Eigenvarieties, families of Galois representations, p-adic L-functions

Incomplete notes from a Course at Brandeis university given in Fall 2010

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Introduction

The aim of this book is to give a gentle but fairly complete introduction to the two interrelated theory of p-adic families of modular forms and of p-adic L-functions of modular forms, for graduate students or researchers from other fields (or from other subfields of the field of number theory and automorphic forms). It grew up from a course I gave at Brandeis during the Fall of 2010. A part III, which is still mainly in preparation, aims to explain the higher rank generalizations of those theories.

The version you have in your hands or on your screen is a preliminary, unfinished version. What I intend to do before realsing it is

(1) Correct every typos.
(2) Finish this introduction.
(3) Write for each chapter a Notes and References section that attributes each result to its real author: Most of the results, and almost all of the early chapters are not new. For many of them, it has revealed a not a too easy task to track down their first appearance in literature. I am still working on that, and this is the main reason I am delaying release.
(4) Write down the last theorem on values of p-adic adjoint L-function. This is just a combination of lemmas done earlier in the book.
(5) Check the internal references and complete the bibliography
(6) Check the consistency of notations across chapters.

There is also a last part (part III) on eigenvariety for unitary groups that needs to be written, but I think I will release the book without it first, delaying the part III for a later long period without teaching.

The expected release of the final versioon is now February 2014 (it has been reported several times).

All questions and remarks are welcome: please email me at jbellaic@brandeis.edu.

Part 1

The 'eigen' construction

CHAPTER I

Construction of eigenalgebras

Except for two sections (§I.6 and §I.8) concerning modular forms, intended as motivations for and illustrations of the general theory, this chapter is purely algebraic. We want to explain a very simple, even trivial, construction that has played an immense role in the arithmetic theory of automorphic forms during the last forty years. This construction attaches to a family of commuting operators acting on some space or module an algberaic object, called the *eigenalgebra*, that parameterizes the systems of eigenvalues for those operators appearing in the given space or module.

In all this chapter, R is a commutative Noetherian ring.

I.1. A reminder on the ring of endomorphisms of a module

Let M be a finite R-module. We shall denote by $\mathrm{End}_R(M)$ the R-algebra of R-linear endomorphisms $\phi : M \to M$. If R' is a commutative R-algebra, and $M' = M \otimes_R R'$, there is a natural morphism of R-algebras $\mathrm{End}_R(M) \to \mathrm{End}_{R'}(M')$ sending $\phi : M \to M$ to $\phi \otimes \mathrm{Id}_{R'} : M' \to M'$. Hence there is a natural morphism of R'-algebras

$$(1) \qquad \mathrm{End}_R(M) \otimes_R R' \to \mathrm{End}_{R'}(M').$$

When R' is a fraction ring of R, and more generally when R' is R-flat, this morphism is an isomorphism. In other words, *the formation of* $\mathrm{End}_R(M)$ *commute with localization*, and more generally, flat base change.

EXERCISE I.1.1. Show this, using that M is finite, hence of finite presentation.

EXERCISE I.1.2. Give an example of R-algebra R' and finite R-module M where this morphism is not an isomorphism.

We now assume that in addition of being a finite R-module, M is flat, or what amounts to the same, projective, or locally free. Recall that the *rank* of M is the locally constant function $\mathrm{Spec}\, R \to \mathbb{N}$ sending $x \in \mathrm{Spec}\, R$ to the dimension over $k(x)$ of $M \otimes_R k(x)$, where $k(x)$ is the residue field of the point x. When this function is constant (which is always the case when $\mathrm{Spec}\, R$ is connected, for example when R is local), we also call *rank* of M its value.

Since the formation of $\mathrm{End}_R(M)$ commutes with localizations, many properties enjoyed by $\mathrm{End}_R(M)$ in the case where M is free (in which case $\mathrm{End}_R(M)$ is just a

matrix algebra $M_d(R)$ if M is of rank d) are still true in the flat case. For example, if M is flat, the natural morphism (1)

$$\mathrm{End}_R(M) \otimes_R R' \to \mathrm{End}_{R'}(M')$$

is an isomorphism for all R'-algebras R. To see this, note that it suffices to check that the morphism (1) is an isomorphism after localization at every prime ideal $\mathfrak{p}'$ of R', and thus after localization of R as well at the prime ideal $\mathfrak{p}$ of R below $\mathfrak{p}'$. Hence we can assume that R is local, so that M is free, in which case the result is clear by the description of $\mathrm{End}_R(M)$ as an algebra of square matrices.

Similarly, we can define the characteristic polynomial $P_\phi(X) \in R[X]$ of an endomorphim $\phi \in \mathrm{End}_R(M)$ of a finite flat module M by gluing the definitions $P_\phi(X) = \det(\phi - X\mathrm{Id})$ in the free case. The Cayley-Hamilton theorem $P_\phi(\phi) = 0 \in \mathrm{End}_R(M)$ holds since it holds locally. Observe that $P_\phi(X)$ is not necessarily monic in general, but is monic of degree d when M has constant rank d. In paticular, $P_\phi(X)$ is monic of $\mathrm{Spec}\, R$ is connected.

EXERCISE I.1.3. Show that the formation of $P_\phi(X)$ commutes with arbitrary base change, that is that if $R \to R'$ is any map, and $\phi' = \phi \otimes \mathrm{Id}_{R'} \in \mathrm{End}_{R'}(M')$, then $P_{\phi'}(X)$ is the image of $P_\phi(X)$ in $R'[X]$.

I.2. Construction of eigenalgebras

As before, let R be a commutative noetherian ring and let M be a finite flat R-module (or equivalently, finite locally free, or finite projective). In the applications, elements of M will be modular forms, or automorphic forms, or families thereof, over R. Finally, we suppose given a commutative ring $\mathcal{H}$ and a morphism of rings $\psi : \mathcal{H} \to \mathrm{End}_R(M)$.

To those data $(R, M, \mathcal{H}, \psi)$, we attach the sub-$R$-module $\mathcal{T} = \mathcal{T}(R, M, \mathcal{H}, \psi)$ of $\mathrm{End}_R(M)$ generated by the image $\psi(\mathcal{H})$. It is clear that $\mathcal{T}$ is a sub-algebra of $\mathrm{End}_R(M)$.

Equivalently, $\mathcal{T}$ is the quotient of $\mathcal{H} \otimes_{\mathbb{Z}} R$ that acts faithfully on M, that is to say, the quotient of $\mathcal{H} \otimes_{\mathbb{Z}} R$ by the ideal annihilator of M.

DEFINITION I.2.1. The R-algebra $\mathcal{T}$ is called the *eigenalgebra* of $\mathcal{H}$ acting on the module M.

Another version of the same beginning is simpler and more direct: we are given R, M as above and we attach to a family of commuting endomorphisms $(T_i)_{i \in I}$ (infinite in general) in $\mathrm{End}_R(M)$, the R-subalgebra $\mathcal{T}$ they generate. This is equivalent to the preceding situation as we can take for $\mathcal{H}$ the ring of polynomials $\mathbb{Z}[(X_i)_{i \in I}]$ with independent variables X_i, and for ψ the map that sends X_i on T_i, giving the same $\mathcal{T}$. Conversely, the situation with $\mathcal{H}$ and ψ can be converted into the situation with the T_i's by choosing a family of generators $(X_i)_{i \in I}$ of $\mathcal{H}$ as a $\mathbb{Z}$-algebra and setting $T_i = \psi(X_i)$.

The aim of this section is to explain the meaning and properties of this simple construction.

I.3. First properties

By construction $\mathcal{T}$ is a commutative R-algebra, and M has naturally a structure of $\mathcal{T}$-module. We record some obvious properties.

LEMMA I.3.1. *As an R-module, $\mathcal{T}$ is finite. If R is a domain, $\mathcal{T}$ is also torsion-free. The ring $\mathcal{T}$ is noetherian. The module M is finite over $\mathcal{T}$.*

Proof — The first two assertions are obvious since $\mathrm{End}_R(M)$ is finite and torsion-free, and R is noetherian. The third follows form the first and Hilbert's theorem. And the fourth is clear since M is already finite as an R-module. $\square$

The finiteness of $\mathcal{T}$ over R implies that it is integral over R, which can be made more precise by the following lemma.

LEMMA I.3.2. *Assume that M has constant rank d over R. Every element of $\mathcal{T}$ is killed by a monic polynomial of degree d with coefficients in R.*

Proof — Every element of $\mathcal{T}$ is killed by its characteristic polynomial, according to the Cayley-Hamilton theorem. $\square$

EXERCISE I.3.3. 1.– Suppose given a projective R-submodule A of M that is $\mathcal{H}$-stable. Then we can also define the eigenalgebra of $\mathcal{H}$ acting on A, and denote it by $\mathcal{T}_A$. Construct a natural surjective map $\mathcal{T} \to \mathcal{T}_A$. Show that if R is a domain, and M/A is torsion, then this map is an isomorphism.

2.– Suppose given two $\mathcal{H}$-stable submodules A and B of M such that $A \cap B = 0$. Show that the map $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$ is neither injective nor surjective in general.

3.– Same hypotheses as in 2. and assume that $M = A \oplus B$, or that R is a domain and $M/(A \oplus B)$ is torsion. Show however that the map is $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$ injective.

4.– Same hypothesis as in 2. and assume there is a $T \in \mathcal{H}$ that acts by multiplicaton by a scalar a on A and by a scalar b and B, and that $b - a$ is invertible in R. Show that the map $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$ is surjective.

EXERCISE I.3.4. Let $R, M, \mathcal{H}, \psi$ be as above. We write $M^\vee$ for the dual module $\mathrm{Hom}_R(M, R)$. There is a natural map $\psi^\vee : \mathcal{H} \to \mathrm{End}_R(M^\vee)$ defined by $\psi^\vee(h) = {}^t\psi(h)$ where ^{t} denotes the transpose of a map. We denote by $\mathcal{T}$ and $\mathcal{T}^\vee$ the eigenalgebras of $\mathcal{H}$ action on M and $M^\vee$.

1.– Show that $\mathcal{T}$ is canonically isomorphic to $\mathcal{T}^\vee$.

2.– Show that if R is a field, and $\mathcal{H}$ is generated by one element, then $M^\vee \simeq M$ as $\mathcal{H}$-modules and as $\mathcal{T}$-modules.

3.– Show that the same results may fail to hold if either R is not a field, or R is a field (even $R = \mathbb{C}$) but $\mathcal{H}$ is not generated by a single element.

Let us assume here that R is a Noetherian domain. Can we say more about the abstract structure of $\mathcal{T}$ than just its being finite and torsion-free over R? Or on the contrary, is any finite and torsion-free R-algebra S an eigenalgebra $\mathcal{T}$ for some $M, \mathcal{H}, \psi$. The question is clearly equivalent to the following:

QUESTION I.3.5. *Is any finite and torsion-free R-algebra S a sub-algebra of some $\mathrm{End}_R(M)$ for some finite flat R-module M?*

The answer obviously depends on the nature of the noetherian domain R.

When R is a Dedekind domain, the answer is yes. Indeed, any finite torsion-free R-algebra S is also a flat R-module, so we can take $M = S$ and see S as a sub-algebra of $\mathrm{End}_R(M)$ by left-multiplication.

A Dedekind domain is regular of dimension 1. It was pointed out to me by Chenevier that the answer to Question I.3.5 is also yes when R is a regular ring of dimension 2. Indeed the bidual of any finite torsion-free module over R is projective by a result of Serre (see e.g. [**S**]), so if we define M as the bidual of S seen as an R-module, then S embeds naturally in $\mathrm{End}_R(M)$.

According to Mel Hochster of Michigan University[1], Question I.3.5 is still open for regular rings R of dimension ≥ 3. In general, the answer is no: there may exist a torsion-free R-algebra S that cannot be a $\mathcal{T}$. Here is an example also due to Mel Hochster: $R = k[[x^2, xy, y^2]]$ and $S = k[[x, y]]$ for k a field.

EXERCISE I.3.6. 1.– If I is an ideal of R, observe that $R \oplus \epsilon I$ is a sub-R-algebra of $R[\epsilon]/(\epsilon^2)$. Show that all those algebras $R \oplus \epsilon I$ are eigenalgebras $\mathcal{T}$. Deduce that there are examples of eigenalgebras that are not flat over R, and not reduced.

2.– Is there an example of eigenalgebra $\mathcal{T}$ that is reduced and non-flat over R?

Anyway, there are properties enjoyed by any finite, torsion-free $\mathcal{T}$-algebras over a commutative noetherian domain R, in particular by eigenalgebras, that are worth noting:

PROPOSITION I.3.7. *The map $Spec\,\mathcal{T} \to Spec\,R$ is surjective. Actually, every irreducible component of $Spec\,\mathcal{T}$ maps surjectively onto $Spec\,R$. In particular if $Spec\,R$ has dimension n, then $Spec\,\mathcal{T}$ is equidimensional of dimension n.*

Proof — Let $\mathfrak{p}$ be a prime ideal of $\mathcal{T}$, $\mathfrak{q} = R \cap \mathfrak{p}$ the corsponding prime ideal of R. Then by construction the localization $\mathcal{T}_{(\mathfrak{p})}$ is still finite torsion-free over $R_{(\mathfrak{q})}$. In $\mathfrak{p}$ is a minimal prime ideal, then $\mathcal{T}_{(\mathfrak{p})}$ is a field, and if a field is finite, torsion-free over a domain, this domain is a field. Hence $\mathfrak{p}$ is a minimal prime ideal of R, that

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[1]I thank Kevin Buzzard for this information.

is $\mathfrak{p} = (0)$ and we have shown that $\operatorname{Spec} \mathcal{T} \to \operatorname{Spec} R$ maps every generic point of an irreducible component of $\operatorname{Spec} \mathcal{T}$ to the generic point of $\operatorname{Spec} R$. Since this map is closed, it follows that every irreducible component of $\operatorname{Spec} \mathcal{T}$ is surjective onto $\operatorname{Spec} R$. $\qquad \square$

REMARK I.3.8. A trivial adaptation of this proof shows that even if R is not a domain, if $\operatorname{Spec} R$ is equidimensional of dimension n, then so is $\operatorname{Spec} T$.

I.4. Behavior under base change

Let us investigate the behavior of the construction of eigenalgebras with respect to base change. Let $R \to R'$ be a morphism of noetherian rings, and define $M' = M \otimes_R R'$, and let $\psi' : \mathcal{H} \to \operatorname{End}_R(M) \to \operatorname{End}_{R'}(M')$ be the obvious composition. We call $\mathcal{T}'$ the R'-eigenalgebra of $\mathcal{H}'$ acting on M'.

Through the isomorphism $\operatorname{End}_R(M) \otimes_R R' \to \operatorname{End}_{R'}(M')$, the image of $\mathcal{T} \otimes_R R'$ in $\operatorname{End}_{R'}(M')$ is precisely $\mathcal{T}'$. Therefore, there is a natural surjective map of R'-algebras

$$(2) \qquad \mathcal{T} \otimes_R R' \to \mathcal{T}'.$$

Then

PROPOSITION I.4.1. *The kernel of the base change map 2 is a nilpotent ideal. This map is an isomorphism when R' is R-flat.*

Proof — We first prove the second assertion. By definition, the map $\mathcal{T} \to \operatorname{End}_R(M)$ is injective. By flatness of R', so is the map $\mathcal{T} \otimes_R R' \to \operatorname{End}_{R'}(M')$. In other words $\mathcal{T} \otimes_R R'$ acts faithfully on M'. Since $\mathcal{T}'$ is the quotient of $\mathcal{H} \otimes R'$ that acts faithfully on M', we have $\mathcal{T}' = \mathcal{T} \otimes_R R'$.

For the first assumption, let us assume first that the map $R \to R'$ is surjective, of kernel I. We may assume in addition that M is free, of rank d. Let $\tilde{\phi}$ in $\mathcal{T} \otimes_R R' = \mathcal{T}/I\mathcal{T}$ be an element whose image $\phi' \in \mathcal{T}' \subset \operatorname{End}_{R'}(M')$ is 0. Let ϕ in $\mathcal{T}$ be an element that lifts $\tilde{\phi}$. We thus have $0 = \phi' = \phi \otimes 1 \in \operatorname{End}_{R'}(M')$. The characteristic polynomial of ϕ' is X^d. Since the formation of characteristic polynomial commutes with base change, we deduce that the characteristic polynomial of ϕ belongs to $X^d + IR[X]$. By Cayley-Hamilton, we thus have $\phi^d \in I\operatorname{End}_R(M)$, and so $\tilde{\phi}^d = 0$. Since $\mathcal{T} \otimes_R R'$ is Noetherian, it follows that the kernel of $\mathcal{T} \otimes_R R' \to \mathcal{T}'$ is nilpotent.

In general, any morphism $R \to R'$ may be factorized at $R \to R'' \to R'$ with $R \to R''$ flat and $R'' \to R'$ surjective. By the second assumption, the map $\mathcal{T} \otimes_R R'' \to \mathcal{T}''$ is an isomorphism and by the first in the surjective case, the map $\mathcal{T} \otimes_R R' = \mathcal{T}'' \otimes_{R''} R' \to \mathcal{T}'$ has nilpotent kernel. $\qquad \square$

EXERCISE I.4.2. Let $R = \mathbb{Z}_p$, $M = R^2$, $T \in \mathrm{End}_R(M)$ given by the matrix $\begin{pmatrix} 1 & p \\ 0 & 1 \end{pmatrix}$. Describe $\mathcal{T}$ in this case. Set $R' = \mathbb{F}_p$. Describe $\mathcal{T}'$ and the morphism $\mathcal{T} \otimes \mathbb{F}_p \to \mathcal{T}'$ in this case.

PROPOSITION I.4.3. *If M is flat as a $\mathcal{T}$-module, then the map (2) is an isomorphism for any R-algebra R'. In particular, this holds if $\mathcal{T}$ is étale over R.*

Proof — If M is flat as a $\mathcal{T}$-module, that is locally free, then $M' = M \otimes_R R'$ is locally free over $\mathcal{T} \otimes_R R'$ and in particular, $\mathcal{T} \otimes_R R'$ acts faithfully on M'. It follows that (2) is an isomorphism. For the second assertion, we just use that if M is a finite $\mathcal{T}$-module and $\mathcal{T}$ an étale finite R-algebra, then M is flat over $\mathcal{T}$ if and only if it is flat over R. This is true because flatness of a module over a ring may be checked on the strict henselianzations of this ring at all prime ideals, and those are the same for $\mathcal{T}$ and R. $\qquad\square$

I.5. Eigenalgebras over a field

In this section we assume that R is a field k.

I.5.1. Structure of Spec $\mathcal{T}$. Since $\mathcal{T}$ is a finite algebra over k, it is an Artinian semi-local ring: $\mathcal{T}$ has only finitely many prime ideals, which are maximal as well, say $\mathfrak{m}_1, \ldots, \mathfrak{m}_l$ and $\mathcal{T}$ is canonically the product of the local Artinian k-algebras $\mathcal{T}_{\mathfrak{m}_i}$. We shall set $M_{\mathfrak{m}_i} = M \otimes_{\mathcal{T}} \mathcal{T}_{\mathfrak{m}_i}$ so that we have a decomposition $M = \oplus_{i=1}^l M_{\mathfrak{m}_i}$. This decomposition is stable by the action of $\mathcal{T}$, each $\mathcal{T}_{\mathfrak{m}_i}$ acting by 0 on the summands other than $M_{\mathfrak{m}_i}$. Since $\mathcal{T}$ acts faithfully on M, $\mathcal{T}_{\mathfrak{m}_i}$ acts faithfully on $M_{\mathfrak{m}_i}$ and in particular those subspace are non-zero. (See e.g. [**E**, §2.4] if any of those results is not clear)

If we write $M[I]$ for the subspace of M of elements killed by an ideal I of $\mathcal{T}$, then $M_{\mathfrak{m}_i}$ can be canonically identified with $M[\mathfrak{m}_i^\infty] := \cup_{n \in \mathbb{N}} M[\mathfrak{m}_i^n]$ which since M is Noetherian is the same as $M[\mathfrak{m}_i^n]$ for n large enough. Indeed, $M[\mathfrak{m}_i^\infty] = \oplus_j M_{\mathfrak{m}_j}[\mathfrak{m}_i^\infty]$ and $M_{\mathfrak{m}_j}[\mathfrak{m}_i^\infty] = 0$ if $j \neq i$ since $M_{\mathfrak{m}_j}$ has a finite composition series with factors $R/\mathfrak{m}_j$ on which for any n some elements of $\mathfrak{m}_i^n$ acts invertibly, and $M_{\mathfrak{m}_i}[\mathfrak{m}_i^\infty] = M_{\mathfrak{m}_i}$ since $M_{\mathfrak{m}_i}$ has a finite composition series with factors $R/\mathfrak{m}_i$). In particular, $M[\mathfrak{m}_i^\infty]$ is non-zero, from which we easily deduce by descending induction that $M[\mathfrak{m}_i]$ is non-zero.

I.5.2. System of eigenvalues, eigenspaces and generalized eigenspaces. We recall here some basic definitions from linear algebra:

DEFINITION I.5.1. A vector $v \in M$ is a common *eigenvector* (resp. *generalized eigenvector*) for $\mathcal{H}$ if for every $T \in \mathcal{H}$, there exists a scalar $\chi(T) \in k$ (resp. and an integer n) such that $\psi(T)v = \chi(T)v$ (resp. $(\psi(T) - \chi(T)\mathrm{Id})^n v = 0$).

Note that if $v \neq 0$ is an eigenvector or a generalized eigenvector, the scalar $\chi(T)$, called the *eigenvalue* or *generalized eigenvalue*, is well-determined, and the map $T \mapsto \chi(T)$, $\mathcal{H} \to k$ is a character (that is a morphism of algebra); we then say that v is an *eigenvector* (resp. *generalized eigenvector*) *for* the character χ. By convention, we shall say that 0 is an eigenvector for any character $\chi : \mathcal{H} \to k$.

DEFINITION I.5.2. If $\chi : \mathcal{H} \to k$ is a character, the set of $v \in M$ that are eigenvectors (resp. generalized eigenvectors) for χ is a vector space called the *eigenspace of χ* and denoted $M[\chi]$ (resp. the generalized eigenspace of χ denoted $M_{(\chi)}$).

DEFINITION I.5.3. A character $\chi : \mathcal{H} \to k$ (or what amounts to the same, $\mathcal{H} \otimes k \to k$) is said to be a *system of eigenvalues appearing in M* if $M[\chi] \neq 0$, or equivalently $M_{(\chi)} \neq 0$.

EXERCISE I.5.4. If k is algebraically closed, show that we have a decomposition $M = \oplus_\chi M_{(\chi)}$ where χ runs among the finite set of all systems of eigenvalues of $\mathcal{H}$ appearing in M.

EXERCISE I.5.5. Let k' be an extension of k and $M' = M \otimes k'$. Let $\chi : \mathcal{H} \to k$ be a character and $\chi' = \chi \otimes 1 : \mathcal{H} \otimes k' \to k'$. Show that $M[\chi] \otimes k' = M'[\chi']$ and $M_{(\chi)} \otimes k' = M'_{(\chi')}$.

EXERCISE I.5.6. Prove that $M[\chi] \neq 0$ if and only if $M_{(\chi)} \neq 0$, as asserted in Definition I.5.3.

EXERCISE I.5.7. Let $0 \to K \to M \to N \to 0$ be an exact sequence of k-vector spaces with actions of $\mathcal{H}$, and let $\chi : \mathcal{H} \to k$ be a character. Show that the sequence $0 \to K_{(\chi)} \to M_{(\chi)} \to N_{(\chi)} \to 0$ is exact, but not necessarily the sequence $0 \to K[\chi] \to M[\chi] \to N[\chi] \to 0$.

EXERCISE I.5.8. Show that $M_{(\chi)}^\vee$ is isomorphic as $\mathcal{H}$-module with $(M^\vee)_{(\chi)}$ where $M^\vee$ is defined as in Exercise I.3.4, but that in general we do not have $\dim M[\chi] = \dim(M^\vee)[\chi]$

I.5.3. Systems of eigenvalues and points of $\operatorname{Spec} \mathcal{T}$.

THEOREM I.5.9. *Let $\chi : \mathcal{H} \to k$ be a character. Then χ is a system of eigenvalues appearing in M if and only if χ factors as a character $\mathcal{T} \to k$. Moroever, if this is the case, then denoting $\mathfrak{m}$ the kernel of $\chi : \mathcal{T} \to k$ one has*

$$M[\chi] = M[\mathfrak{m}] \quad and \quad M_{(\chi)} = M_\mathfrak{m}.$$

Proof — If χ is a system of eigenvalues appearing in M, and v a non-zero eigenvector for χ, the relation $\psi(T)v = \chi(T)v$ shows that $\chi(T)$ depends only of $\psi(T)$, that is that χ factors through $\mathcal{T}$.

Conversely, let $\chi : \mathcal{T} \to k$ be a character, and let $\mathfrak{m}$ be its kernel, which is a maximal ideal of $\mathcal{T}$. We see χ as a character $\chi : \mathcal{H} \to k$, by precomposition with ψ. The ideal $\mathfrak{m}$ of $\mathcal{T}$ is generated, as a k-vector space, by the elements $\psi(T) - \chi(T)1_{\mathcal{T}}$ for $T \in \mathcal{H}$ (indeed, an element m of $\mathfrak{m}$ can be written as a finite sum $m = \sum \lambda_r \psi(T_r)$ for $T_r \in \mathcal{H}$, and since $\chi(m) = 0$, we get $\sum \lambda_r \chi(T_r) = 0$, so $m = \sum \lambda_r (\psi(T_r) - \chi(T_r)1_{\mathcal{T}})$.) It follows that $M[\chi] = M[\mathfrak{m}]$ and that $M_{(\chi)} = M_{\mathfrak{m}}$. Moreover, since $M_{\mathfrak{m}} \neq 0$ (cf. §I.5.1), $M_{(\chi)} \neq 0$ and χ is a system of eigenvalues appearing in M.   □

We can now describe the k'-points of the eigenalgebra $\operatorname{Spec}\mathcal{T}$, for any extension k' of k:

COROLLARY I.5.10. *For any field k' containing k, we have a natural bijection between $(\operatorname{Spec}\mathcal{T})(k')$ and the set of systems of eigenvalues of $\mathcal{H}$ that appear in $M \otimes_k k'$.*

Proof — When $k' = k$, the natural bijection is implicit in the preceding theorem: a system of eigenvalues of $\mathcal{H}$ appearing in M corresponds bijectively to a character $\mathcal{T} \to k$, that is a point of $(\operatorname{Spec}\mathcal{T})(k)$. In general, the same result applied to k' gives a bijection between the set of systems of eigenvalues of $\mathcal{H}$ appearing in $M \otimes_k k'$ and $(\operatorname{Spec}\mathcal{T}')(k')$, where $\mathcal{T}'$ is the eigenalgebra of $\mathcal{H}$ acting on $M \otimes_k k'$, but $\operatorname{Spec}\mathcal{T}'(k') = \operatorname{Spec}\mathcal{T}(k')$ by Proposition I.4.1.   □

REMARK I.5.11. This result explains and justifies the name *eigenalgebra*.

COROLLARY I.5.12. *Let $\bar{k}$ be an algebraic closure of k, and write $G_k = Aut(\bar{k}/k)$. There is a natural bijection between $\operatorname{Spec}\mathcal{T}$ and the set of G_k-orbits of characters $\chi : \mathcal{H} \to \bar{k}$ appearing in $M \otimes \bar{k}$.*

Proof — If k is algebraically closed, since $\mathcal{T}$ is finite over k, there is a natural bijection between $\operatorname{Spec}\mathcal{T}$ and $(\operatorname{Spec}\mathcal{T})(k)$. In the case of a general field k, the result follows because $\operatorname{Spec}\mathcal{T}$ is in natural bijection with $\mathcal{T}(\bar{k})^{\mathbb{G}_k}$.   □

Of course, if k is perfect, then G_k is just the absolute Galois group of k.

COROLLARY I.5.13. *The algebra $\mathcal{T}$ is étale over k if and only if $\mathcal{H}$ acts semi-simply on $M \otimes_k \bar{k}$. If this holds, one has $\dim_k \mathcal{T} \leq \dim_k M$.*

Proof — Let $\mathcal{T}_{\bar{k}}$ be the eigenalgebra generated by $\mathcal{H}$ on $M \otimes_k \bar{k}$. By Prop. I.4.1, $\mathcal{T}_{\bar{k}} = \mathcal{T} \otimes_k \bar{k}$, hence $\mathcal{T}_{\bar{k}}$ is étale over $\bar{k}$ if and only if $\mathcal{T}$ is étale over k. So we may assume that k is algebraically closed.

Then, $\mathcal{T}$ is étale if and only if for every maximal ideal $\mathfrak{m}$ of $\mathcal{T}$, $\mathcal{T}_{\mathfrak{m}} = k$. The later condition is equivalent to $M[\mathfrak{m}] = M_{\mathfrak{m}}$ for every maximal ideal $\mathfrak{m}$ of $\mathcal{T}$ (see §I.5.1) hence using Theorem I.5.9, to $M[\chi] = M_{(\chi)}$ for every system of eigenvalues

appearing in M, which (using Exrecise I.5.4) is equivalent to $\mathcal{H}$ acting semi-simply on M. Moreover, if this holds, $\mathcal{T} = \prod_{\mathfrak{m}} \mathcal{T}_m = k^r$ where r is the number of systems of eigenvalues appearing in M, hence clearly $\dim_k \mathcal{T} = r \leq \dim_k M$.   □

Remember that $\mathcal{T}$ is étale over k is equivalent to $\mathcal{T}$ being a finite product of fields that are finite separable extensions of k. Remember also that when k is perfect, $\mathcal{H}$ acts semi-simply on $M \otimes_k \bar{k}$ if and only if it acts semi-simply on M.

REMARK I.5.14. When $\mathcal{H}$ does not act semi-simply on $M \otimes_k \bar{k}$, the dimension of $\mathcal{T}$ may be larger than the dimension of M. An old result of Schur ([**3**] and [**2**] for a simple proof) states that the maximal possible dimension of $\mathcal{T}$ is $1 + \lfloor (\dim_k M)^2/4 \rfloor$.

EXERCISE I.5.15. Show that there exist commutative subalgebras of $\mathrm{End}_k(M)$ of that dimension.

I.6. The fundamental example of Hecke operators acting on a space of modular forms

The motivating example of the theory above is the action of Hecke operators on spaces of modular forms. We assume that the reader is familiar with the basic theory of modular forms as exposed in many textbooks, e.g. [**Shi**], [**Mi**], or [**D**], but we nevertheless recall the definitions and main results that we will use.

I.6.1. Complex modular forms and diamond operators. We recall the standard action of $\mathrm{GL}_2^+(\mathbb{Q})$ (the $+$ indicates matrices with positive determinant) on the Poincaré upper half-plane $\mathcal{H}$:

$$\gamma \cdot z = (az + b)/(cz + d) \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \; z \in \mathcal{H},$$

and the standard right-action of weight k on the space of functions on $\mathcal{H}$:

(3) $$f_{|_k \gamma}(z) = (\det \gamma)^{k-1}(cz + d)^{-k} f(\gamma \cdot z).$$

Let $k \geq 0$ be an integer and let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, that is a subgroup containing all matrices congruent to Id mod N for a certain integer $N \geq 1$. A *modular form of weight k and level Γ* is an holomorphic function on $\mathcal{H}$ invariant by Γ for that action, satisfying a condition of holomorphy at cusps of $\mathcal{H}/\Gamma$ that we shall not recall here. A modular form is *cuspidal* if it vanishes at all the cusps. We shall denote by $M_k(\Gamma)$ (resp. $S_k(\Gamma)$) the complex vector space of modular forms (resp. cuspidal modular forms) of weight k and level Γ.

The fundamental examples of congruence subgroups are, for $N \geq 1$ an integer, the subgroup $\Gamma_0(N)$ of matrices which are upper-triangular modulo N, and its normal subgroup $\Gamma_1(N)$ of matrices which are unipotent modulo N. Actually any congruence subgroup is conjugate to another one which contains a $\Gamma_1(N)$, so these congruences subgroups are in some sense universal.

Let us identify the quotient $\Gamma_0(N)/\Gamma_1(N)$ with $(\mathbb{Z}/N\mathbb{Z})^*$ by sending a matrix in $\Gamma_0(N)$ to the reduction modulo N of its upper-left coefficient. Then the spaces $M_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N))$ have a natural action of the group $(\mathbb{Z}/N\mathbb{Z})^* = \Gamma_0(N)/\Gamma_1(N)$. We denote by by $\langle a \rangle$ the action of $a \in (\mathbb{Z}/N\mathbb{Z})^*$. For $\epsilon : (\mathbb{Z}/\mathbb{N}\mathbb{Z})^* \to \mathbb{C}^*$ any character, which we shall call a *nebentypus* in this context, we denote by $M_k(\Gamma_1(N), \epsilon)$ the common eigenspace in $M_k(\Gamma)$ for the Diamond operators with system of eigenvalues ϵ, and similarly for S_k. We obviously have

$$M_k(\Gamma_1(N)) = \oplus_\epsilon M_k(\Gamma_1(N), \epsilon)$$

where ϵ runs among the set of nebentypus, and

$$M_k(\Gamma_1(N), 1) = M_k(\Gamma_0(N)).$$

Similar results hold for S_k.

Finally, we call $\mathcal{E}_k(\Gamma_1(N))$ the submodule generated by the Eisenstein series (specifically by the forms $E_{k,\chi,\psi,t}$ as in Proposition I.6.8 below). We have

$$M_k(\Gamma_1(N)) = S_k(\Gamma_1(N)) \oplus \mathcal{E}_k(\Gamma_1(N)).$$

I.6.2. Hecke operators. Since $M_k(\Gamma_1(N))$ is defined as the space of invariants under $\Gamma_1(N)$ in a $\mathrm{GL}_2^+(\mathbb{Q})$-module, it is acted upon by the the Hecke operators $[\Gamma_1(N)g\Gamma_1(N)]$ for $g \in \mathrm{GL}_2^+(\mathbb{Q})$ (see [**Shi**]). For example, the operator $[\Gamma_1(N)g\Gamma_1(N)]$ when g is a matrix in $\Gamma_0(N)$ whose upper-left coefficient is a is the diamond operator $\langle a \rangle$ defined above. More important are the operators T_n defined by

$$(4) \qquad\qquad T_n = [\Gamma_1(N) \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(N)]$$

for any integer $n \geq 1$. Those operators commute with each other and with the Diamond operators. In particular they stabilize the spaces $M_k(\Gamma_1(N), \epsilon)$. They also stabilize the subspaces of cuspidal forms $S_k(\Gamma_1(N))$ and $S_k(\Gamma_1(N), \epsilon)$, and of Eisenstein series $\mathcal{E}_k(\Gamma_1(N))$ and $\mathcal{E}_k(\Gamma_1(N), \epsilon)$.

We recall that

$$(5) \qquad\qquad T_n = T_{l_1^{m_1}} \ldots T_{l_r^{m_r}} \text{ if } n = l_1^{m_1} \ldots l_r^{m_r}$$

where the l_i are distinct primes and the m_i are positive integers (in particular $T_1 = \mathrm{Id}$), and that

$$(6) \qquad\qquad T_{l^{m+1}} = T_l T_{l^m} - l^{k-1} \langle l \rangle T_{l^{m-1}}$$

for l a prime, $m \geq 1$. Because of these formulas, we need only for most problems to consider the T_n when n is a prime l.

To emphasize the difference of behavior of T_l when l divides N or not (and for reasons which will become clear below) we shall use the notation U_l instead of T_l

when l divides N. One has

$$\text{(7)} \qquad T_l f \ = \ \sum_{a=0}^{l-1} f_{|_k \left(\begin{smallmatrix} 1 & a \\ 0 & l \end{smallmatrix} \right)} + \langle l \rangle f_{|_k \left(\begin{smallmatrix} l & 0 \\ 0 & 1 \end{smallmatrix} \right)} \text{ when } l \nmid N$$

$$\text{(8)} \qquad U_l f \ = \ \sum_{a=0}^{l-1} f_{|_k \left(\begin{smallmatrix} 1 & a \\ 0 & l \end{smallmatrix} \right)} \text{ when } l \mid N$$

When $N' \mid N$, $\Gamma_1(N) \subset \Gamma_1(N')$ so we have $M_k(\Gamma_1(N')) \subset M_k(\Gamma_1(N))$. When we consider $M_k(\Gamma_1(N'))$ as a space of modular forms on its own, it gets its own Hecke operator T_l for $l \nmid N'$ and U_l for $l \mid N'$ and also $\langle a \rangle$ for a in $\mathbb{Z}$, a coprime to N'. When we considere $M_k(\Gamma_1(N'))$ as a subspace of $M_k(\Gamma_1(N))$, the formulas above show that it is stable by the T_l, $l \nmid N$, by the U_l, $l \mid N'$ and by the diamond operator $\langle a \rangle$, $(a, N) = 1$, and that each of these operators induces the operator of the same name on $M_k(\Gamma_1(N))$. Note however that when l is a prime dividing N' but not N, the operator T_l on $M_k(\Gamma_1(N))$ does not necessarily stabilize $M_k(\Gamma_1(N'))$.

I.6.3. A brief reminder of Atkin-Lehner's theory. We fix an integer $N \geq 1$. We shall denote by $\mathcal{H}$ (or by $\mathcal{H}(N)$ when this precision is useful) the polynomial ring over $\mathbb{Z}$ in infinitely many variables with names T_l for $l \nmid N$, U_l for $l \mid N$ and $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$. We let $\mathcal{H}$ acts on $M_k(\Gamma_1(N))$ by letting each variable acts by the Hecke operator of the same name. This action stabilizes $S_k(\Gamma_1(N))$ and $\mathcal{E}_k(\Gamma_1(N))$. We shall denote by $\mathcal{H}_0$ (or by $\mathcal{H}_0(N)$) the subring generated by the variables T_l for $l \nmid N$ and $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$. So $\mathcal{H}_0$ acts on $M_k(\Gamma_1(N))$ as well, with the advantage that its action stabilize all the subspaces $M_k(\Gamma_1(N'))$ for $N' \mid N$ (cf. the discussion at the end of the preceding subsection).

DEFINITION I.6.1. Let $\lambda : \mathcal{H}_0 \to \mathbb{C}$ be a system of eigenvalues appearing in $M_k(\Gamma_1(N))$. We shall say that λ is *new* if λ does not appear in any $M_k(\Gamma_1(N'))$ for N' a proper divisor of N, *old* otherwise.

We shall need an obvious refinement of that definition: if l is a prime factor of N, we shall say that λ is *new at l* if it does not appear in $M_k(\Gamma_1(N/l))$. Obviously a system of eigenvalues is new if and only if it is new at every prime factors of N.

The first fundamental result of Atkin-Lehner is

THEOREM I.6.2. *A system of eigenvalues $\lambda : \mathcal{H}_0 \to \mathbb{C}$ appearing in $M_k(\Gamma_1(N))$ is new if and only if $M_k(\Gamma_1(N))[\lambda]$ has dimension 1.*

When λ is new, there is a unique form $f \in M_k(\Gamma_1(N))[\lambda]$ which is normalized (that is its coefficient a_1 is 1). This form is called the *newform* of the system of eigenvalues λ. Since all the Hecke operators commute, we see that f is also an eigenform for $\mathcal{H}$.

DEFINITION I.6.3. The *system of eigenvalues of E_2* is the morphism $\mathcal{H}_0 \to \mathbb{C}$ that sends T_l for $l \nmid N$ to $1 + l$, and all the $\langle a \rangle$ to 1.

THEOREM I.6.4. *Assume that λ is a system of eigenvalues for $\mathcal{H}_0$ appearing in $M_k(\Gamma_1(N))$, different from the system of eigenvalues of E_2. Then there is a divisor N_0 of N such that for every divisor N' of N, λ appears in $M_k(\Gamma_1(N'))$ if and only if N_0 divides N'. For such an N', the dimension of $M_k(\Gamma_1(N'))[\lambda]$ is $\sigma(N'/N_0)$ where $\sigma(n)$ is the number of divisors of n. If $f(z)$ is a generator of the one-dimensional space $M_k(\Gamma_1(N_0))[\lambda]$, then a basis of $M_k(\Gamma_1(N'))[\lambda]$ is given by the forms $f(dz)$, $d \mid N'/N_0$.*

DEFINITION I.6.5. We call N_0 the *minimal level* of λ.

REMARK I.6.6. Obviously λ is *new* if and only if $N = N_0$. In general, any system of eigenvalues for $\mathcal{H}_0$ appearing in $M_k(\Gamma_1(N))$ can be considered new when seen as a system of eigenvalues appearing in $M_k(\Gamma_1(N_0))$. More precisely, the above theorem says that $M_k(\Gamma_1(N_0))[\lambda]$ has dimension 1, where the eigenspace is for the algebra $\mathcal{H}_0(N)$. The algebra $\mathcal{H}_0(N_0)$ may be bigger; nevertheless since all the Hecke operators commute, it stabilize $M_k(\Gamma_1(N_0))[\lambda]$ which, being of dimension 1, is therefore also an eigenspace for $\mathcal{H}_0(N_0)$ of system of eigenvalues some well-defined extension $\tilde{\lambda}$ of λ to $\mathcal{H}_0(N_0)$, and this system of value is new.

REMARK I.6.7. By definition, the T_l, $l \nmid N_0$ acts on $M_k(\Gamma_1(N))[\lambda]$ by the scalar $\lambda(T_l)$. It is also possible to describe the action of certain of the operators U_l. Let as above f be a generator of $M_k(\Gamma_1(N))[\lambda]$. Then an easy computation gives gives for d any positive divisor of N/N_0:

(9) $$U_l(f(dz)) = f(d/l\ z) \text{ if } l \mid d$$

(10) $$U_l(f(dz)) = (T_l f)(dz) - l^{k-1}\langle l \rangle(f(dlz)) \text{ if } l \nmid dN_0$$

In the last formula, $T_l f$ is to be understood as the action of the operator T_l of $M_k(\Gamma_1(N))$ on f. In other words, $T_l f = \tilde{\lambda}(T_l)f$ where $\tilde{\lambda}$ is defined as in the preceding remark.

Let us briefly indicate where the proof of Theorems **??** and I.6.4. All the results above are well known and due to Atkin and Lehner in the case of cuspidal forms, cf. [**Mi**], or [**D**]. For $\mathcal{E}_{k+2}(\Gamma_1(N))$ they follow easily from the explicit description of all Eisenstein series that can be found in the last chapter of [**Mi**]. We recall this description.

Let χ and ψ be two primitive Dirichlet characters of conductors L and R. We assume that $\chi(-1)\psi(-1) = (-1)^k$. Let

$$E_{k,\chi,\psi}(q) = c_0 + \sum_{m \geq 1} q^m \sum_{n|m}(\psi(n)\chi(m/n)n^{k-1})$$

where $c_0 = 0$ if $L > 1$ and $c_0 = -B_{k,\psi}/2k$ if $L = 1$. If t is a positive integer, let $E_{k,\chi,\psi,t}(q) = E_{k,\chi,\psi}(q^t)$ except in the case $k = 2$, $\chi = \psi = 1$, where one sets $E_{2,1,1,t} = E_{2,1,1}(t) - tE_{2,1,1}(q^t)$.

PROPOSITION I.6.8 (Miyake, Stein). *The series $E_{k,\chi,\psi,t}(q)$ are modular forms of level $\Gamma_1(N)$ and character $\chi\psi$ for all positive integers L, R, t such that $LRt|N$ and all primitive Dirichlet character χ of conductor L and ψ of conductor R, satisfying $\chi(-1)\psi(-1) = \epsilon$ (and $t > 1$ in the case $k = 0$, $\chi = \psi = 1$) and moreover they form a basis of the space $\mathcal{E}_{k+2}(\Gamma_1(N))$.*

For every prime l not dividing N, we have

$$T_l E_{k,\chi,\psi,t} = (\chi(l) + \psi(l)l^{k-1})E_{k,\chi,\psi,t}.$$

Proof — See Miyake ([**Mi**]) for the computations leading to those results, Stein ([**St**]) for the results stated as here. □

From this description it follows easily that

COROLLARY I.6.9. *The Eisenstein series in $M_{k+2}(\Gamma_1(N))$ that are new are exactly: the* normal *new Eisenstein series $E_{k+2,\chi,\psi}$ with $N = LR$ (excepted of course $E_{2,1,1,1}$ which is not even a modular form); the* exceptional *new Eisenstein series $E_{2,1,1,l}$, that we shall denote simply by $E_{2,l}$ when $N = l$ is prime.*

The reader may check as an exercise that all the statements of Atkin-Lehner's theory holds for $\mathcal{E}_{k+2}(\Gamma_1(N))$ (and thus for $M_{k+2}(\Gamma_1(N))$) for a system of eigenvalues λ that is different of the one of E_2. Note that when λ is the system of eigenvalues of E_2, then the minimal level of λ is not well-defined anymore: all prime factors l of N are minimal elements of the set of divisors N' of N such that λ appears in $M_2(\Gamma_1(N'))$.

Let us also note the following important corollary:

COROLLARY I.6.10. *The algebra $\mathcal{H}_0$ acts semi-simply on $M_k(\Gamma_1(N))$.*

Proof — We prove separately that $\mathcal{H}_0$ acts semi-simply on $S_k(\Gamma_1(N))$ and on $\mathcal{E}_k(\Gamma_1(N))$. On $\mathcal{E}_k(\Gamma_1(N))$, Proposition I.6.8 provides a basis of eigenform for $\mathcal{H}_0$, proving the semi-simplicity. On $S_k(\Gamma_1(N))$, there exists a natural Hermitian product, the Peterson inner product, for which the adjoint of the Hecke operator T_l, $l \nmid N$ is $\langle l \rangle T_\ell$ and the adjoint of $\langle a \rangle$ is $\langle a^{-1} \rangle$. It follows that all operators in $\mathcal{H}_0$ commute with their adjoints, hence are diagonalizable, and $\mathcal{H}_0$ acts semi-simply on $S_k(\Gamma_1(N))$. □

I.6.4. Hecke eigenalgebra constructed on spaces of complex modular forms. For a choice of a space $M \subset M_k(\Gamma_1(N))$ which is $\mathcal{H}$-stable, set $\mathcal{T}_0 = \mathcal{T}(\mathbb{C}, M, \mathcal{H}_0, \psi)$ and $\mathcal{T} = \mathcal{T}(\mathbb{C}, M, \mathcal{H}, \psi)$.

The great advantage of $\mathcal{T}_0$ is that it is semisimple. Therefore $\mathcal{T}_0$ is a copy of a certain number r of copies of $\mathbb{C}$, one for each system of eigenvalues appearing in M (see Corollary I.5.13). The problem is that in general we do not have multiplicity 1: if χ is a system of eigenvalues $\mathcal{H}_0 \to \mathcal{T}_0 \to \mathbb{C}$, $M[\chi]$ may have dimension greater

than 1, and, what is worse, depending on χ. Actually, we know that the dimension of that space is $\sigma(N/N_0)$ where N_0 is the minimal level of χ_i, at least when χ_i is not the system of eigenvalues of E_2. So M is not, in general, a free module over $\mathcal{T}_0$.

The operators U_p, $p|N$, acting on the space of forms of level N, are not semi-simple in general (but see exercise I.6.16 for a discussion of when they are). Therefore the algebra $\mathcal{T}$ is not semi-simple in general, that is it may have nilpotent elements. However, we shall see that the multiplicity one principle hold and that the structure of the $\mathcal{T}$-modules M and $M^\vee = \operatorname{Hom}_R(M, R)$ are very simple.

For the latter, there is a simple standard argument.

PROPOSITION I.6.11. *Assume $k > 0$. The pairing $\mathcal{T} \times M \to \mathbb{C}$, $(T, f) \mapsto \langle T, f \rangle = a_1(Tf)$ is a perfect $\mathcal{T}$-equivariant pairing.*

Proof — Recall that a simple standard computation gives, for all modular forms $f \in M$

(11) $a_1(T_n f) = a_n(f)$.

That the pairing $\langle T, f \rangle$ is $\mathcal{T}$-equivariant means that for all $T' \in \mathcal{T}$, we have $\langle T'T, f \rangle = \langle T, T'f \rangle$, and this is obvious since $\mathcal{T}$ is commutative.

If $f \in M$ is such that $\langle T, f \rangle = 0$ for all $T \in \mathcal{T}$, then $a_1(T_n f) = 0$ for every integer $n \geq 1$, so $a_n(f) = 0$ for every $n \geq 1$ by (11) and f is a constant. Since the non-zero constant modular forms are of weight 0, our hypothesis implies $f = 0$.

If $T \in \mathcal{T}$ is such that $\langle T, f \rangle = 0$ for every $f \in M$, then for any given $f \in M$ we have $a_1(TT_n f) = 0$, so $a_n(Tf) = 0$ for all $n \geq 1$ by (11), hence $Tf = 0$ by the same argument as above. Since this is true for all f, and $\mathcal{T}$ acts faithfully on M, $T = 0$. Hence the pairing is perfect.   □

For $k = 0$, if $M = M_k(\Gamma_1(N)) = \mathcal{E}_k(\Gamma_1(N)) = \mathbb{C}$, then $\mathcal{T} = \mathbb{C}$, but in this case the pairing $(T, f) \mapsto a_1(Tf)$ is 0.

COROLLARY I.6.12. *In all cases, $M^\vee$ is free of rank one over $\mathcal{T}$.*

Proof — This follows from the above proposition if $k > 0$, and this is trivial if $k = 0$ since in this case $M = M_0(\Gamma_1(N)) = \mathcal{E}_0(\Gamma_1(N))$, M is of dimension 1, and $\mathcal{T} = \mathbb{C}$, and in the case $M = S_0(\Gamma_1(N))$, $m = 0$ and $\mathcal{T}$ is the zero ring.   □

COROLLARY I.6.13. *The multipicity one principle holds, that is for every character $\chi : \mathcal{H} \to \mathcal{T} \to \mathbb{C}$, we have $\dim M[\chi] = 1$.*

Proof — Let $\mathfrak{m}$ be the maximal ideal of $\mathcal{T}$ corresponding to χ. Since $M^\vee \simeq \mathcal{T}$, one has $M \simeq \mathcal{T}^\vee$ and $M[\chi] = M[\mathfrak{m}] \simeq \mathcal{T}^\vee[\mathfrak{m}] = (\mathcal{T}/\mathfrak{m}\mathcal{T})^\vee = \mathbb{C}$.   □

To study M however, we need the full force of the Atkin-Lehner theory.

THEOREM I.6.14. *Let M be either $S_k(\Gamma_1(N))$, $M_k(\Gamma_1(N))$ or $\mathcal{E}_k(\Gamma_1(N))$. Then M is free of rank one as a $\mathcal{T}$-module.*

Proof — Write $M = \oplus_\lambda M[\lambda]$ when λ runs among the finite number of systems of eigenvalues of $\mathcal{H}_0$ that appear in M. Let $\mathcal{T}_\lambda$ be the eigenalgebra attached to the action of $\mathcal{H}$ on $M[\lambda]$. Then $\mathcal{T} = \prod_\lambda \mathcal{T}_\lambda$ and the action of $\mathcal{T}$ on M is the product of the action of $\mathcal{T}_\lambda$ over $M[\lambda]$ (cf. Exercise I.3.3). Hence it is enough to prove that $M[\lambda]$ is free of rank one over $\mathcal{T}_\lambda$. By the corollary above, we know that $\dim M[\lambda] = \dim \mathcal{T}_\lambda$. Therefore it suffices to prove that $M[\lambda]$ is generated by one element over $\mathcal{T}_\lambda$.

Assume first that λ is not the system of eigenvalues of E_2. We use Atkin-Lehner's theory: let N_0 be the minimal level of the character λ of $\mathcal{H}_0$, and f a generator of $M(N_0)[\lambda]$. Then $f\left(\frac{N}{N_0}z\right)$ generates $M(N_0)[\chi]$ under $\mathcal{H}$ since for $d|N/N_0$, writing $N/(N_0 d) = l_1^{a_1} \dots l_m^{a_m}$, we have $f(dz) = U_{l_1}^{a_1} \dots U_{l_m}^{a_m} f\left(\frac{N}{N_0}z\right)$ bu (9) and these forms $f(dz)$ generate $M[\lambda]$ as a vector space (Theorem I.6.4).

The case where λ is the system of eigenvalues of E_2 deserves a special treatment. For $d \mid N$ write $F_d(z) = E_2(z) - dE_2(dz)$, so that $F_1 = 0$ and the forms F_d for $d \mid N$, $d \neq 1$ form a basis of $M[\lambda]$. For a prime $l \mid N$, we have

(12) $$U_l F_d = F_l + l F_{d/l} \text{ if } l \mid d$$

(13) $$U_l F_d = (l+1)F_d - l F_{dl} \text{ if } l \nmid d$$

Choose a prime factor l of N. For $d \mid N$, $d = l^n d'$ with $l \nmid d'$, we have

$$U_{l^k} F_{d'l^n} = l^k F_{d'l^{n-k}} + \frac{l^k - 1}{l - 1} F_l, \text{ for } k = 0, 1, \dots, n$$

by an easy induction using (12), hence in particular $U_{l^n} F_{d'l^n} = l^n F_{d'} + \frac{l^n - 1}{l - 1} F_l$. Applying (13) we get

$$U_{l^{n+1}} F_{d'l^n} = l^n(l+1)F_{d'} - l^{n+1}F_{d'l} + \frac{l^n - 1}{l - 1} F_l.$$

We claim that the forms $F_{d'l^i}$, for $i = 0, \dots, n$, all belong to the module generated by F_d under U_l. To prove the claim, consider first the case $d' = 1$. In this case, the square matrix expressing the n vectors $U_{l^k} F_{l^n}$ for $k = 0, 1, \dots, n-1$ for $k = 0, \dots, n-1$ in terms of the n independent vectors $F_{l^{n-k}}$ is

$$\begin{pmatrix} 1 & & & & \\ & l & & & \\ & & \ddots & & \\ & & & l^{n-2} & \\ 0 & 1 & \dots & \frac{l^{n-2}-1}{l-1} & \frac{l^n-1}{l-1} \end{pmatrix}.$$

As this matrix is invertible, the claim follows in the case $d' = 1$. When $d' \neq 1$ the square matrix of size $n + 2$ expressing the family of $n + 2$ vectors $U_{l^k} F_{d'l^n}$ for

$k = 0, 1, \ldots, n, n+1$ in term of the family of $n+2$ vectors which consists in $F_{d'l^{n-k}}$ for $k = 0, 1, \ldots, n$ and F_l is

$$
\begin{pmatrix}
1 & & & & & \\
 & l & & & & \\
 & & \ddots & & & \\
 & & & l^{n-1} & & -l^{n+1} \\
 & & & & l^n & l^n(l+1) \\
0 & 1 & \frac{l^2-1}{l-1} & \cdots & \frac{l^n-1}{l-1} & \frac{l^n-1}{l-1}
\end{pmatrix}.
$$

By computing the determinant of this matrix (which is $l^{n(n+1)/2+1}$), one sees that it is invertible and the claims follows in the remaining case.

By induction on the number of prime factors of N, we deduce that all F_d for $d \mid N$ are in the $\mathcal{H}$-module generated by F_N, hence F_N generates the $\mathcal{H}$-module $M[\lambda]$. □

COROLLARY I.6.15. *Let M be as in the above theorem. We have $M \simeq M^\vee \simeq \mathcal{T}$ as an $\mathcal{H}$-module. The eigenalgebra $\mathcal{T}$ is a Gorenstein $\mathbb{C}$-algebra.*

The fact that $\mathcal{T}$ is Gorenstein is a serious restriction on its possible structure. For example $\mathbb{C}[X]/X^2$ or $\mathbb{C}[X,Y]/(X^2, Y^2)$ are Gorenstein, but $\mathbb{C}[X,Y]/(X^2, Y^2, XY)$ is not. For an introduction to Gorenstein rings, see [**E**, Chapter 21].

EXERCISE I.6.16. Let λ be a system of eigenvalues of $\mathcal{H}_0$ different of the systen of E_2 that appears on $M_k(\Gamma_1(N))$ and has minimal level N_0. Let f be a generator of the one-dimensional vector space $M_k(\Gamma_1(N_0))[\lambda]$.

For l a prime dividing N, denote by a_l and ϵ_l the eigenvalues of T_l and $\langle l \rangle$ on f in the case $l \nmid N_0$; denote by u_l the eigenvalue of U_l on f in the case $l \mid N_0$.

1.– First assume that $N/N_0 = l^{m_l}$ for $m_l \in \mathbb{N}$. Show that:

(i) If $l \nmid N_0$, U_l acts semi-simply on $M[\lambda]$ if and only if either $m_l = 0$, or $m_l = 1, 2$ and the equation $X^2 - a_l X + l^{k-1}\epsilon_l = 0$ has distinct roots.

(ii) If $l \mid N_0$, U_l acts semi-simply on $M[\lambda]$ if and only if either $m_l = 0$, or $m_l = 1$ and $u_l \neq 0$.

2.– In general, show that $\mathcal{H}$ acts semi-simply on $M[\lambda]$ if and only if, for all prime factor l on N/N_0, with m_l defined so that l^{m_l} is the maximal power of l that divides N/N_0, the same condition for the semi-simplicty of U_l given above holds.

I.6.5. Eigenalgebras and Galois representations. Instead of working over $\mathbb{C}$, we can work rationally. Namely, if K is a subfield of $\mathbb{C}$, we write $M_k(\Gamma_1(N), K)$ for the K-subspace of $M_k(\Gamma_1(N))$ of forms that have a q-expansion at ∞ with coefficients in K, and we define similarly $S_k(\Gamma_1(N), K)$, etc. Those spaces are stable by the Hecke operators, and, provided that K contains the image of ϵ in the case of a space of modular forms with Nebentypus ϵ, their formations commute

with base change $K \subset K'$ for subfields of $\mathbb{C}$ (see [**Shi**]). Hence we can define more generally, if K is any field of characteristic 0, $M_k(\Gamma_1(N), K)$ as $M_k(\Gamma_1(N)) \otimes_{\mathbb{Q}} K$ and $S_k(\Gamma_1(N), K)$ as $S_k(\Gamma_1(N)) \otimes_{\mathbb{Q}} K$.

Let us call M_K any of these K-vector spaces, and M the corresponding $\mathbb{C}$-vector space. Then M_K is stable by $\mathcal{H}$ and $M_K \otimes_K \mathbb{C} = M$. Hence we can define K-algebras $\mathcal{T}_{0,K}$ and $\mathcal{T}_K$ using M_K instead of M, and we have $\mathcal{T}_{0,K} \otimes_K \mathbb{C} = \mathcal{T}_0$, $\mathcal{T}_K \otimes_K \mathbb{C} = \mathcal{T}$. Hence we see easily by descent that the results we proved above also holds over K (semi-simplicity of $\mathcal{T}_{0,K}$, freeness of rank one of M_K over $\mathcal{T}_K$, Gorensteinness of $\mathcal{T}_K$, ...)

Let us assume that $k \geq 2$, p a prime number. A fundamental theorem of Eichler-Shimura and Deligne states the existence of Galois representations attached to eigenform in $M_k(\Gamma)$:

THEOREM I.6.17. *Let f be a normalized eigenform (for $\mathcal{H}_0$) in $M_k(\Gamma_1(N), K)$, where K is any finite extension of $\mathbb{Q}_p$. Then there exists a unique semi-simple continuous Galois representation $\rho_f : G_{\mathbb{Q}, Np} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ such that for all prime number l not dividing Np, $\mathrm{tr}\, \rho(\mathrm{Frob}_l) = a_l$. Here $G_{\mathbb{Q}, Np}$ is the Galois group of the maximal extension of $\mathbb{Q}$ unramified outside Np and Frob_l is any element in the conjugacy class of Frobenius at l.*

Moreover, we also have $\mathrm{tr}\, \rho(c) = 0$ where c is the complex conjugation.

The case $k = 2$ is due to Eicher-Shimura. A modern reference is [**D**] ([**DDT**] also contains a useful sketch). The case $k > 2$ is due to Deligne ([**De**]).

COROLLARY I.6.18. *Let K be any finite extension of $\mathbb{Q}_p$. There exists a unique continuous pseudocharacter[2] of dimension 2*

$$\tau : G_{\mathbb{Q}, Np} \to \mathrm{GL}_2(\mathcal{T}_{0,K})$$

sending Frob_l (for $l \neq Np$) to T_l. Here $\mathcal{T}_{0,K}$ is provided with its natural topology as a finite K-vector space. We also have $\tau(c) = 0$ if c is the complex conjugation.

Proof — The key remark is that it is enough to prove the corollary for K replaced by a finite extension K' of K. Indeed, note that $\mathcal{T}_{0,K}$ is a closed subspace in $\mathcal{T}_{0,K} = \mathcal{T}_{0,K} \otimes_K K'$. So if we have a pseudocharacter as above $\tau : G_{\mathbb{Q}, Np} \to \mathcal{T}_{0,K'}$, since τ sends a dense subset (the Frob_l's) into $\mathcal{T}_{0,K}$ (because $T_l \in \mathcal{T}_{0,K}$), its image is in $\mathcal{T}_{0,K}$. Thus τ can be seen as a pseudocharacter $G_{\mathbb{Q}, Np} \to \mathcal{T}_K$ which obviously satisfies the desired property.

Now if K is large enough, $\mathcal{T}_{0,K} = K^r$, where every factor corresponds to a system of eigenvalues of $\mathcal{H}_0$ appearing in $M_k(\Gamma_1(N), K)$. Let $\chi_i : \mathcal{H} \to \mathcal{T}_{0,K} = K^r \xrightarrow{\text{projection on the } i\text{-th component}} K$ be one of those systems. There exists an eigenform f_i in $M_k(\Gamma_1(N), K)$ such that $\psi(T)f_i = \chi_i(T)f_i$ for every $T \in \mathcal{H}_0$. The

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[2]For the definition of a pseudocharacter, see [**Ro**]. For an introduction to pseudocharacters, see [**B2**].

theorem of Eichler-Shimura and Deligne attaches to f_i a continuous Galois representation $\rho_i : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(K)$ such that $\mathrm{tr}\,\rho_i(\mathrm{Frob}_l) = \chi_i(T_l)$. Therefore, the product $\rho = \prod_{i=1}^r \rho_i : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(K^r) = \mathrm{GL}_2(\mathcal{T}_K)$ is a representation whose character $\tau := \mathrm{tr}\,\rho$ satisfies the required properties. $\qquad\qquad\qquad\square$

I.7. Eigenalgebras over discrete valuation rings

The study of Hecke algebras over a discrete valuation ring R is important to get a better understanding of the general case and is fundamental for the applications to number theory. Hecke algebras over a DVR are the framework in which the the proofs of the Taniyama-Weil conjecture, of the Serre conjecture, of most cases of the Fontaine-Mazur were developped. Actually, in these applications, the discrete valuation ring R is also complete, and this hypothesis simplifies somewhat the theory (cf. §**??**). However, the theory for a general DVR is only slightly harder, and we expose it first, in §I.7.1 and §I.7.2. After a brief discussion of the Deligne-Serre's lemma, which in this point of view is just a simple consequence of the general structure theory of Hecke algebra, we give an exposition of the theory of congruences over a DVR.

We now fix some notations and terminology for all this section: R is a discrete valuation ring. So R is a domain, is principal and local, and has exactly two prime ideals, the maximal ideal $\mathfrak{m}$ and the minimal ideal (0). As usual, we refer to the two corresponding points of $\mathrm{Spec}\,R$ as the *special* point and the *generic* point. A uniformizer of R is chosen and denoted by π. We call $k = R/\mathfrak{m}$ the residue field and K the fraction field of R. As in §I.2, M is a projective module of finite type over R, that is a free module of a module of finite rank since R is principal. We write $M_K := M \otimes_R K$ and $M_k := M \otimes_R k = M/\mathfrak{m}M$. We suppose given a commutative ring $\mathcal{H}$ and a map $\psi : \mathcal{H} \to \mathrm{End}_R(M)$, and we write $\mathcal{T}$, $\mathcal{T}_K$ and $\mathcal{T}_k$ the Hecke algebras constructed on M, M_K and M_k.

I.7.1. Closed and non-closed points of $\mathrm{Spec}\,\mathcal{T}$. In this §, we describe and interpret as systems of eigenvalues the points of $\mathrm{Spec}\,\mathcal{T}$, in other words the prime ideals of $\mathcal{T}$. We shall distinguish between the closed points of $\mathrm{Spec}\,\mathcal{T}$, which are the maximal ideals of $\mathcal{T}$, and the non-closed points, which are the prime ideals of $\mathcal{T}$ that are not maximal.

PROPOSITION I.7.1. *A point of $\mathrm{Spec}\,\mathcal{T}$ is closed (resp. non-closed) if and only if by the structural map $\mathrm{Spec}\,\mathcal{T} \to \mathrm{Spec}\,\mathcal{R}$ it is sent to the special point (resp. the generic point) of $\mathrm{Spec}\,R$. The natural map $\mathrm{Spec}\,\mathcal{T}_k \to \mathrm{Spec}\,\mathcal{T}$ (resp. $\mathrm{Spec}\,\mathcal{T}_K \to \mathrm{Spec}\,\mathcal{T}$) induces a bijection between $\mathrm{Spec}\,\mathcal{T}_k$ (resp. $\mathrm{Spec}\,\mathcal{T}_K$) and the set of closed (resp. non-closed) points of $\mathrm{Spec}\,\mathcal{T}$. Every irreducible component of $\mathrm{Spec}\,\mathcal{T}$ contains at least one closed point and exactly one non-closed point, and every non-closed point is contained in a unique irreducible component, namely its closure.*

Proof — The algebra $\mathcal{T}$ is finite, hence integral, over R, so the so-called *incomparability of prime ideals* apply (cf. [**E**, Cor 4.18]): If $\mathfrak{p} \subset \mathfrak{p}'$ are two disjoint primes in $\mathcal{T}$, then $\mathfrak{p} \cap R \neq \mathfrak{p}' \cap R$. Since $\mathfrak{p} \cap R$ and $\mathfrak{p}' \cap R$ are obviously primes of R, this means that $\mathfrak{p} \cap R = (0)$ and $\mathfrak{p} \cap R' = \mathfrak{m}$. It follows that the prime ideals $\mathfrak{p}$ of $\mathcal{T}$ such that $\mathfrak{p} \cap R = (0)$ (resp. $\mathfrak{p} \cap R = \mathfrak{m}$) are minimal prime ideals (resp. maximal ideals) in $\mathcal{T}$. On the other hand, since R is a discrete valuation ring, $\mathcal{T}$ is free, hence flat, over R, and the going-down lemma holds (cf. [**E**, Lemma 10.11]): for every prime ideal $\mathfrak{p}'$ of $\mathcal{T}$ such that $\mathfrak{p}' \cap R = \mathfrak{m}$, there exists a prime $\mathfrak{p}$ of $\mathcal{T}$, contained in $\mathfrak{p}'$, such that $\mathfrak{p} \cap R = (0)$. It follows that no maximal ideal of $\mathcal{T}$ is a minimal prime ideal. Thus we have the following equivalences: $\mathfrak{p} \cap R = (0)$ if and only if $\mathfrak{p}$ is a minimal prime ideal of $\mathcal{T}$; $\mathfrak{p} \cap R = \mathfrak{m}$ if and only if $\mathfrak{p}$ is a maximal ideal of $\mathcal{T}$. Translated into geometric terms, this gives the first sentence of the proposition. The second sentence follows immediately since by Proposition I.4.1 the points of $\mathrm{Spec}\,\mathcal{T}_k$ (resp $\mathrm{Spec}\,\mathcal{T}_K$) are the same as the points of the special fiber (resp. of the generic fiber) of $\mathrm{Spec}\,\mathcal{T} \to \mathrm{Spec}\,R$. For the last sentence, it is enough to recall that the irreducible components are the closed subsets of $\mathrm{Spec}\,\mathcal{T}$ corresponding to the minimal prime ideals. $\square$

One can reformulate the proposition using the simple concept of *multi-valued map*. If A and B are two sets, by a *multi-valued map from A to B*, denoted $\tilde{f} : A \rightsquigarrow B$, we shall mean a map f in the ordinary sense from A to the set of non-empty parts of B. We shall say that the multi-valued map $\tilde{f}$ is surjective if $\cup_{a \in A}\tilde{f}(a) = B$, and we shall denote such a map $\tilde{f} : A \rightsquigarrow B$.

The proposition allows one to define a multi-valued *specialization* map, $\widetilde{\mathrm{sp}}$, from the set of non-closed points of $\mathrm{Spec}\,\mathcal{T}$ to the set of closed points of $\mathrm{Spec}\,\mathcal{T}$. Indeed, to every non-closed point of $\mathrm{Spec}\,\mathcal{T}$, we attach the set of closed points of the irreducible component it belongs to. For example, in the following pictures, $\mathrm{Spec}\,\mathcal{T}$ has four irreducible components (namely from bottom to top the two straight lines, the oval, and the ugly curve) hence four non-closed points. It has also four closed points, but the multi-valued map from the non-closed points to the closed point is not a bijection. Instead, it sends the two non-closed points corresponding to the two straight lines to the same closed point, the non-closed point corresponding to the oval to a set of two closed points, and the non-closed point corresponding to the ugly curve to one closed point.
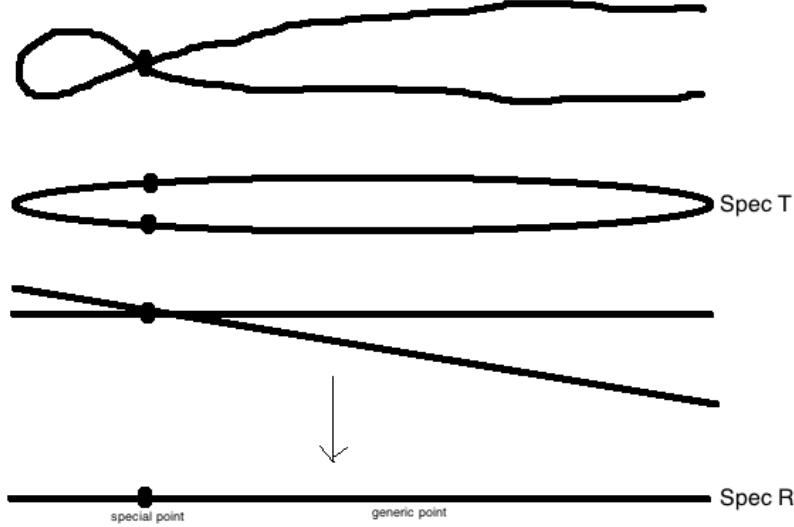
I.7.2. Reduction of characters. Let K' be a finite extension of K and $\chi :$ $\mathcal{T} \to K'$ a character (that is, a morphism of R-algebras). We are going to define a finite sets of characters $\bar{\chi}_1, \ldots, \bar{\chi}_r$ from $\mathcal{T}$ to $k_1, \ldots, k_r$, where the k_i are algebraic extensions of k which depend only on K', not on χ.

Let R' be any sub R-algebra of K', containing $\chi(\mathcal{T})$, and integral over R. Two examples of such sub-algebras are $\chi(\mathcal{T})$ and the integral closure of R in K', and all other such algebras are contained between those two. By the Krull-Akizuki theorem ([**E**, Theorem 11.13]), R' is a noetherian dimension 1 domain and it has only finitely many ideals containing $\mathfrak{m}$. The prime ideals of R' are therefore the minimal prime ideal (0), which of course lies above the ideal (0) of R, and its maximal ideals which (as is easily seen using [**E**, Cor 4.18] using that R' is integral over R, as in the proof of Prop. I.7.1) lies over the maximal ideal $\mathfrak{m}$ of R. Hence R' has only finitely many prime ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$, and they satisfy $\mathfrak{m}_i \cap R = \mathfrak{m}$. Let us call, for $i = 1, \ldots, r$, $k_i = R'/\mathfrak{m}_i$; this is an algebraic extension of k since R' is integral over R. For $i = 1, \ldots r$, we define a character $\bar{\chi}_i : \mathcal{T} \to k_i$ by reducing $\chi : \mathcal{T} \to R'$ modulo $\mathfrak{m}_i$.

DEFINITION I.7.2. The characters $\bar{\chi}_i : \mathcal{T} \to k_i$ for $i = 1, \ldots, r$ are called the *reductions of the character* $\chi : \mathcal{T} \to K'$ *along* R'.

REMARK I.7.3. In the case $K' = K$, taking $R' = R$ is the only choice since R is integrally closed. In this case, R' has only one maximal ideal $\mathfrak{m}_1 = \mathfrak{m}$, and χ has only one reduction $\bar{\chi}_1$ that we denote simply by $\bar{\chi}$.

With R' as above, one can group all the $\bar{\chi}_i$'s into one character $\bar{\chi} : \mathcal{T} \to (R'/\mathfrak{m}R')^{\mathrm{red}}$, the reduction mod $\mathfrak{m}R'$ of χ, since $R'/\mathfrak{m}R'$ is just the profluct of the

k_i. This is particularly convenient to express the (obvious) functoriality of the construction of the reductions of a character, which is as follows: *if K_1 and K_2 are two finite extensions of K, $\sigma : K_1 \to K_2$ a K-morphism, $\chi_i : \mathcal{T} \to K_i$ for $i = 1, 2$ two characters such that $\chi_2 = \sigma \circ \chi_1$, R_i for $i = 1, 2$ two R-subalgebras of K_i containing $\chi_i(\mathcal{T})$ and integral over R, such that $\sigma(R_1) = R_2$ and $\bar{\chi}_i : \mathcal{T} \to (R_i/\mathfrak{m}R_i)^{red}$ the reduction of χ_i along R_i for $i = 1, 2$, then one has*

$$\bar{\chi}_2 = \bar{\sigma} \circ \bar{\chi}_1,$$

where $\bar{\sigma} : (R_1/\mathfrak{m}R_1)^{red} \to (R_2/\mathfrak{m}R_2)^{red}$ is the morphism induced by $\sigma : R_1 \to R_2$.

This functoriality property shows that the reductions of a character χ along $\chi(R)$ are *universal* amongst all reductions modulo an algebra R'. We can use these reductions to define a natural *reduction multi-valued map*

$$\widetilde{\mathrm{red}} : \{\text{characters } \mathcal{T} \to \bar{K}\}/G_K \rightsquigarrow \{\text{characters } \mathcal{T} \to \bar{k}\}/G_k .$$

Here, as in §I.5.3 we have denoted by $\bar{K}$ and $\bar{k}$ some algebraic closures of K and k respectively, and we have set $G_K = \mathrm{Aut}(\bar{K}/K)$ and $G_k = \mathrm{Aut}(\bar{k}/k)$. One proceeds as follows: if $\chi : \mathcal{T} \to \bar{K}$ is a character, let $R' = \chi(\mathcal{T})$. If $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals of R, and $k_i = R'/\mathfrak{m}_i$, the reduction $\bar{\chi}_i : \mathcal{T} \to k_i$ can be seen as a character $\bar{\chi}_i : \mathcal{T} \to \bar{k}$ by choosing a k-embedding $k_i \hookrightarrow \bar{k}$, which is then well-defined up to the action of G_k. Hence a well-defined multi-valued map

$$\{\text{characters } \mathcal{T} \to \bar{K}\} \rightsquigarrow \{\text{characters } \mathcal{T} \to \bar{k}\}/G_k$$

which sends χ to the set of the $\bar{\chi}_i$, for $i = 1, \ldots, r$. One checks easily that this multi-valued map factors through $\{\text{characters } \mathcal{T} \to \bar{K}\}/G_K$, defining $\widetilde{\mathrm{red}}$.

THEOREM I.7.4. *One has the following natural commutative diagram of sets, where the horizontal arrows are bijection, and the vertical arrows are surjective-multivalued maps:*

$$\begin{array}{ccccc}
\{\text{non-closed points of Spec}\,\mathcal{T}\} & \xrightarrow{\simeq} & \{\text{points of Spec}\,\mathcal{T}_K\} & \xrightarrow{\simeq} & \{\text{characters } \mathcal{T} \to \bar{K}\}/G_K \\
\Big\downarrow{\widetilde{sp}} & & & & \Big\downarrow{\widetilde{red}} \\
\{\text{closed points of Spec}\,\mathcal{T}\} & \xrightarrow{\simeq} & \{\text{points of Spec}\,\mathcal{T}_k\} & \xrightarrow{\simeq} & \{\text{characters } \mathcal{T} \to \bar{k}\}/G_k
\end{array}$$

Proof — The morphisms have been constructed above. To check the commutativity of the diagram, start with a character $\chi : \mathcal{T} \to \bar{K}$. Then $\mathfrak{p} = \ker \chi$ is a minimal prime ideal of $\mathcal{T}$, and is the non-closed point of $\mathrm{Spec}\,\mathcal{T}$ corresponding to χ. Since $\chi(\mathcal{T}) \simeq \mathcal{T}/\mathfrak{p}$, the maximal ideals of $\mathcal{T}$ containing $\mathfrak{p}$ are the maximal ideals $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ of $\chi(\mathcal{T})$, and those ideals are the kernel of the reduced character $\bar{\chi}_1, \ldots, \bar{\chi}_r$. Since by definitions $\widetilde{sp}(\mathfrak{p}) = \{\mathfrak{m}_1, \ldots, \mathfrak{m}_r\}$, and $\widetilde{\mathrm{red}}(\mathfrak{p}) = \{\bar{\chi}_1, \ldots, \bar{\chi}_r\}$, the commutativity of the diagram is proved. The surjectivity of $\widetilde{\mathrm{red}}$ then follows from the surjectivity of $\widetilde{sp}$. $\square$

I.7.3. The case of a complete discrete valuation ring. When R is a **complete** discrete valuation ring the situation becomes simpler.

PROPOSITION I.7.5. *If R is complete, every irreducible component of $\operatorname{Spec}\mathcal{T}$ contains exactly one closed point. Moreover, the map from the set of closed points of $\operatorname{Spec}\mathcal{T}$ to its set of connected components, which to a closed point attaches the connected component where it belongs, is a bijection.*

Proof — Since $\mathcal{T}$ is finite over R local complete, by [**E**, Cor. 7.6], one has $\mathcal{T} = \prod_{i=1}^{r} \mathcal{T}_{\mathfrak{m}_i}$ where $\mathfrak{m}_1, \ldots, \mathfrak{m}_r$ are the maximal ideals of $\mathcal{T}$. This means that $\operatorname{Spec}\mathcal{T}$ is the disjoint union of the schemes $\operatorname{Spec}\mathcal{T}_{\mathfrak{m}_i}$, which are connected since $\mathcal{T}_{\mathfrak{m}_i}$ is local. Hence the $\operatorname{Spec}\mathcal{T}_{\mathfrak{m}_i}$ are the connected components of $\operatorname{Spec}\mathcal{T}$, and they obviously contain exactly one closed point, namely $\mathfrak{m}_i$. Since connected components contain at least one irreducible component, hence at least one closed point by Prop. I.7.1, the second assertion follows. But an irreducible component, being contained in a connected component, cannot contain more than one closed point, and the first assertion follows as well. $\square$

The consequeces for our general picture are as follows. First, when R is complete, the multi-valued maps $\widetilde{\operatorname{sp}}$ and $\widetilde{\operatorname{red}}$ becomes ordinary single valued maps and we thus denote them sp and red. Similarly, a character $\chi : \mathcal{T} \to K'$ has only one reduction along any subalgebra R' of K containing $\chi(\mathcal{T})$ and integral over R.

Second, the set of closed (resp. non-closed) points being identified with the set of connected (resp. irreducible) components of $\operatorname{Spec}\mathcal{T}$, we get a new description of the map sp, as the map $\widetilde{\operatorname{incl}}$ that sends an irreducible component of $\operatorname{Spec}\mathcal{T}$ to the connected component that contains it. We summarize this information in the following corollary of Theorem I.7.4.

COROLLARY I.7.6. *If R is complete, one has the following natural commutative diagram of sets, where the horizontal arrows are bijections, and the vertical arrows are surjective maps:*

$$
\begin{array}{ccccc}
\{\text{irreducible comp. of } \operatorname{Spec}\mathcal{T}\} & \xrightarrow{\simeq} & \operatorname{Spec}\mathcal{T}_K & \xrightarrow{\simeq} & \{\text{characters } \mathcal{T} \to \bar{K}\}/G_K \\
\Big\downarrow{\scriptstyle \widetilde{\operatorname{incl}}} & & & & \Big\downarrow{\scriptstyle \operatorname{red}} \\
\{\text{connected comp. of } \operatorname{Spec}\mathcal{T}\} & \xrightarrow{\simeq} & \operatorname{Spec}\mathcal{T}_k & \xrightarrow{\simeq} & \{\text{characters } \mathcal{T} \to \bar{k}\}/G_k
\end{array}
$$

EXERCISE I.7.7. Assume that R is a complete DVR.

1.– Show that there is a connected component in $\operatorname{Spec}\mathcal{T}$ which is not irreducible if and only if there are two systems of eigenvalues $\mathcal{T} \to \bar{K}$, not in the same G_K-orbit, that have the same reduction.

2.– Assume that $\mathcal{T}_K$ is étale over K. Show that $\operatorname{Spec}\mathcal{T}$ is not étale over $\operatorname{Spec}R$ if and only if there are two distinct systems of eigenvalues $\mathcal{T} \to \bar{K}$ that have the same reduction.

EXERCISE I.7.8. Let $R = \mathbb{Z}_p$, $M = \mathbb{Z}_p^2$, and $\mathcal{H} = \mathbb{Z}_p[T]$ with $\psi(T) = \begin{pmatrix} 0 & \pi^a \\ \pi^b & 0 \end{pmatrix}$, for some $a, b \in \mathbb{N}$. Compute $\mathcal{T}$ in this case. Describe prime and maximal ideals of $\mathcal{T}$. When is $\mathcal{T}$ regular ? when is $\mathcal{T}$ irreducible ? When is $\mathcal{T}$ connected ? When is $\mathcal{T}$ étale over R ? When is M free over $\mathcal{T}$?

I.7.4. A simple application: Deligne-Serre's lemma. It is the very useful following simple result. If $m, m' \in M$, we say that $m \equiv m' \pmod{\mathfrak{m}}$ if $m - m' \in \mathfrak{m}M$.

LEMMA I.7.9 (Deligne-Serre). *Assume there is an $m \in M$, $m \not\equiv 0 \pmod{\mathfrak{m}}$ such that for every $T \in \mathcal{H}$, we have $\psi(T)m \equiv \alpha(T)m \pmod{\mathfrak{m}}$ for some $\alpha(T) \in R$. Then there is a finite extension K' of K, such that if R' is the integral closure of R in K', there is a vector $m' \in M \otimes_R R'$ which is a common eigenvector for $\mathcal{H}$ and whose system of eigenvalues χ satisfies $\chi(T) \equiv \alpha(T) \pmod{\mathfrak{m}'}$ for every $T \in \mathcal{H}$.*

In other words, if we have an eigenvector (mod $\mathfrak{m}$) then one can lift its eigenvalues (mod $\mathfrak{m}$) (but maybe not the eigenvector itself) into true eigenvalues (after possibly extending R to R').

The proof is actually contained in what we have said above: it follows from the relation $\psi(T)m \equiv \alpha(T)m \pmod{m}$ that $\alpha(T) \pmod{m}$ depends only of $\psi(T)$ and hence is a character $\mathcal{T} \to k$. We have seen that those characters can be lifted into characters $\mathcal{T} \to R' \subset K'$ for a suitable finite extension K' of $\mathfrak{m}$.

Actually the need to replace K by a finite extension K' follows from the fact that point of $\operatorname{Spec} \mathcal{T}_K$ may not be defined over K, but only on a finite extension. The same proof gives the following variant, which is sometimes useful:

LEMMA I.7.10 (Variant of Deligne-Serre's lemma). *Assume there is an $m \in M$, $m \not\equiv 0 \pmod{m}$ such that for every $T \in \mathcal{H}$, we have $\psi(T)m \equiv \alpha(T)m \pmod{m}$ for some $\alpha(T) \in R$. Also assume that all points of $\mathcal{T}_K$ are defined over K. Then there is a vector $m' \in M$ which is a common eigenvector for $\mathcal{H}$ and whose system of eigenvalues χ satisfies $\chi(T) \equiv \alpha(T) \pmod{\mathfrak{m}}$ for every $T \in \mathcal{H}$.*

The variant of Deligne-Serre's lemma implies the classical version, since there is always a finite extension K' of K such that all points of $\mathcal{T}_{K'}$ are defined over K', and applying the variant to K' gives the classical Deligne-Serre's lemma. The variant has the advantage it gives some control on what extension K' is needed, if any.

EXERCISE I.7.11. It is often said that the Deligne-Serre's lemma does not hold modulo $\mathfrak{m}^2$, or $\mathfrak{m}^c$ for $c > 1$. While technically this is not true for the classical version (any congruence mod $\mathfrak{m}$ becomes a congruence mod $\mathfrak{m}'^c$ if K is replaced by an extension of index of ramification at least c), this is true for the variant of the Deligne-Serre's lemma.

Indeed, prove that a character $\mathcal{T} \to R/\mathfrak{m}^2$ needs not be liftable to a character $\mathcal{T} \to R$ even if every point of $\operatorname{Spec} \mathcal{T}_K$ is defined over K.

I.7.5. The theory of congruences.

I.7.5.1. *Congruences between two submodules.* As usual, M is a finite projective module over R. We write $M_K = M \otimes K$ and we suppose given a decomposition $M_K = A \oplus B$ of K-vector spaces. We write p_A and p_B for the first and second projections of M_K on the factors of that decomopisition, and we define $M_A = p_A(M)$, $M_B = p_B(M)$. We thus have exact sequences

$$0 \longrightarrow M \cap B \longrightarrow M \xrightarrow{p_A} M_A \longrightarrow 0,$$

$$0 \longrightarrow M \cap A \longrightarrow M \xrightarrow{p_B} M_B \longrightarrow 0.$$

In this situation, we define the *congruence module*

$$C = M/((M \cap A) \oplus (M \cap B)).$$

EXERCISE I.7.12. **(easy)** Let $M = \mathbb{Z}_2^2$, and A (resp. B) be the $\mathbb{Q}_2$-subspace of $M_{\mathbb{Q}}$ generated by $(1,1)$ (resp. $(1,-1)$). What is C in this case?

EXERCISE I.7.13. a.– Show that C is a finite torsion module.

b.– Show that the map p_A identifies $M \cap A$ with a submodule of M_A, and show that $C = M_A/(M \cap A)$. By symmetry, $C = M_B/(M \cap B)$.

c.– Show that (p_A, p_B) identifies M with a sub-module of $M_A \oplus M_B$. Show that $C = (M_A \oplus M_B)/M$.

DEFINITION I.7.14. In this situation, C is called the *module of congruences*. Its annihilator is called the *ideal of congruences*.

To explain the name, we shall relate C to actual congruences between elements of A and B. For $c \geq 1$, $f, g \in M$, we shall write $f \equiv g \pmod{\pi^c}$ if $f - g \in \pi^c M$. We define a *congruence in M modulo π^c between A and B* as the data of $f \in M \cap A, g \in M \cap B$, such that $f \equiv g \pmod{\pi^c}$ and, $f \not\equiv 0 \pmod{\pi}$ (which is the same as $g \not\equiv 0 \pmod{\pi}$).

PROPOSITION I.7.15. *There exists a congruence modulo in M between A and B modulo π^c if and only if C contains a submodule isomorphic to R/π^c.*

Proof — Let x be a generator of a sub-module of $C = M/(M \cap A \oplus M \cap B)$ isomorphic to R/π^c. Then we have $\pi^c x \in M \cap A \oplus M \cap B$, so we can write $\pi^c x = f - g$ with $f \in M \cap A$ and $g \in M \cap B$, and we also have $\pi^{c-1} f \notin (M \cap A) \oplus (M \cap B)$ which implies that $f \notin \pi M$ (otherwise, we would have $f = \pi f'$, $g = \pi g'$, with $f' \in M \cap A$, $g' \in M \cap B$ and $\pi^{c-1} x = f' - g'$.) Thus $f \equiv g \pmod{\pi^c}$ while $f \not\equiv 0$.

Conversely, if f, g define a congruence in M between A and B then $f - g = \pi^c x$ for some $x \in M$. Now if $\pi^{c-1} x = f' - g'$ with $f' \in M \cap A$, $g' \in M \cap B$, then $\pi f' - \pi g' = f - g$ which implies $f = \pi f'$, which is absurd. So $\pi^{c-1} x \notin M \cap A \oplus M \cap B$. This shows that x generates a module isomorphic to R/π^c in C. $\qquad\square$

In other words, the ideal of congruences is the maximal ideal modulo which there are congruences between A and B.

I.7.5.2. *Congruences in presence of a bilinear product.* There is a situation, which arises in applications, where it is easy to compute the congruence module. It is the situation where there is a *bilinear product*, on M: $M \times M \to R$, $(x, y) \mapsto \langle x, y \rangle$, which is *non-degenerate* (or *perfect*), which means that the maps $p : x \mapsto (y \mapsto \langle x, y \rangle)$ and $q : y \mapsto (y \mapsto \langle x, y \rangle)$ are isomorphisms of M onto M^*.

PROPOSITION I.7.16. *Assume that M has a non-degenerate bilinear product as above such that $\langle A, B \rangle = 0$ (in other words, $A^\perp = B$). Then there is an iso-morphsim $C = (M \cap A)^*/p(M \cap A)$. In particular, $C = 0$ if and only if the restriction of the bilinear product to $M \cap A$ is still non-degenerate.*

Proof — We consider the composition $r : M \overset{p}{\to} M^* \to (M \cap A)^* \to (M \cap A)^*/p(M\cap A)$, where the second morphism is the restriction map (which is surjective since $M \cap A$ is a direct summand of M). The morphism r is surjective as the composition of three surjective morphisms. By definition, an element $m \in M$ is in $\ker r$ if and only if there exist an element $a \in M \cap A$ such that $\langle m, a' \rangle = \langle a, a' \rangle$. This is equivalent to $m - a \in A^\perp = B$, but since $m - a \in M$, this is also equivalent to $m - a \in M \cap B$. Therefore $\ker r = M \cap A \oplus M \cap B$, and the results follows. $\square$

EXERCISE I.7.17. **(easy)** Assume that M has a non-degenerate bilinear product. Let $f \in M$, $f \not\equiv 0 \pmod M$, and $\langle f, f \rangle \neq 0$. Let $c \in \mathbb{N}$. Show that there exists $g \in M$, $\langle f, g \rangle = 0$ and $g \equiv f \pmod{\mathfrak{m}^c}$ if and only if $\langle f, f \rangle \in \mathfrak{m}^c$.

