

# EQUIVARIANT LOCALIZATION, PARITY SHEAVES, AND CYCLIC BASE CHANGE FUNCTORIALITY

TONY FENG

**ABSTRACT.** Lafforgue and Genestier-Lafforgue have constructed the global and (semisimplified) local Langlands correspondences for arbitrary reductive groups over function fields. We apply equivariant localization arguments, inspired by work of Treumann-Venkatesh, to moduli spaces of shtukas, in order to prove properties of these correspondences regarding functoriality for cyclic base change.

Globally, we establish the existence of functorial transfers of mod  $p$  automorphic forms through  $p$ -cyclic base change. Locally, we prove that Tate cohomology realizes cyclic base change functoriality in the mod  $p$  Genestier-Lafforgue correspondence, verifying a function field version of a conjecture of Treumann-Venkatesh.

The proofs draw upon new tools from representation theory, including parity sheaves and Smith-Treumann theory. In particular, we use these to establish a categorification of the base change homomorphism for Hecke algebras, in a joint appendix with Gus Lonergan.

## CONTENTS

1.	Introduction	1
2.	Generalities on Smith theory	7
3.	Parity sheaves and the base change functor	12
4.	Functoriality and the excursion algebra	20
5.	Cyclic base change in the global setting	25
6.	Cyclic base change in the local setting	40
Appendix A. The base change functor realizes Langlands functoriality by Tony Feng and Gus Lonergan		47
References		48

## 1. INTRODUCTION

**1.1. Global results.** Let  $G$  be a reductive group over a global function field  $F$ , of characteristic  $\neq p$ . Let  $k$  be an algebraic closure of  $\mathbf{F}_p$ . We regard the Langlands dual group  ${}^L G$  over  $k$ . Vincent Lafforgue has constructed in [Laf18a, §13] a global “mod  $p$ ” Langlands correspondence

$$\left\{ \begin{array}{c} \text{irreducible cuspidal} \\ \text{automorphic representations} \\ \text{of } G \text{ over } k \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{Langlands parameters} \\ \text{Gal}(F^s/F) \rightarrow {}^L G(k) \end{array} \right\} / \sim.$$

For split groups  $G$ , Lafforgue’s correspondence has been generalized beyond the case of cusp forms by work of Cong Xue [Xue20, Xuea].<sup>1</sup>

Langlands’ principle of functoriality predicts that given a map of  $L$ -groups  $\phi: {}^L H \rightarrow {}^L G$  and an automorphic form  $f$  for  $H$ , there should be a transfer  $f_\phi$  to  $G$ . In this paper we are concerned with a specific type of functoriality: base change functoriality, arising from the case where  $H$  is a reductive group over  $F$ , and  $G = \text{Res}_{E/F}(H_E)$  for a cyclic  $p$ -extension  $E/F$ . The relevant map  $\phi: {}^L H \rightarrow {}^L G$  is the diagonal embedding on the dual groups. We emphasize that it is crucial for our results that the degree of the extension coincides with the characteristic of our automorphic functions.

**Theorem 1.1** (Existence of global base change). *Assume  $p$  is an odd good prime<sup>2</sup> for  $\widehat{G}$ . Let  $\phi: {}^L H \rightarrow {}^L G$  be as above. Let  $\rho$  be a Langlands parameter arising from an automorphic form on  $H$  by Lafforgue(-Xue)’s correspondence. Then  $\phi \circ \rho$  arises from an automorphic form on  $G$  by Lafforgue(-Xue)’s correspondence.*

**Remark 1.2.** The base change of a cuspidal automorphic representation may no longer be cuspidal, so the theorem really requires Xue’s generalization of Lafforgue’s correspondence. Also because of this, the notion of a Langlands parameter “arising from an automorphic form” is a bit subtle, and is explained in §5.2.4 (it is the analog of footnote 2 below for the excursion algebra instead of the Hecke algebra).

Our proof is inspired by work of Treumann-Venkatesh [TV16]. The analog of [TV16] in the function field context would guarantee that for every Hecke eigensystem “appearing in” the space of automorphic forms<sup>3</sup> for  $H$ , the transferred eigensystem “appears in” the space of automorphic forms for  $G$ . For general groups, our theorem is more refined in view of the failure of Multiplicity One. Indeed, Lafforgue’s correspondence can assign different Langlands parameters to Hecke eigenfunctions with the same unramified eigensystem; in fact, it can even assign different parameters to different automorphic forms generating *isomorphic* automorphic representations, with examples occurring already for  $\text{SL}_n$  when  $n \geq 3$  [Bla94, Lap99]. The reason for this is the failure of local conjugacy to imply global conjugacy; see [Laf18a, §0.7] for more discussion of this phenomenon. Our theorem guarantees a transfer with the correct Langlands parameter, which is a subtler property than *cannot* in general be detected by Hecke operators; the proof thus requires more work.

**Remark 1.3.** In fact, the statement of the theorem is conjecturally true with characteristic zero coefficients. This is already established for  $G = \text{GL}_n$  in which the full global Langlands correspondence is already known, using the trace formula. For general groups it does not seem like trace formula methods can prove the characteristic zero analog of Theorem 1.1, because of the issues mentioned in the previous paragraph.

Moreover, our method can be used to prove analogous base change results for some cohomology classes in the moduli of shtukas, which do not necessarily lift to characteristic zero. At present our results towards this are somewhat messy, so we postpone a precise statement.

---

<sup>1</sup>Our results are conditional on the extension of Xue’s results to non-split  $G$ , which are announced in [Xueb, Slide 30] but have not yet appeared in writing at this time.

<sup>2</sup>Explicitly, this means that we require  $p > 2$  if  $\widehat{G}$  is of type  $A, B, C$  or  $D$ ;  $p > 3$  if  $\widehat{G}$  is of type  $G_2, F_4, E_6, E_7$ ; and  $p > 5$  if  $\widehat{G}$  is of type  $E_8$ .

<sup>3</sup>Here say that a Hecke eigensystem “appears in” the space of automorphic forms for  $H$  if, regarding the space of automorphic forms for  $H$  as a module over the Hecke algebra for  $H$ , the corresponding maximal ideal is in the support of this module. We are not necessarily saying that there is actually a function with that eigensystem.

Still, for some groups such as  $\mathrm{GL}_n$ , our theorem gives no more information than a transfer of Hecke eigenvalues, since two semisimple representations into  $\mathrm{GL}_n$  with the same characteristic polynomials are automatically isomorphic by the Brauer-Nesbitt Theorem. However, even in this case our method has the advantage that it also gives information about the *local* Langlands correspondence, which we explain next. This allows us to prove a conjecture of Treumann-Venkatesh on the behavior of base change functoriality in the local Langlands correspondence, which is one of the main motivations for this paper.

**1.2. Local results.** Genestier-Lafforgue have constructed a semi-simplified form of the Local Langlands correspondence over function fields [GL]. More precisely, let  $F_v$  be a function field of characteristic  $\neq p$  and  $W_v$  the Weil group of  $F_v$ . For any reductive group  $G$  over  $F_v$ , [GL] constructs a map

$$\left\{ \begin{array}{c} \text{irreducible admissible} \\ \text{representations of } G(F_v) \text{ over } k \end{array} \right\} / \sim \longrightarrow \left\{ \begin{array}{c} \text{Langlands parameters} \\ W_v \rightarrow {}^L G(k) \end{array} \right\} / \sim .$$

Now let  $H$  be a reductive group over  $F_v$  and  $G = \mathrm{Res}_{E_v/F_v}(H_{E_v})$ , where  $E_v/F_v$  is a cyclic  $p$ -extension, and take  $\phi: {}^L H \rightarrow {}^L G$  as above. Let  $\sigma$  be a generator of  $\mathrm{Gal}(E_v/F_v)$ ; it acts on  $G$  and its induced action on  $G(F_v) = H(E_v)$  is the Galois action. If the isomorphism class of a  $k$ -representation  $\Pi$  of  $G(F_v)$  is preserved by  $\sigma$ , then it should come from base change. For any irreducible admissible representation  $\Pi$  of  $G(F_v)$  whose isomorphism class is fixed by  $\sigma$ , there is a unique  $\sigma$ -action on  $\Pi$  compatible with the  $G(F_v)$ -action (Lemma 6.1). Hence we can form the *Tate cohomology* groups  $T^0(\Pi)$ ,  $T^1(\Pi)$  with respect to the  $\sigma$ -action, which retain actions of  $H(F_v) = G(F_v)^\sigma$ , and are conjecturally admissible  $H(F_v)$ -representations. We prove:

**Theorem 1.4** (Tate cohomology realizes local functoriality). *Assume  $p$  is an odd good prime for  $\widehat{G}$ . Let  $\Pi$  be as above and  $\Pi^{(p)} := \Pi \otimes_{k, \mathrm{Frob}} k$  the Frobenius twist of  $\Pi$ . Let  $\pi$  be any irreducible admissible subquotient of  $T^*(\Pi)$  as an  $H(F_v)$ -representation and  $\rho_\pi: \mathrm{Weil}(\overline{F}_v/F_v) \rightarrow {}^L H(k)$  be the corresponding Langlands parameter constructed by Genestier-Lafforgue. Then  $\phi \circ \rho_\pi$  is the Langlands parameter attached to  $\Pi^{(p)}$  by Genestier-Lafforgue.*

This verifies, for the Genestier-Lafforgue local Langlands correspondence, a conjecture of Treumann-Venkatesh that “Tate cohomology realizes functoriality”; see §6.1 for more discussion of this.

**Remark 1.5.** Over local fields of characteristic zero, forthcoming work of Fargues-Scholze will construct a semisimplified local Langlands correspondence for all reductive groups. Moreover, their construction seems likely to be compatible with our methods, so we are optimistic that our arguments will generalize to prove the analog of Theorem 1.4 with respect to the Fargues-Scholze correspondence.

**1.3. Elements of the proof.** In this subsection we hint at the ingredients in the proofs of Theorem 1.1 and Theorem 1.4.

**1.3.1. The excursion algebra.** In order to convey the substance of the argument, we need to explain a bit more about the correspondences of Lafforgue and Genestier-Lafforgue. They are based on the notion of the *excursion algebra*. We summarize this very briefly below; a more complete discussion appears in §4.

To abstract the situation a bit, given a group  $\Gamma$  and a reductive group  ${}^L G$  over an algebraically closed field  $k$ , Lafforgue introduces the *excursion algebra*  $\mathrm{Exc}(\Gamma, {}^L G)$  (which is commutative) whose key property is that (see §4.3):

There is a canonical bijection between homomorphisms  $\text{Exc}(\Gamma, {}^L G) \rightarrow k$  and semi-simple Langlands parameters<sup>4</sup>  $\Gamma \rightarrow {}^L G(k)$ .

So, if  $\text{Exc}(\Gamma, {}^L G)$  acts on a vector space, then to each (generalized) eigenvector  $v$  of this action we get a maximal ideal  $\mathfrak{m}_v \subset \text{Exc}(\Gamma, {}^L G)$ , and therefore a “Langlands parameter”  $\rho_v: \Gamma \rightarrow {}^L G(k)$  which is well-defined modulo  $\widehat{G}$ -conjugacy. For  $\Gamma = \text{Gal}(\overline{F}/F)$ , Lafforgue constructs an action of  $\text{Exc}(\Gamma, {}^L G)$  on the space of cuspidal automorphic functions for  $G$ , thus defining a global Langlands correspondence by this mechanism.

For  $\Gamma = \text{Weil}(\overline{F}_v/F_v)$ , Genestier-Lafforgue construct an action of  $\text{Exc}(\Gamma, {}^L G)$  on any irreducible admissible representation of  $G(F_v)$ . Since the action is  $G(F_v)$ -equivariant, the irreducibility forces it to factor through a character of  $\text{Exc}(\Gamma, {}^L G)$ , which gives the local Langlands correspondence of [GL].

**Remark 1.6** (The excursion algebra as functions on the representation stack). The following perspective, due to Drinfeld and explained in [Laf18b], offers a more conceptual way to picture the situation. There is a “representation stack”  $\text{Rep}(\Gamma, {}^L G)$  which parametrizes  ${}^L G$ -valued parameters of  $\Gamma$ , meaning homomorphisms  $\Gamma \rightarrow {}^L G$  modulo the action of  $\widehat{G}$ -conjugation. If  $k$  had characteristic zero then  $\text{Exc}(\Gamma, {}^L G)$  would be the ring of functions on the representation stack  $\text{Rep}(\Gamma, {}^L G)$  [Zhu, Remark 2.1.20]. When  $k$  has positive characteristic (which is the situation in this paper) we speculate that the same is true up to issues of derivedness and reducedness; in any case the interpretation of  $k$ -points remains valid.

The excursion algebra has an explicit presentation with generators  $S_{I,f,(\gamma_i)}$  indexed by:  $I$  a finite set,  $f \in \mathcal{O}(\widehat{G} \backslash ({}^L G)^I / \widehat{G})$ ,  $(\gamma_i)_{i \in I} \in \Gamma^I$ . If we imagine  $S_{\{0, \dots, n\}, f, (\gamma_0, \dots, \gamma_n)}$  as a function on the representation stack, its value on a representation  $\rho: \Gamma \rightarrow {}^L G(k)$  is  $f((\rho(\gamma_0 \gamma_n), \rho(\gamma_1 \gamma_n), \dots, \rho(\gamma_{n-1} \gamma_n), \rho(\gamma_n)))$ .

**1.3.2. Equivariant localization.** We now explain the strategy of our proof. It will be instructive to compare it to work of Treumann-Venkatesh [TV16], so we begin by recalling their setup. Momentarily assuming that  $F$  is a characteristic 0 number field, let  $Y_G, Y_H$  be locally symmetric spaces associated to  $G, H$ , with compatible level structures. Then  $\text{Gal}(E/F)$  acts on  $Y_G$  through its action on  $G$ , and for good choices of level structures  $Y_H$  is a connected component of  $Y_G^{\text{Gal}(E/F)}$ . Treumann-Venkatesh show that for any Hecke eigensystem  $\{h_{v,V} \mapsto \chi(h_{v,V})\}$  occurring in the action of the Hecke algebra for  $H$  acting on  $H^*(Y_H; k)$ , a certain transferred eigensystem  $\{h_{w,W} \mapsto \chi(h_{w,\phi^* W})\}$  occurs in the Hecke algebra for  $G$  acting on  $H^*(Y_G; k)$ .

Now suppose that  $F$  is a global function field, where Lafforgue(-Xue) has constructed an action of the excursion algebra on the space of compactly supported automorphic forms for any reductive  $G$ . We show that for any eigensystem  $\{S_{I,f,(\gamma_i)_{i \in I}} \mapsto \chi(S_{I,f,(\gamma_i)_{i \in I}})\}$  occurring in the action of  $\text{Exc}(\text{Gal}(F^s/F), {}^L H)$  on the space of compactly supported automorphic functions for  $H$ , a transferred eigensystem occurs in the action of  $\text{Exc}(\text{Gal}(F^s/F), {}^L G)$  on the space of compactly supported automorphic functions for  $G$ . This gives control over the Satake parameters because Hecke operators at unramified places are among the excursion operators, but it also gives a lot of additional information. In particular, if one believes in local-global compatibility then taking all the  $\gamma_i$  to be in the Weil group at a particular place  $v$  should give information about the semi-simplified local Langlands correspondence at  $v$ , and this is indeed the source of our traction on the local functoriality.

<sup>4</sup>See §4.1.5 for the precise definition of this.

The method of Treumann-Venkatesh is based on relating  $H^*(Y_G; k)$  and  $H^*(Y_H; k)$  using *equivariant localization* theorems for a space with  $\mathbf{Z}/p\mathbf{Z}$ -action, which fall under the heading of Smith theory. (We note that the core idea first occurs in [Clo14], in the context of quaternion algebras over  $\mathbf{Q}$ , wherein the topological aspect becomes trivial.) In general, this can be phrased as an isomorphism of *Tate cohomology*, which is the composition of Tate’s construction with the usual cohomology, and it says:

$$T^*(X; k) \cong T^*(X^{\mathbf{Z}/p\mathbf{Z}}; k).$$

In the setting of arithmetic manifolds, Treumann-Venkatesh show that these equivariant localization isomorphisms are “sufficiently Hecke-equivariant” to establish a transfer of Hecke eigensystems. We show that in the function field situation, the equivariant localization theorems are similarly “sufficiently equivariant” for the excursion operators.

The proof of this equivariance is very different from that of Treumann-Venkatesh, because the excursion action arises in a much less direct manner than the Hecke action (which is the reason for the name “excursion algebra”). Lafforgue’s construction of the excursion action works by chasing cohomology classes through a plethora of auxiliary cohomology groups, of *moduli spaces of shtukas* with coefficients in perverse sheaves indexed by  $\mathrm{Rep}_k(({}^L G)^I)$  (ultimately coming from the Geometric Satake equivalence). The upshot is that we need to prove compatible equivariant localization theorems for “enough” of these cohomology groups. This resembles the situation of Treumann-Venkatesh, except that we must compare cohomologies not only with constant coefficients, but with coefficients in various perverse sheaves.

The difficulty here is that the theory of perverse sheaves (and consequently the Geometric Satake equivalence) does not interface well with restriction to subvarieties. Because of this, it is very unclear how to even relate the coefficient sheaves whose cohomologies should be compared. The one exception is the constant sheaves on the trivial Schubert strata, which in our context can be thought of as corresponding to the trivial representation of  $({}^L G)^I$ ; this case is the function-field analog of [TV16].

**1.3.3. Smith theory for sheaves.** Our solution to the difficulty raised above hinges on a purely representation-theoretic problem. The Geometric Satake equivalence asserts that<sup>5</sup>  $\mathrm{Rep}_k({}^L G)$  is equivalent to  $P_{G(\mathcal{O})}(\mathrm{Gr}_G; k)$ . Therefore, given a map  $\phi: {}^L H \rightarrow {}^L G$  over  $k$  there is a corresponding functor  $\mathrm{Res}(\phi): P_{G(\mathcal{O})}(\mathrm{Gr}_G; k) \rightarrow P_{H(\mathcal{O})}(\mathrm{Gr}_H; k)$ . To utilize it, we need a “geometric” description of the functor  $\mathrm{Res}(\phi)$  (e.g., which does not pass through the above equivalence).

We solve this problem in the context of  $p$ -cyclic base change functoriality, giving a categorification of the *Brauer homomorphism* of Treumann-Venkatesh. Since it would take much more setup to say anything substantial about the content, let us just touch on some of the novel ingredients. For one, we invoke the theory of *parity sheaves* introduced in [JMW14]. The reason they come up is that we want to employ “sheaf-theoretic Smith-Treumann theory” [Tre19, LL, RW]. This necessitates passing through certain “exotic” categories, which can be interpreted as categories of sheaves on the affine Grassmannian with coefficients in  $E_\infty$ -ring spectra. These are morally derived categories but they have no t-structure; because of this, they interact poorly with the theory of perverse sheaves. However, it turns out that these exotic categories have enough structure to support a well-behaved theory of parity sheaves.

---

<sup>5</sup>For this equivalence, one has to be careful with how the  $L$ -group is defined. See §4.1 for a precise discussion.

**Remark 1.7** (Analogy to the twisted trace formula). For automorphic forms in characteristic 0, cyclic base change is established for some groups by comparison of the trace formula for  $H$  with the twisted trace formula for  $G$ . The idea of the twisted trace formula is that “twisting” an operator by the automorphism  $\sigma$  picks out the contribution from the  $\sigma$ -fixed summands, which should come from base change.

Our argument can, to some extent, be viewed as a categorification of such a comparison. It was modeled on certain trace computations carried out in a very special situation in [Fen20]. Here, instead of relating traces of (Hecke and Frobenius) operators acting on vector spaces of automorphic forms, as one would do in the classical theory, we relate certain cohomology groups of shtukas which can (at least morally) be viewed as traces of (Hecke and Frobenius) operators acting on *categories* of automorphic sheaves by the formalism of [Gai]. The analog of the twisted trace is Tate cohomology, which functions to “pick out” the contribution from  $\sigma$ -fixed isomorphism classes (but also forces us to work modulo  $p$ ).

#### 1.4. Further questions.

- (1) Some version of our story should go through the generality of any group  $G$  with  $\mathbf{Z}/p\mathbf{Z}$ -action, as was treated in [TV16]. Our arguments mostly work at this level of generality; the most serious problem is that the additional examples are nearly all in bad characteristics, and this screws up the representation-theoretic input about parity sheaves – in particular, parity sheaves need no longer be perverse in bad characteristic. A notable exception is a type of automorphic induction studied in [Clo17], which we hope to address in future work.
- (2) Xinwen Zhu has formulated a conjectural description of the cohomology of shtukas in terms of coherent sheaves on the moduli stack of Langlands parameters [Zhu]. Is it possible to view our results in terms of his picture, perhaps as some kind of ( $K$ -theoretic) equivariant localization on this stack of Langlands parameters?

#### 1.5. Organization of the paper.

The structure of this paper is as follows.

- In §2, we review the basic framework of sheaf-theoretic Smith theory from [Tre19]. We introduce the notion of Tate categories, the Smith functor  $\mathrm{Psm}$  and its properties, Tate cohomology, and explain the relation to classical equivariant localization theorems for  $\mathbf{Z}/p\mathbf{Z}$ -actions.
- In §3, we recall the fundamentals of parity sheaves due to Juteau-Mautner-Williamson, and the analogous notion of “Tate-parity sheaves” due to Leslie-Lonergan. We explain how to combine these with the functor  $\mathrm{Psm}$  to construct a base change functor for parity objects in the Satake category. In terms of the analogy between our method and the twisted trace formula (Remark 1.7), this functor plays the categorized role of the base change homomorphism for Hecke algebras.
- In §4, we define *excursion algebras* and recall their relation to Langlands parameters. We explain functoriality from the perspective of excursion algebras.
- In §5, we prove Theorem 1.1. First we recall background on moduli spaces of shtukas and Lafforgue’s global Langlands correspondence in terms of actions of the excursion algebra on the cohomology of shtukas. Then we establish certain equivariant localization isomorphisms for the Tate cohomology of shtukas in the setting of  $p$ -cyclic base change, which gives relations between excursion operators in the context of functoriality.
- In §6 we recall the conjectures of Treumann-Venkatesh, and the relevant aspects of the Genestier-Lafforgue correspondence. Then we use the results established earlier to prove Theorem 1.4.

### 1.6. Notation.

- (Coefficients) We let  $k$  be an algebraic closure of  $\mathbf{F}_p$  (considered with the discrete topology).

In general we will consider geometric objects over fields of characteristic  $\neq p$ , and étale sheaves over  $p$ -adically complete coefficients.

- ( $\sigma$ -actions) Throughout the paper,  $\sigma$  denotes a generator of a group isomorphic to  $\mathbf{Z}/p\mathbf{Z}$ . When we say that a widget has a “ $\sigma$ -action”, what we mean is that the widget has an action of a cyclic group of order  $p$  with chosen generator  $\sigma$ .

Let  $N := 1 + \sigma + \dots + \sigma^{p-1} \in \mathbf{Z}[\sigma]$ . We will also denote by  $N$  the induced operation on any  $\mathbf{Z}[\sigma]$ -module.<sup>6</sup>

If  $A$  is a ring or module, then  $A^\sigma$  denotes the  $\sigma$ -invariants in  $A$ .

- (Reductive groups) For us, reductive groups are connected by definition. The Langlands dual group  $\widehat{G}$  is considered as a split reductive group over  $k$ . For our conventions on the  $L$ -group, see §4.1.

For any group,  $\mathbb{1}$  denotes the trivialization representation (with the group made clear by context).

- (Equivariant derived categories) If a (pro-)algebraic group  $\Sigma$  acts on  $X$ , then we denote by  $D_\Sigma(X)$  or  $D(X)^\Sigma$  the  $\Sigma$ -equivariant derived category of constructible sheaves with coefficients in  $k$ .

**1.7. Acknowledgments.** We thank Gus Lonergan, David Treumann, Geordie Williamson, Zhiwei Yun, and Xinwen Zhu for helpful conversations related to this work. We thank Laurent Clozel and Michael Harris for comments on a draft. During the writing of this paper, the author was supported by an NSF Postdoctoral Fellowship under grant No. 1902927, as well as the Friends of the Institute for Advanced Study.

## 2. GENERALITIES ON SMITH THEORY

We shall require some general formalism from [Tre19], which we recall here. While [Tre19] operates in the setting of complex algebraic varieties in the analytic topology, most of the results generalize in a well-known way to  $\ell$ -adic sheaves on algebraic stacks, as will be formulated here. Much of what we will say is also covered in more detail in [RW, §2,3], which also works in the context of  $\ell$ -adic sheaves.

**2.1. The Tate category.** Let  $\Lambda$  be a  $p$ -adic coefficient ring; we will be interested in the cases where  $\Lambda = k$  or  $W(k)$ . We will denote by  $\Lambda[\sigma]$  the group ring of  $\langle \sigma \rangle$  with coefficients in  $\Lambda$ . Our geometric objects will be over a field of characteristic  $\neq p$  and we will consider  $\Lambda$ -adic sheaves.

For an algebraic stack<sup>7</sup>  $Y$  with a  $\sigma$ -action, there is an equivariant (constructible) derived category  $D_\sigma^b(Y; \Lambda)$ . If  $\sigma$  acts *trivially* on  $Y$ , then we have an equivalence of derived categories

$$D_\sigma^b(Y; \Lambda) \cong D^b(Y; \Lambda[\sigma]). \quad (2.1)$$

We will also be interested in the (full) subcategory  $\text{Perf}(Y; \Lambda[\sigma]) \subset D^b(Y^\sigma; \Lambda[\sigma])$  consisting of complexes whose stalks at all points of  $Y$  are perfect.

<sup>6</sup>This is to be contrasted with the operation  $Nm$ , which will mean  $Nm(a) = a * \sigma(a) * \dots * \sigma^{p-1}(a)$  in the context where there is a monoidal operation  $*$ .

<sup>7</sup>For us, this includes by definition the conditions of being locally finite type and separated.

**Definition 2.1.** The *Tate category of  $Y$*  (with respect to  $\Lambda$ ) is the Verdier quotient category  $D^b(Y; \Lambda[\sigma]) / \text{Perf}(Y; \Lambda[\sigma])$ .

According to [Tre19, Remark 4.1], the category  $D(Y; \Lambda[\sigma]) / \text{Perf}(Y; \Lambda[\sigma])$  can be regarded as a derived category of sheaves for a certain  $E_\infty$ -ring spectrum  $\mathcal{T}_\Lambda$ . So we will denote the corresponding Tate categories by  $\text{Shv}(Y; \mathcal{T}_\Lambda)$ . For our purposes  $\mathcal{T}_\Lambda$  can be thought of as just a notational device.

We denote the tautological projection map from  $D^b(Y; \Lambda[\sigma])$  to  $\text{Shv}(Y; \mathcal{T}_\Lambda)$  by

$$\mathbb{T}: D^b(Y; \Lambda[\sigma]) \rightarrow \text{Shv}(Y; \mathcal{T}_\Lambda).$$

**Example 2.2** ([Tre19, Proposition 4.2]). The Tate category over a point (meaning the spectrum of a separably closed field) is  $D^b(\Lambda[\sigma]) / \text{Perf}(\Lambda[\sigma])$ . In this category the shift-by-2 functor is isomorphic to the identity functor, as one sees by considering the nullhomotopic complex

$$0 \rightarrow V \rightarrow V \otimes \Lambda[\sigma] \xrightarrow{1-\sigma} V \otimes \Lambda[\sigma] \rightarrow V \rightarrow 0$$

whose middle two terms project to 0 in the Tate category.

**2.2. The Smith operation.** Let  $X$  be a stack with an action of  $\mathbf{Z}/p\mathbf{Z} \cong \langle \sigma \rangle$ . The  $\sigma$ -fixed points of  $X$  are defined by the cartesian square

$$\begin{array}{ccc} X^\sigma & \xrightarrow{i} & X \\ \downarrow & & \downarrow \sigma \times \text{Id} \\ X & \xrightarrow{\Delta} & X \times X \end{array}$$

Note that the map  $i: X^\sigma \rightarrow X$  may not necessarily be a closed embedding when  $X$  is not a scheme.

Given a  $\sigma$ -equivariant complex  $\mathcal{F} \in D_\sigma^b(X; \Lambda)$ , we can restrict it (via  $i^*$ ) to  $X^\sigma$  to get an object of  $D_\sigma^b(X^\sigma; \Lambda)$ , putting ourselves in the situation of the previous subsection.

**Definition 2.3** ([Tre19, Definition 4.2]). We define the *Smith operation*

$$\text{Psm} := \mathbb{T} \circ i^*: D_\sigma(X; \Lambda) \rightarrow \text{Shv}(X^\sigma; \mathcal{T}_\Lambda)$$

to be the composition of  $i^*: D_\sigma(X; \Lambda) \rightarrow D_\sigma(X^\sigma; \Lambda) \xrightarrow{(2.1)} D(X^\sigma; \Lambda[\sigma])$  with the projection  $\mathbb{T}$  to the Tate category.

