

Quantum Groups, the loop Grassmannian, and the Springer resolution.

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Abstract

We establish equivalences of the following three triangulated categories:

$$D_{\text{quantum}}(\mathfrak{g}) \longleftrightarrow D_{\text{coherent}}^G(\tilde{\mathcal{N}}) \longleftrightarrow D_{\text{perverse}}(\text{Gr}).$$

Here, $D_{\text{quantum}}(\mathfrak{g})$ is the derived category of the principal block of finite dimensional representations of the quantized enveloping algebra (at an odd root of unity) of a complex semisimple Lie algebra \mathfrak{g} ; the category $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$ is defined in terms of coherent sheaves on the cotangent bundle on the (finite dimensional) flag manifold for G (= semisimple group with Lie algebra \mathfrak{g}), and the category $D_{\text{perverse}}(\text{Gr})$ is the derived category of perverse sheaves on the Grassmannian Gr associated with the loop group LG^\vee , where G^\vee is the Langlands dual group, smooth along the Schubert stratification.

The equivalence between $D_{\text{quantum}}(\mathfrak{g})$ and $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$ is an ‘enhancement’ of the known expression (due to Ginzburg-Kumar) for quantum group cohomology in terms of nilpotent variety. The equivalence between $D_{\text{perverse}}(\text{Gr})$ and $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$ can be viewed as a ‘categorification’ of the isomorphism between two completely different geometric realizations of the (fundamental polynomial representation of the) affine Hecke algebra that has played a key role in the proof of the Deligne-Langlands-Lusztig conjecture. One realization is in terms of locally constant functions on the flag manifold of a p -adic reductive group, while the other is in terms of equivariant K -theory of a complex (Steinberg) variety for the dual group.

The composite of the two equivalences above yields an equivalence between *abelian* categories of quantum group representations and perverse sheaves. A similar equivalence at an even root of unity can be deduced, following Lusztig program, from earlier deep results of Kazhdan-Lusztig and Kashiwara-Tanisaki. Our approach is independent of these results and is totally different (it does not rely on representation theory of Kac-Moody algebras). It also gives way to proving Humphreys’ conjectures on tilting $U_q(\mathfrak{g})$ -modules, as will be explained in a separate paper.

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1 Introduction

1.1 Main players. Most of the contents of this paper may be roughly summed-up in the following diagram of category equivalences (where ‘ Q ’ stands for ”quantum”, and ‘ P ’ stands for

”perverse”):

$$\begin{array}{ccccc}
 & & \text{Lusztig multiplicity} & & \\
 & \nearrow & & \nwarrow & \\
 D^b \text{block}^{\text{mix}}(\mathbf{U}) & \xrightleftharpoons{Q} & D^b \text{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}}) & \xleftarrow[\text{[CG]-conjecture}]{P} & D^b \text{Perv}^{\text{mix}}(\text{Gr})
 \end{array} \quad (1.1.1)$$

In this diagram, G is a connected complex semisimple group of adjoint type with Lie algebra \mathfrak{g} . We fix a Borel subgroup $B \subset G$, write $\mathfrak{b} = \text{Lie } B \subset \mathfrak{g}$ for the corresponding Borel subalgebra, and \mathfrak{n} for the nilradical of \mathfrak{b} . Let $\tilde{\mathcal{N}} := G \times_B \mathfrak{n}$ be the Springer resolution, and $\text{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$ the abelian category of $G \times \mathbb{C}^*$ -equivariant coherent sheaves on $\tilde{\mathcal{N}}$, where the group G acts on $\tilde{\mathcal{N}}$ by conjugation and \mathbb{C}^* acts by dilations along the fibers. Further, let \mathbf{U} be the quantized universal enveloping algebra of \mathfrak{g} specialized at a root of unity. The category $\text{block}^{\text{mix}}(\mathbf{U})$ on the left of (1.1.1) stands for a mixed version, see [BGS, Definition 4.3.1] or Sect. 9.2 below, of the abelian category of finite-dimensional \mathbf{U} -modules in the linkage class of the trivial 1-dimensional module. Finally, we write $D^b \mathcal{C}$ for the bounded derived category of an abelian category \mathcal{C} .

Forgetting part of the structure one may consider, instead of $\text{block}^{\text{mix}}(\mathbf{U})$, the category $\text{block}(\mathbf{U})$ of actual (non-mixed) \mathbf{U} -modules as well. Forgetting the mixed structure on the left of diagram (1.1.1) corresponds to forgetting the \mathbb{C}^* -equivariance in the middle term of (1.1.1), i.e., to replacing $G \times \mathbb{C}^*$ -equivariant sheaves on $\tilde{\mathcal{N}}$ by G -equivariant ones. Although this sort of simplification may look rather attractive, the resulting triangulated category $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$ that will have to replace the middle term in the diagram above will no longer be the derived category of the corresponding abelian category $\text{Coh}^G(\tilde{\mathcal{N}})$ and, in effect, of any abelian category. This subtlety is rather technical; the reader may ignore it at first reading.

Finally, let G^\vee denote the complex connected and simply-connected semisimple group dual to G in the sense of Langlands. We write Gr for the loop Grassmannian of G^\vee . The Grassmannian has a standard stratification by Iwahori (= affine Borel) orbits. The strata, usually called Schubert cells, are isomorphic to finite dimensional affine-linear spaces. We let $\text{Perv}(\text{Gr})$ denote the abelian category of perverse sheaves on Gr which are constructible with respect to this stratification, and we write $\text{Perv}^{\text{mix}}(\text{Gr})$ for its mixed counterpart, the category of mixed ℓ -adic perverse sheaves, see [BBD].

The main result of the paper says that all three categories in (1.1.1) are equivalent as triangulated categories. Furthermore, we show that the composite equivalence $P^{-1} \circ Q$ is compatible with the natural t -structures on the categories on the LHS and RHS of (1.1.1), hence induces equivalences of abelian categories:

$$\text{block}(\mathbf{U}) \simeq \text{Perv}(\text{Gr}). \quad (1.1.2)$$

This yields, in particular, the conjecture formulated in [GK, §4.3], relating quantum group cohomology to perverse sheaves on the loop Grassmannian.

The equivalence in (1.1.2) also provides character formulas, conjectured by Lusztig [L3] and referred to as ”Lusztig multiplicity” formulas in (1.1.1), for simple \mathbf{U} -modules in the principal block in terms of intersection homology sheaves on the loop Grassmannian. Very similar character formulas have been proved earlier by combining several known deep results due to Kazhdan-Lusztig [KL2], Kashiwara-Tanisaki [KT].

1.2 Relation to results by Kazhdan-Lusztig and Kashiwara-Tanisaki. For each negative rational number k (=’level’), Kashiwara and Tanisaki consider an abelian category $\text{Mod}_k^G(\mathcal{D})$ of G -equivariant holonomic modules over \mathcal{D} , a sheaf of twisted differential operators on the affine flag variety, with a certain monodromy determined by (the denominator of) k . This category is equivalent via the Riemann-Hilbert correspondence to $\text{Perv}_k(\text{Gr}^G)$, a category of monodromic

perverse sheaves. The latter category is defined similarly to the category $\mathcal{Perv}(\mathrm{Gr})$ considered in (1.1.2), with the following two differences:

- The Grassmannian Gr^G stands for the loop Grassmannian for the group G rather than for the Langlands dual group G^\vee ; and
- The objects of $\mathcal{Perv}_k(\mathrm{Gr}^G)$ are perverse sheaves not on the Grassmannian Gr^G itself but on the total space of a \mathbb{C}^* -bundle (so-called *determinant bundle*) on the Grassmannian, with monodromy along the fibers (determined by the rational number k).

Further, let $\widehat{\mathfrak{g}}$ be the affine Lie algebra associated with \mathfrak{g} and $\mathrm{Rep}(\mathcal{U}_k\widehat{\mathfrak{g}})$ the category of $\widehat{\mathfrak{g}}$ -integrable highest weight $\widehat{\mathfrak{g}}$ -modules of level $k - h$, where h denotes the dual Coxeter number of the Lie algebra \mathfrak{g} . Kazhdan and Lusztig used a ‘fusion type’ product to make $\mathrm{Rep}(\mathcal{U}_k\widehat{\mathfrak{g}})$ a tensor category. On the other hand, let U be the quantized enveloping algebra with parameter $q := \exp(\pi\sqrt{-1}/d \cdot k)$, where $d = 1$ if \mathfrak{g} is a simple Lie algebra of types **A**, **D**, **E**, $d = 2$ for types **B**, **C**, **F**, and $d = 3$ for type **G**. Let $\mathrm{Rep}(U)$ be the tensor category of finite dimensional U -modules. In [KL2] the authors have established an equivalence of tensor categories $\mathrm{Rep}(U) \xleftarrow{\sim} \mathrm{Rep}(\mathcal{U}_k\widehat{\mathfrak{g}})$. The subcategory $\mathbf{block}(U) \subset \mathrm{Rep}(U)$ goes under the equivalence to the corresponding principal block $\mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}}) \subset \mathrm{Rep}(\mathcal{U}_k\widehat{\mathfrak{g}})$.

Each of the categories $\mathcal{Perv}_k(\mathrm{Gr}^G)$, $\mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}})$ and $\mathrm{Mod}_k^G(\mathcal{D})$, comes equipped with collections $\{\Delta_\mu\}_{\mu \in \mathbb{Y}}$, resp., $\{\nabla_\mu\}_{\mu \in \mathbb{Y}}$, of so-called *standard*, resp., *costandard* objects, all labelled by the same partially ordered set \mathbb{Y} . In each case, one has

$$\mathrm{Ext}^i(\Delta_\lambda, \nabla_\mu) = \begin{cases} \mathbb{C} & \text{if } \lambda = \mu \\ 0 & \text{if } \lambda \neq \mu. \end{cases} \quad (1.2.1)$$

This is essentially well-known: in the case of category $\mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}})$ a proof can be found e.g. [KL2]; in the case of $\mathcal{Perv}_k(\mathrm{Gr}^G)$, isomorphism (1.2.1) follows from a similar formula for the Ext-groups in $D^b(\mathrm{Gr}^G)$, a larger triangulated category containing the abelian category $\mathcal{Perv}_k(\mathrm{Gr}^G)$ as a subcategory, and a result of [BGS, Corollary 3.3.2] saying that the Ext-groups in the two categories are the same. By the equivalence $\mathrm{Mod}_k^G(\mathcal{D}) \cong \mathcal{Perv}_k(\mathrm{Gr}^G)$, the isomorphism in (1.2.1) holds also for the category $\mathrm{Mod}_k^G(\mathcal{D})$. To sum-up, the categories $\mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}})$, $\mathcal{Perv}_k(\mathrm{Gr}^G)$ and $\mathrm{Mod}_k^G(\mathcal{D})$ are *highest weight categories* in the terminology of [CPS].

Kashiwara-Tanisaki consider the global sections functor $\Gamma : \mathrm{Mod}_k^G(\mathcal{D}) \rightarrow \mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}})$, $\mathcal{M} \mapsto \Gamma(\mathcal{M})$. One of the main results of [KT] says that this functor provides a bijection

$$\{\text{standard/costandard objects in } \mathrm{Mod}_k^G(\mathcal{D})\} \longleftrightarrow \{\text{standard/costandard objects in } \mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}})\}$$

which is compatible with the labelling of the objects involved by the set \mathbb{Y} . More recently, Beilinson-Drinfeld [BD] proved that Γ is an exact functor, cf. also [FG]. Now, by an elementary general result (proved using (1.2.1) and ‘devissage’, cf. Lemma 3.9.3), any exact functor between highest weight categories that gives bijections (compatible with labelling) both of the sets of (isomorphism classes of) standard and costandard objects, respectively, must be an equivalence. It follows that the category $\mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}})$ is equivalent to $\mathrm{Mod}_k^G(\mathcal{D})$. Thus, one obtains the following equivalences:

$$\mathcal{Perv}_k(\mathrm{Gr}^G) \xlongequal{\text{Riemann-Hilbert}} \mathrm{Mod}_k^G(\mathcal{D}) \xlongequal{[\text{KT}]} \mathbf{block}(\mathcal{U}_k\widehat{\mathfrak{g}}) \xlongequal{[\text{KL}]} \mathbf{block}(U) \quad (1.2.2)$$

In this paper we consider the special case where q is an odd root of unity of order prime to 3. In that case, the corresponding rational number k , such that $q = \exp(\pi\sqrt{-1}/d \cdot k)$, has a small denominator. Comparing the composite equivalence in (1.2.2) with the one in (1.1.2), we get $\mathcal{Perv}_k(\mathrm{Gr}^G) \cong \mathbf{block}(U) \cong \mathcal{Perv}(\mathrm{Gr})$. Although we do not know how to construct a direct equivalence $\mathcal{Perv}_k(\mathrm{Gr}^G) \xrightarrow{\sim} \mathcal{Perv}(\mathrm{Gr})$ by geometric means, the results of Lusztig [L5] imply that the character formulas for simple objects in these categories are identical. This explains the relation of our results with those of [KL2] and [KT].

1.3 Outline of our strategy. The construction of both equivalences in (1.1.1) is carried out according to the following rather general pattern. Let D denote any of the three triangulated categories in (1.1.1). In each case, we find an appropriate object $P \in D$, and form the differential graded (dg-) algebra $\mathrm{RHom}_D^\bullet(P, P)$. Then, the assignment $F : M \mapsto \mathrm{RHom}_D^\bullet(P, M)$ gives a functor from the category D to the derived category of dg-modules over $\mathrm{RHom}_D^\bullet(P, P)$. We show, as a first step, that the functor F is an equivalence. We express this by saying that the category D is ‘governed’ by the dg-algebra $\mathrm{RHom}_D^\bullet(P, P)$. The second step consists of proving that the dg-algebra $\mathrm{RHom}_D^\bullet(P, P)$ is *formal*, that is, quasi-isomorphic to $\mathrm{Ext}_D^\bullet(P, P)$, the corresponding Ext-algebra under the Yoneda product. The formality implies that the category D is ‘governed’ by the algebra $\mathrm{Ext}_D^\bullet(P, P)$, considered as a graded algebra with trivial differential. The third step consists of an explicit calculation of this Ext-algebra. An exciting outcome of the calculation (Theorem 8.5.2) is that the Ext-algebras turn out to be the same for all three categories in question. Thus, all three categories are ‘governed’ by the same algebra, and we are done.

1.4 The functor Q : $D^b\mathrm{block}^{\mathrm{mix}}(\mathbf{U}) \rightarrow D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$ giving the first equivalence in (1.1.1) is a refinement of a very naive functor introduced in [GK]. Specifically, let \mathfrak{b} be the “Borel part” of the “small” quantum group $\mathfrak{u} \subset \mathbf{U}$, and $H^\bullet(\mathfrak{b}, \mathbb{C})$ the cohomology algebra of \mathfrak{b} with trivial coefficients. Since $\mathfrak{b} \subset \mathbf{U}$, any \mathbf{U} -module may be viewed as a \mathfrak{b} -module, by restriction, and the cohomology $H^\bullet(\mathfrak{b}, M|_{\mathfrak{b}})$ has a canonical graded $H^\bullet(\mathfrak{b}, \mathbb{C})$ -module structure. The following functor has been considered in [GK]:

$$Q_{\mathrm{naive}} : \mathrm{block}(\mathbf{U}) \longrightarrow H^\bullet(\mathfrak{b}, \mathbb{C})\text{-mod}, \quad M \longmapsto H^\bullet(\mathfrak{b}, M|_{\mathfrak{b}}). \quad (1.4.1)$$

Now, we have fixed a Borel subgroup $B \subset G$ with Lie algebra \mathfrak{b} . According to [GK] one has a natural $\mathrm{Ad} B$ -equivariant (degree doubling) algebra isomorphism $H^{2\bullet}(\mathfrak{b}, \mathbb{C}) \simeq \mathbb{C}^\bullet[\mathfrak{n}]$, where the group B acts on \mathfrak{n} , the nilradical of \mathfrak{b} , by the adjoint action. This puts, for any \mathbf{U} -module M , the structure of a B -equivariant graded $H^\bullet(\mathfrak{b}, \mathbb{C})$ -module, hence $\mathbb{C}[\mathfrak{n}]$ -module, on $H^\bullet(\mathfrak{b}, M|_{\mathfrak{b}})$. The module $H^\bullet(\mathfrak{b}, M|_{\mathfrak{b}})$ is finitely generated, provided $\dim M < \infty$, hence, gives rise to an object of $\mathrm{Coh}^{B \times \mathbb{C}^*}(\mathfrak{n})$, the category of $B \times \mathbb{C}^*$ -equivariant coherent sheaves on \mathfrak{n} . Further, inducing sheaves from the vector space \mathfrak{n} up to the Springer resolution $\tilde{\mathcal{N}} = G \times_B \mathfrak{n}$, we obtain from (1.4.1) the following composite functor:

$$Q_{\mathrm{naive}} : \mathrm{block}(\mathbf{U}) \longrightarrow \mathrm{Coh}^{B \times \mathbb{C}^*}(\mathfrak{n}) \xrightarrow{\text{induction}} \mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}}), \quad (1.4.2)$$

where the second arrow denotes the obvious equivalence, whose inverse is given by restricting to the fiber $\mathfrak{n} = \{1\} \times_B \mathfrak{n} \hookrightarrow G \times_B \mathfrak{n} = \tilde{\mathcal{N}}$.

The functor Q_{naive} may be viewed as a “naive” analogue of the functor Q in (1.1.1). In order to construct Q itself, one has to ‘lift’ considerations above to the level of derived categories. To this end, we will prove in §5 that the dg-algebra $\mathrm{RHom}_{\mathfrak{b}}(\mathbb{C}, \mathbb{C})$ is *formal*, that is, we will construct an $\mathrm{Ad} B$ -equivariant (degree doubling) dg-algebra map

$$\mathbb{C}^\bullet[\mathfrak{n}] \longrightarrow \mathrm{RHom}_{\mathfrak{b}}^{2\bullet}(\mathbb{C}, \mathbb{C}), \quad (1.4.3)$$

where $\mathbb{C}^\bullet[\mathfrak{n}] = \mathrm{Sym}(\mathfrak{n}^*[-2])$ is viewed as a dg-algebra (generated by the space \mathfrak{n}^* of linear functions placed in degree 2) and equipped with zero differential. The map in (1.4.3) will be shown to induce the above mentioned isomorphism of cohomology $\mathbb{C}^\bullet[\mathfrak{n}] \xrightarrow{\sim} \mathrm{Ext}_{\mathfrak{b}}^{2\bullet}(\mathbb{C}, \mathbb{C}) = H^\bullet(\mathfrak{b}, \mathbb{C})$ proved in [GK], in particular, it is a quasi-isomorphism.

The main idea of our approach to constructing quasi-isomorphism (1.4.3) is as follows. Recall first a well-known result due to Gerstenhaber saying that any associative algebra \mathfrak{a} and a 1-st order deformation of \mathfrak{a} parametrized by a vector space V , give rise to a canonical linear map

$V \rightarrow HH^2(\mathfrak{a})$, the second Hochschild cohomology group of \mathfrak{a} . The Hochschild cohomology being a commutative algebra, the latter map extends to a unique degree doubling algebra homomorphism $\mathrm{Sym}^\bullet(V[-2]) \rightarrow HH^{2\bullet}(\mathfrak{a})$. We show in [BG] that any extension of the 1-st order deformation to a deformation of infinite order provides a canonical lift of the homomorphism $\mathrm{Sym}^\bullet(V[-2]) \rightarrow HH^{2\bullet}(\mathfrak{a})$ to the dg-level, i.e. to a dg-algebra homomorphism $\mathrm{Sym}^\bullet(V[-2]) \rightarrow \mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$, (where the graded algebra $\mathrm{Sym}(V[-2])$ is viewed as a dg-algebra with zero differential) and such that the induced map on cohomology is the Gerstenhaber map mentioned above, see Theorem 5.7.1 for a precise statement.

Our crucial observation is that the De Concini-Kac version (without divided powers) of the quantum Borel algebra provides a formal (infinite order) deformation of the algebra \mathfrak{b} , with $V = \mathfrak{n}^*$ being the parameter space. Further, the algebra \mathfrak{b} has a natural Hopf algebra structure, hence, the Hochschild cohomology algebra maps naturally to the algebra $H^{2\bullet}(\mathfrak{b}, \mathbb{C})$, the cohomology with trivial coefficients. Adapting the general construction of the dg-algebra homomorphism $\mathrm{Sym}(V[-2]) \rightarrow \mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$ to the Hopf algebra $\mathfrak{a} := \mathfrak{b}$, yields the desired dg-algebra map (1.4.3).

It is worth mentioning perhaps that we actually need a stronger, $\mathcal{U}\mathfrak{b}$ -equivariant version, of quasi-isomorphism (1.4.3). The construction of such an equivariant quasi-isomorphism exploits the existence of *Steinberg representation*, and also a Hopf-adjoint action of the Lusztig version (with divided powers) of the quantum Borel algebra on the De Concini-Kac version (without divided powers) of the same algebra. We refer to §5 for details.

One may compose a quasi-inverse of the equivalence Q on left of (1.1.1) with the forgetful functor $\mathrm{block}^{\mathrm{mix}}(\mathcal{U}) \rightarrow \mathrm{block}(\mathcal{U})$. This way, we obtain the following result involving no mixed categories (see Theorems 3.5.5 and 3.9.1):

Corollary 1.4.4. *There exists a triangulated functor $F : D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}}) \rightarrow D^b\mathrm{block}(\mathcal{U})$ such that the image of F generates $D^b\mathrm{block}(\mathcal{U})$ as a triangulated category and we have:*

- (i) $F(\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = \mathrm{RInd}_{\mathfrak{b}}^{\mathcal{U}}(l\lambda)$, and $F(z^i \otimes \mathcal{F}) = F(\mathcal{F})[i]$, $\forall \lambda \in \mathbb{Y}$, $i \in \mathbb{Z}$, $\mathcal{F} \in D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$.
- (ii) Write $i : \mathfrak{n} = \{1\} \times_{\mathfrak{b}} \mathfrak{n} \hookrightarrow G \times_{\mathfrak{b}} \mathfrak{n} = \tilde{\mathcal{N}}$ for the natural imbedding. Then, cf. (1.4.1)-(1.4.2), we have

$$\mathrm{R}\Gamma(\mathfrak{n}, i^* \mathcal{F}) = \mathrm{RHom}_{\mathfrak{b}\text{-mod}}(\mathbb{C}_{\mathfrak{b}}, F(\mathcal{F})), \quad \forall \mathcal{F} \in D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}}).$$

- (iii) The functor F induces, for any $\mathcal{F}, \mathcal{F}' \in D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$, canonical isomorphisms

$$\bigoplus_{i \in \mathbb{Z}} \mathrm{Hom}_{D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})}^\bullet(\mathcal{F}, z^i \otimes \mathcal{F}') \xrightarrow{\sim} \mathrm{Hom}_{D^b\mathrm{block}(\mathcal{U})}^\bullet(F(\mathcal{F}), F(\mathcal{F}')).$$

Here, given a \mathbb{C}^* -equivariant sheaf (or complex of sheaves) \mathcal{F} , we write $z^i \otimes \mathcal{F}$ for the sheaf (or complex of sheaves) obtained by twisting the \mathbb{C}^* -equivariant structure by means of the character $z \mapsto z^i$, and let $\mathcal{F}[k]$ denote the homological shift of \mathcal{F} by k in the derived category.

1.5 The functor P : $D^b\mathrm{Perv}^{\mathrm{mix}}(\mathrm{Gr}) \rightarrow D^b\mathrm{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$. The point of departure in constructing the functor on the right of (1.1.1) is the fundamental result of geometric Langlands theory saying that there is an equivalence $\mathcal{P} : \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Perv}_{G^{\vee}(\mathcal{O})}(\mathrm{Gr})$, between the tensor category of finite dimensional rational representations of the group G and the tensor category of $G^{\vee}(\mathcal{O})$ -equivariant perverse sheaves on the loop Grassmannian equipped with a convolution-type monoidal structure: $\mathcal{M}_1, \mathcal{M}_2 \mapsto \mathcal{M}_1 \star \mathcal{M}_2$, see [G2], [MV] and also [Ga]. In particular, write $\mathbf{1}_{\mathrm{Gr}} = \mathcal{P}(\mathbb{C}) \in \mathrm{Perv}_{G^{\vee}(\mathcal{O})}(\mathrm{Gr})$ for the sky-scraper sheaf at the base point of Gr that corresponds to the trivial one-dimensional G -module, and write $\mathcal{R} = \mathcal{P}(\mathbb{C}[G])$ for the ind-object in $\mathrm{Perv}_{G^{\vee}(\mathcal{O})}(\mathrm{Gr})$ corresponding to the regular G -representation. The standard algebra structure on the coordinate

ring $\mathbb{C}[G]$, by pointwise multiplication, makes \mathcal{R} a ring-object in $\mathcal{Perv}_{G^{\vee}(\mathcal{O})}(\mathrm{Gr})$. It is easy to see that this gives a canonical commutative graded algebra structure on the Ext-group $\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$, and that the G -action on $\mathbb{C}[G]$ by right translations gives a G -action on the Ext-algebra. Furthermore, for any perverse sheaf \mathcal{M} on Gr , the Ext-group $\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{M} \star \mathcal{R})$ has the natural structure of a G -equivariant finitely-generated graded $\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$ -module, via the Yoneda product.

A crucial Ext-calculation, carried out in section 7, provides a canonical G -equivariant (degree doubling) algebra isomorphism

$$\mathrm{Ext}_{D^b(\mathrm{Gr})}^{2^{\bullet}}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R}) \simeq \mathbb{C}^{\bullet}[\mathcal{N}] \quad \left(\text{and} \quad \mathrm{Ext}_{D^b(\mathrm{Gr})}^{\mathrm{odd}}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R}) = 0 \right), \quad (1.5.1)$$

where \mathcal{N} is the nilpotent variety in \mathfrak{g} . The homomorphism $\mathbb{C}^{\bullet}[\mathcal{N}] \rightarrow \mathrm{Ext}_{D^b(\mathrm{Gr})}^{2^{\bullet}}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$ in (1.5.1) is induced by a morphism of algebraic varieties $\mathrm{Spec}(\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})) \rightarrow \mathcal{N}$. The latter is constructed by means of *Tannakian formalism* as follows.

In general, let Y be an affine algebraic variety. Constructing a map $Y \rightarrow \mathcal{N}$ is equivalent to producing a family $\{\Phi_V : \mathcal{O}_Y \otimes V \rightarrow \mathcal{O}_Y \otimes V\}_{V \in \mathrm{Rep}(G)}$, (endomorphisms of the trivial vector bundle with fiber V) such that, for any $V, V' \in \mathrm{Rep}(G)$, one has $\Phi_{V \otimes V'} \simeq \Phi_V \otimes \mathrm{Id}_{V'} + \mathrm{Id}_V \otimes \Phi_{V'}$.

In the special case $Y = \mathrm{Spec}(\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R}))$, the geometric Satake isomorphism, cf. §6, provides a canonical isomorphism $\Gamma(Y, \mathcal{O}_Y \otimes V) = \mathrm{Ext}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R}) \otimes V \cong \mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{P}V)$. To construct a nilpotent endomorphism $\Phi_V : \mathrm{Ext}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{P}V) \rightarrow \mathrm{Ext}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{P}V)$, consider the first Chern class $c \in H^2(\mathrm{Gr}, \mathbb{C})$ of the standard determinant line bundle on the loop Grassmannian, see [G2]. Cup-product with c induces a morphism¹ $c : \mathcal{P}(V) \rightarrow \mathcal{P}(V)[2]$. We let Φ_V be the map $\mathrm{Id}_{\mathcal{R}} \star c : \mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{P}V) \rightarrow \mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet+2}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{P}V)$, obtained by applying the functor $\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star (-))$ to the morphism above. For further ramifications of this construction see §§7.7, 8.8.

Using (1.5.1), we may view a G -equivariant graded $\mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$ -module as a $\mathbb{C}^{\bullet}[\mathcal{N}]$ -module, equivalently, as a $G \times \mathbb{C}^*$ -equivariant sheaf on \mathcal{N} . This way we obtain a functor:

$$P_{\mathrm{naive}} : \mathcal{Perv}(\mathrm{Gr}) \longrightarrow \mathrm{Coh}^{G \times \mathbb{C}^*}(\mathcal{N}), \quad \mathcal{M} \longmapsto \mathrm{Ext}_{D^b(\mathrm{Gr})}^{\bullet}(\mathbf{1}_{\mathrm{Gr}}, \mathcal{M} \star \mathcal{R}). \quad (1.5.2)$$

The functor thus obtained may be viewed as a "naive" analogue of the functor P in (1.1.1). The actual construction of the equivalence P is more involved: one has to replace \mathcal{N} by the Springer resolution $\tilde{\mathcal{N}}$, and to make everything work on the level of derived categories. This is made possible by the technique of *weights* of mixed ℓ -adic sheaves combined with known results on the purity of intersection cohomology for flag varieties, due to [KL1].

1.6 Relation to affine Hecke algebras. One of the motivations for the present work was an attempt to understand an old mystery surrounding the existence of two completely different realisations of the affine Hecke algebra. The first realisation is in terms of locally constant functions on the flag manifold of a p -adic reductive group, while the other is in terms of equivariant K -theory of a complex variety (Steinberg variety) acted on by the Langlands dual complex reductive group, see [KL3], [CG]. The existence of the two realisations indicates a possible link between *perverse* sheaves on the affine flag manifold, on one hand, and *coherent* sheaves on the Steinberg variety

¹For any variety X and $M \in D^b(X)$, in the derived category, there is a natural map $H^p(X, \mathbb{C}) \rightarrow \mathrm{Ext}_{D^b(X)}^p(M, M) = \mathrm{Hom}_{D^b(X)}(M, M[p])$, cf. e.g., [CG, (8.3.17)]. We apply this to $X = \mathrm{Gr}$, $c \in H^2(\mathrm{Gr}, \mathbb{C})$, and $M = \mathcal{P}V$.

(over \mathbb{C}) for the Langlands dual group, on the other hand. Specifically, it has been conjectured in [CG, p.15] that there should be a functor

$$F_{\text{Hecke alg}} : D^b \mathcal{Perv}^{\text{mix}}(\text{affine flag manifold}) \longrightarrow D^b \mathcal{Coh}^{G \times \mathbb{C}^*}(\text{Steinberg variety}). \quad (1.6.1)$$

For the *finite* Hecke algebra, a functor of this kind has been constructed by Tanisaki, see [Ta], by means of \mathcal{D} -modules: each perverse sheaf gives rise to a \mathcal{D} -module, and taking associated graded module of that \mathcal{D} -module with respect to a certain filtration yields a coherent sheaf on the Steinberg variety. Tanisaki's construction does not extend, however, to the whole *affine* Hecke algebra; also, it by no means explains the appearance of the Langlands dual group. Our equivalence $P^{-1} \circ Q$ in (1.1.1) provides a "correct" construction of a counterpart of the functor (1.6.1) for the fundamental polynomial representation of the affine Hecke algebra instead of the algebra itself.² A complete construction of (1.6.1) in the algebra case will be carried out in a forthcoming paper. Here we mention only that replacing the module by the algebra results, geometrically, in replacing the loop Grassmannian by the affine flag manifold, on one hand, and replacing the Springer resolution by the Steinberg variety, on the other hand. In addition to that, handling the algebra case involves an important extra-ingredient: the geometric construction of the center of the affine Hecke algebra by means of nearby cycles, due to Gaitsgory [Ga].

To conclude the Introduction, the following remark is worth mentioning. None of the equivalences P and Q taken separately, as opposed to the composite $P^{-1} \circ Q$ in (1.1.1), is compatible with the natural t -structures. In other words, the abelian subcategory $Q(\text{block}^{\text{mix}}(\mathbf{U})) = P(\mathcal{Perv}^{\text{mix}}(\text{Gr}))$ of the triangulated category $D^b \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$ does not coincide with $\mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$. The "exotic" t -structure on $D^b \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$ arising, via Q (equivalently, via P), from the natural t -structure on $Q(\text{block}^{\text{mix}}(\mathbf{U}))$ is, in effect, closely related to the *perverse coherent t -structure* studied in [Be2]. Specifically, it will be shown in a subsequent paper that the functor $P : D^b \mathcal{Perv}^{\text{mix}}(\text{Gr}) \longrightarrow D^b \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$ takes indecomposable *tilting*, resp. *simple*, objects of $\mathcal{Perv}^{\text{mix}}(\text{Gr})$ into *simple*, resp. *tilting*, (with respect to perverse coherent t -structure) objects of $\mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$. This, combined with the results of the present paper, implies that the *tilting* \mathbf{U} -modules in the category $\text{block}(\mathbf{U})$ go, under the equivalence $Q : D^b \text{block}^{\text{mix}}(\mathbf{U}) \longrightarrow D^b \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$ to *simple* perverse coherent sheaves on $\tilde{\mathcal{N}}$. Moreover, an additional argument based on results of [AB] shows that the parameters labelling the tilting objects in $\text{block}(\mathbf{U})$ and the simple perverse coherent sheaves on $\tilde{\mathcal{N}}$ correspond to each other, see [B5]. It follows, in particular, that the support of the quantum group cohomology of a tilting \mathbf{U} -module agrees with the one conjectured by Humphreys.

1.7 Organization of the paper. In §2 we recall basic constructions regarding various versions of quantum groups that will be used later in the paper. The main result of this section is Theorem 2.9.4 which is closely related to the De Concini-Kac-Procesi results [DKP] on *quantum coadjoint action*. In §3 we introduce basic categories of \mathbf{U} -modules and state two main results of the algebraic part of the paper. Section 4 is devoted to the proof of the first result, *Induction theorem*, saying that the derived category of \mathbf{U} -modules in the principal block is equivalent to an appropriate derived category of modules over the Borel part of \mathbf{U} . The proof exploits the techniques of *wall-crossing functors*. In §5, we prove the second main result saying that the dg-algebra of derived endomorphisms of the trivial 1-dimensional module over \mathfrak{b} (= Borel part of the "small" quantum group) is formal, i.e., is quasi-isomorphic to its cohomology algebra. In §6 we review the (known) relation between finite dimensional representations of a semisimple group and perverse sheaves on Gr , the loop Grassmannian for the Langlands dual group. We remind also the role of the principal nilpotent element in describing the cohomology of Gr . In §7 we prove an algebra isomorphism that generalizes isomorphism (1.5.1). Section 8 is devoted to the basics of the theory of *Wakimoto*

²cf. also [AB] for an alternative approach which is, in a sense, 'Koszul dual' to ours.

perverse sheaves on the affine flag manifold, due to Mirković (unpublished). The classes of these sheaves in the Grothendieck group correspond, under the standard isomorphism with the affine Hecke algebra, to base elements of an important large commutative subalgebra in the affine Hecke algebra that has been introduced by Bernstein. The main results of the paper are proved in §9 where the functors Q, P are constructed and the category equivalences (1.1.1) are established. The arguments there use both algebraic and geometric results obtained in all the previous sections. In §10 we prove Ginzburg-Kumar conjecture [GK, §4.3] relating quantum group cohomology to perverse sheaves.

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PART I: Algebra

2 Various quantum algebras

2.1 Let \mathbb{k} be an algebraically closed field of characteristic zero, and set $\otimes = \otimes_{\mathbb{k}}$. We write $\mathbb{k}[X]$ for the coordinate ring of an algebraic variety X .

Given a \mathbb{k} -algebra A with an augmentation $\epsilon : A \rightarrow \mathbb{k}$, let A_{ϵ} denote its kernel. Thus, A_{ϵ} is a two-sided ideal of A , called the *augmentation ideal*, and $\mathbb{k}_A := A/A_{\epsilon}$ is a 1-dimensional A -module.

Definition 2.1.1. Given an associative algebra A and a subalgebra $\mathfrak{a} \subset A$ with augmentation $\mathfrak{a} \rightarrow \mathbb{k}$, we say that \mathfrak{a} is a *normal* subalgebra if one has $A \cdot \mathfrak{a}_{\epsilon} = \mathfrak{a}_{\epsilon} \cdot A$. We then write $(\mathfrak{a}) := A \cdot \mathfrak{a}_{\epsilon} \subset A$ for this two-sided ideal.

Given a \mathbb{k} -algebra A , we write either $A\text{-mod}$ or $\text{Mod}(A)$ for the abelian category of left A -modules. The notation $\text{Rep}(A)$ is reserved for the tensor category of finite-dimensional modules over a Hopf algebra A , unless specified otherwise (this convention will be altered slightly in 2.7.1). In case A is a Hopf algebra, we always assume that the augmentation $\epsilon : A \rightarrow \mathbb{k}$ coincides with the counit.

2.2 Let \mathfrak{t} be a finite dimensional \mathbb{k} -vector space, \mathfrak{t}^* the dual space, and write $\langle -, - \rangle : \mathfrak{t}^* \times \mathfrak{t} \rightarrow \mathbb{k}$ for the canonical pairing. Let $R \subset \mathfrak{t}^*$ be a finite reduced root system. From now on we fix the set $R_+ \subset R$ of positive roots of our root system, and write $\{\alpha_i\}_{i \in I}$ for the corresponding set of simple roots (labelled by a finite set I). Let $\check{\alpha}$ denote the coroot corresponding to a root $\alpha \in R$, so that $a_{ij} = \langle \check{\alpha}_i, \alpha_j \rangle$ is the Cartan matrix.

Let W be the Weyl group of our root system, acting naturally on the lattices \mathbb{X} and \mathbb{Y} , see (2.2.1). There is a unique W -invariant inner product $(-, -)_{\mathbb{Y}} : \mathbb{Y} \times \mathbb{Y} \rightarrow \mathbb{Q}$, normalized so that $(\alpha_i, \alpha_i)_{\mathbb{Y}} = 2d_i$, $\forall i \in I$, where the integers $d_i \geq 1$ are mutually prime. It is known further that

$d_i \in \{1, 2, 3\}$ and that $a_{ij} = (\alpha_i, \alpha_j)_{\mathbb{R}} / (\alpha_i, \alpha_i)_{\mathbb{R}}$. In particular the matrix $\|d_i \cdot a_{i,j}\|$ is symmetric.

$\mathbb{X} = \{\mu \in \mathfrak{t}^* \mid \langle \mu, \check{\alpha}_i \rangle \in \mathbb{Z}, \forall i \in I\}$	weight lattice	(2.2.1)
$\mathbb{X}^{++} = \{\mu \in \mathbb{X} \mid \langle \mu, \check{\alpha}_i \rangle \geq 0, \forall i \in I\}$	dominant Weyl chamber	
$\mathbb{Y} = \sum_{i \in I} \mathbb{Z} \cdot \alpha_i \subset \mathbb{X}$	root lattice	
$\mathbb{Y}^\vee = \text{Hom}(\mathbb{Y}, \mathbb{Z}) \subset \mathfrak{t}$	coweight lattice	
$\mathbb{Y}^{++} = \mathbb{Y} \cap \mathbb{X}^{++}$		

Let $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{t} \oplus \bar{\mathfrak{n}}$ be a semisimple Lie algebra over \mathbb{k} with a fixed triangular decomposition, such that R is the root system of $(\mathfrak{g}, \mathfrak{t})$, and such that \mathfrak{n} is spanned by root vectors for R_+ .

2.3 Let $\mathbb{k}(q)$ be the field of rational functions in the variable q . We write $U_q = U_q(\mathfrak{g})$ for the Drinfeld-Jimbo quantized enveloping algebra of \mathfrak{g} . Thus, U_q is a $\mathbb{k}(q)$ -algebra with generators $E_i, F_i, i \in I$, and $K_\mu, \mu \in \mathbb{Y}^\vee$, and with the following defining relations:

$$\begin{aligned}
K_{\mu_1} \cdot K_{\mu_2} &= K_{\mu_1 + \mu_2}, \\
K_\mu \cdot E_i \cdot K_\mu^{-1} &= q^{\langle \mu, \alpha_i \rangle} \cdot E_i, \quad K_\mu \cdot F_i \cdot K_\mu^{-1} = q^{-\langle \mu, \alpha_i \rangle} \cdot F_i \\
E_i \cdot F_j - F_j \cdot E_i &= \delta_{i,j} \cdot \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, \quad \text{where } K_i = K_{d_i \cdot \check{\alpha}_i},
\end{aligned}$$

and some q -analogues of the Serre relations, see e.g. [L2].

We will freely use Lusztig results on quantum groups at roots of unity, see [L2] and also [AP], pp.579-580.

Fix an *odd* positive integer l which is greater than the Coxeter number of the root system R , and which is moreover prime to 3 if our root system has factors of type \mathbf{G}_2 . Fix $\zeta \in \mathbb{k}^\times$, a primitive l -th root of unity, and let $\mathcal{A} \subset \mathbb{k}(q)$ be the local ring at ζ and $\mathfrak{m} \subset \mathcal{A}$ the maximal ideal in \mathcal{A} .

Remark 2.3.1. One may alternatively take $\mathcal{A} = \mathbb{k}[q, q^{-1}]$ as is done in [L2], [AP]; our choice of \mathcal{A} leads to the same theory. We alert the reader that the variable ' q ' that we are using here was denoted by ' v ' in [L2], [AP]. \diamond

2.4 \mathcal{A} -forms of U_q . Let $\mathbb{U}_{\mathcal{A}}$ be the Lusztig's integral form of U_q , the \mathcal{A} -subalgebra in U_q generated by divided powers $E_i^{(n)} = E_i^n / [n]_{d_i}!$, $F_i^{(n)} = F_i^n / [n]_{d_i}!$, $i \in I, n \geq 1$ (where $[m]_d! := \prod_{s=1}^m \frac{q^{d \cdot s} - q^{-d \cdot s}}{q^d - q^{-d}}$), and also various divided powers $\left[\begin{smallmatrix} K_\mu, m \\ n \end{smallmatrix} \right]_{d_i}$, as defined in [L2]. We will also use a different \mathcal{A} -form of U_q , *without divided powers*, introduced by De Concini-Kac, see [DK]. This is an \mathcal{A} -subalgebra $\mathbb{U}_{\mathcal{A}} \subset U_q$ generated by the elements $E_i, F_i, \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, i \in I$, and $K_\mu, \mu \in \mathbb{Y}^\vee$. We set $\mathbb{U} := \mathbb{U}_{\mathcal{A}} / \mathfrak{m} \cdot \mathbb{U}_{\mathcal{A}}$, the specialization of $\mathbb{U}_{\mathcal{A}}$ at $q = \zeta$. Further, the elements $\{K_i^l\}_{i \in I}$ are known to be central in the algebra $\mathbb{U}_{\mathcal{A}} / \mathfrak{m} \cdot \mathbb{U}_{\mathcal{A}}$, see [DK, Corollary 3.1]. Put $\mathbb{U} := \mathbb{U}_{\mathcal{A}} / (\mathfrak{m} \cdot \mathbb{U}_{\mathcal{A}} + \sum_{i \in I} (K_i^l - 1) \cdot \mathbb{U}_{\mathcal{A}})$. Thus, \mathbb{U} and \mathbb{U} are \mathbb{k} -algebras, which are known as, respectively, the Lusztig and the De Concini-Kac quantum algebras at a root of unity.

The algebra U_q has a Hopf algebra structure over $\mathbb{k}(q)$. It is known that both $\mathbb{U}_{\mathcal{A}}$ and $\mathbb{U}_{\mathcal{A}}$ are Hopf \mathcal{A} -subalgebras in U_q . Therefore, \mathbb{U} and \mathbb{U} are Hopf algebras over \mathbb{k} .

By definition, one has $\mathbb{U}_{\mathcal{A}} \subset \mathbb{U}_{\mathcal{A}}$. Hence, the imbedding of \mathcal{A} -forms induces, after the specialization at ζ , a canonical (not necessarily injective) Hopf algebra homomorphism $\mathbb{U} \rightarrow \mathbb{U}$. The image of this homomorphism is a Hopf subalgebra $\mathfrak{u} \subset \mathbb{U}$, first introduced by Lusztig, and referred

to as the *small quantum group*. Equivalently, \mathfrak{u} is the subalgebra in \mathfrak{U} generated by the elements $E_i, F_i, \frac{K_i - K_i^{-1}}{q^{d_i} - q^{-d_i}}, i \in I$, and $K_\mu, \mu \in \mathbb{Y}^\vee$.

2.5 The algebra U_q has a triangular decomposition $U_q = U_q^+ \otimes_{\mathbb{k}(q)} U_q^\circ \otimes_{\mathbb{k}(q)} U_q^-$, where U_q^+, U_q° and U_q^- are the $\mathbb{k}(q)$ -subalgebras generated by the set $\{E_i\}_{i \in I}$, the set $\{K_\mu\}_{\mu \in \mathbb{Y}^\vee}$, and the set $\{F_i\}_{i \in I}$, respectively. Given any subring $A \subset U_q$, we set $A^\pm := A \cap U_q^\pm$, and $A^\circ := A \cap U_q^\circ$. With this notation, both Lusztig and De Concini-Kac \mathcal{A} -forms are known to admit triangular decompositions $\mathfrak{U}_{\mathcal{A}} = \mathfrak{U}_{\mathcal{A}}^+ \otimes_{\mathcal{A}} \mathfrak{U}_{\mathcal{A}}^\circ \otimes_{\mathcal{A}} \mathfrak{U}_{\mathcal{A}}^-$, and $\mathfrak{U}_{\mathcal{A}} = \mathfrak{U}_{\mathcal{A}}^+ \otimes_{\mathcal{A}} \mathfrak{U}_{\mathcal{A}}^\circ \otimes_{\mathcal{A}} \mathfrak{U}_{\mathcal{A}}^-$, respectively. The latter decompositions induce the corresponding decompositions

$$\mathfrak{U} = \mathfrak{U}^+ \otimes_{\mathbb{k}} \mathfrak{U}^\circ \otimes_{\mathbb{k}} \mathfrak{U}^-, \quad \mathfrak{U} = \mathfrak{U}^+ \otimes_{\mathbb{k}} \mathfrak{U}^\circ \otimes_{\mathbb{k}} \mathfrak{U}^-, \quad \mathfrak{u} = \mathfrak{u}^+ \otimes_{\mathbb{k}} \mathfrak{u}^\circ \otimes_{\mathbb{k}} \mathfrak{u}^-. \quad (2.5.1)$$

The subalgebras $B_q := U_q^+ \otimes_{\mathbb{k}(q)} U_q^\circ \subset U_q$, $\mathfrak{B}_{\mathcal{A}} := \mathfrak{U}_{\mathcal{A}}^+ \otimes_{\mathcal{A}} \mathfrak{U}_{\mathcal{A}}^\circ \subset \mathfrak{U}_{\mathcal{A}}$, and $\mathfrak{B}_{\mathcal{A}} := \mathfrak{U}_{\mathcal{A}}^+ \otimes_{\mathcal{A}} \mathfrak{U}_{\mathcal{A}}^\circ \subset \mathfrak{U}_{\mathcal{A}}$, as well as various specializations like

$$\mathfrak{B} := \mathfrak{U}^+ \otimes \mathfrak{U}^\circ \subset \mathfrak{U}, \quad \mathfrak{B} := \mathfrak{U}^+ \otimes \mathfrak{U}^\circ \subset \mathfrak{U}, \quad \mathfrak{b} := \mathfrak{u}^+ \otimes \mathfrak{u}^\circ \subset \mathfrak{u}, \quad (2.5.2)$$

will be referred to as Borel parts of the corresponding algebras. All of these "Borel parts" are known to be Hopf subalgebras in U_q with coproduct and antipode given by the formulas:

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(K_i) = K_i \otimes K_i, \quad S(E_i) = -K_i^{-1} \cdot E_i, \quad S(K_i) = K_i^{-1}. \quad (2.5.3)$$

Note that formulas (2.5.3) show that U_q^+ is *not* a Hopf subalgebra.

Put $\overline{B}_q := U_q^\circ \otimes_{\mathbb{k}(q)} U_q^-$. This is a Hopf subalgebra in U_q , and Drinfeld constructed a perfect pairing:

$$\overline{B}_q \otimes B_q \longrightarrow \mathbb{k}(q), \quad \bar{b} \times b \mapsto \langle \bar{b}, b \rangle. \quad (2.5.4)$$

The Drinfeld's pairing enjoys an invariance property. To formulate it, one first uses (2.5.4) to define a "differentiation-action" of the algebra \overline{B}_q on B_q by the formula

$$\bar{b} : b \longmapsto \partial_{\bar{b}}(b) := \sum_i \langle \bar{b}, b'_i \rangle \cdot b''_i, \quad \text{where} \quad \sum_i b'_i \otimes b''_i := \Delta(b). \quad (2.5.5)$$

The differentiation-action makes B_q a \overline{B}_q -module. The invariance property states, cf. e.g. [L2]:

$$\langle \bar{x} \cdot \bar{y}, z \rangle + \langle \bar{y}, \partial_{\bar{x}}(z) \rangle = \epsilon(\bar{x}) \langle \bar{y}, z \rangle, \quad \forall \bar{x}, \bar{y} \in \overline{B}_q, z \in B_q.$$

2.6 Frobenius functor. Let G be a connected semisimple group of adjoint type (with trivial center) such that $\text{Lie } G = \mathfrak{g}$. Let \tilde{G} be the simply-connected covering of G , and $\mathfrak{Z}(\tilde{G})$ the center of \tilde{G} (a finite abelian group). Thus, we have a short exact sequence

$$1 \longrightarrow \mathfrak{Z}(\tilde{G}) \longrightarrow \tilde{G} \xrightarrow{\pi} G \longrightarrow 1.$$

The pull-back functor $\pi^* : \text{Rep}(G) \rightarrow \text{Rep}(\tilde{G})$ identifies a finite-dimensional G -module with a finite-dimensional \tilde{G} -module, such that the group $\mathfrak{Z}(\tilde{G})$ acts trivially on it.

Let $\mathcal{U}\mathfrak{g}$ denote the (classical) universal enveloping algebra of \mathfrak{g} . Lusztig introduced a certain completion, $\widehat{\mathcal{U}}\mathfrak{g}$, of the algebra $\mathcal{U}\mathfrak{g}$ such that the category $\text{Rep}(\widehat{\mathcal{U}}\mathfrak{g})$ of finite-dimensional $\widehat{\mathcal{U}}\mathfrak{g}$ -modules may be identified with the category $\text{Rep}(G)$. In more detail, one has the canonical algebra map $j : \mathcal{U}\mathfrak{g} \rightarrow \widehat{\mathcal{U}}\mathfrak{g}$, which induces a functor $j^* : \text{Rep}(\widehat{\mathcal{U}}\mathfrak{g}) \rightarrow \text{Rep}(\mathcal{U}\mathfrak{g})$.

On the other hand, any finite-dimensional $\mathcal{U}\mathfrak{g}$ -module may be regarded, via the exponentiation, as a \tilde{G} -module. The completion $\widehat{\mathcal{U}}\mathfrak{g}$ has the property that in the diagram below the images of the two imbeddings π^* and j^* coincide

$$\mathrm{Rep}(G) \xhookrightarrow{\pi^*} \mathrm{Rep}(\tilde{G}) = \mathrm{Rep}(\mathcal{U}\mathfrak{g}) \xleftarrow{j^*} \mathrm{Rep}(\widehat{\mathcal{U}}\mathfrak{g}).$$

Thus, simple objects of the category $\mathrm{Rep}(\widehat{\mathcal{U}}\mathfrak{g})$ are labelled by the elements of \mathbb{Y}^{++} .

Let $\{e_i, f_i, h_i\}_{i \in I}$ denote the standard Chevalley generators of the Lie algebra \mathfrak{g} . Lusztig proved that the assignment: $E_i^{(l)} \mapsto e_i$, $E_i \mapsto 0$, and $F_i^{(l)} \mapsto f_i$, $F_i \mapsto 0$, $i \in I$, can be extended to a well-defined algebra homomorphism $\phi : \mathcal{U} \rightarrow \widehat{\mathcal{U}}\mathfrak{g}$, called the *Frobenius* map. Further, the subalgebra \mathfrak{u} is known to be *normal* in \mathcal{U} , cf. Definition 2.1.1. Moreover, Lusztig has proved that $\mathrm{Ker} \phi = (\mathfrak{u})$, i.e., one has an exact sequence of bi-algebras

$$0 \longrightarrow (\mathfrak{u}) \longrightarrow \mathcal{U} \xrightarrow{\phi} \widehat{\mathcal{U}}\mathfrak{g}. \quad (2.6.1)$$

The pull-back via the Frobenius morphism $\phi : \mathcal{U} \rightarrow \widehat{\mathcal{U}}\mathfrak{g}$ gives rise to an exact tensor functor

$$\phi^* : \mathrm{Rep}(G) = \mathrm{Rep}(\widehat{\mathcal{U}}\mathfrak{g}) \longrightarrow \mathrm{Rep}(\mathcal{U}), \quad V \longmapsto \phi^* V \quad \textbf{(Frobenius functor)}. \quad (2.6.2)$$

Remark 2.6.3. The reader may have observed that the quantum algebras $U_q(\mathfrak{g})$, \mathcal{U} , etc., that we are using are of "adjoint type". The results of the paper can be adapted to "simply-connected" quantum algebras as well. In that case, the group G must be taken to be the simply-connected group with Lie algebra \mathfrak{g} . Therefore, G^\vee , the Langlands dual group, is of adjoint type. Hence, the corresponding loop Grassmannian Gr considered in Part II becomes disconnected; the group $\pi_0(\mathrm{Gr})$ of its connected components is canonically isomorphic to $\pi_0(\mathrm{Gr}) \cong \pi_1(G^\vee) = \mathrm{Hom}(\mathfrak{Z}(G), \mathbb{C}^*)$, the Pontryagin dual of the center of the simply-connected group G . \diamond

Let $e_\alpha \in \mathfrak{n}$ and $f_\alpha \in \bar{\mathfrak{n}}$ denote root vectors corresponding to a root $\alpha \in R_+$ (so, for any $i \in I$, we have $e_{\alpha_i} = e_i$, $f_{\alpha_i} = f_i$). Throughout the paper, we fix a reduced expression for $w_0 \in W$, the element of maximal length. This puts a normal (total) linear order on the set R_+ and, for each $\alpha \in R_+$, gives, via the braid group action on U_q (see [L3] for details), an element $E_\alpha \in U_q^+$ and $F_\alpha \in U_q^-$. The elements $\{E_\alpha^l, F_\alpha^l\}_{\alpha \in R_+}$ are known to be central in \mathfrak{U} , by [DK, Corollary 3.1].

Definition 2.6.4. Let Z denote the central subalgebra in \mathfrak{B} generated by the elements $\{E_\alpha^l\}_{\alpha \in R_+}$.

Set $(Z) := Z_\epsilon \cdot \mathfrak{B} = \mathfrak{B} \cdot Z_\epsilon$, a two-sided ideal in \mathfrak{B} . Part (i) of the following Lemma is due to De Concini-Kac [DK], part (ii) is due to Lusztig, and other statements can be found in De Concini-Lyubashenko [DL, §3] (cf. also [BFS, pp. 12-13, 72-88] for related results on the pairing $\mathfrak{u}^- \otimes \mathfrak{u}^+ \rightarrow \mathbb{k}$).

