

NONLINEAR TRACES

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ABSTRACT. We combine the theory of traces in homotopical algebra with sheaf theory in derived algebraic geometry to deduce general fixed point and character formulas. The formalism of dimension (or Hochschild homology) of a dualizable object in the context of higher algebra provides a unifying framework for classical notions such as Euler characteristics, Chern characters, and characters of group representations. Moreover, the simple functoriality properties of dimensions clarify celebrated identities and extend them to new contexts.

We observe that it is advantageous to calculate dimensions, traces and their functoriality directly in the nonlinear geometric setting of correspondence categories, where they are directly identified with (derived versions of) loop spaces, fixed point loci and loop maps, respectively. This results in universal nonlinear versions of Grothendieck-Riemann-Roch theorems, Atiyah-Bott-Lefschetz trace formulas, and Frobenius-Weyl character formulas. We can then linearize by applying sheaf theories, such as the theories of ind-coherent sheaves and \mathcal{D} -modules constructed by Gaitsgory-Rozenblyum [GR2]. This recovers the familiar classical identities, in families and without any smoothness or transversality assumptions. On the other hand, the formalism also applies to higher categorical settings not captured within a linear framework, such as characters of group actions on categories.

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1. INTRODUCTION

This paper is devoted to traces and characters in homotopical algebra and their application to algebraic geometry and representation theory. We observe that many geometric fixed point and trace formulas can be expressed as linearizations of fundamental nonlinear identities, describing dimensions and traces directly in the setting of correspondence categories of varieties or stacks. This gives a simple uniform perspective on (and useful generalizations of) geometric character and fixed point formulas of Grothendieck-Riemann-Roch and Atiyah-Bott-Lefschetz type. In addition, one can also specialize the universal geometric formulas to higher categorical settings not captured within a linear framework, such as characters of group actions on categories.

The paper is organized as follows: after a brief summary in Section 1.1, we give a detailed overview in Section 2, in three sections: first, the abstract functoriality of traces in higher category theory; second, their calculation in correspondence categories in derived algebraic geometry; and third, their specialization via sheaf theories. The rest of the paper follows the same structure with more details provided. We emphasize the formal nature and appealing simplicity of the constructions in any sufficiently derived setting. For example, in the second part, we work within derived algebraic geometry, but the statements and proofs should hold in any setting (for example, derived manifolds) with a suitable notion of fiber product to handle non-transversal intersections. The main objects appearing in trace formulas are the derived loop space (the self-intersection of the diagonal in its role as the nonlinear trace of the identity map) and more general derived fixed point loci. The importance of a derived setting also appears prominently in the third part, where the sheaf theories we apply must have good functorial properties with respect to fiber products. As a result, the theory of characters in Hochschild and cyclic homology is expressed directly by the geometry, resulting in simpler formulations. For example, the Todd genus in Grothendieck-Riemann-Roch and the denominators in the classical Atiyah-Bott formula arise naturally from derived calculations.

1.1. Summary. We now describe the main theorem, extending classical trace and dimension formulas to a very general setting in derived algebraic geometry (including equivariance for arbitrary Lie algebroids or affine algebraic groups) without any smoothness or transversality assumptions, while emphasizing that the main contribution of the paper is the simple geometric formalism underlying these formulas. For our general and formal nonlinear results, we need not assume anything about what classes of derived stacks and morphisms we work with. For applications, we need to be in a setting in which the powerful mechanism of sheaf theory is fully developed [G1, DG, GR2].

Setting 1.1. *Throughout this paper, we work over a field k of characteristic zero. A stack X connotes either*

- (1) *a QCA derived stack in the sense of [DG]. In other words X is quasicompact with affine diagonal and the underived inertia of X is finite presentation over the underlying underived stack X_{cl} .*
- (2) *an ind-inf-scheme, in the sense of [GR2]. These include (derived) schemes of finite (or locally almost of finite) type and ind-schemes built out of unions of the former along closed embeddings, as well as their quotients by arbitrary Lie algebroids, or equivalently formal groupoids.*

By a proper map we indicate a proper and schematic map, while for ind-proper indicates ind-proper and ind-inf-schematic. All appearances of proper maps in this paper may be replaced by ind-proper ones.

Thus the class of spaces we consider includes all k -derived schemes of finite type and their quotients by either Lie algebroids or finite type affine group schemes.

Given a derived stack $\pi_X : X \rightarrow \mathrm{Spec} k$, we denote by $\pi_{\mathcal{L}X} : \mathcal{L}X = \mathrm{Map}(S^1, X) \rightarrow \mathrm{Spec} k$ its derived loop space. In general, the derived loop space is a derived thickening of the inertia stack. For a map $f : X \rightarrow Y$, we will denote by $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$ the induced map on loops.

Example 1.2. For many applications, the following two special cases are noteworthy.

When X is a smooth scheme, $\mathcal{L}X \simeq \mathbb{T}_X[-1]$ is the total space of the shifted tangent space by the HKR theorem. The same holds for an arbitrary scheme, if we replace the tangent space by the tangent *complex*, see for example [BN10b]. For $f : X \rightarrow Y$ a map of schemes, $\mathcal{L}f : \mathbb{T}_X[-1] \rightarrow \mathbb{T}_Y[-1]$ is (the shift of) the usual tangent map.

When $Y = BG$ is a classifying stack, $\mathcal{L}Y \simeq G/G$ is the adjoint quotient. For X a G -scheme, and $f : X/G \rightarrow BG$ the corresponding classifying map, $\mathcal{L}f : \mathcal{L}(X/G) \rightarrow \mathcal{L}BG \simeq G/G$ is the universal family of derived fixed point loci. More precisely, for any element $g \in G$, the derived fixed point locus $X^g \subset X$ is precisely the derived fiber $X^g \simeq \mathcal{L}(X/G) \times_{G/G} \{g\}$.

We will measure stacks X by differential graded (dg) enhancements of derived categories of sheaves.

The most familiar is the assignment $X \mapsto \mathcal{Q}(X)$ of the (unbounded) category of quasicoherent sheaves. However, we will make essential use of Grothendieck-Serre duality, in the guise of an adjunction $(f_*, f^!)$ between push-forward and extraordinary pullback for proper maps X . This duality is most naturally expressed in the setting of *ind-coherent sheaves* $X \mapsto \mathcal{Q}^!(X)$ as developed in [G1, GR2]. Ind-coherent sheaves agree with quasicoherent sheaves for smooth schemes but differ on singular schemes, where [bounded complexes of] coherent sheaves (the compact objects of $\mathcal{Q}^!$) differ from perfect complexes (the compact objects of \mathcal{Q}). In other words, ind-coherent sheaves are to coherent sheaves and G -theory (the setting of Grothendieck-Riemann-Roch theorems) as quasicoherent sheaves are to perfect complexes and K -theory.

Another sheaf theory to which the general formalism developed in [GR2] applies is the theory of \mathcal{D} -modules $X \rightarrow \mathcal{D}(X)$, which for smooth schemes agrees with the classical notion of quasicoherent complexes of modules for the sheaf of differential operators \mathcal{D}_X (i.e. with compact objects given by bounded coherent complexes of \mathcal{D}_X -modules). In general [GR1, GR2] $\mathcal{D}(X)$ is defined as the category of crystals, i.e., as ind-coherent complexes on the de Rham space of X

$$\mathcal{D}(X) := \mathcal{Q}^!(X_{dR}).$$

The compact objects in $\mathcal{D}(X)$ for X a scheme are the coherent \mathcal{D} -modules, while for X a stack they form a smaller class, the *safe* \mathcal{D} -modules of [DG].

The book [GR2] develops the theories $\mathcal{Q}^!$ and \mathcal{D} in particular as functors out of 2-categories of correspondences of schemes and stacks, with 1-morphisms from X to Y given by correspondences representable over Y and 2-morphisms given by ind-proper ind-schematic morphisms of correspondences. This theory encodes a huge amount of structure, including in particular pull-back and pushforward functors $f^!$ and f_* satisfying base change, as well as the $(f_*, f^!)$ adjunction for proper (or even ind-proper) maps. They also establish symmetric monoidal properties of the sheaf theory.¹ (See Sections 2.3 and 5 for more background and precise statements.)

Let $\mathcal{S} = \mathcal{Q}^!$ or $\mathcal{S} = \mathcal{D}$ denote either of these sheaf theories.

¹We will also need from [DG] the construction of continuous pushforward functors for all maps of QCA stacks (such as the non-representable projection $\pi_X : X \rightarrow pt$ from a stack) and their base-change property.

We let $\omega_X = \pi_X^! \mathcal{O}_{\mathrm{Spec} k} \in \mathcal{S}(X)$ denote the appropriate dualizing sheaf. Thus for ind-coherent sheaves, $\omega_X \in \mathcal{Q}^!(X)$ is the algebraic dualizing sheaf, and for \mathcal{D} -modules, $\omega_X \in \mathcal{D}(X)$ is the Verdier dualizing sheaf. Let $\omega(X) = \pi_{X*} \omega_X$ denote the corresponding complex of global volume forms: for ind-coherent sheaves, $\omega(X) \in k\text{-mod}$ consists of algebraic volume forms, and for \mathcal{D} -modules, $\omega(X) \in k\text{-mod}$ consists of locally constant distributions (Borel-Moore chains) for X a scheme.

For X a stack we use the continuous “renormalized” pushforward functor on \mathcal{D} -modules of [DG], which roughly replaces equivariant cohomology (derived invariants) by a shift of equivariant homology (derived coinvariants, see [DG, Example 9.1.6]), so that $\omega(Y/G)$ for a G -variety Y is given by a shift of G -coinvariants on Borel-Moore chains on Y . (Note that even for $X = BG$ this differs from the standard definition of equivariant Borel-Moore homology, which is identified in this case with equivariant cohomology.)

For a proper (or ind-proper) map $f : X \rightarrow Y$, adjunction provides an integration map $\int_f : \omega(X) \rightarrow \omega(Y)$.

Example 1.3. Let us continue with the special cases of Example 1.2, and focus in particular on algebraic distributions $\omega_{\mathcal{L}X} \in \mathcal{Q}^!(\mathcal{L}X)$ on the loop space.

When X is a smooth scheme, $\mathcal{L}X \simeq \mathbb{T}_X[-1]$ is naturally Calabi-Yau, and its global volume forms are identified with differential forms $\omega(\mathbb{T}_X[-1]) \simeq \mathcal{O}(\mathbb{T}_X[-1]) \simeq \mathrm{Sym}^\bullet(\Omega_X[1])$. The canonical “volume form” on $\mathcal{L}X$ is given by the Todd genus (as explained by Markarian [Ma]): the resulting integration of functions on $\mathcal{L}X$ differs from the integration of differential forms on X by the Todd genus.

When $Y = BG$ is a classifying stack, $\mathcal{L}Y \simeq G/G$ is naturally Calabi-Yau, and its global volume forms are invariant functions $\omega(G/G) \simeq \mathcal{O}(G/G) \simeq \mathcal{O}(G)^G$. If G is reductive with Cartan subgroup $T \subset G$ and Weyl group W , the naive invariants $\mathcal{O}(G)^G \simeq \mathcal{O}(T)^W$ are equivalent to the derived invariants, but in general there may be higher cohomology.

Theorem 1.4. *Let $\mathcal{S} = \mathcal{Q}^!$ or $\mathcal{S} = \mathcal{D}$ denote either the theory of ind-coherent sheaves or \mathcal{D} -modules. Recall our conventions for stacks and morphisms, Setting 1.1.*

- *For a stack X , there is a canonical identification $HH_*(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X)$ of the Hochschild homology of sheaves on X with distributions (or renormalized Borel-Moore chains) on the loop space.*

- *For a proper (or ind-proper) map of stacks $f : X \rightarrow Y$ the induced map $HH_*(\mathcal{S}(X)) \rightarrow HH_*(\mathcal{S}(Y))$ is given by integration along the loop map $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$.*

- **Grothendieck-Riemann-Roch:** *In particular, for any compact object $M \in \mathcal{S}(X)$ (coherent sheaf or safe coherent \mathcal{D} -module) with character $[M] \in HH_*(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X)$, there is a canonical identification*

$$[f_* M] \simeq \int_{\mathcal{L}f} [M] \in HH_*(\mathcal{S}(Y)) \simeq \omega(\mathcal{L}Y)$$

In other words, the character of a pushforward along a proper map is the integral of the character along the induced loop map.

- **Atiyah-Bott-Lefschetz:** *Let G be an affine group, and X a proper stack with G -action, or equivalently, a proper map $f : X/G \rightarrow BG$. Then for any compact object $M \in \mathcal{S}(X/G)$ (G -equivariant coherent sheaf or safely equivariant coherent \mathcal{D} -module on X), and element $g \in G$, there is a canonical identification*

$$[f_* M]|_g \simeq \int_{\mathcal{L}f} [M]|_{X^g}$$

In other words, under the identification of invariant functions and volume forms on the group, the value of the character of an induced representation at a group element is given by the integral of the original character along the corresponding fixed point locus of the group element.

• **Extension to traces:** The trace $\text{Tr}(\mathcal{S}(Z))$ of the endofunctor of $\mathcal{S}(X)$ given by a self-correspondence (e.g. a self-map) $X \leftarrow Z \rightarrow X$ is given by distributions on the fixed points $\omega(Z|_{\Delta})$.

• For a map $f : (X, Z) \rightarrow (Y, W)$ of stacks with self-correspondences², the induced map $\text{Tr}(\mathcal{S}(Z)) \rightarrow \text{Tr}(\mathcal{S}(W))$ is given by integration along fixed points $Z|_{\Delta_X} \rightarrow W|_{\Delta_Y}$.

Example 1.5 (Frobenius-Weyl Character Formula). Here is a reminder of a well-known application of the Atiyah-Bott-Lefschetz formula in representation theory.

• If G is a finite group, and $X = G/K$ is a homogeneous set, and $M = k[G/K]$ the ring of functions, one recovers the Frobenius character formula for the induced representation $k[G/K]$.

• If G is a reductive group, $X = G/B$ is the flag variety, $X/G = pt/B \rightarrow pt/G$. The loop map

$$\mathcal{L}(X/G) = B/B \simeq \tilde{G}/G \rightarrow G/G$$

is the (group) Grothendieck-Springer simultaneous resolution, with fibers giving fixed point loci on the flag variety. For $M = \mathcal{L}$ an equivariant line bundle on G/B , and $g \in G$ runs over a maximal torus, one recovers the Weyl character formula for the induced representation $H^*(G/B, \mathcal{L})$.

Remark 1.6. The reader will note no explicit appearance of the Todd genus in the above formulas. In other words as for K-theory, pushforward of sheaves naturally agrees with the pushforward in Hochschild homology. The Todd genus arises in comparing these natural pushforwards with the pushforward in cohomology, i.e., integration of forms. It arises when one unwinds the integration map $\int_{\mathcal{L}_f} : \omega(\mathcal{L}X) \rightarrow \omega(\mathcal{L}Y)$, given by Grothendieck duality, in terms of functions (or differential forms) using the Hochschild-Kostant-Rosenberg theorem. In particular, the familiar denominators in the Atiyah-Bott formula are implicit in the integration measure on the fixed point locus.

For instance, as mentioned above, when X is a smooth scheme, a geometric version of the HKR theorem asserts that the loop space is the total space of the shifted tangent complex $\mathcal{L}X \simeq \mathbb{T}_X[-1]$, and global volume forms are canonically functions $\omega(\mathcal{L}X) \simeq \mathcal{O}(\mathbb{T}_X[-1]) \simeq \text{Sym}^\bullet(\Omega_X[1])$. Under this identification (as explained by Markarian [Ma]), the resulting integration of functions on $\mathcal{L}X$ differs from the integration of differential forms on X by the Todd genus.

Remark 1.7. The paper [KP] carries out the program described in this paper (i.e. recovering classical identities from non-linear ones) by calculating explicitly the derived contributions in the case of the Atiyah-Bott formula.

