Honors Single Variable Calculus 110.113

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Contents

1 Equivalence Relation

Week 3 Reading: [3, Ch.3.5, Ch.4]*, On the construction of* $\mathbb{Q}$*, see* [1, 2.4].

Learning Objectives

Last few lectures:

- Defined the natural numbers and sets axiomatically.
- Discussed how *cardinality* came up from "counting" sets.

This and next lecture:

- discuss equivalence relation.
- construct $\mathbb{Z}, \mathbb{Q}$. Extend addition and multiplication in this context.

1.1 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 1.1. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1, 2\} = \{2, 1\}$

In ordered pairs

- $(1, 2) \neq (2, 1)$

Definition 1.2. Let X, Y be two sets. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Currently, we can either put the existence of such a set as an axiom, or use the axioms of set theory, this is in hw.

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [2, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 1.3. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$
$$a \sim a$$
- *Transitive*. For all $a, b, c \in A$,
$$a \sim b, b \sim c \Rightarrow a \sim c$$
- *Symmetric*. For all $a, b \in A$,
$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Often times, people do not describe the subset R, but describe it a relation *equivalently* as a rule: saying $a, b \in A$ are related if some property $P(a, b)$ is true. In short hand, one writes

$$a \sim b \text{ iff } \ldots$$

Definition 1.4. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as [1]

$$[x] := \{y \in A : x \sim y\}$$

An element in such an equivalence is called a *representative* of that class.

Definition 1.5. Quotient set. Given an equivalence relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

Example

Consider $\mathbb{N}$ and the equivalence relation that $a \sim b$ iff a and b have the same parity. [a]

- There are two equivalence classes: the odds and evens.
- For the odd class, a *representative*, or an element in the equivalence class, is 3.

[a] i.e. both or odd or even.

There is a relation between equivalence and partition of sets.

Definition 1.6. A *partition* of a set X is a collection ???

1.2 Integers

What are the integers? It consists of the natural numbers and the negative numbers. What is *subtraction*? We do not know yet. Can we define *negative* numbers without referencing minus sign? Yes, we can. Say

$$-1\text{is "}0-1\text{" is } (0,1)$$

Discussion

Let us say we define the integers as pairs (a, b) where $a, b \in \mathbb{N}$. Would this be our desired

$$\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$$

- How many -1s are there?

But we have a problem, there are multiple ways to express -1. Our system cannot have multiple -1s. What are other ways We can also have $1-2$, or the pair $(1, 2)$.

[1] It does not matter if we write $\{y \in A : y \sim x\}$ by symmetry condition.

Discussion

Now that we have our $\mathbb{Z}$, how do we define addition? [a] Can we leverage our understanding?

[a] What is addition abstractly? It is an operation $+ : X \times X \to X$.

Intuitively, the *integers* is an expression [2] of non-negative integers, (a, b), thought of as $a - b$. Two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 1.7. Let
$$R \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$
consists of all pairs (a, b) and (c, d) such that $a + d = b + c$. Equivalently,

$$R := \{(a, b), (c, d) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \, : \, a + d = b + c\}$$

The *integers* is the set
$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

Definition 1.8. Addition, multiplication. [3, 4.1.2] .

We can now finally define negation.

Definition 1.9. [3, 4.1.4].

Proposition 1.10. Algebraic properties. Let $x, y, z \in \mathbb{Z}$.

- Addition
  - Symmetric $x + y = y + x$.
  - Admits identity element.

1.3 Rational nunbers

Reading: [1, 2.4]. Be careful of the notation used! See 1.11.

Definition 1.11. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

$$\mathbb{Z} \setminus \{0\} := \{n \in \mathbb{Z} \, : \, n \neq 0\}$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote *the equivalence class* of pair (a, b) by $[a/b]$

[2] Rather than a pair, as an expression has multiple ways of presentation

Again, we need the notion of addition, multiplication, and negation.

Definition 1.12. Let $[a/b], [c/d] \in \mathbb{Q}$. Then

1. Addition:
$$[a/b] + [c/d] := [(ad + bc)/bd]$$

2. Multiplication
$$[a/b] \cdot [c/d] := [(ac)/(bd)]$$

3. Negation.
$$-[a/b] := [(-a)/b]$$

Discussion

Is this definition well defined? The notation we use here is from 1.11. In 1. we *want* to define a function:

$$+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

which takes as input two equivalence class and outputs a new one. Let us consider two equivalence class

$$x := \left\{a'/b' \, : \, a'/b' \sim a/b\right\} \in \mathbb{Q}$$

$$y := \left\{c'/d' \, : \, c'/d' \sim c/d\right\} \in \mathbb{Q}$$

To add these two classes, we proceed as follows:

1. We pick two representatives from each class, let us say a/b of x and c/d of y.

2. We define
$$x + y := [(ad + bc)/bd]$$

Why can't we say this is the definition of addition? This definition is not complete yet, as $x + y$ can take *more than one possible values* - which is not a function!

