LANGLANDS PARAMETERIZATION OVER FUNCTION FIELDS FOLLOWING V. LAFFORGUE

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Abstract. These are a slightly expanded notes of an expository talk on V. Lafforgue's construction of one direction (called automorphic to Galois) of the Langlands correspondence for function fields.

1. Introduction

These are notes for an overview talk given at VIASM on Vincent Lafforgue's work [8] constructing one direction of the Langlands correspondence over function fields. Let me mention right away that there are already several very nice surveys on Lafforgue's article: his own introduction [9] giving a much more detailed summary of the important ideas and a Bourbaki talk of B. Stroh [17], which in particular contains a detailed explanation for the special and more accessible case of the group $G = \mathsf{GL}_1$ which recovers class field theory. (As we would have little to add to Stroh's notes on this instructive case, we will skip this and refer to [17] for it.) Moreover, lecture notes of a course by B.C. Ngô were also very helpful in preparing this talk. These notes do not contain original material.

The plan of this overview is as follows. In Section 2 we will start by stating the main result after a very brief introduction of the setup. At that point the statement will contain some objects that will only be explained afterwards in Section 3. As this theorem is difficult to digest if you have never seen instances of the Langlands correspondence over function fields before, we also include a brief review of the analogy with the original correspondence that was observed first for modular forms. After this classical aside we will come back to the main theorem and discuss two key ingredients that can be understood independently of the main result. The first (see Section 4) is a very general method to construct representations of a group from a certain type of numerical data, called a pseudo-character. The second (see Section 5) is the geometric Satake isomorphism and its connection to the geometry of the Beilinson–Drinfeld–Grassmannian, which allows to formulate a quite general "factorization"-type construction. Here the geometric setup is essential because it allows to move Frobenius and Hecke operators in families.

Finally (in Section 6) we then try to indicate how these two ingredients are combined in the geometry of moduli of shtukas to construct Lafforgue's excursion operators, which produce the decomposition claimed in the main theorem.

Before we start, let me warn the reader that throughout we try to ignore some basic technical difficulties which lie at the heart of much of Lafforgue's work. Namely, we will not comment on the finiteness problems that occur throughout. For a first overview it seemed worthwhile to try to present the argument as if technical problems would not arise. As Lafforgue explains how to resolve all of these points in a masterly way in [9] we hope that a reader who wants to understand how one can resolve these problems will be able to find all that is needed in the given references.

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2. Statement of the result

Throughout we will fix a smooth projective, geometrically connected curve C over a finite field $k = \mathbb{F}_q$ and a reductive group G over $\mathbb{F}_q$. If you prefer, you can safely assume G to be GL_n or SL_n, as almost all of the phenomena already appear for these groups. Also, even if the main result is a statement over finite fields, the proofs rely on interesting geometric constructions that often work over any field, e.g., the complex numbers.

The set of isomorphism classes of G-bundles on C will be denoted by $\mathrm{Bun}_G(\mathbb{F}_q)$, which for $G = \mathsf{GL}_n$ is the same as the set of isomorphism classes of vector bundles of rank n on C. These are the points of a geometric object, called the stack of G-bundles Bun_G, whose characterizing property is that maps from a scheme T to Bun_G are the same as G-bundles on[1] $C \times T$, which one thinks of as a family of bundles on C parametrized by the points of T. We will see a group theoretic presentation of Bun_G that reveals more of its geometry in Section 3.

For the full strength of the main result one also considers a variant of bundles with so called level structure, i.e., one fixes a finite set of points $S \subset C$ and an integer N and considers $\mathrm{Bun}_{G,NS}$ which is the stack parameterizing G-bundles $\mathcal{E}$ together with a trivialization of the restriction $\mathcal{E}|_{NS}$ of $\mathcal{E}$ to the divisor $N \cdot S$.

We will write $\mathcal{C}(\mathrm{Bun}_G(\mathbb{F}_q))$ for the space of functions on the set $\mathrm{Bun}_G(\mathbb{F}_q)$ with coefficients in $\overline{\mathbb{Q}}_\ell$ (for some $\ell \neq \mathrm{char}(\mathbb{F}_q)$).

The main theorem – which requires some more notation that will be explained afterwards – now reads as follows:

Theorem (V. Lafforgue)**.** *There exists a canonical decomposition*

$$\mathcal{C}(\mathrm{Bun}_{G,NS}(\mathbb{F}_q)/\Xi)^{\mathrm{cusp}} = \bigoplus_{\substack{\sigma\colon \pi_1(C-S)\to \check{G} \\ \textit{semisimple}}} \mathcal{C}_\sigma$$

such that for every point $x \in C - S$ and every $V_\lambda \in \mathrm{Irrep}(\check{G})$ the Hecke operator $T_{V_\lambda,x}$ acts on $\mathcal{C}_\sigma$ and its only eigenvalue on this space is $\mathrm{tr}(\sigma(\mathrm{Frob}_x), V_\lambda)$.

We still need to explain some of the notions appearing in the above theorem and we will do this shortly.

[1] As we work over the fixed base field k, we will write $\times$ for the fiber product $\times_k$.

Remark 2.1. For $G = \mathsf{GL}_2$ this result was proven by Drinfeld [2] and this was generalized to GL_n by L. Lafforgue [5]. Both of these results are of course famous. In these works a more precise statement that also determines the dimensions of the spaces $\mathcal{C}_\sigma$ is proven. Their proofs make essential use of the trace formula, a difficult representation theoretic method. For $G = \mathsf{GL}_n$ it is possible to deduce the more precise version from the above result [8, Section 16]. That V. Lafforgue manages to avoid the trace formula completely is very remarkable.

3. Trying to parse the main result

Let us now try to introduce the missing notations from the main theorem, before giving a short indication of some motivation for it. To simplify notations we will from now on suppose $S = \emptyset$.

3.1. **The right hand side.** First let us look at the right hand side of the formula. The group $\check{G}$ is the so called (Langlands–)dual group of G. It is a reductive group over the field in which we chose our coefficients, which is $\overline{\mathbb{Q}}_\ell$. The easiest way to define it is by saying that its root system is the root system that is dual to the one of G. Thus $\check{G}$ is closely related to G, but usually it is slightly different, e.g., simply connected and adjoint groups get exchanged by this duality ($\check{\mathsf{SL}}_n = \mathsf{PGL}_n$). Fortunately, in the proof of the main theorem the group $\check{G}$ only enters through a geometric result explained in Section 5 and it is remarkable that the proof does not use the concrete description of $\check{G}$.
Next, the group $\pi_1(C)$ is the étale fundamental group of C. The finite quotients of this group classify finite étale coverings of C, thereby combining topological information with the Galois group of the base field k. There is one important special case to keep in mind: Connected étale coverings of the spectrum of a field $\operatorname{Spec} k$ (we usually think of this as a point) are the same as separable field extensions (by the Jacobi criterion for smoothness). For example, for a finite field $k = \mathbb{F}_q$ the Galois group is generated by the Frobenius automorphism $\mathrm{Frob}_q(x) = x^q$, i.e., $\pi_1(\operatorname{Spec} k) = \widehat{\mathbb{Z}} = \langle \mathrm{Frob}_q \rangle$, which is the profinite completion of the fundamental group of an oriented circle. This analogy will be helpful in several places.
In our case for any closed point $x \in C$ the residue field $k(x)$ is a finite field and so the inclusion $x \hookrightarrow C$ gives a conjugacy class of morphisms $\pi_1(x) \to \pi_1(C)$ (as in topology this is only defined up to conjugacy unless we fix a common base point for C and x). The image of the Frobenius automorphism of $k(x)$ is denoted by Frob_x. As the trace appearing in the main theorem is invariant under conjugation, this now defines all notations pertaining to the right hand side of the equation in the theorem.
This also gives a first indication why the theorem is interesting. Namely, the Frobenius elements Frob_x are known to generate the fundamental group of C, but they satisfy complicated relations. In contrast, all the terms occurring on the left hand side of the theorem are of geometric origin.

Remark 3.1 (Functions vs sheaves)**.** As in topology, representations of $\pi_1(C)$ are often considered as locally constant sheaves on C, which are an algebraic analogue of bundles with a flat connection. The Frobenius elements give us an easy way to attach a function to any étale sheaf with finite dimensional fibers, simply by taking the trace of Frob_x on the fiber at x for all $x \in X$.
This will be important for the proof of the main theorem, because for all of the geometric arguments we will work with sheaves and only pass to functions at the very end.

3.2. **The left hand side.** Let us now consider the left hand side of the expression occurring in the main theorem. Here the unexplained extra symbols are introduced only to turn the vector space $\mathcal{C}(\mathrm{Bun}_G(\mathbb{F}_q))$ into a finite dimensional vector space. Let us briefly go through the list — however, as we will try to ignore the problems arising from infinite dimensional spaces, it may also be a good idea to simply ignore the extra notation until the problem of finite dimensionality becomes important to you.

Let us start with Ξ, which is only needed if G is not semisimple: The problem already appears from $G = \mathsf{GL}_1$ the case of line bundles or $G = \mathsf{GL}_n$ the case of vector bundles of rank n. Such bundles $\mathcal{E}$ have a discrete invariant, the degree $d = c_1(\mathcal{E}) \in \mathbb{Z}$. Therefore, the stack of all bundles has connected components $\mathrm{Bun}^d_{\mathsf{GL}_n}$ indexed by the degree. Once the degree is fixed, the stack turns out to be connected. Moreover, given a line bundle $\mathcal{L}$ of degree 1, tensoring with $\mathcal{L}$ gives an isomorphism $\mathrm{Bun}^d_n \cong \mathrm{Bun}^{d+n}_n$, so only finitely many of these components are pairwise non-isomorphic. In this case we would take $\Xi = \langle \mathcal{L} \rangle$ for any line bundle $\mathcal{L}$ of positive degree. In general the group $H^1(C, Z(G))$ (where $Z(G)$ denotes the center of G) acts on Bun_G via a similar twisting construction and $\Xi \subset H^1(C, Z(G))$ is a lattice of maximal rank. Passing to Bun_G / Ξ ensures that the resulting object has only finitely many connected components.

