LANGLANDS, WEIL, AND LOCAL CLASS FIELD THEORY

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Abstract. The local Langlands conjecture predicts a relationship between Galois representations and representations of the topological general linear group. In this talk we explain a special case of the program, local class field theory, with a goal towards its nonabelian generalization. One of our main conclusions is that the Galois group should be replaced by a variant called the Weil group.

As number theorists, we would like to understand the absolute Galois group $\Gamma_{\mathbb{Q}}$ of $\mathbb{Q}$. It is hard to say briefly why such an understanding would be interesting. But for one, if we understood the open, index-n subgroups of $\Gamma_{\mathbb{Q}}$ then we would understand the number fields of degree n. There are much deeper reasons, related to the geometry of algebraic varieties, why we suspect $\Gamma_{\mathbb{Q}}$ carries deep information, though that is more speculative.

One way to study $\Gamma_{\mathbb{Q}}$ is through its n-dimensional complex representations, that is, (conjugacy classes of) homomorphisms

$$\Gamma_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{C}).$$

Starting with a 1967 letter to Weil, Langlands proposed that these representations are related to the representations of a completely different kind of group, the adelic group $\mathrm{GL}_n(\mathbb{A}_{\mathbb{Q}})$. This circle of ideas is known as the Langlands program.

The Langlands program over $\mathbb{Q}$ turned out to be very difficult. Even today, we cannot properly formulate a precise conjecture. So to get a start on the problem, mathematicians restricted attention to the completions of $\mathbb{Q}$, namely $\mathbb{Q}_p$ and $\mathbb{R}$, and studied the problem there. As over $\mathbb{Q}$, Langlands conjectured a relationship between n-dimensional Galois representations

$$\Gamma_{\mathbb{Q}_p} \to \mathrm{GL}_n(\mathbb{C})$$

and representations of the group $\mathrm{GL}_n(\mathbb{Q}_p)$, as well as an analogous relationship over $\mathbb{R}$.

In a non-technical overview, it is difficult to explain anything of substance about the Langlands program. The first reason is that this program draws on several areas of mathematics, including algebraic geometry, number theory, representation theory, and analysis. The second reason is that the historical precursors to the program are rather complicated. The third reason is that the program is incomplete. We cannot even properly formulate the conjectures in the global setting. Nonetheless, it must be possible to say something, because many mathematicians work in this field and our knowledge is rapidly advancing!

The goal of this talk is to say a few words about an important test case for the local Langlands program, class field theory. One of our main conclusions is that the Galois group should be replaced by a closely related group, the Weil group. I had hoped to say much more, and in particular give a broad overview of the local Langlands correspondence for GL_n, but it surpassed my expositional ability to do so without becoming too abstract.

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1. Structure of the p-adic Galois group

The Langlands program studies a specific kind of structure carried by $\Gamma_{\mathbb{Q}_p}$, its finite-dimensional complex representations. But before we discuss the representation theory, it's worth summarizing what is known about the general structure of this profinite group.

There are two ways to study the structure of a group. First, you can describe it in terms of generators and relations, explaining how to multiply elements in the group. Second, you can describe it in terms of a composition series, explaining how the group is built from simpler groups. For the Galois group of $\mathbb{Q}_p$, we have good answers to both of these questions.

1.1. Presentation.

We have known a presentation for $\Gamma_{\mathbb{Q}_p}$ since the 1980's. The following theorem is due to Jannsen and Wingberg [NSW08, Theorem 7.5.14].

Theorem 1. *Let $k \neq \mathbb{Q}_2$ be a finite extension of $\mathbb{Q}_p$ of degree n. There is a presentation for the profinite group Γ_k with $n+3$ generators and with relations analogous to the relations for the fundamental group of a compact surface of genus $\approx n/2$.*

Surprisingly, Jannsen and Wingberg's description of $\Gamma_{\mathbb{Q}_p}$ is not the end of the story. I have never seen it used to prove a technical result of interest to the Langlands program. Nonetheless, Jannsen and Wingberg's theorem is significant for the following two reasons. First, it indicates that the Galois group carries a much more interesting kind of structure than anything that could be captured by group theory alone. Second, the suggested analogy between p-adic fields and compact Riemann surfaces had inspired several fruitful geometric formulations of the Langlands program.

