# PERVERSE SHEAVES

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Abstract. These are notes for a talk given in the MIT Graduate Seminar on D-modules and Perverse Sheaves in Fall 2015. In this talk, I define perverse sheaves on a stratifiable space. I give the definition of t structures, describe the simple perverse sheaves and examine when the 6 functors on the constructibe derived category preserve the subcategory of perverse sheaves. The main reference for this talk is [HTT].

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## 1. Introduction

Let $X$ be a complex algebraic vareity and $D^b_c(X)$ be the constructible derived category of sheaves on $X$. The category of perverse sheaves $P(X)$ is defined as the full subcategory of $D^b_c(X)$ consisting of objects $F^*$ that satifsy two conditions:

1. Support condition: $\dim \operatorname{supp}(\mathcal{H}^j(F^*)) \leq -j$, for all $j \in \mathbb{Z}$.
2. Cosupport condition: $\dim \operatorname{supp}(\mathcal{H}^j(DF^*)) \leq -j$, for all $j \in \mathbb{Z}$.

Here, $D$ denotes the Verdier duality functor. If $F^*$ satisfies the support condition, we say that $F^* \in {}^pD^{\leq 0}(X)$ and if it satisfies the cosupport condition, we say that $F^* \in {}^pD^{\geq 0}(X)$. These conditions actually imply that $P(X)$ is an abelian category sitting inside $D^b_c(X)$ but to see this, we first need to talk about $t$-structures on triangulated categories.

## 2. t-Structures on Triangulated Categories

Let me begin by recalling (part of) the axioms of a triangulated category. The only examples we will really be considering will be derived categories of abelian categories so we could get away with not defining the abstract notion of a triangulated category but sometimes, it's easier to prove things by removing unnecessary properties.

Let $C$ be an additive category equipped with an automorphism $T$ called the translation functor. By a triangle in $C$, we mean a sequence of morphisms

$$F \to G \to H \to TF.$$

**Definition 2.1.** A *triangulated category* is a triple $(C, T, \mathscr{T})$ where $C, T$ is a pair of an additive category and a translation functor as above and $\mathscr{T}$ is a subset of the set of all triangles in $T$ (called the set of distinguished triangles in $C$) such that the following hold:

(TR0) Any triangle isomorphic to a distinguished triangle is distinguished (where isomorphism of triangles means a commutative diagram with the top and bottom rows triangles and the vertical maps isomorphisms).

(TR1) For any $F \in C$,

$$\begin{tikzcd} F \arrow[r, "\mathrm{id}"] & F \arrow[r] & 0 \arrow[r] & TF \end{tikzcd}$$

is distinguished.

(TR2) Any morphism $f : F \to G$ in $C$ can be embedded into a distingusihed triangle

$$F \to G \to H \to TF.$$

(TR3) $F \to G \to H \to TF$ is distinguished if and only if $G \to H \to TF \to TG$ is dinstinguished, where if $f$ is the morphism from $F$ to $G$, then $-Tf$ is the morphism from $TF$ to $TG$.

(TR4) Given two dinstinguished triangles

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] & G_1 \arrow[r] & H_1 \arrow[r] & TF_1 \end{tikzcd}$$

and

$$\begin{tikzcd} F_2 \arrow[r, "f_2"] & G_2 \arrow[r] & H_2 \arrow[r] & TF_2 \end{tikzcd}$$

and a commutative diagram

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] \arrow[d] & G_1 \arrow[d] \\ F_2 \arrow[r, "f_2"] & G_2 \end{tikzcd}$$

we can embed them into a morphism of triangles

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] \arrow[d] & G_1 \arrow[r] \arrow[d] & H_1 \arrow[r] \arrow[d] & TF_1 \arrow[d] \\ F_2 \arrow[r, "f_2"] & G_2 \arrow[r] & H_2 \arrow[r] & TF_2 \end{tikzcd}$$

(TR5) Octahedron axiom (see [HTT, p. 337])

Suppose $A$ is now an abelian category and let $C$ be the derived category of $A$, with $T$ the shift functor $F \to F[1]$. Recall the mapping cone construction: given a morphism of complexes $f : F^* \to G^*$, define $C_f$ as the complex

$$C_f^n = F^{n+1} \oplus G^n$$

with differential given by the matrix

$$\begin{pmatrix} d_F & 0 \\ f & d_G \end{pmatrix}.$$

This has an obvious map to $TF$ given by projection onto the left factor and an obvious map from $G$ given by inclusion of the right factor. We set $\mathscr{T}$ as the set of triangles isomorphic to mapping cone triangles.

**Proposition 2.2.** The triple $(C, T, \mathscr{T})$ as above is a triangulated category.

*Proof.* Axioms (TR0), (TR2) and (TR3) are obvious by construction. Axiom (TR1) follows from the long exact sequence of cohomology associated to the mapping cone construction. We omit the check of the octahedral axiom because it is tedious. We just need to check axiom (TR5) where the two triangles are mapping cone triangles. Given a commutative diagram

$$\begin{tikzcd} F_1 \arrow[r, "f_1"] \arrow[d, "\phi"] & G_1 \arrow[d, "\psi"] \\ F_2 \arrow[r, "f_2"] & G_2 \end{tikzcd}$$

we define a map $\rho : C_{f_1} \to C_{f_2}$ by defining $\rho^n : F^{n+1} \oplus G_1^n \to F_2^{n+1} \oplus G_2^n$ as $\phi^{n+1} \oplus \psi^n$. Checking the resulting diagram commutes is now an easy exercise.

□

So, if we view triangulated categories as generalizations of derived categories, then distinguished triangles are generalizations of mapping cone triangles. Since, cohomology takes mapping cone triangles to long exact sequences, we have the following definition.

**Definition 2.3.** If $A$ is an abelian category and $C$ is triangulated, then a functor $F : C \to A$ is *cohomological* if it converts distinguished triangles into long exact sequences (note the distinguished triangles can be viewed as a doubly infinite sequence of morphisms).

