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Homotopy Exact Sequence for the Pro-Étale Fundamental Group

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The pro-étale fundamental group of a scheme, introduced by Bhatt and Scholze, generalizes the usual étale fundamental group $\pi_1^{\text{ét}}$ defined in SGA1 and leads to an interesting class of "geometric coverings" of schemes, generalizing finite étale covers. We prove exactness of the general homotopy sequence for the pro-étale fundamental group, i.e., that for a geometric point $\bar{s}$ on S and a flat proper morphism $X \to S$ of finite presentation whose geometric fibres are connected and reduced, the sequence $\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S) \to 1$ is "nearly exact." This generalizes a theorem of Grothendieck from finite étale covers to geometric coverings. We achieve the proof by constructing an infinite (i.e., non-quasi-compact) analogue of the Stein factorization in this setting.

1 Introduction

In [2], the authors introduced the pro-étale topology for schemes. Along with the new topology, they defined a new fundamental group—the pro-étale fundamental group. It is defined for a connected locally topologically Noetherian scheme X with a geometric point $\bar{x}$ and is denoted $\pi_1^{\text{proét}}(X, \bar{x})$.

In Grothendieck's approach, one takes the category $\mathrm{F\acute{E}t}_X$ of finite étale covers together with the fibre functor $F_{\bar{x}}$ and obtains an equivalence $\pi_1^{\text{ét}}(X, \bar{x}) - \mathrm{FSets} \simeq \mathrm{F\acute{E}t}_X$, where $G - \mathrm{FSets}$ denotes *finite* sets with a continuous G-action. The pro-étale

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fundamental group generalizes this; there is an equivalence between the category $\pi_1^{\text{proét}}(X,\bar{x})$ − Sets of possibly infinite discrete sets with a continuous $\pi_1^{\text{proét}}(X,\bar{x})$-action and a larger class of coverings, namely "geometric coverings," which are defined to be schemes Y over X such that $Y \to X$:

1. is étale (not necessarily quasi-compact!),
2. satisfies the valuative criterion of properness.

We denote the category of geometric coverings by Cov_X. An example of a nonfinite covering in Cov_X can be obtained by viewing an infinite chain of (suitably glued) $\mathbb{P}^1_k$'s as a covering of the nodal curve $X = \mathbb{P}^1/\{0,1\}$ obtained by gluing 0 and 1 on $\mathbb{P}^1_k$. Let π_1^{SGA3} be the prodiscrete group defined in [4, Chapter X.6]. The groups $\pi_1^{\text{ét}}$ and π_1^{SGA3} can be recovered from $\pi_1^{\text{proét}}$ by the profinite and the prodiscrete completion, respectively. The name "pro-étale" is justified by the fact that there is an equivalence $\mathrm{Cov}_X \simeq \mathrm{Loc}_{X_{\text{proét}}}$, where $\mathrm{Loc}_{X_{\text{proét}}}$ denotes the category of locally constant sheaves of discrete sets in $X_{\text{proét}}$.

The results

In SGA1, Grothendieck proved exactness of the homotopy sequence for the étale fundamental group.

Theorem. ([10, Exp. X, Corollary 1.4 + Corollary 1.8]) Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume S is connected and let $\bar{s}$ be a geometric point of S. Let $\bar{x}$ be a geometric point on $X_{\bar{s}}$. Then the sequence

$$\pi_1^{\text{ét}}(X_{\bar{s}},\bar{x}) \to \pi_1^{\text{ét}}(X,\bar{x}) \to \pi_1^{\text{ét}}(S,\bar{s}) \to 1$$

is exact.

Our main result generalizes this theorem from $\pi_1^{\text{ét}}$ to $\pi_1^{\text{proét}}$.

Theorem. (See Theorem 3.1) Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume that S is locally Noetherian, connected, and such that the normalization morphism $S^\nu \to S$ is finite. Let $\bar{s}$ be a geometric point of S and let $\bar{x}$ be a geometric point on $X_{\bar{s}}$. Then the sequence induced on the pro-étale fundamental groups

$$\pi_1^{\text{proét}}(X_{\bar{s}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(S, \bar{s}) \to 1$$

is nearly exact (see Definition 2.17).

The near-exactness means that one needs to take certain kinds of closures of the images to make them equal the kernels. For exactness in the middle, this cannot be avoided; taking for example $X = \{ZY^2 = W^3 + ZW^2 + Z^3 t\} \subset \mathbb{P}^2_R$, where $R = \mathbb{C}[[t]]$, one sees that the special fibre is the nodal curve described earlier and $\pi_1^{\text{proét}}(X_{\bar{s}}) = \mathbb{Z}$. On the other hand, the scheme X is normal, which implies that $\pi_1^{\text{proét}}(X) = \pi_1^{\text{ét}}(X) = \widehat{\mathbb{Z}}$. Thus, the sequence reads $\mathbb{Z} \to \widehat{\mathbb{Z}} \to 1 \to 1$ (see Remark 3.2).

The need for the notion of near-exactness stems from the fact that the group $\pi_1^{\text{proét}}$ has a more complicated topology than $\pi_1^{\text{ét}}$; the groups $\pi_1^{\text{proét}}$ belong to the class of Noohi groups. For example, they are not necessarily compact in general. However, near-exactness of the homotopy sequence translates to a statement about the geometric coverings (of $X_{\bar{s}}$, X and S), which is a generalization of the theorem of Grothendieck, where one replaces FÉt by Cov.

Besides $\pi_1^{\text{proét}}$ having a more complicated topology, the main difficulties in trying to directly generalize the proof of Grothendieck are as follows:

- geometric coverings of schemes, i.e., elements of Cov_X defined above, are often not quasi-compact, unlike elements of $\mathrm{FÉt}_X$. Some useful constructions that worked for finite étale covers, like the Stein factorization, will fail, unless considerably generalized.
- for a connected geometric covering $Y \in \mathrm{Cov}_X$, there is in general no Galois geometric covering dominating it. Equivalently, there might exist an open subgroup $U < \pi_1^{\text{proét}}(X)$ that does not contain an open normal subgroup. This prevents some proofs that would work for π_1^{SGA3} from carrying over to $\pi_1^{\text{proét}}$.

Indeed, the (near-)exactness in the middle of the homotopy exact sequence boils down to the following statement. For a connected $Y \in \mathrm{Cov}_X$ such that the structure morphism $Y \times_X X_{\bar{s}} \to X_{\bar{s}}$ has a section, there exists $T \in \mathrm{Cov}_S$ such that $Y \simeq T \times_S X$. For Y finite étale, one takes T to be the Stein factorization of $h : Y \to S$, i.e., $Y = \underline{\mathrm{Spec}}_S(h_* \mathcal{O}_Y)$ and checks that it is (finite) étale and has the desired properties. For non-quasi-compact Y, this definition usually does not give the correct answer. When trying to directly write Y

as a union (or a gluing) of quasi-compact subschemes (e.g., open or closed, or unions of irreducible components) and apply the Stein factorization to each, one quickly runs into problems. We construct the desired T in a different way and call it an "infinite Stein factorization." Our method is as follows; by the equivalence $\mathrm{Cov}_X \simeq \mathrm{Loc}_{X_{\text{proét}}}$, such a T should split completely after a base-change to some large pro-étale cover $\widetilde{S}$ of S. Thus, after base-changing to $\widetilde{S}$, there is an obvious candidate for $\widetilde{T} = T \times_S \widetilde{S}$; namely, $\widetilde{T} = \sqcup_{i \in I} \widetilde{S}_i$, i.e., a disjoint union of copies of $\widetilde{S}$ parametrized by some indexing set I. More precisely, $I = \pi_0(Y_{\tilde{S}})$. We show that for $\widetilde{T}$ defined in such a way, there is a map $\widetilde{Y} \to \widetilde{T}$ with the desired properties, and moreover, we have a descent datum with respect to the pro-étale cover $\widetilde{S} \to S$. We do this by first showing the theorem for S equal to the spectrum of a strictly Henselian ring and then generalizing it to $\widetilde{S}$, whose connected components are the spectra of strictly Henselian local rings, but $\pi_0(\widetilde{S})$ might have a complicated profinite topology.

The precise statement of the existence of the infinite Stein factorization is as follows.

Theorem (Theorem 3.4). Let S be a locally Noetherian scheme such that the normalization morphism $S^\nu \to S$ is finite. Let $X \to S$ be as in the theorem above and let $Y \in \mathrm{Cov}_X$ be connected. Then there exists a connected $T \in \mathrm{Cov}_S$ and a morphism $g : Y \to T$ over $X \to S$ such that g has geometrically connected fibres.

Moreover, for any two T_1, T_2 and maps $g_i : Y \to T_i$, $i = 1, 2$, as in the statement, there exists a unique isomorphism $\phi : T_1 \simeq T_2$ in Cov_S such that $g_2 = \phi \circ g_1$.

This article continues the study started in [11], where we dealt with the Künneth formula, general van Kampen theorem and the comparison between "geometric" and "arithmetic" fundamental groups in the case of $\pi_1^{\text{proét}}$. The main result, however, does not rely on the main results of *op. cit.*

1.1 Conventions and remarks

- A "topological group" will mean a "Hausdorff topological group."
- The pro-étale fundamental group is only defined for schemes that are locally topologically Noetherian. This does not cause problems when considering geometric coverings, as a geometric covering of a locally topologically Noetherian scheme is locally topologically Noetherian again—this is [2, Lemma 6.6.10].

- A “G-set” for a topological group G will mean a discrete set with a continuous action of G. We will denote the category of G-sets by $G-\mathrm{Sets}$. We will denote the category of (discrete) sets by Sets.
- We will often omit the base points from the statements and the discussion; by [11, Corollary 3.21], this usually does not change much.

2 Overview of [2] and some results from [11]

We will use the language and results of [2], especially of Chapter 7, as this is where the pro-étale fundamental group was defined. We are going to give a quick overview of some of these results below, but we recommend keeping a copy of [2] at hand.

Definition 2.1. ([2, Definition 7.1.1]) A Hausdorff topological group G is a *Noohi group* if the natural map induces an isomorphism $G \to \mathrm{Aut}(F_G)$ of topological groups. Here, $F_G : G-\mathrm{Sets} \to \mathrm{Sets}$ is the forgetful functor. For a discrete set S, we endow $\mathrm{Aut}(S)$ with the compact-open topology and $\mathrm{Aut}(F_G)$ is topologized using $\mathrm{Aut}(S)$ for $S \in \mathrm{Sets}$. More precisely, the stabilizers $\mathrm{Stab}^{\mathrm{Aut}(F_G)}_{F(S),s}$ for connected $S \in G-\mathrm{Sets}$, $s \in F(S)$, form a basis of neighbourhoods of $1 \in \mathrm{Aut}(F_G)$.

By [2, Proposition 7.1.5], a topological group is Noohi if and only if it satisfies the following conditions:

- its open subgroups form a basis of open neighbourhoods of $1 \in G$,
- it is Raĭkov complete.

A topological group G is Raĭkov complete if it is complete for its two-sided uniformity (see [5] or [1, Chapter 3.6] for an introduction to the Raĭkov completion).

Example 2.2. The following classes of topological groups are Noohi: discrete groups, profinite groups, $\mathrm{Aut}(S)$ with the compact-open topology for S a discrete set (see [2, Lemma 7.1.4]), groups containing an open subgroup, which is Noohi (see [2, Lemma 7.1.8]).

The following groups are Noohi: $\mathbb{Q}_\ell$, $\overline{\mathbb{Q}}_\ell$ for the colimit topology induced by expressing $\overline{\mathbb{Q}}_\ell$ as a union of finite extensions (in contrast with the situation for the ℓ-adic topology), $\mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$ for the colimit topology (see [2, Example 7.1.7]).

The notion of a Noohi group is tightly connected to the notion of an infinite Galois category, defined in [2, Definition 7.2.1]. An *infinite Galois category* is a pair

$(\mathcal{C}, F : \mathcal{C} \to \text{Sets})$ satisfying certain categorical properties, where $\mathcal{C}$ is a category. Such a pair is called *tame* if $\mathrm{Aut}(F)$ acts transitively on $F(Y)$ for any connected $Y \in \mathcal{C}$. Here, connectedness is defined in a category-theoretic sense, see [2, §7.2]. A tame infinite Galois category generalizes the notion of a Galois category, introduced by Grothendieck to define $\pi_1^{\text{ét}}$. The basic example is as follows: if G is a topological group, then $(G-\text{Sets}, F_G)$ is a tame infinite Galois category. It turns out, that any tame infinite Galois category is of this form. For a pair $(\mathcal{C}, F)$, one defines $\pi_1(\mathcal{C}, F) = \mathrm{Aut}(F)$. It is topologized using $\mathrm{Aut}(S)$ with compact-open topology, for $S \in$ Sets. More precisely, the stabilizers $\mathrm{Stab}_{F(S),s}^{\mathrm{Aut}(F_G)}$ for connected $S \in \mathcal{C}$, $s \in F(S)$, form a basis of neighbourhoods of $1 \in \mathrm{Aut}(F_G)$. By [2, Theorem 7.2.5], if $(\mathcal{C}, F)$ is an infinite Galois category, then $\pi_1(\mathcal{C}, F)$ is Noohi and if $(\mathcal{C}, F)$ is moreover tame, then F induces an equivalence

$$\mathcal{C} \simeq \pi_1(\mathcal{C}, F) - \text{Sets}.$$

The above formalism is a corrected version of the theory in [15]. It was also studied in [14, Chapter 4] under different names.

