Introductory talk on Arthur packets

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Abstract

As preparation for the Fields Institute Number Theory Seminar on *Arthur packets for unipotent representations of the p-adic exceptional group* G_2, this introductory talk recalls the main properties of Arthur packets, or A-packets, for quasi-split classical groups G over p-adic fields F, including a review of the local Langlands correspondence and the endoscopic classification of admissible representations of Arthur type for these groups.

Motivation

The main local result from Arthur's book *The Endoscopic Classification of Representations: Orthogonal and Symplectic groups* [Art13]:

THEOREM 1.5.1. *Assume that F is local and that $G \in \widetilde{\mathcal{E}}_{\mathrm{sim}}(N)$.*

(a) *For any local parameter $\psi \in \widetilde{\Psi}(G)$, there is a finite set $\widetilde{\Pi}_\psi$ over $\widetilde{\Pi}_{\mathrm{unit}}(G)$, constructed from ψ by endoscopic transfer, and equipped with a canonical mapping*

$$\pi \longrightarrow \langle \cdot, \pi \rangle, \qquad \pi \in \widetilde{\Pi}_\psi,$$

from $\widetilde{\Pi}_\psi$ into the group $\widehat{\mathcal{S}}_\psi$ of characters on $\mathcal{S}_\psi$ such that $\langle \cdot, \pi \rangle = 1$ if G and π are unramified (relative to K_F).

(b) *If $\phi = \psi$ belongs to the subset $\widetilde{\Phi}_{\mathrm{bdd}}(G)$ of parameters in $\widetilde{\Psi}(G)$ that are trivial on the factor $SU(2)$, the elements in $\widetilde{\Pi}_\phi$ are tempered and multiplicity free, and the corresponding mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is injective. Moreover, every element in $\widetilde{\Pi}_{\mathrm{temp}}(G)$ belongs to exactly one packet $\widetilde{\Pi}_\phi$. Finally, if F is nonarchimedean, the mapping from $\widetilde{\Pi}_\phi$ to $\widehat{\mathcal{S}}_\phi$ is bijective.*

The local Langlands Correspondence/Conjecture

Let G be a connected reductive linear algebraic group over a p-adic field F. The LLC claims that here is a surjection

$$\begin{array}{ccc} \Pi(G(F)) & \rightarrow & \Phi(G(F)) \\ \pi & \mapsto & \phi_\pi \end{array}$$

where $\Pi(G(F))$ is the set of equivalence classes of irreducible admissible representations of $G(F)$ and $\Phi(G(F))$ is the set of equivalence classes of Langlands parameters $\phi : W'_F \rightarrow {}^LG$.

The LLC also includes the demand that this surjection satisfies many pleasing properties, derived mainly from compatibility with class field theory and the principle of functoriality, especially the theory of endoscopy.

L-packets

Among these properties: the L-packet $\Pi_\phi(G(F))$, which is defined as the fibre above ϕ, may be parametrized by irreducible representations of the finite group

$$S_\phi := Z_{\widehat{G}}(\phi)/Z_{\widehat{G}}(\phi)^\circ Z(\widehat{G})^{\Gamma_F}.$$

Consequently it is common to work with "enhanced" Langlands parameters and seek a bijection

$$\begin{array}{rcl} \Pi(G(F)) & \to & \Phi_{\mathrm{enh}}(G(F)) \\ \pi & \mapsto & (\phi_\pi, \rho_\pi) \end{array}$$

where ρ_π is an irreducible representation of S_ϕ and $\Phi_{\mathrm{enh}}(G(F))$ is the set of equivalence classes of such pairs (ϕ, ρ).

L-packet coefficients

Another perspective on the Langlands correspondence is that it determines a bijection

$$\begin{array}{rcl} \Pi_\phi(G(F)) & \to & \widehat{S_\phi} \\ \pi & \mapsto & \langle\ ,\pi\rangle_\phi \end{array}$$

and thus, through the characters of these representations, *L-packet coefficients*

$$\begin{array}{rcl} S_\phi \times \Pi_\phi(G(F)) & \to & \mathbb{C} \\ (s,\pi) & \mapsto & \langle s,\pi\rangle_\phi \end{array}$$

