MOTIVIC ACTION ON COHERENT COHOMOLOGY OF HILBERT MODULAR VARIETIES

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Abstract. We propose an action of a certain motivic cohomology group on coherent cohomology of Hilbert modular varieties, extending conjectures of Venkatesh, Prasanna, and Harris. The action is described in two ways: on cohomology modulo p and over $\mathbb{C}$, and we conjecture that they both lift to an action on cohomology with integral coefficients. The latter is supported by theoretical evidence based on Stark's conjecture on special values of Artin L-functions and by numerical evidence in base change cases.

A surprising property of cohomology of locally symmetric spaces is that Hecke operators can act on multiple cohomological degrees with the same eigenvalues. One can observe this by a standard dimension count, but this does little to explain the phenomenon. In a series of papers, Venkatesh and his collaborators propose an arithmetic reason for this: a hidden degree-shifting action of a certain motivic cohomology group.

Initially, Prasanna–Venkatesh [PV16] and Venkatesh [Ven16] developed these conjectures for singular cohomology of locally symmetric spaces. Later, Harris–Venkatesh [HV17] observed similar behavior for coherent cohomology of the Hodge bundle on the modular curve. Connections to (derived) Galois deformation theory and modularity lifting were also explored by Galatius–Venkatesh [GV18]. For a general introduction to this subject, see [Ven14].

In this paper, we propose analogous conjectures for coherent cohomology of the Hodge bundle on Hilbert modular varieties. To give a more precise statement, we first set up some notation.

Let F be a totally real extension of $\mathbb{Q}$ of degree d and let f be a parallel weight one, cuspidal, normalized Hilbert modular eigenform for F with Fourier coefficients in the ring of integers $\mathcal{O}_{E_f}$ of a number field E_f. One can identify f with a section of the Hodge bundle ω on a Hilbert modular variety X:

$$f \in H^0(X, \omega) \otimes \mathcal{O}_{E_f}.$$

More specifically, we consider an integral model X of the toroidal compactification of the open Hilbert modular variety with good reduction away from primes dividing the discriminant of F and the conductor of f. While this choice is not canonical, the resulting cohomology groups are independent of the choice of X.

The action of the Hecke algebra extends to higher cohomology groups $H^i(X, \omega) \otimes \mathcal{O}_{E_f}$ and we may consider the subspace on which the Hecke algebra acts with the same eigenvalues as on f, which we denote by $H^i(X, \omega)_f$. It follows from [Su18] that

$$\text{rank}\, H^i(X, \omega)_f = \binom{d}{i} \tag{0.1}$$

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(c.f. Corollary 3.12). We conjecture that there is a degree-shifting action of a (dual) motivic cohomology group $U_f^\vee$ attached to f on the cohomology space $H^*(X,\omega)_f$ which makes $H^*(X,\omega)_f$ a module of rank one over the exterior algebra $\bigwedge U_f^\vee$, generated by $f \in H^0(X,\omega)_f$. The group U_f is the Stark unit group [Sta75] associated to the trace zero adjoint representation of f.

We can describe this action in two ways: modulo p and over $\mathbb{C}$. Let $\mathfrak{p}$ be a prime of $\mathcal{O}_{E_f}$ and $\iota\colon E_f \hookrightarrow \mathbb{C}$ be an embedding. We show that there is:

(1) a map $\displaystyle\bigoplus_{j=1}^d U_{f,j}^{\mathfrak{p}^n} \to U_f^\vee \otimes \mathcal{O}_{E_f}/\mathfrak{p}^n$ for some free $\mathcal{O}_{E_f}/\mathfrak{p}^n$-modules $U_{f,j}^{\mathfrak{p}^n}$ of rank one (Proposition 2.5), and an action of $U_{f,j}^{\mathfrak{p}^n}$ on $H^*(X,\omega)_f \otimes \mathcal{O}_{E_f}/\mathfrak{p}^n$ via derived Hecke operators (Section 2.3),

(2) an isomorphism $\displaystyle\bigoplus_{j=1}^d U_{f,j}^{\mathbb{C}} \xrightarrow{\cong} U_f^\vee \otimes \mathbb{C}$ for some one-dimensional $\mathbb{C}$-vector spaces $U_{f,j}^{\mathbb{C}}$ (Proposition 3.2), and an action of $U_{f,j}^{\mathbb{C}}$ on $H^*(X,\omega)_f \otimes \mathbb{C}$ via partial complex conjugation $z_j \mapsto \overline{z_j}$ (Section 3.2).

Conjecture A (Conjectures 2.7, 3.25)**.** *There is a graded action of the exterior algebra $\bigwedge U_f^\vee$ on the cohomology space $H^*(X,\omega)_f$ such that:*

(1) *the action of $\bigwedge U_f^\vee \otimes \mathcal{O}_{E_f}/\mathfrak{p}^n$ is obtained from (1) above, up to $\mathcal{O}_{E_f}^\times$,*
(2) *the action of $\bigwedge U_f^\vee \otimes \mathbb{C}$ is obtained from (2) above, up to $E_f^\times$.*

Moreover, the cohomology space $H^*(X,\omega)_f$ is generated by $f \in H^0(X,\omega)_f$ over $\bigwedge U_f^\vee$.

The conjectures will be stated precisely in the main body of the paper.

Part (1) is a generalization of the main conjecture of Harris and Venkatesh [HV17, Conjecture 3.1]. It should be seen as a first step towards establishing a p-adic conjecture, similar to Venkatesh's conjecture [Ven16]. In fact, our original motivation to study the Stark unit group U_f for Hilbert modular forms was to generalize the conjecture of Darmon–Lauder–Rotger [DLR15] to elliptic curves over totally real fields. A p-adic version of Conjecture A may explain the appearance of p-adic logarithms of Stark units therein.

Part (2) is similar to the main conjecture of Prasanna and Venkatesh [PV16, Conjecture 1.2.1] but in the coherent (as opposed to singular) cohomology setting. As far as we know, it is new even when $F = \mathbb{Q}$. In the Hilbert case, it also provides a definition of period invariants attached to parallel weight one forms at the infinite places. Such period invariants had previously been defined and studied by Shimura [Shi78, Shi88], Harris [Har93, Har90b, Har94], and Ichino–Prasanna [IP16] for Hilbert modular forms at places where the weight of f is at least two. The parallel weight one case is different because the form does not transfer to a quaternion algebra ramified at any infinite place, so the periods at infinite places do not admit a simple interpretation as periods of a holomorphic differential form on a Shimura curve, or even as ratios of periods of holomorphic forms on quaternionic Shimura varieties. Instead, we give a definition in terms of logarithms of units; this is natural because the adjoint L-value is non-critical at $s = 1$ in this case, so one should expect the periods to be of "Beilinson-type".

These conjectures lead to many interesting questions about potential generalizations to partial weight one forms (such as the question about periods mentioned in the previous paragraph) and to other reductive groups which we hope to return to in the future.

We now state part (2) of the conjecture more precisely and summarize our evidence for it. There is an explicit basis ω_f^J of $H^i(X,\omega)_f \otimes \mathbb{C}$ associated to partial complex conjugation at subsets J of size i of the infinite places of F; for example, a basis of $H^1(X,\omega)_f \otimes \mathbb{C}$ is given by

$$\omega_f^{\sigma_j} = f(\epsilon_1 z_1, \ldots, \epsilon_j \overline{z_j}, \ldots, \epsilon_d z_d) y_j \frac{dz_j \wedge d\overline{z_j}}{y_j^2} \quad \text{for } j = 1, \ldots, d, \tag{0.2}$$

where $\epsilon \in \mathcal{O}_F^\times$ satisfies $\epsilon_j < 0$ and $\epsilon_i > 0$ for $i \neq j$.[1] For Hilbert modular forms all of whose weights are at least two, the f-isotypic components of the cohomology groups are one-dimensional, so some multiples of these classes are necessarily E_f-rational. Harris [Har90b] uses this fact to define his period invariants mentioned above. However, the cohomology groups $H^i(X,\omega)_f$ for parallel weight one forms are higher-dimensional (equation (0.1)), so it is not known whether any multiples of the classes (0.2) obtained by partial complex conjugation are E_f-rational.

Conjecture B (Conjecture 3.21)**.**

(a) *For each subset J of the infinite places of F, there exists $\nu^J(f) \in \mathbb{C}^\times$ such that the cohomology class $\dfrac{\omega_f^J}{\nu^J(f)}$ is E_f-rational.*
(b) *More precisely, we may take $\nu^J(f) = \prod_{j\in J} \log|u_{f,\sigma_j}|$, a product of logarithm of Stark units $u_{f,\sigma_j} \in U_{f,j}^{\mathbb{C}} \subseteq U_f$.*

This conjecture implies that the action in Conjecture A(2) descends from $\otimes\mathbb{C}$ to $\otimes E_f$. We prove some theoretical results in this direction by applying results of Stark [Sta75] and Tate [Tat84] on special values of Artin L-functions. For simplicity, we assume throughout the rest of the introduction that the automorphic representation associated to f is not supercuspidal at $p = 2$ (although we expect this assumption to be innocuous; see Remark 4.15).

Since $H^d(X,\omega)_f$ is one-dimensional, part (a) of Conjecture B is immediate when $\#J = d$. Part (b) amounts to the following theorem.

Theorem C (Theorem 4.6)**.** *If the coefficient field E_f is $\mathbb{Q}$, then*

$$\frac{\omega_f^{\sigma_1,\ldots,\sigma_d}}{\prod_{j=1}^d \log|u_{f,\sigma_j}|} \in H^d(X,\omega)_f \otimes E_f.$$

Hence the action of top degree elements in the exterior algebra $\bigwedge U_f^\vee$ in Conjecture A(2) is E_f-rational. For general E_f, the analogous statement follows from Stark's Conjecture 4.2.

[1] Here and elsewhere, we write $\epsilon_i = \sigma_i(\epsilon) \in \mathbb{R}$ for an element $\epsilon \in F$.

In the classical modular forms case ($F = \mathbb{Q}$), this means that Stark's Conjecture implies both Conjectures B and A(2):

Corollary D (Corollary 4.7)**.** *Conjectures A(2) and B hold when $F = \mathbb{Q}$ (assuming Stark's Conjecture 4.2 when $E_f \neq \mathbb{Q}$). The action of $u_f^\vee \in U_f^\vee$ is given by:*

$$\begin{aligned} H^0(X, \omega)_f &\xrightarrow{u_f^\vee} H^1(X, \omega)_f, \\ f &\mapsto \frac{\omega_f^\infty}{\log |u_f|}. \end{aligned}$$

When $d = 2$, it remains to verify the rationality of the classes

$$\frac{\omega_f^{\sigma_1}}{\log |u_{f,\sigma_1}|}, \frac{\omega_f^{\sigma_2}}{\log |u_{f,\sigma_2}|} \in H^1(X, \omega)_f \otimes \mathbb{C}. \tag{0.3}$$

Theorem E (Corollary 4.20)**.** *Suppose $d = 2$ so F is real quadratic.*

(a) *There exist constants $\nu^{\sigma_1}(f), \nu^{\sigma_2}(f) \in \mathbb{C}^\times$ such that the cohomology classes $\frac{\omega_f^{\sigma_1}}{\nu^{\sigma_1}(f)}, \frac{\omega_f^{\sigma_2}}{\nu^{\sigma_2}(f)}$ are E'-rational where E' is at most a quadratic extension of E_f. In particular, Conjecture B(a) is true after extending scalars to E'.*
(b) *Furthermore, assuming Stark Conjecture 4.2, there is a constant $\lambda \in \mathbb{C}$ such that we may take:*

$$\begin{aligned} \nu^{\sigma_1}(f) &= \lambda \cdot \log |u_{f,\sigma_1}|, \\ \nu^{\sigma_2}(f) &= \lambda^{-1} \cdot \log |u_{f,\sigma_2}|. \end{aligned}$$

Next, we provide numerical evidence that suggests $\lambda \in E_f^\times$ and hence Conjectures A(2) and B are true for $d = 2$. The cohomology classes (0.2) are naturally Dolbeault classes, which we identify with sheaf cohomology classes via the Dolbeault and the GAGA theorems. To check that they are E_f-rational is to show that the resulting sheaf cohomology classes come from base change of cohomology classes in $H^1(X, \omega)_f$. The translation between Dolbeault and sheaf cohomology is not explicit enough to yield a satisfactory criterion for rationality. Worse yet, there seems to be no natural automorphic criterion to verify rationality. Indeed, the integral representations of Rankin–Selberg or triple product L-functions for Hilbert modular forms only involve cohomology classes ω_f^J where J is the set of places where f is dominant (see [Har90b] for details). Since parallel weight one forms are never dominant at any place, the cohomology classes we are interested in do not feature in these integral representations.

Instead, we consider an embedded modular curve $\iota \colon Y \hookrightarrow X$ and check computationally in some cases that the restriction of $\frac{\omega_f^{\sigma_j}}{\log |u_{f,\sigma_j}|}$ to Y is rational, i.e.

$$\iota^* \left(\frac{\omega_f^{\sigma_j}}{\log |u_{f,\sigma_j}|} \right) \in H^1(Y, \iota^* \omega) \otimes E_f. \tag{0.4}$$

This requires an explicit computation of the trace of the cohomology class, i.e. an integral on the modular curve $Y(\mathbb{C})$ (see Conjecture 5.7). We use formulas developed by Nelson [Nel15] to derive an explicit expression for this integral (Theorem 5.14) and compute it numerically up to 20 digits of accuracy (Table 5.1). The drawback of this approach is that this restriction is non-zero only if the Hilbert modular form f is the base change of a modular form over $\mathbb{Q}$ (see, for example, Proposition 5.9).

The paper is organized as follows:

- Section 1 introduces the group of Stark units U_f, discusses their rank, relation to a motivic cohomology group, and examples.
- Section 2 discusses a *derived Hecke action* and the direct generalization of the conjecture of Harris and Venkatesh [HV17] in the totally real case (Conjecture A (1)).
- Section 3 introduces partial complex conjugation operators on cohomology and the archimedean conjectures (Conjectures A(2) and B).
- Section 4 discusses how Stark's results [Sta75] give evidence for the archimedean conjecture, proving Theorems C, E.
- Section 5 discusses base change cases and provides numerical evidence for the archimedean conjecture.

Sections 2 and 3 are independent of one another and hence may be read in any order. The reader who wants to understand the full statements of the two conjectures as fast as possible may just skim Section 1.2 and proceed directly to these two sections.

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1. Stark units

The goal of this section is to introduce the unit group U_f mentioned in the introduction, compute its rank, and discuss its relation to motivic cohomology. We start with a slightly more general setup and specialize it to our case.

1.1. **General Stark units.** We follow [Sta75] to introduce the group of *Stark units* associated to an Artin representation.

Consider any Artin representation, i.e. a representation of the absolute Galois group $G_{\mathbb{Q}}$ which factors through a finite Galois extension L of $\mathbb{Q}$:

$$\begin{tikzcd} G_{\mathbb{Q}} \arrow[rr, "\varrho"] \arrow[dr, "\mathrm{Res}_L"'] & & \mathrm{GL}(M) \\ & G_{L/\mathbb{Q}} \arrow[ur, "\varrho"'] & \end{tikzcd}$$

where M is a free $\mathcal{O}_E$-module of rank n and E is a finite extension of $\mathbb{Q}$. We often write G for the Galois group $G_{L/\mathbb{Q}}$ and U_L for the group of units of $\mathcal{O}_L$.

Definition 1.1. The group of *Stark units* associated to $\varrho\colon G_{L/\mathbb{Q}} \to \mathrm{GL}(M)$ is:

$$U_L[\varrho] = \mathrm{Hom}_{\mathcal{O}_E[G]}(M, U_L \otimes_{\mathbb{Z}} \mathcal{O}_E).$$

We will soon check that $U_L[\varrho]$ depends only on ϱ and not on the choice of L. To describe the group $U_L[\varrho]$ in more detail, we first need to understand the structure of U_L as a $G_{L/\mathbb{Q}}$-module.

Fix an embedding $\tau\colon L \hookrightarrow \mathbb{C}$ which induces a complex conjugation c_0 of L. Note that $\operatorname{rank} U_L + 1 = \#(G/\langle c_0\rangle)$ by Dirichlet's units theorem.

Lemma 1.2 ([Sta75, Lemma 2, Minkowski's unit theorem])**.** *There is a unit ϵ of L, fixed by c_0, such that there is only one relation among the* $\operatorname{rank} U_L + 1$ *units $\epsilon^{\sigma^{-1}}$ for $\sigma \in G/\langle c_0\rangle$, and this relation is*

$$\prod_{\sigma\in G/\langle c_0\rangle} \epsilon^{\sigma^{-1}} = \pm 1.$$

Definition 1.3. A unit whose existence is guaranteed by Lemma 1.2 is called a *Minkowski unit* of L with respect to $\tau\colon L \hookrightarrow \mathbb{C}$.

Corollary 1.4. *The* log *map induces a G-equivariant isomorphism:*

$$U_L/U_L^{\mathrm{tors}} \xrightarrow{\cong} \frac{\mathbb{Z}[\log(|\tau(\epsilon^{\sigma^{-1}})|) \mid \sigma \in G/\langle c_0\rangle]}{\left\langle \sum_{\sigma\in G/\langle c_0\rangle} \log(|\tau(\epsilon^{\sigma^{-1}})|) \right\rangle},$$

(the numerator on the right hand side is the free abelian group in those variables) and there is also a G-equivariant isomorphism:

$$\begin{aligned} \mathrm{Ind}_{\langle c_0\rangle}^{G} \mathbb{Z} &\xrightarrow{\cong} \mathbb{Z}[\log(|\tau(\epsilon^{\sigma^{-1}})|) \mid \sigma \in G/\langle c_0\rangle], \\ (f\colon G/\langle c_0\rangle \to \mathbb{Z}) &\mapsto \sum_{\sigma\in G/\langle c_0\rangle} f(\sigma\langle c_0\rangle)[\log(|\tau(\epsilon^{\sigma^{-1}})|)]. \end{aligned}$$

In particular,

$$U_L/U_L^{\mathrm{tors}} \cong \mathrm{Ind}_{\langle c_0\rangle}^G \mathbb{Z} \text{ as a representation of } G = G_{L/\mathbb{Q}}.$$

We now compute the rank of $U_L[\varrho]$ and find a natural basis for $U_L[\varrho] \otimes_{\mathcal{O}_E} E$, given a basis of $M \otimes_{\mathcal{O}_E} E$. Let

$$a = \dim_E M_E^{\langle c_0\rangle}.$$

Note that $a = (\mathrm{Tr}\varrho(1)+\mathrm{Tr}\varrho(c_0))/2$, so since any two complex conjugations of L are conjugate, this number is independent of the choice of c_0. We write $b = n - a$ where $n = \dim_E M_E$.

Proposition 1.5. *Suppose ϱ does not contain a copy of the trivial representation. Then*

$$U_L[\varrho] \otimes E \cong M_E^{\langle c_0\rangle}$$

and hence $\mathrm{rank}\, U_L[\varrho] = a$.

Moreover, if $m_1, \dots, m_a$ is a basis of $M_E^{\langle c_0\rangle}$ and we complete it to a basis $m_1, \dots, m_n$ of M_E such that $\varrho(c_0) = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$ in this basis, then the corresponding basis of $U_L[\varrho] \otimes_{\mathcal{O}_E} E$ consists of the homomorphisms $\varphi_1, \dots, \varphi_a$ defined by:

$$\varphi_i(m_j) = \prod_{\sigma\in G} (\epsilon^{\sigma^{-1}})^{a_{ij}(\sigma)} \in U_L \otimes E, \tag{1.1}$$

where

$$\varrho(\sigma) = (a_{ij}(\sigma))_{i,j} \text{ in the basis } m_1, \dots, m_n.$$

Proof. We have that

$$\begin{aligned} U_L[\varrho] \otimes_{\mathcal{O}_E} E &= \mathrm{Hom}_{E[G]}(M_E, U_L \otimes_{\mathbb{Z}} E) \\ &= \mathrm{Hom}_{E[G]}\left(M_E, \mathrm{Ind}_{\langle c_0\rangle}^G E - E\right) && \text{Corollary 1.4} \\ &= \mathrm{Hom}_{E[G]}\left(M_E, \mathrm{Ind}_{\langle c_0\rangle}^G E\right) && \varrho \text{ does not contain the trivial rep.} \\ &= \mathrm{Hom}_{E[\langle c_0\rangle]}(M_E, E) && \text{Frobenius reciprocity} \\ &= M_E^{\langle c_0\rangle}. \end{aligned}$$

Now, pick a basis $m_1, \dots, m_n$ of M such that $\varrho(c_0) = \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$ in it. By definition of the matrix $(a_{ij}(\sigma))_{i,j}$,

$$\varrho(\sigma)m_j = \sum_{i=1}^n a_{ij}(\sigma)m_i.$$

Hence a map $\varphi \in \mathrm{Hom}_{\mathcal{O}_E}(M, U_L \otimes_{\mathbb{Z}} \mathcal{O}_E)$ is G-equivariant if and only if:

$$(\varphi(m_j))^\tau = \varphi(\varrho(\tau)m_j) = \prod_{k=1}^n \varphi(m_k)^{a_{kj}(\tau)} \tag{1.2}$$

(where the group of units is written multiplicatively).

We check that each φ_i defined above satisfies this equation. Let

$$u_{ij} = \prod_{\sigma\in G} (\epsilon^{\sigma^{-1}})^{a_{ij}(\sigma)} \in U_L \otimes \mathcal{O}_E.$$

Then:

$$\begin{aligned}
u_{ij}^{\tau} &= \left(\prod_{\sigma\in G}(\epsilon^{\sigma^{-1}})^{a_{ij}(\sigma)}\right)^{\tau} \\
&= \prod_{\sigma\in G}(\epsilon^{\sigma^{-1}\tau})^{a_{ij}(\sigma)} \\
&= \prod_{\sigma'\in G}(\epsilon^{(\sigma')^{-1}})^{a_{ij}(\sigma'\tau)} && \text{for } \sigma'=\tau^{-1}\sigma \\
&= \prod_{\sigma'\in G}(\epsilon^{(\sigma')^{-1}})^{\sum\limits_{k=1}^{n} a_{ik}(\sigma')a_{kj}(\tau)} \\
&= \prod_{k=1}^{n}\underbrace{\left(\prod_{\sigma\in G}(\epsilon^{\sigma^{-1}})^{a_{ik}(\sigma)}\right)}_{u_{ik}}^{a_{kj}(\tau)} && \text{for } \sigma'=\sigma \\
&= \prod_{k=1}^{n} u_{ik}^{a_{kj}(\tau)}.
\end{aligned}$$

This shows that the morphisms φ_i given by $\varphi_i(m_j)=u_{ij}$ satisfy the G-equivariance property (1.2). Indeed:

$$\varphi_i(m_j)^{\tau} = u_{ij}^{\tau} = \prod_{k=1}^{n} u_{ik}^{a_{kj}(\tau)} = \prod_{k=1}^{n}\varphi_i(m_k)^{a_{kj}(\tau)}.$$

Hence $\varphi_1,\dots,\varphi_a \in U_L[\varrho]$.

Tracing through the isomorphism

$$U_L[\varrho]\otimes_{\mathcal{O}_E} E \cong M_E^{\langle c_0\rangle}$$

established above, we see that

$$\varphi_i \mapsto m_i \quad \text{for } i=1,\dots,a.$$

Since this is an isomorphism and $m_1,\dots,m_a$ is a basis of $M_E^{\langle c_0\rangle}$, $\varphi_1,\dots,\varphi_a$ is a basis of $U_L[\varrho]\otimes E$. □

Corollary 1.6. *Suppose $\varrho\colon G_{\mathbb{Q}}\to \mathrm{GL}(M)$ is an Artin representation. Then $U_L[\varrho]\otimes E$ is independent of the choice of splitting field $L/\mathbb{Q}$.*

Proof. For an extension L'/L, the natural inclusion $U_L\hookrightarrow U_{L'}$ induces an inclusion $U_L[\varrho]\to U_{L'}[\varrho']$. By Proposition 1.5, $\dim U_L[\varrho]\otimes E = \dim U_{L'}[\varrho']\otimes E$, which completes the proof. □

We will later be interested in the reduction of $U_L[\varrho]$ modulo $\mathfrak{p}^n$ for a prime $\mathfrak{p}$ of E. For now, we just remark that the following follows from Proposition 1.5.

Corollary 1.7. *If a prime p is coprime to $\#U_L^{\mathrm{tors}}$, then $U_L[\varrho]\otimes_{\mathbb{Z}}\mathbb{Z}_p$ is a free $\mathcal{O}_E\otimes_{\mathbb{Z}}\mathbb{Z}_p$-module of rank d. Hence, for a prime $\mathfrak{p}$ of E above p, $U_L[\varrho]\otimes\mathcal{O}_E/\mathfrak{p}^n$ is a free $(\mathcal{O}_E/\mathfrak{p}^n)$-module of rank d.*

Proof. This follows immediately from Proposition 1.5 and the structure theorem of modules over PIDs. □

1.2. **Stark units for weight one Hilbert modular forms.** We now discuss Stark units for Artin representations associated to weight one Hilbert modular forms. Let F be a totally real field. By [RT83], normalized weight one Hilbert modular eigenforms f with Fourier coefficients in $\mathcal{O}_{E_f}$ correspond to 2-dimensional odd irreducible Artin representations

$$\begin{tikzcd} G_F \arrow[rr, "\varrho_F"] \arrow[dr, "\mathrm{Res}_L"'] & & \mathrm{GL}(M) \\ & G_{L/F} \arrow[ur, "\varrho_F"'] & \end{tikzcd}$$

where M is a $\mathcal{O}_{E_f}$-module of rank 2. By enlarging L if necessary, we may assume that L is Galois over $\mathbb{Q}$. We write $G = G_{L/\mathbb{Q}}$ and $G' = G_{L/F}$ for simplicity.

As in the previous section, fix an embedding $\tau\colon L \hookrightarrow \mathbb{C}$ which induces a complex conjugation c_0 of L. Note that c_0 necessarily in G' because F is totally real. Since ϱ_F is an odd representation,

$$\varrho_F(c_0) \text{ is conjugate to } \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Consider the *adjoint representation of* ϱ, i.e.

$$\begin{aligned} \mathrm{Ad}\,\varrho_F\colon G_{L/F} &\to \mathrm{GL}(\mathrm{End}(M)) \\ \sigma &\mapsto (T \mapsto \varrho(\sigma) T \varrho(\sigma)^{-1}). \end{aligned}$$

We note that if T has trace 0, then so does $\varrho(\sigma)T\varrho(\sigma)^{-1}$. The representation is hence reducible, and we define

$$\mathrm{Ad}^0\,\varrho_F\colon G_{L/F} \to \mathrm{GL}(\mathrm{End}^0(M)),$$

where $\mathrm{End}^0(M) = \{T\colon M \to M \mid \operatorname{Tr} T = 0\}$. This is a 3-dimensional representation.