Lemma 2.6.5. (i) Z is a Hopf subalgebra in \mathfrak{B} .

(ii) The projection $\mathfrak{U} \rightarrow \mathfrak{u}$ induces, by restriction, an exact sequence of bi-algebras:

$$0 \longrightarrow (Z) \longrightarrow \mathfrak{B} \xrightarrow{\phi} \mathfrak{b} \longrightarrow 0.$$

(iii) Drinfeld's pairing (2.5.4) restricts to a well-defined \mathcal{A} -bilinear pairing $\mathcal{U}_{\mathcal{A}}^- \otimes \mathcal{U}_{\mathcal{A}}^+ \rightarrow \mathcal{A}$; the latter gives, after specialization at $q = \zeta$, a perfect pairing $\mathfrak{U}^- \otimes \mathfrak{U}^+ \rightarrow \mathbb{k}$.

(iv) The annihilator of the subspace $Z \subset \mathfrak{U}^+$ with respect to the pairing in (iii) is equal to the ideal $(\mathfrak{u}^-) \subset \mathfrak{U}^-$, i.e., we have $(\mathfrak{u}^-) = Z^\perp$. \square

Parts (iii)-(iv) of the Lemma combined with the isomorphism $\mathfrak{U}^-/(\mathfrak{u}^-) = \mathcal{U}\bar{\mathfrak{n}}$, give rise to a perfect pairing

$$\mathcal{U}\bar{\mathfrak{n}} \otimes Z \longrightarrow \mathbb{k}. \quad (2.6.6)$$

2.7 Smooth coinduction. Let $A \subset U_q$ be a subalgebra with triangular decomposition: $A = A^+ \otimes A^\circ \otimes A^-$, where $A^? := U_q^? \cap A$, $? = +, \circ, -$. There is a natural algebra map $A^\pm \otimes A^\circ \rightarrow A^\circ$ given by $a^\pm \otimes a^\circ \mapsto \epsilon(a^\pm) \cdot a^\circ$. We write $\mathbb{k}_{A^\circ}(\lambda)$ for the 1-dimensional A° -module corresponding to an algebra homomorphism $\lambda : A^\circ \rightarrow \mathbb{k}$, and let $\mathbb{k}_{A^\pm \otimes A^\circ}(\lambda)$ denote its pull-back via the projection $A^\pm \otimes A^\circ \rightarrow A^\circ$.

Given a possibly infinite dimensional A -module M , define an A -submodule $M^{\text{alg}} \subset M$ as follows:

$$M^{\text{alg}} := \{m \in M \mid \dim(A \cdot m) < \infty \text{ and } A^\circ\text{-action on } A \cdot m \text{ is diagonalizable}\}.$$

Notation 2.7.1. Let $\text{Rep}(A)$ denote the abelian category of finite dimensional A° -diagonalizable A -modules.

We write $\lim \text{ind Rep}(A)$ for the category of all (possibly infinite dimensional) A -modules M such that $M = M^{\text{alg}}$. Clearly, $\text{Rep}(A) \subset \lim \text{ind Rep}(A)$, and any object of the category $\lim \text{ind Rep}(A)$ is a direct limit of its finite dimensional submodules.

Let $A = A^+ \otimes A^\circ \otimes A^-$ be an algebra as above, which is stable under the antipode (anti-)homomorphism $S : A \rightarrow A$. Given a subalgebra $\mathfrak{a} \subset A$, one has a *smooth coinduction* functor

$$\text{Ind}_{\mathfrak{a}}^A : \text{Rep}(\mathfrak{a}) \longrightarrow \lim \text{ind Rep}(A), \quad \text{Ind}_{\mathfrak{a}}^A N := (\text{Hom}_{\mathfrak{a}}(A, N))^{\text{alg}},$$

where $\text{Hom}_{\mathfrak{a}}(A, N)$ is an infinite dimensional vector space with *left* A -action given by $(x \cdot f)(y) = f(S(x) \cdot y)$, see e.g. [APW₂]. It is clear that the functor $\text{Ind}_{\mathfrak{a}}^A$ is the right adjoint to the obvious restriction functor $\text{Res}_{\mathfrak{a}}^A : \text{Rep}(A) \rightarrow \text{Rep}(\mathfrak{a})$, in other words, for any $M \in \text{Rep}(A)$, $N \in \text{Rep}(\mathfrak{a})$, there is a canonical adjunction isomorphism (Frobenius reciprocity):

$$\text{Hom}_{\mathfrak{a}}(\text{Res}_{\mathfrak{a}}^A M, N) \simeq \text{Hom}_A(M, \text{Ind}_{\mathfrak{a}}^A N). \quad (2.7.2)$$

2.8 Ind- and pro-objects. We view $\mathbb{k}[G]$, the coordinate ring of the algebraic group G , as a \mathfrak{g} -module via the left regular representation. It is clear that $\mathbb{k}[G]$ is a direct limit of its finite dimensional G -submodules, that is an ind-object in the category $\text{Rep}(\mathcal{U}\mathfrak{g})$. Let ${}^\phi\mathbb{k}[G] \in \lim \text{ind Rep}(\mathcal{U})$ be the corresponding Frobenius pull-back.

Let $T \subset B$ be the maximal torus and the Borel subgroup corresponding to the Lie algebras $\mathfrak{t} \subset \mathfrak{b}$, respectively. Given $\lambda \in \mathbb{Y} = \text{Hom}(T, \mathbb{k}^\times)$, we let $\mathbf{I}_\lambda := \text{Ind}_T^B \lambda$ be the induced B -module formed by the regular algebraic functions on B that transform via the character λ under right translations by T . One can also view \mathbf{I}_λ as a locally-finite $\mathcal{U}\mathfrak{b}$ -module, which is smoothly co-induced up to $\mathcal{U}\mathfrak{b}$ from the character $\lambda : \mathcal{U}\mathfrak{t} \rightarrow \mathbb{k}$.

Definition 2.8.1. We introduce the algebra $\mathfrak{p} := \mathfrak{b} \cdot \mathcal{U}^\circ = \mathfrak{u}^+ \otimes \mathcal{U}^\circ \subset \mathcal{U}$.

The algebra \mathfrak{p} is slightly larger than \mathfrak{b} ; it plays the same role as the group scheme $B_1T \subset B$ plays for the Borel group in a reductive group G over an algebraically closed field of positive characteristic, cf. [Ja].

For any $\lambda \in \mathbb{Y}$, the weight $l\lambda$ clearly defines a one-dimensional representation of \mathfrak{p} . The pull-back via the Frobenius morphism yields the following isomorphism

$$\text{Ind}_{\mathfrak{p}}^B(\mathbb{k}_{\mathfrak{p}}(l\lambda)) \simeq {}^\phi(\mathbf{I}_\lambda) \quad (2.8.2)$$

Given a module M over any Hopf algebra, there is a well-defined notion of contragredient module M^\vee constructed using the antipode anti-automorphism. The duality functor $M \mapsto M^\vee$ commutes with the Frobenius functor $M \mapsto {}^\phi M$, and sends ind-objects into pro-objects.

Let $N \subset B$ denote the unipotent radical of B , so that $\text{Lie } N = \mathfrak{n}$. View $\mathbb{k}[N]$ as the left regular $\mathcal{U}\mathfrak{n}$ -representation. We make $\mathbb{k}[N]$ into a $\mathcal{U}\mathfrak{b}$ -module by letting the Cartan algebra \mathfrak{t} act on $\mathbb{k}[N]$ via the adjoint action. We have an $\mathcal{U}\mathfrak{b}$ -module isomorphism $\mathbb{k}[N] \simeq \mathbf{I}_0 = \mathbb{k}[B/T]$. Thus, $\phi(\mathbf{I}_0)$ is a \mathbf{B} -module.

Lemma 2.8.3. *For any $M \in \text{Rep}(\mathbf{U})$, there is a natural isomorphism of \mathbf{U} -modules: $\text{Ind}_{\mathbf{u}}^{\mathbf{U}}(\text{Res}_{\mathbf{u}}^{\mathbf{U}} M) \simeq M \otimes_{\mathbb{k}} \phi\mathbb{k}[G]$. Similarly, for a finite dimensional \mathbf{B} -module M , there is a natural isomorphism of \mathbf{B} -modules $\text{Ind}_{\mathbf{p}}^{\mathbf{B}}(\text{Res}_{\mathbf{p}}^{\mathbf{B}} M) \simeq M \otimes_{\mathbb{k}} \phi\mathbf{I}_0$.*

Proof: We view elements $x \in \mathcal{U}\mathfrak{g}$ as left invariant differential operators on G . One has a perfect pairing: $\mathcal{U}\mathfrak{g} \times \mathbb{k}[G] \rightarrow \mathbb{k}$, given by $(x, f) \mapsto (xf)(\mathbf{1}_G)$. This pairing induces a canonical isomorphism of $\mathcal{U}\mathfrak{g}$ -modules: $\mathbb{k}[G] \xrightarrow{\sim} \text{Hom}_{\mathbb{k}}(\mathcal{U}\mathfrak{g}, \mathbb{k})^{\text{alg}}$. Applying the Frobenius functor ϕ^* , one obtains a natural \mathbf{U} -module isomorphism $\text{Ind}_{\mathbf{u}} \mathbb{k}_{\mathbf{u}} \simeq \phi\mathbb{k}[G]$, (here and below we use shorthand notation $\text{Ind} := \text{Ind}_{\mathbf{u}}^{\mathbf{U}}$ and $\text{Res} := \text{Res}_{\mathbf{u}}^{\mathbf{U}}$).

Now, for any $L, M \in \text{Rep}(\mathbf{U})$ and $N \in \text{Rep}(\mathfrak{u})$, we have:

$$\begin{aligned} \text{Hom}_{\mathbf{U}}(L, \text{Ind}((\text{Res } M) \otimes N)) &= \text{Hom}_{\mathbf{u}}(\text{Res } L, (\text{Res } M) \otimes N) \\ &= \text{Hom}_{\mathbf{u}}(\text{Res}(M^{\vee} \otimes L), N) \\ &= \text{Hom}_{\mathbf{U}}(M^{\vee} \otimes L, \text{Ind } N) = \text{Hom}_{\mathbf{U}}(L, M \otimes \text{Ind } N). \end{aligned}$$

Since L is arbitrary, we get a functorial isomorphism $\text{Ind}(N \otimes \text{Res } M) \simeq M \otimes \text{Ind } N$. The first isomorphism of the Lemma now follows by setting $N = \mathbb{k}_{\mathbf{u}}$, and using the isomorphism $\text{Ind } \mathbb{k}_{\mathbf{u}} \simeq \phi\mathbb{k}[G]$. The second isomorphism is proved similarly. \square

2.9 Hopf-adjoint action. Given a Hopf algebra A , we always write Δ for the coproduct and S for the antipode in A , and use Sweedler notation: $\Delta(a) = \sum a'_i \otimes a''_i$. The Hopf algebra structure on A makes the category of left A -modules, resp. A -bimodules, a monoidal category with respect to the tensor product \otimes (over \mathbb{k}). For each $a \in A$, the map $\text{Ad}_{\text{hopf}}(a) : m \mapsto \sum a'_i \cdot m \cdot S(a''_i)$ defines a Hopf-adjoint A -action on any A -bimodule M , such that the action map: $A \otimes M \otimes A \rightarrow M$, $a_1 \otimes m \otimes a_2 \mapsto a_1 m a_2$, is a morphism of A -bimodules. We call an element $m \in M$ central if $am = ma$, $\forall a \in A$, and we say that m is $\text{Ad}_{\text{hopf}} A$ -invariant if $\text{Ad}_{\text{hopf}} a(m) = \epsilon(a) \cdot m$, $\forall a \in A$. The following result is well-known, see e.g. [APW, Proposition 2.9], [Jo].

Lemma 2.9.1. (i) *For any A -bimodule M , the assignment: $a \mapsto \text{Ad}_{\text{hopf}}(a)$ gives an algebra homomorphism: $A \rightarrow \text{End}_{\mathbb{k}} M$.*

(ii) *An element $m \in M$ is $\text{Ad}_{\text{hopf}} A$ -invariant if and only if it is central.* \square

Proposition 2.9.2. (i) *The Hopf-adjoint $\mathbf{U}_{\mathcal{A}}$ -action on $U_q(\mathfrak{g})$ preserves the \mathcal{A} -module $\mathfrak{U}_{\mathcal{A}}$. Similarly, the Hopf-adjoint $\mathbf{B}_{\mathcal{A}}$ -action preserves the \mathcal{A} -modules: $\mathfrak{U}_{\mathcal{A}}^+ \subset \mathfrak{B}_{\mathcal{A}} \subset B_q$. These actions induce, after specialization, an \mathbf{U} -module structure on \mathfrak{U} , resp. a \mathbf{B} -module structure on \mathfrak{B} .*

(ii) *The subalgebras $Z_{\epsilon} \subset Z \subset \mathfrak{U}^+ \subset \mathfrak{B}$, are all $\text{Ad}_{\text{hopf}} \mathbf{B}$ -stable.*

(iii) *The Hopf-adjoint action of the subalgebra $\mathfrak{b} \subset \mathbf{B}$ on Z is trivial.*

Proof. In U_q , consider the set $C = \{E_i^l, K_i^l - K_i^{-l}, F_i^l\}_{i \in I}$. According to De Concini-Kac [DK], every element $c \in C$ projects to a central element of the algebra $\mathfrak{U}_{\mathcal{A}}/\mathfrak{m} \cdot \mathfrak{U}_{\mathcal{A}}$, where $\mathfrak{m} = (q - \zeta)$ is the maximal ideal corresponding to our root of unity. Hence, Lemma 2.9.1(ii) implies that, for any $c \in C$, in the algebra $\mathfrak{U}_{\mathcal{A}}/\mathfrak{m} \cdot \mathfrak{U}_{\mathcal{A}}$ one has $\text{Ad}_{\text{hopf}} c(u) = 0$. Hence, $\text{Ad}_{\text{hopf}} c(u) \in \mathfrak{m} \cdot \mathfrak{U}_{\mathcal{A}}$, for any $u \in \mathfrak{U}_{\mathcal{A}}$. We deduce that the map

$$\text{Ad}_{\text{hopf}}\left(\frac{c}{q - \zeta}\right) : u \mapsto \frac{1}{q - \zeta} \cdot \text{Ad}_{\text{hopf}} c(u)$$

takes the \mathcal{A} -algebra $\mathfrak{U}_{\mathcal{A}}$ into itself. But the elements of the form $\frac{c}{q-\zeta}$, $c \in C$, generate the \mathcal{A} -algebra $\mathfrak{U}_{\mathcal{A}}$. Thus, we have proved that $\text{Ad}_{\text{hopf}} \mathfrak{U}_{\mathcal{A}}(\mathfrak{U}_{\mathcal{A}}) \subset \mathfrak{U}_{\mathcal{A}}$.

Next, from (2.5.3), for any $i \in I$, we find: $\text{Ad}_{\text{hopf}} E_i(x) = E_i x - K_i x K_i^{-1} E_i$, and $\text{Ad}_{\text{hopf}} K_i(x) = K_i x K_i^{-1}$. We deduce that the subalgebras $U_q^+ \subset B_q \subset U_q$ are stable under the adjoint B_q -action on U_q . Further, a straightforward computation yields that, for any $i \in I$, one has:

$$(\text{Ad}_{\text{hopf}} E_i^l) E_j \subset \mathfrak{m} \cdot \mathfrak{B}_{\mathcal{A}}, \quad \text{and} \quad (\text{Ad}_{\text{hopf}} (K_i^l - 1)) E_j \subset \mathfrak{m} \cdot \mathfrak{B}_{\mathcal{A}}, \quad (2.9.3)$$

It follows that the Hopf-adjoint action of the subalgebra $\mathfrak{B}_{\mathcal{A}} \subset B_q$ preserves the subspaces $\mathfrak{U}_{\mathcal{A}}^+ \subset \mathfrak{B}_{\mathcal{A}} \subset B_q$. Specializing at ζ , we obtain part (i), as well as $\text{Ad}_{\text{hopf}} \mathfrak{B}$ -invariance of \mathfrak{U}^+ . \square

Recall the complex semisimple group G with Lie algebra \mathfrak{g} . Let $B \subset G$, resp. $\overline{B} \subset G$, be the two opposite Borel subgroups with Lie algebras $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$ and $\overline{\mathfrak{b}} = \mathfrak{t} \oplus \overline{\mathfrak{n}}$, respectively. We consider the flag manifold G/B with base point B/B , and the ‘opposite open cell’ $\overline{B} \cdot B/B \subset G/B$. The left G -action on G/B induces a Lie algebra map $\mathfrak{b} = \text{Lie } B \mapsto \text{Vector fields on } \overline{B} \cdot B/B$. This makes the coordinate ring $\mathbb{k}[\overline{B} \cdot B/B]$ a $\mathcal{U}\mathfrak{b}$ -module (via Lie derivative).³

Theorem 2.9.4. *There is an algebra isomorphism $Z \simeq \mathbb{k}[\overline{B} \cdot B/B]$ that intertwines the Hopf-adjoint action of the algebra $\mathcal{U}\mathfrak{b} = \mathfrak{B}/(\mathfrak{b})$ on Z resulting from Proposition 2.9.2(ii) and the $\mathcal{U}\mathfrak{b}$ -action on $\mathbb{k}[\overline{B} \cdot B/B]$ described above.*

Proof. Let $D(B_q)$ be the Drinfeld double of the Hopf $\mathbb{k}(q)$ -algebra B_q . Thus, $D(B_q)$ is a Hopf algebra that contains $\overline{B}_q = U_q^\circ \otimes_{\mathbb{k}(q)} U_q^- \subset D(B_q)$ and $B_q \subset D(B_q)$ as Hopf subalgebras, and is isomorphic to $\overline{B}_q \otimes_{\mathbb{k}(q)} B_q$ as a vector space. We combine differentiation-action, see (2.5.5) of the algebra \overline{B}_q on B_q , and the $\text{Ad}_{\text{hopf}} B_q$ -action on B_q to obtain a map

$$a : (\overline{B}_q \otimes_{\mathbb{k}(q)} B_q) \otimes B_q \longrightarrow B_q, \quad (\bar{b} \otimes b) \otimes b' \longmapsto \partial_{\bar{b}} (\text{Ad}_{\text{hopf}} b(b')).$$

One verifies that the commutation relations in $D(B_q)$ insure that the \overline{B}_q - and $\text{Ad}_{\text{hopf}} B_q$ -actions on B_q fit together to make the map a a $D(B_q)$ -algebra action on B_q such that the multiplication map $B_q \otimes B_q \longrightarrow B_q$ is a morphism of $D(B_q)$ -modules.

By Proposition 2.9.2(ii), the subspace $U_q^+ \subset B_q$ is stable under the $\text{Ad}_{\text{hopf}} B_q$ -action. Further, the inclusion $\Delta(U_q^+) \subset B_q \otimes U_q^+$ implies that the subspace $U_q^+ \subset B_q$ is also stable under the differentiation-action of \overline{B}_q on B_q . Thus, U_q^+ is a $D(B_q)$ -submodule in B_q .

According to Drinfeld, one has an algebra isomorphism $D(B_q) \cong U_q \otimes U_q^\circ$. Therefore, the $D(B_q)$ -action on U_q^+ constructed above gives, by restriction, to the subalgebra $U_q \subset D(B_q)$ an U_q -action on U_q^+ . Further, specializing Drinfeld’s pairing (2.5.4) at $q = \zeta$, that is, using the pairing $U^- \otimes U^+ \rightarrow \mathbb{k}$ and Proposition 2.9.2(i)-(ii), we see that the action of the $\mathbb{k}(q)$ -algebra U_q on U_q^+ defined above can be specialized at $q = \zeta$ to give a well-defined \mathfrak{U} -action on \mathfrak{U}^+ .

From Proposition 2.9.2(ii)-(iii) we deduce

- The subspace $Z \subset \mathfrak{U}^+$ is stable under the \mathfrak{U} -action on \mathfrak{U}^+ , moreover, the subalgebra $\mathfrak{u} \subset \mathfrak{U}$ acts trivially (via the augmentation) on Z .

It follows that the action on Z of the algebra \mathfrak{U} factors through $\mathfrak{U}/(\mathfrak{u})$. Thus, we have constructed an action of the Hopf algebra $\mathcal{U}\mathfrak{g}$ (more precisely, of its completion $\widehat{\mathcal{U}\mathfrak{g}}$) on the algebra Z ; in particular, the Lie algebra $\mathfrak{g} \subset \mathcal{U}\mathfrak{g}$ of primitive elements acts on Z by derivations.

Observe next that both the algebra $\mathcal{U}\mathfrak{g}$ and the space Z have natural \mathbb{Y} -gradings, and the $\mathcal{U}\mathfrak{g}$ -action on Z is clearly compatible with the gradings. Write $\mathbb{Y}^+ \subset \mathbb{Y}$ for the semigroup generated

³Note that since the open cell $\overline{B} \cdot B/B \subset G/B$ is not a B -stable subset in the flag variety, the Lie algebra action of \mathfrak{b} on $\mathbb{k}[\overline{B} \cdot B/B]$ cannot be exponentiated to a B -action.

by the positive roots. Clearly, all the weights occurring in Z belong to \mathbb{Y}^+ , while all the weights occurring in $\mathcal{U}\bar{\mathfrak{n}}$ belong to $-\mathbb{Y}^+$. It follows that the action of $(\mathcal{U}\bar{\mathfrak{n}})_\epsilon$, the augmentation ideal of $\mathcal{U}\bar{\mathfrak{n}}$, on Z is locally nilpotent, i.e., for any $z \in Z$, there exists $k = k(z)$ such that $(\mathcal{U}\bar{\mathfrak{n}})_\epsilon^k z = 0$.

Further, the invariance of the Drinfeld pairing (2.5.4) implies that the perfect pairing $\mathcal{U}\bar{\mathfrak{n}} \otimes Z \rightarrow \mathbb{k}$, see (2.6.6), is a morphism of $\mathcal{U}\bar{\mathfrak{n}}$ -modules. This gives an $\mathcal{U}\bar{\mathfrak{n}}$ -module imbedding $Z \hookrightarrow \text{Hom}_{\mathbb{k}}(\mathcal{U}\bar{\mathfrak{n}}, \mathbb{k})$, where $\text{Hom}_{\mathbb{k}}(\mathcal{U}\bar{\mathfrak{n}}, \mathbb{k})$ is viewed as a contragredient representation to the left regular representation of the algebra $\mathcal{U}\bar{\mathfrak{n}}$ on itself.

Now, let $f \in \mathbb{k}[\bar{N}]$, be a regular function on the unipotent group $\bar{N} \subset G$ corresponding to the Lie algebra $\bar{\mathfrak{n}}$. The assignment $u \mapsto \hat{f}(u) := (uf)(1)$, where u runs over the space of left invariant differential operators on the group \bar{N} , gives a linear function on $\mathcal{U}\bar{\mathfrak{n}}$, hence an element $\hat{f} \in \text{Hom}(\mathcal{U}\bar{\mathfrak{n}}, \mathbb{k})$. The map $f \mapsto \hat{f}$ identifies the coordinate ring $\mathbb{k}[\bar{N}]$ with the subspace

$$\{\psi \in \text{Hom}(\mathcal{U}\bar{\mathfrak{n}}, \mathbb{k}) \mid \exists k = k(\psi) \text{ such that } (\mathcal{U}\bar{\mathfrak{n}})_\epsilon^k \psi = 0\}.$$

But the action of $(\mathcal{U}\bar{\mathfrak{n}})_\epsilon$ on Z being locally-nilpotent, we see that the image of the imbedding $Z \hookrightarrow \text{Hom}(\mathcal{U}\bar{\mathfrak{n}}, \mathbb{k})$ must be contained in the space above, that is, in $\mathbb{k}[\bar{N}]$. This way, we obtain an $\mathcal{U}\bar{\mathfrak{n}}$ -module imbedding $Z \hookrightarrow \mathbb{k}[\bar{N}]$. We claim that $Z = \mathbb{k}[\bar{N}]$. Indeed, the group \bar{N} is isomorphic as an algebraic variety to a vector space V , so that the pairing $\mathcal{U}\bar{\mathfrak{n}} \otimes \mathbb{k}[\bar{N}] \rightarrow \mathbb{k}$ may be identified with the canonical pairing $\text{Sym}(V) \otimes \text{Sym}(V^*) \rightarrow \mathbb{k}$. Hence, if Z were a proper subspace in $\mathbb{k}[\bar{N}] = \text{Sym}(V^*)$ then, the pairing $\mathcal{U}\bar{\mathfrak{n}} \otimes Z \rightarrow \mathbb{k}$ could not have been perfect. Thus, $Z = \mathbb{k}[\bar{N}]$.

Next we use the perfect pairing to identify $\mathcal{U}\bar{\mathfrak{n}}$ with Z^\dagger , the continuous dual (in the adic topology) of $Z = \mathbb{k}[\bar{N}]$. The $\mathcal{U}\mathfrak{g}$ -module structure on Z defined above gives rise to an $\mathcal{U}\mathfrak{g}$ -module structure on Z^\dagger . It is clear that restricting the $\mathcal{U}\mathfrak{g}$ -action to the subalgebra $\mathcal{U}\bar{\mathfrak{n}} \subset \mathcal{U}\mathfrak{g}$ we have

- Z^\dagger is a rank 1 free $\mathcal{U}\bar{\mathfrak{n}}$ -module generated by the element $\epsilon \in Z^\dagger$, and
- The action of the ideal $\mathcal{U}\mathfrak{n}_\epsilon \subset \mathcal{U}\mathfrak{n} \subset \mathcal{U}\mathfrak{g}$ annihilates the element $\epsilon \in Z^\dagger$.

Observe that these two properties, combined with the commutation relations in the algebra $\mathcal{U}\mathfrak{g}$, completely determine the $\mathcal{U}\mathfrak{g}$ -module structure on Z^\dagger .

Now, the Lie algebra \mathfrak{g} acts on $\mathbb{k}[\bar{B} \cdot B/B]$, the coordinate ring of the "big cell". Let $\mathbb{k}[\bar{B} \cdot B/B]^\dagger$ denote the continuous dual equipped with a natural $\mathcal{U}\mathfrak{g}$ -module structure. This latter $\mathcal{U}\mathfrak{g}$ -module also satisfies the two properties above. Therefore, there exists an $\mathcal{U}\mathfrak{g}$ -module isomorphism $Z^\dagger \cong \mathbb{k}[\bar{B} \cdot B/B]^\dagger$. Dualizing, we obtain an $\mathcal{U}\mathfrak{g}$ -module isomorphism $Z \cong \mathbb{k}[\bar{B} \cdot B/B]$, and the Theorem is proved. \square

Remark 2.9.5. Theorem 2.9.4 is closely related to the results of De Concini-Kac-Procesi on "quantum coadjoint action". In particular, it was shown in [DKP, Theorem 7.6] that the isomorphism $Z \cong \mathbb{k}[\bar{B} \cdot B/B]$ is a Poisson algebra isomorphism. \diamond

General properties of Hopf-adjoint actions imply, by part (i) of Proposition 2.9.2, that the product map: $\mathfrak{B} \otimes \mathfrak{B} \rightarrow \mathfrak{B}$ is a morphism of \mathfrak{B} -modules. Therefore, we deduce from (ii) that $(Z_\epsilon)^2 \subset Z_\epsilon$ is again an $\text{Ad}_{\text{hopf}} \mathfrak{B}$ -stable subspace. This makes the finite dimensional vector space $Z_\epsilon/(Z_\epsilon)^2$ into a \mathfrak{b} -module, and we have

Corollary 2.9.6. (i) *There is a \mathfrak{b} -equivariant vector space isomorphism $Z_\epsilon/(Z_\epsilon)^2 \simeq \mathfrak{n}$, where the Lie algebra \mathfrak{b} acts on \mathfrak{n} via the adjoint action.*

(ii) *There is a \mathfrak{b} -equivariant graded algebra isomorphism $\text{Tor}_\bullet^Z(\mathbb{k}_Z, \mathbb{k}_Z) \cong \wedge^\bullet \mathfrak{n}$.*

Proof. The cotangent space to G/B at the base point is the space $(\mathfrak{g}/\mathfrak{b})^* = \mathfrak{b}^\perp \subset \mathfrak{g}^*$, which is \mathfrak{b} -equivariantly isomorphic to $\mathfrak{n} \subset \mathfrak{g}$ via an invariant bilinear form on \mathfrak{g} . Now, by Theorem 2.9.4, one may identify the open cell $\bar{B} \cdot B/B$ with $\text{Spec } Z$. The base point goes under this identification,

to the augmentation ideal $Z_\epsilon \in \text{Spec } Z$. The cotangent space $T_\epsilon^*(\text{Spec } Z)$ at that point equals $Z_\epsilon/(Z_\epsilon)^2$, by definition. This yields part (i) of the Corollary.

To prove (ii) recall that, for any smooth affine variety X and a point $x \in X$, one has a canonical graded algebra isomorphism $\text{Tor}_{\bullet}^{\mathbb{k}[X]}(\mathbb{k}_x, \mathbb{k}_x) \cong \wedge^\bullet(T_x^*X)$, where \mathbb{k}_x denotes the 1-dimensional $\mathbb{k}[X]$ -module corresponding to evaluation at x . Part (ii) now follows from (i). \square

Remark 2.9.7. In this paper we will not use the isomorphism $Z \simeq \mathbb{k}[\overline{B} \cdot B/B]$ itself but only the resulting isomorphism of Corollary 2.9.6. \diamond

2.10 Cross-product construction. In the next section we will use the following general construction, see e.g. [Mo]. Let \mathfrak{a} be an associative algebra and A a Hopf algebra.

Proposition 2.10.1. *Let A act on \mathfrak{a} in such a way that the multiplication map $m : \mathfrak{a} \otimes \mathfrak{a} \rightarrow \mathfrak{a}$ is a morphism of A -modules. Then there is a natural associative algebra structure on the vector space $A \otimes \mathfrak{a}$, to be denoted $A \ltimes \mathfrak{a}$, such that*

- (i) $\mathfrak{a} = 1 \otimes \mathfrak{a}$ and $A = A \otimes 1$ are subalgebras in $A \ltimes \mathfrak{a}$;
- (ii) The \mathfrak{a} -action on itself by left multiplication, and the A -action on \mathfrak{a} can be combined to give a well-defined $A \ltimes \mathfrak{a}$ -action on \mathfrak{a} . Furthermore, the multiplication map $(A \ltimes \mathfrak{a}) \otimes (A \ltimes \mathfrak{a}) \rightarrow (A \ltimes \mathfrak{a})$ is $\text{Ad}_{\text{hopf}} A$ -equivariant with respect to the tensor product A -module structure on $A \ltimes \mathfrak{a}$.
- (iii) For any Hopf algebra A acting on itself by the Hopf-adjoint action, the assignment $a \ltimes a_1 \mapsto a \otimes (a \cdot a_1)$ gives an algebra isomorphism $\gamma : A \ltimes A \xrightarrow{\sim} A \otimes A$. \square

In the special case of a Lie algebra \mathfrak{a} acting by derivations on an associative algebra A , the Proposition reduces, for $A = \mathcal{U}\mathfrak{a}$, to the very well-known construction of the cross product algebra $\mathcal{U}\mathfrak{a} \ltimes \mathfrak{a}$. We will apply Proposition 2.10.1 to the Hopf algebra $A := \mathcal{U}$ acting via the Hopf-adjoint action on various algebras described in Proposition 2.9.2.

2.11 Cohomology of Hopf algebras. Given an augmented algebra \mathfrak{a} and a left \mathfrak{a} -module M , one defines the cohomology of \mathfrak{a} with coefficients in M as $H^\bullet(\mathfrak{a}, M) := \text{Ext}_{\mathfrak{a}\text{-mod}}^\bullet(\mathbb{k}_{\mathfrak{a}}, M)$. In particular, we have $\text{Ext}_{\mathfrak{a}\text{-mod}}^0(\mathbb{k}_{\mathfrak{a}}, M) = M^{\mathfrak{a}} := \{m \in M \mid \mathfrak{a}_e m = 0\}$, is the space of \mathfrak{a} -invariants in M . Further, the space $H^\bullet(\mathfrak{a}, \mathbb{k}) := \text{Ext}_{\mathfrak{a}\text{-mod}}^\bullet(\mathbb{k}_{\mathfrak{a}}, \mathbb{k}_{\mathfrak{a}})$, has a natural graded algebra structure given by the Yoneda product. This algebra structure is made explicit by identifying $\text{Ext}_{\mathfrak{a}\text{-mod}}^\bullet(\mathbb{k}_{\mathfrak{a}}, \mathbb{k}_{\mathfrak{a}})$ with the cohomology algebra of the dg-algebra $\text{Hom}_{\mathfrak{a}}(P, P)$, where $P = (P^i)$ is a projective \mathfrak{a} -module resolution of $\mathbb{k}_{\mathfrak{a}}$.

Let A be an augmented algebra and $\mathfrak{a} \subset A$ a normal subalgebra, see Definition 2.1.1. For any left A -module M , the space $M^{\mathfrak{a}} \subset M$ of \mathfrak{a} -invariants is A -stable. Moreover, the A -action on $M^{\mathfrak{a}}$ descends to the quotient algebra $A/(\mathfrak{a})$ and, clearly, we have $M^A = (M^{\mathfrak{a}})^{A/(\mathfrak{a})}$. According to general principles, this gives rise to a spectral sequence

$$E_2^{p,q} = H^p(A/(\mathfrak{a}), H^q(\mathfrak{a}, M)) \implies E_\infty^{p+q} = \text{gr } H^{p+q}(A, M). \quad (2.11.1)$$

Below, we will use a special case of the spectral sequence where A is a Hopf algebra and \mathfrak{a} is a normal Hopf subalgebra. Fix two left A -modules M, N . The vector space $\text{Hom}_{\mathbb{k}}^\bullet(M, N)$ has a natural structure of A -bimodule, hence of a left A -module, via the Hopf-adjoint action. Observe further that we may identify the subspace $\text{Hom}_{\mathfrak{a}}^\bullet(M, N) \subset \text{Hom}_{\mathbb{k}}^\bullet(M, N)$ with the space $(\text{Hom}_{\mathbb{k}}^\bullet(M, N))^{\mathfrak{a}} = \{f \in \text{Hom}_{\mathbb{k}}^\bullet(M, N) \mid a \circ f = f \circ a, \forall a \in \mathfrak{a}\}$ of central elements of $\text{Hom}_{\mathbb{k}}(M, N)$, viewed as an \mathfrak{a} -bimodule. But any central element of an \mathfrak{a} -bimodule is $\text{Ad}_{\text{hopf}} \mathfrak{a}$ -invariant, by Lemma 2.9.1(ii). Therefore, we deduce that the Hopf-adjoint \mathfrak{a} -action on $\text{Hom}_{\mathfrak{a}}^\bullet(M, N)$ is trivial. Hence, the $\text{Ad}_{\text{hopf}} A$ -action on $\text{Hom}_{\mathfrak{a}}^\bullet(M, N)$ descends to the algebra $A/(\mathfrak{a})$, and the spectral sequence in (2.11.1) yields

Lemma 2.11.2. *For any left A -modules M, N , there is a natural Hopf-adjoint $A/(\mathfrak{a})$ -action on $\mathrm{Ext}_a^\bullet(M, N)$; it gives rise to a spectral sequence*

$$E_2^{p,q} = H^p(A/(\mathfrak{a}), \mathrm{Ext}_a^q(M, N)) \implies E_\infty^{p+q} = \mathrm{gr} \mathrm{Ext}_A^{p+q}(M, N). \quad \square$$

3 Algebraic category equivalences

The goal of the next three sections is to construct a chain of functors that will provide the following equivalences of triangulated categories (undefined notations will be explained later):

$$\begin{aligned} D_{\mathrm{coherent}}^G(\tilde{\mathcal{N}}) &\xrightarrow[\sim]{\text{restriction}} D_{\mathrm{coherent}}^B(\mathfrak{n}) \xrightarrow[\text{sections}]{\text{global}} D(\mathcal{U}\mathfrak{b} \ltimes \mathbb{k}^\bullet[\mathfrak{n}]) \\ &\xrightarrow[\text{duality}]{\text{Koszul}} D(\mathcal{U}\mathfrak{b} \ltimes (\wedge^\bullet \mathfrak{n}[1])) \xrightarrow[\S 5]{\text{formality}} D_{\mathrm{triv}}(\mathcal{B}) \xrightarrow[\S 4]{\text{induction}} D(\mathrm{block}(\mathcal{U})). \end{aligned}$$

The composite of the equivalences above will yield (a non-mixed version of) a quasi-inverse of the functor Q on the left of diagram (1.1.1).

3.1 Reminder on dg-algebras and dg-modules. Given an algebra A , we write $A\text{-bimod}$ for the category of A -bimodules, and \underline{A} for A , viewed either as a rank one free left A -module, or as an A -bimodule. Similar notation will be used below for differential graded (dg-)algebras.

Notation 3.1.1. Let $[n]$ denote the shift functor in the derived category, and also the grading shift by n in a dg-algebra or a dg-module.

Given a dg-algebra $A = \bigoplus_{i \in \mathbb{Z}} A^i$, write $\mathrm{DGM}(A)$ for the homotopy category of all dg-modules $M = \bigoplus_{i \in \mathbb{Z}} M^i$ over A (with differential $d : M^\bullet \rightarrow M^{\bullet+1}$), and $D(\mathrm{DGM}(A))$ for the corresponding derived category. Given two objects $M, N \in \mathrm{DGM}(A)$ and $i \in \mathbb{Z}$, we put $\mathrm{Ext}_A^i(M, N) := \mathrm{Hom}_{\mathrm{DGM}(A)}(M, N[i])$. The graded space $\mathrm{Ext}_A^\bullet(M, M) = \bigoplus_{j \geq 0} \mathrm{Ext}_A^j(M, M)$ has a natural algebra structure, given by the Yoneda product.

An object $M \in \mathrm{DGM}(A)$ is said to be *projective* if it belongs to the smallest full subcategory of $\mathrm{DGM}(A)$ that contains the rank one dg-module A , and which is closed under taking mapping-cones and infinite direct sums. Any object of $\mathrm{DGM}(A)$ is quasi-isomorphic to a projective object, see [Ke] for a proof. (Instead of projective objects, one can use *semi-free objects* considered e.g. in [Dr, Appendix A,B].)

Given $M \in \mathrm{DGM}(A)$, choose a quasi-isomorphic projective object P . The graded vector space $\bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{k}}(P, P[n])$ has a natural algebra structure given by composition. Commutator with the differential $d \in \mathrm{Hom}_{\mathbb{k}}(P, P[-1])$ makes this algebra into a dg-algebra, to be denoted $\mathrm{REnd}_A(M) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{k}}(P, P[n])$. This dg-algebra does not depend, up to quasi-isomorphism, on the choice of projective representative P .

Given a dg-algebra morphism $f : A_1 \rightarrow A_2$, we let $f_* : D(\mathrm{DGM}(A_1)) \rightarrow D(\mathrm{DGM}(A_2))$ be the push-forward functor $M \mapsto f_* M := A_2 \overset{L}{\otimes}_{A_1} M$, and $f^* : D(\mathrm{DGM}(A_2)) \rightarrow D(\mathrm{DGM}(A_1))$ the pull-back functor given by change of scalars. Note that the functor f^* is the right adjoint of f_* and, moreover, if the map f is a dg-algebra quasi-isomorphism then the functors f_* and f^* are triangulated equivalences, quasi-inverse to each other.

3.2 From coherent sheaves on $\tilde{\mathcal{N}}$ to $\mathbb{k}[\mathbf{n}]$ -modules. We say that a linear action of an algebraic group \mathbf{G} on a (possibly infinite-dimensional) vector space M is *algebraic* if M is a union of finite-dimensional \mathbf{G} -stable subspaces M_s and, for each s , the action homomorphism $\mathbf{G} \rightarrow \mathrm{GL}(M_s)$ is an algebraic group homomorphism.

Notation 3.2.1. Given a \mathbb{k} -algebra A and an algebraic group \mathbf{G} , acting *algebraically* on A by algebra automorphisms, we let $\mathrm{Mod}^{\mathbf{G}}(A)$ denote the abelian category of \mathbf{G} -equivariant A -modules, i.e., A -modules M equipped with an algebraic \mathbf{G} -action such that the action-map $A \otimes M \rightarrow M$ is \mathbf{G} -equivariant. If, in addition, A has a \mathbb{Z} -grading preserved by the \mathbf{G} -action, we write $\mathrm{Mod}^{\mathbf{G} \times \mathbb{G}_m}(A)$, where \mathbb{G}_m denotes the multiplicative group, for the abelian category of \mathbf{G} -equivariant \mathbb{Z} -graded A -modules.

If A is noetherian, we let $\mathrm{Mod}_f^{\mathbf{G}}(A)$ be the full subcategory in $\mathrm{Mod}^{\mathbf{G}}(A)$ formed by finitely-generated A -modules. In the case of trivial group $\mathbf{G} = \{1\}$ we drop the superscript \mathbf{G} and write $\mathrm{Mod}_f(A)$ for the corresponding category.

Given an algebraic \mathbf{G} -variety X , we write $\mathrm{Coh}^{\mathbf{G}}(X)$ for the abelian category of \mathbf{G} -equivariant coherent sheaves on X . A quasi-coherent sheaf on X is said to be \mathbf{G} -equivariant if it is a direct limit of its \mathbf{G} -equivariant coherent subsheaves.

Given, a \mathbb{G}_m -equivariant sheaf \mathcal{F} , write $z^k \otimes \mathcal{F}$ for a \mathbb{G}_m -equivariant sheaf obtained by twisting \mathbb{G}_m -equivariant structure on \mathcal{F} by the 1-dimensional character $z \mapsto z^k$ of the group \mathbb{G}_m .

The Borel subgroup $B \subset G$ acts on \mathfrak{n} , the nilradical of $\mathfrak{b} = \mathrm{Lie} B$, by conjugation. Further, the multiplicative group \mathbb{G}_m acts on \mathfrak{n} by dilations: we let $z \in \mathbb{G}_m$ act via multiplication by z^2 . The two actions commute, making \mathfrak{n} a $B \times \mathbb{G}_m$ -variety. This gives a $B \times \mathbb{G}_m$ -action on the polynomial algebra $\mathbb{k}^{\bullet}[\mathbf{n}]$. In particular, the algebra $\mathbb{k}^{\bullet}[\mathbf{n}]$ acquires a natural $\mathbb{Z} \times \mathbb{Y}$ -grading: the \mathbb{Z} -component of the grading is given by *twice* the degree of polynomial (this is consistent with our earlier convention that $z \in \mathbb{G}_m$ act via multiplication by z^2), and the \mathbb{Y} -component of the grading is induced from the natural \mathbb{Y} -grading on \mathfrak{n} .

We also consider the Springer resolution $\tilde{\mathcal{N}} := G \times_B \mathfrak{n}$, which is a $G \times \mathbb{G}_m$ -variety in a natural way.

The closed imbedding $i : \mathfrak{n} = \{1\} \times_B \mathfrak{n} \hookrightarrow G \times_B \mathfrak{n} = \tilde{\mathcal{N}}$ gives rise to a natural restriction functor $i^* : \mathrm{Coh}^G(\tilde{\mathcal{N}}) \rightarrow \mathrm{Coh}^B(\mathfrak{n})$. This functor is an equivalence of categories whose inverse is provided by the induction functor $\mathrm{Ind} : \mathrm{Coh}^B(\mathfrak{n}) \rightarrow \mathrm{Coh}^G(\tilde{\mathcal{N}})$. Further, the variety \mathfrak{n} being affine, we deduce that the functor of global sections yields an equivalence of abelian categories $\Gamma : \mathrm{Coh}^B(\mathfrak{n}) \xrightarrow{\sim} \mathrm{Mod}_f^B(\mathbb{k}^{\bullet}[\mathbf{n}])$, $\mathcal{F} \mapsto \Gamma(\mathfrak{n}, \mathcal{F})$.

We are now going to extend the considerations above to triangulated categories. In general, let \mathbf{G} be an algebraic group and X a $\mathbf{G} \times \mathbb{G}_m$ -variety. We let $\mathrm{DG}_{\mathrm{coherent}}^{\mathbf{G}}(X)$ denote the category whose objects are diagrams $\mathcal{F}_+ \rightrightarrows \mathcal{F}_-$, where $\mathcal{F}_+, \mathcal{F}_-$ are $\mathbf{G} \times \mathbb{G}_m$ -equivariant quasi-coherent sheaves on X , such that

- The arrows in the diagram are morphisms of quasi-coherent sheaves; specifically, we have $\mathbf{G} \times \mathbb{G}_m$ -equivariant morphisms $\partial : \mathcal{F}_+ \rightarrow z \otimes \mathcal{F}_-$, and $\partial : \mathcal{F}_- \rightarrow z \otimes \mathcal{F}_+$, such that $\partial \circ \partial = 0$.
- The cohomology sheaves (with respect to the differential ∂) $\mathcal{H}(\mathcal{F}_{\pm})$ are coherent.

Similarly, let A be a dg-algebra, such that the cohomology $H^{\bullet}(A)$ is a finitely generated Noetherian algebra. Let an algebraic group \mathbf{G} act algebraically on A by algebra automorphisms preserving the grading and commuting with the differential.

Notation 3.2.2. We write $\mathrm{DGM}_f^{\mathbf{G}}(A)$ for a subcategory in $\mathrm{Mod}^{\mathbf{G} \times \mathbb{G}_m}(A)$ formed by differential graded A -modules $M = \bigoplus_{i \in \mathbb{Z}} M^i$ (with differential $d : M^{\bullet} \rightarrow M^{\bullet+1}$) such that the \mathbf{G} -action on M preserves the grading and commutes with d and, moreover,

- The cohomology $H^{\bullet}(M)$ is a finitely-generated $H^{\bullet}(A)$ -module.

The category $\mathrm{DG}_{\mathrm{coherent}}^{\mathbf{G}}(X)$, resp. $\mathrm{DGM}_f^{\mathbf{G}}(A)$, has a natural structure of homotopy category, and we write $D_{\mathrm{coherent}}^{\mathbf{G}}(\mathfrak{n})$, resp. $D_f^{\mathbf{G}}(A)$, for the corresponding derived category obtained by localizing at quasi-isomorphisms. Again, if $\mathbf{G} = \{1\}$, the superscript \mathbf{G} will be dropped.

Notation 3.2.3. Let $\mathbf{S} := \mathbb{k}^\bullet[\mathfrak{n}] = \mathrm{Sym}^\bullet(\mathfrak{n}^*[-2])$, resp., $\mathbf{\Lambda} := \wedge^\bullet(\mathfrak{n}[1])$, be the Symmetric algebra of \mathfrak{n}^* , resp., the exterior algebra of \mathfrak{n} , viewed as a differential graded algebra with zero differential generated by the vector space \mathfrak{n}^* placed in grade degree 2, resp., by the vector space \mathfrak{n} , placed in grade degree (-1) .

Thus, we have triangulated categories $D_{\mathrm{coherent}}^B(\mathfrak{n})$, $D_{\mathrm{coherent}}^G(\tilde{\mathcal{N}})$, and also $D_f^B(\mathbb{k}^\bullet[\mathfrak{n}]) = D_f^B(\mathbf{S})$ and $D_f^B(\mathbf{\Lambda})$. As above, we obtain equivalences

$$D_{\mathrm{coherent}}^G(\tilde{\mathcal{N}}) \xrightleftharpoons[\mathrm{RInd}]{i^*} D_{\mathrm{coherent}}^B(\mathfrak{n}) \xrightarrow[\sim]{(\mathcal{F}_+, \mathcal{F}_-) \mapsto \Gamma(\mathfrak{n}, \mathcal{F}_+ \oplus \mathcal{F}_-)} D_f^B(\mathbf{S}). \quad (3.2.4)$$

3.3 Koszul duality. Let $\underline{\mathbf{S}}$ be a free rank 1 graded \mathbf{S} -module (with generator in degree zero), and \mathbb{k}_\wedge the trivial $\mathbf{\Lambda}$ -module (concentrated in degree zero). We may view $\underline{\mathbf{S}}$ as an object of $D_f^B(\mathbf{S})$, and \mathbb{k}_\wedge as an object of $D_f^B(\mathbf{\Lambda})$. We recall the well-known Koszul duality, cf. [BGG] and [GKM], [BGS], between the graded algebras \mathbf{S} and $\mathbf{\Lambda}$. The Koszul duality provides an equivalence:

$$\kappa : D_f^B(\mathbf{S}) \xrightarrow{\sim} D_f^B(\mathbf{\Lambda}), \quad \text{such that } \kappa(\underline{\mathbf{S}}) = \mathbb{k}_\wedge. \quad (3.3.1)$$

In the *non-equivariant* case, this equivalence amounts, essentially, to a dg-algebra quasi-isomorphism

$$\mathrm{RHom}_{D_f(\mathbf{\Lambda})}^\bullet(\mathbb{k}_\wedge, \mathbb{k}_\wedge) \xrightarrow{\mathrm{qis}} \mathrm{Ext}_{D_f(\mathbf{\Lambda})}^\bullet(\mathbb{k}_\wedge, \mathbb{k}_\wedge) \cong \mathbf{S}, \quad (3.3.2)$$

given by the standard Koszul complex.

The reader should be warned that the Koszul duality we are using here is slightly different from the one used in [BGG] in two ways.

First of all, we consider \mathbf{S} and $\mathbf{\Lambda}$ as dg-algebras, and our Koszul duality is an equivalence of derived categories of the corresponding homotopy categories of dg-modules over \mathbf{S} and $\mathbf{\Lambda}$, respectively. In [BGG], the authors consider \mathbf{S} and $\mathbf{\Lambda}$ as *plain* graded algebras, and establish an equivalence between derived categories of the *abelian* categories of \mathbb{Z} -graded \mathbf{S} -modules and $\mathbf{\Lambda}$ -modules, respectively. In our case, the proof of the equivalence is very similar; it is discussed in [GKM] in detail.

Second, in [BGG] no B -equivariant structure was involved. However, the construction of the equivalence given in [BGG],[GKM], being canonical, it extends in a straightforward manner to the equivariant setting as well.

3.4 The principal block of \mathbf{U} -modules. We keep the notation introduced in §2, in particular, we fix a primitive root of unity of order l . Form the semidirect product $W_{\mathrm{aff}} := W \ltimes \mathbb{Y}$, cf. (2.2.1), to be called the affine Weyl group. Let $\rho = \frac{1}{2} \sum_{i \in I} \alpha_i \in \mathbb{X}$ be the half-sum of positive roots. We define a W_{aff} -action on the lattice \mathbb{X} as follows. We let an element $\lambda \in \mathbb{Y} \subset W_{\mathrm{aff}}$ act on \mathbb{X} by translation: $\tau \mapsto \tau + l\lambda$, and let the subgroup $W \subset W_{\mathrm{aff}}$ act on \mathbb{X} via the ‘dot’-action centered at $(-\rho)$, i.e., for an arbitrary element $w_a = (w \ltimes \lambda) \in W_{\mathrm{aff}}$ we put:

$$w_a = w \ltimes \lambda : \tau \mapsto w_a \bullet \tau := w(l\lambda + \tau - \rho) + \rho, \quad \forall \tau \in \mathbb{X}.$$

Recall that the weight $w_a \bullet 0 \in \mathbb{X}^{++}$ is *dominant*, see (2.2.1), if and only if $w_a \in W_{\mathrm{aff}}$ is the minimal element (relative to the standard Bruhat order) in the corresponding left coset $W \cdot w_a$.

From now on we will identify the set of such minimal length representatives in the left cosets $W \backslash W_{\text{aff}}$ with the lattice \mathbb{Y} using the natural bijection:

$$\mathbb{Y} \xrightarrow{\sim} W \backslash W_{\text{aff}} \ , \ \lambda \longmapsto w_{\min}^\lambda := \text{minimal element in the coset } W \cdot (1 \ltimes \lambda) \subset W_{\text{aff}}. \quad (3.4.1)$$

Thus, at the same time we also obtain a bijection

$$\mathbb{Y} \xrightarrow{\sim} \mathbb{X}^{++} \cap W_{\text{aff}} \bullet 0, \ \lambda \longmapsto w_{\min}^\lambda \bullet 0 \quad (3.4.2)$$

For each $\nu \in \mathbb{X}$ we let L^ν be the simple \mathbf{U} -module with highest weight ν . It is known that L^ν is finite dimensional if and only if $\nu \in \mathbb{X}^{++}$.

For $\lambda \in \mathbb{Y}$, let $\nu = w_{\min}^\lambda \bullet 0 \in \mathbb{X}^{++}$. Writing w_{\min}^λ in the form $w_{\min}^\lambda = w \ltimes \lambda \in W \ltimes \mathbb{Y}$, we have $w_{\min}^\lambda \bullet 0 = w \bullet (l\lambda)$, where $w \in W$ is the unique element such that $w \bullet (l\lambda) \in \mathbb{X}^{++}$.

Definition 3.4.3. For $\lambda \in \mathbb{Y}$, let L_λ denote the simple \mathbf{U} -module with highest weight $\nu = w_{\min}^\lambda \bullet 0$. Thus,

$$L_\lambda = L^\nu \quad \text{has highest weight} \quad \nu = w(l\lambda - \rho) + \rho = w \bullet (l\lambda). \quad (3.4.4)$$

Note that if $V \in \text{Rep}(G)$ is a simple G -module with highest weight $\lambda \in \mathbb{Y}^{++}$, then $w_{\min}^\lambda = 1$, hence, we have ${}^\phi V = L_\lambda$.

In this paper we will be frequently using the following three partially ordered sets:

(\mathbb{Y}, \succeq)	$\lambda \succeq \mu \iff w_{\min}^\lambda \bullet 0 - w_{\min}^\mu \bullet 0 = \text{sum of a number of } \check{\alpha}_i\text{'s}$	(3.4.5)
$(\mathbb{Y}, \triangleright)$	$\lambda \triangleright \mu \iff \lambda - \mu \in \mathbb{Y}^{++} = \mathbb{Y} \cap \mathbb{X}^{++} \text{ is dominant.}$	
(\mathbb{Y}, \geq)	$\lambda \geq \nu \iff \lambda - \nu = \text{sum of a number of } \alpha_i\text{'s}$	

We write $\lambda \succ \mu$ whenever $\lambda \succeq \mu$ and $\lambda \neq \mu$. This order relation corresponds, as will be explained in section 6 below, to the closure relation among Schubert varieties in the loop Grassmannian for the Langlands dual group G^\vee .

Definition 3.4.6. Let $\text{block}(\mathbf{U}) \subset \text{Rep}(\mathbf{U})$ be the ‘principal block’, i.e., the full subcategory of the abelian category $\mathbf{U}\text{-mod}$ formed by finite-dimensional \mathbf{U} -modules M such that all simple subquotients of M are of the form L_λ , $\lambda \in \mathbb{Y}$, see (3.4.4).

The abelian category $\text{block}(\mathbf{U})$ is known to have enough projectives and injectives, and we let $D^b \text{block}(\mathbf{U})$ denote the corresponding bounded derived category.

