The Langlands program and the moduli of bundles on the curve

Laurent Fargues and Peter Scholze

Abstract. This is a review of the work of the authors on the geometrization of the local Langlands correspondence. We explain the geometry of the stack Bun_G of G-bundles on the curve, the structure of the category $D_{lis}(\mathrm{Bun}_G, \Lambda)$, and the construction of local Langlands parameters using the preceding. We finally explain the categorical geometrization conjecture.

1. Introduction

We fix E a local field with finite residue field $\mathbb{F}_q$ of characteristic p and uniformizing element π. We thus have either $E \cong \mathbb{F}_q(\!(\pi)\!)$ (equal charasteristic case) or $[E : \mathbb{Q}_p] < +\infty$ (unequal characteristic case). We fix an algebraic closure $\overline{\mathbb{F}}_q$ of $\mathbb{F}_q$ and denote the completion of the associated maximal unramified extension of E by $\breve{E}$. Its Frobenius is denoted by σ. We fix a prime number $\ell \neq p$.

Let G be a reductive group over E. For $\Lambda \in \{\overline{\mathbb{F}}_\ell, \overline{\mathbb{Q}}_\ell\}$ we explain how to construct the semi-simple local Langlands correspondence

$$\pi \longmapsto \varphi_\pi$$

from irreducible smooth representations of $G(E)$ with coefficients in Λ to semi-simple Langlands parameters. We even further explain how to construct it "in families" over $\mathbb{Z}_\ell$ as a morphism between two categorical centers. For this we use methods of the geometric Langlands program in the context of the moduli of G-bundles on the Fargues-Fontaine curve ([**9**]).

2. The Artin v-stack Bun_G

2.1. Definition and basic properties.

2.1.1. *The relative curve (*[**9**, Chapter II.1]*).* We denote by $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$ the category of $\overline{\mathbb{F}}_q$-perfectoid spaces. We equip it with *the v-topology* ([**23**]). By definition, a colection of morphisms of perfectoid spaces $(S_i \to S)_{i\in I}$ is a v-cover if for any quasi-compact open subset U of S there exists a finite subset J of I, and a collection of quasi-compact open subsets V_j, $j \in J$, of S_j such that $U = \cup_{j\in J} \mathrm{Im}(V_j \to S)$.

2020 *Mathematics Subject Classification.* Primary 11R39; Secondary 14G22, 14D24.

L. Fargues was supported by the ANR project ANR-19-CE40-0015 "COLOSS" and the ERC Advanced Grant 742608 GeoLocLang.

P. Scholze was supported by a DFG Leibniz Prize, and by the DFG under the Excellence Strategy EXC-2047/1 390685813.

This is an analog for perfectoid spaces of the fpqc topology for schemes. We will use frequently another Grothendieck topology on perfectoid spaces: *the pro-étale topology*, the v-topology being finer than the pro-étale one. For this recall that, contrary to the categogry of Noetherian affinoid adic spaces, the category of affinoid perfectoid spaces is complete: for a projective system $(\mathrm{Spa}(R_i,R_i^+))_i$ of affinoid perfectoid spaces

$$\varprojlim_i \mathrm{Spa}(R_i,R_i^+) = \mathrm{Spa}(R_\infty,R_\infty^+)$$

where if ϖ is the image in $\varinjlim_i R_i^+$ of a pseudo-uniformizer of R_{i_0} for some index i_0, R_∞^+ is the ϖ-adic completion of $\varinjlim_i R_i^+$ and $R_\infty = R_\infty^+[\frac{1}{\varpi}]$. By definition, a morphism of perfectoid spaces is pro-étale if locally on the source and the target this can be written as a limit of affinoid perfectoid spaces with étale transition morphisms. The pro-étale topology is the one where the covers are defined in the same way as for the v-topology but we ask that in the family $(S_i \to S)_{i\in I}$ the morphisms are pro-étale (one has to be careful that pro-étale morphisms are not open in general contrary to étale morphisms and, as for the v-topology, asking that $\coprod_{i\in I}|S_i| \to |S|$ is surjective is not sufficient to define a pro-étale cover).

In the following we denote by

$$* = \mathrm{Spa}(\overline{\mathbb{F}}_q)$$

the final object of the v-topos of sheaves on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. This is not representable by a perfectoid space but will be the base point on which we will work. Sometimes we base change the situation to $\mathrm{Spa}(C)$ where $C|\overline{\mathbb{F}}_q$ is an algebraically closed perfectoid field but it is crucial to work "absolutely" over $*$ for some results.

For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ we denote by

$$X_S = Y_S/\varphi^{\mathbb{Z}}$$

the relative curve associated to S. Here

$$Y_S = \mathrm{Spa}(W_{\mathcal{O}_E}(R^+), W_{\mathcal{O}_E}(R^+)) \smallsetminus V(\pi[\varpi]) \tag{2.1}$$

when $S = \mathrm{Spa}(R,R^+)$ is affinoid perfectoid, ϖ is a pseudo-uniformizer in R, and by definition $W_{\mathcal{O}_E}(R^+) = R^+\hat{\otimes}_{\overline{\mathbb{F}}_q}\mathcal{O}_E \cong R^+[\![\pi]\!]$ when E is equal characteristic. The Frobenius φ acting on Y_S is induced by the Frobenius of the ramified Witt vectors $W_{\mathcal{O}_E}(R^+)$. To explain the construction of Y_S together with its action of $\varphi^{\mathbb{Z}}$ for any S we need to recall some basic facts about diamonds.

Recall in fact that if $\mathcal{F}$ is a functor on perfectoid spaces we denote by

$$\mathcal{F}^\diamond$$

the functor on $\mathbb{F}_p$-perfectoid spaces sending $T \in \mathrm{Perf}_{\mathbb{F}_p}$ to

$$\{(T^\sharp,\iota,s)\}/\sim$$

where

- $T^\sharp$ is a perfectoid space,
- $\iota : T \xrightarrow{\sim} (T^\sharp)^\flat$,
- $s \in \mathcal{F}(T^\sharp)$,
- the equivalence relation $\sim$ is given by $(T^\sharp,\iota,s) \sim (T^{\sharp'},\iota',s')$ if the is an isomorphism $f : T^\sharp \xrightarrow{\sim} T^{\sharp'}$ satisfying

– $f^\flat \circ \iota = \iota'$,
– $f^* s' = s$.

The correspondence $\mathcal{F} \mapsto \mathcal{F}^\diamond$ send v-sheaves on Perf to v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$ and the v-sheaf represented by an analytic adic spaces to some particular type of v-sheaves called *diamonds*. Diamonds are the v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$ that are "algebraic spaces for the pro-étale topology" in the sense that they can be written as

$$X/R$$

where

- X is an $\mathbb{F}_p$-perfectoid space
- $R \subset X \times_{\mathrm{Spa}(\mathbb{F}_p)} X$ is an equivalence relation represented by a perfectoid space such that both morphisms $R \rightrightarrows X$ are pro-étale.

For example:

- if X is a perfectoid space then $X^\diamond$ is represented by the perfectoid space $X^\flat$ and is thus a diamond,
- the v-sheaf $\mathrm{Spa}(E)^\diamond$ is a diamond as seen on the formula

$$\mathrm{Spa}(E)^\diamond = \mathrm{Spa}(\widehat{\overline{E}}^\flat)/\underline{\mathrm{Gal}(\overline{E}|E)}.$$

Remark 2.1. Since we used the notation $\underline{\mathrm{Gal}(\overline{E}|E)}$; for any topological space T the functor

$$\underline{T} : X \longmapsto \mathcal{C}(|X|, T)$$

is a v-sheaf on perfectoid spaces that is the sheaf associated to the constant presheaf with value T. This construction typically allows us to give a precise meaning to the notion of pro-étale H-torsor when H is a locally profinite topological group by speaking about $\underline{H}$-torsors for the pro-étale topology on perfectoid spaces. For example, let $\widehat{\mathbb{G}}_m$ be the multiplicative formal group over $\mathbb{Z}_p$ and $\widehat{\mathbb{G}}_{m,\mathbb{Q}_p}$ be its rigid analytic generic fiber over $\mathrm{Spa}(\mathbb{Q}_p)$. The logarithm $\log : \widehat{\mathbb{G}}_{m,\mathbb{Q}_p} \to \mathbb{G}_{a,\mathbb{Q}_p}$ induces a $\underline{\mathbb{Z}_p(1)}$-pro-étale torsor

$$\varprojlim_{n \geq 1} (\widehat{\mathbb{G}}_{m,\mathbb{Q}_p})^\diamond \longrightarrow \mathbb{G}_{a,\mathbb{Q}_p}^\diamond$$

where

- the transition morphism in the preceding projective limit is given by $z \mapsto z^{\frac{m}{n}}$ if $n|m$,
- $\underline{\mathbb{Z}_p(1)}$ is the twisted form of $\underline{\mathbb{Z}_p}$ over $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ isomorphic to $\underline{\mathbb{Z}_p}$ over $\mathrm{Spa}(\mathbb{C}_p^\flat)$ and given by the cyclotomic character.

Remark 2.2. The functor $(-)^\diamond$ from v-sheaves on Perf to v-sheaves on $\mathrm{Perf}_{\mathbb{F}_p}$ is a form of tilting generalizing the tilting procedure for perfectoid spaces. More precisely, the v-topology on perfectoid spaces is sub-canonical and there is an embedding

$$\mathrm{Perf} \subset \widetilde{\mathrm{Perf}}$$

(topos of v-sheaves). The following diagram then commutes

$$\begin{tikzcd} \mathrm{Perf} \arrow[r, hook] \arrow[d, "(-)^\flat"'] & \widetilde{\mathrm{Perf}} \arrow[d, "(-)^\diamond"] \\ \mathrm{Perf}_{\mathbb{F}_p} \arrow[r, hook] & \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}. \end{tikzcd}$$

The tilting equivalence for perfectoid spaces is the fact that if X is a perfectoid space then

$$(-)^\flat : \mathrm{Perf}_X \xrightarrow{\sim} \mathrm{Perf}_{X^\flat}.$$

This extends to an equivalence of topoi for $\mathcal{F}$ a v-sheaf on Perf

$$(-)^\diamond : \widetilde{\mathrm{Perf}}/\mathcal{F} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}/\mathcal{F}^\diamond$$

(localized topoi). Typically, this allows us to see the topos of v-sheaves on $\mathbb{Q}_p$-perfectoid spaces as a characteristic p topos,

$$(-)^\diamond : \widetilde{\mathrm{Perf}_{\mathbb{Q}_p}} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}/\mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$

Let us come back to the definition of X_S for any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. For any $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$, Y_S together with its action of φ is the unique analytic adic space over $\mathrm{Spa}(E)$ satisfying:

(1) its diamond is given by the formula

$$Y_S^\diamond = S \times_{\mathrm{Spa}(\mathbb{F}_q)^\diamond} \mathrm{Spa}(E)^\diamond \tag{2.2}$$

where φ acts as $\mathrm{Frob}_S \times \mathrm{Id}$, $\mathrm{Frob_S}$ being the absolute q-Frobenius of S,

(2) it is given by formula (2.1) for any affinoid perfectoid open subset of S.

This means that formula (2.2) shows that the given construction for S affinoid perfectoid glues to an adic space via the identification $|Y_S^\diamond| = |Y_S|$. In fact, as for "classical algebraic spaces", one can define the topological space of a diamond in a way such that $|X^\diamond| = |X|$ if X is an analytic adic space.

The construction $S \mapsto X_S$ is functorial in $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. In fact we have a collection of "classical curves", the adic version of the one studied in [**8**],

$$(X_{k(s),k(s)^+})_{s\in S},$$

and the adic space X_S is a way to take this collection and build a family out of it.

The spaces Y_S and X_S are sous-perfectoid E-adic spaces that become perfectoid when pulled back to $\mathrm{Spa}(\widehat{\overline{E}})$.

2.1.2. *The stack* Bun_G *(*[**9**, Chapter III.1)]. For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ we define

$$\mathrm{Bun}_G(S)$$

as the groupoid of G-bundles on X_S. Since X_S is sous-perfectoid there is a good notion of vector bundle on X_S, and here a G-bundle is by definition an exact tensor functor $\mathrm{Rep}_E(G) \to \{\text{vector bundles on } X_S\}$.

Let us note that when $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid then there is an associated "algebraic curve" $\mathfrak{X}_S$, an E-scheme like in [**8**] (see [**9**, Chapter II.2.3]) together with a morphism of ringed spaces $X_S \to \mathfrak{X}_S$ inducing a GAGA equivalence ([**9**, Proposition II.2.7])

$$\{\text{vector bundles on } \mathfrak{X}_S\} \xrightarrow{\sim} \{\text{vector bundles on } X_S\}.$$

Then a G-bundle on X_S is the same as an étale G-torsor on the scheme $\mathfrak{X}_S$.

The first basic result says that G-bundles on the curve satisfy descent for the v-topology. This is the following statement.

THEOREM 2.3. *The correspondence* $S \mapsto \mathrm{Bun}_G(S)$ *defines a* v*-stack*

$$\mathrm{Bun}_G \longrightarrow *.$$

Using that $X_S \hat{\otimes}_E \hat{\breve{E}}$ is perfectoid, this is in fact easily reduced to that fact that vector bundles on perfectoid spaces satisfy descent for the v-topology ([**25**, Lemma 17.1.8]).

2.2. Points ([9, Chapter II.2.1]). Recall the following construction. Let $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$. There is a functor

$$\begin{aligned} \text{Isocrystals} &\longrightarrow \{\text{vector bundles on } X_S\} \\ (D,\varphi) &\longmapsto \mathcal{E}(D,\varphi) \end{aligned}$$

given by the formula

$$\mathcal{E}(D,\varphi) = Y_S \overset{\varphi^{\mathbb{Z}}}{\times} D \longrightarrow Y_S/\varphi^{\mathbb{Z}} = X_S.$$

Here an isocrystal is a finite dimensional $\breve{E}$-vector space D together with a σ-semilinear automorphism $\varphi : D \xrightarrow{\sim} D$. We use that the morphism $Y_S \to \mathrm{Spa}(\breve{E})$ is compatible with the action of φ on Y_S and σ on $\breve{E}$. The preceding construction $(D,\varphi) \mapsto \mathcal{E}(D,\varphi)$ upgrades to a map

$$\begin{aligned} G(\breve{E}) &\longrightarrow \{G\text{-bundles on } X_S\} \\ b &\longmapsto \mathcal{E}_b. \end{aligned}$$

Recall that *the Kottwitz set* is

$$B(G) = G(\breve{E})/\sigma\text{-conjugation},\ b \sim gbg^{-\sigma},$$

see [**16**]. We now have the following result.

THEOREM 2.4 ([**6**],[**1**]). *When* $S = Spa(F, F^+)$ *is a geometric point, that is to say* F *is an algebraically closed perfectoid field, then*

$$\begin{aligned} B(G) &\xrightarrow{\sim} \{G\text{-bundles on } X_{F,F^+}\}/\sim \\ [b] &\longmapsto [\mathcal{E}_b]. \end{aligned}$$

In the GL_n-case this result is one of the main results of [**8**]: any vector bundle on the algebraic curve $\mathfrak{X}_F$ is isomorphic to $\oplus_i \mathcal{O}(\lambda_i)$ for some slopes $\lambda_i \in \mathbb{Q}$. We give a new proof of this result in [**9**, Chapter II.2.4] using the theory of diamonds. The proof in [**8**] relied on period domains for Lubin-Tate and Drinfeld spaces.

REMARK 2.5. In the preceding correspondence $(D,\varphi) \mapsto \mathcal{E}(D,\varphi)$, the slopes are reversed in the sense that if (D,φ) is isoclinic with slope λ then $\mathcal{E}(D,\varphi)$ is semi-stable with Harder-Narasimhan slope $-\lambda$. If (D,φ) is a simple isocrystal with slope λ then $\mathcal{E}(D,\varphi) \simeq \mathcal{O}(-\lambda)$.

We thus have an identification of sets

$$B(G) = |\mathrm{Bun}_G|.$$

This equality is the starting point of the study of the geometry of Bun_G.

2.3. Results on Banach-Colmez spaces ([9, Chapter II.2], [18]). Banach-Colmez spaces are in some sense the linear objects in the theory of diamonds. Those are the building blocks we use everywhere to do some geometry. They are obtained as the relative sheaf cohomology of vector bundles on the curve. In the following we speak about *locally spatial diamonds*. In fact, the categroy of diamonds is too large and to obtain a category with similar properties to the one of analytic adic spaces we need to restric it. By definition, a diamond X is spatial if

(1) is is quasi-compact quasi-separated,
(2) its associated topological space is spectral.

Typically, if X is an analytic adic space then $X^\diamond$ is locally spatial. Here are two examples of properties of locally spatial diamonds:

(1) If $f : X \longrightarrow Y$ is a morphism of locally spatial diamonds then $|f| : |X| \longrightarrow |Y|$ is generalizing,
(2) If X is a spatial diamond any pro-constructible generalizing subset Z of $|X|$ defines a spatial diamond $i : Y \hookrightarrow X$ such that i is quasi-compact and $|Y| = Z$.

The first property generalizes the usual one for analytic adic spaces. Here is an application of the second one: if X is a quasi-compact quasi-separated analytic adic space then any pro-constructible generalizing subset of $|X| = |X^\diamond|$ defines a spatial subdiamond of $X^\diamond$ that may not be associated to an analytic adic spacein general. We refer to [23] for more on this.

2.3.1. *Over a point.* Let $C|\overline{\mathbb{F}}_q$ be an algebraically closed perfectoid field. We have the following result about the cohomology sheaves of vector bundles on the curve. Here by slope we mean the Harder-Narasimhan slopes. By sheaf we mean a v-sheaf on Perf_C. When we speak about ℓ-cohomologically smooth morphisms we mean that "relative Poincaré duality is satisfied" in the sense of Verdier.