Some properties of L-packets and L-packet coefficients

1. If $\phi : W_F' \to {}^L G$ is bounded upon restriction to W_F then all the representations in $\Pi_\phi(G(F))$ are tempered and all tempered representations arise in this way.
2. If ϕ is bounded upon restriction to W_F then there is an element $s_\phi \in S_\phi$ such that
$$\Theta_\phi := \sum_{\pi \in \Pi_\phi(G(F))} \langle s_\phi, \pi \rangle_\phi \ \Theta_\pi$$
is a *stable* distribution, where Θ_π is the Harish-Chandra distribution character of π.
3. If π is spherical, *a.k.a.* unramified, then $\langle \ , \pi \rangle_\phi = \mathbb{1}_{S_{\phi_\pi}}$, the trivial character of S_{ϕ_π}.

Non-tempered representations and the *sine qua non* of A-packets

If $\phi : W_F' \to {}^L G$ is not bounded upon restriction to W_F then Θ_ϕ is, typically, not stable and no non-trivial stable distribution may be formed from the characters Θ_π for $\pi \in \Pi_\phi(G(F))$.

A-packets are designed to remedy this defect of L-packets: every A-packet $\Pi_\psi(G(F))$ is an L-packet $\Pi_{\phi_\psi}(G(F))$ together with other irreducible admissible representations, designed so that it's possible to form a non-trivial stable distribution from the characters in the A-packet.

Stabilizing representations

We refer to the extra representations appearing in the A-packet $\Pi_\psi(G(F))$ but not in the L-packet $\Pi_{\phi_\psi}(G(F))$ as *stabilizing representations*.

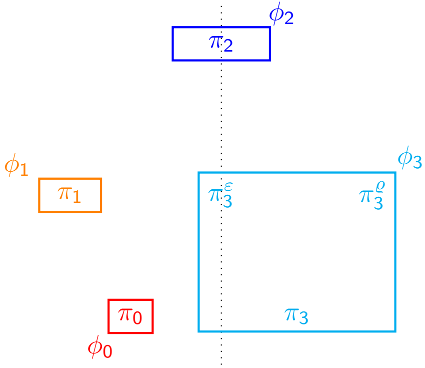
To define the stable distribution attached to an A-packet $\Pi_\psi(G(F))$ it will be necessary to *extend* the definition of L-packet coefficients from $\Pi_{\phi_\psi}(G(F))$ to A-packet coefficients, including the stabilizing representations.

$$\begin{aligned}
\Theta_\psi \quad &:= \sum_{\pi\in\Pi_\psi(G(F))} \langle s_\psi, \pi\rangle_\psi \ \Theta_\pi \\
&= \sum_{\pi\in\Pi_{\phi_\psi}(G(F))} \langle s_{\phi_\psi}, \pi\rangle_{\phi_\psi} \ \Theta_\pi \\
&\qquad + \underbrace{\sum_{\pi\in\Pi_\psi(G(F))\setminus\Pi_{\phi_\psi}(G(F))}}_{\text{stabilizing representations}} \langle s_\psi, \pi\rangle_\psi \ \Theta_\pi .
\end{aligned}$$

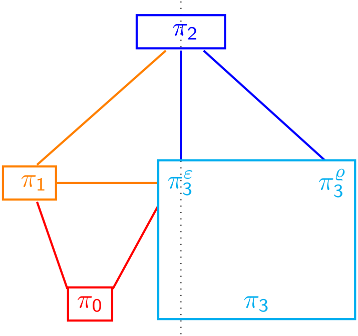
There remains work to be done to make sense of this!

Example: $\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$ partitioned into L-packets

Taken from [CFZa], with the disclaimer that these are *not* examples of Arthur's work.

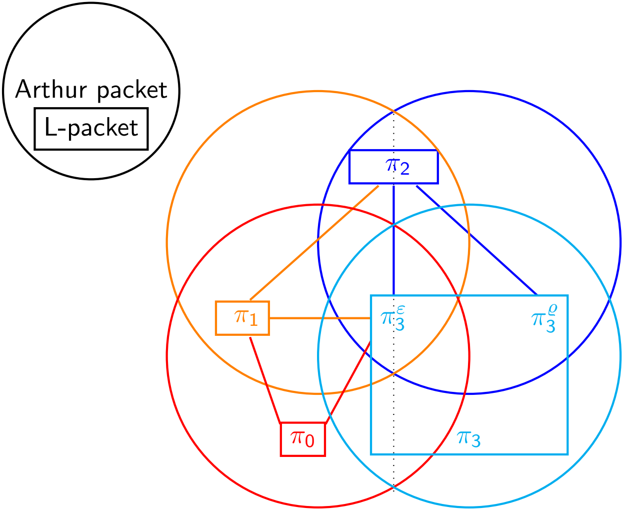


Example: Stabilizing representations for L-packets in $\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$



Remark: π_3^{ε} is depth-zero, supercuspidal and unipotent.