3.5 Induction. Given a subalgebra $A^+ \otimes A^\circ \subset \mathbf{U}$ and an algebra map $\lambda : A^\circ \rightarrow \mathbb{k}$, we write $\text{Ind}_{A^\pm \otimes A^\circ}^{\mathbf{U}}(\lambda)$ instead of $\text{Ind}_{A^\pm \otimes A^\circ}^{\mathbf{U}}(\mathbb{k}_{A^\pm \otimes A^\circ}(\lambda))$.

Using injective resolutions as in [APW], one defines a *derived induction functor* $R\text{Ind}_{\mathbf{B}}^{\mathbf{U}}$ corresponding to a smooth (co)-induction functor $\text{Ind}_{\mathbf{B}}^{\mathbf{U}}$. Let $R^i \text{Ind}_{\mathbf{B}}^{\mathbf{U}}$ denote its i -th cohomology functor.

Let $\ell : W_{\text{aff}} \rightarrow \mathbb{Z}_{\geq 0}$ denote the standard length function on W_{aff} . We recall the following (weak) version of Borel-Weil theorem for quantum groups, proved in [APW].

Lemma 3.5.1 (Borel-Weil theorem). *For $\lambda \in \mathbb{Y}$, let $w \in W$ be an element of minimal length such that $w \bullet (l\lambda) \in \mathbb{X}^{++}$. Then we have:*

- $R^{\ell(w)} \text{Ind}_{\mathbf{B}}^{\mathbf{U}}(l\lambda)$ contains L_λ as a simple subquotient with multiplicity one; Any other simple subquotient of $R^{\ell(w)} \text{Ind}_{\mathbf{B}}^{\mathbf{U}}(l\lambda)$ is isomorphic to L_μ with $\mu \prec \lambda$.
- For any $j \neq \ell(w)$, each simple subquotient of $R^j \text{Ind}_{\mathbf{B}}^{\mathbf{U}}(l\lambda)$ is isomorphic to L_μ with $\mu \prec \lambda$. \square

From Lemma 3.5.1 we immediately deduce

Corollary 3.5.2. (i) For any $\lambda \in \mathbb{Y}$, we have $\mathrm{RInd}_{\mathfrak{b}}^{\mathfrak{u}}(l\lambda) \in D^b\mathrm{block}(\mathfrak{U})$.

(ii) The category $D^b\mathrm{block}(\mathfrak{U})$ is generated, as a triangulated category, by the family of objects $\{\mathrm{RInd}_{\mathfrak{b}}^{\mathfrak{u}}(l\lambda)\}_{\lambda \in \mathbb{Y}}$. \square

Definition 3.5.3. Let $D_{\mathrm{triv}}(\mathfrak{B})$ be a triangulated category whose objects are complexes of \mathbb{Y} -graded \mathfrak{B} -modules $M = \{\dots \rightarrow M_i \rightarrow M_{i+1} \rightarrow \dots\}$, $i \in \mathbb{Z}$, $M_i = \bigoplus_{\nu \in \mathbb{Y}} M_i(\nu)$, such that

- For any $i \in \mathbb{Z}$, we have $M_i = (M_i)^{\mathrm{alg}}$, see Sect. 2.7.
- We have $um = \nu(u) \cdot m$ for any $u \in U^\circ$, $m \in M_i(\nu)$, $i \in \mathbb{Z}$, $\nu \in \mathbb{Y}$;
- The total cohomology module, $H^\bullet(M) = \bigoplus_{i \in \mathbb{Z}} H^i(M)$, has a finite composition series with all subquotients of the form $\mathbb{k}_{\mathfrak{b}}(l\lambda)$, $\lambda \in \mathbb{Y}$.

Remark 3.5.4. Note that the subalgebra $\mathfrak{b} \subset \mathfrak{B}$ acts trivially on the module $\mathbb{k}_{\mathfrak{b}}(l\lambda)$, for any $\lambda \in \mathbb{Y}$.

In section §4 we will prove

Theorem 3.5.5 (Induction theorem). The functor $\mathrm{RInd}_{\mathfrak{b}}^{\mathfrak{u}}$ yields an equivalence of triangulated categories $D_{\mathrm{triv}}(\mathfrak{B}) \xrightarrow{\sim} D^b\mathrm{block}(\mathfrak{U})$.

Remark 3.5.6. An analogue of Theorem 3.5.5 holds also for the principal block of complex representations of the algebraic group $G(\mathbb{F})$ over an algebraically closed field of characteristic $p > 0$. Our proof of the Theorem applies to the latter case as well. \diamond

3.6 Quantum group Formality theorem. The second main result of the algebraic part of this paper is the following theorem that will be deduced from the results of section 5 below. Recall Notation 3.2.3.

Theorem 3.6.1 (Equivariant Formality). There exists a triangulated equivalence $\mathfrak{F} : D_f^{\mathfrak{B}}(\Lambda) \xrightarrow{\sim} D_{\mathrm{triv}}(\mathfrak{B})$, such that $\mathfrak{F}(\mathbb{k}_{\Lambda}) = \mathbb{k}_{\mathfrak{b}}$, and such that $\mathfrak{F}(\mathbb{k}_{\Lambda}(\lambda) \otimes M) \cong \mathbb{k}_{\mathfrak{b}}(l\lambda) \otimes \mathfrak{F}(M)$, for any $\lambda \in \mathbb{Y}$, $M \in D_f^{\mathfrak{B}}(\Lambda)$.

The proof of the theorem is based, as has been already indicated in the Introduction, on a much more general result saying that an infinite order deformation of an associative algebra \mathfrak{a} parametrized by a vector space V yields a homomorphism of dg algebras $\mathrm{def} : \mathrm{Sym}(V[-2]) \rightarrow \mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$. This result will be discussed in detail in the forthcoming paper [BG]; here we only sketch the main idea.

Let A be an infinite order deformation of \mathfrak{a} parametrized by a vector space V , that is, a flat $\mathbb{k}[V]$ algebra such that $A/V^* \cdot A = \mathfrak{a}$. We replace \mathfrak{a} by a quasi-isomorphic dg-algebra R which is a flat $\mathbb{k}[V]$ -algebra such that $H^0(R) = A$. Consider the tensor product $R^e := R \otimes_{\mathbb{k}[V]} R^{\mathrm{op}}$. The tensor category of dg-modules over R^e acts naturally on the category of left dg-modules over R (which is quasi-equivalent to the derived category of \mathfrak{a} -modules). It turns out that this action encodes the desired homomorphism $\mathrm{def} : \mathrm{Sym}(V[-2]) \rightarrow \mathrm{RHom}_{\mathfrak{a}\text{-bimod}}(\mathfrak{a}, \mathfrak{a})$. Specifically, we first construct a quasi-isomorphism $\wedge^\bullet(V^*[1]) = \mathrm{Tor}_{\mathbb{k}[V]}^\bullet(\mathbb{k}, \mathbb{k}) \longrightarrow R^e$, where $\wedge^\bullet(V^*[1])$ is the exterior algebra viewed as a dg-algebra with zero differential. The homomorphism def is then obtained from the latter quasi-isomorphism by Koszul duality, cf. Sect 3.3.

In §5, we carry out this argument for the particular case where $\mathfrak{a} = \mathfrak{b}$, $V = \mathfrak{n}^*$, and $A = \mathfrak{B}$. Additional efforts are required to keep track of the adjoint action of \mathfrak{B} on \mathfrak{b} : the dg-algebra R comes equipped with an action of quantized Borel algebra, while the algebra $\mathrm{Sym}(\mathfrak{n}^*[-2])$ is acted upon by the classical enveloping algebra $\mathcal{U}\mathfrak{b}$. This difficulty is overcome using the Steinberg module. We refer to §5 for full details.

3.7 Digression: Deformation formality. This subsection will not be used elsewhere in the paper. Its sole purpose is to put Theorem 3.6.1 in context.

Recall that the algebra \mathfrak{b} is a normal Hopf subalgebra in \mathcal{B} , and we have $\mathcal{B}/(\mathfrak{b}) = \mathcal{U}\mathfrak{b}$. This gives, by Lemma 2.11.2, a canonical $\mathcal{U}\mathfrak{b}$ -action on the graded algebra $H^\bullet(\mathfrak{b}, \mathbb{k}_{\mathfrak{b}}) = \text{Ext}_{\mathfrak{b}\text{-mod}}^\bullet(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}})$. One of the main results of [GK] says

Proposition 3.7.1. *There is a natural $\mathcal{U}\mathfrak{b}$ -equivariant graded algebra isomorphism $\text{Sym}(\mathfrak{n}^*[-2]) \xrightarrow{\sim} \text{Ext}_{\mathfrak{b}\text{-mod}}^{2\bullet}(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}})$. Moreover, we have $\text{Ext}_{\mathfrak{b}\text{-mod}}^{\text{odd}}(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}}) = 0$.*

Remark 3.7.2. In [GK], we used the standard adjoint (commutator) action $\text{ad } a : x \mapsto ax - xa$, rather than $\text{Ad}_{\text{hopf}} \mathcal{B}$ action on \mathfrak{b} , in order to get an $\mathcal{U}\mathfrak{b}$ -equivariant structure on the Ext-group $\text{Ext}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}})$. Although these two actions are *different* they turn out to induce the *same* $\mathcal{U}\mathfrak{b}$ -equivariant structure on the cohomology.

To explain this, we recall that the construction of [GK] is based on a certain natural transgression map $\tau : H^1(Z, \mathbb{k}_Z) \rightarrow H^2(\mathfrak{b}, \mathbb{k}_{\mathfrak{b}}) := \text{Ext}_{\mathfrak{b}\text{-mod}}^{2\bullet}(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}})$, see [GK, Corollary 5.2]. The isomorphism of Proposition 3.7.1 is obtained by extending the map τ by multiplicativity to an algebra morphism

$$\text{Sym}(\tau) : \text{Sym}^\bullet H^1(Z, \mathbb{k}_Z) \longrightarrow \text{Ext}_{\mathfrak{b}\text{-mod}}^{2\bullet}(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}}), \quad (3.7.3)$$

and then using the canonical vector space isomorphisms $H^1(Z, \mathbb{k}_Z) = Z_\epsilon / (Z_\epsilon)^2 \cong \mathfrak{n}^*$ (the last one is due to Corollary 2.9.6).

Observe that the algebra \mathcal{B} acts on each side of (3.7.3) in two ways: either via the commutator action, or via the Hopf-adjoint action. Furthermore, it has been shown in [GK, Lemma 2.6] that the map τ , hence the isomorphism in (3.7.3), commutes with the commutator action. It is immediate from definition of Hopf-adjoint action that the map τ automatically commutes with the $\text{Ad}_{\text{hopf}} \mathcal{B}$ -actions as well. Now, the point is that although the commutator action of \mathcal{B} on Z *differs* from the $\text{Ad}_{\text{hopf}} \mathcal{B}$ -action on Z , the induced actions on $Z_\epsilon / (Z_\epsilon)^2 \cong \mathfrak{n}^*$ coincide, as can be easily seen from explicit formulas for the two actions. Thus, isomorphism (3.7.3) implies that the two actions on $\text{Ext}_{\mathfrak{b}\text{-mod}}^{2\bullet}(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}})$ are equal, as has been claimed at the beginning of Remark 3.7.2.

In spite of that, it will be essential for us (in the present paper) below to use the Hopf-adjoint action rather than the ordinary commutator. The difference between the two actions becomes important since these actions agree only on the cohomology level, but may not agree at the level of dg-algebras. \diamond

Using (an appropriate version of) the equivalence \mathfrak{F} of Theorem 3.6.1 (more precisely, the equivalence \mathfrak{F}' of Theorem 5.7.1 below), and using that $\mathfrak{F}(\mathbb{k}_\lambda) = \mathbb{k}_{\mathfrak{b}}$, one obtains $\mathcal{U}\mathfrak{b}$ -equivariant dg-algebra quasi-isomorphisms, cf. (3.3.2):

$$\text{REnd}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}}) \simeq \text{REnd}_{\mathfrak{b}}^\bullet(\mathfrak{F}'(\mathbb{k}_\lambda)) \simeq \text{REnd}_{D_{\mathcal{F}}^{\mathcal{B}}(\lambda)}^\bullet(\mathbb{k}_\lambda) = \mathcal{S}. \quad (3.7.4)$$

Our construction of a concrete DG-algebra representing the object $\text{REnd}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}})$ on the left, provides it with additional $\mathcal{U}\mathfrak{b}$ -action, so that all the quasi-isomorphisms in (3.7.4) turn out to be compatible with $\mathcal{U}\mathfrak{b}$ -equivariant structures.

Comparing with Proposition 3.7.1, this yields the following result saying that the dg-algebra $\text{REnd}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}})$ is $\mathcal{U}\mathfrak{b}$ -equivariantly formal:

Theorem 3.7.5. *The dg-algebra $\mathcal{U}\mathfrak{b} \ltimes \text{REnd}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}})$ is quasi-isomorphic to the algebra $\mathcal{U}\mathfrak{b} \ltimes \text{Ext}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}}) = \mathcal{U}\mathfrak{b} \ltimes \text{Sym}^\bullet(\mathfrak{n}^*[-2])$, viewed as a dg-algebra with trivial differential.*

It is not difficult to show that the equivariant formality above is in effect *equivalent* to Theorem 3.6.1. To see this, one has to recall that the very definition of $\mathcal{U}\mathfrak{b}$ -action on $\text{REnd}_{\mathfrak{b}}^\bullet(\mathbb{k}_{\mathfrak{b}})$ appeals to the quantum algebra \mathcal{B} . Hence, the equivariant formality statement must involve in one way or

the other the algebra \mathbf{B} as well. Trying to make this precise leads (as it turns out) inevitably to Theorem 3.6.1.

Remark 3.7.6. A rough idea of our approach to the proof of Theorem 3.7.5 is to replace the algebra \mathbf{b} by a ‘larger’ dg-algebra \mathbf{Rb} , which is quasi-isomorphic to it. We then construct an explicit $\mathcal{U}\mathbf{b}$ -equivariant algebra homomorphism $\mathbf{S} \rightarrow \mathbf{R}\mathrm{End}_{\mathbf{Rb}}^{\bullet}(\mathbb{k}_{\mathbf{Rb}})$ with described properties. The main difficulty in proving Theorem 3.7.5 is in the $\mathcal{U}\mathbf{b}$ -equivariance requirement. Insuring equivariance crucially involves the existence of the Steinberg \mathbf{U} -module, see (5.4.4). Without the equivariance requirement, the result reduces, in view of Proposition 3.7.1, to a special case of the following simple

Proposition 3.7.7. *Let \mathbf{a} be a Hopf algebra such that $\mathrm{Ext}_{\mathbf{a}}^{\bullet}(\mathbb{k}_{\mathbf{a}}, \mathbb{k}_{\mathbf{a}})$ is a free commutative algebra generated by finitely many elements of even degree. Then the dg-algebra $\mathbf{R}\mathrm{End}_{\mathbf{a}}^{\bullet}(\mathbb{k}_{\mathbf{a}})$ is formal.*

Proof. Let $h_1, \dots, h_n \in \mathrm{Ext}_{\mathbf{a}}^{2\bullet}(\mathbb{k}_{\mathbf{a}}, \mathbb{k}_{\mathbf{a}})$ be a finite set of homogeneous generators of the cohomology algebra, and let P be a projective resolution of $\mathbb{k}_{\mathbf{a}}$. The Hopf algebra structure on \mathbf{a} gives rise to a tensor product on the category of complexes of \mathbf{a} -modules. In particular, we may form the complex $P^{\otimes n}$ which is a projective resolution of $\mathbb{k}_{\mathbf{a}}^{\otimes n} \simeq \mathbb{k}_{\mathbf{a}}$ (note that tensor product of any \mathbf{a} -module and a projective \mathbf{a} -module is again a projective \mathbf{a} -module). Thus, the dg-algebra $\mathrm{Hom}_{\mathbf{a}}^{\bullet}(P^{\otimes n}, P^{\otimes n})$ represents $\mathbf{R}\mathrm{Hom}_{\mathbf{a}}^{\bullet}(\mathbb{k}_{\mathbf{a}}, \mathbb{k}_{\mathbf{a}})$. For each $i = 1, \dots, n$, choose $\hat{h}_i \in \mathrm{Hom}_{\mathbf{a}}^{\bullet}(P, P)$ representing the class $h_i \in H^{\bullet}(\mathbf{a}, \mathbb{k}_{\mathbf{a}})$. Then, it is clear that the element

$$\hat{\mathbf{h}}_i := \underbrace{\mathrm{id}_P \otimes \dots \otimes \mathrm{id}_P}_{(i-1) \text{ times}} \otimes \hat{h}_i \otimes \underbrace{\mathrm{id}_P \otimes \dots \otimes \mathrm{id}_P}_{(n-i) \text{ times}} \in \mathrm{Hom}_{\mathbf{a}}^{\bullet}(P^{\otimes n}, P^{\otimes n})$$

also represents the class $h_i \in \mathrm{Ext}_{\mathbf{a}}^{\bullet}(\mathbb{k}_{\mathbf{a}}, \mathbb{k}_{\mathbf{a}})$. Furthermore, for $i \neq j$, the morphisms $\hat{\mathbf{h}}_i$ and $\hat{\mathbf{h}}_j$ act on different tensor factors, hence commute. Therefore, the subalgebra generated by the $\hat{\mathbf{h}}_1, \dots, \hat{\mathbf{h}}_n$ is a commutative subalgebra in the dg-algebra $\mathrm{Hom}_{\mathbf{a}}^{\bullet}(P^{\otimes n}, P^{\otimes n})$ which is formed by cocycles and which maps surjectively onto the cohomology algebra. The latter being free, the map is necessarily an isomorphism, and we are done. \square

An analogue of Theorem 3.7.5 holds for algebraic groups over \mathbb{F} , an algebraically closed field of finite characteristic. Specifically, let $G_{\mathbb{F}}$ be a connected reductive group over \mathbb{F} , let $B_{\mathbb{F}} \subset G_{\mathbb{F}}$ be a Borel subgroup, let $B^{(1)}$ denote the first Frobenius kernel of $B_{\mathbb{F}}$, and write $\mathbb{F}_{B^{(1)}}$ for the trivial $B^{(1)}$ -module. One can see, going through the proof of Theorem 3.6.1, that our argument also may be adapted to prove the following result:

The dg-algebra $\mathbf{R}\mathrm{End}_{B^{(1)}}^{\bullet}(\mathbb{F}_{B^{(1)}})$ is formal as a dg-algebra in the category of $B_{\mathbb{F}}$ -modules.

3.8 Equivariance and finiteness conditions. To prove Theorem 3.6.1 we need to introduce several auxiliary triangulated categories.

Below, we will be considering various dg-algebras $A = \oplus_{i \leq 0} A_i$ (concentrated in non-positive degrees), that will come equipped with the following additional data:

- A natural grading by the root lattice \mathbb{Y} (which is preserved by the differential, as opposed to the \mathbb{Z} -grading): $A = \oplus_{i \in \mathbb{Z}} A_i$, where $A_i = \oplus_{\mu \in \mathbb{Y}} A_i(\mu)$.
- A differential $\mathbb{Z} \times \mathbb{Y}$ -graded subalgebra $C = \oplus_{i \in \mathbb{Z}} C_i \subset A$, $C_i = \oplus_{\mu \in \mathbb{Y}} C_i(\mu)$ such that the cohomology $H^{\bullet}(C)$ is a finitely generated graded Noetherian algebra.
- An \mathbb{Y} -graded subalgebra $U \subset A_0$ equipped with a ‘triangular decomposition’ $U = U^+ \otimes U^{\circ}$, such that U° has $\mathbb{Z} \times \mathbb{Y}$ -degree zero, and $U^+(\mu) = 0$ unless $\mu \in \mathbb{Y}^+$ is a sum of positive roots. Further, we require U to be annihilated by the differential, i.e., that $d(U) \equiv 0$.
- A Hopf algebra structure on U .

Remark 3.8.1. Observe that multiplication map $m : A \otimes A \rightarrow A$ is automatically an $\text{Ad}_{\text{hopf}} U$ -equivariant map. To see this, consider the iterated coproduct $\Delta^3 : U \rightarrow U^{\otimes 4}$. Given $u \in U$, write $\Delta^3(u) = u^{(1)} \otimes u^{(2)} \otimes u^{(3)} \otimes u^{(4)}$. Then, for any $a, \tilde{a} \in A$, we have $m(\text{Ad}_{\text{hopf}} u(a \otimes \tilde{a})) = u^{(1)} \cdot a \cdot S(u^{(2)}) \cdot u^{(3)} \cdot \tilde{a} \cdot S(u^{(4)})$. But the axioms of Hopf algebra imply that, writing $\Delta(u) = u' \otimes u''$ (Sweedler notation), in $U^{\otimes 3}$ one has $u^{(1)} \otimes S(u^{(2)}) \cdot u^{(3)} \otimes u^{(4)} = u' \otimes 1 \otimes u''$. Our claim follows from this equation. \diamond

Let (A, C, U) be a data as above. In what follows, the algebra U will be either classical or quantum enveloping algebra of a Lie subalgebra in our semisimple Lie algebra \mathfrak{g} . So, any weight $\mu \in \mathbb{Y}$ will give rise to a natural algebra homomorphism $\mu : U^\circ \rightarrow \mathbb{k}$.

Let $M = \bigoplus_{i \in \mathbb{Z}, \nu \in \mathbb{Y}} M_i(\nu)$ be a $\mathbb{Z} \times \mathbb{Y}$ -graded A -module, equipped with a differential d such that $d(M_i(\nu)) \subset M_{i+1}(\nu)$, $\forall i \in \mathbb{Z}, \nu \in \mathbb{Y}$. The tensor product of the $\text{Ad}_{\text{hopf}} U$ -action on A and U -action on M obtained by restricting the A -action make $A \otimes M$ an U -module.

We say that the module M is compatible with (A, C, U) -data if the following holds:

- The action map $A \otimes M \rightarrow M$ is an U -module morphism compatible with $\mathbb{Z} \times \mathbb{Y}$ -gradings;
- We have $um = \nu(u) \cdot m$ for any $u \in U^\circ$, $\nu \in \mathbb{Y}$, and $m \in M_i(\nu)$.

Let $\text{DGM}_{\mathbb{Y}}^U(A)$ denote the homotopy category of differential $\mathbb{Z} \times \mathbb{Y}$ -graded A -modules compatible with (A, C, U) -data.

Further, write $H := H^\bullet(C)$. We have a natural graded algebra map $H \rightarrow H^\bullet(A)$, and also an algebra map $U \rightarrow H^0(A)$, since $d(U) = 0$.

Definition 3.8.2. Let $D_{\mathbb{Y}}^U(A, H)$ denote a full subcategory in the triangulated category $D(\text{DGM}_{\mathbb{Y}}^U(A))$ formed by the objects $M \in D(\text{DGM}_{\mathbb{Y}}^U(A))$ such that

- The cohomology module $H^\bullet(M)$ is finitely generated over H ;
- The restriction of the $H^\bullet(A)$ -action on $H^\bullet(M)$ to the subalgebra U is locally finite, i.e., $\dim Um < \infty$, $\forall m \in H^\bullet(M)$.

It is clear that $D_{\mathbb{Y}}^U(A, H)$ is a triangulated category. The objects of $D_{\mathbb{Y}}^U(A, H)$ may be called *U-equivariant, homologically H-finite, dg-modules over A*.

Remark 3.8.3. (i) In the case $U = \mathbb{k}$ we will drop the superscript ‘ U ’ from the notation.

(ii) Observe that in the notations $\text{DGM}_{\mathbb{Y}}^U(A)$ and $D_{\mathbb{Y}}^U(A, H)$ the superscript ‘ U ’ has different meanings: according to our definition, the objects of $\text{DGM}_{\mathbb{Y}}^U(A) = \text{DGM}_{\mathbb{Y}}^{U^\circ}(A)$ are required to have a weight decomposition only with respect to the subalgebra $U^\circ \subset U$, while in the $D_{\mathbb{Y}}^U(A, H)$ -case there is an additional local finiteness condition for the U^+ -action on the cohomology of $M \in D_{\mathbb{Y}}^U(A, H)$.

(iii) Let A be an ordinary (noetherian) algebra, viewed as a dg-algebra concentrated in degree zero and equipped with zero differential, and let $C = A$. Then we have $H = A$, and the category $D_{\mathbb{Y}}(A, H) = D_{\mathbb{Y}}(A, A)$ is in this case (an \mathbb{Y} -graded version of) the full subcategory of the bounded derived category of A -modules, formed by the objects M such that the cohomology $H^\bullet(M)$ is a finitely generated A -module. \diamond

The adjoint action of the group B on the algebra $\Lambda = \wedge^\bullet(\mathfrak{n}[1])$ gives rise to an $\mathcal{U}\mathfrak{b}$ -action on Λ . Therefore, we may perform the cross-product construction of Proposition 2.10.1.

Notation 3.8.4. Let $\mathbf{A} := \mathcal{U}\mathfrak{b} \ltimes \Lambda$ denote the cross-product algebra, viewed as a dg-algebra with zero differential and with the grading given by the natural grading on $\Lambda = \wedge^\bullet(\mathfrak{n}[1]) \subset \mathbf{A}$ and such that the subalgebra $\mathcal{U}\mathfrak{b} \ltimes \{1\} \subset \mathbf{A} = \mathcal{U}\mathfrak{b} \ltimes \Lambda$ is placed in grade degree zero.

Applying Definition 3.8.2 to the triple $A := A = \mathcal{U}\mathfrak{b} \ltimes \Lambda$, $C := \Lambda$, and $U := \mathcal{U}\mathfrak{b}$, one obtains a triangulated category $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$. The algebra Λ being finite dimensional, for any $M \in D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$ one has $\dim H^*(M) < \infty$. In particular, the action on $H^*(M)$ of the Lie algebra \mathfrak{n} is nilpotent, hence, can be exponentiated to an algebraic action of the corresponding unipotent group. Combined with the \mathbb{Y} -grading, this makes $H^*(M)$ a finite-dimensional algebraic B -module.

Now consider B , the quantum Borel subalgebra, as a differential $\mathbb{Z} \times \mathbb{Y}$ -graded algebra concentrated in \mathbb{Z} -degree zero, and equipped with zero differential. The algebra B contains \mathfrak{b} as a subalgebra. Associated to the data $A = U = B$ and $C = H^*(C) = \mathfrak{b}$, we have the triangulated category $D_{\mathbb{Y}}^B(B, \mathfrak{b})$, see Definition 3.8.2. Again, we have $\dim \mathfrak{b} < \infty$, hence, for any $M \in D_{\mathbb{Y}}^B(B, \mathfrak{b})$, the cohomology $H^*(M)$ acquires a natural structure of finite-dimensional B -module.

3.9 Comparison of derived categories. In §5 we will prove the following

Theorem 3.9.1. *There exists a fully faithful triangulated functor $\mathfrak{F} : D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda) \longrightarrow D_{\mathbb{Y}}^B(B, \mathfrak{b})$, such that $\mathfrak{F}(\mathbb{k}_A) = \mathbb{k}_B$, and such that $\mathfrak{F}(\mathbb{k}_A(\lambda) \otimes M) \cong \mathbb{k}_B(\lambda) \otimes \mathfrak{F}(M)$, for any $\lambda \in \mathbb{Y}$, $M \in D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$.*

In order to deduce from this result the equivalence of the Equivariant formality Theorem 3.6.1, we need to replace triangulated categories on each side of the equivalence in Theorem 3.6.1 by larger categories. Specifically, we should replace the category $D_f^B(\Lambda)$, see Sect. 3.2.2, by the category $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$. The objects of the former category are B -equivariant dg-modules over Λ with *algebraic* B -action, while the objects of the latter are dg-modules over Λ with $\mathcal{U}\mathfrak{b}$ -action which is *not* required to be locally-finite, hence, cannot be exponentiated to a B -action, in general. More precisely, for any $M \in \text{DGM}_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$, the action in M of the Cartan subalgebra of \mathfrak{b} is diagonalizable (according to the \mathbb{Y} -grading on M), while the action of the subalgebra $\mathcal{U}\mathfrak{n} \subset \mathcal{U}\mathfrak{b}$ may be arbitrary: only the induced $\mathcal{U}\mathfrak{n}$ -action on the cohomology of M is algebraic. Since any algebraic B -module may be clearly viewed as a Lie B -module, we see that every object of $\text{DGM}_f^B(\Lambda)$ may be also viewed as an object of $\text{DGM}_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$. Thus, we have a natural functor $\mathbf{i}_A : D_f^B(\Lambda) \rightarrow D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$.

Similarly, we would like to replace the category $D_{\text{triv}}(B)$ in Theorem 3.6.1 by a larger category $D_{\mathbb{Y}}^B(B, \mathfrak{b})$ introduced in §3.8. Again, there is a natural triangulated functor $\mathbf{i}_B : D_{\text{triv}}(B) \rightarrow D_{\mathbb{Y}}^B(B, \mathfrak{b})$.

In order to compare Theorem 3.6.1 with Theorem 3.9.1 we are going to prove

Proposition 3.9.2. (i) *The functor $\mathbf{i}_A : D_f^B(\Lambda) \rightarrow D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$ is an equivalence of triangulated categories.*

(ii) *The functor $\mathbf{i}_B : D_{\text{triv}}(B) \rightarrow D_{\mathbb{Y}}^B(B, \mathfrak{b})$ is fully faithful, i.e., makes $D_{\text{triv}}(B)$ a full subcategory in $D_{\mathbb{Y}}^B(B, \mathfrak{b})$.*

The proof of the Proposition exploits the following ‘abstract nonsense’ result that will be also used at several other places.

Lemma 3.9.3. *Let $\mathbf{i} : \mathcal{A} \rightarrow \mathcal{A}'$ be an exact functors between two triangulated categories. Assume given a set S of objects in \mathcal{A} such that*

- *The minimal full triangulated subcategory of \mathcal{A} , resp. of \mathcal{A}' , containing all the objects $M \in S$, resp. all the objects $\mathbf{i}(M)$, $M \in S$, is equal to \mathcal{A} , resp. to \mathcal{A}' .*
- *For any $M_1, M_2 \in S$, the functor \mathbf{i} induces isomorphisms*

$$\text{Hom}_{\mathcal{A}}(M_1, M_2[k]) \xrightarrow{\sim} \text{Hom}_{\mathcal{A}'}(\mathbf{i}(M_1), \mathbf{i}(M_2)[k]), \quad \forall k \in \mathbb{Z}.$$

Then \mathbf{i} is an equivalence of triangulated categories. □

Proof of Proposition 3.9.2. To prove (i), we first show that the functor \mathbf{i}_A induces isomorphisms

$$\mathrm{Ext}_{D_f^B(\Lambda)}^\bullet(\mathbb{k}_\Lambda(\lambda), \mathbb{k}_\Lambda(\mu)) \xrightarrow{\sim} \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda)}^\bullet(\mathbb{k}_\Lambda(\lambda), \mathbb{k}_\Lambda(\mu)) \quad \forall \lambda, \mu \in \mathbb{Y}. \quad (3.9.4)$$

We argue as follows. Write N for the unipotent radical of B and $\mathcal{U}_+ \subset \mathcal{U}\mathfrak{n}$ for the augmentation ideal. For any $k \geq 0$, let $(\mathcal{U}\mathfrak{n}/\mathcal{U}_+^k)^*$ be an $\mathcal{U}\mathfrak{n}$ -module dual to the finite-dimensional left $\mathcal{U}\mathfrak{n}$ -module $\mathcal{U}\mathfrak{n}/\mathcal{U}_+^k$. It is clear that $\lim_{k \rightarrow \infty} \mathrm{ind}(\mathcal{U}\mathfrak{n}/\mathcal{U}_+^k)^*$ is an *injective* $\mathcal{U}\mathfrak{n}$ -module, moreover, we have an imbedding $\mathbb{k}_{\mathcal{U}\mathfrak{n}} \hookrightarrow \lim_{k \rightarrow \infty} \mathrm{ind}(\mathcal{U}\mathfrak{n}/\mathcal{U}_+^k)^*$.

Observe further that the $\mathcal{U}\mathfrak{n}$ -action on $(\mathcal{U}\mathfrak{n}/\mathcal{U}_+^k)^*$ can be exponentiated to give an algebraic representation of the group N . By a standard argument, one therefore obtains a resolution $\mathbb{k}_{\mathcal{U}\mathfrak{n}} \hookrightarrow I_0 \rightarrow I_1 \rightarrow \dots$, where each I_k , $k = 0, 1, \dots$, is an object of $\lim \mathrm{ind} \mathrm{Rep}(N)$ which is injective as an $\mathcal{U}\mathfrak{n}$ -module. Now, for any $\lambda \in \mathbb{Y}$, we may treat each I_k as a $\mathcal{U}^b \ltimes \Lambda$ -module such that $\mathcal{U}\mathfrak{t}$ acts via the character λ and the algebra Λ acts trivially. Since I_k is injective as $\mathcal{U}\mathfrak{n}$ -module, it follows by a standard homological algebra that the morphisms $\mathrm{RHom}_{D_f^B(\Lambda)}^\bullet(I_k, I_l) \rightarrow \mathrm{RHom}_{D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda)}^\bullet(I_k, I_l)$ are isomorphisms, for any $k, l = 0, 1, \dots$. This implies (3.9.4).

We claim next that the objects of the form $\{\mathbb{k}_\Lambda(\lambda)\}_{\lambda \in \mathbb{Y}}$ generate $D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda)$, resp. the objects of the form $\{\mathbb{k}_\Lambda(\lambda)\}_{\lambda \in \mathbb{Y}}$ generate $D_f^B(\Lambda)$, as a triangulated category. This is proved by the standard ‘devissage’, the key point being that $\dim H^\bullet(M) < \infty$ for any $M \in D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda)$. In more detail, let D be the smallest triangulated subcategory in $D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda)$ containing the objects $\{\mathbb{k}_\Lambda(\lambda)\}_{\lambda \in \mathbb{Y}}$. One then shows by descending induction on $\dim H^\bullet(M) < \infty$, that, $M \in D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda) \Rightarrow M \in D$. This is done using standard truncation functors $\tau_{\leq j}$, which take A -modules into A -modules since the algebra A is concentrated in non-negative degrees. This proves our claim for the category $D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda)$; the proof for the category $D_f^B(\Lambda)$ is identical.

The proof of part (i) of the Proposition is now completed, in view of (3.9.4), by Lemma 3.9.3. The proof of part (ii) is entirely similar. \square

To deduce Theorem 3.6.1 from Theorem 3.9.1 we use Proposition 3.9.2 and consider the following diagram

$$D_f^B(\Lambda) \xrightarrow[\mathrm{Prop. 3.9.2}]{\mathbf{i}_A} D_{\mathbb{Y}}^{\mathcal{U}^b}(\Lambda, \Lambda) \xrightarrow[\mathrm{Thm. 3.9.1}]{\mathfrak{F}} D_{\mathbb{Y}}^B(\mathbf{B}, \mathbf{b}) \xleftarrow[\mathrm{Prop. 3.9.2}]{\mathbf{i}_B} D_{\mathrm{triv}}(\mathbf{B}).$$

The functor \mathbf{i}_A in the diagram is an equivalence, and the functors $\mathfrak{F}, \mathbf{i}_B$ are both fully faithful. Further, Theorem 3.9.1 insures that $\mathfrak{F} \circ \mathbf{i}_A(\mathbb{k}_\Lambda(\lambda)) = \mathbf{i}_B(\mathbb{k}_B(l\lambda))$, for any $\lambda \in \mathbb{Y}$. Let $D \subset D_{\mathbb{Y}}^B(\mathbf{B}, \mathbf{b})$ denote the full triangulated category generated by the objects $\{\mathfrak{F} \circ \mathbf{i}_A(\mathbb{k}_\Lambda(\lambda))\}_{\lambda \in \mathbb{Y}}$. It follows from Proposition 3.9.2(ii) and Lemma 3.9.3 that this category is the same as the category generated by the objects $\{\mathbf{i}_B(\mathbb{k}_B(l\lambda))\}_{\lambda \in \mathbb{Y}}$; moreover, our functors induce triangulated equivalences

$D_f^B(\Lambda) \xrightarrow[\sim]{\mathfrak{F} \circ \mathbf{i}_A} D \xleftarrow[\sim]{\mathbf{i}_B} D_{\mathrm{triv}}(\mathbf{B})$. Inverting the equivalence \mathbf{i}_B , we obtain this way an equivalence $\mathfrak{F}: D_f^B(\Lambda) \xrightarrow{\sim} D_{\mathrm{triv}}(\mathbf{B})$, which is by definition the equivalence of Theorem 3.6.1.

Summing-up, we have the following equivalences of triangulated categories

$$\begin{aligned} D_{\mathrm{coherent}}^G(\tilde{\mathcal{N}}) &\xrightarrow[(3.2.4)]{i^*} D_{\mathrm{coherent}}^B(\mathfrak{n}) \xrightarrow{\Gamma} D_f^B(\mathbf{S}) \xrightarrow[(3.3.1)]{} D_f^B(\Lambda) \\ &\xrightarrow[\mathrm{Thm. 3.6.1}]{\mathfrak{F}} D_{\mathrm{triv}}(\mathbf{B}) \xrightarrow[\mathrm{Thm. 3.5.5}]{\mathrm{RInd}_{\mathbf{B}}^{\mathcal{U}}} D^b \mathrm{block}(\mathbf{U}). \end{aligned} \quad (3.9.5)$$

Let $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ be a $G \times \mathbb{G}_{\mathbf{m}}$ -equivariant line bundle on $\tilde{\mathcal{N}}$ obtained by pull-back from the flag manifold G/B of a standard G -equivariant line bundle corresponding to the caharacter $\lambda \in \mathbb{Y}$,

see Notation 8.5.1. Applying the functors i^* and Γ in the top row of (3.9.5), we clearly get $\Gamma(\mathfrak{n}, i^* \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \cong \underline{S}(\lambda)$, where $\underline{S}(\lambda)$ a rank 1 free S -module, viewed as a dg-module over $\mathcal{U}\mathfrak{b} \rtimes S$ with zero differential and with the action of the subalgebra $\mathcal{U}\mathfrak{b} \subset \mathcal{U}\mathfrak{b} \rtimes S$ being the natural one on S twisted by the character λ . Further, twisting the \mathbb{G}_m -equivariant structure on $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ by the character $z \mapsto z^k$ corresponds to degree shift by k in the S -module. Thus, we obtain

Theorem 3.9.6. *The composite functor $Q' : D_{coherent}^G(\tilde{\mathcal{N}}) \xrightarrow{\sim} D^b\text{block}(\mathbf{U})$ in (3.9.5) provides an equivalence of triangulated categories such that $Q'(z^k \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = \text{Ind}_{\mathbf{b}}^{\mathbf{U}}(l\lambda)[k]$, $\forall k \in \mathbb{Z}, \lambda \in \mathbb{Y}$. \square*

The (non-mixed version of the) functor Q in diagram (1.1.1) is defined to be the inverse of the equivalence Q' in the Theorem above.

4 Proof of Induction Theorem.

The goal of this section is to prove Theorem 3.5.5.

4.1 Intertwining functors. For every simple *affine* root $\alpha \in I \cup \{0\}$, let $s_\alpha \in W_{\text{aff}}$ denote the corresponding simple reflection. We partition the lattice \mathbb{X} into alcoves of ‘size’ l in such a way that the ‘base vertex’ of the fundamental alcove is placed at the point $(-\rho)$. Given $\lambda \in \mathbb{Y}$, let C_α denote the unique α -wall of the alcove containing λ , and let λ^{s_α} be the reflection of λ with respect to C_α . The assignment: $\lambda \mapsto \lambda^{s_\alpha}$ extends to a W_{aff} -action, that we call the *right* W_{aff} -action. When restricted to points $\nu \in \mathbb{X}$ of the form $\nu = w_a \bullet 0 \in W_{\text{aff}} \bullet 0$, this action becomes the right multiplication $\nu = w_a \bullet 0 \mapsto \nu^{s_\alpha} = (w_a s_\alpha) \bullet 0$.

It is clear from definition that, for any $\lambda, \mu \in \mathbb{Y}$ and $w \in W_{\text{aff}}$, one has $(\mu + l\lambda)^w = (\mu^w) + l\lambda$.

Below, we will use the following general construction of homological algebra, see e.g. [GM]. Let \mathcal{A}, \mathcal{B} be two abelian categories with enough projectives, and let $F_1, F_2 : \mathcal{A} \rightarrow \mathcal{B}$ be two exact functors. Assume in addition that we have a morphism of functors $\varphi : F_1 \Rightarrow F_2$. Then there is a well-defined *mapping-cone functor* $\text{Cone}(\varphi) : D^b(\mathcal{A}) \rightarrow D^b(\mathcal{B})$, which is a triangulated functor between the corresponding bounded derived categories.

Recall now that there are so-called *reflection functors* $\Xi_\alpha : \text{block}(\mathbf{U}) \rightarrow \text{block}(\mathbf{U})$ defined by composing translation functor ‘to the wall’ C_α , see e.g. [APW, §8], with translation functor ‘out of the wall’ C_α . Translation functors being exact (as direct summands of functors of the form $V \otimes_{\mathbb{k}} (-)$, for a finite dimensional \mathbf{U} -module V), it follows that Ξ_α is an exact functor. Furthermore, there are canonical ‘adjointness’ morphisms $\text{id} \rightarrow \Xi_\alpha$, and $\Xi_\alpha \rightarrow \text{id}$. Applying the above mentioned general construction of the mapping-cone functor to the morphism $\text{id} \rightarrow \Xi_\alpha$, resp. $\Xi_\alpha \rightarrow \text{id}$, one obtains a triangulated functor θ_α^+ , resp. θ_α^- . The functors $\theta_\alpha^\pm : D^b\text{block}(\mathbf{U}) \rightarrow D^b\text{block}(\mathbf{U})$ defined in this way are usually referred to as *intertwining functors*.

Lemma 4.1.1. (i) *In $D^b\text{block}(\mathbf{U})$ we have canonical isomorphisms:*

$$\theta_\alpha^+ \circ \theta_\alpha^- \cong \text{id} \cong \theta_\alpha^- \circ \theta_\alpha^+, \quad \forall \alpha \in I \cup \{0\},$$

in particular, the functors $\theta_\alpha^+, \theta_\alpha^- : D^b\text{block}(\mathbf{U}) \rightarrow D^b\text{block}(\mathbf{U})$ are auto-equivalences.

(ii) *If $\lambda \in W_{\text{aff}} \bullet 0$, and s_α is the reflection with respect to a simple affine root $\alpha \in I \cup \{0\}$ such that $\lambda^{s_\alpha} \geq \lambda$, then $\theta_\alpha^+(\text{RInd}_{\mathbf{b}}^{\mathbf{U}} \lambda) \cong \text{RInd}_{\mathbf{b}}^{\mathbf{U}}(\lambda^{s_\alpha})$.*

Sketch of Proof. Part (ii) of the Lemma follows directly from [APW, Theorem 8.3(i)].

A statement analogous to part (i) of the Lemma is well-known in the framework of the category \mathcal{O} for a complex semisimple Lie algebra, see [Vo]. Specifically, it is clear from the adjunction

properties that the functor $\theta_\alpha^+ \circ \theta_\alpha^-$, resp. $\theta_\alpha^- \circ \theta_\alpha^+$, is quasi-isomorphic to a complex represented by the following commutative square

$$\begin{array}{ccc} \Xi_\alpha & \xrightarrow{\quad} & \text{Id} \\ \downarrow & & \downarrow \\ \Xi_\alpha \circ \Xi_\alpha & \xrightarrow{\quad} & \Xi_\alpha. \end{array}$$

On the other hand, the left vertical and low horizontal arrows of the square form a short exact sequence $0 \rightarrow \Xi_\alpha \rightarrow \Xi_\alpha \circ \Xi_\alpha \rightarrow \Xi_\alpha \rightarrow 0$. This is proved similarly to the corresponding statement for the category \mathcal{O} , with all the necessary ingredients (in our quantum group setting) being provided by [APW, Theorem 8.3]. It follows that the square is quasi-isomorphic to its upper-right corner, and we are done. \square

4.2 Beginning of the proof of Theorem 3.5.5. The functor RInd_B^U takes the set $\{\mathbb{k}_B(l\lambda)\}_{\lambda \in \mathbb{Y}}$ that generates category $D_{\text{triv}}(B)$ as a triangulated category to the set $\{\text{RInd}_B^U(l\lambda)\}_{\lambda \in \mathbb{Y}}$ that generates category $D^b\text{block}(U)$ as a triangulated category, by Corollary 3.5.2. Thus, by Lemma 3.9.3, in order to prove Induction theorem we must show that: *For all $\lambda, \mu \in \mathbb{Y}$, and $i \geq 0$, the canonical morphism, induced by functoriality of induction, gives an isomorphism*

$$\text{Ext}_B^i(\mathbb{k}_B(l\lambda), \mathbb{k}_B(l\mu)) \xrightarrow{\sim} \text{Ext}_{\text{block}(U)}^i(\text{RInd}_B^U(l\lambda), \text{RInd}_B^U(l\mu)). \quad (4.2.1)$$

This isomorphism will be proved in three steps.

Lemma 4.2.2. *Both sides in (4.2.1) have the same dimension*

$$\dim \text{Ext}_B^i(\mathbb{k}_B(l\lambda), \mathbb{k}_B(l\mu)) = \dim \text{Ext}_{\text{block}(U)}^i(\text{RInd}_B^U(l\lambda), \text{RInd}_B^U(l\mu)), \quad \forall i \geq 0.$$

Proof. From Lemma 3.5.1 (quantum version of Borel-Weil theorem [APW]) we obtain

$$\text{R}^0 \text{Ind}_B^U \mathbb{k}_B = \mathbb{k}_U \quad \text{and} \quad \text{R}^i \text{Ind}_B^U \mathbb{k}_B = 0 \quad \text{if} \quad i > 0. \quad (4.2.3)$$

Hence, for any B -module M , we find

$$\text{Hom}_B(\mathbb{k}_B, M) \stackrel{\text{adjunction}}{=} \text{Hom}_U(\mathbb{k}_U, \text{RInd}_B^U M) \stackrel{(4.2.3)}{=} \text{Hom}_U(\text{RInd}_B^U \mathbb{k}_B, \text{RInd}_B^U M).$$

This yields isomorphism (4.2.1) in the special case $\lambda = 0$, and arbitrary $\mu \in \mathbb{Y}$.

The general case will be reduced to the special case above by means of translation functors. Specifically, for any $\lambda, \mu \in \mathbb{Y}$ and $\nu \in \mathbb{Y}^{++}$, we are going to establish an isomorphism

$$\text{RHom}_{\text{block}(U)}(\text{RInd}_B^U(l\lambda), \text{RInd}_B^U(l\mu)) = \text{RHom}_{\text{block}(U)}(\text{RInd}_B^U(l\lambda + l\nu), \text{RInd}_B^U(l\mu + l\nu)). \quad (4.2.4)$$

To prove this isomorphism, we view the root lattice \mathbb{Y} as the subgroup of W_{aff} formed by translations. Let $\nu = s_{\alpha_1} \cdots s_{\alpha_r} \in W_{\text{aff}}$ be a reduced expression of $\nu \in \mathbb{Y}^{++} \subset W_{\text{aff}}$. Using the right W_{aff} -action $\tau \mapsto \tau^w$, we can write $l\nu = (0)^{(s_{\alpha_1} \cdots s_{\alpha_r})}$. Since ν is dominant, and $\tau^{(yw)} = (\tau^y)^w$, $\forall w, y \in W_{\text{aff}}$, we obtain $(0)^{(s_{\alpha_1} \cdots s_{\alpha_r})} = ((0)^{(s_{\alpha_1} \cdots s_{\alpha_{r-1}})})^{s_{\alpha_r}} \geq (0)^{(s_{\alpha_1} \cdots s_{\alpha_{r-1}})} \geq \dots \geq 0$. Therefore, for any $\lambda \in \mathbb{Y}$ and $j = 1, \dots, r-1$, we deduce

$$(l\lambda)^{(s_{\alpha_1} \cdots s_{\alpha_{j+1}})} = l\lambda + (0)^{(s_{\alpha_1} \cdots s_{\alpha_{j+1}})} \geq l\lambda + (0)^{(s_{\alpha_1} \cdots s_{\alpha_j})} = (l\lambda)^{(s_{\alpha_1} \cdots s_{\alpha_j})}.$$

Thus, we obtain

$$l\lambda + l\nu = (l\lambda)^{(s_{\alpha_1} \cdots s_{\alpha_r})} \geq (l\lambda)^{(s_{\alpha_1} \cdots s_{\alpha_{r-1}})} \geq \dots \geq l\lambda, \quad \forall \lambda \in \mathbb{Y}.$$

Hence, Lemma 4.1.1(ii) yields $\text{Ind}_{\mathbb{B}}^{\mathbb{U}}(l\lambda + l\nu) \simeq \theta_{\alpha_r}^+ \circ \theta_{\alpha_{r-1}}^+ \circ \dots \circ \theta_{\alpha_1}^+ \left(\text{Ind}_{\mathbb{B}}^{\mathbb{U}}(l\lambda) \right)$. At this point, isomorphism (4.2.4) follows from part (i) of Lemma 4.1.1, saying that the functor θ_{α} is an equivalence of categories.

To complete the proof, fix an arbitrary pair $\lambda, \mu \in \mathbb{Y}$, and choose $\nu \in \mathbb{Y}^{++}$ sufficiently large, so that $\nu - \lambda \in \mathbb{Y}^{++}$. Then, from (4.2.4) we deduce

$$\begin{aligned} & \text{RHom}_{\text{block}(\mathbb{U})} \left(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l\lambda), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l\mu) \right) \quad \text{shift by } \nu - \lambda \\ & \simeq \text{RHom}_{\text{block}(\mathbb{U})} \left(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l\nu), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l(\mu + \nu - \lambda)) \right) \quad \text{shift by } (-\nu) \\ & \simeq \text{RHom}_{\text{block}(\mathbb{U})} \left(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(0), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l(\mu - \lambda)) \right). \end{aligned} \quad (4.2.5)$$

We deduce that, for fixed i , the corresponding groups $R^i \text{Hom}_{\text{block}(\mathbb{U})}$ in (4.2.5) all have the same dimension. Moreover, by the special case $\lambda = 0$ of isomorphism (4.2.1), that has been already proved, this dimension equals $\dim R^i \text{Hom}_{\mathbb{B}}(\mathbb{k}_{\mathbb{B}}(0), \mathbb{k}_{\mathbb{B}}(l\mu - l\lambda))$. Further, using an obvious isomorphism $\text{RHom}_{\mathbb{B}}(\mathbb{k}_{\mathbb{B}}(0), \mathbb{k}_{\mathbb{B}}(l\mu - l\lambda)) \simeq \text{RHom}_{\mathbb{B}}(\mathbb{k}_{\mathbb{B}}(l\lambda), \mathbb{k}_{\mathbb{B}}(l\mu))$ we conclude

$$\dim \text{Ext}_{\mathbb{B}}^i(\mathbb{k}_{\mathbb{B}}(l\lambda), \mathbb{k}_{\mathbb{B}}(l\mu)) = \dim \text{Ext}_{\text{block}(\mathbb{U})}^i \left(\text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l\lambda), \text{RInd}_{\mathbb{B}}^{\mathbb{U}}(l\mu) \right). \quad (4.2.6)$$

This completes the proof of the Lemma. \square

Remark 4.2.7. Observe that formulas (4.2.3) and (4.2.5) actually produce, for any $\lambda, \mu \in \mathbb{Y}$, a certain map of the form required in (4.2.1). Unfortunately, we were unable to show that the map so constructed is indeed induced via the functor $\text{RInd}_{\mathbb{B}}^{\mathbb{U}}$, by functoriality. Therefore, below we will use an alternative, more round-about, approach.

4.3 A direct limit construction. Let $\bar{\mathfrak{b}} = \mathfrak{t} \oplus \bar{\mathfrak{n}}$ denote the Borel subalgebra opposite to \mathfrak{b} , so that the Chevalley generators $\{f_i\}_{i \in I}$ generate its nilradical $\bar{\mathfrak{n}}$. For any $\mu \in \mathbb{X}^{++}$, the simple \mathfrak{g} -module V_{μ} with highest weight μ is cyclically generated over $\mathcal{U}\bar{\mathfrak{b}}$ by its highest weight vector, i.e., a nonzero vector annihilated by \mathfrak{n} . Specifically, one has a $\mathcal{U}\bar{\mathfrak{n}}$ -module isomorphism $V_{\mu} = \mathcal{U}\bar{\mathfrak{n}} / \langle f_i^{\langle \mu, \bar{\alpha}_i \rangle + 1} \rangle_{i \in I}$. We see that, for any $\nu, \mu \in \mathbb{X}^{++}$ such that $\mu - \nu \in \mathbb{X}^{++}$, there is a unique, up to nonzero factor, map of $\mathcal{U}\bar{\mathfrak{b}}$ -modules $\mathbb{k}_{\mathcal{U}\bar{\mathfrak{b}}}(\nu - \mu) \otimes V_{\mu} \rightarrow V_{\nu}$, sending the highest weight line to the highest weight line.

Dualizing the construction and using the Cartan involution on $\mathcal{U}\mathfrak{g}$ that interchanges $\mathcal{U}\mathfrak{b}$ and $\mathcal{U}\bar{\mathfrak{b}}$, we deduce that there is a unique, up to nonzero factor, map of $\mathcal{U}\mathfrak{b}$ -modules $V_{\nu} \rightarrow V_{\mu} \otimes \mathbb{k}_{\mathcal{U}\mathfrak{b}}(\nu - \mu)$, sending the highest weight line to the highest weight line. For any fixed $\lambda \in \mathbb{Y} \subset \mathbb{X}$ and $\nu, \mu \in \mathbb{Y}$, the induced maps $\varepsilon_{\nu, \mu} : V_{\nu} \otimes \mathbb{k}_{\mathcal{U}\mathfrak{b}}(\lambda - \nu) \rightarrow V_{\mu} \otimes \mathbb{k}_{\mathcal{U}\mathfrak{b}}(\lambda - \mu)$ form a direct system with respect to the partial order $\nu \triangleleft \mu$ on \mathbb{Y} , cf. (3.4.5). We let $\lim_{\nu \in \mathbb{Y}^{++}} (V_{\nu}|_{\mathcal{U}\mathfrak{b}} \otimes \mathbb{k}_{\mathcal{U}\mathfrak{b}}(\lambda - \nu))$

denote the resulting direct limit $\mathcal{U}\mathfrak{b}$ -module. This $\mathcal{U}\mathfrak{b}$ -module is clearly co-free over the subalgebra $\mathcal{U}\mathfrak{n} \subset \mathcal{U}\mathfrak{b}$, and is co-generated by a single vector of weight λ .