Remark 1.8. The main contribution of this paper is hidden in the statement of this theorem: we establish nonlinear versions of character formulas in the setting of derived stacks, and deduce classical formulas and new higher categorical analogues formally by applying suitable sheaf theories. Thanks to the great generality of sheaf theory in derived algebraic geometry [GR2], the resulting applications hold with remarkably few assumptions.

We are particularly interested in the higher categorical variants where one considers sheaves of categories, in particular Frobenius-Weyl character formulas for group actions on categories. Since the requisite foundations are not yet fully developed, we postpone details of this to future

²i.e., we lift f to an identification $Z \simeq X \times_Y W$ of correspondences from X to Y

works. Applications include an identification of the character of the category of \mathcal{D} -modules on the flag variety with the Grothendieck-Springer sheaf, and of the trace of a Hecke functor on the category of \mathcal{D} -modules on the moduli of bundles on a curve with the cohomology of a Hitchin space.

1.2. Inspirations and motivations. This work has many inspirations. It is heavily reliant on the ∞ -categorical foundations of higher algebra, derived algebraic geometry and sheaf theory due to Lurie [L2, L4] and Gaitsgory-Rozenblyum [GR2]. It is also inspired by Lurie’s cobordism hypothesis with singularities [L3], which provides a powerful unifying tool for higher algebra. Already in the setting of one-dimensional field theory, this result can be viewed as a vast generalization of the classical theory Hochschild and cyclic homology and characters therein [Lo], (in particular the natural cyclic symmetry of Hochschild homology is generalized to a circle action on the dimensions of arbitrary dualizable objects). In particular, the formal properties of traces we use are simple instances of the cobordism hypothesis with singularities on marked intervals and cylinders. The work of Toën and Vezzosi [TV] on traces and higher Chern characters of sheaves of categories (and in particular the role of the cobordism hypothesis therein) has also profoundly influenced our thinking.

Another important inspiration is the categorical theory of strong duality, dimensions and traces introduced by Dold and Puppe in [DP] (see [M, PS] for more recent developments) with the express purpose of proving Lefschetz-type formulas. In [DP], dualizability of a space is achieved by linearization (passing to suspension spectra), while our approach is to pass to categories of correspondences (or spans) instead. We were also inspired by the preprint [Ma] and the subsequent work [Cal1, Cal2, Ram, Ram2, Shk]. There have been many recent papers [Pe, Lu, Po, CT] building on related ideas to prove Riemann-Roch and Lefschetz-type theorems in the noncommutative context of differential graded categories and Fourier-Mukai transforms; our work instead places these results in the context of the general formalism of traces in ∞ -categories, and generalizes them to commutative but nonlinear settings.

The Grothendieck-Riemann-Roch type applications in this paper concern the character map taking coherent sheaves to classes in Hochschild homology (or in a more refined version, to cyclic homology). This is significantly coarser than the well established theory of Lefschetz-Riemann-Roch theorems valued in Chow groups (see the seminal [Th], the more recent [Jo] and many references therein). Thus for schemes, the quantities compared are Dolbeault (or de Rham) cohomology classes rather than algebraic cycles or K - (or rather G -)theory classes.

Our primary motivation is the development of foundations for “homotopical harmonic analysis” of group actions on categories, aimed at decomposing derived categories of sheaves (rather than classical function spaces) under the actions of natural operators. This undertaking follows the groundbreaking path of Beilinson-Drinfeld within the geometric Langlands program and is consonant with general themes in geometric representation theory. The pursuit of a geometric analogue of the Arthur-Selberg trace formula by Frenkel and Ngô [FN] has also been a source of inspiration and applications.

Remark 1.9. A companion paper [BN13] presents an alternative approach to Atiyah-Bott-Lefschetz formulas (and in particular a conjecture of Frenkel-Ngô) as a special case of the “secondary trace formula” identifying trace invariants associated to two commuting endomorphisms of a sufficiently dualizable object. This is also applied to establish the symmetry of the 2-class functions on a group constructed as the 2-characters of categorical representations.

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2. OVERVIEW

2.1. Traces in category theory. We highlight structures arising in the general theory of dualizable objects in symmetric monoidal higher categories (see also [DP, M, PS]). For legibility, we suppress all ∞ -categorical notations and complications from the introduction. We rely on [GR2] for the theory of symmetric monoidal $(\infty, 2)$ -categories, though only the formal outline of the theory is in fact needed for this paper. See [TV, HSS] for thorough treatments of the theory of traces in higher category theory.

The basic notion in the theory is that of *dimension* of a dualizable object of a symmetric monoidal category \mathcal{A} . By definition, for such an object A there exists another A^\vee together with a coevaluation map η_A and evaluation map ϵ_A satisfying standard identities. By definition, the dimension of A is the endomorphism of the the unit $1_{\mathcal{A}}$ given by the composition

$$1_{\mathcal{A}} \xrightarrow{\eta_A} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_{\mathcal{A}} \quad \text{dim}(A)$$

Example 2.1. For V a vector space, $V^\vee = \text{Hom}_k(V, k)$ is the vector space of functionals, $\epsilon_V : V \otimes V^\vee \rightarrow k$ is the usual evaluation of functionals, $\eta_V : k \rightarrow \text{End}(V) \simeq V \otimes V^\vee$ is the identity map (which exists only for V finite-dimensional), and $\text{dim}(V)$ can be regarded as an element of the ground field (by evaluating it on the multiplicative unit).

Remark 2.2 (Duality and naiv  t   in ∞ -categories.). It is a useful technical observation that the notion of dualizability in the setting of ∞ -categories is a “naive” one: it is a property of an object that can be checked in the underlying homotopy category. As a result, all of the categorical and 2-categorical calculations in this paper are similarly naive and explicit (and analogous to familiar unenriched categorical assertions), involving only small amounts of data that can be checked by hand.

The notion of dimension is a special case of the *trace* of an endomorphism Φ of a dualizable object A . By definition, the trace of Φ is the endomorphism of the unit $1_{\mathcal{A}}$ given by the composition

$$1_{\mathcal{A}} \xrightarrow{\eta_A} A \otimes A^\vee \xrightarrow{\Phi \otimes \text{id}_{A^\vee}} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_{\mathcal{A}} \quad \text{Tr}(\Phi)$$

which recovers the dimension for $\Phi = \text{id}_A$.

A key feature of dimensions and traces is their *cyclicity*, which at the coarsest level is expressed by a canonical equivalence

$$m(\Phi, \Psi) : \text{Tr}(\Phi \circ \Psi) \xrightarrow{\sim} \text{Tr}(\Psi \circ \Phi),$$

see Proposition 3.15. At a much deeper level, an important corollary of the cobordism hypothesis [L3] is the existence of an S^1 -action on $\text{dim}(A)$ for any dualizable object A (and an analogous structure for general traces, see Remark 3.28).

Remark 2.3 (Dimensions and traces are local). It is useful for applications to note that the notion of dualizability and the definition of dimension and are local in the category \mathcal{A} . Namely, they only require knowledge of the objects $1_{\mathcal{A}}, A, A^\vee, A \otimes A^\vee$, the morphisms η_A, ϵ_A , and

standard tensor product and composition identities among them. Likewise, the notion of trace only requires the additional endomorphism Φ along with a handful of additional identities.

2.1.1. Functoriality of traces. Now suppose the ambient symmetric monoidal category \mathcal{A} underlies a 2-category, so there is the possibility of noninvertible 2-morphisms. This allows for the notion of left and right adjoints to morphisms. Let us say a morphism $A \rightarrow B$ is *continuous*, or *right dualizable*, if it has a right adjoint. (The terminology derives from the setting of presentable categories, where the adjoint functor theorem guarantees the existence of right adjoints for colimit preserving functors.)

Here are natural functoriality properties of dimensions and traces.

Proposition 2.4. *Let A, B denote dualizable objects of \mathcal{A} and $f_* : A \rightarrow B$ a continuous morphism with right adjoint $f^!$.*

- (1) *There is a canonical map on dimensions*

$$\dim(A) \xrightarrow{=} \mathrm{Tr}(\mathrm{Id}_A) \longrightarrow \mathrm{Tr}(f^! f_*) \xrightarrow{\sim} \mathrm{Tr}(f_* f^!) \longrightarrow \mathrm{Tr}(\mathrm{Id}_B) \xrightarrow{=} \dim(B)$$

$\dim(f_*)$

compatible with compositions of continuous morphisms.

- (2) *Given endomorphisms $\Phi \in \mathrm{End}(A)$, $\Psi \in \mathrm{End}(B)$, and a commuting structure*

$$\alpha : f_* \circ \Phi \xrightarrow{\sim} \Psi \circ f_*$$

there is a canonical map on traces

$$\mathrm{Tr}(f_*, \alpha) : \mathrm{Tr}(\Phi) \longrightarrow \mathrm{Tr}(\Psi)$$

compatible with compositions of continuous morphisms with commuting structures.

We refer to the compatibility with compositions stated in the proposition as *abstract Grothendieck-Riemann-Roch*. To see its import more concretely, let us restrict the generality and focus on an *object* of \mathcal{A} in the sense of a morphism $V : 1_{\mathcal{A}} \rightarrow A$.

Corollary 2.5. *Let A, B denote dualizable objects of \mathcal{A} and $f_* : A \rightarrow B$ a continuous morphism. For $V : 1_{\mathcal{A}} \rightarrow A$ an object of \mathcal{A} , we obtain a map on dimensions*

$$\dim(V) : 1_{\mathcal{A}} \simeq \dim(1_{\mathcal{A}}) \longrightarrow \dim(A)$$

called the character of V and alternatively denoted by $[V]$. It satisfies abstract Grothendieck-Riemann-Roch in the sense that the following diagram commutes

$$\begin{array}{ccccc} 1_{\mathcal{A}} & \xrightarrow{[V]} & \dim(A) & \xrightarrow{\dim(f_*)} & \dim(B) \\ & \searrow [f_* V] & & & \nearrow \end{array}$$

Remark 2.6 (Functoriality of dimensions and traces is local). As in Remark 2.3, it is useful to note that the functoriality of dimension is local, depending only on a handful of objects, morphisms and identities, along with the additional adjunction data $(f_*, f^!)$. A similar observation applies to the functoriality of traces.

Remark 2.7. It follows from the cobordism hypothesis with singularities [L3] (see [TV]) that the morphism $\dim(f_*)$ is S^1 -equivariant, and hence the character $[V]$ is S^1 -invariant, though we will not elaborate on this structure here. We refer to [HSS] for a thorough study of the functoriality and cyclicity of traces.

Example 2.8. Let $dgCat_k$ denote the symmetric monoidal ∞ -category of presentable k -linear differential graded categories (or alternatively, stable presentable k -linear ∞ -categories), see e.g. [GR2]. In this setting, any compactly generated category A is dualizable, and its dimension is the Hochschild chain complex $\dim(A) = HH_*(A)$. The S^1 -action on $\dim(A)$ corresponds to Connes' cyclic structure on $HH_*(A)$, so that in particular, the localized S^1 -invariants of $\dim(A)$ form the periodic cyclic homology of A .

More generally, the trace of an endofunctor $\Phi : A \rightarrow A$ is the Hochschild homology $\mathrm{Tr}(\Phi) = HH_*(A, \Phi)$. For example, if $A = R\text{-mod}$ for a dg algebra R , then Φ is represented by an R -bimodule M , and we recover the Hochschild homology $HH_*(R, M)$.

Any compact object $M \in A$ defines a continuous functor

$$1_{dgCat_k} = dgVect_k \xrightarrow{M} A$$

whose character is a vector

$$\dim(M) \in HH_*(A)$$

in Hochschild homology (with refinement in cyclic homology). The abstract Grothendieck-Riemann-Roch theorem expresses the natural functoriality of characters in Hochschild homology (or their refinement in cyclic homology). In fact, the construction of characters factors through the canonical Dennis trace map

$$A_{cpt} \longrightarrow K(A) \longrightarrow HH_*(A)$$

from the space A_{cpt} of compact objects of A .

2.2. Traces in geometry. To apply the preceding formalism to geometry, it is useful to organize spaces and maps within a suitable categorical framework. We then arrive at loop spaces and fixed point loci as nonlinear expressions of dimensions and traces. This simple observation provides the core of the paper. Throughout the discussion, we continue to suppress all ∞ -categorical notations and complications. Our reference for the correspondence 2-category of stacks is Section V of [GR2] (see also [H1]).

To begin, consider the general setup of the symmetric monoidal category $Corr(\mathcal{C})$ of correspondences, or spans, in a category \mathcal{C} such as stacks (or formally a symmetric monoidal ∞ -category with finite limits; see [Ba, H1]). Here the objects $X \in Corr(\mathcal{C})$ are the objects of \mathcal{C} , the morphisms $Corr_{\mathcal{C}}(X, Y)$ are arbitrary spans in \mathcal{C} ,

$$\begin{array}{ccc} & Z & \\ \swarrow & & \searrow \\ X & & Y \end{array}$$

(more generally one can require the left and right legs to live in specified subcategories of \mathcal{C} as in [GR2]). The composition of morphisms $Z \in Corr_{\mathcal{C}}(X, Y)$ and $W \in Corr_{\mathcal{C}}(Y, U)$ is given by fiber product

$$\begin{array}{ccccc} & & Z \times_Y W & & \\ & \swarrow & & \searrow & \\ & Z & & W & \\ \swarrow & & \searrow & & \swarrow \\ X & & Y & & U \end{array}$$

and the symmetric monoidal structure is given in terms of that on \mathcal{C} (Cartesian product in the case of stacks). For the purpose of calculating dimensions and traces, we need not require any further properties of the spaces of $\text{Corr}(\mathcal{C})$, since we need only the modest local data discussed in Remarks 2.3 and 2.6. (See [L3] and [FHLT], where the higher categories Fam_n of iterated correspondences of manifolds are constructed and applied.)

With applications in mind, we will specialize to the correspondence category $\text{Corr}_k = \text{Corr}(\text{St}_k)$ of derived stacks over k . It would also be interesting to work with smooth manifolds instead, for example through the theory of C^∞ -stacks [J] (see Remark 2.20).

It is natural to enhance $\text{Corr}(\mathcal{C})$ to a 2-category $\underline{\text{Corr}}(\mathcal{C})$ by allowing non-invertible maps, or more generally correspondences, between correspondences (see Section V of [GR2] or [H1] for full details), so that maps from X to Y form the category of objects over $X \times Y$. Our constructions naturally fit into the 2-category $\underline{\text{Corr}}_k^{\text{prop}}$ with non-invertible 2-morphisms restricted to be *proper* (or more generally ind-proper) maps of correspondences.

Remark 2.9 (Correspondences are bimodules). It is useful to view the correspondence category Corr within the framework of coalgebras in symmetric monoidal categories. The diagonal map $X \rightarrow X \times X$ makes any space or stack into a cocommutative coalgebra object with respect to the Cartesian product monoidal structure (or commutative coalgebra in the opposite category). Moreover, a map $Z \rightarrow X$ is equivalent to an X -comodule structure on Z . Thus correspondences from X to Y may be interpreted as X – Y -bicomodules, with composition of correspondences given by tensor product of bicomodules.

Furthermore, it is natural to enhance Corr to a 2-category by allowing non-invertible maps between correspondences. This can be viewed as a special case of the Morita category of coalgebras in a symmetric monoidal category. The 2-category $\underline{\text{Corr}}$ of spaces, correspondences, and maps of correspondences is the Morita category on spaces regarded as coalgebra objects. (In particular, the cocommutativity of the coalgebra objects implies they are canonically self-dual, and the transpose of a correspondence is the same correspondence read backwards.) If we further keep track of the E_n -coalgebra structure of spaces and consider the corresponding Morita $(n+1)$ -category, we recover the $(n+1)$ -category of iterated correspondences of correspondences. (See [H1, H2] for a thorough treatment of categories of spans and Morita categories of E_n algebras; see also for example the category Fam_n of [L3] and [FHLT] in the topological setting.)

2.2.1. Geometric dimensions and loop spaces. A crucial feature of the category Corr_k is that any object $X \in \text{Corr}_k$ is dualizable (in fact, canonically self-dual), thanks to the diagonal correspondence.³ Note for this it is crucial that we allow all maps, including the map $\pi_X : X \rightarrow pt$ for any X , as possible legs in a span.