Example: A-packets and their L-packets for $\mathrm{Rep}(G_2(F))_{\mathrm{sub}}$



Arthur parameters

In order to be more precise, it's necessary to define *Arthur parameters*.

$$
\begin{array}{lcl}
W_F & : & \text{Weil group for the } p\text{-adic field } F \\
W'_F & : & \text{a version of the Weil-Deligne group for the } p\text{-adic field } F \\
 & & W_F := W_F \times \mathrm{SL}_2(\mathbb{C}) \\
W''_F & := & W'_F \times \mathrm{SL}_2(\mathbb{C}) = W_F \times \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})
\end{array}
$$

Definition

An Arthur parameter is a group homomorphism $\psi : W''_F \to {}^LG$ such that:

1. $\psi|_{W'_F} : W'_F \to {}^LG$ is a Langlands parameter;
2. $\psi|_{W_F} : W_F \to {}^LG$ is bounded in $\widehat{G}$;
3. $\psi|_{\mathrm{SL}_2(\mathbb{C})} : \mathrm{SL}_2(\mathbb{C}) \to \widehat{G}$ is algebraic.

Langlands parameters of Arthur type

Every Arthur parameter $\psi : W''_F \to {}^LG$ determines a Langlands parameter $\phi_\psi : W'_F \to {}^LG$ by

$$\phi_\psi(w,x) := \psi\left(w, x, \begin{pmatrix} |w|^{1/2} & 0 \\ 0 & |w|^{-1/2} \end{pmatrix}\right).$$

The function

$$\begin{array}{ccc} \Psi(G(F)) & \to & \Phi(G(F)) \\ \psi & \mapsto & \phi_\psi \end{array}$$

is injective but not surjective.

Langlands parameters in the image of this function are said to be of *Arthur type*. An irreducible admissible representation $\pi \in \Pi(G(F))$ is said to be of *Arthur type* if its Langlands parameter ϕ_π is of Arthur type.

Yes, these definitions are highly unsatisfactory!!!

A-packet coefficients

Let $\psi : W''_F \to {}^L G$ be an Arthur parameter. Set

$$S_\psi := Z_{\widehat{G}}(\psi)/Z_{\widehat{G}}(\psi)^\circ Z(\widehat{G})^{\Gamma_F}.$$

Arthur uses the theory of endoscopy and twisted endoscopy from general linear groups to define/characterize a finite set $\Pi_\psi(G(F))$ called an A-packet, containing the L-packet for ϕ_ψ,

$$\Pi_\psi(G(F)) \supseteq \Pi_{\phi_\psi}(G(F))$$

and A-packet coefficients

$$\langle \ , \ \rangle_\psi : S_\psi \times \Pi_\psi(G(F)) \to \mathbb{C}$$

such that, for each $\pi \in \Pi_\psi(G(F))$, the function $s \mapsto \langle s, \pi \rangle_\psi$ is the character of an irreducible representations of S_ψ, and other pleasing properties...

Stable distributions attached to Arthur parameters

Theorem (Arthur, [Art13])

For $s_\psi := \psi(1,1,-1) \in Z_{\widehat{G}}(\psi)$, the following distribution is stable.

$$\Theta_\psi := \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi, \pi \rangle_\psi \ \Theta_\pi$$

With reference to the A-packets in the example [CFZa] above:

$$\begin{aligned}
\Theta_{\psi_0} &= \Theta_{\pi_0} + 2\Theta_{\pi_1} + \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_1} &= \Theta_{\pi_1} - \Theta_{\pi_2} + \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_2} &= \Theta_{\pi_2} - \Theta_{\pi_3^\varrho} - \Theta_{\pi_3^\varepsilon} \\
\Theta_{\psi_3} &= \Theta_{\pi_3} + 2\Theta_{\pi_3^\varrho} + \Theta_{\pi_3^\varepsilon}
\end{aligned}$$

