Honors Single Variable Calculus 110.113

October 13, 2023

Contents

1 The natural numbers

Lecture 1, Monday, August 28th, Last updated: 01/09/23, dmy.
Reading: [12, Ch.2-3]

We assume the notion of *set*, 2, and take it as a primitive notion to mean a "collection of distinct objects."

Learning Objectives

Next eight lectures:

- To construct the objects:

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}$$

  and define the notion of *sets*, 2.

- To prove properties and reason with these objects. In the process, you will learn various proof techniques. Most importantly, *proof by induction* and *proof by contradiction*.

This lecture:

- how to define the natural numbers, $\mathbb{N}$, and appreciate the role of *definitions*.
- how to apply induction. In particular, we would see that even proving statements as associativity of natural numbers is nontrivial!

Pedagogy

1. $\mathbb{N}$ is presented differently in distinct foundations, such as ZFC or type theory. Our presentation is to be *agnostic* of the foundation. From a working mathematician point of view, it *does not matter*, how the natural numbers are constructed, as long as they obey the properties of the axioms, 1.1.
2. We take the point of view that in mathematics, there are various type of objects. Among all objects studied, some happened to be *sets*. Some presentation of mathematics[a] will regard all objects as sets.

The various types of mathematics are more or less equivalent in our context.

[a] such as ZFC

Why should we delve into the foundations? Two reasons:

1. Foundational language is how many mathematicians do new mathematics. One defines new axioms and explores the possibilities.

2. How can we even discuss mathematics without having a rigorous understanding of our objects?

Discussion

A *natural (counting) number*[a], as we conceived informally is an element of

$$\mathbb{N} := \{0, 1, 2, \ldots\}$$

What is ambiguous about this?

- What does "$\cdots$" mean? How are we sure that the list does not cycle back?

- How does one perform operations?

- What *exactly* is a natural number? What happens if I say

$$\{0, A, AA, AAA, AAAA, \ldots\}$$

  are the numbers?

We will answer these questions over the course.

[a] It does not matter if we regard 0 as a natural number or not. This is a convention.

Forget about the natural numbers we love and know. If one were to define the *numbers*, one might conclude that the numbers are about a concept.

Axioms 1.1. The *Peano Axioms*: [1] Guiseppe Peano, 1858-1932.

1. 0 is a natural number.
$$0 \in \mathbb{N}$$

2. if n is a natural number then we have a natural number, called the *successor of* n, denoted $S(n)$.
$$\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$$

[1] In 1900, Peano met Russell in the mathematical congress. The methods laid the foundation of *Principa Mathematica*

3. 0 is not the successor of any natural number.

$$\forall n \in \mathbb{N}, S(n) \neq 0$$

4. If $S(n) = S(m)$ then $n = m$.

$$\forall n, m \in \mathbb{N}, S(n) = S(m) \Rightarrow n = m$$

5. Principle of induction. Let $P(n)$ be any *property* on the natural number n. Suppose that

   a. $P(0)$ is true.

   b. When ever $P(n)$ is true, so is $P(S(n))$.

   Then $P(n)$ is true for all n natural numbers.

Discussion

What could be meant by a *property?* The principle of induction is in fact an *axiom schema*, consisting of a collection of axioms.

- "n is a prime".
- "$n^2 + 1 = 3$".

We have not yet shown that any collection of object would satisfy the axioms. This will be a topic of later lectures. So we will assume this for know.

Axiom 1.2. There exists a set $\mathbb{N}$, whose elements are the *natural numbers*, for which 1.1 are satisfied.

There can be many such systems, but they are all equivalent for doing mathematics.

Discussion

With only up to axiom 4: This can be *not* so satisfying. What have we done? We said we have a collection of objects that satisfy some concept F="natural numbers". But how do we know, Julius Ceasar does not belong to this concept?

Definition 1.3. We define the following natural numbers:

$$1 := S(0), 2 := S(1) = S(S(0)), 3 := S(2) = S(S(S(0)))$$

$$4 := S(3), 5 := S(4)$$

Intuitively, we want to continue the above process and say that whatever created iteratively by the above process are the *natural numbers*.

Discussion

- Give a set that satisfies axioms 1 and 2 but not 3.
- Give a set that satisfies axioms 1,2 and 3 but not 4.
- Give a set satisfying axioms 1,2,3 and 4, but not 5.

$$\{n/2 : n \in \mathbb{N}\} = \{0, 0.5, 1, 1.5, 2, 2.5, \cdots\}$$

Proposition 1.4. 1 is not 0.

Proof. Use axiom 3. □

Proposition 1.5. 3 is not equal to 0.

Proof. $3 = S(2)$ by definition, 1.3. If $S(2) = 0$, then we have a contradiction with Axiom 2, 1.1. □

1.1 Addition

Definition 1.6. (Left) Addition. Let $m \in \mathbb{N}$.

$$0 + m := m$$

Suppose, by induction, we have defined $n + m$. Then we define

$$S(n) + m := S(n + m)$$

In the context of 1.13, for each n, our function is $f_n := S : \mathbb{N} \to \mathbb{N}$ is $a_{S(n)} := S(a_n)$ with $a_0 = m$.

Proposition 1.7. For $n \in \mathbb{N}$, $n + 0 = n$.

Proof. Warning: we cannot use the definition 1.6. We will use the principle of induction. What is the *property* here in Axiom 5 of 1.1? The property $P(n)$ is "$0 + n = n$" for each $n \in \mathbb{N}$. We will also have to check the two conditions 5a. and 5b.

a "$P(0)$ *is true.*". People refer to this as the "base case $n = 0$": $0 + 0 = 0$, by 1.6.

b "*If* $P(m)$ *is true then* $P(m+1)$ *is true*". The statement "*Suppose* $P(m)$ *is true*" is often called the "inductive hypothesis". Suppose that $m+0=m$. We need to show that $P(S(m))$ *is true*, which is

$$S(m)+0=S(m)$$

By def, 1.6,

$$S(m)+0=S(m+0)$$

By hypothesis,

$$S(m+0)=S(m)$$

By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. $\square$

Such proof format is the typical example for writing inductions, although in practice we will often leave out the italicized part.

Example

Prove by induction

$$\sum_{i=1}^{n} i^2 := 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

We observed that we have successfully shown *right* addition with respect to 0 behaves as expected.

Discussion

What should we expect $n+S(m)$ to be?

- Why can't we use 1.6?
- Where would we use 1.7?

Proof is hw.

Proposition 1.8. Prove that for $n, m \in \mathbb{N}$, $n+S(m)=S(n+m)$.

Proof. We induct on n. Base case: $m=0$.

5b. Suppose $n+S(m)=S(n+m)$. We now prove the statement for

$$S(n)+S(m)=S(S(n)+m)$$

by definition of 1.6,

$$S(n)+S(m)=S(n+S(m))$$

which equals to the right hand side by hypothesis.

$\square$

Proposition 1.9. Addiction is commutative. Prove that for all $n, m \in \mathbb{N}$,

$$n + m = m + n$$

Proof. We prove by induction on n. With m fixed. We leave the base case away. $\square$

Proposition 1.10. Associativity of addition. For all $a, b, c \in \mathbb{N}$, we have

$$(a + b) + c = a + (b + c)$$

Proof. hw. $\square$

Discussion

Can we define "+" on any collection of things? What are examples of operations which are noncommutative and associative? For example, the collection of words?

$$+ : (\text{Seq. English words}) \times (\text{Seq. English words}) \to (\text{Seq. English words})$$

$$\text{"a"}, \text{"b"} \mapsto \text{"ab"}$$

This can be a meaningless operation. Let us restrict to the collection of *interpreable* outcomes. In the following examples, there is *structural ambiguity.*

1. (Ice) (cream latte)
2. (British) ((Left) (Waffles on the Falkland Islands))
3. (Local HS Dropouts) (Cut) (in Half)
4. (I ride) (the) (elephant in (my pajamas))
5. (We) ((saw) (the) (Eiffel tower flying to Paris.))

2,3 are actuay news title.

What use is there for addition? We can define the notion of *order* on $\mathbb{N}$. We will see later that this is a *relation* on $\mathbb{N}$.

Definition 1.11. Ordering of $\mathbb{N}$. Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \leq n$ iff there is $a \in \mathbb{N}$, such that $n = m + a$.

1.2 Multiplication

Now that we have addition, we are ready to define multiplication as 1.6.

Definition 1.12.

$$0 \cdot m := 0$$

$$S(n) \cdot m := (n \cdot m) + m$$

1.3 Recursive definition

What does the induction axiom bring us? Please ignore the following theorem on first read.

Theorem 1.13. Recursion theorem. Suppose we have for each $n \in \mathbb{N}$,

$$f_n : \mathbb{N} \to \mathbb{N}$$

Let $c \in \mathbb{N}$. Then we can assign a natural number a_n for each $n \in \mathbb{N}$ such that

$$a_0 = c \quad a_{S(n)} = f_n(a_n) \forall n \in \mathbb{N}$$

Discussion

The theorem seems intuitively clear, but there can be pitfalls.

- When defining $a_0 = c$, how are we sure this is *not* redefined after some future steps? This is Axiom 3. of 1.1
- When defining $a_{S(n)}$ how are we sure this is not redefined? This uses Axiom 4. of 1.1.
- One rigorous proof is in [5, p48], but requires more set theory.

Proof. The property $P(n)$ of 1.1 is "{ a_n is well-defined}". Start with $a_0 = c$.

- Inductive hypothesis. Suppose we have defined a_n - meaning that there is only one value!
- We can now define $a_{S(n)} := f_n(a_n)$.

□

1.4 References and additional reading

- Nice lecture notes by Robert.
- Russell's book [9, 1,2] for an informal introduction to cardinals.

2 Naïve set theory: the axioms

Week 1, Wednesday, August 30th
As in the construction of $\mathbb{N}$, we will define a *set* via axioms.

Discussion

Why put a foundation of sets?

- This is to make rigorous the notion of a "collection of mathematical objects".
- This has its roots in cardinality. How can you "count" a set without knowing how to define a collection?
- The concept of a set can be used - and is till used in practice - as a practical foundation of mathematics. This forms the basis of *classical mathematics*.

Learning Objectives

In this lecture:

- We discuss *set* in detail. We will need this to construct the integers, $\mathbb{Z}$.
- We illustrate what one *can* and *can not* do with sets.

Pedagogy

Again, we don't say what they *are*. This approach is often taken, such as [5].

Discussion

What object can be called a *set*?

A *set* should be

- determined by a *description of the objects* [a] For example, we can consider
$$E := \text{"The set of all even numbers"}$$
$$P := \text{"The set of all primes"}$$
- If x is an object and A is a set, then we can ask whether $x \in A$ or $x \notin A$. *Belonging* is a primitive concept in sets.

[a] this set consists of all objects satisfying this description and *only those objects*.

In this lecture we will discuss some axioms.

Axiom 2.1. If A is a *set* then A is also a *object*.

Axiom 2.2. Axiom of extension. Two sets A, B are equal if and only if (for all objects x , $(x \in A \Leftrightarrow x \in B)$)

Axiom 2.3. There exist a set $\emptyset$ with no elements. I.e. for any object x, $x \notin \emptyset$.

Proposition 2.4 (Single choice)**.** Let A be nonempty. There exists an object x such that $x \in A$.

Proof. Prove by contradiction. Suppose the statement is false. Then for all objects x, $x \notin A$. By axiom of extension, $A = \emptyset$. □

Discussion

How did we use the axiom of extension? Colloquially, some mathematicians would say "trivially true".

2.1 Subcollections

Definition 2.5. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1,2\} \subset \{1,2,3\}$.

2.2 Comprehension axiom

Definition 2.6. Axiom of Comprehension.

Definition 2.7. *General* comprehension principle. (The paradox leading one). For any property φ, one may form the set of all x with property $P(x)$, we denote this set as

$$\{x \big| P(x)\}$$

Proposition 2.8. Russell, 1901. The general comprehension principle cannot work.

Proof. Let

$$R := \{x : x \text{ is a set and } x \notin x\}$$

This is a set. Then

$$R \in R \Leftrightarrow R \notin R$$

□

Discussion

How is this different from the axiom of specification?