I.7.5.3. *Congruences and eigenalgebras.* We place ourselves in the situation of §I.7.5.1: M is a finite free R-module such that $M_K = A \oplus B$. We also assume that we are in a situation which gives rise to an eigenalgebra: we have a commutative ring $\mathcal{H}$, a morphism $\psi : \mathcal{H} \to \mathrm{End}_R(M)$. We assume the following compatibility between those data: $\psi(H)$ stabilizes A and B.

A more natural question in this context is wether there are congruences between **eigenvectors** in A and B, not simply vectors, or better if there are congruences between **system of eigenvalues** appearing in A and B. To discuss those questions, let us introduce some terminology: For any submodule N of M stable by $\psi(\mathcal{H})$, let us call $\mathcal{T}_N$ sub-algebra of $\mathrm{End}_R(N)$ generated by $\psi(\mathcal{H})$. So in particular $\mathcal{T}_M = \mathcal{T}$.

Note that by Exercise I.3.3, $\mathcal{T}_{M \cap A} = \mathcal{T}_{M_A}$ and $\mathcal{T}_{M \cap B} = \mathcal{T}_{M_B}$. We call those algebras $\mathcal{T}_A$ and $\mathcal{T}_B$ for simplicity. Also, by the same exercise, the natural maps $\mathcal{T} \to \mathcal{T}_A$ and $\mathcal{T} \to \mathcal{T}_B$ are surjective, while their product $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$ is injective.

DEFINITION I.7.18. (a) We say that there is a *congruence modulo π^c be-tween eigenvectors of A and B* if there exist $f \in M \cap A$, $g \in M \cap B$ both eigenvectors for $\psi(\mathcal{H})$ (equivalently: for $\mathcal{T}$) such that $f \equiv g \pmod{\pi^c}$ and $f \not\equiv 0 \pmod \pi$

(b) We say that there is a *congruence modulo π^c between system of eigenvalues of A and B* if there exists characters $\chi_A : \mathcal{T} \to \mathcal{T}_A \to R$ and $\chi_B : \mathcal{T} \to \mathcal{T}_B \to R$ such that $\chi_A \equiv \chi_B \pmod{\pi^c}$.

(c) We say that there is an *eigencongruence modulo π^c between A and B* if there is a character $\chi : \mathcal{T} \to R/\pi^c$ that factors both through $\mathcal{T}_A$ and $\mathcal{T}_B$.

Obviously, (a) implies (b) implies (c). However, those inclusions are strict, even if we assume that all points of $\mathrm{Spec}\,\mathcal{T}_K$ are defined over K.

EXAMPLE I.7.19. For an example of situation where (b) holds but not (a), let $R = \mathbb{Z}_p$, $K = \mathbb{Q}_p$ $M = \mathbb{Z}_p^2$, $M_K = \mathbb{Q}_p^2$, $A = \mathbb{Q}_p e_1$, $B = \mathbb{Q}_p e_2$ where (e_1, e_2) is the standard basis of $\mathbb{Z}_p^2$, and let $T \in \mathrm{End}_R(M)$ be the matrix $\begin{pmatrix} 0 & 0 \\ 0 & p \end{pmatrix}$. Then e_1 and e_2 are eigevectors for T of eigenvalues 0 and p, so there is a congruence modulo (p) between systems of eigenvalues appearing in A and B. Yet there is no congruences between A and B, as the $M \cap A = \mathbb{Z}_p e_1$, $M \cap B = \mathbb{Z}_p e_2$, so $M = (M \cap A) \oplus (M \cap B)$ and the module of congruence C is trivial. Note that in this situation the eigenalgebra $\mathcal{T}$ is $\mathbb{Z}_p[X]/X(X-p)$, X acting on M by T. This algebra $\mathcal{T}$ is naturally isomorphic as a $\mathbb{Z}_p$-algebra to $\{(a,b) \in \mathbb{Z}_p^2, a \equiv b \pmod{p}\}$, the isomorphism sendind $P(X)$ to $(P(0), P(p))$. The algebras $\mathcal{T}_A$ and $\mathcal{T}_B$ are just $\mathbb{Z}_p$ and the map $\mathcal{T} \mapsto \mathcal{T}_A \times \mathcal{T}_B$ is just the obvious inclusion $\mathcal{T} \subset \mathbb{Z}_p^2$ (obvious in terms of the second description of $\mathcal{T}$).

EXERCISE I.7.20. Give an example of situation where (c) holds but not (b), even if all points of $\mathrm{Spec}\,\mathcal{T}_K$ are defined over K.

EXERCISE I.7.21. **(easy)** Show that when all points $\mathrm{Spec}\,\mathcal{T}_K$ are defined over K, any eigencongruence modulo $\mathfrak{m}$ between A and B comes form a congruence between system of eigenvalues appearing in A and B. (N.B. we are talking here of congruence modulo $\mathfrak{m}$, not $\mathfrak{m}^c$.)

Among the three notions of congruences that we define, the notion (b) of congruences between system of eigenvalues is the most natural and useful. Unfortunately, it is also the more difficult to detect. Let us just mention the following easy but weak result:

EXERCISE I.7.22. Assume that $\dim_K A = 1$ and that all points $\mathrm{Spec}\,\mathcal{T}_K$ are defined over K. If the module of congruence C is not trivial, show that there is a congruence modulo $\mathfrak{m}$ between system of eigenvalues appearing in A and B. Show that the converse is false.

It is relatively easy, however, to detect eigencongruences. But for that we need a better tool than the ideal of congruence, whose definition does not take into account the action of $\mathcal{H}$ on M.

DEFINITION I.7.23. The *ideal of fusion* F is the conductor of the morphism $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$

Let us recall that the conductor of an injective morphism of rings $S \to S'$ is the set of $x \in S$ such that $xS' \subset S$, that is the annihilator of the S-module S'/S. It is an ideal of S, and also an ideal of S'. Actually, it is easy to see that the conductor is the largest S-ideal which is also an S'-ideal (or which is the same, the largest S'-ideal which is contained in S).

PROPOSITION I.7.24. *Let F_A and F_B be the image of F in $\mathcal{T}_A$ and $\mathcal{T}_B$. Then we have natural isomorphsims of R-algebras $\mathcal{T}_A/F_A \simeq \mathcal{T}/F \simeq \mathcal{T}_B/F_B$*

Proof —   Since F is an ideal of $\mathcal{T}_A \times \mathcal{T}_B$, we have $F = F_A \times F_B$. The map $\mathcal{T} \to \mathcal{T}_A \to \mathcal{T}_A/F_A$ is obviously surjective; let us call I its kernel. We obviously have $F \subset I$. Let $x \in I$, whose image in $\mathcal{T}_A \times \mathcal{T}_B$ is (x_A, x_B). We have $x_A \in F_A$. Therefore there is f in F, whose image (f_A, f_B) in $\mathcal{T}_A \times \mathcal{T}_B$ is such that $f_A = x_A$. Then $x - f = (0, x_B - f_B)$ in $\mathcal{T}_A \times \mathcal{T}_B$. Therefore $x - f \in F$ (the claim here is that any element in $\mathcal{T}$ whose image in $\mathcal{T}_A$ is 0 is in F. For if we have such an element $(0, y) \in \mathcal{T}$, then for any $(u, v) \in \mathcal{T}_A \times \mathcal{T}_B$, v is the image of some element $z \in \mathcal{T}$, and $(0, y)(u, v) = (0, y)z$ is in $\mathcal{T}$ as a product of two elements in $\mathcal{T}$, which shows that $(0, u) \in F$). Hence $x \in F$, so finally $I = F$. Therefore we have an isomorphism $\mathcal{T}/F \simeq \mathcal{T}_{M_A}/F_A$. The other isomorphism is symmetric.   □

What is interesting here is that we get an isomorphism $\mathcal{T}_A/F_A \simeq \mathcal{T}_B/F_B$ that can be interpreted as eigencongruences:

THEOREM I.7.25. *Let c be an integer. Then there is an eigenconguence modulo $\mathfrak{m}^c$ between A and B if and only if there is a surjective map of R-algebras $\mathcal{T}/F \mapsto R/\mathfrak{m}^c$*

Proof —   An eigencongruence modulo $\mathfrak{m}^c$ is a surjective map $\chi : \mathcal{T} \to R/\mathfrak{m}^c$ that factors both through $\mathcal{T}_A$ and $\mathcal{T}_B$. Let $x \in F$, and (x_A, x_B) its image in $\mathcal{T}_A \times \mathcal{T}_B$. Then $(x_A, 0)$ and $(0, x_B)$ are in $\mathcal{T}$, since they are $x(1, 0)$ and $x(0, 1)$, and x is in F. But $\chi(x_A, 0) = \psi_B(0) = 0$ and similarly $\chi(0, x_B) = 0$, so $\chi(x) = 0$. We have just shown $F \subset \ker \chi$, so χ factors into a map $\mathcal{T}/F \to R/\mathfrak{m}^c$, necessarily surjective.
Conversely, a map $\chi : \mathcal{T} \to \mathcal{T}/F \to R/\mathfrak{m}^c$ factors through both $\mathcal{T}_A$ and $\mathcal{T}_B$ by the above proposition.   □

What is the relation between the ideal of fusion and the module of congruence? Let $I_{\mathcal{T}}$ be the annihilator of the module of congruence $C = M/(M \cap A \oplus M \cap B)$ in $\mathcal{T}$ (not to be confused with the ideal of congruence I defined as the annihilator of C in R. We have $I_{\mathcal{T}} \cap R = I$.)

THEOREM I.7.26. *We have an inclusion $F \subset I_{\mathcal{T}}$. If there exists a non-degenerate bilinear product $\mathcal{T} \times M \to R$ satisfying $\langle TT', f \rangle = \langle T', Tf \rangle$ for all $T, T' \in \mathcal{T}$, $f \in M$, then we have $F = I_{\mathcal{T}}$.*

Proof —   By definition, $I_{\mathcal{T}}$ is the set of $T \in \mathcal{T}$ such that $T((M \cap A) \oplus (M \cap B))$ is in M. So $I_{\mathcal{T}} = \{T \in \mathcal{T}, T(\mathcal{T}_A \times \mathcal{T}_B)M \subset M$ since clrarly, $(M \cap A) \oplus (M \cap B)$ is generated by M as a $(\mathcal{T}_A \times \mathcal{T}_B)$-module. So

$$I_{\mathcal{T}} = \{T \in \mathcal{T}, T(\mathcal{T}_A \times \mathcal{T}_B) \subset \mathcal{O}_M\}$$

where $\mathcal{O}_M = \{(T_A, T_B) \in \mathcal{T}_A \times \mathcal{T}_B, (T_A, T_B)M \subset M\}$. On the other hand,

$$F = \{T \in \mathcal{T}, T(\mathcal{T}_A \times \mathcal{T}_B) \subset \mathcal{T}\}$$

Therefore, the inclusion $F \subset I_{\mathcal{T}}$ follows from the trivial inclusion $T \subset \mathcal{O}_M$, which proves the first half of the theorem.

Similarly, $F = I_{\mathcal{T}}$ would follow from $T = \mathcal{O}_M$ which we shall prove assuming the existence of a bilinear product as in the statement. First, because of the perfect pairing $\dim_K \mathcal{T}_K = \dim_K M_K$, and similarly $\dim_K \mathcal{T}_A = \dim_K A$ and $\dim_K \mathcal{T}_B = \dim_K B$ because the pairing induces a perfect pairing between $\mathcal{T}_A \otimes K$ and A, and $\mathcal{T}_B \otimes K$ and B. It follows that the map $\mathcal{T} \hookrightarrow \mathcal{T}_A \times \mathcal{T}_B$ becomes an isomorphism after tensorizing by K. The pairing extends to a pairing $\mathcal{T}_K \times M_K \to K$, and the non-degeneracy of the pairing means that for $T \in \mathcal{T}_K$, we have $T \in \mathcal{T}$ if and only if $\langle T, f \rangle \in R$ for all $f \in M$. If $T \in \mathcal{O}_M \in \mathcal{T}_A \times \mathcal{T}_B \subset \mathcal{T}_K$, and $f \in M$, we have $\langle T, f \rangle = \langle 1, Tf \rangle$ which is in R since $Tf \in M$ by definition of $\mathcal{O}_M$. Hence $T \in \mathcal{T}$ and $\mathcal{O}_M = \mathcal{T}$. $\qquad\square$

EXERCISE I.7.27. Prove that if we release the hypothesis that the pairing is non-degenerate in the above theorem, but we assume instead that it has discriminant of valuation $c \in \mathbb{N}$, then $I_{\mathcal{T}}/F$ is killed by π^c.

I.8. Modular forms with integral coefficients

Let $w \geq 0$ be an integer (the weight). Recall that the $\mathbb{Z}$-submodule $M_w(\Gamma_1(N), \mathbb{Z})$ of $M_w(\Gamma_1(N), \mathbb{Q})$ (same statement with S_w) with integral coefficients is a lattice. So we can extend the preceding definitions by setting $M_w(\Gamma_1(N), R) = M_w(\Gamma_1(N), \mathbb{Z}) \otimes_{\mathbb{Z}} R$ for any commutative ring R. The same holds for S_w.

We now fix a prime number p, a finite extension K of $\mathbb{Q}_p$, of ring of integers R. As above, $\mathfrak{m}$ is the maximal ideal of R, and $k = R/m$. We consider the module $S_w(\Gamma_1(N), R)$ and the eigenalgebras $\mathcal{T}_R$, $\mathcal{T}_{0,R}$ of $\mathcal{H}$, $\mathcal{H}_0$ acting on this module. Similarly, we define the eigenalgebras $\mathcal{T}_K$, $\mathcal{T}_{0,K}$, and $\mathcal{T}_k$, $\mathcal{T}_{0,k}$ for the action of $\mathcal{H}$, $\mathcal{H}_0$ on $S_w(\Gamma_1(N), K)$ and $S_w(\Gamma_1(N), k)$.

We let $\mathcal{T}_k$ be the eigenalgebras generated by all Hecke operators on $S_w(\Gamma_1(N), k)$.

I.8.1. The specialization morphism for Hecke algebras of modular forms.

PROPOSITION I.8.1. *The natural surjective morphism* $\mathcal{T}_R \otimes_R k \to \mathcal{T}_k$ *is an isomorphism, and the module* $S_w(\Gamma_1(N), R)^\vee$ *is free over* $\mathcal{T}_R$.

Proof — Let n be the dimension of $S_w(\Gamma_1(N), K)$. Since $S_w(\Gamma_1(N), R)$ is finite free over R, this is also the dimension of $S_w(\Gamma_1(N), k)$. By Proposition I.6.11 $\dim \mathcal{T}_K = n$. Since $\mathcal{T}_R$ is finite flat oevr R, $\dim \mathcal{T}_R \otimes k = n$. The same argument as in Proposition I.6.11 and its corollary shows that $\dim \mathcal{T}_k = \dim S_w(\Gamma_1(N), k) = n$ (Indeed, the only facts used in Proposition I.6.11 are the fact that an element of $S_w(\Gamma_1(N), k)$ is determined by its q-expansion, which is a tautology with our definition of that space, and the fact that $S_w(\Gamma_1(N), k)$ contains no non-zero element with constant q-expansion, which is clear since the q-expansion of an element of that space has 0 constant term.) It follows by equality of dimensions that the surjective map $\mathcal{T}_R \otimes_R k \to \mathcal{T}_k$ is an isomorphism. Then the pairing $(t, f) \mapsto a_1(tf)$, $\mathcal{T}_R \times S_w(\Gamma_1(N), R)^\vee$ is perfect because its generic and special fibers are perfect. □

EXERCISE I.8.2. Show that both assertions of the theorem may be false when $\mathcal{T}_R$ and $\mathcal{T}_K$ are defined using the full space of modular forms $M_w(\Gamma_1(N), K)$

EXERCISE I.8.3. Let $R = \mathbb{Z}_p$, $w = 12p$ an integer, and define $\mathcal{T}_R^p$ (resp. $\mathcal{T}_k^p$) as the Hecke algebra of all the Hecke operators **excepted** T_p acting on $S_w(\mathrm{SL}_2(\mathbb{Z}), R)$ (resp. on $S_w(\mathrm{SL}_2(\mathbb{Z}), k)$). The specialization map $\mathcal{T}_R^p \otimes_R k \to \mathcal{T}_k^p$ is not an isomorphism

EXERCISE I.8.4. When is the map $\mathcal{T}_{0,R} \otimes_R k \to \mathcal{T}_k$ an isomorphism?

I.8.2. An application to Galois representations. From now on we assume that $p > 2$, because the theory of pseudocharacters of dimension d works well only in rings where $d!$ is not invertible. However, it is possible to include the case $p = 2$ by replacing everywhere pseudocharacter by its generalization by Chenevier called *determinant* ([**C3**]).

THEOREM I.8.5. *There exists a unique continuous pseudocharacter of dimension* 2 $\tau : G_{\mathbb{Q}, Np} \to \mathcal{T}_0$ *sending* Frob_l *(for $l \neq Np$) to T_l. We also have $\tau(c) = 0$ if c is the complex conjugation.*

Proof — We have already seen how to construct a pseudocharacter $\tau : G_{\mathbb{Q}, Np} \to \mathcal{T}_{0,K} = \mathcal{T}_0 \otimes_R K$ satisfying the required properties, and since $\mathcal{T}_{0,R}$ is closed in $\mathcal{T}_{0,K}$ and $\tau(\mathrm{Frob}_l) \in \mathcal{T}_{0,R}$, this pseudocharacter takes values in $\mathcal{T}_{0,R}$. □

Thus we have glued all the Galois representations attached to eigenforms in $M_k(\Gamma_1(N))$ in one unique pseudocharacter, with values in the algebra $\mathcal{T}_{0,R}$.

Let us give one application.

COROLLARY I.8.6. *Let $n \geq 1$ be an integer, $f \in M_k(\Gamma_1(N), \epsilon, R)$ be a normalized form such that for every prime l not dividing Np, $T_l f \equiv \chi(T_l)f \pmod{\mathfrak{m}^n}$ where $\chi(T)$ is some element in $R/\mathfrak{m}^n$. There exists a pseudocharacter $\tau_f : G_{\mathbb{Q},Np} \to R/\mathfrak{m}^n$ such that $\tau(\mathrm{Frob}_l) = \chi(T_l)$ in $R/\mathfrak{m}^n$.*

So we can attach Galois pseudocharacters not only to true eigenforms, but more generally to eigenform modulo $\mathfrak{m}^n$. Of course the pseudocharacter we get takes values in $R/\mathfrak{m}^n$ instead of R.

Had we not the above theorem, the obvious method to prove the result of the corollary would be to replace f by a true eigenform g with eigenvalues congruent to the $\chi(T_l)$ modulo $\mathfrak{m}^n$. Then the would reduce the Eichler-Shimura-Deligne representations ρ_g modulo $\mathfrak{m}^n$ and take the trace. The Deligne-Serre's lemma tells us that we can find such a g when $n = 1$, but in general we cannot. So this approach fails. Yet with what we have done, the proof of the corollary is trivial:

Proof — Since f is normalized, the element $\chi(T_l)$ of $R/\mathfrak{m}^n$ depends only on $T_l \in \mathcal{T}_{0,R}$, hence χ extends to a character $\mathcal{T}_{0,R} \to R/\mathfrak{m}^n$. Composing $\tau : G_{\mathbb{Q},Np} \to \mathcal{T}_{0,R}$ of the above theorem with χ gives the result. $\square$

The drawback of the corollary is that, attached to a modular forms which is an eigenform mofulo $\mathfrak{m}^n$, we get only a pseudocharacter valued in $R/\mathfrak{m}^n$, not a true representation. However, there are many results saying that a pseudocharacter is, in many circumstances, the trace of a representation. Let us recall the most improtant ones: *If $T : G \to k$ is a pseudocharacter of dimension d valued in an algebraically closed field k, then T comes form the trace of a unique semi-simple representation $G \to \mathrm{GL}_d(k)$.* This is a theorem of Taylor [**T**] and Rouquier [**Ro**]. And *if $T : G \to A$ is a pseudocharacter of dimension d valued in an henselian local ring with residue field k, so that the residual pseudocharacter $\bar{T} : G \to k$ is the trace of an absolutely irreducible representation $G \to \mathrm{GL}_d(k)$, then T is the trace of a unique representation $G \to Gl_d(A)$.* This is a theorem of Rouquier [**Ro**] and Nyssen [**Ny**].

For any maximal ideals $\mathfrak{m}_i$ of $\mathcal{T}_0$, the quotient $k_i := \mathcal{T}_0/\mathfrak{m}_i$ is a finite field, so the pseudocharacter $\tau_{\mathfrak{m}_i} : G_{\mathbb{Q},Np} \to \mathcal{T}_0 \to \mathcal{T}_0/\mathfrak{m}_i = k_i$ is the trace of a unique semi-simple representation $\bar{\rho}_{\mathfrak{m}_i} : G_{\mathbb{Q}_p} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\bar{k}_i)$ by the theorem of Taylor and Rouquier. Actually $\bar{\rho}_{\mathfrak{m}_i}$ is defined over k_i (there are many ways to see it: the simplest is too observe that $\bar{\rho}_i(c)$ has distinct eigenvalues).

We call the maximal ideal $\mathfrak{m}_i$ of $\mathcal{T}_0$ (or their connected component in $\mathrm{Spec}\, \mathcal{T}_0$) *non-Eisenstein* if $\bar{\rho}_{\mathfrak{m}_i}$ is absolutely irreducible, and *Eisenstein* otherwise.

THEOREM I.8.7. *Let $Spec\,\mathcal{T}_{0,\mathfrak{m}_i}$ be a non-Eisenstein component. There exists a unique Galois representation $\rho_{\mathfrak{m}_i} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathcal{T}_{0,\mathfrak{m}_i})$ such that $\mathrm{tr}\,\rho_{\mathfrak{m}_i}(\mathrm{Frob}_l) = T_l$ in $\mathcal{T}_{0,m_i}$.*

Indeed, the condition on the trace means that $\mathrm{tr}\,\rho_{\mathfrak{m}_i}$ is the pseudocharacter $\tau_{\mathfrak{m}_i} : G_{\mathbb{Q},Np} \to \mathcal{T}_0 \to \mathcal{T}_{0,\mathfrak{m}_i}$. The residual pseudocharacter is, by assumption, the trace of an absolutely irreducible representation. The result follows from the thorem Rouquier-Nyssen.

The fact that we have a true representation over $\mathcal{T}_{0,\mathfrak{m}_i}$ (assume this component non-Eisenstein) allows us to make a comparison with Mazur's deformation theory. Let $\mathcal{R}_{\mathfrak{m}_i}$ be the universal deformation ring of the residual representation $\bar{\rho}_{\mathfrak{m}_i} : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(k_i)$. SInce $\mathcal{T}_{0,\mathfrak{m}_i}$ is a coefficient ring (that is complete local noetherian of residue field k_i), and since $\rho_{\mathfrak{m}_i}$ obviously deforms $\bar{\rho}_{\mathfrak{m}_i}$, there is a morphism of coefficient rings

$$\mathcal{R}_{\mathfrak{m}_i} \to \mathcal{T}_{0,\mathfrak{m}_i}$$

which sends the universal deformation representation on $\mathcal{R}_{\mathfrak{m}_i}$ to $\rho_{\mathfrak{m}_i}$.

EXERCISE I.8.8. Show that the map $\mathcal{R}_{\mathfrak{m}_i} \to \mathcal{T}_{0,\mathfrak{m}_i}$ is surjective.

The remarkable idea of Wiles leading to the proof of the Shimura-Taniyama-Weil conjecture in the semi-stable case (hence of Fermat's last theorem) was a method to determine the kernel of this map in simple cases (weight $k = 2$, and $\rho_{\mathfrak{m}_i}$ of a very special type). This method has then been generalized to many more cases, leading to the proof of the Shimura-Taniyama-Weil conjecture in general, and even of the Fontaine-Mazur conjecture in many cases.

I.9. A comparison theorem

In this §, we only assume that R is noetherian and reduced. For every point $x \in \mathrm{Spec}\,(R)$, of field $k(x)$, we call $M_x = M \otimes_R k(x)$ and $\mathcal{T}_x$ the subalgebra generated by $\psi(\mathcal{H})$ in $\mathrm{End}_{k(x)}(M_x)$.

LEMMA I.9.1. *Assume that for x in a Zariski-dense subset Z in $Spec\,R$, $\mathcal{H}$ acts semi-simply on M_x. Then $\mathcal{T}$ is reduced.*

Proof — The hypothesis implies that for $x \in Z$, $\mathcal{T}_x$ is reduced.

Let $t \in \mathcal{T}$ such that $t^n = 0$ for some $n \geq 1$. Let t_x be the image of t by the natural map $\mathrm{End}_R(M) \to \mathrm{End}_{k(x)}(M_x)$. We have $t_x^n = 0$, so for $x \in Z$, we get $t_x = 0$.

If M is free over R, that means that every matrix element of $t \in \mathrm{End}_R(M)$ is 0 at every $x \in Z$. Since Z is dense and R is reduced, evert matrix element of t is 0 period, and $t = 0$.

In the general case, M is projective, so $N = M \oplus Q$ say is free. Apply the result in the free case to the endomorphism t' of N such that $t'_{|M} = t$, $t'_{|Q} = 0$. $\square$

Now let us assume that M and M' are two finite projective R-modules with action ψ and ψ' of $\mathcal{H}$, and let $\mathcal{T}$ and $\mathcal{T}'$ be there associated algebras.

THEOREM I.9.2. *Let us assume that for every x in a Zariski-dense subset Z of R, M_x is a semi-simple $\mathcal{H}$-module, and there is an $\mathcal{H}$-injection $M'_x \subset M_x$. Then there is a canonical surjection $\mathcal{T} \to \mathcal{T}'$ (that is, a closed immersion $\operatorname{Spec} \mathcal{T}' \subset \operatorname{Spec} \mathcal{T}$). Moreover, for every $x \in \operatorname{Spec} R$, there is an $\mathcal{H}$-injection $(M'_x)^{ss} \to (M_x)^{ss}$*

Proof — Let $t \in \mathcal{H} \otimes R$. Let $P_t(X)$, $P'_t(X)$ in $R[X]$ be the characteristic polynomials of $\psi(t)$ and $\psi'(t)$. If $x \in \operatorname{Spec} R$, let $P_{t,x}(X)$ and $P'_{t,x}(X)$ be the images of those polynomial in $k(x)[X]$. They are also the characteristic polynomials of $\psi \otimes 1(t)$ and $\psi' \otimes 1(t)$ in $\operatorname{End}_{k(x)}(M_x)$, $\operatorname{End}_{k(x)}(M'_x)$. By hypothesis, for $x \in Z$, $P'_{t,x}$ divides $P_{t,x}$ in $k(x)[X]$ for $x \in Z$. As Z is dense, P_t divides P'_t in $R[X]$, from which it follows that $P_{t,x}|P'_{t,x}$ for every $x \in \operatorname{Spec} R$. By elementary representation theory, this proves the second assertion.

For the second assertion, we can work locally and assume that M is free of rank d. By the lemma, $\mathcal{T}$ and $\mathcal{T}'$ are reduced. Let us call J (resp. J') the kernel J of $\psi : \mathcal{H} \otimes_R \to \operatorname{End}_R(M)$ (resp. $\psi' : \mathcal{H} \otimes R \to \operatorname{End}_R(M')$). If $t \in J$, $\psi(t) = 0$ so $P_t = X^d$. So $P'_t|X^d$, and by Calley-Hamilton $\psi(t')^d = 0$, so $\psi(t') = 0$. Hence $J \subset J'$ and the theorem follows.   □

COROLLARY I.9.3. *Let us assume that for every x in a Zariski-dense subset Z of R, M_x is a semi-simple $\mathcal{H}$-module, and there is an $\mathcal{H}$-isomorphism $M'_x \simeq M_x$. Then there is a canonical isomorphism $\mathcal{T} = \mathcal{T}'$. Moreover, for every $x \in \operatorname{Spec} R$, there is an $\mathcal{H}$-isomorphism $(M'_x)^{ss} \simeq (M_x)^{ss}$*

I.10. Notes and References

In a sense, the idea, in the context of Hecke operators acting on spaces or modules of modular forms, of conisdering the sub-algebra of the endomorphism algebra of that space and moduel generated by those operators – that we call the *eigenalgebra* here – is so tautological that it is impossible to trace back its origin: as soon as we consider operators acting on a space or module, we implicitly or explicitly consider the subalgebra they generate. Yet the realization that this *eigenalgebra* was an important object, even perhaps the central object on the theory, that the algebraic properties of this commutative algebra (which by Grothendieck's theory of schemes can be reformulated into geometric properties of its spectrum) were a way to express and study the properties of eigenforms, this reaization I say was long to come, and a fundamental progress when it did.

Over a field (a subfield of $\mathbb{C}$), this idea certainly occured to Shimura[3], which defines and prove some elementary properties in his book [**Shi**]. He for example

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[3]perhaps even earlier. Eichler? Selberg? If anyone has any information on that -or in general, this history, please contact me.

raised the following question: is the eigenalgebras $\mathcal{T}$ generated by all the Hecke operators acting on the space $S_2(\Gamma_0(\mathbb{N}), \mathbb{Q})$ isomorphic to the algebras of isogenies of the modular Jacobian $J_0(N)$. This question was solved affirmatively by Ribet [**Ri**] in 1975.

Over a ring like the ring of integers of a number field, or a localization or completion of such a ring, the study of the eigenalgebra is intimely related to the study of the arithmetic property of modular forms. The eigenalgebra appears in [**De**], in the second proof of the famous Deligne-Serre's lemma, but it seems that Mazur was the first to study and use the algebraico-geometric propertoes of those eigenalgebras, in its very influential 1978's paper [**Mazur**], to prove deep arithmetic results about modular forms and elliptic curves. Most of the important progresses in the arithmetic theory of modular forms ever since have used and developped that very idea (often named Hecke algebras). To mention a few of them: uses of the Hecke algebras to study congruences between modular forms ([**H1**], [**H2**], [**Ri2**]), use of big Hecke Algebra (Eigenalgebra over Iwasawa's algbra) by Hida to construct the first p-adic modular forms, use of those Hecke algebra to prove the main conjetcures ([**MaW**], [**W**]), use of eigenalgebras (and their fine algebraico-geometric properties) in the proof of Fermat's last theorem ([**W**], [**TW**]), and use of new kind of "big Hecke algebras" by Coleman ([**Col2**] to construct non-ordinary families of modular forms, and more generally by Coleman-Mazur [**CM**] and other to construct the *eigencurve* or *eigenvarieties* (also called *varieties of Hecke*).

In this chapter, we have tried to expose the formal aspects of the construction of eigenalgebras, relying on the work of the aforementioned mathematicians and many other. We know try to give a proper attribution to most of the results that are not standard commutative algebra.

In §I.1, §I.2 and §I.3, almost all results are elementary. Exercise I.3.6 is due to Buzzard. Prop. is due to G. Chenevier ([**C1**]).

Prop. I.4.1 is due to G. Chenevier ([**C1**]), at lest in the case of a surjective map $R \to R'$.

The results of §I.5 are essentially linear algebras, and commutative linear algebra.

Most of the results of §I.6 are due to Shimura ([**Shi**]) though the emphasis on the notion of Gorensteinness in this context seems due to Mazur.

As for eigenalgebras over a base ring which is a D.V.R. (or more generally a Dedekind domain), as we said they appear in [**De**], and [**Mazur**]. The interpretation of those maximal and prime ideals as eigenforms, etc. given in **??**, which is again elementary commutative algebra, is sketched in a few sentences in [**De**]. Each subsequent author stated the part he needed, until all those things became completely standard at the time of the proof of Fermat's Last Theorem – The survey [**DDT**] contains an exposition of that theory, which also mention the link with

Deligne-Serre's lemma. The theory of congruences exposed in §I.7.5 is essentially due to Hida (e.g. [**H1**], [**H2**] and Ribet (e.g. [**Ri2**])). A good survey is [**Gh**].

The results of §I.9 are due to Chenevier ([**C2**]).

CHAPTER II

The Eigenvariety Machine

In this chapter, we explain the construction of eigenvarieties, which is a version at large and in a rigid analytic setting of the construction of eigenalgebras of the previous chapter. We shall not be entirely self-contained for the proofs: the reason is that there is already a very good text explaining the construction of the eigenvarieties, Buzzard's paper [**Bu**]. However we hope that our exposition will allow a reader willing to either admit or look up in [**Bu**] the proofs of a few important technical results along the way, to get a working understanding of what an eigenvariety is and how it is constructed. We also want to improve somewhat on Buzzard's construction by giving a more precise control on the local pieces that are glued to construct the eigenvarieties.

Compared with the setting of a construction of an eigenalgebra of the preceding chapter (a base ring R, a finite flat R-module M, a morphism $\psi : \mathcal{H} \to \mathrm{End}_R(M)$ where $\mathcal{H}$ is some commutative ring of Hecke operators), the setting of eigenvarieties is enlarged in two directions. The base ring R, or rather its spectrum $\mathrm{Spec}\, R$ is replaced by a rigid analytic variety $\mathcal{W}$, which will not be an affinoid in general. Hence it would be natural to replace M by a coherent locally-free sheaf over $\mathcal{W}$, but here comes a second, more important, enlargment: we want to replace the finite module M by a non-finite Banach module. To avoid talking of sheaf of Banach modules (which we could, as in [**CM**] or [**C2**]), we simply define our data as in [**Bu**] by the giving, for every affinoid subdomain $W = \mathrm{Sp}\, R$ of $\mathcal{W}$, of a Banach-module M_W over R (satisfying some technical condition of orthonormability), provided with an action of the ring $\mathcal{H}$. The modules M_W for various W are required to satisfy some compatibility relations, that are explained in section §II.2. To deal with the new infiniteness of our module, we introduce a crucial hypothesis: that our ring $\mathcal{H}$ contains a privileged element $U_p \in \mathcal{H}$ that acts compactly on the various Banach modules M_W.

From those data, our aim is to construct in a canonical way an eigenvariety $\mathcal{E}$, that is a rigid analytic variety together with a locally finite map $\kappa : \mathcal{E} \to \mathcal{W}$, and a morphism of rings $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E})$ allowing us to see element of $\mathcal{H}$ as functions on $\mathcal{E}$. The **eigen**variety should have the fundamental property that its points z (over a given point $w \in W$) classifies the system of $\mathcal{H}$-**eigen**values appearing in the fiber M_w of the modules M_W at w (that fiber does not depend on the chosen affinoid W of $\mathcal{W}$ containing w) that are *of finite slope*, that is for which U_p has a non-zero

eigenvalue. The bijection should be naturally given as follows: to a point z should correspond the system of eigenvalues $T \mapsto \psi(T)(z)$, for any $T \in \mathcal{H}$.

In order to construct the eigenvariety, we glue some local pieces that are rigid analytic spectrum of eigenalgebras constructed as in the precideing chapter. More precisely, if $W = \operatorname{Sp} R$ is an admissible affinoid subdomain of $\mathcal{W}$, and ν is a real number, it is often (but not always!) possible to define in a natural way a finite flat direct summand R-sumboldue $M_W^{\leq \nu}$ of M_W which is morally the "slope less or equal to ν" part of M_W, that is the submodules of vectors on which U_p acts with generalized eigenvalues of valuation lest or equal than ν. When this is possible, we say that W and ν are *adapted.* Those notions are explained in our first section II.1, which also proves that there are enough, in some precise sense, pairs (W, ν) that are adapted.

In section §II.3.2, we explain our variant of [**Bu**]'s eigenvariety machine, constructing the eigenvariety by gluing the spectrum of eigenalgebras $\mathcal{T}_{W,\nu}$ defined by the action of $\mathcal{H}$ on $M_W^{\leq \nu}$ for adpated pair (W, ν).

In section II.4 we list the most important properties of the eigenvarieties, due to Coleman-Mazur ([**CM**]) or Chenevier ([**C1**]), in particular we explain why their points parametrize system of eigenvalues of finite slope.

In the last section II.5 we explain and prove a very useful result of Chenevier (cf. [**C2**]) allowing to compare two eigenvarieties.

II.1. Submodules of slope $\leq \nu$

Let L be a finite extension of $\mathbb{Q}_p$, R be a reduced affinoid algebra over L, that we provide with its supremum norm $|\ |$ extending the norm of L (which itself extends the standard p-adic norm on $\mathbb{Q}_p$). We set as usual $v_p(x) = -\log |x| / \log p$, so $v_p(p) = 1$.

Let us call $R\{\{T\}\}$ the ring of power series $\sum_{n=0}^{\infty} a_n T^n$, with $a_n \in R$, that converge everywhere, that is such that $v_p(a_n) - n\nu \to \infty$ for every $\nu \in \mathbb{R}$.

DEFINITION II.1.1. Let $F(T) \in R\{\{T\}\}$, and $\nu \in \mathbb{R}$. We write $N(F, \nu)$ for the largest integer n such that $v_p(a_n) - n\nu = \inf_m (v_p(a_m) - m\nu)$.

EXERCISE II.1.2. (**easy**) Prove Gauss' lemma: $N(FG, \nu) = N(F, \nu)N(G, \nu)$.

EXERCISE II.1.3. If $F(T) \in L\{\{T\}\}$, and $\nu \in \mathbb{R}$, then $N(F, \nu)$ is the number of zeros z in $\bar{L}$ of $F(T)$ with $v_p(z) \geq -\nu$, counted with multiplicity.

If $x \in \operatorname{Sp} R$ is a closed point of field of definition $L(x)$, we write $F_x(T)$ for the series $\sum_{n=0}^{\infty} a_n(x)T^n$ which belongs to $L(x)\{\{T\}\}$.

EXERCISE II.1.4. (**easy**) If $N(F_x, \nu)$ is a constant N independent of $x \in \operatorname{Sp} R$, then $N(F, \nu) = N$.

DEFINITION II.1.5. A polynomial $Q(T) \in R[T]$ is called ν-*dominant* if

(i) Q has degree $N(Q, \nu)$

(ii) For all $x \in \operatorname{Spec} R$, $N(Q, \nu) = N(Q_x, \nu)$

Note that if a polynomial $Q(T) \in R[T]$ is ν-*dominant*, then its dominant term $a_{N(Q,\nu)}$ is invertible in R. For if it was not, $a_{N(Q,\nu)}(x)$ would be zero for some x, and we would have $N(Q_x, \nu) \leq \deg Q_x < \deg Q = N(Q, \nu)$.

EXERCISE II.1.6. Let Q be a polynomial satsifying (i) of the definition of ν-dominant. Assume that $a_{N(Q,\nu)}$ is invertible, and that $|a_{N(Q,\nu)}||a_{N(Q,\nu)}^{-1}| = 1$. Show that Q is ν-dominant.

Fix an $F \in R\{\{T\}\}$ and assume $F(0) = 1$. As in [**Bu**, §4], let us write $W = \operatorname{Sp} R$, and let Z be the analytic subvariety of $W \times \mathbf{A}_{\operatorname{rig}}^1$ cut out by $F(T)$, where $\mathbf{A}_{\operatorname{rig}}^1$ is the rigid analytic affine line over L. Let f be the projection map $Z \to W$, and let Z_ν be the affinoid of Z of points z whose component in $\mathbf{A}_{\operatorname{rig}}^1$ have $v_p(z) \geq -\nu$. Then the Z_ν are quasi-finite over W according to Exercise II.1.3, and it is not hard to see that they are also flat (cf. [**Bu**, Lemma 4.1]).

PROPOSITION II.1.7. *For $F(T) \in R\{\{T\}\}$ with $F(0) = 1$, and $\nu \in \mathbb{R}$, the following are equivalent:*

(i) *The map $f : Z_\nu \to W$ is finite.*

(ii) *There exists a decomposition $F = QG$ in $R\{\{T\}\}$, with $Q(0) = G(0) = 1$, where Q is a ν-dominant polynomial of degree $N(F, \nu)$.*

(iii) *One has $N(F_x, \nu) = N(F, \nu)$ for all $x \in W$,*

If those properties hold, the decomposition $F = QG$ as in (ii) is unique, $(Q, G) = 1$ in $R\{\{T\}\}$, Z_ν is disconnected from its complement in Z (that is, there is an idempotent $e \in \mathcal{O}(Z)$ such that Z is defined by $e = 1$) and $f : Z_\nu \to B$ is finite flat surjective of degree $\deg Q = N(F, \nu)$.

Proof — Assume (i). Then $f : Z_\nu \to W$, which is always flat, is also surjective. Moreover Z_ν is disconnected from its complement $Z' \subset Z$, since by [**BGR**, 9.6.3/3 and 9.5.3/5], $Z' \subset Z$ is both an open and a closed immersion. The finite flat R-algebra $\mathcal{O}(Z_\nu)$ is generated by the function $t \in \mathcal{O}(Z_\nu)$ defined by the natural map $Z_\nu \to \mathbb{A}^1$. Let $Q \in R[T]$ be the characteristic polynomial of the multiplication by t on $\mathcal{O}(Z_\nu)$. Then the natural surjective morphism $T \mapsto t$, $R[T]/Q(T) \to \mathcal{O}(Z_\nu)$ is an isomorphism be equality of rank. Hence we get a factorization $F = QG$, with $G \in R\{\{T\}\}$ whose zero locus is Z'. For $x \in W$, we have a decomposition $F_x = Q_x G_x$, and the zeros of Q_x are exactly the zeros of F_x in Z_ν, that is such that $v_p(x) \geq -\nu$. It follows that $N(F_x, \nu) = N(Q_x, \nu) = \deg Q_x = \deg Q$. Therefore Q is ν-dominant of degree $N(F, \nu)$ (exercise II.1.4). Moreover since $1 = F(0) = Q(0)G(0)$ we can divides Q by $Q(0)$ and assume that $Q(0) = G(0) = 1$ and we have a decomposition (ii), and (iii) holds.

Conversely, given a decomposition (ii), then the locus $Q = 0$ is finite over W (remember that the dominant term of Q is invertible), and we easily see it is Z_ν.

Also, if (iii) holds, then the fibers of $f : \mathbb{Z}_\nu \to W$ all have the same degrees $N(F, \nu)$. Thus f is finite by [**Bu**, Corollary 4.3], and (i) holds.

So we have proven the equivalence of (i), (ii) and (iii), and that when they hold Z_ν is disconnected from its complement Z'. It follows that $(Q, R) = 1$ in $R\{\{T\}\}$. The uniqueness of the decomposition $F = QR$ follows from the proof since the map $T \mapsto t$, $R[T]/Q(T) \to \mathcal{O}(Z_\nu)$ has to be an isomorphism, which characterizes Q up to an invertible element in R, and that element is fixed by the condition $Q(0) = 0$.
□

DEFINITION II.1.8. A power series $F(T) \in R\{\{T\}\}$ with $F(0) = 1$ being fixed, we say that $\nu \in \mathbb{R}$ is *adapted* to F if the conditions of the above proposition are satisfied.

The main interest of this notion is in the following definition and proposition.

DEFINITION II.1.9. Let M be a Banach R-module satisfying property (Pr) ([**Bu**, §2]), and U_p a compact R-linear operator of M (*loc. cit.*). Let $F(T) \in R\{\{T\}\}$ be the Fredholm power series of U_p (that is $F(T) = \det(1 - TU_p)$ *loc. cit.*). If $\nu \in R$ is adapted for F, we write $M^{\leq \nu}$ for the kernel of the operator $Q^*(U_p)$, where Q is the polynomial appearing in the decomposition of F in (ii) of the above proposition, and Q^* is its reciprocal polynomial, and we call $M^{\leq \nu}$ the *submodule of M of slope less than ν*.

PROPOSITION II.1.10. *If ν is adapted for $F(T)$, then $M^{\leq \nu}$ is a finite flat module over R of rang $N(F, \nu)$. It is a direct summand of M. The formation of $M^{\leq \nu}$ commute with base change, in the sense that if $R \to R'$ is a morphism of affinoid rings over L, then ν is adapted for the image of $F(T)$ in $R'\{\{T\}\}$, and $(M \hat{\otimes}_R R')^{\leq \nu} = M^{\leq \nu} \otimes_R R'$ as submodules of $M \hat{\otimes}_R R'$.*

Proof —   The first sentence is [**Bu**, Theorem 3.3]. For the commutation with base change, note that $U_p' := U_p \hat{\otimes} 1$ is a compact operator of $M \hat{\otimes}_R R'$ whose power series is the image $F'(T)$ of $F(T)$ in $R'\{\{T\}\}$ by [**Bu**, Corollaries 2.9 and 2.10]. The decomposition $F = QG$ in $R\{\{T\}\}$ defines a decomposition $F' = Q'G'$ which obviously still satisfies (ii) of Prop II.1.7. Hence ν is stll adapted for F', and $Q'^*(U_p') = Q^*(U_p) \hat{\otimes} 1$. It follows from the proof of [**Bu**, Theorem 3.3] that $(M \hat{\otimes}_R R')^{\leq \nu} = M^{\leq \nu} \otimes_R R'$.   □

REMARK II.1.11. By its definition as the kernel of a polynomial in U_p, the submodule $M^{\leq \nu}$ of M is stable by any endomorphism of M that commutes with U_p.

For X an affinoid open of W, of affinoid ring R_X, we call $F_X(T)$ the image of $F(T)$ in $R_X\{\{T\}\}$, and define $Z_X = f^{-1}(X)$. Clearly Z_X is also the analytic subvariety in $X \times \mathbf{A}^1_{\mathrm{rig}}$ cut by F_X. We shall say that ν is *adapted* to X (or that X and ν are *adapted*) if ν is adapted to F_X. In this case $Z_{X,\nu} := Z_X \cap Z_\nu$ is finite flat surjective over X, and we have a decomposition $F_X(T) = Q_X(T)G_X(T)$ in $R_X\{\{T\}\}$ as in Prop. II.1.7.

Of course, this notion is useful only if we can prove that there are enough adpated (X, ν).

PROPOSITION II.1.12. *For any real ν, and any $x \in W$, there exists an admissible affinoid neighborhood X of x such that (X, ν) is adapted.*

Proof — Let $N = N(F_x, \nu)$. As $v_p(a_n) - n\nu$ goes to infinity, there exist an n_0 such that for all $n > n_0$, $v_p(a_n) - n\nu > v_p(a_N(x)) - N\nu$. Let X be the subdomain of W such that for all n, $0 \le n \le n_0$, and all $y \in X$, $v_p(a_n(y)) = v_p(a_n(x))$ if $a_n(x) \ne 0$, (resp. $v_p(a_n(y)) > v_p(a_N(x)) - N\nu + n\nu$ if $a_n(x) = 0$). Then X is clearly an affinoid admissible subdomain of W, and it is plain that for all $y \in X$, $N(F_y, \nu) = N(F_x, \nu) = N(F_X, \nu)$. Thus (X, ν) is adapted. $\square$

COROLLARY II.1.13. *The family of affinoid subdomains $Z_{X,\nu}$ of Z for X an affinoid of Z and ν adapted to X is a covering of Z.*

REMARK II.1.14. Actually, we can show using [**Bu**, §5], that this family is an **admissible** covering of Z. This is a slightly more precise version of the main technical result of [**Bu**], namely Theorem 4.6. The covering that Buzzard considers, $\mathcal{C}$ is actually larger, as it is the set of all affinoid subdomain Y of Z such that there exists a subdomain X in W such that $f(Z) = X$ and $f : Z \to X$ is finite. Observe that by definition, an affinoid subdomain of Z of the form $Z_{X,\nu}$ belongs to Buzzard's cover $\mathcal{C}$ if and only if ν is adapted to X. To check that our smaller set of affinoid subdomains is an also admissible cover of Z, we only have to observe that all the affinoid subdomains in $\mathcal{C}$ exhibited by Buzzard in order to prove his Theorem 4.6 are produced in [**Bu**, Lemma 4.5], and are actually (as stated in that lemma) of the form $Z_{X,\nu}$.

II.2. Links

DEFINITION II.2.1. If R is an affinoid ring, M and M' be two Banach R-modules satisfying properties (Pr) with morphisms $\psi : \mathcal{H} \to \mathrm{End}_R(M)$ and $\psi' : \mathcal{H} \to \mathrm{End}_R(M')$ such that $\psi(U_p)$ and $\psi'(U_p)$ are compact, we say that M and M' are *linked* if $\psi(U_p)$ and $\psi'(U_p)$ have the same characteristic power series $F(T)$, and for every monic polynomial $Q \in R[T]$ with invertible constant term, $\ker Q(\psi(U_p))$ and $\ker Q(\psi'(U_p))$ are isomorphic as R-modules with action of $\mathcal{H}$.

It is clear that to be linked is an equivalence relation. Observe that if M and M' are linked, then ν is adapted from M if and only if it is adpated for M', and then the two modules $M^{\leq \nu}$ and $M'^{\leq \nu}$ are isomorphic as R-modules with action of $\mathcal{H}$.

LEMMA II.2.2. *Let M and M' be as in the definition above. Assume that there is two continuous homomorphism of R-modules and $\mathcal{H}$-modules $f : M \to M'$ and $g : M' \to M$ such that $g \circ f = \psi(U_p)$ and $f \circ g = \psi(U'_p)$. If either f or g is compact, then M and M' are linked.*

Proof — That the characteristic power series of $\psi(U_p)$ and $\psi'(U_p)$ are the same results from the equality in $R\{\{T\}\}$: $\det(1 - Tfg) = \det(1 - Tgf)$ proved in [**Bu**, Lemma 2.12]. Moreover f maps $\ker Q(\psi(U_p))$ into $\ker Q(\psi'(U_p))$ since it is an $\mathcal{H}$-homomorphism and g maps $\ker Q(\psi(U_p))$ into $\ker Q(\psi(U_p))$. The compoistion $g \circ f = \psi(U_p)$ on $\ker Q(\psi(U_p))$ is invertible since Q has invertible constant term, and so is $f \circ g$. Thus $\ker Q(\psi(U_p))$ and $\ker Q(\psi'(U_p))$ are isomorphic. $\quad\square$

An interesting special case is the following:

LEMMA II.2.3. *Let M, M' be two R-modules with $\mathcal{H}$-actions as above. Assume that there are two homomorphisms of R-modules and $\mathcal{H}$-modules $f : M \to M'$ and $g : M' \to M$ such that f is injective and continuous, g is compact, and $f \circ g = \psi(U'_p)$. Then M and M' are linked. Moreover, if f factors as a composition $M \hookrightarrow M'' \hookrightarrow M'$, where M'' is as in the definition above and the two morphisms in this factorization are injective continuous morphism of R-modules and $\mathcal{H}$-modules, then M and M' are also linked to M''.*

Proof — For the first assertion, we observe that $f \circ g \circ f = \psi(U'_p) \circ f = f \circ \psi(U_p)$ hence by injectivity of f, $g \circ f = \psi(U_p)$ and we can apply Lemma II.2.2. The second assertion follows form the first by replacing M by M'', f by the injection $M'' \hookrightarrow M'$, and g by the composition of $g : M' \to M$ with the injection $M \to M''$. $\square$

LEMMA II.2.4. *If M and M' are linked, and $Sp\,R' \to Sp\,R$ is a morphism of affinoid, them $M \hat{\otimes}_R R$ and $M' \hat{\otimes}_R R'$ are also linked.*

Proof — This is clear from [**Bu**, Corollary 2.9] $\quad\square$

II.3. The eigenveriety machine

II.3.1. Eigenvariety data. An *eigenvariety data* over a finite extension L of $\mathbb{Q}_p$ is the data of

(ED1) A commutative ring $\mathcal{H}$, with a distinguished element U_p.

(ED2) A reduced rigid analytic variety $\mathcal{W}$ over L, and an admissible covering $\mathfrak{C}$ by admissible affinoid open subsets of $\mathcal{W}$.

(ED3) For every admissible affinoid $W = \operatorname{Sp} R$ of $\mathcal{W}$, $W \in \mathfrak{C}$, a Banach module M_W satisfying property (Pr), and a map $\psi_W : \mathcal{H} \to \operatorname{End}_R(M_W)$.

satisfying the following conditions:

(EC1) For every $W \in \mathfrak{C}$, $\psi_W(U_p)$ is compact on M_W.

(EC2) For every pair of affinoid subdomains $W' = \operatorname{Sp} R' \subset W = \operatorname{Sp} R \subset \mathcal{W}$, $W, W' \in \mathfrak{C}$, the R'-Banach modules $M_W \hat{\otimes}_R R'$ and $M_{W'}$ are *linked* (cf. Definition II.2.1)

In (ED3) it would suffice to give the modules M_W up to a link, as we will only use construction that depends only on the link-equivalence class of M_W

Given an eigenvariety data, and a closed point $w \in \mathcal{W}$ of field of definition $L(w)$, we can define the *fiber* M_w as follows: choose an affinoid admissible subdomain in $\mathfrak{C}$, $W = \operatorname{Sp} R$, containing w, and set $M_w = M_W \otimes_R L(w)$, where the implicit morphism $R \to L(w)$ is the obvious one (the evaluation at w, that is). Then the space M_w inherits a linear action of $\mathcal{H}$ ($\psi_w : \mathcal{H} \to \operatorname{End}_{L(w)}(M_w)$), and it is together with that action, well defined (independent of the choice of W) up to a link (by (ED2) and Lemma II.2.4). In particular, the subspace of finite slope $M_w^\#$ of M_w defined as

$$M_w^\# = \cup_{\nu < \infty} M_w^{\leq \nu},$$

is well-defined.

With no loss of generality, we can and will assume that if $X \subset W$ is an inclusion of affinoid subdomain of $\mathcal{W}$, with $W \in \mathfrak{C}$, then $X \in \mathfrak{C}$ as well. It suffices to define M_X as $M_W \otimes_R R'$ if $W = \operatorname{Sp} R$ and $X = \operatorname{Sp} R'$, which makes M_X well-defined up to a link.

II.3.2. Construction of the eigenvariety. For any admissible affinoid subdomain $W = \operatorname{Sp} R$ in $\mathfrak{C}$, let $F_W \in R\{\{T\}\}$ be the characteristic power series of U_p acting on M_W (that is $F_W(T) = \det(1 - T\psi_W(U_p))$.) To any such W, and $\nu \in \mathbb{R}$ adapted to W (that is, adapted to F_W), we attach a *local piece of the eigencurve* which is the following:

(a) An L-affinoid variety

$$\mathcal{E}_{W,\nu} = \operatorname{Sp} \mathcal{T}_{W,\nu}$$

called *the local piece*.

(b) A finite morphism $\kappa : \mathcal{E}_{W,\nu} \to W$ called *the weight map*.

(c) A morphism of rings $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E}_{W,\nu}) = \mathcal{T}_{W,\nu}$.

The construction is as follows: since ν is adapted, we can definite the finite flat R-submodule $M_W^{\leq \nu}$. Since $\mathcal{H}$ is commutative, it stabilizes $M_W^{\leq \nu}$ by Remark II.1.11. We can therefore let $\mathcal{T}_{W,\nu}$ be the eigenalgebra (cf. Chapter I) of $\mathcal{H}$ acting on $M_W^{\leq \nu}$, that is as the R-subalgebra of $\operatorname{End}_R(M_W)$ generated by $\psi_W(\mathcal{H})$.

Then $\mathcal{T}_{W,\nu}$ is a finite R-module, hence an affinoid algebra since R is, and we define $\mathcal{E}_{W,\nu} = \mathrm{Sp}\,\mathcal{T}_{W,\nu}$. The structural map defines a finite morphism of rigid affinoid variety $\kappa : \mathcal{E}_{W,\nu} = \mathrm{Sp}\,\mathcal{T}_{W,\nu} \to W = \mathrm{Sp}\,R$. And we have by construction a natural map $\psi : \mathcal{H} \to \mathcal{T}_{W,\nu}$.

LEMMA II.3.1. *Let $W' \subset W$ be an inclusion of admissible open affinoid of $\mathcal{W}$, with $W, W' \in \mathfrak{C}$. Let $\nu' \leq \nu$, and assume that (W, ν) and (W', ν') are adapted. Then there exists a unique open immersion $\mathcal{E}_{W',\nu'} \hookrightarrow \mathcal{E}_{W,\nu}$ compatible with the weight maps κ and the maps ψ from $\mathcal{H}$. Its image is the affinoid open subset of $\mathcal{E}_{W,\nu}$ defined by the equations $v_p(U_p(z)) \geq \nu'$, $\kappa(z) \in W'$.*

Proof — Note that since ν is adapted to W, it is also adapted to W'. By considering the intermediate (W', ν), we can reduce to the two cases where $W = W'$ and to the case where $\nu = \nu'$.

In the case $W = W' = \mathrm{Sp}\,R$, $\mathcal{E}_{W,\nu}$ and $\mathcal{E}_{W,\nu'}$ are the Hecke algebras attached to $M^{\leq \nu}$ and $M^{\leq \nu'}$ respectively. But it follows from the definitions that $M^{\leq \nu'}$ is a sub-R-module stable by $\mathcal{H}$ of M^{ν} which is a direct summand as R and $\mathcal{H}$-modules (indeed, $M^{\leq \nu}$ and $M^{\leq \nu'}$ are defined as the kernel of $Q^*(U_p)$ and $Q'^*(U_p)$ and we have that $Q = Q'Q''$ with $(Q', Q'') = 1$ by Prop. II.1.7 applied to Q instead of F). Thus $\mathcal{E}_{W,\nu}$ is the disjoint union of $\mathcal{E}_{W,\nu'}$ and its complement, and the result follows easily in this case.

We now treat the case where $W' = \mathrm{Sp}\,R' \subset W = \mathrm{Sp}\,R$ but $\nu = \nu'$. We have $(M_W \hat{\otimes}_R R')^{\leq \nu} \simeq M_{W'}^{\leq \nu}$ as $\mathcal{H}$-modules, since $M_W \hat{\otimes}_R R'$ and $M_{W'}$ are linked by hypothesis. Therefore their eigenalgebras are isomorphic. On the other hand we also have $(M_W \hat{\otimes}_R R')^{\leq \nu} = M_{\widehat{W}}^{\leq \nu} \otimes_R R'$ by Prop. II.1.10. Then, since R' is R-flat, by proposition I.4.1 we see that the Hecke-algebra of $M_{\widehat{W}}^{\leq \nu} \otimes_R R'$ is $\mathcal{T}_{W,\nu}\hat{\otimes}_R R'$. Combining the above, we have constructed an isomorphism (obviously R-linear and compatible with the morphisms from $\mathcal{H}$):

$$\mathcal{T}_{W,\nu}\hat{\otimes}_R R' = \mathcal{T}_{W',\nu}$$

which shows that $\mathcal{E}_{E',\nu}$ is the open affinoid of $\mathcal{E}_{W,\nu}$ defined by $\kappa(z) \in W'$.

Finally, the isomorphism we have constructed is unique since the eigenalgebras are generated by $\psi(\mathcal{H})$ over R. $\qquad\square$

The eigenvariety is then constructed by gluing the local pieces:

DEFINITION II.3.2. An *eigenveriety* for the above eigenvariety data, consists of

(a) A rigid analytic variety $\mathcal{E}/L$ (the *eigenvariety* proper)
(b) A locally finite map $\kappa : \mathcal{E} \to \mathcal{W}$.
(c) A morphism of rings $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E})$, that sends U_p to an invertible function.

such that for any affinoid subdomain W of $\mathcal{W}$ in $\mathfrak{C}$, and real ν adapted to W, the open subvariety $\mathcal{E}(W, \nu)$ of $\kappa^{-1}(W) \subset \mathcal{E}$ defined by $v_p(\psi(U_p)) \leq \nu$, is isomorphic

(as analytic variety over W with a map ψ form $\mathcal{H}$ to their ring of functions) to the local piece $\mathcal{E}_{W,\nu}$ constructed above, and such that the $\mathcal{E}(W,\nu)$ form an admissible covering of $\mathcal{E}$.

Note that the isomorphisms $\mathcal{E}(W,\nu) \simeq \mathcal{E}_{W,\nu}$ whose existences are required by the definition are necessarily unique.

THEOREM II.3.3. *For any given eigenvariety data, an eigenvariety exists and is unique up to unique isomorphism.*

Proof — We first prove the existence, by gluing, in two steps. First let W be an open admissible affinoid of $\mathcal{W}$, in $\mathfrak{C}$. We will construct an 'eigenvariety' $\mathcal{E}_W$ with the same data and properties as $\mathcal{E}$ but with $\mathcal{W}$ always replaced by W.

We consider the family of affinoid varieties $\mathcal{E}_{X,\nu}$ for $X \subset W$ an open affinoid subdomain, and ν a real number adapted to X. We define a gluing data on this family as follows: given two $\mathcal{E}_{X,\nu}$ and $\mathcal{E}_{X',\nu'}$ then by Lemma II.3.1 $(X \cap X', \min(\nu,\nu'))$ is adapted, and $\mathcal{E}_{X \cap X',\min(\nu,\nu')}$ can be seen as an open affinoid subset of both $\mathcal{E}_{X,\nu}$ and $\mathcal{E}_{X',\nu'}$, in a unique way respecting the maps form $\mathcal{H}$, and the weight maps. Hence we have defined two open affinoids of $\mathcal{E}_{X,\nu}$ and $\mathcal{E}_{X',\nu'}$, and an isomorphism between them. It is clear by uniqueness that those isomorphisms satisfy the cocycle condition. Thus, by [**BGR**, Prop. 9.3.2/1] we can glue the $\mathcal{E}_{X,\nu}$'s to form a rigid analytic variety $\mathcal{E}_W$, of which the $\mathcal{E}_{X,\nu}$'s are an admissible covering by admissible open affinoid subdomains. Then the maps $\kappa : \mathcal{E}_{X,\nu} \to X$ glue into a single map $\kappa : \mathcal{E}_W \to W$ by [**BGR**, Prop. 9.3.1/1]. Also the maps $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E}_{X,\nu})$ glue into a map $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E}_W)$ because the $\mathcal{E}_{X,\nu}$ form an admissible covering. The uniqueness of $\mathcal{E}_W$ up to unique isomorphism (compatible with κ and ψ) is obvious.

Finally, to construct $\mathcal{E}$ we have to glue the $\mathcal{E}_W$'s. The only minor difficulty is that because $\mathcal{W}$ is not itself an affinoid, the intersection of two admissible affinoid subdomains W and W' of $\mathcal{W}$ may not be affinoid. But $W \cap W'$ is an admissible open subdomain and can be admissibly covered by affinoids W_i. Therefore, we can define an isomorphism between the open admissible subdomains $\kappa^{-1}(W \cap W')$ in $\mathcal{E}_W$ and $\mathcal{E}'_W$ by gluing (using again [**BGR**, Prop. 9.3.1/1]) the isomorphism between the $\kappa^{-1}(W_i)$ in $\mathcal{E}_W$ and $\mathcal{E}_{W'}$ obtained by identifying the two $\kappa^{-1}(W_i)$'s considered as a subdomain of $\mathcal{E}_W$ or $\mathcal{E}_{W'}$ obtained from identifying them to the eigenvariety $\mathcal{E}_{W_i}$. Since by uniqueness those isomorphisms obviously satisfy the gluing condition, [**BGR**, Prop 9.2.1/1] and [**BGR**, Prop 9.3.1/1] gives us finally the eigenvariety $\mathcal{E}$ by gluing the $\mathcal{E}_W$, with a map $\kappa : \mathcal{E} \to \mathcal{W}$ gluing the maps $\kappa : \mathcal{E} \to W$ and a morphism of rings $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E})$ gluing the morphism $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{E}_W)$. The eigenvariety $\mathcal{E}$ with those maps obviously is unique up to unique isomorphism and satisfies all the property stated in the theorem. $\qquad\square$

EXERCISE II.3.4. Prove the following easy functoriality results for eigenvarieties.

1.– Show that the eigenvariety (with its maps κ and ψ) depends only of the link class of the modules M_W up to unique isomorphism.

2.– Show that the eigenvariety (with its maps κ and ψ) is not changed (up to unique isomorphsim) if we replace the covering $\mathfrak{C}$ by a finer covering.

3.– Consider two eigenvarety data with same (ED1) and (ED2) but such that for all $W \in \mathfrak{C}$, the modules M_W (for the first data) and M'_W (for the second data) are such that we have an embeeding $M_W \hookrightarrow M'_W$ of Banach R-modules $\mathcal{H}$-modules with closed image.

Let $\mathcal{E}$ and $\mathcal{E}'$ be the eigenvarieties constructed with those data. Show that there is a unique closed immersion $\mathcal{E} \to \mathcal{E}'$ compatible with κ and ψ.

4.– Same as 3. but we assume that for all $W \in \mathfrak{C}$, we have embeedings $M_W \hookrightarrow M'_W \hookrightarrow M_W^2$ of Banach R-modules and $\mathcal{H}$-modules with closed images. Then there is a unique isomorhism (compatible with κ and ψ) between $\mathcal{E}$ and $\mathcal{E}'$.

II.4. Properties of eigenvarieties

The most important property, which is the reason eigenvarieties deserve their name, is the following:

THEOREM II.4.1. *Let w be a closed point of $\mathcal{W}$, of field of definition $L(w)$. For any finite extension L' of $L(w)$, the set of L'-points z of $\mathcal{E}$ such that $\kappa(z) = w$ is in natural bijection with the systems of eigenvalues of $\mathcal{H}$ appearing in $M_w \otimes_{L(w)} L'$ such that the eigenvalues of U_p is non-zero. The bijection attaches to z the system of eigenvalues $\psi_z : \mathcal{H} \to L'$, $T \mapsto \psi(T)(z)$ for $T \in \mathcal{H}$.*

Note that since M_w is well-defined up to a link, the set of systems of eigenvalues of $\mathcal{H}$ appearing in $M_w \otimes_{L(w)} L'$ such that the eigenvalues of U_p is non-zero is (absolutely) well-defined.

Proof — Obviously it is enough to prove that for all $\nu \in \mathbb{R}$, $z \mapsto \psi_z$ is a bijection between the set of L'-points of $\mathcal{E}$ with $\kappa(z) = w$ and $v_p(U_p(z)) \leq \nu$ and systems of eigenvalues of $\mathcal{H}$ appearing in $M_w \otimes_{L(w)} L'$ such that the eigenvalues of U_p have valuation $\leq \nu$. Choose an affinoid admissible neighborhood $W = \mathrm{Sp}\, R$ of $w \in \mathcal{W}$ such that ν is adapted to W. It is enough to prove the last assertion with $\mathcal{E}$ replaced by its local piece $\mathcal{E}_{W,\nu}$, since on the one hand, every L'-point of $\mathcal{W}$ satisfying the desired condition is in $\mathcal{E}_{W,\nu}$, and on the other hand, $(M_w \hat{\otimes}_{L(w)} L')^{\leq \nu} = (M_{\overline{W}}^{\leq \nu}) \otimes_R L'$ (the implied map being $R \to L(w) \hookrightarrow L$) by Prop. II.1.10. Now by Lemma I.4.1, since L' is a field, the L'-points of $\mathcal{E}_{W,\nu}$ above w are the same as the L'-points of the eigenalgebra of $\mathcal{H}$ acting on M_w. The result thus follows from Corollary I.5.10. $\square$

EXERCISE II.4.2. Consider two eigenvarety data with same (ED1) and (ED2) but with different (ED3): for $W \in \mathfrak{C}$ we call M_W and M'_W the $\mathcal{H}$-modules of the data. Show that $W \mapsto M''_W = M_W \oplus M'_W$ defines also an eigenvareity data. Let us call $\mathcal{E}$, $\mathcal{E}'$ and $\mathcal{E}''$ the eigenvarieties defined by those data. There are unique injective map $\mathcal{E} \hookrightarrow \mathcal{E}''$ and $\mathcal{E}' \hookrightarrow \mathcal{E}''$ compatible with κ and ψ by Exercise II.3.4. Show that $\mathcal{E}''$ is the union of the image of $\mathcal{E}$ and $\mathcal{E}'$.

COROLLARY II.4.3. *The image of $\kappa : \mathcal{E} \to \mathcal{W}$ consists of all the w such that the action of U_p on M_w is not nilpotent. It is a Zariski-open subset in $\mathcal{W}$.*

Proof — A point w is in the image if and only if there is a system of $\mathcal{H}$-eigenvalues (over $\bar{L}(w)$) of finite slope in M_w. If there is such a system, there is a non-zero eigenvector for U_p in $M_w \otimes \bar{L}(w)$ with a non-zero eigenvalue, and U_p is not nilpotent. Conversely, if U_p is not nilpotent, it has by Riesz theory a non-zero eigenvalue, say of valuation ν, and after extending the scalar there is an $\mathcal{H}$-eigenvector in $M_w^{\leq \nu}$.

To prove the second assertion, for all $W = \operatorname{Sp} R \in \mathfrak{C}$, write $1 + \sum_{i=1}^{\infty} a_{i,W}$ the Freedholm power series of U_p on M_W. By (EC2), the $a_{i,W}$ glues to defines function $a_i \in \mathcal{O}(W)$. For $w \in \mathcal{W}$, the Freedholm series of U_p on M_w is $1 + \sum_{i=1}^{\infty} a_i(w)T^i$. Hence by the above, w is in the image of κ if and only if at leats one of tha $a_i(w)$ is non-zero. The image of κ is thus Zariski-open, as the complement of the analytic subspace defined by the a_i's $\qquad \square$

We shall give below (Prop. **??**), under some mild hypothesis on $\mathcal{W}$, a result precising the second assertion of that corollary.

To state the next property, we shall need to recall some facts about the delicate notion of irreducible component of a rigid analytic space X. When $X = \operatorname{Sp} R$ is affinoid, one defines naturally its rigid analytic components as the subspaces $X_i = \operatorname{Sp} R/\mathfrak{p}_i$ of X, where the $\mathfrak{p}_i$ are the (finitely many, since R is noetherian) minimal prime ideals of R. In general, irreducible components have been defined by Conrad (cf. [**Con1**]): For every X one defines an irreducible component of X as the image in X of a connected component of the normalization of X. One says that X is irreducible if it has only one irreducible component.

LEMMA II.4.4.   (i) *The space X is the union of its connected components X_i, which may be infinitely many.*
 (ii) *A subset Z of X is Zariski-dense if and only if $Z \cap X_i$ is Zariski-dense in X_i for every i.*
(iii) *Every irreducible component X_i of X is irreducible.*
(iv) *If X is irreducible, it has a dimension, which is the common dimension of all rings of functions R of its admissible open affinoid $U = \operatorname{Sp} R$.*

(v) *The intersection $X_i \cap U$ of an irreducible component X_i with an open admissible affinoid U of X is either empty or a union of irreducible components of U.*

(vi) *If X is irreducible, any non-empty admissible open affinoid U is Zariski-dense in X.*

(vii) *If $f : X \to X'$ is a finite map, the image of an irreducible component of X by f is an irreducible closed analytic subspace of X'.*

(viii) *Let $W = Sp\,R$ is an affinoid, $F \in R\{\{T\}\}$, $F(0) = 1$ and Z be the hypersurface cut out by F in $W \times \mathbf{A}^1_{\mathrm{rig}}$. Then F has a unique decomposition $F = \prod_i F_i(t)^{n_i}$, $F_i(0) = 1$, where the series $F_i(t)$ generate prime ideals in $R\{\{T\}\}$ and the irreducible components of Z are precisely the hypersurface $Z_i \subset W \times \mathbf{A}^1_{\mathrm{rig}}$ cut out by the F_i.*

Proof —   For (v), see [**Con1**, 2.2.9]. For (vii) see [**Con1**, 2.2.3]. For (viii), see [**Bu**, 1.1.10] [**Con1**, 4.3.2] $\square$

One says that a rigid analytic space X is *equidimensional of dimension n* if all its connected component have dimension n. By (iv) and (v), this is equivalent to saying that for all admissible open affinoids $U = \mathrm{Sp}\,R$, R is equi-dimensional of dimension n in the algebraic sense, that is to say all the domains $R/\mathfrak{p}_i$ for $\mathfrak{p}_i$ minimal prime ideals have Krull dimension n.

PROPOSITION II.4.5. *Assume that $\mathcal{W}$ is equidimensional of dimension n. Then so is $\mathcal{E}$.*

Proof —   Since being equidimensional of dimension n is a local property as we just saw, this results from Prop. I.10 and the remark following it. $\square$

The next proposition proves the few properties of eigenvarieties for which we need the full force of Buzzard's theory, namely the result stated in Remark II.1.14 that the admissible pairs (X, ν) defines an *admissible* covering of Z.

PROPOSITION II.4.6.   (i) *The map $\kappa \times \psi(U_p)^{-1} : \mathcal{E} \to \mathcal{W} \times \mathbf{A}^1_{\mathrm{rig}}$ is finite.*

(ii) *The eigenvariety $\mathcal{E}$ is separated.*

(iii) *Assume that $\mathcal{W}$ is equidimensional, and that a subset of $\mathcal{W}$ is Zariski-open in $\mathcal{W}$ is and only if any of its intersection with W (for $W \in \mathfrak{C}$) is Zariski-open in W. If D is an irreducible component of $\mathcal{E}$, then $\kappa(D)$ is Zariski-open in $\mathcal{W}$*

Proof —   For (i) it is enough to prove that for all affinoid subdomains $W = \mathrm{Sp}\,R \subset \mathcal{W}$ the map $\kappa \times \psi(U_p)^{-1} : \mathcal{E}_W \to W \times \mathbf{A}^1_{\mathrm{rig}}$ is finite. This map factors through a map $\mathcal{E}_W \to Z_W$, where Z_W is the closed subvariety of $W \times \mathbf{A}^1_{\mathrm{rig}}$ cut out by the Freedholm power series $F_W(T)$ of U_p acting on M_W and it is enough to show that $\mathcal{E}_W \to Z$

is a finite map. Since $\mathcal{E}_W$ is by definition/construction admissibly covered by the local pieces $\mathcal{E}_{X,\nu}$ for adapted pair $X \subset W, \nu \in \mathbb{R}$, and Z_W is **admissibly** covered by the affinoid $Z_{X,\nu}$ for the same pairs (Remark II.1.14), it suffices to show that the natural maps $\mathcal{E}_{X,\nu} \to Z_{X,\nu}$ are finite. But the composition $\mathcal{E}_{X,\nu} \to Z_{X,\nu} \to X$ is finite by construction of $\mathcal{E}_{X,\nu}$. Assertion (ii) follows from (i) since a finite map is separated and $\mathcal{W} \times \mathbf{A}^1_{\mathrm{rig}}$ is separated.

Let us prove (iii). By the hypothesis made on $\mathcal{W}$, it suffices to show that $\kappa(D \cap \mathcal{E}_W)$ is Zariski-open in W. By Lemma II.4.4, $D \cap \mathcal{E}_W$ is a union of irreducible component of $\mathcal{E}_W$ (or is empty, in which case there is nothing to prove), so we may assume that $D \cap \mathcal{E}_W$ is an irreducible component D_W of $\mathcal{E}_W$. Since $\mathcal{E}_W \to Z_W$ is a finite map, it sends D_W to a closed analytic subspace of Z_W of same dimension $\dim \mathcal{E}_W = \dim W$, irreducible by Lemma II.4.4. Since Z_W is equidimensional of dimension $\dim \mathcal{W} = \dim W$ (same proof as Prop. II.4.5), the image of D_W in Z_W is an irreducible component Z'_W of Z_W, hence is the locus cut out in $W \times \mathbf{A}^1_{\mathrm{rig}}$ by a power series $F'(T) = 1 + \sum_{i=1}^{\infty} a_i T^i \in R\{\{T\}\}$. The image by the first projection map $Z_W \times W$ is the complementary in W of the closed analytic subspaces defined by the equations $a_i = 0$, hence is Zariski-open. But this is also trivially $\kappa(D_W)$. $\square$

REMARK II.4.7. The condition in (c) of the above proposition is satisfied for instance when $\mathcal{W}$ is affinoid, or when $\mathcal{W}$ is an open ball of radius 1 in $\mathbf{A}^1_{\mathrm{rig}}$. For by Lazard's theory ([**L**]), a proper subspace is Zariski-closed in $\mathcal{W}$ if and only if its intersection with each of the closed ball of radius $r < 1$ is finite, that is, Zariski-closed in that ball.

For the definition of *nested*, and the easy proof of the next proposition, we refer the reader to [**BC**, Definition 7.2.10 and Lemma 7.2.11]:

PROPOSITION II.4.8.
(i) *The eigenvariety $\mathcal{E}$ is nested.*
(ii) *If $\mathcal{O}(\mathcal{E})$ is provided with the coarsest locally convex topology such that the restriction maps $\mathcal{O}(\mathcal{E}) \to \mathcal{O}(U)$ for every affinoid subdomain $U \subset \mathcal{E}$, and if $\mathcal{E}$ is reduced, then the subring $\mathcal{O}(\mathcal{E})^0$ of power-bounded functions in $\mathcal{O}(\mathcal{E})$ is compact.*

A criterion for $\mathcal{E}$ to be reduced is given in the next section.

II.5. A comparison theorem for eigenvarieties

In this section we present two very useful theorems of Chenevier: one gives a sufficient condition for an eigenvariety to be reduced, and the other allows to compare two eigenvarieties. Both rely on the notion of *classical structure*.

II.5.1. Classical structures.

DEFINITION II.5.1. Let $\mathcal{W}$ ba a reduced rigid analytic space over $\mathbb{Q}_p$, and $X \subset \mathcal{W}$. We say that X is very Zariski-dense in $\mathcal{W}$ if for every $x \in X$ there is a basis of open affinoid neighborhoods V of x in X such that $V \cap X$ is Zariski-dense in V.

EXERCISE II.5.2. Let $\mathcal{W}$ be the rigid affine line. Show that any set of the form $a + b\mathbb{N}$ with $a, b \in \mathbb{Z}$, $b \neq 0$ is very Zariski-dense

EXERCISE II.5.3. Let $\mathcal{W}$ be the rigid affine plane (dimension 2). Find a set $X \subset \mathcal{W}$ which is Zariski dense but such that for all affinoid open V of $\mathcal{W}$, $V \cap X$ is not Zariski-dense in X.

Now let us fix an eigenvarity data as in §II.3.1, that is (ED1) $\mathcal{H}$ a ring with a distinguished $U_p \in \mathcal{H}$, (ED2) $\mathcal{W}$ a reduced rigid space with an admissible covering $\mathfrak{C}$, and $(ED3)$ Banach modules M_W with an action of $\mathcal{H}$ for every $W \in \mathfrak{C}$, satisfying conditions (EC1) and (EC2).

DEFINITION II.5.4. A *classical structure* on an eigenvarity data is the data of

(CSD1) a very Zariski-dense subset $X \subset \mathcal{W}$;

(CSD2) for every $x \in X$, a finite-dimensional $\mathcal{H}$-module M_x^{cl}.

such that

(CSC1) For every $x \in X$, there exists an $\mathcal{H}$-equivariant injective map $M_x^{\mathrm{cl,ss}} \hookrightarrow M_x^{\#,\mathrm{ss}}$

(CSC2) For every $\nu \in \mathbb{R}$, let X_ν be the set of x in X such that there exist an $\mathcal{H}$-equivariant isomorphism $M_x^{\mathrm{cl}, \leq \nu} \simeq M_x^{\leq \nu}$; then for every x in X, there exists a basis of neighborhoods of x in V such that $X_\nu \cap V$ is Zariski-dense in V.

II.5.2. A reducedness criterion.

THEOREM II.5.5 (Chenevier). *Consider an eigenverity data with a classical structure as above. If for every $x \in X$, M_x^{cl} is semi-simple as an $\mathcal{H}$-module, then the rigenvariety $\mathcal{E}$ is reduced.*

For the proof, see [**C2**]. I'll try to add a proof here later.

II.5.3. A comparison theorem.

THEOREM II.5.6 (Chenevier). *Suppose that we have two eigenvariety datas with the same data (ED1) and (ED2) (that is the same $\mathcal{H}$, $\mathcal{W}$ and $\mathfrak{C}$), but different Banach modules (ED3), that we shall denote by M_W amd M'_W. Let us call $\mathcal{E}$ and $\mathcal{E}'$ the two eigenvarietires attached to those data. Assume that those two eigenvarieties are each provided with a classical structure with the same set X (CSD1), but with different $\mathcal{H}$-modules M_x^{cl} and M'^{cl}_x. Suppose that for every x in X, there exists an $\mathcal{H}$-equivariant injective map:*

$$M_x^{\mathrm{cl},ss} \hookrightarrow M'^{\mathrm{cl},ss}_x.$$

Then there exists a unique closed embeeding $\mathcal{E} \hookrightarrow \mathcal{E}'$ compatible with the weight maps to $\mathcal{W}$ and with tha maps $\mathcal{H} \to \mathcal{E}$ and $\mathcal{H} \to \mathcal{E}$.

For the proof, see [**C2**]. I'll try to add a proof here later.

COROLLARY II.5.7. *Same hypothesis but we assume that for every x in X, there exists an isomorphism of $\mathcal{H}$-modules*

$$M_x^{\mathrm{cl},ss} \simeq M'^{\mathrm{cl},ss}_x.$$

Then there exists a unique isomorphism $\mathcal{E} = \mathcal{E}'$ compatible with the weight maps to $\mathcal{W}$ and with tha maps $\mathcal{H} \to \mathcal{E}$ and $\mathcal{H} \to \mathcal{E}$.

II.6. Notes and references

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Part 2

Modular Symbols, the Eigencurve, and p-adic L-functions

Modular symbols

III.1. Abstract modular symbols

III.1.1. Notion of modular symbols. Let Δ be the abelian group of divisors on $\mathbb{P}^1(\mathbb{Q})$, that is the group of formal sums $\sum_{x \in \mathbb{P}^1(\mathbb{Q})} n_x \{x\}$, with $n_x \in \mathbb{Z}$ and $n_x = 0$ for all x but a finite number. Let Δ_0 be the subgroup of divisors of degree 0, that is such that $\sum_x n_x = 0$. The natural action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$, $\gamma \cdot x = \frac{ax+b}{cx+d}$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, induces an action of $\mathrm{GL}_2(\mathbb{Q})$ on Δ and on Δ_0, respecting the group structures.

Exercise III.1.1. 1.– Prove Manin's lemma: Δ_0 is generated, as an $\mathrm{SL}_2(\mathbb{Z})$-module, by the divisor $\{\infty\} - \{0\}$. You can proceed as follows:

a.– Prove Pick's lemma: a non-flat triangle with vertices in $\mathbb{Z}^2$ has area $\geq 1/2$, with equality if and only if the three vertices are the only points of $\mathbb{Z}^2$ in the triangle.

b.– Let a/c and b/d be two irreducible fractions. Show that there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $\gamma(\{\infty\} - \{0\}) = \{a/c\} - \{b/d\}$ if and only if the triangle of vertices $(0,0), (a,c), (b,d)$ has area $1/2$.

c.– Conclude.

2.– If Γ is a finite index subgroup of $\mathrm{GL}_2(\mathbb{Z})$, then Δ_0 is finitely generated as a $\mathbb{Z}[\Gamma]$-module.

3.– Prove Pick's theorem: If P is a polygon whose all vertices are in $\mathbb{Z}^2$, then the area of P is $I + \frac{F}{2} - 1$ where I is the number of points of $\mathbb{Z}^2$ in the interior of P, and F the number of points of $\mathbb{Z}^2$ on its boundary.

Let Γ be a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$. For V any right Γ-module, that is any abelian group with a Γ-action on the right (denoted bt $v \mapsto v_{|\gamma}$), there is a right action of Γ on $\mathrm{Hom}(\Delta_0, V)$ given by

(14) $$(\phi_{|\gamma})(D) = \phi(\gamma \cdot D)_{|\gamma}$$

for γ in Γ and we define the group of V-valued modular symbols (for Γ),

$$\mathrm{Symb}_\Gamma(V) = \mathrm{Hom}(\Delta_0, V)^\Gamma.$$

Observe that if V is an R-module, for R any commutative ring, with a right Γ-action that preserves its R-module structure, then $\mathrm{Symb}_\Gamma(V)$ has an obvious structure of R-module. The construction $V \mapsto \mathrm{Symb}_\Gamma(V)$ is functorial (from the categories of R-modules with right Γ-actions to the category of R-modules).

LEMMA III.1.2. *The functor* $\mathrm{Symb}_\Gamma(V)$ *is left-exact and commute with flat base change* $R \to R'$ *(that is, if* V *is an* R*-module with right action, there is a natural isomorphism* $\mathrm{Symb}_\Gamma(V) \otimes_R R' \simeq \mathrm{Symb}_\Gamma(V \otimes_R R')$ *when* R' *is* R*-flat).*

Proof — The left-exactness is clear since both $V \mapsto \mathrm{Hom}(\Delta_0, V)$ and the functor of Γ-invariants are left exact. For the second part, first note that $\mathrm{Symb}_\Gamma(V) = \mathrm{Hom}_{\mathbb{Z}[\Gamma]}(\Delta_0, V)$ if we give V its left Γ-structure $\gamma \cdot v = v_{|\gamma^{-1}}$. Now Δ_0 has a finite presentation $\mathbb{Z}[\Gamma]^r \xrightarrow{\mu} \mathbb{Z}[\Gamma]^n \to \Delta_0 \to 0$ as $\mathbb{Z}[\Gamma]$-module (this follows from Exercise III.1.1, point 2 and the fact that $\mathbb{Z}[\Gamma]$ is left-Noetherian; also an explicit such presentation, called *Manin's relations*, can be found in [**PS1**]. The letter μ is supposed to recall Manin's name). Therefore, $\mathrm{Symb}_\Gamma(V) = \ker(V^n \xrightarrow{\mu^*} V^r)$, where μ^* is the obvious map defined from μ. Since the formation of μ^* obviously commutes with arbitrary base change $R \to R'$, the formation of its kernel commutes with any flat base change. $\square$

REMARK III.1.3. Note that there is no hypotheses of finiteness on V nor on the map $R \to R'$ in the above lemma. Indeed we shall have to apply this lemma to huge V's and all kind of flat base changes.

DEFINITION III.1.4. We shall denote by $\mathrm{BSymb}_\Gamma(V)$ the image of the restriction map (called the *boundray map*) $\mathrm{Hom}(\Delta, V)^\Gamma \to \mathrm{Hom}(\Delta_0, V)^\Gamma = \mathrm{Symb}_\Gamma(V)$. An element of $\mathrm{BSymb}_\Gamma(V)$ is called a *boundary modular symbol*.

EXERCISE III.1.5. (**easy**) Show that the kernel of the boundary map is V^Γ. Show that the formation of $\mathrm{BSymb}_\Gamma(V)$ commute with flat base change.

III.1.2. Action of the Hecke operators. Now we suppose given a sub-monoid Σ of $\mathrm{GL}_2(\mathbb{Q})$, containing Γ. If V is a right Σ-module, then $\mathrm{Symb}_\Gamma(V)$ is by definition the group of Γ-invariants in the module $\mathrm{Hom}(\Delta_0, V)$ on which Σ acts by formula (14). Therefore, there is an action (on the right) of the Hecke ring $\mathcal{H}(S, \Gamma)$ of bi-Γ-invariant $\mathbb{Z}$-valued functions on S, and whose support is a finite union of double classes of the form $\Gamma s \Gamma$. Explicitly, the characteristic function of $\Gamma s \Gamma$ (for some $s \in S$), which we note $[\Gamma s \Gamma]$ acts on $\mathrm{Symb}_\Gamma(V)$ by

$$\phi_{|[\Gamma s \Gamma]}(D) = \sum_i \phi(s_i \cdot D)_{|s_i},$$

$$\text{where } \Gamma s \Gamma = \coprod_i \Gamma s_i \quad \text{(finite decomposition)}$$

EXERCISE III.1.6. Prove that this formula is independent of the choice of the s_i's.

The construction $V \mapsto \mathrm{Symb}_\Gamma(V)$ is a functor form the category of right Σ-modules to the category of right-$\mathcal{H}(S, \Gamma)$-modules. When V is an R-module, and

the action of Σ is by R-linear operators, then obviously the Hecke operators are R-linear on $\mathrm{Symb}_\Gamma(V)$.

We use the classical notation T_l for $[\Gamma \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \Gamma]$, replaced by U_l when l divides the level of Γ (that is $l|N$ when $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$).

EXERCISE III.1.7. **(easy)** Let $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$. If V is $\mathbb{Q}$-vector space, with a right action of Σ, and $n \in \mathbb{Z}$, we left $V(n)$ be the group V with action of $s \in \Sigma$ multiplied by $(\det s)^n$. That is, if $v \in V$, $v_{|V(n)s} = (\det s)^n v_{|s}$. Note that the Γ-action on V and $V(n)$ are the same, so $\mathrm{Symb}_\Gamma(V) = \mathrm{Symb}_\Gamma(V(n))$. However, the action of the Hecke operators are different. Show that the action of T_l for $l \nmid N$ (resp. U_l for $l|N$) on $\mathrm{Symb}_\Gamma(V(n))$ is l^n (resp. p^n) times its action on $\mathrm{Symb}_\Gamma(V)$.

III.1.3. Relations with cohomology. Let H be an abelian group. If X is a topological space, the constant sheaf H_X attached to H is defined as the sheaf in abelian groups attached to the presheaf $U \mapsto H$ for any open $U \subset X$.

A *local system* (or *locally constant sheaf*) on a space X is a sheaf in abelian groups G on X such that there exists a group H such that for any $x \in X$, there is an open set U of X containing x such that $G_{|U} \simeq H_U$ as sheaves of abelian group over U.

The abelian group H, if it exists, is obviously unique, up to isomorphism. The fiber of a local system at any point x is isomorphic to H.

EXERCISE III.1.8. Assume that X is path-connected. Let G be a locally free sheaf on X.

1.– If γ is a path in X from x to y, explain how to define an isomorphism $i_\gamma : G_x \to G_y$, and show that it i_γ depends only of the homotopy class in γ.

2.– If γ' is a path from y to z, show that $i_{\gamma'} \circ i_\gamma = i_{\gamma \circ \gamma'}$ where $\gamma \circ \gamma'$ is the concatenation of γ and γ'. In particular, if $x \in X$, we have a natural action of $\pi_1(X, x)$ on G_x called the *monodromy* action.

