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TOKYO LECTURES ON THE GEOMETRY OF DIAMONDS

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CONTENTS

INTRODUCTION

Since its birth in the 60's following the work of Tate, the field of rigid analytic geometry has undergone various mutations. In this text we present some of the latest one. This is not a course in rigid geometry, the purpose is not to be extensive but rather to focus on the key points of the theory that may be well known to the experts but not from everybody. Another purpose of these notes is to "clean up" the theory and organize it in a coherent way with the optimal hypothesis. There is a parti pris: adic spaces, other points of view on rigid geometry being some kind of projection of the adic point of view. To motivate the reader we decided to fix a target: the so-called Artin criterion for spatial diamonds, one of the key tools of the work [28] among many others.

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CHAPTER 1

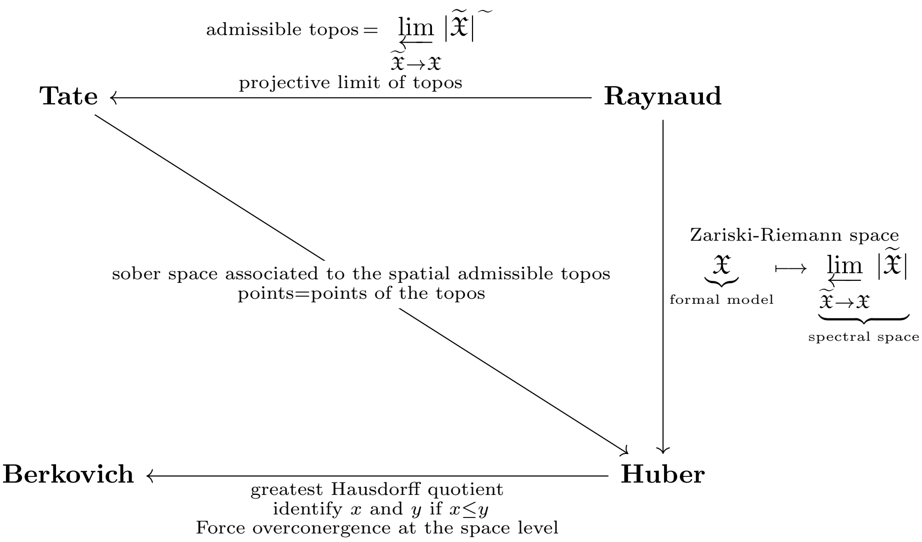
SPECTRAL SPACES

1.1. Background: rigid geometry, 4 points of view

Historically there are 4 points of view on rigid analytic geometry that appeared since the 60's:

1. **Tate ([52])**: the underlying topological space is not a topological space but a site: the site of admissible open subsets and of admissible open coverings. Some good references for this point of view are [10], [30] and [9].
2. **Raynaud ([46]):** this is the point of view of formal Zariski-Riemann spaces, that is to say of admissible formal schemes up to admissible formal blow-ups. The series of articles [11], [12], [14] and [13] and the book [3] are good references.
3. **Berkovich ([6]):** The underlying topological space is (very often) locally compact (and very often) locally contractible. By definition this is an overconvergent theory compared to the preceding one: overconvergence, that is a property of coefficients i.e. sheaves, is forced at the space level . The first two main references are [6] and [7]. This can be completed by [24].
4. **Huber ([37]):** The underlying topological space is a locally spectral space in the sense of Hochster. The two articles [36] and [37] are the starting points. This can be completed by [38].

The link between those theories is summarized in the following diagram



Here for X a quasi-compact quasiseparated rigid analytic space à la Tate over a non-Archimedean field K:

- Raynaud chooses an integral model $\mathfrak{X}$ (a π_K-adic formal scheme over $\mathcal{O}_K$ topologically of finite type) such that $X = \mathfrak{X}_\eta$ (generic fiber) and the admissible topos of X is the projective limit of the topos of all admissible formal blow up $\widetilde{\mathfrak{X}}$ of $\mathfrak{X}$.
- The admissible topos $\widetilde{X}$ is spatial and the associated sober space is Huber's spectral space. Points are given by morphisms of topos $Ens \to \widetilde{X}$ i.e. the points of the admissible topos.
- Huber's spectral space is the projective limit of the spectral spaces associated to all admissible formal blow up.
- The Berkovich space is obtained by taking the quotient of Huber's spectral space given by identifying two point x and y is x is a specialization of y or y is a specialization of x i.e. we identify any point x with its maximal generalization given by a rank 1 valuation (that is to say with values in $\mathbb{R}$).

Huber's adic spaces are the most general objects we can consider, the theory is developed over any base (i.e. without a fixed base). For example:

- The analytic adic spectrum of $\mathbb{Z}_p[\![x_1, \ldots, x_n]\!]$ is a quasi-compact adic space that contains as an open subset the usual rigid analytic open ball
$$\mathring{\mathbb{B}}^n_{\mathbb{Q}_p}$$
and whose complementary $\partial\mathring{\mathbb{B}}^n_{\mathbb{Q}_p}$ is a union of n $(n-1)$-dimensional closed balls over $\mathbb{F}_p(\!(T_1)\!), \ldots, \mathbb{F}_p(\!(T_n)\!)$.
- Another example is the case of tubular neighborhoods. Let X be any Noetherian scheme and $Y \subset X$ be Zariski closed. The formal completion
$$\mathfrak{X} = \widehat{X}_{/Y}$$
is a formal scheme that is a weak approximation to what is a tubular neighborhood of Y in X. One can associate to this formal scheme a much more subtle notion of tubular neighborhood: the adic space $\mathfrak{X}^{ad}$ that contains the special fiber $Y \hookrightarrow \mathfrak{X}^{ad}$ seen as a "discrete" adic space associated to a scheme (something not very different from a scheme) and the generic fiber, the punctured tubular neighborhood $\mathfrak{X}^{ad} \setminus Y$ that is an analytic (the opposite of discrete) adic space,
$$\underbrace{Y}_{\text{discrete}} \hookrightarrow \mathfrak{X}^{ad} \hookleftarrow \underbrace{\mathfrak{X}^{ad} \setminus Y}_{\text{analytic}}.$$

If K is a non-archimedean field, there is a functor $X \mapsto X^{ad}$ that is an equivalence
$$\{\text{quasi-separated } K\text{-rigid spaces}\} \xrightarrow{\sim} \{\text{quasi-separated } K\text{-adic spaces locally of finite type}\}.$$
From this point of view the theory does not seem more powerful than Tate's theory when we work in a topologically of finite type framework over a non-archimedean field. Nevertheless, if $f : X \to Y$ is a separated quasi-compact morphism of quasi-separated rigid spaces Huber constructs a canonical "compactification"
$$\begin{tikzcd} X^{ad} \arrow[r, hook, "j"', "open"] \arrow[d, "f^{ad}"'] & X^{ad}_{/Y^{ad}} \arrow[dl, "f^{ad,c}"] \\ Y^{ad} & \end{tikzcd}$$
that does not exist in Tate's world. The points of the boundary of this compactification $X^{ad}_{/Y^{ad}} \setminus X^{ad}$ are higher rank valuations that do not show up in Tate or Berkovich theory. For example, the compactification of the closed ball $\mathbb{B}^1_K \to \mathrm{Spa}(K)$ is obtained by adding a rank 2 valuation in the boundary that is a specialization of the Gauss norm. This construction is

essential in [38] to define $Rf_!$ as $R(f^{ad,c})_! \circ j_!$ in étale cohomology and is unavailable in Berkovich's theory ([7]) where the compactly supported étale cohomology of an affinoid space is not defined.

Sometimes we will want to take an integral model of an adic space or consider it's Berkovich's spectrum as a locally compact topological space but at the end we will focus on adic spaces.

1.2. Spectral spaces: two equivalent definitions

To any complex analytic space one can associate a locally compact topological space after forgetting the complex analytic structure i.e. the structure sheaf of holomorphic functions. For smooth complex analytic spaces the associated topological space is a topological manifold.

For adic spaces, the associated topological space, after forgetting the structure sheaf of holomorphic functions, is a locally spectral space ([34], [50, Section 08YF]). Let us recall the following definition.

Definition 1.2.1 **(Hochster [34]).** — A topological space X is spectral if

1. It is quasi-compact quasiseparated,
2. It is sober,
3. It has a base of quasi-compact open subsets.

Recall that in this definition:

- X quasiseparated means that the intersection of two quasi-compacts open subsets is quasi-compact,
- sober means that any irreducible closed subset has a unique generic point, that is to say
  — any irreducible closed subset has a generic point,
  — X is $(T0)$.
  The existence of a generic point for Z closed irreducible means that $\bigcap_{\substack{U \subset X \text{ open} \\ U \cap Z \neq \emptyset}} U \cap Z \neq \emptyset$.
- The fact that X is $(T0)$ is equivalent to saying that the specialization relation $\leq$ is an order: $x \leq y$ and $y \leq x$ implies $x = y$.

Remark 1.2.2. — To any (Grothendieck) topos $\mathfrak{X}$ one can associate a sober space X whose points are exactly the set of points of $\mathfrak{X}$, the morphisms of topoi

$$\mathrm{Ens} \longrightarrow \mathfrak{X},$$

and whose open subsets are induced by the subojects of the final object of $\mathfrak{X}$ i.e. if U is a subobject of the final object of $\mathfrak{X}$ we look at the points factorizing through $\mathfrak{X}/U$ via $\mathfrak{X}/U \to \mathfrak{X}$. There is a natural morphism of topoi

$$\widetilde{X} \to \mathfrak{X}.$$

When this is an equivalence we say that $\mathfrak{X}$ is spatial. All of this only depends on the locale of subobjects of the final object of $\mathfrak{X}$ associated to $\mathfrak{X}$ (see [40]). One of the starting points of Huber's work is the discovery that the admissible topos of a qcqs rigid space à la Tate is in fact spatial ([36, Proposition 4.5]).

Remark 1.2.3. — Any Noetherian $(T0)$ topological space, for example the topological space associated to a Noetherian scheme, is spectral. Nevertheless, this is not the type of spaces that we will ultimately focus on in this text.

Remark 1.2.4. — The quasi-separatedness hypothesis is very important. In fact it is "difficult" to fall naturally on quasicompact non-quasiseparated schemes, the first basic example being two copies of $\mathrm{Spec}(k[x_i]_{i\in\mathbb{N}})$ glued along $\mathrm{Spec}(k[x_i]_{i\in\mathbb{N}})\setminus V(x_i)_{i\in\mathbb{N}}$ (it is "easier" to fall on quasicompact non-quasiseparated algebraic spaces like $\mathbb{G}_{a,k}/\mathbb{Z}$ where k is a characteristic zero field). But for adic spaces this can happen naturally. This is for example the case if you glue two closed balls $\mathbb{B}^1_K$ along $\mathbb{B}^1_K \setminus \{0\}$, K a non-archimedean field.

Profinite topological spaces are a particular case of spectral spaces: those are exactly the Hausdorff spectral spaces. They have a purely topological definition: those are the totally disconnected compact topological cases. The definition as a projective limit of finite sets is a combinatorial definition.

The same goes on more generally for spectral spaces.

Topological definition	**Combinatorial definition**
Totally disconnected compact topological spaces	Pro(finite sets)
Spectral topological spaces + qc maps	Pro(finite ordered sets)

Here we use the equivalence between finite ordered sets and finite ($T0$) topological spaces, the order relation corresponding to the specialization relation on the topological space. The finite ($T0$) spaces are exactly the finite spectral spaces. For example, the ordered set $\{s,\eta\}$ with $s\leq\eta$ and $s\neq\eta$ corresponds to the spectral space that is the spectrum of a rank 1 valuation ring. More generally, the ordered set with n elements $\{1\leq 2\leq\cdots\leq n\}$ corresponds to the spectrum of a rank n valuation ring.

When we say that the definitions are equivalent this means that the projective limit functor induces an equivalence of categories between the right hand side of the preceding table and the left hand one: we are speaking of the pro-category of finite sets, resp. finite ordered sets. On the left hand side we have to take quasi-compact morphisms of spaces as morphisms to obtain an equivalence of categories.

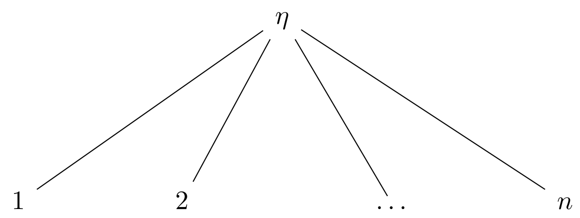
Theorem 1.2.5 **([34, Proposition 10]).** — *There is an equivalence of categories*

$$\varprojlim : \textit{Pro(finite ordered sets)} \xrightarrow{\sim} \{\textit{spectral topological spaces + qc continuous maps}\}.$$

We will give a full proof of this theorem in Section 1.5 and write down an explicit inverse of the functor $\varprojlim$, see Theorem 1.2.5.

Example 1.2.6. — Set $E_n=\{1,\dots,n\}^{disc}\cup\eta$ where $\{1,\dots,n\}^{disc}$ is the discrete ordered set (if $x\neq y$ then $x\not\geq y$ and $x\not\leq y$) with n elements and $\eta\geq 1,\eta\geq 2,\cdots,\eta\geq n$. For $n\leq m$ we define a morphism $E_m\to E_n$ by sending η to η, $i\in\{1,\dots,m\}$ to i if $i\leq n$ and $i\mapsto\eta$ if $i>n$. This forms a projective system $E_1\leftarrow E_2\leftarrow\cdots\leftarrow E_n\leftarrow\cdots$.

Order the prime numbers $p_1,p_2,\dots$. Then $\mathrm{Spec}(\mathbb{Z})$ corresponds to $\varprojlim_{n\geq 1} E_n$ by sending p_i to $i\in E_n$ if $i\leq n$, $p_i\mapsto\eta\in E_n$ if $i>n$ and $\eta\mapsto\eta$.



1.3. The constructible topology

1.3.1. Definition. — Recall the following definition.

Definition 1.3.1. — A *constructible set* in a spectral space X is an element of the Boolean algebra generated by the quasi-compact open subsets.

The quasi-compact assumption is fundamental in the definition of a constructible set. One may be used to work with spectral spaces associated to Noetherian schemes where all open subsets are quasi-compact but when working with adic spaces the most basic objects like the closed ball contains many natural non-quasicompact open subsets like open balls. Moreover, when working with adic spaces, one has to forget at some point the intuition of the Zariski constructible subsets of Noetherian schemes. Typically if Z is a constructible subset of the closed ball $\mathbb{B}^1_K$ as an adic space over the non-archimedean field K, if $0 \in Z$ then $\mathbb{B}^1_K(0,\varepsilon) \subset Z$ for some $\varepsilon > 0$, see Example 1.3.12.

According to the following lemma, the constructible subsets are exactly the finite disjoint unions of subsets of the form $U \cap F$ where U is quasi-compact open and F closed with $X \setminus F$ quasi-compact.

Lemma 1.3.2. — *For a set E and $F \subset P(E)$ containing E and $\emptyset$, the Boolean algebra generated by F, that is to say the smallest subset of $P(E)$ containing F that is stable under complementary and finite intersections, is the set of disjoint unions of sets of the form $S \setminus T$ with S a finite intersections of elements of F and T a finite union of elements of F.*

Proof. — The Boolean algebra generated by F is the union of the Boolean algebras generated by the finite subsets of F. We can thus suppose that F is finite. The Boolean algebra generated by F is seen as a sub-$\mathbb{F}_2$-algebra of $\mathbb{F}_2^E$. More precisely, this is $A = \mathbb{F}_2[\mathbf{1}_S]_{S \in F}$ where $\mathbf{1}_S$ is the characteristic function of S. Now, since for any $a \in A$, $a^2 = a$, and A is of finite type over $\mathbb{F}_2$, A is an Artinian reduced ring with residue fields isomorphic to $\mathbb{F}_2$ at maximal ideals. We thus have, if $\widehat{A} = \mathrm{Hom}(A, \mathbb{F}_2)$,

$$\begin{aligned} A &\xrightarrow{\sim} \mathbb{F}_2^{\widehat{A}} \\ a &\longmapsto \big(\chi(a)\big)_{\chi \in \widehat{A}} \end{aligned}$$

(this is a particular case of the so-called Stone representation theorem). Now, any set in the Boolean algebra generated by F corresponding to a function $f : \widehat{A} \to \mathbb{F}_2$ is written as

$$\coprod_{\substack{\chi \in \widehat{A} \\ f(\chi)=1}} \bigcap_{\substack{S \in F \\ \chi(\mathbf{1}_S)=1}} S \setminus \bigcup_{\substack{S \in F \\ \chi(\mathbf{1}_S)=0}} S.$$

In fact, this union is clearly disjoint and the element

$$a = \sum_{\substack{\chi \in \widehat{A} \\ f(\chi)=1}} \prod_{\substack{S \in F \\ \chi(\mathbf{1}_S)=1}} \mathbf{1}_S . \prod_{\substack{S \in F \\ \chi(\mathbf{1}_S)=0}} (1 + \mathbf{1}_S) \in A$$

is such that for all $\chi \in \widehat{A}$, $\chi(a) = f(\chi)$. □

Let us note that in fact the locally closed constructible subsets of X are exactly the subsets of the form $U \setminus V$ where U and V are quasi-compact open subsets of X, see Lemma 1.3.7.

Definition 1.3.3. — Let X be a spectral space. The topological space

$$X_{cons}$$

is the the set X equipped with the topology that has as a base the set of constructible sets. Its closed subsets are the *pro-constructible* subsets and its open subsets are the *ind-constructible* subsets.

For example:

1. Any closed subsets is pro-constructible.
2. For any $x \in X$, the localization
$$\begin{aligned} X_x &= \{y \in X \mid y \geq x\} \\ &= \bigcap_{\substack{U \ni x \\ \text{qc open}}} U \end{aligned}$$
is pro-constructible inside X.
3. For any $x \in X$, $\{x\}$ is pro-constructible in X since
$$\{x\} = X_x \cap \overline{\{x\}}.$$
4. If X is a finite (T0) space then $X_{cons} = X_{disc}$.

The evident bijective map

$$X_{cons} \longrightarrow X$$

is continuous quasi-compact. Let us moreover remark that the correspondence $X \mapsto X_{cons}$ is functorial with respect to quasi-compact maps of spectral spaces. We will see in Proposition 1.3.8 that it has a definition as an adjoint functor.

Remark 1.3.4. — The geometry of constructible and pro-constructible subsets for quasi-compact quasi-separated schemes goes back to Grothendieck, see [33, Section 1.9]. For example, the compacity of the constructible topology for qc qs schemes is [33, Proposition 1.9.15]. Hochster is the one who realized that this has nothing to do with schemes and brought out the notion of spectral spaces.

1.3.2. Compacity of the constructible topology. — The first result that is used all the time is the following.

Theorem 1.3.5 **([34, Theorem 1]).** — *The topological space X_{cons} is compact totally disconnected.*

Proof. — If $x, y \in X$ are distinct, up to permuting x and y, there exists a quasi-compact open subset U containing x and not y. Then U and $X \setminus U$ are disjoint constructible subsets with $x \in U$ and $y \in X \setminus U$. We deduce that X_{cons} is Hausdorff.

Since a base of the topology of X_{cons} is made of open and closed subsets, X_{cons} is moreover totally discontinuous.

It thus remains to prove that X_{cons} is quasi-compact. For this we use the following version of Alexander's subbase quasi-compacity criterion. Recall that for any set S a subset $\mathscr{F} \subset \mathscr{P}(S)$ is said to have the finite intersection property if for any $A \subset \mathscr{F}$ finite non-empty one has $\bigcap_{E \in A} E \neq \emptyset$.

Let T be a topological space that has a subbase $\mathscr{B}$ satisfying: for any $\mathscr{F} \subset \mathscr{B}$ maximal among the subsets of $\mathscr{B}$ having the finite intersection property, one has $\bigcap\limits_{U \in \mathscr{F}} U \neq \emptyset$. Then T is quasicompact.

We apply this to X_{cons} with $\mathscr{B}$ the set of open quasi-compact subsets of X and their complementarities. Let $\mathscr{F}$ be such a maximal subset. Let us note

$$F = \bigcap_{\substack{Z \in \mathscr{F} \\ \text{closed}}} Z.$$

The quasi-compacity of X implies that the closed set F is non-empty. Let us prove it is irreducible. Suppose

$$F = F_1 \cup F_2$$

with F_1 and F_2 closed strictly contained in F. The open subset $X \setminus F_1$ satisfies $(X \setminus F_1) \cap F_1 = \emptyset$ and $(X \setminus F_1) \cap F_2 \neq \emptyset$. The same type of properties holds for $X \setminus F_2$. From this we deduce that, up to shrinking $X \setminus F_1$ and $X \setminus F_2$, there exists U_1 and U_2 quasi-compact open subsets satisfying

$$U_1 \cap F_1 = \emptyset,\ U_1 \cap F_2 \neq \emptyset,\ U_2 \cap F_2 = \emptyset,\ U_2 \cap F_1 \neq \emptyset.$$

Since $F_2 \not\subset X \setminus U_1$ and $F_1 \not\subset X \setminus U_2$, one has

$$X \setminus U_1 \notin \mathscr{F},\ X \setminus U_2 \notin \mathscr{F}. \tag{1}$$

Moreover if $Z_1, \dots, Z_n \in \mathscr{F}$ and $Z'_1, \dots, Z'_{n'} \in \mathscr{F}$ satisfy

$$Z_1 \cap \dots \cap Z_n \cap X \setminus U_1 = \emptyset \text{ and } Z'_1 \cap \dots \cap Z'_{n'} \cap X \setminus U_2 \neq \emptyset$$

then if $W = Z_1 \cap \dots \cap Z_n \cap Z'_1 \cap \dots Z'_n$ one has

$$W \cap X \setminus U_1 = \emptyset \text{ and } W \cap X \setminus U_2 = \emptyset$$

and thus since $F \subset X \setminus U_1 \cup X \setminus U_2$, $W \cap F = \emptyset$. Now, one has

$$W \cap F = \bigcap_{\substack{Z \in \mathscr{F} \\ \text{closed}}} (W \cap Z)$$

and thus since W is quasicompact as a closed subset of a quasicompact open subset of X, there exists $Z''_1, \dots, Z''_{n''} \in \mathscr{F}$ satisfying

$$Z_1 \cap \dots \cap Z_n \cap Z'_1 \cap \dots Z'_{n'} \cap Z''_1 \cap \dots \cap Z''_{n''} = \emptyset.$$

This is impossible since $\mathscr{F}$ has the finite intersection property. From this we deduce that $\mathscr{F} \cup \{X \setminus U_1\}$ or $\mathscr{F} \cup \{X \setminus U_2\}$ has the finite intersection property. The equation (1) then contradicts the maximality of $\mathscr{F}$. We thus deduce that F is irreducible.

Let now $\xi \in F$ be its generic point. For $U \in \mathscr{F}$ open, $U \cap F \neq \emptyset$ since $\{Z \cap U \mid Z \in \mathscr{F} \text{ closed}\}$ has the finite intersection property and U is quasicompact. One then has

$$\xi \in \bigcap_{\substack{U \in \mathscr{F} \\ \text{open}}} U \cap F = \bigcap_{Z \in \mathscr{F}} Z$$

and this is thus non-empty. □

Example 1.3.6. — The spectral space $\mathrm{Spec}(\mathbb{Z})_{cons}$ is identified with the Alexandroff compactification $\mathbb{N} \cup \{\infty\}$ of the discrete space $\mathbb{N}$. The morphism $\mathbb{N} \cup \{\infty\} \to \mathrm{Spec}(\mathbb{Z})$ sends $n \in \mathbb{N}$ to the $(n+1)$th prime number and ∞ to the generic point η.

For X spectral, *the open/closed subsets of X_{cons} are the constructible subsets of X.* Thus, as a profinite set one has simply

$$X_{cons} = \varprojlim_{\substack{X=\coprod_{i\in I} Z_i \\ \text{finite constructible partition}}} I$$

where the indexing category is the one of finite sets I together with a surjective constructible function $X \twoheadrightarrow I$ and the morphisms are commuting diagrams

$$\begin{tikzcd} & I \arrow[d] \\ X \arrow[ur, two heads] \arrow[dr, two heads] & \\ & J \end{tikzcd}$$

Let us give a simple very first application of the preceding result.

Lemma 1.3.7. — *The locally closed constructible subsets of X are the $U \cap F$ where U is a quasi-compact open subset and F is closed with quasi-compact complementary.*

Proof. — Let Z be a locally closed constructible subset of X. Each point z of Z has a quasi-compact open neighborhood U_z such that $U_z \cap Z$ is closed in U_z. Since Z is compact in X_{cons}, it is quasi-compact and we can thus find a quasi-compact open subset U of X containing Z such that Z is closed in U. Since Z is constructible in X, it is constructible in U and thus $V = U \setminus Z$ is quasi-compact since it is quasi-compact in U_{cons}. We then have $Z = U \setminus V$ and the result is proved. □

Finally, let us note the following categorical characterization of the constructible topology whose proof is elementary.

Proposition 1.3.8. — *The functor*

$$(-)_{cons} : \{\textit{spectral spaces + qc maps}\} \longrightarrow \{\textit{profinite spaces}\}$$

is a right adjoint to the inclusion of the category of profinite spaces in the category of spectral spaces equipped with the quasi-compact maps. The natural map $X_{cons} \to X$ is one of the adjunction maps, the other being the identification between X_{cons} and X for X profinite.

1.3.3. First applications of the compacity of the constructible topology. — Theorem 1.3.5 has plenty of applications that we use frequently. They are the basic tools we use when we work with spectral spaces. Let's cite some of them.

Corollary 1.3.9. —

1. *Let $f : X \to Y$ be a qc (continuous) map of spectral spaces. For any $Z \subset X$ pro-constructible, $f(Z)$ is pro-constructible. In particular, the image of f, $f(X)$, is pro-constructible.*
2. *For Z pro-constructible in X,*
$$\overline{Z} = \bigcup_{z\in Z} \overline{\{z\}}$$
that is to say the specializations of elements of Z. In particular
$$Z \text{ closed} \Longleftrightarrow Z \text{ stable under specialization.}$$
In particular for f as in point (1), its image is closed iff it is stable under specialization.
3. *If Z is pro-constructible inside X then Z (equipped with the induced topology) is spectral.*
4. *Any non-empty spectral space has a closed point.*
5. *Any non-empty spectral space has a maximal point.*

Proof. — Point (1) is evident. Let us note by the way that it gives a canonical factorization

$$X \xrightarrow{f} f(X) \hookrightarrow Y$$

in the category of spectral spaces with qc maps as morphisms.

Let us verify point (2). If $x \in X$ is not a specialization of an element of Z then $X_x \cap Z = \emptyset$. This means

$$\bigcap_{\substack{U \ni x \\ \text{qc open}}} (U \cap Z) = \emptyset.$$

But for such a U, $U \cap Z$ is closed in the compact topological space Z_{cons}. We deduce that a finite sub-intersection is empty and thus there exists such a U as before with $U \cap Z = \emptyset$. Thus $x \notin \overline{Z}$.

Point (3) is easily deduced from the compacity of Z in X_{cons}.

For point (4) we can use Zorn lemma. More precisely, if $(x_i)_{i \in I}$ is a chain of elements of X in the sense that for $i, j \in I$, $x_i \leq x_j$ or $x_j \leq x_i$, then

$$\bigcap_{i \in I} \overline{\{x_i\}} \neq \emptyset.$$

In fact, this is an intersection of closed subsets of the compact topological space X_{cons} and thus if this intersection where empty a finite sub-intersection would be empty. This is impossible because of the chain condition. Such a chain is thus bounded below by an element of X.

For point (5) we gain use Zorn lemma. Let $(x_i)_{i \in I}$ be a chain of elements of X. If

$$\bigcap_{i \in I} X_{x_i} = \emptyset$$

then a finite sub-intersection is empty since for all i, X_{x_i} is a closed subset of X_{cons}. This is impossible because of the chain condition and thus $(x_i)_{i \in I}$ is bounded above. □

Remark 1.3.10. — Points (1) and (2) give rise to the valuation criterion of properness. Point (4) is the analog of the fact that any non-zero ring has a maximal ideal. Point (5) is the analog of the fact that any non-zero ring has a minimal prime ideal.

Example 1.3.11. — For any $x \in X$ a spectral space, the localization of X at x, X_x, is pro-constructible and thus spectral with a unique closed point x.

Example 1.3.12. —

1. Let X be a spectral space, $Z \subset X$ constructible and $x \in Z$. If $X_x \subset Z$ then there exists a neighborhood U of x contained in Z. In fact, $\bigcap_{U \ni x} U \cap (X \setminus Z) = \emptyset$ and using the compactness of $(X \setminus Z)_{cons}$ one concludes.
2. For example, any maximal point lying in Z constructible inside X spectral has a neighborhood contained in Z.
3. For example, let X be a quasi-separated adic space of finite type over $\mathrm{Spa}(K)$ with K a non-archimedean field, i.e. the adic space associated to a quasi-compact quasi-separated Tate rigid space over K. Let $Z \subset |X|$ be constructible. For any classical point $x \in X$, $x \in Z$ implies a neighborhood of x is contained in Z. Typically, if $Z \subset |\mathbb{B}^d_K|$ is constructible, then for any $x \in \mathbb{B}^d(K)$, $x \in Z \Rightarrow \exists \varepsilon > 0$, $\mathbb{B}^d(x, \varepsilon) \subset Z$.

1.4. Projective limits of spectral spaces

When dealing with spectral spaces projective limits of spectral spaces show up very often, already in Theorem 1.2.5. But there is more to it when working with perfectoid spaces and diamonds. In fact, one of the basic notions we will use is the so-called pro-étale topology on perfectoid spaces. This involves projective limits of perfectoid spaces as the name suggests.

1.4.1. Spectrality of the projective limit. — We will use the following elementary result all the time.

Proposition 1.4.1. — *If $(X_i)_{i\in I}$ is a cofiltered projective system of spectral spaces with quasi-compact transition maps then the topological space $\varprojlim_{i\in I} X_i$ is spectral.*

Proof. — From Tychonoff theorem we deduce that

$$\prod_{i\in I} X_i$$

is quasi-compact quasi-separated and has a base of quasi-compact open subsets. It is moreover clearly (T0). Let now $Z \subset \prod_{i\in I} X_i$ be an irreducible closed subset. We have to prove that

$$\bigcap_{\substack{U \text{ qc open} \\ U\cap Z\neq\emptyset}} U\cap Z$$

is non-empty. But such a set U, $U\cap Z$ is closed in the compact topological space $\left(\prod_{i\in I} X_i\right)_{cons}$. Since Z is irreducible, the family $\{U\cap Z \mid U \text{ qc open}, \ U\cap Z\neq\emptyset\}$ has the finite intersection property. We conclude that Z has a generic point. Thus, we have proven that $\prod_{i\in I} X_i$ is spectral.

For $i,j\in I$ with $i\geq j$ let us note $p_{ij}: X_i \to X_j$ the transition map. One has

$$\varprojlim_i X_i = \bigcap_{i\geq j} (p_{ij}\times \mathrm{Id})^{-1}\Delta_{X_j}$$

where $\Delta_{X_j}\subset X_j\times X_j$ is the diagonal. Since $(p_{ij}\times\mathrm{Id})$ is quasi-compact, Lemma 1.4.2 implies that $\varprojlim_i X_i$ is pro-constructible in $\prod_{i\in I} X_i$. Applying point (3) of Corollary 1.3.9 we deduce that $\varprojlim_i X_i$ is spectral. □

Since a spectral space X is quasi-separated, $\Delta : X \to X\times X$ is a quasi-compact map of spectral spaces and thus its image is pro-constructible according to point (1) of Corollary 1.3.9. Nevertheless, we can verify this directly as we do in the following lemma.

Lemma 1.4.2. — *For X a spectral space, the diagonal $\Delta_X\subset X\times X$ is pro-constructible.*

Proof. — One has since X is $(T0)$,

$$X\times X\setminus\Delta_X = \bigcup_{U \text{ open qc}} [(U\times X\setminus U)\cup(X\setminus U\times U)]$$

and this is thus ind-constructible. □

1.4.2. Some properties of the projective limit. — Let us start by giving a simple immediate application of the adjunction property of Proposition 1.3.8.

Proposition 1.4.3. — *For $(X_i)_{i\in I}$ a cofiltered projective system of spectral spaces with quasi-compact transition maps,*

$$\varprojlim_{i\in I} X_{i,cons} \xrightarrow{\sim} \Big(\varprojlim_{i\in I} X_i\Big)_{cons}.$$

The following result is elementary and will be used all the time.

Proposition 1.4.4. — *Let $X=\varprojlim_{i\in I} X_i$ be a qc cofiltered limit of spectral spaces. One has*

$$\varinjlim_{i\in I}\{\textit{qc open subsets of } X_i\} \xrightarrow{\sim} \{\textit{qc open subsets of } X\}$$

$$\varinjlim_{i\in I}\{\textit{constructible subsets of } X_i\} \xrightarrow{\sim} \{\textit{constructible subsets of } X\}.$$

Proof. — The surjectivity of the map is immediate from the quasi-compacity of X, resp. the compacity of X_{cons} and the fact that the open/closed subsets of the constructible topology are the construcible subsets of the original topology + Proposition 1.4.3.

Let us now prove the injectivity. Let us note $p_i : X \to X_i$ and for $j \geq i$, $p_{ji} : X_j \to X_i$. Let $U, V \subset X$ be constructible subsets such that

$$p_i^{-1}(U) = p_j^{-1}(V).$$

Let us set for $j \geq i$, $Z_j = p_{ji}(X_j)$. The sequence $(Z_j)_{j \geq i}$ is a decreasing sequence of pro-constructible subsets of X_i. The formula $p_i^{-1}(U) = p_j^{-1}(V)$ is equivalent to

$$\bigcap_{j \geq i} Z_j \cap U = \bigcap_{j \geq i} Z_j \cap V.$$

We thus have

$$\cap_{j \geq i} Z_j \cap (U \setminus V) = \emptyset, \quad \cap_{j \geq i} Z_j \cap (V \setminus U) = \emptyset.$$

Using the compacity of the constructible topology we deduce that some finite sub-intersections of those two intersections are empty and thus, since $(I, \leq)$ is cofiltered, there exists $j \geq i$ such that

$$(U \setminus V) \cap Z_j = \emptyset, \ (V \setminus U) \cap Z_j = \emptyset.$$

From this we deduce that

$$p_{ji}^{-1}(U) = p_{ji}^{-1}(V). \qquad \square$$

Example 1.4.5. — Let $\mathrm{Spec}(A) = \varprojlim_{i \in I} \mathrm{Spec}(A_i)$ be a cofiltered limit of affine schemes. Any quasi-compact open subset U of $\mathrm{Spec}(A)$ is a finite union of principal open subsets, $U = \cup_{\alpha \in S} D(f_\alpha)$ where the set of indices S is finite. Now for each $\alpha \in S$ we can chose i_α such that f_α is the image of some $g_\alpha \in A_{i_\alpha}$. We can then chose, since S is finite and I cofiltered, some $j \in I$ such that for all $\alpha \in S$, $j \geq i_\alpha$. Let $h_\alpha \in A_j$ be the image of g_α in A_j. One then has: U is the reciprocal image of the quasi-compact open subset $\cup_{\alpha \in S} D(h_\alpha) \subset \mathrm{Spec}(A_j)$.

1.5. The equivalence between spectral spaces and pro-finite (T0) spaces

We can now state and prove the full form of Theorem 1.2.5.

Theorem 1.5.1. — *The projective limit functor induces an equivalence of categories*

$$\varprojlim : \mathit{Pro}(\textit{finite ordered sets}) \xrightarrow{\sim} \{\textit{spectral topological spaces + qc continuous maps}\}.$$

An inverse sends a spectral space X to the pro-finite (T0) space

$$\varprojlim_{A \in \mathcal{I}_X} \mathrm{Im}\big(X \to \{s, \eta\}^A\big)$$

where

- *$\mathcal{I}_X$ is the ordered set of finite subsets of the set of quasi-compact maps $X \to \{s, \eta\}$ i.e. the ordered set of finite subsets of the set of quasicompact open subsets of X,*
- *the map $X \to \{s, \eta\}^A$ sends x to $(f(x))_{f \in A}$.*

Proof. — Proposition 1.4.1 implies that the functor $\varprojlim$ takes values in the category of spectral spaces. Let now X be a spectral space. There is a diagram

$$\begin{tikzcd} X_{cons} \arrow[r, "g"] \arrow[d, "\text{bijective}"] & \left[\varprojlim_{A\in\mathcal{I}_X} \mathrm{Im}\big(X \to \{s,\eta\}^A\big)\right]_{cons} \arrow[r, "h", "\sim"'] \arrow[d, "\text{bijective}"] & \varprojlim_{A\in\mathcal{I}_X} \mathrm{Im}\big(X_{cons} \to \underbrace{(\{s,\eta\}^A)_{disc}}_{\{0,1\}^A}\big) \\ X \arrow[r, "f"] & \varprojlim_{A\in\mathcal{I}_X} \mathrm{Im}\big(X \to \{s,\eta\}^A\big). & \end{tikzcd}$$

According to Proposition 1.4.3 h is an homeomorphism. The set $\mathcal{I}_X$ is identified with a subset of the set of finite subsets of the set of continuous quasi-compact maps $X_{cons} \to \{0,1\}$ that separates the point of X_{cons}. According to Theorem 1.3.5 X_{cons} is profinite and thus $h \circ g$ and thus g are homeomorphisms (see Lemma 1.5.2). We deduce that the quasi-compact continuous map f is bijective. Now, if U is a quasi-compact open subset of X, and $\chi_U : X \to \{s,\eta\}$ is defined by $\chi_U^{-1}(\eta) = U$, then $f(U)$ is the reciprocal image of $\eta \in \mathrm{Im}\big(X \to \{s,\eta\}^{\{\chi_U\}}\big)$ in the factor $A = \{\chi_U\}$. It is thus open and we deduce that f is a homeomorphism.

The full faithfulness of the functor $\varprojlim$ is easy and left to the reader. □

Lemma 1.5.2. — *Let P be a profinite set and $\mathcal{I}$ a subset of the set of finite subsets of the set of continuous quasi-compact maps from P to $\{0,1\}$. Suppose $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ and for $x, y \in P$ there exists $A \in \mathcal{I}$ and $f \in A$ such that $f(x) \neq f(y)$. Then,*

$$P \xrightarrow{\sim} \varprojlim_{A\in\mathcal{I}} \mathrm{Im}\left(P \to \{0,1\}^A\right)$$

is an homeomorphism.

Proof. — The proof is left to the reader. □

Remark 1.5.3. — Let $X = \varprojlim_i X_i$ with X_i a finite (T0) space. As a particular case of Propositions 1.4.3 and 1.4.4 we have the following dictionary between the topological and combinatorial description of spectral spaces:

1. There is an identification
$$X_{cons} = \varprojlim_i X_{i,disc}.$$
2. There is a bijection
$$\varinjlim_{i\in I} \{\text{open subsets of } X_i\} \xrightarrow{\sim} \{\text{qc open subsets of } X\}.$$
3. There is a bijection
$$\varinjlim_{i\in I} \{\text{subsets of } X_i\} \xrightarrow{\sim} \{\text{constructible subsets of } X\}.$$

1.6. Connected components

Let X be a spectral space and write $X = \varprojlim_{i\in I} X_i$ with the X_i finite (T0) spaces and the limit is cofiltered.

The basic results about connected components of spectral spaces are the following (see for example [39, Lemma 2.4.1]).

Proposition 1.6.1. — *1. Every connected component of X is the intersection of the open/closed subsets containing it.*

2. *The set $\pi_0(X)$ equipped with the quotient topology is profinite and $\pi_0(X) \xrightarrow{\sim} \varprojlim_{i\in I} \pi_0(X_i)$ as a profinite set.*
3. *The surjection*
$$X \longrightarrow \pi_0(X)$$
identifies the functor $X \mapsto \pi_0(X)$ as a left adjoint to the inclusion
$$\{\textit{Hausdorff spectral spaces}\} = \{\textit{profinite spaces}\} \hookrightarrow \{\textit{spectral spaces}\}.$$

Proof. — The result is easily deduced from Theorem 1.5.1 once one has verified that if $(X_i)_{i\in I}$ is a cofiltered projective system of connected finite (T0) spaces then $X = \varprojlim_{i\in I} X_i$ is connected. But open/closed subsets of X are quasi-compact and thus $\varinjlim_{i\in I}\{\text{open/closed subsets of } X_i\} \xrightarrow{\sim} \{\text{open/closed subsets of } X\}$. From this we deduce that X is connected. □

Example 1.6.2. — If R is a ring, writing $R = \varinjlim_{i\in I} R_i$, a filtered colimit of Noetherian rings, one has
$$\mathrm{Spec}(R) = \varprojlim_{i\in I} \mathrm{Spec}(R_i)$$
and thus
$$\pi_0(\mathrm{Spec}(R)) = \varprojlim_{i\in I} \underbrace{\pi_0(\mathrm{Spec}(R_i))}_{\text{finite set}}$$
as a profinite set. The Boolean algebra of open/closed subset of the profinite set $\pi_0(\mathrm{Spec}(R))$ is then identified with the one of idempotents of R.

1.7. Quotients of spectral spaces

Taking quotient of spectral spaces is a natural operation when dealing with diamonds that are pro-étale quotients of perfectoid spaces.

Proposition 1.7.1 **([47, Lemma 2.9, Lemma 2.10]).** — *Let X be a spectral space and $R \subset X \times X$ be a pro-constructible equivalence relation such that both maps $R \rightrightarrows X$ are generalizing.*

1. *If the topological space X/R has a basis of open subsets whose preimages in X are quasi-compact then X/R is spectral and the map $X \to X/R$ is quasicompact generalizing.*
2. *If both maps $R \rightrightarrows X$ are open then X/R is spectral and $X \to X/R$ is quasicompact open.*

Proof. — Let us verify point (1). It is clear that X/R is quasicompact and that each point has a basis of neighborhoods made of quasicompact retrocompact open subsets. Let us verify it is $(T0)$. Let $x_1, x_2 \in X$ having distinct images in X/R. Let us note $s, t : R \to X$ for the two maps defining the equivalence relation, and
$$Z_1 = s(t^{-1}(x_1)),\ Z_2 = s(t^{-1}(x_2))$$
the orbits of x and y under R. We thus have $Z_1 \cap Z_2 = \emptyset$. Since R is pro-constructible inside $X \times X$, $s, t : R_{cons} \to X_{cons}$ are continuous maps of compact topological spaces. Since $\{x_1\}$ and $\{x_2\}$ are pro-constructible we deduce that Z_1 and Z_2 are pro-constructible subsets of X. For $i = 1, 2$, since Z_i is pro-constructible, its closure $\overline{Z_i}$ is the set of its specializations. Since t is generalizing we have
$$\overline{t^{-1}(Z_i)} = t^{-1}(\overline{Z_i}).$$
We deduce that
$$s(t^{-1}(\overline{Z_i})) = \overline{Z_i}$$

and thus $\overline{Z_i}$ is stable under R. Now, if

$$Z_1 \cap \overline{Z_2} = \emptyset$$

then $U = X \backslash \overline{Z_2}$ is an open subset stable under R whose image in X/R is an open subset containing x_1 and not x_2. The same goes on if $\overline{Z_1} \cap Z_2 = \emptyset$, the image in X/R of $X \setminus \overline{Z_1}$ is an open subset containing x_2 but not x_1. We are thus reduced to proving that the situation

$$Z_1 \cap \overline{Z_2} = \emptyset \text{ and } \overline{Z_1} \cap Z_2 = \emptyset$$

is impossible. For this consider the pro-constructible subset

$$Z_1 \cup Z_2$$

of X. Being pro-constructible it is a spectral space and thus contains a maximal point (Corollary 1.3.9). Up to permuting x_1 and x_2 and replacing x_1 by this maximal point we can suppose that x_1 is maximal in $Z_1 \cup Z_2$. Since Z_1 is the orbit of one point and $\overline{Z_2}$ is stable under R, the condition $Z_1 \cap \overline{Z_2} \neq \emptyset$ implies that $Z_1 \subset \overline{Z_2}$. This implies that x_1 is a specialization of an element of Z_2 and thus $x_1 \in Z_2$ by maximality of x_1. This is impossible and we conclude that X/R is $(T0)$.

Let now $Z \subset X/R$ be an irreducible closed subset. To prove it has a generic point we have to prove that

$$\bigcap_{\substack{U \subset X/R \text{ open} \\ U \cap Z \neq \emptyset}} U \cap Z \neq \emptyset.$$

Let $\pi : X \to X/R$ be the projection. One has to verify that

$$\bigcap_{\substack{U \subset X/R \text{ open} \\ U \cap Z \neq \emptyset}} \pi^{-1}(U) \cap \pi^{-1}(Z) \neq \emptyset.$$

This is immediately deduced from the fact that $\pi^{-1}(Z)_{cons}$ is compact and the fact that the preceding intersection has the finite intersection property since Z is irreducible.

We have thus proven that X/R is spectral. The fact that $X \to X/R$ is quasicompact generalizing is easy and left to the reader.

Point (2) is deduced from point (1) and the fact that for U a quasicompact open subset of X, its orbit $s(t^{-1}(U))$ is quasicompact open. □

Example 1.7.2. — Let G be a profinite group acting continuously on a spectral space X in the sense that the map

$$G \times X \to X$$

is continuous. Let U be a quasicompact open subset of X. By continuity of the action for any point x of U there exists an open subgroup H of G and a neighborhood V of x such that $H.V \subset U$. Using the quasicompacity of U we deduce the existence of an open subgroup H of G such that $H.U = U$. From this we deduce that $G.U$ is a finite union of translates of U and is thus quasicompact open. We deduce from this that the continuous map

$$\begin{aligned} G \times X &\longrightarrow X \times X \\ (g, x) &\longmapsto (g.x, x) \end{aligned}$$

is quasicompact and thus its image R is pro-constructible. Both projections from R to X are clearly open since the map $(g, x) \mapsto g.x$ is open. Applying Proposition 1.7.1 we deduce that

$$X/G$$

is a spectral space.

1.8. Constructible sheaves

If $X = \varprojlim_{i\in I} X_i$ is a qc cofiltered limit of spectral spaces, then

$$\varinjlim_{i\in I} \{\text{qc open subsets of } X_i\} \xrightarrow{\sim} \{\text{qc open subsets of } X\}.$$

Since quasi-compact open subsets form a base of the topology we deduce an equivalence of topos

$$\widetilde{X} \xrightarrow{\sim} \varprojlim_{i\in I} \widetilde{X_i}.$$

We can then apply [2, Exp.VI Sec.8] to deduce that for any sheaf of abelian groups $\mathscr{F}$ on X with associated $\mathscr{F}_i$ on X_i, $i \in I$, one has

$$\varinjlim_{i\in I} R\Gamma(X_i, \mathscr{F}_i) \xrightarrow{\sim} R\Gamma(X, \mathscr{F}). \tag{2}$$

Example 1.8.1. — Let $f : X \to Y$ be a qc map of spectral spaces, $\mathscr{F}$ a sheaf of abelian groups on X and $y \in Y$. Note $X_y = X \times_Y Y_y$. One has

$$X_y = \varprojlim_{\substack{U \ni y \\ \text{open qc}}} f^{-1}(U).$$

From this we deduce that

$$(Rf_*\mathscr{F})_y = R\Gamma(X_y, \mathscr{F}_{|X_y}).$$

The following proposition is easy.

Proposition 1.8.2. — 1. *When the X_i, $i \in I$, are finite, via pullback to X, there is an equivalence of categories*

$$\varinjlim_{i\in I} \widetilde{X_i} \xrightarrow{\sim} \{\textit{constructible sheaves on } X\}$$

where a constructible sheaf is a sheaf that is constant along a locally closed constructible stratification.

2. *There is an equivalence of categories*

$$\varinjlim : \mathit{Ind}(\textit{constructible sheaves on } X) \xrightarrow{\sim} \widetilde{X}$$

that identifies the category of sheaves on X as the ind-category of constructible sheaves.

3. *The constructible sheaves of abelian groups with finite type stalks are the compact objects of the category of sheaves of abelian groups on X and this last category is compactly generated.*

4. *The category of constructible sheaves of abelian groups on X is the smallest sub-abelian category of sheaves of abelian groups on X stable under extension an containing the sheaves $j_!\underline{\mathbb{Z}}$ for j the inclusion of a locally closed constructible subsets and M.*

1.9. Totally disconnected spectral spaces

Totally disconnected perfectoid spaces are an essential notion in the theory of diamonds. We will often use the slogan "pro-étale locally, any perfectoid space is totally disconnected". Here we first introduce the notion for spectral spaces.

1.9.1. Generalities. — Let X be a spectral space. We say that an open covering

$$X = \bigcup_{i\in I} U_i$$

of X splits if

$$\coprod_{i\in I} U_i \to X$$

has a section. This means that we can find a collection of open subsets $(V_i)_{i\in I}$ with $V_i \subset U_i$ and

$$X = \coprod_{i\in I} V_i.$$

Theorem 1.9.1 **(F.** [47, Lemma 7.2]**).** — *For X a spectral space the following are equivalent:*

1. *Each connected component of X has a unique closed point.*
2. *Each open covering of X splits.*
3. *For any topological space Y and any surjective continuous map $f : Y \to X$ that is a local isomorphism on Y, f has a section.*
4. *For any sheaf of sets $\mathscr{F}$ on X, $\mathscr{F}(X) \neq \emptyset$.*
5. *For any sheaf of groups $\mathscr{G}$ on X, $H^1(X, \mathscr{G}) = \{*\}$.*
6. *For any sheaf of abelian group $\mathscr{F}$ on X and any $i > 0$, $H^i(X, \mathscr{F}) = 0$ i.e. any sheaf of abelian groups is acyclic.*

Proof. — Let us verify the equivalence between points (2) and (6). Suppose point (2) is verified and let $\mathscr{F}$ be a sheaf of abelian groups on X. Point (2) implies immediately the vanishing of the Chech cohomology of X,

$$\forall i > 0,\ \check{H}^i(X, \mathscr{F}) = 0.$$

Since $\check{H}^1(X, \mathscr{F}) \xrightarrow{\sim} H^1(X, \mathscr{F})$ we deduce that $H^1(X, \mathscr{F}) = 0$. This being true for all $\mathscr{F}$, using a standard "décalage cohomologique" method by induction on i starting with $i = 1$, we deduce that

$$\forall i > 0,\ H^i(X, \mathscr{F}) = 0.$$

In the other direction, suppose point (6) is verified. Let $X = \cup_\alpha U_\alpha$ be an open covering of X. Note $j_\alpha : U_\alpha \hookrightarrow X$ the inclusion. There is an epimorphism

$$\bigoplus_\alpha j_{\alpha !}\underline{\mathbb{Z}} \longrightarrow \underline{\mathbb{Z}}.$$

By hypothesis, the global section functor $\Gamma(X; -)$ is exact and we deduce that

$$\bigoplus_\alpha \Gamma(X, j_{\alpha !}\underline{\mathbb{Z}}) \longrightarrow \Gamma(X, \underline{\mathbb{Z}})$$

is surjective. We deduce that one can write

$$\mathbf{1} = \sum_\alpha f_\alpha \tag{3}$$

as locally constant functions from X to $\mathbb{Z}$ where $\mathrm{supp}(f_\alpha) \subset U_\alpha$. Since X is quasicompact and f_α locally constant, $V_\alpha = \mathrm{supp}(f_\alpha) \subset U_\alpha$ is open and closed. The Equation (3) then shows that $X = \bigcup_\alpha V_\alpha$. There is factorization $\coprod_\alpha V_\alpha \to \coprod_\alpha U_\alpha \to X$. Since X is quasi-compact we can suppose that our coverings are finite. Since any finite covering of a topological space by open/closed subsets splits we deduce that $(U_\alpha)_\alpha$ splits and we deduce that point (2) is verified. We have thus proven (1) ⇔ (6).

Let $\mathscr{F}$ be a sheaf of abelian groups on X. Note

$$\pi : X \longrightarrow \pi_0(X).$$

Since $\pi_0(X)$ is profinite any covering of $\pi_0(X)$ splits and thus, as we just saw, any sheaf on $\pi_0(X)$ is acyclic. We thus have for $i \geq 0$,

$$H^i(X, \mathscr{F}) = H^0(\pi_0(X), R^i\pi_*\mathscr{F}). \tag{4}$$

Moreover, for $c \in \pi_0(X)$ one has

$$\pi^{-1}(c) = \varprojlim_{\substack{U \ni c \\ \text{open/closed}}} \pi^{-1}(U).$$

We deduce from this that

$$(R\pi_*\mathscr{F})_c = R\Gamma(\pi^{-1}(c), \mathscr{F}), \tag{5}$$

see Equation (2) in Section 1.8.

Suppose X is connected. Then the equivalence between points (1) and (2) is immediate. In fact, if X has two distinct closed point x_1 and x_2 then the covering $X = X \setminus \{x_1\} \cup X \setminus \{x_2\}$ can not split since X is connected. Reciprocally, if X has a unique closed point x then $X = X_x$ since for any $y \in X$, $\overline{\{y\}}$ has a closed point (see Corollary 1.3.9). Thus, if $X = \cup_i U_i$ is an open covering, as soon as $x \in U_i$ for some index i, $X = U_i$ and thus the covering splits.

Let now X be any spectral space. If point (2) is verified for X then it is verified if we replace X by any closed subspace of X. Since any connected component of X is closed, from the case of a connected space established before, we deduce that point (1) is verified. Reciprocally, suppose point (1) is verified. Let $\mathscr{F}$ be a sheaf of abelian groups on X. Using Equation (5), the case of a connected space treated before and (2) $\Rightarrow$ (6), we deduce that $R^i\pi_*\mathscr{F} = 0$ for $i > 0$. Equation (4) then implies that $H^i(X, \mathscr{F}) = 0$ for $i > 0$. Using (6) $\Rightarrow$ (2) we deduce that (1) is verified. We thus have proven (1) $\Leftrightarrow$ (6).

We have established (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (6). The equivalence (3) $\Leftrightarrow$ (4) is obtained using the "espace étalé" of a sheaf of sets. The implication (4) $\Rightarrow$ (5) is deduced from the fact that $H^1(X, \mathscr{G})$ classifies $\mathscr{G}$-torsors and those are trivial iff they have a section. The implication (5) $\Rightarrow$ (6) is obtained using "décalage cohomologique". Finally, (2) $\Rightarrow$ (3) is deduced from the fact that for such a map $Y \to X$ as in point (3), there exists an open covering $(U_i)_i$ of X such that $\coprod_i U_i \to Y \to X$. □

Definition 1.9.2. — The spectral spaces satisfying the equivalent conditions of theorem 1.9.1 are called totally disconnected.

Those spaces will play a key role in the theory of diamonds where we will use the slogan "pro-étale locally any perfectoid spaces is totally disconnected" very often.

Example 1.9.3. — For X spectral and $x \in X$, X_x is connected and totally disconnected!

1.9.2. The particular case of analytic spectral spaces. — Let us begin with a definition. This is motivated by the nature of spectral spaces associated to analytic adic spaces. Here we enter a world that is very different from the one of Noetherian schemes. It happens a Noetherian topological space is analytic, for example this is the case for the topological space of a dimension 1 Noetherian scheme or the spectrum of a finite height valuation ring, but this is not the case in general.

Definition 1.9.4. — A spectral space X is analytic if for any $x \in X$, the ordered set X_x is a chain i.e. is a totally ordered set for the specialization relation.

The following property is particular to analytic spectral spaces since it is already false for general finite (T0) spaces.

Proposition 1.9.5. — *Let X be an analytic totally disconnected spectral space. Then any pro-constructible subset of X is totally disconnected.*

Proof. — Let $Z \subset X$ be pro-constructible. Let C be a connected component of Z. Let D be the connected component of X containing C. Since D has a unique closed points, $(D, \leq)$ is a chain. Thus, $(C, \leq)$ is a chain and thus has a unique closed point. □

Using this simple result we obtain the more surprising one that follows. We put it to illustrate the difference between Noetherian schemes and analytic adic spaces that will show up later.