We have the following calculations of dimensions and their functoriality. Note that the point $pt = \text{Spec } k$ is the unit of Corr_k . We keep track of properness of maps of correspondences for the later application of sheaf theory.

Proposition 2.10. *Let Corr_k be the category of derived stacks and correspondences, and $\underline{\text{Corr}}_k^{\text{prop}}$ the 2-category of derived stacks, correspondences, and proper maps of correspondences.*

(1) *Any derived stack X is dualizable as an object of Corr_k , and its dimension $\dim(X)$ is identified with the loop space*

$$\mathcal{L}X = X^{S^1} \simeq X \times_{X \times X} X$$

regarded as a self-correspondence of $pt = \text{Spec } k$.

³Likewise, if we wish to make a space n -dualizable for any n we may simply consider it as an object of a higher correspondence category as in Remark 2.9, since E_n -(co)algebras are $n+1$ -dualizable objects of the corresponding Morita category. In other words, a space X defines a topological field theory of any dimension valued in the appropriate correspondence category.

(2) A map $f : X \rightarrow Y$ regarded as a correspondence from X to Y is continuous in $\underline{\text{Corr}}_k^{\text{prop}}$ if and only if f is proper. Given a proper map $f : X \rightarrow Y$, its induced map

$$\dim(f) : \dim(X) \longrightarrow \dim(Y)$$

is identified with the loop map

$$\mathcal{L}f : \mathcal{L}X \longrightarrow \mathcal{L}Y$$

Remark 2.11. All of the objects and maps of the proposition have natural S^1 -actions, on the one hand coming from loop rotation, on the other hand coming from the cyclic symmetry of dimensions. One can check that the identifications of the proposition are S^1 -equivariant (see Remark 4.2).

Remark 2.12. Recall [To2, BN10b] that for a derived scheme X , the loop space $\mathcal{L}X \simeq T_X[-1]$ is the total space of the shifted tangent complex. The action map of the S^1 -rotation action is encoded by the de Rham differential. For an underived stack X , the loop space is a derived enhancement of the inertia stack $IX = \{x \in X, \gamma \in \text{Aut}(x)\}$. The action map of the S^1 -rotation action is manifested by the “universal automorphism” of any sheaf on $\mathcal{L}X$.

Example 2.13. Let G denote an algebraic group and $BG = pt/G$ its classifying space. There is a canonical identification $\mathcal{L}BG \simeq G/G$ of the loop space and adjoint quotient.

Suppose we are given a G -derived stack X , or equivalently a morphism $\pi : X/G \rightarrow BG$, from which one recovers $X \simeq X/G \times_{BG} pt$. (Note that if we want π proper we should take X itself proper.)

Let us explain how the loop map $\mathcal{L}\pi : \mathcal{L}(X/G) \rightarrow \mathcal{L}(BG)$ captures the fixed points of G acting on X . For any self-map $g : X \rightarrow X$, let us write X^g for the derived fixed point locus given by the derived intersection

$$X^g = \Gamma_g \times_{X \times X} X.$$

of the graph $\Gamma_g \subset X \times X$ with the diagonal. Then $\mathcal{L}\pi$ map fits into a commutative square

$$\begin{array}{ccc} \mathcal{L}(X/G) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \\ \sim \downarrow & & \sim \downarrow \\ \{g \in G, x \in X^g\}/G & \xrightarrow{p} & G/G \end{array}$$

where p projects to the group element.

In particular, fix a group element $g \in G$, with conjugacy class $\mathbb{O}_g \subset G$, and centralizer $Z_G(g) \subset G$, so that $\mathbb{O}_g/G \simeq BZ_G(g) \in G/G$. Then the corresponding fiber of $\mathcal{L}\pi$ is the equivariant fixed point locus $X_G^g = X^g/Z_G(g)$, or in other words we have a fiber diagram

$$\begin{array}{ccc} X_G^g & \longrightarrow & \mathbb{O}_g/G \\ \downarrow & & \downarrow \\ \mathcal{L}(X/G) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \end{array}$$

Let us specialize to the case of a subgroup $K \subset G$, and the quotient $X = G/K$, so that we have a map of classifying stacks $\pi : BK \simeq G \backslash (G/K) \rightarrow BG$. Here the loop map $\mathcal{L}\pi$ realizes

the familiar geometry of the Frobenius character formula

$$\begin{array}{ccc} \mathcal{L}(BK) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \\ \sim \downarrow & & \sim \downarrow \\ K/K \simeq \{g \in G, x \in (G/K)^g\}/G & \xrightarrow{p} & G/G \end{array}$$

The equivariant fixed point loci express the equivariant inclusion of conjugacy classes.

Specializing further, for G a reductive group, $B \subset G$ a Borel subgroup, and $X = G/B$ the flag variety, we recover the group-theoretic Grothendieck-Springer resolution

$$\begin{array}{ccc} \mathcal{L}(BB) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \\ \sim \downarrow & & \sim \downarrow \\ B/B \simeq \{g \in G, x \in (G/B)^g\}/G & \xrightarrow{p} & G/G \end{array}$$

2.2.2. Geometric traces of correspondences. More generally, we have the following calculations of traces and their functoriality.

Proposition 2.14. *Let Corr_k be the category of derived stacks and correspondences, and $\underline{\text{Corr}}_k^{\text{prop}}$ the 2-category of derived stacks, correspondences, and proper maps of correspondences.*

(1) *The trace of a self-correspondence $Z \in \underline{\text{Corr}}_k^{\text{prop}}(X, X)$ is its fiber product with the diagonal*

$$\text{Tr}(Z) \simeq Z|_{\Delta} = Z \times_{X \times X} X \simeq Z \times_X \mathcal{L}X$$

In particular, for the graph $\Gamma_f \rightarrow X \times X$ of a self-map $f : X \rightarrow X$, its trace is the fixed point locus of the map

$$\text{Tr}(\Gamma_f) \simeq X^f = \Gamma_f \times_{X \times X} X$$

(2) *Given a proper map $f : X \rightarrow Y$ regarded as a correspondence from X to Y , and self-correspondences $Z \in \underline{\text{Corr}}_k^{\text{prop}}(X, X)$ and $W \in \underline{\text{Corr}}_k^{\text{prop}}(Y, Y)$, together with an identification*

$$\alpha : Z \xrightarrow{\sim} X \times_Y W$$

of correspondences from X to Y , the induced abstract trace map

$$\text{Tr}(f, \alpha) : \text{Tr}(Z) \longrightarrow \text{Tr}(W)$$

is equivalent to the induced geometric map

$$\tau(f, \alpha) : Z|_{\Delta_X} \longrightarrow W|_{\Delta_Y}$$

2.3. Trace formulas via sheaf theories. Given any sufficiently functorial method of measuring derived stacks, the preceding calculations of geometric dimensions, traces and their functoriality immediately lead to trace and character formulas. To formalize the functoriality needed, we will use the language of sheaf theories. Broadly speaking, a sheaf theory is a representation (symmetric monoidal functor out) of a correspondence category in the way a topological field theory is a representation of a cobordism category.⁴ It provides an approach to encoding the standard operations on coherent sheaves and \mathcal{D} -modules, developed by Gaitsgory and Rozenblyum in the book [GR2] (following a suggestion of Lurie and previous versions in [FG, G1, DG, GR1]).

⁴Indeed a typical mechanism to construct “Lagrangian” field theories is as the composition of a sheaf theory with a “classical field theory” as in [FHLT, H1], a symmetric monoidal functor from a cobordism category to a correspondence category.

We will take the target of our sheaf theories to be the linear setting of the symmetric monoidal $(\infty, 2)$ -category \underline{dgCat}_k of presentable k -linear differential graded categories with continuous functors and natural transformations. (Recall from Setting 1.1 that for applications all stacks are assumed to be QCA or ind-inf-schemes. Also all proper maps can be replaced by ind-proper ones at no cost.)

Definition 2.15. A *sheaf theory* is a symmetric monoidal functor of $(\infty, 2)$ -categories

$$\underline{\mathcal{S}} : \underline{Corr}_k^{prop} \longrightarrow \underline{dgCat}_k$$

from correspondences of stacks (with 2-morphisms given by proper maps of correspondences) to dg categories. We denote by

$$\mathcal{S} : Corr_k \longrightarrow dgCat_k$$

the underlying 1-categorical sheaf theory, i.e. the symmetric monoidal functor on $(\infty, 1)$ -categories obtained by forgetting noninvertible morphisms.

Let us first spell out some of the structure encoded in a 1-categorical sheaf theory \mathcal{S} .

The graph of a map of stacks $f : X \rightarrow Y$ provides a correspondence from X to Y and a correspondence from Y to X . We denote the respective induced maps by $f_* : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ and $f^! : \mathcal{S}(Y) \rightarrow \mathcal{S}(X)$. For $\pi : X \rightarrow pt = \text{Spec } k$, we denote by $\omega_X = \pi^! k \in \mathcal{S}(X)$ the \mathcal{S} -analogue of the dualizing sheaf, and by $\omega(X) = \pi_* \omega_X \in \mathcal{S}(pt) = dgVect_k$ the \mathcal{S} -analogue of “global volume forms”. We adopt traditional notations whenever possible, for example writing $\Gamma(X, \mathcal{F}) = \pi_*(\mathcal{F})$, for $\mathcal{F} \in \mathcal{S}(X)$.

The functoriality of \mathcal{S} concisely encodes base change for f_* and $f^!$. Its symmetric monoidal structure provides equivalences

$$\mathcal{S}(X \times Y) \simeq \mathcal{S}(X) \otimes \mathcal{S}(Y),$$

as well as a symmetric monoidal structure on $\mathcal{S}(X)$ for any X (using pullback along diagonal maps). The 2-categorical extension $\underline{\mathcal{S}}$ further encodes an identification of $f^!$ with the right adjoint of f_* for f proper.

Since a sheaf theory \mathcal{S} is symmetric monoidal, it is automatically compatible with dimensions and traces: for any $X \in Corr_k$, and any endomorphism $Z \in Corr_k(X, X)$, we have

$$\dim(\mathcal{S}(X)) \simeq \mathcal{S}(\dim(X)) \quad \text{Tr}(\mathcal{S}(Z)) \simeq \mathcal{S}(\text{Tr}(Z))$$

Let us combine this with the calculation of the right hand sides and highlight specific examples of interest.

Proposition 2.16. *Fix a sheaf theory $\mathcal{S} : Corr_k \rightarrow dgCat_k$.*

(1) *The \mathcal{S} -dimension $\dim(\mathcal{S}(X)) = HH_*(\mathcal{S}(X))$ of any $X \in Corr_k$ is S^1 -equivariantly equivalent with \mathcal{S} -global volume forms on the loop space*

$$\dim(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X).$$

In particular, for G an affine algebraic group, characters of \mathcal{S} -valued G -representations are adjoint-equivariant \mathcal{S} -global volume forms

$$\dim(\mathcal{S}(BG)) \simeq \omega(G/G)$$

(2) *The \mathcal{S} -trace of any endomorphism $Z \in Corr_k(X, X)$ is equivalent to \mathcal{S} -global volume forms on the restriction to the diagonal*

$$\text{Tr}(\mathcal{S}(Z)) \simeq \omega(Z|_{\Delta})$$

In particular, the \mathcal{S} -trace of a self-map $f : X \rightarrow X$ is equivalent to \mathcal{S} -volume forms on the f -fixed point locus

$$\mathrm{Tr}(f_*) \simeq \omega(X^f)$$

Remark 2.17 (Local sheaf theory). To apply this proposition, far less structure than a full sheaf theory is required. We only need the data of the functor \mathcal{S} on the handful of objects and morphisms involved in the construction of dimensions and traces as in Remark 2.3. In particular, we only need base change isomorphisms for pullback and pushforward along specific diagrams, rather than the general base change provided by a functor out of Corr_k . This is often easy to verify in practice, in particular for the examples \mathcal{Q} , $\mathcal{Q}^!$ and \mathcal{D} (see for example [BFN] for the quasicohherent setting).

2.3.1. *Examples of sheaf theories.* As we explain in Section 5, the work [GR2] (combined with essential results from [DG]) construct two sheaf theories $\mathcal{Q}^!$ and \mathcal{D} :

- Theory $\mathcal{Q}^!$: the theory of ind-coherent sheaves $\mathcal{Q}^!(X)$. This is the “large” version $\mathcal{Q}^!(X) = \mathrm{Ind} \, \mathrm{Coh}(X)$ of the category of coherent sheaves, which by definition are the compact objects in $\mathcal{Q}^!(X)$. (For smooth X , ind-coherent and quasicohherent sheaves are equivalent.) Maps are given by the standard pushforward f_* and exceptional pullback $f^!$. The $\mathcal{Q}^!$ -dualizing sheaf is the usual dualizing complex ω_X , and (for X proper) the $\mathcal{Q}^!$ -global volume forms are its sections $R\Gamma(X, \omega_X) = R\Gamma(X, \mathcal{O}_X)^*$. The K -theory of $\mathcal{Q}^!(X)$ is algebraic G -theory $G(X)$, the homological version of algebraic K -theory for potentially singular spaces suited to Grothendieck-Riemann-Roch theorems.

- Theory \mathcal{D} : the theory of \mathcal{D} -modules $\mathcal{D}(X)$ with the standard functors f_* and $f^!$. The compact objects are necessarily coherent \mathcal{D} -modules (this suffices for X a scheme; see [DG] for a characterization in the case of a stack). The \mathcal{D} -dualizing sheaf is the Verdier dualizing complex ω_X , and the \mathcal{D} -global volume forms (for X smooth) are the Borel-Moore homology $R\Gamma(X_{dR}, \omega_X) = H_{dR}(X)^*$.

Remark 2.18. More precisely, [GR2] construct the sheaf theories $\mathcal{Q}^!$ and \mathcal{D} as lax symmetric monoidal functors on a much broader class of stacks, with pullbacks allowed for arbitrary maps but pushforward only for schematic morphisms. The strictness follows from results of [DG], as we explain in Section 5.3.1, as does the definition of pushforwards for arbitrary maps of QCA stacks (without the functorial apparatus of [GR2] but sufficient for all the “local” constructions we need, in the sense of Remarks 2.3, 2.6 and 2.17).

Remark 2.19 (Quasicohherent sheaves). The theory of quasicohherent sheaves $X \mapsto \mathcal{Q}(X)$ behaves similarly with respect to 1-categorical properties. It also defines a symmetric monoidal functor out of the $(\infty, 1)$ -category of correspondences of stacks, using standard pullback f^* and pushforward f_* functors. Assuming X is perfect (in the sense of [BFN]), the compact objects of $\mathcal{Q}(X)$ form the subcategory of perfect complexes $\mathrm{Perf}(X)$, and we have $\mathcal{Q}(X) = \mathrm{Ind} \, \mathrm{Perf}(X)$. The analog of the dualizing sheaf is the structure sheaf \mathcal{O}_X , and the \mathcal{Q} -“global volume forms” are the global functions $R\Gamma(X, \mathcal{O}_X)$. The K -theory of $\mathcal{Q}(X)$ is the usual algebraic K -theory $K(X)$. However while we have (f^*, f_*) adjunction and f^* preserves perfection for arbitrary morphisms, proper pushforward does not preserve perfection and we do not have proper adjunction of the form (f_*, f^*) (unless we add smoothness and twisting by relative dualizing sheaves). In other words, \mathcal{Q} and K -theory are better adapted to pullback, while $\mathcal{Q}^!$ and G -theory are better adapted to integration and character formulas.

Remark 2.20 (Sheaf theories in differential topology and elliptic operators). It is tempting to think of sheaf theories in algebraic geometry as analogues of elliptic operators or complexes

in differential topology. In particular, the theory $\mathcal{Q}^!(X)$ for a smooth variety X is a natural setting for the study of the Dolbeault $\bar{\partial}$ -operator coupled to vector bundles, while the theory $\mathcal{D}(X)$ is similarly a natural setting for the study of the de Rham operator d coupled to vector bundles. The pushforward operation is the analogue of the index. In this direction, it would be interesting to develop sheaf theories on derived manifolds, for example C^∞ -schemes and stacks. Quasicoherent sheaves in the sense of Joyce [J] are a natural candidate. Another interesting setting is categories of elliptic complexes on manifolds. The general results below would then provide an approach to generalizations of the classical Atiyah-Singer and Atiyah-Bott theorems.