THEOREM 2.6. *Let $\mathcal{E}$ be a vector bundle on X_C.*

(1) *If the slopes of $\mathcal{E}$ are > 0 then $H^1(X_S, \mathcal{E}_{|X_S}) = 0$ if S is affinoid perfectoid and $S \mapsto H^0(X_S, \mathcal{E}_{|X_S})$ is a sheaf. This sheaf*

$$\mathcal{BC}(\mathcal{E}) \longrightarrow \mathit{Spa}(C)$$

is a separated ℓ-cohomologically smooth locally spatial diamond of dimension $\deg(\mathcal{E})$.

(2) *If moreover the slopes of $\mathcal{E}$ are in $]0, [E : \mathbb{Q}_p]\,]$ this is represented by an open perfectoid ball, more precisely the universal cover of a formal p-divisible group over $\overline{\mathbb{F}}_q$.*

(3) *If the slopes of $\mathcal{E}$ are < 0 then for any S, $H^0(X_S, \mathcal{E}_{|X_S}) = 0$, and $S \mapsto H^1(X_S, \mathcal{E})$ is a sheaf. This sheaf*

$$\mathcal{BC}(\mathcal{E}[1]) \longrightarrow \mathit{Spa}(C)$$

is a separated ℓ-cohomologically smooth locally spatial diamond of dimension $-\deg(\mathcal{E})$.

Let us explain some particular cases. Suppose $E = \mathbb{Q}_p$. If $\mathcal{G}$ is a formal p-divisible group over $\overline{\mathbb{F}}_q$ with covariant Dieudonné module (D, φ) then

$$\mathcal{BC}(\mathcal{E}(D,\varphi)(1)) \simeq \mathcal{G} \times_{\mathrm{Spa}(\overline{\mathbb{F}}_q)} \mathrm{Spa}(C)$$

where in this formula $\mathcal{G} \simeq \mathrm{Spf}(\overline{\mathbb{F}}_q[\![x_1, \cdots, x_d]\!])$ as a formal scheme that we see as a v-sheaf on $\mathrm{Perf}_{\overline{\mathbb{F}}_q}$. For this recall that if $\mathfrak{X}$ is an $\mathbb{F}_p$-formal scheme then the associated v-sheaf is the sheafification for the analytic topology of the functor $(R, R^+) \mapsto \mathfrak{X}(R^+)$ on $\mathbb{F}_p$-affinoid perfectoid algebras (the fact that this analytic sheafification is a v-sheaf is a consequence of the fact when pulled back to any $\mathrm{Spa}(R, R^+)$ affinoid perfectoid, this is representable by a perfectoid space). In the case of $\mathrm{Spf}(\overline{\mathbb{F}}_q[\![x_1, \cdots, x_d]\!])$ there is no need of such a sheafification for the analytic topology and this is the functor $(R, R^+) \mapsto (R^{\circ\circ})^d$.

For a general $\mathcal{E}$ with > 0 slopes we can find a resolution

$$0 \longrightarrow \mathcal{E}'' \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow 0$$

where $\mathcal{E}''$ is a trivial vector bundle and $\mathcal{E}'$ has slopes in $]0, 1]$. Then $\mathcal{BC}(\mathcal{E})$ is isomorphic to the quotient of $\mathcal{BC}(\mathcal{E}')$ by a pro-étale équivalence relation given by the free action of $\underline{V}$, $V = H^0(X_C, \mathcal{E}'')$ a finite dimensional E-vector space, on this open ball.

In negative slopes we have, after fixing an untilt $C^\sharp$ of C to E, a closed point ∞ on X_C with residue field $C^\sharp$. After fixing an identification $\mathcal{O}_{X_C}(1) \simeq \mathcal{O}_{X_C}(\infty)$ we deduce an isomorphism

$$\mathcal{BC}(\mathcal{O}(-1)[1]) \simeq \mathbb{G}_{a,C^\sharp}^\diamond / \underline{E}$$

which is thus clearly a separated ℓ-cohomologically smooth diamond.

2.3.2. *Absolute versions (*[**9**, Chapter II.2.2]. The following result is new for negative HN slopes but not for positive slopes (see [**7**]). Although the relative cohomology of vector bundles on the curve is not represented by a diamond absolutely over $*$, this is the case after puncture.

THEOREM 2.7. *Let (D, φ) be an isocrystal.*

(1) *If the slopes of (D, φ) are < 0 then the v-sheaf*

$$\mathcal{BC}(\mathcal{E}(D,\varphi)) \longrightarrow *$$

that sends S to $H^0(X_S, \mathcal{E}(D,\varphi))$ satisfies: $\mathcal{BC}(\mathcal{E}(D,\varphi)) \setminus \{0\}$ is a spatial diamond over $*$.

(2) *If the slopes of (D, φ) are > 0 then the v-sheaf*

$$\mathcal{BC}(\mathcal{E}(D,\varphi)[1]) \longrightarrow *$$

that sends S to $H^1(X_S, \mathcal{E}(D,\varphi))$ satisfies: $\mathcal{BC}(\mathcal{E}(D,\varphi)[1]) \setminus \{0\}$ is a spatial diamond over $*$.

The most difficult point is point (2) where we have to use *Artin's criterion for spatial diamonds* ([**23**, Theorem 12.18]) by proving that first $\mathcal{BC}(\mathcal{E}(D,\varphi)[1])$ is a spatial v-sheaf and then exhibiting a stratification of it by locally closed spatial diamonds.

REMARK 2.8. The reason why we call [**23**, Theorem 12.18] Artin's criterion for spatial diamonds is the following. The usual Artin criterion for an fppf sheaf X to be an algebraic space involves some global hypothesis like some finite presentation hypothesis coupled with some local one involving the formal completion of X at some point in $|X|$. Theorem 2.18 of [**23**] works the same way: a small v-sheaf on $\mathbb{F}_p$-perfectoid spaces X is a spatial dialmond if

(1) this is a spatial v-sheaf in the sense that is is quasi-compact quasi-separated and $|X|$ is a spectral space,
(2) for any $x \in |X|$, the localization $X_x = \varprojlim_{\substack{U \ni x \\ \text{nbd. of } x}} U$ is a diamond.

In all applications of this criterion point, Theorem 2.7 or the proof that closed Schubert cells in the B_{dR}-affine Grassmannian are spatial diamonds, point (2) is obtained by exhibiting a stratification of $|X|$ by locally closed generalizing subsets that are representable by diamonds. The way we prove that X is a spatial v-sheaf in Theorem 2.7 is by constructing a v-surjective ℓ-cohomologically smooth morphism $\widetilde{X} \to X$ where $\widetilde{X}$ is a spatial diamond and using that

(1) ℓ-cohomologically smooth morphisms are open,
(2) if Z is a spectral space and $R \subset Z \times Z$ is a pro-constructible equivalence relation such that both maps $R \rightrightarrows Z$ are open then Z/R is a spectral space.

EXAMPLE 2.9. The simplest example is given by the case when the slopes of (D, φ) are in $[-[E : \mathbb{Q}_p], 0[$. We then have

$$\mathcal{BC}(\mathcal{E}(D,\varphi)) \simeq \mathrm{Spa}\big(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]\!], \overline{\mathbb{F}}_q[\![x_1^{1/p^\infty}, \cdots, x_d^{1/p^\infty}]\!]\big)$$

that is not a perfectoid space (this is not an analytic adic space) or a diamond but that becomes a quasicompact perfectoid space after removing the point $V(x_1, \dots, x_d)$.

Let us explain more generally the positive HN slope case,

$$\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\},$$

$d \in \mathbb{N}_{\geq 1}$. For $d = 1$ this is isomorphic to $\mathrm{Spa}(\overline{\mathbb{F}}_q((x^{1/p^\infty})))$. In general let $\Delta = \{(\lambda_1, \dots, \lambda_d) \in (E^\times)^d \mid \prod_i \lambda_i = 1\}$. Then we can prove that the product morphism induces a pro-étale quotient isomorphism

$$\big(\mathcal{BC}(\mathcal{O}(1)) \setminus \{0\} \times \cdots \times \mathcal{BC}(\mathcal{O}(1)) \setminus \{0\}\big)/\underline{\Delta \rtimes \mathfrak{S}_d} \xrightarrow{\sim} \mathcal{BC}(\mathcal{O}(d)) \setminus \{0\}$$

where the action of $\underline{\Delta \rtimes \mathfrak{S}_d}$ is not free. This is anyway sufficient to prove that $\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\}$ is a spatial diamond. We refer to [**7**] for more details about this.

Let us explain the negative HN slope case, $\mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\}$. This classifies extensions of $\mathcal{O}(1)$ by $\mathcal{O}$ on X_S, $0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O}(1) \to 0$, that are fiberwise on S non-split. Thus, geometrically fiberwise on S, $\mathcal{E} \simeq \mathcal{O}(\frac{1}{2})$ (this is a particular case of Theorem 2.4). We deduce that $\mathcal{BC}(\mathcal{O}(-1)[1]) \setminus \{0\} \simeq \mathcal{BC}(\mathcal{O}(\frac{1}{2})) \setminus \{0\}/\underline{(D^\times)^1}$ where D is the quaternion algebra over E. This allows one to checks this is a spatial diamond.

Those absolute Banach-Colmez spaces play a key role in our work, see section 2.8 and the spatial diamond $\widetilde{\mathcal{M}}_b$.

2.3.3. *Families of Banach-Colmez spaces.* In [**9**] we retake the results of [**14**] using the theory of diamonds. Using this we obtain the following result for the relative cohomology of families of vector bundles on the curve and thus families of Banach-Colmez spaces.

THEOREM 2.10. *Let $S \in Perf_{\overline{\mathbb{F}}_q}$ and $\mathcal{E}$ be a vector bundle on X_S.*

(1) *The v-sheaf*
$$T/S \mapsto H^0(X_T, \mathcal{E}_{|X_T})$$
is a locally spatial diamond.

(2) *If fiberwise on S the slopes of $\mathcal{E}$ are > 0 then the sheaf associated to*
$$T/S \mapsto H^1(X_T, \mathcal{E}_{|X_T})$$
is zero. Moreover, $T/S \mapsto H^0(X_T, \mathcal{E}_{|X_T})$ is a separated ℓ-cohomologically smooth locally spatial diamond over S.

(3) *If fiberwise on S the slopes of $\mathcal{E}$ are < 0 then $H^0(X_T, \mathcal{E}_{|X_T}) = 0$ for any T/S and $T/S \mapsto H^1(X_T, \mathcal{E}_{|X_T})$ is a sheaf. This sheaf is moreover a separated ℓ-cohomologically smooth locally spatial diamond over S.*

This gives us a way to construct some nice *linear objects in the category of locally spatial diamonds.* Those are the basic geometric objects we work with.

2.4. Geometric structure ([9, Chapter IV.1]). The v-stack Bun_G is a priori an abstract object but the following result says it has a nice geometric structure.

THEOREM 2.11. *The following is satisfied.*

(1) *The v-stack* Bun_G *is an Artin v-stack:*
   (a) *the diagonal of* Bun_G *is representable in locally spatial diamonds,*
   (b) *there exists a locally spatial diamond U together with a separated surjective ℓ-cohomologically smooth morphism $U \to \mathrm{Bun}_G$.*

(2) $\mathrm{Bun}_G \to *$ *is separated ℓ-cohomologically smooth of dimension 0. Its dualizing complex is isomorphic to Λ.*

REMARK 2.12. One has to be careful that the diagonal morphism of Bun_G is not quasicompact contrary to the "classical stack" of vector bundles on a smooth projective curve. In particular, if $U \subset \mathrm{Bun}_G$ is a quasicompact open subset then the inclusion $j : U \hookrightarrow \mathrm{Bun}_G$ is not quasicompact in general and typically quasicompact base change ([**23**, Corollary 16.10]) can not be applied. In general $R^i j_* \mathbb{F}_\ell$ is non-zero for $i > 0$ contrary to the case of a quasi-compact quasi-separated open immersion.

Point (1)(a) in the preceding theorem is easily deduced from Theorem 2.10. Typically for GL_n, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are vector bundles on X_S then the v-sheaf $\mathrm{Isom}(\mathcal{E}_1, \mathcal{E}_2) \to S$ is open in $\mathcal{BC}(\mathcal{E}_1^\vee \otimes \mathcal{E}_2)$.

One way to prove the other results in this theorem is to use the so-called *Beauville-Laszlo uniformization* ([**9**, Chapter III.3]). More precisely, let
$$\mathrm{Gr}_G \longrightarrow \mathrm{Spd}(\breve{E})$$
be *the B_{dR}-affine Grassmannian* ([**25**, Lecture 19]). If S is an $\overline{\mathbb{F}}_q$-perfectoid space together with a morphism $S \to \mathrm{Spd}(\breve{E})$, that is to say an untilt $S^\sharp$ over $\breve{E}$, then
$$S^\sharp \hookrightarrow X_S$$

is a Cartier divisor. Now, $\mathrm{Gr}_G(S)$ is the set of such untilts $S^\sharp$ together with a G-bundle $\mathcal{E}$ on X_S and an isomorphism

$$\mathcal{E}_{1|X_S\setminus S^\sharp} \xrightarrow{\sim} \mathcal{E}_{|X_S\setminus S^\sharp}$$

that is "meromorphic along the divisor $S^\sharp$", where $\mathcal{E}_1$ is the trivial G-bundle. Here the definition of meromorphic is reduced to the case of vector bundles by asking that for any linear representation ρ of G, the isomorphism of vector bundles $\rho_*\mathcal{E}_{1|X_S\setminus S^\sharp} \xrightarrow{\sim} \rho_*\mathcal{E}_{|X_S\setminus S^\sharp}$ is meromorphic. In the case of vector bundles, that is to say G is GL_n for some integer $n\geq 1$, the isomorphism is meromorphic if, locally on S, it is the restriction to $X_S\setminus S^\sharp$ of a morphism

$$\mathcal{E}_1 \longrightarrow \mathcal{E}(kD)$$

for $k\gg 0$ where D is the Cartier divisor defined by $S^\sharp \hookrightarrow X_S$.

Recall moreover that Gr_G *is representable by an ind-diamond*. To write it as an ind-diamond suppose G is split to simplify and let $T\subset B$ be a maximal torus inside a Borel subgroup. For each $\mu\in X_*(T)^+$ there is a corresponding *Schubert cell* inside a closed Schubert cell

$$\mathrm{Gr}_{G,\mu}\subset \mathrm{Gr}_{G,\leq\mu}$$

where the open Schubert cell is a separated ℓ-cohomologically smooth locally spatial diamond over $\mathrm{Spd}(\breve{E})$ of dimension $\langle\mu,2\rho\rangle$ and the closed cell is a proper spatial diamond over $\mathrm{Spd}(\breve{E})$. One then has

$$\mathrm{Gr}_G = \varinjlim_{\mu\in X_*(T)^+} \mathrm{Gr}_{G,\leq\mu}$$

where by definition $\mu_1\leq\mu_2$ if $\mu_2-\mu_1\in\mathbb{N}.\check{\Phi}$, a positive sum of coroots. We then prove the following result.

Theorem 2.13 ([**9**, Proposition III.3.1, Theorem IV.1.19]). *The following is satisfied:*

(1) *The Beauville-Laszlo morphism* $\mathrm{Gr}_G\to\mathrm{Bun}_G$ *is v-surjective.*
(2) *The morphism*

$$\coprod_{\mu\in X_*(T)^+}\big[\,\underline{G(E)}\backslash \mathrm{Gr}_{G,\mu}\,\big]\longrightarrow \mathrm{Bun}_G$$

is separated ℓ-cohomologically smooth and gives a presentation of Bun_G *by a separated ℓ-cohomologically smooth locally spatial diamond over $*$.*

This chart on Bun_G is simple to construct. Nevertheless this is not the one we will use to analyze sheaves on Bun_G, see section 2.8.

2.5. HN stratification and connected components ([9, Chapter III.2.2, III.2.4, IV.1.2.2]). Let G^* be a quasisplit inner form of G and $A\subset T\subset B$ be a maximal split torus inside a maximal torus inside a Borel subgroup. There is an identification

$$X_*(A)^+_{\mathbb{Q}} = [\mathrm{Hom}(\mathbb{D},G_{\overline{E}})/G(\overline{E})]^\Gamma$$

where $\mathbb{D}$ is the slope pro-torus with $X^*(\mathbb{D})=\mathbb{Q}$. We denote by $\pi_1(G)$ the Borovoi fundamental group.

Theorem 2.14. *The following is satisfied.*

(1) *The map* $|\mathrm{Bun}_G| \to X_*(A)^+_{\mathbb{Q}}$ *given by the Harder-Narasimhan polygon is semi-continuous.*
(2) *The map* $|\mathrm{Bun}_G| \to \pi_1(G)_\Gamma$ *given by the first Chern class is locally constant with connected fibers.*

In terms of the identification $B(G) = |\mathrm{Bun}_G|$, the Harder-Narasimhan polygon of $\mathcal{E}_b$ is $w_0.(-\nu_b)$ where $\nu_b \in X_*(A)^+_{\mathbb{Q}}$ is the Newton point and w_0 the maximal length element in the Weyl group. The first Chern class of $\mathcal{E}_b$ is $-\kappa(b)$ where κ is Kottwitz map $\kappa : B(G) \to \pi_1(G)_\Gamma$.

REMARK 2.15. The local constancy of $|\mathrm{Bun}_G| \to \pi_1(G)_\Gamma$ is easy when the derived subgroup G_{der} is simply connected. In fact this is simply given by $|\mathrm{Bun}_G| \to |\mathrm{Bun}_{G/G_{der}}|$. The difficult case when G_{der} is not simply connected is treated in [**9**, Chapter III.2.4].