**Lemma 2.4** ([Tre19, Theorem 4.1]). *Let  $i: X^\sigma \hookrightarrow X$ . The cone on  $i^! \rightarrow i^*$  belongs to  $\text{Perf}(X^\sigma; \Lambda[\sigma])$ .*

*Proof.* The point is that the stalks of the cone are cohomology of neighborhoods on which  $\sigma$  acts freely, which implies that they are perfect complexes. See [RW, Lemma 3.5].  $\square$

**2.3. Six-functor formalism.** The Tate category enjoys a robust 6-functor formalism. We will just recall what we need; see [Tre19, §4.3] for a more complete discussion. Functors between derived categories, e.g.  $f_!, f_*, f^!, f^*$ , will always denote the derived functors.

Let  $f: X \rightarrow S$  be a  $\sigma$ -equivariant morphism.

- (Pullback) As  $f^*$  preserves stalks, it preserves perfect complexes, and so descends to the Tate category to give  $f^*: \text{Shv}(S; \mathcal{T}_\Lambda) \rightarrow \text{Shv}(X; \mathcal{T}_\Lambda)$ .
- (Pushforward) If  $S$  has the trivial  $\sigma$ -action, then proper pushforward preserves perfect complexes by [Tre19, Proposition 4.3], so it descends to an operation on the Tate category  $f_!: \text{Shv}(X; \mathcal{T}_\Lambda) \rightarrow \text{Shv}(S; \mathcal{T}_\Lambda)$ .



- Verdier duality descends to the Tate category, hence so do the operations  $f^!$  and (if  $S$  has the trivial  $\sigma$ -action)  $f_*$ .

We now list some properties which could be remembered under the slogan<sup>8</sup>, “The Smith operation commutes with all operations”.

2.3.1. *Compatibility with pullback.* If  $f: X \rightarrow S$  is a  $\sigma$ -equivariant map, then the diagrams below commute:

$$\begin{array}{ccc} D_\sigma^b(X; \Lambda) & \xleftarrow{f^*} & D_\sigma^b(S; \Lambda) \\ \downarrow \text{Psm} & & \downarrow \text{Psm} \\ \text{Shv}(X^\sigma; \mathcal{T}_\Lambda) & \xleftarrow{f^*} & \text{Shv}(S^\sigma; \mathcal{T}_\Lambda) \end{array} \qquad \begin{array}{ccc} D_\sigma^b(X; \Lambda) & \xleftarrow{f^!} & D_\sigma^b(S; \Lambda) \\ \downarrow \text{Psm} & & \downarrow \text{Psm} \\ \text{Shv}(X^\sigma; \mathcal{T}_\Lambda) & \xleftarrow{f^!} & \text{Shv}(S^\sigma; \mathcal{T}_\Lambda) \end{array}$$

The proof for the first square is formal; from the second it follows immediately from the first plus Lemma 2.4.

2.3.2. *Compatibility with pushforward.* Let  $f: X \rightarrow S$  be a  $\sigma$ -equivariant map where  $S$  has the trivial  $\sigma$ -action. The following diagrams commute:

$$\begin{array}{ccc} D_\sigma^b(X; \Lambda) & \xrightarrow{f_*} & D_\sigma^b(S; \Lambda) \\ \downarrow \text{Psm} & & \downarrow \text{Psm} \\ \text{Shv}(X^\sigma; \mathcal{T}_\Lambda) & \xrightarrow{f_*} & \text{Shv}(S; \mathcal{T}_\Lambda) \end{array} \qquad \begin{array}{ccc} D_\sigma^b(X; \Lambda) & \xrightarrow{f_!} & D_\sigma^b(S; \Lambda) \\ \downarrow \text{Psm} & & \downarrow \text{Psm} \\ \text{Shv}(X^\sigma; \mathcal{T}_\Lambda) & \xrightarrow{f_!} & \text{Shv}(S; \mathcal{T}_\Lambda) \end{array}$$

(Note that we have used  $S^\sigma = S$ , since the  $\sigma$ -action on  $S$  was trivial by assumption.)

2.4. **Tate cohomology.** Given a bounded-below complex of  $\Lambda[\sigma]$ -modules  $C^\bullet$ , we define its *Tate cohomology* as in [LL, §3.3]. Because of the importance of this notion for us, we will spell out some of the details.

The exact sequence

$$0 \rightarrow \Lambda \rightarrow \Lambda[\sigma] \xrightarrow{1-\sigma} \Lambda[\sigma] \rightarrow \Lambda \rightarrow 0$$

induces a morphism in the derived category of  $\Lambda[\sigma]$ -modules,

$$\Lambda \rightarrow \Lambda[2] \in D^b(\Lambda[\sigma]). \tag{2.2}$$

---

<sup>8</sup>We copied this slogan from Geordie Williamson.

Consider the double complex below, where  $N$  denotes multiplication by  $1 + \sigma + \dots + \sigma^{p-1}$  (cf. §1.6)

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow N & & \uparrow N & & \uparrow N & & \uparrow N \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow N & & \uparrow N & & \uparrow N & & \uparrow N \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 \text{Row } -1 & & & & 0 & & 0 & & 0 & & 0
 \end{array} \tag{2.3}$$

We define  $H^n(\epsilon^! C^\bullet)$  to be the  $n$ th cohomology group of the totalization of this double complex. We define  $T^i(C^\bullet)$  to be  $\varinjlim_n H^{i+2n}(\epsilon^! C^\bullet)$ , where the transition maps are induced by (2.2).

If  $C^\bullet$  is bounded, the double complex (2.3) is eventually periodic, and  $T^i(C^\bullet)$  can be computed as the  $i$ th cohomology group of the totalization of the double complex  $\text{Tate}(C^\bullet)$  below:

$$\begin{array}{ccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots \\
 & & \uparrow N & & \uparrow N & & \uparrow N & & \uparrow N \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow N & & \uparrow N & & \uparrow N & & \uparrow N \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma & & \uparrow 1-\sigma \\
 0 & \longrightarrow & \dots & \longrightarrow & C^0 & \xrightarrow{d} & C^1 & \xrightarrow{d} & \dots & \longrightarrow & C^n & \longrightarrow & \dots \\
 & & & & \uparrow N & & \uparrow N & & \uparrow N & & \uparrow N \\
 \text{Row } -1 & & & & \vdots & & \vdots & & \vdots & & \vdots
 \end{array} \tag{2.4}$$

The formation of Tate cohomology descends to the derived category, so we can view Tate cohomology as a collection of functors

$$T^i: D^b(\Lambda[\sigma]) \rightarrow \Lambda - \text{Mod}.$$

The functors  $T^i$  are evidently 2-periodic, i.e.  $T^i \cong T^{i+2}$ . Since Tate cohomology of perfect  $\Lambda[\sigma]$ -complexes vanishes, this construction further factors through the Tate category.

**Remark 2.5.** There is also a more abstract description of Tate cohomology in terms of “Homs in the Tate category”: [LL, Proposition 4.6] implies that for  $C^\bullet \in D(\Lambda[\sigma])$ , we have

$$T^i(C^\bullet) \cong \mathrm{Hom}_{\mathrm{Shv}(\mathrm{pt}; \mathcal{T}_\Lambda)}(\mathbb{T}\Lambda, \mathbb{T}C^\bullet[i]).$$

**Lemma 2.6.** *Suppose  $C^\bullet \in D^b(\Lambda[\sigma])$  is inflated from  $D^b(\Lambda)$ , i.e.  $\sigma$  acts trivially on  $C^\bullet$ . Then  $T^*C^\bullet \cong H^*(C^\bullet) \otimes T^*(\Lambda)$ , where  $\Lambda$  is equipped with the trivial  $\sigma$ -action in the formation of  $T^*(\Lambda)$ .*

*Proof.* In this case (2.4) decomposes as the tensor product of  $C^\bullet$  and the Tate double complex for  $\Lambda$ ; the result then follows from the Künneth theorem.  $\square$

2.4.1. *The long exact sequence for Tate cohomology.* Given a distinguished triangle  $\mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \in D^b(\Lambda[\sigma])$ , we have a long exact sequence

$$\dots T^{-1}\mathcal{F}'' \rightarrow T^0\mathcal{F}' \rightarrow T^0\mathcal{F} \rightarrow T^0\mathcal{F}'' \rightarrow T^1\mathcal{F}' \rightarrow T^1\mathcal{F} \rightarrow T^1\mathcal{F}'' \rightarrow T^2\mathcal{F}' \rightarrow \dots$$

2.4.2. *Tate cohomology of a space.* Suppose  $X$  is a space with a  $\sigma$ -action, and  $\mathcal{F}$  is a  $\sigma$ -equivariant sheaf on  $X$ , then (picking injective resolutions) we can form the cohomology of  $X$  with coefficients in  $\mathcal{F}$ , as a complex  $C^\bullet(X; \mathcal{F}) \in D^+(\Lambda[\sigma])$ . Then  $T^i C^\bullet(X; \mathcal{F})$  is “the Tate cohomology of  $X$  with coefficients in  $\mathcal{F}$ ”, which we will abbreviate  $T^i(X; \mathcal{F})$ .

**Remark 2.7.** In all our later applications we will take care to only form Tate cohomology of  $\mathcal{F}$  when  $C^\bullet(X; \mathcal{F})$  is bounded.

2.4.3. *Tate cohomology sheaves.* Given  $\mathcal{F} \in \mathrm{Shv}(Y; \mathcal{T}_\Lambda)$ , we have by an analogous construction to (2.4) Tate cohomology sheaves  $T^i\mathcal{F}$  on  $Y$ , which are étale sheaves of  $T^0(\Lambda)$ -modules.

2.4.4. *The Tate cohomology spectral sequence.* If  $C^\bullet$  is bounded, then the double complex (2.4) induces a spectral sequence

$$E_1^{ij} = H^j(C^\bullet) \implies T^{i+j}(C^\bullet).$$

The second page is  $E_2^{ij} = T^i(H^j(C^\bullet))$ . Hence we find that the Tate cohomology of  $C^\bullet$  has a filtration whose graded pieces are subquotients of the ordinary cohomology  $H^j(C^\bullet)$ .

**2.5. Equivariant localization.** We will explain how the six-functor formalism captures equivariant localization theorems. For  $f: X \rightarrow S$  a  $\sigma$ -equivariant map where  $S$  has the trivial  $\sigma$ -action, consider the commutative diagram

$$\begin{array}{ccc} D_\sigma^b(X; \Lambda) & \xrightarrow{f_!} & D_\sigma^b(S; \Lambda) \\ \downarrow \mathrm{Psm} & & \downarrow \mathrm{Psm} \\ \mathrm{Shv}(X^\sigma; \mathcal{T}_\Lambda) & \xrightarrow{f_!} & \mathrm{Shv}(S; \mathcal{T}_\Lambda) \end{array}$$

from §2.3.2. This says that for a sheaf  $\mathcal{F} \in D_\sigma^b(X; \Lambda)$ , we have

$$\mathbb{T}(f_!\mathcal{F}) \cong (f|_{X^\sigma})_! \mathrm{Psm}(\mathcal{F}) \in \mathrm{Shv}(S; \mathcal{T}_\Lambda).$$

In particular, taking  $S = \mathrm{pt}$ , and then applying Tate cohomology, we obtain

$$T^i(X; \mathcal{F}) \cong T^i(X^\sigma; \mathrm{Psm}(\mathcal{F})). \quad (2.5)$$

This is one formulation of classical equivariant localization theorems for  $\mathbf{Z}/p\mathbf{Z}$ -actions, e.g. [Qui71, Theorem 4.2].

### 3. PARITY SHEAVES AND THE BASE CHANGE FUNCTOR

We begin by indicating where this section is headed.

The Geometric Satake equivalence  $P_{L+G}(\mathrm{Gr}_G; k) \cong \mathrm{Rep}_k(\widehat{G})$  provides the link between  $G$  and its Langlands dual group. In the situation of functoriality, we have a map  $\widehat{H} \rightarrow \widehat{G}$  and we would ideally like to describe the induced restriction operation  $\mathrm{Rep}_k(\widehat{G}) \rightarrow \mathrm{Rep}_k(\widehat{H})$  on the other side of the Geometric Satake equivalence, as a geometric operation on perverse sheaves.

In the context of base change it is even the case that there is an embedding  $\mathrm{Gr}_H \hookrightarrow \mathrm{Gr}_G$ , and when seeking to describe functoriality it is natural to look to the Smith operation. (One motivation is that the papers [Tre19, TV16] verify that the *function-theoretic* Smith operation is indeed related to functoriality for Hecke algebras.) However, the Smith operation lands in a Tate category, and in Example 2.2 we saw that in the Tate category, the shift-by-2 functor is isomorphic to the identity functor. This makes it seem unlikely that one can capture the notion of “perverse sheaf” in the Tate category.

Juteau-Mautner-Williamson invented the theory of *parity sheaves*, which could be seen as a variant of perverse sheaves that seems to behave better in the setting of modular coefficients. Parity sheaves are cut out in the derived category by constraints on the parity of cohomological degrees, and can therefore make sense in a context where cohomological degrees are only defined modulo 2. The notion of *Tate-parity sheaves* was introduced in [LL] as an analog of parity sheaves for the Tate category, and found to enjoy analogous properties.

After briefly reviewing the notions of parity and Tate-parity sheaves in §3.1 and §3.2, we will establish that the Smith operation respects the parity property, at least under certain conditions satisfied in our application of interest. Using “coefficient lifting” properties of parity sheaves, this will allow us to ultimately define a functor BC from parity sheaves on  $\mathrm{Gr}_G$  to parity sheaves on  $\mathrm{Gr}_H$ , which realizes base change functoriality on the geometric side.

**3.1. Parity sheaves.** We begin with a quick review of the theory of parity sheaves. We will take coefficients in a ring  $\Lambda$ , which in our applications of interest will be either  $k$  or  $\mathbb{O} := W(k)$ .

Let  $Y$  be a stratified variety over a field of characteristic  $\neq p$ , with stratification  $S = \{Y_\lambda\}$ . For the theory to work, we need to assume that the (induced) stratification on  $Y$  is JMW, meaning:

- for any two local systems  $\mathcal{L}, \mathcal{L}'$  on a stratum  $Y_\lambda$ , we have  $\mathrm{Ext}^i(\mathcal{L}, \mathcal{L}')$  is free over  $\Lambda$  for all  $i$ , and vanishes when  $i$  is odd.

This holds for Kac-Moody flag varieties over separably closed fields, and in particular for affine flag varieties over separably closed fields [JMW14, §4.1].

Fix a *pariversity*  $\dagger: S \rightarrow \mathbf{Z}/2\mathbf{Z}$ . In this paper we will always take the *dimension pariversity*  $\dagger(\lambda) := \dim Y_\lambda \bmod 2$ , so we will sometimes omit the pariversity from the discussion. Recall that [JMW14] define *even* complexes (with respect to the pariversity  $\dagger$ ) to be those  $\mathcal{F} \in D_S^b(Y; \Lambda)$  such that for all  $i_\lambda: Y_\lambda \hookrightarrow X$ , for  $\lambda \in S$ ,  $i_\lambda^* \mathcal{F}$  and  $i_\lambda^! \mathcal{F}$  have cohomology sheaves supported in degrees congruent to  $\dagger(\lambda)$  modulo 2, and *odd* complexes analogously. They define *parity complexes* to be direct sums of even and odd complexes. The full subcategory of ( $S$ -constructible) Tate-parity complexes (with coefficients in  $\Lambda$ ) is denoted  $\mathrm{Parity}_S(Y; \Lambda)$ .

**Theorem 3.1** ([JMW14, Theorem 2.12]). *Let  $\mathcal{F}$  be an indecomposable parity complex. Then:*

- $\mathcal{F}$  has irreducible support, which is therefore of the form  $\overline{Y}_\lambda$  for some  $\lambda \in \Lambda$ ,
- $i_\lambda^* \mathcal{F}$  is a shifted local system  $\mathcal{L}[m]$ , and
- Any indecomposable parity complex supported on  $\overline{Y}_\lambda$  and extending  $\mathcal{L}[m]$  is isomorphic to  $\mathcal{F}$ .

A *parity sheaf* (with respect to  $\dagger$ ) is an indecomposable parity complex (with respect to  $\dagger$ ) with  $Y_\lambda$  the dense stratum in its support and extending  $\mathcal{L}[\dim Y_\lambda]$ . Given  $\mathcal{L}[\dim Y_\lambda]$ , it is not clear in general that a parity sheaf extending it exists. If it does exist, then Theorem 3.1 guarantees its uniqueness, and we denote it by  $\mathcal{E}(\lambda, \mathcal{L})$ . The existence is guaranteed for  $\mathrm{Gr}_G$  with the usual stratification by  $L^+G$ -orbits;  $\mathcal{E}(\lambda, \mathcal{L})$  is moreover  $L^+G$ -equivariant if  $p$  is not a torsion prime for  $G$  [JMW16, Theorem 1.4]. If  $\mathcal{E}(\lambda, \mathcal{L})$  exists for all  $\lambda$  and  $\mathcal{L}$ , we will say that “all parity sheaves exist”.

**3.2. Tate-parity sheaves.** As we have seen, the cohomological grading in the Tate category is only well-defined modulo 2, so it does not seem to make sense to talk about perverse sheaves in the Tate category. However, elements of the Tate category have Tate cohomology sheaves (§2.4.3), which are indexed by  $\mathbf{Z}/2\mathbf{Z}$ , so it *could* make sense to talk about an analog of parity sheaves in the Tate category. As [LL] observed, for this to work we must take coefficients in the *integral* version of the Tate category, meaning  $\Lambda = \mathbb{O} = W(k)$ , because then

$$\mathrm{Ext}_{\mathrm{Shv}(\mathcal{T}_\mathbb{O})}^*(\mathbb{T}(\mathbb{O}), \mathbb{T}(\mathbb{O})) = \bigoplus_{i \in \mathbf{Z}} k[2i] \quad (3.1)$$

is supported in even degrees. This is necessary for the assumption of non-vanishing odd Exts in the definition of the JMW stratification.

For a stratification  $S$  on  $Y$ , we define  $\mathrm{Shv}_S(Y; \mathcal{T}_\mathbb{O}) \subset \mathrm{Shv}(Y; \mathcal{T}_\mathbb{O})$  to be the full subcategory generated by objects in  $D_S^b(Y; \mathbb{O}[\sigma])$ . Letting  $\mathrm{Perf}_S(Y; \mathbb{O}[\sigma]) \subset \mathrm{Perf}(Y; \mathbb{O}[\sigma])$  be the full thick subcategory of  $S$ -constructible objects, we have by [LL, Corollary 4.7] that

$$D_S^b(Y; \mathbb{O}[\sigma]) / \mathrm{Perf}_S(Y; \mathbb{O}[\sigma]) \xrightarrow{\sim} \mathrm{Shv}_S(Y; \mathcal{T}_\mathbb{O}).$$

**Definition 3.2** ([LL, Definition 5.3]). Let  $\mathcal{F} \in \mathrm{Shv}_S(Y; \mathcal{T}_\mathbb{O})$ . Fix a pariversity  $\dagger: S \rightarrow \mathbf{Z}/2\mathbf{Z}$ . Let  $?\in \{*, !\}$ .

- (1) We say  $\mathcal{F}$  is *?-Tate-even* (with respect to  $\dagger$ ) if for each  $\lambda \in S$ , we have

$$T^{\dagger(\lambda)+1}(i_\lambda^? \mathcal{F}) = 0.$$

- (2) We say  $\mathcal{F}$  is *?-Tate-odd* (with respect to  $\dagger$ ) if  $\mathcal{F}[1]$  is *?-Tate-even*.  
(3) We say  $\mathcal{F}$  is *Tate-even* (resp. *Tate-odd*) if  $\mathcal{F}$  is both  $*$ -Tate even (resp. odd) and  $!$ -Tate even (resp. odd).  
(4) We say  $\mathcal{F}$  is *Tate-parity complex* (with respect to  $\dagger$ ), if it is isomorphic within  $\mathrm{Shv}_S(Y; \mathcal{T}_\mathbb{O})$  to the direct sum of a Tate-even complex and a Tate-odd complex.<sup>9</sup>

The full subcategory of ( $S$ -equivariant) Tate-parity complexes (with coefficients in  $\mathcal{T}_\mathbb{O}$ ) is denoted  $\mathrm{Parity}_S(Y; \mathcal{T}_\mathbb{O})$ .

Parallel to Theorem 3.1, we have the following result in this context:

**Proposition 3.3** ([LL, Theorem 4.13]). *Let  $\mathcal{F}$  be an indecomposable Tate-parity complex.*

- (1) *The support of  $\mathcal{F}$  is of the form  $\overline{Y}_\lambda$  for a unique stratum  $Y_\lambda$ .*

<sup>9</sup>This is to be distinguished from the (upcoming) notion of *Tate-parity sheaf*, which is more restrictive.

- (2) Suppose  $\mathcal{G}$  and  $\mathcal{F}$  are two indecomposable Tate-parity complexes such that  $\text{supp}(\mathcal{G}) = \text{supp}(\mathcal{F})$ . Letting  $j_\lambda: Y_\lambda \hookrightarrow Y$  be the inclusion of the unique stratum open in this support, if  $j_\lambda^* \mathcal{G} \cong j_\lambda^* \mathcal{F}$  then  $\mathcal{G} \cong \mathcal{F}$ .

*Proof.* The same argument as in [JMW14, Theorem 2.12] works.  $\square$

We define  $\epsilon_*: D_c^b(Y; \mathbb{O}) \rightarrow D_c^b(Y; \mathbb{O}[\sigma])$  for the inflation through the augmentation  $\mathbb{O}[\sigma] \twoheadrightarrow \mathbb{O}$ . Recall that  $\mathbb{T}: D_c^b(Y; \mathbb{O}[\sigma]) \rightarrow \text{Shv}(Y; \mathcal{T}_\mathbb{O})$  denotes projection to the Tate category. We are interested in Tate complexes that come from the composite functor

$$\mathbb{T}\epsilon_*: D_S^b(Y; \mathbb{O}) \rightarrow D_S^b(Y; \mathbb{O}[\sigma]) \rightarrow \text{Shv}_S(Y; \mathcal{T}_\mathbb{O}).$$

**Definition 3.4.** A Tate-parity sheaf  $\mathcal{F} \in \text{Shv}_S(Y; \mathcal{T}_\mathbb{O})$  is an indecomposable Tate-parity complex with the property that its restriction to the unique stratum  $Y_\lambda$  which is dense in its support is of the form  $\mathbb{T}\epsilon_* \mathcal{L}[\dim Y_\lambda]$  for an indecomposable  $\Lambda$ -free local system  $\mathcal{L}$  on  $Y_\lambda$ . If such an  $\mathcal{F}$  exists then it is unique, and we denote it by  $\mathcal{E}_\mathcal{T}(\lambda, \mathcal{L})$ .

If  $\mathcal{E}_\mathcal{T}(\lambda, \mathcal{L})$  exists for all  $\lambda \in S$  and all  $\mathcal{L}$ , we will say that “all Tate-parity sheaves exist” (for  $Y, S$ ).

**3.3. Modular reduction.** We now explain that the functor  $\mathbb{T}$  has good properties that one would expect from “base change of coefficients” functors for categories of sheaves in classical rings. We will suppression mention of the pariversity  $\dagger$ .

**Proposition 3.5** ([LL, Proposition 5.16, Theorem 5.17]).

- (1) If  $\mathcal{F} \in D_S^b(X; \mathbb{O})$  is even/odd, then  $\mathbb{T}\epsilon_* \mathcal{F} \in \text{Sh}_S(X; \mathcal{T}_\mathbb{O})$  is Tate-even/odd.
- (2) If the parity sheaf  $\mathcal{E} = \mathcal{E}(\lambda, \mathcal{L})$  exists and satisfies  $\text{Hom}_{D^b(Y; \mathbb{O})}(\mathcal{E}, \mathcal{E}[n]) = 0$  for all  $n < 0$  (this holds for example if  $\mathcal{E}$  is perverse<sup>10</sup>) then  $\mathcal{E}_\mathcal{T}(\lambda, \mathcal{L})$  exists and we have

$$\mathbb{T}\epsilon_* \mathcal{E}(\lambda, \mathcal{L}) \cong \mathcal{E}_\mathcal{T}(\lambda, \mathcal{L}).$$

*Proof.* We reproduce the proof because it highlights the importance of using  $\mathbb{O}$ -coefficients instead of  $k$ -coefficients. The operation  $\mathbb{T}\epsilon_*$  is compatible with formation of  $i_\lambda^*$  or  $i_\lambda^!$ . Hence to prove (1) we reduce to examining  $T^i \epsilon_* \mathcal{L}$  for a local system  $\mathcal{L}$  of free  $\mathbb{O}$ -modules, with the trivial  $\sigma$ -action. This reduces to the fact that the Tate cohomology of  $\mathbb{O}$  is supported in even degrees, which is (3.1).

For (2), we just need to check that  $\mathbb{T}\epsilon_* \mathcal{E}(\lambda, \mathcal{L})$  is indecomposable. Since  $\text{Parity}_S(Y; \mathcal{T}_\mathbb{O})$  is Krull-Remak-Schmidt by [LL, Proposition 5.8], the endomorphism ring of  $\mathbb{T}\epsilon_* \mathcal{E}(\lambda, \mathcal{L})$  is local. According to [LL, §4.6], for  $\mathcal{F}, \mathcal{G} \in D^b(Y; \mathbb{O})$  we have

$$\text{Hom}_{\text{Shv}(Y; \mathcal{T}_\Lambda)}(\mathbb{T}\mathcal{F}, \mathbb{T}\mathcal{G}) \cong \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(Y; k)}(\mathbb{F}\mathcal{F}, \mathbb{F}\mathcal{G}[2i]). \quad (3.2)$$

We apply this to  $\mathcal{F} = \mathcal{G} = \epsilon_* \mathcal{E}(\lambda, \mathcal{L})$ . Since  $\mathcal{E}(\lambda, \mathcal{L})$  is indecomposable the subalgebra in (3.2) indexed by  $i = 0$  is local, and the assumption that the summands of (3.2) indexed by negative  $i$  vanish. This implies the desired locality of the graded algebra (3.2).  $\square$

**Remark 3.6.** The Proposition (and its proof) are analogous to the following results of parity sheaves [JMW14, §2.5]. Let  $\mathbb{F}$  denote the base change functor

$$\mathbb{F} = k \otimes_{\mathbb{O}}^L (-): D_S(Y; \mathbb{O}) \rightarrow D_S(Y; k).$$

The functor  $\mathbb{F}$  enjoys following properties.

<sup>10</sup>In fact this is an equivalence by [MR18, Lemma 6.6], which we thank Simon Riche for pointing out to us.

- (1)  $\mathcal{E} \in D_S^b(X; \mathbb{O})$  is a parity sheaf if and only if  $\mathbb{F}(\mathcal{F}) \in D_S^b(X; k)$  is a parity sheaf.
- (2) If  $\mathcal{E}(\lambda, \mathcal{L})$  exists, then  $\mathcal{E}(\lambda, \mathbb{F}\mathcal{L})$  exists and we have

$$\mathbb{F}\mathcal{E}(\lambda, \mathcal{L}) \cong \mathcal{E}(\lambda, \mathbb{F}\mathcal{L}).$$

What we have seen can be summarized by the slogan:

If all parity sheaves exist and have vanishing negative self-Exts, then all Tate-parity sheaves exist and  $\mathbb{T} \circ \epsilon_*$  induces a bijection between parity sheaves and Tate-parity sheaves.

**3.4. The lifting functor.** We will now define a functor lifting Tate-parity sheaves to parity sheaves. In fact the preceding slogan already tells us what to do about objects, so we just need to specify what happens on morphisms.

**Definition 3.7.** A *normalized* (Tate-)parity complex is a direct sum of Tate-parity sheaves *with no shifts*. Hence, under our assumptions, their restrictions to the dense open stratum in their support are isomorphic to  $\mathcal{L}[\dim Y_\lambda]$  (resp.  $\mathbb{T}\epsilon_*\mathcal{L}[\dim Y_\lambda]$ ). We denote the full subcategories of such by  $\text{Parity}_S^0(Y; \mathbb{O}) \subset \text{Parity}_S(Y; \mathbb{O})$  and  $\text{Parity}_S^0(Y; \mathcal{T}_{\mathbb{O}}) \subset \text{Parity}_S(Y; \mathcal{T}_{\mathbb{O}})$ , and called them the *categories of normalized (Tate-)parity sheaves*.