For example, one could have chosen other pair of representatives, a'/b', and c'/d', and obtained $x + y$ as

$$[(a'd' + b'c')/b'd']$$

Thus, we have to check that

$$[(a'd' + b'c')/b'd'] = [(ad + bc)/bd]$$

Similarly, we can define also define order relation.

Definition 1.13. Let $x \in \mathbb{Q}$,

- x is *positive* iff $x = [a/b]$ where a, b are positive integers, we often denote positive integers as $\mathbb{Z}_{>0}$.
- x is *negative* iff $x = -y$ where y is some positive rational.

With the notion of positive rationals[3] from def. 1.13, we can define order relation $<, \leq$ on $\mathbb{Q}$.

Definition 1.14. Let $x, y \in \mathbb{Q}$, then we denote

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is zero or positive.

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 1.15. $\sqrt{2}$ is not rational.

Proof. ??? □

[3] The same trick is used to define order in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

2 Homework for week 3

Due: Week 4, Saturday. You will select 3 problems to be graded.
Problems 1-3 are on cardinality. Problem 4 is on a general construction of equivalence relations. Problems 5-7 is about addition, multiplication, and division on $\mathbb{Z}$ and $\mathbb{Q}$.

1. Show that the relation $\leq$ is transitive, i.e. $|X| \leq |Y|, |Y| \leq |Z|$ then $|X| \leq |Z|$.

2. (**) Prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite. [4] Prove that $\mathbb{Q}$ is countably infinite. *You are free to use results from previous problems and theorems stated in lectures.*

3. (**) Let X be any set. Prove that there is no surjection (hence, bijection) between X and $\{0,1\}^X$. Deduce that $\{0,1\}^{\mathbb{N}}$ is uncountable. Argue the first part by contradiction: suppose there exists a surjection

$$f : X \to \{0,1\}^X$$

   - Consider the set
$$A = \{x \in X \,:\, x \notin f(x)\}$$
   - As f is a surjection (write the general definition) there must exists $a \in X$ such that $f(a) = A$. Do case work on whether $a \in A$ or $a \notin A$.

4. (**) Let X be any set. Recall that a binary relation on X, is any subset $R \subseteq X \times X$. We define $R^{(n)}$ as follows

   - For $n = 0$,
$$R^{(0)} = \{(x,x) \,:\, x \in X\}$$
   - Suppose $R^{(n)}$ has been defined.
$$R^{(n+1)} := \Big\{(x,y) \in X \times X \,:\, \exists z \in X, (x,z) \in R^{(n)}, (z,y) \in R\Big\}$$

   (a) Show that
$$R^t := \bigcup_{n \geq 1} R^{(n)} = R^{(1)} \cup R^{(2)} \cup \cdots$$
   defines a *smallest* transitive relation on X containing R. i.e. if Y is any other transitive relation on X containing R, then $R^t \subseteq Y$.

[4]Knowing the Cartesian product is required for this problem, skip 5. and 6. if unfamiliar.

(b) Show that

$$R^{tr} := \bigcup_{n\geq 0} R^{(n)} = R^{(0)} \cup R^{(1)} \cdots$$

is the *smallest* reflexive and transitive relation on X. i.e. if Y is any other transitive and reflexive relation on X containing R, then $R^{tr} \subseteq Y$.

5. (***) Show that addition, product, and negation are well-defined for rational numbers; see def. 1.11 or [3, 4.2]. You are free to use any facts and properties you know about $\mathbb{Z}$, such as the cancellation law.

6. (*) Let $x, y, z \in \mathbb{Z}$. Use the definition of addition and multiplication from 1.8, or [3, 4.1], show :

   (a) $x(y + z) = xy + xz$.

   (b) $x(yz) = (xy)z$.

   You are free to use any facts and properties you know about $\mathbb{N}$.

7. Let $x, y \in \mathbb{Z}$. You are free to use any facts you know about $\mathbb{N}$, in particular, it would be helpful to use the following the result: [3, 2.3.3]: *Let* $n, m \in \mathbb{N}$. *Then* $n \times m = 0$ *if and only if at least one of* n, m *is equal to zero.* Show that if $xy = 0$ then $x = 0$ or $y = 0$.

2.1 Tri-weekly diary

8. (**) Write a 800-1000 words diary or story. Pen down a diary on your experiences with the course topics and experiences so far, focusing particularly on:

   - Concepts or ideas that you initially found challenging or confusing. For example, the axioms of natural numbers $\mathbb{N}$, set theory, etc.
   - Topics that have piqued (if any, XD) your curiosity.
   - Topics that you wanted to be covered, and why.
   - Topics that you would like further elaboration.
   - People you find fun to be with (or scared of)!

   + (*) points for the best diary.

References

[1] Derek Goldrei, *Propositional and predicate calculus: A model of argument*, 2005.

[2] Paul R. Halmos, *Naive set theory*, 1961.

[3] Terence Tao, *Analysis I, 4th edition*, 2022.