3.– Show that if $x \in X$ is fixed, you have defined a functor $G \mapsto G_x$ from the category of locally constant sheaves to the category of abelian groups with an action of $\pi^1(X, x)$. Explain how this functor depends on the choice of the base point x.

If we assume some relatively mild hypothesis on X, such as its being locally connected, and locally simply connected, then this functor is an equivalence of category. (The inverse functor is essentially the construction $V \mapsto \tilde{V}$ explained below, with Γ replaced by $\pi^1(X, x)$ and $U(\Gamma)$ replaced by the universal cover of X.)

Let us recall that Steenrod ([**ST**]) defined (on some mild assumptions on X that will always be satisfied in our situations) groups of cohomology with value in a local systems $H^i(X, G)$, homology $H_i(X, G)$, and cohomology with compact support $H^i_c(X, G)$

Let us also recall that for any discrete group Γ one can prove the existence of a contractible space $U(\Gamma)$ on which Γ acts freely and discontinuously. We can then form the quotient $B(\Gamma) = U(\Gamma)/\Gamma$ which is unique up to homotopy equivalence: this space is called the *classifying space* of Γ. It is path-connected, its π_1 is Γ while its π_n are trivial for $n \geq 2$. For example, when Γ is a congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ acting **freely** on $\mathcal{H}$, e.g. $\Gamma = \Gamma_1(N)$ with $N > 3$, we can take $U(\Gamma) = \mathcal{H}$ and $B(\Gamma) = \mathcal{H}/\Gamma = Y_\Gamma(\mathbb{C})$.

If V is any Γ-module, we can form a sheaf $\tilde{V}$ on $B(\Gamma)$ as follows: we let Γ acts on $U(\Gamma) \times V$ (product in the category of topologial spaces, with V given the discrete topology) by $\gamma(u, v) = (\gamma(u), \gamma(v))$. There is a projection map $(u, v) \mapsto u$, $(U(\Gamma) \times V)/\Gamma \to B(\Gamma)$, and the sheaf $\tilde{\Gamma}$ is then defined as the sheaf of continuous sections of this map. It is easily seen that locally on $B(\Gamma)$, $\tilde{V}$ is isomorphic to the constant sheaf with fiber V. Of course, $V \mapsto \tilde{V}$ is functorial, and is even an equivalence of categories between the category of Γ-modules V and the categories of local systems $\tilde{V}$ on $B(\Gamma)$. The interest of this construction is that it identifies group cohomology with a special case of sheaf cohomology.

PROPOSITION III.1.9. *We have canonical and functorial isomorphisms $H^i(\Gamma, V) \simeq H^i(B(\Gamma), \tilde{V})$. Those isomorphisms are also compatible with the connection morphisms appearing in long exact sequences for both cohomology theories. The same is true with cohomology replaced with homology.*

Note that the cohomology groups with compact support of a space are not a homotopy invariant of that space, so the group $H^i_c(B(\Gamma), \tilde{D})$ depends on the choice of the space $B(\Gamma)$. However, when Γ is a congruence subgroup acting freely on $\mathcal{H}$, we have a natural choice of $B(\Gamma)$, and we use it to define cohomology group with compact support:

DEFINITION III.1.10. Assume Γ acts freely on $\mathcal{H}$. For any Γ-module V, we set $H^i_c(\Gamma, V) = H^i_c(Y_\Gamma(\mathbb{C}), \tilde{V})$

When 6 acts as an automorphism of V, we can extend in an ad hoc manner this definition for a Γ that does not act freely as follows: the congruence subgroup $\Gamma' = \Gamma \cap \Gamma(3)$ always acts freely on $\mathcal{H}$. Since Γ' is normal in Γ with index dividing the order of $\mathrm{SL}_2(\mathbb{Z}/3\mathbb{Z})$, that is $24 = 2^3 \times 3$, the functor "taking the Γ/Γ' invariant" is exact in the categories of abelian groups on which 6 acts invertible with action of Γ/Γ'. Therefore, the Hochchild-Serre spectral sequence (or the inflation-restriction exact sequence) gives

$$(15) \qquad\qquad H^i_c(\Gamma, V) = H^i_c(\Gamma', V)^{\Gamma/\Gamma'}$$

when Γ acts freely on $\mathcal{H}$. When Γ does not acts freely, we thus can take the right hand side of (15) as the definition of the left hand side.

In the rest of this section, we shall always assume either that Γ acts freely on $\mathcal{H}$ or that 6 acts as an isomorphisms on the Γ-modules involved.

We shall do the proofs only in the case where Γ acts freely, since the other case will always be provable by an application of the free case to Γ' and taking the functor of Γ/Γ'-invariants.

Returning to modular symbols, we have the following theorem.

THEOREM III.1.11. *(Ash-Stevens) Let Γ be a congruence ungroup of $\mathrm{SL}_2(\mathbb{Z})$. Then there are canonical and functorial isomorphisms*

$$\mathrm{Symb}_\Gamma(V) \simeq H^1_c(\Gamma, V)$$

Proof — Setting $\delta(\bar{\mathcal{H}}) = \mathbb{P}^1(\mathbb{Q})$, we endow $\bar{\mathcal{H}} := \mathcal{H} \cup \delta(\bar{\mathcal{H}})$ with its usual compact topology defined for example in Shimura's book ([**Shi**]). We call V the constant sheaf of fiber V over $\mathcal{H} \cup \delta(\mathcal{H})$. The long exact sequence of cohomology attached to a closed subset gives us a short exact sequence (see for example [**G**])

$$0 \to H^0(\bar{\mathcal{H}}, V) \to H^0(\delta(\bar{\mathcal{H}}), V) \to H^1_c(\mathcal{H}, V) \to 0$$

Indeed, the preceding term is $H^0_c(\mathcal{H}, V)$ which is 0 since $\mathcal{H}$ is connected and non-compact, and the next term is $H^1(\bar{\mathcal{H}}, V)$ which is 0 since $\bar{\mathcal{H}}$ is contractible.

On the other hand, the exact sequence of abelian groups $0 \to \Delta_0 \to \Delta \to \mathbb{Z} \to 0$ is split, so it gives rise to a short exact sequence

(16) $$0 \to V \to \mathrm{Hom}(\Delta, V) \to \mathrm{Hom}(\Delta_0, V) \to 0.$$

Actually, the two exact sequences are isomorphic as exact sequence of Γ-modules. To see this, it suffices to construct compatible Γ-equivariant isomorphisms between the first terms and between the second terms of the two exact sequences; but $H^0(\bar{\mathcal{H}}, V) = V$ since $\bar{\mathcal{H}}$ is connected, and $H^0(\delta(\bar{\mathcal{H}}), V) = H^0(\mathbb{P}^1(\mathbb{Q}), V) = \mathrm{Hom}(\Delta, V)$ since Δ is the free group generated by $\mathbb{P}^1(\mathbb{Q})$.

Therefore, the third terms on the two exact sequences are isomorphic (as Γ-modules):

$$\mathrm{Hom}(\Delta_0, V) = H^1_c(\mathcal{H}, V).$$

Taking the Γ-invariants, we get

$$\mathrm{Symb}_\Gamma(V) = H^1_c(\mathcal{H}, V)^\Gamma = H^1_c(\mathcal{H}/\Gamma, \tilde{V}),$$

where the last equality follows from Leray Spectral Sequence $H^i(\Gamma, H^j_c(\mathcal{H}), V) \Rightarrow H^{i+j}_c(\mathcal{H}/\Gamma, \tilde{V})$, noting a second time that $H^0_c(\mathcal{H}, V) = 0$. By definition of $H^1_c(\Gamma, V)$, we are done in the case where Γ acts freely on $\mathcal{H}$. The case when 6 acts invertibly on V is easily reduced to that case. $\qquad\square$

We note a scholium and two corollaries.

SCHOLIUM III.1.12. *We have a long exact sequence*

$$0 \to V^\Gamma \to \mathrm{Hom}(\Delta, V)^\Gamma \to \mathrm{Symb}_\Gamma(V) = H^1_c(\Gamma, V) \to H^1(\Gamma, V),$$

where the last map is the natural map from cohomology with compact support to cohomology.

Proof —   This exact sequence is just a part long exact sequence attached to the short exact sequence of Γ-modules (16). The fact that the last map is the natural map follows from the fact that in the long exact sequence for the functor of Γ-invariants, attached to (16), the connecting homomorphism

$$H_c^1(\Gamma, V) = H_c^1(\mathcal{H}, V)^\Gamma \to H^1(\Gamma, H^0(\bar{\mathcal{H}}, V)) = H^1(\Gamma, V)$$

is the natural map. $\square$

COROLLARY III.1.13. *If* $0 \to V_1 \to V_2 \to V_3 \to 0$ *is an exact sequence of* Γ-*modules, then we have a long exact sequence*

$$0 \to \mathrm{Symb}_\Gamma(V_1) \to \mathrm{Symb}_\Gamma(V_2) \to \mathrm{Symb}_\Gamma(V_3) \to H_c^2(\Gamma, V_1) \to H_c^2(\Gamma, V_2) \to H_c^2(\Gamma, V_3) \to 0.$$

Proof —   Follows from the long exact sequence for H_c^i, using that $H_c^3(\Gamma, V_1) = 0$ since $Y(\Gamma)$ has real dimension 2. $\square$

COROLLARY III.1.14. *Assume that* V *is a torsion-free abelian group. Then* $\mathrm{Symb}_\Gamma(V) = H_c^1(\Gamma, V)$ *is torsion-free.*

Proof —   Let n be an integer ≥ 1, and consider the sequence $0 \to V \overset{\times n}{\to} V \to V/nV \to 0$, which is exact since V is torsion free. The long exact sequence for H_c^i gives an exact sequence

(17) $H_c^i(\Gamma, V) \otimes_{\mathbb{Z}} \mathbb{Z}/N\mathbb{Z} \to H_c^i(\Gamma, V/nV) \to H_c^{i+1}(\Gamma, V)[n] \to 0.$

For $i = 0$, noting that $H_c^0(\Gamma, V/nV) = 0$ since $Y(\Gamma)$ is connected, we get that $H_c^1(\Gamma, V)[n] = 0$, hence the result. $\square$

III.1.4. Duality in algebraic topology and application to modular symbols. We recall some results about duality in homology and cohomology with coefficients in *local systems*. Let X be a real oriented manifold with dimension n. Let G be any local system of X. Then we have a canonical isomorphism, functorial in G:

(18) $H_c^i(X, G) \simeq H_{n-i}(X, G)$

This is one form, very simple and general, of *Poincaré duality*. To get a more familiar form, we need to compare the homology group $H_{n-i}(X, G)$ with a suitable cohomology group. Let G' be another local system, and suppose given a perfect pairing $G \times G' \to (S^1)_X$, that is a morphism of sheaves such that $G(U) \times G'(U) \to (S^1)_X(U) = (S_1)_U(U)$ is a perfect pairing for every open U of X. Then we have a perfect pairing (cf. [**ST**, §12])

(19) $H_i(X, G) \times H^i(X, G') \to S^1.$

Combining this and (18), we get a perfect pairing $H_c^{n-i}(X, G) \times H^i(X, G') \to S^1$, a more standard version of Poincaré duality (cf. [**ST**, §16])

Another scenario is when G is a locally constant sheaf of R-modules, where R is a commutative ring. Set $G^\vee = \mathrm{Hom}_X(G, R_X)$, where Hom_X means morphism of sheaves in R-modules, and assume that the pairing $G \times G^\vee \to R$ is perfect. Then there is a natural R-linear pairing

(20) $$H^i(X, G) \times H_i(X, G^\vee) \to R$$

but it is not necessarily perfect.

EXERCISE III.1.15. Give an example, with $R = \mathbb{Z}$, $G = G' = \mathbb{Z}_X$, where the above a pairing is not perfect.

However, the pairing (20) is perfect when R is a field of characteristic 0 (reference?). Another situation where this pairing is perfect is when $R = \mathbb{Z}/l^n\mathbb{Z}$ and G is finite over R, because in this case $G' = G^\vee$.

In the above situation, let us rewrite the second pairing using (18): we obtain a (not always perfect)

(21) $$H^i(X, G^\vee) \times H_c^{n-i}(X, G) \to R.$$

But we also have a pairing given by the cup-product

(22) $$H_c^i(X, G^\vee) \times H_c^{n-i}(X, G) \to H_c^n(X, R) \simeq R,$$

This pairing is compatible with the pairing (21), that is there is a diagram

(23)
$$
\begin{array}{ccc}
H^i(X, G^\vee) \times H_c^{n-i}(X, G) & \longrightarrow & R \\
{\scriptstyle \beta \times \mathrm{Id}} \uparrow & & {\scriptstyle =} \uparrow \\
H_c^i(X, G^\vee) \times H_c^{n-i}(X, G) & \longrightarrow & R
\end{array}
$$

where β denotes the natural map

(24) $$\beta : H_c^i(X, -) \to H^i(X, -)$$

The *interior cohomology* $H_!^i(X, G)$ is defined as the image of $\beta : H_c^i(X, G) \to H^i(X, G)$ (in the context of modular symbols, where X is a modular curve, it is also called *parabolic cohomology* and denoted H_p^i) The preceding paragraph implies formally that if $x \in H_!^i(X, G^\vee)$, $y \in H^{n-i}(X, G)$ and $x = \beta(x_0)$, $y = \beta(y_0)$ then the image of (x_0, y_0) by the pairing (22) does not depend on x_0 and y_0, but only on x and y.

EXERCISE III.1.16. (**easy**) Prove it.

Therefore we get a pairing

(25) $$H_!^i(X, G^\vee) \times H_!^{n-i}(X, G) \to R.$$

EXERCISE III.1.17. (**easy**) Deduce from what we have said above that this pairing is perfect when R is a field of characteristic 0, or when $R = \mathbb{Z}/l^n\mathbb{Z}$ and G is finite.

We just note one corollary of this discussion that we shall need.

COROLLARY III.1.18. *If $0 \to V_1 \to V_2 \to V_3 \to 0$ is an exact sequence of Γ-modules (and if either Γ acts freely on $\mathcal{H}$ or 6 acts invertibly on V_i, $i = 1, 2, 3$), then we have a long exact sequence*

$$0 \to \mathrm{Symb}_\Gamma(V_1) \to \mathrm{Symb}_\Gamma(V_2) \to \mathrm{Symb}_\Gamma(V_3) \to H_0(\Gamma, V_1) \to H_0(\Gamma, V_2) \to H_0(\Gamma, V_3) \to 0.$$

Proof —   This is Corollary III.1.13 combined with Poincaré's duality (18).   □

III.1.5. Hecke operators on cohomology. Assume that we are in the situation of §III.1.2, that is with a submonoid Σ of $\mathrm{GL}_2(\mathbb{Q})$ acting on the right on V, in a way extending the action of Γ on V. Then there is an action of the Hecke operators $[\Gamma s \Gamma]$ (for $s \in \Sigma$) on the group of invariant $V^\Gamma = H^0(\Gamma, V)$. It is therefore not surprising that we can define in a natural way an action of the same Hecke operators on the higher cohomology groups $H^i(\Gamma, V)$. To see this, we proceed as follows: one choses a free resolution of V as $\mathbb{Z}[S]$-modules $V \to F_0 \to F_1 \to F_2 \to \dots$. The key (though obvious) observation is that the free $\mathbb{Z}[S]$-modules are still free as $\mathbb{Z}[\Gamma]$-modules, since $\mathbb{Z}[S] = \mathbb{Z}[\Gamma]^{(I)}$ as $\mathbb{Z}[\Gamma]$-module, where I is a set of representatives of S/Γ. Because of this we can use that resolution to compute the $H^\bullet(\Gamma, V)$ as the cohomology of the complex $F_0^\Gamma \to F_1^\Gamma \to F_2^\Gamma \to \dots$. As the F_i^Γ are groups of Γ-invariants in S-modules, Hecke operators act on them, and this action is compatible with the map $F_i^\Gamma \to F_{i+1}^\Gamma$. That is to say, Hecke operators acts on the complex $F_0^\Gamma \to F_1^\Gamma \to F_2^\Gamma \to \dots$. Hence they act on its cohomology, and standard arguments in homological algebras show that this action is independent of the chosen resolution. It is also clear with this definition that $V \mapsto H^i(\Gamma, V)$ is functorial from the category of Σ-modules to the category of groups with Hecke actions, and that the connecting homomorphisms (in the long exact sequence attached to a short exact sequence of Σ-modules) are also compatible with the Hecke operators.

This definition is simple and natural, but not very explicit. It is not hard however to give an explicit formula for those actions.

LEMMA III.1.19. *If $x \in H^d(\Gamma, V)$ is represented by a cocycle $u : \Gamma^d \to V$, and if $\Gamma s \Gamma = \coprod_{i=1}^r \Gamma s_i$ then the cocycle $x_{|[\Gamma s \Gamma]}$ is represented by*

$$(\gamma_1, \dots, \gamma_d) \to \sum_{i=1}^r u(\tau_i(\gamma_1), \dots, \tau_i(\gamma_d))_{|s_i}$$

where for $\gamma \in \Gamma$, $\tau_i(\gamma)$ is the unique element in Γ such that $\tau_i(\gamma) s_i = s_j \gamma$ for a $j \in \{1, \dots, r\}$.

For a proof see [**RW**].

It is also possible to define abstractly an action of the Hecke operators on the spaces $H_c^i(\Gamma, V)$, such that the natural morphisms $H_c^i(\Gamma, V) \to H^i(\Gamma, V)$ are compatible with the Hecke operators, as well as the connecting homomorphisms of

the long exact sequence of cohomology with compact support. The details can be found in [**H5**]. We shall mainly need the action of the Hecke operators on $H^1_c(\Gamma, V)$ and in this case we can simply say that we define the action of Hecke operators $[\Gamma s\Gamma]$ by transfer of structure using our canonical isomorphism $H^1_c(\Gamma, V) = \mathrm{Symb}_\Gamma(V)$. The fact that, with this definition, the natural map $H^1_c(\Gamma, V) \to H^1(\Gamma, V)$ is Hecke-compatible follows from the fact that this map is a connecting homorphism in a long exact sequence attached to a short exact sequence of Σ-modules by Scholium III.1.12 and its proof.

It is possible to give a simple geometric description of the action of the Hecke operators $[\Gamma s\Gamma]$ on $H^i(\Gamma, V)$ or $H^i_c(\Gamma, V)$, when $s \in \Sigma$ and $\det s > 0$. We assume for simplicity that Γ acts freely on $\mathcal{H}$. Set $\Phi = \Gamma \cap s\Gamma s^{-1}$ which is of finite index in Γ, and in $s\Gamma s^{-1}$. We write Y_Γ, Y_Φ, etc. for $\mathcal{H}/\Gamma$, etc. On those manifolds we have the sheafs $\tilde{V}_\Gamma$, $\tilde{V}_\Phi$, etc. Using the action of s on $\mathcal{H}$ (since $\det s > 0$) and on V, we can easily define an isomorphism $[s] : Y_{s^{-1}\Phi s} \to Y_\Phi$ which identifies the sheaves $\tilde{V}_\Phi$ with $\tilde{V}_{s^{-1}\Phi s}$, hence a map $[s]^* : H^i(Y_\Phi, \tilde{V}_\Phi) \to H^i(Y_{s^{-1}\Phi s}, \tilde{V}_{s^{-1}\Phi s})$. On the other hand, we have finite maps $\mathrm{pr}_1 : Y_\Phi \to Y_\Gamma$ and $\mathrm{pr}_2 : Y_{s^{-1}\Phi s} \to Y_\Gamma$ given by the inclusions $\Gamma \subset \Phi$ and $\Gamma \subset s^{-1}\Phi s$, and compatible with the various sheaves $\tilde{V}$. Hence we get a pull-back map $\mathrm{pr}_1^* : H^i(Y_\Gamma, \tilde{V}_\Gamma) \to H^i(Y_\Phi, \tilde{V}_\phi)$, and since pr_2 is finite, a push-forward (or trace) map $(\mathrm{pr}_2)_* : H^i(Y_{s^{-1}\Phi s}, \tilde{V}_{s^{-1}\Phi s}) \to H^i(Y_\Gamma, \tilde{V}_\Gamma)$. The composition

$$(\mathrm{pr}_2)_* \circ [s]^* \circ \mathrm{pr}_1^*$$

is an endomorphism of $H^i(Y_\Gamma, \tilde{V}_\Gamma) = H^i(\Gamma, V)$. It is easy to see that this isomorphism is the Hecke operator $[\Gamma s\Gamma]$ defined above.

III.2. Classical modular symbols and their relations to modular forms

Concerning spaces of modular forms and Hecke operators acting on them, we use the same notations and conventions recalled in §I.6.4

III.2.1. The monoid S. We consider the following monoid

$$S = \mathrm{GL}_2(\mathbb{Q}) \cap M_2(\mathbb{Z}).$$

We note that this monoid is provided with an anti-involution

$$\gamma \mapsto \gamma' = \det(\gamma)\gamma^{-1}.$$

The *anti* in anti-involution means $(\gamma_1\gamma_2)' = \gamma_2'\gamma_1'$. Explicitly, $\gamma' = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

This anti-involution allows us to see any left-S-module as a right-S-module, and conversely : if S acts on V of the left, we define an action on the right by letting γ acts on the right on V through the left-action of γ' on V, and conversely: for $v \in V, \gamma \in S$

$$\gamma \cdot v = v_{|\gamma'}.$$

Using this involution, we can also define the *contragredient* of a right (to fix ideas) S-module V (say, over a ring R): it is the right R-module $V^\vee = \mathrm{Hom}_R(V, R)$ provided with the action, for $l \in V^\vee$, and every $\gamma \in S$, $v \in V$, given by:

$$l_{|\gamma}(v) = l(v_{|\gamma'})$$

In terms of the right action of γ attached to the left-action, this formula is simply

$$l_{|\gamma}(v) = l(\gamma \cdot v).$$

Obviously $V \simeq (V^\vee)^\vee$ as S-modules. We note that if we restrict the S-action to the subgroup $\mathrm{SL}_2(\mathbb{Z})$ of S, then $V^\vee$ is the usual contragredient of the representation V.

We also set $S^+ = \mathrm{GL}_2^+(\mathbb{Q}) \cap M_2(\mathbb{Z})$. This submonoid of S is stable by the involution $s \mapsto s'$. If Γ is a congruence subgroup, we shall consider the Hecke algebra $\mathcal{H}(S, \Gamma)$ and its sub-algebra $\mathcal{H}(S^+, \Gamma)$ generated by double classes $\Gamma s \Gamma$ for $s \in S$ (resp. in S^+). Note that for $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$, the operators T_l, U_l and the diamond operators defined in §I.6.4, belong to $\mathcal{H}(S^+, \Gamma)$.

III.2.2. The S-modules $\mathcal{P}_k$ and $\mathcal{V}_k$. Let $k \geq 0$ be an integer. For any commutative ring R, let $\mathcal{P}_k(R)$ be the R-module of polynomials in one variable z with degree at most k. So $\mathcal{P}_k(R)$ is free of rank $k + 1$

There is a well-known right-action of S on $\mathcal{P}_k(R)$, given by a formula similar to the one used in the definition of modular forms (but with k replaced by $-k$):

$$P_{|\gamma}(z) = (cz + d)^k P\left(\frac{az + b}{cz + d}\right).$$

The associated left action has a sligthly less familiar form:

(26) $$\gamma \cdot P = P_{\gamma'}(z) = (a - cz)^k P\left(\frac{dz - b}{a - cz}\right).$$

Note that $\mathcal{P}_k(R') = \mathcal{P}_k(R) \otimes_R R'$ as (left or right) S-modules.

We define $\mathcal{V}_k(R) = \mathcal{P}_k^\vee = \mathrm{Hom}_R(\mathcal{P}_k, R)$ as the contragredient of $\mathcal{P}_k(R)$. Hence the action of $\gamma \in S$ on $l \in \mathcal{V}_k(R)$ is given by

$$l_{|\gamma}(P) = l(\gamma \cdot P) = l(P_{|\gamma'}) \ \forall P \in \mathcal{P}_k(R).$$

There is also, of course, an associated left-action on $\mathcal{V}_k$ but we shall not need it.

Again, $\mathcal{V}_k(R') = \mathcal{V}_k(R) \otimes_R R'$ as right S-modules.

LEMMA III.2.1. *There is an isomorphism of right S-modules $\Theta_k : \mathcal{V}_k(\mathbb{Z}) \to \mathcal{V}_k^\vee(\mathbb{Z}) = \mathcal{P}_k(\mathbb{Z})$, which is unique up to multiplying it by -1. It is defined by sending the linear form $l_i \in \mathcal{V}_k(\mathbb{Z})$, defined by $l_i(z^j) = \delta_{i,j}$ for $j = 0, \ldots, k$, to the polynomial*

$$\Theta_k(l_i) = (-1)^i \binom{k}{i} z^{k-i},$$

or to its opposite. In particular, as right $\mathrm{SL}_2(\mathbb{Z})$-representation, Θ_k is an isomorphism $\Theta_k : \mathcal{V}_k(\mathbb{Z}) \to \mathcal{V}_k^\vee(\mathbb{Z})$. One has:

(27) $${}^t\Theta_k = (-1)^k \Theta_k$$

Proof — We can prove this by explicit computation. Or slightly more conceptually, give $\mathbb{Z}^2$ its natural S-action (that is, seeing an element $v \in \mathbb{Z}^2$ as a row vector, $v_{|\gamma} = v\gamma$.) Then it is easy to see that $\mathbb{Z}^2 \simeq V_1(\mathbb{Z})$ as S-module (an isomorphism is defined for example by sending $[1,0]$ to l_0 and $[0,1]$ to $-l_1$.) The contragredient of $\mathbb{Z}^2$ by definition, is the set of row vectors with action of γ on v giving $v^t\gamma'$. We have an isomorphism between those two action by the matrix $e = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ since $e\gamma e^{-1} = {}^t\gamma'$ for all γ. Hence an isomorphism $\Theta_1 : \mathcal{V}_1(\mathbb{Z}) \to \mathcal{V}_1(\mathbb{Z})^\vee$, which is clearly antisymetric. In general, we can easily check that $\mathcal{V}_k(z) = \mathrm{Sym}^k(\mathcal{V}_1(\mathbb{Z})) = \mathrm{Sym}^k(\mathbb{Z}_2)$ as S-module (be careful that the first isomorphism involves binomial coefficients, if written explicitly), so we can define $\Theta_k = \mathrm{Sym}^k\Theta_1$ (which is clearly antysymmetric if k is odd, symmetric if k is even), and the lemma follows $\qquad \square$

Obviously, by base change, all those results holds with $\mathbb{Z}$ replaced by any ring R.

III.2.3. Classical modular symbols. Again, let R be any commutative ring.

The module of *classical modular symbols* (for Γ, of weight k, over R) is by definition $\mathrm{Symb}_\Gamma(\mathcal{V}_k(R))$. It is an R-module endowed with an action of the Hecke-algebra $\mathcal{H}(S,\Gamma)$. In particular (for $\Gamma = \Gamma_1(N)$ or $\Gamma_0(N)$) we have the action of the usual Hecke operators T_l, U_l and the diamond operators.

Using (22), we have a pairing

$$(28)\mathrm{Symb}_\Gamma(\mathcal{V}_k(R)) \times \mathrm{Symb}_\Gamma(\mathcal{V}_k(R)) \overset{1\times\Theta_k}{\to} \mathrm{Symb}_\Gamma(\mathcal{V}_k(R)) \times \mathrm{Symb}_\Gamma(\mathcal{V}_k(R)^\vee) \to R$$

which we denote by (ϕ, ϕ'). By (27), and the antisymmetry of pairing (22) in our case ($i(n-i) = 1$), this pairing is antisymmetric if k is even and symmetric if k is odd. Note the pairing is defined as well (and we use the same notation) if take $\phi' \in H^1(\Gamma, \mathcal{V}_k(R))$ instead of $\phi' \in H^1_c(\Gamma, \mathcal{V}_k(R)) = \mathrm{Symb}_\Gamma(\mathcal{V}_k(R))$. As a pairing between $\mathrm{Symb}_\Gamma(\mathcal{V}_k(R))$ and $H^1(\Gamma, \mathcal{V}_k(R))$, the pairing is non-degenerate if R is a finite ring and a field of characteristic zero.

The following important result describes the adjoint of Hecke operators for this pairing.

PROPOSITION III.2.2 (Hida). *Let $x \in H^i(\Gamma, \mathcal{V}_k(R))$ and $y \in H^{2-i}_c(\Gamma, \mathcal{V}_k(R))$. If $s \in S^+$, then*

$$(x_{|[\Gamma s\Gamma]}, y) = (x, y_{|[\Gamma s'\Gamma]}),$$

where (we recall) $s' = s^{-1}\det s$.

The proof is formal using the geometric description of $[\Gamma s\Gamma]$, and the functoriality of the cup product. See [**H1**, Prop 3.3] or [**Shi**, §3.4.5] for details. We shall see below that this result is not correct when $s \in S$ but $s \notin S^+$, that is when $\det s < 0$.

We also have a pairing

$$H^1_!(\Gamma, \mathcal{V}_k(R)) \times H^1_!(\Gamma, \mathcal{V}_k(R)) \overset{1\times\Theta_k}{\to} H^1_!(\Gamma, \mathcal{V}_k(R)) \times H^1_!(\Gamma, \mathcal{V}_k(R)^\vee) \to R$$

which we still denote (,). It is symmetric if k is odd and antisymmetric if k is even. By definition, this pairing is compatible with the preceding in the sense that if $\phi, \phi' \in \mathrm{Symb}_\Gamma(\mathcal{V}_k(R)) = H^1_c(\Gamma, \mathcal{V}_k(R))$ and $\beta(\phi), \beta(\phi')$ are the canonical images of ϕ, ϕ' in $H^1_!$, then $(\phi, \phi') = (\beta(\phi), \beta(\phi'))$. Moreover, by what we have seen (exercise III.1.17), this pairing is *perfect* when R is a field of characteristic 0 or when $R = \mathbb{Z}/l^n\mathbb{Z}$. However this pairing is not perfect when $R = \mathbb{Z}$ in general. We note for use in exercises below the following result of Hida:

PROPOSITION III.2.3. *Let $\Gamma = \Gamma_1(N)$ or $\Gamma = \Gamma_0(N)$. If $R = \mathbb{Z}_l$ and assume that $l > k$ and l does not divide $6N$. $H^1_!(\Gamma, \mathcal{V}_k(\mathbb{Z}_l))$ is a free direct summand of $H^1(\Gamma, \mathcal{V}_k(\mathbb{Z}_l))$. Moreover the pairing $H^1_!(\Gamma, \mathcal{V}_k(\mathbb{Z}_l)) \times H^1_!(\Gamma, \mathcal{V}_k(\mathbb{Z}_l)) \to \mathbb{Z}_l$ is perfect.*

For the proof, [**H1**, §2]. The first assertion needs some computation using a purely "group-cohomological" description of $H^1_!$, which is due to Shimura ([**Shi**, chapter 8]).

EXERCISE III.2.4. Show that the second assertion follows from the first.

III.2.4. Modular forms and real classical modular symbols. Now let f be a cuspidal modular form for Γ of weight $k + 2$. We define a homomorphism of groups $\phi_f : \Delta_0 \to \mathcal{V}_k(\mathbb{R})$ by setting

$$\phi_f(D) = \left(P \mapsto \mathrm{Re}\left(\int_D f(z)P(z)dz \right) \right).$$

Some explanation may be necessary. if $D = \{b_1\} + \cdots + \{b_n\} - \{a_1\} - \cdots - \{a_n\}$, then $\int_D f(z)P(z)dz := \int_{a_1}^{b_1} f(z)P(z)dz + \cdots + \int_{a_n}^{b_n} f(z)P(z)dz$. Above $\int_a^b$ means the integral along the geodesic from a to b. That the integrals converge follows from the cuspidality of f, and that their sum is independent of the writing of D as above follows from Cauchy's homotopy independence of path integrals. Since $\mathrm{Re}\left(\int_D f(z)P(z)dz \right)$ is a real number depending linearly of $P \in \mathcal{P}_k(\mathbb{R})$, the map $P \mapsto \mathrm{Re}\left(\int_D f(z)P(z)dz \right)$ is an element of $\mathcal{V}_k(\mathbb{R})$.

LEMMA III.2.5. *The morphism ϕ_f belongs to $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{R}))$. The map $f \mapsto \phi_f : S_{k+2}(\Gamma, \mathbb{C}) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{R}))$ is an $\mathbb{R}$-linear map commuting with the actions of $\mathcal{H}(S^+, \Gamma)$.*

Remember that while $\mathcal{H}(S, \Gamma)$ acts on modular symbols, only its sub-algebra $\mathcal{H}(S^+, \Gamma)$ acts on the spaces of modular forms (see §I.6.4).

Proof — We need to show that for all $D \in \Delta_0$, $\gamma \in \Gamma$, we have $\phi_f(\gamma \cdot D)_{|\gamma} = \phi_f(D)$, We compute, for $P \in \mathcal{P}_k(\mathbb{R})$:

$$\phi_f(\gamma \cdot D)_{|\gamma}(P) = \phi_f(\gamma \cdot D)(\gamma \cdot P)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} f(z)(\gamma \cdot P)(z)dz\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} f(z)(a - cz)^k P(\gamma^{-1} \cdot z)dz\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} (a - cz)^{-k-2} f(\gamma^{-1} \cdot z)(a - cz)^k P(\gamma^{-1} \cdot z)(a - cz)^2 d(\gamma^{-1} \cdot z)\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} f(\gamma^{-1} \cdot z) P(\gamma^{-1} \cdot z) d(\gamma^{-1} \cdot z)\right)$$

$$= \mathrm{Re}\left(\int_D f(z)P(z)dz\right) = \phi_f(D)(P).$$

The fact that the map $f \mapsto \phi_f$ is $\mathbb{R}$-linear is obvious.

To prove that the map commutes with the action of the Hecke operators, we only need to check that for $\gamma \in S^+ := S \cap \mathrm{GL}_2(\mathbb{Q})$, $(\phi_f)_{|\gamma} = \phi_{f_{|k+2\gamma}}$. This is a slightly more general computation that the one we just did:

$$\phi_f(\gamma \cdot D)_{|\gamma}(P) = \phi_f(\gamma \cdot D)(\gamma \cdot P)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} f(z)(\gamma \cdot P)(z)dz\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} f(z)(a - cz)^k P(\gamma' \cdot z)dz\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} (f_{|\gamma})_{|\gamma^{-1}}(z)(a - cz)^k P(\gamma' \cdot z)(a - cz)^2 (\det \gamma)^{-1} d(\gamma' \cdot z)\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} ((\det \gamma)^{-1}(a - cz))^{-k-2} (\det \gamma)^{-k-1} f_{|\gamma}(\gamma'z)\right.$$

$$\left. (a - cz)^k P(\gamma' \cdot z)(a - cz)^2 (\det \gamma)^{-1} d(\gamma' \cdot z)\right)$$

$$= \mathrm{Re}\left(\int_{\gamma \cdot D} f_{|\gamma}(\gamma'z)P(\gamma' \cdot z)d(\gamma' \cdot z)\right)$$

$$= \mathrm{Re}\left(\int_D f_{|\gamma}(z)P(z)d(z)\right) = \phi_{f_{|\gamma}}(D)(P).$$

(In the above computation, we have used that $\gamma^{-1} \cdot z = \gamma' \cdot z$.)

$\square$

Recall that the Peterson Hermitian product is defined on $S_{k+2}(\Gamma, \mathbb{C})$ by

(29)
$$(f, g)_\Gamma = \int_{\mathcal{H}/\Gamma} f(z)\overline{g(z)}y^k \, dx \, dy,$$

where $z = x + iy$.

PROPOSITION III.2.6. *Let* f, g *in* $S_{k+2}(\Gamma, \mathbb{C})$. *We have*

$$(\phi_f, \phi_g) = (2i)^{k-1}[(f, g)_\Gamma + (-1)^{k+1}(g, f)_\Gamma]$$

Proof — See [**H1**, §3]. This is actually a computation (where one needs to follow carefully the various isomorphisms from algebraic topology involved) of Shimura. Let us sketch a proof when $k = 0$, and Γ acts freely on $\mathcal{H}$. In this case the formula to prove is that $(\phi_f, \phi_g) = \mathrm{Im}(f, g)_\gamma$. Now if we follow ϕ_f through the map $\mathrm{Symb}_\Gamma(\mathbb{R}) = H_c^1(\Gamma, \mathbb{R}) = H_c^1(Y(\Gamma), \mathbb{R}) \subset H^1(Y(\Gamma), \mathbb{R})$ we easily see that ϕ_f is the cohomology class that sends a closed path c in $Y(\Gamma)$ onto $\mathrm{Re} \int_c f(z) \, dz$. That cohomology class is represented by the differential form $(\mathrm{Re}f) \, dx - (\mathrm{Im}f) \, dy$. This follows from the computation, if c is the path $x(t) + iy(t)$ for $t \in [0, 1]$:

$$\mathrm{Re} \int_c f(z) \, dz = \mathrm{Re} \int_0^1 f(x(t), y(t))(x'(t) + iy'(t)dt) = \int_0^1 ((Ref)x'(t) - (\mathrm{Im}f)y'(t))dt.$$

Therefore, to compute the pairing (ϕ_f, ϕ_g), we only have to integrate over $\mathcal{H}/\Gamma$ the 2-form $((\mathrm{Re}f) \, dx - (\mathrm{Im}f) \, dy) \wedge ((\mathrm{Re}g) \, dx - (\mathrm{Im}g) \, dy) = (\mathrm{Re}g\mathrm{Im}f - \mathrm{Re}f\mathrm{Im}g)(dx \wedge dy)$, thus we get $(\phi_f, \phi_g) = \int_{\mathcal{H}/\Gamma}(\mathrm{Re}g\mathrm{Im}f - \mathrm{Re}f\mathrm{Im}g)dx \, dy$, but this is exactly $\mathrm{Im}(f, g)_\Gamma$. $\square$

EXERCISE III.2.7. **(easy)** Perform a sanity check relative to the symmetry or antisymmetry of the LHS and RHS in f and g. Also prove the formula

$$(\phi_f, \phi_{i^{k-1}g}) = 2^k \mathrm{Re}(f, g)_\Gamma.$$

COROLLARY III.2.8. *The map* $S_{k+2}(\Gamma, \mathbb{C}) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{R}))$ *is injective. Actually, even the composition* $S_{k+2}(\Gamma, \mathbb{C}) \to H_!^1(\Gamma, \mathcal{V}_k(\mathbb{R}))$ *of this map with* β *is injective*

Proof — Let $f \in S_{k+2}(\Gamma, \mathbb{C})$. Assume $\beta(\phi_f) = 0$. Then for any $g \in S_{k+2}(\Gamma, \mathbb{C})$, we have $\mathrm{Re}(f, g)_\Gamma = 2^{-k}(\phi_f, \phi_{i^{k-1}g}) = 2^{-k}(\beta(\phi_f), \beta(\phi_{i^{k-1}g})) = 0$. Applying this to ig we get that $\mathrm{Im}(f, g)_\Gamma = 0$ so $(f, g)_\Gamma = 0$ for all g. Since the Peterson's product is perfect, $f = 0$. $\square$

Actually we have more.

THEOREM III.2.9. *The map* $f \mapsto \beta(\phi_f)$, $S_{k+2}(\Gamma, \mathbb{C}) \to H_!^1(\Gamma, \mathcal{V}_k(\mathbb{R}))$ *is an* $\mathbb{R}$-*linear isomorphism compatible with action of the Hecke operators*

Since we already know that this map is injective, we only need to prove that

$$\dim_\mathbb{R} H_!^1(\Gamma, \mathcal{V}_k(\mathbb{R})) \leq \dim S_{k+2}(\Gamma, \mathbb{C}).$$

We shall do that only when $k = 0$, where $\mathcal{V}_0(\mathbb{R}) = \mathbb{R}$, referring the reader to [**H1**] or [**H5**] for the general case. In this case, the elements of $S_2(\Gamma, \mathbb{C})$ are the holomorphic differentials 1-forms on $X(\Gamma)$ (the compactification of $Y(\Gamma)$. The complex dimension of this space is the genus g of $X(\Gamma)$, so $2g$ for the real dimension. On the other hand, the morphism β is actually composition $H_c^1(Y(\Gamma), \mathbb{R}) \to H^1(X(\Gamma), \mathbb{R}) \to H^1(Y(\Gamma), \mathbb{R})$ where the first morphism is given by covariant functoriality of $H_c^i(-, \mathbb{R})$, and the second by the contravariant functoriality of $H^i(-, \mathbb{R})$. So $\dim_\mathbb{R} H_!^1(\Gamma, \mathbb{R}) \leq \dim_\mathbb{R} H^1(X(\Gamma), \mathbb{R}) = 2g$ and we are done.

REMARK III.2.10. Note that we have proved $H^1_!(\Gamma, \mathbb{R}) \simeq H^1(X(\Gamma), \mathbb{R})$. The same results holds for $\mathbb{R}$ replaced by $\mathcal{V}_k(\mathbb{R})$.

III.2.5. Modular forms and complex classical modular symbols. Let us first recall the notion of the conjugate $\bar{W}$ of a $\mathbb{C}$-vector space W. As an abelian group, and even as a real vector sapce $\bar{W}$ is equal to W, except that we denote by $\bar{w} \in \bar{W}$ the element $w \in W$ when we want to make clear that we see it as an element of $\bar{W}$. We define a $\mathbb{C}$-scalar multiplication on $\bar{W}$ by the rule $\lambda \bar{w} = \overline{\bar{\lambda} w}$. This provides $\bar{W}$ with a structure of $\mathbb{C}$-vector space. Note that the construction $W \mapsto \bar{W}$ is functorial, as a $\mathbb{C}$-linear map $f : W \to W'$ is also a $\mathbb{C}$-linear map $f : \bar{W} \to \bar{W}'$.

EXERCISE III.2.11. **(easy)** Suppose given a commutative ring $\mathcal{H}$ that acts $\mathbb{C}$-linearly on W. Then $\mathcal{H}$ acts on $\bar{W}$ by functoriality. Let χ be a character $\mathcal{H} \to \mathbb{C}$. Do we have $\dim_{\mathbb{C}} W[\chi] = \dim_{\mathbb{C}} \bar{W}[\chi]$?

Are the $\mathbb{C}$-vector spaces W and $\bar{W}$ isomorphic? Obviously yes, since they have the same real dimension, hence the same complex dimension. But more seriously, are they canonically isomorphic? No, they are not. However, if we are given an $\mathbb{R}$-structure $W_{\mathbb{R}}$ on W, that is a sub-$\mathbb{R}$-vector space such that the natural map $W_{\mathbb{R}} \otimes_R \mathbb{C} \to W$ is an isomorphism, then this $\mathbb{R}$-structre defines a canonical $\mathbb{C}$-linear isomorphism $c : W \to \bar{W}$ defined as follows: any $w \in W$ can be written uniquely $w_1 + i w_2$ with $w_1, w_2 \in W_{\mathbb{R}}$ (we write $w_1 = \mathrm{Re} w$ and $w_2 = \mathrm{Im} w$ in this case) and we send w to $c(w) = \overline{w_1 - i w_2} = \overline{w_1} + i \overline{w_2}$ in $\bar{W}$.

EXERCISE III.2.12. **(easy)** check that c is really a $\mathbb{C}$-linear isomorphism $W \to \bar{W}$.

LEMMA III.2.13.   (i) *Let W be a $\mathbb{C}$-vector space. Then there is a canonical and functorial $\mathbb{C}$-linear isomorphism $W \otimes_{\mathbb{R}} \mathbb{C} \simeq W \oplus \bar{W}$, $w \otimes \lambda \to (\lambda w, \lambda \bar{w})$. (Here the complex structure on $W \otimes_{\mathbb{R}} \mathbb{C}$ is the one coming from the second factor, $\mathbb{C}$.)*

  (ii) *Let X be another $\mathbb{C}$-vector space, with a real structure $X_{\mathbb{R}}$. Let h be a $\mathbb{C}$-linear map $W \to X$, so that $\mathrm{Re}\, h$ is an $\mathbb{R}$-linear map $W \to X_{\mathbb{R}}$. Then $\mathrm{Re}\, h \otimes 1 : W \otimes_{\mathbb{R}} \mathbb{C} \to X_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C}$ is the same, modulo the obvious identifications, as the map $W \oplus \bar{W} \to X$ sending $(w, \overline{w'})$ to $\frac{1}{2}(h(w) + c^{-1}(h(w')))$. (Let us insist on the meaning of $c^{-1}(h(w'))$: by functoriality, h induces (in fact, is) a $\mathbb{C}$-linear map $\bar{W} \to \bar{X}$, and c^{-1} is a $\mathbb{C}$-linear isomorphism $\bar{X} \to X$.)*

Proof — Let's prove (i). The map is obviously $\mathbb{C}$-linear. Let us check that iy is injective. Any element of $W \otimes_{\mathbb{R}} \mathbb{C}$ can be written $w \otimes 1 + w' \otimes i$. Such an element is send to $(w + i w', \bar{w} + i \bar{w}') = (w + i w', \overline{w - i w'})$. If that element is 0, then w and

w' are 0, which shows the injectivity of our map. Since the source and target have clearly the same dimension, this proves (i).

Let us prove (ii): we have $(\operatorname{Re} h \otimes 1)(w \otimes 1) = \operatorname{Re} h(w)$, and $(\operatorname{Re} h \otimes 1)((iw) \otimes i) = i \operatorname{Re} h(iw) = -i \operatorname{Im} h(w)$. That is to say, our map after identification sends $(w, \bar{w})$ to $\operatorname{Re} h(w)$ and $(-w, \bar{w})$ to $-i \operatorname{Im} h(w)$. Therefore it sends $(2w, 0)$ to $\operatorname{Re} h(w) + i \operatorname{Im} h(w) = h(w)$ and $(0, 2w')$ to $\operatorname{Re} h(w') - i \operatorname{Im} h(w') = c^{-1}(h(w'))$.   □

You are now guessing, I suppose, how we shall apply this lemma: Let $W = S_{k+2}(\Gamma)$ and $X = \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$. The space X has a real structure $X_\mathbb{R} = \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{R}))$, and the two maps Re and Im, $X \to X_\mathbb{R} \subset X$ are easily described: If $\Phi \in \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$, then $(\operatorname{Re}\Phi)(D)$ is the $\mathbb{C}$-linear map on $\mathcal{P}_k(\mathbb{C})$ that sends a **real** polynomial $P \in \mathcal{P}_k(\mathbb{R})$ to $\operatorname{Re}(\Phi(D)(P))$, and similarly for Im.

Now we have a map $h : f \mapsto \phi_f^1 : W \to X$ where ϕ_f^1 is defined, for every $D \in \Delta_0$ and every $P \in \mathcal{P}_k(\mathbb{C})$, by

$$\text{(30)} \qquad\qquad \phi_f^1(D)(P) = \int_D f(z) P(z) dz$$

Obviously $\operatorname{Re} h$ is the map $f \mapsto \phi_f$ considered in the above §. Also note that by definition $c^{-1} h : \overline{S_{k+2}(\Gamma, \mathbb{C})} \to \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$ sends $\bar{f}$ to the symbol $\phi_{\bar{f}}^2$ defined by

$$\text{(31)} \qquad \phi_{\bar{f}}^2(D)(P) = \overline{\int_D f(z) P(z) dz} \text{ for all } P \in \mathcal{P}_k(\mathbb{R}), \text{ hence}$$

$$\text{(32)} \qquad \phi_{\bar{f}}^2(D)(P) = \int_D \bar{f}(z) P(\bar{z}) d\bar{z} \text{ for all } P \in \mathcal{P}_k(\mathbb{C})$$

THEOREM III.2.14. *The $\mathbb{C}$-linear map*

$$\text{(33)} \qquad\qquad S_{k+2}(\Gamma, \mathbb{C}) \oplus \overline{S_{k+2}(\Gamma, \mathbb{C})} \to \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$$

$$\text{(34)} \qquad\qquad\qquad (f, \bar{g}) \mapsto \phi_f^1 + \phi_{\bar{g}}^2$$

is an injective Hecke-compatible $\mathbb{C}$-linear map.

The composition of this application with $\beta : \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})) \to H_!^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$ is an isomorphism ($\mathbb{C}$-linear, Hecke-compatible)

$$\text{(35)} \qquad\qquad S_{k+2}(\Gamma, \mathbb{C}) \oplus \overline{S_{k+2}(\Gamma, \mathbb{C})} \simeq H_!^1(\Gamma, \mathcal{V}_k(\mathbb{C})).$$

The injective map (33) can be extended to an isomorphism ($\mathbb{C}$-linear and compatible with the operators $[\Gamma s \Gamma]$, $s \in S$, $\det > 0$)

$$\text{(36)} \qquad S_{k+2}(\Gamma, \mathbb{C}) \oplus \overline{S_{k+2}(\Gamma, \mathbb{C})} \oplus \operatorname{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C})) \simeq \operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})).$$

Proof — By the lemma, the application (33) (resp. (35)) is the same as $2(\operatorname{Re} h) \otimes 1$ (resp. $2\beta(\operatorname{Re} h) \otimes 1$), hence the injectivity (resp. bijectivity) and Hecke-compatibility of this application results from Corollary III.2.8 (resp. from Theorem III.2.9). Then (36) follows from Scholium III.1.12.   □

We know need to keep track of the pairing in our complexification process. First an abstract lemma in the spirit of Lemma III.2.13

LEMMA III.2.15. *Let W be a complex vector space with an hermitian pairing $(\ ,\ )_\Gamma$ (conjugate-linear in the second variable). Let $[\ ,\ ]$ be the symmetric (resp. antisymmetric) pairing $\mathrm{Re}(\ ,\ )_\Gamma$ (resp. $\mathrm{Im}(\ ,\ )_\Gamma$). Then under the identification $W \otimes_{\mathbb{R}} \mathbb{C} = W \oplus \bar{W}$ (cf. III.2.13(i)), the pairing $[\ ,\ ]$ is identified with the pairing $[(w_1, \bar{w}_2), (w_1', \bar{w}_2')] = \frac{1}{2}((w_1, w_2')_\Gamma + (w_1', w_2)_\Gamma)$ (resp. to the pairing $[(w_1, \bar{w}_2), (w_1', \bar{w}_2')] = \frac{-i}{2}((w_1, w_2')_\Gamma - (w_1', w_2)_\Gamma)$*

Proof — Since $w \otimes 1$ in $W \otimes \mathbb{C}$ is identified with $(w, \bar{w})$ in $W \oplus \bar{W}$, we have $[(w, \bar{w}), (w', \overline{w'})] = [w, w']$. The same result applied to iw gives $i[(w, -\bar{w}), (w', \overline{w'})] = [iw, w']$. Adding the first equality and $-i$ times the second gives $2[(w, 0), (w', \overline{w'})] = [w, w'] - i[iw, w'] = \mathrm{Re}(w, w')_\Gamma - i\mathrm{Re}i(w, w')_\Gamma = (w, w')_\Gamma$. The same equality with w' replaced by iw' gives $2[(w, 0), (w', -\overline{w'})] = -(w, w')_\Gamma$. Summing we get $[(w, 0), (w', 0)]) = 0$ and finally $2[(w, 0), (0, \bar{w}')] = (w, w')_\Gamma$. The lemma follows by symmetry of $[\ ,\ ]$. (we leave the resped formula to the reader). $\qquad\square$

PROPOSITION III.2.16. *Let $f, g \in S_{k+2}(\Gamma, \mathbb{C})$. Then*

$$(37) \qquad\qquad (\phi_f^1, \phi_g^1) = 0$$

$$(38) \qquad\qquad (\phi_{\bar{f}}^2, \phi_{\bar{g}}^2) = 0$$

$$(39) \qquad\qquad (\phi_f^1, \phi_{\bar{g}}^2) = (2i)^{k-1}(f, g)_\Gamma.$$

Proof — Since $\phi_1 \oplus \phi_2$ is, up to our identifications $\phi \otimes 1 : W \otimes_R \mathbb{C} \to \mathrm{Symb}(\Gamma, \mathcal{V}_k(\mathbb{C}))$, and since ϕ makes the pairing $2(2i)^{k-1}\mathrm{Re}(\ ,\ )_\Gamma$ if k is odd (resp. $(2i)^k\mathrm{Im}(f, g)_\Gamma$ if k is even) on $S_{k+2}(\Gamma, \mathbb{C})$ compatible with the pairing $(\ ,\ )$ on $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{R}))$, the results follows from the above lemma. $\qquad\square$

III.2.6. The involution ι, and how to get rid of the complex conjugation. In this paragraph, we shall assume that for some $N \geq 1$, $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$. This has two nice consequences on the picture described in the above §.

First, the matrix $e = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ normalizes the group Γ. Since $e \in S$, we have an action of the "Hecke" operator $[\Gamma e \Gamma]$ on $\mathrm{Symb}_\Gamma(V)$ for any S-module V and since e normalizes Γ, ι is an involution (sometines called the 'Hecke operator at infinity' and denoted by T_∞).

LEMMA III.2.17. *Let R be any commutative ring. The involution ι acting on $\mathrm{Symb}_\Gamma(\mathcal{V}_k(R))$ commutes with all the Hecke operators T_l ($l \nmid N$), U_l ($l|N$) and the diamond operators.*

Proof —   This is an easy computation, using the fact that conjugating a matrix by e does not change its diagonal terms.

$\square$

If V is any Γ-module (on which 2 acts invertibly as above), we define $\mathrm{Symb}_\Gamma^\pm(V)$ as the subgroup of $\mathrm{Symb}_\Gamma(V)$ on which ι acts by ± 1 and we have $\mathrm{Symb}_\Gamma(V) = \mathrm{Symb}_\Gamma^+(V) \oplus \mathrm{Symb}_\Gamma^-(V)$.

Second, the space $S_{k+2}(\Gamma, \mathbb{C})$ has a canonical real structure $S_{k+2}(\Gamma, \mathbb{R})$ (the forms with Fourier coefficients in $\mathbb{R}$) which is stable by all the Hecke operators $[\Gamma s\Gamma]$, $s \in S$, $\det s > 0$ (cf. [**Mi**, Theorem 4.5.19]). Using it, we can identify canonically the space $S_{k+2}(\Gamma, \mathbb{C})$ with its conjugate $\overline{S_{k+2}(\Gamma, \mathbb{C})}$ (as a module over the Hecke operators). Explicitely, this identification is as follows:

DEFINITION III.2.18. For $f \in S_{k+2}(\Gamma, \mathbb{C})$ we define an holomorphic function $f_\rho : \mathcal{H} \to \mathbb{C}$ by $f_\rho(z) = \overline{f(-\bar{z})}$.

LEMMA III.2.19. *We have $f_\rho \in S_{k+2}(\Gamma, \mathbb{C})$. if $f(z) = \sum_{n=1}^\infty a_n q^n$, then $f_\rho(z) = \sum_{n=1}^\infty \overline{a_n} q^n$. The map $f \mapsto \overline{f_\rho}$ (where $\overline{f_\rho}$ means the element $f_\rho \in S_{k+2}(\Gamma, \mathbb{C})$ seen as en element of $\overline{S_{k+2}(\Gamma, \mathbb{C})}$) is the $\mathbb{C}$-linear isomorphism $S_{k+2}(\Gamma, \mathbb{C}) \to \overline{S_{k+2}(\Gamma, \mathbb{C})}$ attached to the real structure $S_{k+2}(\Gamma, \mathbb{R})$, and is compatible with all the Hecke operators.*

Proof —   The first assertion is clear. For the second, if $f(z) = \sum a_n e^{2i\pi nz}$, $f(-\bar{z}) = \sum a_n e^{-2i\pi n\bar{z}} = \sum_n a_n \overline{e^{2i\pi nz}}$ and therefore $f_\rho(z) = \sum \overline{a_n} e^{2i\pi nz}$. In particular, $f \mapsto f_\rho$ fixes the real structure $S_{k+2}(\Gamma, \mathbb{R})$ and the last assertion follows. $\square$

Using this identification, our somewhat twisted linear map $\overline{S_{k+2}(\Gamma, \mathbb{C})} \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$, $\bar{g} \mapsto \phi_g^2$ defines a map

$$S_{k+2}(\Gamma, \mathbb{C}) \ \to \ \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$$
$$g \ \mapsto \ \phi_g^3 := -\phi_{g_\rho}^2$$

LEMMA III.2.20.

$$\phi_g^3(D)(P) = \int_{\ominus D} g(z) P(-z) dz \text{ for all } g \in S_{k+2}(\Gamma, \mathbb{C}), \ D \in \Delta_0, \ P \in \mathcal{P}_k(\mathbb{C}).$$

Here, $D \mapsto \ominus D$ is the linear map on Δ_0 sending $\{\infty\} - \{a\}$ to $\{\infty\} - \{-a\}$ for $a \in \mathbb{Q}$.

Proof —   Since $g \mapsto \phi_g^3$ is $\mathbb{C}$-linear, and the Cauchy line integral is also $\mathbb{C}$-linear in its integrand, we only have to prove the formula for $g \in S_{k+2}(\Gamma, \mathbb{R})$ and $P \in \mathcal{P}_k(\mathbb{R})$.

By additivity we may assume that $D = \{\infty\} - \{a\}$

$$
\begin{aligned}
\phi_g^3(D)(P) &= -\phi_{\bar{g}}^2(D)(P) \quad \text{(by definition)} \\
&= \overline{-\int_D g_\rho(z) P(z) dz} \quad \text{(by (31))} \\
&= \overline{-\int_0^\infty g_\rho(a + iy) P(a + iy) i\, dy} \\
&= \int_0^\infty g(-a + iy) P(a - iy) i\, dy \quad \text{using } \overline{g_\rho(z)} = g(-\bar{z}) \text{ and } \overline{P(z)} = P(\bar{z})) \\
&= \int_{\ominus D} g(z) P(-z) dz
\end{aligned}
$$

$\square$

PROPOSITION III.2.21. *For any $f \in S_{k+2}(\Gamma, \mathbb{C})$, we have $(\phi_f^1)_{|\iota} = \phi_f^3$. The maps $f \mapsto \phi_f^1$ commutes with the action of $\mathcal{H}(S^+, \Gamma)$ and the map $f \mapsto \phi_f^3$ commutes with the action of the Hecke operators T_l $(l \nmid N)$, U_l $(l \mid N)$ and the diamond operators. Therefore, we have an injective linear application compatible with the above-mentionnes Hecke operators*

$$
S_{k+2}(\Gamma, \mathbb{C}) \oplus S_{k+2}(\Gamma, \mathbb{C}) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})), \ (f, g) \mapsto \phi_f^1 + \phi_g^3.
$$

Its image admits $\mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$ as an $\mathcal{H}(S^+, \Gamma)$-stable supplementary.

The nice thing with that application then there is no complex conjugation around anymore.

Proof — We have $\phi_{|\iota}(D)(P) = \phi(\ominus D)(P(-z))$ by a trivial computation, hence the first assertion follows from the above Lemma. That $f \mapsto \phi_f^1$ commutes with $\mathcal{H}(S^+, \Gamma)$ can be proved by the exact same proof as Lemma III.2.5 (or alternatively, deduced from it). If $T \in \mathcal{H}(S^+, \Gamma)$ it follows from the above that

$$
(40) \qquad (\phi_f^3)_T = (\phi_f^1)_{|\iota T} = (\phi_f^1)_{|\iota T\iota} = (\phi_{f_{|\iota T\iota}}^1)_{|\iota} = \phi_{f_{|\iota T\iota}}^3
$$

and if T is in the list of Hecke operators of the proposition, $\iota T\iota = T$ by Lemma III.2.17. The injectivity of the displayed aplication follows from Theorem III.2.14 and its compatibility with aforementionned Hecke operators from what we have just proven. $\square$

PROPOSITION III.2.22. *Let $f, g \in S_{k+2}(\Gamma, \mathbb{C})$. Then*

$$
(41) \qquad\qquad\qquad (\phi_f^1, \phi_g^1) = 0
$$

$$
(42) \qquad\qquad\qquad (\phi_f^3, \phi_g^3) = 0
$$

$$
(43) \qquad\qquad\qquad (\phi_f^1, \phi_g^3) = -(2i)^{k-1}(f, g_\rho)_\Gamma.
$$

Proof — This follows immediately from Prop. III.2.22 and the definition of ϕ_3. $\square$

We now consider a basis of the space generated by ϕ_f^1 and ϕ_f^3 on which ι acts diagonally: we let

$$\phi_f^\pm = \frac{1}{2}(\phi_f^1 \pm \phi_f^3) = \frac{1}{2}\left(\int_D f(z)P(z)\ dz \pm \int_{\ominus D} f(z)P(-z)\ dz\right).$$

We can split Theorem III.2.14 or Prop. III.2.21 in half:

THEOREM III.2.23. *The map $f \mapsto \phi_{f\pm}$ is a $\mathbb{C}$-linear, compatible with with the Hecke oprators T_l and U_l and diamondopeators, and injective linear map $S_{k+2}(\Gamma, \mathbb{C}) \to$ $\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{C}))$. Its image admits $\mathrm{BSymb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{C}))$ as an an $\mathcal{H}(S^+, \Gamma)$-stable supplementary.*

III.2.7. The endomorphism W_N and the corrected scalar product. .

We keep assuming that $\Gamma_1(N) \subset \Gamma \subset \Gamma_0(N)$. Let L be any commutative ring. Let us recall the classical theory of the operator W_N, which plays a fundamental role in the functional equation of the complex L-function of a modular form (see e.g. [**Shi**])

Let $W_N = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix} \in S^+ = \mathrm{GL}_2^+(\mathbb{Q}) \cap M_2(\mathbb{Z})$. In particular, we can define an Hecke operator $[\Gamma W_N \Gamma]$ on the spaces of modular forms $M_{k+2}(\Gamma, L)$ and on the spaces of modular symbols $\mathrm{Symb}_\Gamma(V)$ (right V any L-module with a right action of S^+).

Observe that

(44) $$W_N \begin{pmatrix} a & b \\ c & d \end{pmatrix} W_N^{-1} = \begin{pmatrix} d & -c/N \\ -Nb & a \end{pmatrix}.$$

LEMMA III.2.24. *The matrix W_N normalizes Γ. Hence the operator $[\Gamma W_N \Gamma]$ on $M_{k+2}(\Gamma, L)$ or $\mathrm{Symb}_\Gamma(\mathcal{V}_k(L))$ is just the action of the matrix W_N. Its square is the multiplication by $(-N)^k$. In particular, this Hecke operator acts invertibly if N is inverible in L.*

Proof — It is obvious on (44) that W_N normalizes $\Gamma_0(N)$. A subgroup Γ as above is always of the form $\Gamma_H = \{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(p),\quad d \pmod{N} \in H$ for H a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^*$. If $d \pmod{N} \in H$, then $a \pmod{N} = d^{-1} \pmod{N}$ is also in H, hence the result. $\square$

Note in particular that if γ is diagonal, $W_N \gamma W_N^{-1} = \gamma'$ (where $\gamma' = (\det \gamma)\gamma^{-1}$ as always). Hence for $s \in S$ diagonal $W_N[\Gamma s \Gamma]W_N^{-1} = [\Gamma s' \Gamma]$.

DEFINITION III.2.25. We define the *corrected scalar product* $[\ ,\ ]_\Gamma$ on $S_{k+2}(\Gamma, L)$ and $[\ ,\ ]$ on $\mathrm{Symb}_\Gamma(\mathcal{V}_k(L))$ for any ring L by the formulas:

$$[f, g]_\Gamma = (f, g_{|W_N})_\Gamma, \quad [\phi_1, \phi_2] = (\phi_1, (\phi_2)_{|W_N})$$

LEMMA III.2.26. *The operators T_l for $l \nmid Np$, U_q and the diamond operators are all self-adjoint for $[\ ,\ ]_\Gamma$ and $[\ ,\ ]$. That is, for T any of the above operator, one has $[f_{|T}, g]_\Gamma = [f, g_{|T}]_\Gamma$ for all $f, g \in M_{k+2}(\Gamma, L)$ and similarly for modular symbols.*

Proof — We do the proof for the pairing $[\ ,\ ]_\Gamma$ on $M_{k+2}(\Gamma, L)$. Write $T = [\Gamma s \Gamma]$ with $s \in S^+$. Then

$$
\begin{aligned}
[f_{|T}, g]_\Gamma &= (f_{[\Gamma s \Gamma]}, g_{|W_N})_\Gamma \\
&= (f, g_{|W_N [\Gamma s' \Gamma]})_\Gamma \\
&= [f, g_{|W_N [\Gamma s' \Gamma] W_N^{-1}}]_\Gamma.
\end{aligned}
$$

Now if $T = T_l$ or U_q we can take s diagonal, so $W_N [\Gamma s' \Gamma] W_N^{-1} = [\Gamma s \Gamma] = T$. If $T = \langle a \rangle$ is a diamond operator, $a \in (\mathbb{Z}/N\mathbb{Z})^*$, then $T = [\Gamma s \Gamma]$ where s is any matrix in $\Gamma_0(N)$ with lower-right term congruent to $a \pmod{N}$, and $W_N s' W_N^{-1}$ is a matrix in $\Gamma_0(Np)$ has the same diagonal terms than s, hence $W_N [\Gamma s' \Gamma] W_N^{-1} = \langle a \rangle$ and the result follows. $\qquad \square$

PROPOSITION III.2.27. *If f is a newform in $M_{k+2}(\Gamma, \mathbb{C})$ then*

$$
f_{|W_N} = W(f) f_\rho
$$

where $W(f)$ is a non-zero scalar in the number field K_f (the subfield of $\mathbb{C}$ generated by the coefficients of f) satisfying $|W(f)| = N^k$.

If f has moreover trivial nebentypus, then $f_{|W_N} = W(f) f$ and $W(f) = \pm N^{k/2}$.

Proof — Write $f = q + \sum_{n \geq 2} a_n q^n$, andlet ϵ the nebentypus of f. We have seen that if f is an eigenvector with eigenvalue λ for $[\Gamma s \Gamma]$ with $s \in S^+$, s diagonal, then $f_{|W_N}$ is an eigenvector for $[\Gamma s' \Gamma]$ with the same eigenvalue λ. Applyng this for $s = \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$, $l \nmid N$, we see that $f_{|W_n}$ is an eigenvector for $T_l = [\Gamma s' \Gamma]$ with eigenvalue the eigenvalue of f for $[\Gamma s \Gamma]$, that is $a_l \epsilon(l^{-1})$. Now the eigenvalue of T_l for f_ρ is $\bar{a}_l$ wihch is also the eigenvalue of $[\Gamma s \Gamma]$ for f using that $(f_{|T_l}, f)_\Gamma = (f, f_{|[\Gamma s \Gamma]})_\Gamma$. Hence $f_{|W_n} = W(f) f_\rho$ are proportional since that are newforms. We have $(f, f)_\Gamma = (f_\rho, f_\rho)_\Gamma = |W(\rho)|^{-2} (f_{|W_N}, f_{|W_N})_\Gamma = |W(\rho)|^{-2} (f_{|W_N W_N'}, f)_\Gamma = |W(\rho)|^{-2} N^k (f, f)_\Gamma$ using that $W_N W_N' = N \mathrm{Id}$, hence $|W(\rho)| = N^{k/2}$.

Finally, when f has a trivial nebentypus, the eigenvalues for T_l of f_ρ are just the a_l, so $f = f_\rho$, and $f_{|W_N} = W(f) f$. Since $W_N^2 = -N \mathrm{Id}$ acts like the multiplication by $(-N)^k$, that is N^k since k is even because the nebetypus is trivial, hence $W(f) = \pm N^{k/2}$ $\qquad \square$

REMARK III.2.28. The sign $W(f)/N^{k/2}$ is called the *root number of f* and is the sign of the functional equation satisfied by $L(f, s)$. See below.

LEMMA III.2.29. *If $f \in S_{k+2}(\Gamma, \mathbb{C})$, we have $\phi^1_{f_{|W_N}} = (\phi^1_f)_{|W_N}$ and $\phi^3_{f_{|W_N}} = (-1)^k (\phi^3_f)_{|W_N}$, and we have*

$$
[\phi^+_f, \phi^-_f] = i^{k-1} 2^{k-2} \mathrm{Re}(f_{|W_N}, f_\rho)_\Gamma.
$$

When f is a new form,

$$[\phi_f^+, \phi_f^-] = i^{k-1}2^{k-2}\mathrm{Re}(W(f))(f,f)_\Gamma.$$

Proof — The assertion concerning ϕ_1^f follows from Prop. **??** since $W_N \in S^+$, and by formula (40) we have $\phi^3_{f_{|W_N}} = \phi^3_{f|eW_ne} = (\phi_f^3)_{|(-W_N)} = (-1)^k(\phi_f^3)_{|W_N}$. For the second assertion, we compute

$$
\begin{aligned}
[\phi_f^+, \phi_f^-] &= \frac{1}{4}[\phi_f^1 + \phi_f^3, \phi_f^1 - \phi_f^3] \\
&= \frac{1}{4}(\phi_f^1 + \phi_f^3, \phi^1_{f_{|W_N}} - (-1)^k\phi^3_{f_{|W_N}}) \\
&= \frac{1}{4}(-1)^{k+1}(-2i)^{k-1}[(f_{|W_N}, f_\rho)_\Gamma + (f, (f_{|W_N})_\rho)_\Gamma]
\end{aligned}
$$

using Prop. III.2.22 and the fact that $[\,,\,]$ is symmetric when k is odd, antisymetric when k is even. Now a simple change of variable gives, for f and g two modular forms $(f, g_\rho)_\Gamma = (g, f_\rho)_\Gamma = \overline{(f_\rho, g)_\Gamma}$, and applying this to $g = f_{|W_N}$ gives the result. In the case where f is a new form, one applies Prop. III.2.27

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

III.2.8. Boundary modular symbols and Eisenstein series. In this section, we assume for simplicity that Γ is stable by the involution $\gamma \mapsto \gamma'$. This is true for $\Gamma_0(N)$, $\Gamma_1(N)$ and all groups we shall consider.

THEOREM III.2.30. *We have a natural perfect pairing $(\,,\,)$*

$$\mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C})) \times \mathcal{E}_{k+2}(\Gamma, \mathbb{C}) \to \mathbb{C}$$

where $\mathcal{E}_{k+2}(\Gamma, \mathbb{C}) := M_{k+2}(\Gamma, \mathbb{C})/S_{k+2}(\Gamma, \mathbb{C})$ is the space of Eisenstein series of weight $k + 2$ and level Γ which satisfies, for all $s \in S^+$,

$$(x_{|[\Gamma s\Gamma]}, y) = (x, y_{|[\Gamma s'\Gamma]})$$

We begin by constructing a map

$$f \mapsto u_f$$
$$M_{k+2}(\Gamma, \mathbb{C}) \to H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$$

as follows: choose an analytic function $g(z)$ on $\mathcal{H}$ such that $\frac{d^{k+1}g}{dz^{k+1}} = f(z)$. For $\gamma \in \Gamma$, let $u_f(\gamma)(z) = (a-cz)^k g\left(\frac{dz-b}{a-cz}\right) - g(z)$. To simplify notations, we set $g_{|\gamma}(z) = (a-cz)^k g\left(\frac{dz-b}{a-cz}\right)$. Then the formula (63) to be proved below (see Lemma **??**) states

(45) $$\frac{d^{k+1}g_{|\gamma}}{dz^k}(z) = \left(\frac{d^{k+1}g}{dz^k}\right)_{|k+2\gamma'}(z) = f_{|k+2\gamma'}(z),$$

where the $_{|k+2}$-action is the standard one on modular forms, recalled in the beginning of this chapter.

By this formula, the modularity of f and the fact that Γ is stable by $\gamma \mapsto \gamma'$, the $(k + 1)$-th derivative of $u_f(\gamma)(z)$ is 0, so $u_f \in \mathcal{V}_k(\mathbb{C})$. It is clear that $\gamma \mapsto u_f(\gamma)$ is a cocycle of Γ with values in $\mathcal{V}_k(\mathbb{C})$. The construction of u_f depends on the choice

of g, but changing g into $g + P$ where $P \in V_k(\mathbb{C})$ adds to u_f the coboundary of P, so u_f is well defined in $H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$.

LEMMA III.2.31. *The map $f \mapsto u_f$ is $\mathbb{C}$-linear, injective, and commutes with the action of the Hecke operators $[\Gamma s \Gamma]$ for all $s \in S^+$.*

Proof — The $\mathbb{C}$-linearity is obvious. If $u_f = 0$, then g satisfies the functional equation of a modular form of weight $-k$. Since g has at most polynomial growth at cusps, it is actually holomorphic at every cusp, hence it is really a modular form. But there is no non-zero modular forms of weight $-k$, excepted constants if $k = 0$. In any case $f = 0$, which proves the injectivity of our map.

For the Hecke operators, we compute, if $\Gamma s \Gamma = \coprod_{i=1}^{r} \Gamma s_i$ and $\tau_i(\gamma)$ is the unique element of Γ such that $s_i \gamma = \tau_i(\gamma) s_j$ for $j \in \{1, \ldots, r\}$.

$$
\begin{aligned}
(u_f)_{|\Gamma s \Gamma}(\gamma) &= \sum_{i=1}^{r} u_f(\tau_i(\gamma))_{|s_i} \quad \text{(by Lemma III.1.19)} \\
&= \sum_{i=1}^{r} g_{|\tau_i(\gamma)s_i} - \sum_i g_{|s_i} \\
&= \sum_{i=1}^{r} g_{|s_i\gamma} - g_{|s_i} \\
&= \sum_{i=1}^{r} u_{f_{|k+2 s_i}}(\gamma) \quad \text{(by (45))} \\
&= u_{f_{|\Gamma s \Gamma]}}
\end{aligned}
$$

$\square$

Now consider the space $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$ as a subspace of $M_{k+2}(\Gamma, \mathbb{C})$ (the kernel of the Peterson product). The restriction of $f \mapsto u_f$ embeds (Hecke-cmpatibly) $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$ into $H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$. We also have an embedding $H^1_!(\Gamma, \mathcal{V}_k(\mathbb{C})) \subset H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$. The (Hecke-equivariant) sum map

$$H^1_!(\Gamma, \mathcal{V}_k(\mathbb{C})) \oplus \mathcal{E}_{k+2}(\Gamma, \mathbb{C}) \to H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$$

is still injective, as no system of Hecke-eigenvalues of an Eisenstein series may appear in $H^1_!(\Gamma, \mathcal{V}_k(\mathbb{C}))$ by Theorem III.2.14 and the Hecke estimates of eigenvalues of cuspidal modular forms. By equality of dimensions between the target of the source and this map (for example when $k = 0$, both sides have dimension $2g + c - 1$ where g is genus of the Riemann surface $X(\Gamma)$, and c the number of cusps for Γ), it is an isomorphism.

Combining this and Scholium III.1.12, we have an exact sequence of Hecke-modules:

$$0 \to \mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C})) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})) \to H^1(\Gamma, \mathcal{V}_k(\mathbb{C})) \to \mathcal{E}_{k+2}(\Gamma, \mathbb{C}) \to 0$$

The pairing $(\ ,\ )$ between $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$ and $H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$ induces a pairing between $\mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$ and $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$, as follows: for $x \in \mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$

that we see as an element of $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$ and $y \in \mathcal{E}_{k+2}(\Gamma, \mathbb{C})$, the number (x, y') is independent of the lift y' of y in $H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$. To see this, let y'' be an other lift, so that $y' - y''$ is the image of an element z in $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$. But $(x, y - y') = (x, z) = \pm(z, x)$ and the latter is 0 since the image of x in $H^1(\Gamma, \mathcal{V}_k(\mathbb{C}))$ is 0. Obviously then, the pairing between $\mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$ and $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$ is perfect. The proves Theorem III.2.30, since its last assertion follows from Prop. III.2.2.

EXERCISE III.2.32. Describe a basis of eigenforms for all Hecke operators for the space $\mathrm{BSymb}_\Gamma(\mathcal{V}_k)$ when $k = 0$ and $\Gamma = \Gamma_0(N)$ where N is a product of r distinct primes $l_1, \ldots, l_r$.

COROLLARY III.2.33. *Let us assume that $\Gamma = \Gamma_1(N)$ for some N and let $\mathcal{H}$ be the commutative sub-algebra of $\mathcal{H}(S^+, \Gamma)$ generated by the Hecke operators T_l for l a prime not dividing N, U_l for l a prime dividing N and the Diamond operators. Then, as $\mathcal{H}$-modules, $\mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$, $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})^\vee$ and $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})^\vee$ are isomorphic.*

Proof — For l not dividing N, let U'_l be $[\Gamma \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix} \Gamma]$. Let $\mathcal{H}'$ be the commutative subalgebra of $\mathcal{H}(S^+, \Gamma)$ generated by the T_l's, the U'_l and the Diamond operator. Fix an isomorphism $\psi : \mathcal{H} \to \mathcal{H}'$ by sending T_l on T_l, U_l and $U'l$ and $\langle a \rangle$ on $\langle a^{-1} \rangle$. Note that in each case ψ sends $[\Gamma s \Gamma]$ to $[\Gamma s' \Gamma]$. Thus the theorem gives us an isomorphism of vector spaces $\mathcal{E}_{k+2}(\Gamma, \mathbb{C}) \to \mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\vee$ which is compatible with the $\mathcal{H}'$-structure on the source and the $\mathcal{H}$-structure on the target (where $\mathcal{H}'$ is identified with $\mathcal{H}$ using ψ).

Now we claim that $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$ as an $\mathcal{H}$-module is isomorphic to itself as an $\mathcal{H}'$-module, when identify $\mathcal{H}$ and $\mathcal{H}'$ using ψ. Let $W = \begin{pmatrix} 1 & 0 \\ -N & 0 \end{pmatrix}$. Then a straightforward computation shows that $f \mapsto f_{|W}$ is an isomorphism of vector spaces of $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$ onto itself, and that $(Tf)_{|W} = \psi(T)f_{|W}$ for any $T \in \mathcal{H}$ (this is [**Mi**, Theorem 4.5.5]), which proves the claim.

Hence we get a genuine $\mathcal{H}$-module isomorphism $\mathcal{E}_{k+2}(\Gamma, \mathbb{C}) \simeq \mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))^\vee$. Dualizying, we get $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})^\vee \simeq \mathrm{BSymb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$. To conclude, we just recall that by Corollary **??**, $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})^\vee \simeq \mathcal{E}_{k+2}(\Gamma, \mathbb{C})$ as $\mathcal{H}$-modules. $\square$

EXERCISE III.2.34. Check directly that $\mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_k(\mathcal{V})) \simeq \mathcal{E}_{k+2}(\Gamma_0(N), \mathbb{C})$ as $\mathcal{H}$-modules when $k = 0$, N a product of distinct primes, using your description of Exercise III.2.32.

REMARK III.2.35. The reader should be warned that while we have constructed for all N a canonical isomorphism

$$\mathcal{E}_{k+2}(\Gamma_1(N), \mathbb{C}) \to \mathrm{BSymb}_{\Gamma_1(N)}(\mathcal{V}_k(\mathbb{C}))$$

these isomorphisms are **not compatible** for different N in the following natural sense: for $t \geq 1$ an integer, let V_t be the Hecke operator $l^{-1-k}[\Gamma_1(N) \begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix} \Gamma_1(Nt)]$ which maps $\mathcal{E}_{k+2}(\Gamma_1(N), \mathbb{C})$ to $\mathcal{E}_{k+2}(\Gamma_1(Nt), \mathbb{C})$, sending $f(z)$ to $f(tz)$, and $\mathrm{BSymb}_{\Gamma_1(N)}(\mathcal{V}_k(\mathbb{C}))$ to $\mathrm{BSymb}_{\Gamma_1(Nt)}(\mathcal{V}_k(\mathbb{C}))$. Then the commutatiove diagram

$$
\begin{array}{ccc}
\mathcal{E}_{k+2}(\Gamma_1(N), \mathbb{C}) & \xrightarrow{\simeq} & \mathrm{BSymb}_{\Gamma_1(N)}(\mathcal{V}_k(\mathbb{C})) \\
\downarrow{\scriptstyle V_t} & & \downarrow{\scriptstyle V_t} \\
\mathcal{E}_{k+2}(\Gamma_1(Nt), \mathbb{C}) & \xrightarrow{\simeq} & \mathrm{BSymb}_{\Gamma_1(Nt)}(\mathcal{V}_k(\mathbb{C}))
\end{array}
$$

does not commute.

EXERCISE III.2.36. Prove the remark by showing that in the situation of Exercise III.2.32, on has $V_t : \mathrm{BSymb}_{\Gamma_0(N)}(\mathbb{C}) \to \mathrm{BSymb}_{\Gamma_0(Nt)}(\mathbb{C})$ for all t.

DEFINITION III.2.37. If f is a new form in $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$, we denote by $\epsilon(f)$ the eigenvalue of the involution ι on the boundary modular symbol corresponding to f by the isomorphism' III.2.33. We refer to this number as the *sign* of the new Eisenstein series f.

In other words, $\epsilon(f)$ is the ι-eigenvalues of the unique boundary symbols with the same Hecke eigenvalues than f. The followong proposition determines the sign. For the proof, we refer the reader to [**BD**].

PROPOSITION III.2.38. . *The sign of a normal new Eisenstein series (cf. §I.6.3) $E_{k+2,\tau,\psi}$ is $\tau(-1)$. The sign of an excpetional Eisenstein series $E_{2,\ell}$ is 1.*

III.2.9. Summary. We summarize the main results of this section by the following version of the theorem of Eichler-Shimura:

THEOREM III.2.39. *Let $\Gamma_1(N) \subset \Gamma \subset \Gamma_1(N)$ and $\mathcal{H}$ the polynomials ring genrated by the Hecke operators T_l (for $l \nmid N$) U_l (for $l \mid N$), and $\langle a \rangle$ (for $a \in (\mathbb{Z}/N\mathbb{Z})^*$). We have a canonical isomorphism of $\mathcal{H}$-modules*

$$
\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C})) = S_{k+2}(\Gamma, \mathbb{C}) \oplus S_{k+2}(\Gamma, \mathbb{C}) = M_{k+2}(\Gamma, \mathbb{C}) \oplus S_{k+2}(\Gamma, \mathbb{C})
$$

which transform the ι involution into the involution that acts by $+1$ on the first summand $S_{k+2}(\Gamma, \mathbb{C})$, -1 and the second summand, and by some involution on $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})$. We have canonical $\mathcal{H}$-embedding

$$
S_{k+2}(\Gamma, \mathbb{C}) \hookrightarrow \mathrm{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{C})) \hookrightarrow M_{k+2}(\Gamma, \mathbb{C}).
$$

Finally, all those results hold when $\mathbb{C}$ is replaced everywhere by any field of characteristic 0, excepted that the isomorphisms are not anymore canonical.

Proof —— The isomorphisms over $\mathbb{C}$ are obtained trivially by putting together Theorem **??** and Corolarry III.2.33. The embeddings follow. Finally, to prove that there are also such isomorphisms or embeddings over any field L of characteristic

0, we only need to consider the case $N = \mathbb{Q}$. In this case, it follows from the following elementary statement: *if M and N are two finite-dimensional $\mathbb{Q}$-vector spaces privided with an action of a ring $\mathcal{H}$, such that $M \otimes \mathbb{C} \simeq N \otimes \mathbb{C}$ as $\mathbb{C}$-vector spaces and $\mathcal{H}$-modules, then $M \simeq N$ as $\mathbb{Q}$-vector spaces and $\mathcal{H}$-modules.* Indeed, choosing a basis of M and N, a $\mathbb{Q}$-linear application $f : M \to N$ is described by its matrix $(x_{i,j})$. The fact that $f : M \to N$ is an isomorphism, as well as the fact that it commutes with the action of $\mathcal{H}$ are expressed by a system of linear equations satisfied by the $x_{i,j}$ with coefficients in $\mathbb{Q}$. By hypothesis, this systems has solution in $\mathbb{C}$. Hence by Gauss' method it has a solution on $\mathbb{Q}$. $\qquad\square$

Through the Atkin-Lehner's theory does not carry over for modular symbols (cf. Remark **??**), many consequences of it carries trivially using the theorem above:

COROLLARY III.2.40. *Let L be any field of characteristic 0. Let $\mathcal{H}_0$ be the poly-nomlials algebras generated by the T_l ($l \nmid N$) and the diamond operators $\langle a \rangle$ ($a \in (\mathbb{Z}/N\mathbb{Z})^*$). Let λ be a system of eigenvalues $\mathcal{H}_0 \to L$ appearing in $\mathrm{Symb}^{\pm}_{\Gamma_1(N)}(\mathcal{V}_k(L))$. Then the set of divisors of N such that λ appears in $\mathrm{Symb}^{\pm}_{\Gamma_1(N)}(\mathcal{V}_k(L))$ has a minimal element N_0, that we call the* minimal level *of λ as in the classical Atkin-Lehner's tehory (§I.6.3). In particular the notions of λ being new and old, l-new and n-old make sense. Also, $\dim \mathrm{Symb}^{\pm}_{\Gamma_1(N)}(\mathcal{V}_k(L))[\lambda] = \sigma(N/N_0)$.*

III.3. Application of classical modular symbols to L-functions and congruences

III.3.1. Reminder about L-functions. Let $f = \sum_{n \geq 1} a_n q^n \in S_{k+2}(\Gamma_1(N), \mathbb{C})$. Let us recall that the L-function of f is the analytic function $L(f, s) = \sum_{n \geq 1} a_n/n^s$; the sum converges (absolutely uniformly over all compact) if $\mathrm{Re}\, s > k/2 + 2$, using the Hecke's easy estimate on the a_n, and actually if $\mathrm{Re}\, s > k/2 + 3/2$ using the estimate given by Ramanujan's generalized conjecture proved by Deligne.

PROPOSITION III.3.1. *We have, for $\mathrm{Re}\, s > k/2 + 2$,*

$$(46) \qquad \int_0^{\infty} f(iy) y^{s-1} dy = \frac{\Gamma(s)}{(2\pi)^s} L(f, s)$$

Proof — Leaving to the reader the easy arguments justifying the permutation of the sum and the integral on our domain of convergence , we compute

$$
\begin{aligned}
\int_0^{\infty} f(iy) y^{s-1} dy &= \sum_{n>0} \int_0^{\infty} a_n e^{-2\pi n y} y^{s-1} dy \\
&= \sum_{n>0} \int_0^{\infty} \frac{a_n}{(2\pi n)^s} e^{-t} t^{s-1} \, dt \quad \text{(change of variable } t = 2\pi n y) \\
&= \frac{\Gamma(s)}{(2\pi)^s} \sum_{n>0} \frac{a_n}{n^s}
\end{aligned}
$$

$\qquad\square$

COROLLARY III.3.2. *The function $L(f, s)$ has an analytic extension as en entire holomorphic function over $\mathbb{C}$, and the formula (46) is valid for all $s \in \mathbb{C}$.*

Proof — Since $f(iy)$ has exponential decay both when $y \to \infty$ and $y \to 0+$, $\int_0^\infty f(iy)y^{s-1}dy$ converges for all values of s and defines an entire function. Since $\Gamma(s)$ has no zero, the results follows. $\square$

COROLLARY III.3.3 (Functional equation). *later...*

Proof — For the Hecke operator W_N defined in §$III.2.7$, one has $f_{|W_N})(z) = \frac{(-1)^k}{Nz^{k+2}}f\left(\frac{-1}{Nz}\right)$. Hence $\frac{\Gamma(s)}{(2\pi)^s}L(f_{|W_N}, z) = \frac{(-1)^k}{Ni^{k+2}}\int_0^\infty \frac{1}{y^{k+2}}f\left(\frac{i}{Ny}\right)y^{s-1}dy$. Make the change of varables $y' = 1/(Ny)$. We get $\frac{\Gamma(s)}{(2\pi)^s}L(f_{|W_N}, z) = \frac{(-1)^k N^{k+2+s}}{i^{k+2}}\int_0^\infty y^{k+1-s}f(iy)dy = \frac{(-1)^k N^{k+2+s}}{i^{k+2}}\frac{\Gamma(k+1-s)}{(2\pi)(k+1-s)}L(f, k+1-s)$. Since $f_{|W_N}, z) = W(f)L(f_\rho, z)$ **To be finsihed...** $\square$

If f is a normalized eigenform for all Hecke operators in $S_{k+2}(\Gamma_1(N), \epsilon)$, its L-function has an Euler product:

$$(47) \qquad L(f, s) = \prod_{p\mathbb{N}}(1 - a_p p^{-s} + \epsilon(p)p^{1-2s})^{-1} \prod_{p|N}(1 - a_p p^{-s})^{-1}$$

Slightly more generally, let $\chi : \mathbb{Z} \to \mathbb{C}^*$ be a Dirichlet character (let us recall that this means there is a $m > 0$ such that χ factors as an application $\chi : \mathbb{Z}/m\mathbb{Z} \to \mathbb{C}^*$, with $\chi(a) = 0$ if $a \notin (\mathbb{Z}/m\mathbb{Z})^*$, and $\chi_{|(\mathbb{Z}/m\mathbb{Z})^*}$ a character. The smallest such d is called the *conductor* of χ).

We define the *twisted L-function* $L(f, \chi, s) = \sum_{n>0} a_n\chi(n)/n^s$.

Actually, this L-function may also be seen as the (untwisted) L-function of a modular form (hence it has an analytic continuation as en entire function as well). To see this, introduce $f_\chi(z) := \sum_{n \geq 1}\chi(n)a_n e^{2\pi inz}$. Then an easy computation shows that

$$f_\chi(z) = \frac{1}{\tau(\bar{\chi})}\sum_{a \,(\mathrm{mod}\; m)} \bar{\chi}(a)f\left(z + \frac{a}{m}\right),$$

where $\tau(\bar{\chi})$ is the Gauss sum $\sum_{a \,(\mathrm{mod}\; m)}\bar{\chi}(a)e^{2i\pi a/m}$, and that f_χ is a modular form of level $\Gamma_1(Nm)$.

EXERCISE III.3.4. Do that computation.

