Cohomology of the minimal nilpotent orbit

Daniel Juteau*

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Abstract

We compute the integral cohomology of the minimal non-trivial nilpotent orbit in a complex simple (or quasi-simple) Lie algebra. We find by a uniform approach that the middle cohomology group is isomorphic to the fundamental group of the sub-root system generated by the long simple roots. The modulo ℓ reduction of the Springer correspondent representation involves the sign representation exactly when ℓ divides the order of this cohomology group. The primes dividing the torsion of the rest of the cohomology are bad primes.

Introduction

Let G be a quasi-simple complex Lie group, with Lie algebra $\mathfrak{g}$. We denote by $\mathcal{N}$ the nilpotent variety of $\mathfrak{g}$. The group G acts on $\mathcal{N}$ by the adjoint action, with finitely many orbits. If $\mathcal{O}$ and $\mathcal{O}'$ are two orbits, we write $\mathcal{O} \leqslant \mathcal{O}'$ if $\mathcal{O} \subset \overline{\mathcal{O}'}$. This defines a partial order on the adjoint orbits. It is well known that there is a unique minimal non-zero orbit $\mathcal{O}_{\min}$ (see for example [CM93], and the introduction of [KP82]). The aim of this article is to compute the integral cohomology of $\mathcal{O}_{\min}$.

The nilpotent variety $\mathcal{N}$ is a cone in $\mathfrak{g}$: it is closed under multiplication by a scalar. Let us consider its image $\mathbb{P}(\mathcal{N})$ in $\mathbb{P}(\mathfrak{g})$. It is a closed subvariety of this projective space, so it is a projective variety. Now G acts on $\mathbb{P}(\mathcal{N})$, and the orbits are the $\mathbb{P}(\mathcal{O})$, where $\mathcal{O}$ is a non-trivial adjoint orbit in $\mathcal{N}$. The orbits of G in $\mathbb{P}(\mathcal{N})$ are ordered in the same way as the non-trivial orbits in $\mathcal{N}$. Thus $\mathbb{P}(\mathcal{O}_{\min})$ is the minimal orbit in $\mathbb{P}(\mathcal{N})$, and therefore it is closed: we deduce that it is a projective variety. Let $x_{\min} \in \mathcal{O}_{\min}$, and let $P = N_G(\mathbb{C}x_{\min})$ (the letter N stands for normalizer, or setwise stabilizer). Then G/P can be identified to $\mathbb{P}(\mathcal{O}_{\min})$, which is a projective variety. Thus P is a parabolic subgroup of G. Now we have a resolution of singularities (see section 2)

$$G \times_P \mathbb{C}x_{\min} \longrightarrow \overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \{0\}$$

which restricts to an isomorphism

$$G \times_P \mathbb{C}^* x_{\min} \xrightarrow{\sim} \mathcal{O}_{\min}.$$

From this isomorphism, one can already deduce that the dimension of $\mathcal{O}_{\min}$ is equal to one plus the dimension of G/P. If we fix a maximal torus T in G and a Borel subgroup B containing it, we can take for $x_{\min}$ a highest weight vector for the adjoint action on $\mathfrak{g}$. Then P is the standard parabolic subgroup corresponding to the simple roots orthogonal to the highest root, and the dimension of G/P is the number of positive roots not orthogonal to the highest root, which is $2h-3$ in the simply-laced types, where h is the Coxeter number (see [Bou68, chap. VI, §1.11, prop. 32]). So the dimension of $\mathcal{O}_{\min}$ is $2h-2$ is that case. In [Wan99], Wang shows that this formula is still valid if we replace h by the dual Coxeter number $h^\vee$ (which is equal to h only in the simply-laced types).

*UFR de mathématiques, Université Denis Diderot Paris 7

We found a similar generalization of a result of Carter (see [Car70]), relating the height of a long root to the length of an element of minimal length taking the highest root to that given long root, in the simply-laced case: the result extends to all types, if we take the height of the corresponding coroot instead (see Section 1, and Theorem 1.14).

To compute the cohomology of $\mathcal{O}_{\min}$, we will use the Gysin sequence associated to the $\mathbb{C}^*$-fibration $G \times_P \mathbb{C}^* x_{\min} \longrightarrow G/P$. The Pieri formula of Schubert calculus gives an answer in terms of the Bruhat order (see section 2). Thanks to the results of section 1, we translate this in terms of the combinatorics of the root system (see Theorem 2.1). As a consequence, we obtain the following results (see Theorem 2.2):

Theorem *(i) The middle cohomology of $\mathcal{O}_{\min}$ is given by*

$$H^{2h^\vee-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq P^\vee(\Phi')/Q^\vee(\Phi')$$

where Φ' is the sub-root system of Φ generated by the long simple roots, and $P^\vee(\Phi')$ (resp. $Q^\vee(\Phi')$) is its coweight lattice (resp. its coroot lattice).

(ii) If ℓ is a good prime for G, then there is no ℓ-torsion in the rest of the cohomology of $\mathcal{O}_{\min}$.

Part (i) is obtained by a general argument, while (ii) is obtained by a case-by-case analysis (see section 3, where we give tables for each type).

In section 4, we explain a second method for the type A_{n-1}, based on another resolution of singularities: this time, it is a cotangent bundle on a projective space (which is also a generalized flag variety). This cannot be applied to other types, because the minimal class is a Richardson class only in type A.

The motivation for this calculation is the modular representation theory of the Weyl group W. To each rational irreducible representation of W, one can associate, via the Springer correspondence (see for example [Spr76, Spr78, BM81, KL80, Slo80, Lus84, Sho88]), a pair consisting in a nilpotent orbit and a G-equivariant local system on it (or, equivalently, a pair (x, χ) where x is a nilpotent element of $\mathfrak{g}$, and χ is an irreducible character of the finite group $A_G(x) = C_G(x)/C_G^0(x)$, up to G-conjugation). Note that Springer's construction differs from the others by the sign character. All the pairs consisting of a nilpotent orbit and the constant sheaf on this orbit arise in this way. In the simply-laced types, the irreducible representation of W corresponding to the pair $(\mathcal{O}_{\min}, \mathbb{Q})$ is the natural representation tensored with the sign representation. In the other types, we have a surjection from W to the reflection group W' corresponding to the subdiagram of the Dynkin diagram of W consisting in the long simple roots. The Springer correspondent representation is then the natural representation of W' lifted to W, tensored with the sign representation.

We believe that the decomposition matrix of the Weyl group (and, in fact, of an associated Schur algebra) can be deduced from the decomposition matrix of G-equivariant perverse sheaves on the nilpotent variety $\mathcal{N}$. In [Jut], we will use Theorem 2.2 to determine some decomposition numbers for perverse sheaves (which give some evidence for this conjecture). Note that we are really interested in the torsion. The rational cohomology must already be known to the experts (see Remark 2.4).

All the results and proofs of this article remain valid for G a quasi-simple reductive group over $\overline{\mathbb{F}_p}$, with p good for G, using the étale topology. In this context, one has to take $\mathbb{Q}_\ell$ and $\mathbb{Z}_\ell$ coefficients, where ℓ is a prime different from p, instead of $\mathbb{Q}$ and $\mathbb{Z}$.

1 Long roots and distinguished coset representatives

The Weyl group W of an irreducible and reduced root system Φ acts transitively on the set Φ_{lg} of long roots in Φ, hence if α is an element of Φ_{lg}, then the long roots are in bijection with W/W_α, where W_α is the stabilizer of α in W (a parabolic subgroup). Now, if we fix a basis Δ of Φ, and if we choose for α the highest root $\tilde{\alpha}$, we find a relation between the partial orders on W and

$\Phi_{\rm lg}$ defined by Δ, and between the length of a distinguished coset representative and the (dual) height of the corresponding long root. After this section was written, I realized that the result was already proved by Carter in the simply-laced types in [Car70] (actually, this result is quoted in [Spr76]). We extend it to any type and study more precisely the order relations involved. I also came across [BB05, §4.6], where the depth of a positive root β is defined as the minimal integer k such that there is an element w in W of length k such that $w(\beta) < 0$. By the results of this section, the depth of a positive long root is nothing but the height of the corresponding coroot (and the depth of a positive short root is equal to its height).

For the classical results about root systems that are used throughout this section, the reader may refer to [Bou68, Chapter VI, §1]. It is now available in English [Bou02].

1.1 Root systems

Let V be a finite dimensional $\mathbb{R}$-vector space and Φ a root system in V. We note $V^* = \mathrm{Hom}(V, \mathbb{R})$ and, if $\alpha \in \Phi$, we denote by $\alpha^\vee$ the corresponding coroot and by s_α the reflexion $s_{\alpha,\alpha^\vee}$ of [Bou68, chap. VI, §1.1, déf. 1, (SR$_{\rm II}$)]. Let W be the Weyl group of Φ. The perfect pairing between V and V^* will be denoted by $\langle , \rangle$. Let $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. In all this section, we will assume that Φ is *irreducible* and *reduced*. Let us fix a scalar product $(\ |\ )$ on V, invariant under W, such that

$$\min_{\alpha\in\Phi}(\alpha|\alpha) = 1.$$

We then define the integer

$$r = \max_{\alpha\in\Phi}(\alpha|\alpha).$$

Let us recall that, since Φ is irreducible and reduced, we have $r \in \{1, 2, 3\}$ and $(\alpha|\alpha) \in \{1, r\}$ if $\alpha \in \Phi$ (see [Bou68, chap. VI, §1.4, prop. 12]). We define

$$\Phi_{\rm lg} = \{\alpha \in \Phi \mid (\alpha|\alpha) = r\}$$

and

$$\Phi_{\rm sh} = \{\alpha \in \Phi \mid (\alpha|\alpha) < r\} = \Phi \setminus \Phi_{\rm lg}.$$

If α and β are two roots, then

$$\textbf{(1)} \qquad \langle \alpha, \beta^\vee \rangle = \frac{2(\alpha|\beta)}{(\beta|\beta)}.$$

In particular, if α and β belong to Φ, then

$$\textbf{(2)} \qquad 2(\alpha|\beta) \in \mathbb{Z}$$

and, if α or β belongs to $\Phi_{\rm lg}$, then

$$\textbf{(3)} \qquad 2(\alpha|\beta) \in r\mathbb{Z}$$

The following classical result says that $\Phi_{\rm lg}$ is a closed subset of Φ.

Lemma 1.1 *If α, $\beta \in \Phi_{\rm lg}$ are such that $\alpha+\beta \in \Phi$, then $\alpha+\beta \in \Phi_{\rm lg}$.*

Proof : We have $(\alpha+\beta \mid \alpha+\beta) = (\alpha|\alpha)+(\beta|\beta)+2(\alpha|\beta)$. Thus, by (3), we have $(\alpha+\beta \mid \alpha+\beta) \in r\mathbb{Z}$, which implies the desired result. $\square$

1.2 Basis, positive roots, height

Let us fix a basis Δ of Φ and let Φ^+ be the set of roots $\alpha \in \Phi$ whose coefficients in the basis Δ are non-negative. Let $\Delta_{\rm lg} = \Phi_{\rm lg} \cap \Delta$ and $\Delta_{\rm sh} = \Phi_{\rm sh} \cap \Delta$. Note that $\Delta_{\rm lg}$ need not be a basis of $\Phi_{\rm lg}$. Indeed, $\Phi_{\rm lg}$ is a root system of rank equal to the rank of Φ, whereas $\Delta_{\rm lg}$ has fewer elements than Δ if Φ is of non-simply-laced type. Let us recall the following well-known result [Bou68, chap. VI, §1, exercice 20 (a)]:

Lemma 1.2 *Let $\gamma \in \Phi$ and write $\gamma = \sum_{\alpha\in\Delta} n_\alpha \alpha$, with $n_\alpha \in \mathbb{Z}$. Then $\gamma \in \Phi_{\rm lg}$ if and only if r divides all the n_α, $\alpha \in \Delta_{\rm sh}$.*

Proof : Let Φ' be the set of roots $\gamma' \in \Phi$ such that, if $\gamma' = \sum_{\alpha\in\Delta} n'_\alpha \alpha$, then r divides n'_α for all $\alpha \in \Delta_{\rm sh}$. We want to show that $\Phi_{\rm lg} = \Phi'$.

Suppose that r divides all the n_α, $\alpha \in \Delta_{\rm sh}$. Then $n_\alpha^2(\alpha|\alpha) \in r\mathbb{Z}$ for all $\alpha \in \Delta$, and by (2) and (3), we have $2n_\alpha n_\beta(\alpha|\beta) \in r\mathbb{Z}$ for all $(\alpha, \beta) \in \Delta \times \Delta$ such that $\alpha \neq \beta$. Thus $(\gamma|\gamma) \in r\mathbb{Z}$, which implies that $\gamma \in \Phi_{\rm lg}$. Thus $\Phi' \subset \Phi_{\rm lg}$.