Proposition 1.9.6. — *Let $j : Z \hookrightarrow X$ be the inclusion of a pro-constructible set inside an analytic spectral space, for example the inclusion of a quasi-compact open subset. Then for any sheaf of abelian groups $\mathscr{F}$ on Z and $i > 0$,*

$$R^i j_* \mathscr{F} = 0.$$

Proof. — Let $x \in X$. According to Example 1.8.1 one has $(R^i j_* \mathscr{F})_x = H^i(X_x \cap Z, \mathscr{F})$. The spectral space X_x is analytic totally disconnected and thus according to Proposition 1.9.5, $X_x \cap Z$ is totally disconnected. The result follows. □

Typically, if j is the inclusion of a quasi-compact open subset of X an analytic spectral space then $R^i j_* \mathscr{F} = 0$ for $i > 0$. For non-analytic spectral spaces this is evidently false. Typically, if k is a field and $j : \mathbb{A}^2_k \setminus \{(0,0)\} \hookrightarrow \mathbb{A}^2_k$ then $R^1 j_* \mathcal{O} \neq 0$.

1.9.3. w-local spaces. —

1.9.3.1. *Pro-étale maps.* — Start with X a spectral space. We fix a base $\mathscr{B}$ of the topology of X that is

- stable under finite intersections,
- made of quasi-compact open subsets,
- such that if $U \in \mathscr{B}$ any open/closed subset of U is in $\mathscr{B}$.

Let us start with a definition that will take on its full meaning in the context of perfectoid spaces and their pro-étale topology.

Definition 1.9.7. —
1. A map of spectral spaces $Y \to X$ is $\mathscr{B}$-étale if it is isomorphic to a map $\coprod_{i \in I} U_i \to X$ with I finite and for each $i, U_i \in \mathscr{B}$.
2. A morphism $Y \to X$ of spectral spaces is $\mathscr{B}$-pro-étale if it is isomorphic to a cofiltered limit of $\mathscr{B}$-étale spaces over X.

A morphism

$$\begin{tikzcd} \coprod_{i\in I} U_i \arrow[rr] \arrow[dr] & & \coprod_{j \in J} V_j \arrow[dl] \\ & X & \end{tikzcd}$$

with I and J finite, $U_i, V_j \in \mathscr{B}$, is given by a decomposition $U_i = \coprod_{j \in J} U_{ij}$ for each $i \in I$ where $U_{ij} \subset V_j$. From this we deduce the following lemma.

Lemma 1.9.8. —
1. *If $f : Y \to X$ is $\mathscr{B}$-étale then $f^{-1}\mathscr{B} = \{f^{-1}(U) \mid U \in \mathscr{B}\}$ is a base of the topology of Y satisfying analogous assumptions to the one satisfied by $\mathscr{B}$.*
2. *If we have a diagram*

$$\begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr] & & Y' \arrow[dl, "f"] \\ & X & \end{tikzcd}$$

   with f and $f \circ g$ $\mathscr{B}$-étales then g is $f^{-1}\mathscr{B}$-étale and is in particular quasi-compact.
3. *The category of $\mathscr{B}$-étale spaces over X is cofiltered:*

- *the equalizer of a pair of maps between $\mathscr{B}$-étale spaces over X is $\mathscr{B}$-étale over X,*
- *For $Y \to X$ and $Y' \to X$ that are $\mathscr{B}$-étale, $Y \times_X Y' \to X$ is $\mathscr{B}$-étale.*

Example 1.9.9. — For any $x \in X$, $X_x \hookrightarrow X$ is $\mathscr{B}$-pro-étale for any choice of $\mathscr{B}$.

Proposition 1.9.10. — *The projective limit functor induces an equivalence of categories*

$$\varprojlim : \mathrm{Pro}\left(\mathscr{B}\text{-étale spaces over } X\right) \xrightarrow{\sim} \{\mathscr{B}\text{-pro-étale spaces over } X + qc \text{ maps}\}.$$

Proof. — The essential surjectivity is the definition of $\mathscr{B}$-pro-étale spaces over X. For the full faithfulness it suffices to prove that

$$\mathrm{Hom}_X(\varprojlim_{i\in I} Y_i, Z) = \varinjlim_{i\in I} \mathrm{Hom}_X(Y_i, Z)$$

but this is easily deduced from Proposition 1.4.4. □

Remark 1.9.11. — Suppose $X = |\operatorname{Spec}(R)|$ for a ring R. Let us choose for $\mathscr{B}$ the base of principal open subsets $\{D(f)\}_{f\in R}$. Then, if $f : Y \to X$ is $\mathscr{B}$-pro-étale, Y inherits automatically a canonical structure of an affine scheme as a cofiltered limit of affine schemes.

1.9.3.2. *w-localization.* — We now want to construct $f : X' \to X$ such that

1. X' is spectral totally disconnected,
2. f is qc surjective $\mathscr{B}$-pro-étale.

If we can do this we will say that "*pro-étale locally, X is totally disconnected*".

One functorial way to do this is to introduce the category of w-local spectral spaces ([8]). In fact, if we want to functorially associate to X a totally disconnected spectral space X' with a morphism $X' \to X$, the simplest way is to find a subcategory $\mathcal{C}$ of the category of totally disconnected spectral spaces such that the inclusion $\mathcal{C} \hookrightarrow \{\text{spectral spaces}\}$ has a right adjoint. But

1. if we take for $\mathcal{C}$ the category of profinite spaces this is too small and $X_{cons} \to X$ is not pro-étale in general,
2. if we take for $\mathcal{C}$ the full sub-category of totally disconnected spectral spaces there is no such adjoint.

The category of w-local spectral spaces is a nice category in between the two preceding one that works well (but this is a not a full sub-category).

Definition 1.9.12 **([8]).** — We say that X is w-local if it is totally disconnected and the set of closed points X_c of X is closed. The category of w-local spectral spaces has objects the w-local spectral spaces and morphisms the qc maps $f : X \to Y$ such that $f(X_c) \subset Y_c$.

Example 1.9.13. — For X spectral and $x \in X$, X_x is w-local with only one closed point.

For X w-local the bijection $\pi_{|X_c} : X_c \longrightarrow \pi_0(X)$ is in fact an homeomorphism and thus the surjection $\pi : X \to \pi_0(X)$ has a continuous section. It it not satisfied in general for a totally disconnected spectral space.

Example 1.9.14. — Consider the profinite set $\overline{\{\frac{1}{n} \mid n \geq 1\}}$. We add a point η and set $X = \overline{\{\frac{1}{n} \mid n \geq 1\}} \cup \{\eta\}$ where

- a basis of neighborhoods of $\frac{1}{n}$ is given by the set with one element that is $\{\frac{1}{n}, \eta\}$,
- a basis of neighborhoods of 0 is $\overline{\{\frac{1}{n} \mid n \geq N\}} \cup \{\eta\}$ when $N \geq 1$ varies,
- a basis of neighborhoods of η is $\{\frac{1}{n} \mid n \geq N\} \cup \{\eta\}$ when $N \geq 1$ varies.

Then X is spectral totally disconnected with $\pi_0(X) = \overline{\{\frac{1}{n} \mid n \geq 1\}}$ as a profinite set. But this is not w-local.

Proposition 1.9.15. — *The natural functor from the category of w-local spectral spaces to the one of spectral spaces has a right adjoint $X \mapsto X^{wl}$. Moreover, $X^{wl} \to X$ is $\mathscr{B}$-pro-étale.*

Proof. — The construction of the adjoint is easy. In fact, seeing the category of spectral spaces as the pro-category of finite ($T0$) spaces, it suffices to define X^{wl} for X finite (by right adjunction, the w-localization functor has to commute with projective limits). But then one verifies that

$$X^{wl} = \coprod_{x \in X} X_x$$

works for X a finite ($T0$) space. The key point here is that if Y is w-local and $f : Y \to X$ is qc with X finite ($T0$) then $f_{|Y_c} : Y_c \to X$ is a qc map from a profinite space to a finite ($T0$) space, thus a locally constant map. This implies f factorizes through $\coprod_{x \in X} X_x$.

The fact that $X^{wl} \to X$ is pro-étale is more complicated. We refer to [47, Lemma 7.13] for this. $\square$

Finally let us note that via the descriptions of the functors $X \mapsto X_{cons}$ and $X \mapsto X^{wl}$ on Pro(finite ($T0$) spaces), we have

$$\pi_0(X^{wl}) = X_{cons}.$$

We thus have two qc maps

$$\begin{tikzcd} & \pi_0(X^{wl}) = X_{cons} \\ X^{wl} \arrow[ur, "\pi_0"] \arrow[dr, "\text{adjunction map}"'] & \\ & X. \end{tikzcd}$$

Next proposition is left to the reader.

Proposition 1.9.16. — *The map $X^{wl} \to X_{cons} \times X$ identifies X^{wl} with $\{(x, y) \in X_{cons} \times X \mid x \leq y\}$ equipped with the induced product topology.*

Thus, X^{wl} is the set of generalizations

$$\Delta^{gen} = \bigcap_{\substack{U \supset \Delta \\ \text{neighborhood}}} U$$

of the diagonal $\Delta = \{(x, x) \mid x \in X\} \subset X_{cons} \times X$.

1.9.4. Another construction. — The morphism $X^{wl} \to X$ is not as nice as we would want for applications to perfectoid spaces: this is not open in general. For applications to perfectoid spaces we need another construction that provides an open pro-étale morphism $X' \to X$ with X' totally disconnected.

Example 1.9.17. — The map $\mathrm{Spec}(\mathbb{Z}_p)^{wl} \to \mathrm{Spec}(\mathbb{Z})$ is not open. In fact, let $\pi : \mathrm{Spec}(\mathbb{Z})^{wl} \to \pi_0(\mathrm{Spec}(\mathbb{Z}))_{cons} =$ Alexandrov compactification of $\mathrm{Spm}(\mathbb{Z})_{disc}$ where the point at infinity is η (see Example 1.3.6). Then for a prime number p, $\{p\}$ is open in $\mathrm{Spm}(\mathbb{Z})_{disc}$ and $\pi^{-1}(p)$ is the open subset $\{(p \leq p), (p \leq \eta)\}$. Its image in $\mathrm{Spec}(\mathbb{Z})$ is $\{p, \eta\}$ that is not open.

Remark 1.9.18. — The reason why $X^{wl} \to X$ is not open in general is the following. In the limit $X^{wl} = \varprojlim_i X_i \to X$ with $X_i \to X$ $\mathscr{B}$-étale and thus open, the transition maps $X_i \to X_j$ for $i \geq j$ are not surjective and $X^{wl} \to X_i$ is not surjective in general although $X^{wl} \to X$ is surjective.

Proposition 1.9.19. — *There exists an open $\mathscr{B}$-pro-étale map $X' \to X$ with X' totally disconnected.*

Proof. — We are going to define a sequence

$$\cdots \longrightarrow X_n \xrightarrow{p_n} \cdots \xrightarrow{p_2} X_1 \xrightarrow{p_1} X_0 = X$$

where p_n is $\mathscr{B}_{n-1} := (p_1 \circ \cdots \circ p_{n-1})^{-1}\mathscr{B}$-pro-étale. Suppose X_n is defined. Let E be the ordered set of finite subsets of the finite subsets of $\mathscr{B}_n$ that form a cover of X_n. Define

$$X_{n+1} = \varprojlim_{\substack{\mathcal{V} \in E \\ \mathcal{V} = \{V_1, \ldots, V_i\}}} \coprod_{U \in V_1} U \times_{X_n} \cdots \times_{X_n} \coprod_{U \in V_i} U.$$

Now set $X' = \varprojlim_{n \geq 0} X_n$. Since all morphisms in the transition maps in our projective limits are open surjective, $X' \to X$ is open. Any finite cover of X' by quasi-compact open subsets comes by pullback from such a cover of X_n for some $n \geq 0$. Any such cover of X_n is refined by a finite cover consisting of elements of $\mathscr{B}_n$. Let $W \subset \mathscr{B}_n$ be such a finite subset that makes a cover of X_n. As soon as $W \in \mathcal{V} = \{V_1, \ldots, V_i\}$, there is a section

$$\begin{tikzcd}
 & & \coprod_{U \in V_1} U \times_{X_n} \cdots \times_{X_n} \coprod_{U \in V_i} U \arrow[dll, bend right] \arrow[dl] \\
\coprod_{U \in W} U \arrow[r] & X_n &
\end{tikzcd}$$

and there is thus a section

$$\begin{tikzcd}
 & & X_{n+1} \arrow[dll, bend right] \arrow[dl] \\
\coprod_{U \in W} U \arrow[r] & X_n. &
\end{tikzcd}$$

Our cover W thus splits after pullback to X_{n+1}. It thus splits after pullback to X'. Our original cover of X' is thus refined by a cover that splits and it thus splits. □

1.10. The Berkovich spectrum

In this section we dig further into the structure of the spectral topological spaces we will be interested in at the end: the analytic one, see Definition 1.9.4. They happen to have a nice compact (Hausdorff) quotient that corresponds to the Berkovich spectrum $X^B = \mathcal{M}(A)$ when one considers the spectral space $X = \mathrm{Spa}(A, A^+)$ with A a Tate ring. But nevertheless, this quotient exists and is well behaved for more general analytic spectral spaces than the preceding adic spaces, for example for spatial diamonds.

1.10.1. Compactness and adjunction. — Let us give the very general definition of the Berkovich quotient of a spectral space.

Definition 1.10.1. — For X spectral define X^B to be the quotient of X by the equivalence relation generated by the order relation that is the specialization. We equip it with the quotient topology.

The open subsets of X^B are thus in bijection with the the open subsets of X stable under specialization (the so-called partially proper, resp. overconvergent, resp. wide open, subsets in Huber's terminology, resp. classical rigid geometry terminology, resp. Coleman's terminology).

We now make the following assumption: X *is analytic*, see Definition 1.9.4. One has in fact the following elementary result that simplifies the description of the quotient X^B.

Lemma 1.10.2. — *If X is analytic then any $x \in X$ has a unique maximal generalization, x^{max}.*

Proof. — The uniqueness is immediate from the chain property. Now, if $\cap_{y \geq x} X_y = \emptyset$, since for all $y \geq x$, X_y is closed in the compact topological space $(X_x)_{cons}$, a finite sub-intersection is empty. This is impossible using the chain property. One deduces that $\cap_{y \geq x} X_y = \{x^{max}\}$ for x^{max} maximal. □

Now, the equivalence relation defining the quotient X^B of X is simply

$$x \sim y \iff x^{max} = y^{max}.$$

Thus, as a set, X^B is identified with the set of maximal points of X.

Lemma 1.10.3. — *Let X be an analytic spectral space and Z a pro-constructible subset. Let Z^{gen} be the set of generalizations of the elements of Z. Then, Z^{gen} is pro-constructible and the open subset*

$$X \setminus \overline{Z^{gen}}$$

is stable under specialization.

Proof. — The set Z^{gen} is the intersection of all open subsets containing Z. But now, since Z is quasi-compact,

$$Z^{gen} = \bigcap_{\substack{U \supset Z \\ \text{qc open}}} U$$

and it is thus pro-constructible. The closed set $\overline{Z^{gen}}$ is thus the set of specializations of the elements of Z^{gen} (Corollary 1.3.9). Now, if $x \in X \setminus \overline{Z^{gen}}$ and $x \geq y$. If $y \in \overline{Z^{gen}}$ there exists $w \in X$ and $z \in Z$ such that $w \geq y$ and $w \geq z$. Since X_y is a chain we have either $x \geq w$ or $w \geq x$. If $x \geq w$ then $x \geq z$ and thus $x \in Z^{gen}$ which is impossible. If $w \geq x$ then $x \in \overline{Z^{gen}}$ which is impossible. We thus have $y \notin \overline{Z^{gen}}$. □

Proposition 1.10.4. — *If X is analytic, X^B is a compact (Hausdorff) space.*

Proof. — We just have to prove that X^B is Hausdorff. Let $x \neq y$ be two maximal points of X. We have, $\overline{\{x\}} \cap \overline{\{y\}} = \emptyset$. We can thus find U and V two quasi-compact open subsets satisfying

$$\overline{\{x\}} \subset U,\ U \cap \overline{\{y\}} = \emptyset,\ \overline{\{y\}} \subset V,\ V \cap \overline{\{x\}} = \emptyset.$$

Let us now consider

$$U' = X \setminus \big(\overline{(X \setminus U)^{gen}} \cup \overline{V}\big).$$

According to Lemma 1.10.3 applied to $Z = X \setminus U$ and $Z = V$, this is an open subset of X stable under specializations and contained in U. One has $x \notin \overline{V}$ since $\overline{V}$ is the set of specializations of the elements of V, $x \notin V$ and x is maximal. The relation $x \in \overline{(X \setminus U)^{gen}}$ implies x is a specialization of an element of $(X \setminus U)^{gen}$ which implies $x \in (X \setminus U)^{gen}$ by maximality of x. But this last relation implies there exists $y \in X \setminus U$ such that $x \geq y$ which is impossible since $\overline{\{x\}} \subset U$. We thus have proven that $x \in U'$. In the same way, if

$$V' = X \setminus \left(\overline{(X \setminus V)^{gen}} \cup \overline{U}\right),$$

this is an open neighborhood of y stable under specializations. We now have $U' \cap V = \emptyset$ and $V' \subset V$. We deduce that $U' \cap V' = \emptyset$. □

If Y is an Hausdorff topological space we have the adjunction formula

$$\mathrm{Hom}(X, Y) = \mathrm{Hom}(X^B, Y)$$

and *if X is analytic X^B is then the biggest Hausdorff quotient of X.*

1.10.2. Overconvergent sheaves and their cohomology. —

1.10.2.1. *Overconvergent sheaves.* — Suppose X is analytic. Let $\beta : X \to X^B$ be the quotient map to the Berkovich spectrum.

Definition 1.10.5. — A sheaf $\mathscr{F}$ on X is overconvergent if for all $x, y \in X$ satisfying $x \geq y$, $\mathscr{F}_y \xrightarrow{\sim} \mathscr{F}_x$.

The basic properties of overconvergent sheaves are the following.

Proposition 1.10.6. — *Let $\mathscr{F}$ be a sheaf of sets on X. The following are equivalent:*

1. *$\mathscr{F}$ is overconvergent,*
2. *for any maximal point x of X, $\mathscr{F}_{|\overline{\{x\}}}$ is constant,*
3. *for any quasi-compact open subset U of X, if $i : \overline{U} \hookrightarrow X$,*
$$\Gamma(\overline{U}, i^*\mathscr{F}) \xrightarrow{\sim} \Gamma(U, \mathscr{F}),$$
4. *for any quasi-compact open subset U of X,*
$$\varinjlim_{U \subset\subset V} \Gamma(V, \mathscr{F}) \xrightarrow{\sim} \Gamma(U, \mathscr{F})$$
where $U \subset\subset V$ means $\overline{U} \subset V$.

Proof. — (2) ⇔ (1) Is deduced from the equality for any x in X, $\left(\mathscr{F}_{|\overline{\{x^{max}\}}}\right)_x = \mathscr{F}_x$ and the fact that for a sheaf $\mathscr{G}$ on $\overline{\{x^{max}\}}$, if $i : \{x^{max}\} \hookrightarrow \overline{\{x^{max}\}}$ then $\mathscr{G}$ is constant iff $\mathscr{G} \xrightarrow{\sim} i_* i^* \mathscr{G}$.
(2) ⇔ (3) Let U be open qc and set $j : U \hookrightarrow \overline{U}$. There is a morphism

$$i^*\mathscr{F} \longrightarrow j_* j^* i^* \mathscr{F} = j_* \mathscr{F}_{|U}.$$

For $x \in \overline{U}$ i.e. $x^{max} \in U$, if $\mathscr{G} = \mathscr{F}_{|X_x}$ then the induced morphism on the stalk at x is identified with

$$\Gamma(X_x, \mathscr{G}) \longrightarrow \Gamma(X_x \cap U, \mathscr{G}).$$

The equivalence between (2) and (2) is easily deduced.
(3) ⇔ (4) Since X is analytic, $\overline{U}$ is stable under generalizations. One thus has

$$\overline{U} = \varprojlim_{U \subset\subset V} V$$

in the category of spectral spaces with V qc open. From this we deduce that

$$\Gamma(\overline{U}, i^*\mathscr{F}) = \varinjlim_{U \subset\subset V} \Gamma(V, \mathscr{F}),$$

see Equation (2) in Section 1.8. The result is immediately deduced. □

The following is then easy.

Proposition 1.10.7. — *The functor β^* is fully faithful and identifies the category of sheaves on X^B with the category of overconvergent sheaves on X.*

1.10.2.2. *Cohomology.* — Next proposition says that the cohomology of overconvergent sheaves is the same as the cohomology of sheaves on the Berkovich space.

Proposition 1.10.8. — *If $\mathscr{F}$ is an overconvergent sheaf of abelian groups on the analytic spectral space X then*

$$R\Gamma(X, \mathscr{F}) \xrightarrow{\sim} R\Gamma(X^B, \beta_* \mathscr{F}).$$

Proof. — We have to prove that $R^i\beta_*\mathscr{F} = 0$ for $i > 0$. Let x be a maximal point of X. If U is an open neighborhood of $\overline{\{x\}}$ then $U \setminus \overline{(X \setminus U)^{gen}}$, where the upperscript "*gen*" means we take the set of generalizations, is an overconvergent open neighborhood of x contained in U (Lemma 1.10.3). We deduce that $\overline{\{x\}}$ has a basis of open overconvergent neighborhoods. We thus have

$$(R^i\beta_*\mathscr{F})_{\beta(x)} = H^i(\overline{\{x\}}, i^*\mathscr{F})$$

where $i : \overline{\{x\}} \hookrightarrow X$. Since $\mathscr{F}$ is overconvergent the sheaf $i^*\mathscr{F}$ is constant. Now, if M is any abelian group the sheaf $\underline{M}$ on $\overline{\{x\}}$ is flabby in the topological sense. More precisely, for any non-empty open subset V of $\overline{\{x\}}$, $\Gamma(\overline{\{x\}}, \underline{M}) \xrightarrow{\sim} \Gamma(V, \underline{M}) = M$. It is thus acyclic and we deduce the result. □

In practice, when one considers adic spaces and their cohomology, a lot of sheaves we consider are overconvergent. Last proposition says that in this case, à priori, we do not care about the adic space and we should only focus on the Berkovich spaces. This is in fact false for two reasons:

- The category of sheaves of abelian groups on a quasi-compact quasi-separated analytic adic spaces is compactly generated with some explicit compact generators, see Proposition 1.8.2. This is à priori not the case for a compact topological space like X^B.
- Compactly supported cohomology, already for an affinoid space, needs the introduction of Huber's canonical compactifications, see Section 2.12, where non-overconvergent sheaves show up (the extension by zero $j_!\mathscr{F}$, where j is the inclusion inside the canonical compactification, is not overconvergent).

1.11. En résumé

Let X be a spectral space. It gives rise to a diagram

$$
\begin{tikzcd}
\overbrace{X^{wl}}^{\text{totally disconnected}} \arrow[drr, "\text{pro-étale}", "\text{surjective}"'] & & \overbrace{\pi_0(X^{wl}) = X_{cons}}^{\text{profinite}} \arrow[ll, hook', "\text{closed points}"'] \arrow[d, "\text{bijective}"] & & \\
\underbrace{X'}_{\text{totally disconnected}} \arrow[rr, "\text{pro-étale, open}"', "\text{quotient map}"'] & & X \arrow[rr, "\text{quotient map}"'] \arrow[d, "\text{quotient map}"] & & \underbrace{X^B}_{\text{compact if } X \text{ analytic}} \\
& & \underbrace{\pi_0(X)}_{\text{profinite}} & &
\end{tikzcd}
$$

CHAPTER 2

ADIC SPACES

2.1. Huber rings

Huber introduced the most general "reasonable" definition for a topological ring to be a "ring of non-Archimedean holomorphic functions". This contains the "classical case" of Banach algebras over a non-archimedean field and the one of adic rings for a finitely generated ideal that are the topological rings used by Grothendieck in his theory of formal schemes. Given the recent developments of the theory it is essential to work in a context where our topological rings may look like Banach algebras but may not contain a field. Typically, in the work [28] the authors consider an object $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ that interpolates between $\mathbb{F}_p$-perfectoid spaces and $\mathbb{Q}_p$-perfectoid spaces. This allows them to reduce some statements for the so-called B_{dR}-affine Grassmanian over $\mathbb{Q}_p$ to a more "classical one" over $\mathbb{F}_p$ using a degeneracy from $p \neq 0$ to $p = 0$. In this degeneracy process some connected perfectoid $\mathbb{Z}_p$-algebras that do not contain a field appear.

2.1.1. Generalities ([36, Section 1],[37]). — Recall the following terminology for a non-archimedean topological ring A (here by non-archimedean we mean that $\mathbb{Z}.1_A$ is bounded in A):

1. A subset S of A is called bounded if for any neighborhood U of 0 in A there exists a neighborhood V of 0 in A such that $V.S \subset U$.

2. We note A° the subring of A of power bounded elements in A with its ideal $A^{\circ\circ}$ of topologically nilpotent elements. Let us note that A° is integrally closed in A and $\sqrt{A^{\circ\circ}} = A^{\circ\circ}$. We note $\widetilde{A} = A^\circ / A^{\circ\circ}$, a reduced ring.

Huber's definition is the following.

Definition 2.1.1. — A Huber ring or an f-adic ring, is a topological ring A that admits an open bounded subring A_0 whose topology is the I-adic topology for an ideal I of finite type. Morphisms between Huber rings are continuous morphisms.

Thus, to define a Huber ring we need:

- a ring A,
- a subring $A_0 \subset A$
- a finite type ideal $I = (f_1, \dots, f_n) \subset A_0$,
- *coherence condition*: such that for all $a \in A$ and $1 \leq i \leq n$, $f_i^k a \in A_0$ for $k \gg 0$.

Remark 2.1.2. — The coherence condition is equivalent to the existence of a factorization (that is unique if it exists)

$$\begin{tikzcd} & \mathrm{Spec}(A) \arrow[d, "\text{schematically dominant}"] \\ \mathrm{Spec}(A_0) \setminus V(I) \arrow[ur, "\exists"] \arrow[r, hook, "\text{open}"'] & \mathrm{Spec}(A_0) \end{tikzcd}$$

which gives some kind of feeling that $\mathrm{Spec}(A)$ sits in between $\mathrm{Spec}(A_0)$ and "its generic fiber" $\mathrm{Spec}(A_0) \setminus V(I)$. See Remark 2.6.4 for a follow up on this.

Remark 2.1.3. — Since we will use it all the time, one of the many reasons why we use adic topologies associated to *finitely generated ideals* is the following. Let R be a ring and I an ideal of R. Note $\widehat{R} = \varprojlim_{k \geq 0} R/I^k$ its I-adic completion where the topology of $\widehat{R}$ is the projective limit topology with R/I^k discrete for $k \geq 0$. If I is finitely generated then the topology of $\widehat{R}$ is the $\widehat{R}.I$-adic one. This is false in general when I is not finitely generated. This is for example false for the polynomial ring $A[X_j]_{j\in J}$, where A is any ring, J is infinite and the ideal I is $(X_j)_{j\in J}$.

The following lemma tells us that this is a good well behaved definition.

Lemma 2.1.4. — *For a Huber ring A the topology of any open bounded subring of A is the I-adic one for a finitely generated ideal I.*

Proof. — Let A_0 be a bounded open subring whose topology is the I-adic one for a finitely generated ideal I of A_0. Let A_1 be an open bounded subring. Since A_1 is open we can find $n \geq 1$ such that $I^n \subset A_1$. Since A_1 is bounded we can find $m \geq n$ such that $A_1 I^m \subset I$. Let $J = A_1 I^m \subset A_1$, an ideal of finite type in A_1. One then has for $k \geq 1$,

$$I^{km} \subset J^k \subset I^k.$$

From this we deduce that the topology on A_1 is the J-adic topology. □

The preceding lemma leads to the following definition.

Definition 2.1.5. — A ring of definition of a Huber ring is an open bounded subring. An ideal of definition of a ring of definition is a finitely generated ideal whose associated adic topology is the topology of the ring of definition.

Since we will use it all the time let us note the following.

Remark 2.1.6. —
- Any ring of definition of A is contained in A^0.
- A morphism $f : A \to B$ between two Huber rings is continuous iff one can find two rings of definition $A_0 \subset A$ and $B_0 \subset B$ such that $f(A_0) \subset B_0$ and $f_{|A_0} : A_0 \to B_0$ is continuous. The later is equivalent to saying that if A_0 has the I-adic topology and B_0 the J-adic one then $f(I^n) \subset J$ for $n \gg 0$.

Example 2.1.7. —
1. Any discrete ring is a Huber ring.
2. If A is a ring equipped with the I-adic topology for a finitely generated ideal I then A is a Huber ring and A itself is a ring of definition.
3. Let A be any ring and $f \in A$. Equip $A/(f^\infty\text{-torsion})$ with the f-adic topology. Then, any subring of $A[\frac{1}{f}]$ containing the image of A is a Huber ring with ring of definition the image of A.
4. If K is a complete non-Archimedean field and A is an affinoid algebra in the sense of Tate, that is to say topologically finitely generated, then A is a Huber ring. If $K\langle X_1, \dots, X_n\rangle \to A$ is a surjection then the image of $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$ is a ring of definition.

5. If A is a Huber ring and $\mathfrak{a}$ is an ideal of A then $A/\mathfrak{a}$ with the quotient topology is a Huber ring. If A_0 is a ring of definition of A then $A_0/A_0 \cap \mathfrak{a}$ is a ring of definition of $A/\mathfrak{a}$.
6. If A is a Huber ring and $n \geq 1$, the topological ring $A[X_1, \dots, X_n]$, where a basis of neighborhoods of 0 is given by $\{U[X_1, \dots, X_n]\}_U$ with U going through the set of neighborhood of 0 in A, is a Huber ring. If A_0 is a ring of definition of A then $A_0[X_1, \dots, X_n]$ is a ring of definition of $A[X_1, \dots, X_n]$. This Huber rings represents the functor on A-Huber rings
$$B \mapsto (B^\circ)^n.$$
7. More generally, if A is a Huber ring, I is a set and $\mathfrak{a}$ is an ideal of $A[X_i]_{i\in I}$, the Huber ring $A[X_i]_{i\in I}/\mathfrak{a}$ represents the functor on A-Huber rings
$$B \mapsto \big\{(b_i)_{i\in I} \mid b_i \in B^\circ,\ \forall f \in \mathfrak{a},\ f(b_i)_{i\in I} = 0\big\}.$$

We will distinguish two classes of Huber rings that are the one we will work with later.

Definition 2.1.8. — A Huber ring A is called

1. *Tate* if there exists a topologically nilpotent unit in A, an element of $\varpi \in A^{\circ\circ} \cap A^\times$. Those topologically nilpotent units are called *pseudo-uniformizers.*
2. *Uniform* if A^0 is bounded (and is thus a ring of definition since $A^{\circ\circ} \subset A^\circ$ is open).

Let us note the following that clarifies the structure of Tate rings.

Lemma 2.1.9. — *If A is Tate, A_0 is a ring of definition and ϖ a pseudo-uniformizer then*
$$A = A_0[\tfrac{1}{\varpi}]$$
and one can choose ϖ such that $\varpi \in A_0$, in which case the topology on A_0 is the ϖ-adic topology.

Proof. — Let I be a finite type ideal of A_0 such that the topology of A_0 is the I-adic one. Since ϖ is topologically nilpotent and I is open, up to replacing ϖ by a positive power we can assume $\varpi \in I$. For any $f \in I$, since f is topologically nilpotent, $f^n\varpi^{-1} \in A_0$ for $n \gg 0$. Since I is finitely generated we deduce that for $n \gg 0$, $I^n \subset A_0\varpi$. For such an n, we thus have
$$I^n \subset A_0\varpi \subset I.$$
The topology of A_0 is thus the ϖ-adic one. The equality $A = A_0[\frac{1}{\varpi}]$ is immediately deduced from the topological nilpotentcy of ϖ. □

Tate rings are a generalization of Banach algebras (although they may not contain a field). In fact, if A is a Tate ring, fix A_0, $\varpi \in A_0$ as before and $\beta \in]0,1[$. Define
$$\|.\| : A \longrightarrow \mathbb{R}_+$$
by the formula
$$\|a\| = \inf\{\beta^n \mid n \in \mathbb{Z} \text{ and } a \in \varpi^n A_0\}.$$
It is easily verified that
$$\varpi^{-1} \notin A_0 \Leftrightarrow A/\overline{\{0\}} \neq 0 \Leftrightarrow \|\varpi\| = \beta \Leftrightarrow \|\varpi\| \neq 0.$$
Then, $\|.\|$ is a non-Archimedean ring semi-norm:

- $\|x + y\| \leq \sup\{\|x\|, \|y\|\}$,
- $\|xy\| \leq \|x\|.\|y\|$,
- $\|1\| = 1$ if $A/\overline{\{0\}} \neq 0$.

This semi-norm defines the topology of A. In fact we have just proven the following characterization of Tate rings.

Proposition 2.1.10. — *A topological ring A such that $A/\overline{\{0\}} \neq 0$ is a Tate ring if and only if there exists $\varpi \in A^{\circ\circ} \cap A^\times$ and $\|.\| : A \to \mathbb{R}_+$ a ring semi-norm defining its topology satisfying*

- $\|\varpi\| \in]0,1[$,
- $\forall a \in A$ *and* $k \in \mathbb{Z}$, $\|\varpi^k a\| = \|\varpi\|^k \|a\|$.

Example 2.1.11. — 1. In point (3) of Example 2.1.7, $A[\frac{1}{f}]$ is a Tate ring. If moreover the image of A in $A[\frac{1}{f}]$ is integrally closed in $A[\frac{1}{f}]$ then

$$A[\tfrac{1}{f}]^{\circ} \subset f^{-1}\,\mathrm{Im}(A \to A[\tfrac{1}{f}])$$

and thus $A[\frac{1}{f}]$ is uniform.

2. In point (4) of Example 2.1.7, A is a Tate ring. Moreover, A is reduced if and only if it is uniform.

2.1.2. Complete Huber rings. — We will mostly be interested in the case of complete Huber rings. Here by complete we will always mean separated and complete i.e. if A_0 is a ring of definition of A whose topology is the I-adic one then

$$A \xrightarrow{\sim} \varprojlim_{n\geq 0} A/I^n.$$

For a Huber ring A, with the preceding notations, set

$$\widehat{A} := \varprojlim_{n\geq 0} A/I^n$$

equipped with the projective limit topology where $\varprojlim_{n\geq 0} A/I^n$ is discrete.

Lemma 2.1.12. — *1. The topological ring $\widehat{A}$ does not depend on the choice of a ring of definition A_0 and the ideal I; in fact*

$$\widehat{A} = \varprojlim_{U} A/U$$

where U goes through the set of neighborhoods of 0 in A that are additive subgroups.

2. $\widehat{A}$ is a complete Huber ring and the functor

$$A \mapsto \widehat{A}$$

is a left adjoint to the inclusion

$$\{\textit{complete Huber rings}\} \hookrightarrow \{\textit{Huber rings}\}.$$

3. If A_0 is a ring of definition of A, $\widehat{A_0}$ is a ring of definition of $\widehat{A}$ and there is an isomorphism

$$\widehat{A}_0 \otimes_{A_0} A \xrightarrow{\sim} \widehat{A}.$$

4. We have $\widehat{A}^{\circ} = \widehat{A^{\circ}}$ and $\widehat{A}^{\circ\circ} = \widehat{A^{\circ\circ}}$.

Proof. — Point (1) is clear. For point (2) we use the fact that if R is a ring and J a finite type ideal of R then its J-adic completion $\widehat{R}$ is a $\widehat{R}J$-adic ring and $R/J \xrightarrow{\sim} \widehat{R}/\widehat{R}J$.

Let us verify point (3). Fix A_0 a ring of definition and I an ideal of definition. There is a canonical morphism $\widehat{A}_0 \longrightarrow \widehat{A}$ that induces

$$f : \widehat{A_0} \otimes_{A_0} A \longrightarrow \widehat{A}.$$

Let $a = (a_n)_{n\geq 0} \in \widehat{A}$ with $a_n \in A$ and $a_{n+1} - a_n \in I^n$ for $n \geq 0$. Then,

$$\alpha = (a_n - a_0)_{n\geq 0} \in \widehat{A_0}$$

and

$$a = f(\alpha \otimes 1 + 1 \otimes a_0).$$

This proves the surjectivity of f.

Consider now
$$x = \sum_i a_i \otimes b_i \in \ker f.$$
We can choose $l \geq 0$ such that for all i, $b_i I^l \subset A_0$. Write $a_i = \alpha_i.1_{\widehat{A_0}} + \beta_i$ with $\alpha_i \in A_0$ and $\beta_i \in I^l \widehat{A_0}$. Since $f(x) = 0$,
$$z = \sum_i \alpha_i b_i \in A_0.$$
We then have
$$x = z.1 \otimes 1 + \sum_i \beta_i \otimes b_i.$$
Now, write $\beta_i = u_i \gamma_i$ with $u_i \in I^l$ and $\gamma_i \in \widehat{A_0}$. We have
$$\sum_i \beta_i \otimes b_i = \sum_i (u_i\gamma_i) \otimes b_i = \sum_i \gamma_i \otimes u_i b_i = \sum_i (u_i b_i).\gamma_i \otimes 1.$$
At the end we obtain
$$x = (\sum_i \alpha_i b_i) \otimes 1 + \sum_i (u_i b_i).\gamma_i \otimes 1.$$
But
$$\sum_i (\alpha_i b_i + (u_i b_i).\gamma_i) \in \widehat{A_0}$$
is sent to 0 via the injection $\widehat{A_0} \hookrightarrow \widehat{A}$ and we conclude.

Point (4) is left to the reader. □

In the preceding lemma, point (3) tells us that the computation of the completion of a Huber ring is reduced to the computation of a "usual" adic completion with respect to an ideal and a scalar extension.

Example 2.1.13. — With the notations of example 2.1.7:

1. The completion of $A[X_1, \dots, X_n]$ is $\widehat{A}\langle X_1, \dots, X_n\rangle$, the subring of $\widehat{A}[\![X_1, \dots, X_n]\!]$ of power series $\sum_{\alpha=(\alpha_1,\dots,\alpha_n)\in\mathbb{N}^n} a_\alpha X_1^{\alpha_1} \dots X_n^{\alpha_n}$ such that $a_\alpha \underset{|\alpha|\to+\infty}{\longrightarrow} 0$
2. The completion of $A/\mathfrak{a}$ is $\widehat{A}/\overline{\mathfrak{a}}$ where $\overline{\mathfrak{a}}$ is the closure of the image of $\mathfrak{a}$ in $\widehat{A}$.

2.1.3. Uniform Tate rings and the spectral norm after Berkovich. — Let A be a Tate ring and fix a pseudo-uniformizer ϖ in A. We will always suppose that $A/\overline{\{0\}} \neq 0$. Next proposition is an upgrade of Proposition 2.1.10.

Proposition 2.1.14. — *The Tate ring A is uniform if and only its topology is defined by a non-Archimedean ring semi-norm $\|.\| : A \to \mathbb{R}_+$ satisfying $\|a^n\| = \|a\|^n$ for $a \in A$ and $n \in \mathbb{N}$. If this is the case then $A^\circ = \{a \in A \mid \|a\| \leq 1\}$ and $A^{\circ\circ} = \{a \in A \mid \|a\| < 1\}$.*

Proof. — Only one direction is non-evident. Suppose A is Tate uniform. Fix a pseudo-uniformizer ϖ and $\beta \in]0,1[$. For $a \in A$ set
$$\|a\| = \inf\{\beta^{-N} \mid \varpi^N a \in A^\circ\}.$$
This is a non-Archimedean ring semi-norm defining the topology of A and satisfying $\|\varpi^k a\| = \beta^k \|a\|$ for all $k \in \mathbb{Z}$ and $a \in A$. Now set
$$\|a\|_\infty = \lim_{n\to+\infty} \|a^n\|^{1/n}$$
(the limit exists since the sequence $(\|a^n\|)_{n\geq1}$ is submultiplicative). For any $a \in A$ one has
$$\|a\|_\infty \leq \|a\|.$$

Now for a given $a \in A \setminus \overline{\{0\}}$, choose N such that $\varpi^N a \in A^\circ$ and $\varpi^{N-1} a \notin A^\circ$. Since $\varpi^{N-1} a \notin A^\circ$, there exists an infinite number of integers n such that $\|(\varpi^{N-1}a)^n\| \geq 1$ that is to say $\|a^n\| \geq \beta^{n(1-N)}$. From this we deduce $\|a\|_\infty \geq \beta^{1-N}$. We have $\|a\| = \beta^{-N}$. Thus, $\beta.\|a\| \leq \|a\|_\infty \leq \|a\|$ and the power multiplicative norm $\|.\|_\infty$ defines the topology of A. □

In fact we have a uniqueness statement once we fix some simple constraints.

Proposition 2.1.15. — *Let A be a uniform Tate ring. Fix ϖ a pseudo-uniformizer and $\beta \in]0, 1[$. Then there exists a unique power multiplicative ring semi-norm defining the topology of A satisfying $\|\varpi\| = \beta$, $\|\varpi^{-1}\| = \beta^{-1}$.*

Proof. — We already saw the existence in the proof of Proposition 2.1.14. Let $\|.\| : A \to \mathbb{R}_+$ be a semi-norm satisfying the assumptions of the statement. From the inequalities $\|\varpi a\| \leq \|\varpi\|.\|a\|$ and $\|a\| \leq \|\varpi^{-1}\|.\|a\|$ for $a \in A$, we deduce that for any $a \in A$ one has $\|\varpi a\| = \|\varpi\|\|a\|$ and thus for any $k \in \mathbb{Z}$ and $a \in A$,

$$\|\varpi^k a\| = \beta^k \|a\|.$$

Let $\|.\|'$ be another semi-norm satisfying the assumptions of the statement. For $a \in A \setminus \overline{\{0\}}$, if $a \in \varpi^k A^\circ \setminus \varpi^{k+1} A^\circ$ then

$$\beta^{k+1} \leq \|a\| \leq \beta^k \text{ and } \beta^{k+1} \leq \|a\|' \leq \beta^k.$$

From this we deduce that for any $a \in A$ one has

$$\beta\|a\|' \leq \|a\| \leq \beta^{-1}\|a\|'.$$

Replacing a by a^n for all n and taking $1/n$-nth roots we obtain

$$\beta^{1/n}\|a\|' \leq \|a\| \leq \beta^{-1/n}\|a\|'.$$

Taking the limit when $n \to +\infty$ we obtain $\|a\| = \|a\|'$. □

This unique semi-norm is unique for a good reason: it has a description as a *spectral norm* in the following sense. Let

$$\mathcal{M}(A) = \{|.| : A \to \mathbb{R}_+ \text{ continuous multiplicative semi-norms satisfying } |\varpi| = \beta\}$$

as in [6, Chapter 1]. With the notations of sections 2.3 and 1.10 one has $\mathcal{M}(A) = |\operatorname{Spa}(A, A^\circ)|^B$ and this is a compact topological space, see Sections 1.10 and 2.8 for more details. Any $a \in A$ defines a continuous function

$$\begin{array}{ccc} \mathcal{M}(A) & \longrightarrow & \mathbb{R}_+ \\ x & \longmapsto & |a(x)|. \end{array}$$

***Theorem 2.1.16* (Berkovich [6, Theorem 1.3.1]).** — *Let A be a uniform Tate ring and ϖ a pseudo-uniformizer. For the unique power multiplicative ring semi-norm $\|.\|$ defining the topology of A and satisfying $\|\varpi\| = \beta$ and $\|\varpi^{-1}\| = \beta^{-1}$, we have*

$$\forall a \in A, \quad \|a\| = \sup_{x \in \mathcal{M}(A)} |a(x)|.$$

Proof. — If $x \in \mathcal{M}(A)$, the continuity hypothesis together with the formulas $|(\varpi^k a)(x)| = \beta^k |a(x)|$ and $\|\varpi^k a\| = \beta^k \|a\|$ for $k \in \mathbb{Z}$ implies that there exists $C > 0$ such that

$$|a(x)| \leq C\|a(x)\|.$$

Using the power-multiplicativity of $\|.\|$ we deduce that we can take $C = 1$. We thus have the inequality $\sup_{x \in \mathcal{M}(A)} |a(x)| \leq \|a\|$.

Let $a \in A$ and chose

$$\rho > \sup_{x \in \mathcal{M}(A)} |a(x)|.$$

We can suppose that A is complete. Let

$$A\langle \rho T\rangle = \Big\{ \sum_{n\geq 0} a_n T^n \mid a_n \in A,\ \lim_{n\to+\infty} \|a_n\|\rho^{-n} = 0\Big\}.$$

For $f = \sum_{n\geq 0} a_n T^n \in A\langle \rho T\rangle$ we put

$$\|f\| = \sup_{n\geq 0} \|a_n\|\rho^{-n}.$$

This is a non-archimedean ring norm. Equipped with this, $A\langle \rho T\rangle$ is a complete Tate ring. It is moreover power-multiplicative. In fact, if

$$f = \sum_{n\geq 0} a_n T^n \in A\langle \rho T\rangle,$$

let

$$n_0 = \inf\{n \geq 0 \mid \|a_n\|\rho^{-n} = \|f\|\}.$$

One can write

$$f = a_{n_0}T^{n_0} + g + T^{n_0+1}h$$

where $\|g\| < \|f\|$ and $\|T^{n_0+1}h\| \leq \|f\|$. Then for $k \geq 1$ one has

$$f^k = a_{n_0}^k T^{kn_0} + \underbrace{u}_{\|-\|<\|f\|^k} + T^{kn_0+1} \underbrace{v}_{\|-\|\leq \|f\|^k \rho^{kn_0+1}}.$$

One deduces that $\|f^k\| = \|f\|^k$.

Let now $x \in \mathcal{M}(A\langle \rho T\rangle)$. One has

$$|aT(x)| = |a(x)|.|T(x)| \underbrace{\leq}_{\substack{\text{since } \|.\| \\ \text{power-mult. on } A\langle \rho T\rangle}} |a(x)|\rho^{-1} < 1.$$

We thus have

$$|(1-aT)(x)| \neq 0.$$

From this we deduce that

$$1 - aT \in A\langle \rho T\rangle^\times,$$

see Proposition 2.3.9 and Remark 2.3.10. Since $A\langle \rho T\rangle \subset A[\![T]\!]$,

$$(1-aT)^{-1} = \sum_{n\geq 0} a^n T^n$$

and we thus have

$$\lim_{n\to+\infty} \|a^n\|\rho^{-n} = 0.$$

In particular, for $n \gg 0$, $\|a^n\|\rho^{-n} \leq 1$ and thus

$$\|a\| \leq \rho.$$

This being true for all $\rho > \sup_{x\in\mathcal{M}(A)} |a(x)|$, we deduce that

$$\|a\| \leq \sup_{x\in\mathcal{M}(A)} |a(x)|. \qquad \square$$

Remark 2.1.17. — The proof is similar to the one that says that for $(A, \|.\|)$ a Banach $\mathbb{C}$-algebra, the spectral radius $\rho(a)$ is given by $\lim_{n\to+\infty} \|a^n\|^{1/n}$. In fact, if $\sigma(a)$ is the spectrum of a then the resolvent $\mathbb{C} \setminus \sigma(a) \ni z \mapsto (\mathrm{Id} - za)^{-1}$ is holomorphic on $\{z \in \mathbb{C} \mid |z| < \rho(a)^{-1}\}$ and its power series expansion as an holomorphic function of z gives the result.

2.2. Affinoid rings

2.2.1. Generalities. — The following definition may seem strange at first sight: where does this A^+ comes from? Why is it there? In classical rigid geometry we only consider one ring, Banach algebra like A but there is no A^+. The fact is that we implicitly choose $A^+ = A^\circ$ in classical Tate rigid geometry. There are in fact deeper reasons, even when one looks at $\mathrm{Spa}(A, A^\circ)$:

- The local structure around a point of an analytic adic space, even one like $\mathrm{Spa}(A, A^\circ)$, is given by spectra of affinoid fields $\mathrm{Spa}(K, K^+)$ where K^+ may be different from K°, see Proposition 2.6.5 for example.
- Huber's canonical compactifications, see Section 2.12, are an essential tool in étale cohomology of adic spaces.

Definition 2.2.1. — An *affinoid ring or a Huber pair* is a couple (A, A^+) where A is a Huber ring and $A^+ \subset A^\circ$ is an open subring integrally closed in A. Moreover,

1. Such a subring A^+ of A is called a ring of integral element.
2. The affinoid ring (A, A^+) is called Tate if A is a Tate ring.
3. It is called complete if A is complete.
4. A morphism $(A, A^+) \to (B, B^+)$ of affinoid rings is a continuous morphism $f : A \to B$ satisfying $f(A^+) \subset B^+$.

Remark 2.2.2. —

- Since A° is integrally closed in A it is equivalent to ask that A^+ is integrally closed in A°.
- Since A^+ is open integrally closed, $A^{\circ\circ} \subset A^+$ and thus at the end

$$A^{\circ\circ} \subset A^+ \subset A^\circ.$$

- Once A is fixed the choice of A^+ a ring of integral elements is then equivalent to the choice of an integrally closed subring of the reduced ring $\widetilde{A} := A^\circ/A^{\circ\circ}$.

Let us now remark the following easy lemma.

Lemma 2.2.3. — *Let (A, A^+) be an affinoid ring. The topological closure of the image of A^+ in $\widehat{A}$ is integrally closed in $\widehat{A}$.*

This leads to the following evident definition.

Definition 2.2.4. — For (A, A^+) an affinoid ring we define its completion $(\widehat{A}, \widehat{A}^+)$ where $\widehat{A}^+ = \widehat{A^+}$ is the closure of the image of A^+ in $\widehat{A}$.

Let us finally remark that if A is a Huber ring then the choice of a ring of integral element in A is equivalent to the choice of a ring of integral elements in $\widehat{A}$ via the equality

$$A^\circ/A^{\circ\circ} = \widehat{A}^\circ/\widehat{A}^{\circ\circ}.$$

2.3. The adic spectrum

2.3.1. Continuous valuations. —

2.3.1.1. *Valuations* ([16, Chapitre 6]). — Recall some basic definitions and facts about valuations. Let R be a ring and Γ be an ordered abelian group. Here we mean that the order on Γ is total (and thus Γ is torsion free) and compatible with the group law:

$$x \leq y \Rightarrow x + z \leq y + z.$$

We consider valuations $v : R \to \Gamma \cup \{\infty\}$ i.e. applications that satisfy

- $v(xy) = v(x) + v(y)$,
- $v(1) = 0$, $v(0) = +\infty$,
- $v(x + y) \geq \inf\{v(x), v(y)\}$.

We can always suppose that $v(R) \setminus \{\infty\}$ generates Γ. The reciprocal image of $+\infty$ is a prime ideal of R, the support of v. To give oneself a valuation with support $\mathfrak{p}$ is the same as to give oneself a valuation on $\mathrm{Frac}(R/\mathfrak{p})$. This is the same as the datum of a valuation subring $V \subset K = \mathrm{Frac(R}/\mathfrak{p})$.

Recall the following equivalent characterizations of such a $V \subset K$:

- V is such that $\forall x \in K^\times, x \in V$ or $x^{-1} \in V$.
- V is maximal for the following order relation on local rings inside K whose field of fractions is K: $(R, \mathfrak{m}) \leq (R', \mathfrak{m}')$ is $R \subset R'$ and $\mathfrak{m}' \cap R = \mathfrak{m}$.
- For $x, y \in V$ there exists $\lambda \in V$ such that $x = \lambda y$ or $y = \lambda x$.
- V is a Bezout local ring.

For such a V the ordered group Γ is identified with the group of fractionnal ideals of V with the order relation given by the inclusion.

The prime ideals of V form a chain. This is in bijection with the set of *convex subgroups of* Γ (that form a chain too), to $\mathfrak{p} \subset V$ one associates the convex subgroup

$$H = v(V \setminus \mathfrak{p}) \cup -v(V \setminus \mathfrak{p}).$$

The localization $V_\mathfrak{p}$ is a valuation ring with associated ordered group Γ/H. If v is the valuation defined by V we note v/H the corresponding valuation.

By definition, the rank of v is the height of V. We call this the rank of Γ too: the length of the chain of convex subgroups. One has the inequality

$$\mathrm{rk}(\Gamma) \leq \dim_\mathbb{Q} \Gamma \otimes_\mathbb{Z} \mathbb{Q}.$$

Moreover, let us remark that for an ideal I of V, *I is a prime ideal if and only if $\sqrt{I} = I$.* Ideals of V correspond to subsets S of Γ_+ satisfying $x \in S \Rightarrow \forall y,\ y \leq x, y \in S$. If S corresponds to I, the convex subgroup H associated to $\sqrt{I}$ is such that $H_+ = \{\gamma \in \Gamma \mid \forall n \geq 1,\ n\gamma \notin S\}$.

Recall finally that Γ has rank 1 i.e. the only convex subgroups of Γ are (0) and Γ if and only if Γ is isomorphic to a subgroup of $(\mathbb{R}, +)$ as an ordered group. Thus, up to equivalence, rank 1 valuations are given by valuations $v : R \to \mathbb{R} \cup \{\infty\}$. Phrased in another way, those are the same as absolute values i.e. multiplicative semi-norms $|.| : R \longrightarrow \mathbb{R}_+$.

Remark 2.3.1. — More generally, the rank of Γ is $\leq n$ if and only if there is an embedding of Γ inside $\mathbb{R}^n$ equipped with the lexicographic order, see [18].

Example 2.3.2. — 1. Let v be a valuation on the ring R. Equip $\Gamma_v \times \mathbb{Z}$ with the lexicographic order. Then the formulas

$$\begin{array}{ccc} R[T] & \longrightarrow & \Gamma_v \times \mathbb{Z} \\ \displaystyle\sum_{n \geq 0} a_n T^n & \longmapsto & \begin{cases} \inf_{n \in \mathbb{N}}\{(v(a_n), n)\} \\ \inf_{n \in \mathbb{N}}\{(v(a_n), -n)\} \end{cases} \end{array}$$

define two valuations w_1 and w_2 on $R[T]$ of rank $\mathrm{rk}(v)+1$. The subgroup $H = \{0\} \times \mathbb{Z}$ is convex and $w_1/H = w_2/H$ that is the so called Gauss valuation

$$\sum_{n \geq 0} a_n T^n \mapsto \inf_{n \in \mathbb{N}} \{v(a_n)\}.$$

2. Let Γ be an ordered abelian group and R a ring. The ring of Hahn series $R(\!(T^\Gamma)\!)$ is

$$R(\!(T^\Gamma)\!) = \big\{ \sum_{\gamma \in \Gamma} a_\gamma T^\gamma \mid \{\gamma \mid a_\gamma \neq 0\} \text{ is well ordered} \big\}$$

(here T^γ is a formal symbol, we simply consider collections of elements in R^Γ whose support is well ordered). Then if we put

$$v(\sum_\gamma a_\gamma T^\gamma) = \inf\{\gamma \mid a_\gamma \neq 0\}$$

for such a non-zero element of $R(\!(T^\Gamma)\!)$, this defines a valuation

$$v : R(\!(T^\Gamma)\!) \to \Gamma \cup \{+\infty\}.$$

3. Let $(R, \mathfrak{m})$ be a regular local ring of dimension n. Fix a regular sequence of parameters $\mathfrak{m} = (x_1, \dots, x_n)$. Note $k = R/\mathfrak{m}$. Equip $\mathbb{Z}^n$ with the lexicographic order. Then

$$k[X_1, \dots, X_n] \xrightarrow{\sim} \mathrm{gr}_{\mathfrak{m}}(R)$$

via $X_i \mapsto x_i \bmod \mathfrak{m}^2$. There is a valuation

$$v : k[X_1, \dots, X_n] \longrightarrow \mathbb{Z}^n \cup \{+\infty\}$$

defined by

$$v : \sum_{\alpha \in \mathbb{N}^n} a_\alpha T_1^{\alpha_1} \dots T_n^{\alpha_n} \mapsto \begin{cases} \inf\{\alpha \mid a_\alpha \neq 0\} \text{ if } \exists \alpha,\ a_\alpha \neq 0 \\ +\infty \text{ otherwise} \end{cases}.$$

For $x \in R$ non-zero let $k \in \mathbb{N}$ be the integer such that $x \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$ and note $\bar{x} \in \mathfrak{m}^k/\mathfrak{m}^{k+1}$. Then if we set $w(x) = v(\bar{x})$ this defines a valuation

$$w : R \to \mathbb{Z}^n \cup \{+\infty\}.$$

The subgroup $H = (0) \times \mathbb{Z}^{n-1}$ of $\mathbb{Z}^n$ is convex and

$$w/H : R \to \mathbb{Z}$$

is the discrete valuation associated to the regular Cartier divisor $V(x_1) \subset \mathrm{Spec}(R)$. Those are the type of valuations that show up in the theory of Okounkov bodies, see [**15**, Section 2].