**Example 2.4.** The functors $\mathrm{Hom}_C(F, -)$ and $\mathrm{Hom}_C(-, F)$ are cohomological.

**Remark.** The translation functor $T$ is a generalization of the shift functors in derived categories. Hence, we will use the notation $F[n]$ to denote $T^n(F)$.

As we have seen above, we have a way to get a triangulated category from an abelian category, by taking the corresponding derived category. A $t$-structure allows you to make the reverse construction i.e. it gives you an abelian category from a triangulated category. The way in which this works mimics the reconstruction of the abelian category from its derived category as the complexes concentrated in degree 0.

**Definition 2.5.** Let $(C, T, \mathscr{T})$ be a triangulated category. A *$t$-structure* on $C$ is the data of two full subcategories $C^{\leq 0}$ and $C^{\geq 0}$ that satifsfy the following properties:

(T1) If $C^{\leq n} = C^{\leq 0}[-n]$ and $C^{\geq n} = C^{\geq 0}[-n]$, then

$$C^{\leq -1} \subseteq C^{\leq 0}, C^{\geq 1} \subseteq C^{\geq 0}.$$

(T2) For any $F \in C^{\leq 0}, G \in C^{\geq 1}$, $\mathrm{Hom}_C(F, G) = 0$.

(T3) For any $F \in C$, there exists a distinguished triangle

$$F_0 \to F \to F_1 \to F_0[1]$$

such that $F_0 \in C^{\leq 0}$ and $F_1 \in C^{\geq 1}$.

**Definition 2.6.** Given a triangulated category $C$ with $t$-structure, the *heart* of the $t$-structure is the full subcategory

$$C^{\leq 0} \cap C^{\geq 0}.$$

If $C$ is the derived category of $A$, then we get a $t$-structure on $C$ by setting $C^{\leq 0}$ to be the complexes concentrated in degrees less than or equal to 0 (and similarly for $C^{\geq 0}$). Axioms (T1) and (T3) are obvious and (T2) follows by taking truncations. The heart of this $t$-structure is the original abelian category $A$.

It turns out that the heart of a $t$-structure on a triangulated category is always an abelian category. But to see this, we need to do some more work. We first construct truncation functors that generalize the truncation functors in the derived category setting. These will be constructed as adjoints to inclusion functors.

**Proposition 2.7.** Denote by $i : C^{\leq n} \to C$ and $i' : C^{\geq n} \to C$ the inclusions of the full subcategories. Then, there exist functors $\tau^{\leq n} : C \to C^{\leq n}$ and $\tau^{\geq n} : C \to C^{\geq n}$ such that for any $Y \in C^{\leq n}$, $Y' \in C^{\geq n}$ and any $X \in C$, we have isomorphisms

$$\mathrm{Hom}_{C^{\leq n}}(Y, \tau^{\leq n} X) \to \mathrm{Hom}_C(i(Y), X)$$

and

$$\mathrm{Hom}_{C^{\geq n}}(\tau^{\geq n} X, Y) \to \mathrm{Hom}_C(X, i'(Y)).$$

*Proof.* (see [HTT, Prop 8.14]) Let me do the proof for $\leq n$. The proof for $\geq n$ is analogous. We can assume $n = 0$ by using the translation functor. It suffices by the Yoneda lemma, to prove that for each $X \in C$, there exists some $Z \in C^{\leq 0}$ such that for $Y \in C^{\leq n}$

$$\mathrm{Hom}_C(Y, Z) = \mathrm{Hom}_C(Y, X).$$

To construct this $Z$ for fixed $X$, we take $X_0$ as in (T3). Since $\mathrm{Hom}_C(Y, -)$ is cohomological, applying it to the distinguished triangle $X_0 \to X \to X_1 \to X_0[1]$ gives us an exact sequence

$$\mathrm{Hom}_C(Y, X_1[-1]) \to \mathrm{Hom}_C(Y, X_0) \to \mathrm{Hom}_C(Y, X) \to \mathrm{Hom}_C(Y, X_1).$$

But, $X_1, X_1[-1] \in C^{\geq 1}$ and so applying (T2) gives the desired result.

□

**Remark.** The proof shows that for $X \in C$, the $X_0, X_1$ in (T3) are functorial because they are $\tau^{\leq 0} X$ and $\tau^{\geq 1} X$ respectively.

We now prove some properties of the truncation functors that are simple consquences of the construction and the fact that Hom is a cohomological functor.

**Proposition 2.8.** The canonical morphism from $\tau^{\leq n} X \to X$ is embedded into a distinguished triangle

$$\tau^{\leq n} X \to X \to \tau^{\geq n+1} X$$

**Proposition 2.9.** The following conditions on $X \in C$ are equivalent:

(1) $X \in C^{\leq n}$ (resp. $C^{\geq n}$)
(2) The canonical map $\tau^{\leq n} X \to X$ (resp. $X \to \tau^{\geq n} X$) is an isomorphism.
(3) $\tau^{\geq n+1} X = 0$ (resp. $\tau^{\leq n-1} X = 0$.)

**Lemma 2.10.** Let $X' \to X \to X''$ be a distinguished triangle in $C$. If $X', X'' \in \mathbb{C}^{\leq 0}$ (or $\mathbb{C}^{\geq 0}$), then so does $X$.

**Proposition 2.11.** Let $a, b \in \mathbb{Z}$.

(i) If $b \geq a$, then

$$\tau^{\leq b} \circ \tau^{\leq a} \cong \tau^{\leq a} \circ \tau^{\leq b} \cong \tau^{a}$$

and

$$\tau^{\geq b} \circ \tau^{\geq a} \cong \tau^{\geq a} \circ \tau^{\geq b} \cong \tau^{b}.$$

(ii) If $a > b$, then

$$\tau^{\leq b} \circ \tau^{\geq a} \cong \tau^{\geq a} \circ \tau^{\leq b} \cong 0.$$

(iii) $\tau^{\geq a} \circ \tau^{\leq b} \cong \tau^{\leq b} \circ \tau^{\geq a}$.

