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On the Parshin-Beilinson Adeles for Schemes

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In algebraic number theory the notion of adeles has long proved to be useful. It is natural to look for a generalization to algebraic geometry. PARSHIN suggested in [11] a definition of adeles (or distributions) for smooth, proper surfaces over a perfect field. BEILINSON gave in his paper "Residues and Adeles" [3] a definition working for all noetherian schemes. No regularity conditions are necessary, not even seperatedness.

His idea runs as follows: Instead of a single ring, a whole complex of adeles $\mathbb{A}^{\bullet}(X, \mathscr{F})$ is associated to the scheme and a quasicoherent sheaf. In fact the dependence on $\mathscr{F}$ is functorial. So they are rather adeles with coefficients in a sheaf. The groups $\mathbb{A}^n(X, \mathscr{F})$ of this complex are certain subgroups of "products of local factors" $\mathbb{A}_\Delta(\mathscr{F})$. More precisely: In degree n consider $n+1$-tupels $(P_0, \ldots, P_n)$ of points of the scheme ordered by specialization. To any such "simplex", a local factor is associated: The stalk in P_n of the sheaf $\mathscr{F}$ is completed with respect to the prime ideal corresponding to P_n. The resulting module is localized at the prime ideal corresponding to P_{n-1}, completed again etc. The group in degree n of BEILINSON's complex is a subgroup of the direct product of the local factors of all n-dimensional simplices. The condition which characterizes the elements of this subgroup is similar to the one in the formation of a restricted topolocical product.

In the special case of the spectrum of a Dedekind-ring and its structure sheaf, the essential part of the adele complex are the usual finite adeles.

What makes the whole construction useful, is the following theorem:

Theorem. (BEILINSON) *The cohomology of the complex $\mathbb{A}^{\bullet}(X, \mathscr{F})$ of adeles on a noetherian scheme is isomorphic to the cohomology of the quasicoherent sheaf $\mathscr{F}$.*

Hence BEILINSON's construction allows to calculate sheaf cohomology in terms of explicit cocycles and cobounderies. In the case of a proper scheme of finite type over a perfect ground field and the dualizing sheaf ω, BEILINSON uses this to define a residue map from $\mathbb{A}^n(X, \omega)$ into the ground field. It factorizes over the Grothendieck trace map. All in all he generalizes TATE's construction for residues of differentials on curves to the multidimensional case.

Another use of classical adeles or rather ideles is in class field theory. PARSHIN generalized local class field theory to a certain higher dimensional case. His starting point are BEILINSON's local adele factors. In the case he is interested in, they are n-dimensional local fields. He discovered that one may establish a local class field theory for them using MILNOR's K-groups instead of just taking the multiplicative group of the field. KATO and SAITO were able to discuss the global case in [7].

More recently [1] made use of BEILINSON's paper to prove an abstract version of the residue theorem.

We want to stress that at this stage only a generalization of the finite adeles is found. It is not clear what one should take at infinity, or in fact even what the infinite "places" should be.

Unfortunately BEILINSON's article [3] is very short. The definitions he gives are very concise, the proofs of his theorems are omitted. In this paper we want to make part of BEILINSON's work more accessible. The proofs turned out to be rather lengthy and technical. Although we use elementary methods from commutative algebra, they are by no means obvious.

We included a section on what we call reduced adeles. They are the part of the complex which really carries information. PARSHIN uses them implicitly.

The construction of rational (as opposed to BEILINSON's analytic) adeles is new. This variation of BEILINSON's construction leaves out the completions in the local factors. For the complex of rational adeles the same theorem on cohomology holds as above.

For a more detailed version, we refer to [6].

Contents

Notations

The following notations will be adopted throughout the whole article.

Let R be a noetherian ring, $\mathbf{p}$, $\mathbf{q}$ prime ideals of R, $\mathbf{m}$ a maximal ideal, M an R-module.

Let $S_{\mathbf{p}} = R \backslash \mathbf{p}$. Write $M_{\mathbf{p}}$ or $S_{\mathbf{p}}^{-1}M$ for the localisation of M at $S_{\mathbf{p}}$.

For a submodule N of M set $S_{\mathbf{p}}(N) = \mathrm{Ker}(M \longrightarrow S_{\mathbf{p}}^{-1}(M/N))$.

For an ideal $\mathbf{a}$ of R set $C_{\mathbf{a}}M = \varprojlim_{l \in \mathbb{N}} M/\mathbf{a}^l M$.

Let X be a noetherian scheme with structure sheaf $\mathcal{O}$. Sheafs are always quasicoherent, the only exception being the sheaf of adeles defined in the text.

Let $\underline{\mathbf{QS}}$ be the category of quasicoherent sheafs on X, $\underline{\mathbf{CS}}$ the category of coherent sheafs, $\underline{\mathbf{Ab}}$ the category of abelian groups.

Let $\mathcal{O}_P$ be the stalk of $\mathcal{O}$ in the point P with maximal ideal $\mathbf{m}_P$. For the canonical morphism $f : \mathrm{Spec}(\mathcal{O}_P) \longrightarrow X$ and an $\mathcal{O}_P$-module N set $[N]_P = f_*\tilde{N}$.

1 Preliminaries

1.1 Commutative Algebra

All R-modules of this section are finitely generated. Consider a submodule N of M. The radical of N in M is

$$\mathrm{rad}(N, M) = \{a \in R \mid \text{There is } n \in \mathbb{N} \text{ with } a^n M \subset N\}.$$

The annihilator of M is

$$\mathrm{ann}(M) = \{a \in R \mid aM = 0\}.$$

N is called primary in M if $N \neq M$ and $ax \in N$ implies: $x \in N$ or $a \in \mathrm{rad}(N, M)$. This is a generalization of the notion of primary ideals to modules. The theory of primary modules can be found in [15] Ch.IV, appendix. The radical of a primary submodule is a prime ideal. Any submodule of a finitely generated R-module can be represented as an intersection of primary submodules. Certain uniqueness properties hold as they do for ideals. If N is a primary submodule of M with radical $\mathbf{p}$, then there is $l \in \mathbb{N}$ with $S_{\mathbf{p}}(\mathbf{p}^l M) \subset N$. Thus for any (finitely generated) module M there are prime ideals $\mathbf{p}_1, \ldots, \mathbf{p}_n$ of R and $l \in \mathbb{N}$ such that

$$\bigcap_{i=1}^{n} S_{\mathbf{p}_i}(\mathbf{p}_i^l M) = 0.$$

Local rings are Zariski-rings with respect to their maximal ideal, i.e. all ideals are closed in the $\mathbf{m}$-adic topology. $C_{\mathbf{m}}R$ is a faithfully flat R-algebra. In the proofs of the following lemmas, properties of Zariski-rings and their completions will be used. These can be found in [15] Ch.VIII.

Lemma 1.1.1. *Let R be local. Then*

$$C_{\mathbf{m}} S_{\mathbf{p}}(\mathbf{p}^k M) \supset \mathrm{Ker}\left(C_{\mathbf{m}} M \longrightarrow S_{\mathbf{p}}^{-1} C_{\mathbf{m}}(M/\mathbf{p}^k M)\right)$$

Proof. Consider $x \in C_{\mathbf{m}} M$ with $x = 0$ in $S_{\mathbf{p}}^{-1} C_{\mathbf{m}}(M/\mathbf{p}^k M)$. There is $f \in R \backslash \mathbf{p}$ with $fx \in \mathbf{p}^k C_{\mathbf{m}} M$. Thus there is a sequence $(x_l)_{l \in \mathbb{N}}$ with $x_l \in M$ and

$$fx \equiv fx_l \bmod f\mathbf{m}^l C_{\mathbf{m}} M \Rightarrow fx_l \in \left(\mathbf{p}^k C_{\mathbf{m}} M + f\mathbf{m}^l C_{\mathbf{m}} M\right) \cap M = \mathbf{p}^k M + f\mathbf{m}^l M.$$

Hence there is $c_l \in M$ with $fc_l \in \mathbf{p}^k M$, $c_l = x_l \bmod \mathbf{m}^l M$.
But then $x = \lim x_l = \lim c_l \in C_{\mathbf{m}} S_{\mathbf{p}}(\mathbf{p}^k M)$. □

Let $\mathbf{p} \subset \mathbf{q}$ be prime ideals of R. $M \longrightarrow C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M$ induces a map $C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M \longrightarrow C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M$. On the other hand, we have the canonical map $C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M \longrightarrow C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M$. The same letter will be used for an element and its image under such a map.

Lemma 1.1.2. *Let R be local. Given $y \in C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M$ and $x \in C_{\mathbf{m}}M$ with*

$$y = x \quad in \quad C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{m}}M\,,$$

there are $y_l \in M/\mathbf{p}^l M$ for $l \in \mathbb{N}$ with $y = (y_l)_{l\in\mathbb{N}}$ in $C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M$.

Proof. Let $N_l = M/\mathbf{p}^l M$.Then $C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M = \varprojlim_{l} S_{\mathbf{p}}^{-1}R \otimes_R N_l$ implies

$$y = \left(\frac{a_l}{f_l}\right)_{l\in\mathbb{N}} \quad \text{with} \quad f_l \in S_{\mathbf{p}}\,,\ a_l \in N_l$$

On the other hand, $x = \sum_{i=1}^{n} r_i x_i$ with $r_i \in C_{\mathbf{m}}R$ and $x_i \in M$. By assumption

$$\sum_{i=1}^{n} r_i x_i = \left(\frac{a_l}{f_l}\right)_{l\in\mathbb{N}} \quad \text{in} \quad C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{m}}M\,.$$

As $C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{m}}M = \varprojlim S_{\mathbf{p}}^{-1}C_{\mathbf{m}}N_l$, there is $u_l \in S_{\mathbf{p}}$ with

$$u_l(f_l \sum_{i=1}^{n} r_i x_i - a_l) = 0 \quad \text{in} \quad C_{\mathbf{m}}N_l\,. \tag{1}$$

W.l.o.g. $u_l = 1$. For the $C_{\mathbf{m}}R$-submodules generated in $C_{\mathbf{m}}N_l$, this means

$$C_{\mathbf{m}}R(f_l x_1, \ldots, f_l x_n) \supset C_{\mathbf{m}}Ra_l\,. \tag{2}$$

As R is a Zariski-ring, we get by intersecting with N_l

$$N_l \supset R/\mathbf{p}^l(f_l x_1, \ldots, f_l x_n) \supset Ra_l\,. \tag{3}$$

Now $a_l \in R/\mathbf{p}^l(f_l x_1, \ldots, f_l x_n)$, i.e. there is $b_{il} \in R/\mathbf{p}^l$ with

$$a_l = f_l \sum_{i=1}^{n} b_{il} x_i \quad \text{in} \quad N_l\,. \tag{4}$$

Set $y_l = \sum_{i=1}^{n} b_{il} x_i \in N_l$. By construction $y_l = \frac{a_l}{f_l}$ in $S_{\mathbf{p}}^{-1}N_l$. □

Lemma 1.1.3. *Let R be local. Given elements $x_{\mathbf{p}} \in C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M$ for $\mathbf{p} \in \mathrm{Spec}R$ such that*

$$x_{\mathbf{p}} = x_{\mathbf{m}} \quad in \quad C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{m}}M\,, \tag{*}$$

there is $x \in M$ with $x = x_{\mathbf{p}}$ in $C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M$ for all $\mathbf{p} \in \mathrm{Spec}R$.