Since W acts transitively on $\Phi_{\rm lg}$, it suffices to show that W stabilizes Φ'. In other words, it is enough to show that, if $\alpha \in \Delta$ and $\gamma \in \Phi'$, then $s_\alpha(\gamma) \in \Phi'$. But $s_\alpha(\gamma) = \gamma - \langle \gamma, \alpha^\vee \rangle \alpha$. If $\alpha \in \Delta_{\rm lg}$, then $s_\alpha(\gamma) \in \Phi'$ because $\gamma \in \Phi'$. If $\alpha \in \Delta_{\rm sh}$, then $\langle \gamma, \alpha^\vee \rangle = 2(\gamma|\alpha) \in r\mathbb{Z}$ because $\gamma \in \Phi' \subset \Phi_{\rm lg}$ (see (1) and (3)). Thus $s_\alpha(\gamma) \in \Phi'$. $\square$

If $\gamma = \sum_{\alpha\in\Delta} n_\alpha \alpha \in \Phi$, the *height* of γ (denoted by $\mathrm{ht}(\gamma)$) is defined by $\mathrm{ht}(\gamma) = \sum_{\alpha\in\Delta} n_\alpha$. One defines the height of a coroot similarly.

If γ is long, we have

$$\gamma^\vee = \sum_{\alpha\in\Delta_{\rm lg}} n_\alpha \alpha^\vee + \frac{1}{r} \sum_{\alpha\in\Delta_{\rm sh}} n_\alpha \alpha^\vee.$$

Let

$$\mathrm{ht}^\vee(\gamma) := \mathrm{ht}(\gamma^\vee) = \sum_{\alpha\in\Delta_{\rm lg}} n_\alpha + \frac{1}{r} \sum_{\alpha\in\Delta_{\rm sh}} n_\alpha.$$

In particular, the right-hand side of the last equation is an integer, which is also a consequence of Lemma 1.2.

If α and β are long roots such that $\alpha + \beta$ is a (long) root, then $(\alpha+\beta)^\vee = \alpha^\vee + \beta^\vee$, so $\mathrm{ht}^\vee$ is additive on long roots.

1.3 Length

Let $l : W \to \mathbb{N} = \{0, 1, 2, \dots\}$ be the *length* function associated to Δ: if we let

$$N(w) = \{\alpha \in \Phi^+ \mid w(\alpha) \in -\Phi^+\},$$

then we have

(4) $$l(w) = |N(w)|.$$

If $\alpha \in \Phi^+$ and if $w \in W$, then we have

(5) $$l(ws_\alpha) > l(w) \textit{ if and only if } w(\alpha) \in \Phi^+.$$

Replacing w by w^{-1}, and using the fact that an element of W has the same length as its inverse, we get

(6) $$\ell(s_\alpha w) > \ell(w) \textit{ if and only if } w^{-1}(\alpha) \in \Phi^+.$$

More generally, it is easy to show that, if x and y belong to W, then

$$\textbf{(7)} \qquad N(xy) = N(y) \dot{+} {}^{y^{-1}}N(x)$$

where $\dot{+}$ denotes the symmetric difference (there are four cases to consider), and therefore

$$\textbf{(8)} \qquad l(xy) = l(x) + l(y) \textit{ if and only if } N(y) \subset N(xy).$$

Let w_0 be the longest element of W. Recall that

$$\textbf{(9)} \qquad l(w_0 w) = l(w w_0) = l(w_0) - l(w)$$

for all $w \in W$. If I is a subset of Δ, we denote by Φ_I the set of the roots α which belong to the sub-vector space of V generated by I and we let

$$\Phi_I^+ = \Phi_I \cap \Phi^+ \qquad \text{and} \qquad W_I = < s_\alpha \mid \alpha \in I > .$$

We also define

$$X_I = \{ w \in W \mid w(\Phi_I^+) \subset \Phi^+ \}.$$

Let us recall that X_I is a set of coset representatives of W/W_I and that $w \in X_I$ if and only if w is of minimal length in wW_I. Moreover, we have

$$\textbf{(10)} \qquad l(xw) = l(x) + l(w)$$

if $x \in X_I$ and $w \in W_I$. We denote by w_I the longest element of W_I. Then $w_0 w_I$ is the longest element of X_I (this can be easily deduced from (9) and (10)). Finally, if i is an integer, we denote by W^i the set of elements of W of length i, and similarly X_I^i is the set of elements of X_I of length i. To conclude this section, we shall prove the following result, which should be well known:

Lemma 1.3 *If $\beta \in \Phi_{\text{lg}}^+$, then $l(s_\beta) = 2 \ \text{ht}^\vee(\beta) - 1$.*

Proof : We shall prove the result by induction on $\text{ht}^\vee(\beta)$. The case where $\text{ht}^\vee(\beta) = 1$ is clear. Suppose $\text{ht}^\vee(\beta) > 1$ and suppose the result holds for all positive long roots whose dual height is strictly smaller.

First, there exists a $\gamma \in \Delta$ such that $\beta - \gamma \in \Phi^+$ (see [Bou68, chap. VI, §1.6, prop. 19]). Let $\alpha = \beta - \gamma$. There are two possibilities:

• If $\gamma \in \Delta_{\text{lg}}$, then $\alpha = \beta - \gamma \in \Phi_{\text{lg}}$ by Lemma 1.1. Moreover, $\text{ht}^\vee(\alpha) = \text{ht}^\vee(\beta) - 1$. Thus $l(s_\alpha) = 2 \ \text{ht}^\vee(\alpha) - 1$. We have $(\alpha|\gamma) \neq 0$ (otherwise $\beta = \alpha + \gamma$ would be of squared length $2r$, which is impossible). By [Bou68, chap. VI, §1.3], we have $(\alpha|\gamma) = -r/2$. Thus $\beta = s_\gamma(\alpha) = s_\alpha(\gamma)$, and $s_\beta = s_\gamma s_\alpha s_\gamma$. Since $s_\alpha(\gamma) > 0$, we have $l(s_\beta s_\alpha) = l(s_\beta) + 1$ (see (5)). Since $s_\gamma s_\alpha(\gamma) = s_\gamma(\beta) = \alpha > 0$, we have $l(s_\gamma s_\alpha s_\gamma) = l(s_\alpha s_\gamma) + 1 = l(s_\alpha) + 2$ (see (5)), as expected.

• If $\gamma \in \Delta_{\text{sh}}$, then, by [Bou68, chap. VI, §1.3], we have $\alpha = \beta - r\gamma \in \Phi_{\text{lg}}^+$, $(\alpha|\gamma) = -r/2$, and $\text{ht}^\vee(\alpha) = \text{ht}^\vee(\beta) - 1$. As in the first case, we have $\beta = s_\gamma(\alpha)$. Thus $s_\beta = s_\gamma s_\alpha s_\gamma$ and the same argument applies. $\square$

Remark 1.4 By duality, if $\beta \in \Phi_{\text{sh}}^+$, we have

$$l(s_\beta) = 2 \ \text{ht}(\beta) - 1.$$

1.4 Highest root

Let $\tilde{\alpha}$ be *the highest root* of Φ relatively to Δ (see [Bou68, chap. VI, §1.8, prop. 25]). It is of height $h-1$, where h is the Coxeter number of Φ. The *dual Coxeter number* $h^\vee$ can be defined as $1+\mathrm{ht}^\vee(\tilde{\alpha})$. Let us recall the following facts:

(11) $$\tilde{\alpha} \in \Phi_{\mathrm{lg}}$$

and

(12) $$\textit{If } \alpha \in \Phi^+ \setminus \{\tilde{\alpha}\}, \textit{ then } \langle \alpha, \tilde{\alpha}^\vee \rangle \in \{0, 1\}.$$

In particular,

(13) $$\textit{If } \alpha \in \Phi^+, \textit{ then } \langle \tilde{\alpha}, \alpha^\vee \rangle \geqslant 0$$

and

(14) $$\tilde{\alpha} \in \bar{C},$$

where C is the chamber associated to Δ.

From now on, $\tilde{I}$ will denote the subset of Δ defined by

(15) $$\tilde{I} = \{\alpha \in \Delta \mid (\tilde{\alpha}|\alpha) = 0\}.$$

By construction, $\tilde{I}$ is stable under any automorphism of V stabilizing Δ. In particular, it is stable under $-w_0$. By (13), we have

(16) $$\Phi_{\tilde{I}} = \{\alpha \in \Phi \mid (\tilde{\alpha}|\alpha) = 0\}.$$

From (16) and [Bou68, chap. V, §3.3, prop. 2], we deduce that

(17) $$W_{\tilde{I}} = \{w \in W \mid w(\tilde{\alpha}) = \tilde{\alpha}\}.$$

Note that w_0 and $w_{\tilde{I}}$ commute (because $-w_0(\tilde{I}) = \tilde{I}$). We have

(18) $$N(w_0 w_{\tilde{I}}) = \Phi^+ \setminus \Phi^+_{\tilde{I}}.$$

Let us now consider the map $W \to \Phi_{\mathrm{lg}}$, $w \mapsto w(\tilde{\alpha})$. It is surjective [Bou68, chap. VI, §1.3, prop. 11] and thus induces a bijection $W/W_I \to \Phi_{\mathrm{lg}}$ by (17). It follows that the map

(19) $$\begin{array}{ccc} X_{\tilde{I}} & \longrightarrow & \Phi_{\mathrm{lg}} \\ x & \longmapsto & x(\tilde{\alpha}) \end{array}$$

is a bijection. If $\alpha \in \Phi_{\mathrm{lg}}$, we will denote by x_α the unique element of $X_{\tilde{I}}$ such that $x_\alpha(\tilde{\alpha}) = \alpha$. We have

(20) $$x_\alpha s_{\tilde{\alpha}} = s_\alpha x_\alpha.$$

Lemma 1.5 *We have $w_0 w_{\tilde{I}} = w_{\tilde{I}} w_0 = s_{\tilde{\alpha}}$.*

Proof : We have already noticed that w_0 and $w_{\tilde{I}}$ commute.

In view of [Bou68, chap. VI, §1, exercice 16], it suffices to show that $N(w_0 w_{\tilde{I}}) = N(s_{\tilde{\alpha}})$, that is, $N(s_{\tilde{\alpha}}) = \Phi^+ \setminus \Phi^+_{\tilde{I}}$ (see (18)). First, if $\alpha \in \Phi^+_{\tilde{I}}$, then $s_{\tilde{\alpha}}(\alpha) = \alpha$, so that $\alpha \notin N(s_{\tilde{\alpha}})$. This shows that $N(s_{\tilde{\alpha}}) \subset \Phi^+ \setminus \Phi^+_{\tilde{I}}$.

Let us show the other inclusion. If $\alpha \in \Phi^+ \setminus \Phi^+_{\tilde{I}}$, then $\langle \tilde{\alpha}, \alpha^\vee \rangle > 0$ by (13) and (16). In particular, $s_{\tilde{\alpha}}(\alpha) = \alpha - \langle \alpha, \tilde{\alpha}^\vee \rangle \tilde{\alpha}$ cannot belong to Φ^+ since $\tilde{\alpha}$ is the highest root. $\square$

Proposition 1.6 *Let $\alpha \in \Phi^+_{\mathrm{lg}}$. Then we have*

$$l(x_\alpha s_{\tilde\alpha}) = l(s_{\tilde\alpha}) - l(x_\alpha)$$

Proof : We have

$$\begin{array}{rcll} l(x_\alpha s_{\tilde\alpha}) & = & l(x_\alpha w_{\tilde I} w_0) & \text{by Lemma 1.5} \\ & = & l(w_0) - l(x_\alpha w_{\tilde I}) & \text{by (9)} \\ & = & l(w_0) - l(w_{\tilde I}) - l(x_\alpha) & \text{by (10)} \\ & = & l(w_0 w_{\tilde I}) - l(x_\alpha) & \text{by (9)} \\ & = & l(s_{\tilde\alpha}) - l(x_\alpha) & \text{by Lemma 1.5.} \end{array}$$

□

Proposition 1.7 *If $\alpha \in \Phi^+_{\mathrm{lg}}$, then $x_{-\alpha} = s_\alpha x_\alpha$ and $l(x_{-\alpha}) = l(s_\alpha x_\alpha) = l(s_\alpha) + l(x_\alpha)$.*

Proof : We have $s_\alpha x_\alpha(\tilde\alpha) = s_\alpha(\alpha) = -\alpha$, so to show that $x_{-\alpha} = s_\alpha x_\alpha$, it is enough to show that $s_\alpha x_\alpha \in X_{\tilde I}$. But, if $\beta \in \Phi^+_{\tilde I}$, we have (see (20)) $s_\alpha x_\alpha(\beta) = x_\alpha s_{\tilde\alpha}(\beta) = x_\alpha(\beta) \in \Phi^+$. Hence the first result.

