# P-ADIC WHITTAKER PATTERNS

Contents



## 1. Introduction: mixed characteristic Casselman-Shalika formula and Whittaker category

Let $G$ be a split connected reductive algebraic group over the finite field $\mathbb{F}_q$. Let $\mathrm{Sph}^{\heartsuit}_{G,e} := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, e)$ be the *spherical category* of $G$, or the category of $L^+G$ equivariant perverse sheaves on $\mathrm{Gr}_G$ with coefficients in $e$. For

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$e$ a field, this is a *highest weight* category, with standard and costandard objects,

$$j_!(\lambda, e) := \pi_0 j_!^\lambda k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle] \text{ and } j_*(\lambda, e) := \pi_0 j_*^\lambda k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle]$$

If $e$ is of characteristic 0, the category is semisimple, with simple objects

$$\{\mathcal{A}_\lambda := j_{!*}(\lambda, e)\}_{\lambda \in \Lambda_+}$$

By the classical Satake isomorphism, this is isomorphic to

$$\mathrm{Rep}(\widehat{G}, e)$$

algebraic representations of the dual group of $G$ with coefficients in $e$, [21]. The reader is welcome to skip from here to the statement of geometric Casselman-Shalika, 1.2.

## 1.1. **The associated function from Frobenius trace.**

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

defined on the set of $k$ points of $\overline{\mathrm{Gr}^\lambda}$, can be viewed as a function of the unramified Hecke algebra [14], $\mathcal{H}_G$[1]. The constant term map

$$\mathcal{H}_G \to \mathcal{H}_T, f \mapsto f^B$$

has formula given by

$$f^B(t) := \delta_{B(K)}^{1/2}(t) \int_{N(K)} f(tu)\, du$$

The obvious basis elements $\{f_\lambda\}_{\lambda \in X_{\bullet_+}} \subset \mathcal{H}_G$, defined as indicator functions of double cosets, has a surprisingly simple formula, [23], under the constant term map

$$f_\lambda^B(t) = \int_{N(K)} A_\lambda(x\varpi^\nu)\, dx = (-1)^{2\langle\rho,\nu\rangle} q^{\langle \rho,\nu\rangle} m_\lambda(\nu)$$

where $\rho$ is the half sum of positive roots.

## 1.2. **The geometric Casselman-Shalika formula.**

The equal characteristic *geometric* Casselman -Shalika states

**Theorem 1.1.** [11, 8.1.2]

$$H_c^i(S^\mu, j_{!*}(\lambda, e)\Big|_{S^\mu} \otimes_e \chi_\mu^*(\mathcal{L}_\psi)) = \begin{cases} e & \textit{if } \lambda = \mu \textit{ and } \langle 2\check{\rho}, \lambda\rangle = i \\ 0 & \textit{otherwise.}\end{cases}$$

*where $\mathcal{L}_\psi$ is pullback of Artin-Schrier sheaf from a nondegenerate character $\psi : N \to \mathbb{G}_a$.*

[1] compactly supported functions in $G(K)$ this is bi-equivariant with respect to $G(\mathcal{O})$

This is a geometrization of the classical Casselman-Shalika formula described in 1.1. A baby version without the character is used by Lusztig in giving the weight structure of the Satake category. The first goal of the project is therefore to give a mixed characteristic (of the geometry) version. This will make extensive use of recent of results of Fargues and Scholze, [12].

The project's second goal is to set up the foundations of Whittaker category in mixed characteristic, by understanding it as a left module over the spherical Hecke category. This is important in setting up geometric Langlands in the mixed characteristic setting, see 1.3.

By generalizing, suggests a fundamental property of the representation theory of reductive groups over local non-archimedean fields and allows one to import further arithmetic information.

1.3. **Related works.** Beyond its applications in the original paper. [11], the geometric CS formula in equal characteristic has been applied in recent work [2] to give an *Iwahori-Whittaker model* of the Satake category.

The implication of such a geometric model is twofold. Firstly, it gives a geometric description of the representation category.

$$D^b_{\mathrm{IW}}(\mathrm{Gr}_G, e) \simeq D^b(\mathrm{Rep}_e(\check{G})^{\heartsuit})$$

But further shows the derived category is *abelian*, which is much more easy to control.

Secondly, this result fits in the framework of *fundamental local equivalence* (FLE), a program initiated by D. Gaitsgory, [13]. The equivalence is present in [7, Thm. 3]. The Iwahori-Whittaker model is what the Whittaker filtration stabilizes to, see [24].

## 2. Notation

- We fix a local field $E/\mathbb{Q}_p$.
- Let $G$ denote a split connected reductive group over $E$.
- Fix a Borel $B$ with unipotent radical $N$ and (maximal split) torus $T$.
- We always work over the absolute base $* = \mathrm{Spd}\overline{\mathbb{F}}_p$.
- Let Perf denote the category of (affinoid) perfectoid spaces in characteristic $p$.
- If $S \in \mathrm{Perf}$, we let $\mathrm{Perf}_S$ denote the slice category of affinoid perfectoid spaces over $S$.