Recall the $\mathcal{U}\mathfrak{b}$ -module $\mathbf{I}_{\lambda} = \text{Ind}_T^B \lambda$ introduced in §2.8. It is clear that there is a natural $\mathcal{U}\mathfrak{b}$ -module isomorphism

$$\lim_{\nu \in \mathbb{Y}^{++}} (V_{\nu}|_{\mathcal{U}\mathfrak{b}} \otimes \mathbb{k}_{\mathcal{U}\mathfrak{b}}(\lambda - \nu)) \simeq \mathbf{I}_{\lambda} = \text{Ind}_T^B \lambda. \quad (4.3.1)$$

Applying the Frobenius functor to each side of isomorphism (4.3.1) we obtain, for any $\lambda \in \mathbb{Y}$, the following isomorphisms of \mathcal{B} -modules:

$$\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{B}}(l\lambda) \simeq \phi(\mathbf{I}_{\lambda}) \simeq \lim_{\nu \in \mathbb{Y}^{++}} \left(\phi V_{\nu} \Big|_{\mathcal{B}} \otimes_{\mathbb{K}_{\mathcal{B}}} (l\lambda - l\nu) \right), \quad (4.3.2)$$

where the subalgebra $\mathfrak{p} \subset \mathcal{B}$ was defined in section 2.8.

Lemma 4.3.3. *For any $\lambda, \mu \in \mathbb{Y}$, the following canonical morphism, induced by functoriality of induction, is injective:*

$$\mathrm{Ext}_{\mathcal{B}}^i \left(\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{B}}(l\lambda), \mathrm{Ind}_{\mathfrak{p}}^{\mathcal{B}}(l\mu) \right) \hookrightarrow \mathrm{Ext}_{\mathrm{block}(\mathcal{U})}^i \left(\mathrm{RInd}_{\mathfrak{p}}^{\mathcal{U}}(l\lambda), \mathrm{RInd}_{\mathfrak{p}}^{\mathcal{U}}(l\mu) \right) \quad (4.3.4)$$

Proof. Recall that by formula (4.3.2) we have $\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{B}}(l\lambda) \simeq \phi \mathbf{I}_{\lambda}$, and therefore $\mathrm{RInd}_{\mathfrak{p}}^{\mathcal{U}}(l\lambda) = \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}} \left(\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{B}}(l\lambda) \right) = \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(\phi \mathbf{I}_{\lambda})$. To prove injectivity part of Lemma 4.3.3, we rewrite the morphism in (4.3.4) using Frobenius reciprocity as follows:

$$\begin{aligned} \mathrm{Ext}_{\mathfrak{p}}^i \left(\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{B}}(l\lambda), \mathbb{K}_{\mathfrak{p}}(l\mu) \right) &= \mathrm{Ext}_{\mathfrak{p}}^i(\phi \mathbf{I}_{\lambda}, \mathbb{K}_{\mathfrak{p}}(l\mu)) \\ &\xrightarrow{\tilde{\rho}} \mathrm{Ext}_{\mathfrak{p}}^i(\mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(\phi \mathbf{I}_{\lambda}), \mathbb{K}_{\mathfrak{p}}(l\mu)) = \mathrm{Ext}_{\mathfrak{p}}^i \left(\mathrm{Ind}_{\mathfrak{p}}^{\mathcal{U}}(l\lambda), \mathbb{K}_{\mathfrak{p}}(l\mu) \right). \end{aligned}$$

Here the morphism $\tilde{\rho}$ is induced by the canonical \mathcal{B} -module adjunction morphism $\rho : \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(\phi \mathbf{I}_{\lambda}) \rightarrow \phi \mathbf{I}_{\lambda}$, restricted to \mathfrak{p} . Injectivity of (4.3.4) would follow, provided we show that

- (i) the object $\mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(\phi \mathbf{I}_{\lambda}) \in D^b \mathrm{block}(\mathcal{U})$ is concentrated in degree 0, i.e., is an actual \mathcal{U} -module and, moreover;
- (ii) the morphism ρ is a surjection, which is split as a morphism of \mathfrak{p} -modules.

To prove (i), we apply the functor $\mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}$ to isomorphism (4.3.2) and obtain

$$\begin{aligned} \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(\phi \mathbf{I}_{\lambda}) &\simeq \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}} \left(\lim_{\nu \in \mathbb{Y}^{++}} \left(\phi V_{\nu} \Big|_{\mathcal{B}} \otimes_{\mathbb{K}_{\mathcal{B}}} (l\lambda - l\nu) \right) \right) \\ &= \phi V_{\nu} \bigotimes \left(\lim_{\nu \in \mathbb{Y}^{++}} \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(l\lambda - l\nu) \right) \end{aligned} \quad (4.3.5)$$

By the quantum group analogue of the Kempf vanishing, see [APW], for $\lambda \in -\mathbb{Y}^{++}$, the object $\mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(l\lambda) \in D^b \mathrm{block}(\mathcal{U})$ is isomorphic to $R^0 \mathrm{Ind}_{\mathcal{B}}^{\mathcal{U}}(l\lambda)$, an actual \mathcal{U} -module, and all other cohomology groups vanish, i.e., $R^i \mathrm{Ind}_{\mathcal{B}}^{\mathcal{U}}(l\lambda) = 0$, for all $i \neq 0$. Hence, formula (4.3.5) shows that $\mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}} \phi \mathbf{I}_{\lambda}$ is an actual \mathcal{U} -module, because $\lambda - \nu \in -\mathbb{Y}^{++}$, for ν large enough.

To prove property (ii), we use the isomorphism $\mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(l\lambda) = R^0 \mathrm{Ind}_{\mathcal{B}}^{\mathcal{U}}(l\lambda)$, for $\lambda \in -\mathbb{Y}^{++}$. Then, by Frobenius reciprocity one has a canonical \mathcal{B} -module projection $\gamma : \mathrm{RInd}_{\mathcal{B}}^{\mathcal{U}}(l\lambda) = \mathrm{Ind}_{\mathcal{B}}^{\mathcal{U}}(l\lambda) \rightarrow \mathbb{K}_{\mathcal{B}}(l\lambda)$. Furthermore, there is also an \mathcal{U} -module morphism $\mathbb{K}_{\mathcal{U}}(l\lambda) \rightarrow \mathrm{Ind}_{\mathcal{B}}^{\mathcal{U}}(l\lambda)$. The latter morphism provides a $\mathfrak{b} (= \mathcal{B} \cap \mathcal{U})$ -equivariant section of the projection γ that, moreover, respects the \mathbb{Y} -gradings. Hence the projection $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{U}}(l\lambda) \rightarrow \mathbb{K}_{\mathcal{B}}(l\lambda)$ is split as a morphism of \mathfrak{p} -modules. Using formula (4.3.5) we deduce from this, by taking direct limits as in the previous paragraph, that the projection ρ is also split as a morphism of \mathfrak{p} -modules. Therefore, formula (4.3.2) implies that the restriction of ρ to \mathfrak{p} is the projection to a direct summand. It follows that the map $\tilde{\rho}$ in (4.3.4) is injective. \square

Lemma 4.3.6. *The objects of the form $\text{Ind}_{\mathfrak{p}}^{\mathbb{B}}(l\lambda)$, $\lambda \in \mathbb{Y}$, generate $D_{\text{triv}}(\mathbb{B})$ as a triangulated category; moreover, the morphism (4.3.4) is an isomorphism.*

Proof. The first part of the Lemma is clear, since the algebra $\mathcal{U}\mathfrak{n}$ has finite homological dimension (hence, any \mathbb{B} -module in $D_{\text{triv}}(\mathbb{B})$ has a finite resolution by objects of the form $\text{Ind}_{\mathfrak{p}}^{\mathbb{B}}(l\lambda)$). Therefore, it remains to prove that (4.3.4) is an isomorphism. Observe that, for any given i , both sides in (4.3.4) are finite-dimensional vector spaces. This is so because the *Ext*-groups involved are finitely generated graded modules over the corresponding *Ext*-algebra $\text{Ext}^{\bullet}(\mathbb{k}, \mathbb{k})$, and the latter is known to be a finitely-generated graded algebra. Hence, by Lemma 4.3.3, we must only show that, for each $i \geq 0$, both sides in (4.3.4) are of the same dimension.

To prove this we observe that, for any finite dimensional G -module V (viewed as an $\widehat{\mathcal{U}}\mathfrak{g}$ -module), translation functors on $\text{block}(\mathcal{U})$ commute with the functor $M \mapsto M \otimes \phi V$. Hence, tensoring by ϕV , from Lemma 4.2.2 we deduce

$$\dim \text{Ext}_{\mathbb{B}}^i(\mathbb{k}_{\mathbb{B}}(l\lambda), \mathbb{k}_{\mathbb{B}}(l\mu) \otimes \phi V|_{\mathbb{B}}) = \dim \text{Ext}_{\text{block}(\mathcal{U})}^i\left(\text{RInd}_{\mathbb{B}}^{\mathcal{U}}(l\lambda), \text{RInd}_{\mathbb{B}}^{\mathcal{U}}(l\mu) \otimes \phi V\right). \quad (4.3.7)$$

We put $V = V_{\nu}$, a simple module with highest weight ν . The equality of dimensions in (4.3.4) follows from equation (4.3.7) by taking direct limit as $\nu \rightarrow +\infty$ in \mathbb{Y}^{++} , and using formula (4.3.5), in the same way as above. Lemma 4.3.6 is proved. \square

This completes the proof of Theorem 3.5.5.

5 Proof of Quantum group formality theorem

5.1 Constructing an equivariant dg-resolution. In order to begin the proof of Theorem 3.9.1 we recall the central subalgebra $Z \subset \mathfrak{B}$, see Definition 2.6.4.

Lemma 5.1.1. *There exists a (super)commutative dg-algebra $\mathbb{R} = \bigoplus_{i \leq 0} \mathbb{R}^i$, equipped with an $\mathcal{U}\mathfrak{b}$ -action, and such that*

- *The $\mathcal{U}\mathfrak{b}$ -action on \mathbb{R} preserves the grading, moreover, for each i , there is a direct sum decomposition $\mathbb{R}^i = \bigoplus_{\nu \in \mathbb{Y}} \mathbb{R}^i(\nu)$ such that $ur = \nu(u) \cdot r$, $\forall u \in \mathcal{U}\mathfrak{t} \subset \mathcal{U}\mathfrak{b}$, $r \in \mathbb{R}^i(\nu)$.*
- *$\mathbb{R}^0 = Z$, and the graded algebra \mathbb{R} is a free \mathbb{R}^0 -module;*
- *$H^0(\mathbb{R}) = \mathbb{k}$, and $H^i(\mathbb{R}) = 0$, for all $i \neq 0$.*

Proof. The argument is quite standard. We will construct inductively a sequence of \mathbb{Y} -graded $\mathcal{U}\mathfrak{b} \ltimes Z$ -modules R^i , $i = 0, -1, -2, \dots$, starting with $R^0 := Z$, and such that each R^i is free over Z . At every step, we put a differential (of degree +1) on the graded algebra $\text{Sym}(\bigoplus_{-n \leq i \leq 0} R^i)$, referred to as an *n-truncated dg-algebra*. We then set $\mathbb{R} := \text{Sym}(\bigoplus_{i \leq 0} R^i)$.

To do the induction step, assume we have already constructed all the modules R^i , $i = 0, -1, \dots, -n$, and differentials d in such a way that, for the *n-truncated dg-algebra* we have

$$H^j(\text{Sym}(\bigoplus_{-n \leq i \leq 0} R^i)) = \begin{cases} \mathbb{k} & \text{if } j = 0 \\ 0 & \text{if } -n + 1 \leq j < 0. \end{cases}$$

Inside the *n-truncated algebra*, we have the following $\mathcal{U}\mathfrak{b} \ltimes Z$ -submodule, C^{-n} , formed by degree $(-n)$ -cycles:

$$C^{-n} := \text{Ker}(\text{Sym}(\bigoplus_{-n \leq i \leq 0} R^i) \xrightarrow{d} \text{Sym}(\bigoplus_{-n \leq i \leq 0} R^i)),$$

(if $n = -1$ we set $C^{-1} := Z_\epsilon$). We can find an $\mathcal{U}\mathfrak{b} \ltimes Z$ -module surjection $R^{-n-1} \twoheadrightarrow C^{-n}$, such that R^{-n-1} is free as a Z -module. We let R^{-n-1} be the space of degree $(-n-1)$ -generators of our $(n+1)$ -truncated algebra, and define the differential on these new generators to be the map $R^{-n-1} \rightarrow C^{-n}$. This completes the induction step. The Lemma is proved. \square

The dg-algebra R has a natural augmentation given by the composite map

$$\epsilon_R : R = \bigoplus_{i \leq 0} R^i \longrightarrow R / (\bigoplus_{i < 0} R^i) = Z \xrightarrow{\epsilon} \mathbb{k}. \quad (5.1.2)$$

We observe that the last property stated in Lemma 5.1.1 implies that the map ϵ_R is a quasi-isomorphism.

Next, we form the tensor product $R \otimes_Z R$. This is again a Z -free (super-)commutative dg-algebra concentrated in non-positive degrees. We shall see (Lemma 5.1.4(i)) that $H^\bullet(R \otimes_Z R) \cong \Lambda$, where $\Lambda = \wedge^\bullet(\mathfrak{n}[1])$ is the exterior algebra generated by the vector space \mathfrak{n} placed in degree (-1) .

The dg-algebra $R \otimes_Z R$ acquires a natural $\mathcal{U}\mathfrak{b}$ -action, the tensor product of the $\mathcal{U}\mathfrak{b}$ -actions on both factors. We form the cross-product dg-algebra $\mathcal{U}\mathfrak{b} \ltimes (R \otimes_Z R)$, where the subalgebra $\mathcal{U}\mathfrak{b}$ is placed in grade degree zero. Lemma 5.1.4 below implies that we have

$$H^\bullet(\mathcal{U}\mathfrak{b} \ltimes (R \otimes_Z R)) = \mathcal{U}\mathfrak{b} \ltimes H^\bullet(R \otimes_Z R) = \mathcal{U}\mathfrak{b} \ltimes \Lambda = A, \quad (5.1.3)$$

cf., Notation 3.8.4. Thus, we may consider triangulated categories $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathcal{U}\mathfrak{b} \ltimes (R \otimes_Z R), \Lambda)$ and $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$, where in the latter case A and Λ are treated as a dg-algebras with trivial differential.

The R -bimodule \underline{R} may be viewed naturally as an object of $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathcal{U}\mathfrak{b} \ltimes (R \otimes_Z R), \Lambda)$.

Lemma 5.1.4. (i) $H^\bullet(R \otimes_Z R) \cong \Lambda$ (isomorphism of algebras and $\mathcal{U}\mathfrak{b}$ -modules);

(ii) The dg-algebra $R \otimes_Z R$ is $\mathcal{U}\mathfrak{b}$ -equivariantly formal, i.e., there is a dg-algebra quasi-isomorphism $\iota : \mathcal{U}\mathfrak{b} \ltimes \Lambda \xrightarrow{\text{qis}} \mathcal{U}\mathfrak{b} \ltimes (R \otimes_Z R)$.

(iii) The induced equivalence $\iota_* : D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda) \xrightarrow{\sim} D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathcal{U}\mathfrak{b} \ltimes (R \otimes_Z R), \Lambda)$ sends \mathbb{k}_A to $\iota_*(\mathbb{k}_A) = \underline{R}$.

Proof. By construction, the augmentation $\epsilon_R : R \rightarrow \mathbb{k}$ in (5.1.2) gives a free Z -algebra resolution of the trivial Z -module \mathbb{k}_Z . Thus, by definition of derived functors, the dg-algebra $R \otimes_Z R$ represents the object $\mathbb{k}_Z \overset{L}{\otimes}_Z \mathbb{k}_Z$ in the derived category of dg-algebras. Therefore, the cohomology algebra $H^\bullet(R \otimes_Z R)$ is isomorphic to the Tor-algebra $\text{Tor}_\bullet^Z(\mathbb{k}_Z, \mathbb{k}_Z)$. By Proposition 2.9.2, we have $Z \simeq \mathbb{k}[\overline{B} \cdot B / B]$. Hence, by Corollary 2.9.6(ii), we obtain $\mathcal{U}\mathfrak{b}$ -equivariant graded algebra isomorphisms $H^\bullet(R \otimes_Z R) \cong \text{Tor}_\bullet^Z(\mathbb{k}_Z, \mathbb{k}_Z) \cong \Lambda$.

Thus, to prove part (ii) of the Lemma we must construct an $\mathcal{U}\mathfrak{b}$ -equivariant dg-algebra quasi-isomorphism $R \otimes_Z R \xrightarrow{\text{qis}} \Lambda$. We first construct such a map that will only be a morphism of complexes of $\mathcal{U}\mathfrak{b}$ -modules (with the algebra structures forgotten).

To this end, we use the standard (reduced) bar-resolution $(\cdots \rightarrow Z \otimes Z_\epsilon \otimes Z_\epsilon \rightarrow Z \otimes Z_\epsilon \rightarrow Z) \xrightarrow{\text{qis}} \mathbb{k}_Z$, and replace the trivial Z -module \mathbb{k}_Z by a quasi-isomorphic $\mathcal{U}\mathfrak{b}$ -equivariant complex of free Z -modules. Applying the functor $\mathbb{k}_Z \otimes_Z (-)$ to this resolution term by term, we represent the object $\mathbb{k}_Z \overset{L}{\otimes}_Z \mathbb{k}_Z$ by the following complex

$$\text{Bar}^\bullet(Z_\epsilon) : \cdots \longrightarrow Z_\epsilon \otimes Z_\epsilon \otimes Z_\epsilon \longrightarrow Z_\epsilon \otimes Z_\epsilon \longrightarrow Z_\epsilon \rightarrow \mathbb{k}_Z.$$

Now, given $a \in Z_\epsilon$, let $\bar{a} \in Z_\epsilon / Z_\epsilon^2$ denote its image. It is well-known (see e.g. [Lo]) that the assignment

$$a_1 \otimes \cdots \otimes a_n \longmapsto \bar{a}_1 \wedge \bar{a}_2 \wedge \cdots \wedge \bar{a}_n, \quad \text{Bar}^\bullet(Z_\epsilon) \longrightarrow \wedge^\bullet(Z_\epsilon / Z_\epsilon^2) \quad (5.1.5)$$

yields an isomorphism of cohomology. The map (5.1.5) is clearly $\mathcal{U}\mathfrak{b}$ -equivariant, hence, we obtain a chain of $\mathcal{U}\mathfrak{b}$ -equivariant quasi-isomorphisms

$$\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbb{k}_{\mathbf{Z}} \overset{L}{\otimes}_{\mathbf{Z}} \mathbb{k}_{\mathbf{Z}} \cong \text{Bar}^{\bullet}(\mathbf{Z}_{\epsilon}) \xrightarrow[(5.1.5)]{\text{qis}} \wedge^{\bullet}(\mathbf{Z}_{\epsilon}/\mathbf{Z}_{\epsilon}^2) = \Lambda. \quad (5.1.6)$$

We can finally construct a dg-algebra quasi-isomorphism required in (5.1.7) as follows. Equip the vector space $\mathfrak{n}[1]$ with the trivial \mathbf{Z} -action (via the augmentation $\mathbf{Z} \rightarrow \mathbb{k}$) and with the natural adjoint $\mathcal{U}\mathfrak{b}$ -action. Let P^{\bullet} be an $\mathcal{U}\mathfrak{b} \ltimes \mathbf{Z}$ -module resolution of $\mathfrak{n}[1]$ such that each term P^i is free as a \mathbf{Z} -module.

Restricting the quasi-isomorphism $\Lambda \xrightarrow{\text{qis}} \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$ in (5.1.6) to the subspace $\mathfrak{n} \subset \Lambda$, we get a morphism $\mathfrak{n}[1] \rightarrow \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$ in the triangulated category of $\mathcal{U}\mathfrak{b}$ -modules. We can represent this morphism as an actual $\mathcal{U}\mathfrak{b}$ -module map of complexes $f : P^{\bullet} \rightarrow \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$, where P^{\bullet} is the \mathbf{Z} -free resolution of the vector space $\mathfrak{n}[1]$ (with trivial \mathbf{Z} -action) chosen in the previous paragraph. The morphism f can be uniquely extended, by multiplicativity, to a dg-algebra morphism $f_{\text{alg}} : \Lambda \rightarrow \mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$. The latter map gives, by construction, an isomorphism on cohomology: $\mathfrak{n} = \wedge^1 \mathfrak{n} \xrightarrow{\sim} H^{-1}(\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R})$. Therefore, since the cohomology algebra $H^{\bullet}(\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}) \cong \Lambda$ is freely generated by its first component $\wedge^1 \mathfrak{n}$, we deduce that the dg-algebra morphism f_{alg} induces a graded algebra isomorphism $\Lambda \xrightarrow{\sim} H^{\bullet}(\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R})$.

Thus, performing the cross-product construction yields a graded algebra isomorphism

$$\text{id}_{\mathcal{U}\mathfrak{b}} \ltimes f_{\text{alg}} : \mathcal{U}\mathfrak{b} \ltimes (\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}) \xrightarrow{\text{qis}} \mathcal{U}\mathfrak{b} \ltimes \Lambda, \quad (5.1.7)$$

that induces the isomorphism of cohomology constructed at the beginning of the proof. \square

Remark 5.1.8. The dg-algebra $\mathbf{R} \otimes_{\mathbf{Z}} \mathbf{R}$ is likely to be quasi-isomorphic to the bar complex $\text{Bar}(\mathbf{Z}_{\epsilon})$, equipped with the shuffle product algebra structure, cf. [Lo]. \diamond

5.2 DG-resolution of \mathfrak{b} . Recall that the quantum Borel algebra \mathfrak{B} is free over its central subalgebra \mathbf{Z} . We put $\mathbf{R}\mathfrak{b} := \mathbf{R} \otimes_{\mathbf{Z}} \mathfrak{B}$. Thus, $\mathbf{R}\mathfrak{b} = \bigoplus_{i \leq 0} \mathbf{R}^i \otimes_{\mathbf{Z}} \mathfrak{B}$ is a dg-algebra concentrated in non-positive degrees and such that its degree zero component is isomorphic to \mathfrak{B} . Further, the augmentation (5.1.2) induces an algebra map

$$\mathbf{R}\mathfrak{b} = \mathbf{R} \otimes_{\mathbf{Z}} \mathfrak{B} \rightarrow \mathbb{k}_{\mathbf{R}} \otimes_{\mathbf{Z}} \mathfrak{B} = \mathfrak{B}/(\mathbf{Z}) = \mathfrak{b}.$$

We may view the algebra \mathfrak{b} on the right as a dg-algebra with trivial differential, concentrated in degree zero. Then, Lemma 5.1.1 implies that the map above gives a quasi-isomorphism $\pi : \mathbf{R}\mathfrak{b} \xrightarrow{\text{qis}} \mathfrak{b}$.

Recall next the $\text{Ad}_{\text{hopf}} \mathbf{B}$ -action on \mathfrak{B} (see Proposition 2.9.2), and view $\mathbf{R} \otimes_{\mathbf{Z}} \mathfrak{B}$ as a tensor product of \mathbf{B} -modules, where the \mathbf{B} -action on the first factor is obtained from $\mathcal{U}\mathfrak{b}$ -action via the Frobenius functor. Performing the cross-product construction we obtain a dg-algebra quasi-isomorphism $\pi : \mathbf{B} \ltimes \mathbf{R}\mathfrak{b} \xrightarrow{\text{qis}} \mathbf{B} \ltimes \mathfrak{b}$.

Since $\mathfrak{b} = H^{\bullet}(\mathbf{R}\mathfrak{b}) \subset H^{\bullet}(\mathbf{B} \ltimes \mathbf{R}\mathfrak{b})$, we may consider the category $D_{\mathbb{Y}}^{\mathbf{B}}(\mathbf{B} \ltimes \mathbf{R}\mathfrak{b}, \mathfrak{b})$. The quasi-isomorphisms constructed above induce the following category equivalences

$$\pi_* : D_{\mathbb{Y}}(\mathbf{R}\mathfrak{b}, \mathfrak{b}) \xrightarrow{\sim} D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b}), \quad \text{and} \quad \pi_* : D_{\mathbb{Y}}^{\mathbf{B}}(\mathbf{B} \ltimes \mathbf{R}\mathfrak{b}, \mathfrak{b}) \xrightarrow{\sim} D_{\mathbb{Y}}^{\mathbf{B}}(\mathbf{B} \ltimes \mathfrak{b}, \mathfrak{b}). \quad (5.2.1)$$

5.3 Construction of a bi-functor. A key ingredient used to construct the equivalence of Theorem 3.9.1 is the following *bifunctor*:

$$D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathcal{U}\mathfrak{b} \ltimes (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}), \Lambda) \times D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathbb{R}\mathfrak{b}, \mathfrak{b}) \longrightarrow D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathbb{R}\mathfrak{b}, \mathfrak{b}), \quad M, N \longmapsto M \overset{L}{\otimes}_{\mathbb{R}} N. \quad (5.3.1)$$

In this formula, the tensor product $M \overset{L}{\otimes}_{\mathbb{R}} N$ is taken with respect to the action on M of the *second* factor in the algebra $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$ and with respect to the \mathbb{R} -module structure on N obtained by restriction to the subalgebra $\mathbb{R} \subset \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{B} = \mathbb{R}\mathfrak{b}$. The object $M \overset{L}{\otimes}_{\mathbb{R}} N$ thus obtained has an additional \mathbb{R} -action coming from the action of the *first* factor $\mathbb{R} \subset \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$ on M . It is instructive to view the action (on M) of the first factor in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$ as a *left* action, and of the second factor, as a *right* action, and thus regard M as an \mathbb{R} -bimodule. Then, the left \mathbb{R} -action on $M \overset{L}{\otimes}_{\mathbb{R}} N$ commutes with the \mathfrak{B} -action on $M \overset{L}{\otimes}_{\mathbb{R}} N$ induced (via the imbedding $\mathfrak{B} \hookrightarrow \mathbb{R}\mathfrak{b}$) from the one on N . Furthermore, the left \mathbb{R} - and the \mathfrak{B} -actions agree on the subalgebra $\mathbb{Z} \subset (\mathbb{R} \otimes 1) \cap (1 \otimes \mathfrak{B})$ (where the intersection is taken inside $\mathbb{R}\mathfrak{b} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{B}$), and combine together to make $M \overset{L}{\otimes}_{\mathbb{R}} N$ into a $\mathbb{R}\mathfrak{b}$ -module, that is, we put:

$$(r \otimes b) \cdot (m \otimes n) = ((r \otimes 1) \cdot m) \otimes b \cdot n, \quad \forall m \in M, n \in N, r \otimes b \in \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{B} = \mathbb{R}\mathfrak{b}.$$

There is also a \mathbb{B} -action on $M \overset{L}{\otimes}_{\mathbb{R}} N$ defined as the tensor product of \mathbb{B} -action on M induced via the Frobenius functor from the $\mathcal{U}\mathfrak{b}$ -action, and the given \mathbb{B} -action on N . It is straightforward to verify that the above actions provide $M \overset{L}{\otimes}_{\mathbb{R}} N$ with a well-defined $\mathbb{B} \ltimes \mathbb{R}\mathfrak{b}$ -module structure.

Remark 5.3.2. We note that although the tensor product in $M \overset{L}{\otimes}_{\mathbb{R}} N$ is taken over the commutative algebra \mathbb{R} , the resulting \mathbb{R} -action on $M \overset{L}{\otimes}_{\mathbb{R}} N$ is *not* compatible with other structures described above, due to the fact that the subalgebra $\mathbb{R} \subset \mathbb{B} \ltimes \mathbb{R}\mathfrak{b}$ is not *central*. Thus, it is imperative to use the ‘additional’ \mathbb{R} -action on $M \overset{L}{\otimes}_{\mathbb{R}} N$ (arising from the *first* tensor factor in $\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}$) in order to get an $\mathbb{B} \ltimes \mathbb{R}\mathfrak{b}$ -module structure on $M \overset{L}{\otimes}_{\mathbb{R}} N$.

The bifunctor (5.3.1) should be thought of as ‘changing’ the $\mathbb{R}\mathfrak{b}$ -module structure on N via the \mathbb{R} -bimodule M . \diamond

5.4 Main result. We now change our point of view and consider the algebra \mathfrak{b} as a *subalgebra* in \mathbb{B} , rather than a *quotient* of \mathfrak{B} . The imbedding $\mathfrak{b} \hookrightarrow \mathbb{B}$ gives, via multiplication in \mathbb{B} , a morphism of $\text{Ad}_{\text{hopf}} \mathbb{B}$ -modules $\text{mult} : \mathbb{B} \ltimes \mathfrak{b} \longrightarrow \mathbb{B}$, $x \otimes y \mapsto xy$. By Proposition 2.10.1, the map mult is in effect an algebra map. The induced direct image functor $\text{mult}_* : D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathfrak{b}, \mathfrak{b}) \longrightarrow D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b})$ is given by the (derived) tensor product functor $M \mapsto ((\mathbb{B} \ltimes \mathfrak{b}) / \text{Ker mult}) \overset{L}{\otimes}_{\mathbb{B} \ltimes \mathfrak{b}} M$. Thus, we can define the following composite dg-algebra map, and the corresponding direct image functor

$$\alpha : \mathbb{B} \ltimes \mathbb{R}\mathfrak{b} \xrightarrow[\pi]{\text{qis}} \mathbb{B} \ltimes \mathfrak{b} \xrightarrow{\text{mult}} \mathbb{B}, \quad (5.4.1)$$

$$\alpha_* : D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathbb{R}\mathfrak{b}, \mathfrak{b}) \xrightarrow{(5.2.1)} D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathfrak{b}, \mathfrak{b}) \xrightarrow{\text{mult}_*} D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b}).$$

Next, we consider the projection to the first factor $\beta : \mathbb{B} \ltimes \mathbb{R}\mathfrak{b} \longrightarrow \mathbb{B}$, $b \ltimes r \mapsto b \cdot \epsilon_{\mathbb{R}\mathfrak{b}}(r)$, where the augmentation $\epsilon_{\mathbb{R}\mathfrak{b}}$ is given by the tensor product $\epsilon_{\mathbb{R}\mathfrak{b}} := \epsilon_{\mathbb{R}} \otimes \epsilon_{\mathfrak{B}} : \mathbb{R}\mathfrak{b} = \mathbb{R} \otimes_{\mathbb{Z}} \mathfrak{B} \longrightarrow \mathbb{k}_{\mathbb{R}} \otimes \mathbb{k}_{\mathfrak{B}} = \mathbb{k}$. By Proposition 2.10.1, the map β is an algebra morphism that gives rise to a pull-back functor β^* . Thus, we obtain the following diagram of algebras and functors:

$$\begin{array}{ccccc} & \mathbb{B} \ltimes \mathbb{R}\mathfrak{b} & & D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathbb{R}\mathfrak{b}, \mathfrak{b}) & \\ & \swarrow \beta \quad \searrow \alpha & & \swarrow \beta^* \quad \searrow \alpha_* & \\ \mathbb{B} & & D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b}) & & D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b}) \end{array} \quad (5.4.2)$$

Definition 5.4.3. Let \mathfrak{L} be the *Steinberg U-module*, twisted by an appropriate 1-dimensional character $\mathbb{B} \rightarrow \mathbb{k}$ in such a way that its highest weight becomes equal to zero. Specifically, in the notation of sect. 3.4 below, we define \mathfrak{L} as the following \mathbb{B} -module $\mathfrak{L} := L^{(l-1)\rho} \otimes_{\mathbb{k}_B} ((1-l)\rho)$.

It is known, cf. [AP], that the module \mathfrak{L} becomes, when restricted to the subalgebra $\mathfrak{u}^+ \subset \mathbb{B}$, a rank one free \mathfrak{u}^+ -module. We regard \mathfrak{L} as an object of $D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b})$. A key property of this object exploited below, cf. (5.5.6), is that, in $D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b})$, one has a (quasi)-isomorphism

$$\alpha_* \beta^* \mathfrak{L} \simeq \mathbb{k}.$$

We are now in a position to combine the above constructions in order to define a functor $\mathfrak{F} : D_{\mathbb{Y}}^{\mathcal{U}^b}(\mathcal{U}^b \ltimes (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}), \Lambda) \rightarrow D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b})$. To this end, recall the notation $\mathbb{A} := \mathcal{U}^b \ltimes \Lambda$, see (3.8.4). We use the bifunctor (5.3.1), and the two functors in diagram (5.4.2), to introduce the following composite functor:

$$\begin{aligned} \mathfrak{F} : D_{\mathbb{Y}}^{\mathcal{U}^b}(\mathbb{A}, \Lambda) &\xrightarrow{\iota_*} D_{\mathbb{Y}}^{\mathcal{U}^b}(\mathcal{U}^b \ltimes (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}), \Lambda) \xrightarrow[(5.3.1)]{(-) \otimes_{\mathbb{R}}^L \beta^* \mathfrak{L}} D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B} \ltimes \mathbb{R} \mathfrak{b}, \mathfrak{b}) \xrightarrow{\alpha_*} D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b}), \\ M &\mapsto \mathfrak{F}(M) := \alpha_*(M \otimes_{\mathbb{R}}^L \beta^* \mathfrak{L}) = \text{mult}_* \pi_*(\iota_* M \otimes_{\mathbb{R}}^L \beta^* \mathfrak{L}). \end{aligned} \quad (5.4.4)$$

Here is a more precise version of Theorem 3.9.1.

Theorem 5.4.5. *We have $\mathfrak{F}(\mathbb{k}_{\mathbb{A}}) = \mathbb{k}_{\mathbb{B}}$, and $\mathfrak{F}(\mathbb{k}_{\mathbb{A}}(\lambda) \otimes M) = \mathbb{k}_{\mathbb{B}}(l\lambda) \otimes \mathfrak{F}(M)$, for any $\lambda \in \mathbb{Y}$ and $M \in D_{\mathbb{Y}}^{\mathcal{U}^b}(\mathbb{A}, \Lambda)$. Moreover, the functor \mathfrak{F} induces, for any $\lambda, \mu \in \mathbb{Y}$, natural isomorphisms*

$$\text{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}^b}(\mathbb{A}, \Lambda)}^{\bullet}(\mathbb{k}_{\mathbb{A}}(\lambda), \mathbb{k}_{\mathbb{A}}(\mu)) \xrightarrow{\sim} \text{Ext}_{D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b})}^{\bullet}(\mathfrak{F}(\mathbb{k}_{\mathbb{A}}(\lambda)), \mathfrak{F}(\mathbb{k}_{\mathbb{A}}(\mu))) = \text{Ext}_{D_{\mathbb{Y}}^{\mathbb{B}}(\mathbb{B}, \mathfrak{b})}^{\bullet}(\mathbb{k}_{\mathbb{B}}(\lambda), \mathbb{k}_{\mathbb{B}}(l\mu)).$$

The rest of this section is mainly devoted to the proof of the Theorem.

5.5 Comparison of functors. The functor (5.4.4) has a ‘non-equivariant analogue’, obtained by forgetting the Hopf-adjoint actions. Specifically, we form the composite map $\alpha' : \mathfrak{b} \ltimes \mathbb{R} \mathfrak{b} \xrightarrow{\pi} \mathfrak{b} \ltimes \mathfrak{b} \xrightarrow{\text{mult}} \mathfrak{b}$, where $\pi : \mathbb{R} \mathfrak{b} \xrightarrow{\text{qis}} \mathfrak{b}$ is the quasi-isomorphism constructed in §5.2. Let $\alpha'_* : D_{\mathbb{Y}}(\mathfrak{b} \ltimes \mathbb{R} \mathfrak{b}, \mathfrak{b}) \rightarrow D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})$ denote the corresponding functor. Also, we have the following non-equivariant counterpart of the bifunctor (5.3.1):

$$D_{\mathbb{Y}}(\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}, \Lambda) \times D_{\mathbb{Y}}(\mathfrak{b} \ltimes \mathbb{R} \mathfrak{b}, \mathfrak{b}) \rightarrow D_{\mathbb{Y}}(\mathfrak{b} \ltimes \mathbb{R} \mathfrak{b}, \mathfrak{b}), \quad M, N \mapsto M \otimes_{\mathbb{R}}^L N. \quad (5.5.1)$$

We use the equivalence ι_* induced by a non-equivariant analogue of Lemma 5.1.4 (cf. also (5.1.7)), to obtain the following functor

$$\begin{aligned} \mathfrak{F}' : D_{\mathbb{Y}}(\Lambda, \Lambda) &\xrightarrow{\iota_*} D_{\mathbb{Y}}(\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}, \Lambda) \xrightarrow[(5.5.1)]{(-) \otimes_{\mathbb{R}}^L \mathbb{k}} D_{\mathbb{Y}}(\mathfrak{b} \ltimes \mathbb{R} \mathfrak{b}, \mathfrak{b}) \xrightarrow{\alpha'_*} D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b}), \\ M &\mapsto \mathfrak{F}'(M) := \alpha'_*(M \otimes_{\mathbb{R}}^L \mathbb{k}) = \text{mult}_* \pi_*(\iota_* M \otimes_{\mathbb{R}}^L \mathbb{k}). \end{aligned} \quad (5.5.2)$$

In order to prove Theorem 5.4.5, we will need to relate the functors \mathfrak{F} and \mathfrak{F}' . To this end, given an algebra A and a subalgebra $\mathfrak{a} \subset A$, we let $\text{Res}_{\mathfrak{a}}^A : A\text{-mod} \rightarrow \mathfrak{a}\text{-mod}$ denote the obvious restriction functor, and use similar notation for derived categories.

Lemma 5.5.3. *There is an isomorphism of functors: $\text{Res}_{\mathbf{b}}^{\mathbf{B}} \circ \mathfrak{F} \simeq \mathfrak{F}' \circ \text{Res}_{\Lambda}^{\mathbf{A}}$, in other words, the following diagram commutes:*

$$\begin{array}{ccc} D_{\mathbb{Y}}^{\mathcal{U}\mathbf{b}}(\mathbf{A}, \Lambda) & \xrightarrow{\text{Res}_{\Lambda}^{\mathbf{A}}} & D_{\mathbb{Y}}(\Lambda, \Lambda) \\ \mathfrak{F} \downarrow & & \mathfrak{F}' \downarrow \\ D_{\mathbb{Y}}^{\mathbf{B}}(\mathbf{B}, \mathbf{b}) & \xrightarrow{\text{Res}_{\mathbf{b}}^{\mathbf{B}}} & D_{\mathbb{Y}}(\mathbf{b}, \mathbf{b}) . \end{array}$$

Proof. We restrict the algebra morphisms α and β in diagram (5.4.2) to the subalgebra $\mathbf{b} \ltimes \mathbf{Rb} \subset \mathbf{B} \ltimes \mathbf{Rb}$, and consider the following diagram:

$$\begin{array}{ccccc} & & \mathbf{b} \ltimes \mathbf{Rb} = \mathbf{b} \ltimes (\mathbf{R} \otimes_Z \mathfrak{B}) & & \\ & \nearrow \beta|_{\mathbf{b} \ltimes \mathbf{Rb}} & \downarrow \text{Id}_{\mathbf{b}} \ltimes \pi & \nwarrow \alpha|_{\mathbf{b} \ltimes \mathbf{Rb}} & \\ \mathbf{b} & \xleftarrow{\text{Id} \otimes \epsilon_{\mathbf{b}}} \mathbf{b} \otimes \mathbf{b} & \xleftarrow{\sim \gamma} \mathbf{b} \ltimes \mathbf{b} & \xrightarrow{\text{mult}} \mathbf{b} & \end{array} \quad (5.5.4)$$

In this diagram, the map $\pi : \mathbf{Rb} \rightarrow \mathbf{b}$ is the quasi-isomorphism of §5.2, and the map $\text{mult} : \mathbf{b} \ltimes \mathbf{b} \rightarrow \mathbf{b}$ (given by multiplication in the algebra \mathbf{b}) is an algebra morphism, by Proposition 2.10.1. Thus, the right triangle in diagram (5.5.4) commutes, by definition of the map α .

In the left triangle of diagram (5.5.4), we have the map

$$\beta|_{\mathbf{b} \ltimes \mathbf{Rb}} : \mathbf{b} \ltimes \mathbf{Rb} = \mathbf{b} \ltimes (\mathbf{R} \otimes_Z \mathfrak{B}) \longrightarrow \mathbf{b}, \quad \mathbf{b} \ltimes (r \otimes \tilde{b}) \longmapsto \mathbf{b} \cdot \epsilon_{\mathbf{R}}(r) \cdot \pi(\tilde{b}).$$

Further, the map γ in (5.5.4) is the algebra isomorphism of Proposition 2.10.1(iii), which is given by: $\mathbf{b} \ltimes \mathbf{b}' \longmapsto \mathbf{b} \otimes \mathbf{b}'$. Thus, we see that the triangle on the left of diagram (5.5.4) commutes also.

Computing the inverse of γ , we get $\gamma^{-1}(\mathbf{b} \otimes \mathbf{b}') = \mathbf{b} \ltimes (S(\mathbf{b}) \cdot \mathbf{b}')$. Therefore, we find: $\text{mult} \circ \gamma^{-1}(\mathbf{b} \otimes \mathbf{b}') = \epsilon_{\mathbf{b}}(\mathbf{b}) \cdot \mathbf{b}'$. Further, observe that since the $\text{Ad}_{\text{hopf}} \mathbf{b}$ -action on Z is trivial, we have an algebra isomorphism $\mathbf{b} \ltimes (\mathbf{R} \otimes_Z \mathfrak{B}) \simeq \mathbf{R} \otimes_Z (\mathbf{b} \ltimes \mathfrak{B})$. We introduce the composite quasi-isomorphism

$$\begin{aligned} \theta &:= \gamma \circ (\text{Id}_{\mathbf{b}} \ltimes \pi) : \mathbf{b} \ltimes \mathbf{Rb} = \mathbf{R} \otimes_Z (\mathbf{b} \ltimes \mathfrak{B}) \xrightarrow{\text{qis}} \mathbf{b} \otimes \mathbf{b}, \\ r \otimes (\mathbf{b} \ltimes \tilde{b}) &\longmapsto \epsilon_{\mathbf{R}}(r) \cdot (\mathbf{b} \otimes (\mathbf{b} \cdot \pi(\tilde{b}))). \end{aligned}$$

Thus, we can rewrite diagram (5.5.4) in the following more symmetric form:

$$\begin{array}{ccccc} & & \mathbf{R} \otimes_Z (\mathbf{b} \ltimes \mathfrak{B}) & & \\ & \nearrow \beta|_{\mathbf{b} \ltimes \mathbf{Rb}} & \downarrow \theta & \nwarrow \alpha|_{\mathbf{b} \ltimes \mathbf{Rb}} & \\ \mathbf{b} & \xleftarrow{\text{Id} \otimes \epsilon_{\mathbf{b}}} \mathbf{b} \otimes \mathbf{b} & \xrightarrow{\epsilon_{\mathbf{b}} \otimes \text{Id}} \mathbf{b} & & \end{array} \quad (5.5.5)$$

Next, let $\mathfrak{L}' := \text{Res}_{\mathbf{b}}^{\mathbf{B}} \mathfrak{L}$ be the Steinberg module \mathfrak{L} viewed as an object of $D_{\mathbb{Y}}(\mathbf{b}, \mathbf{b})$. Since $\mathfrak{L}|_{\mathbf{u}^+} \simeq \mathbf{u}^+$, we deduce that: $(\epsilon_{\mathbf{b}})_*(\mathfrak{L}') = \mathbb{k}$, where $(\epsilon_{\mathbf{b}})_*$ is the direct image functor corresponding to the augmentation $\epsilon_{\mathbf{b}} : \mathbf{b} \rightarrow \mathbb{k}$. On the other hand, one may also view the \mathbf{b} -module \mathfrak{L}' as a \mathfrak{B} -module via the projection $\mathfrak{B} \twoheadrightarrow \mathfrak{B}/(Z) = \mathbf{b}$. Then, using the left triangle in diagram (5.5.5), we get:

$$\text{Res}_{\mathbf{b} \ltimes \mathbf{Rb}}^{\mathbf{B} \ltimes \mathbf{Rb}}(\beta^* \mathfrak{L}) = \theta^* \circ (\text{Id} \otimes \epsilon_{\mathbf{b}})^* \mathfrak{L}' = \theta^*(\mathfrak{L}' \otimes \mathbb{k}) = \mathfrak{L}' \otimes \mathbb{k}. \quad (5.5.6)$$

Therefore, for any $\tilde{M} \in D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathcal{U}\mathfrak{b} \ltimes (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}), \Lambda)$, applying bifunctors (5.3.1), (5.5.1), we obtain

$$\theta_* (\text{Res}_{\mathfrak{b} \ltimes \mathbb{R} \mathfrak{b}}^{\mathbb{B} \ltimes \mathbb{R} \mathfrak{b}} (\tilde{M} \otimes_{\mathbb{R}}^L \beta^* \mathfrak{L})) = \theta_* ((\text{Res}_{\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R}}^{\mathcal{U}\mathfrak{b} \ltimes (\mathbb{R} \otimes_{\mathbb{Z}} \mathbb{R})} \tilde{M}) \otimes_{\mathbb{R}}^L (\mathfrak{L}' \otimes \mathbb{k})). \quad (5.5.7)$$

Now, let $\tilde{M} := \iota_* M$, for some $M \in D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)$. Then, using the right triangle in (5.5.5) and the definition of the functor \mathfrak{F}' , see (5.5.2), we obtain

$$\begin{aligned} \text{Res}_{\mathfrak{b}}^{\mathbb{B}} \mathfrak{F}(M) &= \text{Res}_{\mathfrak{b}}^{\mathbb{B}} \circ \alpha_* (M \otimes_{\mathbb{R}}^L \beta^* \mathfrak{L}) \text{ by (5.5.7)} \\ &= (\epsilon_{\mathfrak{b}} \otimes \text{Id})_* \circ \theta_* (M \otimes_{\mathbb{R}}^L \beta^* \mathfrak{L}) \\ &= (\epsilon_{\mathfrak{b}} \otimes \text{Id})_* \circ \theta_* (\tilde{M} \otimes_{\mathbb{R}}^L (\mathfrak{L}' \otimes \mathbb{R})) \\ &= (\epsilon_{\mathfrak{b}} \otimes \text{Id})_* (\mathfrak{L}' \otimes \mathfrak{F}'(\text{Res}_{\Lambda}^A M)) \\ &= ((\epsilon_{\mathfrak{b}})_* (\mathfrak{L}')) \otimes \mathfrak{F}'(\text{Res}_{\Lambda}^A M) = \mathfrak{F}'(\text{Res}_{\Lambda}^A M), \end{aligned}$$

and the Lemma is proved. \square

5.6 ‘Deformation’ morphism. Below, we will use a well-known result of Gerstenhaber saying that, for any algebra A , the graded algebra $\text{Ext}_{A\text{-bimod}}^{\bullet}(\underline{A}, \underline{A})$ is always commutative. We also remind the reader that the category $A\text{-mod}$ may be viewed as a *module category* over the category $A\text{-bimod}$, of A -bimodules. This gives, for any $M \in A\text{-mod}$, a canonical graded algebra morphism, to be referred to as *evaluation* at M :

$$\text{ev}_M : \text{Ext}_{A\text{-bimod}}^{\bullet}(\underline{A}, \underline{A}) \longrightarrow \text{Ext}_{A\text{-mod}}^{\bullet}(M, M). \quad (5.6.1)$$

Now, recall that the quantum Borel algebra \mathfrak{B} is a free module over its central subalgebra Z . Let $\epsilon \in \text{Spec } Z$ denote the ‘base-point’ corresponding to the augmentation ideal $Z_{\epsilon} \subset Z$. We will view \mathfrak{B} as a flat family of (non-commutative) algebras over the smooth base $\text{Spec } Z$ whose fiber over the base-point is the algebra $\mathfrak{b} = \mathbb{k}_{\epsilon} \otimes_Z \mathfrak{B} \cong \mathfrak{B}/(Z)$. Otherwise put, the algebra \mathfrak{B} is a multi-parameter deformation of \mathfrak{b} . By the classical work of Gerstenhaber, such a deformation gives a linear map $T_{\epsilon}(\text{Spec } Z) \rightarrow \text{Ext}_{\mathfrak{b}\text{-bimod}}^2(\underline{\mathfrak{b}}, \underline{\mathfrak{b}})$, where $T_{\epsilon}(\text{Spec } Z)$ denotes the tangent space at the point ϵ . By commutativity of the algebra $\text{Ext}_{\mathfrak{b}\text{-bimod}}^{\bullet}(\underline{\mathfrak{b}}, \underline{\mathfrak{b}})$, the linear map above can be uniquely extended, by multiplicativity, to a degree doubling algebra morphism

$$\text{deform} : \text{Sym}^{\bullet}(T_{\epsilon}(\text{Spec } Z)) \longrightarrow \text{Ext}_{\mathfrak{b}\text{-bimod}}^{2\bullet}(\underline{\mathfrak{b}}, \underline{\mathfrak{b}}). \quad (5.6.2)$$

Next, we would like to take the $\text{Ad}_{\text{hopf}} \mathbb{B}$ -action on \mathfrak{b} , cf. Proposition 2.9.2, into considerations. The $\text{Ad}_{\text{hopf}} \mathbb{B}$ -action induces, for each $j \geq 0$, a \mathbb{B} -action on $\text{Ext}_{\mathfrak{b}\text{-bimod}}^j(\underline{\mathfrak{b}}, \underline{\mathfrak{b}})$, that makes $\text{Ext}_{\mathfrak{b}\text{-bimod}}^{\bullet}(\underline{\mathfrak{b}}, \underline{\mathfrak{b}})$ a graded \mathbb{B} -algebra. Enhancing Gerstenhaber’s construction to the equivariant setting, one finds that the algebra map (5.6.2) is actually a morphism of \mathbb{B} -modules.

It turns out that the \mathbb{B} -action on each side of (5.6.2) descends to the algebra $\mathbb{B}/(\mathfrak{b})$, which is isomorphic to $\mathcal{U}\mathfrak{b}$ via the Frobenius map. This follows, for the RHS of (5.6.2), from the general result saying that the Hopf-adjoint action of any Hopf algebra \mathfrak{a} on $\text{Ext}_{\mathfrak{a}\text{-bimod}}^{\bullet}(\underline{\mathfrak{a}}, \underline{\mathfrak{a}})$ is trivial, see §2.11. For the LHS, we use Corollary 2.9.6 saying that there is a canonical $\text{Ad}_{\text{hopf}} \mathbb{B}$ -equivariant isomorphism of vector spaces $T_{\epsilon}(\text{Spec } Z) \simeq Z_{\epsilon}/Z_{\epsilon}^2 \simeq \mathfrak{n}^*$. Thus, the morphism in (5.6.2) becomes the following $\mathcal{U}\mathfrak{b}$ -equivariant graded algebra morphism

$$\text{deform} : \text{Sym}^{\bullet}(\mathfrak{n}^*[-2]) \longrightarrow \text{Ext}_{\mathfrak{b}\text{-bimod}}^{\bullet}(\underline{\mathfrak{b}}, \underline{\mathfrak{b}}). \quad (5.6.3)$$

5.7 General deformation formality theorem. Our proof of Theorem 5.4.5 is based on a much more general Theorem 5.7.1 below, proved in [BG].

To explain the setting of [BG], let \mathfrak{b} be (temporarily) an *arbitrary* associative algebra and \mathfrak{B} an *arbitrary* flat deformation of \mathfrak{b} over a smooth base $\mathrm{Spec} Z$. Choose R , a Z -free dg-resolution of the trivial Z -module \mathbb{k}_Z (as in Lemma 5.1.1, but with $\mathcal{U}\mathfrak{b}$ -action ignored), corresponding to the base point $\epsilon \in \mathrm{Spec} Z$. Put $\mathfrak{n} := T_\epsilon Z$ and $\Lambda := \wedge^\bullet(\mathfrak{n}^*[1])$. We form the dg-algebras $R\mathfrak{b} := R \otimes_Z \mathfrak{B}$, and $R \otimes_Z R$. Then we establish, as we have done in §5.1, dg-algebra quasi-isomorphisms $\iota : \Lambda \xrightarrow{\mathrm{qis}} R \otimes_Z R$, cf. (5.1.7), and $\pi : R\mathfrak{b} \xrightarrow{\mathrm{qis}} \mathfrak{b}$. Thus formula (5.5.2) gives, in our general situation, a well-defined functor $\mathfrak{F}' : D_{\mathbb{Y}}(\Lambda, \Lambda) \longrightarrow D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})$.

In [BG], we prove the following result.

Theorem 5.7.1. *We have $\mathfrak{F}'(\mathbb{k}_\Lambda) = \mathbb{k}_{\mathfrak{b}}$. Furthermore, the induced map \mathfrak{F}'_* makes the following diagram commute*

$$\begin{array}{ccc} \mathrm{Sym}(\mathfrak{n}^*[-2]) & \xrightarrow[\sim]{(3.3.2)} \mathrm{Ext}_{D_{\mathbb{Y}}(\Lambda, \Lambda)}^\bullet(\mathbb{k}_\Lambda, \mathbb{k}_\Lambda) & \xrightarrow[\sim]{\mathfrak{F}'_*} \mathrm{Ext}_{D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})}^\bullet(\mathfrak{F}'(\mathbb{k}_\Lambda), \mathfrak{F}'(\mathbb{k}_\Lambda)) \\ \text{deform} \downarrow (5.6.3) & & \parallel \\ \mathrm{Ext}_{\mathfrak{b}\text{-bimod}}^\bullet(\underline{\mathfrak{b}}, \underline{\mathfrak{b}}) & \xrightarrow[\text{(5.6.1)}]{\mathrm{ev}} \mathrm{Ext}_{D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})}^\bullet(\mathbb{k}_{\mathfrak{b}}, \mathbb{k}_{\mathfrak{b}}). \end{array}$$

5.8 Proof of Theorem 5.4.5. Recall that $A := \mathcal{U}\mathfrak{b} \ltimes \Lambda$. We first prove that

$$\mathfrak{F}(\mathbb{k}_A(\lambda)) = \mathbb{k}_{\mathfrak{B}}(l\lambda), \quad \forall \lambda \in \mathbb{Y}. \quad (5.8.1)$$

To this end, we use Lemma 5.5.3 and Theorem 5.7.1 to deduce that the dg-module $\mathrm{Res}_{\mathfrak{B}}^{\mathfrak{B}} \mathfrak{F}(\mathbb{k}_A) \in D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})$ is quasi-isomorphic to the trivial module \mathbb{k} . In particular, it has a single non-zero cohomology group: $H^0(\mathfrak{F}(\mathbb{k}_A)) \simeq \mathbb{k}$, no matter whether it is considered as a \mathfrak{B} -module, or as a \mathfrak{b} -module. But the action of the augmentation ideal $(\mathcal{U}^+)_\epsilon$ of the (sub)algebra $\mathcal{U}^+ \subset \mathfrak{B}$ on the cohomology of any object of the category $D_{\mathbb{Y}}(\mathfrak{B}, \mathfrak{b})$ is necessarily nilpotent. Hence the subalgebra \mathcal{U}^+ acts trivially (that is, via the augmentation) on the 1-dimensional vector space $H^0(\mathfrak{F}(\mathbb{k}_A)) \simeq \mathbb{k}$. Furthermore, since the module \mathfrak{L} has been normalized so that its highest weight is equal to zero, it immediately follows that the 1-dimensional space $H^0(\mathfrak{F}(\mathbb{k}_A(\lambda)))$ has weight $l\lambda$ with respect to the \mathcal{U}° -action. Thus, we have a $\mathfrak{B} \ltimes \mathfrak{b}$ -module isomorphism $H^0(\mathfrak{F}(\mathbb{k}_A(\lambda))) \simeq \mathbb{k}_{\mathfrak{B}}(l\lambda)$. We conclude that the object $\mathfrak{F}(\mathbb{k}_A(\lambda))$ is quasi-isomorphic to $\mathbb{k}_{\mathfrak{B}}(l\lambda) \in D_{\mathbb{Y}}(\mathfrak{B}, \mathfrak{b})$, and (5.8.1) is proved.