Pro-étale topology and the definition of $\pi_1^{\text{proét}}(X)$

Fix a locally topologically Noetherian scheme X.

Definition 2.3. Let $Y \to X$ be a morphism of schemes such that:

1. it is étale (not necessarily quasi-compact!)
2. it satisfies the valuative criterion of properness.

We will call Y a *geometric covering* of X. We will denote the category of geometric coverings by Cov_X.

As Y is not assumed to be of finite type over X, the valuative criterion does not imply that $Y \to X$ is proper (otherwise we would simply get a finite étale morphism). For example, for an algebraically closed field $\bar{k}$, the category $\mathrm{Cov}_{\mathrm{Spec}(\bar{k})}$ consists of (possibly infinite) disjoint unions of $\mathrm{Spec}(\bar{k})$ and we have $\mathrm{Cov}_{\mathrm{Spec}(\bar{k})} \simeq$ Sets. More generally, one has:

Lemma 2.4. ([2, Lemma 7.3.8]) If X is a Henselian local scheme, then any $Y \in \mathrm{Cov}_X$ is a disjoint union of finite étale X-schemes.

A basic example of a nonfinite connected covering in Cov_X can be obtained by viewing an infinite chain of (suitably glued) $\mathbb{P}^1_k$'s as a covering of the nodal curve $X = \mathbb{P}^1/\{0,1\}$ obtained by gluing 0 and 1 on $\mathbb{P}^1_k$.

Let us choose a geometric point $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$ on X. This gives a fibre functor $F_{\bar{x}} : \mathrm{Cov}_X \to \mathrm{Sets}$. By [2, Lemma 7.4.1]), the pair $(\mathrm{Cov}_X, F_{\bar{x}})$ is a tame infinite Galois category. Then one defines

Definition 2.5. The *pro-étale fundamental group* is defined as

$$\pi_1^{\text{proét}}(X, \bar{x}) = \pi_1(\mathrm{Cov}_X, F_{\bar{x}}).$$

In other words, $\pi_1^{\text{proét}}(X, \bar{x}) = \mathrm{Aut}(F_{\bar{x}})$ and this group is topologized using the compact-open topology on $\mathrm{Aut}(S)$ for any $S \in \mathrm{Sets}$.

One can compare the groups $\pi_1^{\text{proét}}(X, \bar{x})$, $\pi_1^{\text{ét}}(X, \bar{x})$ and $\pi_1^{\text{SGA3}}(X, \bar{x})$, where the last group is the group introduced in Chapter X.6 of [4].

Lemma 2.6. For a scheme X, the following relations between the fundamental groups hold

1. The group $\pi_1^{\text{ét}}(X, \bar{x})$ is the profinite completion of $\pi_1^{\text{proét}}(X, \bar{x})$.
2. The group $\pi_1^{\text{SGA3}}(X, \bar{x})$ is the prodiscrete completion of $\pi_1^{\text{proét}}(X, \bar{x})$.

Proof. This follows from [2, Lemma 7.4.3] and [2, Lemma 7.4.6]. ■

As shown in [2, Example 7.4.9], $\pi_1^{\text{proét}}(X, \bar{x})$ and $\pi_1^{\text{SGA3}}(X, \bar{x})$ are not equal in general. This can be also seen by combining [11, Example 4.5] with [11, Prop. 4.8].

The following lemma is very useful. Recall that, for example, a normal scheme is geometrically unibranch.

Lemma 2.7. ([2, Lemma 7.4.10]) If X is geometrically unibranch, then $\pi_1^{\text{proét}}(X, \bar{x}) \simeq \pi_1^{\text{ét}}(X, \bar{x})$.

There is another way of looking at the pro-étale fundamental group, which justifies the name "pro-étale".

Definition 2.8.

1. A map $f : Y \to X$ of schemes is called *weakly étale* if f is flat and the diagonal $\Delta_f : Y \to Y \times_X Y$ is flat.
2. The pro-étale site $X_{\text{proét}}$ is the site of weakly étale X-schemes, with covers given by fpqc covers.

This definition of the pro-étale site is justified by a foundational theorem—part 3 of the following fact.

Fact 2.9. Let $f : A \to B$ be a map of rings.

1. f is étale if and only if f is weakly étale and finitely presented.
2. If f is ind-étale, i.e., B is a filtered colimit of étale A-algebras, then f is weakly étale.
3. ([2, Theorem 2.3.4]) If f is weakly étale, then there exists a faithfully flat ind-étale $g : B \to C$ such that $g \circ f$ is ind-étale.

Definition 2.10. ([2, Definition 7.3.1.]) We say that $F \in \text{Shv}(X_{\text{proét}})$ is *locally constant* if there exists a cover $\{Y_i \to X\}$ in $X_{\text{proét}}$ with $F|_{Y_i}$ constant. We write Loc_X for the corresponding full subcategory of $\text{Shv}(X_{\text{proét}})$.

We are ready to state the following important result. We assume X to be locally topologically Noetherian.

Theorem 2.11. ([2, Lemma 7.3.9.]) One has $\text{Loc}_X = \text{Cov}_X$ as subcategories of $\text{Shv}(X_{\text{proét}})$.

Remark 2.12. There is also a notion of locally weakly constant sheaves ([2, Definition 7.3.1.]) that is sometimes useful. Its definition uses the notion of a classical sheaf. A sheaf is called *classical* if it lies in the essential image of $\nu^* : \text{Shv}(X_{\text{ét}}) \to \text{Shv}(X_{\text{proét}})$. The pullback $\nu^* : \text{Shv}(X_{\text{ét}}) \to \text{Shv}(X_{\text{proét}})$ is fully faithful. Its essential image consists exactly of those sheaves F with $F(U) = \text{colim}_i F(U_i)$ for any $U = \lim_i U_i$, where $i \mapsto U_i$ is a small cofiltered diagram of affine schemes in $X_{\text{ét}}$. Moreover, if a sheaf $G \in \text{Shv}(X_{\text{proét}})$ is classical when restricted to some pro-étale cover $\{Y_i \to X\}$, then G is classical.

Now, $F \in \text{Shv}(X_{\text{proét}})$ is *locally weakly constant* if there exists a cover $\{Y_i \to X\}$ in $X_{\text{proét}}$ with Y_i qcqs such that $F|_{Y_i}$ is classical and is the pullback via π of a sheaf on the profinite set $\pi_0(Y_i)$. Here, π refers to a map of topoi $\pi : \text{Shv}(Y_{\text{proét}}) \to \text{Shv}(\pi_0(Y)_{\text{proét}})$ discussed in [2]. By [2, Lemma 7.3.9.], one has $\text{Loc}_X = w\text{Loc}_X = \text{Cov}_X$, where $w\text{Loc}_X$ denotes the full subcategory of locally weakly constant sheaves.

Let us gather some notions and results that play an important role in the study of the pro-étale topology. They were introduced in [2, §2]. They are also nicely presented in [19, Chapter 0965].

Definition 2.13.

1. A spectral space X is *w-local* if it satisfies:
   (a) All open covers split, i.e., for every open cover $\{U_i \hookrightarrow X\}$, the map $\sqcup_i U_i \to X$ has a section.
   (b) The subspace $X^c \subset X$ of closed points is closed.
2. Fix a ring A.
   (a) A is *w-local* if $\mathrm{Spec}(A)$ is w-local.
   (b) A is *w-strictly local* if A is w-local, and every faithfully flat étale map $A \to B$ has a section.
3. A ring A is *w-contractible* if every faithfully flat ind-étale map $A \to B$ has a section.
4. A compact Hausdorff space is *extremally disconnected* if the closure of every open is open.

Fact 2.14.

1. A spectral space X is w-local if and only if $X^c \subset X$ is closed, and every connected component of X has a unique closed point. For such X, the composition $X^c \to X \to \pi_0(X)$ is a homeomorphism.
2. A w-local ring A is w-strictly local if and only if all local rings of A at closed points are strictly Henselian.
3. Any ring A admits an ind-étale faithfully flat map $A \to A'$ with A' w-strictly local.
4. ([6]) Extremally disconnected spaces are exactly the projective objects in the category of all compact Hausdorff spaces, i.e., those X for which every continuous surjection $Y \to X$ splits.
5. A w-strictly local ring A is w-contractible if and only if $\pi_0(\mathrm{Spec}(A))$ is extremally disconnected.
6. For any ring A, there is an ind-étale faithfully flat A-algebra A' with A' w-contractible.

Recollection of some facts on exactness of sequences of Noohi groups and descent for $\pi_1^{\text{proét}}$

In [11, §2.2] we defined and discussed properties of a "Noohi completion." For a (Hausdorff) topological group G, it is defined to be $G^{\text{Noohi}} = \text{Aut}(F_G)$. *If the open subgroups of G form a basis of neighbourhoods of* 1_G, then $G^{\text{Noohi}} \simeq \widehat{G}$, where $\widehat{G}$ denotes the Raĭkov completion (see [2, Prop. 7.1.5]). More generally (see [11, Prop. 2.26]), if the intersection of the open subgroups of G is trivial, then $G^{\text{Noohi}} \simeq \widehat{(G, \tau')}$, where τ' is a certain topology on G weaker than the original one. It is a Noohi group and has the following properties:

- any (continuous) morphism from G to a Noohi group factorizes through the natural map $\alpha_G : G \to G^{\text{Noohi}}$;
- F_G induces an equivalence $\tilde{F}_G : G - \text{Sets} \to G^{\text{Noohi}} - \text{Sets}$ and $\alpha_G^* \circ \widetilde{F}_G \simeq \text{id}$.

Let us now recall a part of the dictionary between statements regarding exactness of sequences of Noohi groups and statements about the induced maps on the categories of $G - \text{Sets}$. Before we start, recall ([11, Definition 2.29]) that we defined the "thick closure" $\overline{\overline{H}}$ of a subgroup H of a topological group G to be the intersection of all open subgroups of G containing H, i.e., $\overline{\overline{H}} := \bigcap_{H \subset U < G,\, U \text{ open}} U$. If a subgroup satisfies $H = \overline{\overline{H}}$ we will call it thickly closed in G.

Proposition 2.15. (part of [11, Prop. 2.37]) Let $G'' \xrightarrow{h'} G' \xrightarrow{h} G$ be maps between Noohi groups and $\mathcal{C}'' \xleftarrow{H'} \mathcal{C}' \xleftarrow{H} \mathcal{C}$ the corresponding functors between the infinite Galois categories, i.e., $\mathcal{C} = G - \text{Sets}$, $H = h^*$ and so on. Then the following hold:

1. The following are equivalent
   (a) The morphism $h : G' \to G$ has dense image, i.e., the topological closure of $h(G')$ in G is equal to G;
   (b) The functor H maps connected objects to connected objects.
2. $h'(G'') \subset \text{Ker}(h)$ if and only if the composition $H' \circ H$ maps any object to a completely decomposed object, i.e., to a (possibly infinite) disjoint union of final objects.
3. Assume that $h'(G'') \subset \text{Ker}(h)$ and that $h : G' \to G$ has dense image. Then the following conditions are equivalent:
   (a) $\overline{\overline{\text{Im}(h')}} = \text{Ker}(h)$ and the induced map $(G'/\ker(h))^{\text{Noohi}} \to G$ is an isomorphism;
   (b) for any connected $Y \in \mathcal{C}'$ such that $H'(Y)$ contains the final object of $\mathcal{C}''$ as a subobject, Y is in the essential image of H.

Here, being connected is equivalent to the action of the group being transitive. A subobject W of X means $W \to X$ such that $W \xrightarrow{\sim} W \times_X W$.

Remark 2.16. If G is a Noohi group and $H = \overline{H} \lhd G$ is its closed normal subgroup, then the quotient group G/H still has the property that open subgroups form a basis of open neighbourhoods of the identity. This is because the image of some basis of neighbourhoods of the identity of a Hausdorff group via quotient map is a basis of neighbourhoods of 1 in the quotient group (see [3, §3.2, Prop. 17]). However, the quotient G/H might still fail to be Raĭkov complete, and thus Noohi. The counterexample can be obtained by applying (the proof of) [16, Prop. 11.1]) (which gives a way of producing many examples of Raĭkov complete groups with non-complete quotients) to a non-complete abelian group whose open subgroups form a basis of neighbourhoods of 1, e.g., $(\mathbb{Z}_{(p)}, +)$ with p-adic topology. The basis of open neighbourhoods of the constructed group is uncountable and so the group is non-metrizable. This is necessary, as quotients of metrizable complete groups remain complete, see [3, §IX.3.1, Prop. 4].