I leave the proof of this proposition to be looked up in [HTT, Prop 8.18]. Parts (i) and (ii) are obvious but part (iii) requires the use of the octahedral axiom.

We can now use these propositions to prove that the heart of a $t$-structure is abelian.

**Proposition 2.12.** Let $C$ be a triangulated category with $t$-structure and let $A$ be its heart. Then, $A$ is an abelian category.

*Proof.* Since $C$ is additive, Hom spaces in $A$ are abelian groups with composition bilinear. Additionally, direct sums exist in $C$. We claim that

$$X \to X \oplus Y \to Y \to X[1]$$

is a distinguished triangle (with the last morphism being 0). To do so, we note that we must have a distinguished triangle

$$X \to X \oplus Y \to Z \to X[1]$$

and then, we have a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\mathrm{id}"] & X \oplus Y \arrow[r] \arrow[d, "\mathrm{id}"] & Y \arrow[r] \arrow[d, "\phi"] & X[1] \arrow[d] \\ X \arrow[r] & X \oplus Y \arrow[r] & Z \arrow[r] & X[1] \end{tikzcd}$$

where we define $\phi$ by the composition of $X \oplus Y \to Z$ with the map $Y \to X \oplus Y$. Since the left and middle vertical maps are isomorphisms, so is the right one by the five lemma. Hence,

$$X \to X \oplus Y \to Y$$

is a distinguished triangle and hence $X \oplus Y \in A$ by Lemma 2.10. We are left with the construction of kernels, cokernels and proving that the canonical map $\operatorname{coker} \ker f \to \ker \operatorname{coker} f$ is an isomorphism.

Define the cohomology functors $H^n : C \to A$ as $X \mapsto \tau^{\geq 0}\tau^{\leq 0}X[n]$. Given $f : X \to Y$ for $X, Y \in A$, put $f$ into a distinguished triangle

$$X \to Y \to Z.$$

We claim that $\operatorname{coker} f \cong H^0(Z), \ker(f) \cong H^{-1}(Z)$. Note that $H^0(Z) = \tau^{\geq 0}Z$, since $Z \in C^{\leq 0} \cap C^{\geq -1}$ by Lemma 2.10 and axiom (TR3). Similarly, $H^{-1}(Z) = \tau^{\leq -1}Z$. Let $W \in A$. Since $\mathrm{Hom}_C(-, W)$ and $\mathrm{Hom}_C(W, -)$ are cohomological, we get exact sequences

$$\mathrm{Hom}_C(X[1], W) \to \mathrm{Hom}_C(Z, W) \to \mathrm{Hom}_C(Y, W) \to \mathrm{Hom}_C(X, W)$$

and

$$\mathrm{Hom}_C(W, Y[-1]) \to \mathrm{Hom}_C(W, Z[-1]) \to \mathrm{Hom}_C(W, X) \to \mathrm{Hom}_C(W, Y).$$

Using the adjunction property of truncations and the fact that $\mathrm{Hom}_C(X[1], W) = 0 = \mathrm{Hom}_C(W, Y[-1])$, we get exact sequences

$$0 \to \mathrm{Hom}_A(H^0(Z), W) \to \mathrm{Hom}_A(Y, W) \to \mathrm{Hom}_A(X, W)$$

and

$$0 \to \mathrm{Hom}_A(W, H^{-1}(Z)) \to \mathrm{Hom}_A(W, X) \to \mathrm{Hom}_A(W, Y)$$
which proves that $\ker f = H^{-1}(Z), \operatorname{coker} f = H^0(Z)$.

We are now left with proving that $\operatorname{coker} \ker f \cong \ker \operatorname{coker} f$. This is not difficult to prove but involves the use of the octahedral axiom and so we leave it to the reference [HTT, p. 186-187].

□

We state a few corollaries of this proposition that are often useful in actually computing kernels and cokernels.

**Corollary 2.13.** Let

$$0 \to X \to Y \to Z \to 0$$
be a short exact sequence in $A$. Then, we have a distinguished triangle

$$X \to Y \to Z \to X[1].$$

*Proof.* Embed $X \to Y$ into a distinguished triangle

$$X \to Y \to W \to X[1].$$
Then, $W \in C^{\leq -1} \cap C^{\geq 0}$ and by the proof of the proposition, $\tau^{\leq -1} W = \ker(f) = 0$. Hence, $W \in A$ and then $W \cong \tau^{\geq 0} = Z$.

□

**Remark.** Note, however, that it is not obvious what the map $Z \to X[1]$ is.

**Corollary 2.14.** The functor $H^n : C \to A$ is cohomological.

*Proof.* [HTT, Prop 8.1.11].

□

This corollary is often useful in computing the cohomology with respect to a $t$-structure via a Mayer-Vieitoris argument, since it is usually difficult to compute the truncation functors by hand.

# 3. Perverse t-Structure

Let $X$ be a complex algebraic variety. Recall in the introduction that we defined full subcategories of $D^b_c(X)$

1. ${}^p D_c^{\leq 0}(X)$, consisting of objects $F^*$ such that $\dim \operatorname{supp}(\mathcal{H}^j(F^*)) \leq -j$.
2. ${}^p D_c^{\geq 0}(X)$, consisting of objects $F^*$ such that $\dim \operatorname{supp}(\mathcal{H}^j(DF^*)) \leq -j$.

**Remark.** Note that the support of $\mathcal{H}^j(F^*)$ is the set of points $x$ where $H^j(i_x^* F^*) \neq 0$, because stalks commute with taking cohomology. On the other hand, since $D i_x^* D = i_x^!$, the support of $\mathcal{H}^j(DF^*)$ is the set of points $x$ such that $H^{-j}(i_x^! F^*) \neq 0$.