2.3.1.2. *Continuous valuations.* — Let A be a Huber ring. Consider a valuation $v : A \to \Gamma \cup \{\infty\}$. We say that v is continuous if for any $\gamma \in \Gamma$, $\{a \in A \mid v(a) \geq \gamma\}$ is open in A. If A_0 is a ring of definition of A whose topology is the $(f_1, \dots, f_n)$-adic one, set

$$\gamma = \inf\{v(f_1), \dots, v(f_n)\} \in \Gamma \cup \{+\infty\}.$$

Then, v is continuous if and only if

$$n\gamma \underset{n \to +\infty}{\longrightarrow} +\infty$$

in the sens that for any $\gamma' \in \Gamma, n\gamma \geq \gamma'$ for $n \gg 0$.

2.3.2. The adic spectrum. — Let us recall the following definition.

Definition 2.3.3. — Let (A, A^+) be a Huber pair.

1. The set $\mathrm{Spa}(A, A^+)$ is the set of equivalence classes of continuous valuations x on A satisfying $|f(x)| \leq 1$ for all $f \in A^+$.
2. We equip it with the topology generated by the subsets
$$\{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| \leq |g(x)| \neq 0\}$$
with $f, g \in A$.

Those open subsets that define the topology are not the one we will use.

Definition 2.3.4. — A *rational subset* of $X = \mathrm{Spa}(A, A^+)$ is an open subset of the form
$$X\Big(\frac{f_1, \ldots, f_n}{g}\Big) := \{x \in \mathrm{Spa}(A, A^+) \mid |f_1(x)| \leq |g(x)| \neq 0, \ldots, |f_n(x)| \leq |g(x)| \neq 0\}$$
where $f_1, \ldots, f_n \in A$ generate an open ideal of A and $g \in A$.

Remark 2.3.5. — If A is Tate the only open ideal is A itself and thus
$$X\Big(\frac{f_1, \ldots, f_n}{g}\Big) := \{x \in \mathrm{Spa}(A, A^+) \mid |f_1(x)| \leq |g(x)|, \ldots, |f_n(x)| \leq |g(x)|\}.$$

Lemma 2.3.6. — *A base of the topology of* $\mathrm{Spa}(A, A^+)$ *is given by the rational open subsets.*

Proof. — Let $f, g \in A$ be such that $|f(x)| \leq |g(x)| \neq 0$. Let I be a finitely generated ideal of a ring of definition defining its topology. Since I is finitely generated, using the continuity of the valuation defined by x, there exists $n \geq 1$ such that for any $h \in I^n$, $|h(x)| \leq |g(x)|$. If $I^n = (h_1, \ldots, h_k)$ then x is contained in the rational subset $\mathrm{Spa}(A, A^+)\big(\frac{h_1, \ldots, h_k, f}{g}\big)$. □

All usual statements for rational subsets hold in this context:

- they are stable under finite intersections,
- a rational subset of a rational subset is a rational subset,
- $X\Big(\frac{f_1, \ldots, f_n}{g}\Big) = X\Big(\frac{f'_1, \ldots, f'_n}{g'}\Big)$ when the collection $(f'_1, \ldots, f'_n, g')$ is sufficiently near from $(f_1, \ldots, f_n, g)$ ([36, Lemma 3.10]).

One of the main results of Huber is the following.

Theorem 2.3.7 **([36], Theorem 2.13.1).** — *The topological space* $\mathrm{Spa}(A, A^+)$ *is spectral.*

We will give a proof of this theorem in Section 2.13 using Zariski-Riemann spaces "à la Raynaud" that is different from Huber's one and more geometric.

The proof by Huber is done in four steps:

1. First Huber proves ([36, Proposition 2.2]) that the valuation spectrum of any ring R, $\mathrm{Spv}(R) = \{\text{valuations on } R\}/\sim$ equipped with the topology generated by the subsets $\{x \mid |f(x)| \leq |g(x)| \neq 0\}$, is spectral. This is an easy verification.
2. Then Huber defines for an ideal I of a ring R a subset $\mathrm{Spv}(R, I) \subset \mathrm{Spv}(R)$ that is a retract of $\mathrm{Spv}(R)$ and deduces $\mathrm{Spv}(R, I)$ is spectral ([36, Proposition 2.6]).
3. Suppose now that A is a Huber ring. Huber proves then ([36, Theorem 3.1]) that the set of continuous valuations $\mathrm{Cont}(A)$ is a closed subset of $\mathrm{Spv}(A, AA^{\circ\circ})$.
4. The result is finally easily deduced since $\mathrm{Spa}(A, A^+) = \{x \in \mathrm{Cont}(A) \mid \forall f \in A^+,\ |f(x)| \leq 1\}$ is pro-constructible.

Example 2.3.8. — Let K be a field equipped with a non-trivial valuation v that is *microbial* in the sense that there exists γ in the valuation group such that

$$n\gamma \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Let $K^+ \subset K$ be the valuation subring. Equip K with the valuation topology. Then (K, K^+) is a uniform Tate Huber ring. If $\varpi \in K$ is such that

$$nv(\varpi) \underset{n\to+\infty}{\longrightarrow} +\infty$$

then ϖ is a pseudo-uniformizer. One has K° is a rank 1 valuation ring obtained by localizing K^+ at the prime ideal that corresponds to the convex subgroup H whose positive part is

$$\{\gamma \mid \forall n \geq 1,\ n\gamma < v(\varpi)\}.$$

Moreover $\mathrm{Spa}(K, K^+)$ is a chain (for the specialization order) identified with the open prime ideals of K^+ and homeomorphic to $\mathrm{Spec}(K^+/K^{\circ\circ})$. The maximal point of this chain is $\mathrm{Spa}(K, K^\circ)$, the rank 1 valuation. The closed point is the valuation v.

Let us point that we will give proofs of point (1) and (2) and (4) in Section 2.13 of the next proposition using Zariski-Riemann spaces that are different from Huber's one. Point (3) is an easy exercise.

Proposition 2.3.9 **([36, Proposition 3.3 (i), Proposition 3.6, Proposition 3.9], Corollary 2.13.5, 2.13.8 and 2.13.10)**

1. *We have* $A^+ = \{a \in A \mid \forall x \in \mathrm{Spa}(A, A^+),\ |a(x)| \leq 1\}$.
2. *We have* $\mathrm{Spa}(A, A^+) = \emptyset$ *iff* $\widehat{A} = 0$ *that is to say* $\overline{\{0\}} = A$.
3. *The morphism* $(A, A^+) \to (\widehat{A}, \widehat{A}^+)$ *induces an homeomorphism* $\mathrm{Spa}(\widehat{A}, \widehat{A}^+) \xrightarrow{\sim} \mathrm{Spa}(A, A^+)$.
4. *We have* $A^\times = \{a \in A \mid \forall x \in \mathrm{Spa}(A, A^+),\ |a(x)| \neq 0\}$.

Looking at point (3) of this proposition and the definition of Huber's presheaf one can ask why we did not start with a complete Huber pair? Outside of the fact that the theory works well like that, one of the reasons is that one falls sometimes on non-complete Huber pairs naturally in the theory (for example Henselian Huber pairs). For example if $x \in \mathrm{Spa}(A, A^+)$ and $K = A/\mathrm{supp(x)}$ with its valuation ring K^+ there is a morphism $(K, K^+) \to (k(x), k(x)^+)$ where $k(x)$ is the residue field of the structure presheaf at x. This morphism is an isomorphism on completions and thus $\mathrm{Spa}(k(x), k(x)^+) \xrightarrow{\sim} \mathrm{Spa}(K, K^+)$.

Remark 2.3.10. — When A is a Tate ring, points (2) and (3) of Proposition 2.3.9 can be proven as in [6, Theorem 1.2.1, Corollary 1.2.4] using the Berkovich spectrum $\mathcal{M}(A)$ that is identified as a set with the maximal points of $\mathrm{Spa}(A, A^+)$.

2.4. Affinoid fields

Before going further we need to fix some terminology.

Definition 2.4.1. — An affinoid ring (K, K^+) is an *affinoid field* if K is a field, K^+ is a valuation ring for K and

1. (*discrete case*) the topology of K is the discrete topology
2. (*analytic case*) or
   - there exists $\varpi \in K^+ \setminus \{0\}$ such that $nv(\varpi) \underset{n\to+\infty}{\longrightarrow} +\infty$ where v is the valuation defined by V and the condition means $\forall\gamma, \exists n \geq 0, nv(\varpi) \geq \gamma$,

- and the topology of K is the one defined by the valuation i.e. K^+ is a ring of definition equipped with the ϖ-adic topology.

In the analytic case (K, K^+) is a Tate ring. As explained in Example 2.3.8, if $\mathfrak{p} = \sqrt{(\varpi)} \in \mathrm{Spec}(K^+)$ then $K^+_{\mathfrak{p}} = K^\circ$ is a valuation ring of rank 1 whose valuation with values in $\mathbb{R}$ defines the topology of K.

2.5. The presheaf of holomorphic functions

Let (A, A^+) be a Huber pair. Let $X = \mathrm{Spa}(A, A^+)$ as a spectral topological space.

Definition 2.5.1. — Let A_0 be a ring of definition of A whose topology is the I-adic one. For $f_1, \ldots, f_n \in A$ generating an open ideal in A and $g \in A$ we set

- $A\Big\langle \dfrac{f_1, \ldots, f_n}{g} \Big\rangle$ is the completion of the Huber ring $A[\frac{1}{g}]$ with ring of definition $A_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$ equipped with the $IA_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$-adic topology,
- $A\Big\langle \dfrac{f_1, \ldots, f_n}{g} \Big\rangle^+$ is the integral closure of the topological closure of the image of $A^+[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$.

Let us remark that, in the preceding definition, the given ring is indeed a Huber ring . In fact, chose $k \geq 1$ such that $I^k \subset Af_1 + \cdots + Af_n$. This implies that for $r \geq 1$, $I^{kr}.g^{-r}A \subset A[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$. The coherence condition for $A[\frac{1}{g}]$, $A_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$ and $IA_0[\frac{f_1}{g}, \ldots, \frac{f_n}{g}]$ given after Definition 2.1.1 is thus deduced from the one for A, A_0 and I.

The couple

$$\boxed{\Big(A\Big\langle \frac{f_1, \ldots, f_n}{g} \Big\rangle, A\Big\langle \frac{f_1, \ldots, f_n}{g} \Big\rangle^+ \Big)} \tag{6}$$

is a complete affinoid ring. We can rewrite this ring as

$$A\Big\langle \frac{f_1, \ldots, f_n}{g} \Big\rangle = \widehat{A}\langle T_1, \ldots, T_n \rangle / \overline{(T_1 g - f_1, \ldots, T_n g - f_n)} \Big[\frac{1}{g}\Big]$$

and $A\langle \frac{f_1, \ldots, f_n}{g} \rangle^+$ is the integral closure of the image of $\widehat{A}^+\langle T_1, \ldots, T_n \rangle$.

Remark 2.5.2. — (Follow up to Remark 2.3.5) If A is Tate then

$$A\Big\langle \frac{f_1, \ldots, f_n}{g} \Big\rangle = \widehat{A}\langle T_1, \ldots, T_n \rangle / \overline{(T_1 g - f_1, \ldots, T_n g - f_n)}.$$

Remark 2.5.3. — If R is a ring and I an ideal in R we can form the blow-up

$$B_I(R) = \mathrm{Proj}(\oplus_{k \geq 0} I^k) \xrightarrow{\pi} \mathrm{Spec}(R).$$

If $I = (f_\alpha)_{\alpha \in A}$ this is a union

$$B_I(R) = \cup_\alpha U_\alpha$$

where U_α is the open subset where $\pi^{-1}(f_\alpha)$ generates the exceptional Cartier divisor. One has

$$U_\alpha = \mathrm{Spec}\left(R[\tfrac{I}{f_\alpha}]\right)$$

with $R[\frac{I}{f_\alpha}] \subset R[\frac{1}{f_\alpha}]$ is the sub-R-algebra generated by $(\frac{f_\beta}{f_\alpha})_{\beta \in A}$. The preceding Huber rings defining our rational localizations are thus obtained by completion of those type of open subsets in a blow-up. See Section 2.13 for more on this.

The following lemma is easy.

Lemma 2.5.4. — 1. *Given a complete Huber pair (B, B^+), a morphism*

$$\left(A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle, A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle^+\right) \longrightarrow (B, B^+),$$

is the same as a morphism $u : (A, A^+) \to (B, B^+)$ such that the image of

$$u^* : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$$

is contained in the rational subset

$$\mathrm{Spa}(A, A^+)\Big(\frac{f_1,\dots,f_n}{g}\Big).$$

2. *The morphism*

$$(A, A^+) \longrightarrow \left(A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle, A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle^+\right)$$

induces an homeomorphism

$$\mathrm{Spa}\left(A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle, A\left\langle\frac{f_1,\dots,f_n}{g}\right\rangle^+\right) \xrightarrow{\sim} \mathrm{Spa}(A, A^+)\Big(\frac{f_1,\dots,f_n}{g}\Big).$$

Proof. — This is an application of points (1) and (4) of Proposition 2.3.9. In fact, let $u : (A, A^+) \to (B, B^+)$ be such that the image of u^* is contained in $\mathrm{Spa}(A, A^+)\big(\frac{f_1,\dots,f_n}{g}\big)$. For all $y \in \mathrm{Spa}(B, B^+)$, $u(g)(y) = g(u^*(y)) \neq 0$ since $u^*(y) \in \mathrm{Spa}(A, A^+)\big(\frac{f_1,\dots,f_n}{g}\big)$. From this we deduce that

$$u(g) \in B^\times.$$

The same type of computation shows that for $i = 1, \dots, n$,

$$\frac{u(f_i)}{u(g)} \in B^+.$$

The morphism $u : (A, A^+) \to (B, B^+)$ thus extends to a morphism of Huber pairs

$$\left(A\Big[\frac{f_1}{g},\dots,\frac{f_n}{g}\Big], A\Big[\frac{f_1}{g},\dots,\frac{f_n}{g}\Big]^+\right) \longrightarrow (B, B^+)$$

where $A\big[\frac{f_1}{g},\dots,\frac{f_n}{g}\big]^+$ is the integral closure of $A^+\big[\frac{f_1}{g},\dots,\frac{f_n}{g}\big]$ in $A\big[\frac{f_1}{g},\dots,\frac{f_n}{g}\big] \subset A\big[\frac{1}{g}\big]$. This extends canonically to the completions and we deduce point (1) of the lemma. Point (2) is deduced by looking at morphisms toward affinoid fields (K, K^+). □

Let us note the following immediate corollary.

Corollary 2.5.5. — *Any rational subset is quasi-compact.*

From this we deduce that the complete Huber pair (6) depends only canonically on the rational open subset $\mathrm{Spa}(A, A^+)\big(\frac{f_1,\dots,f_n}{g}\big)$ and not on the choice of $f_1, \dots, f_n, g$. The following definition thus makes sense.

Definition 2.5.6. — For $X = \mathrm{Spa}(A, A^+)$ we define presheaves $\mathcal{O}_X$ and $\mathcal{O}_X^+$ on rational open subsets of X by the formulas

- $\Gamma\big(X\big(\frac{f_1,\dots,f_n}{g}\big), \mathcal{O}_X\big) = A\langle\frac{f_1,\dots,f_n}{g}\rangle$,
- $\Gamma\big(X\big(\frac{f_1,\dots,f_n}{g}\big), \mathcal{O}_X^+\big) = A\langle\frac{f_1,\dots,f_n}{g}\rangle^+$.

The question wether or not $\mathcal{O}_X$ defines a sheaf will be discussed in section 2.9. Let us remark nevertheless the following property.

Remark 2.5.7. — One has (Proposition 2.3.9)

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) \mid \forall x \in U,\ |f(x)| \leq 1\}$$

and thus *if $\mathcal{O}_X$ is a sheaf it is immediate that $\mathcal{O}_X^+$ is a sheaf.*

2.6. Analytic points

2.6.1. Analytic points and their generalizations. — We will mainly be interested in the so-called analytic adic spaces in this text.

Definition 2.6.1. — A point $x \in \mathrm{Spa}(A, A^+)$ is analytic if its support $\{f \in A \mid |f(x)| = 0\}$ in $\mathrm{Spec}(A)$ is not an open ideal. We note $\mathrm{Spa}(A, A^+)_a$ for the set of analytic points.

This forms an open subset of $\mathrm{Spa}(A, A^+)$. In fact, if A_0 is a ring of definition whose topology is the I-adic one

$$\mathrm{Spa}(A, A^+)_a = \mathrm{Spa}(A, A^+) \setminus V(AI)$$

and this is thus Zariski open. If $I = (f_1, \dots, f_n)$ one has

$$\mathrm{Spa}(A, A^+)_a = \bigcup_{i=1}^{n} \mathrm{Spa}(A, A^+)\Big(\frac{f_1, \dots, f_n}{f_i}\Big).$$

which gives a natural expression of the open subset of analytic points as a finite union of rational domains. Let us note this in a lemma.

Lemma 2.6.2. — *The open subset of analytic points* $\mathrm{Spa}(A, A^+)_a$ *is quasi-compact, a finite union of rational subsets that are spectra of Tate affinoid rings.*

Example 2.6.3. —

1. Let A be an I-adic ring with I finitely generated. Then the open subset of analytic points of $\mathrm{Spa}(A, A)$ is

$$\mathrm{Spa}(A, A)_a = \mathrm{Spa}(A, A) \setminus V(I)$$

which gives a meaning to the fact that $\mathrm{Spa}(A, A)_a$ is the "generic fiber" of the formal scheme $\mathrm{Spf}(A)$.

2. For example, if k is a field and $k[\![x_1, \dots, x_n]\!]$ is equipped with the $(x_1, \dots, x_n)$-adic topology then

$$\begin{aligned}
\mathrm{Spa}(k[\![x_1, \dots, x_n]\!], k[\![x_1, \dots, x_n]\!])_a &= \mathrm{Spa}(k[\![x_1, \dots, x_n]\!], k[\![x_1, \dots, x_n]\!]) \setminus V(x_1, \dots, x_n) \\
&= \bigcup_{i=1}^{n} \mathrm{Spa}(k[\![x_1, \dots, x_n]\!], k[\![x_1, \dots, x_n]\!])\Big(\frac{x_1, \dots, x_n}{x_i}\Big) \\
&= \bigcup_{i=1}^{n} \underbrace{\mathrm{Spa}\left(k(\!(x_i)\!)\langle \tfrac{x_1}{x_i}, \dots, \tfrac{x_n}{x_i}\rangle, k[\![x_i]\!]\langle \tfrac{x_1}{x_i}, \dots, \tfrac{x_n}{x_i}\rangle\right)}_{\mathbb{B}^{n-1}_{k(\!(x_i)\!)}}
\end{aligned}$$

a union of n closed balls over the non-Archimedean fields $k(\!(x_i)\!)$, $1 \leq i \leq n$.

3. Consider $\mathbb{Z}_p[\![x_1, \dots, x_n]\!]$ equipped with the $(p, x_1, \dots, x_n)$-adic topology and let $X = \mathrm{Spa}(\mathbb{Z}_p[\![x_1, \dots, x_n]\!], \mathbb{Z}_p[\![x_1, \dots, x_n]\!])_a$. One has

$$X = \underbrace{\mathring{\mathbb{B}}^n_{\mathbb{Q}_p}}_{\{p \neq 0\}} \cup \underbrace{\mathrm{Spa}(\mathbb{F}_p[\![x_1, \dots, x_n]\!], \mathbb{F}_p[\![x_1, \dots, x_n]\!])_a}_{\{p=0\}}$$

and one can think of the quasi-compact X as a "compactification" of the open ball $\mathring{\mathbb{B}}^n_{\mathbb{Q}_p}$ in characteristic zero by adding n closed ball of characteristic p over $\mathbb{F}_p(\!(x_1)\!), \dots, \mathbb{F}_p(\!(x_n)\!)$,

$$\partial \mathring{\mathbb{B}}^n_{\mathbb{Q}_p} = \bigcup_{i=1}^{n} \mathbb{B}^{n-1}_{\mathbb{F}_p(\!(x_i)\!)}.$$

***Remark 2.6.4* (Follow up to Remark 2.1.2).** — One has $\mathrm{Spa}(A, A^+)_a = \mathrm{Spa}(A^+, A^+)_a = \mathrm{Spa}(A^+, A^+) \setminus V(A^+I)$ where I is an ideal of definition of a ring of definition contained in A^+.

Let x be an analytic point of $\mathrm{Spa}(A, A^+)$. We note Γ the group of the valuation, $K = \mathrm{frac(A/supp(x))}$ and $K^+ \subset K$ the valuation ring. Let A_0 be a ring of definition of A equipped with the I-adic topology, $I = (f_1, \dots, f_n)$. The image of I in K^+ generates a non-zero (since x is analytic) ideal (ϖ), $\varpi \in K^+$. Since

$$nv(\varpi(x)) \underset{n\to+\infty}{\longrightarrow} +\infty,$$

$K = K^+[\frac{1}{\varpi}]$. We equip K^+ with the ϖ-adic topology and $K = \varinjlim_{n\geq 0} \varpi^{-n}K^+$ with the inductive limit topology. Then, (K, K^+) is a uniform Tate Huber pair and there is a morphism

$$(A, A^+) \to (K, K^+).$$

The valuation ring K^+ has a smallest open prime ideal $\mathfrak{q}$ that corresponds to the convex subgroup whose intersection with $\Gamma_{\geq 0}$ is

$$\{\gamma \in \Gamma_{\geq 0} \mid \forall n \geq 1,\ n\gamma < v(\varpi)\}.$$

One then has

$$K^\circ = \{x \in K \mid \exists N \geq 1, \forall n \geq 1,\ v(x^n) \geq v(\varpi^{-N})\}.$$

But now, for an element $\gamma \in \Gamma$, the inequality $n(-\gamma) \leq Nv(\varpi)$ for all $n \geq 1$ is equivalent to $n(-\gamma) < v(\varpi)$ for all $n \geq 1$. From this we deduce that

$$K^\circ = K^+_{\mathfrak{q}}$$

is a rank 1 valuation ring. From this considerations we deduce the following result. We note $\kappa(x) := K$.

Proposition 2.6.5. — *Let x be an analytic point of* $\mathrm{Spa}(A, A^+)$*. The set of generalizations of x is a chain that is identified with* $\mathrm{Spa}(\kappa(x), \kappa(x)^+)$ *where* $\mathrm{Spa}(\kappa(x), \kappa(x)^\circ)$ *is the maximal generalization of x, $\kappa(x)^\circ = \mathcal{O}_{\kappa(x)}$ is a rank 1 valuation ring.*

Let us remark the following characterization of analytic points of $\mathrm{Spa}(A, A^+)$.

Lemma 2.6.6. — *A point $x \in$* $\mathrm{Spa}(A, A^+)$ *is analytic if and only if it has an open rational neighborhood that is the spectrum of a Huber Tate ring.*

Proof. — Let A_0 be a ring of definition of A and I an ideal of A_0 defining its topology. If x is analytic then $\exists f \in I$ such that $|f(x)| \neq 0$. Since I is of finite type, by continuity of the valuation associated to x, there exists $n \geq 1$ such that $I^n = (g_1, \dots, g_k)$ and $x \in \mathrm{Spa}(A, A^+)\big(\frac{g_1,\dots,g_k}{f}\big) = \mathrm{Spa}\left(A\langle\frac{g_1,\dots,g_k}{f}\rangle, A\langle\frac{g_1,\dots,g_k}{f}\rangle^+\right)$. One easily conclude. $\square$

Finally let us note the following.

***Proposition 2.6.7* ([36]).** — *One has* $\mathrm{Spa}(A, A^+)_a = \emptyset$ *iff* $A/\overline{\{0\}}$ *is discrete.*

We will give a proof of this in Section 2.13 that is different from Huber's proof, see Corollary 2.13.7.

2.6.2. Some remark about specializations in valuation spectra. — Let A be a ring and $\mathrm{Spv}(A)$ be its valuation spectrum ([36], see point (1) after Theorem 2.3.7). We have seen one tool to construct generalizations in $\mathrm{Spv}(A)$. Namely, for $v \in \mathrm{Spv}(A)$, any convex subgroup of Γ_v defines a generalization

$$v/H \geq v$$

with $\Gamma_{v/H} = \Gamma_v/H$ and the simple formula $v_{/H}(a) = v(a)$ modulo H. At the level of the valuations rings this corresponds to a localization with respect to the prime ideal defined by the convex subgroup.

There is another tool used by Huber to construct specializations this time. Let H be a convex subgroup of Γ_v. Define

$$v_{|H}$$

by the formula $v_{|H}(a) = \begin{cases} v(a) \text{ if } v(a) \in H \\ +\infty \text{ if } v(a) \notin H \end{cases}$. Let V be the valuation ring of v inside $\mathrm{Frac}(A/\mathrm{supp}(v))$. There is a biggest prime ideal $\mathfrak{p} \in \mathrm{Spec}(V)$ such that $A \to V_{\mathfrak{p}}$ i.e. the image of A in $\mathrm{Frac}(V)$ is contained in $V_{\mathfrak{p}}$. This prime ideal corresponds to the convex subgroup $c\Gamma_v$ of Γ_v generated by $\{v(a) \mid a \in A \text{ and } v(a) \leq 0\}$. Then $c\Gamma_v \subset H \Leftrightarrow v|H$ is a valuation. If this is the case then the support of $v_{|H}$ is changed, contrary to $v_{/H}$, and the valuation ring is $V/\mathfrak{p}$ with the valuation on A given by the composite $A \to V_{\mathfrak{p}} \to \mathrm{Frac}(V/\mathfrak{p})$. Moreover one has

$$v_{|H} \leq v.$$

Via the surjective map

$$\begin{tikzcd} \mathrm{Spv}(A) \arrow[d, two heads, "\text{support}"] \\ \mathrm{Spec}(A) \end{tikzcd}$$

the first construction produces a *vertical specialization* $v_{/H} \geq v$ (vertical=in a fiber of the support map). The second construction produces an *horizontal specialization* $v \geq v_{|H}$ (horizontal=we make a specialization in the base of the fibration $\mathrm{supp} : \mathrm{Spv}(A) \to \mathrm{Spec}(A)$).

***Proposition 2.6.8* (Proposition 1.2.4,[39]).** — *Any specialization in* $\mathrm{Spv}(A)$ *is an horizontal specialization of a vertical specialization.*

In this text we focus on analytic adic spaces where as we saw all specializations are vertical. We did not speak about the horizontal one. Nevertheless, they are essential to understand the proofs of Theorem 2.3.7 or Proposition 2.3.9 given by Huber.

2.7. Properties of the local rings and the residue fields

2.7.1. Basic properties. — Let (A, A^+) be a Huber pair and set $X = \mathrm{Spa}(A, A^+)$.

Lemma 2.7.1. — *For any $x \in X$, the stalk $\mathcal{O}_{X,x}$ of the structure presheaf at x is a local ring with maximal ideal*

$$\mathfrak{m}_x = \varinjlim_{\substack{U \ni x \\ U \text{ rat. domain}}} \mathrm{supp}_{\mathcal{O}_X(U)}(x).$$

Proof. — Let $U \subset X$ be a rational domain containing x and $g \in \mathcal{O}_X(U)$ such that $|g(x)| \neq 0$. Let I be a finite type ideal of a ring of definition of $\mathcal{O}_X(U)$ defining its topology. Using the continuity of the valuation defined by x one deduces that there exists $k \geq 1$ such that for all $f \in I^k$, $|f(x)| \leq |g(x)|$. Now, if $I^k = (f_1, \dots, f_n)$ one has $x \in U\big(\frac{f_1,\dots,f_n}{g}\big)$ with

$$g_{|U\big(\frac{f_1,\dots,f_n}{g}\big)} \in \mathcal{O}_X\Big(U\big(\tfrac{f_1,\dots,f_n}{g}\big)\Big)^\times.$$

From this we deduce that $\mathcal{O}_{X,x} \setminus \mathrm{m}_x = \mathcal{O}_{X,x}^\times$ and thus $\mathcal{O}_{X,x}$ is local with maximal ideal $\mathfrak{m}_x$. □

Definition 2.7.2. — For $x \in X$ we note

1. $\kappa(x) = \mathrm{Frac}(A/\mathrm{supp}(x))$ and $\kappa(x)^+$ its valuation subring,
2. $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is the residue field of the structure presheaf at x and $k(x)^+$ is its valuation subring,
3. $(K(x), K(x)^+) = \big(\widehat{k(x)}, \widehat{k(x)}^+\big)$,

where the Huber pair $(\kappa(x), \kappa(x)^+)$, resp. $(k(x), k(x)^+)$, is such that $\kappa(x)^+$, resp. $k(x)^+$, is a ring of definition of $\kappa(x)$, resp. $k(x)$, equipped with the $I.\kappa(x)^+$-adic, resp. $I.k(x)^+$-adic, topology where I is an ideal defining the topology of a ring of definition of A contained in A^+.

Those three affinoid rings are affinoid fields in the sense of Definition 2.4.1. There are two cases:

- If x is not analytic then $(\kappa(x), \kappa(x)^+) = (k(x), k(x)^+) = (K(x), K(x)^+)$ are discrete affinoid fields.
- If x is analytic then those are analytic affinoid fields and one verifies easily that

$$\big(\widehat{\kappa(x)}, \widehat{\kappa(x)}^+\big) \xrightarrow{\sim} \big(\widehat{k(x)}, \widehat{k(x)}^+\big) = (K(x), K(x)^+).$$

2.7.2. Henselian properties. — For any complex analytic space X, the local rings $\mathcal{O}_{X,x}$, $x \in X$, are Henselian rings with algebraically closed residue field. If $X = \mathfrak{X}^{an}$ where $\mathfrak{X}$ is a $\mathbb{C}$-scheme locally of finite type, for any $x \in X = \mathfrak{X}(\mathbb{C})$ the local morphism $\mathcal{O}_{\mathfrak{X},x} \to \mathcal{O}_{X,x}$ induces an isomorphism $\widehat{\mathcal{O}}_{\mathfrak{X},x} \xrightarrow{\sim} \widehat{\mathcal{O}}_{X,x}$. If $Y \to X$ is an étale covering of complex analytic spaces, using that the local rings are Henselian with algebraically closed residue fields, the morphism $Y \to X$ has a section over a topological covering i.e. a covering of $|X|$. Thus, the étale topos of X is the same as the topos of $|X|$.

For schemes the situation is the opposite: the local rings are arbitrary with arbitrary residue fields.

For adic spaces the situation is intermediate between schemes and complex analytic spaces: the local rings are Henselian but the residue fields are arbitrary. Thus, the obstruction to split étale coverings comes from the residue fields.

Example 2.7.3. — Let $Y \to X$ be an étale covering of $\mathbb{C}_p$-rigid spaces “à la Tate”. If x is a classical point of X, $x \in X(\mathbb{C}_p)$, then $Y \to X$ splits in a neighborhood of x. But the obstruction to split the covering over an admissible covering of X comes from the residue fields of the adic space X^{ad} that are not algebraically closed in general.

Let's come to the basic structure results for local rings and residue fields of adic spaces.

Proposition 2.7.4. — *For any $x \in X$,*

1. *The local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$ is Henselian.*
2. *If x is analytic then $(k(x)^+, (\varpi))$ is Henselian.*

Proof. — Let $P \in \mathcal{O}_{X,x}[T]$ be a unitary polynomial and $\alpha \in \mathcal{O}_{X,x}$ satisfying $|P(\alpha)(x)| = 0$ and $|P'(\alpha)(x)| \neq 0$. Then one can find a rational subset U containing x such that P is the image of $Q \in \mathcal{O}_X(U)[T]$ and α the image of $\beta \in \mathcal{O}_X(U)$ such that

$$\tfrac{Q(\beta)}{Q'(\beta)} \in \mathcal{O}_X(U)^{\circ\circ}.$$

Newton's method then shows that Q has a root in $\mathcal{O}_X(U)$ lifting α.

Let x be analytic. The morphism $\mathcal{O}_{X,x}^+ \to k(x)^+$ is surjective. In fact let U be a rational domain containing x and $f \in \mathcal{O}_X(U)$ satisfying $|f(x)| \leq 1$. Then, if $V = U(\frac{f,1}{1})$, $x \in V$ and $f_{|V} \in \mathcal{O}_X(V)^+$. Thus, $(k(x)^+, (\varpi))$ is a quotient of the filtered colimit of Henselian couples (see Lemma 2.7.5)

$$\varinjlim_{\substack{U \ni x \\ \text{rat. domain}}} (\mathcal{O}_X(U)^+, (\varpi)).$$

It is thus Henselian. □

Lemma 2.7.5. — *For (A, A^+) a complete affinoid Tate ring, the couple $(A^+, (\varpi))$ is Henselian.*

Proof. — Let A_0 be a ring of definition of A contained in A^+. Since $A^+ \subset A^\circ$, for any finite collection $x_1, \dots, x_n \in A^+$, $A_0[x_1, \dots, x_n]$ is again a ring of definition. Thus,

$$\varinjlim_{\substack{S \subset A^+ \\ \text{finite}}} (A_0[x]_{x \in S}, (\varpi)) \xrightarrow{\sim} (A^+, (\varpi))$$

and thus $(A^+, (\varpi))$ is Henselian as a filtered colimit of complete (and thus Henselian) couples. □

Corollary 2.7.6. — *For any $x \in X$ there is are equivalence of categories*

$$2\text{-}\varinjlim_{U \ni x} \{\textit{finite étale } \mathcal{O}_X(U)\textit{-algebras}\} \xrightarrow{\sim} \{\textit{finite étale } k(x)\textit{-algebras}\} \xrightarrow{\sim} \{\textit{finite étale } K(x)\textit{-algebras}\}.$$

Proof. — This is simply an application of Elkik's approximation ([25, II Theorem 5] for the Noetherian case, [3, Theorem 1.16.23] for the general case). □

2.8. Overconvergent open subsets: the Berkovich spectrum

Let (A, A^+) be a Tate ring with ϖ a pseudo-uniformizer. Thus, all points of $X = \mathrm{Spa}(A, A^+)$ are analytic. Our goal is to compute the compact topological space

$$X^B$$

that is the Berkovich quotient of X, see Section 1.10. Since all points of X are analytic the generalizations of a point form a chain and this is identified as a set with X^{max}, the set of maximal points of X. Thus, if we fix $\beta \in]0, 1[$, as a set

$$X^B = X^{max} = \{|.| : A \to \mathbb{R}_+ \mid \text{continuous valuations s.t.} |\varpi| = \beta\}.$$

2.8.1. The closure of rational subsets. — Let us begin with a lemma.

Lemma 2.8.1. — *Let $f_1, \dots, f_n \in A$ generate A and $g \in A$. Then,*

$$\overline{X\Big(\frac{f_1, \dots, f_n}{g}\Big)} = \bigcap_{k \geq 0} X\Big(\frac{\varpi f_1^k, \dots, \varpi f_n^k}{g^k}\Big).$$

Proof. — Since such a rational subset is constructible, its closure is the set of its specializations (Corollary 1.3.9).

Let x be in the closure of our rational subset. Let $h_i \in K(x)$ be the image of f_i divided by the image of g in $K(x)$. There exists $y \geq x$ such that $h_i \in K(y)^+$ where we have $K(x)^{\circ\circ} \subset K(x)^+ \subset K(y)^+ \subset K(x)^\circ$. Since $\varpi \in K(x)^{\circ\circ}$ this implies that for all $k \geq 0$, $\varpi h_i^k \in K(x)^{\circ\circ} \subset K(x)^+$. This proves that the left hand set is contained in the right hand one.

Let now x be such that, with the preceding notations, for all $k \geq 0$, $\varpi h_i^k \in K(x)^+$. Thus, for all k, $\varpi h_i^k \in K(x)^\circ$. Since $x^{max} = \mathrm{Spa}(K(x), K(x)^\circ)$ is a rank one valuation this implies $h_i \in K(x)^\circ$ and thus $x^{max} \geq x$ with x^{max} in the rational subset. This proves that the right hand set is contained in the left hand one and finishes the proof. $\square$

Example 2.8.2. — The interior of the closed subset $\{|x| < 1\} \subset \mathrm{Spa}(K\langle x\rangle, \mathcal{O}_X\langle x\rangle) = \mathbb{B}^1_K$ is the open ball $\bigcup_{k\geq 0} \{|x^k| \leq |\varpi|\}$.

Corollary 2.8.3. — *A basis of neighborhoods of $\overline{X\big(\frac{f_1,\dots,f_n}{g}\big)}$ is given by the rational subsets $X\big(\frac{\varpi f_1^k,\dots,\varpi f_n^k}{g^k}\big)$, $k \geq 1$.*

Proof. — Let U be an open subset of X containing $\overline{X\big(\frac{f_1,\dots,f_n}{g}\big)}$. One has

$$\bigcap_{k\geq 0} \Big((X \setminus U) \cap X\Big(\frac{\varpi f_1^k, \dots, \varpi f_n^k}{g^k}\Big)\Big) = \emptyset.$$

Using the compactness of $(X \setminus U)_{cons}$ we conclude that for $k \gg 0$,

$$(X \setminus U) \cap X\Big(\frac{\varpi f_1^k, \dots, \varpi f_n^k}{g^k}\Big) = \emptyset. \qquad \square$$

2.8.2. The Berkovich topology.

— The open subsets of the quotient X^B are in bijection with the *partially proper open subsets* of X that is to say *open subsets stable under specialization*. We call them the *overconvergent open subsets*.

Lemma 2.8.4. — *For an open subset U of X the following are equivalent:*

1. *U is overconvergent,*
2. *if $V \subset U$ is quasi-compact open then $\overline{V} \subset U$,*
3. *if $X\big(\frac{f_1,\dots,f_n}{g}\big) \subset U$ then there exists $k \geq 0$ such that $X\big(\frac{\varpi f_1^k,\dots,\varpi f_n^k}{g^k}\big) \subset U$.*

Proof. — The equivalence between (1) and (2) is deduced from the fact that V quasi-compact, $\overline{V}$ is the set of specializations of V. The equivalence between (2) and (3) is deduced from Corollary 2.8.3. $\square$

Remark 2.8.5. — From point (2) of the preceding lemma we deduce that the quasi-compact overconvergent open subsets are the open and closed subsets of X.

Proposition 2.8.6. — *The topology of X^B is the one defined by Berkovich on $X^{max} = \mathcal{M}(A)$, that is to say the one generated by $\{x \in X^{max} \mid |f(x)| < |g(x)|\}$ for $f, g \in A$.*

Proof. — Let $f_1,\dots,f_n \in A$ that generate A as an ideal. We have for $x\in X$,

$$\begin{aligned}\forall i,\ |f_i(x^{max})|<|g(x^{max})| &\iff \exists l\geq 1,\ x^{max}\in X\Big(\frac{f_1^l,\dots,f_n^l}{\varpi g^l}\Big)\\ &\iff \exists l\geq 1,\ x\in \overline{X\Big(\frac{f_1^l,\dots,f_n^l}{\varpi g^l}\Big)}\\ &\iff \exists l\geq 1,\ \forall k\geq 0,\ x\in X\Big(\frac{\varpi f_1^{kl},\dots,\varpi f_n^{kl}}{\varpi^k g^{kl}}\Big)\\ &\iff x\in \underbrace{\bigcup_{m\geq 0} X\Big(\frac{f_1^m,\dots,f_n^m}{\varpi g^m}\Big)}_{\mathrm{Int}\left(\{y\in X\ \mid\ \forall i,\ |f_i(y)|<|g_i(y)|\}\right)}.\end{aligned}$$

From this and Lemma 2.8.4 we conclude. □

2.8.3. Overconvergent sheaves. — Let $\mathscr{F}$ be a sheaf on X. Recall that we say $\mathscr{F}$ is overconvergent if for $x,y\in X$ with $x\leq y$, $\mathscr{F}_x \xrightarrow{\sim} \mathscr{F}_y$. From the preceding and Proposition 1.10.6 we deduce the following are equivalent for $\mathscr{F}$ a sheaf on X:

1. $\mathscr{F}$ is overconvergent,
2. $\mathscr{F}$ comes via pullback of a sheaf on the Berkovich quotient $X\twoheadrightarrow X^B$,
3. if $U\subset X$ is open qc and $i:\overline{U}\hookrightarrow X$,
$$\Gamma(\overline{U},i^*\mathscr{F})\xrightarrow{\sim}\Gamma(U,\mathscr{F}),$$
4. if $U\subset X$ is open qc then
$$\varinjlim_{U\subset\subset V}\Gamma(V,\mathscr{F})\xrightarrow{\sim}\Gamma(U,\mathscr{F})$$
(where by definition $U\subset\subset V$ means $\overline{U}\subset V$),
5. if $f_1,\dots,f_n$ generate A as an ideal and $g\in A$,
$$\varinjlim_{k\geq 0}\Gamma\Big(X\Big(\frac{\varpi f_1^k,\dots,\varpi f_n^k}{g^k}\Big),\mathscr{F}\Big)\xrightarrow{\sim}\Gamma\Big(X\Big(\frac{f_1,\dots,f_n}{g}\Big),\mathscr{F}\Big).$$

2.9. The sheaf property

2.9.1. General results. — Let us put the following definition. The category of rational open subsets form a site and we can speak about sheaves on this site.

Definition 2.9.1. — We say the pair (A,A^+) is *sheafy* if

1. the preceding presheaf $\mathcal{O}_X$ on the rational open subsets of $X=\mathrm{Spa}(A,A^+)$ is a sheaf,
2. if U is a rational subset of X covered by a collection $(V_i)_i$ of rational subsets then the morphism
$$\mathcal{O}_X(U)\longrightarrow\prod_{i\in I}\mathcal{O}_X(V_i)$$
is strict.

Since the rational subsets form a base of the topology stable under finite intersections this is equivalent to the fact that $\mathcal{O}_X$ extends to a sheaf of topological rings on $\mathrm{Spa}(A,A^+)$ ([1, Exposé III-Théorème 4.1]). More precisely, for any open subset U, let us define

$$\mathcal{O}_X(U)=\varprojlim_{\substack{V\subset U\\ \text{rational subset}}}\mathcal{O}_X(V)$$

equipped with the projective limit topology. Then, the following are equivalent:

1. (A, A^+) is sheafy,
2. $\mathcal{O}_X$ is a sheaf of topological rings on X.

Here by a *sheaf of topological rings* we mean that the correspondence $U \mapsto \mathcal{O}_X(U)$ is a functor from open subsets of X to topological rings

- that is a sheaf of rings after forgetting the topological structure,
- such that if $U = \cup_{i\in I} V_i$ is an open cover then the morphism
$$\mathcal{O}_X(U) \longrightarrow \prod_{i\in I} \mathcal{O}_X(V_i)$$
is strict.

Here are some cases when this is known:

1. When A is Tate strongly Noetherian ([37]) in the sense that for all $n \geq 1$, $\widehat{A}\langle X_1, \dots, X_n\rangle$ is Noetherian. This contains the case of classical rigid spaces à la Tate i.e. adic spaces locally of finite type over $\mathrm{Spa}(K)$ with K a non-archimedean field.
2. When $\widehat{A}$ has a Noetherian ring of definition ([37]).
3. A case more general than the two preceding one is treated in [54].
4. When A is perfectoid ([48]) and more generally sous-perfectoid.
5. When A is Tate stably uniform ([17]). This means that for all $U \subset \mathrm{Spa}(A, A^+)$ a rational open subset, the Huber ring $\mathcal{O}_X(U)$ is uniform. This contains the perfectoid case.
6. In the "discrete case" case: if A is a ring equipped with the discrete topology then (A, A^+) is sheafy for any A^+.

Remark 2.9.2. — (Follow up to remark 2.5.2) When A is Tate stably uniform or Tate strongly Noetherian one has the following simpler formula for holomorphic functions on a rational open subset: $A\langle \frac{f_1,\dots,f_n}{g}\rangle = \widehat{A}\langle T_1, \dots, T_n\rangle/(T_1 g - f_1, \dots, T_n g - f_n)$ i.e. there is no need to take the closure of the ideal defining the quotient, it is already closed.

***Example 2.9.3* (Discrete case).** — Let (A, A^+) be an affinoid ring with A discrete. The continuous map
$$\mathrm{supp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A)$$
is open and surjective,
$$\mathrm{supp}\big(\{x \mid |f(x)| \leq |g(x)| \neq 0\}\big) = D(g).$$
Moreover for any rational subset $U \subset \mathrm{Spa}(A, A)$, $\Gamma(U, \mathcal{O}_{\mathrm{Spa}(A,A)}) = \Gamma(\mathrm{supp}(U), \mathcal{O}_{\mathrm{Spec}(A)})$. We deduce that (A, A^+) is sheafy with
$$\mathcal{O}_{\mathrm{Spa}(A,A^+)} = \mathrm{supp}^{-1}\mathcal{O}_{\mathrm{Spec}(A)}.$$

We will explain the strongly Noetherian and the stable uniform case in the following sections.

2.9.2. A general dévissage. — In this section we explain a general common strategy that allows one to prove sheafiness for Tate Huber pairs. More precisely, we seek to prove the following result.

Theorem 2.9.4. — *Let $\mathcal{C}$ be a class of complete Tate Huber pairs that is stable under rational localizations. Suppose that for all (A, A^+) in $\mathcal{C}$ and $f \in A$ the sequence*
$$0 \longrightarrow A \longrightarrow A\langle f\rangle \oplus A\langle f^{-1}\rangle \longrightarrow A\langle f, f^{-1}\rangle \longrightarrow 0$$
is exact. Then any (A, A^+) in $\mathcal{C}$ is sheafy and $\mathcal{O}_{\mathrm{Spa}(A,A^+)}$ is acyclic.

We will give a proof of this theorem in the following subsections.

2.9.2.1. *About the strictness condition.* — We did not speak about the topology of our rings in the preceding for the following reason. Let us first take a definition.

Definition 2.9.5. — Let A be a complete Tate ring. A Banach A-module is a complete topological A-module M such that there exists A_0 a ring of definition of A equipped with a pseudo-uniformizer $\varpi \in A_0$ such that M admits an open bounded sub-A_0-module whose topology is the ϖ-adic one.

The preceding definition is well behaved in the sense that for M a Banach A-module and A_0 *any* ring of definition of A equipped with $\varpi \in A_0$, there exists an open bounded sub-A_0-module of M whose topology is the ϖ-adic one. For such a sub-A_0-module M_0,

$$M = M_0[\tfrac{1}{\varpi}]$$

since M_0 is open. Moreover, since M_0 is bounded the topology of M is the inductive limit topology via the formula $M = \varinjlim_{\times\varpi} M_0$.

We have the following elementary result.

Proposition 2.9.6. — *Let A be a complete Tate ring.*

1. *Any closed sub-module of a Banach A-module is a Banach A-module.*
2. *If N is a closed sub-module of the Banach A-module M then M/N is a Banach A-module.*
3. *Banach open mapping theorem holds: any continuous surjective map of Banach A-modules is open.*

Proposition 2.9.7. — *Let A be a complete Tate ring and $C^\bullet$ a complex of Banach A-modules. Suppose it is exact as a complex of A-modules. Then any boundary map in $C^\bullet$ is strict.*

Proof. — For any index $i \in \mathbb{Z}$, the continuous map of Banach A-modules $\partial^i : C^i / \ker \partial^i \to \ker \partial^{i+1}$ is bijective and thus an homeomorphism according to Banach open mapping theorem. □

Corollary 2.9.8. — *Let (A, A^+) be a complete Tate affinoid ring. Then (A, A^+) is sheafy if and only if $U \mapsto \mathcal{O}_X(U)$ is a sheaf of rings i.e. if $\mathcal{O}_X$ is a sheaf of rings it is automatically a sheaf of topological rings.*

We thus get rid of the strictness condition in the definition of a sheafy pair: it is automatic. We can focus on the purely algebraic part of the result.

2.9.2.2. *A general lemma: reduction to a sub-covering.* — Let us begin by devising a general strategy that allows us to replace coverings by finer coverings.

Lemma 2.9.9. — *Let $\mathcal{C}$ be a class of complete Huber pairs that is stable under rational localizations. Suppose that for any $(A, A^+) \in \mathcal{C}$ and any finite rational cover $\mathcal{U} = (U_i)_i$ of $\mathrm{Spa}(A, A^+)$ there exists a rational cover $\mathcal{V} = (V_j)_j$ such that $\forall j,\ \exists i,\ V_j \subset U_i$ and $A \to \check{C}^\bullet(\mathcal{V}, \mathcal{O}_X)$ is a resolution. Then any $(A, A^+) \in \mathcal{C}$ is sheafy.*

Proof. — Let (A, A^+) be in the category $\mathcal{C}$. According to the hypothesis, for any rational open subset U of $\mathrm{Spa}(A, A^+)$, if

$$\check{H}^\bullet(U, \mathcal{O}_X) = \varinjlim_{\substack{\mathcal{U} \\ \text{finite rational cover of } U}} \check{H}^\bullet(\mathcal{U}, \mathcal{O}_X)$$

then

$$\mathcal{O}_X(U) \xrightarrow{\sim} \check{H}^0(U, \mathcal{O}_X) \tag{7}$$

and

$$\check{H}^i(U, \mathcal{O}_X) = 0 \tag{8}$$

for $i > 0$. Equip the sets of rational subsets of $\mathrm{Spa}(A, A^+)$ with the Grothendieck topology generated by the covers that are the finite rational covers. The sheaf associated to the presheaf $\mathcal{O}_X$ on this site is obtained by applying the functor $\check{H}^0$ two times to $\mathcal{O}_X$ ([1, Exposé II-Théorème 3.4]). Equation (7) thus shows that $\mathcal{O}_X$ is a sheaf on this site. Equation (8) then shows that $\mathcal{O}_X$ is acyclic on any rational open subset (usual argument by induction on the cohomological degree based on the Cech-cohomology spectral sequence $E_2^{pq} = \check{H}^p(U, \mathcal{H}^q(\mathcal{O}_X)) \Rightarrow H^{p+q}(U, \mathcal{O}_X)$, see [1, Exposé V-Proposition 4.3]). We can now apply the Čech spectral sequence for a fixed cover to obtain that $A \to \check{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ is a resolution for any finite rational cover $\mathcal{U}$ of $\mathrm{Spa}(A, A^+)$. □

2.9.2.3. *Refining a cover by a standard rational cover.* — We now begin to specialize to the case of Tate rings. The following lemma is a key tool.

Lemma 2.9.10. — *Let (A, A^+) be a Tate affinoid ring. For any open cover $(U_i)_{i\in I}$ of $X = \mathrm{Spa}(A, A^+)$, there exists an integer $n \geq 1$ and $f_1, \ldots, f_n \in A$ generating A as an ideal such that for all $k \in \{1, \ldots, n\}$ there exists $i \in I$ such that $X(\frac{f_1,\ldots,f_n}{f_k}) \subset U_i$.*

Proof. — Using the quasi-compacity of $\mathrm{Spa}(A, A^+)$ and the fact that the rational subsets form a base of the topology, we can suppose that I is finite and U_i is a rational subset. Write

$$U_i = X(\tfrac{T_i}{g_i})$$

where $T_i \subset A$ is finite generating A as an ideal. We can suppose, up to replacing T_i by $T_i \cup \{g_i\}$, that

$$g_i \in T_i.$$

Fix some $i \in I$. Let ϖ be a pseudo-uniformizer of A. Choose $N \geq 0$ such that

$$\varpi^N (g_{i|U_i})^{-1} \in \mathcal{O}_X(U_i)^+.$$

Up to replacing T_i by $\varpi^{-N} T_i$ and g_i by $\varpi^{-N} g_i$ we can suppose that $1 \in T_i$.

We now set

$$S = \Big\{ \prod_{i\in I} f_i \mid f_i \in T_i,\ \exists j \in I,\ f_j = g_j \Big\}.$$

Since $1 \in T_i$ for all $i \in I$, $g_i \in S$ for all i. Since $X = \cup_{i\in I} X(\frac{T_i}{g_i})$, for all $x \in X$ there exists $i \in I$ such that $|g_i(x)| \neq 0$. From this we deduce that the ideal generated by S is A. We have

$$X = \bigcup_{s\in S} X(\tfrac{S}{s}).$$

Let now $s = \prod_{i\in I} f_i \in S$ with $f_i \in T_i$ for all $i \in I$ and $f_j = g_j$. One has

$$X(\tfrac{S}{s}) \subset \bigcap_{f\in T_j} \Big\{ x \in X \mid |\prod_{i\neq j} f_i.f(x)| \leq |s(x)| \Big\} = X(\tfrac{T_j}{g_j}).$$

This proves the result. □

For $f_1, \ldots, f_n$ as before we call the cover $\big(X(\frac{f_1,\ldots,f_n}{f_i})\big)_{1\leq i\leq n}$ of X the standard rational cover generated by $f_1, \ldots, f_n$.

2.9.2.4. *Intersection with a cover.* — Let (A, A^+) be a Huber pair and $X = \mathrm{Spa}(A, A^+)$.

Lemma 2.9.11. — *Let $\mathcal{U} = (U_i)_i$ and $\mathcal{V} = (V_j)_j$ be finite open covers of X by rational subsets and let $\mathcal{U} \times_X \mathcal{V} = (U_i \cap V_j)_{i,j}$. Suppose that*

1. *$A \to \check{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ is a resolution,*
2. *For all $i_1, \ldots, i_r$, $\mathcal{O}_X(U_{i_1} \cap \cdots \cap U_{i_r}) \longrightarrow \check{C}^\bullet(\{U_{i_1} \cap \cdots \cap U_{i_r}\} \times_X \mathcal{V}, \mathcal{O}_X)$ is a resolution.*

Then,

$$A \longrightarrow \check{C}^\bullet(\mathcal{U} \times_X \mathcal{V}, \mathcal{O}_X)$$

is a resolution.

Proof. — This is deduced from the spectral sequence associated to a double complex. □

2.9.2.5. *Proof of Theorem 2.9.4.* — We finally give the proof of Theorem 2.9.4. Let (A, A^+) be a complete Tate Huber pair. We note $X = \mathrm{Spa}(A, A^+)$. For $f_1, \ldots, f_n \in A$ we call the cover

$$\big(X(\tfrac{1}{f_1}), X(\tfrac{f_1}{1})\big) \times_X \cdots \times_X \big(X(\tfrac{1}{f_n}), X(\tfrac{f_n}{1})\big)$$

of $\mathrm{Spa}(A, A^+)$ the *Laurent cover associated to* $f_1, \ldots, f_n \in A$.

