MODULI OF LANGLANDS PARAMETERS

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Abstract. Let F be a non-archimedean local field of residue characteristic p, let $\hat{G}$ be a split reductive group scheme over $\mathbb{Z}[\frac{1}{p}]$ with an action of W_F, and let ${}^LG$ denote the semidirect product $\hat{G} \rtimes W_F$. We construct a moduli space of Langlands parameters $W_F \to {}^LG$, and show that it is locally of finite type and flat over $\mathbb{Z}[\frac{1}{p}]$, and that it is a reduced local complete intersection. We give parameterizations of the connected components and the irreducible components of the geometric fibers of this space, and parameterizations of the connected components of the total space over $\overline{\mathbb{Z}}[\frac{1}{p}]$ (under mild hypotheses) and over $\overline{\mathbb{Z}}_\ell$ for $\ell \neq p$. In each case, we show precisely how each connected component identifies with the "principal" connected component attached to a smaller split reductive group scheme. Finally, we study the GIT quotient of this space by $\hat{G}$ and give a description of its fibers up to homeomorphism, and a complete description of its ring of functions after inverting an explicit finite set of primes depending only on ${}^LG$.

Contents

1. Introduction and main results

1.1. **Introduction.** Let F be a local field with residue characteristic p, and G a quasi-split connected reductive group over F. Let ℓ be a prime different from p. A *Langlands parameter* for G is a continuous L-homomorphism $W_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$; that is, an ℓ-adically continuous homomorphism from the Weil group W_F to the group of $\overline{\mathbb{Q}}_\ell$-points of the Langlands dual group ${}^LG := \hat{G} \rtimes W_F$ of G, such that the composition with the natural map ${}^LG(\overline{\mathbb{Q}}_\ell) \to W_F$ is the identity.

When G is the general linear group GL_n, then ${}^LG$ is simply the product $\mathrm{GL}_n \times W_F$, and a Langlands parameter for G is simply a continuous representation: $W_F \to \mathrm{GL}_n(\overline{\mathbb{Q}}_\ell)$. Such representations vary nicely in algebraic families; in particular, given

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a continuous representation $\overline{\varphi} : W_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$, we can associate to it the *universal framed deformation ring* $R^{\square}_{\overline{\varphi}}$. a complete Noetherian local $W(\overline{\mathbb{F}}_\ell)$-algebra that admits a continouous representation $\varphi^{\mathrm{univ}} : W_F \to \mathrm{GL}_n(R^{\square}_{\overline{\varphi}})$ such that the pair $(R^{\square}_{\overline{\varphi}}, \varphi^{\mathrm{univ}})$ is universal for pairs (R, φ), where R is a complete Noetherian local $W(\overline{\mathbb{F}}_\ell)$-algebra and $\varphi : W_F \to \mathrm{GL}_n(R)$ is a lift of $\overline{\varphi}$.

Given the importance of such deformation spaces in the Langlands program, it is natural to attempt to construct corresponding "universal deformation spaces" for Langlands parameters attached to groups G other than GL_n. Indeed, Bellovin and Gee [BG19] and Booher and Patrikis [BP19] independently study a closely related problem. Specifically, (cf. [BP19], Section 2) define an *${}^LG$-Weil-Deligne representation* over a $W(\overline{\mathbb{F}}_\ell)$-algebra A to be a triple (D_A, r, N), where D_A is an ${}^LG$-bundle over $\operatorname{Spec} A$, $r : W_F \to \mathrm{Aut}_{{}^LG}(D_A)$ is a homomorphism *with open kernel*, and N is a nilpotent element of the Lie algebra of $\mathrm{Aut}_{{}^LG}(D_A)$ such that $\mathrm{Ad}_r(w)N = |w|N$ for all $w \in W_F$. Both Bellovin-Gee and Booher-Patrikis construct moduli spaces of such ${}^LG$-Weil-Deligne representations, that are schemes locally of finite type over $W(\overline{\mathbb{F}}_\ell)$, and show that their general fibers are generically smooth and equidimensional of dimension equal to the dimension of ${}^LG$.

When A is complete local, and ℓ is invertible in A, Grothendieck's monodromy theorem gives a natural bijection between ${}^LG$-Weil-Deligne representations with values in A and Langlands parameters with values in A, so the results of Bellovin-Gee and Booher-Patrikis in some sense give a solution to the problem of finding universal families for Langlands parameters over G. Their method relies heavily on the exponential and logarithm maps, which have denominators, and also involves division by the order of the image of an element of inertia. There is thus reason to question whether a naive extension of these constructions to situations where ℓ is not invertible gives the "right" objects, particularly if the prime ℓ is small enough to divide one of these denominators. For instance, when $G = \mathrm{GL}_2$, and ℓ divides $q^2 - 1$ (where q denotes the order of the residue field of F), the analogue of the spaces constructed by Bellovin-Gee and Booher-Patrikis fails to be flat over $W(\overline{\mathbb{F}}_\ell)$. Since universal framed deformation rings are known to be flat over $W(\overline{\mathbb{F}}_\ell)$, this means that when $G = \mathrm{GL}_2$, naive generalization of the constructions of Bellovin-Gee and Booher-Patrikis fails to recover the existing theory in such characteristics. It is reasonable to expect that this failure of flatness persists for more complicated groups. Such a failure makes these spaces unsuitable for formulating analogues of Shotton's "$\ell \neq p$ Breuil-Mezard" results for GL_n [Sho18]. We refer the reader to section 2.4 for further discussion of this point.

In light of these issues, it is tempting to look at alternative characterizations of Langlands parameters over fields of characteristic zero, in the hope that they suggest better behaved moduli problems. There are (at least) three definitions of a "Langlands parameter over $\overline{\mathbb{Q}}_\ell$" common in the literature:

(1) pairs (r, N), where $r : W_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is an L-homomorphism with open kernel and $N \in \mathrm{Lie}(\hat{G}_{\overline{\mathbb{Q}}_\ell})$ a nilpotent element, such that $\mathrm{Ad}_r(w) = |w|N$,
(2) maps $W_F \times \mathrm{SL}_2(\overline{\mathbb{Q}}_\ell) \to {}^LG(\overline{\mathbb{Q}}_\ell)$ whose restriction to the first factor is an L-homomorphism with open kernel and whose restriction to the second factor is algebraic, and
(3) L-homomorphisms ${}^L\varphi : W_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$ that are ℓ-adically continuous.

The first of these definitions generalizes in an obvious way to coefficients in an arbitrary $W(\overline{\mathbb{F}}_\ell)$-algebra R, and considering the associated moduli problem leads to the schemes considered by Bellovin-Gee and Booher-Patrikis. The second likewise generalizes to such algebras R, but the associated moduli space is much less well-behaved. For instance, the moduli space of unramified pairs (r, N) as in (1) is connected over $\overline{\mathbb{Q}}_\ell$, whereas the space of unramified maps $W_F \times \mathrm{SL}_2 \to {}^L G$ as in (2) is, over $\overline{\mathbb{Q}}_\ell$, a disjoint union over the set of conjugacy classes of unipotent elements $u \in \hat{G}(\overline{\mathbb{Q}}_\ell)$, of the loci where the image of the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in the SL_2 factor is conjugate to u.

It is therefore tempting to try to construct a moduli space of ℓ-adically continuous L-homomorphisms from W_F to ${}^L G$ as in (3). The notion of ℓ-adic continuity for L-homomorphisms valued in ${}^L G(\overline{\mathbb{Q}}_\ell)$ generalizes naturally to complete local rings of residue characteristic ℓ; this is sufficient for a well-behaved deformation theory but is insufficient to obtain a moduli space that is locally of finite type. In order to obtain such a space, one would need a broader notion of ℓ-adic continuity.

Our approach to this question is inspired by previous work of the second author in [Hel20]. That paper introduces a notion of ℓ-adic continuity for maps $W_F \to \mathrm{GL}_n(R)$ that makes sense for arbitrary $W(\overline{\mathbb{F}}_\ell)$-algebras R, and constructs universal families of such representations over a suitable $W(\overline{\mathbb{F}}_\ell)$-scheme, which we will denote here by X_n. (This notation differs from that of [Hel20], where what we call the scheme X_n only appears implicitly, as the disjoint union of the schemes denoted $X^\nu_{q,n}$). As with the constructions of Bellovin-Gee and Booher-Patrikis, the scheme X_n is locally of finite type over $W(\overline{\mathbb{F}}_\ell)$, but unlike their construction, the completion of the local ring of X_n at any $\overline{\mathbb{F}}_\ell$-point of X_n, corresponding to a map $\overline{\varphi} : W_F \to \mathrm{GL}_n(\overline{\mathbb{F}}_\ell)$, is the universal framed deformation ring $R^\square_{\overline{\varphi}}$. In other words, X_n is a locally of finite-type $W(\overline{\mathbb{F}}_\ell)$-scheme that "interpolates" the universal framed deformation rings of all n-dimensional mod ℓ representations of W_F.

The schemes X_n constructed in [Hel20] play a central role in the formulation and proof of the "local Langlands correspondence in families" for the group GL_n, now proven by two of the authors in [HM18]. (These results, in turn, imply the existence of the families conjectured by Emerton and the second author in [EH14].) In particular, the subring of functions on X_n that are invariant under the conjugation action on Langlands parameters is naturally isomorphic to the center of the category of smooth $W(\overline{\mathbb{F}}_\ell)[\mathrm{GL}_n(F)]$-modules. Morally, this means that aspects of the geometry of X_n are reflected in the representation theory of $\mathrm{GL}_n(F)$. For instance, the connected components of X_n correspond to the "blocks" of the category of smooth $W(\overline{\mathbb{F}}_\ell)[\mathrm{GL}_n(F)]$-modules.

In this paper our first objective is to generalize the construction of [Hel20] to the setting of Langlands parameters for arbitrary quasi-split, connected reductive groups, with an eye towards formulating a conjectural analogue of the local Langlands correspondence in families for such groups. In a departure from previous work on the subject, we work over the base ring $\mathbb{Z}[\frac{1}{p}]$ rather than over a ring of Witt vectors; this introduces some technical complexity but gives us the smallest possible base ring for such a correspondence. (In particular this allows us to study chains of congruences of Langlands parameters modulo several different primes.) We refer the reader to the next subsection for precise definitions.

Second, we aim to understand the geometry of these moduli spaces of Langlands parameters. Several natural questions arise. It turns out that, as in the setting of local deformation theory of Galois representations, the spaces we obtain have a quite tractable local structure: they are reduced local complete intersections that are flat over $\operatorname{Spec}\mathbb{Z}[\frac{1}{p}]$, of dimension $\dim G$. Moreover, we give descriptions of the connected components of these moduli spaces, both over algebraically closed fields of arbitrary characteristic $\ell \neq p$, and (conjecturally) over $\overline{\mathbb{Z}}[\frac{1}{p}]$.

Finally, we study the rings of functions on these moduli spaces that are invariant under $\hat{G}$-conjugacy (or, equivalently, the GIT quotient of the moduli space of Langlands parameters by the conjugation action of $\hat{G}$.) As in the case of GL_n, the ring of such functions is in general quite complicated, and does not admit an explicit description. (In particular, the corresponding GIT quotients are very far from being normal.) Nonetheless, we show that after inverting an explicit finite set of primes (depending only on G), the GIT quotients are quite nice; indeed, they are disjoint unions of quotients of tori by finite group actions. Over the complex numbers these connected components coincide with varieties studied by Haines [Hai14].

1.2. **The moduli space of Langlands parameters.** We now describe in detail the moduli problem that we study. Following [Hel20], the approach we take is to "discretize" the tame inertia group. Fix an arithmetic Frobenius element Fr in W_F and a pro-generator s of the tame inertia group I_F/P_F. These satisfy the relation $\mathrm{Fr}\, s\, \mathrm{Fr}^{-1} = s^q$. We then consider the subgroup $\langle \mathrm{Fr}, s\rangle = s^{\mathbb{Z}[\frac{1}{q}]} \rtimes \mathrm{Fr}^{\mathbb{Z}}$ of W_F/P_F, we denote by W_F^0 its inverse image in W_F, and we endow it with the topology that extends the profinite topology of P_F and induces the discrete topology on $\langle \mathrm{Fr}, s\rangle$. Note that (in contrast to the subgroup W_F of G_F), the subgroup W_F^0 of W_F very much depends on the choices of Fr and s.

Although the topology on W_F^0 is finer than the one induced from W_F, the relation $\mathrm{Fr}\, s\, \mathrm{Fr}^{-1} = s^q$ implies that a morphism $W_F^0 \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is continuous if and only if it is continuous for the topology induced from W_F. It follows that restriction to W_F^0 induces a bijection between objects of type (3) and the following objects :

(4) continuous morphisms ${}^L\varphi : W_F^0 \to {}^LG(\overline{\mathbb{Q}}_\ell)$ (with either the discrete or the natural topology on ${}^LG(\overline{\mathbb{Q}}_\ell)$).

These objects are now easy to define over any $\mathbb{Z}_\ell$-algebra R since only the discrete topology of ${}^LG(R)$ is needed. Indeed, they are also defined for any $\mathbb{Z}[\frac{1}{p}]$-algebra and their moduli space over $\mathbb{Z}[\frac{1}{p}]$ is already interesting.

We therefore consider the following setting:

- $\hat{G}$ is a split reductive group scheme over $\mathbb{Z}[\frac{1}{p}]$ endowed with a finite action of the absolute Galois group G_F (we do not assume that G_F preserves a pinning).
- W_F^0 is the inverse image in W_F of the subgroup $s^{\mathbb{Z}[\frac{1}{q}]} \rtimes \mathrm{Fr}^{\mathbb{Z}}$ of W_F/P_F, which depends on the choice of a generator s of the tame inertia group I_F/P_F and a lift of Frobenius.
- $(P_F^e)_{e\in\mathbb{N}}$ is a decreasing sequence of open subgroups of P_F that are normal in W_F and whose intersection is $\{1\}$.

Note that for any $\mathbb{Z}[\frac{1}{p}]$-algebra R, there is a natural bijection between the continuous L-homomorphisms ${}^L\varphi : W_F^0 \to {}^LG(R)$ (with respect to the discrete topology on ${}^LG(R)$) and the set of continuous 1-cocycles $Z^1(W_F^0, \hat{G}(R))$ on W_F^0 with values in $\hat{G}(R)$. If, given ${}^L\varphi$, we denote by φ the corresponding cocycle, then this bijection is characterized by the identity ${}^L\varphi(w) = (\varphi(w), w)$ for all $w \in W_F$.

Since the cocycles we consider are continuous with respect to the discrete topology, we have $Z^1(W_F^0, \hat{G}(R)) = \bigcup_{e\in\mathbb{N}} Z^1(W_F^0/P_F^e, \hat{G}(R))$. It is easy to see that the functor $R \mapsto Z^1(W_F^0/P_F^e, \hat{G}(R))$ on $\mathbb{Z}[\frac{1}{p}]$-algebras is represented by an affine scheme of finite presentation over $\mathbb{Z}[\frac{1}{p}]$, that we denote by $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$. It follows that the functor $R \mapsto Z^1(W_F^0, \hat{G}(R))$ is represented by a scheme $\underline{Z}^1(W_F^0, \hat{G})$, in which each $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$ sits as a direct summand, and which is the increasing union of all these subschemes.

As a $\mathbb{Z}[\frac{1}{p}]$-scheme, the scheme $\underline{Z}^1(W_F^0, \hat{G})$ depends on the choices we made defining W_F^0 as a subgroup of W_F. Indeed, if $W_F^{0'}$ is the subgroup arising from a different choice (Fr', s') then there is not typically a canonical isomorphism of $\mathbb{Z}[\frac{1}{p}]$-schemes from $\underline{Z}^1(W_F^0, \hat{G})$ to $\underline{Z}^1(W_F^{0'}, \hat{G})$. However, there are canonical such isomorphisms over $\mathbb{Z}_\ell$ for each ℓ not equal to p (Corollary 4.2). Moreover, we show (Theorem 4.18) that the GIT quotient $\underline{Z}^1(W_F^0, \hat{G}) /\!\!/ \hat{G}$ is, up to *canonical* isomorphism, independent of the choices defining W_F^0. We further suspect, but do not prove, that the corresponding quotient stacks are also canonically isomorphic.

Our study of $\underline{Z}^1(W_F^0, \hat{G})$ relies on the restriction map $\underline{Z}^1(W_F^0, \hat{G}) \longrightarrow \underline{Z}^1(P_F, \hat{G})$. One crucial point is that the scheme $\underline{Z}^1(P_F, \hat{G})$ is particularly well behaved, because p is invertible in our coefficient rings. Indeed, we prove the following result in the appendix.

Proposition 1.1. *The scheme $\underline{Z}^1(P_F, \hat{G})$ is smooth and its base change to $\overline{\mathbb{Z}}[\frac{1}{p}]$ is a disjoint union of orbit schemes. More precisely, there is a set $\Phi \subset Z^1(P_F, \hat{G}(\overline{\mathbb{Z}}[\frac{1}{p}]))$ such that*

(1) *$\underline{Z}^1(P_F, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]} = \coprod_{\phi\in\Phi} \hat{G}\cdot\phi$ and $\hat{G}\cdot\phi$ represents the sheaf-theoretic (fppf or étale) quotient $\hat{G}/C_{\hat{G}}(\phi)$.*
(2) *each centralizer $C_{\hat{G}}(\phi)$ is smooth over $\overline{\mathbb{Z}}[\frac{1}{p}]$ with split reductive neutral component and constant π_0.*

This says in particular that any cocycle $\phi' \in Z^1(P_F, \hat{G}(R))$ is, locally for the étale topology on R, $\hat{G}$-conjugate to a locally unique ϕ in Φ.

Via the restriction morphism $\underline{Z}^1(W_F^0, \hat{G}) \longrightarrow \underline{Z}^1(P_F, \hat{G})$, the proposition induces a decomposition

$$\underline{Z}^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]} = \coprod_{\phi\in\Phi} \hat{G} \times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F^0, \hat{G})_\phi$$

where $\underline{Z}^1(W_F^0, \hat{G})_\phi$ is the closed subscheme of parameters φ such that $\varphi_{|P_F} = \phi$. In Section 3 we further decompose $\underline{Z}^1(W_F^0, \hat{G})_\phi$ as follows.

Proposition 1.2. *For each $\phi \in \Phi$, there is a finite set $\Phi_\phi \subset Z^1(W_F^0, \hat{G}(\overline{\mathbb{Z}}[\frac{1}{p}]))_\phi$, which is a singleton if $C_{\hat{G}}(\phi)$ is connected, with the following properties :*

(1) *$\forall\tilde{\varphi} \in \Phi_\phi$, $\tilde{\varphi}(W_F^0)$ normalizes a Borel pair in $C_{\hat{G}}(\phi)^\circ$*
(2) *$\forall\tilde{\varphi} \in \Phi_\phi$, the map $\eta \mapsto \eta\cdot\tilde{\varphi}$ defines a closed and open immersion*

$$\underline{Z}^1_{\mathrm{Ad}_{\tilde{\varphi}}}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ) \hookrightarrow \underline{Z}^1(W_F^0, \hat{G})_\phi$$

(3) *The collection of these maps defines an isomorphism*

$$\coprod_{(\phi,\tilde{\varphi})} \hat{G} \times^{C_{\hat{G}}(\phi)_{\tilde{\varphi}}} \underline{Z}^1_{\mathrm{Ad}_{\tilde{\varphi}}}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ) \xrightarrow{\sim} \underline{Z}^1(W_F^0, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]} \tag{1.1}$$

where $C_{\hat{G}}(\phi)_{\tilde{\varphi}}$ is the open and closed subgroup scheme of $C_{\hat{G}}(\phi)$ that stabilizes the image of (2).

These results are essentially relative versions of the constuctions of [Dat17, Section 2]. We note that if $\hat{G}$ is a classical group and $p > 2$, then $C_{\hat{G}}(\phi)$ is always connected. Moreover, if the center of $\hat{G}$ is smooth over $\mathbb{Z}[\frac{1}{p}]$, then we show that "Borel pair" can be replaced by "pinning" in (1).

In general, this result shows that the crucial case to study is the space of *tame* parameters for a *tame* action of W_F that preserves a Borel pair of $\hat{G}$. This case is thoroughly studied in Section 2. Using the results of that section and the above decomposition we will get the following result, (Theorem 4.1)

Theorem 1.3. *The scheme $\underline{Z}^1(W_F^0, \hat{G})$ is syntomic (flat and locally a complete intersection) over $\underline{Z}^1(P_F, \hat{G})$, generically smooth, of pure absolute dimension $\dim(\hat{G})$.*

Beware that $\dim \hat{G} = \dim G + 1$ whenever $\hat{G}$ is the Langlands dual group of a reductive group G over F, since the base scheme of $\hat{G}$ has dimension 1.

We further conjecture that the summands appearing in the decomposition of (1.1) are connected. The last proposition reduces this conjecture to proving that for any $\hat{G}'$ with a tame Galois action preserving a Borel pair, the summand in (1.1) corresponding to *tame* parameters is connected. In Theorem 4.29 we prove this under the assumption that the action even preserves a pinning, i.e. when ${}^LG'$ is genuinely the L-group of a tamely ramified reductive group G' over F. This allows us to deduce our conjecture in many cases. In particular, Theorem 4.5 asserts :

Theorem 1.4. *If the center of $\hat{G}$ is smooth, then all the summands in the decomposition of (1.1) are connected.*

For $G = \mathrm{GL}_n$, where all centralizers are connected, this result says that each $\underline{Z}^1(W_F^0, \hat{G})_\phi$ is connected, and may be thought of as the Galois counterpart of the fact, discovered by Sécherre and Stevens [SS19], that two irreducible representations of $\mathrm{GL}_n(F)$ belong to the same endoclass if and only if they are connected by a series of congruences at various primes different from p.

The reduction to tame parameters also allows us to obtain a parameterization of the geometric irreducible components of $\underline{Z}^1(W_F^0, \hat{G})$, that is, the irreducible components of $\underline{Z}^1(W_F^0, \hat{G})_L$ for an algebraically closed field L of characteristic different from p. Such components are characterized by "inertial types" (that is, by specifing the restriction of the parameter to the inertia subgroup of W_F^0), together with some extra data that accounts for disconnectedness of centralizers. In particular, combining Corollary 2.4 with this reduction to tame parameters, we find:

Theorem 1.5. *For any algebraically closed field L of characteristic different from p, there is a natural bijection between the irreducible components of $\underline{Z}^1(W_F^0, \hat{G})_L$ and the set of $\hat{G}(L)$-conjugacy classes of pairs $(\xi, \overline{\mathcal{F}}_0)$, where ξ is an element in the image of the restriction map $\underline{Z}^1(W_F^0, \hat{G}(L)) \to \underline{Z}^1(I_F^0, \hat{G}(L))$, and $\overline{\mathcal{F}}_0$ is an element of $\pi_0(T_{\hat{G}}(\xi^{\mathrm{Fr}}, \xi'))$. [Here ξ^{Fr} is the conjugate of ξ under the action of Fr on $\hat{G}$, ξ' is the composition of ξ with the automorphism "conjugation by Fr" of I_F^0, and $T_{\hat{G}}(\xi^{\mathrm{Fr}}, \xi')$ is the transporter; that is, the subgroup of $\hat{G}_L$ consisting of elements that conjugate ξ^{Fr} to ξ'.]*

Moreover, this bijection is characterized by the property that for a general L-point φ of the irreducible component of $\underline{Z}^1(W_F^0, \hat{G})_L$ corresponding to a pair $(\xi, \overline{\mathcal{F}}_0)$, there *exists a $\hat{G}(L)$-conjugate of ξ whose restriction to I_F^0 is equal to ξ, and value at* Fr *lies in the component of $T_{\hat{G}}(\xi^{\mathrm{Fr}},\xi')$ given by $\overline{\mathcal{F}}_0$.*

1.3. **The space of parameters over $\mathbb{Z}_\ell$.** Let us now fix a prime number $\ell \neq p$. For a $\mathbb{Z}_\ell$-algebra R, we say that a $\hat{G}(R)$-valued cocycle φ is *ℓ-adically continuous* if there is some ℓ-adically separated ring R_0 such that φ comes by pushforward from some $\hat{G}(R_0)$-valued cocycle φ_0, all of whose pushforwards to $\hat{G}(R_0/\ell^n)$ are continuous for the topology inherited from W_F. It is not *a priori* clear that this definition is local for any usual topology. But the following result, extracted from Theorem 4.1, shows it is, and may justify again our approach involving the weird group W_F^0.

Theorem 1.6. *The ring of functions $R^e_{{}^LG}$ of the affine scheme $\underline{Z}^1(W_F^0/P_F^e,\hat{G})$ is ℓ-adically separated and the universal cocycle $\varphi^e_{\mathrm{univ}}$ extends uniquely to an ℓ-adically continuous cocycle*

$$\varphi^e_{\ell-\mathrm{univ}} : W_F/P_F^e \longrightarrow \hat{G}(R^e_{{}^LG}\otimes\mathbb{Z}_\ell)$$

which is universal for ℓ-adically continuous cocycles.

The ℓ-adic continuity property of $\varphi^e_{\ell-\mathrm{univ}}$ and the ℓ-adic separateness of $R^e_{{}^LG}\otimes\mathbb{Z}_\ell$ imply that the restriction of $\varphi^e_{\ell-\mathrm{univ}}$ to the prime-to-ℓ inertia group I_F^ℓ factors over a finite quotient. Since the order of this finite quotient is invertible in $\mathbb{Z}_\ell$, we can use the same strategy as before to decompose $\underline{Z}^1(W_F^0,\hat{G})_{\overline{\mathbb{Z}}_\ell}$ using now restriction of parameters to I_F^ℓ. The upshot is a decomposition similar to (1.1)

$$\underline{Z}^1(W_F^0,\hat{G})_{\overline{\mathbb{Z}}_\ell} = \coprod_{(\phi^\ell,\tilde{\varphi})} \hat{G}\times^{C_{\hat{G}}(\phi^\ell)_{\tilde{\varphi}}} \underline{Z}^1_{\mathrm{Ad}_{\tilde{\varphi}}}(W_F^0/P_F, C_{\hat{G}}(\phi^\ell)^\circ)_{1_{I_F^\ell}} \tag{1.2}$$

Theorem 4.8 asserts that each summand of this decomposition has a geometrically connected special fiber so, in particular, is connected. The collection of all these connectedness results for varying ℓ is used in the proof of the connectedness results over $\overline{\mathbb{Z}}[\frac{1}{p}]$.

1.4. **The categorical quotient over a field.** We now fix an algebraically closed field L of characteristic $\ell \neq p$ but we allow $\ell = 0$. We consider the categorical quotient

$$\underline{Z}^1(W_F^0,\hat{G})_L /\!\!/ \hat{G}_L = \varinjlim \mathrm{Spec}((R^e_{{}^LG}\otimes L)^{\hat{G}_L}).$$

Recall that the closed points of $\underline{Z}^1(W_F^0,\hat{G})_L /\!\!/ \hat{G}_L$ correspond to closed $\hat{G}(L)$-orbits in $Z^1(W_F^0,\hat{G}(L))$. A theorem of Richardson tells us that a cocycle φ has closed orbit if and only if its image in ${}^LG = \hat{G}\rtimes W_F$ is *completely reducible* in the sense that whenever it is contained in a parabolic subgroup of ${}^LG$, it has to be contained in some Levi subgroup of this parabolic subgroup.

When $\ell \neq 0$, we already know from (1.2) how to parametrize its connected components, and we now wish to describe them explicitly, at least up to homeomorphism. In order to give a unified treatment including $\ell = 0$, we (re)label the connected components of $\underline{Z}^1(W_F^0,\hat{G})_L /\!\!/ \hat{G}_L$ by the set $\Psi(L)$ of $\hat{G}(L)$-conjugacy classes of pairs (ϕ,β) consisting of

- a completely reducible inertial cocycle $\phi \in Z^1(I_F,\hat{G}(L))$.
- an element β in $\{\tilde{\beta}\in\hat{G}(L)\rtimes\mathrm{Fr},\ \tilde{\beta}\phi(i)\tilde{\beta}^{-1} = \phi(\mathrm{Fr}\, i\,\mathrm{Fr}^{-1})\}/C_{\hat{G}}(\phi)^\circ$.

For such a pair, the centralizer $C_{\hat{G}}(\phi(I_F))$ is a (possibly disconnected) reductive algebraic group over L. So we fix a Borel pair $(\hat{B}_\phi, \hat{T}_\phi)$ in $C_{\hat{G}}(\phi)^\circ$ and we choose a lift $\tilde{\beta}$ of β that normalizes this Borel pair. The adjoint action of $\tilde{\beta}$ on $\hat{T}_\phi$ only depends on β, and so does its action on the Weyl group $\Omega_\phi = \Omega_\phi^\circ \rtimes \pi_0(C_{\hat{G}}(\phi))$. Now for all $\hat{t} \in \hat{T}_\phi$ we can extend ϕ uniquely to a cocycle $\varphi_{\hat{t}\tilde{\beta}} \in \underline{Z}^1(W_F^0, \hat{G}_L)$ such that $\varphi_{\hat{t}\tilde{\beta}}(\mathrm{Fr}) = \hat{t}\tilde{\beta}$. The following result is Corollary 4.22.

Theorem 1.7. *The collection of maps $\hat{t} \mapsto \varphi_{\hat{t}\tilde{\beta}}$ define a universal homeomorphism*

$$\coprod_{(\phi,\beta)\in\Psi(L)} (\hat{T}_\phi)_\beta /\!\!/ (\Omega_\phi)^\beta \xrightarrow{\approx} \underline{Z}^1(W_F^0, \hat{G}_L) /\!\!/ \hat{G}_L,$$

which is an isomorphism if $\mathrm{char}(L) = 0$.

In particular, we see that each connected component of $\underline{Z}^1(W_F^0, \hat{G}_L) /\!\!/ \hat{G}_L$ is irreducible.

When $L = \mathbb{C}$, this allows us to compare in Section 6.3 our categorical quotient with Haines' algebraic variety constructed in [Hai14].

Corollary 1.8. *The scheme $\underline{Z}^1(W_F^0, \hat{G}_{\mathbb{C}}) /\!\!/ \hat{G}_{\mathbb{C}}$ is canonically isomorphic to Haines' variety.*

When $L = \bar{\mathbb{F}}_\ell$, we give in Theorem 6.8 an explicit condition on ℓ for the homeomorphism of the above theorem to be an isomorphism. This involves the notion of ${}^LG$-banal prime that we now discuss.

1.5. **Reducedness of fibers and ${}^LG$-banal primes.** The obstruction to obtaining a description of the GIT quotients over $\mathbb{Z}[\frac{1}{p}]$ analogous to our description of the GIT quotients over fields comes from non-reducedness of certain fibers of $\underline{Z}^1(W_F, \hat{G})$. In Theorem 5.7 we determine an explicit finite set S of primes, depending only on ${}^LG$, such that the fibers of $\underline{Z}^1(W_F, \hat{G})$ are geometrically reduced outside of S.

The reducedness of the fibers mod ℓ, for ℓ outside S implies in particular that given two distinct irreducible components of the geometric general fiber of $\underline{Z}^1(W_F, \hat{G})$, their reductions mod ℓ remain distinct. Moreover, the reduction of each such component has scheme-theoretic multiplicity one.

When ${}^LG$ is the L-group of a quasi-split connected reductive group G over F, the philosophy underlying Shotton's "$\ell \neq p$ Breuil-Mezard conjecture" suggests that this "multiplicity-preserving" bijection between irreducible components in characteristic zero and characteristic ℓ should correspond, on the representation theoretic side of the local Langlands correspondence, to a lack of congruences between distinct "inertial types" for G. It is well-known that such congruences do not appear when the prime ℓ is *banal* for G; that is, when ℓ does not divide the pro-order of any compact open subgroup of G. We therefore call the set of primes ℓ outside S "${}^LG$-banal" primes, and we show that if G is an unramified group over F with no exceptional factors, then the ${}^LG$-banal primes are precisely the primes that are banal for G, see Corollary 5.29. On the other hand, for certain exceptional groups G there exist primes that are banal for G but not ${}^LG$-banal. It would be an interesting question (which we do not attempt to address in this paper) to find an explanation for this discrepancy in terms of the representation theory of G.

Finally, we exploit the reducedness of fibers at primes away from S to compute the GIT quotient $\underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G}$ over $\overline{\mathbb{Z}}[\frac{1}{Mp}]$ for a suitable M, divisible by all ${}^LG$-banal primes. We refer to Subsection 6.1 for more details on the following statement, which is essentially Theorem 6.7.

Theorem 1.9. *There is a set of triples $(\phi, \tilde{\beta}, T_\phi)$ consisting of a cocycle $\phi \in Z^1(I_F, \hat{G}(\overline{\mathbb{Z}}[\frac{1}{pM}]))$, an element $\tilde{\beta} \in \hat{G}(\overline{\mathbb{Z}}[\frac{1}{pM}]) \rtimes \mathrm{Fr}$ such that $\tilde{\beta}\phi(i)\beta^{-1} = \phi(\mathrm{Fr}\, i\, \mathrm{Fr}^{-1})$ for all $i \in I_F$, and an $\mathrm{Ad}_{\tilde{\beta}}$-stable maximal torus of $C_{\hat{G}}(\phi)^\circ$, such that the collection of embeddings $T_\phi \hookrightarrow C_{\hat{G}}(\phi)$ induce an isomorphism of $\overline{\mathbb{Z}}[\frac{1}{pM}]$-schemes*

$$\coprod_{(\phi,\beta)} (T_\phi)_{\mathrm{Ad}_{\tilde{\beta}}} /\!\!/ (\Omega_\phi)^{\mathrm{Ad}_{\tilde{\beta}}} \xrightarrow{\sim} (\underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{pM}]}.$$

When ${}^LG$ is the Langlands dual group of an unramified group, M can be taken as the product of ${}^LG$-banal primes. In general, a description of the integer M can be extracted from Proposition 6.2.

1.6. **Relation to recent work.** This paper has been a long time coming; many of the main results were already announced at the October 2019 Oberwolfach workshop "New developments in the representation theory of p-adic groups", including in the reports [Kur19, Dat19]. The key idea of discretizing tame inertia first appeared in the 2016 arXiv version of [Hel20].

At a late stage in the preparation of this paper, Xinwen Zhu ([Zhu21], particularly Section 3.1) independently generalized the GL_n-construction of [Hel20] to construct a moduli space of Langlands parameters for a general reductive group. Zhu shows, as we do, that the spaces are flat, reduced local complete intersections, although he does not always use the same techniques. The overlap in results between Zhu's work and our own occurs primarily with results contained in our Section 2 and Appendix A. In particular, our global study of the connected components, including the functoriality principle identifying each connected component with the principal component of a smaller group, our parameterization of the irreducible components, our study of reducedness of the fibers, and our explicit description of the GIT quotients by $\hat{G}$ do not appear in his work.

Even more recently, Laurent Fargues and Peter Scholze have proposed in [FS21, Chapter VIII] a different construction of a moduli space of Langlands parameters over $\mathbb{Z}_\ell$, for $\ell \neq p$, in which the continuity constraints are dealt with via condensed mathematics. However, in order to study the main properties of their space and in particular prove flatness, reducedness and l.c.i., they revert to the same discretization process as ours, and their space turns out to be isomorphic to ours after base change to $\mathbb{Z}_\ell$. There is no further overlap with our paper, but they prove an additional beautiful result (under some mild hypothesis) : that the formation of the GIT quotient commutes with arbitrary base change.

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2. The space of tame parameters

We begin by considering moduli of tame Langlands parameters for tame groups. Let F be a non-archimedean local field of residue characteristic p, and let I_F, P_F denote the inertia group and wild inertia group of F, respectively. Let $\mathcal{O}$ be the ring of integers in a finite extension K of $\mathbb{Q}$, and $\hat{G}$ be a split connected reductive algebraic group over $\mathcal{O}[\frac{1}{p}]$, and let $(\hat{B}, \hat{T})$ be a pair consisting of a Borel subgroup $\hat{B}$ of $\hat{G}$ defined over $\mathcal{O}[\frac{1}{p}]$ and a split maximal torus $\hat{T}$ of $\hat{G}$ contained in $\hat{B}$.

We suppose that $\hat{G}$ is equipped with an action of W_F/P_F that preserves the pair $(\hat{B}, \hat{T})$, and factors through a finite quotient W of W_F/P_F. Regard W as a constant group scheme over $\mathcal{O}[\frac{1}{p}]$, and let ${}^LG$ denote the semidirect product $\hat{G} \rtimes W$; we regard ${}^LG$ as a disconnected algebraic group scheme over $\mathcal{O}[\frac{1}{p}]$.

Remark 2.1. Given our general motivations, the most natural setup would require further that the action of W_F on $\hat{G}$ preserves a pinning of $\hat{G}$, so that ${}^LG$ would be the L-group of a connected, quasi-split reductive F-group G that splits over a tamely ramified extension of F. However, in the next section we will reduce the study of the space of all Langlands parameters to the particular setup above, and at the moment we are not able to reduce to the case where a pinning is fixed.

On the other hand, the results of this section do not need the hypothesis above on W_F preserving a Borel pair of $\hat{G}$; it will be useful later when we study the GIT quotient and parametrize connected components.

Let Fr denote a lift of arithmetic Frobenius to W_F/P_F, and let s be a topological generator of I_F/P_F. We will regard Fr and s as elements of W. We have $\mathrm{Fr}\, s\, \mathrm{Fr}^{-1} = s^q$ in W_F/P_F, where q is the order of the residue field of F.

2.1. **Parameters, L-homomorphisms, and 1-cocycles.** Recall that, in the case where ${}^LG$ is the L-group of a connected, quasi-split, reductive F-group G, a *tame Langlands parameter* for G is a continuous homomorphism $\rho : W_F/P_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$, whose composition with the projection ${}^LG(\overline{\mathbb{Q}}_\ell) \to W$ is the natural quotient map $W_F \to W$. We will often refer to such a homomorphism as an *L-homomorphism.* Note that if ρ is a tame Langlands parameter, there is a unique continuous cocycle ρ° in $Z^1(W_F/P_F, \hat{G}(\overline{\mathbb{Q}}_\ell))$ such that $\rho(w) = (\rho^\circ(w), w)$; this gives a bijection between the set of L-homomorphisms and this set of cocycles.

Let $(W_F/P_F)^0$ denote the subgroup of W_F/P_F generated by the elements Fr and s that we fixed above, regarded as a *discrete* group. Let W_F^0 be the preimage of $(W_F/P_F)^0$ in W_F. (Note that both these groups depend heavily on the choices we made for Fr and s!)

For any $\mathcal{O}[\frac{1}{p}]$-algebra R, the set of L-homomorphisms $(W_F/P_F)^0 \to {}^LG(R)$ is naturally in bijection with the set of cocycles $Z^1(W_F^0/P_F, \hat{G}(R))$. *Unless stated otherwise, we will denote by φ a cocycle, and by ${}^L\varphi$ the associated L-homomorphism.*

The group $\hat{G}(R)$ acts by conjugation on the set of L-homomorphisms $(W_F/P_F)^0 \to {}^LG(R)$. The corresponding action on $Z^1(W_F^0/P_F, \hat{G}(R))$ is sometimes called "twisted conjugation". We will denote by ${}^g\varphi$ the twisted-conjugate of the cocycle φ by g. Explicitly, we have ${}^g\varphi(w) = g\varphi(w)({}^wg)^{-1}$ where ${}^wg$ denotes the given action of w on g.

2.2. **The scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$.** The functor that sends R to $Z^1(W_F^0/P_F, \hat{G}(R))$ is representable by an affine scheme denoted by $\underline{Z}^1(W_F^0/P_F, \hat{G})$. Concretely, a cocycle φ is determined by the two elements $\varphi(\mathrm{Fr})$ and $\varphi(s)$ of $\hat{G}(R)$. Conversely, a pair of elements $\mathcal{F}_0, \sigma_0$ arises in this way if, and only if the following identitiy holds in ${}^LG(R)$

$$(\mathcal{F}_0, \mathrm{Fr})(\sigma_0, s)(\mathcal{F}_0, \mathrm{Fr})^{-1} = (\sigma_0, s)^q.$$

We may thus identify $\underline{Z}^1(W_F^0/P_F, \hat{G})$ with the closed subscheme of $\hat{G} \times \hat{G}$ consisting of pairs $(\mathcal{F}_0, \sigma_0) \in \hat{G} \times \hat{G}$ such that the above identity holds in ${}^LG$. In particular, $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is affine, with coordinate ring $R_{{}^LG}$, and we have a "universal pair" $(\mathcal{F}_0, \sigma_0)$ of elements of $\hat{G}(R_{{}^LG})$ satisfying the above identity. The "universal cocycle" φ_{univ} on $Z^1(W_F^0/P_F, \hat{G}(R_{{}^LG}))$ is then the unique cocycle such that $\varphi_{\mathrm{univ}}(\mathrm{Fr}) = \mathcal{F}_0$ and $\varphi_{\mathrm{univ}}(s) = \sigma_0$. We will also let $\mathcal{F}$ and σ denote the universal elements $(\mathcal{F}_0, \mathrm{Fr})$ and (σ_0, s) of ${}^LG(R_{{}^LG})$, respectively, so that the universal L-homomorphism ${}^L\varphi_{\mathrm{univ}}$ is given by ${}^L\varphi_{\mathrm{univ}}(\mathrm{Fr}) = \mathcal{F}$ and ${}^L\varphi_{\mathrm{univ}}(s) = \sigma$.

Given a $\mathcal{O}[\frac{1}{p}]$-algebra R and an R-valued point x of $\underline{Z}^1(W_F^0/P_F, \hat{G})$, we will let $\mathcal{F}_x$, σ_x, $(\mathcal{F}_0)_x$, $(\sigma_0)_x$ φ_x denote the objects obtained by base change from $\mathcal{F}$, σ, $\mathcal{F}_0$, σ_0, and φ_{univ}, respectively.

Of course, the universal cocycle φ_{univ} cannot possibly extend in any nice way to a cocycle in $Z^1(W_F/P_F, \hat{G}(R_{{}^LG}))$. However, we will later show that if v is any finite place of $\mathcal{O}$ of residue characteristic $\ell \neq p$, then φ_{univ} extends naturally to a cocycle $\varphi_{\mathrm{univ},v}$ in $Z^1(W_F/P_F, \hat{G}(R_{{}^LG,v}))$, where $R_{{}^LG,v}$ denotes the tensor product $R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v$. In order to prove this, we must first understand the geometry of $\underline{Z}^1(W_F^0/P_F, \hat{G})$.

2.3. **Geometry of $\underline{Z}^1(W_F^0/P_F, \hat{G})$.** Let L be an algebraically closed field over $\mathcal{O}[\frac{1}{p}]$. Denote by ℓ its characteristic, and consider the fiber $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ over $\operatorname{Spec} L$. We have a map: $\mathrm{ev}_s : \underline{Z}^1(W_F^0/P_F, \hat{G})_L \to ({}^LG)_L$ that takes a cocycle φ to ${}^L\varphi(s)$ or, in other words, a pair $(\mathcal{F}, \sigma)$ to σ. Let ξ be a point of ${}^LG(L)$ in the image of this map. We denote by X_ξ the scheme-theoretic fiber of this map over ξ; it is a closed subscheme of $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$. Similarly, denote by $X_{(\xi)}$ the locally closed subscheme of $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ that is the preimage in $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ of the $\hat{G}(L)$-conjugacy class of ξ in ${}^LG(L)$. In particular, $\underline{Z}^1(W_F^0/P_F, \hat{G})(L)$ is the (set-theoretic) union of the $X_{(\xi)}(L)$, as ξ runs over a set of representatives for the $\hat{G}(L)$-conjugacy classes of ${}^LG(L)$ in the image of the map ev_s.

Let $\hat{G}_\xi$ be the $\hat{G}$-centralizer of ξ. This is a possibly non-reduced group scheme over $\operatorname{Spec} L$ that acts on X_ξ via $g \cdot (\mathcal{F}_x, \sigma_x) = (\mathcal{F}_x g^{-1}, \sigma_x)$. Moreover, for any L-algebra R and any two points $x = (\mathcal{F}_x, \sigma_x)$ and $y = (\mathcal{F}_y, \sigma_y)$ of $X_\xi(R)$, we have $\sigma_x = \sigma_y = \xi$ and $\mathcal{F}_x^{-1}\mathcal{F}_y \in \hat{G}_\xi(R)$. Thus X_ξ is a $\hat{G}_\xi$-torsor over $\operatorname{Spec} L$.

Now fix an L-point $x = (\mathcal{F}_x, \xi)$ in X_ξ. We then obtain a surjective morphism :

$$\pi_x : \hat{G}_L \times \hat{G}_\xi \to X_{(\xi)}$$

that sends (g, g') to $(g\mathcal{F}_x g' g^{-1}, g\xi g^{-1})$. Moreover, we have an action of $\hat{G}_\xi$ on $\hat{G}_L \times \hat{G}_\xi$ given by $g'' \cdot (g, g') = (g(g'')^{-1}, \mathcal{F}_x^{-1} g'' \mathcal{F}_x g'(g'')^{-1})$. This action commutes with π_x and makes $\hat{G}_L \times \hat{G}_\xi$ into a $\hat{G}_\xi$-torsor over $X_{(\xi)}$. In particular, we deduce that the reduced underlying subscheme of $X_{(\xi)}$ is smooth of dimension $\dim \hat{G}_L$.

Lemma 2.2. *Let x be an L-point of $\underline{Z}^1(W_F^0/P_F, \hat{G})$, and let*

$$\sigma_x = \sigma_x^u \sigma_x^{\mathrm{ss}}$$

be the Jordan decomposition of σ_x; i.e. σ_x^u is a unipotent element of ${}^LG(L)$ and σ_x^{ss} is a semisimple element that commutes with σ_x^u. Then the order of σ_x^{ss} is prime to ℓ and divides $e(q^{fN} - 1)$, where N is the order of the Weyl group of $\hat{G}$, e is the order of s in W, and f is the order of Fr *in W.*

Proof. Let e' be the prime-to-ℓ part of e (or $e' = e$ if $\ell = 0$). The element $(\sigma_x^{\mathrm{ss}})^{e'}$ is then a semisimple element σ'_x of $\hat{G}(L)$. The element $\mathcal{F}_x$ conjugates σ'_x to its qth power. Thus $\mathcal{F}_x^f$ is an element of $\hat{G}(L)$ that conjugates σ'_x to its q^fth power. Since σ'_x is semisimple we may assume (conjugating it and $\mathcal{F}_x$ appropriately) that it lies in $\hat{T}(L)$. Since two elements of $\hat{T}(L)$ that are conjugate under $\hat{G}(L)$ are also conjugate under the normalizer $N_{\hat{G}}(\hat{T})(L)$, there is an element w of the Weyl group of $\hat{G}$ that conjugates σ'_x to its q^fth power. Since w^N is the identity we have $\sigma'_x = (\sigma'_x)^{q^{fN}}$ and the claim follows. ☐

Corollary 2.3. *The image of $\underline{Z}^1(W_F^0/P_F, \hat{G})(L)$ in ${}^LG(L)$ under the evaluation map ev_s is a union of finitely many $\hat{G}(L)$-conjugacy classes in ${}^LG(L)$.*

Proof. Let σ be an L-point in the image of ev_s, and let $\sigma = \sigma^u \sigma^{\mathrm{ss}}$ be the Jordan decomposition of σ. Then σ^{ss} is semisimple with bounded order, so lies in one of finitely many conjugacy classes. Moreover, if we fix σ^{ss}, then σ^u lies in the centralizer ${}^LG_{\sigma^{\mathrm{ss}}}$ of σ^{ss} in ${}^LG$, which has reductive connected component of identity, by [Ste68, Cor. 9.4]. Now, two elements σ, σ' with semisimple part σ^{ss} are $\hat{G}(L)$-conjugate if, and only if, their unipotent parts $\sigma^u, (\sigma')^u$ are $\hat{G}_{\sigma^{\mathrm{ss}}}(L)$-conjugate. But there are only finitely many unipotent conjugacy classes in ${}^LG_{\sigma^{\mathrm{ss}}}(L)$ (see, for instance [FG12], Corollary 2.6, for a proof of this in positive characteristic), and therefore only finitely many $\hat{G}_{\sigma^{\mathrm{ss}}}(L)$-orbits of unipotent elements of ${}^LG_{\sigma^{\mathrm{ss}}}(L)$. The result follows. ☐

From this finiteness result we deduce that the scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ is the (set-theoretic) union of the subschemes $X_{(\xi)}$, as ξ runs over a set of representatives for the $\hat{G}(L)$-conjugacy classes of ${}^LG(L)$ in the image of the map ev_s. In particular, the irreducible components of $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ are the closures of the connected components of the $X_{(\xi)}$.

We can use this to give a parameterization of the irreducible components of $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$. For any ξ, let $T_{\hat{G}}({}^{\mathrm{Fr}}\xi, \xi^q)$ be the subscheme of $\hat{G}$ consisting of elements that conjugate ${}^{\mathrm{Fr}}\xi$ to ξ^q. We then have:

Corollary 2.4. *For any algebraically closed field L of characteristic $\ell \neq p$, the irreducible components of $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ are in bijection with $\hat{G}$-orbits of pairs $(\xi, \overline{\mathcal{F}}_0)$, where ξ is an element of $\hat{G} \rtimes s$ and $\overline{\mathcal{F}}_0$ is an element of $\pi_0(T_{\hat{G}}({}^{\mathrm{Fr}}\xi, \xi^q))$*

Proof. The irreducible components of $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$ are in bijection with the union, over a set of representatives ξ of the $\hat{G}(L)$-conjugacy classes in the image of ev_s, of the connected components of $X_{(\xi)}$. It thus suffices to fix a particular ξ and show that the connected components of $X_{(\xi)}$ are in bijection with the orbits, of the Fr-twisted conjugation action of $\hat{G}_\xi$ on $\pi_0(T_{\hat{G}}({}^{\mathrm{Fr}}\xi, \xi^q))$.

Let $\tilde{X}_{(\xi)}$ be the L-scheme that parameterizes tuples (φ, g), where φ is a cocycle in $\underline{Z}^1(W_F^0/P_F, \hat{G})_L$, and g is an element of $\hat{G}$ that conjugates ${}^L\varphi(s)$ to ξ. We have natural maps:

$$X_{(\xi)} \leftarrow \tilde{X}_{(\xi)} \rightarrow T_{\hat{G}}({}^{\mathrm{Fr}}\xi, \xi^q),$$

where the left-hand map forgets g, and the right-hand map sends (φ, g) to $g\varphi(\mathrm{Fr})^{\mathrm{Fr}}g^{-1}$.

The action $h \cdot (\varphi, g) = (\varphi, hg)$ of $\hat{G}_\xi$ on $\tilde{X}_{(\xi)}$ makes $\tilde{X}_{(\xi)}$ into a $\hat{G}_\xi$-torsor over $X_{(\xi)}$, and thus induces a bijection of $\pi_0(X_{(\xi)})$ with $\pi_0(\tilde{X}_{(\xi)})^{\hat{G}_\xi}$. On the other hand, the action $h' \cdot (\varphi, g) = ({}^{h'}\varphi, g(h')^{-1})$ of $\hat{G}_L$ on $\tilde{X}_{(\xi)}$ makes $\tilde{X}_{(\xi)}$ into a $\hat{G}_L$-torsor over $T_{\hat{G}}({}^{\mathrm{Fr}}\xi, \xi^q)$, and thus induces a bijection of $\pi_0(\tilde{X}_{(\xi)})$ with $\pi_0(T_{\hat{G}}({}^{\mathrm{Fr}}\xi, \xi^q))$. The claim follows. ∎

The fact that the $X_{(\xi)}$ have dimension equal to that of $\dim \hat{G}_L$ also lets us deduce:

Corollary 2.5. *The scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is flat over $\mathcal{O}[\frac{1}{p}]$ of pure absolute dimension $\dim \hat{G}$, and is a local complete intersection.*

Proof. The scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is isomorphic to the fiber over the identity of the map:

$$\hat{G} \times \hat{G} \rightarrow \hat{G}$$

given by $(\mathcal{F}_0, \sigma_0) \mapsto (\mathcal{F}_0, \mathrm{Fr})(\sigma_0, s)(\mathcal{F}_0, \mathrm{Fr})^{-1}(\sigma_0, s)^{-q}$. In particular its irreducible components have dimension at least $\dim \hat{G}$, and $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is a local complete intersection if every irreducible component has dimension exactly $\dim \hat{G}$. Suppose we have an irreducible component Y of larger dimension. Then for some prime v of $\mathcal{O}[\frac{1}{p}]$, of characteristic ℓ, the fiber of Y over v has dimension greater than $\dim \hat{G} - 1$. But $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{F}}_\ell}$ is a set-theoretic union of finitely many locally closed subschemes of dimension $\dim \hat{G}_{\overline{\mathbb{F}}_\ell} = \dim \hat{G} - 1$, so this is impossible. Thus every irreducible component has dimension exactly $\dim \hat{G}$, and in particular cannot be contained in the fiber of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ over ℓ for any prime ℓ. By the unmixedness theorem, every associated prime of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ has characteristic zero, so $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is flat over $\mathcal{O}[\frac{1}{p}]$ as claimed. ∎

Lemma 2.2 is a pointwise result about the order of σ_x^{ss}, but it can be turned into a global statement. Indeed, we will say that an R-point of $\hat{G}$ is unipotent if the corresponding map $\mathrm{Spec}\, R \rightarrow \hat{G}$ factors through the unipotent locus on $\hat{G}$. If R is reduced, one can check this pointwise on $\mathrm{Spec}\, R$.

Proposition 2.6. *There exists an integer M, depending only on ${}^LG$, such that σ^M is a unipotent element of $\hat{G}$. When ${}^LG = \mathrm{GL}_n$, one can take $M = q^{n!} - 1$.*

Proof. We first prove this when ${}^LG = \mathrm{GL}_n$. In this case Lemma 2.2 shows that at each geometric point x of $\underline{Z}^1(W_F^0/P_F, \hat{G})$, the expression $(\sigma_x^{\mathrm{ss}})^{q^{n!}-1}$ is equal to the identity. In particular $\sigma^{q^{n!}-1}$ is an element of ${}^LG(R_{{}^LG})$ whose specialization at every geometric point x of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is unipotent. On the other hand, by [Hel20], Proposition 6.2, when ${}^LG = \mathrm{GL}_n$, $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is reduced. Hence $\sigma^{q^{n!}-1}$, seen as a morphism $\underline{Z}^1(W_F^0/P_F, \hat{G}) \to \hat{G}$, factors through the unipotent locus of $\hat{G}$ as claimed. When ${}^LG$ is arbitrary, the result follows by choosing a faithful representation ${}^LG \to \mathrm{GL}_n$, and noting that the unipotent locus on ${}^LG$ is the preimage of the unipotent locus on GL_n. □

We will see in the next section that in fact $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is reduced for all ${}^LG$; the argument above then shows that in fact $\sigma^{e(q^{Nf}-1)}$ is unipotent.

2.4. **A construction of Bellovin-Gee.** The scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is very closely related to certain affine schemes studied by Bellovin-Gee in section 2 of [BG19]. More precisely, for any finite Galois extension L/F they define a scheme $Y_{L/F,\phi,\mathcal{N}}$ ([BG19], Definition 2.1.2) parameterizing tuples $(\Phi, \mathcal{N}, \tau)$ where Φ is an element of ${}^LG$, $\mathcal{N}$ is a nilpotent element of $\mathrm{Lie}(\hat{G})$, and $\tau : I_{L/F} \to {}^LG$ is a homomorphism, that satisfy:

1. $\mathrm{Ad}(\Phi)\mathcal{N} = q\mathcal{N}$,
2. For all $w \in I_{L/F}$, $\Phi\tau(w)\Phi^{-1} = \tau(w^q)$, and
3. For all $w \in I_{L/F}$, $\mathrm{Ad}(\tau(w))\mathcal{N} = \mathcal{N}$.