1.2. Ramification filtration.

Instead of working with a presentation of a group, we can study it by breaking it into simpler pieces, in other words, by introducing a filtration and studying the subquotients of the filtration. For the group $\Gamma_{\mathbb{Q}_p}$, there are three essential pieces, coming from unramified extensions, tamely ramified extensions, and wildly ramified extensions. The third case is by far the most complicated.

Since the unramified part of $\Gamma_{\mathbb{Q}_p}$ comes from $\mathbb{F}_p$, we start by reviewing the structure of $\Gamma_{\mathbb{F}_p}$. For each $n \geq 1$, the field $\mathbb{F}_p$ admits a unique extension $\mathbb{F}_{p^n}$ of degree n, the cyclotomic extension containing the (p^n-1)th roots of unity. The Galois group of this extension is cyclic with a canonical generator, the Frobenius automorphism $\mathrm{Fr}\colon x \mapsto x^p$. It follows that the Galois group of $\mathbb{F}_p$ is the inverse limit

$$\Gamma_{\mathbb{F}_p} = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \widehat{\mathbb{Z}}.$$

So this group is already abelian and we can completely understand it through its characters.

Let's now turn to the p-adic numbers. The Galois group $\Gamma_{\mathbb{Q}_p}$ is nonabelian because not every finite extension of $\mathbb{Q}_p$ is Galois. For instance, for any fixed p, the field $\mathbb{Q}_p$ contains only finitely many roots of unity. If $\mathbb{Q}_p$ does not contain an nth root of unity then the extension $\mathbb{Q}_p(p^{1/n})$ is not Galois.

The Galois theory of $\mathbb{F}_p$ gives insight into the Galois theory of $\mathbb{Q}_p$ through the following construction. An algebraic extension K of $\mathbb{Q}_p$ is unramified if p is prime in $\mathcal{O}_K$. The unramified extensions are exactly the lifts of extensions of $\mathbb{F}_p$ to extensions of $\mathbb{Q}_p$. More precisely, for each $n \geq 1$, the field $\mathbb{Q}_p$ admits a unique *unramified* extension $\mathbb{Q}_{p^n}$ of degree n, the cyclotomic extension containing the (p^n-1)th roots of unity. The Galois group of this extension is cyclic with canonical generator. The description of unramified extensions, follows from the description of finite field extensions by Hensel's Lemma. The canonical generator of $\mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p)$ is a lift of the Frobenius: it raises the adjoined roots of unity to the pth power.

Consequently, the maximal unramified extension $\mathbb{Q}_p^{\mathrm{nr}}$ of $\mathbb{Q}_p$ has the same (relative) Galois group as the (absolute) Galois group of $\mathbb{F}_p$, namely $\widehat{\mathbb{Z}}$. Hence there is a short exact sequence

$$\begin{tikzcd} 1 \arrow[r] & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p^{\mathrm{nr}}) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p) \arrow[r] \arrow[d, equal] & \mathrm{Gal}(\mathbb{Q}_p^{\mathrm{nr}}/\mathbb{Q}_p) \arrow[r] \arrow[d, equal] & 1 \\ 1 \arrow[r] & I_{\mathbb{Q}_p} \arrow[r] & \Gamma_{\mathbb{Q}_p} \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 1 \end{tikzcd} \tag{1}$$

{thm5}

The group $I_{\mathbb{Q}_p}$ is called the inertia group. Since $\widehat{\mathbb{Z}}$ is topologically cyclic, the short exact sequence above splits, that is, $\Gamma_{\mathbb{Q}_p} \simeq I_{\mathbb{Q}_p} \rtimes \widehat{\mathbb{Z}}$, though the splitting is noncanonical.

