Honors Single Variable Calculus 110.113

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Contents

1 Equivalence Relation

Week 3 Reading: [3, Ch.3.5, Ch.4], *On the construction of* $\mathbb{Q}$*, see* [1, 2.4].

Learning Objectives

Last few lectures:

- Defined the natural numbers and sets axiomatically.
- Discussed how *cardinality* came up from "counting" sets.

This and next lecture:

- discuss equivalence relation.
- construct $\mathbb{Z}, \mathbb{Q}$. Extend addition and multiplication in this context.

1.1 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 1.1. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1, 2\} = \{2, 1\}$

In ordered pairs

- $(1, 2) \neq (2, 1)$

Definition 1.2. Let X, Y be two sets. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Currently, we can either put the existence of such a set as an axiom, or use the axioms of set theory, this is in hw.

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [2, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 1.3. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$
$$a \sim a$$
- *Transitive*. For all $a, b, c \in A$,
$$a \sim b, b \sim c \Rightarrow a \sim c$$
- *Symmetric*. For all $a, b \in A$,
$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Often times, people do not describe the subset R, but describe it a relation *equivalently* as a rule: saying $a, b \in A$ are related if some property $P(a, b)$ is true. In short hand, one writes

$$a \sim b \text{ iff } \ldots$$

Definition 1.4. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as [1]

$$[x] := \{y \in A : x \sim y\}$$

An element in such an equivalence is called a *representative* of that class.

Definition 1.5. Quotient set. Given an equivalence relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

Example

Consider $\mathbb{N}$ and the equivalence relation that $a \sim b$ iff a and b have the same parity. [a]

- There are two equivalence classes: the odds and evens.
- For the odd class, a *representative*, or an element in the equivalence class, is 3.

[a] i.e. both or odd or even.

There is a relation between equivalence and partition of sets.

Definition 1.6. A *partition* of a set X is a collection ???

1.2 Integers

What are the integers? It consists of the natural numbers and the negative numbers. What is *subtraction*? We do not know yet. Can we define *negative* numbers without referencing minus sign? Yes, we can. Say

$$-1\text{is "}0 - 1\text{" is } (0, 1)$$

Discussion

Let us say we define the integers as pairs (a, b) where $a, b \in \mathbb{N}$. Would this be our desired

$$\mathbb{Z} := \{\dots, -1, 0, 1, \dots\}$$

- How many -1s are there?

But we have a problem, there are multiple ways to express -1. Our system cannot have multiple -1s. What are other ways We can also have $1 - 2$, or the pair $(1, 2)$.

[1] It does not matter if we write $\{y \in A : y \sim x\}$ by symmetry condition.

Discussion

Now that we have our $\mathbb{Z}$, how do we define addition? [a]Can we leverage our understanding?

[a]What is addition abstractly? It is an operation $+ : X \times X \to X$.

Intuitively, the *integers* is an expression [2] of non-negative integers, (a, b), thought of as $a - b$. Two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 1.7. Let

$$R \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

consists of all pairs (a, b) and (c, d) such that $a + d = b + c$. Equivalently,

$$R := \{(a, b), (c, d) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \,:\, a + d = b + c\}$$

The *integers* is the set

$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

Definition 1.8. Addition, multiplication. [3, 4.1.2] .

We can now finally define negation.

Definition 1.9. [3, 4.1.4].

Proposition 1.10. Algebraic properties. Let $x, y, z \in \mathbb{Z}$.

- Addition
  - Symmetric $x + y = y + x$.
  - Admits identity element.

1.3 Rational nunbers

Reading: [1, 2.4]
In a similar manner

Definition 1.11. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote a pair (a, b) by a/b.

Again, we need the notion of addition, multiplication, and negation.

[2]Rather than a pair, as an expression has multiple ways of presentation

Definition 1.12. Let $a/b, c/d \in \mathbb{Q}$. Then

1. Addition:
$$a/b + c/d := (ad + bc)/bd$$

2. Multiplication
$$a/b \cdot c/d := (ac)/(bd)$$

3. Negation.
$$-(a/b) := (-a)/b$$

Discussion

Is this definition well defined? What does this mean? This is hw.

Similarly, we can define also define order relation.

Definition 1.13. Let $x \in \mathbb{Q}$,

- x is *positive* iff $x = a/b$ where a, b are positive integers, we often denote positive integers as $\mathbb{Z}_{>0}$.
- x is *negative* iff $x = -y$ where y is some positive rational.

With the notion of positive rationals[3] from def. 1.13, we can define order relation $<, \leq$ on $\mathbb{Q}$.

Definition 1.14. Let $x, y \in \mathbb{Q}$, then we say

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is zero or positive.

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 1.15. $\sqrt{2}$ is not rational.