Let $Y^\circ_{L/F,\phi,\mathcal{N}}$ denote the closed subscheme of $Y_{L/F,\phi,\mathcal{N}}$ for which the images of Φ and $\tau(s)$ under the map ${}^LG \to W$ are Fr and s, respectively. Then $Y^\circ_{L/F,\phi,\mathcal{N}}$ is a union of connected components of $Y_{L/F,\phi,\mathcal{N}}$.

We then have:

Proposition 2.7. *Fix M such that σ^M is unipotent, and let L/F be a finite, tamely ramified Galois extension whose ramification index is divisible by M. Then there is a natural isomorphism $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Q}}_\ell} \to (Y^\circ_{L/F,\phi,\mathcal{N}})_{\overline{\mathbb{Q}}_\ell}$.*

Proof. We give maps in both directions that are inverse to each other. On the one hand, without any hypotheses on L/F, there is always a map $Y^\circ_{L/F,\phi,\mathcal{N}} \to \underline{Z}^1(W_F^0/P_F, \hat{G})$ over $\overline{\mathbb{Q}}_\ell$ that takes a triple $(\Phi, \mathcal{N}, \tau)$ to the L-homomorphism ${}^L\varphi$ defined by ${}^L\varphi(\mathrm{Fr}) = \Phi$ and ${}^L\varphi(s) = \tau(s)\exp(\mathcal{N})$. In the other direction, given a cocycle φ we can set $\Phi = {}^L\varphi(\mathrm{Fr})$, $\mathcal{N} = \frac{1}{M}\log({}^L\varphi(s)^M)$, and let $\tau : I_F \to {}^LG(\overline{\mathbb{Q}}_\ell)$ be the map taking s^a to ${}^L\varphi(s)^a \exp(-a\mathcal{N})$; the latter factors through $I_{L/F}$ under our ramification condition on L. These two maps are clearly inverse to each other. □

As this isomorphism involves exponentiation, and division by M, it does not extend to the special fiber modulo small primes. In fact the space $Y^\circ_{L/F,\phi,\mathcal{N}}$ can be quite badly behaved at small primes: for instance, if $\hat{G} = \mathrm{GL}_2$, and we take L/F to be a finite, tamely ramified Galois extension of ramification index M divisible by $q^2 - 1$ (so that σ^M is unipotent), then at any prime ℓ dividing $q + 1$ the fiber of $Y^\circ_{L/F,\phi,\mathcal{N}}$ has dimension five, whereas the generic fiber has dimension four. That is, $Y^\circ_{L/F,\phi,\mathcal{N}}$ fails to be flat in this setting. One could attempt to remedy this by replacing $Y^\circ_{L/F,\phi,\mathcal{N}}$ by the closure of its generic fiber, but even then, at primes ℓ as above, there is not a bijection between the irreducible components of $(Y^\circ_{L/F,\phi,\mathcal{N}})_{\overline{\mathbb{F}}_\ell}$

and those of $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{F}}_\ell}$. Indeed, one can verify that the irreducible components of the latter behave in a manner consistent with the $\ell \neq p$ Breuil-Mezard conjecture of Shotton [Sho18], whereas those of the former do not.

Bellovin-Gee show ([BG19], Theorem 2.3.6) that $Y_{L/F,\phi,\mathcal{N}}$ (and hence $\underline{Z}^1(W_F^0/P_F, \hat{G})$) is generically smooth, by constructing a smooth point on each irreducible component of $Y_{L/F,\phi,\mathcal{N}}$ in characteristic zero. We sketch their construction here (or rather, its adaptation to $\underline{Z}^1(W_F^0/P_F, \hat{G})$), both in the interests of being self-contained and because we will need it for other purposes.

Fix a prime $\ell \neq p$ and a $\overline{\mathbb{Q}}_\ell$ point ξ of ${}^LG$ in the image of the map $\underline{Z}^1(W_F^0/P_F, \hat{G}) \to {}^LG$ taking φ to ${}^L\varphi(s)$. As $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Q}}_\ell}$ is (set-theoretically) the union of the smooth schemes $X_{(\xi)}$ for such ξ, it suffices to construct a smooth point of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ on each connected component of $X_{(\xi)}$.

Let $\xi = \xi^{\mathrm{ss}}\xi^u$ be the Jordan decomposition of ξ. Since we are in characteristic zero ξ^u is a unipotent element of $\hat{G}$, and we may consider its logarithm $\mathcal{N}$, which is a nilpotent element of the Lie algebra of the centralizer $\hat{G}_{\xi^{\mathrm{ss}}}$ of ξ^{ss}.

Let λ be a cocharacter of $\hat{G}^{\xi^{\mathrm{ss}}}$ that is an associated cocharacter of $\mathcal{N}$, in the sense of [BG19], section 2.3. In particular, for all t we have $\mathrm{Ad}(\lambda(t))\mathcal{N} = t^2\mathcal{N}$. Set $\Lambda = \lambda(q^{\frac{1}{2}})$ for some square root $q^{\frac{1}{2}}$ of q, so that $\mathrm{Ad}(\Lambda)\mathcal{N} = q\mathcal{N}$. Then $\Lambda\xi^u\Lambda^{-1} = (\xi^u)^q$.

Further let H denote the normalizer, in ${}^LG$, of the subgroup of ${}^LG$ generated by ξ^{ss}. Let Y be the set of $g \in H$ such that $g\xi^u g^{-1} = (\xi^u)^q$. Note that in particular the map $\underline{Z}^1(W_F^0/P_F, \hat{G}) \to {}^LG$ that takes φ to ${}^L\varphi(\mathrm{Fr})$ identifies X_ξ with a union of connected components of Y.

On the other hand $Y = \Lambda \cdot (H \cap {}^LG^{\mathcal{N}})$. By [Bel16], Proposition 4.9, the inclusion of $H \cap {}^LG^{\mathcal{N}} \cap {}^LG^{\lambda}$ into $H \cap {}^LG^{\mathcal{N}}$ is a bijection on connected components, and by [Bel16], Lemma 5.3 there is a point of finite order on each connected component of $H \cap {}^LG^{\mathcal{N}} \cap {}^LG^{\lambda}$. Thus on each connected component of Y there is a point of the form Λc, where c has finite order and commutes with Λ. Then Bellovin and Gee show, via a cohomology calculation, that when $(\Lambda c, \xi)$ lies in $\underline{Z}^1(W_F^0/P_F, \hat{G})$ it is a smooth point of $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Q}}_\ell}$. We immediately deduce:

Proposition 2.8. *The scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is generically smooth (and therefore reduced.)*

Proof. Generic smoothness is immediate since there is a point of the form $(\Lambda c, \xi)$ on every connected component of $X_{(\xi)}$ for all ξ. Since $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is a local complete intersection there is no embedded locus; that is, $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is reduced. □

Remark 2.9. We will later give an argument that in fact the fibers $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{F}}_\ell}$ are generically smooth outside of an explicit finite set. This argument is independent of (though partially inspired by) the above argument of Bellovin-Gee, and certainly implies the above proposition, as well as the separatedness results below. We include the Bellovin-Gee argument here for convenience of exposition, and because the comparison with their construction is interesting in its own right.

Proposition 2.10. *For any prime $\ell \neq p$, and any irreducible component Y of $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Q}}_\ell}$, there exists a $\overline{\mathbb{Z}}_\ell$-point of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ on Y.*

Proof. We have shown that Y contains a point of the form $(\Lambda c, \xi)$ constructed above. We must show that this point is conjugate to a $\overline{\mathbb{Z}}_\ell$-point. Note that Λ, c and ξ are contained in ${}^LG(L)$ for some $L \subset \overline{\mathbb{Q}}_\ell$ finite over $\mathbb{Q}_\ell$. Now, since $\Lambda = \lambda(q^{\frac{1}{2}})$ for some cocharacter λ and since $q^{\frac{1}{2}}$ is an ℓ-unit for all $\ell \neq p$, the element Λ is compact in ${}^LG(L)$ (i.e. the subgroup of ${}^LG(L)$ generated by Λ has compact closure). Moreover, since c has finite order and commutes to Λ, the element Λc is also compact. Therefore, since Λc normalizes the subgroup of ${}^LG(L)$ generated by ξ, and some power of ξ is unipotent, the subgroup of ${}^LG(L)$ generated by Λc and ξ has compact closure. Thus it normalizes a facet of the semisimple building $B(\hat{G}, L)$ and fixes its barycenter x. There is a finite extension L' of L such that x becomes an hyperspecial point in $B(\hat{G}, L')$ and is conjugate to the "canonical" hyperspecial point o fixed by $\hat{G}(\mathcal{O}_L)$ under some element $g \in \hat{G}(L')$. The fixator of o in ${}^LG(L)$ is $Z_{\hat{G}}(L).{}^LG(\mathcal{O}_L)$, hence the L-homomorphism ${}^L\varphi : W_F^0/P_F \longrightarrow {}^LG(\overline{\mathbb{Q}}_\ell)$ associated to the pair $({}^g(\Lambda c), {}^g\xi)$ takes values in $Z_{\hat{G}}(\overline{\mathbb{Q}}_\ell).{}^LG(\overline{\mathbb{Z}}_\ell)$. Consider its composition with the quotient map to $(Z_{\hat{G}}(\overline{\mathbb{Q}}_\ell).{}^LG(\overline{\mathbb{Z}}_\ell))/\hat{G}(\overline{\mathbb{Z}}_\ell) = Q \rtimes W$ with $Q := (Z_{\hat{G}}(\overline{\mathbb{Q}}_\ell).\hat{G}(\overline{\mathbb{Z}}_\ell))/\hat{G}(\overline{\mathbb{Z}}_\ell) = Z_{\hat{G}}(\overline{\mathbb{Q}}_\ell)/Z_{\hat{G}}(\overline{\mathbb{Z}}_\ell)$. Since it has relatively compact image and Q is discrete, it factors over a finite quotient W' of W_F^0/P_F. But since Q is a $\mathbb{Q}$-vector space of finite dimension, we have $H^1(W', Q) = \{1\}$, so the above composition is conjugate, under some element of $q \in Q$, to the trivial L-homomorphism $W_F^0 \longrightarrow Q \rtimes W$. So if $\tilde{q}$ is any lift of q in $Z_{\hat{G}}(\overline{\mathbb{Q}}_\ell)$, the conjugate ${}^{\tilde{q}}({}^L\varphi)$ associated to the pair $({}^{\tilde{q}g}(\Lambda c), {}^{\tilde{q}g}\xi)$ is ${}^LG(\overline{\mathbb{Z}}_\ell)$-valued, as desired. $\square$

Corollary 2.11. *For any prime $\ell \neq p$, the ring $R_{{}^LG}$ is ℓ-adically separated.*

Proof. Since $R_{{}^LG}$ is reduced and flat over $\mathcal{O}$, we have an embedding:

$$R_{{}^LG} \to \prod_Y \mathcal{O}_Y,$$

where Y runs over the irreducible components of $R_{{}^LG}$. Each $\mathcal{O}_Y$ is affine, integral, and flat over $\mathcal{O}$, and by Proposition 2.10 contains an integral point. In particular ℓ is not invertible on $\mathcal{O}_Y$. Thus it suffices to show that any noetherian integral flat $\mathbb{Z}_\ell$-algebra A in which ℓ is not invertible is ℓ-adically separated. Indeed, suppose a is a nonzero element of A in the intersection of the ideals generated by ℓ^i. Then for each i, there is an $a_i \in A$ such that $\ell^i a_i = a$. Each a_i is unique since A is integral, so $a_{i-1} = \ell a_i$. Since the ascending chain of ideals generated by the a_i stabilizes, we have $a_i = u a_{i-1}$ for some unit u and integer i. Then, as A is integral, we have $u\ell = 1$, contradicting the fact that ℓ is not invertible in A. $\square$

2.5. **The universal family.** Now that we have shown that $R_{{}^LG}$ is ℓ-adically separated, we return to the question of extending the parameter φ_{univ} to an L-homomorphism defined on all of W_F. As we have already remarked, this is only possible after tensoring with the completed local ring $\mathcal{O}_v$ for some finite place v of $\mathcal{O}$ of residue characteristic $\ell \neq p$. The key point is the following notion of continuity, first introduced in [Hel20] in the case ${}^LG = \mathrm{GL}_n$:

Definition 2.12. Let R be a Noetherian $\mathcal{O}[\frac{1}{p}]$-algebra, and let $\rho : W_F \to {}^LG(R)$ be a group homomorphism. We say that ρ is *ℓ-adically continuous* if one of the following two conditions hold:

(1) The ring R is ℓ-adically separated, and for each $n > 0$, the preimage of U_n under ρ is open in W_F, where U_n is the kernel of the map ${}^LG(R) \to {}^LG(R/\ell^n R)$.
(2) There exists a Noetherian, ℓ-adically separated $\mathcal{O}[\frac{1}{p}]$-algebra R', a map $f : R' \to R$, and an ℓ-adically continuous map $\rho' : W_F \to {}^LG(R')$ such that $\rho = f \circ \rho'$.

If R is ℓ-adically separated and condition (2) in the above definition holds, it is easy to check that condition (1) holds as well, so the two conditions are consistent with each other. We will say that a cocycle $\varphi \in Z^1(W_F, \hat{G}(R))$ is ℓ-adically continuous if its associated L-homomorphism ${}^L\varphi$ is ℓ-adically continuous as in the above definition.

Theorem 2.13. *For each finite place v of $\mathcal{O}$ of residue characteristic $\ell \neq p$, there exists a unique ℓ-adically continuous cocycle*

$$\varphi_{\mathrm{univ},v} : W_F/P_F \to \hat{G}(R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v)$$

whose restriction to $(W_F/P_F)^0$ is equal to φ_{univ}. Moreover, if R is any Noetherian $\mathcal{O}_v$-algebra, and $\varphi : W_F/P_F \to \hat{G}(R)$ is an ℓ-adically continuous cocycle, then there is a unique map: $f : R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v \to R$ such that $\varphi = f \circ \varphi_{\mathrm{univ},v}$.

Proof. When ${}^LG = \mathrm{GL}_n$, this is proved in [Hel20], Proposition 8.2; we reduce to this case. Choose a faithful representation $\tau : {}^LG \to \mathrm{GL}_n$ defined over $\mathcal{O}[\frac{1}{p}]$. Then

$$\tau \circ {}^L\varphi_{\mathrm{univ}} \in \mathrm{Hom}(W_F^0/P_F, \mathrm{GL}_n(R_{{}^LG})) = Z^1(W_F^0/P_F, \mathrm{GL}_n(R_{{}^LG}))$$

where GL_n is equipped with the trivial action of W_F. There is thus a unique map $f : R_{\mathrm{GL}_n} \to R_{{}^LG}$ that takes the universal cocycle on $\underline{Z}^1(W_F^0/P_F, \mathrm{GL}_n)$ (actually a homomorphism) to $\tau \circ {}^L\varphi_{\mathrm{univ}}$. Since this universal cocycle extends to an ℓ-adically continuous cocycle on W_F/P_F, with values in $R_{\mathrm{GL}_n} \otimes_{\mathcal{O}} \mathcal{O}_v$, composing this extension with f gives an extension of $\tau \circ {}^L\varphi_{\mathrm{univ}}$ to an ℓ-adically continuous homomorphism $W_F/P_F \longrightarrow \mathrm{GL}_n(R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v)$. Denote this homomorphism by ${}^L\varphi_{\mathrm{univ},v}$. Its restriction to W_F^0/P_F factors through ${}^LG(R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v)$ and is equal to ${}^L\varphi_{\mathrm{univ}}$, so it only remains to prove that ${}^L\varphi_{\mathrm{univ},v}$ factors through ${}^LG(R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v)$ too. But this follows from the ℓ-adic separatedness of $R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v$ and the fact that for each $n \in \mathbb{N}$, we know that the image of ${}^L\varphi_{\mathrm{univ},v}(W_F)$ in $\mathrm{GL}_n(R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v/(\ell^n))$ coincides with the image of ${}^L\varphi_{\mathrm{univ},v}(W_F^0)$, which is contained in ${}^LG(R_{{}^LG} \otimes_{\mathcal{O}} \mathcal{O}_v/(\ell^n))$. Uniqueness and the universal property are now straightforward. ∎

In light of this, we define a "good coefficient ring" to be a Noetherian ring R that is an $\mathcal{O} \otimes \mathbb{Z}_\ell$-algebra for some $\ell \neq p$, and a "good coefficient field" to be a good coefficient ring that is also a field. Theorem 2.13 then implies that for any good coefficient ring R, and any cocycle $\varphi^0 : (W_F/P_F)^0 \to \hat{G}(R)$, there is a unique ℓ-adically continuous cocycle $\varphi : W_F/P_F \to \hat{G}(R)$ extending φ^0.

In particular, if R is a complete local $\mathcal{O}$-algebra with maximal ideal $\mathfrak{m}$, of residue characteristic $\ell \neq p$, then any ℓ-adically continuous cocycle $\varphi : W_F/P_F \to \hat{G}(R)$ is clearly $\mathfrak{m}$-adically continuous. Conversely, given an $\mathfrak{m}$-adically continous cocycle $\varphi : W_F/P_F \to \hat{G}(R)$, Theorem 2.13 shows that there is a unique ℓ-adically continuous cocycle φ' extending the restriction of φ to $(W_F/P_F)^0$. Then φ' and φ are both $\mathfrak{m}$-adically continuous and agree on $(W_F/P_F)^0$, so φ is also ℓ-adically continuous. Thus the notions of ℓ-adic and $\mathfrak{m}$-adic continuity coincide for cocycles valued in R.

3. Reduction to tame parameters

In this section, we broaden the setting as follows. We consider a split reductive group scheme $\hat{G}$ over $\mathbb{Z}[\frac{1}{p}]$ endowed with a finite action of W_F, but *we no longer assume that this action is tame, nor that it stabilizes a Borel pair.*

For any $\mathbb{Z}[\frac{1}{p}]$-algebra R, we denote by $Z^1(W_F, \hat{G}(R))$ the set of 1-cocycles which are continuous for the natural topology of the source and the discrete topology on the target. We use similar notation for W_F^0 and any closed subgroup thereof. Recall that the topology on W_F^0 is such that P_F, with its natural topology, sits as a closed and open subgroup.

It will be handy to switch between 1-cocycles and their associated L-morphisms. In this regard, we usually denote by ${}^LG$ a group scheme of the form $\hat{G} \rtimes W$ with W any *finite* quotient of W_F through which the given action on $\hat{G}$ factors. Note that W may be allowed to change according to our needs, but we prefer to keep it finite in order to work with algebraic group schemes. For the sake of clarity, we will most often distinguish a 1-cocycle φ from its associated L-homomorphism ${}^L\varphi := \varphi \rtimes \mathrm{id} : W_F \longrightarrow {}^LG(R)$, although occasionally it will be more handy to write φ for the L-homomorphism.

3.1. **Overview.**

Our aim is to show how the study of moduli of 1-cocycles $W_F^0 \longrightarrow \hat{G}$ (and subsequently, moduli of ℓ-adically continuous 1-cocycles $W_F \longrightarrow \hat{G}$) can be reduced to the particular case considered in the previous section, namely the case of tame 1-cocycles valued in a reductive group scheme with a tame Galois action that stabilizes a Borel pair. The principle is very simple ; suppose R is a $\mathbb{Z}[\frac{1}{p}]$-algebra and $\varphi : W_F^0 \longrightarrow \hat{G}(R)$ is a 1-cocycle, and denote by $\phi : P_F \longrightarrow \hat{G}(R)$ its restriction to P_F. Then the conjugation action of W_F^0 on $\hat{G}(R)$ through ${}^L\varphi$ stabilizes the centralizer $C_{\hat{G}(R)}({}^L\phi(P_F))$ and the restricted action on this subgroup factors over W_F^0/P_F. Denoting by Ad_φ this action, an elementary computation shows that the map $\eta \mapsto \eta \cdot \varphi$ sets up a bijection

$$Z^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, C_{\hat{G}(R)}({}^L\phi(P_F))) \xrightarrow{\sim} \{\varphi' \in Z^1(W_F^0, \hat{G}(R)),\, \varphi'_{|P_F} = \phi\}.$$

By Lemma A.1 in the appendix, the functor on R-algebras $R' \mapsto C_{\hat{G}(R')}({}^L\phi(P_F))$ is representable by a smooth group scheme over R that we denote by $C_{\hat{G}}(\phi)$. Moreover, by [PY02, Thm 2.1], its connected geometric fibers are reductive. Therefore, one is tempted to see the set $Z^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, C_{\hat{G}(R)}({}^L\phi(P_F)))$ as an instance of the type of tame parameters that were studied in the previous section. However, making this idea work requires addressing the following issues :

- The group scheme $C_{\hat{G}}(\phi)$ may have non-connected fibers.
- Its neutral component $C_{\hat{G}}(\phi)^\circ$ may not be split.
- The action Ad_φ may neither be finite nor preserve a Borel pair of $C_{\hat{G}}(\phi)^\circ$.

In order to address these issues, the first step is to find a nice set of representatives of conjugacy classes of continuous cocycles with source P_F. Since we prefer to work with finitely presented objects, we choose a decreasing sequence $(P_F^e)_{e\in\mathbb{N}}$ of open normal subgroups of P_F whose intersection is $\{1\}$. Then we fix $e \in \mathbb{N}$ such that P_F^e acts trivially on $\hat{G}$, and we restrict attention to cocycles that are trivial on P_F^e. The following theorem follows from Theorems A.9, A.12 and Proposition A.13 in the appendix.

Theorem 3.1. *There is a number field K_e and a finite set*

$$\Phi_e \subset Z^1\left(P_F/P_F^e, \hat{G}\left(\mathcal{O}_{K_e}[\tfrac{1}{p}]\right)\right), \quad \textit{such that}$$

(1) *For any $\mathcal{O}_{K_e}[\frac{1}{p}]$-algebra R, any cocycle $\phi : P_F/P_F^e \longrightarrow \hat{G}(R)$ is étale-locally $\hat{G}$-conjugate to a locally unique $\phi_0 \in \Phi_e$.*
(2) *For any $\phi \in \Phi_e$, the reductive group scheme $C_{\hat{G}}(\phi)^\circ$ is split over $\mathcal{O}_{K_e}[\frac{1}{p}]$ and the component group $\pi_0(\phi) := \pi_0(C_{\hat{G}}(\phi))$ is constant.*

3.2. **Some definitions and constructions.** Let $\phi \in \Phi_e$. For any $\mathcal{O}_{K_e}[\frac{1}{p}]$-algebra R we denote by $Z^1(W_F^0, \hat{G}(R))_\phi$ the set of 1-cocycles $W_F^0 \longrightarrow \hat{G}(R)$ that extend ϕ. The functor $R \mapsto Z^1(W_F^0, \hat{G}(R))_\phi$ is visibly representable by an affine scheme of finite type over $\mathcal{O}_{K_e}[\frac{1}{p}]$, namely a closed subscheme of $\hat{G} \times \hat{G}$. We denote this scheme by $\underline{Z}^1(W_F^0, \hat{G})_\phi$.

Definition 3.2. An element $\phi \in \Phi_e$ is called *admissible* if the scheme $\underline{Z}^1(W_F^0, \hat{G})_\phi$ is not empty.

In the sequel, it will be convenient to choose our "L-group" ${}^LG$ in the form ${}^LG = \hat{G} \rtimes W_e$ where W_e is a finite quotient of W_F into which P_F/P_F^e maps *injectively.* For example, we may choose our sequence $(P_F^e)_e$ such that $P_F^e = P_{F_e}$ for some Galois extension F_e of F and put $W_e = \mathrm{Gal}(F_e/F)$. Then the L-homomorphism ${}^L\varphi$ associated to $\varphi \in Z^1(W_F^0, \hat{G}(R))_\phi$ factors through the subgroup[1]

$$C_{{}^LG(R)}(\phi) := \left\{(g,w) \in {}^LG(R),\, (g,w){}^L\phi(w^{-1}pw)(g,w)^{-1} = {}^L\phi(p), \forall p \in P_F\right\}.$$

Writing the functor $C_{{}^LG}(\phi) :\ R \mapsto C_{{}^LG(R)}(\phi)$ on $\mathcal{O}_{K_e}[\frac{1}{p}]$-algebras as a disjoint union $\bigsqcup_{w\in W_e} T_{\hat{G}}({}^w\phi, \phi)$ of transporters in $\hat{G}$ (where ${}^w\phi$ is defined by ${}^w\phi(p) = w(\phi(w^{-1}pw))$), we see from Lemma A.1 that this functor is represented by a smooth group scheme that sits in an exact sequence

$$1 \to C_{\hat{G}}(\phi) \to C_{{}^LG}(\phi) \to W_e.$$

Actually, it follows from the uniqueness of ϕ_0 in i) of Theorem 3.1 that $T_{\hat{G}}({}^w\phi, \phi)$ is either empty or is a $C_{\hat{G}}(\phi)$-torsor for the étale topology. Therefore, $C_{{}^LG}(\phi)$ is an extension of the *constant* subgroup $W_{e,\phi} := \{w \in W_e, T_{\hat{G}}({}^w\phi, \phi) \neq \emptyset\}$ of W_e by $C_{\hat{G}}(\phi)$. Since $C_{{}^LG}(\phi)^\circ = C_{\hat{G}}(\phi)^\circ$ is a split reductive group scheme over $\mathcal{O}_{K_e}[\frac{1}{p}]$, we know by general results [Con14, Prop. 3.1.3] that

$$\tilde{\pi}_0(\phi) := \pi_0(C_{{}^LG}(\phi))$$

is a separated étale group scheme over $\mathcal{O}_{K_e}[\frac{1}{p}]$. Since it is an extension of $W_{e,\phi}$ by $\pi_0(\phi)$, we see that $\tilde{\pi}_0(\phi)$ is actually *finite* étale. *Therefore, after maybe enlarging K_e, we may assume that $\tilde{\pi}_0(\phi)$ is constant over $\mathcal{O}_{K_e}[\frac{1}{p}]$.* Now, let us assume that ϕ is admissible. Then we have $W_{e,\phi} = W_e$ and an exact sequence of abstract groups

$$1 \to \pi_0(\phi) \to \tilde{\pi}_0(\phi) \to W_e \to 1.$$

Therefore, the affine scheme $\underline{Z}^1(W_F^0, \hat{G})_\phi$ decomposes as a disjoint union

$$\underline{Z}^1(W_F^0, \hat{G})_\phi = \coprod_{\alpha\in\Sigma(\phi)} \underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha}, \text{ where}$$

[1] Note that, despite the notation, this subgroup is not the centralizer of ${}^L\phi$ in ${}^LG$.

- $\Sigma(\phi)$ denotes the set of homomorphisms $W_F \longrightarrow \tilde{\pi}_0(\phi)$ that extend the map $P_F \longrightarrow \tilde{\pi}_0(\phi)$ given by the composition of ${}^L\phi$ with the projection to $\tilde{\pi}_0(\phi)$, and whose composition with $\tilde{\pi}_0(\phi) \longrightarrow W_e$ is the natural projection $W_F \longrightarrow W_e$.
- $\underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha}(R) = Z^1(W_F^0, \hat{G}(R))_{\phi,\alpha}$ is the subset of extensions φ of ϕ such that the composition of ${}^L\varphi$ with the projection to $\tilde{\pi}_0(\phi)$ is α.

Definition 3.3. We will say that $\alpha \in \Sigma(\phi)$ is *admissible* if the scheme $\underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha}$ is not empty.

Observe that there are only finitely many admissible elements in $\Sigma(\phi)$ since the scheme $\underline{Z}^1(W_F^0, \hat{G})_{\phi}$ has finitely many connected components.

We now note that two elements $\varphi, \varphi' \in Z^1(W_F^0, \hat{G}(R))_{\phi,\alpha}$ differ by a tame cocycle valued in $C_{\hat{G}}(\phi)^\circ$ (beware the $\circ$). More precisely, if we write $\varphi'(w) = \eta(w)\varphi(w)$, then $w \mapsto \eta(w)$ belongs to $Z^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ(R))$. In other words, the map $\eta \mapsto \eta \cdot \varphi$ sets up an isomorphism of R-schemes

$$\underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ)_R \xrightarrow{\sim} \underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha,R}.$$

At this point we have dealt with the first two issues mentioned in the beginning of this section. The next result deals with the third issue and will allow us to reduce to the tame parameters that were studied in the previous section.

Theorem 3.4. *There is a finite extension K'_e of K_e such that for any admissible $\phi \in \Phi_e$ and any admissible $\alpha \in \Sigma(\phi)$, there is some $\varphi_\alpha \in Z^1(W_F^0, \hat{G}(\mathcal{O}_{K'_e}[\frac{1}{p}]))_{\phi,\alpha}$ such that ${}^L\varphi_\alpha(W_F^0)$ is finite and $\mathrm{Ad}_{\varphi_\alpha}$ preserves a Borel pair of the split reductive group scheme $C_{\hat{G}}(\phi)^\circ$.*

Fix ϕ, α and φ_α as in the theorem. Since ${}^L\varphi_\alpha(W_F^0)$ is finite, φ_α extends canonically to W_F with ${}^L\varphi_\alpha(W_F) = {}^L\varphi_\alpha(W_F^0)$. So the conjugation action $\mathrm{Ad}_{\varphi_\alpha}$ of W_F^0 on the reductive group $C_{\hat{G}}(\phi)^\circ$ extends to a finite action of W_F, and it has to be trivial on P_F. Since this action stabilizes a Borel pair, we see that the $\mathcal{O}_{K'_e}[\frac{1}{p}]$-scheme $\underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ)$ is (a base change of) an instance of those tame moduli schemes studied in Section 2.

Remark 3.5. It is natural to ask whether we can find φ_α so that $\mathrm{Ad}_{\varphi_\alpha}$ preserves a pinning of $C_{\hat{G}}(\phi)^\circ$. Our techniques can achieve this when the center of $C_{\hat{G}}(\phi)^\circ$ is smooth, see Remark 3.9. In Theorem 3.12, we give a sufficient condition on $\hat{G}$ for each $C_{\hat{G}}(\phi)^\circ$ to have smooth center.

Before we can prove the theorem, we need some preparation. Let us fix a Borel pair $\mathcal{B}_\phi = (B_\phi, T_\phi)$ in $C_{\hat{G}}(\phi)^\circ$ and let us denote by $\mathcal{T}_\phi$ the normalizer in $C_{{}^LG}(\phi)$ of this Borel pair. By [Con14, Prop. 2.1.2], this is again a smooth group scheme over $\mathcal{O}_{K_e}[\frac{1}{p}]$. Since the normalizer of a Borel pair in a connected reductive group over an algebraically closed field is the torus of the Borel pair, we have $(\mathcal{T}_\phi)^\circ = C_{\hat{G}}(\phi)^\circ \cap \mathcal{T}_\phi = T_\phi$. Since any two Borel pairs in a connected reductive group over an algebraically closed field are conjugate, we also have $\pi_0(\mathcal{T}_\phi) = \pi_0(C_{{}^LG}(\phi)) = \tilde{\pi}_0(\phi)$. Moreover, since T_ϕ is abelian, the conjugation action of $\mathcal{T}_\phi$ on T_ϕ factors through an action

$$\tilde{\pi}_0(\phi) \longrightarrow \mathrm{Aut}_{\mathcal{O}_{K_e}[\frac{1}{p}]-gp.sch.}(T_\phi).$$

In particular, any section $\alpha \in \Sigma(\phi)$ provides us with an action of W_F on the torus T_ϕ. This action has to be trivial on P_F, since ${}^L\phi(P_F)$ centralizes $C_{\hat{G}}(\phi)$, so that ${}^L\phi(P_F) \subset \mathcal{T}_\phi(\mathcal{O}_{K_e}[\frac{1}{p}])$ acts trivially on T_ϕ by conjugation. Therefore, the subset

$$\begin{aligned}\Sigma(W_F^0, \mathcal{T}_\phi(R))_\phi &:= \{\varphi \in Z^1(W_F^0, \hat{G}(R))_\phi,\ {}^L\varphi(W_F^0) \subset \mathcal{T}_\phi(R))\} \\ &= \{\varphi \in Z^1(W_F^0, \hat{G}(R))_\phi,\ \mathrm{Ad}_\varphi \text{ preserves } \mathcal{B}_\phi\}\end{aligned}$$

decomposes as a disjoint union

$$\Sigma(W_F^0, \mathcal{T}_\phi(R))_\phi = \bigsqcup_{\alpha \in \Sigma(\phi)} \Sigma(W_F^0, \mathcal{T}_\phi(R))_{\phi,\alpha}$$

where $\Sigma(W_F^0, \mathcal{T}_\phi(R))_{\phi,\alpha}$ denotes the subset of those $\varphi \in \Sigma(W_F^0, \mathcal{T}_\phi(R))_\phi$ such that the composition $W_F^0 \xrightarrow{{}^L\varphi} \mathcal{T}_\phi(R) \longrightarrow \tilde{\pi}_0(\phi)$ is α. Note that $\Sigma(W_F^0, \mathcal{T}_\phi(R))_{\phi,\alpha}$ is either empty or is a principal homogeneous set under the abelian group $Z^1_\alpha(W_F^0/P_F, T_\phi(R))$. Varying R, we get a closed affine subscheme $\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_\phi$ of $\underline{Z}^1(W_F^0, \hat{G})_\phi$ which decomposes as a coproduct of affine $\mathcal{O}_{K_e}[\frac{1}{p}]$-schemes

$$\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_\phi = \bigsqcup_{\alpha \in \Sigma(\phi)} \underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha}$$

where each $\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha}$ carries an action of the abelian group $\mathcal{O}_{K_e}[\frac{1}{p}]$-scheme $\underline{Z}^1_\alpha(W_F^0/P_F, T_\phi)$, and is a *pseudo-torsor* for this action, in the sense of [The Stacks Project, Tag 0497].

Finally, let W be a finite quotient of W_F such that ${}^L\phi$ factors over the image $P \subset W$ of P_F in W and α factors over W. Then the same definitions as above provide us with a $\mathcal{O}_{K_e}[\frac{1}{p}]$-scheme $\underline{\Sigma}(W, \mathcal{T}_\phi)_{\phi,\alpha}$, which is a pseudo-torsor for the natural action of the group $\mathcal{O}_{K_e}[\frac{1}{p}]$-scheme $\underline{Z}^1_\alpha(W/P, T_\phi)$.

Theorem 3.6. *Suppose ϕ and α are admissible.*

(1) *$\underline{Z}^1_\alpha(W_F^0/P_F, T_\phi)$ is a diagonalisable group scheme over $\mathcal{O}_{K_e}[\frac{1}{p}]$.*

(2) *$\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha}$ is a fppf* torsor *under $\underline{Z}^1_\alpha(W_F^0/P_F, T_\phi)$.*

Moreover, these two statements still hold with W_F^0 replaced by a sufficiently large finite quotient W as above.

Before we prove this result, let us see how it implies Theorem 3.4. The claim in Theorem 3.4 is that there exists an extension K'_e of K_e such that $\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha}$ has an $\mathcal{O}_{K'_e}[\frac{1}{p}]$-point φ with finite image. In other words, we need to show the existence of a finite quotient W of W_F such that $\underline{\Sigma}(W, \mathcal{T}_\phi)_{\phi,\alpha}$ has an $\mathcal{O}_{K'_e}[\frac{1}{p}]$-point. So, from Theorem 3.6, it suffices to show that any fppf torsor under a diagonalisable group over $\mathcal{O}_{K_e}[\frac{1}{p}]$ becomes trivial over $\mathcal{O}_{K'_e}[\frac{1}{p}]$ for some finite extension K'_e. Since a diagonalisable group is a product of copies of $\mathbb{G}_m$ and μ_m's, we may treat each of these groups separately. As long as $\mathbb{G}_m$ is concerned, since any fppf $\mathbb{G}_m$-torsor is also an étale $\mathbb{G}_m$-torsor, it suffices to take K'_e equal to the Hilbert class field K_e^h of K_e. On the other hand, when base changed to $\mathcal{O}_{K_e^h}[\frac{1}{p}]$, a μ_m-torsor is given as the torsor of m-th roots of some element $f \in \mathcal{O}_{K_e^h}[\frac{1}{p}]^\times$, because of the exact sequence

$$\mathcal{O}_{K_e}[\tfrac{1}{p}]^\times \xrightarrow{(.)^m} \mathcal{O}_{K_e}[\tfrac{1}{p}]^\times \longrightarrow H^1_{fppf}(S, \mu_m) \longrightarrow H^1_{fppf}(S, \mathbb{G}_m) \xrightarrow{(.)^m} H^1_{fppf}(S, \mathbb{G}_m)$$

where S denotes $\mathrm{Spec}(\mathcal{O}_{K_e}[\frac{1}{p}])$. Thus we can take for K'_e a splitting field of $X^m - f$ over K_e^h in this case.

Proof. (1) Consider the map $Z^1_\alpha(W_F^0/P_F, T_\phi(R)) \longrightarrow T_\phi(R) \times T_\phi(R)$ that sends a 1-cocycle η to the pair of elements $(\eta(\mathrm{Fr}), \eta(s))$. It identifies $Z^1_\alpha(W_F^0/P_F, T_\phi(R))$ with the subset of elements (F, σ) in $T_\phi(R) \times T_\phi(R)$ defined by the equation

$$F \cdot \alpha(\mathrm{Fr})(\sigma) \cdot \alpha(s)^q(F)^{-1} = \sigma \cdot \alpha(s)(\sigma) \cdots \alpha(s^{q-1})(\sigma).$$

This identifies in turn $\underline{Z}^1_\alpha(W_F^0/P_F, T_\phi)$ with the kernel of the morphism of group schemes $T_\phi \times T_\phi \longrightarrow T_\phi$ defined by the ratio of both sides of the equation. But a kernel of a morphism of diagonalisable groups is diagonalisable. Further, let $W = W_F/W_{F'}$ be a finite quotient of W_F for a Galois extension F' such that ${}^L\phi_{|P_{F'}}$ and $\alpha_{|W_{F'}}$ are trivial. Then $\underline{Z}^1_\alpha(W/P, T_\phi)$ is the kernel of the natural restriction map $\underline{Z}^1_\alpha(W_F^0/P_F, T_\phi) \longrightarrow \underline{Z}^1_\alpha(W_{F'}^0/P_{F'}, T_\phi)$, hence is a diagonalisable group too.

(2) We already know that $\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha}$ is finitely presented over $\mathcal{O}_{K_e}[\frac{1}{p}]$, so it remains to find a faithfully flat $\mathcal{O}_{K_e}[\frac{1}{p}]$-algebra R such that $\Sigma(W_F^0, \mathcal{T}_\phi(R))_{\phi,\alpha}$ is not empty. We will actually exhibit an R and a $\varphi \in \Sigma(W_F^0, \mathcal{T}_\phi(R))_{\phi,\alpha}$ such that ${}^L\varphi$ has finite image. This will also show that the last statement of the theorem holds for any finite quotient W over which this ${}^L\varphi$ factors.

Warning : for the sake of readibility, **we will omit the ^{L} from our usual notation for L-morphisms in the remainder of this proof.** It should not create any ambiguity since we will not have to consider their associated 1-cocycles anyway.

Existence of a point over a closed geometric point. By the admissibility assumption, the scheme $\underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha}$ is not empty. Since it has finite presentation over $\mathcal{O}_{K_e}[\frac{1}{p}]$, Chevalley's constructibility theorem ensures that it has a non-empty closed fiber, which in turn ensures that it has a point with finite residue field k of characteristic $\neq p$. Note that the associated L-morphism $W_F^0 \longrightarrow {}^LG(k)$ has to factor over a finite quotient of W_F^0, hence it is continuous for the topology of W_F^0 induced by the usual topology on W_F, and the discrete topology on ${}^LG(k)$. Therefore, Proposition 3.7 below ensures that $\Sigma(W_F^0, \mathcal{T}_\phi(\bar{k}))_{\phi,\alpha}$ is not empty. Pick a point in this set and let $\bar{\varphi} : W_F^0 \longrightarrow \mathcal{T}_\phi(\bar{k})$ be the L-morphism corresponding to this point. Note that $\bar{\varphi}$ also has to factor through a finite quotient of W_F^0, so it extends uniquely to a continuous morphism from W_F.

Lifting this point to characteristic 0. Let us try to lift $\bar{\varphi}$ to a Witt-vectors valued point $\varphi : W_F \longrightarrow \mathcal{T}_\phi(\mathcal{W}_e(\bar{k}))$. Here $\mathcal{W}_e(\bar{k})$ is the ring of integers of the completed maximal unramified extension of the completion of K_e at the place given by $\mathcal{O}_{K_e}[\frac{1}{p}] \longrightarrow \bar{k}$. By smoothness of $\mathcal{T}_\phi$, the map $\mathcal{T}_\phi(\mathcal{W}_e(\bar{k})) \longrightarrow \mathcal{T}_\phi(\bar{k})$ is surjective, so we may choose lifts $\tilde{\varphi}(w) \in \mathcal{T}_\phi(\mathcal{W}_e(\bar{k}))$ of $\bar{\varphi}(w)$ and we may do it in such a way that

- $\tilde{\varphi}(w)$ only depends on $\bar{\varphi}(w)$ and $\tilde{\varphi}(w) = 1$ if $\bar{\varphi}(w) = 1$.
- $\tilde{\varphi}(pw) = \phi(p)\tilde{\varphi}(w)$ for all $w \in W_F$ and $p \in P_F$.

Note that $\tilde{\varphi}(w)$ belongs to the summand $T_{\hat{G}}(\phi, {}^w\phi) \rtimes w$, so that we also have $\tilde{\varphi}(wp) = \phi(wpw^{-1})\tilde{\varphi}(w) = \tilde{\varphi}(w)\phi(p)$ for all $w \in W_F$ and $p \in P_F$. Moreover, the automorphism $(\mathrm{Ad}_{\tilde{\varphi}(w)})_{|T_\phi}$ only depends on the image of $\tilde{\varphi}(w)$ in $\pi_0(\mathcal{T}_\phi)$, which is the same as that of $\bar{\varphi}(w)$. Hence this automorphism is the one given by the action of $\alpha(w)$. It follows that the map

$$c_2 : (w, w') \in W_F \times W_F \mapsto \tilde{\varphi}(w)\tilde{\varphi}(w')\tilde{\varphi}(ww')^{-1} \in \ker\left(\mathcal{T}_\phi(\mathcal{W}_e(\bar{k})) \longrightarrow \mathcal{T}_\phi(\bar{k})\right)$$

has finite image, factors over $W_F/P_F \times W_F/P_F$, and is a 2-cocycle from W_F/P_F into $A := \ker(T_\phi(\mathcal{W}_e(\bar{k})) \longrightarrow T_\phi(\bar{k}))$ endowed with the action α. Having finite image, it is continuous for the discrete topology on A. If this cocycle is cohomologically trivial, that is, if there is some continuous map $t : W_F/P_F \to A$ such that $c_2(w,w') = t(w)({}^{\tilde{\varphi}(w)}t(w'))t(ww')^{-1}$, then the map $w \mapsto \varphi(w) := t(w)^{-1}\tilde{\varphi}(w)$ is a continuous lift of $\bar{\varphi}$. Now, if ℓ denotes the characteristic of $\bar{k}$, the group A is certainly ℓ'-divisible (i.e. m-divisible for any m prime to ℓ), but not ℓ-divisible, so that $H^2(W_F/P_F, A)$ is not a priori trivial. However, if $\bar{\mathcal{O}}$ denotes the ring of integers of an algebraic closure of $\mathcal{W}_e(\bar{k})$, then the group $A' = \ker(T_\phi(\bar{\mathcal{O}}) \longrightarrow T_\phi(\bar{k}))$ is divisible hence, by Lemma 3.8, c_2 is cohomologically trivial there, and we get a lift φ of $\bar{\varphi}$ valued in ${}^LG(\bar{\mathcal{O}})$.

We now modify this lift φ so that it has finite image. To do so we introduce the maximal subtorus C_ϕ of T_ϕ on which W_F/P_F acts trivially. This is the split torus over $\mathcal{W}_e(\bar{k})$ whose group of characters is the torsion-free quotient of the W_F/P_F-coinvariants of the group of characters of T_ϕ. Now, pick an integer m such that $\bar{\varphi}(\mathrm{Fr}^m) = 1$ and $\varphi(\mathrm{Fr}^m)$ is central in $\varphi(W_F)$ (this is possible since $\varphi(I_F)$ is finite). The element $\varphi(\mathrm{Fr}^m) \in A'$ then belongs to $T_\phi(\bar{\mathcal{O}})^{W_F/P_F}$. Since the group scheme $T_\phi^{W_F/P_F}$ is an extension of a finite diagonalizable group scheme by the torus C_ϕ, some power of $\varphi(\mathrm{Fr}^m)$, say $\varphi(\mathrm{Fr}^{m'})$, belongs to $C_\phi(\bar{\mathcal{O}}) \cap \ker(T_\phi(\bar{\mathcal{O}}) \to T_\phi(\bar{k})) = \ker(C_\phi(\bar{\mathcal{O}}) \to C_\phi(\bar{k}))$. But the latter is a divisible group so we may pick there an element c such that $c^{m'} = \varphi(\mathrm{Fr}^{m'})$. Consider then $\varphi' : w \mapsto c^{-\nu(w)}\varphi(w)$. This is still a ${}^LG(\bar{\mathcal{O}})$-valued lift of $\bar{\varphi}$ and it has finite image.

A section over a quasi-finite flat extension. Now, the existence of such a lift shows that the morphism of finite presentation $\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha} \longrightarrow \mathrm{Spec}(\mathcal{O}_{K_e}[\frac{1}{p}])$ is dominant and, even better, that there is a finite quotient W of W_F such that $\underline{\Sigma}(W, \mathcal{T}_\phi)_{\phi,\alpha} \longrightarrow \mathrm{Spec}(\mathcal{O}_{K_e}[\frac{1}{p}])$ is dominant (with obvious notation). Therefore, we can find a finite extension K of K_e and an integer N such that $\underline{\Sigma}(W_F^0, \mathcal{T}_\phi)_{\phi,\alpha}$ has a section over $\mathcal{O}_K[\frac{1}{N}]$ that corresponds to a morphism $\varphi : W_F \longrightarrow \mathcal{T}_\phi(\mathcal{O}_K[\frac{1}{N}])$ which factors over a finite quotient of W_F.

Sections over the missing points. Let us fix a prime λ of K that divides N but not p, and denote by K_λ the completion of K at λ and by $\mathcal{O}_\lambda$ its ring of integers. Using the inclusion $\mathcal{O}_K[\frac{1}{N}] \hookrightarrow K_\lambda$ we get a morphism $\varphi : W_F \longrightarrow \mathcal{T}_\phi(K_\lambda)$. We would like to conjugate it, so that it factors though $\mathcal{T}_\phi(\mathcal{O}_\lambda)$. We will show that this is possible after maybe passing to a ramified extension of K_λ. Indeed, the problem is to find some $t \in T_\phi(K_\lambda)$ such that $t\varphi(w)t^{-1} \in \mathcal{T}_\phi(\mathcal{O}_\lambda)$ for all $w \in W_F$. Observe that $\mathcal{T}_\phi(K_\lambda) = T_\phi(K_\lambda)\mathcal{T}_\phi(\mathcal{O}_\lambda)$, so that $T_\phi(\mathcal{O}_\lambda)$ is a normal subgroup of $\mathcal{T}_\phi(K_\lambda)$ with quotient of the form

$$\mathcal{T}_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda) = (T_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda)) \rtimes \tilde{\pi}_0(\phi).$$

So we see that the existence of t as above is equivalent to the existence of $\bar{t} \in T_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda)$ such that $\bar{t}\varphi\bar{t}^{-1}$ coincides with the trivial section $W_F \xrightarrow{\alpha} \tilde{\pi}_0(\phi) \longrightarrow \mathcal{T}_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda)$ (we have denoted again by φ the composition of φ with the projection to the above quotient). Therefore, the existence of t as above is equivalent to the vanishing of φ_T in $H^1(W', T_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda))$, where φ_T is defined by $\varphi(w) = \varphi_T(w) \rtimes \alpha(w)$ and W' is any finite quotient of W_F through which φ (hence also α) factors.

Now, let v_λ be the normalized valuation on K_λ and let $X^*(T_\phi)$ be the group of cocharacters of T_ϕ. The pairing $(t,\mu)\mapsto v_\lambda(\mu(t))$ for $t\in T_\phi(K_\lambda)$ and $\mu\in X^*(T_\phi)$ induces an isomorphism of abelian groups

$$T_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda) \xrightarrow{\sim} \mathrm{Hom}(X^*(T_\phi),\mathbb{Z})$$

which shows that $T_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda)$ is a free abelian group of rank $\dim(T_\phi)$ and that $H^1(W', T_\phi(K_\lambda)/T_\phi(\mathcal{O}_\lambda))$ has no reason to vanish. However, let $\bar{K}_\lambda$ be an algebraic closure of K_λ with ring of integers $\bar{\mathcal{O}}_\lambda$ and denote by v_λ the unique extension of v_λ to $\bar{K}_\lambda$. Then the same pairing as above induces an isomorphism

$$T_\phi(\bar{K}_\lambda)/T_\phi(\bar{\mathcal{O}}_\lambda) \xrightarrow{\sim} \mathrm{Hom}(X^*(T_\phi),\mathbb{Q})$$

which shows that $T_\phi(\bar{K}_\lambda)/T_\phi(\bar{\mathcal{O}}_\lambda)$ is a $\mathbb{Q}$-vector space, and therefore that the group $H^1(W', T_\phi(\bar{K}_\lambda)/T_\phi(\bar{\mathcal{O}}_\lambda))$ vanishes. It follows that there is some finite extension K'_λ of K_λ with ring of integers $\mathcal{O}'_\lambda$, and some element $t'\in T_\phi(K'_\lambda)$ such that $\varphi_\lambda := t'\cdot\varphi(w)\cdot {t'}^{-1}$ defines a section of $\underline{\Sigma}(W_F^0,\mathcal{T}_\phi)_{\phi,\alpha}$ over $\mathcal{O}'_\lambda$.

Conclusion. With φ and the φ_λ, we have found a section of $\underline{\Sigma}(W_F^0,\mathcal{T}_\phi)_{\phi,\alpha}$ over the finite fpqc covering $\coprod_{\lambda|N,\lambda\nmid p}\mathrm{Spec}(\mathcal{O}'_\lambda)\cup\mathrm{Spec}(\mathcal{O}_K[\frac{1}{N}])$ of $\mathrm{Spec}(\mathcal{O}_{K_e}[\frac{1}{p}])$. Since $\underline{\Sigma}(W_F^0,\mathcal{T}_\phi)_{\phi,\alpha}$ is finitely presented, there also exists a section over a fppf covering. Moreover, φ and the φ_λ's factor over a finite quotient W of W_F, so they provide a section of $\underline{\Sigma}(W,\mathcal{T}_\phi)_{\phi,\alpha}$ over an fpqc covering, and we also deduce that $\underline{\Sigma}(W,\mathcal{T}_\phi)_{\phi,\alpha}$ has a section over an fppf covering of $\mathrm{Spec}(\mathcal{O}_E[\frac{1}{p}])$. ☐

In the above proof, we have used the following result in order to pass from the non-emptyness of $\underline{Z}^1(W_F^0,\hat{G})_{\phi,\alpha}$ to that of $\underline{\Sigma}(W_F^0,\mathcal{T}_\phi)_{\phi,\alpha}$.

Proposition 3.7. *Let K be an algebraically closed field of characteristic different from p, let $\varphi: W_F \longrightarrow {}^LG(K)$ be a continuous L-morphism, and let $\phi := \varphi_{|P_F}$. Then there is another extension $\varphi' = \eta\cdot\varphi$ of ϕ, with $\eta\in Z^1_{\mathrm{Ad}_\varphi}(W_F/P_F, C_{\hat{G}}(\phi)^\circ(K))$, and whose conjugation action $\mathrm{Ad}_{\varphi'}$ on $C_{\hat{G}}(\phi)$ preserves a Borel pair of $C_{\hat{G}}(\phi)^\circ$.*

Proof. Fix a Borel pair $\mathcal{B}_\phi$ of $C_{\hat{G}}(\phi)^\circ$. Since $C_{\hat{G}}(\phi)^\circ$ acts transitively on its Borel pairs, we may choose for all $\bar{w}\in W_F/P_F$ an element $\alpha(\bar{w})\in C_{\hat{G}}(\phi)^\circ(K)$ such that $\mathrm{Ad}_{\alpha(\bar{w})}\circ\mathrm{Ad}_{\varphi(w)}$ stabilizes $\mathcal{B}_\phi$, where w is any lift of $\bar{w}$ in W_F (note that the restriction of $\mathrm{Ad}_{\varphi(w)}$ to $C_{\hat{G}}(\phi)^\circ$ does not depend on the choice of such a lift). Moreover, we may and will choose $\alpha(\bar{w})$ so that it only depends on $\mathrm{Ad}_{\varphi(w)}$, ensuring in turn that the map $\bar{w}\mapsto\alpha(\bar{w})$ is continuous. Since the stabilizer of $\mathcal{B}_\phi$ in $C_{\hat{G}}(\phi)^\circ$ is T_ϕ, we see that the automorphism $\mathrm{Ad}_{\alpha(\bar{w})}\circ\mathrm{Ad}_{\varphi(w)}$ of T_ϕ does not depend on the choice of $\alpha(\bar{w})$, and this defines an action of W_F/P_F on T_ϕ by algebraic automorphisms. Note that this action is the same as the one given by the image of $\mathrm{Ad}_{\varphi(w)}$ in $\mathrm{Out}(C_{\hat{G}}(\phi)^\circ)$ through the canonical identification of T_ϕ with the "abstract" torus of the root datum of $C_{\hat{G}}(\phi)^\circ$. In particular, this action is finite since it factors through the quotient of the normalizer of $\phi(P_F)$ in ${}^LG$ by $C_{\hat{G}}(\phi)^\circ$, which is a finite group. Now we remark that the map

$$(\bar{w},\bar{w}')\mapsto\alpha(\bar{w})\varphi(w)\alpha(\bar{w}')\varphi(w')(\alpha(\bar{w}\bar{w}')\varphi(ww'))^{-1} = \alpha(\bar{w})\,\mathrm{Ad}_{\varphi(w)}(\alpha(\bar{w}'))\alpha(\bar{w}\bar{w}')^{-1}$$

defines a continuous 2-cocycle from W_F/P_F to $T_\phi(K)$ with respect to the action described above. If this cocycle is a coboundary, that is, if there is a continuous map $\beta: W_F/P_F\longrightarrow T_\phi(K)$ such that

$$\alpha(\bar{w})\,\mathrm{Ad}_{\varphi(w)}(\alpha(\bar{w}'))\alpha(\bar{w}\bar{w}')^{-1} = \beta(\bar{w})(\mathrm{Ad}_{\alpha(\bar{w})}\circ\mathrm{Ad}_{\varphi(w)}(\beta(\bar{w}')))\beta(\bar{w}\bar{w}')^{-1},$$

then the map $\eta : \bar{w} \mapsto \beta(\bar{w})^{-1}\alpha(\bar{w})$ is in $Z^1_{\mathrm{Ad}_\varphi}(W_F/P_F, C_{\hat{G}}(\phi)^\circ(K))$ and the parameter $\varphi' = \eta \cdot \varphi$ normalizes the Borel pair $\mathcal{B}_\phi$ as desired.

