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SCHEMATA OVER LOCAL RINGS

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Introduction

Given an algebraic variety defined over a discrete valuation ring with maximal ideal $\mathfrak{p}$, Shimura [12] has introduced the notion of the reduction of this variety mod $\mathfrak{p}$, which is a variety defined over the residue field k. In this paper we develop a sort of reduction mod $\mathfrak{p}^n$ for any power n of the ideal $\mathfrak{p}$. We then obtain an infinite sequence of varieties with connecting regular maps over k. If the ring is complete, then the integral rational points of the original variety have the structure of projective limit of the rational points of these varieties over k. This provides a method of climbing back to the original variety from its reductions.

The possibility of obtaining these higher reductions is a consequence of the fact that we can introduce coordinates in the valuation ring mod $\mathfrak{p}^n$, making it into an algebraic variety over k in which the ring operations become regular maps. If the ring has characteristic 0 and k has characteristic $p > 0$, these coordinates are just the Witt vectors of length n (cf. [14]); in the equal characteristic case, the coordinates are just coefficients of truncated power series. This leads us to introduce the notion of *algebraic ring* (§ 1). It happens that every Artin local ring has a natural structure of algebraic ring, so that our theory over a discrete valuation ring mod $\mathfrak{p}^n$ can be developed equally well over any Artin local ring. It is therefore in this general context that we work.

The construction of these new varieties in the affine case then proceeds as follows: in each polynomial equation with coefficients in the Artin local ring, we can substitute our coordinates for the coefficients and the variables, obtaining an equivalent system of many polynomial equations in more variables, but now over the residue *field*. (This technique has already appeared in [7]). But the generalization of this construction to arbitrary varieties is exceedingly clumsy and difficult to formulate in this language. At this point, A. Grothendieck [5], supplies the language of *schemata* instead of varieties, and we utilize it in § 2–6.

Our construction is then seen to be equivalent to the solution of a certain universal problem (cf. § 4), and from this all the nice functorial properties can be deduced. Two functors come into play: the functor F takes each schema over the Artin local ring into one over the residue field; the functor G takes each schema over the field into a local ringed space. Then there is a morphism of functors λ returning us to the schema with which we started. The functors F and G are "adjoint" to each other with respect to rational points (see the definition of "realization" in § 4 for the precise statement).

The main application, thus far, of this construction has been the exploitation of the structure of *pro-algebraic group* which it gives to the integral rational subgroup of an algebraic group over the $\mathfrak{p}$-adic integers. Pro-algebraic groups appeared previously in Serre's recognition (cf. [10]) that this structure on the multiplicative group of units in a $\mathfrak{p}$-adic field enabled one to do local class field theory by pro-algebraic isogenies; and Tate has shown (cf. [13]) more generally that in higher dimensions the idèle class group is a projective limit of generalized Picard varieties. The technique of projective limits was used in the author's thesis [4] to prove that if A is an abelian variety over a $\mathfrak{p}$-adic field K with non-degenerate reduction mod $\mathfrak{p}$, then any principal homogeneous space of A over K which has a rational point in an unramified extension of K must already have a rational point in K. This completes one of the results in [8]. The proof is based on the fact that the mapping $x \rightarrow x^{\sigma} - x$ is surjective for the rational points of A in the completion of the maximal unramified extension of K; here σ denotes the Frobenius automorphism. This fact was proved by Lang in [6] for group varieties over a finite field, and we get it in our case by passing to the limit in our sequence of group varieties over the residue field. A new proof of Hensel's lemma in a general form given by Chow [2] is also obtained. For these results and a more detailed study of the functors in this paper, foundation material on simple points is needed. In the cadre of schemata this is not yet available.

In conclusion, I take this opportunity to express my deep appreciation to Serge Lang for his suggestions and continual encouragement of my work.

1. Algebraic Rings

In this section we speak the language of algebraic spaces, more precisely, of (k, Ω)-algebraic spaces in the sense of [1, exposé 1], where k is a given ground field and Ω (which will be understood without repeated mention) an algebraically closed extension field of k.

DEFINITION. An *algebraic ring over* k consists of a unitary ring $\mathfrak{R}$ and a structure of k-algebraic space on the set underlying $\mathfrak{R}$ with respect to which the additive group underlying $\mathfrak{R}$ is a k-algebraic group, multiplication is a k-regular map from $\mathfrak{R} \times \mathfrak{R}$ to $\mathfrak{R}$, 1 is a k-rational point, the units U in $\mathfrak{R}$ form a proper locally k-closed subspace, and the map $x \to x^{-1}$ on U is k-regular with respect to the induced structure of k-algebraic space. In addition, a commutative k-algebraic group M which is a (left, say) module over $\mathfrak{R}$ in such a way that the operation of $\mathfrak{R}$ on M is k-regular from $\mathfrak{R} \times M$ to M will be called an *algebraic module over* $\mathfrak{R}$.