For the proof of the semi-continuity of the Harder-Narasimhan polygon we give a proof using the theory of diamonds. Let us explain the GL_n-case. Let $\mathcal{E}$ be a vector bundle on X_S. The HN polygon of $\mathcal{E}$ at a geometric point of S has its first slope $\geq \lambda$ if and only if at this geometric point there is a non-zero morphism $\mathcal{O}(\lambda) \to \mathcal{E}$. The moduli of non-zero morphisms from $\mathcal{O}(\lambda)$ to $\mathcal{E}$ is $\mathcal{BC}(\mathcal{O}(-\lambda)\otimes\mathcal{E}) \setminus \{0\} \to S$. Now we use that the morphism

$$\mathcal{BC}(\mathcal{O}(-\lambda) \otimes \mathcal{E}) \setminus \{0\}/\pi^{\mathbb{Z}} \longrightarrow S$$

is a proper morphism of locally spatial diamonds. The image in S of this morphism, the locus where the first slope of the polygon of $\mathcal{E}$ is $\geq \lambda$, is thus closed. This argument applied to exterior powers of $\mathcal{E}$ allows us to conclude the semi-continuity of the HN polygon of $\mathcal{E}$.

Finally let us remark that in particular, *the semi-stable locus*

$$\mathrm{Bun}_G^{ss} \subset \mathrm{Bun}_G$$

is open. We will describe in detail the structure of this open substack later.

THEOREM 2.16 ([**26**]). *The topology of* $|\mathrm{Bun}_G|$ *is the one induced by the embedding* $B(G) \hookrightarrow X_*(A)^+_{\mathbb{Q}} \times \pi_1(G)_\Gamma$ *and the order on* $X_*(A)^+_{\mathbb{Q}}$.

For GL_n the set $B(G)$ is described as a set of Newton polygons. The result is then that a point x of $|\mathrm{Bun}_G|$ is a specialization of y if and only if the polygon associated to x is over the one associated to y with the same endpoints. This special case is treated in [**4**].

2.6. HN strata as classifying stacks.

2.6.1. *Semi-stable locus ([**9**, Chapter III.4]).* Kottwitz κ map induces a bijection

$$B(G)_{basic} \xrightarrow{\sim} \pi_1(G)_\Gamma.$$

There is thus one semi-stable point in each connected component. We now have the following result. For b basic the sheaf

$$S \mapsto \mathrm{Aut}(\mathcal{E}_{b/X_S})$$

is the sheaf

$$\underline{G_b(E)}$$

whose value on S is continuous functions $|S| \to G_b(E)$. Here G_b is an inner form of G, a so-called extended pure inner form of G. For example $G_1 = G$. We then have the following result.

THEOREM 2.17. *The stratum attached to b basic is identified with the classifying stack*

$$[*/\underline{G_b(E)}]$$

of pro-étale $\underline{G_b(E)}$*-torsors.*

Here the identification sends a G-bundle $\mathcal{E}$ on X_S that is geometrically fiberwise on S isomorphic to $\mathcal{E}_b$ to the torsor

$$T/S \longmapsto \mathrm{Isom}(\mathcal{E}_b, \mathcal{E}_{|X_T}).$$

Let us remark that, using that étale separated morphisms satisfy v-descent ([**23**, Proposition 9.7]), a $\underline{G_b(E)}$-pro-étale torsor is the same as a $\underline{G_b(E)}$-v-torsor.

EXAMPLE 2.18. For the linear group the preceding says that there is an equivalence between vector bundles $\mathcal{E}$ on X_S that are geometrically fiberwise slope 0 semi-stable and pro-étale locally constant sheaves of $\underline{E}$-vector spaces with finite dimensional geometric stalks. Here the correspondence sends such an $\mathcal{E}$ to the sheaf $T/S \mapsto H^0(X_T, \mathcal{E}_{|X_T})$. In the other direction it sends $\mathcal{F}$ to $\mathcal{F} \otimes_{\underline{E}} \mathcal{O}_{X_S}$.

As a corollary we obtain a decomposition

$$\mathrm{Bun}_G^{ss} = \coprod_{[b] \text{ basic}} [*/\underline{G_b(E)}].$$

2.6.2. *More general strata (*[**9**, Chapter III.5]*).* Fix $[b] \in B(G)$. The structure of the automorphism sheaf of $\mathcal{E}_b$ is in general more complicated than in the basic case in the sense that its connected component of the identity is non-trivial. More precisely, we have the following result.

THEOREM 2.19. *The sheaf* $\widetilde{G}_b$ *of automorphisms of* $\mathcal{E}_b$ *that sends* S *to* $\mathrm{Aut}(\mathcal{E}_{b/X_S})$, *is of the form*

$$\widetilde{G}_b = \widetilde{G}_b^0 \rtimes \underline{G_b(E)}$$

where $\widetilde{G}_b^0$ *is a unipotent group diamond that is a successive extension of positive Banach-Colmez spaces. We moreover have* $\dim(\widetilde{G}_b^0) = \langle \nu_b, 2\rho \rangle$.

Let us look at the linear case. Let us fix some slopes $\lambda_1 > \cdots > \lambda_r$ in $\mathbb{Q}$ with some multiplicities $m_1, \cdots, m_r$. The associated vector bundle is $\mathcal{E} = \mathcal{O}(\lambda_1)^{m_1} \oplus \cdots \oplus \mathcal{O}(\lambda_r)^{m_r}$. Let us denote by D_λ the division algebra with invariant λ over E. The automorphism sheaf of $\mathcal{E}$ is a semi-direct product of $\underline{D_{\lambda_1}^\times} \times \cdots \underline{D_{\lambda_r}^\times}$, its group of connected components, with a unipotent diamond that is a successive extension of the Banach-Colmez spaces associated to $\mathcal{H}om(\mathcal{O}(\lambda_i)^{m_i}, \mathcal{O}(\lambda_j)^{m_j})$, $i > j$. The dimension of the Banach-Colmez space associated to a positive vector bundle is the degree of this vector bundle. The dimension of this unipotent diamond is thus $\sum_{i>j} m_j \deg \mathcal{O}(\lambda_j) - m_i \deg \mathcal{O}(\lambda_i)$.

We now have the following result.

THEOREM 2.20. *The Harder-Narasimhan stratum associated to* $[b]$ *is isomorphic to the classifying stack*

$$[*/\widetilde{G}_b].$$

This means that if $\mathcal{E}$ is a G-bundle on X_S that is geometrically fiberwise on S isomorphic to $\mathcal{E}_b$, then v-locally on S it is isomorphic to $\mathcal{E}_b$. In fact we even prove that this is true pro-étale locally on S. For the linear group this means that we can locally on S split the Harder-Narasimhan filtration of a vector bundle on X_S whose Newton polygon is fiberwise constant on S. This result is proved in [**14**] but we give a new proof using the theory of diamonds.

Corollary 2.21. *The HN stratum associated to $[b] \in B(G)$ is separated ℓ-cohomologically smooth of dimension $-\langle \nu_b, 2\rho\rangle$ over $*$.*

2.7. The Jacobian criterion of smoothness ([9, Chapter IV.4]).

2.7.1. *Statement.* To construct some "nice" charts on Bun_G we will need to use some kind of analog/variant of the so-called Quot schemes in the classical case. For this let $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ and

$$Z \longrightarrow X_S$$

be a smooth morphism of sous-perfectoid adic spaces. Make moreover the following "quasi-projective" assumption: there exists a Zariski closed immersion of Z into an open subset of $\mathbb{P}^n_{X_S}$. The basic example we may want to consider is the following. Suppose $S = \mathrm{Spa}(R, R^+)$ is affinoid perfectoid and let $\mathfrak{X}_S$ be the associated "algebraic curve" as a scheme over $\mathrm{Spec}(E)$. Let $\mathfrak{Z} \to \mathfrak{X}_S$ be a smooth quasi-projective morphism of schemes. Then one can define $\mathfrak{Z}^{ad} \to X_S$ that satisfies the preceding assumption for Z. Moreover, adification defines a bijection

$$\{\text{sections of } \mathfrak{Z} \to \mathfrak{X}_S\} \xrightarrow{\sim} \{\text{sections of } \mathfrak{Z}^{ad} \to X_S\}.$$

Let us now define

$$\mathcal{M}_Z \longrightarrow S$$

to be the "moduli space of sections of $Z \to X_S$" that is to say the functor on S-perfectoid spaces that sends T/S to sections s

$$\begin{tikzcd} & Z \arrow[d] \\ X_T \arrow[ur, "s"] \arrow[r] & X_S \end{tikzcd}$$

which is the same as sections of $Z \times_{X_S} X_T \to X_T$ (the fiber product makes sense as a smooth sous-perfectoid space over X_T). One can define T_{Z/X_S} the tangent bundle of $Z \to X_Z$ as a vector bundle over Z. We then define

$$\mathcal{M}_Z^{sm} \subset \mathcal{M}_Z$$

as the open subfunctor where we ask that via the preceding section s, s^*T_{Z/X_S} is a vector bundle bundle on X_T that has fiberwise on T positive (non-zero) HN slopes. Here is our *Jacobian criterion of smoothness.*

Theorem 2.22.
(1) *The functor $\mathcal{M}_Z$ is represented by a locally spatial diamond.*
(2) *The morphism $\mathcal{M}_Z^{sm} \to S$ is separated ℓ-cohomologically smooth of dimension the degree of s^*T_{Z/X_S} at a point given by a section s.*

Remark 2.23. In the "linear case" that is to say when Z is the geometric realization $\mathbb{V}(\mathcal{E})$ of a vector bundle $\mathcal{E}$ on X_S, then $\mathcal{M}_Z = \mathcal{BC}(\mathcal{E})$ and the preceding result is a basic result in the theory of Banach-Colmez spaces. Thus, the preceding result is an extension of this result to more general "non-linear" algebraic equations over the curve.

EXAMPLE 2.24. Let $n \geq 1$ be an integer and Z be the pullback to X_S of the Fermat curve $\{[x : y : z] \in \mathbb{P}^2_E \mid x^n + y^n = z^n\}$. One has a decomposition $\mathcal{M}_Z = \coprod_{k\in\mathbb{Z}} \mathcal{M}_Z^k$ where $\mathcal{M}_Z^k$ is the open/closed substack where the pullback of the line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ has degree k. Let $\mathbb{B}$ be the v-sheaf of rings $S \mapsto \mathcal{O}(Y_S)$. One the has $\mathcal{M}_Z^k = \emptyset$ if $k < 0$ and

$$\mathcal{M}_Z^k = \{(x, y, z) \in (\mathbb{B}^{\varphi=\pi^k})^3 \setminus \{(0, 0, 0)\} \mid x^n + y^n = z^n\}/\underline{E}^\times$$

for $k \geq 0$.

REMARK 2.25. In the "classical case" of schemes the preceding result is well-known and immediate. More precisely, if X is a smooth projective curve over a field k, S is k-scheme, $Z \to X_S$ is quasiprojective and smooth, then $\mathcal{M}_Z^{sm} \to S$ is by definition the moduli of sections s of $Z \to X_S$ such that the vector bundle s^*T_{Z/X_S} has no H^1 fiberwise on the base. This functor is easily checked to be formally smooth.

Let us for example treat the case of the Quot diamonds. Let $\mathcal{E}$ over X_S be a vector bundle. Let

$$\mathrm{Quot}_{\mathcal{E}} \longrightarrow S$$

be the functor that sends T/S to non-zero locally free quotients of $\mathcal{E}_{|X_T}$. The moduli space $Z \to X_S$ of non-zero locally free quotients of $\mathcal{E}$ is a disjoint union of Grassmannians $\coprod_{r\geq 1} \mathrm{Gr}_r(\mathcal{E}) \to X_S$ where r is the rank of the quotient. We have

$$\mathcal{M}_Z = \mathrm{Quot}_{\mathcal{E}}.$$

Now, $\mathcal{M}_Z^{sm} \to S$ sends T/S to locally free quotients $u : \mathcal{E}_{|X_T} \twoheadrightarrow \mathcal{F}$ such that $(\ker u)^\vee \otimes \mathcal{F}$ (this is s^*T_{Z/X_S} with the preceding notations) has positive HN slopes fiberwise on T i.e. the biggest slope of $\ker u$ is strictly less than the smallest one of $\mathcal{F}$. According to the Jacobian criterion this is a separated ℓ-cohomologically smooth locally spatial diamond.

2.7.2. *Some tools of the proof: ULA sheaves.* Suppose Λ is a torsion $\mathbb{Z}_\ell$-algebra. Let $f : X \to Y$ be a compactifiable morphism of locally spatial diamonds of locally finite dim.trg.. Let $A \in D_{\text{ét}}(X, \Lambda)$ (we refer to the beginning of Section 3.1 for a discussion on the definition of $D_{\text{ét}}(X, \Lambda)$). We define a notion of A to be f-ULA (universally locally acyclic) that satisfies the following properties ([**9**, Chapter IV.2]). It satisfies analogous properties to the one satisfied in the scheme case:

(1) If f is ℓ-cohomologically smooth and A locally constant with perfect fibers then A is f-ULA.
(2) If f is the identify then A is f-ULA if and only if A is locally constant with perfect fibers.
(3) If we have a diagram $X' \xrightarrow{g} X \xrightarrow{f} Y$ with g separated ℓ-cohomologically smooth surjective then A is f-ULA if and only g^*A is $f \circ g$-ULA: the notion of ULA is "smooth local on the origin".
(4) If we have a diagram $X' \xrightarrow{g} X \xrightarrow{f} Y$ with g proper and $A = Rg_*B$ with B $f \circ g$-ULA then A is f-ULA.

Although this is not the definition we take, the quickest formal definition of ULA sheaves is via the analog of the work of Lu-Zheng ([**19**]) on dualizability, see [**9**, Chapter IV.2.3.3], as left adjoints in the 2-category of cohomological correspondences. More precisely, if Y is a fixed locally spatial diamond the 2-category of correspondences $\mathcal{C}_Y$ is the following:

- its objects are morphisms of locally spatial diamonds $X \to Y$ that are compactifiable locally of finite dim.trg.,
- 1-morphisms between $X \to Y$ and $X' \to Y$ are given by objects of $D_{\text{ét}}(X \times_Y X', \Lambda)$ that we see as kernels; for $A \in D_{\text{ét}}(X \times_Y X', \Lambda)$ the associated transformation is
$$\begin{array}{rcl} F_A : D_{\text{ét}}(X,\Lambda) & \longrightarrow & D_{\text{ét}}(X',\Lambda) \\ \mathcal{F} & \longmapsto & \mathrm{pr}_{2!}(\mathrm{pr}_1^* \mathcal{F} \otimes^{\mathbb{L}}_{\Lambda} A), \end{array}$$
- the composiion of 1-morphisms is given by the "convolution of kernels" so that $F_{A_1 * A_2} = F_{A_1} \circ F_{A_2}$,
- 2-morphisms between A_1 and A_2 in $D_{\text{ét}}(X \times_Y X', \Lambda)$ are simply morphisms between A_1 and A_2.

We can now come back to the definition of f-ULA étale complexes. Any $A \in D_{\text{ét}}(X, \Lambda)$ is seen as a 1-morphism from $X \to Y$ to $Y \xrightarrow{\mathrm{Id}} Y$. We say that A is f-ULA if this 1-morphism is a left adjoint in the 2-category $\mathcal{C}_Y$ (the definition of a left adjoint in any 2-category is the evident generalization of the usual definition in the 2-category of categories). One can then prove that this left adjoint, if its exists, is automatically its Verdier dual $\mathbb{D}_{X/Y}(A)$.

ULA sheaves have very nice behavior with respect to Verdier duality and base change. We prove the following:

(1) If A is f-ULA then $\mathbb{D}_{X/Y}(A)$ is f-ULA in which case
$$A \xrightarrow{\sim} \mathbb{D}_{X/Y}(\mathbb{D}_{X/Y}(A)).$$
(2) If $g : Y' \to Y$ is a morphism of locally spatial diamonds with $\tilde{g} : X \times_Y Y' \to X$ then if A is f-ULA,
$$\tilde{g}^* \mathbb{D}_{X/Y}(A) \xrightarrow{\sim} \mathbb{D}_{X \times_Y Y'/Y'}(\tilde{g}^* A).$$
(3) If A is f-ULA then for any $B \in D_{\text{ét}}(Y, \Lambda)$ one has
$$\mathbb{D}_{X/Y}(A) \otimes^{\mathbb{L}}_{\Lambda} f^* B \xrightarrow{\sim} R\mathcal{H}om_{\Lambda}(A, Rf^! B).$$

Remark 2.26. The definition we give for an f-ULA étale complex is in fact constrained by property (3). As a matter of fact, point (3) implies $R\mathrm{Hom}(Rf_! A, B) = R\Gamma(X, \mathbb{D}_{X/Y}(A) \otimes^{\mathbb{L}}_{\Lambda} f^* B)$ for any $B \in D_{\text{ét}}(Y, \Lambda)$. If moreover f is quasicompact this property implies $Rf_! A$ restricted to any quasicompact open subset U of Y is a compact object of $D_{\text{ét}}(U, \Lambda)$. This is equivalent to say ([**23**, Chapter 20]) that $Rf_! A$ is constructible with perfect geometric stalks. This is exactly the definition we take by forcing this constructibility property étale locally on X (since the ULA notion has to be étale local on the source).

From properties (2) and (3) applied to $A = \mathbb{F}_\ell$ we deduce the following.

Theorem 2.27. *The morphism f is ℓ-cohomologically smooth if and only if*

(1) *$\mathbb{F}_\ell$ is f-ULA*

(2) *$Rf^!\mathbb{F}_\ell$ is invertible that is to say locally isomorphic to $\mathbb{F}_\ell[d]$ for some $d \in \mathbb{Z}$.*

This is what we use in the proof of the Jacobian criterion of smoothness by cutting the proof in two parts.