Hence the obstruction to finding η as desired lies in $H^2(W_F/P_F, T_\phi(K))$. However, since $T_\phi(K)$ is a divisible group, the following lemma shows that this cohomology group vanishes. □

Lemma 3.8. *Denote $W = W_F/P_F$ and $I = I_F/P_F$, and let A be an abelian group with a finite action of W. We consider only continuous cohomology of W with respect to the discrete topology on the coefficients.*

(1) *There is a short exact sequence*

$$1 \to H^1(W/I, H^1(I, A)) \to H^2(W, A) \to H^2(I, A)^{W/I} \to 1.$$

(2) *We have $H^2(I, A) = \mathrm{colim}_{(n,p)=1}(A^I/N_M(A)^n)$ where*
- *M is the order of the action of a pro-generator s of I and $N_M(a) = as(a)\cdots s^{M-1}(a)$*
- *$\{n \in \mathbb{N}, (n,p) = 1\}$ is ordered by divisibility and the transition map $A^I/N_M(A)^n \to A^I/N_M(A)^{n'}$ for $n|n'$ is induced by the map $a \mapsto a^{n'/n}$.*

In particular, $H^2(I, A) = \{1\}$ whenever A contains a p'-divisible group of finite index.

(3) *We have $H^1(W/I, H^1(I, A)) = H^1(I, A)_{\mathrm{Fr}} = [N_M^{-1}(A[p'])/A(s)]_{\mathrm{Fr}}$ where*
- *$A[p']$ is the prime-to-p torsion of A and $A(s) = \{as(a)^{-1}, a \in A\}$.*
- *Fr is a Frobenius lift in W. Moreover, if m is the order of the action of Fr on A, then Fr^{-m} acts on $H^1(I, A)$ by raising to the power q^m.*

In particular, $H^1(W/I, H^1(I, A)) = \{1\}$ whenever A is a p'-divisible group.

Proof. (1) follows from the Hochschild-Serre spectral sequence with the facts that $W/I = \mathbb{Z}$ and $H^n(\mathbb{Z}, M) = 1$ for any $n \geq 2$ and any $\mathbb{Z}[\mathbb{Z}]$-module M. Note that the existence of the spectral sequence follows from Proposition 5 and the subsequent Remark (2) of [CW74], but the short exact sequence here can also be simply deduced by taking colimits of similar short exact sequences for discrete quotients W/J with $J \subset I$ open and contained in the kernel of the action of W on A.

(2) By identifying I with the inverse limit of $\mathbb{Z}/nM\mathbb{Z}$ for $(n,p) = 1$, we can write $H^2(I, A)$ as the direct limit of $H^2(\mathbb{Z}/nM\mathbb{Z}, A)$, indexed by integers n coprime to p and ordered by divisibility. The standard formula for the H^2 of a cyclic group tells us that $H^2(\mathbb{Z}/nM\mathbb{Z}, A) = A^s/N_M(A)^n$ and that the transition map $A^s/N_M(A)^n \to A^s/N_M(A)^{n'}$ for $n|n'$ is induced by the map $a \mapsto a^{n'/n}$. Now, suppose B is a p'-divisible subgroup of A such that $(A/B)^N = 1$ for some integer $N \geq 1$. Then $N_M(B)$ is a p'-divisible subgroup of A^s hence it is contained in $N_M(A)^n$ for all $n > 0$ with $(n,p) = 1$. Since $N_M(A)$ contains $(A^s)^M$, we see that each $A^s/N_M(A)^n$ has exponent dividing NM. Moreover this exponent is also prime to p since both M and n are prime to p. It follows that, denoting by N' the prime-to-p part of N, the transition maps $A^s/N_M(A)^n \to A^s/N_M(A)^{n'}$ vanish whenever $nN'M|n'$, showing that the colimit vanishes, whence $H^2(I, A) = 1$.

(3) By the continuity constraint on cocycles, the map $Z^1(I, A) \to A$, $\eta \mapsto \eta(s)$ identifies $Z^1(I, A)$ with the subgroup $\{a \in A, \exists n \in \mathbb{N}, (n,p) = 1, N_n(a) = 1\}$. Since $N_{nM}(a) = N_n(N_M(a)) = N_M(a)^n$, this is also the subgroup $\{a \in A, \exists n \in \mathbb{N}, (n,p) = 1, N_M(a)^n = 1\}$. In other words, with the notation of the lemma we have $Z^1(I, A) = N_M^{-1}(A[p'])$. As a consequence $H^1(I, A)$, being by definition the

quotient of $Z^1(I,A)$ by s-conjugacy under A, is also the quotient by the subgroup $A(s)$, and the formula of (3) follows.

Let us make the action of Fr on $H^1(I,A)$ more explicit. Note first that the action of Fr^{-1} on $Z^1(I,A)$ is given by $\mathrm{Fr}^{-1}(\eta)(s) = \mathrm{Fr}^{-1}(\eta(\mathrm{Fr}\, s\, \mathrm{Fr}^{-1})) = \mathrm{Fr}^{-1}(\eta(s^q)) = \mathrm{Fr}^{-1}(N_q(\eta(s)))$. If $m \in \mathbb{N}$ is such that Fr^m acts trivially on A, then $\mathrm{Fr}^{-m}(\eta)(s) = N_{q^m}(\eta(s))$. But since the image $\overline{N_{q^m}(\eta(s))}$ of $N_{q^m}(\eta(s))$ in $Z^1(I,A)/A(s)$ is $\overline{\eta(s)}^{q^m}$, we see that the action of Fr^{-m} on $H^1(I,A)$ is simply given by the q^m-th power map. Therefore, the space of Fr^m co-invariants is the quotient

$$H^1(I,A)_{\mathrm{Fr}^m} = H^1(I,A)/\left(H^1(I,A)\right)^{q^m-1},$$

which is trivial whenever $H^1(I,A)$ is p'-divisible. The latter holds if $Z^1(I,A)$ is p'-divisible, and this holds in turn if A is p'-divisible. □

Remark 3.9. This lemma is the main point in proving the existence of L-morphisms that preserve a Borel pair. When the center Z_ϕ of $C_{\hat{G}}(\phi)^\circ$ is a torus, and more generally when it is *smooth over* $\mathcal{O}_{K_e}[\frac{1}{p}]$, then $Z_\phi(K)$ is a p'-divisible group for any algebraically closed field K of characteristic not p, so that the same lemma implies that $H^2(W_F/P_F, Z_\phi(K))$ vanishes. In this case, fix a pinning ε_ϕ of $C_{\hat{G}}(\phi)^\circ$ and consider its normalizer $\mathcal{Z}_\phi$ in $C_{{}^LG}(\phi)$, which is an extension of $\pi_0({}^LG)$ by Z_ϕ. Thanks to this vanishing result, the same argument as in Theorem 3.6 shows that $\underline{\Sigma}(W_F^\circ, \mathcal{Z}_\phi)_{\phi,\alpha}$ is a fppf torsor under the diagonalisable group scheme $\underline{Z}^1_\alpha(W_F^0/P_F, Z_\phi)$, and therefore that we can find φ_α as in Theorem 3.4 *with the additional property that* $\mathrm{Ad}_{\varphi_\alpha}$ *preserves the pinning* ε_ϕ. In this case, the group scheme $C_{\hat{G}}(\phi)^\circ \cdot \varphi(W_F)$ is isomorphic to a suitable quotient of the Langlands dual group scheme over $\mathcal{O}_{K'_e}[\frac{1}{p}]$ of some tamely ramified reductive group over F, namely "the" quasi-split reductive group $G_{\phi,\alpha}$ dual to $C_{\hat{G}}(\phi)^\circ$ over $\bar{F}$ and whose F-structure is induced by the outer action

$$W_F \xrightarrow{\alpha} \tilde{\pi}_0(\phi) \longrightarrow \mathrm{Out}(C_{\hat{G}}(\phi)^\circ).$$

In particular, when $C_{\hat{G}}(\phi)$ is connected, $\Sigma(\phi)$ is trivial so we get a single associated quasi-split reductive group G_ϕ over F and, under the hypothesis of this Remark, we have an isomorphism over $\mathcal{O}_{K'_e}[\frac{1}{p}]$

$${}^LG_\phi = C_{\hat{G}}(\phi) \rtimes_{\mathrm{Ad}_\varphi} W_e \xrightarrow{\sim} C_{{}^LG}(\phi).$$

Example 3.10 (Classical groups)**.** Let us assume that $p > 2$ and consider the case where ${}^LG$ is a Langlands dual group of a quasi-split classical group G over F, so that $\hat{G}$ is one of Sp_{2n}, SO_{2n+1} or SO_{2n}. Then the following holds :

(1) $C_{\hat{G}}(\phi)$ *is connected for all* $\phi \in \Phi_e$. *More precisely, it is isomorphic to a product* $\hat{G}' \times \mathrm{GL}_{n_1} \times \cdots \times \mathrm{GL}_{n_r}$ *with* $\hat{G}'$ *of the same type as* $\hat{G}$.

This follows from the fact that the only self-dual irreducible representation of a p-group is the trivial representation. Indeed, decomposing the underlying symplectic or orthogonal space as a sum of $\phi(P_F)$-isotypic components, this fact shows that each pair of dual non-trivial irreducible representations contributes a factor GL to the centralizer, while the trivial representation contributes a classical group of the same sign. We then deduce the following :

(2) G_ϕ *is a (possibly non split) classical group times a product of restrictions of scalars of general linear groups and unitary groups.*

(3) *If G is symplectic, then we can find an extension φ of ϕ such that* Ad_φ *preserves a pinning of* $C_{\hat{G}}(\phi)$ (because $\hat{G}$ is adjoint and thanks to the previous remark). *In particular we get an isomophism* ${}^LG_\phi \xrightarrow{\sim} C_{{}^LG}(\phi)$ *as above.* Recall that even though G is split here, we take ${}^LG = \hat{G} \times W_e$ where P_F/P_F^e *injects* into W_e.

The next lemma provides many examples to which Remark 3.9 applies.

Lemma 3.11. *Let H be a reductive group scheme over $\bar{\mathbb{Z}}[\frac{1}{p}]$ and let P be a finite p-group of automorphisms of H. If the center $Z(H)$ of H is smooth over $\bar{\mathbb{Z}}[\frac{1}{p}]$, then so is the center $Z(H^{P,\circ})$ of the connected centralizer $H^{P,\circ}$ of P.*

Proof. Recall that the center Z of a reductive group scheme is a group of multiplicative type, associated to an étale sheaf $X^*(Z)$ of finitely generated abelian groups. In particular, Z is flat over the base, and it is smooth if and only if the order of the torsion subgroups of all stalks of $X^*(Z)$ are invertible on the base. In our case, since $\mathrm{Spec}(\bar{\mathbb{Z}}[\frac{1}{p}])$ is connected, it suffices to check the $\bar{\mathbb{Q}}$-stalk. Hence we see that Z is smooth if and only if the torsion subgroup of $X^*(Z_{\bar{\mathbb{Q}}})$ has p-power order, if and only if $\pi_0(Z_{\bar{\mathbb{Q}}})$ has p-power order.

As a consequence, we are reduced to prove a statement for reductive groups over $\bar{\mathbb{Q}}$: *if H is a reductive algebraic group over $\bar{\mathbb{Q}}$ with an action of a p-group P and such that $\pi_0(Z(H^\circ))$ has p-power order, then $\pi_0(Z(H^{P,\circ}))$ has also p-power order.*

Note that if P_1 is a normal subgroup of P with quotient $P_2 := P/P_1$, then H^{P_1} is a reductive algebraic group and $H^P = (H^{P_1})^{P_2}$. Therefore, if the above statement is true for the action of P_1 on H and that of P_2 on H^{P_1}, it is true for the action of P on H. By using a central series of P, we may thus argue by induction and we see that it suffices to treat the case where P is cyclic of order p. Moreover, we may also assume that H is connected since only $Z(H^\circ)$ and $H^{P,\circ} = (H^\circ)^{P,\circ}$ appear in the above statement. Now, the quotient morphism $H \longrightarrow H_{\mathrm{ad}}$ induces a surjective morphism $H^{P,\circ} \twoheadrightarrow (H_{\mathrm{ad}})^{P,\circ}$ whose kernel is $K := Z(H)^P \cap H^{P,\circ}$. So $\pi_0(K)$ is dual to the torsion subgroup of $X^*(K)$, which is a quotient of the torsion subgroup of the coinvariants $X^*(Z(H))_P$, which has p-power order. Since $Z(H^{P,\circ})$ is an extension of $Z((H_{\mathrm{ad}})^{P,\circ})$ by K, we see that it suffices to prove that $\pi_0(Z((H_{\mathrm{ad}})^{P,\circ}))$ has p-power order. Note that P permutes the simple factors of H_{ad}, and it suffices to treat the case where this permutation is transitive. If H_{ad} is not simple, then this permutation is also simply transitive (since P is simple), and $(H_{\mathrm{ad}})^P$ is isomorphic to a simple factor of H_{ad} (diagonally embedded in H_{ad}). So we are left with the case where H_{ad} is simple. Let θ be a generator of P. Note that θ is a semi-simple element of $H_{\mathrm{ad}} \rtimes P$, hence it is in particular quasi-semisimple in the sense of Steinberg. Let (B, T) be a Borel pair fixed by θ, and write $\theta = \mathrm{Ad}_t \circ \sigma$ with σ quasi-central (see [DM18, Def. 1.19]) and $t \in T^{\theta,\circ} = T^{\sigma,\circ}$, as per [DM18, Prop. 1.16 (1)]. If θ is inner on H_{ad}, then $\sigma = 1$, hence t has order p. Otherwise, by the classification of quasi-central elements below Proposition 1.22 of [DM94], we must have $p = 2$ or $p = 3$, and σ has always order p, so that t also has order p. In all cases, Theorem 3.11 of [DM18] implies that the order of $\pi_0(Z((H_{\mathrm{ad}})^{P,\circ}))$ divides p^2. □

Using Remark 3.9, we can now strengthen Theorem 3.4 for a certain class of groups, by replacing "Borel pair" by "pinning".

Theorem 3.12. *Suppose that the center of $\hat{G}$ is smooth over $\mathbb{Z}[\frac{1}{p}]$. Then there is a finite extension K'_e of K_e such that for any admissible $\phi \in \Phi_e$ and any admissible $\alpha \in \Sigma(\phi)$, there is some $\varphi_\alpha \in Z^1(W_F^0, \hat{G}(\mathcal{O}_{K'_e}[\frac{1}{p}]))_{\phi,\alpha}$ such that ${}^L\varphi_\alpha(W_F^0)$ is finite and* $\mathrm{Ad}_{\varphi_\alpha}$ *preserves* a pinning *of the split reductive group scheme $C_{\hat{G}}(\phi)^\circ$.*

4. Moduli of Langlands parameters

We maintain the setup and notation of the previous section. In particular, $\hat{G}$ is a split reductive group scheme over $\mathbb{Z}[\frac{1}{p}]$ endowed with a finite action of W_F, and ${}^LG = \hat{G} \rtimes W$ is an adjustable associated "L-group" of finite type.

4.1. **The moduli space of cocycles.** Let us fix a "depth" $e \in \mathbb{N}$ such that the action of P_F^e on $\hat{G}$ is trivial. The functor $R \mapsto Z^1(W_F^0/P_F^e, \hat{G}(R))$ is representable by an affine scheme of finite presentation over $\mathbb{Z}[\frac{1}{p}]$ that we denote by $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$, and whose affine ring we denote by $R^e_{{}^LG}$. By construction, it comes with a universal 1-cocycle

$$\varphi^e_{\mathrm{univ}} : W_F^0/P_F^e \longrightarrow \hat{G}(R^e_{{}^LG}).$$

Restriction to P_F provides us with a morphism of $\mathbb{Z}[\frac{1}{p}]$-schemes

$$\underline{Z}^1(W_F^0/P_F^e, \hat{G}) \longrightarrow \underline{Z}^1(P_F/P_F^e, \hat{G}) \tag{4.1}$$

with the notation of appendix A. Using the notation of Theorem 3.1, we have a decomposition of the right hand side over $\mathcal{O}_{K_e}[\frac{1}{p}]$ as follows

$$\underline{Z}^1(P_F/P_F^e, \hat{G})_{\mathcal{O}_{K_e}[\frac{1}{p}]} = \coprod_{\phi \in \Phi_e} \hat{G} \cdot \phi,$$

where $\hat{G} \cdot \phi$ denotes the orbit of ϕ, which in this context is a smooth affine scheme that represents the quotient sheaf $\hat{G}/C_{\hat{G}}(\phi)$ on the big étale site of $\mathcal{O}_{K_e}[\frac{1}{p}]$ (see Remark A.10). This induces in turn a decomposition

$$\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\mathcal{O}_{K_e}[\frac{1}{p}]} = \coprod_{\phi \in \Phi_e^{\mathrm{adm}}} \hat{G} \times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F^0, \hat{G})_\phi \tag{4.2}$$

with the notation of the last section. Here the summand $\hat{G} \times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F^0, \hat{G})_\phi$ is an affine scheme that represents the quotient sheaf of $\hat{G} \times \underline{Z}^1(W_F^0, \hat{G})_\phi$ by the action of $C_{\hat{G}}(\phi)$ by right translations on $\hat{G}$ and by (twisted) conjugation on $\underline{Z}^1(W_F^0, \hat{G})_\phi$. Recall that ϕ is called "admissible" if this summand is non-empty and we have denoted by Φ_e^{adm} the subset of admissible elements. In terms of rings, we have the decomposition

$$R^e_{{}^LG} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathcal{O}_{K_e}[\frac{1}{p}] = \prod_{\phi \in \Phi_e^{\mathrm{adm}}} R_{{}^LG,[\phi]} = \prod_{\phi \in \Phi_e^{\mathrm{adm}}} \left(\mathcal{O}_{\hat{G}} \otimes_{\mathbb{Z}[\frac{1}{p}]} R_{{}^LG,\phi}\right)^{C_{\hat{G}}(\phi)}. \tag{4.3}$$

The $[\phi]$-part of the universal 1-cocycle

$$\varphi^{[\phi]}_{\mathrm{univ}} : W_F^0/P_F^e \longrightarrow \hat{G}(R_{{}^LG,[\phi]})$$

is universal for 1-cocycles $\varphi : W_F^0 \longrightarrow \hat{G}(R)$ such that $\varphi_{|P_F}$ is étale-locally (over R) $\hat{G}$-conjugate to ϕ. Over $R_{{}^LG,\phi}$ we have an extension of ϕ

$$\varphi^{\phi}_{\mathrm{univ}} : W_F^0/P_F^e \longrightarrow \hat{G}(R_{{}^LG,\phi})$$

which is universal for 1-cocycles $\varphi : W_F^0 \longrightarrow \hat{G}(R)$ such that $\varphi_{|P_F} = \phi$. The 1-cocycles $\varphi_{\rm univ}^{[\phi]}$ and $\varphi_{\rm univ}^{\phi}$ determine each other in the following ways.

- $\varphi_{\rm univ}^{\phi}$ is deduced from $\varphi_{\rm univ}^{[\phi]}$ by pushing out along the morphism
$$\left(\mathcal{O}_{\hat{G}} \otimes_{\mathbb{Z}[\frac{1}{p}]} R_{{}^LG,\phi}\right)^{C_{\hat{G}}(\phi)} \longrightarrow \left(\mathcal{O}_{\hat{G}} \otimes_{\mathbb{Z}[\frac{1}{p}]} R_{{}^LG,\phi}\right) \xrightarrow{\varepsilon_{\hat{G}} \otimes {\rm id}} R_{{}^LG,\phi}$$
- $\varphi_{\rm univ}^{[\phi]}$ is deduced from $\varphi_{\rm univ}^{\phi}$ by the formula
$$\varphi_{\rm univ}^{[\phi]}(w) : \mathcal{O}_{\hat{G}} \xrightarrow{{\rm Ad}_w^*} \mathcal{O}_{\hat{G}} \otimes_{\mathbb{Z}[\frac{1}{p}]} \mathcal{O}_{\hat{G}} \xrightarrow{{\rm id} \otimes \varphi_{\rm univ}^{\phi}(w)} \mathcal{O}_{\hat{G}} \otimes_{\mathbb{Z}[\frac{1}{p}]} R_{{}^LG,\phi}$$
where ${\rm Ad}_w^*$ is induced by the w-twisted conjugation action of $\hat{G}$ on itself, and the composition lands into $\left(\mathcal{O}_{\hat{G}} \otimes_{\mathbb{Z}[\frac{1}{p}]} R_{{}^LG,\phi}\right)^{C_{\hat{G}}(\phi)}$.

We now recall the decomposition of the previous section
$$\underline{Z}^1(W_F^0, \hat{G})_\phi = \coprod_{\alpha \in \Sigma(\phi)^{\rm adm}} \underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha} \tag{4.4}$$
and we fix, for each $\alpha \in \Sigma(\phi)^{\rm adm}$, a 1-cocycle $\varphi_\alpha : W_F^0 \longrightarrow \hat{G}(\mathcal{O}_{K'_e}[\frac{1}{p}])$ as in Theorem 3.4. Then we have an isomorphism $\rho \mapsto \rho \cdot \varphi_\alpha$
$$\underline{Z}^1_{{\rm Ad}_{\varphi_\alpha}}((W_F/P_F)^0, C_{\hat{G}}(\phi)^\circ) \xrightarrow{\sim} \underline{Z}^1(W_F^0, \hat{G})_{\phi,\alpha} \times_{\mathcal{O}_{K_e}[\frac{1}{p}]} \mathcal{O}_{K'_e}[\tfrac{1}{p}] \tag{4.5}$$
where the LHS is a space of tame parameters as studied in Section 2. Accordingly, we have a decomposition of $\mathcal{O}_{K_e}[\frac{1}{p}]$-algebras $R_{{}^LG,\phi} = \prod_\alpha R_{{}^LG,\phi,\alpha}$ and, for each α, the α-component of $\varphi_{\rm univ}^{\phi}$ is given, over $R_{{}^LG,\phi,\alpha} \otimes_{\mathcal{O}_{K_e}[\frac{1}{p}]} \mathcal{O}_{K'_e}[\frac{1}{p}]$ by
$$\varphi_{\rm univ}^{\phi,\alpha} = \rho_{{}^LG_{\varphi_\alpha}} \cdot \varphi_\alpha : W_F^0 \longrightarrow \hat{G}\left(R_{{}^LG,\phi,\alpha} \otimes_{\mathcal{O}_{K_e}[\frac{1}{p}]} \mathcal{O}_{K'_e}[\tfrac{1}{p}]\right) \tag{4.6}$$
where $\rho_{{}^LG_{\varphi_\alpha}}$ is the universal 1-cocycle over $\underline{Z}^1_{{\rm Ad}_{\varphi_\alpha}}((W_F/P_F)^0, C_{\hat{G}}(\phi)^\circ)$. We are now in position to prove :

Theorem 4.1. *i) The scheme $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$ is syntomic (flat and local complete intersection) over $\mathbb{Z}[\frac{1}{p}]$ and generically smooth, of pure absolute dimension $\dim(\hat{G})$.*

ii) For any prime $\ell \neq p$, the ring $R_{{}^LG}^e$ is ℓ-adically separated and the pushforward of ${}^L\varphi_{\rm univ}^e$ to $R_{{}^LG}^e \otimes \mathbb{Z}_\ell$ extends uniquely to a ℓ-adically continuous L-morphism
$${}^L\varphi_{\ell-{\rm univ}}^e : W_F/P_F^e \longrightarrow {}^LG(R_{{}^LG}^e \otimes \mathbb{Z}_\ell)$$
which is universal for ℓ-adically continuous L-morphisms as in Definition 2.12.

Proof. i) Since $\mathcal{O}_{K'_e}[\frac{1}{p}]$ is a syntomic cover of $\mathbb{Z}[\frac{1}{p}]$, it suffices to prove i) after base change to this ring. In what follows, we implicitly base-change $\hat{G}$ and all schemes introduced above to this ring, but we omit it in the notation to keep it readable. So, it suffices to prove i) for each summand $\hat{G} \times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F^0, \hat{G})_\phi$ of the decomposition (4.2). Consider the morphism
$$\hat{G} \times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F^0, \hat{G})_\phi \longrightarrow \hat{G} \cdot \phi \tag{4.7}$$
obtained by restriction of (4.1). Its base change along the orbit morphism $\hat{G} \longrightarrow \hat{G} \cdot \phi$ is the first projection
$$\hat{G} \times \underline{Z}^1(W_F^0, \hat{G})_\phi \longrightarrow \hat{G} \tag{4.8}$$

Thanks to the decomposition (4.4) and the isomorphisms (4.5), we may apply Corollary 2.5 and Proposition 2.8 to deduce that $\underline{Z}^1(W_F^0,\hat{G})_\phi$ is syntomic over $\mathcal{O}_{K'_e}[\frac{1}{p}]$ and generically smooth, of pure absolute dimension $\dim(C_{\hat{G}}(\phi))$. It follows that the morphism (4.8) is syntomic of pure relative dimension $\dim(C_{\hat{G}}(\phi))-1$ and that the source space is generically smooth since the target is smooth. Since the orbit morphism is surjective and smooth (because $C_{\hat{G}}(\phi)$ is smooth), the same property holds for the morphism (4.7) by descent. But the orbit $\hat{G}.\phi$ itself is smooth over $\mathcal{O}_{K'_e}[\frac{1}{p}]$ (since it is a summand of $\underline{\mathrm{Hom}}(P_F/P_F^e,\hat{G})$) and has relative dimension $\dim(\hat{G})-\dim(C_{\hat{G}}(\phi))$. So i) follows.

ii) The ℓ-adic separatedness of $R^e_{\hat{G}}$ follows from Corollary 2.11 and (4.3). Moreover, (4.6) together with Theorem 2.13 show that for each $\phi\in\Phi_e$, the universal L-morphism ${}^L\varphi^{\phi}_{\rm univ}$ extends uniquely and ℓ-adically continuously to an L-morphism ${}^L\varphi^{\phi}_{\ell-{\rm univ}}: W_F\longrightarrow {}^LG(R'_{{}^LG,\phi}\otimes\mathbb{Z}_\ell)$. Here we have written $R'_{{}^LG,\phi}:=R_{{}^LG,\phi}\otimes_{\mathcal{O}_{K_e}[\frac{1}{p}]}\mathcal{O}_{K'_e}[\frac{1}{p}]$, and we have used the fact that the φ_α occuring in (4.6) has finite image, hence extends uniquely to W_F by continuity. Using the relation between $\varphi^{[\phi]}_{\rm univ}$ and $\varphi^{\phi}_{\rm univ}$, we see ultimately that ${}^L\varphi^{e}_{\rm univ}$ extends to an ℓ-adically continuous L-morphism ${}^L\varphi^{e}_{\ell-{\rm univ}}: W_F\longrightarrow {}^LG(R'^e_{{}^LG}\otimes\mathbb{Z}_\ell)$ where $R'^e_{{}^LG}=R^e_{{}^LG}\otimes\mathcal{O}_{K'_e}[\frac{1}{p}]$. We now claim that ${}^L\varphi^{e}_{\ell-{\rm univ}}$ factors through ${}^LG(R^e_{{}^LG}\otimes\mathbb{Z}_\ell)$. Indeed, its pushforward to ${}^LG(R'^e_{{}^LG}\otimes\mathbb{Z}/\ell^n\mathbb{Z})$ has the same image as the pushforward of ${}^L\varphi^{e}_{\rm univ}$ by continuity, hence it factors through ${}^LG(R^e_{{}^LG}\otimes\mathbb{Z}/\ell^n)$ for all n. But since $R'^e_{{}^LG}$ is locally free of finite rank over $R^e_{{}^LG}$, the claim follows. The universal property is straightforward. $\square$

Statement ii) clarifies a bit the dependence of our moduli space $\underline{Z}^1(W_F^0/P_F^e,\hat{G})$ on our initial choices of a topological generator s of I_F/P_F and of a lift of Frobenius Fr in W_F/P_F when defining the subgroup W_F^0 of W_F.

Corollary 4.2. *For any prime $\ell\neq p$, the base change $\underline{Z}^1(W_F^0/P_F^e,\hat{G})_{\mathbb{Z}_\ell}$ is independent of the choices made to define the subgroup W_F^0, up to canonical isomorphism. Namely, let $W_F^{0'}$ be another choice of subgroup, then there is a unique isomorphism of $\mathbb{Z}_\ell$-schemes $\underline{Z}^1(W_F^0/P_F^e,\hat{G})_{\mathbb{Z}_\ell}\xrightarrow{\sim}\underline{Z}^1(W_F^{0'}/P_F^e,\hat{G})_{\mathbb{Z}_\ell}$ compatible with the universal ℓ-adically continuous 1-cocycles on each side.*

Besides the above result, our main conjecture over $\overline{\mathbb{Z}}[\frac{1}{p}]$ states that the decomposition (4.4) is the decomposition into connected components.

Conjecture 4.3. *For any pair (ϕ,α), the $\mathcal{O}_{K_e}[\frac{1}{p}]$-scheme $\underline{Z}^1(W_F^0,\hat{G})_{\phi,\alpha}$ is connected and remains connected after any finite flat integral base change.*

The isomorphisms in (4.5) reduce this conjecture to the following one :

Conjecture 4.4. *For any split $\hat{G}$ over $\mathbb{Z}[\frac{1}{p}]$ with a tamely ramified Galois action that preserves a Borel pair, the tame summand $\underline{Z}^1(W_F^0/P_F,\hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]}$ is connected.*

In Subsection 4.6, we prove the last statement under the additional assumption that $\hat{G}$ has smooth center (Corollary 4.30), or the Galois action preserves a pinning (Theorem 4.29). Thanks to Theorem 3.12, this is enough to get the following result towards Conjecture 4.3 :

Theorem 4.5. *Conjecture 4.3 holds if the center of $\hat{G}$ is smooth over $\mathbb{Z}[\frac{1}{p}]$.*

4.2. **Decomposition after localization at a prime $\ell \neq p$.** For each choice of a prime $\ell \neq p$, statement ii) of Theorem 4.1 allows us to refine the decomposition (4.3) after tensoring by $\mathbb{Z}_\ell$. Indeed, denote by I_F^ℓ the maximal closed subgroup of I_F with prime-to-ℓ pro-order. Then, since ${}^L\varphi^e_{\ell-\mathrm{univ}}$ is ℓ-adically continuous, the kernel $I_F^{\ell,e}$ of $({}^L\varphi^e_{\ell-\mathrm{univ}})_{|I_F^\ell}$ is open in I_F^ℓ. It follows that restriction to I_F^ℓ provides a morphism of $\mathbb{Z}_\ell$-schemes

$$\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\mathbb{Z}_\ell} \longrightarrow \underline{Z}^1(I_F^\ell/I_F^{\ell,e}, \hat{G})_{\mathbb{Z}_\ell}.$$

But since the finite group $I_F^\ell/I_F^{\ell,e}$ has order invertible in $\mathbb{Z}_\ell$, we can apply the results of appendix A. In particular, there is a finite étale extension Λ_e of $\mathbb{Z}_\ell$ and a finite set $\Phi_e^\ell \subset Z^1(I_F^\ell/I_F^{\ell,e}, \hat{G}(\Lambda_e))$ such that

$$\underline{Z}^1(I_F^\ell/I_F^{\ell,e}, \hat{G})_{\Lambda_e} = \coprod_{\phi^\ell \in \Phi_e^\ell} \hat{G} \cdot \phi^\ell,$$

from which we deduce a decomposition similar to (4.2)

$$\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e} = \coprod_{\phi^\ell \in \Phi_e^\ell} \hat{G} \times^{C_{\hat{G}}(\phi^\ell)} \underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e, \phi^\ell},$$

where $\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e,\phi^\ell}$ denotes the closed subscheme of $\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e}$ defined by $(\varphi^e_{\ell-\mathrm{univ}})_{|I_F^\ell} = \phi^\ell$. Then we can play the same game as in Subsection 3.2. Namely, taking an L-group $\hat{G} \rtimes W_e$ such that $I_F^\ell/I_F^{\ell,e}$ injects into W_e, we define $C_{{}^LG}(\phi^\ell)$, $\tilde{\pi}_0(\phi^\ell)$ and $\Sigma(\phi^\ell)$ exactly as in that subsection. This allows us to decompose further

$$\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e,\phi^\ell} = \coprod_{\alpha^\ell \in \Sigma(\phi^\ell)} \underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e,\phi^\ell,\alpha^\ell}.$$

We will say again that ϕ^ℓ and α^ℓ are admissible if the corresponding summand is non empty. Moreover, we have an analogue of Theorem 3.4 with the same proof (actually, the proof simplifies a bit since we work here over a DVR).

Theorem 4.6. *There is an integral finite flat extension Λ_e' of Λ_e such that, for each admissible ϕ^ℓ, α^ℓ, we can find a cocycle $\varphi_{\alpha^\ell} \in Z^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e',\phi^\ell,\alpha^\ell}$ with finite image and such that $\mathrm{Ad}_{\varphi_{\alpha^\ell}}$ normalizes a Borel pair of $C_{\hat{G}}(\phi^\ell)^\circ$.*

As in Remark 3.9, this can be improved in certain circumstances. Namely, if the center $Z(C_{\hat{G}}(\phi^\ell)^\circ)$ is smooth over Λ_e – equivalently, if ℓ does not divide the order of $X^*(Z(C_{\hat{G}}(\phi^\ell)^\circ))_{\mathrm{tors}}$ – then one can find φ_{α^ℓ} such that $\mathrm{Ad}_{\varphi_{\alpha^\ell}}$ stabilizes a pinning of $C_{\hat{G}}(\phi^\ell)^\circ$. Using a version of Lemma 3.11 where $\overline{\mathbb{Z}}[\frac{1}{p}]$ is replaced by $\overline{\mathbb{Z}}_\ell$ and P is replaced by any solvable group of order prime to ℓ, one obtains the following analogue of Theorem 3.12.

Theorem 4.7. *Assume that the center of $\hat{G}$ is smooth over $\mathbb{Z}_{(\ell)}$. Then there is an integral finite flat extension Λ_e' of Λ_e such that, for each admissible ϕ^ℓ, α^ℓ, we can find a cocycle $\varphi_{\alpha^\ell} \in Z^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e',\phi^\ell,\alpha^\ell}$ with finite image and such that $\mathrm{Ad}_{\varphi_{\alpha^\ell}}$ fixes a pinning of $C_{\hat{G}}(\phi^\ell)^\circ$.*

In particular, this result applies to classical groups whenever $\ell \neq 2$.

Fix φ_{α^ℓ} as in one of the above theorems. Having finite image, it extends to W_F and the conjugation action $\mathrm{Ad}_{\varphi_{\alpha^\ell}}$ factors over W_F/I_F^ℓ. Then the usual map $\rho \mapsto \rho\cdot\varphi_{\alpha^\ell}$ provides an isomorphism of Λ'_e-schemes

$$\underline{Z}^1_{\mathrm{Ad}_{\varphi_{\alpha^\ell}}}\left((W_F/P_F)^0, C_{\hat{G}}(\phi^\ell)^\circ\right)_{\Lambda'_e,1^\ell} \xrightarrow{\sim} \underline{Z}^1(W_F^0/P_F^e,\hat{G})_{\Lambda'_e,\phi^\ell,\alpha^\ell}$$

where the subscript 1^ℓ on the left hand side denotes the closed and open subscheme of $\underline{Z}^1_{\mathrm{Ad}_{\varphi_{\alpha^\ell}}}\left((W_F/P_F)^0, C_{\hat{G}}(\phi^\ell)^\circ\right)_{\Lambda'_e}$ where the universal ℓ-adically continuous tame parameter restricts trivially to I_F^ℓ.

Theorem 4.8. *For each pair (ϕ^ℓ,α^ℓ), the Λ_e-scheme $\underline{Z}^1(W_F^0/P_F^e,\hat{G})_{\Lambda_e,\phi^\ell,\alpha^\ell}$ has a geometrically connected special fiber. In particular, it is connected and its base change to any integral finite flat extension of Λ_e remains connected.*

We will prove this result after some preparation on categorical quotients. Meanwhile, we note that the second part of the statement follows from the first one since $\underline{Z}^1(W_F^0/P_F^e,\hat{G})_{\Lambda_e,\phi^\ell,\alpha^\ell}$ is the spectrum of an ℓ-adically separated $\mathbb{Z}_\ell$-algebra. The collection of these results for all $\ell\neq p$ will be the main ingredient in the proof of Theorem 4.5.

4.3. **Quotients, moduli spaces of parameters.** The group scheme $\hat{G}$ acts by (twisted) conjugation on $\underline{Z}^1(W_F^0/P_F^e,\hat{G})$. There are several type of quotients which can be considered here : the stacky quotient, the quotient as fppf sheaves, or the quotient in the category of affine schemes, which is simply $\mathrm{Spec}((R^e_{{}^LG})^{\hat{G}})$. Whatever type of quotient is considered, let us denote it by $\underline{H}^1(W_F^0/P_F^e,\hat{G})$. Then, (4.2) induces a decomposition

$$\underline{H}^1(W_F^0/P_F^e,\hat{G})_{\mathcal{O}_{K_e}[\frac{1}{p}]} = \coprod_{\phi\in\Phi_e^{\mathrm{adm}}} \underline{Z}^1(W_F^0,\hat{G})_\phi/C_{\hat{G}}(\phi),$$

where the quotients on the right hand side are of the same type. Next, (4.4) gives for each ϕ a decomposition

$$\underline{Z}^1(W_F^0,\hat{G})_\phi/C_{\hat{G}}(\phi)^\circ = \coprod_{\alpha\in\Sigma(\phi)^{\mathrm{adm}}} \underline{Z}^1(W_F^0,\hat{G})_{\phi,\alpha}/C_{\hat{G}}(\phi)^\circ$$

(beware the $\circ$) while (4.5) provides for each α an isomorphism

$$\underline{H}^1_{\mathrm{Ad}_{\varphi_\alpha}}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ) \xrightarrow{\sim} \left(\underline{Z}^1(W_F^0,\hat{G})_{\phi,\alpha}/C_{\hat{G}}(\phi)^\circ\right)_{\mathcal{O}_{K'_e}[\frac{1}{p}]}.$$

Now, let us denote by $\Sigma(\phi)_0$ a set of representatives of $\pi_0(\phi)$-orbits in $\Sigma(\phi)$ and by $\pi_0(\phi)_\alpha$ the stabilizer of α in $\pi_0(\phi)$. Let $C_{\hat{G}}(\phi)_\alpha$ be the closed subgroup scheme of $C_{\hat{G}}(\phi)$ inverse image of $\pi_0(\phi)_\alpha$. It stabilizes the summand $\underline{Z}^1(W_F^0,\hat{G})_{\phi,\alpha}$ of $\underline{Z}^1(W_F^0,\hat{G})_\phi$, whence an action of $\pi_0(\phi)_\alpha$ on $\underline{H}^1_{\mathrm{Ad}_{\varphi_\alpha}}(W_{F^0}/P_F, C_{\hat{G}}(\phi)^\circ)$ through the last isomorphism. We thus have obtained an isomorphism

$$\underline{H}^1(W_F^0/P_F^e,\hat{G})_{\mathcal{O}_{K'_e}[\frac{1}{p}]} = \coprod_{\phi\in\Phi_e^{\mathrm{adm}}}\ \coprod_{\alpha\in\Sigma(\phi)_0^{\mathrm{adm}}} \underline{H}^1_{\mathrm{Ad}_{\varphi_\alpha}}(W_F^0/P_F, C_{\hat{G}}(\phi)^\circ)_{/\pi_0(\phi)_\alpha}.$$

In the case of the affine categorical quotient, we will use the familiar notation

$$\underline{Z}^1(W_F^0/P_F^e,\hat{G}) /\!\!/ \hat{G} := \mathrm{Spec}((R^e_{{}^LG})^{\hat{G}}).$$

From the above discussion we deduce :

Proposition 4.9. *The affine categorical quotient $\underline{Z}^1(W_F^0/P_F^e,\hat{G}) /\!\!/ \hat{G}$ is a flat, reduced, ℓ-adically separated affine scheme of finite presentation over $\mathbb{Z}[\frac{1}{p}]$ and its ring of functions decomposes as*

$$(R^e_{{}^LG})^{\hat{G}}\otimes \mathcal{O}_{K'_e}[\tfrac{1}{p}] = \prod_{\phi\in\Phi_e^{\rm adm}}\ \prod_{\alpha\in\Sigma(\phi)_0^{\rm adm}} \left(\left(R_{{}^LG_{\varphi_\alpha}}\right)^{C_{\hat{G}}(\phi)^\circ}\right)^{\pi_0(\phi)_\alpha}.$$

With similar notation, we also have local decompositions for each prime $\ell\neq p$

$$(R^e_{{}^LG})^{\hat{G}}\otimes \Lambda'_e = \prod_{\phi^\ell\in\Phi_e^{\ell,\rm adm}}\ \prod_{\alpha^\ell\in\Sigma(\phi^\ell)_0^{\rm adm}} \left(\left(R_{{}^LG_{\varphi_{\alpha^\ell},1^\ell}}\right)^{C_{\hat{G}}(\phi^\ell)^\circ}\right)^{\pi_0(\phi^\ell)_{\alpha^\ell}}.$$

Proof. The first decomposition has been explained above and the second one is similar, based on section 4.2. The claimed properties of $(R^e_{{}^LG})^{\hat{G}}$ follow from Theorem 4.1 except for its finite generation as a $\mathbb{Z}[\frac{1}{p}]$-algebra, which is a difficult result of Thomason [Tho87, Thm 3.8]. ☐

4.4. **Closed orbits over an algebraically closed field.** Let L be an algebraically closed field of characteristic ℓ different from p ($\ell=0$ is allowed here). Let us consider the affine categorical quotient

$$\underline{Z}^1(W_F^0/P_F^e,\hat{G})_L /\!\!/ \hat{G}_L = \mathrm{Spec}((R^e_{{}^LG}\otimes L)^{\hat{G}_L}).$$

Its relation with the affine quotient over $\mathbb{Z}[\frac{1}{p}]$ can be extracted from Alper's paper [Alp14], which builds on the work of Seshadri [Ses77] and Thomason [Tho87] on Geometric Invariant Theory over arbitrary bases.

Proposition 4.10. *The canonical map $(R^e_{{}^LG})^{\hat{G}}\otimes L \longrightarrow (R^e_{{}^LG}\otimes L)^{\hat{G}_L}$ is injective. It is surjective if $\ell=0$ and, when $\ell>0$, there is an integer r such that its image contains $\{f^{\ell^r}, f\in (R^e_{{}^LG}\otimes L)^{\hat{G}_L}\}$. In particular the canonical morphism of L-schemes*

$$\underline{Z}^1(W_F^0/P_F^e,\hat{G})_L /\!\!/ \hat{G}_L \longrightarrow \left(\underline{Z}^1(W_F^0/P_F^e,\hat{G}) /\!\!/ \hat{G}\right)_L$$

is a universal homeomorphism, and even an isomorphism when $\ell=0$.

Proof. The case $\ell=0$ is easy, so we assume that ℓ is prime. Consider first the map $(R^e_{{}^LG})^{\hat{G}}/\ell(R^e_{{}^LG})^{\hat{G}} \longrightarrow (R^e_{{}^LG}/\ell R^e_{{}^LG})^{\hat{G}}$. It is injective because $R^e_{{}^LG}$ is ℓ-torsion free. Moreover, since $\hat{G}$ is geometrically reductive over $\mathbb{Z}[\frac{1}{p}]$ in the sense of [Alp14, Def. 9.1.1] (by Theorem 9.7.5 of [Alp14]), it follows from [Alp14, Rk 5.2.2] that this map is an "adequate" homeomorphism, in the sense of [Alp14, Def 3.3.1]. In particular it is "universally adequate", hence the map of the proposition is adequate too, and [Alp14, Lemma 3.2.3] insures the existence of r as claimed in the proposition. ☐

Remark 4.11. In Theorem 6.8, we will get an explicit bound on the set of primes ℓ for which the canonical morphism of this proposition is an isomorphism.

By classical Geometric Invariant Theory, we know that the L-points of the affine quotient $\underline{Z}^1(W_F^0/P_F^e,\hat{G})_L /\!\!/ \hat{G}_L$ correspond bijectively to Zariski closed $\hat{G}(L)$-orbits in $Z^1(W_F^0/P_F^e,\hat{G}(L))$. On the other hand, a theorem of Richardson provides a criterion to decide when the $\hat{G}(L)$-orbit of $\varphi\in Z^1(W_F^0/P_F^e,\hat{G}(L))$ is closed.

Definition 4.12. We say that $\varphi \in Z^1(W_F^0/P_F^e, \hat{G}(L))$ is *${}^LG$-semisimple* if the Zariski closure $\overline{{}^L\varphi(W_F^0)}$ of the image of ${}^L\varphi$ in ${}^LG(L)$ is a *completely reducible* subgroup of ${}^LG(L)$ in the sense of [BMR05].

Let us recall the definition from [BMR05] : a closed subgroup Γ of ${}^LG(L)$ is called *completely reducible* if for all R-parabolic subgroups $P(L)$ of ${}^LG(L)$ containing Γ, there exists a R-Levi subgroup of $P(L)$ containing Γ. Here, and as in [BMR05, §6], we use Richardson's definition of parabolic and Levi subgroups via cocharacters, which makes perfect sense for non-connected reductive groups. Actually, the definition applies verbatim to Γ an arbitrary subgroup, see [BMR05, §2.6], and we have that Γ is completely reductible if and only if its closure is completely reductible, so that, in the above definition, we may only require that the image ${}^L\varphi(W_F^0)$ be completely reducible.

It wouldn't be difficult to check directly that for a continuous 1-cocycle $\varphi : W_F^0 \to \hat{G}(L)$, the property of being ${}^LG$-semisimple neither depends on the choice of an integer e such that ${}^L\varphi$ factors through W_F^0/P_F^e, nor on the particular choice of L-group we make. Anyway, this fact is also a consequence of Richardson's theorem that we now state.

Theorem 4.13 (Richardson)**.** *The $\hat{G}(L)$-orbit of $\varphi \in Z^1(W_F^0/P_F^e, \hat{G}(L))$ is closed if and only if φ is ${}^LG$-semisimple.*

Proof. Recall that the map $\varphi \mapsto {}^L\varphi$ identifies the set $Z^1 := Z^1(W_F^0/P_F^e, \hat{G}(L))$ with the set of L-homomorphisms $W_F^0/P_F^e \longrightarrow {}^LG(L)$, which is contained in the set H of all homomorphisms $W_F^0/P_F^e \longrightarrow {}^LG(L)$. Both Z^1 and H have a natural reduced L-scheme structure, and Z^1 is open and closed in H. In particular, the $\hat{G}(L)$-orbit of $\varphi \in Z^1$ is closed in Z^1 if and only the $\hat{G}(L)$-orbit of ${}^L\varphi$ is closed in H. Now, on H the action of $\hat{G}(L)$ extends to ${}^LG(L)$ and, since $\hat{G}$ has finite index in ${}^LG$, we see that the $\hat{G}(L)$-orbit of ${}^L\varphi$ is closed if and only if its ${}^LG(L)$-orbit is closed.

Now, let $w_1, \cdots, w_n$ be a finite set of generators of the group W_F^0/P_F^e. Then the map $\psi \mapsto (\psi(w_1), \cdots, \psi(w_n))$ is an ${}^LG(L)$-equivariant closed embedding of H into ${}^LG_L^n$. So we see that the ${}^LG(L)$-orbit of ${}^L\varphi$ in H is closed if and only if the ${}^LG(L)$-orbit of $({}^L\varphi(w_1), \cdots, {}^L\varphi(w_n)) \in {}^LG(L)^n$ is closed in ${}^LG(L)^n$. Now, Richardson's theorem (see Cor 3.7 and §6.3 of [BMR05]) tells us that the latter orbit is closed if and only if the closure of the subgroup of ${}^LG(L)$ generated by $({}^L\varphi(w_1), \cdots, {}^L\varphi(w_n))$ is completely reducible in the sense recalled above. □

In view of this result, we may drop the ${}^LG$ and simply say that "φ is semisimple". The following result will be crucial in our study of the affine quotient.

Proposition 4.14. *Any semisimple 1-cocycle $\varphi : W_F^0 \to \hat{G}(L)$ extends continuously and uniquely to W_F. Moreover, the prime-to-p part $|{}^L\varphi(I_F)|_{p'}$ of the cardinality of ${}^L\varphi(I_F)$ is bounded independently of φ and of the field L. More precisely, $|{}^L\varphi(I_F)|_{p'}$ divides $e.\chi_{\hat{G},\tilde{\mathrm{Fr}}}(q)^2$ where*

- *e is the tame ramification of the finite extension F' of F given by the kernel of the map $W_F \longrightarrow \pi_0({}^LG)$,*
- *$\tilde{\mathrm{Fr}}$ is any lift of Frobenius in W_F.*
- *$\chi_{\hat{G},\tilde{\mathrm{Fr}}} \in \mathbb{Z}[T]$ is introduced in the appendix B.2.*

Remark 4.15. If we restrict attention to fields of characteristic $\ell > 0$, then the statement that φ extends continuously to W_F is true for all 1-cocycles, by (ii) of Theorem 4.1. However, there is obviously no uniform bound on $|{}^L\varphi(I_F)|_{p'}$ without the semisimplicity hypothesis, when we vary the field L.

Proof of the proposition. Recall from [Iwa55, Thm. 2 (iii)] that there exist lifts $\tilde{s}$ and $\tilde{\mathrm{Fr}}$ of s and Fr in W_F such that $\tilde{\mathrm{Fr}}.\tilde{s}.\tilde{\mathrm{Fr}}^{-1} = \tilde{s}^q$ and that W_F decomposes as a semi-direct product $W_F = P_F \rtimes \overline{\langle \tilde{s}, \tilde{\mathrm{Fr}}\rangle}$. Accordingly, W_F^0 decomposes as $W_F^0 = P_F \rtimes \langle \tilde{s}, \tilde{\mathrm{Fr}}\rangle$. Then we see that a continuous 1-cocycle φ from W_F^0 extends continuously to W_F if and only if ${}^L\varphi(\tilde{s})$ has finite order, in which case this order is prime-to-p, the extension is unique, and it satisfies $\varphi(W_F) = \varphi(W_F^0)$.

Let us now assume that φ is ${}^LG$-semisimple and show that ${}^L\varphi(\tilde{s})$ then has finite order. Let F' and e be as in the statement of the proposition. Note that $\tilde{s}^e \in W_{F'}$ and thus ${}^L\varphi(\tilde{s})^e \in \hat{G}(L)$. Since ${}^L\varphi(P_{F'})$ is finite, there certainly is an integer m such that ${}^L\varphi(\tilde{s})^{em}$ commutes with ${}^L\varphi(P_{F'})$. This means that $\langle {}^L\varphi(\tilde{s})^{em}\rangle$ is a normal subgroup of ${}^L\varphi(I_{F'}^0)$, which is a normal subgroup of ${}^L\varphi(W_F^0)$ (here we have set $I_{F'}^0 = I_{F'} \cap W_{F'}^0$). Taking Zariski closures, we get that $\overline{\langle {}^L\varphi(\tilde{s})^{em}\rangle}$ is a normal subgroup of $\overline{{}^L\varphi(I_{F'}^0)}$, which is a normal subgroup of $\overline{{}^L\varphi(W_F^0)}$. Now recall from [BMR05, Thm 3.10] that any normal closed subgroup of a completely reducible closed subgroup of ${}^LG(L)$ is completely reducible. So we infer that $\overline{\langle {}^L\varphi(\tilde{s})^{em}\rangle}$, hence also $\langle {}^L\varphi(\tilde{s})^{em}\rangle$, is a completely reducible subgroup of ${}^LG(L)$, hence also a completely reducible subgroup of $\hat{G}(L)$. This means that ${}^L\varphi(\tilde{s})^{em}$ is a semi-simple element of $\hat{G}(L)$. Since it is conjugate to its q-power under ${}^L\varphi(\tilde{\mathrm{Fr}}) \in \hat{G}(L)$, Proposition B.3 (2) shows that ${}^L\varphi(\tilde{s})^{em}$ has finite order, and this order divides $\chi_{\hat{G},\mathrm{Ad}_{\varphi(\tilde{\mathrm{Fr}})}}(q) = \chi_{\hat{G},\tilde{\mathrm{Fr}}}(q)$.

It now remains to estimate m and prove that $m = \chi_{\hat{G},\tilde{\mathrm{Fr}}}(q)$ works. For this, we may assume that φ belongs to some $Z^1(W_F^0, \hat{G})_{\phi,\alpha}$ and write $\varphi = \rho \cdot \varphi_\alpha$ as in (4.5). By construction ${}^L\varphi_\alpha$ has finite image in ${}^LG(\overline{\mathbb{Z}}[\frac{1}{p}])$. Let m be the order of the element ${}^L\varphi_\alpha(\tilde{s})^e$, which lies in $\hat{G}(\overline{\mathbb{Z}}[\frac{1}{p}])$. Then we have

$$ {}^L\varphi(\tilde{s})^{em} = \rho(\tilde{s}).\,\mathrm{Ad}_{\varphi_\alpha(\tilde{s})}(\rho(\tilde{s}))\cdots \mathrm{Ad}_{\varphi_\alpha(\tilde{s})^{em-1}}(\rho(\tilde{s})) \in C_{\hat{G}}(\phi)^\circ(L), $$

hence ${}^L\varphi(\tilde{s})^{em}$ commutes with $\varphi(P_{F'})$ as desired. But observe now that ${}^L\varphi_\alpha(\tilde{s})^e$ is also a semisimple element of $\hat{G}(\overline{\mathbb{Q}})$ that is conjugate to its q^{th}-power under ${}^L\varphi_\alpha(\tilde{\mathrm{Fr}})$. Hence, as above, its order m divides $\chi_{\hat{G},\tilde{\mathrm{Fr}}}(q)$. $\square$

Let us denote by $N_{\hat{G}}$ the l.c.m of all $|{}^L\varphi(I_F)|_{p'}$ for φ and L varying as in the proposition, and where ${}^LG$ is the minimal L-group, i.e. ${}^LG = \hat{G} \rtimes \Gamma$ with Γ the image of $W_F \longrightarrow \mathrm{Aut}(\hat{G})$.

Corollary 4.16. *For each "depth" e, there is an open normal subgroup I_F^e of I_F with index dividing $N_{\hat{G}}.[P_F : P_F^e]$ such that any semisimple cocycle $\varphi : W_F^0/P_F^e \longrightarrow \hat{G}(L)$ is trivial on $I_F^e \cap W_F^0$, and therefore extends canonically to W_F/I_F^e.*

Proof. Define I_F^e to be the intersection of the kernels of the L-homomorphisms ${}^L\varphi : W_F^0/P_F^e \longrightarrow {}^LG(L)$ associated to all semisimple cocycles $\varphi : W_F^0/P_F^e \longrightarrow \hat{G}(L)$ with L an algebraically closed field of characteristic $\ell \neq p$. Here ${}^LG$ is the minimal L-group, as above. Then $P_F \cap I_F^e$ contains P_F^e, hence it is open in P_F of index dividing that of P_F^e. Moreover, by the above proposition, the cyclic group

$I_F/(P_F.I_F^e)$ is killed by $N_{\hat{G}}$, hence I_F^e has open image in I_F/P_F. It follows that I_F^e is open in I_F and that its index divides $N_{\hat{G}}.[P_F : P_F^e]$. □

With I_F^e as in this corollary, we may consider the $\hat{G}$-stable closed subscheme $\underline{Z}^1(W_F/I_F^e, \hat{G})$ of $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$ consisting of 1-cocycles that are trivial on I_F^e. We will denote by $S_{^LG}^e$ its affine ring, which is thus a quotient of $R_{^LG}^e$.