Let us now show that the lengths add up. By (8), it is enough to show that $N(x_\alpha) \subset N(s_\alpha x_\alpha)$. Let then $\beta \in N(x_\alpha)$. Since $x_\alpha \in X_{\tilde I}$, β cannot be in $\Phi^+_{\tilde I}$. Thus $\langle \beta, \tilde\alpha^\vee \rangle > 0$. Therefore, $\langle x_\alpha(\beta), \alpha^\vee \rangle > 0$. Now, we have $s_\alpha x_\alpha(\beta) = x_\alpha(\beta) - \langle x_\alpha(\beta), \alpha^\vee \rangle \alpha < 0$ (remember that $x_\alpha(\beta) < 0$ since $\beta \in N(x_\alpha)$). □

Proposition 1.8 *For $\alpha \in \Phi^+_{\mathrm{lg}}$, we have*

$$l(x_\alpha) = \frac{l(s_{\tilde\alpha}) - l(s_\alpha)}{2} = \mathrm{ht}^\vee(\tilde\alpha) - \mathrm{ht}^\vee(\alpha)$$

$$l(x_{-\alpha}) = \frac{l(s_{\tilde\alpha}) + l(s_\alpha)}{2} = \mathrm{ht}^\vee(\tilde\alpha) + \mathrm{ht}^\vee(\alpha) - 1$$

Proof : This follows from Propositions 1.6 and 1.7, and (20). □

1.5 Orders

The choice of Δ determines an order relation on V. For x, $y \in V$, we have $y \leqslant x$ if and only if $y - x$ is a linear combination of the simple roots with non-negative coefficients.

For $\alpha \in \Phi_{\mathrm{lg}}$, it will be convenient to define the *level* $L(\alpha)$ of α as follows:

(21) $$L(\alpha) = \begin{cases} \mathrm{ht}^\vee(\tilde\alpha) - \mathrm{ht}^\vee(\alpha) & \text{if } \alpha > 0 \\ \mathrm{ht}^\vee(\tilde\alpha) - \mathrm{ht}^\vee(\alpha) - 1 & \text{if } \alpha < 0 \end{cases}$$

If i is an integer, let Φ^i_{lg} be the set of long roots of level i. Then Proposition 1.8 says that the bijection (19) maps $X^i_{\tilde I}$ onto Φ^i_{lg}.

For $\gamma \in \Phi^+$, we write

(22) $$\beta \xrightarrow{\gamma} \alpha \text{ if and only if } \alpha = s_\gamma(\beta) \text{ and } L(\alpha) = L(\beta) + 1.$$

In that case, we have $\beta - \alpha = \langle b, \gamma^\vee \rangle \gamma > 0$, so $\beta > \alpha$.

If α and β are two long roots, we say that there is a *path* from β to α, and we write $\alpha \preceq \beta$, if and only if there exists a sequence $(\beta_0, \beta_1, \ldots, \beta_k)$ of long roots, and a sequence $(\gamma_1, \ldots, \gamma_k)$ of positive roots, such that

(23)
$$\beta = \beta_0 \xrightarrow{\gamma_1} \beta_1 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_k} \beta_k = \alpha.$$

In that case, we have $L(\beta_i) = L(\beta) + i$ for $i \in \{0, \ldots, k\}$. If moreover all the roots γ_i are simple, we say that there is a *simple path* from β to α.

On the other hand, we have the Bruhat order on W defined by the set of simple reflections $S = \{s_\alpha \mid \alpha \in \Delta\}$. If w and w' belong to W, we write $w \longrightarrow w'$ if $w' = s_\gamma w$ and $l(w') = l(w)+1$, for some positive root γ. In that case, we write $w \xrightarrow{\gamma} w'$ (the positive root γ is uniquely determined). The Bruhat order $\leqslant$ is the reflexive and transitive closure of the relation $\longrightarrow$. On $X_{\tilde{I}}$, we will consider the restriction of the Bruhat order on W.

Let us now consider the action of a simple reflection on a long root.

Lemma 1.9 *Let $\beta \in \Phi_{\mathrm{lg}}$ and $\gamma \in \Delta$. Let $\alpha = s_\gamma(\beta)$.*

1. *(i) If $\beta \in \Delta_{\mathrm{lg}}$ and $(\beta|\gamma) > 0$, then $\gamma = \beta$, $\alpha = -\beta$ and $\langle \beta, \gamma^\vee \rangle = 2$.*

(ii) If $\beta \in -\Delta_{\mathrm{lg}}$ and $(\beta|\gamma) < 0$, then $\gamma = -\beta$, $\alpha = -\beta$ and $\langle \beta, \gamma^\vee \rangle = -2$.

(iii) Otherwise, α and $\langle \beta, \gamma^\vee \rangle$ are given by the following table:

	$\gamma \in \Delta_{\mathrm{lg}}$		$\gamma \in \Delta_{\mathrm{sh}}$	
$(\beta\|\gamma) > 0$	$\alpha = \beta - \gamma$	$\langle \beta, \gamma^\vee \rangle = 1$	$\alpha = \beta - r\gamma$	$\langle \beta, \gamma^\vee \rangle = r$
$(\beta\|\gamma) = 0$	$\alpha = \beta$	$\langle \beta, \gamma^\vee \rangle = 0$	$\alpha = \beta$	$\langle \beta, \gamma^\vee \rangle = 0$
$(\beta\|\gamma) < 0$	$\alpha = \beta + \gamma$	$\langle \beta, \gamma^\vee \rangle = -1$	$\alpha = \beta + r\gamma$	$\langle \beta, \gamma^\vee \rangle = -r$

2. *(i) If $(\beta|\gamma) > 0$ then $L(\alpha) = L(\beta) + 1$, so that $\beta \xrightarrow{\gamma} \alpha$.*

(ii) If $(\beta|\gamma) = 0$ then $L(\alpha) = L(\beta)$, and in fact $\alpha = \beta$.

(iii) If $(\beta|\gamma) < 0$ then $L(\alpha) = L(\beta) - 1$, so that $\alpha \xrightarrow{\gamma} \beta$.

Proof : Part 1 follows from inspection of the possible cases in [Bou68, Chapitre VI, §1.3].

Part 2 is a consequence of part 1. Note that there is a special case when we go from positive roots to negative roots, and *vice versa*. This is the reason why there are two cases in the definition of the level. □

To go from a long simple root to the opposite of a long simple root, one sometimes needs a non-simple reflection.

Lemma 1.10 *Let $\beta \in \Delta_{\mathrm{lg}}$, $\alpha \in -\Delta_{\mathrm{lg}}$ and $\gamma \in \Phi^+$. Then $\beta \xrightarrow{\gamma} \alpha$ if and only if we are in one of the following cases:*

(i) $\alpha = -\beta$ and $\gamma = \beta$. In this case, $\langle \beta, \gamma^\vee \rangle = 2$.

(ii) $\beta + (-\alpha)$ is a root and $\gamma = \beta + (-\alpha)$. In this case, $\langle \beta, \gamma^\vee \rangle = 1$.

Proof : This is straightforward. □

But otherwise, one can use simple roots at each step.

Proposition 1.11 *Let α and β be two long roots such that $\alpha \leqslant \beta$. Write $\beta = \sum_{\sigma \in J} n_\sigma \sigma$, and $\alpha = \sum_{\tau \in K} m_\tau \tau$, where J (resp. K) is a non-empty subset of Δ, and the n_σ (resp. the m_τ) are non-zero integers, all of the same sign.*

(i) If $0 < \alpha \leqslant \beta$, then there is a simple path from β to α.

(ii) If $\alpha < 0 < \beta$, then there is a simple path from β to α if and only if there is a long root which belongs to both J and K. Moreover, there is a path from β to α if and only if there is a long root σ in J, and a long root τ in K, such that $(\sigma|\tau) \neq 0$.

(iii) If $\alpha \leqslant \beta < 0$, then there is a simple path from β to α.

Proof : We will prove (i) by induction on $m = \mathrm{ht}^\vee(\beta) - \mathrm{ht}^\vee(\alpha)$.

If $m = 0$, then $\beta = \alpha$, and there is nothing to prove.

So we may assume that $m > 0$ and that the results holds for $m - 1$. Thus $\alpha < \beta$ and we have

$$\beta - \alpha = \sum_{\gamma \in J} n_\gamma \gamma$$

where J is a non-empty subset of Δ, and the n_γ, $\gamma \in J$, are positive integers. We have

$$(\beta - \alpha \mid \beta - \alpha) = \sum_{\gamma \in J} n_\gamma(\beta|\gamma) - \sum_{\gamma \in J} n_\gamma(\alpha|\gamma) > 0.$$

So there is a γ in J such that $(\beta|\gamma) > 0$ or $(\alpha|\gamma) < 0$. In the first case, let $\beta' = s_\gamma(\beta)$. It is a long root. If γ is long (resp. short), then $\beta' = \beta - \gamma$ (resp. $\beta' = \beta - r\gamma$), so that $\alpha \leqslant \beta' < \beta$ (see Lemma 1.2). We have $\beta \xrightarrow{\gamma} \beta'$ and $\mathrm{ht}^\vee(\beta') = \mathrm{ht}^\vee(\beta) - 1$, so we can conclude by the induction hypothesis. The second case is similar: if $\alpha' = s_\gamma(\alpha) \in \Phi_{\mathrm{lg}}$, then $\alpha < \alpha' \leqslant \beta$, $\alpha' \xrightarrow{\gamma} \alpha$, $\mathrm{ht}^\vee(\alpha') = \mathrm{ht}^\vee(\alpha) + 1$, and we can conclude by the induction hypothesis. This proves (i).

Now (iii) follows, applying (i) to $-\alpha$ and $-\beta$ and using the symmetry -1.

Let us prove (ii). If there is a long simple root σ which belongs to J and K, we have $\alpha \leqslant -\sigma < \sigma \leqslant \beta$. Using (i), we find a simple path from β to σ, then we have $\sigma \xrightarrow{\sigma} -\sigma$, and using (iii) we find a simple path from $-\sigma$ to α. So there is a simple path from β to α.

Suppose there is a long root γ in J, and a long root γ' in K, such that $(\sigma|\tau) \neq 0$. Then either we are in the preceding case, or there are long simple roots $\sigma \in J$ and $\tau \in K$, such that $\alpha \leqslant -\tau < \sigma \leqslant \beta$ and $\gamma = \sigma + \tau$ is a root. By Lemma 1.10, we have $\sigma \xrightarrow{\gamma} -\tau$. Using (i) and (iii), we can find simple paths from β to σ and from $-\tau$ to α. So there is a path from β to α.

Now suppose there is a path from β to α. In this path, we must have a unique step of the form $\sigma \xrightarrow{\gamma} -\tau$, with σ and τ in Δ_{lg}. We have $\sigma \in J$, $\tau \in K$, and $(\sigma|\tau) \neq 0$. If moreover it is a simple path from β to α, then we must have $\tau = -\sigma$. This completes the proof. $\square$

The preceding analysis can be used to study the length and the reduced expressions of some elements of W.

Proposition 1.12 *Let α and β be two long roots. If x is an element of W such that $x(\beta) = \alpha$, then we have $l(x) \geqslant |L(\alpha) - L(\beta)|$.*

Moreover, there is an $x \in W$ such that $x(\beta) = \alpha$ and $l(x) = |L(\alpha) - L(\beta)|$ if and only if α and β are linked by a simple path, either from β to α, or from α to β. In this case, there is only one such x, and we denote it by $x_{\alpha\beta}$. The reduced expressions of $x_{\alpha\beta}$ correspond bijectively to the simple paths from β to α.

If $\alpha \leqslant \beta \leqslant \gamma$ are such that $x_{\alpha\beta}$ and $x_{\beta\gamma}$ are defined, then $x_{\alpha\gamma}$ is defined, and we have $x_{\alpha\gamma} = x_{\alpha\beta}x_{\beta\gamma}$ with $l(x_{\alpha\gamma}) = l(x_{\alpha\beta}) + l(x_{\beta\gamma})$.

The element $x_{-\alpha,\alpha}$ is defined for all $\alpha \in \Phi^+_{\mathrm{lg}}$, and is equal to s_α.

The element $x_{\alpha,\tilde{\alpha}}$ is defined for all $\alpha \in \Phi_{\mathrm{lg}}$, and is equal to x_α.

Proof : Let $(s_{\gamma_k}, \dots, s_{\gamma_1})$ be a reduced expression of x, where $k = l(x)$. For $i \in \{0, \dots, k\}$, let $\beta_i = s_{\gamma_i} \dots s_{\gamma_1}(\beta)$. For each $i \in \{0, \dots, k-1\}$, we have $|L(\beta_{i+1}) - L(\beta_i)| \leqslant 1$ by Lemma 1.9. Then we have

$$|L(\alpha) - L(\beta)| \leqslant \sum_{i=0}^{k-1} |L(\beta_{i+1}) - L(\beta_i)| \leqslant k = l(x)$$

If we have an equality, then all the $L(\beta_{i+1}) - L(\beta_i)$ must be of absolute value one and of the same sign, so either they are all equal to 1, or they are all equal to -1. Thus, either we have a simple path from β to α, or we have a simple path from β to α.

