P-ADIC WHITTAKER PATTERNS

Contents

1. Introduction

The goal of this article is to prove the following theorem.

Theorem 1.1. *If λ is a dominant coweight and ν and μ are coweights such that $\mu+\nu$ are dominant, then the cohomology*

$$H^i_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\lambda|_{\mathrm{MV}_{\lambda,\nu}} \otimes (h^{\lambda,\nu}_\mu)^*(\mathcal{L}_\psi)) = \begin{cases} 0 & \textit{otherwise} \\ \mathrm{Hom}_{\mathrm{Rep}(\widehat{G})}(V^\lambda \otimes V^\mu, V^{\mu+\nu}) & i=(2\rho,\nu) \textit{ and } \mu \textit{ is dominant} \end{cases}$$

Remark 1.2. When $\mu=0$, we recover [NP01, Thm. 3.2]

$$H^i_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\nu|_{\mathrm{MV}_{\lambda,\nu}} \otimes (h^{\lambda,\nu}_0)^*(\mathcal{L}_\psi)) = \begin{cases} 0 & \text{if } i \neq (2\rho,\nu) \\ \overline{\mathbb{Q}}_\ell \left\langle \rho, \lambda \right\rangle & \text{if } i=(2\rho,\nu) \end{cases}$$

2. Notation

Fix a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O} \subset F$, uniformizer $\varpi \in \mathcal{O}$, and residue field $k=\mathcal{O}/\varpi$. Write $q=|k|$. If R is a perfect k-algebra, write

$$W_{\mathcal{O}}(R) = W(R) \otimes_{W(k)} \mathcal{O}$$

where $W(-)$ denotes the p-typical Witt vectors. We also define the truncated Witt vectors

$$W_{\mathcal{O},h}(R) = W_{\mathcal{O}}(R) \otimes_{W(k)} \mathcal{O}/\varpi^n.$$

Date: February 14, 2024.

We also fix some notation for the reductive group.

- Let G be a split reductive group over F.
- Fix a maximal torus T and a Borel B containing it, and let N denote its unipotent radical.
- Let $\bar{G}, \bar{B}, \bar{T}, \bar{N}$ denote the special fibers over k.
- Let Φ denote the set of all roots, and let Φ_+ denote the set of positive roots corresponding to B.
- Let $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ denote the lattice of cocharacters, and let $X_*(T)^+$ denote the cone of dominant cocharacters corresponding to B.
- Write $\leq$ for the usual Bruhat order with respect to the positive roots.
- If $\nu \in X_*(T)$, we write $\varpi^\nu := \nu(\varpi)$.

3. The Witt vector affine Grassmannian

Definition 3.1 ([Zhu17, Section 1])**.**

- If $\mathcal{X}$ is an affine scheme over $\mathcal{O}$, let $L^+\mathcal{X} \in \mathrm{AlgSpc}_k^{\mathrm{pf}}$ denote the positive loop space. As a consequence of [Gre61], we have
$$L^+\mathcal{X} \simeq \varprojlim_h L^h\mathcal{X}$$
where $L^h\mathcal{X}$ is the perfection of the prestack $L^h_p\mathcal{X} \in \mathrm{Shv}(\mathrm{Aff}_k)$, whose R points are $\mathcal{X}(W_{\mathcal{O},h}(R))$.
- if $X \in \mathrm{Aff}_F$, let LX denote the loop space whose R points, for a perfect k-scheme R, are
$$LX(R) = X(W_{\mathcal{O}}(R)[1/\varpi]).$$
The functor LX is represented by an ind perfect scheme.
- If H is any smooth affine group scheme over $\mathcal{O}$, we write
$$\mathrm{Gr}_H = LH/L^+H$$
for the Witt vector affine Grassmannian for H, where we take the quotient in the étale topology.

Recall that Gr_G can be written as the colimit of perfections of projective varieties, called *(affine) Schubert varieties*:
$$\mathrm{Gr}_G = \mathrm{colim}_{\lambda \in X_*(T)^+} \mathrm{Gr}_{\leq\lambda}$$
and that the Schubert varieties are the closure of their maximal Schubert cells:
$$\mathrm{Gr}_{\leq\lambda} = \overline{\mathrm{Gr}_\lambda} = \bigcup_{\lambda' \leq \lambda} \mathrm{Gr}_{\lambda'},$$
where $\mathrm{Gr}_\lambda \subset \mathrm{Gr}_G$ is locally closed, and such that on k-points we get
$$\mathrm{Gr}_\lambda(k) = G(\mathcal{O})\lambda(\varpi)G(\mathcal{O}),$$

in accordance with the Cartan decomposition. By definition there is a left action of LG on Gr_G. This restricts to an action of L^+G on $\mathrm{Gr}_{\leq\lambda}$.

Lemma 3.2. *The action of L^+G on $\mathrm{Gr}_{\leq\lambda}$ factors through L^hG for h large enough.*

Proof. This is explained in the proof of [Zhu17, Proposition 1.23]. □

For $\lambda \in X_*(T)^+$ we let $\mathcal{A}_\lambda$ denote the intersection cohomology sheaf on $\mathrm{Gr}_{\leq\lambda}$, which is defined as the intermediate extension of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on Gr_λ to all of $\mathrm{Gr}_{\leq\lambda}$. We have

$$\mathcal{A}_\lambda \in P_{L^+G}(\mathrm{Gr}_G).$$

Its restriction is

$$\mathcal{A}_\lambda|_{\mathrm{Gr}_\lambda} = \overline{\mathbb{Q}}_\ell[(2\rho, \mu)].$$

The inclusion $N \hookrightarrow G$ functorially induces an inclusion $\mathrm{Gr}_N \hookrightarrow \mathrm{Gr}_G$. The Iwasawa decomposition gives us the following alternative stratification of Gr_G.

Definition 3.3 ([Zhu17, somewhere])**.** The *semi-infinite orbit* of a cocharacter $\nu \in X_*(T)$ is

$$S_\nu = \varpi^\lambda \,\mathrm{Gr}_N \subset \mathrm{Gr}_G\,.$$

Definition 3.4. Let

$$\mathrm{MV}_{\lambda,\nu} := \mathrm{Gr}_{\leq\lambda} \cap S_\nu,$$

where "MV" is short for "Mirkovic–Vilonen". In the literature a *Mirkovic–Vilonen cycle* is typically an irreducible component of $\mathrm{MV}_{\lambda,\nu}$, but we use MV to denote the whole intersection.

3.5. **Character sheaf.** Fix, once and for all, an additive character

$$\psi : F \to F/\mathcal{O} \to \overline{\mathbb{Q}}_\ell^\times$$

such that $\psi(p^{-1}\mathcal{O}) \neq 1$. Choosing conductor zero will simplify the rest of the arguments, but does not amount to any real loss of generality in Theorem 1.1.

In order to geometrize the additive character and consider Whittaker sheaves, we first consider the natural map

$$h : LN \to LN/[LN, LN] \xrightarrow{\sim} \prod_{\alpha\in\Phi_+} L\mathbb{G}_a \xrightarrow{+} L\mathbb{G}_a \to L\mathbb{G}_a/L^+\mathbb{G}_a.$$

This has a natural descent to S_ν.