Proposition 4.17. *i) The homomorphism of rings*

$$(R_{^LG}^e)^{\hat{G}} \longrightarrow (S_{^LG}^e)^{\hat{G}} \tag{4.9}$$

is injective and its image contains $\{f^N, f \in (S_{^LG}^e)^{\hat{G}}\}$ for some integer $N > 0$.

ii) The corresponding morphism of schemes

$$\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G} \longrightarrow \underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G} \tag{4.10}$$

is a finite universal homeomorphism and becomes an isomorphism after extending scalars to $\mathbb{Q}$.

Proof. i) *Injectivity of (4.9).* For any algebraically closed field L in which p is invertible, the last corollary tells us that all closed orbits of $\underline{Z}^1(W_F^0/P_F^e, \hat{G})(L)$ are contained in $\underline{Z}^1(W_F/I_F^e, \hat{G})(L)$, hence the morphism (4.10) is bijective on L-points. It follows that the kernel of (4.9) is contained in the nilradical of $(R_{^LG}^e)^{\hat{G}}$, which is trivial since $R_{^LG}^e$ is reduced, being syntomic over $\mathbb{Z}[\frac{1}{p}]$ and generically smooth by Theorem 4.1.

Image of (4.9). By [Alp14, Thm 9.7.5], $\hat{G}$ is geometrically reductive in the sense of [Alp14, Def. 9.1.1]. By the characterization of this property given in [Alp14, Lem 9.2.5 (2)'], it follows that the map (4.9) is "universally adequate". Then, Proposition 3.3.5 of [Alp14] provides the desired N (note that the map $(R_{^LG}^e)^{\hat{G}} \longrightarrow (S_{^LG}^e)^{\hat{G}}$ is of finite type since $(S_{^LG}^e)^{\hat{G}}$ is finitely generated over $\mathbb{Z}[\frac{1}{p}]$ by [Tho87, Thm 3.8]).

ii) now follows from Lemmas 3.1.4 and 3.1.5 of [Alp14]. □

Note that the ring $S_{^LG}^e$ may not share the nice properties of $R_{^LG}^e$; it may not be reduced nor be flat over $\mathbb{Z}[\frac{1}{p}]$. However, the last proposition implies that the nilradical of $(S_{^LG}^e)^{\hat{G}}$ coincides with its $\mathbb{Z}[\frac{1}{p}]$-torsion ideal. Moreover, the fact that (4.10) is an isomorphism after tensoring by $\mathbb{Q}$ shows that $(\underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G})_{\mathbb{Q}}$ is canonically independent of our initial choice of subgroup W_F^0 in W_F. Actually, we can do better with a little more work :

Theorem 4.18. *The affine quotient $\underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G}$ is canonically independent of the choice of a topological generator s of I_F/P_F and a lift of Frobenius* Fr *in W_F/P_F in the definition of the subgroup W_F^0.*

Proof. Let (Fr', s') be another choice, leading to a subgroup $W_F^{0'}$ of W_F. Denote by $R_{^LG}^{e'}$ the affine ring of $\underline{Z}^1(W_F^{0'}/P_F^e, \hat{G})$. Denote by ι the embedding (4.9) and by ι' the analogous embedding $(R_{^LG}^{e'})^{\hat{G}} \hookrightarrow (S_{^LG}^e)^{\hat{G}}$. For each prime $\ell \neq p$, we have a canonical isomorphism $(R_{^LG}^{e'})^{\hat{G}} \otimes \mathbb{Z}_\ell \simeq (R_{^LG}^e)^{\hat{G}} \otimes \mathbb{Z}_\ell$ from Corollary 4.2. By construction it commutes with the base changes of ι and ι' to $\mathbb{Z}_\ell$, which means that $\iota((R_{^LG}^e)^{\hat{G}}) \otimes \mathbb{Z}_\ell = \iota'((R_{^LG}^{e'})^{\hat{G}}) \otimes \mathbb{Z}_\ell$ inside $(S_{^LG}^e)^{\hat{G}} \otimes \mathbb{Z}_\ell$. This implies that ℓ is not in the support of the quotient $\mathbb{Z}[\frac{1}{p}]$-module $(\iota((R^e_{{}^LG})^{\hat G})+\iota'((R^{e'}_{{}^LG})^{\hat G}))/\iota((R^e_{{}^LG})^{\hat G})$. Since this is true for all $\ell\neq p$, it follows that this quotient is 0, hence $\iota((R^e_{{}^LG})^{\hat G})=\iota'((R^{e'}_{{}^LG})^{\hat G})$ inside $(S^e_{{}^LG})^{\hat G}$. □

We may wonder whether such an independence result still holds for the stacky quotient. We believe that, at least, the categories of quasi-coherent sheaves on such stacks might be equivalent.

4.5. **Geometric connected components in positive characteristic.** We maintain our setup of an algebraically closed field L of characteristic $\ell\neq p$, and we assume that $\ell>0$. In order to parametrize the connected components of $\underline{Z}^1(W_F^0/P_F^e,\hat G)_L$, we first observe that, since $\hat G$ is connected, the canonical morphism

$$\underline{Z}^1(W_F^0/P_F^e,\hat G)_L\longrightarrow \underline{Z}^1(W_F^0/P_F^e,\hat G)_L/\!\!/\hat G_L$$

induces a bijection on the respective sets of connected components. Hence, Proposition 4.17 invites us to study the connected components of $\underline{Z}^1(W_F/I_F^e,\hat G)_L/\!\!/\hat G_L$.

Note : in order to lighten the notation a bit we will sometimes denote by $\underline{H}^1$ the categorical quotient of cocycles modulo the relevant group action.

Using restriction to I_F^ℓ, we have already obtained a decomposition

$$\underline{H}^1(W_F/I_F^e,\hat G_L)=\coprod_{\phi^\ell\in\Phi_e^{\ell,\mathrm{adm}}}\coprod_{\alpha^\ell\in\Sigma(\phi^\ell)^{\mathrm{adm}}}\left(\underline{H}^1_{\mathrm{Ad}_{\varphi_{\alpha^\ell}}}\left(W_F/I_F^eI_F^\ell,C_{\hat G}(\phi^\ell)^\circ_L\right)\right)_{/\!\!/\pi_0(\phi^\ell)_{\alpha^\ell}}.$$

The following result shows that each summand is connected and will provide a topological description of these summands.

Proposition 4.19. *Assume that the action of W_F on $\hat G$ is trivial on I_F^ℓ and stabilizes a Borel pair $(\hat B,\hat T)$. Then the following holds.*

1. *The reduced fixed-points subgroup $(\hat G_L)^{I_F}$ is a connected reductive subgroup of $\hat G_L$ and the reduced fixed-points subgroup $(\hat T_L)^{I_F}$ is a maximal torus of $(\hat G_L)^{I_F}$ whose Weyl group is the I_F-fixed subgroup Ω^{I_F} of the Weyl group Ω of $\hat T$ in $\hat G$.*
2. *The closed immersion $\underline{Z}^1(W_F/I_F,\hat G_L^{I_F})\hookrightarrow\underline{Z}^1(W_F/I_F^eI_F^\ell,\hat G_L)$ induces an homeomorphism*
$$\underline{Z}^1(W_F/I_F,\hat G_L^{I_F})/\!\!/\hat G_L^{I_F}\longrightarrow\underline{Z}^1(W_F/I_F^eI_F^\ell,\hat G_L)/\!\!/\hat G_L.$$
3. *The map $t\mapsto(\varphi:\mathrm{Fr}\mapsto t\rtimes\mathrm{Fr})$ induces an isomorphism*
$$(\hat T_L^{I_F})_{\mathrm{Fr}}/\!\!/\Omega^{W_F}\xrightarrow{\sim}\underline{Z}^1(W_F/I_F,\hat G_L^{I_F})/\!\!/\hat G_L^{I_F}.$$

Proof. (1) The group I_F acts on $\hat G$ through a cyclic ℓ-power quotient, any generator of which is a quasi-semisimple automorphism of $\hat G$ (in the sense of Steinberg). Therefore, the first assertion follows from Thm 1.8 i) (reductivity) and Cor. 1.33 (connectedness) of [DM94].

(2) We first make the following observation. If $\phi:\,I_F\longrightarrow\hat G(L)$ is a semisimple cocycle trivial on I_F^ℓ, then it is $\hat G(L)$-conjugate to the trivial cocycle (note that the latter is indeed semisimple by the characterization given in [BMR05, Cor 3.5 (v)]). To prove this, let us use the "minimal" L-group $\hat G\rtimes\Gamma$, where Γ is the image of W_F in $\mathrm{Aut}(\hat G)$. Then the image C of I_F in Γ is a cyclic ℓ-group. Let $\bar s$ be the image of the pro-generator s of I_F/P_F in C. By [Ste68, 7.2], the element

${}^L\phi(s) := (\sigma, \bar{s})$ normalizes a Borel subroup of $\hat{G}$. After conjugating by some element of $\hat{G}(L)$ we may assume that it normalizes $\hat{B}$, thus ${}^L\phi$ factors trough the minimal R-parabolic subgroup $\hat{B} \rtimes C$ of $\hat{G} \rtimes C$. Since ϕ is assumed to be semisimple, ${}^L\phi$ should factor through some R-Levi subgroup of $\hat{B} \rtimes C$. But these Levi subgroups are $\hat{B}$-conjugated to $\hat{T} \rtimes C$. Therefore we may conjugate again ${}^L\phi$ so that it factors through $\hat{T} \rtimes C$, which means that $\phi \in Z^1(C, \hat{T}(L))$. But since $\hat{T}(L)$ is a ℓ-torsion free divisible group, we have $H^1(C, \hat{T}(L)) = \{1\}$, which means that ϕ is conjugate to the trivial cocycle.

We deduce that the subset $Z^1(I_F/I_F^e I_F^\ell, \hat{G}(L))^{\mathrm{ss}}$ of $Z^1(I_F/I_F^e I_F^\ell, \hat{G}(L))$ that consists of semisimple cocycles is closed, since it is a single orbit and this orbit is closed by definition of semisimple. Moreover this closed subset identifies with $\hat{G}(L)/\hat{G}(L)^{I_F}$. By pull-back, we deduce that the subset $Z^1(W_F/I_F^e I_F^\ell, \hat{G}(L))^{I_F-\mathrm{ss}}$ of $Z^1(W_F/I_F^e I_F^\ell, \hat{G}(L))$ that consists of all cocycles $\varphi : W_F \longrightarrow \hat{G}(L)$ such that $\varphi_{|I_F}$ is semisimple, is closed and identifies with $\hat{G}(L) \times^{\hat{G}(L)^{I_F}} Z^1(W_F/I_F, \hat{G}(L)^{I_F})$.

Now by [BMR05, Thm 3.10] we know that any semisimple $\varphi : W_F \longrightarrow \hat{G}(L)$ has semisimple restriction to I_F, so that the above closed subset contains all closed orbits of $Z^1(W_F/I_F^e I_F^\ell, \hat{G}(L))$. So denote by $\underline{Z}^1(W_F/I_F^e I_F^\ell, \hat{G}_L)^{I_F-\mathrm{ss}}$ the (reduced) closed subscheme of $Z^1(W_F/I_F^e I_F^\ell, \hat{G}_L)$ associated to this closed subset. Then the same argument as in Proposition 4.17 shows that the canonical morphism

$$\underline{Z}^1(W_F/I_F^e I_F^\ell, \hat{G}_L)^{I_F-\mathrm{ss}} /\!\!/ \hat{G}_L \longrightarrow \underline{Z}^1(W_F/I_F^e I_F^\ell, \hat{G}_L) /\!\!/ \hat{G}_L$$

is a finite universal homeomorphism. Using that

$$\underline{Z}^1(W_F/I_F^e I_F^\ell, \hat{G})_L^{I_F-\mathrm{ss}} = \hat{G}_L \times^{\hat{G}_L^{I_F}} \underline{Z}^1(W_F/I_F, \hat{G}_L^{I_F})$$

we infer statement (2).

(3) This is [DM15, Prop. 7.1] applied with $\mathbf{G}^1 = \hat{G}^{I_F} \rtimes \mathrm{Fr}$ and $\mathbf{T}^1 = \hat{T}^{I_F} \rtimes \mathrm{Fr}$, and $\sigma = t \rtimes \mathrm{Fr}$ for any element $t \in \hat{T}^{I_F}$ such that $t \rtimes \mathrm{Fr}$ is quasi-central (note that $(\hat{T}^{I_F})_\sigma = (\hat{T}^{I_F})_{\mathrm{Fr}}$ and $(\Omega^{I_F})^\sigma = (\Omega^{I_F})^{\mathrm{Fr}}$). ☐

We now use the results and notation of Subsection 4.2 to spread out this result.

Corollary 4.20. *Let $\phi^\ell \in \Phi_e^\ell$ and $\alpha^\ell \in \Sigma(\phi^\ell)_0$ be admissible, and fix $\varphi := \varphi_{\alpha^\ell} \in Z^1(W_F^0/P_F^e, \hat{G}(\Lambda_e'))_{\phi^\ell,\alpha^\ell}$ with finite image and such that $\mathrm{Ad}_{\varphi_{\alpha^\ell}}$ normalizes a Borel pair $(B_{\phi^\ell}, T_{\phi^\ell})$ of $C_{\hat{G}}(\phi^\ell)^\circ$. We denote by $\Omega_{\phi^\ell}^\circ$ the Weyl group of T_{ϕ^ℓ} in $C_{\hat{G}}(\phi^\ell)^\circ$ and by $\Omega_{\phi^\ell} = \Omega_{\phi^\ell}^\circ \rtimes \pi_0(\phi^\ell)$ its "Weyl group" in $C_{\hat{G}}(\phi^\ell)$.*

1. *Let $C_{\hat{G}_L}(\varphi_{|I_F}) = (\hat{G}_L)^{\varphi(I_F)}$ be the reduced centralizer of ${}^L\varphi(I_F)$ in $\hat{G}_L$.*
   - (a) *$C_{\hat{G}_L}(\varphi_{|I_F})^\circ$ is reductive with maximal torus $(T_{\phi^\ell,L})^{\varphi(I_F)}$ and Weyl group $(\Omega_{\phi^\ell}^\circ)^{\varphi(I_F)}$.*
   - (b) *$\pi_0(C_{\hat{G}_L}(\varphi_{|I_F})) = \pi_0(\phi^\ell)^{\varphi(I_F)}$ and the "Weyl group" of $(T_{\phi^\ell,L})^{\varphi(I_F)}$ in $C_{\hat{G}_L}(\varphi_{|I_F})$ is $(\Omega_{\phi^\ell})^{\varphi(I_F)} \simeq (\Omega_{\phi^\ell}^\circ)^{\varphi(I_F)} \rtimes \pi_0(\phi^\ell)^{\varphi(I_F)}$.*
2. *The natural closed immersion*

   $$\underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F/I_F, C_{\hat{G}_L}(\varphi(I_F))^\circ) \hookrightarrow \underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F/I_F^e I_F^\ell, C_{\hat{G}_L}(\phi^\ell)^\circ)$$

   induces an homeomorphism

   $$\underline{H}^1_{\mathrm{Ad}_\varphi}(W_F/I_F, C_{\hat{G}_L}(\varphi(I_F))^\circ) \longrightarrow \underline{H}^1_{\mathrm{Ad}_\varphi}(W_F/I_F^e I_F^\ell, C_{\hat{G}_L}(\phi^\ell)^\circ)$$

   which is equivariant for the natural actions of $\pi_0(\phi^\ell)_{\alpha^\ell} = \pi_0(\phi^\ell)^{\varphi(W_F)}$.

(3) *The map* $t \mapsto (\varphi_t : \mathrm{Fr} \mapsto t \rtimes \mathrm{Fr})$ *induces an isomorphism*

$$(T_{\phi^\ell,L}^{\varphi(I_F)})_{\varphi(\mathrm{Fr})} /\!\!/ (\Omega_{\phi^\ell}^\circ)^{W_F} \xrightarrow{\sim} \underline{H}^1_{\mathrm{Ad}_\varphi}(W_F/I_F, C_{\hat{G}_L}(\varphi_{|I_F})^\circ)$$

and subsequently an isomorphism

$$(T_{\phi^\ell,L}^{\varphi(I_F)})_{\varphi(\mathrm{Fr})} /\!\!/ (\Omega_{\phi^\ell})^{W_F} \xrightarrow{\sim} \left(\underline{H}^1_{\mathrm{Ad}_\varphi}(W_F/I_F, C_{\hat{G}_L}(\varphi_{|I_F})^\circ)\right)_{/\!\!/ \pi_0(\phi^\ell)_{\alpha^\ell}}.$$

Proof. (1)(a) we have $C_{\hat{G}}(\varphi_{|I_F})^\circ = ((C_{\hat{G}}(\phi^\ell)^{L\varphi(I_F)})^\circ = ((C_{\hat{G}}(\phi^\ell)^\circ)^{L\varphi(I_F)})^\circ$ and (1) of the previous proposition applied to $C_{\hat{G}}(\phi^\ell)^\circ$ implies $((C_{\hat{G}}(\phi^\ell)^\circ)^{L\varphi(I_F)})^\circ = (C_{\hat{G}}(\phi^\ell)^\circ)^{L\varphi(I_F)}$. For (1)(b), observe first that the fact that Ω_{ϕ^ℓ} is a split extension $\Omega_{\phi^\ell}^\circ \rtimes \pi_0(\phi^\ell)$ of $\pi_0(\phi^\ell)$ by $\Omega_{\phi^\ell}^\circ$ comes from the fact it contains the subgroup $N_{C_{\hat{G}}(\phi^\ell)}(T_{\phi^\ell}, B_{\phi^\ell})/T_{\phi^\ell} \simeq \pi_0(\phi^\ell)$. Since the action of W_F through Ad_φ on $C_{\hat{G}}(\phi^\ell)$ stabilizes T_{ϕ^ℓ} and B_{ϕ^ℓ}, the induced action on Ω_{ϕ^ℓ} preserves the semidirect product decomposition, hence in particular the $\varphi(I_F)$-invariants are given by $(\Omega_\phi)^{\varphi(I_F)} = (\Omega_\phi^\circ)^{\varphi(I_F)} \rtimes \pi_0(\phi^\ell)^{\varphi(I_F)}$. Moreover we have $H^1(I_F, T_{\phi^\ell}(L)) = 1$ since I_F acts on the uniquely ℓ-divisible abelian group $T_{\phi^\ell}(L)$ through a cyclic ℓ-group, therefore the map $N_{C_{\hat{G}}(\phi^\ell)}(T_{\phi^\ell}, B_{\phi^\ell})^{\varphi(I_F)} \longrightarrow \pi_0(\phi^\ell)^{\varphi(I_F)}$ is surjective, which shows that $\pi_0(C_{\hat{G}_L}(\varphi(I_F))) = \pi_0(\phi^\ell)^{\varphi(I_F)}$.

(2) follows from (2) of the last proposition except for the equivariance under the group $\pi_0(\phi^\ell)_{\alpha^\ell} = \pi_0(\phi^\ell)^{\varphi(W_F)}$ which is straightforward.

The first statement of (3) follows directly from (3) of the last proposition, and we infer the second statement from the equality $(\Omega_\phi)^{\varphi(W_F)} = (\Omega_\phi^\circ)^{\varphi(W_F)} \rtimes \pi_0(\phi^\ell)^{\varphi(W_F)}$, which we already explained above. ☐

Applying this corollary to $L = \bar{\mathbb{F}}_\ell$ we see that the special fiber of the Λ_e-scheme $\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e,\phi^\ell,\alpha^\ell}$ of subsection 4.2 is geometrically connected. This finishes the proof of Theorem 4.8.

Corollary 4.21. *There are natural bijections between the following sets :*

1. *The set of connected components of* $\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\Lambda_e}$
2. *The set of connected components of* $\underline{Z}^1(W_F^0/P_F^e, \hat{G})_{\bar{\mathbb{F}}_\ell} = \underline{Z}^1(W_F/P_F^e, \hat{G})_{\bar{\mathbb{F}}_\ell}$
3. *The set of pairs* $(\phi^\ell, [\alpha^\ell])$ *with* $\phi^\ell \in \Phi_e^{\ell,\mathrm{adm}}$ *and* $[\alpha^\ell]$ *a* $\pi_0(\phi^\ell)$*-conjugacy class in* $\Sigma(\phi^\ell)^{\mathrm{adm}}$.
4. *The set of* $\hat{G}(\bar{\mathbb{F}}_\ell)$*-conjugacy classes of admissible pairs* (ϕ^ℓ, α^ℓ) *where* $\phi^\ell \in Z^1(I_F^\ell/I_F^{\ell,e}, \hat{G}(\bar{\mathbb{F}}_\ell))$ *and* $\alpha^\ell \in \Sigma(\phi^\ell)$.
5. *The set of* $\hat{G}(\bar{\mathbb{F}}_\ell)$*-conjugacy classes of admissible pairs* (ϕ, α) *where* $\phi \in Z^1(I_F/I_F^e, \hat{G}(\bar{\mathbb{F}}_\ell))^{\mathrm{ss}}$ *is* ${}^LG$*-semisimple and* $\alpha \in \Sigma(\phi)$.
6. *The set of* $\hat{G}(\bar{\mathbb{F}}_\ell)$*-conjugacy classes of pairs* (ϕ, β) *where* $\phi \in Z^1(I_F/I_F^e, \hat{G}(\bar{\mathbb{F}}_\ell))^{\mathrm{ss}}$ *is* ${}^LG$*-semisimple and* $\beta \in \tilde{\pi}_0(\phi)$ *is the image of some element in* $C_{{}^LG}(\phi) \cap (\hat{G}(\bar{\mathbb{F}}_\ell) \rtimes \mathrm{Fr}) = \{\tilde{\beta} \in \hat{G}(\bar{\mathbb{F}}_\ell) \rtimes \mathrm{Fr}, \tilde{\beta}^L\phi(i)\tilde{\beta}^{-1} = {}^L\phi(\mathrm{Fr}\,.i.\,\mathrm{Fr}^{-1})\}$.
7. *The set of equivalence classes in* $Z^1(W_F/P_F^e, \hat{G}(\bar{\mathbb{F}}_\ell))$ *for the relation defined by* $\varphi \sim \varphi'$ *if and only if there is* $\hat{g} \in \hat{G}(\bar{\mathbb{F}}_\ell)$ *such that* $\varphi_{|I_F^\ell} = {}^{\hat{g}}\varphi'_{|I_F^\ell}$ *and* $\pi \circ \varphi = \pi \circ {}^{\hat{g}}\varphi'$ *with* π *the map* $C_{{}^LG}(\varphi_{|I_F^\ell}) \twoheadrightarrow \pi_0(C_{{}^LG}(\varphi_{|I_F^\ell}))$.

Moreover, one can replace $\bar{\mathbb{F}}_\ell$ *by any algebraically closed field* L *of characteristic* ℓ.

Proof. The bijections between (1), (2) and (3) follow from Theorem 4.8 which we have just proved. The bijection between (3) and (4) follows from the definitions, and so does the bijection between (4) and (7). We now describe bijections between (4), (5) and (6) in a circular way.

(4)$\rightarrow$(5). Start with an admissible pair, (ϕ^ℓ, α^ℓ). Choose an extension φ of ϕ^ℓ that preserves some chosen Borel pair of $C_{\hat{G}}(\phi^\ell)^\circ$. Then $\phi := \varphi_{|I_F}$ is certainly ${}^LG$-semisimple and $\alpha := \pi \circ \varphi$ is an element of $\Sigma(\phi)$ (here π is the projection $C_{{}^LG}(\phi) \longrightarrow \tilde{\pi}_0(\phi)$ as usual). We need to check that any other choice φ' leads to a conjugate of (ϕ, α). Since all Borel pairs are conjugate, we may assume that φ' and φ fix the same Borel pair, and denote it by $(B_{\phi^\ell}, T_{\phi^\ell})$. Then $\varphi' = \eta \cdot \varphi$ for some $\eta \in Z^1_{\mathrm{Ad}_\varphi}(W_F/I_F^\ell, T_{\phi^\ell})$. Since $H^1_{\mathrm{Ad}_\varphi}(I_F/I_F^\ell, T_{\phi^\ell}) = 0$ (because T_{ϕ^ℓ} is uniquely ℓ-divisible), we have $H^1_{\mathrm{Ad}_\varphi}(W_F/I_F^\ell, T_{\phi^\ell}) = H^1_{\mathrm{Ad}_\varphi}(W_F/I_F, (T_{\phi^\ell})^{I_F})$, which means that we can "Ad_φ-conjugate" η by an element $t \in T_{\phi^\ell}$ so that it factors through a cocycle in $Z^1_{\mathrm{Ad}_\varphi}(W_F/I_F, T_{\phi^\ell}^{I_F})$. So, after conjugating φ' by t, it has the form $\eta \cdot \varphi$ with $\eta \in Z^1_{\mathrm{Ad}_\varphi}(W_F/I_F, T_{\phi^\ell}^{I_F})$. We now certainly have $(\eta \cdot \varphi)_{|I_F} = \varphi_{|I_F}$ and, since $T_{\phi^\ell}^{I_F}$ is connected, we also have $\pi \circ (\eta \cdot \varphi) = \pi \circ \varphi$.

(5)$\rightarrow$(6). To a pair (ϕ, α) we associate (ϕ, β) with $\beta := \alpha(\mathrm{Fr})$.

(6)$\rightarrow$(4). Start with a pair (ϕ, β) and put $\phi^\ell := \phi_{|I_F^\ell}$. Choose a lift $\tilde{\beta}$ of β in $C_{{}^LG}(\phi) \cap (\hat{G}(\bar{\mathbb{F}}_\ell) \rtimes \mathrm{Fr})$. Then there is a unique extension φ of ϕ such that $\varphi(\mathrm{Fr}) = \tilde{\beta}$. This extension certainly factors through $C_{{}^LG}(\phi^\ell)$ and we put $\alpha^\ell := \pi^\ell \circ \varphi$ with $\pi^\ell : C_{{}^LG}(\phi^\ell) \longrightarrow \tilde{\pi}_0(\phi^\ell)$. Note that any other choice of lift of β is of the form $c\tilde{\beta}$ with $c \in C_{\hat{G}}(\phi)^\circ$. Since $C_{\hat{G}}(\phi)^\circ$ is contained in $C_{\hat{G}}(\phi^\ell)^\circ$, such a choice defines the same α^ℓ.

The composition of these three applications, starting from any set (4), (5) and (6) is easily seen to be the identity. $\square$

We finish this paragraph with another view on the topological description of the affine categorical quotient $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L) /\!\!/ \hat{G}_L$ that we have obtained above, which makes it strikingly similar to what we will get over fields of characteristic 0.

For a pair (ϕ, β) as in (6) of the last corollary and a Borel pair (B_ϕ, T_ϕ) of the reductive algebraic group $C_{\hat{G}_L}(\phi)^\circ$ we have an action of β on the torus T_ϕ and on its Weyl group $\Omega_\phi = \Omega_\phi^\circ \rtimes \pi_0(\phi)$ in $C_{\hat{G}}(\phi)$ (namely, the conjugation action of any lift $\tilde{\beta}$ of β in $C_{{}^LG}(\phi)$ that preserves (B_ϕ, T_ϕ)). Putting together the last corollary and the previous proposition, we get :

Corollary 4.22. *Let $\Psi_e(L)$ be a set of representatives of $\hat{G}_L$-conjugacy classes of pairs (ϕ, β) as in (6) of the last corollary. For each such pair, choose a lift $\tilde{\beta}$ of β in $C_{{}^LG}(\phi)$ that normalizes a Borel pair (B_ϕ, T_ϕ) of $C_{\hat{G}}(\phi)^\circ$, and denote by ${}^L\varphi_{\tilde{\beta}} : W_F \longrightarrow {}^LG(L)$ the corresponding extension of ${}^L\phi$. Then the collection of morphisms $\underline{Z}^1_{\mathrm{Ad}_\beta}(W_F/I_F, T_\phi) \longrightarrow \underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$, $\eta \mapsto \eta \cdot \varphi_{\tilde{\beta}}$ induce an homeomorphism*

$$\coprod_{(\phi,\beta)\in\Psi_e(L)} (T_\phi)_\beta /\!\!/ (\Omega_\phi)^\beta \xrightarrow{\approx} \underline{Z}^1(W_F^0/P_F^e, \hat{G}_L) /\!\!/ \hat{G}_L,$$

where $(T_\phi)_\beta$ denotes the β-coinvariants of T_ϕ (i.e. the cokernel of the morphism $T_\phi \longrightarrow T_\phi, t \mapsto t^{-1}\beta(t)$).

In Theorem 6.8 we will see that these homeomorphisms are actually isomorphisms when ℓ is "${}^LG$-banal".

4.6. **Connected components over $\mathbb{Z}[\frac{1}{p}]$.** In this subsection, we assume that the action of W_F on $\hat{G}$ is trivial on P_F and stabilizes a Borel pair $(\hat{B}, \hat{T})$, and we study the connectedness of the depth 0 scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$ considered in Section 2, and of all its base changes to finite flat integral extensions of $\mathbb{Z}[\frac{1}{p}]$. Our general strategy relies on what we already know about the connected components of the base change to $\overline{\mathbb{Z}}_\ell$ for all $\ell \neq p$. The following result implements this strategy under some additional hypothesis, that are fulfilled for example if the action of W_F on $\hat{G}$ is unramified. After proving it, we will show that this additional hypothesis is more generally satisfied when the action of I_F stabilizes a pinning.

Proposition 4.23. *Assume that there is a prime $\ell_0 \neq p$ such that, for each subgroup I of finite index of I_F, the $\mathbb{Z}[\frac{1}{p}]$-group scheme $\hat{G}^I$ has connected geometric fibers, and is smooth over $\mathbb{Z}[\frac{1}{\ell_0 p}]$. Then the $\mathbb{Z}[\frac{1}{p}]$-scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is connected, and so are all its base changes to finite flat integral extensions of $\mathbb{Z}[\frac{1}{p}]$.*

Proof. Let C be a connected component of $\underline{Z}^1(W_F^0/P_F, \hat{G})$. Since C is flat and of finite type over $\mathbb{Z}[\frac{1}{p}]$, we certainly have $C(\overline{\mathbb{Q}}) \neq \emptyset$. Let us consider the set $\mathcal{I}$ of open subgroups I of I_F that contain P_F and such that

$$C(\overline{\mathbb{Q}}) \cap Z^1(W_F/I, \hat{G}^I)(\overline{\mathbb{Q}}) \neq \emptyset.$$

By Proposition 4.17 (ii), the set $\mathcal{I}$ is not empty, so we may pick a maximal $I \in \mathcal{I}$.

We claim that I has ℓ_0-power index in I_F. Indeed, suppose the contrary, let $\ell \neq \ell_0$ be a prime that divides $[I_F : I]$ and let $I' \supset I$ be the unique subgroup of I_F such that I'/I is the ℓ-primary part of I_F/I. Since $I \in \mathcal{I}$, there is a connected component C_I of $\underline{Z}^1(W_F/I, \hat{G}^I)$ contained in C and such that $C_I(\overline{\mathbb{Q}}) \neq \emptyset$. Our hypothesis on $\hat{G}$ and [DM94, Thm 1.8] imply that $\hat{G}^I$ is reductive over $\mathbb{Z}[\frac{1}{\ell_0 p}]$, hence Lemma 4.24 (2) applies and ensures that $C_I(\overline{\mathbb{F}}_\ell)$ is not empty. Moreover, by looking fibrewise and using again [DM94, Thm 1.8], we see that the pair $(\hat{B}^I, \hat{T}^I)$ is a Borel pair of $\hat{G}^I$ over $\mathbb{Z}[\frac{1}{\ell_0 p}]$. Now, since I'/I has ℓ-power order, we can repeat the argument of the proof of (2) of Proposition 4.19 and deduce that the injective map $\underline{Z}^1(W_F/I', \hat{G}^{I'}(\bar{\mathbb{F}}_\ell))^{I_F-\mathrm{ss}} \hookrightarrow \underline{Z}^1(W_F/I, \hat{G}^I(\bar{\mathbb{F}}_\ell))^{I_F-\mathrm{ss}}$ induces a bijection between the respective sets of conjugacy classes, which in turn implies that the morphism $\underline{Z}^1(W_F/I', \hat{G}^{I'})_{\bar{\mathbb{F}}_\ell} /\!\!/ \hat{G}^{I'}_{\bar{\mathbb{F}}_\ell} \longrightarrow \underline{Z}^1(W_F/I, \hat{G}^I)_{\bar{\mathbb{F}}_\ell} /\!\!/ \hat{G}^I_{\bar{\mathbb{F}}_\ell}$ is a homeomorphism, and consequently that the morphism $\underline{Z}^1(W_F/I', \hat{G}^{I'})_{\bar{\mathbb{F}}_\ell} \hookrightarrow \underline{Z}^1(W_F/I, \hat{G}^I)_{\bar{\mathbb{F}}_\ell}$ induces a bijection on π_0. Therefore, there is a component $C_{I'}$ of $\underline{Z}^1(W_F/I', \hat{G}^{I'})$ that maps into C_I, hence also into C, and such that $C_{I'}(\overline{\mathbb{F}}_\ell) \neq \emptyset$. But since the index of I' in I_F is prime to ℓ, Lemma 4.24 (3) ensures that $C_{I'}(\overline{\mathbb{Q}}) \neq \emptyset$, which contradicts the maximality of I unless $I' = I$.

Now that we know that I has ℓ_0-power index in I_F, we shrink it so that it still has ℓ_0-power index in I_F and its image in $\mathrm{Aut}(\hat{G})$ has prime-to-ℓ_0 order. For this new I, Lemma A.1 ensures that the group scheme $\hat{G}^I$ is also smooth at ℓ_0, hence, by Lemma 4.24 (2) again, we have $C_I(\overline{\mathbb{F}}_{\ell_0}) \neq \emptyset$. But the map $\underline{Z}^1(W_F/I_F, \hat{G}^{I_F})_{\overline{\mathbb{F}}_{\ell_0}} \hookrightarrow \underline{Z}^1(W_F/I, \hat{G}^I)_{\overline{\mathbb{F}}_{\ell_0}}$ induces a bijection on π_0, by the same argument as above, hence C_I contains a component C_{I_F} of $\underline{Z}^1(W_F/I_F, \hat{G}^{I_F})$, and so does C. So we have shown that the closed immersion $\underline{Z}^1(W_F/I_F, \hat{G}^{I_F}) \hookrightarrow \underline{Z}^1(W_F^0/P_F, \hat{G})$ is surjective on π_0, and our statement follows from the fact that $\underline{Z}^1(W_F/I_F, \hat{G}^{I_F}) \simeq \hat{G}^{I_F}$ is connected, under our assumption. Moreover, the same argument works similarly after base change to any integral finite flat extension of R of $\mathbb{Z}[\frac{1}{p}]$ by reducing modulo prime ideals of R rather than prime numbers. $\square$

Lemma 4.24. *Let $\ell \neq p$ be a prime, and let $I \subset I_F$ be a subgroup of finite index that contains P_F and such that the group scheme $\hat{G}^I$ is reductive over $\mathbb{Z}_{(\ell)}$. Then, for any connected component C of $\underline{Z}^1(W_F/I, \hat{G}^I)$, we have :*

1. *If L is an algebraically closed field and a $\mathbb{Z}_{(\ell)}$-algebra with $C(L) \neq \emptyset$, then $C(L)$ contains a semisimple cocycle valued in $N_{\hat{G}^I}(\hat{T}^I)(L)$.*
2. *$C(\overline{\mathbb{Q}}) \neq \emptyset \Rightarrow C(\overline{\mathbb{F}}_\ell) \neq \emptyset$.*
3. *If ℓ does not divide the index $[I_F : I]$, then $C(\overline{\mathbb{F}}_\ell) \neq \emptyset \Rightarrow C(\overline{\mathbb{Q}}) \neq \emptyset$.*

Proof. We lighten the notation a bit by putting $\hat{H} := \hat{G}^I$, $B_{\hat{H}} := \hat{B}^I$ and $T_{\hat{H}} := \hat{T}^I$. The action of W_F/I on $\hat{H}$ stabilizes the Borel pair $(B_{\hat{H}}, T_{\hat{H}})$ and factors over some finite quotient W. We also put ${}^LH := \hat{H} \rtimes W$ and we still denote by s the image of s in W.

(1) If $C(L)$ is not empty, then $C(L)$ certainly contains a semisimple 1-cocycle $\varphi \in Z^1(W_F/I, \hat{H}(L))$. As usual, we denote by ${}^L\varphi$ the associated L-homomorphism $W_F/I \longrightarrow {}^LH(L)$. By [Ste68, 7.2] the element ${}^L\varphi(s)$ of ${}^LH(L)$ normalizes a Borel subgroup of $\hat{H}$. Since C is stable under conjugation by $\hat{H}$, we may conjugate φ so that ${}^L\varphi(s)$ normalizes $B_{\hat{H}}$. Then ${}^L\varphi(s)$ belongs to the R-Borel subgroup $B_{\hat{H}}(L) \rtimes W$ of ${}^LH(L)$. Since φ is semisimple, ${}^L\varphi(s)$ generates a completely reducible subgroup of ${}^LH(L)$, hence it belongs to a R-Levi subgroup of $B_{\hat{H}} \rtimes W$. Since all these R-Levi subgroups are $B_{\hat{H}}$-conjugate to $T_{\hat{H}} \rtimes W$, we may conjugate further φ so that ${}^L\varphi(s) \in T_{\hat{H}}(L) \rtimes s$. In this situation, $(T_{\hat{H}}(L)^{{}^L\varphi(s)})^\circ$ is a maximal torus of $(\hat{H}(L)^{{}^L\varphi(s)})^\circ$ whose centralizer in $\hat{H}(L)$ is $T_{\hat{H}}(L)$, [DM94, Thm 1.8 iv)]. Now, ${}^L\varphi(\mathrm{Fr})$ normalizes $(\hat{H}(L)^{{}^L\varphi(s)})^\circ = (\hat{H}(L)^{{}^L\varphi(s)^q})^\circ$, hence it conjugates $(T_{\hat{H}}(L)^{{}^L\varphi(s)})^\circ$ to another maximal torus therein. Pick $c \in (\hat{H}(L)^{{}^L\varphi(s)})^\circ$ that conjugates back this torus to $(T_{\hat{H}}(L)^{{}^L\varphi(s)})^\circ$. So $c.{}^L\varphi(\mathrm{Fr})$ normalizes $(T_{\hat{H}}(L)^{{}^L\varphi(s)})^\circ$, hence also its centralizer $T_{\hat{H}}(L)$ in $\hat{H}(L)$. Hence the unique 1-cocycle $\varphi^c : W_F/I \longrightarrow \hat{H}(L)$ such that

$$\begin{cases} {}^L(\varphi^c)(s) := {}^L\varphi(s) \in T_{\hat{H}}(L) \rtimes s \\ {}^L(\varphi^c)(\mathrm{Fr}) := c.{}^L\varphi(\mathrm{Fr}) \in N_{\hat{H}}(T_{\hat{H}})(L) \rtimes \mathrm{Fr}\,. \end{cases}$$

is valued in $N_{\hat{H}}(T_{\hat{H}})(L)$ as desired, and it remains to prove that φ^c belongs to $C(L)$. But the cocycle φ^c makes sense for any $c \in (\hat{H}(L)^{{}^L\varphi(s)})^\circ$, so it is an element in the image of an algebraic morphism $(\hat{H}(L)^{{}^L\varphi(s)})^\circ \longrightarrow \underline{Z}^1(W_F/I, {}^LH(L))$. Since the source of this morphism is connected, its image is contained in $C(L)$.

(2) Let us first prove that $C(\overline{\mathbb{Q}})$ contains a 1-cocycle φ such that ${}^L\varphi$ has finite image, which is here equivalent to ${}^L\varphi(\mathrm{Fr})$ having finite order. By (1), we may start with φ valued in $N_{\hat{H}}(T_{\hat{H}})(\overline{\mathbb{Q}})$. Then, a convenient power ${}^L\varphi(\mathrm{Fr})^r$ of ${}^L\varphi(\mathrm{Fr})$ belongs to $(T_{\hat{H}}(\overline{\mathbb{Q}})^{{}^L\varphi(s)})^\circ$. But the latter is a divisible group, so it contains an element t such that $t^{-r} = {}^L\varphi(\mathrm{Fr})^r$. Then, the cocycle φ^t defined as above has finite image. Now, we argue as in Proposition 2.10 with the building of $\hat{H}(\overline{\mathbb{Q}}_\ell)$ to see that φ^t can be $\hat{H}(\overline{\mathbb{Q}}_\ell)$-conjugated so that it becomes $\hat{H}(\overline{\mathbb{Z}}_\ell)$-valued. Then its image in $Z^1(W_F/I, \hat{H}(\overline{\mathbb{F}}_\ell))$ belongs to $C(\overline{\mathbb{F}}_\ell)$.

(3) By (1) we may start with $\varphi \in C(\overline{\mathbb{F}}_\ell)$ taking values in $N_{\hat{H}}(T_{\hat{H}})(\overline{\mathbb{F}}_\ell)$, and we will show that it can be lifted to a 1-cocycle $W_F/I \longrightarrow N_{\hat{H}}(T_{\hat{H}})(\overline{\mathbb{Z}}_\ell)$. As in the proof of Theorem 3.6, the obstruction to lifting φ belongs to $H^2(W_F/I, K)$ where K is the kernel of the reduction map $N_{\hat{H}}(T_{\hat{H}})(\overline{\mathbb{Z}}_\ell) \longrightarrow N_{\hat{H}}(T_{\hat{H}})(\overline{\mathbb{F}}_\ell)$, which is also the kernel of the reduction map $T_{\hat{H}}(\overline{\mathbb{Z}}_\ell) \longrightarrow T_{\hat{H}}(\overline{\mathbb{F}}_\ell)$. In particular, K is a uniquely ℓ'-divisible abelian group. Since I_F/I has prime to ℓ order, it follows that $H^1(I_F/I, K) = H^2(I_F/I, K) = \{0\}$. Since we also have $H^2(W_F/I_F, K^{I_F}) = \{0\}$, we see that $H^2(W_F/I, K) = 0$ and there is no obstruction to lift φ. □

Proposition 4.23 shows in particular that $\underline{Z}^1(W_F^0/P_F, \hat{G})$ is connected in the case where I_F acts trivially on $\hat{G}$. We will now show this connectedness property in the more general case where I_F preserves a pinning of $\hat{G}$. The next lemma starts with a particular subcase.

Lemma 4.25. *Assume that $\hat{G}$ is semi-simple and simply connected, and that I_F stabilizes a pinning of $\hat{G}$. Then, for any subgroup I of finite index of I_F, the closed subgroup scheme $\hat{G}^I$ has connected geometric fibers and is smooth over* $\mathrm{Spec}(\mathbb{Z}[\frac{1}{2p}])$.

Proof. The connectedness of geometric fibers is Steinberg's theorem in [Ste68, Thm 8.2], so we focus on the smoothness of $\hat{G}^I$. Consider the action of I on the set of simple factors of $\hat{G}$. By treating distinct I-orbits seperately, we may assume that I has a single orbit, so that $\hat{G} = \mathrm{ind}_{I'}^{I} \hat{G}'$ where $\hat{G}'$ is simple and I' has finite index in I. Then $\hat{G}^I = (\hat{G}')^{I'}$, so we are reduced to the case where $\hat{G}$ is simple. In this case, the image of I in $\mathrm{Aut}(\hat{G})$, which is cyclic since P_F acts trivially, has order either 2 or 3. If it has order 2, the smoothness of $\hat{G}^I$ over $\mathbb{Z}[\frac{1}{2p}]$ follows from Lemma A.1. If it has order 3 then $\hat{G} = \mathrm{Spin}_8$. Over fields of characteristic 0, it is known that the subgroup of Spin_8 fixed by the triality automorphism is G_2. It may be true that $\hat{G}^I = G_2$ in our context too, but we find it easier to argue as follows. The big cell $C = \hat{U}^-\hat{T}\hat{U}$ associated to the Borel pair $(\hat{T}, \hat{B})$ is stable under I, and it suffices to prove smoothness of $C^I = (\hat{U}^-)^I\hat{T}^I\hat{U}^I$. Since $\hat{G}$ is simply connected, $\hat{T}^I$ is a torus by Steinberg's theorem, hence it is smooth. By symmetry, it remains to prove smoothness of $\hat{U}^I$. Choose an ordering of the set Φ^+/I of I-orbits of positive roots and, for each orbit $\bar{\alpha} \in \Phi^+/I$, choose an ordering of this orbit. To these choices is associated a decomposition $\hat{U} = \prod_{\bar{\alpha}\in\Phi^+/I} \hat{U}_{\bar{\alpha}}$ with $\hat{U}_{\bar{\alpha}} = \prod_{\alpha\in\bar{\alpha}} \hat{U}_\alpha$. Now, the point here is that there is no pair of I-conjugate positive roots whose sum is again a root. This implies that $\hat{U}_\alpha$ and $\hat{U}_{\alpha'}$ commute with each other if $\alpha, \alpha' \in \bar{\alpha}$, and it follows that $\hat{U}^I = \prod_{\bar{\alpha}} \hat{U}_{\bar{\alpha}}^I$. This also implies that the I-invariant pinning $(X_\alpha)_{\alpha\in\Delta}$ (with X_α a basis of $\mathrm{Lie}(\hat{U}_\alpha)$) can be extended to an I-invariant pinning $(X_\alpha)_{\alpha\in\Phi^+}$ for all positive roots. Then, to each X_α corresponds an isomorphism $\mathbb{G}_a \xrightarrow{\sim} \hat{U}_\alpha$ and the product of these isomorphisms induces $(\mathbb{G}_a)_{\mathrm{diag}} \xrightarrow{\sim} \hat{U}_{\bar{\alpha}}^I$. Whence the smoothness of $\hat{U}_{\bar{\alpha}}^I$. □

Remark 4.26. (1) The same lemma holds with "adjoint" instead of "simply connected". The reference to Steinberg's result has to be replaced by a reference to [DM94, Remarque 1.30] for example (note that a pinning-preserving automorphism is quasi-central in the sense of [DM94]).

(2) If $\hat{G} = \mathrm{SL}_{2n+1}$ with a topological generator of I_F acting by the non-trivial automorphism that preserves the standard pinning, then $(\mathrm{SL}_{2n+1})^{I_F}$ is *not* smooth

over $\mathbb{Z}_{(2)}$. For example, with standard coordinates $x = x_{12}, y = x_{23}, z = x_{13}$ for the upper unipotent subgroup $\hat{U}$ of SL_3, the invariants $\hat{U}^I$ are given by equations $x = y$ and $xy = 2z$. However, it is likely that in any simple simply connected case not of type A_{2n}, the I-invariants are smooth over $\mathbb{Z}$. During the reviewing process of this paper, this expectation was indeed proved (and the above lemma reproved and generalized) in [ALRR22, Thm 1.1 (3)].

This lemma, together with Proposition 4.23, shows that $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]}$ is connected for $\hat{G}$ as in the lemma. In order to spread a bit this result, we will use the next two lemmas.

Lemma 4.27. *Assume given another split reductive group $\hat{G}'$ over $\mathbb{Z}[\frac{1}{p}]$ equipped with an action of W_F/P_F and with an equivariant surjective morphism $\hat{G}' \xrightarrow{f} \hat{G}$ whose kernel is a torus. Then the morphism $\underline{Z}^1(W_F^0/P_F, \hat{G}') \xrightarrow{f_*} \underline{Z}^1(W_F^0/P_F, \hat{G})$ induces a surjection on π_0.*

Proof. Put $\hat{S} := \ker f$. For any prime $\ell \neq p$ and $\varphi \in Z^1(W_F/P_F, \hat{G}(\bar{\mathbb{F}}_\ell))$, the obstruction to lifting φ to an element of $Z^1(W_F/P_F, \hat{G}'(\bar{\mathbb{F}}_\ell))$ lies in the group $H^2(W_F/P_F, \hat{S}(\bar{\mathbb{F}}_\ell))$, which vanishes by Lemma 3.8 since $\hat{S}(\bar{\mathbb{F}}_\ell)$ is a divisible group. Therefore the map

$$f_* : Z^1(W_F/P_F, \hat{G}'(\bar{\mathbb{F}}_\ell)) \longrightarrow Z^1(W_F/P_F, \hat{G}(\bar{\mathbb{F}}_\ell))$$

is surjective. Since any connected component of $\underline{Z}^1(W_F^0/P_F, \hat{G})$ has $\bar{\mathbb{F}}_\ell$-points for some ℓ (and even for all ℓ), this implies the lemma. $\square$

Lemma 4.28. *There exists a split reductive group $\hat{G}'$ over $\mathbb{Z}[\frac{1}{p}]$ equipped with an action of W_F/P_F and an equivariant surjective morphism $\hat{G}' \longrightarrow \hat{G}$ whose kernel is a torus, and such that, for all open subgroups I of I_F, we have :*

1. *$(\hat{G}')^I$ has geometrically connected fibers, and*
2. *$(\hat{G}')^I$ is smooth over $\mathbb{Z}[\frac{1}{2p}]$ if I_F preserves a pinning of $\hat{G}$.*

Proof. By Theorem 5.3.1 of [Con14] (or [MR070b, Exp XXII, §6.2]), there is a unique closed semi-simple subgroup scheme $\hat{G}_{\mathrm{der}}$ of $\hat{G}$ over $\mathbb{Z}[\frac{1}{p}]$ that represents the fppf sheafification of the set-theoretical derived subgroup presheaf. Further, by Exercise 6.5.2 of [Con14], there is a canonical central isogeny $\hat{G}_{\mathrm{sc}} \longrightarrow \hat{G}_{\mathrm{der}}$ over $\mathbb{Z}[\frac{1}{p}]$, such that all the geometric fibers of $\hat{G}_{\mathrm{sc}}$ are simply connected semi-simple groups. Being canonical, the action of W_F/P_F on $\hat{G}_{\mathrm{der}}$ lifts uniquely to $\hat{G}_{\mathrm{sc}}$ and still preserves a Borel pair or a pinning, depending on the case. Now denote by $R(\hat{G})$ the radical of $\hat{G}$, which is a split torus. Then the natural morphism $R(\hat{G}) \times \hat{G}_{\mathrm{sc}} \longrightarrow \hat{G}$ is a W_F-equivariant central isogeny. We already know that $(\hat{G}_{\mathrm{sc}})^I$ satisfies properties (1) and (2) for finite index any subgroup $I \subset I_F$, by Lemma 4.25. On the other hand, $R(\hat{G})^I$ is the diagonalisable group associated to the abelian group $X^*(R(\hat{G}))_I$ of I-coinvariants in $X^*(R(\hat{G}))$, which may have torsion. Let W be a finite quotient of W_F/P_F through which the action of W_F on $\hat{G}$ factors. Choosing a dual basis of the lattice $X^*(R(\hat{G}))$, we get a W_F-equivariant embedding $X^*(R(\hat{G})) \hookrightarrow \mathbb{Z}[W]^{\dim R(\hat{G})}$ where the target has torsion-free I-coinvariants for all I. Dually we get a surjective W_F-equivariant morphism of tori $\hat{S} \twoheadrightarrow R(G)$ such that $(\hat{S})^I$ is a torus, hence is smooth with connected geometric fibers. Thus we have a W_F-equivariant surjective morphism $\hat{G}'' := \hat{S} \times \hat{G}_{\mathrm{sc}} \twoheadrightarrow \hat{G}$ whose source satisfies both properties (1) and (2), but whose kernel D, a diagonisable subgroup, is not necessarily a torus.

Now let us choose a surjective W_F-equivariant morphism $X \longrightarrow X^*(D)$ such that X is a permutation module (i.e. W_F permutes a $\mathbb{Z}$-basis of X) and such that for every finite index subgroup $I \subset I_F$, the map on I-invariants $X^I \longrightarrow X^*(D)^I$ is surjective. For example, one can take $X = \bigoplus_I \mathbb{Z}[W/I] \otimes Y_I$ where I runs over subgroups of the image of I_F inside W and Y_I is any free abelian group mapping surjectively to $X^*(D)^I$. Dually, we have a W_F-equivariant embedding $D \hookrightarrow \hat{S}'$ of D into the split torus $\hat{S}'$ over $\mathbb{Z}[\frac{1}{p}]$ with character group $X^*(\hat{S}') = X$. Since its character group is a permutation module, the torus $\hat{S}'$ satisfies both properties (1) and (2). Namely, $(\hat{S}')^I$ is a torus for any finite index subgroup $I \subset I_F$. Now, by [MR070b, Exp. XXII §4.3] the quotient $\hat{G}' := \hat{S}' \times^D \hat{G}''$ is representable by a split reductive group scheme, which by construction is a W_F-equivariant extension of $\hat{G}$ by the torus $\hat{S}'$. Let us prove that $\hat{G}'$ has property (1). Fix a prime $\ell \neq p$ and look at the exact sequence

$$(\hat{S}'(\bar{\mathbb{F}}_\ell) \times \hat{G}''(\bar{\mathbb{F}}_\ell))^I \longrightarrow \hat{G}'(\bar{\mathbb{F}}_\ell)^I \longrightarrow H^1(I, D(\bar{\mathbb{F}}_\ell)) \longrightarrow H^1(I, \hat{S}'(\bar{\mathbb{F}}_\ell) \times \hat{G}''(\bar{\mathbb{F}}_\ell)).$$

Since I is procyclic, say with topological generator t, we have $H^1(I, D(\bar{\mathbb{F}}_\ell)) \simeq D(\bar{\mathbb{F}}_\ell)_I = (D/(\mathrm{id} - t)D)(\bar{\mathbb{F}}_\ell) = \mathrm{Hom}(X^*(D)^I, \bar{\mathbb{F}}_\ell^\times)$. Therefore, it follows that the map $H^1(I, D(\bar{\mathbb{F}}_\ell)) \longrightarrow H^1(I, \hat{S}'(\bar{\mathbb{F}}_\ell))$ identifies with the map

$$\mathrm{Hom}(X^*(D)^I, \bar{\mathbb{F}}_\ell^\times) \longrightarrow \mathrm{Hom}(X^*(\hat{S}')^I, \bar{\mathbb{F}}_\ell^\times),$$

hence it is injective since the map $X^*(S')^I \longrightarrow X^*(D)^I$ is surjective. It follows that the map

$$(\hat{S}'(\bar{\mathbb{F}}_\ell) \times \hat{G}''(\bar{\mathbb{F}}_\ell))^I \longrightarrow \hat{G}'(\bar{\mathbb{F}}_\ell)^I$$

is surjective. But the source of this map is a connected variety since $X^*(S')$ is a permutation module, so the target is also connected as desired. This proves the connectedness of the closed geometric fibers of $(\hat{G}')^I$, and that of the generic geometric fiber follows since $(\hat{G}')^I$ is smooth with reductive neutral component, hence étale component group, after restriction to a suitable open subset of $\mathrm{Spec}\mathbb{Z}[\frac{1}{p}]$. Let us now prove that $\hat{G}'$ has property (2). So we assume that I_F preserves a pinning of $\hat{G}$, which implies that it also preserves a pinning of $\hat{G}''$ and $\hat{G}'$, and we shall prove smoothness of $(\hat{G}')^I$ over $\mathbb{Z}[\frac{1}{2p}]$. Since the big cell $(\hat{U}'^{-}\hat{T}'\hat{U}')^I = (\hat{U}'^{-})^I(\hat{T}')^I(\hat{U}')^I$ is non-empty and $(\hat{G}')^I$ is connected, it suffices to prove its smoothness. We already know from Lemma 4.25 that $(\hat{U}'^{-})^I$ and $(\hat{U}')^I$ are smooth over $\mathbb{Z}[\frac{1}{2p}]$ so we may concentrate on the diagonalisable subgroup $(\hat{T}')^I$. But, by the same argument as for property (1) above, this diagonalisable group has geometrically connected fibers, so this is a torus (because the base is connected and has fibers of at least two distinct residual characteristics), and in particular it is smooth. □

Putting the last two lemmas and Proposition 4.23 together, we get the following result.