- If $\mathcal{F}$ is any presheaf on Perf, we let $\mathrm{Perf}_{\mathcal{F}}$ denote the slice category of Perf over $\mathcal{F}$ (identifying $S \in \mathrm{Perf}$ with the presheaf it represents).
- For $S \in \mathrm{Perf}$ we let $X_S$ denote the relative Fargues–Fontaine curve over $S$.
- Let $\mathrm{Div}_X$ denote the usual mirror curve; recall that there is a functorial bijection between $\mathrm{Div}_X(S)$ and the set of closed Cartier divisors on $X_S$.
- If $Z$ is an affine scheme over $E$ we let $L^{(+)}Z$ denote the (positive) loop space of $Z$, defined by

$$\begin{aligned} L^{(+)}N : \mathrm{Perf}_{\mathrm{Div}_X} &\to \mathrm{Grp} \\ S &\mapsto B^{(+)}_{\mathrm{Div}_X}(S) \end{aligned}$$

  This is a $v$-sheaf, and if $Z$ is a group then it is a $v$-sheaf of groups. Here $B^{(+)}_{\mathrm{Div}_X}(S)$ is the completion of $\mathcal{O}_{X_S}$ along the divisor $\mathcal{I}_S$ corresponding to the map $S \to \mathrm{Div}_X$, and is sometimes also denoted $B^{(+)}_{\mathrm{dR}}(S)$.
- If $G$ is an algebraic group over $E$, let $\mathrm{Gr}_G$ denote the $B^+_{\mathrm{dR}}$-affine Grassmannian, defined by letting $\mathrm{Gr}_G(S)$ equal the set of $G$-torsors over $\mathrm{Spec}(B^+_{\mathrm{dR}}(S^\sharp))$ blah blah
- If $G$ is an algebraic group over $E$, let $\mathrm{Bun}_G(S)$ denote the groupoid of $G$-bundles on $X_S$. Note $\mathrm{Bun}_G$ is a small $v$-stack.

## 3. Requisites on Fargues Fontaine Curve

3.0.1. *Introduction.* The Fargue-Fontaine curve exhibit similarities to $\mathbb{P}^1_k$. To motivate, consider encoding $k(X)$, when $X \in \mathrm{SmProj}^{\mathrm{cn},g=1}_{\mathbb{C}}$. This is the "point at $\infty$" perspective, [18], [28].

**Example**

The curve $\mathbb{P}^1_{\mathbb{C}}$ has the following properties.

- $\mathbb{A}^1 := \mathbb{P}^1 \backslash \{\infty\}$ has ring of functions $\mathbb{C}[z]$.

The FF curve has many similarity with $\mathbb{P}^1$ albeit not exactly:

- if $\infty \in |X|$ then $\Gamma(X\backslash \{\infty\}, \mathcal{O}_X)$ is provided with the *almost euclidean* $-\mathrm{ord}_\infty = \deg$ evaluation.

3.1. **de Rham period ring.** Let us recall period rings from the conjectures of Fontaine.

**Conjecture 3.1.**

- *(dR) Ley $Y \in \mathrm{Alg}^{\mathrm{sm,prop}}_K$, where $K = W(k)[1/p]$. Then there is a canonical isomorphisn*

$$\alpha_{\mathrm{dR}} : H^*_{\mathrm{dR}}(Y) \otimes B_{\mathrm{dR}} \simeq H^*_{\acute{e}t}(Y_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{dR}}$$

- *(Crys) Let $\mathfrak{Y} \in \mathrm{Alg}_{W(k)}$ st. $\mathfrak{Y}_\eta \simeq Y$.*

$$\alpha_{\mathrm{cris}} : H^*_{\mathrm{dR}}(Y) \otimes_K B_{\mathrm{cris}} \simeq H^*(Y_{\bar{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{cris}}$$

**Definition 3.2.** We regard the de Rham period ring as a v-sheaf.

$$\theta := W(R^{\flat\circ})[1/p] \twoheadrightarrow R$$

3.1.1. *Adic Spaces.* Adic rings are the basic models

**Definition 3.3.** A *Huber ring* consists of a ring $A$ satisfies the following: there exists an open subring $A_0$ whose topology is induced by a finitely generated ideal $I$ of $A_0$.

Examples:

(1) $K$ be a nonarchimedean field: For instance, $\mathbb{Q}_p$. For a choice $K$, this are equivalence of classes of norms $| \quad |$ associated it, [?Lur205, 4]. [30].
(2) $A = \mathbb{Q}_p[[t]]$.

(3) *Tate algebra* are an important class of examples. Let $k$ be a field with topology induced by a rank 1 valuation $|\quad|$. [20, II.1.4,(8)]. Then

$$T_n := k\langle x_1, \ldots, x_n\rangle$$
$$:= \left\{ f = \sum_{\nu \in \mathbb{N}^n} a_\nu X^\nu \in k[[x_1, \ldots, x_n]] \,:\, |a_\nu| \to 0 \text{ as } |\nu| \to \infty \right\}$$

- $T_n^\circ = k^\circ \langle x_1, \ldots, x_n\rangle$.
- $T_n^{\circ\circ} = k^{\circ\circ} \langle x_1, \ldots, x_n\rangle$.

