THE GALOIS ACTION ON SYMPLECTIC K-THEORY

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ABSTRACT. We study a symplectic variant of algebraic K-theory of the integers, which comes equipped with a canonical action of the absolute Galois group of $\mathbf{Q}$. We compute this action explicitly. The representations we see are extensions of Tate twists $\mathbf{Z}_p(2k-1)$ by a trivial representation, and we characterize them by a universal property among such extensions. The key tool in the proof is the theory of complex multiplication for abelian varieties.

CONTENTS

1. INTRODUCTION

1.1. **Motivation and results.** Let $\mathrm{Sp}_{2g}(\mathbf{Z})$ be the group of automorphisms of $\mathbf{Z}^{2g}$ preserving the standard symplectic form $\langle x, y \rangle = \sum_{i=1}^{g}(x_{2i-1}y_{2i} - x_{2i}y_{2i-1})$. The group homology

$$H_i(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_p) \tag{1.1}$$

with coefficients in the ring of p-adic numbers, carries a natural action of the group $\mathrm{Aut}(\mathbf{C})$ which comes eventually from the relationship between $\mathrm{Sp}_{2g}(\mathbf{Z})$ and $\mathcal{A}_g$, the moduli stack of principally polarized abelian varieties; we discuss this in more detail in §1.2. It is a natural question to understand this action; indeed, studying the actions of Galois automorphisms on (co)homology of arithmetic groups has been a central concern of number theory.

It was proved by Charney ([Cha87, Corollary 4.5]) that the homology groups (1.1) are independent of g, as long as $g \geq 2i + 5$, in the sense that the evident inclusion $\mathrm{Sp}_{2g}(\mathbf{Z}) \hookrightarrow \mathrm{Sp}_{2g+2}(\mathbf{Z})$ induces an isomorphism in group homology. These maps are also equivariant for $\mathrm{Aut}(\mathbf{C})$, and so it is sensible to ask how $\mathrm{Aut}(\mathbf{C})$ acts on the *stable homology*

$$H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p) := \varinjlim_g H_i(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_p).$$

The answer to this question with rational $\mathbf{Q}_p$-coefficients is straightforward. The homology in question has an algebra structure induced by $\mathrm{Sp}_{2g_1} \times \mathrm{Sp}_{2g_2} \hookrightarrow \mathrm{Sp}_{2(g_1+g_2)}$, and is isomorphic

1

to a polynomial algebra:

$$H_*(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Q}_p) \simeq \mathbf{Q}_p[x_2, x_6, x_{10}, \dots]$$

and $\mathrm{Aut}(\mathbf{C})$ acts on x_{4k-2} by the $(2k-1)$st power of the cyclotomic character. The elements $x_2, x_6, \dots$, can be chosen primitive with respect to the coproduct on homology.

With $\mathbf{Z}_p$ coefficients, it is not simple even to describe the stable homology as an abelian group. However situation looks much more elegant after passing to a more homotopical invariant—the *symplectic K-theory* $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$—which can be regarded as a distillate of the stable homology. We recall the definition in §1.2; for the moment we just note that $\mathrm{Aut}(\mathbf{C})$ also acts on the symplectic K-theory and there is an equivariant morphism

$$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) \to H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p) \tag{1.2}$$

which, upon tensoring with $\mathbf{Q}_p$, identifies the left-hand side with the primitive elements in the right-hand side. In particular,

$$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) \otimes \mathbf{Q}_p \cong \begin{cases} \mathbf{Q}_p(2k-1), & i = 4k-2 \in \{2, 6, 10, \dots\}, \\ 0, \text{else.} \end{cases} \tag{1.3}$$

where $\mathbf{Q}_p(2k-1)$ denotes $\mathbf{Q}_p$ with the $\mathrm{Aut}(\mathbf{C})$-action given by the $(2k-1)$st power of the cyclotomic character.

The identification of (1.3) can be made very explicit. The moduli stack of principally polarized abelian varieties carries a Hodge vector bundle (see §1.2) whose Chern character classes induce maps $\mathrm{ch}_{2k-1} : H_{4k-2}(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_p) \to \mathbf{Q}_p$. Passing to the limit $g \to \infty$ and composing with (1.2) gives rise to homomorphisms $c_H : \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \to \mathbf{Z}_p$ for all $k \geq 1$; then $c_H \otimes \mathbf{Q}_p$ recovers (1.3).

1.1.1. *Statement of main results.* For each n, let $\mathbf{Q}(\zeta_{p^n})$ be the cyclotomic field obtained by adjoining p^nth roots of unity, and let H_{p^n} be the maximal everywhere unramified abelian extension of $\mathbf{Q}(\zeta_{p^n})$ of p-power degree; put $H_{p^\infty} = \bigcup H_{p^n}$. We regard these as subfields of $\mathbf{C}$.

Main theorem (see Theorem 7.8). Let p be an odd prime.
 (i) The map $c_H : \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \to \mathbf{Z}_p(2k-1)$ is surjective and equivariant for the $\mathrm{Aut}(\mathbf{C})$ actions;
 (ii) The kernel of c_H is a finite p-group with trivial $\mathrm{Aut}(\mathbf{C})$ action;
 (iii) The action of $\mathrm{Aut}(\mathbf{C})$ factors through the Galois group Γ of H_{p^∞} over $\mathbf{Q}$. The sequence

$$\mathrm{Ker}(c_H) \hookrightarrow \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{c_H} \mathbf{Z}_p(2k-1). \tag{1.4}$$

 is initial among all such extensions of $\mathbf{Z}_p(2k-1)$ by a Γ-module with trivial action (all modules being p-complete and equipped with continuous Γ-action).

In particular, the extension of $\mathrm{Aut}(\mathbf{C})$-modules $\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \to \mathbf{Z}_p(2k-1)$ is not split if $\mathrm{Ker}(c_H)$ is nontrivial, and in this case the $\mathrm{Aut}(\mathbf{C})$-action does *not* factor through the cyclotomic character. In fact $\mathrm{Ker}(c_H)$ is canonically isomorphic to the p-completed algebraic K-theory $K_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$ which, through the work of Voevodsky and Rost, and Mazur and Wiles, we know is non-zero precisely when p divides the numerator of $\zeta(1-2k)$ (see [Wei05, Example 44]). The first example is $k = 6, p = 691$. The group $\Gamma = \mathrm{Gal}(H_{p^\infty}/\mathbf{Q})$ itself is a central object of Iwasawa theory; it surjects onto $\mathbf{Z}_p^\times$ via the cyclotomic character, with abelian kernel. In general Γ is non-abelian, with its size is controlled by the p-divisibility of ζ-values.

Remark 1.1. The theorem addresses degree $4k - 2$; this is the only interesting case. For $i = 4k$ or $4k + 1$ with $k > 0$, we explain in §3 that $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) = 0$. For $i = 4k + 3$, $\mathrm{KSp}_{4k+3}(\mathbf{Z}; \mathbf{Z}_p) \cong K_{4k+3}(\mathbf{Z}; \mathbf{Z}_p)$ is a finite group, and we establish in §7.7 that the $\mathrm{Aut}(\mathbf{C})$-action on $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$ is trivial.

1.1.2. *Other formulations.* There are other, equally reasonable, universal properties that can be formulated. For example—and perhaps more natural from the point of view of number theory—$\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$ can be considered as the fiber, over $\mathrm{Spec}\,\mathbf{C}$, of an étale sheaf on $\mathbf{Z}[1/p]$; then it is (informally) the largest split-at-$\mathbf{Q}_p$ extension of $\mathbf{Z}_p(2k - 1)$ by a trivial étale sheaf. See §7.5 for more discussion of this and other universal properties.

1.1.3. The prediction of the Langlands program is—informally—that "every Galois representation that looks like it could arise in the cohomology of arithmetic groups, in fact does so arise." In the cases at hand there is no more exact conjecture available; but we regard the universality statement above as fulfilling the spirit of this prediction. The occurrence of extensions as in (1.4) is indeed familiar from the Langlands program, where they arise (see e.g. [Rib76]) in the study of congruences between Eisenstein series and cusp forms. They arise in our context in a very direct way, and our methods are also quite different.

It would be of interest to relate our results to the study of the action of Hecke operators on stable cohomology; the latter has been computed for GL_n by Calegari and Emerton [CE16].

1.1.4. *Consequences.* Before we pass to a more detailed account, let us indicate a geometric implication of this result (which is explained in more detail in §8).

If $\mathcal{A} \to S$ is an principally polarized abelian scheme over $\mathbf{Q}$ with fiber dimension g then one has a classifying map $S \to \mathcal{A}_g$. If S is projective over $\mathbf{Q}$ of odd dimension $(2k - 1)$, then we get a cycle class $[S] \in H_{4k-2}(\mathcal{A}_g; \mathbf{Z}_p)$ which transforms according under $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ by the $(2k - 1)$st power of the cyclotomic character. (Examples of this situation can be constructed arising from a Shimura variety, or from the relative Jacobian of a family of curves.) By pairing $[S]$ with the Chern character of the Hodge bundle, we get a characteristic number

$$c_{\mathrm{H}}([S]) \in \mathbf{Q}$$

of the family. If the numerator of $c_{\mathrm{H}}([S])$ is not divisible by p then $[S]$ splits the analogue of the sequence (1.4), but replacing KSp_{4k-2} by H_{4k-2}. Now, in the range when $p > 2k$ we may in fact identify KSp_{4k-2} as a quotient of H_{4k-2} as a Galois module (see §8), and thereby the sequence (1.4) itself splits. Comparing with our theorem, we see that

$$p > 2k \text{ divides numerator of } \zeta(1 - 2k) \implies p \text{ divides numerator of } c_{\mathrm{H}}([S]). \qquad (1.5)$$

In other words, our theorem gives a universal divisibility for characteristic numbers of families of abelian varieties over $\mathbf{Q}$.

1.2. Symplectic K-theory of $\mathbf{Z}$: definition, Galois action, relationship with usual K-theory. We now give some background to the discussion of the previous section, in particular outlining the definition of symplectic K-theory and where the Galois action on it comes from. For the purposes of this section we adopt a slightly *ad hoc* approach to K-theory that differs somewhat from the presentation in the main text (§3), but is implicit in the later discussion where the Galois action is constructed (§6.2 and Appendix B). More detailed explanations are given in the later text.

First let us explain in more detail the Galois action on the *homology* of $\mathrm{Sp}_{2g}(\mathbf{Z})$ with $\mathbf{Z}_p$ coefficients. As usual in topology, the group homology of a discrete group such as $\mathrm{Sp}_{2g}(\mathbf{Z})$ can be computed as the singular homology of its classifying space $B\mathrm{Sp}_{2g}(\mathbf{Z})$, that is to say, the quotient of a contractible $\mathrm{Sp}_{2g}(\mathbf{Z})$-space with sufficiently free action. In the case at hand, there is a natural model for this classifying space that arises in algebraic geometry:

The group $\mathrm{Sp}_{2g}(\mathbf{Z})$ acts on the contractible Siegel upper half plane $\mathfrak{h}_g$ (complex symmetric $g \times g$ matrices with positive definite imaginary part) and uniformization of abelian varieties identifies the quotient $\mathfrak{h}_g /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z})$, as a complex orbifold, with the complex points of $\mathcal{A}_g$ in the analytic topology. Since $\mathfrak{h}_g$ is contractible, we may identify the cohomology group $H^i(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}/p^n\mathbf{Z})$ with the sheaf cohomology of the constant sheaf $\mathbf{Z}/p^n\mathbf{Z}$ on $\mathcal{A}_{g,\mathbf{C}}$, which by a comparison theorem is identified with étale cohomology. The fact that $\mathcal{A}_g$ is defined over $\mathbf{Q}$ associates a map of schemes $\sigma : \mathcal{A}_{g,\mathbf{C}} \to \mathcal{A}_{g,\mathbf{C}}$ to any $\sigma \in \mathrm{Aut}(\mathbf{C})$, inducing a map on (étale) cohomology. This is Pontryagin dualized to an action on $H_i(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}/p^n\mathbf{Z})$, for all n, and hence an action on (1.1) by taking inverse limit. (Here we used that arithmetic groups have finitely generated homology groups, in order to see that certain derived inverse limits vanish.)

1.2.1. *Definition of symplectic K-theory.* Next let us outline one definition of symplectic K-theory. We will do so only with p-adic coefficients, and in a way that is adapted to discussing the Galois action; a more detailed exposition from a more sophisticated viewpoint is given in §3.

The first step is the insight, due to Sullivan, that there is an operation on spaces (or homotopy types) that carries out p-completion at the level of homology. In particular, there is a p-completion map

$$B\mathrm{Sp}_{2g}(\mathbf{Z}) \to B\mathrm{Sp}_{2g}(\mathbf{Z})_p^{\wedge}$$

inducing an isomorphism in mod p homology and hence mod p^n homology, and whose codomain turns out to be simply connected (at least for $g \geq 3$ where $\mathrm{Sp}_{2g}(\mathbf{Z})$ is a perfect group). Moreover, the $\mathrm{Aut}(\mathbf{C})$ action that exists on the mod p^n homology of the left hand side can be promoted to an actual action of $\mathrm{Aut}(\mathbf{C})$ on the space $B\mathrm{Sp}_{2g}(\mathbf{Z})_p^{\wedge}$.

Although the space $B\mathrm{Sp}_{2g}(\mathbf{Z})$ has no homotopy in degrees 2 and higher, its p-completion *does*. As with (1.1), these homotopy groups are eventually independent of g; the resulting stabilized groups are the (p-completed) *symplectic K-theory* groups denoted

$$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) := \varinjlim_g \pi_i(B\mathrm{Sp}_{2g}(\mathbf{Z})_p^{\wedge})$$

in analogy with the p-completed algebraic K-theory groups $K_i(\mathbf{Z}; \mathbf{Z}_p)$, which can be similarly computed as $\mathrm{colim}_g \pi_i(B\mathrm{GL}_g(\mathbf{Z})_p^{\wedge})$.

The action of $\mathrm{Aut}(\mathbf{C})$ on the space $B\mathrm{Sp}_{2g}(\mathbf{Z})_p^{\wedge}$ now gives an action of $\mathrm{Aut}(\mathbf{C})$ on $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$, for which the Hurewicz morphism

$$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) \to H_i(\mathrm{Sp}_{\infty}(\mathbf{Z}); \mathbf{Z}_p) \tag{1.6}$$

is equivariant.

Remark 1.2. Although it is not obvious from the presentation above, these groups $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$ are in fact the p-completions of symplectic K-groups $\mathrm{KSp}_i(\mathbf{Z})$ which are finite generated abelian groups (see §3). However, the $\mathrm{Aut}(\mathbf{C})$ action exists only after p-adically completing.

1.2.2. We also recall what is known about the underlying $\mathbf{Z}_p$-modules (ignoring Galois-action). These results are deduced from Karoubi's work on *Hermitian K-theory* [Kar80], combined with what is now known about algebraic K-theory of $\mathbf{Z}$. The upshot is isomorphisms for $k \geq 1$ and odd primes p

$$\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{(c_B, c_H)} K_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \times \mathbf{Z}_p$$

$$\mathrm{KSp}_{4k-1}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{(c_B, c_H)} K_{4k-1}(\mathbf{Z}; \mathbf{Z}_p)$$

and vanishing homotopy groups in degrees $\equiv 0, 1 \mod 4$. Here:

- The homomorphism c_B arises from the evident inclusion $\mathrm{Sp}_{2g}(\mathbf{Z}) \subset \mathrm{GL}_{2g}(\mathbf{Z})$.

- The homomorphism c_H is obtained as the composite

$$\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \to H_{4k-2}(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Q}_p) \overset{c_H}{\to} \mathbf{Q}_p.$$

Here the final map is the Chern character of the g-dimensional (Hodge) vector bundle arising from $\mathrm{Sp}_{2g}(\mathbf{Z}) \hookrightarrow \mathrm{Sp}_{2g}(\mathbf{R}) \simeq U(g)$; the composite map is valued in $\mathbf{Z}_p$ even though the Chern character involves denominators in general, reflecting one advantage of homotopy over homology.

1.3. **Method of proof and outline of paper.** For the present sketch we consider the reduction of symplectic K-theory modulo $q = p^n$; one recovers the main theorem by passing to a limit over n.

Remark 1.3. Rather than naïvely reducing homotopy groups modulo q, it is better to consider the groups $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$ which sit in a long exact sequence with the multiplication-by-q map $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) \to \mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$. But that is the same in degree $4k - 2$ since $\mathrm{KSp}_{4k-3}(\mathbf{Z}; \mathbf{Z}_p) = 0$.

The basic idea for proving the main theorem is to construct enough explicit classes on which one can compute the Galois action. In more detail, the theory of complex multiplication (CM) permits us to exhibit a large class of complex principally polarized abelian varieties with actions of a cyclic group C with order q. If $A \to \mathrm{Spec}\,(\mathbf{C})$ comes with such an action then the induced action on $H_1(A(\mathbf{C})^{\mathrm{an}}; \mathbf{Z})$, singular homology of the complex points in the analytic topology, gives a homomorphism $C \to \mathrm{Sp}_{2g}(\mathbf{Z})$. This gives a morphism

$$\pi_i^s(BC; \mathbf{Z}/q) \longrightarrow \mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$$

from the stable homotopy groups of the classifying space BC to symplectic K-theory. The left hand side contains a polynomial ring[1] on a degree 2 element, the "Bott element," and the image of this ring produces a supply of classes in KSp.

Let us call temporarily call classes in $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$ arising from this mechanism *CM classes*. We shall then show, on the one hand, that CM classes generate all of $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$. On the other hand the Main Theorem of Complex Multiplication allows us to understand the action of $\mathrm{Aut}(\mathbf{C})$ on CM classes. Taken together, this allows us to compute the $\mathrm{Aut}(\mathbf{C})$ action on $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$.

The contents of the various sections are as follows:

- §2, *K-theory and its relation to algebraic number theory:* We review facts about homotopy groups, Bott elements, K-theory, and the relation of K-theory and étale cohomology. From the point of view of the main proof, the main output here is Proposition 2.17, which identifies the transfer map from the K-theory of a cyclotomic ring to the K-theory of $\mathbf{Z}$ in terms of algebraic number theory: namely, a transfer in the homology of corresponding Galois groups.
- §3, *Symplectic K-theory:* We review the definition of symplectic K-theory, and recall the results of [Kar80] which, for odd p, lets us describe $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$ and $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$ in terms of usual algebraic K-theory. The conclusions we need are summarized in Theorem 3.5.
- §4, *Construction of CM classes in symplectic K-theory:* The point of §4 is to set up the theory of CM in a slightly unconventional form that allows the CM classes to be easily defined. One key output of the section is the sequence (4.6): it formulates the construction of CM abelian varieties as a functor between groupoids. We also prove in Proposition 4.7 a technical result about the existence of "enough" CM abelian varieties associated to cyclotomic fields.

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[1] This is an advantage of stable homotopy over homology: the latter (in even degrees) is a divided power algebra.

- §5, *CM classes exhaust all of symplectic K-theory:* We give the construction of CM classes and prove that that they exhaust all of symplectic K-theory (see in particular Proposition 5.1). To prove the exhaustion one must check both that KSp is not too large and that there are enough CM classes. These come, respectively, from the previously mentioned Proposition 2.17 and Proposition 4.7.
- §6 *Computation of the action of* Aut($\mathbf{C}$) *on CM classes:* The action of Aut($\mathbf{C}$) on CM classes can be deduced from the "Main Theorem of CM," which computes how Aut($\mathbf{C}$) acts on moduli of CM abelian varieties. (In its original form this is due to Shimura and Taniyama; we use the refined form due to Langlands, Tate, and Deligne.) We recall this theorem, in a language adapted to our proof, in §6.3
- §7, *Proof of the main theorem (Theorem 7.1).* The results of the previous sections have already entirely computed the Galois action. More precisely, they allow one to explicitly give a cocycle that describes the extension class of (1.3). In §7.3 we explicitly compare this cocycle to one that describes the universal extension and show they are equal.

  The remainder of §7 describes variants on the universal property (e.g. passing between $\mathbf{Z}/q$ and $\mathbf{Z}_p$ coefficients, or a version for Bott-inverted symplectic K-theory which also sees extensions of *negative* Tate twists).
- §8, *Consequences in homology.* The stable homology $H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p)$ naturally surjects onto $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$, at least for $i \leq 2p - 2$. In this short section we use this to deduce divisibility of certain characteristic numbers of families of abelian varieties defined over $\mathbf{Q}$.

Remark 1.4. Let us comment on the extent to which our result depends on the *norm residue theorem*, proved by Voevodsky and Rost. The p-completed homotopy groups $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}_p)$ in our main theorem may be replaced by groups we denote $\mathrm{KSp}_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$ and call "Bott inverted symplectic K-theory" see Subsection 7.6. They agree with $\pi_*(L_{K(1)}\mathrm{KSp}(\mathbf{Z}))$, the so-called $K(1)$-*local homotopy groups*.

The norm residue theorem can be used to deduce that the canonical map $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) \to \mathrm{KSp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$ is an isomorphism for all $i \geq 2$. Independently of the norm residue theorem, the main theorem stated above may be proved with $\mathrm{KSp}_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$ in place of $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$. Besides the simplification of the proof, this has the advantage of giving universal extensions of $\mathbf{Z}_p(2k-1)$ for all $k \in \mathbf{Z}$, including non-positive integers.

In our presentation we have chosen to work mostly with $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}_p)$ instead of $\mathrm{KSp}_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$, for reasons of familiarity. A more puritanical approach would have compared $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}_p)$ and $\mathrm{KSp}_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$ at the very end, and this would have been the only application of the norm residue theorem.

1.4. **Notation.** For q any *odd* prime power, we denote:

- $\mathcal{O}_q$ the cyclotomic ring $\mathbf{Z}[e^{2\pi i/q}]$ obtained by adjoining a primitive qth root of unity to $\mathbf{Z}$, and $K_q = \mathcal{O}_q \otimes \mathbf{Q}$ its quotient field. For us we shall always regard these as subfields of $\mathbf{C}$. We denote by $\zeta_q \in \mathcal{O}_q$ the primitive qth root of unit $e^{2\pi i/q}$.
- $\mathbf{Z}' := \mathbf{Z}[\frac{1}{p}]$, and $\mathcal{O}'_q := \mathcal{O}_q[\frac{1}{p}]$.
- We denote by H_q the largest algebraic unramified extension of K_q inside $\mathbf{C}$ whose Galois group is abelian of p-power order. Thus H_q is a subfield of the Hilbert class field, and its Galois group is isomorphic to the p-power torsion inside the class group of $\mathcal{O}_q$.
- For a ring R, we denote by $\mathrm{Pic}(R)$ the groupoid of locally free rank one R-modules, and by $\pi_0\mathrm{Pic}(R)$ the group of isomorphism classes, i.e. the class group of R. In particular, the class group of $\mathcal{O}_q$ is denoted $\pi_0\mathrm{Pic}(\mathcal{O}_q)$.

- There are "Hermitian" variants of the Picard groupoid that will play a crucial role for us. For the ring of integers $\mathcal{O}_E$ in a number field E, $\mathcal{P}_E^+$ will denote the groupoid of rank one locally free $\mathcal{O}_E$-modules endowed with a $\mathcal{O}_E$-valued Hermitian form, and $\mathcal{P}_E^-$ will denote the groupoid of rank one locally free $\mathcal{O}_E$-modules endowed with a skew-Hermitian form valued in the inverse different. See §4.2 for details of these definitions.

We emphasize that q is assumed to be *odd*. Many of our statements remain valid for q a power of 2, and we attempt to make arguments that remain valid in that setting, but for simplicity we prefer to impose q odd as a standing assumption.

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2. Recollections on algebraic K-theory

This section reviews algebraic K-theory and its relation with étale cohomology. Since it is somewhat lengthy we briefly outline the various subsections:

- §2.1 is concerned with summarizing facts about stable and mod q homotopy groups; in particular we introduce the Bott element in the stable homotopy of a cyclic group.
- After a brief discussion of infinite loop space machines in §2.2, we review algebraic K-theory in §2.3. In §2.4 we discuss the Picard group and Picard groupoid, which are used to analyze a simple piece of algebraic K-theory.
- A fundamental theorem of Thomason asserts that algebraic K-theory satisfies étale descent after inverting a Bott element (defined in mod q algebraic K-theory in §2.5). We review this theorem and its consequences in §2.6.
- §2.7 uses Thomason's results to compute Bott-inverted K-theory of $\mathbf{Z}$ and of $\mathcal{O}_q$ in terms of étale (equivalently, Galois) cohomology.
- Finally, in Proposition 2.17 we rewrite some of the results of §2.7 in terms of homology of Galois groups, which is most appropriate for our later applications. Specifically, the Proposition identifies the transfer map from the K-theory of a cyclotomic ring to the K-theory of $\mathbf{Z}$ in a corresponding transfer in the group homology.

We refer the reader to the Appendix A for a brief summary of some topological background, including a brief account of the theory of spectra.

2.1. **Recollections on stable and mod q homotopy.** Recall that, for a topological space Y, the notation Y_+ means the space $Y \coprod \{*\}$ consisting of Y together with a disjoint basepoint. Each space gives rise to a spectrum $\Sigma_+^\infty Y$, namely the suspension spectrum on Y_+, and consequently we can freely specialize constructions for spectra to those for spaces. In particular, the *stable homotopy groups* of Y are, by definition, the homotopy groups of the associated spectrum:

$$\pi_k^s(Y) = \pi_k(\Sigma_+^\infty Y) := \varinjlim_n [S^{k+n}, \Sigma^n Y_+],$$

where $[-,-]$ denotes homotopy classes of based maps. We emphasize that π_*^s is defined for an *unpointed* space Y. (In some references, it is defined for a based space, and in those references the definition does not involve an added disjoint basepoint).

Remark 2.1. One could regard $\pi_*^s(Y)$ as being the "homology of Y with coefficients in the sphere spectrum," and it enjoys the properties of any generalized homology theory (see Example A.2 in Appendix A). There is a Hurewicz map $\pi_*^s(Y) \to H_*(Y)$, which is an isomorphism in degree $* = 0$ and a surjection in degree 1.

We will be interested in the corresponding notion with $\mathbf{Z}/q$ coefficients. For any spectrum E, the homomorphisms $\pi_i(E) \to \pi_i(E)$ which multiply by $q \in \mathbf{Z}$ fit into a long exact sequence

$$\cdots \to \pi_i(E) \xrightarrow{q} \pi_i(E) \to \pi_i(E \wedge (\mathbb{S}/q)) \to \pi_{i-1}(E) \xrightarrow{q} \pi_i(E) \to \ldots,$$

where the spectrum $\mathbb{S}/q$ is the mapping cone of a degree-q self map of the sphere spectrum. For $q > 0$ we write

$$\pi_i(E; \mathbf{Z}/q) := \pi_i(E \wedge (\mathbb{S}/q))$$

for these groups, the *homotopy groups of E with coefficients in $\mathbf{Z}/q$.* (For more about this, see §A.5.) Correspondingly, we get stable homotopy groups $\pi_*^s(Y; \mathbf{Z}/q)$ for a space Y. These have the usual properties of a homology theory.

2.1.1. *The Bott element in the stable homotopy of a cyclic group.* The stable homotopy of the classifying space of a cyclic group contains a polynomial algebra on a certain "Bott element" in degree 2. This will be a crucial tool in our later arguments, and we review it now.

We recall (see [Oka84]) that for $q = p^n > 4$ the spectrum $\mathbb{S}/q$ has a product which is unital, associative, and commutative up to homotopy. It makes $\pi_*(E; \mathbf{Z}/q)$ into a graded ring when E is a ring spectrum, graded commutative ring when the product on E is homotopy commutative. In the rest of this section we shall tacitly assume $q > 4$ in order to have such ring structures available. (In fact everything works also in the remaining case $p = q = 3$ with only minor notational updates: see Remark 2.19.)

For the current subsection §2.1.1 set $Y := B(\mathbf{Z}/q)$, the classifying space of a cyclic group of order q. This Y has the structure of H-space, in fact a topological abelian group, and correspondingly $\pi_*^s(Y)$ has the structure of a graded commutative ring.

Recall that q is supposed odd. Then there is a unique element (the "Bott class") $\beta \in \pi_2^s(Y; \mathbf{Z}/q)$ such that, in the diagram

$$
\begin{array}{ccc}
\beta \in \pi_2^s(Y; \mathbf{Z}/q) & \longrightarrow & \pi_1^s(Y; \mathbf{Z}) \\
\sim \downarrow \text{Hur} & & \downarrow \\
0 \longrightarrow H_2(Y; \mathbf{Z}/q) \longrightarrow & H_1(Y; \mathbf{Z})[q] & = \mathbf{Z}/q
\end{array}
\tag{2.1}
$$

the image of β in the bottom right $\mathbf{Z}/q\mathbf{Z}$ is the canonical generator 1 of $\mathbf{Z}/q\mathbf{Z}$. In fact, the Hurewicz map Hur above is an isomorphism.

Remark 2.2. The diagram (2.1) exists for all q, but for q a power of 2 the map $\pi_2^s(Y; \mathbf{Z}/q) \to H_2(Y; \mathbf{Z}/q)$ is not an isomorphism. There is a class in $\pi_2^s(Y; \mathbf{Z}/q)$ fitting into (2.1) when q is a power of 2, but the diagram above does not characterize it. To pin down the correct β in that case, note that the map $S^1 \to Y$ inducing $\mathbf{Z} \to \mathbf{Z}/q$ on π_1 extends to $M(\mathbf{Z}/q, 1) = S^1 \cup_q D^2$, the pointed mapping cone of the canonical degree q map $S^1 \to S^1$, and one can construct β starting from the identification of $\pi_k^s(Y; \mathbf{Z}/q) = \varinjlim_n [\Sigma^n M(\mathbf{Z}/q, k-1), \Sigma^n Y_+].)$

Lemma 2.3. *The induced map $\mathbf{Z}/q[\beta] \to \pi_*^s(Y; \mathbf{Z}/q)$ is a split injection of graded rings.*

Proof. The map $a \to e^{2\pi i a/q}$ is a homomorphism from $\mathbf{Z}/q$ to S^1 and it gives rise to a line bundle $\underline{L}$ on Y. In turn this induces a map from the suspension spectrum of Y_+ to the spectrum ku representing topological K-theory, and thereby induces on homotopy groups a map

$$\pi_*^s(Y; \mathbf{Z}/q) \to \pi_*(ku; \mathbf{Z}/q).$$

This map is in fact a ring map (see Section A.4 for discussion of products in K-theory).

We claim that the class β is sent to the reduction of the usual Bott class $\mathrm{Bott} \in \pi_2(ku)$; this implies that $\mathbf{Z}/q[\beta] \to \pi_*^s(Y; \mathbf{Z}/q)$ is indeed split injective, because $\pi_*(ku; \mathbf{Z}/q)$ is, in non-negative degrees, a polynomial algebra on this reduction. The Bott class in $\pi_2(ku; \mathbf{Z}) \simeq \pi_2(\mathrm{BU}, \mathbf{Z})$ is characterized (at least up to sign, depending on normalizations) by having pairing 1 with the first Chern class of the line bundle $\underline{L}$ arising from $\det : U \to S^1$. It suffices then to show that

$$\langle \bar{\beta}, c_1(\underline{L}) \rangle = 1 \in \mathbf{Z}/q,$$

where $\bar{\beta} \in H_2(Y; \mathbf{Z}/q)$ is the image of β in by the Hurewicz map, and $c_1(\underline{L})$ is the first Chern class of $\underline{L}$ considered as a line bundle on Y.

This Chern class is the image of $j \in H^1(Y; \mathbf{R}/\mathbf{Z})$ by the connecting homomorphism $H^1(Y, \mathbf{R}/\mathbf{Z}) \to H^2(Y, \mathbf{Z})$ arising from the map $\mathbf{Z} \to \mathbf{R} \overset{e^{2\pi i x}}{\to} \mathbf{R}/\mathbf{Z}$. Therefore $c_1(\underline{L}) \in H^2(Y, \mathbf{Z})$ is obtained from the tautological class $\tau \in H^1(Y, \mathbf{Z}/q)$ by the connecting map δ associated to $\mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}/q$, and the reduction of $c_1(\underline{L})$ modulo q is simply the Bockstein of τ. Therefore, the pairing of $c_1(\underline{L})$ with $\overline{\beta}$ is the same as the pairing of τ with the Bockstein of $\overline{\beta}$; this last pairing is 1, by definition of β. $\qquad\square$

Remark 2.4. The reasoning of the proof also shows the following: had we replaced the morphism $j : \mathbf{Z}/q \to S^1$ by j^a (for some $a \in \mathbf{Z}$), then the corresponding element in $\pi_2(ku; \mathbf{Z}/q)$ is also multiplied by a.

2.2. Infinite loop space machines.

Recall that associated to a small category $\mathcal{C}$, there is a *classifying space* $|\mathcal{C}|$, which is the geometric realization of the nerve of $\mathcal{C}$ (a simplicial set). In particular $\pi_0(|\mathcal{C}|)$ is the set of isomorphism classes. A symmetric monoidal structure on $\mathcal{C}$ induces in particular a "product" $|\mathcal{C}| \times |\mathcal{C}| \to |\mathcal{C}|$ which is associative and commutative up to homotopy. The theory of *infinite loop space machines* associates to the symmetric monoidal category $\mathcal{C}$ a *spectrum* $K(\mathcal{C})$ and a map

$$|\mathcal{C}| \to \Omega^\infty K(\mathcal{C}). \tag{2.2}$$

Up to homotopy this map preserves products, and the induced monoid homomorphism $\pi_0(|\mathcal{C}|) \to \pi_0(\Omega^\infty K(\mathcal{C}))$ is the universal homomorphism to a group, namely, the "Grothendieck group" of the monoid. The map (2.2) can be viewed as a derived version of the universal homomorphism from a given monoid to a group.

This useful principle—that small symmetric monoidal categories give rise to spectra—may be implemented in various ways, technically different but resulting in weakly equivalent spectra. We have chosen to work with Segal's *Gamma spaces*, but we shall only need to handle them explicitly for a few technical results, confined to Appendices A and B where we also review some background.

2.3. Algebraic K-theory: definitions.

For a ring R, let $\mathcal{P}(R)$ denote the symmetric monoidal groupoid whose objects are finitely generated projective R-modules, morphisms are R-linear isomorphisms, and with the Cartesian symmetric monoidal structure (i.e., direct sum of R-modules). The set $\pi_0(\mathcal{P}(R))$ of isomorphism classes in $\mathcal{P}(R)$ then inherits a commutative monoid structure. Write $|\mathcal{P}(R)|$ for the associated topological space (i.e. geometric realization of the nerve of $\mathcal{P}(R)$). Direct sum of projective R-modules is a symmetric monoidal structure on $\mathcal{P}(R)$ and induces a map $\oplus : |\mathcal{P}(R)| \times |\mathcal{P}(R)| \to |\mathcal{P}(R)|$.

As recalled above and in Appendix A, there is a canonically associated spectrum $K(R) := K(\mathcal{P}(R))$ and a "group completion" map

$$|\mathcal{P}(R)| \to \Omega^\infty K(R).$$

The algebraic K-groups of R are defined as the homotopy groups of $K(R)$. Alternately, for $i = 0$, it is the projective class group $K_0(R)$ while for $i > 0$ it may be defined as

$$K_i(R) := \pi_i B\mathrm{GL}_\infty(R)^+,$$

the homotopy groups of the Quillen plus construction applied to the commutator subgroup of $\mathrm{GL}_\infty(R) = \varinjlim_n \mathrm{GL}_n(R)$.[2]

When R is commutative, we also have product maps

$$K_i(R) \otimes K_j(R) \to K_{i+j}(R),$$

induced from tensor product of R-modules, making $K_*(R)$ into a graded commutative ring (see Section A.4 for a construction.)

Definition 2.5. We define the *mod q algebraic K-theory groups of R* to be

$$K_i(R; \mathbf{Z}/q\mathbf{Z}) := \pi_i(K(R); \mathbf{Z}/q\mathbf{Z}).$$

In the case $R = \mathbf{Z}$ we define the *p-adic algebraic K-theory groups* via

$$K_i(\mathbf{Z}; \mathbf{Z}_p) := \varprojlim_n K_i(\mathbf{Z}; \mathbf{Z}/p^n).$$

(This is the correct definition because of finiteness properties of $K_*(\mathbf{Z}; \mathbf{Z}/p^n\mathbf{Z})$; in general, we should work with "derived inverse limits.")

2.3.1. *Adams operations.* Finally, let us recall that (again for R commutative, as shall be the case in this paper) there are Adams operations $\psi^k : K_i(R) \to K_i(R)$ for $k \in \mathbf{Z}$ satisfying the usual formulae. We shall make particular use of ψ^{-1}, which in the above model is induced by the functor $\mathcal{P}(R) \to \mathcal{P}(R)$ sending a module M to its dual $D(M) := \mathrm{Hom}_R(M, R)$ and an isomorphism $f : M \to M'$ to the inverse of its dual $D(f) : D(M') \to D(M)$.

2.4. **Picard groupoids.** We now define certain spaces which can be understood explicitly and used to probe algebraic K-theory. They are built out of categories that we call Picard groupoids.

Definition 2.6. (The Picard groupoid.) For a commutative ring R, let $\mathrm{Pic}(R) \subset \mathcal{P}(R)$ be the subgroupoid whose objects are the rank 1 projective modules, with the symmetric monoidal structure given by $\otimes_R$.

The associated space $|\mathrm{Pic}(R)|$ inherits a group-like product $\otimes_R : |\mathrm{Pic}(R)| \times |\mathrm{Pic}(R)| \to |\mathrm{Pic}(R)|$, and there are canonical isomorphisms of abelian groups $\pi_0(|\mathrm{Pic}(R)|) = H^1(\mathrm{Spec}(R); \mathbf{G}_m)$ (the classical Picard group) and $\pi_1(|\mathrm{Pic}(R)|, x) = H^0(\mathrm{Spec}(R); \mathbf{G}_m) = R^\times$ for any object $x \in \mathrm{Pic}(R)$. The higher homotopy groups are trivial.

When R is a ring of integers, $\mathrm{Pic}(R)$ is equivalent to the groupoid whose objects are the invertible fractional ideals $I \subset \mathrm{Frac}(R)$ and whose set of morphisms $I \to I'$ is $\{x \in R^\times \mid xI = I'\}$.

The tensor product of rank 1 projective modules gives a product on the space $|\mathrm{Pic}(R)|$ and makes the stable homotopy groups $\pi_*^s(|\mathrm{Pic}(R)|)$ into a graded-commutative ring. We have a canonical ring isomorphism $\mathbf{Z}[\pi_0(\mathrm{Pic}(R))] \to \pi_0^s(|\mathrm{Pic}(R)|)$ from the group ring of the abelian

---

[2] The group completion theorem can be used to induce a comparison between $K_0(R) \times B\mathrm{GL}_\infty(R)^+$ and $\Omega^\infty K(R)$, roughly speaking by taking direct limit over applying $[R] \oplus - : |\mathcal{P}(R)| \to |\mathcal{P}(R)|$ infinitely many times and factoring over the plus construction.

group $\pi_0(\text{Pic}(R)) \cong H^1(\text{Spec}\,(R); \mathbf{G}_m)$. The fact that stable homotopy (being a homology theory) takes disjoint union to direct sum implies that the product map

$$\pi_*^s(BR^\times) \otimes \mathbf{Z}[\pi_0(\text{Pic}(R))] \xrightarrow{\cong} \pi_*^s(|\text{Pic}(R)|). \qquad (2.3)$$

is an isomorphism.

The inclusion functor induces maps $|\text{Pic}(R)| \to |\mathcal{P}(R)| \to \Omega^\infty K(R)$ preserving $\otimes_R$, at least up to coherent homotopies. The adjoint map $\Sigma_+^\infty |\text{Pic}(R)| \to K(R)$ is then a map of ring spectra, and we get a ring homomorphism

$$\pi_*^s(BR^\times) \otimes \mathbf{Z}[\pi_0(\text{Pic}(R))] \xrightarrow{\cong} \pi_*^s(|\text{Pic}(R)|) \to K_*(R). \qquad (2.4)$$

2.5. Bott elements in K-theory with mod q coefficients.

Definition 2.7. The *algebraic K-theory of R with mod q coefficients* is defined as $K_i(R; \mathbf{Z}/q) := \pi_i(K(R); \mathbf{Z}/q)$.

As discussed earlier, $K_*(R; \mathbf{Z}/q)$ has the structure of a graded-commutative ring for $q = p^n > 4$.