Langlands Correspondence revisited

Theorem

The following diagram commutes,

$$\begin{tikzcd}
\Pi_\psi(G(F)) \arrow[r, "\pi\mapsto\langle\ ,\pi\rangle_\psi"] & \widehat{S_\psi} \\
\Pi_{\phi_\psi}(G(F)) \arrow[u, hook] \arrow[r, "\pi\mapsto\langle\ ,\pi\rangle_{\phi_\psi}", "\text{bijection}"'] & \widehat{S_{\phi_\psi}} \arrow[u]
\end{tikzcd}, \tag{1}$$

Endoscopic transfer

Let $\psi : W''_F \to {}^LG$ be an Arthur parameter. For semisimple $s \in Z_{\widehat{G}}(\psi)$, define

$$\Theta_{\psi,s} := \sum_{\pi \in \Pi_\psi(G(F))} \langle s_\psi s, \pi \rangle_\psi \; \Theta_\pi$$

Let (G', s, ξ) be an endoscopic triple with $\widehat{G'} = Z_{\widehat{G}}(s)^\circ$. Let $\psi' : W''_F \to {}^LG'$ be an Arthur parameter for $G'(F)$.

Theorem ([Art13])

$$\Theta_{\xi\circ\psi',s}(f) = \Theta_{\psi'}(f')$$

where f' is the Langlands-Shelstad transfer of f from $G(F)$ to $G'(F)$.

Endoscopic classification

Theorem ([Art13])

A-packet coefficients

$$\begin{array}{rcl} \Pi_\psi(G(F)) & \rightarrow & \widehat{S_\psi} \\ \pi & \mapsto & \langle\ ,\pi\rangle_\psi \end{array}$$

are uniquely characterized by the endoscopic transfer theorem above.

Tempered representations revisited

Tempered representations are of Arthur type.

If $\psi : W''_F \to {}^LG$ is bounded upon restriction to W_F then

$$\Pi_\psi(G(F)) = \Pi_{\phi_\psi}(G(F)),$$

all the representations in this A/L-packet are tempered and all tempered representations arise appear in such A/L-packets.

1. In general,
$$\Pi_\psi(G(F)) \to \widehat{S_\psi}$$
is neither injective nor surjective.
2. In some sense, most representations are not of Arthur type.

Concluding remarks

We seek a theory of Arthur packets for all representations, not just those of Arthur type and not just for classical groups.

In the next talk we will see how to apply [CFM+21], building on [Vog93], to the case of the p-adic exceptional group G_2 [CFZa] and [CFZb] and define "ABV-packets" for any unramified Langlands parameter

$$\Pi_\phi^{\mathrm{ABV}}(G(F)) \supset \Pi_\phi(G(F)),$$

and ABV-packet coefficients

$$\Pi_\psi(G(F)) \to \widehat{S_\psi^{\mathrm{ABV}}},$$

and distributions

$$\Theta_\phi^{\mathrm{ABV}},$$

so that, conjecturally, these generalize the constructions above:

$$\Pi_{\phi_\psi}^{\mathrm{ABV}}(G(F)) = \Pi_\psi(G(F)),$$

when ϕ is of Arthur type ψ.

References

[Art13] James Arthur, *The endoscopic classification of representations: Orthogonal and symplectic groups*, American Mathematical Society Colloquium Publications, vol. 61, American Mathematical Society, Providence, RI, 2013.

[CFM+21] Clifton Cunningham, Andrew Fiori, Ahmed Moussaoui, James Mracek, and Bin Xu, *Arthur packets for p-adic groups by way of microlocal vanishing cycles of perverse sheaves, with examples*, Memoirs of the American Mathematical Society (in press) (2021). https://arxiv.org/abs/arXiv:1705.01885.

[CFZa] Clifton Cunningham, Andrew Fiori, and Qing Zhang, *Arthur packets for G(2) and perverse sheaves on cubics*. https://arxiv.org/abs/2005.02438.

[CFZb] ______, *Arthur packets and endoscopy for unipotent representations of the p-adic exceptional group G_2*. (in preparation).

[Vog93] David A. Vogan Jr., *The local Langlands conjecture*, Representation theory of groups and algebras, Contemp. Math., vol. 145, Amer. Math. Soc., Providence, RI, 1993, pp. 305–379.