Choosing a basis of M such that $\varrho(c_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, we see that

$$(\mathrm{Ad}\,\varrho)(c) \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} a & -b \\ -c & -a \end{pmatrix}.$$

Hence $\dim\left((\mathrm{Ad}^0\,\varrho_F)^{\langle c_0 \rangle}\right) = 1$.

Definition 1.8. Let U_L be the units of L and $\mathcal{O} = \mathcal{O}_{E_f}$ be the integers of the coefficient field E_f of a Hilbert modular form f of parallel weight one. The group of *Stark units* associated to f is

$$U_f = \mathrm{Hom}_{\mathcal{O}[G_{L/F}]}(\mathrm{Ad}^0\,\varrho_F, U_L \otimes_{\mathbb{Z}} \mathcal{O}).$$

We sometimes write $\mathrm{Ad}^*\,\varrho = \mathrm{Hom}_{\mathcal{O}[G_{L/F}]}(\mathrm{Ad}^0\,\varrho_F, \mathcal{O})$, so that $U_f = \mathrm{Ad}^*\,\varrho \otimes_{\mathbb{Z}} U_L$.

Corollary 1.9. *Suppose that ϱ_F is irreducible. Then:*

$$U_f = U_L[\mathrm{Ind}_{G'}^{G}\,\mathrm{Ad}^0\,\varrho_F],$$

the group of Stark units associated to the $3d$*-dimensional Artin representation* $\operatorname{Ind}_{G'}^{G} \operatorname{Ad}^0 \varrho_F$*. Therefore,*

$$U_f \cong (\operatorname{Ind}_{G'}^{G} \operatorname{Ad}^0 \varrho_F)^{\langle c_0 \rangle} \cong \bigoplus_{\sigma \in G/G'} (\operatorname{Ad}^0 \varrho_F)^{\langle \sigma c_0 \sigma^{-1} \rangle}$$

and hence

$$\operatorname{rank} U_f = d.$$

Proof. We have that:

$$\begin{aligned} U_f &= \operatorname{Hom}_{\mathcal{O}[G']}(\operatorname{Ad}^0 \varrho_F, U_L \otimes_{\mathbb{Z}} \mathcal{O}) \\ &= \operatorname{Hom}_{\mathcal{O}[G]}(\operatorname{Ind}_{G'}^{G} \operatorname{Ad}^0 \varrho_F, U_L \otimes_{\mathbb{Z}} \mathcal{O}) && \text{Frobenius reciprocity} \\ &= \operatorname{Hom}_{\mathcal{O}[G]}(\operatorname{Ad}^0 \operatorname{Ind}_{G'}^{G} \varrho_F, U_L \otimes_{\mathbb{Z}} \mathcal{O}) \\ &= U_L[\operatorname{Ad}^0 \varrho]. \end{aligned}$$

Since ϱ_F is irreducible, $\operatorname{Ad}^0 \varrho_F$ does not contain a copy of the trivial representation. We may hence apply Proposition 1.5 to the Artin representation $\operatorname{Ad}^0 \varrho$ to get the result. Finally:

$$\begin{aligned} (\operatorname{Ad}^0 \varrho)^{c_0} &= (\operatorname{Ad}^0 \operatorname{Ind}_{G'}^{G} \varrho_F)^{c_0} \\ &= \bigoplus_{\sigma \in G/G'} \left(\sigma \operatorname{Ad}^0 \varrho_F\right)^{c_0} \\ &= \bigoplus_{\sigma \in G/G'} \left(\operatorname{Ad}^0 \varrho_F\right)^{\sigma c_0 \sigma^{-1}}, \end{aligned}$$

completing the proof. □

Remark 1.10. The decomposition in Corollary 1.9 generalizes to any *plectic* Artin representation [NS16], i.e. an Artin representation of G_F for a totally real field F. We have not used anything specific to Hilbert modular forms.

Remark 1.11. There is also a description of U_f similar to [DLR15]. For a chosen prime $\mathfrak{p}$ of F, for each φ_σ, we may consider the component of $\varphi_\sigma(\operatorname{Ad}^0 \varrho_F) \subseteq U_L$ on which a chosen Frobenius $\operatorname{Frob}_{\mathfrak{p}} \in G_{L/F}$ acts by α/β where α and β are the ordered eigenvalues $\varrho_F(\operatorname{Frob}_{\mathfrak{p}})$. This description may be useful when considering a p-adic analog of the conjecture, but we omit this here entirely.

1.3. **Examples.** The Stark unit group can be determined explicitly in many cases. We provide a few illustrative examples.

Example 1.12 (Heegner units)**.** The first example of Stark units comes from the theory of elliptic units.

Let $F = \mathbb{Q}$ and $K/\mathbb{Q}$ be an imaginary quadratic extension. For any Dirichlet character $\chi\colon G_{H/K} \to \mathbb{C}^\times$ of K, where H/K is an abelian extension, there is an associated weight one form $f = \theta_\chi$, the *theta function* of χ, such that

$$L(s, \chi) = L(s, f).$$

The Artin representation ϱ associated to f is the 2-dimensional representation:

$$\varrho_f = \operatorname{Ind}_{G_{H/K}}^{G_{H/\mathbb{Q}}} \chi = \{\phi\colon G_{H/\mathbb{Q}} \to \mathbb{C} \mid \phi(\sigma\tau) = \chi(\sigma)\phi(\tau) \quad \text{for } \sigma \in G_{H/K}\}.$$

For the non-trivial element $c \in G_{K/\mathbb{Q}}$, we can define a character $\chi^c(\sigma) = \chi(c\sigma c)$. Writing 1 for the trivial representation and χ_0 for $\chi \cdot (\chi^c)^{-1}$, we see that

$$\mathrm{Ad}^0 \varrho_f \cong 1 \oplus \mathrm{Ind}_{G_{H/K}}^{G_{H/\mathbb{Q}}} \chi_0.$$

Since the unit group does not contain a copy of the trivial representation, this shows that

$$U_f \cong U_H[\chi_0],$$

the χ_0-isotypic component of the units of H. For a unit $u \in \mathcal{O}_H^\times$, we have a natural generator:

$$u_{\chi_0} = \sum_{\sigma \in G_{H/K}} \chi_0(\sigma)^{-1} u^\sigma \in U_H[\chi_0]$$

(writing the group law on the units $\mathcal{O}_H^\times$ additively), since χ_0 is trivial on $G_{K/\mathbb{Q}}$. *Elliptic units*, constructed using singular values of modular functions, provide examples of units $u \in \mathcal{O}_H^\times$.

The *logarithms* of these units appear as special values of the L-function of χ_0, via Kronecker's second limit formula. This also has a p-adic analog: the *p-adic logarithm* of u_{χ_0} accounts for the special value of the Katz p-adic L-function evaluated at the finite order character χ_0^{-1}, which is outside of the range of interpolation [Kat76, 10.4, 10.5]. More generally, Darmon, Lauder, and Rotger conjecture [DLR15, Conjecture ES] that p-adic logarithms of other Stark units associated to weight one modular forms appear in a formula for values of triple product p-adic L-functions outside the range of interpolation.

The following example is suitable for computations in the case $F = \mathbb{Q}$. In fact, it is the example where Harris–Venkatesh [HV17] perform their computations. It is also a simple example where our archimedean conjecture (Conjecture 3.25) can be proved (Theorem 4.6); this realization goes back to [Sta75, pp. 91].

Example 1.13 (Units in cubic fields, $F = \mathbb{Q}$)**.** This example is discussed in [HV17, Sec. 5.6], but we recall it here in detail to provide context for the generalizations to $[F : \mathbb{Q}] = 2$ we make below.

Let K be a cubic field of signature $[1, 1]$ and write L for the Galois closure of K. Then $G_{L/\mathbb{Q}} \cong S_3$ and we may assume that K is the fixed field $L^{(12)}$ of the action of the cycle $(12) \in S_3$ on L.

To give a 2-dimensional representation of $G_{L/\mathbb{Q}}$, we need to give a 2-dimensional representation of S_3. There is a unique irreducible 2-dimensional representation: the *regular representation* $\varrho\colon G_\mathbb{Q} \twoheadrightarrow S_3 \to \mathrm{GL}(M) \cong \mathrm{GL}_2(\mathbb{Z})$, obtained by considering the action of S_3 on

$$M = \left\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \;\middle|\; \sum x_i = 0\right\}$$

by permuting the coordinates.

In the basis $e_1 = (1, 0, -1)$, $e_2 = (0, 1, -1)$ of M, we have that:

$$\sigma = (12) \mapsto S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\tau = (123) \mapsto T = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

Note that ϱ is an *odd* Galois representation since $\det S = -1$. Therefore, there is a weight one modular form f corresponding to ϱ.

Recall that

$$U_f = \mathrm{Hom}_{G_{L/\mathbb{Q}}}(\mathrm{Ad}^0\,\varrho, U_L).$$

Lemma 5.7 in [HV17] shows that

$$\begin{aligned} U_f \otimes \mathbb{Z}\left[\frac{1}{6}\right] &\xrightarrow{\cong} U_K^{(1)} \otimes \mathbb{Z}\left[\frac{1}{6}\right] \\ (\varphi\colon\ \mathrm{Ad}^0\,\varrho \to U_L) &\mapsto \varphi(S), \end{aligned} \tag{1.3}$$

where $U_K^{(1)}$ are the norm 1 units of K.

We recall the proof here. By definition

$$\mathrm{Ad}^0\,\varrho \cong \mathrm{End}^0(M),$$

with the action of S_3 on the right hand side given by conjugation. Note that each element of S_3 gives an element of $\mathrm{End}(M)$ and we may use the S_3-invariant projection

$$\begin{aligned} \mathrm{End}(M) &\to \mathrm{End}^0(M) \\ A &\mapsto A - (1/2)\mathrm{Tr}(A) \end{aligned}$$

to get a spanning set for $\mathrm{Hom}^0(M, M)$ this way. Since the lengths of cycles are conjugation-invariant, we see that

$$\mathrm{Hom}^0(M, M) \cong \mathrm{span}(\text{images of } (123), (132)) \oplus \mathrm{span}(\text{images of } (12), (13), (23)).$$

One checks that $\mathrm{span}(\text{images of } (123), (132)) = \mathbb{Z}[e]$, where $e \in \mathrm{Hom}^0(M, M)$ is the common image of (123) and (132). We write $W = \mathrm{span}(\text{images of } (12), (13), (23))$. Hence

$$\mathrm{Ad}^0\,\varrho \cong \mathbb{Z}[e] \oplus W.$$

Now, for any S_3-representation V:

- $\mathrm{Hom}_{S_3}(\mathbb{Z}[e], V) = V^{\mathrm{sgn}}$, the sgn-isotypic part of V,
- $\mathrm{Hom}_{S_3}(W, V) \cong \{v \in V^{(12)} \mid v + (123)v + (132)v = 0\}$ via $\varphi \mapsto \varphi(S)$.

This shows that:

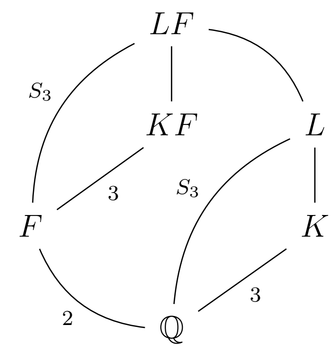
$$U_f \cong U_L^{\mathrm{sgn}} \oplus U_K^{(1)}$$

Since $\mathbb{Q}(\sqrt{\mathrm{disc}(L)}) = L^{\langle (123) \rangle}$, $U_L^{\mathrm{sgn}} = U_{\mathbb{Q}\left(\sqrt{\mathrm{disc}(L)}\right)}$ is a finite group of order at most 6. Hence

$$U_f \otimes \mathbb{Z}\left[\frac{1}{6}\right] \cong U_K^{(1)} \otimes \mathbb{Z}\left[\frac{1}{6}\right].$$

The following is the simplest example of explicit Stark units over real quadratic fields. It is the base change of Example 1.13 to a real quadratic field and the main example in which we will do the numerical computations later on.

Example 1.14 (Units in cubic extensions of F for $[F : \mathbb{Q}] = 2$)**.** Consider K as in Example 1.13 and consider a quadratic extension F of $\mathbb{Q}$. Then KF is a cubic extension of F of signature $[2, 2]$:



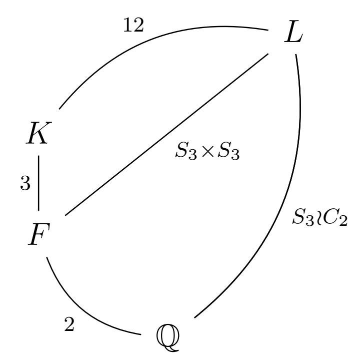
As above, we consider the Galois representation $\varrho\colon G_{LF/F} \cong S_3 \to \mathrm{GL}(M)$. If f is the weight 1 Hilbert modular form associated to ϱ, then one can check that (c.f. Proposition 5.2)

$$\begin{aligned} U_f \otimes \mathbb{Z}[1/6] &\cong \mathrm{Hom}_{G_{LF/F}}(\mathrm{Ad}^0\,\varrho, U_{LF}) \otimes \mathbb{Z}[1/6] \\ &\cong U_{LF}^{\mathrm{sgn}} \otimes \mathbb{Z}[1/6] \oplus \{u \in U_K \mid N_{\mathbb{Q}}^K u = 1\}^{\oplus 2} \otimes \mathbb{Z}[1/6] \\ &\cong \{u \in U_K \mid N_{\mathbb{Q}}^K u = 1\}^{\oplus 2} \otimes \mathbb{Z}[1/6]. \end{aligned}$$

The Hilbert modular form f is the base change of the modular form f_0 associated to K in the previous example.

Finally, we present the "simplest" non base change example where explicit Stark units are available over real quadratic fields. It is a direct analog of Example 1.13, but the Galois theory is more complicated.

Example 1.15 (Units in cubic extensions of F for $[F:\mathbb{Q}] = 2$, non base change)**.** We generalize Example 1.13 to the case $[F:\mathbb{Q}] = 2$ and a cubic extension K of F of signature $[2,2]$:



We may assume that $K = L^{S_3 \times \langle (12) \rangle}$. Consider the representation

$$\begin{aligned} \varrho = \mathrm{sgn} \boxtimes \mathrm{reg}\colon S_3^2 &\to \mathrm{GL}_2(\mathbb{Z}), \\ (\sigma, (12)) &\mapsto \mathrm{sgn}(\sigma) \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ (\sigma, (123)) &\mapsto \mathrm{sgn}(\sigma) \cdot \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned}$$

Then ϱ corresponds to a Hilbert modular form f of parallel weight 1.

As before,

$$\mathrm{Ad}^0\,\varrho \cong \mathbb{Z}[e] \oplus W$$

and for any S_3^2-representation V:

- $\mathrm{Hom}_{S_3^2}(\mathbb{Z}[e], V) \cong V^{\mathrm{sgn}\boxtimes\mathrm{sgn}}$,
- $\mathrm{Hom}_{S_3^2}(W, V) \cong \{v \in V^{S_3\times(12)} \mid v + (1,(123))v + (1,(132))v = 0\}$ with the isomorphism given by sending $\varphi\colon W \to V$ to $\varphi(S)$.

Therefore,

$$\begin{aligned}\mathrm{Ad}^*\,\varrho &= \mathrm{Hom}_{S_3^2}(\mathrm{Ad}^0\,\varrho, U_L)\\ &= \mathrm{Hom}(\mathbb{Z}[e], U_L) \oplus \mathrm{Hom}(W, U_L)\\ &= (U_L^{\mathrm{sgn}\boxtimes\mathrm{sgn}}) \oplus \{u \in U_K \mid N_F^K u = 1\}.\end{aligned}$$

We next check that

$$U_L^{\mathrm{sgn}\boxtimes\mathrm{sgn}} \cong U_{F\left(\sqrt{\mathrm{disc}(L/F)}\right)} \Big/ U_F.$$

First, note that if $u \in U_L^{\mathrm{sgn}\boxtimes\mathrm{sgn}}$, then u is fixed by $\langle(123)\rangle \times \langle(123)\rangle$ and by $((12),(12))$ which together generate a subgroup H of S_3^2 of order 18. Moreover, $L^H = F\left(\sqrt{\mathrm{disc}(L/F)}\right)$ is a CM extension of F and

$$u \in U_{F\left(\sqrt{\mathrm{disc}(L/F)}\right)}.$$

Since $u \in U_L^{\mathrm{sgn}\boxtimes\mathrm{sgn}}$ is not fixed by $S_3 \times S_3$, we have that

$$u \in U_{F\left(\sqrt{\mathrm{disc}(L/F)}\right)} \Big/ U_F.$$

Finally, since we have that:

$$\mathrm{rank}(U_F) = \mathrm{rank}\left(U_{F\left(\sqrt{\mathrm{disc}(L/F)}\right)}\right),$$

the group $U_{F\left(\sqrt{\mathrm{disc}(L/F)}\right)} \Big/ U_F$ is finite. If we let N be the order of this group, then

$$U_f \otimes \mathbb{Z}[1/N] \cong \{u \in U_K \mid N_F^K u = 1\} \otimes \mathbb{Z}[1/N]. \tag{1.4}$$

As expected by Corollary 1.9 this is a group of rank 2.

1.4. **Comparison with motivic cohomology.** This section is not used in the remainder of this paper. The general conjectures of Venkatesh [Ven16] predict the action of the dual of a motivic cohomology group associated to the coadjoint motive of f. We identify this motivic cohomology group with the group of Stark units U_f, analogously to [HV17, Sec. 2.8]. Some of this section is based on standard conjectures.

1.4.1. *Motivic cohomology.* Let k be any number field and $\mathcal{O}_k$ be its ring of integers. (In general, $\mathcal{O}_k$ could be any Dedekind domain and k its field of fractions). Let E be a field of characteristic 0.

For any Chow motive M defined over k with coefficients in E, we may define motivic cohomology groups (cf. [Blo86] or [MVW06, Definition 3.4])

$$H^r_{\mathcal{M}_k}(M, E(n))$$

which are equipped with specialization maps to various cohomology theories, including étale cohomology:

$$H^r_{\mathcal{M}_k}(M, E(n)) \otimes E_{\mathfrak{p}} \to H^r_{\text{ét}}(M, E_{\mathfrak{p}}(n)).$$

Scholl [Sch00, Theorem 1.1.6] proved that these have a subspace of *integral classes*

$$H^r_{\mathcal{M}_{\mathcal{O}_k}}(M, E(n)) \subseteq H^r_{\mathcal{M}_k}(M, E(n)).$$

We will be concerned with the case $r = 1$, $n = 1$. For the trivial motive $M = k$, conjecturally:

$$H^1_{\mathcal{M}_{\mathcal{O}_k}}(k, E(1))_{\text{int}} \cong U_k \otimes E.$$

This statement is certainly true in all realizations; see, for example, [Nek94, 4.3] or [MVW06, Corollary 4.2].

1.4.2. *Motivic cohomology of the coadjoint motive.* Conjecturally, there is a 3-dimensional Chow motive M_{coad} with coefficients in E, the *coadjoint motive of* f, associated to the dual of the trace zero adjoint representation, $\operatorname{Ad}^* \varrho_f$. By definition, for any prime $\mathfrak{p}$ of E, its $\mathfrak{p}$-adic étale realization is isomorphic to:

$$H^\bullet_\lambda(M_{\text{coad}} \times_{\mathbb{Q}} \overline{\mathbb{Q}}, E_{\mathfrak{p}}) \cong \operatorname{Ad}^* \varrho_f \otimes_E E_{\mathfrak{p}}$$

(concentrated in cohomological degree 0). Without loss of generality, we assume that M_{coad} is defined over F (and not just $\overline{F_\lambda}$).

Remark 1.16. Motives associated to Hilbert modular forms were constructed in [BR93] in some cases where the weights are cohomological.

According to the general conjectures of Venkatesh [Ven16], we should consider the motivic cohomology group

$$H^1_{\mathcal{M}_{\mathcal{O}_F}}(M_{\text{coad}}, E(1)).$$

There is a natural map

$$H^1_{\mathcal{M}_{\mathcal{O}_F}}(M_{\text{coad}}, E(1)) \to H^1_{\mathcal{M}_{\mathcal{O}_L}}(M_{\text{coad}}, E(1))^{G_{L/F}}$$

and we will work with the codomain instead. According to [HV17, (2.8)], this map should be an isomorphism. In the proof of Proposition 1.17 below, we check this in the étale realization (the induced map is denoted by i).

For a prime $\mathfrak{p}$ of E, the $\mathfrak{p}$-adic étale realization map:

$$H^1_{\mathcal{M}_{\mathcal{O}_F}}(M_{\text{coad}}, E(1)) \otimes \mathcal{O}_{\mathfrak{p}} \to H^1_f(F, (\operatorname{Ad}^* \varrho_f \otimes \mathcal{O}_{\mathfrak{p}})(1))$$

is conjecturally an isomorphism [BK90, 5.3(ii)]. Here, H^1_f denotes the Bloch–Kato Selmer group [BK90]. (We apologize for the clash of notation with the Hilbert modular form f and hope that this does not cause confusion.) We compute the last group.

Proposition 1.17. *We have that*

$$H^1_f(F, (\mathrm{Ad}^* \varrho_f \otimes \mathcal{O}_\mathfrak{p})(1)) \cong U_f \otimes \mathbb{Q} \otimes \mathcal{O}_\mathfrak{p}$$

for all $\mathfrak{p}$ such that $N\mathfrak{p}$ is coprime to $[L : F]$.

Proof. This argument is adapted from [HV17, Lemma 4.5]. We claim that

$$H^1_f(G_F, \mathrm{Ad}^* \varrho_f \otimes \mathcal{O}_\mathfrak{p}) \cong (U_L \otimes \mathbb{Q} \otimes \mathrm{Ad}^* \varrho_f \otimes \mathcal{O}_\mathfrak{p})^{G_{L/F}}.$$

Recall that $(U_L \otimes \mathrm{Ad}^* \varrho_f)^{G_{L/F}} = U_f$ by definition, so this will prove the proposition.

We write $\mathrm{Ad}^* \varrho_\mathfrak{p}$ for $\mathrm{Ad}^* \varrho_f \otimes \mathcal{O}_\mathfrak{p}$ for simplicity. The (global) Bloch–Kato Selmer group H^1_f is defined by the short exact sequence:

$$0 \longrightarrow H^1_f(F, \mathrm{Ad}^* \varrho_\mathfrak{p}(1)) \longrightarrow H^1(F, \mathrm{Ad}^* \varrho_\mathfrak{p}(1)) \longrightarrow \bigoplus_v \frac{H^1(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}{H^1_f(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}$$

where $H^1_f(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))$ are the local Bloch–Kato Selmer groups. The restriction maps to the subgroup $G_{\overline{L}/L} \subseteq G_{\overline{F}/F}$ give a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & H^1_f(F, \mathrm{Ad}^* \varrho_\mathfrak{p}(1)) \arrow[r] \arrow[d, "i"] & H^1(F, \mathrm{Ad}^* \varrho_\mathfrak{p}(1)) \arrow[r] \arrow[d, "j"] & \bigoplus_v \frac{H^1(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}{H^1_f(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))} \arrow[d, "k"] \\
0 \arrow[r] & \left(H^1_f(L, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))\right)^{G_{L/F}} \arrow[r] & (H^1(L, \mathrm{Ad}^* \varrho_\mathfrak{p}(1)))^{G_{L/F}} \arrow[r] & \left(\bigoplus_w \frac{H^1(L_w, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}{H^1_f(L_w, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}\right)^{G_{L/F}}
\end{tikzcd}$$

with exact rows. Since $\mathrm{Ad}^* \varrho_\mathfrak{p}(1)$ is trivial as a $G_{\overline{L}/L}$-representation, we have that:

$$\begin{aligned}
\left(H^1_f(L, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))\right)^{G_{L/F}} &\cong \left(\mathrm{Ad}^* \varrho_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} H^1(L, \mathcal{O}_\mathfrak{p}(1))\right)^{G_{L/F}} \\
&\cong \left(\mathrm{Ad}^* \varrho_\mathfrak{p} \otimes_{\mathcal{O}_\mathfrak{p}} U_L \otimes \mathcal{O}_\mathfrak{p} \otimes \mathbb{Q}\right)^{G_{L/F}},
\end{aligned}$$

so we just need to show that the map i is an isomorphism.

Since $N\mathfrak{p}$ is coprime to $[L : F]$, the restriction map j is an isomorphism by a general group cohomology result [Ser02, I.2.4]. By the Snake Lemma, this shows that i is also injective.

To show that it is surjective, we must show that k is injective. In fact, for a place w of L above a place v of F, the restriction map

$$\frac{H^1(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}{H^1_f(F_v, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))} \to \frac{H^1(L_w, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}{H^1_f(L_w, \mathrm{Ad}^* \varrho_\mathfrak{p}(1))}$$

is split by the corestriction map divided by $[L_w : F_v]$ (since $[L_w : F_v]$ is invertible in $\mathcal{O}_\mathfrak{p}$). $\square$

2. Derived Hecke operators (on the special fiber)

Let

- f be a normalized Hilbert modular eigenform of parallel weight one, new of level $\mathfrak{N}$, with coefficients in the ring $\mathcal{O} = \mathcal{O}_{E_f}$; it has an associated Artin representation $\varrho = \varrho_F$ and hence an associated $\mathcal{O}$-module U_f of *Stark units*, which has rank $d = [F : \mathbb{Q}]$,
- $\mathfrak{p}$ be a prime of E_f such that $(p) = \mathfrak{p} \cap E_f$ has good reduction in F and p is coprime to $\mathfrak{N}$, and let $k = \mathcal{O}_{E_f}/\mathfrak{p}^n$.

Let $X = X_1(\mathfrak{N})$ be a (smooth, compact) Hilbert modular variety associated to F of level $\mathfrak{N}$ (the level of f). The integral models for the toroidal compactifications with the level structures considered here were developed in [DT04], following the standard methods of Rapoport [Rap78]. See also [Cha90], [Dim13], or [Gor02] for surveys on Hilbert modular varieties and Hilbert modular forms.

Let ω be the Hodge bundle on the integral Hilbert modular surface $X_{\mathbb{Z}[1/N(\mathfrak{N})]}$, so that

$$f \in H^0(X_{\mathbb{Z}[1/N\mathfrak{N}]}, \omega) \otimes_{\mathbb{Z}} \mathcal{O}_{E_f}.$$

In this section we construct an action of $U_f^\vee \otimes_{\mathcal{O}_{E_f}} k$ on the cohomology space

$$(H^*(X_{\mathbb{Z}[1/N\mathfrak{N}]}, \omega) \otimes_{\mathbb{Z}} \mathcal{O}_{E_f})_f \otimes_{\mathcal{O}_{E_f}} k \cong H^*(X_k, \omega)_f$$

via derived Hecke operators on the special fiber and conjecture that it lifts to $\mathcal{O}_{E_f}$. This is an analog of the Harris–Venkatesh conjecture [HV17] for coherent cohomology of the Hodge bundle on Hilbert modular varieties.

