ring with maximal ideal $\mathfrak{m}$, $W(R)$ is local as well with maximal ideal $\{\, (x_0, x_1, \ldots) \in W(R) \mid x_0 \in \mathfrak{m} \,\}$.

(5.2.13)   Now we can use Zink's theory of displays to give a description of special formal p-divisible groups in terms of semi-linear algebra. Let R be a complete local noetherian O_K-algebra with perfect residue field k. We extend σ and τ to $W(R) \otimes_{\mathbb{Z}_p} O_K$ in an O_K-linear way. Then we get using [Zi2]:

Proposition: *The category of special formal p-divisible O_K-modules of height h over R is equivalent to the category of tuples (P, Q, F, V^{-1}) where*

- *P is a finitely generated $W(R) \otimes_{\mathbb{Z}_p} O_K$-module which is free of rank h over $W(R)$,*
- *$Q \subset P$ is a $W(R) \otimes_{\mathbb{Z}_p} O_K$-submodule which contains $I_R P$, and the quotient P/Q is a direct summand of the R-module $P/I_R P$ such that the induced action of $R \otimes_{\mathbb{Z}_p} O_K$ on P/Q factorizes through the multiplication $R \otimes O_K \longrightarrow R$,*
- *$F \colon P \longrightarrow P$ is a σ-linear map,*

- $V^{-1} \colon Q \longrightarrow P$ is a σ-linear map whose image generates P as a $W(R)$-module,

satisfying the following two conditions:

(a) For all $m \in P$ and $x \in W(R)$ we have the relation

$$V^{-1}(\tau(x)m) = xF(m).$$

(b) The unique $W(R) \otimes_{\mathbb{Z}_p} O_K$-linear map

$$V^{\#} \colon P \longrightarrow W(R) \otimes_{\sigma, W(R)} P$$

satisfying the equations

$$V^{\#}(xFm) = p \cdot x \otimes m$$
$$V^{\#}(xV^{-1}n) = x \otimes n$$

for $x \in W(R)$, $m \in P$ and $n \in Q$ is topologically nilpotent, i.e. the homomorphism $V^{N\#} \colon P \longrightarrow W(R) \otimes_{\sigma^N, W(R)} P$ is zero modulo $I_R + pW(R)$ for N sufficiently large.

(5.2.14)   To define the notion of a Drinfeld level structure we need the following definition: Let R be a ring and let $X = \mathrm{Spec}(A)$ where A is finite locally free over R of rank $N \geq 1$. For any R-algebra R' we denote by $X(R')$ the set of R-algebra homomorphisms $A \longrightarrow R'$ (or equivalently of all R'-algebra homomorphisms $A \otimes_R R' \longrightarrow R'$). The multiplication with an element $f \in A \otimes_R R'$ defines an R'-linear endomorphism of $A \otimes_R R'$. As A is finite locally free we can speak of the determinant of this endomorphism which is an element $\mathrm{Norm}(f)$ in R'.

We call a finite family of elements $\varphi_1, \dots, \varphi_N \in X(R')$ a *full set of sections of X over R'* if we have for every R'-algebra T and for all $f \in A \otimes_R T$ an equality in T

$$\mathrm{Norm}(f) = \prod_{i=1}^{N} \varphi_i(f).$$

(5.2.15)   Let R be an O_K-algebra and let G be a special p-divisible O_K-module over R. We assume that its O_K-height h is constant on $\mathrm{Spec}(R)$,

e.g. if R is a local ring (the only case which will be used in the sequel). The O_K-action on G defines for every integer $m \geq 1$ the multiplication with π_K^m

$$[\pi_K^m]: G \longrightarrow G.$$

This is an endomorphism of p-divisible groups whose kernel is a finite locally free group scheme $G[\pi_K^m]$ over $\mathrm{Spec}(R)$ of rank q^{mh}.

Let R' be an R-algebra. A *Drinfeld $\mathfrak{p}_K^m$-structure on G over R'* is a homomorphism of O_K-modules

$$\alpha: (\mathfrak{p}^{-m}/O_K)^h \longrightarrow G[\pi_K^m](R')$$

such that the finite set of $\alpha(x)$ for $x \in (\mathfrak{p}_K^{-m}/O_K)^h$ forms a full set of sections.