Lemma 3.6. *If $\mu \in X_\bullet(T)$ is a character such that $\mu + \nu$ is dominant, then the map h descends to a map*

$$h^\nu_\mu : S_\nu \to L\mathbb{G}_a/L^+\mathbb{G}_a.$$

Proof. Note that $S_\nu = \varpi^\nu \operatorname{Gr}_N = (\varpi^\nu LN)/L^+N$. But this is the étale sheafification of the naïve quotient of presheaves. So for R a perfect k-algebra we define

$$\begin{aligned}(\varpi^\nu LN(R))/L^+N(R) &\to L\mathbb{G}_a(R)/L^+\mathbb{G}_a(R)\\ \varpi^\nu n \mod L^+N(R) &\mapsto h(\operatorname{ad}(\varpi^{\mu+\nu})(n)).\end{aligned}$$

To see that this is well-defined, suppose $\varpi^\nu n L^+N(R) = \varpi^\nu m L^+N(R)$. Then $n^{-1}m \in L^+N(R)$, but $\mu+\nu$ is dominant so $\operatorname{ad}(\varpi^{\mu+\nu})(n^{-1}m) \in L^+N(R)$, which maps to $L^+\mathbb{G}_a(R)$ under the group homomorphism h. This is clearly functorial and extends to a morphism of presheaves, which we then sheafify. □

We then want to turn the nontrivial additive character

$$\psi : F \to F/\mathcal{O} \to \overline{\mathbb{Q}}_\ell$$

into a character sheaf (i.e. a multiplicative rank 1 étale local system) on

$$\operatorname{Gr}_{\mathbb{G}_a} := L\mathbb{G}_a/L^+\mathbb{G}_a$$

(whose k points are exactly $F/\mathcal{O}$) and pull it back along h_μ^ν. However, $\operatorname{Gr}_{\mathbb{G}_a}$ is a group ind-scheme, and a geometric version of ψ on $\operatorname{Gr}_{\mathbb{G}_a}$ would have to be supported everywhere. To formalize this, one would have to define the category of étale sheaves on $\operatorname{Gr}_{\mathbb{G}_a}$ as a *limit* of sheaves on finite pieces of the ind-scheme, as opposed to Definition 7.3, which is defined by taking a colimit. We want to avoid having to make such a definition.

In the existing proofs of geometric Casselman–Shalika in equal characteristic, the character sheaf is induced from residue map h which ends with the residue map $L\mathbb{G}_a \xrightarrow{\sum c_i t^i \mapsto c_{-1}} \mathbb{G}_a$. In mixed characteristics, this cannot work because ψ does not factor through any finite subgroup of $F/\mathcal{O}$.

But Lemma 3.8 below saves us from this predicament.

Definition 3.7. If H is a smooth affine group scheme over $\mathcal{O}$ and $s \in \mathbb{Z}$, we let $L^{\geq s}H$ denote the image of L^+H under the isomorphism

$$LH \xrightarrow{\cdot \varpi^s} LH.$$

For $s > 0$ it's clear that the natural embedding $L^+H \to LH$ factors through $L^{\geq -s}H$, so we can form the quotient

$$L^{\geq s}H/L^+H,$$

which is isomorphic to L^sH.

Lemma 3.8. *If λ is a dominant coweight and and ν is a coweight, there is a factorization*

$$\begin{tikzcd} \operatorname{MV}_{\lambda,\nu} \arrow[r, dashed, "h_\mu^{\lambda,\nu}"] \arrow[d, hook] & L\mathbb{G}_a^{\geq -s}/L^+\mathbb{G}_a \arrow[d, hook] \\ S_\nu \arrow[r, "h_\mu^\nu"'] & L\mathbb{G}_a/L^+\mathbb{G}_a \end{tikzcd}$$

where $s > 0$ is some large enough positive integer.

Proof. Note $\mathrm{MV}_{\lambda,\nu}$ is a subscheme of $\mathrm{Gr}_{\leq\lambda}$, which is the perfection of a projective variety over k, by the results of [BS17], and is therefore quasi-compact over k. So the morphism to the ind-scheme

$$L\mathbb{G}_a/L^+\mathbb{G}_a = \mathrm{colim}_s L\mathbb{G}_a^{\geq -s}/L^+\mathbb{G}_a$$

must factor through one of the $L\mathbb{G}_a^{\geq -s}/L^+\mathbb{G}_a$. □

Lemma 3.9. *The quotient $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ is represented by a perfect group scheme and its k-points are naturally identified with $\varpi^{-s}\mathcal{O}/\mathcal{O}$.*

Proof. We exhibit an isomorphism $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \xrightarrow{\sim} L^s\mathbb{G}_a$. If R is a perfect k-algebra, we can define an isomorphism of presheaves

$$L^{\geq -s}\mathbb{G}_a(R)/L^+\mathbb{G}_a(R) \to L^s\mathbb{G}_a(R)$$
$$\sum_{i=-s}^{-1}[r_i]\varpi^i \mapsto \sum_{i=-s}^{-1}[r_i]\varpi^{i+s}$$

and then take the sheafification. We conclude by noting that $L^s\mathbb{G}_a$ is by definition a perfect group scheme, and has k-points $\mathcal{O}/\varpi^s\mathcal{O}$, which map to $\varpi^{-s}\mathcal{O}/\mathcal{O}$ under the inverse of the isomorphism. □

Proposition 3.10. *The Lang isogeny*

$$L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \xrightarrow{\mathrm{Frob}-\mathrm{id}} L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$$

along with ψ give rise to a nontrivial rank 1 multiplicative ℓ-adic local system $\mathcal{L}_\psi$ on $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$.

Proof. We identified $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \xrightarrow{\sim} L^s\mathbb{G}_a$. The perfect scheme $L^s\mathbb{G}_a$ is the perfection of the finite type commutative group scheme $L^s_p\mathbb{G}_a$, so we can consider the Artin–Schreier sequence

$$0 \to \mathcal{O}/\varpi^s\mathcal{O} \to L^s_p\mathbb{G}_a \xrightarrow{\mathrm{Frob}-\mathrm{id}} L^s_p\mathbb{G}_a \to 0.$$

As sheafification is left exact, the restriction functor from stacks to perfect stacks commutes with both colimits and limits in Equation 7, perfection preserves short exact sequences of group schemes

$$0 \to \mathcal{O}/\varpi^s\mathcal{O} \to L^s\mathbb{G}_a \xrightarrow{\mathrm{Frob}-\mathrm{id}} L^s\mathbb{G}_a \to 0$$

noting that $\mathcal{O}/\varpi^s\mathcal{O}$ is already perfect. This extension of commutative group schemes therefore gives rise to a $\mathcal{O}/\varpi^s\mathcal{O}$-torsor on $L^s\mathbb{G}_a$, which corresponds to a surjective map in $\mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(L^s\mathbb{G}_a, 0), \mathcal{O}/\varpi^s\mathcal{O})$. Composing with

$$\mathcal{O}/\varpi^s\mathcal{O} \to \varpi^{-s}\mathcal{O}/\mathcal{O} \xrightarrow{\psi} \overline{\mathbb{Q}}_\ell^\times$$

which gives rise to a nontrivial rank 1 ℓ-adic local system on $L^s\mathbb{G}_a$. Passing through the isomorphism gives a nontrivial rank 1 ℓ-adic local system on $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$, as desired. By construction $\mathcal{L}_\psi$ is multiplicative, i.e. if

$$a : L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \times L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$$

is the addition map then $a^*\mathcal{L}_\psi = \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi$. □

Remark 3.11. One could also simply define the character sheaf directly on $L_p^s\mathbb{G}_a$, and the use the equivalence of étale sites for $L_p^s\mathbb{G}_a$ and $L^s\mathbb{G}_a$ via perfection.

Moreover, if $t > s$ there is an inclusion

$$\iota : L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \hookrightarrow L^{\geq -t}\mathbb{G}_a/L^+\mathbb{G}_a$$

and it is easy to check that $\iota^*\mathcal{L}_\psi = \mathcal{L}_\psi$.

4. The non-dominant case

In this section, we verify Theorem 1.1 when $\mu \in X_*(T)$ is not dominant.

4.1. **Equivariance.** By Lemma 3.2 the L^+G-action on $\mathrm{Gr}_{\leq\lambda}$ factors through L^hG for some large enough $h > 0$. Therefore, the L^+N-action on $\mathrm{MV}_{\lambda,\mu}$ factors through L^hN as well. A direct computation shows that the map $h_\mu|_{L^+N} : L^+N \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ also factors as

$$h_\mu|_{L^+N} : L^+N \to L^hN \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$$

for large enough h.