Suppose there is a simple path from β to α. Let $(\beta_0, \dots, \beta_k)$ be a sequence of long roots, and $(\gamma_1, \dots, \gamma_k)$ a sequence of simple roots, such that

$$\beta = \beta_0 \xrightarrow{\gamma_1} \beta_1 \xrightarrow{\gamma_2} \dots \xrightarrow{\gamma_k} \beta_k = \alpha.$$

Let $x = s_{\gamma_k} \dots s_{\gamma_1}$. Then we have $x(\beta) = \alpha$, and $l(x) \leqslant k$. But we have seen that $l(x) \geqslant L(\alpha) - L(\beta) = k$. So we have equality. The case where there is a simple path from α to β is similar.

Let $\alpha \in \Phi_{\mathrm{lg}}$. If $\alpha > 0$, we have $0 < \alpha \leqslant \tilde{\alpha}$, so by Proposition 1.11 (i), there is a simple path from $\tilde{\alpha}$ to α. If $\alpha < 0$, we have $\alpha < -\alpha \leqslant \tilde{\alpha}$, so by Proposition 1.11 (i) and (ii), there is also a simple path from $\tilde{\alpha}$ to α in this case. Let x be the product of the simple reflections it involves. Then $l(x) = L(\alpha)$, so x is of minimal length in $xW_{\tilde{I}}$, and $x \in X_{\tilde{I}}$. Thus $x = x_\alpha$ is uniquely determined, and $x_{\alpha,\tilde{\alpha}}$ is defined. It is equal to x_α and is of length $L(\alpha)$.

Let α and β be two long roots such there is a simple path from β to α, and let x be the product of the simple reflections it involves. We have $xx_\beta(\tilde{\alpha}) = x(\beta) = \alpha$, and it is of length $L(\alpha)$, so it is of minimal length in its coset modulo $W_{\tilde{I}}$. Thus $xx_\beta = x_\alpha$, and $x = x_\alpha x_\beta^{-1}$ is uniquely determined. Therefore, $x_{\alpha\beta}$ is defined and equal to $x_\alpha x_\beta^{-1}$. Any simple path from β to α gives rise to a reduced expression of $x_{\alpha\beta}$, and every reduced expression of $x_{\alpha\beta}$ gives rise to a simple path from β to α. These are inverse bijections.

If $\alpha \leqslant \beta \leqslant \gamma$ are such that $x_{\alpha\beta}$ and $x_{\beta\gamma}$ are defined, one can show that $x_{\alpha\gamma}$ is defined, and that we have $x_{\alpha\gamma} = x_{\alpha\beta}x_{\beta\gamma}$ with $l(x_{\alpha\gamma}) = l(x_{\alpha\beta}) + l(x_{\beta\gamma})$, by concatenating simple paths from γ to β and from β to α.

If α is a positive long root, then there is a simple path from α to $-\alpha$. We can choose a symmetric path (so that the simple reflections form a palindrome). So $x_{-\alpha,\alpha}$ is defined, and is a reflection: it must be s_α. It is of length $L(-\alpha) - L(\alpha) = 2\mathrm{ht}^\vee(\alpha) - 1$. $\square$

Remark 1.13 We have seen in the proof that, if $\alpha \in \Phi_{\mathrm{lg}}^+$, then

$$l(s_\alpha) = l(x_{-\alpha,\alpha}) = L(-\alpha) - L(\alpha) = 2\mathrm{ht}^\vee(\alpha) - 1$$

and, if $\alpha \in \Phi_{\mathrm{lg}}$, then

$$l(x_\alpha) = l(x_{\alpha,\tilde{\alpha}}) = L(\alpha)$$

thus we have a second proof of Lemma 1.3 and Proposition 1.8. Similarly, the formulas $x_{\alpha\gamma} = x_{\alpha\beta}x_{\beta\gamma}$ and $l(x_{\alpha\gamma}) = l(x_{\alpha\beta}) + l(x_{\beta\gamma})$, applied to the triple $(-\alpha, \alpha, \tilde{\alpha})$, give another proof of Proposition 1.7.

To conclude this section, let us summarize the results which we will use in the sequel.

Theorem 1.14 *The bijection (19) is an anti-isomorphism between the posets* $(\Phi_{\mathrm{lg}}, \preceq)$ *and* $(X_{\tilde{I}}, \leqslant)$ *(these orders were defined at the beginning of 1.5), and a root of level* i *corresponds to an element of length* i *in* $X_{\tilde{I}}$.

If β and α are long roots, and γ is a positive root, then we have

$$\beta \xrightarrow{\gamma} \alpha \quad \text{if and only if} \quad x_\beta \xrightarrow{\gamma} x_\alpha$$

(these relations have been defined at the beginning of 1.5).

Moreover, in the above situation, the integer $\partial_{\alpha\beta} = \langle \beta, \gamma^\vee \rangle$ is determined as follows:

(i) if $\beta \in \Delta_{\mathrm{lg}}$ and $\alpha \in -\Delta_{\mathrm{lg}}$, then $\partial_{\alpha\beta}$ is equal to 2 if $\alpha = -\beta$, and to -1 if $\beta + (-\alpha)$ is a root;

(ii) otherwise, $\partial_{\alpha\beta}$ is equal to 1 if γ is long, and to r if γ is short (where $r = \max_{\alpha\in\Phi}(\alpha|\alpha)$).

If β and α are two long roots such that $L(\alpha) = L(\beta) + 1$, then we set $\partial_{\alpha\beta} = 0$ if there is no simple root γ such that $\beta \xrightarrow{\gamma} \alpha$.

The numbers $\partial_{\alpha\beta}$ will appear in Theorem 2.1 as the coefficients of the matrices of some maps appearing in the Gysin sequence associated to the $\mathbb{C}^*$-fibration $\mathcal{O}_{\min} \simeq G \times_P \mathbb{C}^* x_{\min}$ over G/P, giving the cohomology of $\mathcal{O}_{\min}$. By Theorem 1.14, these coefficients are explicitly determined in terms of the combinatorics of the root system.

2 Resolution of singularities, Gysin sequence

Let us choose a maximal torus T of G, with Lie algebra $\mathfrak{t} \subset \mathfrak{g}$. We then denote by $X(T)$ its group of characters, and $X^\vee(T)$ its group of cocharacters. For each $\alpha \in \Phi$, there is a closed subgroup U_α of G, and an isomorphism $u_\alpha : \mathbb{G}_a \to U_\alpha$ such that, for all $t \in T$ and for all $\lambda \in \mathbb{C}$, we have $t u_\alpha(\lambda) t^{-1} = u_\alpha(\alpha(t)\lambda)$. We are in the set-up of 1.1, with Φ equal to the root system of (G,T) in $V = X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. We denote $X(T) \times_{\mathbb{Z}} \mathbb{Q}$ by $V_{\mathbb{Q}}$, and the symmetric algebra $S(V_{\mathbb{Q}})$ by S.

There is a root subspace decomposition

$$\mathfrak{g} = \mathfrak{t} \oplus \left(\bigoplus_{\alpha\in\Phi} \mathfrak{g}_\alpha \right)$$

where $\mathfrak{g}_\alpha$ is the (one-dimensional) weight subspace $\{x \in \mathfrak{g} \mid \forall t \in T,\ \mathrm{Ad}(t).x = \alpha(t)x\}$. We denote by e_α a non-zero vector in $\mathfrak{g}_\alpha$. Thus we have $\mathfrak{g}_\alpha = \mathbb{C}e_\alpha$.

Let $W = N_G(T)/T$ be the Weyl group. It acts on $X(T)$, and hence on $V_{\mathbb{Q}}$ and S.

Let us now fix a Borel subgroup B of G containing T, with Lie algebra $\mathfrak{b}$. This choice determines a basis Δ, the subset of positive roots Φ^+, and the height (and dual height) function, as in 1.2, the length function l as in 1.3, the highest root $\tilde{\alpha}$ and the subset $\tilde{I}$ of Δ as in 1.4, and the orders on Φ_{lg} and $X_{\tilde{I}}$ as in 1.5. So we can apply all the notations and results of section 1.

Let H be a closed subgroup of G, and X a variety with a left H-action. Then H acts on $G \times X$ on the right by $(g,x).h = (gh, h^{-1}x)$. If the canonical morphism $G \to G/H$ has local sections, then the quotient variety $(G \times X)/H$ exists (see [Spr98, §5.5]). The quotient is denoted by $G \times_H X$. One has a morphism $G \times_H X \to G/H$ with local sections, whose fibers are isomorphic to X. The quotient is the *fibre bundle over G/H associated to X*. We denote the image of (g,x) in this quotient by $g *_H x$, or simply $g * x$ if the context is clear. Note that G acts on the left on $G \times_H X$, by $g'.g *_H x = g'g *_H x$.

In 2.1, we describe the cohomology of G/B, both in terms Chern classes of line bundles and in terms of fundamental classes of Schubert varieties, and we state the Pieri formula (see [BGG73, Dem73, Hil82] for a description of Schubert calculus). In 2.2, we explain how this generalizes to the parabolic case. In 2.3, we give an algorithm to compute the cohomology of any line bundle minus the zero section, on any generalized flag variety. To do this, we need the Gysin sequence (see for example [BT82, Hus94], or [Mil80] in the étale case). In 2.4, we will see that the computation of the cohomology of $\mathcal{O}_{\min}$ is a particular case. Using the results of section 1, we give a description in terms of the combinatorics of the root system.

2.1 Line bundles on G/B, cohomology of G/B

Let $\mathcal{B} = G/B$ be the flag variety. It is a smooth projective variety of dimension $|\Phi^+|$. The map $G \longrightarrow G/B$ has local sections (see [Spr98, §8.5]). If α is a character of T, one can lift it to B: let $\mathbb{C}_\alpha$ be the corresponding one-dimensional representation of B. We can then form the G-equivariant line bundle

(24)
$$\mathcal{L}(\alpha) = G \times_B \mathbb{C}_\alpha \longrightarrow G/B.$$

Let $c(\alpha) \in H^2(G/B, \mathbb{Z})$ denote the first Chern class of $\mathcal{L}(\alpha)$. Then $c : X(T) \longrightarrow H^2(G/B, \mathbb{Z})$ is a morphism of $\mathbb{Z}$-modules. It extends to a morphism of $\mathbb{Q}$-algebras, which we still denote by $c : S \longrightarrow H^*(G/B, \mathbb{Q})$. The latter is surjective and has kernel $\mathcal{I}$, where $\mathcal{I}$ is the ideal of S generated by the W-invariant homogeneous elements in S of positive degree. So it induces an isomorphism of $\mathbb{Q}$-algebras

(25)
$$\overline{c} : S/\mathcal{I} \simeq H^*(G/B, \mathbb{Q})$$

which doubles degrees.

The algebra $S/\mathcal{I}$ is called the coinvariant algebra. As a representation of W, it is isomorphic to the regular representation. We also have an action of W on $H^*(G/B, \mathbb{Q})$, because G/B is homotopic to G/T, and W acts on the right on G/T by the formula $gT.w = gnT$, where $n \in N_G(T)$ is a representative of $w \in W$, and $g \in G$. One can show that $\overline{c}$ commutes with the actions of W.

On the other hand, we have the Bruhat decomposition [Spr98, §8.5]

(26)
$$G/B = \bigsqcup_{w \in W} C(w)$$

where the $C(w) = BwB/B \simeq \mathbb{C}^{l(w)}$ are the Schubert cells. Their closures are the Schubert varieties $S(w) = \overline{C(w)}$. Thus the cohomology of G/B is concentrated in even degrees, and $H^{2i}(G/B, \mathbb{Z})$ is free with basis $(Y_w)_{w \in W^i}$, where Y_w is the cohomology class of the Schubert variety $S(w_0 w)$ (which is of codimension $l(w) = i$). The object of Schubert calculus is to describe the multiplicative structure of $H^*(G/B, \mathbb{Z})$ in these terms (see [BGG73, Dem73, Hil82]). We will only need the following result (known as the Pieri formula, or Chevalley formula): if $w \in W$ and $\alpha \in X(T)$, then

(27)
$$c(\alpha)\ .\ Y_w \quad = \sum_{w \xrightarrow{\gamma} w'} \langle w(\alpha), \gamma^\vee \rangle\, Y_{w'}$$

2.2 Parabolic invariants

Let I be a subset of Δ. Let P_I be the parabolic subgroup of G containing B corresponding to I. It is generated by B and the subgroups $U_{-\alpha}$, for $\alpha \in I$. Its unipotent radical U_{P_I} is generated by the U_α, $\alpha \in \Phi^+ \setminus \Phi_I^+$. And it has a Levi complement L_I, which is generated by T and the U_α, $\alpha \in \Phi_I$. One can generalize the preceding constructions to the parabolic case.