Under the assumption that all parity sheaves exist and have vanishing negative self-Exts, we then have a lifting functor [LL, Theorem 5.19]

$$L: \text{Parity}_S^0(Y; \mathcal{T}_{\mathbb{O}}) \rightarrow \text{Parity}_S^0(Y; k)$$

sending  $\mathcal{E}_{\mathcal{T}}(\lambda, \mathcal{L})$  to  $\mathcal{E}(\lambda, \mathcal{L} \otimes_{\mathbb{O}} k)$  on objects, and on morphisms inducing projection to the summand indexed by  $i = 0$  under identification (3.2). It can be thought of as an “intermediate” reduction between  $\mathbb{O}$  and  $k$  in the sense that the following diagram commutes:

$$\begin{array}{ccc} \text{Parity}_S^0(Y; \mathbb{O}) & \xrightarrow{\mathbb{T}\epsilon_*} & \text{Parity}_S^0(Y; \mathcal{T}_{\mathbb{O}}) \\ & \searrow \mathbb{F} & \downarrow L \\ & & \text{Parity}_S^0(Y; k) \end{array}$$

**3.5. Parity sheaves on the affine Grassmannian and tilting modules.** We now consider the preceding theory in the context of the affine Grassmannian  $\text{Gr}_G$  over a separably closed field  $\mathbf{F}$ , with the stratification by  $L^+G$ -orbits. Since this is a special case of a Kac-Moody flag variety, the stratification is JMW by [JMW14, §4.1].

If  $p$  is a good prime for  $\widehat{G}$ , [MR18, Corollary 1.6] implies that all parity sheaves exist, and that all normalized parity sheaves are perverse. Therefore, the category of normalized parity sheaves corresponds under the Geometric Satake equivalence to some subcategory of  $\text{Rep}_k(\widehat{G})$ , and it is natural to ask what this is. The answer is given in terms of *tilting modules* for  $\widehat{G}$  (recall that these are the objects of  $\text{Rep}_k(\widehat{G})$  having both a filtration by standard objects, and a filtration by costandard objects). Let  $\text{Tilt}_k(\widehat{G}) \subset \text{Rep}_k(\widehat{G})$  denote the full subcategory of tilting modules.

**Theorem 3.8** ([MR18, Corollary 1.6], generalizing [JMW16, Theorem 1.8]). *If  $p$  is good for  $G$ , then the Geometric Satake equivalence restricts to an equivalence<sup>11</sup>*

$$\text{Parity}_{L+G}^0(\text{Gr}_G; k) \cong \text{Tilt}_k(\widehat{G}).$$

We need a few facts about the representation theory of tilting modules. For our arithmetic applications, the key point is that there are “enough” tilting modules to generate the derived category of  $\text{Rep}_k(\widehat{G})$ , as articulated by the Theorem below.

**Theorem 3.9** ([BBM04]). *The subcategory  $\text{Tilt}_k(\widehat{G})$  generates the bounded derived category of  $\text{Rep}_k(\widehat{G})$ . More precisely, the natural projection from the bounded homotopy category  $K^b(\text{Tilt}_k(\widehat{G}))$  to  $D^b(\text{Rep}_k(\widehat{G}))$  is an equivalence.*

*Proof.* This follows from general highest weight theory. A convenient reference is [Ric, Proposition 7.17].  $\square$

**3.6. Base change functoriality for the Satake category.** We now consider a specific geometric situation relevant to Langlands functoriality for  $p$ -cyclic base change. Let  $\mathbf{F}$  be a field of characteristic  $\neq p$ . We will consider reductive groups, and their affine Grassmannians, over  $\mathbf{F}$ .

**3.6.1. The base change setup.** We now specialize the situation a bit further:  $H$  is any reductive group over a separably closed field  $\mathbf{F}$  and  $G = H^p$ . We let  $\sigma$  act on  $G$  by cyclic rotation, sending the  $i$ th factor to the  $(i+1)$ st (mod  $p$ ) factor. Then it is clear that the stratification on  $\text{Gr}_G$  by  $L^+G$ -orbits induces by restriction the stratification on  $\text{Gr}_H$  by  $L^+H$ -orbits.

Evidently the “diagonal” embedding  $H \hookrightarrow G$  realizes  $H$  as the fixed points of  $G$  under the automorphism  $\sigma$ . This map  $H \hookrightarrow G$  also induces a diagonal map  $\text{Gr}_H \rightarrow \text{Gr}_G$ .

**Lemma 3.10.** *The diagonal map induces an isomorphism  $\text{Gr}_H \cong \text{Gr}_G^\sigma$  as subfunctors of  $\text{Gr}_G$ .*

*Proof.* We have  $\text{Gr}_G \cong (\text{Gr}_H)^p$ , with  $\sigma$  acting by cyclic rotation of the factors, from which the claim is clear.  $\square$

Henceforth we assume that  $p$  is odd and good for  $\widehat{G}$ , so that the results of §3.5 apply.

The restriction functor along the diagonal embedding  $\widehat{H}_k \hookrightarrow \widehat{G}_k$  induces a restriction functor  $\text{Res}_{\text{BC}}: \text{Tilt}_k(\widehat{G}) \rightarrow \text{Tilt}_k(\widehat{H})$ . We aim to give a “geometric” description of the corresponding functor under the Geometric Satake equivalence,  $\text{Parity}(\text{Gr}_G; k) \rightarrow \text{Parity}(\text{Gr}_H; k)$ , in terms of Smith theory. (Of course, one could give an “ad hoc” description using that  $G = H^p$ . The point is to define a functor that does not make reference to this, which will then descend well to the situation where  $G = \text{Res}_{\mathbf{E}/\mathbf{F}}(H)$  where  $\mathbf{E}/\mathbf{F}$  is a non-trivial field extension.)

**Definition 3.11.** Given  $\mathcal{F} \in \text{P}_{L+G}(\text{Gr}_G; k)$ , we define

$$\text{Nm}(\mathcal{F}) := \mathcal{F} * {}^\sigma \mathcal{F} * \dots * {}^{\sigma^{p-1}} \mathcal{F} \in \text{P}_{L+G \rtimes \sigma}(\text{Gr}_G; k),$$

<sup>11</sup>Strictly speaking, the cited references employ the trivial pariversity instead of the dimension pariversity. Since dimensions of Schubert strata in  $\text{Gr}_G$  have constant parity on connected components, the trivial pariversity and dimension pariversity lead to the same notion of parity complexes in this case, so the only difference is in the notion of “normalization”. We follow [LL] in the use of the dimension pariversity so that perverse sheaves are  $\dagger$ -even.



equipped with the  $\sigma$ -equivariant structure coming from the commutativity constraint for  $(P_{L+G}(\mathrm{Gr}_G; k), *)$ :

$${}^\sigma \mathrm{Nm}(\mathcal{F}) = {}^\sigma \mathcal{F} * \dots * {}^{\sigma^{p-1}} \mathcal{F} * \mathcal{F} \xrightarrow{\sim} \mathcal{F} * {}^\sigma \mathcal{F} * \dots * {}^{\sigma^{p-1}} \mathcal{F} = \mathrm{Nm}(\mathcal{F}). \quad (3.3)$$

There is a realization functor  $P_{L+G \rtimes \sigma}(\mathrm{Gr}_G; k) \rightarrow D_{L+G \rtimes \sigma}(\mathrm{Gr}_G; k)$  due to Beilinson, which we will use to view  $\mathrm{Nm}(\mathcal{F}) \in D_{L+G \rtimes \sigma}(\mathrm{Gr}_G; k)$  (so that we may apply the Smith functor, for example). Equipping a general object of  $D_{L+G}(\mathrm{Gr}_G; k)$  with a  $\sigma$ -equivariant structure is much more involved than just specifying isomorphisms (3.3) (satisfying cocycle conditions), so we emphasize that we construct  $\mathrm{Nm}(\mathcal{F})$  first as a  $\sigma$ -equivariant perverse sheaf, and then apply the realization functor to get a  $\sigma$ -equivariant object of  $D_{L+G}(\mathrm{Gr}_G; k)$ .

**Remark 3.12.** In our applications we will assume that  $p$  is large enough so that all parity sheaves are perverse. The properties of being  $L^+G$ -constructible and  $L^+G$ -equivariant are equivalent for perverse sheaves on  $\mathrm{Gr}_G$ . Therefore, we will not need to worry about any extra complications coming from the equivariance. For Tate categories,  $\mathrm{Shv}_{(L^+G)}(\mathrm{Gr}_G; \mathcal{T}_\Lambda)$  means by definition the category of  $L^+G$ -stratified sheaves.

**Lemma 3.13.** *Let  $i: \mathrm{Gr}_H \cong \mathrm{Gr}_G^\sigma \hookrightarrow \mathrm{Gr}_G$ . For  $\mathcal{F} \in D_{L+G}^b(\mathrm{Gr}_G; \mathbb{O})$ , regard  $\mathrm{Nm}(\mathcal{F}) \in D_{L+G \rtimes \sigma}^b(\mathrm{Gr}_G; \mathbb{O})$  as in Definition 3.11 above.*

- (i) *The stalks of  $i^* \mathrm{Nm}(\mathcal{F})$  have Jordan-Hölder constituents being either trivial or free  $\mathbb{O}[\sigma]$ -modules.*
- (ii) *The costalks of  $i^! \mathrm{Nm}(\mathcal{F})$  have Jordan-Hölder constituents being either trivial or free  $\mathbb{O}[\sigma]$ -modules.*

*Proof.* By filtering  $\mathcal{F}$  into its Jordan-Hölder constituents, we may assume that  $\mathcal{F}$  itself is simple. Any simple  $L^+G \approx (L^+H)^p$ -equivariant sheaf  $\mathcal{F}$  on a stratum  $\mathrm{Gr}_G^\Delta$  is of the form  $\mathcal{F} \approx \mathcal{F}_1 \boxtimes \dots \boxtimes \mathcal{F}_p$ , since the stratum is a product of homogeneous spaces for (a finite type quotient of)  $L^+H$ . Then

$$\mathrm{Nm}(\mathcal{F}) \approx (\mathcal{F}_1 * \mathcal{F}_2 * \dots * \mathcal{F}_p) \boxtimes (\mathcal{F}_2 * \dots * \mathcal{F}_p * \mathcal{F}_1) \boxtimes \dots \boxtimes (\mathcal{F}_p * \mathcal{F}_1 * \dots * \mathcal{F}_{p-1}),$$

with  $\sigma$  acting by rotating the tensor factors, and the  $\sigma$ -equivariant structure coming from the commutativity constraint.

Write  $\mathcal{F}' := \mathcal{F}_1 * \mathcal{F}_2 * \dots * \mathcal{F}_p \in P_{L+H}(\mathrm{Gr}_H; \mathbb{O})$ . Since  $i$  may be identified with the diagonal embedding  $\mathrm{Gr}_H \hookrightarrow \mathrm{Gr}_H^p$ , we have  $i^*(\mathrm{Nm} \mathcal{F}) \approx (\mathcal{F}')^{\otimes p}$ , with  $\sigma$ -equivariant structure given by cyclic rotation of the tensor factors. In particular, the stalk of  $i^*(\mathrm{Nm} \mathcal{F})$  at  $x \in \mathrm{Gr}_H$  is the tensor-induction of the stalk of  $\mathcal{F}'_x$  from  $\mathbb{O}$  to  $\mathbb{O}[\sigma]$ .

Hence it suffices to prove that any such tensor induction has Jordan-Hölder constituents being either trivial or free. This is verified by explicit inspection: choosing a basis for  $\mathcal{F}'_x$ , the induced basis of  $(\mathcal{F}'_x)^{\otimes p}$  is grouped into either trivial or free orbits under the  $\sigma$ -action.

The argument for (ii) is completely analogous (we could also apply Verdier duality to (i)).  $\square$

**3.6.2. Smith theory for parity sheaves.** We return momentarily to the general setup for Smith theory:  $X$  has a  $\sigma$ -action and  $Y = X^\sigma$ .

**Proposition 3.14** (Variant of [LL, Theorem 6.3]). *Suppose  $\mathcal{E} \in D_{S, \sigma}^b(X; \mathbb{O})$  is a parity complex satisfying the condition:*

- (\*) *all  $*$  and  $!$ -stalks of cohomology sheaves of  $\mathcal{E}$  at fixed points  $x \in X$  have  $\mathbb{O}[\sigma]$ -module Jordan-Hölder constituents being trivial or free.*

*Then  $\mathrm{Psm}(\mathcal{E}) \in D_S(Y; \mathcal{T}_\mathbb{O})$  is Tate-parity.*

*Proof.* This theorem is closely related to Theorem 6.3 of [LL], except [LL, Theorem 6.3] imposes the stronger condition that the  $\sigma$ -action on all stalks is trivial. This is satisfied in their application (to the loop-rotation action), but not in ours, so we need to re-do the argument in the requisite generality.

Let  $Y = X^\sigma$  and  $i: Y \rightarrow X$ ,  $i_\lambda^X: X_\lambda \hookrightarrow X$ ,  $i_\lambda^Y: Y_\lambda \hookrightarrow Y$ ,  $i^\lambda: Y_\lambda \hookrightarrow X_\lambda$ . Without loss of generality suppose  $\mathcal{E}$  is an even complex on  $X$ . We are given that  $(i_\lambda^X)^? \mathcal{E}$  has  $\mathbb{O}$ -free cohomology sheaves supported in degrees congruent to  $\dagger_X(\lambda) \bmod 2$ , where  $? \in \{*, !\}$ ; we want to show that  $(i_\lambda^Y)^? \text{Psm}(\mathcal{E})$  has Tate-cohomology sheaves supported in degrees congruent to  $\dagger_Y(\lambda) \bmod 2$ . Unraveling the definitions, we have

$$\begin{aligned} (i_\lambda^Y)^* \text{Psm}(\mathcal{E}) &= (i_\lambda^Y)^* \mathbb{T} i^* \mathcal{E} \\ &\cong \mathbb{T} (i_\lambda^Y)^* i^* \mathcal{E} \\ &\cong \mathbb{T} (i^\lambda)^* (i_\lambda^X)^* \mathcal{E}. \end{aligned}$$

Similarly, using Lemma 2.4 we have

$$(i_\lambda^Y)^! \text{Psm}(\mathcal{E}) \cong \mathbb{T} (i^\lambda)^! (i_\lambda^X)^! \mathcal{E}. \quad (3.4)$$

By hypothesis,  $(i_\lambda^X)^* \mathcal{E}$  has its cohomology sheaves supported in degrees congruent to  $\dagger_X(\lambda) \bmod 2$ . Moreover, by assumption (\*), all the stalks and costalks have Jordan-Hölder constituents being even shifts of either trivial or free  $\mathbb{O}[\sigma]$ -modules. So the stalks of  $(i^\lambda)^* (i_\lambda^X)^* \mathcal{E}$  are supported in degrees congruent to  $\dagger_X(\lambda) \bmod 2$ , and we must verify that their Tate cohomology groups are also supported in degrees of a single parity.

For trivial  $\mathbb{O}[\sigma]$ -modules the odd Tate cohomology groups vanish by (3.1), while for free  $\mathbb{O}[\sigma]$ -modules all the Tate cohomology groups vanish. Hence for any  $\mathbb{O}[\sigma]$  whose Jordan-Hölder constituents are all trivial or free, all odd Tate cohomology groups vanish by the long exact sequence for Tate cohomology (§2.4.1). This shows that the Tate cohomology sheaves of  $(i^\lambda)^* (i_\lambda^X)^* \mathcal{E}$  are supported in degrees congruent to  $\dagger_X(\lambda) \bmod 2$ .

A completely analogous argument, using (3.4) instead, shows that  $(i^\lambda)^! (i_\lambda^X)^! \mathcal{E}$  also has Tate cohomology sheaves supported in degrees congruent to  $\dagger_X(\lambda) \bmod 2$ .  $\square$

For an  $\mathbb{O}$ -linear abelian category  $\mathcal{C}$ , with all Hom-spaces being free  $\mathbb{O}$ -modules, we denote by  $\mathcal{C} \otimes_{\mathbb{O}} k$  the  $k$ -linear category obtained by tensoring all Hom-spaces with  $k$  over  $\mathbb{O}$ .

**Lemma 3.15.** *Suppose that all the strata  $X_\lambda$  are simply connected and all parity sheaves  $\mathcal{E}(\lambda, \mathcal{L})$  exist. Then we have that*

$$\text{Parity}_{S, \sigma}^0(X; \mathbb{O}) \otimes_{\mathbb{O}} k \xrightarrow{\sim} \text{Parity}_{S, \sigma}^0(X; k).$$

*Proof.* To see that the functor is well-defined, we note:

- The Hom-spaces of  $\text{Parity}_{S, \sigma}^0(X; \mathbb{O})$  are all free  $\mathbb{O}$ -modules by [JMW14, Remark 2.7], so that the domain is well-defined.
- The functor lands in parity sheaves since the modular reduction of a  $\mathbb{O}$ -parity sheaf is a  $k$ -parity sheaf by Remark 3.6.

It is essentially surjective because every  $k$ -parity sheaf lifts to a  $\mathbb{O}$ -parity sheaf under our assumption that all parity sheaves exist and all strata are simply connected (which implies that all  $k$ -local systems on strata lift to  $\mathbb{O}$ , since they are trivial). The fact that the functor is fully faithful again follows from [JMW14, Remark 2.7].  $\square$

**3.6.3. The base change functor.** We return now to the base change setup of §3.6.1. Let  $\mathcal{F} \in \text{Parity}_{L+G}^0(\text{Gr}_G; \mathbb{O})$ . Then  $\mathcal{F} \in \mathcal{P}_{L+G}(\text{Gr}_G; \mathbb{O})$  is perverse since  $p$  is good for  $\widehat{G}$  (this is a part of Theorem 3.8), and  $\text{Nm}(\mathcal{F}) \in \text{Parity}_{L+G \rtimes \sigma}^0(\text{Gr}_G; \mathbb{O})$  is a parity sheaf by [JMW16, Theorem 1.5]. Furthermore, the  $\sigma$ -equivariant structure on  $\text{Nm}(\mathcal{F})$  satisfies the assumption (\*) of Proposition 3.14 by Lemma 3.13. Hence we may apply Proposition 3.14 to deduce that  $\text{Psm}(\text{Nm}(\mathcal{F})) \in \text{Parity}_{(L+H)}(\text{Gr}_H; \mathcal{T}_{\mathbb{O}})$  is Tate-parity.

We claim that moreover  $\text{Psm}(\text{Nm}(\mathcal{F})) \in \text{Parity}_{(L+H)}^0(\text{Gr}_H; \mathcal{T}_{\mathbb{O}})$ , i.e. is normalized as long as  $p > 2$ . Indeed, suppose  $\text{Gr}_{\lambda}$  is the unique orbit dense in the support of  $\text{Nm}(\mathcal{F})$ . Then  $\text{Gr}_H^{\lambda} = (\text{Gr}_G^{\lambda})^{\sigma}$ , and their dimensions are congruent modulo 2 (since  $[2] \cong \text{Id}$  in the Tate category). To verify this latter claim, writing  $\lambda = (\lambda_1, \dots, \lambda_p)$  for  $\lambda_i \in X_*(H)_+$ , we have

$$\begin{cases} \text{Gr}_G^{\lambda} \cap \text{Gr}_H = \text{Gr}_H^{\lambda_1} & \lambda = (\lambda_1, \dots, \lambda_1), \\ \text{Gr}_G^{\lambda} \cap \text{Gr}_H = \emptyset & \text{otherwise.} \end{cases}$$

By [Zhu17, Proposition 2.1.5] we have  $\dim \text{Gr}_G^{\lambda} = \langle 2\rho_G, \lambda \rangle$ . So we just have to verify that  $\langle 2\rho_G, (\lambda_1, \dots, \lambda_1) \rangle \equiv \langle 2\rho_H, \lambda_1 \rangle \pmod{2}$ . Indeed,  $\rho_G = (\rho_H, \dots, \rho_H)$ , so  $\langle 2\rho_G, (\lambda_1, \dots, \lambda_1) \rangle = p\langle \rho_H, \lambda_1 \rangle$ , and  $p$  is odd.<sup>12</sup>

Thanks to the claim of the preceding paragraph, we can apply the lifting functor  $L$  to  $\text{Psm}(\text{Nm}(\mathcal{F}))$ . At this point we have constructed the diagram

$$\begin{array}{ccc} \text{Parity}_{L+G}^0(\text{Gr}_G; \mathbb{O}) & \xrightarrow{\text{Psm} \circ \text{Nm}} & \text{Parity}_{(L+H)}^0(\text{Gr}_H; \mathcal{T}_{\mathbb{O}}) \\ \downarrow \mathbb{F} & & \downarrow L \\ \text{Parity}_{L+G}^0(\text{Gr}_G; k) & & \text{Parity}_{L+H}^0(\text{Gr}_H; k). \end{array}$$

By Lemma 3.15, the composite functor factors uniquely through a functor  $\text{Parity}_{L+G}^0(\text{Gr}_G; k) \rightarrow \text{Parity}_{L+H}^0(\text{Gr}_H; k)$ .

**Definition 3.16.** We define

$$\text{BC}^{(p)} : \text{Parity}_{L+G}^0(\text{Gr}_G; k) \rightarrow \text{Parity}_{L+H}^0(\text{Gr}_H; k)$$

to be the functor unique filling in the commutative diagram

$$\begin{array}{ccc} \text{Parity}_{L+G}^0(\text{Gr}_G; \mathbb{O}) & \xrightarrow{\text{Psm} \circ \text{Nm}} & \text{Parity}_{(L+H)}^0(\text{Gr}_H; \mathcal{T}_{\mathbb{O}}) \\ \downarrow \mathbb{F} & & \downarrow L \\ \text{Parity}_{L+G}^0(\text{Gr}_G; k) & \xrightarrow{\text{BC}^{(p)}} & \text{Parity}_{L+H}^0(\text{Gr}_H; k). \end{array}$$

One more step is required to obtain the desired base change functor. On a  $k$ -linear additive category there is an auto-equivalence  $\text{Frob}_p$  of the underlying category, which is the identity on objects and the Frobenius automorphism  $(-) \otimes_{k, \text{Frob}_p} k$  on morphisms. We define

$$\text{BC} := \text{Frob}_p^{-1} \circ \text{BC}^{(p)} : \text{Parity}_{L+G}^0(\text{Gr}_G; k) \rightarrow \text{Parity}_{L+H}^0(\text{Gr}_H; k).$$

**Remark 3.17** (Galois equivariance). If  $H$  base changed from some subfield  $\mathbb{F}_0 \subset \mathbb{F}$ , then  $\text{Aut}(\mathbb{F}/\mathbb{F}_0)$  acts on  $H_{\mathbb{F}}, G_{\mathbb{F}}$  and therefore also on  $\text{Gr}_{H_{\mathbb{F}}}, \text{Gr}_{G_{\mathbb{F}}}$ . It will be important for us later that  $\text{BC}$  is equivariant with respect to this action. This is because the constituent functors  $\text{Nm}, i^*, \mathbb{T}, L$ , and  $\mathbb{F}$  all have this property, and  $\mathbb{F}$  is essentially surjective and full.

<sup>12</sup>The use of  $p$  being odd is rather superficial here. We could adjust the definition of normalized complexes in the case  $p = 2$ , but ultimately this only extends the final results in type  $A$  since 2 is a bad prime in all other types.

**Remark 3.18.** The construction of  $\text{BC}$  was motivated by a similar functor “ $LL$ ” appearing in [LL, §6.2], which gives a partial geometric description of the Frobenius contraction functor. Another motivation was the “normalized Brauer homomorphism” of [TV16, §4.3], which our construction categorifies.

**Theorem 3.19.** *Let  $\text{Res}_{\text{BC}}: \text{Rep}_k(\widehat{G}) \rightarrow \text{Rep}_k(\widehat{H})$  be restriction along the diagonal embedding. We also denote by  $\text{Res}_{\text{BC}}$  the same functor restricted to the subcategories of tilting modules.<sup>13</sup> The following diagram commutes:*

$$\begin{array}{ccc} \text{Parity}^0(\text{Gr}_G; k) & \xrightarrow{\text{BC}} & \text{Parity}^0(\text{Gr}_H; k) \\ \downarrow \sim & & \downarrow \sim \\ \text{Tilt}_k(\widehat{G}) & \xrightarrow{\text{Res}_{\text{BC}}} & \text{Tilt}_k(\widehat{H}) \end{array}$$

*Proof sketch.* The argument is given in Appendix A. For now let us just explain the key trick (which we learned from the proof of [LL, Theorem 7.3]): since  $\text{Psm}$  commutes with hyperbolic localization by §2.3, and the restriction functor to a maximal torus  $\text{Rep}(\widehat{H}) \rightarrow \text{Rep}(T_{\widehat{H}})$  is faithful and injective on tilting objects, one can reduce to the case where  $H$  is a *torus*. In this case the functor can be computed explicitly, since the affine Grassmannian of a torus is simply a discrete set.  $\square$

#### 4. FUNCTORIALITY AND THE EXCURSION ALGEBRA

In this section we formalize the *abstract excursion algebra*  $\text{Exc}(\Gamma, {}^L G)$ , a device used to decomposable a space into pieces indexed by Langlands parameters. This notion appears implicitly in [Laf18a], but there it is the image<sup>14</sup> of the abstract excursion algebra in a certain endomorphism group which is emphasized.

Since we work with non-split groups, we first clarify in §4.1 our conventions regarding  $L$ -groups. This is a bit subtle, as one finds (at least) two natural versions of the  $L$ -group in the literature: the “algebraic  $L$ -group”  ${}^L G^{\text{alg}}$ , following Langlands, and the “geometric  $L$ -group”  ${}^L G^{\text{geom}}$ , derived from the Geometric Satake equivalence. The difference between them is parallel to the difference between  $L$ -algebraicity and  $C$ -algebraicity emphasized in [BG14].

We emphasize that the unadorned notation  ${}^L G$  denotes the algebraic  $L$ -group, to be consistent with [Laf18a], although the geometric  $L$ -group is really what appears more naturally in our arguments.

We introduce two explicit presentations for the excursion algebra in §4.2 and §4.4. The first presentation is more natural for making the connection to Langlands parameters, which we recall in 4.3. The second presentation is more amenable to constructing actions of the excursion algebra, which makes it more convenient for our purposes, and it is the only one that will be used in the sequel.

Finally in §4.5 we explain how functoriality is interpreted in terms of excursion algebras.

**4.1. Conventions on  $L$ -groups and Langlands parameters.** For a reductive group  $G$  over a field  $\mathbf{F}$  with separable closure  $\mathbf{F}^s$ , we regard its Langlands dual group  $\widehat{G}$  as a split reductive group over  $k$ . The  $L$ -group is a certain semi-direct product  ${}^L G = \widehat{G} \rtimes$

<sup>13</sup>Note that it is not obvious that  $\text{Res}_{\text{BC}}$  preserves the tilting property, but this follows from the non-trivial theorem (building on work of many authors – see the discussion around [JMW16, Theorem 1.2]) that tensor products of tilting modules are tilting.

<sup>14</sup>This image is denoted  $\mathcal{B}$  in [Laf18a].

$\text{Gal}(\mathbf{F}^s/\mathbf{F})$ . Actually, in the case where  $\mathbf{F}$  is a local field we shall instead work with the “Weil form”  $\widehat{G} \rtimes \text{Weil}(\mathbf{F}^s/\mathbf{F})$ . (This is just for consistency with [GL]; because we consider mod  $p$  representations, in our case it would make no difference to work with the Galois form.)

4.1.1. *Algebraic  $L$ -group.* In fact there are at least two conventions for the definition of the  $L$ -group. The one which is more traditionally used in the literature is what we shall call the *algebraic  $L$ -group*, denoted  ${}^L G^{\text{alg}}$ , defined as in [TV16, §2.5]. The root datum  $\Psi(G)$  of  $G_{\mathbf{F}^s}$  determines a pinning for  $\widehat{G}$ , which in turns gives a splitting  $\text{Out}(\widehat{G}) \rightarrow \text{Aut}(\widehat{G})$  and an identification  $\text{Aut}(\Psi(G)) \cong \text{Out}(\widehat{G})$ . The  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$ -action on  $\Psi(G)$  transports to an action  $\text{act}^{\text{alg}}$  of  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$  on  $\widehat{G}$ , and we define  ${}^L G^{\text{alg}}$  to be the semidirect product

$${}^L G^{\text{alg}} := \widehat{G} \rtimes_{\text{act}^{\text{alg}}} \text{Gal}(\mathbf{F}^s/\mathbf{F}).$$

Since the action  $\text{act}^{\text{alg}}$  factors through a finite quotient, we may regard  ${}^L G^{\text{alg}}$  as a pro-algebraic group over  $k$ .

4.1.2. *Geometric  $L$ -group.* We now make a different construction of the  $L$ -group, using the Tannakian theory, following [Zhu15, Appendix A] and [Zhu17, §5.5]. We begin with the Geometric Satake equivalence,

$$\text{P}_{L+G_{\mathbf{F}^s}}(\text{Gr}_{G,\mathbf{F}^s}; k) \cong \text{Rep}_k(\widehat{G}).$$

The Galois group  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$  acts on  $\text{Gr}_{G,\mathbf{F}^s}$ , inducing an action on the neutralized Tannakian category  $(\text{P}_{L+G_{\mathbf{F}^s}}(\text{Gr}_{G,\mathbf{F}^s}; k), \underbrace{H^*(-)}_{\text{fiber functor}})$ . By [Zhu15, Lemma A.1] this in turn induces an action  $\text{act}^{\text{geom}}$  of  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$  on  $\widehat{G}_k$ . We define

$${}^L G^{\text{geom}} := \widehat{G}_k \rtimes_{\text{act}^{\text{geom}}} \text{Gal}(\mathbf{F}^s/\mathbf{F}).$$

In the case at hand we shall see that  $\text{act}^{\text{geom}}$  also factors through a finite quotient of  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$ , so we may also regard  ${}^L G^{\text{geom}}$  as a pro-algebraic group.