(5.2.16) It follows from the definition (5.2.14) that if $\alpha: (\mathfrak{p}^{-m}/O_K)^h \longrightarrow G[\pi_K^m](R')$ is a Drinfeld $\mathfrak{p}_K^m$-structure over R' then for any R'-algebra T the composition

$$\alpha_T: (\mathfrak{p}^{-m}/O_K)^h \xrightarrow{\alpha} G[\pi_K^m](R') \longrightarrow G[\pi_K^m](T),$$

where the second arrow is the canonical one induced by functoriality from $R' \longrightarrow T$, is again a Drinfeld $\mathfrak{p}_K^m$-structure.

(5.2.17) Being a Drinfeld $\mathfrak{p}_K^m$-structure is obviously a closed property. More precisely: Let $\alpha: (\mathfrak{p}^{-m}/O_K)^h \longrightarrow G[\pi_K^m](R')$ be a homomorphism of abelian groups. Then there exists a (necessarily unique) finitely generated ideal $\mathfrak{a} \subset R'$ such that a homomorphism of O_K-algebras $R' \longrightarrow T$ factorizes over $R'/\mathfrak{a}$ if and only if the composition α_T of α with the canonical homomorphism $G[\pi_K^m](R') \longrightarrow G[\pi_K^m](T)$ is a Drinfeld $\mathfrak{p}_K^m$-structure over T.

It follows that for every special formal p-divisible O_K-module (G, ι) over some O_K-algebra R the functor on R-algebras which associates to each R-algebra R' the set of Drinfeld $\mathfrak{p}_K^m$-structures on $(G, \iota)_{R'}$ is representable by an R-algebra $\mathrm{DL}_m(G, \iota)$ which is of finite presentation as R-module. Obviously $\mathrm{DL}_0(G, \iota) = R$.

(5.2.18) Let $\alpha: (\mathfrak{p}^{-m}/O_K)^h \longrightarrow G[\pi_K^m](R')$ be a Drinfeld $\mathfrak{p}_K^m$-structure over R'. As α is O_K-linear, it induces for all $m' \leq m$ a homomorphism

$$\alpha[\pi_K^{m'}]: (\mathfrak{p}^{-m'}/O_K)^h \longrightarrow G[\pi_K^{m'}](R').$$

Proposition: *This is a Drinfeld $\mathfrak{p}_K^{m'}$-structure.*

For the proof of this non-trivial fact we refer to [HT] 3.2 (the hypothesis in loc. cit. that $\mathrm{Spec}(R')$ is noetherian with a dense set of points with residue field algebraic over κ is superfluous as we can always reduce to this case by [EGA] IV, §8 and (5.2.17)). If R' is a complete local noetherian ring with perfect residue class field (this is the only case which we will use in the sequel) the proposition follows from the fact that we can represent (G, ι) by a formal group law and that in this case a Drinfeld level structure as defined above is the same as a Drinfeld level structure in the sense of [Dr1].

(5.2.19) Let (G, ι) be a special formal p-divisible O_K-module over an O_K-algebra R. By (5.2.18) we get for non-negative integers $m \geq m'$ canonical homomorphisms of R-algebras

$$\mathrm{DL}_{m'}(G, \iota) \longrightarrow \mathrm{DL}_m(G, \iota).$$

It follows from [Dr1] 4.3 that these homomorphisms make $\mathrm{DL}_m(G, \iota)$ into a finite locally free module over $\mathrm{DL}_{m'}(G, \iota)$.

(5.2.20) Example: If R is an O_K-algebra of characteristic p and if G is a special formal p-divisible O_K-module of O_K-height h and of dimension 1, then the trivial homomorphism

$$\alpha^{\mathrm{triv}} \colon (\mathfrak{p}_K^{-m}/O_K)^h \longrightarrow G[\mathfrak{p}_K^m], \qquad x \mapsto 0$$

is a Drinfeld $\mathfrak{p}_K^m$-structure. If R is reduced, this is the only one.

5.3 DEFORMATION OF p-DIVISIBLE O-MODULES

(5.3.1) In this paragraph we fix an algebraically closed field k of characteristic p together with a homomorphism $O_K \longrightarrow k$. Further we fix integers $h \geq 1$ and $m \geq 0$. By (5.2.9) and by (5.2.20) there exists up to isomorphism only one special formal p-divisible O_K-module Σ_h of height h and dimension 1 with Drinfeld $\mathfrak{p}_K^m$-structure α^{triv} over k. We denote the pair $(\Sigma_h, \alpha^{\mathrm{triv}})$ by $\Sigma_{h,m}$.