Proof. There are $\mathbf{p}_1, \ldots, \mathbf{p}_n \in \operatorname{Spec}R$, $k \in \mathbb{N}$ with $0 = \bigcap_{i=1}^{n} S_{\mathbf{p}_i}(\mathbf{p}_i^k M)$.

By lemma 1.1.2 $x_{\mathbf{p}_i} = (x_{il})_{l \in \mathbb{N}}$ with $x_{il} \in M/\mathbf{p}_i^l M$. Choose $b_i \in M$ such that $b_i = x_{ik}$ in $M/\mathbf{p}_i^k M$. By assumption

$$b_i - x_{\mathbf{m}} \in \operatorname{Ker}(C_{\mathbf{m}}M \longrightarrow S_{\mathbf{p}_i}^{-1} C_{\mathbf{m}}(M/\mathbf{p}_i^k M)) \overset{1.1.1}{\subset} C_{\mathbf{m}} S_{\mathbf{p}_i}(\mathbf{p}_i^k M)\,.$$

All modules are finitely generated over a Zariski-ring. We construct inductively $c_j \in M$ with

$$\begin{aligned} c_j &\equiv b_i \bmod S_{\mathbf{p}_i}(\mathbf{p}_i^k M) \quad \text{for} \quad i \le j \quad \text{and} \\ c_j &\equiv x_{\mathbf{m}} \bmod C_{\mathbf{m}} \left(\bigcap_{i=1}^{j} S_{\mathbf{p}_i}(\mathbf{p}_i^k M) \right). \end{aligned}$$

Set $x := c_n \in M$. We have

$$x \equiv x_{\mathbf{m}} \bmod C_{\mathbf{m}} \left(\bigcap_{i=1}^{n} S_{\mathbf{p}_i}(\mathbf{p}_i^k M) \right) = 0\,,$$

i.e. $x = x_{\mathbf{m}}$. Now consider an arbitrary prime ideal $\mathbf{p}$. By 1.1.2 $x_{\mathbf{p}} = (x_{\mathbf{p}l})_{l \in \mathbb{N}}$ with $x_{\mathbf{p}l} \in M/\mathbf{p}^l M$. Now $x = x_{\mathbf{p}l}$ in $S_{\mathbf{p}}^{-1} C_{\mathbf{m}}(M/\mathbf{p}^l M)$ (*) implies $x = x_{\mathbf{p}l}$ in $S_{\mathbf{p}}^{-1}(M/\mathbf{p}^l M)$. □

Lemma 1.1.4. *Given elements $x_{\mathbf{p}} \in S_{\mathbf{p}}^{-1}M$ for $\mathbf{p} \in \operatorname{Spec}R$ such that*

$$x_{\mathbf{p}} = x_{\mathbf{q}} \quad in\ S_{\mathbf{q}}^{-1}M \ if\ \mathbf{q} \subset \mathbf{p}\,, \tag{**}$$

there is $x \in M$ with $x = x_{\mathbf{p}}$ in $S_{\mathbf{p}}^{-1}M$ for all $\mathbf{p} \in \operatorname{Spec}R$.

Proof. Suppose 0 is a primary submodule of M, $\mathbf{n} = \operatorname{rad}(0, M)$ prime, $\mathbf{a} = \operatorname{ann}(M)$. M is an $R/\mathbf{a}$-module in a natural way, $\mathbf{n}/\mathbf{a} = \operatorname{rad}(0, M)$ its nilradical.

Set $x = x_{\mathbf{n}}$. The map $S_{\mathbf{p}}^{-1}M \longrightarrow S_{\mathbf{n}}^{-1}M$ is injective for all $\mathbf{p} \supset \mathbf{n}$. By (**) $x \in S_{\mathbf{p}}^{-1}M$ for all $\mathbf{p} \supset \mathbf{n}$. From [4] II.3 Thm 1 Cor 1 for the ring $R/\mathbf{a}$, we get $x \in M$. By construction $x = x_{\mathbf{p}}$ in $S_{\mathbf{p}}^{-1}M$ for all $\mathbf{p} \supset \mathbf{n}$. As $S_{\mathbf{p}}^{-1}M = 0$ for $\mathbf{p} \not\supset \mathbf{n}$, we have $x = x_{\mathbf{p}} = 0$ in $S_{\mathbf{p}}^{-1}M$ in this case.

Now consider arbitrary M. There is a primary decomposition $0 = N_1 \cap \ldots \cap N_k$ in M. From the first case for M/N_i we get elements $x_i \in M$ with $x_i = x_{\mathbf{p}}$ in $S_{\mathbf{p}}^{-1}(M/N_i)$ for all prime ideals $\mathbf{p}$ of R. By iteration we construct $y_j \in M$ with

$$y_j = x_{\mathbf{p}} \bmod S_{\mathbf{p}}^{-1}(N_1 \cap \ldots \cap N_j) \quad \text{for all} \quad \mathbf{p} \in \operatorname{Spec}R\,.$$

Finally set $x = y_k$. □

1.2 Construction of Functors

For $\mathscr{F} \in \mathrm{Ob}(\underline{\mathbf{QS}})$ and $l' \leq l$ there is the canonical projection between tl $\mathscr{O}_P$-modules $\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P$ and $\mathscr{F}_P/\mathbf{m}_P^{l'}\mathscr{F}_P$. It induces a surjective morphis of sheafs

$$\left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P \longrightarrow \left[\mathscr{F}_P/\mathbf{m}_P^{l'}\mathscr{F}_P\right]_P .$$

These sheafs form a projective system for $l \in \mathbb{N}$.

$\left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P$ can be considered as direct image of a sheaf on the schen $\mathrm{Spec}\mathscr{O}_P/\mathbf{m}_P^l\mathscr{O}_P$. It is a sky-scraper sheaf. Note that the functor $[\cdot]_P$ is exact i this case.

Lemma 1.2.1. *Let $\Phi : \underline{\mathbf{QS}} \longrightarrow \underline{\mathbf{Ab}}$ be an exact additive functor and P a poi of X. Define $C_P\Phi : \underline{\mathbf{CS}} \longrightarrow \underline{\mathbf{Ab}}$ by*

$$C_P\Phi(\mathscr{F}) = \varprojlim_{l \in \mathbb{N}} \Phi\left(\left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P\right)$$

This functor is exact and additive.

Proof. Functoriality and additivity are obvious. Exactness is proved in th same way one proves exactness of completion in the $\mathbf{a}$-adic topology in th category of finitely generated modules over a noetherian ring ([2] Prop. 10.12 The main ingredient is the lemma of Artin-Rees. [

Lemma 1.2.2. *On a noetherian scheme any short exact sequence of quasicc herent sheaves is direct limit of short exact sequences of coherent sheaves. Fc $\phi : \mathscr{F} \longrightarrow \mathscr{G} \in \mathrm{Mor}(\underline{\mathbf{QS}})$, there are $\phi_i : \mathscr{F}_i \longrightarrow \mathscr{G}_i \in \mathrm{Mor}(\underline{\mathbf{CS}})$ with $\varinjlim \phi_i =$ ς $\mathscr{F}_i \subset \mathscr{F}$, $\mathscr{G}_i \subset \mathscr{G}$.*

Proof. [5] II Ex.5.15d. [

Lemma 1.2.3. *Let $\Psi : \underline{\mathbf{CS}} \longrightarrow \underline{\mathbf{Ab}}$ be an exact additive functor. Then Ψ con mutes with direct limits.*

Proof. Consider a direct system $\{\mathscr{F}_i : i \in I ; \phi_{ij} : \mathscr{F}_i \to \mathscr{F}_j (i \leq j)\}$ wit $\varinjlim \mathscr{F}_i = 0$. Let $x \in \varinjlim \Psi(\mathscr{F}_i)$ be represented by $x_i \in \Psi(\mathscr{F}_i)$. There i some $j \in I$ with $\phi_{ij} \equiv 0$. Ψ is additive $\Rightarrow \Psi(\phi_{ij}) \equiv 0 \Rightarrow \Psi(\phi_{ij})(x_i) = 0$ i $\Psi(\mathscr{F}_j) \Rightarrow x = 0$ in $\varinjlim \Psi(\mathscr{F}_j)$. The general case follows easily from this. [

Lemma 1.2.4. *An exact additive functor $\Psi : \underline{\mathbf{CS}} \longrightarrow \underline{\mathbf{Ab}}$ can be uniquely ex tended to a functor $\Psi' : \underline{\mathbf{QS}} \longrightarrow \underline{\mathbf{Ab}}$ which commutes with direct limits. Thi new functor is exact as well.*

Proof. Let $\mathscr{F} \in \mathrm{Ob}(\underline{\mathbf{QS}})$. By lemma 1.2.2 $\mathscr{F} = \varinjlim \mathscr{F}_i$ with $\mathscr{F}_i \in \mathrm{Ob}(\underline{\mathbf{CS}})$. W define

$$\Psi'(\mathscr{F}) = \varinjlim \Psi(\mathscr{F}_i) .$$

We have $\Psi'(\mathscr{F}) = \Psi(\mathscr{F})$ for $\mathscr{F} \in \mathrm{Ob}(\underline{\mathbf{CS}})$ by lemma 1.2.3.