Discussion

How can it even be the case that $x \in x$, for a set? Can this hold for any set x below?

- $\emptyset$
- The set of all primes.
- The set of natural numbers.

The latter two shows that : this set itself is *not even a number*! Indeed, In Zermelo-Frankel set theory foundations it will be proved that $x \notin x$ for all set x. So the set R in 2.8 is the *set of all sets.*

2.3 References

- A nice introduction to set theory is Saltzman's notes [10].
- The relevant section in Tao's notes, [12, 3].
- For the axioms of set theory, an elementary introduction is [5], and also notes by Asaf, [7].

3 Homework for week 1

[2]

In these exercises: our goal is to get familiar with

- manipulating axioms in a definition.
- the notion of the principle of induction.

Problems:

1. Prove 5 is not equal to 2.
2. (*) Prove 1.8.
3. (*) Prove 1.9, assuming 1.8 if necessary.
4. (*) Prove 1.10 assuming 1.8, 1.9 if necessary.
5. (*) $n \in \mathbb{N}$ is *positive* if and only if $n \neq 0$. Prove that if $a, b \in \mathbb{N}$, *a is positive,* then $a + b$ is positive.
6. (***) Let M be a set with 2023 elements. Let N be a positive integer, $0 \leq N \leq 2^{2023}$. Prove that it is possible to color each subset of S so that
   (a) The union of two white subsets is white.
   (b) The union of two black subsets is black.
   (c) There are exactly N white subsets.
7. (**) Integers 1 to n are written ordered in a line. We have the following algorithm:
   - If the first number is k then reverse order of the first k numbers.

   Prove that 1 appears first in the line after a finite number of steps.
8. (**) We defined $\leq$ of natural numbers in 1.11. A finite sequence $(a_i)_{i=1}^n := \{a_1, \ldots, a_n\}$ of natural numbers is *bounded*, if there exists some other natural number M, such that $a_i \leq M$ for all $1 \leq i \leq n$. Show that every finite sequence of natural numbers, $a_1, \ldots, a_n$, is bounded.

[2] *Due: Week 2, Write the numbering of the three questions to be graded clearly on the top of the page. Each unstarred problem worth 12 points. Each star is an extra 5 points.*

Hints for problems

1: prove using Peano's axioms. First prove 3 is not equal to 0.

6: The number 2023 is irrelevant. Induct on the size of the set M. What happens when $M = 1$? For the inductive argument: suppose the statement is true when M has size n. In the case when M has size $n+1$, consider when

- $0 \leq N \leq 2^n$. Use the hypothesis on the first n elements.
- $N \geq 2^n$. Use symmetry here that there was nothing special about "white".

7: Let us consider the inductive scenario. If $n+1$ were in the first position, we are done by induction. Thus, let us suppose $n+1$ never appears in the first position, *and* it is not in the last position, which is given by number $k \neq n+1$.

- Would the story be the same if we switch the position of k and $n+1$?

Discussion

As one observes, both 6 and 7 uses a natural *symmetry* in the problem.

4 Power set construction

Lecture 3: will miss one class due to Labor day.
Reading: [12, Ch.3.1-4], [8, 2].

Learning Objectives

In last lectures, we

- Defined $\mathbb{N}$ axiomatically using the Peano axioms.
- Used induction to prove properties of operations as $+$ and $\times$ on $\mathbb{N}$.

In the next two lectures

- Discuss the remaining axioms of set theory. We begin by discussing new notions: *subsets*, 2.1, We end with the construction of the power set.
- Discuss *equivalence relation*, 7, and *ordered pairs*, 7.1. which constructs the integers and the rationals

4.1 Remaining axioms of set theory

Week 2

In this section we continue from previous lecture and discuss remaining axioms from what is known as the *Zermelo-Fraenkel (ZF) axioms of set theory*, due to Ernest Zermelo and Abraham Fraenkel.

Axiom 4.1. Singleton set axiom. If a is an object. There is a set $\{a\}$ consists of just one element.

Axiom 4.2. Axiom of pairwise union. Given any two sets A, B there exists a set $A \cup B$ whose elements which belong to either A or B or both.

Often we would require a stronger version.

Axiom 4.3. Axiom of union. Let A be a set of sets. Then there exists a set

$$\bigcup A$$

whose objects are precisely the elements of the set.

Example

Let

- $A = \{\{1,2\},\{1\}\}$
- $A = \{\{1,2,3\},\{9\}\}$

Discussion

Using the axioms, can we get from $\{1,3,4\}$ to $\{2,4,5\}$?

We will now state the power set axiom for completeness but revisit again.

Axiom 4.4. Axiom of power set. Let X, Y be sets. Then there exists a set Y^X consists of all functions $f : X \to Y$.

We will review definition of function later, 4.11.

4.2 Replacement

If you are an ordinary mathematician, you will probably never use replacement.

Axiom 4.5. Axiom of replacement. For all $x \in A$, and y any object, suppose there is a statement $P(x, y)$ pertaining to x and y. $P(x, y)$ satisfies the property for a given x, there is a *unique* y. There is a set

$$\{y : P(x, y) \text{ is true for some } x \in A\}$$

Discussion

This can intuitively be thought of as the set

$$\{y : y = f(x) \text{ some } x \in A\}$$

That is, *if* we can define a function, then the range of that function is a set. However, $P(x, y)$ described may *not* be a function, see [4, 4.39].

Example

- Assume, we have the set $S := \{-3, -2, -1, 0, 1, 2, 3, \ldots\}$, $P(x, y)$ be the property that $y = 2x$. Then we can construct the set

$$S' := \{-6, -4, -2, 0, 2, 4, 6, \ldots\}$$

- If x is a set, then so is $\{\{y\} : y \in x\}$. Indeed, we let

$$P(x, y) : \text{"}y = \{x\}\text{"}$$

Again, this is a *schema* as described previously in axiom of comprehension 2.6.

Proposition 4.6. The axiom of comprehension 2.6 follows from axiom of replacement 4.5.

Proof. Let ϕ be a property pertaining to the elements of the set X. We can define the property [3]

$$\psi(x, y) : \begin{cases} y = \{x\} & \text{if } \phi(x) \text{ is true} \\ \emptyset & \text{if } \phi(x) \text{ is false} \end{cases}$$

Let

$$\mathcal{A} := \{y : \exists x, \quad \psi(x, y) \text{ is true}\}$$

be the collection of sets defined by axiom of replacement. Then by union axiom

$$\bigcup \mathcal{A} := \{x \in X : \phi(x) \text{ is true}\}$$

□

4.3 Axiom of regularity (well-founded)

As a first read, you can skip directly and read 4.9. For a set S, and a binary relation, $<$ on S, we can ask if it is *well-founded.* It is well founded when we can do *induction.*

Definition 4.7. A subset A of S is $<$*-inductive* if for all $x \in S$,

$$(\forall t \in S, t < x) \Rightarrow x \in A$$

Definition 4.8. Let X, Y we denote the *intersection of* X *and* Y[4] as

$$X \cap Y := \{x \in X : x \in X \text{ and } x \in Y\} = \{y \in Y : y \in X \text{ and } y \in Y\}$$

X and Y are *disjoint* if $X \cap Y = \emptyset$.

[3] This can be written in the language of "property" via $(\phi(x) \to y = \{x\}) \wedge (\neg\phi(x) \to y = \emptyset)$

[4] which exists, thanks to axiom of comprehension.

One would ould ask the $\in$ relation on all sets to be inductive. Then what would be required for that $A \notin A$?

Axiom 4.9. Axiom of foundation (regularity) The $\in$ relation is "well-founded". That is for all nonempty sets x, there exists $y \in x$ such that either

- y is not a set.
- or if y is a set, $x \cap y = \emptyset$.

An alternative way to reformulate, is that y is a *minimal element* under $\in$ relation of sets.

Example

- $\{\{1\}, \{1,3\}, \{\{1\}, 2, \{1,3\}\}\}$. What are the $\in$-minimal elements?
- Can I say that there is a "set of all sets"? No, see how.

One can use axiom of foundation that we cannot have an infinite descending sequence:

Proposition 4.10. There are no infinite descent $\in$-chains. Suppose that (x_n) is a sequence of nonempty sets. Then we cannot have

$$\cdots \in x_{n+1} \in x_n \cdots \in x_1 \in x_0$$

Similarly one can use axiom of replacement for the product, at p32.

4.4 Function

Discussion

How would you intuitively define a function?

Definition 4.11. Let X, Y be two sets. Let

$$P(x, y)$$

be a *property* pertaining to $x \in X$ and $y \in Y$, such that for all $x \in X$, there *exists a unique* $y \in Y$ such that $P(x, y)$ is true. A *function associated to* P is an object

$$f_P : X \to Y$$

such that for each $x \in X$ assigns an output $f_P(x) \in Y$, to be the unique object such that $P(x, f_P(x))$ is true. [5]

[5] We will often omit the subscript of P.

- X is called the *domain*
- Y is called the *codomain*.

Definition 4.12. The *image*...?

Discussion

What kind of properties P does not satisfy the condition of being function?

- "$y^2 = x$".
- "$y = x^2$".

5 The various sizes of infinity

Lecture 4: for competition. We will use our defined notion of, "counting numbers" or "inductive numbers", $\mathbb{N}$ to *count* other sets. This is *cardinality*. In this section, we fix sets X, Y.

Definition 5.1. A function $f : X \to Y$ is

- *injective* if for all $a, b \in X$, $f(a) = f(b)$ implies $a = b$.
- *surjective* if for all $b \in Y$, exists $a \in X$ st. $f(a) = b$.
- *bijective* if f is both injective and bijective.

Example

- the map from $\emptyset \to X$ an injection. The conditions for injectivity vacuously holds.
- $\mathbb{N}$ is in bijection with the set of even numbers,

$$\mathbb{E} := \{n \in \mathbb{N}\,;\, \exists k \in \mathbb{N} \,:\, n = 2k\}$$

- there is no bijection from an empty set to a nonempty set.

Definition 5.2. Two sets X, Y have *equal cardinality* if there is a bijection

$$X \simeq Y$$

- A set is said to have *cardinality* n if

$$\{i \in \mathbb{N} \,:\, 1 \leq i \leq n\} \simeq X$$

  In this case, we say X is *finite*. Otherwise, X is *infinite*.

- A set X is *countably infinite*[6] if it has same cardinality with $\mathbb{N}$.

Definition 5.3. We denote the *cardinality of a set* X by $|X|$. [7]

[6] Or *countable*. Sometimes countable means (finite *and* countably infinite).

[7] This definition does *not make sense yet!*. What if a set has two cardinality? Let us assume this is well-defined first. See question 2.

Discussion

To think about infinity is an interesting problem. Consider Hilbert's Grand Hotel.

- One new guest.
- 1000 guest.
- Hilbert Hotel 2 move over.
- Hilbert chain. Directs customer m in hotel n to position $3^n \times 5^m$. (This shows that $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$.)

Historically, some take *cardinal numbers* as i.e. the equivalence class of bijective sets as the primitive notion.

Definition 5.4. Let X, Y be sets: We denote

- $|X| \leq |Y|$ if there is an injection from X to Y.
- $|X| = |Y|$ if there is a bijection between X and Y.
- $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

One of the beautiful results in Set theory is the Schroeder Bernstein theorem.

Theorem 5.5. The $\leq$ relation on cardinality, is reflexive: if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$. [8]

Without axiom of choice, one cannot say the following: for all sets X and Y, either $|Y| \leq |X|$ or $|X| \leq |Y|$.

[8] Why is this not obvious? Challenge: google and try to understand the proof.

6 Homework for week 2

Due: Week 3, Friday. All questions in 6.1, Boolean algebra is compulsory. Select 3 other questions to be graded.

Reading: A nice reference in set theory, [3, 4]. We collectively refer to the axioms of set theory we have discussed thus far as the ZF axioms. We did not discuss the axiom of replacement, [12, 3.5] and regularity. This will be left as required reading for certain problems.

Problems

1. Let A, B, C be sets.

   (a) Prove set inclusion, is reflexive and transitive, i.e.

   $$(A \subseteq B \wedge B \subseteq A) \Rightarrow A = B$$

   $$(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$$

   the notation $\wedge$ here reads "and".