Theorem 4.29. *If I_F preserves a pinning of $\hat{G}$, the scheme $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]}$ is connected.*

Corollary 4.30. *If the center of $\hat{G}$ is smooth, then $\underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]}$ is connected.*

Proof. Indeed, in this case there is $\varphi \in \underline{Z}^1(W_F^0/P_F, \hat{G})(\overline{\mathbb{Z}}[\frac{1}{p}])$ such that Ad_φ preserves a pinning (see Remark 3.9), and right multiplication by φ provides an isomorphism $\underline{Z}^1_{\mathrm{Ad}_\varphi}(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]} \xrightarrow{\sim} \underline{Z}^1(W_F^0/P_F, \hat{G})_{\overline{\mathbb{Z}}[\frac{1}{p}]}$. □

5. Unobstructed points

In this section, we fix an algebraically closed field L of characteristic $\ell \neq p$. From 5.2 on, we will further assume that ℓ is finite.

5.1. **Deformation Theory.** Here, for an L-point x of $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$, we are interested in the tangent space $T_x\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ and, in particular, we wish to compute its dimension. We will need the L-linear continuous representation $\mathrm{Ad}\,\varphi_x$ of W_F^0 on the Lie algebra $\mathrm{Lie}(\hat{G}_L)$ obtained by composing ${}^L\varphi_x$ with the adjoint representation of ${}^LG$.

Recall that an element of $T_x\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is given by a map $\tilde{x} : \mathrm{Spec}\, L[\epsilon]/\epsilon^2 \to \underline{Z}^1(W_F^0/P_F^e, \hat{G})$ whose composition with the natural map $\mathrm{Spec}\, L \to \mathrm{Spec}\, L[\epsilon]/\epsilon^2$ is equal to x. In particular, the zero element $\tilde{x}_0$ of $T_x\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is given by the composition of x with the natural map $\mathrm{Spec}\, L[\epsilon]/\epsilon^2 \to \mathrm{Spec}\, L$. Given such a $\tilde{x}$ we form a cocycle for $\mathrm{Ad}\,\varphi_x$ as follows: for each $w \in W_F^0$, the element ${}^L\varphi_{\tilde{x}}(w)\,{}^L\varphi_{\tilde{x}_0}(w)^{-1}$ is a tangent vector to $\hat{G}$ at the identity element of $\hat{G}(L)$; that is, an element of $\mathrm{Lie}(\hat{G}_L)$. In this way one obtains a continuous 1-cocycle $W_F^0 \longrightarrow \mathrm{Lie}(\hat{G}_L)$ that lives in $Z^1(W_F^0, \mathrm{Ad}\,\varphi_x)$, and this sets up an isomorphism

$$T_x\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L) \simeq Z^1(W_F^0, \mathrm{Ad}\,\varphi_x). \tag{5.1}$$

To compute the dimension of this tangent space, we use the following familiar-looking cohomological lemma.

Lemma 5.1. *For any finite dimensional L-vector space V with a continuous linear action of W_F^0, we have :*

1. $H^i(W_F^0, V) = 0$ *for* $i > 2$,
2. $\dim H^2(W_F^0, V) - \dim H^1(W_F^0, V) + \dim H^0(W_F^0, V) = 0$
3. $H^2(W_F^0, V)^* \simeq H^0(W_F^0, V^* \otimes \omega)$ *where ω is the cyclotomic character of W_F and * denotes the L-linear dual.*

Proof. The open compact subgroup P_F of W_F^0 is a pro-p group, hence the functor of P_F-invariants on continuous L-representations is exact and commutes with taking L-linear duals. Hence it suffices to prove (1), (2) and (3) for L-representations of the discrete group $W^0 := W_F^0/P_F = \langle s, \mathrm{Fr}\rangle$. To this aim, observe that the equality

$$(1 - s^q)(1 - \mathrm{Fr}) = (t_q - \mathrm{Fr})(1 - s) \text{ with } t_q = 1 + s + \cdots + s^{q-1}$$

in $L[W^0]$, enables us to define the following complex :

$$0 \longrightarrow L[W^0] \xrightarrow{\delta} L[W^0] \oplus L[W^0] \xrightarrow{\gamma} L[W^0] \xrightarrow{\varepsilon} L \longrightarrow 0,$$

$$\text{where} \begin{cases} \varepsilon \text{ is the augmentation map,} \\ \gamma(f, g) = f(1 - \mathrm{Fr}) - g(1 - s) \\ \delta(h) = (h(1 - s^q), h(t_q - \mathrm{Fr})) \end{cases}$$

We claim that this complex is exact. Admitting this for now, this gives us a projective resolution of the trivial representation, and shows that $H^*(W^0, V)$ is the cohomology of a complex of the form $V \longrightarrow V^{\oplus 2} \longrightarrow V$. This implies (1) and (2). Moreover, this shows that $H^2(W^0, V) = V/((1 - s^q)V + (t_q - \mathrm{Fr})V)$. Observe that the inclusion $(1-s^q)V \subset (1-s)V$ has to be an equality for dimension reasons, since the action of Fr induces an isomorphism $(1-s)V \xrightarrow{\sim} (1-s^q)V$. Similarly, we have $(1-s^q)V = (1-s^{q^{-r}})V$ for all $r \in \mathbb{N}$. Denoting $I^0 := s^{\mathbb{Z}[\frac{1}{q}]}$, this means that the canonical map $V/(1-s^q)V \longrightarrow V_{I^0} = \mathrm{colim}_r V/(1-s^{q^{-r}})V$ is an isomorphism. Since t_q acts as multiplication by q on V_{I^0} this induces in turn an isomorphism

$$H^2(W^0, V) \xrightarrow{\sim} V_{I^0}/(q-\mathrm{Fr})V_{I^0} = (V \otimes \omega^{-1})_{W^0},$$

from which we deduce (3).

Let us now prove the exactness of the above complex. Note first that δ is injective since multiplication by $1-s^q$ is injective, and ε is clearly surjective. To see that $\ker\varepsilon = \operatorname{im}\gamma$, it suffices to see that $1-w \in L[W^0](1-\mathrm{Fr}) + L[W^0](1-s)$ for all $w \in W^0$, which follows from the fact that s and Fr generate W^0. It remains to check that $\ker\gamma = \operatorname{im}\delta$. So let $(f,g) \in \ker\gamma$. If we can prove that f has the form $f = h(1-s^q)$, then $g(1-s) = f(1-\mathrm{Fr}) = h(t_q - \mathrm{Fr})(1-s)$, hence $g = h(t_q - \mathrm{Fr})$ since $1-s$ is not a zero divisor, and $(f,g) \in \operatorname{im}\delta$. Writing $f = \sum_{i,j} a_{i,j}\,\mathrm{Fr}^i s^j$ with $i \in \mathbb{Z}$ and $j \in \mathbb{Z}[\frac{1}{q}]$, it thus suffices to prove that $\sum_{k\in\mathbb{Z}} a_{i,j+qk} = 0$ for all (i,j). In the expansion $f(1-\mathrm{Fr}) = \sum_{i,j} b_{i,j}\,\mathrm{Fr}^i s^j$, we have $b_{i,j} = a_{i,j} - a_{i-1,qj}$. The fact that $f(1-\mathrm{Fr}) \in L[W^0](1-s)$ translates into $\sum_{k\in\mathbb{Z}} b_{i,j+k} = 0$ for all i,j, that is $\sum_{k\in\mathbb{Z}} a_{i,j+k} = \sum_{k\in\mathbb{Z}} a_{i-1,qj+qk}$ for all i,j, which we can rewrite as

$$\forall i,j,\ \sum_{k\in\mathbb{Z}} a_{i,j+qk} = \sum_{k\in\mathbb{Z}} a_{i+1,j/q+k}.$$

Writing $k = r + qk'$ in the right hand sum, we get

$$\sum_{k\in\mathbb{Z}} a_{i+1,j/q+k} = \sum_{r=0}^{q-1}\sum_{k'\in\mathbb{Z}} a_{i+1,j/q+r+qk'} = \sum_{r=0}^{q-1}\sum_{k'\in\mathbb{Z}} a_{i+2,j/q^2+r/q+k'} = \sum_{k\in\mathbb{Z}} a_{i+2,j/q^2+k/q}.$$

Proceeding by induction, we get for any $s \in \mathbb{N}$:

$$\forall i,j,\ \sum_{k\in\mathbb{Z}} a_{i,j+qk} = \sum_{k\in\mathbb{Z}} a_{i+s,(j+qk)/q^s}.$$

But for $s >> 0$, the right hand side vanishes, hence so does the left hand side. This implies that $\ker\gamma = \operatorname{im}\delta$ and completes the proof that the complex above is exact. $\square$

From this lemma and (5.1), we get :

Proposition 5.2. *For an L-valued point $x : \operatorname{Spec} L \to \underline{Z}^1(W_F^0/P_F^e, \hat{G})$, the dimension of $T_x\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ over L is equal to $\dim \hat{G}_L + \dim H^0(W_F^0, (\operatorname{Ad}\varphi_x)^* \otimes \omega)$, where ω is the cyclotomic character of W_F.*

Proof. We have seen that the dimension of $T_x\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is equal to that of $Z^1(W_F^0, \operatorname{Ad}\varphi_x)$. The latter is equal to the dimension of $H^1(W_F^0, \operatorname{Ad}\varphi_x)$ plus the dimension of the space of coboundaries (principal crossed homomorphisms). These are all of the form $w \mapsto wy - y$, where y is an element of $\operatorname{Ad}\varphi_x$. The dimension of $\operatorname{Ad}\varphi_x$ is equal to $\dim \hat{G}_L$, and those y that give the zero element of $Z^1(W_F^0, \operatorname{Ad}\varphi_x)$ are precisely those fixed by W_F^0. Thus we have

$$\dim Z^1(W_F^0, \operatorname{Ad}\varphi_x) = \dim H^1(W_F^0, \operatorname{Ad}\varphi_x) + \dim \hat{G}_L - \dim H^0(W_F^0, \operatorname{Ad}\varphi_x).$$

Hence the proposition follows from the last lemma applied with $V = \operatorname{Ad}\varphi_x$. $\square$

Corollary 5.3. *The point x is a smooth point of $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ if and only if $H^0(W_F^0, (\mathrm{Ad}\,\varphi_x)^* \otimes \omega) = 0$.*

Proof. We know that the algebraic L-scheme $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ has pure dimension $\dim \hat{G}_L$. Therefore the local ring at the closed point x has dimension $\dim \hat{G}_L$, while its tangent space has dimension $\dim \hat{G}_L + \dim H^0(W_F^0, (\mathrm{Ad}\,\varphi_x)^* \otimes \omega)$ by the last proposition. $\square$

Remark 5.4. It is interesting to note that the obstruction theory naturally suggested by the moduli problem is "optimal", in that it faithfully detects smoothness of points. Namely, let A be a finite length local L-algebra with residue field L, $\tilde{x} : \mathrm{Spec}\, A \to \underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ a map whose composition with the map $\mathrm{Spec}\, L \to \mathrm{Spec}\, A$ is equal to x, and let A' be a small extension of A; that is, a finite length local L-algebra with residue field L, and a principal ideal $I \subseteq A'$ such that I is annihilated by the maximal ideal of A', and an isomorphism $A'/I \cong A$. The problem of lifting $\tilde{x}$ to a map $\tilde{x}' : \mathrm{Spec}\, A' \to \underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is equivalent to lifting the 1-cocycle $\varphi_{\tilde{x}} : W_F^0/P_F^e \longrightarrow \hat{G}(A)$ to a 1-cocycle $\varphi_{\tilde{x}'} : W_F^0/P_F^e \longrightarrow \hat{G}(A')$. This problem is standard: let φ' be any lift of $\varphi_{\tilde{x}}$ to a continuous *function* (not necessarily a cocycle): $W_F^0 \to \hat{G}(A')$. Then the map taking $w_1, w_2 \in W_F^0$ to ${}^L\varphi'(w_1w_2)^{-1}\,{}^L\varphi'(w_1)\,{}^L\varphi'(w_2)$ is a 2-cocycle with values in $(\mathrm{Ad}\,\varphi_x) \otimes I$, and we can adjust our choice of φ' to yield a 1-cocycle $\varphi_{\tilde{x}'}$ lifting $\varphi_{\tilde{x}}$ if, and only if, this 2-cocycle is a coboundary. We thus obtain an obstruction theory for $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ in a formal neighborhood of x with values in $H^2(W_F^0, \mathrm{Ad}\,\varphi_x) = H^0(W_F^0, (\mathrm{Ad}\,\varphi_x)^* \otimes \omega)$. Now, since an unobstructed point is smooth, the last corollary says that the obstruction to lifting vanishes if and only if the space which it naturally belongs to vanishes. *For this reason, we will indifferently use the words "unobstructed" or "smooth" to denote these points in the rest of this section.*

5.2. **Existence of unobstructed points.** The primary goal of this section is to show that if the characteristic ℓ of L does not lie in an explicit finite set (depending only on $\hat{G}$ and its W_F-action), the fiber $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is generically smooth. We have already established this smoothness in characteristic zero, *so we assume henceforth that L has finite characteristic ℓ.* In this case, the restriction map $\underline{Z}^1(W_F/P_F^e, \hat{G}_L) \xrightarrow{\sim} \underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is an isomorphism, so there is no need to distinguish between W_F^0 and W_F.

Notation 5.5. Let $\varphi : W_F \to \hat{G}(L)$ be a continuous 1-cocycle. For any $g \in C_{\hat{G}}(\varphi_{|I_F})(L)$ there is a unique continuous 1-cocycle φ^g, whose restriction to I_F is $\varphi_{|I_F}$, and such that $\varphi^g(\mathrm{Fr}) = g\varphi(\mathrm{Fr})$.

The rough version of our main result here is :

Theorem 5.6. *There is a finite set of primes S, depending only on $\hat{G}$ and the image of W_F in $\mathrm{Out}(\hat{G})$, such that, if $\ell \notin S$, then for any continuous 1-cocycle $\varphi : W_F \to \hat{G}(L)$, there exists a $g \in C_{\hat{G}}(\varphi_{|I_F})^\circ(L)$ such that φ^g is unobstructed.*

In order to state a more precise version, we need notations (B.2) and (B.3) of the appendix. In particular, $h_{\hat{G},1}$ is the Coxeter number of the root system of $\hat{G}$.

Theorem 5.7. *Let Fr be a lift of Frobenius in W_F, and denote by e the tame ramification index of the finite extension of F whose Weil group is the kernel of $W_F \longrightarrow \mathrm{Out}(\hat{G})$. Then the set S in Theorem 5.6 can be taken as*

(1) $S = \{$*primes* ℓ *dividing* $e.\chi^*_{\hat{G},\mathrm{Fr}}(q)\}$*, whatever* $\hat{G}$ *is.*

(2) $S = \{$*primes* ℓ *dividing* $e.\chi_{\hat{G},\mathrm{Fr}}(q).(h_{\hat{G},1})!\}$ *if* $\hat{G}$ *has no exceptional factor.*

Here, "exceptional" includes triality forms of D_4. Note also that ℓ not dividing $\chi^*_{\hat{G},\mathrm{Fr}}(q)$ is equivalent to q having order greater than $h_{\hat{G},\mathrm{Fr}}$ in $\mathbb{F}_\ell^\times$, which implies $\ell > h_{\hat{G},\mathrm{Fr}}$ hence also $\ell > h_{\hat{G},1}$. We will also prove that, in the case where $\hat{G}$ has no exceptional factor and the action of W_F is unramified and $\ell > h_{\hat{G},1}$, the condition $\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in $\mathbb{F}_\ell$ is also necessary to have generic smoothness.

We now start the proofs of Theorems 5.6 and 5.7. Fix a φ as in Theorem 5.6; our first step will be to reduce to a setting in which the action of W_F on $\hat{G}$ is unramified and stabilizes a pinning, and the image of $\varphi_{|I_F}$ is unipotent.

Denote by ϕ^ℓ the restriction of φ to I_F^ℓ and by α^ℓ the composition $W_F \xrightarrow{\varphi} C_{{}^LG}(\phi^\ell) \longrightarrow \tilde{\pi}_0(\phi^\ell)$, so that φ lies in the closed subscheme $\underline{Z}^1(W_F/P_F^e, \hat{G}_L)_{\phi^\ell,\alpha^\ell}$, as defined in subsection 4.2. Then, according to Theorem 4.8, the connected component of $\underline{Z}^1(W_F/P_F^e, \hat{G}_L)$ that contains φ has the form

$$\hat{G} \times^{C_{\hat{G}}(\phi^\ell)^\circ} \underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)_{\phi^\ell,\alpha^\ell}.$$

Thus we see that φ is a smooth point of $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ if and only if it is a smooth point of $\underline{Z}^1(W_F/P_F^e, \hat{G}_L)_{\phi^\ell,\alpha^\ell}$.

By Theorem 4.6, there exists $\varphi' \in Z^1(W_F/P_F^e, \hat{G}(L))_{\phi^\ell,\alpha^\ell}$ such that the action of W_F on $C_{\hat{G}}(\phi^\ell)^\circ$ via $\mathrm{Ad}_{\varphi'}$ preserves a Borel pair. Actually, we a have better result in this setting :

Proposition 5.8. *We can choose* $\varphi' \in Z^1(W_F/P_F^e, \hat{G}(L))_{\phi^\ell,\alpha^\ell}$ *so that the action of* W_F *on* $C_{\hat{G}}(\phi^\ell)^\circ$ *via* $\mathrm{Ad}_{\varphi'}$ *preserves a pinning.*

Proof. We take up the proof of Proposition 3.7, replacing P_F by I_F^ℓ and "Borel pair" by "pinning". Since the fixator of a pinning of $C_{\hat{G}}(\phi^\ell)^\circ$ under conjugation is the center Z of $C_{\hat{G}}(\phi^\ell)^\circ$, the argument of that proof shows that the obstruction to the existence of a cocycle φ' as in this proposition lies in the group $H^2(W_F/I_F^\ell, Z(L))$. On the other hand, repeating the proof of Lemma 3.8 for W_F/I_F^ℓ instead of W_F/P_F shows that this cohomology group vanishes if $Z(L)$ can be proved to be ℓ-divisible.

To prove this, recall that the group scheme Z is diagonalisable and let M be its character group. Then we have a (non canonical) decomposition $M \simeq M_{\ell-\mathrm{tors}} \times M_{\ell'-\mathrm{tors}} \times M_{\mathrm{free}}$ which induces a decomposition $Z \simeq Z_\ell \times Z^\ell \times T_Z$, where T_Z is a torus, Z^ℓ is finite smooth and Z_ℓ is finite infinitesimal. Correspondingly we get $Z(L) \simeq Z^\ell(L) \times T_Z(L)$. Now, $T_Z(L)$ is clearly ℓ-divisible since L is algebraically closed and $Z^\ell(L)$ has prime-to-ℓ order hence is also ℓ-divisible. □

Choose φ' as in this proposition and recall that the action $\mathrm{Ad}_{\varphi'}$ on $C_{\hat{G}}(\phi^\ell)$ factors over the quotient W_F/I_F^ℓ. Then we have an isomorphism $\eta \mapsto \eta \cdot \varphi'$

$$\underline{Z}^1_{\mathrm{Ad}_{\varphi'}}(W_F/I_F^\ell, C_{\hat{G}}(\phi^\ell)^\circ) \xrightarrow{\sim} \underline{Z}^1(W_F/P_F^e, \hat{G}_L)_{\phi^\ell,\alpha^\ell},$$

The isomorphism above shows that φ is an unobstructed point of $\underline{Z}^1(W_F/P_F^e, \hat{G}_L)$ if, and only if, $\varphi \cdot \varphi'^{-1}$ is an unobstructed point of $\underline{Z}^1_{\mathrm{Ad}_{\varphi'}}(W_F/I_F^\ell, C_{\hat{G}}(\phi^\ell)^\circ)$. So we are reduced to study unobstructedness in a much simpler case, but in order to make this reduction step effective, we need some control on $\mathrm{Ad}_{\varphi'}$.

Lemma 5.9. *Fix φ' as in Proposition 5.8, let $w \in W_F$ and denote by o_w its order in $\mathrm{Out}(\hat{G})$. Then $\mathrm{Ad}_{\varphi'}(w)$ has order dividing $o_w|\Omega_{\hat{G}}|$ in $\mathrm{Aut}(C_{\hat{G}}(\phi^\ell)^\circ)$, where $\Omega_{\hat{G}}$ denotes the Weyl group of $\hat{G}$.*

Proof. Put $\hat{H} := C_{\hat{G}}(\phi^\ell)^\circ$ and let $T_{\hat{H}}$ be a maximal torus of $\hat{H}$ that is part of a pinning stable under $\mathrm{Ad}_{\varphi'}$. Pick a maximal torus $\hat{T}$ of $\hat{G}$ that contains $T_{\hat{H}}$. Then there is an element m of the centralizer of $T_{\hat{H}}$ in $\hat{G}$ such that $g_w := m^L\varphi'(w)$ normalizes $\hat{T}$. The action of $(g_w)^{o_w}$ on $X^*(\hat{T})$ is the action of an element of $\hat{G}$ that normalizes $\hat{T}$, so its order divides $|\Omega_{\hat{G}}|$. Hence, $\mathrm{Ad}_{\varphi'}(w)^{o_w|\Omega_{\hat{G}}|}$ acts trivially on $T_{\hat{H}}$ and therefore also on $\hat{H}$, since it stabilizes a pinning and fixes the maximal torus of this pinning. $\square$

As in Theorem 5.7, denote by e the tame ramification index of the finite extension of F whose Weil group is the kernel of $W_F \longrightarrow \mathrm{Out}(\hat{G})$. Applying this lemma to a suitable lift of our generator s of tame inertia, we see that, *if we assume that ℓ is prime to $e|\Omega_{\hat{G}}|$* (which is satisfied if ℓ is prime to e and $\ell > h_{\hat{G},1}$), then the action $\mathrm{Ad}_{\varphi'}$ is *unramified*. For this reason, we will now focus on the following particular setting :

$$(5.2)\qquad \begin{cases} \text{– the action of } W_F \text{ on } \hat{G} \text{ is unramified and stabilizes a pinning,} \\ \text{– the restriction of } \varphi \text{ to } I_F^\ell \text{ is trivial.} \end{cases}$$

In this setting, ${}^LG$ is the Langlands dual group of a uniquely determined quasi-split unramified reductive group G over F, and the restriction $\varphi_{|I_F}$ is determined by $u := \varphi(s)$ which is a unipotent element of $\hat{G}(L)$. By definition, the cocycle φ corresponds to an unobstructed point if, and only if, q^{-1} is not an eigenvalue of $(\mathrm{Ad}_\varphi)^*(\mathrm{Fr})$ on $(\mathrm{Lie}\,\hat{G}_L)^{*,\mathrm{Ad}^* u}$.

Lemma 5.10. *Let $\hat{G}$ and φ be as in (5.2), put $u := \varphi(s)$, and assume that ℓ is prime to $|\pi_1(\hat{G}_{\mathrm{ad}})|$. Endow the reductive group $\hat{G}' := \hat{G}_{\mathrm{ad}} \times \hat{G}_{\mathrm{ab}}$ with the unique action of W_F that makes the isogeny $\pi : \hat{G} \longrightarrow \hat{G}'$ equivariant, and put $\varphi' := \pi \circ \varphi$. Then there is $g \in C_{\hat{G}}(u)^\circ(L)$ such that φ^g is unobstructed if, and only if, there is $g' \in C_{\hat{G}'}(\pi(u))^\circ(L)$ such that $\varphi'^{g'}$ is unobstructed.*

Proof. As with any isogeny, π induces an isomorphism from the unipotent subvariety of $\hat{G}$ to that of $\hat{G}'$. In particular, the map $C_{\hat{G}}(u)(L) \xrightarrow{\pi} C_{\hat{G}'}(\pi(u))(L)$ is surjective for all unipotent $u \in \hat{G}(L)$, and so is the map $C_{\hat{G}}(u)^\circ(L) \xrightarrow{\pi} C_{\hat{G}'}(\pi(u))^\circ(L)$. Therefore, it suffices to prove that φ is unobstructed if and only if φ' is unobstructed. Note that $\ker\pi = \ker(\hat{G}_{\mathrm{der}} \longrightarrow \hat{G}_{\mathrm{ad}})$ is a finite diagonalisable group scheme whose order divides the order of $\pi_1(\hat{G}_{\mathrm{ad}})$, hence is prime to ℓ by our assumption. So π is a separable isogeny and $d\pi_L$ induces a $(\mathrm{Ad}\,\varphi, \mathrm{Ad}\,\varphi')$-equivariant isomorphism $\mathrm{Lie}(\hat{G}_L) \xrightarrow{\sim} \mathrm{Lie}(\hat{G}'_L)$, and the desired property follows. $\square$

Remark 5.11. Let $\pi : \hat{G} \longrightarrow \hat{G}'$ be as in the lemma or, more generally, any surjective morphism with central Fr-stable kernel. Then any $\varphi' \in Z^1(W_F/I_F^\ell, \hat{G}')$ lifts through π, in the sense that there is some $\varphi \in Z^1(W_F/I_F^\ell, \hat{G})$ such that $\varphi' = \pi \circ \varphi$. Indeed, let $\varphi(s) \in \hat{G}(L)$ be the unique unipotent element above $\varphi'(s)$, and let $\varphi(\mathrm{Fr}) \in \hat{G}(L)$ be any lift of $\varphi'(\mathrm{Fr})$. Then $(\varphi(\mathrm{Fr}) \rtimes \mathrm{Fr})\varphi(s)(\varphi(\mathrm{Fr}) \rtimes \mathrm{Fr})^{-1}$ is unipotent and above $\varphi'(s)^q$, so it is equal to $\varphi(s)^q$.

This lemma allows us to further reduce the setting (5.2) to the cases where $\hat{G}$ is a torus or an adjoint group. Dealing with tori is quite easy :

Lemma 5.12. *If $\hat{G}$ as in (5.2) is a torus, then the following are equivalent :*

1. *there is an unobstructed φ in $\underline{Z}^1(W_F/I_F^\ell, \hat{G}_L)$,*
2. *any φ in $\underline{Z}^1(W_F/I_F^\ell, \hat{G}_L)$ is unobstructed,*
3. *$\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L.*

Proof. For any φ in $Z^1(W_F/I_F^\ell, \hat{G}(L))$, we have $\varphi(s) = 1$ and ${}^L\varphi(\mathrm{Fr}) \in \hat{G}(L) \rtimes \mathrm{Fr}$, so the condition for φ to be unobstructed is that $H^0(\langle \mathrm{Fr}\rangle, \omega \otimes \mathrm{Lie}(\hat{G}_L)^*) = 0$, which is independent of φ, and equivalent to q^{-1} not being an eigenvalue of $(\mathrm{Ad}_{\mathrm{Fr}})^*$ on $\mathrm{Lie}(\hat{G}_L)^*$. Since $\mathrm{Lie}(\hat{G}_L)^* = X^*(\hat{G}) \otimes L$, we have

$$\det\left(q(\mathrm{Ad}_{\mathrm{Fr}})^* - \mathrm{id} \,|\, \mathrm{Lie}(\hat{G}_L)^*\right) = \det\left(q\,\mathrm{Fr} - \mathrm{id}\,|X^*(\hat{G})\right)_L = \chi_{\hat{G},\mathrm{Fr}}(q^{-1})_L$$

where the subscripts L denote the image of an integer in L. Hence we see that q^{-1} is an eigenvalue of $\mathrm{Ad}_{\mathrm{Fr}}$ if and only if $\chi_{\hat{G},\mathrm{Fr}}(q^{-1}) = 0$ in L, which is equivalent to $\chi_{\hat{G},\mathrm{Fr}}(q) = 0$ in L since $\chi_{\hat{G},\mathrm{Fr}}$ is a product of cyclotomic polynomials. □

Let us now deal with the adjoint part. We have a Fr-equivariant decomposition as a product of simple adjoint groups

$$\hat{G}_{\mathrm{ad}} = \underbrace{\hat{G}_{11} \times \cdots \times \hat{G}_{1f_1}}_{\hat{G}_1} \times \cdots \times \underbrace{\hat{G}_{r1} \times \cdots \times \hat{G}_{rf_r}}_{\hat{G}_r} \tag{5.3}$$

where Fr permutes cyclically $\hat{G}_{i1} \to \hat{G}_{i2} \to \cdots \to \hat{G}_{if_i}$ and Fr^{f_i} restricts to an outer automorphism of $\hat{G}_{i1}$. Accordingly, φ decomposes as a product $\varphi_1 \times \cdots \times \varphi_r$ with $\varphi_i \in Z^1(W_F, \hat{G}_i(L))$ and we see that φ is unobstructed if and only if each φ_i is unobstructed. Denote by F_{f_i} the unramified extension of degree f_i of F. Then we have a Shapiro morphism $\underline{Z}^1(W_F, \hat{G}_i) \longrightarrow \underline{Z}^1(W_{F_{f_i}}, \hat{G}_{i1})$ given by $\varphi_i \mapsto \varphi_i' := \pi_{i1} \circ (\varphi_i)_{|W_{F_{f_i}}}$, where π_{i1} is the projection onto $\hat{G}_{i1}$.

Lemma 5.13. *The Shapiro morphism $\underline{Z}^1(W_F, \hat{G}_i) \longrightarrow \underline{Z}^1(W_{F_{f_i}}, \hat{G}_{i1})$ is smooth.*

Proof. Denote by $\varphi_{ij} := \pi_{ij} \circ \varphi_i$ the j-th component of φ_i and define $\underline{Z}_{i1}$ to be the affine scheme whose R-points are given by

$$\underline{Z}_{1i}(R) := \left\{ f : W_F \longrightarrow \hat{G}_{i1}(R), \forall w' \in W_{F_{f_i}}, f(w'w) = f(w') \cdot {}^{w'}f(w) \right\}$$

for any $\mathbb{Z}[\frac{1}{p}]$-algebra R. We claim that the map $\varphi_i \mapsto \varphi_{i1}$ induces an isomorphism $\underline{Z}^1(W_F, \hat{G}_i) \xrightarrow{\sim} \underline{Z}_{i1}$. Indeed, denoting by $\tilde{\mathrm{Fr}}$ a lift of Fr in W_F, the cocycle condition on φ_i implies that for any integer j we have

$$\varphi_i(w) = {}^{\mathrm{Fr}^j}\left(\varphi_i(\tilde{\mathrm{Fr}}^{-j})^{-1}\varphi_i(\tilde{\mathrm{Fr}}^{-j}w)\right).$$

Taking the j-th component, this shows that φ_{ij} is determined by φ_{i1} and this gives a formula for the putative inverse to $\varphi_i \mapsto \varphi_{i1}$. Namely, given $f \in \underline{Z}_{i1}(R)$, define $\varphi_i : W \longrightarrow \hat{G}_i(R)$ component-wise by $\varphi_{ij}(w) := {}^{\mathrm{Fr}^j}\left(f(\tilde{\mathrm{Fr}}^{-j})^{-1}f(\tilde{\mathrm{Fr}}^{-j}w)\right)$. Then a computation shows that $\varphi_i \in Z^1(W_F, \hat{G}_i(R))$ and that this defines the desired inverse isomorphism. But now, the map $\varphi_{i1} \mapsto ((\varphi_{i1})_{|W_{F_{f_i}}}, \varphi_{i1}(\tilde{\mathrm{Fr}}), \cdots, \varphi_{i1}(\tilde{\mathrm{Fr}}^{f_i-1}))$

defines an isomorphism $\underline{Z}_{i1} \longrightarrow \underline{Z}^1(W_{F_{f_i}}, \hat{G}_{i1}) \times (\hat{G}_{i1})^{f_i - 1}$ and the Shapiro morphism becomes the projection on the first factor. $\square$

Resuming the discussion above the lemma, we see that φ is unobstructed if, and only if, for each $i = 1, \cdots, r$, the cocycle $\varphi'_i : W_{F_{f_i}} \longrightarrow \hat{G}_{i1}(L)$ is unobstructed. On the other hand, it follows from the definitions that $\chi_{\hat{G}_i,\mathrm{Fr}}(T) = \chi_{\hat{G}_{i1},\mathrm{Fr}}(T^{f_i})$ so that, coming back to a general $\hat{G}$, we have the following equality :

$$\chi_{\hat{G},\mathrm{Fr}}(T) = \chi_{\hat{G}_{\mathrm{ab}},\mathrm{Fr}}(T)\chi_{\hat{G}_{11},\mathrm{Fr}^{f_1}}(T^{f_1}) \cdots \chi_{\hat{G}_{r1},\mathrm{Fr}^{f_r}}(T^{f_r}). \tag{5.4}$$

In this way, we are reduced to study the case where $\hat{G}$ is *simple* and *adjoint*.

5.3. **The simple adjoint case.** In light of the above discussion, *we will now focus on the case where $\hat{G}$ is simple adjoint in the setting (5.2).*

In this case, it will come in handy to express the unobstructedness condition on the adjoint representation, as opposed to the coadjoint one. Recall that a prime ℓ is *good* for $\hat{G}$ if it does not divide the coefficient of any root of $\hat{G}$ when expressed as a linear combination of simple roots. Moreover, ℓ is called *very good* if it is good and does not divide the order of the fundamental group of the root system of $\hat{G}$.

Theorem 5.14 (Springer-Steinberg). *Suppose $\hat{G}$ is simple adjoint and ℓ is very good for $\hat{G}$. Then a suitable rational multiple of the Killing form on* $\mathrm{Lie}\,\hat{G}_{\mathrm{sc}}$ *induces a non-degenerate bilinear form on* $\mathrm{Lie}\,\hat{G}_L$.

Proof. According to [SS70, p.180] (see also [GN04, §5]), the discriminant of the Killing form on $\mathrm{Lie}\,\hat{G}_{\mathrm{sc}}$ divided by 2 times the dual Coxeter number of $\hat{G}$ is prime to ℓ. Moreover, since ℓ does not divide the degree of the isogeny $\hat{G}_{\mathrm{sc}} \longrightarrow \hat{G}$, this isogeny induces an isomorphism $\mathrm{Lie}(\hat{G}_{\mathrm{sc}})_L \xrightarrow{\sim} \mathrm{Lie}\,\hat{G}_L$. $\square$

Since the Killing form is invariant under the automorphism group of $\hat{G}$, we see that a cocycle φ corresponds to an unobstructed point if, and only if, q^{-1} is not an eigenvalue of $\mathrm{Ad}_\varphi(\mathrm{Fr})$ on $(\mathrm{Lie}\,\hat{G}_L)^{\mathrm{Ad}\,u}$.

Note that $(\mathrm{Lie}\,\hat{G}_L)^{\mathrm{Ad}\,u}$ is the Lie algebra of the scheme-theoretic centralizer $C_{\hat{G}_L}(u)$ of u in $\hat{G}_L$, which may not be reduced. Following standard notation, we denote by $\hat{G}_u$ the *reduced* centralizer of u, which is a closed smooth algebraic subgroup of $\hat{G}_L$. The following result of Slodowy will be useful in our discussion below.

Theorem 5.15. [Slo80, p.38] *If ℓ is very good for $\hat{G}$, then $C_{\hat{G}_L}(u)$ is smooth for all unipotent elements $u \in \hat{G}(L)$, so that $C_{\hat{G}_L}(u) = \hat{G}_u$ and* $(\mathrm{Lie}\,\hat{G}_L)^{\mathrm{Ad}\,u} = \mathrm{Lie}\,\hat{G}_u$.

Our arguments below will extensively make use of the following tool to construct points in $\underline{Z}^1(W_F/I_F^\ell, \hat{G})$. Assume that we are given

- a homomorphism $\lambda : \mathrm{SL}_2 \to \hat{G}_L$ and
- an element $\mathcal{F} \in (\hat{G}(L) \rtimes \mathrm{Fr})_\lambda$, i.e. an element of ${}^LG(L)$ that centralizes λ and projects to Fr.

Then there is a unique 1-cocycle $\varphi : W_F/I_F^\ell \longrightarrow \hat{G}(L)$ such that

$$\varphi(s) = \lambda(U) \text{ and } {}^L\varphi(\mathrm{Fr}) = \lambda(S)\mathcal{F}, \tag{5.5}$$

where S and U denote the matrices $\begin{pmatrix} q^{\frac{1}{2}} & 0 \\ 0 & q^{-\frac{1}{2}} \end{pmatrix}$ and $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ in $\mathrm{SL}_2(L)$, respectively, and where $q^{\frac{1}{2}}$ is a choice of a square root of q in L.

However, we will need a condition to ensure exhaustivity of this construction. Recall that over characteristic zero fields, for any unipotent element u in $\hat{G}(L)$ there is a homomorphism $\lambda : \mathrm{SL}_2 \to \hat{G}_L$ such that $\lambda(U) = u$ and, moreover, λ is unique up to $\hat{G}_L$-conjugacy. In finite characteristic ℓ, the situation is more subtle. An obvious necessary condition for the existence of λ is that u have order ℓ. When ℓ is good for $\hat{G}$, this was proven to be sufficient by Testerman in [Tes95]. In order to study uniqueness, Seitz [Sei00] has introduced the following notion : a morphism $\lambda : \mathrm{SL}_2 \to \hat{G}_L$ over L is a "good SL_2" if the weights of the conjugation action of the maximal torus $T_2 \subset \mathrm{SL}_2$ on $\mathrm{Lie}(\hat{G})$ are bounded above by $2\ell - 2$ (here we identify T_2 to $\mathbb{G}_m$ via the map $\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right) \mapsto z$).

Theorem 5.16 ([Sei00], Theorems 1.1 and 1.2)**.** *Suppose $\hat{G}$ is simple adjoint and ℓ is a good prime for $\hat{G}$, and let u be a unipotent element of $\hat{G}(L)$ of order ℓ. Then there is a "good SL_2" $\lambda : \mathrm{SL}_2 \to \hat{G}_L$ such that $\lambda(U) = u$. Moreover, any two such λ are conjugate by an L-point of the unipotent radical $R_u(\hat{G}_u)$. Finally, the centralizer $\hat{G}_\lambda$ of λ in $\hat{G}$ is reductive, and $\hat{G}_u = \hat{G}_\lambda R_u(\hat{G}_u)$.*

In order to ensure that all non-trivial unipotent elements of $\hat{G}(L)$ have order ℓ, we will henceforth assume that

$$\ell > h, \text{ where } h = h_{\hat{G},1} \text{ is the Coxeter number of } \hat{G}.$$

Indeed, since h is one plus the height of the highest positive root of $\hat{G}$, it follows from Proposition 3.5 of [Sei00] and the Bala-Carter classification, that any nontrivial unipotent element of $\hat{G}(L)$ has order ℓ under this hypothesis. Moreover, such an ℓ is also automatically good for $\hat{G}$, so that Seitz' theorem applies to any u under this hypothesis, and even very good for $\hat{G}$, so that Slodowy's theorem 5.15 also holds.

Corollary 5.17. *Let $\hat{G}$ and φ be as in (5.2) with $\hat{G}$ simple adjoint, and suppose that $\ell > h_{\hat{G},1}$. Then there is $g \in (\hat{G}_u)^\circ(L)$ such that φ^g is of the form (5.5) associated to a pair $(\lambda, \mathcal{F})$ such that $\mathcal{F}$ normalizes a Borel pair (or even a pinning) of $(\hat{G}_\lambda)^\circ$.*

Proof. Let us choose a "good SL_2" $\lambda : \mathrm{SL}_2 \to \hat{G}_L$ with $\lambda(U) = u := \varphi(s)$. Set $\mathcal{F}_1 := \lambda(S)^{-1}.{}^L\varphi(\mathrm{Fr})$. Then $\mathcal{F}_1 \in \hat{G} \rtimes \mathrm{Fr}$ centralizes u, so ${}^{\mathcal{F}_1}\lambda$ is a second "good SL_2" that takes U to u. Since any two such are conjugate by an element centralizing u, we have a unipotent element $u' \in R_u(\hat{G}_u)$ such that ${}^{u'}\lambda = {}^{\mathcal{F}_1}\lambda$; then $\mathcal{F}_2 = u'^{-1}\mathcal{F}_1$ centralizes λ and, in particular, normalizes $(\hat{G}_\lambda)^\circ$. Choose a pinning ε in $(\hat{G}_\lambda)^\circ$; then there exists $h \in (\hat{G}_\lambda)^\circ(L)$ such that ${}^h\varepsilon = {}^{\mathcal{F}_2}\varepsilon$. Then $\mathcal{F} := h^{-1}\mathcal{F}_2$ still centralizes λ, and preserves ε. Now,

$${}^L\varphi(\mathrm{Fr}) = \lambda(S)u'h\mathcal{F} = (\lambda(S)u'h\lambda(S)^{-1})(\lambda(S)\mathcal{F})$$

with u' in the unipotent radical of $\hat{G}_u$ and h in $(\hat{G}_\lambda)^\circ(L)$. Thus hu' lies in $(\hat{G}_u)^\circ(L)$, and since $\lambda(S)$ normalizes $(\hat{G}_u)^\circ$, it follows that $\lambda(S)\mathcal{F} \in (\hat{G}_u)^\circ(L).{}^L\varphi(\mathrm{Fr})$. ☐

We now consider a particular case, which shows that the condition $\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L is necessary for the existence of unobstructed translates.

Proposition 5.18. *Let $\hat{G}$ be simple adjoint, and assume that $\ell > h_{\hat{G},1}$. Then there exists φ as in (5.2) such that $\varphi(s)$ is regular unipotent. Moreover, the following properties are equivalent :*

1. *There is an unobstructed φ such that $\varphi(s)$ is regular unipotent.*
2. *Any φ with $\varphi(s)$ regular unipotent is unobstructed.*
3. *$\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L.*

Proof. Fix a pinning $(\hat{T}, \hat{B}, (X_\alpha)_{\alpha\in\Delta})$ stable under Fr. The sum $E = \sum_{\alpha\in\Delta} X_\alpha$ is a regular nilpotent element of $\mathrm{Lie}(\hat{G})$, which is fixed by Fr. Moreover, the sum $H = \sum_{\beta\in\Phi^+} \beta^\vee \in \mathrm{Lie}(\hat{T}_L)$ is also fixed by Fr (here Φ^+ denotes the set of positive roots and we denote by $\check{\beta}$ the image of the associated coroot in $\mathrm{Lie}(\hat{T}_L) \simeq X_*(\hat{T}) \otimes L$). Then the pair (H, E) is part of a unique principal $\mathfrak{sl}_2$-triple, which is also fixed under Fr. Now, pick a regular unipotent $u \in \hat{G}(L)$ and a good SL_2, say $\lambda : \mathrm{SL}_2 \longrightarrow \hat{G}_L$, such that $\lambda(U) = u$. Then, evaluating $d\lambda$ on the standard basis $\mathfrak{sl}_2$ yields another principal $\mathfrak{sl}_2$-triple. The latter has to be conjugate to (F, H, E) by some element $g \in \hat{G}(L)$, which means that, after conjugating by g, we may assume that λ (and therefore u) is fixed by Fr. Then we can construct φ as desired by putting $\varphi(s) := \lambda(U)$ and ${}^L\varphi(\mathrm{Fr}) := \lambda(S) \rtimes \mathrm{Fr}$.

If φ' is another cocycle with $\varphi'(s)$ regular unipotent, then we may conjugate it so that $\varphi'(s) = \varphi(s) = u$, and this does not affect the property of being unobstructed. Then ${}^L\varphi'(\mathrm{Fr}) = {}^L\varphi(\mathrm{Fr})g$ for some $g \in \hat{G}_u(L)$, and φ' is unobstructed if and only if q^{-1} is not an eigenvalue of $(\mathrm{Ad}\,\varphi')(\mathrm{Fr})$ on $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}$. But under our running assumption $\ell > h_{\hat{G}}$, which implies that ℓ is very good for $\hat{G}$, Theorem 5.15 implies that $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}\,u} = \mathrm{Lie}(\hat{G}_u)$. Moreover $\hat{G}_u$ is known to be commutative, hence $(\mathrm{Ad}\,\varphi')(\mathrm{Fr}) = (\mathrm{Ad}\,\varphi)(\mathrm{Fr})$ on $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}$ and we have the equivalence of (1) and (2).

It remains to study when q^{-1} is an eigenvalue of $(\mathrm{Ad}\,\varphi)(\mathrm{Fr})$. Observe that $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}$ coincides with the centralizer $\mathrm{Lie}(\hat{G})_E$ of E in $\mathrm{Lie}(\hat{G})$. Moreover, our hypothesis $\ell > h_{\hat{G},1}$ implies that ℓ does not divide the order of the Weyl group $\Omega_{\hat{G}}$. Therefore, we can use Kostant's section theorem as in subsection B.4. In particular, Proposition B.5 tells us that

$$\det\left(q\,\mathrm{Ad}_{\lambda(S)\,\mathrm{Fr}} - \mathrm{id} \mid \mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}\right) = \pm\chi_{\hat{G},\mathrm{Fr}}(q),$$

which shows that φ is unobstructed if and only if $\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L. $\square$

Remark 5.19. Let $\mathcal{F}$ be any automorphism of $\hat{G}$ and suppose λ is a $\mathcal{F}$-invariant good SL_2 such that $u = \lambda(U)$ is regular in $\hat{G}$. Then the same proof shows that $\det\left(q\,\mathrm{Ad}_{\lambda(S)\mathcal{F}} - \mathrm{id} \mid \mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}\right) = \pm\chi_{\hat{G},\mathcal{F}}(q)$.

In order to study more general unipotent classes, the following lemma will allow us to use inductive arguments.

Lemma 5.20. *Let $\hat{G}$ and φ be as in (5.2), and let $\hat{S}$ be a torus in the centralizer $C_{\hat{G}}(\varphi)$ of φ. Then :*

1. *$\exists h \in \hat{G}(L)$ such that $\hat{M} := hC_{\hat{G}}(\hat{S})h^{-1}$ is a Fr-stable Levi subgroup of $\hat{G}$.*
2. *If the h-conjugate ${}^h\varphi$ is unobstructed in $\underline{Z}^1(W_F/I_F^\ell, \hat{M})$, then there is $g \in (\hat{G}_u)^\circ(L)$ such that φ^g is unobstructed.*

Proof. (1) The centralizer $C_{^LG}(\hat{S})$ contains ${}^L\varphi(\mathrm{Fr})$, hence it surjects onto $\pi_0({}^LG)$. By [Bor79, Lemma 3.5], it is a "Levi subgroup" of ${}^LG$ in Borel's sense. It is thus conjugate by some $h \in \hat{G}(L)$ to the standard Levi subgroup of a standard parabolic subgroup of ${}^LG$. Such standard Levi subgroups are of the form ${}^LM = \hat{M} \rtimes \langle \mathrm{Fr} \rangle$.

(2) Since unobstructedness is invariant by conjugacy, we may and will assume that $h = 1$. Then observe that ${}^L\varphi$ factors indeed through ${}^LM$, and also that $\mathrm{Lie}(\hat{M})$ is the weight 0 subspace of $\mathrm{Lie}(\hat{G})$ in the decomposition $\mathrm{Lie}(\hat{G}) = \bigoplus_{\kappa \in X^*(\hat{S})} \mathrm{Lie}(\hat{G})_\kappa$ of $\mathrm{Lie}(\hat{G})$ as a sum of weight spaces for the adjoint action of $\hat{S}$. So, for any element $s \in \hat{S}(L)$, unobstructedness of φ^s is equivalent to q^{-1} not being an eigenvalue of ${}^L\varphi(\mathrm{Fr})s$ on each $\mathrm{Lie}(\hat{G}_u)_\kappa$. For $\kappa = 0$, this property is fulfilled by our hypothesis, since s acts trivially on $\mathrm{Lie}(\hat{G}_u)_0$. For any other κ, this property is fulfilled for s outside a proper Zariski closed subset of $\hat{S}$, because s commutes with ${}^L\varphi(\mathrm{Fr})$. Therefore we can find s that works for all κ. □

Following a standard terminology, we will say that a 1-cocycle φ is *discrete* if $C_{\hat{G}}(\varphi)$ contains no non-central torus of $\hat{G}$. In the case where φ is given by a pair $(\lambda, \mathcal{F})$ as in Lemma 5.17, this is equivalent to $C_{\hat{G}}(\lambda)^{\mathcal{F}}$ not containing any non-central torus of $\hat{G}$, since $C_{\hat{G}}(\lambda) = \hat{G}_\lambda$ is a Levi factor of $C_{\hat{G}}(\varphi(s)) = \hat{G}_u$. In this case we will also say that the pair $(\lambda, \mathcal{F})$ is discrete. If $\varphi(s) = \lambda(U)$ is a *distinguished unipotent* element (meaning that its centralizer does not contain any non-central torus), then φ is certainly discrete. The converse is not always true, but we note that if $C_{\hat{G}}(\lambda)$ has *positive* semisimple rank, then φ is *not* discrete.

In the next proposition, we include triality forms of D_4 (i.e. any group of type D_4 with action of Fr of order 3) in the "exceptional types".

Proposition 5.21. *Let $\hat{G}$ be as in (5.2) with no simple factor of exceptional type. Assume that $\ell > h_{\hat{G},1}$, i.e. ℓ is greater than the Coxeter numbers of the simple factors of $\hat{G}$. Then the following are equivalent :*

1. *For all φ as in (5.2), there exists $g \in (\hat{G}_{\varphi(s)})^\circ$ such that φ^g is unobstructed.*
2. *$\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L.*

Proof. By Lemma 5.10, Lemma 5.12, decomposition (5.3) and equality (5.4), we may assume that $\hat{G}$ is simple and adjoint. In this case, the implication (1)⇒(2) follows from Proposition 5.18. So we now focus on the other implication. Using again Lemma 5.10 together with Remark 5.11 and the fact that $\chi_{\hat{G},\mathrm{Fr}}(q)$ is insensitive to isogenies, we may assume that $\hat{G}$ is either PGL_n, Sp_{2n} or SO_N. Actually, the PGL_n case can be treated on GL_n, since Remark 5.11 also applies to the central morphism $\mathrm{GL}_n \xrightarrow{\pi} \mathrm{PGL}_n$, while $\chi_{\mathrm{GL}_n,\mathrm{Fr}}(T) = \chi_{\mathrm{PGL}_n,\mathrm{Fr}}(T)\chi_{Z(\mathrm{GL}_n),\mathrm{Fr}}(T)$ and $\chi_{Z(\mathrm{GL}_n),\mathrm{Fr}}(T)$ divides $\chi_{\mathrm{PGL_n},\mathrm{Fr}}(T)$ for $n > 1$.

So let $\hat{G}$ be either GL_n, Sp_{2n} or SO_N. Then Fr acts on $\hat{G}$ by an automorphism of order at most 2, and we will let ${}^LG$ denote the minimal form of the L-group. Now let φ be as in (5.2). By Corollary 5.17, we may assume that φ is given by a pair $(\lambda, \mathcal{F})$. Moreover, Lemma 5.20, Proposition B.3 (1) and an inductive argument allow us to restrict attention to *discrete* pairs $(\lambda, \mathcal{F})$.

Case $\hat{G} = \mathrm{GL}_N$ with $\mathrm{Fr} = \mathrm{id}$. Let V be an L-vector space of dimension N, and $\lambda : \mathrm{SL}_2 \longrightarrow \mathrm{GL}(V)$ a morphism. Since $\ell > N$, the SL_2-module V is semi-simple and, for any $d \leq N$, the d-dimensional representation $S_d = \mathrm{Sym}^{d-1}(L^2)$ of $\mathrm{SL}_2(L)$

is irreducible. Let V_d be the S_d-isotypic part of V. We then have decompositions $V = \bigoplus_{d\geq 0} V_d$ and $S_d \otimes W_d \xrightarrow{\sim} V_d$, where $W_d := \mathrm{Hom}_{\mathrm{SL}_2}(S_d, V_d)$. In particular, we get that $\mathrm{GL}(V)_\lambda = \prod_d \mathrm{GL}(W_d)$. Since this is a connected group, we may assume that $\mathcal{F} = 1$. Then we see that $(\lambda, 1)$ is discrete if and only λ is principal, i.e. $u = \lambda(U)$ is regular. In this case we conclude thanks to Proposition 5.18 (note that this proposition applies directly to PGL_n and extends to GL_n thanks to Lemma 5.12 and Remark 5.11).

We now assume that V is endowed with a non-degenerate bilinear form of sign ε and we denote by $I(V)$ the isometry group, so that $I(V) \simeq \mathrm{Sp}_N$ if $\varepsilon = -1$ and $I(V) \simeq \mathrm{O}_N$ if $\varepsilon = 1$. We take up the above notations, assuming that λ factors through $I(V)$. Then each V_d is a non-degenerate subspace of V and the decomposition $V = \bigoplus_d V_d$ is orthogonal. Further, each S_d carries a natural non-degenerate bilinear form of sign $(-1)^{d-1}$ such that SL_2 acts through $I(S_d)$. Then W_d inherits a non-degenerate form of sign $(-1)^{d-1}\varepsilon$ such that the isomorphism $S_d \otimes W_d \xrightarrow{\sim} V_d$ is compatible with the tensor product form. It follows in particular that $I(V)_\lambda = \prod_d I(W_d)$. Writing $r_d := \dim(W_d)$, we have

(1) $I(W_d) \simeq \mathrm{O}_{r_d} \simeq \mathrm{SO}_{r_d} \rtimes \mathbb{Z}/2\mathbb{Z}$ if $(-1)^{d-1}\varepsilon = 1$.
(2) $I(W_d) \simeq \mathrm{Sp}_{r_d}$ if $(-1)^{d-1}\varepsilon = -1$.

In particular, $\pi_0(I(V)_\lambda)$ admits a section into $I(V)_\lambda$, and we may take $\mathcal{F}$ in the image of such a section, so that $\mathcal{F}$ has order at most 2. Moreover, we see that $(I(V)_\lambda)^{\mathcal{F}}$ contains a non-trivial torus whenever there is a symplectic factor (associated to some d such that $(-1)^{d-1}\varepsilon = -1$ and $W_d \neq 0$). Since we may restrict attention to discrete $(\lambda, \mathcal{F})$, we will assume that $I(V)_\lambda$ has no symplectic factor. In particular, this fixes the parity of the d's such that $V_d \neq 0$.

