# ON MODULI STACKS OF $G$-BUNDLES OVER A CURVE

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Abstract. Let $C$ be a smooth projective curve over an algebraically closed field $k$ of arbitrary characteristic. Given a linear algebraic group $G$ over $k$, let $\mathcal{M}_G$ be the moduli stack of principal $G$-bundles on $C$. We determine the set of connected components $\pi_0(\mathcal{M}_G)$ for smooth connected groups $G$.

## 1. Introduction

Let $C$ be a smooth projective algebraic curve over an algebraically closed field $k$. This text explains some basic properties of the moduli stack $\mathcal{M}_G$ of algebraic principal $G$-bundles on $C$, for a linear algebraic group $G$ over $k$. The arguments given are purely algebraic, and valid in any characteristic.

The stack $\mathcal{M}_G$ is algebraic in the sense of Artin, and locally of finite type over $k$. Moreover, $\mathcal{M}_G$ is smooth if $G$ is smooth. The main purpose of this paper is to determine the set of connected components $\pi_0(\mathcal{M}_G)$ if $G$ is smooth and connected. It turns out that the unipotent radical of $G$ doesn't matter for this. In the case where $G$ is reductive, Theorem 5.8 gives a canonical bijection between $\pi_0(\mathcal{M}_G)$ and the fundamental group $\pi_1(G)$, the latter being defined in terms of the root system; cf. Definition 5.4.

This statement is well-established folklore, and thus not a new result. But the published literature seems to contain no proof of it in full generality, covering also the case of positive characteristic $\mathrm{char}(k) = p > 0$. For simply connected $G$, the result is proved in [6]; the general case is treated, from a different point of view, in the apparently unpublished preprint [11].

The proof given here is based on the maps $\mathcal{M}_G \to \mathcal{M}_H$ induced by group homomorphisms $G \to H$. In particular, it uses criteria for lifting $H$-bundles to $G$-bundles if $H$ is a quotient of $G$. Corollary 3.4 states that this is always possible if $G$, $H$, and the kernel are smooth and connected; this little observation might be of independent interest.

After recalling the algebraicity of $\mathcal{M}_G$ in Section 2, these lifting problems are studied in Section 3. Based on them, the standard deformation theory argument for smoothness of $\mathcal{M}_G$ is recalled in Section 4. Finally, Section 5 contains the results mentioned above about connected components of $\mathcal{M}_G$.

## 2. Algebraicity

Throughout this text, we fix an algebraically closed base field $k$ and an irreducible smooth projective curve $C/k$. We denote by $\mathcal{M}_G$ the moduli stack of principal $G$-bundles $E$ on $C$, where $G \subseteq \mathrm{GL}_n$ is a linear algebraic group.

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*Remark* 2.1. More precisely, $\mathcal{M}_G$ is given as a prestack over $k$ by the groupoid $\mathcal{M}_G(S)$ of principal $G$-bundles on $C \times_k S$ for each $k$-scheme $S$. This prestack is indeed a stack: the required descent for $G$-bundles is a special case of the standard descent for affine morphisms since $G$ is affine.

*Remark* 2.2. More generally, one can consider the moduli stack $\mathcal{M}_{\mathcal{G}}$ of principal bundles under a relatively affine group scheme $\mathcal{G}$ over $C$. We will use only the special case where $\mathcal{G} = V$ is (the underlying additive group scheme of) a vector bundle on $C$. Here principal $V$-bundles correspond to vector bundle extensions

$$0 \longrightarrow V \longrightarrow E \longrightarrow \mathcal{O}_C \longrightarrow 0,$$

so their moduli stack $\mathcal{M}_V$ is the stack quotient of the affine space $\mathrm{H}^1_{\mathrm{Zar}}(C, V)$ modulo the trivial action of the additive group $\mathrm{H}^0_{\mathrm{Zar}}(C, V)$. In particular, we see that $\mathcal{M}_V$ is a smooth connected Artin stack in this case.