2.7.3. *Some tools of the proof: formal smoothness and deformation to the normal cone.* The first part of the proof of theorem 2.22 consists in proving that $\mathbb{F}_\ell$ is ULA with respect to $\mathcal{M}_Z^{sm} \to S$. This is achieved using the fact that, contrary to the notion of cohomological smoothness, *the notion of being ULA is "stable under retract"*. Let us explain this with an example. Suppose S is affinoid perfectoid. Let $\mathbb{B}_S^d \to S$ be the closed d-dimensional perfectoid ball over S. Let $Z \subset \mathbb{B}_S^d$ be Zariski closed. Suppose that there is an étale neighborhood $U \to \mathbb{B}_S^d$ of Z and a retraction of the inclusion $Z \times_{\mathbb{B}_S^d} U \hookrightarrow U$. Then $\mathbb{F}_\ell$ is f-ULA with respect to $Z \to S$. In fact we prove the following theorem.

Theorem 2.28. *The morphism $\mathcal{M}_Z^{sm} \to S$ is formally smooth in the sense that for any diagram*

$$\begin{tikzcd} T' \arrow[r] \arrow[d, hook] & \mathcal{M}_Z^{sm} \arrow[d] \\ T \arrow[r] \arrow[ur, dashed, "s"] & S \end{tikzcd}$$

with T affinoid perfectoid and T' Zariski closed in T, up to replacing T by an étale neighborhood $U \to T$ of T' and T' by $T' \times_T U$, there exists a morphism s completing the diagram.

An elementary argument then shows that this implies that $\mathbb{F}_\ell$ is ULA with respect to $\mathcal{M}_Z^{sm} \to S$. We now use that if $\mathbb{F}_\ell$ is f-ULA then the formation of $Rf^!\mathbb{F}$ commutes with base change. Using this we are reduced, up to change S, to prove that if

$$\begin{tikzcd} \mathcal{M}_Z^{sm} \arrow[d, "f"'] \\ S \arrow[u, bend left, "i"] \end{tikzcd}$$

then $i^*Rf^!\mathbb{F}_\ell$ is invertible. Such a section i corresponds to a section

$$\begin{tikzcd} Z \arrow[d] \\ X_S. \arrow[u, bend left, "k"] \end{tikzcd}$$

We then use the deformation to the normal cone of the regular immersion $X_S \hookrightarrow Z$, replacing Z by this deformation to the normal cone C to obtain a diagram

$$\begin{tikzcd} \mathcal{M}_C \arrow[d] \\ S \times \underline{E} \arrow[u, bend left] \end{tikzcd}$$

whose fiber at $0 \in \underline{E}$ is the zero section of $\mathcal{BC}(k^*T_{Z/X_S}) \to S$, and is isomorphic to $\mathcal{M}_Z \times \underline{E}^\times$ outside $0 \in \underline{E}$. This diagram is $\underline{E}^\times$-equivariant and using this action together with the cohomological smoothness of $\mathcal{BC}(k^*T_{Z/X_S}) \to S$ we can conclude.

2.8. Some "nice" charts on Bun_G ([9, Chapter V.3]). Rather than the Beauville-Laszlo uniformization or the preceding Quot diamonds (that work well only in the linear group case) we use other charts in our work to study sheaves on Bun_G. This are given by the following. Suppose G is quasisplit to simplify. Let $[b]\in B(G)$ and let M be the standard Levi subgroup that is the centralizer of $[\nu_b]\in X_*(A)^+_{\mathbb{Q}}$. Up to σ-conjugacy we can suppose that b is a basic element b_M in $M(\breve{E})$. Let P be the standard parabolic subgroup associated to $[\nu_b]$ with standard Levi subgroup M. We define

$$\mathcal{M}_b$$

to be the moduli space given by a P-bundle $\mathcal{E}_P$ on X_S such that $\mathcal{E}_P\overset{P}{\times}M$ (Levi quotient) is geometrically fiberwise on S isomorphic to $\mathcal{E}_{b_M}$. There is thus a diagram

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_b \arrow[r] \arrow[d] & \mathcal{M}_b \arrow[r, "\pi_b"] \arrow[d, "q_b"] & \mathrm{Bun}_G \\
* \arrow[r] \arrow[u, bend left] & {[*/\underline{G_b(E)}]} \arrow[u, bend left, "s_b"] &
\end{tikzcd}$$

where q_b sends $\mathcal{E}_P$ to $\mathcal{E}_P\overset{P}{\times}M$, π_b to $\mathcal{E}_P\overset{P}{\times}G$, and the section s_b sends $\mathcal{E}_M$ to $\mathcal{E}_M\overset{M}{\times}P$. We then have the following result that is at the core of our study of sheaves on Bun_G.

THEOREM 2.29. (1) *The morphism $q_b:\mathcal{M}_b\to[*/\underline{G_b(E)}]$ is representable in locally spatial diamonds and separated ℓ-cohomologically smooth of dimension $\langle\nu_b,2\rho\rangle$. In particular $\mathcal{M}_b$ is a cohomologically smooth Artin v-stack of dimension $\langle\nu_b,2\rho\rangle$.*

(2) *The morphism $\pi_b:\mathcal{M}_b\to\mathrm{Bun}_G$ is partially proper, representable in locally spatial diamonds, separated ℓ-cohomologically smooth with image the set of generalizations of $[b]$ inside $|\mathrm{Bun}_G|$.*

(3) *$\widetilde{\mathcal{M}}_b\setminus\{*\}$ is a spatial diamond.*

Point (1) is deduced from the fact that q_b is an iterated fibration in negative Banach-Colmez spaces, see section 2.3. Point (2) is an immediate application of the Jacobian criterion of smoothness, Theorem 2.22. Given a G bundle $\mathcal{E}$ on X_S that we view as an étale G-torsor $\mathcal{E}\to X_S$, we apply the Jacobian criterion to $Z=P\backslash\mathcal{E}$. The space $\mathcal{M}_Z\to S$ is then the moduli space of reductions of $\mathcal{E}$ to P that is to say $\mathrm{Bun}_P\times_{\mathrm{Bun}_G}S$. Point (3) is similar to the proof that punctured absolute negative Banach-Colmez spaces are spatial diamonds, see section 2.3.2.

REMARK 2.30. The spatialness of $\widetilde{\mathcal{M}}_b\setminus\{*\}$, the fact that it is *quasicompact* (and not only locally spatial), is a key tool in our work. This is one of the main reasons why we work with $\mathrm{Bun}_G\to *$ "absolutely" over $*$ and not with $\mathrm{Bun}_G\times\mathrm{Spa}(C)$ after a scalar extension to some algebraically closed perfectoid field C. In fact, $\widetilde{\mathcal{M}}_b\times\mathrm{Spa}(C)$ is a locally spatial diamond but $(\widetilde{\mathcal{M}}_b\setminus\{*\})\times\mathrm{Spa}(C)$ is not quasicompact anymore. This type of phenomenon arises frequently in our world and a consequence of the fact that *the final object $*$ of the v-topos is not quasiseparated.* For example the v-sheaf

$$D=\mathrm{Spa}\big(\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty},\dots,x_n^{1/p^\infty}]\!],\overline{\mathbb{F}}_q[\![x_1^{1/p^\infty},\dots,x_n^{1/p^\infty}]\!]\big)$$

with its section $\{*\} \hookrightarrow D$ given by $x_1 = \cdots = x_n = 0$ satisfies:

(1) D is not a diamond,
(2) $D \setminus \{*\}$ is a spatial diamond,
(3) after a scalar extension to C, $D \times \mathrm{Spa}(C)$ is a locally spatial diamond but $(D \setminus \{*\}) \times \mathrm{Spa}(C)$ is not spatial.

In our case the situation is even worste since the v-sheaf $\widetilde{\mathcal{M}}_b$ is not even representable by a formal scheme in general.

Let us treat for example the case when $G = \mathrm{GL}_2$ and $b = \begin{pmatrix} 1 & 0 \\ 0 & \pi^{-1} \end{pmatrix}$. The sheaf $\widetilde{\mathcal{M}}_b$ is the moduli of extensions of $\mathcal{O}(1)$ by $\mathcal{O}$ that is to say

$$\widetilde{\mathcal{M}}_b = \mathcal{BC}(\mathcal{O}(-1)[1]) \longrightarrow *.$$

We have

$$\mathcal{M}_b = \big[\mathcal{BC}(\mathcal{O}(-1)[1])/\underline{E}^\times \times \underline{E}^\times\big]$$

where $(a,b) \in E^\times \times E^\times$ acts via ab^{-1} on the Banach-Colmez space. The morphism

$$\big[\mathcal{BC}(\mathcal{O}(-1)[1])/\underline{E}^\times \times \underline{E}^\times\big] \to \mathrm{Bun}_{GL_2}$$

sends the section $[*/\underline{E}^\times \times \underline{E}^\times]$ to the point of $|\mathrm{Bun}_G|$ given by $[b]$. The complementary is sent to the semi-stable point associated to $\begin{pmatrix} 0 & \pi^{-1} \\ 1 & 0 \end{pmatrix}$, this is the locus where the extension of $\mathcal{O}(1)$ by $\mathcal{O}$ is geometrically fiberwise isomorphic to $\mathcal{O}(\frac{1}{2})$.

3. The category $D_{lis}(\mathrm{Bun}_G, \Lambda)$

3.1. The torsion case.

Let Λ be a $\mathbb{Z}_\ell$-algebra. Suppose in this section that Λ is torsion. We define

$$D_{lis}(\mathrm{Bun}_G, \Lambda) := D_{\text{ét}}(\mathrm{Bun}_G, \Lambda).$$

Since we did not explain it let us precise the definition of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ when Λ is torsion. The definition is done in [**23**] in different steps via a descent procedure from the strictly totally disconnected case where there is no choice for the definition:

(1) There is a good notion of étale morphism of perfectoid spaces. For X a perfectoid space one has an étale site $X_{\text{ét}}$ and an associated topos. Then, $D^+_{\text{ét}}(X, \Lambda)$ is the usual derived category of sheaves of Λ-modules on $X_{\text{ét}}$.
(2) Separated étale morphisms of perfectoid spaces descend along pro-étale covers (and even v-covers). We can thus define a notion of (locally separated) morphism of diamonds. For X a diamond this allows us to define an étale site $X_{\text{ét}}$ and we set $D^+_{\text{ét}}(X, \Lambda) = D^+(X_{\text{ét}}, \Lambda)$.
(3) For X a diamond let us note $\mathcal{D}^+_{\text{ét}}(X, \Lambda)$ for the natural stable infinite category whose homotopy category is $D^+_{\text{ét}}(X, \Lambda)$. One proves that $X \mapsto \mathcal{D}^+_{\text{ét}}(X, \Lambda)$ is a v-hypersheaf. As a corollary, we obtain the following description of $D^+_{\text{ét}}(X, \Lambda)$ for X a diamond

$$D^+_{\text{ét}}(X, \Lambda) = \Big\{ A \in \underbrace{D^+(X_v, \Lambda)}_{v\text{-sheaves of } \Lambda\text{-modules}} \ \Big|\ \begin{array}{l} \forall f : S \to X,\ S \text{ strictly tot.disc.}, \\ f^*A \in D^+_{\text{ét}}(S, \Lambda) = D^+(|S|, \Lambda) \end{array} \Big\}.$$

(4) In general for a small v-stack X one defines

$$\mathcal{D}_{\text{ét}}(X,\Lambda) = \varprojlim_{\substack{S\to X \\ S \text{ strict. tot. disc.}}} \underbrace{\mathcal{D}_{\text{ét}}(S,\Lambda)}_{\mathcal{D}(|S|,\Lambda)},$$

where S is a strictly totally disconnected perfectoid space, whose homotopy category is nothing else than

$$D_{\text{ét}}(X,\Lambda) = \left\{ A \in \underbrace{D(X_v,\Lambda)}_{v\text{-sheaves of } \Lambda\text{-modules}} \,\middle|\, \begin{array}{l} \forall f : S \to X,\ S \text{ strictly tot.disc.}, \\ f^*A \in D_{\text{ét}}(S,\Lambda) = D(|S|,\Lambda) \end{array} \right\}.$$

(5) One verifies that for a locally spatial diamond X the preceding definition of $D_{\text{ét}}(X,\Lambda)$ gives the left completion of $D(X_{\text{ét}},\Lambda)$. Under finite cohomological hypothesis, typically the ℓ-cohomological dimension of any quasi-compact open subset of X is finite, one has $D_{\text{ét}}(X,\Lambda) = D(X_{\text{ét}},\Lambda)$ i.e. $D(X_{\text{ét}},\Lambda)$ is left complete but this may not be true in general.

Thus, concretely, if $S_\bullet \to \mathrm{Bun}_G$ is a v-hypercover by disjoint unions of strictly totally disconnected perfectoid spaces then

$$D_{\text{ét}}(\mathrm{Bun}_G,\Lambda) = \underbrace{D(|S_\bullet|,\Lambda)}_{\text{derived cat. of cartesian sheaves}}.$$

REMARK 3.1. We want to deal with unbounded étale complexes on the left and on the right for the following reason. The v-stack Bun_G is not quasi-compact and it is necessary to work with complexes in $D^+_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ that may be unbounded. But we want to study Verdier duality for our complexes and this leads to the introduction and study of $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$.

A way to analyse objects of $D_{\text{ét}}(\mathrm{Bun}_G,\Lambda)$ is the following. For $[b]\in B(G)$ there is the inclusion of the corresponding HN stratum

$$i^b : [*/\widetilde{G}_b] \hookrightarrow \mathrm{Bun}_G.$$

We want to pull back étale sheaves on Bun_G via i^b to understand them. For this we need the following result, see [**9**, Chapter V.2].

THEOREM 3.2. *For $[b]\in B(G)$ there are identifications*

$$D(G_b(E),\Lambda) = D_{\text{ét}}([*/\underline{G_b(E)}],\Lambda) = D_{\text{ét}}([*/\widetilde{G}_b],\Lambda)$$

where the left category is the derived category of smooth representations of $G_b(E)$ with coefficients in Λ.

The first identification is easy. The second one is more subtle and uses in an essential way the fact that $\ell\neq p$. More precisely, let us recall that $\widetilde{G}_b = \widetilde{G}_b^0 \rtimes \underline{G_b(E)}$. Moreover $\widetilde{G}_b^0$ is a successive extension of positive Banach-Colmez spaces and is thus "ℓ-étale-contractible" (we give a precise meaning to this). This is where the second identification comes from.

At the end we have functors for each $[b]\in B(G)$

$$D_{\text{ét}}(\mathrm{Bun}_G,\Lambda) \underset{(i^b)_!}{\overset{(i^b)^*}{\rightleftarrows}} D(G_b(E),\Lambda).$$

that gives a *semi-orthogonal decomposition of the triangulated category* $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$ *in terms of the collection* $\big(D(G_b(E), \Lambda)\big)_{[b]\in B(G)}$.

In particular, via $(i^1)^*$ and $(i^1)_!$ *the category* $D(G(E), \Lambda)$ *is a direct factor of* $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$. This means "classical smooth representation theory of $G(E)$ with coefficients in Λ is a direct factor of $D_{\text{ét}}(\mathrm{Bun}_G, \Lambda)$". This is what we will use for the construction of L-parameters.

3.2. The general case ([9, Chapter VII]). The case of any $\mathbb{Z}_\ell$-algebra Λ is more complicated. We only explain the problem and sketch the solution. First we don't want to suppose Λ to be ℓ-adically complete since this would impose at the end that we would construct a morphism toward the ℓ-adic completion of the Bernstein center of G, and this is not what we want to do. Even if we suppose Λ to be ℓ-adically complete, if we consider ℓ-adically complete sheaves, i.e. limits of torsion étale sheaves on $\Lambda/\ell^n\Lambda$ when n varies, and invert formally ℓ, we will fall at the end on the representation theoric side on continuous representations of $G(E)$ with values in $\mathbb{Q}_\ell$-Banach spaces that have an invariant lattice. This is not what we want. We want to deal with purely "algebraic" smooth representations of $G(E)$.

The solution comes from *the theory of pro-étale solid sheaves*. Let Λ be any $\mathbb{Z}_\ell$-algebra. We define

$$D_{\text{proét}}(X, \Lambda_\blacksquare)$$

for any Artin v-stack X. We develop a theory of solid proétale sheaves of Λ-modules in this context, in particular a formalism of 5 operations $(f^*, Rf_*, f_\natural, R\mathcal{H}om, \overset{\blacksquare}{\otimes}{}^{\mathbb{L}}_{\Lambda})$ where $f_\natural$ is the relative homology functor defined as a left adjoint to f^* (there is no good notion of $Rf_!$ in this context and we use this relative homology as a replacement). The main problem now is the following. There is a functor

$$(i^1)^* : D_{\text{proét}}(\mathrm{Bun}_G, \Lambda_\blacksquare) \longrightarrow D(G(E), \Lambda_\blacksquare)$$

where the right hand side category is the derived category of representations of $G(E)$ as a condensed group in solid Λ-modules. Here Λ is the condensed ring defined as $\Lambda^{disc} \otimes_{\mathbb{Z}_\ell^{disc}} \mathbb{Z}_\ell$. There is an inclusion

$$D(G(E), \Lambda) \subset D(G(E), \Lambda_\blacksquare)$$

sending a smooth representation π to $\pi^{disc} \otimes_{\mathbb{Z}_\ell^{disc}} \mathbb{Z}_\ell$. But the category $D(G(E), \Lambda_\blacksquare)$ is much bigger than $D(G(E), \Lambda)$. Typically, when $\Lambda = \mathbb{Q}_\ell$, any $\mathbb{Q}_\ell$-Fréchet continuous representation of $G(E)$ gives rise to a $\mathbb{Q}_{\ell,\blacksquare}$ representation of $G(E)$ as a condensed group.

The solution to this problem is to define

$$D_{lis}(X, \Lambda) \subset D_{\text{proét}}(X, \Lambda_\blacksquare)$$

as the smallest triangulated subcategory stable under direct sums that contains $f_\natural\Lambda$ for all $f : Y \to X$, which are separated, representable in locally spatial diamonds and ℓ-cohomologically smooth. We prove this gives rise to a "good" triangulated category $D_{lis}(\mathrm{Bun}_G, \Lambda)$ that admits a semi-orthogonal decomposition by the $D(G_b(E), \Lambda)$, $[b] \in B(G)$. In particular, $D(G(E), \Lambda)$ is a direct factor in $D_{lis}(\mathrm{Bun}_G, \Lambda)$.