Let us spell out the main ingredients of Proposition 2.16 for our examples. Recall that for X a smooth scheme, $\mathcal{L}X \simeq \mathrm{Spec}_X \mathrm{Sym}^\bullet(\Omega_X[1])$, and for BG a classifying stack, $\mathcal{L}(BG) \simeq G/G$.

- **Theory \mathcal{Q} :** For X a smooth scheme, we have the HKR identification of functions on the loop space (or the Hochschild chain complex) with differential forms, $\dim(\mathcal{Q}(X)) \simeq \Gamma(X, \mathrm{Sym}^\bullet(\Omega_X[1]))$, or more generally, \mathcal{Q} -global volume forms on X^f are the coherent cohomology $\mathcal{O}(X^f)$. For BG a classifying stack, \mathcal{Q} -global volume forms on $\mathcal{L}(BG)$ are the coherent cohomology $\mathcal{O}(G/G)$, which for G reductive are the underived invariants $\mathcal{O}(T)^W$.

- **Theory $\mathcal{Q}^!$:** For X smooth, we have $\mathcal{Q}(X) \simeq \mathcal{Q}^!(X)$, and so we recover the above descriptions. For X proper, \mathcal{Q} -global volume forms on $\mathcal{L}X$ are the dual of the Hochschild chain complex (see [P]).

- **Theory \mathcal{D} :** For X a smooth scheme, \mathcal{D} -global volume forms on $\mathcal{L}X$ are the de Rham cochains $\dim(\mathcal{D}(X)) \simeq C_{dR}^*(X)$, or more generally, \mathcal{D} -global volume forms on X^f are the de Rham cochains $C_{dR}^*(X^f)$, or equivalently those of the underlying underived scheme of X^f . For BG a classifying stack, \mathcal{D} -global volume forms on $\mathcal{L}(BG)$ are the Borel-Moore homology of G/G .

2.3.2. Integration formulas for traces. Now let us turn to the functoriality of dimensions and traces, which is reflected in integration of volume forms along proper maps.

For a sheaf theory

$$\underline{\mathcal{S}} : \underline{\mathrm{Corr}}_k^{\mathrm{prop}} \rightarrow \underline{\mathrm{dgCat}}_k,$$

the counit of the $(f_*, f^!)$ adjunction for a proper map $f : X \rightarrow Y$ gives rise to a canonical integration map

$$\int_f : \omega(X) \longrightarrow \omega(Y)$$

Theorem 2.21. *Fix a sheaf theory $\underline{\mathcal{S}} : \underline{\mathrm{Corr}}_k \rightarrow \underline{\mathrm{dgCat}}_k$.*

(1) *For any proper map $f : X \rightarrow Y$, the induced map on dimensions*

$$\dim(f_*) : \dim(\mathcal{S}(X)) \longrightarrow \dim(\mathcal{S}(Y))$$

is identified (S^1 -equivariantly) with integration along the loop map

$$\dim(f_*) \simeq \int_{\mathcal{L}f} : \omega(\mathcal{L}X) \longrightarrow \omega(\mathcal{L}Y)$$

(2) *Given a proper map $f : X \rightarrow Y$ regarded as a correspondence from X to Y , and self-correspondences $Z \in \underline{\mathrm{Corr}}_k(X, X)$ and $W \in \underline{\mathrm{Corr}}_k(Y, Y)$, together with an identification*

$$\alpha : Z \xrightarrow{\sim} X \times_Y W$$

of correspondences from X to Y , the induced trace map is identified with integration along the natural map

$$\mathrm{Tr}(f_*, \alpha) \simeq \int_{\tau(f, s)} : \omega(Z|_{\Delta_X}) \longrightarrow \omega(W|_{\Delta_Y})$$

Remark 2.22. Similarly, in the case of the theory \mathcal{Q} of quasicoherent sheaves, the standard adjunction (f^*, f_*) leads to the evident contravariant functoriality of dimensions under arbitrary maps, given by pullback of functions on loop spaces.

Remark 2.23 (Categorified version). For applications to categorical representation theory, in particular the geometric Langlands program, it is interesting to have character formulas for group actions on categories. Such formulas would follow from a good formalism of “stack theories”, the higher unstable analogs of sheaf theories, such as the assignment $X \rightarrow \text{ShvCat}_k(X)$. Such stack theories could be formulated as symmetric monoidal functors $\underline{\mathcal{S}} : \text{Corr}_k \rightarrow \mathcal{A}$ out of a correspondence $(\infty, 2)$ (or more naturally $(\infty, 3)$) category with values in a category \mathcal{A} such as that of module categories for dgCat_k . Namely, we are interested in categorified analogues of \mathcal{D} and \mathcal{Q} , taking values in the ∞ -category Pr^L of presentable ∞ -categories, in which we assign to a scheme or stack X the ∞ -category of quasicoherent sheaves of module categories over \mathcal{D} or \mathcal{Q} . Since such theories have not been fully constructed yet, we will only briefly sketch the idea.

For any stack X and sheaf theory \mathcal{S} , the category of sheaves $\mathcal{S}(X)$ is naturally symmetric monoidal, and so we may consider its ∞ -category of (presentable, stable) module categories $\mathcal{S}(X)\text{-mod}$. To obtain a more meaningful geometric theory we should sheafify this construction. For example, strong or Harish-Chandra G -categories (in other words, module categories over $\mathcal{D}(G)$ with convolution) are identified with sheaves of categories over the de Rham stack of BG . However, in the quasicoherent case, the “1-affineness” theorem of Gaitsgory [G2] identifies $\mathcal{Q}(X)$ -modules with sheaves of categories on X for a large class of stacks (specifically, for X an eventually coconnective quasi-compact algebraic stack of finite type with an affine diagonal over a field of characteristic 0). In particular, $\mathcal{Q}(BG)$ -modules are identified with algebraic G -categories.

In the quasicoherent case, the general formalism of this paper should provide an S^1 -equivariant equivalence $\dim(\mathcal{Q}(X)\text{-mod}) = \mathcal{Q}(\mathcal{L}X)$, identifying the class $[\mathcal{Q}(X)]$ of the structure stack with the structure sheaf $\mathcal{O}(\mathcal{L}X)$. In particular, the characters of quasicoherent G -categories are given by $\mathcal{Q}(G/G)$. The induced map on dimensions $\dim(f_*) : \dim(\mathcal{Q}(X)\text{-mod}) \rightarrow \dim(\mathcal{Q}(Y)\text{-mod})$ is identified S^1 -equivariantly with the morphism given by pushforward along the loop map

$$\dim(f_*) = \mathcal{L}f_* : \mathcal{Q}(\mathcal{L}X) \longrightarrow \mathcal{Q}(\mathcal{L}Y)$$

In particular, for an algebraic group G and G -space X with $\pi : X/G \rightarrow BG$, the character of the G -category $\mathcal{Q}(X/G)$ is given by the pushforward $\mathcal{L}\pi_* \mathcal{O}(\mathcal{L}X/G) \in \mathcal{Q}(G/G)$. Analogous results are expected for strong or Harish-Chandra G -categories (module categories for $\mathcal{D}(G)$ with convolution) using the sheafification of the theory of $\mathcal{D}(X)$ -module categories. We hope to return to these applications in future works.

3. TRACES IN CATEGORY THEORY

3.1. Preliminaries. Our working setting is the higher category theory and algebra developed by J. Lurie [L1, L2, L4], see Chapter I.1 of [GR2] for an excellent overview.

Throughout what follows, we will fix once and for all a symmetric monoidal $(\infty, 2)$ -category \mathcal{A} with unit object $1_{\mathcal{A}}$. By forgetting non-invertible 2-morphisms we obtain a symmetric monoidal $(\infty, 1)$ -category $f(\mathcal{A})$, which we will abusively refer to as \mathcal{A} whenever only invertible higher morphisms are involved. Conversely, given a symmetric monoidal $(\infty, 1)$ -category \mathcal{C} , we can always regard it as a symmetric monoidal $(\infty, 2)$ -category $i(\mathcal{C})$ with all 2-morphisms invertible.⁵

⁵One can understand the above two operations as forming an adjoint pair (i, f) .

Thus developments for higher ∞ -categories equally well apply to the more familiar $(\infty, 1)$ -categories. In what follows, noninvertible 2-morphisms only play a significant role starting with Section 3.4.

We will use \otimes to denote the symmetric monoidal structure of \mathcal{A} . We will write $\Omega\mathcal{A} = \text{End}_{\mathcal{A}}(1_{\mathcal{A}})$ for the “based loops” in \mathcal{A} , or in other words, the symmetric monoidal $(\infty, 1)$ -category of endomorphisms of the monoidal unit $1_{\mathcal{A}}$. Note that the monoidal unit $1_{\Omega\mathcal{A}}$ is nothing more than the identity $\text{id}_{1_{\mathcal{A}}}$ of the monoidal unit $1_{\mathcal{A}}$.

Example 3.1 (Algebras). Fix a symmetric monoidal $(\infty, 1)$ -category \mathcal{C} , and let $\mathcal{A} = \text{Alg}(\mathcal{C})$ denote the Morita $(\infty, 2)$ -category of algebras, bimodules, and intertwiners of bimodules within \mathcal{C} . The forgetful map $\mathcal{A} = \text{Alg}(\mathcal{C}) \rightarrow \mathcal{C}$ is symmetric monoidal, and in particular, the monoidal unit $1_{\mathcal{A}}$ is the monoidal unit $1_{\mathcal{C}}$ equipped with its natural algebra structure. Finally, we have $\Omega\mathcal{A} \simeq \mathcal{C}$.

For a specific example, one could take $\mathcal{C} = k\text{-mod} = \text{dgVect}_k$ the $(\infty, 1)$ -category of complexes of k -modules (with quasi-isomorphisms inverted). Then $\mathcal{A} = \text{Alg}(\mathcal{C})$ is the $(\infty, 2)$ -category of k -algebras, bimodules, and intertwiners of bimodules.

Example 3.2 (Categories). A natural source of $(\infty, 2)$ -categories is given by various theories of $(\infty, 1)$ -categories. For example, one could consider dgCat_k , the $(\infty, 2)$ -category of k -linear stable presentable ∞ -categories (or k -linear presentable dg categories), k -linear continuous functors, and natural transformations.

Observe that $\text{Alg}(k\text{-mod})$ is a full subcategory of dgCat_k , via the functor assigning to a k -algebra its stable presentable ∞ -category of modules. The essential image consists of dg categories admitting a compact generator.

3.2. Dualizability.

Definition 3.3. An object A of the symmetric monoidal $(\infty, 2)$ -category \mathcal{A} is said to be *dualizable* (equivalently, A is dualizable in the $(\infty, 1)$ -category $f(\mathcal{A})$) if it admits a monoidal dual: there is a dual object $A^{\vee} \in \mathcal{A}$ and evaluation and coevaluation morphisms

$$\epsilon_A : A^{\vee} \otimes A \longrightarrow 1_{\mathcal{A}} \quad \eta_A : 1_{\mathcal{A}} \longrightarrow A \otimes A^{\vee}$$

such that the usual compositions are naturally equivalent to the identity morphism

$$A \xrightarrow{\eta_A \otimes \text{id}_A} A \otimes A^{\vee} \otimes A \xrightarrow{\text{id}_A \otimes \epsilon_A} A \quad A^{\vee} \xrightarrow{\text{id}_{A^{\vee}} \otimes \eta_A} A^{\vee} \otimes A \otimes A^{\vee} \xrightarrow{\epsilon_A \otimes \text{id}_{A^{\vee}}} A^{\vee}$$

Example 3.4. Any algebra object $A \in \text{Alg}(\mathcal{C})$ is dualizable with dual the opposite algebra $A^{op} \in \text{Alg}(\mathcal{C})$. The evaluation morphism

$$\epsilon_A : A^{op} \otimes A \longrightarrow 1_{\mathcal{C}}$$

is given by A itself regarded as an A -bimodule. The coevaluation morphism

$$\eta_A : 1_{\mathcal{C}} \longrightarrow A \otimes A^{op}$$

is also given by A itself regarded as an A -bimodule.

3.2.1. Dualizable morphisms. Consider two objects $A, B \in \mathcal{A}$, and a morphism

$$\Phi : A \longrightarrow B.$$

Example 3.5. If $\mathcal{A} = \text{Alg}(\mathcal{C})$, then Φ is simply an $A^{op} \otimes B$ -module.

If B is dualizable with dual B^\vee , we can package Φ in the equivalent form of the morphism

$$e_\Phi : B^\vee \otimes A \rightarrow 1_{\mathcal{A}}$$

defined by

$$\begin{array}{ccc} B^\vee \otimes A & \xrightarrow{e_\Phi} & 1_{\mathcal{A}} \\ \text{id}_{B^\vee} \otimes \Phi \downarrow & \nearrow \epsilon_B & \\ B \otimes B^\vee & & \end{array}$$

If A is dualizable with dual A^\vee , we can package Φ in the equivalent form of the morphism

$$u_\Phi : 1_{\mathcal{A}} \rightarrow B \otimes A^\vee$$

defined by

$$\begin{array}{ccc} & A \otimes A^\vee & \\ \eta_A \nearrow & \downarrow \Phi \otimes \text{id}_{A^\vee} & \\ 1_{\mathcal{A}} \xrightarrow{u_\Phi} & B \otimes A^\vee & \end{array}$$

If both A and B are dualizable, we can also encode Φ by its dual morphism

$$\Phi^\vee : B^\vee \longrightarrow A^\vee$$

defined by

$$\begin{array}{ccccccc} B^\vee & \xrightarrow{\text{id}_{B^\vee} \otimes \eta_A} & B^\vee \otimes A \otimes A^\vee & \xrightarrow{\text{id}_{B^\vee} \otimes \Phi \otimes \text{id}_{A^\vee}} & B^\vee \otimes B \otimes A^\vee & \xrightarrow{\epsilon_{B^\vee} \otimes \text{id}_{A^\vee}} & A^\vee \\ & & & \searrow \Phi^\vee & & & \end{array}$$

There is a natural composition identity

$$(\Phi\Psi)^\vee \simeq \Psi^\vee \Phi^\vee$$

Note that for fixed A, B , the construction $\Phi \mapsto \Phi^\vee$ naturally defines a covariant map

$$(-)^\vee : \text{Hom}(A, B) \longrightarrow \text{Hom}(B^\vee, A^\vee)$$

and in particular a morphism $\Phi_1 \rightarrow \Phi_2$ induces a natural morphism $\Phi_1^\vee \rightarrow \Phi_2^\vee$.

Let us record the canonical equivalences encoded by the following commutative diagrams

$$(3.1) \quad \begin{array}{ccc} & A \otimes A^\vee & \\ \eta_A \nearrow & \downarrow \Phi \otimes \text{id}_{A^\vee} & \\ 1_{\mathcal{A}} \xrightarrow{u_\Phi} & B \otimes A^\vee & \\ \eta_B \searrow & \uparrow \text{id}_B \otimes \Phi^\vee & \\ & B \otimes B^\vee & \end{array} \quad \begin{array}{ccc} & A^\vee \otimes A & \\ \Phi^\vee \otimes \text{id}_A \uparrow & \nwarrow \epsilon_{A^\vee} & \\ B^\vee \otimes A & \xrightarrow{e_\Phi} & 1_{\mathcal{A}} \\ \text{id}_{B^\vee} \otimes \Phi \downarrow & \nearrow \epsilon_B & \\ & B \otimes B^\vee & \end{array}$$

Example 3.6. In the setting of algebras, bimodules and intertwiners, the morphisms Φ , u_Φ , e_Φ and Φ^\vee are all different manifestations of the same bimodule Φ , making their various compatibilities particularly evident.