**Definition 3.4.** A *huber pair* $(R, R^+)$ is a huber ring $R$ with a subring

$$R^+ \hookrightarrow_{\text{open,int.cl}} R^\circ$$

The category of *adic spaces* can be constructed in two steps:

- Define a category with continuous valuations on stalks. Its objects consists of

$$(X, \mathcal{O}_X, |\quad|_{x\in X})$$

  where
  - $|\quad|_x$ is an equivalence class of valuation on $\mathcal{O}_{X,x}$.
  - $\mathcal{O}_X$ is a sheaf of topological rings on $X$.

A particular class of rings of our interests are

**Definition 3.5.** A *perfectoid ring* $A$ is a complete Tate ring $A$ such that

- $A$ is uniform, i.e. $A^\circ$ is bounded.
- Exists psu $\varpi$, such that $\varpi^p | p$ and $A^\circ/\varpi \to A^\circ/\varpi^p$ is an isomorphism.

3.2. **The diamond functor.** Let $\mathrm{Pftd}_{\mathbb{F}_p}$ be the category of perfectoid spaces of char $p$.

**Definition 3.6** (Topologies on Pftd)**.** Ordering by fine-ness of topology, we have

$$v \quad \subset \quad \text{Étale} \quad \subset \quad \text{Analytic} \tag{1}$$

One can access any $v$-sheaf via reversing the following properties

(1) A $v$-sheaf diamond is quotient of perfectoid space under pro-étale equivalence [26, 1.21]

The diamond functor generalizes tilting.

(2)
$$\begin{tikzcd}
\mathrm{AdicSpc}_{\mathrm{Spa}_{\mathbb{Z}_p}} \arrow[r] \arrow[rrdd, dashed, bend left] & \mathrm{AdicSpc}_{\mathbb{F}_p} \arrow[rdd] & \\
\mathrm{Pftd}_{\mathbb{Z}_p} \arrow[r, "(\quad)^\flat"] \arrow[u, hook] & \mathrm{Pftd}_{\mathbb{F}_p} \arrow[u, hook] & \\
\mathrm{AffPftdAlg}_{\mathbb{Z}_p} \arrow[r] \arrow[u, hook] & \mathrm{AffPftd}_{\mathbb{F}_p} \arrow[r, hook] \arrow[u, hook] & \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_p}, v)
\end{tikzcd}$$

Diamonds are the algebraic spaces under the pro-'etale equivalence relation in the characteristic $p$ world. This comes from the phenomena that

**Proposition 3.7.** *If $X \in \mathrm{AdicSpc}_{\mathrm{Spa}\,\mathbb{Z}_p}$ then $X$ is a pro-étale quotient of perfectoid space.*

This implies an intuitive construction of $\Diamond$: given $X \in \mathrm{AdicSpc}_{\mathrm{Spa}_{\mathbb{Z}_p}}$, choose a proétale surjection $\tilde{X} \to X$, such that $\tilde{X}/R \simeq X$ where $R \subset \tilde{X} \times \tilde{X}$ is an equivalence relation. Then
$$X^\Diamond \simeq \tilde{X}^\flat / R^\flat$$

**Definition 3.8.** A *diamond* is $X \in \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_q}, \text{pro-ét})$ such that there eixsts a perfectoid space $\tilde{X}$

Note that one define a proétale sheaf $\mathrm{Spd}\mathbb{Z}_p$ is *is not* a diamond.

3.2.1. *Embedding schemes as $v$-sheaves.* We birefly recall [1, 2.2]. For schemes locally of finite type over $\mathcal{O}$ one has two ways of embedding as $v$-sheaves,

(3)
$$\begin{tikzcd}
 & \mathrm{Sch}^{\mathrm{lft}}_{\mathcal{O}} \arrow[d, shift right=2] \arrow[rd, shift left, "(-)^\diamond"] \arrow[rd, shift right, "(-)^\Diamond"'] \arrow[d, shift left=2] & \\
\mathrm{DiscFmlSch}_{\mathcal{O}} \arrow[r] & \mathrm{DiscAdic}_{\mathcal{O}_{\mathrm{disc}}} \arrow[u] \arrow[r] & v\mathrm{Shv}_{\mathrm{Spd}\mathcal{O}}
\end{tikzcd}$$

There is a 1-morphism
$$(-)^\diamond \to (-)^\Diamond$$
which is equivalent on proper schemes.

We describe the points:
$$\mathrm{Spec}\,A^\diamond : \mathrm{Spa}(R, R^+) \mapsto \left\{ \left(R^\#, i, f^+\right) \right\} / \sim$$
$$\mathrm{Spec}\,A^\Diamond : \mathrm{Spa}(R, R^+) \mapsto \left\{ \left(R^\#, i, f\right) \right\} / \sim$$
where $(R^\#, i)$ is untilt and $f^+ : A \to R^{\#,+}$, and $f : A \to R^\#$ are ring homomorphisms.