Let $\mathcal{C}$ be the category of pairs (R, s) where R is a complete local noetherian O_K-algebra and where s is an isomorphism of the residue class field of R with k. The morphisms in $\mathcal{C}$ are local homomorphisms of O_K-algebras inducing the identity on k.

(5.3.2) Definition: Let $(R, s) \in \mathcal{C}$ be a complete local noetherian O_K-algebra. A triple (G, α, φ) consisting of a formal special p-divisible O_K-module G over R, of a Drinfeld $\mathfrak{p}_K^m$-structure α of G over R and of an isomorphism

$$\varphi \colon \Sigma_{h,m} \overset{\sim}{\longrightarrow} (G \otimes_R k, \alpha_k)$$

is called a *deformation of $\Sigma_{h,m}$ over R*.

A triple $(R_{h,m}, \tilde{\Sigma}_{h,m}, \varphi)$ consisting of a complete local noetherian ring $R_{h,m}$ with residue field k and of a deformation $(\tilde{\Sigma}_{h,m}, \varphi)$ of $\Sigma_{h,m}$ is called *universal deformation of $\Sigma_{h,m}$* if for every deformation (G, α, φ) over some $R \in \mathcal{C}$ there exists a unique morphism $R_{h,m} \longrightarrow R$ in $\mathcal{C}$ such that $(\tilde{\Sigma}_{h,m}, \varphi)_R$ is isomorphic to (G, α, φ).

A universal deformation is unique up to unique isomorphism if it exists.

(5.3.3) Proposition: *We keep the notations of (5.3.2).*
  (1) *A universal deformation $(R_{h,m}, \tilde{\Sigma}_{h,m}, \varphi)$ of $\Sigma_{h,m}$ exists.*
  (2) *For $m = 0$ the complete local noetherian ring $R_{h,0}$ is isomorphic to the power series ring $W_K(k)[\![t_1, \ldots, t_{h-1}]\!]$.*
  (3) *For $m \geq m'$ the canonical homomorphisms $R_{h,m'} \longrightarrow R_{h,m}$ are finite flat. The rank of the free $R_{h,0}$-module $R_{h,m}$ is $\#GL_h(O_K/\mathfrak{p}_k^m)$.*
  (4) *The ring $R_{h,m}$ is regular for all $m \geq 0$.*

Proof: Assertion (1) follows from a criterion of Schlessinger [Sch] using rigidity for p-divisible groups (e.g. [Zi1]) and the fact that the canonical functor from the category of special p-divisible O_K-modules over $\mathrm{Spf}(R_{h,m})$ to the category of special p-divisible O_K-modules over $\mathrm{Spec}(R_{h,m})$ is an equivalence of categories (cf. [Me] II, 4). The second assertion follows easily from general deformation theory of p-divisible groups (for an explicit description of the universal deformation and a proof purely in terms of linear algebra one can use [Zi2] and (5.2.13)). Finally, (3) and (4) are more involved (see [Dr1] §4, note that (3) is essentially equivalent to (5.2.19)).

(5.3.4) Let $D_{1/h}$ be "the" skew field with center K and invariant $1/h$. The ring $R_{h,m}$ has a continuous action of the ring of units $O_{D_{1/h}}^{\times}$ of the integral closure $O_{D_{1/h}}$ of O_K in $D_{1/h}$: Let $\tilde{\Sigma}_{g,m} = (G, \alpha, \varphi)$ be the universal special formal p-divisible O_K-module with Drinfeld $\mathfrak{p}_K^m$-structure over $R_{h,m}$. For $\delta \in O_{D_{1/h}}^{\times}$ the composition

$$\Sigma_{h,m} \xrightarrow{\ \delta\ } \Sigma_{h,m} \xrightarrow{\ \varphi\ } (G, \alpha) \otimes_{R_{h,m}} k$$

is again an isomorphism if we consider δ as an automorphism of $\Sigma_{h,m}$ (which is the same as an automorphism of $\Sigma_{h,0}$) by (5.2.11). Therefore $(G, \alpha, \varphi \circ \delta)$ is a deformation of $\Sigma_{h,m}$ over $R_{h,m}$ and by the definition of a universal deformation this defines a continuous automorphism $\delta \colon R_{h,m} \longrightarrow R_{h,m}$.