4.1.3. *Relation between the two  $L$ -groups.* The relation between these two actions is as follows. We let  $\rho$  be the half sum of positive coroots of  $G^\vee$ , and we denote by  $\rho: \mathbf{G}_m \rightarrow G_{\text{ad}}^\vee$  the corresponding cocharacter. With  $\text{cyc}_p: \text{Gal}(\mathbf{F}^s/\mathbf{F}) \rightarrow \mathbf{F}_p^\times$  denoting the mod  $p$  cyclotomic character, let  $\chi$  denote the composite

$$\text{Gal}(\mathbf{F}^s/\mathbf{F}) \xrightarrow{\text{cyc}_p} \mathbf{F}_p^\times \hookrightarrow k^\times \xrightarrow{\rho} \widehat{G}_{\text{ad}}(k).$$

This induces a homomorphism  $\text{Ad}_\chi: \text{Gal}(\mathbf{F}^s/\mathbf{F}) \rightarrow \text{Aut}(\widehat{G})$ .

**Proposition 4.1** ([Zhu15, Proposition 1.6]). *We have  $\text{act}^{\text{geom}} = \text{act}^{\text{alg}} \circ \text{Ad}_\chi$ .*<sup>15</sup>

Given a choice of lift  $\widetilde{\chi}: \text{Gal}(\mathbf{F}^s/\mathbf{F}) \rightarrow \widehat{G}(k)$  of  $\chi$ , which could for example come from a square root of the mod  $p$  cyclotomic character, we get an isomorphism  ${}^L G^{\text{alg}} \xrightarrow{\sim} {}^L G^{\text{geom}}$  by

$$(g, \gamma) \mapsto (g\widetilde{\chi}(\gamma^{-1}), \gamma). \quad (4.1)$$

<sup>15</sup>The cited reference operates over  $\overline{\mathbf{Q}}_p$  instead of  $k$ . However, the stated result follows by reducing the statement over  $W(k)$  modulo  $p$ . Alternatively, we can apply the same proof as in [Zhu15, Proposition 1.6]; the appearance of the cyclotomic character is based on the fact that the first Chern class of a line bundle lies in  $H^2(\text{Gr}_{G,\mathbf{F}^s}; k(1))^{\text{Gal}(\mathbf{F}^s/\mathbf{F})}$ .

By [Zhu17, Remark 5.5.8], we can always choose a square root of the cyclotomic character when  $\text{char}(\mathbf{F}) > 0$ . However, in general it can happen that  ${}^L G^{\text{alg}}$  and  ${}^L G^{\text{geom}}$  are not isomorphic; for an example see [Zhu17, Example 5.5.9].

At different points we will want to consider both versions of  $L$ -groups. If we write  ${}^L G$  without a superscript, then by default we mean the algebraic  $L$ -group  ${}^L G^{\text{alg}}$ .

**4.1.4. Representation categories.** For any Galois extension  $\mathbf{F}'/\mathbf{F}$  such that  $G_{\mathbf{F}'}$  is split, the analogous construction to §4.1.1 gives a “finite form” algebraic  $L$ -group  $\widehat{G} \rtimes_{\text{act}^{\text{alg}}} \text{Gal}(\mathbf{F}'/\mathbf{F})$ . We define the category of ( $k$ -linear) algebraic representations of  ${}^L G^{\text{alg}}$  to be

$$\text{Rep}_k({}^L G^{\text{alg}}) := \varinjlim_{\mathbf{F}'} \text{Rep}_k(\widehat{G} \rtimes_{\text{act}^{\text{alg}}} \text{Gal}(\mathbf{F}'/\mathbf{F})).$$

Let  $\text{Rep}_k({}^L G^{\text{geom}}) := \text{Rep}_k(\widehat{G})^{\text{Gal}(\mathbf{F}^s/\mathbf{F}), \text{geom}}$  denote the category of continuously  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$ -equivariant objects in  $\text{Rep}_k(\widehat{G})$  with respect to the geometric action. The Geometric Satake equivalence induces by descent an equivalence

$$\text{P}_{L+G}(\text{Gr}_G; k) \cong \text{Rep}_k(\widehat{G})^{\text{Gal}(\mathbf{F}^s/\mathbf{F}), \text{geom}} \quad (4.2)$$

where the action of  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$  on  $\text{Rep}_k(\widehat{G})$  on the right side is via  $\text{act}^{\text{geom}}$ , and on the left hand side,  $\text{Gr}_G$  is considered over  $\mathbf{F}$ . By definition, on the right side we take are taking objects on which  $\text{Gal}(\mathbf{F}^s/\mathbf{F})$  acts *continuously* with its Krull topology. Since  $k$  is algebraic over  $\mathbf{F}_p$ , in this case  $\text{Rep}_k(\widehat{G})^{\text{Gal}(\mathbf{F}^s/\mathbf{F}), \text{geom}}$  can be identified with  $\varinjlim_{\mathbf{F}'/\mathbf{F}} \text{Rep}_k(\widehat{G})^{\text{Gal}(\mathbf{F}'/\mathbf{F}), \text{geom}}$  where the limit runs over finite Galois extensions  $\mathbf{F}'/\mathbf{F}$  over which the geometric action factors.

An isomorphism (4.1) gives an embedding  $\text{Rep}_k({}^L G^{\text{alg}}) \hookrightarrow \text{Rep}_k(\widehat{G})^{\text{Gal}(\mathbf{F}^s/\mathbf{F}), \text{geom}}$ , which as just remarked is an equivalence for our choice of  $k$ . See [Zhu15, Proposition A.10] for a description of the essential image in general.

#### 4.1.5. Langlands parameters.

**Definition 4.2.** Let  $\Gamma$  be a group and  $\overline{\Gamma}$  be a quotient of  $\Gamma$  acting on  $\widehat{G}$ . A *Langlands parameter* from  $\Gamma$  into  $\widehat{G}(k) \rtimes \overline{\Gamma}$  is a  $\widehat{G}(k)$ -conjugacy class of continuous homomorphisms  $\rho: \Gamma \rightarrow \widehat{G}(k) \rtimes \overline{\Gamma}$ , which has the property that the composite map  $\Gamma \rightarrow \widehat{G} \rtimes \Gamma \rightarrow \overline{\Gamma}$  is the given quotient  $\Gamma \twoheadrightarrow \overline{\Gamma}$ .

Equivalently, we may view  $\rho$  as an element of the continuous cohomology group  $H^1(\Gamma, \widehat{G}(k))$ , where the action of  $\Gamma$  on  $\widehat{G}(k)$  is the given one (via  $\Gamma \rightarrow \overline{\Gamma}$ ) in the semi-direct product.

We will consider Langlands parameters with  $\widehat{G}(k) \rtimes \Gamma$  being either  ${}^L \widehat{G}^{\text{alg}}(k)$  or  ${}^L \widehat{G}^{\text{geom}}(k)$ , and  $\Gamma$  being either  $\text{Gal}(F^s/F)$  for a global field  $F$  or  $\text{Weil}(\overline{F}_v/F_v)$  for a local field  $F_v$ .

Note that the algebraic  $\Gamma$ -action on  $\widehat{G}(k)$  factors through a finite quotient  $\Gamma \twoheadrightarrow \text{Gal}(\mathbf{F}'/\mathbf{F})$ . It is clear that Langlands parameters into  ${}^L G^{\text{alg}}(k)$  are in bijection (under restriction) with Langlands parameters into  $\widehat{G}(k) \rtimes \text{Gal}(\mathbf{F}'/\mathbf{F})$  for any such  $\mathbf{F}'$ .

We say that a representation  $\rho: \Gamma \rightarrow {}^L G^{\text{alg}}(k)$  is *semisimple*<sup>16</sup> if the Zariski-closure of the image of  $\rho$  in  $\widehat{G}(k) \rtimes \text{Gal}(\mathbf{F}'/\mathbf{F})$ , for any finite extension  $\mathbf{F}'/\mathbf{F}$  over which the  $\Gamma$ -action factors, has reductive component group.

<sup>16</sup>Also called “completely reducible” in [BHKT19].

**4.2. Presentation of the excursion algebra.** Let  $\Gamma$  be a group, which is either  $\text{Gal}(F^s/F)$  for a global field  $F$  or  $\text{Weil}(F^s/F)$  for a local field  $F$ . Let  $G$  be a reductive group over  $F$  and  ${}^L G^{\text{alg}}$  the algebraic  $L$ -group as defined in §4.1.1.

We will define the *excursion algebra*  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$  to be the commutative algebra over  $k$  presented by explicit generators and relations given below. (The topology on  $\Gamma$  will not be relevant for the definition of  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$ .) For a more conceptual perspective see [Zhu, §2], wherein the excursion algebra is denoted  $k[\mathcal{R}_{\Gamma, {}^L G^{\text{alg}}} // \widehat{G}]$ .

**4.2.1. Generators.** We define  $\mathcal{O}({}^L G_k^{\text{alg}}) := \varinjlim \mathcal{O}(\widehat{G}_k \rtimes \text{Gal}(F'/F))$  where the limit runs over finite extensions  $F'/F$  over which the  $\Gamma$ -action on  $\widehat{G}_k$  factors.

Generators of  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$  will be denoted  $S_{I, f, (\gamma_i)_{i \in I}}$ , where the indexing set  $(I, f, (\gamma_i)_{i \in I})$  consists of:

- (i)  $I$  is a finite (possibly empty) set,
- (ii)  $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^I / \widehat{G}_k) := \mathcal{O}(({}^L G_k^{\text{alg}})^I)^{\widehat{G}_k \times \widehat{G}_k}$ , where the quotient is for the actions of  $\widehat{G}_k$  by diagonal left and right translation, respectively, and
- (iii)  $\gamma_i \in \Gamma$  for each  $i \in I$ .

**4.2.2. Relations.** Next we describe the relations. (Compare [Laf18a, §10].)

- (i)  $S_{\emptyset, f, *} = f(1_G)$ .
- (ii) The map  $f \mapsto S_{I, f, (\gamma_i)_{i \in I}}$  is a  $k$ -algebra homomorphism in  $f$ , i.e.

$$\begin{aligned} S_{I, f+f', (\gamma_i)_{i \in I}} &= S_{I, f, (\gamma_i)_{i \in I}} + S_{I, f', (\gamma_i)_{i \in I}}, \\ S_{I, ff', (\gamma_i)_{i \in I}} &= S_{I, f, (\gamma_i)_{i \in I}} \cdot S_{I, f', (\gamma_i)_{i \in I}}, \end{aligned}$$

and

$$S_{I, \lambda f, (\gamma_i)_{i \in I}} = \lambda S_{I, f, (\gamma_i)_{i \in I}} \text{ for all } \lambda \in k.$$

- (iii) For all maps of finite sets  $\zeta: I \rightarrow J$ , all  $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^I / \widehat{G}_k)$ , and all  $(\gamma_j)_{j \in J} \in \Gamma^J$ , we have

$$S_{J, f^\zeta, (\gamma_j)_{j \in J}} = S_{I, f, (\gamma_{\zeta(i)})_{i \in I}}$$

where  $f^\zeta \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^J / \widehat{G}_k)$  is defined by  $f^\zeta((g_j)_{j \in J}) := f((g_{\zeta(i)})_{i \in I})$ .

- (iv) For all  $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^I / \widehat{G}_k)$  and  $(\gamma_i)_{i \in I}, (\gamma'_i)_{i \in I}, (\gamma''_i)_{i \in I} \in \Gamma^I$ , we have

$$S_{I \sqcup I \sqcup I, \tilde{f}, (\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I}} = S_{I, f, (\gamma_i (\gamma'_i)^{-1} \gamma''_i)_{i \in I}},$$

where  $\tilde{f} \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^{I \sqcup I \sqcup I} / \widehat{G}_k)$  is defined by

$$\tilde{f}((g_i)_{i \in I} \times (g'_i)_{i \in I} \times (g''_i)_{i \in I}) = f((g_i (g'_i)^{-1} g''_i)_{i \in I}).$$

- (v) If  $f$  is inflated from a function on  $\Gamma^I$ , then  $S_{I, f, (\gamma_i)_{i \in I}}$  equals the scalar  $f((\gamma_i)_{i \in I})$ . More generally, if  $J$  is a subset of  $I$  and  $f$  is inflated from a function on  $(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^J / \widehat{G}_k) \times \Gamma^{I \setminus J}$ , then we have

$$S_{I, f, (\gamma_i)_{i \in I}} = S_{J, \tilde{f}, (\gamma_j)_{j \in J}}$$

where  $\tilde{f}((g_j)_{j \in J}) := f((g_j)_{j \in J}, (\gamma_i)_{i \in I \setminus J})$ . (Compare [Laf18a, p. 164].)

**Definition 4.3.** The *excursion algebra*  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$  is the  $k$ -algebra with generators and relations specified as above.

**4.3. Constructing Galois representations.** The following result of Lafforgue (generalized to modular coefficients by Bökke-Harris-Khare-Thorne) explains how to obtain Langlands parameters from characters of  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$ .

**Proposition 4.4** ([BHKT19, Theorem 4.5], [Laf18a, §13]). *For any character  $\nu: \text{Exc}(\Gamma, {}^L G^{\text{alg}}) \rightarrow k$ , there is a semisimple representation  $\rho_\nu: \Gamma \rightarrow {}^L G^{\text{alg}}(k)$ , unique up to conjugation by  $\widehat{G}(k)$ , which is characterized by the following condition:*

*For all  $n \in \mathbb{N}$ ,  $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\text{alg}})^{n+1} / \widehat{G}_k)$ , and  $(\gamma_0, \dots, \gamma_n) \in \Gamma^{n+1}$ , we have*

$$\nu(S_{\{0, \dots, n\}, f, (\gamma_0, \gamma_1, \dots, \gamma_n)}) = f((\rho(\gamma_0 \gamma_n), \rho(\gamma_1 \gamma_n), \dots, \rho(\gamma_{n-1} \gamma_n), \rho(\gamma_n))). \quad (4.3)$$

**4.4. Another presentation for the excursion algebra.** We will now describe a second presentation of  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$ , following [Laf18a, Lemma 0.31], which is more useful for constructing actions of  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$  in practice.

**4.4.1. Generators.** We take a set of generators indexed by tuples of data of the form  $(I, W, x, \xi, (\gamma_i)_{i \in I})$ , where:

- (i)  $I$  is a finite set,
- (ii)  $W \in \text{Rep}_k(({}^L G_k^{\text{alg}})^I)$  (cf. §4.1.4),
- (iii)  $x \in W$  is a vector invariant under the diagonal  $\widehat{G}_k$ -action,
- (iv)  $\xi \in W^*$  is a functional invariant under the diagonal  $\widehat{G}_k$ -action,
- (v)  $\gamma_i \in \Gamma$  for each  $i$ .

The corresponding generator of  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$  will be denoted by  $S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}} \in \text{Exc}(\Gamma, {}^L G^{\text{alg}})$ .

**4.4.2. Relations.** Next we describe the relations.

- (i)  $S_{\emptyset, x, \xi, *} = \langle x, \xi \rangle$ .
- (ii) For any morphism of  $({}^L G_k^{\text{alg}})^I$ -representations  $u: W \rightarrow W'$  and functional  $\xi' \in (W')^*$  invariant under the diagonal  $\widehat{G}_k$ -action, we have

$$S_{I, W, x, {}^t u(\xi'), (\gamma_i)_{i \in I}} = S_{I, W', u(x), \xi', (\gamma_i)_{i \in I}}, \quad (4.4)$$

where  ${}^t u: (W')^* \rightarrow W^*$  denotes the dual to  $u$ .

- (iii) For two tuples  $(I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1})$  and  $(I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2})$  as in §4.4.1, we have

$$S_{I_1 \sqcup I_2, W_1 \boxtimes W_2, x_1 \boxtimes x_2, \xi_1 \boxtimes \xi_2, (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} \circ S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \quad (4.5)$$

Letting  $\Delta: \mathbb{1} \rightarrow \mathbb{1} \oplus \mathbb{1}$  be the diagonal inclusion, and  $\nabla: \mathbb{1} \oplus \mathbb{1} \rightarrow \mathbb{1}$  the addition map, we also have

$$S_{I_1 \sqcup I_2, W_1 \oplus W_2, (x_1 \oplus x_2) \circ \Delta, \nabla \circ (\xi_1 \oplus \xi_2), (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} + S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \quad (4.6)$$

Furthermore, the assignment  $(I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}) \mapsto S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}} \in \text{Exc}(\Gamma, {}^L G^{\text{alg}})$  is  $k$ -linear in  $x$  and  $\xi$ .

- (iv) Let  $\zeta: I \rightarrow J$  be a map of finite sets. Suppose  $W \in \text{Rep}(({}^L G)^I)$ ,  $x: \mathbb{1} \rightarrow W|_{\Delta(\widehat{G})}$ ,  $\xi: W|_{\Delta(\widehat{G})} \rightarrow \mathbb{1}$ , and  $(\gamma_j)_{j \in J} \in \Gamma^J$ . Letting  $W^\zeta$  be the restriction of  $W$  under the functor  $\text{Rep}(({}^L G)^I) \rightarrow \text{Rep}(({}^L G)^J)$  induced by  $\zeta$ , we have

$$S_{J, W^\zeta, x, \xi, (\gamma_j)_{j \in J}} = S_{I, W, x, \xi, (\gamma_{\zeta(i)})_{i \in I}}. \quad (4.7)$$



- (v) Letting  $\delta_W: \mathbb{1} \rightarrow W \otimes W^*$  and  $\text{ev}_W: W^* \otimes W \rightarrow 1$  be the natural counit and unit, we have

$$S_{I,W,x,\xi,(\gamma_i(\gamma'_i)^{-1}\gamma''_i)_{i \in I}} = S_{I \sqcup I \sqcup I, W \boxtimes W^* \boxtimes W, \delta_W \boxtimes x, \xi \boxtimes \text{ev}_W, (\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I}}. \quad (4.8)$$

- (vi) If  $W$  is inflated from a representation of  $({}^L G^{\text{alg}})^J \times \Gamma^{I \setminus J}$ , then we have

$$S_{I,W,x,\xi,(\gamma_i)_{i \in I}} = S_{J,W|_{({}^L G^{\text{alg}})^J}, ((1_j)_{j \in J}, (\gamma_i)_{i \in I \setminus J}) \cdot x, \xi, (\gamma_j)_{j \in J}}.$$

**4.4.3. Relation between the presentations.** The two presentations in §4.2 and §4.4 are related as follows. The generator  $S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}}$  corresponds to  $S_{I, f_{x,\xi}, (\gamma_i)_{i \in I}}$  where  $f_{x,\xi}$  is the function on  $({}^L G_k)^I$  given by  $(g_i)_{i \in I} \mapsto \langle \xi, (g_i)_{i \in I} \cdot x \rangle$ . The assumptions on  $\xi$  and  $x$  imply that  $f_{x,\xi}$  is invariant under the left and right diagonal  $\widehat{G}_k$ -actions. The relations in §4.4.2 imply that  $S_{I,W,x,\xi,(\gamma_i)_{i \in I}}$  depends only on  $f_{x,\xi}$  (and not on the choice of  $x, \xi$ ) [Laf18a, Lemme 10.6].

**4.5. Functoriality for excursion algebras.** A homomorphism of  $L$ -groups  $\phi: {}^L H^{\text{alg}} \rightarrow {}^L G^{\text{alg}}$  is *admissible* if it lies over the identity map on  $\Gamma$ , i.e. the diagram below commutes.

$$\begin{array}{ccc} {}^L H^{\text{alg}} & \xrightarrow{\phi} & {}^L G^{\text{alg}} \\ \downarrow & & \downarrow \\ \Gamma & \xlongequal{\text{Id}} & \Gamma \end{array}$$

**Lemma 4.5.** *Let  $\phi: {}^L H^{\text{alg}} \rightarrow {}^L G^{\text{alg}}$  be an admissible homomorphism. Then there is a homomorphism  $\phi^*: \text{Exc}(\Gamma, {}^L G^{\text{alg}}) \rightarrow \text{Exc}(\Gamma, {}^L H^{\text{alg}})$  which on  $k$ -points sends a parameter  $\rho \in H^1(\Gamma, \widehat{H}(k))$  to  $\phi \circ \rho \in H^1(\Gamma, \widehat{G}(k))$ .*

*Proof.* The map  $\phi$  induces  $\text{Res}_\phi: \text{Rep}_k({}^L G^{\text{alg}}) \rightarrow \text{Rep}_k({}^L H^{\text{alg}})$ . At the level of generators, the map  $\phi^*$  sends

$$S_{V,x,\xi,\{\gamma_i\}_{i \in I}} \mapsto S_{\text{Res}_\phi(V), \text{Res}_\phi(x), \text{Res}_\phi(\xi), \{\gamma_i\}_{i \in I}}.$$

We verify by inspection that this map sends relations to relations. To see that this indeed induces composition with  $\phi$  at the level of Langlands parameters, use (4.3).  $\square$

**Definition 4.6** (Base change). In the base change situation, where  $H$  is a reductive group over  $F$  and  $G = \text{Res}_{E/F}(H_E)$ , the relevant morphism of  $L$ -groups  $\phi_{\text{BC}}: {}^L H^{\text{alg}} \rightarrow {}^L G^{\text{alg}}$  is defined by the formula  $(h, \gamma) \mapsto (\Delta(h), \gamma)$ . In fact this same formula also defines the corresponding map of geometric  $L$ -groups  $\phi_{\text{BC}}^{\text{geom}}: {}^L H^{\text{geom}} \rightarrow {}^L G^{\text{geom}}$ , so  $\phi_{\text{BC}}^{\text{geom}}$  and  $\phi_{\text{BC}}$  are compatible with (4.1) if we use the same choice of square root of the cyclotomic character in the latter to define isomorphisms  ${}^L H^{\text{alg}} \approx {}^L H^{\text{geom}}$  and  ${}^L G^{\text{alg}} \approx {}^L G^{\text{geom}}$ . We denote

$$\phi_{\text{BC}}^*: \text{Exc}(\Gamma, {}^L G^{\text{alg}}) \rightarrow \text{Exc}(\Gamma, {}^L H^{\text{alg}})$$

the induced map of excursion algebras.

## 5. CYCLIC BASE CHANGE IN THE GLOBAL SETTING

In this section we will prove Theorem 1.1. This will require knowledge of how Lafforgue's parametrization works, which we summarize in §5.2. It is based on interpreting the space of automorphic functions as the cohomology of moduli spaces of shtukas, and constructing an action of the excursion algebra on it using geometry. We briefly recall the definitions of the relevant geometric objects in §5.1.

The main work occurs in §5.3, where we use a variant of Lafforgue's ideas to construct and analyze an action of the “ $\sigma$ -equivariant excursion algebra” on the Tate cohomology of

moduli spaces of shtukas. In the base change situation, equivariant localization mediates between the Tate cohomology of shtukas for  $G$  and for  $H$ , allowing us to relate certain excursion operators for the two groups. This is then used in §5.4 to establish the existence of base change for mod  $p$  automorphic forms; it will also be the crucial input for our local results in the next section.

**5.1. Moduli of shtukas.** We will use the theory of moduli stacks of shtukas, due to Drinfeld and generalized by Varshavsky. Here we very briefly recall the relevant definitions in order to set notation. More comprehensive references include [Var04] and [Laf18a].

**5.1.1. Shtukas.** Fix a smooth projective curve  $X$  over a finite field  $\mathbf{F}_\ell$  of characteristic  $\neq p$ . For an affine group scheme  $G \rightarrow X$  and a finite set  $I$ , the stack  $\text{Sht}_{G,I}$  represents the following moduli functor on  $\mathbf{F}_\ell$ -schemes  $S$ :

$$\text{Sht}_{G,I}: S \mapsto \left\{ \begin{array}{l} (x_i)_{i \in I} \in X^I(S) \\ \mathcal{E} = \text{fppf } G\text{-torsor over } X \times S \\ \varphi: \mathcal{E}|_{X \times S - \bigcup_{i \in I} \Gamma_{x_i}} \xrightarrow{\sim} {}^\tau \mathcal{E}|_{X \times S - \bigcup_{i \in I} \Gamma_{x_i}} \end{array} \right\},$$

where  $\tau$  is the Frobenius  $\text{Frob}_\ell$  on the  $S$  factor in  $X \times S$ , and  ${}^\tau \mathcal{E}$  is the pullback of  $\mathcal{E}$  under the map  $1 \times \tau: X \times S \rightarrow X \times S$ .

Geometrically,  $\text{Sht}_{G,I}$  has a Schubert stratification whose strata are Deligne-Mumford stacks locally of finite type. We regard it as an ind-(locally finite type) Deligne-Mumford stack.

**5.1.2. Hecke stack.** The Hecke stack  $\text{Hk}_{G,I}$  classifies

$$\text{Hk}_{G,I}: S \mapsto \left\{ \begin{array}{l} (x_i)_{i \in I} \in X^I(S) \\ \mathcal{E}, \mathcal{E}' = \text{fppf } G\text{-torsors over } X \times S \\ \varphi: \mathcal{E}|_{X \times S - \bigcup \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}'|_{X \times S - \bigcup \Gamma_{x_i}} \end{array} \right\}.$$

The Geometric Satake equivalence provides a functor  $\text{Rep}_k(({}^L G)^I) \rightarrow D(\text{Hk}_{G,I}; k)$ , which we normalize as in [Laf18a, Theorem 0.9].

**5.1.3. Satake sheaves.** There is a map  $\text{Sht}_{G,I} \rightarrow \text{Hk}_{G,I}$  sending  $(\{x_i\}_{i \in I}, \mathcal{E}, \varphi)$  to  $(\{x_i\}_{i \in I}, \mathcal{E}, {}^\tau \mathcal{E}, \varphi)$ . Composing with the  $*$ -pullback through  $\text{Sht}_{G,I} \rightarrow \text{Hk}_{G,I}$  induces a functor

$$\text{Sat}^{\text{geom}}: \text{Rep}_k(\widehat{G}^I)^{\text{Gal}(\mathbf{F}^s/\mathbf{F}), \text{geom}} \rightarrow D^b(\text{Sht}_{G,I}; k).$$

Finally, we may identify  $\text{Rep}_k(({}^L G^{\text{alg}})^I) \xrightarrow{\sim} \text{Rep}_k(\widehat{G}^I)^{\text{Gal}(\mathbf{F}^s/\mathbf{F}), \text{geom}}$  as in §4.1.4, giving a functor (cf. [Laf18a, Theorem 0.11])

$$\text{Sat}: \text{Rep}_k(({}^L G^{\text{alg}})^I) \rightarrow D^b(\text{Sht}_{G,I}; k).$$

The Schubert stratification is defined by the support of the sheaves in the image of  $\text{Sat}$ , with the closure relations corresponding to the Bruhat order. (In particular,  $\text{Sat}$  lands in the derived category of sheaves constructible with respect to the Schubert stratification on  $\text{Sht}_{G,I}$ .)

**5.1.4.** There is a map

$$\pi_I: \text{Sht}_{G,I} \rightarrow X^I$$

projecting a tuple  $(\{x_i\}_{i \in I}, \mathcal{E}, \varphi_i)$  to  $\{x_i\}_{i \in I}$ .

**5.1.5. Level structures.** For  $D \subset X$  a finite-length subscheme, there are level covers  $\text{Sht}_{G,D,I} \rightarrow \text{Sht}_{G,I}|_{(X-D)^I}$  which parametrize the additional datum of a  $\tau$ -equivariant trivialization of  $\mathcal{E}$  over  $S \times D$ . Note that by definition, the “legs”  $\{x_i\}_{i \in I} \in (X - D)(S)^I$  avoid  $D$ .

5.1.6. *Iterated shtukas.* Let  $I_1, \dots, I_r$  be a partition of  $I$ . We define  $\mathrm{Sht}_{G,D,I}^{(I_1, \dots, I_r)}$  (sometimes called a moduli stack of *iterated shtukas*) to be the stack

$$\mathrm{Sht}_{G,D,I}^{(I_1, \dots, I_r)} : S \mapsto \left\{ \begin{array}{l} (x_i)_{i \in I} \in X^I(S) \\ \mathcal{E}_0, \dots, \mathcal{E}_r = \text{fppf } G\text{-torsors over } X \times S \\ \varphi_j : \mathcal{E}_{j-1}|_{X \times S - \bigcup_{i \in I_j} \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_j|_{X \times S - \bigcup_{i \in I_j} \Gamma_{x_i}} \quad j = 1, \dots, r \\ \varphi : \mathcal{E}_r \xrightarrow{\sim} {}^\tau \mathcal{E}_0 \\ \text{trivialization over } D \times S \end{array} \right\}.$$

There is a map  $\nu : \mathrm{Sht}_{G,D,I}^{(I_1, \dots, I_r)} \rightarrow \mathrm{Sht}_{G,D,I}$ . A key property of this morphism is that it is stratified small (with respect to the Schubert stratification), which is a consequence of the same property of the convolution morphism for Beilinson-Drinfeld Grassmannians.

