ON A THEOREM OF BRADEN

V. DRINFELD AND D. GAITSGORY

Dedicated to E. Dynkin

ABSTRACT. We give a new proof of Braden's theorem ([Br]) about *hyperbolic restrictions* of constructible sheaves/D-modules. The main geometric ingredient in the proof is a 1-parameter family that degenerates a given scheme Z equipped with a $\mathbb{G}_m$-action to the product of the attractor and repeller loci.

CONTENTS

INTRODUCTION

0.1. The setting for Braden's theorem.

0.1.1. Given a scheme Z (or algebraic space) of finite type over a field k of characteristic 0, let D-mod(Z) denote the DG category of D-modules on it.

If $f : Z_1 \to Z_2$ is a morphism of such schemes one has the de Rham direct image functor $f_\bullet : \text{D-mod}(Z_1) \to \text{D-mod}(Z_2)$ and the !-pullback functor $f^! : \text{D-mod}(Z_2) \to \text{D-mod}(Z_1)$. One also has the *partially* defined functor $f^\bullet : \text{D-mod}(Z_2) \to \text{D-mod}(Z_1)$, left adjoint to $f_\bullet$.

0.1.2. Suppose now that Z is equipped with an action of the group $\mathbb{G}_m$. Let Z^+ (resp., Z^-) denote the corresponding attractor (resp., repeller) locus, see Sects 1.4 and 1.8 for the definitions. Let Z^0 denote the locus of $\mathbb{G}_m$-fixed points.

Consider the diagram

(0.1)

$$
\begin{array}{ccc}
Z^0 & \xrightarrow{\quad i^+ \quad} & Z^+ \\
\downarrow{\scriptstyle i^-} & & \downarrow{\scriptstyle p^+} \\
Z^- & \xrightarrow{\quad p^- \quad} & Z.
\end{array}
$$

Let $\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \subset \text{D-mod}(Z)$ be the full subcategory consisting of $\mathbb{G}_m$-monodromic[1] objects. In the context of D-modules, Braden's theorem [Br] (inspired by a result[2] from [GM]) says that the composed functors

$$(i^+)^\bullet \circ (p^+)^! \text{ and } (i^-)^! \circ (p^-)^\bullet, \quad \text{D-mod}(Z) \to \text{D-mod}(Z^0)$$

---

[1]The definition of $\mathbb{G}_m$-monodromic object is recalled in Subsect. 3.1.1.

[2]In [GM] M. Goresky and R. MacPherson work in a purely topological setting. They work with correspondences rather than torus actions. According to [GM, Prop. 9.2], under a certain condition (which is satisfied if the correspondence comes from a $\mathbb{G}_m$-action) one has $A_4^\bullet \xrightarrow{\sim} A_5^\bullet$, where $A_i^\bullet$ is defined in [GM, Prop. 4.5]. This is the proptotype of Braden's theorem.

are both defined [3] on objects of D-mod$(Z)^{\mathbb{G}_m\text{-mon}}$ and we have a canonical isomorphism

$$(0.2) \qquad (i^+)^\bullet \circ (p^+)^! |_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \simeq (i^-)^! \circ (p^-)^\bullet |_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \ .$$

0.1.3. In his paper [Br], T. Braden formulated and proved his theorem assuming that Z is a normal algebraic variety. Although his formulation is enough for practical purposes, we prefer to formulate and prove this theorem for *algebraic spaces* of finite type over a field (without any normality or separateness conditions).

In this more general context, the representability of the functors defining the attractor Z^+ (and other related spaces such as $\widetilde{Z}$ from Sect. 0.3.2 below) is no longer obvious; it is established in [Dr].

0.2. **Why should we care?** Braden's theorem is hugely important in geometric representation theory.

0.2.1. Here is a typical application in the context of Lusztig's theory of induction and restriction of character sheaves.

Take $Z = G$, a connected reductive group. Let $P \subset G$ be a parabolic, and let P^- be an opposite parabolic, so that $M := P \cap P^-$ identifies with the Levi quotient of both P and P^-. Denote the corresponding closed embeddings by

$$M \xrightarrow{i^+} P \xrightarrow{p^+} G \text{ and } M \xrightarrow{i^-} P^- \xrightarrow{p^-} G.$$

Then the claim is that we have a canonical isomorphism of functors

$$\text{D-mod}(G)^{\text{Ad}_G\text{-mon}} \to \text{D-mod}(M)^{\text{Ad}_M\text{-mon}}$$

between the corresponding categories of Ad-monodromic D-modules:

$$(0.3) \qquad (i^+)^\bullet \circ (p^+)^! \simeq (i^-)^! \circ (p^-)^\bullet.$$

The proof is immediate from (0.2): the corresponding $\mathbb{G}_m$-action is the adjoint action corresponding to a co-character $\mathbb{G}_m \to M$, which maps to the center of M and is dominant regular with respect to P. [4]

0.2.2. For other applications of Braden's theorem see [Ach], [AC], [AM], [Bi-Br], [GH], [Ly1], [Ly2], [MV], [Nak].

0.3. **The new proof.** The goal of this paper is to give an alternative proof of Braden's theorem. The reason for our decision to publish it is that

(a) the new proof gives another point of view on "what Braden's theorem is really about";

(b) a slight modification of the new proof of Braden's theorem allows to prove a new result in the geometric theory of automorphic forms, see [DrGa3, Thm. 1.2.5].

Let us explain the idea of the new proof.

---

[3]The issue here is that in the context of D-modules the $\bullet$-pullback functor is only partially defined.

[4]Unfortunately, it seems that this particular proof of the isomorphism (0.3), although very simple and well-known in the folklore, does not appear in the published literature.

0.3.1. *Braden's theorem as an adjunction.* Let us complete the diagram (0.1) to

(0.4)

where

$$q^+ : Z^+ \to Z^0 \text{ and } q^- : Z^- \to Z^0$$

are the corresponding contraction maps.

First, we observe that the functors $(i^+)^{\bullet}$ and $(i^-)^!$, when restricted to the corresponding monodromic categories, are isomorphic to $(q^+)_{\bullet}$ and $(q^-)_!$, respectively. Hence, the isomorphism (0.2) can be rewritten as

(0.5) $$(q^+)_{\bullet} \circ (p^+)^!|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \simeq (q^-)_! \circ (p^-)^{\bullet}|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}}.$$

Next we observe that the functor $(q^-)_! \circ (p^-)^{\bullet}$ is the left adjoint functor of $(p^-)_{\bullet} \circ (q^-)^!$. Hence, *the isomorphism* (0.5) *can be restated as the assertion that the functors*

$$(q^+)_{\bullet} \circ (p^+)^!|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \text{ and } (p^-)_{\bullet} \circ (q^-)^!|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}}$$

form an adjoint pair.

0.3.2. *The geometry behind the adjunction.* In turns out that the co-unit for this adjunction, i.e., the map

(0.6) $$(q^+)_{\bullet} \circ (p^+)^! \circ (p^-)_{\bullet} \circ (q^-)^! \to \text{Id}_{\text{D-mod}(Z^0)},$$

is easy to write down (just as in the original form of Braden's theorem, a map in one direction is obvious).

The crux of the new proof consists of writing down the unit for the adjunction, i.e., the corresponding map

(0.7) $$\text{Id}_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \to (p^-)_{\bullet} \circ (q^-)^! \circ (q^+)_{\bullet} \circ (p^+)^!|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}}.$$

The map (0.7) comes from a certain geometric construction described in Sect. 2. Namely, we construct a 1-parameter "family" [5] of schemes (resp., algebraic spaces) $\widetilde{Z}_t$ mapping to $Z \times Z$ (here $t \in \mathbb{A}^1$) such that for $t \neq 0$ the scheme (resp., algebraic space) $\widetilde{Z}_t$ is the graph of the map $t : Z \xrightarrow{\sim} Z$, and $\widetilde{Z}_0$ is isomorphic to $Z^+ \underset{Z^0}{\times} Z^-$.

---

[5]The quotation marks are due to the fact that this "family" is not flat, in general. If Z is affine then each $\widetilde{Z}_t$ is a closed subscheme of $Z \times Z$. If Z is separated, then for each t the map $\widetilde{Z}_t \to Z \times Z$ is a monomorphism (but not necesaarily a locally closed embedding).

0.4. **Other sheaf-theoretic contexts.** This paper is written in the context of D-modules on schemes (or more generally, algebraic spaces of finite type) over a field k of characteristic 0.

However, Braden's theorem can be stated in other sheaf-theoretic contexts, where the role of the DG category D-mod(Z) is played by a certain triangulated category $D(Z)$. The two other contexts that we have in mind are as follows:

(i) k is any field, and $D(Z)$ is the derived category of $\mathbb{Q}_\ell$-sheaves with constructible cohomologies,

(ii) $k = \mathbb{C}$, and $D(Z)$ is the derived category of sheaves of R-modules with constructible cohomologies (where R is any ring).

In these two contexts the new proof of Braden's theorem presented in this article goes through with the following modifications:

First, the functors $f^\bullet$ and $f_!$ are always defined, so one should not worry about pro-categories.[6]

Second, the definition of the G-monodromic category $D(Z)^{G\text{-mon}}$ (where G is any algebraic group, e.g., the group $\mathbb{G}_m$) should be slightly different from the definition of D-mod$(Z)^{G\text{-mon}}$ given in Sect. 3.1.1.

Namely, $D(Z)^{G\text{-mon}}$ should be defined as the full subcategory of $D(Z)$ *strongly generated* by the essential image of the pullback functor $D(Z/G) \to D(Z)$ (i.e., its objects are those objects of $D(Z)$ that can be obtained from objects lying in the image of the above pullback functor by a *finite* iteration of the procedure of taking the cone of a morphism).

0.5. **Some conventions and notation.**

0.5.1. In Sects. 1 and 2 we will work over an arbitrary ground field k, and in Sects. 3-5 we will assume that k has characteristic 0 (because we will be working with D-modules).

0.5.2. In this article all schemes, algebraic spaces, and stacks are assumed to be "classical" (as opposed to derived).

0.5.3. When working with D-modules, our conventions follow those of [DrGa1, Sects. 5 and 6]. The only notational difference is that for a morphism $f : Z_1 \to Z_2$, we will denote the direct image functor D-mod$(Z_1) \to$ D-mod(Z_2) by $f_\bullet$ (instead of $f_{\mathrm{dR},*}$), and similarly for the left adjoint, $f^\bullet$ (instead of f_{dR}^*).

0.5.4. Given an an algebraic space (or stack) Z of finite type over a field k of characteristic 0, we let D-mod(Z) denote the DG category of D-modules on it. Our conventions regarding DG categories follow those of [DrGa1, Sect. 0.6].

On the other hand, the reader may prefer to replace each time the DG category D-mod(Z) by its homotopy category, which is a triangulated category.[7] Then the formulations and proofs of the main results of this article will remain valid. Moreover, once we know that the morphism (0.6) in the triangulated setting is the co-unit of an adjunction, it follows that the same is true in the DG setting.

---

[6]Pro-categories are considered in Appendix A.

[7]However, the most natural approach to *constructing* the triangulated category of D-modules on an algebraic stack is to construct the corresponding DG category first, as is done in [DrGa1].

0.5.5.   In Appendix A we define the notion of pro-completion $\mathrm{Pro}(\mathbf{C})$ of a DG category $\mathbf{C}$.

The reader who prefers to stay in the triangulated world, can replace it by the category of all covariant triangulated functors from the homotopy category $\mathrm{Ho}(\mathbf{C})$ to the homotopy category of complexes of k-vector spaces. (Note that the category of such functors is not necessarily triangulated, but this is of no consequence for us.)

0.6. **Organization of the paper.**

0.6.1.   Sects. 1-2 are devoted to the geometry of $\mathbb{G}_m$-actions on algebraic spaces.

Let Z be an algebraic space of finite type over the ground field k, equipped with a $\mathbb{G}_m$-action. In Sect. 1 we define the attractor Z^+ and the repeller Z^- by

$$(0.8) \qquad\qquad Z^+ := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z), \qquad Z^- := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1_-, Z),$$

where $\mathbf{Maps}^{\mathbb{G}_m}$ stands for the space of $\mathbb{G}_m$-equivariant maps and $\mathbb{A}^1_- := \mathbb{P}^1 - \{\infty\}$ (or equivalently, $\mathbb{A}^1_-$ is the affine line equipped with the $\mathbb{G}_m$-action opposite to the usual one). The basic facts on $Z^\pm$ are formulated in Sect. 1; the proofs of the more difficult statements are given in [Dr].

As was already mentioned in Sect. 0.3.2, in the proof of Braden's theorem we use a certain 1-parameter family of algebraic spaces $\widetilde{Z}_t$, $t \in \mathbb{A}^1$. These spaces are defined and studied in Sect. 2. The definition is formally similar to (0.8): namely,

$$\widetilde{Z}_t := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{X}_t, Z),$$

where $\mathbb{X}_t$ is the hyperbola $\tau_1 \cdot \tau_2 = t$ and the action of $\lambda \in \mathbb{G}_m$ on $\mathbb{X}_t$ is defined by

$$\tilde{\tau}_1 = \lambda \cdot \tau_1, \qquad \tilde{\tau}_2 = \lambda^{-1} \cdot \tau_2.$$

Note that $\mathbb{X}_0$ is the union of the two coordinate axes, which meet at the origin; accordingly, $\widetilde{Z}_0$ identifies with $Z^+ \underset{Z^0}{\times} Z^-$ (as promised in Sect. 0.3.2).

0.6.2.   In Sect. 3 we first state Braden's theorem in its original formulation, and then reformulate it as a statement that certain two functors are adjoint (with the specified co-unit of the adjunction).

In Sect. 4 we carry out the main step in the proof of Theorem 3.3.4 by constructing the unit morphism for the adjunction.

The geometric input in the construction of the unit is the family $t \rightsquigarrow \widetilde{Z}_t$ mentioned above. The input from the theory of D-modules is the *specialization map*

$$\mathrm{Sp}_{\mathcal{K}} : \mathcal{K}_1 \to \mathcal{K}_0,$$

where $\mathcal{K}$ is a $\mathbb{G}_m$-monodromic object in $\mathrm{D\text{-}mod}(\mathbb{A}^1 \times \mathcal{Y})$ (for any algebraic space/stack $\mathcal{Y}$), and where $\mathcal{K}_1$ and $\mathcal{K}_0$ are the !-restrictions of $\mathcal{K}$ to $\{1\} \times \mathcal{Y}$ and $\{0\} \times \mathcal{Y}$, respectively. The map $\mathrm{Sp}_{\mathcal{K}}$ is a simplified version of the specialization map that goes from nearby cycles to the !-fiber.

In Sect. 5 we show that the unit and co-unit indeed satisfy the adjunction property.

In Appendix A we define the notion of pro-completion $\mathrm{Pro}(\mathbf{C})$ of a DG category $\mathbf{C}$.

1. Geometry of $\mathbb{G}_m$-actions: fixed points, attractors, and repellers

In this section we review the theory of action of the multiplicative group $\mathbb{G}_m$ on a scheme or algebraic space Z. Specifically, we are concerned with the fixed-point locus, denoted by Z^0, as well as the attractor/repeller spaces, denoted by Z^+ and Z^-, respectively.

The main results of this section are Proposition 1.3.4 (which says that the fixed-point locus is closed), Theorem 1.5.2 (which ensures representability of attractor/repeller sets), and Proposition 1.9.4 (the latter is used in the construction of the unit of the adjunction given in Sect. 3.3.2).

In the case of a scheme equipped with a locally linear $\mathbb{G}_m$-action these results are well known (in a slightly different language).

1.1. k-spaces.

1.1.1. We fix a field k (of any characteristic). By a k-space (or simply space) we mean a contravariant functor Z from the category of affine schemes to that of sets which is a sheaf for the fpqc topology. Instead of $Z(\mathrm{Spec}(R))$ we write simply $Z(R)$; in other words, we consider Z as a covariant functor on the category of k-algebras.

Note that for any scheme S we have $Z(S) = \mathrm{Maps}(S, Z)$, where Maps stands for the set of morphisms between spaces. Usually we prefer to write $\mathrm{Maps}(S, Z)$ rather than $Z(S)$.

We write $\mathrm{pt} := \mathrm{Spec}(k)$.

1.1.2. General spaces will appear only as "intermediate" objects. For us, the really geometric objects are *algebraic spaces* over k. We will be using the definition of algebraic space from [LM] (which goes back to M. Artin). [8]

Any quasi-separated scheme (in particular, any scheme of finite type) is an algebraic space. The reader may prefer to restrict his attention to schemes, and even to separated schemes, as this will cover most of the cases of interest to which the main the result of this paper, i.e., Theorem 3.1.6, is applied.

Note that in the definition of spaces, instead of considering affine schemes as "test schemes", one can consider algebraic spaces (any fpqc sheaf on the category of affine schemes uniquely extends to an fpqc sheaf on the category of algebraic spaces).

1.1.3. A morphism of spaces $f : Z_1 \to Z_2$ is said to be a *monomorphism* if the corresponding map

$$\mathrm{Maps}(S, Z_1) \to \mathrm{Maps}(S, Z_2)$$

is injective for any scheme S. In particular, this applies if Z_1 and Z_2 are algebraic spaces. It is known that a morphism *of finite type* between schemes (or algebraic spaces) is a monomorphism if and only if each of its geometric fibers is a reduced scheme with at most one point.

A morphism of algebraic spaces is said to be *unramified* if it has locally finite presentation and its geometric fibers are finite and reduced.

1.2. The space of $\mathbb{G}_m$-equivariant maps.

---

[8]In particular, quasi-separatedness is included into the definition of algebraic space. Thus the quotient $\mathbb{A}^1/\mathbb{Z}$ (where the discrete group $\mathbb{Z}$ acts by translations) is *not* an algebraic space.

1.2.1. Let Z_1 and Z_2 be spaces. We define the space $\mathbf{Maps}(Z_1, Z_2)$ by

$$\mathrm{Maps}(S, \mathbf{Maps}(Z_1, Z_2)) := \mathrm{Maps}(S \times Z_1, Z_2)$$

(the right-hand side is easily seen to be an fpqc sheaf with respect to S).

1.2.2. Let Z_1, Z_2 be spaces equipped with an action of $\mathbb{G}_m$. Then we define the space $\mathbf{Maps}^{\mathbb{G}_m}(Z_1, Z_2)$ as follows: for any scheme S,

$$(1.1) \qquad\qquad \mathrm{Maps}(S, \mathbf{Maps}^{\mathbb{G}_m}(Z_1, Z_2)) := \mathrm{Maps}(S \times Z_1, Z_2)^{\mathbb{G}_m}$$

(the right-hand side is again easily seen to be an fpqc sheaf with respect to S).