(5.3.5) Similarly as in (5.3.4) we also get a continuous action of $GL_h(O_K/\mathfrak{p}_K^m)$ on $R_{h,m}$: Again let $\tilde{\Sigma}_{g,m} = (G, \alpha, \varphi)$ be the universal special formal p-divisible O_K-module with Drinfeld $\mathfrak{p}_K^m$-structure over $R_{h,m}$. For $\gamma \in \mathrm{GL}_h(O_K/\mathfrak{p}_K^m)$, $\alpha \circ \gamma$ is again a Drinfeld $\mathfrak{p}_K^m$-structure, hence $(G, \alpha \circ \gamma, \varphi)$ is a deformation of $\Sigma_{h,m}$ and defines a continuous homomorphism

$$\gamma \colon R_{h,m} \longrightarrow R_{h,m}.$$

(5.3.6) By combining (5.3.4) and (5.3.5) we get a continuous left action of

$$\mathrm{GL}_h(O_K) \times O_{D_{1/h}}^{\times} \longrightarrow GL_h(O_K/\mathfrak{p}_K^m) \times O_{D_{1/h}}^{\times}$$

on $R_{h,m}$. Now we have the following lemma ([HT] p. 52)

Lemma: *This action can be extended to a continuous left action of* $GL_h(K) \times D_{1/h}^{\times}$ *on the direct system of the* $R_{h,m}$ *such that for* $m_2 >> m_1$ *and for* $(\gamma, \delta) \in \mathrm{GL}_h(K) \times D_{1/h}^{\times}$ *the diagram*

$$
\begin{array}{ccc}
R_{h,m_1} & \xrightarrow{\ (\gamma,\delta)\ } & R_{h,m_2} \\
\uparrow & & \uparrow \\
W(k) & \xrightarrow{\ \sigma_K^{v_K(\det(\gamma)) - v_K(\mathrm{Nrd}(\delta))}\ } & W(k)
\end{array}
$$

commutes.

5.4 VANISHING CYCLES

(5.4.1) Let W be a complete discrete valuation ring with maximal ideal (π), residue field k and field of fractions L. Assume that k is algebraically closed (or more generally separably closed). The example we will use later on is the ring $W = W_K(k)$ for an algebraically closed field k of characteristic p. Set $\eta = \mathrm{Spec}(L)$, $\bar{\eta} = \mathrm{Spec}(\bar{L})$ and $s = \mathrm{Spec}(k)$.

We will first define vanishing and nearby cycles for an algebraic situation (cf. [SGA 7] exp. I, XIII). Then we will generalize to the situation of formal schemes.

(5.4.2) Let $f \colon X \longrightarrow \mathrm{Spec}(W)$ be a scheme of finite type over W and define $X_{\bar{\eta}}$ and X_s by cartesian diagrams

$$
\begin{array}{ccccc}
X_s & \xrightarrow{\ i\ } & X & \xleftarrow{\ \bar{j}\ } & X_{\bar{\eta}} \\
{\scriptstyle f_s}\downarrow & & {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f_{\bar{\eta}}} \\
s & \longrightarrow & \mathrm{Spec}(W) & \leftarrow \eta \leftarrow & \bar{\eta}.
\end{array}
$$

The formalism of vanishing cycles is used to relate the cohomology of X_s and of $X_{\bar{\eta}}$ together the action of the inertia group on the cohomology of $X_{\bar{\eta}}$.

Fix a prime ℓ different from the characteristic p of k and let Λ be a finite abelian group which is annihilated by a power of ℓ. For all integers $n \geq 0$ the sheaf

$$\Psi^n(\Lambda) = i^* R^n \bar{j}_* \Lambda$$

is called the *sheaf of vanishing cycles* of X over W. Via functoriality it carries an action of $\mathrm{Gal}(\bar{L}/L)$. Note that by hypothesis $\mathrm{Gal}(\bar{L}/L)$ equals the inertia group of L.