Let us next recall the construction of a canonical *Bott element* in $K_2(R; \mathbf{Z}/q)$ associated to a choice of primitive qth root of unity $\zeta_q \in R^\times$. The choice of ζ_q induces a homomorphism $\mathbf{Z}/q \to \text{GL}_1(R)$. Regarding $\text{GL}_1(R)$ as the automorphism group of the object $R \in \text{Pic}(R)$ gives a map $B(\mathbf{Z}/q) \to |\text{Pic}(R)|$. Now we previously produced a "Bott element" $\beta \in \pi_2^s(B(\mathbf{Z}/q))$; under the maps (2.4) we have

$$\beta \in \pi_2^s(B(\mathbf{Z}/q)) \to \pi_2^s(|\text{Pic}(R)|; \mathbf{Z}/q) \to K_2(R; \mathbf{Z}/q).$$

The image is the *Bott element* and shall also be denoted $\beta \in K_2(R; \mathbf{Z}/q)$. More intrinsically, this discussion gives a homomorphism

$$\beta : \mu_q(R) \to K_2(R; \mathbf{Z}/q) \qquad (2.5)$$

which is independent of any choices; since our eventual application is to subrings of $\mathbf{C}$ where we will take $\zeta = e^{2\pi i/q}$, we will not use this more intrinsic formulation.

2.6. Bott inverted K-theory and Thomason's theorem.
The element $\beta \in K_2(R; \mathbf{Z}/q)$ may be inverted in the ring structure (when $q > 8$), leading to a 2-periodic $\mathbf{Z}$-graded ring $K_*(R; \mathbf{Z}/q)[\beta^{-1}]$ called *Bott inverted K-theory* of R, when R contains a primitive qth root of unity. As explained in [Tho85, Appendix A] we can still make sense of this functor when R does not contain primitive qth roots of unity: the power $\beta^{p-1} \in K_{2p-2}(\mathbf{Z}[\mu_p]; \mathbf{Z}/p)$ comes from a canonical element in $K_{2p-2}(\mathbf{Z}; \mathbf{Z}/p)$, also denoted β^{p-1} (even though it is not the $(p-1)$st power of any element of $K_*(\mathbf{Z}; \mathbf{Z}/p)$), whose p^{n-1}st power lifts to an element of $K_{2p^{n-1}(p-1)}(\mathbf{Z}; \mathbf{Z}/p^n)$. Inverting the image of these elements gives a functor

$$X \mapsto K_*(X; \mathbf{Z}/q)[\beta^{-1}]$$

from schemes to $\mathbf{Z}$-graded $\mathbf{Z}/q$-modules (graded commutative $(\mathbf{Z}/q)$-algebras when $q > 4$), where $q = p^n$ as before. For typographical ease, we will denote this via $K^{(\beta)}$:

$$K_*^{(\beta)}(X; \mathbf{Z}/q) = K_*(X; \mathbf{Z}/q)[\beta^{-1}].$$

In the case $X = \text{Spec}\,\mathbf{Z}$, we also define the *$p$-adic Bott-inverted K-theory* groups

$$K_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p) := \varprojlim_n K_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}/p^n).$$

Remark 2.8. As also recalled in [Tho85, Appendix A] this may be implemented on the spectrum level as follows: Adams constructed spectrum maps $\Sigma^m(\mathbb{S}/p^n) \to (\mathbb{S}/p^n)$ for $m = 2p^{n-1}(p-1)$ when p is odd, with the property that it induces isomorphisms $\mathbf{Z}/q = \pi_0(ku; Z/q) \to \pi_m(ku; \mathbf{Z}/q) = \mathbf{Z}/q$, where ku is the topological K-theory spectrum, and we can let T be the homotopy colimit of the infinite iteration $\mathbb{S}/q \to \Sigma^{-m}(\mathbb{S}/q) \to \Sigma^{-2m}(\mathbb{S}/q) \to \dots$. Then $K_*^{(\beta)}(X; \mathbf{Z}/q)$ is canonically the homotopy groups of the spectrum $K(X) \wedge T$. We will on occasion denote this spectrum as $K^{(\beta)}(X; \mathbf{Z}/q)$.

2.6.1. *Étale descent and Thomason's spectral sequence.* The main result of [Tho85] is an étale descent property for the Bott inverted K-theory functor. (Because of this, Bott-inverted K-theory is essentially the same as Dwyer–Friedlander's "étale K-theory" [DF85], at least in positive degrees. See also [CM19] for a recent perspective.)

For a scheme X over $\mathrm{Spec}\,(\mathbf{Z}[1/p])$, Thomason constructs a convergent spectral sequence

$$E_{s,t}^2 = H_{\mathrm{et}}^{-s}(X; \mu_q^{\otimes(t/2)}) \Rightarrow K_{t+s}^{(\beta)}(X; \mathbf{Z}/q), \tag{2.6}$$

concentrated in degrees $s \in \mathbf{Z}_{\leq 0}$ and $t \in 2\mathbf{Z}$. (Existence and convergence of the spectral sequence requires mild hypotheses on X, satisfied in any case we need.) The spectral sequence arises as a hyperdescent spectral sequence for $K^{(\beta)}$, regarded as a sheaf of spectra on the étale site of X.

Since the Adams operations ψ^a act on $K^{(\beta)}$ through maps of sheaves of spectra when $a \not\equiv 0 \mod p$, there are compatible actions of Adams operations on the spectral sequence. The operation ψ^a acts by multiplication by $a^{t/2}$ on $E_{s,t}^2$ and in particular ψ^{-1} acts as $+1$ on the rows with $t/2$ even and as -1 on the rows where $t/2$ is odd.

2.6.2. *Comparison with algebraic K-theory.* This étale descent property makes Bott-inverted K-theory amenable to computation. On the other hand, it is a well known consequence of the *norm residue theorem* (due to Voevodsky and Rost) that when X is a scheme over $\mathrm{Spec}\,(\mathbf{Z}[1/p])$ satisfying a mild hypothesis, the localization homomorphism $K_*(X; \mathbf{Z}/q) \to K_*(X; \mathbf{Z}/q)[\beta^{-1}]$ is an isomorphism in sufficiently high degrees. We briefly spell out how this comparison between K-theory and Bott inverted K-theory follows from the norm residue theorem (see [HW19] for a textbook account of the latter) in the cases of interest:

Proposition 2.9. *For $X = \mathrm{Spec}\,(\mathbf{Z}')$ or $X = \mathrm{Spec}\,(\mathcal{O}_q')$, the localization map*

$$K_i(X; \mathbf{Z}/q) \to K_i^{(\beta)}(X; \mathbf{Z}/q)$$

is an isomorphism for all $i > 0$ and a monomorphism for $i = 0$. (It is in fact also an isomorphism for $i = 0$, as will be proved in §2.7). The same assertion holds for $\mathrm{Spec}\,(\mathbf{Z})$ or $\mathrm{Spec}\,(\mathcal{O}_q)$ if we suppose $i \geq 2$.

Proof sketch. For any field k of finite cohomological dimension (and admitting a "Tate-Tsen filtration", as in [Tho85, Theorem 2.43]), there are spectral sequences converging to both domain and codomain of the map $K_*(k; \mathbf{Z}/p) \to K^{(\beta)}(k; \mathbf{Z}/p)$. In the codomain it is the above-mentioned spectral sequence of Thomason, applied to $X = \mathrm{Spec}\,(k)$, and in the domain it is the motivic spectral sequence. There is a compatible map of spectral sequences, which on the E^2 page is the map from motivic to étale cohomology

$$H_{\mathrm{mot}}^{-s}(\mathrm{Spec}\,(k); (\mathbf{Z}/p)(t/2)) \to H_{\mathrm{et}}^{-s}(\mathrm{Spec}\,(k); \mu_p^{\otimes t/2}).$$

induced by changing topology from the Nisnevich to étale topology. The norm residue theorem implies that this map is an isomorphism for $t/2 \geq -s$. Below this line the motivic cohomology vanishes but the étale cohomology need not. If $\mathrm{cd}_p(k) = d$ we may therefore have non-trivial étale cohomology in $E_{-d,2d-2}^2$ which is not hit from motivic cohomology, and the total degree

$d - 2$ of such elements is the highest possible total degree in which this can happen. By convergence of the spectral sequences, the map

$$K_i(k; \mathbf{Z}/p) \to K_i^{(\beta)}(k; \mathbf{Z}/p)$$

is an isomorphism for $i \geq d - 1$ and an injection for $i = d - 2$; the same conclusion follow with $\mathbf{Z}/q$ coefficients by induction using the long exact sequences.

This applies to $k = \mathbf{Q}$ which has p-cohomological dimension 2 (we use here that p is odd) and $k = \mathbf{F}_\ell$ which has p-cohomological dimension 1 for $\ell \neq p$, as well as finite extensions thereof. Finally, Quillen's localization sequence

$$\bigvee_{\ell \neq p} K(\mathbf{F}_\ell) \to K(\mathbf{Z}') \to K(\mathbf{Q})$$

and its Bott-inverted version imply that $K_i(\mathbf{Z}'; \mathbf{Z}/q) \to K_i^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)$ is an isomorphism for $i \geq 1$ and a monomorphism for $i = 0$, and a similar argument applies when $X = \mathrm{Spec}\,(\mathcal{O}_q')$.

The final assertion results from using Quillen's localization sequence to compare $\mathbf{Z}$ and $\mathbf{Z}'$, plus Quillen's computation of the K-theory of finite fields [Qui72]. For reference we state this as Lemma 2.10, and expand on the proof below. $\square$

Lemma 2.10. *The map* $\mathbf{Z} \to \mathbf{Z}'$ *induces an isomorphism on mod q K-theory in all degrees except 1, where* $K_1(\mathbf{Z}'; \mathbf{Z}/q) \cong \mathbf{Z}/q \oplus K_1(\mathbf{Z}; \mathbf{Z}/q)$. *The same assertion holds true for* $\mathcal{O}_q \to \mathcal{O}_q'$. *In particular, the maps* $K_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q)) \to K_*^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)$ *and* $K_*^{(\beta)}(\mathcal{O}_q) \to K_*^{(\beta)}(\mathcal{O}_q'; \mathbf{Z}/q)$ *are both isomorphisms in all degrees.*

Proof. Quillen's devissage and localization theorems [Qui73, Section 5] gives fiber sequences

$$K(\mathbf{F}_p) \to K(\mathbf{Z}) \to K(\mathbf{Z}')$$
$$K(\mathbf{F}_p) \to K(\mathcal{O}_q) \to K(\mathcal{O}_q').$$

His calculation [Qui72] of K-theory of finite fields implies $K_i(\mathbf{F}_p; \mathbf{Z}/q) = 0$ for $i \neq 0$, while $K_0(\mathbf{F}_p; \mathbf{Z}/q) = \mathbf{Z}/q$. Finall we note the homomorphisms $K_0(\mathbf{Z}) \to K_0(\mathbf{Z}')$ and $K_0(\mathcal{O}_q) \to K_0(\mathcal{O}_q')$ are injective – the latter because the prime above p in $\mathcal{O}_q$ is principal. $\square$

2.7. **Some computations of Bott-inverted K-theory in terms of étale cohomology.** In this section, we shall use Thomason's spectral sequence (2.6)

$$E_{s,t}^2 = H_{\mathrm{et}}^{-s}(X; \mu_q^{\otimes(t/2)}) \Rightarrow K_{t+s}^{(\beta)}(X; \mathbf{Z}/q),$$

to compute Bott-inverted K-theory of number rings in terms of étale cohomology. By Proposition 2.9, many of the results can be directly stated in terms of K-theory. Through this use of Proposition 2.9, our main result – in the form stated in the introduction – depends on the norm residue theorem; but that dependence is easily avoided by replacing $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$ by its Bott-inverted version, see Subsection 7.6.

We recall that we work under the standing assumption that q is odd.

Lemma 2.11. *We have the following isomorphisms, for all $k \in \mathbf{Z}$:*

$$K_{4k-2}^{(\beta)}(\mathcal{O}_q'; \mathbf{Z}/q)^{(+)} \cong H^2(\mathrm{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes 2k})$$
$$K_{4k-2}^{(\beta)}(\mathcal{O}_q'; \mathbf{Z}/q)^{(-)} \cong H^0(\mathrm{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes(2k-1)})$$
$$K_{4k}^{(\beta)}(\mathcal{O}_q'; \mathbf{Z}/q)^{(+)} \cong H^0(\mathrm{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes 2k})$$
$$K_{4k}^{(\beta)}(\mathcal{O}_q'; \mathbf{Z}/q)^{(-)} \cong H^2(\mathrm{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes 2k+1}).$$

In odd degrees we have an isomorphism

$$K_{2k-1}^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q) \cong H_{\text{et}}^1(\text{Spec}\,(\mathcal{O}'_q); \mu_q^{\otimes k}) \tag{2.7}$$

and ψ^{-1} acts by $(-1)^k$.

Finally, the map $K_i(\mathcal{O}'_q; \mathbf{Z}/q) \to K_i^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)$ is an isomorphism for all $i \geq 0$, and the map $K_i(\mathcal{O}_q; \mathbf{Z}/q) \to K_i^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)$ is an isomorphism for $i = 0$ or $i \geq 2$.

Proof. We apply (2.6) to $X = \text{Spec}\,(\mathcal{O}'_q)$. This scheme has étale cohomological dimension 2, so the spectral sequence is further concentrated in the region $-2 \leq s \leq 0$. The spectral sequence must collapse for degree reasons, since no differential goes between two non-zero groups (since only $t \in 2\mathbf{Z}$ appears). Convergence of the spectral sequence gives in odd degrees (2.7).

In even degrees we obtain a short exact sequence

$$0 \to H_{\text{et}}^2(\text{Spec}\,(\mathcal{O}'_q); \mu_q^{\otimes k}) \to K_{2k-2}^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q) \to H_{\text{et}}^0(\text{Spec}\,(\mathcal{O}'_q); \mu_q^{\otimes k-1}) \to 0.$$

For odd q this sequence splits canonically, using the action of the Adams operation ψ^{-1} on the spectral sequence: it acts as $(-1)^k$ on the kernel and as $(-1)^{k-1}$ on the cokernel in the short exact sequence.

For the final assertion for $\mathcal{O}'_q$: by Proposition 2.9 we need only consider $i = 0$, and by injectivity in degree 0 it follows in that case from a computation of orders: both sides have order $q \cdot \#(\text{Pic}(\mathcal{O}_q)/q)$. (Alternatively prove surjectivity as in Corollary 2.12 below.) The version for $\mathcal{O}_q$ follows from Lemma 2.10. □

The isomorphisms in different degrees in Lemma 2.11 are intertwined through the action of β in an evident way; this switches between $+$ and $-$ eigenspaces. For example, the group $K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(-)}$ is isomorphic to $\mathbf{Z}/q$ for any $k \in \mathbf{Z}$, generated by β^{2k-1}. We want to make the isomorphism on the $+$ eigenspace in degree $4k - 2$ more explicit.

Corollary 2.12. *The map*

$$\pi_0(\text{Pic}(\mathcal{O}_q))/q \to K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(+)}$$
$$[L] \mapsto \beta^{2k-1} \cdot ([L] - 1)$$

is an isomorphism of groups (where the group operation is induced by tensor product in the domain and direct sum in the codomain). More invariantly, in the notation of (2.5), the isomorphism may be written

$$\pi_0(\text{Pic}(\mathcal{O}_q)) \otimes \mu_q(\mathcal{O}_q)^{\otimes(2k-1)} \to K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(+)}$$
$$[L] \otimes \zeta^{\otimes(2k-1)} \qquad \mapsto \beta(\zeta)^{2k-1} \cdot ([L] - 1), \tag{2.8}$$

valid for any $L \in \text{Pic}(\mathcal{O}_q)$ and any $\zeta \in \mu_q(\mathcal{O}_q)$. In this formulation the isomorphism is equivariant for the evident action of $\text{Gal}(K_q/\mathbf{Q}) \cong (\mathbf{Z}/q)^\times$ on both sides.

A similar result holds for $K_{4k}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)$, except the roles of positive and negative eigenspaces for ψ^{-1} are reversed.

Proof. Multiplication by $\beta^{2k-1} \colon K_0^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)^{(-)} \to K_{4k-2}^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)^{(+)}$ is an isomorphism which under the isomorphisms of Lemma 2.11 corresponds to multiplication by $\zeta_q^{\otimes(2k-1)} \colon H^2(\text{Spec}\,(\mathcal{O}'_q); \mu_q) \to H^2(\text{Spec}\,(\mathcal{O}'_q); \mu_q^{\otimes(2k-1)})$, so it suffices to prove that the composition

$$\pi_0(\text{Pic}(\mathcal{O}'_q))/q \to K_0(\mathcal{O}'_q; \mathbf{Z}/q)^{(-)} \to K_0^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)^{(-)} \xrightarrow{\text{Lem. 2.11}} H_{\text{et}}^2(\text{Spec}\,(\mathcal{O}'_q); \mu_q)$$
$$[L] \mapsto [L] - 1 \tag{2.9}$$

is an isomorphism. We will do that by identifying it with the usual étale Chern class $[L] \mapsto c_1(L)$, which is an isomorphism. This identification of (2.9) is surely well known to experts, but we were unable to locate a reference so let us outline the proof. (See Remark 2.13 for a shortcut.)

To see that (2.9) agrees with c_1, we inspect the construction of Thomason's spectral sequence, which is where the isomorphisms in Lemma 2.11 came from. As usual we write $K^{(\beta)}(X; \mathbf{Z}/q) \to \tau_{\leq n} K^{(\beta)}(X; \mathbf{Z}/q)$ for the Postnikov truncation, and we shall also write $\mathbb{H}_{\mathrm{et}}(X; \tau_{\leq n} K^{(\beta)}/q)$ for the derived global sections of $U/X \mapsto \tau_{\leq n} K^{(\beta)}(U; \mathbf{Z}/q)$. Under reasonable assumptions on $X \to \mathrm{Spec}\,(\mathbf{Z}[\frac{1}{p}])$, [Tho85, Theorem 2.45] gives that the natural map

$$K^{(\beta)}(X; \mathbf{Z}/q) \to \mathbb{H}_{\mathrm{et}}(X; K^{(\beta)}/q) \simeq \operatorname*{holim}_n \mathbb{H}_{\mathrm{et}}(X; \tau_{\leq n} K^{(\beta)}/q)$$

is a weak equivalence. Viewing this as a filtration of $K^{(\beta)}(X; \mathbf{Z}/q)$ gives the spectral sequence, whose E^2 page is identified in [Tho85, Theorem 3.1].

The canonical group-completion map (see Appendix A.3) restricts to a map $B\mathbf{G}_m(R) = B\mathrm{GL}_1(R) \to \Omega^\infty K(R)$ which lands in the path component of $1 = [R] \in K_0(R)$. Subtracting a constant map leads to a map of presheaves of pointed simplicial sets

$$B\mathbf{G}_m \to \Omega^\infty K \to \tau_{\leq 1} \Omega^\infty K, \tag{2.10}$$

where $\tau_{\leq 1}$ denotes the Postnikov truncation. The first map in (2.10) induces

$$|\mathrm{Pic}(X)| \simeq \mathbb{H}_{\mathrm{Zar}}(X; B\mathbf{G}_m) \to \Omega^\infty K(X),$$

which on the level of π_0 induces $[L] \mapsto [L] - 1$. Here the "-1" corresponds to the constant map we subtracted to obtain (2.10). The composition (2.10) is a based map $B\mathbf{G}_m \to \tau_{\leq 1} \Omega^\infty K$ which may be canonically promoted to an infinite loop map, up to a contractible space of choices (this is simply because the higher homotopy has been killed so there are no obstructions to such a promotion; the first map $B\mathbf{G}_m(R) \to \Omega^\infty K(R)$ is certainly not an infinite loop map). Hence it deloops to a map of presheaves of spectra $\Sigma H\mathbf{G}_m \to \tau_{\leq 1} K$. Taking smash product of this delooping with $\mathbb{S} \to \mathbb{S} \to \mathbb{S}/q$ leads to a map between two fiber sequences of presheaves of spectra, part of which looks like

$$\begin{array}{ccc} \Sigma H\mathbf{G}_m & \longrightarrow & \tau_{\leq 1} K \\ \downarrow & & \downarrow \\ \mathbb{S}/q \wedge \Sigma H\mathbf{G}_m & \longrightarrow & \mathbb{S}/q \wedge \tau_{\leq 1} K. \end{array} \tag{2.11}$$

By the Kummer sequence, the canonical map $\Sigma^2 \mu_q \to \mathbb{S}/q \wedge \Sigma H\mathbf{G}_m$ induces a weak equivalence of derived global sections, and on the level of π_0 the left horizontal map in (2.11) becomes the connecting map $c_1 \colon H^1_{\mathrm{et}}(X; \mathbf{G}_m) \to H^2_{\mathrm{et}}(X; \mu_q)$. Furthermore, [Tho85, Theorem 3.1] implies that the canonical map

$$\tau_{\leq 2}((\mathbb{S}/q) \wedge K) \to (\mathbb{S}/q) \wedge (\tau_{\leq 1} K)$$

also induces a weak equivalence of derived global sections. We deduce a commutative diagram

$$\begin{array}{ccc} |\mathrm{Pic}(X)| \xrightarrow{L-1} \Omega^\infty K(X) & \longrightarrow & \Omega^\infty K^{(\beta)}(X; \mathbf{Z}/q) \\ \downarrow{\scriptstyle c_1} & & \downarrow \\ \Omega^\infty \mathbb{H}(X; \Sigma^2 \mu_q) & \longrightarrow & \tau_{\leq 2} \Omega^\infty K^{(\beta)}(X; \mathbf{Z}/q). \end{array} \tag{2.12}$$

Comparing with the proof of [Tho85, Theorem 3.1], in particular diagram (3.8) of op.cit., we identify the bottom map in (2.12) with Ω^∞ of the homotopy fiber of the truncation map $\tau_{\leq 2} K^{(\beta)}(X; \mathbf{Z}/q) \to \tau_{\leq 0} K^{(\beta)}(X; \mathbf{Z}/q)$, which is part of the filtration of $K^{(\beta)}(X; \mathbf{Z}/q)$ giving

rise to the spectral sequence (2.6). On homotopy groups $\pi_{s-2}(-)$, the bottom map therefore gives rise to the isomorphism $H^s_{\mathrm{et}}(X; \mu_q) \to E^2_{-s,2}$ in the spectral sequence.

We see that if we then start with $[L] \in \pi_0(\mathrm{Pic}(X)) = H^1(X; \mathbf{G}_m)$, map it to $[L] - 1 \in K_0(X; \mathbf{Z}/q)$, then to $K_0^{(\beta)}(X; \mathbf{Z}/q)$ and project to $E^\infty_{-2,2}$ in Thomason's spectral sequence (which we can because its projection to $E^\infty_{0,0}$ is 0), then commutativity of the diagram shows that we obtain the class of $c_1(L) \in H^2_{\mathrm{et}}(X; \mu_q) \cong E^2_{-2,2}$. $\qquad\square$

Remark 2.13. For the reader who prefers to keep *both* the norm residue theorem and Thomason's spectral sequence as black boxes not to be opened, it may be shorter to consider the two maps

$$\pi_0(\mathrm{Pic}(\mathcal{O}'_q))/q \to K_0(\mathcal{O}'_q; \mathbf{Z}/q)^{(-)} \to K_0^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)^{(-)}$$

separately. The first is an isomorphism by the usual splitting $K_0(\mathcal{O}'_q; \mathbf{Z}/q) \cong \mathbf{Z} \oplus \pi_0(\mathrm{Pic}(\mathcal{O}'_q))$ and the second by the final part of Lemma 2.11. That route gives a proof that (2.9) is an isomorphism without inspecting what the map is, at the cost of appealing to the norm residue theorem, thus invalidating Remark 1.4.

Lemma 2.14. *For all $k \in \mathbf{Z}$ we have*

$$K_{4k-2}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)^{(+)} \cong H^2(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes 2k})$$

$$K_{4k-2}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)^{(-)} \cong H^0(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes (2k-1)})$$

$$K_{4k}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)^{(+)} \cong H^0(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes 2k})$$

$$K_{4k}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)^{(-)} \cong H^2(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes 2k+1}).$$

In odd degrees we have $K_{2k-1}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q) \cong H^1(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes k})$ *for all k, on which ψ^{-1} acts as* $(-1)^k$.

Finally, the map $K_i(\mathbf{Z}'; \mathbf{Z}/q) \to K_i^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)$ is an isomorphism for all $i \geq 0$ and the map $K_i(\mathbf{Z}; \mathbf{Z}/q) \to K_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q)$ is an isomorphism for for $i = 0$ or $i \geq 2$.

Recall our standing assumption that q is odd.

Proof. Similarly to the prior analysis we get canonical isomorphisms

$$K_{2k-1}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q) \cong H^1_{\mathrm{et}}(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes k})$$

in odd degrees, and in even degrees we have short exact sequences

$$0 \to H^2_{\mathrm{et}}(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes k}) \to K_{2k-2}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q) \to H^0_{\mathrm{et}}(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes k-1}) \to 0, \qquad (2.13)$$

canonically split into positive and negative eigenspaces for ψ^{-1} when q is odd. The periodicity of these groups has longer period though: multiplying with $\beta^{p^{n-1}(p-1)}$ increases k by $p^{n-1}(p-1)$.

As before the asssertion comparing K-theory and Bott-inverted K-theory of $\mathbf{Z}'$ follows from Proposition 2.9 by computing orders, and the assertion for $\mathbf{Z}$ uses Lemma 2.10. $\qquad\square$

Proposition 2.15. *Suppose that q is odd. Let $\mathrm{Gal}(K_q/\mathbf{Q}) \cong (\mathbf{Z}/q)^\times$ act on $K_*(\mathcal{O}'_q; \mathbf{Z}/q)$ by functoriality of algebraic K-theory. Then the homomorphisms*

$$\left(K_{4k-2}^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)^{(+)}\right)_{\mathrm{Gal}(K_q/\mathbf{Q})} \to K_{4k-2}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)^{(+)}$$

$$\left(K_{4k}^{(\beta)}(\mathcal{O}'_q; \mathbf{Z}/q)^{(-)}\right)_{\mathrm{Gal}(K_q/\mathbf{Q})} \to K_{4k}^{(\beta)}(\mathbf{Z}'; \mathbf{Z}/q)^{(-)},$$

induced by the transfer map $K_*(\mathcal{O}'_q; \mathbf{Z}/q) \to K_*(\mathbf{Z}'; \mathbf{Z}/q)$, are both isomorphisms. (Here $(-)_{\mathrm{Gal}(K_q/\mathbf{Q})}$ denotes coinvariants for $\mathrm{Gal}(K_q/\mathbf{Q})$.)

Remark 2.16. It will follow implicitly from the proof that the transfer map behaves in the indicated way with respect to eigenspaces for ψ^{-1}, but let us give an independent explanation for why the transfer map $K_*(\mathcal{O}_q; \mathbf{Z}/q) \to K_*(\mathbf{Z}; \mathbf{Z}/q)$ commutes with the Adams operation ψ^{-1}. This may seem surprising at first, since the forgetful map from $\mathcal{O}_q$-modules to $\mathbf{Z}$-modules does not obviously commute with dualization. The "correction factor" is the dualizing module ω, isomorphic to the inverse of the *different* $\mathfrak{d}$, which will play an important role later in the paper. In this case the different is principal, and any choice of generator leads to a functorial isomorphism between the $\mathbf{Z}$-dual and the $\mathcal{O}_q$-dual.

Proof. The argument is the same in both cases, and uses naturality of Thomason's spectral sequence with respect to transfer maps: there is a map of spectral sequences which on the E_2 page is given by the transfer in étale cohomology and on the E_∞ page by (associated graded of) the transfer map in K-theory. This naturality is proved in Section 10 of [BM15], the preprint version of [BM20]. In our case the spectral sequences collapse, and identify the two homomorphisms in the corollary with the maps on $E_2^{-2,4k}$ and $E_2^{-2,4k+2}$, respectively. Hence we must prove that the transfer maps

$$\left(H_{\text{et}}^2(\text{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes t}) \right)_{\text{Gal}(K_q/\mathbf{Q})} \to H_{\text{et}}^2(\text{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes t})$$

are isomorphisms for all t or, equivalently, that their Pontryagin duals are isomorphisms. By Poitou–Tate duality, the Pontryagin dual map may be identified with

$$\pi^* : H_c^1(\text{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes(1-t)}) \to \left(H_c^1(\text{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes(1-t)}) \right)^{\text{Gal}(K_q/\mathbf{Q})},$$

where the "compactly supported" cohomology is taken in the sense of [GV18, Appendix], i.e., defined as cohomology of a mapping cone. In this context we may apply a relative Hochschild-Serre spectral sequence[3] to give an exact sequence

$$0 \to H^1((\mathbf{Z}/q)^*; H_c^0(\text{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes(1-t)})) \to H_c^1(\text{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes(1-t)})$$
$$\to H_c^1(\text{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes(1-t)})^{(\mathbf{Z}/q)^*} \to H^2((\mathbf{Z}/q)^*; H_c^0(\text{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes(1-t)})).$$

Now, the compactly supported cohomology group $H_c^0(\text{Spec}\,(\mathcal{O}_q'); \mu_q^{\otimes(1-t)})$ is the kernel of the restriction map $\mu_q(\mathcal{O}_q')^{\otimes(1-t)} \to \mu_q(\mathbf{Q}_p[\mu_q])^{\otimes(1-t)}$, which is an isomorphism. The exact sequence then precisely becomes the desired isomorphism. $\square$

2.8. Bott inverted algebraic K-theory and homology of certain Galois groups.
In this subsection we express Bott inverted algebraic K-theory of cyclotomic rings of integers in terms of certain Galois homology groups. This will be useful later one, when trying to relate K-theory to extensions of Galois modules.

Let $\widetilde{H}_q \subset \mathbf{C}$ be the Hilbert class field of $K_q = \mathbf{Q}[\zeta_q] \subset \mathbf{C}$, the maximal abelian extension unramified at all places. Class field theory asserts an isomorphism $\pi_0(\text{Pic}(\mathcal{O}_q)) \cong \text{Gal}(\widetilde{H}_q/K_q)$, given by the Artin symbol. Let $H_q \subset \widetilde{H}_q$ be the largest extension with p-power-torsion Galois group, so that the Artin symbol factors over an isomorphism

$$\pi_0(\text{Pic}(\mathcal{O}_q)) \otimes \mathbf{Z}_p \xrightarrow{\cong} \text{Gal}(H_q/K_q)$$
$$[\mathfrak{p}] \mapsto \left(\frac{H_q/K_q}{\mathfrak{p}} \right). \tag{2.14}$$

---

[3]The relative Leray spectral sequence is noted in a topological context, for example, in Exercise 5.6 of [McC01]. This implies such a spectral sequence for pairs of finite groups, and then for profinite groups by a limit argument.

It is easy to check that this map is equivariant for the action of $\mathrm{Gal}(K_q/\mathbf{Q})$ which acts in the evident way on the domain, and on the codomain the action is induced by the short exact sequence

$$\mathrm{Gal}(H_q/K_q) \to \mathrm{Gal}(H_q/\mathbf{Q}) \to \mathrm{Gal}(K_q/\mathbf{Q}). \tag{2.15}$$

The following diagram gives the main tool through which we will understand the transfer map $\mathrm{tr} : K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(+)} \to K_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$. To keep typography simple, we write (in the statement and its proof) μ_q for $\mu_q(\mathbf{C})$, and for a Galois extension E/F of fields, we write $H_*(E/F, -)$ for the group homology of the group $\mathrm{Gal}(E/F)$.

Proposition 2.17. *For all $k \in \mathbf{Z}$ there is a commutative diagram, with all horizontal maps isomorphisms*

$$
\begin{array}{ccccc}
H_1(H_q/K_q; \mu_q^{\otimes(2k-1)}) & \xleftarrow[\cong]{\mathrm{Art}} & \pi_0(\mathrm{Pic}(\mathcal{O}_q)) \otimes \mu_q^{\otimes(2k-1)} & \xrightarrow{\cong} & K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(+)} \\
\downarrow{\scriptstyle i_*} & & & & \downarrow{\scriptstyle \mathrm{tr}} \\
H_1(H_q/\mathbf{Q}; \mu_q^{\otimes(2k-1)}) & & \xrightarrow{\hspace{4cm}\cong\hspace{4cm}} & & K_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q)^{(+)},
\end{array}
\tag{2.16}
$$

where:

- *the map denoted i_* is induced by the inclusion $\mathrm{Gal}(H_q/K_q) \subset \mathrm{Gal}(H_q/\mathbf{Q})$;*
- *the map denoted Art is induced by the Artin map (2.14), together with the identification $H_1(H_q/K_q, \mu_q^{\otimes(2k-1)}) \simeq \mu_q^{\otimes(2k-1)} \otimes \mathrm{Gal}(H_q/K_q)$;*
- *the top arrow labeled "$\cong$" is the map of (2.8), i.e. the product of the map $[L] \mapsto [L] - 1 \in K_0(\mathcal{O}_q; \mathbf{Z}/q)^{(-)}$ composed with $K_0(\mathcal{O}_q; \mathbf{Z}/q)^{(-)} \to K_0^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(-)}$ and $\beta^{2k-1} : \mu_q(\mathbf{C})^{\otimes(2k-1)} \to K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(-)}$;*
- *the bottom arrow labeled "$\cong$" is induced by the rest of the diagram.*

The same assertion holds without Bott-inversion of the K-theory for $k \geq 1$.

Proof. That the right top arrow is an isomorphism was already proved in Corollary 2.12. We have also seen that $K_{4k-2}^{(\beta)}(\mathcal{O}_q; \mathbf{Z}/q)^{(+)} \to K_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$ induces an isomorphism from the $\mathrm{Gal}(K_q/\mathbf{Q})$ coinvariants on the source: see Proposition 2.15, Lemma 2.10 and Proposition 2.9.

Therefore, we need only verify the corresponding property for i_*: it induces an isomorphism from the $\mathrm{Gal}(K_q/\mathbf{Q})$-coinvariants on the source. This follows from the Hochschild–Serre spectral sequence for the extension (2.15), which gives an exact sequence

$$H_2(K_q/\mathbf{Q}; \mu_q^{\otimes(2k-1)}) \to H_0(K_q/\mathbf{Q}; H_1(H_q/K_q; \mu_q^{\otimes(2k-1)})) \to H_1(H_q/\mathbf{Q}; \mu_q^{\otimes(2k-1)})$$
$$\to H_1(K_q/\mathbf{Q}; \mu_q^{\otimes(2k-1)}). \tag{2.17}$$

Considering the action of the central element $c \in \mathrm{Gal}(K_q/\mathbf{Q})$ given by complex conjugation we see that the two outer terms vanish (the "center kills" argument).

For the last sentence use Lemma 2.11 and Lemma 2.14. $\qquad\square$

Remark 2.18. Let $c \in \mathrm{Gal}(H_q/\mathbf{Q})$ be complex conjugation. Then $H_0(\langle c \rangle; \mu_q^{\otimes(2k-1)}) = 0 = H_1(\langle c \rangle; \mu_q^{\otimes(2k-1)})$. Therefore the map

$$H_1(H_q/\mathbf{Q}; \mu_q^{\otimes(2k-1)}) \to H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)})$$

is an isomorphism; on the right we have "relative" group homology, i.e. relative homology of classifying spaces. This relative group homology may therefore be substituted in place of the lower left corner of (2.16). This observation will be significant later.

Remark 2.19. The case $p = q = 3$ is anomalous in that the Moore spectrum $\mathbb{S}/3$ does not admit a unital multiplication which is associative up to homotopy. It does admit a unital and homotopy commutative multiplication though, which induces graded commutative—but a priori possibly non-associative—ring structures on $K_*(\mathcal{O}_3; \mathbf{Z}/3)$ and $\mathrm{K}_*(\mathbf{Z}; \mathbf{Z}/3)$. There is no problem in defining Bott inverted K-theory, e.g. as in Remark 2.8, and according to [Tho85, A.11] the construction of the spectral sequence holds also in this case. The Bott element $\beta \in K_2(\mathcal{O}_3; \mathbf{Z}/3)$ is defined as before, and multiplication by β defines an endomorphism of $K_*(\mathcal{O}_3; \mathbf{Z}/3)$. Iterating this endomorphism $2k-1$ times gives a homomorphism $K_0(\mathcal{O}_3; \mathbf{Z}/3) \to K_{4k-2}(\mathcal{O}_3; \mathbf{Z}/3)$, which we use to give meaning to expressions like $\beta^{2k-1}([L] - 1)$ in this section.

In this interpretation the results of this section hold also in the case $p = q = 3$. Multiplication by powers of a Bott element also appear in Section 5, we leave it to the diligent reader to verify that similar remarks apply there.

3. Symplectic K-theory

In this section, we define the symplectic K-theory of the integers. Our main goal is to state and prove Theorem 3.5, which shows that this symplectic K-theory, with $\mathbf{Z}/q$-coefficients, splits into two parts: one arising from the $+$ part of the algebraic K-theory of $\mathbf{Z}$, and the other from the $-$ part of topological K-theory.

3.1. **Definition of symplectic K-theory.**
Just as K-theory arises from the symmetric monoidal category of projective modules, symplectic K-theory arises from the symmetric monoidal category of *symplectic* modules:

Consider the groupoid whose objects are pairs (L, b), where L is a finitely generated free $\mathbf{Z}$-module and $b : L \times L \to \mathbf{Z}$ is a skew symmetric pairing whose adjoint $L \to L^\vee$ is an isomorphism, and whose morphisms are $\mathbf{Z}$-linear isomorphisms $f : L \to L'$ such that $b'(fx, fy) = b(x, y)$ for all $x, y \in L$. This groupoid becomes symmetric monoidal with respect to orthogonal direct sum $(L, b) \oplus (L', b') = (L \oplus L', b + b')$, and we shall denote it $\mathcal{SP}(\mathbf{Z})$. The corresponding space $|\mathcal{SP}(\mathbf{Z})|$ then inherits a product structure, and as before (see again Appendix A for more details) we get a spectrum $\mathrm{KSp}(\mathbf{Z})$ and a group-completion map

$$|\mathcal{SP}(\mathbf{Z})| \to \Omega^\infty \mathrm{KSp}(\mathbf{Z}).$$

The positive degree homotopy groups of $\mathrm{KSp}(\mathbf{Z})$ can be computed via the Quillen plus construction (with respect to the commutator subgroup of $\mathrm{Sp}_\infty(\mathbf{Z}) = \pi_1(B\mathrm{Sp}_\infty(\mathbf{Z}))$).

Definition 3.1. We define the *mod q symplectic K-theory groups of R* to be

$$\mathrm{KSp}_i(R; \mathbf{Z}/q\mathbf{Z}) := \pi_i(\mathrm{KSp}(R); \mathbf{Z}/q\mathbf{Z}).$$

In the case $R = \mathbf{Z}$ we define the *p-adic symplectic K-theory groups* via

$$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p) := \varprojlim_n \mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/p^n).$$

3.2. **Hodge map and Betti map.**
The groups $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q) = \pi_i(\mathrm{KSp}(\mathbf{Z}); \mathbf{Z}/q)$ are described in Theorem 3.5 below. The result is stated in terms of two homomorphisms, the *Hodge map* and the *Betti map*, which we first define.

3.2.1. *The Betti map.*

Definition 3.2. Let $c_B : \mathrm{KSp}(\mathbf{Z}) \to K(\mathbf{Z})$ be the spectrum map defined by the forgetful functor $\mathcal{SP}(\mathbf{Z}) \to \mathcal{P}(\mathbf{Z})$. We shall use the same letter c_B to denote the induced homomorphism on mod q homotopy groups

$$c_B : \mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q) \to K_i(\mathbf{Z}; \mathbf{Z}/q).$$

3.2.2. *The Hodge map.* The Hodge map is more elaborate. It arises from the functors of

$$\underbrace{\text{symplectic } \mathbf{Z}\text{-modules}}_{\mathcal{SP}(\mathbf{Z})} \rightarrow \underbrace{\text{symplectic } \mathbf{R}\text{-modules}}_{\mathcal{SP}(\mathbf{R}^{\text{top}})} \leftarrow \underbrace{\text{Hermitian } \mathbf{C}\text{-vector spaces}}_{\mathcal{U}(\mathbf{C}^{\text{top}})}. \qquad (3.1)$$

where the entries are now regarded as symmetric monoidal groupoids that are enriched in topological spaces. In more detail:

- Let $\mathcal{SP}(\mathbf{R}^{\text{top}})$ be the groupoid (enriched in topological spaces) defined as $\mathcal{SP}(\mathbf{Z})$ but with $\mathbf{R}$-modules L and $\mathbf{R}$-bilinear symplectic pairings $b : L \times L \rightarrow \mathbf{R}$. We regard it as a groupoid enriched in topological spaces, where morphism spaces are topologized in their Lie group topology, inherited from the topology on $\mathbf{R}$ (the superscript "top" signifies that we remember the topology, as opposed to considering $\mathbf{R}$ as a discrete ring).
- Write $\mathcal{U}(\mathbf{C}^{\text{top}})$ for the groupoid (again enriched in topological spaces) whose objects are finite dimensional $\mathbf{C}$-vector spaces L equipped with a positive definite Hermitian form $h : L \otimes_{\mathbf{C}} \overline{L} \rightarrow \mathbf{C}$, and morphisms the unitary maps topologized in the Lie group topology.
- The functor $\mathcal{U}(\mathbf{C}^{\text{top}}) \rightarrow \mathcal{SP}(\mathbf{R}^{\text{top}})$ is obtained by sending a unitary space (L, h), as in (ii), to the underlying real vector space $L_{\mathbf{R}}$, equipped with the symplectic form $\text{Im } h$. This functor induces a bijection on sets of isomorphism classes and homotopy equivalences on all morphisms spaces, because $U(g) \subset \text{Sp}_{2g}(\mathbf{R})$ is a homotopy equivalence.