The action of $\mathbb{G}_m$ on Z_2 induces a $\mathbb{G}_m$-action on $\mathbf{Maps}^{\mathbb{G}_m}(Z_1, Z_2)$.

1.2.3. Note that even if Z_1 and Z_2 are schemes, the space $\mathbf{Maps}^{\mathbb{G}_m}(Z_1, Z_2)$ does not have to be a scheme (or an algebraic space), in general.

1.3. The space of fixed points.

1.3.1. Let Z be a space equipped with an action of $\mathbb{G}_m$. Then we set

$$(1.2) \qquad\qquad\qquad Z^0 := \mathbf{Maps}^{\mathbb{G}_m}(\mathrm{pt}, Z).$$

Note that Z^0 is a subspace of Z because $\mathrm{Maps}(S, Z^0) = \mathrm{Maps}(S, Z)^{\mathbb{G}_m}$ is a subset of $\mathrm{Maps}(S, Z)$.

Definition 1.3.2. Z^0 is called *the subspace of fixed points* of Z.

1.3.3. We have the following result:

Proposition 1.3.4. *If Z is an algebraic space (resp. scheme) of finite type then so is Z^0. Moreover, the morphism $Z^0 \to Z$ is a closed embedding.*

The assertion of the proposition is nearly tautological if Z is separated. This case will suffice for most of the cases of interest to which the main result of this paper applies.

The proof in general is given in [Dr, Prop. 1.2.2]. It is not difficult; the only surprise is that $Z^0 \subset Z$ is closed even if Z is not separated. (Explanation in characteristic zero: Z^0 is the subspace of zeros of the vector field on Z corresponding to the $\mathbb{G}_m$-action.)

Example 1.3.5. Suppose that Z is an affine scheme $\mathrm{Spec}(A)$. A $\mathbb{G}_m$-action on Z is the same as a $\mathbb{Z}$-grading on A. Namely, the n-th component of A consists of $f \in \Gamma(Z, \mathcal{O}_Z)$ such that $f(\lambda \cdot z) = \lambda^n \cdot f(z)$.

It is easy to see that $Z^0 = \mathrm{Spec}(A^0)$, where A^0 is the maximal graded quotient algebra of A concentrated in degree 0 (in other words, A^0 is the quotient of A by the ideal generated by homogeneous elements of non-zero degree).

1.4. Attractors.

1.4.1. Let Z be a space equipped with an action of $\mathbb{G}_m$. Then we set

$$(1.3) \qquad\qquad\qquad Z^+ := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z),$$

where $\mathbb{G}_m$ acts on $\mathbb{A}^1$ by dilations.

Definition 1.4.2. Z^+ is called the *attractor* of Z.

1.4.3. *Pieces of structures on Z^+.*

(i) $\mathbb{A}^1$ is a monoid with respect to multiplication. The action of $\mathbb{A}^1$ on itself induces an $\mathbb{A}^1$-action on Z^+, which extends the $\mathbb{G}_m$-action defined in Sect. 1.2.

(ii) Restricting a morphism $\mathbb{A}^1 \times S \to Z$ to $\{1\} \times S$ one gets a morphism $S \to Z$. Thus we get a $\mathbb{G}_m$-equivariant morphism $p^+ : Z^+ \to Z$.

Note that if Z is *separated* (i.e., the diagonal morphism $Z \to Z \times Z$ is a closed embedding), then $p^+ : Z^+ \to Z$ is a *mono*morphism. To see this, it suffices to interpret p^+ as the composition

$$\mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z) \to \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{G}_m, Z) = Z.$$

Thus if Z is separated then p^+ identifies $Z^+(S)$ with the subset of those points $f : S \to Z$ for which the map $S \times \mathbb{G}_m \to Z$, defined by $(s,t) \mapsto t \cdot f(s)$, extends to a map $S \times \mathbb{A}^1 \to Z$; informally, the limit

(1.4) $$\lim_{t \to 0} t \cdot z$$

should exist.

(iii) Recall that $Z^0 = \mathbf{Maps}^{\mathbb{G}_m}(\mathrm{pt}, Z)$. The $\mathbb{G}_m$-equivariant maps $0 : \mathrm{pt} \to \mathbb{A}^1$ and $\mathbb{A}^1 \to \mathrm{pt}$ induce the maps

$$q^+ : Z^+ \to Z^0 \text{ and } i^+ : Z^0 \to Z^+,$$

such that $q^+ \circ i^+ = \mathrm{id}_{Z^0}$, and the composition $p^+ \circ i^+$ is equal to the canonical embedding $Z^0 \hookrightarrow Z$.

Note that if Z is separated then for $z \in Z^+(S) \subset Z(S)$ the point $q^+(S)$ is the limit (1.4).

1.4.4. *The case of a contracting action.* Let Z be a separated space. Then it is clear that if a $\mathbb{G}_m$-action on Z can be extended to an action of the monoid $\mathbb{A}^1$ then such an extension is unique. In this case we will say that that the $\mathbb{G}_m$-action is *contracting*.

Proposition 1.4.5. *Let Z be a separated space of finite type equipped with a $\mathbb{G}_m$-action. The morphism $p^+ : Z^+ \to Z$ is an isomorphism if and only if the $\mathbb{G}_m$-action on Z is contracting.*

Proof. The "only if" assertion follows from Sect. 1.4.3(i). For the "if" assertion, we note that the $\mathbb{A}^1$-action on Z defines a morphism $g : Z \to Z^+$ such that the composition of the maps

(1.5) $$Z \xrightarrow{g} Z^+ \xrightarrow{p^+} Z$$

equals id_Z. Since the map p^+ is a monomorphism (see Sect. 1.4.3(ii)), the assertion follows. $\square$

Remark 1.4.6. In [Dr, Prop. 1.4.15] it will be shown that if Z is an *algebraic* space of finite type, then the assertion of Proposition 1.4.5 remains valid even if Z is not separated: i.e., p^+ is an isomorphism if and only if the $\mathbb{G}_m$-action on Z can be extended to an $\mathbb{A}^1$-action; moreover, such an extension is unique.

1.4.7. *The affine case.* Suppose that Z is affine, i.e., $Z = \operatorname{Spec}(A)$, where A is a $\mathbb{Z}$-graded commutative algebra. It is easy to see that in this case Z^+ is represented by the affine scheme $\operatorname{Spec}(A^+)$, where A^+ is the maximal $\mathbb{Z}^{\geq 0}$-graded quotient algebra of A (in other words, the quotient of A by the ideal generated by by all homogeneous elements of A of strictly negative degrees).

By Example 1.3.5, $Z^0 = \operatorname{Spec}(A^0)$, where A^0 is the maximal graded quotient algebra of A (or equivalently, of A^+) concentrated in degree 0. Since the algebra A^+ is $\mathbb{Z}^{\geq 0}$-graded, A^0 identifies with the 0-th graded component of A^+. Thus we obtain the homomorphisms $A^0 \hookrightarrow A^+ \twoheadrightarrow A^0$. They correspond to the morphisms

$$Z^0 \xleftarrow{q^+} Z^+ \xleftarrow{i^+} Z^0.$$

1.4.8. *Attractors of open/closed subspaces.* We have:

Lemma 1.4.9. *Let Z be a space equipped with a $\mathbb{G}_m$-action, and let $Y \subset Z$ be a $\mathbb{G}_m$-stable open subspace.*

(i) *Suppose that $Y \to Z$ is an open embedding. Then the subspace $Y^+ \subset Z^+$ equals $(q^+)^{-1}(Y^0)$, where q^+ is the natural morphism $Z^+ \to Z^0$.*

(ii) *Suppose that $Y \to Z$ is a closed embedding. Then the subspace $Y^+ \subset Z^+$ equals $(p^+)^{-1}(Y)$, where q^+ is the natural morphism $Z^+ \to Z$.*

Proof. Let $Y \to Z$ be an open embedding. For any test scheme S, we have to show that if

$$f : S \times \mathbb{A}^1 \to Z$$

is a $\mathbb{G}_m$-equivariant morphism such that $\{0\} \times S \subset f^{-1}(Y)$ then $f^{-1}(Y) = S \times \mathbb{A}^1$. This is clear because $f^{-1}(Y) \subset S \times \mathbb{A}^1$ is open and $\mathbb{G}_m$-stable.

Let $Y \to Z$ be a closed embedding. An S-point of $(p^+)^{-1}(Y)$ is a $\mathbb{G}_m$-equivariant morphism $f : S \times \mathbb{A}^1 \to Y$ such that $S \times \mathbb{G}_m \subset f^{-1}(Y)$. Since $f^{-1}(Y)$ is closed in $S \times \mathbb{A}^1$ this implies that $f^{-1}(Y) = S \times \mathbb{A}^1$, i.e., $f(S \times \mathbb{A}^1) \subset Y$.

$\square$

1.5. **Representability of attractors.**

1.5.1.   We have the following assertion:

Theorem 1.5.2. *Let Z be an algebraic space of finite type equipped with a $\mathbb{G}_m$-action. Then*

(i) Z^+ *is an algebraic space of finite type;*

(ii) *The morphism $q^+ : Z^+ \to Z^0$ is affine.*

The proof of this theorem is given in [Dr, Thm. 1.4.2]. Here we will prove a particular case (see Sect. 1.5.4), sufficient for most of the cases of interest to which the main result of this paper applies.

Combining Theorem 1.5.2 with Proposition 1.3.4, we obtain:

Corollary 1.5.3.

(i) *If Z is a separated algebraic space of finite type then so is Z^+.*

(ii) *If Z is a scheme of finite type then so is Z^+.*

Proof. Follows from Theorem 1.5.2(ii) because by Proposition 1.3.4, Z^0 is a closed subspace of Z.

$\square$

1.5.4. *The case of a locally linear action.*

Definition 1.5.5. An action of $\mathbb{G}_m$ on a scheme Z is said to be *locally linear* if Z can be covered by open affine subschemes preserved by the $\mathbb{G}_m$-action.

Remark 1.5.6. Suppose that Z admits a $\mathbb{G}_m$-equivariant locally closed embedding into a projective space $\mathbb{P}(V)$, where $\mathbb{G}_m$ acts linearly on V. Then the action of $\mathbb{G}_m$ is locally linear.

For this reason, locally linear actions include most of the cases of interest that come up in practice.

Remark 1.5.7. If k is *algebraically closed* and Z_{red} is a *normal* separated[9] scheme of finite type over k, then by a theorem of H. Sumihiro, *any action of $\mathbb{G}_m$ on Z is locally linear.* (The proof of this theorem is contained in [Sum] and also in [KKMS, p.20-23] and [KKLV].)

1.5.8.   Let us prove Theorem 1.5.2 in the locally linear case on a scheme. First, we note that Lemma 1.4.9(i) reduces the assertion to the case when Z is affine. In the latter case, the assertion is manifest from Sect. 1.4.7.

1.6. **Further results on attractors.** The results of this subsection are included for completeness; they will not be used for the proof of the main theorem of this paper.

We let Z be an algebraic space of finite type equipped with a $\mathbb{G}_m$-action.

1.6.1.   We have:

Proposition 1.6.2.

(i) *If Z is separated then $p^+ : Z^+ \to Z$ is a monomorphism.*

(ii) *If Z is an affine scheme then $p^+ : Z^+ \to Z$ is a closed embedding.*

(iii) *If Z is proper then each geometric fiber of $p^+ : Z^+ \to Z$ is reduced and has exactly one geometric point.*

(iv) *The fiber of $p^+ : Z^+ \to Z$ over any geometric point of $Z^0 \subset Z$ is reduced and has exactly one geometric point.*

Proof. Point (i) has been proved in Sect. 1.4.3(ii). Point (ii) is manifest from Sect. 1.4.7. Point (iii) follows from point (i) and the fact that any morphism from $\mathbb{A}^1 - \{0\}$ to a proper scheme extends to the whole $\mathbb{A}^1$.

After base change, point (iv) is equivalent to the following lemma:

Lemma 1.6.3. *If $f : \mathbb{A}^1 \to Z$ is a $\mathbb{G}_m$-equivariant morphism such that $f(1) \in Z^0$ then f is constant.*

$\square$

Proof of Lemma 1.6.3. The map $\mathrm{pt} \to Z$, corresponding to $f(1) \in Z(k)$ is a closed embedding (whether or not Z is separated). Hence, the assertion follows from Lemma 1.4.9(ii). $\square$

Example 1.6.4. Let Z be the projective line $\mathbb{P}^1$ equipped with the usual action of $\mathbb{G}_m$. Then $p^+ : Z^+ \to Z$ is the canonical morphism $\mathbb{A}^1 \sqcup \{\infty\} \to \mathbb{P}^1$. In particular, p^+ is *not a locally closed embedding.*

---

[9]We do not know if separateness is really necessary in Sumihiro's theorem.

Example 1.6.5. Let Z be the curve obtained from $\mathbb{P}^1$ by gluing 0 with ∞. Equip Z with the $\mathbb{G}_m$-action induced by the usual action on $\mathbb{P}^1$. The map $\mathbb{P}^1 \to Z$ induces a map $(\mathbb{P}^1)^+ \to Z^+$. It is easy to see that the composed map

$$\mathbb{A}^1 \hookrightarrow (\mathbb{P}^1)^+ \to Z^+$$

is an isomorphism $\mathbb{A}^1 \simeq Z^+$.

Remark 1.6.6. Suppose that the action of $\mathbb{G}_m$ is locally linear. Then Proposition 1.6.2(ii) and Lemma 1.4.9 imply that the map p^+ is, *Zariski locally on the source*, a locally closed embedding.

Note, however, that is is not the case in general, as can be seen from Example 1.6.5.

1.6.7. In the example of $\mathbb{P}^1$, the restriction of $p^+ : Z^+ \to Z$ to each connected component of Z^+ *is* a locally closed embedding. This turns out to be true in a surprisingly large class of situations (but there are also important examples when this is false):

Theorem 1.6.8. *Let Z be a separated scheme over an algebraically closed field k equipped with a $\mathbb{G}_m$-action. Then each of the following conditions ensures that the restriction of $p^+ : Z^+ \to Z$ to each connected component* [10] *of Z^+ is a locally closed embedding:*

(i) *Z is smooth;*

(ii) *Z is normal and quasi-projective;*

(iii) *Z admits a $\mathbb{G}_m$-equivariant locally closed embedding into a projective space $\mathbb{P}(V)$, where $\mathbb{G}_m$ acts linearly on V.*

Case (i) is due to A. Białynicki-Birula [Bia]. Case (iii) immediately follows from the easy case $Z = \mathbb{P}(V)$. Case (ii) turns out to be a particular case of (iii) because by Theorem 1 from [Sum], if Z is normal and quasi-projective then it admits a $\mathbb{G}_m$-equivariant locally closed embedding into a projective space.

Remark 1.6.9. In case (i) the condition that Z be a scheme (rather than an algebraic space) is essential, as shown by A. J. Sommese [Som].

In case (ii) the quasi-projectivity condition is essential, as shown by J. Konarski [Kon] using a method developed by J. Jurkiewicz [Ju1, Ju2]. In this example Z is a 3-dimensional toric variety which is proper but not projective; it is constructed by drawing a certain picture on a 2-sphere, see the last page of [Kon].

In case (ii) normality is clearly essential, see Example 1.6.5.

1.7. **Differential properties.** The results of this subsection are included for the sake of completeness and will not be needed for the sequel.

We let Z be an algebraic space of finite type, equipped with an action of $\mathbb{G}_m$.

1.7.1. First, we have:

Lemma 1.7.2. *For any $z \in Z^0$ the tangent space* [11] *$T_z Z^0 \subset T_z Z$ equals $(T_z Z)^{\mathbb{G}_m}$.*

Proof. We can assume that the residue field of z equals k (otherwise do base change). Then compute $T_z Z^0$ in terms of morphisms $\operatorname{Spec} k[\varepsilon]/(\varepsilon^2) \to Z^0$. $\square$

---

[10]Using the $\mathbb{A}^1$-action on Z^+, it is easy to see that each connected component of Z^+ is the preimage of a connected component of Z^0 with respect to the map $q^+ : Z^+ \to Z^0$.

[11]We define the tangent space by $T_z Z := (T_z^* Z)^*$, where $T_z^* Z$ is the fiber of $\Omega^1_{Z/k}$ at z. (The equality $T_z^* Z = m_z/m_z^2$ holds *if the residue field of z is finite and separable* over k.)

1.7.3. Next we claim:

Proposition 1.7.4. *Let Z be an algebraic space of finite type equipped with a $\mathbb{G}_m$-action. Then the map $p^+ : Z^+ \to Z$ is unramified.*

Proof. We can assume that k is algebraically closed. Then we have to check that for any $\zeta \in Z^+(k)$ the map of tangent spaces

$$T_\zeta Z^+ \to T_{p^+(\zeta)} Z \tag{1.6}$$

induced by $p^+ : Z^+ \to Z$ is injective. Let $f : \mathbb{A}^1 \to Z$ be the $\mathbb{G}_m$-equivariant morphism corresponding to ζ. Then

$$T_\zeta Z^+ = \mathrm{Hom}_{\mathbb{G}_m}(f^*(\Omega_Z^1), \mathcal{O}_{\mathbb{A}^1}), \tag{1.7}$$

and the map (1.6) assigns to a $\mathbb{G}_m$-equivariant morphism $\varphi : f^*(\Omega_Z^1) \to \mathcal{O}_{\mathbb{A}^1}$ the corresponding map between fibers at $1 \in \mathbb{A}^1$. So the kernel of (1.6) consists of those $\varphi \in \mathrm{Hom}_{\mathbb{G}_m}(f^*(\Omega_Z^1), \mathcal{O}_{\mathbb{A}^1})$ for which $\varphi|_{\mathbb{A}^1 - \{0\}} = 0$. This implies that $\varphi = 0$ because $\mathcal{O}_{\mathbb{A}^1}$ has no nonzero sections supported at $0 \in \mathbb{A}^1$.

$\square$

1.7.5. Finally, we claim:

Proposition 1.7.6. *Suppose that Z is smooth. Then Z^0 and Z^+ are smooth. Moreover, the morphism $q^+ : Z^+ \to Z^0$ is smooth.*

Proof. We will only prove that q^+ is smooth. (Smoothness of Z^0 can be proved similarly, and smoothness of Z^+ follows.)