The dictionary above suggests that we should work with weaker notions of exactness that translate into a statement about $G -$ Sets. Recall from [11, Definition 2.38].

Definition 2.17. Let $G'' \xrightarrow{h'} G' \xrightarrow{h} G \to 1$ be a sequence of topological groups such that $\mathrm{im}(h') \subset \ker(h)$. Then we will say that the sequence is

1. *nearly exact on the right* if h has dense image,
2. *nearly exact in the middle* if $\overline{\overline{\mathrm{im}(h')}} = \ker(h)$, i.e., the thick closure of the image of h' in G' is equal to the kernel of h,
3. *nearly exact* if it is both nearly exact on the right and nearly exact in the middle.

Let us finish this section by mentioning two facts about the geometric coverings. The first one is stated in [11, Prop. 3.12] and [12, Prop. 1.16] and relies on the results of [17].

Fact 2.18. Let $g : S' \to S$ be a proper, surjective morphism of finite presentation, then g is a morphism of effective descent for geometric coverings.

The following fact is a part of [11, Theorem 4.4].

Fact 2.19. Let X be a geometrically connected scheme of finite type over a field k. Let $\bar{k}$ be an algebraic closure of k and $\bar{x}$ be a $\mathrm{Spec}(\bar{k})$-point on X. Then the induced map $\pi_1^{\text{proét}}(X, \bar{x}) \to \mathrm{Gal}_k$ is open and surjective.

3 Homotopy exact sequence

3.1 Statement of the main result, infinite Stein factorization and some examples

Let us state the aim of this section. As we will soon see, the main ingredient of the proof will be the construction of the "infinite Stein factorization" for geometric coverings of Theorem 3.4 below.

Theorem 3.1. Let $f : X \to S$ be a flat proper morphism of finite presentation whose geometric fibres are connected and reduced. Assume that S is locally Noetherian, connected, and such that the normalization morphism $S^\nu \to S$ is finite. Let $\bar{s}$ be a geometric point of S and let $\bar{x}$ be a geometric point on $X_{\bar{s}}$. Then the sequence induced on the pro-étale fundamental groups

$$\pi_1^{\text{proét}}(X_{\bar{s}}, \bar{x}) \to \pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(S, \bar{s}) \to 1$$

is nearly exact (see Definition 2.17).

Moreover, the induced map

$$\left(\pi_1^{\text{proét}}(X, \bar{x}) / \overline{\overline{\mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{s}}, \bar{x}))}}\right)^{\text{Noohi}} \to \pi_1^{\text{proét}}(S, \bar{s})$$

is a homeomorphism. Here, $\overline{\overline{H}}$ denotes the "thick closure" defined earlier.

We will usually omit the base points.

Although it is not a part of the statement, it is true that the group $\pi_1^{\text{proét}}(X_{\bar{s}})$ does not depend on the underlying field of the geometric point of $\bar{s}$. This is by properness of f and [11, Prop. 3.31.].

Remark 3.2. An example that "nearly exact" is needed in the statement, i.e., we need thick closures: let R be a complete dvr with algebraically closed residue field k. Denote by K the field of fractions of R. Let X be a normal scheme proper over R such that X_K is an elliptic curve and X_k is a nodal curve. For example, take $R = \mathbb{C}[[t]]$ and $X = \{ZY^2 = W^3 + ZW^2 + Z^3t\} \subset \mathbb{P}^2_R$. It is a normal scheme. We then have $\pi_1^{\text{proét}}(X_k) = \mathbb{Z}$,

$\pi_1^{\text{proét}}(X) = \pi_1^{\text{ét}}(X) = \pi_1^{\text{ét}}(X_k) = \widehat{\mathbb{Z}}$ and $\pi_1^{\text{proét}}(\mathrm{Spec}(R)) = \pi_1^{\text{ét}}(\mathrm{Spec}(R)) = 0$. We used [10, Exp. X, Théorème 2.1] to identify $\pi_1^{\text{ét}}(X) = \pi_1^{\text{ét}}(X_k)$.

Proof. (of near exactness on the right in Theorem 3.1, i.e., the image of $\pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S)$ is dense): by Prop. 2.15, it is enough to check that the pullback of a connected geometric covering of S remains connected. This follows directly from Lemma 3.28 below. Let us give a slightly different proof. Geometrically connected and reduced fibres imply that $f_*(\mathcal{O}_X) = \mathcal{O}_S$ (see [19, Lemma 0E0L]). Let $u : U \to S$ be a connected geometric covering. As $X \to S$ is in particular quasi-compact and separated and $U \to S$ is flat, [20, Theorem 24.2.8] applies, i.e., "cohomology commutes with flat base change"; we get that in this situation $u^*f_*\mathscr{F} = f_{U*}u_X^*\mathscr{F}$ for a quasicoherent sheaf on X. Applying this to $\mathcal{O}_X$ and using our assumption, we get $f_{U*}\mathcal{O}_{X\times_S U} = \mathcal{O}_U$. Now, if $X \times_S U$ were disconnected, $\mathcal{O}_{X\times_S U}$ could be written as a product of two sheaves of algebras and the same would be true for $f_{U*}\mathcal{O}_{X\times_S U} = \mathcal{O}_U$, which would contradict connectedness of U. ■

Let us remark that if S is normal, we get actual surjectivity on the right, see Lemma 3.11.

Definition 3.3. To make the statements shorter, we will call a morphism of schemes $f : X \to S$ a "*morphism as in h.e.s.*" if S is connected and f is flat proper of finite presentation with geometrically connected and geometrically reduced fibres.

Let us state the main result. Its proof and the preparatory lemmas will occupy the subsections §3.2 and §3.3. It will imply the near exactness in the middle in Theorem 3.1.

Theorem 3.4. Let S be a locally Noetherian scheme such that the normalization morphism $S^\nu \to S$ is finite. Let $X \to S$ be as in h.e.s. and let $Y \in \mathrm{Cov}_X$ be connected. Then there exists a connected $T \in \mathrm{Cov}_S$ and a morphism $g : Y \to T$ over $X \to S$ such that g has geometrically connected fibres.

Moreover, for any two T_1, T_2 and maps $g_i : Y \to T_i$, $i = 1, 2$, as in the statement, there exists a unique isomorphism $\phi : T_1 \simeq T_2$ in Cov_S making the diagram

$$\begin{tikzcd} & T_1 \arrow[d, "\phi"] \\ Y \arrow[ur] \arrow[dr] & \\ & T_2 \end{tikzcd}$$

commute.

The uniqueness part of this theorem is elementary and we will explain it here. Let $\bar{x}$ be a geometric point on X and $\bar{s}$ its image on S. Let $Y \in \mathrm{Cov}_X$ be connected. Given two connected $T_i \in \mathrm{Cov}_S$ and maps $g_i : Y \to T_i$ over $X \to S$ that have geometrically connected fibres, we easily see that the maps g_i induce bijections $b_i : \pi_0(Y_{\bar{s}}) \to (T_i)_{\bar{s}}$. Let "$\phi_{\bar{s}}$" : $(T_1)_{\bar{s}} \xrightarrow{\sim} (T_2)_{\bar{s}}$ be the bijection given by $b_2 \circ b_1^{-1}$. We have to check, that the bijection "$\phi_{\bar{s}}$" is a map of $\pi_1^{\text{proét}}(S, \bar{s})-$Sets, i.e., that it is $\pi_1^{\text{proét}}(S, \bar{s})$-equivariant. It is easy to check that the following diagram of $\pi_1^{\text{proét}}(X, \bar{x}) -$ Sets is commutative

$$\begin{tikzcd} & (T_1)_{\bar{s}} \arrow[d, "\text{"}\phi_{\bar{s}}\text{"}"] \\ Y_{\bar{x}} \arrow[ur, two heads] \arrow[dr, two heads] & \\ & (T_2)_{\bar{s}} \end{tikzcd}$$

Thus, “$\phi_{\bar{s}}$” is $\pi_1^{\text{proét}}(X, \bar{x})$-equivariant and so also $\pi_1^{\text{proét}}(S, \bar{s})$-equivariant, as $\pi_1^{\text{proét}}(X, \bar{x}) \to \pi_1^{\text{proét}}(S, \bar{s})$ has dense image by the part of Theorem 3.1 that was already proven. The equivalence of categories $\mathrm{Cov}_S \simeq \pi_1^{\text{proét}}(S, \bar{s}) -$ Sets gives us ϕ. This finishes the proof of the uniqueness part. It is the proof of the existence part that will be our main concern.

Definition 3.5. In the situation of Theorem 3.4, we will refer to the scheme $T \in \mathrm{Cov}_S$ as the *infinite Stein factorization* of Y.

In the case when $Y \in \mathrm{Cov}_X$ is a finite étale cover, the “infinite Stein factorization” coincides with the usual Stein factorization of the map $Y \to S$. See [19, Lemma 0BUN] or [10, Exp. X, Proposition 1.2].

Proposition 3.6. Assume that Theorem 3.4 holds. Then Theorem 3.1 holds.

Proof. We have already checked above that the image of the map $\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X)$ is dense. We have to prove the remaining statements.

Let $Y \in \mathrm{Cov}_X$ be connected and such that $Y_{\bar{s}}$ has a section $\sigma : X_{\bar{s}} \to Y_{\bar{s}}$. Let $T \in \mathrm{Cov}_S$ be the “infinite Stein factorization” of Y over S constructed in Theorem 3.4. The section σ gives that $Y_{\bar{s}}$ contains a copy $X'_{\bar{s}}$ of $X_{\bar{s}}$ as a connected clopen subset. Let us write $T_{X_{\bar{s}}} = T_{\bar{s}} \times_{\bar{s}} X_{\bar{s}}$. Observe that $T_{\bar{s}} \simeq \sqcup_t \bar{s}$, and so $T_{X_{\bar{s}}} \simeq \sqcup_t X_{\bar{s}}$. As $Y_{\bar{s}} \to T_{\bar{s}}$ has connected fibres, one easily checks that that $X'_{\bar{s}}$ equals one of the fibres and the restriction of $Y_{\bar{s}} \to T_{X_{\bar{s}}}$ to $X'_{\bar{s}}$ is an isomorphism. Thus, one of the geometric fibres of $Y_{\bar{s}} \to T_{X_{\bar{s}}}$ is a singleton and so the same holds for $Y \to T_X$. As Y and T_X are connected geometric coverings of X, we conclude that $Y \to T_X$ is an isomorphism. By Proposition 2.15, the proof is finished. ■

An example of a class of schemes with finite normalization morphism is given by Nagata schemes. Let us recall some facts about those schemes. Firstly, Nagata schemes are locally Noetherian (by definition).

Fact 3.7. ([19, Lemma 035S]) Let X be a Nagata scheme. Then the normalization $\nu : X^\nu \to X$ is a finite morphism.

Fact 3.8. The spectra of the following rings are Nagata schemes: fields, Noetherian complete local rings, Dedekind rings of characteristic zero. Moreover, any scheme locally of finite type over a Nagata scheme is Nagata.

Proof. See [19, Tag 035A] and [19, Tag 035B]. ■

Example in the case S is normal

As an example, let us apply the stated homotopy exact sequence in the case S – normal. The direct analogue of the following result holds for the étale fundamental groups and can be checked using the usual homotopy exact sequence and diagram chasing. The point of the following proof is to show that we can redo this proof even if we have only near exactness.

Corollary 3.9. Let $f : X \to S$ be as in h.e.s. Assume S to be normal and locally Noetherian. Let ξ be its generic point. Then the induced morphism

$$\alpha : \pi_1^{\text{proét}}(X_\xi) \to \pi_1^{\text{proét}}(X)$$

has dense image.

Proof. Let $\bar{\xi}$ be a geometric point over ξ. Denote $K = \kappa(\xi)$. Applying Theorem 3.1, we have the following diagram

$$\begin{tikzcd}
\pi_1^{\text{proét}}(X_{\bar{\xi}}) \arrow[r] \arrow[d, equal] & \pi_1^{\text{proét}}(X_\xi) \arrow[r] \arrow[d, "\alpha"'] & \mathrm{Gal}_K \arrow[r] \arrow[d] & 1 \\
\pi_1^{\text{proét}}(X_{\bar{\xi}}) \arrow[r] & \pi_1^{\text{proét}}(X) \arrow[r, "\pi_1^{\text{proét}}(f)"] & \pi_1^{\text{proét}}(S) \arrow[r] & 1
\end{tikzcd}$$

with nearly exact rows. As S is normal, we have $\pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S)$ and the map $\mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$ is surjective by [19, Prop. 0BQM].