We equip these categories with the symmetric monoidal structures given by direct sum. Then, as discussed in the Appendix, the diagram (3.1) gives rise to a diagram of Γ-spaces and thereby to a diagram of spectra:

$$\text{KSp}(\mathbf{Z}) \rightarrow \text{KSp}(\mathbf{R}^{\text{top}}) \leftarrow ku, \qquad (3.2)$$

where we follow standard notation in using ku (connective K-theory) to refer to the spectrum associated to $\mathcal{U}(\mathbf{C}^{\text{top}})$. The last arrow here is a weak equivalence, i.e. induces an isomorphism on all homotopy groups. Indeed, as noted above, $\coprod_g B\text{U}(g) \simeq \coprod_g B\text{Sp}_{2g}(\mathbf{R})$ is a weak equivalence, therefore the group completions are weakly equivalent, therefore $\Omega^{\infty}(ku) \rightarrow \Omega^{\infty}(\text{KSp}(\mathbf{R}^{\text{top}}))$ is a weak equivalence, and so the map $ku \rightarrow \text{KSp}(\mathbf{R}^{\text{top}})$ of connective spectra is a weak equivalence.

In the homotopy category of spectra, weak equivalences become invertible, and so the diagram (3.2) induces there a map $\text{KSp}(\mathbf{Z}) \rightarrow ku$.

Definition 3.3. The *Hodge map* is the morphism

$$c_H : \text{KSp}(\mathbf{Z}) \rightarrow ku$$

in the homotopy category of spectra that has just been constructed. The map c_H induces a homomorphism

$$\text{KSp}_i(\mathbf{Z}; \mathbf{Z}/q) \rightarrow \pi_i(ku; \mathbf{Z}/q)$$

which we shall also call the Hodge map. By Bott periodicity, the target is $\mathbf{Z}/q$ when i is even and 0 when i is odd.

Remark 3.4 (Explanation of terminology). With reference to the relationship between symplectic K-theory and moduli of principally polarized abelian varieties (§1.2) the Hodge map is related to the Chern classes of the *Hodge bundle* whose fiber over A is $H^0(A, \Omega^1)$, and thus to the "Hodge realization" of A. On the other hand, the Betti map is related to the "Betti realization" $H_1(A, \mathbf{Z})$.

3.3. **Determination of symplectic K-theory in terms of algebraic K-theory.** As explained above, the Adams operation ψ^{-1} induces an involution of $K_*(\mathbf{Z}; \mathbf{Z}/q)$ which gives a splitting for odd q into positive and negative eigenspaces. There are also Adams operations on ku, and their effect on homotopy groups are very easy to understand. In particular, ψ^{-1} acts as $(-1)^k$ on $\pi_{2k}(ku)$. The main goal of this section is to explain the following result.

Theorem 3.5. *For odd $q = p^n$, the homomorphism*

$$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q) \to (K_i(\mathbf{Z}; \mathbf{Z}/q))^{(+)} \oplus (\pi_i(ku; \mathbf{Z}/q))^{(-)} \tag{3.3}$$

defined by the Betti and Hodge maps, composed with the projections onto the indicated eigenspaces for ψ^{-1}, is an isomorphism. (We will refer later to the induced isomorphism as the Betti-Hodge map). In particular we get for $k \geq 1$

$$\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \cong H^2(\mathrm{Spec}\,(\mathbf{Z}'); \mu_q^{\otimes 2k}) \oplus (\mathbf{Z}/q).$$

The latter statement follows from the first using Corollary 2.14. Using the other statements of that Corollary, taking the inverse limit over n, and using that $K_i(\mathbf{Z})$ and $\mathrm{KSp}_i(\mathbf{Z})$ are finitely generated for all i to see that the relevant derived inverse limits vanish, we deduce the following.

Corollary 3.6. *For odd p and $i > 0$, the groups $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$ are as in the following table, with the identifications given explicitly by the Betti-Hodge map:*

$i \mod 4$	0	1	2	3
$\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_p)$	0	0	$K_i(\mathbf{Z}; \mathbf{Z}_p) \oplus \mathbf{Z}_p$	$K_i(\mathbf{Z}; \mathbf{Z}_p)$

Remark 3.7. The relationship between K-theory and Hermitian K-theory is more complicated when the prime 2 is not inverted, and is well understood only quite recently. See [CDH+20a, CDH+20b, CDH+20c][4] as well as [HSV19] and [Sch19], and unpublished work of Hesselholt and Madsen (cf. [HM]).

We consider only odd primes in this paper, and will deduce the isomorphism (3.3) from the work of Karoubi [Kar80].

3.4. **Hermitian K-theory according to Karoubi.** Since we will make crucial use of Karoubi's results on Hermitian K-theory, let us give a review of the main definitions and results in [Kar80].

Following [Kar80], we consider the *Hermitian K-theory* of a ring A with involution $a \mapsto \overline{a}$, i.e. a ring isomorphism $A^{\mathrm{op}} \to A$ satisfying $\overline{\overline{a}} = a$ for all $a \in A$, and a central element $\epsilon \in A$ satisfying $\epsilon\overline{\epsilon} = 1$. The most important example for us will be $A = \mathbf{Z}$ with trivial involution and $\epsilon = \pm 1$, but it seems clearer to review definitions in more generality. (For the purposes of this paper, nothing is lost by assuming all rings in this section are commutative, but for the purposes of this brief survey of Karoubi's result it also seems that nothing is gained.)

3.4.1. *Sesquilinear forms.* A *sesquilinear form* on a projective right A-module M is a $\mathbf{Z}$-bilinear map $\phi : M \times M \to A$ such that $\phi(xa, y) = \overline{a}\phi(x, y)$ and $\phi(x, ya) = \phi(x, y)a$ for all $a \in A$. The set of sesquilinear forms on M forms an abelian group which we shall denote $\mathrm{Sesq}_A(M)$. For $\phi \in \mathrm{Sesq}_A(M)$, setting $\phi^*(x, y) = \overline{\phi(y, x)}$ defines another sesquilinear form ϕ^*, satisfying $\phi^{**} = \phi$. More generally, we may for each central $\epsilon \in A$ satisfying $\epsilon\overline{\epsilon} = 1$ define an action of the cyclic group C_2 of order two on the abelian group $\mathrm{Sesq}_A(M)$, where the generator acts by

$$\phi \mapsto \epsilon\phi^*.$$

A sesquilinear form is ϵ-*symmetric* if it is fixed by this action, in other words if $\phi(x, y) = \epsilon\overline{\phi(y, x)}$, and it is *perfect* if the adjoint

$$M \to M^t$$

---

[4]The second author wishes to thank Fabian Hebestreit, Markus Land, Kristian Moi, and Thomas Nikolaus for helpful conversations.

is an isomorphism, where M^t denotes the set of $\mathbf{Z}$-linear maps $f : M \to A$ satisfying $f(xa) = \overline{a}f(x)$ for all $a \in A$ and $x \in M$, made into a right A-module by setting $(fa)(x) = (f(x))a$. (More concisely, $M^t = \operatorname{Hom}_A(\overline{M}, A)$.)

3.4.2. *Quadratic forms.* As for any action of a finite group on an abelian group, we have the norm (or "symmetrization") map from coinvariants to invariants:

$$\operatorname{Sesq}(M)_{C_2} \to \operatorname{Sesq}(M)^{C_2}$$
$$[\psi] \mapsto \psi + \epsilon\psi^*.$$

An ϵ-*quadratic form* on an A-module M is an element of the coinvariants $\operatorname{Sesq}(M)_{C_2}$ whose norm (a.k.a. symmetrization) is perfect.

Example 3.8. The most important example of an ϵ-quadratic form is the *hyperbolization* of a projective A-module M: the underlying A-module is

$$H(M) = M \oplus M^t,$$

with the quadratic form represented by $\psi((x, f), (y, g)) = f(y)$. The symmetrization is $\phi((x, f), (y, g)) = f(y) + \epsilon g(x)$, which is perfect.

Example 3.9. For example, if we take $A = \mathbf{Z}$ with trivial involution, and L a free A-module,
   (i) A ϵ-symmetric form on L is a symmetric or skew-symmetric bilinear form $L \times L \to \mathbf{Z}$, according to whether $\epsilon = 1$ or $\epsilon = -1$.
  (ii) Symmetrization identifies $+1$-quadratic forms on L with even unimodular $+1$-symmetric forms on L; here "even" means that $\langle x, x \rangle \in 2\mathbf{Z}$.
 (iii) Symmetrization identifies -1-quadratic forms on L with unimodular -1-symmetric forms on L *equipped with* a quadratic refinement $q : L/2 \to \mathbf{Z}/2$; that is to say, $\langle x, y \rangle = q(x + y) - q(x) - q(y)$ modulo 2. Such a refinement is uniquely determined up to a function of the form ℓ^2 for $\ell : L/2 \to \mathbf{Z}/2$ linear.

Definition 3.10. The symmetric monoidal category of quadratic modules, denoted $\mathcal{Q}(A, \epsilon)$, is defined thus:
   - Objects are ϵ-quadratic modules, that is to say, pairs $(M, [\psi])$ where M is a finitely generated projective right A-module and $[\psi]$ is an ϵ-quadratic form on M.
   - A morphism $(M', [\psi']) \to (M, [\psi])$ of ϵ-quadratic modules is an A-linear isomorphism $f : M' \to M$ satisfying $[f^*\psi] = [\psi']$, where $(f^*\psi)(x, y) = \psi(fx, fy)$.
   - The orthogonal direct sum $(M, [\psi]) \oplus (M', [\psi'])$ is defined as $(M \oplus M', [\psi + \psi'])$, where $(\psi + \psi')((x, x'), (y, y')) = \psi(x, y) + \psi'(x', y')$.

3.4.3. *Hermitian K-theory.* As before, the symmetric monoidal category $\mathcal{Q}(A, \epsilon)$ gives rise to a spectrum $KH(A, \epsilon) := K(\mathcal{Q}(A, \epsilon))$ and a group-completion map

$$|\mathcal{Q}(A, \epsilon)| \to \Omega^\infty KH(A, \epsilon),$$

whose homotopy groups are the higher Hermitian K-groups[5] denoted by $KH_i(A, \epsilon)$. A standard lemma (see [Bak81, Lemma 2.9]) shows that any quadratic module is an orthogonal direct summand in $H(A^{\oplus n})$ for some finite n, which implies that KH_i can be obtained using the plus construction applied to the direct limit of automorphism groups of $H(A^{\oplus n})$, for $i > 0$. On the other hand $KH_0(A, \epsilon)$ is the *Grothendieck-Witt group* of ϵ-quadratic forms over A, the Grothendieck group of the monoid $\pi_0 \mathcal{Q}(A, \epsilon)$.

---

[5]Karoubi denotes these as $_\epsilon L_n(A)$, but the letter L seems unfortunate: what is nowadays called "L-theory" is more similar to what Karoubi calls "higher Witt groups", which we denote $W_i(A, \epsilon)$ below.

3.4.4. *Comparison to algebraic K-theory.* There are symmetric monoidal functors

$$\mathcal{Q}(A, \epsilon) \to \mathcal{P}(A) \to \mathcal{Q}(A, \epsilon),$$

where the first map is forgetful, i.e. on objects given by $(M, [\psi]) \mapsto M$, and the second map is the hyperbolization functor, given on objects by $M \mapsto H(M)$. Both promote to symmetric monoidal functors and define spectrum maps

$$KH(A, \epsilon) \xrightarrow{\text{forget}} K(A)$$

$$K(A) \xrightarrow{\text{hyp}} KH(A, \epsilon)$$

and in turn homomorphisms on homotopy groups and mod q homotopy groups. Since the underlying A-module of $H(M) = M \oplus M^t$ is isomorphic to $M \oplus M^\vee$, we see that the map $K(A) \to KH(A, \epsilon)$ is homotopic to its precomposition with ψ^{-1}, and the map $KH(A, \epsilon) \to K(A)$ is homotopic with its post-composition with ψ^{-1}. It follows that the forgetful map on homotopy groups is valued in the subgroup of $K_*(A)$ fixed by ψ^{-1} and the hyperbolic map factors over the coinvariants of ψ^{-1} acting on $K_*(A)$.

Karoubi then defines *higher Witt groups* as

$$W_i(A, \epsilon) = \text{Cok}\big(K_i(A) \xrightarrow{\text{hyp}} KH_i(A, \epsilon)\big)$$

for $n > 0$, so he gets a splitting of $\mathbf{Z}[\frac{1}{2}]$-modules

$$KH_i(A, \epsilon)[\tfrac{1}{2}] = \big(K_i(A)[\tfrac{1}{2}]\big)^{(+)} \oplus \big(W_i(A, \epsilon)[\tfrac{1}{2}]\big). \tag{3.4}$$

The main result of [Kar80] is now a periodicity isomorphism $W_n(A, \epsilon)[\frac{1}{2}] \cong W_{n+2}(A, -\epsilon)[\frac{1}{2}]$, and hence the groups $W_n(A, \epsilon)[\frac{1}{2}]$ are 4-periodic in n.

Remark 3.11. According to [Kar80, §3.14] the higher Witt groups $W_i(A, \epsilon)[\frac{1}{2}]$ may be calculated as follows when A is a ring of integers in a number field, or a localization thereof: to each real embedding we may use the signature to define a homomorphism $W_0(A, 1) \to \mathbf{Z}$; the resulting maps $W_{4n+2}(A, -1)[\frac{1}{2}] \cong W_{4n}(A, 1)[\frac{1}{2}] \to \mathbf{Z}[\frac{1}{2}]^{r_1}$ are isomorphisms, and all other higher Witt groups vanish after inverting 2. We need only the case $A = \mathbf{Z}$, where $W_i(\mathbf{Z})[\frac{1}{2}] = \mathbf{Z}[\frac{1}{2}]$ for $i \equiv 0 \mod 4$, and $W_i(\mathbf{Z})[\frac{1}{2}] = 0$ otherwise.

3.5. **Hermitian K-theory spectra.** The statement proved by Karoubi is about homotopy groups of the Hermitian K-theory spectrum. It is easy to upgrade his statement from isomorphism between homotopy groups to a weak equivalence of spectra, as we shall briefly review in this subsection. The spectrum level upgrade of Karoubi's theorem is more convenient for reading off mod q homotopy groups of Hermitian K-theory.

Let $\mathbb{S}$ denote the sphere spectrum and write $2 : \mathbb{S} \to \mathbb{S}$ for a self-map of degree 2. It induces a self-map of $K(A) = \mathbb{S} \wedge K(A)$ which on homotopy groups induces multiplication by 2. We write $K(A)[\frac{1}{2}]$ for the sequential homotopy colimit of $K(A) \xrightarrow{2} K(A) \xrightarrow{2} \ldots$. In the same way we may invert the self-maps $1 + \psi^{-1}$ and $1 - \psi^{-1}$ of $K(A)$, and the canonical maps $K(A) \to K(A)[(1 \pm \psi^{-1})^{-1}]$ induce a spectrum map

$$K(A)[\tfrac{1}{2}] \xrightarrow{\simeq} \big(K(A)[\tfrac{1}{1+\psi^{-1}}]\big) \times \big(K(A)[\tfrac{1}{1-\psi^{-1}}]\big), \tag{3.5}$$

which is a weak equivalence: on the level of homotopy groups it gives the isomorphism $K_i(A)[\frac{1}{2}] \to K_i(A)[\frac{1}{2}]^{(+)} \oplus K_i(A)[\frac{1}{2}]^{(-)}$.

The formula (3.4) for Hermitian K-theory then upgrades to a spectrum level equivalence

$$KH(A, \epsilon)[\tfrac{1}{2}] \xrightarrow{\simeq} \big(K(A)[\tfrac{1}{1+\psi^{-1}}]\big) \times \big(W(A, \epsilon)[\tfrac{1}{2}]\big), \tag{3.6}$$

whose first coordinate is obtained from the forgetful map $KH(A, \epsilon) \to K(A)$ by inverting 2 and projecting to the first coordinate in (3.5). This projection is split surjective in the homotopy category, by the hyperbolic map restricted to the first factor in (3.5). Hence we may define[6] $W(A, \epsilon)[\frac{1}{2}]$ as the complementary summand.

If we smash with the mod q Moore spectrum $\mathbb{S}/q$ for odd $q = p^n$, multiplication by 2 becomes a weak equivalence, so inverting 2 does not change the homotopy type. Hence (3.5) gives a spectrum level eigenspace splitting

$$(\mathbb{S}/q) \wedge K(A) \xrightarrow{\simeq} \left((\mathbb{S}/q) \wedge K(A)\right)^{(+)} \times \left((\mathbb{S}/q) \wedge K(A)\right)^{(-)},$$

where the superscripts denote inverting $1 \pm \psi^{-1}$ (by taking mapping telescope), and similarly (3.6) gives

$$(\mathbb{S}/q) \wedge KH(A, \epsilon) \xrightarrow{\simeq} \left((\mathbb{S}/q) \wedge K(A)\right)^{(+)} \times \left((\mathbb{S}/q) \wedge W(A, \epsilon)\right).$$

Specializing to $(A, \epsilon) = (\mathbf{Z}, -1)$ and forming smash product with $\mathbb{S}/q$, Remark 3.11 implies that

$$\pi_i(W(\mathbf{Z}, -1); \mathbf{Z}/q) \cong \begin{cases} \mathbf{Z}/q & i \equiv 2 \mod 4 \\ 0 & \text{otherwise.} \end{cases} \tag{3.7}$$

Together with the formulae for $K_*(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$, this completely determines the mod q homotopy groups of $KH(\mathbf{Z}, -1)$.

3.6. **Witt groups, the Hodge map, and the proof of Theorem 3.5.** The proof of the Theorem requires the following non-triviality result about the Hodge map in degrees $\equiv 2$ mod 4, proved in Section 5.1.

Proposition 3.12. *The homomorphism*

$$K\mathrm{Sp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/p) \to \pi_{4k-2}(ku; \mathbf{Z}/p) \cong \mathbf{Z}/p$$

induced by the Hodge map is non-zero for all $k \geq 1$.

This proposition, valid under our standing assumption that p is odd, implies that the homomorphism $K\mathrm{Sp}_{4k-1}(\mathbf{Z}) \to \pi_{4k-2}(ku) \cong \mathbf{Z}$ is non-zero, and in fact that it becomes surjective after inverting 2. The proof (in Proposition 5.2) amounts to constructing a spectrum map $\Sigma_+^\infty B(\mathbf{Z}/p) \to K\mathrm{Sp}(\mathbf{Z})$ whose composition with the Hodge map is nonzero in $\pi_{4k-2}(-; \mathbf{Z}/p)$.

3.6.1. *Comparison between symplectic K-theory and Hermitian K-theory.* The symplectic K-theory groups $K\mathrm{Sp}_*(\mathbf{Z})$ appearing in this paper are very closely related to, but not quite equal to, Karoubi's Hermitian K-groups in the special case where $A = \mathbf{Z}$ with trivial involution, and $\epsilon = -1$. As explained in Example 3.9.(iii), objects of $\mathcal{Q}(\mathbf{Z}, -1)$ are free abelian groups L, equipped with a unimodular (-1)-symmetric form $\omega : L \times L \to \mathbf{Z}$ and a quadratic form $q : L \to \mathbf{Z}/2\mathbf{Z}$ satisfying $q(x+y) - q(x) - q(y) = \omega(x, y) \mod 2$. Forgetting q and remembering L and ω leads to a forgetful functor

$$\mathcal{Q}(\mathbf{Z}, -1) \to \mathcal{SP}(\mathbf{Z})$$

which promotes to a symmetric monoidal functor and induces a spectrum map

$$KH(\mathbf{Z}, -1) \to K\mathrm{Sp}(\mathbf{Z}) \tag{3.8}$$

and on homotopy groups a homomorphism $KH_i(\mathbf{Z}, -1) \to K\mathrm{Sp}_i(\mathbf{Z})$.

---

[6]This agrees with the spectrum defined in [Kar80, §3.10]. When 2 is not inverted, it is better to define $W(A, \epsilon)$ by a cofiber sequence $K(A)_{hC_2} \to KH(A, \epsilon) \to W(A, \epsilon)$, which is the approach taken by many later authors.

Lemma 3.13. *The map of spectra* (3.8) *induces an isomorphism in homotopy groups after inverting 2, and hence in homotopy groups modulo* $q = p^n$ *for any odd prime* p.

This result seems to have been known to experts for a while, but for completeness we provide an elementary proof. Very recently, the surprising fact that (3.8) induces an isomorphism in homotopy groups in degrees ≥ 4 without inverting 2 was proved in [CDH$^+$20c, Theorem 3.2.13].

Proof of surjectivity in Lemma 3.13. It follows from homological stability ([Cha87, Corollary 4.5] with $A = \mathbf{Z}$, $\lambda = -1$, and $\Lambda = \mathbf{Z}$) that the group completion map $B\mathrm{Sp}_{2g}(\mathbf{Z}) \subset |\mathcal{SP}(\mathbf{Z})| \to \Omega^\infty \mathrm{KSp}(\mathbf{Z})$ factors over a map

$$B\mathrm{Sp}_{2g}(\mathbf{Z})^+ \to \Omega^\infty \mathrm{KSp}(\mathbf{Z}) \tag{3.9}$$

which induces an isomorphism in homotopy groups π_i with suitable basepoint, when $g \gg i > 0$. Let us write $\pi_i(E; \mathbf{Z}_{(p)}) = \pi_i(E) \otimes \mathbf{Z}_{(p)}$ for the homotopy groups of a pointed space or a spectrum E, localized at p. We first claim that the homomorphisms

$$\pi_*(KH(\mathbf{Z}, -1); \mathbf{Z}_{(p)}) \to \pi_*(\mathrm{KSp}(\mathbf{Z}); \mathbf{Z}_{(p)}), \tag{3.10}$$

induced by (3.8) are surjective for any odd prime p. We shall prove this claim by factoring the map (3.9) through $KH(\mathbf{Z}, -1)$, after suitably localizing all spectra (by inverting a natural number prime to p, by a colimit as in the previous sections).

Let us consider the hyperbolic object $H(\mathbf{Z}^g) \in \mathcal{Q}(Z, -1)$ and denote[7] its automorphism group $\mathrm{Sp}_{2g}^q(\mathbf{Z}) \subset \mathrm{GL}_{2g}(\mathbf{Z})$. The automorphism group of the image of $H(\mathbf{Z}^g)$ in $\mathcal{SP}(\mathbf{Z})$ is $\mathrm{Sp}_{2g}(\mathbf{Z})$, which acts through $\mathrm{Sp}_{2g}(\mathbf{Z}/2\mathbf{Z})$ on the set of quadratic forms $q : \mathbf{Z}^{2g} \to \mathbf{Z}/2\mathbf{Z}$ refining the symplectic form. There are 2^{2g} many such refinements, all of the form $q(x) = \sum_{i=1}^g (a_{2i-1}x_{2i-1}^2 + a_{2i}x_{2i}^2 + x_{2i-1}x_{2i})$ with no conditions on the $a_i \in \mathbf{Z}/2\mathbf{Z}$. The action has precisely 2 orbits, distinguished by $\mathrm{Arf}(q) = \sum_{i=1}^g a_{2i-1}a_{2i} \in \mathbf{Z}/2\mathbf{Z}$. An easy counting argument shows that there are $2^{g-1}(2^g + 1)$ refinements of Arf invariant 0 and $2^{g-1}(2^g - 1)$ of Arf invariant 1. The subgroup $\mathrm{Sp}_{2g}^q(\mathbf{Z}) < \mathrm{Sp}_{2g}(\mathbf{Z})$ is the stabilizer of $q = \sum_i x_{2i-1}x_{2i}$ and therefore has index $2^{g-1}(2^g + 1)$.

Before considering homotopy groups, we prove the easier statement that the induced map in group homology

$$H_i(B\mathrm{Sp}_{2g}^q(\mathbf{Z}); \mathbf{Z}_{(p)}) \xrightarrow{i_*} H_i(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_{(p)})$$

is surjective for $g \gg i$ for odd primes p. Since the subgroup has finite index, we have a transfer homomorphism

$$H_i(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_{(p)}) \xrightarrow{t_*} H_i(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_{(p)}),$$

and the composition $t_* \circ i_*$ agrees with multiplication by the index $2^{g-1}(2^g + 1)$. Since the odd prime p cannot divide both $2^g + 1$ and $2^{g+1} + 1$, we may choose $g \gg i$ such that $2^{g-1}(2^g + 1) \in \mathbf{Z}_{(p)}^\times$ and therefore i_* is surjective and t_* is injective.

The homomorphism i_* is induced by a map of spaces $B\mathrm{Sp}_{2g}^q(\mathbf{Z}) \to B\mathrm{Sp}_{2g}(\mathbf{Z})$ which up to homotopy equivalence is a $2^{g-1}(2^g + 1)$-sheeted covering map, and the transfer map is induced by a map of suspension spectra $\Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z}) \to \Sigma_+^\infty B\mathrm{Sp}_{2g}^q(\mathbf{Z})$. Inverting the index may be performed on the spectrum level as well, by a suitable colimit as in the previous sections. We obtain spectrum maps

$$\left(\Sigma_+^\infty B\mathrm{Sp}_{2g}^q(\mathbf{Z})\right)\left[\tfrac{1}{2^{g-1}(2^g+1)}\right] \xrightarrow{i} \left(\Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z})\right)\left[\tfrac{1}{2^{g-1}(2^g+1)}\right]$$

$$\left(\Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z})\right)\left[\tfrac{1}{2^{g-1}(2^g+1)}\right] \xrightarrow{t} \left(\Sigma_+^\infty B\mathrm{Sp}_{2g}^q(\mathbf{Z})\right)\left[\tfrac{1}{2^{g-1}(2^g+1)}\right],$$

---

[7]This agrees with the group denoted $_{-1}O_g(\mathbf{Z})$ in [Kar80]. It is known under various other names in the literature. The notation $\mathrm{Sp}_{2g}^q(\mathbf{Z})$ is common in the topology literature, where "q" stands for "quadratic refinement".

with the property that $i \circ t$ is a weak equivalence. We now consider the diagram of spectra

$$
\begin{array}{ccc}
\Sigma_+^\infty B\mathrm{Sp}_{2g}^q(\mathbf{Z}) & \longrightarrow & KH(\mathbf{Z}, -1) \\
{\scriptstyle t}\big\uparrow\,\big\downarrow{\scriptstyle i} & & \big\downarrow \\
\Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z}) & \longrightarrow & \mathrm{KSp}(\mathbf{Z}),
\end{array}
$$

where the horizontal maps are adjoint to the group-completion maps. After inverting $2^{g-1}(2^g + 1)$ on the spectrum level, the composition $i \circ t$ becomes a weak equivalence, so we obtain a right inverse to i as $t \circ (i \circ t)^{-1}$. This implies a homotopy commutative diagram

$$
\begin{array}{ccc}
 & & KH(\mathbf{Z}, -1)\big[\tfrac{1}{2^{g-1}(2^g+1)}\big] \\
 & \nearrow & \big\downarrow \\
\Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z}) & \longrightarrow & \mathrm{KSp}(\mathbf{Z})\big[\tfrac{1}{2^{g-1}(2^g+1)}\big].
\end{array}
$$

The spectrum map $\Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z}) \to \Sigma_+^\infty B\mathrm{Sp}_{2g}(\mathbf{Z})^+$ is a homology equivalence and hence a weak equivalence, so we get a homotopy commutative diagram

$$
\begin{array}{ccc}
 & & \Omega^\infty KH(\mathbf{Z}, -1)[\tfrac{1}{2^{g-1}(2^g+1)}] \\
 & \nearrow & \big\downarrow \\
B\mathrm{Sp}_{2g}(\mathbf{Z})^+ & \longrightarrow & \Omega^\infty \mathrm{KSp}(\mathbf{Z})[\tfrac{1}{2^{g-1}(2^g+1)}],
\end{array}
$$

inducing a commutative diagram on the level of homotopy groups with suitable basepoints. For $g \gg i$, the bottom map induces an isomorphism

$$
\pi_i(B\mathrm{Sp}_{2g}(\mathbf{Z})^+) \otimes \mathbf{Z}[\tfrac{1}{2^{g-1}(2^g+1)}] \xrightarrow{\sim} \mathrm{KSp}_i(\mathbf{Z}) \otimes \mathbf{Z}[\tfrac{1}{2^{g-1}(2^g+1)}],
$$

and since no odd prime can divide both $2^g + 1$ and $2^{g+1} + 1$, this implies surjectivity of (3.10) in positive degrees.

   Strictly speaking we have not yet considered (3.10) in degree 0, but this can be done by hand. Indeed, $\pi_0(KH(\mathbf{Z}, -1)) \cong \mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z}$, given by (rank/2, Arf), and the map to $\pi_0(\mathrm{KSp}(\mathbf{Z}))$ may be identified with the projection $\mathbf{Z} \oplus \mathbf{Z}/2\mathbf{Z} \to \mathbf{Z}$. $\qquad\square$

Proof of injectivity in Lemma 3.13, assuming Proposition 3.12. We now prove that the surjection (3.10) is also injective. Indeed, we have seen that the domain splits as $K_i(\mathbf{Z}; \mathbf{Z}_{(p)})^{(+)} \oplus \pi_i(W(\mathbf{Z}, -1); \mathbf{Z}_{(p)})$, and it is clear that the projection $KH_i(\mathbf{Z}, -1) \to K_i(\mathbf{Z}; \mathbf{Z}_{(p)})^{(+)}$ factors through $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}_{(p)})$. This proves that (3.10) is injective when $i \not\equiv 2 \bmod 4$. For $i \equiv 2 \bmod 4$ it proves injectivity on the torsion subgroup, so since $W_i(\mathbf{Z}, -1) \otimes \mathbf{Z}_{(p)} = \mathbf{Z}_{(p)}$ in such degrees, it must either be injective or have finite image. But $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_{(p)})$ cannot be finite: it follows from Proposition 3.12 that $c_H : \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_{(p)}) \to \pi_{4k-2}(ku; \mathbf{Z}_{(p)}) = \mathbf{Z}_{(p)}$ is surjective. $\qquad\square$

Proof of Theorem 3.5, assuming Proposition 3.12. Isomorphism with p-local coefficients implies isomorphism with mod $q = p^n$ coefficients, so we have isomorphisms

$$
\pi_i(\mathrm{KSp}(\mathbf{Z}); \mathbf{Z}/q) \xleftarrow{\cong} \pi_i(KH(\mathbf{Z}, -1); \mathbf{Z}/q) \xrightarrow{\cong} K_i(\mathbf{Z}; \mathbf{Z}/q)^{(+)} \oplus \pi_i(W(\mathbf{Z}, -1); \mathbf{Z}/q),
$$

while we wish to show that

$$
\pi_i(\mathrm{KSp}(\mathbf{Z}); \mathbf{Z}/q) \xrightarrow{(c_B, c_H)} K_i(\mathbf{Z}; \mathbf{Z}/q)^{(+)} \oplus \pi_i(ku; \mathbf{Z}/q)^{(-)} \tag{3.11}
$$

is an isomorphism. By inspection, the groups $\pi_i(ku; \mathbf{Z}/q)^{(-)}$ and $\pi_i(W(\mathbf{Z}, -1); \mathbf{Z}/q)$ are abstractly isomorphic for all $i \geq 0$, and since all the groups involved here are finite, it suffices to

show that (3.11) is surjective. Composing the hyperbolization map $K_i(\mathbf{Z}; \mathbf{Z}/p) \to K\mathrm{Sp}_i(\mathbf{Z}; \mathbf{Z}/p)$ with the Betti-Hodge map induces

$$(1 + \psi^{-1}, 0) : K_i(\mathbf{Z}; \mathbf{Z}/p) \to K_i(\mathbf{Z}; \mathbf{Z}/p) \times \pi_i(ku; \mathbf{Z}/p)^{(-)},$$

because the composition with the Hodge map may be identified with the map $K(\mathbf{Z}) \to ku$ arising from the inclusion $\mathbf{Z} \to \mathbf{C}^{\mathrm{top}}$ composed with $1 + \psi^{-1}$, which lands in the positive eigenspace. Hence the image of (3.11) contains the first summand. The second summand is nonzero only for i congruent to 2 modulo 4, and the claim follows from Proposition 3.12. $\square$

Remark 3.14. A similar argument shows that on the level of spectra, there is a weak equivalence $K\mathrm{Sp}(\mathbf{Z})[\frac{1}{2}] \xrightarrow{\simeq} K(\mathbf{Z})[\frac{1}{1+\psi^{-1}}] \times ku[\frac{1}{1-\psi^{-1}}]$.

3.7. **Bott inverted symplectic K-theory.** This subsection is in fulfillment of Remark 1.4, but is not logically necessary for the main presentation of our results.

Recall from Remark 2.8 that there is a spectrum T such that $K^{(\beta)}(X; \mathbf{Z}/q) = \pi_*(K(X) \wedge T)$, and that T is defined as a mapping telescope of a self-map $\Sigma^m \mathbb{S}/q \to \mathbb{S}/q$ with $m = 2p^{n-1}(p-1)$, chosen with the property that it induces an isomorphism $\pi_i(K) \to \pi_{i+m}(K)$ for all i when K denotes periodic complex K-theory. Such a map is often called a v_1 *self-map*, and serves as a replacement for multiplication by the Bott element. We may then define "Bott inverted homotopy groups" of any spectrum E as the homotopy groups of $E \wedge T$, although this is more commonly called v_1-*inverted homotopy groups* and denoted

$$\pi_i(E; \mathbf{Z}/q)[v_1^{-1}] = \pi_i(E \wedge T) = \mathrm{colim}\left(\pi_i(E; \mathbf{Z}/q) \to \pi_{i+m}(E; \mathbf{Z}/q) \to \dots\right),$$

where the maps in the direct limit are induced by the chosen v_1 self-map.

For example, the natural map $ku \to K$ from connective to periodic complex K-theory induces isomorphisms

$$\pi_i(ku; \mathbf{Z}/q)[v_1^{-1}] \to \pi_i(K; \mathbf{Z}/q)[v_1^{-1}] \leftarrow \pi_i(K; \mathbf{Z}/q),$$

for all $i \in \mathbf{Z}$. In this notation we have

$$K_i^{(\beta)}(X; \mathbf{Z}/q) = \pi_i(K(X); \mathbf{Z}/q)[v_1^{-1}],$$

and we may completely similarly define

$$K\mathrm{Sp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q) := \pi_i(K\mathrm{Sp}(\mathbf{Z}); \mathbf{Z}/q)[v_1^{-1}]. \tag{3.12}$$

Also, we define the *p-adic Bott-inverted symplectic K-theory groups*

$$K\mathrm{Sp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p) := \varprojlim_n K\mathrm{Sp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}/p^n).$$

Since colimits preserve isomorphisms, we immediately deduce the following.

Corollary 3.15. *The Bott inverted symplectic K-theory groups of $\mathbf{Z}$ are given by isomorphisms*

$$K\mathrm{Sp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q) \xrightarrow{(c_B, c_H)} K_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q)^{(+)} \times \pi_i(K; \mathbf{Z}/q)^{(-)}$$

for all $i \in \mathbf{Z}$. $\square$

These groups are periodic in i and in particular they are likely non-zero in negative degrees. We then obtain isomorphisms

$$K\mathrm{Sp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{(c_B, c_H)} K_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)^{(+)} \times \pi_i(K; \mathbf{Z}_p)^{(-)}$$

of $\mathbf{Z}_p$-modules. These are still non-zero in many negative degrees, but are no longer periodic of any degree.

Remark 3.16. To elaborate upon Remark 1.4, we explain that these inverse limits of Bott inverted mod p^n groups may be re-expressed using $K(1)$-*localization*. The $K(1)$-localization of a spectrum E consists of another spectrum $L_{K(1)}E$ and a map $E \to L_{K(1)}E$ with various good properties. (The functor $L_{K(1)}$ depends on p, which is traditionally omitted from the notation.) The defining properties include that the induced homomorphism in K/p-homology $(K/p)_*(E) \to (K/p)_*(L_{K(1)}E)$ is an isomorphism, where K denotes periodic complex K-theory and $K/p = (\mathbb{S}/p) \wedge K$. More relevant for us is that it "implements inverting v_1", see [Rav84, Theorem 10.12], and we have canonical isomorphisms

$$\mathrm{KSp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q) \cong \pi_i(L_{K(1)}\mathrm{KSp}(\mathbf{Z}); \mathbf{Z}/q)$$

$$\mathrm{KSp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p) \cong \pi_i(L_{K(1)}\mathrm{KSp}(\mathbf{Z})).$$

4. Review of the theory of CM abelian varieties

In the main part of this section, we discuss the theory of abelian varieties with *complex multiplication* (CM). In order to motivate why we are doing this, let us first explain how KSp is related to abelian varieties, and then outline how the theory of complex multiplication can be used to produce classes in KSp($\mathbf{Z}$).

4.1. **Abelian varieties, symplectic K-theory, and the construction of CM classes.** As discussed in the outline in §1.3, we are going to construct certain "CM classes" in the symplectic K-theory of $\mathbf{Z}$. Let us first go from abelian varieties to K-theory, before considering how CM enters the picture.

There is a functor between groupoids

$$\mathcal{A}_g(\mathbf{C}) \to \mathcal{SP}(\mathbf{Z}). \tag{4.1}$$

Here the domain $\mathcal{A}_g(\mathbf{C})$ denotes the groupoid of principally polarized abelian varieties and isomorphisms between such. In particular, we do not take the topology of $\mathbf{C}$ into account at this moment.

An object in $\mathcal{A}_g(\mathbf{C})$ consists of an abelian variety $A \to \mathrm{Spec}\,(\mathbf{C})$ together with a polarization, which is given by a line bundle $\mathcal{L} \to A \times_{\mathrm{Spec}\,(\mathbf{C})} A$, rigidified by a non-zero section of $\mathcal{L}$ over (e, e). The reference map $A \to \mathrm{Spec}\,(\mathbf{C})$ allows us to take "Betti" homology $H_*(A(\mathbf{C}); \mathbf{Z})$ and cohomology, and $c_1(\mathcal{L}) \in H^2(A(\mathbf{C}) \times A(\mathbf{C}); \mathbf{Z})$ defines a skew symmetric pairing on $L = H_1(A(\mathbf{C}); \mathbf{Z})$ which is perfect because the polarization is principal. We therefore have an object $(H_1(A; \mathbf{Z}), c_1(\mathcal{L})) \in \mathcal{SP}(\mathbf{Z})$. Similarly isomorphisms in $\mathcal{A}_g(\mathbf{C})$ are sent to isomorphisms in $\mathcal{SP}(\mathbf{Z})$, so (4.1) induces $|\mathcal{A}_g(\mathbf{C})| \to |\mathcal{SP}(\mathbf{Z})| \to \Omega^\infty \mathrm{KSp}(\mathbf{Z})$ and then by adjunction a map of spectra

$$\Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})| \to \mathrm{KSp}(\mathbf{Z}). \tag{4.2}$$

Next we explain what CM classes are. We take a principally polarized abelian variety A that admits an action of the cyclotomic ring $\mathcal{O}_q = \mathbf{Z}[e^{2\pi i/q}] \subset \mathbf{C}$. In particular, the cyclic group $\mathbf{Z}/q$ acts on A (where $1 \in \mathbf{Z}/q$ acts via $e^{2\pi i/q}$) giving rise to a morphism of groupoids $B(\mathbf{Z}/q) \to \mathcal{A}_g(\mathbf{C})$, whence

$$\Sigma_+^\infty(B(\mathbf{Z}/q)) \to \Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})| \to \mathrm{KSp}(\mathbf{Z}).$$

Now take homotopy with mod q coefficients. On the left, we get the stable homotopy of $B(\mathbf{Z}/q)$ with mod q coefficients; in §2.1.1 we described a polynomial algebra $\mathbf{Z}/q[\beta]$ inside this homotopy ring. The image of powers of β under the composite in $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}/q)$ are, by definition, the "CM classes" of §1.3. (A more precise version of this discussion is given after Proposition 5.1).