**Proposition 3.9.** $X^\diamondsuit \simeq (X_\flat)^\diamondsuit$, [9].

Equivalently (relative to a finite extension $E/\mathbb{Q}_p$)

$$\begin{gathered}\mathbb{G}_a^\diamondsuit : \mathrm{Pftd}_{\mathbb{F}_p,\mathrm{Spd}E} \to \mathrm{Grp} \\ ((R,R^+) \to \mathrm{Spd}E) \mapsto \{\mathrm{Spa}(R^\sharp, R^{\sharp,+}) \to \mathbb{G}_a\} = \mathbb{G}_a(R^\sharp) = R^\sharp\end{gathered}$$

## 4. Fargues Fontaine Curve

Let us fix our base $p$-adic field $K$. In analogy to classical construction of Shtukas.

**Definition 4.1.** For $S = \mathrm{Spa}(R, R^+) \in \mathrm{Pftd}_{\mathbb{F}_q}$

$$\mathcal{Y}_S := \mathrm{Spa}(W_{\mathcal{O}_K}(R^+))\backslash V([\varpi]) \in \mathrm{AdicSpc}_{\mathrm{Spa}\,\mathcal{O}_K}$$

where $\varpi \in R^+$ is a psu. [2]

**Proposition 4.2.** [12, II.1.2]

$$\mathcal{Y}_S^{\diamondsuit} \simeq \mathrm{Spd}\mathcal{O}_E \times S \in \mathrm{Dia}$$

### 4.1. Banach Colmez Space.

4.2. **Perfectoid fields.** In greater generality, we can define *integral perfectoid rings*. A *non-archimedean(narc) field* is a field $K$ equipped with an absolute value $|\quad| : K \to \mathbb{R}_{\geq 0}$, this makes $K$ into a metric space $d(x, y) := |x - y|$. $K$ is *complete* if it is so with respect to this metric. We can always uniquely extend a valuation on $K$, a complete narc field, to $L$, where $L/K$ is a finite extension. This extends to a valuation on $\bar{\mathbb{Q}}_p$, but is no longer complete. Many examples can be derived from [19, 1], [17].

$$\widehat{\mathbb{Q}_p(p^{1/p^\infty})} = \varprojlim \left( \varinjlim \mathbb{Z}_p[x_n]/(x_n^{p^n} = p) \right)$$

$$\widehat{\mathbb{Q}_p(\zeta_p^{1/p^\infty})} = \varprojlim \left( \varinjlim \mathbb{Z}_p[x_n]/(x_n^{p^n} = 1) \right)$$

4.3. **Recollection of affine curves.** Much of the intuition comes from the theory of algebraic curves, in particular *Dedekind schemes*, [15, 7.13].

**Definition 4.3.** $X \in \mathrm{Sch}_{\mathbb{Z}}^{\text{Noet, int}}$ is Dedekind, iff $\Gamma(U, \mathcal{O}_X)$ is a Dedekind domain for all $U \hookrightarrow_{\text{open,aff}} X$

Examples :

- Regular integral curve $C$ over $k$.

4.4. **$p$-adic period domains.** The work of Griffiths is the classical introduction. We list the crucial notions and their $p$-adic counterpart

- Hodge structure. The analogous definition was proposed by Fontaine.

$$\mathrm{IsoCrys}_K \simeq \mathrm{CrysRep}(\Gamma_K)$$

**Definition 4.4.** Let $(V, \Phi) \in \mathrm{IsoCrys}_k$, $\dim_k V = n$.

[2] This is independent of choice of $\varpi$.

In general for a vs and a filtration of type $\nu$, can associate [6, 2.1.3] of Zariski open subset of a flag variety,

$$\mathcal{F}(V,\nu)^{\mathrm{ss}} \subset \mathcal{F}(V,\nu)$$

As a prestack on $\mathrm{Aff}_k$, the $R$-points consists of flags

$$\{\mathcal{V}_0 \subset \cdots \subset \mathcal{V}_r := V \otimes_k R\}$$

4.5. **Fargues Fontaine curves and $p$-divisible groups.**

## 5. Character sheaf

**Lemma 5.1.** *There is an isomorphism*

$$\mathrm{Bun}_N \cong [*/\underline{N(E)}]$$

*where $\underline{N(E)}$ denotes the constant pro-étale sheaf associated with the locally profinite group $N(E)$.*

*Proof.* We prove this by induction on $U$.

First suppose $U \cong \mathbb{G}_a$. By the Tannakian formalism, the data of a $\mathbb{G}_a$-bundle on $X_S$ is the same as a short exact sequence

$$0 \to \mathcal{O}_{X_S} \to \mathcal{V} \to \mathcal{O}_{X_S} \to 0$$

of vector bundles on $X_S$. In other words, it is determined by an element of

$$\mathrm{Ext}^1_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}, \mathcal{O}_{X_S}) = H^1(X_S, \mathcal{O}_{X_S}).$$

By [12, Proposition II.2.5] the pro-étale sheafification of the functor $S \mapsto H^1(X_S, \mathcal{O}_{X_S})$ vanishes so pro-étale locally, the only $\mathbb{G}_a$-bundle is

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S} \oplus \mathcal{O}_{X_S} \to \mathcal{O}_{X_S} \to 0$$

up to isomorphism. An endomorphism of this $\mathbb{G}_a$-bundle is a morphism of short exact sequences which induces identities on the ends, which can be represented as a matrix $\begin{pmatrix} \mathrm{id} & \alpha \\ 0 & \mathrm{id} \end{pmatrix}$ where

$$\alpha \in \mathrm{End}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}) = \mathrm{Hom}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}, \mathcal{O}_{X_S}) = H^0(X_S, \mathcal{O}_{X_S})$$

which is pro-étale locally $\underline{E}(S)$. Therefore the natural map

$$[*/\underline{E}] \to \mathrm{Bun}_{\mathbb{G}_a}$$

given by inclusion of the trivial bundle is an isomorphism of stacks.