Lemma 2.9.12. — *Under the assumptions of Theorem 2.9.4, for any Laurent cover of* $\mathrm{Spa}(A, A^+)$ *the associated Čech complex is a resolution of A.*

Proof. — This is proved by induction on n using Lemma 2.9.11. □

Lemma 2.9.13. — *For any $f_1, \ldots, f_n \in A$ generating A as an ideal, there exists a Laurent cover $\mathcal{V}$ of* $\mathrm{Spa}(A, A^+)$ *such that for any $V \in \mathcal{V}$ the intersection of the rational cover generated by $f_1, \ldots, f_n$ with V is a rational cover generated by units of $\mathcal{O}_X(V)$.*

Proof. — Let ϖ be a pseudo-uniformizer. There exists $N > 0$ such that $\varpi^N \in A^+ f_1 + \cdots + A^+ f_n$. From this we deduce that for all $x \in X$, there exists $i \in \{1, \ldots, n\}$ such that $|\varpi^{N+1}(x)| < |f_i(x)|$. Let now $\mathcal{V}$ be the Laurent cover associated to

$$\varpi^{-N-1} f_1, \ldots, \varpi^{-N-1} f_n.$$

For each element V of $\mathcal{V}$, the rational cover $(V \cap U_i)_{1 \le i \le n}$ of V is generated by units. In fact, if

$$V = \bigcap_{i \notin I} X\Big(\frac{\varpi^{-N-1} f_i}{1}\Big) \ \cap \ \bigcap_{i \in I} X\Big(\frac{1}{\varpi^{-N-1} f_i}\Big)$$

with $I \subset \{1, \ldots, n\}$, $V \neq \emptyset$ implies $I \neq \emptyset$ and moreover $(V \cap U_i)_{1 \le i \le n}$ is generated by the elements $(f_i)_{i \in I}$ that are in $\mathcal{O}_X(V)^\times$. □

We will conclude the proof of Theorem 2.9.4 using the following result.

Lemma 2.9.14. — *Any rational cover generated by units of A can be refined to a Laurent cover.*

Proof. — In fact, the rational cover generated by $f_1, \ldots, f_n \in A^\times$ is refined to the Laurent cover associated to $(f_i f_j^{-1})_{1 \le i < j \le n}$. □

Proof of Theorem 2.9.4. According to Lemmas 2.9.9 and 2.9.10 we are reduced to proving that $A \to \check{C}^\bullet(\mathcal{U}, \mathcal{O}_X)$ is a resolution for $\mathcal{U}$ a standard rational cover. Using Lemmas 2.9.11, 2.9.12 and 2.9.13 we can suppose that our standard rational cover $\mathcal{U}$ is generated by units. According to Lemma 2.9.14 there exists a Laurent cover $\mathcal{V}$ refining $\mathcal{U}$. We can apply Lemmas 2.9.11 and 2.9.12 to deduce that $A \to \check{C}^\bullet(\mathcal{U} \times_X \mathcal{V}, \mathcal{O}_X)$ is a resolution. Moreover, for any $U_1, \ldots, U_r$ in $\mathcal{U}$, Lemma 2.9.12 shows that $\mathcal{O}_X(U_1 \cap \cdots \cap U_r) \to \check{C}^\bullet(\{U_1 \cap \cdots \cap U_r\} \times_X \mathcal{V}, \mathcal{O}_X)$ is a resolution. The spectral sequence associated to a double complex then shows that $\check{C}^\bullet(\mathcal{U}, \mathcal{O}_X) \to \check{C}^\bullet(\mathcal{U} \times_X \mathcal{V}, \mathcal{O}_X)$ is a quasi-isomorphism. We conclude. □

2.9.3. An example: the strongly Noetherian case.

— We are going to prove the following theorem using Theorem 2.9.4.

Theorem 2.9.15 ([37, Theorem 2.5]). — *If (A, A^+) is an affinoid Tate ring with A strongly Noetherian then (A, A^+) is sheafy and moreover if* $X = \mathrm{Spa}(A, A^+)$, $H^i(X, \mathcal{O}_X) = 0$ *for* $i > 0$.

This result contains as a particular case the one of topologically of finite type affinoid algebras over a non-archimedean field i.e. the case of "classical" Tate rigid spaces.

2.9.3.1. *Background on Noetherian Banach modules.* — Let us recall the following. Here A is a complete Tate ring.

Proposition 2.9.16. — *Let M be a Banach-A-module. The following are equivalent:*

1. *M is a Noetherian A-module.*
2. *Any sub-module of M is closed.*

Proof. — Suppose (1) is verified. Let $N \subset M$ be a sub-module. Since $\overline{N}$ is of finite type, there exists a surjection

$$f : A^n \to \overline{N}$$

for some $n \geq 0$. Such a morphism is automatically continuous and we can apply Banach's open mapping theorem to deduce that $W = f(A_0^n)$ is a bounded open sub-A_0-module of $\overline{N}$. Moreover,

$$W = (W \cap N) + \varpi W$$

by density of N in $\overline{N}$. We deduce that the A_0-module of finite type $P = W/W \cap N$ satisfies

$$P = \varpi P.$$

Since $\varpi \in \mathrm{Rad}(A_0)$, Nakayama lemma implies that $P = 0$ and thus $N = \overline{N}$.

Reciprocally, suppose (2) is verified. Let $(M_i)_{i\geq 0}$ be a growing chain of sub-A-modules and note

$$M_\infty = \cup_{i\geq 0} M_i.$$

Bair's theorem implies that there exists $i \geq 0$ such that the interior of M_i in M_∞ is non-empty. This implies, using a translations argument, that M_i is open in M_∞ and thus $M_\infty = M_i$. $\square$

Any finite type A-module M has a canonical topology: the quotient topology defined by a surjective morphism $A^n \twoheadrightarrow M$. One verifies immediately that this topology does not depend on the choice of such a surjection. For this canonical topology, any morphism between finite type A-modules is continuous.

Proposition 2.9.17. — *Suppose A is Noetherian.*

1. *Any finite type A-module is a Banach A-module.*
2. *Any morphism between finite type A-modules is strict.*
3. *Any ideal of A is closed.*

As a consequence of the preceding proposition, if A is a complete Tate strongly Noetherian ring then for $f_1, \ldots, f_n \in A$ generating A as an ideal and $g \in A$,

$$A\Big\langle \frac{f_1, \ldots, f_n}{g} \Big\rangle = A\langle T_1, \ldots, T_n\rangle/(gT_1 - f_1, \ldots, gT_n - f_n)$$

and this ring is strongly Noetherian.

2.9.3.2. *Flatness of $A\langle X\rangle, A\langle f^{-1}\rangle$ and $A\langle f\rangle$.* — Let A be a strongly Noetherian complete Tate ring. For M a Banach A-module,

$$M\hat{\otimes}_A A\langle X\rangle = \Big\{ \sum_{i\geq 0} m_i X^i \mid m_i \in M,\ \lim_{i\to+\infty} m_i = 0 \Big\}. \tag{9}$$

From Proposition 2.9.17 we deduce the following Lemma.

Lemma 2.9.18. — *If M is a finite type A-module then $M \otimes_A A\langle X\rangle = M\hat{\otimes}_A A\langle X\rangle$.*

We can now prove our result.

Proposition 2.9.19. — *For any $f \in A$ the following rings are flat over A:*

$$A\langle X\rangle,\ A\langle X\rangle/(fX - 1),\ A\langle X\rangle/(X - f).$$

Proof. — Let

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

be an exact sequence of finite type A-modules. Using Lemma 2.9.18 the sequence obtained by applying $- \otimes_A A\langle X\rangle$ is en exact sequence

$$M'\hat{\otimes}_A A\langle X\rangle \longrightarrow M\hat{\otimes}_A A\langle X\rangle \longrightarrow M''\hat{\otimes}_A A\langle X\rangle \longrightarrow 0.$$

But now, using the explicit description (9), one verifies immediately that in fact the sequence

$$0 \longrightarrow M'\hat{\otimes}_A A\langle X\rangle \longrightarrow M\hat{\otimes}_A A\langle X\rangle \longrightarrow M''\hat{\otimes}_A A\langle X\rangle \longrightarrow 0$$

is exact. We deduce from this that $A\langle X\rangle$ is flat over A.

For $P \in \{fX-1, X-f\}$ and M a finite type A-module there is an exact sequence

$$M\hat{\otimes}_A A\langle X\rangle \xrightarrow{\times P} M\hat{\otimes}_A A\langle X\rangle \longrightarrow M\hat{\otimes}_A (A\langle X\rangle/(P)) \longrightarrow 0.$$

Let

$$\sum_{i\geq 0} m_i X^i$$

be in the kernel of the left hand map. We use the Equation (9). If $P = fX-1$ we get $m_0 = 0$ and the induction relation $fm_i = m_{i+1}$ for $i \geq 0$. We deduce from this that our element is zero. For $P = X - f$ we get $fm_0 = 0$ and $fm_{i+1} = m_i$ for $i \geq 0$. Let N be the sub-module of M generated by $(m_i)_{i\geq 0}$. It is finitely generated, let's say

$$N = Am_0 + \cdots + Am_d.$$

For any $i \geq 0$ we can write

$$m_{i+d+1} = \sum_{k=0}^{d} a_k m_k.$$

From this we deduce that

$$m_i = f^{d+1} m_{i+d+1} = 0.$$

Our element in the kernel is thus zero.

Thus, for all M of finite type we have an exact sequence

$$0 \longrightarrow M\hat{\otimes}_A A\langle X\rangle \xrightarrow{\times P} M\hat{\otimes}_A A\langle X\rangle \longrightarrow M\hat{\otimes}_A (A\langle X\rangle/(P)) \longrightarrow 0.$$

From this we deduce that

$$\mathrm{Tor}_1^A(M, A\langle X\rangle/(P)) = 0$$

for all such M and thus $A\langle X\rangle/(P)$ is A-flat. □

2.9.3.3. *The sheaf property.* — The complete Tate ring A is assumed to be strongly Noetherian.

Proposition 2.9.20. — *For any $f \in A$, the Cech complex*

$$0 \longrightarrow A \longrightarrow A\langle f\rangle \oplus A\langle f^{-1}\rangle \longrightarrow A\langle f, f^{-1}\rangle \longrightarrow 0.$$

is exact.

Proof. — According to Proposition 2.9.19, the morphism $A \to A\langle f\rangle \times A\langle f^{-1}\rangle$ is flat. According to Corollary 2.13.9, there is a diagram

$$\begin{tikzcd} \mathrm{Spa}(A,A^+)(\frac{1}{f}) \coprod \mathrm{Spa}(A,A^+)(\frac{f}{1}) \arrow[r, "supp"] \arrow[d, two heads] & \mathrm{Spec}(A\langle \frac{1}{f}\rangle) \coprod \mathrm{Spec}(A\langle \frac{f}{1}\rangle) \arrow[d] \\ \mathrm{Spa}(A,A^+) \arrow[r, two heads, "supp"] & \mathrm{Spec}(A). \end{tikzcd}$$

We deduce that $A \to A\langle f\rangle \times A\langle f^{-1}\rangle$ is faithfully flat and thus injective. The exactness in the middle and on the right are elementary computations. □

Using Theorem 2.9.4 we thus have proven Theorem 2.9.15.

2.9.4. Another example: the stably uniform case ([17]). — We are now going to prove the following result using Theorem 2.9.4.

Theorem 2.9.21. — *Let (A, A^+) be a stably uniform affinoid Tate ring. Then (A, A^+) is sheafy.*

2.9.4.1. *Laurent domains computations.* — Let us begin by exploiting the spectral norm of Theorem 2.1.16.

Lemma 2.9.22. — *Let A be a uniform complete Tate ring. For any $f \in A$, the morphisms*

$$A\langle T\rangle \xrightarrow{\times fT-1} A\langle T\rangle$$
$$A\langle T\rangle \xrightarrow{\times T-f} A\langle T\rangle.$$

are strict injections. In particular, the ideals $(fT-1)$ and $(T-f)$ of $A\langle T\rangle$ are closed and $A\langle f\rangle = A\langle T\rangle/(fT-1)$, $A\langle f^{-1}\rangle = A\langle T\rangle/(T-f)$.

Proof. — Fix ϖ a pseudo-uniformizer in A and let $\|.\|$ be a power multiplicative norm defining the topology of A and such that

$$\|\varpi a\| = \|\varpi\|.\|a\| \text{ and } \|\varpi^{-1}a\| = \|\varpi\|^{-1}\|a\|.$$

Since A is uniform, $A\langle T\rangle$ is uniform. In fact, the Gauss norm on $A\langle T\rangle$ associated to $\|.\|$ on A is power multiplicative (see the proof of Theorem 2.1.16) and defines its topology. We can now use the interpretation of the Gauss norm on $A\langle T\rangle$ as a spectral norm, see Theorem 2.1.16. We use the projection map

$$\mathcal{M}(A\langle T\rangle) \to \mathcal{M}(A).$$

In fact, if $K(x)$ is the completed residue field associated to $x \in \mathcal{M}(A)$, for $P \in A\langle T\rangle$ we have

$$\|P\| = \sup_{x\in\mathcal{M}(A)} \sup_{y\in\mathcal{M}(K(x)\langle T\rangle)} |P_x(y)|$$

where P_x is the image of P in $K(x)\langle T\rangle$. Now we use that the Gauss norm of $K(x)\langle T\rangle$ is multiplicative. We thus have for $P \in A\langle T\rangle$

$$\|(fT-1)P\| = \sup_{x\in\mathcal{M}(A)} \left[\sup\{|f(x)|, 1\}.\|P_x\|\ \right]$$

and thus

$$\|P\| \leq \|(fT-1)P\| \leq \sup\{\|f\|, 1\}.\|P\|.$$

The same formula holds for $T-f$ instead of $fT-1$ and this implies the result. □

Lemma 2.9.23. — *Let $(U_i)_{i\in I}$ be a finite rational cover of $X = \mathrm{Spa}(A, A^+)$ with A a complete stably uniform Tate ring. Then the application*

$$A \longrightarrow \bigoplus_{i\in I} \mathcal{O}_X(U_i)$$

is a strict embedding.

Proof. — Fix ϖ and $\beta \in]0,1[$ as in Theorem 2.1.16. For each i, the topology of $O_X(U_i)$ is induced by the supremum norm on $\mathcal{M}(\mathcal{O}_X(U_i)) = U_i^B$. The same goes on for A. The result is therefore deduced from the covering $\mathcal{M}(A) = \mathrm{Spa}(A, A^+)^B = \bigcup_{i\in I} U_i^B$. □

Proposition 2.9.24. — *Let A be a stably uniform complete Tate ring and $f \in A$. The Čech complex*

$$0 \longrightarrow A \longrightarrow A\langle X\rangle/(fX-1) \oplus A\langle Y\rangle/(f-Y) \longrightarrow A\langle X,Y\rangle/(fX-1, f-Y) \longrightarrow 0$$

associated to the Laurent cover $\{|f|\le 1\}, \{|f|\ge 1\}$ is exact and all the maps are strict.

Proof. — Consider the exact sequence of Tate rings

$$0 \longrightarrow A \xrightarrow{u} A[X]/(fX-1) \oplus A[Y]/(f-Y) \xrightarrow{v} A[X,Y]/(fX-1, f-Y) \longrightarrow 0.$$

According to Lemma 2.9.23 the map u is strict. The map v is open for evident reasons and thus strict. The associated sequence obtained by completion is thus exact. $\square$

2.9.4.2. *Sheafiness.* — According to Theorem 2.9.4 we thus have proven Theorem 2.9.21. We can go even further.

Proposition 2.9.25. — *If A is a stably uniform complete Tate ring then for $f_1, \dots, f_n \in A$ generating the unit ideal and $g \in A$ we have*

$$A\Big\langle \frac{f_1,\dots,f_n}{g}\Big\rangle = A\langle T_1,\dots,T_n\rangle/(gT_1 - f_1, \dots, gT_n - f_n).$$

Proof. — One can find $\lambda_1, \dots, \lambda_n \in A^\circ$ such that $\sum_{i=1}^n \lambda_i f_i = \varpi$ for a pseudo-uniformizer ϖ. If U is our rational subset we then have $\varpi g_{|U}^{-1} \in \mathcal{O}(U)^+$. We then have

$$A\Big\langle \frac{f_1,\dots,f_n}{g}\Big\rangle = A\Big\langle \frac{1}{\varpi^{-1}g}\Big\rangle\Big\langle \frac{g^{-1}f_1}{1}\Big\rangle \cdots \Big\langle \frac{g^{-1}f_n}{1}\Big\rangle.$$

From Lemma 2.9.22 we deduce that

$$A\Big\langle \frac{f_1,\dots,f_n}{g}\Big\rangle = A\langle X, Y_1,\dots,Y_n\rangle/(\varpi^{-1}gX - 1, Y_1 - \varpi^{-1}f_1X, \dots, Y_n - \varpi^{-1}f_nX).$$

There are two morphisms

$$\begin{tikzcd}
A\langle T_1,\dots,T_n\rangle/(gT_1 - f_1,\dots,gT_n - f_n) \arrow[d, shift left, "T_i\mapsto Y_i"] \\
A\langle X, Y_1,\dots,Y_n\rangle/(\varpi^{-1}gX - 1, Y_1 - \varpi^{-1}f_1X,\dots,Y_n - \varpi^{-1}f_nX) \arrow[u, shift left, "{\substack{Y_i\mapsto T_i \\ X\mapsto \sum_i \lambda_i T_i}}"]
\end{tikzcd}$$

that are clearly inverse to each other. $\square$

2.10. Adic spaces

We refer to the beginning of Section 2.9.1 for the definition of a sheaf of topological rings.

Definition 2.10.1. — The category of adic spaces is the subcategory of the category of triplets $(X, \mathcal{O}_X, (v_x)_{x\in X})$ that are locally isomorphic to $\mathrm{Spa}(A, A^+)$ with (A, A^+) a sheafy Huber pair and where

- X is a topological space,
- $\mathcal{O}_X$ a sheaf of topological rings such that $\forall x \in X, \mathcal{O}_{X,x}$ is a local ring,
- v_x is a valuation on the residue field $k(x)$ of $\mathcal{O}_{X,x}$,

with morphisms the morphisms of locally ringed spaces $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ satisfying:

- for any open subset V of Y the morphism $f^* : \mathcal{O}_Y(V) \to \mathcal{O}_X(f^{-1}(V))$ is continuous,
- for any $x \in X$ the morphism $k(f(x)) \to k(x)$ is compatible with v_x and $v_{f(x)}$.

We need to verify the following elementary result so that our definition is a good one. This is the analog of the basic result in scheme theory that says that the functor $\mathrm{Spec}(-)$ from rings to locally ringed spaces is fully faithful.

Lemma 2.10.2. — *The functor $(A, A^+) \mapsto \mathrm{Spa}(A, A^+)$ from sheafy Huber pairs to adic spaces is fully faithful.*

The underlying topological space of an adic space is locally spectral. We will mainly be interested in analytic adic spaces.

Definition 2.10.3. — An adic space X is analytic if any point has a neighborhood of the form $\mathrm{Spa}(A, A^+)$ with (A, A^+) a sheafy Tate Huber pair.

This definition is justified by the fact that one verifies immediately that, with the terminology of section 2.6, a point of $\mathrm{Spa}(A, A^+)$ is analytic if and only if it has a neighborhood of the form $\mathrm{Spa}(A, A^+)$ with A a Tate Huber ring. We thus have in particular the following.

Proposition 2.10.4. — *For $x \in X$ an analytic adic space, the set of generalizations of x, X_x, is a chain that is identified with $\mathrm{Spa}(K(x), K(x)^+)$ where the topology on $K(x)$ is the one deduced from the maximal generalization $\mathrm{Spa}(K(x), K(x)^\circ)$ of x, a rank 1 valuation.*

2.11. Analytic adic spaces

Analytic adic spaces have other nice properties.

Lemma 2.11.1. — *For (A, A^+) a Tate Huber ring, $\varpi \in A$ a pseudo-uniformizer, and $x \in \mathrm{Spa}(A, A^+)$ one has*

$$\mathcal{O}^+_{X,x}/\varpi \xrightarrow{\sim} K(x)^+/\varpi.$$

Proof. — For $f \in K(x)^+$ we can lift it to an element $g \in \mathcal{O}_X(U)$ where U is an affinoid neighborhood of x. Now, let $V = U(\frac{g}{1})$. Then $g_{|V} \in \mathcal{O}^+_X(V)$. Its image in $\mathcal{O}^+_{X,x}$ is sent to f. Thus, $\mathcal{O}^+_{X,x} \to K(x)^+$ is surjective. Now, if $f \in \varpi K(x)^+$ this means $|g(x)| \leq |\varpi(x)|$. We can then shrink V to $W = V(\frac{\varpi}{g})$. One has $g_{|W} \in \varpi\mathcal{O}^+_X(W)$ and we conclude for the injectivity. □

Thus, the ϖ-adic completion of $\mathcal{O}^+_{X,x}$ is $K(x)^+$. From this we deduce the following.

Proposition 2.11.2. — *For $x \in X$ an analytic adic space $\mathrm{Spa}(K(x), K(x)^+) \xrightarrow{\sim} \varprojlim_{U \ni x} U$ in the category of adic spaces.*

This sets analytic adic spaces apart from schemes. In fact, if X is a scheme then $\varprojlim_{U \ni x} U = \mathrm{Spec}(\mathcal{O}_{X,x})$, the spectrum of a local ring. Here what shows up is not any local ring but a valuation ring. Let us give an application of this phenomenon that we will use all the time.

Proposition 2.11.3. — *A morphism between analytic adic spaces is generalizing.*

Proof. — Let $f; X \to Y$ be such a morphism with $f(x) = y$. This gives rise to a morphism $\mathrm{Spa}(K(x), K(x)^+) \to \mathrm{Spa}(K(y), K(y)^+)$. Since $K(y)^+$ is a valuation and $K(x)^+$ without torsion as a $K(y)^+$-module, $\mathrm{Spec}(K(x)^+) \to \mathrm{Spec}(K(y)^+)$ is flat and thus generalizing. □

Remark 2.11.4. — Thus, is $f : X \to Y$ is a morphism between qcqs analytic adic spaces then Im(f) ⊂ |Y| is pro-constructible stable under generalizations. For example, if Y is a strongly Noetherian adic space and $X \subset Y$ Zariski closed then $|X| \subset |Y|$ is closed stable under generalizations! We are in a very different situation from the scheme case.

This is an important result because of the following.

Corollary 2.11.5. — *Let $f : X \to Y$ be a qcqs surjective morphism between analytic adic spaces. Then $|f| : |X| \to |Y|$ is a quotient map.*

Proof. — Apply Lemma 2.11.6. $\square$

This last corollary is one of the starting points of the v-topology on perfectoid spaces.

Lemma 2.11.6. — *If $f : X \to Y$ is a surjective generalizing qc map between spectral spaces then f is a quotient map i.e. the topology on Y is the quotient topology on X by the equivalence relation $X \times_Y X \subset X \times X$.*

Proof. — Let $V \subset Y$ be a subset such that $f^{-1}(V)$ is open. One has $Y \setminus V = f(X \setminus f^{-1}(V))$ which is thus pro-constructible. Moreover since $V = f(f^{-1}(V))$ with $f^{-1}(V)$ open, V is stable under generalizations. Thus, $Y \setminus V$ is pro-constructible stable under generalizations and thus closed. $\square$

2.12. Canonical compactifications

Let $f : X = \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+) = Y$ be a morphism of affinoid adic spaces where we suppose that our Huber pairs are sheafy. Define a new Huber pair (C, C^+) by setting $C = B$ and $C^+ = f(A^+) + B^{\circ\circ}$. Set $X^{/Y} = \mathrm{Spa}(C, C^+)$. We then have a diagram

$$\begin{tikzcd} X \arrow[r, hook, "j"] \arrow[d, "f"'] & X^{/Y} \arrow[dl] \\ Y & \end{tikzcd}$$

where the inclusion j is a pro-constructible immersion. In general this is not an open immersion but it happens to be the case in a lot of "standard" situations. For example, if A and B are topologically of finite type over the non-Archimedean field K and $A^+ = A^0$, $B^+ = B^0$ (i.e. we are working with affinoid adic spaces associated to "classical" affinoid Tate spaces). Then, if $f_1, \dots, f_n \in B^\circ$ generate the image of $\widetilde{A} \to \widetilde{B}$, $X = X^{/Y}\left(\frac{f_1,\dots,f_n}{1}\right)$ and j is an open immersion.

The fact now is that $X^{/Y} \to Y$ is proper and when the preceding immersion is open we have thus constructed a canonical compactification of f by changing B^+. In the classical case of affinoid adic spaces associated to affinoid Tate rigid spaces this compactification does not exist in the world of Tate, it exists only in the more general category of adic spaces. This is an occurence where the power of considering rings of integral elements A^+ more general than the case $A^+ = A^\circ$ shows up in the definition of an adic space.

Remark 2.12.1. — The reason why $\mathrm{Spa}(A, A^\circ)$ is smaller than $\mathrm{Spa}(A, A^+)$ is that in general, in a totally ordered abelian group Γ, an element γ is such that $\mathbb{N}.\gamma$ is bounded does not imply $\gamma \geq 0$ unless Γ has rank 1. For example, for $\Gamma = \mathbb{Z} \times \mathbb{Z}$ with the lexicographic order, $\mathbb{N}.(0, -1) > (-1, 0)$.

Example 2.12.2. — Take $\mathbb{B}^1_K = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K\langle T\rangle)$ the classical 1-dimensional ball over K. It represents the functor $(R, R^+) \mapsto R^+$ on K-affinoid rings. Its canonical compactification $\mathbb{B}^{1,c}_K = \mathrm{Spa}(K\langle T\rangle, \mathcal{O}_K + K^{\circ\circ}\langle T\rangle)$ represents the functor $(R, R^+) \mapsto R^\circ$. This is in fact a "one point compactification",

$$\mathbb{B}^{1,c}_K \setminus \mathbb{B}^1_K = \{x\}$$

with x being the valuation with value group $\mathbb{R} \times \mathbb{Z}$ equipped with the lexicographic order given by

$$v\Big(\big(\sum_{n\geq 0} a_n T^n\big)(x)\Big) = \inf_{n\geq 0}\{(v(a_n), -n)\}.$$

2.13. The Zariski-Riemann-Raynaud point of view

The point of view we develop here on adic spectra is the one due to Raynaud. For anyone who really wants to understand well adic spaces, all points of view are important. This part is new and original, not available in the literature.

2.13.1. The adic spectrum as a Zariski-Riemann space. — Suppose (A, A^+) is Tate Huber pair and let ϖ be a pseudo-uniformizer. Consider the diagram

$$\begin{tikzcd} X_s \arrow[r, hook] \arrow[d, equal] & X \arrow[d, equal] & X_\eta \arrow[l, hook'] \arrow[d, equal] \\ \mathrm{Spec}(A^+/A^{\circ\circ}) \arrow[r, hook] & \mathrm{Spec}(A^+) & \mathrm{Spec}(A). \arrow[l, hook'] \end{tikzcd}$$

Consider the topological space

$$\varprojlim_{X' \longrightarrow X} |X'_s|$$

where the indexing category is the one of proper morphisms $X' \to X$ that are isomorphisms outside ϖ i.e.

$$X'_\eta := X' \times_X X_\eta \xrightarrow{\sim} X_\eta,$$

and $X'_s := X' \times_X X_s$. For such an $X' \to X$, its image is closed and contains the dense subset X_η. We deduce that $|X'| \to |X|$ is surjective and thus $|X'_s| \to |X_s|$ is surjective too. The indexing category is cofiltered. In fact for $X' \to X$ and $X'' \to X$ as before, $X' \times_X X''$ is again in this category. If moreover we have a diagram

$$\begin{tikzcd} X'' \arrow[rr, shift left] \arrow[rr, shift right] \arrow[dr] & & X' \arrow[dl] \\ & X & \end{tikzcd}$$

we can form

$$X''' = X'' \times_{X' \times_X X', \Delta_{X'/X}} X',$$

the equalizer of our two morphisms, that is again in our indexing category. Let us finally remark that, since we can replace $X' \to X$ by the schematical closure of $X'_\eta = X_\eta$ inside X', the subcategory of $X' \to X$ such that X'_η is schematically dense in X' is cofinal. This last category is equivalent to a small category and the projective limit makes sense. This projective limit is then a spectral space equipped with a surjective qc map to $|X_s|$. We will always assume from now on that *X'_η is schematically dense in X'.*

We now define a map

$$\mathrm{Spa}(A, A^+) \longrightarrow \varprojlim_{X' \longrightarrow X} |X'_s|.$$

Let $x \in \mathrm{Spa}(A, A^+)$. It is given by a valuation ring V and a morphism $f : \mathrm{Spec}(V) \to X$ such that $f^*\varpi \in V \setminus \{0\}$. We have $\mathrm{Frac}(V) = V[\frac{1}{f^*\varpi}]$ by continuity of the valuation x and after inverting ϖ the morphism f litts to a morphism $\mathrm{Spec}(\mathrm{Frac}(V)) \to X_\eta = X'_\eta \hookrightarrow X'$. We thus have a diagram

$$\begin{tikzcd} \mathrm{Spec}(\mathrm{Frac}(V)) \arrow[r] \arrow[d, hook] & X' \arrow[d] \\ \mathrm{Spec}(V) \arrow[r] \arrow[ur, dashed] & X. \end{tikzcd}$$

The valuation criterion of properness gives a unique morphism $\mathrm{Spec}(V) \to X'$ as in the preceding diagram. This induces a morphism $|\mathrm{Spec}(V/\varpi)| \to |X'_s|$. The image of x is then defined to be the image of the closed point of $\mathrm{Spec}(V/\varpi)$. When $X' \to X$ varies, this defines an element of the projective limit.

Theorem 2.13.1. — *The preceding map is an homeomorphism*

$$\mathrm{Spa}(A, A^+) \xrightarrow{\sim} \varprojlim_{X' \longrightarrow X} |X'_s|$$

and thus $\mathrm{Spa}(A, A^+)$ *is spectral.*

Proof. — Let us begin with the injectivity. Let v and w be two distinct elements of $\mathrm{Spa}(A, A^+)$. There exists $a, b \in A$ such that $v(a) \geq v(b)$ and $w(a) < w(b)$. Let us choose $n \geq 1$ such that $v(\varpi^n) > w(a)$, which is possible since $w(a) \neq +\infty$. Let

$$\widetilde{X} \to X$$

be the blow-up of the ideal (a, b, ϖ^n). The morphism $\widetilde{X} \to X$ is in our category, $\widetilde{X}_\eta \xrightarrow{\sim} X_\eta$. Let $U \subset X_\eta$ be the open subset where (b, ϖ^n) generates the exceptional divisor. The morphism $\mathrm{Spec}(V) \to \widetilde{X}$ defined by v has its image in U. Thus, the image of the closed point of $\mathrm{Spec}(V)$ lies in $U \cap \widetilde{X}_s$. The image of the morphism $\mathrm{Spec}(W) \to \widetilde{X}$ defined by w is not contained in U. Since all points of $\mathrm{Spec}(W)$ are generalizations of its closed point, the image of the closed point of $\mathrm{Spec}(W)$ lies in $\widetilde{X}_s \setminus U$. This prove the injectivity statement.

For the surjectivity, let $\mathfrak{X}$ be the ringed space

$$\varprojlim_{X' \to X} X'.$$

Let $x \in \mathfrak{X}_s$ and consider the local ring $\mathcal{O}_{\mathfrak{X},x}$. For $I \subset \mathcal{O}_{\mathfrak{X},x}$ a finite type ideal that contains a power of ϖ, we can find some $X' \to X$ in our category of modifications, an affine open subset $U \subset X'$ such that x maps to an element of U, and a finite type ideal $J \subset \mathcal{O}_{X'}(U)$ that contains a power of ϖ that generates I when pulled back to $\mathcal{O}_{\mathfrak{X},x}$. We can extend J to a finite type quasi-coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{X'}$ such that $\varpi^n \mathcal{O}_{X'} \subset \mathcal{J}$ when $n \gg 0$. Considering the blow up of $\mathcal{J}$ and pulling back the situation to $\mathfrak{X}$ we deduce that $\mathcal{O}_{\mathfrak{X},x}$ satisfies the hypothesis of Lemma 2.13.3. Thus,

$$V = \mathcal{O}_{\mathfrak{X},x} / \cap_{n \geq 0} (\varpi^n),$$

which is non-zero since $x \in \mathfrak{X}_s$, is a valuation ring in which $v(\varpi^n) \underset{n \to +\infty}{\longrightarrow} +\infty$. The morphism $A^+ \to V$ defines thus a point of $\mathrm{Spa}(A, A^+)$. This proves that the map of the statement is surjective.

We now have to check that this is an homemorphism. For this, consider the subcategory of the indexing category of modifications of X in our projective limit formed by the admissible blow ups. By definition, those are the blow ups $\widetilde{X} \to X$ of a finite type ideal I of A^+ containing a power of ϖ. They have the following property:

- If $\widetilde{X} \to X$ and $\widetilde{\widetilde{X}} \to X$ are two admissible blow ups then $\widetilde{X} \times_X \widetilde{\widetilde{X}} \to X$ is an admissible blow up.
- If $\mathcal{I} \subset \mathcal{O}_{\widetilde{X}}$ is a finite type quasi-coherent sheaf of ideals with $\widetilde{X} \to X$ an admissible blow up then the blow up of $\widetilde{X}$, $\widetilde{\widetilde{X}} \to \widetilde{X}$ is such that the composite $\widetilde{\widetilde{X}} \to \widetilde{X} \to X$ is an admissible blow up.

The preceding proof works as before to prove that

$$\mathrm{Spa}(A, A^+) \longrightarrow \varprojlim_{\widetilde{X} \to X} |\widetilde{X}_s|$$

is bijective, where here is the indexing category is the one of admissible blow ups. The quasi-compact continuous map of spectral spaces

$$\varprojlim_{X'\to X} |X'_s| \longrightarrow \varprojlim_{\widetilde{X}\to X} |\widetilde{X}_s|$$

is thus bijective. This morphism is moreover specializing since if we have a diagram

$$\begin{tikzcd} X' \arrow[rr] \arrow[dr] & & \widetilde{X} \arrow[dl] \\ & X & \end{tikzcd}$$

then the morphism $X' \to \widetilde{X}$ is proper. This is thus an homeomorphism.

Thus, it suffices to prove that

$$\mathrm{Spa}(A, A^+) \to \varprojlim_{\widetilde{X}\to X} |\widetilde{X}_s|$$

is an homeomorphism. But the exceptional divisors of our blow ups are very ample. If $\widetilde{X} \to X$ is the admissible blow up of the ideal $I \subset A^+$ then a basis of the topology of $\widetilde{X}$ is the $D^+(f)$ where $f \in I^n$. For $f \in I^n$, the morphism $\mathrm{Spec}(V) \to \mathrm{Spec}(A^+)$ factorizes through $D^+(f)$ if and only if the image of I^n in V is generated by f. This means exactly that for all $g \in I^n$, $v(g) \geq v(f)$, which means that v is in the rational subset $\mathrm{Spa}(A, A^+)\langle \frac{I^n}{f} \rangle$. The result is easily deduced. □

Remark 2.13.2. — The proof gives in fact that the three projective limits $\varprojlim_{X'\to X} |X'_s|$ where the modification $X' \to X$ is either taken proper, projective or an admissible blow up are homeomorphic. Nevertheless this can be proven directly "in finite level" without taking the projective limit using [22, Corollaire 1.4] for the projective/Noetherian case and [20, Theorem 2.11] for the general case that proves that in fact the category of admissible blow-ups is cofinal among the preceding proper modifications.

***Lemma 2.13.3* (see [29, Section 3]).** — *Let V be a local ring equipped with some element ϖ in its maximal ideal satisfying: any finitely generated ideal of V that contains a power of ϖ is principal generated by a regular element i.e. is a Cartier divisor. Then the separation of V for the ϖ-adic topology, $V/\cap_{n\geq 0}(\varpi^n)$, if non-zero, is a valuation ring in which any non-zero finite type ideal contains a power of ϖ.*

Proof. — The hypothesis implies that ϖ is a regular element. Now for any $a \in V$ and $n \geq 0$, if we write $(a, \varpi^n) = (\alpha)$ with α regular, then we can find $\lambda, \mu, u, v \in V$ such that

$$\begin{cases} \alpha = ua + v\varpi^n \\ a = \lambda\alpha \\ \varpi^n = \mu\alpha. \end{cases}$$

We thus have $\alpha = (u\lambda + v\mu)\alpha$. Since α is regular we deduce $u\lambda + v\mu = 1$. Since V is local this implies either $u, \lambda \in V^\times$ or $v, \mu \in V^\times$. Thus, either $(a, \varpi^n) = (a)$ or $(a, \varpi^n) = \varpi^n$. From this we deduce that if $a \notin \cap_{n\geq 0}(\varpi^n)$ then there exists $n \geq 1$ such that $\varpi^n \in (a)$ and a is regular. Thus, any non-zero finitely generated ideal of $V/\cap_{n\geq 0}(\varpi^n)$ is principal.

Now, if $a, b \notin \cap_{n\geq 0}(\varpi^n)$, write $\varpi^k = \lambda a$ and $\varpi^l = \mu b$ for $k, l \gg 0$ and $\lambda, \mu \in V$. If $ab \in \cap_{n\geq 0}(\varpi^n)$ then we can write $ab = c\varpi^{k+l+1}$ with $c \in V$. Using the regularity of ϖ, we deduce that $\lambda\mu c\varpi = 1$, which is a contradiction. Thus, $V/\cap_{n\geq 0}(\varpi^n)$ is an integral domain. □

Corollary 2.13.4. — *For any Huber pair (A, A^+), $\mathrm{Spa}(A, A^+)_a$ and $\mathrm{Spa}(A, A^+)$ are spectral spaces.*

Proof. — Since one can write

$$\mathrm{Spa}(A, A^+)_a = \cup_{i=1}^n U_i$$

where for all i, j, U_i and $U_i \cap U_j$ are adic spectra of affinoid Tate rings, we deduce that $\mathrm{Spa}(A, A^+)_a$ is spectral. Now, one verifies that if X is a (T0) topological space that admits an open subset $U \subset X$ such that U and $X \setminus U$ are spectral spaces, then X is a spectral space. One thus has to verify that

$$\mathrm{Spa}(A/AA^{\circ\circ}, (A/AA^{\circ\circ})^+)$$

is spectral. We are thus reduced to proving that $\mathrm{Spa}(R, R^+)$ is spectral for R discrete. Since $\mathrm{Spa}(R, R^+)$ is pro-constructible in $\mathrm{Spv}(R) = \mathrm{Spa}(R, \mathbb{Z}.1)$, it suffices to verify that $\mathrm{Spv}(R)$ is spectral. This is [36, Proposition 2.2] whose proof is elementary. □

Corollary 2.13.5. — *For (A, A^+) a Huber pair, one has*

$$A^+ = \{a \in A \mid \forall x \in \mathrm{Spa}(A, A^+),\ |a(x)| \leq 1\}.$$

Proof. — Let us first suppose that A is Tate. Let $f : X' \to X$ be as in the proof of Theorem 2.13.1. Since f is proper of finite presentation $f_*\mathcal{O}_{X'}$ is a quasicoherent $\mathcal{O}_X$-module of finite type. Let us write $f_*\mathcal{O}_{X'} = \widetilde{B}$ with B a finite A^+-algebra. Since f is an isomorphism outside the schematically dense open set $D(\varpi)$, we have $A^+ \subset B \subset A$. Since A^+ is integrally closed in A we deduce that $B = A^+$.

Let

$$\mathfrak{X} = \varprojlim_{X' \to X} X'$$

as a ringed space. For $x \in |\mathfrak{X}|$ that corresponds to the valuation $v_x \in \mathrm{Spa}(A, A^+)$, an element $a \in A = \mathcal{O}_{\mathfrak{X},x}[\frac{1}{\varpi}]$ satisfies $v_x(a) \geq 0$ if and only its image in

$$\mathcal{O}_{\mathfrak{X},x}[\tfrac{1}{\varpi}] / \cap_{n\geq 0} \mathcal{O}_{\mathfrak{X},x}\varpi^n$$

is in

$$\mathcal{O}_{\mathfrak{X},x} / \cap_{n\geq 0} \mathcal{O}_{\mathfrak{X},x}\varpi^n.$$

This is equivalent to saying that $a \in \mathcal{O}_{\mathfrak{X},x}$. Thus, an element $a \in A$ satisfies $v_x(a) \geq 0$ if and only if there exists some $X' \to X$ and $U \subset X'$ whose pullback to $\mathfrak{X}$ is a neighborhood of x such that $a_{|U_\eta}$ extends to an element of $\mathcal{O}_{X'}(U)$. Using the quasi-compacity of $|\mathfrak{X}|$ we deduce that if a satisfies

$$\forall x \in \mathrm{Spa}(A, A^+),\ |a(x)| \leq 1$$

then there exists some $X' \to X$ such that $a \in \Gamma(X', \mathcal{O}_{X'})$. This finishes the proof when A is a Tate ring.

When the ring A is discrete the statement is immediately reduced to the fact that if R is a normal integral domain then R is the intersection of the valuation rings of $\mathrm{Frac}(R)$ containing R.

Let now A be any Huber ring. Let A_0 be a ring of definition contained in A^+ and I an ideal of finite type of A_0 defining its topology. Let $X = \mathrm{Spec}(A^+)$ and

$$\pi : \widetilde{X} \to X$$

be the blow up of the ideal A^+I. Let

$$\sigma : \overline{\widetilde{X}} \to \widetilde{X}$$

be the normalization of $\widetilde{X}$ inside the complementary of the exceptional Cartier divisor, that is to say $X \setminus V(A^+I)$. There is a diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi", "\substack{\text{proper}\\ \text{finite presentation}}"'] & & \overline{\widetilde{X}} \arrow[ll, "\sigma"', "\text{integral}"] & \\ X & & X \setminus V(A^+I) \arrow[ll, hook', "open"] \arrow[u, hook, "\substack{\text{open}\\ \text{schematically dense}}"'] \arrow[r, "f"] & \mathrm{Spec}(A). \end{tikzcd}$$

If $a \in A$ satisfies $|a(x)| \leq 1$ for all $x \in \mathrm{Spa}(A, A^+)_a$ then according to the Tate ring case treated before,

$$f^*a \in \Gamma(\overline{\widetilde{X}}, \mathcal{O}_{\overline{\widetilde{X}}}).$$

But now, since σ is integral $\sigma_*\mathcal{O}_{\overline{\widetilde{X}}}$ is a filtered colimit of quasi-coherent $\mathcal{O}_{\widetilde{X}}$-algebras of finite type. Since π is proper of finite presentation we deduce that $(\pi\sigma)_*\mathcal{O}_{\overline{\widetilde{X}}}$ is an integral quasi-coherent sheaf of $\mathcal{O}_X$-algebras. From this we deduce that f^*a lies in the normalization of X inside $X \setminus V(A^+I)$. This means concretely that there exists $b_1, \dots, b_n \in A^+$ such that

$$f^*(a^n + b_1a^{n-1} + \cdots + b_{n-1}a + b_n) = 0.$$

We deduce from this that $a^n + b_1a^{n-1} + \cdots + b_{n-1}a + b_n$ is killed by a power of I. Now, the case of a discrete Huber ring treated before shows that the image of A in A/AI lies in the integral closure of the image of A^+ in A/AI. There exists thus $c_1, \dots, c_m \in A^+$ such that

$$a^m + c_1a^{m-1} + \cdots + c_{m-1}a + c_m \in AI.$$

Thus, for $k \gg 0$,

$$(a^n + b_1a^{n-1} + \cdots + b_{n-1}a + b_n).(a^m + c_1a^{m-1} + \cdots + c_{m-1}a + c_m)^k = 0.$$

From this we deduce that a is integral over A^+ and thus $a \in A^+$. □

2.13.2. The specialization map and applications. — Let (A, A^+) be a Tate ring and ϖ a pseudo-uniformizer. To any point $x \in \mathrm{Spa}(A, A^+)$ there is associated a morphism $\mathrm{Spec}(V_x) \to \mathrm{Spec}(A^+)$ where V_x is a valuation ring. We note

$$\mathrm{sp}(x) \in \mathrm{Spec}(A^+/\varpi)$$

the image of the closed point of $\mathrm{Spec}(V_x)$.

Proposition 2.13.6. — *The specialization map* $\mathrm{sp} : \mathrm{Spa}(A, A^+) \to \mathrm{Spec}(A^+/\varpi)$ *is a surjective specializing quasi-compact continuous map between spectral spaces.*

Proof. — For $f \in A^+$ with image $\bar{f}$ in A^+/ϖ,

$$\mathrm{sp}^{-1}(D(\bar{f})) = \{x \in \mathrm{Spa}(A, A^+) \mid |f(x)| = 1\}.$$

We deduce that sp is quasi-compact continuous.

The surjectivity is deduced from Theorem 2.13.1 since the transition maps in the projective limit are surjective. Since the transition map in the projective limit are specializing we deduce that sp is specializing. □

Corollary 2.13.7. — *Let* (A, A^+) *be a Huber pair. Then,* $\mathrm{Spa}(A, A^+)_a = \emptyset$ *if and only if* $A/\overline{\{0\}}$ *is discrete.*

Proof. — By surjectivity of the specialization map, if A is Tate then $\mathrm{Spa}(A,A^+)=\emptyset$ if and only if

$$\mathrm{Spec}(A^+/\varpi)=\emptyset.$$

But this is equivalent to $\varpi\in(A^+)^\times$. Since $A^+\subset A^\circ$ this implies $\varpi^{-1}\in A^\circ$. Now, if A_0 is a ring of definition of A, there exists $N\geq 0$ such that for all $n\geq 0$, $\varpi^{-n}\in\varpi^{-N}A_0$. Taking $n=N+1$ we find that $\varpi\in A_0^\times$. Thus, $A_0=A=\overline{\{0\}}$.

In general, if $I=(f_1,\dots,f_n)$ is an ideal of a ring of definition A_0 defining its topology, let

$$\pi:\widetilde{X}\to X=\mathrm{Spec}(A_0)$$

be the blow up of I. If R is a ring and $f\in R$ a regular element, the f-adic completion of R is 0 if and only if $D(f)\subset\mathrm{Spec}(R)$ is closed. Thus, if $\mathrm{Spa}(A,A^+)_a=\emptyset$ then

$$\widetilde{X}\setminus\pi^{-1}(V(I))$$

is closed in $\widetilde{X}$ according to the preceding Tate ring case. Since π is proper we deduce that $\mathrm{Spec}(A_0)\setminus V(I)$ is closed in $\mathrm{Spec}(A_0)$. This is equivalent to saying that there exists an idempotent $a\in A_0$ such that $V(I)=V(e)$ or equivalently $\sqrt{I}=\sqrt{(e)}$. Thus, the topology of A_0 is the e-adic one. Since $e^2=e$, $\overline{\{0\}}=\cap_{n\geq 1}(e^n)=(e)$ is open and we deduce $A/\overline{\{0\}}$ is discrete. □

Corollary 2.13.8. — *Let (A,A^+) be a Huber pair. Then,* $\mathrm{Spa}(A,A^+)=\emptyset$ *if and only if* $\overline{\{0\}}=A$.

Proof. — Suppose $\mathrm{Spa}(A,A^+)=\emptyset$. We can suppose A is separated. According to Corollary 2.13.7, A is discrete. If $A\neq 0$ we can choose a maximal ideal $\mathfrak{m}$ of A. We can moreover choose a valuation ring V of $A/\mathfrak{m}$ containing $A^+/A^+\cap\mathfrak{m}$. Then, $(A,A^+)\to(A/\mathfrak{m},V)$ defines a point in $\mathrm{Spa}(A,A^+)$. We thus deduce that $A=0$. □

Let us point the following immediate corollary.

Corollary 2.13.9. — *For (A,A^+) a Huber pair, the image of the support map* $\mathrm{supp}:\mathrm{Spa}(A,A^+)\to\mathrm{Spec}(A)$ *is the set of closed prime ideals of A.*

Corollary 2.13.10. — *Let (A,A^+) be a complete Huber pair and $a\in A$. One has $a\in A^\times$ if and only if for all $x\in\mathrm{Spa}(A,A^+)$, $|a(x)|\neq 0$.*

Proof. — Let $a\in A$ satisfy $|a(x)|\neq 0$ for all $x\in\mathrm{Spa}(A,A^+)$. Since A is complete, $1+A^{\circ\circ}\subset A^\times$ that is thus open inside A. We deduce that any maximal ideal of A is closed. Let $\mathfrak{m}$ be such a maximal ideal. Let $(A/\mathfrak{m})^+$ be the integral closure of the image of A^+ in $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ is separated, according to Corollary 2.13.8, $\mathrm{Spa}(A/\mathfrak{m},(A/\mathfrak{m})^+)\neq\emptyset$. Using the map

$$\mathrm{Spa}(A/\mathfrak{m},(A/\mathfrak{m})^+)\to\mathrm{Spa}(A,A^+)$$

one can thus find $x\in\mathrm{Spa}(A,A^+)$ such that $\mathrm{supp}(x)=\mathfrak{m}$ and thus $a\notin\mathfrak{m}$ since $|a(x)|\neq 0$. □

2.14. Other structure sheaves on the adic spectrum: Henselian rigid spaces

2.14.1. Henselianization. —

2.14.1.1. *Background on Henselian pairs.* — Let (A, I) be a pair where A is a ring and I an ideal of A. We note $X = \mathrm{Spec}(A)$ and $Y = V(I) \hookrightarrow X$. Recall that we say that (A, I) is Henselian if

$$I \subset \mathrm{Rad}(A)$$

i.e. the only neighborhood of Y in X is X itself, and one of the following equivalent properties is satisfied:

1. For any finite A-algebra B,
$$\big\{\text{open/closed subsets of } \mathrm{Spec}(B)\big\} \xrightarrow{\sim} \big\{\text{open/closed subsets of } \mathrm{Spec}(B/IB)\big\}.$$
2. For any $P \in A[T]$ unitary and $x \in A/I$ satisfying $P(x) = 0$, $P'(x) \in (A/I)^\times$, there exists $a \in A$ such that $a \equiv x$ modulo I and $P(a) = 0$.
3. For any $U \to X$ étale with U affine, the map
$$\big\{\text{sections of } U \to X\big\} \longrightarrow \big\{\text{sections of } U \times_X Y \to Y\big\}$$
is surjective.

If (A, I) is Henselian then one has the following additional property: reduction modulo I induces a bijection

$$\big\{\text{finite étale } A\text{-algebras}\big\} \xrightarrow{\sim} \big\{\text{finite étale } A/I\text{-algebras}\big\}.$$

2.14.1.2. *Zariski localization around a closed subset.* — Let (A, I) be a pair as before. Before looking at étale neighborhoods of Y in X let us look at the Zaiski neighborhoods. For this let $S = 1 + I$, a multiplicative subset of A. One has

$$S^{-1}I \subset \mathrm{Rad}(S^{-1}A)$$

since for $a, f \in I$, $1 + \frac{a}{1+f} = \frac{1+a+f}{1+f} \in (S^{-1}A)^\times$. We deduce from this that the open subsets $D(f), f \in S$, form a basis of neighborhoods of Y in X. We have

$$\begin{aligned} S^{-1}A &= \varinjlim_{f \in S} \Gamma(D(f), \mathcal{O}_X) \\ &= \varinjlim_{\substack{U \supset Y \\ \text{open}}} \Gamma(U, \mathcal{O}_X) \end{aligned}$$

and $\mathrm{Spec}(S^{-1}A) = \bigcap\limits_{U \supset Y} U = Y^{gen}$. We thus have

$$i_* i^{-1} \mathcal{O}_X = \mathscr{S}^{-1} \mathcal{O}_X$$

with $\mathscr{S} = 1 + \widetilde{I}$. For the next lemma we follow [51, Proposition B.1.4]. The statement is simple but the proof is slightly more complicated than what we could expect.

Lemma 2.14.1. — *We have* $\Gamma(Y, i^{-1}\mathcal{O}_X) = S^{-1}A$ *with* $S = 1 + I$.

Proof. — Suppose we have a finite covering $Y = \cup_i D(\bar{g}_i)$ with $g_i \in A$ and $\bar{g}_i$ is its reduction modulo I. Let $S_i = 1 + I[\frac{1}{g_i}]$ and $S_{ij} = 1 + I[\frac{1}{g_i g_j}]$. Suppose given elements $x_i \in S_i^{-1}A[\frac{1}{g_i}]$ such that x_i and x_j have the same image in $S_{ij}^{-1}A[\frac{1}{g_i g_j}]$.

For any index i, we can write x_i under the form

$$x_i = \frac{y_i}{g_i^{n_i} + a_i}$$

with $y_i \in A, a_i \in I$ and $n_i \in \mathbb{N}$ that we can chose as big as we want. We can thus find a collection of elements $(y_i)_i$ of elements of A, $(a_i)_i$ of I and $n \in \mathbb{N}$ such that for all i,

$$x_i = \frac{y_i}{g_i^n + a_i}.$$

Suppose now that A is an integral domain. Since x_i and x_j have the same image in $S_{ij}^{-1}A[\frac{1}{g_ig_j}] \subset \mathrm{Frac}(A)$, we have for all indices i and j,

$$(g_i^n + a_i)y_j = (g_j^n + a_j)y_i.$$

Now, since $Y = \cup_i D(\bar{g}_i)$, we can find a collection $(\lambda_i)_i$ of elements of A such that

$$\sum_i \lambda_i g_i^n \in 1 + I.$$

Let us note

$$\mu = \sum_i \lambda_i(g_i^n + a_i) \in 1 + I.$$

and

$$\xi = \sum_i \lambda_i y_i \in A.$$

We have for all j,

$$\xi(g_j^n + a_j) = \mu y_i.$$

We deduce that

$$\mu^{-1}\xi \in S^{-1}A$$

has image x_i in $S_i^{-1}A[\frac{1}{g_i}]$ for all indices i. This proves the result when A is an integral domain.

Now, for any A, the fact that x_i and x_j have the same image in $S_{ij}^{-1}A[\frac{1}{g_ig_j}]$ is translated into the existence of an integer m and a collection of elements $(b_{ij})_{i,j}$ of A such that the following relations

$$(g_i^m g_j^m + b_{ij})(g_i^n + a_i)y_j = (g_i^m g_j^m + b_{ij})(g_j^n + a_j)y_i$$

are satisfied in A for all indices i, j. We can now replace A by the sub-$\mathbb{Z}$-algebra generated by $(g_i)_i, (b_{ij})_{i,j}, (y_i)_i, (a_i)_i$ and I by its intersection with this sub-ring and suppose that A is Noetherian. Now, since A is Noetherian, it has a finite filtration by ideals

$$(0) = J_r \subset J_{r-1} \subset \cdots \subset J_0 = A$$

with $J_k/J_{k+1} \simeq A/\mathfrak{p}_k$, $0 \leq k \leq r-1$, where $\mathfrak{p}_k$ is a prime ideal of A. The case when A is an integral domain shows that if M is an A-module isomorphic to $A/\mathfrak{p}$. Then,

$$S^{-1}M \xrightarrow{\sim} \Gamma(Y, i^{-1}\widetilde{M}).$$

Now, if we have an exact sequence of A-modules

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

this gives rise to a diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & S^{-1}M' \arrow[r] \arrow[d] & S^{-1}M \arrow[r] \arrow[d] & S^{-1}M'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \Gamma(Y, i^{-1}\widetilde{M'}) \arrow[r] & \Gamma(Y, i^{-1}\widetilde{M}) \arrow[r] & \Gamma(Y, i^{-1}\widetilde{M''}). &
\end{tikzcd}$$

We deduce the result for $(A/J_k)_{0\leq k\leq r}$ by induction on k using the snake lemma. □

2.14.1.3. *Étale localization around a closed subset.* —

Definition 2.14.2. — For (A, I) a couple we define

$$A^h = \Gamma(Y, i^{-1}\mathcal{O}_{X_{\text{ét}}}),$$

the henselization of (A, I), where $i : Y = V(I) \hookrightarrow \operatorname{Spec}(A) = X$.