For $\phi \in \mathrm{Mor}(\underline{\mathbf{QS}})$, lemma 1.2.2 gives $\phi = \varinjlim \phi_i$ where $\phi_i \in \mathrm{Mor}(\underline{\mathbf{CS}})$. Define

$$\Psi'(\phi) = \varinjlim \Psi(\phi_i)\,.$$

We have $\Psi'(\phi) = \Psi(\phi)$ for $\phi \in \mathrm{Mor}(\underline{\mathbf{CS}})$ by lemma 1.2.3. This is obviously the only possible definition, so uniqueness is clear. Using lemma 1.2.3, we can show that it is well defined as well. The properties of the functor are clear. □

Lemma 1.2.5. *Let $\Psi_1 : \underline{\mathbf{CS}} \longrightarrow \underline{\mathbf{Ab}}$ and $\Psi_2 : \underline{\mathbf{CS}} \longrightarrow \underline{\mathbf{Ab}}$ be exact additive functors, Ψ'_1, Ψ'_2 their extensions to $\underline{\mathbf{QS}}$. Further let $H : \Psi_1 \longrightarrow \Psi_2$ be a transformation of functors. It can be uniquely extended to a transformation H' of the functors Ψ'_1 and Ψ'_2. It commutes with direct limits.*

Proof. For $\mathscr{F} \in \mathrm{Ob}(\underline{\mathbf{QS}})$, we have $\mathscr{F} = \varinjlim \mathscr{F}_i$ with $\mathscr{F}_i \in \mathrm{Ob}(\underline{\mathbf{CS}})$. Define

$$H'_{\mathscr{F}} : \Psi'_1(\mathscr{F}) \longrightarrow \Psi'_2(\mathscr{F}) \quad \text{by} \quad H'_{\mathscr{F}} = \varinjlim H_{\mathscr{F}_i}\,.$$ □

1.3 The Simplicial Set S(X)

Definition 1.3.1. Let X be a noetherian scheme, $P(X)$ the set of points of X. Consider $P, Q \in P(X)$. Define $P \geq Q$ if $Q \in \overline{\{P\}}$. $\geq$ is a half ordering on P(X). Let $S(X)$ be the simplicial set induced by $(P(X), \geq)$, i.e.

$$S(X)_n = \{(P_0, \ldots, P_n) \mid P_i \in P(X); P_i \geq P_{i+1}\}$$

with the usual boundary maps δ_i^n and degeneracy maps σ_i^n for $n \in \mathbb{N}$, $0 \leq i \leq n$.

Remark. The elements of $S(X)_n$ are sequences of irreducible subschemes of X ordered by inclusion. In the affine case, they are sequences of prime ideals ordered by inclusion. P_0 belongs to the smallest prime ideal, P_n to the largest.

Notations. For $\Delta = (P_0, \ldots, P_n)$, $K \subset S(X)_n$, $P \in P(X)$ set

$$\begin{aligned} (P, \Delta) &= (P, P_0 \ldots, P_n) \quad \text{and} \quad PK = \{(P, \Delta) \mid \Delta \in K\} \\ \hat{P}K &= \{\Delta \in S(X)_{n-1} \mid (P, \Delta) \in K\} \\ \bar{P}K &= \{\Delta \in K \mid \Delta = (P, \Delta') \text{ for some } \Delta' \in S(X)_{n-1}\} \\ \mathrm{supp}(K) &= \{P \in P(X) \mid P \in \Delta \text{ for some } \Delta \in K\}\,. \end{aligned}$$

Lemma 1.3.2. *Let U and U' be open subsets of X. Then $S(U)_n \cup S(U')_n = S(U \cup U')_n$ and $S(U)_n \cap S(U')_n = S(U \cap U')_n$.*

Proof. $(P_0, \ldots, P_n) \in S(U)_n$ if and only if $P_n \in S(U)_n$. □

Definition 1.3.3. Let $S(X)_n^{(red)}$ be the set of non degenerate n-dimensional simplices.

A simplicial set G is a simplicial group if all G_n are groups and the maps δ σ_i^n are homomorphisms of groups. We shall need the dual concept.

Definition 1.3.4. A cosimplicial group G is a sequence $\{G^n\}_{n\in\mathbb{N}_0}$ of abelia groups together with homomorphisms

$$\begin{array}{llll} d_i^n : G^{n-1} \longrightarrow G^n & \text{for} & 0 \le i \le n & \text{(boundary maps)} \\ s_i^n : G^{n+1} \longrightarrow G^n & \text{for} & 0 \le i \le n & \text{(degeneracy maps)} \end{array}$$

such that

$$\begin{array}{llllll} \text{(i)} & d_j d_i = d_i d_{j-1} & i < j & \text{(iii)} & s_j d_i = d_i s_{j-1} & i < j \\ \text{(ii)} & s_j s_i = s_i s_{j+1} & i \le j & & s_j d_j = \mathrm{id} = s_j d_{j+1} & \\ & & & & s_j d_i = d_{i-1} s_j & i > j+1 \end{array}$$

The elements of G^n are called n-cosimplices.

Proposition 1.3.5. *Define $d^n : G^{n-1} \longrightarrow G^n$ by $d^n = \sum_{j=0}^{n} (-1)^j d_j^n$. This makes* C *into a (cohomological) complex of abelian groups.*

Proof. Direct calculation. [

Definition 1.3.6. The cohomology groups of a cosimplicial group G are th cohomology groups of its associated complex G^*.

2 Definition of Beilinson's Adeles

2.1 The Groups of Adeles

Proposition 2.1.1. *Let $S(X)$ be the simplicial set associated to the noetheria scheme X. Then there exist for $n \in \mathbb{N}_0$, $K \subset S(X)_n$ functors*

$$A(K,\cdot) : \underline{\mathbf{QS}} \longrightarrow \underline{\mathbf{Ab}}$$

uniquely determined by the properties a), b), c), *which are additive and exact.*

a) *$A(K,\cdot)$ commutes with direct limits.*

b) *For $n = 0$, $\mathscr{F} \in \mathrm{Ob}(\underline{\mathbf{CS}})$*

$$A(K,\mathscr{F}) = \prod_{P\in K} \lim_{\substack{\longleftarrow \\ l}} \mathscr{F}_P / \mathbf{m}_P^l \mathscr{F}_P .$$

c) *For $n > 0$, $\mathscr{F} \in Ob(\underline{\mathbf{CS}})$*

$$A(K,\mathscr{F}) = \prod_{P\in P(X)} C_P A\left(\hat{P}K,\cdot\right)(\mathscr{F}) .$$

Proof. Suppose $n = 0$. By lemma 1.2.1, $\varprojlim_{l} \mathcal{F}_P/\mathbf{m}_P^l\mathcal{F}_P$ is a an exact additive functor for all $P \in P(X)$. So $A(K, \cdot)$ is uniquely defined, additive and exact on $\underline{\mathbf{CS}}$. Using lemma 1.2.4, $A(K, \cdot)$ can be uniquely extended to an exact additive functor from $\underline{\mathbf{QS}}$ to $\underline{\mathbf{Ab}}$ commuting with direct limits.

Now suppose $n > 0$, $K \subset S(X)_n$. By inductive hypothesis, $A\left(\hat{P}K, \cdot\right)$ is uniquely defined for all $P \in P(X)$ such that a) and b) resp. c) hold, and exact. By lemma 1.2.1, $C_P\left(A\left(\hat{P}K, \cdot\right)\right)$ is a uniquely defined exact functor $\underline{\mathbf{CS}} \longrightarrow \underline{\mathbf{Ab}}$. $\prod$ is exact on $\underline{\mathbf{Ab}}$ as well. Thus $A(K, \cdot)$ is uniquely defined on $\underline{\mathbf{CS}}$ by c), and exact. Lemma 1.2.4 gives the proposition in the general case. □

Definition 2.1.2. Let the group of n-dimensional adeles of X with coefficients in $\mathcal{F}$ be

$$\mathbb{A}^n(X, \mathcal{F}) = A(S(X)_n, \mathcal{F}) .$$

Let the local factor of $\mathbb{A}^n(X, \mathcal{F})$ in $\Delta \in S(X)_n$ be

$$\mathbb{A}_\Delta(\mathcal{F}) = A\left(\{\Delta\}, \mathcal{F}\right) .$$

Lemma 2.1.3. *For $K \subset S(X)_n$ and $\mathcal{F} \in \underline{\mathbf{CS}}$*

$$A(K, \mathcal{F}) = \prod_{P \in P(X) \text{ with } \bar{P}K \neq \emptyset} A\left(\bar{P}K, \mathcal{F}\right) .$$

Proof. This follows from the definition if one takes into account that $A\left(\emptyset, \mathcal{F}\right) = 0$. □

Proposition 2.1.4. *For $n \in \mathbb{N}_0$, $K \subset S(X)_n$, $\mathcal{F} \in \mathrm{Ob}(\underline{\mathbf{QS}})$*

$$A(K, \mathcal{F}) \subset \prod_{\Delta \in K} \mathbb{A}_\Delta(\mathcal{F}) .$$

The inclusion is a transformation of functors.

Proof. By definition with lemma 1.2.5 and left exactness of $\varprojlim$. □

Proposition 2.1.5. *Let K, L, $M \subset S(X)_n$ such that $K \cup M = L$, $K \cap M = \emptyset$. Then there are natural transformations $\imath$ and π of functors*

$$\imath(\cdot) : A(K, \cdot) \longrightarrow A(L, \cdot)$$
$$\pi(\cdot) : A(L, \cdot) \longrightarrow A(M, \cdot)$$

such that the following diagram is commutative and has split-exact lines for all $\mathcal{F} \in \mathrm{Ob}(\underline{\mathbf{QS}})$:

$$\begin{tikzcd}
0 \arrow[r] & A(K,\mathscr{F}) \arrow[r, "\imath(\mathscr{F})"] \arrow[d] & A(L,\mathscr{F}) \arrow[r, "\pi(\mathscr{F})"] \arrow[d] & A(M,\mathscr{F}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{\Delta\in K} \mathbb{A}_\Delta(\mathscr{F}) \arrow[r] & \prod_{\Delta\in L} \mathbb{A}_\Delta(\mathscr{F}) \arrow[r] & \prod_{\Delta\in M} \mathbb{A}_\Delta(\mathscr{F}) \arrow[r] & 0
\end{tikzcd}$$

Proof. The second line is obviously split-exact.

For $n = 0$ everything is clear by 1.2.5. Let $n > 0$, $\mathscr{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$. Then f all $P \in P(X)$: $\hat{P}K \cup \hat{P}M = \hat{P}L$, $\hat{P}K \cap \hat{P}M = \emptyset$. Set $\left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P = \mathscr{F}_{l,P}$ for short. Suppose the proposition holds in

$$\begin{tikzcd}
0 \arrow[r] & A\left(\hat{P}K,\mathscr{F}_{l,P}\right) \arrow[r, "\imath"] \arrow[d] & A\left(\hat{P}L,\mathscr{F}_{l,P}\right) \arrow[r, "\pi"] \arrow[d] & A\left(\hat{P}M,\mathscr{F}_{l,P}\right) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \prod_{\Delta\in \hat{P}K} \mathbb{A}_\Delta(\mathscr{F}_{l,P}) \arrow[r] & \prod_{\Delta\in \hat{P}L} \mathbb{A}_\Delta(\mathscr{F}_{l,P}) \arrow[r] & \prod_{\Delta\in \hat{P}M} \mathbb{A}_\Delta(\mathscr{F}_{l,P}) \arrow[r] & 0
\end{tikzcd}$$

The application of $\prod \varprojlim$ defines the natural transformations $\imath(\mathscr{F}), \pi(\mathscr{F})$ a gives the diagram of the proposition. $\varprojlim$ is left exact, the situation splits, t implies that the lines are split-exact.