5.1.7. *Partial Frobenius.* There is a partial Frobenius  $F_{I_1} : \mathrm{Sht}_{G,D,I}^{(I_1, \dots, I_r)} \rightarrow \mathrm{Sht}_{G,D,I}^{(I_2, \dots, I_r, I_1)}$  sending

$$\begin{aligned} x_i &\mapsto \begin{cases} {}^\tau x_i & i \in I_1 \\ x_i & \text{otherwise} \end{cases} \\ (\mathcal{E}_0, \dots, \mathcal{E}_r) &\mapsto (\mathcal{E}_1, \dots, \mathcal{E}_r, {}^\tau \mathcal{E}_1) \\ (\varphi_1, \dots, \varphi_r) &\mapsto (\varphi_2, \dots, \varphi_r, {}^\tau \varphi_1). \end{aligned}$$

It lies over the partial Frobenius  $\mathrm{Frob}_{I_1}$  on  $X^I$  (applying  $\mathrm{Frob}_\ell$  to the coordinates indexed by  $i \in I_1$ ), so that the diagram below is commutative (and cartesian up to radicial maps):

$$\begin{array}{ccc} \mathrm{Sht}_{G,D,I}^{(I_1, \dots, I_r)} & \xrightarrow{F_{I_1}} & \mathrm{Sht}_{G,D,I}^{(I_2, \dots, I_r, I_1)} \\ \downarrow \nu & & \downarrow \nu \\ X^I & \xrightarrow{\mathrm{Frob}_{I_1}} & X^I \end{array} \quad (5.1)$$

5.1.8. *Base change setup.* We now consider the following “base-change setup”. Let  $F$  be the function field of  $X$  and  $H_F$  a reductive group over  $F$ . We choose a parahoric extension of  $H_F$  to a smooth affine group scheme  $H$  over  $X$ .

Let  $E/F$  be a cyclic extension of  $F$  having degree  $p$ , so  $E$  corresponds to the function field of a smooth projective curve  $X'$ . Define  $G := \mathrm{Res}_{X'/X}(H_{X'})$ , which is an affine group scheme over  $X$  with generic fiber  $G_F \cong \mathrm{Res}_{E/F}(H_E)$ . The group scheme  $G \rightarrow X$  comes with an induced action of  $\langle \sigma \rangle = \mathrm{Aut}(X'/X)$ .

**5.2. Review of V. Lafforgue’s global Langlands correspondence.** Write  $\Gamma = \mathrm{Gal}(F^s/F)$ . In [Laf18a, §13], Lafforgue constructs an action of  $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$  on the space of cusp forms for  $G$  with coefficients in  $k$ . This has been improved by Cong Xue, who extended the action to all compactly supported functions [Xuea, §7].<sup>17</sup>

We summarize the construction of the excursion action, as we shall make use of some of its internal aspects, and we also need to explain why it can be used to construct some excursion actions on Tate cohomology.

<sup>17</sup>The cited paper is written for split  $G$ , but the argument can be generalized, as will appear in forthcoming work of Xue (announced in [Xueb]).

5.2.1. *Constructing actions of the excursion algebra.* We will explain an abstract setup that gives rise to actions of the excursion algebra.

**Definition 5.1.** Let  $A$  be a (not necessarily commutative) ring. A family of functors  $H_I: \text{Rep}_k(({}^L G)^I) \rightarrow \text{Mod}_A(\Gamma^I)$ , where  $I$  runs over (possibly empty) finite sets, is *admissible* if it satisfies the two conditions below.

- (1) (*Compatibility with fusion*) For all  $\zeta: I \rightarrow J$ , there is a natural isomorphism  $\chi_\zeta$  between the functors  $H_I \circ \text{Res}_\zeta$  and  $\text{Res}_\zeta \circ H_J$  in the diagram:

$$\begin{array}{ccc} \text{Rep}_k(({}^L G)^I) & \xrightarrow{H_I} & \text{Mod}_A(\Gamma^I) \\ \text{Res}_\zeta \downarrow & \searrow \chi_\zeta & \downarrow \text{Res}_\zeta \\ \text{Rep}_k(({}^L G)^J) & \xrightarrow{H_J} & \text{Mod}_A(\Gamma^J) \end{array} \quad (5.2)$$

- (2) (*Compatibility with composition*) For  $I' \xrightarrow{\zeta'} I \xrightarrow{\zeta} J$ , we have  $\chi_{\zeta \circ \zeta'} = \chi_\zeta \circ \chi_{\zeta'}$ .

**Construction 5.2.** Let  $\mathbb{1}$  denote the trivial representation of  ${}^L G$ . Given an admissible family of functors  $H_I: \text{Rep}_k(({}^L G)^I) \rightarrow \text{Mod}_A(\Gamma^I)$ , we get an  $A$ -linear action of  $\text{Exc}(\Gamma, {}^L G)$  on  $H_{\{0\}}(\mathbb{1})$  as follows.

For a tuple  $(I, W, x, \xi, (\gamma_i)_{i \in I})$  we define an endomorphism, which gives the image of  $S_{I, W, x, \xi, (\gamma_i)_{i \in I}}$  in  $\text{End}_A(H_{\{0\}}(\mathbb{1}))$ , by the following composition:

$$H_{\{0\}}(\mathbb{1}) \xrightarrow{H_{\{0\}}(x)} H_{\{0\}}(W^\zeta) \xrightarrow{\sim} H_I(W) \xrightarrow{(\gamma_i)_{i \in I}} H_I(W) \xrightarrow{\sim} H_{\{0\}}(W^\zeta) \xrightarrow{H_{\{0\}}(\xi)} H_{\{0\}}(\mathbb{1}).$$

From the assumptions of admissibility it is straightforward to check the relations in §4.4.2.

**Remark 5.3.** Note that it follows from admissibility that the  $A$ -module underlying  $H_I(\mathbb{1})$  for any  $I$  is identified with  $H_\emptyset(\mathbb{1})$  by  $\chi_{\emptyset \rightarrow \{1\}}$ . Proposition 4.4 then attaches a Galois representation to each generalized eigenvector for the  $\text{Exc}(\Gamma, {}^L G)$ -action on  $H_\emptyset(\mathbb{1})$ . (Of course, such an eigenvector is not guaranteed to exist in general.)

5.2.2. *Excursion action on the cohomology of shtukas.* Let  $\mathcal{H}_G$  be the Hecke algebra acting on  $\text{Sht}_{G, D}$ ; it is the tensor product of local Hecke algebras with the level structure dictated by  $D$ . For any finite set  $I$ , we have a map

$$R\pi_I: \text{Sht}_{G, D, I} \rightarrow (X - D)^I$$

remembering the points of the curve indexed by  $I$  (which avoid  $D$  by definition). Let  $\eta^I$  denote the generic point of  $X^I$  and  $\overline{\eta^I}$  the spectrum of an algebraic closure, viewed as a geometric generic point of  $X^I$ . When  $I$  is a singleton, we will just abbreviate these by  $\eta$  and  $\overline{\eta}$ .

We will define a family of functors indexed by finite sets  $I$ :

$$H_I: \text{Rep}_k(({}^L G^{\text{alg}})^I) \rightarrow \text{Mod}_{\mathcal{H}_G}(\Gamma^I) \quad (5.3)$$

sending  $V \in \text{Rep}_k(({}^L G^{\text{alg}})^I)$  to

$$R^0 \pi_{I!}(\text{Sht}_{G, D, I} |_{\overline{\eta^I}}; \text{Sat}(V)). \quad (5.4)$$

Here and throughout, we use the *perverse* t-structure in formation of  $R^0 \pi_{I!}$ . Note that a priori  $H_I(V)$  has an action of  $\pi_1(\eta^I, \overline{\eta^I})$ , which maps<sup>18</sup> to  $\Gamma^I$  but neither injectively nor surjectively.

<sup>18</sup>The map is non-canonical: it depends on a choice of specialization as in [Laf18a, Remark 8.18].

5.2.3. We explain why the  $\pi_1(\eta^I, \overline{\eta^I})$  extends canonically to an action of  $\Gamma^I$ . Assume  $I$  is non-empty, since otherwise there is nothing to prove. The Satake functor of §5.1.3 admits a generalization  $\text{Sat}^{(I_1, \dots, I_r)}: \text{Rep}_k(({}^L G)^I) \rightarrow D^b(\text{Sht}_{G,D,I}^{(I_1, \dots, I_r)}; k)$ , such that the map

$$\nu: \text{Sht}_{G,D,I}^{(I_1, \dots, I_r)} \rightarrow \text{Sht}_{G,D,I}$$

has the property that  $R\nu_* \text{Sat}^{(I_1, \dots, I_r)}(V) \cong \text{Sat}(V)$ . Furthermore, there are natural isomorphisms  $F_{I_1}^* \text{Sat}^{(I_1, \dots, I_r)}(V) \cong \text{Sat}^{(I_2, \dots, I_r, I_1)}(V)$ , where  $F_{I_1}$  is the partial Frobenius from §5.1.7.

Write  $I = \{1, \dots, n\}$ . Thanks to the above properties and (5.1), the partial Frobenius maps on  $\text{Sht}_{G,D,I}^{(\{1\}, \dots, \{n\})}$  then induce maps

$$\text{Frob}_{\{1\}}^* H_I(V) \xrightarrow{\sim} H_I(V).$$

That equips  $H_I(V)$  with the action of the larger group  $\text{FWeil}(\eta^I, \overline{\eta^I})$  that we now recall, summarizing [Laf18a, Remarque 8.18]. Let  $F^I$  denote the function field of  $X^I$ , so  $\eta^I = \text{Spec } F^I$ , and  $\overline{F^I}$  an algebraic closure, so we may take  $\overline{\eta^I} = \text{Spec } \overline{F^I}$ . Write  $(F^I)^{\text{perf}}$  for the perfect closure of  $F^I$ , and  $\text{Frob}_{\{i\}}$  for the “partial Frobenius” automorphism of  $(F^I)^{\text{perf}}$  induced by  $\text{Frob}_q$  on the  $i$ th factor. We define

$$\text{FWeil}(\eta^I, \overline{\eta^I}) := \{\gamma \in \text{Aut}_{\overline{\mathbf{F}}_q}(\overline{F^I}) : \exists (n_i)_{i \in I} \in \mathbf{Z}^I \text{ such that } \gamma|_{(F^I)^{\text{perf}}} = \prod_{i \in I} (\text{Frob}_{\{i\}})^{n_i}\}.$$

Writing  $\pi_1^{\text{geom}}(\eta^I, \overline{\eta^I}) := \ker(\pi_1(\eta^I, \overline{\eta^I}) \xrightarrow{\deg} \widehat{\mathbf{Z}})$ , this fits into an extension

$$0 \rightarrow \pi_1^{\text{geom}}(\eta^I, \overline{\eta^I}) \rightarrow \text{FWeil}(\eta^I, \overline{\eta^I}) \rightarrow \mathbf{Z}^I \rightarrow 0.$$

Fixing a specialization morphism  $\overline{\eta^I} \rightsquigarrow \Delta(\eta^{\{1\}})$  induces a surjection

$$\text{FWeil}(\eta^I, \overline{\eta^I}) \twoheadrightarrow \text{Weil}(\eta, \overline{\eta})^I.$$

A form of Drinfeld’s Lemma [Xuea, Lemma 7.4.2] is used to show that the action of  $\text{FWeil}(\eta^I, \overline{\eta^I})$  on  $H_I(V)$  factors through  $\text{Weil}(F^s/F)^I$ ; continuity considerations then imply that the action extends uniquely to one of  $\Gamma^I$ .

**Example 5.4.** Let us unravel

$$H_{\{1\}}(\mathbb{1}) = R^0 \pi_{\{1\}}^*(\text{Sht}_{G,D,\{1\}} |_{\overline{\eta^{\{1\}}}}; \text{Sat}(\mathbb{1})). \quad (5.5)$$

By Remark 5.3 the underlying Hecke module of  $H_{\{1\}}(\mathbb{1})$  is isomorphic to  $H_\emptyset(\mathbb{1})$ . According to [Laf18a, Remarque 12.2], this is the space of compactly supported  $k$ -valued functions on the discrete groupoid

$$\text{Bun}_{G,D}(\mathbf{F}_\ell) = \coprod_{\alpha \in \ker^1(F,G)} \left( G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F) / \prod_v K_v \right), \quad (5.6)$$

where  $G_\alpha$  is the pure inner form of  $G$  corresponding to  $\alpha$ ,  $K_v = G(\mathcal{O}_v)$  for  $v \notin D$ , and  $K_v = \ker(G(\mathcal{O}_v) \rightarrow G_D)$ .

The family of functors  $H_I$  is admissible; this is an immediate consequence of the fact that  $\text{Sat}$  is already compatible with composition and fusion. Hence Construction 5.2 applies to define an action of  $\text{Exc}(\Gamma, {}^L G)$  on  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$ . Elements of the image of  $\text{Exc}(\Gamma, {}^L G)$  in  $\text{End}(C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k))$  are called “excursion operators”.

5.2.4. *Xue's generalization.* The subspace  $C_{\text{cusp}}^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k) \subset C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$  of cusp forms is finite-dimensional and stable under the  $\text{Exc}(\Gamma, {}^L G)$ -action, and therefore decomposes into a direct sum of generalized eigenspaces under the action of  $\text{Exc}(\Gamma, {}^L G)$ . Using Proposition 4.4, this decomposition corresponds to a parametrization by Langlands parameters.

We cannot find a larger finite-dimensional subspace of  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$  stable under  $\text{Exc}(\Gamma, {}^L G)$ . However, we *can* find finite-dimensional quotient spaces on which the  $\text{Exc}(\Gamma, {}^L G)$ -action descends.

For example, quotients of the following form arise in [Xue20, Theorem 3.6.7]. Since  $\text{Exc}(\Gamma, {}^L G)$  acts Hecke-equivariantly on  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$ , and the latter is a finite<sup>19</sup>  $\mathcal{H}_{G,u}$ -module for  $u \notin D$ , any finite-codimension ideal  $\mathcal{I} \subset \mathcal{H}_{G,u}$  for such  $u$  gives a (possibly zero) finite-dimensional quotient space  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k) \otimes_{\mathcal{H}_{G,u}} (\mathcal{H}_{G,u}/\mathcal{I})$  which carries a  $\text{Exc}(\Gamma, {}^L G)$ -action.

We will consider any Langlands parameter arising via Proposition 4.4 from the  $\text{Exc}(\Gamma, {}^L G)$ -action on any finite-dimensional  $\text{Exc}(\Gamma, {}^L G)$ -equivariant quotient of  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$  to “arise from an automorphic form” for the purpose of Theorem 1.1.

By the finiteness of  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$  over  $\text{Exc}(\Gamma, {}^L G)$ , we can state this equivalently as: a Langlands parameter  $\rho$  “arises from an automorphic form” if the corresponding maximal ideal  $\mathfrak{m}_\rho \subset \text{Exc}(\Gamma, {}^L G)$  is in the support of  $C_c^\infty(\text{Bun}_{G,D}(\mathbf{F}_\ell); k)$  as an  $\text{Exc}(\Gamma, {}^L G)$ -module.

**5.3. Excursion action on the Tate cohomology of shtukas.** For a category  $\mathcal{C}$  with  $\sigma$ -action, we let  $\mathcal{C}^{\sigma\text{-eq}}$  denote the category of  $\sigma$ -equivariant objects in  $\mathcal{C}$ . This comes equipped with a forgetful functor to  $\mathcal{C}$ .

5.3.1. *Tate cohomology of shtukas.* If  $\sigma$  acts on  $G$ , it induces an action  $V \mapsto \sigma V$  on  $\text{Rep}({}^L G)$ .

Given  $V \in \text{Rep}_k(({}^L G^{\text{alg}})^I)^{\sigma\text{-eq}}$ , we can form  $R\pi_{I!}(\text{Sht}_G|_{\overline{\eta^I}}; \text{Sat}(V))$  as above. The  $\sigma$ -equivariant structure on  $V$  equips this with a  $\sigma$ -equivariant structure; more formally, because  $\text{Sat}$  and  $\pi_I$  are  $\sigma$ -equivariant,  $R\pi_{I!}(\text{Sht}_G, \text{Sat}(-))$  lifts to a functor  $\text{Rep}_k(({}^L G^{\text{alg}})^I)^{\sigma\text{-eq}} \rightarrow D(X^I; k)^{\sigma\text{-eq}}$ . Hence we can form  $T^j(R\pi_{I!})(\text{Sht}_{G,D,I}|_{\overline{\eta^I}}; \text{Sat}(V))$ , the Tate cohomology (§2.4) of  $(R\pi_{I!})(\text{Sht}_G|_{\eta^I}; \text{Sat}(V))$ ; we shall always do this with respect to the *perverse* t-structure. To ease notation, we will abbreviate

$$T^j(\text{Sht}_{G,D,I}; V) := T^j(R\pi_{I!})(\text{Sht}_{G,D,I}|_{\overline{\eta^I}}; \text{Sat}(V)). \quad (5.7)$$

Let us explain in what category we regard (5.7). Since  $(R\pi_{I!})(\text{Sht}_{G,D,I}|_{\overline{\eta^I}}; \text{Sat}(V))$  has commuting actions of  $\text{FWeil}(\eta^I, \overline{\eta^I})$  and the Hecke algebra  $\mathcal{H}_G$  (the former commuting with the  $\sigma$ -action), its Tate cohomology has commuting actions of  $\text{FWeil}(\eta^I, \overline{\eta^I})$  and of  $T^0(\mathcal{H}_G)$ , where Tate cohomology is formed with respect to the  $\sigma$ -action. We regard (5.7) as a  $T^0(\mathcal{H}_G)[\text{FWeil}(\eta^I, \overline{\eta^I})]$ -module, a priori. (Later we will see that the  $\text{FWeil}(\eta^I, \overline{\eta^I})$ -action factors uniquely through a  $\pi_1(\eta, \overline{\eta})^I$ -action, and it will be natural to regard (5.7) as a  $T^0(\mathcal{H}_G)[\pi_1(\eta, \overline{\eta})^I]$ -module.)

**Remark 5.5** (Automorphisms of shtukas). For any  $G$ -torsor  $\mathcal{E}$  on  $X$  and any point  $v \in X$ , we have a restriction map

$$\text{Aut}(\mathcal{E}) \xrightarrow{\text{ev}_v} \text{Aut}(\mathcal{E}|_v) \cong G.$$

The kernel of  $\text{ev}_v$  is unipotent, since  $\text{Aut}(\mathcal{E})$  embeds into the group of automorphisms of  $\mathcal{E}$  restricted to a formal disk around  $v$ , which is  $G(\mathcal{O}_v)$ , and the kernel of the evaluation map  $G(\mathcal{O}_v) \rightarrow G$  is pro-unipotent.

<sup>19</sup>By [Xuea] for split groups, and its forthcoming generalization for non-split groups.

Hence as soon as  $D$  is non-empty, the support of  $\text{Sat}(V)$  in  $\text{Sht}_{G,D,I}$  is locally finite type with stabilizers being finite ( $\text{char}(\mathbf{F}_\ell) \neq p$ )-groups, which therefore have trivial group cohomology with coefficients in  $k$ . Therefore,  $(R\pi_{I!})(\text{Sht}_{G,D,I} |_{\overline{\eta}^I}; \text{Sat}(V))$  lies in the *bounded* derived category  $D^b(\overline{\eta}^I; k)$ , so we may apply the results on Tate cohomology of bounded complexes from §2.4. We will always assume that  $D$  is non-empty so that this holds.

**Lemma 5.6.** *The diagonal map  $H \rightarrow G$  induces an isomorphism  $\text{Sht}_{H,D,I}^{(I_1, \dots, I_r)} \cong (\text{Sht}_{G,D,I}^{(I_1, \dots, I_r)})^\sigma$  as subfunctors of  $\text{Sht}_{G,D,I}^{(I_1, \dots, I_r)}$ .*

*Proof.* For notational convenience we just treat the case of non-iterated shtukas,  $\text{Sht}_{G,D,I}$ ; the general case is essentially the same but with cumbersome extra notation.

The main point that the “diagonal” map  $\text{Bun}_{H,D} \xrightarrow{\sim} \text{Bun}_{G,D}^\sigma$  is already an isomorphism. Indeed, there is an equivalence of categories between  $\text{Res}_{X'/X}(H)$ -bundles on  $X_S$  and  $H$ -bundles on  $X'_S$ , which we denote  $\mathcal{E} \mapsto \mathcal{E}'$ . Then straightforward definition-chasing shows that the datum of an isomorphism of  $\mathcal{E}$  with its  $\sigma$ -twist, exhibiting  $\mathcal{E}$  as a  $\sigma$ -fixed point of  $\text{Bun}_{G,D}$ , translates to a descent datum for  $\mathcal{E}'$  to descend to an  $H$ -bundle on  $X_S$ . This is compatible with level structures: a level structure on  $\mathcal{E}'$  descends to  $\mathcal{E}$  if and only if it is  $\sigma$ -equivariant.

More generally, notate the  $S$ -points of  $\text{Sht}_{G,D,I}$  as (the groupoid)  $\{(\{x_i\}_{i \in I}, \mathcal{E}, \varphi)\}$ . The subfunctor  $(\text{Sht}_{G,D,I})^\sigma$  parametrizes the groupoid of such data where  $\mathcal{E}$  is equipped with a  $\sigma$ -equivariant structure, and  $\varphi: \mathcal{E}|_{X \times S \setminus \bigcup \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}|_{X \times S \setminus \bigcup \Gamma_{x_i}}$  is  $\sigma$ -equivariant. By the preceding paragraph,  $\mathcal{E}$  is induced by an  $H$ -bundle on  $X_S$ . It is similarly straightforward to check that the  $\sigma$ -equivariance of  $\varphi$  is equivalent to it being induced by a map of  $H$ -bundles.  $\square$

From Lemma 2.6 we deduce the following simple but important identity:

**Lemma 5.7.** *Suppose  $\sigma$  acts trivially on  $\text{Sht}_H$  and  $\mathcal{F}$ . Then*

$$T^*(R\pi_{I!})(\text{Sht}_{H,D,I} |_{\overline{\eta}^I}; \mathcal{F}) \cong R^*\pi_{I!}(\text{Sht}_{H,D,I} |_{\overline{\eta}^I}; \mathcal{F}) \otimes T^*(k).$$

### 5.3.2. $\sigma$ -equivariant excursion algebra.

**Definition 5.8.** We define the  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$  to be the algebra on generators  $S_{V,x,\xi,(\gamma_i)_{i \in I}}$  where

- $V \in \text{Rep}_k({}^L G)^{\sigma\text{-eq}}$ ,
- $x: \mathbb{1} \rightarrow V|_{\Delta(\widehat{G})}$  and  $\xi: V|_{\Delta(\widehat{G})} \rightarrow \mathbb{1}$  are  $\sigma$ -equivariant morphisms of  $\widehat{G}$ -representations, and
- $(\gamma_i)_{i \in I} \subset \Gamma^I$ ,

with the following relations.

- $S_{\emptyset,x,\xi,*} = \langle x, \xi \rangle$ .
- For any  $\sigma$ -equivariant morphism of  $\sigma$ -equivariant  $({}^L G)^I$ -representations  $u: W \rightarrow W'$  and functional  $\xi' \in (W')^*$  invariant under the diagonal  $\widehat{G} \rtimes \sigma$ -action, we have

$$S_{I,W,x,tu(\xi'),(\gamma_i)_{i \in I}} = S_{I,W',u(x),\xi',(\gamma_i)_{i \in I}}, \quad (5.8)$$

where  $t u: (W')^* \rightarrow W^*$  denotes the dual to  $u$ .

- For two tuples  $(I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1})$  and  $(I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2})$  as above, we have

$$S_{I_1 \sqcup I_2, W_1 \boxtimes W_2, x_1 \boxtimes x_2, \xi_1 \boxtimes \xi_2, (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} \circ S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \quad (5.9)$$

Letting  $\Delta: \mathbb{1} \rightarrow \mathbb{1} \oplus \mathbb{1}$  be the diagonal inclusion, and  $\nabla: \mathbb{1} \oplus \mathbb{1} \rightarrow \mathbb{1}$  the addition map, we also have

$$S_{I_1 \sqcup I_2, W_1 \oplus W_2, (x_1 \oplus x_2) \circ \Delta, \nabla \circ (\xi_1 \oplus \xi_2), (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} + S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \quad (5.10)$$

Furthermore, the assignment  $(I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}) \mapsto S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}} \in \text{Exc}(\Gamma, {}^L G^{\text{alg}})^{\sigma\text{-eq}}$  is  $k$ -linear in  $x$  and  $\xi$ .

- (iv) Let  $\zeta: I \rightarrow J$  be a map of finite sets. Suppose  $W \in \text{Rep}(({}^L G)^I)^{\sigma\text{-eq}}$ ,  $x: \mathbb{1} \rightarrow W|_{\Delta(\widehat{G})}$ ,  $\xi: W|_{\Delta(\widehat{G})} \rightarrow \mathbb{1}$ , and  $(\gamma_j)_{j \in J} \in \Gamma^J$ . Letting  $W^\zeta$  be the restriction of  $W$  under the functor  $\text{Rep}(({}^L G)^I)^{\sigma\text{-eq}} \rightarrow \text{Rep}(({}^L G)^J)^{\sigma\text{-eq}}$  induced by  $\zeta$ , we have

$$S_{J, W^\zeta, x, \xi, (\gamma_j)_{j \in J}} = S_{I, W, x, \xi, (\gamma_{\zeta(i)})_{i \in I}}. \quad (5.11)$$

- (v) Letting  $\delta_W: \mathbb{1} \rightarrow W \otimes W^*$  and  $\text{ev}_W: W^* \otimes W \rightarrow \mathbb{1}$  be the natural counit and unit, we have

$$S_{I, W, x, \xi, (\gamma_i (\gamma'_i)^{-1} \gamma''_i)_{i \in I}} = S_{I \sqcup I \sqcup I, W \boxtimes W^* \boxtimes W, \delta_W \boxtimes x, \xi \boxtimes \text{ev}_W, (\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I}}. \quad (5.12)$$

- (vi) If  $W$  is inflated from a representation of  $({}^L G^{\text{alg}})^J \times \Gamma^{I \setminus J}$ , then we have

$$S_{I, W, x, \xi, (\gamma_i)_{i \in I}} = S_{J, W|_{({}^L G^{\text{alg}})^J}, ((1_j)_{j \in J}, (\gamma_i)_{i \in I \setminus J}), x, \xi, (\gamma_j)_{j \in J}}.$$

In short,  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$  has the same type of generators and relations as in §4.4, but all data must be  $\sigma$ -equivariant.

**Remark 5.9** ( $\sigma$ -action on the excursion algebra). Since  $\sigma$  acts on  $G$ , it acts on  $\text{Exc}(\Gamma, {}^L G^{\text{alg}})$  by transport of structure. Concretely, we have

$$\sigma \cdot S_{V, x, \xi, (\gamma_i)_{i \in I}} = S_{\sigma(V), \sigma(x), \sigma(\xi), (\gamma_i)_{i \in I}}. \quad (5.13)$$

There is an obvious map  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}} \rightarrow \text{Exc}(\Gamma, {}^L G)$  sending  $S_{V, x, \xi, (\gamma_i)_{i \in I}} \in \text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$  to the element with the same name in  $\text{Exc}(\Gamma, {}^L G)$ .

It seems natural to ask if this map is injective and identifies  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$  with the  $\sigma$ -invariants on  $\text{Exc}(\Gamma, {}^L G)^\sigma \subset \text{Exc}(\Gamma, {}^L G)$ . We believe this is true at least in characteristic 0.

**Lemma 5.10.** Recall the Tate cohomology spectral sequence §2.4.4,

$$E_{ij}^1 = R^j \pi_{I!}(\text{Sht}_{G,D,I} |_{\overline{\eta}}; \text{Sat}(V)) \implies T^{i+j}(\text{Sht}_{G,D,I}; V).$$

- (i) There is an  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$ -action on the Tate cohomology spectral sequence  $E_{ij}^r \implies T^*(\text{Sht}_{G,D,I}; V)$ , such that the diagrams

$$\begin{array}{ccc} & & E_{ij}^r \\ & \nearrow & \\ \ker(d_{ij}^r) & & \\ & \searrow & \\ & \ker(d_{ij}^r) / \text{Im}(d_{i-r,j+r-1}^r) = E_{ij}^{r+1} \end{array}$$

are all  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$ -equivariant. The  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$ -action on every term for  $r \geq 1$  factors through the map  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}} \rightarrow \text{Exc}(\Gamma, {}^L G)$  from Remark 5.9.