Now suppose $\dim N > 1$, so that there is a nontrivial unipotent subgroup $N'$ of $N$ such that $N'/N \cong \mathbb{G}_a$, [25, 14.3.10]. This induces a sequence of maps

$$\begin{tikzcd}
\mathrm{Bun}_{N'} \arrow[r, "\sim"] \arrow[d] & B\underline{N'(E)} \arrow[d] \\
\mathrm{Bun}_N \arrow[r] \arrow[d] & B\underline{N(E)} \arrow[d] \\
\mathrm{Bun}_{\mathbb{G}_a} \arrow[r, "\sim"] & B\underline{E}
\end{tikzcd}$$

Both vertical sequences are fibre sequences; therefore, the middle horizontal map is an isomorphism. ∎

Recall from [12, III.3] that there is a Beauville–Laszlo uniformization map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G$$

which is a surjective morphism of $v$-stacks.

We can use this to construct a map

$$LN \to LN/L^+N = \mathrm{Gr}_G \to \mathrm{Bun}_N \xrightarrow{\sim} B\underline{N(E)} \to B\underline{E}$$

where the last map is induced by

$$N \to N/[N,N] \cong \bigoplus_{\text{simple roots}} \mathbb{G}_a \xrightarrow{+} \mathbb{G}_a$$

But $[*/\underline{E}]$ is the moduli stack of pro-étale $\underline{E}$-torsors on the Fargues–Fontaine curve, so any non-trivial character $\psi : E \to \overline{\mathbb{Q}}_\ell^\times$ corresponds to an $\ell$-adic local system on $B\underline{E}$. We can then pull this back to obtain an $\ell$-adic local system on $LN$.

**Proposition 5.2.** *The map $h$ induces a well-defined map on $h : S_\nu \to \mathbb{G}_a$.*

*Proof.* This is well defined as $h$ is trivial on $L^+N$. ??? □

## 6. Orbit Intersections: Mirkovic-Vilonen Cycles

To compute the Hecke action, we need to understand the intersection of semi-infinite orbits [10, 7]. These played a dominant role in the first complete proof of geometric Langlands [21]. Over $\mathbb{C}$, the statement has already appeared in the work of [16, p282]. In mixed characteristic, this was discussed [31, 2.2]. Let us recall the semi-infinite orbits in the $p$-adic setting from [12, VI.3]. [8, 4.2]. To make the first cohomolgoical computation, we follow the argument of Ngô-Polo [23, 5].

**Definition 6.1.** Let $\Omega_\mu := \{\mu \in X_\bullet : \lambda^+ \leq \mu\}$, where $\lambda^+$ is the unique dominant $W$-translate of $\lambda$. [3]

As [12, VI.6.7] the general spirit of argument follows proving ULA property, which degenerates our study to that of Witt vector Grassmanian. For (possible) future use, we consider the *Beilinson Drinfeld Grassmanian*, which we recall in 6.2. For convenience, we omit the base stack of divisors $\mathrm{Div}^I$.

Set up

- $G$ is a split reductive group over $K$, a $p$-adic field. [4] We thus fix a split reductive model over $\mathcal{O}_K$.

[3] Alternatively, this is $\lambda + \mathbb{Z}\Phi^\vee \cap \mathrm{Conv}(W\lambda)$

[4] One can always base change when necessary.

**Definition 6.2.** Let $I$ be a finite set. For $\nu_\bullet := (\nu_i)_{i\in I} \in (X_\bullet)^I$. The *semi-infinite obrit* associated to $\nu_\bullet$ is the small $v$-sheaf $S_G^{\nu_\bullet} \in \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_q}, v)_{/\mathrm{Div}^I}$ given by the pullback

$$\begin{tikzcd} S_G^{\nu_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_B^I \arrow[d] \\ \mathrm{Gr}_T^{\nu_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_T^I \arrow[d] \\ \mathrm{pt} \arrow[r, "\nu_\bullet"] & (X_\bullet)^I \end{tikzcd}$$

**Proposition 6.3.** [12, IV.3.1] *The inclusion*

$$q : \mathrm{Gr}_B \simeq \bigsqcup_\nu S^\nu \to \mathrm{Gr}_G$$

*is a locally closed immersion on each component, $S^\nu$.*

**Definition 6.4.** For $\lambda \in X^I_{\bullet,+}$, we let $\mathrm{Gr}_G^{\lambda_\bullet}$ be the locally closed subfunctor of $\mathrm{Gr}_G^I$.