By lemma 1.2.5, we get the result for $\mathscr{F} \in \mathrm{Ob}(\underline{\mathrm{QS}})$.

2.2 The Boundary Maps

Definition 2.2.1. Let $K \subset S(X)_0$, $\mathscr{F} \in \mathrm{Ob}(\underline{\mathrm{QS}})$. Let

$$d^0(K,\mathscr{F}) : \Gamma(X,\mathscr{F}) \longrightarrow A(K,\mathscr{F})$$

be the canonical map. It is a natural transformation of functors.

Definition 2.2.2. (Boundary Maps) Let $K \subset S(X)_{n+1}$, $L \subset S(X)_n$, $\delta_i^{n+1}K \subset$ for some $i \in \{0,\ldots,n+1\}$. Transformations of functors

$$d_i^{n+1}(K,L,\cdot) : A(L,\cdot) \longrightarrow A(K,\cdot)$$

are defined by the properties:

a) If $i = 0$ and $\mathscr{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$: Apply the functor $A(L,\cdot)$ to $\mathscr{F} \to \left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P$. Compose this map with the projection of 2.1.5 for $L \supset \hat{P}$. Use the universal property of $\prod_{P\in P(X)} \varprojlim$.

b) If $i = 1$, $n = 0$ and $\mathscr{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$: The injection of 2.1.5 for $L \subset P($ is composed with the following: The maps $d^0(\hat{P}K, \left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P)$ form projective system for $l \in \mathbb{N}$. Apply $\prod_{P\in P(X)} \varprojlim$ to it.

c) If $i > 0$, $n > 0$, $\mathcal{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$: The hypothesis $\delta_i^{n+1} K \subset L$ implies $\delta_{i-1}^{n} \hat{P} K \subset \hat{P} L$ for all $P \in P(X)$. Set

$$d_i^{n+1}(K, L, \mathcal{F}) = \prod_{P \in P(X)} \varprojlim_{l \in \mathbb{N}} d_{i-1}^{n}(\hat{P} K, \hat{P} L, [\mathcal{F}_P / \mathfrak{m}_P^l \mathcal{F}_P]_P) .$$

d) $d_i^{n+1}(K, L, \cdot)$ commutes with direct limits.

Remark. The exceptional case d_1^1 disappears if one thinks of $d^0(K, \cdot)$ as d_0^0 and of $\Gamma(X, \cdot)$ as $A(\cdot, \cdot)$ in degree -1.

Definition 2.2.3. For $K = S(X)_{n+1}$, $L = S(X)_n$, we call $d_i^{n+1}(K, L, \mathcal{F})$ global boundary map

$$d_i^{n+1} : \mathbb{A}^n(X, \mathcal{F}) \longrightarrow \mathbb{A}^{n+1}(X, \mathcal{F}) .$$

For $K = \{\Delta\}$, $L = \{\Delta' = \delta_i^{n+1} \Delta\}$, we call $d_i^{n+1}(K, L, \mathcal{F})$ local boundary map

$$d_i^{n+1} : \mathbb{A}_{\Delta'}(\mathcal{F}) \longrightarrow \mathbb{A}_{\Delta}(\mathcal{F}) .$$

Instead of $d^0(P(X), \mathcal{F})$ we write

$$d^0 : \Gamma(X, \mathcal{F}) \longrightarrow \mathbb{A}^0(X, \mathcal{F}) .$$

For $K \subset S(X)_{n+1}$, $L \subset S(X)_n$ with $\delta_i^{n+1} K \subset L$, we define

$$\begin{aligned} D_i^{n+1}(\mathcal{F}) : \prod_{\Delta \in L} \mathbb{A}_{\Delta}(\mathcal{F}) &\longrightarrow \prod_{\Delta \in K} \mathbb{A}_{\Delta}(\mathcal{F}) \\ (x_\Delta)_{\Delta \in L} &\longmapsto (y_\Delta)_{\Delta \in K} \end{aligned}$$

by $y_\Delta = d_i^{n+1}(x_{\delta_i^{n+1}\Delta})$. Here d_i^{n+1} is the local boundary map.

Proposition 2.2.4. *Let $K \subset S(X)_{n+1}$, $L \subset S(X)_n$ with $\delta_i^{n+1} K \subset L$. The following diagram commutes:*

$$\begin{array}{ccc} A(L, \mathcal{F}) & \xrightarrow{d_i^{n+1}} & A(K, \mathcal{F}) \\ \big\downarrow & & \big\downarrow \\ \prod_{\Delta \in L} \mathbb{A}_{\Delta}(\mathcal{F}) & \xrightarrow{D_i^{n+1}} & \prod_{\Delta \in K} \mathbb{A}_{\Delta}(\mathcal{F}) \end{array}$$

Proof. We have to follow the inductive definitions. □

2.3 The Degeneracy Maps

Proposition 2.3.1. *Let $L \subset S(X)_n$ and $0 \leq i \leq n$. Then*

$$A(L, \mathscr{F}) = A\left(\sigma_i^n L, \mathscr{F}\right) .$$

The equality is functorial.

Proof. W.l.o.g. assume $\mathscr{F}$ coherent. First let $i = 0$. As $\bar{P}\sigma_0^n L = \bar{P}\bar{P}L$

$$A\left(\sigma_0^n L, \mathscr{F}\right) \overset{2.1.3}{=} \prod_{P \in P(X)} A\left(\bar{P}\bar{P}L, \mathscr{F}\right) = \prod_{P \in P(X)} A\left(\bar{P}L, \mathscr{F}\right) = A(L, \mathscr{F}) .$$

Now consider $i > 0$. $\hat{P}\sigma_i^n L = \sigma_{i-1}^{n-1}\hat{P}L$ for all $P \in P(X)$. By inducti hypothesis, we have

$$A\left(\hat{P}\sigma_i^n L, \left[\mathscr{F}_P/\mathbf{m}_P^l \mathscr{F}_P\right]_P\right) = A\left(\hat{P}L, \left[\mathscr{F}_P/\mathbf{m}_P^l \mathscr{F}_P\right]_P\right) .$$

Application of $\prod \varprojlim$ gives the proposition.

Definition 2.3.2. (Degeneracy Maps) Let $K \subset S(X)_{n+1}$ and $L \subset S(X)_n$ wi $\sigma_i^n L \subset K$ for some $i \in \{0, \ldots, n\}$. Then a transformation of functors

$$s_i^n(K, L, \cdot) : A(K, \cdot) \longrightarrow A(L, \cdot)$$

is defined as the composition of the projection 2.1.5 with the identity from 2.3.

$$A(K, \mathscr{F}) \xrightarrow{\pi} A\left(\sigma_i^n L, \mathscr{F}\right) \xrightarrow{\text{id}} A(L, \mathscr{F}) .$$

For $K = S(X)_{n+1}$, $L = S(X)_n$, we call $s_i^n(K, L, \mathscr{F})$ global degeneracy map.

$$s_i^n : \mathbb{A}^{n+1}(X, \mathscr{F}) \longrightarrow \mathbb{A}^n(X, \mathscr{F}) .$$

For $L = \{\Delta\}$, $K = \{\Delta' = \sigma_i^n \Delta\}$, we call $s_i^n(K, L, \mathscr{F})$ local degeneracy map.

$$s_i^n : \mathbb{A}_{\Delta'}(\mathscr{F}) \longrightarrow \mathbb{A}_\Delta(\mathscr{F}) .$$

For $K \subset S(X)_{n+1}$, $L \subset S(X)_n$ with $\sigma_i^n L \subset K$ for some $0 \leq i \leq n$, we define

$$\begin{array}{rcl} S_i^n(\mathscr{F}) : \prod\limits_{\Delta \in K} \mathbb{A}_\Delta(\mathscr{F}) & \longrightarrow & \prod\limits_{\Delta \in L} \mathbb{A}_\Delta(\mathscr{F}) \\ (x_\Delta)_{\Delta \in K} & \longmapsto & (y_\Delta)_{\Delta \in L} \end{array}$$

by $y_\Delta = s_i^n(x_{\sigma_i^n \Delta})$. Here s_i^n is the local degeneracy map.

Remark. By definition, all local degeneracy maps are just equalities.

Proposition 2.3.3. *Let $K \subset S(X)_{n+1}$, $L \subset S(X)_n$ with $\sigma_i^n L \subset K$ for some 0 $i \leq n$. Then the following diagram commutes:*

$$\begin{tikzcd} A(K,\mathcal{F}) \arrow[r, "s_i^n"] \arrow[d] & A(L,\mathcal{F}) \arrow[d] \\ \prod_{\Delta\in K} \mathbb{A}_\Delta(\mathcal{F}) \arrow[r, "S_i^n"] & \prod_{\Delta\in L} \mathbb{A}_\Delta(\mathcal{F}) \end{tikzcd}$$

Proof. This is immediately clear from the definition of s_i^n. □

2.4 The Cosimplicial Group of Adeles

Theorem 2.4.1. *Let X be a noetherian scheme, $\mathcal{F}$ a quasicoherent $\mathcal{O}$-module sheaf on X.*

1. *The sequence of global adele groups $\mathbb{A}^n(X,\mathcal{F})$ together with the global boundary maps d_i^n for $0 \le i \le n$ and the global degeneracy maps s_i^n for $0 \le i \le n$ form a cosimplicial group. It is called the cosimplicial group of adeles of the scheme X with coefficients in $\mathcal{F}$.*

2. *The sequence of products of local factors $\prod_{\Delta\in S(X)_n} \mathbb{A}_\Delta(\mathcal{F})$ together with the boundary maps D_i^n for $0 \le i \le n$ and the degeneracy maps S_i^n for $0 \le i \le n$ form a cosimplicial group.*

3. *They are both covariant additive functors from the categegory of quasicoherent sheaves on X into the category of cosimplicial groups. The inclusion of the global adeles into the product of local factors is a transformation of functors.*

4. *Both functors are exact, the first one commutes with direct limits.*

Proof. We have to show that the commutation relations between the boundary and degeneracy maps hold. We do it exemplary in one case. W.l.o.g. $\mathcal{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$. Let $\Delta \in S(X)_{n+2}$, $\Delta' = \delta_i^{n+1}\delta_j^{n+2}\Delta = \delta_{j-1}^{n+1}\delta_i^{n+2}\Delta \in S(X)_n$ for $i < j$. We show $d_j d_i = d_i d_{j-1} : \mathbb{A}_{\Delta'}(\mathcal{F}) \longrightarrow \mathbb{A}_\Delta(\mathcal{F})$.