   (b) Prove that the union operation $\cup$ on sets 4.2, is associative and commutative:

   $$A \cup (B \cup C) = (A \cup B) \cup C$$
   $$A \cup B = B \cup A$$

2. (**) Let I be a set and that for all $\alpha \in I$, I have a set A_α. [9] Read about the axiom of replacement; see [12, Axiom 3.5] or 4.5.

   (a) Prove that under the ZF axioms, one can form the union of the collection:

   $$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha \,:\, \alpha \in I\}$$

   In particular, explain why the following two objects

   i.
   $$\{A_\alpha \,:\, \alpha \in I\}$$

[9] For example, if $I = \{a, b, c\}$, then I have three sets

$$A_a, A_b, A_c$$

ii.

$$\bigcup \{A_\alpha : \alpha \in I\}$$

are sets.

(b) Give a one line explanation briefly describing why axiom of union 4.3 is insufficient to construct the set $\bigcup_{\alpha \in I} A_\alpha$.

3. The *axiom of regularity* states

Axiom 6.1. [12, 3.9] If A is a nonempty set, then there is at least one element $x \in A$ which is either not a set or, (if it is a set) disjoint from A.

Prove (with singleton set axiom) that for all sets $A, A \notin A$.

4. (***) Let A, B, C, D be sets. This exercise shows that we can actually construct *ordered pairs* using the ZF axioms.[10] Prove

   - We can construct the following set [11]

   $$\langle A, B \rangle := \{A, \{A, B\}\}$$

   from the axioms of set theory.

   - $\langle A, B \rangle = \langle C, D \rangle$ if and only if $A = C, B = D$. For this part you will require the *axiom of regularity.* in problem 3. You are free to use the results there.

5. This is a variation of problem 4 [12]. Suppose for two sets A, B we define

$$[A, B] = \{\{A\}, \{A, B\}\}$$

In this case, the problem is a lot easier. Prove $[A, B] = [C, D]$ if and only if $A = C, B = D$.

6. (***) Show that the collection

$$\{Y : Y \text{ is a subset } X\}$$

is a set using the ZF axioms. We denote this as the power set 2^X, where 2 is regarded as the two elements set $\{0, 1\}$. You will need to use the axiom of replacement.
Here are two important remarks on possible false solutions:

[10] Another definition is discussed in or [12, 3.5.1], where they assume this as an axiom.
[11] RIP. So another model of this is $\langle A, B \rangle := \{\{A\}, \{A, B\}\}$
[12] which is what I should have written

(a) (Ryan's) if your property for axiom of replacement $P(x, y) =$ "y is a subset of x" then this is *not correct*. The condition for replacement is that *there is at most one* y, [12, 3.6].

(b) (Kauí's) You cannot use axiom of comprehension, this is similar to Russell's paradox!

As a hint: $\{0,1\}^X$ is a set, by 4.4. For $Y \subseteq X$, $f \in \{0,1\}^X$, let $P(f, Y)$ be the property that

$$Y = f^{-1}(1) := \{x \in X : f(x) = 1\}$$

6.1 Boolean algebras

This section is compulsory. Boolean algebras form the foundation of probability theory. We will need this later when we get to the projects.

Reading: For some overview of the context, see [2, 1-3], [6, 1], or Tao's Lecture 0 on probability theory.

Definition 6.2. Let Ω be a set. A *Boolean algebra* in Ω is a set $\mathcal{E}$ of subsets of Ω (equivalently, $\mathcal{E} \subseteq 2^\Omega$) satisfying

1. $\emptyset \in \mathcal{E}$
2. closed under unions and intersections.

$$E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$$
$$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$$

3. closed under complements.

A *σ-algebra in* Ω is a Boolean algebra in Ω such that it satisfies

4. Countable[13] closure. If $A_i \in \mathcal{E}$ for $i \in \mathbb{N}$, then $\bigcup A_i \in \mathcal{E}$.

Problems

1. Prove that $\mathcal{E} := \{\emptyset, \Omega\}$ is a σ-algebra.
2. Prove that $2^\Omega := \{E : E \subseteq \Omega\}$ is a σ-algebra.
3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Hints for problems

3. There are 3 cases. What happens $A = \emptyset$ or $A = \Omega$? Now consider the case $A \neq \emptyset$ and $A \neq \Omega$.

[13] A set X is countable if it is in bijection with $\mathbb{N}$. We will explore this word in further detail in the future.

Solutions to Week 2

Featured solutions: Solutions to Q2, by Yvette, Q4, by Sri, Q6, by Tyler, Boolen algebra, by Granger.

Q2:

2) Let I be a set & $\forall \alpha \in I$, there is a set A_α.

a) Prove under ZF axioms, one can form the union of the collection:

$$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha : \alpha \in I\}$$

$\forall \alpha$ in I, there is a property connecting $\alpha \to A_\alpha$ $\to$ Axiom of Replacement

$\bigcup_{\alpha \in I} A_\alpha$ contains the image of the property on $\forall \alpha \in I$ $\to$ Axiom of Union/Collection

By Axiom of Replacement, $\forall \alpha \in I$, $\exists A_\alpha$ s.t. $\exists P(\alpha, A_\alpha)$ pertaining to α & A_α.

Then there is a set s.t. $\bigcup_{i=0}^{\infty} A_\alpha = \bigcup \{A_\alpha : \alpha \in I\}$

Q4

(4) (a) Proposition: we can construct the set $\langle A, B \rangle := \{A, \{A, B\}\}$

Proof: A and B are sets.

sets are objects. by axiom

$\therefore \exists$ Singleton sets : $\{A\}$, $\{B\}$ by singleton set axiom

$\exists$ Pair set : $\{A, B\}$ by pair set axiom

Treating $\{A, B\}$ as an object,

$\exists$ singleton set : $\{\{A, B\}\}$

$\{A\} \cup \{\{A, B\}\} = \{A, \{A, B\}\}$ by pairwise union axiom

$\therefore$ we have constructed a set $\{A, \{A, B\}\}$

(b) Proposition: $\langle A, B\rangle = \langle C, D\rangle$ iff $A = C \land B = D$

Proof: $\langle A, B\rangle := \{A, \{A, B\}\}$, $\langle C, D\rangle := \{C, \{C, D\}\}$

$$\{A, \{A, B\}\} = \{C, \{C, D\}\}$$

$A = C$ or $A = \{C, D\}$

Suppose $A = \{C, D\}$, then $C = \{A, B\}$.

A and C are sets.

Sets are objects. by axiom

$\therefore \exists$ pair set: $\{A, C\}$ by pair set axiom

By axiom of regularity, either A or C is disjoint from $\{A, C\}$.

Case #1: If A is disjoint,

$C \in A$, but $C \in \{A, C\}$.

This is a contradiction.

Case #2: If C is disjoint,

$A \in C$, but $A \in \{A, C\}$.

This is a contradiction.

We have proven that $A \neq \{C, D\}$.

$\therefore \underline{A = C}$

Then, $\{A, B\} = \{C, D\}$.

$\{C, B\} = \{C, D\}$. from $A=C$

$C = C \Rightarrow \underline{B = D}$ or $C = D$, $B = C$

$\Rightarrow B = C = D$

$\Rightarrow \underline{B = D}$

We have proven that $\underline{\langle A, B\rangle = \langle C, D\rangle \Rightarrow A = C \land B = D.}$

If $A = C \land B = D$,

$LHS = \{A, \{A, B\}\} \overset{\text{from } A=C}{=} \{C, \{C, B\}\} \overset{\text{from } B=D}{=} \{C, \{C, D\}\} = RHS$

We have proven that $\underline{A = C \land B = D \Rightarrow \langle A, B\rangle = \langle C, D\rangle.}$

$\underline{\therefore \langle A, B\rangle = \langle C, D\rangle \text{ iff } A = C \land B = D}$. <proven>

Q6:

(6) Prop. the collection $\{Y : Y \text{ is a subset of } X\}$ is a set.

Pf: • X is a set.

• $0, 1$ are objects, by pair set Axiom $\exists \{0,1\}$

• by powerset Ax. $\exists \{0,1\}^X$, which is a set of functions that map X to $\{0,1\}$

- each one of these functions, f, map a different group of X, or a subset, to 1 and the rest of the elements to zero.

• by the axiom of replacement, we can replace each of these elements with a Y such that $Y = f^{-1}(\{1\})$ for that f, replacing each f with the subset that f uniquely correlates to 1.

$$f \in \{0,1\}^X$$

$$p(f, Y) : Y = f^{-1}(\{1\}) := \{x \in X : f(x) = 1\}$$

– by relplacement $\exists \{Y : Y = f^{-1}(\{1\}) \text{ for some } f \text{ in } \{0,1\}^X\}$

• and by replacing each f, with each f mapping a subgroup to 1, with the subgroup they map, we get a set that is all subgroups of X

$\therefore$ or, $\{Y : Y = f^{-1}(\{1\}) \text{ for some } f \text{ in } \{0,1\}^X\} = \{Y : Y \text{ is a subset of } X\}$

$\therefore$ $\{Y : Y \text{ is a subset of } X\}$ is a set.

Boolean algebra solutions

1. Prove that $\varepsilon := \{\varnothing, \Omega\}$ is a $\sigma - algebra$.

   1. $$\varnothing \in \Omega$$

   2. Closed under unions and intersections.

      a. $$\varnothing \cup \Omega = \Omega \in \varepsilon$$

---

      b. $$\varnothing \cap \Omega = \Omega \in \varepsilon$$

   3. Closed under complements.

      a. $$\varnothing^c = \Omega \in \varepsilon$$

      b. $$\Omega^c = \varnothing \in \varepsilon$$

   4. Closed under countable closure.

      a. $\forall i \in N,\ A_i = \varnothing$, then $\bigcup_{i=0}^{\infty} A_i = \varnothing \in \varepsilon$

      b. $\exists i \in N,\ A_i = \Omega$ and for all the rest $A_j = \varnothing$, then $\bigcup_{i=0}^{\infty} A_i = \Omega \in \varepsilon$

   Therefore $\varepsilon := \{\varnothing, \Omega\}$ is a $\sigma - algebra$.

2. Prove that $2^{\Omega} := \{E : E \subseteq \Omega\}$ is a $\sigma - algebra$.

   1. $\emptyset \in 2^{\Omega}$ since $\emptyset$ is defined to contain no elements and thus vacuously satisfies as a subset for any set according to the definition of a subset.
   2. Closed under pairwise unions and intersections.
      a. If we can prove the union of two subsets of any set is another subset of that set, then, since 2^{Ω} contains all subsets of Ω, the union of any two subsets in 2^{Ω} will be closed in 2^{Ω}.

         Accessory Proof 1

         $A \subseteq C$ and $B \subseteq C$, then $A \cup B \subseteq C$.

         We assume by definition, if $x \in A \rightarrow x \in C$ and $x \in B \rightarrow x \in C$.

         By definition, if $x \in A$ or $x \in B \rightarrow x \in A \cup B$.

         Therefore, by our original assumption $x \in A \cup B \rightarrow x \in C$.

      b. If we can prove the intersection of two subsets of any set is another subset of that set, then, since 2^{Ω} contains all subsets of Ω, the intersection of any two subsets in 2^{Ω} will be closed in 2^{Ω}.

         Accessory Proof 2

         $A \subseteq C$ and $B \subseteq C$, then $A \cap B \subseteq C$.

         We assume by definition, if $x \in A \rightarrow x \in C$ and $x \in B \rightarrow x \in C$.

         By definition, if $x \in A$ and $x \in B \rightarrow x \in A \cap B$.

         Therefore, by our original assumption $x \in A \cap B \rightarrow x \in C$.

3. Closed under complements.

    a. Since all subsets of Ω are contained in 2^{Ω} by definition and all complements of an arbitrary subset, A, of Ω will just generate another subset, A^c, of Ω by the definition of complements.

4. Closed under countable union.

    a. If we can prove the countable union of subsets of any set yields another subset of the same set, then, since 2^{Ω} contains all subsets of Ω, the countable union of any combination of subsets in 2^{Ω} will be closed in 2^{Ω}.