One has, by definition,

$$\Gamma(Y, {}^{ps}i^{-1}\mathcal{O}_{X_{\text{ét}}}) = \varinjlim_{\mathcal{C}} \mathcal{O}(U)$$

where ${}^{ps}i^{-1}$ means the reciprocal image as a presheaf on $Y_{\text{ét}}$ and where $\mathcal{C}$ is the category of diagrams

$$\begin{tikzcd} & U \arrow[d, "\pi\ \text{étale}"] \\ Y \arrow[ur, "s"] \arrow[r, hook] & X. \end{tikzcd}$$

Let us rewrite this diagram as

$$\begin{tikzcd} U \times_X Y \arrow[r, hook] \arrow[d, "\text{étale}"] \arrow[dr, phantom, "\square"] & U \arrow[d, "\pi", "\text{étale}"'] \\ Y \arrow[u, bend left, "s"] \arrow[r, hook] & X. \end{tikzcd}$$

Since $U \times_X Y \to Y$ is étale the section s is an open embedding. Thus, up to replacing U by an open subscheme we can suppose that in our colimit we have an isomorphism

$$U \times_X Y \xrightarrow{\sim} Y.$$

We change the definition of the category $\mathcal{C}$ accordingly. Now, the section s is a closed immersion. One has to be careful that, à priori, in the preceding $U \to X$ may not be separated in a neighborhood of Y. Nevertheless, let us recall that we want to compute the sections on Y of the sheaf associated to the presheaf ${}^{ps}p^{-1}\mathcal{O}_{X_{\text{ét}}}$. We are thus allowed to Zariski localize on Y. For each point y of Y we can choose an affine neighborhood U_y of $s(y)$ in U. There is then a diagram

$$\begin{tikzcd} & & \coprod_{y\in Y} U_y \arrow[r] & U \arrow[d, "\pi"] \\ \coprod_{y\in Y} s^{-1}(U_y) \arrow[urr] \arrow[rr, two heads, "\text{cover}"] & & Y \arrow[ur, "s"] \arrow[r, hook] & X. \end{tikzcd}$$

It is thus natural to add the following condition in the definition of $\mathcal{C}$: U is affine. But now we can replace U by

$$\varprojlim_{s(Y)\subset V\subset U} V$$

where V goes through the set of neighborhoods of $s(Y)$ as in Section 2.14.1.2. We have reached the following definition.

Definition 2.14.3 **(**[45, Definition 5-Chapter XI]**).** — A local-étale neighborhood of Y in X is a morphism $U \to X$ with U of the form $\operatorname{Spec}(S^{-1}B)$ where $\operatorname{Spec}(B) \to X$ is étale, $\operatorname{Spec}(B) \times_X Y \xrightarrow{\sim} Y$ and $S = 1 + BI$.

Thus, for $U \to X$ a local-étale neighborhood of Y in X, $U \to X$ is only pro-étale and not étale in general. The advantage of the Zariski localization we did on U is the following.

Lemma 2.14.4. — *The category of local-étale neighborhoods of Y in X is equivalent to a poset i.e. there is at most one morphism between two objects (and it has a collection of objects that from a set such that any object in the category is isomorphic to one of those objects).*

Proof. — Let $U \to X$ and $U' \to X$ be two local-étale neighborhoods of Y. Suppose $U = \varprojlim_{Y \subset W \subset V} W$ and $U' = \varprojlim_{Y \subset W' \subset V'} W'$ with $V \to X$ and $V' \to X$ affine étale satisfying $V \times_X Y \xrightarrow{\sim} Y$ and $V' \times_X Y \xrightarrow{\sim} Y$. Since $V \to X$ and $V' \to X$ are morphisms of finite presentation one has

$$\mathrm{Hom}_X(U', U) = \varprojlim_{Y \subset W \subset V} \varinjlim_{Y \subset W' \subset V'} \mathrm{Hom}_X(W', W).$$

Consider now two morphisms of X-schemes

$$W' \underset{g}{\overset{f}{\rightrightarrows}} W$$

This corresponds to a morphism of étale schemes over X, $W' \to W \times_X W$. When restricted to $Y \subset W'$ this factorizes through $\Delta_{W/X}$ that is open in $W \times_X W$ since $W \to X$ is étale. We deduce that up to shrinking W' one has $f = g$ and we deduce the result. $\square$

We can now prove the main result of this section.

Proposition 2.14.5. — *We have*

$$\Gamma(Y, i^{-1}\mathcal{O}_{X_{\text{ét}}}) = \varinjlim_{\substack{U \to X \\ \textit{local-étale nbd of } Y}} \mathcal{O}(U).$$

Proof. — Let $\mathscr{F}$ be the presheaf on the principal open subsets of Y_{Zar} defined by

$$\mathscr{F}(V) = \varinjlim_{\substack{U \to W \\ \text{local-étale nbd of } V}} \mathcal{O}(U)$$

where W is any principal open subset of X such that $W \cap Y = V$. One clearly has using some finite presentation arguments,

$$\forall y \in Y, \quad \mathscr{F}_y = \mathcal{O}^h_{X,y},$$

the usual henselization of the local ring $\mathcal{O}_{X,y}$ that is equal to

$$(\nu_* i^{-1}\mathcal{O}_{X_{\text{ét}}})_y$$

where $\nu : X_{\text{ét}} \to X_{Zar}$ is the projection. There is in fact an evident natural morphism of presheaves

$$\mathscr{F} \longrightarrow \nu_* i^{-1}\mathcal{O}_{X_{\text{ét}}}.$$

If we can prove that $\mathscr{F}$ is a Zariski sheaf this will prove that $\mathscr{F} = \nu_* i^{-1}\mathcal{O}_{X_{\text{ét}}}$. Using some finite presentation arguments this is reduced to the case when the ring A is a finite type $\mathbb{Z}$-algebra and thus in particular excellent. Let us now remark that since A is Noetherian,

$$\mathscr{F} \subset \mathcal{O}_{\widehat{X}_{/Y}}$$

as presheaves on Y_{Zar}. This is in fact a consequence of Krull's intersection theorem: if R is a Noetherian ring, J an ideal of R, $S = 1 + J$, $\widehat{R}$ the J-adic completion of R then $S^{-1}R \subset \widehat{R}$. We can now use [45, Corollaire 1-Section 3-Chapitre XI]. Let $B = \overline{A}^{\widehat{A}}$ (integral closure). Note $\overline{X} = \mathrm{Spec}(B)$ and $\overline{Y} = V(B \cap I\widehat{A})$. We note $\bar{i} : \overline{Y} \hookrightarrow \overline{X}$ and $\pi : \overline{Y} \to \overline{X}$. Let $Y = \cup_i D(\bar{g}_i)$ be a finite cover of Y by principal open subsets where $g_i \in A$ and $\bar{g}_i$ is its reduction modulo I. Let $(x_i)_i \in \check{H}^0\big((D(\bar{g}_i))_i, \mathscr{F}\big)$ with $x_i \in \mathscr{F}(D(\bar{g}_i))$. Since $\mathcal{O}_{\widehat{X}_{/Y}}$ is a sheaf it gives rise to an element $x \in \widehat{A}$ such that for all index i, $x_i = x$ in $\Gamma\big(D(\bar{g}_i), \mathcal{O}_{\widehat{X}_{/Y}}\big)$. Now we have if

$$C_i = S_i^{-1}\overline{A[\tfrac{1}{g_i}]}^{\widehat{A}\langle \frac{1}{g_i}\rangle}$$

where $S_i = 1 + \big(\overline{A[\frac{1}{g_i}]}^{\widehat{A}\langle \frac{1}{g_i}\rangle} \cap I\widehat{A}\langle \frac{1}{g_i}\rangle\big)$,

$$x_i \in C_i \cap \mathrm{Im}(\widehat{A} \to \widehat{A}\langle \tfrac{1}{g_i}\rangle).$$

This implies that

$$x \in H^0(Y, \pi_* \bar{i}^{-1} \mathcal{O}_{\overline{X}})$$

that is equal to $\mathscr{F}(Y)$ according to Lemma 2.14.1. □

2.14.2. Zariskian and Henselian structure sheaves. —

2.15. Étale morphisms of Noetherian analytic adic spaces

To explain the definition of an étale morphism of perfectoid spaces we need to explain first a basic result by Huber about the structure of étale morphisms of Noeherian analytic adic spaces.

2.15.1. Definition. — Here we work in the framework of what we call *Noetherian analytic adic spaces*. This means that we consider adic spaces that are locally the spectrum of a Tate strongly Noetherian affinoid ring. This contains as a particular case the "classical" Tate quasi-separated rigid spaces.

There are different equivalent definitions of étale morphisms of Noetherian analytic adic spaces. For a morphism of schemes $f : X \to Y$ we have the following equivalent definitions of an étale morphism:

1. f is flat locally of finite presentation and unramified in the sens that for all $x \in X$,
   - $\mathrm{m}_{f(x)} \mathcal{O}_{Y,f(x)} = \mathfrak{m}_x \subset \mathcal{O}_{X,x}$,
   - the extension $k(x)|k(f(x))$ is finite degree separable.
2. f is flat locally of finite presentation, locally quasi-finite with reduced geometric fibers.
3. f is locally of finite presentation and formally étale.
4. If $U = \mathrm{Spec}(B)$ is an affine open subset of X satisfying $f(U) \subset V = \mathrm{Spec}(A)$ an affine open subset of Y then, $f^* : A \to B$ is such that

   $$B \simeq A[X_1, \dots, X_n]/(f_1, \dots, f_n)$$

   for some $n \geq 1$ and $f_1, \dots, f_n \in A[X_1, \dots, X_n]$ such that if

   $$J = \det \left(\frac{\partial f_i}{\partial X_j} \right)_{1 \leq i,j \leq n} \in A[X_1, \dots, X_n]$$

   it satisfies

   $$J \bmod (f_1, \dots, f_n) \in \big(A[X_1, \dots, X_n]/(f_1, \dots, f_n) \big)^{\times}.$$

According to Huber ([38, Section 1.7]), such a type of definition extends to the case of Noetherian analytic adic spaces. More precisely, a morphism of Noetherian analytic adic spaces $f : X \to Y$ is called locally of finite type if locally on X and Y it is of the form $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ with

1. (A, A^+) and (B, B^+) complete Tate strongly Noetherian affinoid rings
2. (B, B^+) is topologically of finite type over (A, A^+) in the sens that there exists a surjection

   $$A\langle X_1, \dots, X_n \rangle \longrightarrow B$$

   such that B^+ is the integral closure of the image of $A^+\langle X_1, \dots, X_n \rangle$.

This is a well behaved definition according to the following result.

Proposition 2.15.1 **([35, Satz 3.8.15]).** — *Suppose (A, A^+) and (B, B^+) are complete Tate strongly Noetherian rings and $f : \mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ is locally of finite type. Then, (B, B^+) is of topologically finite type over (A, A^+) in the sense that there exists a surjection $A\langle X_1, \dots, X_n \rangle \to B$ such that B^+ is the integral closure of the image of $A^+\langle X_1, \dots, X_n \rangle$.*

We then have the following result by Huber.

Proposition 2.15.2 **([38, Section 1.7]).** — *For a locally of finite type morphism $f : X \to Y$ between Noetherian analytic adic spaces the following are equivalent:*

1. *f is locally quasi-finite in the sens that for all $y \in Y$, the topological space $f^{-1}(y)$ is locally discrete on X and for all $x \in X$ the morphism of local rings $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat unramified (i.e. $\mathfrak{m}_{f(x)}\mathcal{O}_{X,x} = \mathfrak{m}_x$ and $k(x)|k(f(x))$ is finite degree separable),*
2. *f is formally étale in the sens that for all (R, R^+) a complete Tate strongly Noetherian affinoid ring equipped with*
   - *a square zero ideal $I \subset R$,*
   - *a morphism $\mathrm{Spa}(R, R^+) \to Y$,*

   one has
$$X(R, R^+) \xrightarrow{\sim} X(R/I, (R/I)^+).$$
3. *If $f(U) \subset V$ with $U = \mathrm{Spa}(B, B^+)$ and $V = \mathrm{Spa}(A, A^+)$, with A and B strongly Noetherian complete Tate rings, one can write $B = A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_n)$ with B^+ the integral closure of the image of $A^+\langle X_1, \dots, X_n\rangle$ and such that the image of*
$$J = \det\Big(\frac{\partial f_i}{\partial X_j}\Big)_{1\le i,j\le n} \in A\langle X_1, \dots, X_n\rangle$$
   in B lies in $B^\times$.

2.15.2. Algebraization of étale morphisms à la Elkik. — The following result is a particular case of a deeper result by Elkik ([25, Théorème 7] in the Noetherian case, see [3, Theorem 1.16.23] for some non-Noetherian case) that can be proven directly in an elementary manner as Huber does in [38, Proposition 1.7.1 (iii)].

Proposition 2.15.3. — *Let (A, A^+) and (B, B^+) be strongly Noetherian complete affinoid Tate rings. Let $\mathrm{Spa}(B, B^+) \to \mathrm{Spa}(A, A^+)$ be an étale morphism. There exists $n \geq 1$ and*
$$f_1, \dots, f_n \in A[X_1, \dots, X_n]$$
such that
$$B \simeq A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_n)$$
with B^+ the integral closure of the image of $A^+\langle X_1, \dots, X_n\rangle$ and such that the image of
$$J = \det\Big(\frac{\partial f_i}{\partial X_j}\Big)_{1\le i,j\le n} \in A[X_1, \dots, X_n]$$
in $A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_n)$ is invertible.

2.15.3. A key result by Huber. — Zariski's main theorem says that any quasi-finite separated morphism of schemes $f : X \to Y$ with Y quasi-compact quasi-separated admits a factorization

$$\begin{tikzcd} X \arrow[r, hook, "\text{qc open}", "\text{immersion}"'] \arrow[d, "f"'] & \overline{X} \arrow[dl, "\text{finite}"] \\ Y & \end{tikzcd}$$

This is in particular the case if f is qc separated and étale. Nevertheless, when f is separated étale one can not in general find such a factorization with $\overline{X} \to Y$ finite étale. For example, let $f : X \to Y$ be a dominant morphism of proper smooth algebraic curves over a field k. Suppose that f is generically étale i.e. $k(X)|k(Y)$ is separable. Let $U \subset X$ be the biggest open subset such that $f_{|U}$ is étale. Then, $f_{|U}$ admits a factorization

$$U \xhookrightarrow[\text{immersion}]{\text{open}} \overline{U} \xrightarrow{\text{finite étale}} f(U)$$

if and only if $f^{-1}(f(U)) = U$.

Nevertheless, recall that if $(A, \mathfrak{m})$ is an Henselian local ring and $X \to \mathrm{Spec}(A)$ is quasi-compact étale then one can split

$$X = U \coprod U'$$

with

$$U \longrightarrow \mathrm{Spec}(A) \setminus \{\mathfrak{m}\}$$

and

$$U' \xrightarrow{\text{finite étale}} \mathrm{Spec}(A).$$

As a consequence, if $f : X \to Y$ is a quasi-compact étale morphism of schemes and $y \in Y$, up to replacing Y by an étale neighborhood of y, we can split $X = X' \coprod X''$ with $y \notin f(X')$ and $X'' \longrightarrow Y$ is finite étale.

The local rings of analytic adic spaces are Henselian (Proposition 2.7.4) and one can go even further for étale morphisms of analytic Noetherian adic spaces.

Proposition 2.15.4 **([38, Lemma 2.2.8]).** — *Let (A, A^+) and (B, B^+) be strongly Noetherian complete affinoid Tate rings and*

$$f : X = \mathrm{Spa}(B, B^+) \longrightarrow \mathrm{Spa}(A, A^+) = Y$$

be an étale morphism. Any point of Y has a neighborhood V such that the étale morphism $f^{-1}(V) \to V$ has a factorization

$$\begin{tikzcd} f^{-1}(V) \arrow[r, hook, "\textit{open immersion}"] \arrow[d, "\textit{étale}"'] & W \arrow[dl, "\textit{finite étale}"] \\ V & \end{tikzcd}$$

Proof. — We apply Proposition 2.15.3. With the notations of this proposition, let

$$B' = A[X_1, \dots, X_n]/(f_1, \dots, f_n)$$

with Jacobian $J \in A[X_1, \dots, X_n]$ whose image in $B = A\langle X_1, \dots, X_n\rangle/(f_1, \dots, f_n)$ is a unit. Let S be the image of $1 + A^{\circ\circ}[X_1, \dots, X_n]$ in B'. The image of J in $S^{-1}B'$ is thus a unit. We deduce that, up to replacing B' by $B'' = B[\frac{1}{s}]$ for some $s \in S$, we can suppose that we have an étale morphism of schemes

$$g : \mathrm{Spec}(B'') \longrightarrow \mathrm{Spec}(A)$$

that induces $f : X \to Y$ in the following sense. There is an analytification functor

$$(-)^{ad} : \{\text{finite type schemes}/\,\mathrm{Spec}(A)\} \longrightarrow \{\text{locally of finite type adic spaces}/\,\mathrm{Spa}(A, A^+)\}$$

that sends $\mathbb{A}^n_{\mathrm{Spec}(A)}$ to the adic affine space $\mathbb{A}^{n,ad}_{\mathrm{Spa}(A,A^+)}$. Let $t_1 \dots, t_n \in B''$ be the image of $T_1, \dots, T_n$. One then has

$$X = \left\{ x \in \mathrm{Spec}(B'')^{ad} \;\middle|\; |t_1(x)| \le 1, \dots, |t_n(x)| \le 1 \right\} \xrightarrow{g^{ad}} \mathrm{Spa}(A, A^+) = Y.$$

Let $y \in Y$. Since $\mathcal{O}_{Y,y}$ is an Henselian local ring, the étale morphism

$$\mathrm{Spec}(B'') \times_{\mathrm{Spec}(A)} \mathrm{Spec}(\mathcal{O}_{Y,y}) \longrightarrow \mathrm{Spec}(\mathcal{O}_{Y,y})$$

splits as a disjoint union of a finite étale morphism and an étale morphism whose image is contained in $\mathrm{Spec}(\mathcal{O}_{Y,y}) \setminus \{\mathfrak{m}_y\}$. Since g is of finite presentation we deduce that, up to replacing $\mathrm{Spa}(A, A^+)$ by a rational localization that is a neighborhood of y, we can suppose that g splits as a disjoint union

$$g = g_1 \sqcup g_2 : \mathrm{Spec}(B_1'') \coprod \mathrm{Spec}(B_2'') \longrightarrow \mathrm{Spec}(A)$$

with $g_1 : \mathrm{Spec}(B_1'') \to \mathrm{Spec}(A)$ finite étale and

$$\mathrm{supp}(y) \notin \mathrm{Im}(g_2).$$

The set $\mathrm{Im}(g_2)$ is Zariski open in $\mathrm{Spec}(A)$. We deduce that $\mathrm{Im}(g_2^{ad}) \subset Y$ is Zariski open. It contains the quasi-compact open subset $V = g_2^{ad}(X \cap \mathrm{Spec}(B_2'')^{ad})$. Since V is quasi-compact open in Y, $\overline{V}$ is its set of specializations. The specializations of a given point of Y have the same support in $\mathrm{Spec}(A)$ and we deduce that $y \notin \overline{V}$. Up to replacing Y by a rational localization that is a neighborhood of y, we can thus suppose that $\mathrm{Spec}(B_2'')^{ad} \cap X = \emptyset$. We thus have a factorization of f as

$$f : X \xhookrightarrow{\text{open immersion}} \mathrm{Spec}(B_1'')^{ad} \xrightarrow{\text{finite étale}} Y. \qquad \square$$

2.16. Vector bundles on analytic adic spaces

CHAPTER 3

PERFECTOID SPACES

3.1. Perfectoid rings

In this text we emphasize perfectoid rings over any base i.e. without a base. They do not contain a field in general contrary to Scholze's original article [48].

3.1.1. Generalities. — Let us start by giving the definition of a perfectoid ring.

Definition 3.1.1. — A perfectoid ring A is a complete Tate ring satisfying

1. A is uniform i.e. A° is bounded,
2. There exists a pseudo-uniformizer ϖ satisfying $\varpi^p|p$ and such that $\text{Frob} : A^\circ/\varpi \to A^\circ/\varpi^p$ is an isomorphism.

An affinoid perfectoid ring is an affinoid ring (A, A^+) such that A is perfectoid.

For A a perfectoid ring, since $p \in A^{\circ\circ}$, A is automatically a $\mathbb{Z}_p$-algebra that is p-adically separated complete.

Example 3.1.2. —

1. Let $K|\mathbb{Q}_p$ be an arithmetically profnite algebraic extension of $\mathbb{Q}_p$, for example $K|\mathbb{Q}_p$ Galois of infinite degree with $\text{Gal}[K|\mathbb{Q}_p)$ a p-adic Lie group ([49]). Then, the main result of Fontaine and Wintemberger ([53]) says that $\widehat{K}$ is a perfectoid field. This is for example the case for $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ or $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$.
2. If K is a complete algebraically closed non-Archimedean field then K is perfectoid.
3. Any characteristic p perfect complete non-Archimedean field like $\mathbb{F}_p(\!(T^{1/p^\infty})\!)$.
4. If K is a perfectoid field then $K\langle X_1^{1/p^\infty}, \dots, X_d^{1/p^\infty}\rangle$, the completion for the Gauss norm of the K-algebra $\cup_{n\geq 0} K\langle X_1^{1/p^n}, \dots, X_d^{1/p^\infty}\rangle$, is perfectoid.
5. More generally, if A is perfectoid and I a set let us consider
$$A\langle X_i^{1/p^\infty}\rangle_{i\in I}$$
that is $A^\circ\langle X_i^{1/p^\infty}\rangle_{i\in I}[\frac{1}{\varpi}]$ where ϖ is a pseudo-uniformizer of A and $A^\circ\langle X_i^{1/p^\infty}\rangle_{i\in I}$ is the ϖ-adic completion of $A^\circ\big[X_i^{1/p^\infty}\big]_{i\in I} = \bigcup_{n\geq 0} A\big[X_i^{1/p^n}\big]_{i\in I}$. This is a perfectoid ring with
$$A\langle X_i^{1/p^\infty}\rangle^\circ_{i\in I} = A^\circ\langle X_i^{1/p^\infty}\rangle_{i\in I}.$$
6. If A is a perfectoid ring and P is a profinite topological space then $\mathscr{C}(P, A)$ is perfectoid with $\mathscr{C}(P, A)^\circ = \mathscr{C}(P, A^\circ)$.

Remark 3.1.3. — Since we teased it to the reader, consider

$$A = \mathbb{Z}_p[\![T^{1/p^\infty}]\!]\langle X^{1/p^\infty}\rangle\left[\tfrac{1}{T}\right]/(TX-p)$$

where $\mathbb{Z}_p[\![T^{1/p^\infty}]\!]\langle X^{1/p^\infty}\rangle$ is the (T,p)-adic completion of $\cup_{n\geq 0}\mathbb{Z}_p[T^{1/p^n}, X^{1/p^n}]$. We will prove later (Example 3.4.11) that this is a perfectoid ring that does not contain a field and whose adic spectrum as a spectral topological space is connected, identified with the one of the adic closed ball $\mathbb{B}^1_{\mathbb{F}_p((T))}$ (if one wants a non-connected example it suffices to consider $K_1 \times K_2$ where K_1, resp. K_2, is a perfectoid field of characteristic p, resp. 0).

The following lemma says that we can in fact shorten the definition of a perfectoid ring and that, hopefully, this does not depend on the choice of the pseudo-uniformizer.

Lemma 3.1.4. — 1. *In Definition 3.1.1 the injectivity of* Frob *is automatic.*

2. *For A perfectoid, the Frobenius of A°/p is surjective.*
3. *For A perfectoid and ϖ any pseudo-uniformizer satisfying $\varpi^p|p$,* $\mathrm{Frob} : A^\circ/\varpi \to A^\circ/\varpi^p$ *is an isomorphism.*

Proof. — Point (1) is deduced from the fact that A° is integrally closed in A.

For point (2), let ϖ be such that $\varpi^p|p$ and $\mathrm{Frob} : A^\circ/\varpi \to A^\circ/\varpi^p$ is bijective. It suffices to prove by induction that for all $n \geq 1$,

$$\mathrm{Frob} : A^\circ/(p,\varpi^n) \to A^\circ/(p,\varpi^{pn})$$

is surjective. The case $n=1$ is immediate. Now if $a \in A^\circ$ satisfies

$$a = b^p + \lambda p + \mu\varpi^{pn}$$

with $\lambda, \mu \in A^\circ$, write $\mu = c^p + \nu\varpi^p$. One obtains

$$a = b^p + (c\varpi^n)^p + \lambda p + \nu\varpi^{p(n+1)}$$

which is congruent to $(b + c\varpi^n)^p$ modulo $(p, \varpi^{p(n+1)})$.

Point (3) is an immediate consequence of points (1) and (2). □

3.1.2. The characteristic p case. — In characteristic p, perfectoid rings are simple to describe. What is remarkable in the next proposition is that the uniformity of our ring is automatic. This is in fact deduced from Banach's open mapping theorem.

Proposition 3.1.5. — *A complete Tate $\mathbb{F}_p$-algebra is perfectoid if and only if it is a perfect ring.*

Proof. — Let A be an $\mathbb{F}_p$-perfectoid algebra. We use point (2) of Lemma 3.1.4 to deduce that for all $n \geq 1$,

$$\mathrm{Frob} : A^\circ/\varpi^n \xrightarrow{\sim} A^\circ/\varpi^{pn}.$$

By taking the projective limit when $n \geq 1$ varies we deduce that A° and thus A is perfect.

Reciprocally, let A be a perfect complete Tate $\mathbb{F}_p$-algebra. Let A_0 be a ring of definition of A and $\varpi \in A_0$ be a pseudo-uniformizer. The Frobenius

$$\mathrm{Frob} : A \xrightarrow{\sim} A$$

is a surjective map of $\mathbb{F}_p((\varpi^{1/p^\infty}))$-Banach spaces. By Banach's open mapping theorem, $\varpi^N A_0 \subset A_0^p$ for some $N \gg 0$. Thus, $A_0^{1/p} \subset \varpi^{-\frac{N}{p}} A_0$. By induction we deduce that

$$A_0^{1/p^k} \subset \varpi^{-N(\frac{1}{p}+\frac{1}{p^2}+\cdots+\frac{1}{p^k})} A_0.$$

From this we deduce that A_0^{1/p^∞}, the perfection of A_0 inside A, is bounded. We can thus suppose, up to replacing A_0 by its perfection, that A_0 is perfect. Now, if $a \in A^\circ$, there exists an integer $n \in \mathbb{Z}$ such that for all $k \geq 0$, $a^{p^k} \in \varpi^n A_0$ and thus, since A_0 is perfect, $a \in \varpi^{\frac{n}{p^k}} A_0$. This being true for all $k \geq 0$, we deduce that $a \in \varpi^{-1}A_0$. Thus, $A^\circ \subset \varpi^{-1}A_0$ and is thus bounded. We deduce that A is uniform and it is immediate to conclude that A is perfectoid. $\square$

Remark 3.1.6. — The proof gives more: it says that if A_0 is a ring of definition of the perfectoid $\mathbb{F}_p$-algebra A, then the inclusion $A_0^{1/p^\infty} \subset A^\circ$ is in fact an almost equality

$$A_0^{1/p^\infty} \underset{\text{almost}}{=} A^\circ.$$

This allows us to construct plenty of affinoid perfectoid $\mathbb{F}_p$-algebras without having to verify that it is uniform and to almost compute their ring of integral elements.

Example 3.1.7. — Let (A, A^+) be an affinoid perfectoid $\mathbb{F}_p$-algebra. Let $f_1, \dots, f_n \in A$ that generate A as an ideal and $g \in A$. Then the ring $A\langle \frac{f_1,\dots,f_n}{g}\rangle$ is perfect since $A\langle \frac{f_1,\dots,f_n}{g}\rangle = A\langle \frac{f_1^p,\dots,f_n^p}{g^p}\rangle$. It is thus perfectoid with

$$A\Big\langle \frac{f_1, \dots, f_n}{g} \Big\rangle^+ \underset{\text{almost}}{=} \bigcup_{k\geq 0} A^+\Big\langle \frac{f_1^{1/p^k}, \dots, f_n^{1/p^k}}{g^{1/p^k}} \Big\rangle.$$

Thus, *any rational subset of* $\mathrm{Spa}(A, A^+)$ *is affinoid perfectoid.*

The following corollary will be used later to see that for any $\mathbb{F}_p$-adic space X, one can define $\varprojlim_{\mathrm{Frob}} X$ as a perfectoid space. This will give us plenty of examples of perfectoid spaces for free.

Corollary 3.1.8. — *Let A be a Tate $\mathbb{F}_p$-algebra. Then the completion of its perfection $\varinjlim_{\mathrm{Frob}} A$,*

$$\widehat{\varinjlim_{\mathrm{Frob}} A_0}[\tfrac{1}{\varpi}],$$

where ϖ is a pseudo-uniformizer in the ring of definition A_0 and the completion is the ϖ-one, is perfectoid.

3.2. Tilting

3.2.1. Any p-adic ring. — In the following, the word p-adic means p-adically separated complete.

Definition 3.2.1. — For A a p-adic $\mathbb{Z}_p$-algebra we set

$$A^\flat = \varprojlim_{x\mapsto x^p} A$$

as a set. For $x \in A^\flat$, $x = (x^{(n)})_{n\geq 0}, (x^{(n+1)})^p = x^{(n)}$, we set $x^\sharp = x^{(0)}$. We equip $A^\flat$ with a multiplicative monoid structure by setting $(xy)^{(n)} = x^{(n)}y^{(n)}$.

The application $x \mapsto x^\sharp$ is multiplicative,

$$(xy)^\sharp = x^\sharp y^\sharp.$$

The following lemma is well known and essentially due to Fontaine.

Lemma 3.2.2. — *The reduction modulo p induces a bijection*

$$A^\flat \xrightarrow{\sim} \varprojlim_{\mathrm{Frob}} A/pA.$$

An inverse sends $(x_n)_{n\geq 0}$ to $\big(\lim_{k\to+\infty} (\widetilde{x_{n+k}})^{p^k}\big)_{n\geq 0}$ where for $x \in A/pA$, $\widetilde{x}$ is any lift to A.

Proof. — Use the fact that for $a, b \in A$, $a \equiv b$ modulo p^k implies $a^p \equiv b^p$ modulo p^{k+1}. $\square$

This gives $A^\flat$ the structure of a perfect $\mathbb{F}_p$-algebra. The multiplication rule in this algebra is simple, $(xy)^{(n)} = x^{(n)}y^{(n)}$, but the addition rule is more complicated, this is given by a renormalization process:

$$(x+y)^{(n)} = \lim_{k\to+\infty} \left(x^{(n+k)} + y^{(n+k)}\right)^{p^k}.$$

which we write with the $\sharp$ notation as

$$\boxed{(x+y)^\sharp = \lim_{k\to+\infty} \left(\left(x^{1/p^k}\right)^\sharp + \left(y^{1/p^k}\right)^\sharp\right)^{p^k}}$$

Example 3.2.3. — Let $A = \mathbb{Z}_p\langle T^{1/p^\infty}\rangle$, the p-adic completion of $\cup_{n\geq 0}\mathbb{Z}_p[T^{1/p^n}]$, one has $A^\flat = \mathbb{F}_p[X^{1/p^\infty}]$ with $X^\sharp = T$. One then has

$$(1+X)^\sharp = \sum_{\alpha\in\mathbb{N}[\frac{1}{p}]\cap[0,1]} \lim_{k\to+\infty} \binom{p^k}{p^k\alpha} T^\alpha.$$

We refer to [21] for a study of the p-adic numbers $\lim_{k\to+\infty}\binom{p^k}{p^k\alpha} \in \mathbb{Z}_p$.

3.2.2. Perfectoid rings. — Let us first verify that we can change the definition of the tilting for a perfectoid ring.

Lemma 3.2.4. — *For A a perfectoid ring and ϖ a pseudo-uniformizer of A satisfying $\varpi|p$, reduction modulo ϖ induces an isomorphism*

$$A^{\circ,\flat} = \varprojlim_{\mathrm{Frob}} A^\circ/p \xrightarrow{\sim} \varprojlim_{\mathrm{Frob}} A^\circ/\varpi.$$

Proof. — There is an exact sequence of projective systems of abelian groups

$$0 \longrightarrow (A^\circ/\varpi^{-1}pA^\circ)_{n\geq 0} \longrightarrow (A^\circ/pA^\circ)_{n\geq 0} \longrightarrow (A^\circ/\varpi A^\circ)_{n\geq 0} \longrightarrow 0$$

where the transitions maps are resp. $\varpi^{p-1}\mathrm{Frob}$, Frob and Frob. The left hand projective system is essentially 0 and thus

$$\varprojlim_{n\geq 0} A^\circ/\varpi^{-1}pA^\circ = 0$$
$$R^1\varprojlim_{n\geq 0} A^\circ/\varpi^{-1}pA^\circ = 0.$$

We deduce that

$$\varprojlim_{\mathrm{Frob}} A^\circ/pA^\circ \xrightarrow{\sim} \varprojlim_{\mathrm{Frob}} A^\circ/\varpi A^\circ.$$ □

Let us now verify that, up to an integral unit, any pseudo-uniformizer is a sharp.

Lemma 3.2.5. — *For A a perfectoid ring, for any pseudo-uniformizer ϖ there exists $\lambda \in (A^\circ)^\times$ such that $\lambda\varpi = x^\sharp$ for some $x \in A^{\circ,\flat}$.*

Proof. — Let us chose ϖ a pseudo-uniformizer satisfying $\varpi^p|p$. From the surjectivity of

$$\mathrm{Frob} : A^\circ/\varpi^p \to A^\circ/\varpi^p$$

and Lemma 3.2.4 we deduce that there exists $x \in A^{\circ,\flat}$ such that $x^\sharp \equiv \varpi$ modulo ϖ^p. This implies the result for ϖ since then $\frac{x^\sharp}{\varpi} \in 1 + \varpi^{p-1}A^\circ \subset (A^\circ)^\times$.

Now for any pseudo-uniformizer ϖ, let us chose $\varpi' \in A^{\circ,\flat}$ such that $\varpi'^\sharp$ is a pseudo-uniformizer. Up to replacing ϖ' by ϖ'^{1/p^n} for $n \gg 0$, we can find $k \in \mathbb{N}$ such that $\varpi'' = \varpi'^{\sharp,-k}\varpi$ satisfies $\varpi'' \in A^\circ$ is a pseudo-uniformizer and $(\varpi'')^p|p$ (this type of result is typically deduced from Proposition 2.1.15). We deduce the assertion for ϖ by application of the preceding case to ϖ''. □

Remark 3.2.6. — In general, the question to say something non-trivial about the image of the application $(-)^\sharp : A^\flat \to A$ for A perfectoid is a difficult one, see Proposition 3.8.8 and Corollary 3.13.2 for example.

Proposition 3.2.7. — *Let A be a perfectoid ring. Define $A^\flat = \varprojlim_{x\mapsto x^p} A$ with the multiplication rule $(xy)^{(n)} = x^{(n)}y^{(n)}$ and equipped with the projective limit topology.*

1. *If $\varpi \in A^{\circ,\flat}$ is such that $\varpi^\sharp$ is a pseudo-uniformizer of A then $A^\flat = A^{\circ,\flat}[\frac{1}{\varpi}]$ and the ring structure on $A^{\circ,\flat}[\frac{1}{\varpi}]$ corresponds to the ring structure on $A^\flat$ defined by the formula $(x+y)^{(n)} = \lim_{k\to+\infty}\left(x^{(n+k)}+y^{(n+k)}\right)^{p^k}$.*
2. *$A^\flat$ is an $\mathbb{F}_p$-perfectoid algebra satisfying $A^{\flat,\circ} = A^{\circ,\flat}$ and $\varpi \in A^\flat$ is a pseudo-uniformizer of $A^\flat$ if and only if $\varpi^\sharp$ is a pseudo-uniformizer of A.*
3. *For $\varpi \in A^\flat$ a pseudo-uniformizer such that $\varpi^\sharp | p$ there is a canonical isomorphism $A^{\flat,\circ}/\varpi \xrightarrow{\sim} A^\circ/\varpi^\sharp$.*

Proof. — Point (1) is easy. For point (2), if $\|.\| : A \to \mathbb{R}_+$ is a power multiplicative norm defining the topology of A then if we set $\|x\| = \|x^{(0)}\|$ for $x \in A^\flat$, $\|.\|$ is a power multiplicative norm defining the topology of $A^\flat$ which is thus a uniform Tate ring satisfying $A^{\flat,\circ} = A^{\circ,\flat}$.

For point (3), the projection $x \mapsto x^{(0)}$ from $A^{\circ,\flat}$ to A°/p induces a surjective morphism

$$A^{\circ,\flat}/\varpi \longrightarrow A^\circ/(p,\varpi^\sharp) = A^\circ/\varpi^\sharp.$$

The injectivity is deduced from the easy fact that if $x \in A^\circ$ satisfies $x^{p^n} \in A^\circ\varpi^\sharp$ then $x \in A^\circ\varpi^{1/p^n,\sharp}$ since A° is integrally closed in A. ☐

3.3. Integral perfectoid rings

There is a notion of integral perfectoid rings that is useful in proofs.

***Proposition 3.3.1* (Integral perfectoid ring).** — *Let R be a ring and $\varpi \in R$. Suppose*

1. *ϖ is a non-zero divisor,*
2. *R is ϖ-adically separated complete,*
3. *$\varpi^p | p$ and $\mathrm{Frob} : \mathrm{R}/\varpi \xrightarrow{\sim} \mathrm{R}/\varpi^{\mathrm{p}}$.*

Then $R[\frac{1}{\varpi}] = \varinjlim_{\times\varpi} R$ equipped with the inductive limit topology is a perfectoid ring such that R is almost equal to $R[\frac{1}{\varpi}]^\circ$.

Proof. — By definition, R is a ring of definition of the Huber ring $R[\frac{1}{\varpi}]$. Let

$$\frac{a}{\varpi^k} \in R\Big[\frac{1}{\varpi}\Big]^\circ$$

with $a \in R$ and $k \geq 1$. There exists $N \geq 1$ such that for all $n \geq 0$, $\left(\frac{a}{\varpi^k}\right)^n \in \varpi^{-N}R$ that is to say $a^n \in \varpi^{kn-N}R$. Replacing n by p^n we obtain for all $n \geq 0$,

$$a^{p^n} \in \varpi^{kp^n-N}R.$$

Thus, for $n \gg 0$,

$$a^{p^n} \in \varpi^{(k-1)p^n}R.$$

The injectivity of

$$\mathrm{Frob} : \mathrm{R}/\varpi \to \mathrm{R}/\varpi^{\mathrm{p}},$$

implies that for any $x \in R$, $(\frac{x}{\varpi})^p \in R \Rightarrow \frac{x}{\varpi} \in R$. From this we deduce by induction that

$$\forall x \in R,\ \forall i,j \geq 0,\ \left(\frac{x}{\varpi^j}\right)^{p^i} \in R \Longrightarrow \frac{x}{\varpi^j} \in R.$$

We can apply this to deduce that

$$a \in \varpi^{k-1} R$$

since $(\frac{a}{\varpi^{k-1}})^{p^n} \in R$. We have thus proven that

$$R\Big[\frac{1}{\varpi}\Big]^\circ \subset \varpi^{-1} R$$

and it is thus bounded.

Now, as in Lemma 3.2.5, we can suppose, up to multiplying ϖ by a unit in R, that there exists $\varpi^\flat \in R^\flat$ such that $(\varpi^\flat)^\sharp = \varpi$. The preceding result is true if replace ϖ by $(\varpi^{\flat,1/p^n})^\sharp$ for all n. We deduce that R is almost equal to $R[\frac{1}{\varpi}]^\circ$.

Finally one has to check that the fact that

$$\mathrm{Frob} : \mathrm{R}[\tfrac{1}{\varpi}]^\circ/\varpi \to \mathrm{R}[\tfrac{1}{\varpi}]^\circ/\varpi^{\mathrm{p}}$$

is almost surjective implies it is surjective. We can suppose that we can write $\varpi = \varpi^{\flat,\sharp}$. Let $x \in R[\frac{1}{\varpi}]^\circ$. One has $\varpi x \in R$ and thus there exists $y \in R$ such that $\varpi x \equiv y^p$ modulo $\varpi^p R$. We deduce that $x \equiv z^p$ modulo $\varpi^{p-1} R$ with $z \in R[\frac{1}{\varpi}]^\circ$ since $z^p \in R$. We have thus proven that is $\varpi' = (\varpi^{\flat,\frac{p-1}{p}})^\sharp$ then

$$\mathrm{Frob} : R[\tfrac{1}{\varpi}]^\circ/\varpi' \to R[\tfrac{1}{\varpi}]^\circ/\varpi'^p$$

is surjective. The result is deduced from the independence of the choice of a pseudo-uniformizer in the definition of a perfectoid ring, see point (3) of Lemma 3.1.4. □

Here is a typical example of an integral perfectoid ring.

Proposition 3.3.2. — *Let (A, A^+) be a Huber pair with A perfectoid. Then for any pseudo-uniformizer ϖ satisfying $\varpi^p|p$,* $\mathrm{Frob} : \mathrm{A}^+/\varpi \xrightarrow{\sim} \mathrm{A}^+/\varpi^{\mathrm{p}}$ *and A^+ is an integral perfectoid ring.*

Proof. — Let us begin with the injectivity. If $x \in A^+$ satisfies $x^p \in A^+\varpi^p$ it then satisfies $x^p \in A^\circ\varpi^p$. This implies $x = \lambda\varpi$ with $\lambda \in A^\circ$. But then, $\lambda^p\varpi^p \in \varpi^p A^+$ which implies $\lambda^p \in A^+$ and thus $\lambda \in A^+$ since A^+ is integrally closed.

For the surjectivity. For any $x \in A^+$ one can write $x = a^p + \lambda\varpi^p$ with $a, \lambda \in A^\circ$. This implies $a^p \in A^+$ and thus $a \in A^+$. We deduce that

$$\mathrm{Frob} : \mathrm{A}^+/\varpi \to \mathrm{A}^+/\varpi^{\mathrm{p}}\mathrm{A}^\circ$$

is surjective. Using that the ring A°/A^+ is perfect we conclude. □

3.4. Untilting

The tilting functor $(-)^\flat$, due to Fontaine, has in fact an adjoint and the adjunction map is Fontaine's θ map. *This adjoint to tilting will allow us to untilt.*

Proposition 3.4.1. — *The functors*

$$\textit{perfect } \mathbb{F}_p\textit{-algebras} \underset{(-)^\flat}{\overset{W(-)}{\rightleftarrows}} p\textit{-adically separated complete rings}$$

are adjoint, where $W(-)$ is a left adjoint of $(-)^\flat$. The adjunction maps are given by

$$\begin{aligned} \theta : W(R^\flat) &\longrightarrow R \\ \sum_{n\geq 0} [a_n] p^n &\longmapsto \sum_{n \geq 0} a_n^\sharp p^n. \end{aligned}$$

and

$$\begin{array}{ccc} R & \longrightarrow & W(R)^\flat \\ x & \longmapsto & \big([x^{p^{-n}}]\big)_{n\geq 0}. \end{array}$$

Thus, for R a perfect $\mathbb{F}_p$-algebra and $x \in R$ seen as $W(R)^\flat$,

$$x^\sharp = [x].$$

Let's dig further the structure of θ for perfectoid rings.

Lemma 3.4.2. — *For A a perfectoid ring, $\theta : W(A^{\flat,\circ}) \to A^\circ$ is surjective.*

Proof. — Since A° is p-adically complete it suffices to check that the reduction modulo p of θ is surjective. But this is identified with the projection onto the first component $\varprojlim_{\text{Frob}} A^\circ/p \to A^\circ/p$. □

We are now going to classify all possible kernels for θ. The following definition showed up in [27] for perfectoid fields. This is inspired by the theory of Weierstrass factorization of power series; we see Witt vectors as "holomorphic functions of the variable p" (see [27, Chapitre 1]).

Definition 3.4.3. — Let A be a characteristic p perfectoid ring. An element

$$\xi = \sum_{n\geq 0}[a_n]p^n \in W(A^\circ)$$

is distinguished of degree 1 if $a_0 \in A^{\circ\circ}$ and $a_1 \in (A^\circ)^\times$. We note $\mathscr{D}_1(A)$ the set of degree 1 distinguished elements in $W(A^\circ)$.

Thus, the element ξ is primitive of degree 1 if and only if ξ mod $W(A^{\circ\circ})$ is in $pW(\widetilde{A})^\times$. From this we deduce that $W(A^\circ)^\times$ acts by multiplication on $\mathscr{D}_1(A)$. Moreover,

$$\mathscr{D}_1(A)/W(A^\circ)^\times \hookrightarrow \big\{\text{Cartier divisors of } \operatorname{Spec}(W(A^\circ))\big\}.$$

Those are in fact a particular type of *Cartier divisors on* $\operatorname{Spec}(W(A^\circ))$ thanks to the following lemma.

Lemma 3.4.4. — *Any degree* 1 *distinguished element $\xi \in W(A^\circ)$ is a non-zero divisor.*

Proof. — Write $\xi = \sum_{n\geq 0}[a_n]p^n$. Let ϖ be a pseudo-uniformizer such that $a_0 \in \varpi A^\circ$. The image of ξ in $W(A^\circ)/[\varpi]$ is of the form $p\times$unit. The result is deduced from the fact that $W(A^\circ)/[\varpi]$ has no p-torsion and $W(A^\circ)$ is $[\varpi]$-adically separated without $[\varpi]$-torsion. □

Remark 3.4.5. — The point of view of distinguished elements comes from [27] where the authors use the terminology "primitive". This point of view is the starting point of the work [23] in the non-perfect prismatic case.

We will need the following result.

Lemma 3.4.6. — *Let A be a characteristic p perfectoid ring, $\varpi \in A$ a pseudo-uniformizer and $\xi \in \mathscr{D}_1(A)$.*

1. *$W(A^\circ)$ is $[\varpi]$-adically separated complete.*
2. *$\times\xi : W(A^\circ) \hookrightarrow W(A^\circ)$ is strict with respect to the $[\varpi]$-adic topology on $W(A^\circ)$.*

Proof. — The $([\varpi], p)$-adic topology on $W(A^\circ)$ is the weak topology of uniform convergence of the Teichmüller coefficients i.e.

$$\begin{array}{ccc} (A^\circ)^{\mathbb{N}} & \xrightarrow{\sim} & W(A^\circ) \\ (a_n)_{n\geq 0} & \longmapsto & \sum_{n\geq 0}[a_n]p^n. \end{array}$$

is an homeomorphism where the left hand side is equipped with the product topology and the right hand one with the $([\varpi], p)$-adic one. We deduce from this observation that $W(A^\circ)$ is $([\varpi], p)$-adically separated complete. It is thus $[\varpi]$-adically separated complete (if a ring is I-adically separated complete for an ideal I then it is J-adically separated complete for any sub-finite type ideal $J \subset I$).

Let us now prove point (2). Write $\xi = \sum_{n\geq 0}[a_n]p^n$. Since $a_0 \in A^{\circ\circ}$, up to changing ϖ we can suppose that $\varpi | a_0$ in A°. We then have $\xi \equiv p \times \text{unit}$ modulo $[\varpi]$. Since $W(A^\circ)/[\varpi]$ is without p-torsion, if for some $f \in W(A^\circ)$, $\xi f \in ([\varpi])$ then $f \in ([\varpi])$. From this we deduce by induction that $(\xi) \cap ([\varpi^n]) = (\xi[\varpi^n])$ as ideals of $W(A^\circ)$. □

Lemma 3.4.7. — *For A a perfectoid ring,*

1. *If $\xi \in W(A^{\flat,\circ})$ is distinguished of degree 1 and satisfies $\theta(\xi) = 0$ then $\ker\theta = (\xi)$.*
2. *The kernel of θ is generated by a degree 1 distinguished element.*

Proof. — Let us first construct a generator of $\ker\theta$. Let $\varpi \in A^\flat$ be a pseudo-uniformizer satisfying $\varpi^\sharp | p$. Using the surjectivity of θ one c"an write

$$p = \theta(a)\varpi^\sharp.$$

We deduce that

$$\xi = p - a[\varpi] \in \ker\theta.$$

Let us prove this is a generator. If we equip $W(A^{\flat,\circ})$ with the $[\varpi]$-adic topology, the inclusion

$$\times\xi : W(A^{\flat,\circ}) \hookrightarrow W(A^{\flat,\circ})$$

is strict (Lemma 3.4.6). Since $W(A^{\flat,\circ})$ is $[\varpi]$-adically separated complete we deduce that $(\xi) \subset W(A^{\flat,\circ})$ is a closed ideal for the $[\varpi]$-adic topology. Thus,

$$W(A^{\flat,\circ})/\xi$$

is $[\varpi]$-adically separated complete. It thus suffices to prove that

$$\theta : W(A^{\flat,\circ})/\xi \to A^\circ$$

is injective when reduced modulo $[\varpi]$. But the reduction modulo $[\varpi]$ of this morphism is

$$A^{\flat,\circ}/\varpi \to A^\circ/\varpi^\sharp$$

that is an isomorphism (Proposition 3.2.7).

Let now $\xi' \in W(A^{\flat,\circ})$ be another degree 1 distinguished element such that $\theta(\xi') = 0$. Write

$$\xi' = x\xi \quad \text{with} \quad x = \sum_{n\geq 0}[x_n]p^n \in W(A^{\flat,\circ}).$$

One can find a pseudo-uniformizer ϖ' of $A^\flat$ such that ξ' and ξ are $p\times$unit in $W(A^{\flat,\circ})/[\varpi']$. Since $W(A^{\flat,\circ})/[\varpi']$ has no p-torsion, we deduce that x modulo $[\varpi']$ is a unit and thus x is a unit since $W(A^{\flat,\circ})$ is $[\varpi']$-adic. □

We can now state the main result.

Theorem 3.4.8. — *Let A be a characteristic p perfectoid ring. There is a bijection*

$$\mathscr{D}_1(A)/W(A^\circ)^\times \xrightarrow{\sim} \{\text{Untilts of } A \text{ over } \mathbb{Z}_p\}$$

that sends (ξ) to $W(A^\circ)[\frac{1}{[\varpi]}]/(\xi)$ where a ring of definition is the image of $W(A^\circ)$. An inverse sends $A^\sharp$ to the kernel of $\theta : W(A^\circ) \to A^{\sharp,\circ}$.

Proof. — It remains to prove that for ξ of degree 1, $W(A^\circ)/\xi$ is integral perfectoid (see Section 3.3). Let $\varpi \in A$ be a pseudo-uniformizer such that if $\xi = \sum_{n\geq 0}[a_n]p^n$ then $\varpi^p | a_0$. We already saw in the proof of Lemma 3.4.7 that $W(A^{\flat,\circ})/\xi$ is $[\varpi]$-adically separated complete. Moreover, $([\varpi], \xi) = ([\varpi], p)$ and $([\varpi^p], \xi) = ([\varpi^p], p)$. The result is easily deduced. □

Corollary 3.4.9. — *The category of perfectoid rings is equivalent to the category of couples (A, I) where A is a characteristic p perfectoid ring and I a Cartier divisor of* $\mathrm{Spec}(W(A^\circ))$ *generated by a degree one distinguished element, $I \in \mathscr{D}_1(A)/W(A^\circ)^\times$.*

Remark 3.4.10. — The subcategory of $\mathbb{Q}_p$-perfectoid rings corresponds to degree 1 distinguished elements $\xi = \sum_{n\geq 0}[a_n]p^n$ such that $a_0 \in A^\times$, that is to say a_0 is a pseudo-uniformizer.

Example 3.4.11. — We can now explain Remark 3.1.3. Let $R = \mathbb{F}_p((U^{1/p^\infty}))\langle Y^{1/p^\infty}\rangle$, an $\mathbb{F}_p$-perfectoid algebra with $\varpi = U$. There is an identification $W(R^\circ) = \mathbb{Z}_p[\![T^{1/p^\infty}]\!]\langle X^{1/p^\infty}\rangle$ since this last ring is p-adic without p-torsion with reduction modulo p the perfect ring R°. Via this identification, $T = [U]$ and $X = [Y]$. Let now $\xi = [\varpi X] - p \in \mathscr{D}_1(R)$. The resulting untilt of R over $\mathbb{Z}_p$ is the ring A of Remark 3.1.3. This perfectoid ring does not contain a field since if this field where of characteristic p then A would be a perfectoid $\mathbb{F}_p$-algebra and thus $\xi \equiv 0$ modulo p, if this where of characteristic 0 then A would be a $\mathbb{Q}_p$-perfectoid algebra and thus ξ modulo p would be in $R^\times$. The identification of $\mathrm{Spa}(A, A^\circ)$ with $\mathrm{Spa}(R, R^\circ)$ as topological spaces is Proposition 3.7.2.

Remark 3.4.12. — With the fancy point of view of [47], the preceding example 3.4.11 proves the existence of a morphism $\mathbb{B}^{1,1/p^\infty}_{\mathbb{F}_p((T^{1/p^\infty}))} \longrightarrow \mathrm{Spa}(\mathbb{Z}_p)^\diamond$ such that the reciprocal image of $\mathrm{Spa}(\mathbb{Q}_p)^\diamond \hookrightarrow \mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is the punctured perfectoid ball $\mathbb{B}^{1,1/p^\infty}_{\mathbb{F}_p((T^{1/p^\infty}))} \setminus \{0\}$ and the reciprocal image of $\mathrm{Spa}(\mathbb{F}_p)^\diamond$ is the origin $\{0\}$. From this we deduce that the v-sheaf $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is spatial with topological space $\{s, \eta\}$ with $s \leq \eta$. If all perfectoid rings contained a field then $|\mathrm{Spa}(\mathbb{Z}_p)^\diamond|$ would be a discrete set with two elements.

***Corollary 3.4.13* (Tilting equivalence).** — *If A is a perfectoid ring, tilting induces an equivalence between the category of perfectoid A-algebras and the category of perfectoid $A^\flat$-algebras.*

Finally, let us make a definition.

Definition 3.4.14. — A perfectoid algebra A is a perfectoid field if A is a field and A° is a rank 1 valuation ring.

The following is easy and left to the reader.

Proposition 3.4.15. — *The tilting equivalence identifies perfectoid fields: A is a perfectoid field if and only if $A^\flat$ is a perfectoid field.*

Example 3.4.16. — Take $K = \mathbb{F}_p((T^{1/p^\infty}))$. The untilt over $\mathbb{Q}_p$ corresponding to $\xi = [T] - p$ is $\widehat{\mathbb{Q}_p(p^{1/p^\infty})}$. Let $\varepsilon = 1 + T$. The one corresponding to $\xi = 1 + [\varepsilon^{\frac{1}{p}}] + \cdots + [\varepsilon^{\frac{p-1}{p}}]$ is $\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$.

For the next example, recall ([10, Section 2.4.4]) that a complete non-archimedean field $(K, |.|)$ is spherically complete if any decreasing set of closed balls has non-empty intersection (if the radius of the balls goes to zero this is automatic by completeness). This is equivalent to saying that if $(\mathfrak{a}_n)_{n\in\mathbb{N}}$ is a decreasing sequence of ideals of $\mathcal{O}_K$ then

$$R^1 \varprojlim_{n\geq 0} \mathfrak{a}_n = 0.$$

According to Kaplansky ([41]) this is equivalent to saying that $(K, |.|)$ is maximally complete: if $(L, |.|)$ is an extension of $(K, |.|)$ with the same residue field and the same value group, i.e. $|L| = |K|$ inside $\mathbb{R}$, then $L = K$.