Obviously, $L(f_\chi, s) = L(f, \chi, s)$. Therefore, by the same computation as above:

LEMMA III.3.5.

$$\frac{\Gamma(s)}{(2\pi)^s}L(f, \chi, s) = \frac{1}{\tau(\bar{\chi})}\sum_{a \,(\mathrm{mod}\; m)}\bar{\chi}(a)\int_0^\infty f(iy + a/m)y^{s-1}dy.$$

As Tate's thesis showed, the right way to think of $L(f, \chi, s)$ is as an extended L-function of f, in the variable (χ, s). Actually, one can think of (χ, s) as a continuous character of the idèle class group $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$, with values in $\mathbb{C}^*$ as follows. First $\chi :$ $(\mathbb{Z}/m\mathbb{Z})^* \to \mathbb{C}^*$ can be seen as a character of $\text{Gal}(\mathbb{Q}(\zeta_m)/\mathbb{Q})$ of $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$, that is by Class Field Theory as a character of $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ trivial moreover on $\mathbb{R}_+^*$. Or more directly, using that the inclusion $\prod_l \mathbb{Z}_l^* \subset \mathbb{A}_{\mathbb{Q}}^*$ realizes an isomorphism $\prod_l \mathbb{Z}_l^* = \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*\mathbb{R}_*^+$, we see χ as a character of $\prod_l \mathbb{Z}_l^*$, hence of the idèle classes, by the natural surjection $\prod_l \mathbb{Z}_l^* \to (\mathbb{Z}/m\mathbb{Z})^*$ whose kernel is $\prod_{l\nmid m} \mathbb{Z}_l^* \times \prod_{l|m}(1 + l^{v_l(m)}\mathbb{Z}_l)$. In any case, we see χ as a character of $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$ trivial on $\mathbb{R}_+^*$. Now we identifies (χ, s) as the character $\chi| \ |^s$ on $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*$. Actually, any continuous character $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \to \mathbb{C}^*$ corresponds this way to one and only one pair (χ, s) where χ is a Dirichlet character and $s \in \mathbb{C}$.

EXERCISE III.3.6. **(easy)** Prove the last assertion.

If f is a normalized eigenform for all Hecke operators in $S_{k+2}(\Gamma_1(N), \epsilon)$, its L-function has an Euler product:

$$(48)\, L(f, \chi, s) = \prod_{p\mathbb{N}}(1 - a_p\chi(p)p^{-s} + \epsilon(p)\chi^2(p)p^{1-2s})^{-1} \prod_{p|N}(1 - a_p\chi(p)p^{-s})^{-1}$$

There is an important basic topic about $L(f, s)$ and $L(f, \chi, s)$ that we do not mention here: the functional equation. See [**Shi**] for a complete treatment.

III.3.2. Modular symbols and L-functions. We can now use what we have done to prove a deep arithmetic results about the values of the L-function of a modular eigenform.

In this §, let $\Gamma = \Gamma_1(N)$ and let $f \in S_{k+2}(\Gamma)$ a normalized eigenform for all the Hecke operators in $\mathcal{H}$. We call $\lambda : \mathcal{H} \to \mathbb{C}$ be the corresponding character: $Tf = \lambda(T)f$ for all $T \in \mathcal{H}$.

We recall the modular symbols $\phi_f^\pm \in \text{Symb}_\Gamma(\mathcal{V}_k(\mathbb{C}))$. Since $f \mapsto \phi_f^\pm$ is $\mathcal{H}$-equivariant (and $\mathbb{C}$-linear) we have $\phi_f^\pm \in \text{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{C}))[\lambda]$. We have more precisely:

LEMMA III.3.7. *The dimension of* $\text{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{C}))[\lambda]$ *is* 1

Proof — We have a natural embedding $S_{k+2}(\Gamma, \mathbb{C})[\lambda] \hookrightarrow \text{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{C}))[\lambda]$ whose cokernel is a direct summand in $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})^\vee[\lambda]$ by Theorem III.2.23 and Corollary **??**. But the space $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})[\lambda]$ is 0 (hence also $\mathcal{E}_{k+2}(\Gamma, \mathbb{C})^\vee[\lambda]$ since by the Petersion estimate, there is no Wisenstein series which is an eigenform with the same eigenvalues as a cuspidal form. And $S_{k+2}(\Gamma, \mathbb{C})[\lambda]$ has dimension 1 by multiplicity one (Prop. **??**). $\qquad\square$

Let K_f be the number field generated by the $\lambda(T)$, $T \in \mathcal{H}$. Then

LEMMA III.3.8. *The dimension over K_f of* $\text{Symb}_\Gamma^\pm(\mathcal{V}_k(K_f))[\lambda]$ *is* 1

Proof — Since λ takes values in K_f, this follows form the preceding lemma and Exercise I.5.5 □

DEFINITION III.3.9. We define *the periods* $\Omega_f^{\pm}$ of f as the non-zero complex numbers such that

$$\phi_f^{\pm}/\Omega_f^{\pm} \in \mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(K_f))[\lambda].$$

Obviously, those numbers exists and are well-determine up to multiplication by an element of K_f^* (each of them).

REMARK III.3.10. Sometimes we want to work integrally rather than rationally, and have a smaller indeterminacy on the periods $\Omega_f^{\pm}$. Let $\mathcal{O}_f$ be the ring of integer of f. We know that $\lambda(\mathcal{H}) \subset \mathcal{O}_f$, so $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_f))[\lambda]$ is an $\mathcal{O}_f$-lattice in the 1-dimensional space $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(K_f))[\lambda]$. The minor irritating issue we are facing is that since $\mathcal{O}_f$ is a Dedekind domain, but not necessarily principal, $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_f))[\lambda]$ might not be free of rank one. It is at any rate projective of rank one.

When it happens that $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_f))[\lambda]$ is free of rank one, we can define $\Omega_f^{\pm}$ by the requirement that $\phi_f^{\pm}/\Omega_f^{\pm}$ is a generator of $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_f))[\lambda]$. Thus the periods $\Omega_f^{\pm}$ are well defined up to a unit in $\mathcal{O}_f^*$. For example, if f has coefficient in $\mathbb{Q}$, then $K_f = \mathbb{Q}$, $\mathcal{O}_f = \mathbb{Z}$ which is principal, so $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathbb{Z}))[\lambda]$ is free of rank one, and the periods $\Omega_f^{\pm}$ are well defined up to a sign.

When $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_f))[\lambda]$ is not free, we have several solutions, depending on what we want to do. For example we can replace K_f by a finite extension K' on which this module becomes free (for example, the Hilbert Class Field of K_f will do), and have $\Omega_f^{\pm}$ well defined up to an element in $\mathcal{O}_{K'}^*$. Or we can work at one prime at a time: for $\mathfrak{p}$ a prime of $\mathcal{O}_f$, the localization $\mathcal{O}_{f,(\mathfrak{p})}$ is principal, so it is possible to require that $\Omega_f^{\pm}$ is chosen such that $\phi_f^{\pm}/\Omega_f^{\pm}$ is in $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_f))[\lambda]$, and generates that module after localizing at $\mathfrak{p}$. Such a requirement fixes $\Omega_f^{\pm}$ up to multiplication by an element of $K_\mathfrak{p}^*$ whose $\mathfrak{p}$-valuation is 0, which is often sufficient in the applications. When $\Omega^{\pm}$ is chosen this way, we shall say that it is $\mathfrak{p}$-*normalized* (that's our terminology, not a standard one).

THEOREM III.3.11 (Manin-Shokurov). *For any Dirichlet character χ, any integer j such that $0 \leq j \leq k$, we have*

$$\frac{L(f,\chi,j+1)}{\Omega_f^{\pm}(i\pi)^{j+1}} \in K_f[\chi],$$

where the sign $\pm$ is chosen such that $\pm 1 = (-1)^j \chi(-1)$, and $K_f[\chi]$ is the extension of K_f generated by the image of χ.

Proof —   We have for j an integer, $0 \leq j \leq k$,

$$\int_0^\infty f(iy + a/m)y^i dy = i^{-j-1} \int_{a/m}^\infty f(z)(z - a/m)^j dz$$
$$= i^{-j-1}\phi_f^1(\infty - \{a/m\})((z - a/m)^j).$$

Hence using Lemma III.3.5,

$$\frac{\Gamma(j+1)i^{j+1}\tau(\bar\chi)}{(2\pi)^{j+1}}L(f, \chi, j+1) = \sum_{a \,(\mathrm{mod}\; m)} \bar\chi(a)\phi_f^1(\{\infty\} - \{a/m\})((z - a/m)^j)$$

We note that the right hand side is multiplied by $(-1)^j\chi(-1)$ if the occurrence of ϕ_f^1 in it is replaced by $(\phi_f^1)_{|\iota}$. Hence we can replace ϕ_f^1 by $\phi_f^\pm$ with $\pm 1 = (-1)^j\chi(-1)$ in that formula. We thus have

$$(49)\frac{j!\tau(\bar\chi)}{(-2i\pi)^{j+1}\Omega_f^\pm}L(f, \chi, j+1) = \sum_{a \,(\mathrm{mod}\; m)} \bar\chi(a)\frac{\phi_f^\pm(\{\infty\} - \{a/m\})((z - a/m)^j)}{\Omega_f^\pm}$$

and the results follows from Definition III.3.9.   □

EXERCISE III.3.12. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_f$ and assume that $\Omega^\pm$ is $\mathfrak{p}$-normalized (notations of Remark III.3.10). Show that the numbers $\frac{j!\tau(\bar\chi)}{(-2i\pi)^{j+1}\Omega_f^\pm}L(f, \chi, s)$ (when $0 \leq j \leq k$, χ is a Dirichlet character, and $\chi(-1)(-1)^j = \pm 1$) are $\mathfrak{p}$-integers if the conductor m of χ is prime to $\mathfrak{p}$. Show also that when $k = 0$, at least one of those number has $\mathfrak{p}$-valuation 0.

III.3.3. Scalar product and congruences. In this paragraph, we present as an application a weak version of a nice theorem of Hida (cf. [**H1**] and [**H2**]). We continue to assume $\Gamma = \Gamma_1(N)$. Let f be a newform in $S_{k+2}(\Gamma, \mathbb{C})$.

PROPOSITION III.3.13. *One has*

$$[\phi_f^+, \phi_f^-] = -4W(f)(-i)^{k-1}(f, f)_\Gamma$$

and the number $(f, f)_\Gamma/(i^{k-1}\Omega_f^+\Omega_f^-)$ *belongs to* K_f. *One has* $[\phi_f^+, \phi_f^+] = [\phi_f^-, \phi_f^-] = 0$

Proof —   By (43), one has

$$(50)\qquad\qquad (\phi_f^1, \phi_{f_\rho}^3) = (\phi_f^1, \phi_{\bar f}^2) = -2i^{k-1}(f, f)_\Gamma.$$

Applying this to f_ρ, one also has

$$(51)\qquad\qquad (\phi_{f_\rho}^1, \phi_f^3) = -2i^{k-1}(f, f)_\Gamma$$

since obviously $(f, f)_\Gamma = (f_\rho, f_\rho)_\Gamma$. Recall also that one has, for any f, g,

$$(52)\qquad\qquad (\phi_f^1, \phi_g^1) = (\phi_f^3, \phi_g^3) = 0.$$

$$
\begin{aligned}
[\phi_f^+, \phi_f^-] &= [\phi_f^1 + \phi_f^3, \phi_f^1 - \phi_f^3] \quad \text{(by def. of } \phi_f^\pm) \\
&= (\phi_f^1 + \phi_f^3, (\phi_f^1 - \phi_f^3)_{|W_N}) \quad \text{(by def. of the corrected pairing)} \\
&= (\phi_f^1 + \phi_f^3, (\phi_{f_{|W_N}}^1 - (-1)^k \phi_{f_{|W_n}}^3) \quad \text{(by Lemma III.2.29)} \\
&= W(f)(\phi_f^1 + \phi_f^3, \phi_{f_\rho}^1 + (-1)^{k+1}\phi_{f_\rho}^3) \quad \text{(by Prop. III.2.27 and (52))} \\
&= W(f)[(\phi_f^1, (-1)^{k+1}\phi_{f_\rho}^3) + (-1)^{k+1}(\phi_{f_\rho}^1, \phi_f^3)] \quad \text{(see below)} \\
&= -4W(f)(-i)^{k-1}(f,f)_\Gamma \quad \text{(by (50) and (51))},
\end{aligned}
$$

the justification for the line missing one being that $(\,,\,)$ is antisymmetric when k is even and symmetric when k is odd. As the symbols $\phi_f^\pm/\Omega_f^\pm$ belong to $\mathrm{Symb}_\Gamma(\mathcal{V}_k(K_f))$, their scalar product belongs to K_f. The computation of $[\phi_f^+, \phi_f^+]$ and $[\phi_f^+, \phi_f^-]$ is exactly similar, with the two terms in the penultiem line being now opposite instead of being equal. Hence those corrected products are 0. $\qquad\square$

THEOREM III.3.14 (Hida). *Let $\mathfrak{p}$ be a prime ideal of $\mathcal{O}_f$, of residual characteristic p. Assume that p does not divides $6Nk!$. Then the following are equivalent:*

(a) *When $\Omega_f^\pm$ are chosen $\mathfrak{p}$-normalized, $v_\mathfrak{p}(W(f)(f,f)_\Gamma/(i^{k-1}\Omega_f^+\Omega_f^-)) > 0$.*

(b) *There exists a cuspidal normalized eigenform $g \in S_{k+2}(\Gamma, \mathbb{C})$ with coefficients in a finite extension K' of K_f, and a prime $\mathfrak{p}'$ of K' above $\mathfrak{p}$, such that $g \neq f$ and*

$$
g \equiv f \pmod{\mathfrak{p}'}.
$$

Proof — Let $\mathcal{O}_\mathfrak{p}$ be the localized ring at $\mathfrak{p}$ of the ring of integers $\mathcal{O}_f$ of K_f. Choosing $\Omega_f^\pm$ $\mathfrak{p}$-normalized means that $\phi_f^\pm/\Omega_f^\pm$ is a generator of $\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathcal{O}_\mathfrak{p}))[\lambda_f]$. Equivalently, since 2 is invertible in $\mathcal{O}_\mathfrak{p}$, ϕ_f^+/Ω_f^+ and ϕ_f^-/Ω_f^- form a basis of $\mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathcal{O}_p))[\lambda_f]$.

Let us write $M = H_!^1(\Gamma, \mathcal{V}_k(\mathcal{O}_\mathfrak{p}))$, so that $M \otimes K_f = H_!^1(\Gamma, \mathcal{V}_k(K_f))$. We provide M with the corrected scalar product $[\,,\,]$ (definition III.2.25). It is a perfect pairing by Prop III.2.3 and Lemma III.2.24 using that $6Nk!$ is invertible in $\mathcal{O}_\mathfrak{p}$, for which the usual Hecke operators are self-adjoint.

Let $A = \mathrm{Symb}_\Gamma(\mathcal{V}_k(K_f))[\lambda_f]$ which is 2-dimensional subspace of $M \otimes K_f = H_!^1(\Gamma, \mathcal{V}_k(K_f))$ since f is cuspidal. Obviously, $M \cap A = \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathcal{O}_\mathfrak{p}))[\lambda]$. Let B be the orthogonal of A in $M \otimes K_f$.

Since ϕ_f^+/Ω_f^+ and ϕ_f^-/Ω_f^- form a basis of $M \cap A$, we can very easily compute the discriminant of the restriction of the scalar product to $M \cap A$, using the above proposition: it is $2^{-2k}(-1)^{1-k}W(f)(f,f)_\Gamma/(\Omega_f^+\,\Omega_f^-)^2$. Since this discriminant is non zero, we see that $M \otimes K_f = A \oplus B$. We can thus define the congruence module $C = M/(M \cap A) \oplus (M \cap B)$ and since the the scalar product on M is a perfect pairing (see Prop III.2.3) we know by Prop I.7.16 that this module C is non 0 if and only if the discriminant of the scalar product on $M \cap A$ has positive $\mathfrak{p}$ valuation, that is if and only if (a) holds.

But on the other hand, the fact that the congruence module C is non-zero is equivalent (cf. Prop. I.7.15) to the existence of a modular symbol ϕ' in B such that $\phi' \equiv a\phi_f^+ + b\phi_f^-$ (mod $\mathfrak{p}$), with $a, b \in \mathcal{O}_{\mathfrak{p}}$, not both in $\mathfrak{p}$. By Deligne-Serre's lemma (up to allowing finite extension K' of K_f and replacement of $\mathfrak{p}$ be a prime $\mathfrak{p}'$ of K' dividing $\mathfrak{p}$) this is equivalent to the same thing with ϕ' assume to be an eigenvector for all Hecke operators. Such an eigenvector ϕ' defines a cuspidal normalized eigenform $g \in S_{k+2}(\Gamma, K')$ with the same Hecke eigenvalues. Hence C is non-zero if and only if there exists a cuspidal normalized eigenform g whose Hecke eigenvalues are congruent to those of f. In our case, this is also equivalent to $g \equiv f$ (mod $\mathfrak{p}$), for the two residual form g mod $\mathfrak{p}$ and f (mod $\mathfrak{p}$) in $S_{k+2}(\Gamma, \mathcal{O}_{\mathfrak{p}}/\mathfrak{p})$ will have the same eigenvalues, hence the same Fourier coefficients, hence will be equal by the q-development principle.

In other words, $C \neq 0$ is equivalent to (b). This proves the theorem. $\qquad\square$

III.4. Distributions

In all this section, L will denote a commutative noetherian $\mathbb{Q}_p$-Banach algebra (as in [**Bu**]) with a norm denoted $|\ |$ extending the usual absolute value on $\mathbb{Q}_p$, so that $|p| = 1/p$.

III.4.1. Some modules of sequences and their dual. For L any commutative $\mathbb{Q}_p$-algebra, and r any positive real number, we shall denote by $c_r(L)$ (resp. $b_r(L)$) the Banach L-module of sequences $(a_n)_{n \geq 0}$, with $a_n \in L$ and $|a_n|r^n \to 0$ when $n \to \infty$ (resp. with $|a_n|r^n$ bounded above), with the norm $|(a_n)| = \sup_{n \geq 0} |a_n|r^n$. When $r = 1$, we shall denote $c(L)$ for $c_1(L)$ (resp. $b(L)$ for $b_1(L)$). Note that the Cauchy product makes $c_r(L)$ and $b_r(L)$ Banach algebras over L.

LEMMA III.4.1. *The natural map $c_r(\mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} L \to c_r(L)$ is an isometry, and is compatible with the L-algbera structure. In particular, for any continuous morphism $L \to L'$ of $\mathbb{Q}_p$-Banach algebras, we have an isomorphism $c_r(L)\hat{\otimes}_L L' = c_r(L')$ of L'-Banach algebras.*

Proof — The first statement is proved exactly as [**BGR**, Prop 2.1.7/8]. The second folloews from the first and the associativity of completed tensor product. $\square$

LEMMA III.4.2. *If there is an R in L^* with $|R| = r$ and $|R^{-1}| = r^{-1}$, then there are isometric isomorphisms of L-algebras $c_r(L) \simeq c(L)$ and $b_r(L) \simeq b(L)$.*

Proof — Since $|R^{-1}| = |R|^{-1}$, the element R is multiplicative ([**BGR**, Prop. 1.2.2/4]), that is $|a_n R^n| = |a_n||R|^n = |a_n|r^n$. Hence we can define a map $c_r(L) \to c(L)$ that sends (a_n) to $(a_n R^n)$. It is obviously an isometric isomorphism of algebras. Same proof for $b_r(L)$ and $b(L)$. $\qquad\square$

LEMMA III.4.3. *The Banach L-module $c_r(L)$ is always potentially ON-able. If there is an element $R \in L^*$ such that $|R| = r$ and $|R^{-1}| = r^{-1}$, then $\mathcal{A}[r](L)$ is actually ON-able.*

Proof — The $\mathbb{Q}_p$-Banach space $c_r(\mathbb{Q}_p)$ is potentially ONable as is any $\mathbb{Q}_p$-Banach space by [**S**]. Then the L-Banach module

$$c_r(L) = c_r(\mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} L$$

is potentially ONable by [**Bu**, Lemma 2.8]. The second assertion follows from Lemma III.4.2 since $c(L)$ is ON-able by definition. $\qquad\square$

EXERCISE III.4.4. **(easy)** Show that $c_{p^{1/2}}(\mathbb{Q}_p)$ is not ON-able.

LEMMA III.4.5. *There are natural isometric isomorphisms of Banach L-modules*

(53) $$b_{\frac{1}{r}}(L) = \mathrm{Hom}_L(c_r(L), L) = \mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$$

Proof — if $b = (b_n)_{n \in \mathbb{N}} \in b_{\frac{1}{r}}(L)$, one defines an L-linear form l_b on $c_r(L)$ by setting $l_b((a_n)) = \sum_n a_n b_n$. The sum converges since $|a_n|r^n$ goes to 0 and $|b_n|/r^n$ is bounded, so $a_n b_n$ goes to 0. The operator norm of l_b is clearly $|b|$. Conversely, any continuous linear form l on $c(L)$ defines a sequence $b = (b_n)$ by setting $b_n = l((\delta_{n,m})_{m \geq 0})$ where $\delta_{m,n}$ is Kronecker's symbol, and that sequence is bounded by the operator of norm of l. One checks immediately that $l = l_b$. This proves the first isomorphisms. The second is proved the same way. $\qquad\square$

Let us define a Banach L-module $b'_r(L)$ as $b_r(\mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} L$. The map $b_r(\mathbb{Q}_p) \subset b(L)$ induces an injective norm-preserving $b'_r(L) \hookrightarrow b_r(L)$, which allows us to see $b'_r(L)$ as a closed submodule of $b_r(L)$ with the induced norm. Using lemma III.4.5, one can also seen $b'_{\frac{1}{r}}(L)$ as a closed submodule of $\mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$.

PROPOSITION III.4.6. *The sub-module $b'_{\frac{1}{r}}(L)$ of $\mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$ contains exactly the linear applications from $c_r(\mathbb{Q}_p)$ to L that are comapct (that is, completely continuous).*

Proof — [**S**, Corollary, page 74] $\qquad\square$

COROLLARY III.4.7. *The inclusion $b'_r(L) \subset b_r(L)$ is proper if and only if L is infinite-dimensional over $\mathbb{Q}_p$.*

Proof — If L is finite-dimensional, every linear map in $\mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$ has finite rank hence is compact. If L is infinite-dimensional, there are linear maps in $\mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$ that are non-compact. $\qquad\square$

If $\tau \in \mathrm{End}_L(c_r(L))$, we denote by ${}^t\tau \in \mathrm{End}_L(\mathrm{Hom}_L(c_r(L), L) = \mathrm{End}_L(b_r(L))$ the transpose map: ${}^\tau(l) = l \circ \tau$.

PROPOSITION III.4.8. *Let $\tau_0 \in \mathrm{End}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p))$, and let $\tau = \tau \hat{\otimes} Id_L \in \mathrm{End}_L(c_r(L))$. Then ${}^t\tau$ stabilizes the submodule $b'_{\frac{1}{r}}(L)$ of $b_{\frac{1}{r}}(L)$.*

Proof — Under the identification $b_{\frac{1}{r}}(L) = \mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$ of Lemma III.4.5, ${}^t\tau$ is identified with the endomorphism of $\mathrm{Hom}_{\mathbb{Q}_p}(c_r(\mathbb{Q}_p), L)$ that sends u to $u \circ \tau_0$. If u is a compact linear map, so is $u \circ \tau_0$ (e.g. by [**Bu**, Lemma 2.7]), hence the result by Prop. III.4.6. $\qquad\square$

PROPOSITION III.4.9. *Let $a \in c_r(L)$, and let $\tau_a \in \mathrm{End}_L(c_r(L))$ be the multiplication by a. Then ${}^t\tau_a$ stabilizes the submodule $b'_{\frac{1}{r}}(L)$ of $b_{\frac{1}{r}}(L)$.*

Proof — By Lemma **??**, a is the limit in $c_r(L)$ of a sequence of elements $a_k \in c_r(\mathbb{Q}_p) \otimes L$ (algebraic tensor product) when $k \to \infty$. Hence ${}^t\tau_a$ is the limit in $\mathrm{End}_L(b_{\frac{1}{r}}(L))$ of the sequence ${}^t\tau_{a_k}$. Since $b'_{\frac{1}{r}}(L)$ is closed in $b_{\frac{1}{r}}(L)$ it suffices to show that ${}^t\tau_{a_k}$ stabilizes $b'_{\frac{1}{r}}(L)$. In other words, we may assume that $a \in c_r(\mathbb{Q}_p) \otimes L$. We may therefore assume that a is of the form $a_0 \otimes x$ with $a_0 \in c_r(\mathbb{Q}_p)$ and $x \in L$. Thus $\tau_a = x\tau_{a_0}$ stabilizes $b'_{\frac{1}{r}}(L)$ by Prop. III.4.8. $\qquad\square$

III.4.2. Modules of functions over $\mathbb{Z}_p$.

DEFINITION III.4.10. For $r \in |\mathbb{C}_p^*| = p^{\mathbb{Q}}$, we define $\mathcal{A}[r](L)$ as the L-module of functions $f : \mathbb{Z}_p \to L$ such that for every $e \in \mathbb{Z}_p$, there exists a power series

$$\sum_{n=0}^{\infty} a_n(e)(z - e)^n,$$

with $a_n(e) \in L$, that converges on the closed ball $\bar{B}(e, r) = \{z \in \mathbb{C}_p, |z - e| < r\}$ in $\mathbb{C}_p$, and that coincides with f on $\bar{B}(e, r) \cap \mathbb{Z}_p$

DEFINITION III.4.11. For $f \in \mathcal{A}[r]$, we set $\|f\|_r = \sup_{e \in \mathbb{Z}_p} \sup_n |a_n(e)| r^n$

It is clear that $\mathcal{A}[r](L)$ with this norm is a Banach-algebra over L. Show that the algebra $L[z]$ of polynomials in z is dense in $\mathcal{A}[r](L)$ for all r.

PROPOSITION III.4.12. *In the definition of $\mathcal{A}[r](L)$, it suffices to check the condition for a set E of $e \in \mathbb{Z}_p$ such that the balls $\bar{B}(e, r)$ cover $\mathbb{Z}_p$, and such a set can be chosen finite. Moreover, the first $\sup$ in the definition on $\|f\|$ can be taken over $e \in E$ where E is as in the above exercise.*

Proof — Saying that the power series $\sum_{n=0}^{\infty} a_n(e)(z - e)^n$ on $\bar{B}(e, r)$, is equivalent to saying that $\lim_{n \to +\infty} |a_n(e)| r^n = 0$. If $e' \in \bar{B}(e, r)$, then $\bar{B}(e', r) = \bar{B}(e, r)$. Set

$$a_n(e') = \sum_{m=0}^{\infty} \frac{(m + n)! a_{m+n}(e)(e' - e)^m}{m! n!}$$

This series converges since $|a_{m+n}(e)(e'-e)^m| \leq |a_{m+n}(e)|r^{m+n}r^{-n}$ and for n fixed, $m \to \infty$, $|a_{m+n}(e)|r^{m+n}$ goes to 0 since f is in $\mathcal{A}[r]$.

We have $|a_n(e')r^n| \leq \sum_m a_{m+n}(e)r^n r^n \leq \sup_m |a_m(e)|r^m$, which shows that the series $\sum a_n(e')(z-e')^n$ converges on $B(e,r)$ (to the same limit as $\sum_n a_n(e)(z-e)^n$), and that $\sup_n |a_n(e)|r^n \geq \sup_n |a_n(e')|r^n$. By symmetry, we have equality. This proves that in the definition of $\mathcal{A}[r]$ and of $\|f\|_r$, it is sufficient to work with one e in each closed ball of radius r. Since $\mathbb{Z}_p$ is compact, it is covered by finitely many such balls (which are also open). It is clear that $\mathcal{A}[r](L)$ is complete for that norm and that polynomials are dense. $\qquad\square$

Before going on, let us record an estimate (essentially the Taylor-Laplace estimate) the we shall need later:

SCHOLIUM III.4.13. *For f in $\mathcal{A}[r](L)$, $e, e' \in \mathbb{Z}_p$ such that $|e'-e| \leq r$, $n, N \in \mathbb{N}$, $N \geq n$ we have*

$$|a_n(e') - \sum_{k=n}^{N} \frac{k! a_k(e)(e'-e)^{k-n}}{(k-n)! n!}| \leq r^{-N-1}|e'-e|^{N+1-n}\|f\|_r.$$

Proof — The displayed equation in the above proof is, after the change of variables $k = m + n$:

$$a_n(e') = \sum_{k=n}^{\infty} \frac{k! a_k(e)(e'-e)^{k-n}}{(k-n)! n!}$$

for $k > N$, one has $|\frac{k! a_k(e)(e'-e)^{k-n}}{(k-n)! n!}| \leq |a_k(e)||e'-e|^{k-N-1}|e'-e|^{N+1-n} \leq |a_k(e)||e'-e|r^{k-N-1} = |a_k(e)|r^k|e'-e|^{N+1-n}r^{-N-1} \leq \|f\|_r|e'-e|^{N+1-n}r^{-N-1}$. $\qquad\square$

COROLLARY III.4.14. *The choice of a finite subset E of $\mathbb{Z}_p$ as in the above proposition determines a natural isometric isomorphism of L-algebra*

(54) $$\mathcal{A}[r](L) = c_r(L)^E.$$

Proof — Send f to $(a_n(e))_{n \in \mathbb{N}, e \in E})$. $\qquad\square$

COROLLARY III.4.15. *The formation of $\mathcal{A}[r](L)$ commutes with base change, in the following sense: $\mathcal{A}[r](L) \hat{\otimes}_L L' = \mathcal{A}[r](L')$ for every L-Banach algebra L'.*

Proof — This follows from the above corollary and Lemma III.4.1 $\qquad\square$

EXERCISE III.4.16. Define $B[\mathbb{Z}_p, r] = \{z \in \mathbb{C}_p, \exists a \in \mathbb{Z}_p, |z - a| \leq r\}$ (a finite union of closed balls of radius r, and even only one ball of center 0 and radius r when $r \geq 1$).

1.– Show that an $f \in \mathcal{A}[r](L)$ defines a continuous application $B[\mathbb{Z}_p, r] \to L$.

2.– Show that $\|f\|_r = \sup_{z \in B[\mathbb{Z}_p, r]} |f(z)|$.

3.– Show that $B[\mathbb{Z}_p, r]$ has a natural structure of affinoid space over $\mathbb{Q}_p$ and that $\mathcal{A}[r](\mathbb{Q}_p)$ is the ring of global analytic functions on this space.

LEMMA III.4.17. *The Banach L-module $\mathcal{A}[r](L)$ is always potentially ON-able.
If there is an element $R \in L^*$ such that $|R| = r$ and $|R| = r^{-1}$, then $\mathcal{A}[r](L)$ is
actually ON-able*

Proof —   This follows from Corollary III.4.14 and Lemma **??**.   □

There are natural restriction maps $\mathcal{A}[r_1](L) \to \mathcal{A}[r_2](L)$ for $r_1 > r_2$, that are
clearly injective, and have dense image since their images contain all polynomials).
Moreover

LEMMA III.4.18. *The restriction maps $\mathcal{A}[r_1](L) \to \mathcal{A}[r_2](L)$ for $r_1 > r_2$ are
compact.*

Proof —   Note first that the assertion makes sense since our module are potentially
ON-able. We observe that the result is true in the case where L is a finite extension
of $\mathbb{Q}_p$ and there exist $R_1, R_2 \in L^*$ such that $|R_1| = r_1$, $|R_1^{-1}| = r_1^{-1}$, and $|R_2| = r_2$,
$|R_2|^{-1} = r_2^{-1}$ for in this case our two module are ON-able and the results follows
from [**?**, Prop. A5.2][1] (see the exercise below for another proof). We next prove the
result for any r_1, r_2, but for $L = \mathbb{Q}_p$. By the above, the result is true for a suitable
finite extension L' of $\mathbb{Q}_p$. Writing the matrix of the restriction map in potentially
ON based over $\mathbb{Q}_p$, we note that the matrix is the same over L' by [**Bu**]. Therefore,
since compactness is read on the matrix ([**Bu**, Prop. 2.4]), our operator is compact
over $\mathbb{Q}_p$. We conclude over any L by applying [**Bu**, Prop. 2.4] again.   □

EXERCISE III.4.19. Prove directly, using the matrix criterion of compactness
([**Bu**, Prop. 2.4]) that for any L, in the case where there exist $R_1, R_2 \in L^*$ such
that $|R_1| = r_1$ and $|R_2| = r_2$ the restriction map is compact.

III.4.3. Modules of convergent distributions.

DEFINITION III.4.20. We denote by $\mathcal{D}[r](\mathbb{Q}_p)$ the continuous dual $\mathrm{Hom}_{\mathbb{Q}_p}(\mathcal{A}[r], \mathbb{Q}_p)$
of $\mathcal{A}[r](\mathbb{Q}_p)$. The space $\mathcal{D}[r](\mathbb{Q}_p)$ is a Banach module over $\mathbb{Q}_p$ for the norm

$$\|\mu\|_r = \sup_{f \in \mathcal{A}[r]} \frac{|\mu(f)|}{\|f\|_r}.$$

For L any $\mathbb{Q}_p$-Banach algebra, we define $\mathcal{D}[r](L) = \mathcal{D}[r](\mathbb{Q}_p) \hat{\otimes} L$

PROPOSITION III.4.21. *The Banach L-module $\mathcal{D}[r](L)$ are potentially ON-able.
The formation of $\mathcal{D}[r](L)$ commutes with base change. The natural restriction maps
$\mathcal{D}[r_2](L) \to \mathcal{D}[r_1](L)$ are injective and compact*

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[1]Actually, Coleman's proposition is stronger, as it works for L any reduced affinoid algebra
and without the hypothesis on r_2.

Proof — The Banach $\mathcal{D}[r](\mathbb{Q}_p)$ is potentially orthonormalizable by [**S**]. It is clear by definition that the formation of $\mathcal{D}[r](L)$ commute with base change. The transposed restriction maps $\mathcal{D}[r_2](\mathbb{Q}_p) \to \mathcal{D}[r_1](\mathbb{Q}_p)$ are obviously injective since the restriction maps $\mathcal{A}[r_1](\mathbb{Q}_p) \to \mathcal{A}[r_2](\mathbb{Q}_p)$ have dense image (since their images contain polynomials). They are compact, since the transpose of a compact map is compact (see [**S**, Lemma 14] or [**Sch**, Lemma 16.4]). The restriction maps $\mathcal{D}[r_2](L) \to \mathcal{D}[r_1](L)$ are defined by base change from the case $L = \mathbb{Q}_p$, so their injectivity and compactness follow. $\qquad\square$

We note that the we have a natural injective norm-preserving map $\mathcal{D}[r](L) = \mathcal{D}[r](\mathbb{Q}_p)\hat{\otimes}L \to \mathrm{Hom}_L(\mathcal{A}[r](L), L)$ which identified $\mathcal{D}[r](L)$ as a closed subspace of $\mathrm{Hom}_L(\mathcal{A}[r](L), L)$. The following lemma is obvious form the definitions.

LEMMA III.4.22. *Let E be as in Prop. III.4.12. Then through the isomorphisms (54) and (??)*

$$\mathrm{Hom}_L(\mathcal{A}[r](L), L) = \mathrm{Hom}_L(c_r(L)^E, L) = b_{\frac{1}{r}}(L)^E$$

the subspace $\mathcal{D}[r](L)$ of $\mathrm{Hom}_L(\mathcal{A}[r](L), L)$ is identified with the subspace $b'_{\frac{1}{r}}(L)^E$ of $b_{\frac{1}{r}}(L)^E$.

In particular, we see that $\mathcal{D}[r](L)$ is strictly smaller than the L-dual of $\mathcal{A}[r](L)$ when L is not finite-dimensional over $\mathbb{Q}_p$. This is the price to pay to have a $\mathcal{D}[r](L)$ whose formation commute with base change, which will be crucial for the application to the eigencurve below.

III.4.4. Modules of overconvergent functions and distributions.

DEFINITION III.4.23. For $r \geq 0$, we define the modules of overconvergent functions

$$\mathcal{A}^\dagger[r](L) = \varinjlim_{r'>r, r' \in p^{\mathbb{Q}}} \mathcal{A}[r'](L) = \bigcup_{r'>r} \mathcal{A}[r'](L).$$

The module $\mathcal{A}^\dagger[0](L)$ is simply the module of L-valued locally analytic functions on $\mathbb{Z}_p$: functions that around every point e admit a converging Taylor expansion on some ball of some positive radius r depending on e. The module $\mathcal{A}^\dagger[1]$ is the module of analytic functions on $\mathbb{Z}_p$ defined by a powers series that converges "a little bit more than necessary", that is on some ball of center 0 and radius $r > 1$.

When L is a finite extension of $\mathbb{Q}_p$, we shall need to give a topology on $\mathcal{A}^\dagger[r](L)$: we give it the *locally convex final topology* (see [**Sch**, §5.E]), that is the finest locally convex topology such that the natural morphism $\mathcal{A}[r'](L) \to \mathcal{D}^\dagger[r](L)$ for $r' > r$ are continuous.

DEFINITION III.4.24. For any $r \geq 0$, we set

$$\mathcal{D}^\dagger[r](L) = \varprojlim_{r'>r,\ r' \in p^{\mathbb{Q}}} \mathcal{D}[r'](L).$$

In other words $\mathcal{D}^\dagger[r](L)$ is the intersection of the $\mathcal{D}[r'](L)$ for $r' > r$.

We give those modules the topology of the projective limit. In other words, the topology of $\mathcal{D}^\dagger[r](L)$ is defined by the family of norms $\|\mu\|_{r'}$ for all $r' > r$, $r' \in p^{\mathbb{Q}}$. It is clear that $\mathcal{D}^\dagger[r](L)$ is complete for that topology.

In proposition below the notion of completed tensor product of two Frechet spaces over $\mathbb{Q}_p$, which is introduced in [**Sch**, §17], cf. in particular [**Sch**, Prop. 17.6]. Let us briefly recall that if V and W are two locally convex spaces over $\mathbb{Q}_p$, then one can define on $V \otimes W$ two natural locally convex topologies: the *inductive topology* (resp. *projective topology*) is the finest one that makes the map $V \times W \to V \otimes W$ separately (resp. jointly) continuous. The projective topology is also the one defines by the semi-norms $p \otimes q$, where p (resp. q) runs among a family of semi-norms defining the topology of V (resp. of W), and $p \otimes q$ is the semi-norm of $V \otimes W$ defined by $p \otimes q(x) = \inf_{x = \sum v_i \otimes w_i} \sup_i p(v_i) q(w_i)$, for all $x \in V \otimes W$, the inf being taken on all the writing $x = \sum_i v_i \otimes w_i$ with the v_i's in V, the w_i's in W. Then [**Sch**, Prop. 17.6] proves that if V and W are Frechet, the completion of $V \otimes W$ for both those topology coincides. This is this completion that is denoted by $V \hat{\otimes} W$.

We shall need repeatedly the following observation, where for V a locally convex space, we denote by $c(V)$ the locally convex space of sequences $(v_n)_{n \geq 0}$ of elements of V that converge to 0: if V is a Frechet, then $c_0(\mathbb{Q}_p) \hat{\otimes} V$ is naturally isomorphic to the Frechet space $c_0(V)$. Indeed, one sees immediately that the obvious map $c_0(\mathbb{Q}_p) \times V \to c_0(V)$ is separately continuous, and has dense image, hence an isomorphism between the completion of $c_0(\mathbb{Q}_p) \otimes V$ and $c_0(V)$.

PROPOSITION III.4.25. *For any Banach algebra L, we have a natural isomorphism of Frechet spaces $\mathcal{D}^\dagger[r](\mathbb{Q}_p) \hat{\otimes} L \to \mathcal{D}^\dagger[r](L)$.*

Proof —   The maps $\mathcal{D}^\dagger[r] \hat{\otimes} L \hookrightarrow \mathcal{D}[r'] \hat{\otimes} L$ for $r' > r$ induces an injective map $\mathcal{D}^\dagger[r] \hat{\otimes} L \hookrightarrow \mathrm{proj} \lim_{r' > r} \mathcal{D}[r'](L) = \mathcal{D}^\dagger[r](L)$. This map is obviously an isomorphism when L is finite over $\mathbb{Q}_p$. Otherwise since L is potentially orthonormalizable, we can choose an isomorphism $L \simeq c(\mathbb{Q}_p)$ and by the above observation, we only need to prove that the natural map $c(\mathcal{D}^\dagger[r]) \to \mathrm{proj} \lim_{r' > r} c(\mathcal{D}[r']) = \cap_{r' > r} c(\mathcal{D}[r'])$ is an isomorphism, which is clear since a sequence (v_n) in $\mathcal{D}^\dagger[r]$ goes to 0 if and only if $\|v_n\|_{r'}$ goes to 0 for all $r' > r$.   □

For the next corollary, we shall need the notion of the completed tensor product, over a Banach algebra L, of a Banach module W and of a module V is defined by a family of semi-norms. Unfortunately, this notion is not covered in either of our two basic references in non-archimedean analysis, [**BGR**] (which consider only Banach modules over a Banach algebra, with topology defined by *one* norm), or [**Sch**] which considers completed tensor products over a field, never over a ring. We shall defined

$V\hat{\otimes}_L W$ as follows: if p is a norm defining the topology of W, and (q_i) are the semi-norms defining the topology of V, we put on $V \otimes_L W$ the family of semi-norms $(q_i \otimes p)$, and we define $V\hat{\otimes}_R W$ as the completion of $V \otimes_L W$ for that topology. It is cleat thai if $L = \mathbb{Q}_p$, the topology we have defines on $V \otimes W$ is the projective topology, hence our notion of $V\hat{\otimes}_{\mathbb{Q}_p}W$ coincide with theone defined in [**Sch**] and recalled above. If $W = L'$ is a Banach algebra, whose L-module structure is a continious morphism of Banach algebra $L \to L'$, then $V\hat{\otimes}_L L'$ is an L'-module and the semi-norms $(q_i \otimes p)$ are norm of L'-modules. We can thus, if $L' \to L''$ is another morphism of Banach algebras, consided $(V\hat{\otimes}_L L')\hat{\otimes}_{L'}L''$. One sees easily that this may be identified with $V\hat{\otimes}_L L''$. This, with Prop. III.4.25, immediately gives:

COROLLARY III.4.26. *For any morphism $L \to L'$ of Banach algebras, we have a natural isomorphism of topological module overL':*

$$\mathcal{D}^\dagger[r](L)\hat{\otimes}_L L' = \mathcal{D}^\dagger[r](L')$$

Let us also note:

PROPOSITION III.4.27. *Assume that L is a finite extension of $\mathbb{Q}_p$. Then that family of norms makes $\mathcal{D}^\dagger[r](L)$ a Frechet vector space over L. Moreover, $\mathcal{D}^\dagger[r](L)$ may be canonically identified, as a topological vector space, to the continuous dual (with its strong topology) of $\mathcal{A}^\dagger[r](L)$ with its locally convex inductive limit topology.*

Proof — Remember that $\mathcal{A}^\dagger[r](L) = \lim_{\substack{\leftarrow \\ r'>r}} \mathcal{A}[r'](L)$ by definition, where the transition maps are injective and compact (Prop. **??**), and that the dual of $\mathcal{A}[r'](L)$ with its operator norm (that is, its strong dual, cf. [**Sch**, Remark 6.7]) is $\mathcal{D}[r'](L)$. We can therefore apply directly [**Sch**, Prop 16.10], whose points ii. and iii. are exactly what we want. $\square$

Let L be a finite extension of $\mathbb{Q}_p$. If $f \in \mathcal{A}[r](L)$ or $f \in \mathcal{A}^\dagger[r](L)$, and $\mu \in \mathcal{D}[r](L)$ or $\mathcal{D}^\dagger[r](L)$, we have by definition or by the above proposition a scalar $\mu(f) \in L$. We will sometimes use the following longer but intuitive notation:

$$\int_{\mathbb{Z}_p} f(z)d\mu(z) := \mu(f).$$

Slightly more generally, for $a \in \mathbb{Z}_p$ and $n \in \mathbb{N}$, we set

$$\int_{a+p^n\mathbb{Z}_p} f(z)d\mu(z) := \int_{\mathbb{Z}_p} f(z)1_{a+p^n\mathbb{Z}_p}(z)d\mu(z) = \mu(f1_{a+p^n\mathbb{Z}_p}).$$

We shall refer to the modules $\mathcal{D}[r]$ and $\mathcal{D}^\dagger[r]$ as *modules of distributions* over $\mathbb{Z}_p$, even if properly speaking only $\mathcal{D}^\dagger[0]$ (the continuous dual of the space $\mathcal{A}^\dagger[0]$ of locally analytic functions) is the real *module of distributions over $\mathbb{Z}_p$*.

To simplify notations, we shall henceforth write $\mathcal{D}$ instead of $\mathcal{D}^\dagger[0]$

III.4.5. Order of growth of a distribution.

DEFINITION III.4.28. Let $\nu \geq 0$ be a real number. We say that a distribution $\mu \in \mathcal{D}$ has *order* $\leq \nu$ if there exists a constant $C \in \mathbb{R}^+$ such that for all a in $\mathbb{Z}_p$, $k, n \in \mathbb{N}$, we have

$$(55) \qquad \left| \int_{a+p^n\mathbb{Z}_p} \left(\frac{z-a}{p^n}\right)^k d\mu(z) \right| < Cp^{n\nu}.$$

A distribution of order ≤ 0 is called *a measure* on $\mathbb{Z}_p$. A distribution which is of order $\leq \nu$ for some ν is called *a tempered distribution*.

REMARK III.4.29. Visik ([**V**]) and Colmez ([**CO1**]) use an intersting equivalent definition for ν an integer: a distribution μ is of order $\nu \in \mathbb{N}$ if, as a continuous linear form on the space $\mathcal{A}^\dagger[0]$ of locally analytic functions on $\mathbb{Z}_p$, it extends to the larger space of class $\mathcal{C}^\nu$, defined as the space of functions that admits N-derivative and such that the last-derivative is Lipschitz. We shall not use this defintion.

There is an easy characterization of the order of growth in terms of norms:

LEMMA III.4.30. *Let $\mu \in \mathcal{D}(L)$ be a distribution. Then μ has order $\leq \nu$ if and only if there exists a real $D > 0$ such that $\|\mu\|_r \leq Dr^{-\nu}$ when $r \to 0^+$. Moreover, we can take $D = Cp$ if C is as in (55).*

Proof — Note that $g_{a,n,k}(z) := (\frac{z-a}{p^n})^k 1_{a+p^n\mathbb{Z}_p}(z)$ satisfies $\|g_{a,n,k}\|_r = 1$ for $r = p^{-n}$. Thus if $\|\mu\|_r \leq Dr^{-\nu}$ for all r, we get for $r = p^{-n}$ that $|\mu(g_{a,n,k})| \leq Dp^{n\nu}$ and μ has order of growth $\leq \nu$.

For the converse, let μ have order of growth $\leq \nu$. Then there is a C such that $\left| \int_{a+p^n\mathbb{Z}_p} (\frac{z-a}{p^n})^k d\mu(z) \right| < Cp^{n\nu}$. Let $f \in \mathcal{A}[p^{-n}]$, and $e \in \mathbb{Z}_p$. Write the power series of f about e as $f(z) = \sum_{k=0}^\infty a_k(e)p^{-nk}(z-e)^k$. Then $\|f\|_{p^{-n}} = \sup_{e \in E} \sup_k |a_k(e)p^{-nk}|p^{-nk} = \sup_{e \in E} \sup_{k \in \mathbb{N}} |a_k(e)|$, where E is a finite subset of $\mathbb{Z}_p$ as in Prop. III.4.12. Then

$$\mu(f) = \sum_{e \in E} \sum_{k=0}^\infty a_k(e) \int_{a+p^n\mathbb{Z}_p} \left(\frac{z-a}{p^n}\right)^k d\mu(z)$$

so $|\mu(f)| \leq \|f\|_{p^{-n}} Cp^{n\nu}$.

Now if r is a positive real number, $r < 1$, there is an $n \in \mathbb{N}$ such that $p^{-n} < r \leq p^{-n+1}$, and we have $|\mu(f)| \leq \|f\|_{p^{-n}} Cp^{n\nu} \leq \|f\|_r Cpr^{-\nu}$, that is $\|\mu\|_r \leq Dr^{-\nu}$ with $D = Cp$. □

THEOREM III.4.31 (Vishik, Amice-Velu). *Let L be a finite extension of $\mathbb{Q}_p$*

　(i) *Let $\mu \in \mathcal{D}(L)$ a distribution of order $\leq \nu$, and let N be an integer grater or equal to the integral part of ν. Then μ is uniquely determined by the*

linear forms for $a \in \mathbb{Z}_p$ and $n \in \mathbb{N}$:

$$i_{\mu, a + p^n \mathbb{Z}_p} : \; \mathcal{P}_N(L) \; \to \; L$$

$$P \mapsto \int_{a + p^n \mathbb{Z}_p} P(z) d\mu(z).$$

Here as above, $\mathcal{P}_N(L)$ is the finite-dimensional space of polynomials of degree less than N over L.

(ii) *Conversely, suppose we are given, for every ball $a + p^n \mathbb{Z}_p$ in $\mathbb{Z}_p$, a linear form $i_{a + p^n \mathbb{Z}_p} : \quad \mathcal{P}_N(L) \to L$ which satisfy the natural additivity relation (for all $a \in \mathbb{Z}_p$, $n \in \mathbb{N}$)*

(56)
$$i_{a + p^n \mathbb{Z}_p} = \sum_{i=0}^{p-1} i_{a + p^n i + p^{n+1} \mathbb{Z}_p},$$

and such that there exist constants $C > 0$ and $\nu \geq 0$ such that for every $a \in \mathbb{Z}_p$, $k, n \in \mathbb{N}$, and $k \leq N$

(57)
$$\left| i_{a + p^n \mathbb{Z}_p} \left(\left(\frac{z - a}{p^n} \right)^k \right) \right| \leq C p^{n\nu}.$$

Then there exists a (unique) distribution μ of order $\leq \nu$ such that $i_{\mu, a + p^n \mathbb{Z}_p} = i_{a + p^n \mathbb{Z}_p}$, and for any $n \in \mathbb{N}$ one has

$$\|\mu\|_{p^{-n}} \leq C p^{n\nu}.$$

This theorem is not hard to prove, just a little bit computational. It says that a distribution on order ν is known when you know how it integrates polynomials of degree less or equal than $\lfloor \nu \rfloor$ on closed balls in $\mathbb{Z}_p$, and conversely, that any way to "integrate" those polynomials satisfying the obvious additivity relation and a growth condition can be extended in a way to integrate all locally analytic functions, that is a distribution, which moreover satisfies the same growth condition. When $\nu = 0$, this takes the very intuitive form: a measure μ on $\mathbb{Z}_p$ is determined by the volume it gives to all balls $a + p^n \mathbb{Z}_p$. Conversely, if we have a way to attributes a volume to all ball $a + p^n \mathbb{Z}_p$ that is additive, and such that *the set of all such volumes is bounded*, then this defines a measure.

Proof — We frist prove (i). Fix a real $D > 0$ such that $\|\nu\|_r \leq D r^{-\nu}$.

For f in $\mathcal{A}[r](L)$ for some $r > 0$, and $e \in \mathbb{Z}_p$, such that $f(z) = \sum_{i=0}^{\infty} a_i(e)(z-e)^i$, and for $n \geq 0$ such that $p^{-n} \leq r$, and $N \geq 0$ an integer, let us define $T_{f,e,n,N} \in \mathcal{P}_N(L)$ as the function with support $e + p^n \mathbb{Z}_p$ defined by

$$T_{f,e,n,N}(z) := \sum_{i=0}^{N} a_i(e)(z-e)^i.$$

That is, $T_{f,e,n,N}(z)$ is the degree N Taylor-approximation of f around e, restricted to $e + p^n \mathbb{Z}_p$. The following states how good is that approximation:

(58)
$$\| f 1_{e + p^n \mathbb{Z}_p} - T_{f,e,n,N} \|_{p^{-n}} \leq (p^n / r)^{N+1} \|f\|_r.$$

Indeed, $f1_{e+p^n\mathbb{Z}_p} - T_{f,e,n,N}$ has Taylor expansion about e as follows: $\sum_{i=0}^{N+1} a_i(e)(z - e)^i$; hence $\|f1_{e+p^n\mathbb{Z}_p} - T_{f,e,n,N}\|_{p^{-n}} = \sup_{i \geq N+1} |a_i(e)|p^{-ni} \leq (p^{-n}/r)^{N+1} \sup_{i \geq N+1} |a_i(e)|r^i \leq (p^{-n}/r)^{N+1}\|f\|_r$, and (58) is proven.

Now let $\mu \in \mathcal{D}(L)$, and $f \in \mathcal{A}[r](L)$. We consider the following approximation of $\mu(f)$, a kind of genralized Riemann sum, depending on the choice of an integer n such that $p^n \leq r$ and of a set of representative E of $\mathbb{Z}_p$ mod p^n

$$S_{\mu,f,n,N,E} := \sum_{e \in E} \mu(T_{f,e,n,N}) = \sum_{e \in E} i_{\mu,e+p^n\mathbb{Z}_p}(T_{f,e,n,N}) \in L$$

We have

$$\begin{aligned}
|\mu(f) - S_{\mu,f,n,N,E}| &= |\sum_{e \in E} \mu(f1_{e+p^n\mathbb{Z}_p} - T_{f,e,n,N})| \\
&\leq (p^{-n}/r)^{N+1}\|\mu\|_{p^{-n}}\|f\|_r. \\
&\leq D(p^{-n}/r)^{N+1}p^{n\nu}\|f\|_r
\end{aligned}$$

Since $\nu < N + 1$ $|\mu(f) - S_{\mu,f,n,N,E}|$ goes to 0 for a fixed f, when n goes to infinity. It follows that $\mu(f)$ depends only on the linear forms $i_{\mu,e+p^n\mathbb{Z}_p}$, which proves (i).

Let us prove (ii). We begin by the following comutation: For $n, n' \in \mathbb{N}$, $n' \geq n$ and $e, e' \in \mathbb{Z}_p$, $e' \in e + p^n\mathbb{Z}_p$ we have for all z such that $|z - e'| \leq p^{-n'}$

$$\begin{aligned}
T_{f,e',n',N}(z) - T_{f,e,n,N}(z) &= \sum_{i=0}^N a_i(e')(z - e')^i - \sum_{k=0}^n a_k(e)(z - e)^k \\
&= \sum_{i=0}^N a_i(e')(z - e')^i - \sum_{k=0}^N a_k(e)\sum_{i=0}^k \frac{k!}{i!(k-i)!}(z - e')^i(e' - e)^{k-i} \\
&= \sum_{i=0}^N (z - e')^i[a_i(e') - \sum_{k=i}^n \frac{k!}{(k-i)!i!}a_k(e)(e' - e)^{k-i}] \\
&= \sum_{i=0}^N (z - e')^i E(i, n, e, e')
\end{aligned}$$

where $E(i, n, e, e')$ is an error term in L equal to the formula between bracket on the line above in L, and such that

$$|E(i, n, e, e')| \leq r^{-N-1}|e' - e|^{N+1-i}\|f\|_r \leq r^{-N-1}p^{-n(N+1-i)}\|f\|_r$$

using Scholium III.4.13.

From a family of linear forms $i_{e+p^n\mathbb{Z}_p}$ as in the statement of the theorem, we shall construct $\mu \in \mathcal{D}(L)$ by defining $\mu(f)$ as a limit of "Riemann sums" of the type considered above. Precisely we proceed as follows: Let $f \in \mathcal{A}[r](L)$ for some $r > 0$; for $n \in \mathbb{N}$ such that $p^{-n} \leq r$, and E a set of representative of $\mathbb{Z}_p$ mod p^n, define

$$S_{f,n,N,E} := \sum_{e \in E} i_{\mu,e+p^n\mathbb{Z}_p}(T_{f,e,n,N}).$$

For $n' \geq n$, E' any set of representative of $\mathbb{Z}_p$ mod $p^{n'}$, we compute

$$
\begin{aligned}
|S_{f,n,N,E} - S_{f,n',N,E'}| &\leq |\sum_{e' \in E'} i_{e'+p^{n'}\mathbb{Z}_p} (T_{f,e,n,N} 1_{e'+p^{n'}\mathbb{Z}_p} - T_{f,e',n',N}) \quad \text{(using (56))} \\
&\leq |\sum_{e' \in E'} i_{e'+p^{n'}\mathbb{Z}'_p} ((z-e')^i E(i,n,e,e'))| \\
&\leq |E(i,n,e,e')| C p^{n(\nu-i)} \quad \text{(using (57))} \\
&\leq C r^{-N-1} \|f\|_r p^{n(\nu-(N+1))}
\end{aligned}
$$

The above computation, using the hypothesis $\nu < N+1$, shows that the sequence $n \mapsto S_{f,n,N,E}$, however we choose our systems of representative E for each n, is a Cauchy sequence. We call $\mu(f)$ its limit, which by the above does not depends of the choices of E's. Thus we have constructed a linear form μ on $\mathcal{A}^\dagger[0](L)$ which satifies by construction $i_{\mu,a+p^n\mathbb{Z}_p} = i_{a+p^n\mathbb{Z}_p}$.

Now fix an n, and let $f \in \mathcal{A}[p^{-n}](L)$ We have

$$
\begin{aligned}
|S_{f,n,N,E}| &= |\sum_{e \in E} i_{e+p^n\mathbb{Z}_p} (\sum_{i=0}^{N} a_i(e)(z-e)^i)| \\
&\leq C \sup_{e \in E} |a_i(e)| p^{n\nu} p^{-ni} p^{n\nu} \quad \text{(using (57))} \\
&\leq C \|f\|_{p^{-n}} p^{n\nu}
\end{aligned}
$$

For any $n' > n$, and E' system of representatives of $\mathbb{Z}_p$ modulo $p^{n'}$, it follows from the estimate of the above paragraph, applied to $r = p^{-n}$, that $|S_{f,n',N',E} - S_{f,n,N,E}| \leq C\|f\|_{p^{-n}} p^{n\nu}|$. We thus get $|S_{f,n',N',E} \leq C\|f\|_{p^{-n}} p^{n\nu}$ for all n', E', hence by passing to the limit

$$
|\mu(f)| \leq C\|f\|_{p^{-n}} p^{n\nu}.
$$

It follows that for every $n \geq 0$, the restriction of μ to $\mathcal{A}[p^{-n}](L)$ is a *continuous* linear form, of norm

$$
\|\mu\|_{p^{-n}} \leq C p^{n\nu}.
$$

That is, μ belongs to $\mathcal{D}[p^{-n}](L)$ for all n, hence to $\mathcal{D}(L)$. $\square$

III.5. The weight space and the Mellin transform

III.5.1. The weight space. The p-adic weight space shall play two different roles, both very important, in what we are going to do. First its elements will generalize the possible weights $k \in \mathbb{N}$ of a modular form, and more generally, the couples (k, ϵ) where ϵ is the nebentypus of a modular form. Second it will be the set in which we will draw the variable of our p-adic L-function, in a way analogous to the fact that the variable of the complex L-function $L(f, s, \chi)$ is a couple (s, χ) (where s is a complex variable and χ a Dirichlet character), or which is equivalent as we have seen (see §III.3.1), a continuous character $\mathbb{A}_{\mathbb{Q}}^* / \mathbb{Q}^* \to \mathbb{C}^*$.

We use this analogy as a motivation for the following definition: a $\mathbb{C}_p$-valued weight is a continuous character $\kappa : \mathbb{A}_{\mathbb{Q}}^* \to \mathbb{C}_p^*$. Some remarks are in order: first, since $\mathbb{R}_+^*$ is connected, and $\mathbb{C}_p^*$ has no connected component, κ will be trivial on $\mathbb{R}_+^*$ hence factor through a character $\kappa : \mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^*\mathbb{R}_*^+ = \prod_l \mathbb{Z}_l^* \to \mathbb{C}_p^*$. Second:

EXERCISE III.5.1. Show that a continuous morphism $\prod_l \mathbb{Z}_l^* \to \mathbb{C}_p^*$ is trivial on some open subgroup of $\prod_{l \neq p} \mathbb{Z}_l^*$.

Now, a basis of open subgroup of $\prod_{l \neq p} \mathbb{Z}_l^*$ is given by the

$$U_M := \{x \in \prod_{l \neq p} \mathbb{Z}_l^*, \ x_{l_k} \equiv 1 \pmod{l_k^{a_k}}\}$$

for M an integer prime to p, with a decomposition $M = \prod_{k=1}^r l_k^{a_k}$. Note that we have $\prod_{l \neq p} \mathbb{Z}_l^*/U_M = (\mathbb{Z}/M\mathbb{Z})^*$. Thus our character κ will factor through $\prod_l \mathbb{Z}_l^*/U_M = \mathbb{Z}_p^* \times (\mathbb{Z}/M\mathbb{Z})^*$. This motivates our definition:

DEFINITION III.5.2. Let M be an integer coprime to p. The (p-adic) *weight space of tame conductor* M is the rigid analytic space $\mathcal{W}_M$ over $\mathbb{Q}_p$ such that, for all complete $\mathbb{Q}_p$-algebra L,

$$\mathcal{W}_M(L) = \mathrm{Hom}(\mathbb{Z}_p^* \times (\mathbb{Z}/M\mathbb{Z})^*, L^*).$$

We write $\mathcal{W}$ instead of $\mathcal{W}_1$.

Most of the times, we shall take $L = \mathbb{C}_p$ (and we have seen that all characters $\mathbb{A}_{\mathbb{Q}}^*/\mathbb{Q}^* \to \mathbb{C}_p^*$ belongs to $\mathcal{W}_M(\mathbb{C}_p)$ for M large enough). But allowing all L in the definition is necessary to ensure the uniqueness of the rigid analytic structure (by Yoneda's lemma). Implicit in the definition is the assertion of existence of a rigid analytic space $\mathcal{W}_M$ having the given functor of points. Let us prove it by describing explicitly this analytic structure:

First, set $q = p$ is p is odd and $q = 4$ if $p = 2$, and remember that that there is a canonical isomorphism $\mathbb{Z}_p^* = (\mathbb{Z}/q\mathbb{Z})^* \times (1+q\mathbb{Z}_p)$, given by $x \mapsto (\tau(x), \langle x \rangle)$, where τx is the Teichmuller lift of $x \mod p$, that is the only $y \in \mathbb{Z}_p^*$ such that $y^{\phi(q)} = 1$ and $y \equiv x \pmod{q}$, and $\langle x \rangle = x/\tau(x)$. Remember also that the multiplicative group $1 + q\mathbb{Z}_p$ is isomorphic to the additive group $\mathbb{Z}_p$ (as follows: choose a generator γ of $1 + q\mathbb{Z}_p$ and send $a \in \mathbb{Z}_p$ to $\gamma^a := \exp_p(a \log_p(\gamma)) \in 1 + p\mathbb{Z}_p$). Therefore, an element $\kappa \in \mathcal{W}_M(L)$ is given by a L-valued character κ_f on the finite group $(\mathbb{Z}/M\mathbb{Z})^* \times (\mathbb{Z}/q\mathbb{Z})^* = (\mathbb{Z}/Mq\mathbb{Z})^*$ and a character κ_p on $1 + q\mathbb{Z}_p$. Such a character is of course completely defined by the image $\kappa_p(\gamma)$ of the chosen generator γ of $1 + q\mathbb{Z}_p$. Since $\gamma^{p^n} \to 1$ when $n \to \infty$, we have $\kappa_p(\gamma)^{p^n} \to 1$, and this easily implies that $|\kappa_p(\gamma) - 1| < 1$. Conversely, by completeness, any element $x \in L^*$ satisfying $|x - 1| < 1$ defines a continuous character $\kappa_p : 1 + q\mathbb{Z}_p \to L^*$ sending γ to x.

The functor $L \mapsto \mathrm{Hom}((\mathbb{Z}/qM\mathbb{Z})^*, L^*)$ for all $\mathbb{Q}_p$-algebra L is obviously representable by a finite scheme (hence a finite rigid analytic space) over $\mathbb{Q}_p$, that we shall denote $((\mathbb{Z}/qM\mathbb{Z})^*)^\vee$. It has $\phi(qM)$ points over $\mathbb{C}_p$, but less over $\mathbb{Q}_p$, since not

all characters may take values in $\mathbb{Q}_p$. When $M = 1$, the $\phi(q)$ points of this schemes are defined over $\mathbb{Q}_p$, and correspond to the power of the Teichmuler character τ

The functor $L \mapsto \{x \in L^*, |x - 1| < 1\}$ for all complete $\mathbb{Q}_p$-algebras is represented by a rigid analytic space, the open ball $B(1,1)$ of center 1 and radius 1.

Since $\mathcal{W}_M(L) = ((\mathbb{Z}/pM\mathbb{Z})^*)^\vee(L) \times B(1,1)(L)$ functorially in L, by the map $\kappa \mapsto (\kappa_f, \kappa_p(\gamma))$, we have shown that $\mathcal{W}_M$ is representable by a rigid analytic scheme $((\mathbb{Z}/qM\mathbb{Z})^*)^\vee \times B(1,1)$. Over $\mathbb{C}_p$, or even an extension of $\mathbb{Q}_p$ containing all $\phi(M)$-th roots of unity, it is the union of $\phi(qM)$ copies of the open unit ball of center 1 and radius 1.

For later use, let us record

SCHOLIUM III.5.3. *Over $\mathbb{C}_p$, or over any extension of $\mathbb{Q}_p$ containing all $\phi(M)$-th roots of unity, $\mathcal{W}_M$ is a finite union of components $\mathcal{W}_{M,\kappa_f}$ indexed by the characters $\kappa_f : (\mathbb{Z}/qM\mathbb{Z})^* \to \mathbb{C}_p^*$, each of which is isomorphic to the open ball $B(1,1)$. A generator γ of $1 + q\mathbb{Z}_p$ being fixed, an isomorphism $\mathcal{W}_{M,\kappa_f}(\mathbb{C}_p) \to B(1,1)(\mathbb{C}_p)$ is given by $\kappa \mapsto \kappa(\gamma)$.*

In view of the scholium, a more explicit description of the character κ_x of $1+q\mathbb{Z}_p$ that sends γ to x would be welcome. Here is one. Recall that $\log_p(x)$ is defined for $x \in \mathbb{C}_p$, $|x - 1| < 1$, while $\exp_p(x)$ is defined for $|x| < p^{\frac{-1}{p-1}}$.

LEMMA III.5.4. *The expression $\exp_p(\log_p(x) \log_p(z)/ \log_p(\gamma))$ defines an analytic function in two variables $x \in L$, $z \in \mathbb{C}_p$ on the domain defined by $|x - 1| < 1$, $|z - 1| < 1$, $|\log_p(x) \log_p(z)| < p^{\frac{-p}{p-1}}$. In particular, for x fixed in L satisfying $|x-1| < 1$, this is an analytic function of z converging for $|z-1| \leq p^{\frac{-p}{p-1}}/|\log_p(x)|$. The restriction of κ_x to that domain coincides to that analytic function.*

Proof — We make the proof when p is odd, leaving the case $p = 2$ to the reader.

The expression $\exp_p(\log_p(x) \log_p(z)/ \log_p(\gamma))$ makes sense for x and z in the given domain since $|\log_p(x)||\log_p(z)|/|\log_p(\gamma))| \leq p^{\frac{-p}{p-1}}/p^{-1} = p^{\frac{-1}{p-1}}$, and we can take the $\exp_p$. This expression is even a jointly analytic function in x and z because $\log_p(x) \log_p(z)$ is jointly analytic, $\exp_p$ is analytic on its domain, and the composition of analytic functions is analytic.

For x fixed, $|x - 1| < 1$, it is thus clear that $z \mapsto \exp_p(\log_p(x) \log_p(z)/ \log_p(\gamma))$ is a continuous character on the subgroup $1 + p^n\mathbb{Z}_p$ of $1 + p\mathbb{Z}_p$ which is the intersection of $1 + p\mathbb{Z}_p$ with the above domain, sending the topological generator $\gamma^{p^{n-1}}$ of that subgroup on $x^{p^{n-1}}$. The character κ_x also sends $\gamma^{p^{n-1}}$ on $x^{p^{n-1}}$, so the two characters coincide. $\square$

PROPOSITION AND DEFINITION III.5.5. Let $\kappa \in \mathcal{W}(L)$. We define $r(\kappa)$ as the supremum of all real numbers r such that the application $\mathbb{Z}_p \to L$, $z \mapsto \kappa(1 + pz)$

belongs to $\mathcal{A}[r](L)(r)$. Then $r(\kappa) > 0$, and for all $0 < r < r(\kappa)$, $\kappa(1 + pz) \in \mathcal{A}[r]$.
Moreover, if $r'(\kappa) = \min(r(\kappa)/p, 1)$, then the function $\kappa : \mathbb{Z}_p \to L$ (extended by 0
on $p\mathbb{Z}_p$) belongs to $\mathcal{A}[r'](\kappa)$

Proof —   For the first assertion it suffice to check that there is an $r > 0$ such
that there is a power series $\sum_{n \geq 0} a_n z^n$ converging to $\kappa(1 + pz)$ on $\bar{B}(0, r)$. For r
sufficiently small, $\kappa(1 + pz) = \kappa_x(1 + pz)$ on $\bar{B}(0, r)$, so the result follows from the
above lemma.

For the second assertion, we have to check that for any a in $\mathbb{Z}_p$, there is a
power series $\sum_{n \geq 0} a_n(z - a)^n$ converging for $|z - a| \leq r'(\kappa)$ which agrees with $\kappa(z)$
on the points of $\mathbb{Z}_p$ in that closed ball. For $a = 1$, this follows from the above
since $r'(\kappa) \leq r(\kappa)/p$. For $a \in \mathbb{Z}_p^*$, this follows form the case $a = 1$ by the formula
$\kappa_x(z) = \kappa_x(z/a)\kappa(a)$. And for $a \in p\mathbb{Z}_p$, the 0 power series agrees with κ on $\bar{B}(a, r)$
for all $r < r'(\kappa) \leq 1$ since $\bar{B}(a, r) \cap \mathbb{Z}_p = p\mathbb{Z}_p$ and κ is 0 on $p\mathbb{Z}_p$. $\square$

EXERCISE III.5.6. a.– let x be a primitive p^n-root of 1 in $\mathbb{C}_p$, and let $r_n = r(\kappa_x)$.
Compute r_n. b.– Deduce that there is no universal $r > 0$ such that for all $\kappa \in \mathcal{W}(L)$,
$\kappa(1 + pz) \in \mathcal{A}[r](L)$

III.5.2. Some remarkable elements in the weight space. Let $k \in \mathbb{Z}$.
Then the maps $z \mapsto z^k$, $\mathbb{Z}_p^* \to \mathbb{Q}_p^*$ is a continuous character, hence an element of
$\mathcal{W}(\mathbb{Q}_p)$ (or of $\mathcal{W}(\mathbb{C}_p)$ if you prefer). We shall also denote by k this element in
$\mathcal{W}(\mathbb{Q}_p)$, and we call those elements the *integral points*, or *integral weights*. Hence
we have an inclusion $\mathbb{Z} \subset \mathcal{W}(\mathbb{Q}_p)$. Note that $r(k) = +\infty$ (cf. proposition and
definition **??**).

LEMMA III.5.7. *The set $\mathbb{Z}$ of integral weights is Zariski-dense in $\mathcal{W}$ (that is to
say, any rigid analytic function on $\mathcal{W}$ that vanishes on $\mathbb{Z}$ is 0.) The p-adic topology
on $\mathcal{W}$ induces by restriction the topology on $\mathbb{Z}$ for which a basis of neighborhood of
$a \in \mathbb{Z}$ is given by the congruences classes of a modulo $p^n(p-1)$ for all n.*

Proof —   Any character κ_f of $\mathbb{Z}_p^*$ factoring through $(\mathbb{Z}/q\mathbb{Z})^*$ has the form $z \mapsto$
$\tau(z)^{k_0}$ for some integer k_0 well determined modulo $\phi(q)$. The intersection of $\mathbb{Z}$ with
the open ball $B(1, 1)$ corresponding to κ_f is the set $\{\gamma^k, \ k \equiv k_0 \pmod{\phi(q)}\}$. Since
this set is infinite and contained in the proper closed ball of center 1 and radius $1/p$
in $B(1, 1)$ it is Zariski-dense, which proves the first assertion. For two integers k, k'
seen as characters to be close, we need their κ_f to be equal, so $k \equiv k' \pmod{\phi(q)}$
and γ^k to be close to $\gamma^{k'}$, which means k close to k' p-adically. This proves the
second assertion. $\square$

Now suppose we have chosen an embedding $\bar{\mathbb{Q}} \to \mathbb{C}_p$, and an embedding $\bar{\mathbb{Q}} \to \mathbb{C}$.
Slightly more generally, let (k, ϵ) be a pair of an integer $k \in \mathbb{Z}$, and a Dirichlet
character ϵ. The character ϵ takes values in $\mathbb{C}^*$, but those values are actually in

$\bar{\mathbb{Q}}$, and well determined there by our chose of emedding, and we can see them as element of $\mathbb{C}_p^*$ if we like by our other embedding. Write the conductor of ϵ as $p^\nu N$, where N is prime to p. If $N|M$, we can see ϵ as a character $\mathbb{Z}_p^* \times (Z/M\mathbb{Z})^* \to \mathbb{C}_p^*$ (factoring through the quotient $(\mathbb{Z}/p^\nu\mathbb{Z})^* \times (\mathbb{Z}/N\mathbb{Z})^*$). To (k,ϵ) we finally attach the character $z \mapsto z^k\epsilon(z)$ of $\mathbb{Z}_p^* \times (\mathbb{Z}/M\mathbb{Z})^*$, that we see as an element of $\mathcal{W}_M(\mathbb{C}_p)$. Those points are called the *arithmetic points* of $\mathcal{W}_M(\mathbb{C}_p)$.

III.5.3. The p-adic Mellin transform. The real Mellin transform takes a function $g(y)$ (with rapid decay near 0 and ∞) over $\mathbb{R}_+^*$ to the analytic function of a complex variable $s \mapsto M(s) = \int_{\mathbb{R}_+^*} g(y)y^s \frac{dy}{y}$. (For example, the L-function of a modular form $f(z)$ is, up to a simple factor, the Mellin transform of $f(iy)$.) We can see the map $y \mapsto y^s$ as a character of $\mathbb{R}_+^*$ (and all characters $\mathbb{R}_+^* \to \mathbb{C}^*$ are of this form, so the variable s can be thought of as a variable character), and $f(iy)\frac{dy}{y}$ as a measure on $\mathbb{R}_+^*$. Thus we can see the Mellin transform as taking a measure μ on $\mathbb{R}_+^*$ (with rapid decay) and giving an analytic function on the set of character $\mathrm{Hom}(\mathbb{R}_+^*, \mathbb{C}^*)$. Actually, since the character $y \mapsto y^s$ are C^∞, we can even define the Mellin transform of a distribution μ with rapid decay at 0 and ∞.

EXERCISE III.5.8. The measure $\frac{dy}{y(e^y-1)}$ is not of rapid decay at 0, so you can only define its Mellin transform for $\mathrm{Re}s > 1$. Show that this Mellin transform is $\Gamma(s)\zeta(s)$.

By analogy, we now define the p-adic Mellin transform over $\mathbb{Q}_p$.

DEFINITION III.5.9. Let us denote by $\mathcal{R}$ the Frechet $\mathbb{Q}_p$-algbera of analytic functions on the weight space $\mathcal{W}$.

PROPOSITION AND DEFINITION III.5.10. For any distribution $\mu \in \mathcal{D}(\mathbb{Q}_p)$, there exists a unique analyatic function $M(\mu) \in \mathcal{R}$ such that for every Banach $\mathbb{Q}_p$-algebra L, and every $\sigma \in \mathcal{W}(L) = \mathrm{Hom}(\mathbb{Z}_p^*, L^*)$,

$$(59) \qquad M(\mu)(\sigma) = \int_{\mathbb{Z}_p^*} \sigma(z)d\mu(z).$$

The function $M(\mu)$ is called the p-adic Mellin transform of the distribution μ.

Proof — Let us first explain the meaning of the RHS of (59). First, $\int_{\mathbb{Z}_p^*} f(z)d\mu(z)$ means of course

$$\int_{\mathbb{Z}_p} f(z)1_{\mathbb{Z}_p^*}(z)d\mu(z) = \mu(f1_{\mathbb{Z}_p^*}).$$

Second, by Prop. III.5.5 any character $\sigma : \mathbb{Z}_p^* \to L^*$ belongs to $\mathcal{A}[r]$ for some $r > 0$, hence to $\mathcal{A}^\dagger[0]$ and $\mu(\sigma 1_{\mathbb{Z}_p^*})$ makes sense. The uniqueness of M_μ is clear. In view of the definition of the rigid analytic structure on $\mathcal{W}$ (see Scholium III.5.3), to prove the existence of M_μ, what we need to show is just that for any character $\kappa_f : (\mathbb{Z}/p\mathbb{Z})^* \to L^*$, the map $x \mapsto M(\mu)(\kappa_f\kappa_x)$ is analytic in x on the open ball $B(1,1)$. It is eneough to prove that the restriction of this function to the open

ball $B(1, R)$ for all $R < 1$ is analytic in x. Let $m > 0$ be an integer, to be chosen later. As $M(\mu)(\kappa_f \kappa_x) = \sum_{a=0}^{p^m-1} \kappa_f(a) \int_{a+p^m \mathbb{Z}_p} \kappa_x(z) d\mu(z)$ it suffices to prove that each term in that sum is analytic in x on $|x - 1| \leq R$, and by translation it clearly suffices to do so for $a = 1$. By Lemma 3.29, if n is chosen big enough with respect to R, the function $\kappa_x(z)$ is jointly analytic on $|x - 1| \leq R$, $|z - 1| \leq p^{-m}$. That is, $\kappa_x(z) = \sum_{m \geq 0} a_n(z) x^n$, where the $a_n(z)$ are analytic functions on the closed ball of center 1 and radius p^{-m} (that we can see as functions on $\mathcal{A}[p^{-m}]$ by extending them by 0 outside) such that $\|a_n\|_{p^{-m}} R^n$ goes to 0 when n goes to infinity. Thus

$$\int_{1+p^m \mathbb{Z}_p} \kappa_x(z) d\mu(z) = \int_{1+p^m \mathbb{Z}_p} \sum_{n \geq 0} a_n(z) x^n d\mu(z) = \sum_{n \geq 0} b_n x^n$$

where $b_n = \int_{1+p^m \mathbb{Z}_p} a_n(z) d\mu(z)$, and $|b_n| R^n \leq \|\mu\|_{p^{-m}} \|a_n\|_{p^{-m}} R^n$ goes to 0 when n goes to infinity. This convergence justifies the permutation of the summation and the integral for any x such that $|x| \leq R$ in the computation above, and shows that $\int_{1+p^m \mathbb{Z}_p} \kappa_x(z) d\mu(z)$ is a powers series in x converging in the open ball $|x - 1| \leq R$.
$\square$

EXERCISE III.5.11. Show that $M : \mu \mapsto M(\mu)$, $\mathcal{D}(\mathbb{Q}_p) \to \mathcal{R}$ is continuous with respect to the Frechet topologies. For those not familiar with Frechet topologies, what you need to show is that for every $r' < 1$, there exist an $r > 0$, and a $C > 0$, such that for all $\mu \in \mathcal{D}(\mathbb{Q}_p)$,

$$\sup_{|x-1| \leq r'} |M_\mu(x)| < C \|\mu\|_r.$$

For U an open subset of $\mathbb{Z}_p$, and μ a distribution over $\mathbb{Z}_p$, we define the restriction $\mu_{|U}$ of μ to U as the distribution $\mu_{|U}(f) = \mu(f 1_U)$. We say that μ has *support in U* if $\mu = \mu_{|U}$. Obviously, the Mellin transform M_μ depends only of the restriction $\mu_{|\mathbb{Z}_p^*}$ of μ to $\mathbb{Z}_p^*$. We call $\mathcal{D}_U(\mathbb{Q}_p)$ the subspace of distribution in $\mathcal{D}(\mathbb{Q}_p)$ with support in U. It is clearly closed, hence inherits from $\mathcal{D}(\mathbb{Q}_p)$ a structure of Frechet space.

DEFINITION III.5.12. Let $\nu > 0$ be a real. We say that an analytic function $h(x) = \sum_{n \geq 0} a_n(x - 1)^n$ converging on the open ball $|x - 1| < 1$ *has order $\leq \nu$* if $|a_n| = O(n^\nu)$. We say that a function on $\mathcal{W}$ *has order $\leq \nu$* if its restriction to the $p - 1$ connected components of $\mathcal{W}$ have order $\leq \nu$.

EXERCISE III.5.13. Show that a function $h(x)$ as in the definition has order 0 if and only if it is bounded on $B(1, 1)$. Show that this is equivalent to saying that $h(x)$ has finitely many 0's on $B(1, 1)$ or is identically 0.

EXERCISE III.5.14. **(easy)** Show that $\log(1 + x)$ has order 1.

EXERCISE III.5.15. Assume that $\nu \in \mathbb{N}$. Show that $h(x)$ has order $\leq \nu$ is equivalent to $\sup_{|x|<r} |h(x)| = O(\sup_{|x|<r} |\log_p(1 + x)^\nu|)$

EXERCISE III.5.16. Give an example of function of order $1/2$.

THEOREM III.5.17 (Visik, Amice-Velu).   (i) *The application $\mu \mapsto M(\mu)$ is an isomorphism of Frechet space from the subspace $\mathcal{D}(\mathbb{Q}_p)_{\mathbb{Z}_p^*}$ of $\mathcal{D}^\dagger[0]$ of distributions with support of $\mathbb{Z}_p^*$ onto $\mathcal{R}$.*

(ii) *Let $\mu \in \mathcal{D}_0$, and $\nu \geq 0$ a real number. Then μ has order $\leq \nu$ if and only if $M(\mu)$ has order $\leq \nu$.*

Proof —   See e.g. [**CO1**]   □

Now, let L be any Banach algebra over $\mathbb{Q}_p$. Recall that $\mathcal{D}(L) = \mathcal{D}(\mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} L$ by Prop. III.4.25.

DEFINITION III.5.18. The Mellin transform over L is the L-linear map

$$M_L : \mathcal{D}(L) = \mathcal{D}(\mathbb{Q}_p)\hat{\otimes}_{\mathbb{Q}_p} L \to \mathcal{R}\hat{\otimes}_{\mathbb{Q}_p} L$$

defin es by $M_L = M\hat{\otimes}\mathrm{Id}_L$.

If L is an affinoid algebra over $\mathbb{Q}_p$, then an element of $\mathcal{R}\hat{\otimes}_{\mathbb{Q}_p} L$ is just an analytic function on the rigid space $\mathcal{W} \times \mathrm{Sp}\,L$. This is by this Mellin transform over L that we dhall construct two-variables p-adic L-function: one variable in the weight space $\mathcal{W}$ and one variable in the rigid analytic space $\mathrm{Sp}\,L$ (an open set in an eigenvariety).

III.6. Rigid analytic modular symbols

III.6.1. The monoid $S_0(p)$ and its action on $\mathcal{A}[r]$, $\mathcal{D}[r]$, $\mathcal{A}^\dagger[r]$, $\mathcal{D}^\dagger[r]$. We define a monoid

$$S_0(p) = \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}),\ p \nmid a, p|c,\ ad - bc \neq 0 \right\}.$$

This is a submonoid of the monoid S defined in §III.2.1. We note that

$$S_0(p) \cap \mathrm{SL}_2(\mathbb{Z}) = \Gamma_0(p),$$

so a congruence subgroup Γ is contained in $S_0(p)$ if and only if it is contained in $\Gamma_0(p)$.

Let L be a commutative Banach algebra over $\mathbb{Q}_p$, and let us choose $\kappa \in \mathcal{W}(L)$ a character $\mathbb{Z}_p^* \to L^*$.

For $f(z)$ an L-valued continuous function on $\mathbb{Z}_p$, and $\gamma \in S_0(p)$ we define a new function

(60)
$$(\gamma \cdot_\kappa f)(z) = \kappa(a - cz)f\left(\frac{dz - b}{a - cz}\right).$$

This formula should look familiar: when κ is an integral weight $k \geq 0$, this is exactly the formula (26) defining the left-action of S on the space $\mathcal{P}_k(L)$. Note however that in our case, this formula makes sense only for $\gamma \in S_0(p)$, not $\gamma \in S$: the hypothesis $\gamma \in S_0(p)$ ensures that $a - cz \in \mathbb{Z}_p^*$, so $\kappa(a - cz)$ is defined, and also that $\frac{dz-b}{a-cz} \in \mathbb{Z}_p$, so that $f\left(\frac{dz-b}{a-cz}\right)$ makes sense. It is clear that we have defined, for

every weight $\kappa \in \mathcal{W}(L)$, a left-action of $S_0(p)$ on the module of continuous L-valued functions on $\mathbb{Z}_p$.

LEMMA III.6.1. *Let $f \in \mathcal{A}[r]$, $\gamma \in S_0(p)$. Assume that $p^n | \det(\gamma)$, and let r' be such that $0 < r' < p$, and $r' \leq p^n r$. Then $f\left(\frac{dz-b}{a-cz}\right) \in \mathcal{A}[r']$. Moreover if p^n divides exactly $\det(\gamma)$, and $p^n r < p$, then $\|f\|_r = \|f\left(\frac{dz-b}{a-cz}\right)\|_{p^n r}$.*

Proof — The derivative of the map $g : z \mapsto \frac{dz-b}{a-cz}$ is $g'(z) = \frac{\det \gamma}{(a-cz)^2}$. For $z \in \mathbb{C}_p$, $|z| < p$, we have $|a - cz| = 1$, and we thus have $|g'(z)| \leq p^{-n}$. Therefore the map $z \mapsto g(z) = \frac{dz-b}{a-cz}$ sends a ball of radius r' (with $r' < p$) centered on an element e of $\mathbb{Z}_p$ into the ball of radius $p^{-n} r' \leq r$ and center $g(e) \in \mathbb{Z}_p$. Since g is analytic, this show that $f(g(z))$ is in $\mathcal{A}[p^n r]$ if f is in $\mathcal{A}[r]$.

If p^n divides r exactly, then the same argument applied to g^{-1} show that g actually is a bijection from the closed ball of center e and radius $p^n r$ to the ball of center $g(e)$ and radius r. The assertion on the norms follows (by Exercise **??**(2)). $\square$

PROPOSITION AND DEFINITION III.6.2. Fix $\kappa \in \mathcal{W}(L)$. For any $0 < r < r(\kappa)$ (cf definition. III.5.5), the formula (60) defines a continuous left action of $S_0(p)$ on $\mathcal{A}[r](L)$, called the action of weight κ. If p^n divides exactly $\det \gamma$, and $p^n r < p$, then $\|\gamma \cdot_\kappa f\|_{p^n r} = \|f\|_r$. In particular, if $\det \gamma$ is prime to p, and $r < p$, then γ acts by isometry.

Those actions are compatible with the restriction maps. Therefore, they define a left-action of weight κ on $\mathcal{A}^\dagger[r](L)$ for $0 \leq r < \kappa(w)$.

Proof — By Prop. III.5.5, $\kappa(a - cz) = \kappa(a)\kappa(1 - c/az)$ is in $\mathcal{A}[r](L)$ for $r < r(\kappa)$, since $p|c/a$ in $\mathbb{Z}_p$. Hence $\kappa(a - cz)f\left(\frac{dz-b}{a-cz}\right) \in \mathcal{A}[r](L)$, and we have defined an action on $\mathcal{A}[r](L)$. The other assertions follow from Lemma III.6.1, noting that $|\kappa(a - cz)| = 1$ in L for all $z \in \mathbb{C}_p$, $|z| < p$. $\square$

We now proceed to define a "dual" right-action of weight κ on $\mathcal{D}[r](L)$, when $r < r(\kappa)$. The difficulty comes from the fact that for L infinite dimensional over $\mathbb{Q}_p$, $\mathcal{D}[r](L)$ is not the L-dual of $\mathcal{A}[r](L)$, but a proper sub-module of it.

On $\mathrm{Hom}_L(\mathcal{A}[r](L), L)$, we can define the dual action of the action on $\mathcal{A}[r]$ by

$$\mu_{|_\kappa \gamma}(f) = \mu(\gamma \cdot_\kappa f).$$

LEMMA III.6.3. *Let $\mu \in \mathcal{D}[r](L) \subset \mathrm{Hom}_L(\mathcal{A}[r](L))$. Then $\mu_{|_\kappa \gamma} \in \mathcal{D}[r](L)$.*

Proof — FIx $\gamma \in S_0(p)$. The map $\tau : \mathcal{A}[r](L) \to \mathcal{A}[r](L), f \mapsto \gamma \cdot_\kappa f$ can be decomposed as $\tau_2 \circ \tau_1$, where $\tau_1(f) = f\left(\frac{dz-b}{a-cz}\right)$ and $\tau_2(f) = \kappa(a - cz)f(z)$. What we need to show is that ${}^t\tau$ staibilized the subspaces $\mathcal{D}[r](L)$ of $\mathrm{Hom}_L(\mathcal{A}[r](L), L)$, and for this wen need to show that ${}^t\tau^1$ and ${}^t\tau^2$ stabilizes this subspace. Note that τ_1 clearly comes by extension of scalal from the map $\mathcal{A}[r](\mathbb{Q}_p) \to \mathcal{A}[r](\mathbb{Q}_p), f \mapsto$

$f\left(\frac{dz-b}{a-cz}\right)$, while τ_2 is the multiplication by an element of $\mathcal{A}[r](L)$. Thus the lemma follows from III.4.8 and **??** via the translaton given by Lemma III.4.22.   □

Hence, we have defined a right action of $S_0(p)$ on $\mathcal{D}[r](L)$ of weight κ, if $r < r(\kappa)$.

PROPOSITION III.6.4. *This action is continuous. Assume $r < p$. Let $\mu \in \mathcal{D}[r](L)$, $\gamma \in S_0(p)$ such that $p^n | \gamma$. Then $\mu_{|_\kappa}(\gamma) \in \mathcal{D}[r/p^n](L)$. If p^n divides exactly $\det \gamma$, then moreover $\|\mu_{|_\kappa \gamma}\|_{r/p^n} = \|\mu\|_r$. In particular, if $\det \gamma$ is prime to p, then γ acts by isometry.*

Proof —   Since the injection $\mathcal{D}[r](L) \subset \mathrm{Hom}_L(\mathcal{A}[r](L), L)$ is norm-preserving, this follows from the analog result for $\mathcal{A}[r]$ (Prop. III.6.2) except for the fact that $\mu_{|_\kappa}(\gamma) \in \mathcal{D}[r/p^n](L)$ which is proven using the same method as in the above lemma. □

Since the action of weight κ on the $\mathcal{D}[r](L)$ for various $r < r(\kappa)$ are clearly compatible, we can use them to define an action of weight κ on $\mathcal{D}^\dagger[r](L)$ for $0 \leq r < r(\kappa)$.