Accessory Proof 3 (Prop: $\forall i \in N,\ A_i \subseteq B \rightarrow \bigcup_{i=0}^{\infty} A_i \subseteq B$).

We assume, by definition subset $\forall i \in N, x \in A_i \rightarrow x \in B$.

By definition of union $\forall i \in N, x \in A_i \rightarrow x \in \bigcup_{i=0}^{\infty} A_i$.

Therefore $x \in \bigcup_{i=0}^{\infty} A_i \rightarrow\ x \in B$.

3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Assume the set ε is the smallest σ-algebra in Ω containing A.

Therefore, to satisfy part 1 of the Boolean algebra, $\emptyset \in \varepsilon$.

Also, to satisfy closed under complement $\emptyset^c = \Omega \in \varepsilon$ and $A^c \in \varepsilon$.

Therefore $\varepsilon = \{\emptyset,\ A, A^c, \Omega\}$, (Special case where $A = \emptyset$ and $A^c = \ \Omega$ then $\varepsilon = \{\emptyset, \Omega\}$)

1. $$\emptyset \in \varepsilon$$

2. Closed under pairwise unions and intersections.

   a. $\emptyset \cup A = A \in \varepsilon, \emptyset \cup \emptyset = \emptyset \in \varepsilon, \emptyset \cup A^c = A^c \in \varepsilon,$

      $\emptyset \cup \Omega = \Omega \in \varepsilon, A \cup A = A \in \varepsilon, A \cup A^c = \Omega \in \varepsilon,$

      $A \cup \Omega = \Omega \in \varepsilon, A^c \cup A^c = A^c \in \varepsilon, \mathrm{A}^c \cup \Omega = \Omega \in \varepsilon,$

      $\Omega \cup \Omega = \Omega \in \varepsilon.$

   b. $\emptyset \cap A = \emptyset \in \varepsilon, \emptyset \cap \emptyset = \emptyset \in \varepsilon, \emptyset \cap A^c = \emptyset \in \varepsilon,$

      $\emptyset \cap \Omega = \emptyset \in \varepsilon, A \cap A = A \in \varepsilon, A \cap A^c = \emptyset \in \varepsilon,$

$A \cap \Omega = \mathrm{A} \in \varepsilon, A^c \cap A^c = A^c \in \varepsilon, \mathrm{A}^c \cap \Omega = \mathrm{A}^c \in \varepsilon,$

$\Omega \cap \Omega = \Omega \in \varepsilon.$

3. Closed under complements.
    a. $\varnothing^c = \Omega \in \varepsilon, \Omega^c = \varnothing \in \varepsilon, (A)^c = A^c \in \varepsilon, (A^c)^c = A \in \varepsilon$
4. Closed under countable union.
    a. $\forall i, A_i = \varnothing \rightarrow \cup_{i=0}^{\infty} = \varnothing \in \varepsilon, \forall i, A_i = A \rightarrow \cup_{i=0}^{\infty} = A \in \varepsilon,$

    $\forall i, A_i = A^c \rightarrow \cup_{i=0}^{\infty} = A^c \in \varepsilon, \forall i, A_i = \Omega \rightarrow \cup_{i=0}^{\infty} = \Omega \in \varepsilon,$

    For all following cases $\forall i \in N$.

    For any case where $\exists i : A_i = \Omega \rightarrow \cup_{i=0}^{\infty} = \Omega \in \varepsilon$.

    For any case where $\exists i: A_i = A^c$ and $\exists j: A_j = A \rightarrow \cup_{i=0}^{\infty} = \Omega \in \varepsilon$.

    For any case where $\forall i: A_i \neq A^c \wedge \Omega$ and $\exists j: A_j = A \rightarrow \cup_{i=0}^{\infty} = \mathrm{A} \in \varepsilon$.

    For any case where $\forall i: A_i \neq A \wedge \Omega$ and $\exists j: A_j = A^c \rightarrow \cup_{i=0}^{\infty} = \mathrm{A}^c \in$.

7 Equivalence Relation

Week 3 Reading: [12, Ch.3.5, Ch.4], *On the construction of* $\mathbb{Q}$, *see* [4, 2.4].

Learning Objectives

Last few lectures:

- Defined the natural numbers and sets axiomatically.
- Discussed how *cardinality* came up from "counting" sets.

This and next lecture:

- discuss equivalence relation.
- construct $\mathbb{Z}, \mathbb{Q}$. Extend addition and multiplication in this context.

7.1 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 7.1. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1, 2\} = \{2, 1\}$

In ordered pairs

- $(1, 2) \neq (2, 1)$

Definition 7.2. Let X, Y be two sets. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Currently, we can either put the existence of such a set as an axiom, or use the axioms of set theory, this is in hw.

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [5, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 7.3. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$

$$a \sim a$$

- *Transitive.* For all $a, b, c \in A$,

$$a \sim b, b \sim c \Rightarrow a \sim c$$

- *Symmetric.* For all $a, b \in A$,

$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Often times, people do not describe the subset R, but describe it a relation *equivalently* as a rule: saying $a, b \in A$ are related if some property $P(a, b)$ is true. In short hand, one writes

$$a \sim b \text{ iff } \ldots$$

Definition 7.4. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as [14]

$$[x] := \{y \in A \, : \, x \sim y\}$$

An element in such an equivalence is called a *representative* of that class.

Definition 7.5. Quotient set. Given an equivalence relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

Example

Consider $\mathbb{N}$ and the equivalence relation that $a \sim b$ iff a and b have the same parity. [a]

- There are two equivalence classes: the odds and evens.
- For the odd class, a *representative*, or an element in the equivalence class, is 3.

[a]i.e. both or odd or even.

There is a relation between equivalence and partition of sets.

Definition 7.6. A *partition* of a set X is a collection ???

7.2 Integers

What are the integers? It consists of the natural numbers and the negative numbers. What is *subtraction*? We do not know yet. Can we define *negative* numbers without referencing minus sign? Yes, we can. Say

$$-1\text{is "}0-1\text{" is } (0,1)$$

Discussion

Let us say we define the integers as pairs (a, b) where $a, b \in \mathbb{N}$. Would this be our desired

$$\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$$

- How many -1s are there?

But we have a problem, there are multiple ways to express -1. Our system cannot have multiple -1s. What are other ways We can also have $1-2$, or the pair $(1, 2)$.

[14]It does not matter if we write $\{y \in A \, : \, y \sim x\}$ by symmetry condition.

Discussion

Now that we have our $\mathbb{Z}$, how do we define addition? [a]Can we leverage our understanding?

[a]What is addition abstractly? It is an operation $+ : X \times X \to X$.

Intuitively, the *integers* is an expression [15] of non-negative integers, (a, b), thought of as $a - b$. Two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 7.7. Let

$$R \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

consists of all pairs (a, b) and (c, d) such that $a + d = b + c$. Equivalently,

$$R := \{(a, b), (c, d) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \,:\, a + d = b + c\}$$

The *integers* is the set

$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

Definition 7.8. Addition, multiplication. [12, 4.1.2] .

We can now finally define negation.

Definition 7.9. [12, 4.1.4].

Proposition 7.10. Algebraic properties. Let $x, y, z \in \mathbb{Z}$.

- Addition
  - Symmetric $x + y = y + x$.
  - Admits identity element.

7.3 Rational nunbers

Reading: [4, 2.4]. Be careful of the notation used! See 7.11.

Definition 7.11. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

$$\mathbb{Z} \setminus \{0\} := \{n \in \mathbb{Z} \,:\, n \neq 0\}$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote *the equivalence class* of pair (a, b) by $[a/b]$

[15]Rather than a pair, as an expression has multiple ways of presentation

Again, we need the notion of addition, multiplication, and negation.

Definition 7.12. Let $[a/b], [c/d] \in \mathbb{Q}$. Then

1. Addition:
$$[a/b] + [c/d] := [(ad + bc)/bd]$$

2. Multiplication
$$[a/b] \cdot [c/d] := [(ac)/(bd)]$$

3. Negation.
$$-[a/b] := [(-a)/b]$$

7.3.1 Is addition well-defined?

This subsection gives an extensive discussion of well-definess. The notation we use here is from 7.11. In 1. we *want* to define a function:

$$+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

which takes as input two equivalence class and outputs a new one. Let us consider two equivalence class

$$x := \left\{a'/b' \,:\, a'/b' \sim a/b\right\} \in \mathbb{Q}$$
$$y := \left\{c'/d' \,:\, c'/d' \sim c/d\right\} \in \mathbb{Q}$$

To add these two classes, we proceeded as follows:

1. We pick two representatives from each class, let us say a/b of x and c/d of y.

2. We define
$$x + y := [(ad + bc)/bd]$$

Why can't we say this is the definition of addition, yet? In the above description, $x + y$ can take *more than one possible value* - which is not a function!
For example, one could have chosen other pair of representatives, a'/b', and c'/d', and obtained $x + y$ as

$$[(a'd' + b'c')/b'd']$$

Thus, we have to check that

$$[(a'd' + b'c')/b'd'] = [(ad + bc)/bd]$$

To check this: by definition, this means we have to show:

$$bd(a'd' + b'c') = (ad + bc)b'd'$$

which is

$$bda'd' + bdb'c' = adb'd' + bcb'd' \tag{1}$$

Now $a'/b' \sim a/b$ and $c/d \sim c'/d'$ means $a'b = ab'$ and $cd' = c'd$, Now using commutativity in $\mathbb{Z}$, and the required two equalities for Eq. 1

$$bda'd' = a'bdd' =^{(a'b=ab')} ab'dd' = adb'd'$$

$$bdb'c' = c'dbb' =^{(cd'=c'd)} cd'bb' = bcb'd'$$

7.4 Order relation

Similarly, we can define also define order relation.

Definition 7.13. Let $x \in \mathbb{Q}$,

- x is *positive* iff $x = [a/b]$ where a, b are positive integers, we often denote positive integers as $\mathbb{Z}_{>0}$.
- x is *negative* iff $x = -y$ where y is some positive rational.

With the notion of positive rationals[16] from def. 7.13, we can define order relation $<, \leq$ on $\mathbb{Q}$.

Definition 7.14. Let $x, y \in \mathbb{Q}$, then we denote

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is zero or positive.

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 7.15. $\sqrt{2}$ is not rational.

Proof. ??? $\square$

[16] The same trick is used to define order in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

8 Homework for week 3

Due: Week 4, Saturday. You will select 3 problems to be graded.
Problems 1-3 are on cardinality. Problem 4 is on a general construction of equivalence relations. Problems 5-7 is about addition, multiplication, and division on $\mathbb{Z}$ and $\mathbb{Q}$.

1. Show that the relation $\leq$ is transitive, i.e. $|X| \leq |Y|, |Y| \leq |Z|$ then $|X| \leq |Z|$.

2. (**) Prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite. [17] Prove that $\mathbb{Q}$ is countably infinite. *You are free to use results from previous problems and theorems stated in lectures.*

3. (**) Let X be any set. Prove that there is no surjection (hence, bijection) between X and $\{0,1\}^X$. Deduce that $\{0,1\}^{\mathbb{N}}$ is uncountable. Argue the first part by contradiction: suppose there exists a surjection

$$f : X \to \{0,1\}^X$$

   - Consider the set
$$A = \{x \in X \,:\, x \notin f(x)\}$$
   - As f is a surjection (write the general definition) there must exists $a \in X$ such that $f(a) = A$. Do case work on whether $a \in A$ or $a \notin A$.

4. (**) Let X be any set. Recall that a binary relation on X, is any subset $R \subseteq X \times X$. We define $R^{(n)}$ as follows

   - For $n = 0$,
$$R^{(0)} = \{(x,x) \,:\, x \in X\}$$
   - Suppose $R^{(n)}$ has been defined.
$$R^{(n+1)} := \left\{(x,y) \in X \times X \,:\, \exists z \in X, (x,z) \in R^{(n)}, (z,y) \in R\right\}$$

   (a) Show that
$$R^t := \bigcup_{n \geq 1} R^{(n)} = R^{(1)} \cup R^{(2)} \cup \cdots$$
   defines a *smallest* transitive relation on X containing R. i.e. if Y is any other transitive relation on X containing R, then $R^t \subseteq Y$.