Proposition 4.2. *Choose s such that $h_\mu|_{L^+N}$ and $h_\mu^{\lambda,\nu}$ both factor through $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \to L\mathbb{G}_a/L^+\mathbb{G}_a$. Then the following diagram commutes:*

$$\begin{tikzcd}
L^+N \times \mathrm{MV}_{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[d] & \mathrm{MV}_{\lambda,\nu} \arrow[d, equal] \\
L^hN \times \mathrm{MV}_{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[d, "h_\mu \times h_\mu^{\lambda,\nu}"'] & \mathrm{MV}_{\lambda,\nu} \arrow[d, "h_\mu^{\lambda,\nu}"] \\
L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \times L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "a"'] \arrow[d] & L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[d] \\
L\mathbb{G}_a/L^+\mathbb{G}_a \times L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "a"] & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

Proof. This is a diagram chase. ☐

Corollary 4.3. *If μ is non-dominant, $\mu+\nu$ is dominant, and λ is dominant, then*

$$R\Gamma_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) = 0.$$

Proof. By Proposition 4.2 and the fact that $\mathcal{A}_\lambda$ is L^+G-equivariant,

$$\begin{aligned}
\mathrm{act}^*(\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) &= \mathrm{act}^*\mathcal{A}_\lambda \otimes \mathrm{act}^*(h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu \times h_\mu^{\lambda,\nu})^* a^*\mathcal{L}_\psi \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu \times h_\mu^{\lambda,\nu})^*(\mathcal{L}_\psi \boxtimes \mathcal{L}_\psi) \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu^*\mathcal{L}_\psi \boxtimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) \\
&= h_\mu^*\mathcal{L}_\psi \boxtimes (\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi),
\end{aligned}$$

so $\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi$ is $(L^hN, h_\mu^*\mathcal{L}_\psi)$-equivariant.

If μ is not dominant, pick a simple root α such that $(\alpha, \mu) < 0$ and let $u_\alpha : \mathbb{G}_a \to N$ denote the inclusion of the root subgroup. Then the composition

$$L^+\mathbb{G}_a \hookrightarrow L\mathbb{G}_a \xrightarrow{u_\alpha} LN \xrightarrow{\operatorname{ad} \varpi^\mu} LN \to LN/[LN, LN] \xrightarrow{+} L\mathbb{G}_a$$

is just the multiplication by $\varpi^{(\alpha,\mu)}$ map. Therefore, $h_\mu|_{L^+N}$ is non-trivial. This implies that $h_\mu^*\mathcal{L}_\psi$ is also non-trivial. To see why, note that the construction of the Lang torsor implies that $h_\mu^*\mathcal{L}_\psi \cong \mathcal{L}_{\psi\circ h_\mu(k)}$. But one can check by hand that $\psi \circ h_\mu(k)$ is a non-trivial character, so we conclude by Proposition 4.4. $\square$

Proposition 4.4. *Suppose $Z \in \mathrm{Sch}_k^{pf}$ over k with an action*

$$\mathrm{act} : G \times Z \to Z$$

of a pfp perfect group scheme G defined over k. If $\mathcal{L}$ is a non-trivial rank 1 local system on G and $\mathcal{F} \in \mathrm{Shv}(Z)$ is $(G, \mathcal{L})$-equivariant, i.e.

$$a^*\mathcal{F} \simeq \mathcal{L} \boxtimes \mathcal{F}$$

then

$$R\Gamma_c(Z, \mathcal{F}) = 0.$$

Proof. This is analogous to [Ngô00, Lemma 3.3]. Consider the diagram

$$\begin{array}{ccc} G \times Z & \xrightarrow{\mathrm{id} \times a} & G \times Z \\ \downarrow & & \downarrow \\ G & \longrightarrow & G \end{array}$$

we obtain

$$(\mathrm{id} \times a)^* (k \boxtimes \mathcal{F}) \simeq \mathcal{L} \boxtimes \mathcal{F}$$

or by adjunction

$$k \boxtimes \mathcal{F} \simeq (\mathrm{id} \times a)_* \mathcal{L} \boxtimes \mathcal{F}$$

suppose $\pi_!\mathcal{F} \in \mathrm{Shv}(k) \simeq \mathrm{Mod}_k$ were non zero. This means there exists $i : z \hookrightarrow Z$, such that[1]

$$\pi_! i^* \mathcal{F} \not\simeq 0$$

In other words, we'd have

$$(\mathrm{id} \times i)^*(k \boxtimes \mathcal{F}) \simeq (\mathrm{id} \times i)^*(\mathrm{id} \times a)_*(\mathcal{L} \boxtimes \mathcal{F})$$

yields

$$k \otimes \pi_! i^* \mathcal{F} \simeq \mathcal{L} \otimes \pi_! i^* \mathcal{F}$$

in $\mathrm{Shv}(k)$, and as both $k, \mathcal{L}$ are irreducible sheaves we have $k \simeq \mathcal{L}$.[2] $\square$

5. The dominant case

Now we treat the case when λ is dominant.

[1]Indeed, for topological spaces, if $\mathcal{F}$ is bdd below complex of sheaves $\pi_! i^* \mathcal{F} \simeq \varinjlim_{Z \in U} H^k(U, \mathcal{F}_U)$. In our setting $\mathcal{F}$ is quasicoherent sheaf, this implies that $\pi_! i^* \mathcal{F} \simeq \varinjlim_D M_D \simeq M_z$ where we localize $M := \Gamma(Z, \mathcal{F})$.

[2]For instance, use semisimplicity representation category.

5.1. **Weyl orbit.** First, we treat the case where $\nu = w\lambda$. We want to show

$$H_c^i(\mathrm{MV}_{\lambda,w\lambda}, \mathcal{A}_\lambda|_{\mathrm{MV}_{\lambda,w\lambda}} \otimes (h_\sigma^{\lambda,w\lambda})^*(\mathcal{L}_\psi)) = \begin{cases} 0 & \text{if } i \neq (\rho, \lambda + w\lambda) \\ \overline{\mathbb{Q}}_\ell & \text{if } i = (\rho, \lambda + w\lambda) \end{cases}$$

For this, we first need to describe $\mathrm{MV}_{\lambda,w\lambda}$.

Lemma 5.2. *For $w \in W$ and $\lambda \in X_*(T)_+$,*

(1) $\mathrm{MV}_{\lambda,w\lambda} = S_{w\lambda} \cap \mathrm{Gr}_\lambda = L^+N\varpi^{w\lambda}$, and

(2) if σ is a dominant cocharacter then the map $h_\sigma^{\lambda,w\lambda}$ takes $\mathrm{MV}_{\lambda,w\lambda}$ to the identity.

Proof. [Zhu17, Corollary 2.8] implies that

$$S_{w\lambda} \cap \mathrm{Gr}_\mu = \varnothing \text{ if and only if } w\lambda \in \Omega(\mu).$$

If $\mu < \lambda$ we cannot have $w\lambda \in \Omega(\mu)$. So since $\mathrm{MV}_{\lambda,w\lambda} = \bigsqcup_{\mu \leq \lambda} S_{w\lambda} \cap \mathrm{Gr}_\mu$, we see that $\mathrm{MV}_{\lambda,w\lambda} = S_{w\lambda} \cap \mathrm{Gr}_\lambda$.