Finally, the exponent $(\quad)^{\mathrm{cusp}}$ denotes the subspace of cuspidal functions. Formally these are defined by the vanishing of constant terms in Fourier expansions, in analogy with cusp forms in geometry. One can define these as follows: For every parabolic subgroup $P \subset G$ with unipotent radical U and Levi quotient $L := P/U$ one has a diagram

$$\begin{tikzcd} \mathrm{Bun}_P \arrow[r, "\mathrm{gr}", "\mathcal{E} \mapsto \mathcal{E}/U"'] \arrow[d, "p", "\mathcal{E} \mapsto \mathcal{E} \times^P G"'] & \mathrm{Bun}_L \,. \\ \mathrm{Bun}_G & \end{tikzcd}$$

A function f on Bun_G is called cuspidal if for all of these diagrams the integration of $p^* f$ over the fibers of gr vanishes. One often thinks of these functions as those f that cannot be constructed from functions on Bun_L for smaller reductive groups L.

Now the only notation left from the theorem are the Hecke operators $T_{V_\lambda, x}$. These are operators acting on the space $\mathcal{C}(\mathrm{Bun}_{G,NS}(\mathbb{F}_q)/\Xi)^{\mathrm{cusp}}$, which are defined geometrically as sums over modifications of bundles at a point x. We will study these more closely in Section 5. To give a first idea, let us recall a better known analogy.

3.3. **Where does the statement come from?** If you have seen statements as the main theorem before, you are strongly advised to skip this section. If not, it may be helpful to recall the setting of modular forms, which you may have seen.

Recall that modular forms are differential forms on quotients $\Gamma \backslash \mathbb{H}$, where $\Gamma \subset \mathsf{SL}_2(\mathbb{Z})$ is a subgroup of finite index and $\mathbb{H} = \{z \in \mathbb{C} | \operatorname{Im}(z) > 0\}$ is the upper half plane. Considering $S = \emptyset$ essentially amounts to choosing $\Gamma = \mathsf{SL}_2(\mathbb{Z})$, which we will now do for simplicity.

There are two helpful ways of thinking about $\Gamma \backslash \mathbb{H}$. First, we can rewrite $\mathbb{H} = \mathsf{SL}_2(\mathbb{R})/\mathsf{SO}_2$, as $\mathsf{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$ and SO_2 is the stabilizer of i. In this way

$$\Gamma \backslash \mathbb{H} = \mathsf{SL}_2(\mathbb{Z}) \backslash \mathsf{SL}_2(\mathbb{R}) / \mathsf{SO}_2.$$

For semisimple groups G, the points of Bun_G have a similar description: Choosing a point $\infty \in C(k)$, we write $\mathcal{O}_\infty$ for the complete local ring at ∞ and K_∞ for its quotient field. Choosing a local parameter t at ∞ we have $\mathcal{O}_\infty = k[[t]]$ and

$K_\infty = k((t))$, which is a completion of the field of rational functions $k(C)$ on C. Then there is a uniformization theorem:

$$\mathrm{Bun}_G(\mathbb{F}_q) = G(\mathcal{O}(C - \{\infty\})\backslash G(K_\infty)/G(\mathcal{O}_\infty),$$

which is formally analogous to the double quotient we saw before, e.g., because $G(\mathcal{O}_\infty)$ turns out to be a maximal compact subgroup in $G(K_\infty)$, just as $\mathsf{SO}_2 \subset SL_2(\mathbb{R})$ is a maximal compact subgroup.
Second, the quotient $\Gamma\backslash\mathbb{H}$ is a moduli space of elliptic curves $E/\mathbb{C}$. On this space one has for every prime p a Hecke correspondence - in which we will use $E[p]$ for the p-torsion points in E:

$$\begin{tikzcd}
& \Big\langle (E' \xrightarrow{f} E) \,|\, E[p]/f(E'[p]) \cong \mathbb{Z}/p\mathbb{Z} \Big\rangle \arrow[dl, "q"'] \arrow[dr, "q'"] & \\
\Gamma\backslash\mathbb{H} & & \Gamma\backslash\mathbb{H}
\end{tikzcd}.$$

Here, as in the following we use the brackets $\langle\,\rangle$ to indicate the moduli stack parameterizing the objects described in the brackets.
On modular forms this diagram induces an operator T_p by pulling back a form via q and then summing over the fibers of the map q'.
For G-bundles (which we will denote $\mathcal{E}$) the Hecke operators will be constructed by the same diagram, which now depends on a point $x \in C$ (instead of a prime p) and $E' \to E$ will be replaced by an isomorphism outside of x, with bounded pole order. For example for $G = \mathsf{GL}_n$ the basic Hecke operators are given by homomorphisms of sheaves $\mathcal{E}' \to \mathcal{E}$ such that the quotient is a 1-dimensional vector space supported at a point $x \in C$. In general, the type of the pole order will turn out to be naturally be given by an element in $\mathrm{Irrep}(\check{G})$ (see Section 5).

Remark 3.2 (Fine print)**.** The above is a first analogy in which we made use of the coincidence that for SL_2 a maximal compact subgroup happens to be a 1-parameter subgroup which we might also view as a cocharacter of G (i.e., a homomorphism from a multiplicative group into G). In Section 6 we will see moduli of shtukas which are constructed from Bun_G but incorporate a choice of several cocharacters of G. These spaces are a closer analog of $\Gamma\backslash\mathbb{H}$ from a cohomological point of view.

Remark 3.3. Although we will not use it in these notes, the theorem is often phrased in terms of adèlic quotients, which for reductive groups give a convenient way to obtain a variant of the uniformization theorem. Namely denote by $k(C)$ the function field of C, i.e., the field of meromorphic functions on C. Then we have:

$$\begin{aligned}
\mathrm{Bun}_G(\mathbb{F}_q) &= \varinjlim_{\substack{I \subset C \\ \text{finite}}} G(\mathcal{O}(C - I)\backslash \prod_{x \in I} G(K_x)/G(\mathcal{O}_x) \\
&= G(k(C))\backslash G(\mathbb{A})/\prod_{x \in C} G(\mathcal{O}_x).
\end{aligned}$$

4. What do we need to find? Constructing representations from numbers

The starting point of the proof of the main result is the following. On the left hand side of the main equation we have a vector space that comes equipped with the action of some operators. We want to use the operators to decompose into summands that give rise to representations, or more precisely group homomorphisms $\pi_1(C-S) \to \check{G}$. It seems that from our operators we will only get numbers (e.g., the eigenvalues of the operators). Thus we have to understand how one could construct a group homomorphism from a collection of numbers only.

To clarify this issue, let us consider the general problem: Given an abstract group Γ (finally we would like to taken it to be $\pi_1(C-S)$, but for the moment we take any Γ) and a reductive group $\check{G}$ over a field E of characteristic 0 (it will be $\overline{\mathbb{Q}}_\ell$ for us), how can one describe representations Γ in $\check{G}$, i.e., conjugacy classes of homomorphisms $\sigma\colon \Gamma \to \check{G}$?
If Γ is finitely generated, the space of all representations is a geometric object. Namely, there is an affine scheme $\mathrm{Rep}(\Gamma,\check{G}) = \mathrm{Hom}(\Gamma,\check{G})//(\check{G},\mathrm{conj})$ whose geometric points correspond to isomorphism classes of semisimple representations. (A representation is called semisimple, if for any parabolic subgroup P that contains the image of σ there also exists a Levi subgroup in P containing σ.)
Thus, to give a semisimple representation, it would suffice to construct an algebra homomorphism

$$\mathcal{O}_{\mathrm{Rep}(\Gamma,\check{G})} \to \overline{\mathbb{Q}}_\ell,$$

where $\mathcal{O}_{\mathrm{Rep}(\Gamma,\check{G})}$ is the ring of functions on $\mathrm{Rep}(\Gamma,\check{G})$.
A lot of functions on $\mathrm{Rep}(\Gamma,\check{G})$ are easy to write down. Given a function f on $\check{G}$ that is invariant under conjugation we get for every $\gamma\in\Gamma$ a function $\sigma \mapsto f(\sigma(\gamma))$, e.g., for $\check{G} = \mathsf{GL}_n$ we get the coefficients of the characteristic polynomial of $\sigma(\gamma)$ in this way.
More generally for any finite set I and function f on $\check{G}^I$ that is invariant under simultaneous conjugation by $\check{G}$ we get a function

$$F_{\underline{\gamma},f}(\sigma) := f(\sigma(\gamma_i)_{i\in I}).$$

Remark 4.1 (Sanity check for finite groups)**.** If $\check{G}$ was a finite group the datum coming from invariant functions on $\check{G}$ is what one would get from the character table of $\check{G}$. As the character table does not determine the group this will not suffice to construct homomorphisms to $\check{G}$ in general. However, it is known that adding information on tensor products of representations (which are encoded in functions on products of $\check{G}$) does allow to recover the finite group, which gives a first indication that the above functions might suffice to construct representations.