Given a morphism of linear algebraic groups $\phi : G \to H$, extending the structure group of principal $G$-bundles to $H$ defines a 1-morphism

$$\phi_* : \mathcal{M}_G \longrightarrow \mathcal{M}_H.$$

**Fact 2.3.** *If $\iota : H \hookrightarrow G$ is a closed embedding, then the* 1*-morphism of stacks $\iota_* : \mathcal{M}_H \to \mathcal{M}_G$ is representable and locally of finite type.*

*Proof.* (cf. [15, 3.6.7]) The homogeneous space $G/H$ exists by Chevalley's theorem [5, III, §3, Thm. 5.4]; more precisely, $G$ is a principal $H$-bundle over some quasiprojective variety $X = G/H$. Given a principal $G$-bundle $\pi : E \to C \times_k S$, reductions of its stucture group to $H$ correspond bijectively to sections of the associated bundle $\pi_X : E \times^G X \to C \times_k S$ with fiber $X$.

This means that the fiber product of $S$ and $\mathcal{M}_H$ over $\mathcal{M}_G$ is the functor from $S$-schemes to sets that sends $f : T \to S$ to the sections of $f^*\pi_X$. This functor is representable by some locally closed subscheme of an appropriate relative Hilbert scheme, which is locally of finite type over $S$. □

By an *algebraic stack* over $k$, we always mean an Artin stack that is locally of finite type over $k$ (but not necessarily quasi-compact). For example, the moduli stack $\mathcal{M}_V$ for a vector bundle $V$ on $C$ is algebraic, according to Remark 2.2.

**Fact 2.4.** *If $G$ is a linear algebraic group, then $\mathcal{M}_G$ is an algebraic stack.*

*Proof.* (cf. [15, 3.6.6.]) In the case $G = \mathrm{GL}_n$, this is well known, cf. [12, 4.14.2.1]. The general case $G \hookrightarrow \mathrm{GL}_n$ then follows from the previous fact. □

## 3. Lifting principal bundles

We say that a short sequence of linear algebraic groups

$$1 \longrightarrow K \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1 \tag{3.1}$$

is *exact* if $\pi$ is faithfully flat and $K$ is the kernel of $\pi$. Then $H$ acts on $K$ by conjugation in $G$. Given a principal $H$-bundle $F$ on $C$, we denote by

$$K^F := K \times^H F := (K \times F)/H$$

the corresponding twisted group scheme over $C$ with fiber $K$.

**Proposition 3.1.** *Suppose that* (3.1) *is a short exact sequence of linear algebraic groups, with $K$ commutative. Let $F$ be a principal $H$-bundle on $C$.*

i) *There is a canonical obstruction class* $\mathrm{ob}_F \in \mathrm{H}^2_{\mathrm{fppf}}(C, K^F)$*, which vanishes if and only if* $F \cong \pi_* E$ *for some principal* $G$*-bundle* $E$ *on* $C$*.*

ii) *If* $\mathrm{ob}_F$ *vanishes, then the fiber of* $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ *over the point* $F$ *is* 1*-isomorphic to the moduli stack* $\mathcal{M}_{K^F}$ *of principal* $K^F$*-bundles.*

$$\begin{tikzcd} \mathcal{M}_{K^F} \arrow[r] \arrow[d] & \mathcal{M}_G \arrow[d, "\pi_*"] \\ \mathrm{Spec}(k) \arrow[r, "F"] & \mathcal{M}_H \end{tikzcd}$$

*Proof.* The lifts of $F$ to $G$-bundles $E$ form a stack $\mathcal{K}_F$ over $C$, which is more precisely given by the following groupoid $\mathcal{K}_F(X)$ for each $C$-scheme $f : X \to C$:

- Its objects are principal $G$-bundles $\mathcal{E}$ on $X$ together with isomorphisms $\pi_*(\mathcal{E}) \cong f^*(F)$ of principal $H$-bundles on $X$.
- Its morphisms are isomorphisms of principal $G$-bundles on $X$ which are compatible with the identity on $f^*(F)$.

If $F$ is trivial, then a lift of $F$ to a principal $G$-bundle is nothing but a principal $K$-bundle, so $\mathcal{K}_F$ is just the classifying stack $BK \times C$ in this case. In any case, $F$ is fppf-locally trivial, so $\mathcal{K}_F$ is an fppf-gerbe over $C$, whose band is the common automorphism group scheme $K^F$ of all (local) lifts of $F$. The class of this gerbe in $\mathrm{H}^2_{\mathrm{fppf}}(C, K^F)$ is the required obstruction $\mathrm{ob}_F$; cf. [7, IV, Thm. 3.4.2].