We put an index κ when we think of $\mathcal{A}[r](L)$, $\mathcal{A}^\dagger[r](L)$, $\mathcal{D}[r](L)$, $\mathcal{D}^\dagger[r](L)$ as provided with that action: $\mathcal{A}_\kappa[r](L)$, $\mathcal{A}_\kappa^\dagger[r](L)$, $\mathcal{D}_\kappa[r](L)$, $\mathcal{D}^\dagger[0]_\kappa[r](L)$. As before, $\mathcal{D}_\kappa(L)$ means $\mathcal{D}_\kappa^\dagger[0](L)$.

III.6.2. The module of locally constant polynomials and its dual. In this section we assume (just for simplicity of the formulation of some results) that L is a finite extension of $\mathbb{Q}_p$.

DEFINITION III.6.5. Let $k \geq 0$ be an integer. For $r > 0$, we shall denote by $\mathcal{P}_k[r](L)$ the subspace of $\mathcal{A}[r](L)$ of functions which on each closed ball $\bar{B}(a, r)$ in $\mathbb{Z}_p$ are polynomials of degree at most k. We shall set $\mathcal{V}_k[r](L) = \mathrm{Hom}_L(\mathcal{P}_k[r](L), L)$.

LEMMA III.6.6. *For $r > 0$, the L-vector spaces $\mathcal{P}_k[r](L)$ and $\mathcal{V}_k[r](L)$ are free of finite rank and their formation commutes with any base change. If we put $\mathcal{A}[r](L)$ its action of weight k, then $\mathcal{P}_k[r](L)$ is $S_0(p)$-stable in $\mathcal{A}_k[r](L)$. Hence we have a dual surjective map $\mathcal{D}_k[r](L) \to \mathcal{P}_k[r](L)$*

Proof —   This is clear.   □

We provided $\mathcal{P}_k[r](L)$ with its left-action of $S_0(p)$ induced by the action of weight k on $\mathbb{A}[r](L)$, and $\mathcal{V}_k[r](L)$ with the dual right-action.

LEMMA III.6.7. *For all $r > 0$, one has a trivial $S_0(p)$-equivariant injective map $\mathcal{P}_k(L) \to \mathcal{P}_k[r](L)$, and a dual trivial $S_0(p)$-equivariant surjective map $\mathcal{V}_k(L) \to \mathcal{V}_k[r](L)$. These maps are isomorphisms if $r \geq 1$.*

Proof —   A polynomial with coefficients in L and degree at most k defines a function $\mathbb{Z}_p \to L$ that lies in $\mathcal{P}_k[r](L)$ for all r. If $r \geq 1$, a function in $\mathcal{P}_k[r](L)$ coincides with a polynomial od degree at most k on $\bar{B}(0,1) \supset \mathbb{Z}_p$, which proves the surjectivity of $\mathcal{P}_k[r](L) \to \mathcal{P}_k(L)$. The fact that this map is $S_0(p)$-equivariant follows from the fact that as we have already noticed, the action of $S_0(p)$ on $\mathcal{A}_k[r](L)$ and on $\mathcal{P}_k[r]$ are given by the same formulas. The assertion for $\mathcal{V}_k$ follows. $\square$

Recall that $\mathcal{P}_k(L)$ and $\mathcal{V}_k(L)$ (with their action of $S \subset S_0(p)$) were defined in §III.2.2.

DEFINITION III.6.8. For any $r > 0$, we denote by ρ_k the $S_0(p)$-equivariant surjective maps $\mathcal{D}_k[r](L) \to \mathcal{V}_k[L]$ obtained by composing the morphism $\mathcal{D}_k[r](L) \to \mathcal{V}_k[r](L)$ and $\mathcal{V}_k[r](L) \to \mathcal{V}_k(L)$ of the two preceding lemmas. For $r \geq 0$, we also denote by ρ_k the map $\mathcal{D}_k^\dagger[r](L) \to \mathcal{V}_k(L)$ obtained by composing $\mathcal{D}_k^\dagger[r](L) \hookrightarrow \mathcal{D}_k[r'](L)$ for some $r' > r$ with $\rho_k : \mathcal{D}_k[r'](L) \to \mathcal{V}_k(L)$.

EXERCISE III.6.9. **(easy)** Show that the maps $\rho_k : \mathcal{D}_k^\dagger[r](L) \to \mathcal{V}_k(L)$ commute with the restriction maps $\mathcal{D}_k[r](L) \to \mathcal{D}_k[r'](L)$ for $r' > r$. Show that $\rho_k : \mathcal{D}_k^\dagger[r](L) \to \mathcal{V}_k(L)$ is independent of the choice of r' used to define it, and is surjective.

III.6.3. The fundamental exact sequence. In this paragraph, let L be a finite extension of $\mathbb{Q}_p$.

Let $k \in \mathbb{N}$. Let us consider the map $(\frac{d}{dz})^{k+1} : \mathcal{A}[r](L) \to \mathcal{A}[r](L)$ or $\mathcal{A}^\dagger[r](L) \to \mathcal{A}^\dagger[r](L)$. The kernel of this map is obviously $\mathcal{P}_k[r](L)$ defined above.

LEMMA III.6.10. *The map $(\frac{d}{dz})^{k+1} : \mathcal{A}^\dagger[r](L) \to \mathcal{A}^\dagger[r](L)$ is surjective, continuous and open (for the locally convex inductive limit topology on $\mathcal{A}^\dagger[r](L)$).*

Proof —   Let E be a finite set as in Prop. III.4.12. If $f \in \mathcal{A}^\dagger[r](L)$, then at any point $e \in E$, $f(z)$ has a Taylor expansion of the form $f(z) = \sum_{n \geq 0} a_n(e)(z - e)^n$, with $|a_n(e)|(r')^n \to 0$ for some $r' > r$. Then

$$f = (\frac{dg}{dz})^{k+1}, \text{ with } g(z) = \sum_{n \geq k+1} \frac{a_{n-k-1}(e)}{n(n-1)\dots(n-k)}(z - e)^n.$$

For any r'' such that $r < r'' < r'$, $|\frac{a_{n-k-1}(e)}{n(n-1)\dots(n-k)}|r''^n \to 0$ for $r < r'' < r')$. Hence $g \in \mathcal{A}^\dagger[r]$. This proves the surjectivity. We observe also that there is a constant $C(r', r'') > 0$ depending on r' and r'', but not on f, such that

(61) $\|g\|_{r''} \leq C\|f\|_{r'}$

Since the map $(\frac{d}{dz})^{k+1} : \mathcal{A}[r'](L) \to \mathcal{A}[r'](L)$ is clearly continuous for all r', and so is the post-composition $(\frac{d}{dz})^{k+1} : \mathcal{A}[r'] \to \mathcal{A}^\dagger[r]$ of this map with the continuous inclusion $\mathcal{A}[r'](L) \to \mathcal{A}^\dagger[r](L)$. Hence the inductive limit of those maps, $(\frac{d}{dz})^{k+1} : \mathcal{A}^\dagger[r] \to \mathcal{A}^\dagger[r]$ is also continuous by [**Sch**, Lemma 5.1.i].

To prove that $(\frac{d}{dz})^{k+1}$ is open, we need only to prove that it sends any open lattice on a neighborhood of 0. An open lattice in $\mathcal{A}^\dagger[r]$ is by definition a lattice Λ such that $\Lambda \cap \mathcal{A}[r''](L)$ is open in $\mathbb{A}[r'']$ for all $r'' > r$. Hence $\Lambda \cap \mathcal{D}[r'']$ contains an open ball of center 0 and some radius R (depending on r'') in $\mathcal{A}[r''](L)$. But by the first paragraph of this proof, and in particular by (61), the image of Λ by $(\frac{d}{dz})^{k+1}$ contains an open ball of radius R/C in $\mathcal{A}[r'](L)$ for all $r' > r''$, hence is open. $\qquad\square$

EXERCISE III.6.11. Prove that the map $(\frac{d}{dz})^{k+1} : \mathcal{A}[r](L) \to \mathcal{A}[r](L)$ is not surjective. (This is one reason we need to work with the Frechet spaces $\mathcal{A}^\dagger$, rather than with the simpler Banach spaces $\mathcal{A}$.)

We have showed the existence of an exact sequence:

(62) $$0 \to \mathcal{P}_k[r](L) \to \mathcal{A}^\dagger[r](L) \xrightarrow{(\frac{d}{dz})^{k+1}} \mathcal{A}^\dagger[r](L) \to 0$$

If put $\mathcal{A}^\dagger[r](L)$ its weight k action of $S_0(p)$, then the natural inclusion $\mathcal{P}_k[r](L) \hookrightarrow \mathcal{A}^\dagger_k[r](L)$ is $S_0(p)$-equivariant. Can we make the second arrow of the exact sequence, $(\frac{d}{dz})^{k+1}$, equivariant as well?

LEMMA III.6.12. *The exact sequence*

$$0 \to \mathcal{P}_k \to \mathcal{A}^\dagger_k[r] \xrightarrow{(\frac{d}{dz})^{k+1}} \mathcal{A}^\dagger_{-2-k}[r](k+1) \to 0$$

is $S_0(p)$-equivariant, where the $(k+1)$ means, as above, that the action of $S_0(p)$ is twisted by $\det^{k+1}$.

Proof — (I owe this proof to John Bergdall) We need to prove that for all $f \in \mathcal{A}^\dagger[r]$, and $\gamma \in S_0(p)$, one has

(63) $$\frac{d^{k+1}}{dz^{k+1}}\left((a-cz)^k f\left(\frac{dz-b}{a-cz}\right)\right) = \det \gamma^{k+1}(a-cz)^{-2-k}\frac{d^{k+1}f}{dz^{k+1}}\left(\frac{dz-b}{a-cz}\right).$$

We prove (63) by induction on k, the formula being clearly true for $k = 0$. Assuming the formula is true for $k' < k$ (for all f), we compute using the product rule

$$\frac{d^{k+1}}{dz^{k+1}}\left((a-cz)^k f\left(\frac{dz-b}{a-cz}\right)\right) = \frac{d^k}{dz^k}\left(-ck(a-cz)^{k-1}f\left(\frac{dz-b}{a-cz}\right)\right)$$
$$+ \frac{d^k}{dz^k}\left((\det \gamma)(a-cz)^{k-2}f'\left(\frac{dz-b}{a-cz}\right)\right)$$

By induction, using (63) for $k' = k - 1$, the first term of the sum is

$$\frac{d^k}{dz^k}\left(-ck(a-cz)^{k-1}f\left(\frac{dz-b}{a-cz}\right)\right) = -ck(a-cz)^{-1-k}(\det \gamma)^k\frac{d^k f}{dz^k}\left(\frac{dz-b}{a-cz}\right).$$

By induction also, using (63) for $k' = k - 2$ and f replaced by f', the second term is

$$\frac{d^k}{dz^k}\left((\det \gamma)(a - cz)^{k-2}f'\left(\frac{dz-b}{a-cz}\right)\right) = \frac{d}{dz}\left(\det \gamma^k(a-cz)^{-k}\frac{d^k}{dz^k}\left(f\left(\frac{dz-b}{a-cz}\right)\right)\right)$$

$$= ck\det \gamma^k(a-cz)^{-k-1}\frac{d^k f}{dz^k}\left(\frac{dz-b}{a-cz}\right)$$

$$+ \det \gamma^{k+1}(a-cz)^{-2-k}\frac{d^{k+1}f}{dz^{k+1}}\left(\frac{dz-b}{a-cz}\right)$$

Adding the two terms, we find (63).   □

Dualizing, we get the fundamental

THEOREM III.6.13. *There is a natural exact sequence*

$$0 \longrightarrow \mathcal{D}^\dagger_{-2-k}[r](L)(k+1) \xrightarrow{\Theta_k} \mathcal{D}^\dagger_k[r](L) \xrightarrow{\rho_k} \mathcal{V}_k[r](L) \longrightarrow 0.$$

Proof — We apply the contravariant functor *topological dual* to the exact sequence (62) getting a complex

$$0 \longrightarrow \mathcal{D}^\dagger_{-2-k}[r](L)(k+1) \xrightarrow{\Theta_k} \mathcal{D}^\dagger_k[r](L) \xrightarrow{\rho_k} \mathcal{V}_k[r](L) \longrightarrow 0.$$

We claim that this complex is exact. The injectivity of Θ_k is obvious. The surjectivity of ρ_k is exactly Hahn-Banach's theorem, applicable here because our spaces are locally convex and the base field L is spherically complete (cf. [**Sch**, Cor. 9.4]). The exactness in the middle follows from the openness assertion in Lemma III.6.10. □

EXERCISE III.6.14. (**easy**) Why is the $(k+1)$ not becoming a $-1-k$ when we dualize ?

EXERCISE III.6.15. We define a *normable group* as a topological abelian group G whose topology can be defined by a ultrametric distance that is invariant by translation (that is $d(x+z, y+z) = d(x,y)$ for all $x, y, z \in G$).

a.– Show that it amounts to the same to ask that G is the underlying topological group of a normed group in the sense of [**BGR**, §1.1], in othwr words that the topology is defined by a norm $\|\| : G \to \mathbb{R}^+$ such that for all x, y in G, $|x| = 0$ if and only if $x = 0$; and $|x - y| \leq \max(|x|, |y|)$.

b.– Let V be a Hausdorf topological space over $\mathbb{Q}_p$ as in [**Sch**], whose topology is defined by a countable family of semi-norms (e.g. a Banach space or a Frechet space). Then the underlying topological abelian group V is a normable group.

EXERCISE III.6.16. Let us recall that a continuous morphism $f : X \to Y$ between two normable group is said *strict* if the induced map $\bar{f} : X/\ker f \to \mathrm{Im}\, f$ is an homeomorphism, where $X/\ker f$ is given the quotien topology, and $\mathrm{Im}\, f$ the topology restricted from Y.

a.– Show that if f is injective, it is strict if and only if $f : X \to \operatorname{Im} f$ is an homeomorphism.

b.– Show that if f is surjective, it is strict if and only if it is open.

c.– Show that if f is an isomorphism of groups, it is strict if and only if it as also an homeomorphism.

d.– Let us denote by $|\ |$ two norms, one on X and one on Y defining the topologies of X and Y. Show that a morphism $f : X \to Y$ is strict if and only if for every real number $\epsilon > 0$, there exists a real number $\delta > 0$ such that for all $x \in X$, $|f(x)| \leq \delta$ implies $|x + k| \leq \epsilon$ for some $k \in \ker f$.

e.– Let us assume that X and Y are Frechet spaces over $\mathbb{Q}_p$. Then a continuous linear map $f : X \to Y$ is strict if and only if $f(X)$ is closed in Y.

f.– Let R be a normed $\mathbb{Q}_p$ algebra, and let X and Y be R-modules with a Hausdorf topology defined by a countable family of R-modules semi-norms (hence X and Y are normable groups by b. of the above exercise) Let R' be a normed R-module which is R-flat. Let $f : X \to Y$ be a strict morphism of R-module. Then $f' = f \otimes 1 : X' = X \otimes_R R' \to Y'Y \otimes_R R'$ is also strict. Here we topologize X' (resp. Y') by the family of semi-norms $p' = p \otimes |\ |_{R'}$ where p run among the family of semi-norms on X (resp. Y) and $|\ |_{R'}$ is the norm on R'.

EXERCISE III.6.17. Show that the map $\Theta_k : \mathcal{D}^\dagger[r](L) \to \mathcal{D}^\dagger[r](L)$ is strict.

III.6.4. Rigid analytic and overconvergent modular symbols. We fix a congruence subgroup $\Gamma \subset \Gamma_0(p)$, that we assume normalized by the diagonal matrix of entries $(1, -1)$.

DEFINITION III.6.18. For L a Banach algebra over $\mathbb{Q}_p$, $\kappa \in \mathcal{W}(L)$, and $0 < r < r(\kappa)$ (resp. $0 \leq r < r(\kappa)$) we define the space of *rigid analytic modular symbols* (resp. *overconvergent modular symbols*) of weight k as $\operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$ (resp. $\operatorname{Symb}_\Gamma(\mathcal{D}_\kappa^\dagger[r](L))$).

The module $\operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$ is naturally a Banach L-module, for the norm $\|\Phi\|_r = \sup_{D \in \Delta_0} \|\Phi(D)\|_r$ for $\Phi \in \operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r])$, which makes sense because of

LEMMA III.6.19. *The set of $\|\Phi(D)\|_r$ for $D \in \Delta_0$ is bounded*

Proof — By Manin's lemma (Exercise III.1.1) , we can choose $(D_i)_{i \in I}$ a finite family of generators of Δ_0 under $\mathbb{Z}[\Gamma]$. That is, any $D \in \Delta_0$ can be written $\sum_{i \in I, j \in J_i} n_{i,j} \gamma_{i,j} \cdot D_i$ where the J_i are finite set, and for $j \in J_i$, $\gamma_{i,j} \in \Gamma$ and $n_{i,j} \in \mathbb{Z}$. It is clear that $\|\Phi(D)\|_r \leq \sup_{i,j} \|\Phi(D_i)_{|\gamma_{i,j}}\|_r = \sup_{i \in I} \|\Phi(D_i)\|_r$ since the $\gamma_{i,j}$ acts by isometries (Prop. III.6.4). The result follows since I is finite.   □

From this proof we have learned more than just what we wanted to show:

SCHOLIUM III.6.20. *We can choose divisors $D_i \in \Delta_0$, for $i \in I$, I finite, such that the map $\Phi \mapsto (\Phi(D_i))_{i \in I}$, $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L)) \to \mathcal{D}[r](L)^I$ is a closed isometric embedding.*

Since $S_0(p)$ acts on $\mathcal{D}_\kappa[r](L)$, $\mathcal{D}_\kappa^\dagger[r](L)$, we get an action of the Hecke operators $T_l = [\Gamma \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \Gamma]$, $U_p = [\Gamma \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} \Gamma]$, the Diamond operators $\langle a \rangle$, and $\iota = [\Gamma \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Gamma]$.

LEMMA III.6.21. *All the Hecke operators above are bounded on $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$ and have norm at most one. For U_p, we have the following more precise result: if $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$, then $\Phi_{|U_p} \in \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r/p](L))$ and*

$$\|\Phi_{|U_p}\|_r \leq \|\Phi_{|U_p}\|_{r/p} \leq \|\Phi\|_r.$$

Proof — If T is any Hecke operator, we have by definition $\Phi_{|T}(D) = \sum \phi(D_{|\gamma})_{|\gamma}$, where the sum is a finite sum for some $\gamma \in S_0(p)$. But if p^n divides exactly $\det \gamma$, we have by Prop. III.6.4:

$$\|\Phi(D_{|\gamma})_\gamma\|_{r/p^n} = \|\Phi(D_{|\gamma})\|_r \leq \|\Phi\|_r.$$

For T an Hecke operator distinct of U_p, all the γ's intervening in the sum have determinant 1, and the result is clear. For $T = U_p$, writing $\gamma_a = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$, we get

$$\begin{aligned} \|\Phi_{|U_p}(D)\|_{r/p} &= \|\sum_{a=0}^{p-1} \phi(D_{|\gamma_a})_{|\gamma_a}\|_{r/p} \\ &\leq \|\Phi(D_{|\gamma_a})\|_r \\ &\leq \|\Phi\|_r \end{aligned}$$

hence the second inequality in the displayed formula. The first inequaity is obvious. $\square$

REMARK III.6.22. It might seem at first glance that in the above proof, the inequality $\|\phi(D_\gamma)\|_{p^{-n}r} \leq \|\phi(D_\gamma)\|_r$ is gross when $n \geq 1$ and can be improved. But this is not true, as for the constant function 1 for example, $\|1\|_{p^{-n}r} = \|1\|_r$. Thus, the upper bound we get for the norm of U_p for example, namely 1, can not be improved, at least this way. This estimate will play a fundamental role in the proof of Stevens' control theorem below.

III.6.5. Compactness of U_p. In all thus §, L will be a finite extension of $\mathbb{Q}_p$, and we shall work with various vector spaces over L, all provided with an operator U_p. If V is Banach, and U_p is compact, we have already defines for each real ν *the part of slope $\leq \nu$, or $< \nu$ of V*, denoted $V^{\leq \nu}$ or $V^{<\nu}$, as the simplest special case of the theory explained in §**??**, but much simpler. We shall need to extend the definition to vector spaces V that are not Banach, as follows

DEFINITION III.6.23. Let ν be a real number.

(i) We say that a monic polynomial $P(X) \in L[X]$ has slope $\leq \nu$ (resp. $< \nu$) if all its root in some extension of L have p-adic valuations $\leq \nu$ (resp. $< \nu$)

(ii) If V is any L-vector space with a linear operator named U_p acting on it, we define the subspace of vectors of slope $\leq \nu$ (resp. $< \nu$) than ν, denoted by $V^{\leq \nu}$ (resp. $V^{<\nu}$), as the sum of the subspaces $\ker P(U_p)$ where P runs among monic polynomials of slope $\leq \nu$ (resp. $< \nu$).

One sees easily that this notion of $V^{\leq \nu}$ extends the one we have already defined.

LEMMA III.6.24.   (i) *The associations $V \mapsto V^{\leq \nu}$ and $V \mapsto V^{<\nu}$ define left exact functors from the category of L-vector spaces with an endomorphism (called U_p) to the category of L-vector spaces.*

(ii) *If V is a Banach space and U_p act compactly on it, $V^{\leq \nu}$ and $V^{<\nu}$ are of finite dimension.*

(iii) *Let V and W be a continuous morphism of Banach space with a compact endomorphism U_p, and $f : V \to W$ be an U_p-equivarainr morphism. Then $f(V^{\leq \nu}) = W^\nu \cap f(V)$.*

(iv) *The associations $V \mapsto V^{\leq \nu}$ and $V \mapsto V^{<\nu}$ exact functors from the category of Banach L-vector space with a compact operators U_p to the category of finite dimensional L-vector spaces.*

Proof — Property (i) is obvious.

Property (ii) follows directly from [**S**].

For (iii), let P be the characteristic polynomial of U_p on $W^{\leq \nu}$. Then P has slope less than ν, and according to Riesz decomposition ([**S**, page 82]), V (resp. W) is the sum of two closed U_p-stable subspace $V(P)$ and $V'(P)$ such that $V(P)$ is finite-dimensional and $P(U_p)$ is nilpotent on it, and $V'(P)$ is such that $P(U_p)$ acts invertibly on it (resp. $W(P)$ and $W'(P)$ etc.) Replacing P by P^n for n big enough, we may assume that $P(U_p)$ is 0 on $W(P)$. Let $w \in W^{\leq \nu}$, and assume that $w \in f(V)$. Write $w = f(v + v')$, with $v \in V(P)$ and $v' \in V'(P)$. Note that $f(v') \in W^{\leq \nu}$ since $f(v') = w - f(v)$ and $f(v) \in W^{\leq \nu}$ since $v \in V(P) \subset V^{\leq \nu}$. Write $v' = P(u_p)v''$ for some $v'' \in V'(P)$. Then $f(v') = P(U_p)f(v'')$. Write $f(v'') = w + w'$ with $w \in W(P)$ and $w' \in W'(P)$. Then $f(v') = P(U_p)f(v'') = P(U_p)w + P(U_p)w'$. Since $f(v')$ and $P(U_p)w$ are in $W(P)$, and $P(U_p)w' \in W'(P)$, we get $P(U_p)w' = 0$ hence $w' = 0$

since $P(U_p)$ is invertible in $W'(P)$. Therefore $f(v'') = w$ and $f(v') = P(U_p)w = 0$ since $P(U_p)$ is 0 on $W(P)$. Finally we get $w = f(v)$, hence $w \in f(V^{\leq \nu})$.

Now (iv) follows from (i), (ii) and (iii). $\qquad\qquad\qquad\qquad\qquad\qquad\quad\square$

EXERCISE III.6.25. **(easy)** Prove that in general, the functor $V \mapsto V^{\leq \nu}$ is not right exact.

THEOREM III.6.26 (Stevens). *Assume that L is a finite extension of $\mathbb{Q}_p$, $\kappa \in \mathcal{W}(L)$. Let $0 < r < \min(r(\kappa), p)$. Let ν be a real number.*

  (i) *The operator U_p acts compactly on $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$*
  (ii) *The space $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))^{\leq \nu}$ is finite-dimensional.*
  (iii) *The natural maps (that is, induced by the transposed restriction maps)*

$$\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_1](L))^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_2](L))^{\leq \nu}$$

  are isomorphisms for any $0 < r_1 \leq r_2 < \min(r(\kappa), p)$. Same result for the maps

$$\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa^\dagger[r_1](L))^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_2](L))^{\leq \nu}$$

  for any $0 \leq r_1 < r_2 \leq \min(r(\kappa), p)$.

Proof — Let $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$ and $D \in \Delta_0$. We have

$$\Phi_{|U_p}(D) = \sum_{a=0}^{p-1} \Phi(\beta(a, p) \cdot D)_{|_\kappa \beta(a,p)}$$

where $\beta(a, p) = \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$. Since $\det(\beta_{a,p}) = p$, it follows from Prop. **??** $\mu_{|_\kappa \beta(a,p)} \in \mathcal{D}[r/p](L)$ for every distribution $\mu \in \mathcal{D}[r](L)$. Hence $\Phi_{|U_p}$ actually belongs to $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r/p](L))$. In other words, the map U_p factors through:

(64) $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L)) \to \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r/p](L)) \overset{\text{restriction}}{\to} \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L)).$

Therefore, to prove (i) it suffices to prove that the second map is compact. But by Scholium III.6.20, this map is the restriction to a closed subspace of the restriction map $\mathcal{D}[r'](L)^I \to \mathcal{D}[r](L)^I$ (for I a finite set), which is compact by Lemma III.4.21.

The assertion (ii) follows from (i).

To prove (iii) we only need to prove that for r, r' such that $0 < r \leq r' \leq rp$, and $r' < p$, the restriction map $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r])^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r'])^{\leq \nu}$ is an isomorphism. The injectivity follows from the injectivity of $\mathcal{D}[r'] \to \mathcal{D}[r]$ and the left exactness of Symb_Γ. Since any vector in $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r'])^{\leq \nu}$ is in the image of U_p, the surjectivity follows from (64). $\qquad\qquad\qquad\square$

The following proposition gives a control how the norms behave in the equality of vector spaces $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_1](L))^{\leq \nu} = \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_2](L))^{\leq \nu}$. It shall not be used before chapter **??**.

PROPOSITION III.6.27. *For any $0 < r_1 < r_2 \leq \min(r(\kappa), p)$ as in (iii), there exists a real constant D, $0 < D < 1$ depending on r_1 and r_2 and of ν, **but not of** κ, such that if $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_1](L))^{\leq \nu} = \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r_2](L))^{\leq \nu}$ we have*

$$D\|\Phi\|_{r_1} \leq \|\Phi\|_{r_2} \leq \|\Phi\|_{r_1}$$

Proof — We obviously have $\|\Phi\|_{r_2} \leq \|\phi\|_{r_1}$. By an easy induction we may assume that Φ is an eigenvector for U_p, that is that $\Phi_{|U_p} = \lambda U_p$ with $v_p(\lambda) \leq \nu$, in other words with $|\lambda^{-1}| \leq p^\nu$. Choose an integer n large enough so that $r_2/p^n \leq r_1$. We have

$$
\begin{aligned}
\|\Phi\|_{r_1} &\leq \|\Phi\|_{r_2/p^n} \\
&= \|\lambda^{-n}\Phi_{|U_p^n}\|_{r_2/p^n} \\
&= \|\lambda^{-n}\Phi\|_{r_2} \quad \text{(by Prop. III.6.21)} \\
&= p^{n\nu}\|\Phi\|_{r_2}.
\end{aligned}
$$

Hence $D\|\Phi\|_{r_1} \leq \|\Phi\|_{r_2}$ with $D = p^{n\nu}$, and clearly D does not depend of κ. $\square$

Let us also note a result similar to but simpler than Theorem III.6.26.

PROPOSITION III.6.28. *For all $r < p$, and all real ν, the natural map*

$$\mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L))^{\leq \nu}$$

is an isomorphism.

Proof — For $r \geq 1$ this is clear because $\mathcal{V}_k[r](L) = \mathcal{V}_k(L)$. Otherwise, it suffices to prove that for r, r' such that $0 < r \leq r' \leq rp$, and $r' < p$, the restriction map $\mathrm{Symb}_\Gamma(V_k[r])^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{V}_k[r'])^{\leq \nu}$ is an isomorphism, which is proved exactly as (iii) of the above proposition. $\square$

Let us finish this § by another simple and nice result of Stevens relating the slope and the rate of growth:

PROPOSITION III.6.29 (Stevens). *Let $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}(L))^{\leq \nu}$ be a modular symbol. Then for all $D \in \Delta_0$, the distribution $\Phi(D)$ has order of growth $\leq \nu$.*

Proof — We may assume that $U_p\Phi = \alpha\Phi$ for some $\alpha \in L^*$. Then $|\alpha|^n\|\Phi\|_r = \|U_p^n\Phi\|_r = \|\Phi\|_{r/p^n}$ so $\|\Phi\|_{r/p^n} = O(p^{-nv_p(\alpha)})$ from which we deduce $\|\Phi\|_{r'} = O(r'^{-v_p(\alpha)})$. The result then follows from Lemma **??** $\square$

III.6.6. The fundamental exact sequence for modular symbols.

LEMMA III.6.30. *Let Δ be the operator of "discrete differentiation" on $\mathcal{A}^\dagger[r](\mathbb{Q}_p)$:*
$\Delta(f) := f(z+1) - f(z)$ *(independently of k). The sequence*

$$0 \to \mathbb{Q}_p \to \mathcal{A}^\dagger[r](\mathbb{Q}_p) \xrightarrow{\Delta} \mathcal{A}^\dagger[r](\mathbb{Q}_p) \to 0$$

is exact, and Δ is continuous and open.

PROOF. The exactness in the middle expresses the fact that if $\Delta f = 0$, i.e. $f(z+1) = f(z)$ for all $z \in \mathbb{Z}_p$, then f is constant; this is clear as we deduce $f(z+n) = f(z)$ for all $n \in \mathbb{Z}$ by induction, and then $f(z+a) = f(z)$ for all $a \in \mathbb{Z}_p$ by continuity, therefore f constant on $\mathbb{Z}_p$. The exactness at the right, that is the surjectivity of Δ is a little harder, and quite analog to Lemma III.6.10: if $f \in \mathcal{A}^\dagger[r](\mathbb{Q}_p)$, then it is in $\mathcal{A}[r'](\mathbb{Q}_p)$ for some $r' > r$, we can write formally for any $e \in \mathbb{Z}_p$, $f(z) = \sum_{n \geq 0} b_n(e)(z-e)^{(n)}$ where $x^{(n)} := x(x-1)\ldots(x-n+1)$. This is easily seen from the writing $f(z) = \sum_{n \geq 0} a_n(e)(z-e)^n$ using that there is a triangular matrix with entries in $\mathbb{Z}_p$ and diagonal terms equal to 1 that transforms the basis $((z-e)^n)$ of the space of polynomials into the basis $((z-e)^{(n)})$, and one sees the same way that for any ρ, $|b_n(e)|\rho^n$ goes to 0 if and only if $|a_n(e)|\rho^n$ goes to 0. In particular, the series $\sum_{n \geq 0} b_n(e)(z-e)^{(n)}$ converges to $f(z)$ for $|z-e| \leq r'$. Let us now define g by the formula (for $e \in \mathbb{Z}_p$) $g(z) = \sum_{n=1}^\infty \frac{b_{n-1}(e)(z-e)^{(n)}}{n}$. It is immediately seen that this series converges for $|z-e| < r''$ where r'' is any real such that $r'' < r'$, and for a given z is independent on e. Hence, choosing an r'' such that $r < r'' < r'$, we have defines a $g \in \mathcal{A}^\dagger[r''](\mathbb{Q}_p) \subset \mathcal{A}^\dagger[r](\mathbb{Q}_p)$ and we see easily that $\Delta(g) = f$ because $\Delta((z-e)^{(n)}) = (z-e)^{(n-1)}$. This proves that Δ is surjective. The same argument than in Lemma III.6.10 also proves that Δ is continuous and open. $\square$

LEMMA III.6.31. *For L any finte extension of $\mathbb{Q}_p$, we have an exact sequence*

$$0 \longrightarrow \mathcal{D}_k^\dagger[r](L) \xrightarrow{\Delta} \mathcal{D}_k^\dagger[r](L) \xrightarrow{\mu \mapsto \mu(1)} L \longrightarrow 0.$$

Proof — We just dualize the exact sequence of the above lemma (cf. the proof of Theorem III.6.13 to see why we still get an exact sequence) and then tensorize by L. $\square$

REMARK III.6.32. The same is also true when L is any $\mathbb{Q}_p$-Banach algebra, as we shall see later.

EXERCISE III.6.33. Show that $\Delta : \mathcal{D}_k^\dagger[r](L) \to \mathcal{D}_k^\dagger[r](L)$ is strict.

LEMMA III.6.34 (Stevens-Pollack). *Let L be a finite extension of $\mathbb{Q}_p$. For every $k \in \mathbb{Z}$, every $r > 0$, $H_0(\Gamma, \mathcal{D}_k^\dagger[r]) = 0$, excepted for $k = 0$ where $H_0(\Gamma, \mathcal{D}_0^\dagger[r]) = L$.*

Proof — Recall that by definition $H_0(\Gamma, V) = V/IV$, where I is the ideal in $\mathbb{Z}[G]$ generated by the element $\gamma - 1$ for $\gamma \in \Gamma$. Let $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$. Then g acts on $\mathcal{D}_k^\dagger[r](L)$ as Δ. From the exact sequence of the above lemma, we see that $\mu \mapsto \mu(1)$ defines an isomorphism $\mathcal{D}_k^\dagger[r](L)/(g-1)\mathcal{D}_k^\dagger[r](L) \simeq L$.

If $k = 0$, one sees at once that for all $\mu \in \mathcal{D}_k^\dagger[r](L)$ and all $\gamma \in \Gamma$, $\mu_{|(\gamma-1)}(1) = \mu(1) - \mu(1) = 0$, hence $\mathcal{D}_k^\dagger[r](L)/I\mathcal{D}_k^\dagger[r](L) \simeq L$.

Now assume that $k \neq 0$. Let us consider the distribution δ_0' in $\mathcal{D}_k^\dagger[r](L)$ which sends $f \in \mathcal{A}_k^\dagger[r](L)$ to $f'(0)$. For $\gamma \in \Gamma$, let $\mu_\gamma = (\delta_0')_{|(\gamma-1)} \in I\mathcal{D}_k^\dagger[r]$. We compute $\mu_\gamma(1) = \delta_0'((a-cz)^k - 1) = kca^{k-1}$. Hence the image of μ_γ in L is non-zero since $k \neq 0$ provided we chose a γ such that $ac \neq 0$, which is always possible. Hence when $k \neq 0$, $\mathcal{D}_k^\dagger[r]/(g-1)\mathcal{D}_k^\dagger[r] = 0$. $\qquad\square$

PROPOSITION III.6.35 (Pollack-Stevens). *Let $k > 0$ be an integer, L a finite extension of $\mathbb{Q}_p$, $r > 0$. We have an exact sequence, compatible with the Hecke operators*

$$0 \to \mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}^\dagger[r](L))(k+1) \to \mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](L)) \to \mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L)) \to 0,$$

where the $(k+1)$ after $\mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}^\dagger[r])$ means that the action of an Hecke operator $[\Gamma s\Gamma]$ for $s \in S_0(p)$ is $(\det s)^{k+1}$ times what it would be otherwise.

Let $\nu > 0$. Assume $r < p$. We have an exact sequence, compatible with the Hecke operators:

$$0 \to \mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}^\dagger[r](L))(k+1)^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](L))^{\leq \nu} \overset{\rho_k}{\to} \mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{\leq \nu} \to 0,$$

and similarly with $\leq \nu$ replaced by $< \nu$, where ρ_k is the map induced by the map $\mathcal{D}_k^\dagger[r](L) \to \mathcal{V}_k(L)$ dual of the natural inclusion $\mathcal{P}_k(L) \subset \mathcal{A}_k[r](L)$

Proof — We apply the long exact sequence of cohomology with compact support to the fundamental exact sequence, obtaining

$$0 \longrightarrow \mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}^\dagger[r](L))(k+1) \longrightarrow \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger[r]_k(L))$$
$$\longrightarrow \mathrm{Symb}_\Gamma(\mathcal{V}_k(L)) \longrightarrow H_c^2(\Gamma, \mathcal{D}_{-2-k}^\dagger(L)(k+1))$$

The last term is by Poincaré's duality (forgetting the Hecke action) $H_0(\Gamma, \mathcal{D}_{-2-k}^\dagger[r](L))$, so Lemma III.6.34 allows us to complete the proof of the first exact sequence.

Applying the left-exact functor $V \mapsto V^{\leq \nu}$ (Lemma III.6.24(i)) to that exact sequence, we get

$$0 \to \mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}^\dagger[r](L))(k+1)^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](L))^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L))^{\leq \nu}.$$

Let us prove that the last map $\mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](L))^{\leq \nu} \to \mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L))^{\leq \nu}$. is surjective. Let $w \in \mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L))^{\leq \nu}$. Then w is in the image of some $v \in \mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](L))$, and for any $r' > r$, such a v belongs to the Banach space $\mathrm{Symb}_\Gamma(\mathcal{D}_k[r'](L))$ on which U_p acts compactly. Then Lemma III.6.24(iii) tells us

that there is a $v' \in \mathrm{Symb}_\Gamma(\mathcal{D}_k[r'](L))^{\leq\nu}$ whose image by the natural map followed by the restriction map is $w \in \mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L))$. By Theorem III.6.26, $v' \in \mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](L))^{\leq\nu}$. This prove the desired surjectivity. To conclude the proof of the proposition one applies Lemma ??, which allows us to replace $\mathrm{Symb}_\Gamma(\mathcal{V}_k[r](L))^{\leq\nu}$ by $\mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{\leq\nu}$. $\square$

III.6.7. Stevens' control theorem.

THEOREM III.6.36. *Let L be any finite extension of $\mathbb{Q}_p$.*

(i) *[Stevens' control theorem] The natural map*
$$\rho_k : \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))^{<k+1} \longrightarrow \mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{<k+1}$$
is an isomorphism.

(ii) *If we provide both $\mathrm{Symb}_\Gamma(\mathcal{D}_k(L))^{<k+1}$ and $\mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{<k+1}$ of the norm $\|\ \|_1$, then ρ_k is an isometry.*

Proof — We shall give two differents proofs of the important result (i). The first one, which is the proof outlined by Stevens in his preprint [**S**] and completed by Pollack-Stevens [**PS2**] is sleek and short, but does not give (ii). From Prop. III.6.35, we have an exact sequence

$$0 \to (\mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}(L))(k+1))^{<k+1} \to \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))^{<k+1} \to \mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{<(k+1)} \to 0,$$

We can rewrite the first term as $\mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}(L))^{<0}$, but since U_p acts on $\mathrm{Symb}_\Gamma(\mathcal{D}_{-2-k}^\dagger(L))$ with norm at most 1 (Lemma III.6.21), this is 0. The result (i) follows.

The second proof is essentially the method of Visik ([**V**] or [**MTT**], and will give (ii) as well as (i). Let $\phi \in \mathrm{Symb}_\Gamma(\mathcal{V}_k(L))^{<k+1}$. We shall prove that there is a unique $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_k(L))^{<k+1}$ such that $\rho_k(\Phi) = \phi$, and moreover that $\|\Phi\|_1 = \|\phi\|_1$. We may clearly assume that ϕ is a generalized eigenvector of U_p, then by inudction a true eigenvctor of U_p, of eigenvalues $\lambda \in L$.

First we prove the uniqueness of Φ. Let $D \in \Delta_0$. Write $\mu = \Phi(D)$. Then μ is a distribution of order $\leq \nu$. Moreover for every $n \in \mathbb{N}$, $a \in \mathbb{Z}_p$, and $0 \leq j \leq k$ we have $\mu((z-a)/p^n)^j 1_{a+p^n\mathbb{Z}_p}) = \lambda^{-n}\Phi(D_{|U_p^n})(z^j) = \lambda^{-n}\phi(D_{|U_p^n})(z^j)$. Hence all the values $\mu((z-a)/p^n)^j 1_{a+p^n\mathbb{Z}_p})$ are determined by ϕ, hence μ itself is determined by ϕ by Theorem III.4.31(i), and so is Φ.

Now we prove the existence of Φ, as follows. For $D \in \Delta_0$ we define $\Phi(D)$ as the unique distribution μ of slope $\leq \nu$ such that for every $n \in \mathbb{N}$, $a \in \mathbb{Z}_p$, and $0 \leq j \leq k = \mu((z-a)/p^n)^j 1_{a+p^n\mathbb{Z}_p}) = \lambda^{-n}\phi(D_{|U_p^n})(z^k)$. Such a distribution exists by Theorem III.4.31(ii), because the additivity (56) hypothesis follows from the fact $\phi_{|U_p} = \lambda U_p$ and hypothesis (57) with $C = \|\phi\|_1$ follows from $|\lambda| \geq p^{-\nu}$; moreover it satisfies $\|\mu\|_1 \leq \|\phi\|_1$. By uniqueness of that μ, it follows easily that Φ is a modular symbol, which clearly satisfies $\rho_k(\Phi) = \phi$ and $\|\Phi\|_1 \leq \|\phi\|_1$. The other equality $\|\|\phi_1 \leq \|\Phi\|_1$ follwos from the fact that ρ_k has norm at most one since the natural

map $\mathcal{D}[1](L) \to \mathcal{V}_k(L)$ has norm one by the theorem of Hahn-Banach, as it is the dual map of the inclusion map $\mathcal{P}_k(L) \to \mathcal{A}[1](L)$ which is clear of norm one.   □

III.7. Applications to the p-adic L-functions of non-critical slope modular forms

Let $f(z) = \sum_{n \geq 1} a_n q^n$, $q = e^{2i\pi z}$ be a cuspidal form of weight $k+2$, level $\Gamma_1(N)$ and character ϵ, which is an eigenform for all Hecke operators T_l ($l \nmid N$). Let p be a prime number.

The idea of Stevens' method to construct a p-adic L-function attached to f is too see f as a symbol in $\mathrm{Symb}_\Gamma(\mathcal{V}_k)$ by Manin-Shokurov's theorem and to lift it uniquely as a symbol in $\mathrm{Symb}_{\Gamma_1(N)}(\mathcal{D}_k)$ by Stevens' control theorem. In order to do the latter, we apparently need two things: that the level $\Gamma_1(N)$ is a subgroup of $\Gamma_0(p)$ (that is that $p|N$), otherwise $\mathrm{Symb}_{\Gamma_1 N}(\mathcal{D}_k^\dagger[0])$ is not even defined; and that the slope of our modular symbol is less than $k+1$. The first condition $p|N$ is exceedingly restrictive (leaving us, for each f, with only a finite number of prime to play with), but fortunately, there is a way to make it true when it is not to begin with. Actually, and interstingly, there is not one way, but two ways to do so:

III.7.1. Refinements. We start with an f as above, and we assume that we are in the "bad" case where p is prime to N. Actually an easy way to remedy to this problem is to consider $f(z)$ as a form of level $\Gamma := \Gamma_1(N) \cap \Gamma_0(p)$ (and same weight $k+2$), which it is. But as is well known, the form $f(pz)$ is also a form of level Γ (and same weight).

EXERCISE III.7.1. **(easy)** check this fact.

The forms $f(z)$ and $f(pz)$ have the inconvenience of not being eigenvectors for U_p. But let's call α and β the two roots of the equations

$$(65) \qquad X^2 - a_p X + p^{k+1} \epsilon(p),$$

and let's set

$$(66) \qquad f_\alpha(z) = f(z) - \beta f(pz)$$
$$(67) \qquad f_\beta(z) = f(z) - \alpha f(pz)$$

LEMMA III.7.2. *The two forms f_α and f_β are eigenforms for U_p of eigenvalues α and β respectively.*

Proof —   Let V_p be the operator $f(z) \mapsto f(pz)$. One has $U_p V_p = \mathrm{Id}$ and $T_p = U_p + p^{k+1}\langle p \rangle V_p$ (see I.6.3 or [**D**, Prop. 5.2.2]). Therefore $U_p f_\alpha = U_p(f - \beta V f) = U_p f - \beta f = a_p f - \epsilon(p)p^{k+1}V f - \beta f = \alpha f - \alpha\beta V f = \alpha f_\alpha$.   □

The forms f_α and f_β are also eigenforms for all the other Hecke operators, with the same eigenvalues as f, and for the Diamond operators: their nebentypus is ϵ seen as a non-primitive character of $(\mathbb{Z}/Np\mathbb{Z})^*$ – that is, it is trivial at p.

DEFINITION III.7.3. The forms f_α and f_β are called the *refinements* of f at p.

REMARK III.7.4. It is conjectured, but not known excepted in weight 2 (that is when $k = 0$), that $\alpha \neq \beta$. cf. [**CE**]

Moreover, by Atkin-Lehner's theory and multiplicity one, if f is an eigenform all Hecke-operators T_l ($l \nmid N$) and U_l ($l|N$), for example if f is new, then $f(z)$ and $f(pz)$ form a basis of the space of all forms of weight $k + 2$, level $\Gamma_1(N) \cap \Gamma_0(p)$, nebentypus ϵ, and with the same eigenvalues as f for operators T_l, for l prime to Np and U_l for $l \neq p$. If $\alpha \neq \beta$, that f_α and f_β are a basis of that sapce, in which U_p is diagonal.

We note that the complex numbers α and β are algebraic by definitions, and actually integral. Choosing an embedding $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ (which determines α and β as elements of $\bar{\mathbb{Q}}$) and then of $\bar{\mathbb{Q}} \to \mathbb{C}_p$ (which allows us to see α and β as elements of $\mathbb{C}_p$), we have by looking at equation (65)

$$(68) \qquad\qquad v_p(\alpha) + v_p(\beta) = k + 1$$

$$(69) \qquad\qquad v_p(\alpha) \geq 0$$

$$(70) \qquad\qquad v_p(\beta) \geq 0$$

Hence $0 \leq v_p(\alpha), v_p(\beta) \leq k + 1$. We also have $v_p(a_p) \geq \min(v_p(\alpha), v_p(\beta))$ with equality when $v_p(\alpha) \neq v_p(\beta)$. In particular, if $v_p(\alpha) = 0$, then $v_p(\beta) = k + 1 \neq 0$ and $v_p(a_p) = 0$.

DEFINITION III.7.5. We say that f is *p-ordinary* if $v_p(a_p) = 0$. If it is, then $v_p(\alpha) = 0$ and $v_p(\beta) = k+1$ up to exchanging α and β, and we call f_α the *ordinary refinement*, and f_β the *critical slope refinement*.

III.7.2. Construction of the p-adic L-function. We shall deal at the same time with the cases $p \nmid N$ and $p|N$. In all case, let $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ which of course is simply $\Gamma_1(N)$ in the case $p|N$. Let us call $g = f_\alpha$ if $p \nmid N$, where α is one of the root of (65) with $v_p(\alpha) < k + 1$. We have two choices of such α when f is not p-ordinary (and $\alpha \neq \beta$), but only one when f is p-ordinary. If $p|N$, then we set $g = f$. The U_p-eigenvalue of g is $\alpha := a_p$ and we shall assume that $v_p(a_p) < k + 1$.

To summarize, in any case we end up with a normalized cuspidal eigenform g of weight $k + 2$, level Γ containing $\Gamma_0(p)$, nebentypus ϵ, and such that $U_p g = \alpha g$ with $\alpha < k + 1$. We now proceed to the consutrution. Remember that we have chosen embeddings $\bar{\mathbb{Q}} \to \bar{\mathbb{C}}$ and $\bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p$.

Step 1: To the cuspidal form g we attach the two modular symbols ϕ_g^+/Ω_g^+ and ϕ_g^-/Ω_g^- in $\mathrm{Symb}_\Gamma(\mathcal{V}_k(K_g))$. (Here $K_g \subset \bar{\mathbb{Q}} \subset \mathbb{C}$ is the number field generated

by the eigenvalues of g, the symbols $\phi_g^{\pm}$ were defined above Theorem III.2.23, the periods $\Omega_g^{\pm}$ are defined in Defnition III.3.9.) Let L be the finite extension of $\mathbb{Q}_p$ generated in $\bar{\mathbb{Q}}_p$ by the image of K_g by our embedding $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$. Then we can see $\phi_g^{\pm}/\Omega_g^{\pm}$ as elements of $\mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{V}_k(L))$.

Note that the periods $\Omega_g^{\pm}$ are defined up to multipliation by an element of K_g^*, but as explained in Remark III.3.10 we can do better. In our context, it makes senses to force our periods $\Omega_f^{\pm}$ to be $\mathfrak{p}$-integer, where $\mathfrak{p}$ is the place of K_g above p defined by our fixed embedding $K_g \subset L$ (hence L is the completion of K_g at $\mathfrak{p}$). Thus our periods are defined up to an element of K_g^* which has $\mathfrak{p}$-valuation 0.

Step 2: The modular symbols $\phi_g^{\pm}/\Omega_g^{\pm}$ have the same eigenvalues for the Hecke operators than g, in particular they have U_p-eigenvalue α. Since $v_p(\alpha) < k+1$, we have

$$\phi_g^{\pm}/\Omega_g^{\pm} \in \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{V}_k(L))^{<k+1}$$

Therefore, applying Steven's control theorem (Theorem III.6.36), we see that there exist unique symbols

$$\Phi_g^{\pm} \in \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{D}_k(L))$$

such that

(71) $$\rho_k(\Phi_g^{\pm}) = \phi_g^{\pm}/\Omega_g^{\pm}$$

Step 3: We set

(72) $$\mu_g^{\pm} = \Phi_g^{\pm}(\{\infty\} - \{0\}) \in \mathcal{D}^{\dagger}[0](\mathbb{C}_p)$$

Step 4: One sees easily that $M_{\mu_g^{\pm}}(\sigma) = 0$ whenever $\sigma(-1) \neq \pm 1$. We thus **define** the p-adic L-function of g by

$$L_p(g, \sigma) = M_{\mu_g^{\pm}}(\sigma), \quad \forall \sigma \in \mathcal{W}(\mathbb{C}_p), \quad \text{where } \pm 1 = \sigma(-1).$$

The construction is over.

REMARK III.7.6.   (i) The construction is canonical, except for the normalization of the period $\Omega_g^{\pm}$.
   (ii) Without the hypothesis $v_p(\alpha) < k+1$, it is still possible to perform each step of the construction, but we don't know in step 2 that the symbols $\Phi_g^{\pm}$ are unique. Instead, we have a whole finite-dimensional space of possible symbols (of unknown dimension), hence a finite-dimensional vector space of possible p-adic L-function, with no way (without further insight) to know which one is better.

III.7.3. Computation of the p-adic L-functions at special characters.

Now we want to compute some of the values of the p-adic L-function we just defined. For that we need to be able to compute some integrals $\mu_g^\pm(h) = \int h(z) d\mu^\pm(z) dz$.

As a warm-up, we begin with the case $h(z) = z^j$ for $0 \le j \le k$.

$$
\begin{aligned}
\mu_g^\pm(z^j) &= \Phi_g^\pm(\{\infty\} - \{0\})(z^j) \\
&= \frac{1}{\Omega_g^\pm} \phi_g^\pm(\{\infty\} - \{0\})(z^j) \quad \text{since } \phi_g^\pm/\Omega_g^\pm = \rho_k(\Phi_g^\pm) \\
&= \frac{\Gamma(j+1)}{(2\pi)^{j+1}\Omega_g^\pm} L(f, j+1) \text{ or } 0 \quad \text{according to } \pm = (-1)^i \text{ or not}
\end{aligned}
$$

The last line is computed using the definition of $\phi_g^\pm$ and formula (46).

EXERCISE III.7.7. **(easy)** where did we use that $0 \le j \le k$ in this computation?

Now, let's try something a little bit more complicated: let $n \ge 0$, $a \in \{0, 1, \ldots, p^n - 1\}$, and as above $0 \le j \le k$. We want to compute $\mu_g^\pm(z^j 1_{a+p^n \mathbb{Z}_p}(z))$.

$$
\begin{aligned}
\mu_g^\pm(z^j 1_{a+p^n \mathbb{Z}_p}(z)) &= \Phi_g^\pm(\{\infty\} - \{0\})(z^j 1_{a+p^n \mathbb{Z}_p}(z)) \\
&= \alpha^{-n}(\Phi_g^\pm)_{|U_p^n}(\{\infty\} - \{0\})(z^j 1_{a+p^n \mathbb{Z}_p}(z)) \quad (\text{since } U_p^n \Phi_g^\pm = \alpha^n \Phi_g^\pm) \\
&= \alpha^{-n} \sum_{b=1-p^n}^{0} \Phi_g^\pm(\{\infty\} - \{b/p^n\})((p^n z - b)^j 1_{a+p^n \mathbb{Z}_p}(p^n z - b)) \\
&\qquad \left(\text{by definition of the action of } U_p^n, \text{ using } \Gamma \begin{pmatrix} 1 & 0 \\ 0 & p^n \end{pmatrix} \Gamma = \coprod_{b=1-p^n}^{0} \Gamma \begin{pmatrix} 1 & b \\ 0 & p^n \end{pmatrix}\right) \\
&= \alpha^{-n} \Phi_g^\pm(\{\infty\} - \{a/p^n\})((a + p^n z)^j) \quad (\text{only the fittest term survives: } b = -a) \\
&= \frac{\alpha^{-n}}{\Omega_g^\pm} \phi_g^\pm(\{\infty\} - \{a/p^n\})((a + p^n z)^j) \quad (\text{since } \phi_g^\pm/\Omega_g^\pm = \rho_k(\Phi_g^\pm)) \\
&= \frac{\alpha^{-n} p^{nj}}{\Omega_g^\pm} \phi_g^\pm(\{\infty\} - \{a/p^n\})((\frac{a}{p^n} + z)^j)
\end{aligned}
$$

Now if χ is a Dirichlet character of conductor p^n, with $n \ge 1$ and $\chi(-1)(-1)^j = \pm 1$, and if we see χ as a character of $\mathbb{Z}_p^*$,

$$
\begin{aligned}
L_p(g, \chi z^j) &= \sum_{a=0}^{p^n-1} \chi(a) \mu_g^\pm(z^j 1_{a+p^n \mathbb{Z}_p}(z)) \text{ (by def., using that } \chi(a) = 0 \text{ if } p|a) \\
&= \frac{\alpha^{-n} p^{nj}}{\Omega_g^\pm} \sum_{a=0}^{p^n-1} \chi(a) \Phi_g^\pm(\{\infty\} - \{a/p^n\})(\frac{a}{p^n} + z)^j) \\
&= \frac{\alpha^{-n} p^{nj}}{\Omega_g^\pm} \frac{j! \tau(\chi)}{(-2i\pi)^{j+1}} L(g, \chi^{-1}, j+1) \quad \text{by (49)} \\
&= \frac{p^{n(j+1)} j!}{\alpha^n (-2i\pi)^{j+1} \tau(\chi^{-1}) \Omega_g^\pm} L(g, \chi^{-1}, j+1) \text{ (since } \tau(\chi)\tau(\chi^{-1}) = p^n)
\end{aligned}
$$

And we have computed our p-adic L-function (at special characters) in terms of the complex L-function.

The computation when χ is of conductor $p^0 = 1$, that is when χ is trivial, is a little different (we still assume that $(-1)^j = \pm 1$): for $0 \le j \le k$,

$$
\begin{aligned}
L_p(g, z^j) &= \Phi_g^\pm(\{\infty\} - \{0\})(z^j 1_{\mathbb{Z}_p^*}(z)) \\
&= \Phi_g^\pm(\{\infty\} - \{0\})(z^j) - \Phi_g^\pm(\{\infty\} - \{0\})(z^j 1_{p\mathbb{Z}_p}(z)) \\
&= (1 - \alpha^{-1} p^j) \Phi_g^\pm(\{\infty\} - \{0\})(z^j) \\
&= (1 - \alpha^{-1} p^j) \frac{\Gamma(j+1)}{(2\pi)^{j+1} \Omega_g^\pm} L(g, j+1)
\end{aligned}
$$

To summarize:

THEOREM III.7.8 (Mazur–Swinnerton-Dyer, Manin, Vishik, Amice–Velu). *Let g be a modular cuspidal normalized eigenform of weight $k + 2$, character ϵ, level $\Gamma_1(N)$. Assume that $p|N$ and that $U_p g = \alpha g$ with $v_p(g) < k + 1$. Then there exists a unique function $L_p(g, \sigma)$ that enjoys the two following properties:*

(interpolation) *For any finite image character $\chi : \mathbb{Z}_p^* \to \mathbb{C}_p^*$ of conductor p^n, and any integer j such that $0 \le j \le k$,*

$$
L_p(g, \chi z^j) = e_p(g, \chi, j) \frac{p^{n(j+1)} j!}{\alpha^n (-2i\pi)^{j+1} \tau(\chi^{-1}) \Omega_g^\pm} L(g, \chi^{-1}, j+1),
$$

where $e_p(g, \chi, j) = 1$ if χ is non-trivial and $e_p(g, 1, j) = (1 - \alpha^{-1} p^j)$

(growth rate) *The order of growth of $L_p(g, \sigma)$ is $\le v_p(\alpha)$.*

Proof — We have just constructed a p-adic L-function $L_p(g)$ that satisfies the given interpolation property. The growth rate of $\mu_g^\pm$ is $\le v_p(\alpha)$ by Prop. III.6.29, so the growth rate of $L_p(g, \sigma)$ is $\le v_p(\alpha)$ by Theorem III.5.17(ii). This shows the existence part of the theorem.

By Theorem III.5.17(ii), the uniqueness of $L_p(g, \sigma)$ is equivalent to the uniqueness of $\mu_g^\pm$ satisfying the corresponding property (i) and (ii). Property (i) means that we can compute $\mu_g^\pm(P)$ for any polynomial P of degree less or equal to k. Since the order of growth of $\mu_g^\pm$ is $\le \alpha < k + 1$, the uniqueness follows from Theorem III.4.31(ii). $\square$

The result is not completely satisfying when we started with an f such that $p \nmid N$. We know that we can apply the above theorem to $g = f_\alpha$ or $g = f_\beta$, you choose (provided g is not of critical slope). But we want to express the interpolation property in terms of f, not of the auxiliary function g.

Observe that $L(f(pz), s) = p^{-s} L(f, s)$ (this is best seen on the integral representation), hence

$$
\begin{aligned}
L(f_\alpha, s) &= (1 - \beta p^{-s}) L(f, s) \\
&= (1 - \alpha^{-1} \epsilon(p) p^{k+1-s}) \text{ using } \alpha\beta = p^{k+1} \epsilon(p)
\end{aligned}
$$

For non trivial χ, applying the same result to the form $f_{\chi^{-1}}$ we get

$$
L(f_\alpha, \chi^{-1}, s) = (1 - \alpha^{-1} \epsilon(p) \chi^{-1}(p) p^{k+1-s}) L(f, \chi^{-1}, s) = L(f, \chi^{-1}, s)
$$

since $\chi^{-1}(p) = 0$. From this, we get

COROLLARY III.7.9 (Same as above, plus Mazur-Tate-Teitelbaum). *Let f be a modular cuspidal normalized eigenform of weight $k + 2$, character ϵ, level $\Gamma_1(N)$. Assume that N is prime to p. Let α, β be the two roots of $X^2 - a_p X + p^{k+1}\epsilon(p)$, and choose one, say α, with $v_p(\alpha) < k + 1$. There exists a unique function $L_p(f_\alpha, \sigma)$ that enjoys the two following properties:*

(interpolation) *For any finite image character $\chi : \mathbb{Z}_p^* \to \mathbb{C}_p^*$ of conductor p^n, and any integer j such that $0 \leq j \leq k$,*

$$L_p(f_\alpha, \chi z^j) = e_p(f, \alpha, \chi, j)\frac{p^{n(j+1)}j!}{\alpha^n(-2i\pi)^{j+1}\tau(\chi^{-1})\Omega_g^\pm}L(f, \chi^{-1}, j + 1),$$

where $e_p(f, \alpha, \chi, j) = 1$ if χ is non trivial and $e_p(f, \alpha, 1, j) = (1-\alpha^{-1}\epsilon(p)p^{k-j})(1-\alpha^{-1}p^j)$

(growth rate) *The order of growth of $L_p(f_\alpha, \sigma)$ is $\leq v_p(\alpha)$.*

REMARK III.7.10.   (i) The construction of the p-adic L-function of a modular form we have given (or for that matter, any other known construction) is quite indirect in the that sense that it uses complex transcendental method. One could expect that to construct from an algebraic object (the sequence of coefficients a_n of f) a p-adic object $L_p(f_\alpha, \sigma)$, it would suffice to use purely algebraic and p-adic means. But we used the modular symbols $\phi_{g^\pm}$, which is defined by complex transcendental means from g.

   (ii) It is expected, but not known in general, that the order of growth of $L_p(f_\alpha, \sigma)$ is exactly $v_p(\alpha)$. The only case where this seems to be known is when $a_p = 0$ (and $p \nmid N$), in which case both $L_p(f_\alpha, \sigma)$ and $L_p(f_\beta, \sigma)$ have the expected order $(k - 1)/2$ (see [**?**]). Also, it is known (*loc. cit.*) and due to Mazur that at least one of the functions $L_p(f_\alpha, \sigma)$ and $L_p(f_\beta, \sigma)$ has positive order.

  (iii) We know little on the values of L_p at characters σ not of the form χz^k for $0 \leq j \leq k$, or for that matter on the derivatives of L_p at characters of that form. They are important conjectures predicting those values, and relating it to arithmetic invariants of f.

  (iv) Fix an M prime to p. One can extend the p-adic L-function of f_α form a function on $\mathcal{W}$ to a function on $\mathcal{W}_M$ (with essentially no change in the proof). The formulas are essentially the same, excepted that now χ is interpreted as a Dirichlet character of conductor $p^n M$ (and accordingly the p^n in the formula has to be replaced by $p^n M$), and that the *p-adic multiplier $e_p(f, \alpha, \chi, j)$* is defined by a slightly more complicated formula. See [**MTT**] for the precise formulas.

   (v) The question of whether the p-adic multiplier $e_p(f, \alpha, \chi, j)$ is 0, and what consequences this vanishing has when it occurs, is discussed extensively in [**MTT**]

(vi) There is an other completely independent way to construct the p-adic L-function of a modular form: it is by using Perrin-Riou's method applied to Kato's Euler system. We shall not discuss this point view here. See [**CO3**] and [**?**] and the refenreces there for more informatiom.

(vii) The p-adic L-function we constructed are never 0. This is easy to see when the form g has weight $k+2 > 2$, since then the interpolation property gives, for χ a non-trivial Dirichlet character $L_p(g, \chi z^k) = C L_p(g, \chi^{-1}, k+1)$ where C is a non-zero constant. If $k + 2 > 2$, that is $k > 0$, then $k + 1 \geq k/2 + 3/2$ so $k+1$ is either in the interior or on the boundary (when $k = 1$, i.e. in weight 3) of the domain of convergence of the Dirichlet Series of $L(g, \chi^{-1}, s)$. In both case, it is known that this functions does not vanish at $k + 1$, because of the Euler product description in the first case, of the Jacquet-Shalika generalization of the theorem of Hadamard and De La Vallée-Poussin in the second case. Proving that the p-adic L-function does not vanish for a form of weight 2 is harder, and requires a difficult thorem of Rohrlich.

(viii) When $v_p(\alpha) = k + 1$, we can still construct a p-adic L-function satisfying (i) and (ii) by choosing for $\Phi_g^{\pm}$ any lift of $\phi_g^{\pm}/\Omega_g^{\pm}$, but then the construction is not canonical, and (i) and (ii) does not determine uniquely the p-adic L-function. It is not possible to remediate this lack of uniqueness (there definitely are several functions satisfying (i) and (ii)), but it is possible to modify the construction so as to get a *canonical, well-defined p-adic L-function*, and to state properties that characterize it uniquely. See below.

III.8. Notes and references

Classical modular symbols appear in [**Man**], in connection with the study of the arithmetic properties of the special values of a modular forms and with the construction of its p-adic L-function. They are studied systematically in [**Sho**].

The eigencurve of modular symbols

Stevens' overconvergent modular symbols can replace Coleman's overconvergent modular forms in the construction of the eigencurve. This was outlined by Stevens himself in a talk in Paris in 2000, but he never released his personal notes on the subject. A very quick sketch of his argument is given in [**K**] or in [**P**] and a detailed construction of the local pieces of Steven's eigencurve is to be found in [**B**] of which the following is a slightly extended global version.

IV.1. Construction of the eigencurve using rigid analytic modular symbols

From now on, we will fix an integer N, an odd prime p not dividing N, and $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$. We will construct in this §the local pieces of the eigencurve *of tame level N*. We call $\mathcal{H}$ the ring generated by symbols called T_l, for l not dividing Np, U_p, and $\langle a \rangle$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$. Note that the meaning of $\mathcal{H}$ has changed since §I.6.3 in that we have dropped out of $\mathcal{H}$ all the operator U_l for $l \mid N$ (we won't need them) keeping only U_p (which is fundamental in the construction of the eigencurve). We could actually work with the full Hecke algebra, including the U_l's but this would lead to a different, non-reduced, eigencurve.

We shall also occasionaly need the Hecke algebra $\mathcal{H}_0$, which has the same meaning as always: it is the polynomial ring in the T_l, $l \nmid N$, and $\langle a \rangle$, $a \in (\mathbb{Z}/N\mathbb{Z})^*$ (no U_p).

IV.1.1. Overconvergent modular symbols over an admissible open affinoid of the weight space. In all this §, let $W = \operatorname{Sp} R$ be an admissible open affinoid of the weight space $\mathcal{W}$. So R is an affinoid algebra over $\mathbb{Q}_p$, in particular a Noetherian commutative $\mathbb{Q}_p$-Banach algebra. The inclusion map $W = \operatorname{Sp} R \in \mathcal{W}$ is a point in $\mathcal{W}(R)$, hence by definition of $\mathcal{W}$ a continuous group homorphism

$$K : \mathbb{Z}_p^* \to R^*$$

The map K should be thought of as a family of characters of $\mathbb{Z}_p^*$ parametrized by the weights $w \in W$. That is, for L a finite extension of $\mathbb{Q}_p$, if w is a point of $\mathcal{W}$ that beongs to $W(L) = \operatorname{Hom}_{\text{cont,ring}}(R, L^*)$, then the character $\kappa : \mathbb{Z}_p^* \to L^*$ attached to w is precisely the composition

(73) $$\kappa : \mathbb{Z}_p^* \overset{K}{\to} R^* \overset{w}{\to} L^*.$$

By the work done in §III.4.2, and we have Banach R-modules $\mathcal{A}[r](R)$, $\mathcal{D}[r](R)$ for $r > 0$, and R-modules $\mathcal{A}^{\dagger}[r](R)$, $\mathcal{D}^{\dagger}[r](R)$ for $r \geq 0$ (the latter denoted just $\mathcal{A}$ and $\mathcal{D}$ when $r = 0$). Let us recall (Prop. III.4.21) that the formation of the Banach-module $\mathcal{D}[r](R)$ commutes with base change $R \to R'$, where R' is any $\mathbb{Q}_p$-Banach algebra (in particular, when R' is an affinoid algebra and $\operatorname{Sp} R'$ an affinoid subdomain of $\operatorname{Sp} R = W$). Here the completed tensor product is the one of normed module over a normed algebra, as defined and used throughout in [**BGR**]. The same is true for $\mathcal{D}^{\dagger}[r](R)$ by Corollary III.4.26, but here we need the notion of completed tensor product defined *loc. cit.*

LEMMA IV.1.1. *f L is any finite extension of $\mathbb{Q}_p$, and if $w \in W(L)$, then we have a natural isomorphsim*

$$\mathcal{D}^{\dagger}[r](R) \otimes_{R,w} L = \mathcal{D}^{\dagger}[r](L).$$

Proof — This is a special case of Corollary III.4.26 using the easy observation that since L is finite over R, our $\hat{\otimes}_R L$ functor is just $\otimes_R L$ □

Now, by the work done in §III.6.1, we can use the "weight" $K : \mathbb{Z}_p^* \to R^*$ to define weight-K action of $S_0(p)$ on $\mathcal{A}[r](R)$, $\mathcal{D}[r](R)$, for all r such that $0 < r < r(K)$, $\mathcal{A}^{\dagger}[r](R)$, $\mathcal{D}^{\dagger}[r](R)$, for all r such that $0 \leq r < r(K)$, where $r(K)$ is a positive real number depending only on K, that is only on W, defined in Definition III.5.5. We shall write belo $r(W)$ instead of $r(K)$. When provided with these actions, these modules are denoted by $\mathcal{A}_K[r](R)$, $\mathcal{D}_K[r](R)$, $\mathcal{A}_K^{\dagger}[r](R)$, $\mathcal{D}_K^{\dagger}[r](R)$.

LEMMA IV.1.2. *Let L be a finite extension of $\mathbb{Q}_p$, $w \in W(L) = \operatorname{Hom}(R, L)$ and $\kappa : \mathbb{Z}_p^* \to L^*$ the corresponding character. The natural isomorphism $\mathcal{D}_K^{\dagger}[r](R) \otimes_{R,w} L = \mathcal{D}_\kappa^{\dagger}[r](L)$ is compatible with the $S_0(p)$-actions (and similarly for the other modules $\mathcal{D}_K[r](R)$ etc.)*

Proof — This is clear from the definitions of those isomorphisms and from (73). □

Hence one should think of the R-module $\mathcal{D}_K^{\dagger}[r](R)$ with its $S_0(p)$-action as the family of $S_0(p)$-modules $\mathcal{D}_\kappa^{\dagger}[r]$ parametrized by points $w \in W$, with κ the character corresponding to w.

Now If $\Gamma \subset \Gamma_0(p)$ is a congruence subgroup, we can form modules of modular symbols, e.g. $\operatorname{Symb}_\Gamma(\mathcal{D}_K[r](R))$ which is a Banach R-module or $\operatorname{Symb}_\Gamma(\mathcal{D}_K^{\dagger}[r](R))$.

From now on we shall make the following assertion on R: R **is a PID, and the residue ring $\tilde{R}$ is a PID** (where $\tilde{R} = R^0/pR^0$, and R^0 is the unit ball for the norm of R). There is one technical point that must be adressed before we carry in this context the work done over field in §III.6.5:

LEMMA IV.1.3. *The R-module $\operatorname{Symb}_\Gamma(\mathcal{D}_K[r](R))$ is potentially ONable*

Proof —   This follows from scholium III.6.20 and from the following abstract lemma.   □

LEMMA IV.1.4. *Let M be any potentially orthonormalizable Banach R-module, and N be any closed sub-module of M. Then N is potentially orthonormalizable.*

Proof —   We can change the norm of M so that M becomes orthonormalizable. In particular, the norm of any element of M is a norm of an element of R, a property that N obviously inherits. Therefore, to prove that N is orthonormalizable it is enough to prove that $\tilde{N}$ is free as an $\tilde{R}$-module, where $\tilde{R}$ (resp. $\tilde{N}$, $\tilde{M}$) is defined as R^0/pR^0 (resp. N^0/pN^0, M^0/pM^0) and R^0 (resp. N^0, M^0) is the closed unit ball of R (resp. N, M). But the natural map of $\tilde{R}$-modules $\tilde{N} \to \tilde{M}$ is injective, since $(pM^0) \cap N = pN^0$ obviously. So $\tilde{N}$ is free as a sub-module of the module $\tilde{M}$ which is free (since M is orthonormalizable) over the **principal** ideal domain $\tilde{R}$.   □

EXERCISE IV.1.5. Is $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](L))$ potentially ONable for all $\mathbb{Q}_p$-Banach algebra L?

With the potential ONability of $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))$ and the above definition, the following analog of Theorem III.6.26 makes sense and is proved word by word as it is.

PROPOSITION IV.1.6.   (i) *Let $0 < r < \min(r(K), p)$. The operator U_p acts compactly on the potentially orthonormalizable R-module $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))$.*

(ii) *Let $0 < r, r' < \min(r(K), p)$. Then as Banach-modules with action of U_p, the modules $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))$ and $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r'](R))$ are linked in the sense of Definition* **??**.

In particular, the operator U_p acting on $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))$ for $0 < r < \min(r(K), p)$ has a Freedholm power series $F(T)$ which is independent of r (hence depends only on $W = \mathrm{Sp}\, R$). If ν is a real number that is *adapted* to $F(T)$ (definition II.1.8), we shall simply say that ν is adapted to W. In this case we can defined the sub-module $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))^{\leq \nu}$ of $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))$, which are finite and projective over R, and independent of r.

The next two subsections are aimed at proving to results concerning the comuptation of the formation of $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))$ with base change.

IV.1.2. The restriction theorem. The aim of this subsection is to prove

THEOREM IV.1.7. *Let $W' = Sp\, R'$ be an affinoid subsomain of $W = Sp\, R$. Then for $0 < r \leq r(W)$, the two Banach-modules with action of $\mathcal{H}$: $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))\hat{\otimes}_R R'$ and $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R'))$ are linked.*

First note that by definition $r(W') \geq r(W)$, so $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R'))$ is defined.

Let us recall (cf. Lemma III.6.19 and above) that if V is an R-module with an R-linear $S_0(p)$-action, and if q is a norm (or even a semi-norm) of R-module on V which is preserved by Γ, then $\mathrm{Symb}_\Gamma(V)$ inherits from q a norm of R-module (or a semi-norm), that we shall still denote by q. If V is a topological module whose topolgy is defined by a family of norms q_i as abobe, then we consder $\mathrm{Symb}_\Gamma(V)$ as provided with the topology defined by the norms q_i. If V is as above, let us define the *difference operator*: $\Delta : V \to V$, $v \mapsto v_{\left|\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right.} - v$.

LEMMA IV.1.8 (Pollack-Stevens). *There exists an integer $t > 0$, an integer $s \geq 0$, and elements $\gamma_i \in \Gamma$ for $i = 1, \dots, t$, all depending only on Γ, such that if $n = t+s+1$, and $\mu_V^* : V^n \to V$ is defined by $(v_i) \mapsto \sum_{i=1}^t (v_i)_{|\gamma_i - Id} + \sum_{i=t+1}^{t+s} v_i + \Delta v_n$, then we have an isomorphism*

$$\mathrm{Symb}_\Gamma(V) = \ker(V^n \xrightarrow{\mu_V^*} V)$$

that respects the topology of $\mathrm{Symb}_\Gamma(V)$ when we have put one on it.

Proof —   This is a refined version of the argument based on Manin's lemma used in the proof of Lemma **??**, obtained by chosing artfully a system of generators of Δ_0 as $\mathbb{Z}[\Gamma]$-module. For the proof, see [**PS1**, Theorem 2.9].   □

LEMMA IV.1.9. *If V is as above, we have a natural injective isometric $\mathcal{H}$-compatible R-linear morphism*

$$\widehat{\mathrm{Symb}_\Gamma(V)} \hookrightarrow \mathrm{Symb}_\Gamma(\hat{V}).$$

If $\mu_V^* : V \to V$ is strict (cf. Exercise III.6.16), then the displayed morphism is an isomorphism.

Proof —   In particular, if V is has in the statement, we hat for $\hat{V}$ the completion of V:

$$\mathrm{Symb}_\Gamma(\hat{V}) = \ker(\hat{V}^n \xrightarrow{\mu_{\hat{V}}^*} \hat{V})$$

and it is clear from the descripton of μ_V^* that $\mu_{\hat{V}}^*$ is just the extension by continuity of μ_V^*. Hence an injective map $\mathrm{Symb}_\Gamma(V) \hookrightarrow \mathrm{Symb}_\Gamma(\hat{V})$ inducing by the universal property of completion a map

$$\widehat{\mathrm{Symb}_\Gamma(V)} \hookrightarrow \mathrm{Symb}_\Gamma(\hat{V}).$$

This proves the first half of the lemma. For the second half, apply [**BGR**, Prop. 1.1.9/5].   □

EXERCISE IV.1.10. Prove directly the second half of the lemma, without looking in [**BGR**].

LEMMA IV.1.11. *The maps* μ_V^* *are strict when* $V = \mathcal{D}_k^\dagger[r](\mathbb{Q}_p)$ $V = \mathcal{D}_k^\dagger[r](R)$ *and* $V = \mathcal{D}_k^\dagger[r](R) \otimes_R R'$.

Proof — Let first $V = \mathcal{D}_k^\dagger[r](\mathbb{Q}_p)$. Then by Lemma **??**, $\Delta(V)$ is a closed hyperplan of Δ. Hence the image of μ_V^*, which contains $\Delta(V)$ is closed in V. It follows that μ_V^* is strict (point e. of Exercise III.6.16). It follows by point f. of Exercise III.6.16 that μ_V^* is also strict when $V = \mathcal{D}_k^\dagger[r](\mathbb{Q}_p) \otimes R$. Since the completion of a strict morphism is strict ([**BGR**, Prop 1.1.9/4]), μ_V^* is also strict for $V = D_k^\dagger[r](\mathbb{Q}_p) \otimes R$ that is $V = \mathcal{D}_k^\dagger[r](R)$ by Prop. III.4.25. And finally, the result for $V = \mathcal{D}_k^\dagger[r](R) \otimes_R R'$ follows by applying point f. of Exercise III.6.16 again.

□

LEMMA IV.1.12. *We have an isometric natural isomorphism of normed R-modules, respecting the action of $\mathcal{H}$*

$$\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R)) \otimes_R R' = \mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R) \otimes_R R')$$

and a natural isomorphsim respecting the norms and the action of $\mathcal{H}$

$$\mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r](R)) \otimes_R R' = \mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r](R) \otimes_R R')$$

Proof — By [**Con2**, page 13], R' is R-flat Hence the lemma follows from Lemma III.1.2. □

PROPOSITION IV.1.13. *We have*

$$\mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](R')) = \mathrm{Symb}_\Gamma(\mathcal{D}_k^\dagger[r](R)) \hat{\otimes}_R R'$$

Proof — One has the following sequence[1] of isomorphisms of R'-module with action of $\mathcal{H}$

$$\mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r](R')) = \mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r](R) \hat{\otimes}_R R') \text{ (by Prop. ??)}$$
$$= \mathrm{Symb}_\Gamma(\overline{\mathcal{D}_K^\dagger[r](R) \otimes_R R'}) \text{ (by def. of } \hat{\otimes})$$
$$= \overline{\mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r](R) \otimes_R R')} \text{ (by Lemma IV.1.9 and Lemma IV.1.11)}.$$
$$= \overline{\mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r](R)) \otimes_R R'} \text{ (by Lemma IV.1.12)}$$
$$= \mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R)) \hat{\otimes}_R R' \text{ (by def. of } \hat{\otimes}).$$

□

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[1]Where I denote by $\overline{A}$ instead of $\widehat{A}$ the completion of A, because I can't obtain a *long enough wide hat.*

To complete the proof of the theorem, observe that one has, as R'-module with action of $\mathcal{H}$-the following isomorphisms and embeddings:

$$\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))\hat{\otimes}_R R' = \overline{\mathrm{Symb}_\Gamma(D_K[r](R)) \otimes_R R'} \text{ (by def. of } \hat{\otimes})$$

$$\hookrightarrow \mathrm{Symb}_\Gamma(\overline{\mathcal{D}_K[r](R) \otimes_R R'}) \text{ (by Lemma IV.1.9)}$$

$$= \mathrm{Symb}_\Gamma(D_K[r](R)\hat{\otimes}_R R') \text{ (by def. of } \hat{\otimes})$$

$$= \mathrm{Symb}_\Gamma(D_K[r](R')) \text{ (by Prop. III.4.21)}.$$

On the other hand, choosing an r' and r'' such that $r < r' < r'' < r(W)$, and remembering that $\mathcal{D}_k[r] \subset \mathcal{D}_k[r'] \subset \mathcal{D}_k[r'']$, one has

$$\mathrm{Symb}_\Gamma(D_K[r](R')) \hookrightarrow \mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r'](R')) \text{ (since } \mathrm{Symb}_\Gamma \text{ is left-exact)}$$

$$= \overline{\mathrm{Symb}_\Gamma(\mathcal{D}_K^\dagger[r'](R)) \otimes_R R'} \text{ (by Prop IV.1.13)}$$

$$\hookrightarrow \overline{\mathrm{Symb}_\Gamma(\mathcal{D}_K[r''](R) \otimes_R R'} \text{ (since } \mathrm{Symb}_\Gamma \text{ is left-exact)}$$

$$= \mathrm{Symb}_\Gamma(\mathcal{D}_K[r''](R))\hat{\otimes}_R R' \text{ (by def. of } \hat{\otimes}).$$

Finally we note that $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r''](R))\hat{\otimes}_R R'$ and $\mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))\hat{\otimes}_R R'$ are linked by prop. IV.1.6, and thus $\mathrm{Symb}_\Gamma(D_K[r](R'))$, which sits in between is also linked to them by Lemma II.2.3. Theorem IV.1.7 is now proved.

REMARK IV.1.14. I don't know if μ_V^* is strict when V is the Banach space $\mathcal{D}_k[r](\mathbb{Q}_p)$. Of course, it is the case if $s > 0$ because then μ_V^* is surjective, but for a torsion-free Γ, one has $s = 0$ unfortunately, Since $\Delta : \mathcal{D}_k[r](\mathbb{Q}_p) \to \mathcal{D}_k[r](\mathbb{Q}_p)$ is not strict, it is very possible that μ_V^* be not strict either.

This forces us to work with the Frechet cousins $\mathcal{D}_k^\dagger[r]$, for which μ_V^* is strict, because it is not true that the functors completion and $\hat{\otimes}_R R'$, commute with taking the kernel of a continuous morphism of Banach (or Frechet) space, when the morphism is not supposed to be strict (See Exercise IV.1.15 below). Otherwise, the theorem would be proved exactly as Lemma III.1.2.

EXERCISE IV.1.15. Let $R = \mathbb{Q}_p\{T\}$ be the Tate algebra in one variable, so $\mathrm{Sp}\,R$ is the closed ball of center 0 and radius 1. Let M be an orthonormalizable Banach module over R of orthonormal basis $(e_i)_{i \geq 1}$, and define a continuous linear map $u : M \to M$ bu $u(e_1) = \sum_{i=1}^\infty p^i T^i e_i$ and $u(e_i) = e_{i-1}$ for $i > 1$. Show that u is injective, but that if $\mathrm{Sp}\,R'$ is the ball of center 0 and radius $r < 1$, $r \in p^\mathbb{Q}$, then $u\hat{\otimes}1 : M\hat{\otimes}_R R' \to M\hat{\otimes}_R R'$ is not injective. (This example is due to V. Lafforgue, cf. [**C2**, footnote 1]).

IV.1.3. The specialization theorem. In all this §, $W = \mathrm{Sp}\,R$ is an admissible affinoid open subset of $\mathcal{W}$.

LEMMA IV.1.16. *Let* $0 \leq r < r(K)$ *If* $0 \notin W(\mathbb{Q}_p) = Sp\,R(\mathbb{Q}_p) = \mathrm{Hom}(R, \mathbb{Q}_p)$, *then* $H_0(\Gamma, \mathcal{D}^\dagger[r](R)) = 0$. *If* $0 \in W(\mathbb{Q}_p) = \mathrm{Hom}(R, \mathbb{Q}_p)$, *then we have a natural isomorphism of* $\mathbb{Q}_p$-*vector spaces* $H_0(\Gamma, \mathcal{D}^\dagger[r](R)) = \mathbb{Q}_p$. *This isomorphism is an*

isomorphism of R-module if we give the RHS an R-module structure using the morphism $R \to \mathbb{Q}_p$ corresponding to the character 0.

Proof — We claim that the sequence of R-modules

$$(74) \qquad 0 \longrightarrow \mathcal{D}^\dagger[r](R) \xrightarrow{\Delta} \mathcal{D}^\dagger[r](R) \xrightarrow{\rho} R \longrightarrow 0$$

is still exact. We know it is exact for $R = \mathbb{Q}_p$ (Lemma III.6.31). It is thus enough to show that he completed tensor product by an orthonormalizable Banach space is exact in the category of Frechet spaces, which amounts to seeing that the functor $V \mapsto c(V)$ is exact, and only the right-exactness is not obvious. So let $f : V \to W$ be a surjective morphism of Frechet spaces. We can assume that the topology of V is defined by semi-norms $(p_k)_{k\in\mathbb{N}}$ such that $p_k \le p_{k+1}$ for all k. By the Open Mapping Theorem ([**Sch**, Prop. 8.6]) the topology of V' is the quotient of the topology of V, hence is defined by the semi-norms $p'_k(v') := \inf_{v\in V, f(v)=v'} p_k(v)$. By [**Sch**, Prop 8.1 and its proof], the topology of V (resp. V') is also defined by a distance d (resp. d') such that $d(v,0) = \sup_k \frac{1}{2^k} \frac{p_k(v)}{1+p_k(v)}$ (resp. $d'(v',0) = \sup_k \frac{1}{2^k} \frac{p'_k(v')}{1+p'_k(v')}$). It follows easily that $d'(v',0) = \inf_{v\in V, f(v)=v'} d(v,0)$. Hence a sequence (v'_n) in V' such that $d'(v'_n, 0)$ goes to 0 can be lifted into a sequence (v_n) in V such that $d(v_n, 0)$ goes to 0, which proves that $c(V) \to c(V')$ is surjective.

Now we need to compute $H^0(\Gamma, \mathcal{D}^\dagger_K[r](R)) = \mathcal{D}^\dagger_K[r](R)/I\mathcal{D}^\dagger_K[r](R)$ where I is the augmentation ideal of $\mathbb{Z}[\Gamma]$. We proceed as in [**PS2**, Lemma 5.2], but "in family".

The ideal I contains $(g-1)$ with $g = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, which acts as Δ. Thus $I\mathcal{D}^\dagger[r](R)$ contains $\ker\rho$, and we have an exact sequence

$$(75) \qquad 0 \to \rho(I\mathcal{D}^\dagger(R)) \to R \to H_0(\Gamma, \mathcal{D}^\dagger[r](R)) \to 0.$$

We now construct two explicit elements in $\rho(I\mathcal{D}^\dagger(R))$.

First, set $\gamma = \begin{pmatrix} 1+pN & pN \\ pN & 1-pN \end{pmatrix} \in \Gamma_1(Np) \subset \Gamma$. Let δ_1 be the distribution that sends f to $f(1)$. We compute

$$\begin{aligned} \rho((\delta_1)_{|\gamma-1}) &= \delta_1(\tilde{K}(1+pN+pNz) - 1) \\ &= \tilde{K}(1+2pN) - 1 \in \rho(I\mathcal{D}^\dagger(R)) \end{aligned}$$

Second, let a be an integer such that $a \pmod p$ is a generator of $(\mathbb{Z}/q\mathbb{Z})^*$, and $a \equiv 1 \pmod N$ (remember that $q = p$ if p is odd, $q = 4$ if $p = 2$). By an easy application of Bezout's lemma there exists integers b, c, d such that $\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in Γ. Let δ_0 be the distribution $f \mapsto f(0)$. Then we compute:

$$\begin{aligned} \rho((\delta_0)_{|\gamma-1}) &= \delta_0(\tilde{K}(a+cz) - 1) \\ &= \tilde{K}(a) - 1 \in \rho(I\mathcal{D}^\dagger(R)) \end{aligned}$$

From the exact sequence (75) we deduce the existence of a surjective map

$$(76) \qquad R/(\tilde{K}(1 + 2pN) - 1, \tilde{K}(a) - 1) \to H_0(\Gamma, \mathcal{D}^\dagger[r](R)).$$

Since $1 + 2pN$ generates $1 + q\mathbb{Z}_p$, and a generates $(\mathbb{Z}/q\mathbb{Z})^*$, we see that $R/(\tilde{K}(1 + 2pN) - 1, \tilde{K}(a) - 1)$ is either $\mathbb{Q}_p$ or 0, according to whether $0 \in W$ or not, and that in the former case, the map $R \to R/(\tilde{K}(1 + 2pN) - 1, \tilde{K}(a) - 1) = \mathbb{Q}_p$ is the one corresponding to the trivial character 0. In this case, we have a natural map

$$(77) \qquad H_0(\Gamma, \mathcal{D}^\dagger[r](R)) \to H_0(\Gamma, \mathcal{D}_0^\dagger[r])$$

surjective by right exactness of H_0, hence $H_0(\Gamma, \mathcal{D}^\dagger[r](R))$ has dimension at least 1 over $\mathbb{Q}_p$, since by [**PS2**, Lemma 5.2 and its proof] $H_0(\Gamma, \mathcal{D}_0^\dagger[r])$ has dimension 1. This shows that the surjective maps (76) and (77) are both isomorphisms and that $H_0(\Gamma, \mathcal{D}^\dagger[r](R))$ has dimension exactly 1 over $\mathbb{Q}_p$. $\qquad \square$

THEOREM IV.1.17. *Let $W = Sp\, R$ be a nice affinoid of $\mathcal{W}$. Let L be a finite extension of $\mathbb{Q}_p$. Let $w \in W(L) = \mathrm{Hom}(R, L)$ which by definition is a character $\mathbb{Z}_p^* \to L^*$. Let $0 \le r < r(K)$ be a real number, then there is a natural injective morphism of L-vector spaces, compatible with the action of $\mathcal{H}$*

$$(78) \qquad \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger[r](R)) \otimes_{R,w} \mathbb{Q}_p \hookrightarrow \mathrm{Symb}_\Gamma(\mathcal{D}_w^\dagger[r]).$$

This map is surjective excepted when $w = 0$. In this case, the cokernel is a space of dimension 1.