Proof. ??? $\square$

[3] The same trick is used to define order in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

2 Homework for week 3

Due: Week 4, Saturday. You will select 3 problems to be graded.
Problems 1-3 are on cardinality. Problem 4 is on a general construction of equivalence relations. Problems 5-7 is about addition, multiplication, and division on $\mathbb{Z}$ and $\mathbb{Q}$.

1. Show that the relation $\leq$ is transitive, i.e. $|X| \leq |Y|, |Y| \leq |Z|$ then $|X| \leq |Z|$.

2. (**) Prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite. [4] Prove that $\mathbb{Q}$ is countably infinite. *You are free to use results from previous problems and theorems stated in lectures.*

3. (**) Let X be any set. Prove that there is no surjection (hence, bijection) between X and $\{0,1\}^X$. Deduce that $\{0,1\}^{\mathbb{N}}$ is uncountable. Argue the first part by contradiction:

   - Consider the set
   $$A = \{x \in X : x \notin f(x)\}$$
   - As f is a surjection (write the general definition) there must exists $a \in X$ such that $f(a) = A$. Do case work on whether $a \in A$ or $a \notin A$. [5]

4. (**) Let X be any set. Recall that a binary relation on X, is any subset $R \subseteq X \times X$. We define $R^{(n)}$ as follows

   - For $n = 0$,
   $$R^{(0)} = \{(x,x) : x \in X\}$$
   - Suppose $R^{(n)}$ has been defined.
   $$R^{(n+1)} := \left\{(x,y) \in X \times X : \exists z \in X, (x,z) \in R^{(n)}, (z,y) \in R\right\}$$

   (a) Show that
   $$R^t := \bigcup_{n \geq 1} R^{(n)} = R^{(1)} \cup R^{(2)} \cup \cdots$$
   defines a *smallest* transitive relation on X containing R. i.e. if Y is any other transitive relation on X containing R, then $R^t \subseteq Y$.

[4] Knowing the Cartesian product is required for this problem, skip 5. and 6. if unfamiliar.
[5] The argument is similar to that of Russell's argument.

(b) Show that

$$R^{tr} := \bigcup_{n\geq 0} R^{(n)} = R^{(0)} \cup R^{(1)} \cdots$$

is the *smallest* reflexive and transitive relation on X. i.e. if Y is any other transitive and reflexive relation on X containing R, then $R^{st} \subseteq Y$.

5. (***) Show that addition, product, and negation are well-defined for rational numbers; see def. 1.11 or [3, 4.2]. You are free to use any facts and properties you know about $\mathbb{Z}$.

6. (*) Let $x, y, z \in \mathbb{Z}$. Use the definition of addition and multiplication from 1.8, or [3, 4.1], show :

   (a) $x(y + z) = xy + xz$.

   (b) $x(yz) = (xy)z$.

   You are free to use any facts and properties you know about $\mathbb{N}$.

7. Let $x, y \in \mathbb{Z}$. You are free to use any facts you know about $\mathbb{N}$, in particular, it would be helpful to use the following the result: [3, 2.3.3]: *Let $n, m \in \mathbb{N}$. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero.* Show that if $xy = 0$ then $x = 0$ or $y = 0$.

2.1 Tri-weekly diary

8. (**) Write a 800-1000 words diary or story. Pen down a diary on your experiences with the course topics and experiences so far, focusing particularly on:

   - Concepts or ideas that you initially found challenging or confusing. For example, the axioms of natural numbers $\mathbb{N}$, set theory, etc.
   - Topics that have piqued (if any, XD) your curiosity.
   - Topics that you wanted to be covered, and why.
   - Topics that you would like further elaboration.
   - People you find fun to be with (or scared of)!

   + (*) points for the best diary.

3 Project Homework

... in mathematics you don't understand things. You just get used to them - von Neumann

Learning Objectives

A recurring theme that you would see throughout your study of more "theoretical" sciences is

- Making *good* definitions.
- *Working* with definitions

This project aims to familiarize you with the foundations of probability theory as set up by A. Komolgorov. In pure mathematics and its applications, it is desirable to have a foundation where one can discuss non deterministic statements, which we will refer as *events*, and non deterministic values, which are *random variables*. The project will proceed in the following order:

1. Probability space, 3.1.
2. Modeling (statistical)
3. You will then have few choices to explore:
   - Probability: we will explore foundational results as the strong law of large numbers.
   - Statistical: this is "more applied; we will explore application in the realm of inference and language models.
   - (?) Suprise.

Current status of the available content:

- 1. is available. The problems there are compulsory.
- 2 is not available. Problems are compulsory.
- 3 is not available. You will only have to pick one option depending on your taste.

3.1 Defining a probability space following A. Kolmogorov

Reading:

Definition 3.1. A *measure space* consists of a pair $(\Omega, \mathcal{E})$ where Ω is a set, and $\mathcal{E}$ is a σ-algebra on Ω.