- (ii) There is an  $\mathrm{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$ -action on  $T^j(\mathrm{Sht}_{G,D,I}; V)$ , which preserves the (increasing) filtration  $F^\bullet T^j(\mathrm{Sht}_{G,D,I}; V)$  induced by the Tate cohomology spectral sequence 2.4.4, so that the diagrams

$$\begin{array}{ccc}
 & & T^j(\mathrm{Sht}_{G,D,I}; V) \\
 & \nearrow & \\
 F^i(T^j(\mathrm{Sht}_{G,D,I}; V)) & & \\
 & \searrow & \\
 & F^i(T^j(\mathrm{Sht}_{G,D,I}; V))/F^{i-1}(T^j(\mathrm{Sht}_{G,D,I}; V)) = E_\infty^{ij} &
 \end{array}$$

are all  $\mathrm{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$ -equivariant, with the action on  $E_\infty^{ij}$  being the same as in part (i).

*Proof.* For part (i), the existence of the action is formal from the fact that the  $\mathrm{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$ -action on  $R\pi_{I!}(\mathrm{Sht}_{G,D,I}; \mathrm{Sat}(V))$  commutes with  $\sigma$ , and the definition of the Tate double complex (2.4). The factorization of the action through  $\mathrm{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}} \rightarrow \mathrm{Exc}(\Gamma, {}^L G)$  follows from the fact that  $E_{ij}^1 = H_c^j(\mathrm{Sht}_{G,D,I} |_{\overline{\eta}^I}; \mathrm{Sat}(V))$ , on which the action factors through  $\mathrm{Exc}(\Gamma, {}^L G)$  by Lafforgue-(Xue)'s construction.

For part (ii), we begin by constructing the action. We will define a family of functors  $T_I^j: \mathrm{Rep}({}^L G)^I)^{\sigma\text{-eq}} \rightarrow \mathrm{Rep}_{T^0 \mathcal{H}_G}(\Gamma^I)$  which is compatible with composition and fusion. From this, the action of  $\mathrm{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$  is defined as in Construction 5.2. We set

$$T_I^j(V) := T^j(\mathrm{Sht}_{G,D,I}; V)$$

regarded a priori as a  $T^0(\mathcal{H}_G)[\mathrm{FWeil}(\eta^I, \overline{\eta}^I)]$ -module. The compatibility with fusion and composition follow formally from these same properties of the functor  $\mathrm{Sat}$ . The extension of the natural  $\pi_1(\eta^I, \overline{\eta}^I)$ -action to an  $\mathrm{FWeil}(\eta^I, \overline{\eta}^I)$ -action using partial Frobenius is the same as in §5.2. The only issue is to check that the  $\mathrm{FWeil}(\eta^I, \overline{\eta}^I)$ -action on  $T^*(\mathrm{Sht}_G; V)$  factors through  $\pi_1(\eta, \overline{\eta})^I$ .

This will follow from Drinfeld's Lemma in the form [Xuea, Lemma 7.4.2] as soon as we establish that  $T^j(\mathrm{Sht}_G; V)$  is a finite module over some  $A$ -algebra such that the  $A$ -action commutes with the action of  $\mathrm{FWeil}(\eta^I, \overline{\eta}^I)$ . We take  $A = T^0(\mathcal{H}_{G,u})$  for some  $u$  where  $G$  is hyperspecial. By the generalization of [Xuea, Theorem 0.0.3] to non-split groups (to appear in [Xueb]), we know that  $R^j \pi_{I!}(\mathrm{Sht}_{G,D,I} |_{\overline{\eta}^I}; \mathrm{Sat}(V))$  is a finite  $\mathcal{H}_{G,u}$ -module. By the Artin-Tate Lemma,  $\mathcal{H}_{G,u}$  is a finite  $\mathcal{H}_{G,u}^\sigma$ -algebra, so  $R^j \pi_{I!}(\mathrm{Sht}_{G,D,I} |_{\overline{\eta}^I}; \mathrm{Sat}(V))$  is also finite over  $\mathcal{H}_{G,u}^\sigma$ . As  $\mathcal{H}_{G,u}^\sigma$  is Noetherian, the subquotient  $T^i(R^j \pi_{I!}(\mathrm{Sht}_{G,D,I} |_{\overline{\eta}^I}; \mathrm{Sat}(V)))$  is also a finite  $\mathcal{H}_{G,u}^\sigma$ -module, and therefore a finite  $T^0(\mathcal{H}_{G,u})$ -module (since the  $\mathcal{H}_{G,u}^\sigma$ -action factors through  $T^0(\mathcal{H}_{G,u})$ ). Finally, each  $E_\infty^{ab}$  is a further  $T^0(\mathcal{H}_{G,u})$ -equivariant subquotient of such a module, therefore also a finite  $T^0(\mathcal{H}_{G,u})$ -module. As these are the subquotients in a finite filtration of  $T^j(\mathrm{Sht}_G; V)$ , the latter is also a finite  $T^0(\mathcal{H}_{G,u})$ -module.

Since the formation of the Tate double complex (2.4) is functorial with respect to the sheaf, the filtration is functorial as well. Therefore we have natural transformations  $F^r T_I^j(V) \rightarrow T_I^j(V)$ , compatible with fusion and composition. This implies the desired equivariance of excursion operators. Concretely, the action of  $S_{V,x,\xi,(\gamma_i)_{i \in I}}$  on  $T^j(\mathrm{Sht}_{G,D,I}; \mathbb{1})$  and

$F^r T^j(\text{Sht}_{G,D,I}; \mathbb{1})$  are given by the two rows in the diagram

$$\begin{array}{ccccccc}
T^j(\text{Sht}_{G,D,I}; \mathbb{1}) & \xrightarrow{x} & T^j(\text{Sht}_{G,D,I}; V) & \xrightarrow{(\gamma_i)_{i \in I}} & T^j(\text{Sht}_{G,D,I}; V) & \xrightarrow{\xi} & T^j(\text{Sht}_{G,D,I}; \mathbb{1}) \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
F^r T^j(\text{Sht}_{G,D,I}; \mathbb{1}) & \xrightarrow{x} & F^r T^j(\text{Sht}_{G,D,I}; V) & \xrightarrow{(\gamma_i)_{i \in I}} & F^r T^j(\text{Sht}_{G,D,I}; V) & \xrightarrow{\xi} & F^r T^j(\text{Sht}_{G,D,I}; \mathbb{1})
\end{array}$$

and the commutativity of the outer rectangle is exactly the desired equivariance.  $\square$

**5.3.3. Equivariant localization for excursion operators.** We define  $\text{Nm}: \text{Rep}_k(({}^L G)^I) \rightarrow \text{Rep}_k(({}^L G)^I)^{\sigma\text{-eq}}$  to be the functor taking a representation  $V$  to  $V \otimes_k {}^\sigma V \otimes_k \dots \otimes_k {}^{\sigma^{p-1}} V$ , with the  $\sigma$ -equivariant structure

$${}^\sigma \text{Nm}(V) = {}^\sigma V \otimes_k {}^{\sigma^2} V \otimes_k \dots \otimes_k {}^{\sigma^{p-1}} V \otimes_k V \xrightarrow{\sim} V \otimes_k {}^\sigma V \otimes_k \dots \otimes_k {}^{\sigma^{p-1}} V = \text{Nm}(V)$$

given by the commutativity constraint for tensor products. It corresponds under Geometric Satake to Definition 3.11. Given  $h: V \rightarrow V' \in \text{Rep}_k(({}^L G)^I)$ , we set

$$\text{Nm}(h) := h \otimes {}^\sigma h \otimes \dots \otimes {}^{\sigma^{p-1}} h: \text{Nm}(V) \rightarrow \text{Nm}(V').$$

Note that  $\text{Nm}$  is *not* an additive functor, nor is it even  $k$ -linear. We linearize it by defining  $\text{Nm}^{(p^{-1})} := \text{Frob}_p^{-1} \circ \text{Nm}$ , where (as in §3.6.3)  $\text{Frob}_p^{-1}$  is the identity on objects and on morphisms it is  $(-) \otimes_{k, \text{Frob}_p^{-1}} k$ . Then  $\text{Nm}^{(p^{-1})}: \text{Rep}_k(({}^L G)^I) \rightarrow \text{Rep}_k(({}^L G)^I)^{\sigma\text{-eq}}$  is  $k$ -linear, although still not additive.

For  $V \in \text{Rep}_k(({}^L G)^I)$ , we denote by  $N \cdot V$  the  $\sigma$ -equivariant representation  $V \oplus {}^\sigma V \oplus \dots \oplus {}^{\sigma^{p-1}} V$ , with  $\sigma$ -equivariant structure

$${}^\sigma(N \cdot V) = {}^\sigma V \oplus {}^{\sigma^2} V \oplus \dots \oplus {}^{\sigma^{p-1}} V \oplus V \xrightarrow{\sim} V \oplus {}^\sigma V \oplus \dots \oplus {}^{\sigma^{p-1}} V = (N \cdot V)$$

given by the commutativity constraint for direct sums. For  $h: V \rightarrow V' \in \text{Rep}_k(({}^L G)^I)$ , we denote  $N \cdot h: N \cdot V \rightarrow N \cdot V'$  the  $\sigma$ -equivariant map  $h \oplus {}^\sigma h \oplus \dots \oplus {}^{\sigma^{p-1}} h$ . Let  $\Delta_p: \mathbb{1} \rightarrow \mathbb{1}^{\oplus p}$  denote the diagonal map and  $\nabla_p: \mathbb{1}^{\oplus p} \rightarrow \mathbb{1}$  denote the sum map.

Our goal in this subsection is to prove the theorem below.

**Theorem 5.11.** (i) The action of  $S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}$  on  $T^*(\text{Sht}_G; \mathbb{1})$  is identified with the action of  $S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}$  on  $T^*(\text{Sht}_H; \mathbb{1})$ .

(ii) The action of  $S_{N \cdot V, (N \cdot x) \circ \Delta_p, \nabla_p \circ (N \cdot \xi), (\gamma_i)_{i \in I}}$  on  $T^*(\text{Sht}_G; \mathbb{1})$  is 0.

We first establish a key technical proposition giving an equivariant localization theorem for shtukas.

**Proposition 5.12.** Let  $V \in \text{Rep}_k(({}^L G)^I)$ . Then we have a natural isomorphism of functors

$$T^*(\text{Sht}_{G,D,I}; \text{Nm}^{(p^{-1})}(V)) \cong T^*(\text{Sht}_{H,D,I}; \text{Res}_{\text{BC}}(V)): \text{Rep}(({}^L G)^I) \rightarrow \text{Mod}_k(\pi(\eta, \bar{\eta})^I) \quad (5.14)$$

which is compatible with fusion and composition.

**Remark 5.13.** Note that for this proposition, we forget the  $T^0(\mathcal{H}_G)$ -action on  $T^*(\text{Sht}_{G,D,I}; -)$ . In fact, the proposition can be enhanced to give a compatible family of natural isomorphisms including the Hecke-module structure, where  $T^*(\text{Sht}_{H,D,I}; \text{Res}_{\text{BC}}(V))$  is regarded as a  $T^0(\mathcal{H}_G)$ -module via the “Brauer homomorphism” (to be defined later in §6.3)  $\text{Br}: T^0(\mathcal{H}_G) \rightarrow \mathcal{H}_H$ . However, this is unnecessary for us and would lengthen the already lengthy argument, so we omit it.

*Proof.* Since the  $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$ -actions on  $T^*(\mathrm{Sht}_G; \mathrm{Nm}^{(p^{-1})}(V))$  and on  $T^*(\mathrm{Sht}_H; \mathrm{Res}_{\mathrm{BC}}(V))$  are determined by their respective  $\pi_1(\eta^I, \overline{\eta^I})$ -actions plus partial Frobenius morphisms, we can and will focus on these two equivariance structures separately, starting with the  $\pi_1(\eta^I, \overline{\eta^I})$ -actions.

The basic idea is that our geometric description of  $V \mapsto \mathrm{Res}_{\mathrm{BC}}(V)$  in Theorem 3.19 implies the statement in the case where  $V$  is a tilting module, after passing to a base extension, by equivariant localization. We will then deduce the full statement using descent and the fact that there are “enough” tilting modules by Theorem 3.9.

Now we begin the argument. Consider the commutative diagram

$$\begin{array}{ccc}
 \mathrm{Rep}_k(\widehat{G}^I) & \xrightarrow[\text{geom. Satake}]{\sim} & \mathrm{P}_{(L+G)_{F^s}}((\mathrm{Gr}_G)_{F^s}; k)^{\otimes I} \\
 \downarrow & & \downarrow \\
 D^b(\mathrm{Rep}_k(\widehat{G}^I)) & \dashrightarrow & D^b(\mathrm{Hk}_{G,I} |_{\overline{\eta^I}}; k) \\
 \uparrow \sim & & \downarrow \\
 K^b(\mathrm{Tilt}_k(\widehat{G}^I)) & & D^b(\mathrm{Sht}_{G,D,I} |_{\overline{\eta^I}}; k) \xrightarrow{T^j} \mathrm{Mod}_k
 \end{array} \tag{5.15}$$

All the geometric objects appearing in the second column of (5.15), as well as the maps between them inducing the functors depicted there, are defined over  $\eta^I$ . Therefore, there is a  $\pi_1(\eta^I, \overline{\eta^I})$ -action on all the categories involved, with the action on  $\mathrm{Rep}_k(\widehat{G}^I)$  factoring through the map  $\pi_1(\eta^I, \overline{\eta^I}) \rightarrow \Gamma^I$ , and the  $\Gamma$ -action on  $\mathrm{Rep}_k(\widehat{G})$  coming from the Geometric Satake equivalence plus descent for sheaves on  $\mathrm{Gr}_G$  (i.e. the “geometric action” of §4.1.2). Furthermore, all the functors in (5.15) are  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant, hence we may consider the  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariantization of (5.15), obtaining the diagram below.

$$\begin{array}{ccc}
 \mathrm{Rep}_k(({}^L G^{\mathrm{geom}})^I) & \xrightarrow{\sim} & \mathrm{P}_{L+G}(\mathrm{Gr}_G; k)^{\otimes I} \\
 \downarrow \sim & & \text{descent} \Rightarrow \downarrow \sim \\
 \mathrm{Rep}_k(\widehat{G}^I)^{\Gamma^I, \mathrm{geom}} & \xrightarrow{\sim} & (\mathrm{P}_{(L+G)_{F^s}}((\mathrm{Gr}_G)_{F^s}; k)^{\otimes I})^{\Gamma^I} \\
 \downarrow & & \downarrow \\
 D^b(\mathrm{Rep}_k(\widehat{G}^I))^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} & \dashrightarrow & D^b(\mathrm{Hk}_{G,I} |_{\overline{\eta^I}}; k)^{\pi_1(\eta^I, \overline{\eta^I})} \\
 \uparrow \sim & & \downarrow \\
 K^b(\mathrm{Tilt}_k(\widehat{G}^I))^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} & & D^b(\mathrm{Sht}_{G,D,I} |_{\overline{\eta^I}}; k)^{\pi_1(\eta^I, \overline{\eta^I})} \xrightarrow{T^j} \mathrm{Mod}_k(\pi_1(\eta^I, \overline{\eta^I}))
 \end{array} \tag{5.16}$$

We emphasize here that  $D^b(-)^{\pi_1(\eta^I, \overline{\eta^I})}$  denotes the equivariant derived category for the action of  $\pi_1(\eta^I, \overline{\eta^I})$ .

The functor  $\mathrm{Res}_{\mathrm{BC}}: \mathrm{Rep}_k(\widehat{G}^I) \rightarrow \mathrm{Rep}_k(\widehat{H}^I)$  extends to the derived category, and then lifts to the  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant derived category and intertwines diagram (5.16) compatibly with the analogous one for  $\widehat{H}$ . The resulting composition of functors

$$\mathrm{Rep}_k(\widehat{G}^I)^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} \xrightarrow{\mathrm{Res}_{\mathrm{BC}}} \mathrm{Rep}_k(\widehat{H}^I)^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} \xrightarrow{(5.16) \text{ for } H} \mathrm{Mod}_k(\pi_1(\eta^I, \overline{\eta^I}))$$

is the rightmost functor of (5.14). Let

$$\mathcal{T}_1^j : D^b(\mathrm{Rep}_k(\widehat{G}^I))^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} \rightarrow \mathrm{Mod}_k(\pi_1(\eta^I, \overline{\eta^I}))$$

be the composite functor

$$D^b(\mathrm{Rep}_k(\widehat{G}))^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} \xrightarrow{\mathrm{Res}_{\mathrm{BC}}} D^b(\mathrm{Rep}_k(\widehat{H}))^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}} \xrightarrow{(5.16) \text{ for } H} \mathrm{Mod}_k(\pi_1(\eta^I, \overline{\eta^I}))$$

so that the rightmost functor of (5.14) is the pullback of  $\mathcal{T}_1^j$  to  $\mathrm{Rep}_k(\widehat{G}^I)^{\pi_1(\eta^I, \overline{\eta^I}), \mathrm{geom}}$ .

Then  $\mathcal{T}_1^j$  is the  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariantization of the functor

$$(\mathcal{T}_1^j)^{\mathrm{de-eq}} : D^b(\mathrm{Rep}_k(\widehat{G})) \rightarrow \mathrm{Mod}_k$$

given by the composition of functors

$$D^b(\mathrm{Rep}_k(\widehat{G})) \xrightarrow{\mathrm{Res}_{\mathrm{BC}}} D^b(\mathrm{Rep}_k(\widehat{H})) \xrightarrow{(5.15) \text{ for } H} \mathrm{Mod}_k.$$

We claim that the  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant functor  $V \in \mathrm{Tilt}_k(\widehat{G}^I) \mapsto T^j(\mathrm{Sht}_{G,D,I} |_{\overline{\eta^I}}; \mathrm{Nm}^{(p^{-1})}(V)) \in \mathrm{Mod}_k$  extends (necessarily uniquely) to a  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant functor

$$(\mathcal{T}_2^j)^{\mathrm{de-eq}} : K^b(\mathrm{Tilt}_k(\widehat{G}^I)) \rightarrow \mathrm{Mod}_k.$$

Note that this is not obvious because  $V \mapsto \mathrm{Nm}^{(p^{-1})}(V)$  is not even additive, and so  $\mathrm{Nm}^{(p^{-1})}$  itself certainly does not extend to a functor out of the homotopy category. Nevertheless, we will see that the composite functor is well-behaved (in particular, composing with Tate cohomology restores the additivity). Indeed, we have<sup>20</sup>

$$\begin{aligned} T^j(\mathrm{Sht}_{G,D,I}; \mathrm{Nm}^{(p^{-1})}(V)) &:= T^j(\mathrm{Sht}_{G,D,I}; \mathrm{Sat}(\mathrm{Nm}^{(p^{-1})}(V))) \\ \text{Lemma 5.6 and §2.5} &\implies \cong T^j(\mathrm{Sht}_{H,D,I}; \mathrm{Frob}_p^{-1} \circ \mathrm{Psm}(\mathrm{Nm}(\mathrm{Sat}(V)))) \\ \text{Theorem 3.19} &\implies \cong T^j(\mathrm{Sht}_{H,D,I}; \mathrm{Sat}(\mathrm{Res}_{\mathrm{BC}}(V))). \end{aligned} \quad (5.17)$$

(Above,  $\mathrm{Frob}_p^{-1}$  is the automorphism of the  $k$ -linear category of sheaves on  $\mathrm{Sht}_{H,D,I}$  obtained by applying  $(-) \otimes_{k, \mathrm{Frob}_p^{-1}} k$  to spaces of morphisms.) Moreover, these isomorphisms are natural in  $V$ , and in particular  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant. Hence we have presented the functor in question as a composition of two functors,  $\mathrm{Sat} \circ \mathrm{Res}_{\mathrm{BC}}$  and  $T^j(\mathrm{Sht}_{H,D,I}, -)$ , which both extend  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariantly to the homotopy categories of their domains.

The upshot is that  $(\mathcal{T}_2^j)^{\mathrm{de-eq}}$  is  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant, and the preceding computation showed that there is a natural (in particular  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant) isomorphism  $(\mathcal{T}_1^j)^{\mathrm{de-eq}} \cong (\mathcal{T}_2^j)^{\mathrm{de-eq}}$  as functors  $K^b(\mathrm{Tilt}(\widehat{G})) \rightarrow \mathrm{Mod}_k$ . By Theorem 3.9 we may equivalently view  $(\mathcal{T}_1^j)^{\mathrm{de-eq}}$  and  $(\mathcal{T}_2^j)^{\mathrm{de-eq}}$  as functors on  $D^b(\mathrm{Rep}_k(\widehat{G}^I))$ , and so we have a natural isomorphism  $(\mathcal{T}_1^j)^{\mathrm{de-eq}} \cong (\mathcal{T}_2^j)^{\mathrm{de-eq}}$  as functors  $D^b(\mathrm{Rep}_k(\widehat{G}^I)) \rightarrow \mathrm{Mod}_k$ . Then their  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariantizations are naturally isomorphic functors  $D^b(\mathrm{Rep}_k(\widehat{G}^I))^{\pi_1(\eta^I, \overline{\eta^I})} \rightarrow \mathrm{Mod}_k(\pi_1(\eta^I, \overline{\eta^I}))$ .

<sup>20</sup>We draw attention to a subtlety in the computation below which is suppressed by the notation. We are using that there is a natural isomorphism between the two functors  $D(\mathrm{Hk}_{G,I}; k) \rightarrow D(\mathrm{Sht}_G; k) \rightarrow D(\mathrm{Sht}_H; k)$  and  $D(\mathrm{Hk}_{G,I}; k) \rightarrow D(\mathrm{Hk}_{H,I}; k) \rightarrow D(\mathrm{Sht}_H; k)$ , coming from the commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_H & \longrightarrow & \mathrm{Hk}_{H,I} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_G & \longrightarrow & \mathrm{Hk}_{G,I} \end{array}$$

in order to identify  $\mathrm{Psm}(\mathrm{Nm}^{(p^{-1})}(\mathrm{Sat}(V)))$  on  $\mathrm{Sht}_{H,D,I}$  with the pullback of the complex with the same name in  $D(\mathrm{Hk}_{H,I}; k)$ .

Finally, the pullbacks of these functors to  $\text{Rep}_k(({}^L G^{\text{geom}})^I) \cong \text{Rep}_k(\widehat{G}^I)^{\Gamma^I, \text{geom}}$  are naturally isomorphic, and these two pullbacks are exactly the two sides of (5.14) after using (4.1) to identify  $\text{Rep}({}^L G) \cong \text{Rep}({}^L G^{\text{geom}})$  and  $\text{Rep}({}^L H) \cong \text{Rep}({}^L H^{\text{geom}})$ , which can be done compatibly as discussed in Definition 4.6.

Finally, we check the compatibility with partial Frobenius. We want to show that the diagram

$$\begin{array}{ccc} F_{\{1\}}^* T^j(\text{Sht}_{G,D,I}; \text{Nm}^{(p^{-1})}(V)) & \xrightarrow{\sim} & T^j(\text{Sht}_{G,D,I}; \text{Nm}^{(p^{-1})}(V)) \\ \downarrow \sim & & \downarrow \sim \\ F_{\{1\}}^* T^j(\text{Sht}_{H,D,I}; \text{Res}_{\text{BC}}(V)) & \xrightarrow{\sim} & T^j(\text{Sht}_{H,D,I}; \text{Res}_{\text{BC}}(V)) \end{array} \quad (5.18)$$

commutes, where the vertical isomorphisms (as  $k$ -modules) have just been established. By Lemma 5.6, the  $\sigma$ -fixed points of

$$F_{\{1\}} : \text{Sht}_{G,D,I}^{\{\{1\}, \dots, \{n\}\}} \rightarrow \text{Sht}_{G,D,I}^{\{\{2\}, \dots, \{n\}, \{1\}\}}$$

are identified with

$$F_{\{1\}} : \text{Sht}_{H,D,I}^{\{\{1\}, \dots, \{n\}\}} \rightarrow \text{Sht}_{H,D,I}^{\{\{2\}, \dots, \{n\}, \{1\}\}}.$$

This implies that the isomorphisms (5.17) are compatible with the maps  $F_{\{1\}}^*$ . This establishes the de-equivariantized version of the desired compatibility with coefficients in tilting modules; the  $\pi_1(\eta^I, \overline{\eta^I})$ -equivariant version then follows by re-running the same argument as for the first part.  $\square$

*Proof of Theorem 5.11.* (i) Proposition 5.12 gives a chain of compatible identifications

$$\begin{array}{ccccccc} T^*(\text{Sht}_{G,D,I}; \mathbb{1}) & \xrightarrow{\text{Nm}^{(p^{-1})}(x)} & T^*(\text{Sht}_{G,D,I}; \text{Nm}^{(p^{-1})}(V)) & \xrightarrow{(\gamma_i)_{i \in I}} & T^*(\text{Sht}_{G,D,I}; \text{Nm}^{(p^{-1})}(V)) & \xrightarrow{\text{Nm}^{(p^{-1})}(\xi)} & T^*(\text{Sht}_{G,D,I}; \mathbb{1}) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ T^*(\text{Sht}_{H,D,I}; \mathbb{1}) & \xrightarrow{x} & T^*(\text{Sht}_{H,D,I}; \text{Res}_{\text{BC}}(V)) & \xrightarrow{(\gamma_i)_{i \in I}} & T^*(\text{Sht}_{H,D,I}; \text{Res}_{\text{BC}}(V)) & \xrightarrow{\xi} & T^*(\text{Sht}_{H,D,I}; \mathbb{1}) \end{array}$$

The operator  $S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}$  on  $T^*(\text{Sht}_G; \mathbb{1})$  is obtained by tracing along the upper row, while the operator  $S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}$  on  $T^*(\text{Sht}_H; \mathbb{1})$  is obtained by tracing along the lower row. Hence they coincide under the vertical identifications.

(ii) By Lemma 5.6 and §2.5 we have a chain of compatible identifications

$$\begin{array}{ccccccc} T^*(\text{Sht}_{G,D,I}; \mathbb{1}) & \xrightarrow{(N \cdot x) \circ \Delta_p} & T^*(\text{Sht}_{G,D,I}; N \cdot V) & \xrightarrow{(\gamma_i)_{i \in I}} & T^*(\text{Sht}_{G,D,I}; N \cdot V) & \xrightarrow{\nabla_p \circ (N \cdot \xi)} & T^*(\text{Sht}_{G,D,I}; \mathbb{1}) \\ \downarrow \sim & & \downarrow \sim & & \downarrow \sim & & \downarrow \sim \\ T^*(\text{Sht}_{H,D,I}; \mathbb{1}) & \xrightarrow{(N \cdot x) \circ \Delta_p} & T^*(\text{Sht}_{H,D,I}; \text{Psm}(N \cdot V)) & \xrightarrow{(\gamma_i)_{i \in I}} & T^*(\text{Sht}_{H,D,I}; \text{Psm}(N \cdot V)) & \xrightarrow{\nabla_p \circ (N \cdot \xi)} & T^*(\text{Sht}_{H,D,I}; \mathbb{1}) \end{array}$$

The operator  $S_{N \cdot V, (N \cdot x) \circ \Delta_p, \nabla_p \circ (N \cdot \xi), (\gamma_i)_{i \in I}}$  on  $T^*(\text{Sht}_{G,D,I}; \mathbb{1})$  is obtained by tracing along the upper row. But the stalks and costalks of  $N \cdot \text{Sat}(V)|_{\text{Gr}_H}$  are all induced  $\mathbb{O}[\sigma]$ -modules, so in particular they are perfect. Hence  $\text{Psm}(N \cdot V)$  is equivalent to 0 in the Tate category, so  $T^*(\text{Sht}_{H,D,I}; \text{Psm}(N \cdot V)) = 0$ . Therefore the endomorphism in question factors through the zero map, hence is itself zero.  $\square$

**5.4. Applications to base change for automorphic forms.** In §5.2 we described Laforgue's action of  $\text{Exc}(\Gamma, {}^L G)$  on  $H_\emptyset(\mathbb{1})$ . By (5.6), we have

$$H_\emptyset(\mathbb{1}) = \bigoplus_{\alpha \in \ker^1(F, G)} C_c^\infty(G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F) / \prod_v K_v; k).$$

Here  $\ker^1(F, G) := \ker(H^1(F, G) \rightarrow \prod_v H^1(F_v, G))$  is the isomorphism class of the generic fiber of the  $G$ -torsor. More generally, this defines a decomposition

$$\mathrm{Sht}_{G,D,I} = \coprod_{\alpha \in \ker^1(F,G)} (\mathrm{Sht}_{G,D,I})_\alpha \quad (5.19)$$

according to the isomorphism class of the generic fiber of  $\mathcal{E}$ . The construction outlined in §5.2 preserves the decomposition (5.19), and so gives an action of  $\mathrm{Exc}(\Gamma, {}^L G)$  on each  $H_c^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})_\alpha := C_c^\infty(G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F) / \prod_v K_v; k)$ .

In the base change situation, the “diagonal embedding” map  $\phi: H \rightarrow G$  induces a map  $\phi_*: \ker^1(F, H) \rightarrow \ker^1(F, G)$ .

Theorem 1.1 is evidently implied by the theorem below, whose proof occupies this subsection.