It suffices to check that q^+ is formally smooth. Let R be a k-algebra and $\bar{R} = R/I$, where $I \subset R$ is an ideal with $I^2 = 0$. Let $\bar{f} : \mathrm{Spec}(\bar{R}) \times \mathbb{A}^1 \to Z$ be a $\mathbb{G}_m$-equivariant morphism and let $\bar{f}_0 : \mathrm{Spec}(\bar{R}) \to Z^0$ denote the restriction of $\bar{f}$ to

$$\mathrm{Spec}(\bar{R}) \stackrel{\mathrm{id} \times \{0\}}{\hookrightarrow} \mathrm{Spec}(\bar{R}) \times \mathbb{A}^1 .$$

Let $\varphi : \mathrm{Spec}(R) \to Z^0$ be a morphism extending $\bar{f}_0$. We have to extend $\bar{f}$ to a $\mathbb{G}_m$-equivariant morphism $f : \mathrm{Spec}(R) \times \mathbb{A}^1 \to Z$ so that $f_0 = \varphi$, where $f_0 := f|_{\mathrm{Spec}(R)}$.

Since Z is smooth, we can find a *not-necessarily equivariant* morphism $f : \mathrm{Spec}(\bar{R}) \times \mathbb{A}^1 \to Z$ extending $\bar{f}$ with $f_0 = \varphi$. Then standard arguments show that the obstruction to existence of a $\mathbb{G}_m$-equivariant f with the required properties belongs to

$$H^1(\mathbb{G}_m , M), \quad M := H^0(\mathrm{Spec}(\bar{R}) \times \mathbb{A}^1 , \bar{f}^*(\Theta_Z) \otimes \mathcal{J}) \underset{\bar{R}}{\otimes} I,$$

where Θ_Z is the tangent bundle of Z and $\mathcal{J} \subset \mathcal{O}_{\mathrm{Spec}(R) \times \mathbb{A}^1}$ is the ideal of the zero section. But H^1 of $\mathbb{G}_m$ with coefficients in any $\mathbb{G}_m$-module is zero. $\square$

1.8. Repellers.

1.8.1. Set $\mathbb{A}^1_- := \mathbb{P}^1 - \{\infty\}$; this is a monoid with respect to multiplication containing $\mathbb{G}_m$ as a subgroup. One has an isomorphism of monoids

$$\mathbb{A}^1 \stackrel{\sim}{\longrightarrow} \mathbb{A}^1_- , \qquad t \mapsto t^{-1}. \tag{1.8}$$

1.8.2. Given a space Z, equipped with a $\mathbb{G}_m$-action, we set

$$Z^- := \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1_- , Z). \tag{1.9}$$

Definition 1.8.3. Z^- is called the *repeller* of Z.

1.8.4. Just as in Sect. 1.4.3 one defines an action of the monoid $\mathbb{A}^1_-$ on Z^- extending the action of $\mathbb{G}_m$, a $\mathbb{G}_m$-equivariant morphism $p^- : Z^- \to Z$, and $\mathbb{A}^1_-$-equivariant morphisms $q^- : Z^- \to Z^0$ and $i^- : Z^0 \to Z^-$ (where Z^0 is equipped with the trivial $\mathbb{A}^1_-$-action). One has $q^- \circ i^- = \mathrm{id}_{Z^0}$, and the composition $p^- \circ i^-$ is equal to the canonical embedding $Z^0 \hookrightarrow Z$.

Using the isomorphism (1.8), one can identify Z^- with the attractor for the inverse action of $\mathbb{G}_m$ on Z (this identification is $\mathbb{G}_m$-*anti*-equivariant). Thus the results on attractors from Sects. 1.4.7 and 1.5 imply similar results for repellers.

In particular, if Z is the spectrum of a $\mathbb{Z}$-graded algebra A then Z^- canonically identifies with $\mathrm{Spec}(A^-)$, where A^- is the maximal $\mathbb{Z}_-$-graded quotient algebra of A.

1.9. **Attractors and repellers.** In this subsection we let Z be an algebraic space of finite type, equipped with an action of $\mathbb{G}_m$.

1.9.1. First, we claim:

Lemma 1.9.2. *The morphisms $i^\pm : Z^0 \to Z^\pm$ are closed embeddings.*

Proof. It suffices to consider i^+. By Theorem 1.5.2(ii), the morphism $q^+ : Z^+ \to Z^0$ is separated. One has $q^+ \circ i^+ = \mathrm{id}_{Z^0}$. So i^+ is a closed embedding. $\square$

1.9.3. Now consider the fiber product $Z^+ \underset{Z}{\times} Z^-$ formed using the maps $p^\pm : Z^\pm \to Z$.

Proposition 1.9.4. *The map*

$$(1.10) \qquad\qquad j := (i^+, i^-) : Z^0 \to Z^+ \underset{Z}{\times} Z^-$$

is both an open embedding and a closed one (i.e., is the embedding of a union of some connected components).

Remark 1.9.5. If Z is affine then j is an isomorphism (this immediately follows from the explicit description of $Z^\pm$ in the affine case, see Sects. 1.4.7 and 1.8).

In general, j is not necessarily an isomorphism. To see this, note that by (1.3) and (1.9), we have

$$(1.11) \qquad\qquad Z^+ \underset{Z}{\times} Z^- = \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{P}^1, Z)$$

(where $\mathbb{P}^1$ is equipped with the usual $\mathbb{G}_m$-action), and a $\mathbb{G}_m$-equivariant map $\mathbb{P}^1 \to Z$ does not have to be constant, in general.

Proof of Proposition 1.9.4. We will give a proof in the case when Z is a scheme; the case of an arbitrary algebraic space is treated in [Dr, Prop. 1.6.2].

Writing j as

$$Z^0 = Z^0 \underset{Z}{\times} Z^0 \overset{(i^+, i^-)}{\longrightarrow} Z^+ \underset{Z}{\times} Z^-,$$

and using Lemma 1.9.2, we see that j is a closed embedding.

To prove that j is an open embedding, we note that the following diagram is Cartesian:

$$
\begin{array}{ccc}
Z^0 & \overset{\sim}{\longrightarrow} \mathbf{Maps}^{\mathbb{G}_m}(\mathrm{pt}, Z) & \longrightarrow \mathbf{Maps}(\mathrm{pt}, Z) \\
j \downarrow & \downarrow & \downarrow \\
Z^+ \underset{Z}{\times} Z^- & \overset{\sim}{\longrightarrow} \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{P}^1, Z) & \longrightarrow \mathbf{Maps}(\mathbb{P}^1, Z).
\end{array}
$$

Now, the required result follows from the next (easy) lemma:

Lemma 1.9.6. *For a scheme Z, the map*

$$Z = \mathbf{Maps}(\mathrm{pt}, Z) \to \mathbf{Maps}(\mathbb{P}^1, Z)$$

induced by the projection $\mathbb{P}^1 \to \mathrm{pt}$ is an open embedding.

$\square$

Corollary 1.9.7.

(i) *If the map $p^+ : Z^+ \to Z$ is an isomorphism then so are the maps $Z^0 \xrightarrow{i^-} Z^- \xrightarrow{q^-} Z^0$.*

(ii) *If the map $p^- : Z^- \to Z$ is an isomorphism then so are the maps $Z^0 \xrightarrow{i^+} Z^+ \xrightarrow{q^+} Z^0$.*

Proof. Let us prove (ii). By Proposition 1.9.4, the morphism $i^+ : Z^0 \to Z^+$ is an open embedding. It remains to show that any point $\zeta \in Z^+$ is contained in $i^+(Z^0)$. Set

$$U_\zeta := \{t \in \mathbb{A}^1 \,|\, t \cdot \zeta \in i^+(Z^0)\}.$$

We have to show that $1 \in U_\zeta$. But U_ζ is an open $\mathbb{G}_m$-stable subset of $\mathbb{A}^1$ containing 0, so $U_\zeta = \mathbb{A}^1$. $\square$

2. Geometry of $\mathbb{G}_m$-actions: the key construction

We keep the conventions and notation of Sect. 1. The goal of this section is, given an algebraic space Z equipped with a $\mathbb{G}_m$-action, to study a certain canonically defined algebraic space $\widetilde{Z}$, equipped with a morphism $\widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$, such that for $t \in \mathbb{A}^1 - \{0\}$ the fiber $\widetilde{Z}_t$ equals the graph of the map $t : Z \xrightarrow{\sim} Z$, and the fiber $\widetilde{Z}_0$, corresponding to $t = 0$, equals $Z^+ \underset{Z^0}{\times} Z^-$.

As was mentioned in Sect. 0.3.2, the space $\widetilde{Z}$ is the main geometric ingredient in the proof of Theorem 3.3.4. However, the reader can skip this section now and return to it when the time comes.

The main points of this section are following. In Sect. 2.2 we define the space $\widetilde{Z}$ and the main pieces of structure on it (e.g., the morphism $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ and the action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\widetilde{Z}$). In Sect. 2.4 we address the question of representability of $\widetilde{Z}$. In Sect. 2.6 we prove Proposition 2.6.3, which plays a key role in Sect. 5.

2.1. A family of hyperbolas.

2.1.1. Set $\mathbb{X} := \mathbb{A}^2 = \mathrm{Spec}\, k[\tau_1, \tau_2]$. Throughout the paper equip $\mathbb{X}$ with the structure of a scheme over $\mathbb{A}^1$, defined by the map

$$\mathbb{A}^2 \to \mathbb{A}^1, \quad (\tau_1, \tau_2) \mapsto \tau_1 \cdot \tau_2.$$

Let $\mathbb{X}_t$ denote the fiber of $\mathbb{X}$ over $t \in \mathbb{A}^1$; in other words, $\mathbb{X}_t \subset \mathbb{A}^2$ is the curve defined by the equation $\tau_1 \cdot \tau_2 = t$. If $t \neq 0$ then $\mathbb{X}_t$ is a hyperbola, while $\mathbb{X}_0$ is the "coordinate cross" $\tau_1 \cdot \tau_2 = 0$.

One has $\mathbb{X}_0 = \mathbb{X}_0^+ \cup \mathbb{X}_0^-$, where

$$(2.1) \qquad \mathbb{X}_0^+ := \{(\tau_1, \tau_2) \in \mathbb{A}^2 \,|\, \tau_2 = 0\}, \quad \mathbb{X}_0^- := \{(\tau_1, \tau_2) \in \mathbb{A}^2 \,|\, \tau_1 = 0\}.$$

2.1.2. *The schemes $\mathbb{X}_S$.* For any scheme S over $\mathbb{A}^1$ set

$$(2.2) \qquad\qquad\qquad \mathbb{X}_S := \mathbb{X} \underset{\mathbb{A}^1}{\times} S.$$

If $S = \mathrm{Spec}(R)$ we usually write $\mathbb{X}_R$ instead of $\mathbb{X}_S$.

2.1.3.   We will need the following pieces of structure on $\mathbb{X}$:

(i) The projection $\mathbb{X} \to \mathbb{A}^1$ admits two canonically defined sections:

$$(2.3) \qquad\qquad \sigma_1(t) = (1, t) \text{ and } \sigma_2(t) = (t, 1).$$

(ii) The scheme $\mathbb{X}$ carries a tautological action of the monoid $\mathbb{A}^1 \times \mathbb{A}^1$:

$$(\lambda_1, \lambda_2) \cdot (\tau_1, \tau_2) = (\lambda_1 \cdot \tau_1, \lambda_2 \cdot \tau_2) \,.$$

This action covers the action of $\mathbb{A}^1 \times \mathbb{A}^1$ on $\mathbb{A}^1$, given by the product map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ and the tautological action of $\mathbb{A}^1$ on itself.

(iii) In particular, we obtain an action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{X}$.

This action covers the action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{A}^1$, given by the product map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ and the tautological action of $\mathbb{G}_m$ on $\mathbb{A}^1$.

2.1.4. *The $\mathbb{G}_m$-action on $\mathbb{X}_S$.* Consider the action of the anti-diagonal copy of $\mathbb{G}_m$ on the scheme $\mathbb{X}$ from Sect. 2.1.3(iii). That is,

$$(2.4) \qquad\qquad \lambda \cdot (\tau_1, \tau_2) := (\lambda \cdot \tau_1, \ \lambda^{-1} \cdot \tau_2).$$

This action preserves the morphism $\mathbb{X} \to \mathbb{A}^1$, so for any scheme S over $\mathbb{A}^1$ one obtains an action of $\mathbb{G}_m$ on $\mathbb{X}_S$.

Remark 2.1.5. If S is over $\mathbb{A}^1 - \{0\}$, then $\mathbb{X}_S$ is $\mathbb{G}_m$-equivariantly isomorphic to $S \times \mathbb{G}_m$ by means of either of the maps σ_1 or σ_2.

2.2. **Construction of the interpolation.**

2.2.1.   Given a space Z equipped with a $\mathbb{G}_m$-action, define $\widetilde{Z}$ to be the following space over $\mathbb{A}^1$: for any scheme S over $\mathbb{A}^1$ we set

$$\mathrm{Maps}_{\mathbb{A}^1}(S, \widetilde{Z}) := \mathrm{Maps}(\mathbb{X}_S, Z)^{\mathbb{G}_m},$$

where $\mathbb{X}_S$ is acted on by $\mathbb{G}_m$ as in Sect. 2.1.4.

In other words, for any scheme S, an S-point of $\widetilde{Z}$ is a pair consisting of a morphism $S \to \mathbb{A}^1$ and a $\mathbb{G}_m$-equivariant morphism $\mathbb{X}_S \to Z$.

Note that for any $t \in \mathbb{A}^1(k)$ the fiber $\widetilde{Z}_t$ has the following description:

$$(2.5) \qquad\qquad\qquad \widetilde{Z}_t = \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{X}_t, Z).$$

2.2.2.   The sections σ_1 and σ_2 (see Sect. 2.1.3(i)) define morphisms

$$\pi_1 : \widetilde{Z} \to Z \text{ and } \pi_2 : \widetilde{Z} \to Z,$$

respectively.

Let

$$(2.6) \qquad\qquad\qquad \widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$$

denote the morphism whose first component is the tautological projection $\widetilde{Z} \to \mathbb{A}^1$, and the second and the third components are π_1 and π_2, respectively.

2.2.3. Note that the action of the group $\mathbb{G}_m \times \mathbb{G}_m$ on $\mathbb{X}$ from Sect. 2.1.3(iii) gives rise to an action of $\mathbb{G}_m \times \mathbb{G}_m$ on $\widetilde{Z}$. This action has the following properties:

(i) It is compatible with the canonical map $\widetilde{Z} \to \mathbb{A}^1$ via the multiplication map $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ and the *inverse* of the canonical action of $\mathbb{G}_m$ on $\mathbb{A}^1$.

(ii) It is compatible with $\pi_1 : \widetilde{Z} \to Z$ via the projection on the first factor $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ and the initial action of $\mathbb{G}_m$ on Z.

(iii) It is compatible with $\pi_2 : \widetilde{Z} \to Z$ via the projection on the second factor $\mathbb{G}_m \times \mathbb{G}_m \to \mathbb{G}_m$ and the *inverse* of the initial action of $\mathbb{G}_m$ on Z.

2.2.4. Restricting to the *anti-diagonal* copy of $\mathbb{G}_m \subset \mathbb{G}_m \times \mathbb{G}_m$ (i.e., $\lambda \mapsto (\lambda, \lambda^{-1})$), we obtain an action of $\mathbb{G}_m$ on $\widetilde{Z}$. (It is the same action as the one induced by the initial action of $\mathbb{G}_m$ on Z). This $\mathbb{G}_m$-action on $\widetilde{Z}$ preserves the projection $\widetilde{Z} \to \mathbb{A}^1$.

Both maps π_1 and π_2 are $\mathbb{G}_m$-equivariant.

2.2.5. For $t \in \mathbb{A}^1$ let

$$(2.7) \qquad \qquad \widetilde{p}_t : \widetilde{Z}_t \to Z \times Z$$

denote the morphism induced by (2.6) (as before, $\widetilde{Z}_t$ stands for the fiber of $\widetilde{Z}$ over t).

It is clear that $(\widetilde{Z}_1, \widetilde{p}_1)$ identifies with $(Z, \Delta_Z : Z \to Z \times Z)$. More generally, for $t \in \mathbb{A}^1 - \{0\}$ the pair $(\widetilde{Z}_t, \widetilde{p}_t)$ identifies with the graph of the map $Z \to Z$ given by the action of $t \in \mathbb{G}_m$.

More precisely, we have:

Proposition 2.2.6. *The morphism* (2.6) *induces an isomorphism between*

$$\mathbb{G}_m \underset{\mathbb{A}^1}{\times} \widetilde{Z}$$

and the graph of the action morphism $\mathbb{G}_m \times Z \to Z$. $\qquad \square$

2.2.7. We are now going to construct a canonical morphism

$$(2.8) \qquad \qquad \widetilde{Z}_0 \to Z^+ \underset{Z^0}{\times} Z^-.$$

Recall that $\widetilde{Z}_0 = \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{X}_0, Z)$ and $\mathbb{X}_0 = \mathbb{X}_0^+ \cup \mathbb{X}_0^-$, where $\mathbb{X}_0^+$ and $\mathbb{X}_0^-$ are defined by formula (2.1). One has $\mathbb{G}_m$-equivariant isomorphisms

$$(2.9) \qquad \mathbb{A}^1 \overset{\sim}{\longrightarrow} \mathbb{X}_0^+, \quad s \mapsto (s, 0); \qquad \mathbb{A}_-^1 \overset{\sim}{\longrightarrow} \mathbb{X}_0^-, \quad s \mapsto (0, s^{-1}).$$

They define a morphism

$$\widetilde{Z}_0 = \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{X}_0, Z) \to \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{X}_0^+, Z) \overset{\sim}{\longrightarrow} \mathbf{Maps}^{\mathbb{G}_m}(\mathbb{A}^1, Z) = Z^+$$

and a similar morphism $\widetilde{Z}_0 \to Z^-$.

By construction, the following diagram commutes:

$$(2.10)$$

$$
\begin{array}{ccc}
\widetilde{Z}_0 & \overset{\widetilde{p}_0}{\longrightarrow} & Z \times Z \\
\downarrow & & \uparrow{\scriptstyle p^+ \times p^-} \\
Z^+ \underset{Z^0}{\times} Z^- & \longrightarrow & Z^+ \times Z^-.
\end{array}
$$

2.2.8.   We now claim:

Proposition 2.2.9. *Let Z be a scheme. Then the map* (2.8) *is an isomorphism.*

Proof. Follows from the fact that for an affine scheme S, the diagram

$$
\begin{array}{ccc}
S \times \mathrm{pt} & \longrightarrow & S \times \mathbb{X}_0^+ \\
\downarrow & & \downarrow \\
S \times \mathbb{X}_0^- & \longrightarrow & S \times \mathbb{X}_0
\end{array}
$$

is a push-out diagram *in the category of schemes.*

$\square$

Remark 2.2.10. In [Dr, Prop. 2.1.11] it is shown that the map (2.8) is an isomorphism more generally when Z is an algebraic space.