1. Case Let $i = 0$, $\Delta = (P, \delta_0\Delta)$. d_{j-1}^{n+1} is a transformation of functors. The diagram

$$\begin{tikzcd} \mathbb{A}_{\Delta'}(\mathcal{F}) \arrow[r, "d_{j-1}"] \arrow[d] & \mathbb{A}_{\delta_0\Delta}(\mathcal{F}) \arrow[d] \\ \mathbb{A}_{\Delta'}\left(\left[\mathcal{F}_P/\mathbf{m}_P^l\mathcal{F}_P\right]_P\right) \arrow[r, "d_{j-1}"] & \mathbb{A}_{\delta_0\Delta}\left(\left[\mathcal{F}_P/\mathbf{m}_P^l\mathcal{F}_P\right]_P\right) \end{tikzcd}$$

commutes. Application of $\varprojlim$ gives the commutative diagram

$$\begin{tikzcd}
\mathbb{A}_{\Delta'}(\mathscr{F}) \arrow[r, "d_{j-1}"] \arrow[d, "d_0"'] & \mathbb{A}_{\delta_0\Delta}(\mathscr{F}) \arrow[d, "d_0"'] \\
\mathbb{A}_{\delta_j\Delta}(\mathscr{F}) \arrow[r, "d_j"] & \mathbb{A}_{\Delta}(\mathscr{F})
\end{tikzcd}$$

2. Case Now let $i = 1$, $n = 0$. This implies $j = 2$. Let $\Delta = (P, Q, R)$, $\Delta' = (P$ $\mathscr{G} = \left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P$, and $\phi : \mathscr{G} \longrightarrow \left[\mathscr{G}_Q/\mathbf{m}_Q^{l'}\mathscr{G}_Q\right]_Q$ the canonical morphis of sheaves. The diagram

$$\begin{tikzcd}
\Gamma(X,\mathscr{G}) \arrow[r, "d_0"] \arrow[d, "\Gamma(\phi)"'] & \mathbb{A}_R(\mathscr{G}) \arrow[d, "\mathbb{A}(\phi)"'] \\
\Gamma(X, \left[\mathscr{G}_Q/\mathbf{m}_Q^{l'}\mathscr{G}_Q\right]_Q) \arrow[r, "d_0"] & \mathbb{A}_R\left(\left[\mathscr{G}_Q/\mathbf{m}_Q^{l'}\mathscr{G}_Q\right]_Q\right)
\end{tikzcd}$$

commutes. By definition, we get in the projective limit

$$\begin{tikzcd}
\Gamma(X,\mathscr{G}) \arrow[r, "d^0"] \arrow[d, "d^0"'] & \mathbb{A}_R(\mathscr{G}) \arrow[d, "d_0^1"'] \\
\mathbb{A}_Q(\mathscr{G}) \arrow[r, "d_1^1"] & \mathbb{A}_{(Q,R)}(\mathscr{G})
\end{tikzcd}$$

Replace $\mathscr{G}$ by $\left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P$. Application of $\varprojlim_l$ gives the proposition.

3. Case The general case is $i > 0$, $n > 0$. By inductive hypothesis, we alreac have $d_{j-1}^{n+1}d_{i-1}^n = d_{i-1}^{n+1}d_{j-2}^n$ for all sheaves $\left[\mathscr{F}_P/\mathbf{m}_P^l\mathscr{F}_P\right]_P$. Application of $\varprojlim$ gives the proposition.

Let $(x_{\Delta'})_{\Delta'\in S(X)_n}$ be an element of $\prod_{\Delta\in S(X)_n} \mathbb{A}_\Delta(\mathscr{F})$.

By definition 2.5.7, we have for $i < j$

$$\begin{aligned}
D_jD_i(x_{\Delta'})_{\Delta'\in S(X)_n} &= (d_jd_ix_{\delta_i\delta_j\Delta})_{\Delta\in S(X)_{n+2}} \\
&= (d_id_{j-1}x_{\delta_{j-1}\delta_i\Delta})_{\Delta\in S(X)_{n+2}} \\
&= D_iD_{j-1}(x_{\Delta'})_{\Delta'\in S(X)_n}\,.
\end{aligned}$$

This way we can prove that the product of local factors is a cosimplici group. The global boundary and degeneracy maps are restrictions of D_i ar S_i. Hence the required commutation relations hold for them as well.

3 Structure

3.1 Some Auxiliary Rules

Lemma 3.1.1. *Let $K \subset S(X)_n$, $\mathscr{F} \longrightarrow \mathscr{G}$ a morphism of quasicoherent sheaves. Suppose $\mathscr{F}_P = \mathscr{G}_P$ for all $P \in \operatorname{supp}(K)$. Then*

$$A(K,\mathscr{F}) = A(K,\mathscr{G})\ .$$

Proof. Induction on n. For coherent sheaves use 2.1.3. □

Proposition 3.1.2. *Let $i : Z \longrightarrow X$ be a closed immersion. Then for $\mathscr{F} \in$* $\mathrm{Ob}(\underline{\mathbf{QS}}(Z))$

$$\mathbb{A}^*(X, i_*\mathscr{F}) = \mathbb{A}^*(Z,\mathscr{F})\ .$$

Proof. By Proposition 2.1.5

$$\mathbb{A}^n(X, i_*\mathscr{F}) = A(S(Z)_n, i_*\mathscr{F}) \times A\left(S(X)_n \setminus S(Z)_n, i_*\mathscr{F}\right)\ ,$$

$A\left(S(X)_n \setminus S(Z)_n, i_*\mathscr{F}\right) = 0$ and $A(S(Z)_n, i_*\mathscr{F}) = \mathbb{A}^n(Z,\mathscr{F})$. Equality as cosimplicial groups follows from the locality of the boundary and degeneracy maps as restrictions of D_i resp. S_i. □

Proposition 3.1.3. *Let $i : U \to X$ be an open immersion. For $\mathscr{F} \in \mathrm{Ob}(\underline{\mathbf{QS}}(X))$ we have*

$$\mathbb{A}^n\left(U, \mathscr{F}\,|_U\right) = A(S(U)_n, \mathscr{F})\ .$$

Let $\Delta = (P_0, \ldots, P_n) \in S(X)_n$. Let U be an open affine subscheme which contains the point P_n and therefore all of Δ. Let $M = \mathscr{F}(U)$. Then

$$\mathbb{A}_\Delta(\mathscr{F}) = \mathbb{A}_\Delta\left(\tilde{M}\right)\ .$$

Proof. The proof runs as the one of 3.1.2. □

Lemma 3.1.4. *Let $X = \operatorname{Spec} R$, M a finitely generated R-module in which the zeromodule is $\mathfrak{n}$-primary. Let $f \in R \setminus \mathfrak{n}$, $K \subset S(X)_n$. Then*

$$A\left(K,\tilde{M}\right) \subset A\left(K,\tilde{M}_{\mathfrak{n}}\right) \quad \textit{and} \quad A\left(K,\left(\frac{1}{f}M\right)^{\sim}\right) \subset A\left(K,\tilde{M}_{\mathfrak{n}}\right)\ ,$$

$$A\left(K,\left(\frac{1}{f}M\right)^{\sim}\right) = \frac{1}{f}A\left(K,\tilde{M}\right) \quad \textit{and} \quad A(K,(fM)^{\sim}) = fA\left(K,\tilde{M}\right)\ .$$

Proof. The map $M \longrightarrow S_{\mathfrak{n}}^{-1}M$ is injective. $\frac{1}{f}M$ is considered as a submodule of $S_{\mathfrak{n}}^{-1}M$. Exactness of $A(K,\cdot)$ gives the inclusions. In $M_{\mathfrak{n}}$ multiplication with f is an isomorphism. Then $A(K,f\cdot)$ is an isomorphism as well. By construction $A(K,f\cdot)$ is multiplication with the element f. Thus $\frac{1}{f}A\left(K,\tilde{M}\right) \subset A\left(K,\tilde{M}_{\mathfrak{n}}\right)$ is well defined. We get the equalities by applying $A(K,\cdot)$ to

$$0 \longrightarrow \frac{1}{f}M \xrightarrow{f\cdot} M \longrightarrow 0 \quad \text{resp.} \quad 0 \longrightarrow M \xrightarrow{f\cdot} fM \longrightarrow 0\ .$$ □

3.2 Structure of the Local Factors

Let $X = \mathrm{Spec} R$ and $\mathscr{F} = \tilde{M}$. Further let $\Delta = (\mathbf{p}_0, \ldots, \mathbf{p}_n) \in S(X)_n$.

Proposition 3.2.1. *It is*

$$\mathbb{A}_\Delta(\mathscr{F}) = C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \ldots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} R \otimes_R M .$$

$C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \ldots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} R$ *is a flat noetherian R-algebra. For finitely generated R-modules*

$$C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \ldots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} R \otimes_R M = C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \ldots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} M .$$

Proof. Induction on n. Suppose first $\Delta = (\mathbf{p})$, M a finitely generated R-module. By definition we have

$$\mathbb{A}_{(\mathbf{p})}\left(\tilde{M}\right) = C_{\mathbf{p}} S_{\mathbf{p}}^{-1} M = C_{\mathbf{p}} S_{\mathbf{p}}^{-1} R \otimes_R M .$$

If M is not finitely generated, it is at least direct limit of finitely generated submodules $\{M_j\}_{j\in J}$. It follows

$$\mathbb{A}_{(\mathbf{p})}\left(\tilde{M}\right) = \varinjlim_{j\in J} \mathbb{A}_{(\mathbf{p})}\left(\tilde{M}_j\right) = C_{\mathbf{p}} S_{\mathbf{p}}^{-1} R \otimes_R (\varinjlim M_j) .$$

It is clear that the resulting algebra is noetherian and flat.