When the objects in question are varieties (i.e., absolutely irreducible algebraic spaces) we will speak of *ring-varieties* and *module-varieties*. It is clear what is meant by homomorphism and isomorphism in a category consisting of such objects.

In this section we are concerned only with the construction of a large class of commutative algebraic local rings over k. We will return to the study of algebraic rings in general in a later paper. Meanwhile, we give some examples of algebraic rings:

(1) The affine line S^1 over k, with the usual addition and multiplication making it into a field.

(2) Assuming k perfect of characteristic $p > 0$, the ring W_n of Witt vectors of length $n+1$ with coordinates in Ω; this is a commutative local ring-variety having affine $(n+1)$-space S^{n+1} as its underlying variety (cf. [14]).

(3) Assuming $\mathfrak{R}$ a commutative algebraic ring over k, the residue ring of the formal power series ring $\mathfrak{R}[[X_1, \cdots, X_n]]$ obtained by cutting off all power series after some fixed term.

(4) As in example (3), with non-commutative formal power series instead.

(5) The ring of matrices of a given degree with coefficients in a commutative algebraic ring over k.

(6) The direct sum of any finite number of k-algebraic rings. Needless to say, these examples do not quite exhaust the field.

Let $\mathfrak{R}$ be an Artin local ring, i.e., a commutative Noetherian ring with only one proper prime ideal $\mathfrak{m}$, also called commutative primary ring with minimum condition. Assume k is the residue field of $\mathfrak{R}$, i.e., assume given as part of its structure a homomorphism ψ of $\mathfrak{R}$ onto k. We distinguish two cases:

Case 1: Equal characteristics. If $\mathfrak{R}$ has the same characteristic as k, the exact sequence

$$0 \to \mathfrak{m} \to \mathfrak{R} \to k \to 0$$

splits, so that there is a monomorphism μ of k into $\mathfrak{R}$ such that $\psi\mu =$ identity. μ is not unique unless k is perfect of characteristic $p > 0$ or is an algebraic extension of the rational numbers. Each such μ endows $\mathfrak{R}$ with a structure of finite-dimensional algebra over k.

Case 2: Unequal characteristics. We deal here only with the case k perfect (of characteristic $p > 0$). Let p^{n+1} be the characteristic of $\mathfrak{R}$, $n > 0$. Then there is a canonical map $\mu: k \to \mathfrak{R}$ such that $\psi\mu =$ identity, $\mu(0) = 0$, and the restriction of μ to the multiplicative group of k is a monomorphism into the group of units of $\mathfrak{R}$. If $\mathfrak{m}^m = 0$, then $\mu(k)$ coincides with the subset $\mathfrak{R}^{p^m}$ of p^m-powers of elements of $\mathfrak{R}$; these are the multiplicative representatives of k (cf. [3]). The elements of $\mathfrak{R}$ of type $a = \mu(a_0) + \mu(a_1)p + \cdots + \mu(a_n)p^n$ form an Artin local subring of $\mathfrak{R}$ having residue field k and maximal ideal generated by p. The map $a \to (a_0, a_1^{p^{-1}}, \cdots, a_n^{p^{-n}})$ is an isomorphism of this subring with the ring $W_n(k)$ of Witt vectors of length $n+1$ over k, so that $\mathfrak{R}$ is canonically endowed with a structure of $W_n(k)$-algebra. As a module over $W_n(k)$, $\mathfrak{R}$ is of finite type–for example, as generators we can take the element 1 and the union of any sets of ideal generators of $\mathfrak{m}, \mathfrak{m}^2, \cdots, \mathfrak{m}^{m-1}$. Furthermore, let $\mathfrak{R}'$ be another Artin local ring with residue field k, and suppose we are given a ring homomorphism f of $\mathfrak{R}$ into $\mathfrak{R}'$. Since $f(a^r) = f(a)^r$ and $f(ra) = rf(a)$ for all $a \in \mathfrak{R}$ and $r > 0$, f sends the maximal ideal of $\mathfrak{R}$ into that of $\mathfrak{R}'$ and preserves p. Assuming f induces the identity on k by passage to the quotient, then f induces on $W_n(k)$ the canonical epimorphism onto the subring $W_{n'}(k)$ of $\mathfrak{R}'$ where $p^{n'+1}$ is the characteristic of $\mathfrak{R}'$ and necessarily $0 \leqq n' \leqq n$. If we regard $\mathfrak{R}'$ as an algebra over $W_n(k)$, f is an algebra homomorphism.