Remark 2.2.11. Combining the isomorphism (2.8) with the isomorphism $\widetilde{Z}_1 \simeq Z$, we can interpret $\widetilde{Z}$ as an $\mathbb{A}^1$-family[12] of spaces interpolating between Z and its "degeneration" $Z^+ \underset{Z^0}{\times} Z^-$. Hence, the title of the subsection.

2.3. **Basic properties of the interpolation.**

2.3.1.   We have:

Proposition 2.3.2.
(i) *Let $Y \subset Z$ be a $\mathbb{G}_m$-stable closed subspace. Then the diagram*

$$
\begin{array}{ccc}
\widetilde{Y} & \longrightarrow & \widetilde{Z} \\
\widetilde{p}_Y \downarrow & & \downarrow \widetilde{p}_Z \\
\mathbb{A}^1 \times Y \times Y & \lhook\joinrel\longrightarrow & \mathbb{A}^1 \times Z \times Z
\end{array}
$$

is Cartesian. In particular, the morphism $\widetilde{Y} \to \widetilde{Z}$ is a closed embedding.

(ii) *Let $Y \subset Z$ be a $\mathbb{G}_m$-stable open subspace. Then the above diagram identifies $\widetilde{Y}$ with an open subspace of the fiber product*

$$
\widetilde{Z} \underset{\mathbb{A}^1 \times Z \times Z}{\times} \left(\mathbb{A}^1 \times Y \times Y \right).
$$

In particular, the morphism $\widetilde{Y} \to \widetilde{Z}$ is an open embedding.

Proof. Set

$$
\overset{\circ}{\mathbb{X}} := \mathbb{X} - \{0\},
$$

where $0 \in \mathbb{X}$ is the zero in $\mathbb{X} = \mathbb{A}^2$. For $S \to \mathbb{A}^1$, set $\overset{\circ}{\mathbb{X}}_S := \mathbb{X}_S \underset{\mathbb{X}}{\times} \overset{\circ}{\mathbb{X}}$.

(i) Let S be a scheme over $\mathbb{A}^1$ and $f : \mathbb{X}_S \to Z$ a $\mathbb{G}_m$-equivariant morphism. Formula (2.3) defines two sections of the map $\mathbb{X}_S \to S$. We have to show that if f maps these sections to $Y \subset Z$ then $f(\mathbb{X}_S) \subset Y$. By $\mathbb{G}_m$-equivariance, we have

$$
f(\overset{\circ}{\mathbb{X}}_S) \subset Y .
$$

Since $\overset{\circ}{\mathbb{X}}_S$ is schematically dense in $\mathbb{X}_S$ this implies that $f(\mathbb{X}_S) \subset Y$.

---

[12]In general, this "family" is not flat, see the example from Remark 2.5.4.

(ii) Just as before, we have a $\mathbb{G}_m$-equivariant morphism $f : \mathbb{X}_S \to Z$ such that $f(\overset{\circ}{\mathbb{X}}) \subset Y$. The problem is now to show that the set

$$\{s \in S \,|\, \mathbb{X}_s \subset f^{-1}(Y)\}$$

is open in S.

The complement of this set equals $\mathrm{pr}_S(\mathbb{X}_S - f^{-1}(Y))$, where $\mathrm{pr}_S : \mathbb{X}_S \to S$ is the projection. The set $\mathrm{pr}_S(\mathbb{X}_S - f^{-1}(Y))$ is closed in S because $\mathbb{X}_S - f^{-1}(Y)$ is a closed subset of $\mathbb{X}_S - \overset{\circ}{\mathbb{X}}_S$, while the morphism $\mathbb{X}_S - \overset{\circ}{\mathbb{X}}_S \to S$ is closed (in fact, it is a closed embedding). $\qquad\square$

2.3.3. Next we claim:

Proposition 2.3.4. *Let Z be separated. Then the map*

$$\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$$

is a monomorphism.

Proof. As before, set $\overset{\circ}{\mathbb{X}} := \mathbb{X} - \{0\}$, where $0 \in \mathbb{X}$ is the zero in $\mathbb{X} = \mathbb{A}^2$. Given a map $S \to \widetilde{Z}$, the corresponding $\mathbb{G}_m$-equivariant map

$$\overset{\circ}{\mathbb{X}}_S \to Z$$

is completely determined by the composition

$$S \to \widetilde{Z} \xrightarrow{\widetilde{p}} \mathbb{A}^1 \times Z \times Z \,.$$

Now use the fact that $\overset{\circ}{\mathbb{X}}_S$ is schematically dense in $\mathbb{X}_S$.

$\qquad\square$

Corollary 2.3.5. *If Z is separated then so is $\widetilde{Z}$.*

2.3.6. *The affine case.* We are going to prove:

Proposition 2.3.7. *Assume that Z is an affine scheme of finite type. Then the morphism $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a closed embedding. In particular, $\widetilde{Z}$ is an affine scheme of finite type.*

Proof. If Z is a closed subscheme of an affine scheme Z' and the proposition holds for Z' then it holds for Z by Proposition 2.3.2(i). So we are reduced to the case that Z is a finite-dimensional vector space equipped with a linear $\mathbb{G}_m$-action.

If the proposition holds for affine schemes Z_1 and Z_2 then it holds for $Z_1 \times Z_2$. So we are reduced to the case that $Z = \mathbb{A}^1$ and $\lambda \in \mathbb{G}_m$ acts on $\mathbb{A}^1$ as multiplication by λ^n, $n \in \mathbb{Z}$.

In this case it is straightforward to compute $\widetilde{Z}$ directly. In particular, one checks that $\widetilde{p}$ identifies $\widetilde{Z}$ with the closed subscheme of $\mathbb{A}^1 \times Z \times Z$ defined by the equation $x_2 = t^n \cdot x_1$ if $n \geq 0$ and by the equation $x_1 = t^{-n} \cdot x_2$ if $n \leq 0$ (here t, x_1, x_2 are the coordinates on $\mathbb{A}^1 \times Z \times Z = \mathbb{A}^3$). $\qquad\square$

2.4. **Representability of the interpolation.**

2.4.1.   We have the following assertion, which is proved in [Dr, Thm. 2.2.2 and Prop. 2.2.3]:

Theorem 2.4.2. *Let Z be an algebraic space (resp., scheme) of finite type equipped with a $\mathbb{G}_m$-action. Then $\widetilde{Z}$ is an algebraic space (resp., scheme) of finite type.*

Below we will give a proof in the case when Z is a scheme and the action of $\mathbb{G}_m$ on Z is locally linear. This case will be sufficient for the applications in this paper.

Proof. By assumption, Z can be covered by open affine $\mathbb{G}_m$-stable subschemes U_i. By Proposition 2.3.7, each $\widetilde{U}_i$ is an affine scheme of finite type. By Proposition 2.3.2(ii), for each i the canonical morphism $\widetilde{U}_i \to \widetilde{Z}$ is an open embedding. It remains to show that $\widetilde{Z}$ is covered by the open subschemes $\widetilde{U}_i$.

It suffices to check that for each $t \in \mathbb{A}^1$ the fiber $\widetilde{Z}_t$ is covered by the open subschemes $(\widetilde{U}_i)_t$. For $t \neq 0$ this is clear from Proposition 2.2.6. It remains to consider the case $t = 0$.

By Proposition 2.2.9, $\widetilde{Z}_0 = Z^+ \underset{Z^0}{\times} Z^-$. So a point of $\widetilde{Z}_0$ is a pair $(z^+, z^-) \in Z^+ \times Z^-$ such that $q^+(z^+) = q^-(z^-)$. The point $q^+(z^+) = q^-(z^-)$ is contained in some U_i. By Lemma 1.4.9(i), we have $z^+, z^- \in U_i$. So our point $(z^+, z^-) \in \widetilde{Z}_0$ belongs to $(\widetilde{U}_i)_0$.

$\square$

2.4.3. *The contracting situation.* Let Z be an algebraic space of finite type, and assume that the $\mathbb{G}_m$-action on Z is contracting, i.e., the $\mathbb{G}_m$-action can be extended to an action of the monoid $\mathbb{A}^1$ (recall that such an extension is unique, see Sect. 1.4.4 including Remark 1.4.6). In this case we claim:

Proposition 2.4.4.
(i) *The morphism $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ identifies $\widetilde{Z}$ with the graph of the $\mathbb{A}^1$-action on Z; in particular, the composition*

$$(2.11) \qquad \widetilde{Z} \xrightarrow{\widetilde{p}} \mathbb{A}^1 \times Z \times Z \to \mathbb{A}^1 \times Z \times \mathrm{pt} = \mathbb{A}^1 \times Z$$

is an isomorphism.
(ii) *The inverse of* (2.11) *is the morphism*

$$(2.12) \qquad Z \times \mathbb{A}^1 \to \widetilde{Z},$$

corresponding to the $\mathbb{G}_m$-equivariant map $Z \times \mathbb{X} \to Z$, defined by

$$(z, \tau_1, \tau_2) \mapsto \tau_1 \cdot z, \qquad (\tau_1, \tau_2) \in \mathbb{X}, \ z \in Z.$$

Proof. Let $\alpha : \widetilde{Z} \to \mathbb{A}^1 \times Z$ denote the composition (2.11) and $\beta : \mathbb{A}^1 \times Z \to \widetilde{Z}$ the morphism (2.12). It is easy to see that $\alpha \circ \beta = \mathrm{id}$.

In order to prove that $\beta \circ \alpha = \mathrm{id}$, it is enough to show that α is a monomorphism. By Theorem 2.4.2, we are dealing with a morphism between algebraic spaces of finite type, so being a monomorphism is a fiber-wise condition. Thus, it suffices to show that α induces an isomorphism between fibers over any $t \in \mathbb{A}^1$.

For $t \neq 0$ this follows from Proposition 2.2.6. If $t = 0$ then by Proposition 2.2.9 (resp., Remark 2.2.10 in the case of algebraic spaces), the morphism in question is the composition

$$Z^+ \underset{Z^0}{\times} Z^- \to Z^+ \xrightarrow{p^+} Z.$$

By Proposition 1.4.5 (resp., Remark 1.4.6 in the non-separated case), p^+ is an isomorphism, and the projection $q^- : Z^- \to Z^0$ is also an isomorphism by Corollary 1.9.7(i).

$\square$

2.4.5. From Proposition 2.4.4 we formally obtain the following one:

Proposition 2.4.6. *Let Z be an algebraic space, and assume that the* inverse *of the $\mathbb{G}_m$-action on Z is contracting. Then:*

(i) *the morphism $\widetilde{p} \colon \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a monomorphism, which identifies $\widetilde{Z}$ with*

$$\{(t, z_1, z_2) \in \mathbb{A}^1 \times Z \times Z \mid z_1 = t^{-1} \cdot z_2 \};$$

in particular, the composition

$$(2.13) \qquad \widetilde{Z} \xrightarrow{\widetilde{p}} \mathbb{A}^1 \times Z \times Z \to \mathbb{A}^1 \times \mathrm{pt} \times Z = \mathbb{A}^1 \times Z$$

is an isomorphism.

(ii) *The inverse of* (2.13) *is the morphism*

$$(2.14) \qquad Z \times \mathbb{A}^1 \to \widetilde{Z},$$

corresponding to the $\mathbb{G}_m$-equivariant map $Z \times \mathbb{X} \to Z$, defined by

$$(z, \tau_1, \tau_2) \mapsto \tau_2^{-1} \cdot z \,, \qquad (\tau_1, \tau_2) \in \mathbb{X} \,, \; z \in Z.$$

2.5. **Further properties of the interpolation.** The material in this subsection is included for completeness and will not be used in the sequel.

Throughout this subsection, Z will be be an algebraic space of finite type equipped with a $\mathbb{G}_m$-action.

2.5.1. We claim:

Proposition 2.5.2. *If Z is smooth then the canonical morphism $\widetilde{Z} \to \mathbb{A}^1$ is smooth.*

Proof. It suffices to check formal smoothness. We proceed just as in the proof of Proposition 1.7.6. Let R be a k-algebra equipped with a morphism $\mathrm{Spec}(R) \to \mathbb{A}^1$. Let $\bar{R} = R/I$, where $I \subset R$ is an ideal with $I^2 = 0$. Let $\bar{f} \in \mathrm{Maps}(\mathbb{X}_{\bar{R}}, Z)^{\mathbb{G}_m}$. We have to show that $\bar{f}$ can be lifted to an element of $\mathrm{Maps}(\mathbb{X}_R, Z)^{\mathbb{G}_m}$. Since $\mathbb{X}_R$ is affine and Z is smooth there is no obstruction to lifting $\bar{f}$ to an element of $\mathrm{Maps}(\mathbb{X}_R, Z)$. The standard arguments show that the obstruction to existence of a $\mathbb{G}_m$-equivariant lift is in $H^1(\mathbb{G}_m, M)$, where $M := H^0(\mathbb{X}_{\bar{R}}, \bar{f}^*(\Theta_Z)) \underset{\bar{R}}{\otimes} I$ and Θ_Z is the tangent bundle of Z. But H^1 of $\mathbb{G}_m$ with coefficients in any $\mathbb{G}_m$-module is zero. $\square$

2.5.3. Let Z be affine. In this case, by Proposition 2.3.7, the morphism $\widetilde{p}$ identifies $\widetilde{Z}$ with the closed subscheme $\widetilde{p}(\widetilde{Z}) \subset \mathbb{A}^1 \times Z \times Z$. By Proposition 2.2.6, the intersection of $\widetilde{p}(\widetilde{Z})$ with the open subscheme

$$\mathbb{G}_m \times Z \times Z \subset \mathbb{A}^1 \times Z \times Z$$

is equal to the graph of the action map $\mathbb{G}_m \times Z \to Z$. Hence, $\widetilde{Z}$ contains the closure of the graph in $\mathbb{A}^1 \times Z \times Z$.

Remark 2.5.4. In general, this containment is not an equality. E.g., this happens if Z is the hypersurface in $\mathbb{A}^{2n}$ defined by the equation $x_1 \cdot y_1 + \dots x_n \cdot y_n = 0$ and the $\mathbb{G}_m$-action on Z is defined by $\lambda(x_1, \dots, x_n, y_1, \dots, y_n) = (\lambda \cdot x_1, \dots, \lambda \cdot x_n, \lambda^{-1} \cdot y_1, \dots, \lambda^{-1} \cdot y_n)$.

However, one has the following:

Proposition 2.5.5. *If Z is affine and smooth then*

$$\widetilde{p}(\widetilde{Z}) = \overline{\Gamma},$$

where $\Gamma \subset \mathbb{G}_m \times Z \times Z$ is the graph of of the action map $\mathbb{G}_m \times Z \to Z$ and $\overline{\Gamma}$ denotes its scheme-theoretical closure in $\mathbb{A}^1 \times Z \times Z$.

Indeed, this immediately follows from Proposition 2.5.2.

2.5.6.   We claim:

Proposition 2.5.7. *The morphism $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is unramified.*

Proof. The morphism $\widetilde{p}$ is of finite presentation (because $\widetilde{Z}$ and $\mathbb{A}^1 \times Z \times Z$ have finite type over k). It remains to check the condition on the geometric fibers of $\widetilde{p}$. Over $\mathbb{A}^1 - \{0\}$, it follows from Proposition 2.2.6. Over $0 \in \mathbb{A}^1$ it follows from Proposition 1.7.4 combined with Proposition 2.2.9 (for schemes) and Remark 2.2.10 (for arbitrary algebraic spaces). $\square$

2.5.8.   Recall that according to Proposition 2.3.7, if Z is affine, the map $\widetilde{p}$ is a closed embedding.

Note, however, that if Z is the projective line $\mathbb{P}^1$ equipped with the usual $\mathbb{G}_m$-action then the map $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is *not a closed* embedding (because, e.g., the scheme $\widetilde{Z}_0 = Z^+ \underset{Z^0}{\times} Z^-$ is not proper).

We have the following assertion:

Proposition 2.5.9. *Let Z be a projective space $\mathbb{P}^n$ equipped with an arbitrary $\mathbb{G}_m$-action. Then the morphism $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a locally closed embedding.*

Proof. For a suitable coordinate system in $\mathbb{P}^n$, the $\mathbb{G}_m$-action is given by

$$\lambda * (z_0 : \ldots : z_n) = (\lambda^{m_0} \cdot z_0 : \ldots : \lambda^{m_n} \cdot z_n), \quad \lambda \in \mathbb{G}_m.$$

Let $U_i \subset Z = \mathbb{P}^n$ denote the open subset defined by the condition $z_i \neq 0$. It is affine, so by Proposition 2.3.7, the canonical morphism $\widetilde{U}_i \to \mathbb{A}^1 \times U_i \times U_i$ is a closed embedding. Thus to finish the proof of the proposition, it suffices to show that $\widetilde{p}^{-1}(\mathbb{A}^1 \times U_i \times U_i) = \widetilde{U}_i$. By Proposition 2.2.6, $\widetilde{p}^{-1}(\mathbb{G}_m \times U_i \times U_i) = \mathbb{G}_m \underset{\mathbb{A}^1}{\times} \widetilde{U}_i$. So it remains to prove that the morphism $\widetilde{p}_0 : \widetilde{Z}_0 \to Z \times Z$ has the following property: $(\widetilde{p}_0)^{-1}(U_i \times U_i) = (\widetilde{U}_i)_0$. Identifying $\widetilde{Z}_0$ with $Z^+ \underset{Z^0}{\times} Z^-$ and using Lemma 1.4.9(i), we see that it remains to prove the following lemma:

Lemma 2.5.10. *Let $z^+, z^- \in \mathbb{P}^n$. Suppose that*

$$\lim_{\lambda \to 0} \lambda * z^+ = \lim_{\lambda \to \infty} \lambda * z^- = \zeta.$$

If $z^+, z^- \in U_i$ then $\zeta \in U_i$.

$\square$

Proof of Lemma 2.5.10. Write $z^+ = (z_0^+ : \ldots : z_n^+)$, $z^- = (z_0^- : \ldots : z_n^-)$, $\zeta = (\zeta_0 : \ldots : \zeta_n)$. We have $z_i^{\pm} \neq 0$, and the problem is to show that $\zeta_i \neq 0$.