The inertia group is quite complicated, and all the complexity comes from wild ramification. Recall that an extension of $\mathbb{Q}_p$ is wildly ramified if it is totally ramified and its degree is a power of p. The inertia group $I_{\mathbb{Q}_p}$ sits at the top of a tower of subgroups $(I^r_{\mathbb{Q}_p})_{r\geq 0}$ of $I_{\mathbb{Q}_p}$ called the higher ramification groups. By a "tower" we mean that

$$I^r_{\mathbb{Q}_p} \supseteq I^s_{\mathbb{Q}_p} \qquad \text{if } r \leq s.$$

The group

$$P_{\mathbb{Q}_p} = I^{0+}_{\mathbb{Q}_p} := \bigcup_{r>0} I^r_{\mathbb{Q}_p}$$

is called the wild inertia group. It is a pro-p group that classifies the wildly ramified extensions. The difference between the inertia group and the wild inertia group is small:

$$I_{\mathbb{Q}_p}/P_{\mathbb{Q}_p} \simeq \prod_{\ell\neq p} \mathbb{Z}_\ell \subsetneq \widehat{\mathbb{Z}}.$$

To define $I^r_{\mathbb{Q}_p}$ precisely takes some work – see, for instance, Chapter IV of Serre's *Local Fields* [Ser79] – but the rough idea is that the deeper an automorphism of $\overline{\mathbb{Q}}_p$ lies in the ramification filtration, the less it can move elements of $\overline{\mathbb{Q}}_p$.

The ramification filtration suggests a strategy for answering questions about $\Gamma_{\mathbb{Q}_p}$: start with the unramified part and work deeper and deeper into the filtration. This strategy has succeeded in proving several cases of a general local Langlands correspondence.

2. Local class field theory and the Weil group

The simplest Galois representations are the characters, homomorphisms $\Gamma_{\mathbb{Q}_p} \to \mathbb{C}^\times$. Since the group $\mathbb{C}^\times$ is abelian, every character of $\Gamma_{\mathbb{Q}_p}$ factors through the abelianization $\Gamma^{\mathrm{ab}}_{\mathbb{Q}_p}$ of $\Gamma_{\mathbb{Q}_p}$, in other words, the maximal abelian quotient. It turns out that the abelianization admits a simple and explicit description, which is roughly summarized by the following slogan:

$$\Gamma^{\mathrm{ab}}_{\mathbb{Q}_p} \approx \mathbb{Q}_p^\times.$$

This approximation cannot be upgraded to an isomorphism, however, because the Galois group is (profinite, hence) compact while the units group is noncompact. In this section we use the ramification filtration of the previous section to give a plausibility argument for a correct version of the approximation above, Artin reciprocity. We then explain how to fix this deficiency by replacing the Galois group with a closely related group called the Weil group.

2.1. **p-adic class field theory.** To understand $\Gamma_{\mathbb{Q}_p}^{\mathrm{ab}}$, use the ramification filtration. We saw earlier that this filtration on $\Gamma_{\mathbb{Q}_p}$ is continuous, in the sense that it has a jump at every positive number. But once we abelianize, the Hasse-Arf theorem states that the ramification filtration on $\Gamma_{\mathbb{Q}_p}^{\mathrm{ab}}$ is discrete: it only jumps at integers. Further, the subquotients of the filtration are $\mathbb{Z}/p\mathbb{Z}$, except from 1 to 0, where the subquotient is $\mathbb{F}_p^\times$. All in all, we have the following picture:

$$\cdots \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \Gamma_{\mathbb{Q}_p}^{\mathrm{ab},n+1} \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \Gamma_{\mathbb{Q}_p}^{\mathrm{ab},n} \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \cdots \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \Gamma_{\mathbb{Q}_p}^{\mathrm{ab},1} \xhookrightarrow{\mathbb{F}_p^\times} \Gamma_{\mathbb{Q}_p}^{\mathrm{ab},0} \xhookrightarrow{\widehat{\mathbb{Z}}} \Gamma_{\mathbb{Q}_p}^{\mathrm{ab}}. \tag{2}$$ {thm3}

