TUTORIAL: GROUP ACTIONS ON CATEGORIES

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Contents

1. Strong and weak actions

Let G be an affine algebraic group. There are two versions of G acting on $\mathcal{C}$. For $\mathcal{C} \in \mathrm{DGCat}$: weak and strong.

An action of $D(G)$ on $\mathcal{C}$ is a "strong action." This is the categorical analogue of $G(\mathbb{F}_q)$ acting on a vector space V.

An action of $\mathrm{QCoh}(G)$ on $\mathcal{C}$ is a "weak action." There is no classical analogue of this notion.

Remark 1.1. There is a monoidal (with respect to convolution) functor $\mathrm{Ind}\colon \mathrm{QCoh}(G) \to D(G)$. This induces a restriction functor from strong G-actions to weak G-actions.

2. Examples

The main constructions involving group actions on categories are taking the invariants or coinvariants. For G acting strongly on $\mathcal{C}$, we write $\mathcal{C}^G, \mathcal{C}_G$ for invariant and coinvariants. Write $\mathcal{C}^{G,w}$ and $\mathcal{C}_{G,w}$ for weak invariants and co-invariants.

Example 2.1. If G acts on X, then G acts weakly on $\mathcal{C} := \mathrm{QCoh}(X)$ or $\mathcal{C} := \mathrm{IndCoh}(X)$. Then $\mathcal{C}^{G,w} = \mathrm{QCoh}(X/G)$ (resp. $\mathrm{IndCoh}(X/G)$) where X/G is the stack quotient.

Example 2.2. In geometric terms, if Y is a prestack then G acts $\mathrm{QCoh}(Y)$ "if and only if" (i.e. morally) $\widehat{G}_e$ acts trivially on Y. If G acts on X then G_{dR} acts on X_{dR}, which is tautologically equivalent to G acting strongly on $\mathrm{QCoh}(X_{\mathrm{dR}})$.

Example 2.3. We have $D(X)^G = D(X/G) = D_G(X)$, the Bernstein-Lunts equivariant derived category.

On the other hand, $D(X)^{G,w}$ is the category of *weakly equivariant* D-modules on X. Note that D_X is weakly equivariant but not strongly so.

Example 2.4. A group G acts strongly on $D(G)$ on both sides. Therefore, G acts strongly $D(G)^{G,w} \cong \mathfrak{g}\text{–}\mathsf{mod}$. This has no classical incarnation.

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Notes by Tony Feng.

3. Invariants and coinvariants

So what is the definition of invariants? Usually if G acts on V, the space of invariants is the equalizer of the diagram

$$V \rightrightarrows V \otimes \mathrm{Fun}(G) = \{G \to V\}$$

where the two maps are the constant and coaction maps. In the categorical situation, there are two functors

$$\mathcal{C} \rightrightarrows \mathcal{C} \otimes D(G).$$

In fact that there is a cosimplicial diagram

$$\mathcal{C} \rightrightarrows \mathcal{C} \otimes D(G) \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots$$

whose limit is, by definition, $\mathcal{C}^G$.

Example 3.1. For $\mathcal{C} = D(X)$. Since $D(-)$ sends products to tensor products, we have

$$D(X)^G := \lim \left(D(X) \rightrightarrows D(X \times G) \substack{\rightarrow\\\rightarrow\\\rightarrow} \ldots \right).$$

Exercise 3.2. For $X = \mathrm{pt}$, consider the trivial D-module $k \in D(\mathrm{pt})^G$. Show that

$$\mathrm{End}_{D(\mathrm{pt})^G}(k) = H^*_G(\mathrm{pt}).$$

Remark 3.3. What is the difference between D-modules and constructible sheaves? Anything making reference to the Lie algebra $\mathfrak{g}$ or weak actions is illegal. Also, the category of sheaves on X needs to remembered along with the category of sheaves on $X \times Y$ for any Y: unlike for D-modules, the category of sheaves on $X \times Y$ cannot be recovered from that on X.

There are adjoint factors

$$\mathrm{triv}\colon \mathrm{DGCat}_{\mathrm{cont}} \rightleftarrows \mathbf{G}\text{–}\mathsf{mod} : \mathcal{C} \mapsto \mathcal{C}^G.$$

This is essentially for formal reasons.

The *coinvariants* of G acting on $\mathcal{C}$ are $\mathcal{C}_G := \mathcal{C} \otimes_{D(G)} \mathrm{Vect}$ where G acts trivially on Vect. This means that

$$\mathrm{colim}(\ldots D(G) \otimes \mathcal{C} \otimes \mathcal{C} \substack{\rightarrow\\\rightarrow\\\rightarrow} D(G) \otimes \mathcal{C} \rightrightarrows \mathcal{C}) =: \mathcal{C}_G.$$

This is left adjoint to triv, again for formal reasons.

4. The averaging functor

Theorem 4.1 (Gaitsgory)**.** *(1) There is a forgetful functor* $\mathrm{Oblv} : \mathcal{C} \to \mathcal{C}^G$*, which admits a continuous right adjoint* $\mathrm{Av}_* = \mathrm{Av}_*^G$.

(2) There is a forgetful functor $\mathrm{Oblv} : \mathcal{C} \to \mathcal{C}^{G,w}$*, which admits a continuous right adjoint* $\mathrm{Av}_*^{G,w}$.

(3) The induced functor θ *as in*

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{C}^G \\ & \mathcal{C}_G \arrow[ur, dashed, "\theta"'] & \end{tikzcd}$$

is an equivalence, and similarly in the weak setting.

Remark 4.2. This is an analogue of the fact that for a finite group G acting on a vector space $V/\mathbf{Q}$, we can average over the group to get a map from V to its invariants.

Here is an important difference between weak invariants and classical invariants:

Theorem 4.3 (Gaitsgory)**.** *The functor*

$$\mathbf{G}\text{–}\mathsf{mod}_{\text{weak}} \xrightarrow{(-)^{G,w}} \mathrm{DGCat}_{\text{cont}}$$

is conservative. This is totally false for strong invariants!

Heuristically, $D_X \in D(X)^{G,w}$ is weakly invariant, while $\omega_X \in D(X)^G$. Now D_X "knows everything" in $D(X)$ for X affine, while ω_X "knows only itself."

Example 4.4. I'll give one example where this formalism helped me understand something.

Let G be reductive and $B \subset G$ be a Borel. Arkhipov defined twisting functors T_w on category $\mathcal{O}$. The T_w are automorphisms of the derived category $\mathcal{O}$. I tried to read his paper but I could not understand the construction.

General setup: $\mathfrak{g}_{\text{mod}}^B$ is the direct sum

$$\mathfrak{g}_{\text{mod}}^B = \bigoplus_{\lambda \in \Lambda/W} \mathcal{O}_{[\lambda]}.$$

It is a general fact that for G acting on $\mathcal{C}$, the Hecke algebra $D(B\backslash G/B)$ acts on $\mathcal{C}^B$. In fact, $\mathcal{H}_{G,B} = \mathrm{End}(\mathcal{C} \mapsto \mathcal{C}^B)$. (This is endomorphisms of the functor sending a category to its B-invariants.) We have an inclusion

$$B\backslash G/B \hookleftarrow X_w = B\backslash BwB/B \colon j_w.$$

Then we get $j_{w,*,\mathrm{dR}}(\omega_{X_w}) \in \mathcal{H}_{G,B}$. These are the twisting operators.

Fact: these are invertible with inverse $j_{w^{-1},!}(k_{X_{w^{-1}}})$ (in $\mathcal{H}_{G,B}$). They correspond to Arkhipov's twisting functors.