Remark 4.2 (Trading conjugation for a $\check{G}\times\check{G}$ action)**.** To understand $\check{G}$-invariant functions on $\check{G}^I$ it is useful to note that these are the same as $\check{G}\times\check{G}$-invariant functions on $\check{G}^{I\sqcup\{0\}}$, where $\check{G}\times\check{G}$ acts by simultaneous multiplication from the left and right.
The Peter-Weyl theorem then says that for any reductive group $\check{G}$ the algebraic functions on $\check{G}$ are

$$\overline{\mathbb{Q}}_\ell[\check{G}] = \bigoplus_{W\in\mathrm{Irrep}(\check{G})} W\otimes W^*.$$

Applying this to $\check{G}^{I\sqcup\{0\}}$ we see that all $\check{G}\times\check{G}$-invariant functions f on $\check{G}^{I\sqcup\{0\}}$ come from representations, i.e., $\overline{\mathbb{Q}}_\ell[\check{G}^{I\sqcup\{0\}}] = \bigoplus_{W\in\mathrm{Irrep}(\check{G}^{I\sqcup\{0\}})} W\otimes W^*$. And an element $w\otimes w^*$ is $\check{G}\times\check{G}$ invariant if $w\in W^{\Delta(\check{G})}, w^*\in W^{*,\Delta(\check{G})}$, where $()^{\Delta(\check{G})}$ denotes invariants with respect to the diagonal action. Taking sums we see that for a suitable choice of $W\in\mathrm{Rep}(\check{G}^{I\sqcup\{0\}}), w\in W^{\Delta(\check{G})}, w^*\in W^{*,\Delta(\check{G})}$ any invariant function f is of the form $f(g) = w^*(g.w)$.
Observe that this reformulation also allows us to repackage invariant functions in terms of representation theory only. Namely, the choice of an Element $w\in W^{\Delta(\check{G})}$ (resp. $w^*\in W^{*,\Delta(\check{G})}$) is the same as the choice of a homomorphism of the 1-dimensional trivial $\check{G}$-representation $\mathbb{1}_{\check{G}}$ to the restriction of W to the diagonal copy $\Delta(\check{G})\subset\check{G}\times\check{G}$ i.e., $w\colon \mathbb{1}_{\check{G}} \to W|_{\Delta(\check{G})}$ (resp. $w^*\colon W|_{\Delta(\check{G})} \to \mathbb{1}_{\check{G}}$).

Back to our problem. Lafforgue shows that the collection of all of the functions $F_{\underline{\ },f}(\sigma)$ suffices to reconstruct a semisimple representation σ for any Γ.

Let us first list the obvious properties of $F_{\underline{\gamma},f} := F_{\underline{\gamma},f}(\sigma)$ for a fixed σ:

(1) (Homomorphism) For any $I, \underline{\gamma} \in \Gamma^I$ we have an algebra homomorphism:
$$F_{\underline{\gamma},\_\_}\colon E[\check{G}^I//(\check{G},\mathrm{conj})] \to \overline{\mathbb{Q}}_\ell$$
(2) (FinSet) For any morphism of finite sets $\phi\colon I \to J$, inducing $\phi\colon \Gamma^J \to \Gamma^I$ and $\phi^*\colon k[\check{G}^I] \to k[\check{G}^J]$ we have
$$F_{\phi(\underline{\gamma}),f} = F_{\underline{\gamma},\phi^* f}$$
(3) (Disjoint union) If $I = I_1 \sqcup I_2$ and $f_i \in k[\check{G}^{I_i}]^{\check{G}}$ we have
$$F_{\phi(\underline{\gamma}_1 \sqcup \underline{\gamma_2}), f_1 \otimes f_2} = F_{\phi(\underline{\gamma}_1),f_1} \cdot F_{\phi(\underline{\gamma}_2),f_2}$$
(4) (Multiplication) For $I_{12} = I' \sqcup \{1,2\}, I_1 = I' \sqcup \{1\}$ denote by $m_{12}\colon \check{G}^{I_{12}} \to \check{G}^{I_1}$ the multiplication of the factors $1,2$ and we will denote the corresponding morphism on $\Gamma^{I_{12}} \to \Gamma^{I_1}$ by the same letter. Then for all $f \in E[\check{G}^{I_1}]^{\check{G}}$ we have
$$F_{m_{12}(\underline{\gamma}),f} = F_{\underline{\gamma},m_{12}^*(f)}.$$
(5) (For topological groups) For all I, f the map
$$F_{\_\_,f}\colon \Gamma^I \to \overline{\mathbb{Q}}_\ell$$
is continuous.

Theorem 4.3 ([8, Proposition 11.7])**.** *Let Γ be an abstract group and $\check{G}$ a reductive group over a field E of characteristic 0. Then for any collection $F_{\underline{\gamma},f}$ satisfying the properties (1)-(4) above there exists a semisimple representation $\sigma\colon \Gamma \to \check{G}$ such that*
$$F_{\underline{\gamma},f} = F_{\underline{\gamma},f}(\sigma).$$
Up to isomorphism σ is determined uniquely by this condition.

Idea of the proof: The key input is a theorem of Richardson [14], identifying the points of the quotient $\check{G}^I//(\check{G},\mathrm{conj})$ (or equivalently the closed orbits of the action given by conjugation of $\check{G}$ on $\check{G}^I$) as those conjugacy classes of $\underline{g} = (g_i)_{i\in I}$ such that the Zariski closure $\overline{\langle \underline{g}\rangle} \subseteq \check{G}$ of the subgroup $\langle \underline{g}\rangle$ generated by the g_i is itself a reductive (possibly disconnected) group. These are called semisimple conjugacy classes.
The given data defines for all I and $\underline{\gamma} \in \Gamma^I$ a closed point in $\check{G}^I//(\check{G},\mathrm{conj})$, which by Richardson's result corresponds to a semisimple conjugacy class $\langle \underline{g}\rangle$. From condition (2) we deduce that the groups $\overline{\langle \underline{g}\rangle}$ get larger if one enlarges $\underline{\gamma}$. We can therefore choose $\underline{\gamma}$ and corresponding elements $\underline{g}$ such that $\overline{\langle \underline{g}\rangle}$ is maximal and such that the centralizer of $\underline{g}$ is minimal.
Fix such $\gamma_1, \dots, \gamma_n$ and a corresponding semisimple conjugacy class $g_1, \dots, g_n$ and let $\gamma \in \Gamma$ be arbitrary.
Then $\gamma, \gamma_1, \dots, \gamma_n$ defines a new semisimple conjugacy class $\langle g, g_1', \dots, g_n'\rangle$.
However, since these define the same reductive subgroup of G and the restriction to the coordinates $1, \dots, n$ defines the same closed orbit as $g_1, \dots, g_n$ we see that after conjugation we may assume that $g_i = g_i'$ for all i. Then g is uniquely determined, because by construction any element centralizing the g_i also centralizes g.
Finally (4) shows that the assignment $\gamma \mapsto g$ is a group homomorphism. If Γ is a topological group, this homomorphism turns out to be continuous by (5). Also by construction for all finitely generated subgroups of Γ that contain $\gamma_1, \dots \gamma_n$ the image defines a semisimple group and from this one deduces that the representation is semisimple. □

Remark 4.4. To construct the data $F_{\underline{\gamma},f}$ in the situation of the main theorem Lafforgue replaces conjugation invariant functions on $\check{G}$ by bi–invariant functions as in Remark 4.2. In this setup condition (4) becomes:

(4') Let $I_{123} := I' \sqcup \{1,2,3\}, I_{12} := I' \sqcup \{1,2\}, I_1 := I' \sqcup \{1\}$ and let

$$m_{+-+} \colon \check{G}^{I_{123}} \to \check{G}^{I_1}$$

the morphism defined on the factors $1,2,3$ by $(g_1, g_2, g_2) \mapsto g_1 g_2^{-1} g_3$. We will denote the corresponding morphism on $\Gamma^{I_{123}} \to \Gamma^{I_1}$ by the same letter. Using $E[\check{G}^{I_{123}}]^{\check{G}\times\check{G}} \cong E[\check{G}^{I_{12}}]^{\check{G},\mathrm{conj}}$ we have for all $f \in E[\check{G}^{I_1}]^{\check{G}}$

$$F_{m_{+-+}(\underline{\gamma}),f} = F_{\underline{\gamma},m^*_{+-+}(f)}.$$

For Lafforgue's argument, it is important to observe that this condition, which involves the group law of $\check{G}$ can again be reformulated in terms of representation theory only, as the group law on $\check{G}$ can also be recovered from the category of representations, using the trace and identity maps $W \otimes W^* \to \mathbb{1}$ and $\mathbb{1} \to W \otimes W^*$ (see e.g., [9, Proof of property (iii) after Definition 5.5.] for a one line argument).

Remark 4.5. Lafforgue obtains the decomposition of $\mathcal{C}(\mathrm{Bun}(\mathbb{F}_q)/\Xi)^{\mathrm{cusp}}$ by constructing a family of operators

$$HL_{\underline{\gamma},W,w,w^*} \in \mathrm{End}(\mathcal{C}(\mathrm{Bun}(\mathbb{F}_q)/\Xi)^{\mathrm{cusp}})$$

for any finite set I, $\underline{\gamma} \in \Gamma^I$, $W \in \mathrm{Rep}(\check{G}^I)$, $w \in W^{\Delta(\check{G})}, w^* \in W^{*,\Delta(\check{G})}$ such that these satisfy conditions (1)-(3),(4'). Note that conditions (2) and (3) imply that these operators will generate a commutative algebra. (This is why we added condition (3) that was not used in the proof of Theorem 4.3.)
Theorem 4.3 then implies that every character of the algebra generated by these operators will define a representation σ. Decomposing $\mathcal{C}(\mathrm{Bun}(\mathbb{F}_q)/\Xi)^{\mathrm{cusp}}$ into generalized eigenspaces for the algebra generated by the operators $HL_{\underline{\gamma},W,w,w^*}$ will then construct the $\mathcal{C}_\sigma$ of the main theorem.
As promised before, in this talk we will ignore the exponent $^{\mathrm{cusp}}$. This is an important technical point of the proof. However, if we ignore this, the construction of such a family of operators on $\mathcal{C}_c(\mathrm{Bun}(\mathbb{F}_q)/\Xi)$ (where $_c$ indicates functions with compact support) will give us the decomposition of Lafforgue's Theorem!