Suppose that $\zeta_i = 0$. Choose j so that $\zeta_j \neq 0$. Then $z_j^{\pm} \neq 0$ and

$$\lim_{\lambda \to 0} \lambda^{m_i - m_j} \cdot (z_i/z_j) = \zeta_i/\zeta_j = 0, \quad \lim_{\lambda \to \infty} \lambda^{m_i - m_j} \cdot (z_i/z_j) = \zeta_i/\zeta_j = 0.$$

This means that $m_i > m_j$ and $m_i < m_j$ at the same time, which is impossible. $\square$

2.5.11. As a corollary of Proposition 2.5.9, combined with Proposition 2.3.2, we obtain that if Z admits a $\mathbb{G}_m$-equivariant locally closed embedding into a projective space $\mathbb{P}(V)$, where $\mathbb{G}_m$ acts linearly on V, then the morphism $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is a locally closed embedding. (Recall, however, that the map $p^{\pm} : Z^{\pm} \to Z$ is typically not a locally closed embedding, see Example 1.6.4.)

2.5.12. More generally, suppose that the $\mathbb{G}_m$-action on Z is locally linear. Then the proof of Theorem 2.4.2 shows that in this case the map $\widetilde{p}$ is, *Zariski locally on the source*, a locally closed embedding.

However, even this is not the case for a general Z:

Consider the curve obtained from $\mathbb{P}^1$ by gluing 0 with ∞. Equip Z with the $\mathbb{G}_m$-action induced by the usual action on $\mathbb{P}^1$. Then $\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z$ is not a locally closed embedding, locally on the source. In fact, already $\widetilde{p}_0 : \widetilde{Z}_0 \to Z \times Z$ is not a locally closed embedding locally on the source (because the maps $p^{\pm} : Z^{\pm} \to Z$ are not).

2.6. **Some fiber products.** In this subsection we let Z be an algebraic space of finite type, equipped with an action of $\mathbb{G}_m$.

2.6.1. In Sect. 2.2.2 we defined morphisms $\pi_1, \pi_2 : \widetilde{Z} \to Z$. In Sect. 5 we will need to consider the fiber product

$$(2.15) \qquad Z^{-} \underset{Z}{\times} \widetilde{Z},$$

formed using $\pi_1 : \widetilde{Z} \to Z$, and the fiber product

$$(2.16) \qquad \widetilde{Z} \underset{Z}{\times} Z^{+},$$

formed using $\pi_2 : \widetilde{Z} \to Z$.

2.6.2. Consider the composition

$$(2.17) \qquad \mathbb{A}^1 \times Z^{+} \to \widetilde{Z^{+}} = \widetilde{Z^{+}} \underset{Z^{+}}{\times} Z^{+} \to \widetilde{Z} \underset{Z}{\times} Z^{+},$$

where the first arrow is the morphism (2.12) for the space Z^{+} and the second arrow is induced by the morphism $p^{+} : Z^{+} \to Z$. Consider also the similar composition

$$(2.18) \qquad \mathbb{A}^1 \times Z^{-} \to \widetilde{Z^{-}} = Z^{-} \underset{Z^{-}}{\times} \widetilde{Z^{-}} \to Z^{-} \underset{Z}{\times} \widetilde{Z},$$

where the first arrow is the morphism (2.14) for the space Z^{-}. In Sect. 5 we will need the following result.

Proposition 2.6.3. *The compositions* (2.17) *and* (2.18) *are open embeddings.*

Note that unlike the situation of Proposition 1.9.4, these embeddings are usually not closed.

Remark 2.6.4. By Propositions 2.4.4 and 2.4.6, the maps $\mathbb{A}^1 \times Z^{+} \to \widetilde{Z^{+}}$ and $\mathbb{A}^1 \times Z^{-} \to \widetilde{Z^{-}}$ are isomorphisms, so Proposition 2.6.3 means that the morphisms

$$\widetilde{Z^{+}} \to \widetilde{Z} \underset{Z}{\times} Z^{+}, \quad \widetilde{Z^{-}} \to Z^{-} \underset{Z}{\times} \widetilde{Z}$$

are open embeddings.

Remark 2.6.5. In the course of the proof of Proposition 2.6.3 we will see that if Z is affine, then the maps (2.17) and (2.18) are isomorphisms.

2.6.6. We will prove Proposition 2.6.3 assuming that the action of $\mathbb{G}_m$ on Z is locally linear. The general case is considered in [Dr, Prop. 3.1.3].

We will show that (2.17) is an open embedding. The case of (2.18) is similar.

Proof. First, Proposition 2.3.2(ii) and Lemma 1.4.9(i) allow to reduce the assertion to the case when Z is affine. In the affine case we will show that the map (2.17) is an isomorphism.

Next, it follows from Proposition 2.3.2(i) and Lemma 1.4.9(ii) that if $Z \to Z'$ is a closed embedding and (2.17) is an isomorphism for Z', then it is also an isomorphism for Z. This reduces the assertion to the case when Z is a vector space equipped with a linear action of $\mathbb{G}_m$.

Third, it is easy to see that if $Z = Z_1 \times Z_2$, and (2.17) is an isomorphism for Z_1 and Z_2, then it is an isomorphism for Z. This reduces the assertion further to the case when either the action of $\mathbb{G}_m$ on Z or its inverse is contracting.

Suppose that the action is contracting. In this case $Z^+ \simeq Z$ by Proposition 1.4.5, and under this identification the map

$$\widetilde{Z^+} \underset{Z^+}{\times} Z^+ \to \widetilde{Z} \underset{Z}{\times} Z^+,$$

appearing in (2.17), is the identity map.

Suppose that the inverse of the given $\mathbb{G}_m$-action on Z is contracting. By Corollary 1.9.7(ii), we can identify $Z^+ \simeq Z^0$, and by Proposition 2.4.6, $\widetilde{Z} \simeq \mathbb{A}^1 \times Z$. Under these identifications, the map (2.17) is the identity map

$$\mathbb{A}^1 \times Z^0 \to (\mathbb{A}^1 \times Z) \underset{Z}{\times} Z^0 \simeq \mathbb{A}^1 \times Z^0 .$$

$\square$

3. Braden's theorem

From now on we will assume that the ground field k has characteristic 0 (because we will be working with D-modules).

The goal of this section is to state Braden's theorem (Theorem 3.1.6) in the context of D-modules, and reduce it to another statement (Theorem 3.3.4) that says that certain two functors are adjoint.

Braden's theorem applies to any algebraic space Z of finite type over k, equipped with an action of $\mathbb{G}_m$. The reader may prefer to restrict his attention to the case of Z being a scheme or even a separated scheme.

Furthermore, because of Remark 1.5.6, for most applications, it is sufficient to consider the case when the $\mathbb{G}_m$-action on Z is locally linear, which would make the present paper self-contained, as the main technical results in Sects. 1-2 were proved only in this case.

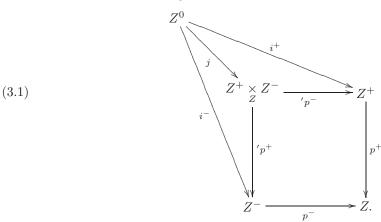3.1. **Statement of Braden's theorem: the original formulation.**

3.1.1. Let G be an algebraic group. If Z is an algebraic space of finite type equipped with a G-action, then

$$\text{D-mod}(Z)^{G\text{-mon}} \subset \text{D-mod}(Z)$$

stands for the full subcategory generated by the essential image of the pullback functor $\text{D-mod}(Z/G) \to \text{D-mod}(Z)$, where Z/G denotes the quotient *stack*. Here one can use either the !- or the •-pullback: this makes no difference as the morphism $Z \to Z/\mathbb{G}_m$ is smooth, and hence the two pullback functors differ by the cohomological shift by $2 \cdot \dim(G)$.

Note that if the G-action is trivial then $\text{D-mod}(Z)^{G\text{-mon}} = \text{D-mod}(Z)$ (because the morphism $Z \to Z/G$ admits a section).

3.1.2. From now on let Z be an algebraic space of finite type equipped with a $\mathbb{G}_m$-action. Consider the commutative diagram

(3.1)

$$
\begin{array}{ccc}
Z^0 & & \\
& \searrow{}^{i^+} & \\
\downarrow{}^{j} & & \\
& Z^+ \underset{Z}{\times} Z^- \xrightarrow{\ 'p^-\ } & Z^+ \\
{}^{i^-}\downarrow & \downarrow{}^{'p^+} & \downarrow{}^{p^+} \\
& Z^- \xrightarrow{\quad p^- \quad} & Z.
\end{array}
$$

(The definitions of Z^0, $Z^{\pm}$, $i^{\pm}$, and $p^{\pm}$ were given in Sects. 1.3, 1.4, and 1.8.)

Recall that by Proposition 1.9.4, the morphism $j : Z^0 \to Z^+ \underset{Z}{\times} Z^-$ is an open embedding (and also a closed one).

We consider the categories

$$\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}, \ \text{D-mod}(Z^+)^{\mathbb{G}_m\text{-mon}}, \ \text{D-mod}(Z^-)^{\mathbb{G}_m\text{-mon}}$$

and

$$\text{D-mod}(Z^0)^{\mathbb{G}_m\text{-mon}} = \text{D-mod}(Z^0).$$

Consider the functors

$(p^+)^! : \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \to \text{D-mod}(Z^+)^{\mathbb{G}_m\text{-mon}}$ and $(i^-)^! : \text{D-mod}(Z^-)^{\mathbb{G}_m\text{-mon}} \to \text{D-mod}(Z^0)$.

The formalism of pro-categories (see Appendix A) also provides the functors

$$(p^-)^{\bullet} : \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \to \text{Pro}(\text{D-mod}(Z^-)^{\mathbb{G}_m\text{-mon}})$$

and

$$(i^+)^{\bullet} : \text{D-mod}(Z^+)^{\mathbb{G}_m\text{-mon}} \to \text{Pro}(\text{D-mod}(Z^0)),$$

left adjoint in the sense of Sect. A.3 to

$$(p^-)_{\bullet} : \text{D-mod}(Z^-)^{\mathbb{G}_m\text{-mon}} \to \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}$$

and

$$(i^+)_{\bullet} : \text{D-mod}(Z^0) \to \text{D-mod}(Z^+)^{\mathbb{G}_m\text{-mon}},$$

respectively.

3.1.3.   Consider the composed functors
$$(i^+)^\bullet \circ (p^+)^! \text{ and } (i^-)^! \circ (p^-)^\bullet, \quad \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \to \text{Pro}(\text{D-mod}(Z^0)).$$

   They are called the functors of *hyperbolic restriction*.

3.1.4.   We claim that there is a canonical natural transformation
$$(3.2) \qquad\qquad\qquad\qquad (i^+)^\bullet \circ (p^+)^! \to (i^-)^! \circ (p^-)^\bullet.$$
Namely, the natural transformation (3.2) is obtained via the $((i^+)^\bullet, (i^+)_\bullet)$-adjunction from the natural transformation
$$(3.3) \qquad\qquad\qquad\qquad (p^+)^! \to (i^+)_\bullet \circ (i^-)^! \circ (p^-)^\bullet,$$
defined in terms of diagram (3.1) as follows.

   Note that since $j : Z^0 \to Z^+ \underset{Z}{\times} Z^-$ is an *open embedding* (see Proposition 1.9.4), the functor $j^!$ is left adjoint to $j_\bullet$. Now define the morphism (3.3) to be the composition
$$(p^+)^! \to (p^+)^! \circ (p^-)_\bullet \circ (p^-)^\bullet \simeq ('p^-)_\bullet \circ ('p^+)^! \circ (p^-)^\bullet \to$$
$$\to ('p^-)_\bullet \circ j_\bullet \circ j^! \circ ('p^+)^! \circ (p^-)^\bullet \simeq (i^+)_\bullet \circ (i^-)^! \circ (p^-)^\bullet,$$
where $(p^+)^! \circ (p^-)_\bullet \simeq ('p^-)_\bullet \circ ('p^+)^!$ is the base change isomorphism and the map
$$\text{Id} \to j_\bullet \circ j^!$$
comes from the $(j^!, j_\bullet)$-adjunction.

3.1.5.   We are now ready to state Braden's theorem:

Theorem 3.1.6. *The functors*
$$(i^+)^\bullet \circ (p^+)^! \text{ and } (i^-)^! \circ (p^-)^\bullet, \quad \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \to \text{Pro}(\text{D-mod}(Z^0))$$
take values in $\text{D-mod}(Z^0) \subset \text{Pro}(\text{D-mod}(Z^0))$ *and the map* (3.2) *is an isomophism.*

Remark 3.1.7. As we will see in Sect. 3.3.1, the fact that the functor $(i^+)^\bullet \circ (p^+)^!$ takes values in $\text{D-mod}(Z^0) \subset \text{Pro}(\text{D-mod}(Z^0))$ is easy to prove. The fact that the functor $(i^-)^! \circ (p^-)^\bullet$ takes values in $\text{D-mod}(Z^0)$ will follow *a posteriori* from the isomorphism with $(i^+)^\bullet \circ (p^+)^!$.

3.2. **Contraction principle.**

3.2.1.   Assume for a moment that the $\mathbb{G}_m$-action on Z extends[13] to an action of the monoid $\mathbb{A}^1$. (Informally, this means that the $\mathbb{G}_m$-action on Z contracts it onto the fixed point locus Z^0.)

Proposition 3.2.2. *In the above situation we have the following:*
(a) *The left adjoint* $i^\bullet : \text{D-mod}(Z) \to \text{Pro}(\text{D-mod}(Z^0))$ *of* $i_\bullet$ *sends* $\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}$ *to* $\text{D-mod}(Z^0)$, *and we have a canonical isomorphism*
$$i^\bullet|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \simeq q_\bullet|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}}.$$
More precisely, for each $\mathcal{F} \in \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}$ *the natural map*
$$q_\bullet(\mathcal{F}) \to q_\bullet \circ i_\bullet \circ i^\bullet(\mathcal{F}) = (q \circ i)_\bullet \circ i^\bullet(\mathcal{F}) = i^\bullet(\mathcal{F})$$
is an isomorphism.

---

[13]By Remark 1.4.6, such extension is unique if it exists.

(b) *The left adjoint* $q_!$: D-mod$(Z) \to$ Pro(D-mod(Z^0))) *of* $q^!$ *sends* D-mod$(Z)^{\mathbb{G}_m\text{-mon}}$ *to* D-mod(Z^0), *and we have a canonical isomorphism*

$$q_!|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} \simeq i^!|_{\text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}}} .$$

More precisely, for each $\mathcal{F} \in$ D-mod$(Z)^{\mathbb{G}_m\text{-mon}}$ *the natural map*

$$i^!(\mathcal{F}) \to i^! \circ q^! \circ q_!(\mathcal{F}) = (q \circ i)^! \circ q_!(\mathcal{F}) = q_!(\mathcal{F})$$

is an isomorphism.

For the proof see [DrGa2, Theorem C.5.3].

3.2.3. Note that we can reformulate point (a) of Proposition 3.2.2 above as the statement that the (iso)morphism

$$q_\bullet \circ i_\bullet \to \text{Id}_{\text{D-mod}(Z^0)}$$

defines the co-unit of an adjunction between

$$q_\bullet : \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightleftarrows \text{D-mod}(Z^0) : i_\bullet.$$

Similarly, point (b) of Proposition 3.2.2 can be reformulated as the statement that the (iso)morphism

$$i^! \circ q^! \to \text{Id}_{\text{D-mod}(Z^0)}$$

defines the co-unit of an adjunction between

$$i^! : \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightleftarrows \text{D-mod}(Z^0) : q^!.$$

3.3. Reformulation of Braden's theorem.

3.3.1. We return to the set-up of Theorem 3.1.6. By Proposition 3.2.2, we obtain canonical isomorphisms

$$(i^+)^\bullet \simeq (q^+)_\bullet \text{ and } (i^-)^! \simeq (q^-)_!.$$

In particular, we obtain that the functor

$$(i^+)^\bullet \circ (p^+)^! \simeq (q^+)_\bullet \circ (p^+)^!$$

sends D-mod$(Z)^{\mathbb{G}_m\text{-mon}}$ to D-mod(Z^0).

In addition, we see that the functor

$$(i^-)^! \circ (p^-)^\bullet \simeq (q^-)_! \circ (p^-)^\bullet$$

is the left adjoint functor to $(p^-)_\bullet \circ (q^-)^!$.

3.3.2. Now define a natural transformation

(3.4) $$\big((q^+)_\bullet \circ (p^+)^!\big) \circ \big((p^-)_\bullet \circ (q^-)^!\big) \to \text{Id}_{\text{D-mod}(Z^0)}$$

to be the composition

$$(q^+)_\bullet \circ (p^+)^! \circ (p^-)_\bullet \circ (q^-)^! \simeq (q^+)_\bullet \circ ('p^-)_\bullet \circ ('p^+)^! \circ (q^-)^! \to$$
$$\to (q^+)_\bullet \circ ('p^-)_\bullet \circ j_\bullet \circ j^! \circ ('p^+)^! \circ (q^-)^! \simeq (q^+)_\bullet \circ (i^+)_\bullet \circ (i^-)^! \circ (q^-)^! \simeq$$
$$\simeq (q^+ \circ i^+)_\bullet \circ (q^- \circ i^-)^! = \text{Id}_{\text{D-mod}(Z^0)} .$$

The above natural transformation corresponds to the diagram

$$Z^0$$

3.3.3. The natural transformation (3.4) gives rise to (and is determined by) a natural transformation

$$(3.5) \qquad\qquad (q^+)_\bullet \circ (p^+)^! \to \big((p^-)_\bullet \circ (q^-)^!\big)^L \simeq (q^-)_! \circ (p^-)^\bullet.$$

Here $\big((p^-)_\bullet \circ (q^-)^!\big)^L$ denotes the left adjoint of $(p^-)_\bullet \circ (q^-)^!$ in the sense of Sect. A.3.

It follows by diagram chase that the following diagram of natural transfomations commutes:

$$
\begin{array}{ccc}
(q^+)_\bullet \circ (p^+)^! & \xrightarrow{\ (3.5)\ } & (q^-)_! \circ (p^-)^\bullet \\
\sim\uparrow & & \sim\uparrow \\
(i^+)^\bullet \circ (p^+)^! & \xrightarrow{\ (3.2)\ } & (i^-)^! \circ (p^-)^\bullet
\end{array}
$$

Hence, the assertion of Theorem 3.1.6 follows from the next one:

Theorem 3.3.4. *The natural transformation* (3.4) *is the co-unit of an adjunction for the functors*

$$(q^+)_\bullet \circ (p^+)^! : \text{D-mod}(Z)^{\mathbb{G}_m\text{-mon}} \rightleftarrows \text{D-mod}(Z^0) : (p^-)_\bullet \circ (q^-)^!$$

3.4. The equivariant version.

3.4.1. Consider now the stacks

$$\mathcal{Z} := Z/\mathbb{G}_m\,,\ \ \mathcal{Z}^0 := Z^0/\mathbb{G}_m\,,\ \ \mathcal{Z}^\pm := Z^\pm/\mathbb{G}_m$$

and the morphisms

$$\mathsf{p}^\pm : \mathcal{Z}^\pm \to \mathcal{Z}\,,\ \ \mathsf{q}^\pm : \mathcal{Z}^\pm \to \mathcal{Z}^0$$

induced by the morphisms

$$p^\pm : Z^\pm \to Z,\ \ q^\pm : Z^\pm \to Z^0$$

from Sects. 1.4.3 and 1.8.