(5.4.3) If $f \colon X \longrightarrow \mathrm{Spec}(W)$ is proper, the functor i^* induces an isomorphism (proper base change)

$$i^* \colon H^p(X, R^q \bar{j}\Lambda) \xrightarrow{\ \sim\ } H^p(X_s, i^* R^q \bar{j}_* \Lambda)$$

and the Leray spectral sequence for $\bar{j}$ can be written as

$$H^p(X_s, \Psi^n(\Lambda)) \Rightarrow H^{p+q}(X_{\bar{\eta}}, \Lambda).$$

This spectral sequence is $\mathrm{Gal}(\bar{L}/L)$-equivariant. This explains, why the vanishing cycles "measure" the difference of the cohomology of the special and the generic fibre.

(5.4.4) Let Λ be as above. If X is proper and smooth over W, there is no difference between the cohomology of the generic and the special fibre. More precisely, we have ([SGA 4] XV, 2.1):

    Proposition: *Let $f\colon X \longrightarrow W$ be a smooth and proper morphism. Then we have for all $n \geq 0$ a canonical isomorphism*

$$H^n(X_{\bar{\eta}}, \Lambda) \xrightarrow{\sim} H^n(X_s, \Lambda).$$

This isomorphism is $\mathrm{Gal}(\bar{L}/L)$-equivariant where the action on the right-hand side is trivial. Further $\psi^n(\Lambda) = 0$ for $n \geq 1$ and $\psi^0 = \Lambda$.

(5.4.5) Now we define vanishing cycles for formal schemes: We keep the notations of (5.4.1). We call a topological W-algebra A *special*, if there exists an ideal $\mathfrak{a} \subset A$ (called an *ideal of definition of A*) such that A is complete with respect to the $\mathfrak{a}$-adic topology and such that $A/\mathfrak{a}^n$ is a finitely generated W-algebra for all $n \geq 1$ (in fact the same condition for $n = 2$ is sufficient ([Ber2] 1.2)). Equivalently, A is topologically W-isomorphic to a quotient of the topological W-algebra

$$W\{T_1, \ldots, T_m\}[\![S_1, \ldots, S_n]\!] = W[\![S_1, \ldots, S_n]\!]\{T_1, \ldots, T_m\},$$

in particular A is again noetherian. Here if R is a topological ring, $R\{T_1, \ldots, T_m\}$ denotes the subring of power series

$$\sum_{\underline{n} \in \mathbb{N}_0^n} c_{\underline{n}} T^{\underline{n}}$$

in $R[\![T_1, \ldots, T_M]\!]$ such that for every neighborhood V of 0 in R there is only a finite number of coefficients $c_{\underline{n}}$ not belonging to V. If R is complete Hausdorff with respect to the $\mathfrak{a}$-adic topology for an ideal $\mathfrak{a}$, we have a canonical isomorphism

$$R\{T_1, \ldots, T_m\} \xrightarrow{\sim} \varprojlim_n (R/\mathfrak{a}^n)[T_1, \ldots, T_m].$$

Note that all the rings $R_{h,m}$ defined in (5.3.3) are special $W_K(k)$-algebras.

(5.4.6) Let A be a special W-algebra and let $\mathcal{X} = \mathrm{Spf}(A)$. Denote by $\mathcal{X}^{\mathrm{rig}}$ its associated rigid space over L. It can be constructed as follows: For

$$A = W\{T_1, \ldots, T_m\}[\![S_1, \ldots, S_n]\!]$$

we have $\mathcal{X}^{\mathrm{rig}} = E^m \times D^n$ where E^m and D^n are the closed resp. open polydiscs of radius 1 with center at zero in L^m resp. in L^n. If now A is some quotient of the special W-algebra $A' = W\{T_1, \ldots, T_m\}[\![S_1, \ldots, S_n]\!]$ with kernel $\mathfrak{a}' \subset A'$, $\mathcal{X}^{\mathrm{rig}}$ is the closed rigid analytic subspace of $\mathrm{Spf}(A')^{\mathrm{rig}}$ defined by the sheaf of ideals $\mathfrak{a}' \mathcal{O}_{\mathrm{Spf}(A')^{\mathrm{rig}}}$. This defines a functor from the category of special W-algebras to the category of rigid-analytic spaces over L (see [Ber2] §1 for the precise definition).