In this set up we have the constant term functor which fits in the following diagram

$$\begin{tikzcd} & \mathrm{Gr}_B \arrow[dl, "q^+"] \arrow[dr, "p^+"] & \\ \mathrm{Gr}_T & & \mathrm{Gr}_G \\ & \mathrm{Gr}_{B^-} \arrow[ul, "q^-"] \arrow[ur] & \end{tikzcd}$$

In the context of geometric Langlands we have

$$\begin{tikzcd} \mathrm{Dmod}(\mathrm{Bun}_G) \arrow[r, "\mathbb{L}_G"] \arrow[d, shift left, "\mathrm{CT}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{G}}) \arrow[d, shift left, "\mathrm{CT}"] \\ \mathrm{Dmod}(\mathrm{Bun}_T) \arrow[r, "\mathbb{L}_T"] \arrow[u, shift left, "\mathrm{Eis}"] & \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\check{T}}) \arrow[u, shift left, "\mathrm{Eis}"] \end{tikzcd}$$

$$\mathrm{Eis} : p_! \circ q^* \quad \mathrm{CT} : q_* \circ p^!$$

**Theorem 6.5.** [12, I.6.3] *For a finite index $I$,*

$$\mathrm{Sat}_G^I \simeq \mathrm{Rep}_\Lambda({}^L G^I)$$

**Proposition 6.6.** [8, 4.4] *For all finite index sets $I$, the followin diagram commutes*

$$\begin{tikzcd} \mathrm{Sat}_G^I \arrow[r, "\mathrm{CT}[\deg]"] \arrow[d, "F_G^I"] & \mathrm{Sat}_T^I \arrow[d, "F_T^I"] \\ \mathrm{Rep}_\Lambda({}^L G) \arrow[r, "\mathrm{res}_T^I"] & \mathrm{Rep}_\Lambda^I({}^L T) \end{tikzcd}$$

*where*

- CT *is the constant term functor.*
- $F_G^I, F_I^T$ *are due to Tannakian equivalence* [12, Thm 1.6.3].

**Proposition 6.7.** *Let $\lambda \in X_{\bullet,+}$. Let $x \to \mathrm{Div}^1$ be a geometric point.*

$$H_c^k({}_xS^\nu \cap \overline{{}_x\mathrm{Gr}^\lambda}, \mathcal{A}_\lambda)$$

*vanishes unless $k = \langle 2\rho, \nu \rangle$, in which case, it is isomorphic to $V^\lambda(\nu)^\vee$.*

*Proof.* Let us consider the following diagram

$$\begin{tikzcd} \mathrm{pt} & S^\lambda \arrow[l, "p"'] \arrow[r, hook, "q"] & \mathrm{Gr} \\ & S^\lambda \cap \overline{\mathrm{Gr}^\mu} \arrow[ul, "p'"] \arrow[u, hook] \arrow[r, hook, "q'"] & \overline{\mathrm{Gr}^\mu} \arrow[u, hook] \\ & & \mathrm{Gr}^\mu \arrow[u, hook] \end{tikzcd}$$

Let $\mathcal{S}_{V^\lambda}$ be the sheaf corresponding to highest weight representation $V^\lambda$, as 6.5. Then by applying 6.6,

$$\begin{aligned} H_c^k({}_xS^\nu \cap \overline{{}_x\mathrm{Gr}^\lambda}, \mathcal{A}_\lambda) &= (p')_!(q')^*(\mathcal{A}_\lambda) \\ &\simeq p_!q^*(\mathcal{S}_{V^\lambda}) \\ &= H_c^{-\langle 2\rho,\nu\rangle}(S^\nu, \mathcal{S}_{V^\lambda}) \\ &\simeq V^\lambda(\nu)^\vee \end{aligned}$$

□

6.1. **Properties of orbit intersection.** Note that the arguments in [23] do not really generalized in the mixed characteristic setting; as pointed in [31, p20], there is no Birkhoff decomposition, hence, it is unclear whether one can construct the "big open cell".

**Definition 6.8.** Let $J$ be the unipotent as [22, 2].

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^+G \arrow[d] \\ N \arrow[r] & G \end{tikzcd}$$

Define

$$J^\lambda := J \cap L^{<\lambda}G$$

**Proposition 6.9.** *Let $\lambda \in X_{\bullet,+}$ then for all $w \in W$*

$$wJ^\lambda w^{-1} \cap LN \xrightarrow{\simeq} S^{w\lambda} \cap \mathrm{Gr}^\lambda$$

**Proposition 6.10.** [4, 5.2], [29, 6.4] *Let $\lambda, \nu \in X_\bullet$ with $\lambda$ dominant, $x \to \mathrm{Div}^1$ be a geometric point.*

(1) *Nonemptiness.*

$$_xS^\nu \cap \overline{_x\mathrm{Gr}^\lambda} \neq \emptyset \Leftrightarrow \nu \in \Omega_\lambda$$

(2) *Dimension.*

$$_xS^\nu \cap {_x\mathrm{Gr}^{\leq\nu}}$$

*is equidimensional of rank $\langle \rho, \nu + \lambda \rangle$.*

(3) *Containment property.*

$$\bigsqcup_{\nu \in \Omega_\lambda} {_xS^\nu} \cap \overline{_x\mathrm{Gr}^\lambda} \xrightarrow{\simeq} {_x\mathrm{Gr}^{\leq\nu}}$$

*of underlying topological spaces.*

6.2. **Recollection on affine Grassmanian.** We will consider the $B_{\mathrm{dR}}^+$ affine Grassmanian. The local definition can be specialized from the global definition. We include the latter when we need to describe the Hecke action. Let $S \in \mathrm{Pftd}_{\mathbb{F}_q}$. Recall in 4, we could construct curves

$$\mathcal{Y}_S, Y_S := \mathcal{Y}_S \backslash V(\pi) \text{ and } X_S = Y_S/\varphi^{\mathbb{Z}}$$

We can define the following stacks of divisors on such curves.