Definition 3.7. (1) A morphism $\Phi : A \rightarrow B$ is said to be *left dualizable* if it admits a left adjoint: there is a morphism $\Phi^\ell : B \rightarrow A$ and unit and counit morphisms

$$\eta_\Phi : \text{id}_B \longrightarrow \Phi \circ \Phi^\ell \quad \epsilon_\Phi : \Phi^\ell \circ \Phi \longrightarrow \text{id}_A$$

satisfying the usual identities.

(2) A morphism $\Phi : A \rightarrow B$ is said to be *right dualizable* if it admits a right adjoint: there is a morphism $\Phi^r : B \rightarrow A$ and unit and counit morphisms

$$\eta_\Phi : \text{id}_A \longrightarrow \Phi^r \circ \Phi \quad \epsilon_\Phi : \Phi \circ \Phi^r \longrightarrow \text{id}_B$$

satisfying the usual identities.

Remark 3.8. If A and B are dualizable, and $\Phi : A \rightarrow B$ is left (resp. right) dualizable, then $\Phi^\vee : B^\vee \rightarrow A^\vee$ is right (resp. left) dualizable with right adjoint $(\Phi^\ell)^\vee : A^\vee \rightarrow B^\vee$ (resp. left adjoint $(\Phi^r)^\vee : A^\vee \rightarrow B^\vee$).

3.3. Traces and dimensions. (We continue to refer to [TV, HSS] for thorough treatments of the theory of traces in higher category theory.)

Let $A \in \mathcal{A}$ be a dualizable object with dual A^\vee . Consider an endomorphism

$$\Phi : A \longrightarrow A$$

Since A is dualizable, Φ has a trace defined as follows.

Definition 3.9. (1) The *trace* of $\Phi : A \rightarrow A$ is the object $\text{Tr}(\Phi) \in \Omega\mathcal{A}$ defined by

$$1_{\mathcal{A}} \xrightarrow{\eta_A} A \otimes A^\vee \xrightarrow{\Phi \otimes \text{id}_A} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_{\mathcal{A}} .$$

$\text{Tr}(\Phi)$

Given a natural transformation $\varphi : \Phi \rightarrow \Psi$, we define the induced morphism

$$\text{Tr}(\varphi) : \text{Tr}(\Phi) \longrightarrow \text{Tr}(\Psi)$$

by applying $\varphi \otimes \text{id}_{A^\vee}$ to the middle arrow above.

(2) The *dimension* (or *Hochschild homology*) of A is the trace of the identity

$$\dim(A) = \text{Tr}(\text{id}_A) \in \Omega\mathcal{A}$$

or in other words, the object defined by

$$1_{\mathcal{A}} \xrightarrow{\eta_A} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_{\mathcal{A}}$$

$\dim(A)$

Remark 3.10. Equivalently, we can describe the trace as the composition

$$1_{\mathcal{A}} \xrightarrow{\Phi} \text{End}(A) \xrightarrow{\sim} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_{\mathcal{A}}$$

where the middle arrow is the identification deduced from the dualizability of A .

Remark 3.11. Observe that for fixed dualizable $A \in \mathcal{A}$, taking traces gives a functor

$$\text{Tr} : \text{End}(A) \longrightarrow \Omega\mathcal{A}$$

Remark 3.12. Observe that for any dualizable endomorphism Φ , the standard identities encoded by Diagrams 3.1 give rise to an identification

$$\text{Tr}(\Phi) \simeq \text{Tr}(\Phi^\vee)$$

Example 3.13. When $A = 1_{\mathcal{A}}$ is the monoidal unit, and $\Phi : 1_{\mathcal{A}} \rightarrow 1_{\mathcal{A}}$ is an endomorphism, we have an evident equivalence of endomorphisms

$$\mathrm{Tr}(\Phi) \simeq \Phi$$

Theorem 3.14 ([L3]). *There is a canonical S^1 -action on the dimension $\dim(A)$ of any dualizable object A of a symmetric monoidal ∞ -category \mathcal{A} .*

3.3.1. Cyclic symmetry.

Proposition 3.15. *Given two morphisms*

$$A \begin{array}{c} \xrightarrow{\Phi} \\ \xleftarrow{\Psi} \end{array} B$$

between dualizable objects $A, B \in \mathcal{A}$, there is a canonical equivalence

$$m(\Phi, \Psi) : \mathrm{Tr}(\Phi \circ \Psi) \xrightarrow{\sim} \mathrm{Tr}(\Psi \circ \Phi)$$

functorial in morphisms of both Φ and Ψ .

Proof. We construct $m(\Phi, \Psi)$ following the commutative diagram below:

$$\begin{array}{ccccccc}
 & & A \otimes A^\vee & \xrightarrow{\Phi \otimes \mathrm{id}_{A^\vee}} & B \otimes A^\vee & \xrightarrow{\Psi \otimes \mathrm{id}_{A^\vee}} & A \otimes A^\vee \\
 & \nearrow \eta_A & & \searrow \mathrm{id}_A \otimes \Psi^\vee & & \searrow \mathrm{id}_B \otimes \Psi^\vee & \searrow \epsilon_A \\
 1_{\mathcal{A}} & & & & & & 1_{\mathcal{A}} \\
 & \searrow \eta_B & & \nearrow \mathrm{id}_A \otimes \Psi^\vee & & \nearrow \mathrm{id}_B \otimes \Psi^\vee & \nearrow \epsilon_B \\
 & & B \otimes B^\vee & \xrightarrow{\Psi \otimes \mathrm{id}_{B^\vee}} & A \otimes B^\vee & \xrightarrow{\Phi \otimes \mathrm{id}_{B^\vee}} & B \otimes B^\vee
 \end{array}$$

Following the top edge, we find the definition of $\mathrm{Tr}(\Psi \circ \Phi)$. Following the bottom edge, we find the definition of $\mathrm{Tr}(\Phi \circ \Psi)$. The identifications filling the left and right diamonds arise from the standard identities encoded by Diagrams 3.1. The identification filling the central square results from the symmetric monoidal structure.

The construction is evidently functorial for morphisms $\Phi \rightarrow \Phi'$. The functoriality for morphisms $\Psi \rightarrow \Psi'$ is similar, once one recalls that the construction $\Psi \mapsto \Psi^\vee$ is covariantly functorial in morphisms of Ψ . \square

Example 3.16. Taking $\Phi = \mathrm{id}_A$ yields a canonical equivalence

$$\gamma' : \mathrm{id}_{\mathrm{Tr}(\Phi')} \xrightarrow{\sim} m(\mathrm{id}_A, \Phi'_A)$$

and likewise, taking $\Phi' = \mathrm{id}_A$ yields a canonical equivalence

$$\gamma : \mathrm{id}_{\mathrm{Tr}(\Phi)} \xrightarrow{\sim} m(\Phi_A, \mathrm{id}_A)$$

Thus taking $\Phi = \Phi' = \mathrm{id}_A$ yields an automorphism of the identity of the Hochschild homology

$$(\gamma')^{-1} \circ \gamma : \mathrm{id}_{\mathrm{Tr}(\mathrm{id}_A)} \xrightarrow{\sim} \mathrm{id}_{\mathrm{Tr}(\mathrm{id}_A)}$$

called the *BV homotopy*.

Remark 3.17. The proposition is only the initial part of the full cyclic symmetry of trace (see Remark 3.28), and the example is the lowest level structure of the S^1 -action on Hochschild homology (see Theorem 3.14) defining cyclic homology.

Lemma 3.18. *Given morphisms*

$$A \xrightarrow{\Phi} B \xrightarrow{\Psi} C \xrightarrow{\Upsilon} A$$

between dualizable objects $A, B, C \in \mathcal{A}$, there is a canonical commutative diagram

$$\begin{array}{ccc} \mathrm{Tr}(\Psi\Phi\Upsilon) & \xrightarrow{m(\Psi, \Phi\Upsilon)} & \mathrm{Tr}(\Phi\Upsilon\Psi) \\ & \searrow m(\Psi\Phi, \Upsilon) & \downarrow m(\Phi, \Upsilon\Psi) \\ & & \mathrm{Tr}(\Upsilon\Psi\Phi) \end{array}$$

Proof. We construct the desired equivalence from the following diagram:

$$\begin{array}{ccccccc} & & C \otimes C^\vee & \xrightarrow{\Upsilon} & A \otimes C^\vee & \xrightarrow{\Phi} & B \otimes C^\vee & \xrightarrow{\Psi} & C \otimes C^\vee & & \\ & \nearrow & \searrow \Psi^\vee & & \searrow \Psi^\vee & & \searrow \Psi^\vee & & \searrow \Psi^\vee & & \\ 1_{\mathcal{A}} & \longrightarrow & B \otimes B^\vee & \xrightarrow{\Psi} & C \otimes B^\vee & \xrightarrow{\Upsilon} & A \otimes B^\vee & \xrightarrow{\Phi} & B \otimes B^\vee & \longrightarrow & 1_{\mathcal{A}} \\ & \searrow & \searrow \Phi^\vee & & \searrow \Phi^\vee & & \searrow \Phi^\vee & & \searrow \Phi^\vee & & \\ & & A \otimes A^\vee & \xrightarrow{\Phi} & B \otimes A^\vee & \xrightarrow{\Psi} & C \otimes A^\vee & \xrightarrow{\Upsilon} & A \otimes A^\vee & & \\ & & & & & & & & & & \end{array}$$

The natural transformations $m(\Psi, \Phi\Upsilon)$ and $m(\Phi, \Upsilon\Psi)$ describe passage from the top row to the middle row and from the middle to the bottom, respectively. The transformation $m(\Psi\Phi, \Upsilon)$ can then be identified with the transformation from the top row to the bottom given by inserting the diagonal morphisms $\mathrm{id} \otimes \Phi^\vee \circ \Psi^\vee$ and using standard composition identities. \square

3.4. Functoriality of dimension. Let $\mathcal{A}^{\mathrm{cont}} \subset \mathcal{A}$ denote the $(\infty, 2)$ -subcategory of dualizable objects and *continuous* or right dualizable morphisms (morphisms that are left duals).

Definition 3.19. Let $\Psi : A \rightarrow B$ denote a morphism in $\mathcal{A}^{\mathrm{cont}}$ with right adjoint $\Psi^r : B \rightarrow A$. We define the induced morphism of dimensions

$$\mathrm{dim}(\Psi) : \mathrm{dim}(A) \longrightarrow \mathrm{dim}(B)$$

to be the composition

$$\mathrm{Tr}(\mathrm{id}_A) \xrightarrow{\eta_\Psi} \mathrm{Tr}(\Psi^r \circ \Psi) \xrightarrow{m(\Psi^r, \Psi)} \mathrm{Tr}(\Psi \circ \Psi^r) \xrightarrow{\epsilon_\Psi} \mathrm{Tr}(\mathrm{id}_B)$$

Remark 3.20. In other words, the morphism $\mathrm{dim}(\Psi)$ is defined by the following diagram

$$\begin{array}{ccc} & A \otimes A^\vee & \\ \eta_A \nearrow & \updownarrow \Psi \otimes \mathrm{id}_{A^\vee} & \searrow \Psi^r \otimes \mathrm{id}_A \\ 1_{\mathcal{A}} & \xrightarrow{u_\Psi} B \otimes A^\vee \xrightarrow{c_\Psi} & 1_{\mathcal{A}} \\ \eta_B \searrow & \updownarrow \mathrm{id}_B \otimes \Psi^\vee & \nearrow \mathrm{id}_{B^\vee} \otimes \Psi^{r^\vee} \\ & B \otimes B^\vee & \end{array} \quad \begin{array}{ccc} & \mathrm{dim}(A) & \\ \curvearrowright & & \curvearrowleft \\ 1_{\mathcal{A}} & \xrightarrow{\mathrm{Tr}(\Psi^r \Psi) \simeq \mathrm{Tr}(\Psi \Psi^r)} & 1_{\mathcal{A}} \\ \curvearrowleft & & \curvearrowright \\ & \mathrm{dim}(B) & \end{array}$$

Following the top and bottom edge, we find the respective definitions of $\dim(A)$ and $\dim(B)$. The unit η_Ψ defines a morphism from the top edge to the top zig-zag. The counit ϵ_Ψ defines a morphism from the bottom zig-zag to the bottom edge. The passage from the top to bottom zig-zag is given by the construction $m(\Psi^r, \Psi)$ and the identification

$$\mathrm{Tr}(\Psi^{r\vee} \circ \Psi^\vee) \simeq \mathrm{Tr}((\Psi \circ \Psi^r)^\vee) \simeq \mathrm{Tr}(\Psi \circ \Psi^r)$$

Proposition 3.21. *For a diagram*

$$A \xrightarrow{\Phi} B \xrightarrow{\Psi} C$$

within \mathcal{A}^{cont} , there is a canonical equivalence

$$\dim(\Psi \circ \Phi) \simeq \dim(\Psi) \circ \dim(\Phi) : \dim(A) \longrightarrow \dim(C)$$

Proof. The equivalence is given by filling in the following diagram

$$\begin{array}{ccccccc} \dim(A) & \xrightarrow{\eta_\Phi} & \mathrm{Tr}(\Phi^r \Phi) & \xrightarrow{m} & \mathrm{Tr}(\Phi \Phi^r) & \xrightarrow{\epsilon_\Phi} & \dim(B) \\ & \searrow \eta_\Psi & \downarrow \eta_\Psi & & \downarrow \eta_\Psi & & \downarrow \eta_\Psi \\ & & \mathrm{Tr}(\Phi^r \Psi^r \Psi \Phi) & \xrightarrow{m} & \mathrm{Tr}(\Psi^r \Psi \Phi \Phi^r) & \xrightarrow{\epsilon_\Phi} & \mathrm{Tr}(\Psi^r \Psi) \\ & & & \searrow m & \downarrow m & & \downarrow m \\ & & & & \mathrm{Tr}(\Psi \Phi \Phi^r \Psi^r) & \xrightarrow{\epsilon_\Phi} & \mathrm{Tr}(\Psi \Psi^r) \\ & & & & & \searrow \epsilon_{\Psi \Phi} & \downarrow \epsilon_\Psi \\ & & & & & & \dim(C) \end{array}$$

Along the three boundary edges, we find the definitions of $\dim(\Phi)$, $\dim(\Psi)$ and $\dim(\Psi \Phi)$ respectively.

The two corner triangles are given by the composition identities for adjoints (for example, at the top left, relating the adjoint of $\Phi \Psi$ with the composition of adjoints of Ψ and Φ).

The middle triangle is given by the identity of Lemma 3.18.

The top right square is given by taking traces of the evident commutative diagram of endomorphisms

$$\begin{array}{ccc} \Phi \Phi^r \otimes \mathrm{Id}_{B^\vee} & \longrightarrow & \mathrm{Id}_B \otimes \mathrm{Id}_{B^\vee} \\ \downarrow & & \downarrow \\ \Phi \Phi^r \otimes (\Psi^r \Psi)^\vee & \longrightarrow & \mathrm{Id}_B \otimes (\Psi^r \Psi)^\vee \end{array}$$

and using the canonical identification $\mathrm{Tr}(F) = \mathrm{Tr}(F^\vee)$ for any dualizable morphism.

Finally, the two remaining commuting squares are given by the functoriality of the cyclic rotation of the trace in its two arguments. For instance, in the top left square, we may either rotate $\mathrm{Tr}(\Phi^r \circ (\mathrm{Id}_A \circ \Phi))$ and then apply the unit $\eta_\Psi : \mathrm{Id}_A \rightarrow \Psi^r \Psi$ or first apply the unit and then rotate.

This concludes the construction. \square

Since we have an evident equivalence $\dim(1_A) \simeq 1_A$ for the unit $1_A \in \mathcal{A}$, we have the following specialization of Proposition 3.21 in which we adopt suggestive notation.