Let $U \subset \pi_1^{\text{proét}}(X)$ be an open subgroup and let $g \in \pi_1^{\text{proét}}(X)$. We need to show, that $\mathrm{im}(\alpha) \cap gU \neq \emptyset$. By Theorem 3.1, $\left(\pi_1^{\text{proét}}(X)/\ker(\pi_1^{\text{proét}}(f))\right)^{\text{Noohi}} \simeq \pi_1^{\text{proét}}(S)$, and thus there is $h \in \mathrm{im}(\alpha) \cap (\ker(\pi_1^{\text{proét}}(f)) \cdot gU)$. It can be also seen more directly, using Lemma 3.11: it implies that the morphism $\pi_1^{\text{proét}}(f)$ is open. It is also surjective, and thus it is a quotient map and $\pi_1^{\text{proét}}(X)/\ker(\pi_1^{\text{proét}}(f)) \simeq \pi_1^{\text{proét}}(S)$ as topological groups. We do not need to pass to Noohi completions in this case. So, from the diagram, we know that α is surjective modulo $\ker(\pi_1^{\text{proét}}(f))$. But from the near exactness of $\pi_1^{\text{proét}}(X_{\bar{\xi}}) \to \pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S)$, we have that $\ker(\pi_1^{\text{proét}}(f)) \cdot gU = \mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{\xi}})) \cdot gU$ and so $h \in \mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{\xi}})) \cdot gU$ implies that $h = xgu$ with $x \in \mathrm{im}(\pi_1^{\text{proét}}(X_{\bar{\xi}}))$, $u \in U$ and so $x^{-1}h \in \mathrm{im}(\alpha) \cap gU$. ■

Remark 3.10. Using Theorem 3.4 directly, one can give a short alternative proof of the above Corollary. Indeed, let $Y \in \mathrm{Cov}_X$ be connected. Let $T \in \mathrm{Cov}_S$ be the scheme obtained by applying Theorem 3.4. It is connected, and thus finite, as $\pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S)$. The scheme T_{ξ} is connected by the surjectivity of $\mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$. The morphism $Y \to T$ has geometrically connected fibres. Thus, the same holds also for $Y_{\xi} \to T_{\xi}$. As $Y_{\xi} \to T_{\xi}$ is open and surjective, this implies that Y_{ξ} is connected, as desired.

The following lemma is independent, i.e., we do not assume Theorem 3.1 in the proof.

Lemma 3.11. Let S be a locally Noetherian normal domain and ξ its generic point. Let $f : X \to S$ be a quasi-separated morphism of finite type. Assume X is connected and the fibre X_{ξ} is geometrically connected. Then the induced morphism

$$\pi_1^{\text{proét}}(f) : \pi_1^{\text{proét}}(X) \to \pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S)$$

is open and surjective.

Proof. We have a following diagram

$$\begin{tikzcd} \pi_1^{\text{proét}}(X_{\xi}) \arrow[r, two heads] \arrow[d] & \mathrm{Gal}_K \arrow[d, two heads] \\ \pi_1^{\text{proét}}(X) \arrow[r] & \pi_1^{\text{ét}}(S) \end{tikzcd}$$

As $\pi_1^{\text{proét}}(X)$ is Noohi, it is enough to show that the image of any open subgroup $U \subset \pi_1^{\text{proét}}(X)$ is open. Fix such U. Let V be the preimage of U in $\pi_1^{\text{proét}}(X_{\xi})$. Then the image $\pi_1^{\text{proét}}(f)(U)$ contains the image of V via $\pi_1^{\text{proét}}(X_{\xi}) \to \mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$. But both $\pi_1^{\text{proét}}(X_{\xi}) \to \mathrm{Gal}_K$ and $\mathrm{Gal}_K \to \pi_1^{\text{ét}}(S)$ are open; the first one by Fact 2.19 and the second one follows from Fact 3.12. Thus, $\pi_1^{\text{proét}}(f)(U)$ contains an open subgroup and so is open as desired. ■

Fact 3.12. Let $f : G \to H$ be a surjective morphism of Hausdorff topological groups. Assume G compact. Then f is open.

As stated, this fact can be easily checked by hand. This is also a special case of a more general "Open mapping theorem," see [5, Theorem 7.2.8].

3.2 Preliminary results on connected components of schemes

π_0 of a qcqs scheme

We found Appendix A of [18] to be a handy reference for dealing with connected components of qcqs schemes. We include two useful statements below.

Let X be a topological space and $a \in X$ a point. The *connected component* containing a, denoted C_a, is the union of all connected subsets containing a. This is the largest connected subset containing a, and it is closed. In contrast, the *quasicomponent* $Q = Q_a$ is defined as the intersection of all clopen neighborhoods of a, which is also closed. We list some handy facts below.

The following result is stated as [18, Lemma A.1]. As remarked there, it is due to Ferrand in the affine case, and Lazard in the general case (see [13, Prop. 6.1], [13, Corollary 8.5]).

Lemma 3.13. Let X be a qcqs scheme and $a \in X$ be a point. Then we have an equality $C_a = Q_a$ between the connected component and the quasicomponent containing a.

By [19, Tag 0900], each quasi-compact space X satisfying the assertion of the lemma above has a profinite set of connected components $\pi_0(X)$.

Corollary 3.14. Let X be a qcqs scheme. Then $\pi_0(X)$ is profinite.

Let us mention a lemma on pro-étale covers that fits the discussion.

Lemma 3.15. Let S be an affine scheme and $\widetilde{S} \to S$ a pro-étale cover by a w-strictly local affine scheme. Then

1. Each connected component of $\widetilde{S}$ is the strict henselization of the local ring at a certain point of S.

2. If S is moreover Noetherian, connected and normal and η denotes its generic point, then each connected component $c \subset \widetilde{S}$ is Noetherian, normal and its generic point is the unique point of c lying over η. The map

$$\pi_0(\widetilde{S}_\eta) \to \pi_0(\widetilde{S})$$

is a homeomorphism.

Proof. The scheme $\widetilde{S}$ is pro-étale over S and c is a pro-Zariski localization of $\widetilde{S}$. This is because a connected component of a qcqs scheme can be seen as an inverse limit of clopen subschemes containing it, see Lemma 3.13. In particular, $c \to S$ is weakly étale. By [19, Lemma 094Z], $c \to S$ induces an isomorphism on the strict henselizations of the local rings. As $\widetilde{S} \to S$ is w-strictly local, c is itself a strictly Henselian local ring. It follows that c is the strict henselization of a local ring at some point of S. As the strict henselization of a Noetherian ring is Noetherian, it follows that c is Noetherian. Being weakly étale over a normal scheme, c is normal (see [19, Tag 0950]). The scheme c is local, Noetherian and normal, thus integral. By [9, Corollary 18.8.14], the fibre c_η contains only one point—the generic point of c. We are using here that the associated primes of a reduced Noetherian ring are precisely the generic points of the irreducible components (see [19, Lemma 0EME] and [19, Lemma 05AR]). It follows that $\pi_0(\widetilde{S}_\eta) \to \pi_0(\widetilde{S})$ is a (continuous) bijection. As $\widetilde{S}$ is qcqs, $\pi_0(\widetilde{S})$ is compact (see Corollary 3.14) and so $\pi_0(\widetilde{S}_\eta) \to \pi_0(\widetilde{S})$ is in fact a homeomorphism. ■

However, to deal with (infinite) geometric coverings, one has to be more careful. The proof of Lemma 3.13 relies on a useful fact on the behaviour of connected components under cofiltered limits. It is essentially [8, Proposition 8.4.1 (ii)], but as explained in [18], in [8] the scheme is only assumed to be quasi-compact, while one needs to assume qcqs.

Fact 3.16. ([18, Proposition A.2]) Let X_0 be a quasi-compact and quasi-separated scheme, and X_λ a filtered inverse system of affine X_0-schemes, and $X = \varprojlim_{\lambda \in \Lambda} X_\lambda$. If $X = X' \sqcup X''$ is a decomposition into disjoint open subsets, then there is some $\lambda \in \Lambda$ and a decomposition $X_\lambda = X'_\lambda \sqcup X''_\lambda$ into disjoint open subsets so that $X', X'' \subset X$ are the respective preimages.

As explained below the proof of [18, Proposition A.2], both assumptions, quasi-compact and quasi-separated, are needed in general. All the spaces we are going to deal with will be quasi-separated. But nonfinite geometric coverings are not quasi-compact, and thus some extra care is needed when dealing with them. Thus, we devote some time to study connected components of (often) non-quasi-compact schemes.

Some aspects of Galois action on π_0

The following lemma is used a couple of times throughout the text. Its proof is based on the results in [19, Tag 0361] and [19, Tag 038D].

Lemma 3.17. Let X be a connected scheme over a field k with an l'-rational point with l'/k a finite field extension. Fix a separable closure k^{sep} of k. Then $\pi_0(X_{k^{\mathrm{sep}}})$ is finite, the Gal_k action on $\pi_0(X_{k^{\mathrm{sep}}})$ is continuous and there exists a finite separable extension l/k such that the induced map $\pi_0(X_{k^{\mathrm{sep}}}) \to \pi_0(X_l)$ is a bijection. Moreover, there exists the smallest field (contained in k^{sep}) with this property and it is Galois over k.

Proof. This is [11, Lemma 4.3]. ■

Connected components, fibres and geometric coverings

Lemma 3.18. Let X be a topologically Noetherian scheme and $Y \to X$ be in Cov_X. Let Z be an irreducible component of Y. Then Z is quasi-compact.

Proof. The image of Z in X sits in an irreducible component of X. We can base-change the situation to that component and assume that X is irreducible. Let $\eta \in Z \subset Y$ be the generic point of Z. Let $\widetilde{X} \to X$ be a cover in $X_{\mathrm{pro\acute{e}t}}$ by a qcqs scheme such that $\widetilde{Y} = Y \times_X \widetilde{X}$ represents a constant sheaf, i.e., $\widetilde{Y} \simeq \sqcup_{i\in I}\widetilde{X}$, where the indexing set I is possibly infinite. The morphism $\widetilde{X} \to X$ is qcqs. Thus, the same is true for $\widetilde{Y} \to Y$ and in turn the preimage $\tilde{E}$ of η in $\widetilde{Y}$ is quasi-compact. So there is a finite subset $I' \subset I$ such that $\tilde{E} \subset \sqcup_{i\in I'}\widetilde{X} \subset \widetilde{Y}$. Let $\tilde{Z}$ be the preimage of Z in $\widetilde{Y}$. Any point of Z generalizes to η and so, by flatness of $\widetilde{Y} \to Y$, the going-down property implies that any point of $\tilde{Z}$ generalizes to a point in $\tilde{E}$. It follows that the closure of $\tilde{E}$ in $\widetilde{Y}$ contains $\tilde{Z}$. But this closure is contained in $\sqcup_{i\in I'}\widetilde{X} \subset \widetilde{Y}$. This last set is quasi-compact. As $\widetilde{Z}$ is closed in $\widetilde{Y}$, it is quasi-compact as well. As $\widetilde{Z} \to Z$ is surjective, Z is quasi-compact, as desired. ■

Remark 3.19. An alternative proof of the last lemma can be given if the normalization X^ν of X is topologically Noetherian. Let $X^\nu \to X$ be the normalization map. Then the base-change $Y \times_X X^\nu$ is the normalization Y^ν of Y (see [19, Lemma 03GV]) and we have a diagram

$$\begin{array}{ccc} Y^{\nu} & \longrightarrow & Y \\ \downarrow & & \downarrow \\ X^{\nu} & \longrightarrow & X. \end{array}$$

Each irreducible component of Y is the image of a connected component of Y^{ν}. Thus, it is enough to show that the connected components of Y^{ν} are quasi-compact. As X^{ν} is topologically Noetherian, we can apply Lemma 2.7 to get that each connected component of Y^{ν} is finite (étale) over X^{ν}, and thus quasi-compact.

Lemma 3.20. Let X be a connected reduced topologically Noetherian scheme and let $Y \in \mathrm{Cov}_X$ be connected. Then there exist open immersions $U_n \overset{i_n}{\to} Y$ and closed immersions $Z_n \overset{j_n}{\to} Y$, $n \in \mathbb{Z}_{\geq 0}$ such that:

1. U_n, Z_n are of finite type over X, j_n factorizes through i_n and i_n factorizes through j_{n+1}, i.e., we have $Z_n \to U_n \to Z_{n+1} \to U_{n+1} \to \ldots \to Y$,
2. $\bigcup_n U_n = Y$,
3. Each Z_n is a finite union of irreducible components of Y.