To complete the proof the Theorem, we must show that the functor \mathfrak{F} induces a graded algebra isomorphism

$$\begin{aligned} \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)}^\bullet(\mathbb{k}_A(\lambda), \mathbb{k}_A(\mu)) &\xrightarrow{\sim} \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathfrak{B}}(\mathfrak{B}, \mathfrak{b})}^\bullet(\mathfrak{F}(\mathbb{k}_A(\lambda)), \mathfrak{F}(\mathbb{k}_A(\mu))) \\ &= \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathfrak{B}}(\mathfrak{B}, \mathfrak{b})}^\bullet(\mathbb{k}_{\mathfrak{B}}(l\lambda), \mathbb{k}_{\mathfrak{B}}(l\mu)). \end{aligned} \quad (5.8.2)$$

To compare the Ext-groups on the LHS and on the RHS, we use the spectral sequence provided by Lemma 2.11.2. Specifically, since $A/(\Lambda) = \mathcal{U}\mathfrak{b}$ and $\mathfrak{B}/(\mathfrak{b}) = \mathcal{U}\mathfrak{b}$ we have the following two spectral sequences, see (2.11.1):

$$\begin{array}{ccc} H^p(\mathcal{U}\mathfrak{b}, \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)}^q(\mathbb{k}_A(\lambda), \mathbb{k}_A(\mu))) = E_2^{p,q} & \Longrightarrow & E_\infty^{p,q} = \mathrm{gr} \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(A, \Lambda)}^{p+q}(\mathbb{k}_A(\lambda), \mathbb{k}_A(\mu)) \\ \downarrow \mathfrak{F}'_* & & \downarrow \mathfrak{F}_* \\ H^p(\mathcal{U}\mathfrak{b}, \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathfrak{B}}(\mathfrak{B}, \mathfrak{b})}^q(\mathbb{k}_{\mathfrak{B}}(l\lambda), \mathbb{k}_{\mathfrak{B}}(l\mu))) = E_2^{p,q} & \Longrightarrow & E_\infty^{p,q} = \mathrm{gr} \mathrm{Ext}_{D_{\mathbb{Y}}^{\mathfrak{B}}(\mathfrak{B}, \mathfrak{b})}^{p+q}(\mathbb{k}_{\mathfrak{B}}(l\lambda), \mathbb{k}_{\mathfrak{B}}(l\mu)). \end{array} \quad (5.8.3)$$

The vertical arrow on the left of the diagram is induced by the map $\mathfrak{F}'_* : \text{Ext}_{D_{\mathbb{Y}}(\Lambda, \Lambda)}^\bullet(\mathbb{k}_\Lambda, \mathbb{k}_\Lambda) \rightarrow \text{Ext}_{D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})}^\bullet(\mathfrak{F}'(\mathbb{k}_\Lambda), \mathfrak{F}'(\mathbb{k}_\Lambda))$. Theorem 5.7.1 implies that this map may be identified with the composite map

$$\text{deform} \circ \text{ev}_{\mathbb{k}_\mathfrak{b}} : \text{Ext}_{D_{\mathbb{Y}}(\Lambda, \Lambda)}^\bullet(\mathbb{k}_\Lambda, \mathbb{k}_\Lambda) \cong \text{Sym}(\mathfrak{n}^*[-2]) \longrightarrow \text{Ext}_{D_{\mathbb{Y}}(\mathfrak{b}, \mathfrak{b})}^\bullet(\mathfrak{F}'(\mathbb{k}_\Lambda), \mathfrak{F}'(\mathbb{k}_\Lambda)).$$

But the latter map is exactly the map that was used in [GK] to construct the $\mathcal{U}\mathfrak{b}$ -equivariant isomorphism in Proposition 3.7.1. Thus, we conclude that the functor \mathfrak{F}'_* induces an isomorphism between the E_2 -terms of the two spectral sequences in (5.8.3).

Further, the vertical map between the E_∞ -terms on the right of diagram (5.8.3) is induced by the functor \mathfrak{F} . This map coincides, by Lemma 5.5.3, with the map induced by the isomorphism between the E_2 -terms of the spectral sequences. Hence, it is itself an isomorphism. It follows that morphism (5.8.2) is an isomorphism. \square

PART II: Geometry.

Throughout Part II (with the exception of §9) we let $\mathbb{k} = \mathbb{C}$.

6 The loop Grassmannian and the Principal nilpotent.

In this section we recall a connection, discovered in [G2], between the cohomology of a loop Grassmannian, to be introduced below, and the principal nilpotent element in the Lie algebra \mathfrak{g} .

Let $D^b(X)$ be the bounded derived category of constructible complexes on an algebraic variety X , cf. [BBD]. Given an algebraic group \mathbf{G} and a \mathbf{G} -action on X , we let $D_{\mathbf{G}}^b(X)$ denote the \mathbf{G} -equivariant bounded derived category on X ; see [BL] for more information on the equivariant derived category. We write $D_{\mathbf{G}\text{-mon}}^b(X)$, for the full subcategory of $D^b(X)$ formed by " \mathbf{G} -monodromic" complexes, that is, formed by complexes whose cohomology sheaves are locally constant along \mathbf{G} -orbits.

We let $\text{Perv}_{\mathbf{G}}(X) \subset D_{\mathbf{G}}^b(X)$, resp., $\text{Perv}_{\mathbf{G}\text{-mon}}(X) \subset D_{\mathbf{G}\text{-mon}}^b(X)$ stand for the abelian category of \mathbf{G} -equivariant, resp. \mathbf{G} -monodromic, perverse sheaves on X .

6.1 The loop group. Let G^\vee be a complex connected semisimple group with maximal torus $T^\vee = \mathbb{C}^* \otimes_{\mathbb{Z}} \mathbb{Y}$, which is dual to (G, T) in the sense of Langlands. Thus, G^\vee is a simply-connected group such that the root system of (G^\vee, T^\vee) is dual to that of $(\mathfrak{g}, \mathfrak{t})$. Let $\check{\mathfrak{g}} = \text{Lie } G^\vee$ be the Lie algebra of G^\vee . The Lie algebra of the maximal torus $T^\vee \subset G^\vee$ gives a distinguished Cartan subalgebra: $\text{Lie } T^\vee = \check{\mathfrak{t}} = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Y} = \mathfrak{t}^*$ in $\check{\mathfrak{g}}$.

Let $\mathcal{K} = \mathbb{C}((z))$ be the field of formal Laurent power series, and $\mathcal{O} = \mathbb{C}[[z]] \subset \mathcal{K}$ its ring of integers, that is, the ring of formal power series regular at $z = 0$. Write $G^\vee(\mathcal{K})$, resp. $G^\vee(\mathcal{O})$, for the set of \mathcal{K} -rational, resp. \mathcal{O} -rational, points of G^\vee . The coset space $\text{Gr} := G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ is called the *loop Grassmannian*. It has the natural structure of an ind-scheme, more precisely, Gr is a direct limit of a sequence of $G^\vee(\mathcal{O})$ -stable projective varieties of increasing dimension, see e.g. [BD], [Ga], [G2] or [L1], such that the action of $G^\vee(\mathcal{O})$ on any such variety factors through a finite-dimensional quotient of $G^\vee(\mathcal{O})$.

An Iwasawa decomposition for $G^\vee(\mathcal{K})$, see [G2], [PS], implies that the loop Grassmannian is isomorphic, as a topological space, to the space of based loops into a compact form of the complex group G^\vee . It follows that Gr is an H -space, hence, the cohomology $H^\bullet(\text{Gr}, \mathbb{C})$ has the natural structure of a graded commutative and cocommutative Hopf algebra. Further, the group G^\vee being simply-connected, we deduce that the loop Grassmannian Gr is connected.

6.2 Cohomology of the loop Grassmannian. We recall that the group $G^\vee(\mathcal{O})$ is homotopy equivalent to G^\vee , hence for the $G^\vee(\mathcal{O})$ -equivariant cohomology of a point we have

$$H_{G^\vee(\mathcal{O})}^\bullet(pt) = H_{G^\vee}^\bullet(pt) = H^\bullet(BG^\vee) = \mathbb{C}[\mathfrak{t}]^W = \mathbb{C}[\mathfrak{t}^*]^W = \mathbb{C}[\mathfrak{g}^*]^G = (\mathrm{Sym} \mathfrak{g})^G, \quad (6.2.1)$$

where BG^\vee stands for the classifying space of the group G^\vee . Thus $H_{G^\vee(\mathcal{O})}^\bullet(\mathrm{Gr})$, the $G^\vee(\mathcal{O})$ -equivariant cohomology of the Grassmannian, has a natural $H_{G^\vee(\mathcal{O})}^\bullet(pt)$ -module structure, hence, a $\mathbb{C}[\mathfrak{t}^*]^W$ -module structure.

Next, we introduce the notation $\mathfrak{g}^x \subset \mathfrak{g}$ for the Lie algebra of the isotropy group of an element $x \in \mathfrak{g}^*$ under the coadjoint action (thus, if one uses the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ provided by an invariant form, then \mathfrak{g}^x becomes the centralizer of x in \mathfrak{g}). Let $\mathfrak{g}^{\mathrm{reg}} \subset \mathfrak{g}^*$ be the Zariski open dense subset of regular (not necessarily semisimple) elements in \mathfrak{g}^* . The family of spaces $\{\mathfrak{g}^x, x \in \mathfrak{g}^{\mathrm{reg}}\}$, gives an $\mathrm{Ad} \, G$ -equivariant vector bundle on $\mathfrak{g}^{\mathrm{reg}}$. We let \mathcal{G} denote the corresponding vector bundle on the adjoint quotient space $\mathfrak{g}^{\mathrm{reg}}/\mathrm{Ad} \, G$. The latter space is isomorphic, due to Kostant [Ko], to \mathfrak{t}^*/W , hence is a smooth affine variety. We will often regard \mathcal{G} as a vector bundle on \mathfrak{t}^*/W via the Kostant isomorphism. The fibers of \mathcal{G} are abelian Lie subalgebras in \mathfrak{g} , and we let \mathcal{UG} denote the vector bundle on $\mathfrak{g}^{\mathrm{reg}}/\mathrm{Ad} \, G$ with fibers $\mathcal{U}(\mathfrak{g}^x)$, $x \in \mathfrak{g}^{\mathrm{reg}}$. Let $\Gamma(\mathfrak{t}^*/W, \mathcal{UG})$ be the commutative algebra of global regular sections of \mathcal{UG} .

Fix a principal \mathfrak{sl}_2 -triple $\langle t, e, f \rangle \subset \mathfrak{g}$, such that $t = \sum_{\alpha \in R_+} \check{\alpha} \in \mathfrak{t}$, and such that the principal nilpotent $e \in \mathfrak{g}$ is a linear combination of simple root vectors with non-zero coefficients. The Lie algebra \mathfrak{g}^e is an abelian Lie subalgebra in \mathfrak{g} of dimension $rk \, \mathfrak{g}$. The ad-action of t puts a grading on \mathfrak{g} , and we endow \mathfrak{g}^e and $\mathcal{U}(\mathfrak{g}^e)$, the enveloping algebra of \mathfrak{g}^e , with induced gradings.

The natural \mathbb{C}^* -action on \mathfrak{t}^* by dilations makes \mathfrak{t}^*/W a \mathbb{C}^* -variety. Moreover, the grading on \mathfrak{g} considered above, gives a \mathbb{C}^* -action on \mathfrak{g} , hence, makes \mathcal{G} a \mathbb{C}^* -equivariant vector bundle on \mathfrak{t}^*/W . Thus, $\Gamma(\mathfrak{t}^*/W, \mathcal{UG})$ acquires a grading compatible with the algebra structure. In [G2] we have proved the following

Lemma 6.2.2 ([G2]). *There is a natural graded Hopf algebra isomorphism $\varphi_{\mathcal{U}} : H_{G^\vee(\mathcal{O})}^\bullet(\mathrm{Gr}) \simeq \Gamma(\mathfrak{t}^*/W, \mathcal{UG})$.* \square

Observe that the fiber over $0 \in \mathfrak{t}^*/W$ of the vector bundle \mathcal{G} clearly identifies with \mathfrak{g}^e . Hence, from Lemma 6.2.2 we get

Corollary 6.2.3. [G2, Proposition 1.7.2] *There is a natural graded Hopf algebra isomorphism $\varphi : H^\bullet(\mathrm{Gr}, \mathbb{C}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g}^e)$.* \square

6.3 Geometric Satake equivalence. Let $D^b(\mathrm{Gr})$ denote the bounded derived category of constructible complexes on Gr , to be understood as a direct limit of the corresponding bounded derived categories on finite dimensional projective subvarieties that exhaust Gr . One similarly defines $\mathrm{Perv}(\mathrm{Gr}) \subset D^b(\mathrm{Gr})$, the abelian category of perverse sheaves.

Definition 6.3.1. Let $\mathcal{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$ be the (full) abelian subcategory in $\mathrm{Perv}(\mathrm{Gr})$ formed by semisimple⁴ $G^\vee(\mathcal{O})$ -equivariant perverse sheaves on Gr .

For any $\mathcal{L} \in \mathcal{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$, there is a standard convolution functor: $D^b(\mathrm{Gr}) \rightarrow D^b(\mathrm{Gr})$, $\mathcal{M} \mapsto \mathcal{M} \star \mathcal{L} := a_*(\mathcal{M} \tilde{\boxtimes} \mathcal{L})$, where $a : G^\vee(\mathcal{K}) \times_{G^\vee(\mathcal{O})} \mathrm{Gr} \rightarrow \mathrm{Gr}$ is the action-map, and $\mathcal{M} \tilde{\boxtimes} \mathcal{L}$ stands for a twisted version of external tensor product, see [MV] or [G2] for more details. A fundamental result due to Gaitsgory says that this functor takes perverse sheaves into perverse sheaves, that is, we have the following

⁴Any $G^\vee(\mathcal{O})$ -equivariant perverse sheaf on Gr is, in effect, automatically semisimple, cf. e.g. [MV].

Theorem 6.3.2 ([Ga]). *The convolution gives an exact bifunctor*

$$\mathrm{Perv}(\mathrm{Gr}) \times \mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr}) \longrightarrow \mathrm{Perv}(\mathrm{Gr}), \quad \mathcal{M}, \mathcal{L} \longmapsto \mathcal{M} \star \mathcal{L}. \quad \square$$

Remark 6.3.3. In the special case, where $\mathcal{M} \in \mathrm{Perv}(\mathrm{Gr})$ is a perverse sheaf which is constant along the Schubert cell stratification (by Iwahori orbits) of the loop Grassmannian, cf. §9.1 below, the Theorem above has been first conjectured in [G2, p.22], and proved by Lusztig [L4] shortly after that. This special case of Theorem 6.3.2 is the only case that will be actually used in the present paper. \diamond

For each $g \in G^\vee(\mathcal{K})$, the double coset $G^\vee(\mathcal{O}) \cdot g \cdot G^\vee(\mathcal{O})$ contains an element $\lambda \in \mathrm{Hom}(\mathbb{C}^*, T^\vee)$, viewed as a loop in G^\vee . Moreover, such an element is unique up to the action of W , the Weyl group. This gives a parametrization of $G^\vee(\mathcal{O})$ -orbits in Gr by dominant (co)weights $\lambda \in \mathrm{Hom}(\mathbb{C}^*, T^\vee)^{++} = \mathbb{Y}^{++}$. We write $\overline{G^\vee(\mathcal{O})} \cdot \lambda$ for the $G^\vee(\mathcal{O})$ -orbit corresponding to a dominant (co)weight λ . The closure, $\overline{G^\vee(\mathcal{O})} \cdot \lambda \subset \mathrm{Gr}$ is known to be a finite dimensional projective variety, singular in general. Let IC_λ denote the intersection complex on $\overline{G^\vee(\mathcal{O})} \cdot \lambda$ corresponding to the constant sheaf on $G^\vee(\mathcal{O}) \cdot \lambda$ (extended by zero on $\mathrm{Gr} \setminus \overline{G^\vee(\mathcal{O})} \cdot \lambda$, and normalized to be a perverse sheaf). The IC_λ , $\lambda \in \mathbb{Y}^{++}$, are the simple objects of the category $\mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$. Theorem 6.3.2 puts on $\mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$ the structure of a monoidal category, via the convolution product.

Recall the tensor category $\mathrm{Rep}(G)$ of finite dimensional rational representations of G . For each $\lambda \in \mathbb{Y}^{++}$, let $V_\lambda \in \mathrm{Rep}(G)$ denote an irreducible representation with highest weight λ . The proof of the following fundamental result, inspired by Lusztig [L1], can be found in [G2, Theorem 1.4.1] (following an idea of Drinfeld); a more geometric proof (involving a different commutativity constraint, also suggested by Drinfeld) has been found later in [MV]; the most conceptual argument is given in [Ga].

Theorem 6.3.4. *There is an equivalence $\mathcal{P} : \mathrm{Rep}(G) \xrightarrow{\sim} \mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$, of monoidal categories which sends V_λ to IC_λ , for any $\lambda \in \mathbb{Y}^{++}$.* \square

6.4 Fiber functors. We will need a more elaborate ‘equivariant’ version of Theorem 6.3.4, established in [G2]. To formulate it, identify T^\vee with the subgroup in $G^\vee(\mathcal{K})$ formed by constant loops into T^\vee . Thus, any object $A \in \mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$ may be regarded as a T^\vee -equivariant perverse sheaf, hence there are well defined T^\vee -equivariant (hyper)-cohomology groups, $H_{T^\vee}^\bullet(\mathrm{Gr}, A)$. Given $s \in \mathrm{Lie} T^\vee$, we write $H_s(A) = H_{T^\vee}^\bullet(\mathrm{Gr}, A)|_s$ for the T^\vee -equivariant (hyper)-cohomology of A specialized at s , viewed as a point in $\mathrm{Spec} H_{T^\vee}^\bullet(pt)$. For $s = 0$, we have $H_s(A) = H^\bullet(\mathrm{Gr}, A)$, is the ordinary cohomology of A (due to the collapse of the spectral sequence for equivariant intersection cohomology).

Observe that, for s regular, the s -fixed point set in Gr is the lattice \mathbb{Y} , viewed as a discrete subset in Gr via the natural imbedding $i : \mathbb{Y} = \mathrm{Hom}(\mathbb{C}^*, T^\vee) \hookrightarrow \mathrm{Gr}$. For each $\lambda \in \mathbb{Y}$, let $i_\lambda : \{\lambda\} \hookrightarrow \mathrm{Gr}$ denote the corresponding one point imbedding. By the Localisation theorem in equivariant cohomology, the map: $H_s(i^!A) \rightarrow H_s(A)$, induced by the adjunction morphism: $i_!i^!A \rightarrow A$ yields, see [G2, (3.6.1)], the following direct sum

Fixed point decomposition:

$$H_s(A) = \bigoplus_{\lambda \in \mathbb{Y}} H_s(i_\lambda^!A), \quad \forall A \in \mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr}). \quad (6.4.1)$$

Recall the principal \mathfrak{sl}_2 -triple $\langle t, e, f \rangle \subset \mathfrak{g}$. Observe that the element $t+e$ is $\mathrm{Ad} G$ -conjugate to t , hence is a regular semisimple element in \mathfrak{g} . Thus, $\mathfrak{h} := \mathfrak{g}^{t+e}$ is a Cartan subalgebra. Furthermore, the fiber of the vector bundle \mathcal{UG} over the $\mathrm{Ad} G$ -conjugacy class $\mathrm{Ad} G(t+e) = \mathrm{Ad} G(t) \subset \mathfrak{g}^{\mathrm{reg}}$ gets identified with $\mathcal{U}(\mathfrak{g}^{t+e}) = \mathcal{U}\mathfrak{h}$.

Proposition 6.4.2 ([G2]). (i) For any $s \in \text{Lie } T^\vee$, the assignment $A \mapsto H_s(A)$ gives a fiber functor on the tensor category $\text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$.

(ii) There is an isomorphism of the functor $H_s(-)$ on $\text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$ with the forgetful functor on $\text{Rep}(G)$ (i.e., a system of isomorphisms $\varphi_V : H_s(\mathcal{P}V) \xrightarrow{\sim} V$, $\forall V \in \text{Rep}(G)$, compatible with morphisms in $\text{Rep}(G)$ and with the tensor structure) such that:

For any $u \in H_s(\text{Gr})$, the natural action of u on the hyper-cohomology $H_s(\mathcal{P}V)$ corresponds, via φ_V and the isomorphism $\varphi_u : H_s(\text{Gr}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{h})$ of Lemma 6.2.2, to the natural action of $\varphi_u(u) \in \mathcal{U}(\mathfrak{g}^{t+e})$ in the G -module V . \square

6.5 Equivariant and Brylinski's filtrations. The standard grading on the equivariant cohomology $H_{T^\vee}^\bullet(\text{Gr}, A)$ induces, after specialization at a point $s \in \text{Spec } H_{T^\vee}^\bullet(\text{Gr}) = \check{\mathfrak{t}}$, a canonical increasing filtration, $W_\bullet H_s(\text{Gr}, A)$, on the specialized equivariant cohomology. Furthermore, the collapse of the spectral sequence for equivariant intersection cohomology yields a natural isomorphism $H^\bullet(\text{Gr}, A) \cong H_{\{o\}}^\bullet(A)$, where the LHS stands for the (non-equivariant) cohomology of A and the RHS stands for the specialization of equivariant cohomology of A at the zero point: $o \in \check{\mathfrak{t}} = \text{Spec } H_{T^\vee}^\bullet(\text{Gr})$. On the other hand, for any $s \in \check{\mathfrak{t}} = \text{Spec } H_{T^\vee}^\bullet(\text{Gr})$, one has a canonical graded space isomorphism $\text{gr}_\bullet^W H_s(A) \simeq H_{\{o\}}^\bullet(A)$, by the definition of filtration W_\bullet . Thus, composing the two isomorphisms we obtain, for any $s \in \text{Spec } H_{T^\vee}^\bullet(\text{Gr})$, a natural graded space isomorphism $\text{gr}_\bullet^W H_s(A) \simeq H^\bullet(\text{Gr}, A)$.

From now on we will make a particular choice of the point $s \in \text{Spec } H_{T^\vee}^\bullet(\text{Gr}, A) = \check{\mathfrak{t}}$. Specifically, we let $s = \sum_{\alpha \in R_+} \alpha \in \check{\mathfrak{t}}$ be the element "dual", in a sense, to $t = \sum_{\alpha \in R_+} \check{\alpha} \in \mathfrak{t}$. Further, the eigenvalues of the t -action in any finite dimensional G -module V are known to be integral.

Definition 6.5.1 (Brylinski filtration). We define an increasing filtration $W_\bullet V$ on $V \in \text{Rep}(G)$ by letting $W_k V$ be the direct sum of all eigenspaces of t with eigenvalues $\leq k$.

Further, given a finite dimensional \mathfrak{g} -module V , and a weight $\mu \in \mathfrak{h}^*$, write $V(\mu)$ for the corresponding weight space of V (with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^{e+t}$, not \mathfrak{t}).

Theorem 6.5.2 ([G2], Thm. 5.3.1). If $s = \sum_{\alpha \in R_+} \check{\alpha}$ then the isomorphisms $\varphi_V : H_s(\mathcal{P}V) \xrightarrow{\sim} V$ (of Proposition 6.4.2(ii)) can be chosen so that, in addition to claims of Proposition 6.4.2, one has:

- The canonical filtration $W_\bullet H_s(\mathcal{P}V)$ goes, under the isomorphism φ_V , to the filtration $W_\bullet V$;
- The fixed point decomposition (6.4.1) corresponds, under the isomorphism φ_V , to the weight decomposition: $V = \bigoplus_{\mu \in \mathbb{Y}} V(\mu)$ with respect to the Cartan subalgebra \mathfrak{h} . \square

To replace equivariant cohomology by the ordinary cohomology in the Theorem above, note first that, for a G -module V and for $\mu = 0$ we have $V(0) = V^{\mathfrak{h}}$ (note that the weights of any finite-dimensional G -module belong to the root lattice \mathbb{Y}). The filtration $W_\bullet V$ induces by restriction a filtration on $V^{\mathfrak{h}}$, and R. Brylinski [Br] proved

Proposition 6.5.3. For any $V \in \text{Rep}(G)$ there is a canonical graded space isomorphism $\text{gr}_\bullet^W(V^{\mathfrak{h}}) \simeq V^{\mathfrak{g}^e}$. \square

In particular, for $V = \mathfrak{g}$, the adjoint representation, the Proposition yields a canonical graded space isomorphism $\text{gr}(\mathfrak{h}) \simeq \mathfrak{g}^e$ (which has been constructed earlier by Kostant), hence, a graded algebra isomorphism $\text{gr}(\mathcal{U}\mathfrak{h}) \simeq \mathcal{U}(\mathfrak{g}^e)$. Thus, passing to associated graded objects in Theorem 6.5.2, and using the canonical isomorphisms: $\text{gr}_\bullet^W H_s(\text{Gr}) \simeq H^\bullet(\text{Gr}, \mathbb{C})$ and $\text{gr}_\bullet^W H_s(A) \simeq H^\bullet(\text{Gr}, A)$, $\forall A \in \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$, yields

Corollary 6.5.4 ([G2], **Theorems 1.6.3, 1.7.6**). *The isomorphism of functors $\varphi_V : H_s(\mathcal{P}V) \xrightarrow{\sim} V$, of Theorem 6.5.2, gives an isomorphism of tensor functors $\text{gr}(\varphi_V) : H^\bullet(\text{Gr}, \mathcal{P}V) \xrightarrow{\sim} V$, $\forall V \in \text{Rep}(G)$, such that*

- *The grading on $H^\bullet(\text{Gr}, \mathcal{P}V)$ goes, under the isomorphism φ_V , to the grading on V by the eigenvalues of the t -action.*
- *For any $u \in H^\bullet(\text{Gr}, \mathbb{C})$, the natural action of u on the hyper-cohomology $H^\bullet(\text{Gr}, \mathcal{P}V)$ corresponds, via $\text{gr}(\varphi_V)$ and the isomorphism $\varphi_u : H^\bullet(\text{Gr}, \mathbb{C}) \xrightarrow{\sim} \mathcal{U}(\mathfrak{g}^e)$ of Corollary 6.2.2, to the natural action of $\varphi_u(u) \in \mathcal{U}(\mathfrak{g}^e)$ in the G -module V . \square*

We have used here an obvious canonical identification of V , viewed as graded space, with $\text{gr}_\bullet^W V$, the associated graded space corresponding to the filtration $W_\bullet V$.

7 Self-extensions of the Regular sheaf

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent variety in \mathfrak{g} , and $e \in \mathcal{N}$ a fixed regular element. The results of [G2] outlined in the previous section allow to ‘see’ the element e , as well as its centralizer \mathfrak{g}^e , in terms of perverse sheaves on the loop Grassmannian $\text{Gr} = G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$. One of the goals of this section is to show how to reconstruct the whole nilpotent variety \mathcal{N} , not just the principal nilpotent conjugacy class $\text{Ad } G \cdot e \subset \mathcal{N}$, from the category of perverse sheaves on Gr .

7.1 The regular perverse sheaf \mathcal{R} . Let $\widehat{\mathcal{U}\mathfrak{g}} := \mathbb{C}[G]^\vee$ be the continuous dual of the Hopf algebra $\mathbb{C}[G]$, viewed as a topological algebra with respect to the topology induced by the augmentation. Thus, $\widehat{\mathcal{U}\mathfrak{g}}$ is a topological Hopf algebra equipped with a canonical continuous perfect Hopf pairing $\widehat{\mathcal{U}\mathfrak{g}} \times \mathbb{C}[G] \rightarrow \mathbb{C}$. The pairing yields, cf. the proof of Lemma 2.8.3, a canonical Hopf algebra imbedding $j : \mathcal{U}\mathfrak{g} \hookrightarrow \widehat{\mathcal{U}\mathfrak{g}}$, thus identifies $\widehat{\mathcal{U}\mathfrak{g}}$ with a completion of the enveloping algebra $\mathcal{U}\mathfrak{g}$. Similarly to the situation considered in §2.6, we have a natural equivalence $\text{Rep}(\widehat{\mathcal{U}\mathfrak{g}}) \cong \text{Rep}(G)$, so that the (isomorphism classes of) simple objects of $\text{Rep}(\widehat{\mathcal{U}\mathfrak{g}})$ are labelled by the set \mathbb{Y}^{++} . We will view the left regular representation of the algebra $\widehat{\mathcal{U}\mathfrak{g}}$ as a projective (pro-)object in the category $\text{Rep}(\widehat{\mathcal{U}\mathfrak{g}})$.

Let $\mathcal{R} := \mathcal{P}(\mathbb{C}[G])$ be the ind-object of the category $\mathcal{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$ corresponding to the regular representation $\mathbb{C}[G]$, viewed as an ind-object of the category $\text{Rep}(G)$. Then, $\mathcal{R}^\vee = \mathcal{P}(\mathbb{C}[G]^\vee) = \mathcal{P}(\widehat{\mathcal{U}\mathfrak{g}})$, is the dual pro-object. Explicitly, applying the functor \mathcal{P} to the G -bimodule direct sum decomposition of the regular representation on the left (below), and using Theorem 6.3.4(i) we deduce:

$$\mathbb{C}[G] = \bigoplus_{\lambda \in \mathbb{Y}^{++}} V_\lambda \otimes_{\mathbb{C}} V_\lambda^*, \quad \mathcal{R} = \bigoplus_{\lambda \in \mathbb{Y}^{++}} IC_\lambda \otimes_{\mathbb{C}} V_\lambda^*, \quad \mathcal{R}^\vee = \prod_{\lambda \in \mathbb{Y}^{++}} IC_\lambda \otimes_{\mathbb{C}} V_\lambda^*.$$

Observe that right translation by an element $g \in G$ gives a morphism $R_g : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$ of left G -modules. Hence, applying the functor $\mathcal{P}(-)$ we get, for any $g \in G$, a morphism $R_g : \mathcal{R} \rightarrow \mathcal{R}$, that corresponds to the g -action on the factor V_λ^* in the above decomposition $\mathcal{R} = \bigoplus IC_\lambda \otimes_{\mathbb{C}} V_\lambda^*$. The collection of morphisms R_g , $g \in G$, satisfies an obvious associativity. Therefore, for any objects $M, N \in D^b(\text{Gr})$, these morphisms induce, by functoriality, a G -action on the graded vector space $\text{Ext}_{D^b(\text{Gr})}^\bullet(M, N \star \mathcal{R})$. This is the G -module structure on the various Ext-groups that will be considered below.

7.2 Two Ext-algebras. The first Ext-algebra that we are going to consider is the space $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}, \mathcal{R})$, equipped with Yoneda product. More explicitly, write the ind-object \mathcal{R} as $\mathcal{R} = \varinjlim \mathcal{R}_\alpha$, and accordingly write the dual pro-object as $\mathcal{R}^\vee = \varprojlim \mathcal{R}_\alpha^\vee$. Then, we have

$$\begin{aligned} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}, \mathcal{R}) &:= \varprojlim_{\beta} \varinjlim_{\alpha} \text{Ext}_{D^b(\text{Gr})}(\mathcal{R}_\beta, \mathcal{R}_\alpha) \\ &\simeq \varprojlim_{\beta} \varinjlim_{\alpha} \text{Ext}_{D^b(\text{Gr})}((\mathcal{R}_\alpha)^\vee, (\mathcal{R}_\beta)^\vee) =: \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}^\vee, \mathcal{R}^\vee), \end{aligned} \quad (7.2.1)$$

where we have used the canonical isomorphism $\text{Ext}^\bullet(\mathcal{L}, \mathcal{M}) = \text{Ext}^\bullet(\mathcal{M}^\vee, \mathcal{L}^\vee)$.

To define the second algebra observe first that multiplication of functions makes $\mathbb{C}[G]$ a ringed ind-object of the category $\text{Rep}(G)$, that is, the product map induces the following morphisms

$$m_{\mathbb{C}[G]} : \mathbb{C}[G] \otimes \mathbb{C}[G] \longrightarrow \mathbb{C}[G], \quad \text{resp.}, \quad \mathfrak{m} = \mathcal{P}(m_{\mathbb{C}[G]}): \mathcal{R} \star \mathcal{R} \longrightarrow \mathcal{R}, \quad (7.2.2)$$

in the categories $\lim \text{ind Rep}(G)$ and $\lim \text{ind Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$, respectively.

Write $\mathbf{1}_{\text{Gr}} = IC_0 = \mathcal{P}\mathbb{C}$ for the sky-scraper sheaf (corresponding to the trivial 1-dimensional G -module) at the base point of Gr , and let $\mathcal{R}[i]$ denote the shift of \mathcal{R} in the derived category.

We define an associative graded algebra structure on the vector space $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) := \bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, IC_\lambda) \otimes_{\mathbb{C}} V_\lambda^*$ as follows. Let $x \in \text{Ext}_{D^b(\text{Gr})}^i(\mathbf{1}_{\text{Gr}}, \mathcal{R}) = \text{Hom}_{D^b(\text{Gr})}(\mathbf{1}_{\text{Gr}}, \mathcal{R}[i])$. Taking convolution of the identity morphism $\text{id}_{\mathcal{R}} : \mathcal{R} \rightarrow \mathcal{R}$ with x , viewed as a "derived morphism", gives a morphism $\mathcal{R} = \mathcal{R} \star \mathbf{1}_{\text{Gr}} \xrightarrow{\star x} \mathcal{R} \star \mathcal{R}[i]$. Given $y \in \text{Ext}_{D^b(\text{Gr})}^j(\mathbf{1}_{\text{Gr}}, \mathcal{R})$, we define $y \cdot x \in \text{Ext}_{D^b(\text{Gr})}^{j+i}(\mathbf{1}_{\text{Gr}}, \mathcal{R})$ to be the composite:

$$y \cdot x : \mathbf{1}_{\text{Gr}} \xrightarrow{y} \mathcal{R}[j] = (\mathcal{R} \star \mathbf{1}_{\text{Gr}})[j] \xrightarrow{\star x} \mathcal{R} \star \mathcal{R}[i+j] \xrightarrow{\mathfrak{m}} \mathcal{R}[i+j]. \quad (7.2.3)$$

Similarly, for any $\mathcal{M} \in \text{Perv}_1(\text{Gr})$, the following maps make $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{M} \star \mathcal{R})$ a graded $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$ -module: $\mathbf{1}_{\text{Gr}} \longrightarrow \mathcal{M} \star \mathcal{R}[j] \xrightarrow{\star x} \mathcal{M} \star \mathcal{R} \star \mathcal{R}[i+j] \xrightarrow{\mathfrak{m}} \mathcal{M} \star \mathcal{R}[i+j]$.

Note that the complex $\mathbf{1}_{\text{Gr}}$ is the unambiguously determined direct summand of \mathcal{R} , and the corresponding projection: $\mathcal{R} \twoheadrightarrow \mathbf{1}_{\text{Gr}}$ induces an imbedding $\varepsilon_{\text{geom}} : \text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) \hookrightarrow \text{Ext}^\bullet(\mathcal{R}, \mathcal{R})$. It is easy to check that this imbedding becomes an algebra homomorphism, provided the Ext-group on the right is equipped with the Yoneda product, and the Ext-group on the left is equipped with the product defined in (7.2.3). Dually, there is a map $\varepsilon_{\text{geom}} : \text{Ext}^\bullet(\mathcal{R}^\vee, \mathbf{1}_{\text{Gr}}) \hookrightarrow \text{Ext}^\bullet(\mathcal{R}^\vee, \mathcal{R}^\vee)$ induced by the imbedding: $\mathbf{1}_{\text{Gr}} \hookrightarrow \mathcal{R}^\vee$.

7.3 Main result. The adjoint \mathfrak{g} -action on \mathcal{N} makes the coordinate ring $\mathbb{C}[\mathcal{N}]$ a locally finite $\mathcal{U}\mathfrak{g}$ -module, hence an $\widehat{\mathcal{U}\mathfrak{g}}$ -module. Let $\widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathcal{N}]$ be the corresponding cross-product algebra. There is an obvious algebra imbedding $\varepsilon_{\text{alg}} : \mathbb{C}[\mathcal{N}] \hookrightarrow \widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathcal{N}]$. We put a grading on $\widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathcal{N}]$ by taking the natural grading on the subalgebra $\mathbb{C}[\mathcal{N}] \subset \widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathcal{N}]$, and by placing $\widehat{\mathcal{U}\mathfrak{g}}$ in grade degree zero.

Theorem 7.3.1. *There are natural G -equivariant graded algebra isomorphisms Ψ and ψ making the following diagram commute:*

$$\begin{array}{ccc} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}^\vee, \mathbf{1}_{\text{Gr}}) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) & \xrightarrow{\varepsilon_{\text{geom}}} & \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}, \mathcal{R}) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}^\vee, \mathcal{R}^\vee) \\ \downarrow \psi & & \downarrow \Psi \\ \mathbb{C}[\mathcal{N}] & \xrightarrow{\varepsilon_{\text{alg}}} & \widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathcal{N}]. \end{array}$$

The rest of this section is devoted to the proof of the Theorem.

7.4 Some general results. According to the well-known results of Kostant, the centralizer in G (an adjoint group) of the principal nilpotent element e is a connected unipotent subgroup $G^e \subset G$ with Lie algebra \mathfrak{g}^e . Furthermore, the $\text{Ad } G$ -conjugacy class of e is known to be the open dense subset $\mathcal{N}^{\text{reg}} \subset \mathcal{N}$ formed by the regular nilpotents. Thus, we get: $\mathcal{N}^{\text{reg}} = G^e \backslash G$, where $\text{Ad } G$ -action is viewed as a right action. Moreover, Kostant has shown in [Ko] that the natural imbedding: $\mathcal{N}^{\text{reg}} \hookrightarrow \mathcal{N}$ induces an isomorphism of the rings of regular functions. Thus we obtain a chain of natural algebra isomorphisms, where the superscripts stand for invariants under left translation

$$\mathbb{C}[G]^{\mathfrak{g}^e} = \mathbb{C}[G]^{G^e} = \mathbb{C}[G^e \backslash G] = \mathbb{C}[\mathcal{N}^{\text{reg}}] = \mathbb{C}[\mathcal{N}]. \quad (7.4.1)$$

Next we consider the vector space $\text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G])$ of \mathbb{C} -linear *continuous* maps $\mathbb{C}[G] \rightarrow \mathbb{C}[G]$, that is a pro-ind limit of finite dimensional $\text{Hom}_{\mathbb{C}}$ -spaces, defined the same way as we have earlier defined the Ext -spaces between ind-objects, see (7.2.1). The space $\text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G])$ has a natural G -action by conjugation; furthermore, it has the structure of topological algebra via composition. We have

Lemma 7.4.2. *There is a natural topological algebra isomorphism*

$$(\text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G]))^{\mathfrak{g}^e} \simeq \widehat{\mathcal{U}\mathfrak{g}} \rtimes \mathbb{C}[\mathcal{N}].$$

Proof. Write $\mathcal{D}(G)$ for the algebra of regular algebraic differential operators on the group G . The action of any such differential operator $u \in \mathcal{D}(G)$ clearly gives a *continuous* \mathbb{C} -linear map $u : \mathbb{C}[G] \rightarrow \mathbb{C}[G]$. This way one obtains a G -equivariant imbedding: $\mathcal{D}(G) \hookrightarrow \text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G])$ with dense image. View the algebra $\mathcal{U}\mathfrak{g}$ as left invariant differential operators on G , and the algebra $\mathbb{C}[G]$ as multiplication-operators. Then the algebra $\mathcal{D}(G)$ is isomorphic to the cross-product: $\mathcal{D}(G) \simeq \mathcal{U}\mathfrak{g} \rtimes \mathbb{C}[G]$. One can show that the composition: $\mathcal{U}\mathfrak{g} \rtimes \mathbb{C}[G] \xrightarrow{\sim} \mathcal{D}(G) \hookrightarrow \text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G])$ extends by continuity to a G -equivariant *topological algebra* isomorphism $\widehat{\mathcal{U}\mathfrak{g}} \rtimes \mathbb{C}[G] \xrightarrow{\sim} \text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G])$. Observe that the group G acts *trivially* on the space $\mathcal{U}\mathfrak{g}$ formed by left-invariant differential operators, and acts on $\mathbb{C}[G]$ via left translations. Hence, taking G^e -invariants on each side of the isomorphism above, we obtain

$$\left(\text{Hom}_{\text{cont}}(\mathbb{C}[G], \mathbb{C}[G]) \right)^{\mathfrak{g}^e} = (\widehat{\mathcal{U}\mathfrak{g}} \rtimes \mathbb{C}[G])^{\mathfrak{g}^e} = \widehat{\mathcal{U}\mathfrak{g}} \rtimes (\mathbb{C}[G]^{\mathfrak{g}^e}) \stackrel{(7.4.1)}{=} \widehat{\mathcal{U}\mathfrak{g}} \rtimes \mathbb{C}[\mathcal{N}]. \quad \square$$

Notation 7.4.3. Given two graded objects L_1, L_2 , we set $\text{Hom}^\bullet(L_1, L_2) := \bigoplus_i \text{Hom}^i(L_1, L_2)$, where $\text{Hom}^i(L_1, L_2)$ stands for the space of morphisms shifting the grading by i .

Next, we remind the reader that, for any $\mathcal{L} \in D^b(\text{Gr})$, the hyper-cohomology group $H^\bullet(\text{Gr}, \mathcal{L})$ has a natural structure of graded $H^\bullet(\text{Gr}, \mathbb{C})$ -module. Further, for any $\mathcal{L}_1, \mathcal{L}_2 \in D^b(\text{Gr})$, there is a functorial linear map of graded vector spaces:

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{L}_1, \mathcal{L}_2) \longrightarrow \text{Hom}_{H(\text{Gr})}^\bullet(H^\bullet(\text{Gr}, \mathcal{L}_1), H^\bullet(\text{Gr}, \mathcal{L}_2)). \quad (7.4.4)$$

The main result of [G1], in the special case of a \mathbb{C}^* -action on Gr implies

Proposition 7.4.5. *If $\mathcal{L}_1, \mathcal{L}_2 \in D^b(\text{Gr})$ are semisimple perverse sheaves constructible relative to a Bialinicki-Birula stratification of Gr (cf. [G1] for more details), Then the map (7.4.4) is an isomorphism. \square*

We note that since any $G^\vee(\mathcal{O})$ -equivariant perverse sheaf on Gr is constructible relative to the Schubert stratification, cf. [PS] or §8 below, the map (7.4.4) is an isomorphism for any $\mathcal{L}_1, \mathcal{L}_2 \in \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$.

The action of the semisimple element $t \in \mathfrak{t}$ (from the principal \mathfrak{sl}_2 -triple) puts a grading on the underlying vector space of any representation $V \in \text{Rep}(G)$. Further, by Corollary 6.2.3 we have an algebra isomorphism $H^\bullet(\text{Gr}) \simeq \mathcal{U}(\mathfrak{g}^e)$. Hence, from Corollary 6.5.4 and Proposition 7.4.5 we deduce

Corollary 7.4.6. *For any $V_1, V_2 \in \text{Rep}(G)$, there is a functorial isomorphism*

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{P}V_1, \mathcal{P}V_2) \xrightarrow{\sim} \text{Hom}_{\mathcal{U}(\mathfrak{g}^e)}^\bullet(V_1, V_2) = \left(\text{Hom}_{\mathbb{C}}^\bullet(V_1, V_2) \right)^{\mathfrak{g}^e}. \quad \square$$

7.5 Proof of Theorem 7.3.1: Let $r \in \text{Hom}_{D^b(\text{Gr})}(\mathbb{C}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) = \text{Ext}_{D^b(\text{Gr})}^0(\mathbb{C}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$ be the natural restriction morphism. Composing with r yields a canonical map

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{L}) \longrightarrow \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbb{C}_{\text{Gr}}, \mathcal{L}) = H^\bullet(\text{Gr}, \mathcal{L}), \quad \forall \mathcal{L} \in D^b(\text{Gr}). \quad (7.5.1)$$

This map can also be identified with the map of Corollary 7.4.6, for $\mathbf{1}_{\text{Gr}} = \mathcal{P}\mathbb{C}$ and $\mathcal{L} = \mathcal{P}V_2$. Hence, Corollary 7.4.6 says that (7.5.1) gives an isomorphism

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{L}) \xrightarrow{\sim} \text{Hom}_{H(\text{Gr})}^\bullet(\mathbb{C}, H^\bullet(\text{Gr}, \mathcal{L})) = H^\bullet(\text{Gr}, \mathcal{L})^{\mathfrak{g}^e}, \quad \forall \mathcal{L} \in \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr}).$$

The isomorphism above holds, in particular, for $\mathcal{L} := \mathcal{R}$, an ind-object of $\text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})$. We use this isomorphism to define a linear isomorphism ψ as the following composite:

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) \xrightarrow[\sim]{\text{Cor. 7.4.6}} H^\bullet(\text{Gr}, \mathcal{R})^{\mathfrak{g}^e} \xrightarrow[\sim]{\text{Cor. 6.5.4 for } \mathcal{R}=\mathcal{P}\mathbb{C}[G]} \mathbb{C}[G]^{G^e} \xrightarrow{(7.4.1)} \mathbb{C}[\mathcal{N}]. \quad (7.5.2)$$

To complete the proof of the theorem, it suffices to verify that the chain of isomorphisms (7.5.2) transports the above defined algebra structure on $\text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$ to the standard algebra structure on $\mathbb{C}[G]^{G^e} = \mathbb{C}[\mathcal{N}^{\text{reg}}] \simeq \mathbb{C}[\mathcal{N}]$. To this end, we start with the canonical identification $\mathbb{C}[G] = H^\bullet(\text{Gr}, \mathcal{R})$. Since $H^\bullet(-)$ is a fiber functor on $\text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})$, we also have $H^\bullet(\text{Gr}, \mathcal{R} \star \mathcal{R}) = \mathbb{C}[G] \otimes \mathbb{C}[G]$. By construction, the natural imbedding $\mathbb{C}[G]^{G^e} \hookrightarrow \mathbb{C}[G]$ corresponds, via the identification $\text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) = \mathbb{C}[G]^{G^e}$ in (7.5.2), to the morphism (7.5.1). Similarly, writing $G_{\text{diag}}^e \subset G^e \times G^e$ for the diagonal, we may identify $(\mathbb{C}[G] \otimes \mathbb{C}[G])^{G_{\text{diag}}^e} \subset \mathbb{C}[G] \otimes \mathbb{C}[G]$ with $\text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R} \star \mathcal{R}) \subset H^\bullet(\text{Gr}, \mathcal{R} \star \mathcal{R})$.

Observe further that the multiplication $m_{\mathbb{C}[G]}$ in the coordinate ring $\mathbb{C}[G]$ is recovered from the morphism $m : \mathcal{R} \star \mathcal{R} \rightarrow \mathcal{R}$, cf. (7.2.2), as the induced morphism of hyper-cohomology:

$$H^\bullet(\text{Gr}, \mathcal{R}) \otimes H^\bullet(\text{Gr}, \mathcal{R}) = H^\bullet(\text{Gr} \times \text{Gr}, \mathcal{R} \boxtimes \mathcal{R}) \xrightarrow{\sim} H^\bullet(\text{Gr}, \mathcal{R} \star \mathcal{R}) \xrightarrow{m} H^\bullet(\text{Gr}, \mathcal{R}). \quad (7.5.3)$$

Restricting the map $m_{\mathbb{C}[G]}$, resp., m , to G^e -invariants yields the corresponding map in the top, resp., bottom, row of the following diagram:

$$\begin{array}{ccccc} \mathbb{C}[G]^{G^e} \otimes \mathbb{C}[G]^{G^e} & \xhookrightarrow{i} & (\mathbb{C}[G] \otimes \mathbb{C}[G])^{G_{\text{diag}}^e} & \xrightarrow{m_{\mathbb{C}[G]}} & \mathbb{C}[G]^{G^e} \\ \parallel & & \parallel & & \parallel \\ \text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) \otimes \text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) & \xrightarrow{(7.2.3)} & \text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R} \star \mathcal{R}) & \xrightarrow{m} & \text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}). \end{array}$$

As has been explained, the identifications we have made insure that the diagram commutes. Further, it is clear that the composite map in the top row of the diagram is the multiplication map $\mathbb{C}[G]^{G^e} \otimes \mathbb{C}[G]^{G^e} \rightarrow \mathbb{C}[G]^{G^e}$. It follows that the latter map corresponds, geometrically, to the

composite map in the bottom row. This proves that the map ψ of the Theorem is an algebra isomorphism.

We now similarly construct the map Ψ . By Corollary 7.4.6, we have the following isomorphisms

$$\mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathcal{R}, \mathcal{R}) = \mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathcal{P}(\mathbb{C}[G]), \mathcal{P}(\mathbb{C}[G])) = (\mathrm{Hom}_{\mathrm{cont}}(\mathbb{C}[G], \mathbb{C}[G]))^{\mathfrak{g}^e}. \quad (7.5.4)$$

Hence, from Lemma 7.4.2 we deduce

$$\mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathcal{R}, \mathcal{R}) = (\mathrm{Hom}_{\mathrm{cont}}^\bullet(\mathbb{C}[G], \mathbb{C}[G]))^{\mathfrak{g}^e} = \widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathcal{N}].$$

Compatibility of the maps that we have constructed above with algebra, resp. module, structures is verified in a similar way, as we did for $\mathrm{Ext}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$. We leave details to the reader. \square

7.6 Equivariant version. Write $\mathrm{Ext}_{G^\vee(\mathcal{O})}^\bullet$ for the Ext-groups in $D_{G^\vee(\mathcal{O})}^b(\mathrm{Gr})$, $G^\vee(\mathcal{O})$ -equivariant derived category of constructible complexes in the sense of [BL]. These Ext-groups have canonical $H_{G^\vee(\mathcal{O})}^\bullet(pt)$ -module structure, cf. (6.2.1). Clearly, $\mathcal{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$ is an abelian subcategory of $D_{G^\vee(\mathcal{O})}^b(\mathrm{Gr})$, hence \mathcal{R} is an ind-object of $D_{G^\vee(\mathcal{O})}^b(\mathrm{Gr})$. As in (7.2.3), the map $m : \mathcal{R} \star \mathcal{R} \rightarrow \mathcal{R}$ makes $\mathrm{Ext}_{G^\vee(\mathcal{O})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$ into a graded algebra, and we have the following equivariant analogue of Theorem 7.3.1.

Theorem 7.6.1. *There are natural algebra isomorphisms Ψ and ψ making the following diagram commute:*

$$\begin{array}{ccc} \mathrm{Ext}_{G^\vee(\mathcal{O})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R}) & \xrightarrow{\varepsilon_{geom}} & \mathrm{Ext}_{G^\vee(\mathcal{O})}^\bullet(\mathcal{R}, \mathcal{R}) \\ \downarrow \psi & & \downarrow \Psi \\ \mathbb{C}[\mathfrak{g}^*] & \xrightarrow{\varepsilon_{alg}} & \widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathfrak{g}^*]. \end{array}$$

Remark 7.6.2. It is tempting to use a kind of *delocalized* equivariant cohomology, see [BBM], in order to be able to replace the algebra $\widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[\mathfrak{g}]$ above (where we identify \mathfrak{g} with \mathfrak{g}^* via an invariant form) by the algebra $\widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[G]$. Recall that both $\widehat{\mathcal{U}\mathfrak{g}}$ and $\mathbb{C}[G]$ have natural structures of Hopf algebras, topologically dual to each other, i.e., $\widehat{\mathcal{U}\mathfrak{g}}$ is an inverse limit, while $\mathbb{C}[G]$ is a direct limit. The algebra $\widehat{\mathcal{U}\mathfrak{g}} \ltimes \mathbb{C}[G]$ is the Drinfeld's *double* Hopf algebra. \diamond

In order to begin the proof of Theorem 7.6.1 we need to introduce more notation. Given any vector space E we will write \underline{E} for the quasi-coherent sheaf (trivial vector bundle) on \mathfrak{t}^*/W with geometric fiber E . Thus, $\Gamma(\mathfrak{t}^*/W, \underline{E}) = \mathbb{C}[\mathfrak{t}^*]^W \otimes E$.

The result below is an equivariant version of the last statement of Corollary 6.5.4, applied to the regular perverse sheaf \mathcal{R} .

Lemma 7.6.3. *There is a natural $\mathbb{C}[\mathfrak{t}^*]^W$ -module isomorphism*

$$H_{G^\vee(\mathcal{O})}^\bullet(\mathrm{Gr}, \mathcal{R}) \simeq \mathbb{C}[\mathfrak{t}^*]^W \otimes_{\mathbb{C}} \mathbb{C}[G] = \Gamma(\mathfrak{t}^*/W, \underline{\mathbb{C}[G]});$$

The canonical $H_{G^\vee(\mathcal{O})}^\bullet(\mathrm{Gr})$ -module structure on $H_{G^\vee(\mathcal{O})}^\bullet(\mathrm{Gr}, \mathcal{R})$ corresponds, under the isomorphism of Lemma 6.2.2, to the natural left $\Gamma(\mathfrak{t}^/W, \mathcal{U}\mathfrak{g})$ -action on $\Gamma(\mathfrak{t}^*/W, \underline{\mathbb{C}[G]})$. \square*

With the results of [G2] mentioned in §6, the proof of the Lemma is straightforward and will be omitted. \square

PROOF OF THEOREM 7.6.1. We will only establish the isomorphism ψ ; once it is understood, the construction of Ψ is entirely similar to that in the proof of Theorem 7.3.1.