As for the units group $\mathbb{Q}_p^\times$, it fits into a short exact sequence

$$1 \longrightarrow \mathbb{Z}_p^\times \longrightarrow \mathbb{Q}_p^\times \xrightarrow{\mathrm{ord}_p} \mathbb{Z} \longrightarrow 1.$$

Furthermore, $\mathbb{Q}_p^\times$ is filtered by the higher unit groups with $\mathbb{Q}_p^{\times,0} := \mathbb{Z}_p^\times$ and

$$\mathbb{Z}_p^{\times,n} := \ker(\mathbb{Z}_p^\times \longrightarrow (\mathbb{Z}/p^n\mathbb{Z})^\times)$$

for $n \geq 1$. The subquotient for the top of the filtration is

$$\mathbb{Q}_p^\times/\mathbb{Z}_p^\times \simeq \mathbb{Z}$$

by the p-adic order map. The second subquotient $\mathbb{Z}_p^\times/\mathbb{Z}_p^{\times,1}$ is isomorphic to $\mathbb{F}_p^\times$ by the map sending a p-adic number to its first digit. The other quotients in the filtration are isomorphic to $\mathbb{Z}/p\mathbb{Z}$ since

$$(1+ap^n)(1+bp^n) = 1 + (a+b)p^n + O(p^{n+1}).$$

All in all, we have the following picture:

$$\cdots \hookrightarrow \mathbb{Z}_p^{\times,n+1} \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}_p^{\times,n} \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \cdots \xhookrightarrow{\mathbb{Z}/p\mathbb{Z}} \mathbb{Z}_p^{\times,1} \xhookrightarrow{\mathbb{F}_p^\times} \mathbb{Z}_p^\times \xhookrightarrow{\mathbb{Z}} \mathbb{Q}_p^\times \tag{3}$$ {thm4}

Diagrams (2) and (3) are very similar, and match up under Artin reciprocity, but there is one slight difference. On the Galois side, the top quotient is $\widehat{\mathbb{Z}}$; on the units-group side, the top quotient is $\mathbb{Z}$. There are two ways to fix this discrepancy, by changing either the units-group side or the Galois side.

On the units-group side, we can form the profinite completion $\mathbb{Q}_{p,\mathrm{pro}}^\times$. After completion, there is an isomorphism

$$\Gamma_{\mathbb{Q}_p}^{\mathrm{ab}} \simeq \mathbb{Q}_{p,\mathrm{pro}}^\times.$$

However, profinitely completing $\mathbb{Q}_p^\times$ is somewhat unnatural. For one, it shrinks the size of the character group: the characters of $\mathbb{Q}_{p,\mathrm{pro}}^\times$ are precisely the unitary characters of $\mathbb{Q}_p^\times$, but the latter group has non-unitary characters, namely $x \mapsto |x|_p^s$ for $s \in \mathbb{C}$.

On the Galois side, we can shrink the Galois group by replacing $\widehat{\mathbb{Z}}$ with $\mathbb{Z}$ in the short-exact sequence (1):

$$\begin{tikzcd}
1 \arrow[r] & I_{\mathbb{Q}_p} \arrow[r] \arrow[d, equal] & W_{\mathbb{Q}_p} \arrow[r] \arrow[d, "\text{dense}"] & \mathbb{Z} \arrow[r] \arrow[d, "\text{dense}"] & 1 \\
1 \arrow[r] & I_{\mathbb{Q}_p} \arrow[r] & \Gamma_{\mathbb{Q}_p} \arrow[r] & \widehat{\mathbb{Z}} \arrow[r] & 1
\end{tikzcd}$$

Formally, the diagram above is a pullback of group extensions, and the resulting group $W_{\mathbb{Q}_p} := \mathbb{Z} \times_{\widehat{\mathbb{Z}}} \Gamma_{\mathbb{Q}_p}$ is known as the Weil group of $\mathbb{Q}_p$. With this reformulation, the one that we prefer, Artin reciprocity states that

$$W_{\mathbb{Q}_p}^{\mathrm{ab}} \simeq \mathbb{Q}_p^\times.$$

The Weil group is very nearly the right form of the Galois group for the Langlands program.