5. Where do we find such data? The affine Grassmannian and factorization

In the previous section we saw that in order to construct homomorphisms

$$\sigma \colon \pi_1(C - S) \to \check{G}$$

that appear on the right hand side of the main theorem we will need a collection of operators indexed by finite sets of representations of $\check{G}$, invariant vectors and elements of $\pi_1(C - S)$, satisfying an amazingly long list of compatibilities.
Luckily, the geometric interpretation of Hecke operators will show that these – although they are defined in terms of the group G – are indeed naturally indexed by representations of $\check{G}$. This takes care of the first index of the operators. Moreover, this even allows us to get a geometric interpretation of the categories $\mathrm{Rep}(\check{G})$ and $\mathrm{Rep}(\check{G}^I)$. Lafforgue observed that since the long list of compatibilities can be described in terms of morphisms in these categories, the categorical statement provides us with a machine producing operators that automatically satisfy the whole list of compatibilities.
To explain how this works, let us start from the classical point of view. As mentioned before, the Langlands correspondence is supposed to be constructed through

Hecke operators, which are defined by averaging functions over modifications of bundles at a given point $x \in C$. Let us recall this in more detail.

5.1. **Modifications of bundles.** Given a point $x \in C$ and G-bundles $\mathcal{E}, \mathcal{E}'$ we will write

$$\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}' \text{ or } \mathcal{E} \overset{\phi,x}{\dashrightarrow} \mathcal{E}'$$

for an isomorphism $\phi \colon \mathcal{E}|_{C-x} \xrightarrow{\cong} \mathcal{E}'|_{C-x}$ outside of x. We will also use this notation when we replace x by a finite subset of C.

Example.

(1) For $G = \mathsf{GL}_1$ giving a G-bundle $\mathcal{E}$ is equivalent to giving a line bundle $\mathcal{L}$ and giving $\phi \colon \mathcal{L} \overset{x}{\dashrightarrow} \mathcal{L}'$ is the same as identifying $\mathcal{L}' = \mathcal{L}(d \cdot x)$ for some $d \in \mathbb{Z}$.

(2) For $G = \mathsf{GL}_n$ we can think of GL_n-bundles $\mathcal{E}$ as vector bundles $\mathcal{E}_n$. Again a morphism of sheaves $\phi \colon \mathcal{E}_n \to \mathcal{E}'_n$ such that $\mathcal{E}'_n/\mathcal{E}_n$ is supported at a point x defines a modification $\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}'$. Note that if $\mathcal{E}'_n/\mathcal{E}_n \cong k(x)$ is the skyscraper sheaf supported at x, such a modification is a close analog of the objects considered in Section 3.3.

Finally, we would like to observe that in this case for any $\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}'$ there exists a $d \in \mathbb{Z}$ such that ϕ induces a morphism of the corresponding sheaves $\mathcal{E}_n \to \mathcal{E}'_n(d \cdot x)$, that is an isomorphism over $C - x$.

Remark 5.1. Often one thinks of the map $\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}'$ as the datum needed to construct $\mathcal{E}'$ from $\mathcal{E}$. As ϕ identifies the two bundles on $C - x$ one can view it as a recipe on how to change $\mathcal{E}$ locally around the point x in order to obtain $\mathcal{E}'$.
In other words, the datum $\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}'$ is equivalent to giving a G-bundle on the space $C \cup_{C-x} C$ obtained from C by doubling the point x.

The space of modifications is an object with remarkable symmetry properties. The (big) Hecke stack is defined as

$$\mathrm{Hecke}_G := \left\langle (x, \mathcal{E}, \mathcal{E}', \phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}') \right\rangle$$

It comes equipped with forgetful maps

$$\begin{tikzcd} & \mathrm{Hecke}_G \arrow[dl, "(x,\mathcal{E},\mathcal{E}')\mapsto\mathcal{E}"', "q"] \arrow[dr, "(x,\mathcal{E},\mathcal{E}')\mapsto(\mathcal{E}',x)", "q'"'] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \times C. \end{tikzcd} \tag{5.1}$$

It will also be convenient to consider modifications at several points indexed by a finite set I:

$$\mathrm{Hecke}_{G,I} := \left\langle (\underline{x} = (x_i)_{i \in I}, \mathcal{E}, \mathcal{E}', \phi \colon \mathcal{E} \overset{\{\underline{x}\}}{\dashrightarrow} \mathcal{E}') | \underline{x} \in C^I; \mathcal{E}, \mathcal{E}' \in \mathrm{Bun}_G \right\rangle,$$

which now maps to $\mathrm{Bun}_G \times C^I$.
As we saw in the examples, the fibers of q' are infinite and in analogy with Section 3.3 we will need to restrict to substacks where the type of the singularity of $\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}'$ at x is "bounded". Let us define this notion more precisely. As it should be a local notion at x let us choose a local coordinate t at x and denote $k = k_x$ the residue field at x. The algebraist's version of the (formal) disc around x is $\mathbb{D}_x := \operatorname{Spec} \hat{\mathcal{O}}_x \cong \operatorname{Spec} k[[t]]$. Choosing a trivialization of $\mathcal{E}, \mathcal{E}'$ over the disc $\mathbb{D}_x$ a morphism $\phi \colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}'$ becomes an element of $LG := G(k((t))) = G(\mathbb{D}_x - x)$ — the algebraic geometer's version of the loop group of G. Let us write $L^+G := G(k[[t]])$, which acts on LG by multiplication from the left and the right. Different choices

of trivializations on $\mathcal{E}_{\mathbb{D}_x}, \mathcal{E}_{\mathbb{D}_x}$ will change the element in LG by multiplication of an element in $L^+G \times L^+G$ and therefore the type of the singularity of ϕ will be defined as the corresponding element of $L^+G\backslash LG/L^+G$. To summarize:

$$\phi\colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}' \text{ defines an element } [\phi] \in L^+G\backslash LG/L^+G,$$

the element $[\phi]$ is called the *type of the modification* ϕ.

Example. For $G = \mathsf{GL}_n$ the elementary divisor theorem says that the double cosets $L^+\mathsf{GL}_n\backslash L\mathsf{GL}_n/L^+\mathsf{GL}_n$are indexed by diagonal matrices

$$\begin{pmatrix} t^{d_1} & & \\ & \vdots & \\ & & t^{d_n} \end{pmatrix} \text{ with } d_1 \geq \cdots \geq d_n.$$

This holds generally, i.e., choose a split maximal torus $T \subset G$ and a Borel subgroup $T \subset B$ then the double cosets $G(k[[t]])\backslash G(k((t)))/G(k[[t]])$ are given by evaluating dominant cocharacters $X_*(T)_+ \subset X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ at t.

Remark 5.2. Note that irreducible representations of G are indexed by dominant characters $X^*(T)_+ \subset X^*(T) = \mathrm{Hom}(T, \mathbb{G}_m)$. This already indicates that in the statement of the theorem the reductive group $\check{G}$ whose root system is dual to the one of G should show up.

Remark 5.3. We can use the quotient LG/L^+G to describe the fibers of

$$p\colon \mathrm{Hecke}_G \to \mathrm{Bun}_G .$$

Given $\mathcal{E}'$, a trivialization of $\mathcal{E}'$ over $\mathbb{D}_x$ and $g \in G(K_x)$ we can reconstruct $\mathcal{E}$ by choosing g as new transition function on the disc. Forgetting the trivialization we see that the points of the fibers of p can be parametrized by the quotient LG/L^+G.

Fact 5.4. The set LG/L^+G is the set of k-points of the affine Grassmannian Gr_G, the moduli space parameterizing modifications of the trivial G-bundle $\mathcal{E}_0$. This is a limit (union) of proper schemes.
The space Gr_G is infinite dimensional, but for each cocharacter $\lambda \in X_*(T)$ the closure of the corresponding double coset $\overline{\mathrm{Gr}^\lambda} \subset \mathrm{Gr}_G$ is a projective scheme. (See e.g., [21, Thm 1.1.3].)