[17] Knowing the Cartesian product is required for this problem, skip 5. and 6. if unfamiliar.

(b) Show that
$$R^{tr} := \bigcup_{n\geq 0} R^{(n)} = R^{(0)} \cup R^{(1)} \cdots$$
is the *smallest* reflexive and transitive relation on X. i.e. if Y is any other transitive and reflexive relation on X containing R, then $R^{tr} \subseteq Y$.

5. (***) Show that addition, product, and negation are well-defined for rational numbers; see def. 7.11 or [12, 4.2]. You are free to use any facts and properties you know about $\mathbb{Z}$, such as the cancellation law.

6. (*) Let $x, y, z \in \mathbb{Z}$. Use the definition of addition and multiplication from 7.8, or [12, 4.1], show :

   (a) $x(y+z) = xy + xz$.

   (b) $x(yz) = (xy)z$.

   You are free to use any facts and properties you know about $\mathbb{N}$.

7. Let $x, y \in \mathbb{Z}$. You are free to use any facts you know about $\mathbb{N}$, in particular, it would be helpful to use the following the result: [12, 2.3.3]: *Let $n, m \in \mathbb{N}$. Then $n \times m = 0$ if and only if at least one of n, m is equal to zero.* Show that if $xy = 0$ then $x = 0$ or $y = 0$.

8.1 Tri-weekly diary

8. (**) Write a 800-1000 words diary or story. Pen down a diary on your experiences with the course topics and experiences so far, focusing particularly on:

   - Concepts or ideas that you initially found challenging or confusing. For example, the axioms of natural numbers $\mathbb{N}$, set theory, etc.
   - Topics that have piqued (if any, XD) your curiosity.
   - Topics that you wanted to be covered, and why.
   - Topics that you would like further elaboration.
   - People you find fun to be with (or scared of)!

   + (*) points for the best diary.

9 The real numbers

Week 3, Reading: [12, 5]*, notes by Todd, Cauchy's construction. Goldrei's textbook gives another construction of* $\mathbb{R}$ *using Dedekind cuts,* [4, 2.2].

Learning Objectives

We have defined $\mathbb{Q}$. To define $\mathbb{R}$.
- We use Cauchy sequence.

Pedagogy

We can define real numbers geometrically, adopted by Euclid, and mostly between 1500-1850, or as presented in [11]
- This ultimately leads to Dedekind's picture of how an irrational number sits among the rational.

9.1 Characterizing properties of $\mathbb{R}$: the completeness properrty

As with construction of $\mathbb{N}$, ultimately for $\mathbb{R}$, we are interested in the structural properties they have. The essential properties of $\mathbb{R}$ can be described by Thm. 9.1. If you have learned any algebra, this is also known as a complete ordered field.

Theorem 9.1. Properties of $\mathbb{R}$, this is a rehash of the list in [4, 2.3]. $\mathbb{R}$ is a set with

- operations $+$ and $\cdot$
- relations $=$ and $\leq$
- special elements $0, 1$ with $0 \neq 1$.

such that

1. $\leq$ is a reflexive and transitive relation.
2. $\leq$ behaves well under addition and multiplication : If $x \leq y$ and $z \geq 0$.
   - then $x + z \leq y + z$
   - $x \cdot z \leq y \cdot z$.
3. The operation $+$, def. is commutative and associative, admits inverses and admits identity 0. In other words:

- Associativity: for all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
- Commutativity: for all $x, y \in \mathbb{R}$, $x + y = y + x$.
- Admits inverse: for all $x \in \mathbb{R}$, there exists y such that

$$x + y = y + x = 0$$

- Admits identity 0: for all $x \in \mathbb{R}$,

$$x + 0 = 0 + x = x$$

4. The operation $\cdot$ is commutative and associative, admits inverses and identity 1:

5. Completeness: for any $A \subseteq \mathbb{R}$, $A \neq \emptyset$ which is bounded above has a least in upper bound *in* $\mathbb{R}$.

Proof. Properties of $+$ is left as homework. $\square$

Worthy of distinction is the last axiom.

Definition 9.2. A *partial order* on a set X, is a relation $\leq$ on X which is

- reflexive
- transitive: for all $a, b, c \in X$, if $a \leq b$, $b \leq c$, then $a \leq c$.
- antisymmetric: for all $a, b \in X$, $a \leq b$ and $b \leq a$ implies $a = b$.

Example

$(\mathbb{N}, \leq), (\mathbb{Q}, \leq), (\mathbb{Z}, \leq)$ are all partial orders. However $<$ is *not.*

We will apply the following definitions to the case of $X = \mathbb{R}$.

Definition 9.3. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation.

- $M \in X$ is a *upper bound* iff for all $x \in E$, $x \leq M$.
- $M \in X$ is a *lower bound* iff for all $x \in E$ $x \geq M$.

Definition 9.4. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation. $M \in X$ is a *least upper bound for E* if

1. M is an upper bound for E.
2. any other upper bound M' on E must satisfy $M \leq M'$.

Definition 9.5. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation. $M \in X$ is a *least upper bound for* E if

1. M is anlower bound for E.
2. any other lowerbound M' on E must satisfy $M \geq M'$.

Example

Let us consider $(\mathbb{Q}, \leq)$. What is the order relation here? see 7.14. Discuss the upper bound and least upper bound for the following sets.

- $E := \{x \in \mathbb{Q} : x > 0\}$.
- $E := \left\{x \in \mathbb{Q} : x^2 < 2\right\}$
- $E := \emptyset$

Definition 9.6. Upper bounds in reals. This is the case of 9.3 with $(X, \leq) = (\mathbb{R}, \leq)$.

1. if $E \neq \emptyset$ and has an upper bound, we denote
$$\sup E$$
as its least upper bound. This exists in $\mathbb{R}$ by 15.11.
2. If $E \neq \emptyset$ it has no upper bound, we write
$$\sup E := +\infty$$
3. If $E = \emptyset$ we set.
$$\sup E = -\infty$$

we have a similar definition for greatest lower bound. We can easily extend this to the case for extended reals, see def. 15.9.

Example

What is
$$\sup E := \left\{x \in \mathbb{R} : x^2 \leq 2\right\}?$$

9.2 Cauchy sequences

Let us start by constructing $\sqrt{2}$ using $\mathbb{Q}$. The idea is to represent such a number using sequence. All inequalities and numbers discussed in this section will be rationals.

Discussion

- If a "real" number x grows continually, but is bounded, does it approach a limiting value?

Definition 9.7. Let $m \in \mathbb{Z}$. A *sequence of rational numbers* denoted $(a_n)_{n=m}^{\infty}$ is a function

$$\{n \in \mathbb{Z} : n \geq m\} \to \mathbb{Q}$$

We can slowly increase our level of "closeness" of *a sequence* to *a point* via these three definitions.

Definition 9.8. We can slowly increase our level of "closeness" of *a sequence* to *a point* via these three definitions. Let $x \in \mathbb{Q}$, a sequence $(a_n)_{n=0}^{\infty}$ of rationals

1. Let $\varepsilon \in \mathbb{Q}_{>0}$. $(a_n)_{n=0}^{\infty} = \{a_0, a_1, \ldots\}$ is *ε-adherent* to x if exists $N \in \mathbb{N}$ st. $|a_N - x| < \varepsilon$.
2. Let $\varepsilon \in \mathbb{Q}_{>0}$ we say $(a_n)_{n=0}^{\infty}$, is *ε-close to x* if $|a_n - x| < \varepsilon$ for all $n \geq 0$.
3. Let $\varepsilon \in \mathbb{Q}_{>0}$ we say $(a_n)_{n=0}^{\infty}$ is *eventually ε-close to x* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|a_n - x| < \varepsilon$$
4. *converges to x* iff it is eventually ε-close to x for all $\varepsilon \in \mathbb{Q}_{>0}$.

We will give the same define for real sequences, see **??**

Definition 9.9. A sequence is $(x_n)_0^{\infty}$,

- *eventually ε-steady*, if exists some N such that for all $n, m \geq N$,
$$|x_n - x_m| < \varepsilon$$
- a *Cauchy sequence* iff for all $\varepsilon > 0$, $(x_n)_{n=0}^{\infty}$ is eventually ε-steady.

Example

Proofs using quantifiers. Prove for all positive rationals, ε, there exists a positive rational δ such that $\delta < \varepsilon$.
Mathematicians often translate this to notation

$$\forall \varepsilon \in \mathbb{Q}_{>0}, (\exists \delta \in \mathbb{Q}_{>0}, \delta < \varepsilon)$$

but this is up to taste.

Proof. ??? □

Proposition 9.10. Prove that $(a_n)_{n=1}^{\infty} := (1/n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. See text [12]. □

Example

- $(n)_{n=0}^{\infty}, (\sqrt{n})_{n=0}^{\infty}$ are not Cauchy.

Discussion

We want to use a Cauchy sequence to represent the real numbers. However, two sequences can represent the same number. Consider

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

$$1.5, 1.42, 1.4143, 1.41422, \ldots$$

Definition 9.11. Two sequences $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}$ are *eventually ε-close.* if there exists some N, such that for all $n \geq N$,

$$|a_n - b_n| < \varepsilon$$

Discussion

Are the following two sequences Cauchy equivalent?

- $(10^{10}, 10^{1}000, 1, 1, \ldots)$ and $(1, 1, \ldots,)$

Definition 9.12. Let $\mathcal{C}$ denote the set of cauchy sequences.[18] Then we set

$$\mathbb{R} := \mathcal{C}/\sim$$

where $\sim$ is the equivalence relation that

$$(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty} \text{ if and only if } (x_n)_{n=0}^{\infty} \text{ and } (y_n)_{n=0}^{\infty} \text{ are eventually } \varepsilon\text{-close}$$

We denote the equivalence of $(x_n)_{n=0}^{\infty}$ as $[(x_n)]$. Note that in [12], Tao denotes the class as $\mathrm{LIM}_{n\to\infty} x_n$.

Definition 9.13. Let $x, y \in \mathbb{R}$. Choose two representatives[19], say $(x_n)_{n=0}^{\infty} \in x$ and $(y_n)_{n=0}^{\infty} \in y$, then

- the sum of x and y is defined as

  $$x + y := [(x_n + y_n)_{n=0}^{\infty}]$$

  Addition is well-defined. [12, 5.3.6, 5.3.7].

- the product of x and y is defined as

  $$x \cdot y := [(x_n \cdot y_n)_{n=0}^{\infty}]$$

Now we can define the order relation on $\mathbb{R}$, compare to def. 7.13

Definition 9.14. $x \in \mathbb{R}$ is

- *positive* iff there exists a positive rational $c \in \mathbb{Q}_{>0}$, and $(x_n)_{n=0}^{\infty} \in x$ such that $x_n \geq c$ for all $n \geq 1$.
- *negative* iff $-(x_n)_{n=0}^{\infty} := (-x_n)_{n=0}^{\infty}$ is positive.

Definition 9.15. Let $x, y \in \mathbb{R}$, we say

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is positive or $x = y$.

[18] This is a subset of $\mathbb{Q}^{\mathbb{N}}$.

[19] an element of the equivalence class

10 More on Sequences

Reading: [12, 6].
Previously, we have worked with Cauchy sequences of rational numbers, see def 9.9, these were used to define $\mathbb{R}$. Now let us work with Cauchy sequences of real numbers:

Definition 10.1. A sequence $(x_n)_0^\infty$ of real numbers, i.e. a map $\mathbb{N} \to \mathbb{R}$, is

- *eventually ε-steady*, if exists some N such that for all $n, m \geq N$,

$$|x_n - x_m| < \varepsilon$$

- a *Cauchy sequence* iff for all $\varepsilon > 0$, $(x_n)_{n=0}^\infty$ is eventually ε-steady.

Learning Objectives

- Understand the notion of supremum and infima.
- Note that all convergent sequence is bounded, but is the bounded sequences convergent? This is the monotone convergence theorem. [12, 6.3.8].