More specifically, consider the following field diagram:

$$\begin{tikzcd}
& \mathfrak{Q} \arrow[d, dash] & L \arrow[d, dash] & & & \\
N\mathfrak{q} \equiv 1\ (p^n) & \mathfrak{q} \arrow[dr, dash] & F \arrow[dr, dash] & & E_f \arrow[dl, dash] & \mathfrak{p} \arrow[dl, dash] \\
& & q & \mathbb{Q} & p &
\end{tikzcd}$$

For each prime $\mathfrak{q}$ of F such that $N\mathfrak{q} \equiv 1\ (p^n)$ and a choice of prime $\mathfrak{Q}$ above $\mathfrak{q}$, we will describe:

- a *decomposition*
$$\bigoplus_{\sigma \in G/G'} U_{f,\sigma}^\vee \otimes (\mathbb{F}_{\sigma\mathfrak{q}}^\times \otimes k)^\vee \to U_f^\vee \otimes_{\mathcal{O}_{E_f}} k$$
in Section 2.1 (Proposition 2.5),
- an action of this group via *derived Hecke operators*:
$$T_{\mathfrak{q},z} \colon H^q(X_k, \omega)_f \to H^{q+1}(X_k, \omega)_f$$
associated to $z \in (\mathbb{F}_{\mathfrak{q}}^\times \otimes k)^\vee$ in Sections 2.2, 2.3,

and conjecture that the action lifts to characteristic 0 in Section 2.4.

2.1. **Dual Stark units mod $\mathfrak{p}^n$.** We start by describing the group $U_f^\vee \otimes_{\mathcal{O}_{E_f}} k$. The description will depend on a choice of a Taylor–Wiles prime $\mathfrak{q}$ of F.

2.1.1. *Taylor–Wiles primes.* Suppose $\mathfrak{p}$ is a prime of E_f above p and for any n, consider
$$k = \mathcal{O}_{E_f}/\mathfrak{p}^n.$$

Definition 2.1. A *Taylor–Wiles* prime for f of level $n \geq 1$ consists of the following data:

(1) a prime $\mathfrak{q}$ of of F, relatively prime to the level of f, such that $N\mathfrak{q} \equiv 1$ (p^n),
(2) a choice $(\alpha, \beta) \in \mathbb{F}_{\mathfrak{p}}^2$ with $\alpha \neq \beta$ such that
$$\overline{\varrho}(\mathrm{Frob}_{\mathfrak{q}}) = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix},$$
where $\overline{\varrho}$ is the reduction of ϱ modulo $\mathfrak{p}$.

Remark 2.2. The second condition amounts to ordering the Frobenius eigenvalues; it will not be needed to reduce the units or to define the derived Hecke action, but we will need it to pin down the action in our conjecture.

If $\mathfrak{q}$ is a Taylor–Wiles prime, $(\mathcal{O}_F/\mathfrak{q})^\times$ contains a subgroup $\Delta \cong \mathbb{Z}/p^n\mathbb{Z}$ of size p^n. We frequently denote it by $(\mathcal{O}_F/\mathfrak{q})^\times_{p^n}$.

We also write
$$k\langle 1\rangle_{\mathfrak{q}} = k \otimes (\mathcal{O}_F/\mathfrak{q})^\times_{p^n}, \quad k\langle -1\rangle_{\mathfrak{q}} = \mathrm{Hom}((\mathcal{O}_F/\mathfrak{q})^\times_{p^n}, k), \tag{2.1}$$
both non-canonically isomorphic to k. When the underlying prime $\mathfrak{q}$ is clear, we drop it from the notation.

Finally, for any $\mathbb{Z}$-module M, we write
$$M\langle n\rangle = M \otimes_{\mathbb{Z}} k\langle n\rangle \quad \text{for } n = \pm 1. \tag{2.2}$$
For example, $\mathbb{F}_p\langle 1\rangle$ is canonically identified with a quotient of $(\mathcal{O}_F/\mathfrak{q})^\times$ of size p.

2.1.2. *Reduction of dual Stark units at a Taylor–Wiles prime.* Let $\mathfrak{Q}$ be a prime of L above a Taylor–Wiles prime $\mathfrak{q}$ of F. Let
$$\mathrm{Frob}_{\mathfrak{Q}} = \mathrm{Frob}_{\mathfrak{Q}/\mathfrak{q}} \in G_{L/F} \subseteq G_{L/\mathbb{Q}}$$
be the Frobenius automorphism associated to the prime $\mathfrak{Q}$ above $\mathfrak{q}$.

Lemma 2.3. *For any Artin representation $\varrho_0 \colon G_{L/\mathbb{Q}} \to \mathrm{GL}(M_0)$ where M_0 is an $\mathcal{O}_{E_f}$-module, there is a natural pairing*
$$\begin{aligned} (U_L[\varrho_0] \otimes k) \times (M_0^{\mathrm{Frob}_{\mathfrak{Q}}} \otimes k) &\to k\langle 1\rangle \\ (\varphi, m) &\mapsto \text{reduction of } \varphi(m). \end{aligned}$$

Proof. For $\varphi \in U_L[\varrho_0]$ and $m \in M_0^{\mathrm{Frob}_{\mathfrak{Q}}}$, we obtain
$$\varphi(m) \in (U_L \otimes k)^{\mathrm{Frob}_{\mathfrak{Q}}}.$$
The composition
$$U_L \hookrightarrow U_{L_{\mathfrak{Q}}} \twoheadrightarrow U_{L_{\mathfrak{Q}}}/(1+\mathfrak{Q}) \cong \mathbb{F}_{\mathfrak{Q}}^\times$$

induces a reduction map

$$(U_L \otimes k)^{\mathrm{Frob}_{\mathfrak{Q}}} \to (\mathbb{F}_{\mathfrak{Q}}^\times \otimes k)^{\mathrm{Frob}_{\mathfrak{Q}}} \cong k\langle 1\rangle,$$

where we recall that $k\langle 1\rangle = k \otimes (\mathcal{O}_F/\mathfrak{q})^\times_{p^n}$. □

Remark 2.4. We think of the *reduction map* as a *discrete logarithm*. Then this lemma is the discrete analog of Lemma 3.1, where the actual logarithm will be used. To generalize this result p-adically, one would use a p-adic logarithm.

Proposition 2.5. *Let $\varrho\colon G' = G_{L/F} \to \mathrm{GL}(M)$ be the Artin representation associated to f. Recall that $G = G_{L/\mathbb{Q}}$. Then there is a natural map:*

$$\theta^\vee\colon \bigoplus_{\sigma\in G/G'} (\mathrm{Ad}^0 M \otimes k)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}}} \otimes k\langle -1\rangle \to U_f^\vee \otimes k$$

where the domain is a direct sum of free k-modules of rank 1.

We will later use the shorthand $U_{f,\sigma}^\vee = (\mathrm{Ad}^0 M \otimes k)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}}} \otimes k\langle -1\rangle$. In the notation of the introduction, $U_{f,\sigma_i}^\vee = U_{f,i}^{\mathfrak{p}^n}$ if we label the representatives of G/G' by $\sigma_1, \dots, \sigma_d$.

Proof. Applying Lemma 2.3 to $\varrho_0 = \mathrm{Ind}_{G'}^G \mathrm{Ad}^0 \varrho$, we see that there is a pairing:

$$(U_f \otimes k) \times (M_0^{\mathrm{Frob}_{\mathfrak{Q}}} \otimes k\langle -1\rangle) \to k,$$

which induces a map

$$(M_0^{\mathrm{Frob}_{\mathfrak{Q}}} \otimes k\langle -1\rangle) \to (U_f^\vee \otimes k).$$

Then

$$\begin{aligned} M_0^{\mathrm{Frob}_{\mathfrak{Q}}} &= (\mathrm{Ind}_{G'}^G \mathrm{Ad}^0 M)^{\mathrm{Frob}_{\mathfrak{Q}}} \\ &= \left(\bigoplus_{\sigma\in G/G'} \sigma \,\mathrm{Ad}^0 M\right)^{\mathrm{Frob}_{\mathfrak{Q}}} \\ &= \bigoplus_{\sigma\in G/G'} (\mathrm{Ad}^0 M)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}}} \end{aligned}$$

because $\sigma \,\mathrm{Frob}_{\mathfrak{Q}/\mathfrak{q}}\, \sigma^{-1} = \mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}} \in G'$.

Finally, using the basis such that $\varrho(\mathrm{Frob}_{\mathfrak{Q}}) = \begin{pmatrix} \alpha & \\ & \beta \end{pmatrix}$ for $\alpha \neq \beta$, we have that

$$\mathrm{Ad}^0 \varrho(\mathrm{Frob}_{\mathfrak{Q}}) = \begin{pmatrix} \frac{\alpha}{\beta} & & \\ & \frac{\beta}{\alpha} & \\ & & 1 \end{pmatrix}.$$

Since $\alpha \neq \beta$, this shows that $(\mathrm{Ad}^0 M)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma q}}$ has rank 1. □

We finally recast this in the language of [HV17, Section 2.9]. For any $\mathfrak{Q}$, we may consider the element

$$e_{\mathfrak{Q}} = \varrho(\mathrm{Frob}_{\mathfrak{Q}}) - (1/2)\mathrm{Tr}\varrho(\mathrm{Frob}_{\mathfrak{Q}}) \in \mathrm{Ad}^0 \varrho.$$

Note that for all $g \in G_{L/F}$,

$$e_{g\mathfrak{Q}} = \mathrm{Ad}(\varrho(g))e_{\mathfrak{Q}}.$$

Therefore:

$$\mathrm{Ad}^0(\mathrm{Frob}_{\mathfrak{Q}})e_{\mathfrak{Q}} = e_{\mathrm{Frob}_{\mathfrak{Q}}\,\mathfrak{Q}} = e_{\mathfrak{Q}},$$

showing that

$$e_{\mathfrak{Q}} \in (\mathrm{Ad}^0\,\varrho)^{\mathrm{Frob}_{\mathfrak{Q}}}.$$

By Proposition 2.5, this choice defines a map

$$\theta^\vee(e_{\mathfrak{Q}} \otimes -)\colon k\langle -1\rangle \to U_f^\vee \otimes k. \tag{2.3}$$

When $F = \mathbb{Q}$, this recovers the map $\theta_q^\vee$ from [HV17, Section 2.9].

2.2. **The Shimura class.** We consider two level structures: for an ideal $\mathfrak{N} \subseteq \mathcal{O}_F$,

$$\Gamma_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \;\middle|\; c \in \mathfrak{N} \right\},$$
$$\Gamma_1(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_F) \;\middle|\; c, a-1, d-1 \in \mathfrak{N} \right\}.$$

Note that $\Gamma_1(\mathfrak{N}) \subseteq \Gamma_0(\mathfrak{N})$ and the quotient is isomorphic to $(\mathcal{O}/\mathfrak{N})^\times$. We let

$$X_0(\mathfrak{N}), X_1(\mathfrak{N}) = \text{Hilbert modular variety with } \Gamma_0(\mathfrak{N}), \Gamma_1(\mathfrak{N})\text{-level structure, respectively.}$$

Both of these are schemes over $\mathbb{Z}[1/N_{F/\mathbb{Q}}\mathfrak{N}]$ (c.f. [DT04]) and they have good reduction modulo $\mathfrak{p}$ for $\mathfrak{p}$ not dividing $\mathfrak{N}$. The covering

$$X_1(\mathfrak{N}) \to X_0(\mathfrak{N})$$

descends to a covering

$$X_1(\mathfrak{N})_k \to X_0(\mathfrak{N})_k$$

with Galois group $(\mathcal{O}/\mathfrak{N})^\times$.

Then

$$X_1(\mathfrak{q}) \to X_0(\mathfrak{q})$$

is a $(\mathcal{O}/\mathfrak{q})^\times$-covering. We may pass to the unique subcovering with Galois group $\Delta = (\mathcal{O}/\mathfrak{q})^\times_{p^n}$:

$$X_1(\mathfrak{q})^\Delta \to X_0(\mathfrak{q}).$$

This extends to an étale covering of schemes over $\mathbb{Z}[1/q]$, and hence induces an étale covering

$$X_1(\mathfrak{q})^\Delta_k \to X_0(\mathfrak{q})_k$$

(cf. for example [Dim13, Prop. 3.4]). We hence get a class

$$\mathfrak{S}_k \in H^1_{\text{ét}}(X_0(\mathfrak{q})_k, k\langle 1\rangle), \tag{2.4}$$

where we recall that $k\langle 1\rangle \cong k \otimes \Delta$. Using the natural map $k \to \mathbb{G}_a$ of étale sheaves over $X_0(\mathfrak{q})_k$, we obtain a class:

$$\mathfrak{S}_{\mathbb{G}_a} \in H^1_{\text{ét}}(X_0(\mathfrak{q})_k, \mathbb{G}_a\langle 1\rangle). \tag{2.5}$$

Finally, using Zariski–étale comparison, we have an isomorphism:

$$H^1(X_0(\mathfrak{q})_k, \mathcal{O}\langle 1\rangle) \to H^1(X_0(\mathfrak{q})_k, \mathbb{G}_a\langle 1\rangle)$$

and hence $\mathfrak{S}_{\mathbb{G}_a}$ defines a class

$$\mathfrak{S} \in H^1(X_0(\mathfrak{q})_k, \mathcal{O}\langle 1\rangle). \tag{2.6}$$

Definition 2.6. The *Shimura class* is the cohomology class $\mathfrak{S} \in H^1(X_0(\mathfrak{q})_k, \mathcal{O}\langle 1\rangle)$ obtained above (2.6).

We will use it next to construct a mod $\mathfrak{p}^n$ derived Hecke operator.

2.3. **Construction of derived Hecke operators.** Let $\mathfrak{N}$ be the level of f and recall that we consider $X = X_1(\mathfrak{N})$ over $\mathbb{Z}[1/N\mathfrak{N}]$.

Write $X_{0,1}(\mathfrak{q}, \mathfrak{N})$ for X with added $\Gamma_0(\mathfrak{q})$-level structure at $\mathfrak{q}$. This is a Hilbert modular variety for the group $\Gamma_1(\mathfrak{q}, \mathfrak{N})$ in the notation of [DT04], and hence also has a smooth, projective, integral model.

Then the Shimura class $\mathfrak{S}$ pulls back to a class

$$\mathfrak{S}_X \in H^1(X_{0,1}(\mathfrak{q}, \mathfrak{N})_k, \mathcal{O}\langle 1\rangle).$$

Cupping with this class gives a map

$$H^0(X_{0,1}(\mathfrak{q}, \mathfrak{N})_k, \omega) \xrightarrow{\cup \mathfrak{S}_X} H^1(X_{0,1}(\mathfrak{q}, \mathfrak{N})_k, \omega)\langle 1\rangle. \tag{2.7}$$

Classically, Hecke operators are defined as operators on cohomology induced by certain correspondences:

$$\begin{tikzcd} & X_{0,1}(\mathfrak{q}, \mathfrak{N}) \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ X & & X \end{tikzcd}$$

We define the *derived Hecke operator* by the same push-pull procedure but cupping with $\mathfrak{S}_X$ in the middle:

$$H^0(X_k, \omega) \xrightarrow{\pi_1^*} H^0(X_{0,1}(\mathfrak{q}, \mathfrak{N})_k, \omega) \xrightarrow{\cup \mathfrak{S}_X} H^1(X_{0,1}(\mathfrak{q}, \mathfrak{N})_k, \omega)\langle 1\rangle \xrightarrow{\pi_{2,*}} H^1(X_k, \omega)\langle 1\rangle.$$

Finally, for any $z \in k\langle -1\rangle$, we define

$$T_{\mathfrak{q},z} \colon H^0(X_k, \omega) \to H^1(X_k, \omega) \tag{2.8}$$

by composing the above map with multiplication by z.

More generally, for each $z \in k\langle -1\rangle$, there is an operator

$$T_{\mathfrak{q},z} \colon H^q(X_k, \omega) \to H^{q+1}(X_k, \omega), \tag{2.9}$$

defined analogously.

2.4. **The conjecture.** We conjecture there is an action of $U_f^\vee$ on the f-isotypic component of the cohomology space $H^*(X, \omega)_f$ which reduces modulo $\mathfrak{p}^n$ to the action of the operators $T_{\mathfrak{q},z}$ from equation (2.9).

For $h \in H^*(X_{\mathcal{O}[1/N(\mathfrak{N})]}, \omega)$, we write $\overline{h} \in H^*(X_k, \omega)$ for its reduction. In Proposition 2.5, we described a map

$$\theta^\vee \colon \bigoplus_{\sigma \in G/G'} U_{f,\sigma}^\vee = \bigoplus_{\sigma \in G/G'} (\mathrm{Ad}^0 M \otimes k)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}}} \otimes k\langle -1\rangle \to U_f^\vee \otimes k \tag{2.10}$$

associated to a Taylor–Wiles primes $\mathfrak{q}$ of F and a prime $\mathfrak{Q}$ above it.

We may then define an action of $U_f^\vee \otimes k$ by derived Hecke operators. Note that the choice of $e_{\sigma\mathfrak{Q}} \in (\mathrm{Ad}^0 M)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}}}$ gives a k-basis of:

$$(\mathrm{Ad}^0 M \otimes k)^{\mathrm{Frob}_{\sigma\mathfrak{Q}/\sigma\mathfrak{q}}} \cong k.$$

For $\sigma \in G/G'$, $e_{\sigma\mathfrak{Q}}$, and $z \in k\langle -1\rangle_{\sigma\mathfrak{q}}$, the corresponding element $\theta^\vee(e_{\sigma\mathfrak{Q}} \otimes z) \in U_f^\vee \otimes k$ acts on $H^*(X_k, \omega)$ by

$$T_{\sigma\mathfrak{q},z} \colon H^*(X_k, \omega) \to H^{*+1}(X_k, \omega).$$

This naturally extends to an action of $\bigwedge U_f^\vee \otimes k$ on $H^*(X_k, \omega)$.

Conjecture 2.7. *There is an action $\star$ of the exterior algebra $\bigwedge(U_f^\vee)$ on $H^*(X_{\mathcal{O}[1/N(\mathfrak{N})]}, \omega)_f$ such the induced action of $\bigwedge(U_f^\vee) \otimes k$ on $H^*(X_{\mathcal{O}[1/N(\mathfrak{N})]}, \omega)_f \otimes k$ is the one described above, i.e. for every $(\mathfrak{p}, n, \sigma, \mathfrak{q}, z)$ with*

- $\mathfrak{p}$ *a prime of E_f satisfying the above conditions,*
- $n \geq 1$ *an integer,*
- $\sigma \in G/G'$,
- $\mathfrak{q}$ *a Taylor–Wiles primes of level n; in particular $N\mathfrak{q} \equiv 1$ (p^n),*
- $z \in (\mathcal{O}/\mathfrak{p}^n)\langle -1\rangle_{\sigma\mathfrak{q}}$,

and every $\omega_f \in H^q(X, \omega)_f$:

$$T_{\sigma\mathfrak{q},z}\overline{\omega_f} = \alpha \cdot \overline{\theta^\vee(e_{\sigma\mathfrak{Q}} \otimes z)^\sim \star \omega_f}$$

where $\theta^\vee(e_{\sigma\mathfrak{Q}} \otimes z)^\sim \in U_f^\vee$ is an arbitrary lift of $\theta^\vee(e_{\sigma\mathfrak{Q}} \otimes z) \in U_f^\vee \otimes k$, and the bar refers to reduction modulo $\mathfrak{p}^n$.

Remark 2.8. It is clear that the prescribed mod $\mathfrak{p}^n$ operators define an action.

Remark 2.9. Harris and Venkatesh provide numerical evidence for this conjecture for $n = 1$, in Example 1.13. To do that, they first perform an *explication* ([HV17, Section 5]), putting the conjecture in a much more computable form. They relate it to a question about a pairing considered by Mazur [Maz77] and then rely on a computation of this pairing due to Merel [Mer96].

While the initial steps of the explication can be performed in our case, putting Conjecture 2.7 in a similar framework, we have not been able to obtain analogs of Merel's computations.

When $F = \mathbb{Q}$, Harris–Venkatesh [HV17, Section 4] prove the following result when $n = 1$:

$$\text{vanishing of } T_{q,z}\overline{f} \quad\Longrightarrow\quad \text{vanishing of the map } \theta_q^\vee \colon k\langle -1\rangle \to U_f^\vee \otimes k,$$

assuming an "$R = T$" theorem. It would be interesting to study an analog of this result in our case. We expect that the rank r of the map

$$\theta^\vee \colon \bigoplus_{\sigma \in G/G'} U_{f,\sigma}^\vee \to U_f^\vee \otimes k$$

from equation (2.10) can be any number $0 \leq r \leq d$. Hence the strongest analog of the above result should be:

$$\mathrm{rank}\langle T_{\sigma\mathfrak{q},z}\overline{f} \mid \sigma \in G/G'\rangle = \mathrm{rank}(\theta^\vee).$$

A weaker version simply states:

$$\text{vanishing of } T_{\sigma\mathfrak{q},z}\overline{f} \text{ for all } \sigma \in G/G' \quad \Longrightarrow \quad \text{vanishing of the map } \theta^\vee.$$

Note that the proof in the case $F = \mathbb{Q}$ relies on the approach of Calegari–Geraghty [CG18] to modularity lifting. Since their results apply to general F, one could hope to prove the above results in a similar way, but we have not explored this further yet.

Since we expect that the map $\theta^\vee$ may sometimes have rank d, we want to make sure that we can produce a rank d group of operators $T_{\mathfrak{q},z}$ in order to pin down the conjectural action.

Lemma 2.10. *For any $\mathfrak{p}$ and n, there is a prime $q \equiv 1$ (p^n) which splits completely in F and the primes $\mathfrak{q}_1, \dots, \mathfrak{q}_d$ above q are Taylor–Wiles primes for f of level n.*

Proof. We first show that there exists a positive density of primes q of $\mathbb{Q}$ which splits completely in F such that $q \equiv 1$ (p^n). Consider the field $F(\zeta_{p^n})$ for a primitive p^nth root of unity and a prime q of $\mathbb{Q}$ in the field diagram:

$$\begin{tikzcd}[column sep=small]
 & \mathfrak{Q} \arrow[dl, dash] & & F(\zeta_{p^n}) \arrow[dl, dash] \arrow[dr, dash] & \\
\mathfrak{q} \arrow[dr, dash] & & F \arrow[dr, dash] & & \mathbb{Q}(\zeta_{p^n}) \arrow[dl, dash] \\
 & q & & \mathbb{Q} &
\end{tikzcd}$$

Since we assume that p has good reduction in F, the fields $\mathbb{Q}(\zeta_{p^n})$ and F have disjoint ramification, and hence we have isomorphisms:

$$\begin{tikzcd}
G_{F(\zeta_{p^n})/F} \arrow[r, "\cong"] & G_{\mathbb{Q}(\zeta_{p^n})/\mathbb{Q}} \arrow[r, "\cong"] & (\mathbb{Z}/p^n\mathbb{Z})^\times \\
D(\mathfrak{Q}/\mathfrak{q}) \arrow[r, "\cong"] \arrow[u, hook] & D(\mathfrak{Q} \cap \mathbb{Q}(\zeta_{p^n})/q) \arrow[r, "\cong"] \arrow[u, hook] & \langle q \rangle \arrow[u, hook]
\end{tikzcd}$$

via the restriction map. By Cheboratev density theorem, there is a positive density of primes q of $\mathbb{Q}$ that splits completely in $F(\zeta_{p^n})$. Then q also splits completely in $\mathbb{Q}(\zeta_{p^n})$ which shows that

$$q \equiv 1 \mod p^n$$

using the above diagram, and q splits completely in F.

Since there is a positive density of primes q with the above property, there exists a positive density for which $\mathfrak{q}_1, \dots, \mathfrak{q}_d$ are Taylor–Wiles primes for f of level n. $\square$

In this case, we have d derived Hecke operators $T_{\mathfrak{q}_1,z_1}, \dots, T_{\mathfrak{q}_d,z_d}$ and we expect that if they are linearly independent, then the map $\theta^\vee$ is an isomorphism.

3. Archimedean realization of the motivic action

We conjectured above (Conjecture 2.7) an action of the (motivic) exterior algebra $\bigwedge(U_f^\vee)$ on the f-isotypic component $H^*(X,\omega)_f$ of the coherent cohomology group of the Hodge bundle ω for a parallel weight one Hilbert modular form f.

We now describe an action of $\bigwedge(U_f^\vee)\otimes\mathbb{C}$ on $H^*(X_{\mathbb{C}},\omega)_f$ and conjecture that $\bigwedge(U_f^\vee)\otimes\mathbb{Q}$ preserves the natural rational structure $H^*(X_{\mathbb{Q}},\omega)_f$. This is an analog of the Prasanna–Venkatesh conjecture [PV16] for coherent cohomology of the Hodge bundle on Hilbert modular varieties.

More specifically, for embeddings $E_f\hookrightarrow\mathbb{C}$ and $L\hookrightarrow\mathbb{C}$, we will describe:

- a decomposition
$$\bigoplus_{\sigma\in G/G'} U_{f,\sigma}^\vee\otimes\mathbb{C}\xrightarrow{\cong} U_f^\vee\otimes_{E_f}\mathbb{C}$$
in Section 3.1 (Proposition 3.2),
- an action of this group via *complex conjugation operators*
$$H^q(X_{\mathbb{C}},\omega)_f\to H^{q+1}(X_{\mathbb{C}},\omega)_f$$
in Sections 3.2 3.3 (equation (3.16)),

and conjecture the action descends to rational structures in Section 3.3.

3.1. **Dual Stark units over $\mathbb{C}$.** We start by describing the dual unit group $U_f^\vee\otimes_\iota\mathbb{C}$ for an embedding $\iota\colon E_f\to\mathbb{C}$. We first introduce a pairing induced by applying a logarithm map to units.