Proof — The last isomorphism of Lemma **??** may be reformulated as the exact sequence

$$0 \longrightarrow \mathcal{D}^\dagger[r](R) \xrightarrow{\times u} \mathcal{D}^\dagger[r](R) \xrightarrow{w} \mathcal{D}_w^\dagger[r] \longrightarrow 0.$$

where u is a generator of the ideal $\ker w$ in R. The long exact sequence of cohomology with compact support attached to this short exact sequence is

$$0 \to H_c^1(\Gamma, \mathcal{D}^\dagger[r](R)) \xrightarrow{\times u} H_c^1(\Gamma, \mathcal{D}^\dagger[r](R)) \to H_c^1(\Gamma, D_w^\dagger[r]) \to H_c^2(\Gamma, \mathcal{D}^\dagger[r](R)) \xrightarrow{\times u} H_c^2(\Gamma, \mathcal{D}^\dagger[r](R))$$

Using Poincaré duality for Γ, which is functorial and compatible with the Hecke operators in $\mathcal{H}$ we see that the last morphism is the same as $H_0(\Gamma, \mathcal{D}^\dagger[r](R)) \xrightarrow{\times u} H_0(\Gamma, \mathcal{D}^\dagger[r](R))$. The kernel A of this map is, by the lemma above, 0 excepted when κ is trivial in which case it is $\mathbb{Q}_p$. Using the functorial isomorphism $\mathrm{Symb}_\Gamma(V) = H_c^1(Y(\Gamma), V)$ of Ash-Stevens ([**AS**]), we get an exact sequence

$$0 \to \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger[r](R)) \xrightarrow{\times u} \mathrm{Symb}_\Gamma(\mathcal{D}^\dagger[r](R)) \to \mathrm{Symb}_\Gamma(\mathcal{D}_w^\dagger[r]) \to A \to 0,$$

and the result follows. $\qquad \square$

LEMMA AND DEFINITION IV.1.18. *The image of the map* (78) *is independent of the choice of the affinoid W containing w. We call it* $\mathrm{Symb}_\Gamma(\mathcal{D}_w^\dagger[r])_g$.

Proof — There is nothing to prove in the case $w \neq 0$. When $w = 0$, consider the image for an affinoid W and for another affiinod W'. If $W \subset W'$, the image for the former is included into the image of the latter, and since both have codimension 1 in $\mathrm{Symb}_\Gamma(\mathcal{D}_w^\dagger[r])$, they are equal. In general, consider a affinoid $W'' \subset W \cap W'$. $\square$

The g should make think of "global", as $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w)_g$ is the space of modular symbols in $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w)$ that comes by specialization from "global" modular symbols, i.e. defined over W.

We want to determine the action of $\mathcal{H}$ and ι on the line $\mathrm{Symb}_\Gamma(\mathcal{D}_0^\dagger[r])/\mathrm{Symb}_\Gamma(\mathcal{D}_0^\dagger[r])_g$. We shall not be able to do that before the next section, but as a preparation:

DEFINITION IV.1.19. *The system of eigenvalues of E_2^{crit} is the character $\mathcal{H} \to L$ (L any ring) that sends T_l to $1 + l$, U_p to p, $\langle a \rangle$ to 1.*

LEMMA IV.1.20. *There exists a non-zero boundary modular symbol $\Phi_{E_2^{\mathrm{crit}}}$ in $\mathrm{BSymb}_\Gamma(\mathcal{D}_0[r](\mathbb{Q}_p)) \subset \mathrm{Symb}_\Gamma(\mathcal{D}_0[r](\mathbb{Q}_p))$ (for any $r > 0$) which is an eigenvector for $\mathcal{H}$ with the system of eigenvalues of E_2^{crit} and for ι with sign -1.*

Proof — We can assume $\Gamma = \Gamma_0(p)$. We will define $\Phi_{E_2^{\mathrm{crit}}}$ as the restriction to Δ_0 of an element Φ of $\mathrm{Hom}(\Delta, \mathcal{D}_0[r](\mathbb{Q}_p))^\Gamma$. Since every cusp is $\Gamma_0(p)$-equivalent to 0 or ∞, in order to define ϕ it is sufficient (Lemma **??**) to give two distributions $\Phi(0)$ and $\Phi(\infty)$ in $\mathcal{D}_0[r]$ invariant by the stabilizers of 0 and ∞ in $\Gamma_0(p)$ respectively. We set $\Phi(\infty) = 0$, and $\Phi(0)(f) = f'(0)$ for all $f \in \mathbb{A}[r](\mathbb{Q}_p)$. Checking the required invariance of the latter distribution amounts to checking that $f(z)$ and $f(z/(1 + pz))$ have the same derivative at 0 for every test function $f \in \mathbb{A}[r](\mathbb{Q}_p)$, which is obvious. To see that Φ is an eigenvector for $\mathcal{H}$ we compute for example: $\Phi_{|U_p}(0) = \sum_{a=0}^{p-1} \Phi(a/p) \begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix}$ and in this sum, only the term $a = 0$ matters since for $a \neq 0$, a/p is the $\Gamma_0(p)$-class of ∞. So $\Phi_{|U_p}(0)(f(z)) = \left(\frac{df(pz)}{dz} \right)_{|z=0} = pf'(0) = p\Phi(0)(f)$.

On the other hand $\Phi_{|U_p}(\infty) = 0$ since $\begin{pmatrix} 1 & a \\ 0 & p \end{pmatrix} \cdot \infty = \infty$ for all a. Hence we see that $\Phi_{|U_p} = p\Phi$. A similar computation, left to the reader, shows that $\Phi_{|T_l} = (1 + l)\Phi$, and that $\Phi_{|\langle a \rangle} = \Phi$. Finally, to compute the sign of Φ, we just observe that $\Phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(0)(f(z)) = \Phi(0)(f(-z)) = \left(\frac{df(-z)}{dz} \right)_{|z=0} = -f'(0)$, and $\Phi \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}(\infty) = 0$ so $\Phi_{|\iota} = -\Phi$. $\square$

IV.1.4. Construction. We fix as above $\Gamma = \Gamma_1(N) \cap \Gamma_0(p)$ (with $p \nmid N$).

We are ready to apply the eigenveriety machine in order to construct the eigencurve of modular symbols. Actually we shall not construct only one eigencurves, but two, denoted $\mathcal{C}^+$, $\mathcal{C}^-$. To unify the two construction, let $\pm$ mean either $+$ or $-$.

We describe the eigenvariety data that will feed the machine.

(ED1) The ring $\mathcal{H}$ is the ring generated by symbols T_l ($l \nmid Np$), U_p and the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/p\mathbb{Z})^*$.

(ED2) For $\mathcal{W}$ we take the usal weight space defined in §**??**. For $\mathfrak{C}$ we take the following asmissible covering: each connected component of $\mathcal{W}$ is isomorphic, after the choice of a generator of $1 + q\mathbb{Z}_p$, to the open ball of center 1 and radius 1: we define $\mathfrak{C}$ as the set of closed open ball in thsoe open ball of center an integer and radius $\rho \in p^{\mathbb{Q}}$, $\rho < 1$.

(ED3) If $W = \operatorname{Sp} R$ is in $\mathfrak{C}$, we choose an r such that $0 < r < \max(r(W), p)$

$$M_W := \operatorname{Symb}_\Gamma^{\pm}(\mathcal{D}_K[r](R)).$$

Note that M_W is a direct summand of $\operatorname{Symb}_\Gamma^{\pm}(\mathcal{D}_K[r](R))$ hence satisfies property (PR) by Lemma IV.1.3. We have a natural action of the Hecke operators $\psi : \mathcal{H} \to \operatorname{End}_R(M_W)$.

We check the conditions:

(ED1) The action of U_p on M_W is compact by Prop. **??**.

(ED2) If $W' = \operatorname{Sp} R' \subset W = \operatorname{Sp} R$, the module $M_W \hat{\otimes}_R R'$ and $M_{W'}$ are linked by the retsriction theorem (Theorem IV.1.7).

Hence the eigenvariety machine provides us with an eigenvarity $\mathcal{C}^{\pm}$, which is an equidimensional of dimension 1, separated, rigid analytic space over $\mathbb{Q}_p$ with

(i) A rigid $\mathbb{Q}_p$-analytic map $\kappa : \mathcal{C}^{\pm} \to \mathcal{W}$, which is locally finite and flat.

(ii) Analytic functions T_l (for $l \nmid N$), U_p and $\langle a \rangle$ on $\mathcal{C}^{\pm}$ (that is, the analytic function T_l is defined as the image of $T_l \in \mathcal{H}$ by the map $\psi : \mathcal{H} \to \mathcal{O}(\mathcal{C}^{\pm})$).

Note that the flatness of κ does not follow from general properties of eigenvariety, but follows in our case from the fact for $W = \operatorname{Spec} R$ in $\mathfrak{C}$, R is PID, hence any torsion free module is flat over R, in particular the algebra $\mathcal{T}_{W,\nu}$, so $\kappa : \mathcal{C}_{W,\nu} \to W$. Since flat is a local property, the flatness of κ follows.

LEMMA IV.1.21. *The functions T_l (for $l \nmid N$), U_p and $\langle a \rangle$ on $\mathcal{C}^{\pm}$ are power-bounded.*

Proof — It is enough to prove that for any affinoid of the form $\mathcal{C}_{W,\nu} = \operatorname{Sp} \mathcal{T}_{W,\nu}$ of $\mathcal{C}^{\pm}$, the image of the opertors T_l, U_p, etc. in $\mathcal{T}_{W,\nu}$ for one norm of algebra on this Banach algebra is bounded by 1. Let us chose the operator norm on $\mathcal{T}_{W,\nu} \in \operatorname{End}_R(M_{\overline{W}}^{\leq \nu})$ attched to the norm $\| \ \|_r$ on $M_{\overline{W}}^{\leq \nu} = \operatorname{Symb}_\Gamma(\mathcal{D}^*[r](R))$. This is lemma III.6.21. $\qquad \square$

IV.2. Comparison with the Coleman-Mazur eigencurve

We keep the same notation and hypotheses as above. We have constructed two eigencurves $\mathcal{C}^+$ and $\mathcal{C}^-$ with modular symbols and we want to the classical eigencurves of Coleman-Mazur and Buzard $\mathcal{C}$ (the full eigencurve) and $\mathcal{C}^0$ (the cuspidal eigencurve). Those eigencurves are constructed with the same data (ED1), that is the ring $\mathcal{H}$, and (ED2) that is the same weight space $\mathcal{W}$, but with different data (ED3): For $W = \mathrm{Sp}\, R$ a suitable affinoid of the weight space, and for $r > 0$ a sufficiently small real number (w.r.t to W), Coleman defines a Banach R-module of overconvergent modular forms (resp. cuspidal modular forms) $M^\dagger[r](\Gamma, R)$ (resp. $S^\dagger[r](\Gamma, R)$), with action of $\mathcal{H}$, such that U_p acts compactly. Those modules are linked for various r and their formations commute with restriction of affinoids. Moreover, one has specialization $\mathcal{H}$-equivariant isomorphisms, for $k \in \mathbb{Z} \subset W(\mathbb{Q}_p)$

$$(79) \qquad M^\dagger[r](\Gamma, R) \otimes_{R,k} \mathbb{Q}_p \;=\; M^\dagger[r]_{k+2}(\Gamma, \mathbb{Q}_p)$$

$$(80) \qquad S^\dagger[r](\Gamma, R) \otimes_{R,k} \mathbb{Q}_p \;=\; S^\dagger[r]_{k+2}(\Gamma, \mathbb{Q}_p)$$

where for L any finite extension of $\mathbb{Q}_p$, $M^\dagger[r]_{k+2}(\Gamma, L)$ (resp. $S^\dagger[r]_{k+2}(\Gamma, Ł)$ is the space of r-overconvergent modular forms (resp. cuspidal modular forms) with coefficients in L defined by Katz. (Note the unusual shift by 2 on the wight k). Since the $M^\dagger[r]_{k+2}(\Gamma, L)$ (resp. $S^\dagger[r]_{k+2}(\Gamma, L)$ are linked for various r, $M^\dagger[r](\Gamma, \mathbb{Q}_p)^\#$ or $M^\dagger[r](\Gamma, \mathbb{Q}_p)^{\leq \nu}$ (resp. etc.) are independent of r and we remove the $[r]$ from the notations. Furthermore, for $k \geq 0$, we have natural inclusions $M_{k+2}(\Gamma, \mathbb{Q}_p) \subset M^\dagger[r]_{k+2}(\Gamma, \mathbb{Q}_p)$ and $M^\dagger_{k+2}(\Gamma, \mathbb{Q}_p) \subset S^\dagger[r]_{k+2}(\Gamma, \mathbb{Q}_p)$, which induce, according to Coleman's control theorem:

$$(81) \qquad M_{k+2}(\Gamma, \mathbb{Q}_p)^{<k+1} \;=\; M^\dagger_{k+2}(\Gamma, \mathbb{Q}_p)^{<k+1}$$

$$(82) \qquad S_{k+2}(\Gamma, \mathbb{Q}_p)^{<k+1} \;=\; S^\dagger_{k+2}(\Gamma, \mathbb{Q}_p)^{<k+1}$$

The first one is proved in [**Col**] and the second can be easily deduced from the first one and the theory of Eisenstein series (cf. [**B**]). In the construction of $\mathcal{C}$, (resp. $\mathcal{C}^0$), one takes for M_W the module $M[r](\Gamma, R)$ (resp. $S[r](\Gamma, R)$) for some arbitrary sufficiently small $r > 0$.

THEOREM IV.2.1.   (i) *There exist unique closed immersions*

$$\mathcal{C}^0 \hookrightarrow \mathcal{C}^\pm \hookrightarrow \mathcal{C}$$

which are compatible with the weight morphisms κ to $\mathcal{W}$ and the maps $\mathcal{H} \to \mathcal{O}(\mathcal{C}^0), \mathcal{O}(\mathcal{C}^\pm), \mathcal{O}(\mathcal{C})$. Moreover, these eigencurves are all reduced, and $\mathcal{C} = \mathcal{C}^+ \cup \mathcal{C}^-$.

(ii) *For $k \in \mathbb{Z} \subset \mathcal{W}(\mathbb{Q}_p)$, L a finite extension of $\mathbb{Q}_p$ there exist $\mathcal{H}$-injections between the $\mathcal{H}$-modules*

$$S_{k+2}(\Gamma)(L)^{ss,\#} \subset \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))_g^{ss,\#} \subset M^\dagger_{k+2}(\Gamma)^{ss,\#},$$

where ss means semi-simplification as an $\mathcal{H}$-module, and $\mathcal{H}$ (resp. ι) acts on the line $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))/\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))_g$ by the system of eigenvalue of E_2^{crit} (resp. -1).

(iii) *A system of $\mathcal{H}$-eigenvalues of finite slope (that is, with a non-zero U_p-eigenvalue) appears in $\mathrm{Symb}_\Gamma(\mathcal{D}_k)$ if and only if it appears on $M_{k+2}^\dagger(\Gamma)$ except when $N = 1$ (that is, $\Gamma = \Gamma_0(p)$ for the system of eigenvalues of E_2^{crit}) which appears in $\mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_0)$ but not in $M_2^\dagger(\Gamma_0(p))$.*

Proof —

We shall aplly Theorem II.5.6 repeatedly.

For $x \in \mathcal{W}(L)$, we shall denote by $(M_x^0)^\#$, resp. $(M_x^\pm)^\#$, resp. $(M_x)^\#$, the finite slope part of the fiber at x of the eigenverity data used to construct $\mathcal{C}^0$, $\mathcal{C}^\pm$, $\mathcal{C}^-$ respectively. Hence we have, if $x = k \in \mathbb{Z}$, $(M_k^0)^\# = S_{k+2}^\dagger(\Gamma, \mathbb{Q}_p)^\#$ and $(M_k)^\# = M_{k+2}^\dagger(\Gamma, \mathbb{Q}_p)^\#$ by (80) and (79), while $(M_k^\pm)^\# = \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(\mathbb{Q}_p))_g$ by definition.

We know define classical structures for the eigenverity data used to construct $\mathcal{C}^0$, $\mathcal{C}^\pm$, $\mathcal{C}^-$. In all cases we take for $X = \mathbb{N} \subset \mathcal{W}$ (CSD1) the set of integers $k \geq 0$. It is very Zariski-dense. For $x = k \in \mathbb{N}$, we take for $(M_k^0)^{\mathrm{cl}}$, resp. $(M_k)^{\mathrm{cl}}$ the finite dimensional $\mathcal{H}$-modules $S_{k+2}(\Gamma, \mathbb{Q}_p)^\#$, resp. $M_{k+2}(\Gamma, \mathbb{Q}_p)^\#$, while for $(M_k^\pm)^{\mathrm{cl}}$ we take $\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{Q}_p))^\#$ which defines the data (CSD2). As $\mathcal{H}$-modules $(M_k^0)^{\mathrm{cl}}$ and $(M_k)^{\mathrm{cl}}$ (resp. $(M_k^\pm)^{\mathrm{cl}}$ are sub-modules (resp. quotient) of $(M_k^0)^\#$ and $(M_k)^\#$ (resp. $(M_k^\pm)^\#$); after semi-simplification, quotients become sub-modules as well, hence condition (CSC1) is satisfied. For (CSC2), fix a $\nu \in \mathbb{R}$. The set of $k \in X$ such that there exists an $\mathcal{H}$-isomorphism $(M_k^*)^{\mathrm{cl}, \leq \nu} \simeq (M_k^*)^{\leq \nu}$ contains all the k such that $k + 1 > \nu$ either by Coleman's or by Stevens' control Theorem, hence condition (CSC2) is clearly satisfied in all cases.

Having defined those classical structures, we check the hypothesis of Theorem **??**: We observe that we have $\mathcal{H}$-equivariant map $S_{k+2}(\Gamma, \mathbb{Q}_p) \hookrightarrow \mathrm{Symb}_\Gamma^\pm(\mathcal{V}_k(\mathbb{Q}_p)) \hookrightarrow M_{k+2}(\Gamma, \mathbb{Q}_p)$. Hence four applications of Theorem II.5.5 shows that there are unique closed immersions

$$\mathcal{C}^0 \hookrightarrow \mathcal{C}^\pm \hookrightarrow \mathcal{C}$$

compatible with the map to $\mathcal{W}$ and from $\mathcal{H}$. Moreover it gives us $\mathcal{H}$-equivariant inclusion

$$S_{k+2}(\Gamma)(L)^{\mathrm{ss}, \#} \subset \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))_g^{\mathrm{ss}, \#} \subset M_{k+2}^\dagger(\Gamma)^{\mathrm{ss}, \#}.$$

That $\mathcal{C}$ is the union of $\mathcal{C}^+$ and $\mathcal{C}^-$ is a consequence of Exercise II.4.2 since we have $M_{k+2}(\Gamma, \mathbb{Q}_p) \hookrightarrow \mathrm{Symb}_\Gamma^+(\mathcal{V}_k(\mathbb{Q}_p)) \oplus \mathrm{Symb}_\Gamma^-(\mathcal{V}_k(\mathbb{Q}_p)) = \mathrm{Symb}_\Gamma(\mathcal{V}_k(\mathbb{Q}_p))$.

We know that $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))_g^{\mathrm{ss}, \#}$ is the same as $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_k(L))^{\mathrm{ss}, \#}$ except perhaps for $k = 0$ and at most one value of the sign $\pm$, where it might be of codimension one. Let us assume that $N = 1$, that is $\Gamma = \Gamma_0(p)$ for a minute: we know by Lemma IV.1.20 that the system of eigenvalues of E_2^{crit} appears in

$\mathrm{Symb}^-_{\Gamma_0(p)}(\mathcal{D}_0(L))^{\mathrm{ss},\#}$, but it does not appear in $\mathrm{Symb}^-_{\Gamma_0(p)}(\mathcal{D}_0(L))^{\mathrm{ss},\#}_g \subset M^\dagger_2(\Gamma_0(p))^{\mathrm{ss},\#}$ by the main result of [**CGJ**]. Hence, we see that $\mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_0(L))/\mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_0(L))_g$ is a line on which $\mathcal{H}$ acts by the system of eigenvalues of E^{crit}_2 and ι acts by -1. Now, for $N > 1$, we have a natural inclusion $\mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_0(L))/\mathrm{Symb}_{\Gamma_0(p)}(\mathcal{D}_0(L))_g \subset$ $\mathrm{Symb}_\Gamma(\mathcal{D}_0(L))/\mathrm{Symb}_\Gamma(\mathcal{D}_0(L))_g$ which is an equality since both spaces have dimension 1. Note however that the system of eigenvalues od E^{crit}_2 does appear in $\mathrm{Symb}^\pm_\Gamma(\mathcal{D}_0(L))$ for both values of $\pm$ when $N > 1$ because for any prime factor l of N the critical refinement $E^{\mathrm{crit}}_{2,l}$ of the new exceptional Eisenstein series $E_{2,l}$ belongs to $S^\dagger_2(\Gamma, L)^{\mathrm{ss},\#} \subset \mathrm{Symb}^\pm_\Gamma(\mathcal{D}_0(L))$. This completes the proof of (ii) and (iii).

It remains to show that the eigencurves are reduced. It suffices to do that for $\mathcal{C}$. Let us check the hypothesis of Theorem II.5.5. FIx $\nu \in \mathbb{R}$. For $k \in X$ such that $k > 2\nu - 1$, we claim that $M_{k+2}(\Gamma, \mathbb{Q}_p)^{\leq\nu}$ is semi-simple as an $\mathcal{H}$-module. All the Hecke operators in $\mathcal{H}$ excepted U_p are normal, hence semi-simple. It is enough to prove that U_p acts semi-simply on every generalized $\mathcal{H}$-eigenspace of $M_{k+2}(\Gamma)^{\leq\nu}$. For a generalized eigenspace which is new at p, this results from Atkin-Lehner's theory. Consider therefore a generalized eigenspace $M_{k+2}(\Gamma)^{\leq\nu}_{(f)}$ which is old at p, corresponding to a newform f for $\Gamma_1(N')$ for some N' dividing N (possibly $N' = N$), of T_p-eigenvalue a_p and nebentypus ϵ. Since $X^2 - a_pX + p^{k+1}\epsilon(p)$ has a unique root α such that $v_p(\alpha) \leq \nu$ (because $\nu < (k+1)/2$), we have that $g \mapsto g_\alpha$ is an isomorphism $M_{k+2}(\Gamma_1(N))_{(f)} \simeq M_{k+2}(\Gamma)^{\leq\nu}_{(f)}$, and that U_p acts by the scalar α on our generalized eigenspace. This complete the proof of the claim. Since the set of $k > 2\nu - 1$ is obviously Zariski-dense in any ball of center k' and some radius, we can apply Theorem II.5.5 which tells us that $\mathcal{C}$ is reduced.

$\square$

REMARK IV.2.2. As we saw in the proof, the spaces of classical modular symbols $\mathrm{Symb}_\Gamma(\mathcal{V}_k(L))$ tend to be semi-simple as $\mathcal{H}$-module, except perhaps for the operator U_p on the subspace of old symbols of slope $(k+1)/2$ – and even this exception is conjectured not to happen (and known not to happen when $k = 2$).

In contrast, the space of overconvergent modular symbols $\mathrm{Symb}^\pm_\Gamma(\mathcal{D}_k(L))^{\leq\nu}$ are in general not semi-simple as $\mathcal{H}$-module (except of course of $\nu < k+1$ since in this case they are isomorphic to submodule of the module of classical modular symbols). This non semi-simplicity happens already for $\nu = k+1$, as we shall see.

Similar remarks can be made with modular symbols replaced by modular forms.

It is therefore a natural question whether we can remove the semi-simplification in (ii) of the theorem, that is roughly speaking if spaces of overconvergent modular symbols and corresponding modules of over convergent modular forms are isomorphic as $\mathcal{H}$-module, and not only after semi-simplification. This question is open and I have personally no opinion whether the answer is yes or no. (Fabrizio Andreatta, reporting on a joint work with Glenn Stevens and Adrian Iovita in Goa in August 2010 stated this question as a conjecture). We shall see below that this question

has a positive answer at most classical points (of slope $k+1$, otherwise this is trivial), in the sense that $S^\dagger_{k+2}(\Gamma))_{(f)} \simeq \mathrm{Symb}^\pm_\Gamma(\mathcal{D}_k))_{(f)}$ as $\mathcal{H}$-module for f any decent classical eigenform of weight $k + 2 \geq 2$ and slope $k + 1$. (Perhaps surprisingly, it is not necessary to assume f cuspidal here).

IV.3. Points of the eigencurve

We keep the notations of the preceding section.

IV.3.1. Interpretations of the points as systems of eigenvalues of overconvergent modular symbols.
Let $x \in \mathcal{C}^\pm(L)$, L any finite extension of $\mathbb{Q}_p$. To x we can attach a character $\lambda_x = \mathcal{H} \to L$, $\lambda_x(T) = \psi(T)(x)$ for $T \in \mathcal{H}$, that is what we have called a *system of eigenvalues of $\mathcal{H}$*

THEOREM IV.3.1. *The map $x \mapsto \lambda_x$ is a bijection from $\mathcal{C}^\pm(L)$ to the set of all system of eigenvalues appearing in a space $\mathrm{Symb}^\pm_\Gamma(\mathcal{D}_w(L))^\#$ for some $w \in \mathcal{W}(L)$ (minus the system of eigenvalue of E_2^{crit} if it belongs to that set).*

Proof —   This is just Theorem **??** using (ii) of Theorem **??**).   □

IV.3.2. Very classical points.

DEFINITION IV.3.2. Let L be a finite extension of $\mathbb{Q}_p$. An L-point x in $\mathcal{C}^\pm(L)$ is called *very classical*[2] of weight $k \in \mathbb{N}$ if $\kappa(x) = k$ and the system of eigenvalues λ_x appears in $\mathrm{Symb}^\pm_\Gamma(\mathcal{V}_k(L))$.

A point x in $\mathcal{C}^\pm$ is said *very classical* if it is very classical at a point of $\mathcal{C}^\pm(L(x))$ where $L(x)$ is the field of definition of x. We shall denote by $\lambda_{x,0}$ the restriction of λ_x to $\mathcal{H}_0$ (that is forgetting U_p. We define the *minimal level* of x as the minimal level $M_0 \mid Np$ of $\lambda_{x,0}$ in $\mathrm{Symb}^\pm_\Gamma(\mathcal{V}_k(L))$, when $\lambda_{x,0}$ is not the system of eigenvalues of E_2, and by convention as being p when $\lambda_{x,0}$ is the system of eigenvalues of E_2.

In particular, we say that x is *p-new* if that minimal level is divisible by p, *p-old* if it is not.

PROPOSITION IV.3.3. *The set of very classical points in $\mathcal{C}^\pm$ is very Zariski-dense.*

Proof —   First we prove that this set is Zariski-dense. It is enough to show (by Lemma II.4.4) that very classical points are Zariski-dense in any irreducible component D of $\mathcal{C}^\pm$. By Prop. II.4.6(iii), $\kappa(D)$ is Zariski open in $\mathcal{W}$, hence there is a $k \in \mathbb{N}$ that belongs to $\kappa(D)$. Let us fix $x \in D$ such that $\kappa(D) = x$ and $\nu \in \mathbb{R}$ such that $\nu > v_p(U_p(x))$. There is an affinoid $W = \mathrm{Sp}\,R$ in $\mathfrak{C}$ containing k such that (W, ν) is adapted, $x \in \mathcal{C}_{W, \leq \nu}$, and $Z := \{n \in \mathbb{N}, n > \nu - 1\} \cap W$ is Zariski dense in W. By Steven's control theorem, every point in $\mathcal{C}_{W, \leq \nu}$ with weight in Z is very

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[2]French *très classique.* This terminology seems to be due to Chenevier.

classical. It follows, by the following lemma, hence very classical points are dense in $\mathcal{C}_{W,\leq\nu}$, hence in $\mathcal{C}_{W,\leq\nu} \cap D$ because it is a union of irreducible components of $\mathcal{C}_{W,\leq\nu}$ (Lemma II.4.4). Since $\mathcal{C}_{W,\leq\nu} \cap D$ is a non-empty affinoid (it contains x) of the irreducible space D, it is Zariski-dense in D (Lemma II.4.4), and this completes the proof that Z is Zariski-dense. Actually this also shows that very classical points are very Zariski-dense, because for x a very classical point of weight k, the above argument proved an affinoid neighborhood of x on which very classical points are Zariski-dense. $\square$

LEMMA IV.3.4. *Let $f :\ X \to Y$ be a finite flat morphism of noetherian scheme, and $Z \subset Y$ a Zarsiki-dense subset. Then $f^{-1}(Z)$ is Zariski-dense in X.*

Proof — Since f is finite flat, and the schemes notherian, f maps irreduciblie compoeent X_i of X surjectively to irreducible component $Y_{j(i)}$ of Y, and the generic point of $X_{(i)}$ is the only point of $X_{(i)}$ that maps to the generic point of $Y_{j(i)}$. In particular, one may assume that X and Y are irreducible. Let T be the Zariski-closure of $f^{-1}(Z)$. Since f is closed, $f(T)$ is closed in Y, but it also contains Z which is dense, so $f(T) = Y$. hence, there is a point in T that maps to the generic of Y, and this point necessarily is the generic point of X. Since T is closed, $T = X$. $\square$

LEMMA IV.3.5. *If x is a p-new very classical point, with $\kappa(x) = k \in \mathbb{Z}$ then $v_p(U_p(x)) = k/2$.*

Proof — By Theorem III.2.39, the system of eigenvalues λ_x appear in $M_{k+2}(\Gamma, L)$ but not in $M_{k+2}(\Gamma_1(N), L)$. This system is either cuspidal or Eisenstein. If it is cuspidal, the result folllows from [**Mi**, Theorem 4.6.17/2]. If it is Eisenstein, then according to the classification of new Eisenstein series (see §I.6.3) the system has to be the one of the exceptional Eisenstein series $E_{2,p}$ (since the p-order of the level od normal new Eisenstein form $E_{k,\psi,\tau}$ with trivial nebentypus at a prime p is always even, because the level of the form is the product of the conductors of the two primitive Dirichlet character ψ and τ, while the nebentypus is $\psi\tau$.) In this case, ve have $v_p(U_p(x)) = 0$ (exceptional new Eisenstein series are ordinary), and $k = 0$, hence the result. $\square$

PROPOSITION IV.3.6. *The set of very classical p-old points in $\mathcal{C}^{\pm}$ is very Zariski-dense.*

Proof — The proof is exactly the same as the proof of Proposition IV.3.3, just replacing the set $Z = \{n \in \mathbb{N}, n > \nu - 1\} \cap W$ by its subset $Z' = \{n \in \mathbb{N}, n > \nu - 1\ n > \nu/2\} \cap W$: the lemma above ensures that a point in $\mathcal{C}_{W,\nu}$ with weight in Z' is not only very classical, but also p-old. $\square$

IV.3.3. Classical points.

DEFINITION IV.3.7. Let L be a complete field containing $\mathbb{Q}_p$. An L-point x in $\mathcal{C}^{\pm}(L)$ is said *classical* if the system of eigenvalues λ_x appears in $\mathrm{Symb}^{\pm}_{\Gamma_1(M)}(\mathcal{V}_k(L))$ for some $k \in \mathbb{N}$, $M \in \mathbb{N}$, $M \geq 1$.

Similar definition for a point in $\mathcal{C}^{\pm}$. It is not hard to see, using Theorem III.2.39 and the Hecke estimate on coefficients of modular forms, that the integer k is uniquely determined by x. We shall call it the *weight* of the classical point x, but not without warning the reader that it is not part of the definition, nor, as we shall see, always true that $\kappa(x) = k$. Also, as for very classical points, we define the minimal level M_0 of x as the minimal level of the system of eigenvalues $\lambda_{x,0}$ of $\mathcal{H}_0$ in $\mathrm{Symb}^{\pm}_{\Gamma_1(M)}(\mathcal{V}_k(L))$. Writing $M_0 = N_0 p^{t_0}$ with $p \nmid N_0$, we call N_0 the *minimal tame level* and p^{t_0} the *minimal wild level*.

While the very classical points appear on the eigencurve, in a sense, *by construction*, it is not clear a priori that there are any classical points on the eigencurve beyond the very classical points. The aim of this § is to produce a good supply of classical points. Later, with help of the Galois representations, we shall prove that this supply together with the very classical points exhausts all classical points.

LEMMA IV.3.8. *Let L be a finite extension of $\mathbb{Q}_p$, W an L-vector space with a structure of right $S_0(p)$-module.*

    (i) *If $w \in W^{\Gamma_0(p^t) \cap \Gamma_1(N)}$ with $p \nmid N$, for some $t \geq 1$, and $U_p w = aw$ with $a \neq 0$, then $c \in W^{\Gamma_0(p) \cap \Gamma_1(N)}$.*

    (ii) *Let $w \in W^{\Gamma_1(p^t N)} - W^{\Gamma_1(p^{t-1}M)}$ with $p \nmid N$, for some $t \geq 1$, and $U_p w = aW$, the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/p^t\mathbb{Z})^*$ acting on w through a character ϵ acts on w. Then $a \neq 0$ if and and only if ϵ is primitive.*

Proof —   An easy matrix computation (cf. e.g. [**Mi**, Lemma 4.5.11]) show that for $t > 1$, $(\Gamma_0(p^t) \cap \Gamma_1(N)) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} (\Gamma_0(p^t) \cap \Gamma_1(N)) = (\Gamma_0(p^t) \cap \Gamma_1(N)) \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix} (\Gamma_0(p^{t-1}) \cap \Gamma_1(N))$. Since the first double class describes the action of U_p, one sees that $aw = U_p w \in W^{\Gamma_0(p^{t-1}) \cap \Gamma_1(N)}$, hence w is in this space, so (i) follows by induction on t. For (ii) see [**Mi**, theorem 4.6.17]   □

Now let us fix an integer $k \geq 0$, an integer $t \geq 1$, and a Dirichlet character $\epsilon : (\mathbb{Z}/p^t\mathbb{Z})^* \to L^*$, where L is some finite extension of $\mathbb{Q}_p$. We see ϵ as an element of $\mathcal{W}(L)$ by precomposing it with the canonical surjection $\mathbb{Z}_p^* \to (\mathbb{Z}/p^t\mathbb{Z})^*$. Let us call $w \in \mathcal{W}(L)$ the character $z \mapsto z^k \epsilon(k)$. We clearly have $r(\epsilon) = r(w) = p^{-\nu}$ (see Definition III.5.5) if p^ν is the conductor of ϵ. For $0 < r < r(\epsilon)$, consider the inclusion $\mathcal{P}_k[r](L) \subset \mathcal{A}_w[r](L)$. An easy but important observation is that the left-action of $S_0(p)$ on $\mathcal{A}_w[r]$ leaves stable $\mathcal{P}_k[r]$: this is because, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in S_0(p)$, and P in $\mathcal{P}_k[r]$, $P_{|w\gamma}(z) = \epsilon(a - cz) P_{|k\gamma}(z)$ and $\epsilon(a - cz) = \epsilon(a)\epsilon(1 - \frac{c}{a}z)$ is constant on

every closed ball of radius r since $p | \frac{c}{a}$. We denote by $\mathcal{P}_w[r](L)$ the space $\mathcal{P}_k[r](L)$ with the action of $S_0(p)$ induced by its action on $\mathcal{A}_w[r]$, and by $\mathcal{V}_w[r](L)$ its L-dual with its dual right $S_0(p)$-action. Hence we get a surjective $S_0(p)$-equivariant map $\rho_w : \mathcal{D}_w[r](L) \to \mathcal{V}_w[r](L)$. The results of the following proposition are proved exactly as in the case $\epsilon = 1$ (cf. Proposition [**?**] and Theorem III.6.36). We leave the details to the reader.

PROPOSITION IV.3.9. *Let N be any positive integer not divisible by p. The map*

$$\rho_w : \mathrm{Symb}_\Gamma(\mathcal{D}_w[r](L))^\# \to \mathrm{Symb}_\Gamma(\mathcal{V}_w[r](L))^\#$$

is surjective, and induces an isomorphism

$$\rho_w : \mathrm{Symb}_\Gamma(\mathcal{D}_w[r](L))^{<k+1} \to \mathrm{Symb}_\Gamma(\mathcal{V}_w[r](L))^{<k+1}.$$

PROPOSITION IV.3.10. *We have a natural isomorphism, compatible with the action of $\mathcal{H}$*

$$\mathrm{Symb}_\Gamma(\mathcal{V}_w[r](L))^\# = \mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\epsilon]^\#.$$

Proof — First, by Lemma IV.3.8(i), we have

$$\mathrm{Symb}_\Gamma(\mathcal{V}_w[r](L))^\# = \mathrm{Symb}_{\Gamma_0(p^t) \cap \Gamma_1(N)}(\mathcal{V}_w[r](L))^\#.$$

Next, observe that the action of $\Gamma_1(Np^t)$ on $\mathcal{V}_w[r](L)$ is the same as its action on $\mathcal{V}_k[r](L)$ since $\epsilon(a - cz) = 1$ for $|z| \le 1$ and $a \equiv 1 \pmod{p^t}$, $p^t \mid c$. Hence an obvious inclusion map

$$\mathrm{Symb}_{\Gamma_0(p^t) \cap \Gamma_1(N)}(\mathcal{V}_w[r](L)) \subset \mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k[r](L)).$$

The right hand side has an action of the diamonds operators $\langle a \rangle$ for $a \in (\mathbb{Z}/p^t\mathbb{Z})^*$, and one sees easily that the above inclusion induces an equality

(83) $$\mathrm{Symb}_{\Gamma_0(p^t) \cap \Gamma_1(N)}(\mathcal{V}_w[r](L)) = \mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k[r](L))[\epsilon].$$

Finally, $\mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k[r](L))^\# = \mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))^\#$ by a trivial generalization of Prop. III.6.28. The proposition follows. □

PROPOSITION IV.3.11. *There exists a natural $\mathcal{H}$-equivariant surjective map*

$$\mathrm{Symb}_\Gamma(\mathcal{D}_w(L))^\# \to \mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\epsilon]^\#$$

which induces an isomorphism

$$\mathrm{Symb}_\Gamma(\mathcal{D}_w(L))^{<k+1} \to \mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\epsilon]^{<k+1}$$

Proof — We get this map by combining the two propositions above, using that $\mathrm{Symb}_\Gamma(\mathcal{D}_w[r](L))^\# = \mathrm{Symb}_\Gamma(\mathcal{D}_w(L))^\#$ (Theorem III.6.26).

The surjectivity and bijectivity properties of the above defined map directly follows from the same property for ρ_w (Prop. **??**). □

PROPOSITION AND DEFINITION IV.3.12. Let L be a finite extension of $\mathbb{Q}_p$. For $t \geq 1$, and ϵ a primitive character of $(\mathbb{Z}/p^t\mathbb{Z})^* \to L^*$, every system of $\mathcal{H}$-eigenvalues λ appearing in $\mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\epsilon]^{\#}$ defines a unique point $x \in \mathcal{C}(L)$ such that $\lambda_x = \lambda$ and $\kappa(x)$ is the character $z \mapsto z^k\epsilon(z)$. This point x is classical of weight k and level Np^t. We shall call those classical points *Hida classical points.* In other words a point of the eiegncurve is a Hida classical point if $\kappa(x)$ has the form $z \mapsto z^k\epsilon(z)$ for a (clearly unique) integer k and a (clearly unique) non-trivial finite order character ϵ, and its system of eigenvalues λ_x appears in $\mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\epsilon]^{\#}$ for k and ϵ determined by $\kappa(x)$ as above, and for the integer $t \geq 1$ such that p^t is the conductor of ϵ.

Proof — The first assertion follows from Prop. IV.3.11. The rest is obvious. □

REMARK IV.3.13. The reader should note that for a Hida classical point x of weight k and level Np^t, one does **not** have $\kappa(x) = k$. Instead, $\kappa(x)$ is the character $z \mapsto z^k\epsilon(z)$, where ϵ is a primitive character of conductor p^t.

PROPOSITION IV.3.14. *Let $x \in \mathcal{C}(L)$ be a point such that $\kappa(x)$ is of the form $z \mapsto z^k\epsilon(z)$ with ϵ a finite order character, and assume that $v_p(U_p(x)) < k + 1$. Then x is either a very classical point or a Hida classical point. In particular, it is a classical point.*

Proof — If $\epsilon = 1$, then we already know that x is very classical by Theorem III.6.36. If $\epsilon \neq 1$, it is Hida classical by Prop. IV.3.11. □

REMARK IV.3.15. At this stage, it is not clear that every classical point is either a Hida classical point or a very classical point. We don't even know that there are no classical points whose minimal tame level is not a divisor of N, or such that $\kappa(x)$ is not of the form $z \mapsto z^k\epsilon(z)$ for $k \in \mathbb{N}$ and ϵ a finite order character. With the help of the family of Galois representations carried by the eigencurve, it will be easy to prove these results: see below.

IV.4. The family of Galois representations carried by $\mathcal{C}^{\pm}$

THEOREM IV.4.1. *There exists a unique continuous pseudocharacter*

$$\tau : G_{\mathbb{Q},Np} \to \mathcal{O}(\mathcal{C}^{\pm})$$

of dimension 2 such that for every l prime to Np,

$$\tau(\mathrm{Frob}_l) = T_l.$$

Moreover we have $\tau(c) = 0$ where c is a complex conjugation in $G_{\mathbb{Q},Np}$.

Note that if L is any finite extension of $\mathbb{Q}_p$, and $x \in \mathcal{C}^{\pm}(L)$, we can compose τ with the *evaluation at x* morphism $\mathrm{ev}_x : \mathcal{O}(\mathcal{C}) \to L$, getting a continuous pseudocharacter of dimension 2:

$$\tau_x = \mathrm{ev}_x \circ \tau : G_{\mathbb{Q},Np} \to L.$$

By Taylor's theorem, over a suitable finite extension L' of L, τ_x is the trace of a unique (up to isomorphim) semi-simple continuous representation

$$\rho_x : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(L').$$

However, in this situation one can show that ρ_x is defined over L. Indeed, the theory of descent of representations (cf.[**?**] or [**Ro**]) shows that there exist a central simple algebra D of dimension 4 of L, and a representation $G_{\mathbb{Q}_p} \to D^*$ whose reduced trace is T_x. The image c' of c by this map is an element of D of square 1 and reduced trace 0. If D is a division algebra, the L-algebra generated by c' in D is a field L', and since $c'^2 = 1$ in the fie;d L', $c = \pm 1$ and $L' = L$. But the reduced trace of an element c' of L is $2c'$ and we get $c' = 0$, a contradiction. Hence D is a matrix algebra $M_2(L)$. Hence:

COROLLARY IV.4.2. *For every $x \in \mathcal{C}^{\pm}(L)$, there exists a unique continuous semi-simple Galois representation*

$$\rho_x : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\bar{L})$$

such that for every prime l not dividing Np,

$$\mathrm{tr}\, \rho_x(\mathrm{Frob}_l) = T_l(x).$$

Proof — We know prove Theorem IV.4.1 by an abstract ane beautiful interpolation argument due to Chenevier. Suppose we have a Zariski-dense set of points $Z \in \mathcal{C}$ such that for $z \in Z$ there exists a Galois representation $\rho_z : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\bar{\mathbb{Q}}_p)$ satisfying $\mathrm{tr}\, \rho_z(\mathrm{Frob}_l) = T_l(z)$. for all l not dividing Np. Consider the map

$$\mathrm{ev}_Z : \mathcal{O}(\mathcal{C})^0 \overset{\prod_{z \in Z} \mathrm{ev}_z}{\longrightarrow} \prod_{z \in Z} \bar{\mathbb{Q}}_p.$$

This is obviously a continuous morphism of algebra for the product topology on the target; it is injective by density of Z and reducedness of $\mathcal{C}_{W,\leq\nu}$; and since $\mathcal{O}(\mathcal{C})^0$ (the set of power-bounded elements in $\mathcal{T}_{W,\leq\nu}^{\pm}$) is compact by Prop. II.4.8, the range of ev_Z is compact (hence closed in $\prod_{z \in Z} \bar{\mathbb{Q}}_p$) and ev_Z is an homeomorphsim of its source over its image.

Now consider the map

$$\tau_Z : G_{\mathbb{Q},Np} \to \prod_{z \in Z} \bar{\mathbb{Q}}_p.$$

This is obviously a continuous pseudocharacter of dimension 2. We have $\tau_Z(\mathrm{Frob}_l) = (T_l(z))_{z \in Z} = \mathrm{ev}_Z(T_l)$ for $l \nmid Np$, hence $\tau_Z(\mathrm{Frob}_l)$ lies in the image of ev_Z. Since the Frob_l are dense in $G_{\mathbb{Q},Np}$, τ_Z is continous, and the image of ev_Z is closed,

we deduce that $\tau_Z(G_{\mathbb{Q},Np})$ entirely lies in the image of ev_Z. We can this define $\tau = \mathrm{ev}_Z^{-1} \circ \tau_Z : G_{\mathbb{Q},Np} \to \mathcal{T}^{\pm}$, which is a continuous pseudocharacter of dimension 2. Moreover $\mathrm{ev}_Z(\tau(\mathrm{Frob}_l)) = \tau_Z(\mathrm{Frob}_l) = (T_l(z))_{z \in Z}$, while $\mathrm{ev}_Z(T_l) = (T_l(z))_{z \in Z}$. Hence $\tau(\mathrm{Frob}_l) = T_l$ by injectivity of ev_Z.

To apply the above, take for Z the set of very classical points in $\mathcal{C}^{\pm}$. The existence of a suitable ρ_z for $z \in Z$ is known by Eichler-Shimura (in weight $k+2 = 2$) and Deligne (in weight $k + 2 > 2$) and the density of Z is Prop **??**. $\qquad\square$

We shall prove some important local properties of the Galois representations ρ_x. Before stating them, a few notations and reminders: For any prime l, we call $G_{\mathbb{Q}_l}$ the absolute Galois group of $\mathbb{Q}_l$. We have a natural $G_{\mathbb{Q},Np}$-conjugacy class of maps $i_l : G_{\mathbb{Q}_l} \to G_{\mathbb{Q},Np}$ whose image is a decomposition group at l (these maps are injective if $N > 1$ and $l \mid Np$ according to [**?**], but we shall not use this beautiful result) and for ρ a representation of $G_{\mathbb{Q},Np}$ we denote by $\rho_{|G_{\mathbb{Q}_l}}$ the composition $\rho \circ i_l$ which is a a representation well-defined up to isomorphism. We shall need the rather subtle notion of tame conductor $N(\rho)$ of a representation ρ, due to Serre (using work of Swan and Grothendieck): It is a product $N(\rho) = \prod_{l \neq p} l^{n_l(\rho_{|G_{\mathbb{Q}_l}})}$, where $n_l(\rho_{|G_{\mathbb{Q}_l}})$, the *conductor exponent* at l is defined in [**?**, §2.1] or in a different, but equivalent, way in [**Li**, §1]. We have $n_l(\rho) = 0$ if and only if ρ is unramified at l. We recall the following facts about conductors:

LEMMA IV.4.3.   (i) *Let $T : G_{\mathbb{Q},Np} \to \mathcal{O}(X)$ be a pseudocharacter, where X is a rigid analytic space, and for x in X defined over L denote by ρ_L the semisimple representation over L of trace T_x. Let $M > 1$ be an integer. Then the set of $x \in X$ such that $N(\rho_x)|M$ is Zariski-closed in X.*

  (ii) *Same assumtions as in (i) but assume moreover than T has dimension 2. Let x in X and $\ell|N$. Assume that $(\rho_x)_{|Q_l}$ is not, up to an unramified twist, the direct sum $1 \oplus \omega_p$, where ω_p is the cyclotomic character. Then for y in a suitable neighborhood of x, $n_l(\rho_y)$ is constant, equal to $n_l(\rho_x)$.*

 (iii) *Let M be a free $\mathcal{O}(X)$-module of rank 2 with a continuous action of $G_{\mathbb{Q},Np}$, and let us call ρ_x the (possibly non semi-simple) representation of dimension 2 on the fiber at a point $x \in X$. Assume that $(\rho_x)_{|G_{\mathbb{Q}_l}}$ is not, up to an unramified twist, the direct sum $1 \oplus \omega_p$. Then for y in a suitable neighborhood of x, $n_l(\rho_y)$ is constant, equal to $n_l(\rho_x)$.*

 (iv) *Let f be a modular eigenform of minimal level Np^t (we assume that f is not an exceptional Eisenstein series), N prime to p, then one has $N(\rho_f) = N$.*

Proof —   The proof of (i) is an exercise left to the reader using either of the two definitions of the tame conductor.

Assertion (ii) is easily reduced to assertion (iii) using [**BC**, Lemma 7.8.11].

To prove (iii), we need to recall more about Livne's definition of $n_l(\rho_{|G_{\mathbb{Q}_l}})$: $n_l(\rho_{|G_{\mathbb{Q}_l}}) = \dim \rho - \dim \rho^{I_l} + \mathrm{sw}(\mathrm{gr}\rho)$. Here I_l is the inertia subgroup of $G_{\mathbb{Q}_l}$ and $\mathrm{gr}\rho$ is a the representation of I_l on the filtration of ρ with respect to the Grothendieck filtration at l, so that I_l acts on $\mathrm{gr}\rho$ through a finite quotient. In particular, the action of I_l on $\mathrm{gr}\rho$ on a neighborhood of x is constant, and so is $\mathrm{sw}(\mathrm{gr}\rho)$. We thus need to show $y \mapsto \dim(\rho_y)^{I_l}$ is locally constant near x, excepted when ρ_x is of the special form given in the statement. According to [**BC**, §7.8] the inertia subgroup I_l acts through a finite quotient on M (in a neighborhhod of X) excepted if the monodromy operator N admits no Jordan form, which implies that it is zero at x and conjugate to $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ generically on a neighborhood of x. In particlar $\rho_x = 1 \oplus \omega_p$ up to a twist. If this twist is by a ramified character, then $\dim(\rho_x)^{I_l} = 0$, which obviously implies the same result on a neighborhood. Therefore, the only case where $n_l(\rho_y)$ may not be constant around x is for ρ_x an unramified twist of $1 \oplus \omega_p$.

Let us prove (iv). Let f' be the new form attached to f. For a cuspidal form f', (iv) is a famous theorem of Carayol (cf. [**Cara**] and [**Li**, Lemma 4.1]). If $f = E_{k+2,\tau,\psi}$ with τ, ψ primitive of conductor Q and R, and minimal level QR, the Galois representation ρ_f is $\tau \oplus \psi\omega_p^{k+1}$ so $N(\rho_f)$ is the prime-to-p part of QR. $\quad\square$

PROPOSITION IV.4.4. *Let x be a point of the eigencurve $\mathcal{C}^{\pm}$, defined over a finite extension L. Then*

(i) *The tame conductor $N(\rho_x)$ of ρ_x divides N.*

(ii) *Let us see $\det \rho_x$ as a character of $\mathbb{A}_{\mathbb{Q}}^*$ by class field theory (arithmetic normalization) and restrict it to $\mathbb{Z}_p^* \subset \mathbb{A}_{\mathbb{Q}}^*$. Then it is the character $z \mapsto z\kappa(x)(z)$, $\mathbb{Z}_p^* \to L^*$.*

(iii) *The (Hodge-Tate-)Sen weights of $(\rho_x)_{|G_{\mathbb{Q}_p}}$ are 0 and $-d\kappa(x) - 1$, where $d\kappa(x)$ is the derivative at 1 of the character $\kappa(x) : \mathbb{Z}_p^* \to L^*$.*

(iv) *The space $D_{crys}((\rho_x)_{|G_{\mathbb{Q}_p}})^{\phi=U_p(x)}$ has dimension at least 1 over L.*

Proof — The idea of proof of those four properties is similar: we prove that they hold at a very Zariski-dense set of points, and prove that they are "closed".

For (i), if x is very classical, the Galois representation ρ_x is the one attached to a modular form of level Np, hence the tame conductor of ρ_x divides N (in fact, is N if the form is new at primes dividng N, less otherwise) by a famous result of Carayol ([**Li**], [**Cara**]) for a cuspidal form, and by direct inspection for an Eisenstein series. The definition of the tame conductor (e.g. [**Li**]) makes clear that it is semi-continuous for the Zariski topology, hence the result follows.

For (ii), if x is very classical of weight $\kappa(x) = k$, then we know that $\det \rho_x = \omega_p^{k+1} \times \epsilon$ where $\epsilon : (\mathbb{Z}/N\mathbb{Z})^* \to L^*$ is the nebentypus of the modular form comrresponding to x and ω_p is the cyclotomic character. The restriction of ϵ seen as

a character $\mathbb{A}_{\mathbb{Q}}^* \to L^*$ to $\mathbb{Z}_p^*$ is trivial because N is not divisible by p and the restriction of ω_p^{k+1} to $\mathbb{Z}_p^*$ is $z \mapsto z^{k+1} = \kappa(x)(z)z$. Hence the formula to be proved is true for x a very classical point. The result follows by the density of very classical points.

For (iii), if x is very classical and $\kappa(x) = k \in \mathbb{N}$, the Galois representation ρ_x is the one attached to a modular form of weight $k + 2$, hence the Hodge-Tate weights of $(\rho_x)_{|G_{\mathbb{Q}_p}}$ are (according to our normalization) 0 and $-k - 1$, that is 0 and $-d\kappa(x) + 1$ since $\kappa(x) : z \mapsto z^k$ has derivative k at 1. Then (ii) follows from the density of very classical points again and Sen's theory (the precise argument can be found in [**BC**, Lemma 4.3.3(i)].)

For (iv), if x is very classical and p-old, then $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is crystalline and its ϕ-eigenvalues are $U_p(x)$ and $p^{k+1}U_p(x)^{-1}$. Then the result follows from Kisin's theorem ([**Ki**], [**?**, Theorem 3.3.3(i)]). $\square$

COROLLARY IV.4.5. *The only classical points on $\mathcal{C}^{\pm}$ are the very classical points and the Hida classical points.*

Proof — If x is a classical point of weight k and minimal level $N_0 p^t$ with N_0 not divsible by p, then by definition ρ_x is the representation attached to the system of eigenvalues λ_x which appears in $\mathrm{Symb}_{\Gamma_1(N_0 p^t)}(\mathcal{V}_k(L))$ and is new. Since the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/p^t\mathbb{Z})^*$ commutes with $\mathcal{H}$, λ_x must appear in $\mathrm{Symb}_{\Gamma_1(N_0 p^t)}(\mathcal{V}_k(L))[\epsilon]$ for some character ϵ of $(\mathbb{Z}/p^t\mathbb{Z})^*$. Hence the restriction to $\mathbb{Z}_p^*$ of the determinant of ρ_x (seen as a character of $\mathbb{A}_{\mathbb{Q}}^*$ by class field theory) is $z \mapsto z^{k+1}\epsilon(z)$. By (ii) of the above theorem, $\kappa(x)(z) = z^k\epsilon(z)$. Also, by (i) of the above theorem, $N_0 \mid N$.

Now we distinguish two cases: the first case is when $t = 0$ or $t = 1$, and the character ϵ is trivial. In this case λ_x appears in $\mathrm{Symb}_{\Gamma_0(p) \cap \Gamma_1(N_0)}(\mathcal{V}_k(L))$, hence in $\mathrm{Symb}_{\Gamma}(\mathcal{V}_k(L))$ and we have $\kappa(x) = k$ by the above paragraph. Hence x is a very classical point (p-new if $t = 1$ and p-old if $t = 0$). The second case is when $t \geq 2$ or ϵ is non-trivial. Since $t = 0$ implies obviously $\epsilon = 1$, this means either that $t = 1$ and ϵ non-trivial, or $t \geq 2$. Note that in both case, ϵ is primitive: for $t = 1$, any non-trivial character if $(\mathbb{Z}/p^t\mathbb{Z})^*$ is primitive, and for $t > 1$, this follows from Lemma **??**(ii). Summarizaing, λ_x appears in $\mathrm{Symb}_{\Gamma_1(Np^t)}(\mathcal{V}_k(L))[\epsilon]$ and $\kappa(x)$ is the character $z \mapsto z^k\epsilon(z)$ with ϵ primitive: this is exactly the definition of x being Hida-classical. $\square$

We end up with a lemma proving that the minimal tame level is locally constant around a classi

DEFINITION IV.4.6. Let x be a classical point on the eigencurve $\mathcal{C}^{\pm}$. We shall say thet x is *abnormal* (resp. *strongly abnormal*), if we have either:

(ABN1) The system of eigenvalues $\lambda_{x,0}$ of $\mathcal{H}_0$ is the system of eigenvalues of E_2.

(ABN2) The system of eigenvalues $\lambda_{x,0}$ of $\mathcal{H}_0$ is the system of eigenvalues of a normal new Eisenstein series $E_{2,\psi,\tau}$ of weight 2 with τ and ψ primivitive Dirichlet characters of conductor Q and R, and there is a prime factor l of N (resp. that does not divide the minimal level N_0 of $\lambda_{x,0}$, or equivalently, that does not divides QR) such that such that $\tau_{|G_{\mathbb{Q}_l}} = \psi_{|G_{\mathbb{Q}_l}}$.

LEMMA IV.4.7. *Let x be classical point on the eigencurve of tame minimal level N_0. If x is not strongly abnormal, then in a sufficiently small neighborhood of x all classical points have minimal level N_0.*

Proof —   If x is not strongly abnormal, there exists by Atkin-Lehner's theory a new form f of tame level N_0 whose system of eigenvalues for $\mathcal{H}_0$ is λ_x, and $\rho_x = \rho_f$. For l a prime different from p, we need to show that $n_l(\rho_y)$ is constant on a neighborhoof of x. There is nothing to prove for $l \nmid N$. For $l \mid N$, the result follows from Lemma **??**(ii), excepted when $(\rho_f)_{|G_{\mathbb{Q}_l}}$ is $1 \oplus \omega_p$ up to an unramfied twist. We shall prove that this cannot happen under our hypothesis that x is not strongly abnormal. If f is cuspidal, this follows from the local-global compatibility between automorphic forms for GL_2 and their Galois representation (cf. [**?**]), for the representations of $GL_2(\mathbb{Q}_l)$ whose attached Galois-representation of $G_{\mathbb{Q}_l}$ is up to a twist, $1 \oplus \omega_p$ are the one-dimensional representations, which can never be a component of a cuspidal automorphic representation of GL_2. If f is Eisenstein, then $(\rho_f)_{|G_{\mathbb{Q}_l}}$ is $1 \oplus \omega_p$ up to an unramfied twist only if f is exceptional (excluded by (ABN1), or if $f = E_{2,\tau,\psi}$, with $\tau_{|G_{\mathbb{Q}_l}} = \psi_{|G_{\mathbb{Q}_l}}$ and those two characters are unramfied, which means that l does not divides the tame level N_0 of f. $\square$

EXERCISE IV.4.8. Assume that $N = l$ is a prime. By Theorem IV.2.1 and its proof, there is a classical point x in $\mathcal{C}^\pm$ such that λ_x is the system of eigenvalues of E_2^{crit}. Show that there exists an affinoid neighboroood of x in $\mathcal{C}^\pm$ in which every classical point y has minimal tame level l.

IV.5. The ordinary locus

DEFINITION IV.5.1. The *ordinary locus* $\mathcal{C}_{\mathrm{ord}}^\pm$ of the eigencurve $\mathcal{C}^\pm$ is the locus of points x such that $v_p(U_p(x)) = 0$.

PROPOSITION IV.5.2. *The ordinary locus $\mathcal{C}_{\mathrm{ord}}^\pm$ is a union af connected components of $\mathcal{C}^\pm$. In particular, it is equi-dimensional of dimension 1. The very classical points are very Zariski-dense in it. The restriction $\kappa : \mathcal{C}_{\mathrm{ord}}^\pm \to \mathcal{W}$ of the weight map κ is finite.*

Proof —   Recall that $U_p \in \mathcal{O}(\mathcal{C}^\pm)$ is bounded by 1, and that the ball of functions bounded by 1 in $\mathcal{O}(\mathcal{C}^\pm)$ is compact (Prop. II.4.8). Thus every subsequence of the sequence $U_p^{n!}$ has a converging subsequence, and all those converging subsequence

have the same limit since these limits obviously assume the value 1 on x such that $|U_p(x)| = 1$ and 0 elsewehre. It follows that $e = \lim_{n \to \infty} U_p^{n!}$ exists in $\mathcal{O}(C)$ and take the value 1 on $\mathcal{C}_{\mathrm{ord}}^{\pm}$, 0 elsewhere. In particular e is an idempotent in $\mathcal{O}(\mathcal{C}^{\pm})$, and $\mathcal{C}_{\mathrm{ord}}^{\pm}$ is defined by $e = 1$, hence is a union of connected components. The other assertions are clear. $\square$

REMARK IV.5.3. Well before the eigencurve was constructed, Hida has defined the so-called Hida-family, as some eigenalgebras over the Iwasawa algebra Λ of function bounded by 1 in $\mathcal{O}(\mathcal{W})$. One can show that $\mathcal{C}_{\mathrm{ord}}^{\pm}$ is the base change to $\mathcal{W}$ of the corresponding Hida family.

Here we see the trade-off between Hida families and eigenvarieties. Hida families only contains the ordinary modular forms, but they are defined over the Iwasawa algebra, which allows ones to reason modulo p when we need to.

LEMMA IV.5.4. *If x is a very classical point of weight k such that $v_p(U_p(x)) = 0$ or $v_p(U_p(x)) = k + 1$, then $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is an extension of an unramfied character χ_1 by an Hodge-Tate character χ_2 of weight $-k - 1$.*

Proof —   In both cases, x is old, and corresponds to a refinement of an ordinary modular form. The result is thus known by a deep result of Hida and Wiles. $\square$

PROPOSITION IV.5.5. *If $x \in \mathcal{C}_{\mathrm{ord}}^{\pm}$, then $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is an extension of an unramified character χ_1 (in particular with Hodge-Tate weight 0) by a character χ_2 whose Hodge-Tate weight is $-d\kappa(x)$.*

Proof —   For x very classical on $\mathcal{C}^{\pm}$, this is the preceding lemma, and one concludes by Prop. IV.5.2. $\square$

IV.6. Local geometry of the eigencurve

IV.6.1. Clean neighborhoods. Let x be a closed point of the eigencurve $\mathcal{C}^{\pm}$ of field of definition L. We want to study the geometry of the eigencurve near x. In order to avoid the minor but annoying complications due to rationality question, we shall make a base change to L. That is, we consider in this section both the eigencurve and the weight space as rigid sapces over L istead of $\mathbb{Q}_p$ (without changing notations). In particular, all affinoids of $\mathcal{C}_{\pm}$ and $\mathcal{W}$ will be L-affinoids without further notices. Let us call $w = \kappa(x)$, which is a closed point of $\mathcal{W}$ of field of definiton L. After we make this base change, saying that κ is étale at x is the same as saying that it is an isomorphism on its image over any small enough neighborhood of x.

LEMMA AND DEFINITION IV.6.1. There exists a basis of affinoid admissible neighborhoods U of $x \in \mathcal{C}^{\pm}$ such that

(a) There exist an admissible open affinoid $W = \operatorname{Sp} R$ of $\mathcal{W}$ (so R is an affinoid L-algebra), and $\nu \in \mathbb{R}$ adpated to W, such that U is the connected component of x in $\mathcal{C}^{\pm}_{W,\nu}$.

(b) The point x is the only point in U above w.

(c) The map $\kappa : U \to W$ is étale at every point of U except perhaps x

We shall call such a neighborhood a *clean* neighborhood of x.

Proof — By construction, the $\mathcal{C}^{\pm}_{W,\nu}$ are a basis of affinoid neighborhoods of x in $\mathcal{C}^{\pm}$. Fix such a neighborhood Choose two admissible open disjoint subsets U_1 and U_2 such that U_1 contains x and U_2 contains every points above w excepted x (this is possible since $\kappa^{-1}(w)$ is a finite set). Since $\kappa : \mathcal{C}^{\pm}_{W,\nu} \to W$ is finite flat, the $\kappa^{-1}(W')$ for $w \in W' \subset W$ form a basis of neighborhood of $\kappa^{-1}(w)$ ([**?**, 2.1.6]), so it is possible to find a $W' \in \mathfrak{C}$ such that $\kappa^{-1}(W') \subset U_1 \times U_2$. For such a W', the connected component U of x in $\mathcal{C}^{\pm}_{W'\nu} = \kappa^{-1}(W')$ is contained in U_1, hence contains no point in $\kappa^{-1}(w)$ other than x. By the openness (already for the Zariski topology) of the étale locus of a finite flat map, if W' is small enough then $\kappa : U \to W'$ is étale excepted perhaps at x. $\qquad\square$

If U is a clean neighborhood of x, it follows from the definition that U is a connected component of a local piece $\mathcal{C}^{\pm}_{W,\nu} = \operatorname{Sp} \mathcal{T}^{\pm}_{W,\nu}$ where $\mathcal{T}^{\pm}_{W,\nu}$ is the eigenalgebra corresponding to the action of $\mathcal{H}$ on the finite projective (actually free since R is a PID) $\operatorname{Symb}^{\pm}_{\Gamma}(\mathcal{D}_K[r])^{\leq \nu}$ which is defined for $r > 0$ small enough, and independent of r. That is to say, there is an idempotent $\varepsilon \in \mathcal{T}^{\pm}_{W,\nu}$ such that U is defined in $\mathcal{C}^{\pm}_{W,\nu}$ by the equation $\varepsilon = 1$, and $U = \operatorname{Sp}(\varepsilon \mathcal{T}^{\pm}_{W,\nu})$. Also, the module $\varepsilon \operatorname{Symb}^{\pm}_{\Gamma}(\mathcal{D}_K[r])^{\leq \nu}$ is a direct summand of $\operatorname{Symb}^{\pm}_{\Gamma}(\mathcal{D}_K[r])^{\leq \nu}$, hence is finite projective (free) over R, and is stable by $\mathcal{H}$ since $\varepsilon \in \mathcal{T}^{\pm}_{W,\nu}$. Clearly (see Exercise **??**), $\varepsilon \mathcal{T}^{\pm}_{W,\nu}$ is the eigenalgebra defined by the action of $\mathcal{H}$ on the module.

Since the geometry of clean neighborhoods of x will be the main object of study in this section, we shall introduce shorter notations for the objects of the above paragraph. For U a clean neighborhhod of x :

(i) We shal write $M := \epsilon \operatorname{Symb}^{\pm}_{\Gamma}(\mathcal{D}_K[r])^{\leq \nu}$. It is a finite free module over R. We call d its rank.

(ii) We shall write $\mathcal{T}$ for the eigenalgebra defined by the action of $\mathcal{H}$ on M. It is a finite free module over R. We call e its rank. We have $U = \operatorname{Sp} \mathcal{T}$.

(iii) We still write $\kappa : U \to W$ for the restriction of $\kappa : \mathcal{C}^{\pm} \to \mathcal{W}$ to U. It is a finite flat map of degree e.

(iv) For $w' \in W(L)$ we write $M_{w'} := M \otimes_{R,w'} L$ and $\mathcal{T}_{w'} = \mathcal{T} \otimes_{R,w'} L$. The space $M_{w'}$ is of dimension d, independently of w', equal to the rank of M and gets an action of $\mathcal{H}$. There is a surjective morphism with nilpotent kernel from $\mathcal{T}_{w'}$ to the eigenalgbra defined by that action (see Prop. I.4.1), and this morphism is an isomorphism when $w' \neq w$ by Prop. I.4.3 since κ

is étale at any point over w'. (N.B. at w, the algebra $\mathcal{T}_w$ may be strictly larger that the eigenalgebra on M_w).

A simple but important observation is that $\mathcal{T}_w$ is a local L-algebra (since x is the only point above w in U) and that, as an $\mathcal{H}$-module

$$M_w = \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w)_{(x)},$$

where the subscript $_{(x)}$ indicates as usual that the we take the generalized eigenspace of $\mathcal{H}$ for the system of eigenvalues λ_x in the space $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w)$ (or what amounts to the same, in the space $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w[r])$). This observation is just a special case of Theorem **??**.

LEMMA IV.6.2. *Assume that x is a classical point that is not strongly abnormal and which has minimal tame level N_0 (so $N_0 \mid N$). Then $d = \sigma(N/N_0)e$ where $\sigma(n)$ is the number of positive divisors of a positive integer n.*

Proof — By Cor. IV.4.5, the weight $w = \kappa(x)$ has the form $z \mapsto z^k\epsilon(z)$ for some $k \geq 0$ and some finite order character ϵ of conductor p^t. We can choose a non-negative integer k' such that $k' > 2\nu - 1$, $k' > \nu - 1$, $k' \neq k$ and the character $w'(z) = z^{k'}\epsilon(z)$ is in W, and moreover such that all points x' above k have minimal tame level N_0 (by Lemma IV.4.7).

In particular $M_{w'} = \varepsilon\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_{w'}(L))^{\leq\nu} = \varepsilon\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_{w'}(L))^{\leq\nu}$ by Stenvens' control theorem (Theorem III.6.36 and Prop. IV.3.11). Since κ is étale over w', after extending the field of scalars L if necessary, we have $\mathcal{T}_{w'} = L^e$ as L-algebra, which means that the eiegnalgebra of the action of $\mathcal{H}$ on $M_{w'}$ is L^e. Therefore, $M_{w'}$ is the direct sum of the eigenspaces $M_{w'}(x') = \mathrm{Symb}_\Gamma^\pm(\mathcal{V}_{w'}(L))[x']$ for x' running among the e points of the fiber of κ at k'. It therefore suffices to prove that $\dim_L \mathrm{Symb}_\Gamma^\pm(\mathcal{V}_{w'}(L))(x') = \sigma(N/N_0)$ for each such x'. This is done in the following lemma (of which the assumption on $\lambda(U_p)$ is satisfied because $\nu < (k'+1)/2$). $\square$

LEMMA IV.6.3. *Let $w \in W(L)$ be of the form $w(z) = z^k\epsilon(z)$ with $k \geq 0$ and ϵ a finite order character of $\mathbb{Z}_p^*$. Let λ be an $\mathcal{H}$-system of eigenvalues appearing in $\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_w(L))$, which is not the system of eigenvalues of E_2, and let N_0 be its minimal tame level. If $\epsilon = 1$, assume moreover that $\lambda(U_p)^2 \neq p^{k+1}\lambda(\langle p \rangle)$. Then the $\mathcal{H}$-eigenspace $\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_w(L))[\lambda]$ and generalized eigenspace $\mathrm{Symb}_\Gamma^\pm(\mathcal{V}_w(L))_{(\lambda)}$ are equal and both have dimension $\sigma(N/N_0)$.*

Proof — By Theorem **??**, this amounts to proving that in the case where $\epsilon = 1$

$$(84) \qquad \dim_L M_{k+2}(\Gamma, L)_{(\lambda)} = \dim_L M_{k+2}(\Gamma, L)[\lambda] = \sigma(N/N_0),$$

and in the case where $\epsilon \neq 1$, using also Prop. IV.3.10, and denoting the conductor of ϵ by p^t,

$$(85) \dim_L M_{k+2}(\Gamma_1(Np^t), L)[\epsilon][\lambda] = \dim_L M_{k+2}(\Gamma_1(Np^t), L)[\epsilon]_{(\lambda)} = \sigma(N/N_0).$$

We first prove (85), by appliying Atkin-Lehner theory, cf §I.6.3, for the level Np^t. We shall denote in this paragraph by $\mathcal{H}_0'$ the polynomial denoted *loc. cit.* by $\mathcal{H}_0$ for that level Np^t, that is the polynomial ring generated by the T_l for $l \nmid Nt$ and the diamond operators a for $\langle a \rangle$ for $a \in (\mathbb{Z}/p^t N\mathbb{Z})^*$. Thus $\mathcal{H}_0'$ contains the polynomial ring we currently denotw by $\mathcal{H}_0$ and in addition variables corresponding to the diamond operators $\langle a \rangle$ for $a \in (\mathbb{Z}/p^t\mathbb{Z})^*$. Thus, ϵ and the restriction λ_0 of λ form $\mathcal{H}$ to $\mathcal{H}_0$ together define a character $\lambda_0' : \mathcal{H}_0' \to L$. The minimal level of this character is $N_0 p^t$ since we already know its minimal tame level is N_0 and its minimal wild level is p^t because that's the conductor of ϵ. Hence $\dim M_{k+2}(\Gamma_1(Np^t), L)[\lambda_0'] = \sigma(Np^t/N_0 p^t) = \sigma(N/N_0)$ by Atkin-Lehner theory. Finally, we note that $\dim M_{k+2}(\Gamma_1(Np^t), L)[\epsilon][\lambda] = \dim M_{k+2}(\Gamma_1(Np^t), L)[\lambda_0']$: the difference between the two eigenspaces is that in the LHS the operator U_p is included, but not in the LHS. But since λ is new at p, including U_p does not change the eigenspace by Atkin-Lehner theory. Moreover, for the same reason, U_p acts semi-simply on $\dim M_{k+2}(\Gamma_1(Np^t), L)[\epsilon]_{(\lambda)}$, and since all the other operator in $\mathcal{H}_0$ are semi-simple, we see that this generalized eigenspace is actually equal to the corresponding eigenspace.

We now turn to (84). The case where λ is p-new is done exactly as (85), so let us assume that λ is p-old. In this case, the minimal level of x is N_0, hence, if λ_0 denotes the restriction of the system of eigenvalues x to $\mathcal{H}_0$, we have $\dim M_{k+2}(\Gamma_1(N), L)[\lambda_0] = \sigma(N/N_0)$ and $\dim M_{k'+2}(\Gamma, L)[\lambda_0] = \sigma(pN/N_0) = 2\sigma(N/N_0)$ by Atkin-Lehner's theory. Now consider the two "refinement" maps $M_{k+2}(\Gamma_1(N), L)[\lambda_0] \to M_{k+2}(\Gamma, L)[\lambda_0]$, defined by $f(z) \mapsto f(z) - \frac{p^{k+1}\lambda(\langle p \rangle)}{\lambda(U_p)} f(pz)$ and $f(z) \mapsto f(z) - \lambda(U_p)f(pz)$. It is easy to see that these maps are injective (e.g. by looking at q-expansions). By Lemma III.7.2, the image of the first map is included in $M_{k+2}(\Gamma, L)[\lambda]$ (that is the $\mathcal{H}$-eigenspace for λ), hence *a fortiori* in $M_{k+2}(\Gamma, L)_{(\lambda)}$, and the image of the second is included in $M_{k'+2}(\Gamma, L)[\lambda']$, hence in $M_{k+2}(\Gamma, L)_{(\lambda')}$ where λ' is the system of eigenvalues of $\mathcal{H}$ that restricts to λ_0 on $\mathcal{H}_0$ and sends U_p to $\frac{p^{k+1}\lambda(\langle p \rangle)}{\lambda(U_p)}$. Since those two generalized eigenspaces are in direct sum because of our hypothesis $\lambda(U_p) \neq \frac{p^{k+1}\lambda(\langle p \rangle)}{\lambda(U_p)}$, the sum of their dimension is at most $2\sigma(N/N_0)$, but as we just saw each of those spaces has dimension at least $\sigma(N/N_0)$. Hence they both have dimension $\sigma(N/N_0)$, and are equal to the corresponding eigenspaces, which proves (84), hence the lemma. $\qquad\square$

IV.6.2. Etaleness of the eigencurve at non-critical slope classical points.

THEOREM IV.6.4. *Let x be a normal classical point on $\mathcal{C}^\pm$ of weight k, and such that $U_p(x)^2 \neq p^{k+1}\langle p \rangle(x)$ If x is of* non-critical slope, *that is if $v_p(U_p(x)) < k + 1$, then κ is étale at x. In particular, $\mathcal{C}^\pm$ is smooth at x.*

Proof — We choose a clean neighborhood on x and keep all the notations used above. Let N_0 be the minimal tame level of x and write $w = \kappa(x)$. We have $d = \dim M_w = \dim \operatorname{Symb}_\Gamma^\pm(\mathcal{D}_w)_{(x)}$ The latter is, by Stevens' control theorem (using that x is of non-critical slope), $\dim \operatorname{Symb}_\Gamma^\pm(\mathcal{V}_w)_{(x)}$. Hence, $d = \sigma(N/N_0)$ by Lemma IV.6.3. But by Lemma IV.6.2, $d = e\sigma(N/N_0)$. Hence $e = 1$. $\square$

REMARK IV.6.5. In this theorem, $v_p(U_p(x)) < k + 1$ is the serious hypothesis, while x normal and $U_p(x)^2 \neq p^{k+1}\langle p\rangle(x)$ are the technical hypothesis. We discuss wether those technical hypothesis are really necessary.

The hypothesis $U_p(x)^2 \neq p^{k+1}\langle p\rangle(x)$ when x is very classical is necessary because when $U_p(x)^2 = p^{k+1}\langle p\rangle(x)$, the eigenalgebra defined by the action of $\mathcal{H}$ on M_w is non semi-simple (cf. Exercise I.6.16), hence a fortiori κ is not étale at x. On the other hand, it is conjectured, as already noted, that $U_p(x)^2 = p^{k+1}\langle p\rangle(x)$ never happens for a classical point, and this is known for $k = 0$.

About the hypothesis that x is not strongly abnormal, there is one noteworthy strongly abnormal case where the conclusion of Theorem IV.6.4 still holds and can be proved with only a slight modification of the proof: it is when $N = 1$ and x is the point corresponding to the ordinary Eisenstein series $E_{2,p}$ (actually, this point lies on the ordinary Eisenstein line in the eigencurve of tame level 1, which is isomorphic to the weight space $\mathcal{W}$ through the weight map κ.) I ignore if the eigencurve is étale at the other abnormal classical points x of non-critical slope, but it is unlikely. It is perhaps possible to prove that several components of the eigencurve meet at such a strongly abnormal classical point x using a generalization of the criterions of [**?**] and [**?**].

IV.6.3. Geometry of the eigencurve at critical slope very classical points. In this section, we study very classical points x in $\mathcal{C}^\pm$ of weight $\kappa(x) = k \in \mathbb{N}$ that are of *critical slope* (that is, $v_p(U_p(x)) = k+1$). Note that by Lemma IV.3.5 thoses points are always p-old. Most of the section is devoted to the case where x is not abnormal, with some complements about the abnormal cases at the end.

LEMMA AND DEFINITION IV.6.6. Let x be a very classical point of weight k and critical slope. Assume that x is not abnormal. There exists a unique (up to isomorphism) Galois representation $\rho_x^P : G_{\mathbb{Q},Np} \to \operatorname{GL}_2(\bar{\mathbb{Q}}_p)$ such that

  (i) The representation ρ_x^P satisfies the Eichler-Shimura relation $\operatorname{tr}(\rho_f(\operatorname{Frob}_l)) = T_l(x)$ for all l prime to Np.
  (ii) The restriction $(\rho_x^P)_{|G_{\mathbb{Q}_p}}$ is crystalline at p.
  (iii) The representation ρ_x^P is indecomposable.

Moreover, the Hodge-Tate weights of $(\rho_x)_{|G_{\mathbb{Q}_p}}$ are 0 and $-k - 1$ and $U_p(x)$ is an eigenvalue of the crystalline Frobenius φ on $D_{\operatorname{crys}}((\rho_f)_{|G_{\mathbb{Q}_p}})$.

We call ρ_x^P the *preferred Galois representation* attached to x.

Proof — When x is cuspidal, ρ_x is irreducible, hence satisfies (iii). It also satisfies (i), (ii) (since x is p-old), and the "moreover" by the known property of Galois representations attached to modular forms. Any reprersentation that satisfies (i) is isomorhic to $\rho_x^{\text{ss}} = \rho_x$, hence the uniqueness.

When x is Eisensetein, it is attached to a new Eisenstein series $E_{k+2,\psi,\tau}$ and we have $\rho_x = \tau\omega_p^{k+1} \oplus \psi$. A representation of ρ_x^P satisfying (i) and (iii) is either a non-trivial extension (in the category of $\mathbb{G}_{\mathbb{Q},Np}$-representations of $\tau\omega^p$ by ψ or a non-trivial extension of ψ by $\tau\omega^p$. We will show that there is one and only one (up to isomorphism of $G_{\mathbb{Q},Np}$-representations) non-trivial extension of ψ by $\tau\omega_p^{k+1}$ and none of $\tau\omega_p^{k+1}$ by ψ satisfying (ii).

First consider a non-trivial extension of ψ by $\tau\omega_p^{k+1}$ and twist it by ψ^{-1}. It becomes a non-trivial extension V of 1 by $\psi^{-1}\tau\omega_p^{k+1}$. Property (ii) is equivalent to $V_{|G_{\mathbb{Q}_p}}$ being cristalline (using that τ and ψ are crystalline). Such extensions V are parametrized by the Bloch-Kato style Selmer group,

$$(86) \qquad \ker\left(H^1(G_{\mathbb{Q},Np},\psi^{-1}\tau\omega_p^{k+1}) \to H^1_{/f}(G_{\mathbb{Q}_p},\psi^{-1}\tau\omega_p^{k+1})\right),$$

where the notation $H^1_{/f}$ means H^1/H^1_f. We claim that this space has dimension 1. This space is actually the same as

$$H^1_f(\mathbb{Q},\psi^{-1}\tau\omega_p^{k+1}) = \ker\left(H^1(G_{\mathbb{Q},Np},\psi^{-1}\tau\omega_p^{k+1}) \to \prod_{l|Np} H^1_{/f}(G_{\mathbb{Q}_l},\psi^{-1}\tau\omega_p^{k+1})\right)$$

because for $l \mid N$ we have $H^1_{/f}(G_{\mathbb{Q}_l},\psi^{-1}\tau\omega_p^{k+1}) = 0$ excepted if $\psi^{-1}\tau\omega_p^{k+1} = \omega_p$ on $G_{\mathbb{Q}_l}$, which means $\psi = \tau$ on $G_{\mathbb{Q}_l}$ and $k = 0$, which is excluded by out hypothesis that x is not abnormal. To conclude this case, we observe that teh dimension $H^1_f(\mathbb{Q},\epsilon\omega_p^n)$ is known (by the work of Soulé ([**?**], using some arguments of [**?**]) to be 1 when $n \geq 1$, $\epsilon(-1)(-1)^n = -1$, and $\epsilon\omega_p^n \neq \omega_p$, and 0 in all other cases. So $H^1_f(\mathbb{Q},\psi^{-1}\tau\omega_p^{k+1})$ has dimension 1, because $\psi^{-1}\tau\omega_p^{k+1} = \omega_p$ implies that x is abnormal, as either N is different from 1 and any prime factor l of N satsifies (ABN2) or $N = 1$, which implies $\psi = \tau = 1$ and $k = 0$ is excluded by (ABN1). Hence we have proved the existence and uniqueness of an extension of ψ by $\tau\omega_p^{k+1}$ satisfyingthe required conditions.

For the extensions in the other direction, we are reduced to compute, by similar agument, $H^1_f(\mathbb{Q},\psi\tau^{-1}\omega_p^{-k-1})$ which is 0 by the result of Soulé quoted above. $\qquad\square$

DEFINITION IV.6.7. We say that a point x on $\mathcal{C}^{\pm}$ with $\kappa(x) = k \in \mathbb{N}$ has a *companion* point y if there exists a point $y \in \mathcal{C}^{\pm(-1)^{k+1}}$ such that $\kappa(y) = -2 - k$, $T_l(y) = l^{-k-1}T_l(x)$ for all $l \nmid N$, $U_p(y) = p^{-k-1}U_p(x)$ and $\langle a \rangle(y) = \langle a \rangle(x)$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$. The point y, if it exists, is necessarily unique and is called the *companion* of x.

REMARK IV.6.8. Our definition of a companion is slightly different from the standard one, which wold be phased exactly the same but for $\mathcal{C}^{\pm}$ and $\mathcal{C}^{\pm(-1)^{k+1}}$

both replaced by the full Coleman-Mazur-Buzzard eigencurve $\mathcal{C}$. Our definition is clearly more natural in our context of modular symbols, the requirement that the companion lies on the eigencurve of sign $\pm(-1)^{k+1}$ being the counterpart for the ι-involution of the twist by l^{-k-1} for the T_l-operators. We think that our definition is *absolutely* more natural. Actually, using theorem IV.2.1, it is easy to see that for x cuspidal, the classical notion of a companion and ours are the same. In the Eisenstein case, for the traditional definition. a point x always have a companion, but this situation leads to unnatural exceptions in most theorems on companion forms, for example [**?**]; also the equivalence between (b) and (c) in the theorem below would be false in the Eisenstein case with the traditional notion of companion).

THEOREM IV.6.9. *Let x be a very classical point on $\mathcal{C}^{\pm}$ of weight k. We assume that x is not abnormal, and that it is of critical slope, that is $v_p(U_p(x)) = k + 1$. The following are equivalent:*

(a) *The weight map κ is étale at x.*
(b) *The map $\rho_k : \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{D}_k)_{(x)} \to \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{V}_k)_{(x)}$ is an isomorphism.*
(c) *The point x has no companion point y on the eigencurve $\mathcal{C}^{\pm}(-1)^{k+1}$, that is to say there exist no point $y \in \mathcal{C}^{\pm}(-1)^{k+1}$ such that $\kappa(y) = -2 - k$, $T_l(y) = l^{-k-1}T_l(x)$ for all $l \nmid N$, $U_p(y) = p^{-k-1}U_p(x)$ and $\langle a \rangle(y) = \langle a \rangle(x)$ for $a \in (\mathbb{Z}/N\mathbb{Z})^*$,*
(d) *The restriction $(\rho_x^P)_{G_{|\mathbb{Q}_p}}$ is not the direct sum of two characters.*

THEOREM IV.6.10. *Let x be as above, and if x is cuspdial, assume moreover than $H_g^1(\mathbb{Q}, ad\rho_x) = 0$. Then the eigencurve $\mathcal{C}^{\pm}$ is smooth at x.*

We shall prove together Theorems IV.6.9 and IV.6.10.

REMARK IV.6.11. Excepted for the restriction to forms that are refinement of forms of level prime to p, the equivalence between (c) and (d) above is the famous result of Breuil and Emerton [**1**] (which was before a conjecture of Gross) on the equivalence between having a companion form and having a Galois representations splits at p. The proof we give here is completely different (and arguably, more elementary) than the one given in [**1**], as we don't unse any p-adic local langlands theory for instance. This proof is a joint work with John Bergdal. For a proof in the general case along the same lines, see a forthcoming work of John Bergdal.

Proof — We write by N_0 the minimal tame level of x. It is also the minimal level of x since as we have already observed, a critical point x is p-old.

Equivalence between (a) and (b): Using notations and results of §IV.6.1, we have $d = e\sigma(N/N_0)$, $\dim \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{V}_k)_{(x)} = \sigma(N/N_0)$, and $d = \mathrm{Symb}_{\Gamma}^{\pm}(\mathcal{D}_k)_{(x)}$. The map ρ_k is surjective, so it is an isomorphism if and only if the dimension of

its source and target are the same, that is if and only if $d = \sigma(N/N_0)$, which is equivalent to $e = 1$. (Remark: the proof of that equivalence uses only that x is not strongly abnormal: the stronger hypothesis that x is not abnormal is not used in §IV.6.1).

Equivalence between (b) and (c): By Prop. III.6.35, we have an $\mathcal{H}$-equivariant exact sequence

$$0 \to \mathrm{Symb}_\Gamma^{\pm(-1)^{k+1}}(\mathcal{D}_{-2-k}(L))_{(y)} \to \mathrm{Symb}_\Gamma^{\pm}(\mathcal{D}_k(L))_{(x)} \to \mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(L))_{(x)} \to 0$$

where we denote by y the system of $\mathcal{H}$-eigenvalues described in the theorem. Hence (b) is equivalent to $\mathrm{Symb}_\Gamma^{\pm(-1)^{k+1}}(\mathcal{D}_{-2-k}(L))_{(y)} = 0$ which by Theorem **??** is equivalent to the fact that x has no companion point y.

Proof of "(d) implies (c)" when x is cuspidal: Assume non-(c), so x has a companion point y. Since $U_p(y) = p^{-k-1}U_p(x)$, we have $v_p(U_p(y)) = 0$, so y is in the ordinary locus (Definition). Hence $(\rho_y)_{|G_{\mathbb{Q}_p}}$ is an extension of a character of Hodge-Tate weight 0 by a character of Hodge-Tate $k + 1$. Since $\rho_x = \rho_y(k + 1)$ by Cebotarev's density theorem, we deduce that $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is an extension of a character of Hodge-Tate weight $-k - 1$ by a character of Hodge-Tate weight 0. On the other hand, by Lemma IV.5.4, $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is an extension of a character of Hodge-Tate weight 0 by a character of Hodge-Tate weight $-k - 1$. Hence $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is the sum of two characters, hence non-(d).

Proof of Theorem IV.6.10 when x is cuspidal: Since x is cuspidal, the Galois representation ρ_x of $G_{\mathbb{Q},Np}$ is irreducible. Therefore, by a well-known theorem of Rouquier and Nyssen (cf. [**Ro**] or [**Ny**]), the Galois pseudocharacter $T : G_{\mathbb{Q},Np} \to \mathbb{T}$, where $\mathbb{T}$ is the completed local ring of the eigencurve $\mathcal{C}$ at the point x is the trace of a unique representation $\rho : G_{\mathbb{Q},Np} \to \mathrm{GL}_2(\mathbb{T})$ whose residual representation is ρ_x. We consider the following deformation problem: for all Artinian local algebra A with residue field L, we define $D(A)$ as the set of strict isomorphism classes of representations $\rho_A : G_{\mathbb{Q}} \to \mathrm{GL}_2(A)$ deforming ρ_x such that

(i) The restriction $(\rho_A)_{|G_{\mathbb{Q}_p}}$ has a constant weight equal to 0;

(ii) the A-module $D_{\mathrm{crys}}((\rho_A)_{|G_{\mathbb{Q}_p}})^{\varphi=\tilde\beta}$ is free of rank one for some $\tilde\beta$ lifting β;

(iii) for $l|N$, the restriction $(\rho_A)_{|I_l}$ to the inertia subgroup at l is constant.

By [**Ki**, Prop. 8.7] (using the fact that $\beta \neq \alpha$ and that $0 \neq k+1$), and [**BC**, Prop. 7.6.3(i)] (for condition (iii)), we see that D is pro-representable, say by a complete Noetherian local ring R.