We have the following hierarchy of sequences in reals:

$$\{\text{Convergent Seq in } \mathbb{R}\} \subseteq \{\text{Cauchy Seq in } \mathbb{R}\} \subseteq \{\text{Bounded Seq in } \mathbb{R}\}$$

which we will short hand denote as

$$\text{CvgSeq}(\mathbb{R}) \subseteq \text{CcSeq}(\mathbb{R}) \subseteq \text{BddSeq}(\mathbb{R})$$

We may ask: what bounded sequence are convergent?

Theorem 10.2. Let $(a_n)_{n=0}^\infty$

Now that we have defined $\mathbb{R}$, we will review again the notion of convergence.

Definition 10.3. Sequences of real numbers. Same as 9.7 but with $\mathbb{R}$ instead of $\mathbb{Q}$.

Definition 10.4. Same as 9.8 but with real sequences and converging to real number. Let $x \in \mathbb{R}$.

1. Let $\varepsilon \in \mathbb{R}_{>0}$. $(a_n)_{n=0}^\infty = \{a_0, a_1, \ldots\}$ is *ε-adherent* to x if exists $N \in \mathbb{N}$ st. $|a_N - x| < \varepsilon$.

2. Let $\varepsilon \in \mathbb{R}_{>0}$ we say $(a_n)_{n=0}^{\infty}$, is *ε-close to x* if $|a_n - x| < \varepsilon$ for all $n \geq 0$.

3. Let $\varepsilon \in \mathbb{R}_{>0}$ we say $(a_n)_{n=0}^{\infty}$ is *eventually ε-close to x* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|a_n - x| < \varepsilon$$

4. *converges to x* iff it is eventually ε-close to x for all $\varepsilon \in \mathbb{R}_{>0}$. IN this case we denote
$$\lim_{n\to\infty} (a_n) = a$$

Discussion

Consider our favourite sequence of 1.

$$0.9, 0.99, 0.999$$

- What are choices of x that satisfy 1?

Discussion

- In 1. what if $n = 0$? For instance
$$1, 0, 0, 0, 0, 0, \ldots$$
is ε close to 1. This wouldn't be a nice definition of the sequence "converging to x".

- In 2. This may be too much of demand? What about the sequence
$$1, 1/2, 1/3, \ldots, 1/n, \ldots$$

Proposition 10.5. Uniqueness of limits of sequences. [12, 6.1.7]. Let (a_n) be a sequence of real numbers. Let $L \neq L'$ be distinct real numbers. Such that we cannot have both
$$\lim_{n\to\infty} a_n = L \text{ and } \lim_{n\to\infty} a_n = L'$$
The notation means that $\lim_n a_n = L$ means "a_n converges to L "

The limit operation behaves well for convergent sequences.

11 Homework for week 4

Due: Week 5, Wednesday. You will select 3 problems to be graded.
References: [4, 2], [12, 5].

You are free to assume anything you know about $\mathbb{Q}$. The problem on Dedekind construction is one problem it self. It has extended number of points not because of its difficulty, but because of its length.

Problems

1. (**) Prove that the relation defined in def. 9.12, is an equivalence relation.

2. Review the definition of addition on $\mathbb{R}$, 9.13. Prove that addition, $+$, on $\mathbb{R}$ satisfies properties from 9.1. That is, prove :

   - Associativity: for all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
   - Commutativity: for all $x, y \in \mathbb{R}$, $x + y = y + x$.
   - Admits identity 0: for all $x \in \mathbb{R}$,

   $$x + 0 = 0 + x = x$$

3. (*) Review the definition of multiplication on $\mathbb{R}$, def. 9.13. Prove that any $x \in \mathbb{R}$ where $x \neq 0$ [20] admits a multiplicative inverse y, i.e. exists $y \in \mathbb{R}$ such that

   $$x \cdot y = y \cdot x = 1$$

4. Let $E \subseteq \mathbb{Q}$. Prove that under the order relation $\leq$, least upper bound is unique if exists

5. (**) Here we discuss some conditions to see whether a sequence of rationals $(a_n)_{n=0}^{\infty}$ is Cauchy:

   (a) Suppose that for all $n \in \mathbb{N}$,

   $$|a_{n+1} - a_n| < 2^{-n}$$

   prove that (a_n) is Cauchy.

   (b) if we replace the condition in a. as

   $$|a_{n+1} - a_n| < 1/(n + 1)$$

   for all $n \in \mathbb{N}$, give an example where (a_n) is not Cauchy.

[20] here $0 := (0)_{n=0}^{\infty}$ is the Cauchy sequence consisting of 0s

6. (***) How can we construct $\sqrt{2}$ using Cauchy sequence? Consider the following three sequence $(a_n), (b_n), (x_n)$ defined as follows

$$a_0 = 1, b_0 = 2$$

for each $n \geq 0$,

$$x_n = \frac{1}{2}(a_n + b_n)$$

$$a_{n+1} = \begin{cases} x_n & x_n^2 < 2 \\ a_n & \text{otherwise} \end{cases}$$

$$b_{n+1} = \begin{cases} b_n & x_n^2 < 2 \\ x_n & \text{otherwise} \end{cases}$$

   (a) Prove that all sequences are Cauchy.
   (b) Prove that all sequences are Cauchy equivalent.
   (c) Prove $[(a_n)_{n=0}^{\infty}] \cdot [(a_n)_{n=0}^{\infty}] = 2$.

7. Show that a Cauchy sequence is bounded.

Week 4 solutions

Remaining available upon request
Featured solutions: Q1, by Ethan, Q3, by Kauí, Q5a, by Ethan.
Q1:

1. Prove the relation $\mathbb{R} := {}^{c}/_{\sim}$ is an equivalence relation

NTS: if $\forall \varepsilon \in \mathbb{Q}_{>0}, \exists N_\varepsilon \; \forall n \geq N_\varepsilon, |a_n - b_n| < \varepsilon$ is equivalent

Reflexsive: $\{a_n\} \sim \{a_n\}$

$|a_n - a_n| < \varepsilon$

$a_n - a_n = 0$

b/c $\varepsilon \in \mathbb{Q}_{>0}, \varepsilon > 0$

Therefore $|a_n - a_n| < \varepsilon$

Symmetric: $\{a_n\} \sim \{b_n\} \implies \{b_n\} \sim \{a_n\}$

$|a_n - b_n| < \varepsilon \implies |b_n - a_n| < \varepsilon$

$a_n - b_n = -(b_n - a_n)$

proving

$|a_n - b_n| = |b_n - a_n| < \varepsilon$

transitive: $\{a_n\} \sim \{b_n\}, \{b_n\} \sim \{c_n\} \implies \{a_n\} \sim \{c_n\}$

$\forall \varepsilon \in \mathbb{Q}_{>0}, \exists N, \forall n \geq N_1, |a_n - b_n| < \varepsilon/2$

$\forall \varepsilon \in \mathbb{Q}_{>0}, \exists N, \forall n \geq N_2, |b_n - c_n| < \varepsilon/2$

$\{a_n\} \sim \{c_n\} \to \forall \varepsilon \in \mathbb{Q}_{>0}, \exists N = N_1 + N_2 \; \forall n \geq N$

$|a_n - c_n| = |a_n - b_n + b_n - c_n|$

$\downarrow$ by triangle inequality def'n

$|a_n - b_n| + |b_n - c_n|$

$\frac{\varepsilon_1}{2} + \frac{\varepsilon_2}{2} = \varepsilon$

Therefore $|a_n - c_n| < \varepsilon$.

Q3:

Question 3: Prove that any $x \in R$ where $x \neq 0$ admits a multiplicative inverse y.

We need to show $\forall [X] \in R$, I can find $[Y] \in R$ such that a $Xn \in [X]$ multiplied by a $Yn \in [Y]$ will be equivalent to [1].
So, we need to show that $|Xn*Yn - 1| < \varepsilon,\ \forall \varepsilon \in Q_{>0}$.

3.1 Let (Xn) be a Cauchy sequence such that Xn = f(n), where f(n) is a function f: $N \to Q_{\neq 0}$.
The function f(n) is defined as follows: choose a representative $(An) \in [X]$. Xn = f(n) equals:

1. An, if $An \neq 0$.
2. 2^{-n}, if An = 0. .

We now have to show that such a function really yields $(Xn) \in [X]$. For that, we have to see if they are Cauchy equivalent:
$|An - Xn| < \varepsilon,\ \forall \varepsilon \in Q_{>0}$.
We have two options:

1. Xn = An, in which case we have $0 < \varepsilon$ being true.
2. Xn - An = 2^{-n}. As 2^{-n} is eventually smaller than any ε, we see that we can find an N such that $\forall n > N$, $|An - Xn| < \varepsilon$

Thus, (Xn) such as Xn = f(n) is equivalent to An, meaning $Xn \in [X]$.

Since we have $Xn \neq 0$, there exists the inverse of Xn, 1/Xn = 1/f(n).
Then, let $(Yn) \in [Y]$ be a Cauchy sequence such that Yn = g(n), where g(n) is such that:
g: $N \to Q_{\neq 0}$, such that g(n) = 1/f(n).
By properties of rationals, we know Xn*Yn = f(n)*(1/f(n)) = 1.

3.2 Now, we can show $|Xn*Yn - 1| < \varepsilon,\ \forall \varepsilon \in Q_{>0}$.
Since Xn*Yn = 1, we have $|Xn*Yn - 1| = 0 < \varepsilon,\ \forall \varepsilon \in Q_{>0}$.
Thus, we prove that all $x \in R$ where $x \neq 0$ admits a multiplicative inverse.

Q5a:

5. a) for $\forall n \in \mathbb{N}$, $|a_{n+1} - a_n| < 2^{-n}$
Prove that (a_n) is cauchy.

by def'n $|a_{n+1} - a_n|$ is cauchy if it is $< \varepsilon$

Fix $\varepsilon \in \mathbb{Q}_{>0}$, Then we show (a_n) is eventual ε-steady

$\exists S$ st. $\forall N, M \geq S$ where $|a_M - a_N| < \varepsilon$

$$|a_M - a_N| = |a_M - a_{M-1} + a_{M+1} - a_{M-2} + \ldots + a_{n+1} - a_n|$$

With the triangle inequality

$$|a_M - a_N| \leq |a_M - a_{M-1}| + |a_{M+1} - a_{M-2}| + \ldots |a_{n+1} - a_n| < \varepsilon$$

If we sub 2^{-n} back in...

$$|a_M - a_N| < 2^{-n} \rightarrow < 2^{(-(M-1))} + \overbrace{2^{(-(M-2))} + \ldots + 2^{(-N)}}^{\text{converges to } 2}$$

$$|a_M - a_N| < 2^{(-N)} \cdot 2 = 2^{1-N} \text{ which must } < \varepsilon$$

let $a, b \in \mathbb{N}_{>0}$, $2^{1-N} < \varepsilon = \frac{a}{b}$

$$2 \cdot 2^{-N} < \frac{a}{b} \Rightarrow \frac{2}{2^N} < \frac{a}{b} \quad \text{let } N = b$$

$$\frac{2}{2^b} < \frac{1}{b} < \frac{a}{b}$$

Proving $|a_{n+1} - a_n| < 2^{-n}$, (a_n) is cauchy.

12 Continuity

Week5, Reading [12, 9.3].
Previously we have been dealing with sequences, 10.

Learning Objectives

In the next two lectures:

- Understand the underlying algebra
- State the Intermediate Value Theorem.

Define the exponential function exp, or $e^{(-)}$. To do this we need.

- Continuity.
- Formal power series.

12.1 Subsets in analysis

Reading: [12, 9.1].
In analysis, we often work with certain subsets of $\mathbb{R}$. To define these, we need to know the partial order $\leq$ on $\mathbb{R}$, see def. 9.15.

Definition 12.1. Let $a, b \in \mathbb{R}$. We can construct

- We define the closed interval.
$$[a, b] := \{x \in \mathbb{R} \,:\, a \leq x \leq b\}$$
- The *half open* intervals
$$[a, b) := \{x \in \mathbb{R} \,:\, a \leq x < b\} \quad (a, b] := \{x \in \mathbb{R} \,:\, a < x \leq b\}$$
- The open intervals
$$(a, b) := \{x \in \mathbb{R} \,:\, a < x < b\}$$
  We will also let $\mathbb{R}$ be an open interval as a special case.