Proof. Observe, that as Y is locally topologically Noetherian and locally of finite type over X, it is enough to ensure that Z_n and U_n are quasi-compact, to obtain that they are of finite type over X. By Lemma 3.18, every irreducible component of Y is quasi-compact. Let us define Z_1 to be any irreducible component of Y and U_1 to be a connected quasi-compact open neighbourhood of Z_1. It exists as Z_1 is quasi-compact and connected and Y is locally topologically Noetherian and so locally connected. Now, let

$$Z_2 = \bigcup_{\text{irr. comp. } Z \text{ of } Y \text{ s.t. } Z \cap U_1 \neq \emptyset} Z$$

As U_1 is a quasi-compact subset of a locally Noetherian space it is Noetherian and we see that the indexing set in the above sum is finite. By Lemma 3.18, we see that Z_2 is quasi-compact. Z_2 is a closed subset of Y and we put the reduced induced structure on it. Moreover, $U_1 \subset Z_2$ and Z_2 is connected. We can now take U_2 to be a connected quasi-compact open containing Z_2. Repeating this procedure we produce connected schemes Z_n, U_n satisfying (1) and (3). To check (2) we need to show that $U_\infty = \bigcup_n U_n$ is equal to Y. From connectedness of Y, it is enough to show that U_∞ is clopen. It is obviously open. In particular constructible. Thus, it is enough to show that it is closed under specialization (see [19, Tag 0542]. It is stated for Noetherian topological spaces but clearly locally Noetherian is enough, as for any $y \in \overline{U_\infty}$ we can check whether $y \in U_\infty$ by restricting to a topologically Noetherian neighbourhood V of y and working with the intersection $U_\infty \cap V$). Let $\xi \in U_\infty$ and assume ξ specializes to a point $y \in Y$. Let m be such that $\xi \in U_m \subset U_\infty$. There exist an irreducible component Z of Y containing ξ. It is closed and so $y \in Z$. But $Z \subset Z_{m+1}$ by construction. Thus, $y \in Z_{m+1} \subset U_{m+1} \subset U_\infty$ as desired. ■

Remark 3.21. One can use the above result to check that the closure of a quasi-compact subset of Y remains quasi-compact.

Observation 3.22. Let Y be a scheme and let $U_1 \subset U_2 \subset U_3 \subset \ldots \subset Y$ be an increasing sequence of open subschemes such that $\bigcup_n U_n = Y$. Then, directly from the sheaf property, it follows that

$$\Gamma(Y, \mathcal{O}_Y) = \varprojlim \Gamma(U_n, \mathcal{O}_{U_n}).$$

Lemma 3.23. Let Y be a reduced scheme over an algebraically closed field k having a filtration $Z_0 \subset U_0 \subset Z_1 \subset U_1 \subset \ldots \subset Y$ with U_i open and Z_i connected and proper over k. Then $\Gamma(Y, \mathcal{O}_Y) = k$

Proof. By Obs. 3.22, $\Gamma(U, \mathcal{O}_U) = \varprojlim \Gamma(U_n, \mathcal{O}_{U_n})$. But every map $\Gamma(U_{n+1}, \mathcal{O}_{U_{n+1}}) \to \Gamma(U_n, \mathcal{O}_{U_n})$ factorizes through $\Gamma(Z_{n+1}, \mathcal{O}_{Z_{n+1}}) = k$ (from the properness of Z_n and $k = \bar{k}$). Thus, $\varprojlim \Gamma(U_n, \mathcal{O}_{U_n}) = \varprojlim \Gamma(Z_n, \mathcal{O}_{Z_n}) = k$. ■

The following lemma makes precise the statement that "a flat degeneration of a disconnected scheme is either disconnected or nonreduced."

Lemma 3.24 Let R be a dvr and let X be a connected scheme flat over R. If the special fibre X_s is reduced, then the generic fibre X_ξ is connected.

Proof. This is [19, Tag 055J]. ■

The following remark is not used later and can be skipped, but gives some extra intuition.

Remark 3.25. Let X be a connected Noetherian scheme such that the normalization $X^\nu \to X$ is finite (e.g., X Nagata). Let $Y \in \mathrm{Cov}_X$ be connected and $\bar{x}$ be a geometric point on X. Then $Y_{\bar{x}}$ is countable. Indeed, using the van Kampen theorem in [11], we can write $\pi_1^{\text{proét}}(X, \bar{x})$ as the Noohi completion of a quotient of $*^{\text{top}}_v G_v * D$, v running over a finite set, G_v - profinite and $D \simeq \mathbb{Z}^{*r}$ a discrete and countable group. Fix some v_0. Then, with the notation as in the proof of [11, Theorem 4.14], the sets $O_{v_0}^N$ are finite and $Y_{\bar{x}} = \bigcup_{N>0} \bigcup_{o \in O_{v_0}^N} o$, which finishes the proof, as the G_{v_0}-orbits o are finite.

Some topology involving π_0's of non-Noetherian schemes

Lemma 3.26. Let $f : W \to T$ be a qcqs morphism of schemes. Assume that each connected component of T is locally connected (e.g., each connected component is locally topologically Noetherian). Assume that the image of W is dense in every connected component of T. Then the induced map $\pi_0(f) : \pi_0(W) \to \pi_0(T)$ is a topological quotient map.

Proof. The map $\pi_0(f)$ is surjective by the assumption of dense images. Let $U_0 \subset \pi_0(T)$. Assume that $\pi_0(f)^{-1}(U_0)$ is open. We want to show that U_0 is open. As the topology on $\pi_0(T)$ is the quotient topology from T, it is enough to show that $U = \pi^{-1}(U_0) \subset T$ is open. We have a commutative diagram

$$\begin{tikzcd} W \arrow[r, "f"] \arrow[d, "\pi"'] & T \arrow[d, "\pi"'] \\ \pi_0(W) \arrow[r, "\pi_0(f)"] & \pi_0(T). \end{tikzcd}$$

Thus, $f^{-1}(U) = \pi^{-1}(\pi_0(f)^{-1}(U_0))$ is open. To prove that U is open, it is enough to show that, for each affine open V of T, the intersection $U \cap V$ is open in V. Fix such V and denote $W_V = f^{-1}(V)$. Observe that $f|_{W_V}^{-1}(U \cap V) = f^{-1}(V) \cap W_V$ is open in W_V. Consider the commutative diagram of topological spaces

$$\begin{tikzcd} W_V \arrow[r, "f|_{W_V}"] \arrow[d, "\pi"'] & V \arrow[r, "\subset"] \arrow[d, "\pi"'] & T \arrow[d, "\pi"'] \\ \pi_0(W_V) \arrow[r, "\pi_0(f|_{W_V})"] & \pi_0(V) \arrow[r] & \pi_0(T). \end{tikzcd}$$

It follows from the diagram that there exists a subset $U_0' \subset \pi_0(V)$ such that $V \cap U = \pi^{-1}(U_0')$. Moreover, as V is affine and f is qcqs, $\pi_0(f|_{W_V})$ is a (continuous) surjective map of compact spaces and so a quotient map. Surjectivity of $\pi_0(f|_{W_V})$ follows from the assumptions: local connectedness of connected components of T implies that each connected component of V is an open subset of a connected component of T and by the assumption that the image of f is dense in every connected component of T, we get the desired surjectivity. As $\pi^{-1}(\pi_0(f|_{W_V})^{-1}(U_0')) = f^{-1}(V \cap U)$ is open and both π and $\pi_0(f|_{W_V})$ are quotient maps, we conclude that U_0' is open, and thus $V \cap U$ is open as desired. ■

Lemma 3.27. Let $f : W \to T$ be a continuous map of topological spaces. Assume that f is a topological quotient map (e.g., surjective and open or surjective and closed). Then $\pi_0(f)$ is a topological quotient map.

Proof. Let $U_0 \subset \pi_0(T)$ be such that $\pi_0(f)^{-1}(U_0)$ is open. We want to show that U_0 is open. It is equivalent to checking that $U = \pi^{-1}(U_0)$ is open. But $f^{-1}(U) = \pi^{-1}(\pi_0(f)^{-1}(U_0))$ and so is open. As f is a quotient map, U is open as well, which finishes the proof. ■

Lemma 3.28. Let $f : X \to S$ be a universally open and surjective morphism of schemes (e.g., f faithfully flat locally of finite presentation) with geometrically connected fibres. Then for any morphism of schemes $\widetilde{S} \to S$, the map induced on π_0's by the base-change of f to $\widetilde{S}$

$$\pi_0(f) : \pi_0(\widetilde{X}) \to \pi_0(\widetilde{S})$$

is a homeomorphism.

Proof. We can obviously assume $\widetilde{S} = S$. Let $t \in \pi_0(S)$. We can see it as a closed subscheme $t \hookrightarrow S$ and obtain $f_t : X_t \to t$ via base-change. Let us first show that $\pi_0(f)$ is bijective. As f is surjective, $\pi_0(f)$ is surjective as well, and thus we only need to show that X_t is connected. But this follows easily from the fact that t is connected, f_t is open, surjective and has connected fibres. Thus, $\pi_0(f)$ is a continuous bijection and by Lemma 3.27, it is a homeomorphism. ■

3.3 Proof of Theorem 3.4

Let us start with two results that essentially give the homotopy exact sequence in the case S equal to a spectrum of a strictly Henselian ring. Recall that in this case $\pi_1^{\text{proét}}(S) = \pi_1^{\text{ét}}(S) = 1$ by [2, Lemma 7.3.8].

Proposition 3.29. Let S be a spectrum of a strictly Henselian Noetherian ring, let $X \to S$ be proper with X connected and let $Y \in \mathrm{Cov}_X$ be connected. Let $\bar{s}$ be a geometric point over the closed point s of S. Then the geometric fibre $Y_{\bar{s}}$ is connected. In other words, by Prop. 2.15, the morphism

$$\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X)$$

has dense image.

As a result, for any $Y \in \mathrm{Cov}_X$, the natural map $\pi_0(Y_{\bar{s}}) \to \pi_0(Y)$ is a bijection (of discrete sets).

Proof. As the residue field $\kappa(s)$ is separably closed, it is enough to show that Y_s is connected ([19, Tag 0387]). Assume the contrary and write $Y_s = W_1 \sqcup W_2$, where W_1, W_2 are clopen subsets of Y_s. Apply Lemma 3.20 to Y and produce a sequence of connected quasi-compact closed $Z_n \subset Y$ such that $\bigcup_n Z_n = Y$. There exist $N \geq 0$ such that $Z_N \cap W_1 \neq \emptyset$ and $Z_N \cap W_2 \neq \emptyset$. Thus, the fibre $Z_{N,s}$ is not connected. But Z_N is of finite type over X and satisfies the valuative criterion of properness (as $Y \in \mathrm{Cov}_X$). Thus, Z_N is proper over X and so also over S. By a special case of the proper base-change theorem (see e.g., [19, Lemma 0A3S] or [19, Lemma 0A0B]), the induced map $\pi_0(Z_{N,s}) \to \pi_0(Z_N)$ is bijective. This is a contradiction. ■

Lemma 3.30. Let S be the spectrum of a strictly Henselian dvr R and let $X \to S$ be a morphism as in h.e.s. Let $\bar{\xi}$ be a geometric point over the generic point ξ of S. Then the morphism

$$\pi_1^{\text{proét}}(X_{\bar{\xi}}) \to \pi_1^{\text{proét}}(X)$$

has dense image. In other words, by Prop. 2.15, for a connected $Y \in \mathrm{Cov}_X$, $Y_{\bar{\xi}}$ remains connected. As a result, for any $Y \in \mathrm{Cov}_X$, the natural map $\pi_0(Y_{\bar{\xi}}) \to \pi_0(Y)$ is a bijection (of discrete sets).

Proof. Denote by s the closed point of S. Let $Y \in \mathrm{Cov}_X$ be connected. Using Prop. 3.29, we know that Y_s is connected. The scheme Y_ξ is locally of finite type over $K = \kappa(\xi)$. Thus, it has an L'-point with L'/K a finite field extension. Applying Lemma 3.17, we get that there exists a finite separable extension L/K such that $Y_{\xi,L}$ has a finite number of connected components and each of them is geometrically connected. Let R' be an integral closure of R in L. As R is Henselian and the extension L/K is separable, R' is a local algebra finite over R. The ring R' is thus a dvr. Let π' be its uniformizer. The field $k' = R'/\pi'$ is a finite (purely inseparable) extension of $R/\pi = k$. Thus, we can assume $k' \subset \kappa(\bar{s})$. Thus, we can see $\bar{s}$ as a geometric point of $\mathrm{Spec}(R')$ lying over the special point s'. Denoting $Y' = Y \times_S \mathrm{Spec}(R')$, we have $Y'_{\bar{s}} = Y_{\bar{s}}$ and so it is reduced and connected. As $Y'_{\bar{s}} \to Y'_{s'}$ is faithfully flat, we conclude that the fibre $Y'_{s'}$ is connected and reduced as well. Thus, also Y' is connected; this is clear when thinking of $Y'_{s'}$ and Y' as elements of $\pi_1^{\text{proét}}(X'_{s'}) - \text{Sets}$ and $\pi_1^{\text{proét}}(X') - \text{Sets}$. By Lemma 3.24, we conclude that the generic fibre $Y'_{\mathrm{Frac}(R')}$ is connected. But $Y'_{\mathrm{Frac}(R')} = Y' \times_{\mathrm{Spec}(R')} \mathrm{Spec}(L) = Y \times_{\mathrm{Spec}(R)} \mathrm{Spec}(L) = Y_{\xi,L}$.