We now need to investigate the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}}\,\mathrm{Ad}_{\lambda(S)}$ on $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}$, where $u = \lambda(U)$. We have decompositions

$$\mathrm{End}_u(V) = \prod_{d,d'} \mathrm{Hom}_u(V_d, V_{d'}) \simeq \prod_{d,d'} \mathrm{Hom}_U(S_d, S_{d'}) \otimes \mathrm{Hom}_L(W_d, W_{d'}).$$

The weights of $\lambda(T_2)$ on V_d are the weights of S_d, i.e. $d-1, d-3, \cdots, 1-d$, hence the weights of $\mathrm{Ad}_{\lambda(T_2)}$ on $\mathrm{Hom}_u(V_d, V_{d'})$ are the same as those on $\mathrm{Hom}_U(S_d, S_{d'})$, i.e. $d + d' - 2i$ for $1 \leq i \leq \min(d, d')$, each one occurring with multiplicity $r_d r_{d'}$. In particular, these weights are bounded above by $N-2$ if $d \neq d'$ (because then $d + d' \leq N$) or if $d = d' \leq \frac{N}{2}$. On the other hand, there is at most one $d > \frac{N}{2}$ with $W_d \neq 0$ and in this case $r_d = 1$. So any weight $k > N-2$ of $\lambda(T_2)$ on $\mathrm{End}_u(V)$ is even and occurs with multiplicity 1. Actually, it is easy to exhibit a weight vector. Namely, put $e := d(\lambda_{|\mathbb{G}_a})(1)$, which is a nilpotent endomorphism of V. We also have $e = \log(u)$ since the logarithm is well defined under our hypothesis $\ell > h_{\hat{G},1}$. Then e is a weight 2 element of $\mathrm{End}_u(V) = \mathrm{End}_e(V)$ and for any $k = 2k' > N-2$, the element $e^{k'}$ generates the subspace of weight k whenever it is non-zero. In other words, we have a decomposition

$$\mathrm{End}_u(V) = \left\langle e^{k'} \right\rangle_{k' \geq \lfloor \frac{N}{2} \rfloor} \oplus \mathrm{End}_u(V)_{\leq N-2}$$

where the last term is the sum of weight spaces of weight $\leq N-2$.

Now, let τ denote the involution $\psi \mapsto -\psi^*$ of $\mathrm{End}(V)$ associated with the bilinear form on V. We have $\mathrm{Lie}(I(V))^{\mathrm{Ad}_u} = \mathrm{End}_u(V)^\tau$ and $\tau(\mathrm{Hom}(V_d, V_{d'})) =$

$\mathrm{Hom}(V_{d'}, V_d)$. Using the fact that $\tau(e^{k'}) = (-1)^{k'+1}e^{k'}$, we get :

$$\mathrm{Lie}(I(V))^{\mathrm{Ad}_u} = \mathrm{End}_u(V)^\tau = \left\langle e^{k'} \right\rangle_{k' \geq \lfloor \frac{N}{2} \rfloor, \mathrm{odd}} \oplus \mathrm{End}_u(V)^\tau_{\leq N-2}.$$

Case $\hat{G}$ is symplectic or odd orthogonal. In this case, Fr acts trivially and we have $\chi_{\hat{G},\mathrm{Fr}}(T) = \prod_{d=1}^{\lfloor \frac{N}{2} \rfloor}(T^{2d}-1)$. The eigenvalues of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})} = q\,\mathrm{Ad}_{\mathcal{F}}\,\mathrm{Ad}_{\lambda(S)}$ on $\mathrm{End}_u(V)^\tau_{\leq N-2}$ are of the form $\pm q^k$ for k such that $0 < 2k \leq N$. For such an eigenvalue to be equal to 1, we need that q be a root of $T^{2k}-1$, which is a factor of $\chi_{\hat{G},\mathrm{Fr}}(T)$. On the other hand each non-zero e^k is an eigenvector of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ with eigenvalue q^{k+1}. Of course $e^N = 0$, so $k \leq N-1$ and we have seen that k must be odd. So $k+1$ is even, between 2 and N. Therefore, an eigenvalue q^{k+1} is 1 in L only if q is a root of $\chi_{\hat{G},\mathrm{Fr}}(T)$, as desired.

Case $\hat{G}$ is even orthogonal. Here we set $\hat{G} = \mathrm{SO}(V)$, endowed with an outer action of Fr of order $f = 1$ or 2. In these cases, setting $N = 2n$, we have

$$\chi_{\hat{G},\mathrm{Fr}}(T) = (T^n + (-1)^f)\prod_{d=1}^{n-1}(T^{2d}-1).$$

We will take advantage of the fact that, when $f = 2$, we have $\mathrm{O}(V) \simeq {}^L\mathrm{SO}(V)$. A pair $(\lambda, \mathcal{F})$ thus defines a L-homomorphism ${}^L\varphi$ for $\mathrm{SO}(V)$ endowed with a trivial, resp. quadratic, action of Fr if $\det\mathcal{F} = 1$, resp. if $\det\mathcal{F} = -1$.

As above, each e^k is an eigenvector of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ with eigenvalue q^{k+1} with $k+1$ even. Moreover we have $e^{N-1} = 0$ (no Jordan matrix of rank N is orthogonal), so $k+1 \leq N-2$ and we see that $q^{k+1} = 1$ only if q is a root of $\chi_{\hat{G},\mathrm{Fr}}(T)$.

Next, the weight spaces with weight $< N-2$ are treated exactly as in the previous case, but the weight space $\mathrm{End}_u(V)^\tau_{N-2}$ of weight $N-2$ needs more attention. Indeed, we already know that the eigenvalues of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ on this weight space are of the form $\pm q^n$, but we need more precise information since, for example, T^n+1 does not divide $\chi_{\hat{G},\mathrm{Fr}}(T)$ when $f = 1$, and $T^n - 1$ may not divide $\chi_{\hat{G},\mathrm{Fr}}(T)$ when $f = 2$. Since $\mathrm{Hom}_u(V_d, V_{d'})_{N-2}$ is zero unless $d + d' = N$, we have to consider two cases.

(1) $V = V_d \oplus V_{d'} \simeq S_d \oplus S_{d'}$, with (necessarily) d and d' odd and, say $d > d'$. In this setting, $\mathcal{F}$ belongs to the center $\{\pm 1\} \times \{\pm 1\}$ of $\mathrm{O}(V_d) \times \mathrm{O}(V_{d'})$. Writing $\mathcal{F} = (\varepsilon_d, \varepsilon_{d'})$, we see that $\mathcal{F}$ acts on $\mathrm{Hom}_u(V_d, V_{d'})$ and $\mathrm{Hom}_u(V_{d'}, V_d)$ by multiplication by $\varepsilon_d\varepsilon_{d'}$, and since d and d' are odd, we have $\varepsilon_d\varepsilon_{d'} = \det\mathcal{F}$. Now we have

$$\mathrm{End}_u(V)^\tau_{N-2} = \langle e^{n-1}\rangle^\tau \oplus (\mathrm{Hom}_u(V_d, V_{d'}) \oplus \mathrm{Hom}_u(V_{d'}, V_d))^\tau_{N-2}.$$

So if $\det\mathcal{F} = 1$, the action of $\mathcal{F}$ on $\mathrm{End}_u(V)^\tau_{N-2}$ is trivial, hence the eigenvalue of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ is q^n and we are done, since $T^n - 1$ divides $\chi_{\hat{G},\mathrm{Fr}}(T)$ when $f = 1$. Suppose now $\det\mathcal{F} = -1$. Then $\mathcal{F}$ acts on the second summand of $\mathrm{End}_u(V)^\tau_{N-2}$ by -1, so the eigenvalue of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ is $-q^n$, which is fine since T^n+1 divides $\chi_{\hat{G},\mathrm{Fr}}(T)$ when $f = 2$. On the other hand, $\mathcal{F}$ acts trivially on the first summand, but the latter is non-zero only if n is even, in which case $T^n - 1$ also divides $\chi_{\hat{G},\mathrm{Fr}}(T)$.

(2) $V = V_n$. Then we may decompose V as an orthogonal sum of two $\lambda(\mathrm{SL}_2)$-stable non-degenerate subspaces $V = V_n^1 \oplus V_n^2$. Moreover, since n has to be odd, $(e_{|V_n^i})^{n-1}$ is not in $\mathrm{End}_u(V)^\tau$ so we have

$$\mathrm{End}_u(V)^\tau_{N-2} = \left(\mathrm{Hom}_u(V_n^1, V_n^2) \oplus \mathrm{Hom}_u(V_n^2, V_n^1)\right)^\tau_{N-2}.$$

On the other hand, $\mathrm{O}(V)_\lambda \simeq \mathrm{O}_2$ acts on this space through its component group $\{\pm 1\}$ with the non trivial element acting as $\psi \mapsto \psi^*$. So, in particular, $\mathcal{F}$ acts by multiplication by $\det \mathcal{F}$. The eigenvalue of $q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ is thus $\det \mathcal{F}.q^n$ and it equals 1 only if $q^n - \det \mathcal{F} = 0$, hence also only if $\chi_{\hat{G},\mathrm{Fr}}(q) = 0$.

Case $\hat{G} = \mathrm{GL}_N$ *and* $\mathrm{Fr} \neq \mathrm{id}$. Here we have $\chi_{\hat{G},\mathrm{Fr}}(T) = \prod_{d=1}^N (T^d - (-1)^d)$. We continue with the same notations V, λ, u etc, and we assume that there is $\mathcal{F} \in (\hat{G} \rtimes \mathrm{Fr})_\lambda$ that fixes a Borel pair of $\hat{G}_\lambda$. Using the explicit description $\hat{G}_\lambda = \prod_d \mathrm{GL}(W_d)$, we see that $(\lambda, \mathcal{F})$ is discrete if and only if $r_d = 1$ for all d (so that $\hat{G}_\lambda = \mathbb{G}_m \times \cdots \times \mathbb{G}_m$ is the center of $\prod_d \mathrm{GL}(V_d)$) and $(\hat{G}_\lambda)^{\mathcal{F}} = \{\pm 1\} \times \cdots \times \{\pm 1\}$. This implies that $\mathcal{F}$ normalizes each $\mathrm{GL}(V_d)$ and induces the non-trivial element α_d of $\mathrm{Out}(\mathrm{GL}(V_d))$. Since $u_{|V_d}$ is regular, it follows from Proposition 5.18 and the subsequent remark that no eigenvalue of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ on $\mathrm{End}_u(V_d)$ equals 1 unless $\chi_{\mathrm{GL}(V_d),\alpha_d}(q) = 0$ in L, in which case we also have $\chi_{\hat{G},\mathrm{Fr}}(q) = 0$ by Proposition B.3 (1). Let us now focus on the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ on each $\mathrm{Hom}_u(V_d, V_{d'})$ for $d \neq d'$. As we have already seen, the eigenvalues of $q\,\mathrm{Ad}_{\lambda(S)}$ are of the form $q^{\frac{1}{2}(d+d')-i}$ with $0 \leq i < \min(d, d')$. So it remains to understand how $\mathcal{F}$ acts. Note that $\mathcal{F}^2 \in (\hat{G}_\lambda)^{\mathcal{F}}$ so at least we know that $\mathcal{F}^4 = 1$. We distinguish two cases.

(1) Suppose that all the d's occuring have the same parity. Then there is a non-degenerate bilinear form on V (symplectic if the d's are even, orthogonal if they are odd) such that $u \in I(V)$, see [LS12, Cor. 3.6 (2)] for example. We may then conjugate λ so that it factors through $I(V)$. But $I(V)$ is the fixed-point subgroup of an involution given by conjugation by an element of the form $g \rtimes \mathrm{Fr}$. So we may set $\mathcal{F}$ to this element and we have achieved $(\mathrm{Ad}_{\mathcal{F}})^2 = 1$. It follows that the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ on each $\mathrm{Hom}_u(V_d, V_{d'})$ are of the form $\pm q^k$ for some integer $k \leq \frac{N}{2}$. Should such an eigenvalue be equal to 1, we would have $q^{2k} - 1 = 0$, hence a fortiori $\chi_{\hat{G},\mathrm{Fr}}(q) = 0$.

(2) Suppose there are both even and odd d's. Write $(\mathcal{F}^2)_d$ for the component of $\mathcal{F}^2$ in $\mathrm{GL}(V_d)$. This is a central element of $\mathrm{GL}(V_d)$ equal to ± 1. We then decompose $V = V_+ \oplus V_-$ where $V_\pm = \bigoplus_{d,\,(\mathcal{F}^2)_d = \pm 1} V_d$. We have $\mathrm{GL}(V)^{\mathcal{F}^2} = \mathrm{GL}(V_+) \times \mathrm{GL}(V_-)$, hence also $\mathrm{GL}(V)^{\mathcal{F}} = \mathrm{GL}(V_+)^{\mathcal{F}} \times \mathrm{GL}(V_-)^{\mathcal{F}}$. But $\mathcal{F}$ acts on both $\mathrm{GL}(V_-)$ and $\mathrm{GL}(V_+)$ as an involution that induces the non trivial outer automorphism. So each $\mathrm{GL}(V_\pm)^{\mathcal{F}}$ is an orthogonal or symplectic group. This implies that all d's occurring in the decomposition of V_+, resp. V_-, have the same parity (because all multiplicities r_d are 1). As a consequence, we see that $\mathcal{F}^2$ acts on $\mathrm{Hom}_u(V_d, V_{d'})$ by multiplication by $(-1)^{d+d'}$. So we now have two subcases :

- if d, d' have the same parity, the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ on $\mathrm{Hom}_u(V_d, V_{d'})$ are of the form $\pm q^{\frac{1}{2}k}$ for some *even* integer $k \leq d + d' \leq N$. As before, should such an eigenvalue be equal to 1, we would have $q^k - 1 = 0$, hence a fortiori $\chi_{\hat{G},\mathrm{Fr}}(q) = 0$.
- if d, d' have different parities, the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ on $\mathrm{Hom}_u(V_d, V_{d'})$ are of the form $\zeta q^{\frac{1}{2}k}$ for some *odd* integer $k \leq d + d' \leq N$ and a primitive 4^{th}-root of unity ζ in L. This time, should such an eigenvalue be equal to 1, we would have $q^k + 1 = 0$ hence, again, $\chi_{\hat{G},\mathrm{Fr}}(q) = 0$.

□

It may be tempting to believe that the nice equivalence of Proposition 5.21 holds in general. However, it fails in the case of triality, i.e. a group of type D_4 with Frobenius acting with order 3. In this case, the irreducible factors of $\chi_{\hat{G},\mathrm{Fr}}(T) = (T^2-1)(T^6-1)(T^8+T^4+1)$ are $\Phi_n(T)$ for $n = 1,2,3,6,12$. But to get an equivalence, we need also $\Phi_4(T)$:

Lemma 5.22. *Assume that $\hat{G} = \mathrm{PSO}_8$ with Fr of order 3 (triality), and $\ell > h_{\hat{G}} = 6$. Then the following are equivalent :*

1. *For all φ as in (5.2), there exists $g \in (\hat{G}_{\varphi(s)})^\circ$ such that φ^g is unobstructed.*
2. *$\chi'_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L, where $\chi'_{\hat{G},\mathrm{Fr}}(T) = T^{12}-1$.*

Proof. As in the proof of Proposition 5.21, we may focus on discrete pairs $(\lambda, \mathcal{F})$ with $\lambda : \mathrm{SL}_2 \longrightarrow \hat{G}$ and $\mathcal{F} \in (\hat{G} \rtimes \mathrm{Fr})_\lambda$ (where ${}^LG$ is the minimal L-group, so that $\pi_0({}^LG) = \mathbb{Z}/3\mathbb{Z}$). We still denote by λ the unique lift $\mathrm{SL}_2 \longrightarrow \mathrm{SO}_8$ and see SO_8 as $\mathrm{SL}(V) \cap I(V)$ for an 8-dimensional vector space with a non-degenerate symmetric bilinear form. With the notation of the proof of Proposition 5.21, there are only three possible types of decomposition of V associated to such a λ. Either $V = V_7 \oplus V_1$, or $V = V_5 \oplus V_3$ or, $V = V_3 \oplus V_1$ with $V_3 = S_3^2$ and $V_1 = S_1^2$.

(1) Type $(7,1)$. This is the regular orbit, so it is covered by Proposition 5.18.

(2) Type $(5,3)$. This is the only distinguished non-regular orbit, so it is stable under Fr. Therefore, for λ of type $(5,3)$, there exists $\mathcal{F} \in (\hat{G} \rtimes \mathrm{Fr})_\lambda$. Since $\hat{G}_\lambda = Z(\hat{G}) = \{1\}$, we have $(\mathrm{Ad}_\mathcal{F})^3 = \mathrm{id}$. On the other hand, the $\lambda(T_2)$-weights on $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u} = \mathrm{End}_u(V)^\tau$ are 2, 4 and 6, so the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ are respectively of the form ζq^2, q^3 or ζq^4 for some 3^{rd}-root of unity ζ. If any of these numbers equals 1 in L, then $q^{12} = 1$, hence $\chi'_{\hat{G},\mathrm{Fr}}(q) = 0$. However, it is actually possible to prove that q^4 is not an eigenvalue, so that the polynomial $\chi_{\hat{G},\mathrm{Fr}}$ is still good for this orbit.

(3) Type $(3,3,1,1)$. This orbit intersects $G_2 = \hat{G}^{\mathrm{Fr}}$ along its non-regular distinguished orbit. So we may pick a relevant λ that is centralized by Fr. Then $\pi_0({}^LG_\lambda)$ is isomorphic to $\mathbb{Z}/6\mathbb{Z}$ and contains two elements such that $(\lambda, \mathcal{F})$ is discrete : Fr of order 3, and $c\,\mathrm{Fr}$ of order 6, where c is the image in $\hat{G}_\lambda$ of a reflection that generates $\pi_0(I(V)_\lambda \cap \mathrm{SL}(V))$. The weights are $0,2$ and 4. Hence the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ on weight 0 and 2 spaces are of the form ζq^2 or ζq for a sixth root of unity ζ, so they are different from 1 unless $q^{12} = 1$. On the other hand, the weight 4 space has dimension 1 and comes from G_2. So Fr acts trivially on it, and $c\,\mathrm{Fr}$ acts by ± 1. Hence the corresponding eigenvalue is $\pm q^3$ and is also different from 1 unless $q^{12} = 1$. Now, the computation in G_2 of Remark 5.24 shows that -1 is an eigenvalue of $\mathcal{F} = c\,\mathrm{Fr}$ on the weight 2 space, so $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$ has eigenvalue $-q^2$, and it is different from 1 if and only if $\Phi_4(q) \neq 0$. ☐

We now turn to the exceptional groups. Recall the polynomials $\chi^*_{\hat{G},\mathrm{Fr}}$ from (B.3).

Proposition 5.23. *Suppose that $\hat{G}$ is simple of exceptional type. If $\chi^*_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L, then for all φ as in (5.2), there exists $g \in (\hat{G}_{\varphi(s)})^\circ$ such that φ^g is unobstructed.*

Recall that $\chi^*_{\hat{G},\mathrm{Fr}}(q) \neq 0$ is equivalent to "q has order greater than $h_{\hat{G},\mathrm{Fr}}$ in $L^\times$", which implies $\ell > h_{\hat{G},\mathrm{Fr}}$, hence also $\ell > h_{\hat{G},1}$.

Proof. Thanks to Lemma 5.10 and equality (5.4), we may assume $\hat{G}$ is adjoint. We will use the tables in Chapter 11 of [LT11]. These tables cover all the nilpotent classes of exceptional groups, including a description of the reductive quotient C of the centralizers (both the neutral component, denoted there by C° and the π_0, denoted there by C/C°), and the weights of an associated cocharacter τ on the Lie algebra centralizer (denoted by m there). Actually, they even describe the weights on each subquotient of the central series of the nilpotent part of the Lie algebra centralizer (with integer n denoting the n^{th} step of the central series).

Using a Springer isomorphism $e \leftrightarrow u$ between the nilpotent cone and the unipotent variety, we get a table of unipotent classes, and we may identify the centralizers $\hat{G}_u = \hat{G}_e$. Then for any good λ associated to u, we have identifications $\hat{G}_\lambda \simeq C$, hence the table provides us with descriptions of $(\hat{G}_\lambda)^\circ = C^\circ$, $\pi_0(\hat{G}_\lambda) = C/C^\circ$ and the weights m of $\lambda(T_2)$ on $\mathrm{Lie}(R_u(\hat{G}_u))$.

Thanks to Corollary 5.17 we may focus on φ associated to a pair $(\lambda, \mathcal{F})$. Then, using Lemma 5.20 (together with Proposition 5.21 an inductive argument for the E series), we may restrict attention to *discrete* $(\lambda, \mathcal{F})$. In the case where Fr acts trivially on $\hat{G}$, this means that, in the tables of *loc. cit.*, we may restrict to classes such that C° is a torus and consider all $\mathcal{F} \in C$ such that $(C^\circ)^{\mathcal{F}}$ is finite. In this setting, the exponent of $(C^\circ)^{\mathcal{F}}$ divides the order f of the image $\bar{\mathcal{F}}$ of $\mathcal{F}$ in the component group C/C° (since the endomorphism $t \mapsto t({}^{\bar{\mathcal{F}}}t)\cdots({}^{\bar{\mathcal{F}}^{f-1}}t)$ of C vanishes), hence $\mathcal{F}$ has finite order dividing f^2, and this order does only depend on the connected component of C that contains $\mathcal{F}$. So the eigenvalues of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)} = q\,\mathrm{Ad}_{\varphi(\mathrm{Fr})}$ on $\mathrm{Lie}(\hat{G})^{\mathrm{Ad}_u}$ are of the form $\zeta q^{\frac{m}{2}+1}$ for some root of unity ζ whose order t divides the order of $\mathcal{F}$. Therefore, what we have to check is that, in all cases, we have $t(\frac{m}{2}+1) \leq h_{\hat{G},\mathrm{Fr}}$ when $t(\frac{m}{2}+1)$ is an integer, or $t(m+2) \leq h_{\hat{G},\mathrm{Fr}}$ else. In the only twisted case ${}^2E_6$ where Fr has order 2, we will apply the same strategy except that here $\mathcal{F} \in \hat{G} \rtimes \mathrm{Fr}$.

Below we list all "discrete" orbits except the regular ones, which are treated in Proposition 5.18. The numbering is that of [LT11, §11].

G_2, orbit 3. Here $h = 6$, $m = 2$ or 4, and $C = S_3$, so that $t = 1, 2$ or 3. Hence the desired inequalities $t(\frac{m}{2}+1) \leq h = 6$ always hold except if $t = 3$ and $m = 4$. But this case doesn't happen since the weight 4 subspace is 1 dimensional, so $C = S_3$ acts on it via a character, hence via an element of order 2.

F_4, orbit 10. Here $h = 12$, $C = S_4$ hence $t \leq 4$, and the desired inequality holds trivially for all weights except possibly for weight 6. But the weight 6 subspace has dimension 2, so the action of $C = S_4$ factors over a quotient isomorphic to S_3, hence $t \leq 3$ on this subspace.

F_4, orbit 13. Here $h = 12$, $C = S_2$ hence $t \leq 2$ and weights m are even and ≤ 10, hence the desired inequality holds.

F_4, orbit 14. Here $h = 12$ and $C = S_2$, so only the weight $m = 14$ space might contradict the desired inequality, but the columns $\mathcal{Z}^\natural$ and $\mathcal{Z}$ of the table show that C acts trivially on this space, so $t = 1$ and $t(\frac{m}{2}+1) = 8 \leq 12$.

E_6, orbits 17 and 19. Here again, $h = 12$ and $C = S_2$ or $\{1\}$. In each case, the desired inequalities follow directly from the list of weights.

E_6, orbit 11. Our source here is section 9.3.4 of *loc.cit.* This orbit comes from the distinguished non regular orbit of a Levi subgroup H of root system D_4 and C°

is the two-dimensional connected center of H, while C normalizes a Borel pair of H and has component group $C/C^\circ = S_3$. The action of S_3 on $X^*(C^\circ)$ is the standard representation, as can be seen by embedding E_6 as a Levi subgroup of E_7 and using the description of the reductive centralizer C_7 of this orbit from *loc.cit.* Therefore, an element of order 1 or 2 of C/C° fixes a subtorus, and we see that if $(\lambda, \mathcal{F})$ is to be discrete, then $\bar{\mathcal{F}}$ should have order 3 in C/C°. Then $\mathcal{F}$ itself has order 3 or 9 in C. To prove it has order 3, we embed E_6 as a Levi subgroup of E_8 and consider the reductive centralizer C_8 there. Then section 9.3.4 of *loc.cit* exhibits two elements c_1 and c_2 of order 2 in C, whose images generate $C/C^\circ = C_8/C_8^\circ$, and that act on C_8° by fixing a pinning. But C_8° is a simple group of type D_4, so its center has exponent dividing 2. Hence the element $(c_1c_2)^3$, which belongs to C° and fixes a pinning of C_8° is central in C_8°, hence has order dividing 2. Since we have seen that c_1c_2 has order 3 or 9, we conclude it has order 3. Hence $\mathcal{F}$ has order 3, and all desired inequalities follow from the list of weights.

${}^2E_6$, orbit 11. Here, $h_{\hat{G},\mathrm{Fr}} = 18$. Again, we refer to section 9.3.4 of *loc.cit.*, except that we find it easier to argue with the nilpotent representative $e' := \mathrm{Ad}_g(e)$ in their notation (and the same cocharacter τ described in table 3 of their chapter 6). Indeed, for the λ corresponding to (e', τ), we easily see that $h_2(-1) \rtimes \mathrm{Fr} \in {}^LG_\lambda$. Then, ${}^LG_\lambda/({}^LG_\lambda)^\circ$ is an extension of $\mathbb{Z}/2\mathbb{Z}$ by $\hat{G}_\lambda/(\hat{G}_\lambda)^\circ = C/C^\circ = S_3$. Such an extension has to be split and contains a central element of order 2. In the present case, using notations $c_2' := gc_2g^{-1}$, the element $\mathcal{F}_0 := c_2'h_2(-1) \rtimes \mathrm{Fr}$ belongs to $(\hat{G} \rtimes \mathrm{Fr})_\lambda$ and its image in ${}^LG_\lambda/({}^LG_\lambda)^\circ$ is the central element of order 2. A computation shows that $\mathcal{F}_0$ acts on $({}^LG_\lambda)^\circ = C^\circ$ by inversion. It follows that the pair $(\lambda, \mathcal{F}_0)$ is discrete and that, more generally, a pair $(\lambda, \mathcal{F})$ is discrete if, and only if, writing $\mathcal{F} = c\mathcal{F}_0$ for some $c \in C$, the image of c in C/C° has order 1 or 3. Putting $\mathcal{F}_1 := c_1'c_2'\mathcal{F}_0$, this means that any $\mathcal{F}$ such that the pair $(\lambda, \mathcal{F})$ is discrete is a C°-translate of $\mathcal{F}_0$ or $\mathcal{F}_1^{\pm 1}$. Let us compute their orders. Since $\mathcal{F}_0^2 \in C^\circ$ is fixed by inversion, we have $\mathcal{F}_0^4 = 1$. On the other hand, since $\mathcal{F}_0$ is central in $\pi_0({}^LG_\lambda)$, there is some $c \in C^0$ such that $(c_1'c_2')\mathcal{F}_0(c_1'c_2')^{-1} = \mathcal{F}_0c$, which yields $(c_1'c_2')\mathcal{F}_0^2(c_1'c_2')^{-1} = \mathcal{F}_0c\mathcal{F}_0c = \mathcal{F}_0^2c^{-1}c = \mathcal{F}_0^2$. Hence $\mathcal{F}_0^2$ belongs to the $c_1'c_2'$-fixed subgroup of C°, which has order 3. This implies $\mathcal{F}_0^2 = 1$. On the other hand, there is some $c' \in C^\circ$ such that $\mathcal{F}_1^3 = \mathcal{F}_0c'$, which, as above, implies $\mathcal{F}_1^6 = (\mathcal{F}_0c')^2 = \mathcal{F}_0^2 = 1$. Having computed the orders 2 and 6, we now see that the desired inequalities follow from the list of weights, except maybe for weight 6 when $\mathcal{F} = \mathcal{F}_1^{\pm 1}$. However,an easy computation shows that the action of $\mathcal{F}_0$ on the weight 6 space is trivial, so that the action of $\mathcal{F}_1$ on it actually has order 3, and the desired inequalities hold too.

${}^2E_6$, orbits 14, 16, 18. Here $\hat{G}_\lambda = C = \mathbb{G}_m$, and $\mathcal{F}$ should map to the non-trivial element of $\pi_0({}^LG_\lambda) = \pi_0({}^LG)$, so $\mathcal{F}$ has order dividing 4. The desired inequalities are then straightforward for weights ≤ 7 since $h = 18$. For the weight 8, 10 or 14 spaces, we use the fact that C° acts trivially on them, so $\mathrm{Ad}_{\mathcal{F}}$ has order $t \leq 2$ there, whence the wanted inequality.

${}^2E_6$, orbits 17. Here $\hat{G}_\lambda = C = \{\pm 1\}$, hence $\mathcal{F}^2 = \pm 1$, and $\mathcal{F}$ has a priori order 2 or 4. But the representative e of the table is visibly invariant under Fr, which means that we can pick u and λ invariant under Fr, and set $\mathcal{F} = \mathrm{Fr}$ or $\mathcal{F} = (-1).\mathrm{Fr}$. In each case, $\mathcal{F}$ has order 2 and the desired inequalities follow from the list of weights.

${}^2E_6$, orbit 19. Here $\hat{G}_\lambda = C = \{1\}$, so $\mathcal{F}^2 = 1$, and the desired inequalities follow from the list of weights (recall $h = 18$).

E_7, orbit 24. Here $h = 18$, C° is a torus and $\pi_0(C) = S_2$. So $\mathcal{F}$ has order dividing 4. The desired inequality $4(\frac{m}{2}+1) \leq h = 18$ holds for all weights, except weight 8, but C° acts trivially on this weight space, so $\mathrm{Ad}_\mathcal{F}$ has order $t = 2$ there and the inequality holds too.

E_7, orbits 33, 37,41,42,43. In these cases $C = S_3$ or S_2 or $\{1\}$, and the inequalities are straightforward, except for the weight 18 space in orbit 41, where we need to use the column $\mathcal{Z}$ to ensure C acts trivially on this weight space.

E_7, orbit 39. Here $C^\circ = \mathbb{G}_m$ and $C/C^\circ = S_2$ acts non trivially on C°, but C is not a semi-direct product of C° by S_2, so $\mathcal{F}$ has order 4 with $\mathcal{F}^2 = -1 \in (C^\circ)^\mathcal{F}$. However, the explicit form of C° given in the table shows that it acts with even weights on all root subgroups (trivially on the simple roots of the Levi subsystem E_6 and with weight 2 on the remaining simple root). Hence $\mathrm{Ad}_\mathcal{F}$ has order 2, and the desired inequalities follow since all weights are even and less than 16.

E_8, orbit 41. Here $h = 30$ and $C = S_5$, so $t \leq 6$. Hence the desired inequalities are at least satisfied for all weights ≤ 8. This leaves us with the 4-dimensional weight 10 space Z_{10}, which is stable under $C = S_5$. We claim that Z_{10} is isomorphic to the standard representation of S_5. This implies that the eigenvalues of the elements of order 6 of S_5 have order $1, 2$ or 3, and not 6. So $t \leq 5$ on this space, and the desired inequalities still hold. To justify the claim, we use the notation of 9.3.17 of [LT11]. There, the authors exhibit three elements c_1, c_2 and c_3 that generate $C = S_5$, as well as a basis $z_{10}^1, \cdots, z_{10}^4$ of Z_{10}. The element c_1 is a 5-cycle, and all z_{10}^i are eigenvectors of $\mathrm{Ad}(c_1)$, with respective eigenvalues $\zeta, \zeta^3, \zeta^4, \zeta^2$ where ζ is a primitive 5^{th}-root of unity. This implies that Z_{10} is either the standard representation or its twist by the sign character. To show it is the untwisted standard representation, it suffices to show that the trace of a transposition is 2. According to *loc.cit.* the element c_2c_3 is a transposition. The action of $\mathrm{Ad}(c_2)$ and $\mathrm{Ad}(c_3)$ on Z_{10} is not made explicit in *loc.cit.* but according to the authors (private communication), they are given by matrices

$$\mathrm{Ad}(c_2) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathrm{Ad}(c_3) = \tfrac{1+2\phi}{5} \begin{pmatrix} -1 & \phi & 1 & 1+\phi \\ \phi & 1 & 1+\phi & -1 \\ 1 & 1+\phi & -1 & \phi \\ 1+\phi & -1 & \phi & 1 \end{pmatrix},$$

where $\phi = \zeta^2 + \zeta^3$. Thus the trace of $\mathrm{Ad}(c_2c_3)$ is $\frac{(1+2\phi)(2+4\phi)}{5} = 2$, as desired.

E_8, orbits 47, 50, 52. Here $h = 30$ and $C = \mathbb{G}_m$ and $C/C^\circ = S_2$. So $\mathcal{F}$ has order dividing 4, which makes directly all desired inequalities hold except for weights 14 and 16 subspaces, but the latter are fixed by C° according to column $\mathcal{Z}^\sharp$, so $\mathrm{Ad}_\mathcal{F}$ has order $t \leq 2$ there, and the inequalities hold too.

E_8, orbits 54, 58, 60, 62, 63, 65, 66, 67, 68. Here $C = S_3$, S_2 or $\{1\}$, and all inequalities are straightforward, except for the one dimensional weight 22 space in orbit 60 and weight 34 space in orbit 66. But the latter are fixed by C in each case according to column $\mathcal{Z}$, so $t = 1$ there and the inequalities still hold.

E_8, orbit 55. Here $C^\circ = \mathbb{G}_m$ and $C/C^\circ = S_2$. The table features a lift c in C of the non-trivial element of C/C°. One can compute that $c^2 = 1$ (e.g. by using the list of positive roots of E_8 in Bourbaki). So we can take $\mathcal{F} = c$ and the desired inequalities follow from the list of weights.

$\square$

Remark 5.24. Consider the orbit 3 of G_2, on page 73 of [LT11]. The reflection c_2 exchanges the two root vectors e_{11} and e_{21}, which have both weight 2. So -1 is an eigenvalue of $\mathcal{F} := c_2$ on the space generated by these vectors, hence $-q^2$ is an eigenvalue of $q\,\mathrm{Ad}_{\mathcal{F}\lambda(S)}$. But $\Phi_4(T)$ does not divide $\chi_{G_2,1}(T) = (T^2-1)(T^6-1)$, so we see that in this case the equivalence of Proposition 5.21 with the polynomial $\chi_{\hat{G},\mathrm{Fr}}$ really fails, just as for ${}^3D_4$. It fails also for F_4 due to the weight 8 space of orbit 13, which requires Φ_5 and Φ_{10} (depending on $\mathcal{F}$) although none of these polynomials divides $\chi_{F_4,1}$. The same orbit and the same weight space viewed in E_6 and ${}^2E_6$ through the identification of F_4 with the fixed points of the outer involution (orbit 17 in *loc.cit.*) again requires Φ_5 and Φ_{10}, although Φ_{10} does not divide $\chi_{E_6,1}$ and Φ_5 does not divide $\chi_{{}^2E_6,\mathrm{Fr}}$. In orbit 33 of E_7, taking $\mathcal{F} = c_1$, the weight 8 space requires Φ_{15}, which does not divide $\chi_{E_7,1}$. Finally, in orbit 66 of E_8, taking $\mathcal{F} = c$, the weight 26 space requires Φ_{28}, which does not divide $\chi_{E_8,1}$. Note it is certainly possible in each case to compute explicitly a polynomial χ' dividing χ^* for which equivalence between $\chi'(q) \neq 0$ and generic smoothness holds.

For convenience of the reader, we include a table showing the prime factors of $\chi_{\hat{G},\beta}$ in $\mathbb{Z}[T]$ for the exceptional types.

type of $\hat{G},\beta$	${}^3D_4$	G_2	F_4	E_6	${}^2E_6$	E_7	E_8
$\left\{n, \Phi_n \mid \chi_{\hat{G},\beta}\right\}$	$\left\{\begin{matrix}1,2,3\\6,12\end{matrix}\right\}$	$\left\{\begin{matrix}1,2\\3,6\end{matrix}\right\}$	$\left\{\begin{matrix}1,\ldots,4,\\6,8,12\end{matrix}\right\}$	$\left\{\begin{matrix}1,\ldots,6,\\8,9,12\end{matrix}\right\}$	$\left\{\begin{matrix}1,\ldots,4\\6,8,10\\12,18\end{matrix}\right\}$	$\left\{\begin{matrix}1,\ldots,10,\\12,14,18\end{matrix}\right\}$	$\left\{\begin{matrix}1,\ldots,10,12\\14,15,18\\20,24,30\end{matrix}\right\}$

Corollary 5.25. *Let G be as in (5.2). If $\chi^*_{\hat{G},\mathrm{Fr}}(q) \neq 0$ in L, then for any φ as in (5.2), there is $g \in \hat{G}^\circ_{\varphi(s)}$ such that φ^g is unobstructed.*

Proof. This follows from Lemma 5.10, Lemma 5.12, decomposition (5.3), Proposition 5.21, Lemma 5.22 and Proposition 5.23. Note again that $\chi^*_{\hat{G},\mathrm{Fr}}(q) \neq 0$ is equivalent to q having order greater than $h_{\hat{G},\mathrm{Fr}}$, which implies $\ell > h_{\hat{G},\mathrm{Fr}}$ hence also $\ell > h_{\hat{G}}$. It also implies that $\chi_{\hat{G},\mathrm{Fr}}(q) \neq 0$. $\square$

Proof of Theorem 5.7. (1) We assume that ℓ does not divide $e\chi^*_{\hat{G},\mathrm{Fr}}(q)$. Fix $\varphi \in Z^1(W_F, \hat{G}(L))$ and choose φ' as in Proposition 5.8. By Lemma 5.9, the action $\mathrm{Ad}_{\varphi'}$ of W_F on $\hat{H} := C_{\hat{G}}(\varphi(I_F^\ell))^\circ$ is unramified and $\eta := \varphi \cdot (\varphi')^{-1} \in Z^1(W_F/I_F^\ell, \hat{H}(L))$. By Proposition B.3, we have $\chi^*_{\hat{H},\mathrm{Ad}_{\varphi(\mathrm{Fr})}}(q) \neq 0$ in L, so the last Corollary gives us an element $h \in (\hat{H}_{\eta(s)})^\circ$ such that η^h is unobstructed in $Z^1(W_F/I_F^\ell, \hat{H}(L))$. We have explained after Proposition 5.8 that $\eta^h \cdot \varphi'$ is then unobstructed in $Z^1(W_F, \hat{G}(L))$, but we have $(\hat{H}_{\eta(s)})^\circ = (\hat{G}_\tau)^\circ$ and $\eta^h \cdot \varphi' = (\eta \cdot \varphi')^h = \varphi^h$.

(2) We assume here that $\hat{G}$ has no exceptional factor, that $\ell > h_{\hat{G},1}$, and that ℓ does not divide $\chi_{\hat{G},\mathrm{Fr}}(q)$. Then we repeat the above argument, observing that $\hat{H} = C_{\hat{G}}(\varphi(I_F^\ell))$ is again a group with no exceptional component. Indeed, it suffices to check this in a classical group where it is fairly standard. However, the action of $\varphi'(\mathrm{Fr})$ on $\hat{H}$ may feature instances of triality. Fortunately, this is harmless because

the modified polynomial $\chi'_{\hat{H},\mathrm{Fr}}$ still divides $\chi_{\hat{G},\mathrm{Fr}}$. Indeed, if $\Phi_{12}(T)$ divides $\chi_{\hat{G},\mathrm{Fr}}$ for $\hat{G}$ a classical group, then so does $\Phi_4(T)$.

□

5.4. ${}^LG$**-banal primes.** We keep the general setup of this section.

Proposition 5.26. *Let $\ell \neq p$ be a prime. Then the following are equivalent :*

(1) For every algebraically closed field of characteristic ℓ, and every continuous L-homomorphism $\varphi : W_F \to {}^LG(L)$, there is $g \in C_{\hat{G}}(\varphi(I_F))^\circ$ such that φ^g is unobstructed.

(2) For any $e \in \mathbb{N}$ and any finite place v of $\mathcal{O}_{K_e}[\frac{1}{p}]$ such that the residue field k_v has characteristic ℓ, the fiber $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_{k_v})$ of $\underline{Z}^1(W_F^0/P_F^e, \hat{G})$ is reduced.

Proof. Assume (1). It suffices to prove reducedness for k_v replaced by its algebraic closure L. Let x be an L-point of $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ contained on exactly one irreducible component of $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$. Let τ be the restriction of φ_x to I_F; there then exists a $g \in \hat{G}_\tau^\circ(L)$ such that φ_x^g is unobstructed. The corresponding L-point y of $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is smooth and lies in the same irreducible component of $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ as x, so that irreducible component is generically reduced. Since x was arbitrary, we deduce that $\underline{Z}^1(W_F^0/P_F^e, \hat{G}_L)$ is generically reduced; since it is also a local complete intersection, it must be reduced.

Now assume (2). Following the same reduction process as above Proposition 5.8, we may assume that ${}^LG$ and φ are as in (5.2). Recall from the discussion above Lemma 2.2 the map

$$\hat{G}_L \times \hat{G}^\circ_{\varphi(s)} \longrightarrow \underline{Z}^1(W_F/I_F^\ell, \hat{G})_L,\ (h,g) \mapsto {}^h(\varphi^g).$$

We have shown there that $\underline{Z}^1(W_F/I_F^\ell, \hat{G})_L$ is covered by the images of finitely many of these maps, and that these images all have the same dimension. This implies that the closure of these images are the irreducible components of $\underline{Z}^1(W_F/I_F^\ell, \hat{G})_L$. In particular, the image of the above map is dense in one of the components that contain φ. Therefore, since reducedness implies generic smoothness of all components, we get (1). □

Definition 5.27. A prime $\ell \neq p$ is ${}^LG$-banal if the properties of the last proposition hold for ℓ.

One way to view this reducedness of fibers from a philosophical standpoint is to say that there are no nontrivial "congruences" between Langlands parameters modulo an ${}^LG$-banal prime ℓ: the closures of distinct irreducible components in characteristic zero remain distinct modulo ℓ. One expects that this should correspond, on the other side of the local Langlands correpondence, to a lack of nontrivial congruences between admissible smooth representations of the reductive group G over F whose L-group is ${}^LG$. So this should be related to the representation theoretic notion of "banal". Recall indeed that, for a reductive group G over F, a prime $\ell \neq p$ is called *banal* if it does not divide the order of a torsion element of $G(F)$. For the sake of precision, we will reterm this as "G-banal".

Lemma 5.28. *Suppose that G is a reductive group over $\mathcal{O}_F$.*

1. *A prime $\ell \neq p$ is G-banal if and only if it does not divide the order of $G(k_F)$.*

(2) *The set of G-banal primes only depends on the isogeny class of G.*
(3) *We have $|G(k_F)| = q^N \cdot \chi_{G,\mathrm{Fr}}(q)$ where N is the dimension of a maximal unipotent subgroup of G.*

Proof. (1) Let $g \in G(F)$ have finite order prime to p. Then it stabilizes a facet of the Bruhat-Tits building of $G(F)$, and fixes its barycenter. This barycenter becomes a hyperspecial point in the building of $G(F')$ for some totally ramified extension of F. So the order of g divides $|G(k_{F'})|$, but $k_{F'} = k_F$. Conversely, let ℓ be a prime that divides $|G(k_F)|$ and pick an element $\bar{g} \in G(k_F)$ with order ℓ. Choose a lift $g \in G(\mathcal{O}_F)$ of $\bar{g}$ and consider the topological Jordan decomposition $g = g_{as}g_{tu}$ of g as in [Spi08, Thm 2.38]. Then $g_{as} \in G(\mathcal{O}_F)$ and it has order ℓ.

(2) This follows from (1), see the proof of Theorem B.4.

(3) This is the Chevalley-Steinberg formula, see Theorem B.4. $\square$

Corollary 5.29. *Suppose G is an unramified group over F with no exceptional factor, denote by ${}^LG = \hat{G} \rtimes \langle \mathrm{Fr} \rangle$ its Langlands dual group, and let ℓ be a prime greater than the Coxeter number of G. Then ℓ is ${}^LG$-banal if and only if it is G-banal.*

Proof. This follows from the above lemma together with Proposition 5.21, and the equality $\chi_{\hat{G},\mathrm{Fr}}(T) = \chi_{G,\mathrm{Fr}}(T)$. $\square$

It is a bit surprising that our results in Lemma 5.22 and Remark 5.24 show that this equivalence does not hold for exceptional groups.

6. The GIT quotient in the banal case

Our aim in this section is to get a complete description of the affine quotient $\underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G}$ after base change to $\overline{\mathbb{Z}}[\frac{1}{N}]$ for some sufficiently well controlled integer N. Our strategy rests on the universal homeomorphism (4.10)

$$\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G} \longrightarrow \underline{Z}^1(W_F^0/P_F^e, \hat{G}) /\!\!/ \hat{G}.$$

We have already singled out the so-called ${}^LG$-banal primes, which are particularly well behaved for the RHS. On the other hand, the integer $N_{\hat{G}}$ defined above Corollary 4.16 plays a particular role regarding the LHS:

Lemma 6.1. *The structural morphism $\underline{Z}^1(W_F/I_F^e, \hat{G}) \longrightarrow \mathrm{Spec}(\mathbb{Z}[\frac{1}{p}])$ is smooth over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{pN_{\hat{G}}}])$.*

Proof. Since the finite group I_F/I_F^e has invertible order in $\mathbb{Z}[\frac{1}{pN_{\hat{G}}}]$, Lemma A.1 tells us that $\underline{Z}^1(I_F/I_F^e, \hat{G})$ is smooth over $\mathrm{Spec}(\mathbb{Z}[\frac{1}{pN_{\hat{G}}}])$. Let ϕ_{univ} denote the universal 1-cocycle $I_F/I_F^e \longrightarrow \hat{G}(\mathcal{O}_{\underline{Z}^1(I_F/I_F^e,\hat{G})})$. Then the map $\varphi \mapsto \varphi(\mathrm{Fr})$ identifies $\underline{Z}^1(W_F/I_F^e, \hat{G})$ with the $\hat{G}$-transporter from ${}^{\mathrm{Fr}}\phi_{\mathrm{univ}}$ to ϕ_{univ}, as a scheme over $\underline{Z}^1(I_F/I_F^e, \hat{G})$. Hence, by Lemma A.1 again, the restriction map $\underline{Z}^1(W_F/I_F^e, \hat{G}) \longrightarrow \underline{Z}^1(I_F/I_F^e, \hat{G})$ is also smooth, and the lemma follows. $\square$

Recall that the universal homeomorphism (4.10) becomes an isomorphism after tensoring by $\mathbb{Q}$. The next result gives a bound on the set of integers that actually need to be inverted.

Proposition 6.2. *The morphism* $\underline{Z}^1(W_F/I_F^e,\hat{G}) /\!\!/ \hat{G} \longrightarrow \underline{Z}^1(W_F^0/P_F^e,\hat{G}) /\!\!/ \hat{G}$ *of (4.10) is an isomorphism after inverting* $N_{\hat{G}}$ *and the non* ${}^LG$*-banal primes.*

Proof. Consider the dual map (4.9) on rings of functions $(R^e_{{}^LG})^{\hat{G}} \longrightarrow (S^e_{{}^LG})^{\hat{G}}$. We already know it is injective and its cokernel is a torsion abelian group. Let $\ell \neq p$ be an associated prime of this cokernel. If ℓ does not divide $N_{\hat{G}}$, there is no ℓ-torsion in $S^e_{{}^LG}$ (by the last lemma), hence the reduced map $(R^e_{{}^LG})^{\hat{G}}\otimes\mathbb{F}_\ell \longrightarrow (S^e_{{}^LG})^{\hat{G}}\otimes\mathbb{F}_\ell$ is not injective. But this map induces a bijection on $\overline{\mathbb{F}}_\ell$-points, so its kernel lies in the Jacobson radical, and we deduce that $(R^e_{{}^LG})^{\hat{G}}\otimes\mathbb{F}_\ell$ is not reduced. On the other hand, $(R^e_{{}^LG})^{\hat{G}}$ is an ℓ-adically saturated submodule of $R^e_{{}^LG}$, so that the map $(R^e_{{}^LG})^{\hat{G}}\otimes\mathbb{F}_\ell \longrightarrow (R^e_{{}^LG}\otimes\mathbb{F}_\ell)^{\hat{G}}$ is actually injective. So we infer that $R^e_{{}^LG}\otimes\mathbb{F}_\ell$ is not reduced, hence ℓ is not ${}^LG$-banal. ☐

Remark 6.3. When ${}^LG$ is the Langlands dual group of an unramified group, Proposition 5.18 and the estimate of Proposition 4.14 show that the prime divisors of $N_{\hat{G}}$ are non ${}^LG$-banal. We believe this is true in general.

In view of the last proposition, we focus in the next subsection on the explicit description of $\underline{Z}^1(W_F/I_F^e,\hat{G}) /\!\!/ \hat{G}$, over $\overline{\mathbb{Z}}[\frac{1}{pN_{\hat{G}}}]$. The description that we obtain in Theorem 6.7 bears a striking analogy with the usual description of the Bernstein center. Actually, in Subsection 6.3, we extend scalars to $\mathbb{C}$ and we show that our description gives back Haines' definition of a structure of algebraic variety on the set of semisimple complex Langlands parameters.

6.1. **Description of $\underline{Z}^1(W_F/I_F^e,\hat{G}) /\!\!/ \hat{G}$ over $\overline{\mathbb{Z}}[\frac{1}{pN_{\hat{G}}}]$.** Since the order of I_F/I_F^e is invertible in $\mathbb{Z}[\frac{1}{pN_{\hat{G}}}]$, we can obtain decompositions of $\underline{Z}^1(W_F/I_F^e,\hat{G})_{\overline{\mathbb{Z}}[\frac{1}{pN_{\hat{G}}}]}$ similar to (4.2) and (4.4) by restricting cocycles to I_F instead of restricting to P_F. Indeed, we first infer the following results from Theorems A.7, A.9, A.12 and A.13 in the appendix.

Proposition 6.4. *There is a finite extension* $\tilde{K}_e$ *of* K_e *and a set*

$$\tilde{\Phi}_e \subset Z^1\left(I_F/I_F^e,\hat{G}\left(\mathcal{O}_{\tilde{K}_e}[\tfrac{1}{pN_{\hat{G}}}]\right)\right), \textit{ such that}$$

- *for each* $\phi\in\tilde{\Phi}_e$*, the group scheme* $C_{\hat{G}}(\phi)^\circ$ *is split reductive and* $\pi_0(\phi) := \pi_0(C_{\hat{G}}(\phi))$ *is constant over* $\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]$ *and*
- *we have an orbit decomposition*

$$\underline{Z}^1(I_F/I_F^e,\hat{G})_{\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]} = \coprod_{\phi\in\tilde{\Phi}_e} \hat{G}\cdot\phi \simeq \coprod_{\phi\in\tilde{\Phi}_e} \hat{G}/C_{\hat{G}}(\phi)$$

where each summand represents the corresponding étale sheaf quotient.

The above decomposition induces in turn the following ones :

$$\underline{Z}^1(W_F/I_F^e,\hat{G})_{\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]} = \coprod_{\phi\in\tilde{\Phi}_e^{\mathrm{adm}}} \hat{G}\times^{C_{\hat{G}}(\phi)} \underline{Z}^1(W_F,\hat{G})_\phi.$$

$$(\underline{Z}^1(W_F/I_F^e,\hat{G}) /\!\!/ \hat{G})_{\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]} = \coprod_{\phi\in\tilde{\Phi}_e^{\mathrm{adm}}} \underline{Z}^1(W_F,\hat{G})_\phi /\!\!/ C_{\hat{G}}(\phi). \tag{6.1}$$

Here, $\underline{Z}^1(W_F,\hat{G})_\phi$ is the affine scheme over $\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]$ that classifies all 1-cocycles $\varphi : W_F \longrightarrow \hat{G}$ such that $\varphi_{|I_F} = \phi$ and, as usual, we say that ϕ is admissible if this scheme is not empty.

Define the Fr-twist of ϕ by ${}^{\mathrm{Fr}}\phi(i) := \mathrm{Fr}(\phi(\mathrm{Fr}^{-1}\, i\, \mathrm{Fr}))$. Then we have an isomorphism $\varphi \mapsto \varphi(\mathrm{Fr})$

$$\underline{Z}^1(W_F,\hat{G})_\phi \xrightarrow{\sim} T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)$$

where the RHS denotes the transporter in $\hat{G}$ from ${}^{\mathrm{Fr}}\phi$ to ϕ for the natural action of $\hat{G}$ on $\underline{Z}^1(I_F,\hat{G})$. This isomorphism is $C_{\hat{G}}(\phi)$-equivariant if we let $C_{\hat{G}}(\phi)$ act on the transporter by Fr-twisted conjugation $c\cdot t := ct\,\mathrm{Fr}(c)^{-1}$. On the other hand, $T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)$ is also a left pseudo-torsor over $C_{\hat{G}}(\phi)$ under composition $(c,t)\mapsto ct$. When ϕ is admissible, $T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)$ is actually a $C_{\hat{G}}(\phi)$-torsor for the étale topology, and the étale sheaf quotient $\pi_0({}^{\mathrm{Fr}}\phi,\phi) := T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)/C_{\hat{G}}(\phi)^\circ$ is a $\pi_0(\phi)$-torsor. Therefore $\pi_0({}^{\mathrm{Fr}}\phi,\phi)$ is representable by a finite étale $\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]$-scheme and, after maybe enlarging $\tilde{K}_e$, we may and will assume that it is *constant*. Then we get a further decomposition

$$T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi) = \coprod_{\beta\in\pi_0({}^{\mathrm{Fr}}\phi,\phi)} T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)_\beta$$

which is nothing but the decomposition into connected components, and where each component is a left $C_{\hat{G}}(\phi)^\circ$-torsor. Moreover, the Fr-twisted conjugation action of $C_{\hat{G}}(\phi)$ on $T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)$ induces an action of $\pi_0(\phi)$ on $\pi_0({}^{\mathrm{Fr}}\phi,\phi)$. Denote by $\pi_0(\phi)_\beta$ the stabilizer of β for this action, and by $\pi_0({}^{\mathrm{Fr}}\phi,\phi)_0$ a set of representatives of orbits. Then we get

$$T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)/\!\!/ C_{\hat{G}}(\phi) = \coprod_{\beta\in\pi_0({}^{\mathrm{Fr}}\phi,\phi)_0} \left(T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi)_\beta /\!\!/ C_{\hat{G}}(\phi)^\circ\right)_{/\pi_0(\phi)_\beta}. \tag{6.2}$$

Our next result will allow us to compute each term of this decomposition. Before we can state it, note that if R is an $\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]$-algebra and $\tilde{\beta}\in T_{\hat{G}(R)}({}^{\mathrm{Fr}}\phi,\phi)$, then conjugation by $\tilde{\beta}\rtimes\mathrm{Fr}$ in $\hat{G}(R)\rtimes W_F$ normalizes $C_{\hat{G}(R)}(\phi)$. We denote the automorphism thus induced by $\mathrm{Ad}_{\tilde{\beta}}$.