3.3. Compact generators ([9, Chapter V.4]). We suppose here that Λ is torsion to simplify the exposition. *The category $D(G(E),\Lambda)$ is well-known to be compactly generated.* A set of generators is given by the smooth representations

$$c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda$$

where K is a compact open pro-p subgroup of $G(E)$. From the geometric point of view, for such a K, there is a morphism of Artin v-stacks

$$f_K : [*/\underline{K}] \longrightarrow [*/\underline{G(E)}].$$

Then,

$$(f_K)_!\Lambda \in D_{lis}([*/\underline{G}],\Lambda) = D(G(E),\Lambda)$$

corresponds to $c\text{-}\mathrm{Ind}_K^{G(E)}\Lambda$.

This construction extends using the charts $\pi_b : \mathcal{M}_b \to \mathrm{Bun}_G$ from section 2.8. Recall that $\mathcal{M}_b = [\widetilde{\mathcal{M}}_b/\underline{G_b(E)}]$. Define for K compact open pro-p inside $G_b(E)$,

$$f_{b,K} : [\widetilde{\mathcal{M}}_b/\underline{K}] \longrightarrow \mathrm{Bun}_G.$$

According to theorem 2.29 this is separated ℓ-cohomologically smooth. Now, define

$$A_K^b = Rf_{b,K!}Rf_{b,K}^!\Lambda \in D_{lis}(\mathrm{Bun}_G,\Lambda).$$

Using the fact that $\widetilde{\mathcal{M}}_b \smallsetminus \{*\}$ is a spatial diamond we can prove the following theorem.

THEOREM 3.3. *For $A \in D_{\text{ét}}(\widetilde{\mathcal{M}}_b,\Lambda)$, if $i : * \hookrightarrow \widetilde{\mathcal{M}}_b$ is the closed point, then $R\Gamma(\widetilde{\mathcal{M}}_b, A) \xrightarrow{\sim} i^*A \in D(\Lambda)$.*

This allows us to prove the following key result.

THEOREM 3.4. *The set of objects $(A_K^b)_{[b]\in B(G), K\subset G_b(E)}$ is a set of compact generators of $D_{lis}(\mathrm{Bun}_G,\Lambda)$.*

In fact, for $B \in D_{lis}(\mathrm{Bun}_G,\Lambda)$, since f_K^b is ℓ-cohomologically smooth,

$$\begin{aligned} R\mathrm{Hom}(A_K^b, B) &= R\mathrm{Hom}(Rf_{b,K}^!\Lambda, Rf_{b,K}^!B) \\ &= R\Gamma(\widetilde{\mathcal{M}}_b, Rf_{b,K}^*B) \\ &= [(i^b)^*B]^K \in D(\Lambda) \end{aligned}$$

using theorem 3.3, theorem 3.4 is then easily deduced from this formula.

Those compact generators are the main tool we use for the following.

3.4. Finite type, admissible, Berstein-Zelevinsky involution ([9, Chapter V.4, V.5, V.7]). *One of the motto of this work is that at the end the natural objects involved in the local Langlands correspondence are not smooth representations of $G(E)$ but rather objects of $D_{lis}(\mathrm{Bun}_G,\Lambda)$.* This is supported by the following result that says that the notions of finite type, admissible, and Bernstein-Zelevinsky involution extend to $D_{lis}(\mathrm{Bun}_G,\Lambda)$.

THEOREM 3.5. *Let $A \in D_{lis}(\mathrm{Bun}_G,\Lambda)$.*

(1) *A is compact if and only if it has finite support and for all $[b] \in B(G)$,*
$$(i^b)^* A \in D(G_b(E), \Lambda)$$
is compact.
(2) *A is ULA for the morphism $\mathrm{Bun}_G \to *$ if and only if for all $[b] \in B(G)$ and all compact pro-p open subgroup $K \subset G_b(E)$,*
$$[(i^b)^* A]^K \in D(\Lambda)$$
is a perfect complex.
(3) *There exists an involution*
$$\mathbb{D}_{BZ} : \left(D_{lis}(\mathrm{Bun}_G, \Lambda)^\omega\right)^{op} \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G, \Lambda)^\omega$$
extending the usual Bernstein-Zelevinsky involution on $D(G(E), \Lambda)$.

Here compactness in $D(G_b(E), \Lambda)$ is equivalent to lying in the thick triangulated subcategory generated by the c-$\mathrm{Ind}_K^{G_b(E)}\Lambda$ as K runs through the pro-p compact open subgroups of $G_b(E)$.

Remark 3.6. Suppose Λ is $\overline{\mathbb{Q}}_\ell$ or $\mathbb{Q}_\ell$.
(1) The category of smooth representations of $G_b(E)$ has finite cohomological dimension (this is due to Bernstein, see [**22**] for example). Thus, in this case, compact objects are objects of $D^b_{ft}(G_b(E), \Lambda)$ (finite type cohomology). Thus, A is compact if and only if it has finite support and for all $[b]$, $(i^b)^* A$ is a bounded complex with finite type cohomology.
(2) A is ULA if and only if for all $[b]$, $(i^b)^* A$ is a complex with admissible cohomology such that for all K compact open pro-p, $[(i^b)^* A]^K$ is bounded.

The key tool in the preceding theorem is the explicit set of compact generators A_K^b of the preceding section.

Example 3.7. Suppose π is a smooth admissible representation of $G(E)$ with coefficients in $\overline{\mathbb{Q}}_\ell$. Let $\mathcal{F}_\pi$ be the associated sheaf on $[*/\underline{G(E)}]$. Then $(i^1)_! \mathcal{F}_\pi$ is ULA and thus its Verdier dual is too. We deduce from this that the stalks of $R(i^1)_* \mathcal{F}_\pi$ at all $[b]$ are complexes with admissible cohomology, a non-trivial finiteness statement.

4. The geometric Satake correspondence ([9, Chapter VII])

4.1. The local Hecke stack. Fix an integer $d \geq 1$. We let
$$\mathrm{Div}^d \longrightarrow *$$
be the sheaf of degree d effective Cartier divisors on the curve. More precisely, $\mathrm{Div}^d(S)$ is the set of equivalence classes of couples $(\mathcal{L}, u)$ where $\mathcal{L}$ is a fiberwise on S degree d line bundle on X_S and $u \in H^0(X_S, \mathcal{L})$ is fiberwise on S non-zero. One has
$$\mathrm{Div}^1 = \mathrm{Spd}(\breve{E})/\varphi^{\mathbb{Z}}.$$
This identification is deduced from the morphism $\mathrm{Spd}(\breve{E}) \to \mathrm{Div}^1$ sending an until $S^\sharp$ of S to the associated divisor Cartier $S^\sharp \hookrightarrow X_S$. Moreover one has an isomorphism
$$(\mathrm{Div}^1)^d / \mathfrak{S}_d \xrightarrow{\sim} \mathrm{Div}^d$$

where the quotient is a pro-étale quotient and the morphism is given by summing d degree 1 divisors to a degree d divisor. Another way to see Div^d is as a quotient of a punctured absolute Banach-Colmez space (see Section 2.3.2 and [**7**])

$$\mathcal{BC}(\mathcal{O}(d)) \setminus \{0\}/\underline{E}^\times.$$

There is a *Beilinson-Drinfeld type affine Grassmannian*, a v-sheaf

$$\mathrm{Gr}_{G,\mathrm{Div}^d} \longrightarrow \mathrm{Div}^d$$

whose value on S is given by a degree d divisor $D \subset X_S$, a G-bundle $\mathcal{E}$ on X_S and an isomorphism between the trivial G-bundle and $\mathcal{E}$ on $X_S \setminus D$ that is meromorphic along D. The "usual" B_{dR} affine Grassmannian of section 2.4 and [**25**] is

$$\mathrm{Gr}_{G,\mathrm{Div}^1} \times_{\mathrm{Div}^1} \mathrm{Spd}(\breve{E}).$$

The Grassmannian $\mathrm{Gr}_{G,\mathrm{Div}^d} \to \mathrm{Div}^d$ is equipped with an action of

$$L^+_{\mathrm{Div}^d} G \to \mathrm{Div}^d$$

the associated positive loop group. By definition, for S affinoid perfectoid, $L^+_{\mathrm{Div}^d}G(S)$ is given by $D \in \mathrm{Div}^d(S)$ and an element of

$$\varprojlim_{k\geq 1} G(\mathcal{O}_{X_S}/\mathcal{O}_{X_S}(-kD)).$$

We can then define *a local Hecke stack*

$$\mathcal{H}ck_{G,\mathrm{Div}^d} = \left[L^+_{\mathrm{Div}^d} G \backslash \mathrm{Gr}_{G,\mathrm{Div}^d}\right] \longrightarrow \mathrm{Div}^d$$

as a v-stack. For I a finite set with $|I| = d$ we define

$$\mathcal{H}ck^I_G = \mathcal{H}ck_{G,\mathrm{Div}^d} \times_{\mathrm{Div}^d} (\mathrm{Div}^1)^I \longrightarrow (\mathrm{Div}^1)^I$$

where $(\mathrm{Div}^1)^I \to \mathrm{Div}^d$ is the sum map. Its value on $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ is the groupoid given by the following datum:

(1) a collection of degree 1 relative Cartier divisors $(D_i)_{i\in I}$ on X_S,
(2) two G-bundles $\mathcal{E}$ and $\mathcal{E}'$ on X_S,
(3) a meromorphic isomorphism $\mathcal{E}_{|X_S \setminus \prod_{i\in I} D_i} \xrightarrow{\sim} \mathcal{E}'_{|X_S \setminus \prod_{i\in I} D_i}$.

4.2. The Satake category ([9, Chapter VI.7.1]). We suppose that Λ is torsion. The Satake category

$$\mathrm{Sat}^I_G(\Lambda)$$

is defined as the subcategory of complexes $A \in D_{\text{ét}}(\mathcal{H}ck^I_G, \Lambda)$ that satisfy

(1) A has bounded support on the Hecke stack
(2) A is ULA over $(\mathrm{Div}^1)^I$,
(3) A is flat perverse over $(\mathrm{Div}^1)^I$.

The last condition means that for any Λ-module M, $A \otimes^{\mathbb{L}}_\Lambda M$ is perverse. The perversity condition means here that

(1) for any morphism $x : \mathrm{Spa}(C, C^+) \to \mathcal{H}ck^I_G$ given by r distinct points on the curve, and sitting in the Schubert cell associated to $\mu_1, \dots, \mu_r \in X_*(T)^+$ at those r-distinct points (those Schubert cells are given by relative positions of B^+_{dR} lattices as usual), $x^*A \in D^{\leq -\sum_{i=1}^r \langle \mu_i, 2\rho\rangle}$.
(2) The same holds for $\mathbb{D}(A)$ its Verdier dual.

One of the first results is the following. Let

$$R\pi_{G*} : \mathrm{Sat}_G^I(\Lambda) \longrightarrow D_{\text{ét}}((\mathrm{Div}^1)^I, \Lambda)$$

be the pullback to $\mathrm{Gr}_{G,\mathrm{Div}^d} \times_{\mathrm{Div}^d} (\mathrm{Div}^1)^I$ composed with the push forward to $(\mathrm{Div}^1)^I$.

Theorem 4.1. *The functor $R\pi_{G*}$ takes values in complexes $C \in D_{\text{ét}}((\mathrm{Div}^1)^I, \Lambda)$ such that for all $i \in \mathbb{Z}$, $\mathcal{H}^i(C)$ is a local system of finite projective Λ-modules. The functor*

$$\bigoplus_{i\in\mathbb{Z}} \mathcal{H}^i(R\pi_{G*}) : \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{LocSys}((\mathrm{Div}^1)^I, \Lambda)$$

is exact, faithfull, and conservative.

One has $\mathrm{Div}^1 = \mathrm{Spd}(\widehat{\overline{E}})/\underline{W_E}$. This gives rise to a morphism

$$\mathrm{Div}^1 \longrightarrow [*/\underline{W_E}].$$

We can prove the following result, see [**9**, Chapter IV.7].

Theorem 4.2 (Drinfeld lemma). *For any small v-stack X the pullback functor*

$$D_{\text{ét}}(X \times [*/\underline{W_E^I}], \Lambda) \longrightarrow D_{\text{ét}}(X \times (\mathrm{Div}^1)^I, \Lambda).$$

is

(1) *fully faithful and restrict to an equivalence on local systems,*
(2) *an equivalence when I has one element.*

The proof is surprisingly "simple" compared to the classical case for usual moduli spaces of shtukas. This is one of the rare cases where our work is simpler than the "classical case" for function fields over a finite field. We will apply this theorem later to $X = \mathrm{Bun}_G$ but let us note the following corollary now.

Corollary 4.3. *There is an exact, faithful and conservative functor*

$$F^I : \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{Rep}_\Lambda(W_E^I)$$

where $\mathrm{Rep}_\Lambda(W_E^I)$ is the category of continuous representations of W_E on finite type projective Λ-modules.

4.3. Convolution and fusion.

4.3.1. *Convolution (*[**9**, Chapter VI.8]*).* There is a usual convolution diagram

$$\begin{tikzcd} & L^+G\backslash LG \overset{L^+G}{\times} LG/L^+G \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \mathcal{H}ck_{G,\mathrm{Div}^d} \times_{\mathrm{Div}^d} \mathcal{H}ck_{G,\mathrm{Div}^d} & & \mathcal{H}ck_{G,\mathrm{Div}^d} \end{tikzcd}$$

where we use the formula

$$\mathcal{H}ck_{G,\mathrm{Div}^d} = [L^+G\backslash LG/L^+G]$$

with LG and L^+G the loop group and positive loop group over Div^d. Here the action of L^+G on $L^+G\backslash LG \times LG/L^+G$ that defines the twisted stacky product is given for $x, y \in LG$ and $g \in L^+G$ by $g.([x],[y]) = ([xg^{-1}],[gy])$. The morphism a is given by $a\big([([x],[y])]\big) = ([x],[y])$. The morphism b is given by $b\big([([x],[y])]\big) = [xy]$.

After pullback to $(\mathrm{Div}^1)^I$ via $(\mathrm{Div}^1)^I \to \mathrm{Div}^d$ this gives rise to two morphisms a', b' and one can define for $A_1, A_2 \in D_{\text{ét}}(\mathcal{H}ck_G^I, \Lambda)^{bd}$, where the upper script "bounded" means with quasi-compact support, their convolution

$$A_1 \star A_2 = Rb'_* a'^*(A_1 \boxtimes A_2) \in D_{\text{ét}}(\mathcal{H}ck_G^I, \Lambda)^{bd}.$$

The basic result now is the following.

Theorem 4.4. *The operation $\star$ preserves $\mathrm{Sat}_G^I(\Lambda)$ and defines a convolution product*

$$\star : \mathrm{Sat}_G^I(\Lambda) \times \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{Sat}_G^I(\Lambda).$$

4.3.2. *Fusion ([**9**, Chapter VI.9]).* The main problem with the convolution product is to prove that this defines *a symmetric monoidal structure on the Satake category.* This is achieved using the fusion product as in [**20**]. Here we use in an essential way the ULA property in the definition of the Satake category. More precisely, suppose that $I = I_1 \coprod \cdots \coprod I_k$. We define a monoidal functor

$$\mathrm{Sat}_G^{I_1}(\Lambda) \times \cdots \times \mathrm{Sat}_G^{I_k}(\Lambda) \longrightarrow \mathrm{Sat}_G^I(\Lambda)$$

in the following way. For this consider the open subset

$$j : (\mathrm{Div}^1)^{I; I_1, \dots I_k} \hookrightarrow (\mathrm{Div}^1)^I$$

where $x_i \neq x_{i'}$ (as points on the curve) when $i, i' \in I$ lie in different I_j's. Define

$$\mathrm{Sat}_G^{I; I_1, \dots, I_k}(\Lambda) \subset D_{\text{ét}}(\mathcal{H}ck_G^I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I; I_1, \dots I_k}, \Lambda)$$

in the same way as we defined Sat_G^I as bounded ULA flat perverse sheaves.

Theorem 4.5. *The open immersion $j : \mathcal{H}ck_G^I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I; I_1, \dots I_k} \hookrightarrow \mathcal{H}ck_G^I$ induces a fully faithful functor*

$$j^* : \mathrm{Sat}_G^I(\Lambda) \longrightarrow \mathrm{Sat}_G^{I; I_1, \dots, I_k}(\Lambda).$$

The full faithfulness is essential here to force the commutativity constraint later. The next result is then the following. There are morphisms for $1 \leq r \leq k$

$$\mathcal{H}ck_G^I \times_{(\mathrm{Div}^1)^I} (\mathrm{Div}^1)^{I; I_1, \dots I_k} \longrightarrow \mathcal{H}ck_G^{I_r}.$$

This allows us to define for $A_1 \in \mathrm{Sat}_G^{I_1}(\Lambda), \dots, A_k \in \mathrm{Sat}_G^{I_k}(\Lambda)$

$$A_1 \boxtimes \cdots \boxtimes A_k \in \mathrm{Sat}_G^{I; I_1, \dots, I_k}(\Lambda).$$

Theorem 4.6. *The image of the exterior tensor product*

$$\mathrm{Sat}_G^{I_1}(\Lambda) \times \cdots \times \mathrm{Sat}_G^{I_k}(\Lambda) \longrightarrow \mathrm{Sat}_G^{I; I_1, \dots, I_k}(\Lambda)$$

lies in $\mathrm{Sat}_G^I(\Lambda)$ via the fully faithful functor j^* of Theorem 4.5.