Fix an embedding $\tau\colon L\hookrightarrow\mathbb{C}$ and let c_0 be the complex conjugation associated to τ. Define
$$\begin{aligned}\log\colon \mathbb{C}\cong L\otimes_\tau\mathbb{C}&\to\mathbb{R}\\ z&\mapsto\log|z|\end{aligned}$$
and extend it linearly to
$$\begin{aligned}\log\colon (L\otimes_\tau\mathbb{C})\otimes(E_f\otimes_\iota\mathbb{C})&\to\mathbb{C}\\ z\otimes\lambda&\mapsto\lambda\log|z|.\end{aligned}$$
Thus for $x\otimes y\in L\otimes E_f$, we write
$$\log|\tau\otimes\iota(x\otimes y)|=\iota(y)\cdot\log|\tau(x)|\in\mathbb{C}. \tag{3.1}$$

Lemma 3.1. *For any Artin representation $\varrho_0\colon G_{L/\mathbb{Q}}\to\mathrm{GL}(M_0)$ where M_0 is an E_f-vector space, there is a natural perfect pairing*
$$\begin{aligned}(U_L[\varrho_0]\otimes_\iota\mathbb{C})\times(M_0^{c_0}\otimes_\iota\mathbb{C})&\to\mathbb{C}\\ (\varphi,m)&\mapsto\log(|(\tau\otimes\iota)(\varphi(m))|)\end{aligned}$$
which induces an isomorphism
$$M_0^{c_0}\otimes\mathbb{C}\xrightarrow{\cong}U_L[\varrho_0]^\vee\otimes\mathbb{C}.$$

Proof. This is immediate using Proposition 1.5. □

Proposition 3.2. *Let $\varrho\colon G' = G_{L/F} \to \mathrm{GL}_2(M)$ be the Artin representation associated to a Hilbert modular newform of weight* 1*. Recall that $G = G_{L/\mathbb{Q}}$. Then there is a natural isomorphism:*

$$\bigoplus_{\sigma \in G/G'} (\mathrm{Ad}^0 M \otimes_\iota \mathbb{C})^{\sigma c_0 \sigma^{-1}} \xrightarrow{\cong} U_f^\vee \otimes_\iota \mathbb{C}.$$

We will later use the shorthand $U_{f,\sigma}^\vee = (\mathrm{Ad}^0 M)^{\sigma c_0 \sigma^{-1}}$. In the notation of the introduction $U_{f,\sigma_i}^\vee \otimes_\iota \mathbb{C} = U_{f,i}^{\mathbb{C}}$ if we label the representatives of G/G' by $\sigma_1, \ldots, \sigma_d$.

Proof. The result is obtained by applying Lemma 3.1 to $\varrho_0 = \mathrm{Ad}^0 \operatorname{Ind}_{G'}^G \varrho$ and recalling that $M_0^{c_0} \cong \bigoplus_{\sigma \in G/G'} \left(\mathrm{Ad}^0 M\right)^{\sigma c_0 \sigma^{-1}}$ by the proof of Corollary 1.9. $\square$

Remark 3.3. Note that both $U_f^\vee \otimes_\iota \mathbb{C}$ and $(\mathrm{Ad}^0 M \otimes_\iota \mathbb{C})^{\sigma c_0 \sigma^{-1}}$ have natural E_f rational structures $U_f^\vee \otimes E_f$ and $(\mathrm{Ad}^0 M \otimes E_f)^{\sigma c_0 \sigma^{-1}}$ but the above pairing does not respect them. The rational structures differ by logarithms of a units.

3.2. **Partial complex conjugation and Harris' periods.** Following [Har90b], we define *partial complex conjugation* operators on Hilbert modular forms. We first use the automorphic language but later translate it to classical language.

3.2.1. *Automorphic forms as sections of line bundles.* Let $G = R_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$ be the Weil restriction of scalars of $\mathrm{GL}_{2,F}$, and Z_G be its center.

Definition 3.4. An (automorphic) *Hilbert modular form of weight $(\underline{k}, r)$ for F*, where $\underline{k} = (k_1, \ldots, k_d) \in \mathbb{N}_{>0}$ and $r \in \mathbb{Z}$ satisfies $k_i \equiv r$ (2) for $1 \leq i \leq d$, is an automorphic form φ on $G(\mathbb{Q})\backslash G(\mathbb{A})$ such that

(1) $\varphi(z_\infty g) = N_{F/\mathbb{Q}}(z_\infty)^r \varphi(g)$ for $g \in G(\mathbb{A})$, $z_\infty \in Z_G(\mathbb{R})$,
(2) $\varphi(g r_j(\theta)) = e^{ik_j\theta} \cdot \varphi(g)$ for $g \in G(\mathbb{A})$ and $r_j(\theta)$ the rotation by θ matrix in the jth coordinate,
(3) $R(\mathfrak{p}^-)\varphi = 0$, where $\mathfrak{p}^-$ is the antiholomorphic tangent space to $\mathcal{H}^d$ at $(i, \ldots, i)$ and $R(\cdot)$ is the right regular action on functions of the universal enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$.

In sections to follow, we will occasionally drop the r from the notation and just write $\underline{k}$ for the weight for simplicity. Note that each index j corresponds to a place $\sigma = \sigma_j$ of F.

We consider the Shimura variety $\mathcal{M}$ associated to G, which is the *Hilbert modular variety.* It is defined over $\mathbb{Q}$ and its $\mathbb{C}$-points are

$$\mathcal{M}(\mathbb{C}) = \varprojlim_{K_f} G(\mathbb{Q})\backslash G(\mathbb{A})/K_\infty^+ \times K_f$$

where K_f runs over the set of open compact subgroups of $G(\mathbb{A}_{\mathrm{fin}})$, and $K_\infty^+ = \mathrm{SO}(2)^d$. We previously denoted an integral model of a toroidal compactification (with fixed level structure) by X. While this choice of X is not canonical, its cohomology groups will be independent of the choice of X (for a more precise statement, confer Remark 3.9).

Modular forms of weight $(\underline{k}, r)$ with $k_j \geq 2$ are *cohomological* (c.f. [Lan19]), i.e. they appear in the de Rham cohomology of a local system on the Hilbert modular variety. One may then define period invariants using rational de Rham cohomology, making them much easier to study (see, for example, [TU18]).

However, if at least one weight k_j is 1, they cease to be cohomological and only appear in the coherent cohomology of line bundles on the Shimura variety $\mathcal{M}$. This theory was developed in more generality in [Har90a] and specialized to the case of Hilbert modular forms in [Har90b]. We mainly follow the second reference here.

Consider the representation

$$Z_G(\mathbb{R}) \cdot K_\infty^+ \to \mathbb{C},$$
$$z_\infty(r_1(\theta), \ldots, r_d(\theta)) \mapsto N_{F/\mathbb{Q}}(z_\infty)^{-r} \prod_{j=1}^{d} e^{-ik_j\theta},$$

which naturally extends to any conjugate P of $B = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ in $G(\mathbb{C})$. We hence have a 1-dimensional representations

$$\rho_{\underline{k},r} \colon P \to \mathbb{C}.$$

Let M be the homogeneous space G/B with the natural $\mathbb{Q}$-rational structure. The $G(\mathbb{R})$-orbit of the point of G/B corresponding to P is isomorphic to $(\mathbb{C} \setminus \mathbb{R})^d$ (with P mapping to $(i, \ldots, i)$).

We define the homogeneous line bundle $\mathcal{L}_{\underline{k},r}$ as

$$\begin{array}{ccc} (g, v) & & G \times_{\rho_{\underline{k},r}} \mathbb{C} \cong (G \times \mathbb{C})/\sim \\ \big\downarrow & & \big\downarrow \\ g & & G/P \cong G/B = M \end{array}$$

where $\sim$ is given by $(gp, v) \sim (g, \rho_{\underline{k},r}(p)v)$. Any homogeneous section of this line bundle is a map

$$\varphi \colon M \to \mathbb{C}$$

such that

$$\begin{aligned} (z_\infty r(\theta) g, \varphi(z_\infty r(\theta) g)) &= \left(g, N_{F/\mathbb{Q}}(z_\infty)^{-r} \prod_{j=1}^{d} e^{-ik_j\theta} \varphi(z_\infty r(\theta) g) \right) \\ &= (g, \varphi(g)) \end{aligned}$$

which shows that φ satisfies conditions (1) and (2) of the definition of a Hilbert modular forms of weight $(\underline{k}, r)$.

Next, we descend the line bundle $\mathcal{L}_{\underline{k},r}$ on M to a line bundle

$$\mathcal{E}_{\underline{k},r} = \varinjlim_{U} G(\mathbb{Q})^+ \backslash \mathcal{L}_{\underline{k},r}|_{G^+} \times G(\mathbb{A}_{\mathrm{fin}})/U$$

on $\mathcal{M}(\mathbb{C})$.

Remark 3.5. We identify here the two notions of line bundle on a manifold: a vector bundle of rank 1 and an invertible sheaf.

The global section of this line bundle are Hilbert modular forms of weight $(\underline{k}, r)$.

Theorem 3.6. *Let $\mathcal{A}(\underline{k}, r)$ be the space of Hilbert modular forms of weight $(\underline{k}, r)$ for F. Then*

$$H^0(\mathcal{M}(\mathbb{C}), \mathcal{E}_{\underline{k},r}) \cong \mathcal{A}(\underline{k}, r).$$

We first note that

$$\Omega^1_{\mathcal{M}} \cong \bigoplus_{j=1}^{d} \mathcal{E}_{\underline{2}_j,0} \tag{3.2}$$

where $\underline{2}_j = (0, \dots, 0, 2, 0 \dots, 0)$ with the 2 at the jth place. In particular,

$$\Omega^d_{\mathcal{M}} \cong \bigwedge^{d} \Omega^1_{\mathcal{M}} \cong \mathcal{E}_{\underline{2},0} \tag{3.3}$$

where $\underline{2} = (2, \dots, 2)$. In fact, there is a natural decomposition

$$\Omega^1_{\mathcal{M}} = \bigoplus_{j=1}^{d} \Omega^1_j \tag{3.4}$$

and $\Omega_{\mathcal{M},j} \cong \mathcal{E}_{\underline{2}_j,0}$.

Note that dz_j is a $K_G(\mathbb{R}) \cdot K_\infty^+$ eigenvector with eigenvalue corresponding to the sheaf $\mathcal{L}_{2_j,0}$. This choice determines an isomorphism

$$\begin{aligned} \underline{\mathrm{Lift}}_{\underline{k},r} \colon H^0(\mathcal{M}(\mathbb{C}), \mathcal{E}_{\underline{k},r}) &\to \mathcal{A}(\underline{k}, r) \\ dz_j &\mapsto 1. \end{aligned}$$

Theorem 3.7. *The $G(\mathbb{A}_{\mathrm{fin}})$-equivariant bundle $\mathcal{E}_{\underline{k},r}$ is rational over $F(\underline{k}) = F^{\Gamma(\underline{k})}$, where $\Gamma(\underline{k}) = \{\sigma \in G_{\mathbb{Q}} \mid \underline{k}^\sigma = \underline{k}\}$.*

Let (π, H_π) be an irreducible cuspidal automorphic representation of $G(\mathbb{A})$, generated by a Hilbert modular form φ of weight $(\underline{k}, r)$. Let π_{fin} be the finite component of π, which can be defined over a field $E = \mathbb{Q}(\pi)$. Let $H_\pi^{\mathrm{hol}} \subseteq H_\pi$ denote the subspace of homomorphic vectors so that the $G(\mathbb{A}_{\mathrm{fin}})$-action on H_π^{hol} is isomorphic to π_{fin}. Since $H_\pi^{\mathrm{hol}} \subseteq \mathcal{A}(\underline{k}, r)$, it is isomorphic to a $G(\mathbb{A}_{\mathrm{fin}})$-submodule:

$$H^0(\mathcal{M}, \mathcal{E}_{\underline{k},r})_\pi \subseteq H^0(\mathcal{M}, \mathcal{E}_{\underline{k},r}).$$

One can show that $F(\underline{k}) \subseteq E$ and hence $\mathcal{E}_{\underline{k},r}$ is defined over E. Then we can let

$$H^0(\mathcal{M}, \mathcal{E}_{\underline{k},r})_\pi(E)$$

denote the set of E-rational elements and $H_\pi^{\mathrm{hol}}(E)$ denote the corresponding E-subspace of H_π^{hol}.

3.2.2. *Higher sheaf cohomology.* We now discuss forms that are not holomorphic. In particular, we define for a subset $I \subseteq \Sigma_F = \{\sigma_1, \dots, \sigma_d\}$:

$$H_\pi^I = \{\varphi \in H_\pi \mid R(\mathfrak{p}_j^+)\varphi = 0 \text{ for } \sigma_j \in I,\ R(\mathfrak{p}_j^-)\varphi = 0 \text{ for } \sigma_j \notin I\}.$$

Equivalently, if we let

$$\varphi^I(g) = \varphi(gJ^I),$$

where $J^I = (J_1^I, \dots, J_d^I) \in G(\mathbb{R})$ is given by

$$J_j^I = \begin{cases} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } j \in I, \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \text{if } j \notin I, \end{cases}$$

then

$$H_\pi^I = (H_\pi^{\mathrm{hol}})^I = \{\varphi^I \mid \varphi \in H_\pi^{\mathrm{hol}}\}.$$

Remark 3.8. Thinking about these as functions on $\mathcal{H}^d$, this roughly corresponds to applying the automorphism $z_j - \overline{z_j}$ at all places $\sigma_j \in I$. We discuss the precise translation at the end of this section.

We next discuss which cohomology groups these non-holomorphic subspaces of H_π live in. An element $\varphi \in H_\pi^I$ may be regarded as global C^∞-section of the line bundle $\mathcal{E}_{\underline{k}(I)',r}$, where

$$\underline{k}(I)'_j = \begin{cases} k_j & \text{if } j \notin I, \\ -k_j & \text{if } j \in I. \end{cases}$$

To express it as a holomorphic section, note that we have an isomorphism:

$$H^0(\mathcal{M}(\mathbb{C})^{\mathrm{an}}, (\mathcal{E}_{\underline{k}(I)',r})^\infty) \to \Gamma\left(\mathcal{M}(\mathbb{C})^{\mathrm{an}}, (\mathcal{E}_{\underline{k}(I)',r})^\infty \otimes \bigwedge_{j\in I}(\Omega^1_{\mathcal{M},j})^\infty \wedge \bigwedge_{j\in I}\overline{(\Omega^1_{\mathcal{M},j})}\right),$$

$$\varphi \mapsto \varphi \cdot \bigwedge_{j\in I} dz_j \wedge d\overline{z_j},$$

and

$$(\mathcal{E}_{\underline{k}(I)',r})^\infty \otimes \bigwedge_{j\in I}(\Omega^1_{\mathcal{M},j})^\infty \cong (\mathcal{E}_{\underline{k}(I)',r})^\infty \otimes \left(\bigwedge_{j\in I}\mathcal{E}_{2_j,0}\right)^\infty \cong (\mathcal{E}_{\underline{k}(I),r})^\infty,$$

where

$$k(I)_j = \begin{cases} k_j & \text{if } j \notin I, \\ 2 - k_j & \text{if } j \in I. \end{cases}$$

We hence get a map:

$$H_\pi^I \cong H^0(\mathcal{M}(\mathbb{C})^{\mathrm{an}}, (\mathcal{E}_{\underline{k}(I)',r})^\infty) \to H^0(\mathcal{M}(\mathbb{C})^{\mathrm{an}}, \Omega_{\mathcal{M}}^{0,|I|} \otimes \mathcal{E}_{\underline{k}(I),r})$$

$$\varphi \mapsto \omega_\varphi = \varphi \cdot \bigwedge_{j\in I} dz_j \wedge d\overline{z_j}$$

where $\Omega^{0,|I|}_{\mathcal{M}}$ denotes the sheaf of $(0,|I|)$-forms on $\mathcal{M}(\mathbb{C})^{\mathrm{an}}$. Finally, we note that

$$\begin{aligned}
\overline{\partial}\omega_\varphi &= \overline{\partial}\left(\varphi \cdot \bigwedge_{j\in I} dz_j \wedge d\overline{z_j}\right) \\
&= (\overline{\partial}\varphi) \wedge \bigwedge_{j\in I} dz_j \wedge d\overline{z_j} \pm \varphi \cdot \underbrace{\overline{\partial}\bigwedge_{j\in I} dz_j \wedge d\overline{z_j}}_{=0} \\
&= \left(\sum_{j\in I} \frac{\partial\varphi}{\partial\overline{z_j}} d\overline{z_j}\right) \wedge \left(\bigwedge_{j\in I} dz_j \wedge d\overline{z_j}\right) && \varphi \text{ holomorphic at } j \notin I \\
&= 0.
\end{aligned}$$

Thus ω_φ is a closed $(0,|I|)$-form and defines a Dolbeault cohomology class

$$[\omega_\varphi] \in H^{0,|I|}(\mathcal{M}(\mathbb{C}), \mathcal{E}_{\underline{k}(I),r}) \cong H^{|I|}(\mathcal{M}(\mathbb{C}), \mathcal{E}_{\underline{k}(I),r})$$

with the last isomorphism given by Dolbeault Theorem. We identify the last group with the Zariski cohomology of a sheaf on $\mathcal{M}_{\mathbb{C}}$ via the GAGA Theorem.

Remark 3.9. In [Har90a], Harris extends this cohomology group to a toroidal compactification of the Hilbert modular variety $\mathcal{M}$ and checks that the cohomology is independent on the choice of the compactification. The vector bundle $\mathcal{E}_{\underline{k}(I),r}$ has a subcanonical and a canonical extension to the toroidal compactification. We henceforth write $\overline{H}^{\bullet}(\mathcal{E}_{\underline{k}(I),r})$ for the image of the cohomology of the subcanonical extension in the cohomology of the canonical extension and refer to loc. cit. for the full details. Later, we will sometimes write $H^{\bullet}(X_{\mathbb{Q}}, \mathcal{E}_{\underline{k}(I),r})$ to mean $\overline{H}^{\bullet}(\mathcal{E}_{\underline{k}(I),r})$ for simplicity.

Because the variety $X_{\mathbb{Q}}$ is defined over $\mathbb{Q}$ and sheaf $\mathcal{E}_{\underline{k}(I),r}$ is defined over $F(\underline{k}(I))$, the cohomology group has a natural $G(\mathbb{A}_{\mathrm{fin}})$-invariant $F(\underline{k}(I))$-rational structure. Therefore, we may now define $\overline{H}^{|I|}(E(I))$ as the $E(I) = EF(\underline{k}(I))$-rational cohomology classes in $\overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})$.

The above discussion is summarized in the following lemma.

Lemma 3.10 ([Har90b, Lemma 1.4.3])**.** *Let $I \subseteq \Sigma$ be any subset. There is a natural embedding*

$$H^I_\pi \hookrightarrow \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi$$

of $G(\mathbb{A}^f)$-modules, and the image of H^I_π is a $E(I)$-rational subspace of $\overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})$.

The key to the definition of the period $\nu^I(\pi)$ is the fact that there is another way to put a rational structure on H^I_π. Indeed, we may consider the rational structure obtained by applying the operator $\varphi \mapsto \varphi^I$ to rational cohomology classes in H^{hol}_π:

$$\overline{H}^I_\pi(E(I))^{\mathrm{hol}} = \overline{H}^{\mathrm{hol}}_\pi(E(I))^I.$$

It will also be important to know the dimensions of the cohomology groups.

Lemma 3.11. *If $I_1 = \{j \mid k_j = 1\}$, then:*

$$\dim \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi = \binom{|I_1|}{|I \cap I_1|}$$

and it is spanned by the images of the 1-dimensional spaces H_π^J for subsets $J \subseteq \Sigma$ such that $|J| = |I|$ and $\underline{k}(I) = \underline{k}(J)$.

Proof. For $k_j > 1$ for all j, this follows from results of Harris [Har90a] (see, for example, the proof of [Har90b, Lemma 2.4.5]). For general weights, this follows from the main theorem of [Su18] and an analogous $(\mathfrak{P}, K)$-cohomology computation. □

For parallel weight one forms, this specializes to the following.

Corollary 3.12. *If $\underline{k} = \underline{1}$, then*

$$\dim \overline{H}^q(\mathcal{E}_{\underline{k},r})_\pi = \binom{d}{q}$$

for $q = 0, \ldots, d$. Moreover, $\overline{H}^q(\mathcal{E}_{\underline{k},r})_\pi$ is spanned by

$$\{\omega_{\varphi^J} \mid J \subseteq \Sigma,\ \#J = q\}.$$

It will also be important to know when the cohomology groups are 1-dimensional.

Corollary 3.13. *For any $I \subseteq \Sigma$, $\dim \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi > 1$ if and only if both I and $\Sigma \setminus I$ contain a place such that f has weight 1 at that place.*

Proof. For the 'if' implication, take $\sigma \in I \cap I_1$ and $\sigma' \in (\Sigma \setminus I) \cap I_1$, and define

$$I' = (I \setminus \{\sigma\}) \cup \{\sigma'\}.$$

Then $\#I' = \#I$ and $\underline{k}(I') = \underline{k}(I)$, so both H_π^I and $H_\pi^{I'}$ inject into $\overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})$ by Lemma 3.10 and are linearly independent by Lemma 3.11.

Conversely, suppose $\dim \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi > 1$. Then there exists $I' \neq I$ such that $H_\pi^{I'} \hookrightarrow \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi$, i.e. $\#I' = \#I$ and $(I \cup I') \setminus (I \cap I') \subseteq I_1$. Then $\sigma \in I \setminus I'$ belongs to $I \cap I_1$ and $\sigma' \in I' \setminus I$ belongs to $(\Sigma \setminus I) \cap I_1$. □

The following lemma defines Harris' period invariant.

Lemma 3.14 ([Har90b, Lemma 1.4.5])**.** *Let I be a set of places which contains either all of the weight one places of f or none of the weight one places of f. Then there is a number $\nu^I(\pi) \in \mathbb{C}^\times$, well-defined up to multiplication by elements in $E(I)^\times$, such that*

$$\nu^I(\pi) \cdot \overline{H}_\pi^I(E(I)) = \overline{H}_\pi^I(E(I))^{\mathrm{hol}}.$$

Clearly, when $I = \emptyset$, we may take $\nu^I(\pi) = 1$.

Proof. We only give a sketch of the proof. Under the first assumption, $\dim \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi = 1$ by Corollary 3.13. We can get two *primitive* vectors in H_π^I:

(1) by considering an E-rational primitive vector in H_π^{hol} and applying $(\bullet)^I$ to it (this one is in $H_\pi^I(E(I))^{\mathrm{hol}}$),
(2) by considering $H_\pi^I(E(I))$ as an $G(\mathbb{A}_{\mathrm{fin}})$-module with coefficients in $E(I)$ (c.f. [Del73, pp. 82–83]).

Since both of these are primitive vectors in H_π^I, they must differ by a constant in $\mathbb{C}^\times$. $\square$

Definition 3.15. Let π be an automorphic representation associated to a Hilbert modular form f and I be a set of infinite places such that either:

- I contains all the weight one places of f,
- I contains no weight one places of f.

Then the complex number $\nu^I(\pi)$ defined by Lemma 3.14 is the *period* or *period invariant* associated to π and I. It is well-defined up to $E(I)^\times$.

Shimura defines periods by considering Petersson inner products on Shimura varieties associated to quaternion algebras over F. Harris' definition is much less explicit, but it is related to Petersson inner products as follows.

Proposition 3.16 ([Har90b, Prop. 1.5.6])**.** *For a suitably defined cup product pairing (which is a Tate twist of the Serre duality pairing):*

$$\cup\colon \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r}) \otimes \overline{H}^{|\Sigma\setminus I|}(\mathcal{E}_{\underline{k}(\Sigma\setminus I),r}) \to \mathbb{C},$$

we can define the normalized Petersson inner product as $\langle\varphi,\psi\rangle = \varphi \cup \overline{\psi}$ *for* $\varphi,\psi \in H_\pi^{\mathrm{hol}}$*. Then for a form* $0 \neq \varphi \in H^0(\mathcal{M},\mathcal{E}_{\underline{k},r})_\pi(E)$ *and any* $I \subseteq \Sigma$ *for which* $\nu^I(\pi)$ *is defined, we have that:*

$$\nu^I(\pi)\nu^{\Sigma\setminus I}(\pi^c) \sim_{E(I)^\times} \langle\varphi,\varphi\rangle,$$

where π^c *denotes the contragredient representation. In particular, we may take* $\nu^\Sigma(\pi) = \langle\varphi,\varphi\rangle$*.*

Therefore, we may think of $\nu^I(\pi)$ as a certain *factor* of the Petersson inner product $\langle\varphi,\varphi\rangle$.

Remark 3.17. The proof in loc. cit. is based on the rationality of the Serre duality pairing [Har90b, (1.5.4)]:

$$\overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_\pi(E(I)) \otimes \overline{H}^{|\Sigma\setminus I|}(\mathcal{E}_{\underline{k}(\Sigma\setminus I),r})_{\pi^c}(E(I)) \to E(I) \tag{3.5}$$

and the identity

$$\varphi^I \cup (\varphi^c)^{\Sigma\setminus I} = \langle\varphi,\varphi\rangle \tag{3.6}$$

for any $\varphi \in H_\pi^{\mathrm{hol}}$ [Har90b, (1.5.5.2)] where $\varphi \mapsto \varphi^c$ is Shimura's complex conjugation map.

We note that $\pi \cong \pi^c$ if π is unitary but we are also interested in cases where π is not unitary. For example, when f has weight one, it has a non-trivial character, and hence π is not unitary.

Remark 3.18. In this extended remark, we discuss the relation of Harris' periods to other periods attached to Hilbert modular forms. The study of period invariants was initiated by Shimura [Shi83, Shi88], who studied the case when the weights at all places are at least two. In this case, Shimura conjectured the existence of a set of period invariants c_σ, one attached to each infinite place σ of F; moreover, he conjectured that if B is any quaternion algebra over F such that f transfers to a form f_B on $B^\times$, then the Petersson norm of f_B (if f_B is chosen to be algebraic) is essentially a product of some of the c_σ up to algebraic factors. More precisely, defining

$$q_B(\pi) := \langle f_B, f_B\rangle,$$

Shimura conjectured that

$$q_B(\pi) \sim_{\overline{\mathbb{Q}}^\times} \prod_{\sigma \in \Sigma_{B,\infty}} c_\sigma, \tag{3.7}$$

where $\Sigma_{B,\infty}$ is the set of infinite places where B is split. This conjecture was proved by Harris [Har93], using the theta correspondence for unitary groups. In this work, the periods c_σ are essentially *defined* as suitable ratios of periods on quaternion algebras. The fact that the definition of the periods does not depend on choices of quaternion algebras boils down to proving relations between periods on different quaternion algebras, which provides the main thread of Harris' argument. This work admits an integral refinement which is studied in the ongoing work of Ichino–Prasanna (for example, [IP16]).

In related work [Har90b, Har94], Harris gave another definition of such period invariants using rational structures on coherent cohomology. This is what was recalled in Definition 3.15. The advantage of this definition is that it does not require working with quaternion algebras; rather everything happens on the Hilbert modular variety attached to the group $\mathrm{GL}_{2,F}$. This also makes it easy to see the relations between these periods and the transcendental factors of Rankin–Selberg and triple product L-functions attached to two (respectively, three) Hilbert modular forms.

The point of our work is to define periods attached to parallel weight one forms, and relate them to rational structures on coherent cohomology. For dimension reasons, one cannot simply use these rational structures directly to define periods. Indeed, the proof of Lemma 3.14 relies on higher cohomology groups being 1-dimensional whereas the dimensions are greater than 1 for weight one forms (c.f. Corollary 3.12). Instead, we give an ad hoc definition using logarithms of units, and conjecture (Conjecture 3.21) a relationship to rational structures on coherent cohomology, similar to Lemma 3.14.

It would be interesting to define periods in other cases where Harris' periods are not available (for example, define periods at weight one places when the Hilbert modular form has partial weight one).