**Definition 6.11.** We have the following small $v$-sheaves $\mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_q}, v)$

$$\mathrm{Div}^1_{\mathcal{Y}} := \mathrm{Spd}(\mathcal{O}_K)$$

$$\mathrm{Div}^1_X := \mathrm{Div}^1 := \mathrm{Spd} K/\varphi^{\mathbb{Z}}$$

where $\mathrm{Div}^1$ is the *mirror curve* [5] For a finite set $I$ with $|I| = d$, we will denote

$$\mathrm{Div}^I_{\mathcal{Y}} := (\mathrm{Div}^1_{\mathcal{Y}})^d$$

[5] Its $S$ points are the degree 1 Cartier divisors on $X_S$, where one has $\pi_1(\mathrm{Div}^1) = W_K$.

**Definition 6.12.** Let $I$ be a finite set.

$$\mathrm{Gr}^I_{G,\mathrm{Div}^1_{\mathcal{Y}}} \to \mathrm{Div}^I_{\mathcal{Y}}$$

$$\mathrm{Gr}^I_{G,\mathrm{Div}^1} \to \mathrm{Div}^I$$

be the *Beilinson-Drinfeld* Grassmanian [12, VI.1.8]. This is a small $v$-sheaf. Unless stated otherwise, will omit the $\mathrm{Div}^I$. For $S \to \mathrm{Div}^d_{\mathcal{Y}}$ we denote

$$\mathrm{Gr}_{G,S} := \mathrm{Gr}_G \times_{\mathrm{Div}^d_{\mathcal{Y}}} S$$

**Definition 6.13.** [29, 6.1]. Let $\mathrm{Gr}_G^{B^+_{\mathrm{dR}}} \in \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_p}, v)_{/\mathrm{Spd}\mathbb{Q}_p}$ be the small $v$-sheaf over $\mathrm{Spd}\mathbb{Q}_p$ such that for each $S := \mathrm{Spa}(R, R^+) \to \mathrm{Spa}\,\mathbb{Q}_p$

$$\mathrm{Gr}_G(S) = \{(\mathcal{E}, \beta)\} \,/ \sim$$

- The structure map $S \to \mathrm{Spd}\mathbb{Q}_p$ is the data of an until $S^\# := \mathrm{Spa}(R^\#, R^{\#+})$ of characteristic 0.
- $\mathcal{E} \in G\mathrm{Tors}(S \times \mathrm{Spa}\,\mathbb{Q}_p)$
- $\beta$ is trivialization on $\mathcal{P}\Big|_{S\times \mathrm{Spa}\,\mathbb{Q}_p \setminus S^\#}$

As in the classical setting, to define the Schubert stratification, one uses the Cartan decomposition [27, 19.2.1]

$$\mathrm{Gr}_G^{B^+_{\mathrm{dR}}}(C) \simeq \bigsqcup_{\mu \in X_{\bullet,+}} G(B^+_{\mathrm{dR}}(C))\xi^\mu G(B^+_{\mathrm{dR}}(C))$$

where $C$ is an algebraically closed field containing $\mathbb{Q}_p$. For each cocharacter $\mu \in X_{\bullet,+}$ with field of definition $E := E(G, \{\mu\})$ [6], We let $\mathrm{Gr}_{G,\mathrm{Spd}E}$ be the base change of the functor.

- $\mathrm{Gr}_{\leq\mu}$ is a spatial diamond, proper over $\mathrm{Spd}E$. [27, 19.2.4] , [12, VI.1.2]

Our constructions would live over these stacks, but we shall omit them for convenience.

6.3. **At the special fiber: In the context of Witt vector.** Once base changed to the special fiber we have the Witt vector Grassmanian $\mathrm{Gr}^{\mathrm{Witt}}_{G,k}$, [5], where $k = \bar{\mathbb{F}}_q$. Following notation as [31, A] we will use the perfection function

$$\mathrm{Aff}^{\mathrm{perf}}_k \hookrightarrow \mathrm{Aff}_k \hookrightarrow \mathrm{Sch}_k \hookrightarrow \mathrm{AlgSpc}_k$$

[6] Fill in example.