Now let us review how principally polarized abelian varieties with an action of $\mathcal{O}_q$ are parameterized. We will work a little more generally: for any $2g$-dimensional abelian variety A, the dimension of any commutative $\mathbf{Q}$-subalgebra of $\mathrm{End}(A) \otimes \mathbf{Q}$ is at most g. If equality holds,

then A is said to have "complex multiplication" (or CM for short), and the ring $\mathrm{End}(A)$ is necessarily a *CM order*:

Definition 4.1. (1) A *CM field* is, by definition, a field extension of $\mathbf{Q}$ which is a totally imaginary extension of a totally real field E^+, i.e. $E \cong E^+(\sqrt{d})$ where all embeddings $E^+ \to \mathbf{C}$ have real image, and all take d to negative real numbers.[8] A *CM algebra* is a product of CM fields.

(2) A *CM order* is an order in a CM algebra E stable by conjugation.

Here "order" means a subring $\mathcal{O} \leqslant E$ which is free as a $\mathbf{Z}$-module and for which $\mathcal{O} \otimes_{\mathbf{Z}} \mathbf{Q} \to E$ is an isomorphism; and "conjugation" is the unique automorphism $x \mapsto \bar{x}$ of E which induces conjugation in any homomorphism $E \hookrightarrow \mathbf{C}$.

We can construct CM abelian varieties as follows: taking $\mathcal{O}$ a CM-order, let $\mathfrak{a} \leqslant \mathcal{O}$ be an ideal, and $\Phi : \mathcal{O} \otimes \mathbf{R} \simeq \mathbf{C}^g$ an isomorphism. Then $(\mathcal{O} \otimes \mathbf{R})/\mathfrak{a}$ has the structure of complex analytic torus. To give it an algebraic structure, one must polarize the resulting torus: one needs a symplectic $\mathbf{Z}$-valued pairing on the first homology group $\mathfrak{a}$. To get it one chooses a suitable purely imaginary element $u \in \mathcal{O} \otimes \mathbf{Q}$ and considers the symplectic form $(x,y) \in \mathfrak{a} \mapsto \mathrm{Tr}(xu\bar{y})$. All CM abelian varieties over $\mathbf{C}$ arise from this construction.

The resulting construction produces a complex abelian variety A from the data $\mathcal{O}, \mathfrak{a}, \Phi, u$. For any automorphism $\sigma \in \mathrm{Aut}(\mathbf{C})$ the twist $\sigma(A)$, i.e. the abelian variety obtained by applying σ to a system of equations defining A, necessarily arises from some other data $(\mathcal{O}', \mathfrak{a}', \Phi', u')$. The *Main Theorem of Complex Multiplication* in its sharpest form, describes how to compute this new data. This theorem (in a slightly weaker form) is due to Shimura and Taniyama, and it will eventually be used by us to compute the action of $\mathrm{Aut}(\mathbf{C})$ on CM classes in $\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}/q)$.

In our presentation – designed to simplify the interface with algebraic K-theory – we will regard the basic object as the $\mathcal{O}$-module $\mathfrak{a}$ together with the skew-Hermitian form $x, y \mapsto xu\bar{y}$, valued in $\mathfrak{a}u\bar{\mathfrak{a}}$. We will interpret the construction sketched above as a functor of groupoids

$$\mathcal{P}_E^- = \text{groupoid of skew-Hermitian } \mathcal{O}\text{-modules} \xrightarrow{\mathsf{ST}} \mathcal{A}_g(\mathbf{C}).$$

The composition of this functor with (4.1) $\mathcal{A}_g(\mathbf{C}) \to \mathcal{SP}(\mathbf{Z})$ associates to a skew-Hermitian module an underlying symplectic $\mathbf{Z}$-module.

Remark 4.2. The appearance of Hermitian forms is quite natural from the point of view of the theory of Shimura varieties: indeed, the set of abelian varieties with CM by a given field E is related to the Shimura variety for an associated unitary group.

4.2. **Picard groupoids.** Recall from Section 2.6 that for a commutative ring R we have defined $\mathrm{Pic}(R)$ as the groupoid whose objects are rank 1 projective R-modules and whose morphisms are R-linear isomorphisms between them. For a ring with involution, there is a version of this groupoid where the objects are equipped with perfect sesquilinear forms.

Definition 4.3. For a commutative ring $\mathcal{O}$ with involution $x \mapsto \overline{x}$ and an $\mathcal{O}$-module L we shall write $\overline{L}$ for the module with the same underlying abelian group but $\mathcal{O}$-action changed by the involution. For a rank 1 projective $\mathcal{O}$-module ω equipped with an $\mathcal{O}$-linear involution $\iota : \omega \to \overline{\omega}$ we shall write $\mathcal{P}(\mathcal{O}, \omega, \iota)$ for the following groupoid:

- Objects are pairs (L, b) where L is a rank 1 projective $\mathcal{O}$-module and $b : L \otimes_{\mathcal{O}} \overline{L} \to \omega$ an isomorphism satisfying $b(x \otimes y) = \iota(b(y \otimes x))$.

We may equivalently view b as a function $L \times L \to \omega$ which is $\mathcal{O}$-linear in the first variable and conjugate $\mathcal{O}$-linear in the second variable, and we will frequently do this below.

---

[8]In the literature one sometimes sees a slightly broader definition of CM fields, including totally real fields.

- Morphisms $(L, b) \to (L', b')$ are $\mathcal{O}$-linear isomorphisms $\phi : L \to L'$ such that $b'(\phi(x), \phi(y)) = b(x, y)$ for all $x, y \in L$.

There are some instances of this construction of particular interest for us. Take E to be a CM field and $\mathcal{O}$ to be its ring of integers (i.e., the integral closure of $\mathbf{Z}$ in E), with involution the conjugation $x \mapsto \bar{x}$.

(i) $\mathcal{P}_E^+$, the groupoid of Hermitian forms on $\mathcal{O}$:

Take $\omega = \mathcal{O}$ with the conjugation involution and set $\mathcal{P}_E^+ = \mathcal{P}(\mathcal{O}, \omega, \iota)$.

(ii) $\mathcal{P}_{E \otimes \mathbf{R}}^+$, the groupoid of Hermitian forms on $E \otimes \mathbf{R}$:

As in (i), but now replacing $\mathcal{O}$ by $\mathcal{O} \otimes \mathbf{R}$ and ω by $\mathcal{O} \otimes \mathbf{R} = \omega \otimes \mathbf{R}$, i.e. $\mathcal{P}_{E \otimes \mathbf{R}} = \mathcal{P}(\mathcal{O} \otimes \mathbf{R}, \mathcal{O} \otimes \mathbf{R}, \iota \otimes \mathbf{R})$.

(iii) $\mathcal{P}_E^-$, the groupoid of skew-Hermitian forms on $\mathcal{O}$ valued in the inverse different:

Take $\omega = \mathfrak{d}^{-1}$ the inverse different[9] for E, with the *negated* conjugation involution $-\iota : z \mapsto -\bar{z}$ and set $\mathcal{P}_E^- = \mathcal{P}(\mathcal{O}, \mathfrak{d}^{-1}, -\iota)$.

(iv) $\mathcal{P}_{E \otimes \mathbf{R}}^-$, the groupoid of skew-Hermitian forms on $E \otimes \mathbf{R}$:

As in (iii), but tensoring with $\mathbf{R}$.

Now, given $(L, b) \in \mathcal{P}_E^-$, we shall write $L_{\mathbf{Z}}$ for the $\mathbf{Z}$-module underlying L. It is a free $\mathbf{Z}$-module of rank $2g = \dim_{\mathbf{Q}}(E)$, and inherits a bilinear pairing

$$
\begin{aligned}
L_{\mathbf{Z}} \times L_{\mathbf{Z}} &\to \mathbf{Z} \\
(x, y) &\mapsto -\operatorname{Tr}_{\mathbf{Q}}^E(b(x, y)).
\end{aligned} \tag{4.3}
$$

This pairing is readily verified to be skew-symmetric and perfect (i.e., the associated map $L_{\mathbf{Z}} \to L_{\mathbf{Z}}^\vee$ is an isomorphism) so that associating to $(L, b) \in \mathcal{P}_E^-$ the free $\mathbf{Z}$-module with the pairing above defines a *functor*

$$
\mathcal{P}_E^- \to \mathcal{SP}(\mathbf{Z}). \tag{4.4}
$$

We shall return to this in Section 4.3 below.

Finally, we comment on monoidal structure. Unlike $\operatorname{Pic}(R)$, we do not have a symmetric monoidal structure on $\mathcal{P}(\mathcal{O}, \omega, \iota)$ in general. However, if we take $\omega = \mathcal{O}$ equipped with the involution on $\mathcal{O}$, then $\mathcal{P}_E^+ = \mathcal{P}(\mathcal{O}, \mathcal{O}, \iota)$ has the structure of a symmetric monoidal groupoid, and more generally:

Definition 4.4. Let $(\mathcal{O}, \omega, \iota)$ and $(\mathcal{O}, \omega', \iota')$ be as above (same underlying ring with involution, two different invertible modules with involution). Define a functor

$$
\mathcal{P}(\mathcal{O}, \omega, \iota) \times \mathcal{P}(\mathcal{O}, \omega', \iota') \xrightarrow{\otimes} \mathcal{P}(\mathcal{O}, \omega \otimes_{\mathcal{O}} \omega', \iota \otimes \iota') \tag{4.5}
$$

as $(L, b) \otimes (L', b') = (L \otimes_{\mathcal{O}} L', b \otimes b')$, where $(b \otimes b')(x \otimes x', y \otimes y') = b(x, y) b(x', y')$.

In particular this construction gives a symmetric monoidal structure on $\mathcal{P}_E^+$ and an "action" bifunctor $\mathcal{P}_E^+ \times \mathcal{P}_E^- \to \mathcal{P}_E^-$.

4.3. **Construction of CM abelian varieties.** Let E be a CM field. We will now construct the map $\mathsf{ST} : \mathcal{P}_E^- \to \mathcal{A}_g(\mathbf{C})$ promised in §4.1. In fact this factors the functor $\mathcal{P}_E^- \to \mathcal{SP}(\mathbf{Z})$ of (4.4).

$$
\mathcal{P}_E^- \xrightarrow{\mathsf{ST}} \mathcal{A}_g(\mathbf{C}) \xrightarrow{(4.1)} \mathcal{SP}(\mathbf{Z}). \tag{4.6}
$$

---

[9]The inverse different $\mathfrak{d}^{-1}$ is, by definition,

$$
\mathfrak{d}^{-1} = \{y \in E \mid \operatorname{Tr}_{\mathbf{Q}}^E(xy) \in \mathbf{Z} \text{ for all } x \in \mathcal{O}\}.
$$

which is canonically isomorphic to $\operatorname{Hom}_{\mathbf{Z}}(\mathcal{O}, \mathbf{Z})$, with module structure defined by $(a.f)(x) = f(ax)$; under this identification the trace $\mathfrak{d}^{-1} \to \mathbf{Z}$ is sent to the functional on $\operatorname{Hom}_{\mathbf{Z}}(\mathcal{O}, \mathbf{Z})$ given by precomposition with $\mathbf{Z} \to \mathcal{O}$.

To construct the functor ST, start with an object $(L, b) \in \mathcal{P}_E^-$. We shall equip $L_\mathbf{R}/L_\mathbf{Z}$ with the structure of a principally polarized abelian variety. In order to do so it is necessary to specify, firstly, a complex structure J on $L_\mathbf{R}$, and secondly a Hermitian form on $L_\mathbf{R}$ whose imaginary part is a perfect symplectic pairing $L_\mathbf{Z} \times L_\mathbf{Z} \to \mathbf{Z}$. (This data can be used, as in [Mum08, Section I.2], to construct an explicit ample line bundle on $L_\mathbf{R}/L_\mathbf{Z}$ whose first Chern class is the specified symplectic pairing.)

We begin by specifying the symplectic pairing: it is given by the expression of (4.3), i.e.

$$L_\mathbf{Z} \times L_\mathbf{Z} \to \mathbf{Z}$$
$$(x, y) \mapsto -\mathrm{Tr}_\mathbf{Q}^E b(x, y). \tag{4.7}$$

(The sign is a purely a convention—the opposite convention would lead to other signs elsewhere, e.g. the inequality in (4.10) below would be the other way around.) The definition of $\mathfrak{d}^{-1}$ makes this form $\mathbf{Z}$-valued and perfect, by the corresponding properties of b. The real-linear extension of this symplectic form is the imaginary part of a Hermitian form on $L_\mathbf{R}$ in a complex structure; we specify this complex structure and Hermitian form next.

A *CM type* Φ for E is, by definition, a subset $\Phi \subset \mathrm{Hom}_{\mathrm{Rings}}(E, \mathbf{C})$ with the property that the induced map

$$E \otimes \mathbf{R} \to \mathbf{C}^\Phi \tag{4.8}$$

is an isomorphism; equivalently, Φ contains precisely one element in each conjugacy class $\{j, \overline{j}\}$. Such a Φ determines a complex structure on $L_\mathbf{R}$, for (4.8) gives $E \otimes \mathbf{R}$ the structure of $\mathbf{C}$-algebra.

If Φ is a CM type, then

$$\mathrm{Tr}_\mathbf{Q}^E b(x, y) = 2 \mathrm{Re}\bigg(\sum_{j \in \Phi} j(b(x, y)) \bigg),$$

where we used that $\mathrm{Tr}_\mathbf{Q}^E(x) = \sum_{j : E \to \mathbf{C}} j(x) \in \mathbf{Q} \subset \mathbf{C}$, where the sum is over all ring homomorphisms $E \to \mathbf{C}$. In particular, the function $L_\mathbf{R} \times L_\mathbf{R} \to \mathbf{C}$ given by

$$\langle x, y \rangle_b = -2i \bigg(\sum_{j \in \Phi} j(b(x, y)) \bigg), \tag{4.9}$$

has (4.7) for imaginary part. Moreover, $\langle -, - \rangle_b$ is Hermitian with respect to the complex structure on $L_\mathbf{R}$ induced by Φ. Finally, $\langle -, - \rangle_b$ is positive definite precisely for the unique CM-type $\Phi = \Phi_{(L,b)}$, defined as

$$\Phi_{(L,b)} := \{j : E \to \mathbf{C} \mid \mathrm{Im}(jb(x,x)) \geq 0 \text{ for all } x \in L_\mathbf{R}\} \tag{4.10}$$

i.e. the embeddings sending $b(x, x) \in E \otimes \mathbf{R}$ to the upper half-plane for all $x \in L_\mathbf{R}$. We shall say that $\Phi_{(L,b)}$ is the CM structure on E *associated* to the object $(L, b) \in \mathcal{P}_E^-$. Evidently, it depends only on the image of (L, b) under the base change functor

$$\mathcal{P}_E^- \xrightarrow{-\otimes_\mathbf{Z}\mathbf{R}} \mathcal{P}_{E\otimes\mathbf{R}}^-$$

To summarize, to (L, b) we have associated:

- a complex structure on $L_\mathbf{R}$ (the one induced from $\Phi_{(L,b)}$ via (4.8));
- a positive definite Hermitian form $\langle -, - \rangle_b$ on this complex vector space; the imaginary part of this form restricts to the symplectic form (4.7).

The quotient $L_\mathbf{R}/L_\mathbf{Z}$ thus has the structure of a principally polarized abelian variety over $\mathbf{C}$; we denote it by $\mathsf{ST}(L, b)$.

The $\mathcal{O}$-module structure on L gives a homomorphism

$$\mathcal{O} \to \mathrm{End}\, \mathsf{ST}(L, b),$$

which is a homomorphism of rings with involution when the target is given the Rosatti involution induced by the polarization of $\mathsf{ST}(L, b)$. Acting by an element of $a \in \mathcal{O}$ gives an endomorphism of $\mathsf{ST}(L, b)$, which will be an automorphism if $a \in \mathcal{O}^\times$, but not necessarily one that preserves the polarization: the polarization is given by a line bundle $\mathscr{L}$ on $\mathsf{ST}(L, b) \times \mathsf{ST}(L, b)$, and the correct statement is that

$$(a, 1)^*(\mathscr{L}) = (1, \overline{a})^*(\mathscr{L}). \tag{4.11}$$

However, acting by an element of the subgroup $\{a \in \mathcal{O}^\times \mid x\overline{x} = 1\}$ does preserve the polarization.

The association $(L, b) \mapsto \mathsf{ST}(L, b)$ defines the desired functor

$$\mathsf{ST} : \mathcal{P}_E^- \to \mathcal{A}_g(\mathbf{C}). \tag{4.12}$$

Remark 4.5. In fact, the association $(L, b) \mapsto \mathsf{ST}(L, b)$ can be made into an equivalence by modifying the target category. Namely, consider principally polarized abelian variety A over the complex numbers together with a map $\iota : \mathcal{O} \to \mathrm{End}(A)$ that respects the polarization in the sense of (4.11). Such form a groupoid in an evident way (the morphisms being isomorphisms of abelian varieties respecting polarization and $\mathcal{O}$-action); call this groupoid $\mathcal{A}_g^{\mathcal{O}}(\mathbf{C})$. The functor ST defined in (4.12) factors through

$$\mathcal{P}_E^- \longrightarrow \mathcal{A}_g^{\mathcal{O}}(\mathbf{C})$$

and this is an equivalence: an inverse functor sends (A, ι) to (L, b), where $L = H_1(A(\mathbf{C}); \mathbf{Z})$, and $b : L \times L \to \mathfrak{d}^{-1}$ is uniquely specified by the requirement that $-\mathrm{Tr}_{\mathbf{Q}}^E b(x, y)$ coincides with the skew-symmetric pairing on L induced by the principal polarization.

Remark 4.6. As in (4.5), there is a tensor bifunctor $\mathcal{P}_E^- \times \mathcal{P}_E^+ \to \mathcal{P}_E^-$.

If $(X, q) \in \mathcal{P}(\mathcal{O}, \mathcal{O}, \iota_+)$ is positive definite – that is, $q(x, x) \in \mathcal{O}^+$ is totally positive for all $x \in X$ – then this tensor operation can be described algebraically via "Serre's tensor construction" [AK]: if $(L', b') = (X, q) \otimes (L, b)$ then

$$\mathsf{ST}(L', b') \cong X \otimes_{\mathcal{O}} \mathsf{ST}(L, b), \tag{4.13}$$

the abelian variety representing the functor $R \mapsto X \otimes_{\mathcal{O}} \mathrm{Hom}(\mathrm{Spec}\,(R), \mathsf{ST}(L, b))$, equipped with a polarization induced by that of $A(L, b)$ and q.

If q is not positive definite it seems difficult to give an explicit description such as (4.13). For example, tensoring with $(X, q) = (\mathcal{O}, -1)$, where "$-1$" denotes the form $x \otimes y \mapsto -x\overline{y}$, sends $A = \mathsf{ST}(L, b)$ to its "complex conjugate" variety $\overline{A}$. (In the discussion above, it replaces the CM type $\Phi_{(L,b)}$ with its complement.)

4.4. Construction of enough objects of $\mathcal{P}_E^-$ for a cyclotomic field.

We now specialize to the case when $E = K_q \subset \mathbf{C}$, the cyclotomic field generated by the qth roots of unity. We shall prove a slightly technical result about the existence of enough objects in the groupoid $\mathcal{P}_{K_q}^-$; this is the key setup in our later verification (Proposition 5.1) that CM classes exhaust symplectic K-theory.

Recall that a CM structure on $\mathcal{O}_q$, the ring of integers of $\mathbf{Q}(\mu_q)$, may be defined either as an $\mathbf{R}$-algebra homomorphism $\mathbf{C} \to \mathcal{O}_q \otimes \mathbf{R}$, or as a set of embeddings $\mathcal{O}_q \to \mathbf{C}$ containing precisely one element in each equivalence class $\{j, \overline{j}\}$ under conjugation. As in (4.10) each object (L, b) of the groupoid $\mathcal{P}_{K_q}^-$ picks out a CM type, which we denote as $\Phi_{(L,b)}$; explicitly, $(L \otimes \mathbf{R}, b \otimes \mathbf{R})$ is isomorphic to $\mathcal{O}_q \otimes \mathbf{R}$ with Hermitian form given by $(x, y) \mapsto xu\overline{y}$ for some $u \in \mathcal{O}_q \otimes \mathbf{R}$ purely imaginary, and the CM type is given by those embeddings for which the imaginary part of $j(u)$ is positive.

Proposition 4.7. *Let Φ be a CM structure on $\mathcal{O}_q$ and let $L \in \mathrm{Pic}(\mathcal{O}_q)$. Then there exist objects (B_1, b_1) and (B_2, b_2) of $\mathcal{P}_{K_q}^-$ such that*

(i) $[B_1][B_2] = [L][\overline{L}]^{-1} \in \pi_0(\mathrm{Pic}(\mathcal{O}_q))$,

(ii) $\Phi_{(B_1,b_1)} = \Phi_{(B_2,b_2)} = \Phi$.

Proof. Let $\zeta_q = e^{2\pi i/q} \in \mathcal{O}_q$ as usual, and recall that the different $\mathfrak{d} \subset \mathcal{O}_q$ is principal and generated by $q/(\zeta_q^{q/p} - 1)$ (see [Was97, Proposition 2.7] for the calculation of the discriminant, from which it's easy to deduce the statement about the different, using that $K_q/\mathbf{Q}$ is totally ramified over p). The element $w = (1 - \zeta^2)/(1 - \zeta) = (1 + \zeta)$ is a unit in $\mathcal{O}_q$ and has the property that $\overline{w} = \zeta_q^{-1} w$. If we set

$$\delta := w^{q/p} \frac{q}{\zeta_q^{q/p} - 1},$$

it follows that $(\delta) = \mathfrak{d}$ and $\overline{\delta} = -\delta$, i.e. δ is purely imaginary. The inverse different ideal $\mathfrak{d}^{-1} \subset K_q$ is therefore also principal, generated by the purely imaginary element δ^{-1}.

It is now easy to satisfy (i): set

$$B_1' = \mathcal{O}_q \qquad\qquad b_1'(x,y) = \delta^{-1} x \overline{y}$$
$$B_2' = L \otimes \overline{L}^{-1} \quad b_2'(x \otimes \phi, y \otimes \psi) = \delta^{-1} \cdot \psi(x) \cdot \phi(y)$$

where in the first line $x \in \mathcal{O}_q$ and $y \in \overline{\mathcal{O}}_q$, and in the second line $x \otimes \phi \in B_2' = L \otimes \overline{L}^{-1}$ and $y \otimes \psi \in \overline{B_2'} \cong \overline{L} \otimes L^{-1}$ (and the evaluation pairing between L and L^{-1} comes from viewing L^{-1} as the dual of L). It is clear that the pairings $B_i' \otimes_{\mathcal{O}_q} \overline{B}_i' \to K_q$ defined by the two formulae give isomorphisms onto $(\delta^{-1}) = \omega_{\mathcal{O}_q}$, and the fact that δ is totally imaginary implies that $b_i(x,y) = -\overline{b_i(y,x)}$, so that we indeed have two objects $(B_i, b_i) \in \mathcal{P}_{K_q}^-$. It is also obvious that $[B_1'][B_2'] = [L][\overline{L}]^{-1} \in \pi_0 \mathrm{Pic}(\mathcal{O}_q)$. These objects do not necessarily satisfy (ii) though: any complex embedding $j : K_q \to \mathbf{C}$ will take $b_1(x,x)$ and $b_2(x,x)$ to a non-negative real multiple of the imaginary number $j(\delta^{-1})$, so in fact $\Phi_{(B_1',b_1')} = \Phi_{(B_2',b_2')} = \Phi_0$, where

$$\Phi_0 = \{ j : K_q \to \mathbf{C} \mid \mathrm{Im}(j(\delta^{-1})) > 0 \}.$$

To realize other CM structures we shall use the tensor product (4.5) and set

$$(B_1, b_1) = (B_1', b_1') \otimes (X, q)$$
$$(B_2, b_2) = (B_2', b_2') \otimes (X, q)^{-1}$$

for a suitable object $(X, q) \in \mathcal{P}_{K_q}^+$. We shall choose (X, q) using the following Lemma:

Lemma 4.8. *Let $\mathcal{O}_q^+ \subset K_q^+$ denote the ring of integers in $K_q^+ = \mathbf{Q}[\cos(2\pi/q)]$, the maximal totally real subfield of K_q, and let $S = \mathrm{Hom}(\mathcal{O}_q^+, \mathbf{R})$ be the set of real embeddings of $\mathcal{O}_q$. For any function $f : S \to \{\pm 1\}$ there exists a non-zero prime element $t \in \mathcal{O}_q^+$ such that*

- $\mathrm{sgn}(j(t)) = f(j)$ *for all $j \in S$,*
- $t\mathcal{O}_q = \mathfrak{r}\overline{\mathfrak{r}}$ *for a (prime) ideal $\mathfrak{r} \subset \mathcal{O}_q$.*

We give the proof of the lemma below, but let us first explain why it permits us to conclude the proof. Let X be the $\mathcal{O}_q$-module underlying $\mathfrak{r}$ and define a sesquilinear pairing on X by

$$q(x,y) = t^{-1} x \overline{y}.$$

This defines an isomorphism $q : X \otimes_{\mathcal{O}_q} \overline{X} \to \mathcal{O}_q$, and $q(x,y) = \overline{q(y,x)}$ since t is totally real. Hence we have an object $(X, q) \in \mathcal{P}_{K_q}^+$. Now the difference between $\Phi_{(L,b) \otimes (X,q)}$ and $\Phi_{(L,b)}$ is precisely determined by the signs of t under the real embeddings of K_q^+, which are controlled by the function f in the lemma, which may be arbitrary. $\qquad\square$

Proof of Lemma 4.8. This will be a consequence of the Chebotarev density theorem in algebraic number theory, which produces a prime ideal with a specified splitting behavior in a field extension; for us the extension is $H_q^+ K_q / K_q$, where H_q^+ is the narrow Hilbert class field of K_q^+, that is, the largest abelian extension of K_q^+ that is unramified at all finite primes.

Restriction defines an isomorphism

$$\operatorname{Gal}(H_q^+ K_q / K_q^+) \xrightarrow{\sim} \operatorname{Gal}(H_q^+ / K_q^+) \times \operatorname{Gal}(K_q / K_q^+), \qquad (4.14)$$

(the map is surjective because K_q / K_q^+ is totally ramified at the unique prime above q and H_q^+ / K_q^+ is unramified, so the inertia group at q maps trivially to the first factor and surjects to the second factor). Now class field theory defines an isomorphism

$$\operatorname{Art} : \frac{\{\pm 1\}^S \times \text{fractional ideals}}{\text{principal signed ideals}} \xrightarrow{\sim} \operatorname{Gal}(H_q^+ / K_q^+) \qquad (4.15)$$

where the principal signed ideals are elements of the form $(\operatorname{sign}(\lambda), \lambda)$ for λ a nonzero element of K_q^+. The map from left to right is the Artin map on fractional ideals, and sends the -1 factor indexed by $j \in S$ to the complex conjugation above j.

By the Chebotarev density theorem, there exists a prime ideal $\mathfrak{t}$ of K_q^+ whose image under (4.14) is trivial in the second factor, and, in the first factor, coincides with $\operatorname{Art}(f \times \text{trivial})$. Triviality in the second factor forces $\mathfrak{t}$ to be split in K_q / K_q^+; the condition on the first factor forces $\mathfrak{t} = t\mathcal{O}_q^+$ where the sign of $j(t)$ is given by $f(j)$, for each $j \in S$. $\qquad\square$

5. CM CLASSES EXHAUST SYMPLECTIC K-THEORY

The primary goal of this section is to verify that the construction of classes in symplectic K-theory sketched in §4.1 in fact produces all of symplectic K-theory in the degrees of interest.

In more detail: we have constructed a sequence (4.6) $\mathcal{P}_E^- \xrightarrow{\mathsf{ST}} \mathcal{A}_g(\mathbf{C}) \to \mathcal{SP}(\mathbf{Z})$ associated to a CM field E; the functor ST produces a CM abelian variety from a skew-Hermitian module over the ring of integers of E. There are induced maps of spaces $|\mathcal{P}_E^-| \to |\mathcal{A}_g| \to |\mathcal{SP}(\mathbf{Z})| \to \Omega^\infty \mathrm{KSp}(\mathbf{Z})$, where the last map is the group completion map. By adjunction there are associated map of spectra

$$\Sigma_+^\infty |\mathcal{P}_E^-| \to \Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})| \to \mathrm{KSp}(\mathbf{Z}). \qquad (5.1)$$

We emphasize that $\mathcal{A}_g(\mathbf{C})$ is a discretely topologized groupoid, that is to say, the topology on $\mathbf{C}$ plays no role. This makes the middle term of (5.1) rather huge. In this section we show that the composition of (5.1) is surjective on homotopy, in the degrees of interest:

Proposition 5.1. *Take $E = K_q$, the cyclotomic field. The composition*

$$\pi_{4k-2}^s(|\mathcal{P}_{K_q}^-|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q). \qquad (5.2)$$

is surjective for all $k \geq 1$.

More precisely, we show that a certain natural supply of classes in the source already surject on the target. All objects $(L, b) \in \mathcal{P}_E^-$ have automorphism group the unitary group $U_1(\mathcal{O}) = \{x \in \mathcal{O} \mid x\overline{x} = 1\}$, so we get a homotopy equivalence $|\mathcal{P}_E^-| \simeq BU_1(\mathcal{O}) \times \pi_0(\mathcal{P}_E^-)$ and since stable homotopy takes disjoint union to direct sum we get isomorphisms analogous to (2.3)

$$\pi_*^s(|\mathcal{P}_E^-|) \cong \pi_*^s(BU_1(\mathcal{O})) \otimes \mathbf{Z}[\pi_0(\mathcal{P}_E^-)]$$
$$\pi_*^s(|\mathcal{P}_E^-|; \mathbf{Z}/q) \cong \pi_*^s(BU_1(\mathcal{O}); \mathbf{Z}/q) \otimes \mathbf{Z}[\pi_0(\mathcal{P}_E^-)] \qquad (5.3)$$

In the case $E = K_q$ with ring of integers $\mathcal{O}_q$, we get a map $\mathbf{Z}/q \to \mathcal{O}_q^\times$ sending a to $e^{2\pi i a/q}$, and thereby

$$\pi_2^s(B(\mathbf{Z}/q); \mathbf{Z}/q) \to \pi_2^s(BU_1(\mathcal{O}_q); \mathbf{Z}/q),$$

The left-hand side contains a distinguished "Bott element" β, which generates a polynomial algebra in $\pi_2^s(B(\mathbf{Z}/q); \mathbf{Z}/q)$, as discussed in (2.1.1). We denote by the same letter its image inside the right-hand side.
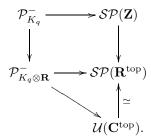
What we shall show, in fact, is that elements of the form $\beta^{2k-1} \otimes [(L,b)] \in \pi_{4k-2}^s(|\mathcal{P}_{K_q}^-|, \mathbf{Z}/q)$, with $(L,b) \in \pi_0 \mathcal{P}_{K_q}^-$, generate the image of (5.2). To show this, we use Theorem 3.5, which provides a sufficient supply of maps *out of* KSp, namely the Hodge map c_H and the Betti map c_B. In §5.1, we compute $c_H \circ (5.1)$, and in §5.2 we compute $c_B \circ (5.1)$. We them assemble the results in the final section §5.3.

5.1. **Hodge map for CM abelian varieties.**
We first describe the composition

$$B\mathbf{Z}/q \times \pi_0(\mathcal{P}_{K_q}^-) \simeq |\mathcal{P}_{K_q}^-| \to \Omega^\infty \mathrm{KSp}(\mathbf{Z}) \xrightarrow{c_H} \mathbf{Z} \times BU,$$

which is most conveniently expressed one path component at a time.

5.1.1. *Reminders on the Hodge map.* Recall that the Hodge map $\mathrm{KSp}(\mathbf{Z}) \to ku$ arose from a zig-zag of functors $\mathcal{SP}(\mathbf{Z}) \to \mathcal{SP}(\mathbf{R}^{\mathrm{top}}) \xleftarrow{\simeq} \mathcal{U}(\mathbf{C}^{\mathrm{top}})$, as in (3.1). Understanding the Hodge map $\mathrm{KSp}(\mathbf{Z}) \to ku$ therefore involves inverting the weak equivalence, which informally amounts reducing a structure group from $\mathrm{Sp}_{2g}(\mathbf{R})$ to $U(g)$. Roughly speaking, for a symplectic real vector space we must choose compatible complex structures and Hermitian metrics with the given symplectic form as imaginary part.

5.1.2. *Computation of the Hodge map for $\mathcal{P}_{K_q}^-$.* For the symplectic vector spaces arising from objects $(L,b) \in \mathcal{P}_E^-$ by the construction in §4.3 above we already produced such a choice. Indeed, the Hermitian inner product $\langle -, - \rangle_b$ from (4.9) and the CM structure $\Phi_{(L,b)}$ on E induces exactly this structure on $L_\mathbf{R} = L \otimes \mathbf{R}$. This observation gives the diagonal arrow in the following diagram

$$
\begin{array}{ccc}
\mathcal{P}_{K_q}^- & \longrightarrow & \mathcal{SP}(\mathbf{Z}) \\
\downarrow & & \downarrow \\
\mathcal{P}_{K_q \otimes \mathbf{R}}^- & \longrightarrow & \mathcal{SP}(\mathbf{R}^{\mathrm{top}}) \\
& \searrow & \uparrow{\scriptstyle \simeq} \\
& & \mathcal{U}(\mathbf{C}^{\mathrm{top}}).
\end{array}
$$

Restricting the composition $\mathcal{P}_{K_q}^- \to \mathcal{U}(\mathbf{C}^{\mathrm{top}})$ to the object (L,b) and its automorphism group $\mu_q = \mathrm{Aut}(L,b)$, we may describe the composition

$$\{(L,b)\} /\!\!/ \mu_q \hookrightarrow \mathcal{P}_{K_q}^- \to \mathcal{U}(\mathbf{C}^{\mathrm{top}}), \tag{5.4}$$

(where $\{(L,b)\} /\!\!/ \mu_q$ is shorthand for the full sub-groupoid of $\mathcal{P}_{K_q}^-$ on the object (L,b)) as follows. Giving a functor $\{(L,b)\} /\!\!/ \mu_q \to \mathcal{U}(\mathbf{C}^{\mathrm{top}})$ is equivalent to giving a unitary representation of μ_q, and in these terms the composition (5.4) corresponds to the unitary representation

$$\bigoplus_{j \in \Phi_{(L,b)}} j|_{\mu_q} \tag{5.5}$$

where we recall that $\Phi_{L,b}$ consists of various complex embeddings $K_q \hookrightarrow \mathbf{C}$, and we may therefore regard each restriction $j|_{\mu_q} : \mu_q \to U_1(\mathbf{C}) \subset \mathbf{C}^\times$ as a 1-dimensional (unitary) representation of μ_q. The CM structure $\Phi_{(L,b)}$ depends only on the image of (L,b) under the base change functor $\mathcal{P}_{K_q}^- \xrightarrow{- \otimes_\mathbf{Z} \mathbf{R}} \mathcal{P}_{K_q \otimes \mathbf{R}}^-$ so the same is true for the functor (5.4), up to natural isomorphism.

Finally, we use this discussion to compute the image of Bott elements under the Hodge map. The embeddings $\mathcal{O} \to \mathbf{C}$ are parameterized by $s \in (\mathbf{Z}/q)^{\times}$: the sth embedding j_s satisfies $j_s(e^{2\pi i/q}) = e^{2\pi is/q}$. As discussed in §2.1.1, j_1 induces a homomorphism of graded rings

$$(j_1)_* : \pi_*^s(B\mathbf{Z}/q; \mathbf{Z}/q) \to \pi_*(ku, \mathbf{Z}/q),$$

and this sends the Bott element $\beta \in \pi_2^s((B\mathbf{Z}/q); \mathbf{Z}/q)$ to the mod q reduction of the usual Bott element – we denote this by Bott. The powers of Bott generate the mod q homotopy groups of ku. More generally we have

$$(j_a)_*(\beta) = a \cdot \mathrm{Bott} \in \pi_2(ku; \mathbf{Z}/q),$$

and in particular $(j_a)_*(\beta^i) = a^i \cdot (j_1)_*(\beta^i) = a^i \cdot \mathrm{Bott}^i$ for any $a \in (\mathbf{Z}/q)^{\times}$ (cf. Remark 2.4). Combining with (5.5) we arrive at the following formula:

Proposition 5.2. *As above, take* $(L, b) \in \pi_0 \mathcal{P}_{K_q}^-$, *giving a class* $\beta^i[L, b] \in \pi_{2i}^s(|\mathcal{P}_{K_q}^-|; \mathbf{Z}/q)$. *The image of* $\beta^i[L, b]$ *under the map of homotopy groups induced by (cf. (5.1))* $\Sigma_+^{\infty}|\mathcal{P}_{K_q}^-| \to \Sigma_+^{\infty}|\mathcal{A}_g(\mathbf{C})| \to \mathrm{KSp}(\mathbf{Z}) \to ku$ *is given by*

$$\left(\sum_{a \in (\mathbf{Z}/q)^* : j_a \in \Phi} a^i \right) \mathrm{Bott}^i \in \pi_{2i}(ku; \mathbf{Z}/q) \tag{5.6}$$

Moreover, for any odd i there exists a CM structure Φ on $K_q = \mathcal{O}_q \otimes \mathbf{Q}$ for which the element (5.6) is a generator for $\pi_{2i}(ku; \mathbf{Z}/q) \cong \mathbf{Z}/q$.

Proof. The previous discussion already established (5.6), so we turn our attention to the last assertion. Since Bott^i generates, we must find a CM structure satisfying

$$\sum_{a \in (\mathbf{Z}/q)^* : j_a \in \Phi} a^i \in (\mathbf{Z}/q)^{\times}.$$

Equivalently, we must find a subset $X \subset (\mathbf{Z}/q)^{\times}$ containing precisely one element from each subset $\{a, -a\} \subset (\mathbf{Z}/q)^{\times}$, such that

$$\sum_{a \in X} a^i \in (\mathbf{Z}/q)^{\times}.$$

Choose such a set X arbitrarily, and let X' be obtained from X by switching the element in which X intersects $\{1, -1\}$. Then $\sum_{a \in X} a^i$ and $\sum_{a \in X'} a^i$ differ by $(-1)^i - 1 = -2$, and so at least one is a unit in $\mathbf{Z}/q$. $\square$

Remark 5.3. For $p = 2$ it seems a similar argument will show that there exists a Φ for which (5.6) is twice a generator.

5.2. **Betti map for CM abelian varieties.** Next we treat the composition of (5.1) with the Betti map. The map $\mathcal{P} \to \mathcal{A}_g(\mathbf{C})$ sends (L, b) to an abelian variety with underlying space $A = L_{\mathbf{R}}/L_{\mathbf{Z}}$, from which we read off $H_1(A; \mathbf{Z}) = L_{\mathbf{Z}}$. This implies a diagram of functors

$$
\begin{array}{ccc}
\mathcal{P}_E^- & \longrightarrow & \mathcal{SP}(\mathbf{Z}) \\
\downarrow & & \downarrow \\
\mathrm{Pic}(\mathcal{O}) \lhook\joinrel\longrightarrow \mathcal{P}(\mathcal{O}) & \xrightarrow{\mathrm{forget}} & \mathcal{P}(\mathbf{Z}),
\end{array}
$$

commuting up to natural isomorphism, where the vertical maps are induced by forgetting the pairings, i.e., $(L, b) \mapsto L$. Passing to the associated spaces and composing with group

completion maps we get a diagram of spectra

$$
\begin{array}{ccc}
\Sigma_+^\infty |\mathcal{P}_E^-| & \longrightarrow & \mathrm{KSp}(\mathbf{Z}) \\
\downarrow & & \downarrow \\
\Sigma_+^\infty |\mathrm{Pic}(\mathcal{O})| & \longrightarrow K(\mathcal{O}) \xrightarrow{\ \mathrm{tr}\ } & K(\mathbf{Z}),
\end{array}
\tag{5.7}
$$

where the map $\mathrm{tr} : K(\mathcal{O}) \to K(\mathbf{Z})$ is the "transfer map" induced by the functor $\mathcal{P}(\mathcal{O}) \to \mathcal{P}(\mathbf{Z})$ sending a projective $\mathcal{O}$-module to its underlying (projective) $\mathbf{Z}$-module.