**Remark.** The dimension of the support and cosupport of $F^*$ can also be computed by taking a stratification of $X$ with respect to which $F^*$ is smooth and then looking at to which strata $F^*$ and $DF^*$ restrict in a nonzero manner.

We further defined $P(X)$ as the intersection of these subcategories. We want to show that the two subcategories above define a $t$-structure on $D^b_c(X)$. Hence, $P(X)$ will be a full abelian subcategory of $D^b_c(X)$ that is preserved by the Verdier duality functor.
Let us first prove that axiom (T2) holds.

**Lemma 3.1.** [HTT, Prop 8.1.24] Suppose $G^*$ satisfies the cosupport conditions. Then, for any locally closed $S \in X$, we have

$$\mathcal{H}^j(i_S^! G^*) = 0$$

for $j < -\dim(S)$.

*Proof.* First note that we can assume $S$ is closed because shriek-pullbacks by open immersions are the same as star-pullbacks which commute with taking cohomology. So, assume $S$ is closed. Then, it suffices to prove that

$$\mathcal{H}^j(R\Gamma_S G^*) = 0$$

for $j < -\dim(S)$. Let us take a stratification

$$X = X_n \supseteq \cdots X_k \cdots \supseteq X_{-1} = \emptyset$$

with respect to which $G^*$ is smooth. We prove the statement by proving by induction on $k$ that

$$\mathcal{H}^j(R\Gamma_{S\cap X_k} G^*) = 0$$

for $j < -\dim(S)$. The case of $k = -1$ is the base case and is trivial. So, suppose this holds for $k-1$. Then, since we have a distinguished triangle

$$R\Gamma_{S\cap X_{k-1}} G^* \to R\Gamma_{S\cap X_k} G^* \to R\Gamma_{S\cap (X_k \backslash X_{k-1})}(G^*),$$

it suffices to prove that

$$\mathcal{H}^j(R\Gamma_{S\cap (X_k\backslash X_{k-1})}(G^*)) = 0$$

for $j < -\dim(S)$. Now, $X_k \backslash X_{k-1}$ is a disjoint union of strata $X_\alpha$. Hence,

$$\mathcal{H}^j(R\Gamma_{S\cap (X_k\backslash X_{k-1})}(G^*)) = \bigoplus_{X_\alpha} \mathcal{H}^{-j}(R\Gamma_{S\cap X_\alpha} G^*)$$

so it suffices to prove that

$$\mathcal{H}^j(R\Gamma_{S\cap X_\alpha} G^*) = 0$$

for $j < -\dim S$ if $X_\alpha$ is a stratum in $X$. Now,

$$R\Gamma_{S\cap X_\alpha} G^* = R\Gamma_{S\cap X_\alpha} i_{X_\alpha}^! G^*.$$

The result now follows from the following lemma:

**Lemma 3.2.** [HTT, Lemma 8.1.25] Let $Y$ be a smooth complex variety and $F^* \in D_c^b(Y)$. Assume that all the cohomology sheaves of $G^*$ are locally constant on $Y$ and for an integer $d \in \mathbb{Z}$, we ahve

$$H^j F^* = 0$$

for $j < d$. Then, for any locally closed subvariety $Z$ of $Y$, we have

$$H_Z^j(F^*) = 0$$

for $j < d + 2\mathrm{codim}_Y Z$.

From the cosupport conditions, we get that $F^* = i_{X_\alpha}^! G^*$ satisfies the conditions of the lemma with $d = -\dim(X_\alpha)$, which gives us the desired bound for $S$.

$\square$

**Proposition 3.3.** [HTT, Prop 8.1.26] Let $F^* \in {}^pD_c^{\leq 0}(X)$ and $G^* \in {}^pD_c^{\geq 0}(X)$. Then,

$$\mathcal{H}^j(R\mathcal{H}om(F^*, G^*)) = 0$$

for $j < 0$.

*Proof.* Let

$$S = \bigcup_{j<0} \operatorname{supp}(\mathcal{H}^j R\mathcal{H}om(F^*, G^*)).$$

Note that since $F^*, G^*$ have bounded cohomology, $S$ is closed. Let $i_S$ be the closed embedding. For $j < 0$, we have

$$\operatorname{supp}(\mathcal{H}^j R\mathcal{H}om(F^*, G^*)) \subseteq S$$

and hence

$$\begin{aligned}
\mathcal{H}^j R\mathcal{H}om(F^*, G^*) &= \mathcal{H}^j(R\Gamma_S R\mathcal{H}om(F^*, G^*)) \\
&= \mathcal{H}^j((i_S)_* i_S^! R\mathcal{H}om(F^*, G^*)) \\
&= (i_S)_* \mathcal{H}^j(R\mathcal{H}om(i_S^* F^*, i_S^! G^*))
\end{aligned}$$

Note that since $F^*$ satisfies te support conditions,

$$\dim \mathcal{H}^j(i_S^* F^*) \leq -j.$$

Hence,

$$Z := \bigcup_{k > -\dim(S)} \operatorname{supp}(\mathcal{H}^k(i_S^{-1} F^*))$$

is a closed proper subset of $S$. Let $S^0$ be its complement. Then, $\mathcal{H}^j(i_{S^0}^* F^*) = 0$ for $j > -\dim(S) = \dim(S^0)$. On the other hand, since $G^*$ satisfies the cosupport condition, $\mathcal{H}^j(i_S^! G^*) = 0$ for $j < -\dim(S)$ by the Lemma above. Thus, the same holds for $\mathcal{H}^j(i_{S^0} i_S^! G^*)$. Now, we can compute

$$\mathcal{H}^j(R\mathcal{H}om(i_S^* F^*, i_S^! G^*))|_{S^0}$$

by a spectral sequence whose $E^2_{p,q}$-term is

$$\mathcal{H}om(\mathcal{H}^{-q} i_{S^0}^* F^*, \mathcal{H}^p i_{S^0}^* i_S^! G^*)$$

(because all complexes are bounded). Thus, the $E^2$-page of the spectral sequences is concentrated in non-negative degrees and hence

$$\mathcal{H}^j(R\mathcal{H}om(i_S^* F^*, i_S^! G^*))|_{S^0} = 0$$

for any $j < 0$. But this contradicts the assumption that

$$S = \bigcup_{j<0} \operatorname{supp}(\mathcal{H}^j R\mathcal{H}om(F^*, G^*)).$$

□

Note that axiom (T2) is a trivial consequence of this proposition (take global sections after applying translation). We will now prove axiom (T3) (modulo an application of the octahedron axiom).