2.2. **Real local class field theory.** What can be said at the real place? Here the absolute Galois group $\Gamma_\mathbb{R}$ is cyclic of order two, and there is not much substance in studying its abelianization. However, in analogy with the p-adic case, we would like to define some kind of real Weil group whose abelianization is $\mathbb{R}^\times$. We define $W_\mathbb{R}$ as the nonsplit extension

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_\mathbb{R} \longrightarrow \Gamma_\mathbb{R} \longrightarrow 1$$

Explicitly, $W_\mathbb{R}$ is the group obtained from $\mathbb{C}^\times$ by adjoining an element j such that

$$j^2 = -1 \qquad \text{and} \qquad jzj^{-1} = \bar{z} \quad \text{for } z \in \mathbb{C}.$$

Since $[j, z] = \bar{z}/z$, the commutator subgroup of $W_\mathbb{R}$ is the circle group S^1. Hence

$$W_\mathbb{R}^{\mathrm{ab}} = (\mathbb{C}^\times/S^1) \times \Gamma_\mathbb{R} \simeq \mathbb{R}^\times,$$

the archimedean reformulation of local class field theory that uses the Weil group.

The definitions of the real and p-adic Weil groups seem very different from each other. Nonetheless, there is a uniform definition using group cohomology [Tat79, Section 1].

2.3. **Beyond class field theory.** One final summary of local class field theory is in order, a reformulation that suggests the shape of the general local Langlands correspondence. For $k = \mathbb{R}$ or $k = \mathbb{Q}_p$, the local Langlands correspondence for GL_1 is the canonical bijection between the following two objects.

(1) One-dimensional representations $W_k \to \mathrm{GL}_1(\mathbb{C})$ (Galois side)
(2) Irreducible complex representations of $\mathrm{GL}_1(k)$ (Automorphic side)

To generalize the correspondence to GL_n, we need only replace GL_1 above by GL_n and add a few more words. The local Langlands correspondence for GL_n, which is a theorem, states that there is a canonical bijection between the following two objects.

(1) n-dimensional representations $W_k' \to \mathrm{GL}_n(\mathbb{C})$ (Galois side)
(2) Irreducible admissible complex representations of $\mathrm{GL}_n(k)$ (Automorphic side)

Here W_k' is the Weil-Deligne group, a mild enhancement of the Weil group when $k = \mathbb{Q}_p$.

Appendix A. Real local Langlands for GL_2

In this appendix we explain the local Langlands correspondence for the group GL_2 and the field $\mathbb{R}$. Here is the statement of the correspondence.

Theorem 2. *There is a bijection between*

(1) two-dimensional representations $W_\mathbb{R} \to \mathrm{GL}_2(\mathbb{C})$ and *(Galois side)*
(2) irreducible admissible representations of $\mathrm{GL}_2(\mathbb{R})$. *(automorphic side)*

In this section we explain the two sides of the bijection. Our treatment of the subject follows Tate [Tat79, (2.2.2)] and Kudla [Kud94].

A.1. **Representations of $W_{\mathbb{R}}$.** The representations that interest us are semisimple, meaning they decompose as a direct sum of irreducible representations. Classifying the (finite-dimensional, complex) semisimple representations of $W_{\mathbb{R}}$, amounts to classifying the irreducible representations. We have already implicitly discussed the characters of $W_{\mathbb{R}}$. They are the same as the characters of $\mathbb{R}^\times$, hence of the form

$$z \mapsto |z|^t, \quad j \mapsto \pm 1 \qquad \text{(type } (\pm, t))$$

for $t \in \mathbb{C}$.

As for the two-dimensional irreducible representations of $W_{\mathbb{R}}$, they are all induced from characters of the subgroup $\mathbb{C}^\times$. Recall that every character of $\mathbb{C}^\times$ is of the form

$$z \mapsto z^\ell |z|^t \qquad \text{(type } (\ell, t))$$

for $t \in \mathbb{C}$ and $\ell \in \mathbb{Z}$. The representation of $W_{\mathbb{R}}$ induced from this character is irreducible as long as $\ell \neq 0$, and it depends only on $|\ell|$ and t up to isomorphism, so we may assume that $\ell \geq 1$.