As explained in Section 2.6, the homotopy groups of the spectrum in the lower left corner are $\pi_*^s(|\mathrm{Pic}(\mathcal{O})|) = \pi_*^s(BO^\times) \otimes \mathbf{Z}[\pi_0(\mathrm{Pic}(\mathcal{O}))]$ and with mod q coefficients $\pi_*^s(BO^\times; \mathbf{Z}/q) \otimes \mathbf{Z}[\pi_0(\mathrm{Pic}(\mathcal{O}))]$, similar to (5.3). Commutativity of the induced diagram on homotopy groups gives the following, after we specialize to the case of $E = K_q, \mathcal{O} = \mathcal{O}_q$:

Corollary 5.4. *Notation as above. The composition*

$$
\pi_{4k-2}^s(|\mathcal{P}_E^-|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \xrightarrow{c_B} K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}
$$

sends the element $\beta^{2k-1} \cdot [(L, b)]$ (in the notation of (5.3)) to the element $\mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$. Here $[(L, b)] \in \pi_0(\mathcal{P}_{K_q}^-)$ is any element, and $[L] - 1 \in K_0(\mathcal{O})$ is the projective class associated to $[L] \in \mathrm{Pic}(\mathcal{O})$. $\qquad\square$

Proof. Commutativity of the diagram (5.7) yields

$$
\beta^{2k-1} \cdot [(L, b)] \mapsto \mathrm{tr}(\beta^{2k-1} \cdot [L]) = \mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) + \mathrm{tr}(\beta^{2k-1}).
$$

Now $\mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$ and $\mathrm{tr}(\beta^{2k-1}) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(-)}$, cf. Remark 2.16. Since the image of $c_B : \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \to K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ is contained in the $(+1)$-eigenspace, commutativity of the diagram (5.7) implies that $\mathrm{tr}(\beta^{2k-1}) = 0$, which proves the claim. $\qquad\square$

Remark 5.5. Alternatively, the vanishing of $\mathrm{tr}(\beta^{2k-1}) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ may be seen by identifying tr with the transfer map in étale cohomology

$$
H^0(\mathcal{O}_q'; \mu_q^{2k-1}) \xrightarrow{\ \mathrm{tr}\ } H^0(\mathbf{Z}'; \mu_q^{2k-1}),
$$

which sends $\beta^{2k-1} \in \mu_q(\mathcal{O}_q')^{\otimes(2k-1)}$ to the sum of all its Galois translates. This vanishes for the same reason as

$$
\sum_{a \in (\mathbf{Z}/q)^\times} a^i = 0 \in \mathbf{Z}/q
$$

when $p - 1$ does not divide i, and in particular for any odd i.

5.3. **Surjectivity.** Recall that in Theorem 3.5 we proved that the combination of the Hodge and Betti maps define an isomorphism

$$
\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \xrightarrow{(c_H, c_B)} \pi_{4k-2}(ku; \mathbf{Z}/q)^{(-)} \times K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}.
\tag{5.8}
$$

Proof of Proposition 5.1. The coordinates of $\beta^{2k-1}[L, b]$, under the map above, have been computed in Proposition 5.2 and Corollary 5.4. They are given by:

$$
\begin{cases}
\text{Hodge: } c_H(\beta^{2k-1} \cdot [L, b]) = \mathrm{Bott}^{2k-1} \sum_{a \in \Phi_{L,b}} a^{2k-1} & \in \pi_{4k-2}(ku; \mathbf{Z}/q), \\
\text{Betti: } c_B(\beta^{2k-1} \cdot [L, b]) = \mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) & \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q).
\end{cases}
$$

By Proposition 5.2, there exists a CM structure Φ_0 for which $\sum_{a \in \Phi} a^{2k-1}$ is invertible in $(\mathbf{Z}/q)$. It therefore suffices to see that

$$
\{\mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) \mid \Phi_{(L,b)} = \Phi_0\} \text{ generates } K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}
\tag{5.9}
$$

and indeed our proof will show this is valid for any CM structure Φ_0.

For any $[L] \in \mathrm{Pic}(\mathcal{O}_q)$, there exist by Proposition 4.7 two objects $(L_1, b_1), (L_2, b_2) \in \mathcal{P}_{K_q}^-$ satisfying $\Phi_{(L_1,b_1)} = \Phi_{(L_2,b_2)} = \Phi_0$, and whose images in $\pi_0(\mathrm{Pic}(\mathcal{O}_q))$ satisfy

$$[L_1][L_2] = [L][\overline{L}]^{-1}.$$

The corresponding elements in $K_0(\mathcal{O}_q)$ then satisfy $[L_1] + [L_2] = [L] - [\overline{L}] + 2$. Applying the same Proposition with $[L] = 1 = [\mathcal{O}_q]$ gives $(L_3, b_3), (L_4, b_4) \in \mathcal{P}_{K_q}^-$ with $[L_3] + [L_4] = 2 \in K_0(\mathcal{O}_q)$. We then have

$$\mathrm{tr}\big(\beta^{2k-1} \cdot ([L_1] + [L_2] - [L_3] - [L_4])\big)$$
$$= \mathrm{tr}\big(\beta^{2k-1} \cdot ([L] - 1)\big) - \mathrm{tr}\big(\beta^{2k-1} \cdot ([\overline{L}] - 1)\big)$$
$$= \mathrm{tr}\big(\beta^{2k-1} \cdot ([L] - 1)\big) + \mathrm{tr}\big(\beta^{2k-1} \cdot ([L] - 1)\big) = 2\mathrm{tr}\big(\beta^{2k-1} \cdot ([L] - 1)\big)$$

where the last line used that the automorphism of $K(\mathcal{O}_q)$ induced by the involution on $\mathcal{O}_q$ sends $\beta \mapsto -\beta$ and $[L] \mapsto [\overline{L}]$, and that the transfer map is invariant under this automorphism (as the underlying $\mathbf{Z}$-modules of M and $\overline{M}$ are equal).

Proposition 2.17 implies that the elements $\mathrm{tr}\big(\beta^{2k-1} \cdot ([L] - 1)\big) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$ generate as $[L]$ range over all of $\pi_0(\mathrm{Pic}(\mathcal{O}_q))$, and since q is odd the factor of 2 does not matter for surjectivity. □

Remark 5.6. The method used here to produce elements of $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ is very similar to the method used by Soulé [Sou81] to produce elements in algebraic K-theory of rings of integers. In our notation the elements he constructs in $K_{4k+1}(\mathbf{Z}; \mathbf{Z}/q)^{(-)}$ are of the form $\mathrm{tr}(\beta^{2k} \cdot u)$ with $u \in \mathcal{O}_q^\times / q = K_1(\mathcal{O}_q; \mathbf{Z}/q)^{(-)}$. By a compactness argument he lifts his elements from the mod $q = p^n$ theory to the p-adic groups, which can also be done here.

Related ideas were also used by Harris and Segal [HS75].

6. The Galois action on KSp and on CM abelian varieties

Now that we understand the abstract $(\mathbf{Z}/q)$-module $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ and how to produce elements in it, we will study the Galois action on it.

The first task is to define it properly. We outline the construction in §6.2, referring to Appendix B for more technical aspects. In §6.3 we compute the action of the $\mathrm{Aut}(\mathbf{C})$ on CM classes.

6.1. **Galois conjugation of complex varieties.** Given a $\mathbf{C}$-scheme X, we obtain a $\mathbf{C}$-scheme σX by "applying σ to all the coefficients of the equations defining X." More formally, we are given a pair (X, ϕ) consisting of an underlying scheme X and a reference map $\phi : X \to \mathrm{Spec}\,(\mathbf{C})$, and we define

$$\sigma(X, \phi) = (X, \mathrm{Spec}\,(\sigma^{-1}) \circ \phi),$$

i.e. we simply postcompose the reference map with the map $\mathrm{Spec}\,(\sigma^{-1}) : \mathrm{Spec}\,(\mathbf{C}) \to \mathrm{Spec}\,(\mathbf{C})$ while the underlying schemes are equal (not just isomorphic). The resulting $\mathbf{C}$-scheme $\sigma(X, \phi) =: (\sigma X, \sigma\phi)$ fits in a cartesian square

$$\begin{array}{ccc}
\sigma X & \xrightarrow{\mathrm{Id}} & X \\
\downarrow{\scriptstyle \sigma\phi} & & \downarrow{\scriptstyle \phi} \\
\mathrm{Spec}\,\mathbf{C} & \xrightarrow{\sigma} & \mathrm{Spec}\,\mathbf{C}.
\end{array} \qquad (6.1)$$

The rule $(X, \phi) \mapsto (\sigma X, \sigma\phi)$ extends to a functor from $\mathbf{C}$-schemes to $\mathbf{C}$-schemes in an evident way.

Applying this construction when $X = A \to \mathrm{Spec}\,(\mathbf{C})$ is a complex abelian variety gives a new complex abelian variety, which inherits a principal polarization from that of A. We arrive at a functor

$$\mathcal{A}_g(\mathbf{C}) \xrightarrow{\sigma} \mathcal{A}_g(\mathbf{C}),$$

which agrees up to natural isomorphism with applying the "functor of points" $\mathcal{A}_g$ to $\mathrm{Spec}\,(\sigma)$: $\mathrm{Spec}\,(\mathbf{C}) \to \mathrm{Spec}\,(\mathbf{C})$, because coordinates on $\mathcal{A}_g$ are coefficients of the equations defining the abelian varieties (e.g. using the Hilbert scheme atlas on $\mathcal{A}_g$ as in [MFK94a, Section 6]). In this way we get an action[10] of $\mathrm{Aut}(\mathbf{C})$ on the groupoid $\mathcal{A}_g(\mathbf{C})$ and hence on the space $|\mathcal{A}_g(\mathbf{C})|$.

6.2. **Construction of the Galois action on homotopy of** KSp**.** Recall from Section 4.1 that we consider the functor $\mathcal{A}_g(\mathbf{C}) \to \mathcal{SP}(\mathbf{Z})$ induced by sending a principally polarized abelian variety A to $\pi_1(A(\mathbf{C})^{\mathrm{an}}, e)$, equipped with the symplectic form induced from the polarization. We emphasize that we here regard $\mathcal{A}_g(\mathbf{C})$ as just a groupoid in sets, so the domain of this spectrum map is rather huge: for example $\pi_0^s(|\mathcal{A}_g(\mathbf{C})|)$ is the free abelian group generated by $\pi_0(|\mathcal{A}_g(\mathbf{C})|)$, the (uncountable) set of isomorphism classes of complex principally polarized abelian varieties.

Proposition 6.1. *For all $k \geq 1$ and odd $q = p^n$, the map*

$$\pi_{4k-2}^s(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$$

induced by (4.2) is surjective, when $g \geq \varphi(q) = p^{n-1}(p-1)$.

Proof. It suffices to consider $g = \phi(q)$ since otherwise we may use any $A_0 \in \mathcal{A}_{g-\phi(q)}(\mathbf{Q})$ to define a map $A_0 \times - : \mathcal{A}_{\phi(q)} \to \mathcal{A}_g$. We consider the spectrum maps of (5.1)

$$\Sigma_+^\infty |\mathcal{P}_{K_q}^-| \to \Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})| \to \mathrm{KSp}(\mathbf{Z}).$$

Since the composition induces a surjection on mod q stable homotopy, by Proposition 5.1, the same must be true for the second map alone. $\square$

As in §6.1 any $\sigma \in \mathrm{Aut}(\mathbf{C})$ induces a functor $\mathcal{A}_g(\mathbf{C}) \to \mathcal{A}_g(\mathbf{C})$ and hence an automorphism of the spectrum $\Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})|$ and in turn an action of $\mathrm{Aut}(\mathbf{C})$ on $\pi_*^s(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q)$. The following proposition characterizes the Galois action on symplectic K-theory.

Proposition 6.2. *For any $k \geq 1$ and odd prime power $q = p^n$, there is a unique action of $\mathrm{Aut}(\mathbf{C})$ on $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ for which the homomorphisms*

$$\pi_{4k-2}^s(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$$

are equivariant for all g.

Proof sketch. We have seen that these homomorphisms are surjective for sufficiently large g, so for $\sigma \in \mathrm{Aut}(\mathbf{C})$ there is at most one homomorphism

$$
\begin{array}{ccc}
\pi_{4k-2}^s(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) & \longrightarrow & \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \\
\downarrow{\scriptstyle \sigma_*} & & \Big\downarrow{\scriptstyle \sigma_*} \\
\pi_{4k-2}^s(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) & \longrightarrow & \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)
\end{array}
\tag{6.2}
$$

making the diagram commute. If these exist for all σ, uniqueness guarantees that composition is preserved, inducing an action. It remains to see existence. This is quite technical, and we

---

[10]We prefer not to take the cartesian square (6.1) as the definition of $\sigma(X, \phi)$: with our definitions $(\sigma \circ \sigma')(X, \phi)$ is *equal* to $\sigma(\sigma'(X, \phi))$, which ensures we get an actual action on $|\mathcal{A}_g(\mathbf{C})|$. This issue is mostly cosmetic, and could presumably alternatively be handled by "keeping track of higher homotopies".

refer to Appendix B for details and a recollection of the relevant theory. Let us explain the idea and some of the key ingredients.

Let X_g be a simplicial variety defined over $\mathbf{Q}$, representing the stack $\mathcal{A}_g$. Let us write $X_g(\mathbf{C})^{\mathrm{an}}$ for the complex points $X_g(\mathbf{C})$, equipped with the analytic topology. This is a simplicial topological space, in fact a simplicial object in complex manifolds, and it follows from uniformization of complex abelian varieties that the geometric realization $|X_g(\mathbf{C})^{\mathrm{an}}|$ is a model for $B\mathrm{Sp}_{2g}(\mathbf{Z})$. Let us write $X_{g,\mathbf{C}} = \mathrm{Spec}\,(\mathbf{C}) \times_{\mathrm{Spec}\,(\mathbf{Q})} X_g$ for the base change in each simplicial degree. The theory of étale homotopy type associates a space (pro-simplicial set, in fact) $\mathrm{Et}(X_{g,\mathbf{C}})$ and a comparison map

$$B\mathrm{Sp}_{2g}(\mathbf{Z}) \simeq |X_g(\mathbf{C})^{\mathrm{an}}| \dashrightarrow \mathrm{Et}(X_{g,\mathbf{C}}),$$

inducing an isomorphism in cohomology with finite coefficients. We write $\mathrm{Et}_p(X_{g,\mathbf{C}})$ for a suitable p-completion of $\mathrm{Et}(X_{g,\mathbf{C}})$, and get an induced weak equivalence of p-completed spaces

$$B\mathrm{Sp}_{2g}(\mathbf{Z})_p^\wedge \simeq \mathrm{Et}_p(X_{g,\mathbf{C}}).$$

Up to weak equivalence the discussion so far does not depend on the choice of X_g, which arises from the choice of an atlas $U \to \mathcal{A}_g$, so we shall henceforth write $\mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}})$ instead of $\mathrm{Et}_p(X_{g,\mathbf{C}})$

The fact that $H^1(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}/p) = 0$ for $g \geq 3$ implies that these spaces are simply connected, and hence there is an induced map from the plus-construction

$$B\mathrm{Sp}_{2g}(\mathbf{Z})^+ \to B\mathrm{Sp}_{2g}(\mathbf{Z})_p^\wedge \simeq \mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}}),$$

identifying $\mathrm{Et}_p(X_{g,\mathbf{C}})$ up to equivalence with the p-completion of the plus-construction of $B\mathrm{Sp}_{2g}(\mathbf{Z})$. (In this discussion we implicitly used that $B\mathrm{Sp}_{2g}(\mathbf{Z})$ is homotopy equivalent to a CW complex with finitely many cells in each degree.)

Any $\sigma \in \mathrm{Aut}(\mathbf{C})$ induces an automorphism of $X_{g,\mathbf{C}} = \mathrm{Spec}\,(\mathbf{C}) \times_{\mathrm{Spec}\,(\mathbf{Q})} X_g$ in the category of simplicial schemes, and the point is now that etale homotopy type is functorial with respect to all maps of simplicial schemes. Therefore σ induces a map of spaces $\mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}}) \to \mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}})$, which we also denote σ, and therefore an automorphism of the homotopy groups

$$\pi_i(\mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}}); \mathbf{Z}/q) \cong \pi_i(B\mathrm{Sp}_{2g}(\mathbf{Z})^+; \mathbf{Z}/q),$$

which may be identified with $\mathrm{KSp}_i(\mathbf{Z}; \mathbf{Z}/q)$ when $g \gg i$, by homological stability. This describes an action of $\mathrm{Aut}(\mathbf{C})$ on $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}/q)$.

Similar ideas may be employed to give an action on the level of *spectra*: we give a model for the p-completed spectrum $\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p)$ on which $\mathrm{Aut}(\mathbf{C})$ acts by spectrum maps. Morally, this comes from $\mathrm{Aut}(\mathbf{C})$-equivariance of the maps

$$\mathcal{A}_{g,\mathbf{C}} \times \mathcal{A}_{g',\mathbf{C}} \to \mathcal{A}_{g+g',\mathbf{C}}$$
$$(A, A') \mapsto A \times A',$$

since these maps model the symmetric monoidal structure $\oplus : \mathcal{SP}(\mathbf{Z}) \times \mathcal{SP}(\mathbf{Z}) \to \mathcal{SP}(\mathbf{Z})$, which is responsible for the infinite loop space structure on symplectic K-theory. The details of this argument are a bit lengthy however, and we have postponed them to Appendix B, using background about stable homotopy theory and "infinite loop space machines" in Appendix A. It will follow from the construction that the action of $\sigma \in \mathrm{Aut}(\mathbf{C})$ fits into a homotopy commutative diagram

$$
\begin{array}{ccccc}
\Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})| & \longrightarrow & \Sigma_+^\infty \mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}}) & \longrightarrow & \mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p) \\
\downarrow{\scriptstyle \Sigma_+^\infty \sigma} & & \downarrow{\scriptstyle \Sigma_+^\infty \sigma} & & \vdots \\
\Sigma_+^\infty |\mathcal{A}_g(\mathbf{C})| & \longrightarrow & \Sigma_+^\infty \mathrm{Et}_p(\mathcal{A}_{g,\mathbf{C}}) & \longrightarrow & \mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p),
\end{array}
\tag{6.3}
$$

from which commutativity of (6.2) is deduced. □

Remark 6.3. The spectrum level action constructed above, with details in Appendix B, should probably be viewed as more intrinsic than the particular statement of the proposition. From an expositional point of view, the main advantage of the statement of the proposition is that it uniquely characterizes the action on homotopy groups which we are studying, at least in degrees 2 mod 4, while not making explicit reference to étale homotopy type. This allows us to quarantine the fairly technical theory of étale homotopy type to the proof of Proposition 6.2.

It will also be clear from the spectrum level construction that the actions of $\mathrm{Aut}(\mathbf{C})$ on $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/p^n)$ are compatible over varying n, including in the inverse limit $n \to \infty$, so that the universal property for each n also determines the action on the p-complete symplectic K-theory groups $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$. The spectrum level action also induces an action on homotopy groups in degrees $4k - 1$, which by Corollary 3.6 is the only other interesting case when p is odd. In Subsection 7.7 we prove that the action on $\mathrm{KSp}_{4k-1}(\mathbf{Z}; \mathbf{Z}_p)$ is trivial.

Lemma 6.4. *The Betti map*

$$\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \xrightarrow{c_B} K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$$

is equivariant for the subgroup $\langle c \rangle \subset \mathrm{Aut}(\mathbf{C})$, *where* c *denotes complex conjugation, and* $K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ *is given the trivial action.*

Proof. The composite $\pi^s_{4k-2}(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ is induced from the functor $\mathcal{A}_g(\mathbf{C}) \to \mathcal{P}(\mathbf{Z})$ sending an abelian variety $A \to \mathrm{Spec}(\mathbf{C})$ to $H_1(A(\mathbf{C})^{\mathrm{an}}; \mathbf{Z})$. Complex conjugation induces a functor $\mathcal{A}_g(\mathbf{C}) \to \mathcal{A}_g(\mathbf{C})$ which we'll denote $A \mapsto A^c$ on objects. The fact that complex conjugation is continuous on $\mathbf{C}$ implies that the induced bijection $A(\mathbf{C}) \to A^c(\mathbf{C})$ is continuous in the analytic topology, and hence induces a canonical isomorphism $H_1(A(\mathbf{C})^{\mathrm{an}}; \mathbf{Z}) \to H_1(A^c(\mathbf{C})^{\mathrm{an}}; \mathbf{Z})$. Therefore the diagram

$$
\begin{array}{ccc}
|\mathcal{A}_g(\mathbf{C})| & \longrightarrow & |\mathcal{P}(\mathbf{Z})| \\
\downarrow{\scriptstyle c} & \nearrow & \\
|\mathcal{A}_g(\mathbf{C})| & &
\end{array}
$$

commutes up to homotopy (see also Subsection B.1. It follows that the homomorphism

$$\pi^s_{4k-2}(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) \to \pi^s_{4k-2}(B\mathrm{GL}_{2g}(\mathbf{Z}); \mathbf{Z}/q) \to K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$$

coequalizes c_* and the identity. The claim is then deduced from surjectivity of $\pi^s_{4k-2}(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$. □

Remark 6.5. It may be deduced from our main theorem that $c_B \colon \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \to K_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$ is also equivariant for $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subset \mathrm{Aut}(\mathbf{C})$ for suitable isomorphisms $\mathbf{C} \cong \overline{\mathbf{Q}}_p$, see Subsection 7.5. It would be interesting to understand whether that equivariance could be seen more geometrically.

6.3. **Galois conjugation of CM abelian varieties.** Fix a CM field E. It follows from Remark 4.5 that there exists a functor of groupoids making the following diagram commutative:

$$
\begin{array}{ccc}
\mathcal{P}_E^- & \xrightarrow{\mathrm{ST}} & \mathcal{A}_g(\mathbf{C}) \\
{\scriptstyle F_\sigma}\downarrow & & \downarrow{\scriptstyle \sigma} \\
\mathcal{P}_E^- & \xrightarrow{\mathrm{ST}} & \mathcal{A}_g(\mathbf{C}),
\end{array}
$$

The main theorem of complex multiplication, originally due to Shimura and Taniyama for automorphisms fixing the reflex field, and extended to the general case by Deligne and Tate, effectively provides a formula for F_σ.

Let H be the Hilbert class field of the CM field E. We will formulate the result only when E (so also H) is Galois over $\mathbf{Q}$. Let $\Phi = \Phi(L, b) \subset \mathrm{Emb}(E, \mathbf{C})$ be the CM structure on E determined by $(L, b) \in \mathcal{P}_E^-$. Let c denote the complex conjugation on E and choose for each $\tau \in \mathrm{Emb}(E, \mathbf{C})$ an extension $w_\tau : H \to \mathbf{C}$ to a complex embedding of H, such that

$$w_{\tau c} = w_{c\tau} = cw_\tau.$$

Then for each $\sigma \in \mathrm{Gal}(H/\mathbf{Q})$ and $\tau \in \mathrm{Emb}(E, \mathbf{C})$, both σw_τ and $w_{\sigma\tau}$ give embeddings $H \to \mathbf{C}$ extending $\sigma\tau$ and, therefore, $w_{\sigma\tau}^{-1}\sigma w_\tau \in \mathrm{Gal}(H/K)$.

The following theorem computes much of the action of F_σ on the homotopy of $\mathcal{P}_E^-$, in the cases of interest.

Theorem 6.6.   *(i) The map $\pi_0(F_\sigma) \colon \pi_0(\mathcal{P}_E^-) \to \pi_0(\mathcal{P}_E^-)$ is given on each fiber of $\pi_0\mathcal{P}_E^- \to$ $\pi_0\mathcal{P}_{E\otimes\mathbf{R}}^-$ (i.e., upon fixing the CM type) by tensoring, as in (4.5), with a certain $[(X, q)] \in \pi_0\mathcal{P}_E^+$ determined by σ and the CM type.*

Moreover, the class of $[X]$ under the Artin map $\pi_0(\mathrm{Pic}(\mathcal{O}_E)) \xrightarrow{\mathrm{Art}} \mathrm{Gal}(H/K)^{\mathrm{ab}}$ is given by

$$\mathrm{Art}\,(X) = \;\; \text{class of } \left[\sum_{\tau \in \Phi} w_{\sigma\tau}^{-1}\sigma w_\tau\right] \;\; \text{in } \mathrm{Gal}(H/K)^{\mathrm{ab}}. \tag{6.4}$$

(ii) In the case $E = K_q$ the map on higher homotopy groups

$$\pi_*(F_\sigma) \colon \pi_*^s(|\mathcal{P}_E^-|, \mathbf{Z}/q) \to \pi_*^s(|\mathcal{P}_E^-|, \mathbf{Z}/q) \tag{6.5}$$

is $\mathbf{Z}/q[\beta]$-linear, that is to say, it sends $[\beta^j(L, b)]$ to $\beta^j\sigma([L, b])$, with notation as described after Proposition 5.1.

For example, $(X, q) = (\mathcal{O}_q, -1)$ when $\sigma = c$ is complex conjugation, see Remark 4.6.

Proof. We defined in Remark 4.6 a tensoring bifunctor $\mathcal{P}_E^- \times \mathcal{P}_E^+ \to \mathcal{P}_E^-$, and for each *positive definite* $(Y, h) \in \mathcal{P}_E^+$ we have

$$F_\sigma((L, b) \otimes (Y, h)) \cong F_\sigma(L, b) \otimes (Y, h),$$

naturally in (L, b) and (Y, h). This is just the statement that applying σ commutes with the Serre tensor construction. The first assertion of (i) follows from this; the explicit formula is that given in [Mil07, Theorem 4.2], except he has replaced the Artin map by its refinement $\mathbf{A}_{f,E}^\times/E^\times \to \mathrm{Gal}(\overline{\mathbf{Q}}/E)^{\mathrm{ab}}$.

Writing $\mathcal{P}_E^{+\,\mathrm{pos.def.}} \subset \mathcal{P}_E^+$ for the full subgroupoid on the positive definite (Y, h), naturality implies that (6.5) is linear over the graded ring $\pi_*^s(|\mathcal{P}_E^{+\,\mathrm{pos.def.}}|; \mathbf{Z}/q)$, which contains $\mathbf{Z}/q[\beta]$ because $\mathbf{Z}/q \cong U_1(\mathcal{O}_q)$ is the automorphism group of $(Y_0, h_0) = (\mathcal{O}_q, 1) \in \mathcal{P}_E^{+\,\mathrm{pos.def.}}$. $\square$

7. The main theorem and its proof

Recall from Theorem 3.5 that there is an isomorphism

$$\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \xrightarrow[\sim]{(c_H, c_B)} \pi_{4k-2}(ku; \mathbf{Z}/q) \times K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}.$$

Let us recall that $\pi_{4k-2}(ku; \mathbf{Z}/q)$ is a cyclic of order q, generated by the $2k - 1$st power of the Bott class $\mathrm{Bott} \in \pi_2(ku; \mathbf{Z}/q)$. For purposes of making Galois equivariance manifest, we will in the current section identify

$$\pi_{4k-2}(ku; \mathbf{Z}/q) \xrightarrow{\sim} \mu_q^{\otimes 2k-1}$$

via $\mathrm{Bott}^{2k-1} \mapsto \zeta_q^{\otimes 2k-1}$. By means of this identification, the target of the map c_H can be considered to be $\mu_q^{\otimes 2k-1}$.

Theorem 7.1. *Let $H_q \subset \mathbf{C}$ be the largest unramified extension of K_q with abelian p-power Galois group. Let $G = \mathrm{Gal}(H_q/\mathbf{Q})$, and let $\langle c \rangle \leqslant G$ be the order 2 subgroup generated by complex conjugation.*

(i) *The Galois action on $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ factors through G.*

(ii) *The sequence*

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \overset{c_H}{\to} \mu_q^{\otimes 2k-1} \tag{7.1}$$

is a short exact sequence of G-modules, where the G-action on $\mathrm{Ker}(c_H)$ is understood to be trivial, and the action on μ_q is via the cyclotomic character.

(iii) *The sequence (7.1) is universal for extensions of $\mu_q^{\otimes(2k-1)}$ by a trivial $\mathbf{Z}/q[G]$-module.*

In detail, the final assertion (iii) means that the sequence is the initial object of a category $\mathcal{C}_{\mathbf{Z}/q}(G; \mu_q^{\otimes(2k-1)})$ of extensions of G-modules of $\mu_q^{\otimes(2k-1)}$ by a trivial G-module. This category and its basic properties are discussed in §7.1.

Remark 7.2. It turns out to be technically more convenient to work in a more rigid category of sequences equipped with splitting, and we will in fact prove the following statements:

(ii') *There is a unique splitting of the sequence that is equivariant for the action of $\langle c \rangle$; explicitly the kernel of c_B maps isomorphically to $\mu_q^{\otimes 2k-1}$ under c_H and yields such a splitting.*

(iii') *The sequence (7.1) is universal for extensions of $\mu_q^{\otimes 2k-1}$ by a trivial $\mathbf{Z}/q[G]$-module that are equipped with a $\langle c \rangle$-equivariant splitting.*

We will first give some generalities on universal extensions in §7.1. We then verify (i) and (ii) in §7.2, and then (iii) in §7.3. Finally, we give a number of related universal properties in §7.4, 7.5.

7.1. Cocycles and universal extensions.

Definition 7.3. Let G be a discrete group, $H \leqslant G$ a subgroup, and M a $\Lambda[G]$-module for some coefficient ring Λ. We consider a category $\mathcal{C}_\Lambda(G, H; M)$ of "extensions of M by a trivial G-module, equipped with an H-equivariant splitting". More precisely, the objects of $\mathcal{C}_\Lambda(G, H; M)$ are triples (V, π, s) where V is a $\Lambda[G]$-module, $\pi \in \mathrm{Hom}_{\Lambda[G]}(V, M)$ and $s \in \mathrm{Hom}_{\Lambda[H]}(M, V)$ satisfy $s \circ \pi = \mathrm{id}_M$, and the $\Lambda[G]$-module $T = \mathrm{Ker}(\pi)$ has trivial G-action; the morphisms $(V, \pi, s) \to (V', \pi', s')$ are those $\phi \in \mathrm{Hom}_{\Lambda[G]}(V, V')$ for which $\pi' \circ \phi = \pi$ and $\phi \circ s = s'$.

We also consider the variant $\mathcal{C}_\Lambda(G; M)$ where there is only given (V, π) with $\Lambda[G]$-linear $\pi : V \to M$ and morphisms satisfy only $\pi' \circ \phi = \pi$. (As a warning, this is not the same category as $\mathcal{C}_\Lambda(G, \{e\}; M)$.)

Objects of $\mathcal{C}_\Lambda(G, H; M)$ may be depicted as short exact sequences of $\Lambda[G]$-modules

$$T \longrightarrow V \underset{s}{\overset{\pi}{\longrightarrow}} M. \tag{7.2}$$

equipped with $\Lambda[H]$-equivariant splittings. The identity map of M evidently gives a terminal object in this category.

We will show that the category $\mathcal{C}_\Lambda(G, H; M)$ always has an initial object, which we call the *universal extension* and denote

$$T^{\mathrm{univ}} \longrightarrow V^{\mathrm{univ}} \underset{s}{\overset{\pi}{\longrightarrow}} M, . \tag{7.3}$$

We will also see that there is a canonical isomorphism $H_1(G, H; M) \cong T^{\mathrm{univ}}$ (where $H_1(G, H; -)$ is *relative* group homology). Any other object of $\mathcal{C}_\Lambda(G, H; M)$ arises by pushout from the universal extension, so we think of (7.3) as being the "most non-trivial" object in $\mathcal{C}_\Lambda(G, H; M)$.

To an object (7.2) of $\mathcal{C}_\Lambda(G, H; M)$, as above, we associate a function $\alpha : G \times M \to T$, by

$$\alpha(g, m) = g.(s(m)) - s(g.m). \tag{7.4}$$

This function satisfies

  (i) for every $g \in G$, the function $\alpha(g, m)$ is Λ-linear in $m \in M$,
 (ii) $\alpha(g, m) = 0$ when $g \in H$,
(iii) the cocycle condition

$$\alpha(gg', m) = \alpha(g, g'.m) + \alpha(g', m). \tag{7.5}$$

Now, the rule $(m, t) \mapsto s(m) + t$ defines a $\Lambda[H]$-linear isomorphism $M \times T \xrightarrow{\sim} V$, with respect to which

$$g.(m, t) = (g.m, t + \alpha(g, m)), \tag{7.6}$$

so the object (7.2) is described uniquely by the Λ-module T and the function α. This defines an equivalence of categories between $\mathcal{C}_\Lambda(G, H; M)$ and a category whose objects are pairs (T, α) and whose morphisms are Λ-linear maps $f : T \to T'$ such that $f(\alpha(g, m)) = \alpha'(g, m)$.

Recall also that group homology $H_*(G; M)$ is calculated by a standard "bar" complex with

$$C_i(G; M) = \mathbf{Z}[G^i] \otimes_{\mathbf{Z}} M \cong \Lambda[G^i] \otimes_\Lambda M.$$

The inclusion $i : H \subset G$ gives an injection $C_*(H; \mathrm{Res}_H^G M) \to C_*(G; M)$ and we let $C_*(G, H; M)$ be the quotient; in particular $C_0(G, H; M) = 0$. Its homology is the relative group homology $H_*(G, H; M)$, which sits in a long exact sequence with $i_* : H_*(H; \mathrm{Res}_H^G M) \to H_*(G; M)$. For an object (7.2) the cocycle α defines a Λ-linear map

$$C_1(G; M) = \mathbf{Z}[G] \otimes M \cong \Lambda[G] \otimes_\Lambda M \xrightarrow{\alpha} T, \tag{7.7}$$

and the conditions (ii) and (iii) say that this map has both $C_1(H; \mathrm{Res}_H^G M)$ and $\partial C_2(G; M)$ in its kernel. Therefore α gives a Λ-linear map

$$H_1(G, H; M) \xrightarrow{[\alpha]} T. \tag{7.8}$$

Lemma 7.4. *The rule associating $[\alpha]$ of (7.8) to the extension (7.2) defines an equivalence of categories between $\mathcal{C}_\Lambda(G, H; M)$ and the category of Λ-modules under $H_1(G, H; M)$. In particular, the category $\mathcal{C}(G, H; M)$ has an initial object*

$$T^{\mathrm{univ}} \to V^{\mathrm{univ}} \twoheadrightarrow M$$

wherein $T^{\mathrm{univ}} \cong H_1(G, H; M)$, the relative group homology.

Proof. Indeed, the functor in the other direction is described as follows: given $f : H_1(G, H; M) \to T$, define $\alpha : G \times M \to T$ by composing f with the canonical maps $G \times M \to C_1(G, H; M) \to H_1(G, H; M)$, and set $V = M \times T$ with G-action given by (7.6). The identity map of $H_1(G, H; M)$ then corresponds to an initial object with $T^{\mathrm{univ}} = H_1(G, H; M)$. $\square$

Remark 7.5. The proof above also gives an explicit description of the map $T^{\mathrm{univ}} \to T$ arising from the universal map to another object (7.2): first extract $\alpha : G \times M \to T$ as in (7.4), extend to an additive map (7.7) and factor as in (7.8).

There are natural situations where one can drop H.

Lemma 7.6. (a) *If $H_0(H; M) = 0 = H_1(H; M)$, then the forgetful functor*

$$\mathcal{C}_\Lambda(G, H; M) \to \mathcal{C}_\Lambda(G; M)$$

is an equivalence. In particular the image of the initial object of $\mathcal{C}_\Lambda(G, H; M)$ is initial in $\mathcal{C}_\Lambda(G; M)$.

(b) *If $H_0(H, M) = 0$, then the forgetful functor*

$$\mathcal{C}_\Lambda(G, H; M) \to \mathcal{C}_\Lambda(G; M)^{H\text{-}split}$$

is an equivalence, where, on the right, we take the full subcategory of $\mathcal{C}_\Lambda(G; M)$ consisting of sequences which admit splittings as sequences of H-modules.

Proof. If we regard an object (7.2) as an extension of $\Lambda[H]$-modules, it is classified by an element of $\mathrm{Ext}^1_{\Lambda[H]}(M, T)$ and two splittings differ by an element of $\mathrm{Hom}_{\Lambda[H]}(M, T)$. Under the vanishing assumption of (a), both these groups vanish; so the splitting is unique and hence the forgetful functor is an equivalence. In the setting of (b) only the latter group vanishes, which still implies that the stated forgetful functor is an equivalence. $\square$

In the absense of a specified H, it can be shown that $\mathcal{C}_\Lambda(G; M)$ admits an initial object if and only if $H_0(G; M) = 0$, and in this case the kernel is $H_1(G; M)$. (Note that $\mathcal{C}_\Lambda(G; M)$ is not the same as $\mathcal{C}_\Lambda(G, \{e\}; M)$.)

7.2. **Proof of (i) and (ii) of the main theorem.**
We briefly recall some of the prior results before proceeding to the proof. We have constructed maps

$$\pi^s_{4k-2}(|\mathcal{P}^-_{K_q}|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \overset{(c_H, c_B)}{\longrightarrow} \pi_{4k-2}(ku; \mathbf{Z}/q) \times K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}.$$

Recall from (5.3) that each class $[(L, b)] \in \pi_0(\mathcal{P}^-_{K_q})$ gives a class $\beta^{2k-1}[(L, b)] \in \pi^s_{4k-2}(|\mathcal{P}^-_{K_q}|; \mathbf{Z}/q)$; here $\beta \in \pi^s_2(|\mathcal{P}^-_{K_q}|; \mathbf{Z}/q)$ is a Bott element induced by the primitive root of unity $e^{2\pi i/q} = \zeta_q \in \mathcal{O}_q$. With this notation, we have previously verified:

(a) Under the composite map, the images of elements $\beta^{2k-1}[(L, b)]$ generate the codomain KSp_{4k-2} (proof of Proposition 5.2).

(b) (Proposition 5.2 and Corollary 5.4) Explicitly, the image of $\beta^{2k-1}[(L, b)]$ is

$$\left(\textstyle\sum_{a: j_a \in \Phi} a^{2k-1}\right) \mathrm{Bott}^{2k-1} \in \pi_{4k-2}(ku; \mathbf{Z}/q)$$

$$\beta^{2k-1}[(L, b)] \qquad\qquad\qquad\qquad\qquad\qquad\qquad (7.9)$$

$$\mathrm{tr}(\beta^{2k-1}([L] - 1)) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}.$$

where:
- $\mathrm{tr}: K_*(\mathcal{O}_q; \mathbf{Z}/q) \to K_*(\mathbf{Z}; \mathbf{Z}/q)$ is the transfer,
- $\Phi = \Phi_{(L, b)} \subset \mathrm{Hom}(K_q, \mathbf{C})$ is the CM type associated to (L, b) by (4.10), and $j_a \in \mathrm{Hom}(K_q, \mathbf{C})$ is the embedding that sends $\zeta_q \mapsto e^{2\pi i a/q}$, for $a \in (\mathbf{Z}/q)^\times$,
- On the right, $\mathrm{Bott} \in \pi_2(ku; \mathbf{Z}/q)$ is the mod q reduction of the Bott element.

(c) (By (6.5) and surrounding discussion): The action of $\sigma \in \mathrm{Aut}(\mathbf{C})$ on $\pi^s_{4k-2}(|\mathcal{P}^-_{K_q}|; \mathbf{Z}/q)$ sends

$$\beta^{2k-1}[(L, b)] \mapsto \beta^{2k-1}[(L, b) \otimes (X, q)] \qquad\qquad (7.10)$$

for a certain $(X, q) \in \mathcal{P}_{K_q}^+$ depending only on the CM type $\Phi_{(L,b)}$ and σ; the image of X under the Artin map was described in Theorem 6.6 and depends only on the restriction of σ to H_q.

Lemma 7.7. *In the extension*

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \xrightarrow{c_H} \mu_q^{\otimes(2k-1)},$$

the action of $\mathrm{Aut}(\mathbf{C})$ *factors through* $\mathrm{Gal}(H_q/\mathbf{Q})$. *The map* c_H *is equivariant for this action, and the action on* $\mathrm{Ker}(c_H)$ *it is trivial. Finally, the kernel of* c_B *maps isomorphically to* $\mu_q^{\otimes(2k-1)}$ *under* c_H, *splitting the above sequence equivariantly for* $\langle c \rangle$.