We end this section by describing the non-holomorphic automorphic forms φ^I and the associated differential forms ω_{φ^I} in the classical language. We assume the class number of F is 1 for simplicity.

For $\tau = (\tau_1, \dots, \tau_d) \in (\mathrm{GL}_2(\mathbb{R})^+)^d$, if $\tau_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}$, there is a natural action of τ on $\mathcal{H}^d$ given by

$$\tau(z_1, \dots, z_d) = \left(\frac{\alpha_1 z_1 + \beta_1}{\gamma_1 z_1 + \delta_1}, \dots, \frac{\alpha_n z_n + \beta_n}{\gamma_n z_n + \delta_n} \right).$$

Given a matrix $\tau = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$, put

$$j(\tau, z) = (\gamma z + \delta)(\det \tau)^{-1/2}.$$

Given a vector $\underline{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and $\tau \in \mathrm{GL}_2(F)^+$, put:

$$j_{\underline{k}}(\tau, \underline{z}) = \prod_{i=1}^{d} j(\sigma_i(\tau), z_i)^{k_i}.$$

Finally, for $f \colon \mathcal{H}^d \to \mathbb{C}$, define the *slash operator* to be

$$(f|_{\underline{k}}\tau)(\underline{z}) = j_{\underline{k}}(\tau, \underline{z})^{-1} f(\tau \underline{z}).$$

We sometimes write $f|[\tau]_{\underline{k}}$ for $f|_{\underline{k}}\tau$ later.

Definition 3.19. A *(classical) Hilbert modular form* of weight $(\underline{k}, r)$ where $\underline{k} = (k_1, \dots, k_d) \in \mathbb{Z}^d$, level $\mathfrak{N}$, character ω such that

$$\omega(\xi) = \mathrm{sgn}(\xi)^r \text{ for } \xi \equiv 1 \mod^{\times} \mathfrak{N}$$

is a holomorphic function $f \colon \mathcal{H}^d \to \mathbb{C}$ such that

$$(f|_{\underline{k}}\tau)(\underline{z}) = \omega(a) f(\underline{z}) \quad \text{for all } \tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N}).$$

The translation of the automorphic Definition 3.4 of a Hilbert modular form to the classical one is standard (see, for example, [Gar90]), so we omit it here.

We finally interpret Harris' operators classically. For any I, we assume that there exists a fundamental unit ϵ_I such that

$$\begin{cases} \sigma(\epsilon_I) > 0 & \text{if } \sigma \notin I, \\ \sigma(\epsilon_I) < 0 & \text{if } \sigma \in I. \end{cases}$$

When $d = 2$, this amounts to the standard assumption (e.g., [Oda82]) that $\mathcal{O}_F$ has a fundamental unit of negative norm.

Then the non-holomorphic automorphic form φ^I corresponds to

$$f^I(\underline{z}) = f(\underline{z}^I) \cdot \prod_{j \in I} \mathrm{Im}(z_j)^{k_j}$$

where

$$(\underline{z}^I)_j = \begin{cases} (\epsilon_I)_j z_j & \text{if } \sigma_j \notin I, \\ (\epsilon_I)_j \overline{z}_j & \text{if } \sigma_j \in I. \end{cases}$$

We then define:

$$\omega_f^I = \left[f^I(\underline{z}) \cdot \bigwedge_{j \in I} \frac{dz_j \wedge d\overline{z_j}}{y_j^2} \right] \in H^{0,|I|}(\Gamma \backslash \mathcal{H}^d, \mathcal{E}_{\underline{k}(I),r}) = H^{|I|}(\Gamma \backslash \mathcal{H}^d, \mathcal{E}_{\underline{k}(I),r}). \tag{3.8}$$

If f is a cusp form, this cohomology class extends to the toroidal compactification and defines a class in $\overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})$ as before.

Remark 3.20. For classical forms, we use Shimura's normalization of Petersson inner products [Shi78, (2.27, 2.28)]. This is also the normalization in [Hid91], [HT93], which we refer to later. According to [Har90b, (1.6.3)], the normalization differs from Harris' normalization by a factor of $(2\pi i)^{-dr}$, i.e.

$$\langle \varphi, \varphi \rangle \sim_{\mathbb{Q}^\times} (2\pi i)^{-dr} \langle f, f \rangle. \tag{3.9}$$

3.3. **The conjectures.** Recall that Proposition 3.2 provided an identification:

$$U_f^\vee \otimes_\iota \mathbb{C} \cong \bigoplus_{\sigma \in G/G'} (\mathrm{Ad}^0 M_0 \otimes_\iota \mathbb{C})^{\sigma c_0 \sigma^{-1}} \tag{3.10}$$

and we sometimes write $U_{f,\sigma}^\vee = (\mathrm{Ad}^0 M_0)^{\sigma c_0 \sigma^{-1}}$ for simplicity.

Also, by Corollary 1.9 given a basis of $\mathrm{Ad}^0 M_0 \otimes_\iota \mathbb{C}$, there is an associated basis:

$$\{\varphi_\sigma \mid \sigma \in G/G'\}$$

of $U_f \otimes_\iota \mathbb{C}$. For each $m_\sigma \in (\mathrm{Ad}^0 M \otimes_\iota \mathbb{C})^{\sigma c_0 \sigma^{-1}}$, we complete it to a basis of $\mathrm{Ad}^0 M \otimes_\iota \mathbb{C}$ and define the unit

$$u_{f,\sigma} = \varphi_\sigma(m_\sigma) \in U_L \otimes (\mathcal{O}_{E_f} \otimes_\iota \mathbb{C}). \tag{3.11}$$

We can consider its logarithm

$$\log(|\tau \otimes \iota(u_{f,\sigma})|) \in \mathbb{C}, \tag{3.12}$$

as explained in equation (3.1). We sometimes make the choices of embeddings implicit and write simply $u_{f,\sigma}$ or $\iota(u_{f,\sigma})$ to mean $(\tau \otimes \iota)(u_{f,\sigma})$.

Conjecture 3.21. *Let $F/\mathbb{Q}$ be a totally real field of degree d. Let ϱ be an Artin representation $G_{L/F} \to \mathrm{GL}_2(\mathcal{O}_{E_f})$ corresponding to a normalized Hilbert modular eigenform f of parallel weight one with coefficient in $\mathcal{O}_{E_f}$.*

Consider a subset $I \subseteq \Sigma_F \cong G/G'$. For each $\sigma \in I$, choose $m_\sigma \in (\mathrm{Ad}^0 M_0)^{\sigma c_0 \sigma^{-1}}$ and consider the unit $u_{f,\sigma}$ defined above. Then the Dolbeault class

$$(2\pi i)^{|I|} \frac{\omega_f^I}{\prod_{\sigma \in I} \log(|\tau \otimes \iota(u_{f,\sigma})|)} \in H^{|I|}(X_{\mathbb{C}}^{\mathrm{an}}, \mathcal{E}_{\underline{1},1}) \cong H^{|I|}(X_{\mathbb{C}}, \mathcal{E}_{\underline{1},1})$$

defines a rational cohomology class, i.e. belongs to $H^{|I|}(X_{\mathbb{Q}}, \mathcal{E}_{\underline{1},1}) \otimes_{\mathbb{Q}} E_f$.

Note that our conjecture gives an explicit definition of period invariants $\nu^I(\pi)$ as a product of the logarithms of units associated to π for sets of places I where Harris' periods are not defined (see Remark 3.18).

Definition 3.22. Let f be a Hilbert modular form of parallel weight one and π be the corresponding automorphic representation. The *period invariant* associated to π and $I \subseteq \Sigma$ is:

$$\nu^I(\pi) = \prod_{\sigma \in I} \frac{\log(\tau \otimes \iota(u_{f,\sigma}))}{2\pi i}.$$

As in Definition 3.15, these are only well-defined up to $E_f^\times$, because they depend on the choices of m_σ as above.

Remark 3.23. Since $\dim \overline{H}^q(\mathcal{E}_{\underline{1},1})_\pi = \binom{d}{q}$, it is not clear that any multiple of ω_f^I is a rational cohomology class. When $d = 2$ and $q = 1$, we prove this in Corollary 4.20. More generally, when $d = 2n$ and $q = n$, we provide some evidence that suggests that this is true. We also mention that whenever the representation π is a base change of a representation of $\mathrm{GL}_{2,\mathbb{Q}}$, the presence of the extra automorphisms of F over $\mathbb{Q}$ implies that multiples of ω_f^I are rational.

Remark 3.24. Note that the periods $\nu^I(\pi)$ automatically satisfy the analog of Shimura's factorization (3.7). We will see (Theorem 4.6) that Stark's Conjecture 4.2 implies that

$$\langle \varphi, \varphi \rangle \sim \nu^\Sigma(\pi),$$

as in Proposition 3.16 for Harris' periods.

Furthermore, clearly (3.1) implies that:

$$\log(|\tau \otimes \iota(u_{f,\sigma})|) = \overline{\log(|\tau \otimes \overline{\iota}(u_{f,\sigma})|)}. \tag{3.13}$$

Hence for any $I \subseteq \Sigma$:

$$\nu^I(\pi) \cdot \nu^{\Sigma \setminus I}(\pi^c) \sim_{E_f^\times} \langle \varphi, \varphi \rangle. \tag{3.14}$$

Therefore, the periods are compatible with Serre duality (c.f. Proposition 3.16).

We will now state an archimidean analog of Conjecture 2.7, implied by Conjecture 3.21. Roughly, it will say that the exterior algebra $\bigwedge U_f^\vee$ acts on the cohomology space $H^*(X_\mathbb{Q}, \mathcal{E}_{\underline{1},1})_f$ such that the induced action of $\bigwedge U_f^\vee \otimes \mathbb{C}$ on $H^*(X_\mathbb{C}, \mathcal{E}_{\underline{1},1})_f$ is given by partial complex conjugation divided by periods corresponding to the units. We define this action next.

Note that for each Dolbeault cohomology class $\omega \in H^q(X_\mathbb{C}, \mathcal{E}_{\underline{1},1})_\pi$, we may use the same process as above to define $\omega^I \in H^{q+|I|}(X_\mathbb{C}, \mathcal{E}_{\underline{1},1})_\pi$ for each $I \subseteq \Sigma$. Indeed, by Corollary 3.12, we may write $\omega = \sum_{\#J=q} \lambda_J \omega_f^J$ for some $\lambda_J \in \mathbb{C}$ and we define

$$\omega^I = \sum_{\substack{\#J=q \\ I\cap J=\emptyset}} \lambda_J \omega_f^{I\cap J}. \tag{3.15}$$

The action of a wedge of $|I|$ dual units is then defined this way after dividing by a product of logarithms of units as in Conjecture 3.21.

Specifically, consider a subset $I \subseteq \Sigma$ and label its elements $I = \{\sigma_1, \dots, \sigma_{|I|}\}$. For elements $m_j \in U_{f,\sigma_j}^\vee$ for $j = 1, \dots, |I|$, consider

$$m_1 \wedge \cdots \wedge m_{|I|} \in \bigwedge^{|I|} U_f^\vee \otimes \mathbb{C}$$

via identification (3.10). For $\omega \in H^q(X_\mathbb{C}, \mathcal{E}_{\underline{1},1})_\pi$, we define the action by:

$$(m_1 \wedge \cdots \wedge m_{|I|}) \star \omega = \pm(2\pi i)^{|I|} \frac{\omega^I}{\prod_{j=1}^{|I|} \log(|(\tau \otimes \iota)(\varphi_{\sigma_j}(m_j))|)} \in H^{q+|I|}(X_\mathbb{C}, \mathcal{E}_{\underline{1},1})_\pi \tag{3.16}$$

where φ_σ corresponds to m_σ as above. In terms of the periods $\nu^I(\pi)$ from Definition 3.22, we can write more concisely:

$$(m_1 \wedge \cdots \wedge m_{|I|}) \star \omega = \pm \frac{\omega^I}{\nu^I(\pi)}. \tag{3.17}$$

The sign $\pm$ is fixed once we order the elements of Σ.

Conjecture 3.25. *There is an action $\star$ of the exterior algebra $\bigwedge(U_f^\vee) \otimes E_f$ on $H^*(X_\mathbb{Q}, \mathcal{E}_{\underline{1},1})_\pi$ such that for any embeddings $\iota\colon E_f \hookrightarrow \mathbb{C}$ and $\tau\colon L \hookrightarrow \mathbb{C}$ the induced action of $\bigwedge(U_f^\vee) \otimes_\iota \mathbb{C}$ on $H^*(X, \mathcal{E}_{\underline{1},1})_\pi \otimes_\iota \mathbb{C}$ is given by equation (3.16).*

Of course, Conjecture 3.21 implies Conjecture 3.25. Also, note that Corollary 3.12 shows that $H^*(X_\mathbb{C}, \mathcal{E}_{(\underline{1},1)})_f$ is generated by $f \in H^0(X_\mathbb{Q}, \omega)_f$ over $\bigwedge U_f^\vee \otimes \mathbb{C}$.

Remark 3.26. We point out a few logical consistencies in the conjecture.

(1) If m_1 and m_2 both belong to the same 1-dimensional space $U_{f,\sigma}^\vee$, $m_1 \wedge m_2 = 0$. This is consistent with the fact that

$$(\omega^I)^J = 0 \quad \text{if } I \cap J \neq \emptyset,$$

because then

$$\bigwedge_{j\in I} \frac{dz_j \wedge \overline{dz_j}}{y_j^2} \wedge \bigwedge_{j \in J} \frac{dz_j \wedge \overline{dz_j}}{y_j^2} = 0.$$

(See equation (3.15) for the definition of ω^I.)

(2) For $\lambda \in \mathbb{C}$, consider

$$\lambda(m_1 \wedge \cdots \wedge m_\alpha) = (\lambda m_1) \wedge m_2 \wedge \cdots \wedge m_\alpha.$$

Replacing m_1 by λm_1 changes the corresponding map φ_{σ_1} to $\frac{1}{\lambda}\varphi_{\sigma_1}$. It acts by

$$(\lambda m_1 \wedge \cdots \wedge m_\alpha) \star \omega = (2\pi i)^\alpha \frac{\omega^I}{\frac{1}{\lambda}\prod_{j=1}^{\alpha} \log(\iota(u_j))} = \lambda \cdot (2\pi i)^\alpha \frac{\omega^I}{\prod_{j=1}^{\alpha} \log(\iota(u_j))},$$

where $u_j = \varphi_{\sigma_j}(m_j)$. The same holds for any m_j in place of m_1. Hence the action is linear.

(3) By Corollary 3.12 each space $\overline{H}^q(X_\mathbb{C}, \mathcal{E}_{\underline{1},1})_\pi$ has a basis given by

$$\{\omega_f^J/\nu^J(\pi) \mid \#J = q\}.$$

According to Conjecture 3.21, this is a basis of $\overline{H}^q(X, \mathcal{E}_{\underline{1},1})$. Therefore, any element ω in $H^q(X, \mathcal{E}_{\underline{1},1})$ can be written as

$$\omega = \sum_{\#J=q} \lambda_J(\omega_f^J/\nu^J(\pi))$$

for $\lambda_J \in E_f$. As in (1), $(\omega_f^J)^I = 0$ if $I \cap J \neq \emptyset$, so (equation (3.15))

$$\omega^I = \sum_{\substack{\#J=q \\ I\cap J=\emptyset}} \lambda_J(\omega_f^{(I\cup J)}/\nu^J(\pi)),$$

showing that

$$\frac{\omega^I}{\nu^I(\pi)} = \sum_{\substack{\#J=q \\ I\cap J=\emptyset}} \lambda_J(\omega_f^{(I\cup J)}/\nu^J(\pi)\nu^I(\pi)) = \sum_{\substack{\#J=q \\ I\cap J=\emptyset}} \lambda_J(\omega_f^{(I\cup J)}/\nu^{I\cup J}(\pi)).$$

Both sides are predicted to be rational in cohomology according to the conjecture.

Remark 3.27. Even in the case $F=\mathbb{Q}$, considered in [HV17], as far as we know, this conjecture is new. As we will see (Corollary 4.7), in that case it follows from Stark's Conjecture 4.2, and hence is true when $E_f=\mathbb{Q}$.

The goal of the next two sections is to provide evidence for Conjecture 3.21. The simplest case to consider is $\Sigma = I$. Then we know by Proposition 3.16 that Harris' period $\nu^\Sigma(\pi)$ exists and satisfies

$$\nu^\Sigma(\pi) = \langle\varphi,\varphi\rangle.$$

We need to show that:

$$\langle\varphi,\varphi\rangle \sim_{E_f^\times} \frac{1}{(2\pi i)^d}\prod_{\sigma\in G/G'} \log(\iota(u_{f,\sigma})|). \tag{3.18}$$

Since the left hand side is related to special values of the adjoint L-function, this assertion will follow from Stark's Conjecture 4.2. We discuss this in Section 4.

The next simplest case is $d=2$ and $I=\{\sigma\}$. Here, we would like to establish rationality of a cohomology class

$$\frac{\omega_f^{\sigma_1}}{\nu^{\sigma_1}(\pi)} \in H^1(X_\mathbb{C},\mathcal{E}_{\underline{1},1})_\pi,$$

where $X_\mathbb{C}$ is the Hilbert modular surface. This seems quite difficult in general and there seem to be few tools available to approach this problem. Our strategy is to consider an embedded modular curve $\iota\colon Y_\mathbb{Q}\hookrightarrow X_\mathbb{Q}$ and show that

$$\frac{\iota^*(\omega_f^{\sigma_1})}{\nu^{\sigma_1}(\pi)} \in H^1(Y_\mathbb{Q},\mathcal{E}_2)\otimes E \subseteq H^1(Y_\mathbb{C},\mathcal{E}_2).$$

We give some theoretical evidence for the veracity of this statement and verify this numerically in many cases. However, $\iota^*(\omega_f^{\sigma_1})=0$ if f is not a *base change form*, so we can only obtain evidence for base change forms this way. Therefore, Section 5 is devoted to base change forms.

4. Evidence: Stark's results

The main piece of evidence for Conjecture 3.21 are the results of Stark [Sta75] and Tate [Tat84] on special values of Artin L-functions. By providing an explicit expression for Stark's regulator (Proposition 4.12), we can show (Theorem 4.6) that

$$\langle f,f\rangle \sim_{E_f^\times} \prod_{\sigma\in G/G'} \log(|\iota(u_{f,\sigma})|), \tag{4.1}$$

conditional upon Stark's Conjecture 4.2, up to a potential factor of $\sqrt{2}$. When $E_f=\mathbb{Q}$, Stark's Conjecture is known (Theorem 4.4) and hence the result is unconditional.

Note that this is equivalent to equation (3.18), since we are using Shimura's normalization of the Petersson inner product of classical forms (c.f. Remark 3.20).

We start by recalling Stark's Conjecture and results, and then specialize them to our case. Dasgupta's senior thesis [Das99] gives an excellent survey of Stark's Conjecture.

4.1. **Stark's Conjecture** [Sta75, Tat84]**.** We give a summary of the results and conjectures on special values of Artin L-functions.

For any Artin representation $\varrho\colon G_{L/\mathbb{Q}} \to \mathrm{GL}(M)$ where M is an n-dimensional E-vector space and an embedding $E \hookrightarrow \mathbb{C}$, we consider the L-function $L(s,\varrho)$ of ϱ. If we need to make the embedding $\iota\colon E \hookrightarrow \mathbb{C}$ explicit, we write $L(s,\varrho,\iota)$ for $L(s,\varrho)$.

The completed L-function is then:

$$\Lambda(s,\varrho) = \left(\frac{f_\varrho}{\pi^n}\right)^{s/2} \Gamma(s/2)^a \Gamma((s+1)/2)^b L(s,\varrho) \tag{4.2}$$

where:

$$f_\varrho = \text{Artin conductor of } \varrho, \tag{4.3}$$

$$a = \frac{1}{2}(\mathrm{Tr}\varrho(1) + \mathrm{Tr}\varrho(c_0)), \tag{4.4}$$

$$b = \frac{1}{2}(\mathrm{Tr}\varrho(1) - \mathrm{Tr}\varrho(c_0)) = n - a. \tag{4.5}$$

In other words, $\varrho(c_0) \sim \begin{pmatrix} I_a & 0 \\ 0 & -I_b \end{pmatrix}$ for any complex conjugation c_0 of L.

It satisfies a functional equation of the form:

$$\Lambda(1-s,\overline{\varrho}) = W(\varrho)\Lambda(s,\varrho)$$

where $|W(\varrho)| = 1$.

Stark gives a formula for the special value of L at $s=1$ (or, equivalently, the residue of the pole at $s=0$). In order to state the theorem, we need to introduce a few quantities.

Definition 4.1. Recall that Proposition 1.5 defined units

$$u_{ij} = \prod_{\sigma \in G} (\epsilon^{\sigma^{-1}})^{a_{ij}(\sigma)} \in U_L \otimes E.$$

The *Stark regulator* associated to ϱ (and the embeddings $\tau\colon L \hookrightarrow \mathbb{C}$ and $\iota\colon E \hookrightarrow \mathbb{C}$) is

$$R(\varrho) = \det\left((|\log(\tau \otimes \iota(u_{ij}))|)_{1 \le i,j \le a}\right).$$

Abstractly, there is a perfect pairing

$$\begin{aligned} U_L[\varrho] \times M^{c_0} &\to \mathbb{C} \\ (\varphi, m) &\mapsto \log(|\tau \otimes \iota(\varphi(m))|) \end{aligned}$$

defined in Lemma 3.1 and $R(\varrho)$ is the determinant of this pairing.

Conjecture 4.2 (Stark, [Sta75, Tat84]). *If ϱ does not contain the trivial representation, then*

$$L(1,\varrho) = \frac{W(\overline{\varrho})2^a\pi^b}{f_\varrho^{1/2}} \cdot \theta(\overline{\varrho}) \cdot R(\overline{\varrho}),$$

for some $\theta(\overline{\varrho}) \in E^\times$.

Remark 4.3. The assumption that ϱ does not contain the trivial representation is completely innocuous. Indeed, $L(s,\chi_{1,L}) = \zeta_L(s)$, so the value at $s=1$ is given by the class number formula for L. Moreover, $L(s,\varrho_1 \oplus \varrho_2) = L(s,\varrho_1) \cdot L(s,\varrho_2)$.

Theorem 4.4 (Stark, [Sta75, Theorem 1]). *Conjecture 4.2 is true for rational representations ϱ (i.e., when $E = \mathbb{Q}$), up to an algebraic number $\theta(\varrho)$. This number is explicitly given by*

$$\theta(\varrho) = \prod_{P \subseteq L} \left[\frac{-h(P)e(P)}{w(P)n(P)i(P,\epsilon)}\right]^{b(P)}$$

where we decompose ([Tat84, Theorem II.1.2]*)*

$$\mathrm{Tr}\varrho = \sum_{P \subseteq L} b(P)\chi_1(G_{K/P})^*$$

for rational numbers $b(P)$, where $\chi_1(H)$ is the trivial character of H, and

$$\begin{aligned} h(P) &= \textit{class number of } P, \\ e(P) &= \begin{cases} 1 & \textit{if } P \textit{ is totally real}, \\ 2 & \textit{otherwise}, \end{cases} \\ w(P) &= \textit{number of roots of unity in } P, \\ n(P) &= [P:\mathbb{Q}], \\ i(P,\epsilon) &= \frac{R(P,\{\epsilon_i\})}{R(P)}. \end{aligned}$$

Theorem 4.5 (Tate, [Tat84, Corollary II.7.4]). *Under the assumptions of Stark's Theorem 4.4, $\theta(\varrho) \in \mathbb{Q}^\times$.*

4.2. **Consequences of Stark's results.** We show that Stark's Conjecture 4.2 implies the following consequence of Conjecture 3.21. In particular, Theorem 4.5 implies this consequence when f has rational Fourier coefficients.

Recall that for each $\sigma \in \Sigma$, we defined a unit $u_{f,\sigma} \in U_L \otimes E_f$ as $\varphi_\sigma(m_\sigma)$ where $m_\sigma \in (\mathrm{Ad}^0\,\varrho)^{\langle \sigma c_0 \sigma^{-1}\rangle}$ and $\varphi_\sigma \in U_f \otimes E_f$ is defined by Proposition 1.5.

Theorem 4.6. *Stark's Conjecture 4.2 implies that for a parallel weight one Hilbert modular form:*

$$\langle f,f\rangle \sim_{E_f^\times} f_{\varrho,2}^{1/2} \prod_{\sigma \in G/G'} \log|\tau \otimes \iota(u_{f,\sigma})|,$$

where $f_{\varrho,2} = 2^{a(\varrho,2)}$ is the Artin conductor at $p=2$ of the trace 0 adjoint representation. In particular, this is true unconditionally (and constant in $E_f^\times$ is explicit) if f has rational Fourier coefficients.

Corollary 4.7. *Conjecture 3.21 is implied by Stark's Conjecture 4.2 when $F=\mathbb{Q}$, up to a possible factor of $\sqrt{2}$. This is hence true unconditionally when $F=\mathbb{Q}$ and $E_f=\mathbb{Q}$.*

Remark 4.8. We expect that the factor $f_{\varrho,2}^{1/2}$ is rational; see the extended Remark 4.15 for more details.

Remark 4.9. We checked computationally (using the methods of Collins [Col18], based on [Nel15]) that for a few modular forms f of weight 1 from Example 1.13, we have that $\langle f,f\rangle = 3\log(|\iota(u)|)$. This was already observed by Stark [Sta75, pp. 91].

The proof of Theorem 4.6 requires 3 steps:

(1) relating $L(1,\mathrm{Ad}^0\varrho_f)$ to $\langle f,f\rangle$,
(2) computing the Stark regulator in the Hilbert modular case,
(3) showing that f_ϱ is a square when $\varrho=\mathrm{Ind}_{G_F}^{G_\mathbb{Q}}\mathrm{Ad}^0(\varrho_f)$, so that $f_\varrho^{1/2}\in\mathbb{Q}^\times$.

The relation of the adjoint L-value to the Petersson inner product was first observed by Hida, based on the work of Shimura [Shi76]. He also related the prime factors of the quotient $\frac{L(1,\mathrm{Ad}(f))}{\langle f,f\rangle}$ to congruence primes of the modular form f [Hid81, Hid81, Hid88]. This work was later generalized to Hilbert modular forms [Hid91, HT93, Gha02]. An integral refinement of Conjecture 3.21 would hence have to account for congruence primes.

For an automorphic proof relating $L(1,\mathrm{Ad}(f))$ to $\langle f,f\rangle$, see [IP16, Prop. 6.6].