Classically, Hecke operators $T^{cl}_{\lambda,x}\colon \mathcal{C}(\mathrm{Bun}_G(\mathbb{F}_q)) \to \mathcal{C}(\mathrm{Bun}_G(\mathbb{F}_q))$ are defined on the level of functions by summing over all modifications of a given type:

$$T^{cl}_{\lambda,x}(f)(\mathcal{E}') = \sum_{\substack{\phi\colon \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}' \\ [\phi]=\lambda}} f(\mathcal{E}). \tag{5.2}$$

For a representation $V \in \mathrm{Rep}(\check{G})$ one analogously defines an operator

$$T^{cl}_{V_\lambda,x} := \sum_{\substack{\mu \text{ dominant} \\ \text{occuring in } V}} h(V,\mu) T^{cl}_{\mu,x},$$

where the constants $h(V,\mu)$ are given by the Satake isomorphism that we will encounter geometrically in the next section.
The reason why one should consider these strange linear combinations instead of the $T^{cl}_{\mu,x}$ which are much easier to define, is explained by the Satake isomorphism. Roughly this says that for every x the collection of the $(T_{\lambda,x})_{\lambda\in X_*(T)}$ defines an algebra that is isomorphic to the representation ring of $\check{G}$. The operators $T^{cl}_{V_\lambda,x}$ will correspond to irreducible representations, whereas the $T^{cl}_{\lambda,x}$ correspond to much more complicated sums. In practice this means that if one computes the eigenvalues of the $T^{cl}_{\lambda,x}$ for different λ it is hard to see a pattern, but the collection of eigenvalues

of $T^{cl}_{V_\lambda,x}$ will be the collection of traces of an element of $\check{G}$ on the representations V_λ.

5.2. **Statement of the geometric Satake isomorphism.** It turns out to be useful to replace functions by sheaves in the above formula. We will denote by

$$\mathrm{Sat}_G := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G)$$

the category of L^+G equivariant perverse sheaves (with $\overline{\mathbb{Q}}_\ell$ coefficients) on Gr_G, which we will think of a sheaves on the double quotient $L^+G\backslash LG/L^+G$. Here perverse sheaves can be thought of as cleverly chosen combinations of characteristic functions of the double cosets, that make their cohomology look as if the support of the sheaf was a smooth projective variety.

Remark 5.5. Another technical point that we want to ignore in these notes is that the double quotient $L^+G\backslash LG/L^+G$ is not known to be an object with nice properties, because the group $L^+G = G(k[[t]])$ acts with infinite dimensional stabilizers on Gr_G. There is a by now standard way (explained in [3, Appendix]) to circumvent this by passing to the finite dimensional subspaces $\overline{\mathrm{Gr}^\lambda}$ throughout the construction, on which $G(k[[t]])$ acts through a finite dimensional quotient.
We will therefore sometimes write $f\colon M \to L^+G\backslash LG/L^+G$ for a map that we define on geometric points only and use the functor $f^*\colon \mathrm{Sat}_G \to D^b(M)$, which would formally have to be constructed through the approximation procedure indicated above.

Iteration of modifications defines a convolution $\star$ on these sheaves as follows: Consider the space

$$\widetilde{Gr}^2_{G,x} := \left\langle (\mathcal{E}_2 \overset{x}{\dashrightarrow} \mathcal{E}_1 \overset{x}{\dashrightarrow} \mathcal{E}_0) | \mathcal{E}_i \in \mathrm{Bun}_G, \mathcal{E}_0 \text{ trivial} \right\rangle.$$

This has a natural map

$$m\colon \left\langle (\mathcal{E}_2 \overset{x}{\dashrightarrow} \mathcal{E}_1 \overset{x}{\dashrightarrow} \mathcal{E}_0) \right\rangle \to \mathrm{Gr}_G$$

given by composition $m(\mathcal{E}_2 \overset{x}{\dashrightarrow} \mathcal{E}_1 \overset{x}{\dashrightarrow} \mathcal{E}_0) := (\mathcal{E}_2 \overset{x}{\dashrightarrow} \mathcal{E}_0) \in \mathrm{Gr}_G$.
Also, for every $(\phi_2\colon \mathcal{E}_2 \overset{x}{\dashrightarrow} \mathcal{E}_1, \phi_1\colon \mathcal{E}_1 \overset{x}{\dashrightarrow} \mathcal{E}_0)$ the classes $[\phi_1], [\phi_2]$ define maps

$$p_i\colon \widetilde{Gr}^2_{G,x} \to L^+G\backslash LG/L^+G.$$

One defines

$$F_1 \star F_2 := m_!(F_1 \tilde{\boxtimes} F_2) := m_!(p_1^* F_1 \otimes p_2^* F_2).$$

Using the description of bundles in terms of adéles as indicated in Remark 3.3 it is not hard to see that on the level of functions this gives the standard convolution operation that defines the Hecke algebra of L^+G-biinvariant functions on LG (see e.g., [3, Section 1.1]). However, it seems magical that the operation $m_!$ turns out to produce perverse sheaves. We will comment on this below after stating the second main ingredient of Lafforgue's proof:

Theorem 5.6 (Geometric Satake Isomorphism, [10])**.** *The convolution $\star$ equips Sat_G with the structure of a $\otimes$-category. Moreover, the functor*

$$H^*(\mathrm{Gr}_G, \underline{})\colon \mathrm{Sat}_G \to \overline{\mathbb{Q}}_\ell\text{-vector spaces}$$

defines an equivalence of $\otimes$-categories:

$$(\mathrm{Sat}_G, \star) \cong (\mathrm{Rep}(\check{G}), \otimes).$$

Remark 5.7. For us the dual group $\check{G}$ will only appear through this theorem. As any algebraic group is determined by its $\otimes$-category of representations one can also view the theorem as a construction of $\check{G}$. In the rest of the proof we will only use that the $\check{G}$ satisfies the above theorem and not that this is the dual group as described before. In particular, we will use the inverse functor $\mathrm{Sat}^{-1}\colon \mathrm{Rep}(\check{G}) \to \mathrm{Sat}_G$, which we will denote by $W \mapsto \mathrm{IC}_W$.

Notation. For any $W \in \mathrm{Rep}(\check{G})$ we will denote by $\mathrm{Gr}_{G,W} \subset \mathrm{Gr}_G$ the finite dimensional support of the sheaf IC_W.

Remark 5.8. To ensure that cohomology defines a symmetric $\otimes$-functor one has to make a careful normalization for the cohomological degrees (see e.g., [10, Proposition 6.3]). This is possible because the cohomology groups turn out to appear (after a suitable normalization) in even dimensions only.

The history of this theorem is long. It was proven in [10] by Mirkovic and Vilonen using important previous contributions from Lusztig, Drinfeld and Ginzburg. A slightly different argument can be found in [15]. There are several versions and generalizations of the theorem. For example, in the orginial article [10] the ground field k is assumed to be of characteristic 0, but the authors are careful to allow more general coefficients than $\overline{\mathbb{Q}}_\ell$. A nice recent overview on the result can be found in [21].

5.3. **Ideas of the proof that reappear in Lafforgue's construction.** As the Hecke correspondence (5.3) indicates, the affine Grassmannian comes in a family, in which we allow the point x to vary. In particular, it would be easy to see that Hecke operators at distinct points commute with each other. As we are working with sheaves one can try to pass from information over distinct points to information at a single point by letting the points collide in a family. More precisely, considering modifications at several points gives an object called the Beilinson–Drinfeld–Grassmannian which has very remarkable properties: For any finite set I let us denote by

$$\mathrm{GR}_I := \left\langle (\underline{x}, \mathcal{E}, \phi\colon \mathcal{E} \overset{\{\underline{x}\}}{\dashrightarrow} G \times C) | \underline{x} \in C^I, \mathcal{E} \in \mathrm{Bun}_G \right\rangle,$$

the space of modifications of the trivial bundle at a set of points on C.
This space has a forgetful map p_I to C^I, which again turns out to be an inductive limit of projective morphisms and moreover this morphism is formally smooth. The fibers of p_I are products of Gr_G, but note that if the points $\underline{x} = (x_i)$ are distinct then the fiber is a product of $|I|$ copies of Gr_G while over the points on the diagonal of $C \subset C^I$ a single copy appears. For proper morphisms this is of course only possible for fibers that are either infinite or 0-dimensional.
If we try to compute the convolution operation $\star$ in families we need to consider a variant of this space for an ordered finite set $I = \{1, \dots n\}$

$$\widetilde{\mathrm{GR}}_I := \left\langle ((x_i)_{i \in I}, \mathcal{E}_i, \phi_i\colon \mathcal{E}_i \overset{x_i}{\dashrightarrow} \mathcal{E}_{i+1}) | \mathcal{E}_{n+1} = G \times C \right\rangle$$

that comes with a composition morphism

$$m\colon \widetilde{\mathrm{GR}}_I \to \mathrm{GR}_I,$$

and projections p_i to $L^+G \backslash \mathrm{Gr}_G$ taking the class $[\phi_i]$. This defines a natural functor

$$\pi^*\colon (\mathrm{Sat}_G)^I \to \mathrm{Perv}(\widetilde{\mathrm{GR}}_I).$$

We can use these maps to define a new convolution functor:

$$\mathrm{Rep}(\check{G}^I) \to \mathrm{Perv}(\mathrm{GR}_I),$$

as $m_!(p_1^*(\mathrm{IC}_{W_1})\otimes\ldots\otimes p_n^*(\mathrm{IC}_{W_n}))$, which over the diagonal computes the convolution $\mathrm{IC}_{W_1}\star\cdots\star\mathrm{IC}_{W_n}$. This description allows to explain the miraculous fact that convolution indeed produces perverse sheaves. Over the subset $C^I-\Delta^{\mathrm{big}}\subset C^I$ that consists of pairwise distinct points the map m is an isomorphism and therefore preserves all properties of sheaves, e.g., perversity. Expressing the image over the diagonal in terms of nearby cycles one can then conclude that the property is also preserved on the diagonal (see [3]).