**Theorem 5.14.** *Let  $[\rho] \in H^1(\Gamma_F, \widehat{H}(k))$  be a Langlands parameter appearing in the action of  $\mathrm{Exc}(\Gamma, {}^L H)$  on  $H_c^0(\mathrm{Sht}_{H,D,I}; \mathrm{Sat}(\mathbb{1}))_\alpha$  in the sense of §5.2.4. Then the image of  $[\rho]$  in  $H^1(\Gamma_F, \widehat{G}(k))$  appears in the action of  $\mathrm{Exc}(\Gamma, {}^L G)$  on  $H_c^0(\mathrm{Sht}_{G,D,I}; \mathrm{Sat}(\mathbb{1}))_{\phi(\alpha)}$  in the sense of §5.2.4.*

**Definition 5.15.** For an algebra  $A$  in characteristic  $p$  with  $\sigma$ -action, we denote by  $N \cdot A$  the subset consisting of elements of the form  $(1 + \sigma + \dots + \sigma^{p-1})a$  for  $a \in A$ . One easily checks that  $N \cdot A$  is an ideal in  $A^\sigma$ .

We denote by  $\mathrm{Nm}: A \rightarrow A^\sigma$  the set map sending  $a \mapsto a \cdot \sigma(a) \cdot \dots \cdot \sigma^{p-1}(a)$ . It is multiplicative but not additive. It is an exercise to verify that the composition of  $\mathrm{Nm}$  with the quotient  $A^\sigma \twoheadrightarrow A^\sigma / N \cdot A$  is an algebra homomorphism.

**Lemma 5.16.** *Let  $A$  be a commutative ring over  $\mathbf{F}_p$ . Let  $A' \subset A^\sigma$  be a subring containing  $\mathrm{Nm}(A)$  and  $N \cdot A$ . (Since  $N \cdot A$  is an ideal in  $A^\sigma$ , it is also an ideal in any such  $A'$ .) Any character  $\chi: A' \rightarrow k$  factoring through  $A'/N \cdot A$  extends uniquely to a character  $\tilde{\chi}: A \rightarrow k$ , which is given by*

$$\tilde{\chi}(a) = \chi(\mathrm{Nm}(a))^{1/p}. \quad (5.20)$$

*Proof.* The same proof as that of [TV16, §3.4] works, but since our situation is a little more general we reproduce it. One easily checks that the given formula (5.20) works (it is a ring homomorphism since  $k$  is in characteristic  $p$ , and it clearly extends  $\chi$ ).

Next we check that it is the unique extension. Note that  $\sigma$  acts on characters of  $A$  by pre-composition; we denote this action by  $\tilde{\chi} \mapsto \sigma \cdot \tilde{\chi}$ . Clearly (5.20) is the unique  $\sigma$ -fixed extension, so we will show that any extension  $\tilde{\chi}'$  must be  $\sigma$ -fixed. Indeed, since any extension  $\tilde{\chi}'$  is trivial on  $N \cdot A$  by the assumption that  $\chi$  factors through  $A'/N \cdot A$ , we have

$$\sum_{i=0}^{p-1} \sigma^i \cdot \tilde{\chi}' = 0.$$

By linear independence of characters [Sta20, Tag 0CKK] we must have  $\sigma^i \cdot \tilde{\chi}' = \tilde{\chi}'$  for all  $i$ , i.e.  $\tilde{\chi}'$  is  $\sigma$ -fixed.  $\square$

**Lemma 5.17.** *Inside  $\mathrm{Exc}(\Gamma, {}^L G)$  we have*

$$\mathrm{Nm}(S_{V,x,\xi,(\gamma_i)_{i \in I}}) = S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}}$$

and

$$N \cdot S_{V,x,\xi,(\gamma_i)_{i \in I}} = S_{N \cdot V, (N \cdot x) \circ \Delta_p, \nabla_p \circ (N \cdot \xi), (\gamma_i)_{i \in I}}.$$

*Proof.* The first equality follows from repeated application of the relations (4.7), (4.5) and the explicit description of the  $\sigma$ -action in (5.13). The second equality follows from repeated application of relations (4.7), (4.6) and the explicit description of the  $\sigma$ -action in (5.13).  $\square$

*Proof of Theorem 5.14.* The Langlands parameter  $[\rho] \in H^1(\Gamma, \widehat{H}(k))$  corresponds to a character  $\chi_\rho: \text{Exc}(\Gamma, {}^L H) \rightarrow k$  under Proposition 4.4. The assumption that  $\chi_\rho$  appears in the action of  $\text{Exc}(\Gamma, {}^L H)$  on  $H_c^0(\text{Sht}_{H,D,\emptyset}; \mathbb{1})_\alpha$  implies that there is a vector  $v_\rho$  in a finite-dimensional quotient of  $H_c^0(\text{Sht}_{H,D,I}; \mathbb{1})_\alpha$  on which  $S \in \text{Exc}(\Gamma, {}^L H)$  acts as multiplication by  $\chi_\rho(S) \in k$ . Since the  $\text{Exc}(\Gamma, {}^L H)$ -action on  $H_c^0(\text{Sht}_{H,D,\emptyset}; \mathbb{1})_\alpha$  is defined over  $\mathbf{F}_p$ , the image of  $S \otimes 1$  under  $\text{Exc}(\Gamma, {}^L H) \otimes_{k, \text{Frob}_p} k \xrightarrow{\sim} \text{Exc}(\Gamma, {}^L H)$  acts on the image – call it  $v_\rho^{(p)}$  – of  $v_\rho \otimes 1$  under  $H_c^0(\text{Sht}_{H,D,\emptyset}; \mathbb{1})_\alpha \otimes_{k, \text{Frob}_p} k \xrightarrow{\sim} H_c^0(\text{Sht}_{H,D,\emptyset}; \mathbb{1})_\alpha$  as multiplication by  $\chi_\rho(S)^p$ .

The decomposition (5.19) induces a compatible direct sum decomposition of Tate cohomology and the Tate spectral sequence, and we denote by a subscript  $\alpha \in \ker^1(F, G)$  the summand indexed by  $\alpha$ . By Lemma 5.7 this eigenvector  $v_\rho^{(p)}$  maps to a *non-zero*  $\overline{v}_\rho^{(p)}$  in some  $\text{Exc}(\Gamma, {}^L H)$ -equivariant finite-dimensional quotient of  $(E_\infty^{ij})_\alpha$ , and the latter is itself a subquotient of  $T^*(\text{Sht}_{H,D,I}; \mathbb{1})_\alpha$ . By Lemma 5.10,  $\overline{v}_\rho^{(p)}$  is also an eigenvector for  $\text{Exc}(\Gamma, {}^L H)$  with the same eigensystem as  $v_\rho^{(p)}$ , namely  $\chi_\rho^p$ .

By Theorem 5.11 and Lemma 5.10(ii),  $\text{Exc}(\Gamma, {}^L G)^{\sigma\text{-eq}}$  acts on  $\overline{v}_\rho$  with eigensystem

$$S_{\text{Nm}(p^{-1})(V), \text{Nm}(p^{-1})(x), \text{Nm}(p^{-1})(\xi), (\gamma_i)_{i \in I}} \cdot \overline{v}_\rho = \chi_\rho(S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}) \overline{v}_\rho,$$

and (using Lemma 5.17)  $N \cdot S$  acts by 0 for any  $S \in \text{Exc}(\Gamma, {}^L G)$ .

Note that  $S_{\text{Nm}(V), \text{Nm}(x), \text{Nm}(\xi), (\gamma_i)_{i \in I}}$  is the image of  $S_{\text{Nm}(p^{-1})(V), \text{Nm}(p^{-1})(x), \text{Nm}(p^{-1})(\xi), (\gamma_i)_{i \in I}}$  under the map  $\text{Exc}(\Gamma, {}^L G) \otimes_{k, \text{Frob}} k \xrightarrow{\sim} \text{Exc}(\Gamma, {}^L G)$ . Let  $\text{Exc}(\Gamma, {}^L G)' \subset \text{Exc}(\Gamma, {}^L G)$  be the subalgebra generated by  $N \cdot \text{Exc}(\Gamma, {}^L G)$  and all elements of the form  $\text{Nm}(S_{V, x, \xi, (\gamma_i)_{i \in I}}) = S_{\text{Nm}(V), \text{Nm}(x), \text{Nm}(\xi), (\gamma_i)_{i \in I}}$  (the equality by Lemma 5.17). Then the preceding discussion shows that  $\overline{v}_\rho^{(p)}$  is an eigenvector for  $\text{Exc}(\Gamma, {}^L G)'$  with eigensystem  $\chi'_\rho: \text{Exc}(\Gamma, {}^L G)' \rightarrow k$  given by

$$\begin{aligned} S_{\text{Nm}(V), \text{Nm}(x), \text{Nm}(\xi), (\gamma_i)_{i \in I}} &\mapsto \chi_\rho(S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}})^p \\ N \cdot S &\mapsto 0 \end{aligned} \tag{5.21}$$

This defines a certain maximal ideal  $\mathfrak{m}_\rho$  of  $\text{Exc}(\Gamma, {}^L G)'$ . The existence of  $\overline{v}_\rho^{(p)}$  implies that  $\mathfrak{m}_\rho$  appears in the support of some  $(E_\infty^{ij})_\alpha$  as an  $\text{Exc}(\Gamma, {}^L G)'$ -module. By Lemma 5.10(i),  $(E_\infty^{ij})_\alpha$  is an  $\text{Exc}(\Gamma, {}^L G)'$ -module subquotient of  $H_c^0(\text{Sht}_{G,D,I}; \mathbb{1})_{\phi(\alpha)}$ , so  $\mathfrak{m}_\rho$  is also in the support of  $H_c^*(\text{Sht}_{G,D,\emptyset}; \mathbb{1})_{\phi(\alpha)}$  as an  $\text{Exc}(\Gamma, {}^L G)'$ -module.

Furthermore, we claim that  $H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})$  is a finite module over  $\text{Exc}(\Gamma, {}^L G)'$ . Indeed, by [Xuea, Theorem 0.0.3] (and its generalization to non-split groups to appear in [Xueb]),  $H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})$  is a finite module over  $\mathcal{H}_{G,u}$  for any  $u \in X \setminus D$ . According to the “ $S = T$ ” Theorem [Laf18a, equation (12.16)], the action of the Hecke operator  $h_{V,u}$  at  $u \in X$  indexed by  $V \in \text{Rep}({}^L G)$  agrees with the action of the particular excursion operator  $S_{\{1,2\}, V, \text{unit}, \text{counit}, \{F_u, 1\}}$  where  $F_u$  is any lift of the Frobenius at  $u$  to  $\pi_1(\eta, \overline{\eta})$ . We choose  $u$  so that  $G$  is reductive and hyperspecial at  $u$ , and so that the extension  $E/F$  is split at  $u$ . In this case  $\mathcal{H}_{G,u} \cong \mathcal{H}_{H,u}^{\otimes p}$ , and the subalgebra  $\mathcal{H}'_{G,u} \subset \mathcal{H}_{G,u}$  generated by all elements of the form  $h_{u, \text{Nm}(V)}$  and  $h_{u, N \cdot V}$  coincides with  $\mathcal{H}_{G,u}^\sigma$ . So by the Artin-Tate Lemma,  $H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})$  is also a finite  $\mathcal{H}'_{G,u}$ -module. Since the endomorphisms in the image of  $\mathcal{H}'_{G,u}$  are contained in the endomorphisms in the image of  $\text{Exc}(\Gamma, {}^L G)'$  by the “ $S = T$ ” Theorem, we conclude that  $H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})$  is also a finite  $\text{Exc}(\Gamma, {}^L G)'$ -module, as desired.

Now, we have established that  $\mathfrak{m}_\rho$  is in the support of  $H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})_{\phi(\alpha)}$  as an  $\text{Exc}(\Gamma, {}^L G)'$ -module. The claim implies that the localization  $(H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})_{\phi(\alpha)})_{\mathfrak{m}_\rho}$  is a finite  $\text{Exc}(\Gamma, {}^L G)'_{\mathfrak{m}_\rho}$ -module, so then  $(H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})_{\phi(\alpha)})/\mathfrak{m}_\rho$  is finite-dimensional and (by Nakayama's Lemma) non-zero over  $k$ . Since the  $\text{Exc}(\Gamma, {}^L G)$ -action obviously commutes with the  $\text{Exc}(\Gamma, {}^L G)'$ -action, it descends to this finite-dimensional  $k$ -vector space  $(H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1})_{\phi(\alpha)})/\mathfrak{m}_\rho$  and therefore has an eigenvector. Then Lemma 5.16 plus Lemma 5.17 show that the only possible eigensystem for this eigenvector is

$$\begin{aligned} S_{V,x,\xi,(\gamma_i)_{i \in I}} &\mapsto = \chi'_\rho(S_{\text{Nm}(V), \text{Nm}(x), \text{Nm}(\xi), (\gamma_i)_{i \in I}})^{1/p} \\ &= \chi_\rho(S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}). \end{aligned}$$

This is precisely the composition  $\chi \circ \phi_{\text{BC}}^*$ , as desired.  $\square$

## 6. CYCLIC BASE CHANGE IN THE LOCAL SETTING

In this section we will prove Theorem 1.4. We begin by formulating a precise version of the Treumann-Venkatesh conjecture in §6.1. Any formulation depends on a “local Langlands correspondence mod  $p$ ”; we use the Genestier-Lafforgue correspondence [GL]. This is our only option at the generality of an arbitrary irreducible admissible mod  $p$  representation of an arbitrary reductive group, but for  $\text{GL}_n$  there are more refined correspondences for non-supercuspidal representations [Vig01, EH14, KM20], and it would be interesting to know what happens in those contexts as well.

We review the relevant aspects of the Genestier-Lafforgue correspondence in §6.2. In §6.3 we recall the *Brauer homomorphism* introduced in [TV16]. Finally, in §6.4 we combine these with earlier global results to conclude the proof of Theorem 1.4.

**6.1. Conjectures on local base change functoriality.** We recall a conjecture of Treumann-Venkatesh that “Tate cohomology realizes base change functoriality” in the mod  $p$  Local Langlands correspondence. We shall prove a form of this conjecture, formulated precisely below, for cyclic base change in the function field setting.

Let  $F_v$  be a local function field with ring of integers  $\mathcal{O}_v$  and characteristic  $\neq p$ . Let  $G_v$  be a reductive group over  $F_v$  with a  $\sigma$ -action. Let  $\Pi$  be a smooth irreducible representation of  $G_v$  with coefficients in  $k$ . Let  $\Pi^\sigma$  be the representation of  $G_v$  obtained by composing  $\Pi$  with  $\sigma: G_v \rightarrow G_v$ . We say that  $\Pi$  is  $\sigma$ -fixed if  $\Pi \approx \Pi^\sigma$  as  $G_v$ -representations.

**Lemma 6.1** ([TV16, Proposition 6.1]). *If  $\Pi$  is  $\sigma$ -fixed, then the  $G_v$ -action on  $\Pi$  extends uniquely to an action of  $G_v \rtimes \langle \sigma \rangle$ .*

Let  $H_v = G_v^\sigma$ . Using Lemma 6.1 we can form the Tate cohomology groups  $T^0(\Pi)$  and  $T^1(\Pi)$  with respect to the  $\sigma$ -action, which are then representations of  $H_v$ . Treumann-Venkatesh conjecture that they are in fact admissible representations of  $H_v$ , but we do not prove or use this.

**Definition 6.2** (Linkage). An irreducible admissible representation  $\pi$  of  $H_v$  is *linked* with an irreducible admissible representation  $\Pi$  of  $G_v(F_v)$  if  $\pi^{(p)}$  appears in  $T^0(\Pi)$  or  $T^1(\Pi)$ , where  $\pi^{(p)}$  is the Frobenius twist

$$\pi^{(p)} := \pi \otimes_{k, \text{Frob}} k.$$

**Conjecture 6.3** ([TV16, Conjecture 6.3]). *Linkage is compatible with functorial transfer: if  $\pi$  is linked to  $\Pi$ , then  $\pi$  transfers to  $\Pi$  under the Local Langlands correspondence.*



**Example 6.4.** The need for the Frobenius twist can be seen in a simple example. Suppose  $G = H^p$  and  $\sigma$  acts by cyclic permutation. Then  $G^\sigma$  is the diagonal copy of  $H$ . In this case a representation  $\pi$  of  $H_v$  should transfer to  $\pi^{\boxtimes p}$  of  $G_v$ . And indeed,

$$T^0(\pi^{\boxtimes p}) = \frac{\ker(1 - \sigma \mid \pi^{\boxtimes p})}{N \cdot \pi^{\boxtimes p}} \cong \pi^{(p)}.$$

Let  $W_v = \text{Weil}(\overline{F}_v/F_v)$ . To give Conjecture 6.3 a precise meaning, we need a precise map

$$\left\{ \begin{array}{c} \text{irreducible admissible} \\ \text{representations of } G_v \text{ over } k \end{array} \right\} / \sim \longrightarrow \left\{ \begin{array}{c} \text{Langlands parameters} \\ W_v \rightarrow {}^L G(k) \end{array} \right\} / \sim$$

as a candidate for “the Local Langlands correspondence”. In the function field setting, Genestier-Lafforgue have constructed such a map, which is expected to be the semi-simplification of the local Langlands correspondence. To any irreducible admissible representation  $\Pi$  of  $G_v$ , it assigns a semi-simple local Langlands parameter, i.e. a  $\widehat{G}(k)$ -conjugacy class of continuous homomorphisms  $\rho_\Pi: W_v \rightarrow {}^L G(k)$  which is continuous and semi-simple [GL, Theorem 8.1]. Equivalently, we may view  $\rho_\Pi \in H^1(W_v, \widehat{G}(k))$ .

Now, if  $\pi$  is a representation of  $H_v$  and  $\Pi$  is a representation of  $G_v$ , we say that  $\pi$  *transfers to*  $\Pi$  *under the Genestier-Lafforgue correspondence* if the image of  $\rho_\pi$  under  $H^1(W_v, \widehat{H}(k)) \rightarrow H^1(W_v, \widehat{G}(k))$  coincides with  $\rho_\Pi$ .

Base change supplies many examples of the situation in Conjecture 6.3. (Most of the other examples are particular to  $p = 2, 3, 5$ ; the relevant maps of  $L$ -groups are “exceptional” homomorphisms that do not lift to characteristic 0.) Let  $H$  be a reductive group over  $F_v$ ,  $E_v/F_v$  a cyclic extension of degree  $p$ , and  $G = \text{Res}_{E_v/F_v}(H_{E_v})$ . Then  $G$  has a  $\sigma$ -action,  $H = G^\sigma$ , and the induced  $\sigma$ -action on  $G(F_v) = H(E_v)$  has  $H(F_v)$  as its fixed point subgroup. We call this the “base change situation”. We prove Conjecture 6.3 in the base change situation, with respect to the Genestier-Lafforgue correspondence.

**Theorem 6.5.** *Let  $H_v, G_v$  be as in the base change situation. Let  $\Pi$  be a  $\sigma$ -fixed representation of  $G_v$ . Let  $\pi^{(p)}$  be an irreducible admissible representation of  $H_v$  appearing as a subquotient of  $T^0(\Pi)$  or  $T^1(\Pi)$ . Then  $\pi$  transfers to  $\Pi$  under the Genestier-Lafforgue correspondence.*

**Remark 6.6.** A special case of the conjecture in the base change situation is proved in [Ron16], for depth-zero supercuspidal representations of  $\text{GL}_n$  compactly induced from cuspidal Deligne-Lusztig representations. Despite the very explicit nature of the Local Langlands Correspondence for such representations, the proof involves rather hefty calculations.

Furthermore, the unramified and tamely ramified base change are handled completely differently in [Ron16], whereas our proof will be completely uniform in the field extension, the reductive group, and the irreducible representation.

**6.2. Review of Genestier-Lafforgue’s local Langlands correspondence.** We briefly summarize the aspects of the Genestier-Lafforgue correspondence that we will need.

**6.2.1. The Bernstein center.** We begin by recalling the formalism of the Bernstein center [Ber84]. Let  $K_v \subset G_v$  be an open compact subgroup. The Hecke algebra of  $G$  with respect to  $K_v$  is

$$\mathcal{H}(G, K_v) := C_c^\infty(K_v \backslash G(F_v)/K_v; k).$$

This forms an algebra under convolution, where we use (for all  $K_v$ ) the left Haar measure on  $G_v$  for which  $G(\mathcal{O}_v)$  has volume 1. We let  $\mathfrak{z}(G, K_v) := Z(\mathcal{H}(G, K_v))$  be the center of

$\mathcal{H}(G, K_v)$ . The *Bernstein center* of  $G$  is

$$\mathfrak{z}(G) := \varprojlim_{K_v} \mathfrak{z}(G, K_v)$$

where the transition maps to  $\mathfrak{z}(G, K_v)$  are given by convolution with  $\mathbb{1}_{K_v}$ , the unit of  $\mathcal{H}(G, K_v)$ , viewed as an idempotent in the full Hecke algebra of compactly supported smooth functions on  $G_v$ .

The Bernstein center of  $G_v$  is isomorphic to the endomorphisms of the identity functor of the category of smooth  $k$ -representations of  $G_v$ . Explicitly, smoothness implies that  $\Pi = \varinjlim_{K_v} \Pi^{K_v}$ , and  $\mathfrak{z}(G, K_v)$  acts on  $\Pi^{K_v}$  as an  $\mathcal{H}(G, K_v)$ -module; this assembles into action of  $\mathfrak{z}(G)$  on  $\Pi$ . In particular, any irreducible admissible representation  $\Pi$  of  $G_v$  induces a character of  $\mathfrak{z}(G)$ .

**6.2.2. Action of the excursion algebra.** The main result of [GL] is the construction of a homomorphism

$$Z_G: \text{Exc}(W_v, {}^L G) \rightarrow \mathfrak{z}(G). \quad (6.1)$$

Extend  $G$  to a parahoric group scheme over  $\mathcal{O}_v$ . For a positive integer  $m$ , let  $K_{mv} := \ker(G(\mathcal{O}_v) \rightarrow G(\mathcal{O}_v/t_v^m))$  be the “ $m$ th congruence subgroup”. We write  $U_{mv} := K_{mv} \cap H(\mathcal{O}_v)$  for the  $m$ th congruence subgroup of  $H$ . We write  $Z_{G,m}: \text{Exc}(W_v, {}^L G) \rightarrow \mathfrak{z}(G, K_{mv})$  for the composition of  $Z_G$  with the tautological projection to  $\mathcal{H}(G, K_{mv})$ , and similarly  $Z_{H,m}: \text{Exc}(W_v, {}^L H) \rightarrow \mathfrak{z}(H, U_{mv})$ .

We will shortly give a characterization of (6.1). First let us indicate how (6.1) defines the correspondence  $\Pi \mapsto \rho_\Pi$ . An irreducible admissible  $\Pi$  induces a character of  $\mathfrak{z}(G)$ , as discussed above. Composing with  $Z_G$  then gives a character of  $\text{Exc}(W_v, {}^L G)$ , which by Proposition 4.4 gives a semisimple Langlands parameter  $\rho_\Pi \in H^1(W_v, \widehat{G}(k))$ .

**Remark 6.7.** In fact the homomorphism (6.1) is defined over  $\mathbf{F}_p$  (with the obvious  $\mathbf{F}_p$ -structures on both sides). This implies the following relation with the Frobenius twist, which will be needed later: if  $\chi_\Pi$  is the character giving the action of  $\text{Exc}(W_v, {}^L G)$  on an irreducible  $G_v$ -representation  $\Pi$ , then the character  $\chi_{\Pi^{(p)}}$  giving the action of  $\text{Exc}(W_v, {}^L G) \otimes_{k, \text{Frob}_p} k \xrightarrow{\sim} \text{Exc}(W_v, {}^L G)$  on  $\Pi^{(p)} := \Pi \otimes_{k, \text{Frob}_p} k$  satisfies

$$\chi_{\Pi^{(p)}}\left(\underbrace{S \otimes 1}_{\in \text{Exc}(W_v, {}^L G) \otimes_{k, \text{Frob}_p} k}\right) = \chi_\Pi(S)^p \quad \text{for all } S \in \text{Exc}(W_v, {}^L G).$$

**6.2.3. Local-global compatibility.** Choose a smooth projective curve  $X$  over  $\mathbf{F}_\ell$  and a point  $v \in X$  so that  $X_v = \text{Spec } \mathcal{O}_v$ , such that  $G$  extends to a reductive group over the generic point of  $X$ . Then choose a further extension of  $G$  to a parahoric group scheme over all of  $X$ .

The map (6.1) is characterized by local-global compatibility with the global excursion action. The idea is that for  $(\gamma_i)_{i \in I} \subset W_v^I$ , the action of the excursion operator  $S_{I, f, (\gamma_i)_{i \in I}}$  on  $H_c^0(\text{Sht}_{G, D, \emptyset}; \mathbb{1})$  is local at  $v$ , i.e. it acts through a Hecke operator for  $G_v$ . Moreover, it commutes with other Hecke operators because all excursion operators commute with all Hecke operators, hence it must actually be in the center of the relevant Hecke algebra. This idea is affirmed by the Proposition below.

**Proposition 6.8** (Genestier-Lafforgue Prop 1.3). *For  $(\gamma_i)_{i \in I} \subset W_v^I$ , the operator  $S_{I, f, (\gamma_i)_{i \in I}}$  acts on  $H_c^0(\text{Sht}_{G, D, \emptyset}; \mathbb{1})$  as convolution by  $Z_{G,m}(S_{I, f, (\gamma_i)_{i \in I}}) \in \mathfrak{z}(G_v, K_{mv})$ .*

**Remark 6.9.** By [GL, Lemme 1.4], for large enough  $D^v$  the action of  $Z_{G,m}(S_{I, f, (\gamma_i)_{i \in I}})$  on  $H_c^0(\text{Sht}_{G, D, \emptyset}; \mathbb{1})$  is faithful. Therefore, Proposition 6.8 certainly characterizes the map (6.1).

What is not clear is that the resulting  $Z_{G,m}(S_{I,f,(\gamma_i)_{i \in I}})$  is independent of choices (of the global curve, or the integral model of the affine group scheme). In [GL] this is established by giving a purely local construction of (6.1) in terms of “restricted shtukas”, but for our purposes it will be enough to accept Proposition 6.8 as a black box.

**6.3. The Brauer homomorphism.** We introduce the notion of the Brauer homomorphism from [TV16], whose utility for our purpose is to capture the relationship between  $\Pi$  and its Tate cohomology from the perspective of Hecke algebras.

Let  $K_v \subset G_v$  be an open compact subgroup, and let  $U_v = K_v^\sigma \subset H_v$ . We say that  $K_v \subset G_v$  is a *plain subgroup* if  $(G_v/K_v)^\sigma = H_v/U_v$ .

We can view  $\mathcal{H}(G, K_v)$  as the ring of  $G_v$ -invariant (for the diagonal action) functions on  $(G_v/K_v) \times (G_v/K_v)$  under convolution. We claim that if  $K_v \subset G_v$  is a plain subgroup, then the restriction map

$$\begin{aligned} \mathcal{H}(G, K_v)^\sigma &= \text{Fun}_{G_v}((G_v/K_v) \times (G_v/K_v), k)^\sigma \\ &\xrightarrow{\text{restrict}} \text{Fun}_{H_v}((H_v/U_v) \times (H_v/U_v), k) = \mathcal{H}(H_v, U_v) \end{aligned} \quad (6.2)$$

is an *algebra homomorphism*. It is called the *Brauer homomorphism* and denoted

$$\text{Br}: \mathcal{H}(G, K_v)^\sigma \rightarrow \mathcal{H}(H, U_v).$$

*Proof of claim.* What we must verify is that for  $x, z \in H_v/U_v$ , and  $f, g \in \mathcal{H}(G_v, K_v)^\sigma$ , we have

$$\sum_{y \in G_v/K_v} f(x, y)g(y, z) = \sum_{y \in H_v/U_v} f(x, y)g(y, z). \quad (6.3)$$

Since  $f$  and  $g$  are  $\sigma$ -invariant, we have

$$f(x, y) = f(\sigma x, \sigma y) = f(x, \sigma y) \text{ and } g(y, z) = g(\sigma y, \sigma z) = g(\sigma y, z).$$

If  $y \notin H_v/U_v$ , then the plain-ness assumption implies that  $y$  is not fixed by  $\sigma$ . Therefore the contribution from the orbit of  $\sigma$  on  $y$  to (6.3) is divisible by  $p$ , hence is 0.  $\square$

**Lemma 6.10.** *If  $K_v \subset \ker(G(\mathcal{O}_v) \rightarrow G(\mathcal{O}_v/\mathfrak{m}_v))$ , then  $K_v$  is plain.*

*Proof.* By the long exact sequence for group cohomology, the plain-ness is equivalent to the map on non-abelian cohomology  $H^1(\langle \sigma \rangle; K_v) \rightarrow H^1(\langle \sigma \rangle; G_v)$  having trivial fiber over the trivial class. But the assumption implies that  $K_v$  has a filtration, e.g. the restriction of the lower central series on  $\ker(G(\mathcal{O}_v) \rightarrow G(\mathcal{O}_v/\mathfrak{m}_v))$ , with subquotients being abelian  $\text{char}(\mathbf{F}_\ell)$ -groups, so that they are acyclic for  $H^1(\langle \sigma \rangle, -)$  as  $\sigma$  has order  $p$ . Therefore  $H^1(\langle \sigma \rangle, K_v)$  vanishes for such  $K_v$  as in the statement of the Lemma.  $\square$

**Lemma 6.11** (Relation to the Brauer homomorphism). *Assume  $K_v \subset G_v$  is plain. Suppose  $\Pi$  is a  $\sigma$ -fixed representation of  $G_v$ . Then the map of Tate cohomology groups  $T^*(\Pi^{K_v}) \rightarrow T^*(\Pi)$  lands in the  $U_v$ -invariants, and for any  $h \in \mathcal{H}(G_v, K_v)^\sigma$  we have the commutative diagram below.*

$$\begin{array}{ccc} T^*(\Pi^{K_v}) & \longrightarrow & T^*(\Pi)^{U_v} \\ \downarrow T^0 h & & \downarrow \text{Br}(h) \\ T^*(\Pi^{K_v}) & \longrightarrow & T^*(\Pi)^{U_v} \end{array}$$

(Here  $T^0 h$  is the element of  $T^0(\mathcal{H}(G_v, K_v))$  represented by  $h$ .)