Theorem 4.10 ([HT93, Theorem 7.1])**.** *Let f is a primitive Hilbert modular form of weight $(\underline{k},r)$, level $\mathfrak{N}$. Then*

$$\langle f,f\rangle = |D_F|^{m-1}\Gamma_F(k)N_{F/\mathbb{Q}}(\mathfrak{N})2^{-2\{\underline{k}\}+1}\pi^{-d-\{k\}}L_S(1,\mathrm{Ad}(f)),$$

where

$$L_S(1,\mathrm{Ad}(f)) = \prod_{\mathfrak{q}\in S} L_\mathfrak{q}(N_{F/\mathbb{Q}}(\mathfrak{q})^{-s})L(s,\mathrm{Ad}(f)),$$

S is a set of bad places, $L_\mathfrak{q}(N_{F/\mathbb{Q}}(\mathfrak{q})^{-s})$ are bad local factors, $\{k\}=\sum_j k_j$, and m is an explicit integer which accounts for Hida's unitarization [Hid91, (4.2a), (7.1)].

For parallel weight one Hilbert modular forms, this specializes to the following result we will use.

Corollary 4.11. *Suppose $(\underline{k},r)=(\underline{1},1)$. Then:*

$$\langle f,f\rangle \sim_{E_f^\times} \pi^{-2d}L(1,\mathrm{Ad}(f)).$$

We now proceed to step (2) above.

As in Section 1, let $\mathrm{Ad}^0\varrho_f$ be the 3-dimensional trace 0 adjoint representation of $G_{L/F}$ associated to the Artin representation of f:

$$\varrho_f\colon G_{L/F}\to \mathrm{GL}_2(\mathcal{O}_{E_f}).$$

Consider the $(3d)$-dimensional Artin representation

$$\varrho=\mathrm{Ind}_G^{G'}\mathrm{Ad}^0\varrho_f\colon G_{L/\mathbb{Q}}\to\mathrm{GL}_{3d}(\mathcal{O}_{E_f}),$$

where we again write $G = G_{L/\mathbb{Q}} \supseteq G' = G_{L/F}$. Then $a = d$, $b = 2d$.

The following result provides an explicit expression for the Stark regulator in this case.

Proposition 4.12. *Stark's Conjecture 4.2 implies that*

$$L(1, \mathrm{Ad}^0 \varrho_f) = \frac{W(\varrho)2^d\pi^{2d}}{f_\varrho^{1/2}} \cdot \theta(\overline{\varrho}) \cdot R(\overline{\varrho}),$$

where f_ϱ is the Artin conductor of $\varrho = \mathrm{Ind}_{G_F}^{G_\mathbb{Q}} \mathrm{Ad}^0 \varrho_f$ and

$$R(\overline{\varrho}) = \prod_{\sigma \in G/G'} \log(|\tau \otimes \overline{\iota}(u_{f,\sigma})|).$$

Proof. Since Artin L-functions are inductive, we also know that

$$L(1, \mathrm{Ad}^0 \varrho_f) = L(1, \varrho).$$

We just need to check the assertion about $R(\overline{\varrho})$. Recall (cf. Definition 4.1) that $R(\overline{\varrho})$ is the determinant of the pairing from Lemma 3.1. If the elements φ_σ are associated to m_σ via Proposition 1.5, the matrix of the pairing is diagonal in this basis, with entries $\log(|\tau \otimes \overline{\iota}(u_{f,\sigma})|)$ for $\sigma \in G/G'$. □

To finish the proof of Theorem 4.6, we need to check that f_ϱ is a square (away from $p = 2$).

Proposition 4.13. *Let π_v be the local representation of $\mathrm{GL}_2(F_v)$ associated to f at a finite place v of F. When v lies above 2, assume that π_v is not a theta lifts from a ramified quadratic extension. Then the adjoint conductor of π_v is a square.*

Proof. It is enough to prove that the analytic conductors of the Rankin–Selberg L-functions $L(\pi_v \otimes \pi_v^\vee, s)$ are squares. When π_v is not supercuspidal, Jacquet's results [Jac72] give explicit formulas for the local conductors (see, for example, [Col18, Section 4.2]) and they are visibly squares.

We hence just need to show the conductor is a square at places v where π_v is supercuspidal. Suppose throughout the rest of the proof that F is a finite extension of $\mathbb{Q}_p$ and π is a supercuspidal representation of $\mathrm{GL}(2, F)$. We write $a(-)$ for the valuation of the conductor of a representation and prove that $a(\pi \times \pi^\vee)$ is even.

Since π is supercuspidal, it is a theta lift of a character ξ of a quadratic extension K/F [Gel75, Theorem 7.4]. Then:

$$a(\pi \times \pi^\vee) = 2v_F(d_{K/F}) + f_{K/F} \cdot a(\xi(\xi^\varrho)^{-1}) \tag{4.6}$$

where $d_{K/F}$ is the discriminant of K/F, $f_{K/F}$ is the residue degree of K/F, and ϱ is the non-trivial element of $\mathrm{Gal}(K/F)$. Indeed, if ϱ is the Galois representation corresponding to π via the local Langlands correspondence, then $\varrho = \mathrm{Ind}_K^F(\chi)$ where χ corresponds to ξ via class field theory, and hence

$$\begin{aligned}
a(\pi \times \pi^\vee) &= a(\varrho \otimes \varrho^\vee) \\
&= a(\operatorname{Ind}_K^F \chi \otimes \operatorname{Ind}_K^F \chi^{-1}) \\
&= a(\operatorname{Ind}_K^F \mathbb{1} \oplus \operatorname{Ind}_K^F \chi(\chi^\varrho)^{-1}) \\
&= a(\operatorname{Ind}_K^F \mathbb{1}) + a(\operatorname{Ind}_K^F \chi(\chi^\varrho)^{-1}) \\
&= 2v_F(d_{K/F}) + f_{K/F} \cdot a(\chi(\chi^\varrho)^{-1}) && \text{[Ser79, pp. 101]} \\
&= 2v_F(d_{K/F}) + f_{K/F} \cdot a(\xi(\xi^\varrho)^{-1}).
\end{aligned}$$

When K/F is unramified, $f_{K/F} = 2$, so $a(\pi\times\pi^\vee)$ is even by equation (4.6). Suppose that K/F is ramified and has residue characteristic different than 2. Let $\varpi = \varpi_K$, ϖ_F be uniformizers of K, F, respectively. Then $\varpi_K^\varrho = -\varpi_K$. Also, since $f_{K/F} = 1$, $\mathcal{O}_K/\varpi_K \cong \mathcal{O}_F/\varpi_F$. There is a filtration on the unit group U_K

$$U_K^0 = U_K, \quad U_K^i = 1 + \varpi_K^i \mathcal{O}_K \quad \text{for } i \geq 1$$

with quotients:

$$U_K^0/U_K^1 \cong (\mathcal{O}_K/\varpi_K)^\times, \quad U_K^i/U_K^{i+1} \cong \mathcal{O}_K/\varpi_K. \tag{4.7}$$

We show that if $\xi(\xi^\varrho)^{-1}|_{U_K^i} = \mathbb{1}$ for i odd, then $\xi(\xi^\varrho)^{-1}|_{U_K^{i-1}} = \mathbb{1}$.

For $i = 1$, if $\xi(\xi^\varrho)^{-1}|_{U_K^1} = \mathbb{1}$, then $\xi(\xi^\varrho)^{-1}(x)$ for $x \in U_K$ depends only on the residue class of x (equation (4.7)). We may hence assume $x \in \mathcal{O}_F$ since $\mathcal{O}_K/\varpi_K \cong \mathcal{O}_F/\varpi_F$. Then

$$\xi(\xi^\varrho)^{-1}(x) = \xi(x)\xi(x^\varrho)^{-1} = 1.$$

Similarly, for $i > 1$ odd, if $(\xi(\xi^\varrho)^{-1})|_{U_K^i} = \mathbb{1}$, then $\xi(\xi^\varrho)^{-1}(1+\varpi^{i-1}x)$ for $x \in \mathcal{O}_K$ depends only on the residue class of x (equation (4.7)). We may hence assume $x \in \mathcal{O}_F$ since $\mathcal{O}_K/\varpi_K \cong \mathcal{O}_F/\varpi_F$. Then

$$\xi(\xi^\varrho)^{-1}(1 + \omega_K^{i-1}x) = \xi(1 + \omega_K^{i-1}x)\xi(1 + (-\omega_K)^{i-1}x^\varrho)^{-1} = 1.$$

Therefore, $a(\xi(\xi^\varrho)^{-1})$ is even, which completes the proof. □

Remark 4.14. The strategy in the proof of Proposition 4.13 gives an explicit formula for $a(\pi \times \pi^\vee)$ in terms of $a(\xi)$ when $p \neq 2$. For example, when K/F is ramified:

$$a(\pi \times \pi^\vee) = \begin{cases} a(\xi) + 2 & \text{if } a(\xi) \text{ is even} \\ a(\xi) + 1 & \text{if } a(\xi) \text{ is odd.} \end{cases}$$

A similar result was obtained by Nelson–Pitale–Saha [NPS14, Proposition 2.5] when $F = \mathbb{Q}$ and the central character of π_v is trivial.

It would be interesting to compare these formulas with the ones given in [BHK98], but we have not attempted to do this.

Remark 4.15. In fact, Nelson–Pitale–Saha [NPS14] prove that the adjoint conductor is always a square when $F = \mathbb{Q}$ and f has trivial Nebentypus. We expect that the adjoint conductor is a square also in this more general setting. However, proving this would require a careful analysis of dyadic representations [BH06, Chapter 12] and we decided not to pursue it here.

We are finally ready to prove Theorem 4.6.

Proof of Theorem 4.6. By construction of ϱ_f [RT83],

$$L(1, \mathrm{Ad}^0 \varrho_f) = L(1, \mathrm{Ad}(f)).$$

Then, by Corollary 4.11, we have that

$$\langle f, f\rangle = \langle f^c, f^c\rangle \sim_{E_f^\times} \pi^{-2d} L(1, \mathrm{Ad}(f), \overline{\iota}) = \pi^{-2d} L(1, \mathrm{Ad}^0 \varrho_f, \overline{\iota}).$$

By Proposition 4.12,

$$L(1, \mathrm{Ad}^0 \varrho_f, \overline{\iota}) = \frac{W(\varrho) 2^d \pi^{2d}}{f_\varrho^{1/2}} \cdot \theta(\varrho) \cdot \prod_{\sigma\in\Sigma} \log(|\tau \otimes \iota(u_{f,\sigma})|)$$

for some $\theta(\varrho) \in E_f^\times$. Putting these together and noting that $W(\varrho) = \pm 1$ and f_ϱ is a square away from $p = 2$ (Proposition 4.13) gives the result. □

Remark 4.16. Recall that (Remark 3.24):

$$\prod_{\sigma\in\Sigma} \log(|\tau \otimes \iota(u_{f,\sigma})|) = \prod_{\sigma\in\Sigma} \overline{\log(|\tau \otimes \overline{\iota}(u_{f,\sigma})|)}$$

which agree with the identity $\langle f, f\rangle = \langle f^c, f^c\rangle$ via Theorem 4.6.

4.3. **Archimedean splitting respects rational structure.** Note that Conjecture 3.21 implies that some multiple of $\omega_f^I \in H^{|I|}(X_{\mathbb{C}}, \mathcal{E}_{\underline{1},1})$ is rational. In other words, this would mean that the splitting coming from archimedean representation theory respects the rational structure on coherent cohomology groups. Since the π-isotypic components $H^i(X_{\mathbb{Q}}, \mathcal{E}_{\underline{1},1})_\pi$ are $\binom{d}{i}$-dimensional, this is not automatic.

We give some evidence for this when d is even and $i = \frac{d}{2}$ (Proposition 4.19) and prove it when $d = 2$ (Corollary 4.20).

The vector space $H^{|I|}(X_{\mathbb{Q}}, \mathcal{E}_{\underline{k}(I),r})$ is defined over the field $F(I)$ which is totally real, and hence $H^{|I|}(X_{\mathbb{Q}}, \mathcal{E}_{\underline{k}(I),r}) \otimes_{F(I)} \mathbb{C} \cong H^{|I|}(X_{\mathbb{C}}, \mathcal{E}_{k(I),r})$ has an action of complex conjugation F_∞. By definition, it preserves the rational structure $H^{|I|}(X_{\mathbb{Q}}, \mathcal{E}_{\underline{k}(I),r})$.

Lemma 4.17. *The complex conjugation $F_\infty\colon H^{|I|}(X_{\mathbb{C}}, \mathcal{E}_{k(I),r}) \to H^{|I|}(X_{\mathbb{C}}, \mathcal{E}_{k(I),r})$ is given on the basis ω_f^J where $|J| = |I|$ and $k(J) = k(I)$ by*

$$\omega_f^J \mapsto \omega_{f^c}^J,$$

where $f^c(z) = \overline{f(-\overline{z})}$ is Shimura's complex conjugation. In particular, on Hecke-isotypic subspaces, it defines a map:

$$F_\infty\colon H^{|I|}(X_{\mathbb{Q}}, \mathcal{E}_{k(I),r})_\pi \to H^{|I|}(X_{\mathbb{Q}}, \mathcal{E}_{k(I),r})_{\pi^c}.$$

Proof. This is a paraphrase of an observation of Harris [Har90b, pp. 164]. □

Now, recall that we have a Serre duality pairing (3.5)

$$\langle -,-\rangle_{\mathrm{SD}}\colon \overline{H}^{|I|}(\mathcal{E}_{\underline{k}(I),r})_{\pi}(E_f(I))\otimes \overline{H}^{|\Sigma\setminus I|}(\mathcal{E}_{\underline{k}(\Sigma\setminus I),r})_{\pi^c}(E_f(I))\to E_f(I) \tag{4.8}$$

which is $E_f(I)$-rational.

Definition 4.18. Suppose $d=2n$ is even. We have a symmetric bilinear pairing

$$\langle -,-\rangle\colon H^n(X_{\mathbb{Q}},\mathcal{E}_{\underline{1},1})_{\pi}\otimes H^n(X_{\mathbb{Q}},\mathcal{E}_{\underline{1},1})_{\pi}\to E_f$$

given by $\langle \omega_1,\omega_2\rangle=\langle \omega_1,F_\infty(\omega_2)\rangle_{\mathrm{SD}}$.

Proposition 4.19. *Suppose $d=2n$ is even. Consider the basis*

$$\left\{\frac{\omega_f^I}{\nu^I(f)}\;\middle|\;|I|=n\right\}$$

of $H^n(X_{\mathbb{C}},\mathcal{E}_{\underline{1},1})_{\pi}$, ordered so that the pairs ω_f^I, $\omega_f^{\Sigma\setminus I}$ are consecutive. The matrix of the pairing $\langle -,-\rangle\otimes\mathbb{C}$ is block-diagonal with 2×2 blocks given by

$$\begin{pmatrix}0 & *\\ * & 0\end{pmatrix}.$$

Moreover, assuming Stark's Conjecture 4.2, we have that $*\in E_f[\sqrt{2}]^\times$.

Proof. We have that $\langle \omega_f^I,\omega_f^J\rangle=0$ unless $J=\Sigma\setminus I$ and

$$\begin{aligned}
\langle \omega_f^I,\omega_f^{\Sigma\setminus I}\rangle &= \langle \omega_f^I,\omega_{f^c}^{\Sigma\setminus I}\rangle_{\mathrm{SD}} && \text{Lemma 4.17}\\
&=(2\pi i)^{-d}\langle f,f\rangle && \text{equation (3.6), Remark 3.20}\\
&\sim_{E_f^\times}(2\pi i)^{-d}f_{\varrho,2}^{1/2}\prod_{\sigma\in G/G'}\log|\tau\otimes\iota(u_{f,\sigma})| && \text{Theorem 4.6}\\
&\sim_{E_f[\sqrt{2}]^\times}\nu^I(f)\cdot\nu^{\Sigma\setminus I}(f).
\end{aligned}$$

This gives the desired result. ∎

Corollary 4.20. *When $d=2$, there is an extension E'/E_f with $[E':E_f]\le 2$ and scalars $\lambda_i\in\mathbb{C}$ such that for $\omega_i=\frac{\omega_f^{\sigma_i}}{\nu^{\sigma_i}(f)}$, we have:*

$$\lambda_i\cdot\omega_i\in H^1(X_{\mathbb{Q}},\mathcal{E}_{\underline{1},1})\otimes_{\mathbb{Q}}E'[\sqrt{2}].$$

Furthermore, assuming Stark's Conjecture 4.2, we may take $\lambda_2=\lambda_1^{-1}$, i.e. $\lambda\cdot\omega_1$ is defined over $E'[\sqrt{2}]$ if and only if $\lambda^{-1}\cdot\omega_2$ is defined over $E'[\sqrt{2}]$.

Proof. We write V for the 2-dimensional E_f-vector space $H^1(X_{\mathbb{Q}},\mathcal{E}_{\underline{1},1})_{\pi}$. Consider two anisotropic vectors $v,v'\in V$ and write $\langle v,v\rangle=a\in E_f^\times$, $\langle v,v'\rangle=b\in E_f$, $\langle v',v'\rangle=c\in E_f^\times$. A vector $w=v+\lambda v'$ is isotropic if and only if λ is a solution to the equation

$$a+2b\lambda+c\lambda^2=0.$$

Since $V\otimes\mathbb{C}$ has two isotropic vectors, this equation must have two solutions $\lambda_1,\lambda_2\in E'$, and hence we have two linearly independent vectors:

$$w_i=v+\lambda_i v'\in V\otimes E'.$$

If we write

$$\omega_i = \nu_1 w_1 + \nu_2 w_2 \qquad \nu_i \in \mathbb{C},$$

then

$$0 = \langle \omega_i, \omega_i \rangle = 2\nu_1\nu_2 \langle w_1, w_2 \rangle,$$

so $\nu_1 = 0$ or $\nu_2 = 0$. Without loss of generality, this shows that

$$\omega_i = \nu_i w_i \qquad \nu_i \in \mathbb{C}.$$

Moreover, assuming Stark's Conjecture 4.2,

$$\nu_1\nu_2 = \frac{\langle \omega_1, \omega_2 \rangle}{\langle w_1, w_2 \rangle} \in E'[\sqrt{2}]$$

showing the last assertion. □

Remark 4.21. When $d > 2$ is even, Proposition 4.19 also suggests that multiples of ω_f^I are algebraic, but we have not been able to deduce that directly.

Remark 4.22. For cohomological degrees $q \neq \frac{d}{2}$ (for example, when d is odd), the Serre duality pairing (4.8) is between cohomological degrees q and $d-q$ and hence we do not have a pairing on one vector space $H^q(X_{\mathbb{C}}, \mathcal{E}_{\underline{1},1})_\pi$, but rather between two different vector spaces $H^q(X_{\mathbb{C}}, \mathcal{E}_{\underline{1},1})_\pi$ and $H^{d-q}(X_{\mathbb{C}}, \mathcal{E}_{\underline{1},1})_\pi$. For this reason, it seems difficult to extract from this rationality of multiples of ω_f^I just from Serre duality in general.

5. Evidence: base change forms

Let F_0 be a totally real number field and consider a totally real extension F of F_0. Any Galois representation of $G_{\overline{\mathbb{Q}}/F_0}$ may be restricted to a Galois representation $G_{\overline{\mathbb{Q}}/F}$. Hence, according to Langlands' functoriality conjecture, for any automorphic representation π_0 of $\mathrm{Res}_{F_0/\mathbb{Q}} \mathrm{GL}_{2,F_0}$, there exists an associated *base change* representation π of $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{GL}_{2,F}$, written $\pi = \mathrm{BC}_{F_0}^F \pi_0$. This is discussed in detail and proved when F/F_0 is a cyclic Galois extension in [Lan80]. See also [AC89].

We now make the following definition.

Definition 5.1. A Hilbert modular form f for F is a *base change form* from F_0, if the associated automorphic representation π is equal to $\mathrm{BC}_{F_0}^F \pi_0$ for some automorphic representation π_0.

For a base change Hilbert modular form f for F, we can find a Hilbert modular form f_0 for F_0 such that its associated representation π_0 satisfies $\pi = \mathrm{BC}_{F_0}^F \pi_0$. In this generality, there is no explicit construction of f from f_0.

When F is a real quadratic extension of $F_0 = \mathbb{Q}$ and the weight of f_0 is at least 2, one can define f as a theta lift of f_0, called the *Doi–Naganuma lift*. The reader can consult [DN70], [Nag73], [Zag75] for the original results and [Oda82, Ch. III] or [vdG88, Ch. VI.4] for an overview.

The goal of this section is to consider Conjecture 3.21 for base change forms. We discuss Stark units for base change forms, a consequence of the conjecture in this case (Conjecture 5.6), and provide numerical evidence for the conjecture in the case of real quadratic extensions.

5.1. **Stark units for base change forms.** For a Hilbert modular form f which is the base change of f_0, we want to relate the unit groups U_f and U_{f_0}. We fix a common splitting field L which is Galois over $\mathbb{Q}$. We denote the three Galois groups by:

$$G = G_{L/\mathbb{Q}} \supseteq G_0' = G_{L/F_0} \supseteq G' = G_{L/F}.$$

If ϱ_0 is the Artin representation associated to f_0, then the Artin representation ϱ associated to f is $\varrho = \operatorname{Res}_{G'}^{G_0'} \varrho_0$.

Proposition 5.2. *We have a natural isomorphism:*

$$U_f \otimes E_f \cong \bigoplus_{\sigma_0 \in G_0'/G'} U_{f_0} \otimes E_f \cong (U_{f_0} \otimes E_f)^{\oplus [F:F_0]}$$

which is compatible with the decomposition in Corollary 1.9.

Proof. We have that

$$\begin{aligned}
U_f \otimes E_f &= \operatorname{Hom}_{G'}(\operatorname{Ad}^0 \varrho, \operatorname{Res}_{G'}^{G} U_L \otimes E_f) \\
&= \operatorname{Hom}_{G'}(\operatorname{Res}_{G'}^{G_0'} \operatorname{Ad}^0 \varrho_0, \operatorname{Res}_{G'}^{G} U_L \otimes E_f) \\
&= \operatorname{Hom}_{G'}\left(\operatorname{Res}_{G'}^{G_0'} \operatorname{Ad}^0 \varrho_0, \bigoplus_{\sigma \in G/G'} \operatorname{Ind}_{\langle \sigma c_0 \sigma^{-1}\rangle}^{G'} E_f - E_f\right) && \text{Corollary 1.4} \\
&= \bigoplus_{\sigma \in G/G'} \operatorname{Hom}_{G'}\left(\operatorname{Res}_{G'}^{G_0'} \operatorname{Ad}^0 \varrho_0, \operatorname{Ind}_{\langle \sigma c_0 \sigma^{-1}\rangle}^{G'} E_f\right) \\
&= \bigoplus_{\sigma \in G/G'} \operatorname{Hom}_{G_0'}\left(\operatorname{Ad}^0 \varrho_0, \operatorname{Ind}_{G'}^{G_0'} \operatorname{Ind}_{\langle \sigma c_0 \sigma^{-1}\rangle}^{G'} E_f\right) && \text{Frobenius reciprocity} \\
&= \bigoplus_{\sigma \in G/G'} \operatorname{Hom}_{G_0'}\left(\operatorname{Ad}^0 \varrho_0, \operatorname{Ind}_{\langle \sigma c_0 \sigma^{-1}\rangle}^{G_0'} E_f\right) \\
&= \bigoplus_{\sigma_0 \in G_0'/G'} \bigoplus_{\sigma \in G/G_0'} \operatorname{Hom}_{G_0'}\left(\operatorname{Ad}^0 \varrho_0, \operatorname{Ind}_{\langle \sigma c_0 \sigma^{-1}\rangle}^{G_0'} E_f\right) \\
&= \bigoplus_{\sigma_0 \in G_0'/G'} U_{f_0} \otimes E_f && \text{Corollary 1.9}
\end{aligned}$$

since $\langle \sigma c_0 \sigma^{-1}\rangle \subseteq G_0'$ depends only on the coset of σ modulo G_0'. □

Corollary 5.3. *Let $\pi = \operatorname{BC}_{F_0}^{F} \pi_0$. Then the period invariants in Definition 3.22 satisfy the following compatibility: for any $\sigma \in \Sigma_F$,*

$$\nu^{\sigma}(\pi) \sim_{E_f^\times} \nu^{\sigma|_{F_0}}(\pi_0).$$

Proof. This follows immediately from Proposition 5.2 after unwinding definitions. □

Remark 5.4. This is compatible with Theorem 4.6. Indeed, in the base change case, one can check that

$$\langle f_0, f_0 \rangle^{[F:F_0]} \sim_{E_f^\times} \langle f, f \rangle$$

by applying Theorem 4.10.

5.2. **Embedded Hilbert modular varieties.** To check if Conjecture 3.21 is compatible with base change, we consider the Hilbert modular variety for F_0 embedded in the Hilbert modular variety for F.

We will write $d = [F : F_0]$ and $d' = [F_0 : \mathbb{Q}]$. Let $\tau_1, \ldots, \tau_{d'}$ be the infinite places of F_0. Above each place τ_i, there are d infinite places $\sigma_{i,j}$ for $j = 1, \ldots, d$ of F. We write ζ_i, $i = 1, \ldots, d'$, for the variables on $\mathcal{H} \otimes F_0$ and $z_{i,j}$, $i = 1, \ldots, d'$, $j = 1, \ldots, d$ for the variables on $\mathcal{H} \otimes F$. Here ζ_i corresponds to τ_i and $z_{i,j}$ corresponds to $\sigma_{i,j}$.

We write X_0 and X for the Hilbert modular varieties associated to F_0 and F, respectively. There is a natural embedding

$$\iota \colon X_0 \hookrightarrow X.$$

Over $\mathbb{C}$, it descends from the map

$$\begin{aligned} \mathcal{H} \otimes F_0 &\hookrightarrow \mathcal{H} \otimes F \\ (\zeta_1, \ldots, \zeta_{d'}) &\mapsto (\zeta_1, \ldots, \zeta_1, \zeta_2, \ldots, \zeta_2, \ldots, \zeta_{d'}), \end{aligned}$$

i.e. the subvariety is given by the equation $z_{i,j} = \zeta_i$ for all i, j.

We are interested in the restriction map

$$H^i(X, \mathcal{E}_{\underline{1},1}) \xrightarrow{\iota^*} H^i(X_0, \mathcal{E}_{\underline{d},d}).$$

Particularly, we defined a class $\omega_f^I \in \overline{H}^I(X, \mathcal{E}_{\underline{1},1})$ associated to $f \in \overline{H}^0(X, \mathcal{E}_{\underline{1},1})$ which has the representation

$$\omega_f^I(\underline{z}) = f(\underline{z}^I) \cdot y^I \cdot \bigwedge_{\sigma_{i,j} \in I} \frac{dz_{i,j} \wedge d\overline{z_{i,j}}}{y_{i,j}^2} \tag{5.1}$$

as a Dolbeault class, and we consider $\iota^*(\omega_f^I)$.

Lemma 5.5. *If I contains $\sigma_{i,j}$ and $\sigma_{i,j'}$ for $j \neq j'$,*

$$\iota^*(\omega_f^I) = 0.$$

Proof. This follows immediately from the expression (5.1) and the identity $z_{i,j} = \zeta_i$ on X_0. □

Let us assume that I only contains at most one $\sigma_{i,j}$ for each i, so that it is possible that $\iota^*(\omega_f^I)$ is non-zero.