Proof. By point (a) above, the action of $\mathrm{Aut}(\mathbf{C})$ is determined by its action on classes $\beta^{2k-1}[(L, b)]$ and by point (c) this action indeed factors through $\mathrm{Gal}(H_q/\mathbf{Q})$.

Morally speaking, the equivariance of c_H arises simply from the fact that one can define the Hodge class via algebraic geometry. We give a formal argument by a direct computation, using the explicit formula in (b) above. By Corollary 3.6 and point (a) above, we know the images of $\beta^{2k-1}[(L, b)]$ under $\pi_{4k-2}^s(|\mathcal{P}_{K_q}^-|; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ generate all of $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$. Therefore, it suffices to check the equivariance for $\mathrm{Aut}(\mathbf{C})$ acting on $\beta^{2k-1}[(L, b)]$.

According Proposition 5.2, c_H sends

$$\beta^{2k-1}[(L, b)] \mapsto \left(\sum_{a: j_a \in \Phi} a^{2k-1} \right) \mathrm{Bott}^{2k-1} \in \pi_{4k-2}(ku; \mathbf{Z}/q).$$

Evidently this only depends on the CM type of (L, b), which can be described as the set of characters by which $K_q^\times$ acts on the tangent space of the associated abelian variety $\mathsf{ST}(L, b)$ (cf. §4.3).

Now, consider the action of $K_q^\times$ on the tangent space of $\sigma\mathsf{ST}(L, b)$. This is the same underlying scheme as $\mathsf{ST}(L, b)$ but with its structure map to $\mathrm{Spec}\,(\mathbf{C})$ twisted by $\mathrm{Spec}\,(\sigma^{-1})$, so that as a $\mathbf{C}$-vector space,

$$T_e(\sigma\mathsf{ST}(L, b)) = T_e(\mathsf{ST}(L, b)) \otimes_{\mathbf{C}, \sigma} \mathbf{C}.$$

Therefore, $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ acts on Φ by post-composition with σ. Under the identification $\mathrm{Emb}(K_q, \mathbf{C}) \cong (\mathbf{Z}/q)^\times$, this is identified with multiplication by $\chi_{\mathrm{cyc}}(\sigma) \in (\mathbf{Z}/q)^\times$, the cyclotomic character of σ. Hence we find that

$$c_H(\sigma \cdot \beta^{2k-1}[(L, b)]) = \left(\sum_{a: j_a \in \sigma\Phi} a^{2k-1} \right) \mathrm{Bott}^{2k-1}$$

$$= \left(\sum_{a: j_a \in \Phi} \chi_{\mathrm{cyc}}(\sigma)^{2k-1} a^{2k-1} \right) \mathrm{Bott}^{2k-1}$$

$$= \chi_{\mathrm{cyc}}(\sigma)^{2k-1} c_H(\beta^{2k-1}[(L, b)]).$$

This shows the equivariance of c_H, as desired.

Now we check that the Galois action on $\ker(c_H)$ is trivial. In the course of proving Proposition 5.1 we have seen – see (5.9) – that, as (L, b) ranges over objects in $\mathcal{P}_{K_q}^-$ inducing a fixed CM structure $\Phi_{(L,b)} = \Phi \subset \mathrm{Emb}(K_q, \mathbf{C})$, the values of $c_B([(L, b)]) = \mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) \in K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)}$ exhaust that group. Therefore it suffices to see that if $[(L, b)]$ and $[(L', b')]$ induce the same CM structure on K_q, then $\sigma \in \mathrm{Aut}(\mathbf{C})$ acts trivially on the element

$$\mathrm{tr}(\beta^{2k-1} \cdot ([L] - 1)) - \mathrm{tr}(\beta^{2k-1} \cdot ([L'] - 1)). \tag{7.11}$$

According to point (c) above, $\sigma \in \mathrm{Aut}(\mathbf{C}/\mathbf{Q})$ takes $\beta^{2k-1}[(L,b)] \mapsto \beta^{2k-1}[(L,b) \otimes (X,q)]$ where (X,q) depends on (L,b) only through the CM type $\Phi_{(L,b)}$. Using the formula $([L \otimes_{\mathcal{O}_q} X] - 1) = ([L] - 1) + ([X] - 1) \in K_0(\mathcal{O}_q)$ (which is seen by noting that both sides having same rank and determinant) we get an equality inside $K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$

$$c_B(\beta^{2k-1} \cdot [(L,b) \otimes (X,q)]) = c_B(\beta^{2k-1} \cdot [(L,b)]) + \mathrm{tr}(\beta^{2k-1}([X] - 1)), \qquad (7.12)$$

where $[X]$ depends on (L,b) only through its CM type. Therefore, $\mathrm{Aut}(\mathbf{C})$ acts trivially on the expression (7.11) in which (L,b) and (L',b') have the same CM type, as desired.

The last part, about the equivariance of the splitting for the subgroup $\langle c \rangle$ generated by conjugation, follows from Lemma 6.4.   □

This concludes the proof of parts (i) and (ii) of Theorem 7.1, as well as the statements of (ii') about splitting.

7.3. **Proof of (iii) of the main theorem.** It remains to prove (iii) of Theorem 7.1. The properties verified in Lemma 7.7 show that in the sequence

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q) \stackrel{c_H}{\to} \mu_q^{\otimes(2k-1)} \qquad (7.13)$$

defines an object of $\mathcal{C}_{\mathbf{Z}/q}(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)})$. Our final task is to prove that it is an initial object in this category. This will prove (iii') of Theorem 7.1, from which (iii) follows by Lemma 7.6.

Let us denote "the" initial object of $\mathcal{C}_{\mathbf{Z}/q}(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)})$ by $T^{\mathrm{univ}} \to V^{\mathrm{univ}} \to \mu_q^{\otimes(2k-1)}$. Now Lemma 7.4 gives an abstract isomorphism of $\mathrm{Ker}(c_H)$ with T^{univ}, via the isomorphisms:

$$\mathrm{Ker}(c_H) \stackrel{c_B}{\to} K_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)^{(+)} \stackrel{(2.16)}{\longrightarrow} H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)}). \qquad (7.14)$$

(In the case at hand $H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes 2k-1}) = H_1(\mathrm{Gal}(H_q/\mathbf{Q}); \mu_q^{\otimes 2k-1})$ since the homology of $\langle c \rangle$ on $\mu_q^{\otimes 2k-1}$ is trivial in all degrees.) We shall show that, with reference to this identification, the 1-cocycle

$$\alpha : \mathrm{Gal}(H_q/\mathbf{Q}) \times \mu_q^{\otimes(2k-1)} \to \mathrm{Ker}(c_H) \qquad (7.15)$$

(arising from (7.13) and its splitting via c_B) is identified with the tautological 1-cocycle valued in $H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)})$. This will complete the proof of Theorem 7.1 (iii') by Lemma 7.4 and the discussion preceding it.

Denote by Pr the projection of $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ to $\mathrm{Ker}(c_H)$ with kernel $\mathrm{Ker}(c_B)$. For $\sigma \in \mathrm{Gal}(H_q/\mathbf{Q})$ and $m \in \mu_q^{\otimes 2k-1}$, we have in the notation of §7.1 the equality $\alpha(\sigma, m) = \mathrm{Pr} \circ (g - \mathrm{id})(\widetilde{m})$ where $\widetilde{m} \in \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ is any element with $c_H(\widetilde{m}) = m$. Therefore, the value of the cocycle α on $\sigma \in \mathrm{Gal}(H_q/\mathbf{Q})$ and $c_H(\beta^{2k-1}[(L,b)]) \in \mu_q^{\otimes(2k-1)}$ is given by

$$\alpha(\sigma, c_H(\beta^{2k-1}[(L,b)])) = \mathrm{Pr} \circ (\sigma - \mathrm{id}) \circ \beta^{2k-1}[(L,b)] \in \mathrm{Ker}(c_H).$$

By Theorem 6.6, we have $\sigma(\beta^{2k-1}[(L,b)]) = \beta^{2k-1} \cdot [(L,b) \otimes (X,q)]$ where (X,q) is determined explicitly by the CM type of (L,b). Hence

$$\mathrm{Pr} \circ (\sigma - \mathrm{id}) \circ \beta^{2k-1}[(L,b)] = \beta^{2k-1}[(X,q)].$$

Under the identification (7.14), the class $\beta^{k-1}[(X,q)]$ is sent to the Artin class of X pushed forward via $\mathrm{Gal}(H_q/K_q) \to \mathrm{Gal}(H_q/\mathbf{Q})$. In detail, there is a diagram:

$$\mathrm{Pr} \circ (\sigma - \mathrm{id}) \circ \beta^{2k-1}[(L,b)] \longmapsto \mathrm{tr}(\beta^{2k-1}([X]-1)) \longmapsto \iota_*(\mathrm{Art}(X) \otimes \zeta_q^{2k-1})$$

$$\cap \qquad\qquad\qquad\qquad \cap \qquad\qquad\qquad\qquad\qquad \cap$$

$$\mathrm{Ker}(c_H) \xrightarrow[\sim]{c_B} K_{4k-2}(\mathbf{Z};\mathbf{Z}/q)^{(+)} \xrightarrow[\sim]{(2.16)} H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)}).$$

- In the middle, we used (7.10) and (7.12); tr is the K-theoretic trace from $\mathcal{O}_q$ to $\mathbf{Z}$;
- On the right, we used Proposition 2.17; $\mathrm{Art}(X)$ is the Artin map applied to the class of X in the Picard group of $\mathcal{O}_q$, and ι_* is induced on homology by the inclusion $\iota : \mathrm{Gal}(H_q/K_q) \to \mathrm{Gal}(H_q/\mathbf{Q})$.

Therefore, using the explicit formula in the Main Theorem of CM given in (6.4), we find:

$$\alpha(\sigma, c_H(\beta^{2k-1}[(L,b)])) = \left(\prod_{\varphi \in \Phi(L,b)} w_{\sigma\varphi}^{-1} \sigma w_\varphi \right) \otimes \zeta_q^{\otimes(2k-1)}$$
$$\in C_1(\mathrm{Gal}(H_q/\mathbf{Q}); \mu_q^{\otimes(2k-1)}) \twoheadrightarrow H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)}).$$

Now we manipulate this expression using that we are allowed to change this expression by elements of $C_1(\langle c \rangle; \mu_q^{\otimes(2k-1)})$ and $\partial C_2(\mathrm{Gal}(H_q/\mathbf{Q}); \mu_q^{\otimes(2k-1)})$, without affecting the homology class. The latter gives a relation (cf. (7.5))

$$(g_1 g_2) \otimes m \sim g_1 \otimes g_2 m + g_2 \otimes m, \quad g_1, g_2 \in \mathrm{Gal}(H_q/\mathbf{Q}),\ m \in \mu_q^{\otimes(2k-1)} \qquad (7.16)$$

from which we also deduce

$$0 = (gg^{-1}) \otimes m = g \otimes g^{-1}m + g^{-1} \otimes m, \quad g \in \mathrm{Gal}(H_q/\mathbf{Q}),\ m \in \mu_q^{\otimes(2k-1)}. \qquad (7.17)$$

By repeated application of (7.16) we get

$$\left(\prod_{\varphi \in \Phi} w_{\sigma\varphi}^{-1} \sigma w_\varphi \right) \otimes \zeta_q^{\otimes(2k-1)} = \sum_{\varphi \in \Phi} (w_{\sigma\varphi}^{-1} \sigma w_\varphi \otimes \zeta_q^{\otimes(2k-1)}) \in H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)}).$$
$$(7.18)$$

Similarly, by combining (7.16) and (7.17) we get

$$w_{\sigma\varphi}^{-1} \sigma w_\varphi \otimes \zeta_q^{\otimes(2k-1)} = w_{\sigma\varphi}^{-1} \otimes \sigma\varphi(\zeta_q^{\otimes(2k-1)}) + \sigma \otimes \varphi(\zeta_q^{\otimes(2k-1)}) + w_\varphi \otimes \zeta_q^{\otimes(2k-1)}$$
$$= -w_{\sigma\varphi} \otimes \zeta_q^{\otimes(2k-1)} + \sigma \otimes \varphi(\zeta_q^{\otimes(2k-1)}) + w_\varphi \otimes \zeta_q^{\otimes(2k-1)}. \qquad (7.19)$$

Finally, observe that

$$\sum_{\varphi \in \Phi} w_{\sigma\varphi} \otimes \zeta_q^{\otimes(2k-1)} = \sum_{\varphi \in \Phi} w_\varphi \otimes \zeta_q^{\otimes(2k-1)} \in C_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)})$$

for q odd. Indeed, $\sigma\Phi$ is another CM type, which contains exactly one representative from each conjugate pair of embeddings $E \hookrightarrow \mathbf{C}$, and also, by construction $w_{c\varphi} = cw_\varphi$, and $cw_\varphi \otimes \zeta_q^{\otimes(2k-1)} - w_\varphi \otimes \zeta_q^{\otimes(2k-1)}$ belongs to $C_1(\langle c \rangle; \mu_q^{\otimes(2k-1)})$.

We deduce the formula

$$\alpha(\sigma, c_H(\beta^{2k-1}[(L,b)])) = \sum_{\varphi \in \Phi(L,b)} \sigma \otimes \varphi(\zeta_q^{\otimes(2k-1)})$$

$$= \sigma \otimes \Big(\sum_{\varphi \in \Phi(L,b)} \varphi(\zeta_q^{\otimes(2k-1)}) \Big)$$

$$= \sigma \otimes c_H(\beta^{2k-1}[(L,b)]).$$

Since this holds for any $(L,b) \in \mathcal{P}_{K_q}^-$, which generate under c_H by Proposition 5.2, we deduce the simple formula

$$\alpha(\sigma, x) = [\sigma \otimes x] \in H_1(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)}),$$

which verifies the claim made after (7.15) and thereby completes the proof. $\square$

7.4. **Universal property of symplectic K-theory with $\mathbf{Z}_p$ coefficients.** We have finished the proof of the universal property characterizing the $\mathrm{Aut}(\mathbf{C})$-action on $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q)$ for all k and all odd prime powers $q = p^n$. By taking inverse limit over n, this also determines the action on $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$. We shall now formulate a universal property adapted to this limit.

We need some generalities on profinite group homology. This has no real depth in our case as it only serves as a notation to keep track of inverse limits of *finite* group homology. Let G be a profinite group. Let Λ be a coefficient ring, complete for the p-adic topology. A topological Λ-module will be a Λ-module M such that each M/p^n is finite and the induced map $M \to \varprojlim M/p^n$ is an isomorphism; we always regard M as being endowed with the p-adic topology. These assumptions are not maximally general, cf. [RZ10]: among profinite abelian groups our assumptions on M are equivalent to it being a finitely generated $\mathbf{Z}_p$-module.

Define the completed group algebra

$$\Lambda[[G]] := \varprojlim_{n,U} \frac{\Lambda}{p^n \Lambda}[G/U]$$

the limit ranging over open subgroups U and positive integers n. If M is a topological Λ-module with a continuous action of G, then M carries a canonical structure of $\Lambda[[G]]$ module, since the G-action on each $M/p^n M$ factors through the quotient by some open subgroup U_n.

We define the profinite group homology with Λ coefficients by tensoring M with the bar complex $(\Lambda[[G^m]])_m$, where the tensor product is now completed tensor product, and taking homology. That is to say:

$$m\text{-chains for } (G,M) = \varprojlim_{U,n:U \subset U_n} \frac{M}{p^n M}[(G/U)^m]$$

(we refer to §5–6 of [RZ10] for a more complete discussion).

Since this complex is the inverse limit of the complexes computing homology of G/U_n acting on M/p^n, and taking an inverse limit of a system of *profinite groups* preserves exactness, we have

$$H_*(G,M) = \varprojlim_{U,n:U \subset U_n} H_*(G/U, M/p^n). \tag{7.20}$$

Finally, we can similarly define relative group homology $H_1(G,H;M)$ for $H \leqslant G$ using the induced map on chain complexes.

One verifies that the contents of §7.1 go through when G, H, Λ are as just described. Namely, one has a category $\mathcal{C}_\Lambda(G,H;M)$ of extensions $T \longrightarrow V \xrightarrow[s]{\pi} M.$ where:

- T, V, M are topological Λ-modules with continuous G-action (the maps are automatically continuous by definition of the topology).
- s is equivariant for H.

For any object in this category, there is a map $H_1(G, H; M) \to T$ which may be constructed in a similar fashion to (7.8) (although the kernel of $V/p^n \to M/p^n$ need not be T/p^n, this becomes true after passing to the inverse limit). One verifies, as before, that an object is universal if and only if this map $H_1(G, H; M) \to T$ is an isomorphism.

Theorem 7.8. *Let* $\Gamma = \mathrm{Gal}(H_\infty/\mathbf{Q})$ *be the Galois group of* $H_\infty = \bigcup H_{p^n}$ *over* $\mathbf{Q}$, *and* $c \in \mathrm{Gal}(H_\infty/\mathbf{Q})$ *the conjugation. The sequence*

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{c_H} \mathbf{Z}_p(2k-1) \tag{7.21}$$

of G-*modules is uniquely split equivariantly for* $\langle c \rangle \subset G$. *The resulting sequence is initial both in the category* $C_{\mathbf{Z}_p}(\Gamma, \langle c \rangle; \mathbf{Z}_p(2k-1))$ *and in the category* $C_{\mathbf{Z}_p}(\Gamma; \mathbf{Z}_p(2k-1))$.

Proof. The induced map

$$H_1(\Gamma, \langle c \rangle; \mathbf{Z}_p(2k-1)) \to \mathrm{Ker}(c_H)$$

is an isomorphism, because it is an inverse limit of corresponding isomorphisms for the sequences (7.23). That we can ignore c follows from (the profinite group analogue of) Lemma 7.6. $\square$

7.5. Universal property using full unramified Galois group.

We now reformulate the universal property of the extension with reference to the étale fundamental group of $\mathbf{Z}[1/p]$, or, in Galois-theoretic terms,

$$G := \text{Galois extension of largest algebraic extension } \mathbf{Q}^{(p)} \text{ of } \mathbf{Q} \text{ unramified } p.$$

For the results that involve explicit splittings of the sequence, we need to carefully choose a decomposition group for G. Let

$$\wp(q) = \{\text{prime ideals of } H_q \text{ above } p\}.$$

The subset $\wp(q)^c$ fixed by complex conjugation is nonempty, because $\wp(q)$ has odd cardinality $[H_q : K_q]$. The sets $\wp(q)^c$ form an inverse system of nonempty finite sets as one varies q through powers of p; since the inverse limit of such is nonempty, there exists a prime $\mathfrak{p}$ of $H_\infty = \bigcup_q H_q$ inducing an element of $\wp(q)^c$ on each H_q. We extend $\mathfrak{p}$ as above to $\mathbf{Q}^{(p)}$ in an arbitrary way. Let

$$G_{\mathfrak{p}} \leqslant G$$

be the decomposition group at $\mathfrak{p}$.

Remark 7.9. In fact, Vandiver's conjecture is *equivalent* (for any n) to the statement that $\wp(q)$ is a singleton. Indeed, if c fixes two different primes $\mathfrak{p}$ and $\mathfrak{p}'$ in H_q lying over p, then conjugation by c preserves the subset $\mathrm{Trans}(\mathfrak{p}, \mathfrak{p}') \subset \mathrm{Gal}(H_q/K_q)$ which transports $\mathfrak{p}$ to $\mathfrak{p}'$. But since p is totally split in H_q/K_q, $\mathrm{Trans}(\mathfrak{p}, \mathfrak{p}')$ consists of a single element, so conjugation by c must fix a nontrivial element of $\mathrm{Gal}(H_q/K_q)$. Equivalently, by class field theory, c must fix a non-trivial element of the p-part of $\mathrm{Pic}(K_q)$, i.e. the p-part of $\mathrm{Pic}(K_q^+)$ is non-trivial. But Vandiver's conjecture predicts exactly that the p-part of $\mathrm{Pic}(K_p^+)$ is trivial, which is equivalent to the statement that the p-part of $\mathrm{Pic}(K_q^+)$ is trivial for all $q = p^n$ by [Was97, Corollary 10.7].

Theorem 7.10. *Let* $\mathfrak{p}$ *be chosen as above. The exact sequence*

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p) \xrightarrow{c_H} \mathbf{Z}_p(2k-1) \tag{7.22}$$

of G-*modules is uniquely split for* $G_{\mathfrak{p}}$; *the kernel of the Betti map maps isomorphically to* $\mathbf{Z}_p(2k-1)$ *and furnishes this unique splitting. The resulting sequence is initial in the category* $C_{\mathbf{Z}_p}(G, G_{\mathfrak{p}}; \mathbf{Z}_p(2k-1))$ *and in the category* $C_{\mathbf{Z}_p}(G, \mathbf{Z}_p(2k-1))^{G_{\mathfrak{p}}\text{-}split}$ *(see Lemma 7.6).*

Remark 7.11. Let us rephrase this in geometric terms. By virtue of its Galois action $\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p)$ can be considered as (the **C**-fiber of) an étale sheaf over $\mathbf{Z}[1/p]$. This structure arises eventually from the fact that the moduli space of abelian varieties has a structure of $\mathbf{Z}[1/p]$-scheme. The last assertion of the Theorem can then be reformulated:

> The étale sheaf on $\mathbf{Z}[1/p]$ defined by $\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p)$ is the universal extension of $\mathbf{Z}_p(2k-1)$ by a trivial étale sheaf which splits when restricted to the spectrum of $\mathbf{Q}_p$.

More formally we consider the category whose objects are étale sheaves $\mathcal{F}$ over $\mathbf{Z}[1/p]$ equipped with $\pi : \mathcal{F} \twoheadrightarrow \mathbf{Z}_p(2k-1)$ whose kernel is a trivial sheaf, and with the *property* that π splits when restricted to Spec $\mathbf{Q}_p$. Our assertion is that the sheaf defined by KSp, together with its Hodge morphism to $\mathbf{Z}_p(2k-1)$, is initial in this category.

We deduce Theorem 7.10 from Theorem 7.1 in stages. First (Lemma 7.12) we replace the role of complex conjugation by a decomposition group. Next (Lemma 7.13) we pass from $\mathrm{Gal}(H_q/\mathbf{Q})$ to G. Finally we pass from $\mathbf{Z}/q$ coefficients to $\mathbf{Z}_p$.

Lemma 7.12. *The sequence*

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q\mathbf{Z}) \xrightarrow{c_H} \mu_q^{\otimes(2k-1)}$$

of $\mathrm{Gal}(H_q/\mathbf{Q})$*-modules is uniquely split for the decomposition group* $\mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}$*, where* $\mathfrak{p} \in \wp(q)^c$ *is any prime fixed by complex conjugation. The resulting sequence with splitting is universal in* $\mathcal{C}_{\mathbf{Z}/q}(\mathrm{Gal}(H_q/\mathbf{Q}), \mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}; \mu_q^{\otimes(2k-1)})$*.*

Note that, in particular, any splitting that is invariant by this decomposition group is also invariant by $\langle c \rangle$; so the unique splitting referenced in the Lemma is in fact provided by the Betti map.

Proof. The cyclotomic character $\mathrm{Gal}(H_q/\mathbf{Q}) \to (\mathbf{Z}/q)^{\times}$ restricts to an isomorphism on the decomposition group at $\mathfrak{p}$, and in particular this decomposition group is abelian; thus $c \in \mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}$ is central, and so $H_0(\mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}; \mu_q^{\otimes(2k-1)}) = H_1(\mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}; \mu_q^{\otimes(2k-1)}) = 0$, which permits us to apply Lemma 7.6. $\square$

Lemma 7.13. *The sequence*

$$\mathrm{Ker}(c_H) \to \mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}/q\mathbf{Z}) \xrightarrow{c_H} \mu_q^{\otimes(2k-1)} \tag{7.23}$$

now considered as G*-modules, is uniquely split for* $G_{\mathfrak{p}}$ *by the kernel of the Betti map. The resulting sequence with splitting is universal in* $\mathcal{C}_{\mathbf{Z}/q}(G, G_{\mathfrak{p}}; \mu_q^{\otimes(2k-1)})$*, defined as in §7.4.*

Proof. That the sequence is uniquely split follows from the same property for $\mathrm{Gal}(H_q/\mathbf{Q})$, and that this unique splitting comes from $\ker(c_B)$ is as argued after Lemma 7.12. As in (7.8) one gets $H_1(G, G_{\mathfrak{p}}; \mu_q^{\otimes(2k-1)}) \to \mathrm{Ker}(c_H)$ which we must prove to be an isomorphism. This map factors through the similar map for $\mathrm{Gal}(H_q/\mathbf{Q})$, and so it is enough to show that the natural map f of pairs of groups:

$$(G, G_{\mathfrak{p}}) \xrightarrow{f} (\mathrm{Gal}(H_q/\mathbf{Q}), \mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}). \tag{7.24}$$

induces an isomorphism on relative H_1 with coefficients in $\mu_q^{\otimes(2k-1)}$.

The action on $\mathbf{Q}(\zeta_q)$ gives a surjection $G \twoheadrightarrow (\mathbf{Z}/q)^{\times}$, which factors through $\mathrm{Gal}(H_q/\mathbf{Q})$ and restricts there to an isomorphism $\mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}} \cong (\mathbf{Z}/q)^{\times}$. Write G^0 for the kernel, and similarly define

$$G_{\mathfrak{p}}^0 \leqslant G^0, \quad \{e\} = \mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}^0 \leqslant \mathrm{Gal}(H_q/\mathbf{Q})^0.$$

From the morphism to $(\mathbf{Z}/q)^{\times}$ we obtain (as in the proof of Proposition 2.15) compatible spectral sequences computing $H_*(G, G_{\mathfrak{p}})$ in terms of $H_*(G^0, G_{\mathfrak{p}}^0)$ and similarly for $H_*(\mathrm{Gal}(H_q/\mathbf{Q})^0, \mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}^0)$.

By the same argument as in (2.17) the maps

$$(\mathbf{Z}/q)^\times \text{ coinvariants on } H_1(G^0, G_{\mathfrak{p}}^0; \mu_q^{\otimes(2k-1)}) \to H_1(G, G_{\mathfrak{p}}; \mu_q^{\otimes(2k-1)})$$

is an isomorphism, and the same for $\mathrm{Gal}(H_q/\mathbf{Q})$. (Here the flanking terms of (2.17) vanish for even simpler reasons, because relative group H_0 always vanishes.)

Therefore, it is sufficient to verify that

$$f^0 : (G^0, G_{\mathfrak{p}}^0) \to (\mathrm{Gal}(H_q/\mathbf{Q})^0, \mathrm{Gal}(H_q/\mathbf{Q})_{\mathfrak{p}}^0)$$

induces an isomorphism on first homology with $\mu_q^{\otimes(2k-1)}$ coefficients. The coefficients have trivial action by definition of the groups, and it suffices to consider $\mathbf{Z}_p$ coefficients because relative H_0 vanishes. But $H_1(G^0, G_{\mathfrak{p}}^0) \otimes \mathbf{Z}_p$ is the Galois group of the maximal abelian p-power extension of K_q that is unramified everywhere and split at $\mathfrak{p}$; this coincides with H_q because H_q/K_q is already split at $\mathfrak{p}$. $\square$

Proof of of Theorem 7.10. The sequence (7.22) is the inverse limit of the sequences (7.23), and the existence of a splitting follows from this. Uniqueness follows from the fact that $G_{\mathfrak{p}}$ surjects to $(\mathbf{Z}_p)^\times$, and thus contains an element acting by -1 on $\mathbf{Z}_p(2k-1)$ and trivially on $\mathrm{Ker}(c_H)$. Finally the induced map

$$H_1(G, G_{\mathfrak{p}}; \mathbf{Z}_p(2k-1)) \to \mathrm{Ker}(c_H)$$

is an isomorphism, because it is an inverse limit of corresponding isomorphisms for the sequences (7.23). $\square$

7.6. Universal properties of Bott-inverted K-theory.

We have seen that symplectic K-theory realizes certain universal extensions of $\mu_q^{\otimes 2k-1}$ as Galois modules, for k a positive integer. It is natural to ask if the universal extensions of other cyclotomic powers is realized in a similar way. Here we explain that for negative k, the Bott-inverted symplectic K-theory provides such a realization.

By Corollary 3.15, we have short exact sequences for Bott-inverted symplectic K-theory (discussed in 3.7):

$$0 \to K_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q) \to \mathrm{KSp}_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}/q) \to \mu_q(\mathbf{C})^{\otimes 2k-1} \to 0 \tag{7.25}$$

and

$$0 \to K_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p) \to \mathrm{KSp}_{4k-2}^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p) \to \mathbf{Z}_p(2k-1) \to 0. \tag{7.26}$$

Our main theorems have analogues for Bott inverted symplectic K-theory:

Theorem 7.14. *Let k be any (possibly negative!) integer.*

(1) *Let notation be as in Theorem 7.1. The extension (7.25) is initial in $\mathcal{C}_{\mathbf{Z}/q}(\mathrm{Gal}(H_q/\mathbf{Q}), \langle c \rangle; \mu_q^{\otimes(2k-1)})$.*
(2) *Let notation be as in Theorem 7.8. The extension (7.26) is initial in both $C_{\mathbf{Z}_p}(\Gamma, \langle c \rangle; \mathbf{Z}_p(2k-1))$ and in the category $C_{\mathbf{Z}_p}(\Gamma, \mathbf{Z}_p(2k-1))$.*
(3) *Let notation be as in Theorem 7.10. The extension (7.26) is initial in $C_{\mathbf{Z}_p}(G, G_{\mathfrak{p}}; \mathbf{Z}_p(2k-1))$.*

Proof. As above, parts (2) and (3) follow formally from (1) by an inverse limit argument, so it suffices to prove (1). By Proposition 2.9, for positive k these short exact sequences agree with the ones where β is not inverted, and hence of course enjoys the same universal property. For non-positive k the universal property for the short exact sequence (7.25) is deduced immediately by periodicity in k. $\square$

Remark 7.15. The natural map

$$\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p) \to \mathrm{KSp}_i^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$$

is an isomorphism whenever $i \geq 0$, but from a conceptual point of view it may be preferable to work entirely with $\mathrm{KSp}_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$. For one thing, the universal property for (7.26) is in some ways more interesting, in that we see universal extensions of $\mathbf{Z}_p(2i-1)$ for all $i \in \mathbf{Z}$, not only $i > 0$. Secondly, the relationship between $K_*^{(\beta)}(\mathbf{Z}; \mathbf{Z}_p)$ and étale cohomology of $\mathrm{Spec}\,(\mathbf{Z}')$ does not depend on the work of Voevodsky and Rost, and therefore not on any motivic homotopy theory.

7.7. **Degree** $4k - 1$. For odd p the homotopy groups of $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}_p)$ are non-zero only in degrees $* \equiv 2 \mod 4$ and $* \equiv 3 \mod 4$. We shall prove that the Galois action is trivial in the latter case.

Proof. There is a homomorphism

$$\pi_*^s(\mathrm{point}; \mathbf{Z}_p) \to K_*(\mathbf{Z}; \mathbf{Z}_p) \tag{7.27}$$

induced by the natural functor $S \mapsto \mathbf{Z}[S]$ from the symmetric monoidal category of sets (under disjoint union) to the symmetric monoidal category of free $\mathbf{Z}$-modules (under direct sum).

The work of Quillen in [Qui76] implies that this map is surjective in degree $4k - 1$. More precisely, if we choose an auxiliary prime ℓ for which the class of ℓ topologically generates $\mathbf{Z}_p^\times$, then Quillen's work implies that the composite map

$$\pi_*^s(\mathrm{point}; \mathbf{Z}_p) \to K_*(\mathbf{Z}; \mathbf{Z}_p) \to K_*(\mathbf{F}_\ell; \mathbf{Z}_p)$$

is surjective; on the other hand, the latter map is an isomorphism by the norm residue theorem[11].

There is also a natural map

$$\pi_*^s(\mathrm{point}) \to \mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}_p) \tag{7.28}$$

arising from the functor of symmetric monoidal categories sending a finite set S to $\mathbf{Z}[S] \otimes (\mathbf{Z}e \oplus \mathbf{Z}f)$, equipped with the symplectic form $\langle s \otimes e, s' \otimes f \rangle = \delta_{ss'}$. When composed with c_B, the map (7.28) recovers twice (7.27); but p is odd and c_B is injective by Theorem 3.3, so it follows that (7.28) is also surjective in degree $4k - 1$.

The proof may now be finished by showing that $\pi_{4k-1}^s(\mathrm{point}) \to \mathrm{KSp}_{4k-1}(\mathbf{Z}; \mathbf{Z}_p)$ is equivariant for the trivial action on the domain. Indeed, it is induced by $\{*\} \to \mathcal{A}_1(\mathbf{C})$, sending the point to some chosen elliptic curve $E \to \mathrm{Spec}\,(\mathbf{C})$. Since we may choose E to be defined over $\mathbf{Q}$, the map is indeed equivariant for the trivial action on $\pi_{4k-1}^s(\mathrm{point})$. $\qquad\square$

Remark 7.16. Let us sketch an alternative argument: we will show that the map $\mathrm{KSp}_{4k-1}(\mathbf{Z}; \mathbf{Z}_p) \to K_{4k-1}(\mathbf{F}_\ell; \mathbf{Z}_p)$ (which is a group isomorphism by the Norm Residue Theorem, as in the first proof, plus Theorem 3.5) is equivariant for the trivial action on $K_{4k-1}(\mathbf{F}_\ell; \mathbf{Z}_p)$.

This map comes from the functor

$$\mathcal{A}_g(\mathbf{C}) \to \mathcal{SP}(\mathbf{Z}) \to \mathcal{P}(\mathbf{F}_\ell)$$

sending a complex abelian variety $A \to \mathrm{Spec}\,(\mathbf{C})$ to the $\mathbf{F}_\ell$-module $H_1(A; \mathbf{F}_\ell)$. The latter is canonically identified with $A[\ell] \subset A(\mathbf{C})$, the ℓ-torsion points. These are defined purely algebraically and hence the ℓ-torsion points of A and of σA are *equal* $\mathbf{F}_\ell$-modules for $\sigma \in \mathrm{Aut}(\mathbf{C})$. Therefore this composite functor intertwines the natural action of $\mathrm{Aut}(\mathbf{C})$ on the étale homotopy type of $\mathcal{A}_{g,\mathbf{C}}$ with the *trivial* action on $|\mathcal{P}(\mathbf{F}_\ell)|$, from which it may be deduced that $\mathrm{KSp}_*(\mathbf{Z}; \mathbf{Z}_p) \to K_*(\mathbf{F}_\ell; \mathbf{Z}_p)$ is indeed equivariant for the trivial action on $K_*(\mathbf{F}_\ell; \mathbf{Z}_p)$.

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[11]Appealing to the norm residue theorem reduces this to checking that the restriction map $H^1(\mathbf{Z}'; \mathbf{Z}_p(2k)) \to H^1(\mathbf{F}_\ell; \mathbf{Z}_p(2k))$ in étale cohomology is an isomorphism. Since these groups are finite, this map is identified via the connecting homomorphism with the map $H^0(\mathbf{Z}', \mathbf{Q}_p/\mathbf{Z}_p(2k)) \to H^0(\mathbf{F}_\ell, \mathbf{Q}_p/\mathbf{Z}_p(2k))$. Now, $H^0(\mathbf{Z}', \mathbf{Q}_p/\mathbf{Z}_p(2k))$ is invariants of $\mathbf{Q}_p/\mathbf{Z}_p(2k)$ for the $\mathbf{Z}_p^\times$-action via $\chi_{\mathrm{cyc}}^{2k-1}$, and $H^0(\mathbf{F}_\ell, \mathbf{Q}_p/\mathbf{Z}_p(2k))$ is the invariants for the subgroup of $\mathbf{Z}_p^\times$ generated by the element ℓ.

Remark 7.17. Poitou–Tate duality implies the isomorphism

$$H_2(G, G_\mathfrak{p}; \mathbf{Z}_p(2k-1)) \cong K_{4k-1}(\mathbf{Z}; \mathbf{Z}_p) = \mathrm{KSp}_{4k-1}(\mathbf{Z}; \mathbf{Z}_p).$$

One may wonder whether the homotopy groups $\mathrm{KSp}_{4k-2}(\mathbf{Z}; \mathbf{Z}_p)$ and $\mathrm{KSp}_{4k-1}(\mathbf{Z}; \mathbf{Z}_p)$ are shadows of one "derived universal extension" of $\mathbf{Z}_p(2k-1)$ as a continuous $\mathbf{Z}_p[[G]]$-module split over $G_\mathfrak{p}$. Or better yet, whether there is a ($G_\mathfrak{p}$-split) sequence of spectra

$$L_{K(1)}K(\mathbf{Z})^{(+)} \to L_{K(1)}\mathrm{KSp}(\mathbf{Z}) \to K^{(-)}$$

in a suitable category of spectra with continuous G-actions, characterized by a universal property. Here $K = L_{K(1)}ku$ denotes the p-completed periodic complex K-theory spectrum.

8. Families of abelian varieties and stable homology

In this section, we give a more precise version of (1.5) from the introduction. The proof also illustrates a technique of passing to homology from homotopy.

Suppose $\pi : A \to X$ is a principally polarized abelian scheme over a smooth, n-dimensional, projective complex variety X. The Hodge bundle $\omega = \mathrm{Lie}(A)^*$ defines a vector bundle ω_X on X, and we obtain characteristic numbers of the family by integrating Chern classes of this Hodge bundle. In particular, if $\dim X = n$ then for any partition $\underline{n} = (n_1, \ldots, n_r)$ of n with each n_i odd, we have a Chern number

$$s_{\underline{n}}(A/X) := \int_X \mathrm{ch}_{n_1}(\omega_X) \smile \ldots \smile \mathrm{ch}_{n_r}(\omega_X) \in \mathbf{Q}.$$

Our results then imply divisibility constraints for the characteristic numbers of such families where A/X is defined over $\mathbf{Q}$ (that is: A, X and the morphism $A \to X$ are all defined over $\mathbf{Q}$):

Theorem 8.1. *Suppose that A/X is defined over $\mathbf{Q}$. For each partition $\underline{n}$ of n as above, the characteristic number $s_{\underline{n}}(A/X)$ is divisible by each prime $p \geq \max_j(n_j)$ such that, for some i, p divides the numerator of the Bernoulli number B_{n_i+1} .*

Proof. (Outline): In what follows we shall freely make use of étale *homology* of varieties and stacks, which can be made sense of, for example, using the setup of §B.1.

The assumption $p \geq n_i$ implies that ch_{n_i} lifts to a $\mathbf{Z}_{(p)}$-integral class. In particular we have universal $\mathrm{ch}_{n_i}(\omega) \in H^{2n_i}(\mathcal{A}_{g,\mathbf{C}}; \mathbf{Z}_p(n_i))$. The family A/X induces a classifying map $f : X \to \mathcal{A}_{g,\mathbf{C}}$, hence a cycle class in $H_{2n}(\mathcal{A}_{g,\mathbf{C}}; \mathbf{Z}_p)$ transforming according to the nth power of the cyclotomic character; more intrinsically we get an equivariant $\mathbf{Z}_p(n) \to H_{2n}(\mathcal{A}_{g,\mathbf{C}}; \mathbf{Z}_p)$. Taking cap product with $\prod_{j \neq i} \mathrm{ch}_{n_j}(\omega)$ gives $\mathbf{Z}_p(n_i) \longrightarrow H_{2n_i}(\mathcal{A}_{g,\mathbf{C}}; \mathbf{Z}_p)$, whose pairing with $\mathrm{ch}_{n_i}(\omega)$ is the Chern number $s_{\underline{n}}(A/X) \in \mathbf{Z}_{(p)} = \mathbf{Q} \cap \mathbf{Z}_p$. Assuming for a contradiction that this number is not divisible by p, the morphism

$$H_{2n_i}(\mathcal{A}_{g,\mathbf{C}}, \mathbf{Z}_p) \xrightarrow{\mathrm{ch}_{n_i}(\omega)} \mathbf{Z}_p(n_i) \tag{8.1}$$

splits as a morphism of Galois modules.