Consider now $n > 0$, $\Delta = (\mathbf{p}_0, \Delta')$. Set $\mathbb{A} = \mathbb{A}_{\Delta'}\left(\tilde{R}\right)$ for short. Let M be finitely generated. Then

$$\begin{aligned}
\mathbb{A}_\Delta\left(\tilde{M}\right) &= \varprojlim \mathbb{A}_{\Delta'}\left(\left(M_{\mathbf{p}_0}/\mathbf{p}_0^l M_{\mathbf{p}_0}\right)^{\sim}\right) \\
&= \varprojlim \left(\mathbb{A} \otimes S_{\mathbf{p}_0}^{-1} R \otimes (M/\mathbf{p}_0^l)\right) \\
&= \varprojlim \left(S_{\mathbf{p}_0}^{-1}(\mathbb{A} \otimes M)/\mathbf{p}_0^l S_{\mathbf{p}_0}^{-1}(\mathbb{A} \otimes M)\right) \\
&= \begin{cases} C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \mathbb{A} \otimes_R M \\ C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \ldots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} M \end{cases}
\end{aligned}$$

To be noetherian is preserved in localisation and completion. Arbitrary R-modules are treated as in the case $n = 0$. Flatness of $C_{\mathbf{p}_0} S_{\mathbf{p}_0}^{-1} \mathbb{A}$ is exactness of $\mathbb{A}_\Delta(\cdot)$. □

Proposition 3.2.2. *Let $\Delta' = \delta_i^n \Delta$. Let ϕ be the canonical map*

$$\phi : C_{\mathbf{p}_{i-1}} S_{\mathbf{p}_{i-1}}^{-1} \cdots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} R \otimes M \longrightarrow C_{\mathbf{p}_i} S_{\mathbf{p}_i}^{-1} \cdots C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} R \otimes M .$$

Then for the local boundary map $d_i^n : \mathbb{A}_{\Delta'}\left(\tilde{M}\right) \longrightarrow \mathbb{A}_\Delta\left(\tilde{M}\right)$

$$d_i^n = C_{\mathbf{p}_n} S_{\mathbf{p}_n}^{-1} \cdots C_{\mathbf{p}_{i+1}} S_{\mathbf{p}_{i+1}}^{-1}(\phi) .$$

Proof. Induction as for 3.2.1. □

Remark. The operator $S_{\mathbf{p}}^{-1}$ means localisation at the set $R\backslash\mathbf{p}$. Hence the algebras $\mathbb{A}_\Delta(\mathcal{O})$ are not local in general!

3.3 Global Structure

Lemma 3.3.1. *Let* $K \subset S(X)_n$ *and* $K = \bigcup_{i=1}^{k} K_i$. *Let* $(x_\Delta)_{\Delta \in K}$ *be an element of* $\prod_{\Delta \in K} \mathbb{A}_\Delta(\mathscr{F})$. *Equivalent are:*

(i) $(x_\Delta)_{\Delta \in K} \in A(K, \mathscr{F})$.

(ii) $(x_\Delta)_{\Delta \in K_i} \in A(K_i, \mathscr{F})$ *for all* $1 \leq i \leq k$.

Proof. (i)$\Rightarrow$(ii) follows from Proposition 2.1.5. The inverse conclusion follows by induction on k from Proposition 2.1.5. □

Definition 3.3.2. Set

$$\mathbb{A}^n_{(red)}(X, \mathscr{F}) = A\left(S(X)^{(red)}_n, \mathscr{F}\right) .$$

The elements of $\mathbb{A}^n_{(red)}(X, \mathscr{F})$ are called reduced n-dimensional adeles.

Proposition 3.3.3.

$$\mathbb{A}^n(X, \mathscr{F}) = \mathbb{A}^n_{(red)}(X, \mathscr{F}) \times \prod_{k=0}^{n-1} \mathbb{A}^k_{(red)}(X, \mathscr{F})^{\binom{n}{k}} .$$

Proof. Elementary combinatorics and 2.3.1. □

Curves. Let X be an integral one dimensional noetherian scheme. Let P_0 be the generic point, $K(X)$ the function field of X. Then

$$\mathbb{A}^0_{(red)}(X, \mathcal{O}) = K(X) \times \prod_{P \text{ cl.}} C_P \mathcal{O}_P .$$

$S(X)^{(red)}_1$ contains all pairs(P_0, P) where P is a closed point. By 3.2.1 we have

$$\mathbb{A}_{(P_0,P)}(\mathcal{O}) = C_{P_0} S^{-1}_{P_0} C_P \mathcal{O}_P .$$

If $0 = \bigcap_{i=1}^{n} \mathbf{p}_i$ in $C_P \mathcal{O}_P$, then $\mathbb{A}_{(P_0,P)}(\mathcal{O})$ is direct sum of the fields $S^{-1}_{\mathbf{p}_i} C_P \mathcal{O}_P$. In the affine case a tupel (x_P) of elements of $\mathbb{A}_{(P_0,P)}(\mathcal{O})$ is a reduced adele if and only if there is $f \in R \setminus 0$ such that $f x_P \in C_{\mathbf{p}} \mathcal{O}_P$. But f is invertible in almost all $S^{-1}_{\mathbf{p}} R$, the condition can be translated into (also in the non affine case): x_P almost always in $C_{\mathbf{p}} \mathcal{O}_P$.

$$\mathbb{A}^1_{(red)}(X, \mathcal{O}) = \left\{ (x_P)_P \in \prod_{P \text{ cl.}} \mathbb{A}_{(P_0,P)}(\mathcal{O}) \mid x_P \text{ almost always in } C_P \mathcal{O}_P \right\} .$$

Global Fields. Consider the spectrum of the ring of integers $\mathcal{O}$ of a global field K. For all maximal prime ideals $\mathbf{p}$ the ring $\mathcal{O}_{\mathbf{p}}$ is a discrete valuation ring. By the preceding example and the definitions of algebraic number theory, we get

$$\begin{aligned} \mathbb{A}^0_{(red)}(X, \mathcal{O}) &= K \times \{ \text{ integral finite adeles of K } \}, \\ \mathbb{A}^1_{(red)}(X, \mathcal{O}) &= \{ \text{ finite adeles of K } \}. \end{aligned}$$

4 Adeles and Cohomology

4.1 The Affine Case

Theorem 4.1.1. *Let $X = \operatorname{Spec} R$ be an affine noetherian scheme, $\mathcal{F} \in \mathrm{Ob}(\underline{\mathrm{QS}}$ Then*

$$H^i(\mathbb{A}^\bullet(X,\mathcal{F})) = \begin{cases} \Gamma(X,\mathcal{F}) & i=0 \\ 0 & i>0 \end{cases}.$$

Proposition 4.1.2. *Let $X = \operatorname{Spec} R$ be an affine noetherian scheme, $\mathcal{F}$ $\mathrm{Ob}(\underline{\mathrm{CS}})$. Then*

$$H^0(\mathbb{A}^\bullet(X,\mathcal{F})) = \Gamma(X,\mathcal{F}).$$

Proof. $\mathcal{F} = \tilde{M}$, where $M = \Gamma(X,\mathcal{F})$ is a finitely generated R-module. I definition we have

$$\mathbb{A}^0(X,\mathcal{F}) = \prod_{\mathbf{p}\in P(X)} C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M \quad \text{and} \quad \mathbb{A}^1(X,\mathcal{F}) \subset \prod_{\mathbf{p}\subset\mathbf{q}} C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M.$$

Now consider the sequence

$$0 \longrightarrow M \xrightarrow{d^0} \prod_{\mathbf{p}\in P(X)} C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M \xrightarrow{D_0^1-D_1^1} \prod_{\mathbf{p}\subset\mathbf{q}} C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M \qquad (**$$

Exactness of the sequence $(***)$ in M is clear. By the proof of theorem 2.4. $d^0(M) \subset \operatorname{Ker}(D_0^1 - D_1^1)$.

Let $(x_{\mathbf{p}})_{\mathbf{p}\in P(X)} \in \operatorname{Ker}(D_0^1 - D_1^1)$, i.e. we have for all $\mathbf{p}\subset\mathbf{q}$

$$d_0^1 x_{\mathbf{q}} = d_1^1 x_{\mathbf{p}} \quad \text{in} \quad C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M.$$

In the conventions of paragraph **1** this means

$$x_{\mathbf{p}} = x_{\mathbf{q}} \quad \text{in} \quad C_{\mathbf{p}}S_{\mathbf{p}}^{-1}C_{\mathbf{q}}S_{\mathbf{q}}^{-1}M \quad \text{for all} \quad \mathbf{p}\subset\mathbf{q} \qquad ($$

Fix $\mathbf{q}$ and consider all $\mathbf{p}\subset\mathbf{q}$. Apply lemma 1.1.3 to the $S_{\mathbf{q}}^{-1}R$-module $S_{\mathbf{q}}^{-1}M$ Hence there is

$$y_{\mathbf{q}} \in S_{\mathbf{q}}^{-1}M \text{ with } y_{\mathbf{q}} = x_{\mathbf{p}} \text{ in } C_{\mathbf{p}}S_{\mathbf{p}}^{-1}S_{\mathbf{q}}^{-1}M = C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M \text{ for all } \mathbf{p}\subset\mathbf{q}.$$

Hence $x_{\mathbf{q}} \in S_{\mathbf{q}}^{-1}M$ for all $\mathbf{q}$, and for all $\mathbf{p}\subset\mathbf{q}$

$$x_{\mathbf{q}} = x_{\mathbf{p}} \quad \text{in} \quad C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M.$$

As $S_{\mathbf{p}}^{-1}M \subset C_{\mathbf{p}}S_{\mathbf{p}}^{-1}M$ and $x_{\mathbf{p}}$, $x_{\mathbf{q}} \in S_{\mathbf{p}}^{-1}M$, this actually means

$$x_{\mathbf{q}} = x_{\mathbf{p}} \quad \text{in} \quad S_{\mathbf{p}}^{-1}M \quad \text{for all} \quad \mathbf{q} \quad \text{and all} \quad \mathbf{p}\subset\mathbf{q} \qquad (*$$

By lemma 1.1.4 there is $x \in M$ with

$$x = x_{\mathbf{p}} \quad \text{in} \quad S_{\mathbf{p}}^{-1}M \quad \text{for all prime ideals} \quad \mathbf{p}.$$

Hence we have $d^0(x) = (x_{\mathbf{p}})_{\mathbf{p}\in P(X)}$. Therefore the sequence $(***)$ is exact. follows

$$M = \operatorname{Ker}(D_0^1 - D_1^1) = H^0(\mathbb{A}^\bullet(X,\mathcal{F})).$$

Lemma 4.1.3. *Suppose the sequence of the theorem is exact in $\mathbb{A}^i\left(X, \tilde{M}'\right)$ for all finitely generated R-modules M' and all $0 \leq i < k$. Let M be a finitely generated R-module in which the zeromodule is primary. Then the sequence of the theorem is exact in $\mathbb{A}^k\left(X, \tilde{M}\right)$.*

Proof. Let $\mathfrak{n} = \mathrm{rad}(0, M)$, $\mathfrak{a} = \mathrm{ann}(M)$. Then $\mathfrak{a}$ is a primary ideal with radical $\mathfrak{n}$. Let $X' = \mathrm{Spec}R/\mathfrak{a}$. This is a closed subscheme of X. By proposition 3.1.2, it is w.l.o.g. enough to prove the proposition for $\mathfrak{a} = 0$ (M torsion free). Then $\mathfrak{n}$ is the nilradical of R (prime in R). Consider arbitrary $x \in \mathrm{Ker}d^k$: $\mathbb{A}^k\left(X, \tilde{M}\right) \longrightarrow \mathbb{A}^{k+1}\left(X, \tilde{M}\right)$. We want to construct

$$w \in \mathbb{A}^{k-1}\left(X, \tilde{M}\right) \quad \text{with} \quad d^{k-1}w = x\,.$$

By definition

$$\mathbb{A}^k\left(X, \mathscr{F}\right) = \varprojlim_{l} A\left(\hat{\mathfrak{n}}S(X)_k, \left[\mathscr{F}_{\mathfrak{n}}/\mathfrak{m}_{\mathfrak{n}}^l \mathscr{F}_{\mathfrak{n}}\right]_{\mathfrak{n}}\right) \times \prod_{P \in P(X)\setminus\{\mathfrak{n}\}} A\left(\bar{P}S(X)_k, \mathscr{F}\right)\,.$$

$\mathfrak{n} \geq P$ for all $P \in P(X)$ implies $\hat{\mathfrak{n}}S(X)_k = S(X)_{k-1}$. The nilradical is nilpotent in a noetherian ring, thus

$$\varprojlim_{l} A\left(\hat{\mathfrak{n}}S(X)_k, (M_{\mathfrak{n}}/\mathfrak{n}^l M_{\mathfrak{n}})^{\sim}\right) = A\left(S(X)_{k-1}, \tilde{M}_{\mathfrak{n}}\right) = \mathbb{A}^{k-1}\left(X, \tilde{M}_{\mathfrak{n}}\right)\,.$$

Write $x = (x_{\mathfrak{n}}, x_0)$ with

$$x_{\mathfrak{n}} \in \mathbb{A}^{k-1}\left(X, \tilde{M}_{\mathfrak{n}}\right) \quad \text{and} \quad x_0 \in \prod_{P \in P(X)\setminus\{\mathfrak{n}\}} A\left(\bar{P}S(X)_k, \mathscr{F}\right)\,.$$

This notation will be used repeatedly for elements of adele groups.