If $\mathrm{ob}_F$ vanishes, then the gerbe $\mathcal{K}_F \to C$ admits a section, so $\mathcal{K}_F$ is the classifying stack $B(K^F)$ over $C$ by [12, Lemme 3.21]. Thus sections $C \to \mathcal{K}_F$ are nothing but principal $K^F$-bundles on $C$; this implies ii. □

*Remark* 3.2. In the above situation, suppose that $K$ is central in $G$. Given a principal $G$-bundle $E$ with $\pi_* E \cong F$, we can explicitly describe a 1-isomorphism between $\mathcal{M}_{K^F} = \mathcal{M}_K$ and the fiber of $\pi_*$ over $[F]$ as follows:

The multiplication $\mu : K \times G \to G$ is a group homomorphism, so it induces a 1-morphism $\mu_* : \mathcal{M}_K \times \mathcal{M}_G \to \mathcal{M}_G$. Its restriction $\mu_*(\_, [E]) : \mathcal{M}_K \to \mathcal{M}_G$ is then a 1-isomorphism onto the fiber of $\pi_*$ over $[F]$.

*Remark* 3.3. Up to now, we have not used the assumption $\dim(C) = 1$. Using it, one can show that the obstruction $\mathrm{ob}_F$ vanishes in the following two cases:

i) Assume $K \cong \mathbb{G}_a^r$, and that the action $H \to \mathrm{Aut}(K)$ factors through $\mathrm{GL}_r$. (The latter is automatic for $K \cong \mathbb{G}_a$, since $\mathrm{Aut}(K) \cong \mathbb{G}_m$ in this situation. But for $r > 1$ and $\mathrm{char}(k) = p > 0$, this is actually a condition.) Then $K^F$ is a vector bundle on $C$, and

$$\mathrm{H}^2_{\mathrm{fppf}}(C, K^F) = \mathrm{H}^2_{\mathrm{\acute{e}t}}(C, K^F) = \mathrm{H}^2_{\mathrm{Zar}}(C, K^F) = 0$$

due to [8, Thm. 11.7], [10, Exp. VII, Prop. 4.3], and the assumption $\dim(C) = 1$.

ii) Assume $K \cong \mathbb{G}_m^r$, and that $H$ is connected. Then $\mathrm{Aut}(K) \cong \mathrm{GL}_r(\mathbb{Z})$ is discrete, so the action of $H$ on $K$ is trivial. Thus $K^F$ is just the split torus $\mathbb{G}_m^r$ over $C$, and $\mathrm{H}^2_{\mathrm{fppf}}(C, K^F) = \mathrm{H}^2_{\mathrm{\acute{e}t}}(C, K^F) = 0$ by Tsen's theorem.

**Corollary 3.4.** *If* $1 \to K \longrightarrow G \xrightarrow{\pi} H \to 1$ *is a short exact sequence of smooth connected linear algebraic groups, then* $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ *is surjective.*

*Proof.* Choose a Borel subgroup $B_G$ in $G$. Then $B_H := \pi(B_G)$ is a Borel subgroup in $H$ due to [3, Proposition (11.14)]. Every principal $H$-bundle $F$ on $C$ admits a reduction of its structure group to $B_H$ by [6, Theorem 1 and Remark 2.e].

The identity component $B_K^0 \subseteq B_K$ of the intersection $B_K := K \cap B_G$ is a Borel subgroup in $K$ due to [3, Proposition (11.14)] again. As $B_K^0$ is normal in $B_K$, it follows that $B_K$ is contained in the normalizer of $B_K^0$ in $K$, which is just $B_K^0$ itself by [3, Theorem (11.15)]. Thus $B_K^0 = B_K$, and the sequence $1 \to B_K \to B_G \to B_H \to 1$ is again exact. Replacing the given exact sequence by this one, we may assume without loss of generality that the three groups $G$, $H$ and $K$ are all solvable.

Using induction on $\dim(K)$, we may then assume $\dim(K) = 1$, which means $K \cong \mathbb{G}_a$ or $K \cong \mathbb{G}_m$. In this situation, the obstruction against lifting principal $H$-bundles on $C$ to principal $G$-bundles vanishes by Remark 3.3. This shows that the induced 1-morphism $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ is indeed surjective. □

## 4. Smoothness

From now on, we will concentrate on *smooth* linear algebraic groups $G$ over $k$. Then every principal $G$-bundle is étale-locally trivial.