Using Lemma 6.2.2, we perform the following calculation, similar to (7.5.2):

$$\begin{aligned}
\text{Ext}_{G^{\vee}(\mathcal{O})}^{\bullet}(\mathbf{1}_{\text{Gr}}, \mathcal{R}) &= \text{Hom}_{H_{G^{\vee}(\mathcal{O})}(\text{Gr})}^{\bullet} \left(H_{G^{\vee}(\mathcal{O})}^{\bullet}(\mathbf{1}_{\text{Gr}}), H_{G^{\vee}(\mathcal{O})}^{\bullet}(\text{Gr}, \mathcal{R}) \right) \quad (\text{by [G1]}) \\
&= \text{Hom}_{H_{G^{\vee}(\mathcal{O})}(\text{Gr})}^{\bullet} \left(H_{G^{\vee}(\mathcal{O})}^{\bullet}(pt), H_{G^{\vee}(\mathcal{O})}^{\bullet}(\text{Gr}, \mathcal{R}) \right) \\
(\text{by Lemma 7.6.3}) \quad &= \text{Hom}_{\Gamma(\mathfrak{t}^*/W, \mathcal{U}\mathcal{G})}^{\bullet} (\mathbb{C}[\mathfrak{t}^*]^W, \mathbb{C}[\mathfrak{t}^*]^W \otimes_{\mathbb{C}} \mathbb{C}[G]) \quad (7.6.4) \\
&= \Gamma(\mathfrak{t}^*/W, \mathcal{H}om_{\mathcal{U}\mathcal{G}}(\mathbb{C}, \mathbb{C}[G])) \\
&= \Gamma(\mathfrak{t}^*/W, \mathbb{C}[G]^{\mathcal{U}\mathcal{G}}) = \Gamma(\mathfrak{g}^{\text{reg}}/\text{Ad } G, \mathbb{C}[G]^{\mathcal{U}\mathcal{G}}).
\end{aligned}$$

The last expression may be identified naturally with the algebra of regular functions on the total space of the canonical fibration $p : \mathfrak{g}^{\text{reg}} \rightarrow \mathfrak{g}^{\text{reg}}/\text{Ad } G$, because for any $x \in \mathfrak{g}^{\text{reg}}$, we have $p^{-1}(p(x)) \simeq G^x \backslash G$. We see that the algebra in question equals $\mathbb{C}[\mathfrak{g}^{\text{reg}}]$. Since the complement $\mathfrak{g}^* \setminus \mathfrak{g}^{\text{reg}}$ is known to have codimension ≥ 2 in \mathfrak{g}^* , we conclude that $\mathbb{C}[\mathfrak{g}^{\text{reg}}] = \mathbb{C}[\mathfrak{g}^*]$. The result is proved. \square

7.7 A fiber functor on perverse sheaves. In this subsection, we will make a link of our results with general Tannakian formalism.

Let \mathcal{C} be an abelian category, which is a (right) *module category* over the tensor category $\text{Rep}(G)$. This means that we are given an exact bifunctor $\mathcal{C} \times \text{Rep}(G) \rightarrow \mathcal{C}$, $(M, V) \mapsto M \otimes V$, satisfying a natural associativity constraint: $(M \otimes V) \otimes V' \cong M \otimes (V \otimes V')$.

Let \mathcal{C} be a module category over $\text{Rep}(G)$. View $\mathbb{C}[G]$ as an ind-object of $\text{Rep}(G)$ and, given $M \in \mathcal{C}$, form an ind-object $M \otimes \mathbb{C}[G]$ in \mathcal{C} . Now, fix $M, N \in \mathcal{C}$. As we have explained (in a special case) at the end of section 7.1, the action of G on $\mathbb{C}[G]$ by *right* translations gives rise to a natural G -module structure on the vector space $\text{Ext}_{\mathcal{C}}^{\bullet}(M, N \otimes \mathbb{C}[G])$.

Now, given $L \in \mathcal{C}$ and $V \in \text{Rep}(G)$, we may apply the above construction to the objects $N = L$ and $N = L \otimes V$, respectively. The proof of the following ‘abstract nonsense’ result is left for the reader.

Lemma 7.7.1. *For any $M, L \in \mathcal{C}$, there is a natural functorial G -module isomorphism*

$$\text{Ext}_{\mathcal{C}}^{\bullet}(M, (L \otimes V) \otimes \mathbb{C}[G]) \simeq V \otimes \text{Ext}_{\mathcal{C}}^{\bullet}(M, L \otimes \mathbb{C}[G]) \quad \forall V \in \text{Rep}(G). \quad \square$$

By Gaitsgory theorem 6.3.2, convolution of perverse sheaves gives an exact bifunctor $\text{Perv}(\text{Gr}) \times \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr}) \rightarrow \text{Perv}(\text{Gr})$, $\mathcal{M}, \mathcal{A} \mapsto \mathcal{M} \star \mathcal{A}$. This way, the category $\text{Perv}(\text{Gr})$ becomes a module category over $\text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})$. Transporting the module structure by means of Satake equivalence $\mathcal{P} : \text{Rep}(G) \xrightarrow{\sim} \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})$, we thus make $\text{Perv}(\text{Gr})$ a module category over $\text{Rep}(G)$. Applying Lemma 7.7.1 to the module category $\text{Perv}(\text{Gr})$ we get, for any $\mathcal{M}, \mathcal{L} \in \text{Perv}(\text{Gr})$, the following natural G -module isomorphism

$$\text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathcal{M}, \mathcal{L} \star \mathcal{P}V \star \mathcal{R}) \simeq V \otimes \text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathcal{M}, \mathcal{L} \star \mathcal{R}), \quad \forall V \in \text{Rep}(G). \quad (7.7.2)$$

Remark 7.7.3. Having in mind further applications of isomorphism (7.7.2), we have written the Ext-groups in the triangulated category $D^b(\text{Gr})$ rather than in the abelian category $\text{Perv}(\text{Gr})$. These two Ext-groups are known (due to Beilinson) to be actually the same. The Beilinson's result is not however absolutely necessary for justifying (7.7.2); indeed, the formal nature of the setup of Lemma 7.7.1 makes it applicable, in effect, to a more general setup of a triangulated ‘module category’. \diamond

Recall now the graded algebra $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$, that comes equipped with a natural G -action. The isomorphism $\psi : \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}) \xrightarrow{\sim} \mathbb{C}[\mathcal{N}]$, of Theorem 7.3.1, induces an equivalence of categories

$$\psi_* : \text{Mod}^{G \times \mathbb{C}^*}(\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})) \xrightarrow{\sim} \text{Mod}^{G \times \mathbb{C}^*}(\mathbb{C}[\mathcal{N}]) = \text{Coh}^{G \times \mathbb{C}^*}(\mathcal{N}),$$

where the category of $G \times \mathbb{C}^*$ -equivariant coherent sheaves on \mathcal{N} is identified naturally with $\text{Mod}^{G \times \mathbb{C}^*}(\mathbb{C}[\mathcal{N}])$, see 3.2.1.

Next we observe that, for any $\mathcal{L} \in \text{Perv}(\text{Gr})$ one can define, by modifying appropriately formula (7.2.3), a natural pairing

$$\text{Ext}_{D^b(\text{Gr})}^i(\mathbf{1}_{\text{Gr}}, \mathcal{L} \star \mathcal{R}) \otimes \text{Ext}_{D^b(\text{Gr})}^j(\mathbf{1}_{\text{Gr}}, \mathcal{R}) \longrightarrow \text{Ext}_{D^b(\text{Gr})}^{i+j}(\mathbf{1}_{\text{Gr}}, \mathcal{L} \star \mathcal{R}).$$

This pairing makes the space $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{L} \star \mathcal{R})$ a graded $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$ -module. We also have the G -action on the Ext-groups involved. Therefore, we obtain an exact functor

$$\text{Perv}(\text{Gr}) \longrightarrow \text{Mod}^{G \times \mathbb{C}^*}(\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})), \quad \mathcal{L} \longmapsto \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{L} \star \mathcal{R}). \quad (7.7.4)$$

We now use the category equivalence ψ_* , mentioned above, together with the Satake equivalence, and form the following composite functor:

$$\mathbb{S} : \text{Rep}(G) \xrightarrow{\mathcal{P}} \text{Perv}(\text{Gr}) \xrightarrow{(7.7.4)} \text{Mod}^{G \times \mathbb{C}^*}(\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})) \xrightarrow{\psi_*} \text{Coh}^{G \times \mathbb{C}^*}(\mathcal{N}). \quad (7.7.5)$$

The canonical isomorphism in (7.7.2) then translates into the following result.

Proposition 7.7.6. *The functor \mathbb{S} is isomorphic to the functor $V \longmapsto V \otimes \mathcal{O}_{\mathcal{N}}$, that assigns to any $V \in \text{Rep}(G)$ the free $\mathcal{O}_{\mathcal{N}}$ -sheaf with fiber V (and equipped with the tensor product G -equivariant structure). \square*

8 Wakimoto sheaves

8.1 We have shown in the previous section that the coordinate ring $\mathbb{C}[\mathcal{N}]$, together with the Ad G -action on it, can be reconstructed as an Ext-group between certain perverse sheaves on the loop Grassmannian.

The goal of this section is to give a similar construction for $\tilde{\mathcal{N}} = G \times_B \mathfrak{n}$, the Springer resolution of \mathcal{N} , see e.g. [CG, ch.3] for a survey. One obstacle for doing so is that $\tilde{\mathcal{N}}$ is *not* an affine variety, and it is known that $\mathbb{C}[\tilde{\mathcal{N}}] = \mathbb{C}[\mathcal{N}]$. Therefore, the variety $\tilde{\mathcal{N}}$ is not determined by the ring of its global regular functions. Instead, we will consider the affine cone over a kind of Plücker imbedding of $\tilde{\mathcal{N}}$. Our main result shows how to reconstruct the (multi)-homogeneous coordinate ring of that cone as an Ext-algebra of perverse sheaves on the loop Grassmannian.

8.2 The affine flag manifold. Let $T^\vee \subset B^\vee$ denote the maximal torus and the Borel subgroup in G^\vee corresponding to our choice of positive roots. We write $\mathbf{I} = \{f \in G^\vee(\mathcal{O}) \mid f(0) \in B^\vee\}$ for the corresponding Iwahori (=affine Borel) subgroup in $G^\vee(\mathcal{K})$. We let $\mathcal{B} = G^\vee(\mathcal{K})/\mathbf{I}$, be the *affine Flag variety*; it has a natural ind-scheme structure. Since $\mathbf{I} \subset G^\vee(\mathcal{O})$, the projection $\varpi : G^\vee(\mathcal{K})/\mathbf{I} \rightarrow G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$ gives a smooth and proper morphism of ind-schemes $\varpi : \mathcal{B} \rightarrow \text{Gr}$, whose fiber is isomorphic to the finite dimensional flag manifold G^\vee/B^\vee .

The left $G(\mathcal{K})$ -action on \mathcal{B} gives rise to the following convolution diagram:

$$G^\vee(\mathcal{K}) \times_{\mathbf{I}} \mathcal{B} \xrightarrow{a} \mathcal{B}, \quad (g, x) \mapsto gx. \quad (8.2.1)$$

Given $\mathcal{A}, \mathcal{M} \in D_{\mathbf{I}}^b(\mathcal{B})$, one defines, using \mathbf{I} -equivariance of \mathcal{M} , (cf. e.g., [G2], [MV] or [Ga]), an object $\mathcal{A} \tilde{\boxtimes} \mathcal{M} \in D^b(G^\vee(\mathcal{K}) \times_{\mathbf{I}} \mathcal{B})$. The assignment $(\mathcal{A}, \mathcal{M}) \mapsto \mathcal{A} \star \mathcal{M} := a_*(\mathcal{A} \tilde{\boxtimes} \mathcal{M})$ gives the standard convolution bifunctor $\star : D_{\mathbf{I}}^b(\mathcal{B}) \times D_{\mathbf{I}}^b(\mathcal{B}) \rightarrow D_{\mathbf{I}}^b(\mathcal{B})$.

Each \mathbf{I} -orbit on either \mathcal{B} or Gr is isomorphic to a finite dimensional vector space \mathbb{C}^n . Moreover, each \mathbf{I} -orbit on Gr , contains a unique coset $\lambda \cdot G^\vee(\mathcal{O})/G^\vee(\mathcal{O})$, where $\lambda \in \text{Hom}(\mathbb{C}^*, T^\vee)$ is viewed as an element of $G^\vee(\mathcal{K})$. This way one gets a natural bijection between the set of \mathbf{I} -orbits in Gr and the set $\text{Hom}(\mathbb{C}^*, T^\vee) = \mathbb{Y}$. We will write Gr_λ for the \mathbf{I} -orbit corresponding to a weight $\lambda \in \mathbb{Y}$. Such an \mathbf{I} -orbit, Gr_λ , is open dense in the $G^\vee(\mathcal{O})$ -orbit of $\lambda \in \text{Gr}$, if and only if $\lambda \in \mathbb{Y}^{++}$ is a dominant weight. Similarly, each \mathbf{I} -orbit in \mathcal{B} contains a unique coset $y \cdot \lambda \cdot \mathbf{I}/\mathbf{I}$, where $y \cdot \lambda \in W_{\text{aff}}$. We let $j_w : \mathcal{B}_w \hookrightarrow \mathcal{B}$ denote the corresponding \mathbf{I} -orbit imbedding. The orbits form a stratification: $\mathcal{B} = \coprod_{w \in W_{\text{aff}}} \mathcal{B}_w$.

Notation 8.2.2. For any $w \in W_{\text{aff}}$ we set: $\mathcal{M}_w := (j_w)_! \mathbb{C}_w[\dim \mathcal{B}_w]$ and $\mathcal{M}_w^\vee := (j_w)_* \mathbb{C}_w[\dim \mathcal{B}_w]$, where \mathbb{C}_w stands for the constant sheaf on the cell \mathcal{B}_w .

The imbedding j_w being affine, it is known that $\mathcal{M}_w, \mathcal{M}_w^\vee \in \text{Perv}_{\mathbf{I}}(\mathcal{B})$. It is known further that, for any $y, w \in W_{\text{aff}}$, such that $\ell(y) + \ell(w) = \ell(y \cdot w)$, one has:

$$\mathcal{M}_y \star \mathcal{M}_w = \mathcal{M}_{yw}, \quad \mathcal{M}_y^\vee \star \mathcal{M}_w^\vee = \mathcal{M}_{yw}^\vee, \quad \text{and} \quad \mathcal{M}_w^\vee \star \mathcal{M}_{w^{-1}} = \mathcal{M}_e, \quad (8.2.3)$$

where $e \in W_{\text{aff}}$ denotes the identity.

The following result will play a crucial role in our approach.

Proposition 8.2.4. (I. Mirković) *For any $w, y \in W_{\text{aff}}$ we have (i) $\mathcal{M}_y^\vee \star \mathcal{M}_w \in \text{Perv}_{\mathbf{I}}(\mathcal{B})$.*

(ii) $\text{supp}(\mathcal{M}_y^\vee \star \mathcal{M}_w) = \overline{\mathcal{B}}_{yw}$, moreover: $(\mathcal{M}_y^\vee \star \mathcal{M}_w)|_{\mathcal{B}_{yw}} = \mathbb{C}_{yw}[-\dim \mathcal{B}_{yw}]$.

Proof. To prove (i), fix any $w \in W_{\text{aff}}$. According to the definition of convolution, for any complex \mathcal{M} supported in $\overline{\mathcal{B}}_w$, and any $\mathcal{A} \in \text{Perv}_{\mathbf{I}}(\mathcal{B})$, we have $\mathcal{A} \star \mathcal{M} = \overline{a}_*(\mathcal{A} \tilde{\boxtimes} \mathcal{M})$. Here $\overline{a}_* = \overline{a}_!$ is the direct image with respect to a natural *proper* action-morphism $\overline{a} : G^\vee(\mathcal{K}) \times_{\mathbf{I}} \overline{\mathcal{B}}_w \rightarrow \mathcal{B}$, and " $\tilde{\boxtimes}$ " stands for a twisted version of external tensor product that has been already mentioned earlier.

In the special case $\mathcal{M} = (j_w)_! \mathbb{C}_w[\dim \mathcal{B}_w]$, the same result is obtained by replacing \overline{a} by the *non-proper* map $a : G^\vee(\mathcal{K}) \times_{\mathbf{I}} \mathcal{B}_w \rightarrow \mathcal{B}$, and taking $a_!(\mathcal{A} \tilde{\boxtimes} \mathbb{C}_w[\dim \mathcal{B}_w])$. But the morphism a being *affine*, by [BBD] one has: $a_!(D^{\leq 0}(G^\vee(\mathcal{K}) \times_{\mathbf{I}} \mathcal{B}_w)) \subset D^{\leq 0}(\mathcal{B})$. Hence, $\mathcal{A} \star \mathcal{M}_w \in D^{\leq 0}(\mathcal{B})$, for any $\mathcal{A} \in \text{Perv}_{\mathbf{I}}(\mathcal{B})$. Dually, one obtains: $\mathcal{M}_y^\vee \star \mathcal{A} \in D^{\geq 0}(\mathcal{B})$, for any $\mathcal{A} \in \text{Perv}_{\mathbf{I}}(\mathcal{B})$. It follows that: $\mathcal{M}_y^\vee \star \mathcal{M}_w \in D^{\leq 0}(\mathcal{B}) \cap D^{\geq 0}(\mathcal{B})$. Thus, $\mathcal{M}_y^\vee \star \mathcal{M}_w$ is a perverse sheaf, and part (i) is proved.

To prove (ii), let $K_{\mathbf{I}}(\mathcal{B})$ denote the Grothendieck group of the category $D_{\mathbf{I}}^b(\mathcal{B})$. The classes $\{[\mathcal{M}_x]\}_{x \in W_{\text{aff}}}$ form a natural \mathbb{Z} -basis of $K_{\mathbf{I}}(\mathcal{B})$. The convolution on $D_{\mathbf{I}}^b(\mathcal{B})$ makes $K_{\mathbf{I}}(\mathcal{B})$ into an associative ring, and the assignment: $x \mapsto [\mathcal{M}_x]$ is well-known (due to Beilinson-Bernstein, Lusztig, MacPherson, and others) to yield a ring isomorphism $\mathbb{Z}[W_{\text{aff}}] \xrightarrow{\sim} K_{\mathbf{I}}(\mathcal{B})$, where $\mathbb{Z}[W_{\text{aff}}]$ denotes the group algebra of the group W_{aff} . Further, in $K_{\mathbf{I}}(\mathcal{B})$, one has: $[\mathcal{M}_x] = [\mathcal{M}_x^\vee]$. Hence the

class $[\mathcal{M}_y^\vee \star \mathcal{M}_w]$ corresponds, under the isomorphism above, to the element $y \cdot w \in W_{\text{aff}} \subset \mathbb{Z}[W_{\text{aff}}]$, in other words, in $K_{\mathbf{I}}(\mathcal{B})$ we have an equation: $[\mathcal{M}_y^\vee \star \mathcal{M}_w] = [\mathcal{M}_{yw}]$. But since $K_{\mathbf{I}}(\mathcal{B}) = K(\text{Perv}_{\mathbf{I}}(\text{Gr}))$ may be identified with the Grothendieck group of the abelian category $\text{Perv}_{\mathbf{I}}(\text{Gr})$, and since $\mathcal{M}_y^\vee \star \mathcal{M}_w$ is a perverse sheaf by (i), the latter equation in $K(\text{Perv}_{\mathbf{I}}(\text{Gr}))$ yields: $\text{supp } (\mathcal{M}_y^\vee \star \mathcal{M}_w) = \text{supp } \mathcal{M}_{yw} = \overline{\mathcal{B}}_{yw}$. \square

8.3 Wakimoto sheaves. Given any $\lambda \in \mathbb{Y}$, one can find a pair of dominant weights $\mu, \nu \in \mathbb{Y}^{++}$ such that $\lambda = \mu - \nu$. Imitating Bernstein's well-known construction of a large commutative subalgebra in the affine Hecke algebra, see e.g. [CG, ch.7], I. Mirković introduced the following objects

$$\mathcal{W}_\lambda := \mathcal{M}_\mu^\vee \star \mathcal{M}_{-\nu}, \quad (8.3.1)$$

where μ, ν are viewed as elements of $\mathbb{Y} \subset W_{\text{aff}}$. Now, a standard argument due to Bernstein implies that the above definition of \mathcal{W}_λ is independent of the choice of presentation: $\lambda = \mu - \nu$. Indeed, if $\mu', \nu' \in \mathbb{Y}^{++}$ is another pair such that $\lambda = \mu' - \nu'$, then in W_{aff} we have $\ell(\mu + \nu') = \ell(\mu) + \ell(\nu')$, $\ell(\mu' + \nu) = \ell(\mu') + \ell(\nu)$, and $\mu + \nu' = \mu' + \nu$. Independence of the presentation now follows from formulas (8.2.3). Further, Proposition 8.2.4(i) guarantees that $\mathcal{W}_\lambda \in \text{Perv}_{\mathbf{I}}(\text{Gr})$. These objects are called *Wakimoto sheaves*.

Corollary 8.3.2. (I. Mirković) (i) $\mathcal{W}_\lambda \star \mathcal{W}_\mu = \mathcal{W}_{\lambda+\mu}$, for any $\lambda, \mu \in \mathbb{Y}$.

(ii) $\text{supp } (\varpi_* \mathcal{W}_\lambda) = \overline{\text{Gr}}_\lambda$; If $w \in W$ is an element of minimal length such that $w(\lambda) \in \mathbb{Y}^{++}$, then: $(\varpi_* \mathcal{W}_\lambda)|_{\text{Gr}_\lambda} = \mathbb{C}_\lambda[-\dim \text{Gr}_\lambda - \ell(w)]$.

Proof. Part (i) is a straightforward application of formulas (8.2.3). To prove (ii), choose dominant weights $\mu, \nu \in \mathbb{Y}^{++}$, such that $\lambda = \mu - \nu$. Observe that, for any $w \in W$, in W_{aff} we have $\ell(-\nu) = \ell((-\nu) \cdot w^{-1}) + \ell(w)$. Hence, equations (8.2.3) yield: $\mathcal{M}_{-\nu} = \mathcal{M}_{(-\nu) \cdot w^{-1}} \star \mathcal{M}_w$. Thus, since $\lambda = \mu - \nu$, we find:

$$\mathcal{W}_\lambda = \mathcal{M}_\mu^\vee \star \mathcal{M}_{-\nu} = \mathcal{M}_\mu^\vee \star \mathcal{M}_{(-\nu) \cdot w^{-1}} \star \mathcal{M}_w. \quad (8.3.3)$$

We now let $w \in W$ be as in part (ii) of the Corollary. Then, the element $\lambda \cdot w^{-1}$ is minimal in the right coset $\lambda \cdot W \subset W_{\text{aff}}$, and we put $x = \lambda \cdot w^{-1}$. Geometrically, this means that the Bruhat cell $\mathcal{B}_x = \mathbf{I} \cdot x \subset \mathcal{B}$ is closed in $\varpi^{-1}(\varpi(\mathcal{B}_x))$. Further, let $A = \mathcal{M}_\mu^\vee \star \mathcal{M}_{(-\nu) \cdot w^{-1}}$. Part (ii) of Proposition 8.2.4 implies that $\text{supp } A = \overline{\mathcal{B}}_x$ and, moreover, $A|_{\mathcal{B}_x} = \mathbb{C}_{\mathcal{B}_x}[-\dim \mathcal{B}_x]$. It follows that $\text{supp } \varpi_*(A \star \mathcal{M}_w) = \varpi(\text{supp } A) = \overline{\text{Gr}}_\lambda$, and that $(\varpi_*(A \star \mathcal{M}_w))|_{\text{Gr}_\lambda} = \mathbb{C}_{\text{Gr}_\lambda}[-\dim \text{Gr}_\lambda - \ell(w)]$. Part (ii) of the Corollary now follows from formula (8.3.3). \square

8.4 An Ext-composition. Recall the setup of equation (8.2.3). In addition to the convolution-product functor $\star : D_{\mathbf{I}}^b(\mathcal{B}) \times D_{\mathbf{I}}^b(\mathcal{B}) \rightarrow D_{\mathbf{I}}^b(\mathcal{B})$, there is also a well-defined convolution functor $\star : D_{\mathbf{I}}^b(\mathcal{B}) \times D_{\mathbf{I}}^b(\text{Gr}) \rightarrow D_{\mathbf{I}}^b(\text{Gr})$ arising from the convolution-diagram $G^\vee(\mathcal{K}) \times_{\mathbf{I}} \text{Gr} \rightarrow \text{Gr}$, $(g, x) \mapsto gx$. Moreover, for any $A \in D_{\mathbf{I}}^b(\mathcal{B})$ and $M \in D_{\mathbf{I}}^b(\text{Gr})$, one has $A \star M = (\varpi_* A) \star M$. In particular, for any $\mu \in \mathbb{Y}^{++}$ and $\lambda \in \mathbb{Y}$, there is a well-defined object $\mathcal{W}_\lambda \star IC_\mu \in D_{\mathbf{I}}^b(\mathcal{B})$.

For any $M \in D_{\mathbf{I}}^b(\text{Gr})$, and $\lambda \in \mathbb{Y}^{++}$, one may form an ind-object $\mathcal{W}_\lambda \star M \star \mathcal{R} \in \lim \text{ind } D_{\mathbf{I}}^b(\text{Gr})$. Therefore, given two objects $M_1, M_2 \in D_{\mathbf{I}}^b(\text{Gr})$, we may consider the following $\mathbb{Z} \times \mathbb{Y}$ -graded vector space

$$\begin{aligned} \mathbf{E}(M_1, M_2) &= \bigoplus_{\lambda \in \mathbb{Y}^{++}} \mathbf{E}^\bullet(M_1, M_2)_\lambda, \\ \mathbf{E}^\bullet(M_1, M_2)_\lambda &:= \text{Ext}_{D^b(\text{Gr})}^\bullet(M_1, \mathcal{W}_\lambda \star M_2 \star \mathcal{R}). \end{aligned} \quad (8.4.1)$$

Next we introduce, for any three objects $M_1, M_2, M_3 \in D_1^b(\text{Gr})$, a "composition-type" pairing of \mathbf{E} -groups, similar to the one defined in §7.2 on the vector space $\text{Ext}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$. Specifically, given any $\mu, \lambda \in \mathbb{Y}^{++}$ and $i, j \in \mathbb{Z}$, we define a pairing

$$\mathbf{E}^i(M_1, M_2)_\lambda \otimes \mathbf{E}^j(M_2, M_3)_\mu \longrightarrow \mathbf{E}^{i+j}(M_1, M_3)_{\lambda+\mu} \quad (8.4.2)$$

as follows. Let $x \in \text{Ext}_{D^b(\text{Gr})}^i(M_2, \mathcal{W}_\lambda \star M_3 \star \mathcal{R})$, $y \in \text{Ext}_{D^b(\text{Gr})}^j(M_1, \mathcal{W}_\mu \star M_2 \star \mathcal{R})$. We may view x as a morphism $x : M_2 \rightarrow \mathcal{W}_\lambda \star M_3 \star \mathcal{R}[i]$. Applying convolution, we get a morphism

$$\mathcal{W}_\mu \star x \star \mathcal{R} : \mathcal{W}_\mu \star M_2 \star \mathcal{R} \longrightarrow \mathcal{W}_\mu \star (\mathcal{W}_\lambda \star M_3 \star \mathcal{R}[i]) \star \mathcal{R} = (\mathcal{W}_\mu \star \mathcal{W}_\lambda) \star M_3 \star (\mathcal{R} \star \mathcal{R})[i].$$

Note that $\mathcal{W}_\mu \star \mathcal{W}_\lambda \cong \mathcal{W}_{\lambda+\mu}$, furthermore, the ring-object structure on \mathcal{R} yields a morphism $\mathbf{m} : \mathcal{R} \star \mathcal{R} \rightarrow \mathcal{R}$. We now define $y \cdot x \in \mathbf{E}^{i+j}(M_1, M_3)_{\lambda+\mu} = \text{Ext}_{D^b(\text{Gr})}^{i+j}(M_1, \mathcal{W}_{\lambda+\mu} \star M_3 \star \mathcal{R})$ as the following composite morphism

$$\begin{aligned} y \cdot x : M_1 &\xrightarrow{y} \mathcal{W}_\mu \star M_2 \star \mathcal{R}[j] \xrightarrow{\mathcal{W}_\mu \star x \star \mathcal{R}} (\mathcal{W}_\mu \star \mathcal{W}_\lambda) \star M_2 \star (\mathcal{R} \star \mathcal{R})[i][j] = \\ &= \mathcal{W}_{\lambda+\mu} \star M_2 \star (\mathcal{R} \star \mathcal{R})[i+j] \xrightarrow{\mathbf{m}} \mathcal{W}_{\lambda+\mu} \star M_3 \star \mathcal{R}[i+j]. \end{aligned}$$

The product $(x, y) \mapsto y \cdot x$ thus defined is associative in a natural way.

8.5 Homogeneous coordinate ring of $\tilde{\mathcal{N}}$ as an \mathbf{Ext} -algebra. Let $\mathcal{F}\ell$ be the (finite dimensional) flag variety for the group G , i.e., the variety of all Borel subalgebras in \mathfrak{g} . We write $\pi : \tilde{\mathcal{N}} = T^*\mathcal{F}\ell \rightarrow \mathcal{F}\ell$ for the cotangent bundle on $\mathcal{F}\ell$. Recall that G is of adjoint type, hence, $\text{Hom}(B, \mathbb{C}^\times) = \text{Hom}(T, \mathbb{C}^\times) = \mathbb{Y}$.

For each $\lambda \in \mathbb{Y}$, we write $\mathcal{O}_{\mathcal{F}\ell}(\lambda) = G \times_B \mathbb{C}(\lambda)$ for the standard G -equivariant line bundle on $\mathcal{F}\ell$ induced from λ , the latter being viewed as a character $B \rightarrow \mathbb{C}^\times$.

Notation 8.5.1. For any $\lambda \in \mathbb{Y}$, we put: $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) = \pi^* \mathcal{O}_{\mathcal{F}\ell}(\lambda)$.

The natural \mathbb{C}^* -action on $T^*\mathcal{F}\ell$ by dilations commutes with the G -action, making $T^*\mathcal{F}\ell$ into a $\mathbb{C}^* \times G$ -variety. Clearly, $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ is a $\mathbb{C}^* \times G$ -equivariant line bundle on $T^*\mathcal{F}\ell$. The \mathbb{C}^* -structure on $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ gives a \mathbb{Z} -grading on $\Gamma^*(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))$, the space of global sections. Further, the obvious canonical isomorphism $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\mu) \xrightarrow{\sim} \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda + \mu)$ induces, for any $\lambda, \mu \in \mathbb{Y}$, a bilinear pairing of the spaces of global sections:

$$\Gamma^*(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \otimes \Gamma^*(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\mu)) \longrightarrow \Gamma^*(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda + \mu)).$$

These pairings make $\bigoplus_{\lambda \in \mathbb{Y}^{++}} \Gamma^*(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))$ a $\mathbb{Z} \times \mathbb{Y}^{++}$ -graded algebra.

Now, the construction of §8.4, applied to the special case $M_1 = M_2 = \mathbf{1}_{\text{Gr}}$, yields a $\mathbb{Z} \times \mathbb{Y}^{++}$ -graded algebra $\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R})$. This algebra comes equipped with a G -action, as has been explained at the end of §7.1. It is known further that the \mathbf{Ext} -group above has no odd degree components, due to a standard parity-type vanishing result for IC-sheaves on the loop Grassmannian (that is, for the affine Kazhdan-Lusztig polynomials).

The main result of this section is the following

Theorem 8.5.2. *There is a canonical G -equivariant $(\mathbb{Z} \times \mathbb{Y}^{++})$ -graded algebra isomorphism*

$$\bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^{2\bullet}(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \simeq \bigoplus_{\lambda \in \mathbb{Y}^{++}} \Gamma^*(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)).$$

To prove the Theorem recall the principal \mathfrak{sl}_2 -triple $\langle t, e, f \rangle \subset \mathfrak{g}$, and the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^{t+e}$, introduced in §6. Associated to this \mathfrak{sl}_2 -triple, is the Brylinski filtration, $W_\bullet V$, cf. Definition 6.5.1, on any G -representation $V \in \text{Rep}(G)$.

Our proof of Theorem 8.5.2 exploits two totally different geometric interpretations of Brylinski filtration. The first one is in terms of Intersection cohomology, due to [G2], and the second one is in terms of equivariant line bundles on the flag manifold for G , due to [Br]. The compatibility of the two interpretations, which is crucial for the proof below, is yet another manifestation of Langlands duality.

We begin by reviewing the main construction of [Br].

8.6 Brylinski filtration in terms of Springer resolution (after [Br]). Write $H \subset G$ for the maximal torus corresponding to the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^{t+e}$. Given a weight $\lambda \in \mathbb{Y}$ let $e^\lambda : H \rightarrow \mathbb{C}^*$ denote the corresponding homomorphism. Put

$$\mathbb{C}[G](-\lambda) = \{f \in \mathbb{C}[G] \mid f(h \cdot g) = e^{-\lambda}(h) \cdot f(g), \forall g \in G, h \in H\}. \quad (8.6.1)$$

Observe that the G -action on $\mathbb{C}[G]$ on the *right* by the formula $(R_g f)(y) = f(y \cdot g^{-1})$ makes the space $\mathbb{C}[G](-\lambda)$ above into a *left* G -module, isomorphic (by means of the anti-involution: $y \mapsto y^{-1}$ on G) to the induced (from character e^λ) representation

$$\text{Ind}_H^G e^\lambda = \{f \in \mathbb{C}[G] \mid f(g \cdot h) = e^\lambda(h) \cdot f(g)\}.$$

Consider the Borel subgroup B corresponding to the Lie algebra $\text{Lie } B = \mathfrak{n} + \mathfrak{t}$. Since $e, t \in \text{Lie } B$, we have $H \subset B$. We identify the flag manifold $\mathcal{F}\ell$ with G/B . The projection: $\text{Ad } g(t+e) \mapsto gB/B$ makes the conjugacy class $\text{Ad } G(t+e) \subset \mathfrak{g}$ an affine bundle over $\mathcal{F}\ell$ relative to the underlying vector bundle $\pi : T^*\mathcal{F}\ell \rightarrow \mathcal{F}\ell$, the cotangent bundle.

Observe next that the map $g \mapsto \text{Ad } g(t+e)$, descends to a G -equivariant isomorphism $G/H \xrightarrow{\sim} \text{Ad } G(t+e)$. We view the space $\text{Ind}_H^G e^\lambda$, cf (8.7.5), as the space of global regular sections of the G -equivariant line bundle on the conjugacy class $\text{Ad } G(t+e)$, corresponding to the character e^λ . Following [Br, §5], consider the natural filtration F_\bullet^{fib} on $\text{Ind}_H^G e^\lambda$ by *fiber degree*, see [Br, Theorem 4.4]. Write $\text{gr}_\bullet^{\text{fib}}(\text{Ind}_H^G e^\lambda)$ for the corresponding associated graded space. By [Br, Theorem 5.5] we have a \mathbb{Z} -graded G -module isomorphism

$$\text{gr}_\bullet^{\text{fib}}(\text{Ind}_H^G e^\lambda) = \Gamma(T^*\mathcal{F}\ell, \pi^*\mathcal{O}(\lambda)) = \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)). \quad (8.6.2)$$

The main idea of Brylinski is that the fiber degree filtration on the line bundle corresponds to a ‘principal’ filtration on (the λ -weight space of) the regular representation $\mathbb{C}[G]$, see [Br, Theorem 5.8], whose k -th term is defined by the formula: $\text{Ker}(e^{k+1}) \cap \mathbb{C}[G] = \{\varphi \in \mathbb{C}[G] \mid e^{k+1}(\varphi) = 0\}$, where $e(\varphi)$ denotes the infinitesimal e -action on φ on the right.

In more detail, for any $V \in \text{Rep}(G)$, we consider the weight decomposition $V = \bigoplus_{\mu \in \mathbb{Y}} V(\mu)$, with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^{t+e}$, and also the Brylinski filtration $W_\bullet V$, cf. Definition 6.5.1. We will use the notation $W_k V(\mu) := V(\mu) \cap W_k V$, and $\text{gr}_k^W V(\mu) := W_k V(\mu) / W_{k-1} V(\mu)$. Write $\text{ht}(\lambda) := \sum_{i \in I} \langle \lambda, \check{\alpha}_i \rangle$ for the *height* of $\lambda \in \mathbb{Y}$.

By [G2, Lemma 5.5.1] we know that, for any $V \in \text{Rep}(G)$, $\lambda \in \mathbb{Y}^{++}$ and $k \in \mathbb{Z}$, one has: $\text{Ker}(e^{k+1}) \cap V(-\lambda) = W_{k+\text{ht}(\lambda)} V \cap V(-\lambda)$. This equation, applied for $V = \mathbb{C}[G]$, and Brylinski’s Theorem [Br, Thm. 5.8] cited above imply that the graded space on the LHS of formula (8.6.2) coincides with the graded space $\text{gr}_{\bullet+\text{ht}(\lambda)}^W(\text{Ind}_H^G e^\lambda)$. Thus, for any $\lambda \in \mathbb{Y}^{++}$, we obtain a chain of isomorphisms

$$\text{gr}_{\bullet+\text{ht}(\lambda)}^W(\text{Ind}_H^G e^\lambda) = \text{gr}_{\bullet+\text{ht}(\lambda)}^{\text{fib}}(\text{Ind}_H^G e^\lambda) = \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)). \quad (8.6.3)$$

8.7 Proof of Theorem 8.5.2. We first fix $\lambda \in \mathbb{Y}^{++}$, and construct a $\mathbb{Z} \times \mathbb{Y}$ -graded vector space isomorphism $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \simeq \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda))$. We will then verify compatibility of the constructed isomorphisms for different λ 's with graded algebra structures on both sides of the isomorphism of Theorem 8.5.2.

Notation 8.7.1. Given $\lambda \in \mathbb{Y}$, let $j_\lambda : \text{Gr}_\lambda \hookrightarrow \text{Gr}$ be the Bruhat cell imbedding, and $\Delta_\lambda := (j_\lambda)_! \mathbb{C}_{\text{Gr}_\lambda}[\dim \text{Gr}_\lambda]$, the corresponding ‘standard’ perverse sheaf on Gr .

We will use below that, for $\lambda \in \mathbb{Y}^{++}$, the restriction of the projection $\varpi : \mathcal{B} \rightarrow \text{Gr}$ to the Bruhat cell $\mathcal{B}_{-\lambda}$ is a fibration $\mathcal{B}_{-\lambda} \rightarrow \text{Gr}_{-\lambda}$ with fiber $\mathbb{C}^{\ell(w)}$, where $w \in W$ is the element of minimal length such that $w(-\lambda) \in \mathbb{Y}^{++}$. It follows that $\varpi_* \mathcal{M}_{-\lambda} = \Delta_{-\lambda}[-\ell(w)]$. Further, for $\lambda \in \mathbb{Y}^{++}$ we have by definition, $\mathcal{W}_{-\lambda} = \mathcal{M}_{-\lambda}$, hence, $\mathcal{W}_{-\lambda} \star \mathbf{1}_{\text{Gr}} = \varpi_* \mathcal{W}_{-\lambda} = \varpi_* \mathcal{M}_{-\lambda} = \Delta_{-\lambda}[-\ell(w)]$.

Note that, since convolving with $(\mathcal{W}_{-\lambda} \star \mathcal{W}_\lambda)$ acts as the identity functor on $D_1^b(\text{Gr})$, the functor $\mathcal{W}_{-\lambda} \star (-)$ is an equivalence. Therefore, we find:

$$\begin{aligned} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) &= \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{W}_{-\lambda} \star \mathbf{1}_{\text{Gr}}, \mathcal{W}_{-\lambda} \star \mathcal{W}_\lambda \star \mathcal{R}) \\ &= \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{W}_{-\lambda} \star \mathbf{1}_{\text{Gr}}, \mathcal{R}) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\varpi_* \mathcal{W}_{-\lambda}, \mathcal{R}) \\ &= \text{Ext}_{D^b(\text{Gr})}^\bullet(\varpi_* \mathcal{M}_{-\lambda}, \mathcal{R}) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\Delta_{-\lambda}[-\ell(w)], \mathcal{R}). \end{aligned}$$

Further, view an element $\nu \in \mathbb{Y}$ as a point in Gr , the center of the cell Gr_ν , and write $i_\nu : \{\nu\} \hookrightarrow \text{Gr}$ for the corresponding imbedding. Then, for any object $N \in D^b \text{Perv}_1(\text{Gr})$, we clearly have $\text{Ext}_{D^b(\text{Gr})}^\bullet(\Delta_\nu, N) = H^\bullet(i_\nu^! N[\dim \text{Gr}_\nu]) = H^{\bullet + \dim \text{Gr}_\nu}(i_\nu^! N)$. Hence, for $\lambda \in \mathbb{Y}^{++}$, we get: $\text{Ext}_{D^b(\text{Gr})}^\bullet(\Delta_{-\lambda}[-\ell(w)], N) = H^{\bullet - \ell(w) + \dim \text{Gr}_{-\lambda}}(i_{-\lambda}^! N)$. Combining all the observations above, we obtain a canonical isomorphism

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) = H^{\bullet + \text{ht}(\lambda)}(i_{-\lambda}^! \mathcal{R}), \quad \text{ht}(\lambda) := \text{height}(\lambda), \quad \forall \lambda \in \mathbb{Y}^{++}, \quad (8.7.2)$$

where we have used that if λ is dominant, and $w \in W$ is an element of minimal length such that $-w(\lambda) \in \mathbb{Y}^{++}$, then $\dim \text{Gr}_{-\lambda} = \ell(w) + \text{ht}(\lambda)$.

We will express the RHS of (8.7.2) in representation theoretic terms. To this end, we recall first that there is a canonical filtration, $W_\bullet H_s(M)$, on the specialized equivariant cohomology of any object $M \in D_{T^\vee}^b(\text{Gr})$. Now, let $\mu \in \mathbb{Y} \subset \text{Gr}$ be a T^\vee -fixed point, and $M = i_\mu^! A$, for some $A \in \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$. By [G2, Proposition 5.6.2], the filtration W_\bullet is strictly compatible with the fixed point decomposition (6.4.1), i.e., we have

$$W_k H_s(i_\mu^! A) = H_s(i_\mu^! A) \cap W_k H_s(A), \quad \forall A \in \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr}), \quad k \in \mathbb{Z}. \quad (8.7.3)$$

Further, for any $V \in \text{Rep}(G)$, the fixed point decomposition on $H_s(\mathcal{P}V)$ corresponds, by Theorem 6.5.2, to the weight decomposition on $V = \bigoplus_{\mu \in \mathbb{Y}} V(\mu)$, with respect to the Cartan subalgebra $\mathfrak{h} = \mathfrak{g}^{t+e}$ and, moreover, the filtration $W_\bullet H_s(\mathcal{P}V)$ corresponds to Brylinski filtration $W_\bullet V$. Hence, (8.7.3) implies that, for any $k \in \mathbb{Z}$, the subspace $W_k H_s(i_\mu^! (\mathcal{P}V)) \subset H_s(\mathcal{P}V)$ corresponds to the subspace $V(\mu) \cap W_k V \subset V$.

It is known, cf. e.g. [G1] that, for any $A \in \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$, the complex $i_\mu^! A$ is *pure* in the sense of [BBD]. Hence, the spectral sequence for equivariant hyper-cohomology of $i_\mu^! A$ collapses. It follows, see [G2, Proposition 5.6.2], that, for any $A \in \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$, one has a canonical isomorphism $H^\bullet(i_\mu^! A) = \text{gr}_\bullet^W(H_s(i_\mu^! A))$. Thus, we obtain canonical isomorphisms

$$H^k(i_\mu^! (\mathcal{P}V)) = \text{gr}_k^W H_s(i_\mu^! (\mathcal{P}V)) = \text{gr}_k^W V(\mu). \quad (8.7.4)$$

Since $\mathcal{R} = \mathcal{P}\mathbb{C}[G]$, the isomorphism (8.7.4) yields: $H^k(i_{-\lambda}^! \mathcal{R}) = \mathrm{gr}_k^W(\mathbb{C}[G](-\lambda))$, where $\mathbb{C}[G](-\lambda)$ is the $(-\lambda)$ -weight space of the *left* regular representation of G . Thus, the considerations above and formulas (8.7.2) and (8.6.3) yield isomorphisms

$$\mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \simeq \mathrm{gr}_{+\mathrm{ht}(\lambda)}^W(\mathrm{Ind}_H^G e^\lambda) \simeq \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)), \quad \forall \lambda \in \mathbb{Y}^{++}. \quad (8.7.5)$$

The composite isomorphism gives the isomorphism claimed in Theorem 8.5.2.

To complete the proof, we must show that, for any $\lambda, \mu \in \mathbb{Y}^{++}$, the isomorphisms (8.7.5) and (8.6.3) transport the product map, see (8.4.2):

$$\mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \otimes \mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{W}_\mu) \longrightarrow \mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{W}_{\lambda+\mu}) \quad (8.7.6)$$

to the natural product-pairing: $\Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \otimes \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\mu)) \longrightarrow \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda + \mu))$.

To this end, consider the action-map $a : G^\vee(\mathcal{K}) \times_{G^\vee(\mathcal{O})} \mathrm{Gr} \longrightarrow \mathrm{Gr}$, see Sect. 6.3. For any $\chi \in \mathbb{Y}$, set $\mathrm{Gr}_\chi^2 := a^{-1}(\{\chi\}) \subset G^\vee(\mathcal{K}) \times_{G^\vee(\mathcal{O})} \mathrm{Gr}$, and write $j_\chi : \mathrm{Gr}_\chi^2 \hookrightarrow G^\vee(\mathcal{K}) \times_{G^\vee(\mathcal{O})} \mathrm{Gr}$ for the imbedding. By base change, we have $i_\chi^! \circ a_* = a_* \circ j_\chi^!$. Further, recall the morphism $m : \mathcal{R} \star \mathcal{R} = a_*(\mathcal{R} \tilde{\boxtimes} \mathcal{R}) \rightarrow \mathcal{R}$ corresponding to the algebra structure on $\mathbb{C}[G]$, see (7.2.2). We form the composite morphism

$$a_* \circ j_\chi^!(\mathcal{R} \tilde{\boxtimes} \mathcal{R}) = i_\chi^! \circ a_*(\mathcal{R} \tilde{\boxtimes} \mathcal{R}) \xrightarrow{m} i_\chi^! \mathcal{R} \quad (8.7.7)$$

Now, given $\nu, \eta \in \mathbb{Y}$, we have an obvious imbedding $i_\nu \boxtimes i_\eta : \{\nu\} \times \{\eta\} \hookrightarrow \mathrm{Gr}_{\nu+\eta}^2$. Composing the maps of cohomology induced by the latter imbedding and by morphism (8.7.7) for $\chi = \nu + \eta$, one obtains natural maps

$$\begin{aligned} H^\bullet(i_\nu^! \mathcal{R}) \otimes H^\bullet(i_\eta^! \mathcal{R}) &= H^\bullet(i_\nu^! \mathcal{R} \boxtimes i_\eta^! \mathcal{R}) \rightarrow H^\bullet(j_{\nu+\eta}^!(\mathcal{R} \tilde{\boxtimes} \mathcal{R})) \\ &= H^\bullet(a_* \circ j_{\nu+\eta}^!(\mathcal{R} \tilde{\boxtimes} \mathcal{R})) \xrightarrow{(8.7.7)} H^\bullet(i_{\nu+\eta}^! \mathcal{R}). \end{aligned} \quad (8.7.8)$$

It is a matter of routine argument involving base change, cf. e.g. proof of [G2, Proposition 3.6.2], to show that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \otimes \mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{W}_\mu) & \xrightarrow[(8.7.2)]{=} & H^{\bullet+\mathrm{ht}(\lambda)}(i_{-\lambda}^! \mathcal{R}) \otimes H^{\bullet+\mathrm{ht}(\mu)}(i_{-\mu}^! \mathcal{R}) \\ \downarrow (8.7.6) & & \downarrow (8.7.8) \\ \mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R} \star \mathcal{W}_{\lambda+\mu}) & \xrightarrow[(8.7.2)]{=} & H^{\bullet+\mathrm{ht}(\lambda+\mu)}(i_{-\lambda-\mu}^! \mathcal{R}). \end{array}$$

Thus, completing the proof of the Theorem amounts to verifying that: *For any ν, η , and $V = \mathbb{C}[G]$, the diagram below (arising from (8.6.1) by means of isomorphisms (8.7.4)) commutes*

$$\begin{array}{ccccc} H^\bullet(i_\nu^! \mathcal{R}) \otimes H^\bullet(i_\eta^! \mathcal{R}) & \xlongequal{\quad} & \mathrm{gr}_\bullet^W H_s(i_\nu^! \mathcal{R}) \otimes \mathrm{gr}_\bullet^W H_s(i_\eta^! \mathcal{R}) & \xlongequal{\quad} & \mathrm{gr}_\bullet^W(\mathbb{C}[G](\nu)) \otimes \mathrm{gr}_\bullet^W(\mathbb{C}[G](\eta)) \\ \downarrow (8.7.8) & & \downarrow (8.7.8) & & \downarrow \mathrm{mult} \\ H^\bullet(i_{\nu+\eta}^! \mathcal{R}) & \xlongequal{\quad} & \mathrm{gr}_\bullet^W H_s(i_{\nu+\eta}^! \mathcal{R}) & \xlongequal{\quad} & \mathrm{gr}_\bullet^W(\mathbb{C}[G](\nu + \eta)). \end{array}$$

The square on the left of the last diagram commutes because $H_s(-)$ is a tensor functor on $\mathrm{Perv}_{G^\vee(\mathcal{O})}(\mathrm{Gr})$, and the canonical filtration W_\bullet on the specialized equivariant cohomology is compatible with convolution. The square on the right commutes, because the filtration on \mathfrak{g} -representations by the eigenvalues of the t -action is obviously compatible with tensor products. This completes the proof of the Theorem. \square

Remark 8.7.9. Write: $\tilde{\mathfrak{g}} = G \times_B \mathfrak{b}$ for Grothendieck's simultaneous resolution, see e.g., [CG, ch.3]. Similarly to Theorem 7.3.1, one also has an equivariant version of Theorem 8.5.2, saying that there is a canonical $\mathbb{Z} \times \mathbb{Y}^{++}$ -graded algebra isomorphism

$$\bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D_{\mathbf{I}}^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \simeq \bigoplus_{\lambda \in \mathbb{Y}^{++}} \Gamma^\bullet(\tilde{\mathfrak{g}}, \mathcal{O}_{\tilde{\mathfrak{g}}}(\lambda)). \quad \diamond$$

8.8 Another fiber functor on $\mathcal{Perv}_{G^\vee(\mathcal{O})}(\text{Gr})$. Following §8.5, we consider a $\mathbb{Z} \times \mathbb{Y}^{++}$ -graded algebra

$$\Gamma[\tilde{\mathcal{N}}] := \bigoplus_{\lambda \in \mathbb{Y}^{++}} \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)).$$

The action of G on $\tilde{\mathcal{N}}$ induces a natural G -action on $\Gamma[\tilde{\mathcal{N}}]$ by graded algebra automorphisms.

By construction, the algebra $\Gamma[\tilde{\mathcal{N}}]$ may be viewed as a multi-graded homogeneous coordinate ring of the variety $\tilde{\mathcal{N}}$. In particular, any object $M \in \text{Mod}^{G \times \mathbb{C}^*}(\Gamma[\tilde{\mathcal{N}}])$ gives rise to a $G \times \mathbb{C}^*$ -equivariant coherent sheaf $\mathcal{F}(M) \in \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$. The assignment $M \mapsto \mathcal{F}(M)$ gives an exact functor $\mathcal{F} : \text{Mod}^{G \times \mathbb{C}^*}(\Gamma[\tilde{\mathcal{N}}]) \rightarrow \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}})$.

Next, recall the notation (8.4.1) and observe that, for any $M \in D_{\mathbf{I}^{\text{mon}}}^b(\text{Gr})$, the space $\mathbf{E}(\mathbf{1}_{\text{Gr}}, M)$ has a natural graded $\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$ -algebra structure, via the pairing defined in (8.4.2).

Now, the graded algebra isomorphism $\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) \cong \Gamma[\tilde{\mathcal{N}}]$, of Theorem 8.5.2, gives rise to an equivalence $\text{Mod}^{G \times \mathbb{C}^*}(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \xrightarrow{\sim} \text{Mod}^{G \times \mathbb{C}^*}(\Gamma[\tilde{\mathcal{N}}])$. We consider the composite functor $\hat{\mathbb{S}} : V \mapsto \mathcal{F}(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathcal{P}V))$; explicitly, our functor is the following composite:

$$\begin{aligned} \hat{\mathbb{S}} : \text{Rep}(G) &\xrightarrow{\mathcal{P}} \mathcal{Perv}_{G^\vee(\mathcal{O})}(\text{Gr}) \hookrightarrow D_{\mathbf{I}^{\text{mon}}}^b(\text{Gr}) \rightarrow \text{Mod}^{G \times \mathbb{C}^*}(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \\ &\xrightarrow{\sim} \text{Mod}^{G \times \mathbb{C}^*}(\Gamma[\tilde{\mathcal{N}}]) \xrightarrow{\mathcal{F}} \mathcal{Coh}^{G \times \mathbb{C}^*}(\tilde{\mathcal{N}}). \end{aligned} \quad (8.8.1)$$

Very similar to Proposition 7.7.6, using the isomorphism in (7.7.2), one proves

Proposition 8.8.2. *The functor $\hat{\mathbb{S}}$ is isomorphic to the functor $V \mapsto V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}$.* □

We will not go into more details here, since a much more elaborate version of this result will be proved in section 9.

8.9 Monodromic sheaves and extended affine flag manifold. At several occasions, we will need to extend various constructions involving convolution of \mathbf{I} -equivariant sheaves to the larger category of \mathbf{I} -monodromic sheaves. In particular, we would like to extend the bifunctor $M_1, M_2 \mapsto \mathbf{E}(M_1, M_2)$, from the category $D_{\mathbf{I}}^b(\text{Gr})$ to the category $D_{\mathbf{I}^{\text{mon}}}^b(\text{Gr})$. The construction of such an ‘extension’ is analogous to a similar construction in the framework of \mathcal{D} -modules on the (finite dimensional) flag manifold, given in [BlG, §5] (see esp. [BlG, (5.6), (5.9.1)-(5.9.2)]). We proceed to more details.

In $G^\vee(\mathcal{K})$, we consider the following subgroup $\mathbf{I}_1 := \{\gamma \in G^\vee(\mathcal{K}) \mid \gamma(0) = 1\}$. Thus, \mathbf{I}_1 is a pro-unipotent algebraic group that may be thought of as the (pro)-unipotent radical of the Iwahori group \mathbf{I} . We have $\mathbf{I} = T^\vee \cdot \mathbf{I}_1$. The coset space $\tilde{\mathcal{B}} := G^\vee(\mathcal{K})/\mathbf{I}_1$ has a natural ind-scheme structure, and will be called the *extended affine flag manifold*. The torus T^\vee normalizes the group \mathbf{I}_1 . Hence, there is a natural T^\vee -action on $\tilde{\mathcal{B}}$ on the right, making the canonical projection $\pi : \tilde{\mathcal{B}} = G^\vee(\mathcal{K})/\mathbf{I}_1 \rightarrow G^\vee(\mathcal{K})/\mathbf{I} = \mathcal{B}$ a principal T^\vee -bundle.

Let X be an (ind-) \mathbf{I} -variety. We recall that the notions of being \mathbf{I}_1 -monodromic and of being \mathbf{I}_1 -equivariant are known to be equivalent, since the group \mathbf{I}_1 is pro-unipotent. In particular, we have $D_{\mathbf{I}^{\text{mon}}}^b(X) \subseteq D_{\mathbf{I}_1^{\text{mon}}}^b(X) = D_{\mathbf{I}_1}^b(X)$.