(5.4.7) Let A be a special W-algebra and let $\mathfrak{a} \subset A$ be its largest ideal of definition. A topological A-algebra B which is special as W-algebra (or shorter a special A-algebra - an abuse of language which is justified by [Ber2] 1.1) is called *étale over A* if B is topologically finitely generated as A-algebra (which is equivalent to the fact that for every ideal of definition $\mathfrak{a}'$ of A, $\mathfrak{a}'B$ is an ideal of definition of B) and if the morphism of commutative rings $A/\mathfrak{a} \longrightarrow B/\mathfrak{a}B$ is étale in the usual sense.

The assignment $B \mapsto B/\mathfrak{a}B$ defines a functor from the category of étale special A-algebras to the category of étale $A/\mathfrak{a}$-algebras which is an equivalence of categories and hence we get an equivalence of étale sites

$$(A)_{\mathrm{ét}} \sim (A/\mathfrak{a})_{\mathrm{ét}}.$$

Note that for every ideal of definition $\mathfrak{a}'$ of A the étale sites $(A/\mathfrak{a})_{\mathrm{ét}}$ and $(A/\mathfrak{a}')_{\mathrm{ét}}$ coincide. We just chose the largest ideal of definition to fix notations.

On the other hand, if we write $\mathcal{X} = \mathrm{Spf}(A)$ and $\mathcal{Y} = \mathrm{Spf}(B)$ for an étale special A-algebra B we get a (quasi-)étale morphism of rigid analytic spaces

$$\mathcal{Y}^{\mathrm{rig}} \longrightarrow \mathcal{X}^{\mathrm{rig}}.$$

By combining this functor with a quasi-inverse of $B \mapsto B/\mathfrak{a}B$ we get a morphism of étale sites

$$s \colon (\mathcal{X}^{\mathrm{rig}} \otimes_L \bar{L})_{\text{ét}} \longrightarrow (\mathcal{X}^{\mathrm{rig}})_{\text{ét}} \longrightarrow (\mathcal{X})_{\text{ét}} \xrightarrow{\sim} (\mathcal{X}_{\mathrm{red}})_{\text{ét}}$$

with $\mathcal{X}_{\mathrm{red}} = \mathrm{Spec}(A/\mathfrak{a})$.

Let Λ be a finite abelian group which is annihilated by a power of ℓ. For $n \geq 0$ the sheaves

$$\psi^n(\Lambda) := R^n s_* \Lambda$$

are called *vanishing cycle sheaves*.

(5.4.8) Let A be a special W-algebra with largest ideal of definition $\mathfrak{a}$, $\mathcal{X} = \mathrm{Spf}(A)$. Then the group $\mathrm{Aut}_W(\mathcal{X})$ of automorphisms of $\mathcal{X}$ over W (i.e. of continuous W-algebra automorphisms $A \to A$) acts on $\psi^n(\Lambda)$. Further we have the following result of Berkovich [Ber2]:

Proposition: *Assume that $\psi^i(\mathbb{Z}/\ell\mathbb{Z})$ is constructible for all i. Then there exists an integer $n \geq 1$ with the following property: Every element $g \in \mathrm{Aut}_W(\mathcal{X})$ whose image in $\mathrm{Aut}_W(A/\mathfrak{a}^n)$ is the identity acts trivially on $\psi^i(\mathbb{Z}/\ell^m\mathbb{Z})$ for all integers $i, m \geq 0$.*

5.5 Vanishing cycles on the universal deformation of special p-divisible O-modules

(5.5.1) Let k be an algebraic closure of the residue field κ of O_K. Then $W = W_K(k)$ is the ring of integers of $\hat{K}^{\mathrm{nr}}$, the completion of the maximal unramified extension of K. Further denote by I_K the inertia group and by W_K the Weil group of K.