For the second equality, recall that $h_\sigma^{\lambda,w\lambda}$ is defined on an element $n\varpi^{w\lambda}L^+G \in S_{w\lambda} \cap \mathrm{Gr}_\lambda$ as $h(\mathrm{ad}(\sigma) \cdot n)$. Since σ is dominant, $\mathrm{ad}(\sigma)L^+N \subset L^+N$. Since h acts trivially on L^+N, we are therefore done if we can show that $L^+N\varpi^{w\lambda} = S_{w\lambda} \cap \mathrm{Gr}_\lambda$. The inclusion $L^+N\varpi^{w\lambda} \subset S_{w\lambda} \cap \mathrm{Gr}_\lambda$ follows since $L^+N \subset LN$ and $W \subset L^+G$, so we conclude by showing that this inclusion is a closed embedding of irreducible perfect schemes of the same dimension. By [Zhu17, Corollary 2.8] $S_{w\lambda} \cap \mathrm{Gr}_\lambda$ is irreducible of dimension $(\rho, w\lambda + \lambda)$. But

$$L^+N\varpi^{w\lambda} \simeq \frac{L^+N}{L^+N \cap \mathrm{ad}(\varpi^{w\lambda})L^+N},$$

which is irreducible of dimension $(\rho, w\lambda + \lambda)$ as argued in [Zhu17, Section 1.2]. It remains to show that $L^+N\varpi^{w\lambda} \subset S_{w\lambda} \cap \mathrm{Gr}_\lambda$ is a closed embedding. □

As a direct corollary, we obtain the following.

Corollary 5.3. *Let $\sigma \in X_*(T)_+$.*

$$R\Gamma_c(\mathrm{MV}_{\nu,\nu}, \mathcal{A}_\nu \otimes (h_\sigma^{\nu,\nu})^*(\mathcal{L}_\psi)) = R\Gamma_c(\mathrm{MV}_{\nu,\nu}, \mathcal{A}_\nu) \tag{1}$$

Then [Zhu17, Corollary 2.7, Proposition 2.8, and Corollary 2.9] implies that $R\Gamma_c(\mathrm{MV}_{\nu,\nu}, \mathcal{A}_\nu)$ is concentrated in degree $(2\rho, \nu)$, and is one-dimensional in that degree. This completes the proof of Theorem 1.1 when $\lambda = \nu$.

Now we bootstrap from this case to prove the general case of $\lambda \neq \nu$. For this, we mimic the strategy of [NP01]; in particular, we exploit the fact that the geometry of the $\mathrm{MV}_{\lambda,\nu}$ becomes simpler when λ is quasi-minuscule. We have the following geometric version of the PRV conjecture:

Lemma 5.4 ([Zhu17, Lemma 2.16]). *There exists a sequence of quasi-minuscule coweights $\lambda_\bullet = (\lambda_1, \dots, \lambda_m)$ such that $W^\lambda_{\lambda_\bullet} \neq 0$ in the decomposition*

$$\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_m} = \bigoplus_{\substack{\xi \in X_*(T)_+, \\ \xi \leq |\lambda_\bullet|}} \mathcal{A}_\xi \otimes W^\xi_{\lambda_\bullet}.$$

in the spherical category $P_{L^+G}(\mathrm{Gr}_G)$. Here, the dimension of $W^\xi_{\lambda_\bullet}$ is equal to the multiplicity of $\mathcal{A}_\xi$ in the convolution.

Pick a sequence $\lambda_\bullet = (\lambda_1, \dots, \lambda_m)$ as in Lemma 5.4. This decomposition induces an isomorphism

$$\begin{aligned} &R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) \\ &= \bigoplus_{\substack{\xi \in X_*(T)_+, \\ \xi \leq |\lambda_\bullet|}} R\Gamma_c(\mathrm{MV}_{\xi,\nu}, \mathcal{A}_\xi \otimes (h_0^{\xi,\nu})^*(\mathcal{L}_\psi)) \otimes V^\xi_{\lambda_\bullet} \end{aligned}$$

So we're done if we can show that the direct factor map

$$\begin{aligned} &R\Gamma_c(\mathrm{MV}_{\nu,\nu}, \mathcal{A}_\nu \otimes (h_0^{\nu,\nu})^*(\mathcal{L}_\psi)) \otimes V^\nu_{\lambda_\bullet} \\ &\to R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) \end{aligned}$$

is a quasi-isomorphism.

But by Equation 1 we are done if we can show that

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) = V^\nu_{\lambda_\bullet}[\langle 2\rho, \nu \rangle] (-\langle \rho, \nu \rangle).$$

Now the left hand side decomposes as follows.

Proposition 5.5. *Let $\sigma_i = \nu_1 + \cdots + \nu_i$ for $i = 1, \dots, m$, then*

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) = \bigoplus_{|\nu_\bullet| = \nu} \bigotimes_{i=1}^m R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes h^{\lambda_i,\nu_i}_{\sigma_{i-1}} \mathcal{L}_\psi)$$

Proof. In contrast to the proof of [NP01, p31], which just passes to the convolution Grassmannian, we need to further resolve by using the $\prod L^{r_i} N$-torsors constructed in Lemma 6.4. This yields a diagram

$$\begin{tikzcd} \bigcup_{|\nu_\bullet|=\nu} \prod_{i=1}^m \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i} \arrow[d, "q_\bullet"] & \\ m^{-1}(\mathrm{MV}_{|\lambda_\bullet|,\nu}) = \bigcup_{|\nu_\bullet|=\nu} \widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{\leq\lambda_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\leq\lambda_n} \arrow[d, "m"] \\ \mathrm{MV}_{|\lambda_\bullet|,\nu} \arrow[r, hook] & \mathrm{Gr}_{\leq|\lambda_\bullet|} \end{tikzcd}$$

where the first map m, as **??**,

$$\widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet} := \mathrm{MV}_{\lambda_1,\nu_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{MV}_{\lambda_n,\nu_n}$$

where by Lemma 6.4, each component splits as a *direct product* in the second resolution.

$$\begin{aligned} R\Gamma_c(\mathrm{MV}_{\lambda_1,\nu_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{MV}_{\lambda_n,\nu_n}, \mathcal{A}_{\mu_\bullet} \otimes h_\bullet^* \mathcal{L}_\psi) &\simeq R\Gamma_c\left(\prod_{i=1}^m \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i}, q_\bullet^* \mathcal{A}_{\lambda_\bullet}\right)\left[2\dim N \cdot \sum_{i=1}^m r_i\right] \\ &\simeq \bigotimes_{i=1}^m \left(R\Gamma_c(\mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i}, p_i^* \mathcal{A}_{\lambda_i} \otimes h_{\sigma_i}^* \mathcal{L}_\psi)[2\dim N \cdot r_i]\right) \\ &\simeq \bigotimes_{i=1}^m R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes h_{\sigma_i}^* \mathcal{L}_\psi) \end{aligned}$$

□

It suffices to compute each individual component for a fixed $\nu \in X_\bullet$.

$$\left\{R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes h_{\sigma_i}^* \mathcal{L}_\psi)\right\}_{\{\nu_i\}_{i=1}^n, \sum \nu_i = \nu} \tag{2}$$

We may suppose

- The σ_i are dominant, otherwise all of these terms vanish, by Corollary 4.3.
- Recall from [NP01, p. 1.1], that as λ_i are minuscule, we have $\Omega\lambda_i = W\lambda_i \cup \{0\}$. We maythus suppose $\nu_i \in \Omega(\lambda_i) = W\lambda_i \cup \{0\}$, for otherwise $\mathrm{MV}_{\lambda_i,\nu_i} = \varnothing$ by **??**.

We now split into cases based on whether $\nu_i = w\lambda_i$ or $\nu_i = 0$.