**Proposition 4.1.** *If the group $G$ is smooth, then the stack $\mathcal{M}_G$ is also smooth.*

*Proof.* (See [1, 4.5.1 and 8.1.9] for a different presentation of similar arguments.) We verify that $\mathcal{M}_G$ satisfies the infinitesimal criterion for smoothness.

Let a pair $(A, \mathfrak{m})$ and $(\tilde{A}, \tilde{\mathfrak{m}})$ of local artinian $k$-algebras with residue field $k$ be given, such that $A = \tilde{A}/(\nu)$ for some $\nu \in \tilde{A}$ with $\tilde{\mathfrak{m}} \cdot \nu = 0$. We have to show that every principal $G$-bundle $\mathcal{E}$ on $C \otimes_k A$ can be extended to $C \otimes_k \tilde{A}$.

We define a functor $G_A$ from $k$-schemes to groups by $G_A(S) := G(S \otimes_k A)$. Then $G_A$ is a smooth linear algebraic group, and the infinitesimal theory of group schemes [5, II, §4, Thm. 3.5] yields an exact sequence

$$1 \longrightarrow \mathfrak{g} \longrightarrow G_{\tilde{A}} \longrightarrow G_A \longrightarrow 1$$

where $\mathfrak{g}$ is (the underlying additive group of) the Lie algebra of $G$.

As $C$ and $C \otimes_k A$ are homeomorphic for the étale topology, the étale-locally trivial principal $G$-bundle $\mathcal{E}$ on $C \otimes_k A$ corresponds to a principal $G_A$-bundle $\mathcal{E}$ on $C$. Using Proposition 3.1 and Remark 3.3.i, we can lift this $G_A$-bundle to a principal $G_{\tilde{A}}$-bundle on $C$. This yields the required $G$-bundle on $C \otimes_k \tilde{A}$. □

**Corollary 4.2.** *If $1 \to K \longrightarrow G \xrightarrow{\pi} H \to 1$ is a short exact sequence of smooth linear algebraic groups, then $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ is also smooth.*

*Proof.* We know already that $\mathcal{M}_G$ and $\mathcal{M}_H$ are smooth over $k$, so it suffices to show that the 1-morphism $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ is submersive.

Let $E$ be a principal $G$-bundle on $C$, with induced $H$-bundle $F := \pi_*(E)$. Given an extension of $F$ to a principal $H$-bundle $\mathcal{F}$ on $C \otimes_k k[\varepsilon]$ with $\varepsilon^2 = 0$, we have to extend $E$ to a principal $G$-bundle $\mathcal{E}$ on $C \otimes_k k[\varepsilon]$ such that the identity $\pi_*(E) = F$ can be extended to an isomorphism $\pi_*(\mathcal{E}) \cong \mathcal{F}$.

The given datum $(E, F, \mathcal{F})$ corresponds to a principal $(G \times_H H_{k[\varepsilon]})$-bundle on $C$. Using the exact sequence of groups

$$1 \longrightarrow \mathfrak{k} := \mathrm{Lie}(K) \longrightarrow G_{k[\varepsilon]} \longrightarrow G \times_H H_{k[\varepsilon]} \longrightarrow 1,$$

we can lift it to a principal $G_{k[\varepsilon]}$-bundle on $C$, according to Proposition 3.1 and Remark 3.3.i. This extends $E$ to a $G$-bundle $\mathcal{E}$ on $C \otimes_k k[\varepsilon]$, as required. □

## 5. Connected components

In this section, we suppose that the linear algebraic group $G$ is smooth and connected. The aim is to describe the set of connected components $\pi_0(\mathcal{M}_G)$.

**Proposition 5.1.** *If $1 \to U \to G \to H \to 1$ is a short exact sequence of smooth connected linear algebraic groups with $U$ unipotent, then $\pi_0(\mathcal{M}_G) = \pi_0(\mathcal{M}_H)$.*

*Proof.* The induced 1-morphism $\mathcal{M}_G \to \mathcal{M}_H$ is smooth by Corollary 4.2, and surjective by Corollary 3.4. We have to show that its fibers are connected.

Let $B_H \subseteq H$ be a Borel subgroup. Every principal $H$-bundle on $C$ admits a reduction of its structure group to $B_H$ by [6, Theorem 1 and Remark 2.e]. Replacing $H$ by $B_H$ and $G$ by the inverse image $B_G$ of $B_H$ if necessary, we may thus assume that $G$ and $H$ are solvable.