Example 3.4.17 **(Untilting spherically complete fields ([43]))**
Let Γ be a non-trivial subgroup of $(\mathbb{R}, +)$. Let k be a characteristic p field. To this datum one associates

$$k(\!(\Gamma)\!) = \Big\{ \sum_{\gamma\in\Gamma} a_\gamma T^\gamma \mid a_\gamma \in k,\ \{\gamma \mid a_\gamma \neq 0\} \text{ is well ordered} \Big\}.$$

This is a spherically complete valued field with residue field k and valuation group Γ. If k is perfect of characteristic p and Γ is divisible by p this is a perfectoid field. When $\Gamma = \mathbb{Z}[\frac{1}{p}]$ this is the spherical closure of the perfectoid field $k(\!(T^{1/p^\infty})\!)$. If moreover k is algebraically closed and Γ is divisible i.e. is a sub-$\mathbb{Q}$-vector space of $\mathbb{R}$, this is algebraically closed. For k perfect and Γ divisible by p one has

$$W(k[\![\Gamma]\!]) = \Big\{ \sum_{\gamma\in\Gamma_+} a_\gamma X^\gamma \mid a_\gamma \in W(k),\ \forall n\ \{\gamma \mid v_p(a_\gamma) = n\} \text{ is well ordered} \Big\}$$

where $X^\gamma = [T^\gamma]$. Now, the untilt $K|\mathbb{Q}_p$ corresponding to $\xi = [T] - p$ is the maximally complete value field extension of $\mathbb{Q}_p$ with residue field k and valuation group Γ described in [43]. Noting $p^\gamma = (T^\gamma)^\sharp$ one has

$$\mathcal{O}_K = \Big\{ \sum_{\gamma\in\Gamma_+} [a_\gamma] p^\gamma \mid a_\gamma \in k,\ \{\gamma \mid a_\gamma \neq 0\} \text{ is well ordered} \Big\}$$

where $[a_\gamma] = a_\gamma^\sharp$. For example, when $\Gamma = \mathbb{Q}$ and $k = \overline{\mathbb{F}}_p$ this is the spherical completion of $\mathbb{C}_p$.

3.5. Affinoid perfectoid rings

In this section we upgrade the preceding results for perfectoid rings to affinoid perfectoid rings i.e. we replace A° by A^+. Let us begin by an evident definition.

Definition 3.5.1. — An affinoid perfectoid ring is a Huber pair (A, A^+) where A is a perfectoid ring. It is called an affinoid perfectoid field if A is a field and A^+ a valuation subring.

Given a perfectoid ring A, the different possible A^+ are in bijection with the integrally closed subrings of $\widetilde{A} = A^\circ/A^{\circ\circ}$. Since $\widetilde{A} = \widetilde{A^\flat}$, we deduce that the tilting equivalence extends to affinoid perfectoid rings. Using Proposition 3.3.2 and the preceding results one deduces the following where we replace A° by any A^+ and we have $A^{\flat,+} = A^{+,\flat}$.

Theorem 3.5.2. — *1. Let (A, A^+) be an affinoid perfectoid ring. The tilting functor induces an equivalence between (A, A^+)-affinoid perfectoid algebras and $(A^\flat, A^{\flat,+})$-affinoid perfectoid algebras.*

2. For (A, A^+) an $\mathbb{F}_p$-affinoid perfectoid algebra let us note

$$\mathscr{D}_1(A, A^+) = \Big\{ \sum_{n\geq 0} [a_n] p^n \in W(A^+) \mid a_0 \in A^{\circ\circ} \text{ and } a_1 \in (A^+)^\times \Big\}.$$

Then the tilting equivalence equivalence induces an equivalence between

- *triples (A, A^+, I) where (A, A^+) is affinoid perfectoid of characteristic p and $I \subset W(R^+)$ an ideal generated by an element of $\mathscr{D}_1(A, A^+)$ i.e. $I \in \mathscr{D}_1(A, A^+)/W(A^+)^\times$,*

- *the category of affinoid perfectoid algebras.*

3. The equivalence puts in correspondence affinoid perfectoid fields.

Proof. — It essentially suffices to verify that $\mathscr{D}_1(A,A^+)/W(A^+)^\times \xrightarrow{\sim} \mathscr{D}_1(A,A^\circ)/W(A^\circ)^\times$. This is an easy exercise. □

3.6. The cotangent complex point of view

In this section we explain the cotangent complex proof of the tilting equivalence as originally presented in [48].

3.6.1. Vanishing of the cotangent complex.

— Let us begin with a lemma.

Lemma 3.6.1. — *Let A be an $\mathbb{F}_p$-algebra and B an A-algebra. Then, the morphism induced by the Frobenius,* $\mathrm{Frob} : (A \to B) \to (A \to B)$, *on the complex $\mathbb{L}_{B/A}$ is homotopic to zero.*

Proof. — According to Lemma A.2.13, $\mathrm{Frob} : (A \to B) \to (A \to B)$ induces a morphism of complexes $P_{\bullet,B/A} \to P_{\bullet,B/A}$ that is homotopic to $\mathrm{Frob}_{P_{\bullet,B/A}}$. The result is deduced since by applying Ω^1, $d(x^p) = px^{p-1}dx$ when $x \in P_{\bullet,B/A}$ and this is zero since $P_{\bullet,B/A}$ is an A-module. □

Proposition 3.6.2. — *For any morphism $A \to B$ of perfect $\mathbb{F}_p$-algebras, $\mathbb{L}_{B/A} \simeq 0$ in the derived category of B-modules.*

Proof. — By functoriality of the cotangent complex, the Frobenius of B induces a semi-linear automorphism of $\mathbb{L}_{B/A}$. But this automorphism is zero according to Lemma 3.6.1. □

Proposition 3.6.3. — *For any morphism $(A,A^+) \to (B,B^+)$ of affinoid perfectoid algebras and ϖ a pseudo-uniformizer of A satisfying $\varpi | p$,*

$$\mathbb{L}_{(B^+/\varpi)/(A^+/\varpi)} \simeq 0$$

in $D(B^+/\varpi)$.

Proof. — Suppose first that $\varpi^p | p$. The isomorphism

$$\begin{tikzcd} B^+/\varpi \arrow[r, "\mathrm{Frob}", "\sim"'] & B^+/\varpi^p \\ A^+/\varpi \arrow[u] \arrow[r, "\mathrm{Frob}", "\sim"'] & A^+/\varpi^p \arrow[u] \end{tikzcd}$$

induces an isomorphism

$$\mathbb{L}_{(B^+/\varpi)/(A^+/\varpi)} \xrightarrow{\sim} \mathbb{L}_{(B^+/\varpi^p)/(A^+/\varpi^p)}.$$

Since this isomorphism is homotopic to 0 we deduce that $\mathbb{L}_{(B^+/\varpi^p)/(A^+/\varpi^p)}$ is acyclic.

For any ϖ satisfying $\varpi | p$, up to multiplying ϖ by a unit in A^+ one can suppose that it has a p-root. This implies the result in general. □

3.6.2. The tilting equivalence via the cotangent complex.

—

3.7. Identification of the adic spectra as topological spaces

Let (A, A^+) be affinoid perfectoid.

Proposition 3.7.1. — *1. For $x \in \mathrm{Spa}(A, A^+)$ and $f \in A^\flat$, the formula*

$$|f(x^\flat)| = |f^\sharp(x)|$$

defines an element $x^\flat \in \mathrm{Spa}(A^\flat, A^{\flat,+})$.

2. This induces a bijection

$$(-)^\flat : |\mathrm{Spa}(A, A^+)| \xrightarrow{\sim} |\mathrm{Spa}(A^\flat, A^{\flat,+})|.$$

Proof. — The only thing to verify in point (1) is that

$$|(f+g)(x^\flat)| \leq \sup\{|f(x^\flat)|, |g(x^\flat)|\}.$$

This is deduced from the continuity of the valuation $x^\flat$ that implies

$$|(f+g)(x^\flat)| = \lim_{k \to +\infty} |(f^{1/p^k,\sharp} + g^{1/p^k,\sharp})(x)|^{p^k}$$

where here the limit is taken in the valuation group associated to x.

For point (2) we need to define an inverse. Let $x \in \mathrm{Spa}(A^\flat, A^{\flat,+})$. The field $K = \mathrm{Frac}(A^\flat/\mathrm{supp}(x))$ is perfect and its valuation subring K^+ too. The completion $(\widehat{K}, \widehat{K}^+)$ is an affinoid perfectoid field and there is a natural morphism $(A^\flat, A^{\flat,+}) \to (\widehat{K}, \widehat{K}^+)$. Using the tilting correspondence we deduce an affinoid perfectoid field (L, L^+) together with a morphism $(A, A^+) \to (L, L^+)$ that gives the preceding by tilting. Let us note $x^\sharp$ for the image of the closed point of $\mathrm{Spa}(L, L^+)$ via $\mathrm{Spa}(L, L^+) \to \mathrm{Spa}(A, A^+)$. One easily verifies that $x \mapsto x^\sharp$ defines an inverse to $x \mapsto x^\flat$. □

Proposition 3.7.2. — *The preceding bijection*

$$|\mathrm{Spa}(A, A^+)| \xrightarrow{\sim} |\mathrm{Spa}(A^\flat, A^{\flat,+})|$$

is in fact an homeomorphism.

Proof. — Let $f_1, \ldots, f_n \in A^\flat$ generate $A^\flat$ as an ideal and $g \in A^\flat$. For each $x \in \mathrm{Spa}(A, A^+)$, there exists $i \in \{1, \ldots, n\}$ such that $|f_i(x^\flat)| \neq 0$ and thus $|f_i^\sharp(x)| \neq 0$. We deduce that $f_1^\sharp, \ldots, f_n^\sharp$ do not vanish simultaneously on $\mathrm{Spa}(A, A^+)$ and thus generate A as an ideal. Now one verifies immediately that the pullback via our bijection of the rational subset $\mathrm{Spa}(A^\flat, A^{\flat,+})\big(\frac{f_1,\ldots,f_n}{g}\big)$ is

$$\mathrm{Spa}(A, A^+)\big(\tfrac{f_1^\sharp,\ldots,f_n^\sharp}{g^\sharp}\big).$$

Our map is thus continuous. It is moreover generalizing. In fact, for $x \in \mathrm{Spa}(A^\flat, A^{\flat,+})$, the affinoid field $(K(x), K(x)^+)$ is perfectoid. The morphism

$$(A^\flat, A^{\flat,+}) \longrightarrow (K(x), K(x)^+)$$

untilts to a morphism

$$(A, A^+) \longrightarrow (K(x)^\sharp, K(x)^{\sharp,+})$$

where $(K(x)^\sharp, K(x)^{\sharp,+})$ is an affinoid perfectoid field. The image of the closed point of $\mathrm{Spa}(K(x)^\sharp, K(x)^{\sharp,+})$ is $x^\sharp \in \mathrm{Spa}(A, A^+)$. Using the identifications

$$|\mathrm{Spa}(A^\flat, A^{\flat,+})|_x = |\mathrm{Spa}(K(x), K(x)^+)| = |\mathrm{Spa}(K(x)^\sharp, K(x)^{\sharp,+})|$$

we conclude. We can now apply Lemma 2.11.6 to conclude that our map is an homeomorphism. □

3.8. Sheafiness and acyclicity

We now come to one of the main results of perfectoid spaces that allows us to glue affinoid perfectoid rings and work locally. This localization process is at the heart of the proof of the purity theorem 3.11.1 by Scholze and was typically missing in the work of Faltings in relative p-adic Hodge theory. It was missing in the work [26] too where in fact the constructed uniform Tate rings are perfectoid.

Theorem 3.8.1. — *Let (A, A^+) be an affinoid perfectoid ring.*

1. *The pair (A, A^+) is sheafy.*
2. *Any rational subset is affinoid perfectoid.*
3. *For $X = \mathrm{Spa}(A, A^+)$ one has*
   - *$H^i(X, \mathcal{O}_X) = 0$ for $i > 0$,*
   - *$H^i(X, \mathcal{O}_X^+)$ is almost zero for $i > 0$.*

3.8.1. The characteristic p case. — Let us prove Theorem 3.8.1 when A is an $\mathbb{F}_p$-perfectoid algebra. For any U a rational subset of $X = \mathrm{Spa}(A, A^+)$, $(\mathcal{O}_X(U), \mathcal{O}_X^+(U))$ is affinoid perfectoid (see Example 3.1.7). Let us fix ϖ a pseudo-uniformizer of A. Now, we can write

$$A^+ = \widehat{\varinjlim_i A_i^+}$$

where the colimit is filtered and (A_i, A_i^+) is an affinoid Tate $\mathbb{F}_p(\!(\varpi)\!)$-algebra such that A_i is of topological finite type over $\mathbb{F}_p(\!(\varpi)\!)$. Since A^+ is perfect we can suppose that for any index i, there exists $j \geq i$ such that $A_i \to A_j$ factorizes through Frob.

Let $(U_\alpha)_\alpha$ be a finite covering of $\mathrm{Spa}(A, A^+)$ by rational subsets. There exists an index i_0 and a rational covering $(U_{i_0,\alpha})_\alpha$ of $\mathrm{Spa}(A_{i_0}, A_{i_0}^+)$ whose pull back to $\mathrm{Spa}(A, A^+)$ is $(U_\alpha)_\alpha$ (use [36, Proposition 3.10]). We note $(U_{i,\alpha})_\alpha$ the pullback of this covering to $X_i = \mathrm{Spa}(A_i, A_i^+)$ when $i \geq i_0$. We then have, using the universal property of rationally localized affinoid rings,

$$\mathcal{O}_X^+(U_\alpha) = \widehat{\varinjlim_{i \geq i_0} \mathcal{O}_{X_i}^+(U_{i,\alpha})}$$

where the completion is the ϖ-adic one. The same goes on for the $U_{\alpha_1} \cap \cdots \cap U_{\alpha_r}$, the finite intersections of elements of our covering. Let $C_i^\bullet$ be the Čech complex of the sheaf $\mathcal{O}_{X_i}^+$ relative to the covering $(U_{i,\alpha})_\alpha$. We have $H^0(C_i^\bullet) = A_i^+$ and $H^q(C_i^\bullet)$, $q > 0$, is ϖ^∞-torsion. We deduce that $\varinjlim_i H^0(C_i^\bullet) = \varinjlim_i A_i^+$ and $\varinjlim_i H^q(C_i^\bullet)$, $q > 0$, is almost zero i.e. killed by ϖ^{1/p^n} for all $n \geq 0$.

Let $D^\bullet$ be the Cech complex of the presheaf $\mathcal{O}_X^+$ with respect to the covering $(U_\alpha)_\alpha$. We have

$$D^\bullet = \widehat{\varinjlim_{i \geq i_0} C_i^\bullet}.$$

From this we deduce the existence of exact sequences for $q \geq 0$

$$0 \longrightarrow R^1 \varprojlim_{n \geq 0} \Big[H^{q-1}\big(\varinjlim_{i \geq i_0} C_i^\bullet \big) \Big] / \varpi^n \longrightarrow H^q(D^\bullet) \longrightarrow \varprojlim_{n \geq 0} \Big[H^q\big(\varinjlim_{i \geq i_0} C_i^\bullet \big) \Big] / \varpi^n \longrightarrow 0.$$

The result is deduced from the fact that $\varprojlim_{n \geq 0}$ and $R^1 \varprojlim_{n \geq 0}$ send projective systems of almost zero modules to almost zero modules.

3.8.2. Some integral perfectoid rings and applications. — Before proving Theorem 3.8.1 we need some preliminary result about untilted rational domains.

Proposition 3.8.2. — *Let A be a perfectoid ring. Let $f_1, \ldots, f_n \in A^{\flat,\circ}$ that generate an open ideal and $g \in A^{\flat,\circ}$. Then the ϖ-adic completion of the sub-ring*

$$\bigcup_{k\geq 0} A^\circ \left[\frac{(f_1^{1/p^k})^\sharp}{(g^{1/p^k})^\sharp}, \ldots, \frac{(f_n^{1/p^k})^\sharp}{(g^{1/p^k})^\sharp} \right]$$

of $\bigcup_{k\geq 0} A \left[\frac{1}{(g^{1/p^k})^\sharp} \right]$ *is integral perfectoid.*

Proof. — Let $\varpi \in A^\flat$ be a pseudo-uniformizer satisfying $(\varpi^\sharp)^p | p$ (and thus we take the $\varpi^\sharp$-adic completion in the statement of the proposition). Let R be the ring in the statement. Since $f_1, \ldots, f_n$ generate an open ideal, for $k \gg 0$, $\varpi \in A^{\flat,\circ} f_1^{1/p^k} + \cdots + A^{\flat,\circ} f_n^{1/p^k}$. Using the identification $A^{\flat,\circ}/\varpi^p \xrightarrow{\sim} A^\circ/(\varpi^\sharp)^p$, we deduce that for $k \gg 0$, $(f_1^{1/p^k})^\sharp, \ldots, (f_n^{1/p^k})^\sharp$ generate an open ideal and thus this is true for any $k \geq 0$. Thus, for $k \geq 0$,

$$A^\circ \left[\frac{(f_1^{1/p^k})^\sharp}{(g^{1/p^k})^\sharp}, \ldots, \frac{(f_n^{1/p^k})^\sharp}{(g^{1/p^k})^\sharp} \right] = \varinjlim_{k\geq 0} A^\circ[T_1, \ldots, T_n] / \big((g^{1/p^k})^\sharp T_1 - (f_1^{1/p^k})^\sharp, \ldots, (g^{1/p^k})^\sharp T_n - (f_n^{1/p^k})^\sharp \big)$$

where the transition maps are given by $T_i \mapsto T_i^p, 1 \leq i \leq n$. The ring R is without $\varpi^\sharp$-torsion, and thus its $\varpi^\sharp$-adic completion is without $\varpi^\sharp$-torsion too. Moreover,

$$R/\varpi^\sharp R = \varinjlim_{k\geq 0} (A^{\flat,\circ}/\varpi)[T_1, \ldots, T_n] / (g^{1/p^k} T_1 - f_1^{1/p^k}, \ldots, g^{1/p^k} T_n - f_n^{1/p^k})$$

where the transition maps are given by $T_i \mapsto T_i^p, 1 \leq i \leq n$. This can be rewritten as

$$(A^{\flat,\circ}/\varpi)[T_1^{1/p^\infty}, \ldots, T_n^{1/p^\infty}] / (T_1^{1/p^k} g^{1/p^k} - f_1^{1/p^k}, \ldots, T_n g^{1/p^k} - f_n^{1/p^k})_{k\geq 0}.$$

The same type of formula holds for the reduction of R modulo $(\varpi^\sharp)^p$. One easily deduces that $\mathrm{Frob} : \mathrm{R}/\varpi^\sharp \xrightarrow{\sim} \mathrm{R}/(\varpi^\sharp)^{\mathrm{p}}$. □

Corollary 3.8.3. — *Let $X = \mathrm{Spa}(A, A^+)$ with (A, A^+) affinoid perfectoid.*

1. *For $f_1, \ldots, f_n \in A^\flat$ that generate the unit ideal and $g \in A^\flat$, the rational subset*
$$X\Big(\frac{f_1^\sharp, \ldots, f_n^\sharp}{g^\sharp}\Big)$$
is affinoid perfectoid.
2. *Via the homeomorphism $|\mathrm{Spa}(A, A^+)| \xrightarrow{\sim} |\mathrm{Spa}(A^\flat, A^{\flat,+})|$, if U is a rational subset of $|\mathrm{Spa}(A^\flat, A^{\flat,+})|$ and $U^\sharp$ the corresponding one in $|\mathrm{Spa}(A, A^+)|$ then*
   (a) *$U^\sharp$ is affinoid perfectoid with tilting U,*
   (b) *If $\varpi^\sharp | p$ there is an identification*
$$\mathcal{O}(U)^+/\varpi = \mathcal{O}(U^\sharp)^+/\varpi^\sharp.$$

We will prove later that any rational domain is affinoid perfectoid but let us already note the following corollary.

Corollary 3.8.4. — *For (A, A^+) affinoid perfectoid,*

1. *$\mathrm{Spa}(A, A^+)$ has a base of its topology made of affinoid perfectoid subsets.*
2. *The completed residue fields $(K(x), K(x)^+)$ for $x \in \mathrm{Spa}(A, A^+)$ are affinoid perfectoid fields.*

3.8.3. Stable uniformity of affinoid perfectoid rings and applications. —

3.8.3.1. *A general result.* —
Lemma 3.8.5. — *Let (A, A^+) be an affinoid Tate ring and let $X = \mathrm{Spa}(A, A^+)$. Suppose that is has a base $\mathscr{B}$ of rational subsets such that for each $U \in \mathscr{B}$, $\mathcal{O}_X(U)$ is uniform. Then (A, A^+) is stably uniform.*

Proof. — This is immediately reduced to proving that (A, A^+) is uniform. Let A_0 be a ring of definition of A and $\varpi \in A_0$ a pseudo-uniformizer. Let $(U_i)_i$ be a finite rational cover of $\mathrm{Spa}(A, A^+)$ such that for all i, $\mathcal{O}(U_i)$ is uniform. According to Lemma 2.9.10 one can find $f_1, \dots, f_n \in A$ generating the unit ideal and such that the associated standard rational cover satisfies

$$\forall j \in \{1, \dots, n\},\ \exists i,\ \mathrm{Spa}(A, A^+)\Big(\frac{f_1, \dots, f_n}{f_j}\Big) \subset U_i.$$

We can suppose that $f_1, \dots, f_n \in A_0$. Consider now

$$\pi : \widetilde{X} \longrightarrow X = \mathrm{Spec}(A_0)$$

the blow up of the ideal $(f_1, \dots, f_n)$. We note $X_\eta = \mathrm{Spec}(A)$. There is a diagram

$$\begin{tikzcd} \widetilde{X} \arrow[d, "\pi"'] & \\ X & X_\eta = \{\varpi \neq 0\}. \arrow[l, hook'] \arrow[ul, hook'] \end{tikzcd}$$

For each $j \in \{1, \dots, n\}$, the morphism $A^\circ \longrightarrow A\langle \frac{f_1, \dots, f_n}{f_j} \rangle^\circ$ factorizes through $\mathcal{O}(U_i)^\circ$ for some index i. From this we deduce that there exists $N \geq 0$ such that for all $j \in \{1, \dots, n\}$, $A^\circ \to \varpi^{-N} A_0\langle \frac{f_1, \dots, f_n}{f_j} \rangle$ and thus

$$A^\circ \longrightarrow \varpi^{-N} A_0\Big[\frac{f_1}{f_j}, \dots, \frac{f_n}{f_j}\Big].$$

We thus have

$$A^\circ \subset \varpi^{-N} H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}}).$$

Now, since $\widetilde{X} \to X$ is proper of finite presentation, $H^0(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$ is a finite A_0-module and it is thus bounded. We deduce the result. □

3.8.3.2. *Applications.* — We can now put together Corollary 3.8.4 and Lemma 3.8.5 to obtain the following result.

Theorem 3.8.6. — *For (A, A^+) affinoid perfectoid and $X = \mathrm{Spa}(A, A^+)$:*

1. *A is stably uniform and thus sheafy with $\mathcal{O}_X$ acyclic.*
2. *For $i > 0$, $H^i(X, \mathcal{O}_X^+)$ is almost zero.*

Proof. — It only remains to prove the second point. Via the homeomorphism $|X| \xrightarrow{\sim} |X^\flat|$ this is reduced to $H^i(X^\flat, \mathcal{O}_X^+) = 0$. But we have, as a sheaf on $X^\flat$, $\mathcal{O}_X^+/\varpi^\sharp = \mathcal{O}_{X^\flat}^+/\varpi$ if $\varpi^\sharp | p$. We thus have for $i > 0$, $H^i(X, \mathcal{O}_X^+/\varpi^\sharp)$ is almost zero. Replacing ϖ by ϖ^{1/p^n} for all $n \geq 0$ one easily deduces that $H^i(X, \mathcal{O}_X^+)$ is almost zero. □

3.8.4. Approximation of rational open subsets. — We have proven everything in Theorem 3.8.1 but point (2).

Proposition 3.8.7. — *Let (A, A^+) affinoid perfectoid. Then, for any rational open subset $U \subset \mathrm{Spa}(A, A^+)$ there exists $f_1, \dots, f_n \in A^\flat$ that generate A and $g \in A^\flat$ such that $U = \mathrm{Spa}(A, A^+)(\frac{f_1^\sharp, \dots, f_n^\sharp}{g^\sharp})$.*

Proof. — Write $U = \mathrm{Spa}(A, A^+)\big(\frac{a_1,\dots,a_n}{b}\big)$ where $Aa_1 + \cdots + Aa_n = A$. Lets us chose a pseudo-uniformizer $\varpi \in A$ such that $\varpi b_{|U}^{-1} \in \mathcal{O}(U)^+$. Then,

$$U = \bigcap_{i=1}^{n} \mathrm{Spa}(A, A^+)\big(\frac{a_i, \varpi}{b}\big).$$

Since rational open subsets are stable under finite intersections, it is sufficient to prove the result for a rational domain of the form $\mathrm{Spa}(A, A^+)\big(\frac{a,\varpi}{b}\big)$. We now use Proposition 3.8.8. Up to multiplying ϖ by a unit we can suppose $\varpi = (\varpi^\flat)^\sharp$ for some $\varpi^\flat \in A^\flat$. Using the proposition we can chose $\alpha \in A^\flat$ such that forall $x \in \mathrm{Spa}(A, A^+)$,

$$|a(x) - \alpha^\sharp(x)| < \sup\{|a(x)|, |\varpi(x)|\}$$

and thus

$$\sup\{|a(x)|, |\varpi(x)|\} = \sup\{|\alpha^\sharp(x)|, |\varpi(x)|\}.$$

Now, if we chose $\beta \in A^\flat$ such that for all x,

$$|b(x) - \beta^\sharp(x)| < \sup\{|b(x), |\varpi|\},$$

we find that

$$\mathrm{Spa}(A, A^+)\Big(\frac{a, \varpi}{b}\Big) = \mathrm{Spa}(A, A^+)\Big(\frac{\alpha^\sharp, (\varpi^\flat)^\sharp}{\beta^\sharp}\Big). \qquad \square$$

For the proof of the next proposition we follow [42, Lemma 5.5] rather than [48, Corollary 6.7]. This approximation result is a key point in the theory.

Proposition 3.8.8. — *For $f \in A^\circ$ and ϖ a pseudo-uniformizer, there exists $h \in A^{\flat,\circ}$ such that for all $x \in \mathrm{Spa}(A, A^+)$,*

$$|f(x) - h^\sharp(x)| < \sup\{|f(x)|, |\varpi(x)|\}.$$

Proof. — We can suppose that $\varpi = \varpi^{\flat,\sharp}$ for some pseudo-uniformizer $\varpi^\flat$ of $A^\flat$. Let $\xi \in W(A^{\flat,+})$ be a degree 1 distinguished element generating the kernel of $\theta : W(A^{\flat,+}) \to A^+$. Up to multiplying ξ by a unit of $W(A^{\flat,+})$ we can suppose

$$\xi \in p + [a]W(A^{\flat,+})$$

with $a \in A^{\circ\circ}$. Write $f = \theta(g)$ with

$$g = \sum_{k\geq 0} [g_k]p^k \in W(A^{\flat,+}).$$

For $h = \sum_{k\geq 0}[h_k]p^k \in W(A^{\flat,+})$, we note

$$\tau(h) = \frac{h - [h_0]}{p} \in W(A^{\flat,+}).$$

Let us now define by induction the sequence $(g_i)_{i\geq 0}$ by setting

$$\begin{aligned} g_0 &= g \\ g_{i+1} &= g_i - \tau(g_i)\xi. \end{aligned}$$

Let $n \in \mathbb{N}$ be such that $\varpi^\flat | a^n$ in $A^{\flat,+}$. Write for any $i \geq 0$,

$$g_i = \sum_{k\geq 0} [g_{i,k}]p^k.$$

Let now $x \in \mathrm{Spa}(A^\flat, A^{\flat,+})$. Let

$$N = \inf\{\ i \geq 0 \mid |g_{i,0}(x)| > |a(x)|^{i+1}\ \} \in \mathbb{N} \cup \{+\infty\}.$$

Let us prove by induction on i that for $0 \leq i \leq N$ one has

$$|g_i(x_G)| \leq |a(x)|^i.$$

The case $i = 0$ is evident. Now if we suppose proven the inequality for $i < N$ we have

$$g_{i+1} \in [g_{i,0}] + [a] \sum_{k \geq 0} [g_{i,k+1}] p^k W(A^{\flat,+})$$

and thus

$$|g_{i+1}(x_G)| \leq |a(x)|^{i+1}$$

which completes the induction.

Suppose $N < +\infty$. From the inequality $|g_N(x_G)| \leq |a(x)|^N$ and $|g_{N,0}(x)| > |a(x)|^{N+1}$ we get

$$\forall k > 0,\ |g_{N+1,k}(x)| < |g_{N+1,0}(x)|.$$

By induction on $i > N$, starting with $i = N + 1$, we find that the preceding inequality extends to

$$\forall i > N,\ \forall k > 0,\ |g_{i,k}(x)| < |g_{i,0}(x)|.$$

Now we have

$$|f(x^\sharp) - g^\sharp_{n,0}(x^\sharp)| \leq \sup_{k>0} |g_{n,k}(x)|.|p(x^\sharp)|^k.$$

If $n \leq N$ this is strictly less than

$$|a(x)|^{n+1} < |\varpi(x)|$$

since $\varpi | a^n$. If $n > N$ this is strictly less than

$$|g_{n,0}(x)| = |g^\sharp_{n,0}(x^\sharp)|.$$

We can thus take $h = g_{n,0}$ and this concludes the proof. □

Corollary 3.8.9. — *For (A, A^+) affinoid perfectoid, any rational subset of* $\mathrm{Spa}(A, A^+)$ *is affinoid perfectoid. The tilting correspondence induces a bijection between rational subsets of* $\mathrm{Spa}(A, A^+)$ *and rational subsets of* $\mathrm{Spa}(A^\flat, A^{\flat,+})$.

We have thus proven Theorem 3.8.1 in its entirety.

3.9. Perfectoid spaces

By definition a perfectoid space is an adic space that is locally isomorphic to the adic spectrum of an affinoid perfectoid ring, the so-called affinoid perfectoid spaces. We now focus on giving examples of perfectoid spaces and some constructions we can do with them that may not exist in the category of Noetherian adic spaces.

3.9.1. The tilting equivalence. — We can now upgrade the tilting equivalence of Corollary 3.4.13 to spaces.

Proposition 3.9.1. — *For X be a perfectoid space tilting induces an equivalence*

$$(-)^\flat : \textit{Perfectoid spaces}/X \xrightarrow{\sim} \textit{Perfectoid spaces}/X^\flat.$$

3.9.2. The perfectoid space associated to an $\mathbb{F}_p$-adic analytic space. — Let us now collect a natural consequence of the results of Section 3.1.2.

Proposition 3.9.2. — *The inclusion of the category of $\mathbb{F}_p$-perfectoid spaces in the category of $\mathbb{F}_p$-adic analytic spaces has a right adjoint,* $X \longmapsto \varprojlim_{\mathrm{Frob}} X$.

Proof. — This is an easy consequence of Corollary 3.1.8. More precisely, for (A, A^+) an $\mathbb{F}_p$-affinoid Tate sheafy ring,

$$\varprojlim_{\text{Frob}} \operatorname{Spa}(A, A^+) = \operatorname{Spa}(R, R^+)$$

where $R^+ = \widehat{\varinjlim_{\text{Frob}} A^+}$ (ϖ-adic completion) and $R = R^+[\frac{1}{\varpi}]$. In fact, if A_0 is a ring of definition of A contained in A^+, for any $x \in A^+$, since x is power bounded there exists an integer $N \geq 1$ such that for all $k \geq 0$, $x^{p^k} \in \varpi^{-N} A_0$. From this we deduce that $\varinjlim_{\text{Frob}} A_0$ is almost equal to $\varinjlim_{\text{Frob}} A^+$ and thus their ϖ-adic completions are almost equal. □

3.9.3. Cofiltered limits of affinoid perfectoids. — The next operation on perfectoid spaces does not exists in general in the category of Noetherian adic spaces. This will be a basic element in the definition of the pro-étale topology on perfectoid spaces.

Proposition 3.9.3. — *In the category of perfectoid spaces cofiltered limits of affinoid perfectoid spaces exist and are affinoid perfectoid.*

Proof. — Consider $\varprojlim_{i \in I} \operatorname{Spa}(A_i, A_i^+)$ a cofiltered limit of affinoid perfectoids. Fix $i_0 \in I$ and $\varpi \in A_{i_0}$ a pseudo-uniformizer. We still note ϖ for its image in A_i when $i \geq i_0$. Then one verifies that if

$$R^+ = \widehat{\varinjlim_{i} A_i^+}$$

(ϖ-adic completion) then R^+ is integral perfectoid and

$$\varprojlim_{i \in I} \operatorname{Spa}(A_i, A_i^+) = \operatorname{Spa}(R^+[\tfrac{1}{\varpi}], R^+).$$

This proves the statement. □

Example 3.9.4. — Let X be a perfectoid space. Then for $x \in X$,

$$\begin{aligned} X_x &= \varprojlim_{\substack{U \ni x \\ \text{nbd of } x}} U \\ &= \operatorname{Spa}(K(x), K(x)^+) \end{aligned}$$

that is perfectoid.

3.9.4. The example of profinite sets. — The possibility to include profinite sets in the theory of perfectoid spaces allows us to speak about pro-étale torsors under a profinite group.

Lemma 3.9.5. — *Let P be a profinite set. Note $\underline{P}$ for the functor on perfectoid spaces that sends X to $\mathscr{C}(|X|, P)$. Then for any perfectoid space X, $X \times \underline{P}$ is representable by a perfectoid space.*

Proof. — This is reduced to the affinoid perfectoid case. In this case, if $X = \operatorname{Spa}(A, A^+)$ and $P = \varprojlim_i P_i$ with P_i finite, then

$$\begin{aligned} X \times \underline{P} &= \varprojlim_i X \times \underline{P_i} \\ &= \varprojlim_i \operatorname{Spa}\left(\mathscr{C}(P_i, A), \mathscr{C}(P_i, A^+)\right) \\ &= \operatorname{Spa}\left(\mathscr{C}(P, A), \mathscr{C}(P, A^+)\right). \end{aligned}$$

This proves the result. □

3.9.5. The example of perfect formal schemes. — Let us here give an example that again relies on Proposition 3.1.5.

Proposition 3.9.6. — *Let R be a perfect $\mathbb{F}_p$-algebra that is I-adic with I an ideal of finite type. Then,* $\mathrm{Spa}(R,R)_a = \mathrm{Spa}(R,R) \setminus V(I)$ *is a perfectoid space.*

3.9.6. Zariski closed subsets. — If X is a perfectoid space and $Z \subset |X|$ is a subset we can look at the functor on perfectoid spaces that sends Y to the set of morphisms $Y \to X$ such that $|Y| \to Z$. If this is representable by a perfectoid space we will say that Z is represented by a perfectoid space.

Let $X = \mathrm{Spa}(A, A^+)$ be affinoid perfectoid. Let I be an ideal of A.

Lemma 3.9.7. — *The closed subset* $V(I) = \{x \in X \mid \forall f \in I,\ |f(x)| = 0\} \subset |X|$ *is represented by an affinoid perfectoid ring.*

Proof. — Fix ϖ a pseudo-uniformizer. We have

$$V(I) = \varprojlim_{\substack{\{f_1,\dots,f_k\}\subset I,\ \text{finite subset} \\ n\geq 1}} X\Big(\frac{f_1,\dots,f_k}{1}\Big)$$

and is thus a cofiltered limit of affinoid perfectoids. □

The characteristic p case is simpler for the following reason. Suppose $X = \mathrm{Spa}(A, A^+)$ is an $\mathbb{F}_p$-affinoid perfectoid space and $I \subset A$ is an ideal. Then, if $V(I) = \mathrm{Spa}(R, R^+)$ one has

$$R = \widehat{\varinjlim_{\mathrm{Frob}} A/I},$$

see Section 3.1.2. Let I^{1/p^∞} be the perfection of I, an ideal of A. One has moreover, see Remark 3.1.6,

$$R^\circ \underset{\text{almost}}{=} \widehat{A^\circ/A^\circ \cap I^{1/p^\infty}}$$

where the completion if the ϖ-adic one. Thus, at the end $A^+ \to R^+$ is almost surjective. This is still the case in general but the proof is more complicated, see Section 3.13.

3.9.7. Connected components. — If X is a qcqs perfectoid space then $\pi : |X| \to \pi_0(X)$. Then any connected component $\pi^{-1}(c)$, $c \in \pi_0(X)$, is affinoid perfectoid. In fact,

$$\pi^{-1}(c) = \varprojlim_{U \ni c} \pi^{-1}(U)$$

where U goes through the set of open/closed neighborhoods of c in the profinite space $\pi_0(X)$. For such a U, $\pi^{-1}(U)$ is open/closed in X and thus this limit exists.

3.9.8. Fiber products. — One of the reasons perfectoid spaces are good objects is the next proposition.

Proposition 3.9.8. — *The category of perfectoid spaces has fiber products. Moreover fiber products commute with tilting and the fiber product of affinoid perfectoid spaces is affinoid perfectoid.*

Proof. — Via the tilting equivalence of Proposition 3.9.1 it is enough to prove the existence of $X \times_Y Z$ for X, Y, Z $\mathbb{F}_p$-perfectoid spaces. Now, for $\mathbb{F}_p$-perfectoid spaces the result is a consequence of Proposition 3.1.5. □

Remark 3.9.9. — Let $\mathbb{Q}_p^{cyc}$ be the completion of the cyclotomic extension of $\mathbb{Q}_p$, a perfectoid field. *The topological ring* $\mathbb{Q}_p^{cyc}\hat{\otimes}_{\mathbb{Q}_p}\mathbb{Q}_p^{cyc}$ *is not perfectoid.* Nevertheless, the fiber product $\mathrm{Spa}(\mathbb{Q}_p^{cyc}) \times \mathrm{Spa}(\mathbb{Q}_p^{cyc})$ exists in the category of $\mathbb{Q}_p$-perfectoid spaces and is equal to

$$\mathrm{Spa}(\mathbb{Q}_p^{cyc}) \times \underline{\mathrm{Gal}(\mathbb{Q}_p^{cyc}|\mathbb{Q}_p)} = \mathrm{Spa}\left(\mathscr{C}(\mathrm{Gal}(\mathbb{Q}_p^{cyc}|\mathbb{Q}_p), \mathbb{Q}_p^{cyc}), \mathscr{C}(\mathrm{Gal}(\mathbb{Q}_p^{cyc}|\mathbb{Q}_p), \mathbb{Z}_p^{cyc})\right)$$

but this is not a fiber product in the category of adic spaces. One has to be careful in which category one takes the fiber products.

Next result is more surprising but an essential tool in [28].

Proposition 3.9.10. — *Let k be a characteristic p discrete perfect field. Let X and Y be k-perfectoid spaces. Then the fiber product $X \times_{\mathrm{Spa}(k)} Y$ exists in the category of adic spaces and is a perfectoid space.*

Proof. — Suppose $X = \mathrm{Spa}(A, A^+)$ and $Y = \mathrm{Spa}(B, B^+)$ are affinoid perfectoid. Consider $A^+\hat{\otimes}_k B^+$ equipped with the $(\varpi_A \otimes 1, 1 \otimes \varpi_B)$-adic topology. We can apply Proposition 3.9.6 to conclude that

$$X \times_k Y = \mathrm{Spa}(A^+\hat{\otimes}_k B^+, A^+\hat{\otimes}_k B^+) \setminus V(\varpi_A \otimes \varpi_B)$$

is perfectoid. □

One has to be careful that in the preceding situation, the fiber product of affinoid perfectoid spaces is not affinoid and not even quasi-compact anymore!

Example 3.9.11. — One has

$$\mathrm{Spa}\left(k((X^{1/p^\infty}))\right) \times_{\mathrm{Spa}(k)} \mathrm{Spa}(k((Y^{1/p^\infty}))) = \mathrm{Spa}\left(k[\![X^{1/p^\infty}, Y^{1/p^\infty}]\!]\right) \setminus V(XY).$$

3.10. Étale morphisms of perfectoid spaces

3.11. Purity

The purity theorem is inspired by the following evident result: let X be a perfect $\mathbb{F}_p$-scheme and $Y \to X$ be an étale morphism. Then Y is a perfect scheme.

Theorem 3.11.1. — *For (A, A^+) affinoid perfectoid, a Huber pair (B, B^+) and a morphism $(A, A^+) \to (B, B^+)$ such that $A \to B$ is finite étale and B^+ is the integral closure of A^+ in B, (B, B^+) is affinoid perfectoid.*

Proof. — The case of a perfectoid field is treated in [27]. Let $X = \mathrm{Spa}(A, A^+)$. The characteristic p-case is deduced from Proposition 3.1.5.

Since B is a projective A-module of finite type, it defines a sheaf of $\mathcal{O}_X$-algebras $\mathscr{B} = B \otimes_A \mathcal{O}_X$. Now, we use the equivalence of 2.7.6, the case of a perfectoid field, the characteristic p case and the tilting equivalence to deduce that over a rational covering $(U_i)_i$ of X, $\mathscr{B}(U_i)$ is a perfectoid ring. According to Proposition 4.1.3, B is stably uniform and thus sheafy. Thus, $\mathrm{Spa}(B, B^+)$ is a perfectoid space that is affinoid. But the morphism $\mathrm{Spa}(B, B^+)^\flat \to \mathrm{Spa}(A^\flat, A^{\flat,+})$ is finite locally free étale and thus $\mathrm{Spa}(B, B^+)^\flat$ is affinoid perfectoid. Via the tilting equivalence we deduce that (B, B^+) is affinoid perfectoid. □

Corollary 3.11.2. — *Let X be an Noetherian analytic adic space and $X' \to X$ be a morphism with X' perfectoid. For any $U \to X$ étale one can define $U \times_X X'$ as an étale perfectoid space over X'. This defines a continuous morphism of sites*

$$X'_{\text{ét}} \longrightarrow X_{\text{ét}}.$$

3.12. A criterion for a morphism to be an isomorphism

Proposition 3.12.1. — *Let $f : X \to Y$ be a qc qs morphism of perfectoid spaces. The following are equivalent:*

1. *f is an isomorphism.*
2. *For all affinoid perfectoid fields (K, K^+), the map $X(K, K^+) \to Y(K, K^+)$ is bijective.*
3. *For all algebraically closed affinoid perfectoid fields (K, K^+), the map $X(K, K^+) \to Y(K, K^+)$ is bijective.*

Proof. — It is clear that (1) implies (2) and (3) and that (2) implies (3). Let us verify (2) $\Rightarrow$ (1). Let us first verify that $|f| : |X| \to |Y|$ is bijective. For this it is enough to verify that for all $y \in Y$,

$$X_y := X \times_Y \mathrm{Spa}(K(y), K(y)^+) \to \mathrm{Spa}(K(y), K(y)^+)$$

is an isomorphism. By hypothesis this morphism has a section $\mathrm{Spa}(K(y), K(y)^+) \to X_y$. Let $x \in X_y$ be the image via this section of the closed point y of $\mathrm{Spa}(K(y), K(y)^+)$. Consider now the morphism $\mathrm{Spa}(K(x), K(x)^+) \hookrightarrow X_y$. The composite

$$\mathrm{Spa}(K(x), K(x)^+) \hookrightarrow X_y \longrightarrow \mathrm{Spa}(K(y), K(y)^+)$$

has a section (the preceding section sends y to x and thus the generalizations of y to the generalizations of x). We deduce that $K(y) = K(x)$ and since $x \mapsto y$, $K(x)^+ = K(y)^+$. We thus have

$$\mathrm{Spa}(K(x), K(x)^+) \xrightarrow{\sim} \mathrm{Spa}(K(y), K(y)^+).$$

But now the morphism

$$X_y \longrightarrow \mathrm{Spa}(K(y), K(y)^+)$$

induces a bijection for all affinoid fields (K, K^+)

$$X_y(K, K^+) \xrightarrow{\sim} \mathrm{Spa}(K(y), K(y)^+)(K, K^+).$$

We deduce that for all (K, K^+),

$$\mathrm{Spa}(K(x), K(x)^+)(K, K^+) \xrightarrow{\sim} X_y(K, K^+).$$

But for such an affinoid field (K, K^+) and an element of $\mathrm{Spa}(K(x), K(x)^+)$, the image of the morphism

$$\mathrm{Spa}(K, K^+) \longrightarrow \mathrm{Spa}(K(x), K(x)^+) \longrightarrow X_y$$

is contained in the generalizations of x in X_y. Since

$$\bigcup_{\substack{(K,K^+) \\ u:\mathrm{Spa}(K,K^+)\to X_y}} \mathrm{Im}(u) = |X_y|$$

we deduce that $|X_y|$ is the set of generalizations of x i.e.

$$\mathrm{Spa}(K(x), K(x)^+) \xrightarrow{\sim} X_y.$$

We thus have a bijection $|f| : |X| \xrightarrow{\sim} |Y|$ satisfying: for all $x \in X$,

$$(K(f(x)), K(f(x))^+) \xrightarrow{\sim} (K(x), K(x)^+).$$

We can now apply Corollary 2.11.5 to deduce that $|f|$ is an homeomorphism. Let us now replace Y by an affinoid perfectoid open subset and fix ϖ a pseudo-uniformizer of $\mathcal{O}(Y)$. According to Lemma 2.11.1,

$$f^{-1}(\mathcal{O}_Y^+/\varpi) \longrightarrow \mathcal{O}_X^+/\varpi$$

is an isomorphism. We can now replace ϖ by ϖ^n for all n to deduce that

$$f^{-1}\mathcal{O}_Y^+ = \varprojlim_{n\geq 0} f^{-1}(\mathcal{O}_Y^+/\varpi^n) \xrightarrow{\sim} \varprojlim_{n\geq 0} \mathcal{O}_X^+/\varpi^n = \mathcal{O}_X^+.$$

This implies that f is an isomorphism and we have proven (2) $\Rightarrow$ (1).

Point (3) is deduced from point (2) using the equality for any affinoid perfectoid field (K, K^+),

$$X(K, K^+) = X\left(\widehat{\overline{K}}, \widehat{\overline{K}}^+\right)^{\mathrm{Gal}(\overline{K}|K)}. \qquad \square$$

3.13. André lemma and Zariski closed subsets

***Lemma 3.13.1* (André).** — *Let (A, A^+) be an affinoid perfectoid ring and $f \in A$. There exists a morphism $(A, A^+) \to (B, B^+)$ of affinoid perfectoid rings such that*

1. *The image of f in B is of the form $g^\sharp$ for some $g \in B^\flat$.*
2. *The morphism $A^+/\varpi \to B^+/\varpi$ is almost faithfully flat.*

More precisely, we can take $\mathrm{Spa}(B, B^+) = V(T - f) \subset \mathrm{Spa}(A\langle T^{1/p^\infty}\rangle, A^+\langle T^{1/p^\infty}\rangle)$.

Proof. — $\square$

The following corollary is worth noting although we won't use it.

Corollary 3.13.2. — *For (A, A^+) an affinoid perfectoid ring there exists a morphism $(A, A^+) \to (B, B^+)$ of affinoid perfectoid rings such that*

1. *$A^+/\varpi \to B^+/\varpi$ is almost faithfully flat.*
2. *For any $f \in B$ there exists $g \in B^\flat$ such that $f = g^\sharp$.*

Proof. — For any finite subset $S = \{f_1, \dots, f_n\} \subset A$ let

$$\mathrm{Spa}(A_S, A_S^+) = V(T_1 - f_1, \dots, T_n - f_n) \subset \mathrm{Spa}\left(A\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle, A^+\langle T_1^{1/p^\infty}, \dots, T_n^{1/p^\infty}\rangle\right).$$

We can now consider

$$\mathrm{Spa}(A_1, A_1^+) = \varprojlim_S \mathrm{Spa}(A_S, A_S^+).$$

This construction gives by induction a projective system

$$\mathrm{Spa}(A, A^+) \leftarrow \mathrm{Spa}(A_1, A_1^+) \leftarrow \cdots \leftarrow \mathrm{Spa}(A_n, A_n^+) \leftarrow \mathrm{Spa}(A_{n+1}, A_{n+1}^+) \leftarrow \cdots.$$

One verifies easily that

$$\mathrm{Spa}(B, B^+) = \varprojlim_{n \geq 1} \mathrm{Spa}(A_n, A_n^+)$$

satisfies the expected properties. $\square$

CHAPTER 4

THE PRO-ÉTALE TOPOLOGY

4.1. Pro-étale perfectoid covers before Scholze

4.1.1. Colmez and Faltings remarks. — The construction of pro-étale perfectoid covers in relative p-adic Hodge theory goes back to the work of numerous authors. Let us cite some constructions.

Proposition 4.1.1 **(Colmez, [19, Section 4.4]).** — *Let A be a complete uniform Tate $\mathbb{Q}_p$-algebra. There exists a filtered system of finite étale uniform A-algebras $(A_i)_{i\in I}$ such that*

$$\widehat{\varinjlim_{i\in I} A_i}$$

is perfectoid.

Proof. — Consider the $\mathbb{F}_p$-vector space

$$V = 1 + A^{\circ\circ}/(1+A^{\circ\circ})^p.$$

For $S \subset V$ a finite dimension sub-vector space we note $\widetilde{S}$ for its reciprocal image inside $1 + A^{\circ\circ}$. For such an S,

$$A_S = A \otimes_{\mathbb{Z}[\widetilde{S}],[s]\mapsto[s^p]} \mathbb{Z}[\widetilde{S}].$$

If $x_1, \ldots, x_d \in \widetilde{S}$ induce a basis of S then

$$A[X_1, \ldots, X_d]/(X_1^p - x_1, \ldots, X_d^p - x_d) \xrightarrow{\sim} A_S$$

and thus $A \to A_S$ is finite étale. According to Proposition 4.1.3, A_S is Tate uniform. We can then look at

$$A_1 = \varinjlim_{S} A_S$$

Let us note that A_1 is uniform. In fact, if we fix ϖ and $\beta \in]0,1[$ then the transition maps in the inductive limit $\varinjlim_S A_S$ are isometric for the spectral norm associated to ϖ and β, see Theorem 2.1.16 and thus the topology of A_∞ is induced by a power multiplicative norm. We now reiterate the operation to obtain a sequence

$$A = A_0 \longrightarrow A_1 \longrightarrow \cdots \longrightarrow A_n \longrightarrow \cdots.$$

We note

$$A_\infty = \widehat{\varinjlim_{n\geq 0} A_n}$$

that is the completion of $\varinjlim_{n\geq 0} A_n$ for the spectral norm or, said in another way, $\widehat{\varinjlim_{n\geq 0} A_n^\circ}[\frac{1}{\varpi}]$ where the completion is the ϖ-adic one. We now have

$$1 + A_\infty^{\circ\circ}/(1+A_\infty^{\circ\circ})^p = \varinjlim_{n\geq 0} 1 + A_n^{\circ\circ}/(1+A_n^{\circ\circ})^p.$$

But the transition maps in this inductive limit are zero and thus any element of $1 + A_\infty^{\circ\circ}$ is a p-power. Now, if ϖ is a pseudo-uniformizer and x_n a p^n-th root of $1+\varpi$, for $n \gg 0$, $x_n - 1$ is a pseudo-uniformizer satisfying $(x_n-1)^p|p$. It is moreover immediate that for such a uniformizer ϖ, the Frobenius $A_\infty^\circ/\varpi \to A_\infty^\circ/\varpi^p$ is surjective. □

Remark 4.1.2. — The C-perfectoid rings A, $C|\mathbb{Q}_p$ complete algebraically closed, for which any element of $1 + A^{\circ\circ}$ has a p-power are called sympathetic by Colmez. Those are exactly those for which the logarithm $\log : 1 + A^{\circ\circ} \longrightarrow A$ is surjective. Another characterization is that those are exactly the one satisfying $H^1_{\text{ét}}(\mathrm{Spa}(A, A^\circ), \mathbb{Q}_p) = 0$.

Proposition 4.1.3. — *Let A be a uniform Tate ring and B a finite étale A-algebra. Then B is a uniform Tate ring and B° is the integral closure of A° in B.*

Proof. — Let R be the integral closure of A° in B. We can find a finite étale B-algebra B' such that $A \to B'$ is Galois with Galois group G and $B \to B'$ is Galois with Galois group H. One can for example take

$$\mathrm{Spec}(B') = \underline{\mathrm{Isom}}_{\mathrm{Spec}(A)}\big(\mathrm{Spec}(A) \times \{1, \dots, n\}, \mathrm{Spec}(B)\big)$$

if the rank of B is constant equal to n, $G = \mathfrak{S}_n$ and $H = \mathfrak{S}_{n-1}$. For any $x \in R$,

$$\mathrm{Tr}_{B/A}(x) = \sum_{[\tau] \in G/H} \tau(x)$$

which is thus integral over A° and thus in A° since A° is integrally closed in A. We thus have

$$\mathrm{Tr}_{B/A}(R) \subset A^\circ.$$

Let M be a finite type sub-A°-module of B such that $M[\frac{1}{\varpi}] = B$. Up to replacing M by $\varpi^k M$ with $k \gg 0$, we can suppose that $M \subset R$. Now, if

$$M^\vee = \{x \in B \mid \forall m \in M,\ \mathrm{Tr}_{B/A}(xm) \in A^\circ\},$$

we have

$$R \subset M^\vee.$$

Now, if we fix a diagram of morphisms of A-modules

$$\begin{tikzcd} A^d \arrow[r, two heads, "\varepsilon"'] & B \arrow[l, bend right, "s"'] \end{tikzcd}$$

with $\varepsilon s = \mathrm{Id}$ and $\varepsilon((A^\circ)^d) = M$, we have $M^\vee \subset s^\vee((A^\circ)^d)$ where $s^\vee : A^d = (A^d)^* \xrightarrow{s^*} B^* \xrightarrow[\sim]{\mathrm{Tr}_{B/A}^{-1}} B$. We deduce from this that R is contained in a finite type sub-A°-module of A. We have $B = R[\frac{1}{\varpi}]$ and B with its canonical topology as a finite type A-module is a Tate ring with R as a ring of definition. Now, if $x \in B^\circ$, there exists $N \geq 0$ such that for all $k \geq 0$, $x^k \in \varpi^{-N}R$. But since R is contained in a finite type A°-module, $\varpi^{-N}R$ is contained in a finite type A°-module and thus $A^\circ[x]$ too. We deduce that $x \in R$. □

Here is an example of uniform Tate algebras where one can construct some more explicit pro-étale perfectoid covers.

Example 4.1.4. — Let K be a charasteristic zero non-Archimedean perfectoid field. Let X be a K-adic space locally of finite type. Suppose X is smooth. Then, up to replacing X by an affinoid cover we can find an étale morphism

$$X \longrightarrow \mathbb{T}_n = \mathrm{Spa}(K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$$

that is an open rational subset inside a finite étale morphisme toward an open rational subset of $\mathbb{T}_n$. Let $\mathbb{T}_n^\infty = \varprojlim_{k\geq 0} \mathbb{T}_n$ where the transition map between level k and l with $l \leq k$ is the Kummer map $t_i \mapsto t_i^{p^{k-l}}, 1 \leq i \leq n$. This is the affinoid perfectoid space

$$\mathbb{T}_n^\infty = \mathrm{Spa}(K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1}\rangle, \mathcal{O}_K\langle T_1^{\pm 1/p^\infty}, \dots, T_n^{\pm 1}\rangle).$$

Then, using the purity theorem,

$$X \times_{\mathbb{T}_n} \mathbb{T}_n^\infty = \varprojlim_{k\geq 0}(X \times_{\mathbb{T}_n,\, t\mapsto t^{p^k}} \mathbb{T}_n)$$

is a perfectoid space pro-étale over X.

In another situation Faltings used the following elementary result.

***Proposition 4.1.5* (Faltings).** — *Let R be an integral normal p-torsion free noetherian $\mathbb{Z}_{(p)}$-algebra. Fix an algebraic closure $\overline{K}$ of $K = \mathrm{Frac}(R)$ and let $L|K$ inside $\overline{K}$ be the union of all finite degree extensions K' of K such that $\overline{R[\frac{1}{p}]}^{K'}$ is an étale $R[\frac{1}{p}]$-algebra. This means that if $\bar{x} : \mathrm{Spec}(\overline{K}) \to \mathrm{Spec}(R[\frac{1}{p}])$,*

$$Gal(L|K) = \pi_1\big(\mathrm{Spec}(R[\tfrac{1}{p}]), \bar{x}\big).$$

Let $\overline{R}$ be the integrale closure of R in L. Then, the p-adic completion $\widehat{\overline{R}}$ is integral perfectoid and thus $\widehat{\overline{R}}[\frac{1}{p}]$ is perfectoid with $\widehat{\overline{R}}[\frac{1}{p}]^\circ$ almost equal to $\widehat{\overline{R}}$.