Combining it with what we observed at the beginning of the proof, we conclude that $Y_{\xi,L}$ is geometrically connected. This finishes the proof. ∎

Theorem 3.31. Let S be a connected Noetherian scheme. Let $X \to S$ be as in h.e.s. Let $\bar{\xi}$ and $\bar{s}$ be two geometric points on S with images $\xi, s \in S$. Then, for any $Y \in \mathrm{Cov}_X$, there is a bijection

$$\pi_0(Y_{\bar{\xi}}) \simeq \pi_0(Y_{\bar{s}}).$$

It depends on the choice of a "path" between $\bar{s}$ and $\bar{\xi}$, i.e., chain of maps from strictly Henselian dvrs (see the proof). When S is the spectrum of a strictly Henselian dvr and $\bar{\xi}$ and $\bar{s}$ lie over the special and the generic point, the bijection is the obvious one given by combining Proposition 3.29 with Lemma 3.30.

Proof. As S is connected and Noetherian, we can join s with ξ by a finite sequence of specializations and generizations of points on S. Indeed, every point lies on one of the finitely many irreducible components $Z_1, \ldots, Z_m$ of S and within a fixed irreducible component every point is a specialization of the generic point. It follows that the set of points reachable via sequence of specializations and generizations from a given point is a union of some irreducible components, and thus closed. There is only finitely many of such "path components," and thus it is also open. Thus, we can and do reduce to the case where s is a specialization of ξ. By [19, Tag 054F] we can find a dvr R and a morphism $\mathrm{Spec}(R) \to S$ such that the generic point of $\mathrm{Spec}(R)$ maps to ξ and the special point maps to s. The strict henselization R^{sh} of R is a strictly Henselian dvr by [19, Tag 0AP3]. Let ξ', s' be the generic and the special point of $\mathrm{Spec}(R^{sh})$, respectively, and let $\bar{\xi}'$ and $\bar{s}'$ be some geometric points over them. By Lemma 3.30 and Proposition 3.29, we conclude that $\pi_0(Y_{\bar{s}'}) \simeq \pi_0(Y_{\mathrm{Spec}(R^{sh})}) \simeq \pi_0(Y_{\bar{\xi}'})$. Choosing geometric points $\bar{s}''$ and $\bar{\xi}''$ on S such that $\bar{s}''$ factors both through $\bar{s}$ and $\bar{s}'$ and $\bar{\xi}''$ factors both through $\bar{\xi}$ and $\bar{\xi}'$ and using the fact that the base-change of a connected scheme over an algebraically closed field to another algebraically closed field remains connected (see [19, Lemma 037R]), we finish the proof. ∎

Corollary 3.32. Let S be the spectrum of a strictly Henselian Noetherian ring and let $X \to S$ be as in h.e.s. Let $\bar{s}$ be any geometric point on S. Then the morphism

$$\pi_1^{\text{proét}}(X_{\bar{s}}) \to \pi_1^{\text{proét}}(X)$$

has dense image. In other words, by Proposition 2.15, for a connected $Y \in \mathrm{Cov}_X$, the base-change $Y_{\bar{s}} \in \mathrm{Cov}_{X_{\bar{s}}}$ remains connected. As a result, for any $Y \in \mathrm{Cov}_X$, the natural map $\pi_0(Y_{\bar{s}}) \to \pi_0(Y)$ is a bijection (of discrete sets).

Proof. If $\bar{s}$ lies over the closed point of S, then the statement was proven in Prop. 3.29. But now Theorem 3.31 tells us that the statement holds for any $\bar{s}$. ■

Let us also mention a technical lemma used later in the proof.

Lemma 3.33. Let X be a compact topological space. Let $W = \sqcup_i X_i$ be a disjoint union of copies of X, indexed by $i \in I$. Let $g : W \to X$ be the obvious structural map. Let $g_i : X_i \subset W \to X$ be the structural (iso-)morphism. Let $\phi \in \mathrm{Aut}_X(W)$ be an automorphism of W over X. Then

1. there exist two decompositions of W into clopen subsets, $W^1_{ij_i}$ and $W^2_{ij'_i}$, $i \in I, j_i \in J_i, j'_i \in J'_i$, such that:
   - $W^1_{ij_i}, W^2_{ij'_i} \subset X_i$,
   - $X_i = \cup_{j_i \in J_i} W^1_{ij_i} = \cup_{j'_i \in J'_i} W^2_{ij'_i}$ and the sets J_i and J'_i are finite for every $i \in I$,
   - ϕ maps the first decomposition onto the second, i.e., for each $i_1 \in I, j_{i_1} \in J_{i_1}$, there exist $i_2 \in I, j'_{i_2} \in J'_{i_2}$ such that $\phi(W^1_{i_1 j_{i_1}}) = W^2_{i_2 j'_{i_2}}$. This gives a bijection $\theta : \sqcup_{i \in I} J_i \to \sqcup_{i \in I} J'_i$.
   - the map ϕ can be recovered from $\{g_{i,j_i} = g_{i|W^1_{ij_i}} : W^1_{ij_i} \to X\}$, $\{g_{i,j'_i} = g_{i|W^2_{ij'_i}} : W^2_{ij'_i} \to X\}$ and θ by defining $\phi|_{W^1_{ij_i}} = g^{-1}_{i',\theta(j_i)} \circ g_{i,j_i} : W^1_{ij_i} \to W^2_{i',\theta(j_i)} \subset W$, where i' is such that $\theta(j_i) \in J'_{i'}$. Here, $g^{-1}_{i',\theta(j_i)}$ is defined on the image of $W^2_{i',\theta(j_i)}$ via $g_{i',\theta(j_i)}$ in X.
2. Assume that $X = \pi_0(X')$ for some topological space X'. Let $W' = \sqcup_i X'_i$ be the disjoint union of copies of X'. Then ϕ lifts uniquely to an automorphism of W' over X', i.e., there exists an automorphism $\phi' \in \mathrm{Aut}_{X'}(W')$ of the X'-topological space W' such that $\phi = \pi_0(\phi')$.
3. Assume that $X = \pi_0(X')$ for some scheme X'. Let $W' = \sqcup_i X'_i$ be the disjoint union of copies of X'. Then ϕ lifts uniquely to an automorphism of W' over X', i.e., there exists an automorphism $\phi' \in \mathrm{Aut}_{X'}(W')$ of the X'-scheme W' such that $\phi = \pi_0(\phi')$.

Proof. The first part follows by defining $W^1_{ij_i}$, $j_i \in J_i$, to be the finitely many non-empty intersections $X_i \cap \phi^{-1}(X_{i'})$, where i' runs through I. Similarly, $W^2_{ij'_i}$'s are defined to be the non-empty intersections $X_i \cap \phi(X_{i'})$. The claim about recovering ϕ from θ and g_{i,j_i}'s follows from the fact that $\phi \in \mathrm{Aut}_X(W)$, and thus respects the structure maps g_i.

To see the second second part, define $(W^1_{ij_i})' = \pi_0^{-1}(W^1_{ij_i})$ and similarly for $(W^1_{ij_{i'}})'$. Let $g'_i : X'_i \to X'$ be the structure isomorphisms. Denote $g'_{i,j_i} = g'_i|_{(W^1_{ij_i})'} : (W^1_{ij_i})' \to X'$ and $g'_{i,j'_i} = g_i|_{(W^2_{ij'_i})'} : (W^2_{ij'_i})' \to X'$. Then $\phi'|_{(W^1_{ij_i})'}$ is defined to be $(g'_{i',\theta(j_i)})^{-1} \circ g'_{i,j_i}$ and $\{\phi'|_{(W^1_{ij_i})'}\}_{i,j_i}$ glue to an (auto)morphism of the topological space W' over X', as g'_{i,j_i} and $(g'_{i',\theta(j_i)})^{-1}$ are morphisms of topological spaces. The uniqueness of this choice follows again from the fact that $\phi' \in \mathrm{Aut}_{X'}(W')$ has to respect the structure isomorphisms g'_i.

The third part follows analogously; in this case g'_{i,j_i} and $(g'_{i',\theta(j_i)})^{-1}$ are morphisms of schemes. ∎

Proof of Theorem 3.4:

Proof. Uniqueness was proven right under the statement of the theorem. Let us proceed with the proof of existence. It proceeds in the following steps:

1. Reduction to the case S - normal.
2. Introduction of the scheme $\widetilde{S}$ and the corresponding base-changes $\widetilde{X}$, $\widetilde{Y}$. Proof of the motivating result: there exists a non-canonical bijection $\pi_0(\widetilde{Y}) \simeq \pi_0(Y_{\bar{s}}) \times \pi_0(\widetilde{S})$. It is not enough to proceed, however; we need to construct a homeomorphism.
3. Proof of Claim 1: there exists a homeomorphism $\pi_0(\widetilde{Y}) \simeq \pi_0(Y_{\bar{s}}) \times \pi_0(\widetilde{S})$. This is the biggest step; it includes two unnumbered lemmas.
4. Definition of the scheme $\widetilde{T}$ and the map $\widetilde{Y} \to \widetilde{T}$ over $\widetilde{S}$.
5. Definition of the descent datum $\phi : p^*\widetilde{T} \to q^*\widetilde{T}$ over $\widetilde{S}_2 = \widetilde{S} \times_S \widetilde{S}$, checking the cocycle condition and finishing the construction of T by descent. This includes Claims 2 and 3.

Step 1: Reduction of the proof to the case S - normal. Let $S^\nu \to S$ be the normalization morphism. It is finite by the assumption. Thus, it is a morphism of effective descent for Cov_S by Fact 2.18. Assume that the statement of the theorem holds for normal schemes. Let $Y \in \mathrm{Cov}_X$ be connected. Denote by Y' and X' the base-changes of Y and X to S^ν (we do not denote it by Y^ν and X^ν to avoid the confusion with the normalizations). Applying the theorem to each element of the discrete set of connected components of S^ν, we obtain $T' \in \mathrm{Cov}_S$ and a surjective morphism $Y' \to T'$ over $X' \to S$. The proof will be finished if we equip $Y' \to T'$ with a descent datum with respect to

$S^\nu \to S$. Let $p, q : S'_2 = S^\nu \times_S S^\nu \to S^\nu$ be the two projections. We get two morphisms $Y'_2 = Y \times_S S'_2 \to p^*T'$ and $Y'_2 = Y \times_S S'_2 \to q^*T'$ over $X'_2 \to S'_2$ that are surjective with geometrically connected fibres. Thus, by the uniqueness part of the theorem applied to each connected component of S'_2, we get an isomorphism $\phi : p^*T' \to q^*T'$. Then the cocycle condition holds by applying the uniqueness statement again, this time over $S'_3 = S^\nu \times_S S^\nu \times_S S^\nu$. Observe that the morphism $Y' \to T'$ descends as well by the following reasoning: the schemes Y', T' descend to Y and T and then we look at Y and $T_X = T \times_S X \in \mathrm{Cov}_X$, and see that $Y' \to T' \times_{S^\nu} X'$ descends to a morphism $Y \to T_X$. This finishes the reduction. Similarly, we see that the problem is Zariski local on S and we can assume S to be affine.

Thus, we can and do assume that S is normal and affine in the rest of the proof.