Using induction on $\dim(U)$, we may then moreover assume $U \cong \mathbb{G}_a$. In this situation, the fibers in question have the form $\mathcal{M}_L$ for line bundles $L$ on $C$, according to Proposition 3.1.ii; see also Remark 3.3.i. Hence these fibers are connected due to Remark 2.2. $\square$

In particular, $\pi_0(\mathcal{M}_G) = \pi_0(\mathcal{M}_{G/G_u})$, where $G_u \subseteq G$ denotes the unipotent radical. Thus it suffices to determine the set $\pi_0(\mathcal{M}_G)$ for reductive groups $G$.

Given any torus $T \cong \mathbb{G}_m^r$ over $k$, we denote its cocharacter lattice by

$$X_*(T) := \mathrm{Hom}(\mathbb{G}_m, T) \cong \mathbb{Z}^r.$$

Sending line bundles to their degree defines a bijection $\pi_0(\mathcal{M}_{\mathbb{G}_m}) \xrightarrow{\sim} \mathbb{Z}$, since the Jacobian $\mathrm{Pic}^0(C)$ is connected. Thus we obtain an induced canonical bijection

$$\pi_0(\mathcal{M}_T) \xrightarrow{\sim} X_*(T).$$

If $T$ appears in a central extension of smooth connected linear algebraic groups

$$1 \longrightarrow T \longrightarrow G \xrightarrow{\pi} H \longrightarrow 1,$$

then the multiplication $\mu : T \times G \to G$ is a group homomorphism, and

$$\mu_* : \pi_0(\mathcal{M}_T) \times \pi_0(\mathcal{M}_G) \longrightarrow \pi_0(\mathcal{M}_G)$$

is an action of the group $\pi_0(\mathcal{M}_T)$ on the set $\pi_0(\mathcal{M}_G)$.

*Remark* 5.2. Actually the group stack $\mathcal{M}_T$ acts on $\mathcal{M}_G$, and $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ is a torsor under this action; see [2, Section 5.1]. But we won't use these stack notions here, since all we need can readily be said in more elementary language.

**Proposition 5.3.** *In the above situation, $\pi_0(\mathcal{M}_H) = \pi_0(\mathcal{M}_G)/\pi_0(\mathcal{M}_T)$.*

*Proof.* The induced 1-morphism $\pi_* : \mathcal{M}_G \to \mathcal{M}_H$ is surjective by Corollary 3.4, and smooth by Corollary 4.2. In particular, $\pi_*$ is open; its fibers are all isomorphic to $\mathcal{M}_T$ by Proposition 3.1.ii. These properties imply the proposition:

Since $\pi_*$ is surjective, it induces a surjective map $\pi_0(\mathcal{M}_G) \to \pi_0(\mathcal{M}_H)$. As it is invariant under the action of $\pi_0(\mathcal{M}_T)$, it descends to a surjective map

$$\pi_0(\mathcal{M}_G)/\pi_0(\mathcal{M}_T) \longrightarrow \pi_0(\mathcal{M}_H).$$

To check that this map is also injective, let $\pi_0(\mathcal{M}_G) = \coprod_i X_i$ be the decomposition into $\pi_0(\mathcal{M}_T)$-orbits. It correspond to a decomposition $\mathcal{M}_G = \coprod_i \mathcal{U}_i$ into open substacks. Due to Remark 3.2, each fiber of $\pi_*$ is contained in a single $\mathcal{U}_i$, so the images $\pi_*(\mathcal{U}_i) \subseteq \mathcal{M}_H$ are still disjoint. As $\pi_*$ is open, $\pi_*(\mathcal{U}_i)$ is open in $\mathcal{M}_H$. They form a decomposition of $\mathcal{M}_H$, since $\pi_*$ is surjective. Hence different $\pi_0(\mathcal{M}_T)$-orbits in $\pi_0(\mathcal{M}_G)$ map to different components of $\mathcal{M}_H$. □

Now suppose that the smooth and connected linear algebraic group $G$ over $k$ is reductive. Choosing a maximal torus $T_G \subseteq G$, let

$$X_{\mathrm{coroots}} \subseteq X_*(T_G)$$

denote the subgroup generated by the coroots of $G$.