3.4.2. The construction of the natural transformation (3.4) can be rendered verbatim to produce a natural transformation

$$(3.6) \qquad \left((\mathsf{q}^+)_\bullet \circ (\mathsf{p}^+)^!\right) \circ \left((\mathsf{p}^-)_\bullet \circ (\mathsf{q}^-)^!\right) \to \mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z}^0)} \,.$$

We will prove the following version of Theorem 3.3.4:

Theorem 3.4.3. *The natural transformation* (3.6) *is the co-unit of an adjunction for the functors*

$$(\mathsf{q}^+)_\bullet \circ (\mathsf{p}^+)^! : \mathrm{D\text{-}mod}(\mathcal{Z}) \rightleftarrows \mathrm{D\text{-}mod}(\mathcal{Z}^0) : (\mathsf{p}^-)_\bullet \circ (\mathsf{q}^-)^!$$

Let us prove that Theorem 3.4.3 implies Theorem 3.3.4.

Proof. We need to show that for $\mathcal{M} \in \mathrm{D\text{-}mod}(Z)^{\mathbb{G}_m\text{-mon}}$ and $\mathcal{N} \in \mathrm{D\text{-}mod}(Z^0)^{\mathbb{G}_m\text{-mon}}$, the map

$$\mathrm{Hom}_{\mathrm{D\text{-}mod}(Z)^{\mathbb{G}_m\text{-mon}}} \left(\mathcal{M}, (p^-)_\bullet \circ (q^-)^!(\mathcal{N})\right) \to \mathrm{Hom}_{\mathrm{D\text{-}mod}(Z^0)^{\mathbb{G}_m\text{-mon}}} \left((q^+)_\bullet \circ (p^+)^!(\mathcal{M}), \mathcal{N}\right),$$

induced by (3.4), is an isomorphism.

By the definition of $\mathrm{D\text{-}mod}(Z)^{\mathbb{G}_m\text{-mon}}$, we can assume that $\mathcal{M}$ is the $\bullet$-pullback of some $\mathcal{M}' \in \mathrm{D\text{-}mod}(\mathcal{Z})$. Let $\mathcal{N}'$ denote the $\bullet$-direct image of $\mathcal{N}$ under the canonical map $Z^0 \to \mathcal{Z}^0$.

Since all the maps $Z \to \mathcal{Z}$, $Z^0 \to \mathcal{Z}^0$ and $Z^\pm \to \mathcal{Z}^\pm$ are smooth, we have the following commutative diagram (with the vertical arrows being isomorphisms by adjunction):

$$
\begin{array}{ccc}
\mathrm{Hom}\left(\mathcal{M}, (p^-)_\bullet \circ (q^-)^!(\mathcal{N})\right) & \longrightarrow & \mathrm{Hom}\left((q^+)_\bullet \circ (p^+)^!(\mathcal{M}), \mathcal{N}\right) \\
\sim \downarrow & & \downarrow \sim \\
\mathrm{Hom}\left(\mathcal{M}', (\mathsf{p}^-)_\bullet \circ (\mathsf{q}^-)^!(\mathcal{N}')\right) & \longrightarrow & \mathrm{Hom}\left((\mathsf{q}^+)_\bullet \circ (\mathsf{p}^+)^!(\mathcal{M}'), \mathcal{N}'\right).
\end{array}
$$

Hence, if the bottom horizontal arrow is an isomorphism, then so is the top one.

$\square$

4. CONSTRUCTION OF THE UNIT

In this section we will perform the main step in the proof of Theorem 3.4.3; namely, we will construct the *unit* for the adjunction between the functors $(\mathsf{q}^+)_\bullet \circ (\mathsf{p}^+)^!$ and $(\mathsf{p}^-)_\bullet \circ (\mathsf{q}^-)^!$.

4.1. **The specialization map.** In this subsection we describe the general set-up for the specialization map. The concrete situation in which this set-up will be applied is described in Sects. 4.4.1-4.4.3 below.

4.1.1. Let $\mathcal{Y}$ be an algebraic *stack* [14] of finite type. Consider the stack $\mathbb{A}^1 \times \mathcal{Y}$, and let ι_1 and ι_0 be the maps $\mathcal{Y} \to \mathbb{A}^1 \times \mathcal{Y}$ corresponding to the points 1 and 0 of $\mathbb{A}^1$, respectively. Let π denote the projection $\mathbb{A}^1 \times \mathcal{Y} \to \mathcal{Y}$.

Let $\mathcal{K}$ be an object of $\mathrm{D\text{-}mod}(\mathbb{A}^1 \times \mathcal{Y})^{\mathbb{G}_m\text{-mon}}$, where

$$\mathrm{D\text{-}mod}(\mathbb{A}^1 \times \mathcal{Y})^{\mathbb{G}_m\text{-mon}} \subset \mathrm{D\text{-}mod}(\mathbb{A}^1 \times \mathcal{Y})$$

is the full subcategory generated by the essential image of the pullback functor

$$\mathrm{D\text{-}mod}\left((\mathbb{A}^1/\mathbb{G}_m) \times \mathcal{Y}\right) \to \mathrm{D\text{-}mod}(\mathbb{A}^1 \times \mathcal{Y}).$$

---

[14]We use the conventions from [DrGa1, Sect. 1.1] for algebraic stacks. We refer the reader to [DrGa1, Sect. 6] for a review of the DG category of D-modules on algebraic stacks of finite type.

Set

$$\mathcal{K}_1 := \iota_1^!(\mathcal{K}), \qquad \mathcal{K}_0 := \iota_0^!(\mathcal{K}).$$

We are going to construct a canonical map

(4.1) $$\mathrm{Sp}_{\mathcal{K}} : \mathcal{K}_1 \to \mathcal{K}_0 \,,$$

which will depend functorially on $\mathcal{K}$. We will call it the *specialization map*.

Remark 4.1.2. The map (4.1) is a simplified version of the specialization map that goes from the nearby cycles functor to the !-fiber.

4.1.3. First, note that Proposition 3.2.2(b) and the definition of the category D-mod($-$) for an algebraic stack [15] imply that the functor $\pi_!$, left adjoint to $\pi^! : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathbb{A}^1 \times \mathcal{Y})$, is defined on the subcategory D-mod($\mathbb{A}^1 \times \mathcal{Y})^{\mathbb{G}_m\text{-mon}}$, and the natural transformation

$$\iota_0^! \to \iota_0^! \circ \pi^! \circ \pi_! \simeq \pi_!$$

is an isomorphism.

Now, we construct the natural transformation (4.1) as

$$\iota_1^!(\mathcal{K}) \simeq \pi_! \circ (\iota_1)_! \circ \iota_1^!(\mathcal{K}) \to \pi_!(\mathcal{K}) \simeq \iota_0^!(\mathcal{K}),$$

where the morphism $\pi_! \circ (\iota_1)_! \circ \iota_1^!(\mathcal{K}) \to \pi_!(\mathcal{K})$ comes from the $((\iota_1)_!, \iota_1^!)$-adjunction. Note that the functor $(\iota_1)_!$ is well-defined because ι_1 is a closed embedding.

4.1.4. It is easy to see that if $\mathcal{K} = \omega_{\mathbb{A}^1} \times \mathcal{K}_{\mathcal{Y}}$ for some $\mathcal{Y} \in \text{D-mod}(\mathcal{Y})$, then the map (4.1) is the identity endomorphism of

$$\iota_1^!(\mathcal{K}) \simeq \mathcal{K}_{\mathcal{Y}} \simeq \iota_0^!(\mathcal{K}).$$

4.1.5. It is also easy to see from the construction that the natural transformation (4.1) is functorial with respect to maps between algebraic stacks in the following sense.

Let $f : \mathcal{Y}' \to \mathcal{Y}$ be a map. Then for $\mathcal{K}' := (\mathrm{id}_{\mathbb{A}^1} \times f)^!(\mathcal{K})$ the diagram

$$\begin{array}{ccc} \mathcal{K}'_1 & \xrightarrow{\mathrm{Sp}_{\mathcal{K}'}} & \mathcal{K}'_0 \\ \sim\uparrow & & \uparrow\sim \\ f^!(\mathcal{K}_1) & \xrightarrow{f^!(\mathrm{Sp}_{\mathcal{K}})} & f^!(\mathcal{K}_0) \end{array}$$

commutes.

Let now f be representable and quasi-compact. Then for $\mathcal{K}' \in \text{D-mod}(\mathbb{A}^1 \times \mathcal{Y}')^{\mathbb{G}_m\text{-mon}}$ and

$$\mathcal{K} := (\mathrm{id}_{\mathbb{A}^1} \times f)_\bullet(\mathcal{K}'),$$

the diagram

$$\begin{array}{ccc} \mathcal{K}_1 & \xrightarrow{\mathrm{Sp}_{\mathcal{K}}} & \mathcal{K}_0 \\ \sim\downarrow & & \downarrow\sim \\ f_\bullet(\mathcal{K}'_1) & \xrightarrow{f_\bullet(\mathrm{Sp}_{\mathcal{K}'})} & f_\bullet(\mathcal{K}'_0) \end{array}$$

also commutes.

4.2. **Digression: functors given by kernels.**

---

[15]According to [DrGa1, Sect. 6.1.1], an object of D-mod($\mathcal{Y}$) is a "compatible collection" of objects of D-mod(S) for all schemes S of finite type mapping to $\mathcal{Y}$.

4.2.1. According to [DrGa1, Definition 1.1.8], an algebraic stack of finite type over k is said to be QCA if the automorphism groups of its geometric points are affine.

If $f : \mathcal{Y} \to \mathcal{Y}'$ is a morphism between QCA stacks then one has a canonically defined functor

$$f_{\blacktriangle} : \text{D-mod}(\mathcal{Y}) \to \text{D-mod}(\mathcal{Y}')$$

defined in [DrGa1, Sect. 9.3].

Remark 4.2.2. The functor $f_{\blacktriangle}$ is a "renormalized version" of the usual functor $f_{\bullet}$ of de Rham direct image (see [DrGa1, Sect. 7.4]). The problem with the functor $f_{\bullet}$ is that it is very poorly behaved unless the morphism f is representable [16]. For example, it fails to satisfy the projection formula and *is not compatible with compositions*, see [DrGa1, Sect. 7.5] for more details. The functor $f_{\blacktriangle}$ cures all these drawbacks, and it equals the usual functor $f_{\bullet}$ if f is representable.

4.2.3. Let $\mathcal{Y}_1$ and $\mathcal{Y}_2$ be QCA algebraic stacks. For an object $\mathcal{Q} \in \text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2)$, consider the functor

$$\mathsf{F}_{\mathcal{Q}} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2), \quad \mathcal{M} \mapsto (\text{pr}_2)_{\blacktriangle}(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}),$$

where $\text{pr}_i : \mathcal{Y}_1 \times \mathcal{Y}_2 \to \mathcal{Y}_i$ are the two projections, and $\overset{!}{\otimes}$ is the usual tensor product on the category of D-modules.

We will refer to $\mathcal{Q}$ as the *kernel* of the functor $\mathsf{F}_{\mathcal{Q}}$.

In fact, it follows from [DrGa1, Corollary 8.3.4] that the assignment $\mathcal{Q} \rightsquigarrow \mathsf{F}_{\mathcal{Q}}$ defines an equivalence between the category $\text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2)$ and the DG category of *continuous* [17] functors $\text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2)$.

For example, if $\mathcal{Y}_1 = \mathcal{Y}_2 = \mathcal{Y}$, then for

$$\mathcal{Q} := (\Delta_{\mathcal{Y}})_{\blacktriangle}(\omega_{\mathcal{Y}}) \in \text{D-mod}(\mathcal{Y} \times \mathcal{Y})$$

the corresponding functor $\mathsf{F}_{\mathcal{Q}}$ is the identity functor on $\text{D-mod}(\mathcal{Y})$. Here $\omega_{\mathcal{Y}} \in \text{D-mod}(\mathcal{Y})$ denotes the dualizing complex on a stack $\mathcal{Y}$.

4.2.4. More generally, let

$$
\begin{array}{ccc}
 & \mathcal{Y}_0 & \\
{\scriptstyle f_1}\swarrow & & \searrow{\scriptstyle f_2} \\
\mathcal{Y}_1 & & \mathcal{Y}_2
\end{array}
$$

(4.2)

be a diagram of QCA algebraic stacks. Set

$$\mathcal{Q} := (f_1 \times f_2)_{\blacktriangle}(\omega_{\mathcal{Y}_0}) \in \text{D-mod}(\mathcal{Y}_1 \times \mathcal{Y}_2).$$

Then, by the projection formula, the functor $\mathsf{F}_{\mathcal{Q}}$ identifies with $(f_2)_{\blacktriangle} \circ (f_1)^!$.

---

[16]Or, more generally, *safe* in the sense of [DrGa1, Definition 10.2.2].

[17]Recall that a functor between cocomplete DG categories is said to be continuous if it commutes with arbitrary direct sums.

4.2.5.   The reader who is reluctant to use the (potentially unfamiliar) functor $f_\blacktriangle$ can proceed along either of the following two routes:

(i) The usual functor of direct image $f_\bullet$ is well-behaved when restricted to the subcategory D-mod$(\mathcal{Y})^+$ of bounded below (=eventually coconnective) objects. It is easy to see that working with this subcategory would be sufficient for the proof of Theorem 3.4.3. [18]

This strategy can be used in order to adapt the proof of Theorem 3.4.3 to the context of ℓ-adic sheaves.

(ii) One can use the following assertion.

Lemma 4.2.6. *Suppose that the morphism $f_2 : \mathcal{Y}_0 \to \mathcal{Y}_2$ is representable. Then*

(i) *The kernel $\mathcal{Q} := (f_1 \times f_2)_\blacktriangle(\omega_{\mathcal{Y}_0})$ is canonically isomorphic to $(f_1 \times f_2)_\bullet(\omega_{\mathcal{Y}_0})$;*

(ii) *The functor*

$$\mathsf{F}_{\mathcal{Q}} : \text{D-mod}(\mathcal{Y}_1) \to \text{D-mod}(\mathcal{Y}_2), \quad \mathcal{M} \mapsto (\text{pr}_2)_\blacktriangle(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}) \simeq (f_2)_\blacktriangle \circ (f_1)^!(\mathcal{M})$$

is canonically isomorphic to the functor

$$\mathcal{M} \mapsto (\text{pr}_2)_\bullet(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}).$$

Proof. Since $f_2 : \mathcal{Y}_0 \to \mathcal{Y}_2$ is representable, so is the morphism $f_1 \times f_2 : \mathcal{Y}_0 \to \mathcal{Y}_1 \times \mathcal{Y}_2$. This implies (i).

We have canonical isomorphisms

$$\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q} \simeq (f_1 \times f_2)_\blacktriangle(f_1^!(\mathcal{M})) \simeq (f_1 \times f_2)_\bullet(f_1^!(\mathcal{M}))$$

(the first one holds by projection formula and the second because $f_1 \times f_2$ is representable). So

$$(\text{pr}_2)_\bullet(\text{pr}_1^!(\mathcal{M}) \overset{!}{\otimes} \mathcal{Q}) \simeq ((\text{pr}_2)_\bullet \circ (f_1 \times f_2)_\bullet)(f_1^!(\mathcal{M})).$$

One also has

$$\mathsf{F}_{\mathcal{Q}}(\mathcal{M}) \simeq (f_2)_\blacktriangle \circ (f_1)^!(\mathcal{M}) \simeq (f_2)_\bullet(f_1^!(\mathcal{M})) = ((\text{pr}_2) \circ (f_1 \times f_2))_\bullet(f_1^!(\mathcal{M})).$$

Finally, the fact that $f_1 \times f_2$ is representable (see [DrGa1, Proposition 7.5.7] [19]) implies that

$$((\text{pr}_2) \circ (f_1 \times f_2))_\bullet \simeq (\text{pr}_2)_\bullet \circ (f_1 \times f_2)_\bullet.$$

$\square$

4.3. **The unit of adjunction: plan of the construction.**

---

[18]Note, however, that if one redefines the assignment $\mathcal{Q} \rightsquigarrow \mathsf{F}_{\mathcal{Q}}$ using $(\text{pr}_2)_\bullet$ instead of $(\text{pr}_2)_\blacktriangle$ then one obtains a *different* functor, even when evaluated on D-mod$(\mathcal{Y}_1)^+$.

[19]For any composable morphisms g, g' between stacks one has a morphism $g_\bullet \circ g'_\bullet \to (g \circ g')_\bullet$, which is *not necessarily an isomorphism*. However, it is an isomorphism if g' is representable. In [DrGa1, Proposition 7.5.7] this is proved if g' is schematic, but the same proof applies if g' is only representable.

4.3.1. In Sect. 3.4.1 we introduced the stacks

$$\mathcal{Z} := Z/\mathbb{G}_m\,,\ \mathcal{Z}^0 := Z^0/\mathbb{G}_m\,,\ \mathcal{Z}^{\pm} := Z^{\pm}/\mathbb{G}_m$$

and the morphisms $\mathsf{p}^{\pm} : \mathcal{Z}^{\pm} \to \mathcal{Z}\,$, $\mathsf{q}^{\pm} : \mathcal{Z}^{\pm} \to \mathcal{Z}^0$. Now consider the diagram

(4.3)

Our goal is to construct a canonical morphism from $\mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z})}$ to the composed functor

$$(4.4) \qquad\qquad \big((\mathsf{p}^-)_{\bullet} \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_{\bullet} \circ (\mathsf{p}^+)^!\big) : \mathrm{D\text{-}mod}(\mathcal{Z}) \to \mathrm{D\text{-}mod}(\mathcal{Z}).$$

The good news is that all morphisms in diagram (4.3) are representable. In particular, p^- and q^+ are representable, so $(\mathsf{p}^-)_{\bullet} = (\mathsf{p}^-)_{\blacktriangle}$ and $(\mathsf{q}^+)_{\bullet} = (\mathsf{q}^+)_{\blacktriangle}\,$.