5.5.1. *Weyl orbit.* If $\nu_i = w\lambda_i$ for some $w \in W$. Using Lemma 5.2

$$R\Gamma_c(\mathrm{MV}_{\lambda_i,w\lambda_i}, \mathcal{A}_{\lambda_i} \otimes h_{\sigma_i}^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_\ell[\langle 2\rho, w\lambda_i\rangle][(\rho, w\lambda_i)]$$

as from Lemma 5.2

$$\mathrm{MV}_{\lambda,w\lambda} = S_{w\lambda} \cap \mathrm{Gr}_\lambda \hookrightarrow \mathrm{Gr}_\lambda$$

with h_σ trivial, and $\mathcal{A}_\lambda|_{\mathrm{MV}_{\lambda,w\lambda}} \simeq \bar{\mathbb{Q}}_\ell[\langle 2\rho, w\lambda\rangle][\langle \rho, w\lambda\rangle]$. This boils down to the fact that $\mathrm{MV}_{\lambda_i,w\lambda_i} \subset S_{\lambda_i} \cap \mathrm{Gr}_{\lambda_i}$ is an affine bundle over an affine space. If $\nu_i = 0$, we will use the computation in Section 5.6. Combining these two, we deduce that

$$H_c^i(\mathrm{MV}_{\mu_\bullet,\nu_\bullet}, \mathcal{A}_{\mu_\bullet} \otimes h^* \mathcal{L}_\psi) = \begin{cases} 0 & i \neq 2\langle \rho, \nu\rangle \\ |\{\text{dominant } \mu_\bullet \text{ paths from 0 to } \nu\}| & i = 2\langle \rho, \nu\rangle \end{cases}$$

5.6. **Zero orbit.** we will prove the following:

Theorem 5.7.
$$R\Gamma_c(\mathrm{MV}_{\lambda,0},\mathcal{A}_\lambda\otimes h_\sigma^*\mathcal{L}_\psi)=\bar{\mathbb{Q}}_\ell^{|\Delta^\sigma_{\lambda^\vee}|}$$

To prove this we will first consider a resolution of $\mathrm{MV}_{\lambda,0}$ induced by one on $\mathrm{Gr}_{\leq\lambda}$, which we denote by $\pi:\widetilde{\mathrm{Gr}}_\lambda\to\mathrm{Gr}_{\leq\lambda}$, further $\widetilde{\mathrm{Gr}}_\lambda$ is itself a $\mathbb{P}^1$-bundle over G/P_λ. [3] P_λ denotes the parabolic subgroup (defined over $\mathcal{O}$) generated by T and the root subgroups U_α for $\alpha\in\Phi$ satisfying $(\alpha,\mu)\leq 0$. In [Zhu17, Section 2.2.2], Zhu defines a smooth resolution

$$\begin{tikzcd}
\pi^{-1}(\mathrm{Gr}_\lambda) \arrow[r, hook] \arrow[d, "\sim"] & \widetilde{\mathrm{Gr}}_{\leq\lambda} \arrow[d, "\pi"] & G/P_\lambda \arrow[l, hook'] \arrow[d] \\
\mathrm{Gr}_\lambda \arrow[r, hook, "j"] & \mathrm{Gr}_{\leq\lambda} & \mathrm{Gr}_0 \arrow[l, hook', "j_0"]
\end{tikzcd} \tag{3}$$

which restricts to an isomorphism over Gr_λ and to a contraction $(\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}$ over the point Gr_0. After restriction to S_0 we obtain

$$\begin{tikzcd}
\pi^{-1}(S_0\cap\mathrm{Gr}_\lambda) \arrow[r, hook] \arrow[d, "\sim"] & \pi^{-1}(S_0\cap\mathrm{Gr}_{\leq\lambda}) \arrow[d, "\pi"] & G/P_\lambda \arrow[l, hook'] \arrow[d] \\
S_0\cap\mathrm{Gr}_\lambda \arrow[r, hook, "j"] & S_0\cap\mathrm{Gr}_{\leq\lambda} & \mathrm{Gr}_0 \arrow[l, hook', "i"]
\end{tikzcd} \tag{4}$$

To compute the twisted cohomology of $\mathrm{MV}_{\lambda,0}$, against the restricted resolution π, to obtain
$$R\Gamma_c(\pi^{-1}(\mathrm{MV}_{\lambda,0}),\pi^*(\mathcal{A}_\lambda\otimes h_\sigma^*\mathcal{L}_\psi))[d]\simeq R\Gamma_c(\mathrm{MV}_{\lambda,0},\mathcal{A}_\mu\otimes h_\sigma^*\mathcal{L}_\psi)\oplus R\Gamma_c(\mathcal{C})$$
where the complex $\mathcal{C}$, is explained in Lemma 5.12. To compute the left-hand side we consider the open-closed decomposition

$$\pi^{-1}(S_0\cap\mathrm{Gr}_\lambda)\xhookrightarrow{j}\pi^{-1}(\mathrm{MV}_{\lambda,0})\xhookleftarrow{i}\pi^{-1}(\mathrm{Gr}_0)=G/P_\lambda$$

This induces long exact sequence
$$\cdots\to H^i_c(\pi^{-1}(S_0\cap\mathrm{Gr}_\lambda))\to H^i_c(\pi^{-1}(S_0\cap\mathrm{Gr}_{\leq\lambda}))\to H^i_c(G/P_\lambda)\to H^{i+1}_c(\pi^{-1}(S_0\cap\mathrm{Gr}_\lambda))\to\cdots \tag{5}$$
By the dimension count from Proposition 5.8 and Lemma 5.12, the result follows.

Proposition 5.8. *(1)* $i>0$ *:* $H_c^{i+d}(\pi^{-1}(\mathrm{MV}_{\lambda,0}))\simeq H_c^{i+d}(G/P_\lambda)$

(2) $i=0$ *:* $\dim H^d_c(\pi^{-1}(\mathrm{MV}_{\lambda,0}))=2|\Delta_{\lambda^\vee}|$

(3) $i<0$*:* $\dim H_c^{i+d-2}(\pi^{-1}(\mathrm{MV}_{\lambda,0})=\dim H_c^{i+d-2}(G/P)$

[3]The strategy here is the same as [NP01, p. 8]

Proof. To compute $H^i_c(\pi^{-1}(S_0 \cap \mathrm{Gr}_\lambda))$, we identify $S_0 \cap \mathrm{Gr}_\lambda$ the total space of a $\mathbb{G}_m$ bundle $\mathcal{L}^\times$. $\widetilde{\mathrm{Gr}}_\lambda$ admits the structure of a $(\mathbb{P}^1)^{\mathrm{pf}}$-bundle over G/P_λ, which restricts (the total space) to a line bundle over G/P_λ, we denote the restriction of this line bundle to $(G/P_\lambda)_- := \bigcup_{w\lambda<0} UwP_\lambda/P_\lambda$ as ϕ_-.

$$\begin{tikzcd}
\mathcal{L}^\times := S_0 \cap \mathrm{Gr}_\lambda \arrow[r, hook] \arrow[dr] & \mathcal{L} := \pi^{-1}\mathrm{Gr}_\lambda \Big|_{(G/P)_-} \arrow[d, "\phi_-"] \\
& (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_-
\end{tikzcd}$$

$S_0 \cap \mathrm{Gr}_\lambda$ is identified as the complement of a section of line bundle ϕ_-. [Milton: how do we know this is true?] This fits to the original diagram as

$$\begin{tikzcd}
\mathcal{L}^\times \arrow[r] \arrow[d, "\simeq"] & \mathcal{L} \arrow[d, hook] & (G/P_\lambda)_- \arrow[l, hook'] \arrow[d, hook] \\
\pi^{-1}(S_0 \cap \mathrm{Gr}_\lambda) \arrow[r, hook, "j"] & \pi^{-1}(\mathrm{MV}_{\lambda,0}) & \pi^{-1}(\mathrm{Gr}_0) = G/P_\lambda \arrow[l, hook', "i"']
\end{tikzcd} \tag{6}$$

(1) When $i > 0$: we observe that
$$\begin{aligned}\dim \mathcal{L}^\times &= \dim (G/P_\lambda)_- + 1 \\ &= \max_{w\lambda<0} \langle \rho, w\mu + \mu\rangle + 1 \\ &= \frac{d}{2} + \max_{w\lambda<0}\langle \rho, w\mu\rangle + 1 \le \frac{d}{2}\end{aligned}$$
where we observe that
$$\max_{w\lambda<0} \langle \rho, w\lambda\rangle \le -1$$
and the maximum is attained iff $w\lambda$ is a simple root. [4] Thus, we deduce that $H^{i+d}(\mathcal{L}^\times)$ and $H^{i+d+1}(\mathcal{L}^\times)$ vanishes in Equation 5.