We now unify these two cases by the following device: In *Case* 1, we choose a particular μ and thereby endow $\mathfrak{R}$ with a structure of $W_0(k)$-algebra, with $W_0(k) = k$. We restrict the word "homomorphism" to mean "algebra homomorphism." In *Case* 2, every extension field k' of k mentioned will be assumed perfect. *These conventions will remain in force for the rest of this section.*

PROPOSITION 1. *Let k' be an extension field of k. For any module M of finite type over $W_n(k)$, let $M_{k'} = M \otimes_{W_n(k)} W_n(k')$ be the module over $W_n(k')$ obtained from M by extension of scalars. Then the canonical homomorphism $x \to x \otimes 1$ of M into $M_{k'}$ is injective.*

PROOF. If $n = 0$ (vector space over k), the statement is well known. For $n > 0$, M can be regarded as a module over the principal ideal domain of Witt vectors of infinite length. Applying the structure theory of such modules ([cf., 15]), we find M isomorphic to a direct sum $W_{n_1}(k) \oplus \cdots \oplus W_{n_r}(k)$, $0 \leqq n_i \leqq n$, $i = 1, \cdots, r$. Clearly $W_{n_i}(k') = W_{n_i}(k) \otimes_{W_n(k)} W_n(k')$. Since extension of scalars is an additive functor, we see that $M_{k'} \cong W_{n_1}(k') \oplus \cdots \oplus W_{n_i}(k')$ and M is imbedded in $M_{k'}$.

PROPOSITION 2. *Let $\mathfrak{R}$ be a local algebra over $W_n(k)$, k' an extension field of k. Let $\mathfrak{R}_{k'}$ be the algebra over $W_n(k')$ obtained by extension of scalars, and identify $\mathfrak{R}$ with a subring of $\mathfrak{R}_{k'}$* (Proposition 1). *Then $\mathfrak{R}_{k'}$ is a local ring with residue field k' and maximal ideal generated by the maximal ideal of $\mathfrak{R}$.*

PROOF. The extension of the epimorphism $\psi : \mathfrak{R} \to k$ to $\mathfrak{R}_{k'} \to k' = k' \otimes_{W_n(k)} W_n(k')$ is again an epimorphism with kernel generated by the kernel of ψ; this by the general theory of tensor products. We then have a maximal ideal in $\mathfrak{R}_{k'}$ all of whose elements are nilpotent. It must therefore be the unique prime ideal of $\mathfrak{R}_{k'}$.

PROPOSITION 3. *Let M be a module of finite type over $W_n(k)$, $\bar{M}$ its extension to a module over $W_n = W_n(\Omega)$. Then*

(1) *There is a unique structure of module-variety over W_n on $\bar{M}$ defined over k (we call this the maximal structure because of* (2)) *such that M is the subset of k-rational points and the map $x \to xa$ is separable for all a in a k-open subspace of $\bar{M}$.*

(2) *Any other structure of module-variety over W_n on $\bar{M}$ is a purely inseparable regular image of the maximal structure (so the maximal structure is unique, period, in characteristic* 0).

(3) *The group variety underlying $\bar{M}$ is isomorphic to a product of Witt groups (product of G_a's in characteristic* 0).

(4) *The dimension of $\bar{M}$ as a variety is equal to its length as a module over W_n, which equals the length of M over $W_n(k)$.*

(5) *Every submodule of $\bar{M}$ spanned by elements of M is a k-closed subvariety.*

(6) *If M', $\bar{M}'$ is another such situation, $f : \bar{M} \to \bar{M}'$* a *linear transformation which is the extension of a linear transformation $M \to M'$ then f is k-regular (with respect to the maximal structure of module-variety on $\bar{M}$). If f is surjective then it is separable,* i.e., *$\bar{M}'$ is isomorphic to the quotient module-variety of $\bar{M}$ by f.*