Proof. — It it clearly p-adic without p-torsion. One can take $p^{1/p}$, a p-root of p in the fixed algebraic closure of $\mathrm{Frac}(R)$, as a pseudo-uniformizer. Since $\overline{R}$ is integrally closed in $\overline{R}[\frac{1}{p}]$,

$$\mathrm{Frob} : \overline{R}/p^{1/p} \to \overline{R}/p$$

is injective. For the surjectivity of the preceding Frobenius, let $a \in \overline{R}$. Then the polynomial

$$P(X) = X^p + pX - a$$

satisfies $(P(X), P'(X)) = \overline{R}[\frac{1}{p}][X]$. Thus, $\overline{R}[X]/(P(X))$ is a finite $\overline{R}$-algebra that is étale outside p and we can find $x \in \overline{R}$ such that $P(x) = 0$. □

4.1.2. The universal cover of a p-divisible group. —

Proposition 4.1.6. — *Let $C|\mathbb{Q}_p$ be an algebraically closed complete non-archimedean field. Let $\mathscr{G}$ be a p-divisible formal group over $\mathcal{O}_C$. Let $\mathscr{G}_\eta$ be its generic fiber as an adic space. Then,*

$$\varprojlim_{\times p} \mathscr{G}_\eta$$

is perfectoid with tilting isomorphic to

$$\varprojlim_{\mathrm{Frob}} \mathscr{G}_{k_C} \times_{k_C} \mathrm{Spa}(C^\flat).$$

Let us explain the meaning of this result. Here $\mathscr{G}$ is a formal Lie group,

$$\mathscr{G} \simeq \mathrm{Spf}(\mathcal{O}_C[\![x_1, \dots, x_d]\!])$$

for some integer d, the dimension of our formal group. Then,

$$\mathscr{G}_\eta \simeq \mathring{\mathbb{B}}^d_C,$$

the d-dimensional open ball as an adic space over $\mathrm{Spa}(C)$. Multiplication by p,

$$\mathscr{G} \xrightarrow{\times p} \mathscr{G}$$

is finite locally free of rank p^h where h is by definition the height of $\mathscr{G}$. This is in fact an fppf torosor under the finite locally free group scheme $\mathscr{G}[p]$ over $\mathrm{Spec}(\mathcal{O}_C)$. In generic fiber,

$$\mathscr{G}_\eta \xrightarrow{\times p} \mathscr{G}_\eta$$

is finite étale, a torsor under the finite étale group $\mathscr{G}[p]_\eta$.

The formal group $\mathscr{G}_{k_C}$ is isomorphic to $\mathrm{Spf}(k_C[\![x_1,\dots,x_d]\!])$ and

$$\varprojlim_{\mathrm{Frob}} \mathscr{G}_{k_C} \simeq \mathrm{Spf}\big(k_C[\![x_1^{1/p^\infty},\dots,x_d^{1/p^\infty}]\!]\big)$$

is a formal $\mathbb{Q}_p$-vector space. Now, in the statement we have in fact previously fixed a section of the projection $\mathcal{O}_C/p \to k_C$ so that $k_C \hookrightarrow C^\flat$. Then,

$$\varprojlim_{\mathrm{Frob}} \mathscr{G}_{k_C} \times_{k_C} \mathrm{Spa}(C^\flat) \simeq \mathring{\mathbb{B}}_{C^\flat}^{d,1/p^\infty},$$

a perfectoid open ball with a structure of $\mathbb{Q}_p$-vector space.

4.2. Definition and elementary properties

Recall that the category of affinoid perfectoid spaces admits cofiltered limits.

Definition 4.2.1. — Let $f : X \to Y$ be a morphism of perfectoid spaces.

1. f is affinoid pro-étale if X and Y are affinoid perfectoid and $Y \to X$ is a cofiltered limit of étale affinoid perfectoid spaces over X.
2. f is pro-étale if locally on X and Y it is affinoid pro-étale.

Let us begin with some examples.

Example 4.2.2. — For X perfectoid and $x \in X$, $\mathrm{Spa}(K(x), K(x)^+) = X_x \to X$ is pro-étale, $X_x = \varprojlim_{U \ni x} U$ with U going through the set of affinoid neighborhoods of x.

Example 4.2.3. — If $X = \mathrm{Spa}(A, A^+)$ and I is an ideal of A then $V(I) \hookrightarrow X$ is affinoid pro-étale: any Zariski closed subset of X is affinoid pro-étale inside X !

Example 4.2.4. — If X is a perfectoid space and P a locally profinite set then $X \times \underline{P} \longrightarrow X$ is pro-étale. If $P = \varprojlim_i P_i$ with P_i finite and the limit is cofiltered then $X \times \underline{P} = \varprojlim_i (X \times P_i)$.

Example 4.2.5. — Let X be a qc qs perfectoid space and $P \to \pi_0(X)$ be a continuous map of profinite spaces. Then, $X \times_{\underline{\pi_0(X)}} \underline{P} \to X$ is pro-étale. For example, if $P = \{c\}$ where $c \in \pi_0(X)$ and $P = \{c\} \hookrightarrow \pi_0(X)$ is the evident inclusion then one finds that any connected component of X is pro-étale inside X. In general, $\pi_0(X \times_{\underline{\pi_0(X)}} P) = P$.

In particular, let us note that if $Y \to X$ is a morphism of qc qs perfectoid spaces, there is a canonical factorization

$$Y \xrightarrow{\substack{\text{iso. on}\\ \text{connected comp.}}} X \underset{\underline{\pi_0(X)}}{\times} \underline{\pi_0(Y)} \xrightarrow{\text{pro-étale}} X.$$

Definition 4.2.6. — A family $(Y_i \xrightarrow{f_i} X)_{i \in I}$ of pro-étale morphisms toward the perfectoid space X is a pro-étale covering if for any quasi-compact open subset U of X there exists $I' \subset I$ finite and for each $i \in I'$ a quasi-compact open subset $V_i \subset Y_i$ such that

$$\bigcup_{i \in I'} f_i(V_i) = U.$$

Example 4.2.7. — 1. For any perfectoid space X, the family of pro-étale morphisms

$$\big(\mathrm{Spec}(K(x), K(x)^+) \to X\big)_{x \in X}$$

is surjective but not a covering in general.

2. For a perfectoid field K, if $\mathbb{B}_K^{1,1/p^\infty}$ is the closed perfectoid ball over K, the pro-étale morphism

$$\{0\} \coprod \mathbb{B}_K^{1,1/p^\infty} \setminus \{0\} \longrightarrow \mathbb{B}_K^{1,1/p^\infty}$$

is surjective but not a covering.

Although Point (1) of the preceding example is not a pro-étale cover, there is a way to "amalgamate" all $\mathrm{Spa}(K(x), K(x)^+)$, $x \in X$, in a pro-étale cover of X when X is affinoid perfectoid. This is the following example.

Example 4.2.8. — If $X = \mathrm{Spa}(A, A^+)$ is affinoid perfectoid, ϖ is a pseudo-uniformizer of A, $R^+ = \prod_{x \in X} K(x)^+$ and $R = R^+[\frac{1}{\varpi}]$ then $\mathrm{Spa}(R, R^+) \to \mathrm{Spa}(A, A^+)$ is a pro-étale cover. In fact, one has

$$\mathrm{Spa}(R, R^+) = \varprojlim_{\substack{|X| = \coprod_{i \in I} Z_i \text{ constructible partition} \\ (U_i)_{i \in I} \\ U_i \text{ nbd. of } Z_i}} \coprod_{i \in I} U_i \longrightarrow X.$$

4.3. Affinoid pro-étale morphisms

Proposition 4.3.1. — *Let $X_\infty = \varprojlim_{i \in I} X_i$ be a cofiltered limit of affinoid perfectoid spaces.*

1. *There is an equivalence*

$$2 - \varinjlim_{i \in I} \{\textit{finite étale}/X_i\} \xrightarrow{\sim} \{\textit{finite étale}/X_\infty\}.$$

2. *There is an equivalence*

$$2 - \varinjlim_{i \in I} \{\textit{qc qs étale }/X_i\} \xrightarrow{\sim} \{\textit{qc qs étale}/X_\infty\}.$$

Proposition 4.3.2. — *Let X be an affinoid perfectoid space. The functor*

$$\varprojlim : \mathrm{Pro}\big(\textit{affinoid perfectoid, étale}/X\big) \longrightarrow \{\textit{affinoid pro-étale}/X\}$$

is an equivalence of categories.

4.4. Totally disconnected perfectoid spaces

Definition 4.4.1. — A perfectoid space is totally disconnected if it is qc qs and $|X|$ is totally disconnected.

Thus, a perfectoid space is totally disconnected if and only if for each connected component C of X there exists a closed point $c \in X$ such that

$$C = \mathrm{Spa}(K(c), K(c)^+) \hookrightarrow X.$$

One can think of a totally disconnected perfectoid spaces as an "amalgamation" of $\mathrm{Spa}(K(c), K(c)^+)$ when c varies in a profinite set, its set of connected components.

Lemma 4.4.2. — *Any totally disconnected perfectoid space is affinoid perfectoid.*

Proof. — Let X be totally disconnected. We can fix a finite open covering of X by affinoid perfectoid spaces $X = \cup_i U_i$. It splits, $X = \coprod_i V_i$ with $V_i \subset U_i$. Since V_i is open/closed in U_i affinoid perfectoid, it is affinoid perfectoid. □

The following result is inspired by the following remark. Let X be a qc qs perfectoid space. If Y is a qc qs perfectoid space together with a morphism $Y \to X$ then the image of $|Y|$ in $|X|$ is a pro-constructible generalizing subset. In general this subset does not have a structure of perfectoid space. Nevertheless suppose X is totally disconnected perfectoid space with $\pi_0(X)$ finite. One has

$$X = \coprod_{\substack{c \in X \\ \text{closed point}}} \mathrm{Spa}(K(c), K(c)^+).$$

Let $Z \subset |X|$ be a pro-constructible generalizing subset. For any $c \in X$ a closed point one has $\mathrm{Spa}(K(c), K(c)^+) \cap Z$ is pro-constructible generalizing. It thus has a closed point z (see Corollary 1.3.9). Being stable under generalizations we deduce that

$$\mathrm{Spa}(K(c), K(c)^+) \cap Z = X_z = \mathrm{Spa}(K(z), K(z)^+)$$

with $K(z) = K(c)$. We thus have

$$Z \cap \mathrm{Spa}(K(c), K(c)^+) = \big\{ x \in \mathrm{Spa}(K(c), K(c)^+) \mid \forall f \in K(c)^+,\ |f(x)| \leq 1 \big\}.$$

We have $\mathcal{O}(X) = \prod_{c \text{ closed pt.}} K(c)$ and we deduce that there exists a subset $S \subset \mathcal{O}(X)$ such that

$$Z = \big\{ x \in X \mid \forall f \in S,\ |f(x)| \leq 1 \big\}.$$

In particular this is an intersection of Laurent domains and it has a structure of a totally disconnected perfectoid space pro-étale inside X. In fact, this extends from the case when the set of connected components is finite to any profinite set.

Theorem 4.4.3. — *Let X be a totally disconnected perfectoid space and $Z \subset |X|$ pro-constructible generalizing. Then, Z is represented by a totally disconnected perfectoid space such that the morphism $Z \hookrightarrow X$ is pro-étale.*

Proof. — We prove that

$$Z = \varprojlim_{\substack{S \subset \mathcal{O}(X) \text{ finite} \\ Z \subset \{|f| \leq 1 \ \mid\ f \in S\}}} \{x \in X \mid \forall f \in S,\ |f(x)| \leq 1\}.$$

Let $x \in X \setminus Z$. We are going to construct $f \in \mathcal{O}(X)$ satisfying $|f(x)| > 1$ and for all $z \in Z$, $|f(z)| \leq 1$. This will prove the theorem.

For this let C be the connected component of x. One has $C = \mathrm{Spa}(K(c), K(c)^+)$ where c is the unique closed point of C. Let us look at $C \cap Z$. Suppose first it is empty. Then,

$$\bigcap_{U \supset C} U \cap Z = \emptyset,$$

where U is an open/closed neighborhood of C, and by compacity of the constructible topology there exists such a U satisfying $U \cap Z = \emptyset$. The function $f \in \mathcal{O}(X)$ whose value on U is ϖ^{-1} and 1 on $X \setminus U$ satisfies $|f(x)| > 1$ and $\forall z \in Z,\ |f(z)| \leq 1$.

Suppose now that $C \cap Z \neq \emptyset$. Since Z is pro-constructible, $C \cap Z$ is pro-constructible in C and thus has a closed point z. Since Z is generalizing and C is a chain, we have

$$C \cap Z = \mathrm{Spa}(K(c), K(z)^+)$$

with $K(c) = K(z)$. We thus have

$$C \cap Z = \bigcap_{f \in K(z)^+} \{y \in C \mid |f(y)| \leq 1\}.$$

Since $x \in C \setminus C \cap Z$, there exists $f \in K(z)^+$ such that $|f(x)| > 1$. The local ring $\mathcal{O}_{X,c}$ is the colimit of $\mathcal{O}_X(U)$ where U goes through the set of open/closed neighborhoods of C. There thus exists U

an open/closed neighborhood of C and $g \in \mathcal{O}_X(U)$ satisfying $|g(x)| > 1$ and for all $y \in C \cap Z$, $|g(y)| \leq 1$. Let

$$V = \{y \in U \mid |g(y)| \leq 1\},$$

an open quasi-compact subset of U. One has

$$\bigcap_{\substack{C \subset U' \subset U \\ U' \text{open/closed}}} U' \cap (Z \cap (U \setminus V)) = \emptyset.$$

By compacity of the constructible topology we can thus suppose, up to shrinking U, that for all $y \in Z \cap U$, $|g(y)| \leq 1$. We can now define $h \in \mathcal{O}(X)$ be setting $h_{|U} = g$ and $h_{|X \setminus U} = 0$. It satisfies $|h(x)| > 1$ and for all $y \in Z$, $|h(y)| \leq 1$. $\square$

Thus, if $f : Y \longrightarrow X$ is a morphism of qcqs perfectoid spaces with X totally disconnected, there is a canonical factorization

$$Y \xrightarrow{\text{surjective}} \underbrace{Z}_{\text{aff. perf.}} \xhookrightarrow[\text{immersion}]{\text{pro-étale}} X.$$

Proposition 4.4.4. — *For any qc qs perfectoid space X, there exists a pro-étale open surjective morphism $\widetilde{X} \to X$ with $\widetilde{X}$ a totally disconnected perfectoid space.*

4.5. Strictly totally disconnected perfectoid spaces

Proposition 4.5.1. — *For X a perfectoid space the following are equivalent:*

1. *X is strictly totally disconnected*
2. *X is totally disconnected and for any $x \in X$, $K(x)$ is algebraically closed.*

Proof. — (1) $\Rightarrow$ (2) The only thing to verify is that the residue fields are algebraically closed. We use the equivalence of Corollary 2.7.6. Since X is totally disconnected, the residue field of $x \in X$ is the same as the one of c, the unique closed point of the connected component C of x. Now,

$$2 - \varinjlim_{\substack{U \subset C \\ \text{open/closed}}} \{\text{étale finite}/U\} \xrightarrow{\sim} \{\text{étale finite}/K(c)\}.$$

But now is $V \to U$ with U open and closed containing C, $V \to U$ extends to an étale cover $V \coprod X \setminus U \to X$ of X. This étale cover has a section and we deduce the result.

(2) $\Rightarrow$ (1) Let $Y \to X$ be an étale cover. Using the quasi-compacity of X and the definition of an étale morphism, it can be refined to a cover $\coprod_{i \in I} V_i \longrightarrow X$ with I finite and $V_i \to U_i$ étale finite where $U_i \subset X$ is open. The topological cover $\coprod_{i \in I} U_i \to X$ splits and we can thus suppose that $Y \to X$ is étale finite. We can now use the equivalence of Corollary 2.7.6 to deduce that any point of X has a neighborhood U such that $Y \times_X U \to U$ splits. Thus, over a topological cover of X the covering splits and we deduce the result by splitting this last topological cover. $\square$

4.6. Quasi-pro-étale morphisms

4.6.1. Scholze's Berkley example. — We will prove the following result explicitly. Let K be a perfectoid field of characteristic different from 2.

Proposition 4.6.1. — *Let $\mathbb{B} = \mathrm{Spa}(K\langle T^{1/p^\infty}\rangle, \mathcal{O}_K\langle T^{1/p^\infty}\rangle)$ be the one dimensional closed perfectoid ball over K. Consider the Kummer morphism $\mathbb{B} \to \mathbb{B}$ defined by $T^{1/p^k} \mapsto T^{2/p^k}$ for all $k \geq 0$. There exists an explicit pro-étale covering $\widetilde{\mathbb{B}} \to \mathbb{B}$ such that the pullback of the Kummer map via this pro-étale cover is pro-étale.*

Proof. — Fix an absolute value $|.|$ on K defining its topology and a radius $\rho \in]0,1[\cap |K|$. We note

$$f : \mathbb{B} \longrightarrow \mathbb{B}$$

our Kummer map. It is étale finite outside the origin $V(T)$. For $n \geq 1$ let

$$U_n = \mathbb{B}(\rho^n) \amalg \Big(\coprod_{i=0}^{n-1} \mathcal{C}(\rho^{i+1}, \rho^i) \Big)$$

where if $\rho = |\varpi|$, $\mathbb{B}(\rho^n) = \{x \in \mathbb{B} \mid |T(x)| \leq |\varpi(x)|^n\}$ is the closed ball with radius ρ^n and

$$\mathcal{C}(\rho^{i+1}, \rho^i) = \{x \in \mathbb{B} \mid |\varpi(x)|^{i+1} \leq |T(x)| \leq |\varpi(x)|^i\}$$

is the annulus with radii $[\rho^{i+1}, \rho^i]$. This forms a projective system of covers of $\mathbb{B}$ and we set

$$U_\infty = \varprojlim_{n \geq 0} U_n \longrightarrow \mathbb{B}$$

that is a pro-étale cover of $\mathbb{B}$. Let

$$V_n = \mathbb{B}(\rho^{n/2}) \amalg \Big(\coprod_{i=0}^{n-1} \mathcal{C}(\rho^{\frac{i+1}{2}}, \rho^{\frac{i}{2}}) \Big)$$

and

$$V_\infty = \varprojlim_{n \geq 0} V_n.$$

There is then a cartesian diagram

$$\begin{tikzcd} V_\infty \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_\infty \arrow[d] \\ \mathbb{B} \arrow[r, "f"'] & \mathbb{B}. \end{tikzcd}$$

We are going to prove that $V_\infty \to U_\infty$ is pro-étale. For this note

$$W_n = \mathbb{B}(\rho^n) \amalg \Big(\coprod_{i=0}^{n-1} \mathcal{C}(\rho^{\frac{i+1}{2}}, \rho^{\frac{i}{2}}) \Big).$$

Define the following transitions morphisms

$$\alpha_{n+1} : W_{n+1} \longrightarrow W_n$$

using the formulas:

- $\alpha_{n+1|\mathbb{B}(\rho^{n+1})} : \mathbb{B}(\rho^{n+1}) \hookrightarrow \mathbb{B}(\rho^n)$ followed by the evident inclusion of $\mathbb{B}(\rho^n)$ in W_n
- $\alpha_{n+1|\mathcal{C}(\rho^{\frac{n+1}{2}}, \rho^{\frac{n}{2}})} : \mathcal{C}(\rho^{\frac{n+1}{2}}, \rho^{\frac{n}{2}}) \xrightarrow{z \mapsto z^2} \mathcal{C}(\rho^{n+1}, \rho^n) \hookrightarrow \mathbb{B}(\rho^n)$ followed by the evident inclusion.
- $\alpha_{n+1|\mathcal{C}(\rho^{\frac{i+1}{2}}, \rho^{\frac{i}{2}})}$ is the evident inclusion of $\mathcal{C}(\rho^{\frac{i+1}{2}}, \rho^{\frac{i}{2}})$ in W_n if $0 \leq i \leq n-1$.

Define now

$$W_\infty = \varprojlim_{n \geq 0} W_n.$$

There is an étale morphism of projective systems

$$g_n : W_n \longrightarrow U_n$$

defined by:

- $g_{n|\mathbb{B}(\rho^n)}$ is the evident inclusion in U_n
- $g_{n|\mathcal{C}(\rho^{\frac{i+1}{2}}, \rho^{\frac{i}{2}})} : \mathcal{C}(\rho^{\frac{i+1}{2}}, \rho^{\frac{i}{2}}) \xrightarrow{z \mapsto z^2} \mathcal{C}(\rho^i, \rho^{i+1})$ followed by the evident inclusion inside U_n, for $0 \leq i \leq n-1$.

This defines a pro-étale morphism
$$g : W_\infty \longrightarrow U_\infty.$$
Define now a morphism of projective systems
$$h_n : V_n \longrightarrow W_n$$
using the formulas:

- $h_{n|\mathbb{B}(\rho^{\frac{n}{2}})} : \mathbb{B}(\rho^{\frac{n}{2}}) \xrightarrow{z\mapsto z^2} \mathbb{B}(\rho^n)$ followed by the evident inclusion in W_n
- $h_{n|\mathcal{C}(\rho^{\frac{i+2}{2}},\rho^{\frac{i}{2}})}$ is the evident inclusion in W_n for $0 \leq i \leq n-1$.

The following diagram commutes for all n
$$\begin{tikzcd} V_n \arrow[rr, "h_n"] \arrow[dr] & & W_n \arrow[dl, "g_n"] \\ & U_n & \end{tikzcd}$$
and we thus have a diagram
$$\begin{tikzcd} V_\infty \arrow[rr, "h_\infty"] \arrow[dr] & & W_\infty \arrow[dl, "g_\infty"] \\ & U_\infty. & \end{tikzcd}$$
We now say that $h_\infty : V_\infty \to W_\infty$ is an isomorphism. For this we apply Proposition 3.12.1. In fact, this is an evident isomorphism outside the origin of $\mathbb{B}$ since for any radius $R \in]0,1[\cap|K|$
$$V_n \times_{\mathbb{B}} \mathbb{B}(R) \longrightarrow W_n \times_{\mathbb{B}} \mathbb{B}(R)$$
is an isomorphism for $n \gg 0$. Moreover, computing the fibers at the origin using Proposition 2.11.2 we find
$$V_\infty \times_{\mathbb{B}} \{0\} = \varprojlim_{n\geq 0} \mathbb{B}(\rho^{n/2}) = \mathrm{Spa}(K, \mathcal{O}_K)$$
and
$$W_\infty \times_{\mathbb{B}} \{0\} = \varprojlim_{n\geq 0} \mathbb{B}(\rho^{n}) = \mathrm{Spa}(K, \mathcal{O}_K).$$
This is thus an isomorphism over the origin of the ball $\mathbb{B}$. We thus have a diagram
$$\begin{tikzcd} & W_\infty \arrow[d, "\text{pro-étale}"] \\ V_\infty \arrow[ur, "\sim"] \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_\infty \arrow[d, "\substack{\text{pro-étale}\\ \text{cover}}"] \\ \mathbb{B} \arrow[r, "f"'] & \mathbb{B} \end{tikzcd}$$
and $V_\infty \to U_\infty$ is pro-étale. □

4.6.2. Structure over a strictly totaly disconnected space. —

Theorem 4.6.2. — *Let $f : Y \longrightarrow X$ be a separated morphism of qcqs perfectoid spaces. Suppose*

1. *X is strictly totally disconnected,*
2. *The fibers of f at maximal points are profinite i.e. for any maximal point x of X, $x = \mathrm{Spa}(C, \mathcal{O}_C)$, there exists a profinite set P_x such that*
$$Y \times_X \mathrm{Spa}(C, \mathcal{O}_C) \simeq \mathrm{Spa}(C, \mathcal{O}_C) \times \underline{P_x}.$$

Then, f is affinoid pro-étale and Y is strictly totally disconnected.

Proof. — We use the factorization

$$Y \longrightarrow \underline{X \underset{\pi_0(X)}{\times} \pi_0(Y)} \longrightarrow X.$$

Using this we can suppose that $\pi_0(Y) \xrightarrow{\sim} \pi_0(X)$. Let $Z = \mathrm{Im}(|f|) \subset |X|$, a pro-constructible generalizing subset that is represented by an affinoid perfectoid space pro-étale over X, see Theorem 4.4.3. Let us note, by abuse of notation, this perfectoid space Z. We are going to prove that $Y \xrightarrow{\sim} Z$. For this, according to Proposition 3.12.1, we have to prove that for any affinoid perfectoid field (K, K^+) the map $Y(K, K^+) \to X(K, K^+)$ is injective. The valuation criterion of separatedness implies it is sufficient to prove that $Y(K, K^\circ) \to X(K, K^\circ)$ is injective. Up to replacing X by a connected component C and Y by $Y \times_X C$, we can suppose that X and thus Y is connected. We have now, $X = \mathrm{Spa}(C, C^+)$.

Let now $y_1, y_2 \in Y$ be two maximal points that have the same image $x = \mathrm{Spa}(C, \mathcal{O}_C)$, the unique maximal point of X, in X. We have $y_1, y_2 \in P_x$ a profinite set. Suppose $y_1 \neq y_2$. We can thus find a decomposition $f^{-1}(x) = U_1 \coprod U_2$ in open/closed subsets U_1 and U_2 of the profinite set $f^{-1}(x)$ such that $y_1 \in U_1$ and $y_2 \in U_2$. Since U_1 and U_2 are closed in the pro-constructible subset $f^{-1}(x)$ of Y, they are pro-constructible in Y. Let us look at $\overline{U_1}$ and $\overline{U_2}$ their closure in Y that are thus the set of their specializations. Since $Y \to X = \mathrm{Spa}(C, C^+)$ is separated, any point of Y has a unique generalization in $f^{-1}(x)$. We deduce that $Y = \overline{U_1} \coprod \overline{U_2}$ which contradicts the connectedness of Y. □

Definition 4.6.3. — A morphism $Y \to X$ of perfectoid spaces is quasi-pro-étale if there exists a pro-étale cover $X' \to X$ such that $Y \times_X X' \to X'$ is pro-étale.

Corollary 4.6.4. — *For $f : Y \to X$ a morphism of perfectoid spaces the following are equivalent:*

1. *f is quasi-pro-étale*
2. *There exists an open cover $Y = \cup_{i \in I} V_i$, such that $f_{|V_i} : V_i \to X$ has profinite geometric fibers at all maximal points of the target.*
3. *For any strictly totally disconnected perfectoid space X' with a morphism $X' \to X$, $Y \times_X X' \to X'$ is pro-étale.*

4.7. The structure pro-étale sheaf

Proposition 4.7.1. — *The presheaf $\mathcal{O} : X \mapsto \mathcal{O}(X)$ on the big pro-étale perfectoid site is a sheaf.*

Proof. — Let us fix a base affinoid perfectoid space X and $\varpi \in \mathcal{O}(X)$ a pseudo-uniformizer. One has $\mathcal{O} = \mathcal{O}^+[\frac{1}{\varpi}]$ and $\mathcal{O}^+ = \varprojlim_{n \geq 1} \mathcal{O}^+/\varpi^n$ as presheaves on affinoid perfectoid spaces/X. It thus suffices to prove that $\mathcal{O}^+/\varpi$ is a pro-étale sheaf. But if $\nu : X_{\text{proét}} \to X_{\text{ét}}$ then

$$\mathcal{O}^+_{X_{\text{proét}}}/\varpi = \nu^{-1}(\mathcal{O}^+_{X_{\text{ét}}}/\varpi).$$

□

CHAPTER 5

THE v-TOPOLOGY

5.1. Definition and basic properties

5.2. The v-sheaf $\mathscr{O}$

Theorem 5.2.1. — *The correspondence $X \mapsto \mathscr{O}(X)$ is a sheaf for the v-topology on perfectoid spaces.*

Proof. — Since for any perfectoid space X, $\mathscr{O}(X) \subset \prod_{x\in X} K(x)$ (use Theorem 2.1.16), this is a separated presheaf. Let $Y \to X$ be a v-cover. We want to prove that $\mathscr{O}(X)$ surjects to $\ker(\mathscr{O}(Y) \rightrightarrows \mathscr{O}(Y \times_X Y))$. Using the separatedness, we can suppose that X is affinoid perfectoid. Using the quasi-compacity assumption in the definition of a v-cover, one can suppose, up to replacing Y by a finite disjoint union of affinoid perfectoid opensubsets of Y, that Y is affinoid perfectoid too. Using the sheaf property for the pro-étale topology we can suppose, up to replacing X by a totally disconnected pro-étale cover $\widetilde{X} \to X$ of X and Y by a totally disconnected pro-étale cover of $Y \times_X \widetilde{X}$, that Y and X are totally disconnected. The result is then a consequence of Proposition 5.2.2. $\square$

Proposition 5.2.2. — *Let $X = \mathrm{Spa}(A, A^+)$ be a totally disconnected perfectoid space and $Y \to X$ with $Y = \mathrm{Spa}(B, B^+)$ affinoid perfectoid. Then*

1. *B^+/ϖ is a flat A^+/ϖ-algebra.*
2. *If $Y \to X$ is surjective then B^+/ϖ is a faithfully flat A^+/ϖ-algebra.*

Proof. — Let $\pi : |X| \to \pi_0(X)$ and note $f : Y \to X$. Consider the sheaf of $\mathscr{O}_X^+/\varpi$-algebras $(f_*\mathscr{O}_Y^+)/\varpi$ on $|X|$. We push it forward to $\pi_0(X)$ and look at

$$\mathscr{F} := \pi_*\big(f_*(\mathscr{O}_Y^+)/\varpi\big)$$

as a sheaf of $\mathscr{A} := \pi_*(\mathscr{O}_X^+/\varpi)$-algebras on the profinite set $\pi_0(X)$. For $c \in \pi_0(X)$, if $x \in X$ is the closed point of the connected component $\pi^{-1}(c)$,

$$\mathscr{A}_c = \mathscr{O}_{X,x}^+/\varpi.$$

One has $R^1\pi_* = 0$ since X is totally disconnected and thus

$$\mathscr{F} = \big((\pi f)_*\mathscr{O}_Y^+\big)/\varpi.$$

We deduce that

$$\mathscr{F}_c = \big((\pi f)_*\mathscr{O}_Y^+\big)_c\,/\varpi.$$

Since $\big((\pi f)_*\mathscr{O}_Y^+\big)_c$ has no ϖ-torsion and $\mathscr{O}_{X,x}^+$ is a valuation ring, $\mathscr{F}_c$ is a flat $\mathscr{A}_c$-module. Now, if

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of $\Gamma(\pi_0(X), \mathscr{A})$-modules, it defines an exact sequence of $\mathscr{A}$-modules

$$0 \longrightarrow M' \otimes \mathscr{A} \longrightarrow M \otimes \mathscr{A} \longrightarrow M'' \otimes \mathscr{A} \longrightarrow 0.$$

If we apply $- \otimes_{\mathscr{A}} \mathscr{F}$ to this sequence the obtained sequence

$$0 \longrightarrow M' \otimes \mathscr{F} \longrightarrow M \otimes \mathscr{F} \longrightarrow M'' \otimes \mathscr{F} \longrightarrow 0$$

is exact since it is exact stalkwise thanks to the preceding flatness result. Taking global sections we obtains that the sequence

$$0 \longrightarrow M' \otimes \Gamma(\pi_0(X), \mathscr{F}) \longrightarrow M \otimes \Gamma(\pi_0(X), \mathscr{F}) \longrightarrow M'' \otimes \Gamma(\pi_0(X), \mathscr{F}) \longrightarrow 0$$

is exact and thus $\Gamma(\pi_0(X), \mathscr{F}) = B^+/\varpi$ is a flat $\Gamma(\pi_0(X), \mathscr{A}) = A^+/\varpi$-modules.

The faithful flatness result is obtained using the surjectivity of the specialization map, see Proposition 2.13.6. $\square$

5.3. Subcanonicity

5.4. A descent result

CHAPTER 6

DIAMONDS

6.1. Definition and first properties

In the the following we note $\mathrm{Perf}_{\mathbb{F}_p}$ for the category of $\mathbb{F}_p$-perfectoid spaces.

Definition 6.1.1. — A pro-étale sheaf X on $\mathrm{Perf}_{\mathbb{F}_p}$ is a diamond if there exists a perfectoid space $\widetilde{X}$ and an equivalence relation $R \subset \widetilde{X} \times \widetilde{X}$ represented by a perfectoid space such that

1. both morphisms $R \rightrightarrows \widetilde{X}$ are pro-étale i.e. R is a pro-étale equivalence relation,
2. $X \simeq \widetilde{X}/R$ as pro-étale sheaves.

6.2. $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$

We note $\mathrm{Perf}_{\mathbb{Q}_p}$ for the category of perfectoid spaces over $\mathrm{Spa}(\mathbb{Q}_p)$.

Definition 6.2.1. — We note $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ the functor on $\mathrm{Perf}_{\mathbb{F}_p}$ defined by the formula

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond(S) = \big\{(S^\sharp, \iota) \mid S^\sharp \in \mathrm{Perf}_{\mathbb{Q}_p},\ \iota : S \xrightarrow{\sim} S^{\sharp,\flat}\big\}/\sim .$$

One of the first basic results is the following.

Proposition 6.2.2. — *The functor* $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ *is a* v*-sheaf.*

Proof. — We prove more generally that $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is a v-sheaf. We have to prove that the functor on $\mathbb{F}_p$-affinoid perfectoid rings

$$(A, A^+) \longmapsto \mathscr{D}_1(A, A^+)/W(A^+)^\times$$

is a v-sheaf. We can rewrite this functor as

$$(A, A^+) \longmapsto \varinjlim_{\varpi \in A^{\circ\circ} \cap A^\times} \big([\varpi]W(A^+) + p\big)/1 + [\varpi]W(A^+).$$

where $\varpi \leq \varpi'$ if $\varpi\varpi'^{-1} \in A^+$. Now the result is an easy consequence of the following two facts:

- $\mathcal{O}^+$ and thus $(A, A^+) \mapsto \varinjlim_{\varpi \in A^{\circ\circ} \cap A^\times} p + [\varpi]W(A^+)$ is a v-sheaf.
- If (R, R^+) is an $\mathbb{F}_p$-perfectoid ring then

$$\varinjlim_{\varpi \in A^{\circ\circ} \cap A^\times} H^1_v(\mathrm{Spa}(R, R^+), 1 + [\varpi]W(\mathcal{O}^+)) = 0.$$

This last fact is an easy consequence of the almost vanishing of $H^1_v(\mathrm{Spa}(R, R^+), \mathcal{O}^+)$ using some standard dévissages. $\square$

Proposition 6.2.3. — *The* v*-sheaf* $\mathrm{Spa}(\mathbb{Q}_p)^\diamond$ *is a diamond.*

Proof. — The tilting equivalence defines a morphism of v-sheaves

$$\mathrm{Spa}(\mathbb{C}_p^\flat) \longrightarrow \mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$

This is an epimorphism of pro-étale sheaves. In fact, if $X^\sharp$ is an untilt over $\mathbb{Q}_p$ of $X \in \mathrm{Perf}_{\mathbb{F}_p}$, for any $K|\mathbb{Q}_p$ of finite degree contained in $\overline{\mathbb{Q}}_p$, according to the purity theorem (Theorem 3.11.1), $X^\sharp \otimes_{\mathbb{Q}_p} K$ is perfectoid with $X^\sharp \otimes_{\mathbb{Q}_p} K \to X$ is étale finite. Over the pro-étale cover

$$(X^\sharp \hat{\otimes}_{\mathbb{Q}_p} \mathbb{C}_p)^\flat = \varprojlim_{K|\mathbb{Q}_p} (X^\sharp \otimes_{\mathbb{Q}_p} K)^\flat \longrightarrow X$$

there is a morphism to $\mathrm{Spa}(\mathbb{C}_p^\flat)$. We have thus proven the pro-étale surjectivity assertion.

It remains to compute the diagram

$$\mathrm{Spa}(\mathbb{C}_p)^\flat \times_{\mathrm{Spa}(\mathbb{Q}_p)^\diamond} \mathrm{Spa}(\mathbb{C}_p^\flat) \rightrightarrows \mathrm{Spa}(\mathbb{C}_p^\flat).$$

This is equivalent to computing

$$\mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{C}_p) \rightrightarrows \mathrm{Spa}(\mathbb{C}_p).$$

where the fiber product is taken in the category of functors on $\mathrm{Perf}_{\mathbb{Q}_p}$. But as functors on $\mathrm{Perf}_{\mathbb{Q}_p}$, $\mathrm{Spa}(\mathbb{C}_p) = \varprojlim_{K|\mathbb{Q}_p} \mathrm{Spa}(K)$ where as before $K|\mathbb{Q}_p$ is finite degree. We then have

$$\mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{C}_p) = \varprojlim_{K|\mathbb{Q}_p} \mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(K).$$

We deduce that

$$\mathrm{Spa}(\mathbb{C}_p) \times_{\mathrm{Spa}(\mathbb{Q}_p)} \mathrm{Spa}(\mathbb{C}_p) = \mathrm{Spa}(\mathbb{C}_p) \times \underline{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}$$

and thus our equivalence relation is pro-étale with

$$\mathrm{Spa}(\mathbb{Q}_p)^\diamond = \mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)}. \qquad \square$$

The proof gives formulas like

$$\begin{aligned}\mathrm{Spa}(\mathbb{Q}_p)^\diamond &= \mathrm{Spa}(\mathbb{C}_p^\flat)/\underline{\mathrm{Gal}(\overline{\mathbb{Q}}_p|\mathbb{Q}_p)} \\ &= \mathrm{Spa}\,\Big(\underbrace{\widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^\flat}_{\mathbb{F}_p((T^{1/p^\infty}))}\Big)/\underline{\mathbb{Z}_p^\times}\end{aligned}$$

where if $\varepsilon = (\zeta_{p^n})_{n\geq 0} \in \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}^\flat$ then $T = \varepsilon + 1$. Here the action of $\underline{\mathbb{Z}_p^\times}$ is given by the usual formula

$$T^a = \sum_{k\geq 0} \binom{a}{k} (T-1)^k$$

for $a \in \mathbb{Z}_p^\times$.

Remark 6.2.4. — The v-sheaf $\mathrm{Spa}(\mathbb{Z}_p)^\diamond$ is not a diamond. This still has the geometric structure of a spatial v-sheaf and one can do some geometry with it. For example (???) one can prove that $\mathrm{Spa}(\mathbb{Z}_p)^\diamond \to *$ is ℓ-cohomologically smooth for $\ell \neq p$.

Proposition 6.2.5. — *There is an equivalence of categories*

$$\mathrm{Perf}_{\mathbb{Q}_p} \xrightarrow{\sim} \mathrm{Perf}_{\mathbb{F}_p}/\,\mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$

This induces an isomorphism of the v, resp. pro-étale, resp. étale, topoi

$$\widetilde{\mathrm{Perf}_{\mathbb{Q}_p}} \xrightarrow{\sim} \widetilde{\mathrm{Perf}_{\mathbb{F}_p}}/\,\mathrm{Spa}(\mathbb{Q}_p)^\diamond.$$

6.3. The diamond associated to a $\mathbb{Q}_p$-adic space

6.3.1. Background on the Shilov boundary. — Let K be a complete non-archimedean field. Let A be a K-affinoid algebra (by this we mean "à la Tate" in the sense that it is topologically of finite type. Let

$$\widetilde{A} = A^\circ / A^{\circ\circ}$$

that is a reduced k_K-algebra of finite type ([10, Corollary 3-Section 6.3.4]). Let $\|.\|_\infty : A \to \mathbb{R}_+$ be the spectral norm. One has

$$A^\circ = \{f \in A \mid \|f\|_\infty \leq 1\}.$$

Let us look at the specialization map

$$\mathrm{sp} : \mathcal{M}(A) \longrightarrow \mathrm{Spec}(\widetilde{A}).$$

This is surjective (Proposition 2.13.6). Moreover, for each minimum prime ideal $\mathfrak{p}$ of $\widetilde{A}$,

$$\mathrm{sp}^{-1}(\mathfrak{p}) = \{\xi_{\mathfrak{p}}\}$$

is one point.

6.4. The main result

Theorem 6.4.1. — *1. Let X be a $\mathrm{Spa}(\mathbb{Q}_p)$-adic space. The functor*

$$\begin{aligned} X^\diamond : \mathrm{Perf}_{\mathbb{F}_p} &\longrightarrow \mathit{Sets} \\ S &\longmapsto \{(S^\sharp, \iota, f) \mid S^\sharp \in \mathrm{Perf}_{\mathbb{Q}_p},\ \iota : S \xrightarrow{\sim} S^{\sharp,\flat},\ f : S^\sharp \to X\}/\sim \end{aligned}$$

is represented by a diamond.

2. If $K|\mathbb{Q}_p$ is a complete non-archimedean field, this defines a fully faithful functor

$$(-)^\diamond : \{\textit{locally of finite type normal adic spaces}/\,\mathrm{Spa}(K)\} \hookrightarrow \{\textit{diamonds}\}/\,\mathrm{Spa}(K)^\diamond.$$

Proof. — Everything is easily reduced to the case $X = \mathrm{Spa}(A, A^+)$ with (A, A^+) a complete sheafy affinoid Tate ring. Let (B, B^+) be the completion of (A, A^+) with respect to the spectral norm associated to a choice of a pseudo-uniformizer and some number in $]0,1[$ as in Theorem 2.1.16. There is a morphism $(A, A^+) \to (B, B^+)$ inducing an isomorphism of functors on $\mathrm{Perf}_{\mathbb{F}_p}$, $\mathrm{Spa}(B, B^+)^\diamond \xrightarrow{\sim} \mathrm{Spa}(A, A^+)^\diamond$. We can thus suppose that A is uniform. We can then apply Proposition 4.1.1 to deduce the existence of a filtered inductive system $(A_i, A_i^+)_{i \in I}$ of uniform affinoid finite étale (A, A^+)-algebras such that

$$(A_\infty, A_\infty^+) = \widehat{\varinjlim_{i \in I} (A_i, A_i^+)}$$

is perfectoid and for all indices $i \geq j$, $|\mathrm{Spa}(A_i, A_i^+)| \to |\mathrm{Spa}(A_j, A_j^+)|$ is surjective. Let

$$X_\infty = \mathrm{Spa}(A_\infty, A_\infty^+) \in \mathrm{Perf}_{\mathbb{Q}_p}.$$

There is a morphism

$$X_\infty^\flat \longrightarrow X^\diamond. \tag{10}$$

It is easily deduced from the fact that $\mathcal{O}$ is a v-sheaf on $\mathrm{Perf}_{\mathbb{Q}_p}$, see Theorem 5.2.1, that the functor $S \mapsto X(S)$ is a v-sheaf on $\mathrm{Perf}_{\mathbb{Q}_p}$. Proposition 6.2.2 coupled with the preceding fact implies that $X^\diamond$ is a v-sheaf on $\mathrm{Perf}_{\mathbb{F}_p}$. Let us prove that the morphism (10) is an epimorphism of pro-étale sheaves. □

6.5. Diamonds as v-sheaves

Recall the following result by Gabber ([50, Proposition 0APL]): every algebraic space is an fpqc sheaf. This result has the following analog.

Theorem 6.5.1. — *Diamonds are v-sheaves.*

6.6. Subdiamonds

The following result says that a lot of objects are diamonds.

Proposition 6.6.1. — *Let Y be a diamond and $X \subset Y$ a sub-v-sheaf. Then X is a diamond.*

Proof. — Let $\widetilde{Y}$ be a perfectoid space and $R \subset \widetilde{Y} \times \widetilde{Y}$ be a pro-étale equivalence relation such that

$$Y \simeq \widetilde{Y}/R$$

as pro-étale sheaves. $\square$

CHAPTER 7

ARTIN'S CRITERION FOR SPATIAL DIAMONDS

APPENDIX A

THE COTANGENT COMPLEX AND DEFORMATION THEORY

A.1. Background on homotopical algebra

A.1.1. Simplicial objects. —

A.1.1.1. *Definition and basic properties.* — Simplicial sets have been introduced, mainly under the influence of Daniel Kan ([5]), as a combinatorial model for the homotopy type of topological spaces.

We note Δ for the category of finite ordered sets. For $n \geq 0$ we note

$$[n] = \{0 < 1 < \cdots < n\} \in \operatorname{Ob} \Delta.$$

Let $\mathcal{C}$ be a category. By definition a simplicial object X in $\mathcal{C}$ is a functor

$$X : \Delta^{op} \longrightarrow \mathcal{C}.$$

For such an X and $n \geq 0$ we note

$$X_n$$

for the value of the functor on $[n]$. When $\mathcal{C}$ is the category of sets we call X_n the set of n-simplices.

Definition A.1.1. — Let $n \geq 1$. For $0 \leq i \leq n$ the inclusion $[n-1] \to [n]$ obtained by "missing i" gives rise to what we call a *face morphism*

$$\boxed{d_{n,i} : X_n \longrightarrow X_{n-1}}$$

For $0 \leq i \leq n-1$, the surjection $[n] \to [n-1]$ obtained by "identifying i and $i+1$" gives rise to a *degeneracy morphism*

$$\boxed{s_{n-1,i} : X_{n-1} \longrightarrow X_n}$$

We send to Remark A.1.6 for the terminology face and degeneracy morphism. Those satisfy the relations

1. $d_i d_j = d_{j-1} d_i$ if $i < j$
2. $d_i s_j = s_{j-1} d_i$ if $i < j$
3. $d_i s_j = \mathrm{Id}$ if $i = j$ or $i = j+1$
4. $d_i s_j = s_j d_{i-1}$ if $i > j+1$
5. $s_i s_j = s_{j+1} s_i$ if $i \leq j$.

Reciprocally, a collection of objects $(X_n)_{n \geq 0}$ together with the face and degeneracy morphisms satisfying the preceding relations defines a simplicial object. We note

$$s\mathcal{C}$$

for the category of simplicial objects in $\mathcal{C}$.

Example A.1.2. — The truncated object $X_1 \overset{d_0}{\underset{d_1}{\overset{\xleftarrow{s_0}}{\rightrightarrows}}} X_0$ is defined by the relations $d_1 s_0 = \mathrm{Id} = d_0 s_0$.

Example A.1.3. — The functor S that sends a finite ordered set E to the topological space $\{(\lambda_e)_{e\in E} \in \mathbb{R}_+^E \mid \sum_e \lambda_e = 1\}$ is a simplicial topological space with S_n the standard simplex. The face maps are projections to faces of the simplex and the degeneracy maps are inclusions of the faces.

Example A.1.4. — Let X be a topological space. We can compose the preceding functor $\Delta \to \mathrm{Top}$ with the functor $\mathscr{C}(-, X)$ of continuous functions toward X. We obtain the singular set

$$Sing(X) \in s\,\mathrm{Sets}$$

whose n-simplices, $Sing_n X$, are continuous maps from the standard simplex $\{(t_0, \dots, t_n) \in \mathbb{R}_+^n \mid \sum_i t_i = 1\}$ to X. This defines a functor

$$\mathrm{Top} \longrightarrow s\,\mathrm{Sets}\,.$$

A.1.1.2. *Reconstructing a simplicial set from its simplices.* — We are going to see that one can reconstruct a simplicial set as a gluing of its n-simplices when n varies in a canonical way.

Definition A.1.5. — For any integer $n \geq 0$ define

$$\Delta_n \in s\,\mathrm{Sets}$$

as $\mathrm{Hom}(-, [n])$.

Yoneda lemma shows that for any simplicial set X,

$$\boxed{\mathrm{Hom}(\Delta_n, X) = X_n}$$

For $n \geq 1$ and $0 \leq i \leq n-1$ there is a morphism

$$\sigma_{n-1,i} : \Delta_n \longrightarrow \Delta_{n-1}$$

that corresponds to the projection $[n] \to [n-1]$ identifying i and $i+1$. The degeneracy map $s_{n-1,i} : X_{n-1} \to X_n$ is then given by

$$\boxed{s_{n-1,i} = \sigma^*_{n-1,i}} \tag{11}$$

that is to say it sends the $n-1$-simplex $\Delta_{n-1} \xrightarrow{u} X$ to

$$\Delta_n \xrightarrow[\text{projection}]{\sigma_{n-1,i}} \Delta_{n-1} \xrightarrow{u} X.$$

There is a morphism for $0 \leq i \leq n$,

$$\delta_{n,i} : \Delta_{n-1} \longrightarrow \Delta_n$$

that corresponds to the injection $[n-1] \to [n]$ missing i. One then has

$$\boxed{d_{n,i} = \delta^*_{n,i}} \tag{12}$$

that is to say it sends the n-simplex $\Delta_n \xrightarrow{u} X$ to

$$\Delta_{n-1} \xrightarrow[\text{inclusion of the } i\text{-th face}]{\delta_{n,i}} \Delta_n \xrightarrow{u} X.$$

Remark A.1.6. — Equations (11) and (12) show that, in the case of simplicial sets, one can interpret the face and degeneracy maps as "real" face and degeneracy maps after interpreting X_n as a set of "n-simplices". The strength of the simplicial objects approach is that one can use the intuition of algebraic topology and simplicial sets in a more general context of simplicial objects in any category where this interpretation does not exist but the intuition remains.

Let $\mathscr{S}_X$ be the category whose objects are couples (E, x) where $E \in \mathrm{Ob}\,\Delta$ and $x \in X(E)$, and

$$\mathrm{Hom}((E, x), (E', x')) = \{f : E \to E' \mid f^* x' = x\}.$$

This is equivalent to the category of couples (n, x) where $n \in \mathbb{N}$, $x \in X_n$ and

$$\mathrm{Hom}((n, x), (n', x')) = \{f : [n] \to [n'] \mid f^* x' = x\}.$$

We note $\Delta_n \to X$ for an object of this last category. Its morphisms are thus commutative diagrams

$$\begin{tikzcd} \Delta_n \arrow[d] \arrow[dr] & \\ \Delta_m \arrow[r] & X \end{tikzcd}$$

via the identification $\mathrm{Hom}(\Delta_n, \Delta_m) = \mathrm{Hom}([n], [m])$. There is a functor

$$\begin{aligned} F : \mathscr{S}_X &\longrightarrow s\,\mathrm{Sets} \\ (E, x) &\longmapsto \mathrm{Hom}(-, E) \end{aligned}$$

and a morphism of functors, where X is the constant functor with values X,

$$\begin{aligned} F &\Longrightarrow X \\ F(E, x) &\longmapsto x. \end{aligned}$$

This induces an isomorphism of simplicial sets

$$\varinjlim_{\mathscr{S}_X} F \xrightarrow{\sim} X.$$

We note this as

$$\boxed{\varinjlim_{\Delta_n \to X} \Delta_n \xrightarrow{\sim} X}$$

This gives a meaning to the fact that X is obtained by gluing

$$\coprod_{X_n} \Delta_n$$

when n varies using the face and degeneracy maps, like a CW complex together with its cell decomposition. More precisely, X is the quotient of

$$\coprod_{n \geq 0} \coprod_{X_n} \Delta_n$$

by the equivalence relation in the category of simplicial sets generated by

$$(x, \delta_{n,i}(a)) \sim (d_{n,i}(x), a) \text{ for } x \in X_n \text{ and } a \in \Delta_{n-1}$$

and

$$(x, \sigma_{n,i}(a)) \sim (s_{n,i}(x), a) \text{ for } x \in X_n \text{ and } a \in \Delta_{n+1}.$$

A.1.1.3. *Geometric realization of a simplicial set.* — There is a geometric realization functor

$$
\begin{aligned}
|-| : s\,\mathrm{Sets} &\longrightarrow \mathrm{Top} \\
X &\longmapsto \varinjlim_{\Delta_n \to X} |\Delta_n|
\end{aligned}
$$

where $|\Delta_n|$ is the usual simplex in $\mathbb{R}_+^{n+1}$. This is a left adjoint functor to the singular set functor

$$\boxed{\mathrm{Hom}(|X|, T) = \mathrm{Hom}(X, Sing(T))}$$

Example A.1.7. — 1. The degeneracy map

$$|\sigma_{n-1,i}| : |\Delta_n| \longrightarrow |\Delta_{n-1}|$$

of Equation (12) is given by, for $(\lambda_0, \dots, \lambda_n) \in \mathbb{R}_+^n$ satisfying $\sum_{i=0}^n \lambda_i = 1$,

$$(\lambda_0, \dots, \lambda_n) \longmapsto (\lambda_0, \dots, \lambda_{i-1}, \lambda_i + \lambda_{i+1}, \lambda_{i+2}, \dots, \lambda_n).$$

2. The face map

$$|\delta_{n,i}| : |\Delta_{n-1}| \longrightarrow |\Delta_n|$$

of Equation (11) is given by

$$(\lambda_0, \dots, \lambda_{n-1}) \longmapsto (\lambda_0, \dots, \lambda_{i-1}, 0, \lambda_i, \dots, \lambda_{n-1}).$$

The topological space $|X|$ is naturally a CW complex. More precisely, let

$$X_n^{ndeg} = X_n \setminus \cup_{i=0}^{n-1} \mathrm{Im}(s_{n-1,i})$$

be the set of "non-degenerate" simplices. There is a quotient map

$$\coprod_{n \geq 0} \coprod_{X_n} |\Delta_n| \longrightarrow |X|.$$

For $k \geq 0$ let

$$|X|_k = \mathrm{Im}\Big(\coprod_{n \leq k} \coprod_{X_n} |\Delta_n| \longrightarrow |X| \Big).$$

This is the k-skeleton of $|X|$ and

$$|X|_k \setminus |X|_{k-1} = \coprod_{X_k^{ndeg}} |\mathring{\Delta}_k|$$

where

$$|\mathring{\Delta}_k| = \big\{ (\lambda_0, \dots, \lambda_k) \in \mathbb{R}_{>0}^k \bigm| \sum_{i=0}^k \lambda_i = 1 \big\}.$$

A.1.1.4. *Augmented simplicial objects.* —

A.1.2. Homotopies. —

A.1.2.1. *Two equivalent formulations.* — Let X and Y be simplicial objects in a category $\mathcal{C}$. Let

$$f, g : X \longrightarrow Y$$

be two morphisms in $s\mathcal{C}$.

Definition A.1.8. — An homotopy between f and g is the data of morphisms $(h_{n,i})_{n \geq 0, 0 \leq i \leq n}$, where $h_{n,i} : X_n \to Y_{n+1}$, satisfying

- $d_0 h_0 = f_n$
- $d_{n+1} h_n = g_n$

- $d_i h_j = \begin{cases} h_{j-1} d_i \text{ if } i < j \\ d_i h_{i-1} \text{ if } i = j \\ h_j d_{i-1} \text{ if } i > j+1 \end{cases}$
- $s_i h_j = \begin{cases} h_j s_{i-1} \text{ if } i > j \\ h_{j+1} s_i \text{ if } i \leq j \end{cases}$

This definition is justified by the following results whose proof are elementary computation (as usual in homotopical algebra, a lot of notions come from natural notions issued from simplicial sets and classical algebraic topology, see Remark A.1.6).

Proposition A.1.9 **([4, Proposition 3.1]).** — *For $h = (h_{n,i})_{0 \leq i \leq n}$ an homotopy between f and g define $(r_{n,i})_{0 \leq i \leq n+1}$ where $r_{n,i} : X_n \to Y_n$ is defined by the formula*

$$r_{n,0} = f_n, \quad r_{n,n+1} = g_n, \quad r_{n,i} = d_{n+1,i}\, h_{n,i} \text{ if } 1 \leq i \leq n.$$

The correspondence $h \mapsto r$ induces a bijection between homotopies between f and g and collections of maps $r_{n,i} : X_n \to Y_n$, $0 \leq i \leq n+1$ satisfying

- $r_{n,0} = f_n$, $r_{n,n+1} = g_n$
- $d_{n,i}\, r_{n,j} = \begin{cases} r_{n-1,j-1}\, d_{n,i} \text{ if } i < j \\ r_{n-1,j}\, d_{n,i} \text{ if } i \geq j \end{cases}$
- $s_{n,i}\, r_{n,j} = \begin{cases} r_{n+1,j+1}\, s_{n,i} \text{ if } i < j \\ r_{n+1,j}\, s_{n,i} \text{ if } i \geq j. \end{cases}$

The inverse to this bijection if given by the formula

$$h_{n,i} = r_{n+1,i}\, s_{n,i}.$$

The preceding proposition allows us to recast the definition of an homotopy in its classical framework of algebraic topology.