Finally, every irreducible representation of $W_{\mathbb{R}}$ is one- or two-dimensional. So the two-dimensional irreducible representations of $W_{\mathbb{R}}$ are

(1) type $(\varepsilon_1, t_1) \oplus (\varepsilon_2, t_2)$ with $t \in \mathbb{C}$ and $\varepsilon_i = \pm$, or
(2) type (ℓ, t) with $\ell \in \mathbb{Z}$ and $t \in \mathbb{C}$.

A.2. **Representations of $\mathrm{GL}_2(\mathbb{R})$.** Let's match up the two-dimensional representations of $W_{\mathbb{R}}$ with the irreducible admissible representations of $\mathrm{GL}_2(\mathbb{R})$.

Start with the irreducible two-dimensional representations of type (ℓ, t). These correspond to certain discrete series representations. Specifically, let $\mathrm{SL}_2(\mathbb{R})^\pm$ be the two-by-two matrices of determinant ± 1, so that

$$\mathrm{GL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})^\pm \cdot \mathbb{R}_{>0}.$$

For each integer $\ell \geq 1$ there is a representation D_ℓ^+ of $\mathrm{SL}_2(\mathbb{R})$. The underlying vector space of D_ℓ^+ is the space of analytic functions $f : \mathbb{H} \to \mathbb{C}$ such that

$$\|f\|_\ell := \int_{\mathbb{H}} |f(z)|^2 y^{\ell-1} \, \mathrm{d}x \, \mathrm{d}y < \infty.$$

The group action on D_ℓ^+ is

$$D_\ell^+(g) f(z) = (bz+d)^{-(\ell+1)} f\left(\tfrac{az+c}{bz+a}\right)$$

where $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.[1] It should not come as a surprise that this representation is closely related to holomorphic modular forms! Now form the induced representation D_ℓ of D_ℓ^+ to $\mathrm{SL}_2(\mathbb{R})^\pm$. The representation

$$D_\ell \otimes |\det(\cdot)|^t$$

of $\mathrm{GL}_2(\mathbb{R})$ corresponds to the representation of $W_{\mathbb{R}}$ of type (t, ℓ).

As for the reducible representations of type $(t_1, \varepsilon_1) \oplus (t_2, \varepsilon_2)$, we can match each of the factors with a character χ_i of $\mathbb{R}^\times$ using the local Langlands correspondence for GL_1. To complete the process, we need to transform these two characters into a representation of $\mathrm{GL}_2(\mathbb{R})$. The procedure to do so is known as parabolic induction. The characters χ_i define a

[1]The formula for the action of the group of Möbius transformations differs from the usual one here because have to take the transpose to ensure that the group acts on the right on the upper half plane, and hence on the left on functions.

character of the (Borel) subgroup $B \subset \mathrm{GL}_2(\mathbb{R})$ of matrices that vanish below the diagonal, by the formula

$$(\chi_1 \otimes \chi_2)\left(\begin{smallmatrix} a & b \\ 0 & d \end{smallmatrix}\right) = \chi_1(a)\chi_2(d).$$

Now form the representation $I(\chi_1, \chi_2)$ of $\mathrm{GL}_2(\mathbb{R})$ induced from the character $\chi_1 \otimes \chi_2$ of B.[2] It turns out that if $\mathrm{Re}\, t_1 \geq \mathrm{Re}\, t_2$ then the representation $I(\chi_1, \chi_2)$ has a unique irreducible quotient, denoted by $J(\chi_1, \chi_2)$. This quotient is the representation of $\mathrm{GL}_2(\mathbb{R})$ that corresponds to the representation of $W_{\mathbb{R}}$ of type $(t_1, \varepsilon_1) \oplus (t_2, \varepsilon_2)$.

References

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[2] It is customary here to multiply before the induction by a certain modulus character to ensure that unitarity is preserved. We omit this character for brevity of exposition.