The proof of Theorem 4.6 relies on an auxiliary ad-hoc construction: a local Hecke stack

$$\mathcal{H}ck^{I, I_1, \dots, I_k} \longrightarrow (\mathrm{Div}^1)^I$$

equipped with projections

$$p_j : \mathcal{H}ck^{I, I_1, \dots, I_k} \longrightarrow \mathcal{H}ck^{I_j}, \quad j = 1, \dots, k$$

and a morphism

$$m : \mathcal{H}ck^{I, I_1, \dots, I_k} \longrightarrow \mathcal{H}ck^I.$$

This is defined in the following way: $\mathcal{H}ck^{I,I_1,\dots,I_k}(S)$ is the datum of

- a collection of degree 1 relative Cartier divisors $(D_i)_{i\in I}$,
- a collection of G-bundles $\mathcal{E}_0,\dots,\mathcal{E}_k$ on X_S,
- isomorphisms
$$\mathcal{E}_{j-1|X_S\smallsetminus\prod_{i\in I_j}D_i} \xrightarrow{\sim} \mathcal{E}_{j|X_S\smallsetminus\prod_{i\in I_j}D_i},\quad 1\le j\le k$$
that are meromorphic along the Cartier divisors.

The projection p_j sends the preceding datum to the couple $(\mathcal{E}_{j-1},\mathcal{E}_j)$ together with the modification $\mathcal{E}_{j-1|X_S\smallsetminus\prod_{i\in I_j}D_i} \xrightarrow{\sim} \mathcal{E}_{j|X_S\smallsetminus\prod_{i\in I_j}D_i}$. The morphism m sends the preceding datum to the couple $(\mathcal{E}_0,\mathcal{E}_k)$ together with the modification obtained by composing the k-modifications restricted to $X_S\smallsetminus\prod_{i\in I}D_i$. One then verifies that
$$B=Rm_*(p_1^*A_1\otimes^{\mathbb{L}}_\Lambda\cdots\otimes^{\mathbb{L}}_\Lambda p_k^*A_k)\in D_{\text{ét}}(\mathcal{H}ck^I,\Lambda)$$
is in fact in $\mathrm{Sat}^I_G(\Lambda)$. Since its pullback via j is $A_1\boxtimes\cdots\boxtimes A_k$ this proves Theorem 4.6.

We can now construct the fusion product. The correspondence $I\mapsto\mathrm{Sat}^I_G(\Lambda)$ is functorial in the sense that for a map of finite sets $I\to J$ there is a functor $\mathrm{Sat}^I_G(\Lambda)\to\mathrm{Sat}^J_G(\Lambda)$. We can thus compose
$$\mathrm{Sat}^I_G(\Lambda)\times\cdots\times\mathrm{Sat}^I_G(\Lambda)\longrightarrow\mathrm{Sat}^{I\coprod\cdots\coprod I}_G(\Lambda)\longrightarrow\mathrm{Sat}^I_G(\Lambda).$$
using the map $I\coprod\cdots\coprod I\to I$. This defines the fusion product $*$ as a symmetric monoidal refinement of the convolution product $\star$ (the fact that this refines the convolution product is part of the proof of the preceding theorem). This defines a functor
$$I\longmapsto(\mathrm{Sat}^I_G(\Lambda),*)$$
from finite sets to symmetric monoidal categories. Moreover one proves that the functor
$$F^I:\mathrm{Sat}^I_G(\Lambda)\longrightarrow\mathrm{Rep}_\Lambda(W_E^I)$$
is symmetric monoidal.

4.4. Tannakian reconstruction ([9, Chapter VI.10]). Let ${}^LG=\widehat{G}\rtimes W_E$ be the Langlands dual of G over Λ. We can define $\mathrm{Rep}_{{}^LG}(\Lambda)$ to be the category of representations of ${}^LG$ on projective Λ-modules of finite type. Here the representations are algebraic when restricted to $\widehat{G}$ and trivial on an open subgroup of W_E. The correspondence
$$I\longmapsto\mathrm{Rep}_\Lambda({}^LG^I)$$
is a functor from finite sets to symmetric monoidal categories.

Theorem 4.7. *For Λ a $\mathbb{Z}_\ell[\sqrt{q}]$-algebra there is an equivalence of functors from finite sets to symmetric monoidal categories*
$$\mathrm{Rep}_\Lambda({}^LG^I)\xrightarrow{\sim}\mathrm{Sat}^I_G(\Lambda)$$
Through this equivalence the functor $F^I:\mathrm{Sat}^I_G(\Lambda)\to\mathrm{Rep}_\Lambda W_E^I$ is identified with the restriction to W_E^I. The natural inclusion $\mathrm{Rep}_\Lambda({}^LG^I)\to\mathrm{Sat}^I_G$ induced by the pullback functor from $\mathcal{H}ck^I_G\to(\mathrm{Div}^1)^I$ corresponds to the pullback via $({}^LG^I)\to W_E^I$.

4.5. The tools we use.

4.5.1. *Hyperbolic localization.* The main tool we use is hyperbolic localization ([**9**, Chapter IV.6]), following the work [**21**]. The proof of Theorem 4.1 relies on this. Hyperbolic localization allows us to prove the following, see [**9**, Chapter VI.3]. Suppose G is split. Let $T \subset B$ be a maximal torus inside a Borel subgroup in G. Let us consider the affine Grassmannian associated to B

$$\mathrm{Gr}_{B,\mathrm{Div}^d} \longrightarrow \mathrm{Div}^d.$$

The quotient morphism $B \to T$ induces a morphism

$$\mathrm{Gr}_{B,\mathrm{Div}^d} \longrightarrow \mathrm{Gr}_{T,\mathrm{Div}^d} = \coprod_{X_*(T)} \mathrm{Div}^d.$$

This defines a decomposition by pullback

$$\mathrm{Gr}_{B,\mathrm{Div}^d} = \coprod_{\lambda \in X_*(T)} \mathrm{Gr}^{\lambda}_{B,\mathrm{Div}^d}.$$

There is then a natural morphism

$$\mathrm{Gr}_{B,\mathrm{Div}^d} \longrightarrow \mathrm{Gr}_{G,\mathrm{Div}^d}$$

that is bijective on geometric points and a locally closed immersion on each $\mathrm{Gr}^{\lambda}_{B,\mathrm{Div}^d}$, $\lambda \in X_*(T)$. The image $\mathrm{Gr}^{\lambda}_{B,\mathrm{Div}^d} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Div}^d}$ is a so called semi-infinite orbit. Let us define

$$\begin{array}{ccc} \mathrm{Gr}_{B,\mathrm{Div}^d} & \xrightarrow{q} & \mathrm{Gr}_{G,\mathrm{Div}^d} \\ {\scriptstyle p}\big\downarrow & & \\ \mathrm{Gr}_{T,\mathrm{Div}^d}. & & \end{array}$$

We can then define a *constant term functor*

$$\mathrm{CT}_B : D_{\text{ét}}(\mathcal{H}ck^I_G, \Lambda)^{bd} \longrightarrow D_{\text{ét}}(\mathcal{H}ck^I_T, \Lambda)^{bd}$$

by applying $Rp_! q^*$. The following result uses heavily hyperbolic localization. We refer to [**9**, Chapter VI.3] for all of this. The shift "deg" is an explicit locally constant function $\mathrm{Gr}_{T,\mathrm{Div}^{\mathrm{d}}} \to \mathbb{Z}$ that is there only to ensure that the constant term of a perverse sheaf is perverse.

THEOREM 4.8. *The constant term functor satisfies the following*

$$\mathrm{CT}_B[\deg] : \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{Sat}^I_T(\Lambda),$$

moreover

$$R\pi_{G*} = R\pi_{T*} \circ \mathrm{CT}_B[\deg] : \mathrm{Sat}^I_G(\Lambda) \longrightarrow \mathrm{LocSys}((\mathrm{Div}^1)^I, \Lambda).$$

This is the main tool we use to analyse the category $\mathrm{Sat}^I_G(\Lambda)$. In fact, $\mathrm{CT}_B[\deg]$ geometrizes the restriction functor $\mathrm{Rep}_\Lambda({}^L G^I) \longrightarrow \mathrm{Rep}_\Lambda({}^L T^I)$. This explains its importance in the proof of the Satake equivalence.

4.5.2. *Degeneration to the Witt vector Grassmannian.* Suppose G is split. We now see it as a reductive group over $\mathcal{O}_E$. For $S \in \mathrm{Perf}_{\overline{\mathbb{F}}_q}$ there is sous-perfectoid space

$$\mathcal{Y}_S \longrightarrow \mathrm{Spa}(\mathcal{O}_{\breve{E}})$$

such that

$$\mathcal{Y}_S^\diamond = S \times \mathrm{Spd}(\mathcal{O}_{\breve{E}}).$$

One has $\{\pi \neq 0\} = Y_S \subset \mathcal{Y}_S \supset S = \{\pi = 0\}$. We can define

$$\mathrm{Div}^1_{\mathcal{Y}} = \mathrm{Spd}(\mathcal{O}_{\breve{E}})$$

and a corresponding v-sheaf

$$\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \longrightarrow \mathrm{Div}^1_{\mathcal{Y}}.$$

Let $\mathrm{Gr}^{\mathrm{Witt}}_{G,\overline{\mathbb{F}}_q} \to \mathrm{Spec}(\overline{\mathbb{F}}_q)$ be Zhu's Witt vector affine Grassmannian ([**28**], [**3**]). We have $\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \times_{\mathrm{Div}^1_{\mathcal{Y}}} \mathrm{Spd}(\overline{\mathbb{F}}_q) = (\mathrm{Gr}^{\mathrm{Witt}}_{G,\overline{\mathbb{F}}_q})^\diamond$. This is used in the proof of the reconstruction theorem 4.7 to prove that for $\mu \in X_*(T)^+$, if $j_\mu : \mathcal{H}ck_{G,\mu} \hookrightarrow \mathcal{H}ck_G$ is the inclusion of the corresponding affine Schubert cell of dimension d_μ, then

$${}^p j_{\mu!}\mathbb{Q}_\ell[d_\mu] \longrightarrow {}^p Rj_{\mu*}\mathbb{Q}_\ell[d_\mu]$$

is an isomorphism. This is transfered to the same type of statement on $\mathrm{Gr}^{\mathrm{Witt}}_{G,\overline{\mathbb{F}}_q}$ where the proof uses at the end *the decomposition theorem applied to a Demazure resolution*, see [**9**, Proposition VI.7.5].

5. Langlands parameters and the spectral action

5.1. Moduli of Langlands parameters ([9, Chapter VIII.1], [5]).

5.1.1. *Existence and singularities.* We can view any $\mathbb{Z}_\ell$-algebra Λ as a condensed ring by setting $\Lambda := \Lambda^{disc} \otimes_{\mathbb{Z}_\ell^{disc}} \mathbb{Z}_\ell$. We define

$$Z^1(W_E, \widehat{G})$$

as the functor on $\mathbb{Z}_\ell$-algebras whose value on Λ is condensed 1-cocycles $W_E \to \widehat{G}(\Lambda)$ where $\widehat{G}(\Lambda)$ is naturally a condensed group. There is an action by conjugation of $\widehat{G}$ on $Z^1(W_E, \widehat{G})$. The first main result about this is the following.

THEOREM 5.1. *The functor $Z^1(W_E, \widehat{G})$ is represented by a $\mathbb{Z}_\ell$-scheme that is an infinite disjoint union of finite type affine schemes. Moreover the algebraic stack $\big[Z^1(W_E, \widehat{G})/\widehat{G}\big]$ is a locally complete intersection over Spec$(\mathbb{Z}_\ell)$ of dimension* 0.

We use the notation

$$\mathrm{LocSys}_{\widehat{G}} = \big[Z^1(W_E, \widehat{G})/\widehat{G}\big].$$

REMARK 5.2. Our moduli of parameters is locally complete intersection of dimension 0. *We don't need to upgrade it to a derived stack* as done in the case of compact Riemann surfaces where the naive underived moduli space of local systems does not satisfy the preceding property. We could consider the derived version of our moduli space but at the end we would prove it is in fact underived. We are thus, from this point of view, in a better situation compared to the geometric Langlands program on a compact Riemann surface. All of this is due to the presence of the Frobenius and its action on the moderate inertia, the presence of $\widehat{\mathbb{Z}}^p(1)$ that shows up as I_E/P_E.

5.1.2. *The coarse moduli space.* The coarse moduli space is

$$Z^1(W_E, \widehat{G}) /\!\!/ \widehat{G}$$

an infinite disjoint union of finite type $\mathbb{Z}_\ell$-affine schemes. Its algebra of functions is

$$\mathcal{O}(Z^1(W_E, \widehat{G}))^{\widehat{G}}.$$

Its geometric points can be described in terms of geometric invariant theory using *the Hilbert–Mumford–Kempf's numerical criterion*, see [**9**, Chapter VIII.3.1].

THEOREM 5.3. *For an algebraically closed field L over $\mathbb{Z}_\ell$, the L-points of the coarse moduli space of* $\mathrm{LocSys}_{\widehat{G}}$ *are given by $\widehat{G}(L)$-conjugacy classes of semi-simple Langlands parameters $W_E \to \widehat{G}(L) \rtimes W_E$.*

By definition here, a parabolic subgroup of $\widehat{G}_L \rtimes W_E$ is a subgroup that surjects onto W_E and whose intersection with $\widehat{G}$ is a parabolic subgroup of $\widehat{G}$. Those are exactly up to $\widehat{G}(L)$-conjugation the $\widehat{P}_L \rtimes W_E$ where P is a parabolic subgroup of G^* the quasi-split inner form of G. The same goes on for Levi-subgroups. Recall then that a semi-simple Langlands parameter is a parameter $\varphi : W_E \to \widehat{G}(L) \rtimes W_E$ such that if its image is contained in a parabolic subgroup then its image is contained in a Levi subgroup of this parabolic subgroup.

If $L = \mathbb{Q}_\ell$ or $\overline{\mathbb{Q}}_\ell$ parameters are given by representations of the *Weil-Deligne group*. This consists of a couple (ρ, N) where $\rho : W_E \to \widehat{G}(L)$ is a parameter that is trivial on an open subgroup of I_E and $N \in \mathrm{Lie}\,\widehat{G} \otimes L$ satisfies

$$\mathrm{Ad}(\rho(\tau)).N = q^{v(\tau)} N$$

for all $\tau \in W_E$ where τ acts by $\mathrm{Frob}_q^{v(\tau)}$ on the reisude field. Then if φ corresponds to (ρ, N), it is semi-simple if and only if $N = 0$ and φ is Frobenius semi-simple.

5.1.3. *Infinitesimal properties (*[**9**, Chapter VII.2]*).* Let us finally look at the infinitesimal properties of this moduli space that is to say the cotangent complex

$$\mathbb{L}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} \in \mathrm{Perf}^{[-1,1]}(\mathrm{LocSys}_{\widehat{G}})$$

using that it is locally complete intersection over $\mathbb{Z}_\ell$. We have the following description.

THEOREM 5.4. *Let x : Spec$(\Lambda) \to \mathrm{LocSys}_{\widehat{G}}$ that corresponds to $\varphi : W_E \to \widehat{G}(\Lambda)$. Then*

$$x^*\mathbb{L}^\vee_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} = R\Gamma\big(W_E, (\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda)_\varphi(1)\big)[1]$$

where $(\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda)_\varphi$ is $\widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda$ equipped with the twisted action of W_E deduced from $\mathrm{Ad} \circ \varphi$.

We need this explicit description later for the Arinkin-Gaitsgory singular support condition. Let

$$\mathrm{Sing}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} := \mathcal{H}^{-1}\big(\mathbb{L}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell}\big) \longrightarrow \mathrm{LocSys}_{\widehat{G}}$$

be the stack of singularities. Here by $\mathcal{H}^{-1}$ we mean that this is the group scheme over $\mathrm{LocSys}_{\widehat{G}}$ representing the functor that sends a scheme S together with a morphism $f : S \longrightarrow \mathrm{LocSys}_{\widehat{G}}$ to $H^{-1}(S, f^*\mathbb{L}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell})$. Locally on such a S, if we writte $f^*\mathbb{L}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} \simeq [\mathcal{E}^{-1} \to \mathcal{E}^0 \to \mathcal{E}^1]$ with $\mathcal{E}^{-1}, \mathcal{E}^0, \mathcal{E}^1$ vector bundles, this is represented by $\ker(\mathbb{V}(\mathcal{E}^{-1}) \to \mathbb{V}(\mathcal{E}^0))$. This group scheme is equipped with an action of $\mathbb{G}_m$ and we can speak about homogeneous Zariski closed subsets inside it.

Corollary 5.5. *For $x : Spec(\Lambda) \to \mathrm{LocSys}_{\widehat{G}}$ we have*

$$x^*\mathrm{Sing}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} = \left\{ v \in \widehat{\mathfrak{g}}^* \otimes_{\mathbb{Z}_\ell} \Lambda \mid \forall \tau \in W_E,\ q^{v(\tau)}\mathrm{Ad}(\varphi(\tau))(\tau.v) = v \right\}.$$

We thus have a natural embedding

$$\mathrm{Sing}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} \hookrightarrow \left[\widehat{\mathfrak{g}}^*/\widehat{G}\right] \times_{B\widehat{G}} \mathrm{LocSys}_{\widehat{G}/\mathbb{Z}_\ell}.$$

Let $\mathcal{N}^*_{\widehat{G}} \subset \mathfrak{g}^*$ be the nilpotent cone. From corollary 5.5 we deduce that after inverting ℓ

$$\mathrm{Sing}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \hookrightarrow \left[\mathcal{N}^*_{\widehat{G}_{\mathbb{Q}_\ell}}/\widehat{G}_{\mathbb{Q}_\ell}\right] \times_{B\widehat{G}_{\mathbb{Q}_\ell}} \mathrm{LocSys}_{\widehat{G}_{\mathbb{Q}_\ell}/\mathbb{Q}_\ell}.$$

Remark 5.6 (Follow up to remark 5.2). As a consequence, after inverting ℓ Arinkin-Gaitsgory singular support condition (see section 6.1) becomes automatic. This again is a simplification compared to the geometric Langlands program for a compact Riemann surface. The reason is again the presence of the Frobenius, in the same way the relation $\varphi N \varphi^{-1} = qN$ implies that the monodromy operator N is automatically nilpotent.