Step 2: Introduction of the schemes $\widetilde{S}, \widetilde{X}, \widetilde{Y}$ and the motivating result – there exists a non-canonical bijection $\pi_0(\widetilde{Y}) \simeq \pi_0(Y_{\bar{s}}) \times \pi_0(\widetilde{S})$. Let $\widetilde{S} \to S$ be a pro-étale cover such that $\widetilde{S}$ is affine w-strictly local (as defined in [2, Definition 2.2.1]) and such that the map $\mathcal{O}_S(S) \to \mathcal{O}_{\widetilde{S}}(\widetilde{S})$ is ind-étale, i.e., $\mathcal{O}_{\widetilde{S}}(\widetilde{S})$ is a filtered colimit of étale $\mathcal{O}_S(S)$-algebras. This is possible by [2, Corollary 2.2.14] or [19, Tag 097R]). All local rings at closed points of $\widetilde{S}$ are strictly Henselian by [2, Lemma 2.2.9]. By Lemma 3.15, they are equal to the strict henselizations of the local rings at the corresponding (geometric) points of S, and thus are Noetherian. By w-locality of $\widetilde{S}$ and [2, Lemma 2.1.4], if $\widetilde{S}^c$ denotes the set of closed points of $\widetilde{S}$, the composition map $\widetilde{S}^c \to \widetilde{S} \to \pi_0(\widetilde{S})$ is a homeomorphism. From this we see that each localization at a closed point of $\widetilde{S}$ is equal to a connected component of $\widetilde{S}$ containing this point. Let us denote by $\widetilde{X}$ and $\widetilde{Y}$ the base-changes of X and Y to $\widetilde{S}$. By Corollary 3.32, if $s' \in \widetilde{S}$ is a closed point and if $c_{s'}$ denotes the connected component of s', then $Y_{c_{s'}}$ is a disjoint union of connected schemes and $\pi_0(Y_{c_{s'}})$ can be canonically identified with $\pi_0(Y_{s'})$. Observe that $\pi_0(Y_{c_{s'}})$ is a discrete set, as $Y_{c_{s'}}$ is locally Noetherian and so the connected components are open. The following diagram shows the schemes and points discussed and should make the argument easier to follow.

$$
\begin{tikzcd}
\tilde{Y} \arrow[d] \arrow[dr] & & & \\
Y \arrow[dr] & \tilde{X} \arrow[r] \arrow[d] & \tilde{S} \arrow[d] & \supset c_{s'} \ni s' \arrow[d, mapsto] \\
& X \arrow[r] & S & \ni s
\end{tikzcd}
$$

The field $\kappa(s')$ is separably closed. Choose a geometric point $\bar{s}$ over s'. As $\bar{s} \to s'$ is a universal homeomorphism, we can identify $\pi_0(Y_{\bar{s}}) = \pi_0(Y_{s'})$. From the discussion above, it follows that $\pi_0(Y_{\bar{s}}) \simeq \pi_0(Y_{c_{s'}})$. By Theorem 3.31, we have a bijection $\pi_0(Y_{\bar{s}_1}) \simeq \pi_0(Y_{\bar{s}_2})$ for any two geometric points $\bar{s}_1$ and $\bar{s}_2$ on S. Thus, we see that π_0 of restrictions of $\widetilde{Y}$ to two connected components of $\widetilde{S}$ can be identified, i.e., if $c_1, c_2 \subset \widetilde{S}$ are two connected components, there is a bijection of discrete sets $\pi_0(\widetilde{Y}_{c_1}) \simeq \pi_0(\widetilde{Y}_{c_2})$. It follows that, as sets, we can write $\pi_0(\widetilde{Y}) \simeq \sqcup_{t \in \pi_0(Y_{\bar{s}})} \pi_0(\widetilde{S})_t$ and this identification is compatible with the natural maps $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{S})$ and $\pi_0(\widetilde{S})_t \xrightarrow{\mathrm{id}} \pi_0(\widetilde{S})$. Here $\bar{s}$ is some fixed (arbitrarily chosen) geometric point of S and the subscript notation in $\pi_0(\widetilde{S})_t$ denotes different copies of the set $\pi_0(\widetilde{S})$. Caution: with this notation, $\pi_0(\widetilde{S})_t$ does *not* denote the fibre over some t. As we explain below, the identification above can be upgraded to a homeomorphism. Observe that $\pi_0(\widetilde{S})$ is a profinite set (as $\widetilde{S}$ is qcqs), but usually it will not be finite.

Claim 1. There is a homeomorphism $\sqcup_{t \in \pi_0(Y_{\bar{s}})} \pi_0(\widetilde{S})_t \to \pi_0(\widetilde{Y})$ over $\pi_0(\widetilde{S})$.

Step 3: Proof of Claim 1. It includes two lemmas. Let η be the generic point of S (recall that S is now assumed to be normal). By Lemma 3.15, we can identify $\pi_0(\widetilde{S}_\eta) = \pi_0(\widetilde{S})$. Let us show that we can identify $\pi_0(\widetilde{Y}_\eta) = \pi_0(\widetilde{Y})$ as well. Here $\widetilde{Y}_\eta = \widetilde{Y} \times_S \eta = \widetilde{Y} \times_{\widetilde{S}} \widetilde{S}_\eta = Y_\eta \times_S \widetilde{S}$.

Lemma. *The map $\pi_0(\widetilde{Y}_\eta) \to \pi_0(\widetilde{Y})$ is a homeomorphism.*

■

To show that $\pi_0(\widetilde{Y}_\eta) \to \pi_0(\widetilde{Y})$ is bijective, it is enough to look fibre by fibre over $\pi_0(\widetilde{S})$. Thus, we can fix a connected component $c \in \pi_0(\widetilde{S})$ and base-change to c. Keep in mind that, as a morphism of schemes, $c \to \widetilde{S}$ is (among other properties) a closed immersion. In particular, $\pi_0(\widetilde{Y} \times_{\widetilde{S}} c)$ is equal to the preimage of c under $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{S})$. Similarly for $\widetilde{Y}_\eta$. The component c is a strict henselization at a (geometric) point on S (Lemma 3.15) and so is Noetherian. As c is a normal connected scheme, the fibre c_η consists of a single point: the generic point of c. Let us call it ξ. Let the subscript c denote the base-change from $\widetilde{S}$ to c. Then the map $(\widetilde{Y}_\eta)_c \to \widetilde{Y}_c$ is equal to the embedding of the fibre over ξ into $\widetilde{Y}_c$ (i.e., the base-change of $\xi \to c$ to $\widetilde{Y}_c$). By Corollary 3.32, $\pi_0((\widetilde{Y}_c)_{\bar{\xi}}) \to \pi_0(\widetilde{Y}_c)$ is a bijection. But we have a factorization $\pi_0((\widetilde{Y}_c)_{\bar{\xi}}) \twoheadrightarrow \pi_0((\widetilde{Y}_c)_\xi) \to \pi_0(\widetilde{Y}_c)$, where the first map is surjective. It follows that $\pi_0((\widetilde{Y}_c)_\xi) \to \pi_0(\widetilde{Y}_c)$ is a bijection. But $(\widetilde{Y}_c)_\xi \simeq (\widetilde{Y}_c)_\eta$ and the proof of bijectivity is finished. By Lemma 3.26, $\pi_0(\widetilde{Y}_\eta) \to \pi_0(\widetilde{Y})$ is a homeomorphism and *the proof of the lemma is finished*.

Thus, we can focus on understanding $\pi_0(\widetilde{Y}_\eta)$. The scheme Y_η is a disjoint union of connected components belonging to Cov_{X_η}. These components are clopen and so $\pi_0(Y_\eta)$

and $\pi_0(\widetilde{Y}_\eta)$ split accordingly. We can thus restrict attention to one connected component and assume Y_η connected in the proof of the claim. Let $c \in \pi_0(\widetilde{S}_\eta)$. The component c is the spectrum of a separable algebraic field extension L_c of $K = \kappa(\eta)$. By Corollary 3.32 and Lemma 3.17, the base-change $\widetilde{Y}_c$ is a disjoint union of finitely many components and each of them is geometrically connected. Thus, Y_{L_c} has geometrically connected components. Moreover, the number of these connected components is constant when c varies (by Theorem 3.31), let us say equal M. As $\mathcal{O}_S(S) \to \mathcal{O}_{\widetilde{S}}(\widetilde{S})$ is ind-étale, the scheme $\widetilde{S}_\eta$ is a cofiltered limit $\lim S_\lambda$ of finite unions of spectra of finite separable extensions of K. Here, $\Lambda \ni \lambda \mapsto S_\lambda$, where Λ is the indexing cofiltered category.

Lemma. *For some $\lambda_0 \in \Lambda$, there is $S_{\lambda_0} = \sqcup_i \mathrm{Spec}(L_i)$ with L_i/K finite separable and such that the connected components of Y_{L_i} are geometrically connected; in other words, Y_{L_i} has precisely M connected components.*

Indeed, for each $\lambda \in \Lambda$, let $W_\lambda \subset S_\lambda$ be the union of those $\mathrm{Spec}(L_i) \subset S_\lambda$ that do not have this property. This forms a sub-inverse system of S_λ. We want to show that, for some λ_0, W_{λ_0} is empty. Assume the contrary. The maps between W_λ's are affine, and thus $\widetilde{W} = \lim W_\lambda$ exists in the category of schemes and moreover $\widetilde{W}_{\mathrm{top}} = \lim W_{\lambda,\mathrm{top}}$ ([19, Tag 0CUF]), where W_{top} denotes the underlying topological space of a scheme W. As $W_{\lambda,\mathrm{top}}$ are finite and non-empty, the inverse limit is nonempty as well (see [19, Lemma 086J]). The image of any point $w \in \widetilde{W}$ in $\widetilde{S}$ gives a point of $\widetilde{S}$, which has as the residue field a separable extension L/K such that Y_L is a disjoint union of geometrically connected components, but L can be written as a filtered colimit of fields L_α with L_α/K finite separable and such that the connected components of Y_{L_α} are not all geometrically connected. But L must contain the smallest field L_{smallest} of Lemma 3.17; consequently, using that L_{smallest}/K is finite, one of L_α must contain L_{smallest} as well. By Lemma 3.17 again, this contradicts the fact that Y_{L_α} has a component that is not geometrically connected. Thus, we proved that there exists λ_0 such that $S_{\lambda_0} = \sqcup_{i=1}^{m_0} \mathrm{Spec}(L_i)$ with L_i/K finite separable and such that the connected components of Y_{L_i} are geometrically connected. *This finishes the proof of the lemma.*

Now, we have an equality $\pi_0(Y_{L_i}) = \pi_0(Y_{\bar{\eta}})$. Taking the preimages of the connected components of each Y_{L_i} in $\widetilde{Y}_\eta$, we see that $\widetilde{Y}_\eta$ decomposes as a disjoint union of clopen subsets $\widetilde{Z}_t$, parametrized by $t \in \pi_0(Y_{\bar{\eta}})$, such that each $\widetilde{Z}_t$ maps surjectively onto $\widetilde{S}_\eta$ and induces a continuous bijection $\pi_0(\widetilde{Z}_t) \to \pi_0(\widetilde{S}_\eta)$. More precisely, we have a diagram

$$\begin{tikzcd}
\widetilde{Y}_\eta \arrow[r] \arrow[d] & \widetilde{S}_\eta \arrow[d] \\
\sqcup_i Y_{L_i} \arrow[r] \arrow[d] & S_{\lambda_0} = \sqcup_i \mathrm{Spec}(L_i) \arrow[d] \\
Y_\eta \arrow[r] & \eta
\end{tikzcd}$$

and we know that for each i, $Y_{L_i} = \sqcup_{t\in\pi_0(Y_{\bar{\eta}})} Z_{i,t}$, with $Z_{i,t} \to \mathrm{Spec}(L_i)$ geometrically connected. We define $\widetilde{Z}_{i,t}$ to be the preimage of $Z_{i,t}$ in $\widetilde{Y}_\eta$ and put $\widetilde{Z}_i = \sqcup_{t\in\pi_0(Y_{\bar{\eta}})}\widetilde{Z}_{i,t}$. As the map $\widetilde{Z}_t \subset \widetilde{Y}_\eta \to \widetilde{S}_\eta$ is open, we get by Lemma 3.28 that $\pi_0(\widetilde{Z}_t) \to \pi_0(\widetilde{S}_\eta)$ is actually a homeomorphism. Thus, we get a homeomorphism $\pi_0(\widetilde{Y}_\eta) \simeq \bigsqcup_{t\in\pi_0(Y_{\bar{\eta}})} \pi_0(\widetilde{S}_\eta)$ as desired. *This finishes the proof of Claim 1.*

Step 4: Definition of the scheme $\widetilde{T}$ and the map $\widetilde{Y} \to \widetilde{T}$ over $\widetilde{S}$. By the last claim, there is an isomorphism $\widetilde{Y} \simeq \bigsqcup_{t\in\pi_0(Y_{\bar{s}})} \widetilde{Y}_t$, i.e., $\widetilde{Y}$ splits as a union of clopen subsets parametrized by $t \in \pi_0(Y_{\bar{s}})$. Define $\widetilde{T} = \bigsqcup_{t\in\pi_0(Y_{\bar{s}})} \widetilde{S}_t$, where $\widetilde{S}_t$ is a copy of $\widetilde{S}$. There is an obvious morphism $\widetilde{Y} \to \widetilde{T}$, which restricted to a fixed $\widetilde{Y}_t$ factorizes through $\widetilde{S}_t$. The scheme $\widetilde{T}$ is in $\mathrm{Cov}_{\widetilde{S}}$ and we want to show that it descends to a covering of T. The morphism $\widetilde{Y} \to \widetilde{T}$ is surjective and has geometrically connected fibres. Indeed, to see the surjectivity, observe that by Lemma 3.28 and by the construction of $\widetilde{T}$, $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{T}_{\widetilde{X}})$ is a homeomorphism and the connected components of $\widetilde{T}_{\widetilde{X}}$ are isomorphic to connected components of $\widetilde{X}$, and thus are Noetherian. Restricting to such a component, $\widetilde{Y} \to \widetilde{T}_{\widetilde{X}}$ becomes a geometric covering with nonempty image, and thus is surjective, e.g., by [2, Lemma 7.3.9]. To see the connectedness of geometric fibres, observe that, by the construction, $\pi_0(\widetilde{Y}) \to \pi_0(\widetilde{T})$ is a homeomorphism and $\widetilde{T} = \pi_0(Y_{\bar{\eta}}) \times \widetilde{S}$. We can restrict the situation to a fixed connected component c of $\widetilde{S}$. Identifying $Y_{\bar{\eta}} \simeq \widetilde{Y}_c \times_c \bar{\eta}$ (using some lift of $\bar{\eta}$ from S to c), we see that the $\bar{\eta}$-fibre of $\widetilde{Y} \to \widetilde{T}$ is connected. But now it follows quite easily from Corollary 3.32 that in fact every geometric fibre is connected. The proof of existence of the descent datum and checking the cocycle condition follows essentially from the uniqueness of the infinite Stein factorization. We cannot, however, simply apply the uniqueness statement proven above; as $\pi_0(\widetilde{S})$ is not discrete, we cannot argue by directly restricting to the connected components. Thus, we have to be slightly more careful.