*Proof.* A direct computation similar to the proof of the claim; see [TV16, §6.2].  $\square$

**6.4. Tate cohomology realizes base change functoriality.** Let  $E_v/F_v$  be a cyclic extension of order  $p$ . Let  $H$  be a reductive group over  $F_v$  and  $G = \text{Res}_{E_v/F_v}(H_{E_v})$ . We shall prove:

**Theorem 6.12.** *Let  $\Pi$  be an irreducible admissible representation of  $G(F_v)$  and let*

$$\chi_{\Pi^{(p)}} : \text{Exc}(W_v, {}^L G) \rightarrow k$$

*the associated character of  $\Pi^{(p)}$ . Form  $T^*(\Pi)$  as above, viewed as a smooth  $H(F_v)$ -representation. Then for any irreducible character  $\chi : \text{Exc}(W_v, {}^L H) \rightarrow k$  appearing in the action on  $T^*(\Pi)$  via  $Z_H$ , the composite character*

$$\text{Exc}(W_v, {}^L G) \xrightarrow{\phi_{\text{BC}}^*} \text{Exc}(W_v, {}^L H) \xrightarrow{\chi} k$$

*agrees with  $\chi_{\Pi^{(p)}}$ .*

It is clear that Theorem 6.12 implies Theorem 6.5. The rest of the section is devoted towards proving Theorem 6.12.

6.4.1. The maps

$$\text{Exc}(W_v, {}^L G) \xrightarrow{Z_{G,m}} \mathfrak{z}(G, K_{mv}) \rightarrow \text{End}_{\mathcal{H}_G}(H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1}))$$

induce upon applying Tate cohomology,

$$T^0 \text{Exc}(W_v, {}^L G) \xrightarrow{T^0 Z_{G,m}} T^0 \mathfrak{z}(G, K_{mv}) \rightarrow \text{End}_{T^0 \mathcal{H}_G}(T^0(H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1}))).$$

For each  $m$  we choose  $D^v$  large enough and non-empty so that the map  $\mathfrak{z}(G, K_{mv}) \rightarrow \text{End}_{\mathcal{H}_G}(H_c^0(\text{Sht}_{G,D,\emptyset}; \mathbb{1}))$  is injective, and Remark 5.5 applies. (Of course, we do not claim that  $D^v$  can be so chosen independently of  $m$ .)

6.4.2. Theorem 5.11 implies that

$$\left( \begin{array}{c} \text{the action of} \\ S_{\text{Nm}^{(p-1)}(V), \text{Nm}^{(p-1)}(x), \text{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}} \\ \text{on } T^0(\text{Sht}_{G, mv+D^v, \emptyset}; \mathbb{1}) \end{array} \right) = \left( \begin{array}{c} \text{the action of } S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}} \\ \text{on } T^0(\text{Sht}_{H, mv+D^v, \emptyset}; \mathbb{1}) \end{array} \right).$$

The latter action factors through the action of  $S_{\text{Res}_{\text{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}$  on  $H_c^0(\text{Sht}_{H, mv+D^v, \emptyset}; \mathbb{1})$ , as Lemma 5.7 implies that  $T^0(\text{Sht}_{H, mv+D^v, \emptyset}; \mathbb{1}) \cong H_c^0(\text{Sht}_{H, mv+D^v, \emptyset}; \mathbb{1})$ .

6.4.3. For any set  $S$ , we let  $k[S]$  denote the  $k$ -vector space of  $k$ -valued functions on  $S$ .

Now suppose  $\tilde{S}$  is a set with an action of  $G_v \rtimes \langle \sigma \rangle$ , such that  $K_v \subset G_v$  acts freely. Then  $\mathcal{H}(G_v, K_v)$  acts on  $k[S := \tilde{S}/K_v]$  since we may view  $\mathcal{H}(G_v, K_v) = \text{Hom}_{G_v}(k[G_v/K_v], k[G_v/K_v])$  and  $k[S] = \text{Hom}_{G_v}(k[G_v/K_v], k[\tilde{S}])$ . This induces an action of  $T^0(\mathcal{H}(G_v, K_v))$  on  $T^0(k[S]) \cong k[S^\sigma]$ , and then by inflation an action of  $\mathcal{H}(G_v, K_v)^\sigma$  on  $k[S^\sigma]$ .

By the same mechanism, there is an induced action of  $\mathcal{H}(H_v, U_v)$  on  $k[\tilde{S}^\sigma/K_v^\sigma] = k[\tilde{S}^\sigma/U_v]$ .

**Lemma 6.13.** *Assume  $K_v \subset G_v$  is a plain subgroup. Then  $k[\tilde{S}^\sigma/U_v]$  is a  $\mathcal{H}(G_v, K_v)^\sigma$ -direct summand of  $k[S^\sigma]$ , and for all  $h \in \mathcal{H}(G_v, K_v)^\sigma$  we have*

$$\left( \text{the action of } h \text{ on } k[\tilde{S}^\sigma/U_v] \right) = \left( \text{the action of } \text{Br}(h) \in \mathcal{H}(H_v, U_v) \text{ on } k[\tilde{S}^\sigma/U_v] \right).$$

*Proof.* See [TV16, equation (4.2.2)]. □

From §6.4.1 we have the diagram

$$\begin{array}{ccc}
T^0 \text{Exc}(W_v, {}^L G) & \xrightarrow{Z_{G,m}} T^0 \mathfrak{z}(G, K_{mv}) & \longrightarrow \text{End}_{T^0 \mathcal{H}_G}(T^0(\text{Sht}_{G,mv+D^v,\emptyset}; \mathbb{1})) \\
& & \downarrow \\
\text{Exc}(W_v, {}^L H) & \xrightarrow{Z_{H,m}} \mathfrak{z}(H, U_{mv}) & \hookrightarrow \text{End}_{\mathcal{H}_H}(T^0(\text{Sht}_{H,mv+D^v,\emptyset}; \mathbb{1}))
\end{array} \tag{6.4}$$

**Corollary 6.14.** *For all  $m \geq 1$ , the action of  $z \in T^0 \mathfrak{z}(G, K_{mv})$  on  $T^0(\text{Sht}_{G,mv+D^v,\emptyset}; \mathbb{1})$  in (6.4) agrees with the action of  $\text{Br}(z)$  on  $T^0(\text{Sht}_{H,mv+D^v,\emptyset}; \mathbb{1})$  in (6.4) under the identification  $T^0(\text{Sht}_{G,mv+D^v,\emptyset}; \mathbb{1}) \cong T^0(\text{Sht}_{H,mv+D^v,\emptyset}; \mathbb{1})$  from §2.5.*

*Proof.* Each  $\text{Sht}_{G,mv+D^v,\emptyset}$  is a discrete groupoid with finite stabilizers. As a special case of Remark 5.5, for all positive  $m$  the automorphisms of  $\text{Sht}_{G,mv+D^v,\emptyset}$  are finite unipotent groups, which therefore have no cohomology. Hence we may apply the preceding discussion with  $S := [\text{Sht}_{G,mv+D^v,\emptyset}]$  the set of isomorphism classes in  $\text{Sht}_{G,mv+D^v,\emptyset}$ , and  $\tilde{S} := [\text{Sht}_{G,\infty v+D^v,\emptyset}] = \varprojlim_{j \geq 0} [\text{Sht}_{G,(m+j)v+D^v,\emptyset}]$ . Then  $k[S]$  is identified with the cochains on  $[\text{Sht}_{G,mv+D^v,\emptyset}]$ , and Lemma 5.6 plus §2.5 identify  $k[\tilde{S}^\sigma / K^\sigma]$  with the cochains on  $[\text{Sht}_{H,mv+D^v,\emptyset}]$ . The assertions for compactly supported cochains then follows by duality.  $\square$

**Corollary 6.15.** *For all  $m \geq 1$ , for all  $\{V, x, \xi, (\gamma_i)_{i \in I}\}$  as in §4.4, the Brauer homomorphism sends the element  $Z_{G,m}(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})) \in \mathfrak{z}(G, K_{mv}) \subset \mathcal{H}(G_v, K_{mv})$  to the element  $Z_{H,m}(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}) \in \mathfrak{z}(H, U_{mv}) \subset \mathcal{H}(H_v, U_{mv})$ .*

*Proof.* The discussion of §6.4.2 shows that

$$\left( \begin{array}{c} \text{the image of} \\ S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}} \\ \in T^0 \text{Exc}(W_v, {}^L G) \\ \text{in } \text{End}_{\mathcal{H}_H}(T^0(\text{Sht}_{H,mv+D^v,\emptyset}; \mathbb{1})) \\ \text{via (6.4)} \end{array} \right) = \left( \begin{array}{c} \text{the image of} \\ S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}} \in \text{Exc}(W_v, {}^L H) \\ \text{in } \text{End}_{\mathcal{H}_H}(T^0(\text{Sht}_{H,mv+D^v,\emptyset}; \mathbb{1})) \\ \text{via (6.4)} \end{array} \right). \tag{6.5}$$

On the other hand, the discussion of §6.4.3 shows that the left hand side of (6.5) agrees with the image of  $\text{Br}(Z_{G,m}(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})))$  via (6.4), for all  $m \geq 1$ . We conclude by using injectivity of  $\mathfrak{z}(H, U_{mv}) \hookrightarrow \text{End}_{\mathcal{H}_H}(T^*(\text{Sht}_{H,mv+D^v,\emptyset}; \mathbb{1}))$  in (6.4).  $\square$

*Conclusion of the proof of Theorem 6.12.* Let  $\Pi$  be a representation of  $G_v$ . Then  $\mathfrak{z}(G)$  acts  $G(F_v)$ -equivariantly on  $\Pi$ , inducing an  $H(F_v)$ -equivariant action of  $\mathfrak{z}(G)^\sigma$  on  $T^*(\Pi)$ . In particular, as  $Z_G$  maps the image of  $\text{Exc}(W_v, {}^L G)^{\sigma\text{-eq}} \rightarrow \text{Exc}(W_v, {}^L G)$  (cf. Remark 5.9) into  $\mathfrak{z}(G)^\sigma$ , we get an  $H(F_v)$ -equivariant action of  $\text{Exc}(W_v, {}^L G)^{\sigma\text{-eq}}$  on  $T^*(\Pi)$ .

By Lemma 6.10,  $K_{mv}$  is plain as soon as  $m \geq 1$ , so in particular the Brauer homomorphism is defined on  $\mathcal{H}(G_v, K_{mv})$  as soon as  $m \geq 1$ . Taking the (filtered) colimit over  $m$  in Lemma 6.11, we find that for all  $S \in \text{Exc}(W_v, {}^L G)^{\sigma\text{-eq}}$ , we have

$$\left( \begin{array}{c} \text{the action on } T^*(\Pi) \text{ of} \\ Z_G(S) \end{array} \right) = \left( \begin{array}{c} \text{the action on } T^*(\Pi) \text{ of} \\ \text{Br}(Z_G(S)) \end{array} \right).$$

In other words, the diagram below commutes:

$$\begin{array}{ccccc}
\text{Image}(Z_G|_{\text{Exc}(W_v, {}^L G)^{\sigma\text{-eq}}}) & \hookrightarrow & \mathfrak{z}(G) & \longrightarrow & \text{End}_{G_v}(\Pi) \\
\downarrow \text{Br} & & & & \downarrow \\
\text{Image}(Z_H) & \hookrightarrow & \mathfrak{z}(H) & \longrightarrow & \text{End}_{H_v}(T^*\Pi)
\end{array} \tag{6.6}$$

On the other hand, taking the inverse limit over  $m$  in Corollary 6.15 yields that

$$\text{Br}(Z_G(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}))) = Z_H(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}). \tag{6.7}$$

Combining (6.6) and (6.7) shows that

$$\left( Z_G(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})) \right) = \left( \text{the action on } T^*(\Pi) \text{ of } Z_H(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}) \right). \tag{6.8}$$

From now on, assume  $\Pi$  is an irreducible admissible representation of  $G(F_v)$ . Then  $\text{End}_{G(F_v)}(\Pi) \cong k$  (by Schur's Lemma applied to the Hecke action on the invariants of  $\Pi$  for every compact open subgroup of  $G_v$ ). The Langlands parameter attached to  $\Pi$  corresponds under Proposition 4.4 to the character

$$\chi_\Pi: \text{Exc}(W_v, {}^L G) \xrightarrow{Z_G} \mathfrak{z}(G) \rightarrow \text{End}_{G_v}(\Pi) \cong k.$$

This induces

$$T^0 \chi_\Pi: T^0 \text{Exc}(W_v, {}^L G) \xrightarrow{T^0 Z_G} T^0 \mathfrak{z}(G) \rightarrow T^0 \text{End}_{G_v}(\Pi) \cong k.$$

Let  $\iota$  denote the natural map  $T^0 \text{End}_{G_v}(\Pi) \rightarrow \text{End}_{H_v}(T^*\Pi)$ . We also consider the homomorphism

$$\chi_{T^0 \Pi}: \text{Exc}(W_v, {}^L H) \xrightarrow{Z_H} \mathfrak{z}_H \rightarrow \text{End}_{H_v}(T^0 \Pi).$$

We have just seen in (6.8) that

$$\iota \circ T^0 \chi_\Pi(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})) = \chi_{T^0 \Pi}(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}). \tag{6.9}$$

Note that the fact that the right hand side of (6.9) lies in  $k$  is already non-obvious. In particular, (6.9) implies that for any irreducible subquotient  $\pi$  of  $T^0 \Pi$ , we have

$$\begin{aligned}
\chi_\pi(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}) &= \chi_{T^0 \Pi}(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}) \\
&= (T^0 \chi_\Pi)(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})) \\
&= \chi_\Pi(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})).
\end{aligned} \tag{6.10}$$

Using Remark 6.7, the same reasoning as in the proof of Theorem 5.14 shows that

$$\chi_{\Pi^{(p)}}(S_{\text{Nm}(V), \text{Nm}(x), \text{Nm}(\xi), (\gamma_i)_{i \in I}}) = \chi_\Pi(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}))^p. \tag{6.11}$$

By Lemma 5.16 and Lemma 5.17, we have

$$\begin{aligned}
\chi_{\Pi^{(p)}}(S_{V, x, \xi, (\gamma_i)_{i \in I}}) &= \chi_{\Pi^{(p)}}(S_{\text{Nm}(V), \text{Nm}(x), \text{Nm}(\xi), (\gamma_i)_{i \in I}})^{1/p} \\
[(6.11) \implies] &= \chi_\Pi(S_{\text{Nm}^{(p^{-1})}(V), \text{Nm}^{(p^{-1})}(x), \text{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}})) \\
[(6.10) \implies] &= \chi_\pi(S_{\text{ResBC}(V), x, \xi, (\gamma_i)_{i \in I}}) \\
&= \chi_\pi \circ \phi_{\text{BC}}^*(S_{V, x, \xi, (\gamma_i)_{i \in I}}).
\end{aligned}$$

This shows that  $\chi_{\Pi^{(p)}} = \chi_\pi \circ \phi_{\text{BC}}^*$  for any irreducible subquotient  $\pi$  of  $T^*(\Pi)$ , which completes the proof.  $\square$

APPENDIX A. THE BASE CHANGE FUNCTOR REALIZES LANGLANDS FUNCTORIALITY  
BY TONY FENG AND GUS LONERGAN

In this section we prove Theorem 3.19. We keep the setup of §3.6.1:  $H$  is any reductive group over a separably closed field  $\mathbf{F}$  of characteristic  $\neq p$ , and  $G = H^p$ . We let  $\sigma$  act on  $G$  by cyclic rotation, sending the  $i$ th factor to the  $(i+1)$ st (mod  $p$ ) factor.

**A.1. Proof of linearity.** We first prove that BC is additive, i.e. we exhibit a natural isomorphism  $\mathrm{BC}(\mathcal{F} \oplus \mathcal{F}') \cong \mathrm{BC}(\mathcal{F}) \oplus \mathrm{BC}(\mathcal{F}')$ . We have

$$\begin{aligned} \mathrm{Nm}(\mathcal{F} \oplus \mathcal{F}') &= (\mathcal{F} \oplus \mathcal{F}') * (\sigma \mathcal{F} \oplus \sigma \mathcal{F}') * \dots * (\sigma^{p-1} \mathcal{F} \oplus \sigma^{p-1} \mathcal{F}') \\ &= \mathrm{Nm}(\mathcal{F}) \oplus \mathrm{Nm}(\mathcal{F}') \oplus (\text{direct sum of free } \sigma\text{-orbits}). \end{aligned}$$

Therefore, the restrictions of  $\mathrm{Nm}(\mathcal{F} \oplus \mathcal{F}')$  and  $\mathrm{Nm}(\mathcal{F}) \oplus \mathrm{Nm}(\mathcal{F}')$  to  $X^\sigma$  differ by a perfect complex of  $\mathbb{O}[\sigma]$ -modules, and hence project to isomorphic objects in the Tate category  $\mathrm{Shv}(X^\sigma; \mathcal{T}_\mathbb{O})$ . This shows that  $\mathrm{Psm} \circ \mathrm{Nm}$  is additive. We conclude by using that the modular reduction functor  $\mathbb{F}$  and the lifting functor  $L$  are both additive.  $\square$

**A.2. Reduction to the case of a torus.** Let  $T_H$  be a maximal torus of  $H$ . Recall that the restriction functor  $\mathrm{Rep}(\widehat{H}) \rightarrow \mathrm{Rep}(T_{\widehat{H}})$  is intertwined under the Geometric Satake equivalence with the *hyperbolic localization* functor [BR18, §5.3].

Since  $*/!$ -restriction and  $*/!$ -pushforward all commute with  $\mathrm{Psm}$  by §2.3, the hyperbolic localization functor commutes with  $\mathrm{Psm}$ . As the restriction functor  $\mathrm{Rep}(\widehat{H}) \rightarrow \mathrm{Rep}(T_{\widehat{H}})$  is faithful and injective on tilting objects (i.e. “tilting modules are determined by their characters”) by [Don93, p. 46], it suffices to prove Theorem 3.19 in the special case where  $H$  is a *torus*.

**A.3. Proof in the case of a torus.** Finally, we examine the case when  $H$  is a torus. Since the theorem is compatible with products, we can even reduce to the case  $H = \mathbf{G}_m$ . For  $H = \mathbf{G}_m$  the underlying reduced scheme of  $\mathrm{Gr}_H$  is a disjoint union of points labeled by the integers.

The irreducible algebraic representations of  $\widehat{H}$  are indexed by  $n \in \mathbf{Z}$ , with  $V_n$  corresponding to the constant sheaf supported on the component  $\mathrm{Gr}_H^n$  labeled by  $n$ . The irreducible algebraic representations of  $\widehat{G}$  are then labeled by  $p$ -tuples of integers  $(n_1, \dots, n_p) \in \mathbf{Z}^p$ . By the linearity of BC established in §A.1 and the complete reducibility of algebraic representations of tori, we may assume that  $\mathcal{F}$  is irreducible, say  $\mathcal{F} = \mathcal{F}(n_1, \dots, n_p)$  is the constant sheaf supported on  $\mathrm{Gr}_G^{(n_1, \dots, n_p)}$ .

The  $\sigma$ -equivariant sheaf  $\mathrm{Nm}(\mathcal{F})$  is then the constant sheaf  $\underline{k}$  supported on the component  $\mathrm{Gr}_G^{(n_1 + \dots + n_p, \dots, n_1 + \dots + n_p)}$ . Its restriction to the diagonal copy of  $\mathrm{Gr}_H$  is the constant sheaf with value  $k$  supported on  $\mathrm{Gr}_H^{n_1 + \dots + n_p}$ . This is already an indecomposable  $k$ -parity sheaf, which tautologically lifts its own image in the Tate category. Hence we have shown that

$$\underline{k}_{\mathrm{Gr}_H^{n_1 + \dots + n_p}} = \mathrm{BC}^{(p)}(V_{n_1, \dots, n_p}).$$

And indeed, this is precisely the sheaf which corresponds under geometric Satake to  $\mathrm{Res}_{\mathrm{BC}}(V_{n_1} \boxtimes V_{n_2} \boxtimes \dots \boxtimes V_{n_p}) \cong V_{n_1 + n_2 + \dots + n_p}$ . This confirms the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Parity}^0(\mathrm{Gr}_G; k) & \xrightarrow{\mathrm{BC}} & \mathrm{Parity}^0(\mathrm{Gr}_H; k) \\ \downarrow \sim & & \downarrow \sim \\ \mathrm{Tilt}_k(\widehat{G}) & \xrightarrow{\mathrm{Res}_{\mathrm{BC}}} & \mathrm{Tilt}_k(\widehat{H}) \end{array}$$

at the level of objects. Our final step is to verify the commutativity on morphisms. Since (as  $H$  is a torus) the categories involved are all semi-simple, the commutativity at the level of morphisms reduces to examining a scalar endomorphism of the simple object  $\mathcal{F}$  above, which corresponds to the simple representation  $V_{n_1, \dots, n_p}$ . The restriction functor  $\text{Res}_{\text{BC}}$  is  $k$ -linear, so what we have to check is that  $\text{BC}$  sends multiplication by  $\lambda$  on  $\mathcal{F}$  to multiplication by  $\lambda$  on  $\text{BC}(\mathcal{F})$ . Now, multiplication by  $\lambda$  on  $\mathcal{F}$  is sent under  $\text{Nm}$  to multiplication by  $\lambda^p$  on  $\text{Nm}(\mathcal{F})$ , which restricts to multiplication by  $\lambda^p$  on  $\text{BC}^{(p)}(\mathcal{F})$ . Then the inverse Frobenius twist  $\text{Frob}_p^{-1}$  sends it to multiplication by  $\lambda$ , so  $\text{BC} := \text{Frob}_p^{-1} \circ \text{BC}^{(p)}$  behaves as desired.  $\square$

## REFERENCES

- [BBM04] A. Beilinson, R. Bezrukavnikov, and I. Mirković, *Tilting exercises*, Mosc. Math. J. **4** (2004), no. 3, 547–557, 782. MR 2119139
- [Ber84] J. N. Bernstein, *Le “centre” de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32. MR 771671
- [BG14] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187. MR 3444225
- [BHKT19] Gebhard Böckle, Michael Harris, Chandrashekar Khare, and Jack A. Thorne,  *$\hat{G}$ -local systems on smooth projective curves are potentially automorphic*, Acta Math. **223** (2019), no. 1, 1–111. MR 4018263
- [Bla94] Don Blasius, *On multiplicities for  $\text{SL}(n)$* , Israel J. Math. **88** (1994), no. 1–3, 237–251. MR 1303497
- [BR18] Pierre Baumann and Simon Riche, *Notes on the geometric Satake equivalence*, Relative aspects in representation theory, Langlands functoriality and automorphic forms, Lecture Notes in Math., vol. 2221, Springer, Cham, 2018, pp. 1–134. MR 3839695
- [Clo14] Laurent Clozel, *Formes modulaires sur la  $\mathbb{Z}_p$ -extension cyclotomique de  $\mathbb{Q}$* , Pacific J. Math. **268** (2014), no. 2, 259–274. MR 3227435
- [Clo17] ———, *Sur l’induction automorphe pour des  $p$ -extensions radicielles et quelques autres opérations fonctorielles (mod  $p$ )*, Doc. Math. **22** (2017), 1149–1180. MR 3690271
- [Don93] Stephen Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993), no. 1, 39–60. MR 1200163
- [EH14] Matthew Emerton and David Helm, *The local Langlands correspondence for  $\text{GL}_n$  in families*, Ann. Sci. Éc. Norm. Supér. (4) **47** (2014), no. 4, 655–722. MR 3250061
- [Fen20] Tony Feng, *Nearby cycles of parahoric *shtukas*, and a fundamental lemma for base change*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 21, 59. MR 4073972
- [Gai] Dennis Gaitsgory, *From geometric to function-theoretic langlands (or how to invent *shtukas*)*, Available at <https://arxiv.org/pdf/1606.09608.pdf>.
- [GL] Alain Genestier and Vincent Lafforgue, *Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale*, Available at <https://arxiv.org/abs/1709.00978>.
- [JMW14] Daniel Juteau, Carl Mautner, and Geordie Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), no. 4, 1169–1212. MR 3230821
- [JMW16] ———, *Parity sheaves and tilting modules*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 2, 257–275. MR 3481350
- [KM20] Robert Kurinczuk and Nadir Matringe, *A characterization of the relation between two  $\ell$ -modular correspondences*, C. R. Math. Acad. Sci. Paris **358** (2020), no. 2, 201–210. MR 4118176
- [Laf18a] Vincent Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, J. Amer. Math. Soc. **31** (2018), no. 3, 719–891. MR 3787407
- [Laf18b] ———, *Shtukas for reductive groups and Langlands correspondence for function fields*, Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018. Vol. I. Plenary lectures, World Sci. Publ., Hackensack, NJ, 2018, pp. 635–668. MR 3966741
- [Lap99] Erez M. Lapid, *Some results on multiplicities for  $\text{SL}(n)$* , Israel J. Math. **112** (1999), 157–186. MR 1714998
- [LL] Spencer Leslie and Gus Lonergan, *Parity sheaves and Smith Theory*, Available at <https://arxiv.org/abs/1708.08174>.



- [MR18] Carl Mautner and Simon Riche, *Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković-Vilonen conjecture*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 9, 2259–2332. MR 3836847
- [Qui71] Daniel Quillen, *The spectrum of an equivariant cohomology ring. I, II*, Ann. of Math. (2) **94** (1971), 549–572; *ibid.* (2) **94** (1971), 573–602. MR 298694
- [Ric] Simon Riche, *Geometric representation theory in positive characteristic*, Habilitation thesis, available at <https://tel.archives-ouvertes.fr/tel-01431526/document>.
- [Ron16] Niccolò Ronchetti, *Local base change via Tate cohomology*, Represent. Theory **20** (2016), 263–294. MR 3551160
- [RW] Simon Riche and Geordie Williamson, *Smith-Treumann theory and the linkage principle*, Available at <https://arxiv.org/pdf/2003.08522.pdf>.
- [Sta20] The Stacks Project Authors, *Stacks Project*, <https://stacks.math.columbia.edu>, 2020.
- [Tre19] David Treumann, *Smith theory and geometric Hecke algebras*, Math. Ann. **375** (2019), no. 1-2, 595–628. MR 4000251
- [TV16] David Treumann and Akshay Venkatesh, *Functoriality, Smith theory, and the Brauer homomorphism*, Ann. of Math. (2) **183** (2016), no. 1, 177–228. MR 3432583
- [Var04] Yakov Varshavsky, *Moduli spaces of principal  $F$ -bundles*, Selecta Math. (N.S.) **10** (2004), no. 1, 131–166. MR 2061225
- [Vig01] Marie-France Vignéras, *Correspondance de Langlands semi-simple pour  $GL(n, F)$  modulo  $\neq p$* , Invent. Math. **144** (2001), no. 1, 177–223. MR 1821157
- [Xuea] Cong Xue, *Cohomology with integral coefficients of stacks of shtukas*, Available at <https://arxiv.org/pdf/2001.05805.pdf>.
- [Xueb] ———, *To be announced*, In preparation.
- [Xue20] ———, *Finiteness of cohomology groups of stacks of shtukas as modules over Hecke algebras, and applications*, Épijournal de Géométrie Algébrique **4** (2020), no. 6, 1–42.
- [Zhu] Xinwen Zhu, *Coherent sheaves on the stack of langlands parameters*, Preprint available at <https://arxiv.org/abs/2008.02998>.
- [Zhu15] ———, *The geometric Satake correspondence for ramified groups*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 2, 409–451. MR 3346175
- [Zhu17] ———, *An introduction to affine Grassmannians and the geometric Satake equivalence*, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 59–154. MR 3752460