One can extend the construction on rings:

$$\begin{tikzcd} \mathrm{Alg}_k^{\mathrm{perf}} \arrow[r, hook] & \mathrm{Alg}_k \arrow[l, bend right, "\varinjlim_{x\mapsto x^p}"'] \end{tikzcd}$$

which extends to an adjunction in the level of algebraic spaces. For any affine scheme $X \in \mathrm{Aff}_{\mathcal{O}_K}$, we define the $p$-adic loop space functor

$$L : \mathrm{Aff}_{\mathcal{O}_K} \to \mathrm{Shv}(\mathrm{Alg}_k^{\mathrm{perf}}, \text{ét})$$

$$LX : R \mapsto X\left(W_{\mathcal{O}}(R)[\frac{1}{p}]\right)$$

as a prestack on $\mathrm{Alg}_k^{\mathrm{perf}}$, of perfect $k$-algebras, where $k$ is a field of characteristic $p$. Under this language , we can also describe the $\mathbb{G}_m^{p^{-\infty}}$ action on $L^{\geq 0}G_k$ as

$$\mathbb{G}_m^{p^{-\infty}} \to L^{\geq 0}\mathbb{G}_m \xrightarrow{2\check{\rho}} L^{\geq 0}T \subset L^{\geq 0}G$$

As $N\backslash G$ is quasi-affine $\mathrm{Gr}_N \hookrightarrow \mathrm{Gr}_G$ is a locally closed embedding [31, 1.20].

**Proposition 6.14.** $\left(\mathrm{Gr}^{\mathrm{Witt}}_{G,\mathrm{Spec}\,k}\right)^{\diamondsuit} \simeq \mathrm{Gr}_{G,\mathrm{Spd}k/\mathrm{Div}^1_{\mathcal{Y}}}$

*Proof.* By 2, A map $S = \mathrm{Spa}(R, R^+) \to \mathrm{Spd}k$ is equivalent to a map $S^\sharp \to \mathrm{Spa}\,k$, or equivalent a $k$ algebra $R$. □

**Theorem 6.15.** [4, 5.2].

(1) *Nonemptiness.*

$$S^\mu \cap \overline{\mathrm{Gr}^\lambda} \neq \emptyset \Leftrightarrow \mu \in \Omega_\lambda$$

(2) *Dimension.* [12, VI.3.8]

$$S^\lambda \cap \mathrm{Gr}_G^{\mathrm{Witt},\leq\mu}$$

*is equidimensional of dimension* $\langle \rho, \mu + \lambda \rangle$.

(3) *Relative property.*

*Proof. The proof in text is incomprehensible.* □

6.4. **The IC sheaves.** Let $(X, \mathcal{T}) \in \mathrm{StrSpc}$. [3, 4.3.1], then $\mathrm{Perv}_{\mathcal{T}}(X, e)$ has finite length and we have the classification of simple objects. If char $e = 0$, then the category is semisimple. As remarked in [12, I.6], there is no general theory of perverse sheaves, however, one can resort to use of relative perversity. Following previous notations, we fix a small $v$-stack $S \to \mathrm{Div}^d_{\mathcal{Y}}$. Recall, the Hecke stack, **??**.

**Definition 6.16.** Let

$$\begin{tikzcd} \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \arrow[d] \\ \mathrm{Hck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \arrow[r, hook] & \mathrm{Hck}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \end{tikzcd}$$

be the inclusion of open cells, [12, IV.7.5], and denote

$$\mathcal{A}_\mu := j_{\mu !}\Lambda[d_\mu]$$

as the IC sheaves.[7]

Recall one has the commutative diagram [12, p216]

$$\begin{tikzcd} \mathrm{Perv}_{L^{\geq 0}G}(\mathrm{Gr}^{\mathrm{Witt}}_{G,k}, \Lambda) \arrow[r] \arrow[d, hook] & \mathrm{Perv}(\mathrm{Hck}_{G,\mathrm{Spd}d/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda) \arrow[d, hook] \\ D_{\acute{e}t}(\mathrm{Gr}^{\mathrm{Witt}}_{G,k}, \Lambda) \arrow[r] & D_{\acute{e}t}(\mathrm{Hck}_{G,\mathrm{Spd}k/\mathrm{Div}^1_{\mathcal{Y}}}, \Lambda)^{bd} \end{tikzcd}$$

**Definition 6.17.** Let

$$\mathrm{Sat}^I_G(\Lambda) \hookrightarrow D_{\acute{e}t}(\mathrm{Hck}^I_G, \Lambda)^{\mathrm{bd}}$$

be a subcategory of sheaves that are perverse, flat, and ULA over $\mathrm{Div}^I$.

**Theorem 6.18.** [12, I.6.3] *For a finite index $I$, we have the tensor equivalence*

$$(\mathrm{Sat}^I_G(\Lambda), \star) \simeq (\mathrm{Rep}_\Lambda({}^LG^I, \otimes)$$

*further satisfying the compatibility* [8, 4.4]

## 7. Cohomological Computation

Recall the construction of $h$, 5.2.

**Theorem 7.1.** [23, 3.1] *For $\lambda \in X_{\bullet,+}$*

$$R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^*\mathcal{L}) = \begin{cases} & \nu = \lambda \\ & \nu \neq \lambda \end{cases}$$

*Proof.* The case when $\nu = \lambda$ follows from the fact that $h$ is trivial on $S_\lambda \cap \bar{\mathrm{Gr}}^\lambda$, so that $h^*\mathcal{L}$ is constant, and we are reduced the case in Prop. 6.7. □

### 7.1. The case when $\nu \neq \lambda$.

**Proposition 7.2.** *For $\nu \neq \lambda$.*

[7]The typical analysis of such sheaves on Hck stack pullback further to the Demazure resolution.

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