If $\alpha \in X(T)^{W_I}$ (that is, if α is a character orthogonal to I), then we can form the G-equivariant line bundle

(28)
$$\mathcal{L}_I(\alpha) = G \times_{P_I} \mathbb{C}_\alpha \longrightarrow G/P_I$$

because the character α of T, invariant by W_I, can be extended to L_I and lifted to P_I.

We have a surjective morphism $q_I : G/B \longrightarrow G/P_I$, which induces an injection

$$q_I^* : H^*(G/P_I, \mathbb{Z}) \hookrightarrow H^*(G/B, \mathbb{Z})$$

in cohomology, which identifies $H^*(G/P_I, \mathbb{Z})$ with $H^*(G/B, \mathbb{Z})^{W_I}$.

The isomorphism $\overline{c}$ restricts to

$$(S/\mathcal{I})^{W_I} \simeq H^*(G/P_I, \mathbb{Q}) \tag{29}$$

We have cartesian square

$$\begin{tikzcd} \mathcal{L}(\alpha) \arrow[r] \arrow[d] & \mathcal{L}_I(\alpha) \arrow[d] \\ G/B \arrow[r, "q_I"'] & G/P_I \end{tikzcd}$$

That is, the pullback by q_I of $\mathcal{L}_I(\alpha)$ is $\mathcal{L}(\alpha)$. By functoriality of Chern classes, we have $q_I^*(c_I(\alpha)) = c(\alpha)$.

We still have a Bruhat decomposition

$$G/P_I = \bigsqcup_{w \in X_I} C_I(w) \tag{30}$$

where $C_I(w) = BwP_I/P_I \simeq \mathbb{C}^{l(w)}$ for w in X_I. We note

$$S_I(w) = \overline{C_I(w)} \text{ and } Y_{I,w} = [\overline{Bw_0wP_I/P_I}] = [\overline{Bw_0ww_IP_I/P_I}] = [S_I(w_0ww_I)] \text{ for } w \text{ in } X_I$$

Note that

$$\text{if } w \text{ is in } X_I, \text{ then } w_0ww_I \text{ is also in } X_I \tag{31}$$

since for any root β in Φ_I^+, we have $w_I(\beta) \in \Phi_I^-$, hence $ww_I(\beta)$ is also negative, and thus $w_0ww_I(\beta)$ is positive. Moreover, we have

$$\text{if } w \in X_I, \text{ then } l(w_0ww_I) = l(w_0) - l(ww_I) = l(w_0) - l(w_I) - l(w) = \dim G/P_I - l(w) \tag{32}$$

so that $Y_{I,w} \in H^{2l(w)}(G/P_I, \mathbb{Z})$. We have $q_I^*(Y_{I,w}) = Y_w$.

The cohomology of G/P_I is concentrated in even degrees, and $H^{2i}(G/P_I, \mathbb{Z})$ is free with basis $(Y_{I,w})_{w \in X_I^i}$. The cohomology ring $H^*(G/P_I, \mathbb{Z})$ is identified *via* q_I^* to a subring of $H^*(G/B, \mathbb{Z})$, so the Pieri formula can now be written as follows. If $w \in X_I$ and $\alpha \in X(T)^{W_I}$, then we have

$$c_I(\alpha)\ .\ Y_{I,w} \quad = \sum_{w \xrightarrow{\gamma} w' \in X_I} \langle w(\alpha), \gamma^\vee \rangle \, Y_{I,w'} \tag{33}$$

2.3 Cohomology of a $\mathbb{C}^*$-fiber bundle on G/P_I

Let I be a subset of Δ, and α be a W_I-invariant character of T. Let us consider

$$\mathcal{L}_I^*(\alpha) = G \times_{P_I} \mathbb{C}_\alpha^* \longrightarrow G/P_I, \tag{34}$$

that is, the line bundle $\mathcal{L}_I(\alpha)$ minus the zero section. In the sequel, we will have to calculate the cohomology of $\mathcal{L}_I^*(\tilde{\alpha})$, but we can explain how to calculate the cohomology of $\mathcal{L}_I^*(\alpha)$ for any given I and α (the point is that the answer for the middle cohomology will turn out to be nicer in our particular case, thanks to the results of section 1).

We have the Gysin exact sequence

$$H^{n-2}(G/P_I, \mathbb{Z}) \xrightarrow{c_I(\alpha)} H^n(G/P_I, \mathbb{Z}) \longrightarrow H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \longrightarrow H^{n-1}(G/P_I, \mathbb{Z}) \xrightarrow{c_I(\alpha)} H^{n+1}(G/P_I, \mathbb{Z})$$

where $c_I(\alpha)$ means the multiplication by $c_I(\alpha)$, so we have a short exact sequence

$$0 \longrightarrow \text{Coker } (c_I(\alpha) : H^{n-2} \to H^n) \longrightarrow H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \longrightarrow \text{Ker } (c_I(\alpha) : H^{n-1} \to H^{n+1}) \longrightarrow 0$$

where H^j stands for $H^j(G/P_I, \mathbb{Z})$.

Moreover, by the hard Lefschetz theorem, $c_I(\alpha) : \mathbb{Q} \otimes_{\mathbb{Z}} H^{n-2} \to \mathbb{Q} \otimes_{\mathbb{Z}} H^n$ is injective for $n \leqslant d_I = \dim \mathcal{L}_I^*(\alpha) = \dim G/P_I + 1$, and surjective for $n \geqslant d_I$. By the way, we see that we could immediately determine the rational cohomology of $\mathcal{O}_{\min}$, using only the results in this paragraph and the cohomology of G/P_I.

But we can say more. The cohomology of G/P_I is free and concentrated in even degrees. In fact, $c_I(\alpha) : H^{n-2} \to H^n$ is injective for $n \leqslant d_I$, and has free kernel and finite cokernel for $n \geqslant d_I$.

We have

(35) $$\text{if } n \text{ is even, then } H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \simeq \text{Coker } (c_I(\alpha) : H^{n-2} \to H^n)$$

which is finite for $n \geqslant d_I$, and

(36) $$\text{if } n \text{ is odd, then } H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \simeq \text{Ker } (c_I(\alpha) : H^{n-1} \to H^{n+1})$$

which is free (it is zero if $n \leqslant d_I - 1$).

Thus all the cohomology of $\mathcal{L}_I^*(\alpha)$ can be explicitly computed, thanks to the results of 2.2.

2.4 Resolution of singularities

Let $\tilde{I}$ be the subset of Δ defined in (15). There is a resolution of singularities (see for example the introduction of [KP82])

(37) $$\begin{array}{ccc} \mathcal{L}_{\tilde{I}}(\tilde{\alpha}) = G \times_{P_{\tilde{I}}} \mathbb{C}_{\tilde{\alpha}} & \longrightarrow & \overline{\mathcal{O}_{\min}} = \mathcal{O}_{\min} \cup \{0\} \\ g * \lambda & \longmapsto & \operatorname{Ad} g.(\lambda.e_{\tilde{\alpha}}) \end{array}$$

It is the one mentioned in the introduction, with $P = P_{\tilde{I}}$ and $x_{\min} = e_{\tilde{\alpha}}$. It induces an isomorphism

(38) $$\mathcal{L}_{\tilde{I}}^*(\tilde{\alpha}) = G \times_{P_{\tilde{I}}} \mathbb{C}_{\tilde{\alpha}}^* \simeq \mathcal{O}_{\min}$$

Set $d = d_{\tilde{I}} = 2h^\vee - 2$. For all integers j, let H^j denote $H^j(G/P_{\tilde{I}}, \mathbb{Z})$. For $\alpha \in \Phi_{\text{lg}}$, we have $x_\alpha \in X_{\tilde{I}}$, so $Z_\alpha := Y_{\tilde{I}, x_\alpha}$ is an element of $H^{2i}(G/P_{\tilde{I}}, \mathbb{Z})$, where $i = l(x_\alpha) = L(\alpha)$. Then $H^*(G/P_{\tilde{I}}, \mathbb{Z})$ is concentrated in even degrees, and $H^{2i}(G/P_{\tilde{I}}, \mathbb{Z})$ is free with basis $(Z_\alpha)_{\alpha \in \Phi_{\text{lg}}^i}$. Combining Theorem 1.14 and the analysis of 2.3, we get the following description of the cohomology of $\mathcal{O}_{\min}$.

Theorem 2.1 *We have*

(39) $$\textit{if } n \textit{ is even, then } H^n(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \text{Coker } (c_{\tilde{I}}(\tilde{\alpha}) : H^{n-2} \to H^n)$$

which is finite for $n \geqslant d$, and

(40) $$\textit{if } n \textit{ is odd, then } H^n(\mathcal{L}_I^*(\alpha), \mathbb{Z}) \simeq \text{Ker } (c_{\tilde{I}}(\tilde{\alpha}) : H^{n-1} \to H^{n+1})$$

which is free (it is zero if $n \leqslant d - 1$).

Moreover, if $\beta \in \Phi_{\text{lg}}^i$, then we have

(41) $$c_{\tilde{I}}(\tilde{\alpha}).Z_\beta = \sum_{\beta \xrightarrow{\gamma} \alpha} \langle \beta, \gamma^\vee \rangle \, Z_\alpha = \sum_{\alpha \in \Phi_{\text{lg}}^{i+1}} \partial_{\alpha\beta} Z_\beta$$

where the $\partial_{\alpha\beta}$ are the integers defined in Theorem 1.14.

As a consequence, we obtain the following results.

Theorem 2.2 *(i) The middle cohomology of $\mathcal{O}_{\min}$ is given by*

$$H^{2h^\vee-2}(\mathcal{O}_{\min},\mathbb{Z}) \simeq P^\vee(\Phi')/Q^\vee(\Phi')$$

where Φ' is the sub-root system of Φ generated by Δ_{lg}, and $P^\vee(\Phi')$ (resp. $Q^\vee(\Phi')$) is its coweight lattice (resp. its coroot lattice).

(ii) The rest of the cohomology of $\mathcal{O}_{\min}$ is as described in section 3. In particular, if ℓ is a good prime for G, then there is no ℓ-torsion in the rest of the cohomology of $\mathcal{O}_{\min}$.

Proof : The map $c_{\tilde{I}}(\tilde{\alpha}) : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ is described as follows. By Theorem 2.1, the cohomology group $H^{2h^\vee-4}$ is free with basis $(Z_\beta)_{\beta\in\Delta_{\mathrm{lg}}}$ (the long roots of level $h^\vee-2$ are of the long roots dual height 1, so they are the long simple roots). Similarly, $H^{2h^\vee-4}$ is free with basis $(Z_{-\alpha})_{\alpha\in\Delta_{\mathrm{lg}}}$. Besides, the matrix of $c_{\tilde{I}}(\tilde{\alpha}) : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ in these bases is $(\partial_{-\alpha,\beta})_{\alpha,\beta\in\Delta_{\mathrm{lg}}}$. We have $\partial_{-\alpha,\alpha} = 2$ for $\alpha\in\Delta_{\mathrm{lg}}$, and for distinct α and β we have $\partial_{-\alpha,\beta} = 1$ if $\alpha+\beta$ is a (long) root, 0 otherwise.

Thus the matrix of $c_{\tilde{I}}(\tilde{\alpha}) : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ is the Cartan matrix of Φ' without minus signs. This matrix is equivalent to the Cartan matrix of Φ': since the Dynkin diagram of Φ' is a tree, it is bipartite. We can write $\Delta_{\mathrm{lg}} = J \cup K$, where no element of J is linked to an element of K in the Dynkin diagram of Φ'. If we replace the $Z_{\pm\alpha}$, $\alpha\in J$, by their opposites, then the matrix of $c_{\tilde{I}}(\tilde{\alpha}) : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$ becomes the Cartan matrix of Φ'.

Now, the Cartan matrix of Φ' is transposed to the matrix of the inclusion of $Q(\Phi')$ in $P(\Phi')$ in the bases Δ_{lg} and $(\varpi_\alpha)_{\alpha\in\Delta_{\mathrm{lg}}}$ (see [Bou68, Chap. VI, §1.10]), so it is in fact the matrix of the inclusion of $Q^\vee(\Phi')$ in $P^\vee(\Phi')$ in the bases $(\beta^\vee)_{\beta\in\Delta_{\mathrm{lg}}}$ and $(\varpi_{\alpha^\vee})_{\alpha\in\Delta_{\mathrm{lg}}}$.

The middle cohomology group $H^{2h^\vee-2}(\mathcal{O}_{\min},\mathbb{Z})$ is isomorphic to the cokernel of the map $c_{\tilde{I}}(\tilde{\alpha}) : H^{2h^\vee-4} \longrightarrow H^{2h^\vee-2}$. This proves (i).