$\varinjlim_{f \in S_{\mathfrak{n}}} \frac{1}{f}M = M_{\mathfrak{n}}$ implies $\mathbb{A}^{k-1}\left(X, \tilde{M}_{\mathfrak{n}}\right) = \varinjlim_{f \in S_{\mathfrak{n}}} \mathbb{A}^{k-1}\left(X, \left(\frac{1}{f}M\right)^{\sim}\right)$. Therefore there is some $f \in S_{\mathfrak{n}}$ with

$$x_{\mathfrak{n}} \in \mathbb{A}^{k-1}\left(X, \left(\frac{1}{f}M\right)^{\sim}\right) = \frac{1}{f}\mathbb{A}^{k-1}\left(X, \tilde{M}\right) \Longleftrightarrow f x_{\mathfrak{n}} \in \mathbb{A}^{k-1}\left(X, \tilde{M}\right)\,. \quad (1)$$

Let $y \in \mathbb{A}^{k-1}\left(X, (M/fM)^{\sim}\right)$ be the image of fx_n. We shall see presently that $d^k y = 0$.

We calculate in components. By hypothesis $d^{k+1}x = 0$. Write $x = (x_\Delta)_{\Delta \in S(X)_k}$. In the $(\mathfrak{n}, \Delta)$-component ($\Delta \in S(X)_k$) of $\mathbb{A}^{k+1}\left(X, \mathscr{F}\right)$ this means:

$$0 = \sum_{i=0}^{k+1} (-1)^i x_{\delta_i^{k+1}(\mathfrak{n},\Delta)} = x_\Delta - \sum_{j=0}^{k} (-1)^j x_{(\mathfrak{n}, \delta_j^k \Delta)} \quad (j = i-1)\,.$$

This was calculated in $\mathbb{A}_{(\mathfrak{n},\Delta)}\left(\tilde{M}\right)$. Hence

$$0 = f x_\Delta - \sum_{j=0}^{k} (-1)^j f x_{(\mathfrak{n}, \delta_j^k \Delta)} \quad \text{in} \quad \mathbb{A}_{(\mathfrak{n},\Delta)}\left(\tilde{M}\right)\,. \quad (2)$$

$\mathbb{A}_\Delta\left(\tilde{M}\right) \subset \mathbb{A}_\Delta\left(\tilde{M}_{\mathbf{n}}\right) = \mathbb{A}_{(\mathbf{n},\Delta)}\left(\tilde{M}_{\mathbf{n}}\right)$, besides $fx_{(\mathbf{n},\delta^k_j\Delta)} = \left(d^k_j f x_{\mathbf{n}}\right)_\Delta \in \mathbb{A}_\Delta\left(\tilde{M}\right)$. x_Δ is in $\mathbb{A}_\Delta\left(\tilde{M}\right)$ anyway. So the equation (2) already holds in $\mathbb{A}_\Delta\left(\tilde{M}\right)$. Further we have $fx_\Delta \in f\mathbb{A}_\Delta\left(\tilde{M}\right) = \mathbb{A}_\Delta\left((fM)^\sim\right)$ i.e. $fx_\Delta = 0$ in $\mathbb{A}_\Delta\left((M/fM)^\sim\right)$. From (2) follows

$$0 = -(d^k y)_\Delta . \tag{3}$$

This holds for all $\Delta \in S(X)_k$, hence $d^k y = 0$.

The assumption of our lemma can be applied to M/fM. Hence there is

$$z' \in \mathbb{A}^{k-2}\left(X, (M/fM)\right) \quad \text{with} \quad d^{k-1}z' = y .$$

For $k = 1$ the elements z' lies in the module M/fM, which can be treated as $\mathbb{A}^{-1}\left(X, (M/fM)^\sim\right)$.

There is a preimage $z'' \in \mathbb{A}^{k-2}\left(X, \tilde{M}\right)$ of z' under the projection.

For $k > 1$

$$\mathbb{A}^{k-1}\left(X, \left(\frac{1}{f}M\right)^\sim\right) = \mathbb{A}^{k-2}\left(X, \tilde{M}_{\mathbf{n}}\right) \times \prod_{P \in P(X)\setminus\{\mathbf{n}\}} A\left(\bar{P}S(X)_{k-1}, \left(\frac{1}{f}M\right)^\sim\right) .$$

Let $x_{\mathbf{nn}}$ be the component of $x_{\mathbf{n}}$ in $\mathbb{A}^{k-2}\left(X, \tilde{M}_{\mathbf{n}}\right)$. Then we have

$$\left(d^{k-1}x_{\mathbf{nn}}\right)_\Delta = (x_{\mathbf{n}})_\Delta - (x_{\mathbf{n}})_\Delta + \left(d^{k-1}x_{\mathbf{nn}}\right)_\Delta = \left(d^{k+1}x\right)_{(\mathbf{n},\mathbf{n},\Delta)} = 0 .$$

By assumption of the lemma, there is $a \in \mathbb{A}^{k-3}\left(X, \tilde{M}_{\mathbf{n}}\right)$ with $d^{k-2}a = x_{\mathbf{nn}}$. (a lies in $M_{\mathbf{n}}$ in the special case k=2.)

Define $z \in \mathbb{A}^{k-2}\left(X, \tilde{M}\right)$ by

$$\begin{aligned} k = 1 : \quad z &= z'' \in \mathbb{A}^{-1}\left(X, \tilde{M}\right) = M \\ k > 1 : \quad z &= (z_{\mathbf{n}}, z_0) \in \mathbb{A}^{k-2}\left(X, \tilde{M}\right) \quad \text{with} \quad z_{\mathbf{n}} = -fa, z_0 = (z'')_0 \end{aligned} \tag{4}$$

There is no need to treat the case $k = 1$ seperatedly in the sequel. Just leave out the consideration concerning the $\mathbf{n}$-component of z.

Now consider $y - d^{k-1}z \in \mathbb{A}^{k-1}\left(X, (M/fM)^\sim\right)$:

$$\begin{aligned} (y - d^{k-1}z)_0 &= (y - d^{k-1}z')_0 = 0 \\ (y - d^{k-1}z)_{\mathbf{n}} &\in \mathbb{A}^{k-2}\left(X, (M/fM)^\sim_{\mathbf{n}}\right) = \mathbb{A}^{k-2}(X, 0) = 0 . \\ \Rightarrow \qquad y &= d^{k-1}z \quad \text{in} \quad \mathbb{A}^{k-1}\left(X, (M/fM)^\sim\right) . \end{aligned} \tag{5}$$

Using z we can finally construct w. With the same convention as before, let $w = (w_{\mathbf{n}}, w_0) = (w_\Delta)_{\Delta \in S(X)_{k-1}}$ where

$$w_{\mathbf{n}} = -\frac{z}{f} \quad \text{in} \quad \frac{1}{f}\mathbb{A}^{k-2}\left(X, \tilde{M}\right) \tag{6}$$

$$w_\Delta = x_{(\mathbf{n},\Delta)} - \frac{1}{f}(d^{k-1}z)_\Delta \quad \text{in} \quad \frac{1}{f}\mathbb{A}_\Delta\left(\tilde{M}\right) \tag{7}$$

$$\text{for} \quad \Delta \in S(X)_{k-1} \setminus (\mathbf{n}, S(X)_{k-2}) .$$

We still have to show that w actually is an element of $\mathbb{A}^{k-1}\left(X,\tilde{M}\right)$. Obviously $w_{\mathbf{n}} \in \mathbb{A}^{k-2}\left(X,\tilde{M}_{\mathbf{n}}\right)$. By choice of f (1), we have $x_{\mathbf{n}} \in \frac{1}{f}\mathbb{A}^{k-1}\left(X,\tilde{M}\right)$. As $z \in \mathbb{A}^{k-2}\left(X,\tilde{M}\right)$, we know that $\frac{1}{f}d^{k-1}z$ is in the same module. Note that

$$\mathbb{A}^{k-1}\left(X,\tilde{M}\right) \subset \frac{1}{f}\mathbb{A}^{k-1}\left(X,\tilde{M}\right) .$$

By construction of y and (5), we have in $\mathbb{A}^{k-1}\left(X,(M/fM)^{\sim}\right)$:

$$f x_{\mathbf{n}} - d^{k-1}z = y - d^{k-1}z = 0 .$$

This implies $fx_{\mathbf{n}}-d^{k-1}z \in f\mathbb{A}^{k-1}\left(X,\tilde{M}\right)$, f divides $fx_{\mathbf{n}}-d^{k-1}z$ in $\mathbb{A}^{k-1}\left(X,\tilde{M}\right)$. Hence as required

$$w_0 \in \prod_{P\in S(X)_{k-1}\setminus\{\mathbf{n}\}} A\left(\bar{P}S(X)_{k-1},\tilde{M}\right) .$$

Now calculate the components of $d^k w$:

1.Case: $\Delta \in S(X)_k$, $\Delta = (P_0,\dots,P_n)$ with $P_0 \neq \mathbf{n}$:

$$\left(d^k w\right)_\Delta \overset{(7)}{=} \sum_{i=0}^{k}(-1)^i x_{(\mathbf{n},\delta_i^k\Delta)} - \frac{1}{f}\sum_{i=0}^{k}(-1)^i (d^{k-1}z)_{\delta_i^k\Delta} = \sum_{i=0}^{k}(-1)^i x_{(\mathbf{n},\delta_i^k\Delta)} .$$

From $(d^{k+1}x)_{(\mathbf{n},\Delta)} = 0$ follows $x_\Delta = \sum_{i=0}^{k}(-1)^i x_{(\mathbf{n},\delta_i^k\delta)}$ in $\mathbb{A}_{(\mathbf{n},\Delta)}\left(\tilde{M}\right)$. As $\mathbb{A}_\Delta\left(\tilde{M}\right) \subset \mathbb{A}_{(\mathbf{n},\Delta)}\left(\tilde{M}\right)$, we get $(d^k w)_\Delta = x_\Delta$ in $\mathbb{A}_\Delta\left(\tilde{M}\right)$.