**Theorem 3.4.** The support and cosupport conditions define a $t$-structure on $D^b_c(X)$.

*Proof.* The axioms (T1) is obvious and (T2) was just proved. We prove (T3). Pick $F^* \in D^b_c(X)$ and take a stratification $X = \sqcup_\alpha X_\alpha$ such that both $i_{X_\alpha}^* F^*, i_{X_\alpha}^! F^*$ are locally constant cohomologically. Set $X_k$ to be the union of all strata of dimension $\leq k$. We prove the following statement by descending induction on $k$:

There exist $F_0^* \in {}^p D_c^{\leq 0}(X \backslash X_k), F_1^* \in {}^p D_c^{\geq 0}(X \backslash X_k)$ such that

$$F_0^* \to F^*|_{X \backslash X_k} \to F_1^*$$

is a distinguished triangle in $D^b_c(X \backslash X_k)$. The base case is $k = \dim X$ which is trivial. Our eventual goal is to prove the statement for $k = -1$. So, suppose the statement holds for $k$. We show it holds for $k - 1$.

Let $j : X\backslash X_k \to X\backslash X_{k-1}$ be the open immersion and let $i : X_k\backslash X_{k-1} \to X\backslash X_{k-1}$ be the closed immersion. Consider the distinguished triangle

$$F_0^* \to F^*|_{X\backslash X_k} \to F_1^*$$

given by the induction hypothesis. Since $F^*|_{X\backslash X_k} = j^! F^*|_{X\backslash X_{k-1}}$, the adjuntcion gives us a map $j_! F_0^* \to F^*|_{X\backslash X_{k-1}}$. Let us embed this morphism into a distinguished triangle

$$j_! F_0^* \to F^*|_{X\backslash X_{k-1}} \to G^* \tag{1}$$

This is a good first approximation to what we want. The problem is that while $G^*$ has the correct definition on $X\backslash X_k$, its costalks on $X_k\backslash X_{k-1}$ might not satisfy the correct conditions i.e. there might be costalkts in degree $-j$ (because the stalks of $j_! F_0^*$ is zero on $X_k$.) So, as a second approximation, we peel off these costalks by taking the composite of the morphism

$$\tau^{\leq -k} i_! i^! G^* \to i_! i^! G^* \to G^*$$

and embedding it into a distinguished triangle

$$\tau^{\leq -k} i_! i^! G^* \to G^* \to \tilde{F}_1^* \tag{2}$$

This $\tilde{F}_1^*$ will not only have the cosupport conditions but its costalks will not be too small on the boundary $X_k\backslash X_{k-1}$. So, we combine these two triangles by taking the composite of the morphism $F^*|_{X\backslash X_{k-1}} \to G^* \to \tilde{F}_1^*$ and embedding it into a distinguished triangle

$$\tilde{F}_0^* \to F^*|_{X\backslash X_{k-1}} \to \tilde{F}_1^*. \tag{3}$$

This is the triangle we want. By construction, both will have locally constant cohomology sheaves when restricted to any $X_\alpha \subseteq X\backslash X_{k-1}$. We need to show that $\tilde{F}_0^*$ satisfies the support conditions and $\tilde{F}_1^*[1]$ satisfies the cosupport conditions (on $X\backslash X_{k-1}$). Note that the leftmost complex in triangle (2) is supported on $X_k\backslash X_{k-1}$ and is hence killed by $j^*$. Hence, applyiing $j^*$ to triangle (2) gives an isomorphism $j^* G^* \to j^* \tilde{F}_1^*$. Applying $j^*$ to triangle (1) thus shows that $j^* \tilde{F}_1^* \cong F_1^*$. Applying $j^*$ to triangle (3) now also gives us an isomorphism $j^* \tilde{F}_0^* \cong F_0^*$. Thus, we see that $\tilde{F}_0^*$ and $\tilde{F}_1^*$ satisfy the correct conditions outside of $X_k\backslash X_{k-1}$. Since $X_k\backslash X_{k-1}$ consists only of $k$-dimensional strata, we are left with checking the following two conditions:

1. $H^j(i^* \tilde{F}_0^*) = 0$ for $j > -k$.
2. $H^j(i^! \tilde{F}_1^*) = 0$ for $j < -k + 1$.

Property 1 can be checked by feeding triangles (1), (3) and (2) into the octahedral axiom. What this axiom gives us is a distinguished triangle

$$j_! F_0^* \to \tilde{F}_0^* \to \tau^{\leq -k} i_! i^! G^*$$

and applying $i^*$ gives us the desired result (because $i^*$ commutes with truncation.) To prove property (2), we apply $i^!$ to triangle (2) (and note that this preserves distinguished triangles in the case of a closed immersion).

□

Let us give some examples of perverse sheaves.

**Example 3.5.** If you go back and look at the notes for lecture 1, you will see that the $\mathcal{IC}$ sheaves satisfy the support conditions. Additionally, the Verdier dual of an $\mathcal{IC}$ sheaf is the $\mathcal{IC}$ sheaf associated to the dual local system. Hence, the $\mathcal{IC}$-sheaves on $X$, defined with respect to an irreducible local system on some smooth open subvartiety $X^0$ of $X$, lie in $P(X)$.