Theorem 6.5. *Fix a pinning $\varepsilon_\phi = (B_\phi, T_\phi, (X_\alpha)_\alpha)$ of $C_{\hat{G}}(\phi)^\circ$ over $\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]$. Then, after maybe enlarging the finite extension $\tilde{K}_e$, we can find for each $\beta\in\pi_0({}^{\mathrm{Fr}}\phi,\phi)$, a lift $\tilde{\beta}\in T_{\hat{G}(\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}])}({}^{\mathrm{Fr}}\phi,\phi)$ of β such that $\mathrm{Ad}_{\tilde{\beta}}$ normalizes ε_ϕ.*

Proof. The proof goes along the same argument as for Theorem 3.4. Let us first do the translation to the notation of Subsection 3.2. To this aim, choose an L-group such that I_F/I_F^e embeds into $\pi_0({}^LG)$. Then we can form the subgroup scheme $C_{{}^LG}(\phi)$ as in Subsection 3.2 and, denoting by $\overline{\mathrm{Fr}}$ the image of Fr in $\pi_0({}^LG)$, we see that the map $\varphi\mapsto {}^L\varphi(\mathrm{Fr})$ defines an isomorphism

$$\underline{Z}^1(W_F,\hat{G})_\phi \xrightarrow{\sim} C_{{}^LG}(\phi)\cap(\hat{G}\rtimes\overline{\mathrm{Fr}}),$$

and that the map $\tilde{\beta}\mapsto\tilde{\beta}\rtimes\overline{\mathrm{Fr}}$ defines a second isomorphism

$$T_{\hat{G}}({}^{\mathrm{Fr}}\phi,\phi) \xrightarrow{\sim} C_{{}^LG}(\phi)\cap(\hat{G}\rtimes\overline{\mathrm{Fr}})$$

whose composition with the previous one is the isomorphism introduced just above. Now, define $\tilde{\pi}_0(\phi)$ and $\Sigma(\phi)$ as above Definition 3.3, so that we have for each admissible ϕ a further decomposition

$$\underline{Z}^1(W_F, \hat{G})_\phi = \coprod_{\alpha\in\Sigma(\phi)} \underline{Z}^1(W_F, \hat{G})_{\phi,\alpha}.$$

Then we have a bijection $\alpha \mapsto \alpha(\mathrm{Fr})$ between $\Sigma(\phi)$ and the fiber of the map $\tilde{\pi}_0(\phi) \to \pi_0({}^LG)$ over $\overline{\mathrm{Fr}}$. On the other hand, we have a natural injection $\pi_0({}^{\mathrm{Fr}}\phi, \phi) \hookrightarrow \tilde{\pi}_0(\phi)$ whose image is precisely the said fiber. So we get a bijection $\beta \leftrightarrow \alpha$ between $\pi_0({}^{\mathrm{Fr}}\phi, \phi)$ and $\Sigma(\phi)$, and it is easily checked that the map $\varphi \mapsto \varphi(\mathrm{Fr})$ identifies $\underline{Z}^1(W_F, \hat{G})_{\phi,\alpha}$ with $T_{\hat{G}}({}^{\mathrm{Fr}}\phi, \phi)_\beta$.

Now the same proof as that of Theorem 3.4 applies, and actually the stronger variant of Remark 3.9 applies too, because what is needed from Lemma 3.8 in the proof of this variant is now trivial : since $W_F/I_F \simeq \mathbb{Z}$, we have $H^2(W_F/I_F, A) = \{0\}$ for any abelian group A with action of W_F/I_F.

So we get the existence of a finite extension $\tilde{K}_e$ and, for each α, a cocycle

$$\varphi_\alpha : W_F \longrightarrow \hat{G}\left(\mathcal{O}_{\tilde{K}_e}[\tfrac{1}{pN_{\hat{G}}}]\right) \tag{6.3}$$

that restricts to ϕ, induces α, normalizes ε_ϕ and has finite image. Writing $\varphi_\alpha(\mathrm{Fr}) = \tilde{\beta} \rtimes \overline{\mathrm{Fr}}$ provides us with the desired element $\tilde{\beta}$. $\square$

With the notation of this theorem we now have an identification $c \mapsto c\tilde{\beta}$

$$C_{\hat{G}}(\phi)^\circ \xrightarrow{\sim} T_{\hat{G}}({}^{\mathrm{Fr}}\phi, \phi)_\beta$$

and the Fr-twisted conjugation action of $C_{\hat{G}}(\phi)^\circ$ on $T_{\hat{G}}({}^{\mathrm{Fr}}\phi, \phi)_\beta$ corresponds to the $\mathrm{Ad}_{\tilde{\beta}}$-twisted conjugation action of $C_{\hat{G}}(\phi)^\circ$ on itself. We thus get

$$T_{\hat{G}}({}^{\mathrm{Fr}}\phi, \phi)_\beta /\!\!/ C_{\hat{G}}(\phi)^\circ = (C_{\hat{G}}(\phi)^\circ \rtimes \mathrm{Ad}_{\tilde{\beta}}) /\!\!/ C_{\hat{G}}(\phi)^\circ$$

where the notation on the right hand side is meant to emphasize that $C_{\hat{G}}(\phi)^\circ$ acts via $\mathrm{Ad}_{\tilde{\beta}}$-twisted conjugation.

Now, denote by Ω_ϕ° the Weyl group of the maximal torus T_ϕ of $C_{\hat{G}}(\phi)^\circ$, and denote by $\Omega_\phi := N_{C_{\hat{G}}(\phi)}(T_\phi)/T_\phi$ its "Weyl group" in $C_{\hat{G}}(\phi)$. The natural map $N_{C_{\hat{G}}(\phi)}(T_\phi, B_\phi) \longrightarrow \pi_0(\phi)$ induces an isomorphism $N_{C_{\hat{G}}(\phi)}(T_\phi, B_\phi)/T_\phi \simeq \pi_0(\phi)$, hence $\Omega_\phi = \Omega_\phi^\circ \rtimes \pi_0(\phi)$ is a split extension of $\pi_0(\phi)$ by Ω_ϕ°. Since the automorphism $\mathrm{Ad}_{\tilde{\beta}}$ of $C_{\hat{G}}(\phi)$ stabilizes T_ϕ and B_ϕ, it acts on Ω_ϕ and preserves the semi-direct product decomposition. Note that the actions of $\mathrm{Ad}_{\tilde{\beta}}$ on T_ϕ and Ω_ϕ only depend on β and not on the choice of $\tilde{\beta}$ as in the theorem. We will thus denote these actions simply by Ad_β. Observe that the invariant subgroup $(\Omega_\phi)^{\mathrm{Ad}_\beta}$ of Ω_ϕ decomposes as

$$(\Omega_\phi)^{\mathrm{Ad}_\beta} = (\Omega_\phi^\circ)^{\mathrm{Ad}_\beta} \rtimes \pi_0(\phi)_\beta$$

and acts naturally on the coinvariant torus $(T_\phi)_{\mathrm{Ad}_\beta}$.

Proposition 6.6. *The inclusion $T_\phi \hookrightarrow C_{\hat{G}}(\phi)^\circ$ induces an isomorphism*

$$(T_\phi)_{\mathrm{Ad}_\beta} /\!\!/ (\Omega_\phi^\circ)^{\mathrm{Ad}_\beta} \xrightarrow{\sim} \left(C_{\hat{G}}(\phi)^\circ \rtimes \mathrm{Ad}_{\tilde{\beta}}\right) /\!\!/ C_{\hat{G}}(\phi)^\circ$$

Proof. Consider the inclusion $T_\phi \rtimes \mathrm{Ad}_{\tilde{\beta}} \hookrightarrow C_{\hat{G}}(\phi)^\circ \rtimes \mathrm{Ad}_{\tilde{\beta}}$. Under the conjugation action of $C_{\hat{G}}(\phi)^\circ$ on the RHS, the LHS is stable by the subgroup scheme $N_{C_{\hat{G}}(\phi)^\circ}(T_\phi)_\beta$ of $N_{C_{\hat{G}}(\phi)^\circ}(T_\phi)$ given as the inverse image of $(\Omega_\phi^\circ)^{\mathrm{Ad}_\beta}$. Whence a morphism

$$\left(T_\phi \rtimes \mathrm{Ad}_{\tilde{\beta}}\right) /\!\!/ N_{C_{\hat{G}}(\phi)^\circ}(T_\phi)_\beta \longrightarrow \left(C_{\hat{G}}(\phi)^\circ \rtimes \mathrm{Ad}_{\tilde{\beta}}\right) /\!\!/ C_{\hat{G}}(\phi)^\circ.$$

Now observe that $(T_\phi)_{\mathrm{Ad}_\beta} = \left(T_\phi \rtimes \mathrm{Ad}_{\tilde\beta}\right) /\!\!/ T_\phi$, so that the above morphism induces in turn a morphism

$$(T_\phi)_{\mathrm{Ad}_\beta} /\!\!/ (\Omega_\phi^\circ)^{\mathrm{Ad}_\beta} \longrightarrow (C_{\hat G}(\phi)^\circ \rtimes \mathrm{Ad}_{\tilde\beta}) /\!\!/ C_{\hat G}(\phi)^\circ.$$

Now, for any algebraically closed field L over $\mathcal{O}_{\tilde K_e}[\frac{1}{pN_{\hat G}}]$, Lemma 6.5 of [Bor79] tells us that this morphism induces a bijection on L-points. In particular the corresponding map on rings of functions is injective since the source is reduced. Its surjectivity can be proved as in [Bor79, Prop. 6.7], which deals with complex coefficients. Namely, put $R := \mathcal{O}_{\tilde K_e}[\frac{1}{pN_{\hat G}}]$ and let X denote the character group of T_ϕ. Then the ring of functions of $(T_\phi)_{\mathrm{Ad}_\beta} /\!\!/ (\Omega_\phi^\circ)^{\mathrm{Ad}_\beta}$ is $R[X^{\mathrm{Ad}_\beta}]^{(\Omega_\phi^\circ)^{\mathrm{Ad}_\beta}}$, hence has a natural R-basis given by $(\Omega_\phi^\circ)^{\mathrm{Ad}_\beta}$-orbits in X^{Ad_β}. Any such orbit has a unique representative in the antidominant cone of X with respect to B_ϕ. So let $\lambda \in X^{\mathrm{Ad}_\beta}$ be antidominant in X and let $\mathcal{L}_\lambda$ be the corresponding invertible sheaf on the flag variety $C_{\hat G}(\phi)^\circ/B_\phi$. Then $M_\lambda := H^0(C_{\hat G}(\phi)^\circ/B_\phi, \mathcal{L}_\lambda)$ is a free R-module of finite rank with an algebraic action of $C_{\hat G}(\phi)^\circ$. Actually, since λ is Ad_β-invariant, it defines a character of the group scheme $T_\phi \rtimes \langle \mathrm{Ad}_\beta\rangle$, and since $C_{\hat G}(\phi)^\circ/B_\phi = (C_{\hat G}(\phi)^\circ \rtimes \langle \mathrm{Ad}_\beta\rangle)/(B_\phi \rtimes \langle \mathrm{Ad}_\beta\rangle)$, we see that M_λ is actually a $C_{\hat G}(\phi)^\circ \rtimes \langle \mathrm{Ad}_\beta\rangle$-module. In particular, the map $g \mapsto \mathrm{tr}(g \rtimes \mathrm{Ad}_\beta | M_\lambda)$ is in the ring of functions of $(C_{\hat G}(\phi)^\circ \rtimes \mathrm{Ad}_{\tilde\beta}) /\!\!/ C_{\hat G}(\phi)^\circ$. Its restriction to T_ϕ factors over $(T_\phi)_{\mathrm{Ad}_\beta}$ and is of the form

$$c\left(\sum_{\lambda' \in (\Omega_\phi^\circ)^{\mathrm{Ad}_\beta}.\lambda} \lambda'\right) + \sum_{\mu > \lambda} a_\mu \mu, \;\; a_\mu \in \mathbb{N},$$

where c denotes the eigenvalue of Ad_β on the λ-eigenspace of T_ϕ in M_λ (which is a free direct factor of rank 1). So we deduce inductively the desired surjectivity. ☐

Eventually, after choosing a pinning ε_ϕ for each $\phi \in \tilde\Phi_e$ and inserting the result of the above proposition inside decompositions (6.1) and (6.2), we get our desired description of the affine quotient over $\mathcal{O}_{\tilde K_e}[\frac{1}{pN_{\hat G}}]$.

Theorem 6.7. *The collection of embeddings $T_\phi \hookrightarrow C_{\hat G}(\phi)$ induce an isomorphism of $\mathcal{O}_{\tilde K_e}[\frac{1}{pN_{\hat G}}]$-schemes*

$$\coprod_{\phi \in \tilde\Phi_e^{\mathrm{adm}}} \coprod_{\beta \in \pi_0({}^{\mathrm{Fr}}\phi, \phi)_0} (T_\phi)_{\mathrm{Ad}_\beta} /\!\!/ (\Omega_\phi)^{\mathrm{Ad}_\beta} \xrightarrow{\sim} (\underline{Z}^1(W_F/I_F^e, \hat G) /\!\!/ \hat G)_{\mathcal{O}_{\tilde K_e}[\frac{1}{pN_{\hat G}}]}.$$

We note that the LHS does not depend on the choices of elements $\tilde\beta$ as in Theorem 6.5. But the maps from the LHS to the RHS a priori depend on these choices.

6.2. **The GIT quotient over a banal algebraically closed field.** With Theorem 6.7 and Proposition 6.2, we now have a description of the affine quotient $\underline{Z}^1(W_F^0/P_F^e, \hat G) /\!\!/ \hat G$ after inverting $N_{\hat G}$ and the non ${}^LG$-banal primes. Let us now consider affine quotients over algebraically closed fields.

Theorem 6.8. *Let L be an algebraically closed field over $\mathcal{O}_{\tilde{K}_e}[\frac{1}{pN_{\hat{G}}}]$ and of ${}^LG$-banal characteristic. Then the natural maps induce isomorphisms*

$$\begin{aligned}\coprod_{\phi\in\tilde{\Phi}_e^{\mathrm{adm}}}\coprod_{\beta\in\pi_0({}^{\mathrm{Fr}}\phi,\phi)_0}(T_{\phi,L})_{\mathrm{Ad}_\beta}/\!\!/(\Omega_\phi)^{\mathrm{Ad}_\beta} &\xrightarrow{\sim} \underline{Z}^1(W_F/I_F^e,\hat{G}_L)/\!\!/\hat{G}_L\\ \underline{Z}^1(W_F/I_F^e,\hat{G}_L)/\!\!/\hat{G}_L &\xrightarrow{\sim} \underline{Z}^1(W_F^0/P_F^e,\hat{G}_L)/\!\!/\hat{G}_L\\ \underline{Z}^1(W_F^0/P_F^e,\hat{G}_L)/\!\!/\hat{G}_L &\xrightarrow{\sim} (\underline{Z}^1(W_F^0/P_F^e,\hat{G})/\!\!/\hat{G})_L.\end{aligned}$$

Proof. The first isomorphism holds without the ${}^LG$-banal hypothesis, and is proved exactly as the isomorphism of Theorem 6.7. We then have a commutative diagram

$$\begin{tikzcd}\coprod_{(\phi,\beta)}(T_{\phi,L})_{\mathrm{Ad}_\beta}/\!\!/(\Omega_\phi)^{\mathrm{Ad}_\beta} \arrow[r, "\sim"] \arrow[d] & \underline{Z}^1(W_F/I_F^e,\hat{G}_L)/\!\!/\hat{G}_L \arrow[d]\\ \coprod_{(\phi,\beta)}\left((T_\phi)_{\mathrm{Ad}_\beta}/\!\!/(\Omega_\phi)^{\mathrm{Ad}_\beta}\right)_L \arrow[r, "\sim"] & \left(\underline{Z}^1(W_F/I_F^e,\hat{G})/\!\!/\hat{G}\right)_L\end{tikzcd}$$

If, in addition, the order of each $(\Omega_\phi)^{\mathrm{Ad}_\beta}$ is invertible in L, which is certainly the case if $\mathrm{char}(L)$ is ${}^LG$-banal, then the left vertical map is an isomorphism, and it follows that the right vertical map is also an isomorphism. So we now have a commutative square involving the analogous map for W_F^0/P_F^e, which, in terms of rings, reads

$$\begin{tikzcd}(R^e_{{}^LG})^{\hat{G}}\otimes L \arrow[r] \arrow[d, "\sim"] & (R^e_{{}^LG}\otimes L)^{\hat{G}} \arrow[d]\\ (S^e_{{}^LG})^{\hat{G}}\otimes L \arrow[r, "\sim"] & (S^e_{{}^LG}\otimes L)^{\hat{G}}\end{tikzcd}$$

The right vertical map is surjective since the bottom map is surjective, and it is also injective since it induces a bijection on L-points and $(R^e_{{}^LG}\otimes L)^{\hat{G}}$ is reduced. Therefore it is an isomorphism, and so is the upper map. $\square$

Remark 6.9. Let L be an algebraically closed field as in the theorem.

i) The index set in the first isomorphism can be replaced by any set $\Psi_e(L)$ of representatives of $\hat{G}(L)$-conjugacy classes of pairs (ϕ,β) consisting of a cocycle $\phi: I_F/I_F^e \longrightarrow {}^LG(L)$ and an element $\beta\in\pi_0({}^{\mathrm{Fr}}\phi,\phi)$. Since any ϕ as above is automatically semisimple, this is the same set as in Corollary 4.22.

ii) As usual, the set of L-points of $\underline{Z}^1(W_F/I_F^e,\hat{G}_L)/\!\!/\hat{G}_L$ is the set of closed $\hat{G}_L$-orbits in $Z^1(W_F/I_F^e,\hat{G}(L))$. For a cocycle $\varphi: W_F/I_F^e \longrightarrow \hat{G}(L)$ in some $\underline{Z}^1(W_F/I_F^e,\hat{G}_L)_\phi$, we claim that the following statements are equivalent, *provided L has characteristic* 0 :

(1) its $\hat{G}_L$-orbit is closed in $\underline{Z}^1(W_F/I_F^e,\hat{G}_L)$,
(2) its $C_{\hat{G}}(\phi)_L$-orbit is closed in $\underline{Z}^1(W_F/I_F^e,\hat{G}_L)_\phi$,
(3) ${}^L\varphi(\mathrm{Fr})$ is a semisimple element of ${}^LG(L)$.
(4) ${}^L\varphi(W_F)$ consists of semisimple elements.

Indeed, (1) $\Rightarrow$ (2) since the small orbit is the intersection of the big one with the closed subset $\underline{Z}^1(W_F/I_F^e,\hat{G}_L)_\phi$. Moreover, (2) is equivalent to the orbit of ${}^L\varphi(\mathrm{Fr})$ being closed in $C_{{}^LG}(\phi)(L)$, which in turn is equivalent to ${}^L\varphi(\mathrm{Fr})$ being a semisimple element of $C_{{}^LG}(\phi)(L)$, hence also of ${}^LG(L)$. Further, (3), being equivalent to (2), applies to any lift of Frobenius, so implies (4). Eventually, (4) implies that the $\hat{G}(L)$-orbits of a finite set of generators of ${}^L\varphi(W_F)$ are closed, which implies (1).

The cocycles that satisfy property (3) are often called "Frobenius semi-simple" in the literature. When L has positive characteristic, a cocycle with closed orbit may not be Frobenius semi-simple. For example suppose $q = q_F$ has prime order $\ell \neq p$ in some $(\mathbb{Z}/n\mathbb{Z})^\times$ with n prime to both p and ℓ, and consider the character $\theta : I_F \twoheadrightarrow \mu_n \hookrightarrow \overline{\mathbb{F}}_\ell^\times$. Extend this character to $I_F \cdot \mathrm{Fr}^{\ell\mathbb{Z}}$ by setting $\theta(\mathrm{Fr}^\ell) = 1$ and induce to W_F. We obtain an irreducible representation $\varphi : W_F \longrightarrow \mathrm{GL}_\ell(\overline{\mathbb{F}}_\ell)$ such that $\varphi(\mathrm{Fr})$ has order ℓ.

6.3. **Comparison with the Haines variety.** In this subsection, we assume that the action of W_F stabilizes a pinning of $\hat{G}$, so that ${}^LG$ is an L-group associated to some reductive group G over F. As noted in point ii) of the last remark, the set of $\mathbb{C}$-points of the affine categorical quotient

$$\underline{Z}^1(W_F/I_F^e, \hat{G})_{\mathbb{C}} /\!\!/ \hat{G}_{\mathbb{C}}$$

is the set of $\hat{G}(\mathbb{C})$-conjugacy classes of Frobenius semisimple L-homomorphisms $W_F/I_F^e \longrightarrow {}^LG(\mathbb{C})$. In [Hai14], Haines endows this set with the structure of a complex affine variety that mimics Bernstein's description of the center of the category of complex representations of $G(F)$. We will denote by $\Omega_e(\hat{G})$ the Haines variety and we wish to compare his construction to ours. Note that, in the notation of [Hai14], $\Omega_e(\hat{G})$ is a summand of $\mathfrak{Y}$ and is the union of all components $\mathfrak{Y}_{\mathfrak{t}}$ corresponding to inertial classes of parameters that are trivial on P_F^e. In a rather abstract form, the main result of this section is the following.

Theorem 6.10. *The set-theoretic identification between $(\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G})(\mathbb{C})$ and $\Omega_e(\hat{G})$ is induced by an isomorphism of varieties*

$$\Omega_e(\hat{G}) \simeq \underline{Z}^1(W_F/I_F^e, \hat{G})_{\mathbb{C}} /\!\!/ \hat{G}_{\mathbb{C}}.$$

We need to recall some features of Haines' construction in Section 5 of [Hai14]. Let ${}^L\varphi : W_F/I_F^e \to {}^LG(\mathbb{C})$ be a Frobenius-semisimple L-morphism (called an "infinitesimal character" by Haines and Vogan), and choose a Levi subgroup $\mathcal{M}$ of ${}^LG$ that contains ${}^L\varphi(W_F)$ and is minimal for this property. Here we consider Levi subgroups in the sense of Borel [Bor79, §3]. In particular, $\mathcal{M}^\circ = \mathcal{M} \cap \hat{G}$ is a Levi subgroup of $\hat{G}$ and $\pi_0(\mathcal{M}) \xrightarrow{\sim} \pi_0({}^LG)$ is a quotient of W_F. As a consequence, the action of $\mathcal{M}$ by conjugation on the center $Z(\mathcal{M}^\circ)$ of $\mathcal{M}^\circ$ factors through an action of $\pi_0({}^LG)$, and provides thus a canonical action of W_F on $Z(\mathcal{M}^\circ)$. We may then consider the torus $(Z(\mathcal{M}^\circ)^{I_F})^\circ$ given by the neutral component of the I_F-invariants, and which still carries an action of $W_F/I_F = \langle \mathrm{Fr} \rangle$. To any $z \in (Z(\mathcal{M}^\circ)^{I_F})^\circ$, Haines associates a new parameter $z \cdot {}^L\varphi$ defined by

$$(z \cdot {}^L\varphi)(w) := z^{\nu(w)}\, {}^L\varphi(w), \quad \text{with } \nu : W_F \twoheadrightarrow \mathbb{Z} \text{ defined by } w\,\mathrm{Fr}^{-\nu(w)} \in I_F.$$

The conjugacy class $(z \cdot {}^L\varphi)_{\hat{G}}$ only depends on the image of z in the Fr-coinvariants $(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}}$, hence we get a map

$$(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}} \longrightarrow \Omega_e(\hat{G}) = (\underline{Z}^1(W_F/I_F^e, {}^LG) /\!\!/ \hat{G})(\mathbb{C}) \tag{6.4}$$

By Haines' definition of the variety structure on $\Omega_e(\hat{G})$, this map is a morphism of algebraic varieties $(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}} \longrightarrow \Omega_e(\hat{G})$. Even better, there is a finite group $W_{\mathcal{M},\varphi}$ of algebraic automorphisms of $(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}}$, whose precise definition is not needed here, such that the map (6.4) factors over an injective map $(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}}/W_{\mathcal{M},\varphi} \hookrightarrow \Omega_e(\hat{G})$. Then, the corresponding morphism of varieties $(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}} /\!\!/ W_{\mathcal{M},\varphi} \longrightarrow \Omega_e(\hat{G})$ is an isomorphism onto a connected component of $\Omega_e(\hat{G})$, by Haines' construction. Moreover, all connected components are obtained in this way.

At this point, we have recalled enough to prove one direction.

Lemma 6.11. *The set-theoretic identification between $\Omega_e(\hat{G})$ and $(\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G})(\mathbb{C})$ is induced by a morphism of varieties*

$$\Omega_e(\hat{G}) \longrightarrow \underline{Z}^1(W_F/I_F^e, \hat{G})_{\mathbb{C}} /\!\!/ \hat{G}_{\mathbb{C}}.$$

Proof. By the foregoing discussion, it now suffices to prove that each map (6.4) is induced by a morphism of schemes

$$(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}} \longrightarrow \underline{Z}^1(W_F/I_F^e, \hat{G})_{\mathbb{C}} /\!\!/ \hat{G}_{\mathbb{C}}.$$

By construction, the map (6.4) is part of a commutative diagram

$$\begin{tikzcd}
(Z(\mathcal{M}^\circ)^{I_F})^\circ \arrow[r] \arrow[d] & \underline{Z}^1(W_F/I_F^e, \hat{G})(\mathbb{C}) \arrow[d] \\
(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}} \arrow[r] & (\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G})(\mathbb{C})
\end{tikzcd}$$

where the top map is given by $z \mapsto z \cdot {}^L\varphi$. Denote by $\zeta \in {}^LG(\mathbb{C}[(Z(\mathcal{M}^\circ)^{I_F})^\circ])$ the element corresponding to the closed immersion $(Z(\mathcal{M}^\circ)^{I_F})^\circ \hookrightarrow {}^LG$. Then $\zeta \cdot {}^L\varphi$ is an element of $Z^1(W_F/I_F^e, \hat{G}(\mathbb{C}[(Z(\mathcal{M}^\circ)^{I_F})^\circ]))$, hence corresponds to a morphism $(Z(\mathcal{M}^\circ)^{I_F})^\circ \longrightarrow \underline{Z}^1(W_F/I_F^e, \hat{G})$. By definition, this morphism induces the top map of the above diagram on the respective sets of $\mathbb{C}$-points. Moreover, the composition of this morphism with the morphism underlying the right vertical map of the diagram is Fr-equivariant for the trivial action of Fr on the target, so it has to factor over a morphism which induces the bottom map of the diagram, as desired. $\square$

We now go in the other direction.

Lemma 6.12. *The set-theoretic identification between $(\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G})(\mathbb{C})$ and $\Omega_e(\hat{G})$ is induced by a morphism of varieties*

$$\underline{Z}^1(W_F/I_F^e, \hat{G})_{\mathbb{C}} /\!\!/ \hat{G}_{\mathbb{C}} \longrightarrow \Omega_e(\hat{G}).$$

Proof. The description of Theorem 6.7 shows that it suffices to prove that for any pair (ϕ, β) as in Theorem 6.7 and any choice of $\tilde{\beta} \in T_{\hat{G}}({}^{\mathrm{Fr}}\phi, \phi)_\beta$ as in Theorem 6.5, the map

$$(T_\phi)_{\mathrm{Ad}_\beta}(\mathbb{C}) \longrightarrow (\underline{Z}^1(W_F/I_F^e, \hat{G}) /\!\!/ \hat{G})(\mathbb{C}) = \Omega_e(\hat{G}) \tag{6.5}$$

is induced by a morphism of algebraic varieties $(T_\phi)_{\mathrm{Ad}_\beta} \longrightarrow \Omega_e(\hat{G})$.

To prove this, we will identify the maps (6.5) to instances of maps (6.4). Let us thus fix a pair (ϕ, β) and $\tilde{\beta}$ as in the statement, so that $\mathrm{Ad}_{\tilde{\beta}}$ fixes a pinning ε_ϕ of $C_{\hat{G}}(\phi)^\circ$ with maximal torus T_ϕ. We denote by $\varphi_{\tilde{\beta}}$ the unique extension of ϕ such that $\varphi_{\tilde{\beta}}(\mathrm{Fr}) = \tilde{\beta}$. Consider the torus $(T_\phi)^{\mathrm{Ad}_{\tilde{\beta}},\circ}$. Its centralizer $\mathcal{M}$ in ${}^LG$ contains

${}^L\varphi_{\tilde\beta}(W_F)$, hence maps onto $\pi_0({}^LG)$ and is thus a Levi subgroup in the sense of Borel. Moreover, the canonical action of W_F on $Z(\mathcal{M}^\circ)$ is induced by $\mathrm{Ad}_{\varphi_{\tilde\beta}}$. We claim that $\mathcal{M}$ is minimal among Levi subgroups of ${}^LG$ that contain ${}^L\varphi_{\tilde\beta}(W_F)$. Indeed, if ${}^L\varphi_{\tilde\beta}(W_F)\subset\mathcal{M}'\subset\mathcal{M}$, then $(T_\phi)^{\mathrm{Ad}_{\tilde\beta},\circ}\subset Z(\mathcal{M})^\circ\subset Z(\mathcal{M}')^\circ\subset C_{\hat G}(\varphi_{\tilde\beta})^\circ$. But $(T_\phi)^{\mathrm{Ad}_{\tilde\beta},\circ}$ is a maximal torus of $C_{\hat G}(\varphi_{\tilde\beta})^\circ=C_{\hat G}(\phi)^{\mathrm{Ad}_{\tilde\beta},\circ}$ by [DM94, Thm 1.8 iii)], so all inclusions above have to be equalities and in particular $\mathcal{M}'=\mathcal{M}$ since a Levi subgroup of ${}^LG$ is the centralizer in ${}^LG$ of its connected center by [Bor79, Lem. 3.5]. Now, observe that

$$(Z(\mathcal{M}^\circ)^{I_F})^\circ=Z(\mathcal{M}^\circ)^{\mathrm{Ad}_{\phi(I_F)},\circ}\subset\mathcal{M}^\circ\cap C_{\hat G}(\phi)^\circ=T_\phi.$$

Indeed, the last equality comes from [DM94, Thm 1.8 iv)] which implies that the centralizer of $(T_\phi)^{\mathrm{Ad}_{\tilde\beta},\circ}$ in $C_{\hat G}(\phi)^\circ$ is T_ϕ. Taking ${}^L\varphi_{\tilde\beta}(\mathrm{Fr})$-invariants, we get

$$Z(\mathcal{M})^\circ=(Z(\mathcal{M}^\circ)^{I_F})^{\mathrm{Ad}_{\tilde\beta},\circ}\subset(T_\phi)^{\mathrm{Ad}_{\tilde\beta},\circ},$$

from which we deduce that $(Z(\mathcal{M}^\circ)^{I_F})^{\mathrm{Ad}_{\tilde\beta},\circ}=(T_\phi)^{\mathrm{Ad}_{\tilde\beta},\circ}$ since $(T_\phi)^{\mathrm{Ad}_{\tilde\beta},\circ}\subset Z(\mathcal{M})$. Since the order of $\mathrm{Ad}_{\tilde\beta}$ on T_ϕ is finite, it follows that the inclusion of tori $(Z(\mathcal{M}^\circ)^{I_F})^\circ\subset T_\phi$ induces an isogeny

$$(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}}\twoheadrightarrow(T_\phi)_{\mathrm{Ad}_{\tilde\beta}}. \tag{6.6}$$

Here, note that the action of $\varphi_{\tilde\beta}(\mathrm{Fr})$ on $Z(\mathcal{M}^\circ)$ is just the action of Fr since $\tilde\beta\in\mathcal{M}^\circ$. Now, by construction, the map (6.5) is part of a commutative diagram

$$\begin{tikzcd}
T_\phi(\mathbb{C}) \arrow[r] \arrow[d] & \underline{Z}^1(W_F/I_F^e,\hat G)(\mathbb{C}) \arrow[d] \\
(T_\phi)_{\mathrm{Ad}_{\tilde\beta}}(\mathbb{C}) \arrow[r] & (\underline{Z}^1(W_F/I_F^e,\hat G)/\!\!/\hat G)(\mathbb{C})
\end{tikzcd}$$

where the top map is given by

$$t\mapsto\left(w\mapsto t^{\nu(w)}\varphi_{\tilde\beta}(w)\right),$$

where ν is the projection $W_F\longrightarrow W_F/I_F=\mathrm{Fr}^{\mathbb{Z}}$. This shows that the composition of the bottom map of the diagram (i.e. the map of the lemma) with (6.6) is an instance of (6.4), which is a morphism of algebraic varieties $(Z(\mathcal{M}^\circ)^{I_F})^\circ_{\mathrm{Fr}}\longrightarrow\Omega_e(\hat G)$ according to Haines' construction. This morphism is constant along the fibers of (6.6), so it has to factor over (6.6) since the latter is a quotient morphism. Hence the bottom map of the diagram is a morphism of varieties $(T_\phi)_{\mathrm{Ad}_{\tilde\beta}}\longrightarrow\Omega_e(\hat G)$. ☐

Proof of Theorem 6.10. It follows from the two above lemmas. ☐

Remark 6.13. The isomorphism of Theorem 6.10 induces of course a bijection between the sets of connected components on both sides. This bijection is easily described as follows :

- $\pi_0(\Omega_e(\hat G))$ is the set of “inertial classes” of Frobenius-semisimple cocycles φ, as defined in [Hai14, §5.3, Def. 4.15], that are trivial on I_F^e.
- $\pi_0(\underline{Z}^1(W_F/I_F^e,\hat G)_{\mathbb{C}}/\!\!/\hat G_{\mathbb{C}})$ is the set of conjugacy classes of pairs (ϕ,β) as in ii) of Remark 6.9.
- The bijection takes φ to $(\varphi_{|I_F},p(\varphi(\mathrm{Fr})))$ with p the projection $T_{\hat G}({}^{\mathrm{Fr}}\phi,\phi)\longrightarrow\pi_0({}^{\mathrm{Fr}}\phi,\phi)$.

Now, we wish to compare more explicitly Haines' construction with our description. This can be done component-wise, so let us fix data $(\phi, \beta, \varepsilon_\phi, \tilde{\beta})$ and put $\varphi = \varphi_{\tilde{\beta}}$ as in the last proof. Recall also the Levi subgroup $\mathcal{M} = C_{{}^LG}((T_\phi)^{\mathrm{Ad}_{\tilde{\beta}},\circ})$ of ${}^LG$ that appeared in the last proof. So we have an inclusion $Z(\mathcal{M}^\circ)^{I_F,\circ} \subset T_\phi$ that induces an isogeny $\pi : (Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}} \twoheadrightarrow (T_\phi)_{\mathrm{Ad}_{\tilde{\beta}}}$ as in (6.6). The associated connected component in Theorem 6.7 is $(T_\phi)_{\mathrm{Ad}_{\tilde{\beta}}} /\!\!/ (\Omega_\phi)^{\mathrm{Ad}_{\tilde{\beta}}}$, which we may also write as a two-steps quotient :

$$\left((Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}} / \ker(\pi)\right) / (\Omega_\phi)^{\mathrm{Ad}_{\tilde{\beta}}},$$

while the same component is described as a two-steps quotient

$$\left((Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}} / \mathrm{Stab}(\varphi)\right) / W_{\varphi,\mathcal{M}^\circ}$$

in Lemmas 4.19 and 4.20 of Section 5.3 in Haines' paper [Hai14]. The relation between these two presentations can be summarized as follows :

Lemma 6.14. *Using the notation right above,*

1. *we have* $\ker(\pi) \subset \mathrm{Stab}(\varphi)$ *as subgroups of* $(Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}}$.
2. *there is a normal subgroup* $K \subset (\Omega_\phi)^{\mathrm{Ad}_\beta}$ *whose action on* $(T_\phi)_{\mathrm{Ad}_\beta}$ *factors over that of* $\mathrm{Stab}(\varphi)/\ker(\pi)$ *through a surjective map* $K \twoheadrightarrow \mathrm{Stab}(\varphi)/\ker(\pi)$.
3. *there is a natural isomorphism* $(\Omega_\phi)^{\mathrm{Ad}_\beta}/K \xrightarrow{\sim} W_{\varphi,\mathcal{M}^\circ}$ *compatible with the respective actions on* $(Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}}/\mathrm{Stab}(\varphi)$.

Moreover, when $C_{\hat{G}}(\phi)$ *is connected, we have* $\ker(\pi) = \mathrm{Stab}(\varphi)$, *while* $(\Omega_\phi)^{\mathrm{Ad}_{\tilde{\beta}}}$ *identifies with* $W_{\varphi,\mathcal{M}^\circ}$ *compatibly with the action on* $(T_\phi)_{\mathrm{Ad}_\beta} = (Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}}/\mathrm{Stab}(\varphi)$ *(i.e., the group* K *above is trivial).*

Proof. Let us simplify the notation by putting $Z := (Z(\mathcal{M}^\circ)^{I_F,\circ})_{\mathrm{Fr}}$ (denoted $Y(\mathcal{M}^\circ)$ in Haines' paper) and $\tilde{Z} := Z(\mathcal{M}^\circ)^{I_F,\circ}$. Then Haines' definition of $\mathrm{Stab}(\varphi)$ (denoted stab_λ there) is

$$\mathrm{Stab}(\varphi) = \{z \in Z, \exists \tilde{z} \mapsto z, \exists m \in \mathcal{M}^\circ, \tilde{z} \cdot \varphi = \mathrm{Ad}_m(\varphi)\}.$$

Here, we use our notation $c \cdot \varphi$ for the unique 1-cocycle that restricts to ϕ on inertia and takes value $c\tilde{\beta}$ on Fr (this makes sense for any $c \in C_{\hat{G}}(\phi)$). Note that if $\tilde{z} \cdot \varphi = \mathrm{Ad}_m(\varphi)$, then in particular $\phi = \mathrm{Ad}_m(\phi)$, i.e. $m \in C_{\hat{G}}(\phi)$, hence we also have $\mathrm{Ad}_m(\varphi) = (m\mathrm{Ad}_{\tilde{\beta}}(m)^{-1}) \cdot \varphi$. Moreover, since m centralizes $(T_\phi)^{\mathrm{Ad}_{\tilde{\beta}},\circ}$, it also normalizes the centralizer of this torus in $C_{\hat{G}}(\phi)^\circ$, which is T_ϕ. So we see that

$$\mathrm{Stab}(\varphi) = \left\{z \in Z, \exists \tilde{z} \mapsto z, \exists m \in \mathcal{M}^\circ \cap N_{C_{\hat{G}}(\phi)}(T_\phi), \tilde{z} = m\mathrm{Ad}_{\tilde{\beta}}(m)^{-1}\right\}.$$

This certainly contains

$$\ker(\pi) = \left\{z \in Z, \exists \tilde{z} \mapsto z, \exists t \in T_\phi, \tilde{z} = t\mathrm{Ad}_{\tilde{\beta}}(t)^{-1}\right\}.$$

We have seen in the last proof that $\mathcal{M}^\circ \cap C_{\hat{G}}(\phi)^\circ = T_\phi$, therefore, when $C_{\hat{G}}(\phi)$ is connected, we have $\ker(\pi) = \mathrm{Stab}(\varphi)$. In general, we have

$$\begin{aligned}
\mathrm{Stab}(\varphi)/\ker(\pi) &\simeq \left\{ \begin{array}{ll} t \in (T_\phi)_{\mathrm{Ad}_\beta}, & \exists \tilde{z} \in \tilde{Z} \mapsto t, \exists m \in \mathcal{M}^\circ \cap N_{C_{\hat{G}}(\phi)}(T_\phi) \\ & \tilde{z} = m\mathrm{Ad}_{\tilde{\beta}}(m)^{-1} \end{array} \right\} \\
&= \left\{ \begin{array}{ll} t \in (T_\phi)_{\mathrm{Ad}_\beta}, & \exists \tilde{t} \in T_\phi \mapsto t, \exists m \in \mathcal{M}^\circ \cap N_{C_{\hat{G}}(\phi)}(T_\phi) \\ & \tilde{t} = m\mathrm{Ad}_{\tilde{\beta}}(m)^{-1} \end{array} \right\}.
\end{aligned}$$

To see the last equality, start with t in the last set and pick $(\tilde{t}, m)$ with $\tilde{t} \mapsto t$ and $\tilde{t} = m\mathrm{Ad}_{\tilde{\beta}}(m)^{-1}$. By surjectivity of (6.6), there is $s \in T_\phi$ such that $s\tilde{t}\,\mathrm{Ad}_\beta(s)^{-1} =: \tilde{z} \in \tilde{Z}$. Then we have $\tilde{z} \mapsto t$ and $\tilde{z} = (sm)\,\mathrm{Ad}_\beta(sm)^{-1}$ with $sm \in \mathcal{M}^\circ \cap N_{C_{\hat{G}}(\phi)}(T_\phi)$.

Now recall that $\Omega_\phi^{\mathrm{Ad}_\beta} = N_\beta / T_\phi$ where

$$N_\beta = \left\{ n \in N_{C_{\hat{G}}(\phi)}(T_\phi), n\,\mathrm{Ad}_{\tilde{\beta}}(n)^{-1} \in T_\phi \right\}.$$

From the description above, we see that $\mathrm{Stab}(\varphi)/\ker(\pi)$ is the image of the map $N_\beta \cap \mathcal{M}^\circ \longrightarrow (T_\phi)_{\mathrm{Ad}_\beta}$ given by $n \mapsto n\,\mathrm{Ad}_{\tilde{\beta}}(n)^{-1}$, and that this map factors over the subgroup $K := (N_\beta \cap \mathcal{M}^\circ)/T_\phi$ of $(\Omega_\phi)^{\mathrm{Ad}_\beta}$. Moreover, the action of $n \in N_\beta \cap \mathcal{M}^\circ$ on $(T_\phi)_{\mathrm{Ad}_\beta}$ is given by $t \mapsto nt\,\mathrm{Ad}_\beta(n)^{-1} = ntn^{-1}(n\,\mathrm{Ad}_\beta(n)^{-1}) = t(n\,\mathrm{Ad}_\beta(n)^{-1})$ because n centralizes $(T_\phi)^{\mathrm{Ad}_\beta,\circ}$. So this action factors through the above morphism.

On the other hand, Haines' definition of $W_{\varphi,\mathcal{M}^\circ}$ (denoted $W^{\hat{G}}_{[\lambda]_{\mathcal{M}^\circ}}$ there) is of the form $W_{\varphi,\mathcal{M}^\circ} = N/\mathcal{M}^\circ$ with

$$N = \left\{ n \in N_{\hat{G}}(\mathcal{M}), \exists m \in \mathcal{M}^\circ, \exists \tilde{z} \in \tilde{Z}, \mathrm{Ad}_n(\varphi) = \mathrm{Ad}_m(\tilde{z} \cdot \varphi) \right\}.$$

We claim that $N_\beta \subset N$. Indeed, note first that the conjugation action of an element $n \in N_\beta$ on T_ϕ commutes with Ad_β, so n normalizes $T_\phi^{\mathrm{Ad}_\beta,\circ}$ hence it normalizes also $\mathcal{M}$. Moreover, writing $\tilde{t} := n\,\mathrm{Ad}_{\tilde{\beta}}(n)^{-1} \in T_\phi$, we have $\mathrm{Ad}_n(\varphi) = \tilde{t} \cdot \varphi$. Finally, since $\tilde{Z}$ surjects onto $(T_\phi)_{\mathrm{Ad}_\beta}$, there are $\tilde{z} \in \tilde{Z}$ and $m \in T_\phi$ such that $\tilde{t} = m\,\mathrm{Ad}_\beta(m)^{-1}\tilde{z}$, hence also $\tilde{t} \cdot \varphi = \mathrm{Ad}_m(\tilde{z} \cdot \varphi)$. So we have $N_\beta \subset N$, and since $T_\phi \subset \mathcal{M}^\circ$, we get a map

$$(\Omega_\phi)^{\mathrm{Ad}_\beta} \longrightarrow W_{\varphi,\mathcal{M}^\circ}. \tag{6.7}$$

We now claim that this map is surjective. Indeed, let $n \in N$ and pick $\tilde{z}$ and m such that $\mathrm{Ad}_n(\varphi) = \mathrm{Ad}_m(\tilde{z} \cdot \varphi)$. The element $n' := m^{-1}n$ has the same image as n in $W_{\varphi,\mathcal{M}^\circ}$, and we have $\mathrm{Ad}_{n'}(\varphi) = \tilde{z} \cdot \varphi$, hence also $\mathrm{Ad}_{n'}(\phi) = \phi$, i.e. $n' \in C_{\hat{G}}(\phi)$. Moreover, n' normalizes $Z(\mathcal{M})^\circ = (T_\phi)^{\mathrm{Ad}_\beta,\circ}$ (see the proof of the last lemma above), hence it normalizes the connected centralizer of $(T_\phi)^{\mathrm{Ad}_\beta,\circ}$ in $C_{\hat{G}}(\phi)$, which is T_ϕ. Hence we see that $n' \in N_\beta$ and we get the surjectivity of (6.7). We also see that

$$\ker((\Omega_\phi)^{\mathrm{Ad}_\beta} \longrightarrow W_{\varphi,\mathcal{M}^\circ}) = \mathrm{im}(N_\beta \cap \mathcal{M}^\circ \longrightarrow (\Omega_\phi)^{\mathrm{Ad}_\beta}) = K.$$

In particular, when $C_{\hat{G}}(\phi)$ is connected, we have $N_\beta \cap \mathcal{M}^\circ \subset C_{\hat{G}}(\phi)^\circ \cap \mathcal{M}^\circ = T_\phi$, so the map (6.7) is bijective in this case. □

Appendix A. Moduli of cocycles

A.1. **Schemes of cocycles.** Let H be an affine group scheme over a noetherian ring R and let Γ be a finite group. Consider the functor $\underline{\mathrm{Hom}}(\Gamma, H)$, which to any R-algebra R' associates the set of homomorphisms $\mathrm{Hom}(\Gamma, H(R'))$. It is represented by a closed and finitely presented R-subscheme of the affine R-scheme $H^{(\Gamma)}$, since it is the inverse image of the closed subscheme $\{1_H\}^{(\Gamma\times\Gamma)}$ of $H^{(\Gamma\times\Gamma)}$ by the R-morphism $H^{(\Gamma)} \longrightarrow H^{(\Gamma\times\Gamma)}$ defined by $(h_\gamma)_{\gamma\in\Gamma} \mapsto (h_\gamma h_{\gamma'} h_{\gamma\gamma'}^{-1})_{(\gamma,\gamma')\in\Gamma\times\Gamma}$.

The group scheme H acts by conjugation on $\underline{\mathrm{Hom}}(\Gamma, H)$. Given an R-algebra R' and an homomorphism $\phi \in \mathrm{Hom}(\Gamma, H(R'))$, the orbit maps $g \mapsto \mathrm{Ad}_g \circ \phi$, $H(R'') \longrightarrow \mathrm{Hom}(\Gamma, H(R''))$ define an R'-morphism $H_{R'} \longrightarrow \underline{\mathrm{Hom}}(\Gamma, H)_{R'}$ of finite presentation, that we call an orbit morphism (here R'' runs over R'-algebras). The fiber over any other homomorphism $\phi' \in \mathrm{Hom}(\Gamma, H(R'))$ of this morphism is the transporter $T_H(\phi, \phi')$ of ϕ, which to any R'' over R' associates the set-theoretic transporter from ϕ to ϕ' in $H(R'')$.

Lemma A.1. *Assume that H is smooth and that Γ has order invertible in R. Then $\underline{\mathrm{Hom}}(\Gamma, H)$ is smooth over R, all the orbit morphisms are smooth and all transporters are smooth.*

Proof. By finite presentation, to prove smoothness it suffices to prove formal smoothness. Let R' be an R-algebra and let I be an ideal of R' of square 0. We need to show that the map $\mathrm{Hom}(\Gamma, H(R')) \longrightarrow \mathrm{Hom}(\Gamma, H(R'/I))$ is surjective. So let $\phi_0 : \Gamma \longrightarrow H(R'/I)$ be a group homomorphism. By smoothness of H we may lift ϕ_0 to a map $h : \Gamma \longrightarrow H(R')$. Consider the map $\Gamma \times \Gamma \longrightarrow \ker(H(R') \longrightarrow H(R'/I))$ that takes $(\gamma, \gamma') \in \Gamma \times \Gamma$ to $h(\gamma)h(\gamma')h(\gamma\gamma')^{-1}$. Note that conjugation by $h(\gamma)$ endows the abelian group $\ker(H(R') \longrightarrow H(R'/I))$ with an action of Γ that actually only depends on ϕ_0. In fact, if we identify $\ker(H(R') \longrightarrow H(R'/I))$ with the R'/I-module $\mathrm{Lie}(H) \otimes_R I$ then this action is induced by the R'/I-linear action of Γ on $\mathrm{Lie}(H) \otimes_R R'/I$ given by the adjoint representation composed with the homomorphism ϕ_0. Now the map defined above is a 2-cocycle, hence since $|\Gamma|$ is invertible in R, it has to be cohomologically trivial, so there is a map $k : \Gamma \longrightarrow \ker(H(R') \longrightarrow H(R'/I))$ such that $h(\gamma)h(\gamma')h(\gamma\gamma')^{-1} = k(\gamma)({}^{h(\gamma)}k(\gamma'))k(\gamma\gamma')^{-1}$. Then the map $\gamma \mapsto \phi(\gamma) := k(\gamma)^{-1}h(\gamma)$ is a group homomorphism $\phi : \Gamma \longrightarrow H(R')$ that lifts ϕ_0, and the smoothness of $\underline{\mathrm{Hom}}(\Gamma, H)$ follows.

Now fix a homomorphism $\phi : \Gamma \longrightarrow H(R)$ and let us show that the corresponding orbit morphism is smooth, by using the infinitesimal criterion. Let again R' be an R-algebra together with an ideal I of square 0, and let ϕ' be another homomorphism $\Gamma \longrightarrow H(R')$ whose image ϕ'_0 in $\mathrm{Hom}(\Gamma, H(R'/I))$ is conjugate to ϕ_0 by an element h_0 in $H(R'/I)$. We must find an element $h \in H(R')$ that conjugates ϕ' to ϕ. By smoothness of H we can pick an element $h' \in H(R)$ that maps to h_0. Then the map $\gamma \mapsto \phi(\gamma)(h'\phi'(\gamma)^{-1}h'^{-1})$ defines a 1-cocycle of Γ in $\ker(H(R') \longrightarrow H(R'/I))$ endowed with the action associated with ϕ_0 as above. By the same argument as above, this cocycle is a coboundary, so there is some $k \in \ker(H(R') \longrightarrow H(R'/I))$ such that $\phi(\gamma)h'\phi'(\gamma)^{-1}h'^{-1} = (\phi(\gamma)k\phi(\gamma)^{-1})k^{-1}$, from which we get an element $h = k^{-1}h'$ as desired. Hence the orbit morphism is smooth. By base change, the centralizers and the transporters are therefore smooth too. □

Suppose now that we are given an action of Γ on H by automorphisms of group schemes over R. Identifying 1-cocycles $\Gamma \longrightarrow H(R')$ with cross-section homomorphisms $\Gamma \longrightarrow H \rtimes \Gamma$ (i.e. homomorphisms whose composition with the projection to Γ is the identity), we see that the functor $R' \mapsto Z^1(\Gamma, H(R'))$ is represented by an R-scheme that is a direct summand of $\underline{\mathrm{Hom}}(\Gamma, H \rtimes \Gamma)$. We denote this scheme by $\underline{Z}^1(\Gamma, H)$. It is stable under the conjugation action of $H \rtimes \Gamma$ *restricted* to H.

When H is smooth, so is $H \rtimes \Gamma$, hence the above lemma implies :

Corollary A.2. *Assume that H is smooth and that Γ has order invertible in R. Then $\underline{Z}^1(\Gamma, H)$ is smooth over R, all the H-orbit morphisms are smooth and all transporters are smooth.*

A.2. **The sheafy quotient.** We henceforth assume that H *is smooth and* Γ *has order invertible in* R, and we are now interested in the quotient object $\underline{H}^1(\Gamma, H)$ of $\underline{Z}^1(\Gamma, H)$ by the conjugation action of H. As for now, we define it as the quotient sheaf, say for the étale topology, that is, the sheaf associated to $R \mapsto H^1(\Gamma, H(R))$.

Corollary A.3. *Assume that R is a local Henselian ring, and denote by k its residue field. Then the map $H^1(\Gamma, H(R)) \longrightarrow H^1(\Gamma, H(k))$ is a bijection.*

Proof. By smoothness of $\underline{Z}^1(\Gamma, H)$ and [Gro67, Thm 18.5.17], any k-point of $\underline{Z}^1(\Gamma, H)$ extends to a section over R, that is, the map $Z^1(\Gamma, H(R)) \longrightarrow Z^1(\Gamma, H(k))$ is surjective. Hence the map of the lemma is surjective too. To prove injectivity, let ϕ, $\phi' : \Gamma \longrightarrow H(R)$ be two 1-cocycles whose images ϕ_0, ϕ'_0 are H-conjugate in $\mathrm{Hom}(\Gamma, H(k) \rtimes \Gamma)$ by some $h_0 \in H(k)$. By the previous lemma, the transporter scheme $T_H(\phi, \phi')$ is smooth over R. Hence, by [Gro67, Thm 18.5.17] again, its k-point h_0 extends to an R-section h that conjugates ϕ to ϕ'. □

Lemma A.4. *Assume that H is reductive, that R is a strictly Henselian local ring, and denote by R' any non-zero R-algebra.*

(i) the map $H^1(\Gamma, H(R)) \longrightarrow H^1(\Gamma, H(R'))$ is injective.

(ii) it is surjective if R is a d.v.r. or a field and R' is local strictly Henselian.

Proof. We adapt the proof of Thm 4.8 of [BHKT19].

(i) We need to prove that if two cocycles ϕ, ϕ' in $Z^1(\Gamma, H(R))$ get $H(R')$-conjugate in $\mathrm{Hom}(\Gamma, H(R') \rtimes \Gamma)$, then they are $H(R)$-conjugate. By the last corollary, it suffices to prove that their images $\phi_0, \phi'_0 \in Z^1(\Gamma, H(k))$ are $H(k)$-conjugate. We will need V. Lafforgue's theory of pseudocharacters for the group $H \rtimes \Gamma$. This notion is introduced without name nor formal definition in the preamble of Proposition 11.7 of [Laf18]. A formal definition is given in [BHKT19, Def 4.1] where the name "pseudocharacter" is also introduced. Unfortunately, unlike Lafforgue, these authors restrict attention to connected (split reductive) groups. However, one has merely to replace $\mathbb{Z}[\hat{G}^n]^{\hat{G}}$ by $\mathbb{Z}[(H \rtimes \Gamma)^n]^H$ in [BHKT19, Def 4.1] to get the correct definition for the non-connected group $H \rtimes \Gamma$ (note that H is a split reductive group over R, since R is strictly Henselian). Then, as in [BHKT19, Lemma 4.3], it follows from the definition that any homomorphism $\phi : \Gamma \longrightarrow H(R) \rtimes \Gamma$ defines a "$H \rtimes \Gamma$-pseudocharacter of Γ over R" denoted by Θ_ϕ. Moreover, if ϕ, ϕ' in $Z^1(\Gamma, H(R))$ become $H(R')$-conjugate in $\mathrm{Hom}(\Gamma, H(R') \rtimes \Gamma)$, then $\Theta_\phi \equiv \Theta_{\phi'} [\mathrm{mod} I]$ where $I = \ker(R \longrightarrow R')$ (as in lemmas 4.3 and 4.4.i of [BHKT19]). Therefore we get $\Theta_{\phi_0} = \Theta_{\phi'_0}$. Then, the main result on pseudocharacters asserts that *the semi-simplifications* of ϕ_0 and ϕ'_0 are conjugate under $H(k)$. Here, the notion of semi-simplicity is the notion of $H \rtimes \Gamma$-complete reducibility of [BMR05, §6]. We note actually that this result is proven in [Laf18, Prop. 11.7] when k has characteristic 0 and in [BHKT19, Thm 4.5] in any characteristic, but in the *connected* case. We leave it to the reader to convince themselves that their argument can be adapted to the non-connected case in any characteristic.