**Definition 5.4.** The fundamental group of $G$ is $\pi_1(G) := X_*(T_G)/X_{\mathrm{coroots}}$.

Note that the Weyl group of $(G, T_G)$ acts trivially on $\pi_1(G)$. Hence this fundamental group does not depend on the choice of the maximal torus $T_G$, up to a *canonical* isomorphism. $G$ is called *simply connected* if $\pi_1(G)$ is trivial.

*Remark* 5.5. If $k = \mathbb{C}$, then $\pi_1(G)$ coincides with the usual topological fundamental group $\pi_1^{\mathrm{top}}(G)$ of $G$ as a complex Lie group. If more generally $\mathrm{char}(k) = 0$, then $\pi_1(G)$ coincides with $\pi_1^{\mathrm{top}}(G \otimes_k \mathbb{C})$ for every embedding $k \hookrightarrow \mathbb{C}$.

*Remark* 5.6. i) Due to [4], each finite quotient $\pi_1(G) \twoheadrightarrow \mathbb{Z}/n_1 \times \cdots \times \mathbb{Z}/n_r$ corresponds to a central isogeny $\widetilde{G} \twoheadrightarrow G$. Its kernel is isomorphic to $\mu_{n_1} \times \cdots \times \mu_{n_r}$.

ii) In particular, étale isogenies $\widetilde{G} \twoheadrightarrow G$ correspond to finite quotients of $\pi_1(G)$ whose order is not divisible by the characteristic of $k$.

iii) If $G$ is semisimple, then $\pi_1(G)$ itself is finite. The corresponding central isogeny $\widetilde{G} \twoheadrightarrow G$ is called the *universal covering* of $G$.

*Remark* 5.7. i) Denote by $\pi_1^{\text{ét}}(G)$ the étale fundamental group of $G$, and by $\hat{\pi}_1(G)$ the profinite completion of $\pi_1(G)$. Let $\pi_1^{\text{ét}}(G) \twoheadrightarrow \pi_1^{\text{ét}}(G)'$ and $\hat{\pi}_1(G) \twoheadrightarrow \hat{\pi}_1(G)'$ be identities if $\mathrm{char}(k) = 0$, and the largest prime-to-$p$ quotients if $\mathrm{char}(k) = p > 0$. Then Remark 5.6.ii implies that $\pi_1^{\text{ét}}(G)'$ is canonically isomorphic to $\hat{\pi}_1(G)'$.

To verify this, one has to show, for every connected scheme $X$ together with a finite étale morphism $\pi : X \to G$ such that $\deg(\pi)$ is not divisible by $\mathrm{char}(k)$, that there is a group structure on $X$ such that $\pi$ is an isogeny. This can be checked like the analogous statement in topology, using the Künneth formula

$$\pi_1^{\text{ét}}(G \times G)' = \pi_1^{\text{ét}}(G)' \times \pi_1^{\text{ét}}(G)'$$

proved in [9, Exp. XIII, Prop. 4.6] and [13, Prop. 4.7].

ii) Suppose $\mathrm{char}(k) = p > 0$. Then each finite quotient of $\pi_1(G)$ which is a $p$-group corresponds to a purely inseperable central isogeny $\widetilde{G} \twoheadrightarrow G$. On the other hand, the $p$-part of $\pi_1^{\text{ét}}(G)$ is huge and in particular non-abelian; cf. for example [14]. Thus the $p$-parts of $\hat{\pi}_1(G)$ and of $\pi_1^{\text{ét}}(G)$ don't seem to be related.

**Theorem 5.8.** *If the linear algebraic group $G$ over $k$ is smooth, connected, and reductive, then one has a canonical bijection $\pi_0(\mathcal{M}_G) \cong \pi_1(G)$.*

*Proof.* We partly follow [6, Proposition 5], where the connectedness of $\mathcal{M}_G$ for simply connected $G$ is proved. Another reference is [11, Proposition 3.15].

Let $B_G \subseteq G$ be a Borel subgroup containing the maximal torus $T_G$. Then $\pi_0(\mathcal{M}_{T_G}) = \pi_0(\mathcal{M}_{B_G})$ by Proposition 5.1. The inclusion $B_G \hookrightarrow G$ induces a map

$$X_*(T_G) = \pi_0(\mathcal{M}_{T_G}) = \pi_0(\mathcal{M}_{B_G}) \longrightarrow \pi_0(\mathcal{M}_G). \tag{5.1}$$

This map is surjective, because every principal $G$-bundle on $C$ admits a reduction of its structure group to $B_G$ by [6, Theorem 1 and Remark 2.e].