Corollary 3.22 (Abstract Grothendieck-Riemann-Roch). *Let $A, B \in \mathcal{A}^{cont}$ and $V : 1_{\mathcal{A}} \rightarrow A$ and $\pi_* : A \rightarrow B$ morphisms in \mathcal{A}^{cont} . Then the following diagram naturally commutes*

$$\begin{array}{ccc} 1_{\mathcal{A}} & \xrightarrow{\dim(V)} & \dim(A) \\ & \searrow \dim(\pi_* V) & \downarrow \dim(\pi_*) \\ & & \dim(B) \end{array}$$

Remark 3.23. One can show along the same lines as the proposition that taking dimensions extends to a symmetric monoidal functor

$$\dim : \mathcal{A}^{cont} \longrightarrow \Omega\mathcal{A}.$$

3.5. Functoriality of traces. We would like to capture the functoriality for traces of arbitrary endomorphisms of dualizable objects. For this purpose we define a morphism between pairs

$$A \in \mathcal{A}^{cont} \quad \Phi_A \in \text{End}_{\mathcal{A}}(A)$$

of an object and an endomorphism to consist of a pair

$$\Psi \in \text{Hom}_{\mathcal{A}^{cont}}(A, B) \quad \psi : \Psi \circ \Phi_A \xrightarrow{\simeq} \Phi_B \circ \Psi$$

of a morphism and a *commuting structure*.

Definition 3.24. For a morphism

$$(\Psi, \psi) : (A, \Phi_A) \rightarrow (B, \Phi_B)$$

as above, we define the induced morphism of traces

$$\text{Tr}(\Psi, \psi) : \text{Tr}(\Phi_A) \longrightarrow \text{Tr}(\Phi_B)$$

to be the composition

$$\text{Tr}(\Phi_A) \xrightarrow{\eta_{\Psi}} \text{Tr}(\Psi^r \Psi \Phi_A) \xrightarrow{\psi} \text{Tr}(\Psi^r \Phi_B \Psi) \xrightarrow{m(\Psi^r, \Phi_B \Psi)} \text{Tr}(\Phi_B \Psi \Psi^r) \xrightarrow{\epsilon_{\Psi}} \text{Tr}(\Phi_B)$$

Remark 3.25. Note that we could alternatively define a morphism $\text{Tr}(\Psi, \psi)$ by applying the unit η_{Ψ} to the right of Φ_A , rotating the trace in the opposite direction, and again using the counit on the right. It is elementary to give a natural equivalence of the two constructions using nothing more than the dualizability of A .

Remark 3.26. In parallel with Remark 3.20 about the functoriality of dimensions, it is enlightening to realize the functoriality of traces as a chase through the following diagram

$$\begin{array}{ccccc} & A \otimes A^{\vee} & \xrightarrow{\Phi_A \otimes \text{id}_{A^{\vee}}} & A \otimes A^{\vee} & \\ & \uparrow \eta_A & \Psi \otimes \text{id}_{A^{\vee}} & \Psi^r \otimes \text{id}_{A^{\vee}} & \downarrow \epsilon_A \\ 1_{\mathcal{A}} & \xrightarrow{u_{\Psi}} & B \otimes A^{\vee} & \xrightarrow{\Phi_B \otimes \text{id}_{A^{\vee}}} & B \otimes A^{\vee} \xrightarrow{c_{\Psi}} 1_{\mathcal{A}} \\ & \downarrow \eta_B & \text{id}_B \otimes \Psi^{\vee} & \text{id}_B \otimes \Psi^r & \downarrow \epsilon_B \\ & B \otimes B^{\vee} & \xrightarrow{\Phi_B \otimes \text{id}_{B^{\vee}}} & B \otimes B^{\vee} & \end{array}$$

$$\begin{array}{ccc} & \text{Tr}(\Phi_A) & \\ & \curvearrowright & \\ 1_{\mathcal{A}} & \xrightarrow[\text{Tr}(\Psi^r \Phi_B \Psi) \simeq \text{Tr}(\Phi_B \Psi \Psi^r)]{\text{Tr}(\Psi^r \Psi \Phi_A) \simeq} & 1_{\mathcal{A}} \\ & \curvearrowleft & \\ & \text{Tr}(\Phi_B) & \end{array}$$

Proposition 3.27. *Suppose given objects $A, B, C \in \mathcal{A}^{cont}$, endomorphisms Φ_A, Φ_B, Φ_C , a commutative diagram of continuous morphisms*

$$\begin{array}{ccccc} & & \Psi_{AC} & & \\ & \swarrow & & \searrow & \\ A & \xrightarrow{\Psi_{AB}} & B & \xrightarrow{\Psi_{BC}} & C \end{array}$$

and commuting structures

$$s_{AB} : \Psi_{AB}\Phi_A \xrightarrow{\sim} \Phi_B\Psi_{AB} \quad s_{BC} : \Psi_{BC}\Phi_B \xrightarrow{\sim} \Phi_C\Psi_{BC} \quad s_{AC} : \Psi_{AC}\Phi_A \xrightarrow{\sim} \Phi_C\Psi_{AC}$$

with an identification $s_{AC} \simeq s_{BC}s_{AB}$. Then there is a canonical equivalence

$$\mathrm{Tr}(\Psi_{AC}, s_{AC}) \simeq \mathrm{Tr}(\Psi_{BC}, s_{BC}) \circ \mathrm{Tr}(\Psi_{AB}, s_{AB}) : \mathrm{Tr}(\Phi_A) \longrightarrow \mathrm{Tr}(\Phi_C)$$

Proof. The construction is obtained from following a minor expansion of the diagram proving Proposition 3.21. The additional moves needed are commuting the commuting structures past the symmetry m of the trace and the unit and counits of the adjunctions. These all follow immediately from the 2-categorical interchange law for natural transformations. \square

Remark 3.28. The full functoriality of the trace Tr takes roughly the following form, see [HSS] for a detailed treatment. Define the *loop category* $\mathcal{L}^{cont}\mathcal{A}$ to be the symmetric monoidal ∞ -category with objects consisting of pairs (A, Φ_A) of a dualizable object $A \in \mathcal{A}$ equipped with a (not necessarily continuous) endomorphism Φ_A , and morphisms given by pairs (Ψ, ψ) as above with Ψ continuous. Taking traces to extend to a symmetric monoidal functor

$$\mathrm{Tr} : \mathcal{L}^{cont}\mathcal{A} \longrightarrow \Omega\mathcal{A}$$

extending the dimension functor

$$\mathrm{dim} : \mathcal{A}^{cont} \longrightarrow \Omega\mathcal{A}$$

for constant loops $\Phi_A = \mathrm{id}_A$, and trivial commuting structures $\psi = \mathrm{id}_\Psi$.

In order to capture the full cyclic symmetry of the trace Tr , one should further extend it to a homotopical trace valued in $\Omega\mathcal{A}$, or in other words, to the appropriate full cyclic bar construction (of which the above forms only the one-simplices).

4. TRACES IN GEOMETRY

4.1. Categories of correspondences. For concreteness, we fix a base commutative ring, and work in the symmetric monoidal $(\infty, 1)$ -category Stacks_k of derived stacks over $\mathrm{Spec} k$. It is worth pointing out that the constructions of this section apply in any presentable ∞ -category with the Cartesian symmetric monoidal structure.

Let Corr_k denote the symmetric monoidal ∞ -category of correspondences in Stacks_k . Thus morphisms are given by the classifying space of correspondences

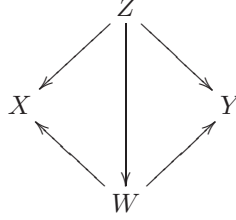
$$X \longleftarrow Z \longrightarrow Y$$

so all higher morphisms are isomorphisms. Composition of correspondences is given by the derived fiber product. The based loop category

$$\Omega\mathrm{Corr}_k = \mathrm{End}_{\mathrm{Corr}_k}(\mathrm{Spec} k) \simeq \mathrm{Stacks}_k$$

is again derived stacks, regarded as self-correspondences of the point $\mathrm{Spec} k$.

We will also enhance $Corr_k$ to the symmetric monoidal $(\infty, 2)$ -category \underline{Corr}_k where we now allow noninvertible maps of correspondences



In other words, the morphisms $\underline{Corr}_k(X, Y)$ now form the ∞ -category $Stacks_{/X \times Y}$ of stacks over $X \times Y$ with arbitrary morphisms rather than isomorphisms as in $Corr_k(X, Y)$.

We will also have need to restrict the class of morphisms of correspondences to some subcategory of $Stacks_{/X \times Y}$. In particular, we will consider the subcategory $\underline{Corr}_k^{prop}$ in which we only allow proper maps of correspondences.

4.2. Traces of correspondences. Given a map $Z \rightarrow X$, it is convenient to introduce the symmetric presentation of the based loop space

$$\mathcal{L}_Z X = Z \times_{Z \times X} Z$$

Note the two natural identification with the traditional based loop space

$$\mathcal{L}X \times_X Z \simeq X \times_{X \times X} Z \xleftarrow{\sim} Z \times_{Z \times X} Z \xrightarrow{\sim} Z \times_{X \times X} X \simeq Z \times_X \mathcal{L}X$$

There is a natural rotational equivalence $\mathcal{L}X \times_X Z \simeq Z \times_X \mathcal{L}X$ that makes the above two identifications coincide. (It does not preserve base points and is not given by swapping the factors). Thus we can unambiguously identify all of the above versions of the based loop space.

Proposition 4.1. (1) Any derived stack X is dualizable as an object of $Corr_k$, with dual X^\vee identified with X itself, and dimension $\dim(X)$ identified with the loop space

$$\mathcal{L}X = X^{S^1} \simeq X \times_{X \times X} X$$

regarded as a self-correspondence of $pt = \text{Spec } k$.

(2) The transpose of any correspondence $X \leftarrow Z \rightarrow Y$ is identified with the reverse correspondence $Y \leftarrow Z \rightarrow X$. The trace of a self-correspondence $X \leftarrow Z \rightarrow X$ is identified with the based loop space

$$\text{Tr}(Z) \simeq Z|_{\Delta_X} = Z \times_{X \times X} X \simeq \mathcal{L}_Z X$$

regarded as a self-correspondence of $pt = \text{Spec } k$.

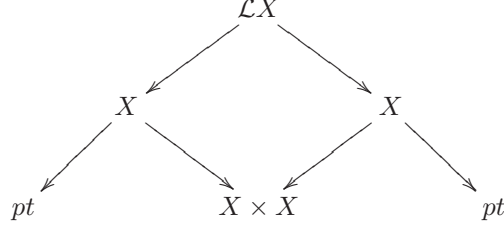
In particular, the trace of the graph $\Gamma_f \rightarrow X \times X$ of a self-map $f : X \rightarrow X$ is identified with the fixed point locus

$$\text{Tr}(f) \simeq \Gamma_f|_{\Delta_X} = \Gamma_f \times_{X \times X} X \simeq X^f$$

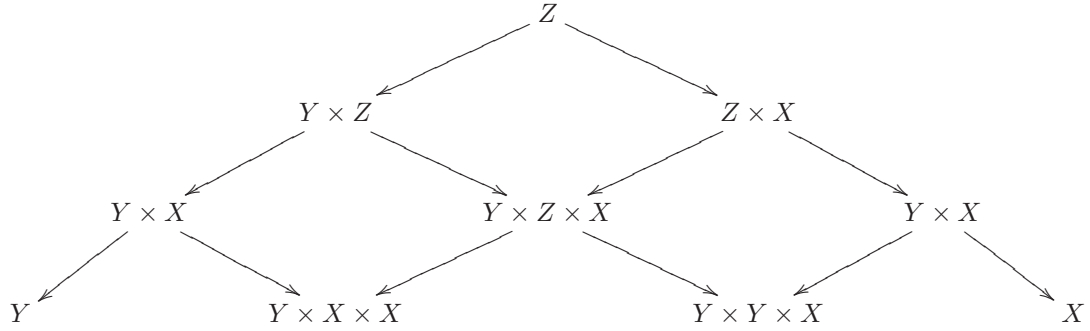
Proof. The evaluation and coevaluation presenting the self-duality of X are both given by X itself as a correspondence between $pt = \text{Spec } k$ and $X \times X$ via the diagonal map. The standard identities follow from the calculation of the fiber product of the two diagonal maps

$$X_{\Delta_{12}} \times_{X \times X \times X} X_{\Delta_{23}} \simeq X$$

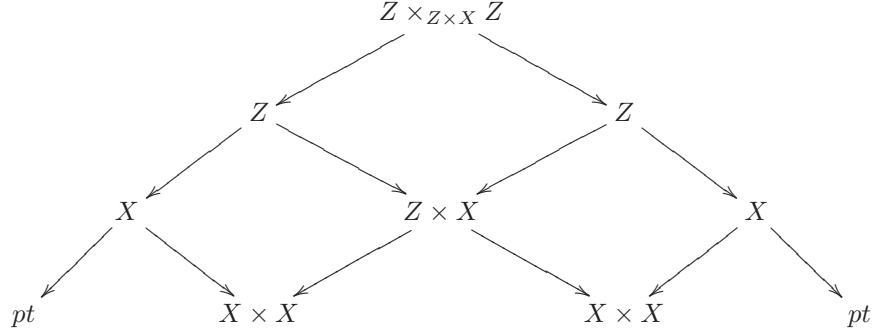
Thus the dimension of X is the loop space



By definition, the transpose of a correspondence $X \leftarrow Z \rightarrow Y$ is identified with $Y \leftarrow Z \rightarrow X$ by checking the definition



The trace of a self-correspondence $X \leftarrow Z \rightarrow X$ is then calculated by the composition



Finally, the case of the graph $Z = \Gamma_f$ of a self-map gives the fixed point locus by definition. \square

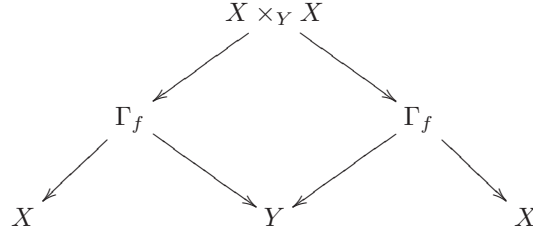
Remark 4.2 (Cyclic version). The identification $\dim(X) \simeq \mathcal{L}X$ above is naturally S^1 -equivariant for the standard loop rotation on $\mathcal{L}X$ and the cyclic symmetry of $\dim(X)$ provided by the cobordism hypothesis. To see this it is useful to consider X as an E_∞ -algebra object in $Stacks_k^{op}$ via the diagonal map (or as an E_n -object for any n). In other words, for $n = 1$ we identify stacks and correspondences with objects and morphisms in the Morita category $Alg(Stacks_k^{op})$. It follows from the properties of topological chiral homology [L2, Theorem 5.3.3.8] that for a (constant) commutative algebra A its topological chiral homology over a manifold is given by the tensoring of commutative algebras over simplicial sets $\int_M A = M \otimes A$. In particular (passing back from the opposite category to stacks) we have $\int_{S^1} X = X^{S^1} = \mathcal{L}X$. Moreover this identification holds not just for a fixed circle but over the moduli space of circles $B\mathcal{D}iff(S^1) \sim BS^1$, i.e. equivariantly for rotation. We also know from [L2, Example 5.3.3.14] or [L3, Example

4.2.2] that the S^1 -action on the dimension of an associative algebra A (given classically by the cyclic structure on the Hochschild chain complex) is given by the rotation S^1 -action on the topological chiral homology $\int_{S^1} A$, i.e., the family of topological chiral homologies over the moduli space of circles. In our case this recovers the rotation action on the loop space.

4.3. Geometric functoriality of dimension.

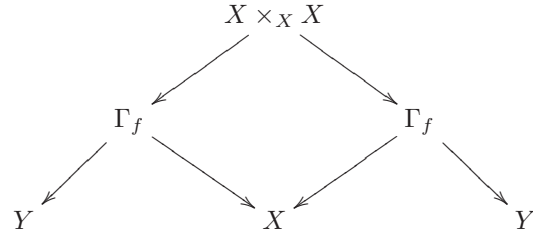
Proposition 4.3. *The graph $X \leftarrow \Gamma_f \rightarrow Y$ of any proper morphism $f : X \rightarrow Y$ gives a continuous morphism $F : X \rightarrow Y$ in $\underline{\text{Corr}}_k^{\text{prop}}$, with right adjoint $F^r : Y \rightarrow X$ identified with the opposite correspondence $Y \leftarrow \Gamma_f \rightarrow X$.*

Proof. We construct the unit and counit of the adjunction as follows. Consider the composition $F^r F : X \rightarrow X$ of correspondences



The unit $\eta_f : \text{id}_X = X \rightarrow F^r F \simeq X \times_Y X$ is given by the relative diagonal map.