Assume now X is an (ind-) $G^\vee(\mathcal{K})$ -variety. The $G(\mathcal{K})$ -action on X gives rise to the following *convolution diagram*: $G^\vee(\mathcal{K}) \times_{\mathbf{I}_1} X \longrightarrow X$, $(g, x) \mapsto gx$, cf., (8.2.1). For any $\mathcal{A} \in D_{\mathbf{I}\text{-mon}}^b(G(\mathcal{K})/\mathbf{I}_1) = D_{\mathbf{I}\text{-mon}}^b(\tilde{\mathcal{B}})$ and $\mathcal{M} \in D_{\mathbf{I}\text{-mon}}^b(X) \subseteq D_{\mathbf{I}_1}^b(X)$, there is a well-defined object $\mathcal{A} \boxtimes \mathcal{M} \in D^b(G^\vee(\mathcal{K}) \times_{\mathbf{I}_1} X)$. We put $\mathcal{A} \star \mathcal{M} := a_*(\mathcal{A} \boxtimes \mathcal{M})$, an \mathbf{I} -monodromic complex on X . This way, one defines a convolution bifunctor $\star : D_{\mathbf{I}\text{-mon}}^b(\tilde{\mathcal{B}}) \times D_{\mathbf{I}\text{-mon}}^b(X) \longrightarrow D_{\mathbf{I}\text{-mon}}^b(X)$. This applies, in particular, for $X = \mathcal{B}$ and $X = \text{Gr}$.

For each $w \in W_{\text{aff}}$, we put $\tilde{\mathcal{B}}_w := \pi^{-1}(\mathcal{B}_w)$, a T^\vee -bundle over the cell \mathcal{B}_w , which is (non-canonically) isomorphic to the trivial T^\vee -bundle $T^\vee \times \mathcal{B}_w \rightarrow \mathcal{B}_w$. Write $\tilde{j}_w : \tilde{\mathcal{B}}_w \hookrightarrow \tilde{\mathcal{B}}$ for the imbedding.

Let \mathcal{E} be a "universal" pro-unipotent local system on T^\vee ; specifically, if Π denotes the group algebra of the fundamental group of the torus T^\vee with respect to a base point $* \in T^\vee$, then the fiber of the local system \mathcal{E} at $* \in T^\vee$ equals the completion of Π at the augmentation ideal. Given $w \in W_{\text{aff}}$, let \mathcal{E}_w denote the pull-back of \mathcal{E} via the projection $\tilde{\mathcal{B}}_w \rightarrow T^\vee$ arising from a (non-canonically) chosen T^\vee -bundle trivialization $\tilde{\mathcal{B}}_w \simeq T^\vee \times \mathcal{B}_w$. We put $\tilde{\mathcal{M}}_w := (\tilde{j}_w)_! \mathcal{E}_w[\dim \tilde{\mathcal{B}}_w]$, and $\tilde{\mathcal{M}}_w^\vee := (\tilde{j}_w)_* \mathcal{E}_w[\dim \mathcal{B}_w]$. These are pro-objects in $\text{Perv}_{\mathbf{I}\text{-mon}}(\tilde{\mathcal{B}})$.

Let $w \in W_{\text{aff}}$. It is straightforward to verify that, for any $M \in D_{\mathbf{I}\text{-mon}}^b(\mathcal{B})$, resp., $M \in D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$, the pro-objects $\tilde{\mathcal{M}}_w \star M$, $\tilde{\mathcal{M}}_w^\vee \star M$ are indeed actual objects of $D_{\mathbf{I}\text{-mon}}^b(\mathcal{B})$, respectively, of $D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$. Moreover, if $M \in D_{\mathbf{I}}^b(\mathcal{B})$, resp., $M \in D_{\mathbf{I}}^b(\text{Gr})$, then one has canonical isomorphisms

$$\tilde{\mathcal{M}}_w \star M \simeq \mathcal{M}_w \star M \quad \text{and} \quad \tilde{\mathcal{M}}_w^\vee \star M \simeq \mathcal{M}_w^\vee \star M. \quad (8.9.1)$$

Remark 8.9.2. We warn the reader that, for $M \in D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$ and $\lambda \in \mathbb{Y}$, the convolution $\mathcal{W}_\lambda \star M$ is *not* defined, in general. \diamond

We are now able to extend the constructions of §8.4, and assign E-groups to objects of the category $D_{\mathbf{I}\text{-mon}}^b(\text{Gr}) \supset D_{\mathbf{I}}^b(\text{Gr})$, as follows. Given $\lambda \in \mathbb{Y}$, choose $\mu, \nu \in \mathbb{Y}^{++}$, such that $\lambda = \mu - \nu$, and consider the objects $\tilde{\mathcal{M}}_\mu^\vee, \tilde{\mathcal{M}}_{-\nu} \in D_{\mathbf{I}\text{-mon}}^b(\tilde{\mathcal{B}})$. Although the convolution of two objects of the category $D_{\mathbf{I}\text{-mon}}^b(\tilde{\mathcal{B}})$ has not been defined, we can define an object

$$\langle \mathcal{W}_\lambda \star M \rangle := \tilde{\mathcal{M}}_\mu^\vee \star (\tilde{\mathcal{M}}_{-\nu} \star M) \in D_{\mathbf{I}\text{-mon}}^b(\text{Gr}), \quad \text{for any } M \in D_{\mathbf{I}\text{-mon}}^b(\text{Gr}),$$

that involves only the convolution $\star : D_{\mathbf{I}\text{-mon}}^b(\tilde{\mathcal{B}}) \times D_{\mathbf{I}\text{-mon}}^b(\text{Gr}) \longrightarrow D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$. One verifies that the object $\langle \mathcal{W}_\lambda \star M \rangle$ thus defined is independent of the choice of the presentation $\lambda = \mu - \nu$. Moreover, it follows readily from (8.9.1) that the functor $\langle \mathcal{W}_\lambda \star (-) \rangle : D_{\mathbf{I}\text{-mon}}^b(\text{Gr}) \longrightarrow D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$, when restricted to the subcategory $D_{\mathbf{I}}^b(\text{Gr}) \subset D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$, agrees with the functor $\mathcal{W}_\lambda \star (-)$ given by the genuine convolution. The functor $\langle \mathcal{W}_\lambda \star (-) \rangle$ also has all the other expected properties, in particular, we have $\langle \mathcal{W}_\mu \star \langle \mathcal{W}_\lambda \star M \rangle \rangle = \langle \mathcal{W}_{\mu+\lambda} \star M \rangle$.

Now, for any $M_1, M_2 \in D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$, we put

$$\text{E}(M_1, M_2) := \bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^\bullet(M_1, \langle \mathcal{W}_\lambda \star M_2 \rangle \star \mathcal{R}),$$

where the ind-object $\langle \mathcal{W}_\lambda \star M_2 \rangle \star \mathcal{R}$ is well-defined since \mathcal{R} is clearly an \mathbf{I} -equivariant ind-sheaf on Gr .

From now on, we will make no distinction between the functors $\mathcal{W}_\lambda \star (-)$ and $\langle \mathcal{W}_\lambda \star (-) \rangle$ and, abusing the notation, write simply $\mathcal{W}_\lambda \star (-)$.

9 Geometric Equivalence theorems

In this section (only) we will be working over the ground field $\mathbb{k} = \overline{\mathbb{Q}_\ell}$, an algebraic closure of the field of ℓ -adic numbers. We write \mathbb{G}_m for the multiplicative group, viewed as a 1-dimensional algebraic group (a torus) over \mathbb{k} .

9.1 Equivalence theorem for perverse sheaves. For any $\lambda \in \mathbb{Y}$, the closure, $\overline{\text{Gr}}_\lambda = \overline{\mathbf{I} \cdot \lambda} \subset \text{Gr}$ is known to be a finite dimensional projective variety, an affine Schubert variety. Let $IC_\lambda = IC(\overline{\text{Gr}}_\lambda)$ denote the corresponding intersection cohomology complex, the Deligne-Goresky-MacPherson extension of the constant sheaf on Gr_λ (shifted by $\dim \text{Gr}_\lambda$). For $\lambda \in \mathbb{Y}^{++}$, we have $\overline{\text{Gr}}_\lambda = \overline{G^\vee(\mathcal{O}) \cdot \lambda}$, hence the notation above agrees with the one used in §§6-8.

Notation 9.1.1. We write $\mathcal{Perv}(\overline{\text{Gr}}_\lambda) := \mathcal{Perv}_{\mathbf{I}\text{-mon}}(\overline{\text{Gr}}_\lambda)$ for the abelian category of \mathbf{I} -monodromic ℓ -adic perverse sheaves on $\overline{\text{Gr}}_\lambda$, and let $\mathcal{Perv}(\text{Gr}) := \lim_{\lambda \in \mathbb{Y}^{++}} \text{ind} \mathcal{Perv}_{\mathbf{I}\text{-mon}}(\overline{\text{Gr}}_\lambda)$, be a direct limit of these categories.

Clearly, $\mathcal{Perv}(\text{Gr})$ is an abelian subcategory in $D^b(\text{Gr})$. The Ext-groups in the categories $\mathcal{Perv}(\text{Gr})$ and $D^b(\text{Gr})$ turn out to be the same. Specifically, one has

Proposition 9.1.2 ([BGS], Corollary 3.3.2). *There is a canonical isomorphism*

$$\text{Ext}_{\mathcal{Perv}(\text{Gr})}^\bullet(\mathcal{L}_1, \mathcal{L}_2) \xrightarrow{\sim} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{L}_1, \mathcal{L}_2), \quad \forall \mathcal{L}_1, \mathcal{L}_2 \in \mathcal{Perv}(\text{Gr}). \quad \square$$

Recall that, for each $\lambda \in \mathbb{Y}^{++}$, the finite-dimensional projective variety $\overline{\text{Gr}}_\lambda$ admits a (finite) algebraic stratification by Schubert cells. Therefore, each category $\mathcal{Perv}(\overline{\text{Gr}}_\lambda)$ has finitely many simple objects, and the simple objects of the category $\mathcal{Perv}(\text{Gr})$ are parametrized by elements of the root lattice $\text{Hom}(\mathbb{G}_m, T^\vee) = \mathbb{Y}$. For each $\lambda \in \mathbb{Y}$, one also has the standard perverse sheaf $\Delta_\lambda = (j_\lambda)_! \mathbb{k}_{\text{Gr}_\lambda}[-\dim \text{Gr}_\lambda] \in \mathcal{Perv}(\text{Gr})$, and the costandard perverse sheaf $\nabla_\lambda = (j_\lambda)_* \mathbb{k}_{\text{Gr}_\lambda}[-\dim \text{Gr}_\lambda] \in \mathcal{Perv}(\text{Gr})$, where $j_\lambda : \text{Gr}_\lambda \hookrightarrow \text{Gr}$ denotes the Schubert cell imbedding.

Notation 9.1.3. Given $\lambda \in \mathbb{Y}$, we put $\overline{\mathcal{W}}_\lambda := \mathcal{W}_\lambda \star \mathbf{1}_{\text{Gr}} = \varpi_*(\mathcal{W}_\lambda) \in \mathcal{Perv}(\text{Gr})$, where $\varpi : \mathcal{B} \rightarrow \text{Gr}$ is the standard projection, cf. §8.2.

If $\lambda \in \mathbb{Y}^{++}$, resp., $\lambda \in -\mathbb{Y}^{++}$, then from the definition of Wakimoto sheaves we get $\overline{\mathcal{W}}_\lambda = \nabla_\lambda$, resp., $\overline{\mathcal{W}}_\lambda = \Delta_\lambda$.

Recall the category $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$ introduced in §3.1, and invertible coherent sheaves $\mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$ on the Springer resolution $\tilde{\mathcal{N}}$, see Notation 8.5.1. Later in this section we are going to prove the following result that yields (a non-mixed counterpart of) the equivalence P on the right of diagram (1.1.1).

Theorem 9.1.4. *There is an equivalence of triangulated categories $P' : D^b \mathcal{Perv}(\text{Gr}) \xrightarrow{\sim} D_{\text{coherent}}^G(\tilde{\mathcal{N}})$, such that $P'(\mathbf{1}_{\text{Gr}}) = \mathcal{O}_{\tilde{\mathcal{N}}}$, and, for any $M \in D^b \mathcal{Perv}(\text{Gr})$, one has*

$$P'(\mathcal{W}_\lambda \star M \star \mathcal{P}V) = V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \otimes P'(M), \quad \text{for all } \lambda \in \mathbb{Y}, V \in \text{Rep}(G).$$

9.2 Mixed categories. An abelian \mathbb{k} -category \mathcal{C}^{mix} is called a *mixed* (abelian) category, cf. [BGS, Definition 4.1.1], provided a map $w : \text{Irr}(\mathcal{C}) \rightarrow \mathbb{Z}$ (called *weight*) is given, such that for any two simple objects $M, N \in \text{Irr}(\mathcal{C}^{\text{mix}})$ with $w(M) \leq w(N)$, one has $\text{Ext}_{\mathcal{C}^{\text{mix}}}^1(M, N) = 0$.

Let \mathcal{C}^{mix} be a mixed category with degree one *Tate twist*, i.e., with an auto-equivalence $M \rightarrow M\langle 1 \rangle$ of \mathcal{C}^{mix} , such that $w(M\langle 1 \rangle) = w(M) + 1$ for any $M \in \text{Irr}(\mathcal{C}^{\text{mix}})$. Every object $M \in \mathcal{C}^{\text{mix}}$ comes equipped with a canonical increasing ‘weight’ filtration $W_\bullet M$, cf. [BGS, Lemma 4.1.2]; the Tate twist shifts the weight filtration by 1.

Remark 9.2.1. In [BGS, Definition 4.1.1], an additional requirement that \mathcal{C}^{mix} is an artinian category is included in the definition of a mixed category. This requirement may be replaced, without affecting the theory, by the following two weaker conditions:

- For any $M \in \mathcal{C}^{\text{mix}}$ and $i \in \mathbb{Z}$, the object $W_i M / W_{-i} M$ has finite length, and
- For any $M, N \in \mathcal{C}^{\text{mix}}$ and $i \geq 0$, we have $\dim \text{Ext}_{\mathcal{C}^{\text{mix}}}^i(M, N) < \infty$.

These two conditions hold for all mixed categories considered below. \diamond

We recall the notion of a ‘mixed version’ of an abelian category, alternatively called a ‘grading’ on the category, see [BGS, §4.3]. Let \mathcal{C} be an abelian \mathbb{k} -category, \mathcal{C}^{mix} a mixed abelian category, and $v : \mathcal{C}^{\text{mix}} \rightarrow \mathcal{C}$ an exact faithful functor.

Definition 9.2.2 ([BGS], Definition 4.3.1). The pair $(\mathcal{C}^{\text{mix}}, v)$ is said to be a *mixed version* of \mathcal{C} (with forgetful functor v) if the following holds:

- The functor v sends semisimple objects into semisimple objects, and any simple object of \mathcal{C} is isomorphic to one of the form $v(M)$, $M \in \text{Irr}(\mathcal{C}^{\text{mix}})$;
- There is a natural isomorphism $\varepsilon : v(M) \xrightarrow{\sim} v(M\langle 1 \rangle)$, $\forall M \in \mathcal{C}^{\text{mix}}$;
- For any $M, N \in \mathcal{C}^{\text{mix}}$, the functor v induces an isomorphism

$$\bigoplus_{n \in \mathbb{Z}} \text{Ext}_{\mathcal{C}^{\text{mix}}}^\bullet(M, N\langle n \rangle) \xrightarrow{\sim} \text{Ext}_{\mathcal{C}}^\bullet(v(M), v(N)). \quad (9.2.3)$$

For example, the category $\text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ has a natural structure of mixed abelian category. It is a mixed version of the abelian category $\text{Coh}^G(\tilde{\mathcal{N}})$, with the functor $v : \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \rightarrow \text{Coh}^G(\tilde{\mathcal{N}})$ forgetting the \mathbb{G}_m -equivariant structure.

Similarly, the abelian category $\text{Mod}_f^{B \times \mathbb{G}_m}(\Lambda)$, see Notation 3.2.1, is a mixed abelian category. The set $\text{Irr}(\text{Mod}_f^{B \times \mathbb{G}_m}(\Lambda))$ of (isomorphism classes of) simple objects of this category consists 1-dimensional graded modules: $\mathbb{k}_\Lambda(\lambda)\langle i \rangle$, $\lambda \in \mathbb{Y}, i \in \mathbb{Z}$, (here $\langle i \rangle$ indicates that the vector space $\mathbb{k}_\Lambda(\lambda)$ is placed in \mathbb{Z} -grade degree i). The weight function on $\text{Irr}(\text{Mod}_f^{B \times \mathbb{G}_m}(\Lambda))$ is given by $w : \mathbb{k}_\Lambda(\lambda)\langle i \rangle \mapsto i$. Further, the natural functor $\text{Mod}_f^{B \times \mathbb{G}_m}(\Lambda) \rightarrow \text{Mod}_f^B(\Lambda)$ forgetting the \mathbb{Z} -grading makes the category $\text{Mod}_f^{B \times \mathbb{G}_m}(\Lambda)$ a mixed version of the category $\text{Mod}_f^B(\Lambda)$, see Notation 3.2.1.

We will also use the notion of a mixed *triangulated* category, for which we refer the reader to [BGSh]. The derived category of a mixed abelian category is a mixed triangulated category. Given a triangulated category D , a mixed triangulated category D^{mix} , and a triangulated functor $v : D^{\text{mix}} \rightarrow D$, one says that the pair (D^{mix}, v) is a mixed version of D if conditions similar to those of Definition 9.2.2 hold.

For example, we have a natural functor $\text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \rightarrow D_{\text{coherent}}^G(\tilde{\mathcal{N}})$, which extends to a triangulated functor $\mathbf{f} : D^b \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \rightarrow D_{\text{coherent}}^G(\tilde{\mathcal{N}})$. We claim that the latter functor makes the category $D^b \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ a mixed version of $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$. To see this, we first apply the equivalences of diagram (3.2.4). This way, we reduce the claim to the statement that a similarly defined functor $D^b \text{Mod}_f^{B \times \mathbb{G}_m}(\mathbf{S}) \rightarrow D_f^B(\mathbf{S})$ makes the category $D^b \text{Mod}_f^{B \times \mathbb{G}_m}(\mathbf{S})$ a mixed version of $D_f^B(\mathbf{S})$. We leave the proof of this latter statement to the reader (one can either compare the Ext-groups in the two categories directly, or else use Koszul duality to eventually reduce the comparison of Ext’s to Proposition 3.9.2(i)).

This way, Theorem 3.9.6 may be suggestively reinterpreted as follows

Corollary 9.2.4. *The pair $(D^b\text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}), v)$, where the functor ‘ v ’ is defined as a composite*

$$v : D^b\text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}) \xrightarrow{\mathbf{f}} D_{\text{coherent}}^G(\tilde{\mathcal{N}}) \xrightarrow{Q'} D^b\text{block}(\mathcal{U}),$$

is a mixed version of the triangulated category $D^b\text{block}(\mathcal{U})$. \square

9.3 Mixed perverse sheaves. In the rest of this section we will be working with algebraic varieties over \mathbb{F} , an algebraic closure of a finite field \mathbb{F}_q , where q is prime to ℓ . Thus, the loop Grassmannian Gr is viewed as an ind-scheme defined over \mathbb{F}_q . Let $D^b(\text{Gr})$ be the constructible derived category of (compactly supported) ℓ -adic sheaves on Gr . Following Deligne [De], we also consider $D_{\text{mixed}}^b(\text{Gr})$, the category of *mixed* ℓ -adic constructible complexes on Gr , cf. [BBD] for more references. We have an obvious forgetful functor $v : D_{\text{mixed}}^b(\text{Gr}) \rightarrow D^b(\text{Gr})$.

For any $M, N \in D_{\text{mixed}}^b(\text{Gr})$, the vector space $\text{Ext}_{D^b(\text{Gr})}^\bullet(vM, vN)$ comes equipped with a natural (geometric) Frobenius action, preserving the Ext-degree. Below, we will be abusing the notation slightly and will be writing $\text{Ext}_{D^b(\text{Gr})}^\bullet(M, N)$ instead of $\text{Ext}_{D^b(\text{Gr})}^\bullet(vM, vN)$; this cannot lead to a confusion since the category we are dealing with at any particular moment is always indicated in the subscript of the Ext-group in question.

Deligne results from [De] imply

Proposition 9.3.1. *For any $M_1, M_2 \in D_{\text{mixed}}^b(\overline{\text{Gr}}_\lambda)$, all the absolute values of eigenvalues of the (geometric) Frobenius action on the spaces $\text{Ext}_{D^b(\text{Gr})}^\bullet(M_1, M_2)$ are integral powers of $q^{1/2}$.* \square

According to [BGS, p. 519], there is a well-defined abelian subcategory $\text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda) \subset D_{\text{mixed}}^b(\overline{\text{Gr}}_\lambda)$ (it is the category denoted by $\tilde{\mathcal{P}}$ in [BGS, above Theorem 4.4.4]), which is a mixed abelian category. Moreover, by [BGS, Lemmas 4.4.1(2), 4.4.6, 4.4.8], one has

Proposition 9.3.2. *For each $\lambda \in \mathbb{Y}^{++}$, the following holds:*

- (i) *The forgetful functor $v : D_{\text{mixed}}^b(\overline{\text{Gr}}_\lambda) \rightarrow D^b(\overline{\text{Gr}}_\lambda)$ restricts to a functor $v : \text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda) \rightarrow \text{Perv}(\overline{\text{Gr}}_\lambda)$, that makes the category $\text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda)$ a mixed version of $\text{Perv}(\overline{\text{Gr}}_\lambda)$.*
- (ii) *The category $\text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda)$ has enough projectives. Moreover, for any projective $P \in \text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda)$, the object vP is projective in $\text{Perv}(\overline{\text{Gr}}_\lambda)$.* \square

It was further shown in [BGS] that, for each $\mu \leq \lambda$, the standard perverse sheaf Δ_μ and the costandard perverse sheaf ∇_μ admit canonical lifts to $\text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda)$. Abusing the notation, we will denote these lifts by Δ_μ and ∇_μ again.

Below, we will make use of the following

Proposition 9.3.3. (i) *Any projective $P \in \text{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda)$ has a Δ -flag.*

(ii) *For any $\lambda, \mu \in \mathbb{Y}$ we have $\mathcal{M}_\lambda^\vee \star \Delta_\mu \in \text{Perv}(\text{Gr})$, is a perverse sheaf.*

Proof. Part (i) is standard, cf. e.g. [BGS, Theorem 3.2.1]. The proof of part (ii) is entirely similar to the proof of Proposition 8.2.4(i), and is left to the reader. \square

Next, we put $\text{Perv}^{\text{mix}}(\text{Gr}) := \lim_{\lambda \in \mathbb{Y}^{++}} \text{ind} \text{Perv}(\overline{\text{Gr}}_\lambda)$. From Proposition 9.3.2 we deduce that $\text{Perv}^{\text{mix}}(\text{Gr})$ is a mixed abelian category, moreover, it is a mixed version of the category $\text{Perv}(\text{Gr})$. In particular, the simple objects of $\text{Perv}^{\text{mix}}(\text{Gr})$ are parametrized by elements of $\mathbb{Z} \times \mathbb{Y}$.

The category $\mathcal{Perv}^{\text{mix}}(\text{Gr})$ does not *a priori* have enough projectives (although, it turns out, *a posteriori* that it does). Nonetheless, given a simple object $L \in \mathcal{Perv}^{\text{mix}}(\text{Gr})$ and any $\lambda \in \mathbb{Y}^{++}$, such that $\text{supp } L \subset \overline{\text{Gr}}_\lambda$, we can use Proposition 9.3.2(ii) to find an indecomposable projective cover $P^\lambda \twoheadrightarrow L$, in the category $\mathcal{Perv}^{\text{mix}}(\overline{\text{Gr}}_\lambda)$. The uniqueness of projective covers yields, for any pair $\lambda < \mu$ (in \mathbb{Y}^{++}), a morphism $P^\mu \rightarrow P^\lambda$. We conclude that the objects $\{P^\lambda\}_{\lambda \in \mathbb{Y}^{++}}$ form an inverse system, and we put $P(L) := \limproj P^\lambda$. This is a pro-object in the category $\mathcal{Perv}^{\text{mix}}(\text{Gr})$, which is a projective cover of L . If L is pure of weight $w(L) = n$, then the weight filtration $W_\bullet P(L)$ is well-defined, moreover, we have $W_n P(L) = P(L)$. This way one proves

Lemma 9.3.4. *Any object of $M \in \mathcal{Perv}^{\text{mix}}(\text{Gr})$ is a quotient of a projective pro-object P such that*

- (i) $W_i P / W_{-i} P \in \mathcal{Perv}^{\text{mix}}(\text{Gr})$, for any $i \in \mathbb{Z}$, and
- (ii) If $W_n M = M$, then we have $W_n P = P$. □

Remark 9.3.5. Note that in the situation of the Lemma one typically has $W_i P \neq 0$ for all $i \ll 0$, in general. ◇

An important role in the construction below will be played by the following

Lemma 9.3.6. *For any projective objects $P_1, P_2 \in \limproj \mathcal{Perv}^{\text{mix}}(\text{Gr})$, and $\lambda, \mu \in \mathbb{Y}^{++}$, we have*

$$\text{Ext}_{D^b(\text{Gr})}^n(P_1, W_\lambda \star P_2 \star IC_\mu) = 0 \quad \text{for any } n \neq 0.$$

Proof. Fix λ and μ as above. We claim first that $W_\lambda \star vP_2 \star IC_\mu$ is a perverse sheaf. To see this, we note that for $\lambda \in \mathbb{Y}^{++}$, we have $W_\lambda = \mathcal{M}_\lambda^\vee$. Further, by Proposition 9.3.3(i), the projective P_2 admits a Δ -flag. Hence, we are reduced to proving that $\mathcal{M}_\lambda^\vee \star \Delta_\nu \star IC_\mu$ is a perverse sheaf, for any $\nu \in \mathbb{Y}$. But, part (ii) of Proposition 9.3.3 says that $\mathcal{M}_\lambda^\vee \star \Delta_\nu \in \mathcal{Perv}(\text{Gr})$. Hence, $W_\lambda \star vP_2 = \mathcal{M}_\lambda^\vee \star vP_2 \in \mathcal{Perv}(\text{Gr})$, by Proposition 9.3.3(ii), and therefore $(W_\lambda \star vP_2) \star IC_\mu$ is a perverse sheaf by Gaitsgory's Theorem 6.3.2.

To complete the proof, we use Proposition 9.1.2 to obtain

$$\text{Ext}_{D^b(\text{Gr})}^n(P_1, W_\lambda \star P_2 \star IC_\mu) = \text{Ext}_{\mathcal{Perv}(\text{Gr})}^n(P_1, W_\lambda \star P_2 \star IC_\mu).$$

Now, since vP_1 is a projective, the Ext-group on the RHS above vanishes for all $n \neq 0$, and we are done. □

Definition 9.3.7. Let $D_{\text{proj}}^{\text{mix}}(\text{Gr})$, resp., $D_{\text{proj}}(\text{Gr})$, be the full subcategory in the homotopy category of complexes in $\limproj \mathcal{Perv}^{\text{mix}}(\text{Gr})$, resp., in $\limproj \mathcal{Perv}(\text{Gr})$, whose objects are complexes $C^\bullet = (\dots \rightarrow C^i \rightarrow C^{i+1} \rightarrow \dots)$ such that

- vC^i is a projective pro-object in $\mathcal{Perv}(\text{Gr})$, for any $i \in \mathbb{Z}$, and $C^i = 0$ for $i \gg 0$;
- $H^i(C^\bullet) = 0$ for $i \ll 0$, moreover, $vH^i(C^\bullet) \in \mathcal{Perv}(\text{Gr})$, for any $i \in \mathbb{Z}$.

In the $D_{\text{proj}}(\text{Gr})$ -case, one has to replace vC^\bullet by C^\bullet in the two conditions above.

From Lemma 9.3.4 one derives the following

Corollary 9.3.8. (i) *The natural functor $\Theta^{\text{mix}} : D_{\text{proj}}^{\text{mix}}(\text{Gr}) \longrightarrow D^b \mathcal{Perv}^{\text{mix}}(\text{Gr})$, resp. the functor $\Theta : D_{\text{proj}}(\text{Gr}) \longrightarrow D^b \mathcal{Perv}(\text{Gr})$, is an equivalence.* □

9.4 From perverse sheaves on Gr to coherent sheaves on $\tilde{\mathcal{N}}$. The rest of this section is mostly devoted to constructing the equivalence P on the right of diagram (1.1.1). Our strategy will be as follows.

First, we consider the $\mathbb{Z} \times \mathbb{Y}^{++}$ -graded algebra $\Gamma[\tilde{\mathcal{N}}]$, introduced in §8.8, and the abelian category $\text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$. We will construct a functor $D_{\text{proj}}^{\text{mix}}(\text{Gr}) \rightarrow D^b \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$. Then, we will invert the equivalence of Corollary 9.3.8, and form the following composite functor

$$P : D^b \text{Perv}^{\text{mix}}(\text{Gr}) \xrightarrow{(\Theta^{\text{mix}})^{-1}} D_{\text{proj}}^{\text{mix}}(\text{Gr}) \rightarrow D^b \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) \xrightarrow{\mathcal{F}} D^b \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}), \quad (9.4.1)$$

where $\mathcal{F} : \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) \rightarrow \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})$ is the natural functor that has been considered in §8.8.

To proceed further, we need the following

Definition 9.4.2. An object $M = \bigoplus_{(i, \lambda) \in \mathbb{Z} \times \mathbb{Y}} M_\lambda^i \in \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$ is said to be *thin* if there exists $\mu \in \mathbb{Y}$ such that $M_\lambda^i = 0$ for all $\lambda \in \mu + \mathbb{Y}^{++}$.

Thin objects form a Serre subcategory in $\text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$, which will be denoted $\text{Mod}_{\text{thin}}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$. The functor \mathcal{F} sends any object of the subcategory $\text{Mod}_{\text{thin}}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$ to zero, and, for any $M \in \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$, one clearly has

$$\bigoplus_{\lambda \in \mathbb{Y}^{++}} \Gamma(\tilde{\mathcal{N}}, \mathcal{F}(M) \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \cong M \quad \text{in} \quad \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) / \text{Mod}_{\text{thin}}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]).$$

Furthermore, an equivariant version of the classical result of Serre implies that the functor \mathcal{F} induces an equivalence

$$\mathcal{F} : \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) / \text{Mod}_{\text{thin}}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) \xrightarrow{\sim} \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}).$$

Our goal in this section is to prove the following result which is, essentially, a mixed analogue of Theorem 9.1.4.

Theorem 9.4.3. *The functor P in (9.4.1) is an equivalence of triangulated categories, such that $P(M \star PV) = V \otimes P(M)$, $\forall V \in \text{Rep}(G)$, moreover, for any $\lambda \in \pm \mathbb{Y}^{++}$, we have $P(\overline{W}_\lambda) = \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$.*

Remark 9.4.4. The reason we restrict, in the last statement of the Theorem, to the case of $\lambda \in \pm \mathbb{Y}^{++}$ is that we do not know, for general $\lambda \in \mathbb{Y}$, whether the object $\overline{W}_\lambda \in \text{Perv}(\text{Gr})$ admits a natural lifting to $\text{Perv}^{\text{mix}}(\text{Gr})$ (this would follow from ‘standard conjectures’ saying that the property that the Frobenius action be semisimple is preserved under direct and inverse image functors). In the special case of $\lambda \in \mathbb{Y}^{++}$, resp. $\lambda \in -\mathbb{Y}^{++}$, we have $\overline{W}_\lambda = \nabla_\lambda$, resp. $\overline{W}_\lambda = \Delta_\lambda$, and such a lifting is then afforded by [BGS]. This is the lifting that we are using in the Theorem above. \diamond

9.5 An Ext-formality result. We view the sky-scraper sheaf $\mathbf{1}_{\text{Gr}}$ as a simple object (of weight zero) in $\text{Perv}^{\text{mix}}(\text{Gr})$. Using Lemma 9.3.4, we construct a projective resolution $\dots \rightarrow P^{-1} \rightarrow P^0 \rightarrow \mathbf{1}_{\text{Gr}}$, where each P^i is a pro-object in $\text{Perv}^{\text{mix}}(\text{Gr})$. Moreover, since $\text{Perv}^{\text{mix}}(\text{Gr})$ is known by [BGS] to be a *Koszul category*, one may choose the resolution in such a way that $W_i P^i = P^i$, for all $i = 0, -1, -2, \dots$. Thus, the direct sum $P := \bigoplus_{i \leq 0} P^i$ is a well-defined pro-object in $\text{Perv}^{\text{mix}}(\text{Gr})$. The differential in the resolution makes P a dg-object, equipped with a quasi-isomorphism $P \xrightarrow{\text{qis}} \mathbf{1}_{\text{Gr}}$. Of course, P is a projective, hence, vP is a projective pro-object in $\text{Perv}(\text{Gr})$, by Proposition 9.3.2(ii).

Recall that for any pair of objects $M_1, M_2 \in D_{\mathbf{I}\text{-mon}}^b(\text{Gr})$, we have defined a vector space $E(M_1, M_2)$, see (8.4.1). Applying the construction to $M_1 = vP^i$ and $M_2 = vP^j$, $i, j = 0, -1, -2, \dots$, we thus get a \mathbb{Z} -graded algebra $E^\bullet(P, P) := \bigoplus_{n \in \mathbb{Z}} E^n(P, P)$, where

$$E^n(P, P) := \bigoplus_{\{(i,j) \in \mathbb{Z}^2 \mid i-j=n, \lambda \in \mathbb{Y}^{++}\}} \text{Ext}_{D^b(\text{Gr})}^0(P^i, \mathcal{W}_\lambda \star P^j \star \mathcal{R}). \quad (9.5.1)$$

Notice, that the compatibility of formulas (9.5.1) and (8.4.1) is insured by Lemma 9.3.6, which yields

$$\text{Ext}_{D^b(\text{Gr})}^k(P^i, \mathcal{W}_\lambda \star P^j \star \mathcal{R}) = 0 \quad \text{for all } k \neq 0.$$

Further, the commutator with the differential $d : P^\bullet \rightarrow P^{\bullet+1}$ induces a differential $d : E^n(P, P) \rightarrow E^{n+1}(P, P)$, thus makes $E(P, P)$ a dg-algebra.

Recall next that by Proposition 9.3.1, all the absolute values of egenvalues of the (geometric) Frobenius action on $E(P, P)$ are integral powers of $q^{1/2}$. These integral powers give rise to an additional \mathbb{Z} -grading $E^i(P, P) = \bigoplus_{j \in \mathbb{Z}} E_j^i(P, P)$ which is preserved by the differential and is compatible with the algebra structure. Thus, the object $E(P, P)$ becomes a differential *bi-graded* algebra. We write $H(E(P, P)) = \bigoplus_j H_j^i(E(P, P))$ for the corresponding cohomology algebra considered as a differential bi-graded algebra with induced bi-grading, and with zero differential.

Proposition 9.5.2. (i) *For any $i \neq j$, we have $H_j^i(E(P, P)) = 0$. Furthermore, there is a canonical graded algebra isomorphism*

$$\bigoplus_{i \in \mathbb{Z}} H_i^i(E(P, P)) \simeq E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) \left(= \bigoplus_{i \in \mathbb{Z}} \left(\bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^i(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \right) \right).$$

(ii) *The dg-algebra $E(P, P)$ is formal, that is, there exists a bi-graded algebra quasi-isomorphism $\sigma : E(P, P) \xrightarrow{\text{qis}} E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$.*

To prove the Proposition, we will use some standard yoga from [De].

Let $A = \bigoplus_i A^i$ be a dg-algebra equipped with an additional ‘weight’ grading $A^i = \bigoplus_k A_k^i$ which is preserved by the differential, i.e., is such that $d : A_k^i \rightarrow A_k^{i+1}$. Thus, the ‘weight’ grading on A induces a grading $H^i(A) = \bigoplus_k H_k^i(A)$, on each cohomology group. The following formality criterion is (implicitly) contained in [De, 5.3.1] (cf. also the proof of [De, Corollary 5.3.7]).

Lemma 9.5.3. *If $H_k^i(A) = 0$ for all $i \neq k$, then the dg-algebra A is formal, i.e., is quasi-isomorphic to $(H(A), d = 0)$ as a bigraded algebra.* \square

Proof of Proposition 9.5.2. From isomorphisms (8.7.2) we find

$$E^\bullet(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} H^{\bullet+\text{ht}(\lambda)}(i_{-\lambda}^! \mathcal{R}).$$

By the pointwise purity of the intersection cohomology of affine Schubert varieties (proved at the end of [KL1]) the space on the right is pure, that is, for any j , all the eigenvalues of the geometric Frobenius action on $H^j(i_{-\lambda}^! \mathcal{R})$ have absolute value $q^{j/2}$. Hence, we obtain $E_j^i(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) = 0$ unless $i = j$.

To complete the proof of part (i), we observe that $P \xrightarrow{\text{qis}} \mathbf{1}_{\text{Gr}}$ is a projective resolution and there are no non-zero Ext’s between the objects P^i and $\mathcal{W}_\lambda \star P^j \star \mathcal{R}$ (by Lemma 9.3.6). It follows that the cohomology groups of the dg-object $\text{Hom}(P, \mathcal{W}_\lambda \star P \star \mathcal{R})$ are by definition, the Ext-groups in the category $\text{Perv}(\text{Gr})$. Further, Proposition 9.1.2 says that we may replace Ext-groups in the category $\text{Perv}(\text{Gr})$ by those in the ambient category $D^b(\text{Gr})$. The isomorphism of part (i) follows.

We deduce from the isomorphism and from the first paragraph of the proof that $H_j^i(\mathbf{E}(\mathbf{P}, \mathbf{P})) = 0$ for any $i \neq j$. This completes the proof of part (i). Part (ii) is an immediate consequence of Lemma 9.5.3. \square

9.6 Re-grading functor. For any $M \in \mathcal{Perv}(\text{Gr})$, the group G acts naturally on the algebra $\mathbf{E}(M, M)$ by graded algebra automorphisms, as explained in §7.1. So, we may consider the category $\text{Mod}^G(\mathbf{E}(M, M))$ of G -equivariant algebraic $\mathbf{E}(M, M)$ -modules, see Notation 3.2.1. In particular, let $\text{Comp}^G(\mathbf{E}(\mathbf{P}, \mathbf{P}))$ be the homotopy category of G -equivariant algebraic differential *bi-graded* finitely-generated $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -modules (the group G preserves each bi-graded component and the differential acts on an object $K \in \text{Comp}^G(\mathbf{E}(\mathbf{P}, \mathbf{P}))$ as follows $d : K_j^i \rightarrow K_j^{i+1}$). The following is a variation of [BGG].

Let $K = (K_j^i) \in \text{Comp}^G(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}))$. Then, for any element $a \in \mathbf{E}^n(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$, the a -action sends K_j^i to K_{j+n}^{i+n} , since $\mathbf{E}^n(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) = \mathbf{E}_n^n(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$, by Proposition 9.5.2. Therefore, for each integer $m \in \mathbb{Z}$, the subspace $\mathbf{G}^m(K) := \bigoplus_{i \in \mathbb{Z}} K_{i+m}^i \subset \bigoplus_{i,j \in \mathbb{Z}} K_j^i$ is $\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$ -stable. We put a \mathbb{Z} -grading $\mathbf{G}^m(K) = \bigoplus_{i \in \mathbb{Z}} \mathbf{G}^m(K)_i$ on this subspace by $\mathbf{G}^m(K)_i := K_{i+m}^i$. Further, the differential $d : K_i^{i+m} \rightarrow K_i^{i+m+1}$, on K , gives rise to $\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$ -module morphisms $\mathbf{G}^m(K) \rightarrow \mathbf{G}^{m+1}(K)$ which preserve the above defined gradings. This way, an object $K = (K_j^i) \in \text{Comp}^G(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}))$ gives rise to a complex $\mathbf{G}(K) = (\dots \rightarrow \mathbf{G}^m(K) \rightarrow \mathbf{G}^{m+1}(K) \rightarrow \dots)$, of graded $\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})$ -modules. The assignment $K \mapsto \mathbf{G}(K)$ thus defined yields a functor

$$\mathbf{G} : \text{Comp}^G(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \longrightarrow D^b \text{Mod}^{G \times \mathbf{G}^{\mathbf{m}}}(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})).$$

9.7 Construction of the functors P and P' . We begin with constructing a functor

$$\Psi : D_{\text{proj}}^{\text{mix}}(\text{Gr}) \longrightarrow \text{Comp}^G(\mathbf{E}(\mathbf{P}, \mathbf{P})) \quad (9.7.1)$$

as follows.

View an object of $D_{\text{proj}}^{\text{mix}}(\text{Gr})$ represented by a complex $\dots \rightarrow C^i \rightarrow C^{i+1} \rightarrow \dots$ as a dg-sheaf $C = \bigoplus_i C^i \in \lim \text{proj } \mathcal{Perv}^{\text{mix}}(\text{Gr})$. To such a C , we associate a graded vector space $\mathbf{E}(\mathbf{P}, C) = \bigoplus_{n \in \mathbb{Z}} \mathbf{E}^n(\mathbf{P}, C)$, where

$$\mathbf{E}^n(\mathbf{P}, C) := \bigoplus_{\{(i,j) \in \mathbb{Z}^2 \mid j-i=n, \lambda \in \mathbb{Y}^{++}\}} \text{Ext}_{D^b(\text{Gr})}^0(\mathbf{P}^i, \mathcal{W}_\lambda \star C^j \star \mathcal{R}).$$

(By Lemma 9.3.6 we know that $\text{Ext}_{D^b(\text{Gr})}^k(\mathbf{P}^i, \mathcal{W}_\lambda \star C^j \star \mathcal{R}) = 0$, for all $k \neq 0$, since C^j is a projective). Further, the formula $u \mapsto d_C \circ u - u \circ d_{\mathbf{P}}$, where $d_C, d_{\mathbf{P}}$ are the differentials on C and on \mathbf{P} , respectively, induces a differential $\mathbf{E}^\bullet(\mathbf{P}, C) \rightarrow \mathbf{E}^{\bullet+1}(\mathbf{P}, C)$. There is also a natural $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -module structure on the space $\mathbf{E}(\mathbf{P}, C)$ coming from the pairing (8.4.2). Finally, the geometric Frobenius action makes $\mathbf{E}(\mathbf{P}, C)$ a bigraded vector space. Summarizing, the space $\mathbf{E}(\mathbf{P}, C)$ has a natural differential bigraded $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -module structure. The assignment $\Psi : C \mapsto \mathbf{E}(\mathbf{P}, C)$ thus obtained gives the desired functor Ψ in (9.7.1).

Further, we compose Ψ with the pull-back functor induced by the quasi-isomorphism σ of Proposition 9.5.2(ii). In view of part (i) of Proposition 9.5.2 we thus obtain a functor

$$D_{\text{proj}}^{\text{mix}}(\text{Gr}) \xrightarrow{\Psi} \text{Comp}^G(\mathbf{E}(\mathbf{P}, \mathbf{P})) \xrightarrow{\sigma^*} \text{Comp}^G(\mathbf{E}(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})). \quad (9.7.2)$$

Next, we exploit Theorem 8.5.2 that provides a G -equivariant $(\mathbb{Z} \times \mathbb{Y}^{++})$ -graded algebra isomorphism, cf. §8.8,

$$\Gamma[\tilde{\mathcal{N}}] \cong E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}}) \left(= \bigoplus_{\lambda \in \mathbb{Y}^{++}} \text{Ext}_{D^b(\text{Gr})}^{2\bullet}(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{R}) \right). \quad (9.7.3)$$

This isomorphism induces a category equivalence $\tau : \text{Mod}^{G \times \mathbb{G}_m}(E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \xrightarrow{\sim} \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}])$. We define a functor Φ , cf. (9.4.1), as the following composite functor:

$$\begin{aligned} \Phi : D_{\text{proj}}^{\text{mix}}(\text{Gr}) &\xrightarrow{\Psi} \text{Comp}^G(E(\text{P}, \text{P})) \xrightarrow{\sigma^*} \text{Comp}^G(E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \\ &\xrightarrow{\mathbb{G}} D^b \text{Mod}^{G \times \mathbb{G}_m}(E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \xrightarrow{\tau} D^b \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]). \end{aligned} \quad (9.7.4)$$

This is the functor that gives the middle arrow in diagram (9.4.1).

Finally, we define $P := \mathcal{F} \circ \Phi \circ (\Theta^{\text{mix}})^{-1}$, the functor used in the statement of Theorem 9.4.3; explicitly, this is the following composite functor:

$$\begin{aligned} P : D^b \mathcal{Perv}^{\text{mix}}(\text{Gr}) &\xrightarrow{(\Theta^{\text{mix}})^{-1}} D_{\text{proj}}^{\text{mix}}(\text{Gr}) \xrightarrow{\Psi} \text{Comp}^G(E(\text{P}, \text{P})) \xrightarrow{\sigma^*} \text{Comp}^G(E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \\ &\xrightarrow{\mathbb{G}} D^b \text{Mod}^{G \times \mathbb{G}_m}(E(\mathbf{1}_{\text{Gr}}, \mathbf{1}_{\text{Gr}})) \xrightarrow{(9.7.3)} D^b \text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) \xrightarrow{\mathcal{F}} D^b \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}}). \end{aligned} \quad (9.7.5)$$

Recall next that we have introduced in Definition 9.3.7 two homotopy categories, $D_{\text{proj}}^{\text{mix}}(\text{Gr})$ and $D_{\text{proj}}(\text{Gr})$. So far, we have only worked with the category $D_{\text{proj}}^{\text{mix}}(\text{Gr})$, since that category serves, by Corollary 9.3.8, as a replacement of $D^b \mathcal{Perv}^{\text{mix}}(\text{Gr})$, the category that we are interested in.

Now, however, it will be convenient for us to start working with the category $D_{\text{proj}}(\text{Gr})$ instead. We observe that the construction of the functor Φ given in §9.7 applies with obvious modifications, such as replacing double gradings by single gradings, to produce a functor

$$\Phi' : D_{\text{proj}}(\text{Gr}) \longrightarrow D_f^G(\Gamma[\tilde{\mathcal{N}}]). \quad (9.7.6)$$

Remark 9.7.7. As opposed to the construction of Φ , in the construction of Φ' the step involving the "re-grading" functor \mathbb{G} should be skipped. Note also that we still use (as we may) the formality statement in Proposition 9.5.2(ii), since we may exploit the mixed structure on P . \diamond

Further, inverting the equivalence $D_{\text{proj}}(\text{Gr}) \cong D^b \mathcal{Perv}(\text{Gr})$ of Corollary 9.3.8 and mimicking (9.4.1), cf. also (9.7.5), we define the following composite functor

$$P' : D^b \mathcal{Perv}(\text{Gr}) \xrightarrow{\Theta^{-1}} D_{\text{proj}}(\text{Gr}) \xrightarrow{\Phi'} D_f^G(\Gamma[\tilde{\mathcal{N}}]) \xrightarrow{\mathcal{F}} D_{\text{coherent}}^G(\tilde{\mathcal{N}}).$$

9.8 Properties of the functor P' . An advantage of considering the "non-mixed" setting is that, for any $\lambda \in \mathbb{Y}$, there is a well-defined functor $\mathcal{W}_\lambda \star (-)$ on $D^b \mathcal{Perv}(\text{Gr})$.

Proposition 9.8.1. *We have $P'(\mathbf{1}_{\text{Gr}}) = \mathcal{O}_{\tilde{\mathcal{N}}}$; Furthermore, for any $M \in D^b \mathcal{Perv}(\text{Gr})$, there is a natural isomorphism*

$$P'(\mathcal{W}_\mu \star M \star \mathcal{P}V) = V \otimes_{\tilde{\mathcal{N}}}(\mu) \otimes P'(M), \quad \forall \mu \in \mathbb{Y}, V \in \text{Rep}(G).$$

Proof. The isomorphism $P'(\mathbf{1}_{\text{Gr}}) = \mathcal{O}_{\tilde{\mathcal{N}}}$ is immediate from the definition.

To prove the second statement, we fix $\mu \in \mathbb{Y}$ and an object of $D_{\text{proj}}(\text{Gr})$ represented by a single pro-object $M \in \mathcal{P}erv(\text{Gr})$, such that vM is projective in $\mathcal{P}erv(\text{Gr})$. To compute $P'(M)$, we have to consider the $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -module $\mathbf{E}(\mathbf{P}, M) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} \mathbf{E}(\mathbf{P}, M)_{\lambda}$, where $\mathbf{E}(\mathbf{P}, M)_{\lambda} = \text{Ext}_{D^b(\text{Gr})}^0(\mathbf{P}, \mathcal{W}_{\lambda} \star M \star \mathcal{R})$.

Now, let $V \in \text{Rep}(G)$. To compute $P'(\mathcal{W}_{\mu} \star M \star \mathcal{P}V)$, we must replace the object $\mathcal{W}_{\mu} \star M \star \mathcal{P}V$ by a projective resolution $C \xrightarrow{\text{qis}} \mathcal{W}_{\mu} \star M \star \mathcal{P}V$, viewed as a dg-object $C = \bigoplus_i C^i$, and set

$$\mathbf{E}(\mathbf{P}, C) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} \mathbf{E}(\mathbf{P}, C)_{\lambda}, \text{ where } \mathbf{E}(\mathbf{P}, C)_{\lambda} = \bigoplus_{(i,j) \in \mathbb{Z}^2} \text{Ext}_{D^b(\text{Gr})}^0(\mathbf{P}^i, \mathcal{W}_{\lambda} \star C^j \star \mathcal{R}) \quad (9.8.2)$$

Thus, $\mathbf{E}(\mathbf{P}, C)$ is a dg-module over the dg-algebra $\mathbf{E}(\mathbf{P}, \mathbf{P})$. The differential on $\mathbf{E}(\mathbf{P}, C)$ is equal to $d = d_p + d_C$, a sum of two (anti-)commuting differentials, the first being induced from the differential on \mathbf{P} , and the second from the differential on C . Each of the two differentials clearly preserves the weight decomposition on the left of (9.8.2).

In order to compare the objects $\mathbf{E}(\mathbf{P}, M)$ and $\mathbf{E}(\mathbf{P}, C)$, we now prove the following

Claim 9.8.3. For any $\lambda \in \mathbb{Y}^{++}$ such that $\lambda + \mu \in \mathbb{Y}^{++}$, the dg-vector space $(\mathbf{E}(\mathbf{P}, C)_{\lambda}, d)$ is canonically quasi-isomorphic to the dg-vector space $(V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_{\mu} \star M)_{\lambda}, d_p)$.

Proof of Claim. We observe first that the functor $\text{RHom}_{D^b(\text{Gr})}(\mathbf{P}, \mathcal{W}_{\lambda} \star (-) \star \mathcal{R})$ applied to the complex C , gives rise to a standard spectral sequence

$$E_2 = \text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathbf{P}, \mathcal{W}_{\lambda} \star H^{\bullet}(C) \star \mathcal{R}) \implies H^{\bullet}(\text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathbf{P}, \mathcal{W}_{\lambda} \star C \star \mathcal{R}), d_C). \quad (9.8.4)$$

Since $C \xrightarrow{\text{qis}} \mathcal{W}_{\mu} \star M \star \mathcal{P}V$, is a resolution, we get $H^{\bullet}(C) = \mathcal{W}_{\mu} \star M \star \mathcal{P}V$. Therefore, the E_2 -term on the left of (9.8.4) equals $\text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathbf{P}, \mathcal{W}_{\lambda} \star \mathcal{W}_{\mu} \star M \star \mathcal{P}V \star \mathcal{R})$. This last Ext-group is canonically isomorphic to $V \otimes \text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathbf{P}, \mathcal{W}_{\lambda} \star \mathcal{W}_{\mu} \star M \star \mathcal{R})$, due to (7.7.2).