(2) When $i = 0$: we count the number of irreducible components of $\mathcal{L}^\times$, which gives the dimension of $H^d(\mathcal{L}^\times)$. This is precisely equal to $|\Delta_\lambda|$, using the observation from the case $i > 0$. Laslty, as $\lambda^\vee$ is a root, we know $d = 2\langle \rho, w\lambda^\vee\rangle \in 2\mathbb{Z}$, Equation 5 reduces to [5]
$$0 \to H^d_c(\mathcal{L}^\times) \to H^d_c(\pi^{-1}(\mathrm{MV}_{\lambda,0}) \to H^d_c(G/P_\lambda) \to 0$$

(3) When $i < 0$: We the Gysin sequence to obtain that in
$$\cdots \to H^{i-2}((G/P_\lambda)_-) \to H^i((G/P_\lambda)_-) \to H^i(\mathcal{L}^\times) \to H^{i-1}((G/P_\lambda)_-) \to H^{i+1}((G/P_\lambda)_-) \to \cdots$$
where we used the identification $H^i(\mathcal{L}) \simeq H^{i-2}((G/P)_-)$. Now Now we split into two cases:

[4] Indeed, $\langle \rho, \mu\rangle > 0$ if $\mu \in \Phi_+$, and minum is attained iff $\mu \in \Delta$. We can similarly argue that $\langle \rho, \mu\rangle \in \mathbb{Z}$, if $\mu \in \mathbb{Z}\Phi^\vee$, the coroot lattice.

[5] the canonical projection $G/B \to G/P$ induces an injection $H^*(G/P) \hookrightarrow H^*(G/B)$, which shows the cohomology of G/P_λ is concentrated in even degrees.

(a) When i is odd: we have $H^{d+i}(G/P) = 0$ at Equation 5

$$\operatorname{coker}(H^{d+i-1}(G/P) \to H^{d+i}(\mathcal{L}^\times)) \simeq H^{d+i}\left(\pi^{-1}(\mathrm{MV}_{\lambda,0})\right)$$

but then by the diagram 6 the map this factors through the restriction map

$$H^{d+i-1}(G/P) \to H^{d+i-1}((G/P)_-)$$

since these are affinely stratified space, as [Hai]. Thus,

$$H^{d+i}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) = 0 = H^{d+i-2}(G/P)$$

(b) When i is even: we have the short exact sequence

$$0 \to H^{i+d-2}(\mathcal{L}^\times) \to H^{i+d-2}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) \to H^{i+d-2}(G/P) \to H^{i+d-1}(\mathcal{L}^\times) \to H^{i+d-1}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) \to 0$$

and

$$0 \to H^{i+d-4}((G/P)_-) \to H^{i+d-2}((G/P)_-) \to H^{i+d-2}(\mathcal{L}^\times) \to 0$$

and

$$0 \to H^{i+d-1}(\mathcal{L}^\times) \to H^{i+d}((G/P)_-) \to H^{i+d+2}((G/P)_-) \to 0$$

The Euler characteristic of this sequence is 0. Now we apply the result of [NP01] as follows: we have diagram 6 and hence the same long exact sequence, as above, the result of [NP01, p. 8], thus gives us

$$\dim H^{i+d-2}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) = \dim H^{i+d-2}(G/P)$$

which is what we wanted

□

Lemma 5.9. *Basis of Schubert cohomology.*

Lemma 5.10.

$$H^*(\mathcal{L}^\times) \simeq \begin{cases} \operatorname{coker}(H^{*-2}((G/P_\lambda)_-) \to H^*((G/P_\lambda)_-)) & \text{if } * \text{ is even} \\ \ker\left(H^{*-1}((G/P_\lambda)_-) \to H^{*+1}((G/P_\lambda)_-)\right) & \text{if } * \text{ is odd} \end{cases}$$

by substituting

Note that the connecting maps here are explicitly given by the Pieri or Chevellay formula.

Proposition 5.11.

$$S_\nu \cap \mathrm{Gr}_{\leq\lambda} = S_\nu \cap \mathrm{Gr}_\lambda = \begin{cases} & \text{if } \nu = w\lambda \in \Phi^\vee_+ \\ UwP_\lambda/P_\lambda & \text{if } \nu = w\lambda \in \Phi^\vee_- \\ \emptyset & \text{otherwise} \end{cases}$$

Thus we have

$$\mathrm{MV}_{\lambda,0} = \pi\left(\phi^{-1}\left(\bigcup_{w\lambda\in\Phi^\vee_-} UwP_\lambda/P_\lambda\right) \setminus \bigcup_{w\lambda\in\Phi^\vee_-} (S_{w\lambda} \cap \mathrm{Gr}_{\leq\lambda})\right)$$

Proof. Consider the stratification of $\mathrm{Gr} = \bigsqcup S_\nu$, intersected with $\mathrm{Gr}_{\leq\lambda}$. □

Lemma 5.12. *With the notation in Equation 3,*

$$\pi_*(h_\sigma \circ \pi)^*\mathcal{L}_\psi \simeq (\mathcal{A}_\mu \otimes h_\sigma^*\mathcal{L}_\psi) \oplus \mathcal{C}$$

where $\mathcal{C}$ satisfies the property

$$H^i(\mathcal{C}) = \begin{cases} H^{i+d}(G/P_\lambda) & i \geq 0 \\ H^{i+d-2}(G/P_\lambda) & i < 0 \end{cases}$$

Proof. As in [Zhu17, Section 2.2.2] we use the decomposition theorem to obtain

$$\pi_*\overline{\mathbb{Q}}_\ell[d] = \mathcal{A}_\mu \oplus \mathcal{C}.$$

Then by the projection formula we deduce that

$$\begin{aligned}\pi_*(h_\sigma \circ \pi)^*\mathcal{L}_\psi &\simeq (\mathcal{A}_\mu \otimes h_\sigma^*\mathcal{L}_\psi) \oplus (\mathcal{C} \otimes j_0^* h_\sigma^*\mathcal{L}_\psi) \\ &\simeq (\mathcal{A}_\mu \otimes h_\sigma^*\mathcal{L}_\psi) \oplus \mathcal{C}.\end{aligned}$$ □

5.13. **Recovering classical Casselman Shalika.** The proof follows that explained [Fre+98, p. 5.4]. Let $\widehat{G}$ denote the Tannakian dual group.

Theorem 5.14. *Let $\gamma \in \widehat{G}$. There exist as unique*

$$W_\gamma \in Fct(G(K), \bar{\mathbb{Q}}_l)$$

satisfying the following property.

- $W_\gamma(gh) = W_\gamma(h)$.
- $W_\gamma(ug) = \Psi^{-1}(u) W_\gamma(g)$.

Furhter for $\lambda \in X_\bullet$,

$$W_\gamma(\varpi^\lambda) = q^{-(\rho,\mu)}\, Tr(\gamma, V(\lambda))$$

These are the *Whittaker functions* which induces a map

$$\begin{aligned} s_\gamma : \mathrm{Fct}(G/K)^{N,\psi} &\to \bar{\mathbb{Q}}_{l,\psi} \\ \phi &\mapsto \int_{N\backslash G} W_\gamma \cdot \phi \end{aligned}$$

in $\mathrm{Mod}_{\mathrm{cHk}(G,K)}$.

Proposition 5.15. *Let $\mathcal{F} \in \mathrm{Shv}_{cstr}(X, \tau_{\acute{e}t})$*

6. Cohomological computation

Recall that the *right* multiplication action of L^+G on LG makes

$$\begin{tikzcd} \mathrm{LG} \arrow[d] \\ \mathrm{Gr} \end{tikzcd}$$

a right L^+G-torsor, and this canonically descends to an L^+N-torsor

$$\begin{aligned} \varpi^\nu LN &\to S_\nu \\ \varpi^\nu n &\mapsto \varpi^\nu n \mod L^+G. \end{aligned}$$

This map is L^+N-equivariant from the left.