2.Case: $\Delta \in S(X)_k$, $\Delta = (\mathbf{n},P_1,\dots,P_n)$ with $P_1 \neq \mathbf{n}$, $\Delta' = \hat{\mathbf{n}}\Delta$:

$$\begin{aligned}\left(d^k w\right)_\Delta &= w_{\Delta'} + \sum_{i=1}^{k}(-1)^i w_{(\mathbf{n},\delta_{i-1}^{k-1}\Delta')} \\ &\overset{(7),(6)}{=} x_{(\mathbf{n},\Delta')} - \frac{1}{f}\left(d^{k-1}z\right)_{\Delta'} - \sum_{j=0}^{k-1}(-1)^j\left(-\frac{z}{f}\right)_{\delta_j^{k-1}\Delta'} = x_\Delta\end{aligned}$$

3.Case: $\Delta \in S(X)_k$, $\Delta = (\mathbf{n},\mathbf{n},\Delta')$ (This case does not exist for $k = 1$.)

$$\begin{aligned}\left(d^k w\right)_\Delta &= w_{(\mathbf{n},\Delta')} - w_{(\mathbf{n},\Delta')} + \sum_{i=2}^{k}(-1)^i (d_i^k w)_\Delta \\ &= \left(d^{k-2}w_{\mathbf{nn}}\right)_{\Delta'} \overset{(6)}{=} \left(d^{k-2}\left(-\frac{z_{\mathbf{n}}}{f}\right)\right)_{\Delta'} \overset{(4)}{=} (d^{k-2}a)_{\Delta'} = x_{(\mathbf{n},\mathbf{n},\Delta')} .\end{aligned}$$

On the whole $d^k w = x$, and the lemma is proved. □

Lemma 4.1.4. *Let M be a finitely generated R-module, $N, N' \subset M$. Suppo: the sequence of the theorem is exact in $k-1$ for the module $M/(N+N')$ ar exact in k for M/N and M/N'. Then it is exact in degree k for the modu $M/(N \cap N')$.*

Proof. Apply the exact functor $\mathbb{A}^{*}(X, \cdot)$ to the exact sequence

$$0 \longrightarrow M/N \cap N' \longrightarrow M/N \times M/N' \longrightarrow M/N + N' \longrightarrow 0 .$$

We get the lemma by diagram chasing. |

Proof of the Theorem. Any R-module M is direct limit of finitely generate ones. $\varinjlim$ is exact and commutes with our adele functors. Hence it suffices t consider finitely generated modules.

Exactness of the sequence in M and in $\mathbb{A}^{0}\left(X, \tilde{M}\right)$ is the assertion of 4.1.: The proof of the general case is done by induction on k. Suppose tl sequence is exact in degree $k-1$ for all modules. The zeromodule has primary decomposition in M.

$$0 = N_1 \cap \ldots \cap N_m .$$

M/N_i fulfills the assumptions of lemma 4.1.3. Hence the sequence is exa(in k for M/N_i. Using lemma 4.1.4 for $N = N_1 \cap \ldots \cap N_i$, $N' = N_{i+1}$, we g(inductively exactness in k for $M/(N_1 \cap \ldots \cap N_{i+1})$, finally for $M/(N_1 \cap \ldots \cap N_m)$: M. [

4.2 The Main Theorem

Definition 4.2.1. The n-dimensional sheaf of adeles $\underline{\mathbb{A}}^{n}(X, \mathscr{F})$ is given by

$$U \longmapsto \mathbb{A}^{n}\left(U, \mathscr{F}|_{U}\right) .$$

Proposition 4.2.2. *$\underline{\mathbb{A}}^{n}(X, \mathscr{F})$ is a flasque sheaf. The boundary and degenerac maps induce morphisms of sheafs.*

Proof. Clearly $S(U)_n \subset S(X)_n$. By proposition 2.1.5, the map

$$\mathbb{A}^{n}(X, \mathscr{F}) \longrightarrow A(S(U)_n, \mathscr{F}) \stackrel{3.1.3}{=} \mathbb{A}^{n}\left(U, \mathscr{F}|_{U}\right)$$

is defined. Obviously we get a presheaf. The map is even surjective, flasquenes is clear.

Let $U = \bigcup_{i \in I} U_i$ be an open covering of U. Then $S(U)_n = \bigcup_{i \in I} S(U_i)_{\imath}$ Application of 3.3.1 proves the sheaf properties.

Because of the local character of the boundary and degenaracy maps it i clear that they induce morphisms of sheafs. [

Theorem 4.2.3. *Let X be a noetherian scheme and $\mathcal{F}$ a quasicoherent $\mathcal{O}$-module sheaf on X. Then*

$$H^i\left(\mathbb{A}^\bullet(X,\mathcal{F})\right) = H^i(X,\mathcal{F}).$$

The cosimplicial group of adeles calculates cohomology of sheafs.

Proof. Consider the sequence of sheafs:

$$0 \longrightarrow \mathcal{F} \longrightarrow \underline{\mathbb{A}}^0(X,\mathcal{F}) \longrightarrow \underline{\mathbb{A}}^1(X,\mathcal{F}) \longrightarrow \cdots$$

By theorem 4.1.1, the sequence

$$0 \longrightarrow \Gamma(U,\mathcal{F}) \longrightarrow \mathbb{A}^0\left(U,\mathcal{F}\mid_U\right) \longrightarrow \mathbb{A}^1\left(U,\mathcal{F}\mid_U\right) \longrightarrow \cdots$$

is exact of all affine $U \subset X$. Therefore the sequence of sheafs is exact. By proposition 4.2.2, the sheafs $\underline{\mathbb{A}}^\bullet(X,\mathcal{F})$ are flasque. Hence we have an acyclic resolution of $\mathcal{F}$. Application of the global section functor to

$$0 \longrightarrow \underline{\mathbb{A}}^\bullet(X,\mathcal{F})$$

gives the complex

$$0 \longrightarrow \mathbb{A}^\bullet(X,\mathcal{F})$$

whose cohomology is cohomology of sheafs on X with coefficients in $\mathcal{F}$. □

5 Miscellaneous

5.1 Reduced Adeles

Definition 5.1.1. Let G be a cosimplicial group. The complex $D(G)^\bullet$ is given by

$$D(G)^n = \bigcap_{i=0}^{n-1} \operatorname{Ker} s_i^{n-1}.$$

The boundary maps are induced by G.

Proposition 5.1.2. *Let G be a cosimplicial group. Then*

$$H^i\left(D(G)^\bullet\right) = H^i\left(G^\bullet\right).$$

This is a dualisation of the corresponding theorem for simplicial groups. ([9] Ch.V Cor.22.3.

Proposition 5.1.3. *Let X be a noetherian scheme, $\mathcal{F}$ a quasicoherent sheaf on X. The complex of reduced adeles $\mathbb{A}^\bullet_{(red)}(X,\mathcal{F})$ has the same cohomology as $\mathbb{A}^\bullet(X,\mathcal{F})$.*

Proof. Show $(\mathbb{A}^\bullet(X,\mathcal{F})_{red}) = \mathbb{A}^\bullet_{(red)}(X,\mathcal{F})$. By proposition 2.3.3, the global degeneracy maps can be calculated by local ones, which are isomorphisms. □

Corollary 5.1.4. *Let X be an n-dimensional noetherian scheme. Then the cohomological dimension of X is at most n.*

Proof. In degrees greater than n all simplices in $S(X)_n$ are degenerate, hence $\mathbb{A}^i_{(red)}(X,\mathcal{F}) = 0$ for $i > n$. □

5.2 Rational Adeles

Proposition 5.2.1. *Let $S(X)$ be the simplicial set associated to the noetheria scheme X. Then there is for all $n \in \mathbb{N}_0$, $K \subset S(X)_n$ a functor*

$$a(K,\cdot) : \underline{\mathbf{QS}} \longrightarrow \underline{\mathbf{Ab}}$$

which is uniquely defined by a), b) *and* c), *additive and exact.*

a) *$a(K,\cdot)$ commutes with direct limits.*

b) *For $n = 0$, $\mathcal{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$, we have*

$$a(K,\mathcal{F}) = \prod_{P \in K} \mathcal{F}_P .$$

c) *For $n > 0$, $\mathcal{F} \in \mathrm{Ob}(\underline{\mathrm{CS}})$, we have*

$$a(K,\mathcal{F}) = \prod_{P \in P(X)} a\left(\hat{P}K, [\mathcal{F}_P]_P\right) .$$

Proof. This follows in the same way 2.1.1 did from lemma 1.2.4 and lef exactness of the direct image functor if one shows at the same time: If $\mathcal{F}$ i a sheaf with $\mathcal{F}_P = 0$ for all $P \in \mathrm{supp}(K)$, then $a(K,\mathcal{F}) = 0$. [

Definition 5.2.2. The group of rational n-dimensional adeles of X with coeffi cients in $\mathcal{F}$ is

$$a^n(X,\mathcal{F}) = a(S(X)_n,\mathcal{F}) .$$

The local factor of $a^n(X,\mathcal{F})$ in $\Delta \in S(X)_n$ is

$$a_\Delta(\mathcal{F}) = a(\{\Delta\},\mathcal{F}) .$$

The rational adeles form a cosimplicial group as well. The construction of th boundary and degeneracy maps in 2.2 and 2.3 was founded on combinatoria arguments, which remain valid.

All results on adeles carry over after having made the obvious changes. I particular

$$H^i\left(a^*(X,\mathcal{F})\right) = H^i(X,\mathcal{F}) .$$

For R-modules, there is a natural map $M \longrightarrow C_{\mathbf{p}}M$. If R is a Zariski-rin with respect to $\mathbf{p}$, then this is an inclusion. For all $\mathcal{O}$-module sheaves $\mathcal{F}$, thi generates an inclusion

$$a^*(X,\mathcal{F}) \longrightarrow \mathbb{A}^*(X,\mathcal{F}) .$$

It is a map of cosimplicial groups, natural in $\mathcal{F}$.

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