For us the main point of the above construction is that it gives us geometric objects that come equipped with compatibilities that are close to the properties that we encountered in Section 4, i.e., we get a family of spaces mapping to finite products C^I of our curve. These spaces come equipped with (perverse) sheaves that are indexed by families of representations of $\check{G}$ such that morphisms of representations give rise to morphisms of the corresponding sheaves. Finally, these objects are compatible with morphisms of finite sets $I\to J$. As this is the starting point for Lafforgue's construction, let us state a slight generalization:

Remark 5.9. In the above construction we could replace GR_I and $\widetilde{GR}_I$ by

$$\mathrm{Hecke}_I := \left\langle (\underline{x},\phi\colon \mathcal{E}\overset{\{\underline{x}\}}{\dashrightarrow}\mathcal{E}')|\underline{x}\in C^I,\mathcal{E},\mathcal{E}'\in\mathrm{Bun}_G\right\rangle \text{ and}$$

$$\widetilde{\mathrm{Hecke}}_I := \left\langle ((x_i)_{i\in I},\mathcal{E}_i,\phi_i\colon \mathcal{E}_i\overset{x_i}{\dashrightarrow}\mathcal{E}_{i+1})|\mathcal{E}_i\in\mathrm{Bun}_G\right\rangle,$$

to obtain a functor

$$\mathrm{Sat}_I^{-1}\colon \mathrm{Rep}(G^I)\to\mathrm{Perv}(\mathrm{Hecke}_I), W=(W_1\boxtimes\cdots\boxtimes W_n)\mapsto\mathrm{IC}_W=\mathrm{IC}_{W_1\boxtimes\cdots\boxtimes W_n}$$

that has symmetries similar to the properties (2),(3) from Section 4.

6. Putting everything together: G-Shtukas

6.1. **Definition of Shtukas as Frobenius fixed points of Hecke correspondences.** In the previous section we saw how one can construct geometric objects described in terms of G-bundles that are related to the category $\mathrm{Rep}(\check{G}^I)$. However, recall from Remark 4.5 that in the end we are trying to construct operators $HL_{\underline{\gamma},W,w,w^*}$ on the vector space $\overline{\mathbb{Q}}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]=\oplus_{\mathcal{E}\in\mathrm{Bun}_G(\mathbb{F}_q)}\overline{\mathbb{Q}}_\ell$.

Our next task is therefore to explain how to get back from geometric objects to linear operators on spaces of functions. This should also involve the elements $\underline{\gamma}\in\pi_1(C)^I$ that have been absent from our discussion so far. The key to this is the Frobenius endomorphism.

Over finite fields one can always pass from a variety to its set of $\mathbb{F}_q$-points by noting that a $\mathbb{F}_q$-point is a geometric point that is fixed by the Frobenius automorphism. For bundles one often abbreviates $\mathrm{Frob}^*(\mathcal{E}):={}^\tau\mathcal{E}$. Then we find

$$\mathrm{Bun}_G(\mathbb{F}_q)=\left\langle(\mathcal{E},\mathcal{E}\xrightarrow{\cong}{}^\tau\mathcal{E})|\mathcal{E}\in\mathrm{Bun}_G\right\rangle,$$

i.e., the discrete stack $\mathrm{Bun}_G(\mathbb{F}_q)$ solves the moduli problem formulated on the right hand side of the equation.

Recall that in the main theorem Hecke operators (acting on the left hand side) are related to images of Frobenius elements under the representation σ that appear on the right hand side.

To construct the Langlands correspondence for $G=\mathsf{GL}_2$ Drinfeld introduced shtukas, which combine the above construction of taking fixed points under Frob^* with Hecke modifications. He showed that this combination produces spaces such that Frobenius elements act on their cohomology through Hecke operators. In the generality needed for Lafforgue's construction these stacks were first considered by

Varshavsky [19]. The general definition is the following:

$$\mathrm{Cht}_{G,I} := \left\langle (\underline{x}, \mathcal{E}, \phi\colon \mathcal{E} \overset{\underline{x}}{\dashrightarrow} {}^{\tau}\mathcal{E}) | \underline{x} \in C^I, \mathcal{E} \in \mathrm{Bun}_G \right\rangle.$$

Alternatively, we can rewrite this definition as taking Frobenius-fixed points of the Hecke correspondence, that is $\mathrm{Cht}_{G,I}$ makes the following diagram cartesian:

$$\begin{tikzcd} \mathrm{Cht}_{G,I} \arrow[r] \arrow[d, "(\underline{x},\mathcal{E},\phi)\mapsto\mathcal{E}"] & \mathrm{Hecke}_I \arrow[d, "(\phi\colon \mathcal{E}\overset{\underline{x}}{\dashrightarrow}\mathcal{E}')\mapsto(\mathcal{E},\mathcal{E}')"] \\ \mathrm{Bun}_G \arrow[r, "(\mathrm{id},\mathrm{Frob})"] & \mathrm{Bun}_G \times \mathrm{Bun}_G . \end{tikzcd}$$

As in Section 5.2, in order to get objects of more reasonable size, we can bound the singularities of the modification ϕ by $W \in \mathrm{Rep}(\check{G}^I)$ to obtain $\mathrm{Cht}_{G,I,W} \subset \mathrm{Cht}_{G,I}$. (We note that Lafforgue prefers to use the underlying reduced stacks.)
The forgetful maps

$$p\colon \mathrm{Cht}_{G,I} \to C^I \text{ and } p\colon \mathrm{Cht}_{G,I,W} \to C^I$$

are called the *legs* of the shtuka.
Lafforgue noticed that as $\mathrm{Cht}_{G,I}$ comes equipped with a forgetful map

$$\mathit{forget}\colon \mathrm{Cht}_{G,I} \to \mathrm{Hecke}_I$$

we can pull back the perverse sheaves IC_W from Remark 5.9 to obtain a functor

$$\mathit{forget}^* \circ \mathrm{Sat}_I^{-1}\colon \mathrm{Rep}(\check{G}^I) \to \mathrm{Perv}(\mathrm{Cht}_{G,I}),$$

that we will still denote by $W \mapsto \mathrm{IC}_{\underline{W}}$.

Remark 6.1. The construction of the functor Sat_I^{-1} implicitly uses the quotient $L^+G\backslash \mathrm{Gr}_G$. Ignoring the technical problems with this quotient the forgetful map

$$\mathrm{Cht}_{G,I} \to L^+G\backslash \mathrm{GR}_I$$

mapping a shtuka $(\underline{x}, \mathcal{E}, \phi)$ to the type of the modification $[\phi]$ around the x_i is a nice map, i.e., it turns out to be (formally) smooth. This allows to show that pull-back does preserve perverse sheaves (up to a cohomological shift). To show smoothness of the map one uses the description

$$\mathrm{Cht}_{G,I} = \mathrm{Bun}_G \times_{\mathrm{Bun}_G \times \mathrm{Bun}_G} \mathrm{Hecke}_I$$

as a fiber product taken via the morphism

$$(\mathrm{Id}, \mathrm{Frob})\colon \mathrm{Bun}_G \to \mathrm{Bun}_G \times \mathrm{Bun}_G$$

and then argues that the map $(\mathrm{Id}, \mathrm{Frob})$ is transverse to the map

$$\mathrm{Hecke}_I \to \mathrm{Bun}_G \times \mathrm{Bun}_G,$$

because the derivative of Frob vanishes.
This argument is problematic for the infinite dimensional space Hecke_I, but it can again be saved by approximating the double quotient by objects of finite type. However, as these are singular spaces the proof requires more care.

The functor $W \mapsto \mathrm{IC}_W$ allows us to define perverse sheaves

$$\mathcal{H}_{I,W} := {}^p\mathbb{R}^0 p_! \, \mathrm{IC}_W$$

on C^I, wehere ${}^p\mathbb{R}^0$ is the functor taking the middle (perverse) cohomology group, which in the language of perverse sheaves happens to be normalized to appear in cohomological degree 0. In other words, we take the cohomology of the sheaves $IC_{\underline{W}}$ along the fibers of the map $p\colon \mathrm{Cht}_{G,I} \to C^I$ and then forget everything but the middle (perverse) cohomology group.

Remark 6.2.

(1) For any finite set I if $W = \mathbb{1} \in \mathrm{Rep}(\check{G}^I)$ is the trivial representation, the space $\mathrm{Cht}_{I,W}$ parametrizes trivial modifications, so that the sheaf $\mathcal{H}_{I,\mathbb{1}} = \overline{\mathbb{Q}}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]$ is simply the constant sheaf with fiber $\overline{\mathbb{Q}}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]$ on C^I. The fibers of this sheaf are the vector space on which we want to construct the operators $HL_{\underline{\gamma},W,w,w^*}$.
(2) The functor $\mathbb{R}p_!$ commutes with arbitrary base change. From this one can deduce that the construction of the sheaves $\mathcal{H}_{I,W}$ is still compatible with morphisms of finite sets $\phi\colon I \to J$.
(3) *Warning:* As is apparent from (1) where $\mathrm{Bun}_G(\mathbb{F}_q)$ is infinite, one needs to truncate the stacks at this point in order to assure finiteness properties and obtain constructible sheaves. Again, in order to simplify the presentation of the basic idea we will ignore this point.