The following conjecture is a consequence of Conjecture 3.21.

Conjecture 5.6. *We have that:*

$$\iota^*(\omega_f^I)/\nu^I(\pi) \in H^{|I|}(X_0, \mathcal{E}_{\underline{d},d}) \otimes E_f \subseteq H^{|I|}((X_0)_{\mathbb{C}}, \mathcal{E}_{\underline{d},d}) \otimes E_f,$$

i.e. the Dolbeault cohomology class $\iota^*(\omega_f^I)/\nu^I(\pi)$ belongs to the subspace of E_f-rational cohomology classes.

Note that it is possible that $\iota^*(\omega_f^I) = 0$ in which case this conjecture is void. We expect that $\iota^*(\omega_f^I) = 0$ if f is not a base change form from F_0 (see Proposition 5.9 for an example of this phenomenon).

5.3. **The case of real quadratic extensions.** We finally restrict our attention to real quadratic extensions $F/\mathbb{Q}$. In the previous notation, $F_0 = \mathbb{Q}$ and $d = 2$. We denote by z_1, z_2 (instead of $z_{1,1}, z_{1,2}$) the variables on $X_{\mathbb{C}}$ and by z (instead of ζ_1) the variable on $(X_0)_{\mathbb{C}}$.

Let f be a holomorphic Hilbert modular form of parallel weight (k, k) and consider $\omega_f^{\sigma_1} \in H^1(X_{\mathbb{C}}^{\mathrm{an}}, \mathcal{E}_{(2-k,k)}^{\mathrm{an}})$, locally given by:

$$\omega_f^{\sigma_1}(z_1, z_2) = f(\epsilon_1 \overline{z_1}, \epsilon_2 z_2) y_1^k \frac{dz_1 \wedge d\overline{z_1}}{y_1^2}. \tag{5.2}$$

There are embedded modular curves $\iota \colon Y = X_0 \hookrightarrow X$ in the Hilbert modular surface. The simplest example is obtained by considering the map:

$$\begin{aligned} \iota \colon Y_{\mathbb{C}}^{\mathrm{an}} &\hookrightarrow X_{\mathbb{C}}^{\mathrm{an}} \\ z &\mapsto (z, z) \end{aligned}$$

over $\mathbb{C}$ which descends to varieties over $\mathbb{Q}$. Via this map,

$$\iota^*(\mathcal{E}_{(2-k,k)}^{\mathrm{an}}) \cong \mathcal{E}_2^{\mathrm{an}} \cong \Omega_Y^{1,\mathrm{an}}$$

where the last isomorphism is the Kodaira–Spencer isomorphism (3.3). Hence:

$$\iota^*(\omega_f^{\sigma_1})(z) = f(\epsilon_1 \overline{z}, \epsilon_2 z) y^k \frac{dz \wedge d\overline{z}}{y^2}$$

defines a class in $H^1(Y_{\mathbb{C}}^{\mathrm{an}}, \Omega_Y^{1,\mathrm{an}})$. Via the trace map, we have:

$$\begin{aligned} H^1(Y_{\mathbb{C}}^{\mathrm{an}}, \Omega_Y^{1,\mathrm{an}}) &\xrightarrow{\cong} \mathbb{C}, \\ \iota^*(\omega_f^{\sigma_1})(z) &\mapsto \int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1 \overline{z}, \epsilon_2 z) y^k \frac{dz \wedge d\overline{z}}{y^2}, \end{aligned}$$

and the isomorphism respects rational structures.

For $k = 1$, Conjecture 5.6 hence specializes to the following. We write $\log(|u_{f,\sigma_1}|) = \log(|(\tau \otimes \iota)(u_{f,\sigma_1})|)$ for brevity, keeping the embeddings τ and ι implicit.

Conjecture 5.7. *If a Hilbert modular form f is the base change for a weight one modular form f_0, then:*

$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1 \overline{z}, \epsilon_2 z) y \frac{dz \wedge d\overline{z}}{y^2} \sim_{E_f^\times} \log(|u_{f,\sigma_1}|).$$

We note that this conjecture is equivalent to our original Conjecture 3.21.

Lemma 5.8. *Assuming Stark's Conjecture 4.2, Conjecture 5.7 is equivalent to Conjecture 3.21, up to at most a quartic extension E'/E_f.*

Proof. This is clear by putting together Theorem 4.6 and Corollary 4.20. □

For $k \geq 2$ and full level, these integrals were considered by Asai [Asa78]. The following result was also obtained by Oda [Oda82].

Proposition 5.9 ([Oda82, Theorem 16.5])**.** *If f is not a base change form, then*

$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1\overline{z}, \epsilon_2 z) y^k \frac{dz \wedge d\overline{z}}{y^2} = 0.$$

Otherwise, if f is the base change of a modular form f_0 of weight k, then there is a rational constant $c \neq 0$, independent of f, such that

$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1\overline{z}, \epsilon_2 z) y^k \frac{dz \wedge d\overline{z}}{y^2} = c\frac{\langle f, f\rangle}{\langle f_0, f_0\rangle}.$$

Remark 5.10. The proof of Proposition 5.9 in loc. cit. uses the explicit realization of f as a Doi–Naganuma lift of f_0, which is currently not available in the literature for weight one forms.

If the analog of Proposition 5.9 holds for a weight one form, then Stark's Conjecture 4.2 for f_0 implies Conjecture 5.7 for base change forms from f_0 to a real quadratic field. More specifically, let us assume the identity:

$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1\overline{z}, \epsilon_2 z) y^k \frac{dz \wedge d\overline{z}}{y^2} = c\frac{\langle f, f\rangle}{\langle f_0, f_0\rangle}.$$

Assuming Stark's Conjecture 4.2, Theorem 4.6 shows that

$$\langle f, f\rangle \sim_{E_f^\times} \log(|u_{f,\sigma_1}|) \cdot \log(|u_{f,\sigma_2}|) = \log(|u_{f,\sigma_1}|) \cdot (2\pi i)\nu^{\sigma_2}(\pi)$$

and

$$\langle f_0, f_0\rangle \sim_{E_f^\times} \log(|u_{f_0}|) = \nu^{\{\infty\}}(\pi_0).$$

By Corollary 5.3, $\nu^{\sigma_2}(\pi) \sim_{E_f^\times} \nu^{\{\infty\}}(\pi_0)$, completing the proof.

Verifying the details of this would take us too far afield, so we will pursue this elsewhere. Instead, in the next section we describe some explicit numerical computations that support Conjecture 5.7.

5.4. **Computing the integrals numerically.** The next goal is to provide a numerical verification of Conjecture 5.7, i.e. check that

$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1\overline{z}, \epsilon_2 z) y \frac{dz \wedge d\overline{z}}{y^2} \sim_{E_f^\times} \log(u_{f,\sigma_1}). \tag{5.3}$$

We first derive a formula (Theorem 5.14) for the integral on the left hand side using Nelson's technique [Nel15] for evaluating integrals on modular curves.

Let $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ be a finite index subgroup and let $F\colon \Gamma\backslash\mathcal{H} \to \mathbb{C}$ be a Γ-invariant function on the upper half plane $\mathcal{H}$. Suppose we have its q-expansions, i.e. for all $\tau \in \mathrm{SL}_2(\mathbb{Z})$, we have

$$F(\tau z) = \sum_{n\in\mathbb{Q}} a_F(n, y; \tau) e(nx) \tag{5.4}$$

where $e(nx) = e^{2\pi i n x}$.

Theorem 5.11 ([Nel15, Theorem 5.6]). *Suppose F is bounded, measurable, and satisfies $F(\tau z) \ll y^{-\alpha}$ for some fixed $\alpha > 0$, almost all $z = x + iy$ with $y \geq 1$, and all $\tau \in \mathrm{SL}_2(\mathbb{Z})$. Then for $0 < \delta < \alpha$ we have that:*

$$\int\limits_{\Gamma\backslash\mathcal{H}} F(z)\frac{dxdy}{y^2} = \int\limits_{(1+\delta)} (2s-1)2\xi(2s) \sum_{\tau\in\Gamma\backslash\mathrm{SL}_2(\mathbb{Z})} a_F(0,\cdot;\tau)^\wedge(1-s)\,\frac{ds}{2\pi i}$$

where

$$\xi(2s) = \frac{\Gamma(s)}{\pi^s}\zeta(2s),$$
$$a_F(0,\cdot;\tau)^\wedge(1-s) = \int\limits_0^\infty a_F(0,y;\tau)y^{s-1}\,\frac{dy}{y}.$$

Applying this to $F(z) = f_0(z)\cdot\overline{f_0(z)}\cdot y^k$ gives an explicit expression for the Petersson inner product $\langle f_0, f_0\rangle$.

Corollary 5.12 (Nelson, [Col18, Theorem 4.2]). *Suppose f_0 is a cusp form in $S_k(N,\chi)$. For a cusp s, let $\sum_n a_{n,s}q^n$ be the q-expansion at ∞ of $f_0|[\tau_{s,h}]_k$, where $\tau_{s,h} = \tau_s\begin{pmatrix} h_s & 0 \\ 0 & 1\end{pmatrix}$ and $\tau_s s = \infty$. Then we have that:*

$$\langle f_0, f_0\rangle = \frac{4}{\mathrm{vol}(\Gamma\backslash\mathcal{H})}\sum_{s\in\Gamma\backslash\mathbb{P}^1(\mathbb{Q})}\frac{h_{s,0}}{h_s}\sum_{n=1}^\infty\frac{|a_{n,s}|^2}{n^{k-1}}\sum_{m=1}^\infty\left(\frac{x}{8\pi}\right)^{k-1}(xK_{k-2}(x)-K_{k-1}(x)),\quad x = 4\pi m\sqrt{\frac{n}{h_s}},$$

where K_v is a K-Bessel function, $h_{s,0}$ is the classical width of the cusp s, and h_s is the width described in [Col18, Lemma 2.1].

Remark 5.13. An algorithm to compute these Petersson inner products was developed and implemented by Collins [Col18, Algorithm 4.3].

The goal for this section is to prove the following theorem, which is an explicit form of Theorem 5.11 in our case.

Recall that for $\alpha \in \mathrm{SL}_2(\mathcal{O}_F)$, we write $\alpha_i = \sigma_i(\alpha)$ and

$$f|[\alpha]_{\underline{k}}(z_1,z_2) = f(\alpha_1 z_1, \alpha_2 z_2)j(\alpha_1,z_1)^{-k_1}j(\alpha_2,z_2)^{-k_2}$$

where

$$j(g,z) = \det(g)^{-1/2}(cz+d).$$

By definition, if f is a Hilbert modular form of weight (k_1,k_2) and level Γ and character χ, then $f|[\alpha]_{\underline{k}} = \chi(d)\cdot f$ for $\alpha = \begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \Gamma$.

Theorem 5.14. *Let f be a normalized parallel weight k Hilbert modular newform of level $\mathfrak{N}$ and character χ. For each cusp $s \in \mathbb{P}^1(\mathbb{Q})/\Gamma_0(N)$, let $\tau \in \mathrm{SL}_2(\mathbb{Z})$ satisfy $\tau s = \infty$. Let h_s be* *the width of the cusp as described in* [Col18, Lemma 2.1]*, and*

$$\tau^\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \tau \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}^{-1},$$

$$\tau_h^\epsilon = \tau^\epsilon \begin{pmatrix} h_s & 0 \\ 0 & 1 \end{pmatrix}.$$

If $\sum_{m \gg 0} a_{(m),s} q^m$ *is the* q*-expansion of* $f[\tau_h^\epsilon]_k$ *at* ∞*, then*

$$\int_{\Gamma_0(N)\backslash\mathcal{H}} f(\epsilon_1 \overline{z}, \epsilon_2 z) y^k \frac{dz \wedge d\overline{z}}{y^2} = 4 \sum_s \frac{h_{s,0}}{h_s} \sum_{m=1}^{\infty} \frac{a_{(m),s}}{(m/\sqrt{d})^{k-1}} \sum_{n=1}^{\infty} \left(\frac{x}{2^{3-i}\pi}\right)^{k-1} (x K_{k-2}(x) - K_{k-1}(x))$$

where $x = 2^{2-i/2}\pi n \sqrt{\frac{m}{h_s\sqrt{d}}}$ *and* $h_{s,0}$ *is the classical width of the cusp* s*, and* $i = 0$ *if* $d \equiv 1$ (4) *or* $i = 1$ *if* $d \equiv 3$ (4)*.*

Remark 5.15. This formula is very similar to the formula for $\langle f_0, f_0 \rangle$ in Corollary 5.12. We can hence adapt the algorithm [Col18, Algorithm 4.3] to compute the integral. The computation of q-expansions of f at other cusps given the q-expansion at ∞ is discussed in the next section (5.5).

We devote the rest of this section to the proof of this theorem. We want to apply Theorem 5.11 to the function

$$F(z) = F_f^{\sigma_1}(z) = f(\epsilon_1 \overline{z}, \epsilon_2 z) \cdot y^k \tag{5.5}$$

where f is a Hilbert modular form of parallel weight k.

We will need q-expansions of $F(z)$ at other cusps, i.e. q-expansions of $F(\tau z)$ for $\tau \in \mathrm{SL}_2(\mathbb{Z})$, as in equation (5.4). The idea is to express them in terms of q-expansions at ∞ of another Hilbert modular form.

Lemma 5.16. *Suppose* f *is a Hilbert modular form of weight* (k, k)*. For a cusp* s*, let* $\tau \in \mathrm{SL}_2(\mathcal{O}_F)$ *be such that* $\tau s = \infty$ *and set*

$$\tau^\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \tau \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}^{-1}.$$

Then we have that:

$$F_f^{\sigma_1}(\tau z) = F^{\sigma_1}_{f|[\tau^\epsilon]_k}(z).$$

Proof. For $\tau \in \mathrm{SL}_2(\mathbb{Z})$, we have that:

$$\begin{aligned} f(\epsilon_1(\tau z_1), \epsilon_2(\tau z_2)) &= f \left| \left[\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \tau \right]_k \right. (z_1, z_2) \cdot (N_{F/\mathbb{Q}}(\epsilon))^{-k/2} \cdot j(\tau, z_1)^k j(\tau, z_2)^k \\ &= f \left| \left[\tau^\epsilon \begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix} \right]_k \right. (z_1, z_2) \cdot (N_{F/\mathbb{Q}}(\epsilon))^{-k/2} \cdot j(\tau, z_1)^k j(\tau, z_2)^k \\ &= f|[\tau^\epsilon]_k(\epsilon_1 z_1, \epsilon_2 z_2) \cdot j(\tau, z_1)^k j(\tau, z_2)^k. \end{aligned}$$

Therefore,

$$\begin{aligned} F_f^{\sigma_1}(\tau z) &= f(\epsilon_1(\tau\overline{z}), \epsilon_2(\tau z)) \cdot (\mathrm{Im}(\tau z))^k \\ &= f|[\tau^\epsilon]_k(\epsilon_1\overline{z}, \epsilon_2 z) \cdot |j(\tau, z)|^{2k} \cdot (\mathrm{Im}(\tau z))^k \\ &= f|[\tau^\epsilon]_k(\epsilon_1\overline{z}, \epsilon_2 z) \cdot \mathrm{Im}(z)^k \\ &= F_{f|[\tau^\epsilon]_k}(z), \end{aligned}$$

since $\mathrm{Im}(\tau z) = |j(\tau, z)|^{-2} y$. □

Lemma 5.17. *For a cusp s, consider $\tau \in \mathrm{SL}_2(\mathbb{Z})$ such that $\tau s = \infty$. Let h_s be the width of cusp s (as in* [Col18, Lemma 2.1]*) and*

$$\tau_h^\epsilon = \tau^\epsilon \begin{pmatrix} h_s & 0 \\ 0 & 1 \end{pmatrix}.$$

The q-expansion coefficients of $F(\tau z)$ (as in equation (5.4)) are given by

$$a_F(n/h_s, y; \tau) = (y/h_s)^k \cdot \sum_{\substack{m \gg 0 \\ \mathrm{Tr}(\epsilon m) = n}} a_{(m),s} \cdot e^{-2\pi(\epsilon_2 m_2/\delta_2 - \epsilon_1 m_1/\delta_1) y/h_s},$$

where $a_{(m),s}$ are Fourier coefficients of $f|[\tau_h^\epsilon]_k$. In particular,

$$a_F(0, y; \tau) = (y/h_s)^k \cdot \sum_{m=1}^{\infty} a_{(m),s} \cdot e^{-2\pi \frac{2^{1-i} m}{\sqrt{d}} (y/h_s)}$$

where $i = 0$ if $d \equiv 1$ (4) and $i = 1$ if $d \equiv 3$ (4).

Proof. We write $h = h_s$ for simplicity. Suppose the q-expansion of $f|[\tau_h^{(\epsilon)}]_k$ is:

$$f|[\tau_h^\epsilon]_k(z_1, z_2) = \sum_{m \gg 0} a_{(m),s} q^{m/\delta}.$$

Then:

$$f|[\tau^\epsilon]_k(z_1, z_2) = h^{-k} \sum_{m \gg 0} a_{(m),s} q^{m/(\delta h)}.$$

By Lemma 5.16,

$$\begin{aligned} F(\tau z) &= f|[\tau^\epsilon]_k(\epsilon_1\overline{z}, \epsilon_2 z) \cdot y^k \\ &= (y/h)^k \sum_{m \in \mathcal{O}_F^+} a_{(m),s} \cdot e^{2\pi i(\epsilon_1 m_1/\delta_1(\overline{z}/h) + \epsilon_2 m_2/\delta_2(z/h))} \\ &= (y/h)^k \sum_{m \in \mathcal{O}_F^+} a_{(m),s} \cdot e^{-2\pi(\epsilon_2 m_2/\delta_2 - \epsilon_1 m_1/\delta_1)(y/h)} e^{2\pi i(\mathrm{Tr}\epsilon m/\delta)(x/h)} \\ &= (y/h)^k \sum_{n \in \mathbb{Z}} \left(\sum_{\substack{m \in \mathcal{O}_F^+ \\ \mathrm{Tr}(\epsilon m/\delta) = n}} a_{(m),s} \cdot e^{-2\pi(\epsilon_2 m_2/\delta_2 - \epsilon_1 m_1/\delta_1)(y/h)} \right) e((n/h)x). \end{aligned}$$

Hence

$$a_F(n/h, y; \tau) = (y/h)^k \cdot \sum_{\substack{m \gg 0 \\ \mathrm{Tr}(\epsilon m/\delta)=n}} a_{(m),s} \cdot e^{-2\pi(\epsilon_2 m_2/\delta_2 - \epsilon_1 m_1/\delta_1)(y/h)},$$

and in particular,

$$a_F(0, y; \tau) = (y/h)^k \cdot \sum_{\substack{m \gg 0 \\ \mathrm{Tr}(\epsilon m/\delta)=0}} a_{(m),s} e^{-2\pi(\epsilon_2 m_2/\delta_2 - \epsilon_1 m_1/\delta_1)(y/h)}.$$

To make this last formula more explicit, we write $m = \alpha + \beta\sqrt{d}$. We may choose $\delta = 2^i\sqrt{d} \cdot \epsilon$ to be the totally positive generator of the different ideal. Then

$$\epsilon m/\delta = \frac{\beta}{2^i} + \frac{\alpha}{2^i d}\sqrt{d}.$$

If $\mathrm{Tr}(\epsilon m/\delta) = 0$, then $\beta = 0$, so $m = \alpha \in \mathbb{Z}_{>0}$. Moreover:

$$\epsilon_2 m_2/\delta_2 - \epsilon_1 m_1/\delta_1 = \frac{2^{1-i} m}{\sqrt{d}}.$$

We may hence rewrite the above sum as

$$a_F(0, y; \tau) = (y/h)^k \cdot \sum_{m=1}^{\infty} a_{(m),s} \cdot e^{-2\pi \frac{2^{1-i} m}{\sqrt{d}}(y/h)},$$

proving the lemma. □

We finally complete the proof of Theorem 5.14.

Proof of Theorem 5.14. We will apply Theorem 5.11 to the invariant function $F(z) = F_f^{\sigma_1}(z)$. By Lemma 5.17,

$$a_F(0, y; \tau) = (y/h_s)^k \cdot \sum_{m=1}^{\infty} a_{(m),s} \cdot e^{-2\pi \frac{2^{1-i} m}{\sqrt{d}}(y/h_s)}.$$

Hence:

$$
\begin{aligned}
a_F(0,\cdot;\tau)^\wedge(1-t) &= \int_0^\infty a_F(0,y;\tau)\, y^{t-1}\,\frac{dy}{y}. \\
&= \sum_{m=1}^\infty a_{(m),s} \int_0^\infty e^{-2\pi \frac{2^{1-i}m}{\sqrt{d}}(y/h_s)}\, y^{t-1}\,(y/h_s)^k\,\frac{dy}{y} \\
&= \sum_{m=1}^\infty a_{(m),s} h_s^{-k} \int_0^\infty e^{-2\pi \frac{2^{1-i}m}{h_s\sqrt{d}}y}\, y^{t+k-1}\,\frac{dy}{y} \\
&= \sum_{m=1}^\infty a_{(m),s} h_s^{-k} \frac{\Gamma(t+k-1)}{(2\pi \frac{2^{1-i}m}{h_s\sqrt{d}})^{t+k-1}} \\
&= \sum_{m=1}^\infty \frac{a_{(m),s}}{(2^{2-i}\pi m/\sqrt{d})^{k-1} h_s} \frac{\Gamma(t+k-1)}{(2^{2-i}\pi \frac{m}{h_s\sqrt{d}})^{t}}.
\end{aligned}
$$

According to [Nel15, Lemma A.4]:

$$
\int_{(1+\delta)} (t-1/2)\frac{\Gamma(t)\Gamma(t+\nu)}{(x/2)^{2t+\nu}}\,\frac{dt}{2\pi i} = xK_{\nu-1}(x) - K_\nu(x)
$$

for $\nu \in \mathbb{C}$ with $\mathrm{Re}(\nu) \geq 0$.

By Theorem 5.11,

$$
\begin{aligned}
\int_{\Gamma\backslash\mathcal{H}} F(z)\frac{dxdy}{y^2} &= \int_{(1+\delta)} (2t-1)2\xi(2t)\sum_\tau a_F(0,\cdot;\tau)^\wedge(1-t)\,\frac{dt}{2\pi i} \\
&= 4\sum_s h_{s,0}\sum_{m=1}^\infty \frac{a_{(m),s}}{(2^{2-i}\pi m/\sqrt{d})^{k-1}h_s}\sum_{n=1}^\infty \int_{(1+\delta)} (t-1/2)\frac{\Gamma(t)\Gamma(t+k-1)}{(2^{2-i}\pi^2\frac{m}{h_s\sqrt{d}})^t}\frac{1}{n^{2t}}\frac{dt}{2\pi i} \\
&= 4\sum_s \frac{h_{s,0}}{h_s}\sum_{m=1}^\infty \frac{a_{(m),s}}{(2^{2-i}\pi m/\sqrt{d})^{k-1}}\sum_{n=1}^\infty \int_{(1+\delta)} (t-1/2)\frac{\Gamma(t)\Gamma(t+k-1)}{(2^{2-i}\pi^2\frac{mn^2}{h_s\sqrt{d}})^t}\frac{ds}{2\pi i} \\
&= 4\sum_s \frac{h_{s,0}}{h_s}\sum_{m=1}^\infty \frac{a_{(m),s}}{(m/\sqrt{d})^{k-1}h_s}\sum_{n=1}^\infty \left(\frac{x}{2^{3-i}\pi}\right)^{k-1}(xK_{k-2}(x) - K_{k-1}(x))
\end{aligned}
$$

where we set $x = 2^{2-i/2}\pi n\sqrt{m/h_s\sqrt{d}}$ in the last line. $\square$

In order to use Theorem 5.14, we need to compute the q-expansions of the Hilbert modular form f at other cusps, i.e. q-expansions of $f||[\alpha]_k$ at ∞ for a matrix α. We discuss this problem in the next section.

5.5. **q-expansions at other cusps.** In this section, we address the following question: given the q-expansion of a Hilbert modular form $f(z)$ at the cusp ∞, what is the q-expansion of $f(z)$ at any cusp of $\Gamma_0(N)\backslash\mathcal{H}^2$?

We take two methods available for modular forms and discuss their generalization to Hilbert modular forms:

- Asai's explicit formula [Asa76] (Theorem 5.19),
- Collins computational method based on a least-squares algorithm [Col18] (Algorithm 5.21).

The first one is much faster in practice but only works for square-free level. The second one works for any level, but our implementation is too slow in practice to compute the above integrals. We include it here since it might be of independent interest.

Collins also introduces an improved computational method for modular forms using twists of eigenforms [Col18, Algorithm 2.6]. This is also discussed Chen's thesis [Che16, Chapter 4].

An alternative approach is to use the adelic language. The Fourier coefficients of a modular form are given by value of the Whittaker newform of f at certain matrices. Loeffler and Weinstein [LW12] give an algorithm to compute the local representations, so one just needs an algorithm to compute the local newforms. For more details, see [CS16, Section 3] .

5.5.1. *Explicit formula, following* [Asa76]. Let F be a totally real field of narrow class number 1 (of arbitrary degree d). Suppose f is a Hilbert modular eigenform of level $\mathfrak{N}$ with character $\chi\colon (\mathcal{O}_F/\mathfrak{N}) \to \mathbb{C}^\times$ and parallel weight k. Suppose the level $\mathfrak{N}$ is square-free. We write $\Gamma = \Gamma_0(\mathfrak{N})$ throughout this section.

The goal is to prove an explicit formula (Theorem 5.19) for the q-expansion of a Hilbert modular form f at a cusp $C = a/b \in F$ in terms of the q-expansion at ∞, generalizing the main result of [Asa76] to the Hilbert modular case.

Since $\mathfrak{N}$ is square-free, the cusps $C = a/b$ of $\Gamma\backslash\mathcal{H}^2$ are in bijection with decompositions $\mathfrak{N} = \mathfrak{A} \cdot \mathfrak{B}$, where $\mathfrak{B} = ((b), \mathfrak{N})$. For each divisor $\mathfrak{A}$, we consider the matrix

$$W_{\mathfrak{A}} = \begin{pmatrix} A\alpha & \beta \\ N\gamma & A\delta \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ B\gamma & A\delta \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}$$

such that:

- A, N are totally positive generators of $\mathfrak{A}$, $\mathfrak{N}$, respectively; then $B = N/A$ is a totally positive generator of $\mathfrak{B}$,
- $\det W_{\mathfrak{A}} = A$,
- $\alpha, \beta, \gamma, \delta \in \mathcal{O}_F$.