We claim that $\rho \otimes \mathbb{T}/I$ is, for all cofinite length ideal I in $\mathbb{T}$, an element of $D(\mathbb{T}/I)$: property (i) follows from Sen's theory, see e.g. [**BC**, Lemma 4.3.3(i)]; property (ii) is Kisin's theorem (see [**Ki**] or [**BC**, chapter 3]). For property (iii), let $l|N$. Let N_x and $N_{s(x)}^{\mathrm{gen}}$ be the special monodromy operator of $\rho_{|I_l}$ at x and the generic monodromy operator of $T_{|I_l}$ at a component $s(x)$ of the eigencurve through

x (cf. [**BC**, definition 7.8.16]). To prove (iii), by [**BC**, Prop. 7.8.19 and Lemma 7.8.17] it is sufficient to prove that $N_x \sim N_{s(x)}^{\mathrm{gen}}$ (see [**BC**, §7.8.1] for the definition of the equivalence relation $\sim$ and the pre-order $\prec$ on the set of nilpotent matrices). Assume first that th cuspidal modular form f attached to x is not special at l, so the monodromy operator N_x of $(\rho_x)_{|I_l}$ is trivial. Then in a neighborhood of x, there is a dense set of classical points that are newforms of the same level (Lemma **??**) and same nebentypus, hence that are not special either, and thus have a trivial monodromy operator. Hence by [**BC**, Prop. 7.8.19(2)], $N_x = N_{s(x)}^{\mathrm{gen}} = 0$. If f is special, then N_x is non-zero, but since by [**BC**, Prop. 7.8.19(3)] $N_x \prec N_{s(x)}^{\mathrm{gen}}$ and we are in dimension 2, $N_x \sim N_{s(x)}^{\mathrm{gen}}$ in this case as well. This completes the proof of the claim.

Thus, ρ defines a morphism of algebras $R \to \mathbb{T}$. A standard argument shows that this morphism is surjective. Since $\mathbb{T}$ has Krull dimension 1, if we prove that the tangent space of R has dimension at most 1, it would follow that the map $R \to \mathbb{T}$ is an isomorphism and that R is a regular ring of dimension 1. This would complete our proof.

The tangent space of R, $t_D := D(L[\varepsilon])$, lies inside the tangent space of the deformation ring of ρ_x without local condition, which is canonically identified with $H^1(G_{\mathbb{Q}}, \mathrm{ad}\rho_f)$. Since $H^1_f(G_{\mathbb{Q}}, \mathrm{ad}\rho_x) = H^1_g(G_{\mathbb{Q}}, \mathrm{ad}\rho_x) = 0$ by hypothesis, this space injects into $\prod_{l|Np} H^1_{/f}(G_{\mathbb{Q}_l}, \mathrm{ad}\rho_x)$ (where $H^1_{/f}$ means H^1/H^1_f) and so does t_D. The image of t_D in $H^1_{/f}(G_{\mathbb{Q}_l}, \mathrm{ad}\rho_x)$ is 0 for $l|N$ by (iii). Hence t_D injects in $H^1_{/f}(G_{\mathbb{Q}_p}, \mathrm{ad}\rho_x)$.

If $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is not the direct sum of two characters, that is if (d) holds, then by what we have already proved (a) holds, that is κ is étale at x which in particular is smooth. So let us assume henceforth that $(\rho_x)_{|G_{\mathbb{Q}_p}} = \chi_1 \oplus \chi_2$. Both χ_1 and χ_2 are crystalline, and say χ_1 has weight 0 while χ_2 has weight $k + 1$. Since $v_p(U_p(x)) = k+1$, $U_p(x)$ is the eigenvalue of the crystalline Frobenius on $D_{\mathrm{crys}}(\chi_2)$. Then we compute:

$$H^1_{/f}(G_{\mathbb{Q}_p}, \mathrm{ad}\rho_f) = H^1_{/f}(G_{\mathbb{Q}_p}, \chi_1\chi_1^{-1}) \oplus H^1_{/f}(G_{\mathbb{Q}_p}, \chi_2\chi_1^{-1}) \oplus H^1_{/f}(G_{\mathbb{Q}_p}, \chi_1\chi_2^{-1}) \oplus H^1_{/f}(G_{\mathbb{Q}_p}, \chi_2\chi_2^{-1}).$$

The condition on $D_{\mathrm{crys}}(-)^{\varphi=\tilde{\beta}}$ implies that the image of t_d in the third and fourth factors are 0 (the third factor is 0 anyway). In particular any deformation of ρ_x in $D(L[\varepsilon])$ is triangular with diagonal terms a deformation $\tilde{\chi}_1$ of χ_1 and the constant deformation χ_2 of χ_2, and the Sen weight of this deformation are thus the Sen weight of $\tilde{\chi}_1$ and $k + 1$. The condition on the constant weight 0 in our deformation problem D thus implies that $\tilde{\chi}_1$ has constant weight 0, hence that it is constant: thus, the image of t_D in the first factor is 0. Therefore t_D injects in $H^1_{/f}(G_{\mathbb{Q}_p}, \chi_2\chi_1^{-1})$. Since $\chi_2\chi_1^{-1}$ is not trivial (its Hodge-Tate weight is not 0), local Tate duality and Euler characteristic formula implies that $H^1(G_{\mathbb{Q}_p}, \chi_2\chi_1^{-1})$ has dimension 1, excepted if $\chi_2\chi_1^{-1}$ is the cyclotomic character of $G_{\mathbb{Q}_p}$, in which case

$H^1(G_{\mathbb{Q}_p}, \chi_2\chi_1^{-1})$ has dimension 2 but $H^1_{/f}(G_{\mathbb{Q}_p}, \chi_2\chi_1^{-1})$ has dimension 1. Therefore, t_D has dimension at most one, which is what remained to prove.

Proof of "(a) implies (d)" when x is cuspidal: Assume non-(d), that is $(\rho_x)_{|G_{\mathbb{Q}_p}}$ is the sum of two characters. Then as we have seen in the above proof, any deformation in $D(L[\varepsilon])$ has a constant Sen weight $k + 1$. Since $R = \mathbb{T}$, this means that one Sen weight of the restrction to any tangeant vector at x of $\mathcal{C}^{\pm}$ is constant $k + 1$. In other words, the tangent map of κ maps any tangent vector at x to 0, that is, κ is not étale at x, which is non-(a).

We have thus finished the proof of both theorems when x is cuspidal. When x is Eisenstein, it remains to prove that (d) is equivalent to (a), (b), and (c), and that $\mathcal{C}^{\pm}$ is smooth at x. With our hypotheses that x is not abnormal, the proof is a straightforward generalization of [**?**]. We quickly remind the main steps for the convenience of the reader.

Proof of Theorem IV.6.10 when x is Eisenstein:

Then the system of $\mathcal{H}_0$-eigenvalues $\lambda_{x,0}$ is is the same as the one of a unique new Eisenstein series $E_{2,\psi,\tau}$. Let $K = \mathrm{Frac}(\mathbb{T})$ which is a finite product of fields. Then there is a representation $\rho : G_{\mathbb{Q}_{Np}} \to \mathrm{GL}_2(K)$ with $\mathrm{tr}\,\rho = T$ and $\rho(c) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ (use the argument given after Theorem IV.4.1). If $\psi(c) = -1$ we conjugate ρ by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ so that we have in any case:

$$\rho(c) = \begin{pmatrix} \tau(c)\omega_p^{k+1}(c) & 0 \\ 0 & \psi(c) \end{pmatrix}.$$

Writing

$$\rho(g) = \begin{pmatrix} a(g) & b(g) \\ c(g) & d(g) \end{pmatrix}$$

we define B (resp. C, resp. B_p, resp. C_p) as the $\mathbb{T}$-submodule of K generated by the $b(g)$, $g \in G_{\mathbb{Q},Np}$ (resp. $c(g)$, $g \in G_{\mathbb{Q},Np}$, resp. $b(g)$, $g \in G_{\mathbb{Q}_p}$, resp. $c(g)$, $g \in G_{\mathbb{Q}_p}$). Then we have natural injective L-linear maps ι_B (resp. etc.) that fits into commutative diagrams (cf [**BC**, chapter1] or [**?**]):

(87)
$$
\begin{array}{ccc}
(B/\mathfrak{m}B)^* & \xrightarrow{\iota_B} & \mathrm{Ext}^1_{G_{\mathbb{Q},Np}}(\psi, \tau\omega_p^{k+1}) \\
\downarrow & & \downarrow \\
(B_p/\mathfrak{m}B_p)^* & \xrightarrow{\iota_{B_p}} & \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\psi, \tau\omega_p^{k+1})
\end{array}
$$

(88)
$$
\begin{array}{ccc}
(C/\mathfrak{m}C)^* & \xrightarrow{\iota_C} & \mathrm{Ext}^1_{G_{\mathbb{Q},Np}}(\tau\omega_p^{k+1}, \psi) \ , \\
\downarrow & & \downarrow \\
(C_p/\mathfrak{m}C_p)^* & \xrightarrow{\iota_{C_p}} & \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\tau\omega_p^{k+1}, \psi)
\end{array}
$$

where the left vertical maps are dual of the inclusion $B_p \subset B$ and $C_p \subset C$, and the top vertical maps are restriction maps. Moreover, by Kisin's lemma the imagoe of ι_B lies in the subspace of $\mathrm{Ext}^1_{G_{\mathbb{Q},Np}}(\psi, \tau\omega_p^{k+1})$ parametrizing extensions that are crystalline at p. As seen is the proof of Lemma IV.6.6, this subspace has dimension 1. It follows that $B/\mathfrak{m}B$ has dimension at most 1, and by Nakayama that B is a principal ideal. Also, from the proof of Lemma IV.6.6 we get that the restriction map $\mathrm{Ext}^1_{G_{\mathbb{Q},Np}}(\tau\omega_p^{k+1}, \psi) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\tau\omega_p^{k+1}, \psi)$ is an isomorphism between spaces of dimension 1 and we conclude as above that C and C_p are principal, and even that they are equal provided that $C_p \neq 0$.

The ideal BC (resp. B_pC_p) is the reducibility ideal of T (resp. $T_{|G_{\mathbb{Q}_p}}$) as defined in [**BC**, chapter 1] or [**?**]. It is proved in [**?**] that B_pC_p is the ideal of the schematic fiber of κ at k and that BC is the maximal ideal $\mathfrak{m}$ of $\mathbb{T}$. By the above $\mathfrak{m}$ is principal, which shows that $\mathbb{T}$ is a discrete valuation ring, hence Theorem IV.6.10 in the Eisenstein case.

Equivalence between (a) and (d) in the Eisenstein case: We keep the notations of the above paragraph. Since $BC = \mathfrak{m}$ and B_pC_p is the ideal of the fiber of κ, we see that (a) is equivalent to $B_pC_p = BC$. By what we have seen, $C_p = C$ (C being clearly non-zero since $BC = \mathfrak{m}$), so the later is equivalent to $B_p = B$. By the commutative diagram (87) this is equivalent to restriction map $\mathrm{Ext}^1_{G_{\mathbb{Q},Np}}(\psi, \tau\omega_p^{k+1}) \to \mathrm{Ext}^1_{G_{\mathbb{Q}_p}}(\psi, \tau\omega_p^{k+1})$ being injective, which by construction of ρ_x^P is clearly equivalent to (d). $\hfill\square$

There are two noteworthy abnormal cases where the same (and actually even stronger) results hold, with some modification of the proof:

THEOREM IV.6.12. *Assume that x is a very classical point of weight $\kappa(x) = 0$, of critical slope $v_p(U_p(x)) = 1$, and such that the system of eigenvalues of x restricted to $\mathcal{H}_0$, $\lambda_{x,0} : \mathcal{H}_0 \to L$ is either the system of Eigenvalues of E_2 or the system of eigenvalues of $E_{2,\tau,\tau}$ where τ is a non-trivial Dirichlet character (note that such very classical points x exist on $\mathcal{C}^{\pm}$ if and only if $\pm = +1$). Then the weight map κ is étale at x, and in particular*

conclusions of Lemma IV.6.6, TheoremIV.6.10, and Theorem IV.6.9, hold. Moreover, the condition (a) to (d) of Theorem IV.6.9 are true.

PROPOSITION IV.6.13. *Assume that the tame level is a prime power $N = l^\nu$, and that x is a very classical point of weight $\kappa(x) = 0$, of critical slope $v_p(U_p(x)) = 1$. Assume that the system of eigenvalues of x restricted to $\mathcal{H}_0$, $\lambda_{x,0} : \mathcal{H}_0 \to L$ is either the system of Eigenvalues of $E_{2,l}$ or the system of eigenvalues of $E_{2,\tau,\tau}$ where τ is a primitive non-trivial Dirichlet character, necessarily of conductor a power of l. (Note that such very classical points x exists on $\mathcal{C}^{\pm}$ if and only if $\pm = +1$). Then the conclusions of Lemma IV.6.6, TheoremIV.6.10, and Theorem IV.6.9, hold. Moreover, the condition (a) to (d) id Theorem IV.6.9 are true.*

Proof — In the proof of Lemma IV.6.6, the only thing that changes is the proof that the space (86) has dimension 1. Namely, we now have to prove that

$$\ker\left(H^1(G_{\mathbb{Q},Np},\omega_p) \to H^1_{/f}(G_{\mathbb{Q}_p},\omega_p)\right)$$

is of dimension 1. By Kummer theory, this dimension of this space is the rank of the group of N-units in $\mathbb{Q}$, so is 1 if (and only if) N is a prime power. For the extensions in the other direction, nothing changes.

In the proof of the theorems, we see that nothing changes until the computation of the rank of ι_B, but the result is the same (that is 1) by the above paragrap. The computation of the rank of ι_C is not changed. Hence the theorems hold. Assertion (d) of Theorem IV.6.9 hold since a N-unit of $\mathbb{Q}$ of infinite order obviously is also of infinite order when seen in $\mathbb{Q}_p$. $\qquad\square$

REMARK IV.6.14. For the other abnormal very classical critical-slope points, Lemma IV.6.6 is false, as there are in this case several independent non-isomorphic extension of ψ by $\tau\omega_p$ in the category of $G_{\mathbb{Q},Np}$-representation that are crystalline at p. I do not know if IV.6.10, and Theorem IV.6.9 hold in this case, but it is unlikely. For example, if $\mathcal{C}^{\pm}$ was smooth at x in this case, then Ribet's lemma would single out a canonical non-trivial extension of ψ by $\tau\omega_p$ as above, and I don't see why any of those extensions would deserve such a privilege.

IV.7. Notes and References

The two-variables p-adic L-function on the eigencurve

V.1. Abstract construction of an $n+1$-variables L-function

Let us place ourselves in the situation of the construction of the eigenalgebra: $\mathcal{H}$ is a commutative ring, R is a commutative Noetherian ring, M a finite flat R-module with an R-linear action of $\mathcal{H}$, so the eigenalgebra $\mathcal{T}$ is defined as the sub-algebra of $\mathrm{End}_R(M)$ generated by the action of $\mathcal{H}$; the algebra $\mathcal{T}$ is finite flat over R and M is naturally a finite $\mathcal{T}$-module. To help the geometric intuition, let us call $X = \mathrm{Spec}\,\mathcal{T}$, $W = \mathrm{Spec}\,R$, and let denote by κ the structural map $X \to W$. If x is a closed point in X corresponding to a maximal ideal $\mathfrak{p}_x$ of $\mathcal{T}$, then $w := \kappa(x)$ is a closed point of W, corresponding to the maximal ideal $\mathfrak{p}_w = \mathfrak{p}_x \cap R$ of R. Let $L(x) = \mathcal{T}/\mathfrak{p}_x$ and $L(w) = R/\mathfrak{p}_w$ be the field of definition of x and w, so that $L(x)$ contains $L(w)$ and is of dinite degree over it. Let us call M_w the $L(w)$-vector space with $\mathcal{H}$-action $M \otimes_{R,w} L(w) = M/\mathfrak{p}_w M$, and $M_w[x]$ the $L(x)$-vector space defined as the eigenspace for $\mathcal{H}$ with tha same eigenvalue as x in $M_w \otimes_{L(w)} L(x)$.

We write $M^\vee = \mathrm{Hom}_R(M, R)$ and provide it with its natural structure of $\mathcal{T}$-module (hence of $\mathcal{H}$-module).

LEMMA V.1.1.   (i) *We have a natural isomorphism of $L(x)$-vector space*

$$M^\vee/\mathfrak{p}_x M^\vee \xrightarrow{\simeq} M_w[x]^\vee$$

(ii) *Assume $M^\vee$ is free of rank one over $\mathcal{T}$, and fix a generator u of $M^\vee$. Call u_x the image of u by the isomorphism (i). Then u_x is a generator of the one-dimensional $L(x)$-vector space $M_w[x]^\vee$.*

Proof — Sine M is finite flat over R, we have a natural isomorphism $M^\vee/\mathfrak{p}_w M^\vee = (M/\mathfrak{p}_w M)^\vee = M_w^\vee$. Thus $M^\vee/\mathfrak{p}_x M^\vee$ is the quotient of $M^\vee/p_w M^\vee = M_w^\vee$ by the sum of the images of the maps $T - x(T)$. Hence by elementary duality theory, it is naturally isomorphic to the dual of the intersection of the kernel of these maps in $M_w \otimes_{L(w)} L(x)$, that is to $M_w[x]^\vee$. This proves (i). If $M^\vee$ is free of rank one over $\mathcal{T}$, generated by u, then $M^\vee/\mathfrak{p}_x M^\vee$ is of dimension 1 over $L(x)$ and generated by the image of u, and (ii) follows. $\qquad\square$

Now assume that R is a $\mathbb{Q}_p$-affinoid reduced algebra provided with the supremum norm $\|\ \|_{\mathrm{sup}}$, and that M is provided with some norm $\|\ \|$ which takes values in $p^{\mathbb{Z}} \cup \{0\}$ and makes M a normed R-module. We provide $M_w = M/\mathfrak{p}_w M$ of the quotient norm $\|\ \|_w$, which makes it a normed finite-dimensional $L(w)$-vector space

where $L(w)$ is provided with the unique norm that extends $\mathbb{Q}_p$. We provide $L(x)$ with its unique norm extending the norm of $\mathbb{Q}_p$, $M_w \otimes_{L(w)} L(x)$ with the tensor norm, $M_w[x]$ with the restriction $\| \ \|_x$ of the latter norm, and finally $M_w[x]^\vee$ of the dual norm, denoted $\| \ \|_x'$. By the theorem of Hahn-Banach, $\| \ \|_x'$ is also the quotient norm on $M_w[x]^\vee$ of the extension to $L(x)$ of the norm $\| \ \|_w'$ on $M_w^\vee$ which is dual of $\| \ \|_w$.

LEMMA V.1.2. *Assume that $\mathcal{T}$ is reduced, that $M^\vee$ is free of rank one over T, and that u is a generator of $M^\vee$. Then there exist two real numbers $0 < c < C$ such that for every closed point of T, one has*

$$c \le \|u_x\|_w' \le C.$$

Proof — From the very nature of the result we want to prove, we can replace the given norm on M by an equivalent one. Since M is free, choose an isomormorphism $M \simeq R^n$ and give M the norm that makes this isomorphism an isometry. This norm is equivalent to the given one by [**BGR**, Prop. 3.7.3/3]. Choosing this norm $\| \ \|$ ensures that if we provide $M^\vee$ with the dual norm $\| \ \|'$, then the norm $\| \ \|_w'$ on $M_w^\vee$ is the quotient norm of $\| \ \|'$. Therefore the norm $\| \ \|_x'$ on $M^\vee/\mathfrak{p}_x M^\vee$ is also the quotient norm of the norm $\| \ \|'$ on $M^\vee$.

We thus are reduced to show that the norm of u_x in $M^\vee/\mathfrak{p}_w M^\vee$ lies between two positive constants c and C, and again, it is enough to prove this after changing the norm on $M^\vee$ by an equivalent one. Consider the $\mathcal{T}$-isomorphism $M^\vee \to \mathcal{T}$ that sends u to 1. Provide $\mathcal{T}$ with its supremum norm (it is a norm because $\mathcal{T}$ is reduced) and $M^\vee$ with the norm that makes this isomorphism an isometry. Now our problem is just to show that for the supremum norm of $\mathcal{T}$, the quotient norm of the unity 1 in any quotient $T/\mathfrak{p}$ is bounded between two positive constants. But thoses quotients norms are always 1. $\qquad\square$

We keep the above hypothesis and suppose given, in addition, an R-linear map $h : M \to \mathcal{D}(R)$, where $\mathcal{D}(R)$ is the R-module of (families of) distributions (cf §III.4.4). This map induces, for every finite extension of $\mathbb{Q}_p$ and $w \in W(L)$, L-linear maps $h_w : M_w \to \mathcal{D}(L)$ defined by $h_w = h \otimes 1 : M_w = M \otimes_{r,w} L \to \mathcal{D}(R) \otimes_{R,w} L = D(L)$, the last equality being Cor. III.4.26 using that the completed tensor product by L over R is the same as the usual tensor product. (In the application, elements of M and M_w shall be distribution-valued modular symbol and h, or h_w, shall be eveluation at the divisor $\{0\} - \{\infty\}$)

DEFINITION V.1.3. Assume that $M^\vee$ is free of rank one over $\mathcal{T}$.

(i) For each closed point x of X, of field of definition L, a distribution $\mu_x \in \mathcal{D}(L)$, well-defined up to multiplication by an element of $\mathcal{O}_L^*$

(ii) A "family of distributions" $\tilde{\mu} \in \mathcal{D}(\mathcal{T})$ well-defined up to a multiplication by an element of $\mathcal{T}^*$.

For (i), we proceed as follows: we choose a generator v_x of the one-dimensional $M_w[x]$-space, of norm 1. Then v_x is well-defined up to an element of $\mathcal{O}_L^*$. We define $\mu_x = h_w(v_x)$.

For (ii), note that the morphism of R-modules $h : M \to \mathcal{D}(R)$ defines an element $\tilde{h}$ in $M^\vee \otimes_R \mathcal{D}(R)$. Choosing an isomorphism $M^\vee \simeq \mathcal{T}$ as $\mathcal{T}$-modules, we can see $\tilde{h}$ as an element of $\mathcal{T} \otimes_R \mathcal{D}(R)$. Since $\mathcal{T}$ is finite over R, $\mathcal{T} \otimes_R \mathcal{D}(R) = \mathcal{T} \hat{\otimes}_R \mathcal{D}(R) = \mathcal{D}(\mathcal{T})$ by Cor. III.4.26, and we define $\tilde{\mu}$ as the element $\tilde{h}$ seen in $\mathcal{D}(T)$ usingthat isomorphism.

In this definition, the distribution μ_x should be thought of as the one variable p-adic L-function attached to (the $\mathcal{H}$-system of eigenvalue of) x. The distribution μ should be thought of as an an $n+1$-variables p-adic L-function, n being the dimension of $\mathcal{T}$ (that is of X).

Finally, we call ev_x the map "evaluation at x" $\mathcal{D}(\mathcal{T}) \to \mathcal{D}(L)$

PROPOSITION V.1.4. *For every closed point x in X of field of definition L, $ev_x(\tilde{\mu}) = c(x)\mu_x$ for $c(x)$ some element in L^*. There are two real constant $0 < c < C$ such that $c \le |c(x)| \le C$ for every closed point x in X.*

Proof — Let us pick a generator u of $M^\vee$ as a $\mathcal{T}$-module. Definition (ii) asks us to choose an isomorphism $\mathcal{T} \to M^\vee$ as $\mathcal{T}$-module, so let us use the one that sends u to 1. Consider the diagram

$$
\begin{array}{ccccc}
\mathcal{D}(\mathcal{T}) & \xrightarrow{\;=\;} & \mathcal{T} \otimes_R \mathcal{D}(R) & \xrightarrow{(1 \mapsto u) \otimes \mathrm{Id}_{\mathcal{D}(R)}} & M^\vee \otimes_R \mathcal{D}(R) \\
\Big\downarrow{\scriptstyle \mathrm{ev}_x} & & \Big\downarrow & & \Big\downarrow \\
& & (\mathcal{T}/\mathfrak{p}_w\mathcal{T}) \otimes_{L(w)} \mathcal{D}(L) & \xrightarrow{\hspace{2cm}} & M_w^\vee \otimes_{L(w)} \mathcal{D}(L) \\
\Big\downarrow & & \Big\downarrow & & \Big\downarrow \\
\mathcal{D}(L(x)) & \xrightarrow{\;=\;} & (\mathcal{T}/\mathfrak{p}_x\mathcal{T}) \otimes_L \mathcal{D}(L) & \xrightarrow{(1 \mapsto u_x) \otimes \mathrm{Id}_{\mathcal{D}(L)}} & (M_w^\vee/\mathfrak{p}_x M_w^\vee) \otimes_L \mathcal{D}(L)
\end{array}
$$

where the short vertical arrows are defined as follows (note that all other arrow are clear or explicitly given). the short vertical maps from the first to the second row are the canonical surjection from $\mathcal{T}$-modules to their quotient by $\mathfrak{p}_w$, and the short vertical maps from the second to the third row are the canonical map to their extensions of scalsr from $L(w)$ to L and then canonical surjection to their quotient by $\mathfrak{p}_x$. All horizontal arrows are isomorphisms. Each of the three small rectangles of this diagramm are commutative by definition, thus so is the full diagram. Consider the element $\tilde{h}$ in the top-right corner of the diagram $M^\vee \otimes_R \mathcal{D}(R)$. The element corresponding to $\tilde{h}$ in the top-left corner (through the upper horizontal isomorphisms) is $\tilde{\mu} \in \mathcal{D}(\mathcal{T})$ by definition (ii). On the other hand, the image of $\tilde{h}$ by the first right-top vertical arrow is the element $\tilde{h}_w \in M_w^\vee \otimes L(w)\mathcal{D}(L)$ corresponding to the morphism $h_w : M_w \to \mathcal{D}(L)$, and the image of that element by the second top-right vertical arrow is the element of $(M_w^\vee/\mathfrak{p}_x M_w^\vee) \otimes_L \mathcal{D}(L) = M_w[x]^\vee \otimes_{L_w} \mathcal{D}(L)$ corresponding

to the restriction of h_w to $M_w[x]$. The element in $\mathcal{D}(L)$ corresponding to that element through the (through the lowwer horizontal isomorphisms) is clearly $h_w(u'_x)$, where u'_x is the unique element of $M_w[x]$ such that $u_x(u'_x) = 1$. For v_x any generator of $M_w[x]$ of norm 1, as in definition (i), we have $u'_x = c(x)v_x$ with $c(x) \in L^*$ and $|c(x)| = \|u'_x\|_w = (\|u_x\|'_w)^{-1}$ is bounded between two positive constants by Lemma V.1.2. By commutativity of the diagram, we thus have shown

$$\mathrm{ev}_x(\tilde{\mu}) = h_w(u'_x) = c(x)h_w(v_x) = c(x)\mu_x$$

and the proposition is proven. $\qquad\square$

V.2. Good points on the eigencurve

LEMMA V.2.1. *Let x be a point on $\mathcal{C}^\pm$ (of field of definition L), with $\kappa(x) = w \in \mathcal{W}$. The following statements are equivalent:*

(i) *The dual of generalized eigenspace $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))_{(x)}^\vee$ is free of rank 1 over the algebra $\mathcal{T}_x$ of the schematic fiber of κ at x.*

(ii) *For any clean neighborhood of x, the module $M^\vee$ is flat of rank one over $\mathcal{T}$ at x (we use the notations of §IV.6.1).*

(iii) *For any sufficiently small clean neighborhood of x, the module $M^\vee$ is free of rank one over $\mathcal{T}$ at x.*

Proof — The equivalence between (ii) and (iii) is clear. If U is a clean neighborhood, x is the only point of U above w, so $\mathcal{T}_x$ is the fiber $\mathcal{T}_w$ of $\mathcal{T}$ at w and the fiber M_w of M at w is $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w)_{(x)}$ (cf. §IV.6.1), hence $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w)_{(x)}^\vee$ is the fiber of $M^\vee$ at w. Since $M^\vee$ and $\mathcal{T}$ are finitie flat over R, Nakayama's lemma implies that (i) is equivalent to (ii). $\qquad\square$

DEFINITION V.2.2. A point $x \in \mathcal{C}^\pm$ satisfying the conditions of the above lemma is called a *good* point.

If x is a good point, there exists a clean neighborhood of x where all points are good (by (iii) of the Lemma).

THEOREM V.2.3. *If x is a good point of field of definition L, $\kappa(x) = w$, then the eigenspace $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))[x]$ has dimension 1*

Proof — Let $U = \mathrm{Sp}\,\mathcal{T}$ be a clean neighborhod as in (ii) of Lemma V.2.1. Since $\mathcal{T}$ is the eigenalgebra of $\mathcal{H}$ acting on M, it is aso the eigenalgebra of $\mathcal{H}$ acting on $M^\vee$ (cf. exercise I.3.4). Hence since $M^\vee$ is flat over $\mathcal{T}$, by Prop. I.4.3 the fiber $\mathcal{T}_w$ is the eigenalgebra of $\mathcal{H}$ acting on $(M^\vee)_w = M_w^\vee$. Thus $\mathcal{T}_w = \mathcal{T}_x$ is the eigenalgeba of $\mathcal{H}$ acting on $M_w^\vee = \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))_{(x)}^\vee$.

On the other hand, since $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))_{(x)}$ is free of rank 1 over $\mathcal{T}_x$, then $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))_{(x)}^\vee / \mathfrak{p}_x \mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))_{(x)}^\vee$ where $\mathfrak{p}_x$ is the maximal ideal of the local

artinian ring $\mathcal{T}_x$, generated by the $T - x(T)$ for $T \in \mathcal{T}_x$. By elementary duality tehory this quotient is isomorphic to the $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))[x]^\vee$, where $[x]$ here means first the eigenspace for the $\mathcal{T}_x$-action, but also for the $\mathcal{H}$-action by the result of the first paragraph. $\qquad\square$

PROPOSITION V.2.4. *Let x be a point of $\mathcal{C}^\pm$ defined over L, $\kappa(x) = w$. We assume that the restriction $\lambda_{x,0}$ to $\mathcal{H}_0$ of the system of eigenvalues of x is not the system of E_2.*

  (i) *If x is a good point, then x is new away from p, that is the minimal tame level of x is N*

  (ii) *If x is a classical point which is new away from p, and x is smooth, then x is a good point.*

Proof —   Assertion (i) follows immediately from the theorem.

For assertion (ii), choosing a clean neighborhood $U = \mathrm{Sp}\,\mathcal{T}$ of x, our hypothesis implies that the local ring of $\mathcal{T}$ at x is a discrete valuation ring, hence that M is finite flat over $\mathcal{T}$ at x since M is finite torsion-free over $\mathcal{T}$ by the following lemma. Calling r the rank of M over $\mathcal{T}$ at x, we see by the proof of the above theorem that r is the dimension of the eigenspace $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))[x]$. On the other hand, calling as in IV.6.1 e the R-rank of $\mathcal{T}$ and d the R-rank of M, we clearly have $e = dr$. Since x is new away from p, it is not strongly abnormal, and Lemma **??** gives $e = d$. Hence $r = 1$. $\qquad\square$

QUESTIONS V.2.5. Is there a point $x \in \mathcal{C}^\pm$ which is not good? Is there a point $x \in \mathcal{C}$ such that x^+ and x^- both exist, but one is good and the other is not?

V.3. The p-adic L-function of a good point of the eigencurve

DEFINITION V.3.1. Let x be a good point of the eigencurve $\mathcal{C}^\pm$, of field of definition L with $\kappa(x) = w$. We shall denote by μ_x the distribution $\Phi_x(\{\infty\} - \{0\})$ where Φ_x is a generator of the eigenspace $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}_w(L))[x]$, and by $\sigma \to L_p(x, \sigma)$ the Mellin transform $M_L(\mu_x)$ of μ_x, an L-analytic function on the weight space $\mathcal{W}$. Hence μ_x and $s\sigma \mapsto L_p(x, \sigma)$ are only defined up to a multiplication by a non-zero scalar in L^*. We refer to the function $\sigma \mapsto L_p(x, \sigma)$, and sometimes by abuse also to the distribution μ_x as the p-adic L-function attached to x.

It doesn't seem possible to define in general the p-adic L-function of a good point x more precisely that up to a scalar in L^*. However, see next section.

The distribution μ_x satisfies $\mu_x(f(-z)) = \pm f(-z)$, where $\pm$ is the same sign as the sign of the eigencurve $\mathcal{C}^\pm$ from which x is taken. In particular, the function $\sigma \to L_p(x, \sigma)$ is 0 on all σ with $\sigma(-1) \neq \pm1$; that is, this function has support on $\mathcal{W}^\pm$.

The following proposition shows that our notion of p-adic L-function extends the classical notion of Visik and Amice-Vélu of p-adic L-function of non-critical slope modular form.

PROPOSITION V.3.2. *Let $g = \sum_{n=1} a_n q^n \in S_{k+2}(\Gamma, L)$ (L a finite extension of $\mathbb{Q}_p$). Assume that g is an eigenform for $\mathcal{H}$ of nebentypus ϵ, that g is new outside p, and that $v_p(a_p) < k + 1$ (non-critical slope). Also assume that $a_p^2 \neq p^{k+1}\epsilon(g)$ though it is conjectured to be always true. For any choice of the sign $\pm$, let $\mu_g^\pm$ be the distribution attached to g as in §?? (step 3), and let x be the unique point in $\mathcal{C}^\pm(L)$ with the same system of $\mathcal{H}$-eigenvalues as x and such that $\kappa(x) = k+1$. The point x is good, and we have*

$$\mu_x^\pm = \mu_g^\pm$$

where the equality is understood with the same indeterminacy as in the definition of $\mu_x^\pm$, that is up to multiplication by en element of L^* (one for each choice of the sign $\pm$) As a consequence, we have

$$L_p(x, \sigma) = L_p(g, \sigma)$$

where the equality of function is understood as the equality above.

Proof —   The fact that x is good follows from Theorem ?? and Prop. V.2.4. From Theorem ??, the space $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}[L])[x]$ has dimension 1. Let $\Phi_x^\pm$ be a generator of that space. By consutruction $\mu_g^\pm = \Phi_g^\pm(\{0\} - \{\infty\})$, where Φ_g is the unique element of $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}[L])$ such that $\rho_k(\Phi_g^\pm)$ is the classical modular symbol $\phi_g^\pm$ of sign $\pm$ attach to g, which by Theorem ?? exists, is unique, is non-zero, and is an eigenvector for $\mathcal{H}$ with the same system of eigenvalue as x. Hence $\Phi_g^\pm$ is a non-zero element in $\mathrm{Symb}_\Gamma^\pm(\mathcal{D}[L])[x]$, so $\Phi_g^\pm = \Phi_x^\pm$ up to a sclar. The proposition follows. □

Let us finish this section by a point of terminology: Let x be a point of the Coleman-Mazur eigencurve $\mathcal{C}$. Then for both values of $\pm$ we write $\mu_x^\pm$ the distribution $\mu_x^\pm$, and we define a L-analytic function on $\mathcal{W}$: $\sigma \mapsto L_p(x, \sigma)$ by $L_p(x, \sigma) = L_p(\mu_x^+, \sigma)$ if $\sigma(-1) = 1$, and $L_p(x, \sigma) = L_p(\mu_x^-, \sigma)$ if $\sigma(-1) = -1$. We refer to $\sigma \mapsto L_p(x, \sigma)$ (or sometimes, by abuse, to the pair (μ_x^+, μ_x^-)) as the *p-adic L-function* of the good point x. N.B. The p-adic L function of a good point is well-defined only up to multiplication by two scalars in L^*, one for the restriction ot $\mathcal{W}^+$ and one for the restriction to $\mathcal{W}^-$.

V.4. Two-variables p-adic L-function in neghborhoods of good points

THEOREM V.4.1. *Let $x \in \mathcal{C}^\pm$ be a good point. Fix $U = Sp\,\mathcal{T}$ a sufficiently small clean neighborhood of x whose every point is good. Then there exists a measure $\tilde{\mu} \in \mathcal{D}(\mathcal{T})$, unique up to multiplication by an element in $\mathcal{O}(U)^* = \mathcal{T}^*$, such that for all $y \in U$,*

$$ev_y(\tilde{\mu}) = \mu_y$$

where μ_y is the distribution attached to y as in Definition **??**, well-defined up to an element of L^*, and the equality is up to that indeterminacy.

Moreover, a choice of $\tilde{\mu}$ as above being fixed, there exists two constants $0 < c < C$ such that if y is a point of U with $\kappa(y) = k \in \mathbb{Z}$ with the same system of eigenvalues as a cusipidal eigenform $g \in S_{k+2}(\Gamma, L)$, then we have $ev_y(\tilde{\mu}) = c(y)\mu_g^{\pm}$ where $c(y)$ is an element of L^* so that $c < |c(y)| < C$.

Proof — Since x is good, choosing the clean neighborhood U small enough ensures that M is free of rank one over $\mathcal{T}$. We recall that $M = \varepsilon\mathrm{Symb}_\Gamma(\mathcal{D}_K(R))^{\leq\nu}$ for some real ν, and some idempotent ε. Let us fix a positive real r such that $r < r(K)$. Then $\mathrm{Symb}_\Gamma(\mathcal{D}_K(R))^{\leq\nu} = \mathrm{Symb}_\Gamma(\mathcal{D}_K[r](R))^{\leq\nu}$ and thus we can provide $\mathrm{Symb}_\Gamma(\mathcal{D}_K(R))^{\leq\nu}$ with the norm $\| \ \|_r$. We provide M with the restriction of that norm.

We are now in posiition to apply the formalism of section V.1. For $w \in W$, we have $M_w = \varepsilon\mathrm{Symb}_\Gamma^{\pm}(\mathcal{D}_K[r](L))_g^{\leq\nu}$ and the quotient norm of the norm $\| \ \|_r$ is still the norm $\| \ \|_r$ by ...

We define an R-linear map $h : M \to \mathcal{D}(R)$ as $h(\Phi) = \Phi(\{\infty\} - \{0\})$. Then clearly the maps $h_w : M_w = \varepsilon\mathrm{Symb}_\Gamma^{\pm}(\mathcal{D}_w(L))^{\leq\nu} \to \mathcal{D}(L)$ defined from h as in §**??** are also the evaluation at $\{\infty\} - \{0\}$.

Definition V.1 gives us a distribution $\tilde{\mu} \in \mathcal{D}(\mathcal{T})$, and for every $y \in U(L)$ with $\kappa(y) = w$ a distribution $\mu_y \in \mathcal{D}(L)$, defined as $h_w(v_y)$ where v_y is a generator of $M_w[y]$ such that $\|v_y\|_r = 1$. This definition of μ_y (up to an element of $\mathcal{O}_L^*$) is obviously compatible with the definition (which is less precise: up to an element of L^*) hence the first part of the theorem follows from the first part of Prop. V.1.4.

For the second part, let us assume that y corresponds to an eingenform g as in the statement of the theorem. Then we need to compare $\mu_y = h_w(v_y)$ with $\mu_g^{\pm} = \phi_g(\Phi_g^{\pm})$, where $\Phi_g^{\pm}$ is defined (up to $\mathcal{O}_L^*$) as an element of $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{D}_k(L))[g]$ which satisfies $\rho_k^*(\Phi_g^{\pm})$ is a generator of $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{V}_k(\mathcal{O}_L))[g]$, in other words such that $\|\rho_k^*(\Phi_g^{\pm})\| = 1$. Since ρ_k is an isometry for $\| \ \|_1$ on $\mathrm{Symb}_\Gamma^{\pm}(\mathcal{D}_k(L))^{\leq k+1}$, the later condition of equivalent to $\|\Phi_g^{\pm}\|_1 = 1$. By Prop. III.6.27, there exists therefore some constant $D > 0$ depending of our clean neighbrohood U and of the choice of r, but **independent** of the choice of the classical point y or of the corresponding modualr form g such that

$$D \leq \|\phi_g^{\pm}\|_r \leq 1.$$

Writing $v_y = c'(y)\Phi_g^{\pm}$ with $c'(y) \in L^*$, we see that $1 \leq |c'(y)| \leq D^{-1}$ since by defintion $\|v_y\|_r = 1$. Since $\mu_y = c'(y)\mu_g^{\pm}$ and $\mathrm{ev}_y(\tilde{\mu}) = c(y)\mu_y = c(y)c'(y)\mu_g^{\pm}$ with $c(y)$ bounded between two positive constants by Prop. V.1.4, the theorem follows. $\square$

COROLLARY V.4.2. *With U as in the above theorem, there exist a two-variable analytic function $\tilde{L}$ on $U \times \mathcal{W}$, well-defined up to an element of $\mathcal{O}(U)^*$, such that*

for all $y \in U$, and all $\sigma \in \mathcal{W}$

$$\tilde{L}(y, \sigma) = L(y, \sigma)$$

where the equality is up to a scalar in L^*. Moreover, there exist two constant $0 < c < C$ such that if y corresponds to a modular eigenform g of non-critical slope, we have

$$\tilde{L}(y, \sigma) = c(y) L(g, \sigma)$$

with $c \le |c(y)| \le C$.

V.5. Notes and References

Adjoint p-adic L-function and the ramification locus of the eigencurve

This chapter is largely inspired by Kim's 2006 Berkeley thesis [**K**]. Indeed the core result, namely the construction of a scalar product on the module of families of modular symbols interpolating the classical scalar product on classical modular symbols is taken directly from [**K**], with only minor modifications. However, we improve on Kim's treatment by proving an explicit and more general result of perfectness on this scalar product (such a result is implicitly stated in [**K**, Theorem 11.2]), and more importantly by changing the way the adjoint p-adic L-function is constructed from the scalar product, and the way this scalar product is proved to be related to the ramification locus of the eigencurve. Actually, we propose a conceptual treatment of those two aspects together in subsection VI.1 below. It seems to us that there is an gap in the argument of [**K**] concerning the descent of the adjoint p-adic L-function to the eigencurve from its normalization) that our method allows to circumvent. At any rate, we obtain more precise results on the relation between the geometry of the weight map and the adjoint p-adic L-function.

Let us now describe in more details these results. As usual, we denote by $\mathcal{C}^0$ the cuspidal eigencurve on tame level N. We construct, using Kim's scalar product on modular symbols, a canonical sheaf of ideals $\mathcal{L}^{\mathrm{adj}}$ of $\mathcal{O}(\mathcal{C}^0)$, and we prove (Theorem VI.4.2) that its closed locus contains all points of $\mathcal{C}^0$ where the weight map κ is not étale. We ignore if the iseal $\mathcal{L}^{\mathrm{adj}}$ is everywhere locally principal, but it is so in a neighborhood of most points of interest, including all classical points that are of minimal tame level N (except possibly for a few Eisenstein series of weight 2 when N is not square-free) and all their companion points when they have one. In a suitable neighborhood U of such a *good* point, we can define the p-adic adjoint L_U^{adj} as a generator of the ideal $\mathrm{Ł}^{\mathrm{adj}}$, and we can determine exactly where this function vanishes (see Theorem VI.4.7):

(i) If $x \in U$ is a point that either has non-integral weight $\kappa(x) \notin \mathbb{N}$, or is cuspidal classical, then $\mathrm{Ł}_U^{\mathrm{adj}}(x) = 0$ if and only if teh weight map κ is étale at x.

(ii) Otehrwise, that is if $x \in U$ has $\kappa(x) = k \in \mathbb{N}$ and x is not a cuspidal classical point (such points x are necessarily slope $\geq k+1$), then $\mathrm{Ł}_U^{\mathrm{adj}}(x) = 0$ anyway.

Note that in case (ii) it may or may not happen that κ is étale at x. For example, if x corresponds to an evil Eisenstein series of minimal tame level N, then it is expected that κ is étale at x, and this is known for example for the Eisenstein series E_k^{crit} (for any even $k > 4$, so for $N = 1$) when p is a regular prime. On the other hand, if x corresponds to the Eisenstein series $E_{2,\ell}^{\mathrm{crit}}$ (where $\ell \neq p$ is an auxiliary prime, and $E_{2,\ell}(z) = E_{(}z) - \ell E_2(\ell z)$, and $E_{2,\ell}^{\mathrm{crit}}$ is the criticial refinement at p of $E_{2,\ell}^{\mathrm{crit}}$) of an eigencurve of tame level N where N is divisible by at least two primes, then it is not hard to prove (see a work of D. Majumdar for details) that several components of the eigencurve $\mathcal{C}^0$ meet at x, and therefore that κ is not étale at x.

When x is a smooth point of the eigencurve (again this includes almost all classical points), we can define two invariants at x: the degree of ramification $e(x)$ of κ at x (so κ is étale at x if and only if $e(x) = 1$), and the order of vanishing of L_U^{adj} at x. It is natural to ask how these invariants are related : an answer is given in Theorem VI.4.8.

Finally we need to justify the name of the adjoint p-adic L-function L_U^{adj} by proving that at cuspdial classical point of non-critical slope x corresponding to the form f_x, the value $L_U^{\mathrm{adj}}(x)$ is related to the value of the archimedean adjoint p-adic L-function of f_x at the near central point. Kim ([**K**, Theorem 12.3]) proves such a formula in the case f_x has wild level p^r with $r \geq 1$ and primitive nebemtypus. We extend this formula to the case of a form of level p and trivial nebentypus, in particular a form f_x which is a refinement of a form of level prime to p (note typsetted yet).

VI.1. The L-ideal of a scalar product

In all this §, R is a Noetherian domain (of fraction field K), and T is a finite R-algebra.

VI.1.1. The Noether different of T/R. Let $m : T \otimes_R T \to T$ denotes the multiplication map: $m(t \otimes u) = tu$. Let I be the kernel of this map.

LEMMA VI.1.1. *As an ideal of $T \otimes_R T$ the ideal I is generated by the elements of the form $t \otimes 1 - 1 \otimes t$ for $t \in T$.*

Proof — It is clear that $t \otimes 1 - 1 \otimes t \in I$. Conversely, if $\sum_k t_k \otimes u_k \in I$ then $\sum t_k u_k = 0$, hence $\sum_k t_k \otimes u_k = \sum (1 \otimes u_k)(t_k \otimes 1 - 1 \otimes t_k)$ which shows the result. $\square$

A trivial but important remark is the following. If M is a $T \otimes_R T$-module, it has two natural structures of T-modules, using the two natural morphisms of algebras $T \to T \otimes_R T$: $t \mapsto t \otimes 1$, $t \mapsto 1 \otimes t$. On the sub-module $M[I] = \{m \in M, Im = 0\}$, the two T-structures coincide, and in fact, $M[I]$ is the largest sub-module of M on

which the two T-structures coincide. In particular, we can see $M[I]$ as a T-module unambiguously.

DEFINITION VI.1.2 ([**N**]). The *Noether's different* $\mathfrak{d}_N(T/R)$ is the ideal of T which is the image of $(T \otimes_R T)[I]$ by m.

The Noether's different is also called *homological different*.

The main interest of this notion is in the following theorem (due, it seems, in this generality to Alexander-Buchsbaum). Recall that for any extension T/R of algebras, a prime $\mathfrak{P}$ of T is said *unramified* over R if for $\mathfrak{p} = \mathfrak{P} \cap R$, we have $\mathfrak{p}T_{\mathfrak{P}} = \mathfrak{P}T_{\mathfrak{P}}$, and $T_{\mathfrak{P}}/\mathfrak{P}T_{\mathfrak{P}}$ is a finite separable extension of $R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}}$. It is said *ramified* if it is not unramified.

THEOREM VI.1.3. *A prime ideal $\mathfrak{P}$ of S is ramified over R if and only if $\mathfrak{d}_N(T/R) \subset \mathfrak{P}$. In other words, the closed subset of $\operatorname{Spec} T$ defined by the Noether's different is the ramification locus of $\operatorname{Spec} T \to \operatorname{Spec} R$.*

For the proof, see [**AB**, Theorem 2.7]. Their theorem is even more general, as they do not actually assume that T/R is finite, but the much weaker assumption that T is Noetherian and I is finitely generated. We shall apply this result in a situation where T/R is not only finite but also flat. In this case, $\mathfrak{P}$ is unramified over R if and only if it is étale, and if and only if it is smooth. Hence the closed set of $\operatorname{Spec} T$ defined by the Noether's different is also the set of points where $\operatorname{Spec} T \to \operatorname{Spec} R$ is not étale.

There are at least two others notions of *different*. We shall not need them, but for the sake of completeness, let us recall them quickly:

The oldest notion is *Dedekind's different* which is defined only under supplementary hypotheses. Namely one assumes that T *is reduced*, and calling K the fraction ring of R and L the total fraction ring of T (which is finite product of fields L_i, each of them finite over K), we assume that L is étale over K (that is, each of the L_i is a separable extension of K). In Dedekind's definition as in most treatments of this notion in the literature, it is moreover assumed that R is Dedekind, but this is not actually necessary. Here is how the Dedekind different is defined: the trace map $\operatorname{tr}_{L/K}$ defines a non-degenerate K-bilinear pairing on L: $(x, y) \mapsto \operatorname{tr}(xy)$. Let T^* be the set of $x \in L$ such that $\operatorname{tr}_{L/K}(xy) \in R$ for all y in T. It is clear that T^* is a T-module, and that it is actually the largest T-submodule of L such that $\operatorname{tr}_{L/K}(T^*) \subset R$. In particular, since $\operatorname{tr}_{L/K}(T) = R$, we have $T \subset T^*$. The *Dedekind's different* $\mathfrak{d}_D(T/R)$ is the ideal $[T : T^*]$ of T, that is the set of $y \in T$ such that $xT^* \subset T$. This definition is the one given in all textbooks on algebraic number theory (where R is invariably assumed to be Dedekind). See e.g. [**S**, Chapter III] for a fairly complete treatment in this case.

The most modern notion is *Kähler's different* $\mathfrak{d}_K(T/R)$ which is defined as the 0-th fitting ideal of the module of differential forms $\Omega_{T/R}$ (cf. [**Ka**]). More

concretely, we can compute Kähler's different as follows: choose $x_1, \ldots, x_n$ a family of generators of T as R-algebra, and let $(P_1, \ldots, P_m)$ in $R[X_1, \ldots, X_n]$ the ideals of their algebraic relations over R. In other words $R[X_1, \ldots, X_n]/(P_1, \ldots, P_m) \simeq T$, the map sending X_i to x_i. Note that we have $m \geq n$ since T is finite over R by assumption but that it is not always possible to choose $m = n$ (when it is, T is said to be *complete intersection* over R). Form the $m \times n$-Jacobian matrix in $R[X]$ whose (i, j) coefficients is $\frac{\partial P_i}{\partial X_j}$. The Kähler's different is then the T-ideal generated by the image in T of the $(n \times n)$-minors of that matrix. This is very easy from the definitions of the module of differentials and of the Fitting ideal, both given in [**E**].

Recall that we always assume that R is a noetherian domain of fraction field K, and that T/R is finite. The following result summarizes the known (to the author) relations between those three notions.

THEOREM VI.1.4. *When T/R is flat, the ideal $\mathfrak{d}_N(T/R)$ and $\mathfrak{d}_K(T/R)$ define the same closed subset of $\operatorname{Spec} T$ (that is, have the same radical), namely the set of points where $\operatorname{Spec} T \to \operatorname{Spec} R$ is non-étale. The same is true if instead of assuming that T/R is flat, we assume that R is integrally closed, that T is reduced, and that the total fraction ring L of T is étale over K.*

If T/R is flat and the Dedekind's different is defined, it is equal to Noether's different $\mathfrak{d}_D(T/R) = \mathfrak{d}_N(T/R)$.

If T/R is flat and locally complete intersection, then $\mathfrak{d}_N(T/R) = \mathfrak{d}_K(T/R)$.

Proof — Since the 0-th Fitting ideal of a module defines the same closed subset as the annihilator of that module, the closed set defined by the Kahler's different is the support of $\Omega_{T/R}$ which is by definition the non-étale locus of $\operatorname{Spec} T \to \operatorname{Spec} R$. When T/R is finite and flat, the non-étale locus is the same as the ramification locus of $\operatorname{Spec} T \to \operatorname{Spec} R$ since étale = flat and unramifed. The non-étale locus is also the same as the ramification locus in the case we assume that R is integrally closed that the total fraction ring of T is a product of finite separable extensions of $\frac{(}{R})$, for in this case, an unramified point of $\operatorname{Spec} T$ is flat over $\operatorname{Spec} R$ by [**AB**, §4]. This completes the proof of the first paragraph of the theorem.

For the second assertion, see [**N**] or [**AB**, §3] (which contain more results along this lines)

For the third assertion, see e.g. [**Ku**, Theorem 8.15]. $\qquad\qquad \square$

VI.1.2. Duality. If M is a T-module which is finite and flat as an R-module, the module $M^\vee := \operatorname{Hom}_R(M, R)$ is also finite and flat over R, and has also a T-module structure. Moreover we have a canonical isomorphism $(M^\vee)^\vee = M$ as R-modules and even as T-modules, and for N a second T-module which is finite and flat as an R-module the transposition defines a canonical isomorhism $\operatorname{Hom}_T(M, N) = \operatorname{Hom}_T(N^\vee, M^\vee)$.

Note that the formation of $\operatorname{Hom}_R(M,N)$ and in particular of the T-module $M^\vee$ commutes with any base change $R \to R'$ in the obvious sense (This is proved exactly as the special case $M = N$ in §I.1.) The formation of $\operatorname{Hom}_T(M,N)$ however only commutes in general with localizations of R.

If T is finite and flat over R, then the preceding apply to $M = T$ and we can define the T-module $T^\vee$, whose formation commutes with arbitrary base change. We say that the R-algebra T is *Gorenstein* if $T^\vee$ is flat of rank one over T. Locally, this is same as assuming the existence of an isomorphism of T-modules $T^\vee \simeq T$.

PROPOSITION VI.1.5. *The natural isomorphism* $\operatorname{Hom}_R(M,N) = M^\vee \otimes_R N$ *restricts to an isomorphism* $\operatorname{Hom}_T(M,N) = (M^\vee \otimes_R N)[I]$.

Proof —   Recall that the natural isomorphism is the one that sends $\sum_i l_i \otimes n_i \in M^\vee \otimes_R N$ to the morphism $f : M \to N$ defined by $f(m) = \sum_i l_i(m)n_i$ (one checks locally that it is an isomorphism). It is sufficient to observe then that $\sum_i l_i \otimes n_i$ is in $(M^\vee \otimes_R N)[I]$ if and only if $\sum_i(tl_i) \otimes n_i = \sum_i(l_i \otimes tn_i)$ for all $t \in T$, if and only if f is T-equivariant. $\square$

COROLLARY VI.1.6. *Assume that* T/R *is flat. The* R-*algebra* T *is Gorenstein if and only if* $(T \otimes_R T)[I]$ *is flat of rank one as* T-*module.*

Proof —   We may assume that R is local, and we have to prove in the case that T/R is Gorenstein if and only if $(T \otimes_R T)[I]$ is free of rank one as T-module.

We have $(T \otimes_R T)[I] = \operatorname{Hom}_T(T^\vee, T)$. So if T/R is Gorenstein, $T^\vee \simeq T$ and $(T \otimes_R T)[I] \simeq \operatorname{Hom}_T(T,T) = T$ as T-modules. Conversely, if $(T \otimes_R T)[I]$ is isomorphic to T, then let $f \in \operatorname{Hom}_T(T^\vee, T)$ corresponding to a generator of that module. Then for $t \in T$, $tf = 0$ implies that $t = 0$. Therefore for $t \in T$, $tf(T) = 0$ implies that $t = 0$. Since T is finite over R, it has no proper ideal having this property, and f is surjective. By equality of R-rank, f is an isomorphism. $\square$

COROLLARY VI.1.7. *If* T/R *is Gorenstein,* $\mathfrak{d}_N(T/R)$ *is a locally principal ideal.*

Proof —   Indeed, locally, this ideal is generated by $m(g)$ where g is a generator of $(T \otimes_R T)[I]$ $\square$

VI.1.3. The L-ideal of a scalar product. Now let M and N be two T-modules, which as R-modules are finite flat. Let $b : M \times N \to R$ be an R-bilinear scalar product which is T-equivariant (that is $b(tm,n) = b(m,tn)$ for $t \in T$, $m \in M$, $n \in N$).

DEFINITION VI.1.8. We call L_b the ideal of T defined by the following construction: extend by linearity the R-linear scalar product b to a T-bilinear scalar product $b_T : (M \otimes_R T) \times (N \otimes_R T) \to T$. Set

$$L_b = b_T((M \otimes_R T)[I], (N \otimes T)[I]) \subset T.$$

Some clarifications may help: $(M \otimes_R T)$ and $(N \otimes_R T)$ have two T-modules structures, or what amounts to the same, a $T \otimes_R T$-module structure. The first T-structure is given by the action of T on M or N, the second by the action of T on T. When we say that b_T is T-bilinear, we are referring to the second T-structure.

We note that the formation of L_b commute with localizations on R. (In general, it doesn't commute with arbitrary base change, but see below).

PROPOSITION VI.1.9. *We have*

$$L_b \subset \mathfrak{d}_N(T/R)$$

Proof — Consider the map $B : M \otimes_R T \otimes_R N \otimes_R T \to T, \, m \otimes t \otimes n \otimes u \mapsto tub(m,n)$. This maps clearly factors through the quotient $(M \otimes_R T) \otimes_T (N \otimes_R T) \to T$ and induces b_T on that quotient. Hence

$$L_b = B((M \otimes_R T)[I]), (N \otimes_R T)[I].$$

There is an other way to factorize B:

$$B : M \otimes_R T \otimes_R N \otimes_R T \to T \xrightarrow{c} (M \otimes_R N) \otimes_R (T \otimes T) \xrightarrow{b \otimes 1} (T \otimes T) \xrightarrow{m} T.$$

where the first map is just the isomorphism c given by commutativity of tensor product.

We claim that $(b \otimes 1) \circ c$ sends $(M \otimes_R T)[I] \otimes_R (N \otimes_R T)[I])$ into $(T \otimes_R T)[I]$. Indeed if $x = \sum_k m_k \otimes t_k \in (M \otimes_R T)[I]$, and $y = \sum_l n_l \otimes u_l \in (N \otimes_R T)[I]$ then $z := (b \otimes 1)(c(x,y)) = \sum_{k,l} b(m_k, n_l) t_k \otimes u_l$ and we compute, for $t \in T$

$$
\begin{aligned}
(1 \otimes t)z &= \sum_{k,l} b(m_k, n_l) t_k \otimes (t u_l) \\
&= \sum_{k,l} b(m_k, t n_l) t_k \otimes u_l \quad \text{(using that } Iy = 0\text{)} \\
&= \sum_{k,l} b(t m_k, n_l) t_k \otimes u_l \quad \text{(using the } T\text{-equivariance of } b\text{)} \\
&= \sum_{k,l} b(m_k, n_l)(t t_k) \otimes u_l \quad \text{(using that } Ix = 0\text{)} \\
&= (t \otimes 1)z
\end{aligned}
$$

Hence we see that $I_b = B((M \otimes_R T)[I] \otimes_R (N \otimes_R T)[I])) \subset m((T \otimes_R T)[I]) = \mathfrak{d}_N(T/R)$. $\qquad \square$

PROPOSITION VI.1.10. *Assume that $M \simeq N \simeq T^\vee$ as T-modules. Then the ideal L_b of T is principal.*

More precisely, the scalar product b can be seen as a linear map $b : T^\vee \otimes_R T^\vee \to R$. By duality, this is the same as an element $\tilde{b}$ in $T \otimes_R T$. We have $L_b = m(\tilde{b})T$.

Proof —   We have $(M \otimes T)[I] \simeq (T^\vee \otimes T)[I] = \mathrm{Hom}_T(T,T) = T$. Let g be a generator of this module. Similarly, let h be a generator of $(N \otimes T)[I]$. Then $b_T(g,h)$ is a generator of L_b.

For the "more precisely", let us identify M and N with $T^\vee$. Let e_i be a basis of T over R and $e_i^\vee$ a basis of $T^\vee$ over R. Take $g = h = \sum_i e_i^\vee \otimes e_i$; it is a generator of $(T^\vee \otimes T)[I]$, since it corresponds to the identity in $\mathrm{Hom}_R(T,T)$. Then by definition: $b_T(g,h) = \sum_{i,j} b(e_i^\vee, e_j^\vee) e_i e_j b(e_i^\vee, e_j^\vee)$. On the other hand, as an element of $T \otimes_R T$, $\tilde{b}$ is by definition $\sum_{i,j} b(e_i^\vee, e_j^\vee) e_i \otimes e_j$. Hence $m(\tilde{b}) = b_T(g,h)$ is a generator of L_b. $\square$

COROLLARY VI.1.11. *Keep the assumptions of the preceding proposition. Then the formation of L_b commutes with arbitrary base changes. (For the sake of precision, this means the following: if R' is an R-algebra, let $M' = M \otimes_R R'$, $N' = N \otimes_R R'$ and $T' = R \otimes_R R'$. Let $b' : M' \otimes N' \to R'$ be the R'-bilinear extension of b, which is clearly T'-equivariant. Then the ideal $L_{b'}$ of T' is the ideal generated by the image of L_b by the morphism $T \to T'$)*

Proof —   The assumption $M \simeq N \simeq T^\vee$ as T-modules implies similar assertions over R': $M' \simeq N' \simeq T'^\vee$ as T'-modules. The element $\tilde{b}'$ in $T' \otimes_{R'} T'$ is just the natural image of $\tilde{b}$, and thus $L_{b'}$, which is the ideal generated by $m(\tilde{b}')$ is generated by the image of $m(\tilde{b})$. The result follows. $\square$

COROLLARY VI.1.12. *Assume that T/R is Gorenstein, and that M and N are flat of rank one over T. Then L_b and $\mathfrak{d}_N(T/R)$ are locally principal, and $L_b = \mathfrak{d}_N(T/R)$ if and only if b is non-degenerate. More precisely, if $b = tb_0$ where $b_0 : M \times N \to R$ is a non-degenerate T-equivariant scalar product, then $L_b = t\mathfrak{d}_N(T/R)$.*

Proof —   Note that since T/R is Gorenstein, locally $T^\vee \simeq T$ as T-module, so we have $M \simeq N \simeq T^\vee$ and we are in a special case of the above proposition. In particular, L_b is principal (locally), and we already know that the Noether's different is principal (locally) since T/R is Gorenstein.

For the rest of the proof we assume that R is local.

Clearly b is non-degenerate if and only if for all $t \in T$, $tb = 0$ implies $t = 0$, which is equivalent to the same assertion for $\tilde{b}$. Since T is Gorenstein, $(T \otimes T)[I] = T$, and we see that b is non-degnerate if and only if $\tilde{b}$ is a generator of $(T \otimes T)[I]$.

Writing $\tilde{b} = t\tilde{b}_0$ with $t \in T$, and where $\tilde{b}_0$ is a generator of $(T \otimes T)[I]$, we have $m(\tilde{b}) = tm(\tilde{b}_0)$ hence $L_b = t\mathfrak{d}_N(T/R)$ with equality if and only if t is invertible, i.e. if and only if b is non-degenerate. $\qquad\qquad\square$

REMARK VI.1.13. Let us go back to the hypothesis of Prop VI.1.3: $M^\vee$ and $N^\vee$ free of rank one over T. If b is non-degenerate, then b defines an isomorphism $M \simeq N^\vee$ as T-modules, so $T^\vee \simeq T$ and T is Gorenstein, and the corollary implies that $L_b = \mathfrak{d}_N(T/R)$. Conversely, if $L_b = \mathfrak{d}_N(T/R)$, is it necessarily true that b is non-degenerate (and therefore T Gorenstein)? I'd like to know the answer.

VI.2. Kim's scalar product

Let us use the same notations of §IV.1: $W = \operatorname{Sp} R$ is an open affinoid subset of $\mathcal{W}$, which belongs to the covering $\mathfrak{C}$ (in particular R and its residue rong $\tilde{R}$ are PID, and $\mathbb{N} \cap W$ is very Zarsiki-dense in W), $K : \mathbb{Z}_p^* \to R^*$ the canonical character, and $\nu \geq 0$ is a real number, adapted to W. Let us fix a real r such that $0 < r < \max(r(W), p)$.

The aim of this section is to construct, if possible, an R-bilinear scalar product b on $\operatorname{Symb}_\Gamma(\mathcal{D}_K[r](R))^{\leq \nu}$ that *interpolate* the corrected scalar products on $\operatorname{Symb}_\Gamma(\mathcal{V}_k[r](\mathbb{Q}_p))^{\leq \nu}$, for $k \in \mathbb{N} \cap W(\mathbb{Q}_p)$ that we denoted by $[\,,\,]$ (def. III.2.25), corrected from the classical pairing defined using Poincarés duality, cf. (28), Let us precise what we mean by "interpolate". For any $\kappa \in W(L)$, where L is a finite extension of $\mathbb{Q}_p$, by extension of scalars $\kappa : R \to L$ the scalar product b gives rises to a L-bilinear scalar product b_κ on $\operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r](R))^{\leq \nu} \otimes_{R,\kappa} L = \operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))_{\tilde{g}}^{\leq \nu} \subset \operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))^{\leq \nu}$ (cf. Theorem IV.1.17 and the definition following it). If κ is character $z \mapsto z^k$, and $k \in \mathbb{N}$, $\operatorname{Symb}_\Gamma(\mathcal{D}_\kappa[r](\mathbb{Q}_p))^{\leq \nu}$ admits the space of classical modular symbols $\operatorname{Symb}_\Gamma(\mathcal{V}_k(\mathbb{Q}_p))^{\leq \nu}$ as a quotient. By "interpolating", we mean that the scalar product b_k factors through that quotient and is equal, on that quotient, to our old good corrected scalar product.

VI.2.1. A bilinear product on the space of overconvergent modular symbols of weight k. Let $k \geq 0$ be an integer. Assume that L is a finite extension $\mathbb{Q}_p$. Remember the surjective map $\rho_k : \mathcal{D}_k[1](L) \to \mathcal{V}_k(L)$, defined as the dual of the inclusion $\mathcal{P}_k(L) \to \mathcal{A}_k[1](L)$ that we shall here denote by i_k. Those maps ρ_k and i_k induce maps on modular symbols, e.g. $\rho_k : \operatorname{Symb}_\Gamma(\mathcal{D}_k[1](L)) \to \operatorname{Symb}_\Gamma(\mathcal{V}_k(L))$.

DEFINITION VI.2.1. We define a scalar product on $\operatorname{Symb}_\Gamma(\mathcal{D}_k[1](L))$, denoted by $[\,,\,]_k$ by

$$[\Phi_1, \Phi_2]_k = [\rho_k(\Phi_1), \rho_k(\Phi_2)],$$

where $[\,,\,]$ is the corrected scalar product on classical modular symbols, defined in Def. III.2.25. We use the same notation $[\,,\,]_k$ for the restriction of this scalar product to the finite-dimensional subspace $\operatorname{Symb}_\Gamma(\mathcal{D}_k[1](L))^{\leq \nu} = \operatorname{Symb}_\Gamma(\mathcal{D}_k[r](L))^{\leq \nu}$ (where r is any real number such that $0 < r < p$).

We shall now give an explicit description of this scalar product. Below, we consider $\mathcal{P}_k(L)$ with its **right** action of S, so that it is the contragredient of the right S-module $\mathcal{V}_k(L)$. We denote by δ the tautological product $\mathcal{P}_k(L) \times \mathcal{V}_k(L) \to L$. Similarly we shall consider $\mathcal{A}_k[r](L)$ with its natural **right** action of $S_0(p)' = \{\gamma \in S, \ p|c, \ p \nmid d\}$:

$$f_{|_k\gamma}(z) = (\gamma' \cdot_k f)(z) = (cz+d)^k f\left(\frac{az+b}{cz+d}\right)$$

The inclusion $i_k : \mathcal{P}_k(L) \to \mathcal{A}_k[r](L)$ evidently respects our right action. We have a natural pairing $\Delta : \mathcal{D}_k[r](L) \times \mathcal{A}_k[r](L) \to L$ defined by $\Delta(\mu, f) = \mu(f)$, which is obviously compatible with δ in the sense that

(89) $$\Delta(\mu, i_k(P)) = \delta(\rho_k(\mu), P).$$

We see by an easy computation

$$\Delta(\mu_{|_k\gamma}, f_{|_k\gamma}) = \det(\gamma)^k \Delta(\mu, f)$$

for all $\gamma \in S_0(p) \cap S_0'(p)$, and a similar formula holds for δ. In particular, the pairings Δ and δ are pairings of $\Gamma_0(p)$-modules to L, so they define pairings, also denoted Δ and δ

$$\Delta : \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L)) \times \mathrm{Symb}_\Gamma(\mathcal{A}_k[r](L)) \to L,$$

$$\delta : \mathrm{Symb}_\Gamma(\mathcal{V}_k(L)) \times \mathrm{Symb}_\Gamma(\mathcal{P}_k(L)) \to L,$$

We now can compute:

$$
\begin{aligned}
[\Phi_1, \Phi_2]_k &= [\rho_k(\Phi_1), \rho_k(\Phi_2)] \quad \text{(by def. VI.2.1)} \\
&= (\rho_k(\Phi_1), \rho_k(\Phi_2)_{|W_{Np}}) \quad \text{(by def. III.2.25)} \\
&= \delta(\rho_k(\Phi_1), \theta_k(\rho_k(\Phi_2)_{|W_{Np}}))) \quad \text{(by def. 28)} \\
&= \Delta(\Phi_1, i_k\theta_k(\rho_k(\Phi_2)_{|W_{Np}}) \quad \text{by (89)} \\
&= \Delta(\Phi_1, V_k(\Phi_2)),
\end{aligned}
$$

where V_k is the map

$$V_k : \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L)) \to \mathrm{Symb}_\Gamma(\mathcal{A}_k[r](L))$$
$$\Phi \mapsto i_k\theta_k(\rho_k(\Phi)_{|W_{Np}}).$$

Now V_k is easily described using the definitions of its constituents: for any $D \in \Delta_0$, $\rho_k(\Phi)(D)(z^i) = \Phi(D)(z^i)$ provided that $0 \le i \le k$. Then $\rho_k(\Phi)_{|W_{Np}}(D)(z^i) = \Phi(W_{Np} \cdot D)(z^i_{|_k W_{Np}}) = \Phi(W_{Np} \cdot D)((-Npz)^{k-i})$. In other words,

$$\rho_k(\Phi)_{|W_{Np}}(D) = \sum_{i=0}^{k} \Phi(W \cdot D)((-Npz)^{k-j})l_i,$$

where $l_i \in \mathcal{V}_k(L)$ is the linear form on $\mathcal{P}_k(L)$ sending z^j to $\delta_{i,j}$.

Using the formula for the operator θ_k given in Lemma III.2.1, we see that

$$\theta_k(\rho_k(\Phi)_{|W_{Np}})(D)(z) = \sum_{i=0}^{k} \Phi(W_{Np} \cdot D)((-Npz)^{k-i})(-1)^i \binom{k}{i} z^{k-i}$$

$$= (-1)^k \sum_{i=0}^{k} (Np)^i \binom{k}{i} \Phi(W_{Np} \cdot D)(z^i) \, z^i$$

Slightly more generally, we also have for any $e \in \mathbb{Z}$

$$\theta_k(\rho_k(\Phi)_{|W_{Np}})(D)(z) = (-1)^k \sum_{i=0}^{k} (Np)^i \binom{k}{i} \Phi(W_{Np} \cdot D)(z^i(1 - Npez)^{k-i}) \, (z - e)^i$$

This follows by applying the above equality to Φ replaced by $\Phi_{|\gamma}$ with $\gamma = \begin{pmatrix} 1 & 0 \\ Npe & 1 \end{pmatrix} \in$ S, using that γ commutes with ρ_k and θ_k and that $\gamma W_{Np} = W_{Np} \begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$.

Applying i_k amounts to seeing that polynomial in z as an element of $\mathcal{A}[r](L)$. Such an element is described by its Taylor series about any $e \in \mathbb{Z}_p$, which is just

$$\iota_k \theta_k(\rho_k(\Phi)_{|W_{Np}})(D)(z) = ((-1)^k \sum_{i=0}^{k} (Np)^i \binom{k}{i} \Phi(W_{Np} \cdot D)(z^i(1 - Npez)^{k-i}) \, (z - e)^i$$

Note that in the above sum, we can replace $\sum_{i=0}^{k}$ by $\sum_{i=0}^{\infty}$ as the new terms are 0 because of their factor $\binom{k}{i}$.

To summarize,

PROPOSITION VI.2.2. *For any $\Phi_1, \Phi_2 \in \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L))$, we have*

(90) $$[\Phi_1, \Phi_2]_k = \Delta(\Phi_1, V_k(\Phi_2)),$$

where $V_k(\Phi) \in \mathrm{Symb}_\Gamma(\mathcal{A}_k[r](L))$ is defined, for any $D \in \Delta_0$, $e \in \mathbb{Z}_p$, and $z \in \mathbb{Z}_p$, by

(91) $$V_k(\Phi)(D) = (-1)^k \sum_{i=0}^{\infty} (Np)^i \binom{k}{i} \Phi(W_{Np} \cdot D)(z^i(1 - Npez)^{k-i}) \, (z - e)^i.$$

VI.2.2. Interpolation of those scalar products. Let L be any $\mathbb{Q}_p$-Banach algebra with a norm $||$ extending the p-adic norm of $\mathbb{Q}_p$, and $\kappa \in \mathcal{W}(L)$ be a continuous character $\mathbb{Z}_p^* \to L^*$.

DEFINITION VI.2.3.   (i) If $y \in L$, $i \in \mathbb{N}$, we define $\binom{y}{i} = \frac{y(y-1)\ldots(y-i+1)}{i!}$, if $i \geq 0$, and $\binom{y}{0} = 1$.
(ii) If $\kappa \in \mathcal{W}(L)$, we define $\binom{\kappa}{i} = \binom{\log_p \kappa(\gamma)/\log_p(\gamma)}{i}$, where γ is a generator of $1 + p\mathbb{Z}_p$

The first definition is standard and due to Isaac Newton. The second is easily seen to be independent of the choice of γ, and to be compatible with the first one, on the sense that $\binom{\kappa}{i} = \binom{k}{i}$ when κ is an integer k, that is to say the character $z \mapsto z^k$,

EXERCISE VI.2.4. Prove that $|\binom{\kappa}{i}| < p^{\frac{ip}{p-1}}$

The key observation is that formula (91) makes sense when k is replaced by an arbitrary character $\kappa \in \mathcal{W}(L)$. Indeed, for any $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L))$, $D \in \Delta_0$, $e \in \mathbb{Z}_p$, we can form the formal Taylor expansion at e:

$$(92) \quad T_{\kappa,\Phi,D,e}(z) := \kappa(-1) \sum_{i=0}^{\infty} (Np)^i \binom{\kappa}{i} \Phi(W \cdot D)(z^i(1 - eNz)^{k-i})(z - e)^i.$$

which when $\kappa = k$ is exactly the RHS of (91).

LEMMA VI.2.5. *Assume that* $r < p^{\frac{-1}{p-1}}$. *The series in the RHS of (92) converges on the closed ball* $|z - e| < r$.

Proof — We have $|(Np)^i| = p^{-i}$ since N is prime to p, $\binom{\kappa}{i}_{|<p^{\frac{ip}{p-1}}}$ by Exercise VI.2.4, and $|\Phi(W_{Np} \cdot D)(z^i(1 - eNz)^{k-i}) \leq \|\Phi(W_{Np} \cdot D)\|_r$ since $\|z^i(1 - eNz)^{k-i}\|_r \leq \sup_{z \in \mathcal{O}_{\mathbb{C}_p}} |z^i(1 - eNz)^{k-i}| \leq 1$. Hence the coefficient of $(z - e)^i$ has norm at most $p^{\frac{i}{p-1}}$, which means that the series converges on disc any disc $|z - e| < p^{\frac{-1}{p-1}}$, hence the lemma. $\square$

PROPOSITION VI.2.6. *Let L be an affinoid algebra, $\kappa \in \mathcal{W}(L)$. We assume that* $r < \min(p^{\frac{-1}{p-1}}, r(\kappa))$.
   (i) *Fix* $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))$, *and* $D \in \Delta_0$. *Then the convergent series* $T_{\kappa,\Phi,D,e}(z)$ *for* $e \in \mathbb{Z}_p$ *define an element in* $\mathcal{A}_k[r](L)$.
   (ii) *Fix* $\Phi \in \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L))$, *and define* $V_\kappa(\Phi) \in \mathrm{Hom}(\Delta_0, \mathcal{A}_k[r](L))$ *by sending* $D \in \Delta_0$ *to the element of* $\mathcal{A}_k[r](L)$ *defined in (i). Then* $V_\kappa(\Phi)$ *lies in* $\mathrm{Symb}_\Gamma(\mathcal{A}_\kappa[r](L))$

Proof — This can probably be shown by brutal computation, but there is a more clever way to do it.

We first consider a particular case: let us assume that $L = R$ when $\mathrm{Sp}\, R = W$ is an affinoid of $\mathcal{W}$ belonging to $\mathfrak{C}$, and $\kappa = K$ is the canonical character $\mathbb{Z}_p^* \to R^*$. To prove (i), we need to prove that for e, e' in $\mathbb{Z}_p$, $T_{K,\phi,D,e}(z)$ and $T_{K,\phi,D,e'}(z)$ have the same values for any $z \in \mathbb{Z}_p$ such that $|z - e| \leq r$, $|z - e'| \leq r$. To prove that those two elements of R are equal, it suffices to prove that there image by any morphisms $R \to \mathbb{Q}_p$ corresponding to an element $k \in \mathbb{N} \cap W$ are equal, since $\mathbb{N} \cap W$ is Zariski-dense in W. But those images are just $T_{k,\phi,D,e}(z)$ and $T_{k,\phi,D,e'}(z)$, and we know from the last paragraph that those two series actually define the same polynomial independent of e, namely $(-1)^k \sum_{i=0}^{k} (Np)^i \binom{k}{i} \Phi(W_{Np} \cdot D)(z^i) z^i$. Hence (i) is done in this case. For (ii), we need to prove that $V_K(\Phi)_{|K\gamma} = V_K(\Phi)$ for $\gamma \in \Gamma$. Again, this reduces to proving that $V_k(\Phi)_{|k\gamma} = V_K(\Phi)$ for all integer $k \in \mathbb{N} \cap W$, which we know since by definition $V_k(\Phi) \in \mathrm{Symb}_\Gamma(\mathcal{D}_k[r])$.

The general case (L, κ) can be reduced to the the case (R, K) as follows. The character $\kappa \in \mathcal{W}(L)$ defines a morphism $\mathrm{Sp}\, L \to \mathcal{W}$ whose image is in some $W = \mathrm{Sp}\, R$ in $\mathfrak{C}$, since those W's form an admissible covering of $\mathcal{W}$. For such a $W = \mathrm{Sp}R$,

we have a morphism of algebra $R \to L$ and the character κ is just the composition of K with this morphism. Therefore, the case (L, κ) follows from (R, K) by base change $R \to L$ (using that the formation of $\mathcal{A}[r](L)$ commute with arbitrary base change, cf. Corollary III.4.15). $\qquad\square$

DEFINITION VI.2.7. Let L be any $\mathbb{Q}_p$-affinoid algebra, $\kappa \in W(L)$, and $r < \min(p^{\frac{-1}{p-1}}, r(\kappa))$ We define a scalar product $[\ ,\ ]_\kappa$ on $\mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L))$ by the formula

$$[\Phi_1, \Phi_2]_\kappa = \Delta(\Phi_1, V_\kappa \Phi_2).$$

THEOREM VI.2.8.

(i) *The formation of this scalar product commutes with any affinoid base change $L \mapsto L'$, in the sense that if $\Phi_1, \Phi_2 \in \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L))$, and Φ'_1, Φ'_2 are the image of $\Phi_1 \hat\otimes 1, \Phi_2 \hat\otimes 1$ by the natural morphisms (which are not always isomorphisms!) $\mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L)) \hat\otimes_L L' \to \mathrm{Symb}_\Gamma(\mathcal{D}_k[r](L'))$, one has*

$$[\Phi'_1, \Phi'_2] = [\Phi_1, \Phi_2]$$

(ii) *When $L = \mathbb{Q}_p$, $\kappa = k \in \mathbb{N}$, this scalar product coincides with the one denoted $[\ ,\ ]_k$ above.*

(iii) *All the Hecke operators in $\mathcal{H}$ are self-adjoint for $[\ ,\ ]_\kappa$.*

Proof — (i) is clear since the "Poincaré duality" pairing Δ is natural, and since the formation of V_κ commute with base change by construction. (ii) is also clear since $V_\kappa = V_k$ when $\kappa = k$ by construction. Finally (iii) is first proved for the case (R, K) where $W = \mathrm{Sp}\, R$ is an affinoid of $\mathcal{W}$ in $\mathfrak{C}$ and K the canonical weight by density from the case of integral weight (which is Lemma **??**) and then for any (L, κ) using (i). $\qquad\square$

Using $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))^{\leq \nu}$ is independent of r when it is defined (that is for ν adpated to L and $r < \min(p, r(\kappa))$), the above scalar product define by restriction a scalar product on $\mathrm{Symb}_\Gamma(\mathcal{D}_\kappa[r](L))^{\leq \nu}$. We note also that by the formal property of Δ, the scalar product $[\ ,\ ]_\kappa$ factors through the quotient $H^1_!(\Gamma, \mathcal{D}_k[r](L))$.

THEOREM VI.2.9. *Let $\nu \geq 0$, and assume that L is a finite extension of $\mathbb{Q}_p$. The restriction of $[\ ,\ ]_\kappa$ to $H^1_!(\Gamma, \mathcal{D}_k[r](L))^{\leq \nu}$ is non-degenerate if $\kappa \notin \mathbb{N}$ or if $\kappa = k \in \mathbb{N}$ and $\nu < k + 1$.*

Proof — Assume that $\Phi_1 \in H^1_!(\Gamma, \mathcal{D}_k[r](L))^{\leq \nu}$ is such that $[\Phi_1, \Phi_2]_\kappa = 0$ for all $\Phi_2 \in H^1_!(\Gamma, \mathcal{D}_k[r](L))^{\leq \nu}$. Then since U_p is self-adjoint, we have $[\Phi_1, \Phi_2]_\kappa = 0$ for all $\Phi_2 \in H^1_!(\Gamma, \mathcal{D}_k[r](L))$. In view of the non-degeneracy of Δ, this means that $V_\kappa(\Phi_1) = 0$. By the definition of V_κ, this means in particular that $T_{\kappa, \Phi_1, D, e}(z)$ is 0

for every $e \in \mathbb{Z}_p$, $D \in \Delta_0$. Applying this to the divisor $W_{Np}^{-1} \cdot D$ and to $e = 0$, we get that for every $D \in \Delta_0$ and every integer i:

$$\binom{\kappa}{i} \Phi_1(D)(z^i) = 0.$$

In the case $\kappa \notin \mathbb{N}$, then $\binom{\kappa}{i}$ is never 0. Therefore for all $D \in \Delta_0$, the distribution $\Phi_1(D)$ which belongs to $\mathcal{D}[1](L)$ satisfies $\Phi(D)(z^i) = 0$. Hence $\Phi_1(D) = 0$ and $\Phi_1 = 0$.

In the case $\kappa = k \in \mathbb{N}$, then at least $\binom{k}{i} \neq 0$ for $0 \leq i \leq k$ and we have

$$\Phi_1(D)(z^i) = 0$$

for all $D \in \Delta_0$ and $0 \leq i \leq k$. This means that $\rho_k(\Phi_1) = 0$. If $\nu < k+1$, then this implies again $\Phi_1 = 0$ by Stevens' control theorem (Theorem III.6.36). $\qquad\square$

VI.3. Good points on the cuspidal eigencurve

We first need to define a notion of good point on the cuspidal eigencurve $\mathcal{C}^0$ analog to the one we defined on the eigencurves $\mathcal{C}^{\pm}$ (cf. Definition **??**). Recall that the cuspidal eigencurve can be constructed by applying the eigenvariety machine to three different eigenvariety data: to the modules of cuspidal overconvergent forms, or to the modules of overconvergent modular symbols $H^1_!(\Gamma, \mathcal{D}_K[r](R))^{\pm}$ for both values of the sign $\pm$. That the three eigenvarieties obtanied these ways are isomorphic (with unique isomorphisms respecting the Hecke operators and the weight maps) was proved in Theorem **??**. We shall use only the two definitions with modular symbols. If $x \in \mathcal{C}(L)$, and $U = \operatorname{Sp} \mathcal{T}$ is a clean neighborhood of x, let as in §IV.6.1 $W = \operatorname{Sp} R$ the image $\kappa(U)$, and, for $r > 0$ sufficiently small, let $M^{\pm}$ be the finite free R-module $\epsilon H^1_!(\mathcal{D}_K[r](R))^{\pm}$ (it is independent of r) where ϵ is the idempotent corresponding to the connected component U of $\kappa^{-1}(W)$.

The following lemma is proven exactly as Lemma V.2.1

LEMMA VI.3.1. *Let x be a point on $\mathcal{C}^0$ (of field of definition L), with $\kappa(x) = w \in \mathcal{W}$. The following statements are equivalent:*

  (i) *For both choices of the sign $\pm$, the dual of the generalized eigenspace $H^1_!(\mathcal{D}_w(L))^{\pm}_{(x)}$ is free of rank 1 over the algebra $\mathcal{T}_x$ of the schematic fiber of κ at x.*

  (ii) *For both choices of the sign $\pm$, and for any clean neighborhood of x, the modules $(M^{\pm})^{\vee}$ are flat of rank one over $\mathcal{T}$ at x*

  (iii) *For both choices of the sign $\pm$, and for any sufficiently small clean neighborhood of x, the modules $(M^{\pm})^{\vee}$ are free of rank one over $\mathcal{T}$ at x.*

DEFINITION VI.3.2. A point $x \in \mathcal{C}^0$ is called good if it satisfies the proposition above.

Exactly as in Prop. V.2.4 and in the, we can prove that if x is a smooth point of $\mathcal{C}^0$ which is classical and new away from p, then x is good.

We shall need the following lemma, which summarizes results obtained earlier:

LEMMA VI.3.3. *If x is a good point, then the eigenspace $H^1_!(\mathcal{D}_w(L))^{\pm}[x]$ has dimension 1. If x is a good and of weight $w(x) = k$ and $H^1_!(\mathcal{V}_k(L))^{\pm}[x]$ has dimension 1 if x is very classical cuspidal, 0 otherwise, and the same is true for the generalized eigenspaces $H^1_!(\mathcal{V}_k(L))^{\pm}_{(x)}$ provided $U_p(x)^2 \neq p^{k+1}\langle p\rangle(x)$. In the former case, the map between one-dimensional spaces $\rho_k : H^1_!(\mathcal{D}_k(L))^{\pm}[x] \to H^1_!(\mathcal{V}_k(L))^{\pm}[x]$ is an isomorphism if κ is étale at x, and is 0 if κ is not. If x is a good point of weight $w(x) = k$ and $v_p(U_p(x)) < k+1$, then x is cuspidal and very classical and κ is étale at x.*

Proof — The first assertion follows by duality from the definition of a good point (see Lemma VI.3.2(i)). One has $H^1_!(\mathcal{V}_k(L))^{\pm}[x]$ is non-zero if and only if x is a cuspidal very classical point by Theorem III.2.39. If x is a cuspidal very classical point, it is new away from p since x is good and $H^1_!(\mathcal{V}_k(L))^{\pm}[x]$ has dimension 1, hence the second assertion. The fact that ρ_k is an isomorphism if and only if κ is étale at x is part of Theorem IV.6.9, noting that x, being cuspidal, is not abnormal. Finally, if x is a good point of weight $w(x) = k$ and $v_p(U_p(x)) < k+1$ by Stevens' control theorem, and it can't be Eisenstein since only critical slope Eisenstein points appear on the cuspidal eigencurve $\mathcal{C}^0$, and they satisfy $v_p(U_p(x)) = k+1$. Moreover κ is étale at x by Theorem **??**. □

VI.4. Construction of the adjoint p-adic L-function on the cuspidal eigencurve

DEFINITION VI.4.1. Let L be a finite extension of $\mathbb{Q}_p$ and $x \in \mathcal{C}^0(L)$ a point. Let $U = \operatorname{Sp}\mathcal{T}$ be a clean neighborhood of x. We define the *adjoint L-ideal* $\mathcal{L}^{\mathrm{adj}}_U$ on U as the L-ideal (cf. Def. VI.1.8) of the scalar product $[\ ,\ ]_K : M^+ \times M^- \to R$ (cf. Def. VI.2.7).

Hence $\mathcal{L}^{\mathrm{adj}}_U$ is an ideal of $\mathcal{T}$. The definition makes sense because the modules M^+ and M^- are finite flat over R and scalar product $[\ ,\ ]_K$ is $\mathcal{T}$-equivariant (cf. Theorem **??**(iii)). Note that the formation of the ideal $\mathcal{L}^{\mathrm{adj}}_U$ commutes with the resriction of the clean affinoid U to a smaller clean affinoid: this follows from the commutation of the formation of $[\ ,\ ]_K$ with base change (Theorem VI.2.8(i)) and the remarks following definition. VI.1.8. In particular, since the there is an admissible covering of the eigencurve $\mathcal{C}^0$ of clean affinoids, we can glue the adjoint $\mathcal{L}$-ideal $\mathcal{L}^{\mathrm{adj}}_U$ to define a global coherent sheaf of ideal $\mathcal{L}^{\mathrm{adj}}$ on the eigencurve $\mathcal{C}^0$. The following result follows directly from Prop. VI.1.9.

THEOREM VI.4.2. *The closed analytic subspace defined by the sheaf of ideal $\mathcal{L}^{\mathrm{adj}}$ is contained in the locus of non-etaleness of the weight map κ.*

If we are willing to restrict ourselves to good points of $\mathcal{C}^0$, and to work locally, it is possible to replace the adjoint L-ideal L^{adj} by a true p-adic adjoint L-function which satisfies an interpolation property relating it to the special value of the archimdean adjoint L-function, and to prove much more precise result on itz zero locus.

PROPOSITION VI.4.3. *Let L be a finite extension of $\mathbb{Q}_p$ and $x \in \mathcal{C}^0(L)$ be a good point. If $U = Sp\,\mathcal{T}$ is a sufficiently small clean neighborhood of x, then the ideal $\mathcal{L}_U^{\mathrm{adj}}$ of $\mathcal{T}$ is principal.*

Proof — This follows from the definition of a good point and Prop. VI.1.3. □

DEFINITION VI.4.4. *If $U = \mathrm{Sp}\,\mathcal{T}$ is as in the above proposition* We define "the" p-*adic adjoint L-function on U*, L_U^{adj} as a generator of the ideal $\mathcal{L}_U^{\mathrm{adj}}$. Thus the p-adic adjoint L-function on U is defined up to multiplication by an element of $\mathcal{T}^*$.