It suffices now to show that ϕ_0 and ϕ'_0 are actually $H \rtimes \Gamma$-completely reducible. Choose an R-parabolic subgroup P of $H \rtimes \Gamma$ containing $\phi_0(\Gamma)$ and minimal for this property. Let $P \xrightarrow{\pi} L_P$ be its Levi quotient and let $L_P \xrightarrow{\iota} P$ be a Levi section of P. Then $\phi_0^{ss} := \iota \circ \pi \circ \phi_0$ is by definition a semisimplification of ϕ_0. If we denote by U_P the unipotent radical of P, the map $\Gamma \longrightarrow U_P$, $\gamma \mapsto \phi_0^{ss}(\gamma)\phi_0(\gamma)^{-1}$ is a 1-cocycle for the action of Γ by conjugation on U_P through ϕ_0. The descending central series of U_P is a Γ-stable descending filtration of U_P by smooth unipotent subgroup schemes whose successive quotients are k-vector space schemes. Since $|\Gamma|$ is invertible in k, we know that $H^1(\Gamma, V)$ is trivial for any $k\Gamma$-module V. Therefore the above 1-cocycle is a coboundary, and we can find some $u \in U_P$ such that $\phi_0^{ss}(\gamma)\phi_0(\gamma)^{-1} = u^{-1}\phi_0(\gamma)u\phi_0(\gamma)^{-1}$. So u^{-1} conjugates ϕ_0 to ϕ_0^{ss} and ϕ_0 is semisimple (ie $H \rtimes \Gamma$-completely reducible) as claimed.

(ii) Denote by k' the residue field of R' and by $\bar{K}$ an algebraic closure of the fraction field of R. In the case where R is a d.v.r, either the composition $R \longrightarrow k'$ factors as $R \longrightarrow k \longrightarrow k'$ or as $R \longrightarrow \bar{K} \longrightarrow k'$. Applying the last corollary to both R and R', we are thus reduced to showing the special case (a) of statement (ii) where $R' = \bar{K}$, and its variant (b) where R and R' are algebraically closed fields.

(a) The case where $R' = \bar{K}$ is an algebraic closure of the fraction field K of R. This case will follow from the following facts of Bruhat-Tits theory :

- BT1. any vertex of the semi-simple building $B(H, K)$ and, more generally, the barycenter of any facet of $B(H, K)$ becomes a hyperspecial point in $B(H, K')$ for a suitable finite extension K' of K
- BT2. two hyperspecial points in $B(H, K')$ become $H(K'')$-conjugate in $B(H, K'')$ for some further finite extension K''.

Note that, here, H is split over K (since R is strictly henselian), so these facts are quite elementary, even in our setting where the discretely valued field K is Henselian but not necessarily complete. For example, BT2 follows from Corollary 7.11.5 of the forthcoming book [KP22]. As for BT1, here is a sketch of the argument. Choose a splitting (B, T, X) of H over R and denote by A the appartment of $B(H, K)$ associated to T. It contains the hyperspecial point o corresponding to the integral model H over R. The pinning defines a "Chevalley valuation" of the root system of T in H, and then an affine root system on A. By [KP22, Prop 6.4.1], we know that, taking o as an origin of the $\mathbb{R}$-affine space A, the affine roots on A are translates of ordinary roots by integers. It then follows that a point $y \in A$ is (hyper)special if $\alpha(y) \in v(K^\times)$ for all roots α, and where v is the valuation of K. Now, let x be the barycenter of some facet of $B(H, K)$. After translating x by an element of $G(K)$ we may assume that x lies in A. Being the barycenter of a facet, there is an integer N such that $\alpha(x) \in \frac{1}{N}v(K^\times)$ for all roots α. So, if K' is any extension whose ramification index is a multiple of N, then x becomes (hyper)special in $B(H, K')$.

Observe also that Γ acts on the building $B(H, K)$ and fixes the hyperspecial point o. Now let $\phi \in Z^1(\Gamma, H(\bar{K}))$. Then ϕ belongs to $Z^1(\Gamma, H(K_1))$ for some finite extension K_1 of K. Pick a point x of $B(H, K_1)$ fixed by ${}^L\phi(\Gamma) \subset H(K_1) \rtimes \Gamma$. Up to replacing x by the barycenter of the facet that contains x, we may assume x is the barycenter of this facet. So it becomes hyperspecial over some finite extension K_2 of K_1 and we may even assume that there is some $h \in H(K_2)$ such that $hx = o$. Then ${}^h({}^L\phi)(\Gamma)$ fixes o so, writing ${}^h({}^L\phi)(\gamma) = ({}^h\phi(\gamma), \gamma) \in H(K_2) \rtimes \Gamma$, we see that ${}^h\phi(\gamma)$ fixes o hence belongs to $H(K_2)_o = H(R_2)Z_H(K_2)$ for all $\gamma \in \Gamma$, *i.e.* ${}^h\phi \in Z^1(\Gamma, H(R_2)Z_H(K_2))$, where R_2 is the normalization of R in K_2. Now, note that $H^1(\Gamma, Z_H(K_2)/Z_H(R_2))$ may not be trivial, but maps trivially in $H^1(\Gamma, Z_H(K_3)/Z_H(R_3))$ for any further finite extension K_3 such that $|\Gamma|$ divides the exponent of $Z_H(K_3)/Z_H(R_3)Z_H(K_2)$. This means that there is $z \in Z_H(K_3)$ such that ${}^{zh}\phi \in Z^1(\Gamma, H(R_3))$. But R_3 is an Henselian local R-algebra with the same residue field as R, so by the previous corollary there is $h' \in H(R_3)$ such that ${}^{h'zh}\phi \in Z^1(\Gamma, H(R))$. So the class $[\phi]$ in $H^1(\Gamma, H(\bar{K}))$ is the image of $[{}^{h'zh}\phi] \in H^1(\Gamma, H(R))$, as desired.

(b) The case where R and R' are algebraically closed fields. This case can certainly be handled via pseudocharacters. Namely, using [BHKT19, Thm 4.5] and the fact that all morphisms $\Gamma \longrightarrow H(R) \rtimes \Gamma$ are $H \rtimes \Gamma$-semisimple (as proved above), we see that it suffices to prove that any $H \rtimes \Gamma$-pseudocharacter of Γ over R' is actually R-valued. However, the result is true under the much more general assumption that H is smooth over R. Indeed, since the orbit morphisms are smooth, the $H(R)$-orbits in $Z^1(\Gamma, H(R))$ are open for the Zariski topology. Since two orbits are either equal or disjoint, there are only finitely many of them. Let $\phi_1, \cdots, \phi_n$ be representatives. The orbit morphisms yield a smooth surjective morphism $(\sqcup_{i=1}^n H) \longrightarrow \underline{Z}^1(\Gamma, H)$ which induces in turn a surjection on R'-points $(\sqcup_{i=1}^n H(R')) \longrightarrow \underline{Z}^1(\Gamma, H)(R')$ since R' is algebraically closed. So we see that each $H(R')$-orbit in $Z^1(\Gamma, H(R'))$ comes from an $H(R)$-orbit in $Z^1(\Gamma, H(R))$. □

Recall now the étale sheafification $\underline{H}^1(\Gamma, H)$ of the functor $R' \mapsto H^1(\Gamma, H(R'))$ on R-algebras. Here we consider the "big" site of affine schemes of finite presentation over R with the étale topology. The maps $H^1(\Gamma, H(R)) \longrightarrow H^1(\Gamma, H(R'))$ define a morphism from the constant presheaf associated to the set $H^1(\Gamma, H(R))$ to the presheaf $R' \mapsto H^1(\Gamma, H(R'))$. It induces in turn a morphism of sheaves

$$\underline{H^1(\Gamma, H(R))} \longrightarrow \underline{H}^1(\Gamma, H)$$

where the left hand side is a "constant" sheaf.

Proposition A.5. *Suppose that H is reductive over a strictly Henselian discrete valuation ring R in which the order of Γ is invertible. Then the above morphism of sheaves is an isomorphism. In particular, $\underline{H}^1(\Gamma, H)$ is representable by a product of finitely many copies of R.*

Proof. We first note that the functor $R' \mapsto H^1(\Gamma, H(R'))$ defined over all R-algebras commutes with filtered colimits. Indeed, this property is certainly true for the functors $R' \mapsto Z^1(\Gamma, H(R'))$ and $R' \mapsto H(R')$ since both these functors are represented by finitely presented R-algebras. Elementary formal nonsense shows that this property holds in turn for the quotient functor $R' \mapsto H^1(\Gamma, H(R'))$.

Therefore, if A is any R-algebra and x is a geometric point of $\mathrm{Spec}(A)$ then, writing A_x^{sh} for the strict henselization of A at x, the set $H^1(\Gamma, H(A_x^{sh}))$ is the stalk of the sheaf $\underline{H}^1(\Gamma, H)$ at x. So by the last lemma, the map $H^1(\Gamma, H(R)) \longrightarrow H^1(\Gamma, H(A_x^{sh}))$ is bijective. This means that the morphism of sheaves under consideration is an isomorphism on stalks. Thus it is an isomorphism.

It remains to justify the finiteness of the set $H^1(\Gamma, H(R))$. But it follows from Corollary A.3 and the last paragraph of the proof of Lemma A.4. □

Remark A.6. Here is a concrete paraphrase of the proposition. First note that the map $Z^1(\Gamma, H(R)) \longrightarrow \underline{H}^1(\Gamma, H)(R)$ is surjective since R is strictly Henselian, so that we can pick a finite subset $\Phi_0 \subset Z^1(\Gamma, H(R))$ mapping bijectively to $\underline{H}^1(\Gamma, H)(R)$. Now, suppose that A is an integral finitely presented R-algebra and let ϕ be a 1-cocycle $\Gamma \longrightarrow H(A)$. Then there is a unique cocycle $\phi_0 \in \Phi_0$ and a faithfully étale map $A \longrightarrow A'$ such that ϕ is $H(A')$-conjugate to the "constant" cocycle ϕ_0.

We now globalize a bit the previous proposition.

Theorem A.7. *Suppose that H is reductive over a Dedekind G-ring R in which the order of Γ is invertible. Then $\underline{H}^1(\Gamma, H)$ is representable by a finite étale R-algebra.*

Proof. Let $\bar{K}$ be an algebraic closure of the fraction field K of R. For a closed point s of $\mathrm{Spec}(R)$, denote by R_s^{sh} a strict henselization of R at s (depending on a choice of geometric point over s) and by $\bar{K}_s$ an algebraic closure of its fraction field. Let us choose a set of representatives $\Phi_s \subset Z^1(\Gamma, H(R_s^{sh}))$ of $H^1(\Gamma, H(R_s^{sh}))$. Since $\underline{Z}^1(\Gamma, H)$ is finitely presented, these representatives are defined over some étale R-domain R', so that Φ_s comes from a subset $\Phi_{R'} \subset Z^1(\Gamma, H(R'))$. Now if s' is another closed point of $\mathrm{Spec}(R)$ in the image of $\mathrm{Spec}(R')$ and if we choose an R-morphism $R' \longrightarrow R_{s'}^{sh}$, then we claim that the natural map $\Phi_{R'} \to H^1(\Gamma, H(R_{s'}^{sh}))$ is also a bijection. Indeed, this follows from the following commutative diagram

$$\begin{tikzcd}
 & H^1(\Gamma, H(R_s^{sh})) \arrow[r, "\sim"] & H^1(\Gamma, H(\bar{K}_s)) & \\
\Phi_{R'} \arrow[ur] \arrow[rrr] \arrow[dr] & & & H^1(\Gamma, H(\bar{K})) \arrow[ul, "\sim"'] \arrow[dl, "\sim"] \\
 & H^1(\Gamma, H(R_{s'}^{sh})) \arrow[r, "\sim"] & H^1(\Gamma, H(\bar{K}_{s'})) &
\end{tikzcd}$$

where we have chosen an R-embedding $R' \hookrightarrow \bar{K}$ and two R'-embeddings $\bar{K} \hookrightarrow \bar{K}_s$ and $\bar{K} \hookrightarrow \bar{K}_{s'}$, and where the $\sim$ denote bijections granted by Lemma A.4. As a consequence, denoting by $\underline{\Phi}_{R'}$ the constant sheaf on R'-algebras associated to the set $\Phi_{R'}$, we see as in the last proof that the natural morphism of sheaves $\underline{\Phi}_{R'} \longrightarrow \underline{H}^1(\Gamma, H)_{R'}$ is an isomorphism.

Now, varying the point s and using the quasicompacity of $\mathrm{Spec}(R)$ we get a faithfully étale morphism $R \hookrightarrow R'' = R'_1 \times \cdots \times R'_n$ and a set $\Phi_{R''} = \Phi_{R'_1} \times \cdots \times \Phi_{R'_n}$ such that the natural morphism of sheaves on R''-algebras $\underline{\Phi}_{R''} \longrightarrow \underline{H}^1(\Gamma, H)_{R''}$ is an isomorphism. In particular, the sheaf $\underline{H}^1(\Gamma, H)$ is representable after base change to R'' by a sum of copies of R''. Since the map of $(\mathrm{Spec}\, R)_{\text{ét}}$-sheaves $\underline{H}^1(\Gamma, H) \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, R'' \longrightarrow \underline{H}^1(\Gamma, H)$ is visibly representable, étale and surjective, it follows that $\underline{H}^1(\Gamma, H)$ is an algebraic space over $(\mathrm{Spec}\, R)_{\text{ét}}$. This algebraic space has to be finite étale (and in particular separated) over R since it is so after base change to R''. Hence by Corollary II.6.17 of [Knu71], this algebraic space is actually a scheme, and it is finite étale over R. □

A.3. **Relation with the affine GIT quotient.** Let us investigate the relationship between $\underline{H}^1(\Gamma, H)$ and another natural quotient of $\underline{Z}^1(\Gamma, H)$ by H. Namely, denote by $\mathcal{O}$ the R-algebra such that $\underline{Z}^1(\Gamma, H) = \mathrm{Spec}(\mathcal{O})$. The action of H on $\underline{Z}^1(\Gamma, H)$ translates into a comodule structure $\mathcal{O} \xrightarrow{\rho} \mathcal{O} \otimes_R R[H]$ on $\mathcal{O}$ under the Hopf R-algebra $R[H]$ corresponding to H. As usual, put

$$\mathcal{O}^H := \ker(\rho - \mathrm{id} \otimes \varepsilon)$$

where ε is the unit of $R[H]$. Then the morphism $\mathrm{Spec}(\mathcal{O}) \longrightarrow \mathrm{Spec}(\mathcal{O}^H)$ is a categorical quotient of $\underline{Z}^1(\Gamma, H)$ by H in the category of *affine* R-schemes.

Note that $\underline{H}^1(\Gamma, H)$ is a categorical quotient in the much larger category of sheaves on the big étale site of $\mathrm{Spec}(R)$. However, under suitable assumptions, Theorem A.7 shows that it is actually represented by an affine R-scheme. So, by uniqueness of categorical quotients, we conclude that up to a unique isomorphism, we have

$$\underline{H}^1(\Gamma, H) = \mathrm{Spec}(\mathcal{O}^H),$$

which we summarize in the following corollary.

Corollary A.8. *Suppose that H is reductive over a Dedekind G-ring R in which the order of Γ is invertible. Then $\mathcal{O}^H$ is a finite étale R-algebra and represents the sheaf $\underline{H}^1(\Gamma, H)$. In particular, its formation commutes with any change of rings $R \longrightarrow R'$.*

A.4. **Representatives.** Suppose that H is reductive over a Dedekind G-ring R in which the order of Γ is invertible. Theorem A.7 ensures that after replacing R by a finite étale extension, $\underline{H}^1(\Gamma, H)$ is a constant sheaf (associated to the set $\underline{H}^1(\Gamma, H)(R)$). The map $Z^1(\Gamma, H(R)) \longrightarrow \underline{H}^1(\Gamma, H)(R)$ need not be surjective, but if R_0 is any R-algebra such that $\underline{H}^1(\Gamma, H)(R)$ is in the image of the map $Z^1(\Gamma, H(R_0)) \longrightarrow \underline{H}^1(\Gamma, H)(R_0)$, then for any finite set $\Phi_0 \subset Z^1(\Gamma, H(R_0))$ mapping bijectively to $\underline{H}^1(\Gamma, H)(R)$, the constant sheaf property ensures that : *for any connected R_0-algebra A and any $\phi \in Z^1(\Gamma, H(A))$, there is a unique $\phi_0 \in \Phi_0$ such that ϕ and ϕ_0 become $H(A')$-conjugate in $Z^1(\Gamma, H(A'))$ for some faithfully étale A-algebra A'.*

By definition of $\underline{H}^1(\Gamma, H)$, we certainly can find a R_0 as above that is faithfully étale over R. However in general, it is not clear whether we can find R_0 *finite* étale over R. The following result uses the strong approximation property to prove that, if R is a localization of a ring of integers in a number field, then we can at least find R_0 finite (not necessarily étale) over R.

Theorem A.9. *Assume that H is reductive over a normal subring R of some number field K, and that Γ has invertible order in R. Then there is a finite extension K_0 of K and a finite set $\Phi_0 \subset Z^1(\Gamma, H(R_0))$ (with R_0 the normalization of R in K_0) such that for any connected R_0-algebra A and any $\phi \in Z^1(\Gamma, H(A))$, there is a unique $\phi_0 \in \Phi_0$ such that ϕ and ϕ_0 becomes $H(A')$-conjugate in $Z^1(\Gamma, H(A'))$ for some faithfully étale A-algebra A'.*

Proof. As we have just argued, we may assume that $\underline{H}^1(\Gamma, H)$ is a constant sheaf, and the problem boils down to finding K_0 such that the map

$$Z^1(\Gamma, H(R_0)) \longrightarrow \underline{H}^1(\Gamma, H)(R_0) = \underline{H}^1(\Gamma, H)(R)$$

is surjective. We certainly can find a faithfully étale R' over R such that any $[\phi] \in \underline{H}^1(\Gamma, H)(R)$ has a representative $\phi \in Z^1(\Gamma, H(R'))$. Let us choose such data, and assume further that H is split over R'. Let $R' = \prod_{i=1}^n R'_i$ be the decomposition of R' in connected components and let $\phi = (\phi_i)_{i=1,\cdots,n}$ be the corresponding decomposition of ϕ. Replacing R and all R'_i by their respective normalizations in the residue field at some generic point of $R'_1 \otimes_R \cdots \otimes_R R'_n$, we may assume that each R'_i is a localization of R (i.e. $\mathrm{Spec}(R') \longrightarrow \mathrm{Spec}(R)$ is a Zariski cover). Then we simplify the notation and write $R_i := R'_i$. Since all ϕ_i map to the same element $[\phi] \in H^1(\Gamma, H(K))$, they become pairwise H-conjugate over some finite extension of K. Replacing R by its normalization in this finite extension, we may thus assume that they are $H(K)$-conjugate in $Z^1(\Gamma, H(K))$. Actually we may, and we will, even assume that they are pairwise $Z(H)^\circ(K) \times H_{\mathrm{sc}}(K)$-conjugate through the canonical isogeny $Z(H)^\circ \times H_{\mathrm{sc}} \longrightarrow H$, where H_{sc} denotes the simply connected covering group of the adjoint group H_{ad}. We now try to construct a $\phi \in Z^1(\Gamma, H(K))$ that is $H(K)$-conjugate to each ϕ_i, and such that $\phi(\Gamma) \subset H(R)$.

If $n = 1$, we are obviously done. Otherwise, start with ϕ_1 and pick elements $(z_i, h_i) \in Z(H)^\circ(K) \times H_{\mathrm{sc}}(K)$ such that ${}^{z_i h_i}\phi_1 = \phi_i$ in $Z^1(\Gamma, H(K))$, for all $i = 2, \cdots, n$. For any prime $\mathfrak{p} \in S := \mathrm{Spec}(R) \setminus \mathrm{Spec}(R_1)$ there is some $i \geq 2$ such that

$\mathfrak{p} \in \operatorname{Spec}(R_i)$. Pick such an i and put $(z_\mathfrak{p}, h_\mathfrak{p}) := (z_i, h_i)$. Since $H_{\rm sc}$ is a split simply connected semisimple group over K, the strong approximation theorem with respect to the finite set of archimedean places ensures the existence of an element $h \in H_{\rm sc}(R_1)$ such that $h \in H_{\rm sc}(R_\mathfrak{p})h_\mathfrak{p}$ for all $\mathfrak{p} \in S$. Then we have $({}^h\phi_1)(\Gamma) \subset H(R_1)$ and $({}^{z_\mathfrak{p}h}\phi_1)(\Gamma) \subset H(R_\mathfrak{p})$ for all $\mathfrak{p} \in S$. Now, since $Z(H)^\circ$ is a split torus, say of dimension d, the obstruction to finding $z \in Z(H)^\circ(R_1) \cap \bigcap_\mathfrak{p} Z(H)^\circ(R_\mathfrak{p})z_\mathfrak{p}$ lies in the d^{th} self-product of the ideal class group $\mathcal{C}\ell(K)^d$. Hence it vanishes over the Hilbert class field K^h of K and we can at least find $z \in Z(H)^\circ(R_1^h) \cap \bigcap_\mathfrak{p} Z(H)^\circ(R_\mathfrak{p}^h)z_\mathfrak{p}$, where the superscript h denotes normalization in K^h. Then we see that ${}^{zh}\phi_1(\Gamma) \subset H(R_1^h)$ and ${}^{zh}\phi_1(\Gamma) \subset H(R_\mathfrak{p}^h)$ for all $\mathfrak{p}$. Therefore we have $({}^{zh}\phi_1)(\Gamma) \subset H(R^h)$ as desired. ☐

Remark A.10 (Orbits)**.** With the notation of the theorem, the morphism

$$\underline{Z}^1(\Gamma, H)_{R_0} \xrightarrow{\pi} \underline{H}^1(\Gamma, H)_{R_0} = \{\pi(\phi), \phi \in \Phi_0\}$$

provides a decomposition as a disjoint union of affine R_0-schemes

$$\underline{Z}^1(\Gamma, H)_{R_0} = \bigsqcup_{\phi \in \Phi_0} \pi^{-1}(\pi(\phi))$$

Moreover, the action $h \mapsto h\cdot\phi$ of H_{R_0} provides a surjective morphism of R_0-schemes $H_{R_0} \longrightarrow \pi^{-1}(\pi(\phi))$, which at the level of étale sheaves identifies $\pi^{-1}(\pi(\phi))$ with the quotient $H_{R_0}/C_H(\phi)$ with $C_H(\phi)$ denoting the centralizer of ϕ. In particular, we see that this quotient sheaf is representable by an affine scheme which identifies with the orbit $H \cdot \phi := \pi^{-1}(\pi(\phi))$ of ϕ. To put it in different words, the natural map $H \longrightarrow H \cdot \phi$, $h \mapsto h \cdot \phi$ is a $C_H(\phi)$-torsor for the étale topology.

A.5. **Centralizers.** Our next task is to study the centralizer $C_H(\phi)$ of a cocycle $\phi \in Z^1(\Gamma, H(R))$. We have seen in Lemma A.1 that this is a smooth group scheme over R. Moreover, by [PY02, Thm 2.1], its geometric fibers have reductive neutral components. In other words, the "neutral" component $C_H(\phi)^\circ$ is a reductive group scheme over R. Thus it follows from Prop 3.1.3 of [Con14] that the quotient sheaf $\pi_0(C_H(\phi)) := C_H(\phi)/C_H(\phi)^\circ$ is representable by a separated étale group scheme over R. Our aim here is to prove that $\pi_0(C_H(\phi))$ is actually *finite* over R, at least when R is a Dedekind G-ring and Γ is a solvable group.

Note that $C_H(\phi)$ is also the subgroup of Γ-fixed points in H for the Ad_ϕ-twisted action of Γ on H. So, up to changing the action of Γ on H, it suffices to study the finiteness of $\pi_0(H^\Gamma)$ as an R-scheme.

Lemma A.11. *As above, assume Γ has invertible order in R.*

i) Let $H' \longrightarrow H$ be a Γ-equivariant central isogeny of reductive group schemes over R. If $\pi_0(H'^\Gamma)$ is finite over R, then so is $\pi_0(H^\Gamma)$.

ii) Let Γ' be a normal subgroup of Γ. If $\pi_0(H^{\Gamma'})$ and $\pi_0((H^{\Gamma',\circ})^{\Gamma/\Gamma'})$ are finite, then so is $\pi_0(H^\Gamma)$.

Proof. i) Let Z be the kernel of the isogeny, which is a finite central subgroup scheme of H' of multiplicative type over R. *We claim that the sheaf $\underline{H}^1(\Gamma, Z)$ is representable by a finite étale group scheme over R.* Indeed, since the category of finite group schemes of multiplicative type over R is abelian ([MR070a, IX.2.8]), the sheaves $\underline{Z}^1(\Gamma, Z)$, $\underline{B}^1(\Gamma, Z)$ and, consequently, $\underline{H}^1(\Gamma, Z)$ are finite group schemes of multiplicative type over R. Let us decompose $Z = \prod_p Z_p$ into a finite product of its p-primary components. Then $\underline{H}^1(\Gamma, Z)$ decomposes accordingly as a product of $\underline{H}^1(\Gamma, Z_p)$. But $\underline{H}^1(\Gamma, Z_p)$ is trivial unless p divides the order of Γ. Since this order is invertible in R, so is the rank of $\underline{H}^1(\Gamma, Z)$, which is therefore étale over R.

Let us now look at the following exact sequence of sheaves of groups on the big étale site of $\mathrm{Spec}(R)$.

$$1 \longrightarrow Z^\Gamma \longrightarrow H'^\Gamma \longrightarrow H^\Gamma \longrightarrow \underline{H}^1(\Gamma, Z) \longrightarrow \underline{H}^1(\Gamma, H').$$

In this sequence, we now know that all terms are R-schemes. Since $\underline{H}^1(\Gamma, Z)$ is finite étale, the morphism $H^\Gamma \longrightarrow \underline{H}^1(\Gamma, Z)$ has to be trivial on the reductive subgroups $(H^\Gamma)^\circ$, so that we deduce the following exact sequence :

$$Z^\Gamma \longrightarrow \pi_0(H'^\Gamma) \longrightarrow \pi_0(H^\Gamma) \longrightarrow \underline{H}^1(\Gamma, Z) \longrightarrow \underline{H}^1(\Gamma, H').$$

Now assume that $\pi_0(H'^\Gamma)$ is finite over R, and therefore finite étale. Since Z^Γ is finite, its image in $\pi_0(H'^\Gamma)$ is closed, hence is finite étale. Therefore $\pi_0(H^\Gamma)$ appears as the middle term of a five terms exact sequence in which all the four remaining terms are finite étale group schemes (the last one is only a pointed scheme and is étale by theorem A.7). Going to a finite étale covering R' of R over which all these étale groups become constant, we see that $\pi_0(H^\Gamma)$ also becomes constant and finite over R', hence is already finite over R.

ii) Put $H' := (H^{\Gamma'})^\circ$. Applying the Γ/Γ'-invariants functors to the exact sequence $H' \hookrightarrow H^{\Gamma'} \twoheadrightarrow \pi_0(H^{\Gamma'})$, we get an exact sequence

$$1 \longrightarrow (H')^{\Gamma/\Gamma'} \longrightarrow H^\Gamma \longrightarrow \pi_0(H^{\Gamma'})^{\Gamma/\Gamma'} \longrightarrow \underline{H}^1(\Gamma/\Gamma', H').$$

By assumption, $\pi_0(H^{\Gamma'})$ is finite étale, so the invariant subgroup $\pi_0(H^{\Gamma'})^{\Gamma/\Gamma'}$ is also étale and finite since it is closed. Therefore the map from H^Γ factors over $\pi_0(H^\Gamma)$. Since $((H')^\Gamma)^\circ = (H^\Gamma)^\circ$, we thus get an exact sequence

$$1 \longrightarrow \pi_0((H')^{\Gamma/\Gamma'}) \longrightarrow \pi_0(H^\Gamma) \longrightarrow \pi_0(H^{\Gamma'})^{\Gamma/\Gamma'} \longrightarrow \underline{H}^1(\Gamma/\Gamma', H').$$

All terms but possibly the middle one are finite étale (by Theorem A.7 for the last one). Therefore, the middle one is also finite étale, as desired. □

Theorem A.12. *Assume that H is reductive over a Dedekind G-ring R and is acted upon by a solvable finite group Γ with invertible order in R. Then $\pi_0(H^\Gamma)$ is a finite étale group scheme over R.*

Proof. As already mentioned in the beginning of this subsection, the problem is to prove finiteness. Thanks to item ii) of the last lemma, we can use induction to reduce the case of a solvable Γ to the case of an abelian Γ, and then further reduce to the case of a cyclic Γ. So let us assume that Γ is cyclic.

By Theorem 5.3.1 of [Con14], there is a unique closed semi-simple subgroup scheme H_{der} of H over R that represents the sheafification of the set-theoretical derived subgroup and such that the quotient H/H_{der} is a torus. Then the natural morphism $Z(H)^\circ \times H_{\mathrm{der}} \longrightarrow H$ is a central isogeny by the fibrewise criterion, and moreover is Γ-equivariant (here $Z(H)^\circ$ denotes the maximal central torus of H). Further, by Exercise 6.5.2 of [Con14], there is a canonical central isogeny $H_{\mathrm{sc}} \longrightarrow H_{\mathrm{der}}$ over R, such that all the geometric fibers of H_{sc} are simply connected semi-simple groups. Being canonical, the action of Γ on H_{der} lifts uniquely to H_{sc}. Let us now consider the Γ-equivariant central isogeny $Z(H)^\circ \times H_{\mathrm{sc}} \longrightarrow H$. By item i) of the previous lemma, it suffices to prove the finiteness of $\pi_0((Z(H)^\circ)^\Gamma)$ and that of $\pi_0((H_{\mathrm{sc}})^\Gamma)$. The first one is clear since $(Z(H)^\circ)^\Gamma$ is smooth and of multiplicative type. For the second one, we use Steinberg's theorem [Ste68, Thm 8.2], which can be applied here since a generator of Γ induces a semisimple automorphism of each geometric fiber of H_{sc}, and which ensures that $(H_{\mathrm{sc}})^{\Gamma}$ has connected fibers, so that $\pi_0((H_{\mathrm{sc}})^{\Gamma})$ is even the trivial group. □

A.6. **Splitting a reductive group scheme over a finite flat extension.** A reductive group scheme over any ring R is known to split over a faithfully étale extension of R. However, in general it won't split over a finite étale extension. Already over $R = \mathbb{Z}$, there are examples where a non-trivial Zariski localization is needed. Here we use a similar argument as in the proof of Theorem A.9 in order to prove that if R is a localization of a ring of integers, then a reductive group scheme over R splits over a suitable finite flat extension of R.

Proposition A.13. *Assume that H is reductive over a normal subring R of a number field K. Then there is a finite extension K_0 of K such that H splits over the normalization R_0 of R in K_0.*

Proof. Pick a faithfully étale R' over R such that H splits over R'. Let $R' = \prod_{i=1}^{n} R'_i$ be the decomposition of R' in connected components. Of course, if $n = 1$ we are done, so we assume $n > 1$. Replacing R and all R'_i by their normalization in the residue field at some generic point of $R'_1 \otimes_R \cdots \otimes_R R'_n$, we may assume that each R'_i is a localization of R (i.e. $\mathrm{Spec}(R') \longrightarrow \mathrm{Spec}(R)$ is a Zariski cover). Let $T_i \subset H_{R'_i}$ be a split maximal torus defined over R'_i. The generic fibers $T_{i,K}$ are split maximal tori in H_K, hence are conjugate under $H(K)$. After replacing K by a finite extension, we may assume that they are conjugate under $H_{\mathrm{sc}}(K)$. So there are elements $h_i \in H_{\mathrm{sc}}(K)$, $i > 1$, such that ${}^{h_i}T_{1,K} = T_{i,K}$. Put $S := \mathrm{Spec}(R) \backslash \mathrm{Spec}(R'_1)$ (a finite set) and for $\mathfrak{p} \in S$, pick a $i \geq 2$ such that $\mathfrak{p} \in \mathrm{Spec}(R'_i)$ and put $h_{\mathfrak{p}} = h_i$. Then by the strong approximation theorem, there is some $h \in H_{\mathrm{sc}}(R'_1)$ such that $h \in H_{\mathrm{sc}}(R_{\mathfrak{p}})h_{\mathfrak{p}}$ for all $\mathfrak{p} \in S$. We claim that the K-torus $T_K := {}^{h}T_{1,K}$ of H_K extends (canonically) to a an R-subtorus of H. Indeed, recall that the functor $\mathrm{Tor}_{H/R}$ which to any R-algebra R' associates the set of maximal subtori of $H_{R'}$ is known to be representable by a smooth quasi-affine, hence in particular separated, scheme over R, see e.g. [Con14, Thm 3.2.6]. By construction, T_K comes from a R'_1-torus T'_1 of $H_{R'_1}$, which is unique by separateness of $\mathrm{Tor}_{H/R}$. Similarly for each $\mathfrak{p} \in S$, there is a unique extension of T_K to a $R_{\mathfrak{p}}$-torus $T_{\mathfrak{p}}$ of $H_{R_{\mathfrak{p}}}$ and the latter is actually defined over a Zariski open neighbourhood of $\mathfrak{p}$. This means that the K-section of $\mathrm{Tor}_{H/R}$ given by T_K extends uniquely to a Zariski covering of $\mathrm{Spec}\, R$, hence extends to $\mathrm{Spec}\, R$ itself, whence a maximal torus T_R of H extending T_K. Since T_K is split and since tori are known to split over finite étale coverings of the base, T_R is split too.

Now, the root subspaces of T_R in $\mathrm{Lie}(H)$ are rank 1 locally free R-modules. Replacing K by its Hilbert class field, we may assume that they are actually free. Since R is connected, this is enough for H to split over R, *cf* the paragraph below Definition 5.1.1 of [Con14]. □

Remark A.14. Exercise 7.3.9 of [Con14] provides another proof that does not use strong approximation. Namely, start by enlarging R so that H_K splits. So H_K contains a Borel subgroup B_K, which extends uniquely to a Borel subgroup scheme B of H by the properness of the scheme of Borel subgroups. Let (H', B') be the constant split pair over R that extends (H_K, B_K). Then the functor $\mathcal{I}$ of isomorphisms between (H', B') and (H, B) is a torsor over the automorphism group $\mathcal{A} = B'_{\mathrm{ad}} \rtimes \mathrm{Out}(H)$ of the pair (H', B'). Its class in $H^1_{\text{ét}}(\operatorname{Spec} R, \mathcal{A})$ has trivial image in $H^1_{\text{ét}}(\operatorname{Spec} R, \mathrm{Out}(H))$ since H is split over K. On the other hand $H^1_{\text{ét}}(\operatorname{Spec} R, B'_{\mathrm{ad}})$ is isomorphic to a sum of copies of $\mathrm{Pic}(R) = H^1_{\text{ét}}(\operatorname{Spec} R, \mathbb{G}_m)$. So let K_0 be the Hilbert class field of K. Since $\mathrm{Pic}(R) \longrightarrow \mathrm{Pic}(R_0)$ has trivial image, $\mathcal{I}$ becomes a trivial $\mathcal{A}$-torsor over R_0, hence H splits over R_0.

Appendix B. Twisted Poincaré polynomials

B.1. **Some characteristic polynomials attached to root data.** Let $\Sigma = (\mathbb{X}, \mathbb{X}^\vee, \Delta, \Delta^\vee)$ be a based root datum with Weyl group Ω and group of automorphisms $\mathrm{Aut}(\Sigma)$. Both Ω and $\mathrm{Aut}(\Sigma)$ embed as groups of linear automorphisms of $\mathbb{X}$ and $\mathbb{X}^\vee$, and $\mathrm{Aut}(\Sigma)$ normalizes Ω. In particular $\mathrm{Aut}(\Sigma)$ acts on the ring of Ω-invariant polynomials $\mathrm{Sym}^\bullet(\mathbb{X})^\Omega$ on $\mathbb{X}^\vee$ and on the conormal module $\mathbb{M}^\bullet$ of $\mathbb{X}^\vee/\Omega$ along the zero section

$$\mathbb{M}^\bullet := \mathrm{Sym}^{\bullet>0}(\mathbb{X})^\Omega / (\mathrm{Sym}^{\bullet>0}(\mathbb{X})^\Omega)^2.$$

For any $\alpha \in \mathrm{Aut}(\Sigma)$ we consider its weighted characteristic polynomial on $\mathbb{M}^\bullet_{\mathbb{Q}}$

$$\chi_{\alpha|\mathbb{M}^\bullet}(T) := \prod_{d>0} \det\left(T^d - \alpha\,|\mathbb{M}^d_{\mathbb{Q}}\right) \in \mathbb{Z}[T]. \tag{B.1}$$

A priori $\mathbb{M}^\bullet$ may have torsion, but a result of Demazure [Dem73, Thm 3] shows that $\mathbb{M}^\bullet \otimes \mathbb{Z}[\frac{1}{|\Omega|}]$ is torsion free, so we deduce the following

Remark B.1. If ℓ does not divide the order of Ω, the image of $\chi_{\alpha|\mathbb{M}^\bullet}$ in $\mathbb{F}_\ell[T]$ is the weighted characteristic polynomial of α on $\mathbb{M}^\bullet_{\mathbb{F}_\ell}$.

Since Ω is a reflection subgroup of $\mathrm{Aut}(\mathbb{X}^\vee)$, the $\bar{\mathbb{Q}}$-algebra $\mathrm{Sym}^\bullet(\mathbb{X})^\Omega_{\bar{\mathbb{Q}}}$ is known to be a weighted polynomial algebra. More precisely, any graded section $\mathbb{M}^\bullet_{\bar{\mathbb{Q}}} \hookrightarrow \mathrm{Sym}^{\bullet>0}(\mathbb{X})^\Omega_{\bar{\mathbb{Q}}}$ induces a graded isomorphism $\mathrm{Sym}(\mathbb{M}^\bullet_{\bar{\mathbb{Q}}}) \xrightarrow{\sim} \mathrm{Sym}^\bullet(\mathbb{X})^\Omega_{\bar{\mathbb{Q}}}$ for the unique ring grading on $\mathrm{Sym}(\mathbb{M}^\bullet_{\bar{\mathbb{Q}}})$ such that $\mathbb{M}^d$ is in degree d for all d. In particular, for $\alpha = 1$, we have $\chi_{1|\mathbb{M}^\bullet}(T) = \prod_{i=1}^r (T^{d_i} - 1)$ where $r = \mathrm{rk}_{\mathbb{Z}}(\mathbb{X})$ and $d_1 \leq \cdots \leq d_r$ are the so-called *fundamental degrees* of Ω acting on $\mathbb{X}^\vee$. Here d_r is known as the *Coxeter number* of Σ and is the maximal $n \in \mathbb{N}$ such that $\Phi_n(T)$ divides $\chi_{1|\mathbb{M}^\bullet}(T)$.

More generally, using an $\mathrm{Aut}(\Sigma)$-equivariant section $\mathbb{M}^\bullet_{\bar{\mathbb{Q}}} \hookrightarrow \mathrm{Sym}^{\bullet>0}(\mathbb{X})^\Omega_{\bar{\mathbb{Q}}}$, we see that, at least when α has finite order, $\chi_{\alpha|\mathbb{M}^\bullet}(T) = (T^{d_1} - \varepsilon_{1,\alpha}) \cdots (T^{d_r} - \varepsilon_{r,\alpha})$ where the $\varepsilon_{i,\alpha}$ are as in Lemma 6.1 of [Spr74]. Note that in this case, $\chi_{\alpha|\mathbb{M}^\bullet}$ is a product of cyclotomic polynomials and we have $\chi_{\alpha|\mathbb{M}^\bullet} = \chi_{\alpha^{-1}|\mathbb{M}^\bullet}$. The maximal $n \in \mathbb{N}$ such that $\Phi_n(T)$ divides $\chi_{\alpha|\mathbb{M}^\bullet}(T)$ has been known in the literature as the *twisted Coxeter number* associated to α. Now, a fundamental consequence of Springer's work in this setup is the following result.

Proposition B.2. *$\chi_{\alpha|\mathbb{M}^\bullet}(T)$ is the lowest common multiple in $\bar{\mathbb{Q}}[T]$ of the characteristic polynomials $\chi_{\omega\alpha|\mathbb{X}}(T)$ of $\omega\alpha$ on $\mathbb{X}_{\bar{\mathbb{Q}}}$, where ω runs over Ω.*

Proof. When α has finite order, this is a reformulation of Theorem 6.2 (i) of [Spr74]. In general, this follows from the decompositions $\mathbb{X}_{\mathbb{Q}} = \mathbb{X}^\Omega_{\mathbb{Q}} \oplus \mathbb{Q}\langle\Delta\rangle$ and $\mathbb{M}^\bullet_{\mathbb{Q}} = \mathbb{X}^\Omega_{\mathbb{Q}} \oplus \mathrm{Sym}^{\bullet>0}(\mathbb{Q}\langle\Delta\rangle)^\Omega / (\mathrm{Sym}^{\bullet>0}(\mathbb{Q}\langle\Delta\rangle)^\Omega)^2$ and the fact that $\alpha_{|\mathbb{Q}\langle\Delta\rangle}$ has finite order, while $\omega\alpha_{|\mathbb{X}^\Omega_{\mathbb{Q}}} = \alpha_{|\mathbb{X}^\Omega_{\mathbb{Q}}}$ for all $\omega \in \Omega$. $\square$

B.2. **Application to reductive groups.** Let $\hat{G}$ be a connected reductive group over an algebraically closed field L of characteristic ℓ. Attached to $\hat{G}$ is a root datum $\Sigma = (\mathbb{X}_{\hat{G}}, \mathbb{X}_{\hat{G}}^\vee, \Delta_{\hat{G}}, \Delta_{\hat{G}}^\vee)$ as above, that comes with an identification $\mathrm{Aut}(\Sigma) = \mathrm{Out}(\hat{G})$. Here Σ denotes the limit over all Borel pairs $(\hat{B}, \hat{T})$ of $\hat{G}$ of the root data $(X^*(\hat{T}), X_*(\hat{T}), \Delta(\hat{B}), \Delta(\hat{B})^\vee)$. Now, let β be an automorphism of $\hat{G}$ with image α in $\mathrm{Out}(\hat{G})$. Using the notation of the last subsection, we put

$$\chi_{\hat{G},\beta}(T) := \chi_{\alpha|\mathbb{M}^\bullet}(T) \in \mathbb{Z}[T]. \tag{B.2}$$

Further, we denote by $h_{\hat{G},\beta}$ the twisted Coxeter number of Σ associated to α and we put

$$\chi^*_{\hat{G},\beta}(T) := \prod_{n \leq h_{\hat{G},\beta}} \Phi_n(T) \ \in \mathbb{Z}[T]. \tag{B.3}$$

The following result is crucial to track the "banal" primes in this paper.

Proposition B.3. *Let β be an automorphism of $\hat{G}$.*

(1) If $\hat{H}$ is a reductive subgroup of $\hat{G}$ stable under β, then $\chi_{\hat{H},\beta}$ divides $\chi_{\hat{G},\beta}$, $h_{\hat{H},\beta} \leq h_{\hat{G},\beta}$ and $\chi^*_{\hat{H},\beta}$ divides $\chi^*_{\hat{G},\beta}$.

(2) Let t be a semi-simple element of $\hat{G}(L)$ such that $\beta(t) = t^q$. Then t has finite order, and this order divides $\chi_{\hat{G},\beta}(q)$.

Proof. (1) As above, we denote by α the image of β in $\mathrm{Out}(\hat{G}) = \mathrm{Aut}(\Sigma_{\hat{G}})$, which acts on the "abstract root lattice" $\mathbb{X}_{\hat{G}}$. Similarly, we denote by $\alpha_{\hat{H}}$ the image of β in $\mathrm{Aut}(\Sigma_{\hat{H}})$, which acts on $\mathbb{X}_{\hat{H}}$. Let $(B_{\hat{H}}, T_{\hat{H}})$ be a Borel pair of $\hat{H}$, so that we have an identification $\Sigma_{\hat{H}} = \Sigma(B_{\hat{H}}, T_{\hat{H}})$, and in particular $\mathbb{X}_{\hat{H}} = X^*(T_{\hat{H}})$. Through this identification, the action of $\alpha_{\hat{H}}$ on $\mathbb{X}_{\hat{H}}$ corresponds to the action of $\mathrm{Ad}_h \circ \beta$ on $X^*(T_{\hat{H}})$ for any $h \in \hat{H}$ such that $\mathrm{Ad}_h \circ \beta$ stabilizes the pair $(B_{\hat{H}}, T_{\hat{H}})$. More generally, for $\omega_{\hat{H}} \in \Omega_{\hat{H}}$ (the "abstract" Weyl group of $\hat{H}$), the action of $\omega_{\hat{H}}\alpha_{\hat{H}}$ on $\mathbb{X}_{\hat{H}}$ corresponds to the action of $\mathrm{Ad}_{nh} \circ \beta$ on $X^*(T_{\hat{H}})$, where h is as above, and $n \in N_{\hat{H}}(T_{\hat{H}})$ is a lift of $\omega_{\hat{H}}$.

Now, let $(\hat{B}, \hat{T})$ be a Borel pair in $\hat{G}$ that induces $(B_{\hat{H}}, T_{\hat{H}})$ on $\hat{H}$. As above, we have an identification $\Sigma_{\hat{G}} = \Sigma(\hat{B}, \hat{T})$, and in particular $\mathbb{X}_{\hat{G}} = X^*(\hat{T})$, from which we deduce a surjective morphism $\mathbb{X}_{\hat{G}} \twoheadrightarrow \mathbb{X}_{\hat{H}}$. With h and n as above, pick also $m \in C_{\hat{G}}(T_{\hat{H}})$ such that $\mathrm{Ad}_{mnh} \circ \beta$ stabilizes $\hat{T}$. Observe that the action of this automorphism on $X^*(\hat{T})$ induces the action of $\mathrm{Ad}_{nh} \circ \beta$ on $X^*(T_{\hat{H}})$. On the other hand, there is a unique $\omega \in \Omega_{\hat{G}}$ such that, for any $n' \in N_{\hat{G}}(\hat{T})$ above ω^{-1}, the automorphism $\mathrm{Ad}_{n'mnh} \circ \beta$ stabilizes also $\hat{B}$. Then the action of this automorphism on $\mathbb{X}_{\hat{G}}$ is α. Hence it follows that the action of $\omega\alpha$ on $\mathbb{X}_{\hat{G}}$ induces the action of $\omega_{\hat{H}}\alpha_{\hat{H}}$ on $\mathbb{X}_{\hat{H}}$. Therefore the characteristic polynomial $\chi_{\omega_{\hat{H}}\alpha_{\hat{H}}|\mathbb{X}_{\hat{H}}}(T)$ divides the characteristic polynomial $\chi_{\omega\alpha|\mathbb{X}_{\hat{G}}}(T)$. By Proposition B.2, we deduce that $\chi_{\alpha_{\hat{H}}|\mathbb{M}^\bullet_{\hat{H}}}$ divides $\chi_{\alpha|\mathbb{M}^\bullet_{\hat{G}}}$, as desired.

(2) The connected centralizer $\hat{H} := C_{\hat{G}}(t)^\circ$ contains t and is stable under β, since $\beta(\hat{H}) = C_{\hat{G}}(t^q)^\circ$ contains $\hat{H}$ and has same dimension as $\hat{H}$. Hence by (1) it suffices to prove the statement when t is central in $\hat{G}$. Then we may compose β with some Ad_g so that it fixes a pinning of $\hat{G}$, with maximal torus $\hat{T}$. Now, consider t as a homomorphism $X^*(\hat{T}) \longrightarrow L^\times$. Since $\beta(t) = t^q$, we see that this homomorphism factors over the cokernel of the endomorphism $\beta - q$ of $X^*(\hat{T})$. But this cokernel

is finite of order $\chi_{\hat{T},\beta}(q)=\det(q-\beta)$. So t has order dividing $\chi_{\hat{T},\beta}(q)$, hence also dividing $\chi_{\hat{G},\beta}(q)$. $\square$

B.3. **The Chevalley-Steinberg formula.** Let now G be a reductive group over $\mathbb{F}_q$. Let G^* be a split form of $G_{\bar{\mathbb{F}}_q}$ over $\mathbb{F}_q$ and pick an isomorphim $\psi : G_{\bar{\mathbb{F}}_q} \xrightarrow{\sim} G^*_{\bar{\mathbb{F}}_q}$. Then $\mathrm{Fr} := {}^{\mathrm{Frob}}\psi^{-1}\circ\psi$ is an automorphism of $G_{\bar{\mathbb{F}}_q}$ (where Frob denotes the Frobenius automorphism of $\bar{\mathbb{F}}_q$), and we have the following Chevalley-Steinberg formula for the number of $\mathbb{F}_q$-rational points of G.

Theorem B.4 (Chevalley-Steinberg)**.** $|G(\mathbb{F}_q)| = q^N.\chi_{G,\mathrm{Fr}}(q)$, *where N is the dimension of a maximal unipotent subgroup of $G_{\bar{\mathbb{F}}_q}$.*

Proof. This formula is stated for absolutely simple adjoint groups in Theorems 25 and 35 of [Ste16]. It is also true for a torus S, since we have an isomorphism $X_*(S)/(q\,\mathrm{Fr}-1)X_*(S) \xrightarrow{\sim} S(\mathbb{F}_q)$ [DL76, (5.2.3)], from which it follows that $|S(\mathbb{F}_q)| = |\det(q\,\mathrm{Fr}-1)| = |\chi_{S,\mathrm{Fr}^{-1}}(q)| = \chi_{S,\mathrm{Fr}}(q)$.

To prove the formula in general, we first observe that if $G \xrightarrow{\pi} G'$ is an isogeny, then $|G(\mathbb{F}_q)| = |G'(\mathbb{F}_q)|$. Indeed, the kernel $H := \ker(\pi)(\bar{\mathbb{F}}_q)$ is a finite group with an action of the arithmetic Frobenius Frob and we have an exact sequence

$$1 \longrightarrow H^{\mathrm{Frob}} \longrightarrow G(\mathbb{F}_q) \longrightarrow G'(\mathbb{F}_q) \longrightarrow H^1(\mathbb{F}_q, H) = H_{\mathrm{Frob}} \longrightarrow 1$$

where the last map is surjective because $H^1(\mathbb{F}_q, G) = 1$. But we also have an exact sequence $H^{\mathrm{Frob}} \hookrightarrow H \xrightarrow{\mathrm{Frob}-\mathrm{id}} H \twoheadrightarrow H_{\mathrm{Frob}}$ which shows that $|H^{\mathrm{Frob}}| = |H_{\mathrm{Frob}}|$, so we get $|G(\mathbb{F}_q)| = |G'(\mathbb{F}_q)|$.

Now we deduce the formula for general G by applying this observation to the isogeny $G \longrightarrow G_{\mathrm{ab}} \times G_{\mathrm{ad}}$ and decomposing G_{ad} as a product of restriction of scalars of absolutely simple groups. $\square$

B.4. **Kostant's section theorem.** We return to the setting of a reductive group $\hat{G}$ over an algebraically closed field L and, for simplicity, we assume that $\hat{G}$ is simple adjoint. We also assume that the characteristic ℓ of L does not divide the order of the Weyl group $\Omega_{\hat{G}}$.

Let us fix a pinning $\varepsilon = (\hat{T}, \hat{B}, (X_\alpha)_{\alpha\in\Delta})$ of $\hat{G}$. The sum $E = \sum_{\alpha\in\Delta} X_\alpha$ is then a regular nilpotent element of $\mathrm{Lie}(\hat{G})$. The sum $H = \sum_{\beta\in\Phi^+} \beta^\vee \otimes 1 \in X_*(\hat{T}) \otimes L = \mathrm{Lie}(\hat{T})$ is a regular semisimple element of $\mathrm{Lie}(\hat{G})$ and the pair (H, E) is part of a unique "principal" $\mathfrak{sl}_2$-triple (F, H, E). Denote by $\mathrm{Lie}(\hat{G})_E$ the centralizer of E in $\mathrm{Lie}(\hat{G})$. Under our assumption on ℓ, Veldkamp has proved that Kostant' section theorem still holds, [Vel72, Prop 6.3]. This states that the map

$$\mathrm{Lie}(\hat{G})_E \longrightarrow \mathrm{Lie}(\hat{G}) /\!\!/ \hat{G}, \quad X \mapsto (F + X) \bmod \hat{G}$$

is an isomorphism of varieties. Moreover, seeing $\lambda := \sum_{\beta\in\Phi^+} \beta^\vee$ as a cocharacter of $\hat{G}$, this map is $\mathbb{G}_m$-equivariant for the action $(t, y) \mapsto t\cdot y := t^2\,\mathrm{Ad}_{\lambda(t)}(y)$ on the LHS and the action $(t, x) \mapsto t^2 x$ on the RHS. Composing with the Chevalley isomorphism (which also holds in this context) yields an isomorphism of $\mathbb{G}_m$-varieties

$$\pi :\ \mathrm{Lie}(\hat{G})_E \xrightarrow{\sim} \mathbb{X}^\vee_L/\Omega_{\hat{G}}.$$

Now let $\mathrm{Aut}(\hat{G})_\varepsilon$ be the group of automorphisms of $\hat{G}$ that preserve the pinning ε. This group fixes E, so it acts on $\mathrm{Lie}(\hat{G})_E$. It also acts on $\mathrm{Lie}(\hat{G}) /\!\!/ \hat{G}$ and $\mathbb{X}^\vee_L/\Omega_{\hat{G}}$, and both the Chevalley map and the Kostant map are equivariant for these actions.

Identifying $\mathrm{Out}(\hat{G})$ with $\mathrm{Aut}(\hat{G})_\varepsilon$, we thus get on conormal modules at the origin an isomorphism

$$\mathbb{M}_L^\bullet \xrightarrow{\sim} (\mathrm{Lie}(\hat{G})_E)^*$$

which is $\mathrm{Out}(\hat{G})$-equivariant, as well as $\mathbb{G}_m$-equivariant for the (dual) action described above on the RHS and the action associated with "twice the $\bullet$-grading" on the LHS. So we deduce the following result.

Proposition B.5. *For $t \in L^\times$ and $\beta \in \mathrm{Aut}(\hat{G})_\varepsilon$ of finite order, we have*

$$\det\left(t^2 \mathrm{Ad}_{\lambda(t)} \mathrm{Ad}_\beta - \mathrm{id} \mid \mathrm{Lie}(\hat{G})_E\right) = \pm \chi_{\hat{G},\beta}(t^2).$$

Proof. Indeed, by the foregoing discussion, the LHS equals

$$\prod_d \det\left(t^{2d}\beta^{-1} - \mathrm{id}\,|\mathbb{M}_L^d\right) = \det(\beta)^{-1} \prod_d \det\left(t^{2d} - \beta|\mathbb{M}_L^d\right).$$

But $\det(\beta) = \pm 1$ since it is a root of unity in $\mathbb{Q}$, while Remark B.1 ensures that $\det\left(t^{2d} - \beta|\mathbb{M}_L^d\right) = \chi_{\hat{G},\beta}(t^2)$ in L. □

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