In fact, the notes show that the $\mathcal{IC}$-sheaves satisfy stronger support conditions which we will later see imply that the $\mathcal{IC}$-sheaves are simple objects in $P(X)$. A complete list of simples in $P(X)$ is obtained by taking $\mathcal{IC}(S,\mathcal{L})$, where $S$ is a closed subvariety of $X$ and $\mathcal{L}$ is some irreducible local system on a smooth open subvariety of $S$.

We end this section by giving a way to construct functors on the category of perverse sheaves.

**Definition 3.6.** Given a functor $F : D^b_c(X) \to D^b_c(Y)$, we define ${}^pF : P(X) \to P(Y)$ as the composite

$$P(X) \longrightarrow D^b_c(X) \xrightarrow{F} D^b_c(Y) \xrightarrow{{}^pH^0} P(Y)$$

We will examine the properties of the functors associated to the 6 functors in the next section and in the next talk. We state some important properties here (proofs in [HTT, 8.1])

**Definition 3.7.** A covariant functor $F : C \to D$ between triangulated categories with $t$-structure is left $t$-exact if $F(C^{\geq 0}) \subseteq D^{\geq 0}$ and is right $t$-exact if $F(C^{\leq 0}) \subseteq D^{\leq 0}$.

1. If $F$ is left (resp. right) $t$-exact, then ${}^pF$ is left (resp. right) exact. In fact, for left $t$-exact $F$,

$$\tau^{\geq 0}F(\tau^{\geq 0}X) \cong \tau^{\geq 0}F(X)$$

   and hence

$${}^pF({}^pH^0(X)) = {}^pH^0(F(X)).$$

2. Let $f : Y \to X$ be a morphism of algebraic varieties such that $\dim f^{-1}(x) \leq d$ for every $x \in X$. Then:
   (a) For $F^* \in {}^pD^{\leq 0}_c(X), f^*F^* \in {}^pD^{\leq d}_c(Y)$.
   (b) For $F^* \in {}^pD^{\geq 0}_c(X)$, $f_*(F^*) \in {}^pD^{\geq -d}_c(Y)$.
   Thus, if $f$ is a locally closed embedding, then $f^*, f_!$ are right $t$-exact and $f^!, f_*$ are left $t$-exact (using the fact the duality swaps the support and cosupport conditions). If $f$ is an open immersion, then $f^*$ is $t$-exact and if $f$ is a closed immersion, $f_*$ is $t$-exact.

3. If $i$ is a closed embedding of $Z$ into $X$, then ${}^pi^* = {}^pi^!$ and $i_*$ and ${}^pi^!$ are mutually inverse functors $P(Z) \to P_Z(X)$ (perverse sheaves supported on $Z$).

# 4. Properties of the Category of Perverse Sheaves

The first important property of the category of perverse sheaves on a finite dimensional complex variety is that is Artinian and Noetherian. Thus, every object has finite length, i.e., it has a filtration with simple quotients. So, one big step in understanding $P(X)$ is to classify the simple objects. We claim that the simples are all of the form $\mathcal{IC}(S,\mathcal{L})$ as in the previous example. To do so, we will give some alternative constructions and characterizations of $\mathcal{IC}$ sheaves and simple perverse sheaves.

4.1. **Minimal Extensions.** Let us first give the construction of $\mathcal{IC}(S,\mathcal{L})$ as an extension of a local system on $S^0$. We do so now using the language of the derived category, rather than using an explicit complex. Let $S$ be a closed subvariety of $X$ and $S^0$ be a Zariski dense open subvariety of the smooth part of $S$. Stratify $S$ by first stratifying $S\backslash S^0$, extending this to a stratification of $S$ by adding $S^0$ as stratum, and then refining further if necessary to get a Whitney stratification of $S$. Write this stratification as

$$S = S_{\dim(S)} \supseteq S_{\dim(S)-1} \supseteq \cdots \supseteq S_{-1} = \emptyset$$

with $S\backslash S_{\dim(S)-1} = S^0$. Let $j_k$ be the inclusion of $X\backslash X_{k-1} \to X\backslash X_{k-2}$.

**Definition 4.1.** Let $\mathcal{L}$ be an irreducible local system on $S^0$. Then, we define $\mathcal{IC}(S,\mathcal{L})$ as

$$(i_S)_*(\tau^{\leq -1}(j_1)_*) \circ (\tau^{\leq -2}(j_2)_*) \circ \cdots \circ (\tau^{\leq -\dim(X)}(j_{\dim(S)})_*)(\mathcal{L}[\dim S]).$$

This definition is due to Deligne and it involves essentially the same idea as was used in the proof that the support conditions and cosupport conditions satisfy (T3). You extend stratum by stratum. The only problem happens at the dimension of the complement so you truncate to peel it off. To show that this construction is independent of the stratification chosen, we will instead construct it as a minimal extension.

The setup for the minimal extension is as follows. Let $X$ be an algebraic variety and $U$ a dense Zariski open subset with complement $Z$. Let $i$ be the inclusion of $Z$ and $j$ be the inclusion of $U$.

**Definition 4.2.** We say that a stratification of $X$ is compatible with $F \in D_c^b(U)$ is $U$ is a union of strata and for each strata contained in $U$, $F$ and $DF$ have locally constant cohomology on the strata.

Such a stratification always exists if $X$ is an equidimensional algebraic variety and we can take the stratification to be Whitney. In this case, we will have $j_*F^*, j_!F^* \in D_c^b(X)$, as they will be smooth with respect to the same stratification. Thus, as long as there exists a stratification of $X$ compatible with $F^*$, we get a canonical morphism

$$j_!F^* \to j^*F^*$$

in $D_c^b(X)$ and taking ${}^pH^0$ gives us a morphism

$${}^pj_!F^* \to {}^pj_*F^*$$

in $P(X)$.