Lastly, in any of the above cases we call:

- a, b to be the boundary points.
- any point in (a, b) as an *interior point.*

We will revise the above definition once we have defined extended reals, def. 15.7.

Example

What is

- $(2, 2)$
- $[2, 2)$
- $(4, 3)$.
- $[3, 3]$.

12.2 Working with real valued functions

In this section we study real valued functions

$$f : \mathbb{R} \to \mathbb{R}$$

$$x \mapsto f(x)$$

Example

1. Characteristic functions. Important for measure theory.
2. Polynomial functions.

We will denote the collection of functions from $\mathbb{R}$ to $\mathbb{R}$ as

$$\mathrm{Fct}(\mathbb{R}, \mathbb{R})$$

Throughout, we will attempt to understand the following types of functions

$$C^\infty(\mathbb{R}, \mathbb{R}) \subseteq C^k(\mathbb{R}, \mathbb{R}) \subseteq \mathrm{Cts}(\mathbb{R}, \mathbb{R}) \subseteq \mathrm{Fct}(\mathbb{R}, \mathbb{R})$$

Whenever you have a collection of objects you can always ask what structure/operations it has.

Definition 12.2. [12, 9.2.1] Structure on $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$. This is what algebraist refer as *composition rings.*

1. Composition.
2. Multiplication.
3. Addition.
4. Negation.

Except the compositional structure, all such structures exist on *function algebras.* These are sets of the form $\mathrm{Fct}(X, \mathbb{R})$ for X any set. For example, when $X = \mathbb{N}$,

$$\mathrm{Fct}(\mathbb{N}, \mathbb{R}) = \{(x_n)_{n=0}^\infty : x_n \in \mathbb{R}\}$$

This set of functions is the set of real sequences starting at 0. The goal now is to study $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$ generalizing what we know about $\mathrm{Fct}(\mathbb{N}, \mathbb{R})$

Discussion

Which of the following are true?

1. $(f+g)\circ h = (f\circ h)+(g\circ h)$.
2. $(f+g)\cdot h = (f\cdot h)+(g\cdot h)$.

History

In the realm of geometry, there is a duality between spaces and their algebra of functions, [1].

In the context of sequences, we were able to make sense of "limit" to a point, "∞"

$$\lim_{n\to\infty} x_n = L$$

[21] Similarly, in the context $\mathrm{Fct}(\mathbb{R},\mathbb{R})$ we would like to consider limit to points $a \in \mathbb{R}$, writing

$$\lim_{x\to a} f(x) = L$$

We first introduction a new notion:

Definition 12.3. The restriction operation: let $E \subseteq X \subseteq \mathbb{R}$ be subsets of $\mathbb{R}$. The restriction map is defined as

$$\mathrm{Fct}(X,\mathbb{R}) \to \mathrm{Fct}(E,\mathbb{R})$$

$$f \mapsto f|_E$$

where $f|_E(x) := f(x)$.

[21] in fact, this is the limit of $\mathbb{N}$, when phrased correctly.

12.3 Limiting value of functions

Reading, [12, 9.3]. We know what it means for a sequence to converge. Now we understand what it means for a function defined on an *interval* to converge.

Definition 12.4. Converging function.

1. Let $X \subseteq \mathbb{R}$ be an interval. $f \in \mathrm{Fct}(X, \mathbb{R})$ *is ε close to L* if for all $x \in X$,
$$|f(x) - L| < \varepsilon$$
2. [12, 9.3.3]. Let $X \subseteq \mathbb{R}$ be an interval. $f \in \mathrm{Fct}(X, \mathbb{R})$ is *local ε-close to L at a* iff there exists $\delta > 0$ such that
   (a) $(a - \delta, a + \delta) \subseteq X$ [22]
   (b) $f|_{(a-\delta,a+\delta)}$ is ε-close to L.
3. Let $L \in \mathbb{R}$, and $a \in X$, then we say *$f(x)$ converges to L as x approaches a* or *f converges to L at a*, iff for all $\varepsilon \in \mathbb{R}_{>0}$, f is local ε-close to L at a. In which case we denote
$$\lim_{x \to a} f(x) = L$$

Example

In 1. Let $f(x) = x^2$.

- 4-close to 2?
- 1-close to 1?

$g(x) = x^3$. $g_1 := g|_{[0,1]}$ and $g_2 := g|_{[1,2]}$.

- 4-close to 2?
- 1-close to 1?

It is *not necessary* that X is an interval and that $a \in X$. The definition can easily be generalized

	Sequences (x_n)	f converging to L at a.
Domain	$\mathbb{N}$	$X \subset \mathbb{R}$ contains a
ε-close	$\forall n \in \mathbb{N}\ \lvert x_n - L\rvert < \varepsilon$.	$\forall x \in X, \lvert f(x) - L\rvert < \varepsilon$.
eventually ε-close local ε-close at a	$\exists N$, for all $n \geq N\ \lvert x_n - L\rvert < \varepsilon$	$\exists \delta > 0, \forall x \in (a - \delta, a + \delta)$. $\lvert f(x) - L\rvert < \varepsilon$,
Converges	$\forall \varepsilon > 0$, (x_n) is ev' ε-close	$\forall \varepsilon > 0$, (x_n) is local ε-close

[22] Note that replacing any of the brackets here with a squared one yields the same definition.

Convergence can in fact be replaced by sequential convergence.

Proposition 12.5. Let $X \subseteq \mathbb{R}$ be an interval. Let $a \in X$ be an interior point as def 12.1. [23] Let $L \in \mathbb{R}$. Then the following are equivalent:

1. f converges to L at a.
2. For every sequence $(a_n)_{n=0}^{\infty}$ where $a_n \in X$ where $\lim_{n\to\infty}(a_n) = a$, def 10.4, we have
$$\lim_{n\to\infty} (f(a_n))_{n=0}^{\infty} = L$$

Proof. Exercise.

1. Choose a decreasing sequence
$$\delta_n > \delta_{n+1} > \cdots$$
such that if
   (a) $x \in [a - \delta_n, a + \delta_n]$
   $$|f(x) - L| < 1/2n$$
   Thus, for any $x, y \in [a - \delta_n, a + \delta_n]$,
   $$|f(x) - f(y)| < \frac{1}{n}$$
   (b) $\lim \delta_n = 0$.
2. Choose any sequence $x_n \in [a - \delta_n, a + \delta_n]$. By hypothesis, $f(x_n) \to L$ as $n \to \infty$.

∎

This is an important result as this shows that many results on continuity can be reduced to the case of sequences.

Note that if $\lim_{x\to a} f(x) = L$ then L is unique, as 10.5.

[23] This is so that we don't have to discuss boundary cases.

12.4 Continuous functions

Definition 12.6. Let $X \subset \mathbb{R}$ be an interval, let $a \in X$ be an interior point, def. 12.1. f is *continuous at* a if f is converges to $f(a)$ as x approaches a.

Example

Continuous functions:

1. Polynomial functions.
2. Linear functions.
3. The constant function $f(x) = c$ for some $c \in \mathbb{R}$, is continuous everywhere.

1. sgn is not continuous at 0.
2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as follows

$$x \mapsto \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$$

Definition 12.7. Let $X \subset \mathbb{R}$ be an open interval. $f : X \to \mathbb{R}$ is continuous at $x_0 \in X$ if for all $\varepsilon \in \mathbb{R}_{>0}$ f is local ε-close to $f(x_0)$. Explicitly, for each $\varepsilon > 0$, there exists $\delta_\varepsilon > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \forall x \in (x_0 - \delta_\varepsilon, x_0 + \delta_\varepsilon)$$

where $(x_0 - \delta_\varepsilon, x_0 + \delta_\varepsilon) \subseteq X$.

Definition 12.8. If f is continuous at all $x \in X$, we say f is *continuous*. We denote the set of all continuous functions on X as

$$\mathrm{Cts}(X, \mathbb{R})$$

We may ask which structures/operations, as 12.2, on $\mathrm{Fct}(X, \mathbb{R})$ which extends to $\mathrm{Cts}(X, \mathbb{R})$.

Proposition 12.9. Let $X \subseteq \mathbb{R}$ be an open set. $f, g : X \to \mathbb{R}$ are functions which are continuous at $a \in X$. Then the following functions are continuous at a:

1. $f + g$
2. $f \cdot g$
3. $\max(f, g)$

4. $\min(f, g)$

Example

1. $f(x) = |x|$ is continuous on $\mathbb{R}$. We will later see that this is not differentiable at 0.

13 Homework for week 5

Due: Week 6, Wednesday. You will select 3 problems to be graded.

1. Which of the following are true on $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$: let $f, g, h \in \mathrm{Fct}(\mathbb{R}, \mathbb{R})$:

   (a) Composition $\circ$ is associativity :
   $$f \circ (g \circ h) = (f \circ g) \circ h$$

   (b) Composition distributes over multiplications:
   $$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

   (c) Composition distributes over addition:
   $$(f + g) \circ h = f \circ h + g \circ h$$

2. (**) Let $(x_n)_{n=0}^{\infty}$ be a sequence of real numbers, *assume that x_n converges to some real number L.* Let $x_1 = 2$,
   $$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$$
   show that $L^2 = 2$. Note that the definition of convergence here is the same as rational number, 10.4. (***) Extra credit if you can show that $(x_n)_{n=0}^{\infty}$ converges.

3. (***) Prove 12.5.

4. (**) In set up of 12.9 prove

   (a) $f \cdot g$

   (b) $\max(f, g)$

   are continuous at a.

5. (*) In set up of 12.9 with $X = \mathbb{R}$, prove $f \circ g$, is continuous at a.

6. (***) Show by definition that the following functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous:

   (a) $f(x) = c$ for all $x \in \mathbb{R}$.

   (b) $f(x) = x$ for all $x \in \mathbb{R}$.

   (c) $f(x) = \sum_{i=0}^{n} c_i x^i$, where c_i is a real number for $i = 1, \ldots, n$. E.g. $f(x) = x^2 + x + \sqrt{2}$. You are free to use any results stated in notes or in previous problems.

14 Main results of continuity

We will consider two fundamental results in continuity of functions, [11, 7].

1. Maximum principle, thm 14.2, see also [12, 9.6]. For this, we would have to review the notion of limsup.

2. Intermediate value theorem.

Together these two results would imply that if f is a continuous function on $[a, b]$, then

$$f([a, b]) = [e, f] \quad a, b, e, f \in \mathbb{R}$$

The notation means the image of f.

Definition 14.1. Let $X \subset \mathbb{R}$ be any subset. Then the *image* of f,

$$\text{im} f := f(X) := \{y \in \mathbb{R} : \exists x \in X f(x) = y\}$$

14.1 Maximum principle

Theorem 14.2. Let $a < b$ be real numbers. Let f be a continuous function on an open interval containing $[a, b]$. $f : [a, b] \to \mathbb{R}$, then f attains its maximum at some point.

The proof of maximum principle, thm. 14.2, breaks down into the following steps:

1. Show that f is bounded, def 14.4. Suppose not, then exists a sequence $(x_n)_{n=0}^{\infty}$ such that $f(x_n) \to +\infty$. Each x_n lies in the same bounded interval. By Bolzano-Weirstrass, 14.6, we can find a convergent subsequence, this is a contradiction. (Why?)

2. Let $E := \sup f(X) \in \mathbb{R}$ by part 1 and completeness property of reals. We find a sequence of elements $x_n \in X$ such that $f(x_n) \to E$.

3. We find a converging subsequence $(x_{n_k})_{k=0}^{\infty}$, def 14.5, of (x_n), using Bolzano-Weirestrass, them. 14.6, such that $\lim_k (x_{n_k}) = x_{\max}$. Then by definition of continuity
$$f(x_{\max}) = L$$

Proposition 14.3. Let $a < b$ be real numbers. Let f be a continuous function on an open interval containing $[a, b]$. $f : [a, b] \to \mathbb{R}$, then f is bounded.

Definition 14.4. Let $X \subseteq \mathbb{R}$ be a subset, $f : X \to \mathbb{R}$ be any function. f is *bounded* if exists $M \in \mathbb{R}$

$$|f(x)| \leq M$$

The definition function bounded above (below and bounded) is a special case of 9.3. It is equivalent to saying that the image of f, def 4.12, is bounded above (below and bounded.)