5.2. The Hecke action ([9, Chapter IX.2]).

5.2.1. *Definition.* We suppose here that Λ is a torsion $\mathbb{Z}_\ell[\sqrt{q}]$-algebra to simplify the exposition. For a finite set I there is a diagram

$$\begin{tikzcd} & \mathrm{Hck}^I_G \arrow[r] \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \mathcal{H}ck^I_G \arrow[dr] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times (\mathrm{Div}^1)^I \arrow[r] & (\mathrm{Div}^1)^I \end{tikzcd}$$

where Hck^I_G is the global Hecke stack that is sent to the local one $\mathcal{H}ck^I_G$ from section 4.1. Using the geometric Satake correspondence we deduce by pullback via $\mathrm{Hck}^I_G \to \mathcal{H}ck^I_G$, for each $V \in \mathrm{Rep}_{^LG^I}(\Lambda)$, a complex

$$\mathcal{S}_V \in D_{lis}(\mathrm{Hck}^I_G, \Lambda).$$

This defines a Hecke action

$$T_V = Rp_{2!}(p_1^*(-) \otimes^{\mathbb{L}}_\Lambda \mathcal{S}_V) : D_{lis}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{lis}(\mathrm{Bun}_G \times (\mathrm{Div}^1)^I, \Lambda).$$

Using Drinfeld's lemma (Theorem 4.2) we obtain a functor

$$T_V : D_{lis}(\mathrm{Bun}_G, \Lambda) \longrightarrow D_{lis}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda).$$

This works again with more work when Λ is not torsion.

5.2.2. *Factorization property.*

(1) The properties of the Satake correspondence show that the construction of T_V is functorial in I in the sense that if $\alpha : I \to J$ and $V \in \mathrm{Rep}_{^LG^I}(\Lambda)$ this defines $\alpha_*V \in \mathrm{Rep}_{^LG^J}(\Lambda)$ and T_{α_*V} is T_V composed with the functor

$$D_{lis}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda) \to D_{lis}(\mathrm{Bun}_G \times [*/\underline{W_E^J}], \Lambda)$$

that is the pullback deduced from $\alpha^* : (\mathrm{Div}^1)^J \to (\mathrm{Div}^1)^I$,

$$\begin{tikzcd}
D_{lis}(\mathrm{Bun}_G, \Lambda) \arrow[d, "T_V"] \arrow[dr, "T_{\alpha_* V}"] & \\
D_{lis}(\mathrm{Bun}_G \times [*/\underline{W_E^I}], \Lambda) \arrow[r, "(\mathrm{Id}\times\alpha^*)^*"'] & D_{lis}(\mathrm{Bun}_G \times [*/\underline{W_E^J}], \Lambda).
\end{tikzcd}$$

(2) If $\rho \in \mathrm{Rep}_{W_E^I}(\Lambda)$ it defines a local system $\mathcal{F}_\rho$ on $[*/\underline{W_E^I}]$. Then we have an identification

$$T_{V\otimes\rho} = [(-) \otimes^{\mathbb{L}}_{\Lambda} (\Lambda \boxtimes \mathcal{F}_\rho)] \circ T_V.$$

(3) $T_{V_1\otimes V_2}$ is given by $T_{V_1}(-) \otimes^{\mathbb{L}}_{\Lambda} T_{V_2}(-)$.

Let us denote by $\mathcal{C} = D_{lis}(\mathrm{Bun}_G, \Lambda)$ as a $\mathrm{Rep}_{W_E^I}(\Lambda)$-linear monoidal category. Let

$$\mathrm{End}(\mathcal{C})^{BW_E^I}$$

be the category of functors $F \in \mathrm{End}(\mathcal{C})$ equipped with a morphism $W_E^I \to \mathrm{Aut}(F)$. This is again a monoidal category. If $f : * \to [*/\underline{W_E^I}]$, there is a morphism $W_E^I \to \mathrm{Aut}(f)$. This implies that the pullback functor from $\mathrm{Bun}_G \to \mathrm{Bun}_G \times [*/\underline{W_E^I}]$ has such automorphisms and we deduce from the preceding T_V an element again denoted

$$T_V \in \mathrm{End}(\mathcal{C})^{BW_E^I}.$$

The preceding three properties prove the following.

Theorem 5.7. *The correspondence $V \mapsto T_V$ defines functorially in I a monoidal $\mathrm{Rep}_{W_E^I}(\Lambda)$-linear functor*

$$\mathrm{Rep}_\Lambda({}^LG)^I \longrightarrow \mathrm{End}(\mathcal{C})^{BW_E^I}.$$

This is the data we use to construct the morphism between the centers.

5.3. The morphism between the centers.

5.3.1. *The morphism between the centers.* Let $\Lambda \in \{\mathbb{Z}_\ell[\sqrt{q}], \mathbb{Q}_\ell[\sqrt{q}]\}$. In the case of $\mathbb{Z}_\ell[\sqrt{q}]$ we will moreover assume that $\ell \gg 0$ with an explicit bound (a "very good prime for $\widehat{G}$"; all primes for GL_n, and $\ell \neq 2$ for classical groups, see the introduction to [**9**, Chapter VII] for an explicit definition).

We base change $\mathrm{LocSys}_{\widehat{G}}$ from $\mathbb{Z}_\ell$ to Λ. Consider the Λ-algebra

$$\mathfrak{Z}^{spec}(G, \Lambda) = \mathcal{O}(Z^1(W_E, \widehat{G}))^{\widehat{G}}$$

(spectral stable Bernstein center) that is the center of the category of coherent sheaves on $\mathrm{LocSys}_{\widehat{G}}$ and the categorical center

$$\mathfrak{Z}^{geo}(G, \Lambda) = \mathfrak{Z}\big(D_{lis}(\mathrm{Bun}_G, \Lambda)\big)$$

(geometric stable Bernstein center). We explain in this section how to construct from theorem 5.7 a morphism

$$\mathfrak{Z}^{spec}(G, \Lambda) \longrightarrow \mathfrak{Z}^{geo}(G, \Lambda).$$

Since $D(G(E),\Lambda)$ is a direct factor of $D_{lis}(\mathrm{Bun}_G,\Lambda)$ there is a morphism

$$\mathfrak{Z}^{geo}(G,\Lambda) \longrightarrow \mathfrak{Z}(G(E),\Lambda)$$

toward the usual *Bernstein center*. Composed with the preceding morphism we obtain a morphism

$$\mathfrak{Z}^{spec}(G,\Lambda) \longrightarrow \mathfrak{Z}(G(E),\Lambda).$$

Using theorem 5.3 we deduce the announced construction $\pi \mapsto \varphi_\pi$ of semi-simple parameters, see Theorem 5.10.

5.3.2. *The algebra of excursion operators.* Here we work over $\mathbb{Z}_\ell$. Let us fix an open subgroup P of the wild inertia of W_E that acts trivially on $\widehat{G}$. We consider the open/closed subscheme

$$Z^1(W_E/P,\widehat{G}) \subset Z^1(W_E,\widehat{G}).$$

The proof of theorem 5.1 shows that we can replace W_E/P by a finite type discrete subgroup W (we essentially replace $I_E/P_E = \widehat{\mathbb{Z}}^p(1)$ by $\mathbb{Z}[\frac{1}{q}](1)$) so that

$$Z^1(W_E/P,\widehat{G}) = Z^1(W,\widehat{G}).$$

To make this explicit we consider the small category $\mathfrak{F}$ whose objects are couples $(n, F_n \to W)$ where $n \in \mathbb{N}_{\geq 1}$, F_n is the free group on n-elements, and $F_n \to W$ is a morphism. Morphisms between $(n, F_n \to W)$ and $(m, F_m \to W)$ are given by morphisms $F_n \to F_m$ such that the diagram $\begin{array}{ccc} F_n & \longrightarrow & W \\ \downarrow & \nearrow & \\ F_m & & \end{array}$ commutes. Then one has for evident reasons an isomorphism of $\mathbb{Z}_\ell$-algebras equipped with an algebraic action of $\widehat{G}$

$$\varinjlim_{(n,F_n)\in\mathfrak{F}} \mathcal{O}(Z^1(F_n,\widehat{G})) \xrightarrow{\sim} \mathcal{O}(Z^1(W,\widehat{G})).$$

Let us define

$$\mathrm{Exc}(W,\widehat{G}) := \varinjlim_{(n,F_n)\in\mathfrak{F}} \mathcal{O}(Z^1(F_n,\widehat{G}))^{\widehat{G}}.$$

The category $\mathfrak{F}$ is not cofiltered but only sifted (colimits indexed by this category commute with finite product but not with finite limits) and the morphism

$$\mathrm{Exc}(W,\widehat{G}) \longrightarrow \mathcal{O}(Z^1(W,\widehat{G}))^{\widehat{G}}$$

is à priori only an isomorphism after inverting ℓ since then taking $\widehat{G}$-invariants is an exact functor. Haboush's theorem on $\widehat{G}$-invariants says that *this is a universal homeomorphism between $\mathbb{Z}_\ell$-algebras.* Nevertheless, we prove the following result using results and methods of modular representation theory for the algebraic group $\widehat{G}$. This is the consequence of a more important result we will explain later.

THEOREM 5.8. *If ℓ is a very good prime then* $\mathrm{Exc}(W,\widehat{G}) \xrightarrow{\sim} \mathcal{O}(Z^1(W,\widehat{G}))^{\widehat{G}}$.

5.3.3. *Excursion operators and the center ([**9**, Chapter VIII.4]).* Let us fix a finite quotient Q of W_E through which the action on $\widehat{G}$ factorizes. We work here over any $\mathbb{Z}_\ell$-algebra Λ.

THEOREM 5.9. *Let $\mathcal{C}$ be any Λ-linear idempotent complete category. Suppose given functorially in the finite set I a monoidal $\mathrm{Rep}_\Lambda Q^I$-linear functor*

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \longrightarrow \mathit{End}(\mathcal{C})^{BW^I}.$$

We can then construct a morphism

$$\mathrm{Exc}(W, \widehat{G}) \longrightarrow \mathfrak{Z}(\mathcal{C})$$

where $\mathfrak{Z}(\mathcal{C}) = \mathit{End}(\mathrm{Id}_\mathcal{C})$ is the Bernstein center of $\mathcal{C}$.

In fact, given an element of $\mathfrak{F}$, we can rewrite $\mathcal{O}(Z^1(F_n, \widehat{G}))^{\widehat{G}}$ as

$$\mathcal{O}\big((\widehat{G})^n \mathbin{/\!\!/} \widehat{G}\big)$$

where the action of $\widehat{G}$ on $(\widehat{G})^n$ is diagonal twisted, $g.(g_1, \ldots, g_n) = (gg_1g^{-\tau_1}, \ldots, gg_ng^{-\tau_n})$ if $F_n \to W$ is given by $(\tau_1, \ldots, \tau_n)$. Let A be any Λ-algebra. To give oneself a morphism

$$\mathcal{O}\big((\widehat{G})^n \mathbin{/\!\!/} \widehat{G}\big) \longrightarrow A$$

is the same as to give a morphism

$$\mathcal{O}\big((\widehat{G} \rtimes Q)^n \mathbin{/\!\!/} \widehat{G}\big) \longrightarrow \mathrm{Map}(W^n, A)$$

linear over $\mathcal{O}(Q^n) \to \mathrm{Map}(W^n, A)$. Now we add one more variable to rewrite this: the pullback via $(g_1, \ldots, g_n) \mapsto (1, g_1, \ldots, g_n)$ is a morphism $\mathcal{O}\big((\widehat{G} \rtimes Q)^{n+1}\big) \to \mathcal{O}\big((\widehat{G} \rtimes Q)^n\big)$ that induces an isomorphism

$$\mathcal{O}\big(\widehat{G}\backslash(\widehat{G} \rtimes Q)^{n+1}/\widehat{G}\big) \otimes_{\mathcal{O}(Q^{n+1})} \mathcal{O}(Q^n) \xrightarrow{\sim} \mathcal{O}\big((\widehat{G} \rtimes Q)^n \mathbin{/\!\!/} \widehat{G}\big).$$

We are now reduced to give a morphism

$$\mathcal{O}\big(\widehat{G}\backslash(\widehat{G} \rtimes Q)^{n+1}/\widehat{G}\big) \longrightarrow \mathrm{Map}(W^{n+1}, A)$$

linear over $\mathcal{O}(Q^{n+1})$.

Denote by $V \mapsto T_V$ the monoidal functor from theorem 5.9. Now, suppose given a quintuplet $\mathcal{D} = (I, V, \alpha, \beta, \gamma)$ as in [**17**] where

(1) I is a finite set,
(2) $V \in \mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I$,
(3) $\gamma \in W^I$,
(4) $\mathbf{1} \xrightarrow{\alpha} V_{|\widehat{G}}$ (diagonal restriction via $\widehat{G} \subset \widehat{G}^I \subset (\widehat{G} \rtimes Q)^I$),
(5) $V_{|\widehat{G}} \xrightarrow{\beta} \mathbf{1}$.

We then define the excursion operator associated to $\mathcal{D}$ as

$$\mathrm{Id}_\mathcal{C} = T_\mathbf{1} \xrightarrow{T_\alpha} T_V \xrightarrow{\gamma} T_V \xrightarrow{T_\beta} T_\mathbf{1} = \mathrm{Id}_\mathcal{C}$$

where we use the functoriality $\emptyset \to I$, resp. $I \to \emptyset$, for T_α, resp. T_β. Varying γ, the quadruple (I, V, α, β) gives rise to an application

$$S(V, \alpha, \beta) : W^I \to \mathrm{End}(\mathrm{Id}).$$

Let us note now if $g \in (\widehat{G} \rtimes Q)^I$ it defines a scalar

$$\mathbf{1} \xrightarrow{\alpha} V \xrightarrow{g} V \xrightarrow{\beta} \mathbf{1}.$$

Varying g this defines an element of $f(I, V, \alpha, \beta) \in \mathcal{O}(\widehat{G} \backslash (\widehat{G} \rtimes Q)^I / \widehat{G})$. The morphism from theorem 5.9 is then constructed by setting

$$f(V, \alpha, \beta) \longmapsto S(V, \alpha, \beta)$$

and verifying different compatibilities.

5.3.4. *L-parameters.* We thus construct morphisms over $\Lambda = \mathbb{Q}_\ell[\sqrt{q}]$, resp. $\Lambda = \mathbb{Z}_\ell[\sqrt{q}]$ for ℓ very good for $\widehat{G}$,

$$\underbrace{\mathfrak{Z}^{spec}(G, \Lambda)}_{\text{spectral stable center}} \longrightarrow \underbrace{\mathfrak{Z}^{geo}(G, \Lambda)}_{\text{geometric stable center}} \longrightarrow \underbrace{\mathfrak{Z}(G(E), \Lambda)}_{\text{Bernstein center}}.$$

If ℓ is not very good for $\widehat{G}$ this is only defined up to a universal homeomorphism. This is a generalization of the work of Helm and Moss about the local Langlands correspondence in families ([**12**]). Using the preceding morphisms bwetween centers we prove the following theorem.

THEOREM 5.10. *Let L be either $\overline{\mathbb{Q}}_\ell$ or $\overline{\mathbb{F}}_\ell$. Let π be a smooth irreducible representation of $G(E)$ with coefficients in L. We can construct its semi-simple Langlands parameter φ_π. It satisfies moreover:*

1. *It is compatible with parabolic induction.*
2. *Is is compatible with Weil restriction of scalars.*
3. *It is compatible with products, $\varphi_{\pi_1 \boxtimes \pi_2} = \varphi_{\pi_1} \times \varphi_{\pi_2}$.*
4. *It is given by local class field for tori if G is a torus.*
5. *It coincides with the semi-simple local Langlands correspondence for GL_n (*[**11**], [**13**], [**27**], [**24**]*).*

Point (5) is checked using the compatibility with the cohomology of Lubin-Tate spaces. This is where the connection with the work of Harris and Taylor ([**11**]) is done. Let us note moreover that the compatibility with the Gan-Takeda local Langlands correspondence for Gsp_4 has been checked in [**10**].

5.3.5. *Independence of ℓ.* The following conjecture is natural. Let us note that both $\mathfrak{Z}^{\mathrm{Spec}}(G)$ (and even $\mathrm{LocSys}_{\widehat{G}}$) and the usual Bernstein center $\mathfrak{Z}(G(E))$ are naturally defined as flat $\mathbb{Z}[\frac{1}{p}]$-algebras.

CONJECTURE 5.11. *Let N be the product of p and the primes ℓ that are not a very good prime for $\widehat{G}$. There is a morphism of $\mathbb{Z}[\frac{1}{N}]$-algebras*

$$\mathfrak{Z}^{spec}(G, \mathbb{Z}[\tfrac{1}{N}]) \longrightarrow \mathfrak{Z}(G(E), \mathbb{Z}[\tfrac{1}{N}])$$

inducing the preceding morphisms between centers for all $\ell \neq p$ a very good prime for $\widehat{G}$.

5.4. The spectral action. Let $\Lambda \in \{\mathbb{Z}_\ell[\sqrt{q}], \mathbb{Q}_\ell[\sqrt{q}]\}$. If $\Lambda = \mathbb{Z}_\ell[\sqrt{q}]$ suppose ℓ is a very good prime. We explain now how to upgrade the construction of the morphism

$$\mathfrak{Z}^{spec}(G, \Lambda) \longrightarrow \mathfrak{Z}^{geo}(G, \Lambda)$$

that allows us to construct the semi-simple L-parameters. This will take into account automorphisms of parameters and for this we will work in a higher categorical framework.

5.4.1. *Modular representation theory (*[**9**, Chapter VIII.5]. The main result here is the following. Here we work over $\mathbb{Z}_\ell$.