We have the diagram

$$\begin{tikzcd} \varpi^\nu LN \arrow[r, hook] \arrow[d] & LG \arrow[d] \\ S_\nu \arrow[r, hook] & \mathrm{Gr} \end{tikzcd}$$

Definition 6.1. Let $r \in \mathbb{N}_{\geq 0} \cup \{\infty\}$. We can form L^rN-torsors over S^ν and $\mathrm{MV}_{\lambda,\nu}$ using the following pullback diagram:

$$\begin{tikzcd} \mathrm{MV}_{\lambda,\nu}^{(r)} \arrow[r] \arrow[d, "p_r"] & S_\nu^{(r)} := \varpi^\nu LN \times^{L^+N} L^rN \arrow[d] \\ \mathrm{MV}_{\lambda,\nu} \arrow[r, hook] & S_\nu \end{tikzcd}$$

We adopt the convention $L^\infty N := L^+N$. Note that $S_\nu^{(0)} = S_\nu$ and $S_\nu^{(\infty)} = \varpi^\nu LN$.

Lemma 6.2. *For $r \geq 0$, the left action of L^+N on $\mathrm{MV}_{\lambda,\nu}^{(r)}$ factors through $L^{r'}N$ for some $r' > 0$.*

Proof. If we write

$$\mathrm{MV}_{\lambda,\nu}^{(r)} = \mathrm{MV}_{\lambda,\nu} \times_{S_\nu} (\varpi^\nu LN \times^{L^+N} L^rN),$$

then the left L^+N-action is just the diagonal action, which descends to the fiber product. Thus, it suffices to check individually on each component that L^+N factors through some L^fN, $f \in \mathbb{N}$.

For the first factor, the left action of L^+G on $\mathrm{Gr}_{\leq\lambda}$ factors through $L^{r'}G$ for some $r > 0$ (which depends on λ), so the left L^+N-action on $\mathrm{MV}_{\lambda,\nu}$ factors through $L^{r'}N$ as well.

For the second factor, note that an arbitrary element of $\varpi^\nu LN \times^{L^+N} L^rN$ is of the form $(\varpi^\nu n, LN^{(r)})$ for some $n \in LN$. We want to show that there exists some large enough $r'' > r'$ such that if $h \in LN^{(r'')}$ then

$$(h\varpi^\nu n, LN^{(r)}) \sim (\varpi^\nu n, LN^{(r)})$$

Since $\varpi^\nu n L^+G \in \mathrm{MV}_{\lambda,\nu}$, if $h \in LN$, then h fixes $\varpi^\nu n L^+G \in \mathrm{MV}_{\lambda,\nu}$, so

$$h\varpi^\nu n = \varpi^\nu n g$$

for some $g \in L^+G$. In fact $g \in LN$, since $g = \mathrm{ad}((\varpi^\nu n)^{-1})(h)$, so $g \in L^+N = LN \cap L^+G$. Then

$$(h\varpi^\nu n, LN^{(r)}) = (\varpi^\nu n g, LN^{(r)}) \sim (\varpi^\nu n, gLN^{(r)}),$$

so we are done if $g \in LN^{(r)}$.

Since $\varpi^\nu n L^+G \in \mathrm{Gr}_{\leq\lambda}$, there exists some $x, g' \in L^+G$ and some dominant $\lambda' \leq \lambda$ such that $\varpi^\nu n = x\varpi^{\lambda'}g'$. Thus

$$n = \varpi^{-\nu}x\varpi^{\lambda'}g'.$$

We conclude by proving two facts:

(1) For any cocharacter ν and any $r' \in \mathbb{N}$, there exists $s \in \mathbb{N}$ so that

$$\mathrm{ad}(\varpi^\nu)(LN^{(s)}) \subseteq LN^{(r')}.$$

(2) For any $x \in L^+G$, $r' \in \mathbb{N}$, there exists $s \in \mathbb{N}$ so that

$$\mathrm{ad}(x)(LN^{(s)}) \subseteq LG^{(r')}.$$

Indeed by 1 and 2 we can show that there exists s such that $g \in LG^{(r)}$. As $LG^{(r)} \cap L^+N$, from the diagram

$$\begin{tikzcd}
LN^{(r)} \arrow[r, hook] \arrow[d] & LG^{(r)} \arrow[d] \\
L^+N \arrow[r, hook] \arrow[d] & L^+G \arrow[d] \\
L^rN \arrow[r, hook] & L^rG
\end{tikzcd}$$

we have $g \in LN^{(r)}$. □

Lemma 6.3. *For any cocharacter ν and any $r' \in \mathbb{N}$, there exists $s \in \mathbb{N}$ so that*

$$\mathrm{ad}(\varpi^\nu)(LN^{(s)}) \subseteq LN^{(r')}.$$

Proof. The case for GL_n is clear. The general case follows from embedding into GL_n and the diagram :

$$\begin{tikzcd}
L^+N^{(s)} \arrow[r, hook] \arrow[d] & LU^{(s)} \arrow[d] \\
L^+N \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^+U \arrow[d] \\
L^+G \arrow[r, hook] \arrow[d, two heads] & L^+\mathrm{GL}_n \arrow[d, two heads] \\
L^{r'}G \arrow[r, hook] & L^r\mathrm{GL}_n
\end{tikzcd}$$

and the fact that being unipotent for an element is an intrinsic property. □

Now pick $\nu_1, \dots, \nu_m$ such that $\nu_1 + \cdots + \nu_m = \nu$.

By the lemma we can choose integers $r_1, \dots, r_m \geq 0$ such that $r_m = 0$ and such that the action of L^+N on $\prod_{k=i}^m \mathrm{MV}_{\lambda_k,\nu_k}^{(r_k)}$ factors through $L^{r_{i-1}}N$ for $i = 2, \dots, m$.

Lemma 6.4. *There are two $\prod_i L^{r_i}N$ torsors $p_\bullet = \prod p_i$ and $q_\bullet$*

$$\begin{tikzcd} & \prod_{i=1}^m (\mathrm{MV}_{\lambda_i,\nu_i})^{(r_i)} \arrow[dl, "p_\bullet"'] \arrow[dr, "q_\bullet"] & \\ \prod_{i=1}^m \mathrm{MV}_{\lambda_i,\nu_i} & & \widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet} \end{tikzcd}$$

such that

$$q_\bullet^* \mathcal{A}_{\lambda_\bullet} \cong p_1^* \mathcal{A}_{\lambda_1} \boxtimes \cdots \boxtimes p_m^* \mathcal{A}_{\lambda_m}.$$

Proof. The torsor $p_\bullet$ is just the product of each individual $L^{r_i}N$-torsor $\mathrm{MV}_{\lambda_i,\nu_i}^{(r_i)} \to \mathrm{MV}_{\lambda_i,\nu_i}$. If $m = 1$ there is nothing to do, so suppose $m > 1$. Since the L^+N-action on $\mathrm{MV}_{\lambda_m,\nu_m}$ factors through $L^{(r_{m-1})}N$, we can form the diagram

$$\begin{tikzcd} & \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}}^{(\infty)} \times \mathrm{MV}_{\lambda_m,\nu_m} \arrow[d, "p_\infty"'] \arrow[ddl, "p"'] \arrow[ddr, "q"] & \\ & \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}}^{(r_{m-1})} \times \mathrm{MV}_{\lambda_m,\nu_m} \arrow[dl, "p_{r_{m-1}} \times \mathrm{id}"] \arrow[dr, "q_{r_{m-1}}"'] & \\ \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}} \times \mathrm{MV}_{\lambda_m,\nu_m} & & \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}} \tilde{\times} \mathrm{MV}_{\lambda_m,\nu_m} \end{tikzcd}$$

in which q is an L^+N-torsor and q_r is an $L^{r_{m-1}}N$-torsor. The morphism p_∞ is just the pushout along the morphism $L^+N \to L^rN$ in the first slot and the identity in the second. The point now is that there is a unique perverse sheaf $\mathcal{A}_{\lambda_{m-1}} \tilde{\boxtimes} \mathcal{A}_{\lambda_m}$ on $\mathrm{MV}_{\lambda_{m-1},\nu_{m-1}} \tilde{\times} \mathrm{MV}_{\lambda_m,\nu_m}$ satisfying

$$p^*(\mathcal{A}_{\lambda_{m-1}} \boxtimes \mathcal{A}_{\lambda_m}) \cong q^*(\mathcal{A}_{\lambda_{m-1}} \tilde{\boxtimes} \mathcal{A}_{\lambda_m}).$$