PROPOSITION VI.4.5. *Let L be a finite extension of $\mathbb{Q}_p$ and $x \in \mathcal{C}^0(L)$ be a good point. Fix $U = Sp\,\mathcal{T}$ a sufficiently small clean neighborhood of x. Then for every sufficently small real $r > 0$ there exist two real constants $0 < c < C$ such that for every point $y \in U$, with $\kappa(y) = w$ one has*

$$c|[\Phi_y^+, \Phi_y^-]_w| \leq |L^{\mathrm{adj}}(y)| \leq C|[\Phi_y^+, \Phi_y^-]_w|,$$

where $\Phi_y^{\pm}$ are generators of the spaces $H_!^1(\Gamma, \mathcal{D}_w[r](L))^{\pm}[y]$ such that $\|\Phi_y^{\pm}\|_r = 1$. Moroever, the adjoint p-adic L-function is characterized (up to an element of $\mathcal{T}^*$) by the above property.

Proof — For $0 < r < \min(p, r(\kappa))$, let us provide M^+ and M^- with the norm $\|\ \|_r$, and $M^+ \otimes_R M^-$ with the tensor norm, also denoted $\|\ \|_r$. Consider the scalar product b as an element of $(M^+ \otimes_R M^-)^{\vee}$. By Lemma V.1.2, there exists real constants $0 < c < C$ such that for every $y \in \mathrm{Spec}\,\mathcal{T}(L) = U(L)$, one has $c < (\|b_y\|_r)' < C$, where b_y is the image of b in $(M^+ \otimes_R M^-) \otimes_{\mathcal{T}} \mathcal{T}/\mathfrak{p}_y = H_!^1(\Gamma, \mathcal{D}_w(L))^+[y] \otimes H_!^1(\Gamma, \mathcal{D}_w(L))^+[y]$, $\mathfrak{p}_y$ being the maximal ideal of $\mathcal{T}$ corresponding to y (and $\mathcal{T}/\mathfrak{p}_y = L$) and $\|\ \|_r'$ is the dual norm of the norm $\|\ \|_r$ on those spaces.

Fix an isomorphism $\psi^+ : (M^+)^{\vee} \to \mathcal{T}$ and $\psi_x : (M^-)^{\vee} \to \mathcal{T}$ as $\mathcal{T}$-module, so that $b \in (M^+ \otimes M^-)^{\vee}$ can be seen as en element of $\tilde{b} \in \mathcal{T} \otimes \mathcal{T}$, and $m(b) \in \mathcal{T}$ is by definition the p-adic L-function L_U. There exists constants $0 < c' < C'$ such that for every y in U, the generator of $\mathcal{T}/\mathfrak{p}_y$ is sent to an element $e_y^{\pm} \in H_!^1(\Gamma, \mathcal{D}_w(L))^{\pm}[y]$ of norm between c' and C' by the isomorphisms $\psi_y^{\pm}$ induced by ψ^+ and ψ^-. Since y is a good point, $H_!^1(\Gamma, \mathcal{D}_w[r](L))^{\pm}[y])$ has L-dimension 1. Therefore $e_y^+ \otimes e_y^-$ has

norm between c'^2 and c^2, and we have $\tilde{b}_y = a_y e_y^+ \otimes e_y^-$ where $a_y \in L$ is such that $c/C'^2 < |a_y| < C/c'^2$. Since $\mathcal{L}_U(y) = m(\tilde{b}_y) = a_y$, one gets the first result.

The fact that the function $\mathcal{L}_U$ is characterized by the above property up to an elment of $\mathcal{T}^*$, because a meromorphic function on U (that is an element of the form f/g, for $f, g \in \mathcal{T}$, g not a zero divisor) whose norm at every closed point of $\operatorname{Spec} \mathcal{T}$ is bounded above and below away from 0 is an element of $\mathcal{T}^*$. $\qquad\square$

COROLLARY VI.4.6. *Let L be a finite extension of $\mathbb{Q}_p$ and $x \in \mathcal{C}^0(L)$ be a good point. Fix $U = Sp\,\mathcal{T}$ a sufficiently small clean neighborhood of x. Then there exists two real constant $0 < c < C$ such that for every very classical point $y \in U$ such that κ is étale at y, one has*

$$c|[\phi_y^+, \phi_y^-]_k| \le |L^{\mathrm{adj}}(y)| \le C|[\phi_y^+, \phi_y^-]_k|,$$

where $k = \kappa(y) \in Z$ is the weight of y and $\phi_y^\pm$ are generators of the one-dimensional spaces $H_!^1(\Gamma, \mathcal{V}_k(L))^\pm[y]$ (cf. Lemma VI.3.3), normalized so that $\|\phi_y^\pm\| = 1$. Observe that the condition on y is satisfied as soon as $\kappa(y)$ is an integer $k \in \mathbb{N}$ and one has $v_p(U_p(y)) < k + 1$.

Proof — By shrinking U if necessary, we can and do assume that for every $y \in U(L) - \{x\}$ such that $\kappa(y) \in \mathbb{N}$, we have $v_p(U_p(y)) < \kappa(y) + 1$. Take $0 < r < \min(p, r(\kappa))$, $r \le 1$ and $0 < c < C$ as in the preceding proposition. For $y \in U(L) - \{x\}$ a classical point of weight $\kappa(y) = k \in \mathbb{N}$, one has that ρ_k is an isomorphism $H_!^1(\Gamma, \mathcal{D}_k[r'](L))^\pm[y]) \to H_!^1(\Gamma, \mathcal{V}_k(L))^\pm[y])$, isometric when $r' = 1$ (cf. Theorem III.6.36) since $v_p(U_p(y)) < k + 1$. Since y is a good point, the spaces $H_!^1(\Gamma, \mathcal{V}_k(L))^\pm[y])$ are one dimensional (cf. Lemma! VI.3.3). Let $\phi_y^\pm$ be generators of those spaces of norm 1. Then by Prop. III.6.27, there exists a real constant $0 < D \le 1$ depending only on r, not on y, such that $D \le \|\Phi_y^\pm\|_1 \le 1$. It follows that $\rho_k(\Phi_y^\pm) = a_y^\pm \phi_y^\pm$ where $D \le |a_y^\pm| \le 1$ hence $|[\Phi_y^+, \Phi_y^-]| \le |[\phi_y^+, \phi_y^-]| \le D^2|[\Phi_y^+, \Phi_y^-]|$ using Definition VI.2.1 and the corollary follows for every y except perhaps for x. Assuming x is very classical and that κ is étale at x, we see that $[\Phi_x^+, \Phi_x^] = b[\phi_x^+, \phi_x^-]$, where b is some non-zero scalar, and by enlarging C or diminishing c, we can assume that $0 < c \le |b| \le C$ which concludes the proof. $\qquad\square$

Note that if is x is a good point, then the condition $L_U^{\mathrm{adj}}(x) = 0$ clearly does not depend on the choice of a sufficiently small clean neighborhood U of x nor of the adjoint p-adic L-function L_U^{adj}. In thie case, we shall simply write $L^{\mathrm{adj}}(x) = 0$. When x is also smooth point of $\mathcal{C}^0$, then it makes sense to talk of the order of vanishing of L_U^{adj} at x, which again does not depend on the choice of the small clean affinoid neighborhood U of x nor of the adjoint L-function. We shall denote this integer by $\operatorname{ord}_x L_U^{\mathrm{adj}}$.

In the following two results, we determine the zero locus, and when it makes sense, the order of vanishing, of the p-adic adjoint L-function L^{adj}.

THEOREM VI.4.7. *Let L be a finite extension of $\mathbb{Q}_p$ and $x \in \mathcal{C}^0(L)$ be a good point. If either $\kappa(x) \notin \mathbb{N}$ or x is a very classical cuspidal point (that is the system of eigenvalues of x is the one of a classical cuspidal modular form of level Γ and weight $k + 2$ where $k = \kappa(x) \in \mathbb{N}$), then we have*

$$L^{\mathrm{adj}}(x) = 0 \text{ if and only if } \kappa \text{ is étale at } x.$$

Otherwise, that is when $\kappa(x) \in \mathbb{N}$ and x is not a very classical cuspidal point, we always have $L^{\mathrm{adj}}(x) = 0$.

Proof — If either $\kappa(x) \notin \mathbb{N}$ or $\kappa(x) = k \in \mathbb{N}$ and $v_p(U_p(x)) < k+1$, then the scalar product on $M^+ \times M^-$ is non-degenerate by Theorem VI.2.9 and the equivalence κ étale at $x \Leftrightarrow L^{\mathrm{adj}}(x) = 0$ follows from VI.1.12. Now assume $\kappa(x) = k \in \mathbb{N}$, and let Φ_x^+ and Φ_x^- be generators of $H_!^1(\Gamma, \mathcal{D}_w[r](L))^{\pm}[x]$. Then by Prop. VI.4.5, one ahs up to a non-zero scalar $L^{\mathrm{adj}}(x) = [\Phi_x^+, \Phi_x^-]_k = [\rho_k(\Phi_x^-), \rho_k(\Phi_x^-)]_k$, the second equality being by definition of the scalar product. When x is a cuspidal classical point, $\rho_k(\Phi_x^-)$ and $\rho_k(\Phi_x^+)$ are non-zero if and only if κ is étale at x (lemma VI.3.3) and when this happens $[\rho_k(\Phi_x^+), \rho_k(\Phi_x^-)]_k \neq 0$ since the scalar product $[\ ,\ ]_k$ on $H_!^1(\Gamma, \mathcal{V}_k)^+[x] \times H_!^1(\Gamma, \mathcal{V}_k)^+[x]$ is non-degenerate and those spaces have dimension 1. When x is not a cuspidal classical point, $\rho_k(\Phi_x^{\pm}) = 0$ by Lemma VI.3.3 and $L^{\mathrm{adj}}(x) = 0$. □

THEOREM VI.4.8. *Let L be a finite extension of $\mathbb{Q}_p$ and $x \in \mathcal{C}^0(L)$ be a smooth good point. Let us call $e(x)$ the degree of the map κ at x (so $\kappa(x) = 1$ if and only if κ is étale at x):*

  (i) *If $\kappa(x) \neq \mathbb{N}$, or if x is a very classical point of non-critical slope (that is $v_p(U_p(x)) < k + 1$ where $\kappa(x) = k \in \mathbb{N}$), then $\mathrm{ord}_x L^{\mathrm{adj}}(x) = e(x) - 1$.*
  (ii) *If $\kappa(x) = k \in \mathbb{N}$ and x is a very classical cuspidal point of critical slope (that is $v_p(U_p(x)) = k + 1$), then $\mathrm{ord}_x L^{\mathrm{adj}}(x) = 2e(x) - 2$*
  (iii) *If $\kappa(x) = k \in \mathbb{N}$ and x is not a very classical cuspidal point, then $\mathrm{ord}_x L^{\mathrm{adj}}(x) \geq 2e(x) - 1$.*

The proof of this theorem will occupy the rest of this §.

Let $U = \mathrm{Sp}\,\mathcal{T}$ be a clean neighborhood of x with $\kappa(U) = W = \mathrm{Sp}\,R$. Let us call $\mathbb{T}$ the the completed local ring of the eigencurve $\mathcal{C}^0$ at x, that is the completed local ring of $\mathcal{T}$ at the maximal ideal corresponding of x, and A the completed local ring of the weight space at $\kappa(x)$, that is the completed local ring of R at the maximal ideal corresponding of $\kappa(x)$. Since $\mathcal{T}$ is finite over R and x is the only point of U above $\kappa(x)$, the natural morphism $\mathcal{T} \otimes_R A \to \mathbb{T}$ is an isomorphism. Let us set $\mathcal{M}^+ = M^+ \otimes_{\mathcal{T}} \mathbb{T}$ and $\mathcal{M}^- = M^- \otimes_{\mathcal{T}} \mathbb{T}$. It follows that the modules $\mathcal{M}^+$ and $\mathcal{M}^-$ are free R-module of rank $e = e(x)$.

Since both A and $\mathbb{T}$ are discrete valuation ring (because x is smooth) and since κ has degree e at x, we can chose an uniformizer u of A and t of T so that we

have isomorphisms $A \simeq L[[u]]$ and $\mathbb{T} \simeq A[[t]]/(t^e - u) = L[[t]]$ combatible with the natural weight map $A \to \mathbb{T}$. Let $b_0 : \mathcal{T} \times \mathcal{T} \to A$ be the bilinear A-form $b_0(t^i, t^j) = 0$ if $i + j \not\equiv e - 1 \pmod{e}$, and $b_0(t^i, t^j) = u^{(i+j-(e-1))/e}$ if $i + j \equiv \pmod{A}$. One easily sees that b_0 is $\mathcal{T}$-equivariant and perfect.

We claim that the modules $\mathcal{M}^+$ and $\mathcal{M}^-$ are free of rank one over $\mathbb{T}$. Indeed, thay are obviously finite, and they are torsion-free over $\mathbb{T}$ since they are so over A and $\mathbb{T}$ is finite over A (for if $m \in \mathcal{M}^+$ is torsion, with $\tau m = 0$ for some $0 \neq \tau \in \mathbb{T}$, then if $P(X) = a_0 + a_1 X + \cdots + X^n$ is a polynomial in $A[X]$ of minimal degree killing τ, then $a_0 \neq 0$ and one sees $0 = P(\tau)m = a_0 m$, which shows that m is a torsion element of $\mathcal{M}^+$ as an A-module, hence is 0.) Since $\mathbb{T}$ is a d.v.r., this implies that $\mathcal{M}^+$ and $\mathcal{M}^-$ are free, and their rank has to be one over $\mathbb{T}$ since they are of rank e over A and $\mathbb{T}$ is also of rank e over A.

The scalar product $[\ ]_K$ define by extension of scalars a scalar product $b : \mathcal{M}^+ \times \mathcal{M}^- \to A$, which is $\mathcal{T}$-equivariant, or choosing an isomorphism $\mathcal{M}^+ \simeq \mathcal{M}^- \simeq \mathbb{T}$ of $\mathcal{T}$-module, a scalar product $b : \mathbb{T} \times \mathbb{T} \to \mathbb{T}$ and we have $\mathcal{L}_U^{\mathrm{adj}} \mathbb{T} = \mathcal{L}_b$ since the formation of the $\mathcal{L}$-ideal commutes with flat base change. One has $b = \tau b_0$ for some $\tau \in \mathcal{T}$ since the $\mathcal{T}$-module of such scalar product is free of rank one over $\mathbb{T}$ (the module structure here is $(\tau b_0)(m, m') := b_0(\tau m, m') = b_0(m, \tau m')$). By Corollary VI.1.12, one has $\mathcal{L}_b = \tau \mathfrak{d}_N(\mathbb{T}/A)$ and the different of $\mathcal{T}$ over A is $\mathfrak{d}_N(\mathbb{T}/A) = t^{e-1}\mathbb{T}$ by an easy and standard computation. Hence

$$t^{\mathrm{ord}_x \mathcal{L}^{\mathrm{adj}}} \mathbb{T} = \mathcal{L}_U^* \mathbb{T} = \tau t^{e-1} \mathbb{T}.$$

To determine τ we seperate the three case of the theorem. In case (i), b is non-degenerate so $\tau \in \mathcal{T}^*$ and we get $\mathrm{ord}_x \mathcal{L}^{\mathrm{adj}} = e - 1$. In case (ii) and (iii) we consider the scalar product $b \pmod{u} : \mathcal{M}^+/u \times \mathcal{M}^-/u \to A/u = L$ induced by b, which is the same as the scalar product $[\ ,\ ]_k : H^1_!(\Gamma, \mathcal{D}_k(L))^+_{(x)} \times H^1_!(\Gamma, \mathcal{D}_k(L))^+_{(x)} \to L$. By definition $[\Phi_1, \Phi_2]_k = [\rho_k(\Phi_1), \rho_k(\Phi_2)]_k$ where the second scalar product is the one on $H^1_!(\Gamma, \mathcal{V}_k(L))^+_{(x)} \times H^1_!(\Gamma, \mathcal{V}_k(L))^-_{(x)} \to L$. The spaces $H^1_!(\Gamma, \mathcal{V}_k(L))^+_{(x)}$ are of dimension 1 in case (ii) and are 0 in case (iii) (cf. Lemma VI.3.3). In case (ii), ρ_k sends a generator of $H^1_!(\Gamma, \mathcal{D}_k(L))^\pm_{(x)}$ as $\mathcal{T}/u$-module on a generator of $H^1_!(\Gamma, \mathcal{V}_k(L))^\pm_{(x)}$ since it is surjective, and it follows that $[\Phi^+, \Phi^-]_k \neq 0$ in L when $\Phi^\pm$ are generators of $H^1_!(\Gamma, \mathcal{V}_k(L))^\pm_{(x)} = \mathcal{M}\mathcal{M}^\pm/u$. Therefore, $b(1, 1)$ is a unit in A, that is $(\tau b_0)(1, 1) = b_0(1, \tau)$ is a unit in A, and this implies $\tau \mathcal{L} = t^{e-1}\mathbb{T}$. We conclude that $\mathrm{ord}_x \mathcal{L}^{\mathrm{adj}} = e - 1 + e - 1 = 2e - 2$ in this case. In case (iii), the scalar product $[\ ,\ ]_k : H^1_!(\Gamma, \mathcal{D}_k(L))^+_{(x)} \times H^1_!(\Gamma, \mathcal{D}_k(L))^+_{(x)} \to L$ is zero, so b takes values in the maximal ideal uA of A. In particular, $b(t^a, 1) = b_0(t^a, \tau) \in uA$. This implies $\tau \mathbb{T} \in t^e \mathbb{T}$ (for if $\tau \mathbb{T} = t^n \mathbb{T}$ with $0 \leq n \leq e - 1$, $b_0(t^{e-1-n}, \tau)$ is a unit in A). One deduces that $\mathrm{ord}_x \mathcal{L}^{\mathrm{adj}} \geq e - 1 + e = 2e - 1$ in case (iii). This completes the proof of the theorem.

VI.5. Relation between the adjoint p-adic L-function and the classical adjoint L-function

To be written.

Part 3

Eigenvarieties for definite unitary groups and a p-adic L-function on them

CHAPTER VII

Automorphic forms and representations in a simple case

VII.1. Reminder on smooth and admissible representations

VII.1.1. lctd groups.

DEFINITION VII.1.1. A topological group G is said to be *lctd* if there is basis of neighborhood of 1 made of compact open subgroups U.

EXERCISE VII.1.2. Prove that a group is lctd if and only if locally compact and totally disconnected.

In the remainder of this subsection, G is an lctd group.

VII.1.2. Smooth and admissible representation. Let k be a field of characteristic 0. By a *smooth representation V of G over k* we mean a k-vector space (not finite dimensional in general) with a linear action of G such that for every $v \in V$, there exists a compact open subgroup U in G such that v is invariant by U.

EXERCISE VII.1.3. 1.– Let V be any representation of G over k. We say that $v \in V$ is a *smooth vector* if there is an open compact subgroup U of G such that $v \in V^U$. Show that the subset of smooth vectors V^{sm} in V is a sub-representation of V, and is actually the largest smooth sub-representation.

A smooth representation is *admissible* if for every open subgroup U of G, the subspace V^U of vectors invariant by U is finite dimensional over k.

VII.1.3. Hecke algebras. Let us fix a (left invariant, but since in practice our G will be *unimodular*, this doesn't really matter) Haar measure dg on G, normalized such that the measure of some open compact subgroup U_0 is 1. Then is is easily seen that the measure of any other compact open subgroup will be a rational, hence an element in k.

We denote by $\mathcal{H}(G, k)$ the spaces of function from G to k that are locally constant and have compact support. This space as a natural product : the convolution of functions $(f_1 * f_2)(g) = \int_G f_1(x) f_2(x^{-1}g) \, dx$. Indeed, since f_1 and f_2 are in $\mathcal{H}(G, k)$, the integral is actually a finite sum, an what is more a finite sum of terms that are products of a value of f_1, a value of f_2, and the measure of a compact open subgroup. So the integral really defines a k-valued function, and it is easy to check that $f_1 * f_2 \in \mathcal{H}(G, k)$. This product makes $\mathcal{H}(G, k)$ a k-algebra, which is not

commutative if G is not, and which in general has no unity (the unity would be a Dirac at 1, which is not a function on G if G is not discrete). If V is a smooth representation of G it has a natural structure of $\mathcal{H}(G, k)$-module by $f.v = \int_G f(g)g.v \, dg$. The algebra $\mathcal{H}(G, k)$ is called the *algebra of the group G* over k.

Let U be a compact open subgroup of G. Let $\mathcal{H}(G, U, k)$ be the set of functions form G to k that have compact support and that are both left and right invariant by U. This is easily seen to be a subalgebra of $\mathcal{H}(G, k)$, with unity (the unity is the characteristic function of U times a normalization factor depending on the Haar measure). The algebra $\mathcal{H}(G, U, k)$ is called the *Hecke algebra of G w.r.t U* over k. We have $\mathcal{H}(G, k) = \cup_U \mathcal{H}(G, U, k)$ and in particular we see that if G is not commutative, the $\mathcal{H}(G, U, k)$ are not for U small enough. The Hecke algebra are important because of the following trivial property:

LEMMA VII.1.4. *Let V be a smooth representation of G (over k). Then the spaces of invariant V^U has a natural structure of $\mathcal{H}(G, U, k)$-modules.*

Indeed, if $v \in V^U \subset$, and $f \in \mathcal{H}(G, U, k) \subset \mathcal{H}(G, k)$, $f.v$ is easily seen to be in V^U.

If there is one thing to remember about Hecke algebras it is this lemma, not the definition : the Hecke algebra is what acts on V^U when V is a (smooth) representation of G and U sub-group of G.

We shall apply this theory to groups $G(F_v)$ when G is a unitary group as above and v is a finite place of F.

EXERCISE VII.1.5. **(easy)** Assume that G is finite. Show that $\mathcal{H}(G, k)$ is naturally isomorphic with the algebra $k[G]$. Assume that V is is a finite dimensional algebra. Show that the structures of $\mathcal{H}(G, k)$-module and $k[G]$-module on V are compatible through that isomorphism.

Hence $\mathcal{H}(G, K)$ is a natural generalization of the classical group algebra of G in the case of a finite G.

EXERCISE VII.1.6. **(easy)** Let U be a normal open subgroup of G, and V be an admissible representation of U. Show that V^U is naturally an admissible representation of G/U. Hence $\mathcal{H}(G/U, k)$ acts on V^U. Show that $\mathcal{H}(G/U, k) = \mathcal{H}(G, U, k)$ and those two algebras have the same action on V^U.

Hence for a non-normal U, the Hecke algebra $\mathcal{H}(G, U, k)$ is a generalization of the group algebra of G/U for U normal. In other words, when U is non normal, we can not define G/U as a group, but we do have a substitute for the group algebra of this non-existing group: $\mathcal{H}(G, U, k)$.

EXERCISE VII.1.7. Let k' be an extension of k.

a.— Show that if V is a smooth representation of G over k, then $V \otimes_k k'$ is a smooth representation of G over k'.

b.– Show that the formation of V^U commutes to the extensions k'/k.

c.– Show that $\mathcal{H}(G, U, k) \otimes_k k' = \mathcal{H}(G, U, k')$.

DEFINITION VII.1.8. If V is a smooth representation of G over a field k', and k is a subfield of k', we say that V is *defined over* k if there a smooth representation V_k of G such that $V_k \otimes_k k' \simeq V$.

EXERCISE VII.1.9. Assume V is an admissible representation of G over a field k, and k' is an extension of k. Let W be a sub-representation of V. Show that W is defined over a finite extension of k.

EXERCISE VII.1.10. Let $\mathcal{C}(G, k)$ and $\mathcal{C}(G/U, k)$ be the space of smooth functions from G and G/U to k.

a.– Show that they both are smooth representation of G (for the left translations)

b.– Show $\mathcal{C}(G, k)^U = \mathcal{C}(G/U, k)$. Therefore $\mathcal{H}(G, U, k)$ acts on $\mathcal{C}(G, U, k)$. Show that this action commutes to the G-action.

c.– Show that $\mathcal{H}(G, U, k)$ is actually naturally isomorphic to $\mathrm{End}_G(\mathcal{C}(G/U, k))$.

EXERCISE VII.1.11. a.– Show that $V \mapsto V^U$ is an exact functor form the category from smooth representation of G over k to $\mathcal{H}(G, U, k)$-module. This functor takes admissible representations to $\mathcal{H}(G, U, k)$ that are finite dimensional.

b.– Let $W = V^U$, $W' \subset W$ a sub-$\mathcal{H}(G, U, k)$-module, and $V' \subset V$ the sub-representation of V generated by W'. Show that $V'^U = W$.

c.– Deduce that if V is irreducible as a representation of G, than V^U is irreducible as a $\mathcal{H}(G, U, k)$-module.

Note that in general the functor defined in a.– above is not fully faithful (even on admissible rep.) and that the converse of c.– is false.

VII.1.4. Complete reducibility of admissible semi-simple representations. We recall that a representation V is *semi-simple* if any stable subspace W has a stable direct summand W'. If V is semi-simple, then all its sub-representations are as well.

LEMMA VII.1.12. *Any non-zero admissible semi-simple representation has an irreducible sub-representation.*

Proof — Let U be a compact open subgroup of G such that $V^U \neq 0$. By admissibility, V^U is finite-dimensional. Thus we can choose M a non-zero subspace of V^U which is of minimal dimension among the subspaces of the form W^U, for W a sub-representation of V.

Now consider the set of sub-representations W of V such that $W^U = M$. This set has a minimal element (since an intersection of sub-representations W satisfying $W^U = M$ stills satisfy that condition). Let W_0 be such a minimal element. We claim that W_0 is irreducible. For if we write $W_0 = W_1 \oplus W_2$ as representations, then $M = W_0^U = W_1^U \oplus W_2^U$ which implies that $W_1^U = M$ and $W_2^U = 0$ (or the other way around), and then $W_0 = W_1$ by minimality of W_0.   □

PROPOSITION VII.1.13. *Any semi-simple admissible representation is a direct sum of irreducible representations, each of them appearing with finite multiplicity.*

Proof — The argument is standard: the set of sub-representations of an admissible representation V that are direct sum of irreducible representations has an obvious inductive order, so has a maximal element W by Zorn's lemma. Writing $V = W \oplus W'$ as representation, and using that W' is still admissible and semi-simple, we see that W' has an irreducible subrepresentation if non-zero (by the lemma), hence $W \oplus W''$ contradicts the fact that W is maximal. Hence $W' = 0$, that is $V = W$ and V is a direct sum of irreducible multiplication. The finite multiplicity is trivial in this case.   □

VII.2. Automorphic forms and automorphic representations

The natural context for the theory of automorphic forms is that of a connected reductive algebraic group over $\mathbb{Q}$ (or a number field). To avoid most of the analytic difficulties, as well as any difficult argument in the theory of algebraic group, we will work with a very special group, yet general enough to show the general flavor of the theory, and to allow for many deep arithmetic applications:

We shall in all this section work with an affine algebraic group G over $\mathbb{Q}$, on which we will eventually make the following hypothesis:

   (a) The group $G(\mathbb{R})$ is compact.
   (b) The group G is a a connected reductive group.

Recall that by definition G is *reductive* if and only if $G_{\mathbb{C}} = G \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\mathbb{C}$ is, and this happens if and only if all algebraic complex representations of $G_{\mathbb{C}}$ are semi-simple. Examples of reductive groups over $\mathbb{C}$ includes GL_n (or variants like PGL_n, SL_n), the orthogonal groups $\emptyset(n)(\mathbb{C})$, the symplectic groups $\mathrm{Sp}(2n)(\mathbb{C})$, as well as a few *exceptional groups*. In this subsection we shall only use hypothesis (b) a couple of time, just as hypothesis on theorems that can be taken as black boxes. In the next subsection, we shall replace hypothesis (b) by the more specific but more concrete assumption that $G_{\mathbb{C}}$ is GL_n. So the reader who ignores anything about reductive group should not have any serious problems reading these notes – which of course should not be taken as a reason not to learn the theory of reductive groups.

VII.2.1. Adelic points of G. .

Since the ring of adèles $\mathbb{A}$ is a $\mathbb{Q}$-algebra, it makes to speak of the group $G(\mathbb{A})$. We provide $G(\mathbb{A})$ with a group topology as follows. Since G is affine, it has a faithful representation $G \to \mathrm{GL}_m$. We thus have an embedding $G(\mathbb{A}) \to \mathrm{GL}_m(\mathbb{A}) \to M_m(\mathbb{A}) = \mathbb{A}^{m^2}$. We provide $G(\mathbb{A})$ with the coarsest topology that makes the inclusion $G(\mathbb{A}) \subset \mathbb{A}^{m^2}$ and the inverse map $g \mapsto g^{-1}$ continuous.

Note that when G is GL_1 (which does not satisfy our assumption (a), but allow me to remove it for a minute) and when we use the obvious faithful representation $\mathrm{GL}_1 \to \mathrm{GL}_1$, this is exactly the usual definition of the topology on the group of idèles $\mathrm{GL}_1(\mathbb{A}) = \mathbb{A}^*$.

LEMMA VII.2.1. *This topology makes $G(\mathbb{A})$ a locally compact topological group. It is independent of the choice of the faithful representation used to define it. The subgroup $G(\mathbb{Q})$ of $G(\mathbb{A})$ is discrete.*

EXERCISE VII.2.2. Prove this lemma

Since $\mathbb{A} = \mathbb{R} \times \mathbb{A}_f$ we have $G(\mathbb{A}) = G(\mathbb{R}) \times G(\mathbb{A}_f)$. We can also write $G(\mathbb{A}_f)$ as a restricted product. To do so, choose a model $\mathcal{G}$ of G over $\mathbb{Z}[1/N]$ for some N, that is a flat group scheme over $\mathbb{Z}[1/N]$ whose generic fiber is isomorphic to G (to be precise, the isomorphism $\mathcal{G} \times_{\mathbb{Q}} \simeq G$ is part of the data defining the model). A model obviously exists by chasing the denominators in a set of equations for G, its multiplication map, and its inverse map. Then $\mathcal{G}(\mathbb{Z}_l)$ is a compact subgroup of the locally compact group $G(\mathbb{Q}_l)$ for all prime l not dividing N. Furthermore, if we choose an other model $\mathcal{G}'$ over $\mathcal{Z}[1/N']$, then there is an isomorphism $\mathcal{G} \to \mathcal{G}'$ over $\mathcal{Z}[1/M]$ for some M divisible by N and N' inducing the identity on the generic fiber. In particular, $\mathcal{G}(\mathbb{Z}_l) = \mathcal{G}'(\mathbb{Z}_l)$ for almost all l. We can then form the restricted product $\prod'_l G(\mathbb{Q}_l)$ with respect to the subgroup $\mathcal{G}(\mathbb{Z}_l)$. It does nor matter for the definition of restricted product that the $\mathcal{G}(\mathbb{Z}_l)$ are not defined for a finite number of l, and the result is clearly independent of the chosen model.

LEMMA VII.2.3. *There is a natural isomorphism of locally compact groups $G(\mathbb{A}_f) = \prod'_l G(\mathbb{Q}_l)$. Thus, $G(\mathbb{A}_f)$ has a basis of neighborhood of 1 made of compact open groups $U = \prod_l U_l$ where U_l is a compact open subgroup of $G(\mathbb{A}_f)$ for all l, and equal to $\mathcal{G}(\mathbb{Z}_l)$ for almost all l (this last condition being independent of the choice of the model by what we have seen). In particular, $G(\mathbb{A}_f)$ is an lctd group.*

EXERCISE VII.2.4. Prove this lemma

VII.2.2. Unramified representations and decomposition of representations of $G(\mathbb{A}_f)$ as tensor products. Here we shall need to present a big black box from the theory of representation of the local groups $G(\mathbb{Q}_l)$. When $G(\mathbb{Q}_l)$ is $\mathrm{GL}_n(\mathbb{Q}_l)$ (which will happen often in our situation), we shall open the black box and see what's inside later.

Assume that G satisfies (b). Then it is known (but non-trivial) that $G(\mathbb{Q}_l)$ always has a maximal compact subgroup, which is always open, and that it actually has only finitely many conjugacy class of compact maximal subgroup.

EXERCISE VII.2.5. If $G(\mathbb{Q}_l) = \mathbb{G}_n(\mathbb{Q}_l)$, show that there is only one conjugacy class of maximal compact subgroups, namely the one of $\mathrm{GL}_n(\mathbb{Z}_l)$.

There is a property, *to be hyperspecial*, which is to long to be defined here (cf [**T**], of the classes of maximal compact subgroups of $G(\mathbb{Q}_l)$. At most one classes of maximal compact subgroup can be *hyperspecial*, and when there is one we say that $G(\mathbb{Q}_l)$ is *unramified*. A difficult theorem of Tits states

THEOREM VII.2.6. *For almost any l, $G(\mathbb{Q}_l)$ is unramified. More precisely, if $\mathcal{G}$ is any model of G over $\mathbb{Z}[1/N]$ for some N, then $\mathcal{G}(\mathbb{Z}_l)$ is an hyperspecial maximal compact subgroup for almost all l.*

Maximal compact special subgroup are important in the theory of representations of $(\mathbb{Q}_l)$.

PROPOSITION AND DEFINITION VII.2.7. (See e.g. [**Ca**].)Let K_l be an hyperspecial maximal compact subgroup of $G(\mathbb{Q}_l)$. Let π_l be an admissible irreducible representation of $G(\mathbb{Q}_l)$. Then $\pi_l^{K_l}$ has dimension 1 or 0. When it has dimension 1, one says that π_l is *unramified*

In particular, saying that π_l is unramified supposes that $G(\mathbb{Q}_l)$ is already unramified.

We shall prove this proposition and more precise results in the case $G(\mathbb{Q}_l) = \mathrm{GL}_n(\mathbb{Q}_l)$ below.

We shall fix an hyperspecial maximal compact open subgroup K_l of $G(\mathbb{Q}_l)$ for all l such that $G(\mathbb{Q}_l)$ are unramified, such that $\mathcal{G}(\mathbb{Z}_l) = K_l$ for almost all l ($\mathcal{G}$ being any model).

Recall that if $(V_i)_{i \in I}$ is a family of vector spaces, with $W_i \subset V_i$ a given dimension 1 subspace defined for almost all i (that is for all i except for a finite set J_0 of I), then the restricted tensor product $\bigotimes'_{i \in I} V_i$ is defined as the inductive limit of $\bigotimes'_{i \in J} V_i \otimes \bigotimes_{i \in I - J} W_i$ over the filtering set ordered by inclusion, of finite subsets J (containing J_0) of I.

We apply this to a family (π_l, V_l) of admissible irreducible representation of $G(\mathbb{Q}_l)$, such that π_l is unramified for almost all l – say all l that are not in a finite set of primes J_0. When π_l is unramified, then $W_l := \pi_l^{K_l}$ has dimension 1, so we can form

$$V := \bigotimes_l{}' V_l$$

with respect to those W_l. It is not difficult to let act on V the group $G(\mathbb{A}_f) = \prod'_{G(\mathbb{Q}_l)}$, and to see that the resulting representation, called $\pi := \otimes'_l \pi_l$ is admissible

irreducible, and satisfies, if $U = \prod_l U_l$ where U_l is a compact open subgroup of $G(\mathbb{Q}_l)$:

(93) $$\pi^U = \otimes_l \pi_l^{U_l}$$

Conversely:

PROPOSITION AND DEFINITION VII.2.8. *Every admissible irreducible representation π of $G(\mathbb{A}_f)$ can be written in a unique way as a restricted tensor product $\pi = \otimes_l' \pi_l$ where π_l is a irreducible admissible representation of $G(\mathbb{Q}_l)$, and π_l is unramified for almost all l.*

The representation π_l are called the *local components* of π.

VII.2.3. Finiteness results.

PROPOSITION VII.2.9. *The subgroup $G(\mathbb{Q})$ is discrete and cocompact in $G(\mathbb{A}_f)$.*

Proof — We already know that $G(\mathbb{Q})$ is discrete in $G(\mathbb{A})$. Since $G(\mathbb{A}) = G(\mathbb{A}_f) \times G(\mathbb{R})$, and $G(\mathbb{R})$ is compact, $G(\mathbb{Q})$ is discrete. The compactness of $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$ is the special form taken by a fundamental result of Borel ([**Bo**]) for reductive groups in the case $G(\mathbb{R})$ compact. □

COROLLARY VII.2.10. *For each open compact subgroup $U \subset G(\mathbb{A}_f)$, the group $U \cap G(\mathbb{Q})$ is finite.*

COROLLARY VII.2.11. *For each open compact subgroup $U \subset G(\mathbb{A}_f)$, the set $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$ is finite.*

We also note that since $G(\mathbb{A}_f)$ and $G(\mathbb{A})$ are locally compact, they have (left, say) Haar measures. Since G is reductive, a well known result says that those Haar-measure are both left and right invariant. A consequence of the proposition is

VII.2.4. Automorphic forms. We shall use the following notation: If $f : H \to L$ is a function on a group H (L being any set), then $r_g(f)$ is the function $H \to L$ which sends h to $f(hg)$. In other word, it is the right translation by g of f.

DEFINITION VII.2.12. An *automorphic form* for G is a function $f : G(\mathbb{A}) \to \mathbb{C}$ such that:

   (i) The function f is left-invariant under $G(\mathbb{Q})$.
   (ii) The function f is right-invariant by some open compact subgroup U of $G(\mathbb{A}_f)$ (U depends on f).
   (iii) The function f is $G(\mathbb{R})$-finite on the right: that is, the space of function on $G(\mathbb{A}_f)$ generated by the $r_g(f)$ for $g \in G(\mathbb{R})$ is finite-dimensional.

We call $A(G)$ the complex space of all automorphic forms. Note that if f is in $A(G)$, and if $g \in G(\mathbb{A})$, then $r_g(f)$ is also in $A(G)$ (the compact open subgroup U in condition (ii) is changed in $g^{-1}Ug$). Thus, there is a (right) representation of $G(\mathbb{A})$ on $A(G)$.

There is an hermitian product of $A(G)$ defined by $(f_1, f_2) = \int_{G(\mathbb{Q})\backslash G(\mathbb{A})} f_1(g)\bar{f}_2(g)\, dg$. since $G(\mathbb{Q})\backslash G(\mathbb{A})$ has a finite measure. This product is preserved by the right representation of $G(\mathbb{A})$, which is thus pre-hermitian. It is not Hermitian because $A(G)$ is not complete with respect to the norm $\sqrt{(f, f)}$. But the pre-hermitianness is sufficient to ensure that $A(G)$ is semi-simple as a $G(\mathbb{A})$-representation.

EXERCISE VII.2.13. Show that the completion of $A(G)$ is $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.

VII.2.5. Automorphic forms of weight W. To analyze $A(G)$, we shall first decompose it as a $G(\mathbb{R})$-representation.

LEMMA VII.2.14. *The representation $A(G)$ is completely reducible as a representation of $G(\mathbb{R})$, that is a direct sum of irreducible (finite-dimensional) representation of $G(\mathbb{R})$.*

Proof —   This is clear since any vector in $A(G)$ is $G(\mathbb{R})$-finite and $A(G)$ is semi-simple. □

PROPOSITION VII.2.15. *There is a natural isomorphism of $G(\mathbb{A}_f) \times G(\mathbb{R})$-representation*

$$A(G) = \oplus_W (A(G) \otimes W^\vee)^{G(\mathbb{R})} \otimes W,$$

where the sums run among representative of isomorhism classes of irreducible representation of W, and where the action of $G(\mathbb{A}_f)$ on the right-hand-side is on the factor $A(G)$ in the first factor, and the action of $G(\mathbb{R})$ is on the second factor W.

Proof —   The natural $G(\mathbb{R})$-equivariant map $W^\vee \otimes W \to \mathbb{C}$ induces a map $(A(G) \otimes W^\vee) \otimes W \to A(G)$ which is obviously $G(\mathbb{A}_f) \times G(\mathbb{R})$-equivariant. By restriction we obtain a map $(A(G) \otimes W^\vee)^{G(\mathbb{R})} \otimes W \to A(G)$, and if we sum over W, a map

$$\bigoplus_W \left((A(G) \otimes W^\vee)^{G(\mathbb{R})} \otimes W \right) \to A(G).$$

The injectivity of this map is formal, and the surjectivity follows from the fact the above lemma. □

REMARK VII.2.16. Of course, the proof above is a general (and standard) argument of representation theory: If V is a representation of $G \times H$, there is a natural injective $G \times H$-map $\oplus_W (V \otimes W^\vee)^H \otimes W \to V$ (the sum over classes of irreducible representation of H) which is an isomorphism when V is totally decomposable as H-representations. In particular, when V is irreducible, the sum contains only one

non-zero term, and $V = Z \otimes W$, where W is an irreducible representation of H and $Z = (V \otimes W^\vee)^H$ is an irreducible representation of G.

We know analyze the $G(\mathbb{A}_f)$-representation $(A(G) \otimes W^\vee)^{G(\mathbb{R})}$.

DEFINITION VII.2.17. We denote $A(G, W)$ the space of functions $f : G(\mathbb{A}_f) \to W^*$ such that

(i) The function f satisfies, for all $\gamma \in G(\mathbb{Q})$ and all $g \in G(\mathbb{A}_f)$, $f(\gamma g) = f(g)_{|\gamma_\infty^{-1}}$, where γ_∞, the canonical image of γ in $G(\mathbb{R})$ acts on $f(g) \in W^\vee$ using the right representation of $G(\mathbb{R})$ on $W^\vee$.

(ii) The function f is right-invariant by some open compact subgroup U of $G(\mathbb{A}_f)$ (U depends on f).

Such functions are called *automorphic forms of weight W*.

Obviously $A(G, W)$ is a right smooth $G(\mathbb{A}_f)$-representation

PROPOSITION VII.2.18. *There is a natural isomorphism of $G(\mathbb{A}_f)$-representation (induced by the restriction from $G(\mathbb{A})$ to $G(\mathbb{A}_f)$)*

$$(A(G) \otimes W)^{G(\mathbb{R})} = A(G, W)$$

Proof — First, $(A(G) \otimes W^\vee)^{G(\mathbb{R})}$ is trivially identified with the space of functions $f : G(\mathbb{A}) \to W^\vee$ satisfying:

(i) The function f is left-invariant under $G(\mathbb{Q})$.

(ii) The function f is right-invariant by some open compact subgroup U of $G(\mathbb{A}_f)$ (U depends on f).

(iii) The function f is $G(\mathbb{R})$-covariant on the right: that is, $f(gh) = f(g)_{|h}$ for $g \in G(\mathbb{A})$, $h \in G(\mathbb{R})$.

Start with an f as above and define $f' : G(\mathbb{A}_f) \to W^*$ by restriction. Then we have for $\gamma \in G(\mathbb{Q})$ and $g \in G(\mathbb{A}_f)$,

$$
\begin{aligned}
f'(\gamma g) &= f((\gamma g, 1_\infty)) \quad \text{(by definition of } f') \\
&= f(\gamma(g, 1)\gamma_\infty^{-1}) \\
&= f((g, 1))_{|\gamma_\infty^{-1}} \quad \text{using (i) and (iii)} \\
&= f'(g)_{|\gamma_\infty^{-1}}
\end{aligned}
$$

Hence $f' \in A(G, W)$. Conversely, given an $f' \in A(G, W)$, there is a unique way to extend it to an map $f : G(\mathbb{A}) \to W^\vee$ satisfying (i), (ii) and (iii) above, namely by setting $f(g, g\infty) = f'(g)_{|g_\infty^{-1}}$. Checking that f actually is in $(A(G) \otimes W)^{G(\mathbb{R})}$ is the same computation as above. $\square$

REMARK VII.2.19. The proof of this proposition is a very standard trick in the theory of automorphic forms. It is important to understand it completely.

PROPOSITION VII.2.20. *For every compact open subgroup $U \subset G(\mathbb{A}_f)$, the space $A(G, W)^U$ is finite dimensional. In other words, the representation $A(G, W)$ is admissible.*

Proof —   A function $f \in A(G, W)^U$ is completely determined by its value in W^* on a set of representatives in $G(\mathbb{A}_f)$ of the quotient $G(\mathbb{Q})\backslash G(\mathbb{A}_f)/U$. This set is finite by Cor. **??**. Since W is finite dimensional the result follows. $\square$

There is also a pre-hermitian product on $A(G, W)$. Choose $(,)_{W^\vee}$ a $G(\mathbb{R})$-invariant hermitian product on $W^\vee$ (such an Hermitian product exists since $G(\mathbb{R})$ is compact, and is unique up to multiplication by a scalar since W^* is irreducible). The we define, for $f.f' \in A(G, W)$:

$$(f, f') = \int_{G(\mathbb{Q})\backslash G(\mathbb{A}_f)} (f(g), f'(g))_{W^\vee} \, dg$$

which makes sense since $(f(\gamma g), f'(\gamma g))_{W^\vee} = (f(g)_{|\gamma_\infty^{-1}}, f'(g)_{|\gamma_\infty^{-1}})_{W^\vee} = (f(g), f'(g))_{W'}$ so the integrand is a function on $G(\mathbb{Q})\backslash G(\mathbb{A}_f)$. Hence $A(G, W)$ is semi-simple as a $G(\mathbb{A}_f)$-representation.

VII.2.6. Automorphic representations.

THEOREM VII.2.21. *The representation $A(G, W)$ is a direct sum of irreducible representation of $G(\mathbb{A}_f)$ with finite multiplicity.*

Proof —   This follows from the semi-simplicty of $A(G, W)$ just seen and its admissibility (Prop. VII.2.20) in view of Prop VII.1.13. $\square$

COROLLARY VII.2.22. *There is a decomposition*

$$A(G) = \oplus m(\pi)\pi$$

as $G(\mathbb{A}) = G(\mathbb{A}_f) \times G(\mathbb{R})$-representations, where the $\pi = \pi_f \otimes \pi_\infty$ are irreducible representations of $G(\mathbb{A})$, with pi_f an irreducible admissible representation of $G(\mathbb{A}_f)$ and π_∞ being a (finite-dimensional) irreducible representation of $G(\mathbb{R})$. Moreover the multipliicty $m(\pi)$ are finite.

This follows from the Theorem and Prop. VII.2.15.

DEFINITION VII.2.23. An irreducible representation π of $G(\mathbb{A})$ that appears in the decomposition of $A(G)$ is said *automorphic*. The number $m(\pi)$ is called its (automorphic) *multiplicity*. The component π_l of π_f (for l a prime), and π_∞ are called the *local component of π*. The equivalence class of the component π_∞ is also called the *weight* of π.

VII.2.7. The automorphic representations are algebraic. Let π be an automorphic representation and set $W = \pi_\infty$ be its weight, a representation of $G(\mathbb{R})$. Let us recall an elementary lemma of representation theory:

LEMMA VII.2.24. *Assume $G(\mathbb{R})$ is compact, and G is reductive.*

(i) *The restriction by the map $G(\mathbb{R}) \hookrightarrow G(\mathbb{C})$ induces a bijection between the equivalence classes of complex algebraic representations of $G(\mathbb{C})$ and complex irreducible continuous representations of $G(\mathbb{R})$.*

(ii) *If W is any irreducible finite-dimensional representation of $G(\mathbb{R})$, the restriction of $G(\mathbb{R})$ to $G(\mathbb{Q})$ has a model $W_{\bar{\mathbb{Q}}}$ over $\bar{\mathbb{Q}}$.*

From this we deduce easily

PROPOSITION VII.2.25. *The admissible $G(\mathbb{A}_f)$-representation $A(G, W)$ is defined over $\bar{\mathbb{Q}}$. So are the π_f for π any automorphic representations.*

Proof — Just define $A(G, W)_{\bar{\mathbb{Q}}}$ as the space of functions $f : G(\mathbb{A}_f) \to W_{\bar{\mathbb{Q}}}^*$ such that

(i) The function f satisfies, for all $\gamma \in G(\mathbb{Q})$ and all $g \in G(\mathbb{A}_f)$, $f(\gamma g) = f(g)_{|\gamma^{-1}}$.

(ii) The function f is right-invariant by some open compact subgroup U of $G(\mathbb{A}_f)$ (U depends on f).

This is clearly an admissible representation over $\bar{\mathbb{Q}}$, and tensorizing it by $\mathbb{C}$ gives back $A(G, W)$, which proves the first assertion. The second follows form the first by Exercise VII.1.9 □

VII.2.8. Levels.

DEFINITION VII.2.26. A level U is

VII.3. A fragment of the theory of admissible representation of GL_n of a local field

In this subsection, let F be an extension of $\mathbb{Q}_l$, and $G = \mathrm{GL}_n(F)$, which is a lctd group.

VII.3.1. Some algebraic subgroups, and the Bruhat decomposition. First do the following easy exercise in group theory:

EXERCISE VII.3.1. Let G be a group, and A and B be two subgroups (not necessarily normal). Show that the following are equivalent:

(i) $AB = G$

(ii) $BA = G$

(iii) $A'B' = G$ where A' (resp. B') is a conjugate of A (resp. of B) in G.

We let B the subgroup of upper triangular matrices in G (a *Borel*), N the subgroup of upper triangular matriceswhose diagonal coefficients are 1 (the *unipotent radical of B*) and T the subgroup of diagonal matrices (a *maximal split torus*)

EXERCISE VII.3.2. **(easy)** Show that $N = [B, B]$; in particular N is normal in B. Show that $B = NT$, and $N \cap T = \{1\}$. In particular, there is a natural split exacts equence $1 \to N \to B \to T \to 1$ and T is abelianization of B.

Let W be the group of matrices whose coefficients are only 1 or 0, and with exactly one 1 on each row and each column. Obviously W acts by permutation on the standard basis of F^n and is thus naturally isomorphic to S_n. Also W acts by conjugation on T.

PROPOSITION VII.3.3 (Bruhat's decomposition). *One has*

$$G = \coprod_{w \in W} BwB.$$

In other words, W is a set of representatives of $B \backslash G / B$.

Proof —   To give a conceptual proof, let's introduce some terminology:

Let $V = F^n$. A *flag* in F^n is a family $0 = V_0 \subset V_1 \subset \cdots \subset V_{n-1} \cdots \subset V_n = n$ where each V_i is a subspace of V of dimension i. For example, taking V_i the space generated by the i first elements of the canonical basis of F^n defines a flag $\mathcal{F}_0$. The group G has a left t action on the sets of flags, defined by $g(V_0, V_1, \ldots, V_n) = (gV_0, gV_1, \ldots, gV_m)$ and B is just the stabilizer of $\mathcal{F}_0$ for that action action. Moreover it is easy to see that G acts transitively on the set of flags so the map $g \mapsto g\mathcal{F}_0$ is easily seen to be an isomorphism of G-sets from G/B onto the set of flags.

A *lines decomposition* is a set of n lines in V whose sum is V. For example, the set of lines generated by a vector of the canonical basis is a *lines decomposition* $\mathcal{L}_0$.

We say that a lines decomposition *belongs* to a flag $(V_0, V_1, \ldots, V_n)$ if every of the V_i is a sum of (necessarily i) lines of the line decomposition. For example $\mathcal{L}_0$ belongs to $\mathcal{F}_0$, and more generally, the lines decomposition that belongs to $\mathcal{L}_0$ are the lines decomposition $b\mathcal{L}_0$ for $b \in B$.

The flags in which a line decomposition belong are clearly in bijection with the ordering of the n lines of the line decomposition (the flag attached to an ordering $L_1, \ldots, L_n$ of lines is $0 \subset L_1 \subset L_1 \oplus L_2 \subset \ldots$). In particular there are $n!$ of them.

Now the key and easy fact is that for any two flags $\mathcal{F}$ and $\mathcal{F}'$ there exists a line decomposition that belongs to both of them. This is left as an exercise.

In particular, if $g \in G$, then there is a line decomposition $\mathcal{L}$ belongings to both $\mathcal{F}_0$ and $g\mathcal{F}_0$. Since $\mathcal{L}$ belongs to $\mathcal{F}_0$, it is of the form $b\mathcal{L}_0$ ome $b \in B$. The flags $\mathcal{F}_0$ and $\mathcal{F}$ defines two ordering of the lines of $\mathcal{L}$, hence of $\mathcal{L}_0$, and there is a permutation matrix $w \in W$ that transform the first ordering into the second : it follows that $bw\mathcal{F}_0 = g\mathcal{F}_0$ and $bwb' = g$ for some $b' \in B$. Thus $G = \cup_{w \in W} BwB$. That the union is disjoint is easy. $\square$

VII.3.2. Normalized induction.

VII.3.3. The Jacquet functor and decomposition of principal series.

VII.3.4. Maximal compact subgroups, Iwahoris, and the Iwasawa's decomposition.

VII.3.5. The spherical Hecke algebras, the Iwahori-Hecke algebra, and the Atkin-Lehner algebra.

VII.3.6. Unramified representation.

VII.3.7. Refinements of unramifed representations.

VII.4. Automorphic representations for a form of GL_n that is compact at infinity

In this subsection, we assume that G is an algebraic group over $\mathbb{Q}$ such that

(a) $G(\mathbb{R})$ is compact.
(b) $G_{\mathbb{C}} \simeq \mathrm{GL}_n$.

Hypothesis (b) implies that G is reducible and connected, hence the results of the above subsection still apply here. There is a classification of the group satisfying (b) (cf [**?**] or [**?**]).

VII.4.1. Classification of weights. As we have seen

Chenevier's eigenvarieties

CHAPTER IX

A p-adic L-function on unitary eigenvariety and its zero locus

Solution to exercises

Solution to Exercise I.1.1   See [**E**, Proposition 2.10].

Solution to Exercise I.1.2   See the solution to Exercise I.4.2.

Hint to Exercise I.1.3   By construction of the characteristic polynomial, this question reduces to the case where M is free, in which case it is trivial.

Solution to Exercise I.3.3   The surjective tautological map $\mathcal{H} \otimes R \to \mathcal{T}_A$ factors through $\mathcal{T}$ since any element $h \in \mathcal{H} \otimes R$ such that $\psi(h) = 0$ obviously acts by 0 on A. Hence we have a natural surjective map $\mathcal{T} \to \mathcal{T}_A$. Its kernel is by construction the ideal generated by the elements $\psi(h)$ of $\mathrm{End}_R M$ than vanish on A. If M/A is torsion, such an endomorphism $\psi(h)$ is torsion in $\mathrm{End}_R(M)$, hence is 0 if R is a domain because in this case $\mathrm{End}_R(M)$ is torsion free. This proves 1.

For 2. and an example of non-injectivity, takes $A = B = 0$. For an example of non-surjectivity, take $A = B = R$ and $M = A \oplus B$ with $\mathcal{H}$ generated by one element T which acts as the identity on M. Then the map $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$ is the diagonal embedding $R \to R^2$ which is not surjective.

3. is proved exactly as the second part of 1.

For 4. let $e = (T - a)/(b - a)$ in $\mathcal{T}$, so e acts by 0 on A and 1 and B, that is the image of $e \in \mathcal{T}$ is 0 in $\mathcal{T}_A$ and 1 in $\mathcal{T}_B$. If $t_A \in \mathcal{T}_A$ and $t_B \in \mathcal{T}_B$, there exists $u_A \in \mathcal{T}$ that maps on t_A in $\mathcal{T}_A$ (since the map $\mathcal{T} \to \mathcal{T}_A$ is surjective by 1.) and $u_B \in \mathcal{T}$ that maps on t_B in $\mathcal{T}_B$ (same reason). Then $t := (1 - e)t_A + et_B \in \mathcal{T}$ maps to (t_A, t_B) in $\mathcal{T}_A \times \mathcal{T}_B$.

Hint to Exercise I.3.4   For 1. consider the map $\mathcal{T} \to \mathcal{T}^\vee$ that sends an endomorphism of M to its transpose.

2. is just a reformulation of the classical result that over any field a square matrix is conjugate to its transpose.

For 3. and R not a field, check that one just need to find a matrix in $M_n(R)$ which is not conjugate (in $\mathrm{GL}_n(R)$) to its transpose. Then check that for $R = \mathbb{Z}$, the matrix $T = \begin{pmatrix} 1 & -5 \\ 3 & -1 \end{pmatrix}$ is not similar to its transpose over $\mathbb{Z}$.

For 3. and $R = \mathbb{C}$, take $M = \mathbb{C}^3$ and $\mathcal{H}$ generated by two elements X and Y acting on M by the matrices $X = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $Y = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$.

Solution to Exercise I.3.4   1. and 2. should be clear after the hint.

For 3. and the case of a ring, checking that the matrix $T = \begin{pmatrix} 1 & -5 \\ 3 & -1 \end{pmatrix} \in M_2(\mathbb{Z})$ is not conjugate to its transpose is a straightforward but tedious computation. For a conceptual proof, and an explanation of how this example was found and how to find many other ones, see [**ConK**], especially Example 15.

For 3. and the case of a field, considering the example given in the hint we observe that $\operatorname{Im} Y \oplus \operatorname{Im} X$ has dimension 2 while $\operatorname{Im} {}^t Y \oplus \operatorname{Im} {}^t X$ has dimension 1. There can therefore be no matrices $P \in \operatorname{GL}_3(\mathbb{C})$ such that $PXP^{-1} = {}^t X$ and $PYP^{-1} = {}^t Y$, which shows that as an $\mathcal{H}$-module, M is not isomorphic to $M^\vee$.

Solution to Exercise I.3   For 1. let $M = R^2$, and $\psi : a + b\epsilon \mapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$ for $a \in R, b \in I$. For 2. take $R = \mathbb{C}[X,Y]$. Since R is regular of dimension 2, any finite torsion-free R-algebra is an eigenalgebra, and there are plenty of them that are non-flat.

Solution to Exercise I.4.2   The eigenalgebra $\mathcal{T}$ is the quotient of $\mathbb{Z}_p[T]$ by the ideal of operators that act trivially on M. Let $P(X) \in \mathbb{Z}_p[X]$. If $P(T) = 0$ as an operator on M, then P is divisible by the minimal polynomial $(X - 1)^2$ of T in $\mathbb{Q}_p[X]$, so is divisible by $(X-1)^2$ in $\mathbb{Z}_p[X]$ (since the algorithm of euclidean division by a monic polynomial like $(X - 1)^2$ is identical in $\mathbb{Z}_p[X]$ and $\mathbb{Q}_p[X]$). It follows that $\mathcal{T} \simeq \mathbb{Z}_p[X]/(X - 1)^2$ with the isomorphism sending T on X.

The eigenalgebra $\mathcal{T}'$ is simply $\mathbb{F}_p$ since T acts as the identity on $M \otimes \mathbb{F}_p = \mathbb{F}_p^2$. The map $\mathcal{T} \otimes_{\mathbb{Z}_p} \mathbb{F}_p = \mathbb{F}_p[X]/(X - 1)^2 \to \mathcal{T}' = \mathbb{F}_p$ is the map that sends X on 1. Of course this map is not an isomorphism.

Solution to Exercise I.5.4   This is standard: by induction on the number of generators of $\psi(\mathcal{H})$, one reduces to the case of one generator T of $\psi(H)$. In this case let $P(X)$ be the characteristic (or minimal) polynomial of T and write $P(X) = (X - a_1)^{n_1} \ldots (X - a_r)^{n_r}$ where the a_i are distinct. Let $P_i(X)$ be the same polynomial with the factor $(X - a_i)^{n_i}$ removed. Then the P_i's are relatively prime, and a Bezout relation between them shows that $M = \sum_i \ker(T - a_i)^{n_i}$, a sum which is obviously direct.

Solution to Exercise I.5.5   We obviously have $M[\chi] \otimes k' \subset M'[\chi']$ in M' and the equality follows from the equality of dimensions, since the rank of a system of linear equations over k does not change when we extend the scalar to k'. Same argument for $M_{(\chi)}$.

Solution to Exercise I.5.6 By Exercise I.5.5, we may assume that k is algebraically closed. The operators $\psi(T)$, $T \in \mathcal{H}$, on the non-zero space $M_{(\chi)}$ commute so they have a common non-zero eigenvector v, whose system of eigenvalue is χ. So $M[\chi] \neq 0$.

Solution to Exercise I.5.7 The only non-trivial part is the surjectivity of $M_{(\chi)} \to N_{(\chi)}$ but it can be checked when the field is assumed to be algebraically closed by Exercise I.5.5 and then it follows easily from Exercise I.5.4.

Solution to Exercise I.5.8 We may assume that k is algebraically closed. Since $M = \oplus_\chi M_{(\chi)}$ by Exercise I.5.4, we get a decomposition $M^\vee = \oplus_\chi M^\vee_{(\chi)}$, and by applying directly Exercise I.5.4 an other decomposition $M^\vee = \oplus_\chi (M^\vee)_{(\chi)}$. Since $M^\vee_{(\chi)} \subset (M^\vee)_{(\chi)}$ the two decompositions are the same.

For an example where $\dim M[\chi] = \dim(M^\vee)[\chi]$, consider the case $M = \mathbb{C}^3$ with the action of $\mathcal{H} = \mathcal{Z}[X,Y]$ introduced in the solution of Exercise I.3.4.

Solution to Exercise I.5.15 Setting $d = \dim_k M$, the algebra of matrices that are upper unipotent by block $(d/2, d/2)$ when d is even, or $((d-1)/2, (d+1)/2)$ when d is odd, is commutative and has the required dimension.

Solution to Exercise I.6.16 For 1., $M[\lambda]$ has a basis $f(z), f(lz), \ldots, f(l^{m_l}z)$. Assume $l \nmid N_0$ first. For $m = 1, 2, 3$ respectively, the matrix of U_l in this basis is

$$\begin{pmatrix} a_l & 1 \\ p^{l-1}\epsilon_l & 0 \end{pmatrix}, \quad \begin{pmatrix} a_l & 1 & 0 \\ p^{l-1}\epsilon_l & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} a_l & 1 & 0 & 0 \\ p^{l-1}\epsilon_l & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

For $m > 3$, the matrix is the same as in the case $m = 3$, extended with entries just above the diagonal equal to 1 and 0 elsewhere.

If $m \geq 3$, this matrix is not semi-simple since 0 is a root of multiplicity $m-1$ of its characteristic polynomial, but the kernel of this matrix has dimension 1. If $m = 1$ (resp. $m = 2$), the characteristic polynomial is $(X^2 - a_l X + l^{k-1}\epsilon_l)$ (resp. $X(X^2 - a_l X + l^{k-1}\epsilon_l)$). If $X^2 - a_l X + l^{k-1}$ has two distinct roots (necessarily non-zero), then the characteristic polynomial has simple roots and U_l is diagonalizable. If this polynomial has a double root, then we see for $m = 1$ that U_l is not diagonalizable as it is not scalar. For $m = 2$, U_l is not diagonalizable either, because by looking at the restriction of U_l on the stable space genarated by $f(z), f(lz)$ we are reduced to the case $m = 1$.

If $l \mid N_0$, then the matrix of U_l is for $m = 3$ (for instance)

$$\begin{pmatrix} u_l & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

We leave the analysis of this case to the reader.

For 2., note that $\mathcal{H}$ acts semi-simply on $M[\lambda]$ if and only if each of the U_l, for $l \mid N/N_0$ acts semi-simply (since those operators commute and the others act by scalars). So fix an $l \mid N/N_0$. For d a divisor of N/N_0 not divisble by l, call $M_d \subset M[\lambda]$ the subspace generated by $f(dz), f(dlz), \ldots, f(dl^{m_l}z)$. Then M_d is stable by U_l and the matrix of U_l in the above basis is the same as written above (in particular is independent on d). Since $M[\lambda]$ is the sum of the M_d for d as above, U_l acts semi-simply on $M[\lambda]$ under the same conditions as in 1, and the result follows.

Solution to Exercise I.7.7   Point 1. is a direct application of the commutative diagrams of sets we have written.

Let K' be a finite Galois extension such that $G_{K'}$ acts trivially on the set of points of non-closed points of $\operatorname{Spec}\mathcal{T}$ (this is possible sine $\mathcal{T} \otimes K$ is étale over K), and define R', $\mathfrak{m}'$, k' as usual. Recall that $\mathcal{T}_{R'} = \mathcal{T} \otimes_R R'$ is étale over R' if and only if $\mathcal{T}$ is étale over R, since R'/R is faithfully flat. Observe that every irreducible component of $\operatorname{Spec}\mathcal{T}_{R'}$ has generic degree 1 over R' since its generic point is defined over K'. Thus every irreducible component of $\operatorname{Spec}\mathcal{T}_{R'}$ is isomorphic (through the structural map) to $\operatorname{Spec} R'$. We thus see that $\operatorname{Spec}\mathcal{T}_{R'}$ is non-etale if and only if it has more irreducible component that connected component, and 2. follows from 1. applied to $\mathcal{T}_{R'}$.

Solution to Exercise I.7.3 $\mathcal{T} = \mathbb{Z}_p[X]/(X^2 - p^{a+b})$ where the X corresponds to $\psi(T)$. Let $n = a + b$. It is connected iff $n \geq 1$, irreducible iff n is odd, regular iff $n \leq 1$, and étale over R iff $n = 0$. The module M is free over $\mathcal{T}$ iff $ab = 0$.

Hint to Exercise I.7.4   Take $R = k[[X]]$ (k any field) and $\mathcal{T} = \{(P,Q) \in R^2, \ P(0) = Q(0)\}$. Note that $\mathcal{T}$ is a finite torsion-free R-algebra over R which is a PID, so is really an eigenalgebra. Observe that $\mathcal{T}_K \simeq K^2$, where $K = \operatorname{Frac}(R) = K((X))$. Consider the map $\chi : \mathcal{T} \mapsto k[[X]]/(X^2), (P,Q) \mapsto P(0)+(P'(0)+Q'(0))X$. Check that this is a morphism of algebra which is not liftable.

Solution to Exercise I.7.5.1 $C = \mathbb{Z}/2\mathbb{Z}$.

Solution to Exercise I.7.5.1   The inclusion between finite R-modules $(M \cap A) \oplus (M \cap B) \hookrightarrow M$ becomes after tensorizing by K the isomorphism $A \oplus B \xrightarrow{\sim} M$. This implies that the cokernel C of this inclusion is torsion. It is finite as a quotient of M. This proves a.

The map p_A restricted to $M \cap A$ has kernel $M \cap A \cap B = 0$, so this map identifies $M \cap A$ with a submodule of M_A. Now consider the composition $M \xrightarrow{p_A} M_A \to M_A/p_A(M \cap A) = M_A/(M \cap A)$, which is surjective as a composition of surjections. Its kernel is the set of $m \in M$ such that $p_A(m) \in p_A(M \cap A)$ that is such that m differs from an element of $M \cap A$ by an element of $\ker p_A = M \cap B$. In

other words, the kernel of this map is $(M \cap A) \oplus (M \cap B)$, and this maps realizes
an isomorphism $M/((M \cap A) \oplus (M \cap B)) \to M_A/(M \cap A)$, which proves b.

The kernel of the map (p_A, p_B) is $(M \cap B) \cap (M \cap A) = 0$ so this map may
be used to identify M to a submodule of $M_A \oplus M_B$. The composition $M_A \hookrightarrow$
$M_A \oplus M_B \to (M_A \oplus M_B)/M$ has for kernel the set of $m \in M_A$ such that there
exists $m' \in M$ satisfying $p_A(m') = m$, $p_B(m') = 0$. The latter condition on m' is
equivalent to $m' \in M \cap A$, so the condition on m is equivalent to $m \in p_A(M \cap A)$
and our map realizes an injection $C = M_A/(M \cap A) \to (M_A \oplus M_B)/M$. This map
is easily seen to be surjective, which proves c.

Solution to Exercise I.7.5.2   Let $A = Kf$, B the orthogonal of A for the
bilinear product. The non-vanishing of $\langle f, f \rangle$ ensures that $B \cap A = 0$, and since
$\dim B + \dim A = \dim M_K$ by the non-degeneracy of the bilinear product, $M_K = $
$A \oplus B$. So we can apply Prop. I.7.16 which gives the result.

Hint to Exercise I.7.5.3   Let $R = k[[X]]$, k a field. Let $\mathcal{T} = \{(p, q, r) \in$
R^3, $p(0) = q(0) = r(0), p'(0) = q'(0) + r'(0)\} \subset R^3$. Check that $\mathcal{T}$ is a torsion-
free subalgebra of R^3. Let $\mathcal{T}_A = \{p \in R\} = R$ and $\mathcal{T}_B = \{(q, r) \in R^2, \ q(0) =$
$r(0)\} \subset R^2$, which are also R-algebras. The obvious maps $\mathcal{T} \to \mathcal{T}_A$ and $\mathcal{T} \to \mathcal{T}_B$
are clearly surjective, while their product $\mathcal{T} \to \mathcal{T}_A \times \mathcal{T}_B$ is injective. Check that
there exists a module M of rank 3 over R, with a action of an operator T, and a
T-stable decomposition $M_K = A \oplus B$ such that $\mathcal{T}, \mathcal{T}_A, \mathcal{T}_B$ are the eigenalgebras of
M, $M \cap A$, $M \cap B$ respectively, and the natural maps $\mathcal{T} \to \mathcal{T}_A$ and $\mathcal{T} \to \mathcal{T}_B$ are
the ones we define. Also check that $\mathcal{T}_K = K^3$, so every point of Spec $\mathcal{T}_K$ is defined
over K.

Consider the character $\chi : \mathcal{T} \to R/\mathfrak{m}^2$ sending (p, q, r) to $p \mod \mathfrak{m}^2$. This char-
acters obviously factors through $\mathcal{T}_A$. Show that this characters also factors through
$\mathcal{T}_B$. So it is an *eigencongruence* between A and B modulo $\mathfrak{m}^2$. However, show that
there is no congruence modulo $\mathfrak{m}^2$ between system of eigenvalues appearing in A
and B.

Hint to Exercise I.7.5.3   Apply the variant of the Deligne-Serre lemma.

Solution to Exercise I.7.5.3   By Prop I.7.15 there is $f \in A$, $g \in B$ such that
$f \equiv g \pmod{\mathfrak{m}}$ but $f \not\equiv 0 \pmod{\mathfrak{m}}$. Since $\dim A = 1$, f is an eigenvetor for $\mathcal{T}$,
defining a character $\phi_f : \mathcal{T} \to R$, which obviously factors through $\mathcal{T}_A$. Since $f \equiv g$,
we have $Tg \equiv \phi_f(T)g \pmod{m}$ for $T \in \mathcal{T}$, shich shows that the character ϕ_f
$\pmod{m} : \mathcal{T} \to R/\mathfrak{m}$ factors through $\mathcal{T}_B$. Lifting this character into a character
$\phi' : \mathcal{T}_B \to R$ (using the variant of Deligne-Serre's lemma) gives the congruence
$\phi' \equiv \phi_f \pmod{\mathfrak{m}}$ we were looking for.

A counter-example to the converse is provided by Example I.7.19

Hint to Exercise I.7.5.3   Prove that $\mathcal{O}_M/\mathcal{T}$ is killed by π^v.

Hint to Exercise I.8.1   Consider the case $w = 12$, $k = 1$, $R = \mathbb{Z}_2$.

Solution to Exercise I.8.1 $M_12(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Z})$ has (E_{12}, Δ). The reduction modulo 2 of the q-expansions of those forms are $\tilde{E}_{12} = 1$ and $\tilde{\Delta} = \sum_{n \text{ odd}} q^{n^2}$. In particular, all Hecke operators T_p sends both forms to 0, on the Hecke algebra $\mathcal{T}_{F_2}$ in this case is just $\mathbb{F}_2$. Howewer, the rank of $\mathcal{T}_{\mathbb{Z}_2}$, that is the dimension of $\mathcal{T}_{\mathbb{Q}}$ is 2 because the eigenform E_{12} and Δ have distinct eigenvalues (or because of Proposition I.6.11). This shows that the map $\mathcal{T}_R \otimes_R k \to \mathcal{T}_k$ is not an isomorphism.

Hint to Exercise I.8.1   By looking on the formula defining Hecke operator on q-expansion, show that for $f \in M_k(\mathrm{SL}_2(\mathbb{Z}), \mathbb{F}_p)$, one has $T_p f^p = f$ and for m an integer not divisible by p, $T_m f^p = (T_m f)^p$.

Solution to Exercise I.8.1   We now form the proof of the proposition above that the rank of $\mathcal{T}_R$ and the dimension of $\mathcal{T}_k$ are equal (and equal to $n = \dim S_w(\mathrm{SL}_2(\mathbb{Z}), \mathbb{Q})$, which is the number of system of eigenvalues appearing in $S_w(\mathrm{SL}_2(\mathbb{Z}), \bar{\mathbb{Q}})$). The rank of $\mathcal{T}_R^p$, that is the dimension of $\mathcal{T}_K^p$ is also the number of system of eigenvalues for all Hecke operators excepted T_p, in $S_w(\mathrm{SL}_2(\mathbb{Z}), \bar{\mathbb{Q}})$, but by the strong multiplicity one theorem, that's the same as the number n of systems of eigenvalues for all Hecke operators. Hence $\dim \mathcal{T}_R^p \otimes_R k = n$. Therefore it suffices to prove that $\dim \mathcal{T}_k^p < n$, or equivalently, that the inclusion $\mathcal{T}_k^p \subset \mathcal{T}_k$ is strict.

Suppose the contrary. Then as an operator of $S_w(\mathrm{SL}_2(\mathbb{Z}), F_p)$, T_p is a linear combination $\sum a_i T_{m_i}$ where the a_i are in $k = \mathbb{F}_p$, and the m_i are integers not divisible by p. Thus $\tilde{\Delta} = T_p \tilde{\Delta}^p = \sum a_i T_{m_i} \Delta^p = (\sum a_i T_{m_i} \tilde{\Delta})^p$ using the hint (which is an easy computation), and this is a contradiction since $\tilde{\Delta}$ is not a p-power since its q-expansions begins with q.

Solution to Exercise II.1   cf. [**L**, Proposition 1]

Solution to Exercise II.1   cf. [**L**, Proposition 2]

Solution to Exercise II.1   The hypothesis means that $v_p(a_n(x)) - n\nu \geq v_p(a_N(x)) - N\nu$ for all n, the inequality being strict if $n > N$. Taking the inf on $x \in X$ gives $v_p(a_n) - n\nu \geq v_p(a_N) - N\nu$, the inequality being still strict if $n > N$ because of the principle of maximum. This means that $N(F, \nu) = N$.

Hint to Exercise II.1   With the assumption we see that $|a_{N(Q,\nu)}(x)| = |a_{N(Q,\nu)}|$ for all $x \in \mathrm{Sp}\, R$, and the result easily follows.

Hint to Exercise II.3.2   1. is already clear at the level of local pieces. 2. is easy. For 3., observe that a ν which is adapted to M_W' is also adapted for M_W, using that the characteristic power series of U_p on M_W divides the one of M_W'. This reduces 3. to proving the existence of the closed immersion for the local pieces (

$\mathcal{E}_{W,\nu} \hookrightarrow \mathcal{E}'_{W,\nu}$, which is easy. For 4., apply 3. twice and observe that the eigenvariety for M_W^2 is the same as for M_W.

Solution to Exercise II.4 A system of eigenvalues appearing in M''_W apears either in M_W or in M'_W. Hence the exercise follows from the above theorem.

Solution to Exercise II.5.1 In an irreducible one-dimensional rigid analytic subspace, any set with an accumulation point is Zariski-dense. In particular $a + b\mathbb{N}$ is Zariski dense. If $x \in a + b\mathbb{N}$, and V is an affinoid closed ball of radius $r > 0$ around a, then for n big enough $x + p^n b\mathbb{N} \in V \cap X$ and those set has x as accumulation point.

Solution to Exercise III.1.1 1.a. Let A, B, C be such a triangle. Let D be the point such that $(ABDC)$ is a parallelogram. Then there is no points of $\mathbb{Z}^2$ in that paralellogram, except A, B, C and D since any point in $(ABDC)$ either lies on the triangle (ABC) or on the triangle (BCD) in which case the symmetric of P with respect to the middle of (BC) would be an integral point in (ABC).

We can assume that that A is the origin $(0,0)$ by translation. Let Λ be the lattice in $\mathbb{R}^2$ generated by the vectors $\vec{AB}$, $\vec{AC}$. Clearly $\Lambda \subset \mathbb{Z}^2$. We claim that this inclusion is an equality. Indeed, if $P \in \mathbb{Z}^2$, there is $v \in \Lambda$ such that $p - v$ lies in the parallelogram $(ABCD)$. Since $p - v$ is in $\mathbb{Z}^2$, it is by the above either A, B, C or D. In any case $p - v \in \Lambda$ so $p \in \Lambda$.

It follows that the vectors $\vec{AB}$, $\vec{AC}$ generates $\mathbb{Z}^2$. Hence their determinant is ± 1, and the area of the triangle (ABC) is $1/2$,

1.b. Let $A = (0,0)$, $B = (a,c)$, $C = (b,d)$. Assume that the angle $\vec{AB}$, $\vec{AC}$ is less than 180 degrees by exchanging B and C if necessary. If the triangle (A, B, C) has area $1/2$, the matrix $\gamma \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has determinant ± 1, and actually 1 by our angle hypothesis. Thus $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ sends $\{\infty\} - \{0\}$ to $\{a/c\} - \{b/d\}$

1.c. Let $A = (0,0)$, $B = (a,c)$, $C = (b,d)$. Assume that the angle $\vec{AB}$, $\vec{AC}$ is less than 180 degrees by exchanging B and C if necessary. We prove by induction on the number of points of $\mathbb{Z}^2$ in the triangle (ABC) that $\{a/c\} - \{b/d\}$ lies in the $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$-module generated by $\{\infty\} - \{0\}$. Since those divisors clearly generate Δ_0 as $\mathbb{Z}$-module, this would be sufficient to prove Manin's lemma.

To start the induction, note that when (ABC) has only three points in $\mathbb{Z}^2$, we know that it has area $1/2$ by 1.a. So by 1.b. we are done.

In general, if (ABC) has more than three points in $\mathbb{Z}^2$, let $D = (e, f)$ be a fourth point. By replacing D by an other point of $\mathbb{Z}^2$ in the segment $[AD]$ (which lies inside the triangle (ABC)), we can assume that e and f are relatively prime integers. Note that D is not on $[AB]$ (resp. nor on $[AC]$) since we have assumed that a and c (resp. b and d) are relatively prime. Therefore the triangles (ABD) and (ADC) are non-flat, and have strictly less integral points that (ABC). By

induction hypothesis, we know that $\{a/c\} - \{e/f\}$ and $\{e/f\} - \{b/d\}$ both belong to the $\mathbb{Z}[\mathrm{SL}_2(\mathbb{Z})]$-module generated by $\{\infty\} - \{0\}$). So their sum also does, which proves the induction step.

2. follows from easily from 1. using that $\mathrm{SL}_2(\mathbb{Z})/\Gamma$ is finite.

3. By induction on the number of integral points in a polygon, any (reasonable) polygon can be covered by triangles whose vertices are in $\mathbb{Z}^2$ and that contain no other point of $\mathbb{Z}^2$. Since we know by a. that Pick's theorem is true for any of those triangles, it is sufficient to check that Pick's theorem is additive, that is holds for any polygon with vertices in $\mathbb{Z}^2$ that is the union of two smaller polygon with vertices in $\mathbb{Z}^2$, disjoint interior, and an edge in common, provided Pick's theorem is true for those two smaller polygons. This is an easy computation. We leave the details to the reader.

Hint to Exercise III.1.2   If this is difficult, read or re-read the chapter on Hecke operators in [**Shi**]

Solution to Exercise III.1.2   We have

$$\Gamma \begin{pmatrix} 1 & 0 \\ 0 & l \end{pmatrix} \Gamma = \coprod_{a=0}^{l-1} \begin{pmatrix} 1 & a \\ 0 & l \end{pmatrix} \coprod \begin{pmatrix} l & 0 \\ 0 & 1 \end{pmatrix}$$

for $l \nmid N$ and the same without the last matrix if $l \mid N$. Since all those matrices have determinant l, the result follows from the definition of the Hecke operators.

Hint to Exercise III.1.3   Look up a good reference in algebraic topology.

Solution to Exercise III.1.4   The image of (x_0, y_0) by (22) only depends on $x = \beta(x_0)$ by the commutativity of (23). To see it only depends on $y = \beta(y_0)$, exchange G with $G^\vee$ and i with $n - i$.

Solution to Exercise III.1.4   By exhanging G with $G^\vee$, we only need to show that for $x \in H_!^i(X, G^\vee)$, there is a $y \in H_!^{n-i}(X, G)$ such that $(x, y) \neq 0$. Seeing x as an element of $H^i(X, G^\vee)$, there is by the non-degeneracy of (21) a $y_0 \in H_c^1(X, G^\vee)$ such that $(x, y_0) \neq 0$. Setting $y = \beta(y_0)$, we have $(x, y) \neq 0$.

Solution to Exercise III.2.5   No, we have $\dim_{\mathbb{C}} W[\chi] = \dim_{\mathbb{C}} \bar{W}[\bar{\chi}]$.

Solution to Exercise III.2.5   The map c is clearly $\mathbb{R}$-linear, so we only need to check that $c(iw) = ic(w)$. If $w = w_1 + iw_2$ with $w_1, w_2 \in W_{\mathbb{R}}$, then $iw = -w_2 + iw_1$, so $c(iw) = \overline{-w_2} + i\overline{w_1} = i(\overline{w_1} + i\overline{w_2}) = ic(w)$.

Hint to Exercise III.2.8 $\Gamma_0(N)$ has 2^r cusps, which are parametrized by the application from $a : \{l_1, \ldots, l_r\} \to \{0, 1\}$; if c is such an application, the set c_a of rational numbers which are l-integral (resp. not l-integral) if $a(l) = 1$ (resp. $a(l) = 0$) is a $\Gamma_0(N)$-class in $\mathbb{P}^1(\mathbb{Q})$.)

Solution to Exercise III.3.1   In the definition of $f_\chi(z)$, introducing the easy equality $\chi(n) = \frac{\sum_{a \,(\mathrm{mod}\ m)} \bar{\chi}(a) e^{2i\pi na/m}}{\tau(\bar{\chi})}$ gives the formula for $f_\chi(z)$. For the fact that $f_\chi(z)$ is a modular form, see [**Shi**, Prop. 3.64]

Solution to Exercise III.3.1   if ψ is any continuous character $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^* \to \mathbb{C}^*$, then $\psi_{|\mathbb{R}_+^*}$ as the form $x \mapsto x^s$ for some (unique) $s \in \mathbb{C}$. Then $\psi|\ |^{-s}$ is a continuous character on $\mathbb{A}_\mathbb{Q}^*/\mathbb{Q}^*\mathbb{R}_*^+$ and we conclude by Class Field Theory.

Solution to Exercise III.3.2   When m is prime to $\mathfrak{p}$, the polynomials $(z - a/m)^j$ are $\mathfrak{p}$-integral, so the right hand side of (**??**) is clearly $\mathfrak{p}$-integral. Let us assume that $k = 0$. Then since $\phi_f^\pm$ are $\mathfrak{p}$-normalized, there is an irreducible fraction a/m such that $\phi_f(\{\infty\} - \{a/m\})/\Omega_f^\pm$ has $\mathfrak{p}$-valuation 0. Thus by linear independence of characters, there is a character χ of conductor m such that $\sum_{a \,(\mathrm{mod}\ m)} \bar{\chi}(a) \frac{\phi_f^\pm(\{\infty\}-\{a/m\})}{\Omega_f^\pm}$ also has $\mathfrak{p}$-valuation zero.

Hint to Exercise III.4.2   Consult [**BGR**]

Solution to Exercise III.4.2   Choose a set E that satisfies the condition of Prop. III.4.12 for both r_1 and r_2. Through the identifications $\mathcal{A}[r_1](L) = c_{r_1}(L)^E \simeq c(L)^E$ and $\mathcal{A}[r_2](L) = c_{r_2}(L)^E \simeq c(L)^E$ given by Corollary and ..., the restriction map becomes the diagonal map $c(L)^E \to c(L)^E$ that on each component is the map $c(L) \to c(L)$, $(a_n) \mapsto ((r_2/r_1)^n a_n)$. SInce $(r_2/r_1)^n$ goes to 0 when n goes to ∞, this map is compact by [**Bu**, Prop. 2.4].

Solution to Exercise III.5.1

Hint to Exercise III.5.3   See [**CO2**]

Hint to Exercise III.5.3   See [**CO1**]

Hint to Exercise III.5.3   This follows from the main results of [**L**, §3].

Hint to Exercise III.5.3   See [**?**]

Hint to Exercise III.5.3   The functions $\log_p^+(1+x)$ and $\log_p^-(1+x)$ defined in [**?**] are good examples.

Solution to Exercise III.6.2   This is trivial.

Solution to Exercise III.6.3   Let $a_n = p^m = n+1$ if $n = p^m - 1$ for an integer m, $a_n = 0$ otherwise. Then $\lim a_n = 0$ so $f(z) = \sum a_n z^n \in \mathcal{A}[1]$. However, the antiderivative $g(z) = \sum b_n z^n$ of f has $b_{p^m} = 1$ for all m so $g \notin \mathcal{A}[1]$.

Solution to Exercise III.6.3   Why should it?

Solution to Exercise III.6.3   For a., if d is an ultramteric distance that is invariant by translation, then $|x| := d(x, 0)$ is a norm: the first property is obvious, and we have $|x - y| = d(x - y, 0) = d(x, y) \leq \max(d(x, 0), d(0, y)) = \max(|x|, |y|)$. Conversely, if $|\ |$ is a norm, then $|-x| = |0 - x| \leq |x|$ and by symmetry $|x| = |-x|$ for all x. If then we set $d(x, y) := |x - y|$, then clearly one has $d(x, y) = 0$ if and only if $x = y$; $d(x, y) = d(y, x)$; $d(x + z, y + z) = d(x, y)$; and $d(x, z) = |x - z| = |((x - y) - (z - y)| \leq \max(|x - y|, |z - y|) = \max(d(x, y), d(z, y)) = \max(d(x, y), d(y, z))$.

For b., see [**Sch**, Prop. 8.1]

Solution to Exercise III.6.3   Points a. and c. are clear from the definition. For point b., the canononical surjection $\pi : X \to X/\ker f$ is open: if U is open, we need to show that $\pi(U)$ is open which means that $\pi^{-1}(\pi(U)) = U + Kerf$ is open, which is clear. Hence $f = \bar{f} \circ \pi$ is open if and only if $\bar{f}$ is open. Since $\bar{f}$ is a continuous isomorphism, it is open if and only if it is an homeomorphism. (After [**BGR**, §1.1.9].)

For d., cf [**BGR**, §1.1.9]

For e., note that both X and Y are normable group by the above exercise. If f is strict, $f(X) \simeq X/\ker f$ is complete, hence closed in X. Connversely, if $f(X)$ is closed, it inherits a structure of Frechet, and the isomorphism $\bar{f} : X/\ker f \to f(X)$ is thus an homeomorphism by the open mapping theorem ([**?**, Prop. 8.6]).

For f., recall that f strict means that the continuous isomorphism $\bar{f} : X/\ker f \to f(X)$ is an homeomorphism. The map $\overline{f'} : X'/\ker f' \to f'(X')$ is just $(\bar{f}) \otimes 1$ by flatness of R' over R. Hence we are reduced to the case where f is an isomorphism. In this case, that f is an homeomorphism means that for every semi-norm q in our familiy of semi-norms on Y, one has two semi-norms p_1 and p_2 in our family on X and two positive constant C_1 and C_2 such that $C_1 p_1 \leq f^* q \leq C_2 p_2$. But then clearly $C_1 p'_1 \leq f'^* q' \leq C_2 p'_2$ and it follows that f' is an homeomorphism.

Solution to Exercise III.6.3   This map is a morphism of Frechet space and has a closed image by Theorem III.6.13. Therefore it is strict by point e. of Exercise III.6.16.

Solution to Exercise III.6.5   Consider the following example: $V = \mathbb{Q}_p^{(\mathbb{N})}$ and U_p is the operator sending $(u_0, u_1, u_2, u_3, u_4, \dots)$ to $(u_0, u_0, u_1, u_2, u_3, \dots)$. Let $W = \mathbb{Q}_p$ with U_p acting on it as the identity. Then the map $V \to W$, $(u_0, u_1, u_2, \dots) \mapsto u_0$ is U_p-equivariant and surjective, but $V^{\leq 0} \to W^{\leq 0}$ is not surjective.

Solution to Exercise III.6.6   Similar to Exercise III.6.17.

Solution to Exercise III.7.1   One has $f(pz) = p^{-1-k} f_{|\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}}(z)$, and the results follows from the fact that $\begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \Gamma \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}^{-1} \subset \Gamma_1(N)$.

Solution to Exercise III.7.3   To get from the first to the second line: we need $z^j \in \mathcal{P}_k$.

Solution to Exercise IV.1.1   There is no solution: I don't know the answer.

Solution to Exercise IV.1.2   After all, look in [**BGR**].

Hint to Exercise IV.4   Look up the appendix of [**?**].

Solution to Exercise IV.4   We have $\rho_x = 1 \oplus \omega_p$. Since $U_p(x) = p$, in a neighborhood of x all classical point y are such that ρ_y is irreducible (cf. the appendix of [**?**].) Hence by Ribet's lemma ([**?**]) we can construct a non trivial extension, in the category of $G_{\mathbb{Q},lp}$-representations, of 1 by ω_p, which is crystalline at p by (iii) of Prop. IV.4.4. If there was infinitely many classical points y in the given neighborhood of minimal tame level 1 (instead of l), then that extension would also be unramified at l: but there is non-trivial extension of 1 by ω_p in the categroy of $G_{\mathbb{Q}}$-representations that is unramfied everywhere and crystalline at p, because $\mathbb{Q}$ has only finitely many units (cf. the appendix of [**?**]).

Solution to Exercise VI.2.2   One has $|\log_p \kappa(\gamma)| < 1$ and $|\log_p(\gamma)| = 1/p$, so $|\log_p \kappa(\gamma)/\log_p(\gamma)| < p$. Hence $\left| \binom{\kappa}{i} \right| < p^i |i!|^{-1}$. Since $v_p(i!) < i/(p-1)$, $|i!|^{-1} < p^{i/(p-1)}$ and the result follows.

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