Consider the opposite composition of correspondences



The counit $\epsilon_f : F F^r \simeq X \rightarrow \text{id}_Y = Y$ is given by f itself.

The standard identities are easily verified by identifying the resulting composite map

$$\Gamma_f \longrightarrow \Gamma_f \times_Y \Gamma_f \times_X \Gamma_f \longrightarrow \Gamma_f$$

of correspondences with the identity. □

Lemma 4.4. *Let $F_Z : X \rightarrow Y$ and $F_W : Y \rightarrow X$ be morphisms in Corr_k given by respective correspondences $X \leftarrow Z \rightarrow Y$ and $Y \leftarrow W \rightarrow X$. Then the canonical equivalence*

$$m(F_W, F_Z) : \text{Tr}(F_W \circ F_Z) \xrightarrow{\sim} \text{Tr}(F_Z \circ F_W)$$

is given by the composition of evident geometric identifications

$$(Z \times_Y W) \times_{X \times X} X \xrightarrow{\sim} W \times_{X \times Y} Z \xrightarrow{\sim} Z \times_{Y \times X} W \xrightarrow{\sim} (W \times_X Z) \times_{Y \times Y} Y$$

Proof. Returning to the definition and using our previous identifications, observe that $m(F_Z, F_W)$ is calculated by commutativity of the diagram of correspondences

$$\begin{array}{ccccc}
 & X \times X & \xrightarrow{Z \times X} & Y \times X & \xrightarrow{W \times X} & X \times X \\
 \nearrow X & & & & & \searrow X \\
 pt & & & & & pt \\
 \searrow Y & & & & & \nearrow Y \\
 & Y \times Y & \xrightarrow{W \times Y} & X \times Y & \xrightarrow{X \times Y} & Y \times Y
 \end{array}$$

$\begin{array}{ccc} & X \times W & \\ & \searrow & \\ & Y \times W & \end{array}$

Following the top edge, we see $\text{Tr}(F_W \circ F_Z) \simeq (Z \times_Y W) \times_{X \times X} X$. Following the bottom edge, we see $\text{Tr}(F_Z \circ F_W) \simeq (W \times_X Z) \times_{Y \times Y} Y$. Moving from the top to bottom edge via the successive equivalences of the three commuting squares, one finds the three successive equivalences in the assertion of the lemma. \square

Proposition 4.5. *Suppose $f : X \rightarrow Y$ is a proper morphism, and $F : X \rightarrow Y$ denotes the induced morphism in $\underline{\text{Corr}}_k^{\text{prop}}$ given by the graph $X \leftarrow \Gamma_f \rightarrow Y$. Then $\dim(F) : \dim(X) \rightarrow \dim(Y)$ is canonically identified with the S^1 -equivariant morphism $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$.*

Proof. Denote by $F^r : Y \rightarrow X$ the right adjoint to F . We must calculate

$$\dim(X) \longrightarrow \text{Tr}(F^r F) \xrightarrow{m(F^r, F)} \text{Tr}(F F^r) \longrightarrow \dim(Y)$$

We have seen that the first and third morphisms correspond to the natural geometric maps

$$\mathcal{L}X \simeq X \times_{X \times X} X \longrightarrow (X \times_Y X) \times_{X \times X} X \quad X \times_{Y \times Y} Y \longrightarrow Y \times_{Y \times Y} Y \simeq \mathcal{L}Y$$

induced by the relative diagonal $X \rightarrow X \times_Y X$ and given map $f : X \rightarrow Y$ respectively. Furthermore, by Lemma 4.4, the middle map is the natural geometric identification

$$(X \times_Y X) \times_{X \times X} X \xrightarrow{\sim} X \times_{Y \times Y} Y$$

Altogether, the composition is easily identified with the loop map $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$. \square

Remark 4.6. It follows from the proposition that the loop map $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$ must be proper when the given map $f : X \rightarrow Y$ is proper. Let us note why this is true geometrically from the factorization $\mathcal{L}X \rightarrow \mathcal{L}_X Y \rightarrow \mathcal{L}Y$ appearing in the proof.

First, the natural morphism $\mathcal{L}X \rightarrow \mathcal{L}_X Y$ is the restriction along the diagonal $X \rightarrow X \times_Y X$ of the relative diagonal $X \rightarrow X \times_Y X$. The relative diagonal is a closed embedding since f is proper, and hence the natural morphism $\mathcal{L}X \rightarrow \mathcal{L}_X Y$ is as well. Second, the natural morphism $\mathcal{L}_X Y \rightarrow \mathcal{L}Y$ is the restriction along the diagonal $Y \rightarrow Y \times_Y Y$ of the proper morphism $f : X \rightarrow Y$ and thus is proper as well. Altogether, we see that $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$ is itself proper.

Remark 4.7. One can invoke the cobordism hypothesis with singularities to endow the morphism $\dim(F) : \dim(X) \rightarrow \dim(Y)$ with a canonical S^1 -equivariant structure, and it will agree with the canonical geometric S^1 -equivariant structure on the map $\mathcal{L}f : \mathcal{L}X \rightarrow \mathcal{L}Y$ under the identification of the proposition.

4.4. Geometric functoriality of trace. Consider a proper morphism $f : X \rightarrow Y$ and endomorphisms $F_Z : X \rightarrow X$ and $F_W : Y \rightarrow Y$ in Corr_k given by respective self-correspondences $X \leftarrow Z \rightarrow X$ and $Y \leftarrow W \rightarrow Y$.

By an f -morphism from the pair (X, F_Z) to the pair (Y, F_W) , we mean an identification

$$s : Z \xrightarrow{\sim} X \times_Y W$$

of correspondences from X to Y . This in turn induces an identification of what might be called relative traces

$$Z \times_{Y \times_Y Y} Y \xrightarrow{\sim} X \times_{Y \times_Y Y} W$$

generalizing the relative loop space $\mathcal{L}_X Y$ from the case of the identity correspondences $Z = X$, $W = Y$. We thus obtain a map of traces

$$\tau(f, s) : Z|_{\Delta_X} = Z \times_{X \times X} X \longrightarrow Z \times_{Y \times_Y Y} Y \xrightarrow{\sim} X \times_{Y \times_Y Y} W \longrightarrow Y \times_{Y \times_Y Y} W = W|_{\Delta_Y}$$

Proposition 4.8. *With the preceding setup, the trace map $\text{Tr}(f, s) : \text{Tr}(F_Z) \rightarrow \text{Tr}(F_W)$ is canonically identified with the geometric map*

$$\tau(f, s) : Z|_{\Delta_X} \longrightarrow W|_{\Delta_Y}$$

Proof. Denote by $F : X \rightarrow Y$ the morphism given by the graph $X \leftarrow \Gamma_f \rightarrow Y$, and by $F^r : Y \rightarrow X$ its right adjoint. We must calculate

$$\text{Tr}(F_Z) \longrightarrow \text{Tr}(F^r F F_Z) \xrightarrow{s} \text{Tr}(F^r F_W F) \xrightarrow{m(F^r, F_W F)} \text{Tr}(F_W F F^r) \longrightarrow \text{Tr}(F_W)$$

We have seen that the first and fourth morphisms correspond to the natural geometric maps

$$\begin{aligned} Z|_{\Delta_X} &= Z \times_{X \times X} X \longrightarrow Z \times_{X \times X} (X \times_Y X) \\ X \times_{Y \times_Y Y} W &\longrightarrow Y \times_{Y \times_Y Y} W = W|_{\Delta_Y} \end{aligned}$$

induced by the relative diagonal $X \rightarrow X \times_Y X$ and given map $f : X \rightarrow Y$ respectively.

Using associativity, the second map, induced by s , is the natural geometric identification

$$Z \times_{X \times X} (X \times_Y X) \simeq Z \times_{Y \times_Y Y} Y \xrightarrow{\sim} W \times_{Y \times_Y Y} X$$

By Lemma 4.4, the third map, given by the cyclic symmetry, is nothing more than the natural identification

$$W \times_{Y \times_Y Y} X \xrightarrow{\sim} X \times_{Y \times_Y Y} W$$

Thus assembling the above maps we arrive at the composition defining $\tau(f, s)$. \square

5. TRACES FOR SHEAVES

In this section, we spell out how to apply the abstract formalism of traces of Section 3 and its geometric incarnation of Section 4 to categories of sheaves. Recall Definition 2.15:

Definition 5.1. A *sheaf theory* is a symmetric monoidal functor of $(\infty, 2)$ -categories

$$\underline{\mathcal{S}} : \underline{\text{Corr}}_k^{\text{prop}} \longrightarrow \underline{\text{dgCat}}_k$$

from correspondences of stacks (with 2-morphisms given by proper maps of correspondences) to dg categories. We denote by

$$\mathcal{S} : \text{Corr}_k \longrightarrow \text{dgCat}_k$$

the underlying 1-categorical sheaf theory, i.e. the symmetric monoidal functor on $(\infty, 1)$ -categories obtained by forgetting noninvertible morphisms.

Applying a sheaf theory to the geometric descriptions of traces of correspondences, one immediately deduces trace formulas for dg categories. We first spell out the consequences of the 1-categorical structure of a sheaf theory \mathcal{S} , then the trace formulae arising from its 2-categorical enhancement $\underline{\mathcal{S}}$, and finally in Section 5.3 explain how to use the results of [DG, GR2] to deduce applications of this formalism.

5.1. Dimensions and traces of sheaf categories: 1-categorical consequences. The graph of a map of derived stacks $f : X \rightarrow Y$ provides a correspondence from X to Y and a correspondence from Y to X . We denote the respective induced maps by $f_* : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ and $f^! : \mathcal{S}(Y) \rightarrow \mathcal{S}(X)$. The functoriality of \mathcal{S} concisely encodes base change for f_* and $f^!$. For $\pi : X \rightarrow pt = \text{Spec } k$, we denote by $\omega_X = \pi^! k \in \mathcal{S}(X)$ the \mathcal{S} -analogue of the dualizing sheaf, and by $\omega(X) = \pi_* \omega_X \in \mathcal{S}(pt) = dgVect_k$ the \mathcal{S} -analogue of “global volume forms”.

Next we will record formal consequences of our prior calculations deduced from the fact that a sheaf theory is symmetric monoidal.

Proposition 5.2. *Fix a sheaf theory $\mathcal{S} : Corr_k \rightarrow dgCat_k$, and $X, Y \in Corr_k$.*

- (1) $\mathcal{S}(X) \in dgCat_k$ is canonically self-dual, and for any $f : X \rightarrow Y$, $f^! : \mathcal{S}(Y) \rightarrow \mathcal{S}(X)$ and $f_* : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ are canonically transposes of each other.
- (2) $\mathcal{S}(X)$ is canonically symmetric monoidal with tensor product

$$\mathcal{F} \otimes^! \mathcal{G} = \Delta^!(\pi_1^! \mathcal{F} \otimes \pi_2^! \mathcal{G}) \quad \mathcal{F}, \mathcal{G} \in \mathcal{S}(X)$$

- (3) For any $f : X \rightarrow Y$, the projection formula holds:

$$f_* \mathcal{F} \otimes^! \mathcal{G} \simeq f_*(\mathcal{F} \otimes^! f^! \mathcal{G}) \quad \mathcal{F} \in \mathcal{S}(X), \mathcal{G} \in \mathcal{S}(Y)$$

- (4) There is a canonical equivalence of functors and integral kernels

$$\text{Hom}_{dgCat_k}(\mathcal{S}(X), \mathcal{S}(Y)) \simeq \mathcal{S}(X \times Y)$$

- (5) The functor $q_* p^! : \mathcal{S}(X) \rightarrow \mathcal{S}(Y)$ associated to a correspondence

$$X \xleftarrow{p} Z \xrightarrow{q} Y$$

is represented by the integral kernel $(p \times q)_* \omega_Z \in \mathcal{S}(X \times Y)$.

Proof. (1) Follows immediately from Proposition 4.1.

(2) Follows immediately from the commutative algebra structure on $X \in Corr_k$ (in fact commutative coalgebra structure on $X \in Stacks_k$) provided by the diagonal map.

- (3) Follows from base change for the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ id \times f \downarrow & & \downarrow \Delta \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \end{array}$$

- (4) Since \mathcal{S} is monoidal, we have

$$\mathcal{S}(X) \otimes \mathcal{S}(Y) \simeq \mathcal{S}(X \times Y)$$

The self-duality of $\mathcal{S}(X)$ provides

$$\text{Hom}_{dgCat_k}(\mathcal{S}(X), \mathcal{S}(Y)) \simeq \mathcal{S}(X)^\vee \otimes \mathcal{S}(Y) \simeq \mathcal{S}(X) \otimes \mathcal{S}(Y)$$

By construction, the composite identification assigns the functor

$$F_K(\mathcal{F}) = \pi_{2*}(\pi_1^! \mathcal{F} \otimes^! K) \quad K \in \mathcal{S}(X \times Y)$$

(5) Follows from the projection formula: consider the diagram

$$\begin{array}{ccccc} & & Z & & \\ & p \swarrow & \downarrow \Pi & \searrow q & \\ X & \xleftarrow{\pi_1} & X \times Y & \xrightarrow{\pi_2} & Y \end{array}$$

where $\Pi = p \times q$. Then we have

$$q_* p^!(-) \simeq \pi_{2*} \Pi_* \Pi^! \pi_1^!(-) \simeq \pi_{2*} \Pi_* (\omega_Z \otimes^! \Pi^! \pi_1^!(-)) \simeq \pi_{2*} (\Pi_* \omega_Z \otimes^! \pi_1^!(-))$$

□

Proposition 5.3. *Fix a sheaf theory $\mathcal{S} : \text{Corr}_k \rightarrow \text{dgCat}_k$.*

(1) *The \mathcal{S} -dimension $\dim(\mathcal{S}(X)) = HH_*(\mathcal{S}(X))$ of any $X \in \text{Corr}_k$ is S^1 -equivariantly equivalent with \mathcal{S} -global volume forms on the loop space*

$$\dim(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X).$$

In particular, for G an affine algebraic group, characters of \mathcal{S} -valued G -representations are adjoint-equivariant \mathcal{S} -global volume forms

$$\dim(\mathcal{S}(BG)) \simeq \omega(G/G)$$

(2) *The \mathcal{S} -trace of any endomorphism $Z \in \text{Corr}_k(X, X)$ is equivalent to \mathcal{S} -global volume forms on the restriction to the diagonal*

$$\text{Tr}(\mathcal{S}(Z)) \simeq \omega(Z|_\Delta)$$

In particular, the \mathcal{S} -trace of a self-map $f : X \rightarrow X$ is equivalent to \mathcal{S} -global volume forms on the f -fixed point locus

$$\text{Tr}(f_*) \simeq \omega(X^f)$$

Proof. (1) Follows immediately from Proposition 4.1(1). To spell this out, using the previous proposition and base change, $\dim(\mathcal{S}(X))$ results from applying the composition

$$\pi_* \Delta^! \Delta_* \pi^! \simeq \pi_* p_{2*} p_1^! \pi^! \simeq \mathcal{L}\pi_* \mathcal{L}\pi^! : \text{dgVect}_k \longrightarrow \text{dgVect}_k$$

to the unit $1_{\text{dgVect}_k} = k$. Here $\pi : X \rightarrow pt$ and $\mathcal{L}\pi : \mathcal{L}X \rightarrow pt$ are the maps to the point, and $p_1, p_2 : \mathcal{L}X \simeq X \times_{X \times X} X \rightarrow X$ are the two natural projections. Thus we find $\dim(\mathcal{S}(X)) \simeq \mathcal{L}\pi_* \mathcal{L}\pi^!(k) \simeq \omega(\mathcal{L}X)$. Furthermore, the S^1 -equivariance results from the one-dimensional cobordism hypothesis: the one-dimensional topological field theory defined by the dualizable object $\mathcal{S}(X) \in \text{dgCat}_k$ factors through that defined by the dualizable object $X \in \text{Corr}_k$. Moreover, we identified the S^1 -action on the dimension $\mathcal{L}X$ with loop rotation.⁶

(2) Similarly follows immediately from Proposition 4.1(2). □

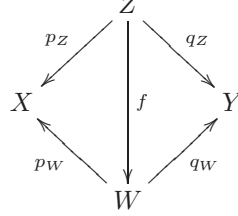
5.2. Integration formulas for traces: 2-categorical consequences. Now we turn to the functoriality of dimensions and traces. For this we require the 2-categorical enhanced version $\underline{\mathcal{S}}$ of a sheaf theory, so as to take advantage of the resulting functorial adjunction

$$\mathcal{S}(X) \xrightleftharpoons[f^!]{f_*} \mathcal{S}(Y)$$

for $f : X \rightarrow Y$ proper. In particular, applying the functor $\underline{\mathcal{S}}$ on two-morphisms we find that

⁶One can also check directly that the cyclic structure on the cyclic bar construction of the dg category $\mathcal{S}(X)$ is induced by the cyclic structure of the loop space $\mathcal{L}X$ under the identification $\omega(\mathcal{L}X) \simeq \dim(\mathcal{S}(X))$.