A key point is, that our assumption: $\lambda + \mu \in \mathbb{Y}^{++}$ (combined with the fact that M , being projective, has a Δ -flag) implies, by Proposition 9.3.3, that $\mathcal{W}_{\lambda+\mu} \star M$ is a *perverse sheaf*. Hence, $\mathcal{W}_{\lambda} \star \mathcal{W}_{\mu} \star M \star \mathcal{R} = (\mathcal{W}_{\lambda+\mu} \star M) \star \mathcal{R}$ is also a perverse sheaf, by Gaitsgory's theorem. Further, since \mathbf{P} is a projective, we obtain

$$\text{Ext}_{D^b(\text{Gr})}^n(\mathbf{P}, \mathcal{W}_{\lambda} \star \mathcal{W}_{\mu} \star M \star \mathcal{R}) = \text{Ext}_{\mathcal{P}erv(\text{Gr})}^n(\mathbf{P}, \mathcal{W}_{\lambda+\mu} \star M \star \mathcal{R}) = 0 \text{ for all } n \neq 0.$$

Thus, for the E_2 -term in (9.8.4), we obtain

$$E_2 = V \otimes \text{Ext}_{D^b(\text{Gr})}^0(\mathbf{P}, \mathcal{W}_{\lambda} \star \mathcal{W}_{\mu} \star M \star \mathcal{R}) = V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_{\mu} \star M)_{\lambda}.$$

This implies that the spectral sequence in (9.8.4) degenerates, i.e., reduces to the following long exact sequence

$$\dots \xrightarrow{d_C} \mathbf{E}(\mathbf{P}, C^2)_{\lambda} \xrightarrow{d_C} \mathbf{E}(\mathbf{P}, C^1)_{\lambda} \xrightarrow{d_C} \mathbf{E}(\mathbf{P}, C^0)_{\lambda} \xrightarrow{d_C} V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_{\mu} \star M)_{\lambda} \longrightarrow 0.$$

The long exact sequence yields a canonical quasi-isomorphism $(\mathbf{E}(\mathbf{P}, C)_{\lambda}, d_C) \xrightarrow{\text{qis}} (V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_{\mu} \star M)_{\lambda}, d_p)$. Claim 9.8.3 now follows from another standard spectral sequence, the one for a bicomplex, in which $\mathbf{E}(\mathbf{P}, C)_{\lambda}$ is viewed as a bicomplex with two differentials, d_p and d_C . \square

Next, recall that we are given a weight $\mu \in \mathbb{Y}^{++}$, and put

$$\begin{aligned} \mathbf{E}(\mathbf{P}, C)^\circ &:= \bigoplus_{\{\lambda \in \mathbb{Y}^{++} \mid \lambda + \mu \in \mathbb{Y}^{++}\}} \mathbf{E}(\mathbf{P}, C)_\lambda, \quad \text{and} \\ \mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)^\circ &:= \bigoplus_{\{\lambda \in \mathbb{Y}^{++} \mid \lambda + \mu \in \mathbb{Y}^{++}\}} \mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)_\lambda. \end{aligned}$$

It is clear that $\mathbf{E}(\mathbf{P}, C)^\circ$ is a $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -submodule in $\mathbf{E}(\mathbf{P}, C)$, and $\mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)^\circ$ is a $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -submodule in $\mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)$. By Claim 9.8.3, we have a quasi-isomorphism $\mathbf{E}(\mathbf{P}, C)^\circ \xrightarrow{\text{qis}} V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)^\circ$. Further, the quotients $\mathbf{E}(\mathbf{P}, C)/\mathbf{E}(\mathbf{P}, C)^\circ$ and $\mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)/\mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)^\circ$, are both *thin* $\mathbf{E}(\mathbf{P}, \mathbf{P})$ -modules, by Definition 9.4.2. Thus, we have established the following quasi-isomorphism

$$\mathbf{E}(\mathbf{P}, C) \simeq V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M) \quad \text{in } D^b \left(\text{Mod}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) / \text{Mod}_{\text{thin}}^{G \times \mathbb{G}_m}(\Gamma[\tilde{\mathcal{N}}]) \right).$$

To complete the proof of the Proposition, assuming that $\lambda + \mu \in \mathbb{Y}^{++}$, we compute $\mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)_\lambda = \mathbf{E}(\mathbf{P}, \mathcal{W}_\lambda \star \mathcal{W}_\mu \star M) = \mathbf{E}(\mathbf{P}, M)_{\lambda + \mu}$. Thus, the graded space $\mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} \mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)_\lambda$ is isomorphic, up to a thin subspace, to the space $\mathbf{E}(\mathbf{P}, M) = \bigoplus_{\lambda \in \mathbb{Y}^{++}} \mathbf{E}(\mathbf{P}, M)_\lambda$, with the \mathbb{Y} -grading being shifted by μ . But shifting by μ is the same as tensoring by $\mathbb{k}(\mu)$. Therefore, in $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$, we have

$$\begin{aligned} \mathcal{F}(\mathbf{E}(\mathbf{P}, C)) &\simeq \mathcal{F}(V \otimes \mathbf{E}(\mathbf{P}, \mathcal{W}_\mu \star M)) \simeq V \otimes \mathcal{F}(\mathbb{k}(\mu) \otimes \mathbf{E}(\mathbf{P}, M)) \\ &= V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\mu) \otimes \mathcal{F}(\mathbf{E}(\mathbf{P}, M)), \end{aligned}$$

and the Proposition is proved. \square

Proposition 9.8.5. *For any $\lambda \in \mathbb{Y}^{++}$ and $\mathcal{L} \in \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})$, the functor P' induces an isomorphism*

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{L}) \xrightarrow{\sim} \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^\bullet(P'(\mathbf{1}_{\text{Gr}}), P'(\mathcal{W}_\lambda \star \mathcal{L})).$$

Proof. We may write $\mathcal{L} = \mathcal{P}V$, for some $V \in \text{Rep}(G)$. Then, using formulas (8.7.2) and (8.7.4), we obtain

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{W}_\lambda \star \mathcal{P}V) = H^{\bullet + \text{ht}(\lambda)}(i_{-\lambda}^! \mathcal{P}V) = \text{gr}_{k + \text{ht}(\lambda)}^W V(-\lambda). \quad (9.8.6)$$

On the other hand, Proposition 9.8.1 yields $P'(\mathcal{W}_\lambda \star \mathcal{P}V) = V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda) \otimes P'(\mathbf{1}_{\text{Gr}}) = V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)$, and also $P'(\mathbf{1}_{\text{Gr}}) = \mathcal{O}_{\tilde{\mathcal{N}}}$. Since $\lambda \in \mathbb{Y}^{++}$, we have

$$\text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^k(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = H^k(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = 0 \quad \forall k > 0.$$

Hence, for $\mathcal{L} = \mathcal{P}V$, we find

$$\begin{aligned} \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^\bullet(P'(\mathbf{1}_{\text{Gr}}), P'(\mathcal{W}_\lambda \star \mathcal{P}V)) &= \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^\bullet(\mathcal{O}_{\tilde{\mathcal{N}}}, V \otimes \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \\ &= \text{Hom}_G \left(\mathbb{k}, V \otimes \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^\bullet(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \right) \\ &= \left(V \otimes \Gamma^\bullet(\tilde{\mathcal{N}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) \right)^G \quad (9.8.7) \\ \text{(by second equality in (8.6.3))} &= \left(V \otimes (\text{gr}_{\bullet + \text{ht}(\lambda)}^{\text{fib}}(\text{Ind}_H^G e^\lambda)) \right)^G \\ \text{(by first equality in (8.6.3))} &= \text{gr}_{\bullet + \text{ht}(\lambda)}^W \left((V \otimes (\text{Ind}_H^G e^\lambda))^G \right). \end{aligned}$$

Further, using Frobenius reciprocity, we obtain $(V \otimes (\text{Ind}_H^G e^\lambda))^G = V(-\lambda)$. Hence, the last line in (9.8.7) equals $\text{gr}_{+\text{ht}(\lambda)}^W V(-\lambda)$, which is exactly the RHS of (9.8.6).

We leave to the reader to check that the isomorphism between the LHS and RHS of (9.8.6) that we have constructed above is the same as the one given by the map in (9.8.6). The Proposition is proved. \square

Lemma 9.8.8. *Let $\lambda \in \mathbb{Y}$ be such that $\lambda \not\geq 0$, i.e., λ does not belong to the semi-group in \mathbb{Y} generated by positive roots. Then, we have*

$$(i) \text{ RHom}_{D^b(\text{Gr})}(\mathbf{1}_{\text{Gr}}, \overline{\mathcal{W}}_\lambda) = 0; \quad \text{and} \quad (ii) \text{ RHom}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = 0.$$

Proof. To prove (i) recall that, for any $\mu, \nu \in \mathbb{Y}$, we have $\mathcal{W}_\mu \star \overline{\mathcal{W}}_\nu = \overline{\mathcal{W}}_{\mu+\nu}$. It follows, since $(\mathcal{W}_{-\mu} \star \mathcal{W}_\mu) \star (-)$ is the identity functor that, for any $\mu \in \mathbb{Y}$, we have $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \overline{\mathcal{W}}_\lambda) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\overline{\mathcal{W}}_\mu, \overline{\mathcal{W}}_{\mu+\lambda})$. We choose μ to be anti-dominant and such that $\mu + \lambda$ is also anti-dominant. Then we know that $\overline{\mathcal{W}}_\mu = \Delta_\mu$ and $\overline{\mathcal{W}}_{\mu+\lambda} = \Delta_{\mu+\lambda}$. Hence, writing $j_\mu : \text{Gr}_\mu \hookrightarrow \text{Gr}$ for the imbedding, we get

$$\begin{aligned} \text{RHom}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \overline{\mathcal{W}}_\lambda) &= \text{RHom}_{D^b(\text{Gr})}^\bullet(\overline{\mathcal{W}}_\mu, \overline{\mathcal{W}}_{\mu+\lambda}) \\ &= \text{RHom}_{D^b(\text{Gr})}^\bullet(\Delta_\mu, \Delta_{\mu+\lambda}) = j_\mu^! \Delta_{\mu+\lambda}, \quad \forall \mu \ll 0. \end{aligned}$$

Now, the condition $\lambda \not\geq 0$ implies that $\text{Gr}_\mu \not\subset \overline{\text{Gr}}_{\mu+\lambda}$, for all sufficiently anti-dominant μ . This forces $j_\mu^! \Delta_{\mu+\lambda} = 0$, and part (i) is proved.

To prove (ii), we first use the chain of equivalences in the first line of (3.9.5) to reduce the Ext-vanishing in the category $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$ by a similar Ext-vanishing in the category $D_f^B(\Lambda)$. Further, recall the algebra $\mathbf{A} = \mathcal{U}\mathfrak{b} \ltimes \Lambda$ and the triangulated category $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathbf{A}, \Lambda)$, see Notation 3.2.1 and definitions following it. By Proposition 3.9.2, the Ext-groups are unaffected if category $D_f^B(\Lambda)$ is replaced by $D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathbf{A}, \Lambda)$. This way, we obtain natural isomorphisms

$$\text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^\bullet(\mathcal{O}_{\tilde{\mathcal{N}}}, \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = \text{Ext}_{D_f^B(\Lambda)}^\bullet(i^* \mathcal{O}_{\tilde{\mathcal{N}}}, i^* \mathcal{O}_{\tilde{\mathcal{N}}}(\lambda)) = \text{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathbf{A}, \Lambda)}^\bullet(\mathbb{k}_{\mathbf{A}}, \mathbb{k}_{\mathbf{A}}(\lambda)). \quad (9.8.9)$$

Thus, we are reduced to showing that, for all $i \in \mathbb{Z}$, one has: $\text{Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathbf{A}, \Lambda)}^i(\mathbb{k}_{\mathbf{A}}, \mathbb{k}_{\mathbf{A}}(\lambda)) = 0$.

To this end, we use the standard spectral sequence (2.11.1) for the cohomology of a semidirect product:

$$\text{Ext}^p(\mathbb{k}_{\mathcal{U}\mathfrak{b}}, \text{Ext}_\Lambda^q(\mathbb{k}_{\mathbf{A}}, \mathbb{k}_{\mathbf{A}}(\lambda))) = E_2^{p,q} \implies E_\infty^{p,q} = \text{gr Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathbf{A}, \Lambda)}^{p+q}(\mathbb{k}_{\mathbf{A}}, \mathbb{k}_{\mathbf{A}}(\lambda)),$$

where the group $\text{Ext}^p(\mathbb{k}_{\mathcal{U}\mathfrak{b}}, -)$ on the left is computed in the category of \mathbb{Y} -graded $\mathcal{U}\mathfrak{b}$ -modules such that the action of the Cartan subalgebra $\mathfrak{t} \subset \mathfrak{b}$ is compatible with the \mathbb{Y} -grading. By (an appropriate version of) formula (3.3.2) we have $\text{Ext}_\Lambda^\bullet(\mathbb{k}_{\mathbf{A}}, \mathbb{k}_{\mathbf{A}}(\lambda)) = \underline{\mathbf{S}}(\lambda)$, furthermore, the above spectral sequence collapses. Thus we get

$$\text{gr Ext}_{D_{\mathbb{Y}}^{\mathcal{U}\mathfrak{b}}(\mathbf{A}, \Lambda)}^p(\mathbb{k}_{\mathbf{A}}, \mathbb{k}_{\mathbf{A}}(\lambda)) = [H^p(\mathfrak{n}, \mathbf{S}(\lambda))]^{(0)},$$

where we write $[\dots]^{(0)}$ for the zero-weight component with respect to the \mathbb{Y} -grading.

Recall now that the Lie algebra cohomology on the RHS above are given by the cohomology of the Koszul complex $\wedge^\bullet \mathfrak{n}^* \otimes \mathbf{S}(\lambda)$. Observe that any weight in $\wedge^\bullet \mathfrak{n}^* \otimes \mathbf{S} = \wedge^\bullet \mathfrak{n}^* \otimes \text{Sym}(\mathfrak{n}^*[-2])$ is clearly a sum of *negative* roots. Therefore, the zero-weight component of the cohomology in

the RHS above vanishes, since λ is not a sum of positive roots, by our assumptions. Thus, $\text{gr Ext}_{D_{\mathbb{A}}^{\text{b}}(\mathbb{A}, \mathbb{A})}^p(\mathbb{k}_{\mathbb{A}}, \mathbb{k}_{\mathbb{A}}(\lambda)) = 0$, for all $p \in \mathbb{Z}$, and (ii) is proved. \square

The proof of the next Lemma is easy and will be left to the reader.

Lemma 9.8.10. *The smallest triangulated subcategory in $D_{\text{proj}}(\text{Gr})$ that contains the following set of objects*

$$X := \{\overline{\mathcal{W}}_{\lambda}, \forall \lambda \not\geq 0; \quad \mathcal{W}_{\nu} \star \mathcal{L}, \forall \nu \in \mathbb{Y}^{++}, \mathcal{L} \in \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})\}$$

coincides with the category $D_{\text{proj}}(\text{Gr})$ itself. \square

9.9 Proof of Theorems 9.1.4 and 9.4.3. We claim first that, for any $M \in D^b \text{Perv}(\text{Gr})$, the functor P' induces an isomorphism

$$\text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathbf{1}_{\text{Gr}}, M) \xrightarrow{\sim} \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^{\bullet}(P'(\mathbf{1}_{\text{Gr}}), P'(M)). \quad (9.9.1)$$

Due to Lemma 9.8.10, it suffices to prove (9.9.1) for all objects $M \in X$. For the objects $M = \overline{\mathcal{W}}_{\lambda}$, $\lambda \not\geq 0$, equation (9.9.1) is insured by Lemma 9.8.8. For the objects $M = \mathcal{W}_{\lambda} \star \mathcal{L}$, where $\lambda \in \mathbb{Y}^{++}$ and $\mathcal{L} \in \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr})$, equation (9.9.1) follows from Proposition 9.8.5. Thus, (9.9.1) is proved.

Now, let $\mu \in \mathbb{Y}$ and $M \in D^b \text{Perv}(\text{Gr})$. From Proposition 9.8.1, using that $P(\mathbf{1}_{\text{Gr}}) = \mathcal{O}_{\tilde{\mathcal{N}}}$, we obtain a natural commutative diagram

$$\begin{array}{ccc} \text{Ext}_{D^b(\text{Gr})}^{\bullet}(\overline{\mathcal{W}}_{\mu}, M) & \xrightarrow{P'} & \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^{\bullet}(P'(\overline{\mathcal{W}}_{\mu}), P'(M)) \\ \Downarrow \mathcal{W}_{-\mu} \star (-) & & \Downarrow \mathcal{O}_{\tilde{\mathcal{N}}}(-\mu) \otimes (-) \\ \text{Ext}_{D^b(\text{Gr})}^{\bullet}(\mathbf{1}_{\text{Gr}}, \mathcal{W}_{-\mu} \star M) & \xrightarrow{P'} & \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^{\bullet}(P'(\mathbf{1}_{\text{Gr}}), P'(\mathcal{W}_{-\mu} \star M)) \end{array}$$

The vertical maps in the diagram are isomorphisms since the functors $\mathcal{W}_{-\mu} \star (-)$, resp. $\mathcal{O}_{\tilde{\mathcal{N}}}(-\mu) \otimes (-)$, are equivalences of derived categories. The map in the bottom line of this diagram is already known to be an isomorphism, by (9.9.1). Hence, the map in the top line is also an isomorphism. But, the set of objects $\{\overline{\mathcal{W}}_{\mu}\}_{\mu \in \mathbb{Y}}$ clearly generates the category $D^b \text{Perv}(\text{Gr})$. Therefore, we deduce that, for any $N, M \in D^b \text{Perv}(\text{Gr})$, the functor P' induces an isomorphism

$$\text{Ext}_{D^b(\text{Gr})}^{\bullet}(N, M) \xrightarrow{\sim} \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^{\bullet}(P'(N), P'(M)).$$

Thus, we have proved that the functor P' is fully faithful.

To prove that P is fully faithful, we recall that the category $\text{Perv}^{\text{mix}}(\text{Gr})$, resp., $D^b \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})$, is a mixed version of $\text{Perv}(\text{Gr})$, resp., of $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$. It is clear from the construction of the functors P and P' that one has an isomorphism of functors $P' \circ v = v \circ P$. Thus, using (9.2.3), we obtain, for any $M, N \in D^b \text{Perv}^{\text{mix}}(\text{Gr})$, a natural commutative square

$$\begin{array}{ccc} \bigoplus_{n \in \mathbb{Z}} \text{Ext}_{D^b \text{Perv}^{\text{mix}}(\text{Gr})}^{\bullet}(M, N\langle n \rangle) & \xrightarrow[\sim]{v} & \text{Ext}_{D^b \text{Perv}(\text{Gr})}^{\bullet}(M, N) \\ \downarrow P & & \downarrow P' \\ \bigoplus_{n \in \mathbb{Z}} \text{Ext}_{D^b \text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})}^{\bullet}(P(M), P(N)\langle n \rangle) & \xrightarrow[\sim]{v} & \text{Ext}_{D_{\text{coherent}}^G(\tilde{\mathcal{N}})}^{\bullet}(P'(M), P'(N)) \end{array}$$

We have already proved that the vertical map on the right is an isomorphism. It follows from the diagram, that the vertical map on the left is also an isomorphism.

The set of objects $\{\mathcal{O}_{\tilde{\mathcal{N}}}(\mu) = P'(\overline{W}_\mu)\}_{\mu \in \mathbb{Y}}$ clearly generates the category $D_{\text{coherent}}^G(\tilde{\mathcal{N}})$. Thus, from Lemma 3.9.3 we deduce, using Proposition 9.8.1, that the functor P' is an equivalence. That completes the proof of Theorem 9.1.4.

Finally, any simple object of $\text{Coh}^G(\tilde{\mathcal{N}})$ has the form $v(\mathcal{F})$, where \mathcal{F} is a simple object of $\text{Coh}^{G \times \mathbb{G}_m}(\tilde{\mathcal{N}})$. We deduce, using that P' is an equivalence and applying Lemma 3.9.3 once again, that P is also an equivalence. Theorem 9.4.3 is proved. \square

9.10 Equivalence of abelian categories. We now combine Theorem 9.4.3 and Theorem 3.9.6 together, and compose the inverse of the equivalence $Q' : D_{\text{coherent}}^G(\tilde{\mathcal{N}}) \xrightarrow{\sim} D^b\text{block}(\mathbf{U})$ with the inverse of the equivalence $P' : D^b\text{Perv}(\text{Gr}) \xrightarrow{\sim} D_{\text{coherent}}^G(\tilde{\mathcal{N}})$. This way, we obtain the following composite equivalence of triangulated categories

$$\Upsilon : D^b\text{block}(\mathbf{U}) \xrightarrow{(Q')^{-1}} D_{\text{coherent}}^G(\tilde{\mathcal{N}}) \xrightarrow{(P')^{-1}} D^b\text{Perv}(\text{Gr}). \quad (9.10.1)$$

The triangulated categories $D^b\text{block}(\mathbf{U})$ and $D^b\text{Perv}(\text{Gr})$, each has a natural t -structure, with cores $\text{block}(\mathbf{U})$ and $\text{Perv}(\text{Gr}) = \text{Perv}_{\mathbf{I}\text{-mon}}(\text{Gr})$, respectively. Below, we will prove the following result.

Theorem 9.10.2. *The equivalence $\Upsilon : D^b\text{block}(\mathbf{U}) \xrightarrow{\sim} D^b\text{Perv}(\text{Gr})$ respects the t -structures, hence induces an equivalence of abelian categories $\mathcal{P} : \text{Perv}(\text{Gr}) \xrightarrow{\sim} \text{block}(\mathbf{U})$, such that $\mathcal{P}L_\lambda = IC_\lambda$, for any $\lambda \in \mathbb{Y}$.*

Remark 9.10.3. Recall that functor $\phi^* : \text{Rep}(G) \rightarrow \text{block}(\mathbf{U})$, $M \mapsto \phi M$, see (2.6.1), identifies $\text{Rep}(G)$ with the full subcategory $\phi^*(\text{Rep}(G)) \subset \text{block}(\mathbf{U})$. Further, it follows from the properties of the functors P' and Q' proved in the previous sections that, for any $V \in \text{Rep}(G)$, the functor Υ in (9.10.1) sends the \mathbf{U} -module ϕV to $\mathcal{P}V$, the perverse sheaf corresponding to V via the Satake equivalence. Therefore, the functor Υ maps the subcategory $\phi^*(\text{Rep}(G))$ into the subcategory $\text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr}) \subset \text{Perv}_{\mathbf{I}\text{-mon}}(\text{Gr})$. Thus, the restriction of Υ to the abelian category $\text{block}(\mathbf{U})$ may be regarded, in view of Theorem 9.10.2, as a natural ‘extension’ of the functor $\mathcal{P} : \text{Rep}(G) \rightarrow \text{Perv}(\text{Gr})$ to the larger category $\text{block}(\mathbf{U})$, i.e., one has the following commutative diagram:

$$\begin{array}{ccc} \text{Rep}(G) & \xhookrightarrow{\text{Frobenius } \phi^*} & \text{block}(\mathbf{U}) \\ \text{Satake} \downarrow \mathcal{P} & & \downarrow \text{equivalence} \\ \text{Perv}_{G^{\vee}(\mathcal{O})}(\text{Gr}) & \xhookrightarrow{\text{inclusion}} & \text{Perv}_{\mathbf{I}\text{-mon}}(\text{Gr}) \end{array} \quad \begin{array}{l} \\ \Upsilon = P^{-1} \circ Q, \text{ see (1.1.2)} \end{array} \quad (9.10.4)$$

For this reason, we use the notation \mathcal{P} for the functor $\Upsilon|_{\text{block}(\mathbf{U})}$. \diamond .

To prove Theorem 9.10.2, we will use filtrations on triangulated categories $D^b\text{block}(\mathbf{U})$ and $D^b\text{Perv}(\text{Gr})$, defined as follows. For any $\lambda \in \mathbb{Y}^{++}$, let $D_{\leq \lambda}^b\text{block}(\mathbf{U})$ be the smallest full triangulated subcategory of $D^b\text{block}(\mathbf{U})$ that contains all the simple objects $L_\mu \in \text{block}(\mathbf{U})$ with $\mu \leq \lambda$. Clearly, we have $D_{\leq \lambda}^b\text{block}(\mathbf{U}) \subset D_{\leq \nu}^b\text{block}(\mathbf{U})$, whenever $\lambda \leq \nu$. Moreover, the category $D_{\leq \lambda}^b\text{block}(\mathbf{U})/D_{< \lambda}^b\text{block}(\mathbf{U})$ is semisimple and is formed by direct sums of objects of the type $L_\lambda[k]$, $k \in \mathbb{Z}$.

Similarly, let $D_{\leq \lambda}^b\text{Perv}$ be the full triangulated subcategory of $D^b\text{Perv}(\text{Gr})$ formed by the objects supported on $\overline{\text{Gr}}_\lambda$.

Lemma 9.10.5. *For any $\lambda \in \mathbb{Y}^{++}$, we have*

- (i) *The functor Υ induces an equivalence $D_{\leq \lambda}^b \mathcal{Perv} \xrightarrow{\sim} D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$. Moreover,*
- (ii) *The induced functor:*

$$D_{\leq \lambda}^b \mathcal{Perv} / D_{< \lambda}^b \mathcal{Perv} \longrightarrow D_{\leq \lambda}^b \mathbf{block}(\mathbf{U}) / D_{< \lambda}^b \mathbf{block}(\mathbf{U})$$

sends the class of IC_λ to the class of L_λ .

Proof. We know that, for any $\lambda \in \mathbb{Y}^{++}$, the functor Υ sends, by construction, the object $\mathrm{RInd}_{\mathbf{B}}^{\mathbf{U}}(l\lambda)$ into $\overline{\mathbf{W}}_\lambda$. Further, fix $\lambda \in \mathbb{Y}$ and let $w \in W$ be the element of minimal length such that the weight $w(l\lambda)$ is dominant. Then by Lemma 3.5.1 we have $R^{\ell(w)} \mathrm{Ind}_{\mathbf{B}}^{\mathbf{U}}(l\lambda) = \mathrm{Weyl}_{w(l\lambda)}$, is a Weyl module, and for any $j \neq \ell(w)$, each simple subquotient of $R^j \mathrm{Ind}_{\mathbf{B}}^{\mathbf{U}}(l\lambda)$ is isomorphic to L_μ with $\mu < \lambda$. It follows that the category $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ is generated by the objects $\{\mathrm{RInd}_{\mathbf{B}}^{\mathbf{U}}(l\nu)\}_{\nu \leq \lambda}$. Hence, by Corollary 8.3.2(ii), we get: $\mathrm{supp} \Upsilon(\mathrm{RInd}_{\mathbf{B}}^{\mathbf{U}}(l\nu)) = \mathrm{supp} \mathcal{W}_\nu \subset \overline{\mathbf{Gr}}_\lambda$. Thus, the functor Υ takes $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$ into $D_{\leq \lambda}^b \mathcal{Perv}$. Similarly, the functor Υ^{-1} takes $D_{\leq \lambda}^b \mathcal{Perv}$ into $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U})$, and part (i) of the Lemma follows.

To prove (ii), we use Lemma 3.5.1, that has been already exploited above. The Lemma says in particular that in $D_{\leq \lambda}^b \mathbf{block}(\mathbf{U}) / D_{< \lambda}^b \mathbf{block}(\mathbf{U})$ one has an isomorphism $\mathrm{RInd}_{\mathbf{B}}^{\mathbf{U}}(l\lambda) \simeq L_{w(l\lambda)}[-\ell(w)]$. Similarly, in $D_{\leq \lambda}^b \mathcal{Perv} / D_{< \lambda}^b \mathcal{Perv}$ one has an isomorphism $\mathcal{W}_\lambda \simeq IC_\lambda[-\ell(w)]$, by Corollary 8.3.2(ii). Statement (ii) now follows from (i) and the equation $\Upsilon(\mathrm{RInd}_{\mathbf{B}}^{\mathbf{U}}(l\lambda)) = \mathcal{W}_\lambda$. \square

We will need the following result on the gluing of t -structures, proved in [BBD].

Lemma 9.10.6. *Let $(D, D_{\leq \lambda})$ be a triangulated category equipped with filtration such that each quotient category $D_{\leq \lambda} / D_{< \lambda}$ is a semisimple category generated by one object. Then there exists a unique t -structure on D compatible with given t -structures on each category $D_{\leq \lambda} / D_{< \lambda}$. \square*

Proof of Theorem 9.10.2. Lemma 9.10.5(i) implies that the functor Υ is an equivalence of triangulated categories which is, moreover, compatible with filtrations. Thus, the part of Theorem 9.10.2 concerning the t -structures follows from Lemma 9.10.6.

Alternatively, the same thing can be proved as follows. The t -structure on $D = D^b \mathcal{Perv}(\mathbf{Gr})$ is characterized by the property that, for any $\lambda \in \mathbb{Y}$, the objects $\Delta_\lambda := (j_\lambda)_! \mathbb{C}_{\mathbf{Gr}_\lambda}[-\dim \mathbf{Gr}_\lambda]$ and $\nabla_\lambda := (j_\lambda)_* \mathbb{C}_{\mathbf{Gr}_\lambda}[-\dim \mathbf{Gr}_\lambda]$, where $j_\lambda : \mathbf{Gr}_\lambda \hookrightarrow \mathbf{Gr}$ is the imbedding of the Bruhat cell \mathbf{Gr}_λ , belong to the core of the t -structure. But the object Δ_λ is completely determined by the following conditions formulated entirely in terms of the filtration $D_{\leq \lambda}^b \mathcal{Perv}$, and the t -structures on $D_{\leq \lambda}^b \mathcal{Perv} / D_{< \lambda}^b \mathcal{Perv}$:

- $\Delta_\lambda \in D_{\leq \lambda}^b \mathcal{Perv}$;
- $\mathrm{Hom}_{D^b \mathcal{Perv}(\mathbf{Gr})}(\Delta_\lambda, D_{< \lambda}^b \mathcal{Perv}) = 0$;
- $\Delta_\lambda = IC_\lambda \bmod D_{< \lambda}^b \mathcal{Perv}$.

Similarly, there is a dual characterisation of the ∇_λ 's. The above characterizations yield the compatibility of the t -structures. They imply also that $\mathcal{P}L_\lambda = IC_\lambda$, for any $\lambda \in \mathbb{Y}$. This completes the proof of Theorem 9.10.2. \square

Recall that convolution of perverse sheaves gives an exact bi-functor $\star : \mathcal{Perv}_{\mathbf{I}^{\mathrm{mon}}}(\mathbf{Gr}) \times \mathcal{Perv}_{\mathbf{G}^{\vee}(\mathcal{O})}(\mathbf{Gr}) \rightarrow \mathcal{Perv}_{\mathbf{I}^{\mathrm{mon}}}(\mathbf{Gr})$. On the other hand, note that if $V \in \mathrm{Rep}(G)$ and $M \in \mathbf{block}(\mathbf{U})$, then $M \otimes_{\mathbb{C}} \phi V \in \mathbf{block}(\mathbf{U})$. Thus the tensor product of \mathbf{U} -modules gives an exact bi-functor $\otimes : \mathbf{block}(\mathbf{U}) \times \mathrm{Rep}(G) \rightarrow \mathbf{block}(\mathbf{U})$.

From Theorem 9.10.2 one derives the following result.

Proposition 9.10.7. *The above bi-functors $\otimes : \text{block}(\mathbf{U}) \times \text{Rep}(G) \rightarrow \text{block}(\mathbf{U})$, and $\star : \text{Perv}(\text{Gr}) \times \text{Perv}_{G^\vee(\mathcal{O})}(\text{Gr}) \rightarrow \text{Perv}(\text{Gr})$ correspond to each other under the equivalences of Theorems 9.10.2 and 6.3.4, that is, for any $V \in \text{Rep}(G)$ and $M \in \text{block}(\mathbf{U})$, we have:*

- (i) *Functorial isomorphism $\mathcal{P}(M \otimes {}^\phi V) \simeq \mathcal{P}M \star \mathcal{P}({}^\phi V)$.*
- (ii) *Functorial vector space isomorphism $H^\bullet(\text{Gr}, \mathcal{P}(M \otimes {}^\phi V)) \simeq V \otimes H^\bullet(\text{Gr}, \mathcal{P}M)$.*

9.11 Duality and convolution. Recall that $\text{Gr} = G^\vee(\mathcal{K})/G^\vee(\mathcal{O})$. By an Iwasawa decomposition, we may identify Gr with a based loop group Ω , cf. [PS]. The inversion map $\sigma : g \mapsto g^{-1}$ on that based loop group induces an auto-equivalence: $\mathcal{M} \mapsto \sigma^* \mathcal{M}$, on $D^b(\text{Gr})$. Let also: $\mathcal{M} \mapsto \mathbf{D}\mathcal{M}$ denote the Verdier duality functor on $D^b(\text{Gr})$. The above two functors obviously commute, and we will write: $\mathcal{M} \mapsto \mathcal{M}^\vee := \sigma^*(\mathbf{D}\mathcal{M}) = \mathbf{D}(\sigma^* \mathcal{M})$ for the composition, an involutive *contravariant* auto-equivalence on $D^b(\text{Gr})$.

The functor: $\mathcal{M} \mapsto \mathcal{M}^\vee$ corresponds, via the equivalence of Theorem 6.3.4, to the duality: $V \mapsto V^\vee$ on the category $\text{Rep}(G)$, i.e., there is a functor isomorphism $\mathcal{P}({}^\phi V^\vee) = (\mathcal{P}({}^\phi V))^\vee$. In particular, for any $\lambda \in \mathbb{Y}^{++}$, we have $(IC_\lambda)^\vee = IC_{-w_\circ(\lambda)}$, where w_\circ is the longest element in the Weyl group W .

One has the standard isomorphism

$$\text{Hom}(\mathcal{L}_1 \star \mathcal{M}, \mathcal{L}_2) \simeq \text{Hom}(\mathcal{L}_1, \mathcal{L}_2 \star (\mathcal{M}^\vee)), \quad \forall \mathcal{L}_1, \mathcal{L}_2, \mathcal{M} \in D^b(\text{Gr}). \quad (9.11.1)$$

10 Quantum group cohomology and the loop Grassmannian

The goal of this section is to prove the conjecture given in [GK, §4.3]. We put $\mathbb{k} = \mathbb{C}$.

10.1 Equivariant coherent sheaves on \mathcal{N} . The group $G \times \mathbb{G}_m$ acts naturally on the nil-cone $\mathcal{N} \subset \mathfrak{g}$ (the group G acts by conjugation and the group \mathbb{G}_m by dilations). Therefore the coordinate ring $\mathbb{k}^\bullet[\mathcal{N}]$ comes equipped with a natural grading, and with a \mathfrak{g} -module structure.

We consider the abelian category of G -equivariant \mathbb{Z} -graded finitely generated $\mathbb{k}[\mathcal{N}]$ -modules. It may be identified, since the nil-cone \mathcal{N} is an affine variety, with $\text{Coh}^{G \times \mathbb{G}_m}(\mathcal{N})$, the category of $G \times \mathbb{G}_m$ -equivariant coherent sheaves on \mathcal{N} .

Let $e \in \mathcal{N}$ be a fixed principal nilpotent element, and $\mathfrak{z} \subset \mathfrak{g} \oplus \mathbb{k}$ be the Lie algebra of the isotropy group of the point $e \in \mathcal{N}$ in $G \times \mathbb{G}_m$. Given a $G \times \mathbb{G}_m$ -equivariant coherent sheaf \mathcal{F} on \mathcal{N} , let $\mathcal{F}|_e$ denote its geometric fiber at e . Thus, $\mathcal{F}|_e$ is a finite dimensional vector space with a natural \mathfrak{z} -action. Clearly, $\mathfrak{g}^e = \mathfrak{g}^e \oplus \{0\} \subset \mathfrak{z}$, hence the space $\mathcal{F}|_e$ comes equipped with a \mathfrak{g}^e -module structure.

Further, we choose an \mathfrak{sl}_2 -triple for e , and let t denote the corresponding semisimple element in the triple. Then the element $\hat{t} := (t, -2) \in \mathfrak{g} \oplus \mathbb{k}$ belongs to \mathfrak{z} , since $\text{ad } t(e) = 2e$. The action of the semisimple element \hat{t} (or rather of the one-parameter multiplicative subgroup \mathbb{G}_m generated by \hat{t}) puts a \mathbb{Z} -grading on $\mathcal{F}|_e$. The action-map $\mathfrak{g}^e \otimes \mathcal{F}|_e \rightarrow \mathcal{F}|_e$ preserves the gradings.

Recall that G^e is the connected unipotent group with Lie algebra \mathfrak{g}^e . Any finite-dimensional \mathfrak{g}^e -module has a canonical G^e -module structure, by exponentiation. We write $\text{Rep}^\bullet(G^e)$ for the abelian category of finite-dimensional \mathbb{Z} -graded G^e -modules equipped with a grading compatible with *Lie* G^e -action. Thus, in the notations above, for any $\mathcal{F} \in \text{Coh}^{G \times \mathbb{G}_m}(\mathcal{N})$, one has: $\mathcal{F}|_e \in \text{Rep}^\bullet(G^e)$.

10.2 Quantum group cohomology and the principal nilpotent. According to Proposition 2.9.2, we have a Hopf-adjoint action of the quantum algebra \mathcal{U} on \mathfrak{u} . The action preserves the kernel of the Frobenius homomorphism $\phi : \mathfrak{u} \rightarrow \mathcal{U}$, hence induces an \mathcal{U} -action on \mathfrak{u} , the image of the Frobenius homomorphism. This gives rise to an \mathcal{U} -action on the cohomology $H^\bullet(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}})$. As we have already mentioned in §2.11, the Hopf-adjoint action of any algebra on its own cohomology is trivial. Thus, the Hopf-adjoint action of \mathcal{U} on $H^\bullet(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}})$ descends to the quotient algebra $\mathcal{U}/(\mathfrak{u})$, which is isomorphic to $\mathcal{U}\mathfrak{g}$ via the Frobenius homomorphism. This makes $H^\bullet(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}})$ a \mathfrak{g} -module, and we have

Theorem 10.2.1 ([GK]). *There is a natural \mathfrak{g} -equivariant graded algebra isomorphism $H^{2\bullet}(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}}) \simeq \mathbb{k}^\bullet[\mathcal{N}]$. Moreover, all odd cohomology groups vanish: $H^{\text{odd}}(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}}) = 0$.*

For any \mathfrak{u} -module M , the cohomology $H^\bullet(\mathfrak{u}, M) = \text{Ext}_{\mathfrak{u}}^\bullet(\mathbb{k}_{\mathfrak{u}}, M)$ has a natural graded $H^\bullet(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}})$ -module structure via the Yoneda product: $\text{Ext}_{\mathfrak{u}}^i(\mathbb{k}_{\mathfrak{u}}, \mathbb{k}_{\mathfrak{u}}) \times \text{Ext}_{\mathfrak{u}}^j(\mathbb{k}_{\mathfrak{u}}, M) \rightarrow \text{Ext}_{\mathfrak{u}}^{i+j}(\mathbb{k}_{\mathfrak{u}}, M)$. One can show that $H^\bullet(\mathfrak{u}, M)$ is a finitely generated $H^\bullet(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}})$ -module provided M is finite dimensional over \mathbb{k} .

Assume now that M is a \mathcal{U} -module, and view it as an \mathfrak{u} -module, by restriction. Then, there is a natural \mathcal{U} -module structure on each cohomology group $H^i(\mathfrak{u}, M)$, that descends to $\mathcal{U}\mathfrak{g}$, see [GK, §5.2]. If M is, in addition, finite dimensional then $H^\bullet(\mathfrak{u}, M)$ is finitely generated over $H^\bullet(\mathfrak{u}, \mathbb{k}_{\mathfrak{u}}) \simeq \mathbb{k}[\mathcal{N}]$, hence each cohomology group $H^i(\mathfrak{u}, M)$ is finite dimensional over \mathbb{k} . It follows that the \mathfrak{g} -action on $H^i(\mathfrak{u}, M)$ may be exponentiated canonically to an algebraic action of the group G . Thus, we may (and will) view $H^\bullet(\mathfrak{u}, M)$ as an object of the category $\text{Coh}^{G \times G_m}(\mathcal{N})$. This way we obtain, following [GK, §4.1], a functor

$$F : \text{Rep}(\mathcal{U}) \rightarrow \text{Coh}^{G \times G_m}(\mathcal{N}), \quad M \mapsto H^\bullet(\mathfrak{u}, \text{Res}_{\mathfrak{u}}^{\mathcal{U}} M). \quad (10.2.2)$$

Taking the geometric fiber at e , and restricting attention to \mathcal{U} -modules that belong to the principal block, we thus obtain the functor

$$F_e : \text{block}(\mathcal{U}) \rightarrow \text{Rep}^\bullet(G^e), \quad M \mapsto F(M)|_e = H^\bullet(\mathfrak{u}, M)|_e.$$

The Theorem below is the main result of this section; it has been conjectured in [GK, §4.3].

Theorem 10.2.3. *There is a canonical isomorphism of the following two functors $\text{block}(\mathcal{U}) \rightarrow \text{Rep}^\bullet(G^e)$:*

$$M \mapsto H^\bullet(\text{Gr}, \mathcal{P}M) \quad \text{and} \quad M \mapsto F_e(M).$$

The rest of this section is devoted to the proof of this Theorem, which requires some preparations.

10.3 Induction. We are going to relate cohomology over the algebras \mathfrak{u} and \mathcal{U} . To this end, consider the smooth-coinduction functor $\text{Ind} = \text{Ind}_{\mathfrak{u}}^{\mathcal{U}}$. By [APW₂, Theorem 2.9.1] we have

Theorem 10.3.1. (i) *Each of the categories $\text{Rep}(\mathcal{U})$ and $\text{Rep}(\mathfrak{u})$, has enough projectives. Moreover, in each category, projective and injective objects coincide.*

(ii) *Furthermore, the functor $\text{Res} : \text{Rep}(\mathcal{U}) \rightarrow \text{Rep}(\mathfrak{u})$ takes projectives into projectives, and the functor $\text{Ind} : \text{Rep}(\mathfrak{u}) \rightarrow \lim \text{ind} \text{Rep}(\mathcal{U})$ is exact.* \square

Next, given $M_{\mathfrak{u}} \in \text{Rep}(\mathcal{U})$, choose a projective resolution of $M_{\mathfrak{u}}$, and apply the adjunction isomorphism (2.7.2) to that resolution term by term. This way, the result of Andersen-Polo-Wen above yields, for any $N_{\mathfrak{u}} \in \text{Rep}(\mathfrak{u})$, a canonical graded space isomorphism

$$\text{Ext}_{\mathfrak{u}}^\bullet(\text{Res } M_{\mathfrak{u}}, N_{\mathfrak{u}}) \simeq \text{Ext}_{\mathfrak{u}\text{-mod}}^\bullet(M_{\mathfrak{u}}, \text{Ind } N_{\mathfrak{u}}). \quad (10.3.2)$$

We can now express the functor $M \mapsto H^\bullet(\mathfrak{u}, M)$ in terms of cohomology of \mathcal{U} rather than \mathfrak{u} .

Lemma 10.3.3. *For any $M \in \text{Rep}(\mathbf{U})$, there is a natural isomorphism $H^\bullet(\mathbf{u}, M) = \text{Ext}_{\mathbf{U}\text{-mod}}^\bullet(\mathbb{k}_{\mathbf{U}}, M \otimes_{\mathbb{k}} \phi_{\mathbb{k}}[G])$.*

Proof: Write $\text{Ind} := \text{Ind}_{\mathbf{U}}^{\mathbf{U}}$ and $\text{Res} := \text{Res}_{\mathbf{U}}^{\mathbf{U}}$. For any $M \in \text{Rep}(\mathbf{U})$, we calculate:

$$\begin{aligned} H^\bullet(\mathbf{u}, M) &= \text{Ext}_{\mathbf{U}}^\bullet(\text{Res } \mathbb{k}_{\mathbf{U}}, \text{Res } M) \\ \text{by (10.3.2)} \quad &= \text{Ext}_{\mathbf{U}\text{-mod}}^\bullet(\mathbb{k}_{\mathbf{U}}, \text{Ind}(\text{Res } M)) \\ \text{by Lemma 2.8.3} \quad &= \text{Ext}_{\mathbf{U}\text{-mod}}^\bullet(\mathbb{k}_{\mathbf{U}}, M \otimes \phi_{\mathbb{k}}[G]). \quad \square \end{aligned}$$

10.4 Quantum group cohomology via the loop Grassmannian. We are going to re-interpret the functor $F : \text{Rep}(\mathbf{U}) \rightarrow \text{Coh}^{G \times \mathbb{G}_m}(\mathcal{N})$, $M \mapsto H^\bullet(\mathbf{u}, M)$ geometrically, in terms of perverse sheaves on Gr .

Recall the setup of Theorem 7.3.1. We will prove the following

Proposition 10.4.1. (i) *There is a natural graded algebra isomorphism*

$$H^\bullet(\mathbf{u}, \mathbb{k}_{\mathbf{U}}) \simeq \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R}).$$

(ii) *For any $M \in \text{block}(\mathbf{U})$, there is a graded module isomorphism compatible with the algebra isomorphism of part (i):*

$$H^\bullet(\mathbf{u}, M) \simeq \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{M} \star \mathcal{R}) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathcal{R}^\vee, \mathcal{M}).$$

Remark 10.4.2. It would be very interesting to give an algebraic interpretation of the space $\text{Ext}_{G^{\vee}(\mathcal{O})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$ in terms of the quantum group \mathbf{u} , that is to find an equivariant analogue of the isomorphism of part (i) of the Proposition above. \diamond

Proof of Proposition 10.4.1. The category $\text{block}(\mathbf{U})$ is known [APW] to be a direct summand of the category $\text{Rep}(\mathbf{U})$. Hence, the Ext-groups in these two categories coincide and, for any $M \in \text{block}(\mathbf{U})$, we compute

$$\begin{aligned} H^\bullet(\mathbf{u}, M) &= \text{Ext}_{\mathbf{U}\text{-mod}}^\bullet(\mathbb{k}_{\mathbf{U}}, M \otimes_{\mathbb{k}} \phi_{\mathbb{k}}[G]) \quad (\text{by Lemma 2.8.3}) \\ &= \text{Ext}_{\text{block}(\mathbf{U})}^\bullet(\mathbb{k}_{\mathbf{U}}, M \otimes_{\mathbb{k}} \phi_{\mathbb{k}}[G]) \\ (\text{by Theorem 9.10.2}) \quad &= \text{Ext}_{\text{Perv}_{\mathbf{I}^{\text{mon}}}(\text{Gr})}^\bullet(\mathcal{P}(\mathbb{k}_{\mathbf{U}}), \mathcal{P}(M \otimes_{\mathbb{k}} \phi_{\mathbb{k}}[G])) \quad (10.4.3) \\ (\text{by Proposition 9.10.7}) \quad &= \text{Ext}_{\text{Perv}_{\mathbf{I}^{\text{mon}}}(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{P}M \star \mathcal{P}(\phi_{\mathbb{k}}[G])) \\ &= \text{Ext}_{\text{Perv}_{\mathbf{I}^{\text{mon}}}(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{P}M \star \mathcal{R}) \stackrel{(9.1.2)}{=} \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{P}M \star \mathcal{R}). \end{aligned}$$

This proves the isomorphism in (ii) which, in the special case $M = \mathbb{k}_{\mathbf{U}}$, reduces to (i).

To see that the isomorphism in (i) is an *algebra* isomorphism we put $M := \mathbb{k}$ in (10.4.3). Recall that the cup product on $H^\bullet(\mathbf{u}, \mathbb{k}_{\mathbf{U}}) (\cong \text{Ext}_{\mathbf{U}\text{-mod}}^\bullet(\mathbb{k}_{\mathbf{U}}, \phi_{\mathbb{k}}[G]))$ is known, see e.g. [GK, §5.1], to be induced by the algebra structure on $\mathbb{k}[G]$. It follows that the cup-product on $\text{Ext}_{\mathbf{U}\text{-mod}}^\bullet(\mathbb{k}_{\mathbf{U}}, \phi_{\mathbb{k}}[G])$ can be defined in a way very similar to the one we have defined a product on $\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{R})$. Specifically, given $x \in \text{Ext}_{\mathbf{U}\text{-mod}}^i(\mathbb{k}_{\mathbf{U}}, \phi_{\mathbb{k}}[G])$, we view it as a "derived morphism" $x : \mathbb{k}_{\mathbf{U}} \rightarrow \phi_{\mathbb{k}}[G][i]$.

Tensoring (over \mathbb{k}) with this "derived morphism" gives, for any $y \in \text{Ext}_{\mathcal{U}\text{-mod}}^j(\mathbb{k}_{\mathcal{U}}, {}^\phi\mathbb{k}[G])$, a composition:

$$y \cdot x : \mathbb{k}_{\mathcal{U}} \xrightarrow{y} {}^\phi\mathbb{k}[G][j] = ({}^\phi\mathbb{k}[G] \otimes_{\mathbb{k}} \mathbb{k}_{\mathcal{U}})[j] \xrightarrow{\otimes x} {}^\phi(\mathbb{k}[G] \otimes_{\mathbb{k}} \mathbb{k}[G])[i+j] \xrightarrow{\text{product}} {}^\phi\mathbb{k}[G][i+j].$$

Comparison with formula (7.2.3) yields compatibility of the algebra structures on the LHS and RHS of the isomorphism in part (i). The rightmost isomorphism in (ii) is due to (9.11.1). The Proposition is proved. \square

10.5 Given any finite-dimensional graded G^e -module E , we form the algebraic G -equivariant vector bundle $\text{Ind}_{G^e}^G E := E \times_{G^e} G$ on $G^e \backslash G = \mathcal{N}^{\text{reg}}$. It is clear from definitions that the space of global regular sections of the corresponding locally free sheaf on \mathcal{N}^{reg} is given by the formula: $\Gamma(\mathcal{N}^{\text{reg}}, \text{Ind}_{G^e}^G E) = (E \otimes_{\mathbb{k}} \mathbb{k}[G])^{G^e}$. Further, let $j : \mathcal{N}^{\text{reg}} \hookrightarrow \mathcal{N}$ denote the open imbedding, and $j_*(\text{Ind}_{G^e}^G E)$ denote the direct image, which is a coherent sheaf on \mathcal{N} since the complement $\mathcal{N} \setminus \mathcal{N}^{\text{reg}}$ has codimension ≥ 2 in \mathcal{N} .

Proposition 10.5.1. (i) *There is a morphism between the following two functors $\text{Rep}(\mathcal{U}) \rightarrow \text{Coh}^{G \times \mathbb{G}_m}(\mathcal{N}) (= \text{category of } G \times \mathbb{G}_m\text{-equivariant } \mathbb{k}[\mathcal{N}]\text{-modules})$:*

$$F(M) = \text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{P}M \star \mathcal{R}) \longrightarrow \Gamma\left(\mathcal{N}, j_*(\text{Ind}_{G^e}^G H^\bullet(\text{Gr}, \mathcal{P}M))\right). \quad (10.5.2)$$

This isomorphism is compatible, via the isomorphism of Proposition 10.4.1, with the algebra actions on each side.

(ii) *For any semisimple object $M \in \text{block}(\mathcal{U})$ the morphism (10.5.2) is an isomorphism.*

Proof. Let E be a finite dimensional G^e -module, and $\mathcal{E} = j_*(\text{Ind}_{G^e}^G E)$ the corresponding coherent sheaf on \mathcal{N} . We observe that by definition one has a canonical isomorphism of the spaces of global sections: $\Gamma(\mathcal{N}^{\text{reg}}, \text{Ind}_{G^e}^G E) = \Gamma(\mathcal{N}, \mathcal{E})$. Therefore, since $\mathbb{k}[\mathcal{N}^{\text{reg}}] \simeq \mathbb{k}[\mathcal{N}]$, one can alternatively *define* the sheaf \mathcal{E} as the sheaf on \mathcal{N} whose space of global sections is the vector space $\Gamma(\mathcal{N}^{\text{reg}}, \text{Ind}_{G^e}^G E) = (E \otimes_{\mathbb{k}} \mathbb{k}[G])^{G^e}$, viewed as a $\mathbb{k}[\mathcal{N}]$ -module.

Recall next that, for any $M \in \text{block}(\mathcal{U})$, the hyper-cohomology $H^\bullet(\text{Gr}, \mathcal{P}M)$ has a natural graded G^e -module structure. Thus, we may view this space as a finite dimensional G^e -module, and perform the construction of the previous paragraph. We get

$$\begin{aligned} \Gamma\left(\mathcal{N}, j_* \text{Ind}_{G^e}^G H^\bullet(\text{Gr}, \mathcal{P}M)\right) &= \Gamma\left(\mathcal{N}^{\text{reg}}, \text{Ind}_{G^e}^G H^\bullet(\text{Gr}, \mathcal{P}M)\right) \\ &= \left(\mathbb{k}[G] \otimes_{\mathbb{k}} H^\bullet(\text{Gr}, \mathcal{P}M)\right)^{G^e} \\ (\text{since } \mathcal{R} = \mathcal{P}(\mathbb{k}[G])) &= \left(H^\bullet(\text{Gr}, \mathcal{R}) \otimes_{\mathbb{k}} H^\bullet(\text{Gr}, \mathcal{P}M)\right)^{G^e} \\ (\text{by Proposition 9.10.7}) &= \left(H^\bullet(\text{Gr}, \mathcal{R} \star \mathcal{P}M)\right)^{G^e} \\ (\text{by Theorem 9.10.2}) &= \text{Hom}_{H(\text{Gr})}^\bullet\left(H^\bullet(\text{Gr}, \mathbf{1}_{\text{Gr}}), H^\bullet(\text{Gr}, \mathcal{R} \star \mathcal{P}M)\right). \end{aligned} \quad (10.5.3)$$

Notice that the expression in the bottom line coincides with a special case of the RHS of formula (7.4.4). Hence, the formula gives a canonical map:

$$\text{Ext}_{D^b(\text{Gr})}^\bullet(\mathbf{1}_{\text{Gr}}, \mathcal{P}M \star \mathcal{R}) \longrightarrow \text{Hom}_{H(\text{Gr})}^\bullet\left(H^\bullet(\mathbf{1}_{\text{Gr}}), H^\bullet(\mathcal{R} \star \mathcal{P}M)\right) = \Gamma\left(\mathcal{N}, j_* \text{Ind}_{G^e}^G H^\bullet(\mathcal{P}M)\right).$$

It is seen easily that this map intertwines the $\mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R})$ -module structure on the RHS with the $\mathbb{K}[\mathcal{N}]$ -module structure on the LHS (using the algebra isomorphism $\mathrm{Ext}_{D^b(\mathrm{Gr})}^\bullet(\mathbf{1}_{\mathrm{Gr}}, \mathcal{R}) \simeq \mathbb{K}[\mathcal{N}]$, see §7). This yields part (i) of the Proposition.

To prove part (ii) observe that, for any simple module $M \in \mathbf{block}(\mathbf{U})$, the perverse sheaf $\mathcal{P}M \star \mathcal{R}$ is semisimple, by Proposition 9.10.7 and Decomposition theorem [BBD]. The result now follows from the chain of isomorphisms in (10.5.3), and Proposition 7.4.5 applied to the semisimple object $\mathcal{L}_2 = \mathcal{P}M \star \mathcal{R}$. \square

10.6 Proof of Theorem 10.2.3. Formula (10.5.2) gives, by adjunction, a morphism: $j^*F(M) \longrightarrow \mathrm{Ind}_{G^e}^G H^\bullet(\mathrm{Gr}, \mathcal{P}M)$. This is a morphism of $\mathrm{Ad} G$ -equivariant coherent sheaves on the orbit $\mathcal{N}^{\mathrm{reg}}$. Giving such a morphism is equivalent to giving a G^e -equivariant linear map between the geometric fibers at e of the corresponding (locally free) sheaves. Thus, we obtain a canonical morphism of functors $\vartheta : F_e(M) = F(M)|_e \longrightarrow H^\bullet(\mathrm{Gr}, \mathcal{P}M)$.

It remains to prove that, for any $M \in \mathbf{block}(\mathbf{U})$, the morphism ϑ is an isomorphism. For M simple this is insured by Lemma 10.5.1. In the general case, choose a Jordan-Hölder filtration: $0 = M_0 \subset M_1 \subset \dots \subset M_n = M$ with simple subquotients M_i/M_{i-1} , $i = 1, \dots, n$, and put $\mathrm{gr} M = \bigoplus M_i/M_{i-1}$. We have the standard convergent spectral sequence: $H^\bullet(\mathbf{u}, \mathrm{gr} M) = E_2 \implies E_\infty = \mathrm{gr} H^\bullet(\mathbf{u}, M)$. The restriction functor of the sheaf $F(M) = H^\bullet(\mathbf{u}, M)$ to the point e being *exact* (since the G -orbit of e is open in \mathcal{N} , and $F(M)$ is G -equivariant), the spectral sequence above induces a spectral sequence of the fibers: $F_e(\mathrm{gr} M) \implies \mathrm{gr} F_e(M)$.

Similar arguments apply to the functor: $M \longmapsto H^\bullet(\mathrm{Gr}, \mathcal{P}M)$, and yield a spectral sequence: $E_2 = H^\bullet(\mathrm{Gr}, \mathcal{P}(\mathrm{gr} M)) \implies \mathrm{gr} H^\bullet(\mathrm{Gr}, \mathcal{P}M)$. Now the canonical morphism ϑ gives a morphism of the two spectral sequences. This morphism induces an isomorphism between the E_2 -terms, due to Lemma 10.5.1. It follows that ϑ is itself an isomorphism. (Equivalently, instead of the spectral sequence argument above, one can use that, for each i , the map ϑ gives a morphism of two long exact sequences of derived functors corresponding to the short exact sequence: $0 \rightarrow M_{i-1} \rightarrow M_i \rightarrow M_i/M_{i-1} \rightarrow 0$. The result then follows by induction on i , by means of the *five-Lemma*.) \square

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