There is also a unique perverse sheaf $\mathcal{L}$ satisfying

$$q_{r_{m-1}}^* \mathcal{L} \cong p_{r_{m-1}}^* (\mathcal{A}_{\lambda_{m-1}} \boxtimes \mathcal{A}_{\lambda_m})$$

But pulling back by p_∞ gives $q^*\mathcal{L} \cong p^*(\mathcal{A}_{\lambda_{m-1}} \boxtimes \mathcal{A}_{\lambda_m})$ so we must have $\mathcal{L} \cong \mathcal{A}_{\lambda_{m-1}} \tilde{\boxtimes} \mathcal{A}_{\lambda_m}$ by uniqueness.

If $m > 2$, one can repeat the same process as above inductively. For example, first replace $\mathrm{MV}_{\lambda_m,\nu_m}$ with $\mathrm{MV}_{\lambda_{m-1},\nu_{m-1}}^{(r_{m-1})} \times \mathrm{MV}_{\lambda_m,\nu_m}$ and run the same argument. $\square$

Lemma 6.5. *The following diagram commutes:*

$$\begin{tikzcd}
\prod_{i=1}^{m} \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i} \arrow[r, "q_\bullet"] \arrow[d, "p_\bullet"'] & \widetilde{\prod}_{i=1}^{m} \mathrm{MV}_{\lambda_i,\nu_i} \arrow[d, "m"] \\
\prod_{i=1}^{m} \mathrm{MV}_{\lambda_i,\nu_i} \arrow[d, "\prod_{i=1}^m h^{\lambda_i,\nu_i}_{\sigma_{i-1}}"'] & \mathrm{MV}_{|\lambda_\bullet|,\nu} \arrow[d, "h^{|\lambda_\bullet|,\nu}"] \\
\prod_{i=1}^{m} L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "+"] & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

As a direct consequence,

$$(h^{|\lambda_\bullet|,\nu} \circ m \circ q_\bullet)^*\mathcal{L}_\psi \simeq (h^{\lambda_1,\nu_1} \circ p_1)^*\mathcal{L}_\psi \boxtimes (h^{\lambda_2,\nu_2}_{\sigma_1} \circ p_2) \boxtimes \cdots \boxtimes (h^{\lambda_m,\nu_m}_{\sigma_{m-1}} \circ p_m)^*\mathcal{L}_\psi.$$

Proof. The following diagram commutes

$$\begin{tikzcd}
\prod_{i=1} \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i} \arrow[d] & & \\
\widetilde{\prod}_{i=1} \mathrm{MV}_{\lambda_i,v_i} \arrow[r, "\simeq"] & S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet} \arrow[r, hook] \arrow[d] & S_{\nu_\bullet} \arrow[d, "m"] \\
& S_{\sigma_n} \cap \mathrm{Gr}_{\leq |\mu_\bullet|} \arrow[dr] & S_\nu \arrow[d, "h^\nu"] \\
& & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

where the map m is defined as the composition of the identification in **??** and the projection:

$$S_{\nu_\bullet} \xrightarrow{\simeq} S_{\sigma_1} \times \cdots \times S_{\sigma_n} \longrightarrow S_{\sigma_n} = S_\nu$$

One can check that a general element

$$(\varpi^{\nu_1}x_1, \ldots, \varpi^{\nu_n}x_n) \in \prod_{i=1} \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i}$$

which, since ϖ^ν normalizes LN, can also be written as

$$(y_1\varpi^{\nu_1}, \ldots, y_n\varpi^{\nu_n}) \in \prod_{i=1}^{n} \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i}$$

where

$$y_i = \mathrm{ad}(\varpi^{\nu_i})x_i \in LN \quad i = 1, \ldots, n,$$

thus maps to

$$\mathrm{ad}(\varpi^{\sigma_1})x_1 \cdots \mathrm{ad}(\varpi^{\sigma_n})x_n\varpi^{\sigma_n} \in S_\nu$$

under the composition. Thus, the right hand side computes as

$$h^{\nu}(\mathrm{ad}(\varpi^{\sigma_1})x_1 \cdots \mathrm{ad}(\varpi^{\sigma_n})x_n\varpi^{\nu}) = \sum_{i=1}^{m} h_{\sigma_i}(x_i) = \sum_{i=1}^{m} h^{\nu_i}_{\sigma_{i-1}}(y_i\varpi^{\nu_i}L^+G) = \sum_{i=1}^{m}(h^{\lambda_i,\nu_i}_{\sigma_{i-1}}\circ p_i)(y_i\varpi^{\nu_i}).$$

□

7. Appendix:perfect geometry

We have the following categories

$$
\begin{tikzcd}
\mathrm{Aff}_k^{\mathrm{pf}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Aff}_k \arrow[d, hook] \\
\mathrm{Sch}_k^{\mathrm{pf}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Sch}_k \arrow[d, hook] \\
\mathrm{AlgSpc}_k^{\mathrm{pf}} \arrow[r, hook] \arrow[d, hook] & \mathrm{AlgSpc}_k^{\mathrm{pf}} \arrow[d, hook] \\
\mathrm{Stk}_k^{\mathrm{pf}} := \mathrm{Shv}(\mathrm{Aff}_k^{\mathrm{pf}}, \tau) \arrow[r, hook] \arrow[d] & \mathrm{Stk}_k := \mathrm{Shv}(\mathrm{Aff}_k, \tau) \arrow[d] \arrow[l, bend left] \\
\mathrm{PShv}(\mathrm{Aff}_k) \arrow[r] & \mathrm{PShv}(\mathrm{Aff}_k) \arrow[l, bend left, "\mathrm{res}"]
\end{tikzcd}
\tag{7}
$$

where the last functor corresponds to the restriction of sheaves from $i : \mathrm{Aff}_k^{\mathrm{pf}} \hookrightarrow \mathrm{Aff}_k$.

Proposition 7.1. *Let $X \in AlgSpc_k$, there is an equivalence of sites,* [6]

$$(X, \tau_{ét}) \underset{\varepsilon_*}{\overset{\varepsilon^*}{\rightleftarrows}} \left(X^{pf}, \tau_{ét}\right)$$

Our main geometric object of interest is the affine Grassmanin and this an ind-scheme, [CW24]. These are of the form

(8) $\quad X = \varinjlim X_i$, where $X_i \in \mathrm{Stk}_k^{\mathrm{Art,lft}}$ with closed immersions $t_{ij} : X_i \to X_j$ as transitions.

Note that we can construct the category

$$\mathrm{Shv} : \mathrm{Stk}_k^{\mathrm{pf}} \to \mathrm{DGCat}$$

Proposition 7.2. *sheaves on ind-schemes of ind-finite types satisfies*

(1) f^* is defined.

Our geometric objects

Definition 7.3. Let $X = \varinjlim X_i$ be of form described Equation 8

$$\mathrm{Shv}^!(X) := \varinjlim_{t^!} \mathrm{Shv}(X_i)$$

where the colimit takes place in DGCat.

Theorem 7.4. *[RS21, Thm. 2.6]* $\mathrm{Shv}^!$ *restricts to a six functor formalism.*

[6]The maps written in topological setting

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