**Definition 4.3.** We denote by $j_{!*}F^*$ the image of the above morphism in $P(X)$ and call it the *minimal extension* of $F^* \in P(U)$.

Note that this is a functorial construction. In the algebraic case, given $F^*, G^* \in P(U)$, we always get a morphism $j_!F^* \to j_*G^*$ and it is easy to check using the universal property of the image that this descends to a morphism $j_{!*}F^* \to j_{!*}G^*$.

We state some properties of the minimal extension. Proofs and/or details can be found in [HTT, 8.2].

**Proposition 4.4.** For $F^* \in P(U)$, we have

$$\mathcal{D}_X(j_{!*}F^*) \cong j_{!*}(\mathcal{D}_U F^*).$$

*Proof.* This uses the following facts: $\mathcal{D}$ preserves perversity and swaps injections and surjections; $\mathcal{D}_X j_! \mathcal{D}_U = j_*, \mathcal{D}_X j_* \mathcal{D}_U = j_!$. □

**Proposition 4.5.** Let $U'$ be a Zariski open susbet of $X$ containing $U$. Let $j_1$ be the inclusion of $U$ into $U'$ and $j_2$ the inclusion of $U'$ into $X$. Let $j$ be the composite. Then,

1. We have canonical isomorphisms
$${}^pj_*F^* \cong {}^p(j_2)_*{}^p(j_1)_*F^*$$
and similarly for $j_!$.
2. $j_{!*}F^* \cong (j_2)_{!*}(j_1)_{!*}F^*$.

**Remark.** The proof of this proposition uses the fact that $j_!$ is right $t$-exact (i.e. sends ${}^pD_c^{\leq 0}(U)$ into ${}^pD_c^{\leq 0}(X)$) and hence the corresponding perverse functor preserves surjection and that $j_!$ is left $t$-exact and hence the corresponding perverse functor preserves injections (in the category of perverse sheaves). These are not difficult to prove but we omit the proof for now since we don't want to spend time talking about the properties of the functors.

We now come to an equivalent way to characterize the minimal extension of $F^*$.

**Proposition 4.6.** Let $F^* \in P(U)$. Then, the minimal extension $G^*$ of $F^*$ is the unique object in $P(X)$ that satisfies

(i) $G^*|_U \cong F^*$.
(ii) $i^*G^* \in {}^pD_c^{\leq -1}(Z)$.
(iii) $i^!G^* \in {}^pD_c^{\geq 1}(Z)$

*Proof.* Let us first prove that the minimal extension of $F^*$ satisfies these properties. Since $j^*$ is $t$-exact, it commutes with taking images and perverse cohomology and hence

$$(j_{!*}F^*)|_U = j^*\, Im[{}^pj_!F^* \to {}^pj_*F^*] = F^*.$$

To prove (ii), note that $i^*$ is right $t$-exact and hence $i^*(j_{!*}F^*) \in pD_c^{\leq 0}(Z)$. So, we just need to check that the 0th-perverse cohomology of $i^*j_{!*}F^*$ is 0. Define $G^* = j_{!*}F^*$ and consider the canonical distinguished triangle

$$j_!j^!G^* \to G^* \to I_*i^*G^*.$$

Since ${}^pH^0$ is cohomological, we get an exact sequence

$${}^pH^0(j_!F^*) \to G^* \to i_*({}^pH^0(i^*G^*)) \to {}^pH^1(j_!F^*)$$

By definition of $G^*$, the left map is surjective. Moreover, $j_!$ is right $t$-exact and hence the rightmost term is 0. Thus, we get the desired result for (ii). (iii) can be proved similarly using the dual distinguished triangle.

We now prove that if $G^*$ satisfies the three properties, then it is canonically isomorphic to $j_{!*}F^*$. First, since we have an isomorphism $F^* \cong j^!G^* = j^*G^* \cong F^*$, then using the adjunction properties, we get a sequence of morphisms

$$j_!F^* \to G^* \to j_*F^*.$$

Taking perverse cohomology gives us a canonical sequence of morphisms

$${}^pj_!F^* \to G^* \to {}^pj_*F^*.$$

We need to prove that the first morphism has no cokernel and the second has no kernel. We only show the first property as the second is analogous. Both maps are isomorphisms on restriction to $U$ so a cokernel of the first map or kernel of the second would be supported on $Z$. Now, since perverse sheaves on $X$ supported on $Z$ are precisely the pushforwards of perverse sheaves on $Z$, we get an exact sequence

$${}^pj_!F^* \to G^* \to i_*H^* \to 0$$

in $P(X)$, with $H^* \in P(Z)$. We want to show that $H^* = 0$. Since ${}^pi^*$ forms an inverse for $i_*$, this follows from property (iii) and the fact that ${}^pi^*$ is right $t$-exact.

□

**Remark.** Hence, we see that minimal extension is characterized by its restriction to $U$ and the fact that it satisfies the stronger support and cosupport conditions

$$\dim \operatorname{supp}(\mathcal{H}^j(G^*)) < -j$$
$$\dim \operatorname{supp}(\mathcal{H}^j(\mathcal{D}_X G^*)) < -j$$

for $j \neq -\dim(X)$, with equality for $j = -\dim(X)$

We can now prove that the minimal extension in the world of perverse sheaves has the same properties as the minimal extension in the world of $D$-modules.

**Proposition 4.7.** Let $U, X, Z$ be as in the previous proposition and let $F^* \in P(U)$. Then,
(a) ${}^pj_*F^*$ has no nontrivial subobject with support contained in $Z$.
(b) ${}^pj_!F^*$ has no nontrivial quotient with support in $Z$.
(c) $j_{!*}F^*$ has no nontrivial subobject or quotient supported in $Z$.