THEOREM 5.12. *Assume ℓ is a very good prime for $\widehat{G}$. Then the morphism*

$$\varinjlim_{(n,F_n)\in\mathfrak{F}} \mathcal{O}(Z^1(F_n,\widehat{G})) \longrightarrow \mathcal{O}(Z^1(W_E,\widehat{G}))$$

is an isomorphism in the presentable stable ∞-category $\mathrm{Ind}\,\mathrm{Perf}(B\widehat{G})$.

Here $\mathrm{Perf}(B\widehat{G})$ is the ∞-category of perfect complexes on the algebraic stack $B\widehat{G}$. Its homotopy category is the one of bounded complexes of algebraic representations of $\widehat{G}$ on finite free $\mathbb{Z}_\ell$-modules. Both objects $\mathcal{O}(Z^1(F_n,\widehat{G}))$ and $\mathcal{O}(Z^1(W_E,\widehat{G}))$ can be seen as Ind-perfect complexes in a canonical way by writing them as inductive limits of their sub-$\widehat{G}$-stable $\mathbb{Z}_\ell$-modules of finite type. The theorem says that there are "no higher derived functors" of $\varinjlim_{\mathfrak{F}}$ when applied to $((\mathcal{O}(F_n,\widehat{G}))_{(n,F_n)\in\mathfrak{F}}$. It implies immediately theorem 5.8. In fact, this implies that for all $i\geq 0$,

$$\varinjlim_{(n,F_n)\in\mathfrak{F}} H^i(\widehat{G},\mathcal{O}(Z^1(F_n,\widehat{G})) \xrightarrow{\sim} H^i(\widehat{G},\mathcal{O}(Z^1(W_E,\widehat{G}))).$$

This allows us to prove that

$$H^i(\widehat{G},\mathcal{O}(Z^1(W_E,\widehat{G}))) = 0 \text{ for } i>0$$

too, the result for $\mathcal{O}(Z^1(F_n,\widehat{G})$ being easily deduced from some already known results of modular representation theory.

The preceding result is even straightened in the following way.

THEOREM 5.13. *Suppose either we work over $\mathbb{Q}_\ell$, or over $\mathbb{Z}_\ell$ and ℓ is a very good prime for $\widehat{G}$. The ∞-category* $\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$ *is generated under cone and retracts by* $\mathrm{Perf}(B\widehat{G})$.

This is what we use to define the spectral action.

5.4.2. *The spectral action.* Using higher categorical methods, theorems 5.12 and 5.13, we can strengthen theorem 5.9 in the following way. Let Λ be the ring of integers in a finite degree extension of $\mathbb{Q}_\ell[\sqrt{q}]$. We fix a finite quotient of W_E through which the action on $\widehat{G}$ factorizes.

THEOREM 5.14. *Assume ℓ is a very good prime for $\widehat{G}$. Let $\mathcal{C}$ be a small idempotent complete Λ-linear ∞-category. Then the following data are equivalent:*

(1) *To give oneself functorially in the finite set I an exact* $\mathrm{Rep}_\Lambda(Q^I)$*-linear monoidal functor*

$$\mathrm{Rep}_\Lambda((\widehat{G}\rtimes Q)^I) \longrightarrow \mathrm{End}_\Lambda(\mathcal{C})^{BW_E^I}.$$

(2) *A Λ-linear action of* $\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$ *on $\mathcal{C}$ such that for each $X\in\mathcal{C}$ the associated action on X factorizes through* $\mathrm{Perf}(Z^1(W_E/P,\widehat{G})_\Lambda/\widehat{G}_\Lambda)$ *for some open subgroup P of the wild inertia of W_E .*

The same statement holds over a finite degree extension of $\mathbb{Q}_\ell[\sqrt{p}]$ without any restriction on ℓ.

In the preceding $[Z^1(W_E/P, \widehat{G})_\Lambda/\widehat{G}\,]$ is an open/closed quasicompact substack of $\mathrm{LocSys}_{\widehat{G}}$. We use the terminology "compactly supported action" for the associated condition about the action on X in the theorem.

In this theorem the correspondence from (2) to (1) is given by the (evident) $\mathrm{Rep}(Q^I)$-linear monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G} \rtimes Q)^I \longrightarrow \mathrm{Perf}(Z^1(W_E, \widehat{G})^I_\Lambda/\widehat{G}_\Lambda)^{BW_E^I}$$

composed with the spectral action.

All the objects appearing in theorem 5.7 have a natural ∞-categorical upgrade. In particular we define a stable Λ-linear ∞-category

$$\mathcal{D}_{lis}(\mathrm{Bun}_G, \Lambda)$$

whose homotopy category is $D_{lis}(\mathrm{Bun}_G, \Lambda)$. This allows us to prove the following theorem.

Theorem 5.15 (spectral action). *Suppose either Λ is an extension of $\mathbb{Q}_\ell[\sqrt{q}]$ or the ring of integers in a finite degree extension of $\mathbb{Q}_\ell[\sqrt{q}]$, in which case we suppose ℓ is very good for $\widehat{G}$. There is then a natural Λ-linear compactly supported action of* $\mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}}/\Lambda)$ *on* $\mathcal{D}_{lis}(\mathrm{Bun}_G, \Lambda)^\omega$.

As before this action is uniquely constrained by its compatibility with the action of the Hecke operators. If we forget the W_E-action the Hecke operator T_V associated to $V \in \mathrm{Rep}_\Lambda(\widehat{G})$ is deduced from the morphism

$$\mathrm{LocSys}_{\widehat{G}} \longrightarrow B\widehat{G}$$

which induces a monoidal functor

$$\mathrm{Rep}_\Lambda(\widehat{G}) \longrightarrow \mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}}).$$

5.4.3. *Application to cuspidal parameters.* Let us give an example of application of the spectral action. We place ourselves over $\overline{\mathbb{Q}}_\ell$. By definition, an L-parameter $\varphi : W_E \to {}^LG(\overline{\mathbb{Q}}_\ell)$ is *cuspidal* if it is semi-simple and

$$S_\varphi/Z(\widehat{G})^\Gamma$$

is finite. One verifies that this defines a connected component

$$C_\varphi \subset \mathrm{LocSys}_{\widehat{G}}$$

which is the open/closed substack of unramified twists of φ. There is a morphism

$$[\mathrm{Spec}(\overline{\mathbb{Q}}_\ell)/S_\varphi] \longrightarrow C_\varphi \subset \mathrm{LocSys}_{\widehat{G}}$$

that is a closed immersion. The morphism $\mathfrak{Z}^{spec} \to \mathfrak{Z}^{geo}$ sends the idempotent associated to C_φ to an idempotent in $\mathfrak{Z}^{geo}$. This defines a direct summand

$$D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega[\varphi] \subset D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega.$$

Let us analyse this. For any $A \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)[\varphi]$ Schur irreducible (i.e. $\mathrm{End}(A) = \overline{\mathbb{Q}}_\ell$), the excursion operators act via scalars on A as determined by an unramified twist of φ. They act via the same character of the excursion algebra on $(i^b)^*A$ for all $[b] \in B(G)$. By compatibility of the construction of the Langlands parameters with parabolic induction coupled with the cuspidality of φ (it does not factorizes through any parabolic subgroup of ${}^LG$), we deduce that $(i^b)^*A = 0$ if b is not basic.

From this argument let's notice we already get the following result.

THEOREM 5.16 (cleanliness of cuspidal parameters). *Let π be an irreducible representation of $G(E)$ such that φ is cuspidal. Then π is supercuspidal and*

$$(i^1)_!\mathcal{F}_\pi = R(i^1)_*\mathcal{F}_\pi.$$

Suppose now $Z(\widehat{G})^\Gamma$ is finite to simplify. From the preceding argument we can deduce that, via $(i^1)_!$,

$$D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)[\varphi] = \bigoplus_{[b] \text{ basic}} \bigoplus_{\substack{\pi \text{ supercusp. of } G_b(E) \\ \varphi_\pi = \varphi}} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell) \otimes \pi.$$

We now use the spectral action: *there is a monoidal action of* $\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\varphi)$ *on* $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)[\varphi]$. For any $\rho \in \mathrm{Irr}_{\overline{\mathbb{Q}}_\ell}(S_\varphi)$ one has

$$\rho_{|Z(\widehat{G})^\Gamma} \in X^*(Z(\widehat{G})^\Gamma) = \pi_1(G)_\Gamma.$$

By construction of the spectral action, for $[b]$ basic if $[b']$ is basic with

$$\kappa(b') = \kappa(b) + \rho_{|Z(\widehat{G})^\Gamma}$$

one has

$$\rho * (-) : \bigoplus_{\substack{\pi \text{ supercusp. of } G_b(E) \\ \varphi_\pi = \varphi}} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell) \otimes \pi \longrightarrow \bigoplus_{\substack{\pi \text{ supercusp. of } G_{b'}(E) \\ \varphi_\pi = \varphi}} \mathrm{Perf}(\overline{\mathbb{Q}}_\ell) \otimes \pi.$$

This shift is a form of Jacquet-Langlands correspondence.

Inspired by this we formulate the following conjecture.

CONJECTURE 5.17. *Suppose G is quasisplit and fix a Whittaker datum (B, ψ).*

(1) *There is a unique irreducible representation π of $G(E)$ with parameter φ that is generic with respect to (B, ψ).*
(2) *The monoidal action*

$$\begin{aligned} \mathrm{Perf}(\mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\varphi)) = \bigoplus_{n\in\mathbb{Z}} \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(S_\varphi)[n] &\longrightarrow D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega[\varphi] \\ \rho[n] &\longmapsto \rho * \pi[n] \end{aligned}$$

is an equivalence.

In the next section we extend this conjecture from $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega[\varphi]$, φ cuspidal, to the entire category $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)^\omega$.

6. The categorical geometrization conjecture

In this section we explain the main conjecture of [**9**] and give some of its consequences.

6.1. Arinkin-Gaitsgory singular support condition ([9, Chapter VIII.2.2]). Let $\mathfrak{X} \to S$ be a locally complete intersection algebraic stack over the regular scheme S. We can look at its cotangent complex

$$\mathbb{L}_{\mathfrak{X}/S} \in \mathrm{Perf}^{[-1,1]}(\mathcal{O}_{\mathfrak{X}}).$$

Arinkin and Gaitsgory ([2]) define the stack of singularities

$$\mathrm{Sing}_{\mathfrak{X}/S} := \mathcal{H}^{-1}(\mathbb{L}_{\mathfrak{X}/S}) \longrightarrow \mathfrak{X}.$$

This is a commutative group scheme over $\mathfrak{X}$ equipped with an action of $\mathbb{G}_m$.

For $\mathcal{E} \in D^b_{coh}(\mathcal{O}_{\mathfrak{X}})$ they define its singular support

$$\mathrm{SingSupp}(\mathcal{E}) \subset \mathrm{Sing}_{\mathfrak{X}/S}$$

a Zariski closed $\mathbb{G}_m$-invariant, i.e. conical, subset of the stack of singularities. This is some kind of "microsupport" in the coherent context. Its image in $\mathfrak{X}$ is the support of $\mathcal{E}$. They prove the following result.

THEOREM 6.1 (Arinkin-Gaitsgory). *The singular support* $\mathrm{SingSupp}(\mathcal{E})$ *is contained in the zero section* $\{0\} \subset \mathrm{Sing}_{\mathfrak{X}/S}$ *if and only if* $\mathcal{E}$ *is a perfect complex.*

This is a coherent analog of the fact that the characteristic cycle of a perverse sheaf is contained in the zero section if and only if it is a local system.

Now, see section 5.1.3, we have

$$\mathrm{Sing}_{\mathrm{LocSys}_{\widehat{G}}/\mathbb{Z}_\ell} \subset [\mathfrak{g}^*/\widehat{G}] \times_{B\widehat{G}} \mathrm{LocSys}_{\widehat{G}}.$$

Let us define

$$\mathrm{Coh}_{Nilp}(\mathrm{LocSys}_{\widehat{G}}) \subset D^b_{coh}(\mathrm{LocSys}_{\widehat{G}}, \mathcal{O})$$

to be the subcategory of complexes whose support is quasicompact, i.e. supported on a finite set of connected components of $\mathrm{LocSys}_{\widehat{G}}$, and whose singular support is contained in the nilpotent cone

$$[\mathcal{N}^*_{\widehat{G}}/\widehat{G}] \times_{B\widehat{G}} \mathrm{LocSys}_{\widehat{G}}$$

Let us notice that this condition is automatic after inverting ℓ.

6.2. The conjecture ([9, Chapter X.3]). Suppose G is quasisplit. Let U be the unipotent radical of a Borel subgroup of G and

$$\psi : U(E) \to \overline{\mathbb{Z}}_\ell^\times$$

be a non-degenerate character. Consider the Whittaker sheaf

$$\mathcal{W}_\psi = (i^1)_!\big(\text{c-Ind}_{U(E)}^{G(E)}\psi\big).$$

This is not a compact object of $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$. Nevertheless we can still define for $\mathcal{F} \in \mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}})$, the spectral action of $\mathcal{F}$ against this object $\mathcal{F} * \mathcal{W}_\psi \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)$ by writing $\text{c-Ind}_{U(E)}^{G(E)}\psi$ as a colimit of finite type representation.

The following conjecture is an upgrade of conjecture 5.17 that was some kind of "toy model" for this one. We work integrally and thus suppose ℓ is a very good prime for $\widehat{G}$. We make the same conjecture over $\overline{\mathbb{Q}}_\ell$ without this restriction on the prime ℓ.

Conjecture 6.2 (Categorical geometrization conjecture). *The functor*

$$\begin{aligned} \mathrm{Perf}(\mathrm{LocSys}_{\widehat{G}}) &\longrightarrow D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell) \\ \mathcal{F} &\longmapsto \mathcal{F} * \mathcal{W}_\psi \end{aligned}$$

takes values in compact objects when restricted to perfect complexes with quasicompact support and extends to an equivalence

$$\mathrm{Coh}_{Nilp}(\mathrm{LocSys}_{\widehat{G}}) \xrightarrow{\sim} D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Z}}_\ell)^\omega$$

compatible with the spectral action.

This is the ultimate form of the local Langlands correspondence as we envision it.

6.3. Some consequences. Let us note now some consequences of the categorical geometrization conjecture.

6.3.1. *Identification between the stable centers.* Let $\Lambda \in \{\mathbb{Z}_\ell[\sqrt{q}], \mathbb{Q}_\ell[\sqrt{q}]\}$. In the integral case we moreover suppose that ℓ is a very good prime for $\widehat{G}$. The full faithfulness part of the geometrization conjecture implies that the composite

$$\begin{tikzcd} \mathfrak{Z}^{spec}(G,\Lambda) \arrow[r] \arrow[rr, bend right, "\simeq"'] & \mathfrak{Z}^{geo}(G,\Lambda) \arrow[r] & \mathrm{End}\big(c\text{--}\mathrm{Ind}_{U(E)}^{G(E)}\psi\big) \end{tikzcd}$$

is an isomorphism. One can moreover hope to describe this center in terms of stable distributions but this is not linked to our work.

6.3.2. *Kernel of functoriality (*[**9**, Chapter X.1]*).* Here we work over $\overline{\mathbb{Q}}_\ell$. Let H and G be quasi-split reductive groups over E. We fix Whittaker data for both groups. Suppose given an L-morphism

$$f : {}^L H \longrightarrow {}^L G.$$

The categorical conjecture implies that the functoriality given by the morphism $f_* : \mathrm{LocSys}_{\widehat{H}} \to \mathrm{LocSys}_{\widehat{G}}$ on the spectral side would give rise to a functor on the geometric side

$$D_{lis}(\mathrm{Bun}_H, \overline{\mathbb{Q}}_\ell) \longrightarrow D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell).$$

We prove that such a functor is automatically given by a kernel

$$A_f \in D_{lis}(\mathrm{Bun}_H \times \mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$$

that is a kernel of functoriality. Reprojecting to the "classical representation theoretic part" we obtain moreover a functor $D(H(E), \overline{\mathbb{Q}}_\ell) \to D(G(E), \overline{\mathbb{Q}}_\ell)$.

Theorem 6.3. *The categorical geometrization conjecture implies the local Langlands functoriality for quasi-split reductive groups as a functor $D(H(E), \overline{\mathbb{Q}}_\ell) \to D(G(E), \overline{\mathbb{Q}}_\ell)$ associated to an L-morphism ${}^L H \to {}^L G$. This is given by a "natural" kernel $A \in D_{lis}(\mathrm{Bun}_H \times \mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ associated to such an L-morphism.*

7. Some final thoughts

At the end, it looks like the natural objects to which the local Langlands program applies are not smooth representations of $G(E)$ but rather objects of $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. Typically, to $A \in D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ Schur irreducible we can attach its semi-simple Langlands parameter

$$\varphi_A : W_E \to {}^L G(\overline{\mathbb{Q}}_\ell).$$

Moreover, see section 3.4, the notions of finite type/admissible/Zelevinsky involution extend naturally to geometric notions in $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$. As seen before, local Langlands functoriality is naturally defined by a kernel at the level of $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$.

This asks the following question: *are automorphic representations the natural objects to which the global Langlands program applies ?* As we already saw in the local case, from the representation theoretic point of view the natural objects are not representations of $G(E)$ but rather of all $G_b(E)$, $[b] \in B(G)$, together simultaneously. A global Kottwitz set exists ([**15**]) and it is natural to ask if we should not consider automorphic representations of all the associated G_b's simultaneously ?

From the $D_{lis}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ point of view, a global curve does not exist and the situation is more mysterious. Nevertheless let us point that it still remains to find an archimedean analog of the preceding work.

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Insitut de Mathématiques de Jussieu, 4 place Jussieu, 75252 Paris Cedex 05
Email address: `laurent.fargues@imj-prg.fr`

MPI für Mathematik, Vivatsgasse 7, 53111 Bonn, Germany
Email address: `scholze@math.uni-bonn.de`