Thus, the problem is to construct a canonical morphism from $\mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z})}$ to the composed functor

$$(4.5) \qquad\qquad \big((\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_{\blacktriangle} \circ (\mathsf{p}^+)^!\big) : \mathrm{D\text{-}mod}(\mathcal{Z}) \to \mathrm{D\text{-}mod}(\mathcal{Z}).$$

Using base change[20], we further identify the functor (4.5) with

$$(4.6) \qquad\qquad (\mathsf{p}^- \circ {}'\mathsf{q}^+)_{\blacktriangle} \circ (\mathsf{p}^+ \circ {}'\mathsf{q}^-)^!,$$

where ${}'\mathsf{q}^+$ and ${}'\mathsf{q}^-$ are as in diagram (4.3).

4.3.2. Set

$$\mathcal{Q}_0 := (\mathsf{p}^+ \times \mathsf{p}^-)_{\blacktriangle}(\omega_{\mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^-}) \in \mathrm{D\text{-}mod}(\mathcal{Z} \times \mathcal{Z}).$$

Then the functor (4.6) (and, hence, (4.4)) is canonically isomorphic to $\mathsf{F}_{\mathcal{Q}_0}\,$.

The identity functor $\mathrm{D\text{-}mod}(\mathcal{Z}) \to \mathrm{D\text{-}mod}(\mathcal{Z})$ equals $\mathsf{F}_{\mathcal{Q}_1}$, where

$$\mathcal{Q}_1 := (\Delta_{\mathcal{Z}})_{\blacktriangle}(\omega_{\mathcal{Z}}) \in \mathrm{D\text{-}mod}(\mathcal{Z} \times \mathcal{Z}).$$

4.3.3. In Sect. 4.4 we will construct a canonical map

$$(4.7) \qquad\qquad\qquad\qquad \mathcal{Q}_1 \to \mathcal{Q}_0\,.$$

By Sects. 4.3.1-4.3.2 and 4.2.3, the map of kernels (4.7) induces a natural transformation

$$(4.8) \qquad\qquad \mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z})} \to \big((\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_{\blacktriangle} \circ (\mathsf{p}^+)^!\big)$$

between the corresponding functors.

In Sect. 5 we will prove that the natural transformations (4.8) and (3.6) satisfy the properties of unit and co-unit of an adjunction between the functors $(\mathsf{q}^+)_{\blacktriangle} \circ (\mathsf{p}^+)^!$ and $(\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!$.

---

[20]Since we have switched to the renormalized direct images, we can apply base change and do other standard manipulations.

4.4. **Constructing the morphism** (4.7). We will first define an object

$$\mathcal{Q} \in \text{D-mod}(\mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z})^{\mathbb{G}_m\text{-mon}},$$

which "interpolates" between $\mathcal{Q}_1$ and $\mathcal{Q}_0$. We will then define (4.7) to be the specialization morphism $\text{Sp}_{\mathcal{Q}}$.

4.4.1. Recall the algebraic space $\widetilde{Z}$ from Sect. 2 and set

$$\widetilde{\mathcal{Z}} := \widetilde{Z}/\mathbb{G}_m, \quad \widetilde{\mathcal{Z}}_t := \widetilde{Z}_t/\mathbb{G}_m \simeq \widetilde{\mathcal{Z}} \underset{\mathbb{A}^1}{\times} \{t\}$$

(the action of $\mathbb{G}_m$ on $\widetilde{\mathcal{Z}}$ was defined in Sect. 2.2.4).

Consider the morphisms

$$\widetilde{\mathsf{p}} : \widetilde{\mathcal{Z}} \to \mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z} \quad \text{and} \quad \widetilde{\mathsf{p}}_t : \widetilde{\mathcal{Z}}_t \to \mathcal{Z} \times \mathcal{Z}$$

induced by the maps (2.6) and (2.7), respectively.

Set

$$\mathcal{Q} := \widetilde{\mathsf{p}}_{\blacktriangle}(\omega_{\widetilde{\mathcal{Z}}}) \in \text{D-mod}(\mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z}).$$

4.4.2. We claim that $\mathcal{Q}$ belongs to the subcategory $\text{D-mod}(\mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z})^{\mathbb{G}_m\text{-mon}}$.

In fact, we claim that $\mathcal{Q}$ is the pullback of a canonically defined object of the category $\text{D-mod}(\mathbb{A}^1/\mathbb{G}_m \times \mathcal{Z} \times \mathcal{Z})$. Indeed, this follows from the existence of the Cartesian diagram

$$
\begin{array}{ccccc}
\widetilde{\mathcal{Z}} & \xrightarrow{\quad = \quad} & \widetilde{Z}/\mathbb{G}_m & \longrightarrow & \widetilde{Z}/\mathbb{G}_m \times \mathbb{G}_m \\
\widetilde{\mathsf{p}}\downarrow & & \widetilde{p}/\mathbb{G}_m\downarrow & & \downarrow \\
\mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z} & \xrightarrow{\quad = \quad} & \mathbb{A}^1 \times Z/\mathbb{G}_m \times Z/\mathbb{G}_m & \longrightarrow & \mathbb{A}^1/\mathbb{G}_m \times Z/\mathbb{G}_m \times Z/\mathbb{G}_m,
\end{array}
$$

where $\mathbb{G}_m \times \mathbb{G}_m$ acts on $\widetilde{Z}$ as in Sect. 2.2.3.

4.4.3. Recall that the pair $(\widetilde{Z}_1, \widetilde{p}_1)$ identifies with (Z, Δ_Z), and the pair $(\widetilde{Z}_0, \widetilde{p}_0)$ identifies with $(Z^+ \underset{Z^0}{\times} Z^-, p^+ \times p^-)$.

Therefore, the pair $(\widetilde{\mathcal{Z}}_1, \widetilde{\mathsf{p}}_1)$ identifies with $(\mathcal{Z}, \Delta_{\mathcal{Z}})$, and the pair $(\widetilde{\mathcal{Z}}_0, \widetilde{\mathsf{p}}_0)$ identifies with $(\mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^-, \mathsf{p}^+ \times \mathsf{p}^-)$.

Hence, by base change, the objects $\mathcal{Q}_1$ and $\mathcal{Q}_0$ from Sect. 4.3.2 identify with the !-restrictions of $\mathcal{Q}$ to

$$\{1\} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z} \text{ and } \{0\} \times \mathcal{Z} \times \mathcal{Z} \to \mathbb{A}^1 \times \mathcal{Z} \times \mathcal{Z},$$

respectively.

Now, the sought-for map (4.7) is given by the map $\text{Sp}_{\mathcal{Q}}$ of (4.1).

5. Verifying the adjunction properties

In Sect. 3.4.1 we introduced the stacks

$$\mathcal{Z} := Z/\mathbb{G}_m \, , \ \mathcal{Z}^0 := Z^0/\mathbb{G}_m \, , \ \mathcal{Z}^\pm := Z^\pm/\mathbb{G}_m$$

and the morphisms $\mathsf{p}^\pm : \mathcal{Z}^\pm \to \mathcal{Z} \, , \ \mathsf{q}^\pm : \mathcal{Z}^\pm \to \mathcal{Z}^0$.

In Sects. 3.3-3.4 we constructed a natural transformation

$$\big((\mathsf{q}^+)_\bullet \circ (\mathsf{p}^+)^!\big) \circ \big((\mathsf{p}^-)_\bullet \circ (\mathsf{q}^-)^!\big) \to \mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z}^0)} \, ,$$

see formula (3.6). Since the morphisms p^- and q^+ are representable we have $(\mathsf{p}^-)_\bullet = (\mathsf{p}^-)_\blacktriangle$ and $(\mathsf{q}^+)_\bullet = (\mathsf{q}^+)_\blacktriangle$. So the above natural transformation (3.6) can be rewritten as a natural transformation

$$(5.1) \qquad \big((\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!\big) \circ \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \to \mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z}^0)} \, .$$

In Sect. 4 we constructed a natural transformation

$$(5.2) \qquad \mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z})} \to \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!\big) \, ,$$

see formula (4.8).

To prove Theorem 3.4.3, it suffices to show that the compositions

$$(5.3) \quad (\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^! \to \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!\big) \circ \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \to (\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!$$

and

$$(5.4) \quad (\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^! \to \big((\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!\big) \circ \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!\big) \to (\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!$$

corresponding to (5.1) and (5.2) are isomorphic to [21] the identity morphisms.

We will do so for the composition (5.3). The case of (5.4) is similar and will be left to the reader.

The key point of the proof is Sect. 5.4.1, which relies on the geometric Proposition 2.6.3. More precisely, we use the part of Proposition 2.6.3 about Z^-. To treat the composition (5.4), one has to use the part of Proposition 2.6.3 about Z^+.

5.1. **The diagram describing the composed functor.**

5.1.1. *The big diagram.* We will use the notation

$$(5.5) \qquad \Phi := \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \circ \big((\mathsf{q}^+)_\blacktriangle \circ (\mathsf{p}^+)^!\big) \circ \big((\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!\big) \, .$$

By base change, Φ is given by pull-push along the following diagram:

---

[21]In the future we will skip the words "isomorphic to" in similar situations. (This is a slight abuse of language since we work with the DG categories of D-modules rather than with their homotopy categories.)

$$\mathcal{Z}^- \underset{\mathcal{Z}}{\times} \mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^-$$

$$\mathcal{Z}^- \underset{\mathcal{Z}}{\times} \mathcal{Z}^+ \qquad\qquad \mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^-$$

$$\mathcal{Z}^- \qquad\qquad \mathcal{Z}^+ \qquad\qquad \mathcal{Z}^-$$

$$\mathsf{q}^- \qquad \mathsf{p}^- \qquad \mathsf{p}^+ \qquad \mathsf{q}^+ \qquad \mathsf{q}^- \qquad \mathsf{p}^-$$

(5.6) $\qquad \mathcal{Z}^0 \qquad\qquad\qquad \mathcal{Z} \qquad\qquad\qquad \mathcal{Z}^0 \qquad\qquad\qquad \mathcal{Z}\,.$

5.1.2. *Some notation.* Set

(5.7)
$$\widetilde{Z}^- := Z^- \underset{Z}{\times} \widetilde{Z}\,,$$

where the fiber product is formed using the composition

$$\widetilde{Z} \xrightarrow{\widetilde{p}} \mathbb{A}^1 \times Z \times Z \to Z \times Z \xrightarrow{\mathrm{pr}_1} Z$$

(i.e., the morphism $\pi_1 : \widetilde{Z} \to Z$ from Sect. 2.2.2).

For $t \in \mathbb{A}^1$ set $\widetilde{Z}^-_t := Z^- \underset{Z}{\times} \widetilde{Z}_t$.

Let $\widetilde{p}^- : \widetilde{Z}^- \to \mathbb{A}^1 \times Z^- \times Z$ denote the map obtained by base change from

$$\widetilde{p} : \widetilde{Z} \to \mathbb{A}^1 \times Z \times Z\,.$$

Let

$$r : \widetilde{Z}^- \to \mathbb{A}^1 \times Z^0 \times Z$$

denote the composition of $\widetilde{p}^- : \widetilde{Z}^- \to \mathbb{A}^1 \times Z^- \times Z$ with the morphism

$$\mathrm{id}_{\mathbb{A}^1} \times q^- \times \mathrm{id}_Z : \mathbb{A}^1 \times Z^- \times Z \to \mathbb{A}^1 \times Z^0 \times Z.$$

Let $r_t : \widetilde{Z}^-_t \to Z^0 \times Z$ denote the morphism induced by $r : \widetilde{Z}^- \to \mathbb{A}^1 \times Z^0 \times Z$.

5.1.3. *More notation.* Recall that

$$\widetilde{\mathcal{Z}} := \widetilde{Z}/\mathbb{G}_m, \quad \widetilde{\mathcal{Z}}_t := \widetilde{Z}_t/\mathbb{G}_m \simeq \widetilde{\mathcal{Z}} \underset{\mathbb{A}^1}{\times} \{t\}\,,$$

where $\widetilde{Z}$ is the algebraic space from Sect. 2 (the action of $\mathbb{G}_m$ on $\widetilde{\mathcal{Z}}$ was defined in Sect. 2.2.4).

Set

$$\widetilde{\mathcal{Z}}^- := \mathcal{Z}^- \underset{\mathcal{Z}}{\times} \widetilde{\mathcal{Z}} = \widetilde{Z}^-/\mathbb{G}_m\,, \qquad \widetilde{\mathcal{Z}}^-_t := \mathcal{Z}^- \underset{\mathcal{Z}}{\times} \widetilde{\mathcal{Z}}_t = \widetilde{Z}^-_t/\mathbb{G}_m\,.$$

Let

$$\mathsf{r} : \widetilde{\mathcal{Z}}^- \to \mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}\,, \quad \mathsf{r}_t : \widetilde{\mathcal{Z}}^-_t \to \mathcal{Z}^0 \times \mathcal{Z}$$

be the morphisms induced by $r : \widetilde{Z}^- \to \mathbb{A}^1 \times Z^0 \times Z$ and $r_t : \widetilde{Z}^-_0 \to Z^0 \times Z$, respectively. In particular, we have the morphisms r_0 and r_1 corresponding to $t = 0$ and $t = 1$.

5.1.4. *A smaller diagram describing the functor* Φ. By Proposition 2.2.9, we have an isomorphism $\widetilde{Z}_0 \xrightarrow{\sim} Z^+ \underset{Z^0}{\times} Z^-$. The corresponding isomorphism

$$\widetilde{Z}_0^- := Z^- \underset{Z}{\times} \widetilde{Z}_0 \xrightarrow{\sim} Z^- \underset{Z}{\times} Z^+ \underset{Z^0}{\times} Z^-$$

induces an isomorphism

$$\widetilde{\mathcal{Z}}_0^- \simeq \mathcal{Z}^- \underset{\mathcal{Z}}{\times} \mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^-.$$

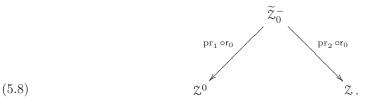Thus the upper term of diagram (5.6) is $\widetilde{\mathcal{Z}}_0^-$. The compositions

$$\mathcal{Z}^- \underset{\mathcal{Z}}{\times} \mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^- \to \mathcal{Z}^- \underset{\mathcal{Z}}{\times} \mathcal{Z}^+ \to \mathcal{Z}^- \xrightarrow{\mathsf{q}^-} \mathcal{Z}^0 \quad \text{and} \quad \mathcal{Z}^- \underset{\mathcal{Z}}{\times} \mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^- \to \mathcal{Z}^+ \underset{\mathcal{Z}^0}{\times} \mathcal{Z}^- \to \mathcal{Z}^- \xrightarrow{\mathsf{p}^-} \mathcal{Z}$$

from diagram (5.6) are equal, respectively, to the compositions

$$\widetilde{\mathcal{Z}}_0^- \xrightarrow{\mathsf{r}_0} \mathcal{Z}^0 \times \mathcal{Z} \xrightarrow{\mathrm{pr}_1} \mathcal{Z}^0 \quad \text{and} \quad \widetilde{\mathcal{Z}}_0^- \xrightarrow{\mathsf{r}_0} \mathcal{Z}^0 \times \mathcal{Z} \xrightarrow{\mathrm{pr}_2} \mathcal{Z}$$

(the morphism r_0 was defined in Sect. 5.1.3).

Hence, the functor Φ is given by pull-push along the diagram

$$
\begin{array}{ccc}
 & \widetilde{\mathcal{Z}}_0^- & \\
 {\scriptstyle \mathrm{pr}_1 \,\circ\, \mathsf{r}_0} \swarrow & & \searrow {\scriptstyle \mathrm{pr}_2 \,\circ\, \mathsf{r}_0} \\
\mathcal{Z}^0 & & \mathcal{Z}\,.
\end{array}
$$

(5.8)

5.2. The natural transformations at the level of kernels. The goal of this subsection is to describe the natural transformations

$$\Phi \to (\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^! \text{ and } (\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^! \to \Phi$$

at the level of kernels.

5.2.1. *The kernel corresponding to* Φ. Set

$$\mathcal{S} := \mathsf{r}_\blacktriangle(\omega_{\widetilde{\mathcal{Z}}^-}) \in \text{D-mod}(\mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}),$$

where $\mathsf{r} : \widetilde{\mathcal{Z}}^- \to \mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}$ was defined in Sect. 5.1.3.

As in Sect. 4.4.2, one shows that

$$\mathcal{S} \in \text{D-mod}(\mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z})^{\mathbb{G}_m\text{-mon}}.$$

Set also

$$\mathcal{S}_0 := (\mathsf{r}_0)_\blacktriangle(\omega_{\widetilde{\mathcal{Z}}_0^-}) \in \text{D-mod}(\mathcal{Z}^0 \times \mathcal{Z}), \qquad \mathcal{S}_1 := (\mathsf{r}_1)_\blacktriangle(\omega_{\widetilde{\mathcal{Z}}_1^-}) \in \text{D-mod}(\mathcal{Z}^0 \times \mathcal{Z}).$$

By Sect. 5.1.4, the functor Φ identifies with $\mathsf{F}_{\mathcal{S}_0}$.

5.2.2. *The kernel corresponding to* $(\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^!$. Now set

$$\mathcal{T} := (\mathsf{q}^- \times \mathsf{p}^-)_\blacktriangle(\omega_{\mathcal{Z}^-}).$$

We have

$$(\mathsf{p}^-)_\blacktriangle \circ (\mathsf{q}^-)^! \simeq \mathsf{F}_{\mathcal{T}}.$$

5.2.3.   Recall the open embedding

$$j : Z^0 \hookrightarrow Z^- \underset{Z}{\times} Z^+,$$

see Proposition 1.9.4.

Let j^- denote the corresponding open embedding

$$Z^- \hookrightarrow Z^- \underset{Z}{\times} Z^+ \underset{Z^0}{\times} Z^- \simeq Z^- \underset{Z}{\times} \widetilde{Z}_0 =: \widetilde{Z}_0^-,$$

obtained by base change.

Let j^- denote the corresponding open embedding

$$\mathcal{Z}^- \hookrightarrow \widetilde{\mathcal{Z}}_0^-.$$

Note that the composition

$$\mathcal{Z}^- \overset{\mathsf{j}^-}{\hookrightarrow} \widetilde{\mathcal{Z}}_0^- \overset{\mathsf{r}_0}{\longrightarrow} \mathcal{Z}^0 \times \mathcal{Z}$$

equals $\mathsf{q}^- \times \mathsf{p}^-$.