Such a matrix always exists: since $\mathfrak{A} = (A)$ and $\mathfrak{B} = (B)$ are coprime, we have that $1 = \lambda A + \mu B$, for some $\lambda, \mu \in \mathcal{O}_F$, so $A = \lambda A^2 + \mu N$, and we may take $\alpha = \beta = 1$ and $\gamma = -\mu$, $\delta = \lambda$ to obtain such a matrix:

$$W_{\mathfrak{A}} = \begin{pmatrix} A & 1 \\ -N\mu & A\lambda \end{pmatrix}.$$

Conversely, for a matrix $W_{\mathfrak{A}}$,

$$W_{\mathfrak{A}}^{-1}\infty = \frac{\delta}{-B\gamma}$$

is a cusp with $((B\gamma), \mathfrak{N}) = \mathfrak{B}$, because

$$1 = A\alpha\delta - B\beta\gamma \equiv -B\beta\gamma \mod \mathfrak{A},$$

so (γ) is coprime to $\mathfrak{A}$.

Such a matrix $W_{\mathfrak{A}}$ associated to $\mathfrak{A}$ is well-defined up multiplication by elements of Γ. Moreover, $W_{\mathfrak{A}}$ normalizes Γ and $A^{-1}W_{\mathfrak{A}}^2 \in \Gamma$.

The q-expansion of f at the cusp corresponding to $\mathfrak{N} = \mathfrak{A}\mathfrak{B}$ is the q-expansion of the Hilbert modular form $f_{\mathfrak{A}} = f|W_{\mathfrak{A}}$ at ∞.

For a prime ideal $\mathfrak{p} = (\varpi)$ of $\mathcal{O}_F$, coprime to $\mathfrak{N}$, with totally positive generator ϖ, the action of the Hecke operator $T(\mathfrak{p})$ on the space of cusp forms $S_k(\mathfrak{N}, \chi)$ is given by

$$f|T(\mathfrak{p}) = N_{F/\mathbb{Q}}(\mathfrak{p})^{k/2-1}\left(\chi(\varpi) f\Big|_k \begin{pmatrix} \varpi & 0 \\ 0 & 1 \end{pmatrix} + \sum_{\nu \in \mathcal{O}_F/\mathfrak{p}} f\Big|_k \begin{pmatrix} 1 & \nu \\ 0 & \varpi \end{pmatrix}\right). \tag{5.6}$$

For example, when $d = 2$, this simplifies to the more familiar expression:

$$f|T(\mathfrak{p}) = N_{F/\mathbb{Q}}(\mathfrak{p})^{k-1}\left(\chi(\varpi) f(\varpi_1 z_1, \varpi_2 z_2) + N_{F/\mathbb{Q}}\mathfrak{p}^{-k} \sum_{\nu \in \mathcal{O}_F/\mathfrak{p}} f\left(\frac{z_1 + \nu_1}{\varpi_1}, \frac{z_2 + \nu_2}{\varpi_2}\right)\right).$$

We will write $T(\mathfrak{p}, \chi)$ for the action of the Hecke operator $T(\mathfrak{p})$ on $S_k(\mathfrak{N}, \chi)$.

Remark 5.18. This normalization of Hecke operators is consistent with $T'(\mathfrak{p})$ in [Shi78].

For simplicity, whenever we write down a generator of an ideal, it is assumed to be totally positive. The main result of this section is the following.

Theorem 5.19. *Let f be a newform in $S_k(\mathfrak{N}, \chi)$ and $f|T(\mathfrak{p}, \chi) = a_{\mathfrak{p}} f$. For each decomposition $\mathfrak{N} = \mathfrak{A}\mathfrak{B}$, let $f_{\mathfrak{A}} = f|W_{\mathfrak{A}}$. Then $f_{\mathfrak{A}}$ is a newform in $S_k(\mathfrak{N}, {}^{\mathfrak{A}}\chi)$ and*

$$f_{\mathfrak{A}}|T(\mathfrak{p}, {}^{\mathfrak{A}}\chi) = a_{\mathfrak{p}}^{(\mathfrak{A})} f_{\mathfrak{A}}$$

for every prime $\mathfrak{p} = (\varpi)$, where

$$a_{\mathfrak{p}}^{(\mathfrak{A})} = \begin{cases} \overline{\chi_{\mathfrak{A}}}(\varpi) a_{\mathfrak{p}} & \text{if } \mathfrak{p} \nmid \mathfrak{A}, \\ \chi_{\mathfrak{B}}(\varpi) \overline{a_{\mathfrak{p}}} & \text{if } \mathfrak{p} \nmid \mathfrak{B}, \end{cases}$$

and

$$\begin{aligned} \chi_{\mathfrak{A}} \colon (\mathcal{O}_F/\mathfrak{A}\mathcal{O}_F)^\times &\to \mathbb{C}^\times, \\ m &\mapsto \chi((-B\beta\gamma)m + (A\alpha\delta)), \\ \chi_{\mathfrak{B}} \colon (\mathcal{O}_F/\mathfrak{B}\mathcal{O}_F)^\times &\to \mathbb{C}^\times, \\ m &\mapsto \chi((A\alpha\delta)m + (-B\beta\gamma)), \\ {}^{\mathfrak{A}}\chi \colon (\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^\times &\to \mathbb{C}^\times, \\ m &\mapsto \chi((A\alpha\delta)m + (-B\beta\gamma)m^{-1}). \end{aligned}$$

Proof. The proof is a straightforward generalization of [Asa76, Theorem 1], so we just give a sketch.

We first check that $f_{\mathfrak{A}}$ has character ${}^{\mathfrak{A}}\chi$ described above. Write

$$\begin{aligned} d\colon \Gamma = \Gamma_0(\mathfrak{N}) &\to (\mathcal{O}_F/\mathfrak{N}\mathcal{O}_F)^\times, \\ g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto d \mod \mathfrak{N}. \end{aligned}$$

Then we just need to check that

$$d\left(W_{\mathfrak{A}} g W_{\mathfrak{A}}^{-1}\right) = \gamma_{\mathfrak{A}}(d(g)),$$

where

$$\gamma_{\mathfrak{A}}(m) \equiv (A\alpha\delta)m + (-B\beta\gamma)m^{-1} \mod \mathfrak{N}.$$

For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have that

$$\begin{aligned} W_{\mathfrak{A}}\gamma W_{\mathfrak{A}}^{-1} &= \begin{pmatrix} A\alpha & \beta \\ N\gamma & A\delta \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \delta & -\beta/A \\ -B\gamma & \alpha \end{pmatrix} \\ &= \begin{pmatrix} A\alpha & \beta \\ N\gamma & A\delta \end{pmatrix} \begin{pmatrix} a\delta - bB\gamma & -a\beta/A + b\alpha \\ c\delta - dB\gamma & -c\beta/A + d\alpha \end{pmatrix} \end{aligned}$$

so

$$\begin{aligned} d(W_{\mathfrak{A}}\gamma W_{\mathfrak{A}}^{-1}) &= -a\beta\gamma B + bN\alpha\gamma - c\beta\delta + dA\alpha\delta \\ &\equiv (-\beta\gamma B)a + (A\alpha\delta)d \mod \mathfrak{N} \qquad \text{since } c \equiv 0 \mod \mathfrak{N} \end{aligned}$$

which proves the above result, since $ad \equiv 1 \mod \mathfrak{N}$.

One then computes a formula for how the Hecke operator $T(\mathfrak{p}, \chi)$ commutes with $W_{\mathfrak{A}}$ using the above expression for Hecke operators (c.f. [Asa76, Lemma 2]). To check that $f_{\mathfrak{A}}$ is a newform, one shows that $W_{\mathfrak{A}}$ preserves oldforms (c.f. [Asa76, Lemma 1]). ☐

The Hecke eigenvalues $a_{\mathfrak{n}}$ of $T(\mathfrak{n})$ may be computed from the eigenvalues $a_{\mathfrak{p}}$ of $T(\mathfrak{p})$ in the standard way [Shi78, (2.26)]. For $\mathfrak{n}$ coprime to $\mathfrak{m}$, we have that

$$a_{\mathfrak{n}\mathfrak{m}} = a_{\mathfrak{n}} \cdot a_{\mathfrak{m}}$$

and for $\mathfrak{n} = \mathfrak{p}^r$, we have that

$$\sum_{r=0}^{\infty} a_{\mathfrak{p}^r} N(\mathfrak{p})^{-rs} = [1 - a_{\mathfrak{p}} N(\mathfrak{p})^{-s} + \chi(\mathfrak{p}) N(\mathfrak{p})^{k_0 - 1 - 2s}]^{-1} \tag{5.7}$$

where $k_0 = \max\{k_1, \ldots, k_n\}$.

We can then recover the q-expansion of $f_{\mathfrak{A}}$, up to a constant λ, from the Hecke eigenvalues $a_{\mathfrak{p}}^{(\mathfrak{A})}$ given by Theorem 5.19. There is an explicit expression for λ, described in the next theorem.

Theorem 5.20. *Let f be a normalized Hilbert newform with character χ and level $\mathfrak{N}$. Then there is a constant λ such that*

$$f_{\mathfrak{A}} = \lambda \cdot \underbrace{\sum_{\nu \gg 0} a_{(\nu)}^{(\mathfrak{A})} q^\nu}_{f^{(\mathfrak{A})}}$$

where we define:

$$\begin{aligned} a^{(\mathfrak{A})}_{(1)} &= 1 & & \\ a^{(\mathfrak{A})}_{(\nu)} &= \overline{\chi_{\mathfrak{A}}(\nu)} a_{(\nu)} & & \text{if } ((\nu), \mathfrak{A}) = \mathcal{O}_F, \\ a^{(\mathfrak{A})}_{(\nu)} &= \chi_{\mathfrak{B}}(\nu) \overline{a_{(\nu)}} & & \text{if } ((\nu), \mathfrak{B}) = \mathcal{O}_F, \\ a^{(\mathfrak{A})}_{(\nu\mu)} &= a^{(\mathfrak{A})}_{(\nu)} a^{(\mathfrak{A})}_{(\mu)} & & \text{if } (\nu, \mu) = \mathcal{O}_F. \end{aligned}$$

Moreover, there is an explicit formula for λ, analogous to [Asa76, Theorem 2]*. First, for a decomposition $\mathfrak{N} = \mathfrak{p}\mathfrak{B}$ for a prime ideal $\mathfrak{p} = (\varpi)$, let*

$$W_{\mathfrak{p}} = \begin{pmatrix} \varpi & 1 \\ N\gamma & \varpi\delta \end{pmatrix}$$

be a matrix of determinant ϖ with $\gamma, \delta \in \mathcal{O}_F$. Then

$$f|W_{\mathfrak{p}} = \lambda_{\mathfrak{p}} f^{(\mathfrak{p})}$$

with

$$\lambda_{\mathfrak{p}} = \begin{cases} \chi_{\mathfrak{p}}(\delta) \cdot C(\chi_{\mathfrak{p}}) \cdot N\mathfrak{p}^{-k/2} \cdot \overline{a_{\mathfrak{p}}} & \text{if } \mathfrak{p} \text{ divides } \mathrm{cond}(\chi), \\ -N\mathfrak{p}^{1-k/2} \cdot \overline{a_{\mathfrak{p}}} & \text{otherwise,} \end{cases}$$

where

$$C(\chi_{\mathfrak{p}}) = \sum_{h \mod \mathfrak{p}} \chi_{\mathfrak{p}}(h) \cdot e^{2\pi i \mathrm{Tr}(h/\varpi)}$$

is a Gauss sum associated to $\chi_{\mathfrak{p}}$.

In general, for any $\mathfrak{N} = \mathfrak{A}\mathfrak{B}$ with an associated matrix $W_{\mathfrak{A}} = \begin{pmatrix} A\alpha & \beta \\ N\gamma & A\delta \end{pmatrix}$, we have that

$$\lambda = \chi(A\delta - B\gamma) \prod_{(\varpi)=\mathfrak{p}|\mathfrak{A}} \chi_{\mathfrak{p}}(A/\varpi)\lambda_{\mathfrak{p}}.$$

Proof. Once again, the proof generalizes the proof of [Asa78, Theorem 2]. Since for $\mathfrak{A}$ coprime to $\mathfrak{A}'$, we may take $W_{\mathfrak{A}\mathfrak{A}'} = W_{\mathfrak{A}} W_{\mathfrak{A}'}$, it is enough to check the assertion for a prime ideal $\mathfrak{A} = \mathfrak{p}$.

By definition of $a^{(\mathfrak{p})}_{(\nu)}$ and $\lambda_{\mathfrak{p}}$, we have that:

$$f|T(\mathfrak{p}) \circ W_{\mathfrak{p}} = a_{\mathfrak{p}} f|W_{\mathfrak{p}} = a_{\mathfrak{p}} \lambda_{\mathfrak{p}} \sum_{\nu \gg 0} a^{(\mathfrak{p})}_{(\nu)} q^{\nu/\delta}. \tag{5.8}$$

We compute the left hand side in another way to get the result.

Since $\det W_{\mathfrak{p}} = \varpi$, we have that

$$B\gamma \equiv B\gamma - \varpi\delta = -1 \mod \mathfrak{p}.$$

Hence for $j \not\equiv 1 \mod \mathfrak{p}$,

$$1 + B\gamma j \equiv 1 - j \not\equiv 0 \mod \mathfrak{p},$$

so there exists $\ell \not\equiv 0 \mod \mathfrak{p}$ such that

$$(1 + B\gamma j)\ell \equiv 1 \mod \mathfrak{p}.$$

Moreover, this defines a bijection

$$\{j \in \mathcal{O}_F/\mathfrak{p} \mid j \not\equiv 1 \mod \mathfrak{p}\} \leftrightarrow \{\ell \in \mathcal{O}_F/\mathfrak{p} \mid j \not\equiv 0 \mod \mathfrak{p}\}.$$

One can then check that for $j \not\equiv 1 \bmod \mathfrak{p}$

$$\begin{pmatrix} 1 & j \\ & \varpi \end{pmatrix} W_{\mathfrak{p}} = \sigma_1 \begin{pmatrix} 1 & \ell \\ & \varpi \end{pmatrix} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$$

for some $\sigma_1 \in \Gamma_0(\mathfrak{N})$ such that $\chi(d(\sigma_1)) = \chi_{\mathfrak{p}}(\ell)$.

For $j = 1$, we have that:

$$\begin{pmatrix} 1 & 1 \\ & \varpi \end{pmatrix} W_{\mathfrak{p}} = \sigma_2 W_{\mathfrak{p}} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$$

for some $\sigma_2 \in \Gamma_0(\mathfrak{N})$ such that $\chi(d(\sigma_2)) = \chi_{\mathfrak{B}}(\varpi)$.

Using the expression (5.6) for $T(\mathfrak{p})$:

$$\begin{aligned} f|T(\mathfrak{p}) \circ W_{\mathfrak{p}} &= (N_{F/\mathbb{Q}}\mathfrak{p})^{k/2-1} \left(\sum_{j \in \mathcal{O}_F/\mathfrak{p}} f \bigg|_k \begin{pmatrix} 1 & j \\ & \varpi \end{pmatrix} W_{\mathfrak{p}} \right) \\ &= (N_{F/\mathbb{Q}}\mathfrak{p})^{k/2-1} \left(\sum_{\ell \not\equiv 0} \chi_{\mathfrak{p}}(\ell) f \bigg|_k \begin{pmatrix} 1 & \ell \\ & \varpi \end{pmatrix} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} \right) + \chi_{\mathfrak{B}}(\varpi) f \bigg|_k W_{\mathfrak{p}} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}. \end{aligned}$$

Using the q-expansions:

$$f = \sum_{\nu \gg 0} a_{(\nu)} q^{\nu/\delta}, \quad f|_k W_{\mathfrak{p}} = \lambda_{\mathfrak{p}} \sum_{\nu \gg 0} a_{(\nu)}^{(\mathfrak{p})} q^{\nu/\delta},$$

we have that

$$\begin{aligned} f \bigg|_k \begin{pmatrix} 1 & \ell \\ & \varpi \end{pmatrix} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} &= \sum_{\nu \gg 0} a_{(\nu)} e^{2\pi i \operatorname{Tr}(\nu\ell/\delta\varpi)} q^{\nu/\delta}, \\ f \bigg|_k W_{\mathfrak{p}} \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix} &= (N_{F/\mathbb{Q}}\mathfrak{p})^{k/2} \lambda_{\mathfrak{p}} \sum_{\nu \gg 0} a_{(\nu)}^{(\mathfrak{p})} q^{\nu\varpi/\delta}. \end{aligned}$$

Hence

$$\begin{aligned} f|T(\mathfrak{p}) \circ W_{\mathfrak{p}} = (N_{F/\mathbb{Q}}\mathfrak{p})^{k/2-1} \sum_{\nu \gg 0} a_{(\nu)} \left(\sum_{\ell \not\equiv 0} \chi_{\mathfrak{p}}(\ell) e^{2\pi i \operatorname{Tr}(\nu\ell/\delta\varpi)} \right) q^{\nu/\delta} \\ + (N_{F/\mathbb{Q}}\mathfrak{p})^{k-1} \chi_{\mathfrak{B}}(\varpi) \lambda_{\mathfrak{p}} \sum_{\nu \gg 0} a_{(\nu)}^{(\mathfrak{p})} q^{\nu\varpi/\delta}. \end{aligned}$$

If $\chi_{\mathfrak{p}}$ is primitive, then

$$\sum_{\ell \not\equiv 0} \chi_{\mathfrak{p}}(\ell) e^{2\pi i \operatorname{Tr}(\nu\ell/\delta\varpi)} = \overline{\chi_{\mathfrak{p}}(\nu)} \chi_{\mathfrak{p}}(\delta) C(\chi_{\mathfrak{p}})$$

since δ is coprime to ϖ, and hence

$$f|T(\mathfrak{p}) \circ W_{\mathfrak{p}} = (N_{F/\mathbb{Q}}\mathfrak{p})^{k/2-1} \chi_{\mathfrak{p}}(\delta) C(\chi_{\mathfrak{p}}) \sum_{\nu \gg 0} \overline{\chi_{\mathfrak{p}}(\nu)} a_{(\nu)} q^{\nu/\delta} + (N_{F/\mathbb{Q}}\mathfrak{p})^{k-1} \chi_{\mathfrak{B}}(\varpi) \lambda_{\mathfrak{p}} \sum_{\nu \gg 0} a_{(\nu)}^{(\mathfrak{p})} q^{\nu\varpi/\delta}.$$

If $\chi_{\mathfrak{p}}$ is not primitive, then $\chi_{\mathfrak{p}} = \mathbb{1}_{\mathfrak{p}}$ is the trivial character modulo $\mathfrak{p}$. Then, since ϖ is coprime to δ,

$$\sum_{\ell \not\equiv 0} \chi_{\mathfrak{p}}(\ell) e^{2\pi i \operatorname{Tr}(\nu\ell/\delta\varpi)} = \sum_{\ell \not\equiv 0} e^{2\pi i \operatorname{Tr}(\nu\ell/\delta\varpi)} = \begin{cases} N(\mathfrak{q}) - 1 & \mathfrak{p}|(\nu), \\ -1 & \text{otherwise.} \end{cases}$$

Hence:

$$f|T(\mathfrak{p})\circ W_{\mathfrak{p}} = -(N_{F/\mathbb{Q}}\mathfrak{p})^{k/2-1} \sum_{\nu \gg 0} a_{(\nu)} q^{\nu/\delta} + \sum_{\nu \gg 0} \left((N\mathfrak{p})^{k/2} a_{(\nu\varpi)} + (N_{F/\mathbb{Q}}\mathfrak{p})^{k-1} \chi_{\mathfrak{B}}(\varpi) \lambda_{\mathfrak{p}} a^{(\mathfrak{p})}_{(\nu)} \right) q^{\nu\varpi/\delta}.$$

Comparing the expression for $f|T(\mathfrak{p})\circ W_{\mathfrak{p}}$ in each case with equation (5.8) gives the result. □

5.5.2. *Numerical method, following* [Col18]. The explicit formulas above only apply to Hilbert modular forms of square-free level. We discuss how one could generalize a method of Collins to compute q-expansions at other cusps for general levels.

As in [Col18, Section 2], we consider a matrix α which takes infinity to the cusp and

$$\alpha_h = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix}.$$

For $f \in S_k(\Gamma_0(\mathfrak{N}))$,

$$f|[\alpha_h]_k \in S_k(\Gamma_0(\mathfrak{N}h))$$

and we want to compute its q-expansion:

$$f|[\alpha_h]_k = \sum_{\nu \gg 0} a_{(\nu),\alpha} q^m = \sum_{\mathfrak{n}} a_{\mathfrak{n},\alpha} \left(\sum_{m \in \mathbb{Z}} q^{u^m \nu} \right) \tag{5.9}$$

where $q^m = e^{2\pi i \operatorname{Tr}(m/\delta)}$ and $u \in (\mathcal{O}_F)^\times_+$ is a fundamental unit.

The idea of Collins [Col18, Section 2.3] is to sample points $z_1, \dots, z_M \in \mathcal{H}^2$ and use the q-expansion at ∞ of f to compute $f[\alpha_h]_k(z)$ for these values. Then to use a least squares algorithm to approximate the constants $a_{\mathfrak{n},\alpha}$ which satisfy

$$f[\alpha_h]_k \approx \sum_{\mathfrak{n}} a_{\mathfrak{n},\alpha} \left(\sum_{m \in \mathbb{Z}} q^{u^m \nu} \right).$$

Algorithm 5.21 (q-expansion at other cusps, adapted from [Col18, Algorithm 2.3])**.** Given:

- a Hilbert modular form f of level $\mathfrak{N}$, weight (k,k), with an algorithm to compute its Fourier coefficients $a_{\mathfrak{n}}$ for arbitrarily large $\mathfrak{n}$,
- a cusp $a/c \in \mathbb{Q}$ of width h,
- a maximal norm K of Fourier coefficients needed,
- a desired accuracy 10^{-E},
- an exponential decay factor e^{-C_0},

we can compute the Fourier coefficients $a_{\mathfrak{n},\alpha}$ for $\mathrm{Norm}(\mathfrak{n}) < N$, accurate up to 10^{-E} as follows:

(1) Either increase $K = K_0$ or decrease $C = C_0$ so that $KC \approx \log(10)E$ and work with interpolating
$$\sum_{\substack{\mathfrak{n} \\ N\mathfrak{n} \leq K}} a_{\mathfrak{n},\alpha} \left(\sum_{m \in \mathbb{Z}} q^{u^m \nu} \right).$$
(2) Choose M (for example, $2K_0$) and pick points $z_1, \ldots, z_M \in \mathcal{H}^2$ with both imaginary parts equal to $C/2\pi$ and $\mathrm{Re}(z_j)$ randomly in $(-d/ch - 1/2, -d/ch + 1/2)^2$.
(3) Numerically compute the values $f||[\alpha_h](z_j) = h^{k/2}(ch(z_{j,1})+d)^{-k}(ch(z_{j,2})+d)^{-k} f(\alpha_h z_j)$ using the q-expansion of f, truncating until we have reached an accuracy a little greater than 10^{-E}, and fill these into a vector b.
(4) Numerically compute the values $\sum_{m \in \mathbb{Z}} q^{u^m \nu}$ for each $z = z_1, \ldots, z_M$ with an accuracy a little greater than 10^{-E}, and store them in a matrix A.
(5) Numerically find the least squares solution to $Ax = b$ as the exact solution to $(A^*A)x = A^*b$. The solution vector is our approximation to the coefficients $a_{\mathfrak{n},\alpha}$ for each $\mathfrak{n}$ of norm at most K.

We implemented this algorithm, but step (3) is very slow in practice. Since we need a lot of Fourier coefficients in our case, it is not realistic to apply this algorithm for our purposes.

5.6. **Numerical evidence.** We can use Theorems 5.14, 5.19, and 5.20 to compute the integral and verify that:
$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1 \overline{z}, \epsilon_2 z) y \frac{dz \wedge d\overline{z}}{y^2} = c \cdot \log(u_{f,\sigma_1})$$
for some $c \in \mathbb{Q}^\times$. This numerically verifies Conjecture 5.6, a consequence of Conjecture 3.21.

We performed the computations in Example 1.14 because the unit group U_{f_F} is easily described in that case. This is the base change of Example 1.13 to a real quadratic extension $F = \mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$.

We briefly recall Example 1.13 to set up the notation. Let $K = \mathbb{Q}(\alpha)$ be a cubic field of signature $[1, 1]$, obtained by adjoining a root α of a cubic polynomial $P(x)$. The splitting field L of $P(x)$ is the Galois closure of K and $G_{L/\mathbb{Q}} \cong S_3$. We consider the irreducible odd Artin representation
$$G_{L/\mathbb{Q}} \cong S_3 \to \mathrm{GL}_2(\mathbb{Z}).$$
It has an associated modular form f_0 and we consider its base change f to $F = \mathbb{Q}(\sqrt{d})$. The associated unit group is $U_{f_0} \cong U_K^{(1)}$, the norm 1 units of K, and we consider a generator $u = u_{f,\sigma_1} = u_{f_0}$ of this group.

Table 5.1 shows constants $c \in \mathbb{Q}$ such that the equality
$$\int_{Y_{\mathbb{C}}^{\mathrm{an}}} f(\epsilon_1 \overline{z}, \epsilon_2 z) y \frac{dz \wedge d\overline{z}}{y^2} = c \cdot \log(u) \tag{5.10}$$
holds up to at least 20 digits. The computations were performed on the High Performance Computing cluster Great Lakes at the University of Michigan.

d	polynomial $P(x)$	level N	unit u	constant c	time taken
5	$x^3 - x^2 + 1$	23	$\alpha^2 - \alpha$	2	00:09:34
5	$x^3 + x - 1$	31	α	-4	00:13:36
5	$x^3 + 2x - 1$	59	α^2	-8	01:56:22
13	$x^3 - x^2 + 1$	23	$\alpha^2 - \alpha$	8	00:10:19
13	$x^3 + x - 1$	31	α	-2	00:49:47
13	$x^3 + 2x - 1$	59	α^2	-22	29:47:44
17	$x^3 - x^2 + 1$	23	$\alpha^2 - \alpha$	14	00:16:52
17	$x^3 + x - 1$	31	α	-18	01:01:15
29	$x^3 - x^2 + 1$	23	$\alpha^2 - \alpha$	4	00:32:08
29	$x^3 + x - 1$	31	α	-14	02:38:12
37	$x^3 - x^2 + 1$	23	$\alpha^2 - \alpha$	10	00:25:45
37	$x^3 + x - 1$	31	α	-6	01:41:38

TABLE 5.1. This table presents constants c such that equation (5.10) holds for the unit u and the base change to $\mathbb{Q}(\sqrt{d})$ of the modular form of level N associated to the polynomial $P(x)$. The time taken to perform the computation with at least 20 digits of accuracy is displayed in the format hh:mm:ss.

It is quite remarkable that all the constants c are even integers and not just rational numbers. Rubin's integral refinement of Stark's Conjecture [Rub96] could provide an explanation. Understanding this phenomenon may also be related to studying congruence numbers for f [DHI98] and a potential integral refinement of Conjecture 3.21 would have to take them into account.

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