Example

Which of the functions are bounded?

- $f(x) = 1/x - a$. $X = \mathbb{R}\setminus\{0\}$.
- $f(x) \in \text{Poly}(X, \mathbb{R})$ with $X = \mathbb{R}$ and $X = (0, 1)$.

14.2 Bolzano-Weierstrass Theorem

We will now study a new collection of sequences: those sequences which have converging subsequences. They fit in the following hierarchy:

$$\text{CvgSeq}(\mathbb{R}) \subseteq \text{CcSeq}(\mathbb{R}) \subseteq \text{BddSeq}(\mathbb{R}) \subseteq \text{CvgSubSeq}(\mathbb{R})$$

Definition 14.5. Subsequence. Let (a_n) be a sequence of reals. Then *a subsequence of* (a_n) is a sequence $(b_k) = (a_{f(k)})_{k=0}^{\infty}$ given by the datum of a function

$$f : \mathbb{N} \to \mathbb{N}$$

which is strictly increasing : for all $i, j \in \mathbb{N}$

$$f(i) > f(j) \text{ if } i > j$$

Often times, people don't say the function f and write instead

$$(b_k)_{k=0}^{\infty} = (a_{n_k})_{k=0}^{\infty}$$

Example

Consider

- $f(x) = 2^x$.
- $f(x) = x + 1$
- $f(x) = x^2$.

What are the subsequences associated to these functions when $(a_n) = ((-1)^n)_{n=0}^{\infty}$.

We begin with the following famous theorem, which is equivalent to the completeness property (or axiom) of the real numbers.

Theorem 14.6. Bolzano-Weirestrass. Let (a_n) be a bounded sequences, then there is at least one subsequence (a_n) which converges.

Recall the definition of limit points, 15.2.

Proof. The conditions implies that $\limsup a_n < \infty$ of which is a limit point. $\square$

Example

Consider the following sequences:

- $(x_n)_{n=0}^{\infty} = (-1)^{n+1}n$. Does this have a converging subsequence.?
- $(x_n)_{n=0}^{\infty} = (n \bmod 5)$. Does this have a converging subsequence?
- $(x_n)_{n=1}^{\infty}$ be a sequence such that for each $x_n \in [0, 100]$: does this have a convergent subsequence?

15 $\limsup$ and $\liminf$

Reading: [12, 6.4].

Learning Objectives

- Limit points.
- Find the sup and inf of he set of all limit points of a sequence (a_n). i.e. For all $L \in \text{LimitPoint}(a_n)$,

$$\liminf a_n \leq L \leq \limsup a_n$$

  see Prop. 15.6.

In general sequences we consider do not converge.

Definition 15.1. $L \in \mathbb{R}$ is a *limit point* of $(a_n)_{n=0}^{\infty}$ if there exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.

An equivalent characterization is

Definition 15.2. $L \in \mathbb{R}$ is a *limit point* of (a_n) if for every $\varepsilon > 0$, for every N, there exists $n \geq N$, such that $|a_n - L| < \varepsilon$.

Proposition 15.3. The two definitions 15.1 and 15.2 are equivalent.

We have many limit points. Consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.00001, \dots \quad (*) \tag{2}$$

This is the sequence $((-1)^n(1+10^{-n})_{n=1}^{\infty}$.
What two limits do you see? It is a combination of two sequences:

- $1.1, 1.001, 1.0001, 1.00001, \ldots$.
- $-1.01, -1.00001, -1.000001, \ldots$.

Example

Let (a_n) be a sequence which converges to L. What are its limit points?

Definition 15.4. Let $(a_n)_{n=m}^{\infty}$ be a sequence. Then for : all $N \in \mathbb{N}$ we set

$$a_N^+ := \sup_n \left((a_n)_{n=N}^{\infty}\right)$$

$$a_N^- := \inf_n (a_n)_{n=N}^{\infty}$$

With the new sequence $(a_N^+)_{N=0}^{\infty}, (a_N^-)_{N=0}^{\infty}$, define

$$\limsup_n a_n := \inf_N \left((a_N^+)_{N=0}^{\infty}\right)$$

$$\liminf a_n = \sup_N \left((a_N^-)_{N=0}^{\infty}\right)$$

where sup is the least upper bound 9.4, and inf, is the greatest lower bound.

This definition is not exactly ideal; some of the terms may be undefined if we use the definition of sup for reals, def. 9.4. For example, if $(a_n) := (n) = (0, 1, 2, 3, \ldots)$, then for all N,

$$a_N^+ = +\infty$$

We would then have to compute
This is *not* a number, and we previously have only defined sup on subsets $E \subseteq \mathbb{R}$, def 9.4. The convention is to formally include $+\infty$ as a new symbol into our number system, as def. 15.7. We need rules on how to work with taking supremum of sets containing $-\infty, +\infty$. 15.9. Under this convention

$$\sup\{+\infty\} = +\infty$$

$$\sup\{-\infty\} = \sup\{-\infty\} \setminus \{-\infty\} = \sup \emptyset = -\infty$$

Note that the sequence $(a_N^+)_{N=0}^{\infty}$ is decreasing.

Example

If the sequence (a_n) where bounded below, and at least one $(a_N^+) \in \mathbb{R}$, then by MCT, the sequence limits to some real number.

Definition 15.5.

Example

In (*) of sequence

- $(a_n^+) = (a_0^+, a_1^+, \dots)$ is the sequence

$$(1.1, 1.01, 1.001)$$

- $\inf(a_n^+) = 1$.

Proposition 15.6. [12, 6.4.12] Properties of limsup and liminf. Let $L^+ := \limsup_n a_n, L^- := \liminf_n a_n$.

1. Elements of (a_n) are eventually less than x for every $x > L^+$: i.e.
2. if c is any limit point of $(a_n)_{n=0}^{\infty}$ we have $L^- \leq c \leq L^+$.
3. Suppose $L^+ < \infty$, then it is a limit point.

Proof. 3. Let us show that L^+ is a limit point. Fix $\varepsilon > 0$ and N. Then $\limsup = \inf(\sup a_N^+)$, choose $N_1 \geq N$ such that $L^+ \leq \sup(a_n)_{n=N_1}^{\infty} \leq L^+ + \varepsilon$. Then choose $N_2 \geq N_1$[24] such that

$$L^+ - \varepsilon < a_{N_2} < L^+ + \varepsilon$$

□

15.1 Extending the number system

We will begin by defining the *suprema* and *infima* of sets. We may or may not work with an extended number system. But we include it here to show how one could extend a number system.

Definition 15.7. The *extended number system* consists of

$$\bar{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

Let $x, y, z \in \bar{\mathbb{R}}$. Define the order relation, 7.3 $x \leq y$ if and only if one of the following holds.

[24] why can you do this?

1. If $x, y \in \mathbb{R}$, $x \leq y$.
2. $x = -\infty$
3. $y = \infty$.

Thus, we have artificially add in new terms.

- We do not include any operations. This can be dangerous. Of course, this can be done. We can set
$$c + (+\infty) = (+\infty) + c := +\infty \quad \forall c \in \mathbb{R}$$
$$c + (-\infty) = (-\infty) + c =: -\infty \quad \forall c \in \mathbb{R}$$
but requires a lot of care.
- We can define order and negation.

This is a common practice for mathematics, in order for one to make better statements.

Definition 15.8. Negation of extended reals.

Example

What is the supremum of the set

- $$\{0, 1, 2, 3, 4, 5, \ldots\}$$
- $$\{1 - 2, 3, -4, 5, -6, \ldots\}$$
- $(a_n) = ((-1)^n 1/n)$.

Definition 15.9. Least upper bound. See the usual definition of upper bound 9.3. Let $E \subseteq \bar{\mathbb{R}}$. Then $\sup E$, the least upper bound [12, 6.2.6] is defined by the following rule:

- Let $E \subseteq \mathbb{R}$. So $\infty, -\infty \notin E$. Then $\sup E$ is as 9.6.
- If $\infty \in E$, then $\sup(E) = \infty$.

We can define the infimum without the use of another definition. [25]

[25] although, in practice, we *think* of inf as we did for defining 9.4.

Definition 15.10. We let

$$\inf E := -\sup(-E)$$
$$-E := \{-x : x \in E\}$$

Example

Let E be negative integers.

$$\inf(E) = -\sup(-E) = -\infty$$

15.2 Completness axiom

Let us recall one of the key properties of real numbers, the least upper bound property. 9.3.

Theorem 15.11. Completeness axiom. [12, 5.5.9]. Let E be a nonempty subset of $\mathbb{R}$. If E has an upper bound, then it has *exactly one least upper bound* in $\mathbb{R}$. Thus, $\sup E \in \mathbb{R}$.

Proof. The hard part is *existence*. Uniqueness was done in hw...??? □

Example

Give an example in $\mathbb{Q}$ which does not satisfy this property.

What are the consequences? It says something about convergence of sequences.

Proposition 15.12. Least upper bound. [12, 6.3.6].

Proof. This boils down to [12, 5.5.9]. □

Proposition 15.13. MCT [12, 6.3.8]. Every monotone bounded sequence converges. Let (a_n) be a bounded sequence of real numbers, which is also increasing. Then limit exists and

$$\lim a_n = \sup(a_n)_{n=0}^{\infty} \leq M$$

Proof. By 15.11, $\sup a_n$ exists and is unique. Let us pick an $\varepsilon > 0$. Then by definition there exist n...??? □

16 Homework for week 6

Due: Week 7, Friday. You will select 3 problems to be graded. Note that when we are arguing with the real numbers, we forget that they are a "Cauchy equivalence class of rationals" and work with them by their properties, see subsec. 9.1. Q1-5, Q9 reviews class content on Bolzano-Weierestrass. Q6-8 are problems on limsup and liminf.

1. Prove 15.3.
2. Write a proof of completeness property 15.11.
3. (**) Write out the full proof Bolzano-Weierestrass's Thm. 14.6. You have to write proofs of any result used.
4. Deduce the boundedness of functions on closed interval Prop. 14.3, using Bolzano-Weierestrass's Thm. 14.6.
5. Sandwiching sequences. Suppose that $(a_n), (b_n), (c_n)$ are three sequences of real numbers satisfying
$$a_n \leq b_n \leq c_n \quad \forall n \in \mathbb{N}$$
and $a_n \to l$ and $c_n \to l$. Show that $b_n \to l$. (**) Give an alternative prove using the comparison principle for liminf and limsup.
6. (**) Using the completeness property 15.11, show that there exists a positive real number x such that $x^2 = 2$ by considering the set
$$E := \left\{x \in \mathbb{R} : x \geq 0 \text{ and } x^2 \leq 2\right\}$$
7. (**) Comparison principle for liminf and limsup. Let $(a_n), (b_n)$ be two sequences of reals, such that
$$a_n \leq b_n \forall n \in \mathbb{N}$$
Show that
$$\liminf a_n \leq \limsup b_n$$
To begin: fix $N, M \in \mathbb{N}$. Show that
$$a_N^- \leq b_M^+$$
Hence, by varying M, show that
$$a_N^- \leq \inf_M (b_M^+)_{M=0}^\infty = \limsup b_n$$
8. (***) Let (a_n) be a sequence of real numbers. Then (a_n) converges iff it is bounded and $\liminf a_n = \limsup a_n$.
9. (*) Show that every cauchy sequence converges using Bolzano Weirestrass, Thm. 14.6.

17 Additional problems

These problems will appear in future homework.

1. (**) In this exercise we prove the MCT theorem: Hints:

   (a) Let (a_n) be sequence of real and $x \in \bar{\mathbb{R}}$ such that $x = \sup(a_n)_{n=0}^{\infty}$. Show that $a_n \leq x$ for all n.

   (b) x is the limit.

2. (***) Let $a < b$, and $f : [a, b] \to \mathbb{R}$ be a continuous and strictly increasing[26] function. Then f restricts to a bijection $f : [a, b] \to [f(a), f(b)]$. This implies there exists an inverse, f^{-1}. Show that f^{-1} is

   (a) continuous

   (b) strictly increasing.

[26] $f(x) > f(y)$ if $x > y$ for all $x, y \in [a, b]$.

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