We wish to pass from this homological statement to a K-theoretic one. To do so we use some facts about stable homology

$$H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p) = \varinjlim_g H_i(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_p)$$

(the limit stabilizes for $i < (g-5)/2$ by [Cha87, Corollary 4.5]) that will be explained in the next subsection. This stable homology carries a Pontryagin product, arising from the natural maps $\mathrm{Sp}_{2a} \times \mathrm{Sp}_{2b} \to \mathrm{Sp}_{2a+2b}$; in particular we can define the "decomposable elements" of H_i as the $\mathbf{Z}_p$-span of all products $x_1 \cdot x_2$ where x, y have strictly positive degree, and a corresponding quotient space of "indecomposables."

We will be interested in a variant that is better adapted to $\mathbf{Z}_p$ coefficients. Define *integral decomposables* in $H_*(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p)$ as the $\mathbf{Z}_p$-span of all $x_1 \cdot x_2$ *and* $\beta(x_1' \cdot x_2')$ where $x_i \in H_*(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p), x_i' \in H_*(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}/p^k)$ have positive degree and β is the Bockstein induced from the sequence $0 \to \mathbf{Z}_p \to \mathbf{Z}_p \to \mathbf{Z}/p^k\mathbf{Z} \to 0$. Correspondingly this permits us to define an indecomposable quotient

$$H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p)_{\mathrm{Ind}} = \frac{H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p)}{\text{integral decomposables}}.$$

Then the fact that we shall use (generalizing a familiar property of *rational K*-theory, see [Nov66, Theorem 1.4])) is that the composite of the Hurewicz map and the quotient map

$$\mathrm{KSp}_i(\mathbf{Z}) \otimes \mathbf{Z}_p \longrightarrow H_i(\mathrm{Sp}_\infty(\mathbf{Z}); \mathbf{Z}_p)_{\mathrm{Ind}} \tag{8.2}$$

is an *isomorphism* for $i \leq 2p - 2$. We sketch the proof of this fact in §8.1 below. In particular, there is a map $H_{2n_i}(\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{Z}_p) \to \mathrm{KSp}_{2n_i}(\mathbf{Z}; \mathbf{Z}_p)$ (by mapping to stable homology followed by $(8.2)^{-1}$); this map is Galois equivariant and intertwines (8.1) with the Hodge map c_H, and therefore the sequence (7.21) is also split.

But this gives a contradiction: By Theorem 7.8, the sequence (7.21) is non-split as long as $\ker(c_H)$ is nonzero, which by Theorem 3.5 is the case precisely when $H^2_{\text{ét}}(\mathbf{Z}[1/p]; \mathbf{Z}_p(n_i + 1)) \neq 0$, which by Iwasawa theory (see [KNQDF96, Cor 4.2]) is the case precisely when p divides the numerator of B_{n_i+1}. $\qquad\square$

Remark 8.2. One can explicitly construct various examples of this situation, e.g.:
  (i) We can take X to be a projective Shimura variety of PEL type; the simplest example is a Shimura curve parameterizing abelian varieties with quaternionic multiplication.
  (ii) There exist many such families of curves, i.e. embeddings of smooth proper X into $\mathcal{M}_g$, and then the Jacobians form a (canonically principally polarized) abelian scheme over X.
  (iii) The (projective) Baily-Borel compactification of $\mathcal{A}_g$ has a boundary of codimension g; consequently, thus, a generic $(g-1)$-dimensional hyperplane section gives a variety X as above.

It should be possible to dirctly verify the divisibility at least in examples (i) and (ii), where it is related to (respectively) divisibility in the cohomology of the Torelli map $\mathcal{M}_g \to \mathcal{A}_g$ and the occurrence of ζ-values in volumes of Shimura varieties.

8.1. **Stable homology and its indecomposable quotient.** The proof of (8.2) is a consequence of a more general fact about infinite loop spaces, formulated and proved in Theorem 8.4. Let E be a p-complete connected spectrum (all homotopy in strictly positive degree) and let $X = \Omega^\infty E$ be the corresponding infinite loop space. We consider the composition

$$\pi_i(E) = \pi_i(X) \to H_i(X; \mathbf{Z}_p) \to H_i(X; \mathbf{Z}_p)_{\mathrm{Ind}} \tag{8.3}$$

of the Hurewicz homomorphism and the quotient by "integral indecomposables." As above, the latter space is defined as $I^2 + \sum \beta_k I_k^2$ where:
  • I is the kernel of the augmentation $H_*(X; \mathbf{Z}_p) \to \mathbf{Z}_p = H_*(\mathrm{pt}; \mathbf{Z}_p)$;
  • I_k is the kernel of the similarly defined $H_*(X; \mathbf{Z}/p^k\mathbf{Z}) \to \mathbf{Z}/p^k\mathbf{Z}$;
  • $\beta_k \colon H_*(X; \mathbf{Z}/p^k\mathbf{Z}) \to H_{*-1}(X; \mathbf{Z}_p)$ is the Bockstein operator associated to the short exact sequence $0 \to \mathbf{Z}_p \to \mathbf{Z}_p \to \mathbf{Z}/p^k\mathbf{Z} \to 0$.

Example 8.3. Let E be the Eilenberg–MacLane spectrum with $E_k = K(\mathbf{Z}/2\mathbf{Z}, k+1)$, so that $X = \mathbf{RP}^\infty = K(\mathbf{Z}/2\mathbf{Z}, 1)$. As is well known, $H_*(X; \mathbf{Z}/2\mathbf{Z})$ is a divided power algebra over $\mathbf{Z}/2\mathbf{Z}$ and $H_*(X; \mathbf{Z}_2)$ is additively $\mathbf{Z}/2\mathbf{Z}$ in each odd degree. Hence $I^2 = 0$ for degree reasons so $I/I^2 = I$ is the entire positive-degree homology. In contrast, $I_1 = H_{>0}(X; \mathbf{Z}/2\mathbf{Z})$ is

one-dimensional in all positive degrees whereas $I_1^2 \subset I_1$ is one-dimensional when the degree is not a power of two, but zero when the degree is a power of two.

Since $\beta_1 : H_*(X; \mathbf{Z}/2\mathbf{Z}) \to H_*(X; \mathbf{Z}_2)$ is surjective in positive degrees we may deduce that in this case the integral indecomposables

$$I/(I^2 + \beta_1(I_1^2))$$

is $\mathbf{Z}/2\mathbf{Z}$ in degrees of the form $2^i - 1$ and vanishes in all other degrees.

Theorem 8.4. *For $X = \Omega^\infty E$ as above, the homomorphism (8.3) is an isomorphism in degrees $* \leq 2p - 2$.*

Proof of Theorem 8.4. Let us first sketch why the result is true when X is a connected Eilenberg–MacLane space, i.e. $X = K(\mathbf{Z}_p, n)$ or $K(\mathbf{Z}/p^k\mathbf{Z}, n)$ for $n \geq 1$. In that case X has the structure of a topological abelian group, and the singular chains $C_*(X; \mathbf{Z}_p)$ form a graded-commutative differential graded algebra (cdga). In the case $X = K(\mathbf{Z}_p, n)$ there is a cdga morphism

$$A = \mathbf{Z}_p[x] \to C_*(X; \mathbf{Z}_p)$$

from the free cdga on a generator x in degree n, and in the case $X = K(\mathbf{Z}/p^k\mathbf{Z}, n)$ a cdga morphism

$$A = \mathbf{Z}_p[x, y \mid \partial y = x] \to C_*(X; \mathbf{Z}_p)$$

where x has degree n and y has degree $n + 1$, in both cases inducing an isomorphism $H_n(A) \to H_n(X; \mathbf{Z}_p)$. In both cases the mapping cone is acyclic in degrees $* \leq 2p - 1$, as follows either by inspecting the explicitly known $H^*(X; \mathbf{F}_p)$, see [Car54, Theorems 4,5,6] (also stated in e.g. [McC01, Theorem 6.19]), or by an induction argument (the case $n = 1$ is a calculation of mod p group homology of $\mathbf{Z}_p$ or $\mathbf{Z}/p^k$, the induction step is by the Serre spectral sequence). The map therefore induces an isomorphism in homology in degrees $* \leq 2p - 2$, and it is clear that $H_*(A)_{\mathrm{Ind}}$ is generated by the class of x.

A general $X = \Omega^\infty E$ may be replaced by its Postnikov truncation $\tau_{\leq 2p-2}X$, which splits as a product of Eilenberg–MacLane spaces, up to homotopy equivalence of loop spaces. Indeed, the deloop $\Omega^\infty \Sigma E$ can have non-vanishing homotopy in degrees $* \geq 2$ only, and the shortest possible k-invariant is $\mathcal{P}^1 : K(\mathbf{Z}/p, 2) \to K(\mathbf{Z}/p, 2p)$. Hence $\tau_{\leq 2p-2}X$ splits as a product of Eilenberg–MacLane spaces, and this splitting respects the H-space structure. Hence the result follows by induction from the final Lemma 8.5. $\qquad\square$

Lemma 8.5. *If X and Y are connected H spaces of finite type, then the natural map*

$$H_*(X; \mathbf{Z}_p)_{\mathrm{Ind}} \oplus H_*(Y; \mathbf{Z}_p)_{\mathrm{Ind}} \to H_*(X \times Y; \mathbf{Z}_p)_{\mathrm{Ind}}$$

is an isomorphism.

Proof. There are "natural maps" induced by $X \to X \times Y$, $Y \to X \times Y$, and the two projections, all of which are H-space maps. A formal argument shows that one composition gives the identity map of $H_*(X; \mathbf{Z}_p)_{\mathrm{Ind}} \oplus H_*(Y; \mathbf{Z}_p)_{\mathrm{Ind}}$ and hence that $H_*(X; \mathbf{Z}_p)_{\mathrm{Ind}} \oplus H_*(Y; \mathbf{Z}_p)_{\mathrm{Ind}} \to H_*(X \times Y; \mathbf{Z}_p)_{\mathrm{Ind}}$ is injective. (This much would also be true if we only take quotient by I^2 and not all the $\beta_k(I_k^2)$.)

The algebra $H_*(X \times Y)$ may be calculated additively by the Kunneth formula, and the main issue in this lemma is to deal with non-vanishing Tor terms. By the finite type assumption, the homology of X and Y will be direct sums of groups of the form $\mathbf{Z}_p$ and $\mathbf{Z}/p^k\mathbf{Z}$. Pick such a direct sum decomposition. Then each $\mathbf{Z}/p^k\mathbf{Z}$ summand in $H_*(X)$ and $\mathbf{Z}/p^l\mathbf{Z}$ summand in $H_*(Y)$ pair to give a $\mathbf{Z}/p^d\mathbf{Z}$ summand in the Tor term, where $d = \min(k, l)$. It is not hard to see that this summand must be in the $\beta_d(I_d^2)$. Indeed, a generator may be chosen as $\beta_d(xy)$, where $\beta_d(x)$ and $\beta_d(y)$ are generators for the p^d torsion in the $\mathbf{Z}/p^k\mathbf{Z}$ and $\mathbf{Z}/p^l\mathbf{Z}$ summands of $H_*(X)$ and $H_*(Y)$ respectively.

We have shown that the multiplication map $H_*(X) \otimes H_*(Y) \to H_*(X \times Y)$ becomes surjective after taking quotient by $\sum \beta_k(I_k^2)$, but then it must remain surjective after passing to augmentation ideals and taking further quotients. $\square$

Appendix A. Stable homotopy theory recollections

In this section we review some rudiments of the stable homotopy theory used in this paper, hopefully sufficient for a first reading, including how to construct symplectic K-theory in the category of spectra.

A.1. **Spectra and stable homotopy groups.** We recall a pedestrian approach to some standard definitions. By *space* in the following we mean a compactly generated Hausdorff space.

Definition A.1. A *spectrum* is a sequence $E = (E_n, \epsilon_n)_{n \in \mathbf{Z}}$ where E_n is a pointed space and $\epsilon_n : E_n \to \Omega E_{n+1}$ is a map, for all n. The *homotopy groups* of E are defined as

$$\pi_k(E) = \operatorname*{colim}_{n \to \infty} \pi_{n+k}(E_n)$$

for $k \in \mathbf{Z}$.

A *map of spectra* $f : E \to E'$ consists of pointed maps $f_n : E_n \to E'_n$ such that $\Omega f_{n+1} \circ \epsilon_n = f_n \circ \epsilon'_n$. It is a *weak equivalence* if the induced map $\pi_k(E) \to \pi_k(E')$ is an isomorphism for all $k \in \mathbf{Z}$.

Example A.2. For any space X we have a suspension spectrum $\Sigma_+^\infty X$, with nth space $S^n \wedge X_+$, the n-fold reduced suspension of X_+, which denotes X with a disjoint basepoint added. The structure maps $S^n \wedge X_+ \to \Omega(S^{n+1} \wedge X_+)$ are adjoint to the canonical homeomorphisms $S^1 \wedge (S^n \wedge X_+) \to S^{n+1} \wedge X_+$. The homotopy groups of $\Sigma_+^\infty X$ are the *stable homotopy groups*

$$\pi_k^s(X) = \pi_k(\Sigma_+^\infty X) = \operatorname*{colim}_{n \to \infty} \pi_{n+k}(S^n \wedge X_+),$$

These groups could be regarded as homology groups of X with coefficients in the sphere spectrum. For example, $X \mapsto \pi_*^s(X)$ has a Mayer–Vietoris sequence.

There are bilinear homomorphisms

$$\pi_k^s(X) \times \pi_{k'}^s(X') \to \pi_{k+k'}^s(X \times X'), \tag{A.1}$$

defined by taking smash product of a map $S^{n+k} \to S^n \wedge X_+$ with a map $S^{n'+k'} \to S^{n'} \wedge X'_+$ and using $(X_+) \wedge (X'_+) \cong (X \times X')_+$. In particular if X is an H-space, i.e. comes with a map $\mu : X \times X \to X$, or at least a zig-zag

$$X \times X \xleftarrow{\simeq} \dots \xrightarrow{\mu} X,$$

which is unital and associative up to homotopy, then the stable homotopy groups

$$\pi_*^s(X) = \bigoplus_k \pi_k^s(X)$$

inherit the structure of a *graded ring*, by composing the exterior products (A.1) with $\pi_*^s(\mu)$. This ring is a stable homotopy analogue of the Pontryagin product on singular homology $H_*(X)$. It is associative and unital because μ is associative and unital up to homotopy, it is graded-commutative if μ is commutative up to homotopy. Similarly, if X is a homotopy associative H-space and $X \times Y \to Y$ satisfies the axioms of a monoid action up to homotopy, then $\pi_*^s(Y)$ acquires the structure of a graded module over $\pi_*^s(X)$.

The *spectrum homology* groups $H_k(E)$ of a spectrum E are defined similarly, for $k \in \mathbf{Z}$: The maps $S^1 \wedge E_n \to E_{n+1}$ adjoint to ϵ_n induce suspension maps $\widetilde{H}_{k+n}(E_n) \cong \widetilde{H}_{k+n+1}(S^1 \wedge E_n) \to \widetilde{H}_{k+n+1}(E_{n+1})$ and one sets

$$H_k(E) = \operatorname*{colim}_n \widetilde{H}_{k+n}(E_n).$$

For $E = \Sigma_+^\infty X$ this agrees with the usual singular homology group $H_k(X; \mathbf{Z})$ up to canonical isomorphism. Spectrum homology $H_k(E; \mathbf{F}_p)$ with mod p coefficients is defined similarly, as are $H_k(E; \mathbf{Q})$ and $H_k(E; \mathbf{Z}/p^k)$.

There are stable Hurewicz homomorphisms

$$\pi_k(E) \to H_k(E) \tag{A.2}$$

for $k \in \mathbf{Z}$ which become isomorphisms after tensoring with $\mathbf{Q}$, for any E. In particular it induces a natural isomorphism

$$\pi_k^s(X) \otimes \mathbf{Q} \xrightarrow{\cong} H_k(X; \mathbf{Q}) \tag{A.3}$$

for all k, for any space X. If $x \in X$ is a basepoint, there are also stabilization maps

$$\pi_k(X, x) \to \pi_k^s(X) \tag{A.4}$$

for $k \in \mathbf{Z}_{\geq 0}$, and the composition of (A.4) and (A.2) for $E = \Sigma_+^\infty X$ is the usual Hurewicz homomorphism $\pi_k(X, x) \to H_k(X)$.

A.2. Infinite loop spaces.

Definition A.3. The spectrum E is an Ω-*spectrum* provided the structure maps $E_n \to \Omega E_{n+1}$ are weak equivalences, for all n.

An Ω-spectrum E is *connective* if $\pi_k(E) = 0$ for $k < 0$, or, equivalently, if the space E_n is $(n-1)$-connected for all $n \in \mathbf{Z}_{\geq 0}$.

The zero space E_0 of a Ω-spectrum contains all the homotopy groups of E in non-negative degree: it a pointed space with $\pi_k(E_0) = \pi_k(E)$, since the maps in the colimit defining $\pi_k(E)$ are all isomorphisms. The homotopy groups of a connective Ω-spectrum may therefore all be calculated from the pointed space E_0. An arbitrary spectrum E may be converted to an equivalent Ω-spectrum, which has zero space

$$\Omega^\infty E = \operatorname*{hocolim}_{n \to \infty} \Omega^n E_n.$$

The phrase "X is an infinite loop space" means that the space X is given a weak equivalence $X \simeq E_0$ for a connective Ω-spectrum E. One consequence of being an infinite loop space is that the stabilization homomorphisms (A.4) are canonically split injective, by evaluation homomorphisms $\pi_k^s(X) = \pi_k(\Sigma_+^\infty \Omega^\infty E) \to \pi_k(E) = \pi_k(X)$. Under the isomorphism (A.3), this splitting is for $k > 0$ an integral refinement of the inclusion of the primitives in the Hopf algebra $H_*(X; \mathbf{Q})$, split by the projection onto the indecomposables.

A.3. Gamma spaces and deloopings of algebraic K-theory spaces. We summarize a convenient formalism for constructing infinite loop structures on certain spaces, and to promote certain maps to infinite loop maps, introduced by G. Segal ([Seg74]) and further developed by Bousfield–Friedlander ([BF78]) and others.

Definition A.4. Let Γ^{op} denote a skeleton of the category whose objects are finite pointed sets and whose morphisms are pointed maps. Let $s\mathrm{Sets}_*$ denote the category of pointed simplicial sets. A Γ-*space* is a functor $X : \Gamma^{\mathrm{op}} \to s\mathrm{Sets}_*$ sending the terminal object $\{*\}$ to a terminal simplicial set (one-point set in each simplicial degree). A morphism of Γ-spaces is a natural transformation of such functors.

There is then a functor

$$B^\infty : \Gamma\text{-spaces} \to \text{connective spectra}. \tag{A.5}$$

Under extra assumptions on the Γ-space X, there is also a way to recognize $\Omega^\infty B^\infty X$ in terms of $X(S^0)$, the value of the functor X on the pointed set $S^0 := \{0, \infty\}$ with basepoint ∞.

The "infinite delooping" functor B^∞ is easy to define. Following [BF78], we first extend $X : \Gamma^{\text{op}} \to s\text{Sets}_*$ to a functor

$$X : s\text{Sets}_* \to s\text{Sets}_*$$

which preserves filtered colimits and geometric realization. Such an extension is unique up to unique isomorphism, and automatically preserves pointed weak equivalences. There are canonical maps

$$X(S^n) \to \Omega X(S^{n+1}),$$

and hence

$$|X(S^n)| \to \Omega|X(S^{n+1})|, \tag{A.6}$$

where S^1 denotes the simplicial circle, and $S^n = (S^1)^{\wedge n}$ the simplicial n-sphere. See e.g. [BF78, Section 4] for more details. These maps let us functorially associate a spectrum to each Γ-space X, and the spectra arising this way are automatically connective.

Definition A.5. The coproduct of two pointed sets S and T is denoted $S \vee T$ and traditionally called the wedge sum. $\vee$ gives a symmetric monoidal structure on Γ^{op}, and any object is isomorphic to a finite wedge sum $S^0 \vee \cdots \vee S^0$.

The Γ-space X is *special* if for any two objects S, T the canonical map

$$X(S \vee T) \to X(S) \times X(T),$$

is a weak equivalence.

When X is a special Γ-space, the pointed simplicial set $X(S^0)$ may be thought of as the underlying space of X. The fold map $S^0 \vee S^0 \to S^0$ induces a diagram

$$X(S^0) \times X(S^0) \xleftarrow{\simeq} X(S^0 \vee S^0) \to X(S^0), \tag{A.7}$$

which makes $|X(S^0)|$ into an H-space, which is unital, associative, and commutative up to homotopy. In particular the pointed set $\pi_0(|X(S^0)|)$ inherits the structure of a commutative monoid. As shown by Segal, the maps (A.6) are weak equivalences for $n \geq 1$ when X is special, so in that case $B^\infty X$ is equivalent to an Ω-spectrum with 0th space $\Omega|X(S^1)|$ and nth space $|X(S^n)|$ for $n \geq 1$. We then have a map of H-spaces

$$|X(S^0)| \to \Omega|X(S^1)| \xrightarrow{\simeq} \Omega^\infty B^\infty X \tag{A.8}$$

which is a "group completion", in the sense that it induces an isomorphism

$$H_*(X(S^0))[\pi_0(X(S^0))^{-1}] \xrightarrow{\sim} H_*(\Omega|X(S^1)|),$$

whose domain is $H_*(X(S^0))$, made into graded-commutative ring using (A.7), and localized at the multiplicative subset $\pi_0(X(S^0))$. A similar localization holds with (local) coefficients in any $\mathbf{Z}[\pi_2(X(S^1))]$-module.

Many spectra may be constructed this way. We list some examples relevant for this paper.

Example A.6. For any pointed simplicial set M, consider the Γ-space

$$S \mapsto S \wedge (M_+),$$

where M_+ denotes M with a disjoint basepoint added. The corresponding spectrum is then the (unbased) suspension spectrum $\Sigma_+^\infty M$ mentioned earlier.

There is a natural map of spectra

$$\Sigma_+^\infty |X(S^0)| \to B^\infty X, \tag{A.9}$$

natural in the Γ-space X, constructed as follows. For any finite pointed set S and any $s \in S$ we have a map $S^0 \to \{s, *\}$ sending the non-basepoint to s. If X is a Γ-space we may apply X to the composition $S^0 \to \{s, *\} \subset S$ to get a map $\{s\} \times X(S^0) \to X(S)$ for each $s \in S$. These assemble to a canonical map from $S \times X(S^0)$ which factors as

$$S \wedge X(S^0)_+ \to X(S).$$

This map is natural in $S \in \Gamma^{\mathrm{op}}$, i.e., defines a map of Γ-spaces and hence gives rise to a map of spectra. On homotopy groups it induces a map from the stable homotopy groups of $|X(S^0)|$ to the homotopy groups of $B^\infty X$.

Example A.7 (Constructing the algebraic K-theory spectrum). Following Segal, let us explain how to use Γ-space machinery to construct algebraic K-theory spectra $K(R)$ for a ring R. The idea is to construct a special Γ-space whose value on S^0 is equivalent to $|\mathcal{P}(R)|$, the classifying space of the groupoid of finitely generated projective R-modules. Its value on $\{*, 1, \ldots, n\}$ should be a classifying space for a groupoid of finitely generated projective modules equipped with a splitting into n many direct summands.

Let $S \in \Gamma^{\mathrm{op}}$ and let R^S denote the ring of all functions $f : S \to R$ under pointwise ring operations. The diagonal $R \to R^S$ makes any R^S-module into an R-module. Let us for $s \in S$ write $e_s \in R^S$ for the idempotent with $e_s(s) = 1$ and $e_s(S \setminus \{s\}) = \{0\}$. Then for projective R^S-module M has submodules $e_s M \subset M$ and the canonical map $\oplus_{s \in S} M_S \to M$ is an isomorphism. Hence each M_s is a projective R-module (for the diagonal R-structure). Let us write $e = 1 - e_* = \sum_{s \in S \setminus \{*\}} e_s \in R^S$ so that $eM = \sum_{s \in S \setminus \{*\}} e_s M$, and let $\mathcal{P}_S(R)$ be the category whose objects are pairs (n, ϕ) with $n \in \mathbf{N}$ and $\phi : R^S \to M_n(R)$ an R-algebra homomorphism, and whose morphisms $(n, \phi) \to (n', \phi')$ are R^S-linear isomorphisms $\phi(e) R^n \to \phi'(e) R^{n'}$. The forgetful functor

$$\mathcal{P}_{S^0}(R) \to \mathcal{P}(R)$$
$$(n, \phi) \mapsto \phi(e) R^n$$

is then an equivalence of categories, since any finitely generated projective module is isomorphic to a retract of R^n for some n. Moreover the association

$$S \mapsto \mathcal{P}_S(R)$$

extends to a functor from Γ^{op} to groupoids: a morphism $f : S \to T$ is sent to the functor $\mathcal{P}_S(R) \to \mathcal{P}_T(R)$ which on objects sends $(n, \phi) \to (n, \phi \circ (f^*))$, where $f^* : R^T \to R^S$ is precomposing with f. We emphasize that composition of morphisms in Γ^{op} is carried to composition of functors *on the nose* (not just up to preferred isomorphism). That is, $S \mapsto \mathcal{P}_S(R)$ is a functor to the 1-category of small groupoids.

For $S = \{*, 1, \ldots, n\}$ the restriction functors induce an equivalence of groupoids

$$\mathcal{P}_S(R) \to \prod_{i=1}^n \mathcal{P}_{\{*, i\}}(R) \simeq \big(\mathcal{P}(R)\big)^n.$$

It follows that $S \mapsto N(\mathcal{P}_S(R))$ is a special Γ-space and the corresponding spectrum is a model for $K(R)$. The map (A.8) is a model for the canonical group-completion map

$$|\mathcal{P}(R)| \to \Omega^\infty K(R),$$

mentioned in Subsection 2.3.

Example A.8 (Constructing the symplectic K-theory spectrum). Finally, let us discuss the spectrum $\mathrm{KSp}(\mathbf{Z})$, where we are looking for a Γ-space with $X(S^0) \simeq N(\mathcal{SP}(\mathbf{Z}))$. The idea is similar to $S \mapsto \mathcal{P}_S(\mathbf{Z})$. Recall that the objects of $\mathcal{P}_S(\mathbf{Z})$ are $\mathbf{Z}^S$-modules M whose underlying $\mathbf{Z}$-module is equal to $\mathbf{Z}^n$ for some $n \in \mathbf{N}$. Let objects of $\mathcal{SP}_S(\mathbf{Z})$ be pairs of an object $M \in \mathcal{P}_S(\mathbf{Z})$ and a symplectic form $b : M \times M \to \mathbf{Z}$ for which $e_s M \subset M$ is a symplectic submodule for each $s \in S$.

This defines a functor from Γ^{op} to the 1-category of small groupoids, as before. We obtain a Γ-space $S \mapsto N(\mathcal{SP}_S(\mathbf{Z}))$, whose associated spectrum is $\mathrm{KSp}(\mathbf{Z})$ and infinite loop space is a model for $\mathbf{Z} \times B\mathrm{Sp}_\infty(\mathbf{Z})^+$.

Remark A.9 (The K-theory spectrum of a small symmetric monoidal category). The examples above are related to a more general phenomenon that we already alluded to in the main text. To any small category $\mathcal{C}$ equipped with a symmetric monoidal structure, we may associate a functor $S \mapsto \mathcal{C}^\otimes(S)$ from finite pointed sets to small categories. The idea is that $S = \{*, 1, \ldots, n\}$ is sent to $\mathcal{C}^\otimes(S) \simeq \mathcal{C}^n = \mathcal{C} \times \cdots \times \mathcal{C}$, and that a morphism $f : S = \{*, 1, \ldots, n\} \to T = \{*, 1, \ldots, m\}$ is sent to a functor $\mathcal{C}^\otimes(S) \to \mathcal{C}^\otimes(T)$ which up to equivalence is given on objects as

$$(X_1, \ldots, X_n) \mapsto \left(\bigotimes_{f(i)=1} X_i, \ldots, \bigotimes_{f(i)=m} X_i \right),$$

where $\otimes$ denotes the given symmetric monoidal structure. This does not quite define a functor into small categories (composition is not preserved on the nose) but can be rectified to one that does.

Composing with the nerve functor from small categories to simplicial sets then gives a special Γ-space

$$N\mathcal{C}^\otimes : S \mapsto N(\mathcal{C}^\otimes(S)),$$

and a group completion map

$$|\mathcal{C}| \simeq |\mathcal{C}^\otimes(S^0)| \to \Omega^\infty B^\infty(N\mathcal{C}^\otimes).$$

In this way the Γ-space machinery can be used to associate a spectrum to any small symmetric monoidal category. In the examples above we have spelled this out in more detail, in the cases of relevance to this paper.

A.4. **Products.** When R is a commutative ring the K-groups $K_*(R)$ form a graded ring. Let us briefly recall how the product on homotopy groups may be produced using Γ-spaces.

If X, Y, and Z are Γ-spaces and there is given a natural transformation

$$\mu : X(S) \wedge Y(T) \to Z(S \wedge T)$$

of functors of $(S, T) \in (\Gamma^{\mathrm{op}})^2$ then maps $f : S^{n+k} \to X(S^n)$ and $g : S^{m+l} \to Y(S^m)$ may be smashed together to form $S^{(n+m)+(k+l)} \to X(S^n) \wedge Y(S^m)$ which may be composed with $\mu : X(S^n) \wedge Y(S^m) \to Z(S^n \wedge S^m)$. Taking colimit over $n, m \to \infty$ we get a bilinear pairing

$$\pi_k(B^\infty X) \otimes \pi_l(B^\infty Y) \xrightarrow{\mu} \pi_{k+l}(B^\infty Z),$$

which is symmetric up to a sign $(-1)^{kl}$ arising from swapping some of the spheres in a smash product. In particular this may be applied with $X = Y = Z$, in which case the homotopy groups $\pi_*(B^\infty X)$ become graded rings, provided there are given "multiplication" maps $\mu : X(S) \wedge X(T) \to X(S \wedge T)$ which are natural in $S, T \in \Gamma^{\mathrm{op}}$ and are unital and associative in the appropriate sense. Such multiplication maps in particular induce a product

$$X(S^0) \times X(S^0) \twoheadrightarrow X(S^0) \wedge X(S^0) \to X(S^0 \wedge S^0) \cong X(S^0), \tag{A.10}$$

in turn inducing a product on the graded abelian groups $\pi_*^s(|X(S^0)|)$. It is an exercise to verify that in this situation the homomorphism

$$\pi_*^s(|X(S^0)|) \to \pi_*(B^\infty X)$$

induced by (A.9) is a homomorphism of graded rings.

The domain may be restricted to any collection of path components of $|X(S^0)|$ which is closed under multiplication. In particular we could let $\pi_0(|X(S^0)|)^\times \subset \pi_0(|X(S^0)|)$ denotes the group of invertible elements in the monoid structure induced by (A.10) and define a space $|X(S^0)|^\times$ as the pullback

$$
\begin{array}{ccc}
|X(S^0)|^\times & \longrightarrow & |X(S^0)| \\
\downarrow & & \downarrow \\
\pi_0(|X(S^0)|)^\times & \longrightarrow & \pi_0(|X(S^0)|).
\end{array}
$$

Then we get a homomorphism of graded rings

$$\pi_*^s(|X(S^0)|^\times) \to \pi_*(B^\infty X), \tag{A.11}$$

natural in the Γ-space X equipped with the associative and unital product maps μ.

When R is commutative, tensor products of R-modules may be used to define natural multiplications $X(S) \wedge X(T) \to X(S \wedge T)$ on the Γ-space $X : S \mapsto N(\mathcal{P}_S(R))$. The finitely generated projective R-modules which are invertible with respect to $\otimes_R$ are precisely those of rank 1, so the space $|X(S^0)|^\times$ is precisely $|\mathrm{Pic}(R)|$ and the ring homomorphism (A.11) is identified with (2.4) from Section 2.

A.5. **Mod p^k homotopy groups of a spectrum, and p-completion of spectra.** The *Moore space* $M(\mathbf{Z}/p^k, 1)$ is defined as the mapping cone of the map $S^1 \to S^1$ given by $z \mapsto z^{p^k}$, and the *Moore spectrum* $\mathbb{S}/p^k$ is the spectrum whose nth space is the reduced suspension $S^{n-1} \wedge M(\mathbf{Z}/p^k, 1)$.

For any other spectrum E we can associate a spectrum $E \wedge (\mathbb{S}/p^k)$, or E/p^k for brevity, as the derived smash product of E with the Moore spectrum. Two explicit models for this space are:

- the spectrum whose nth space is $\mathrm{Map}_*(M(\mathbf{Z}/p^k, 1), E_{n+2})$, the space of pointed maps $M(\mathbf{Z}/p^k, 1) \to E_{n+2}$.
- the spectrum whose nth space is the (derived) smash product $M(\mathbf{Z}/p^k, 1) \wedge E_{n-1}$.

In any case, the mod p^k homotopy groups of E are then defined as

$$\pi_n(E; \mathbf{Z}/p^k) = \pi_n(E/p^k),$$

and they fit into long exact sequences

$$\to \pi_n(E) \xrightarrow{p^k} \pi_n(E) \to \pi_n(E; \mathbf{Z}/p^k) \to \pi_{n-1}(E) \xrightarrow{p^k} \dots$$

and

$$\to \pi_n(E; \mathbf{Z}/p^j) \to \pi_n(E; \mathbf{Z}/p^{j+k}) \to \pi_n(E; \mathbf{Z}/p^k) \to \pi_{n-1}(E; \mathbf{Z}/p^j) \to$$

analogous to the usual "Bockstein" long exact sequences in homology and cohomology. It follows by induction and the 5-lemma that if a map of spectra $E \to E'$ induces isomorphisms $\pi_*(E; \mathbf{Z}/p) \to \pi_*(E'; \mathbf{Z}/p)$ in all degrees, then it also induces isomorphisms in mod p^k homotopy, for all k.

In spectrum homology we have

$$H_*(E/p^k) \cong H_*(E; \mathbf{Z}/p^k).$$

It is not possible to reconstruct the spectrum E from the spectra E/p^k, even up to weak equivalence. The "closest approximation" is a map of spectra

$$E \to \operatorname*{holim}_{k \to \infty} E/p^k.$$

This homotopy limit is a model for the *p-completion* of E and is denoted $E_p^{\wedge}$. If $f : E \to E'$ is a map of spectra such that $f_* : \pi_*(E; \mathbf{Z}/p) \to \pi_*(E'; \mathbf{Z}/p)$ is an isomorphism in all degrees, then the induced map of completions $f_p^{\wedge} : E_p^{\wedge} \to (E')_p^{\wedge}$ is a weak equivalence. If E and E' are connective it suffices that the induced map $H_*(E; \mathbf{Z}/p) \to H_*(E'; \mathbf{Z}/p)$ is an isomorphism.

If E has finite p-type, i.e. if the multiplication map $p : \pi_n(E) \to \pi_n(E)$ has finite kernel and cokernel for all n, or equivalently if the group $\pi_n(E; \mathbf{Z}/p)$ is finite for all n, then the effect of p-completion is simply to p-complete the homotopy groups

$$\pi_k(E)_p^{\wedge} \cong \pi_k(E_p^{\wedge}).$$

Most of the spectra appearing in this paper satisfy the stronger condition that $\pi_n(E)$ is a finitely generated abelian group for all n. For general E the relationship between $\pi_k(E)$ and $\pi_k(E_p^{\wedge})$ is more complicated and involves derived inverse limits.

APPENDIX B. CONSTRUCTION OF THE GALOIS ACTION ON SYMPLECTIC K-THEORY

The goal of this Appendix is to supply details for an argument sketched in the main text, viz. the construction of the Galois action on symplectic K-theory in the proof of Proposition 6.2.

In Subsection B.1 we review two ways to to extract a space from a simplicial scheme quasiprojective over $\operatorname{Spec}(\mathbf{C})$, one might be called "Betti realization" and the other "étale realization". Then in §B.1.2 we explain how to relate Betti realization with étale realization after completing at a prime p. As usual, the point is that the étale realization of objects base changed from $\operatorname{Spec}(\mathbf{Q})$ inherits an action of the group $\operatorname{Aut}(\mathbf{C})$ of all field automorphisms of the complex numbers.

The main construction happens in §B.3, where a certain Γ-object Z in simplicial schemes quasi-projective over $\operatorname{Spec}(\mathbf{Q})$ is constructed. We prove that the Γ-space resulting from base changing Z from $\mathbf{Q}$ to $\mathbf{C}$ and taking Betti realization gives a model for $\operatorname{KSp}(\mathbf{Z})$, the symplectic K-theory spectrum studied in this paper. This eventually boils down to the Betti realization of $\mathcal{A}_g(\mathbf{C})^{\mathrm{an}}$ being a model for $B\operatorname{Sp}_{2g}(\mathbf{Z})$ "in the orbifold sense", is deduced from uniformization of principally polarized abelian varieties over $\mathbf{C}$ and the contractibility of Siegel upper half-space $\mathbb{H}_g$, as we discuss in§B.2. The result is a model for the p-completion of the spectrum $\operatorname{KSp}(\mathbf{Z})$ on which $\operatorname{Aut}(\mathbf{C})$ acts by spectrum maps, as we conclude in §B.4.

B.1. **Homotopy types of complex varieties.** Let us review various "realization functors" assigning a complex scheme $X \to \operatorname{Spec}(\mathbf{C})$. We shall mostly assume that X is a *variety*, which we define as follows.

Definition B.1. Let $\operatorname{Var}_{\mathbf{C}}$ denote the category of schemes over $\operatorname{Spec}(\mathbf{C})$ which are coproducts of quasi-projective schemes.

The realization functors we need may be summarized in a diagram of simplicial sets

$$X(\mathbf{C}) = \operatorname{Sing}_0^{\mathrm{an}}(X) \longrightarrow \operatorname{Sing}^{\mathrm{an}}(X) \dashrightarrow \operatorname{\acute{E}t}_p(X), \qquad (\mathrm{B.1})$$

where the dashed arrow indicates a zig-zag of the form

$$\operatorname{Sing}^{\mathrm{an}}(X) \xleftarrow{\simeq} \cdots \to \operatorname{\acute{E}t}_p(X).$$

As we shall explain in more detail below, the "Betti realization" has n-simplices $\operatorname{Sing}_n^{\mathrm{an}}(X)$ the set of maps $\Delta^n \to X(\mathbf{C})$ which are continuous in the analytic topology on $X(\mathbf{C})$. Therefore

the homotopy type of $\mathrm{Sing}^{\mathrm{an}}(X)$ encodes the weak homotopy type of the space $X(\mathbf{C})$ equipped with its *analytic topology*. Less interestingly, $X(\mathbf{C}) = \mathrm{Sing}_0^{\mathrm{an}}(X)$ is the set of complex points regarded as a constant simplicial set, encoding the homotopy type of $X(\mathbf{C})$ in the discrete topology. Finally, the "p-completed étale realization" $\mathrm{Ét}_p(X)$ is a model for the *étale homotopy type* of X, introduced by Artin and Mazur [AM69], or rather its p-completion.

We obtain similar realization functors when $X \in s\mathrm{Var}_{\mathbf{C}}$ is a *simplicial* complex variety, i.e. a functor $\Delta^{\mathrm{op}} \to \mathrm{Var}_{\mathbf{C}}$. We will make use of the following properties of these realization functors.

(i) $\mathrm{Sing}_0^{\mathrm{an}}(X)$ and $\mathrm{Ét}_p(X)$ are functorial with respect to commutative diagrams

$$
\begin{array}{ccc}
X & \xrightarrow{\;\;f\;\;} & X' \\
\downarrow & & \downarrow \\
\mathrm{Spec}\,(\mathbf{C}) & \xrightarrow{\;\;\sigma\;\;} & \mathrm{Spec}\,(\mathbf{C}),
\end{array}
\tag{B.2}
$$

in which σ is any automorphism of $\mathbf{C}$, and the composition (B.1) is a natural transformation of such functors.

(ii) $\mathrm{Sing}^{\mathrm{an}}(X)$ is functorial with respect to diagrams of the form (B.2) where $\sigma \in \mathrm{Aut}(\mathbf{C})$ is a *continuous* field automorphism (that is, σ is either the identity or complex conjugation), and all arrows in (B.1) are natural transformation of such functors.

(iii) The map $\mathrm{Sing}^{\mathrm{an}}(X) \to \mathrm{Ét}_p(X)$ induces an isomorphism in mod p homology, at least when $H^1(\mathrm{Sing}^{\mathrm{an}}(X); \mathbf{F}_p) = 0$.

