A short story

How to read this? There is much more to fit in two pages, so I will provide references and omit details. I leave the technicalities of my research to the research statement.

For me, mathematics plays a role closer to novels or poetry. Many aspects of it are fictional, requiring imagination and effort to appreciate. Nonetheless, they often deeply reflect the authors' emotions and the greater community.

1. Duality

I am interested in a particular symmetry known as *duality*: when there are two related ways to see the same concept. For example, there is Maxwell's equations in physics. In the absence of charge and currents, it can be written as

$$\nabla \cdot E = 0 \quad \nabla \cdot B = 0$$
$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t} \quad \partial \times B = \frac{1}{c}\frac{\partial E}{\partial t}$$

By changing the roles of (E, B) to $(-B, E)$, the equations are the same!
The *Langland's program* hopes to uncover a duality between *number theory* and *harmonic analysis*. The modern approach to finding such a bridge is by reformulating the problem in terms of *geometry*, which broadly construed is the study of shapes.

$$\begin{tikzcd}
& \{\text{Geometric Langlands}\} & \\
\{\text{Harmonic Analysis}\} \arrow[ur, bend left] \arrow[rr, dashed, leftrightarrow, "?" description] & & \{\text{Number Theory}\} \arrow[ul, bend right]
\end{tikzcd}$$

Let us see how Number theory is related to *geometry*. Number theory studies Diophantine equations, the integer solution set, $X(\mathbb{Z})$, to a set of polynomial equations

$$X := \{f_i \in \mathbb{Z}[t_1, \ldots, t_n] \quad i = 1, \ldots, r\}$$

One can consider the same problem of describing solution set $X(R)$, for different "coefficient" rings R. An insight from A. Grothendieck, is that the aggregate of all this data is described by a geometric object called *scheme*.
Further reading: An overview of various dualites, [?Ati-duality]. A slightly more techincal introduction to Langlands, [?Intro-ll].
To go from geometry to analysis, one packages this data using a generating function. For instance, the *Zeta function*

$$\exp\left(\sum_{n>0} \frac{|X(\mathbb{F}_{p^n})|}{n} t^n\right) =: \zeta_X(t)$$

Already, this says something about the distribution of primes using Euler's discovery on prime factorization![1]

How does this appear in analysis? One looks at harmonic analysis to a particular class of spaces referred as *locally symmetric spaces*. A family of examples can be constructed as follows. Let Γ be a *congruence subgroup*[2] of $\mathrm{SL}_2(\mathbb{Z})$,

$$\mathcal{M}_\Gamma := \Gamma\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2 = \Gamma\backslash\mathbb{H}$$

where $\mathbb{H} := \{z = x + iy : x, y, \in \mathbb{R}, y > 0\}$ is the complex upper half plane. A function f [3] on such space, is *modular*: they have many symmetries. The action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}$, is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z := \frac{az+b}{cz+d}$$

Thus, if $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma$, then f is 1-periodic: $f(z) = f(z+1)$, hence admitting Fourier expansion. The collection of a special subclass of such modular functions, admit natural symmetries called *Hecke operators*. The joint eigenvalues of the operators encode the same data as the Zeta function!

Further reading: More details is in [?Sut-EC, 24]. This is related to Fermat's last theorem. Modular forms and its relation to cryptography, [?Was08].

2. Modern approaches: higher category theory

To express the duality, it is indispensable to use the language of *higher categories*. Homotopy theory studies a world where *deformations* are natural. *Category theory* studies families of objects formally. Higher category theory is the modern language which combine these two concepts. It has been a recent success of J. Lurie, [?HTT] , E. Riehl and D. Verity, [?RV19] - and many more - to set up a concrete foundation.

[1] The uniqueness and existence of prime factorization.
[2] In particular, we can have $\Gamma = \mathrm{SL}_2(\mathbb{Z})$
[3] Here we would be consider *weight* 0 *modular forms level* $\mathrm{SL}_2(\mathbb{Z})$.

Higher category theory has proven successful in various areas of physics, [?Bae06], algebraic topology, [?HA], particularly algebraic K-theory. In context of Langlands duality, the pioneering work started from D.Gaitsgory et al, [?2014-geollc], which generalized works of field medalist, V. Drinfeld back in the 80s.

p-adic geometry is a deeply studied field that studies a *different* geometry that we are accustomed to: most mathematics working with $\mathbb{R}$ or $\mathbb{C}$ work with the equlidean square-norm. In p-adic geometry, the norm is *non-archimedian*, i.e.

$$|x+y| \leq \max |x|, |y|$$

This lead to both similar and surprising phenomena, for instance, the recent work of *prismatic cohomology*, [?BS19]. In this geometric setting, the field medalist P. Scholze, [?FS21] extended geometric Langlands duality.

My interests thus revolves around the following two flow

$$\begin{tikzcd}
\{\text{Higher Category Theory}\} \arrow[r, bend left] & \{\text{p-adic Geometric Langlands}\} \arrow[l, bend left]
\end{tikzcd}$$

Can we use higher category to better explain duality phenomena? Conversely, could duality in Langlands reflect deeper structures in other fields of mathematics?
Further reading: for an introduction to archimedean geometry, see [?Ach20].