We still have one more obstruction to get the data we need. Recall that we are still looking for representations of $\pi_1(C)^I$ (or at least $\mathrm{Gal}(\overline{k(C)}/k(C))^I$) for all finite sets I. We would like to get these from the sheaves $\mathcal{H}_{I,W}$. Locally constant sheaves on C^I only give us representations of $\pi_1(C^I)$. As $C^I = C \times_{\mathbb{F}_q} C \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} C$ and $\pi_1(\mathbb{F}_q) = \langle \mathrm{Frob}_q \rangle \cong \hat{\mathbb{Z}}$ we have a cartesian diagram

$$\begin{tikzcd} \pi_1(C^I) \arrow[r] \arrow[d] & \pi_1(C)^I \arrow[d] \\ \pi_1(\mathbb{F}_q) \arrow[r, "\Delta"] & \pi_1(\mathbb{F}_q)^I \end{tikzcd}$$

where Δ is the diagonal morphism. Drinfeld's lemma says that the difference in the upper row is governed by the lower row, i.e., to get a representation of $\pi_1(C)^I$ one has to specify commuting morphisms F_i such that their product gives the usual Frobenius on C^I. More precisely, denoting by Frob_i the Frobenius on the i-th factor of C^I we have:

Lemma 6.3 (Drinfeld)**.** *Let F be a locally constant (lisse) sheaf F with profinite coefficients, defined on an open subset $U \subset C^n$ that is equipped with commuting isomorphisms $F_i\colon \mathrm{Frob}_i^* F \to F$ such that $\prod F_i$ is the natural Frobenius isomorphism. Then there exists an open subset $V \subset C$ such that $F = \boxtimes F_i$ where F_i are locally constant sheaves on V, given representations of $\pi_1(V)$.*

The proof of this Lemma for $I = \{1, 2\}$ is contained in [2] and the general case can be found in [6, Théorème IV.2] and [7, Theorem 8.1.4 and Lemma 9.2.1].

Aside. It may be instructive to consider the following topological analog of the Lemma. The fundamental group of the circle $\pi_1(S^1) = \mathbb{Z}$ is close to $\pi_1(\mathbb{F}_q) = \hat{\mathbb{Z}}$. If one takes a topological space X that fibers over the circle $X \to S^1$ one can similarly compare the fundamental groups of the I-th fibered product $X \times_{S^1} \cdots \times_{S^1} X$ with the fundamental group of the product $X \times \cdots \times X$. In our setup we always take fiber products over $\mathbb{F}_q$, so there is no evident analog of the direct product. Drinfeld's lemma tells us that nevertheless we can find a categorical description of $\pi_1(C)^I$ in terms of sheaves on C^I.

The sheaves $\mathcal{H}_{I,W}$ come equipped with partial Frobenius automorphisms, coming from geometry: Recall that the sheaves $\mathrm{IC}_{\underline{W}}$ on GR_I could be defined in terms of $\widetilde{\mathrm{GR}}_I$. We can do the same for $\mathrm{Cht}_{I,W}$ and compute the sheaves $\mathcal{H}_{I,W}$ as cohomology of the stack $\widetilde{\mathrm{Cht}}_{I,W}$ that parametrizes chains

$$\mathcal{E} = \mathcal{E}_0 \overset{x_1}{\dashrightarrow} \mathcal{E}_1 \overset{x_2}{\dashrightarrow} \ldots \overset{x_n}{\dashrightarrow} \mathcal{E}_n = {}^\tau\mathcal{E}_0.$$

We can consider any such chain of modifications as an infinite chain by extending it to the right by applying Frobenius:

$$\mathcal{E} = \mathcal{E}_0 \overset{x_1}{\dashrightarrow} \mathcal{E}_1 \overset{x_2}{\dashrightarrow} \dots \overset{x_n}{\dashrightarrow} \mathcal{E}_n = {}^\tau\mathcal{E}_0 \overset{{}^\tau x_1}{\dashrightarrow} {}^\tau\mathcal{E}_1 \overset{{}^\tau x_2}{\dashrightarrow} \dots .$$

In this description we see that Frobenius acts on $\widetilde{\mathrm{Cht}}_{I,W}$ by shifting the sequence n steps to the right. The partial Frobenius Frob_1 is defined by shifting the sequence one step to the right:

$$\mathcal{E}_1 \overset{x_2}{\dashrightarrow} \dots \overset{x_n}{\dashrightarrow} \mathcal{E}_n = {}^\tau\mathcal{E}_0 \overset{{}^\tau x_1}{\dashrightarrow} {}^\tau\mathcal{E}_1.$$

At first sight it is not so clear, why this should allow us to construct n commuting operators, but trom the proof of the commutativity of the $\star$ operation on the Hecke algebra Sat_G we already know that to compute $\mathcal{H}_{I,W}$ we can use $\widetilde{\mathrm{Cht}}_{I,W}$ and we are allowed to permute the order of the x_i, because this is true over the open subset of points C^I that have pairwise distinct coordinates. Thus we can similarly construct $\mathrm{Frob}_2, \dots, \mathrm{Frob}_n$ by permutation of the x_i.

Conclusion. We have a functor $\mathrm{Rep}(\check{G}^I) \to \mathrm{Perv}(C)^I$ given by $W \to \mathcal{H}_{I,W}$ that is compatible with maps of finite sets $\phi\colon I \to J$.

6.2. **Definition of the excursion operators** $HL_{\underline{\gamma},W,w,w^*}$**.** Using the sheaves $\mathcal{H}_{I,W}$ we can now define Lafforgue's excursion operators. Let us denote the stalk of $\mathcal{H}_{I,W}$ over the diagonal of the (geometric) generic point of $\overline{\eta} \to C \overset{\Delta}{\hookrightarrow} C^I$ by

$$H_{I,W} := \mathcal{H}_{I,W}\,|_{\Delta(\overline{\eta})}.$$

Lafforgue defines for any $\underline{\gamma} \in \pi_1(C)^I$, $W \in \mathrm{Rep}(G^I), w \in W^{\Delta(G)}, w^* \in W^{*,\Delta(G)}$ the excursion operator: $HL_{\underline{\gamma},W,w,w^*}$ as the composition:

$$H_{\{0\},\mathbb{1}} \xrightarrow{w} H_{\{0\},W|_{\Delta(G)}} \xrightarrow{bc} H_{I,W} \xrightarrow{\underline{\gamma}} H_{I,W} \xrightarrow{bc^{-1}} H_{\{0\},W|_{\Delta(G)}} \xrightarrow{w^*} H_{\{0\},\mathbb{1}}.$$

Let us go through this construction step by step and explain the notation. From Remark 6.2 we know that the sheaf $\mathcal{H}_{\{0\},\mathbb{1}}$ associated to the trivial representation $\mathbb{1}$ is the constant sheaf with fiber $\overline{\mathbb{Q}}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]$, i.e., $H_{\{0\},\mathbb{1}} = \overline{\mathbb{Q}}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]$.
Also, we decided to view the invariant vector w as a morphism $\mathbb{1} \to W|_{\Delta(\check{G})}$. As our construction of $\mathcal{H}_{I,W}$ was functorial, this defines the map w.
Next, the symbols bc denote the base change maps comparing $(\mathcal{H}_{I,W})|_{\Delta(\overline{\eta})}$ and $(\mathcal{H}_{\{0\},W|_{\Delta(\check{G})}})_{\overline{\eta}}$. These maps use the compatibility of the construction of $\mathcal{H}_{I,W}$ with morphisms of finite sets, i.e., the morphism $I \to \{0\}$. Note restricting a representation of $\check{G}^I$ to the diagonal $\Delta(\check{G})$ is the same as taking the tensor product of I representations of $\check{G}$. This is reflected in the construction of bc as the functoriality was deduced from the properties of the convolution operation appearing in the Satake isomorphism.
Finally, the element $\underline{\gamma}$ is acting on $H_{I,W} = \mathcal{H}_{I,W}\,|_{\overline{\eta}}$ because with the help of Drinfeld's lemma we can (as in Section 3.1) consider the sheaf $\mathcal{H}_{I,W}$ as a representation of $\pi_1(\eta)^I$.

Remark 6.4 (Technical aside)**.** Above we mixed $\pi_1(C)$ and $\pi_1(\eta)$, which is dangerous as we only constructed $\mathcal{H}_{I,W}$ as perverse sheaves and not as locally constant sheaves. However, we know that the shaves $\mathcal{H}_{I,\mathbb{1}}$ are constant and this can ultimately be used to see that although the action of $\underline{\gamma} \in \pi_1(C)^I$ on the middle term of the above sequence may depend on the choice of a preimage under the surjection $\pi_1(\eta)^I \twoheadrightarrow \pi_1(C)^I$, this choice does not affect the action on those sections that are obtained from $bc \circ w$. ([8, Théorème 11.11])

From the functoriality of the construction and the corresponding properties of the sheaves obtained from the cohomology of the Beilinson–Drinfeld Grassmannian it is not hard to check that these operators satisfy the properties (1)-(3) that we need from our operators on $\mathbb{Q}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]$. Property (4′) also follows from this as in Remark 4.4, by observing that multiplication $G \times G$ can be interpreted via the natural trace and identity morphisms on products of representations $W \otimes W^*$. As we remarked in Section 2, this then allows to deduce the first part of the main theorem, i.e., these operators define a decomposition:

$$\mathcal{C}(\mathrm{Bun}_G(\mathbb{F}_q)/\Xi)^{\mathrm{cusp}} = \bigoplus_{\substack{\sigma\colon \pi_1(C)\to\check{G} \\ \text{semisimple}}} \mathcal{C}_\sigma.$$

This was the main result that we wanted to prove. Note that the space on the left hand side of this decomposition appeared as fiber of a constant local system on C in the above construction, but the construction did relate this space to non-constant representations of $\pi_1(C)$. We will come back to this in the next section, in which we identify the elements $\sigma(\mathrm{Frob}_x)$ with geometrically defined operators.