Part (ii) follows from a case-by-case analysis which will be done in section 3. □

Remark 2.3 Besides, we have $\partial_{\alpha\beta} = \partial_{-\beta,-\alpha}$, so the maps "multiplication by $c_{\tilde{I}}(\tilde{\alpha})$" in complementary degrees are transposed to each other. This accounts for the fact that $\mathcal{O}_{\min}$ satisfies Poincaré duality, since $\mathcal{O}_{\min}$ is homeomorphic to $\mathbb{R}_*^+$ times a smooth compact manifold of (real) dimension $2h^\vee-5$ (since we deal with integral coefficients, one should take the derived dual for the Poincaré duality).

Remark 2.4 For the first half of the rational cohomology of $\mathcal{O}_{\min}$, we find

$$\bigoplus_{i=1}^{k} \mathbb{Q}[-2(d_i-2)]$$

where k is the number of long simple roots, and $d_1 \leqslant \ldots \leqslant d_k \leqslant \ldots \leqslant d_n$ are the degrees of W (n being the total number of simple roots). This can be observed case by case, or related to the corresponding Springer representation. The other half is determined by Poincaré duality.

3 Case-by-case analysis

In the preceding section, we have explained how to compute the cohomology of the minimal class in any given type in terms of root systems, and we found a description of the middle cohomology with a general proof. However, for the rest of the cohomology, we need a case-by-case analysis. It will appear that the primes dividing the torsion of the rest of the cohomology are bad. We have no *a priori* explanation for this fact. Note that, for the type A, we have an alternative method, which will be explained in the next section.

For all types, first we give the Dynkin diagram, to fix the numbering $(\alpha_i)_{1 \leqslant i \leqslant r}$ of the vertices, where r denotes the semisimple rank of $\mathfrak{g}$, and to show the part $\tilde{I}$ of Δ (see (15)). The corresponding vertices are represented in black. They are exactly those that are not linked to the additional vertex in the extended Dynkin diagram.

Then we give a diagram whose vertices are the positive long roots; whenever $\beta \xrightarrow{\gamma} \alpha$, we put an edge between β (above) and α (below), and the multiplicity of the edge is equal to $\partial_{\alpha\beta} = \langle \beta, \gamma^\vee \rangle$. In this diagram, the long root $\sum_{i=1}^r n_i \alpha_i$ (where the n_i are non-negative integers) is denoted by $n_1 \dots n_r$. The roots in a given line appear in lexicographic order.

For $1 \leqslant i \leqslant d-1$, let $\mathcal{D}_i$ be the matrix of the map $c_{\tilde{I}}(\tilde{\alpha}) : H^{2i-2} \to H^{2i}$ in the bases Φ_{lg}^{i-1} and Φ_{lg}^{i} (the roots being ordered in lexicographic order, as in the diagram). We give the matrices $\mathcal{D}_i$ for $i = 1 \dots h^\vee - 2$. The matrix $\mathcal{D}_{h^\vee - 1}$ is equal to the Cartan matrix without minus signs of the root system Φ' (corresponding to Δ_{lg}). The last matrices can be deduced from the first ones by symmetry, since (by Remark 2.3) we have $\mathcal{D}_{d-i} = {}^t\mathcal{D}_i$.

Then we give the cohomology of the minimal class with $\mathbb{Z}$ coefficients (one just has to compute the elementary divisors of the matrices $\mathcal{D}_i$).

It will be useful to introduce some notation for the matrices in classical types. Let k be an integer. We set

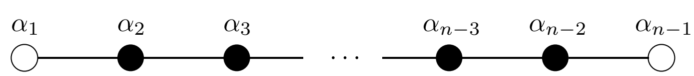
$$M(k) = \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & 1 & \ddots & \vdots & \vdots \\ 0 & 1 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 1 & 1 \end{pmatrix} \qquad N(k) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 1 & 1 & \ddots & \vdots \\ 0 & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 \\ 0 & \dots & 0 & 1 \end{pmatrix}$$

where $M(k)$ is a square matrix of size k, and $N(k)$ is of size $(k+1) \times k$.

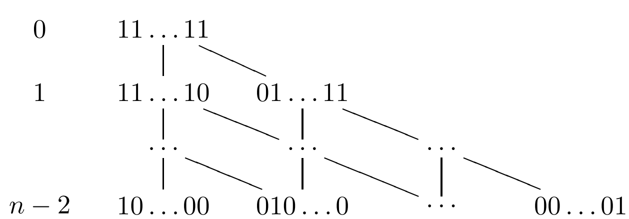
Now let k and l be non-negative integers. For i and j any integers, we define a $k \times l$ matrix $E_{i,j}(k,l)$ as follows. If (i,j) is not in the range $[1,k] \times [1,l]$, then we set $E_{i,j}(k,l) = 0$, otherwise it will denote the $k \times l$ matrix whose only non-zero entry is a 1 in the intersection of line i and column j. If the size of the matrix is clear from the context, we will simply write $E_{i,j}$.

First, the calculations of the elementary divisors of the matrices $\mathcal{D}_i$ were done with GAP3 (see [S+97]). We used the data on roots systems of the package CHEVIE. But actually, all the calculations can be done by hand.

3.1 Type A_{n-1}



We have $h = h^\vee = n$ and $d = 2n - 2$.



The odd cohomology of $G/P_{\tilde{I}}$ is zero, and we have

$$H^{2i}(G/P_{\tilde{I}}) = \begin{cases} \mathbb{Z}^{i+1} & \text{if } 0 \leqslant i \leqslant n-2 \\ \mathbb{Z}^{2n-2-i} & \text{if } n-1 \leqslant i \leqslant 2n-3 \\ 0 & \text{otherwise} \end{cases}$$

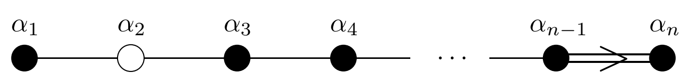
For $1 \leqslant i \leqslant n-2$, we have $\mathcal{D}_i = N(i)$; the cokernel is isomorphic to $\mathbb{Z}$. We have

$$\mathcal{D}_{n-1} = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 \\ 1 & 2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & 2 & 1 \\ 0 & \dots & 0 & 1 & 2 \end{pmatrix}$$

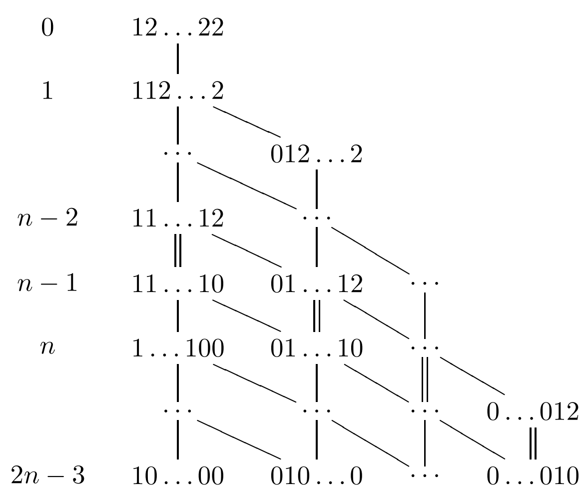
Its cokernel is isomorphic to $\mathbb{Z}/n$. The last matrices are transposed to the first ones, so the corresponding maps are surjective. From this, we deduce the cohomology of $\mathcal{O}_{\min}$ in type A_{n-1}. We will see another method in section 4.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) = \begin{cases} \mathbb{Z} & \text{if } 0 \leqslant i \leqslant 2n-4 \text{ and } i \text{ is even,} \\ & \text{or } 2n-1 \leqslant i \leqslant 4n-5 \text{ and } i \text{ is odd} \\ \mathbb{Z}/n & \text{if } i = 2n-2 \\ 0 & \text{otherwise} \end{cases}$$

3.2 Type B_n



We have $h = 2n$, $h^\vee = 2n-1$, and $d = 4n-4$.



There is a gap at each even line (the length of the line increases by one). The diagram can be a little bit misleading if n is even: in that case, there is a gap at the line $n-2$. Let us now describe the matrices $\mathcal{D}_i$.

First suppose $1 \leqslant i \leqslant n-2$. If i is odd, then we have $\mathcal{D}_i = M\left(\frac{i+1}{2}\right)$ (an isomorphism); if i is even, then we have $\mathcal{D}_i = N\left(\frac{i}{2}\right)$ and the cokernel is isomorphic to $\mathbb{Z}$.

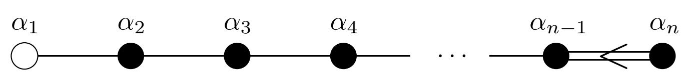
Now suppose $n-1 \leqslant i \leqslant 2n-3$. If i is odd, then we have $\mathcal{D}_i = M\left(\frac{i+1}{2}\right) + E_{i+2-n,i+2-n}$ and the cokernel is isomorphic to $\mathbb{Z}/2$. If i is even, then we have $\mathcal{D}_i = N\left(\frac{i}{2}\right) + E_{i+2-n,i+2-n}$ and the cokernel is isomorphic to $\mathbb{Z}$.

The long simple roots generate a root system of type A_{n-1}. Thus the matrix $\mathcal{D}_{2n-2}$ is the Cartan matrix without minus signs of type A_{n-1}, which has cokernel $\mathbb{Z}/n$.

So the cohomology of $\mathcal{O}_{\min}$ is described as follows.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } 0 \leqslant i \leqslant 4n-8 \text{ and } i \equiv 0 \mod 4, \\ & \text{or } 4n-1 \leqslant i \leqslant 8n-9 \text{ and } i \equiv -1 \mod 4 \\ \mathbb{Z}/2 & 2n-2 \leqslant i \leqslant 6n-6 \text{ and } i \equiv 2 \mod 4 \\ \mathbb{Z}/n & \text{if } i = 4n-4 \\ 0 & \text{otherwise} \end{cases}$$

3.3 Type C_n



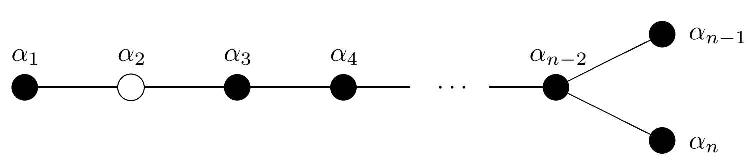
We have $h = 2n$, $h^\vee = n+1$, and $d = 2n$. The root system Φ' is of type A_1. Its Cartan matrix is (2).

$$\begin{array}{cc} 0 & 22\ldots21 \\ & \| \\ 1 & 02\ldots21 \\ & \| \\ & \vdots \\ & \| \\ n-2 & 0\ldots021 \\ & \| \\ n-1 & 00\ldots01 \end{array}$$

The matrices $\mathcal{D}_i$ are all equal to (2).

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } 4n-1 \\ \mathbb{Z}/2 & \text{if } 2 \leqslant i \leqslant 4n-2 \text{ and } i \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

3.4 Type D_n

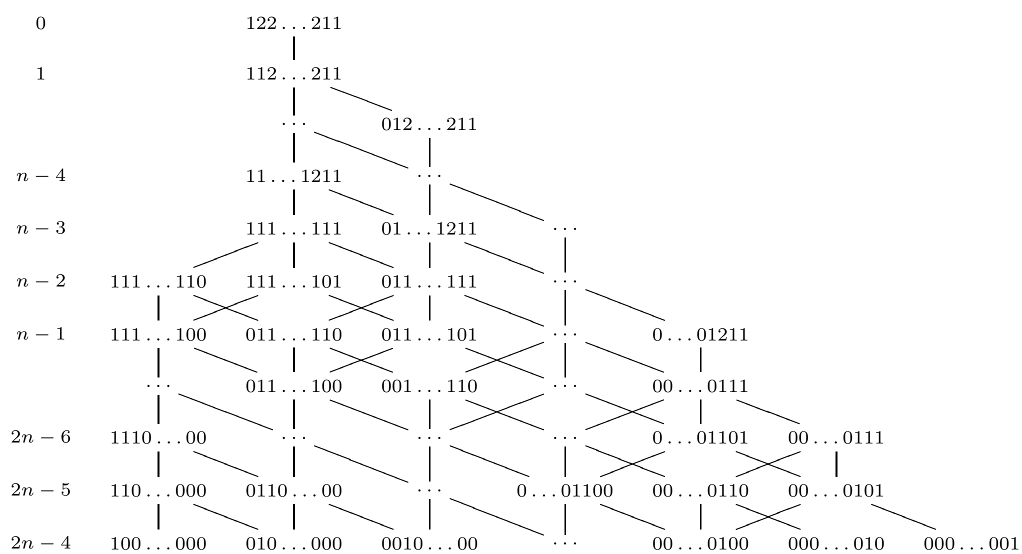


We have $h = h^\vee = 2n-2$, and $d = 4n-6$. We have

$$\mathcal{D}_{2n-3} = \begin{pmatrix} 2 & 1 & 0 & \dots & 0 & 0 \\ 1 & 2 & \ddots & \ddots & \vdots & \vdots \\ 0 & \ddots & \ddots & 1 & 0 & 0 \\ \vdots & \ddots & 1 & 2 & 1 & 1 \\ 0 & \dots & 0 & 1 & 2 & 0 \\ 0 & \dots & 0 & 1 & 0 & 2 \end{pmatrix}$$

Its cokernel is $(\mathbb{Z}/2)^2$ when n is even, $\mathbb{Z}/4$ when n is odd.