- A proper map $f : Z \rightarrow W$ of correspondences



induces a canonical integration morphism of integral transforms

$$\int_f : q_{Z*}p_Z^! \longrightarrow q_{W*}p_W^!$$

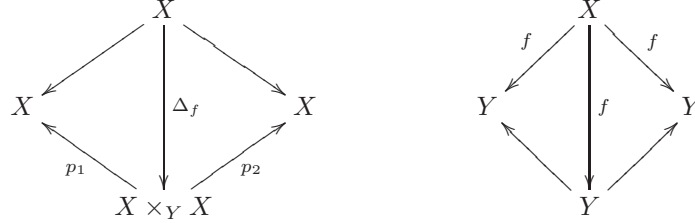
- In particular, when $X = Y = pt$, it induces a map of global volume forms

$$\int_f : \omega(Z) \longrightarrow \omega(W)$$

- There is a canonical composition identity

$$\int_g \circ \int_f \simeq \int_{g \circ f}$$

Proposition 5.4. *For $f : X \rightarrow Y$ proper, the unit and counit of the $(f_*, f^!)$ adjunction are given respectively by integration along the proper maps of self-correspondences $\Delta_{/Y} : X \rightarrow X \times_Y X$ of X and $f_{/Y} : X \rightarrow Y$ of Y :*



Proof. The assertion follows immediately from the geometric description of the unit and counit in the correspondence category, Proposition 4.3, upon applying the functor $\underline{\mathcal{S}}$. \square

Proposition 5.5. *For any proper map $f : X \rightarrow Y$, the induced map on dimensions*

$$\dim(f_*) : \dim(\mathcal{S}(X)) \longrightarrow \dim(\mathcal{S}(Y))$$

is identified (S^1 -equivariantly) with integration along the loop map

$$\dim(f_*) \simeq \int_{\mathcal{L}f} : \omega(\mathcal{L}X) \longrightarrow \omega(\mathcal{L}Y)$$

Proof. According to Definition 3.19, we must calculate the composition

$$\dim(f_*) : \mathrm{Tr}(\mathrm{id}_{\mathcal{S}(X)}) \xrightarrow{\mathrm{Tr}(\eta_f)} \mathrm{Tr}(f^! f_*) \xrightarrow{\sim} \mathrm{Tr}(f_* f^!) \xrightarrow{\mathrm{Tr}(\epsilon_f)} \mathrm{Tr}(\mathrm{id}_{\mathcal{S}(Y)})$$

The equivalence of the middle arrow is given by the canonical identifications

$$\mathrm{Tr}(f^! f_*) \simeq \omega((X \times_Y X) \times_{X \times X} X) \simeq \omega(X \times_{Y \times Y} Y) \simeq \mathrm{Tr}(f_* f^!)$$

By Proposition 5.4, the unit $\eta_f : \mathrm{id}_{\mathcal{S}(X)} \rightarrow f^! f_*$ is given by the integration morphism

$$\int_{\Delta_f} : \Delta_{f*} \omega_X \longrightarrow \omega_{X \times_Y X}$$

and hence its trace $\mathrm{Tr}(\eta_f) : \mathrm{Tr}(\mathrm{id}_{\mathcal{S}(X)}) \rightarrow \mathrm{Tr}(f^! f_*)$ is given by the induced integration map

$$\int_{\Delta_f} : \omega(\mathcal{L}X) \longrightarrow \omega((X \times_Y X) \times_{X \times X} X)$$

Likewise, the counit $\epsilon_f : f_* f^! \rightarrow \mathrm{id}_{\mathcal{S}(Y)}$ is given by the integration morphism

$$\int_f : f_* \omega_X \longrightarrow \omega_Y$$

and hence its trace $\mathrm{Tr}(\epsilon_f) : \mathrm{Tr}(f_* f^!) \rightarrow \mathrm{Tr}(\mathrm{id}_{\mathcal{S}(Y)})$ is given by the induced integration map

$$\int_f : \omega(X \times_{Y \times Y} Y) \longrightarrow \omega(\mathcal{L}Y)$$

Finally, by functoriality, their composition is given by the integration map

$$\int_{\mathcal{L}f} : \omega(\mathcal{L}X) \longrightarrow \omega(\mathcal{L}Y)$$

□

Finally, we have the functoriality of traces in parallel with the previous theorem on the functoriality of dimensions. Let us recall the relevant setup. Consider a proper morphism $f : X \rightarrow Y$ and endomorphisms $F_Z : X \rightarrow X$ and $F_W : Y \rightarrow Y$ in Corr_k given by respective self-correspondences $X \leftarrow Z \rightarrow X$ and $Y \leftarrow W \rightarrow Y$.

By an f -morphism from the pair (X, F_Z) to the pair (Y, F_W) , we mean an identification

$$s : Z \xrightarrow{\sim} X \times_Y W$$

of correspondences from X to Y . This in turn induces an identification of what might be called relative traces

$$Z \times_{Y \times Y} Y \xrightarrow{\sim} X \times_{Y \times Y} W$$

generalizing the relative loop space $\mathcal{L}_X Y$ from the case of the identity correspondences $Z = X$, $W = Y$. We thus obtain a map of traces

$$\tau(f, s) : Z|_{\Delta_X} = Z \times_{X \times X} X \longrightarrow Z \times_{Y \times Y} Y \xrightarrow{\sim} X \times_{Y \times Y} W \longrightarrow Y \times_{Y \times Y} W = W|_{\Delta_Y}$$

Proposition 5.6. *With the preceding setup, the trace map $\mathrm{Tr}(f_*, s) : \mathrm{Tr}(F_{X*}) \rightarrow \mathrm{Tr}(F_{Y*})$ is canonically identified with the integration map*

$$\int_{\tau(f, s)} : \omega(Z|_{\Delta_X}) \longrightarrow \omega(W|_{\Delta_Y})$$

Proof. The argument is parallel to the proof of Proposition 5.5. One calculates $\mathrm{Tr}(f_*, \alpha)$ from Definition 3.24 using Proposition 4.8 and the compatibility of Proposition 5.2 and the integration morphism for integral transforms from Section 2.3.2. □

5.3. Ind-coherent sheaves and \mathcal{D} -modules. We now apply the results of Gaitsgory-Rozenblyum [GR2] and Drinfeld-Gaitsgory [DG] establishing functoriality properties of categories of ind-coherent sheaves and \mathcal{D} -modules.

We first state a fundamental result of Gaitsgory-Rozenblyum [GR2]:

Theorem 5.7. [GR2, Theorem III.3.5.4.3, III.3.6.3] *There is a uniquely defined right-lax symmetric monoidal functor $\mathcal{Q}^!$ from the $(\infty, 2)$ -category whose objects are laft (locally almost of finite type) prestacks, morphisms are correspondences with vertical arrow ind-inf-schematic, and 2-morphisms are ind-proper and ind-inf-schematic, to the $(\infty, 2)$ category dgCat_k of k -linear presentable dg categories with continuous morphisms. The functor $\mathcal{Q}^!$ is strictly symmetric monoidal on the full subcategory of laft ind-inf-schemes.*

The theorem encodes a tremendous amount of structure in great generality. Let us highlight some salient features useful in practice. The theorem assigns a symmetric monoidal dg category $\mathcal{Q}^!(X)$ to any stack satisfying a reasonable finite type assumption. The symmetric monoidal structure, the $!$ -tensor product, is induced by $!$ -pullback along diagonal maps. For an arbitrary morphism $p : X \rightarrow Y$ there is a continuous symmetric monoidal pullback functor $f^! : \mathcal{Q}^!(Y) \rightarrow \mathcal{Q}^!(X)$, while for f schematic (or ind-schematic) there is a continuous pushforward $f_* : \mathcal{Q}^!(X) \rightarrow \mathcal{Q}^!(Y)$, which satisfies base change with respect to $!$ -pullbacks. Moreover for f proper (or ind-proper), $(f_*, f^!)$ form an adjoint pair.

The theorem goes much further through the powerful formalism of *inf-schemes*: prototypical inf-schemes are quotients of schemes by infinitesimal equivalence relation. Thus one can treat on an equal footing ind-coherent sheaves that are equivariant for any formal groupoid. The most important example is the de Rham space X_{dR} of a scheme, and one recovers \mathcal{D} -modules on X as ind-coherent sheaves on the de Rham functor of X , $\mathcal{D}(X) = \mathcal{Q}^!(X_{dR})$. Thus by first applying the functor $(-)_dR$ the theorem encodes the theory of \mathcal{D} -modules, as a functor out of the correspondence 2-category of stacks (or *laft* prestacks) with pullbacks for arbitrary maps and pushforward for (ind-)schematic maps.

Corollary 5.8. *The functors $\mathcal{Q}^!$ and \mathcal{D} define sheaf theories on laft ind-inf-schemes, i.e., define symmetric monoidal functors*

$$\mathcal{Q}^!, \mathcal{D} : \underline{\text{Corr}}(\text{ind} - \text{inf} - \text{Sch}_k)^{\text{ind-prop}} \longrightarrow \underline{\text{dgCat}}_k.$$

Thus the conclusions of Theorem 1.4 apply in this setting (in particular the Grothendieck-Riemann-Roch theorem for ind-proper maps of ind-inf-schemes).

5.3.1. The QCA setting. We are mostly interested in applications of sheaf theory on stacks, e.g. in an equivariant setting. That requires two features of the theories $\mathcal{Q}^!$ and \mathcal{D} that are not encoded in Theorem 5.7.

Theorem 5.7 produces in general a *right-lax* symmetric monoidal functor – in other words, we have the natural map

$$\mathcal{Q}^!(X) \otimes \mathcal{Q}^!(Y) \longrightarrow \mathcal{Q}^!(X \times Y)$$

satisfying the expected coherences, but it is not an equivalence in general (though it is for schemes).

Also, while the theorem encodes arbitrary pullbacks, it does not encode a continuous pushforward functor $p_* : \mathcal{Q}^!(X) \rightarrow \mathcal{Q}^!(Y)$ for non-schematic morphisms (though a generalization to include QCA morphisms has been announced by the authors). This precludes an immediate application of our formalism to traces on stacks.

However, since the full structure of a sheaf theory is far stronger than is needed for the “local” statements we discuss in this paper, we can get around this issue, by taking advantage of the following compilation of results of Drinfeld and Gaitsgory (specifically, see Section 3.6.1, Corollaries 3.7.14, 4.2.3, and 4.4.7, Proposition 4.4.11, Corollaries 8.3.4 and 8.4.3, Definition 9.3.2 and Proposition 9.3.12).

Theorem 5.9. [DG]

- (1) *For a QCA stack X , the categories $\mathcal{Q}^!(X)$ and $\mathcal{D}(X)$ are dualizable and canonically self-dual.*
- (2) *The canonical functors define equivalences*

$$\mathcal{Q}^!(X) \otimes \mathcal{Q}^!(Y) \simeq \mathcal{Q}^!(X \times Y), \quad \mathcal{D}(X) \otimes \mathcal{D}(Y) \simeq \mathcal{D}(X \times Y)$$

(3) For a morphism $f : X \rightarrow Y$ of QCA stack, the “renormalized pushforwards”⁷

$$f_{\bullet} : \mathcal{Q}^!(X) \rightarrow \mathcal{Q}^!(Y), \quad f_{\bullet} : \mathcal{D}(X) \rightarrow \mathcal{D}(Y)$$

defined as the transpose of $f^!$ (by the self-duality of (1)) are continuous functors satisfying base-change and the projection formula with respect to pullback.

Let us now briefly indicate how the results of the previous two sections carry over to QCA stacks in the absence of a fully fledged sheaf theory, where we use the renormalized pushforward functors f_{\bullet} provided by Theorem 5.9 to carry out non-representable pushforwards. In particular, for X a general QCA stack, so that $pi_X : X \rightarrow pt$ is not representable, this means that the notation $\omega(X)$ has to be taken in a renormalized fashion,

$$\omega(X) := \pi_{X,\bullet} \pi_X^! k = \pi_{X,\bullet} \omega_X.$$

For the theory of ind-coherent sheaves this produces the usual notion of derived global sections of the dualizing complex, but for the theory of \mathcal{D} -modules this will differ in general from the nonrenormalized version, namely Borel-Moore chains on X :

$$\omega(X)_{non-renorm} = R\Gamma_{dR}(\omega_X) = C_*^{BM}(X).$$

In Proposition 5.2, the self-duality in assertion (1) for QCA stacks is the content of Theorem 5.9(1), and f_{\bullet} is defined so as to make it the transpose of $f^!$. Assertion (2) follows from the sheaf theory construction (i.e. is independent of non-representable morphisms), the projection formula is asserted in item (3) of the theorem and the last two assertions are deduced from the first three. Proposition 5.3 is also deduced directly from Proposition 5.2.

The general trace construction \int_f discussed in Section 5.2 depends only on the functoriality of proper adjunction (which is part of the [GR2] formalism) and the definition of pullback and (renormalized) pushforward. The identities needed to verify Propositions 5.4, 5.5 and 5.6 only depend on base change, which is guaranteed by Theorem 5.9. We therefore get as a payoff that the conclusions of Theorem 1.4 hold for QCA stacks, in particular:

Theorem 5.10. *Let $\mathcal{S} = \mathcal{Q}^!$ or $\mathcal{S} = \mathcal{D}$ denote either ind-coherent sheaves or \mathcal{D} -modules. Let $f : X \rightarrow Y$ denote a proper morphism of QCA stacks.*

• *For any compact object $M \in \mathcal{S}(X)$ (coherent sheaf or safe coherent \mathcal{D} -module) with character $[M] \in HH_*(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X)$, there is a canonical identification*

$$[f_* \mathcal{M}] \simeq \int_{\mathcal{L}f} [M] \in HH_*(\mathcal{S}(Y)) \simeq \omega(\mathcal{L}Y)$$

• *Assume $Y = BG$ for an affine group and $X = Z/G$ for Z a proper QCA stack. Then for any compact object $M \in \mathcal{S}(Z/G)$ (G -equivariant coherent sheaf or safely equivariant coherent \mathcal{D} -module on X), and element $g \in G$, there is a canonical identification*

$$[f_* M]|_g \simeq \int_{\mathcal{L}f} [M]|_{X^g}$$

• *For a map $f : (X, Z) \rightarrow (Y, W)$ of QCA stacks with self-correspondences, the induced map $Tr(\mathcal{S}(Z)) \rightarrow Tr(\mathcal{S}(W))$ is given by integration along fixed points $Z|_{\Delta_X} \rightarrow W|_{\Delta_Y}$.*

⁷We note that for $\mathcal{Q}^!$ the “renormalized” pushforward is the standard pushforward functor, while for \mathcal{D} it differs for non-safe objects from the more familiar, but discontinuous, de Rham pushforward.

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