(5.5.2) Consider the system P of special formal schemes

$$\ldots \longrightarrow \mathrm{Spf}(R_{m,h}) \longrightarrow \mathrm{Spf}(R_{m-1,h}) \longrightarrow \ldots \longrightarrow \mathrm{Spf}(R_{0,h}).$$

By applying the functor ( )$^{\mathrm{rig}}$ we get a system P^{rig} of rigid spaces

$$\ldots \longrightarrow \mathrm{Spf}(R_{m,h})^{\mathrm{rig}} \longrightarrow \mathrm{Spf}(R_{m-1,h})^{\mathrm{rig}} \longrightarrow \ldots \longrightarrow \mathrm{Spf}(R_{0,h})^{\mathrm{rig}}.$$

these systems P and P^{rig} have an action by the group $\mathrm{GL}_h(O_K) \times O_{D_{1/h}}^\times$ (5.3.4) and (5.3.5). Denote by $\Psi_m^i(\Lambda)$ the vanishing cycle sheaf for $\mathrm{Spf}(R_{h,m})$ with coefficients in some finite abelian ℓ-primary group Λ and set

$$\Psi_m^i = (\varprojlim_n \Psi_m^i(\mathbb{Z}/\ell^n\mathbb{Z})) \otimes_{\mathbb{Z}_\ell} \bar{\mathbb{Q}}_\ell.$$

Note that we have

$$\varprojlim_n \Psi_m^i(\mathbb{Z}/\ell^n\mathbb{Z}) = H^i((\mathrm{Spf}\, R_m)^{\mathrm{rig}} \otimes_{\hat{K}^{\mathrm{nr}}} \hat{\bar{K}}, \mathbb{Z}_\ell),$$

in particular these $\mathbb{Z}_\ell$-modules carry an action by $I_K = \mathrm{Gal}(\hat{\bar{K}}/\hat{K}^{\mathrm{nr}})$. Further write

$$\Psi^i = \varinjlim_m \Psi_m^i.$$

Via our chosen identification $\bar{\mathbb{Q}}_\ell \cong \mathbb{C}$ we can consider Ψ_m^i and Ψ^i as $\mathbb{C}$-vector spaces which carry an action of

$$\mathrm{GL}_h(O_K) \times O_{D_{1/h}}^\times \times I_K.$$

(5.5.3) Lemma: *We have the following properties of the $(\mathrm{GL}_h(O_K) \times O_{D_{1/h}}^\times \times I_K)$-modules Ψ_m^i and Ψ^i.*

(1) *The Ψ_m^i are finite-dimensional $\mathbb{C}$-vector spaces.*
(2) *We have $\Psi_m^i = \Psi^i = 0$ for all $m \geq 0$ and for all $i > h - 1$.*
(3) *The action of $GL_h(O_K)$ on Ψ^i is admissible.*
(4) *The action of $O_{D_{1/h}}^\times$ on Ψ^i is smooth.*
(5) *The action of I_K on Ψ^i is continuous.*

Proof: For the proof we refer to [HT] 3.6. We only remark that (3) – (5) follow from general results of Berkovich [Ber2] and [Ber3] if we know (1). To show (1) one uses the fact that the system of formal schemes P comes from an inverse system of proper schemes of finite type over W (cf. the introduction) and a comparison theorem of Berkovich which relates the vanishing cycles of a scheme of finite type over W with the vanishing cycle sheaves for the associated formal scheme.

(5.5.4) Let A_h be the group of elements $(\gamma, \delta, \sigma) \in \mathrm{GL}_h(K) \times D_{1/h}^\times \times W_K$ such that

$$v_K(\det(\gamma)) = v_K(\mathrm{Nrd}(\delta)) + v_K(\mathrm{Art}_K^{-1}(\sigma)).$$

The action of $\mathrm{GL}_h(K) \times D_{1/h}^\times$ on the system $(R_{h,m})_m$ (5.3.6) gives rise to an action of A_K on Ψ^i.

Moreover, if (ρ, V_ρ) is an irreducible admissible representation of $D_{1/h}^\times$ over $\mathbb{C}$ (and hence necessarily finite-dimensional (2.1.20)) then we set

$$\Psi^i(\rho) = \mathrm{Hom}_{O_{1/h}^\times}(\rho, \Psi^i).$$

This becomes naturally an admissible $\mathrm{GL}_h(K) \times W_K$-module if we define for $\phi \in \Psi^i(\rho)$ and for $x \in V_\rho$

$$((\gamma, \sigma)\phi)(x) = (\gamma, \delta, \sigma)\phi(\rho(\delta)^{-1}x)$$

where $\delta \in D_{1/h}^\times$ is some element with $v_K(\mathrm{Nrd}(\delta)) = v_K(\det(\gamma)) - v_K(\mathrm{Art}_K^{-1}(\sigma))$.

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