As in the B_n case, the reader should be warned that there is a gap at line $n-4$ if n is even. Besides, not all dots are meaningful. The entries $0\dots01211$ and $00\dots0111$ are on the right diagonal, but usually they are not on the lines $n-1$ and n.



First suppose $i \leqslant i \leqslant n-3$. We have

$$\mathcal{D}_i = \begin{cases} M\left(\frac{i+1}{2}\right) & \text{if } i \text{ is odd} \\ N\left(\frac{i}{2}\right) & \text{if } i \text{ is even} \end{cases}$$

Then the cokernel is zero if i is odd, $\mathbb{Z}$ if i is even.

Let V be the $1 \times \frac{n-1}{2}$ matrix $(1, 0, \dots, 0)$. We have

$$\mathcal{D}_{n-2} = \begin{cases} \begin{pmatrix} V \\ N\left(\frac{n-2}{2}\right) \end{pmatrix} & \text{if } n \text{ is even} \\ \\ \begin{pmatrix} V \\ M\left(\frac{n-1}{2}\right) \end{pmatrix} & \text{if } n \text{ is odd} \end{cases}$$

The cokernel is $\mathbb{Z}^2$ if n is even, $\mathbb{Z}$ if i is odd.

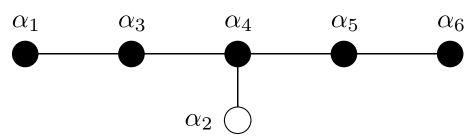
Now suppose $n-1 \leqslant i \leqslant 2n-4$. We have

$$\mathcal{D}_i = \begin{cases} M\left(\frac{i+3}{2}\right) + E_{i+2-n,i+3-n} - E_{i+3-n,i+3-n} + E_{i+3-n,i+4-n} & \text{if } i \text{ is odd} \\ N\left(\frac{i+2}{2}\right) + E_{i+2-n,i+3-n} - E_{i+3-n,i+3-n} + E_{i+3-n,i+4-n} & \text{if } i \text{ is even} \end{cases}$$

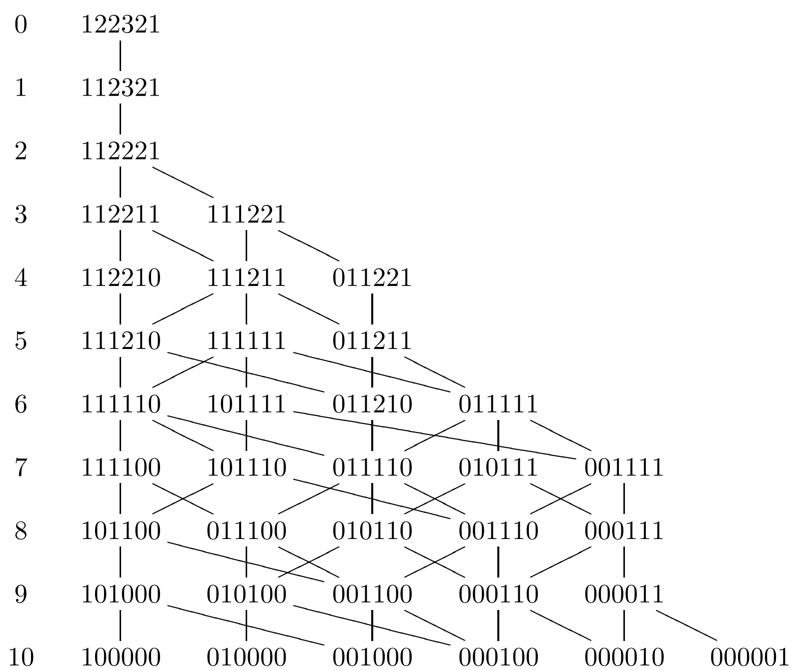
Then the cokernel is $\mathbb{Z}/2$ if i is odd, $\mathbb{Z}$ if i is even.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{if } 0 \leqslant i \leqslant 4n-8 \text{ and } i \equiv 0 \mod 4, \\ & \text{or } 4n-5 \leqslant i \leqslant 8n-13 \text{ and } i \equiv -1 \mod 4; \\ \mathbb{Z}/2 & \text{if } 2n-4 < i < 4n-6 \text{ and } i \equiv 2 \mod 4; \\ & \text{or } 4n-6 < i < 6n-8 \text{ and } i \equiv 2 \mod 4; \\ (\mathbb{Z}/2)^2 & \text{if } i = 4n-6 \text{ and } n \text{ is even;} \\ \mathbb{Z}/4 & \text{if } i = 4n-6 \text{ and } n \text{ is odd;} \\ 0 & \text{otherwise.} \end{cases} \quad \oplus\, \mathbb{Z} \;\; \begin{array}{l} \text{if } i = 2n-4 \\ \text{or } i = 6n-7. \end{array}$$

3.5 Type E_6



We have $h = h^\vee = 12$, and $d = 22$. The Cartan matrix has cokernel isomorphic to $\mathbb{Z}/3$.



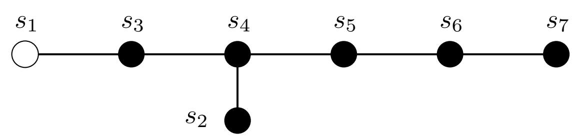
$$\mathcal{D}_1 = \mathcal{D}_2 = \begin{pmatrix} 1 \end{pmatrix} \quad \mathcal{D}_3 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathcal{D}_4 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathcal{D}_5 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{D}_6 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_8 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

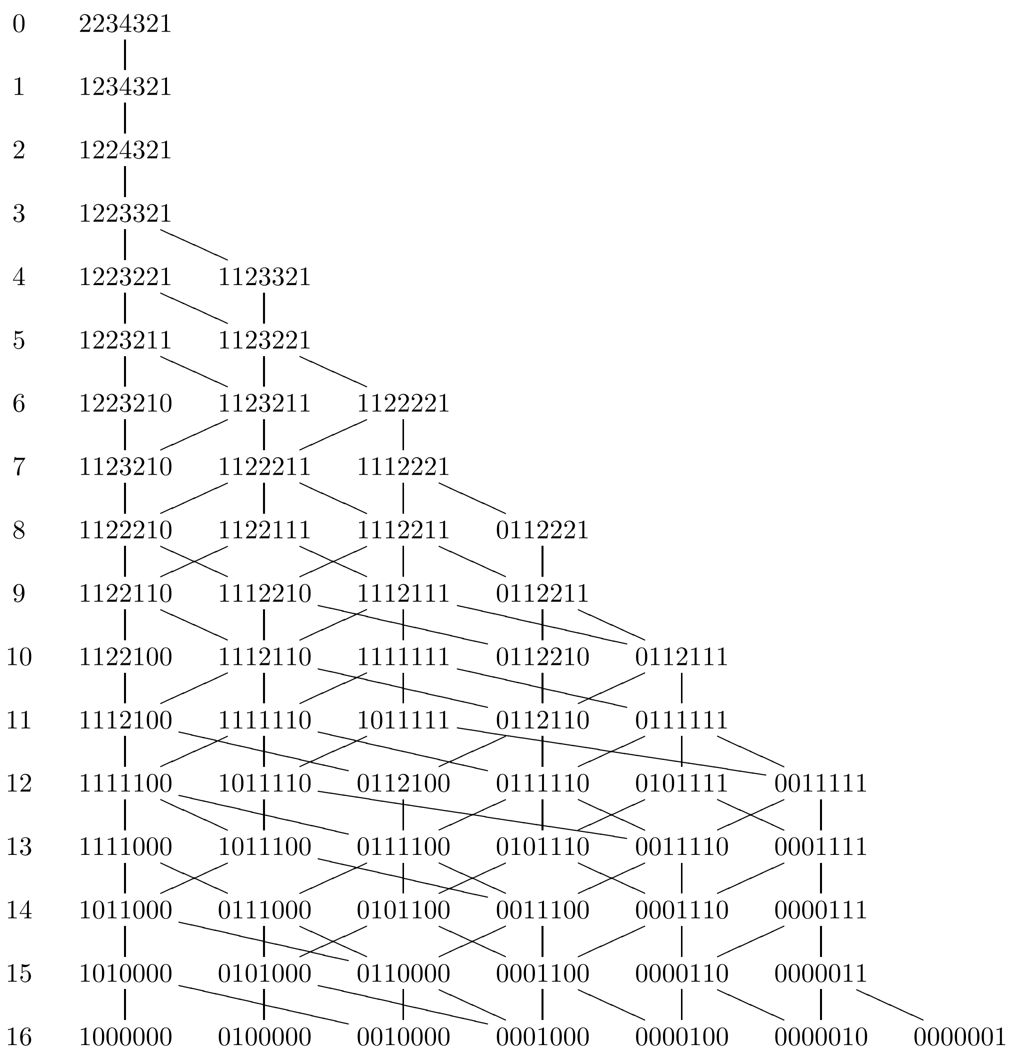
$$\mathcal{D}_9 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0,\ 6,\ 8,\ 12,\ 14,\ 20,\ 23,\ 29,\ 31,\ 35,\ 37,\ 43 \\ \mathbb{Z}/3 & \text{for } i = 16,\ 22,\ 28 \\ \mathbb{Z}/2 & \text{for } i = 18,\ 26 \\ 0 & \text{otherwise} \end{cases}$$

3.6 Type E_7



We have $h = h^\vee = 18$, and $d = 34$. The Cartan matrix has cokernel isomorphic to $\mathbb{Z}/2$.


0 2234321
1 1234321
2 1224321
3 1223321
4 1223221 1123321
5 1223211 1123221
6 1223210 1123211 1122221
7 1123210 1122211 1112221
8 1122210 1122111 1112211 0112221
9 1122110 1112210 1112111 0112211
10 1122100 1112110 1111111 0112210 0112111
11 1112100 1111110 1011111 0112110 0111111
12 1111100 1011110 0112100 0111110 0101111 0011111
13 1111000 1011100 0111100 0101110 0011110 0001111
14 1011000 0111000 0101100 0011100 0001110 0000111
15 1010000 0101000 0110000 0001100 0000110 0000011
16 1000000 0100000 0010000 0001000 0000100 0000010 0000001

$$\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \begin{pmatrix} 1 \end{pmatrix} \quad \mathcal{D}_4 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathcal{D}_5 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \mathcal{D}_6 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_7 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_8 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_9 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{10} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

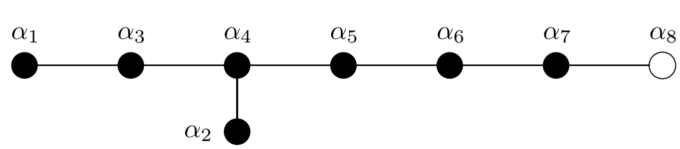
$$\mathcal{D}_{11} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{12} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_{13} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{14} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