6.3. **The analog of the Eichler-Shimura relations.** The last step is to show that Hecke operators act on $\mathcal{C}_\sigma$ and that their eigenvalues on these spaces are indeed determined by the images of Frobenius elements $\sigma(\mathrm{Frob}_x)$ as claimed in the main theorem.
In order to do this, we need to rephrase the definition of the classical Hecke operators (5.2) in geometric terms in order to extend them to operators on the spaces $H_{I,W}$ where the excursion operators are defined. As in 5.2 let us fix a point $x \in C$ and a dominant coweight λ. Denote by $C^\circ := C - x$, similarly

$$\mathrm{Cht}^\circ_{G,I,W} := \left\langle (\underline{x}, \mathcal{E}, \phi\colon \mathcal{E} \overset{x}{\dashrightarrow} {}^\tau\mathcal{E}) | \underline{x} \in {C^\circ}^I, \mathcal{E} \in \mathrm{Bun}_G \right\rangle = p^{-1}({C^\circ}^I)$$

will denote the stack of shtukas with legs different from x and $\mathcal{H}^\circ_{I,W} := \mathcal{H}_{I,W}|_{{C^\circ}^I}$. The analog of the space of Hecke modifications for $\mathrm{Cht}^\circ_{G,I,W}$ is then given by

$$\mathrm{HeckeCht}^\lambda_{G,I,W} := \left\langle \begin{array}{ccc} \mathcal{E}' & \overset{I}{\underset{\phi'}{\dashrightarrow}} & {}^\tau\mathcal{E}' \\ {\scriptstyle\psi}\Big\downarrow{\scriptstyle x} & & {\scriptstyle{}^\tau\psi}\Big\downarrow{\scriptstyle x} \\ \mathcal{E} & \overset{I}{\underset{\phi}{\dashrightarrow}} & {}^\tau\mathcal{E} \end{array} | \phi, \phi' \in \mathrm{Cht}^\circ_{G,I,W}, [\psi] = \lambda \in L^+G\backslash Gr_G \right\rangle.$$

This again defines a correspondence

$$\begin{tikzcd} & \mathrm{HeckeCht}^\lambda_{G,I,W} \arrow[dl, "(\mathcal{E},\mathcal{E}')\mapsto\mathcal{E}"', "p"] \arrow[dr, "(\mathcal{E},\mathcal{E}')\mapsto(\mathcal{E}')", "p'"'] & \\ \mathrm{Cht}^\circ_{G,I,W} & & \mathrm{Cht}^\circ_{G,I,W}. \end{tikzcd}$$

Note that here the compatibility of the modification ψ with Frobenius pull back ensures that the projections define a finite étale correspondence and it is therefore reasonable to ask for equality $[\psi] = \lambda \in L^+G\backslash Gr_G$ instead of considering the closure $\overline{\mathrm{Gr}^\lambda}$ that we used in Section 5.2.
Taking cohomology this correspondence defines Hecke operators

$$T_{\lambda,x}\colon \mathcal{H}^\circ_{I,W} \to \mathcal{H}^\circ_{I,W}$$

that for $W = \mathbb{1}$ reproduce the operators $T^{cl}_{x,\lambda}$ on $\mathbb{Q}_\ell[\mathrm{Bun}_G(\mathbb{F}_q)]$.

As before one defines for any $V \in \mathrm{Rep}(\check{G})$ one defines

$$T_{V,x} = \sum_{\substack{\mu \text{ dominant} \\ \text{occuring in } V}} h(V,\mu) T_{\mu,x}.$$

Lafforgue identifies these operators as particular excursion operators as follows: For every $V \in \mathrm{Rep}(\check{G})$ we have canonical morphisms of representations

$$\mathbb{1} \xrightarrow{\delta} V \boxtimes V^* \xrightarrow{\epsilon} \mathbb{1}.$$

By functoriality of the assignment $V \mapsto \mathcal{H}_{I,V}$ this induces for every $x \in C(\mathbb{F}_q)$ a morphism $HL_{V,x}$ of sheaves on $C^I \times \{x\}$:

$$\mathcal{H}_{I,W} \boxtimes \overline{\mathbb{Q}}_\ell \xrightarrow{\delta} \mathcal{H}_{I \sqcup \{1,2\}, W \boxtimes V \boxtimes V^*}|_{C^I \times \{(x,x)\}} \xrightarrow{\mathrm{Frob}_1} \mathcal{H}_{I \sqcup \{1,2\}, W \boxtimes V \boxtimes V^*}|_{C^I \times \{(x,x)\}} \xrightarrow{\epsilon} \mathcal{H}_{I,W} \boxtimes \overline{\mathbb{Q}}_\ell.$$

Proposition 6.5 ([8, Proposition 6.2]). *The restriction of $HL_{V,x}$ to ${C^\circ}^I$ coincides with the action of the Hecke operator $T_{V,x}$.*

Although the proof of this proposition requires work, on a naive level the statement is quite plausible. Let us try to explain this by going through the definition of the operator $HL_{V,x}$.

We will restrict ourselves to the main case where $W = \mathbb{1}$, so that the shtukas appearing in the definition of $\mathcal{H}_{I,W}$ have trivial modifications at I only and we can therefore drop the corresponding maps $\overset{I}{\dashrightarrow}$ from our notation. We begin with the partial Frobenius operator Frob_1 occurring in the definition of $HL_{V,x}$. This was induced from the morphism on $\mathrm{Cht}_{\{1,2\},V\boxtimes V^*}$ defined by shifting our sequence of modifications

$$(\mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}_1 \overset{x}{\dashrightarrow} {}^\tau\mathcal{E}) \mapsto (\mathcal{E}_1 \overset{x}{\dashrightarrow} {}^\tau\mathcal{E} \overset{{}^\tau x}{\dashrightarrow} {}^\tau\mathcal{E}_1).$$

Note that we can also consider this shift as replacing $\mathcal{E}$ by $\mathcal{E}_1$, i.e, as applying a modifications of type V at x to $\mathcal{E}$.

The morphisms δ and ϵ are induced from $\mathrm{IC}_1 \to \mathrm{IC}_V \star \mathrm{IC}_{V^*} \to \mathrm{IC}_1$, where IC_1 is the skyscraper sheaf supported at trivial modifications and the convolution $\mathrm{IC}_V \star \mathrm{IC}_{V^*}$ was defined as taking cohomology along the map

$$\langle \mathcal{E}_1 \overset{\phi_1,x}{\dashrightarrow} \mathcal{E}_2 \overset{\phi_2,x}{\dashrightarrow} \mathcal{E}_3 \rangle \to \langle \mathcal{E}_1 \overset{\phi_2\circ\phi_1,x}{\dashrightarrow} \mathcal{E}_3 \rangle$$

given by composition of modifications. The fiber over the trivial modification therefore is the space of modifications, in which the first one ϕ_1 is the inverse of the second $\phi_2 = \phi_1^{-1}$.

Therefore the operator $HL_{V,x}$ can also be described as a composition of correspondences:

$$\begin{tikzcd}[column sep=tiny]
& \left\langle \begin{matrix} \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}_1 \overset{x}{\dashrightarrow} {}^\tau\mathcal{E} \\ \text{composition iso} \end{matrix} \right\rangle \arrow[dl] \arrow[dr, hook] & & \left\langle \begin{matrix} \mathcal{E}_1 \overset{x}{\dashrightarrow} {}^\tau\mathcal{E} \overset{x}{\dashrightarrow} {}^\tau\mathcal{E}_1 \\ \text{composition iso} \end{matrix} \right\rangle \arrow[dl, hook'] \arrow[dr] & \\
\langle \mathcal{E} \xrightarrow{\sim} {}^\tau\mathcal{E} \rangle & & \langle \mathcal{E} \overset{x}{\dashrightarrow} \mathcal{E}_1 \overset{x}{\dashrightarrow} {}^\tau\mathcal{E} \rangle & & \langle \mathcal{E}_1 \xrightarrow{\sim} {}^\tau\mathcal{E}_1 \rangle
\end{tikzcd}$$

where in the right correspondence we denoted the middle bundle ${}^\tau\mathcal{E}$ to make all the maps into forgetful maps. Now the fiber product of the middle maps gives the stack:

$$\left\langle \begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d, dashed, "\phi \mid x"'] & {}^\tau\mathcal{E} \arrow[d, dashed, "{}^\tau\phi \mid x"] \\ \mathcal{E}_1 \arrow[r] & {}^\tau\mathcal{E}_1 \end{tikzcd} \right\rangle$$

which is indeed the correspondence used to define $T_{V,x}$.

Of course, to make this geometric computation into a formal proof requires some care. For example one needs to take into account that the sheaves $\mathcal{H}_{I,W}$ were defined from the intersection cohomology sheaves IC_W instead of constant sheaves and one also needs to check that the maps δ, ϵ induced from the functor $W \mapsto \mathrm{IC}_W$ correspond to the obvious maps that appear in the geometric description of the correspondences. This is explained in detail in Lafforgue's article [8].

7. Outlook

Since the appearance of Lafforgue's article the methods and results have led to further results and we would like to mention at least two of them. Results on the Langlands correspondence are always expected to have local counterparts. The decomposition of the main theorem provides us with representations of the fundamental group $\pi_1(C-S)$ and it is of course interesting to know how the monodromy of the representations around the punctures S which we can view as representation of $\pi_1(\mathbb{F}_q((t)))$ is related to the automorphic forms appearing in C_σ. That the global result indeed has a local counterpart is explained in work of Genestier and Lafforgue [4].

Also we cannot finish this overview without mentioning the work of Scholze [16] that was inspired by Lafforgue's construction. In these lectures Scholze makes an attempt to construct a counterpart of the above correspondence for local fields of mixed characteristic starting from a surprising analog of Drinfeld's lemma (Lemma 6.3) for such fields.

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