Lemma A.1.10. — *Suppose that $\mathcal{C}$ is the category of Sets. Then, an homotopy between f and g is the same as a morphism*

$$h : X \times \Delta_1 \longrightarrow Y$$

such that the following diagram

$$\begin{tikzcd} X \arrow[r, shift left, "\mathrm{Id} \times \{0\}"] \arrow[r, shift right, "\mathrm{Id} \times \{1\}"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & X \times \Delta_1 \arrow[r, "h"] & Y \end{tikzcd}$$

commutes.

A.1.2.2. *The Kan extension property.* —

A.1.3. Homotopy groups. —

A.1.3.1. *Connected components.* — Let X be a simplicial set. Consider the relation on X_0 generated by the relation

$$x \sim y \Longleftrightarrow \exists z \in X_1,\ d_1(z) = x \text{ and } d_0(z) = y.$$

One has $x \sim x$ and we still denote $\sim$ for the equivalence relation generated by the preceding. We then define

$$\pi_0(X) = X_0/\sim\, = \mathrm{coeq}\left(X_1 \rightrightarrows X_0 \right).$$

A.1.3.2. *Homotopy groups.* — Let $n \geq 1$. Define the simplicial set

$$\partial\Delta_n = \bigcup_{0\leq i\leq n} \delta_{n,i}(\Delta_{n-1}) \subset \Delta_n.$$

One has

$$(\partial\Delta_n)_m = \{f : [m] \to [n] \mid f \text{ is not surjective}\}.$$

Its geometric realization is

$$|\partial\Delta_n| = \Big\{(\lambda_0, \dots, \lambda_n) \in \mathbb{R}_+^n \mid \sum_{i=0}^{n} \lambda_i = 1 \text{ and } \exists i,\ \lambda_i = 0\Big\}.$$

Let X be a simplicial set and fix a base point $x \in X_0$. For any n, the unique map $[n] \to [0]$ defines a map $X_0 \to X_n$ that is a composite of degeneracy maps. The sphere

$$S^n = \Delta_n/\partial\Delta_n$$

is pointed too since $(S^n)_0$ has only one element. Let us note 0 this point. Let us now look at the morphisms of pointed simplicial sets

$$\mathrm{Hom}((S^n, 0), (X, x)).$$

This is identified with

$$\big\{y \in X_n \mid d_{n,0}(y) = \cdots = d_{n,n}(y) = x_{n-1}\big\}$$

where $x_{n-1} \in X_{n-1}$ is obtained from x via $[n-1] \to [0]$. We define a relation $\sim$ on $\mathrm{Hom}((S^n, 0), (X, x))$ in the following way. For two morphisms

$$f, g : (S^n, 0) \longrightarrow (X, x)$$

we say that $f \sim g$ if there exists a morphism

$$h : S^n \times \Delta_1 \longrightarrow X$$

such that

$$\begin{tikzcd} S^n \arrow[r, shift left, "\mathrm{Id}\times\{0\}"] \arrow[r, shift right, "\mathrm{Id}\times\{1\}"'] \arrow[rr, bend left, "f"] \arrow[rr, bend right, "g"'] & S^n \times \Delta_1 \arrow[r, "h"] & X \end{tikzcd}$$

and

$$h_{|0\times\Delta_1} = x.$$

A.1.4. The Dold-Kan correspondence. —

A.1.5. Simplicial rings. —

A.1.6. Bisimplicial sets and the Eilenberg-Zilber theorem. —

A.1.7. The Hurewicz theorem. —

A.2. Adjoint functors, comonads and the Bar construction

A.2.1. Construction of the simplicial object. — Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a couple of adjoint functors where F is left adjoint to G. Let

$$\begin{aligned} u : F \circ G &\Longrightarrow \mathrm{Id}_{\mathcal{D}} \\ v : \mathrm{Id}_{\mathcal{C}} &\Longrightarrow G \circ F \end{aligned}$$

be the natural transformations associated with the adjunction. Let us note

$$T = F \circ G$$

and

$$\begin{aligned} \eta : T &\overset{u}{\Longrightarrow} \mathrm{Id}_{\mathcal{D}} \\ \mu : T &\overset{FvG}{\Longrightarrow} T^2. \end{aligned}$$

The triple (T, η, μ) is what is called a *comonad*, the comonad property being the relations

$$\begin{aligned} T\mu \circ \mu &= \mu T \circ \mu : T \Longrightarrow T^3 \\ T\eta \circ \mu &= \eta T \circ \mu = \mathrm{Id}_T : T \Longrightarrow T. \end{aligned}$$

Remark A.2.1. — If we have a diagram of functors between categories

$$\begin{tikzcd} \mathcal{C}_1 \arrow[r, "I"] & \mathcal{C}_2 \arrow[r, bend left, "J"{name=J}] \arrow[r, bend right, "K"'{name=K}] & \mathcal{C}_3 \arrow[r, "L"] & \mathcal{C}_4 \arrow[from=J, to=K, Rightarrow, "u"] \end{tikzcd}$$

with u a natural transformation, we note

$$LuI : LJI \Longrightarrow LKI$$

the natural transformation defined by the following rule: for $X \in \mathcal{C}_1$,

$$(LuI)_X = L(u_{I(X)}) : L(J(I(X))) \longrightarrow L(K(I(X))).$$

Starting from such a comononad we can construct a functor

$$\mathcal{D} \longrightarrow \{\text{augmented simplicial objects in } \mathcal{D}\}$$

via the following rule. Let X be an object of $\mathcal{D}$. For $n \geq 0$ set

$$X_n = T^{n+1}(X)$$

with the face and degeneracy maps

$$\begin{aligned} d_{n,i} &= T^i \eta T^{n+1-i} : X_n \longrightarrow X_{n-1} \\ s_{n,i} &= T^i \mu T^{n+1-i} : X_n \longrightarrow X_{n+1}. \end{aligned}$$

This is augmented toward X via

$$X_\bullet \xrightarrow{\eta} X.$$

The simplicial identities are deduced from the comonad relations. In low degree this is the following diagram

$$\begin{tikzcd}[column sep=large] T^3(X) \arrow[r, shift left=6, "T^2\eta"] \arrow[r, shift left=3, leftarrow, "T\mu"] \arrow[r, "T\eta T"] \arrow[r, shift right=3, leftarrow, "\mu T"] \arrow[r, shift right=6, "\eta T^2"'] & T^2(X) \arrow[r, shift left=3, "T\eta"] \arrow[r, leftarrow, "\mu"] \arrow[r, shift right=3, "\eta T"'] & T(X) \arrow[r, "\eta"] & X. \end{tikzcd}$$

This is the so-called *Bar construction.*

A.2.2. Contractibility and acyclicity. — One then has the following result: the augmented simplicial object

$$G(X_\bullet) \longrightarrow G(X)$$

is contractible (see [4] for the different notions of contractible simplicial objects). In fact, the augmented simplicial complex $G(X_\bullet) \to G(X)$ comes with an extra map

$$t_n : G(X_n) \longrightarrow G(X_{n+1})$$

(where we note $X_{-1} = X$ and $t_{-1} : G(X) \to G(X_0)$) defined by the formula

$$G(X_n) \xrightarrow{v} GFG(X_n) = G(X_{n+1}).$$

Let $G(X)_{cons}$ be now the constant simplicial object with value $G(X)$ and face/degeneracy maps given by the identity. There are then two maps of simplicial objects

$$G(X_\bullet) \underset{b}{\overset{a}{\rightleftarrows}} G(X)_{cons}$$

where $G(X_\bullet) \to G(X)_{cons}$ is given by the augmentation and the other one by

$$G(X) \xrightarrow{t_{n-1}\circ\cdots\circ t_{-1}} G(X_n).$$

In fact, one verifies immediately that for $n \geq -1$,

$$\begin{cases} d_{n+1,0} \circ t_n = \mathrm{Id} \\ d_{n+1,i} \circ t_n = t_{n-1} \circ d_{n,i-1} \ \text{ for } 0 < i \leq n+1 \end{cases} \tag{13}$$

which implies in particular that the preceding map is a morphism of simplicial objects. Recall now the following definition.

Definition A.2.2. — Let $f, g : X \to Y$ be two maps of simplicial objects in a category. An homotopy between f and g is the data of maps $(h_{n,i})_{0\leq i\leq n}$, where $h_{n,i} : X_n \to Y_{n+1}$, satisfying

- $d_0 h_0 = f_n$
- $d_{n+1} h_n = g_n$
- $d_i h_j = \begin{cases} h_{j-1} d_i \text{ if } i < j \\ d_i h_{i-1} \text{ if } i = j \\ h_j d_{i-1} \text{ if } i > j+1 \end{cases}$
- $s_i h_j = \begin{cases} h_j s_{i-1} \text{ if } i > j \\ h_{j+1} s_i \text{ if } i \leq j \end{cases}$

For simplicial objects in the category of sets or in an abelian category this is the same as a morphism $X \times \Delta_1 \to Y$ giving f via the inclusion $X \times \{0\} \hookrightarrow X \times \Delta_1$ and g via the inclusion $X \times \{1\} \hookrightarrow X \times \Delta_1$. This is in fact deduced from the following proposition whose proof is a simple computation.

Proposition A.2.3 **([4, Proposition 3.1]).** — *For $h = (h_{n,i})_{0\leq i\leq n}$ an homotopy between f and g define $(r_{n,i})_{0\leq i\leq n+1}$ where $r_{n,i} : X_n \to Y_n$ is defined by the formula*

$$r_{n,0} = f_n, \quad r_{n,n+1} = g_n, \quad r_{n,i} = d_{n+1,i}\, h_{n,i} \ \textit{if}\ 1 \leq i \leq n.$$

The correspondence $h \mapsto r$ induces a bijection between homotopies between f and g and collections of maps $r_{n,i} : X_n \to Y_n$, $0 \leq i \leq n+1$ satisfying

- $r_{n,0} = f_n$, $r_{n,n+1} = g_n$
- $d_{n,i}\, r_{n,j} = \begin{cases} r_{n-1,j-1}\, d_{n,i} \ \textit{if}\ i < j \\ r_{n-1,j}\, d_{n,i} \ \textit{if}\ i \geq j \end{cases}$
- $s_{n,i}\, r_{n,j} = \begin{cases} r_{n+1,j+1}\, s_{n,i} \ \textit{if}\ i < j \\ r_{n+1,j}\, s_{n,i} \ \textit{if}\ i \geq j. \end{cases}$

The inverse to this bijection if given by the formula

$$h_{n,i} = r_{n+1,i}\, s_{n,i}.$$

If the ambient category is abelian such an homotopy induces an homotopy of the associated morphisms between the (unnormalized) chain complexes:

$$\widetilde{h} := \sum_{i=0}^{n} (-1)^i h_{n,i} : X_n \longrightarrow Y_{n+1}$$

satisfies

$$\widetilde{h}\partial + \partial\widetilde{h} = f - g$$

where $\partial_n = \sum_{i=0}^{n} (-1)^i d_{n,i}$ is the usual boundary map.

Remark A.2.4. — One has to be careful that in general the relation "f is homotpic to g" is not an equivalence relation. This is the case if Y is a Kan simplicial set but not in general.

Remark A.2.5. — Both definitions of an homotopy, via h or r, are useful. The point of view of h as a collection of morphisms $X_n \to Y_{n+1}$ gives rise to the homotopy of the associated chain complexes if the category is abelian. The point of view of r as a collection of morphisms $X_n \to Y_n$ is typically useful for the proof of Lemma A.2.13.

One then has the following result.

Proposition A.2.6. — *One has $a \circ b = \mathrm{Id}$ and $b \circ a \sim \mathrm{Id}$ (homotopic to the identity).*

Proof. — The equality $a \circ b = \mathrm{Id}$ is immediate. Define now for $0 \leq i \leq n$, $h_{n,i} : G(X_n) \to G(X_{n+1})$ as the composite

$$G(X_n) \xrightarrow{d_{n,0}} G(X_{n-1}) \xrightarrow{d_{n-1,0}} \cdots \xrightarrow{d_{n-i+1,0}} G(X_{n-i}) \xrightarrow{t_{n-i}} G(X_{n-i+1}) \xrightarrow{t_{n-i+1}} \cdots \xrightarrow{t_n} G(X_{n+1})$$

that is to say

$$h_{n,i} = t_n \circ t_{n-1} \circ \cdots \circ t_{n-i} \circ d_{n-i+1,0} \circ \cdots \circ d_{n-1,0} \circ d_{n,0}.$$

One verifies using Equation (13) that this satisfies the axioms of Definition A.2.2 of an homotopy between the identity and $b \circ a$. $\square$

As a particular case, if $\mathcal{C}$ is an abelian category then the chain complex associated with $G(X_\bullet)$ is homotopy equivalent to the zero complex. For $n \geq 0$, if

$$\partial_n = \sum_{i=0}^{n} (-1)^i d_{n,i} : G(X_n) \longrightarrow G(X_{n-1})$$

one has, using Equation (13),

$$\partial_{n+1} t_n + t_{n-1} \partial_n = \mathrm{Id}\,.$$

This allows us to construct plenty of canonical resolutions when the functor G is conservative. This is summarized in the following proposition.

***Proposition A.2.7* (Bar resolution).** — *Let* $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ *be a couple of adjoint functors where F is left adjoint to G. Let $T = FG : \mathcal{D} \to \mathcal{D}$.*

1. *There is a functor*

$$\begin{array}{ccc} \mathcal{D} & \longrightarrow & \{\textit{augmented toward } \mathrm{Id}_{\mathcal{D}} \textit{ simplicial objects in } \mathcal{D}\} \\ X & \longmapsto & [T^{\bullet+1}(X) \to X]. \end{array}$$

2. *The composite of this functor with G is an homotopy equivalence*

$$G(T^{\bullet+1}(X)) \longrightarrow G(X)_{cons}$$

where $G(X)_{cons}$ is the constant simplicial object with value $G(X)$.

Sometimes we use the same formalism as before but in the opposite categories. In this case the preceding structures are called monads. Let us give some examples.

Example A.2.8. — If X is a topos and $f : U \to e$ is a covering of the final object of X the couple of adjoint functors (f^*, f_*) defines for any $\mathscr{F}$ an abelian group in X an augmented cosimplicial complex $\mathscr{F} \to \mathscr{F}_\bullet$ with $\mathscr{F}_n = (f_* f^*)^{n+1}\mathscr{F}$. After applying f^* to this augmented cosimplical abelian groups in X it becomes contractible. Since f is a cover this implies that the chain complex associated to $\mathscr{F} \to \mathscr{F}_\bullet$ is a resolution of $\mathscr{F}$. This gives rise to the Čech cohomology spectral sequence associated to the cover $U \to e$.

Example A.2.9. — Using the notations of the preceding example, the couple of adjoint functors $(f_!, f^*)$ gives rise to a resolution $(f_! f^*)^{\bullet+1}\mathscr{F} \to \mathscr{F}$. When X is the topos associated to a topological space and U is associated to an open cover of this topological space, this gives rise to the compactly supported Čech cohomology spectral sequence.

Example A.2.10. — If X is a topological space and $f : \coprod_{x\in X}\{x\} \to X$ then the couple of adjoint functors (f^*, f_*) gives rise to the Godement resolution.

Example A.2.11. — If $A \to B$ is a morphism of rings the couple of adjoint functors $(B \otimes_A -, \mathrm{Res}_{B/A})$ gives rise for any A-module M to an augmented cosimplicial A-module $M \longrightarrow B^{\otimes(\bullet+1)} \otimes_A M$. If $A \to B$ is faithfully flat this becomes contractible after applying $B \otimes_A -$ and is thus acyclic as a chain complex. This is the standard Čech resolution of M associated to the fpqc cover $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$.

Example A.2.12. — With the notations of the preceding example, for any B-module M there is an augmented simplicial B-module $B^{\otimes(\bullet+1)} \otimes_A M \to M$. Since the associated restriction of scalars functor from B to A is conservative, this is a resolution. For example, for $M = B$, this is the usual Bar resolution

$$\cdots B \otimes_A B \otimes_A B \xrightarrow{\partial} B \otimes_A B \xrightarrow{\partial} B$$

with $\partial(b_1 \otimes b_2 \otimes b_3) = b_1b_2 \otimes b_3 - b_1 \otimes b_2b_3$ and $\partial(b_1 \otimes b_2) = b_1b_2$ that gives rise to the standard resolution in group cohomology when we take $A = \mathbb{Z} \to \mathbb{Z}[G] = B$.

We will need the following lemma later.

Lemma A.2.13. — *Let $f : \mathrm{Id}_{\mathcal{D}} \Longrightarrow \mathrm{Id}_{\mathcal{D}}$ be an endomorphism of the identity functor. Then the two maps*

$$\begin{tikzcd} T^{\bullet+1}(X) \arrow[r] \arrow[d, shift right, "fT^{\bullet+1}"'] \arrow[d, shift left, "T^{\bullet+1}f"] & X \arrow[d, "f"] \\ T^{\bullet+1}(X) \arrow[r] & X \end{tikzcd}$$

are homotopic maps of simplicial complexes.

Proof. — The fact that those are morphisms of simplicial objects is easy. We use the definition of an homotopy of Proposition A.2.3. It suffices in fact to set

$$r_{n,i} = T^i f T^{n+1-i}$$

to obtain the result. ∎

A.3. The cotangent complex

A.3.1. General construction. — Let A be a ring. We consider the two adjoint functors

$$A\text{-algebras} \underset{A[-]}{\overset{\text{forget}}{\rightleftarrows}} \text{Sets}$$

where for a set X, $A[X]$ is the symmetric algebra on the free A-module $A^{(X)}$ which is a polynomial algebra. We then have a canonical polynomial resolution for any A-algebra B,

$$\cdots \rightrightarrows A\Big[A\big[A[B]\big]\Big] \rightrightarrows A\big[A[B]\big] \underset{d_1}{\overset{d_0}{\rightrightarrows}} A[B] \longrightarrow B$$

where for $b_1, \ldots, b_n \in B$ distinct and $P \in A[X_1, \ldots, X_n]$ one has

$$\begin{aligned} d_0\Big(\,[\,P(\,[b_1], \ldots, [b_n]\,)\,]\,\Big) &= P(\,[b_1], \ldots, [b_n]\,) \\ d_1\Big(\,[\,P(\,[b_1], \ldots, [b_n]\,)\,]\,\Big) &= [\,P(b_1, \ldots, b_n)\,]. \end{aligned}$$

and the degeneracy map $s_0 : A[B] \to A\big[A[B]\big]$ is given by

$$s_0(\,[\,b\,]\,) = \big[\,[\,b\,]\,\big].$$

This is a simplicial A-algebra $P_{B/A,\bullet}$ satisfying:

- for each $n \geq 0$, $P_{B/A,n}$ is a polynomial A-algebra on a set X_n,
- one has $\pi_0(P_{B/A,\bullet}) = B$ and $\pi_i(P_{B/A,\bullet}) = 0$ for $i > 0$ by applying point (2) of Proposition A.2.7 that says that the augmented simplicial set $P_{B/A,\bullet} \to B$ is contractible,
- for each n and $0 \leq i \leq n$, the degeneracy morphism $s_{n,i}$ satisfies $s_{n,i}(X_n) \subset X_{n+1}$.

Definition A.3.1. — The cotangent complex $\mathbb{L}_{B/A}$ of $A \to B$ is the chain complex of B-modules associated to the simplicial B-module

$$\Omega^1_{P_{B/A,\bullet}/A} \otimes_{P_{B/A,\bullet}} B.$$

We will often see it as an object of $D(B)$, the derived category of B-modules, but nevertheless let us remark it is well defined canonically as a complex. It is concentrated in ≤ 0 degrees. The natural augmentation $\mathbb{L}_{B/A} \to \Omega^1_{B/A}$ induces an isomorphism

$$H^0(\mathbb{L}_{B/A}) \xrightarrow{\sim} \Omega^1_{B/A}.$$

A.3.2. Definition as a derived functor. — Let us begin with a definition.

Definition A.3.2. — Let $\mathscr{B}_\bullet$ be a simplicial A-algebra augmented toward the A-algebra B

$$\mathscr{B}_\bullet \to B.$$

We say that $\mathscr{B}_\bullet$ is a polynomial resolution of B is

- $\pi_n(\mathscr{B}_\bullet) = 0$ for $n > 0$,
- $\pi_0(\mathscr{B}_\bullet) \xrightarrow{\sim} B$,
- for all $n \geq 0$, $\mathscr{B}_n$ is a polynomial A-algebra on a $X_n \subset \mathscr{B}_n$ i.e. $A[X_n] \xrightarrow{\sim} \mathscr{B}_n$,
- the degeneracy maps $s_{n,i} : \mathscr{B}_n \to \mathscr{B}_{n+1}$, $0 \leq i \leq n$, satisfy $s_{n,i}(X_n) \subset X_{n+1}$,

Typically, $P_{B/A,\bullet} \to B$ is a polynomial resolution of B. We now have the following deep result by Quillen.

Theorem A.3.3 **(Quillen** [44, Chapter II-Section 4], [31, Theorem 4.17]**)**
The category of simplicial A-algebras has the structure of a model category in the sense of Quillen where a morphism $f : \mathscr{B}_\bullet \to \mathscr{C}_\bullet$ is

1. *a weak equivalence if for all $n \geq 0$, $\pi_n(\mathscr{B}_\bullet) \xrightarrow{\sim} \pi_n(\mathscr{C}_\bullet)$,*
2. *a fibration if the induced map $\mathscr{B}_\bullet \to \pi_0(\mathscr{B}_\bullet) \times_{\pi_0(\mathscr{C}_\bullet)} \mathscr{C}_\bullet$ is surjective.*

Moreover, if $\mathscr{B}_\bullet \to B$ is a polynomial resolution then $\mathscr{B}_\bullet \to B$ is a cofibration.

We can now see the cotangent complex as derived functor i.e. a Kan extension. In fact, we have

$$\varinjlim_{\mathscr{B}_\bullet \to B} \Omega_{\mathscr{B}_\bullet/A} \otimes_{\mathscr{B}_\bullet} B \xrightarrow{\sim} \mathbb{L}_{B/A} \in D(B)$$

where $\mathscr{B}_\bullet \to B$ goes through the set of resolutions of B by simplicial A-algebras and where the colimit is essentially constant. Here by a resolution we mean $\pi_i(\mathscr{B}_\bullet) = 0$ if $i > 0$ and $\pi_0(\mathscr{B}_\bullet) \xrightarrow{\sim} B$. The limit becomes constant as soon as $\mathscr{B}_\bullet \to B$ is a polynomial resolution, in which case

$$\Omega_{\mathscr{B}_\bullet/A} \otimes_{\mathscr{B}_\bullet} B \xrightarrow{\sim} \mathbb{L}_{B/A} \in D(B).$$

A.4. Basic properties of the cotangent complex

A.4.1. The fundamental triangle. — For $A \to B \to C$ morphisms of rings we obtain the usual exact sequence of C-modules

$$\Omega^1_{B/A} \otimes_B C \longrightarrow \Omega^1_{C/A} \longrightarrow \Omega^1_{C/B} \longrightarrow 0.$$

We are going to see that this extends to a long exact sequence and even an exact triangle in $D(C)$.

Proposition A.4.1. — *For $A \longrightarrow B \longrightarrow C$ morphisms of rings there is an exact triangle in the triangulated category $D(C)$*

$$\begin{tikzcd} \mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C \arrow[rr] & & \mathbb{L}_{C/A} \arrow[dl] \\ & \mathbb{L}_{C/B} \arrow[ul, "+1"] & \end{tikzcd}$$

Proof. — Consider the polynomial resolution

$$\mathscr{B}_\bullet = P_{B/A,\bullet} \longrightarrow B.$$

Via the projection to B and the morphism $B \to C$, there is a morphism of simplicial rings $\mathscr{B}_\bullet \to C$ where C is seen as a constant simplicial ring. Consider now the bi-simplicial ring

$$P_{C/\mathscr{B}_\bullet,\bullet}.$$

Let us form its diagonal

$$\mathscr{C}_\bullet = {}^\Delta P_{C/\mathscr{B}_\bullet,\bullet}.$$

It is augmented toward C and this is again a polynomial resolution of C thanks to [32, Proposition 1.9-Chapter IV] (since we work with simplical abelian groups, one can use alternatively the generalized Eilenberg-Zilber theorem, [32, Theorem 2.5-Chapter IV]). We now have a diagram of simplicial rings

$$\begin{tikzcd} & \mathscr{B}_\bullet \arrow[r] \arrow[d] & \mathscr{C}_\bullet \arrow[d] \\ A \arrow[r] \arrow[ur] & B \arrow[r] & C \end{tikzcd}$$

where $\mathscr{B}_\bullet$ is a polynomial resolution of B as an A-algebra and for each n, $\mathscr{C}_n$ is a polynomial $\mathscr{B}_n$-algebra. From this we deduce an exact sequence of simplicial $\mathscr{C}_\bullet$-modules

$$0 \longrightarrow \Omega^1_{\mathscr{B}_\bullet/A} \otimes_{\mathscr{B}_\bullet} \mathscr{C}_\bullet \longrightarrow \Omega^1_{\mathscr{C}_\bullet/A} \longrightarrow \Omega^1_{\mathscr{C}_\bullet/\mathscr{B}_\bullet} \longrightarrow 0.$$

In fact, for each n the morphisms $A \to \mathscr{B}_n \to \mathscr{C}_n$ are of the form $A \to A[X_i]_{i\in I} \to A[X_i, Y_j]_{i\in I, j\in J}$ for some sets I and J. The exactness of the preceding sequence is thus immediate. We can now apply $-\otimes_{\mathscr{C}_\bullet} C$ to obtain an exact sequence of simplicial C-modules

$$0 \longrightarrow \Omega^1_{\mathscr{B}_\bullet/A} \otimes_{\mathscr{B}_\bullet} C \longrightarrow \Omega^1_{\mathscr{C}_\bullet/A} \otimes_{\mathscr{C}_\bullet} C \longrightarrow \Omega^1_{\mathscr{C}_\bullet/\mathscr{B}_\bullet} \otimes_{\mathscr{C}_\bullet} C \longrightarrow 0.$$

We now have

$$\begin{aligned} \Omega^1_{\mathscr{B}_\bullet/A} \otimes_{\mathscr{B}_\bullet} C &= \Omega^1_{\mathscr{B}_\bullet/A} \otimes_{\mathscr{B}_\bullet} B \otimes_B C \\ &\simeq \mathbb{L}_{B/A} \otimes^{\mathbb{L}}_B C. \end{aligned}$$

We have moreover, since $\mathscr{C}_\bullet$ is a polynomial resolution of C as an A-algebra,

$$\Omega^1_{\mathscr{C}_\bullet/A} \otimes_{\mathscr{C}_\bullet} C \simeq \mathbb{L}_{C/A}.$$

Moreover, there is an isomorphism of $\mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B$-modules

$$\Omega^1_{\mathscr{C}_\bullet/\mathscr{B}_\bullet} \otimes_{\mathscr{B}_\bullet} B \xrightarrow{\sim} \Omega^1_{\mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B/B}$$

induced by the morphism of pairs $(\mathscr{B}_\bullet \to \mathscr{C}_\bullet) \longrightarrow (B \to \mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B)$ (the fact that this induces the announced isomorphism after applying Ω^1 is deduced from writing explicitly $\mathscr{C}_n$ as a polynomial algebra over $\mathscr{B}_n$). Now,

$$\mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B \to C$$

is a polynomial resolution of C as a B-algebra. We thus have

$$\begin{aligned} \Omega^1_{\mathscr{C}_\bullet/\mathscr{B}_\bullet} \otimes C &= \big(\Omega^1_{\mathscr{C}_\bullet/\mathscr{B}_\bullet} \otimes_{\mathscr{B}_\bullet} B\big) \otimes_{\mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B} C \\ &\xrightarrow{\sim} \Omega^1_{\mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B/B} \otimes_{\mathscr{C}_\bullet \otimes_{\mathscr{B}_\bullet} B} C \\ &\simeq \mathbb{L}_{C/B}. \end{aligned}$$

□

A.4.2. Computation via smooth resolutions. — Let $\mathscr{B}_\bullet \to B$ be a simplicial A-algebra resolution such that for all $n \geq 0$, $\mathscr{B}_n$ is a smooth A-algebra.

A.5. Infinitesimal extensions of algebras

A.5.1. The Picard groupoid of infinitesimal extensions of an algebra. — Let $A \to B$ be a morphism of rings. Recall the following ([33, Section 18.4]). If $E \twoheadrightarrow B$ is a surjection of rings with square zero kernel I then I is an $E/I = B$-module. For a given B-module M we can now look at the extensions of commutative A-algebras of A by M, that is to say exact sequences of A-modules

$$0 \longrightarrow M \longrightarrow E \longrightarrow B \longrightarrow 0$$

where E is an A-algebra, $E \to B$ is a surjection of A-algebras with square zero kernel the ideal M as a B-module. A morphism between two such extensions is given by a commutative diagram

$$\begin{tikzcd} & & E' \arrow[d, "f"] \arrow[dr] & & \\ 0 \arrow[r] & M \arrow[ur] \arrow[dr] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

where f is a morphism of A-algebras. The corresponding category

$$\underline{\mathrm{Exalg}}_A(B, M)$$

is a groupoid. The group of automorphisms of any object in this groupoid is identified with

$$\mathrm{Der}_A(B, M) = \mathrm{Hom}_B(\mathbb{L}_{B/A}, M).$$

To any derivation $D \in \mathrm{Der}_A(B, M)$ one associated the automorphism $\mathrm{Id} + D$ of the extension E. Let us note

$$\mathrm{Exalg}_A(B, M) = \pi_0\big(\underline{\mathrm{Exalg}}_A(B, M)\big)$$

the set of automorphism classes of such extensions. The groupoid $\underline{\mathrm{Exalg}}_A(B, M)$ is functorial in B and M. If $f : B' \to B$ is a morphism of A-algebras the pull back via this morphism induces a functor

$$f^* : \underline{\mathrm{Exalg}}_A(B, M) \longrightarrow \underline{\mathrm{Exalg}}_A(B', M)$$

where the pullback by f is defined by the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, equal] & E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & B' \arrow[r] \arrow[d, "f"] & 0 \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

i.e. $E' = E \times_B B'$. For $u : M \to M'$ a morphism of B-modules, it induces a functor

$$u_* : \underline{\mathrm{Exalg}}_A(B, M) \longrightarrow \underline{\mathrm{Exalg}}_A(B, M')$$

where the pushforward by u is defined

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "u"'] \arrow[dr, phantom, "\bigcirc"] & E \arrow[r] \arrow[d] & B \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & M' \arrow[r] & E' \arrow[r] & B \arrow[r] & 0 \end{tikzcd}$$

i.e. $E' = E \coprod_M M'$. Both operations commute: there is a natural isomorphism of functors $f^* u_* \xrightarrow{\sim} u_* f^*$. We now use the language of Picard groupoids ([1, Exposé XVIII-Section 1.4]). The preceding two fonctorialities defines a (strictly commutative) Picard groupoid structure on $\underline{\mathrm{Exalg}}_A(B, M)$

$$\underline{\mathrm{Exalg}}_A(B, M) \times \underline{\mathrm{Exalg}}_A(B, M) \xrightarrow{+} \underline{\mathrm{Exalg}}_A(B, M)$$

via the following rule

$$\xi + \xi' = \Sigma_* \Delta^*(\xi \times \xi')$$

where

- if ξ is the extension $0 \to M \to E \to B \to 0$ and ξ' is the extension $0 \to M \to E' \to B \to 0$, $\xi \times \xi'$ is the extension

$$0 \longrightarrow M \times M \longrightarrow E \times E' \longrightarrow B \times B \longrightarrow 0,$$

- $\Sigma : M \times M \to M$ is the morphism $(m_1, m_2) \mapsto m_1 + m_2$,
- $\Delta : B \to B \times B$ is the diagonal morphism $b \mapsto (b, b)$.

This is moreover equipped with a "B-linear structure". This means that for $\lambda \in B$, there is a multiplication by λ functor

$$\boldsymbol{m}_\lambda : \underline{\mathrm{Exalg}}_A(B, M) \longrightarrow \underline{\mathrm{Exalg}}_A(B, M)$$

that is simply the pushforward via $M \xrightarrow{\times\lambda} M$. It is equipped with compatibilities: natural isomorphisms

$$c_{\lambda,\mu} : \boldsymbol{m}_\lambda \circ \boldsymbol{m}_\mu \xrightarrow{\sim} \boldsymbol{m}_{\lambda\mu}$$

and

$$a_\lambda : \boldsymbol{m}_\lambda \circ + \xrightarrow{\sim} + \circ (\boldsymbol{m}_\lambda \times \boldsymbol{m}_\lambda).$$

All those natural transformations satisfy some evident compatibility relations (that are left to the reader) that equip $\underline{\mathrm{Exalg}}_A(B, M)$ with what we call a B-linear Picard groupoid structure.

A.5.2. Expression in terms of the cotangent complex. — Now, there is a correspondence

$$[M^{-1} \to M^0] \longmapsto gp(M^{-1} \to M^0)$$

that associated to a length 2 complex of B-modules a B-linear Picard groupoid. The objects of $gp(M^{-1} \xrightarrow{\partial} M^0)$ are M^0 and

$$\mathrm{Hom}(m, m') = \{n \in M^{-1} \mid m' = \partial(n) + m\}.$$

The composition rule $\mathrm{Hom}(m, m') \times \mathrm{Hom}(m', m'') \to \mathrm{Hom}(m, m'')$ is given by the addition law of M^{-1}. The B-linear groupoid structure is simply given by the B-module structure of M^0.

This correspondence $gp(-)$

- transforms morphisms of complexes into B-linear functors,
- sends homotopies between morphisms of complexes to B-linear natural transformations of B-linear functors,
- is such that a morphism of complexes induces an equivalence if and only if it is a quasi-isomorphism.

Proposition A.5.1. — *There is an equivalence of B-linear Picard groupoids*

$$gp\big(\tau_{\leq 1} R\,\mathrm{Hom}_B(\mathbb{L}_{B/A}, M)[1]\big) \xrightarrow{\sim} \underline{\mathrm{Exalg}}_A(B, M).$$

Proof. — Consider the surjection

$$\pi : A[B] \longrightarrow B$$

and note I its kernel. This defines a length 2-complex of B-modules

$$I/I^2 \xrightarrow{\delta} \Omega^1_{A[B]/A} \otimes_{A[B]} B$$

where for $x \in I$, $\delta(x \bmod I^2) = dx \otimes 1$. Let us define a B-linear functor

$$gp\big(\mathrm{Hom}_B(\Omega^1_{A[B]/A} \otimes_{A[B]} B, M) \longrightarrow \mathrm{Hom}_B(I/I^2, M)\big) \longrightarrow \underline{\mathrm{Exalg}}_A(B, M).$$

For this let us look at the extension

$$0 \longrightarrow I/I^2 \longrightarrow A[B]/I^2 \longrightarrow B \longrightarrow 0. \tag{14}$$

Pushforward of this extension via a morphism $I/I^2 \to M$ defines a map

$$\mathrm{Ob}\left(gp\big(\mathrm{Hom}_B(\Omega^1_{A[B]/A} \otimes_{A[B]} B, M) \longrightarrow \mathrm{Hom}_B(I/I^2, M)\big)\right) \longrightarrow \mathrm{Ob}\left(\underline{\mathrm{Exalg}}_A(B, M)\right).$$

This map extends to a functor in the following way. The Equation (14) fits into a bigger diagram

$$\begin{tikzcd}
0 \arrow[r] & I/I^2 \arrow[r] \arrow[d, "\delta"] & A[B]/I^2 \arrow[r] \arrow[dl, "f"] & B \arrow[r] & 0 \\
& \Omega^1_{A[B]/A} \otimes_{A[B]} B & & &
\end{tikzcd}$$

where for $x \in A[B]$, $f(x \bmod I^2) = dx \otimes 1$. If ε is the upper extension of B by I/I^2, f induces an isomorphism of extensions of B by $\Omega^1_{A[B]/A} \otimes_{A[B]} B$,

$$\mathbf{0} \xrightarrow{\sim} \delta_* \varepsilon$$

where $\mathbf{0}$ is the trivial extension, de dual numbers $B[I/I^2]$. Now, if $u : I/I^2 \to M$ and $v : \Omega^1_{A[B]/A} \otimes_{A[B]} B \to M$, we have a canonically defined isomorphism

$$\mathbf{0} \xrightarrow{\sim} v_* \delta_* \varepsilon \xrightarrow{\sim} (v\delta)_* \varepsilon.$$

Thus, for $u : I/I^2 \to M$, there is an isomorphism

$$u_* \varepsilon \xrightarrow{\sim} u_* \varepsilon \mathbin{\boldsymbol{+}} \mathbf{0} \xrightarrow{\sim} u_* \varepsilon \mathbin{\boldsymbol{+}} v_* \delta_* \varepsilon \xrightarrow{\sim} (u + v\delta)_* \varepsilon.$$

This defines our functor

$$gp\big(\operatorname{Hom}_B(\Omega^1_{A[B]/A} \otimes_{A[B]} B, M) \longrightarrow \operatorname{Hom}_B(I/I^2, M)\big) \longrightarrow \underline{\operatorname{Exalg}}_A(B, M).$$

For the essential surjectivity, given an extension of B by M, α, the choice of a set-theoretical section of $E \twoheadrightarrow B$ defines a morphism g as in the following diagram

$$\begin{tikzcd} & & & A[B] \arrow[dl, "\exists g"'] \arrow[d, "\pi"] & \\ 0 \arrow[r] & M \arrow[r] & E \arrow[r] & B \arrow[r] & 0. \end{tikzcd}$$

Now, $g_{|I}$ defines a B-linear morphism $u = (g_{|I} \bmod I^2) : I/I^2 \to M$. One verifies immediately that g modulo I^2 defines an isomorphism.

$$u_*\varepsilon \xrightarrow{\sim} \alpha.$$

For the full faithfulness we have to verify the two following points:

1. $\ker\big(\operatorname{Hom}_B(\Omega^1_{A[B]/A} \otimes_{A[B]} B, M) \longrightarrow \operatorname{Hom}_B(I/I^2, M)\big) \xrightarrow{\sim} \operatorname{Aut}(\mathbf{0})$,
2. $u_*\varepsilon \simeq \mathbf{0}$ if and only if u factorizes through δ.

Point (1) is an immediate consequence of the usual exact sequence

$$I/I^2 \xrightarrow{\delta} \Omega^1_{A[B]/A} \otimes_{A[B]} B \longrightarrow \Omega^1_{B/A} \longrightarrow 0.$$

For point (2), this is simply that in the diagram

$$\begin{tikzcd} 0 \arrow[r] & I/I^2 \arrow[r] \arrow[d] & A[B]/I^2 \arrow[r] \arrow[dl, dashed, "\exists ?"] & B \arrow[r] & 0 \\ & M & & & \end{tikzcd}$$

there exists an A-derivation $A[B]/I^2 \to M$ making the diagram commute if and only if there exists a B-linear map making the following diagram commute

$$\begin{tikzcd} I/I^2 \arrow[r, "\delta"] \arrow[d] & \Omega^1_{A[B]/A} \otimes_{A[B]} B \arrow[r] \arrow[dl, dashed, "\exists ?"] & \Omega^1_{B/A} \arrow[r] & 0 \\ M. & & & \end{tikzcd}$$

It remains to prove that

$$\big[\operatorname{Hom}_B(\Omega^1_{A[B]/A} \otimes_{A[B]} B, M) \longrightarrow \operatorname{Hom}_B(I/I^2, M)\big] \simeq \tau_{\leq 1} R\operatorname{Hom}_B(\mathbb{L}_{B/A}, M)$$

in $D(B)$. For this let us look at the truncated simplicial augmented A-algebra

$$P_1 \underset{\partial_1}{\overset{\partial_0}{\rightrightarrows}} P_0 \xrightarrow{\pi} B.$$

Since it is a resolution, $\operatorname{Im}(\partial_0 - \partial_1) = I$. The B-linear morphism

$$\Omega^1_{P_1/A} \otimes_{P_1} B \longrightarrow \Omega^1_{P_0/A} \otimes_{P_0} B$$

sends $dx \otimes 1$ to $d\partial_0 x \otimes 1 - d\partial_1 x \otimes 1 = d(\partial_0 - \partial_1)(x) \otimes 1$. Moreover, $x \mapsto (\partial_0 - \partial_1)(x) \bmod I^2$ is an A-derivation from P_1 to I/I^2; this is a consequence of the formula

$$\forall x, y \in P_1, \quad \partial_0(x)(\partial_0(y) - \partial_1(y)) + \partial_1(y)(\partial_0(x) - \partial_1(x)) = \partial_0(xy) - \partial_1(xy).$$

From this we deduce a morphism of complexes

$$\begin{array}{ccccc} \mathbb{L}^{-2}_{B/A} & \xrightarrow{\partial} & \mathbb{L}^{-1}_{B/A} & \longrightarrow & \mathbb{L}^{0}_{B/A} \\ \downarrow & & \twoheaddownarrow & & \| \\ 0 & \longrightarrow & I/I^2 & \xrightarrow{\delta} & \Omega^1_{A[B]/A} \otimes_{A[B]} B. \end{array}$$

In fact, one verifies by an explicit computation that the composite $\mathbb{L}^{-2}_{B/A} \to \mathbb{L}^{-1}_{B/A} \to I/I^2$ is zero. We thus have a surjective B-linear morphism

$$\nu : \mathbb{L}^{-1}_{B/A}/\operatorname{Im}\partial \longrightarrow I/I^2.$$

This is in fact an isomorphism. We are going to construct an explicit inverse μ. For this let $x \in I$. Write $x = \partial_0(y) - \partial_1(y)$ with $y \in P_1$ and set

$$\mu(x) = dy \otimes 1 \in \mathbb{L}^{-1}/\operatorname{Im}\partial.$$

This is well defined since if $x = \partial_0(y') - \partial_1(y')$ then $y - y' \in \ker\partial$ and thus $y - y' = \partial z$ with $z \in P_2$ (where here $\partial = \partial_0 - \partial_1 + \partial_2$ is the usual boundary map from P_2 to P_1). We thus have $dy \otimes 1 - dy' \otimes 1 \in \operatorname{Im}\partial$. Let $s : P_0 \to P_1$ be the degeneracy morphism. Recall that we have $\partial_0 s = \mathrm{Id}$ and $\partial_1 s = \mathrm{Id}$. Let now $x \in P_0$, $y \in I$ and write $y = \partial(z)$ with $z \in P_1$. We have

$$\mu(xy) = d(s(x)z) \otimes 1 \mod \operatorname{Im}\partial.$$

But now,

$$\begin{aligned} d(s(x)z) &= s(x)dz \otimes 1 + zd(\underbrace{s(x)}_{\substack{\in\ker\partial \\ \text{and thus } \in\operatorname{Im}\partial}}) \otimes 1 \\ &\equiv dz \otimes \pi(x) \mod \operatorname{Im}\partial. \end{aligned}$$

We thus have a factorization as a B-linear morphism

$$\mu : I/I^2 \longrightarrow \mathbb{L}^{-1}/\operatorname{Im}\partial.$$

It is immediate that μ and ν are inverse to each other. At the end we have constructed a quasi-isomorphism

$$\tau_{\geq -1}\mathbb{L}_{B/A} \xrightarrow{\sim} \big[I/I^2 \xrightarrow{\delta} \Omega^1_{A[B]/A} \otimes_{A[B]} B\big]. \qquad \square$$

Corollary A.5.2. — *There is a canonical isomorphism of B-modules for any B-module M,*

$$\operatorname{Ext}^1_B(\mathbb{L}_{B/A}, M) \xrightarrow{\sim} \operatorname{Exalg}_A(B, M).$$

A.6. Deformation theory

We now have all the tools to do some deformation theory.

Theorem A.6.1. — *Let $A \to B$ be a morphism of rings. Let $A' \twoheadrightarrow A$ be a surjection with square zero kernel I. Suppose given a B-module N and a morphism of A-modules $I \to N$. Consider the category of couples (B', u)*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & B' & \longrightarrow & B & \longrightarrow & 0 \\ & & \uparrow & & \uparrow{\scriptstyle u} & & \uparrow & & \\ 0 & \longrightarrow & I & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

where B' is a ring, $B' \twoheadrightarrow B$ is a surjection with square zero kernel N and u is a morphism of rings.

1. *This category is a groupoid.*
2. *There is an obstruction in $\operatorname{Ext}^2_B(\mathbb{L}_{B/A}, N)$ so that this groupoid is non-empty.*

3. *If this is non-empty then its isomorphism classes, the π_0 of this groupoid, is a torsor under* $\mathrm{Ext}^1_B(\mathbb{L}_{B/A}, N)$.
4. *The automorphism group of any object is identified with* $\mathrm{Ext}^0_B(\mathbb{L}_{B/A}, N)$.

Proof. — We only verify point (3), the other assertions being more elementary. The extension ε, $0 \to I \to A' \to A \to 0$, has its isomorphism class that is an element

$$[\varepsilon] \in \mathrm{Ext}^1_A(\mathbb{L}_{A/\mathbb{Z}}, I).$$

The pushforward $v_*\varepsilon$ of ε by $v : I \to N$ has an isomorphism class that is

$$v_*[\varepsilon] \in \mathrm{Ext}^1_A(\mathbb{L}_{A/\mathbb{Z}}, N).$$

Our groupoid is non-empty if and only if

$$v_*[\varepsilon] \in \mathrm{Im}\left(\mathrm{Ext}^1_B(\mathbb{L}_{B/\mathbb{Z}}, N) \longrightarrow \underbrace{\mathrm{Ext}^1_A(\mathbb{L}_{A/\mathbb{Z}}, N)}_{\mathrm{Ext}^1_B(\mathbb{L}_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A B, N)} \right).$$

We now use the exact triangle

$$\begin{tikzcd} \mathbb{L}_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A B \arrow[rr] & & \mathbb{L}_{B/\mathbb{Z}} \arrow[dl] \\ & \mathbb{L}_{B/A} \arrow[ul, "+1"] & \end{tikzcd}$$

to deduce that the obstruction we are looking for is

$$\partial(v_*[\varepsilon])$$

where

$$\mathrm{Ext}^1_B(\mathbb{L}_{B/\mathbb{Z}}, N) \longrightarrow \mathrm{Ext}^1_B(\mathbb{L}_{A/\mathbb{Z}} \otimes^{\mathbb{L}}_A B, N) \xrightarrow{\partial} \mathrm{Ext}^2_B(\mathbb{L}_{B/A}, N).$$

This prove point (3). □

Corollary A.6.2. — *If $A' \twoheadrightarrow A$ is a surjection with square zero kernel I and B is a flat A-algebra there is an obstruction in*

$$\mathrm{Ext}^2_B(\mathbb{L}_{B/A}, I \otimes_A B)$$

to lift B to a flat A'-algebra.

BIBLIOGRAPHY

[1] *Théorie des topos et cohomologie étale des schémas. Tome 1: Théorie des topos* – Lecture Notes in Mathematics, Vol. 269, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck, et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[2] *Théorie des topos et cohomologie étale des schémas. Tome 2* – Lecture Notes in Mathematics, Vol. 270, Springer-Verlag, Berlin-New York, 1972, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de N. Bourbaki, P. Deligne et B. Saint-Donat.

[3] A. Abbes – *Éléments de géométrie rigide. Volume I*, Progress in Mathematics, vol. 286, Birkhäuser/Springer Basel AG, Basel, 2010, Construction et étude géométrique des espaces rigides. [Construction and geometric study of rigid spaces], With a preface by Michel Raynaud.

[4] M. Barr, J. F. Kennison & R. Raphael – "Contractible simplicial objects", *Comment. Math. Univ. Carolin.* **60** (2019), no. 4, p. 473–495.

[5] C. Barwick, M. Hopkins, H. Miller & I. Moerdijk – "Daniel M. Kan (1927–2013)", *Notices Amer. Math. Soc.* **62** (2015), no. 9, p. 1042–1045.

[6] V. G. Berkovich – *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, RI, 1990.

[7] ______, "Étale cohomology for non-archimedean analytic spaces", *Inst. Hautes Études Sci. Publ. Math.* **78** (1993), p. 5–161.

[8] B. Bhatt & P. Scholze – "The pro-étale topology for schemes", *Astérisque* (2015), no. 369, p. 99–201.

[9] S. Bosch – *Lectures on formal and rigid geometry*, Lecture Notes in Mathematics, vol. 2105, Springer, Cham, 2014.

[10] S. Bosch, U. Güntzer & R. Remmert – *Non-archimedean analysis*, Grundlehren der Mathematischen Wissenschaften, vol. 261, Springer-Verlag, Berlin.

[11] S. Bosch & W. Lütkebohmert – "Formal and rigid geometry. I. Rigid spaces", *Math. Ann.* **295** (1993), no. 2, p. 291–317.

[12] ______, "Formal and rigid geometry. II. Flattening techniques", *Math. Ann.* **296** (1993), no. 3, p. 403–429.

[13] S. Bosch, W. Lütkebohmert & M. Raynaud – "Formal and rigid geometry. IV. the reduced fibre theorem", *Invent. Math.* **119**, no. 2, p. 361–398.

[14] ______, "Formal and rigid geometry. III. The relative maximum principle", *Math. Ann.* **302** (1995), no. 1, p. 1–29.

[15] S. Boucksom – "Corps d'Okounkov (d'après Okounkov, Lazarsfeld-Mustaţă et Kaveh-Khovanskii)", *Astérisque* (2014), no. 361, p. Exp. No. 1059, vii, 1–41.

[16] N. Bourbaki – *Éléments de mathématique. Fasc. XXX. Algèbre commutative. Chapitre 5: Entiers. Chapitre 6: Valuations*, Actualités Scientifiques et Industrielles [Current Scientific and Industrial Topics], No. 1308, Hermann, Paris, 1964.

[17] K. Buzzard & A. Verberkmoes – "Stably uniform affinoids are sheafy", *J. Reine Angew. Math.* **740** (2018), p. 25–39.

[18] A. H. Clifford – "Note on Hahn's theorem on ordered abelian groups", *Proc. Amer. Math. Soc.* **5** (1954), p. 860–863.

[19] P. Colmez – "Espaces de Banach de dimension finie", *J. Inst. Math. Jussieu* **1** (2002), no. 3, p. 331–439.

[20] B. Conrad – "Deligne's notes on Nagata compactifications", *J. Ramanujan Math. Soc.* **22** (2007), no. 3, p. 205–257.

[21] D. M. Davis – "Binomial coefficients involving infinite powers of primes", *Amer. Math. Monthly* **121** (2014), no. 8, p. 734–737.

[22] P. Deligne – "Le théorème de plongement de Nagata", *Kyoto J. Math.* **50** (2010), no. 4, p. 661–670.

[23] V. Drinfeld – "A 1-dimensional formal group over the prismatization of spf $\mathbb{Z}_p$", https://arxiv.org/abs/2107.11466.

[24] A. Ducros, C. Favre & J. Nicaise (eds.) – *Berkovich spaces and applications*, Lecture Notes in Mathematics, vol. 2119, Springer, Cham, 2015.

[25] R. Elkik – "Solutions d'équations à coefficients dans un anneau hensélien", *Ann. Sci. École Norm. Sup.* **6** (1974), no. (4), p. 553–603.

[26] L. Fargues – "L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques", in *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in math., 262, Birkhäuser, 2008, p. 1–325.

[27] L. Fargues & J.-M. Fontaine – "Courbes et fibrés vectoriels en théorie de Hodge p-adique", *Astérisque* (2018), no. 406, p. xiii+382, With a preface by Pierre Colmez.

[28] L. Fargues & P. Scholze – "Geometrization of the local langlands correspondence", https://arxiv.org/abs/2102.13459.

[29] K. Fujiwara – "Theory of tubular neighborhood in étale topology", *Duke Math. J.* **80** (1995), no. 1, p. 15–57.

[30] K. Fujiwara & F. Kato – *Foundations of rigid geometry. I*, EMS Monographs in Mathematics, European Mathematical Society (EMS), Zürich, 2018.

[31] P. Goerss & K. Schemmerhorn – "Model categories and simplicial methods", in *Interactions between homotopy theory and algebra*, Contemp. Math., vol. 436, Amer. Math. Soc., Providence, RI, 2007, p. 3–49.

[32] P. G. Goerss & J. F. Jardine – *Simplicial homotopy theory*, Modern Birkhäuser Classics, Birkhäuser Verlag, Basel, 2009, Reprint of the 1999 edition [MR1711612].

[33] A. Grothendieck – "Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas. I", *Inst. Hautes Études Sci. Publ. Math.* (1964), no. 20, p. 259.

[34] M. Hochster – "Prime ideal structure in commutative rings", *Trans. Amer. Math. Soc.* **142** (1969), p. 43–60.

[35] R. Huber – "Bewertungsspektrum und rigide geometrie", http://www2.math.uni-wuppertal.de/~huber/preprints/Bewertungsspektrum%20und%20Rigide%20Geometrie.pdf.

[36] ———, "Continuous valuations", *Math. Z.* **212** (1993), no. 3, p. 455–477.

[37] ———, "A generalization of formal schemes and rigid analytic varieties", *Math. Z.* **217** (1994), no. no. 4, p. 513–551.

[38] ———, *Étale cohomology of rigid analytic varieties and adic spaces.*, Aspects of Mathematics., Friedr. Vieweg and Sohn, Braunschweig, 1996.

[39] R. Huber & M. Knebusch – "On valuation spectra".

[40] P. T. Johnstone – *Stone spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1982.

[41] I. Kaplansky – "Maximal fields with valuations", *Duke Math. J.* **9** (1942), p. 303–321.

[42] K. S. Kedlaya – "Nonarchimedean geometry of Witt vectors", *Nagoya Math. J.* **209** (2013), p. 111–165.

[43] B. Poonen – "Maximally complete fields", *Enseign. Math. (2)* **39** (1993), no. 1-2, p. 87–106.

[44] D. G. Quillen – *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967.

[45] M. Raynaud – *Anneaux locaux henséliens*, Lecture Notes in Mathematics, Vol. 169, Springer-Verlag, Berlin-New York, 1970.

[46] M. Raynaud – "Géométrie analytique rigide d'après Tate, Kiehl,$\cdots$", in *Table Ronde d'Analyse non archimédienne (Paris, 1972)*, Bull. Soc. Math. France, Mém. No. 39–40, Soc. Math. France, Paris, 1974, p. 319–327.

[47] P. Scholze – “étale cohomology of diamonds”, https://arxiv.org/abs/1709.07343.

[48] P. Scholze – “Perfectoid spaces”, *Publ. Math. Inst. Hautes Études Sci.* **116** (2012), p. 245–313.

[49] S. Sen – “Ramification in p-adic Lie extensions”, *Invent. Math.* **17** (1972), p. 44–50.

[50] T. Stacks project authors – “The stacks project”, https://stacks.math.columbia.edu, 2022.

[51] H. Tanaka – “Zariskian adic spaces”, *Kodai Math. J.* **41** (2018), no. 3, p. 652–695.

[52] J. Tate – “Rigid analytic spaces”, *Invent. Math.* **12** (1971), p. 257–289.

[53] J.-P. Wintenberger – “Le corps des normes de certaines extensions infinies de corps locaux; applications”, *Ann. Sci. École Norm. Sup. (4)* **16** (1983), no. 1, p. 59–89.

[54] B. Zavyalov – “Sheafiness of strongy rigid-noetherian huber pairs”, *Math Annalen* **207** (2022).