5.2.4. *The morphism $\Phi \to (\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!$ at the level of kernels.*

Recall that the morphism $\Phi \to (\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!$ comes from the morphism

$$\big((\mathsf{q}^+)_{\blacktriangle} \circ (\mathsf{p}^+)^!\big) \circ \big((\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!\big) \to \mathrm{Id}_{\mathrm{D\text{-}mod}(\mathcal{Z}^0)}$$

constructed in Sects. 3.3-3.4. By construction, the natural transformation

$$\Phi \to (\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^!$$

corresponds to the map of kernels

(5.9) $\mathcal{S}_0 \to \mathcal{T}$

equal to the composition

$$\mathcal{S}_0 := (\mathsf{r}_0)_{\blacktriangle}(\omega_{\widetilde{\mathcal{Z}}_0^-}) \to (\mathsf{r}_0)_{\blacktriangle} \circ \mathsf{j}^-_{\blacktriangle}(\omega_{\mathcal{Z}^-}) \overset{\sim}{\longrightarrow} (\mathsf{q}^- \times \mathsf{p}^-)_{\blacktriangle}(\omega_{\mathcal{Z}^-}) =: \mathcal{T},$$

where the first arrow comes from

$$\omega_{\widetilde{\mathcal{Z}}_0^-} \to \mathsf{j}^-_\bullet \circ (\mathsf{j}^-)^\bullet(\omega_{\widetilde{\mathcal{Z}}_0^-}) \simeq \mathsf{j}^-_\bullet(\omega_{\mathcal{Z}^-}) \simeq \mathsf{j}^-_{\blacktriangle}(\omega_{\mathcal{Z}^-}).$$

5.2.5. *The isomorphism $\mathcal{T} \simeq \mathcal{S}_1$.*

The (tautological) identification $\widetilde{Z}_1 \simeq Z$ defines an identification

(5.10) $\widetilde{\mathcal{Z}}_1^- \simeq \mathcal{Z}^-,$

so that the morphism $\mathsf{r}_1 : \mathcal{Z}_1^- \to \mathcal{Z}^0 \times \mathcal{Z}$ identifies with $\mathsf{q}^- \times \mathsf{p}^-$.

Hence, we obtain a tautological identification

(5.11) $\mathcal{T} \simeq \mathcal{S}_1.$

5.2.6. *The morphism $(\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^! \to \Phi$ at the level of kernels.*

The map $\mathrm{Sp}_{\mathcal{S}}$ of (4.1) defines a canonical map

(5.12) $\mathcal{S}_1 \to \mathcal{S}_0.$

By Sect. 4.1.5, the natural transformation $(\mathsf{p}^-)_{\blacktriangle} \circ (\mathsf{q}^-)^! \to \Phi$ comes from the map

(5.13) $\mathcal{T} \to \mathcal{S}_1 \to \mathcal{S}_0,$

equal to the composition of (5.11) and (5.12).

5.2.7. *Conclusion.* Thus, in order to prove that the composition (5.3) is the identity map, it suffices to show that the composed map

(5.14) $$\mathfrak{T} \to \mathfrak{S}_1 \to \mathfrak{S}_0 \to \mathfrak{T}$$

is the identity map on $\mathfrak{T}$.

5.3. Passing to an open substack.

5.3.1. Recall the open embedding

$$j^- : Z^- \hookrightarrow \widetilde{Z}_0^-$$

introduced in Sect. 5.2.3.

Let $\overset{\circ}{\widetilde{Z}}{}^-$ denote the open subset of $\widetilde{Z}^-$ obtained by removing the closed subset

$$\left(\widetilde{Z}_0^- - Z^-\right) \subset \widetilde{Z}_0^- \subset \widetilde{Z}^-.$$

Let $\overset{\circ}{\widetilde{\mathcal{Z}}}{}^-$ denote the corresponding open substack of $\widetilde{\mathcal{Z}}^-$. Let $\overset{\circ}{\widetilde{\mathcal{Z}}}{}_t^-$ denote the fiber of $\overset{\circ}{\widetilde{\mathcal{Z}}}{}^-$ over $t \in \mathbb{A}^1$.

By definition, the open embedding

$$\mathsf{j}^- : \mathcal{Z}^- \hookrightarrow \widetilde{\mathcal{Z}}_0^-$$

defines an *isomorphism*

(5.15) $$\mathcal{Z}^- \overset{\sim}{\longrightarrow} \overset{\circ}{\widetilde{\mathcal{Z}}}{}_0^-.$$

Note that the isomorphism $\mathcal{Z}^- \overset{\sim}{\longrightarrow} \widetilde{\mathcal{Z}}_1^-$ of (5.10) still defines an isomorphism

(5.16) $$\mathcal{Z}^- \overset{\sim}{\longrightarrow} \overset{\circ}{\widetilde{\mathcal{Z}}}{}_1^-.$$

5.3.2. Let

$$\overset{\circ}{\mathsf{r}} : \overset{\circ}{\widetilde{\mathcal{Z}}}{}^- \to \mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z} \quad \text{and} \quad \overset{\circ}{\mathsf{r}}_t : \overset{\circ}{\widetilde{\mathcal{Z}}}{}^- \to \mathcal{Z}^0 \times \mathcal{Z}$$

denote the morphisms induced by the maps r and r_t from Sect. 5.1.3.

Set

$$\overset{\circ}{\mathcal{S}} := \overset{\circ}{\mathsf{r}}_{\blacktriangle}(\omega_{\overset{\circ}{\widetilde{\mathcal{Z}}}{}^-}),$$

and also

$$\overset{\circ}{\mathcal{S}}_0 := (\overset{\circ}{\mathsf{r}}_0)_{\blacktriangle}(\omega_{\overset{\circ}{\widetilde{\mathcal{Z}}}{}_0^-}) \quad \text{and} \quad \overset{\circ}{\mathcal{S}}_1 := (\overset{\circ}{\mathsf{r}}_1)_{\blacktriangle}(\omega_{\overset{\circ}{\widetilde{\mathcal{Z}}}{}_1^-}).$$

The open embedding $\overset{\circ}{\widetilde{\mathcal{Z}}}{}^- \hookrightarrow \widetilde{\mathcal{Z}}^-$ gives rise to the maps

$$\mathcal{S} \to \overset{\circ}{\mathcal{S}}, \quad \mathcal{S}_0 \to \overset{\circ}{\mathcal{S}}_0, \quad \mathcal{S}_1 \to \overset{\circ}{\mathcal{S}}_1.$$

As in Sects. 5.2.4-5.2.6, we have the natural transformations

(5.17) $$\mathfrak{T} \to \overset{\circ}{\mathcal{S}}_1 \to \overset{\circ}{\mathcal{S}}_0 \to \mathfrak{T}.$$

Moreover, the diagram

$$
\begin{array}{ccccccc}
\mathcal{T} & \longrightarrow & \mathcal{S}_1 & \longrightarrow & \mathcal{S}_0 & \longrightarrow & \mathcal{T} \\
\text{id} \downarrow & & \downarrow & & \downarrow & & \downarrow \text{id} \\
\mathcal{T} & \longrightarrow & \overset{\circ}{\mathcal{S}}_1 & \longrightarrow & \overset{\circ}{\mathcal{S}}_0 & \longrightarrow & \mathcal{T}
\end{array}
$$

commutes.

Hence, in order to show that the composed map (5.14) is the identity map, *it suffices to show that the composed map* (5.17) *is the identity map.* We will do this in the next subsection.

5.4. The key argument.

5.4.1. Recall now the open embedding

$$
\mathbb{A}^1 \times Z^- \to Z^- \underset{Z}{\times} \widetilde{Z} =: \widetilde{Z}^-
$$

of (2.18).

By definition, it induces an isomorphism

$$
\mathbb{A}^1 \times Z^- \simeq \overset{\circ}{\widetilde{Z}}{}^-.
$$

Dividing by the action of $\mathbb{G}_m$, we obtain an isomorphism

(5.18)
$$
\mathbb{A}^1 \times \mathcal{Z}^- \simeq \overset{\circ}{\widetilde{\mathcal{Z}}}{}^-.
$$

Under this identification, we have:

- The map $\overset{\circ}{\mathsf{r}} : \overset{\circ}{\widetilde{\mathcal{Z}}}{}^- \to \mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}$ identifies with the map $\mathbb{A}^1 \times \mathcal{Z}^- \to \mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}$ induced by $\mathrm{id}_{\mathbb{A}^1} : \mathbb{A}^1 \to \mathbb{A}^1$ and $(\mathsf{q}^- \times \mathsf{p}^-) : \mathcal{Z}^- \to \mathcal{Z}^0 \times \mathcal{Z}$.

- The isomorphism $\mathcal{Z}^- \overset{\sim}{\longrightarrow} \overset{\circ}{\widetilde{\mathcal{Z}}}{}^-_1$ of (5.16) corresponds to the identity map
$$
\mathcal{Z}^- \to (\mathbb{A}^1 \times \mathcal{Z}^-) \underset{\mathbb{A}^1}{\times} \{1\} \simeq \mathcal{Z}^-.
$$

- The isomorphism $\mathcal{Z}^- \overset{\sim}{\longrightarrow} \overset{\circ}{\widetilde{\mathcal{Z}}}{}^-_0$ of (5.15) corresponds to the identity map
$$
\mathcal{Z}^- \to (\mathbb{A}^1 \times \mathcal{Z}^-) \underset{\mathbb{A}^1}{\times} \{0\} \simeq \mathcal{Z}^-.
$$

5.4.2. Hence, we obtain that the composition (5.17) identifies with

$$
\mathcal{T} \simeq \iota_1^!(\omega_{\mathbb{A}^1} \boxtimes \mathcal{T}) \xrightarrow{\mathrm{Sp}} \iota_0^!(\omega_{\mathbb{A}^1} \boxtimes \mathcal{T}) \simeq \mathcal{T},
$$

where $\mathrm{Sp} := \mathrm{Sp}_{\omega_{\mathbb{A}^1} \boxtimes \mathcal{T}}$ is the specialization map (4.1) for the object

$$
\omega_{\mathbb{A}^1} \boxtimes \mathcal{T} \in \mathrm{D\text{-}mod}(\mathbb{A}^1 \times \mathcal{Z}^0 \times \mathcal{Z}).
$$

The fact that the above map is the identity map on $\mathcal{T}$ follows from Sect. 4.1.4.

Appendix A. Pro-categories

A.1. For a DG category $\mathbf{C}$ let $\mathrm{Pro}(\mathbf{C})$ denote its pro-completion, thought of as the DG category opposite to that of covariant exact functors $\mathbf{C} \to \mathrm{Vect}$, where Vect denotes the DG category of complexes of k-vector spaces. [22]

Yoneda embedding defines a fully faithful functor $\mathbf{C} \to \mathrm{Pro}(\mathbf{C})$. Any object in $\mathrm{Pro}(\mathbf{C})$ can be written as a filtered limit (taken in $\mathrm{Pro}(\mathbf{C})$) of co-representable functors.

A.2. A functor $\mathsf{F} : \mathbf{C}' \to \mathbf{C}''$ between DG categories induces a functor denoted also by F

$$\mathrm{Pro}(\mathbf{C}') \to \mathrm{Pro}(\mathbf{C}'')$$

by applying the *right Kan extension* of the functor

$$\mathbf{C}' \xrightarrow{\mathsf{F}} \mathbf{C}'' \hookrightarrow \mathrm{Pro}(\mathbf{C}'')$$

along the embedding $\mathbf{C}' \to \mathrm{Pro}(\mathbf{C}')$.

The same construction can be phrased as follows: for $\widetilde{\mathbf{c}}' \in \mathrm{Pro}(\mathbf{C}')$, thought of as a functor $\mathbf{C}' \to \mathrm{Vect}$, the object $\mathsf{F}(\widetilde{\mathbf{c}}')$, thought of as a functor $\mathbf{C}'' \to \mathrm{Vect}$, is the *left Kan extension* of $\widetilde{\mathbf{c}}'$ along the functor $\mathsf{F} : \mathbf{C}' \to \mathbf{C}''$.

Explicitly, if $\widetilde{\mathbf{c}} \in \mathrm{Pro}(\mathbf{C}')$ is written as $\varprojlim_{i \in I} \mathbf{c}_i$ with $\mathbf{c}_i \in \mathbf{C}'$, then

$$\mathsf{F}(\widetilde{\mathbf{c}}) \simeq \varprojlim_{i \in I} \mathsf{F}(\mathbf{c}_i),$$

as objects of $\mathrm{Pro}(\mathbf{C}'')$ and

$$\mathsf{F}(\widetilde{\mathbf{c}}) \simeq \varinjlim_{i \in I} \mathcal{M}aps_{\mathbf{C}''}(\mathsf{F}(\mathbf{c}_i), -),$$

as functors $\mathbf{C}'' \to \mathrm{Vect}$.

A.3. Let $\mathsf{G} : \mathbf{C}' \to \mathbf{C}''$ be a functor between DG categories. We can speak of its left adjoint G^L as a functor $\mathbf{C}'' \to \mathrm{Pro}(\mathbf{C}')$. Namely, for $\mathbf{c}'' \in \mathbf{C}''$ the object $\mathsf{G}^L(\mathbf{c}'') \in \mathrm{Pro}(\mathbf{C}')$, thought of as a functor $\mathbf{C}' \to \mathrm{Vect}$ is given by

$$(\mathsf{G}^L(\mathbf{c}''))(\mathbf{c}') = \mathcal{M}aps_{\mathbf{C}''}(\mathbf{c}'', \mathsf{G}(\mathbf{c}')).$$

A.4. We let the same symbol G^L also denote the functor $\mathrm{Pro}(\mathbf{C}'') \to \mathrm{Pro}(\mathbf{C}')$ obtained as the right Kan extension of $\mathsf{G}^L : \mathbf{C}'' \to \mathrm{Pro}(\mathbf{C}')$ along $\mathbf{C}'' \hookrightarrow \mathrm{Pro}(\mathbf{C}'')$.

The functor G^L is the left adjoint of the functor $\mathsf{G} : \mathrm{Pro}(\mathbf{C}') \to \mathrm{Pro}(\mathbf{C}'')$.

We can also think of G^L as follows: for $\widetilde{\mathbf{c}}'' \in \mathrm{Pro}(\mathbf{C}'')$, thought of as a functor $\mathbf{C}'' \to \mathrm{Vect}$, the object $\mathsf{G}^L(\widetilde{\mathbf{c}}'')$, thought of as a functor $\mathbf{C}' \to \mathrm{Vect}$ is given by

$$(\mathsf{G}^L(\widetilde{\mathbf{c}}''))(\mathbf{c}') = \widetilde{\mathbf{c}}''(\mathsf{G}(\mathbf{c}')).$$

---

[22] A way to deal with set-theoretical difficulties is to require that our functors commute with κ-filtered colimits for some cardinal κ, see [Lur, Def. 5.3.1.7].

References

[Ach] P. N. Achar, *Green functions via hyperbolic localization*, Doc. Math. **16** (2011), 869–884.

[AC] P. N. Achar and C. L. R. Cunningham, *Toward a Mackey formula for compact restriction of character sheaves*, arXiv:1011.1846.

[AM] P. N. Achar and C. Mautner, *Sheaves on nilpotent cones, Fourier transform, and a geometric Ringel duality*, arXiv:1207.7044.

[Bi-Br] S. C. Billey and T. Braden, *Lower bounds for Kazhdan-Lusztig polynomials from patterns*, Transform. Groups **8** (2003), no. 4, 321–332.

[Bia] A. Białynicki-Birula. *Some Theorems on Actions of Algebraic Groups*, Ann. of Math. (2), **98** (1973), 480–497.

[Br] T. Braden, *Hyperbolic localization of Intersection Cohomology*, Transformation Groups **8** (2003), no. 3, 209–216. Also: arXiv:math/020225.

[Dr] V. Drinfeld, *On algebraic spaces with an action of* $\mathbb{G}_m$, arXiv:math/1308.2604.

[DrGa1] V. Drinfeld and D. Gaitsgory, *On some finiteness questions for algebraic stacks*, Geom.and Funct. Analysis **23** (2013), 149–294.

[DrGa2] V. Drinfeld and D. Gaitsgory, *Compact generation of the category of D-modules on the stack of G-bundles on a curve*, arXiv:math/1112.2402.

[DrGa3] V. Drinfeld and D. Gaitsgory, *Geometric constant term functor(s)*, arXiv: 1311.2071.

[GM] M. Goresky and R. MacPherson, *Local contribution to the Lefschetz fixed point formula*, Invent. Math. **111** (1993), 1–33.

[GH] I. Grojnowski and M. Haiman, *Affine Hecke algebras and positivity of LLT and Macdonald polynomials*, available at http://math.berkeley.edu/~mhaiman

[Ju1] J. Jurkiewicz, *An example of algebraic torus action which determines the nonfiltrable decomposition*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. **25** (1977), no. 11, 1089–1092.

[Ju2] J. Jurkiewicz, *Torus embeddings, polyhedra, k*-actions and homology*, Dissertationes Math. (Rozprawy Mat.) **236** (1985), 64 pp.

[KKMS] G. Kempf, F. F. Knudsen, D. Mumford, and B. Saint-Donat, Toroidal embeddings. I. Lecture Notes in Math. **339**, Springer-Verlag, Berlin-New York, 1973.

[KKLV] F. Knop, H. Kraft, D. Luna, and T. Vust, *Local properties of algebraic group actions*. In: Algebraische Transformationsgruppen und Invariantentheorie, p. 63–75, DMV Seminar **13**, Birkhäuser, Basel, 1989.

[Kn] D. Knutson, Algebraic spaces. Lecture Notes in Math. **203**, Springer-Verlag, Berlin-New York, 1971.

[Kon] J. Konarski, *A pathological example of an action of k**. In: Group actions and vector fields (Vancouver, B.C., 1981), p. 72–78, Lecture Notes in Math. **956**, Springer, Berlin, 1982.

[LM] G. Laumon and L. Moret-Bailly, *Champs algébriques*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3 Folge, A Series of Modern Surveys in Mathematics), **39**, Springer-Verlag, Berlin, 2000.

[Lur] J. Lurie, *Higher Topos Theory*, Annals of Mathematics Studies, **170**, Princeton University Press, Princeton, NJ, 2009.

[Ly1] S. Lysenko, *Moduli of metaplectic bundles on curves and theta- sheaves*, Ann. Sci. École Norm. Sup. (4) **39** (2006), no. 3, 415–466.

[Ly2] S. Lysenko, *Geometric theta-lifting for the dual pair SO(2m),Sp(2n)*, Ann. Sci. École Norm. Sup. (4) **44** (2011), no. 3, 427–493.

[MV] I. Mirković and K. Vilonen, *Geometric Langlands duality and representations of algebraic groups over commutative rings*, Ann. of Math. (2) **166** (2007), no. 1, 95–143.

[Nak] H. Nakajima, *Quiver varieties and tensor products II*, arXiv:1207.0529

[Som] A. J. Sommese, *Some examples of* $\mathbb{C}^*$ *actions*. In: Group actions and vector fields (Vancouver, B.C., 1981), p. 118–124, Lecture Notes in Math. **956**, Springer, Berlin, 1982.

[Sum] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto Univ. **14** (1974), 1–28.