(iv) If $X_{g,\mathbf{C}} \to \mathrm{Spec}\,(\mathbf{C})$ is the simplicial variety arising from an atlas $U \to \mathcal{A}_{g,\mathbf{C}}$, then $\mathrm{Sing}^{\mathrm{an}}(X_g) \simeq B\mathrm{Sp}_{2g}(\mathbf{Z})$. Moreover, under this equivalence the maps $\mathcal{A}_g \times \mathcal{A}_{g'} \to \mathcal{A}_{g+g'}$ defined by taking product of principally polarized abelian varieties correspond to the symmetric monoidal structure on $\mathcal{SP}(\mathbf{Z})$ given by orthogonal direct sum.

It is essentially well known that realization functors with these properties exist. In particular, the isomorphism between mod p cohomology of $\mathrm{Sing}^{\mathrm{an}}(X)$ and $\mathrm{Ét}_p(X)$ is a combination of Artin's comparison theorem relating étale cohomology with finite constant coefficients to Cech cohomology with finite constant coefficients, and the isomorphism between Cech cohomology and singular cohomology. We shall use two aspects which are perhaps slightly less standard, so we outline the constructions in subsection B.1 below. Firstly, the étale homotopy type usually outputs a pro-object, but it is convenient for us to have a genuine simplicial set. Secondly, as stated in (i), we shall make a point of $X(\mathbf{C}) = \mathrm{Sing}_0^{\mathrm{an}}(X)$ being more functorial than the entire $\mathrm{Sing}^{\mathrm{an}}(X)$. This last property is used only for the verification of commutativity of (6.2).

The reader willing to accept on faith (or knowledge) that realization functors with these properties exist may skip ahead to B.3 to see how to complete the proof of Proposition 6.2.

B.1.1. *Betti realization.* A complex scheme $X \to \mathrm{Spec}\,(\mathbf{C})$ is *quasi-projective* if it is isomorphic (as a scheme over $\mathrm{Spec}\,(\mathbf{C})$) to an intersection of a Zariski open and a Zariski closed subset of $\mathbf{P}_{\mathbf{C}}^N$ for some N. The resulting embedding $X \to \mathbf{P}_{\mathbf{C}}^N$ induces an injection of complex points

$$
X(\mathbf{C}) \hookrightarrow \mathbf{P}_{\mathbf{C}}^N(\mathbf{C}) = \mathbf{C}P^N,
$$

and the set of complex points $X(\mathbf{C})$ inherits the *analytic topology* as a subspace of $\mathbf{C}P^N$, itself the quotient topology from the Euclidean topology on $\mathbf{C}^{N+1} \setminus \{0\}$. We shall write $X(\mathbf{C})^{\mathrm{an}}$ for this topological space, which is Hausdorff and locally compact, and also locally contractible (as follows from Hironaka's theorem that it is triangulable [Hir75]).

For any compact Hausdorff space Δ we have have a $\mathbf{C}$-algebra $\mathbf{C}^\Delta$ of functions $\Delta \to \mathbf{C}$ that are continuous in the Euclidean topology. There is a canonical function

$$
e_\Delta : \Delta \hookrightarrow \mathrm{Spec}\,(\mathbf{C}^\Delta)(\mathbf{C}),
$$

sending a point of Δ to the point corresponding to the evaluation homomorphism $\mathbf{C}^\Delta \to \mathbf{C}$.

Lemma B.2. *For any scheme X over $\mathbf{C}$, and any compact Hausdorff space Δ, the map*

$$\text{maps Spec}\,(\mathbf{C}^\Delta) \to X \text{ of schemes over Spec}\,(\mathbf{C})$$
$$\downarrow$$
$$\text{maps } \Delta \to X(\mathbf{C}) \text{ continuous in the analytic topology}$$

induced by precomposition with e_Δ is a bijection.

Proof. We describe the inverse. Take $f : \Delta \to X(\mathbf{C})$ and choose an affine cover U_i of X, and take $V_i = f^{-1}(U_i)$; choose a partition of unity $1 = \sum g_i$ on Δ where $\text{supp}(g_i) \subset V_i$. The g_i generate the unit ideal of $\mathbf{C}^\Delta$, i.e. the spectrum of $\mathbf{C}^\Delta$ is the union of the open affines corresponding to rings $\mathbf{C}^\Delta[g_i^{-1}]$. We obtain

$$\text{regular functions on } U_i \to \text{continuous functions on } V_i \to \mathbf{C}^\Delta[g_i^{-1}].$$

where the last map sends a continuous function h on V_i to $(hg_i) \cdot g_i^{-1}$, where we extend by zero off V_i to make hg_i a function on Δ. Dually we obtain

$$\text{Spec}\,\mathbf{C}^\Delta[g_i^{-1}] \longrightarrow U_i,$$

These morphisms glue to the desired map $\text{Spec}\,\mathbf{C}^\Delta \to X$. $\qquad\square$

In particular, the simplicial set $\text{Sing}(X(\mathbf{C})^{\text{an}})$ may be written in terms of the functor $X : \mathbf{C}$-algebras $\to$ Sets as

$$\text{Sing}_n(X(\mathbf{C})) = X(\mathbf{C}^{\Delta^n}) = \text{Maps}_{\mathbf{C}\text{-schemes}}(\text{Spec}\,(\mathbf{C}^{\Delta^n}), X).$$

where Δ^n is as usual the (topological) n-simplex. Motivated by this observation, we make the following more general definition.

Definition B.3. Let X be a simplicial complex variety, or more generally any functor from $\mathbf{C}$-algebras to simplicial sets. The *analytic homotopy type* (or "Betti realization") of X is the simplicial set $\text{Sing}^{\text{an}}(X)$ defined by

$$\text{Sing}_n^{\text{an}}(X) = X(\mathbf{C}^{\Delta^n})_n.$$

In other words, $\text{Sing}^{\text{an}}(X)$ is the diagonal of the simplicial set $([n], [m]) \mapsto \text{Sing}_n(X_m(\mathbf{C})^{\text{an}})$.

B.1.2. *Etale homotopy type and p-adic comparison.* The theory of étale homotopy type assigns a pro-simplicial set[12] $\text{Ét}(X)$ functorially to any (locally Noetherian) scheme X, where

$$H^*(\text{Ét}(X), A) \simeq H^*_{\text{ét}}(X, A) \tag{B.3}$$

for finite abelian groups A. We will outline how to modify this construction so as to assign an actual simplicial set $\text{Ét}_p(X)$ to such a scheme, maintaining the validity of (B.3) for p-torsion A. We shall also make the zig-zag of (B.1).

Let $s\text{Sets}^{(p)}$ be the category of p-finite simplicial sets: those simplicial sets X where $\pi_0(X)$ is a finite set, and $\pi_i(X, x)$ is a finite p-group for all $x \in X_0$ and all $i > 0$, which is trivial for sufficiently large i. We define Ét_p as the composition of three functors: the étale homotopy type, p-completion, and homotopy limit, each of which we review in turn:

$$s\text{Var}_\mathbf{C} \xrightarrow{\text{Ét}} \text{pro-}s\text{Sets} \xrightarrow{\text{pro-}p \text{ completion}} \text{pro-}s\text{Sets}^{(p)} \xrightarrow{\text{holim}} s\text{Sets}, \tag{B.4}$$

---

[12]The original approach of Artin and Mazur [AM69] assigns to X a pro-object in the homotopy category of simplicial sets, which was rigidified in later approaches [Fri82] to output a pro-object in simplicial sets.

As the first functor

$$sVar_{\mathbf{C}} \xrightarrow{\text{Ét}} \text{pro-}sSets \tag{B.5}$$

we shall take Friedlander's rigid étale homotopy type. To each hypercover $U_{\bullet} \to X$ of a locally Noetherian schemes X one gets a simplicial set $[n] \mapsto \pi_0(U_n)$, where $\pi_0(U_n)$ denotes the set of connected components of U_n (denoted just $\pi(U_n)$ in [Fri82]). If $X_{\bullet}$ is a simplicial object in locally Noetherian scheme, we may similarly consider bisimplicial object $U_{\bullet,\bullet}$ forming a hypercover $U_{s,\bullet} \to X_s$ for each s: to this situation we associate the diagonal simplicial set $[n] \mapsto \pi_0(U_{n,n})$. Friedlander then defines the étale homotopy type of $X_{\bullet}$ as a functor

$$\text{HRR}(X_{\bullet})^{\text{op}} \to sSets$$
$$(U_{\bullet,\bullet} \to X) \mapsto ([n] \mapsto \pi_0(U_{n,n})),$$

where $\text{HRR}(X_{\bullet})$ is a suitable category of *rigid hypercovers*. These are actually hypercovers equipped with extra data, making $\text{HRR}(X_{\bullet})$ into a filtered category. The details of how $\text{HRR}(X_{\bullet})$ is defined shall not matter for our applications. Let us emphasize however, that this construction outputs an inverse system of simplicial sets functorially in X, which is slightly stronger than outputting a pro-object[13].

The p-profinite completion was introduced in [Mor96], see also [Isa05]. It associates to an inverse system $Y : j \mapsto Y_j$ in the category of simplicial sets another inverse system $Y_p^{\wedge}$ in the category of p-finite simplicial sets, and a map

$$Y \to Y_p^{\wedge}$$

inducing an isomorphism in "continuous" mod p cohomology, defined as $H^*(Y; \mathbf{F}_p) = \text{colim}_j H^*(Y_j; \mathbf{F}_p)$. There is again an explicit construction which outputs an inverse system of p-finite simplicial sets, for instance one can for each j consider all quotients $Y_j \to Z$ which are finite sets in each simplicial degree, then take a Postnikov truncation of a stage in the totalization of the Bousfield–Kan cosimplicial resolution of Y. The pro-object $Y_p^{\wedge}$ is obtained by letting these stages vary over the natural numbers, Z vary over finite quotients of Y_j, and j vary over the indexing category of Y.

Combining these two constructions assigns to any $X \in sVar_{\mathbf{C}}$ an $(\text{Ét}(X))_p^{\wedge}$ which is an inverse system of p-finite simplicial sets, together with a canonical isomorphism

$$H_{\text{et}}^*(X; \mathbf{F}_p) \cong \text{colim} \, H^*(\text{Ét}(X)_p^{\wedge}; \mathbf{F}_p),$$

where the left hand side is étale cohomology of X with coefficients in the constant sheaf X, and the right hand side is the colimit of cohomology of the levels in the inverse system $(\text{Ét}(X))_p^{\wedge}$.

The last step is to replace the inverse system by its homotopy limit, which is a more subtle thing to do. For any inverse system $j \mapsto Y_j$ of simplicial sets, there is a canonical map

$$\text{colim}_j H^*(Y_j; \mathbf{F}_p) \to H^*(\text{holim}_j Y; \mathbf{F}_p),$$

but there is no formal reason for this map to be an isomorphism, and in general it may well not be. It is known to be an isomorphism however, when the domain is finite-dimensional in each cohomological degree and vanishes in degree 1. A recent reference for this statement in the form used here is [Lur11, Proposition 3.3.8 and Theorem 3.4.2], but the insight that such pro-objects may sometimes be replaced with their homotopy limits without too much loss of information goes back to Sullivan's "MIT notes", see for instance [Sul05, Theorem 3.9] for a profinite version.

---

[13]The point of this sentence is that the last step in (B.4), taking homotopy limit, is not strictly functorial in pro-objects, only "functorial up to weak equivalence". This objection is dismissed by upgrading to preferred inverse systems.

We therefore define

$$\mathrm{\acute{E}t}_p(X) := \mathrm{holim}(\mathrm{\acute{E}t}(X)_p^\wedge),$$

and have a canonical map

$$H^*_{\mathrm{et}}(X; \mathbf{F}_p) \to H^*(\mathrm{\acute{E}t}_p(X); \mathbf{F}_p)$$

which is an isomorphism when the domain is finite-dimensional in each degree and vanishes in degree 1.

Remark B.4. Presumably the explicit construction given here could be replaced with any of the recent constructions leading to a pro-space in the ∞-categorical sense, e.g. [BS16], [Hoy18], [Car15], or [BGH18, Section 12].

In particular, some readers may prefer an approach based on the notion of the "shape of an ∞-topos", assigning a pro-space to any ∞-topos and hence to any site in the usual sense. When X is a scheme, Friedlander's explicit construction would then be replaced by the shape of the étale site of X, and the comparison maps constructed below should come from morphisms of sites

$$X(\mathbf{C})^{\mathrm{disc}} \to X(\mathbf{C})^{\mathrm{an}} \to X_{\mathrm{et}},$$

where $X(\mathbf{C})^{\mathrm{disc}}$ denotes the site corresponding to the set $X(\mathbf{C})$ in the discrete topology.

B.1.3. *Comparison map.* When $X \in s\mathrm{Var}_\mathbf{C}$, the Artin comparison gives a canonical isomorphism between $H^*_{\mathrm{et}}(X; \mathbf{F}_p)$ and the Cech cohomology of $X(\mathbf{C})^{\mathrm{an}}$, the complex points in the analytic topology. Since complex varieties are paracompact and locally contractible in the analytic topology (since they are triangulable), Cech cohomology with constant coefficients is also isomorphic to singular cohomology. In total we obtain an isomorphism

$$H^*(\mathrm{Sing}^{\mathrm{an}}(X); \mathbf{F}_p) \cong H^*_{\mathrm{et}}(X; \mathbf{F}_p).$$

Above we explained how étale cohomology is calculated by the space $\mathrm{\acute{E}t}_p(X)$ in good cases, we now finally explain how to define a comparison map $\mathrm{Sing}^{\mathrm{an}} \to \mathrm{\acute{E}t}_p(X)$, or at least a zig-zag.

Let $U_{\bullet,\bullet}$ be a levelwise hypercover as after (B.5). The scheme $\mathrm{Spec}\,(\mathbf{C}^{\Delta^n})$ is connected, so that all maps to $U_{s,t}$ land in the same connected component. Therefore we obtain well defined maps $\mathrm{Sing}^{\mathrm{an}}_n(U_{s,t}) \to \pi_0(U_{s,t})$ which are invariant under simplicial operations in the n-direction, and hence induce continuous maps

$$|\mathrm{Sing}^{\mathrm{an}}(U_{s,t})| \to \pi_0(U_{s,t})$$

for all s, t. Moreover $U^{\mathrm{an}}_{s,\bullet} \to X^{\mathrm{an}}_s$ is a topological hypercover, which implies that $|U^{\mathrm{an}}_{s,\bullet}| \to X^{\mathrm{an}}_s$ is a weak equivalence. Therefore the natural map

$$\mathrm{Sing}^{\mathrm{an}}(U_{s,\bullet}) \to \mathrm{Sing}^{\mathrm{an}}(X_s)$$

is a weak equivalence of simplicial sets for all s, where in the domain we implicitly pass to diagonal simplicial set. Combining all this, and taking geometric realization in the s-direction, we obtain a zig-zag of maps of simplicial sets

$$\mathrm{Sing}^{\mathrm{an}}(X_\bullet) \overset{\simeq}{\Leftarrow} \mathrm{Sing}^{\mathrm{an}}(U_{\bullet,\bullet}) \to \big([n] \mapsto \pi_0(U_{n,n})\big),$$

natural in the hypercover $U_{\bullet,\bullet} \to X_\bullet$. Composing with the canonical map to the p-completion and taking homotopy limit over hypercovers of X, we obtain the desired zig-zag as

$$\mathrm{Sing}^{\mathrm{an}}(X) \overset{\simeq}{\Leftarrow} \left(\underset{U \in \mathrm{HRR}(X)}{\mathrm{holim}}\ \mathrm{Sing}^{\mathrm{an}}(U) \right) \longrightarrow \mathrm{\acute{E}t}_p(X).$$

Together with the canonical map $\mathrm{Sing}^{\mathrm{an}}_0(X) \to \mathrm{Sing}^{\mathrm{an}}(X)$, this finishes the construction of the diagram (B.1) of realization functors.

B.2. **Betti realization of $\mathcal{A}_{g,\mathbf{C}}$.** Let us finally establish the last desideratum, item (iv), asserting that the Betti realization of the simplicial variety arising from an atlas $U \to \mathcal{A}_{g,\mathbf{C}}$ is a model for $B\mathrm{Sp}_{2g}(\mathbf{Z})$.

Example B.5. Let $U, V \in \mathrm{Var}_{\mathbf{C}}$ and let $f : U \to V$ be a smooth surjection. Then $U(\mathbf{C})^{\mathrm{an}}$ and $V(\mathbf{C})^{\mathrm{an}}$ are smooth manifolds and $f^{\mathrm{an}} : U(\mathbf{C})^{\mathrm{an}} \to V(\mathbf{C})^{\mathrm{an}}$ is a surjective submersion in the differential geometric sense. Then we can form an object $U_\bullet \in s\mathrm{Var}_{\mathbf{C}}$ by letting U_n be the n-fold fiber product of U over V. Taking analytic space commutes with fiber products, so $(U_\bullet(\mathbf{C}))^{\mathrm{an}} \to V(\mathbf{C})^{\mathrm{an}}$ is also the simplicial object arising from iterated fiber products of the surjective submersion $U(\mathbf{C})^{\mathrm{an}} \to V(\mathbf{C})^{\mathrm{an}}$. It follows that

$$|U_\bullet(\mathbf{C})^{\mathrm{an}}| \to V(\mathbf{C})^{\mathrm{an}}$$

has contractible point fibers, and standard arguments show that it is a Serre fibration. Hence

$$\mathrm{Sing}^{\mathrm{an}}(U_\bullet) \to \mathrm{Sing}^{\mathrm{an}}(V)$$

is a weak equivalence, too.

Example B.6. Let X_g be the simplicial variety arising from an atlas $U \to \mathcal{A}_g$, or even just a smooth surjective map, i.e. $X_g([n])$ is the $(n+1)$-fold iterated fiber product of U over $\mathcal{A}_g$. If $U' \to \mathcal{A}_g$ is another smooth surjection, then they may be compared using the bisimplicial variety $([n], [m]) \mapsto X_g([n]) \times_{\mathcal{A}_g} X_g'([m])$. By Example B.5, the projection

$$\mathrm{Sing}^{\mathrm{an}}(X_g \times_{\mathcal{A}_g} X_g'([m])) \to \mathrm{Sing}^{\mathrm{an}}(X_g'([m]))$$

is a weak equivalence, and hence the same holds after taking geometric realization in the m-direction. We deduce

$$\mathrm{Sing}^{\mathrm{an}}(X_g) \xleftarrow{\simeq} \mathrm{Sing}^{\mathrm{an}}(X_g'') \xrightarrow{\simeq} \mathrm{Sing}^{\mathrm{an}}(X_g'),$$

where X_g'' is the simplicial variety obtained by iterated fiber products of $U_g \times_{\mathcal{A}_g} U_g' \to \mathcal{A}_g$.

Then $\mathrm{Sing}^{\mathrm{an}}(X_g)$ is a model for $B\mathrm{Sp}_{2g}(\mathbf{Z})$. Indeed, we may use the quasiprojective variety $\mathcal{A}_g(N)$ (the $\Gamma_g(N) := \ker(\mathrm{Sp}_{2g}(\mathbf{Z}) \to \mathrm{Sp}_{2g}(\mathbf{Z}/N))$-cover of $\mathcal{A}_g$, which parametrizes a trivialization of the N-torsion) as atlas for $N \geq 4$. The simplicial variety arising from the atlas $\mathcal{A}_g(N) \to \mathcal{A}_g$ is isomorphic to the Borel construction of $\mathrm{Sp}_{2g}(\mathbf{Z}/N)$ acting on $\mathcal{A}_g(N)$. The action of $\mathrm{Sp}_{2g}(\mathbf{Z}/N)$ on the space $(\mathcal{A}_g(N))^{\mathrm{an}} \cong \mathbb{H}_g/\Gamma_g(N)$ is the canonical one arising from the extension of the action of $\Gamma_g(N) < \mathrm{Sp}_{2g}(\mathbf{Z})$, so we get

$$\mathrm{Sing}^{\mathrm{an}}(X_g) = \mathrm{Sing}^{\mathrm{an}}(\mathcal{A}_g(N) /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z}/N)) = \mathrm{Sing}^{\mathrm{an}}(\mathcal{A}_g(N)) /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z}/N)$$
$$= (\mathrm{Sing}(\mathbb{H}_g)/\mathrm{Sp}_{2g}(\mathbf{Z}, N)) /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z}/N).$$

Here "$/\!\!/$" denotes the Borel construction (homotopy orbits): explicitly, when group G acts on a X, we write $X /\!\!/ G$ for the usual simplicial object with n-simplices $G^n \times X$. At the last step we used the fact that, since the quotient $\mathbb{H}_g \to \mathcal{A}_g(N)$ is a covering map, there is an isomorphism of simplicial sets $\mathrm{Sing}^{\mathrm{an}}(\mathcal{A}_g(N)) \cong (\mathrm{Sing}(\mathbb{H}_g))/\Gamma_g(N)$.

We may then finally use that $\mathbb{H}_g$ is contractible, replace it by a point and the quotient by $\Gamma_g(N)$ by the homotopy quotient:

$$(\mathrm{Sing}(\mathbb{H}_g)/\Gamma_g(N)) /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z}/n) \xleftarrow{\simeq} (E\mathrm{Sp}_{2g}(\mathbf{Z}) \times \mathrm{Sing}(\mathbb{H}_g)) /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z}) \xrightarrow{\simeq} B\mathrm{Sp}_{2g}(\mathbf{Z}).$$

Example B.7. For later use, let us also remark that the same map induces an equivalence of groupoids

$$\mathrm{Sing}_0^{\mathrm{an}}(X_g) \xrightarrow{\simeq} N(\mathcal{A}_g(\mathbf{C})), \tag{B.6}$$

where the domain is the simplicial set obtained by taking $\mathbf{C}$ points levelwise in the simplicial variety X_g, and the codomain denotes the nerve of the groupoid whose objects are rank g principally polarized abelian varieties $(A, \mathcal{L})$ over $\mathrm{Spec}(\mathbf{C})$ and whose morphisms are isomorphisms

of such. By uniformization, we also have an equivalence

$$\mathbb{H}_g^\delta /\!/ \mathrm{Sp}_{2g}(\mathbf{Z}) \xrightarrow{\simeq} N(\mathcal{A}_g(\mathbf{C})),$$

where $\mathbb{H}_g^\delta$ denotes the Siegel upper half space in the discrete topology. The equivalence is induced by the usual construction, sending a symmetric matrix Ω with positive imaginary part to the abelian variety $\mathbf{C}^g/(\mathbf{Z}^g + \Omega \mathbf{Z}^g)$ in the usual principal polarization.

To summarize, the diagram (B.1) for $X = X_g$ becomes a model for the evident maps

$$\mathbb{H}_g^\delta /\!/ \mathrm{Sp}_{2g}(\mathbf{Z}) \to \mathbb{H}_g /\!/ \mathrm{Sp}_{2g}(\mathbf{Z}) \to \left(\mathbb{H}_g /\!/ \mathrm{Sp}_{2g}(\mathbf{Z})\right)_p^\wedge,$$

where the first map is induced by the identity map of Siegel upper half space, from the discrete to the Euclidean topology. The composition is our $\mathrm{Aut}(\mathbf{C})$-equivariant model for

$$|N(\mathcal{A}_g(\mathbf{C}))| \to (B\mathrm{Sp}_{2g}(\mathbf{Z}))_p^\wedge.$$

Example B.6 shows that we may use $\mathcal{A}_g$ to realize $B\mathrm{Sp}_{2g}(\mathbf{Z})$ as the Betti realization of a simplicial variety defined over $\mathbf{Q}$, and hence construct an $\mathrm{Aut}(\mathbf{C})$-action on its p-completion (at least for $g \geq 3$ where $\mathrm{Sp}_{2g}(\mathbf{Z})$ is perfect). It remains to see that this structure is compatible with the structure which constructs the spectrum $\mathrm{KSp}(\mathbf{Z})$ out of the $B\mathrm{Sp}_{2g}(\mathbf{Z})$, i.e. the Γ-space structure.

B.3. **A Gamma-object in simplicial varieties.** In this section we use the moduli stacks $\mathcal{A}_g$ to define a functor from Γ^{op} to simplicial complex varieties, such that the composition

$$Z \colon \Gamma^{\mathrm{op}} \to s\mathrm{Var}_{\mathbf{C}} \xrightarrow{\mathrm{Sing}^{\mathrm{an}}} s\mathrm{Sets}$$

is naturally homotopy equivalent to $T \mapsto |\mathcal{SP}_T(\mathbf{Z})|$. We first discuss how to construct a functor $T \mapsto \mathcal{A}(T) \simeq (\coprod_{g \geq 0} \mathcal{A}_g)^{T\setminus\{*\}}$ from Γ^{op} to groupoids, modeled on how we defined $T \mapsto \mathcal{SP}_T(\mathbf{Z})$.

To avoid excessive notation, let us agree that for a scheme S we denote objects of $\mathcal{A}_g(S)$ like $(A, \mathcal{L})$, where A is an abelian scheme over S and $\mathcal{L}$ is a principal polarization. On the set level, A is an abbreviation for a scheme A and maps of schemes $\pi : A \to S$ and $e : S \to A$, with the property that they make A into a rank g abelian scheme over S with identity section e. Similarly, $\mathcal{L}$ is an abbreviation for a line bundle $\mathcal{L}$ on $A \times_S A$, rigidified by non-zero section i of $\mathcal{L}$ over $A \times_S \{e\} \hookrightarrow A \times_S A$ and i' over $\{e\} \times_S A \hookrightarrow A \times_S A$ agreeing with i over $(e, e) : S \to A \times_S A$, with the property that $(\mathcal{L}, i, i')$ is symmetric under swapping the two factors of A, the restriction $\Delta^* \mathcal{L}$ along the diagonal $\Delta : A \to A \times_S A$ is ample, and the morphism $A \to A^\vee$ induced by $\mathcal{L}$ is an isomorphism. We shall say "$(A, \mathcal{L})$ is a principally polarized abelian variety over S" to mean that we are given all this data for some $g \geq 0$.

For each finite pointed set T we let $\mathcal{A}(T)$ denote the category whose objects are $(A, \mathcal{L}, \phi)$ where $(A, \mathcal{L})$ is a principally polarized abelian variety over S, which is a scheme over $\mathrm{Spec}\,(\mathbf{Q})$, and $\phi : \mathbf{Z}^T \to \mathrm{End}(A)$ is a ring homomorphism, with the property that $\mathcal{L}$ restricts to a principal polarization on the abelian subvarieties $A_t \subset A$, defined as $A_t = \mathrm{Ker}(1 - \phi(e_t)) \subset A$ for all $t \in T$. For $e = \sum_{t \in T\setminus\{*\}} e_t$ we similarly have $\mathrm{Ker}(1 - \phi(e)) \subset A$, which we shall denote eA. Addition in the group structure on A defines an isomorphism of abelian varieties $\oplus_{t \neq *} A_t \to eA$. We now define morphisms in $\mathcal{A}(T)$ to be isomorphisms of abelian schemes $eA \to eA'$ restricting to isomorphisms between the A_t for all $t \in T$ and preserving polarizations. Forgetting everything but S makes this category $\mathcal{A}(T)$ fibered in groupoids over the category of schemes over $\mathrm{Spec}\,(\mathbf{Q})$, and the forgetful map

$$\mathcal{A}(T) \to \prod_{t \in T\setminus\{*\}} \left(\coprod_{g=0}^\infty \mathcal{A}_g \right) \tag{B.7}$$

$$(A, \mathcal{L}, \phi) \mapsto \left((A_t, \mathcal{L}_{|A_t \times A_t}) \right)_{t \in T\setminus\{*\}}$$

defines an equivalence of stacks over $\mathrm{Spec}\,(\mathbf{Q})$. Moreover, the association $T \mapsto \mathcal{A}(T)$ defines a functor from Γ^{op} to (the 1-category of) such fibered categories: functoriality is again by precomposing the map $\mathbf{Z}^T \to \mathrm{End}(A)$.

To turn $\mathcal{A}(T)$ into a simplicial scheme we rigidify the objects. To be specific, let us take $U(T)$ to be a scheme classifying the functor which sends $(S \to \mathrm{Spec}\,(\mathbf{Q}))$ to the set of tuples $(A, \mathcal{L}, \phi, j)$, where $(A, \mathcal{A}, \phi) \in \mathcal{A}(T)$ as above, and $j : A \hookrightarrow \mathbb{P}_S^{N-1}$ is an embedding such that $\mathcal{O}(1)$ restricts to $3\Delta^*(\mathcal{L})$ on A. This functor is represented by a locally closed subscheme of a finite product of Hilbert schemes, and hence is quasi-projective over $\mathrm{Spec}\,(\mathbf{Q})$, as in [MFK94b, Chapter 6]. Finally, we extract a simplicial scheme $Z(T)$ from the map $U(T) \to \mathcal{A}(T)$ by taking iterated fiber products. Then nth space classifies $(n+1)$-tuples $(A_0, \ldots, A_n)$ of abelian schemes over S, each equipped principal polarizations $\mathcal{L}_i$ and with embeddings $j_i : A_i \subset \mathbf{P}_S^{N_i-1}$ as above and ring homomorphisms $\phi_i : \mathbf{Z}^T \to \mathrm{End}(A_i)$, defining principally polarized abelian subvarieties $eA_i = \mathrm{Ker}(1 - \phi_i(e)) \subset A_i$, as well as isomorphisms of abelian varieties

$$eA_0 \xrightarrow{\cong} eA_1 \xrightarrow{\cong} \ldots \xrightarrow{\cong} eA_n$$

preserving polarizations (but no compatibility imposed on projective embeddings).

Proposition B.8. *There is a zig-zag of weak equivalences of simplicial sets*

$$\mathrm{Sing}^{\mathrm{an}}(\mathrm{Spec}\,(\mathbf{C}) \times_{\mathrm{Spec}\,(\mathbf{Q})} Z(T)) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} N_\bullet(\mathcal{SP}_T(\mathbf{Z}))$$

natural in $T \in \Gamma^{\mathrm{op}}$.

In particular, $\mathrm{Sing}^{\mathrm{an}}(\mathrm{Spec}\,(\mathbf{C}) \times_{\mathrm{Spec}\,(\mathbf{Q})} Z(S^0)) \simeq \coprod_g N\mathrm{Sp}_{2g}(\mathbf{Z})$.

Proof sketch. We have explained a smooth surjection $U(T) \to \mathcal{A}(T) \xrightarrow{\simeq} (\coprod_g \mathcal{A}_g)^{T \setminus \{*\}}$, which up to equivalence may be rewritten as a coproduct of smooth surjections into stacks of the form $\mathcal{A}_{g_1} \times \cdots \times \mathcal{A}_{g_m}$. After base changing to $\mathrm{Spec}\,(\mathbf{C})$ all simplicial varieties arising are quasi-projective over $\mathrm{Spec}\,(\mathbf{C})$. The weak equivalence now follows by an argument similar to Example B.6, which can also be used to produce an explicit zig-zag. Since all constructions are strictly functorial in $T \in \Gamma^{\mathrm{op}}$, so is the resulting $T \mapsto Z(T)$. $\qquad\square$

Taking complex points (in the discrete topology) of $\mathcal{A}_g$ gives the groupoid $\mathcal{A}_g(\mathbf{C})$ whose objects are $(A, \mathcal{L})$, principally polarized abelian varieties over $\mathrm{Spec}\,(\mathbf{C})$, and whose morphisms are isomorphisms of such. In this groupoid all automorphism groups are finite, but it has continuum many isomorphism classes of objects for $g > 0$. Hence $|\mathcal{A}_g(\mathbf{C})|$ is a disjoint union of continuum many $K(\pi, 1)$'s for finite groups. The equivalence (B.7) for $T = S^0$ implies a weak equivalence of simplicial sets

$$\mathrm{Map}(\mathrm{Spec}\,(\mathbf{C}), Z(S^0)) \xrightarrow{\simeq} \coprod_{g \geq 0} N(\mathcal{A}_g(\mathbf{C})) \tag{B.8}$$

as in Example B.7. This, and the $\mathrm{Aut}(\mathbf{C})$-equivariant map to $\acute{\mathrm{E}}\mathrm{t}_p(Z(S^0))$, will eventually lead to commutativity of the diagram (6.2).

B.4. **Galois action on symplectic K-theory.** We finally construct the promised action of the group $\mathrm{Aut}(\mathbf{C})$ on the spectrum $\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p)$. For simplicity we first assume $p > 3$, which has the convenient effect that $H_{\mathrm{et}}^1(\mathcal{A}_g; \mathbf{F}_p) = H^1(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{F}_p) = 0$ for all g. (For $p = 3$ this fails for $g = 1$ and for $p = 2$ it fails for $g = 1$ and $g = 2$. A mild variation of the argument applies also in those two cases; see below.)

First, recall that we described a composite functor

$$\Gamma^{\mathrm{op}} \xrightarrow{Z} s\mathrm{Var}_{\mathbf{Q}} \xrightarrow{-\otimes_{\mathbf{Q}} \mathbf{C}} s\mathrm{Var}_{\mathbf{C}},$$

whose composition with $\mathrm{Sing}^{\mathrm{an}}$ is equivalent to the Γ-space delooping $\mathrm{KSp}(\mathbf{Z})$. We get a zig-zag of maps between Γ-spaces

$$\mathrm{Sing}^{\mathrm{an}}(Z) \xleftarrow{\simeq} \ldots \xrightarrow{\mathrm{comparison}} \acute{\mathrm{E}}\mathrm{t}_p(Z)$$

which has the property that evaluated on any object $S \in \Gamma$ it induces an isomorphism in mod p cohomology. In addition to vanishing $H^1(-; \mathbf{F}_p)$, this requires that $H^*(B\mathrm{Sp}_{2g}(\mathbf{Z}); \mathbf{F}_p)$ is finite-dimensional in each degree, which is well known.

Now choose a simplicial set modeling a Moore space $M(\mathbf{Z}/p^k, 2)$ for all k, and choose maps $M(\mathbf{Z}/p^{k+1}, 2) \to M(\mathbf{Z}/p^k, 2)$ corresponding to reduction modulo p^k. Then we get an induced functor

$$\Gamma^{\mathrm{op}} \xrightarrow{M(\mathbf{Z}/p^k, 2) \wedge Z} s\mathrm{Var}_{\mathbf{Q}} \xrightarrow{-\otimes_{\mathbf{Q}} \mathbf{C}} s\mathrm{Var}_{\mathbf{C}},$$

giving rise to two Γ-spaces by applying $\mathrm{Sing}^{\mathrm{an}}$ or $\acute{\mathrm{E}}\mathrm{t}_p$, and a zig-zag

$$B^\infty(M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Sing}^{\mathrm{an}}(Z \otimes_{\mathbf{Q}} \mathbf{C})) \xleftarrow{\simeq} \ldots \xrightarrow{\mathrm{comparison}} B^\infty(M(\mathbf{Z}/p^k, 2) \wedge \acute{\mathrm{E}}\mathrm{t}_p(Z \otimes_{\mathbf{Q}} \mathbf{C})).$$

where B^∞ is as in (A.5). Since any mod p homology isomorphism becomes a weak equivalence after smashing with $M(\mathbf{Z}/p^k, 2)$, and since the comparison map comes from a map of Γ-spaces which is a mod p homology equivalence when evaluated on any $S \in \Gamma^{\mathrm{op}}$ as long as $p > 3$, we have produced a weak equivalence of spectra

$$M(\mathbf{Z}/p^k, 2) \wedge \mathrm{KSp}(\mathbf{Z}) \simeq B^\infty(M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Sing}^{\mathrm{an}}(Z \otimes_{\mathbf{Q}} \mathbf{C})) \xleftarrow{\simeq} \ldots \xrightarrow{\simeq} B^\infty(M(\mathbf{Z}/p^k, 2) \wedge \acute{\mathrm{E}}\mathrm{t}_p(Z \otimes_{\mathbf{Q}} \mathbf{C})).$$

Desuspending twice and taking homotopy inverse limit over k, we get

$$\mathrm{KSp}(\mathbf{Z}; \mathbf{Z}_p) \simeq \operatorname*{holim}_k (\mathbb{S}/p^k) \wedge \mathrm{KSp}(\mathbf{Z}) \simeq \operatorname*{holim}_k M(\mathbf{Z}/p^k) \wedge B^\infty(\acute{\mathrm{E}}\mathrm{t}_p(Z \otimes_{\mathbf{Q}} \mathbf{C})).$$

But by functoriality of the delooping machine, the group $\mathrm{Aut}(\mathbf{C})$ manifestly acts by spectrum maps on $B^\infty(\acute{\mathrm{E}}\mathrm{t}_p(Z \otimes_{\mathbf{Q}} \mathbf{C}))$. Hence this equivalence can be viewed as a homotopy action on the p-completed symplectic K-theory spectrum, and in particular it constructs an action on homotopy groups

$$\mathrm{KSp}_n(\mathbf{Z}; \mathbf{Z}/p^k) \cong \pi_n((\mathbb{S}/p^k) \wedge B^\infty(\acute{\mathrm{E}}\mathrm{t}_p(Z \otimes_{\mathbf{Q}} \mathbf{C})))$$
$$\mathrm{KSp}_n(\mathbf{Z}; \mathbf{Z}_p) \cong \varprojlim_k \pi_n((\mathbb{S}/p^k) \wedge B^\infty(\acute{\mathrm{E}}\mathrm{t}_p(Z \otimes_{\mathbf{Q}} \mathbf{C}))).$$

Remark B.9. For $p \leq 3$ a mild variant of the argument works. The only problem with those small primes was that $\mathcal{A}_g$ has non-trivial mod p cohomology for small g, which prevents us from controlling the mod p cohomology of the inverse limit involved in forming $\acute{\mathrm{E}}\mathrm{t}_p$. But smashing with $M(\mathbf{Z}/p^k, 2)$ makes any simplicial set be simply connected, so if we do that operation before taking $\acute{\mathrm{E}}\mathrm{t}_p$ there is nothing special about the small primes. As a side-effect, this version of the argument will not make use of the calculation $H_1(B\mathrm{Sp}_{2g}(\mathbf{Z}))$.

Let us also discuss a certain compatibility of actions which was used in the proof of Proposition 6.2. Namely, as an instance of the spectrum map (A.9) we have

$$\Sigma^\infty(M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Et}_p(Z(S^0) \otimes_{\mathbf{Q}} \mathbf{C})) \to B^\infty(M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Et}_p(Z \otimes_{\mathbf{Q}} \mathbf{C})),$$

extracted functorially from the Γ-space $T \mapsto M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Et}_p(Z(T) \otimes_{\mathbf{Q}} \mathbf{C})$, and hence equivariant for the $\mathrm{Aut}(\mathbf{C})$-action. By the argument of Example B.7 above, the map of simplicial sets

$$\mathrm{Sing}^{\mathrm{an}}_0(Z(S^0) \otimes_{\mathbf{Q}} \mathbf{C}) \to \mathrm{Et}_p(Z(S^0) \otimes_{\mathbf{Q}} \mathbf{C})$$

is also equivariant. Hence we get an equivariant map of spectra

$$\Sigma^\infty(M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Sing}^{\mathrm{an}}_0(Z(S^0) \otimes_{\mathbf{Q}} \mathbf{C})) \to B^\infty(M(\mathbf{Z}/p^k, 2) \wedge \mathrm{Et}_p(Z \otimes_{\mathbf{Q}} \mathbf{C})).$$

Shifting degrees by 2 and taking homotopy groups we get a homomorphism

$$\pi_n^s(\mathrm{Sing}_0^{\mathrm{an}}(Z(S^0) \otimes_{\mathbf{Q}} \mathbf{C})); \mathbf{Z}/p^k) \to \mathrm{KSp}_n(\mathbf{Z}/\mathbf{Z}/p^k),$$

which is equivariant for the action constructed above. Now finally, the equivalence (B.8) is also $\mathrm{Aut}(\mathbf{C})$-equivariant for the evident action on $\mathcal{A}_g(\mathbf{C})$, i.e. the one changing reference maps $\pi : A \to \mathrm{Spec}\,(\mathbf{C})$ of abelian schemes over $\mathrm{Spec}\,(\mathbf{C})$. Restricting attention to the path component corresponding to abelian varieties of rank g, we have shown that the homomorphism

$$\pi_n^s(|\mathcal{A}_g(\mathbf{C})|; \mathbf{Z}/p^k) \to \mathrm{KSp}_n(\mathbf{Z}/\mathbf{Z}/p^k),$$

induced from mapping $N(\mathcal{A}_g(\mathbf{C})) \simeq \mathbb{H}_g^\delta /\!\!/ \mathrm{Sp}_{2g}(\mathbf{Z}) \to B\mathrm{Sp}_{2g}(\mathbf{Z})$, is equivariant for $\mathrm{Aut}(\mathbf{C})$. This is the commutativity of the diagram (6.2) as claimed in the proof of Proposition 6.2.

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