Step 5: Definition of the descent datum, checking the cocycle condition and finishing the construction of T by descent. This includes Claims 2 and 3. We need to equip $\widetilde{T}$ with a descent datum. Denote for brevity $\widetilde{S}_2 = \widetilde{S} \times_S \widetilde{S}$, $\widetilde{Y}_2 = \widetilde{Y} \times_Y \widetilde{Y}$, $\widetilde{X}_2 = \widetilde{X} \times_X \widetilde{X}$ and let $p, q : \widetilde{S}_2 \to \widetilde{S}$ be the canonical projections. We need to define an isomorphism over $\widetilde{S}_2$ between the two base-changes $p^*\widetilde{T}$ and $q^*\widetilde{T}$. We have diagrams

$$\begin{tikzcd}
\widetilde{Y}_2 \arrow[r, "\alpha"] \arrow[d, "p"'] & p^*\widetilde{T} \arrow[r] \arrow[d] & \widetilde{S}_2 \arrow[d, "p"'] \\
\widetilde{Y} \arrow[r] & \widetilde{T} \arrow[r] & \widetilde{S}
\end{tikzcd}
\qquad
\begin{tikzcd}
\widetilde{Y}_2 \arrow[r, "\beta"] \arrow[d, "q"'] & q^*\widetilde{T} \arrow[r] \arrow[d] & \widetilde{S}_2 \arrow[d, "q"'] \\
\widetilde{Y} \arrow[r] & \widetilde{T} \arrow[r] & \widetilde{S}
\end{tikzcd}$$

with all squares Cartesian. We claim that: α and β induce homeomorphisms on π_0's. Indeed, $\widetilde{Y} \to \widetilde{T}$ is universally open, surjective with geometrically connected fibres. We are using [7, Theorem 2.4.6] here to check openness. To check that $\widetilde{Y} \to \widetilde{T}$ is flat and locally of finite presentation, it is enough to check these properties for $\widetilde{Y} \to \widetilde{T}_{\widetilde{X}} = \widetilde{T} \times_{\widetilde{S}} \widetilde{X}$, as $\widetilde{T}_{\widetilde{X}} \to \widetilde{T}$ has the desired properties. But $\widetilde{Y} \to \widetilde{T}_{\widetilde{X}}$ is a morphism of étale $\widetilde{X}$-schemes, so the properties follow. The surjectivity was proven above. Thus, the same is true for α and β. These assumptions imply that $\pi_0(\alpha)$ and $\pi_0(\beta)$ are continuous bijections, and in fact homeomorphisms by Lemma 3.28. From this, we obtain a homeomorphism $\phi_0 = \pi_0(\beta) \circ \pi_0(\alpha)^{-1} : \pi_0(p^*\widetilde{T}) \to \pi_0(q^*\widetilde{T})$ over $\pi_0(\widetilde{S}_2)$.

Claim 2. ϕ_0 lifts uniquely to an isomorphism $\phi : p^*\widetilde{T} \to q^*\widetilde{T}$ over $\widetilde{S}_2$.

Proof of the claim: $p^*\widetilde{T}$ and $q^*\widetilde{T}$ are both isomorphic over $\widetilde{S}_2$ to a disjoin union $\sqcup_{t \in \pi_0(Y_{\bar{s}})} \widetilde{S}_2$. Fixing these isomorphisms, we can view ϕ_0 as a homeomorphism of $\sqcup_{t \in \pi_0(Y_{\bar{s}})} \pi_0(\widetilde{S}_2)$ with itself, and we want to show that it lifts to an isomorphism of $\sqcup_{t \in \pi_0(Y_{\bar{s}})} \widetilde{S}_2$. This follows by Lemma 3.33, as $\pi_0(\widetilde{S}_2)$ is compact and ϕ_0 is over the base $\pi_0(\widetilde{S}_2)$.

We need to show that ϕ satisfies the cocycle condition. Let $\widetilde{S}_3 = \widetilde{S} \times_S \widetilde{S} \times_S \widetilde{S}$ and analogously for $\widetilde{Y}$. For $i \in \{1, 2, 3\}$ let $p_i : \widetilde{S}_3 \to \widetilde{S}_2$ be the projection forgetting the i-th factor and for $i \neq j$ in $\{1, 2, 3\}$ denote $p_{ij} : \widetilde{S}_3 \to \widetilde{S}$ the projection forgetting the i-th and j-th factors. As in the case of double products, there are morphisms $\widetilde{Y}_3 \to p_{ij}^*\widetilde{T}$ fitting into suitable Cartesian diagrams. Denoting by a and b the morphisms $\widetilde{Y}_3 \to p_{13}^*\widetilde{T}$ and $\widetilde{Y}_3 \to p_{12}^*\widetilde{T}$ respectively, we have a commutative diagram

$$\begin{tikzcd}
 & & & p_1^*(p^*\widetilde{T}) = p_{13}^*\widetilde{T} \arrow[dr] \arrow[ddd, "p_1"] & \\
\widetilde{Y}_3 \arrow[urrr, "a"] \arrow[dr, "b"'] \arrow[ddd, "p_1"'] & & & & \widetilde{S}_3 \arrow[ddd, "p_1"] \\
 & p_1^*(q^*\widetilde{T}) = p_{12}^*\widetilde{T} \arrow[urrr] \arrow[ddd, "p_1"'] & & & \\
 & & & p^*\widetilde{T} \arrow[dr] & \\
\widetilde{Y}_2 \arrow[urrr, "\alpha"] \arrow[dr, "\beta"'] & & & & \widetilde{S}_2 \\
 & q^*\widetilde{T} \arrow[urrr] & & &
\end{tikzcd}$$

and analogous diagrams for the projections p_2 and p_3.

Claim 3. The induced maps $\pi_0(a) : \pi_0(\widetilde{Y}_3) \to \pi_0(p_{13}^*\widetilde{T})$ and $\pi_0(b) : \pi_0(\widetilde{Y}_3) \to \pi_0(p_{12}^*\widetilde{T})$ are homeomorphisms. The homeomorphism $\psi_0 = \pi_0(b) \circ \pi_0(a)^{-1}$ lifts uniquely to an isomorphism $\psi : p_{13}^*\widetilde{T} \to p_{12}^*\widetilde{T}$ over $\widetilde{S}_3$ and is equal to $p_1^*(\phi)$. Analogous statements hold respectively for the diagrams involving projections p_2 and p_3.

Proof of the claim: The proofs that $\pi_0(a), \pi_0(b)$ are homeomorphisms and that ψ_0 lifts canonically is virtually the same as the proof of the Claim 2 above. To see the last part of the claim, we use the commutativity of the last diagram and the fact that $\widetilde{T} = \sqcup_t \widetilde{S}$, from which we easily conclude that $\psi_0 = \pi_0(p_1)^*(\phi_0)$ and from the definitions of ϕ and ψ we see that also $p_1^*(\phi) = \psi$.

Having the claim, the cocycle condition for ϕ follows, and thus we have constructed a descent datum on $\widetilde{T}$. Moreover, by construction it is compatible with $\widetilde{Y} \to \widetilde{T}$. Thus, by fpqc descent, we obtain a sheaf T on S_{fpqc} that becomes constant on $\widetilde{S}$. Thus, we can view T as an element of Loc_S ([2, Definition 7.3.1]) and by the equivalence $\mathrm{Loc}_S = \mathrm{Cov}_S$ of [2, Lemma 7.3.9], T is representable by a geometric covering of S. The descent datum on $\widetilde{Y} \to \widetilde{T}$ gives a morphism $Y \to T$ over S by fpqc descent for morphisms of schemes (by [19, Remark 040L] and [19, Lemma 02W0]). Let $\bar{t} \in T$ be a geometric point. As $\widetilde{T} \to T$ is a base-change of $\widetilde{S} \to S$, and so weakly étale, the point $\bar{t}$ lifts to a geometric point on $\widetilde{T}$ and $Y_{\bar{t}} = \widetilde{Y} \times_{\widetilde{T}} \bar{t}$. Thus, $Y_{\bar{t}}$ is connected, as $\widetilde{Y} \to \widetilde{T}$ had geometrically connected fibres. Similarly, $Y \to T$ is surjective because $\widetilde{Y} \to \widetilde{T}$ was. Thus, T is is connected as an image of Y.

Remark 3.34. Having finished the above proof, one can use Corollary 3.9 together with Lemma 3.17 to conclude that, when S is normal, the indexing set that appeared many times in the proof of Theorem 3.4, namely $\pi_0(Y_{\bar{\eta}})$, is finite.

Remark 3.35. Assume S to be Nagata. One is then tempted to use [2, Remark 7.3.10] to simplify the topological part of the proof (of having to deal with non-discrete π_0's) at the expense of working with henselizations along more general closed subschemes. However, as the case when S is normal is already non-trivial, this does not seem to be a more efficient approach.

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References

[1] Arhangel'skii, A. and M. Tkachenko. *Topological groups and related structures, volume 1 of Atlantis Stud*. Hackensack, NJ: Math. Atlantis Press, Paris; World Scientific Publishing Co. Pte. Ltd., 2008.

[2] Bhatt, B. and P. Scholze. "The pro-étale topology for schemes." *Astérisque* 369 (2015): 99–201.

[3] Bourbaki, N. *Elements of mathematics. General topology. Part 2*. Hermann, Paris: Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont, 1966.

[4] Demazure, M. and A. Grothendieck. *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3). Lecture Notes in Math*, Vol. 152. Berlin-New York: Springer-Verlag, 1970.

[5] Dikranjan, D. (2013). *Introduction to topological groups*. http://users.dimi.uniud.it/~dikran.dikranjan/ITG.pdf.

[6] Gleason, A. M. "Projective topological spaces." *Illinois J. Math.* 2 (1958): 482–9.

[7] Grothendieck, A. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. II." *Inst. Hautes études Sci. Publ. Math.* 24 (1965): 231.

[8] Grothendieck, A. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. III." *Inst. Hautes études Sci. Publ. Math.* 28 (1966): 255.

[9] Grothendieck, A. "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV." *Inst. Hautes études Sci. Publ. Math.* 32 (1967): 361.

[10] Grothendieck, A. and M. Raynaud. *Revêtements étales et groupe fondamental* Lecture Notes in Math., *Topological groups and related structures, volume 1 of Atlantis Stud*, Vol. 224. Berlin-New York: Springer-Verlag, 1971.

[11] Lara, M. "Homotopy exact sequence for the pro-étale fundamental group I arXiv preprint arXiv:1910.14015." (2019) The submitted version was renamed to "Fundamental exact sequence for the pro-étale fundamental group".

[12] Lavanda, E. "Specialization map between stratified bundles and pro-étale fundamental group." *Adv. Math.* 335 (2018): 27–59.

[13] Lazard, D. "Disconnexités des spectres d'anneaux et des préschémas." *Bull. Soc. Math. France* 95 (1967): 95–108.

[14] Lepage, E. "Géométrie anabélienne tempéré." (2010) PhD thesis arXiv:1004.2150.

[15] Noohi, B. "Fundamental groups of topological stacks with the slice property." *Algebraic & Geometric Topology* 8, no. 3 (2008): 1333–70.

[16] Roelcke, W. and S. Dierolf. *Uniform structures on topological groups and their quotients*. New York: McGraw-Hill International Book Co., 1981 Advanced Book Program.

[17] Rydh, D. "Submersions and effective descent of étale morphisms." *Bull. Soc. Math. France* 138, no. 2 (2010): 181–230.

[18] Schröer, S. "Geometry on totally separably closed schemes." *Algebra & Number Theory* 11, no. 3 (2017): 537–82.

[19] Stacks Project Authors (2020). *The stacks project*. https://stacks.math.columbia.edu.

[20] Vakil, R. (2017). *Foundations of algebraic geometry*. http://math.stanford.edu/~vakil/216blog/FOAGjun0417public.pdf. June 4, 2017 draft.