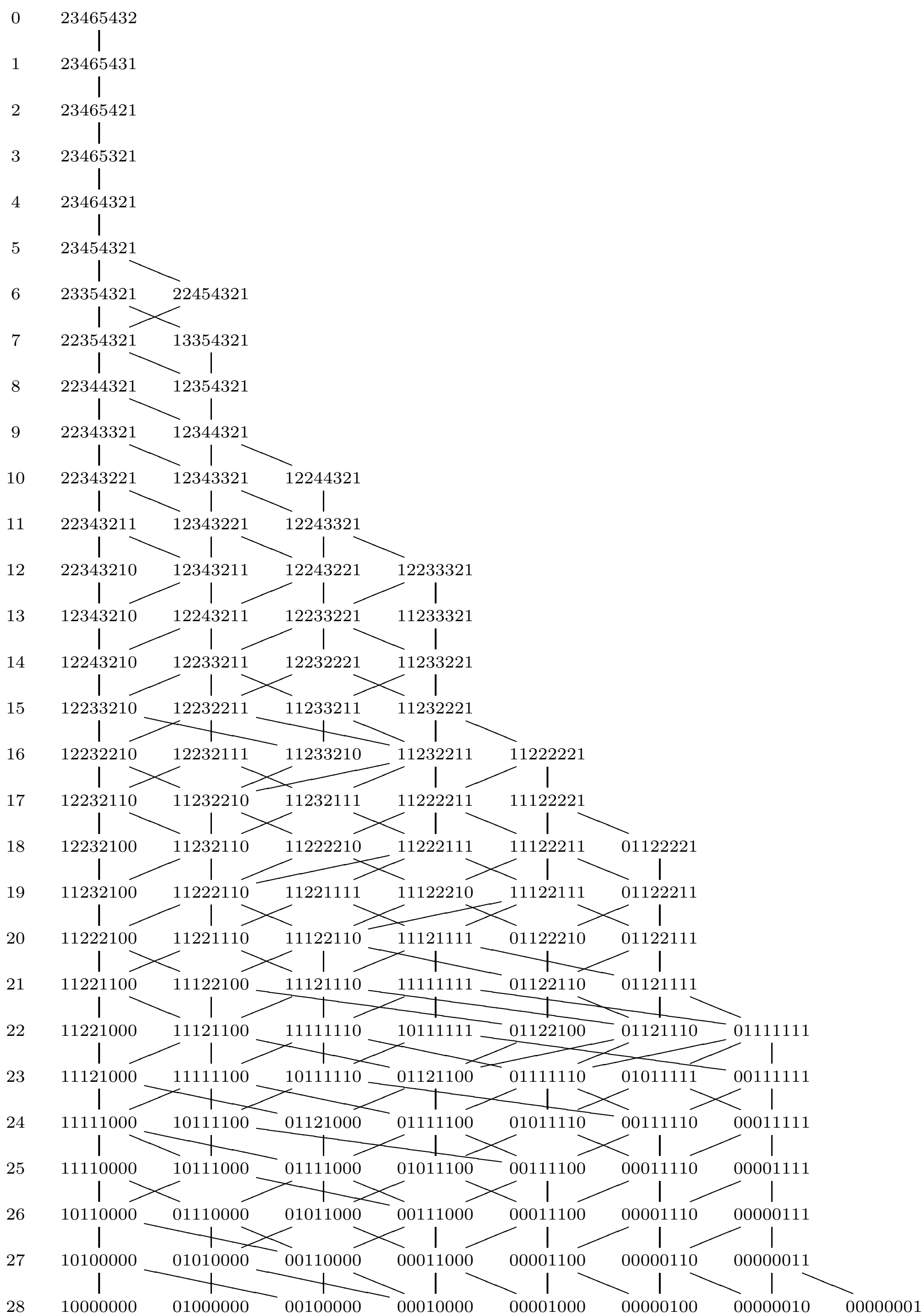
$$\mathcal{D}_{15} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{16} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0,\ 8,\ 12,\ 16,\ 20,\ 24,\ 32, \\ & \qquad 35,\ 43,\ 47,\ 51,\ 55,\ 59,\ 67 \\ \mathbb{Z}/2 & \text{for } i = 18,\ 26,\ 30,\ 34,\ 38,\ 42,\ 50 \\ \mathbb{Z}/3 & \text{for } i = 28,\ 40 \\ 0 & \text{otherwise} \end{cases}$$

3.7 Type E_8



We have $h = h^\vee = 30$, and $d = 58$. The Cartan matrix is an isomorphism.


0 23465432
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3 23465321
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7 22354321 13354321
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9 22343321 12344321
10 22343221 12343321 12244321
11 22343211 12343221 12243321
12 22343210 12343211 12243221 12233321
13 12343210 12243211 12233221 11233321
14 12243210 12233211 12232221 11233221
15 12233210 12232211 11233211 11232221
16 12232210 12232111 11233210 11232211 11222221
17 12232110 11232210 11232111 11222211 11122221
18 12232100 11232110 11222210 11222111 11122211 01122221
19 11232100 11222110 11221111 11122210 11122111 01122211
20 11222100 11221110 11122110 11121111 01122210 01122111
21 11221100 11122100 11121110 11111111 01122110 01121111
22 11221000 11121100 11111110 10111111 01122100 01121110 01111111
23 11121000 11111100 10111110 01121100 01111110 01011111 00111111
24 11111000 10111100 01121000 01111100 01011110 00111110 00011111
25 11110000 10111000 01111000 01011100 00111100 00011110 00001111
26 10110000 01110000 01011000 00111000 00011100 00001110 00000111
27 10100000 01010000 00110000 00011000 00001100 00000110 00000011
28 10000000 01000000 00100000 00010000 00001000 00000100 00000010 00000001

$$\mathcal{D}_1 = \mathcal{D}_2 = \mathcal{D}_3 = \mathcal{D}_4 = \mathcal{D}_5 = (1) \quad \mathcal{D}_6 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \mathcal{D}_7 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathcal{D}_8 = \mathcal{D}_9 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

$$\mathcal{D}_{10} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \mathcal{D}_{11} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{12} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{13} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_{14} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{15} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{16} = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_{17} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{18} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{19} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{D}_{20} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad \mathcal{D}_{21} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{22} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

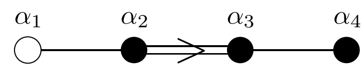
$$\mathcal{D}_{23} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{24} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$\mathcal{D}_{25} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{26} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\mathcal{D}_{27} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \mathcal{D}_{28} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0,\ 12,\ 20,\ 24,\ 32,\ 36,\ 44,\ 56, \\ & \qquad\quad 59,\ 71,\ 79,\ 83,\ 91,\ 95,\ 103, 115 \\ \mathbb{Z}/2 & \text{for } i = 30,\ 42,\ 50,\ 54,\ 62,\ 66,\ 74,\ 86 \\ \mathbb{Z}/3 & \text{for } i = 40,\ 52,\ 64,\ 76 \\ \mathbb{Z}/5 & \text{for } i = 48,\ 68 \\ 0 & \text{otherwise} \end{cases}$$

3.8 Type F_4



We have $h = 12$, $h^\vee = 9$ and $d = 16$.

$$\begin{tikzcd}[row sep=small]
0 & 2342 \arrow[d, dash] & \\
1 & 1342 \arrow[d, dash] & \\
2 & 1242 \arrow[d, equal] & \\
3 & 1222 \arrow[d, equal] \arrow[dr, dash] & \\
4 & 1220 \arrow[d, dash] & 1122 \arrow[dl, equal] \arrow[d, dash] \\
5 & 1120 \arrow[d, equal] \arrow[dr, dash] & 0122 \arrow[d, equal] \\
6 & 1100 \arrow[d, dash] \arrow[dr, dash] & 0120 \arrow[d, equal] \\
7 & 1000 & 0100
\end{tikzcd}$$

We have

$$\mathcal{D}_1 = \mathcal{D}_2 = (1) \quad \mathcal{D}_3 = (2) \quad \mathcal{D}_4 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \mathcal{D}_5 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \mathcal{D}_6 = \begin{pmatrix} 2 & 0 \\ 1 & 2 \end{pmatrix} \quad \mathcal{D}_7 = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$$

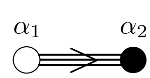
The type of Φ' is A_2, so we have

$$\mathcal{D}_8 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

The matrices of the last differentials are transposed to the first ones.

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0,\ 8,\ 23,\ 31 \\ \mathbb{Z}/2 & \text{for } i = 6,\ 14,\ 18,\ 26 \\ \mathbb{Z}/4 & \text{for } i = 12,\ 20 \\ \mathbb{Z}/3 & \text{for } i = 16 \\ 0 & \text{otherwise} \end{cases}$$

3.9 Type G_2



We have $h = 6$, $h^\vee = 4$, and $d = 6$. The root system Φ' is of type A_1. Its Cartan matrix has cokernel $\mathbb{Z}/2$.

$$\begin{tikzcd}
0 & 23 \arrow[d, dash] \\
1 & 13 \arrow[d, Rrightarrow, no head] \\
2 & 10
\end{tikzcd}$$

We have

$$\mathcal{D}_1 = (1) \quad \mathcal{D}_2 = (3) \quad \mathcal{D}_3 = (2) \quad \mathcal{D}_4 = (3) \quad \mathcal{D}_5 = (1)$$

$$H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \begin{cases} \mathbb{Z} & \text{for } i = 0,\ 11 \\ \mathbb{Z}/3 & \text{for } i = 4,\ 8 \\ \mathbb{Z}/2 & \text{for } i = 6 \\ 0 & \text{otherwise} \end{cases}$$

4 Another method for type A

Here we will explain a method which applies only in type A. This is because the minimal class is a Richardson class only in type A.

So suppose we are in type A_{n-1}. We can assume $G = GL_n$. The minimal class corresponds to the partition $(2, 1^{n-2})$. It consists of the nilpotent matrices of rank 1 in $\mathfrak{gl}_n$, or, in other words, the matrices of rank 1 and trace 0.

Let us consider the set E of pairs $([v], x) \in \mathbb{P}^{n-1} \times \mathfrak{gl}_n$ such that $\mathrm{Im}(x) \subset \mathbb{C}v$ (so x is either zero or of rank 1). Together with the natural projection, this is a vector bundle on $\mathbb{P}^{n-1}$, corresponding to the locally free sheaf $\mathcal{E} = \mathcal{O}(-1)^n$ (we have one copy of the tautological bundle for each column).

There is a trace morphism $\mathrm{Tr} : \mathcal{E} \to \mathcal{O}$. Let $\mathcal{F}$ be its kernel, and let F be the corresponding sub-vector bundle of E. Then F consists of the pairs $([v], x)$ such that x is either zero or a nilpotent matrix of rank 1 with image $\mathbb{C}v$. The second projection gives a morphism $\pi : E \to \overline{\mathcal{O}_{\min}}$, which is a resolution of singularities, with exceptional fiber the null section. So we have an isomorphism from F minus the null section onto $\mathcal{O}_{\min}$.

As before, we have a Gysin exact sequence

$$H^{i-2n+2} \xrightarrow{c} H^i \longrightarrow H^i(\mathcal{O}_{\min}, \mathbb{Z}) \longrightarrow H^{i-2n+3} \xrightarrow{c} H^{i+1}$$

where H^j stands for $H^j(\mathbb{P}^{n-1}, \mathbb{Z})$ and c is the multiplication by the last Chern class c of F. Thus $H^i(\mathcal{O}_{\min}, \mathbb{Z})$ fits in a short exact sequence

$$0 \longrightarrow \mathrm{Coker}(c : H^{i-2n+2} \to H^i) \longrightarrow H^i(\mathcal{O}_{\min}, \mathbb{Z}) \longrightarrow \mathrm{Ker}(c : H^{i-2n+3} \to H^{i+1}) \longrightarrow 0$$

We denote by $y \in H^2(\mathbb{P}^{n-1}, \mathbb{Z})$ the first Chern class of $\mathcal{O}(-1)$. We have $H^*(\mathbb{P}^{n-1}, \mathbb{Z}) \simeq \mathbb{Z}[y]/(y^n)$ as a ring. In particular, the cohomology of $\mathbb{P}^{n-1}$ is free and concentrated in even degrees.

For $0 \leqslant i \leqslant 2n-4$, we have $H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq H^i$ which is isomorphic to $\mathbb{Z}$ if i is even, and to 0 if i is odd. We have $H^{2n-3}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \mathrm{Ker}(c : H^0 \to H^{2n-2})$ and $H^{2n-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \mathrm{Coker}(c : H^0 \to H^{2n-2})$. For $2n-1 \leqslant i \leqslant 4n-5$, we have $H^i(\mathcal{O}_{\min}, \mathbb{Z}) \simeq H^{i-2n+3}$ which is isomorphic to $\mathbb{Z}$ if i is odd, and to 0 if i is even.

We have an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{E} = \mathcal{O}(-1)^n \longrightarrow \mathcal{O} \longrightarrow 0$$

The total Chern class of F is thus

$$(1+y)^n = \sum_{i=0}^{n-1} \binom{n}{i} y^i$$

by multiplicativity (remember that $y^n = 0$). So its last Chern class c is ny^{n-1}.

In fact, F can be identified with the cotangent bundle $T^*(G/Q)$, where Q is the parabolic subgroup which stabilizes a line in $\mathbb{C}^n$, and $G/Q \simeq \mathbb{P}^{n-1}$; then we can use the fact that the Euler characteristic of $\mathbb{P}^{n-1}$ is n.

We can now determine the two remaining cohomology groups.

$$H^{2n-3}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \mathrm{Ker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = 0$$

and

$$H^{2n-2}(\mathcal{O}_{\min}, \mathbb{Z}) \simeq \mathrm{Coker}(\mathbb{Z} \xrightarrow{n} \mathbb{Z}) = \mathbb{Z}/n$$

Thus we find the same result as in section 3 for the cohomology of $\mathcal{O}_{\min}$ in type A_{n-1}.

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Daniel Juteau
Institut de Mathématiques de Jussieu
Université Denis Diderot - Paris VII
2 place Jussieu, 75251 Paris Cedex 05, France
juteau@math.jussieu.fr