We claim that this map (5.1) is constant on cosets modulo $X_{\mathrm{coroots}}$. Given a coroot $\alpha \in X_*(T_G)$ of $G$ and a cocharacter $\delta \in X_*(T_G)$, it suffices to show that $\delta$ and $\delta + \alpha$ have the same image in $\pi_0(\mathcal{M}_G)$. As the inclusion $T_G \hookrightarrow G$ factors through the subgroup of semisimple rank one $G_\alpha \subseteq G$ given by $\alpha$, we may assume without loss of generality that $G$ has semisimple rank one. Splitting off any direct factor $\mathbb{G}_m$ of $G$ reduces us to the cases $G \cong \mathrm{SL}_2$, $G \cong \mathrm{GL}_2$, or $G \cong \mathrm{PGL}_2$.

To deal with these three cases, we choose a closed point $P \in C(k)$. Let $L$ and $L'$ be invertible sheaves on $C$; in the case $G \cong \mathrm{SL}_2$, we assume $L \otimes L' \cong \mathcal{O}_C(P)$. For every line $\ell$ in the two-dimensional vector space $L_P \oplus L'_P$, its inverse image subsheaf $E_\ell \subseteq L \oplus L'$ defines a $G$-bundle on $C$; thus we obtain a $\mathbb{P}^1$-family of $G$-bundles on $C$. This family connects the two $G$-bundles defined by $L(-P) \oplus L'$ and by $L \oplus L'(-P)$, which come from the maximal torus $T_G \subseteq G$. Thus we see that the elements $\delta$ and $\delta + \alpha$ of $X_*(T_G) = \pi_0(\mathcal{M}_{T_G})$ indeed have the same image in $\pi_0(\mathcal{M}_G)$. Hence the above map (5.1) descends to a surjective map

$$\pi_1(G) = X_*(T_G)/X_{\mathrm{coroots}} \longrightarrow \pi_0(\mathcal{M}_G). \tag{5.2}$$

Note that this map does not depend on the choice of the maximal torus $T_G \subseteq G$. Thus it is functorial in $G$, in the sense that the diagram

$$\begin{array}{ccc} \pi_1(G) & \longrightarrow & \pi_0(\mathcal{M}_G) \\ \Big\downarrow{\scriptstyle \varphi_*} & & \Big\downarrow{\scriptstyle \varphi_*} \\ \pi_1(H) & \longrightarrow & \pi_0(\mathcal{M}_H) \end{array}$$

commutes for every homomorphism $\varphi : G \to H$ of smooth, connected, reductive algebraic groups.

Finally, we have to show that this canonical map (5.2) is injective. We first consider the case where the commutator subgroup $[G, G] \subseteq G$ is simply connected. Then $\pi_1(G) = \pi_1(G/[G, G])$, so the required injectivity for $G$ follows by functoriality from the already verified injectivity for the torus $G/[G, G]$.

Next we consider the case where $G$ is semisimple, so $\pi_1(G)$ is finite. Let $\mu$ be the kernel of the universal covering $\widetilde{G} \twoheadrightarrow G$. We choose an embedding $\mu \hookrightarrow T$ into a torus $T$, and denote by $\widehat{G}$ the pushout of linear algebraic groups

$$\begin{array}{ccc} \mu & \longrightarrow & \widetilde{G} \\ \Big\downarrow & & \Big\downarrow \\ T & \longrightarrow & \widehat{G}. \end{array}$$

By construction, $\widehat{G}$ is smooth, connected, reductive, and $[\widehat{G}, \widehat{G}] = \widetilde{G}$ is simply connected. Moreover, we have an exact sequence

$$1 \longrightarrow T \longrightarrow \widehat{G} \longrightarrow G \longrightarrow 1.$$

Using Proposition 5.3, the injectivity for $G$ follows from the injectivity for $\widehat{G}$, which has already been proved in the previous case.

Finally, we consider the case where $G$ is reductive. If $\pi : G \twoheadrightarrow H$ is a central isogeny, then the induced map $\pi_1(G) \to \pi_1(H)$ is injective; hence we may replace $G$ by $H$ without loss of generality. We take $H := G/[G, G] \times G/Z_G$, where $Z_G \subseteq G$ is the center. Splitting off the torus $G/[G, G]$ reduces us to the case where $G$ is of adjoint type. This is covered by the previous case. $\square$

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