- elements $E \in \mathcal{E}$ are referred as *events* or *measurable sets.*

Definition 3.2. Let $(\Omega, \mathcal{E})$ be a measure space. A *(finite) probability measure* is a map $\mathbb{P} : \mathcal{E} \to \mathbb{R}_{\geq 0}$ satisfying

1. $\mathbb{P}(\Omega) = 1$

2. Finitely additivity. Let $\{A_i\}_{i \in I}$ be a finite (that is $|I| = n$ for some $n \in \mathbb{N}$) collection of disjoint (def. **??**) elements in $\mathcal{E}$ [6]. Then

$$\mathbb{P}\left(\bigcup_{i=0}^{N} A_i\right) = \sum_{i=0}^{N} \mathbb{P}(A_i)$$

Once we have learnt the definition of series, we will add in another axiom called *countable additivity.*

Definition 3.3. A *probability space* is the datum of $(\Omega, \mathcal{E}, \mathbb{P})$, where $\mathbb{P}$ is a probability measure.

Example

The discrete case. Let Ω be a finite discrete set.

1. $\mathcal{E} := 2^{\Omega}$ is the set of all subsets of Ω. This is a σ-algebra.

2. Let p_w be any finite collection of real numbers such that $\sum_{w \in \Omega} p_w = 1$. Then there is a map

$$\mathbb{P} : 2^{\Omega} \to [0, 1]$$

uniquely extending the condition

$$\mathbb{P}(\{w\}) = p_w \quad w \in \Omega$$

[6] Remember, these are subsets of 2^{Ω}.

Example

Modeling n tosses of a fair coin. We define $(\Omega_n, \mathcal{E}, \mathbb{P})$.

- Ω_n is the set of all n consecutive ordered sets of letters which are either H or T.[a]
- $\mathcal{E}$ is the set of all subsets of Ω_n. One event can be

$$E_{\geq k} := \{\omega \in \Omega_n \ : \text{ at least } k \text{ heads appear in the } n \text{ toss}\}$$

  This is the set of all sequences with at least k Hs.

- Set $\mathbb{P}(\{\omega\}) = \frac{1}{2^n}$ for all singleton subsets $\{\omega\} \in \mathcal{E}$ where $\omega \in \Omega_n$. This uniquely extends to a function (why?)

$$\mathbb{P} : \mathcal{E} \to \mathbb{R}_{\geq 0}$$

[a] Of course, from our language of set theory, this is not a valid set. But we can equally use 0 or 1 to model this, in this case, this follows from the axioms.

3.1.1 Problems on n toss of a fair coin

The following problems are related to the model described above on n-tosses of a fair coin.

1. List out the elements in the event space of

$$\Omega_i, E_{\geq i}$$

   for $i = 1, 2$ and 3. Prove Ω_n has 2^n elements for $n \in \mathbb{N}_{\geq 1}$[7]

2. For a fix choice of n, give a formula for

$$\mathbb{P}(E_{\geq i})$$

   for $0 \leq i \leq n$.

3. Consider now Ω_{2n}. How many elements are the event

$$E := \{\text{exactly } n \text{ heads appear}\}$$

   Prove that

$$\mathbb{P}(E) = \frac{1}{2^{2n}}\binom{2n}{n}$$

   [8]

[7] This will be a short hand for positive integer.
[8] One can apply *Stirling's* formula to show that this is $\sim \frac{1}{\sqrt{\pi n}}$ as $n \to \infty$.

3.2 Conditional Expectation

Let us consider the discrete case for warm-up, once we have learnt integration, we will repeat the same story for density functions. The result below is referred as Baye's rule.

Definition 3.4. A *partition of a* X is a collection of subsets X_i, indexed by a set $i \in I$ such that

1. $\bigcup_{i \in I} X_i = X$
2. The sets X_is are pairwise disjoint: for any $i, j \in I$, the intersection (Def. **??**) of X_i and X_j is empty, $X_i \cap X_j = \emptyset$.

3.2.1 Problems

1. (**) Let I be a finite set. Let $\{B_1, B_2, \ldots\}_{i \in I}$ be a finite partition, 3.4, of Ω and $\mathbb{P}(B_i) > 0$ for all $i \in I$. Prove that

$$\mathbb{P}(A) = \sum_{i \in I} \mathbb{P}(A|B_i)\mathbb{P}(B_i)$$

using the additivity axiom.

2. (**) By conditioning on something, we would expect that we get a *new* probability space. If $B \in \mathcal{E}$ such that $\mathbb{P}(B) > 0$ show that $\mathbb{Q} : \mathcal{E} \to \mathbb{R}$ given by $\mathbb{Q}(A) = \mathbb{P}(A|B)$ defines a probability space $(\Omega, \mathcal{E}, \mathbb{Q})$.

References

[1] Derek Goldrei, *Propositional and predicate calculus: A model of argument*, 2005.

[2] Paul R. Halmos, *Naive set theory*, 1961.

[3] Terence Tao, *Analysis I, 4th edition*, 2022.