*Proof.* (c) follows immediately from (a) and (b). We prove (a). The proof of (b) is analogous. Let $G^*$ be a subobject of ${}^p j_* F^*$ supported on $Z$. Then, to show $G^*$ is 0, it suffices to show that ${}^p i^! G^* = 0$ (as $i_*$ is an inverse for this). Since ${}^p i^!$ is left $t$-exact, ${}^p i^! G^*$ injects into

$$ {}^p i^! {}^p j_* F^* = {}^p i^! j_* F^* = 0. $$

Here, the first equality uses the fact that both functors are left $t$-exact.

□

As an immediate consequence of this proposition, we see that $j_{!*}$ preserves injections and surjections. But it is important to note that it is neither left nor right exact. However, we do have the following important corollary.

**Corollary 4.8.** If $F^*$ is simple in $P(U)$, then $j_{!*}$ is simple in $P(X)$.

*Proof.* Suppose $G^*$ is a subobject of $j_{!*}F^*$ and let $H^*$ be the quotient. Applying the $t$-exact functor $j^*$, we see by simplicity of $F^*$, that either $j^*G^* = 0$ ot $j^*H^* = 0$. The above proposition then implies that either $G^*$ or $H^*$ is 0.

□

We can now show that the $\mathcal{IC}(S, \mathcal{L})$ defined earlier are minimal extensions from $S^0$ to $S$ of $\mathcal{L}[\dim(S)]$ (pushed forward to $X$). Since pushing forward does nothing, we assume $S = X$.

**Theorem 4.9.** Let $j : X^0 \to X$ be an open immersion with $X^0$ dense and smooth. Let $\mathcal{L}$ be a local system on $X^0$. Then, the $\mathcal{IC}(X, \mathcal{L})$ defined with respect to any suitable stratification of $X$ is isomorphic to $j_{!*}(\mathcal{L}[\dim X])$.

*Proof.* Let

$$ X \supseteq X_{\dim(X)-1} \supseteq \cdots \supseteq X_{-1} = \emptyset $$

be the stratification used to define $\mathcal{IC}(X, \mathcal{L})$. Inductively, it suffices to show that

$$ (j_k)_{!*} F^* \cong \tau^{-k} (j_k)_* F^* $$

for any perverse sheaf $F^*$ on $X \backslash X_{k-1} = U_k$ whose restriction to each stratum $X_\alpha$ in $U_k$ has locally consant cohomology. To do so, we show that it satisfies properties (i), (ii), (iii) characterizing minimal extensions. Since $U_k$ consists of strata of dimension $\geq k$, $\mathcal{H}^r(F^*) = 0$ if $r > -k$ and hence property $(i)$ holds i.e. $G^* = \tau^{-k}(j_k)_* F^*|_{U_k} \cong F^*$. Let $Z = U_{k-1} \backslash U_k$. Then, $Z$ is a disjoint union of dimension $k-1$ strata. Let $i$ be the embedding of $Z$ into $U_{k-1}$. Then, $H^r i^* G^*) = 0$ for $r > -k$ by the right $t$-exactness of $i^*$. Hence, $i^* G^* \in {}^p D_c^{\leq -1}(Z)$ and hence condition $(ii)$ is satisfied. Condition (iii) follows in a similar manner (using the truncation distinguished triangle). See [HTT, Prop 8.2.11].

□

We end the talk by listing all the equivalent characterization of simple perverse sheaves.

**Theorem 4.10.** Let $X$ be a complex algebraic variety. Let $F^* \in P_{\mathbb{S}}(X)$, the category of perverse sheaves smooth with respect to some fixed stratification $\mathbb{S}$ of $X$. Let $j : X^0 \to X$ be the smooth stratum. Assume $F^*$ has full support. Then, the following are equivalent:

1. $F^*$ is simple in $P_{\mathbb{S}}(X)$.
2. $F^* \cong \mathcal{IC}(X, \mathcal{L})$ for some irreducible local system $\mathcal{L}$ on $P_{\mathbb{S}}(X)$.
3. $F^* \cong j_{!*}\mathcal{L}[\dim(X)]$ for some irreducible local system $\mathcal{L}$ on $P_{\mathbb{S}}(X)$.
4. $F^*|_{X^0}$ is simple and $F^*$ has no nontrivial subobjects or quotients supported on $X \backslash X^0$.
5. $F^*|_{X^0}$ is simple and $F^*$ satisfies the stronger support and cosupport conditions (properties (ii), (iii)).

*Proof.* We have shown that $2 \Leftrightarrow 3$ via the proposition above. $5 \Rightarrow 4$ follows from Proposition 4.7. $4 \Rightarrow 1$ follows from the proof of Corollary 4.8.

We show that $1 \Rightarrow 3$. Assume $F^*$ is simple. Since $F^*$ is smooth with respect to $\mathbb{S}$, $F^*|_{X^0} \cong \mathcal{L}[\dim X]$ for some local system $\mathcal{L}$ on $X^0$. Thus, we have

$$j^* F^* \cong \mathcal{L}[\dim X] = G^* \cong j^! F^*$$

and hence by adjunction and taking perverse cohomology we get maps

$${}^p j_! G^* \to F^* \to {}^p j_* G^*$$

with the composite map the same as the map used to define the minimal extension of $G^*$. Neither map is 0, and hence by simplicity of $F^*$, the first map is surjective and the second is injective. Hence, $F^*$ is isomorphic to the image of the composite map, which is precisley $j_{!*} G^*$. It suffices now to show that $G^*$ is simple. If it is not, then there exists some surjection $G^* \to H^*$ with $H^*$ nontrivial. Since minimal extensions preserve surjectivity, $F^*$ surjects onto $j_{!*} H^* \neq 0$ and hence by simplicity, $j_{!*} H^* = F^*$. But then

$$G^* = j^* F^* = j^* j_{!*} H^* = H^*.$$

Finally, we just need to prove that $3 \Rightarrow 5$ but this follows from Proposition 4.6.

$\square$

## References


[HTT] Hotta, Takeuchi, and Taniski. *D-Modules, Perverse Sheaves and Representation Theory.*