PROOF. Choose an isomorphism of $\bar{M}$ with a direct sum $W_{n_1} \oplus \cdots \oplus W_{n_r}$ which extends a similar isomorphism for M over $W_n(k)$. Transport the structure of module-variety on the direct sum to $\bar{M}$ by means of this isomorphism. This structure certainly satisfies the conditions of (1). For any other structure, this module isomorphism would be a regular map (because as part of the definition of module-variety the operation of W_n on $\bar{M}$ is assumed regular), hence purely inseparable since it is bijective. It is easy to see that a linear transformation of a direct sum of Witt rings must be regular. In particular, the structure on $\bar{M}$ does not depend on the particular isomorphism chosen. A submodule of $\bar{M}$ spanned by elements $a_1, \cdots, a_s$ of M is k-closed, since it is the kernel of a k-regular homomorphism. It is irreducible because it is the image of the product W_n^s under the regular map $(x_1, \cdots, x_s) \rightarrow x_1a_1 + \cdots + x_sa_s$. From the representation $\bar{M} \cong W_{n_1} \oplus \cdots \oplus W_{n_r}$ it is easy to exhibit a chain of submodules of length $(n_1 + 1)(n_2 + 1) \cdots (n_r + 1)$, whence the dimension of $\bar{M}$ equals its length (which is the same as the length of M since we can choose these submodules from M). Finally, if $f: \bar{M} \rightarrow \bar{M}'$ is surjective, it induces a regular bijective homomorphism of the quotient onto $\bar{M}'$; since the inverse of this map is linear it must be regular, hence the map is an isomorphism.

Examples of structures of module-variety for $n = 0$ which do not satisfy the separability condition are given in [9]. Also, it should be remarked that in (5), the induced structure of module-variety on the submodule need not be its maximal structure. If the submodule is a direct summand, the two structures coincide.

PROPOSITION 4. *Let $\mathfrak{R}$ be a local algebra over $W_n(k)$, $\bar{\mathfrak{R}}$ its extension to a local algebra over W_n, $\psi: \bar{\mathfrak{R}} \rightarrow \Omega$, $\mu: \Omega \rightarrow \bar{\mathfrak{R}}$ with $\psi\mu =$ identity. If $\bar{\mathfrak{R}}$ is given the maximal structure of module-variety over W_n defined in Proposition 3, then*

(1) *$\bar{\mathfrak{R}}$ is a ring variety defined over k.*

(2) *$\mathfrak{R}$ is the subring of k-rational points of $\bar{\mathfrak{R}}$.*

(3) *Every ideal in $\bar{\mathfrak{R}}$ generated by elements of $\mathfrak{R}$ is a k-closed subvariety.*

(4) *The dimension of $\bar{\mathfrak{R}}$ as a variety is equal to its length as an Artin ring.*

(5) *If $\mathfrak{R}'$, $\bar{\mathfrak{R}}'$ is another such situation, every homomorphism $\bar{\mathfrak{R}} \rightarrow \bar{\mathfrak{R}}'$ which is the extension of a homomorphism $\mathfrak{R} \rightarrow \mathfrak{R}'$ is k-regular and separable if it is surjective.*

(6) *Assuming W_n operates faithfully on $\bar{\mathfrak{R}}$, the map $x \rightarrow x \cdot 1$ is a biregular isomorphism of W_n on a k-closed ring subvariety of $\bar{\mathfrak{R}}$.*

(7) *If $n = 0$, μ is an isomorphism of the field variety $\Omega = S^1$ with a k-closed subvariety of $\bar{\mathfrak{R}}$. If $n > 0$, μ is a biregular map of S^1 onto a k-closed subvariety of $\bar{\mathfrak{R}}$ such that $\mu(0) = 0$ and the restriction of μ to the multiplicative group G_m of S^1 is a monomorphism into the group variety $\bar{U}$ of units in $\bar{\mathfrak{R}}$.*

(8) *$\bar{U}$ is a k-open subvariety of $\bar{\mathfrak{R}}$ isomorphic to the product of G_m and a unipotent group variety. If $\bar{\mathfrak{R}}'$ is a homomorphic image of $\bar{\mathfrak{R}}$, the units in $\bar{\mathfrak{R}}'$ have the structure of quotient variety of $\bar{U}$ (assuming maximal structure).*

PROOF. Multiplication, being bilinear, is certainly k-regular. Similarly homomorphisms, since linear, are k-regular. Ideals, being submodules, are closed. $\bar{U}$ is k-open since it is the complement of the maximal ideal $\bar{\mathfrak{m}}$. $\bar{U}$ is isomorphic to the direct product $G_m \times \bar{V}$, where $\bar{V}$ is the group of units congruent to 1 mod $\bar{\mathfrak{m}}$. The map $x \to 1 + x$ is a biregular correspondence between the variety underlying $\bar{\mathfrak{m}}$ and that underlying $\bar{V}$, hence $\bar{V}$ is unipotent. Assertion (8) follows from the separability of the map concerned (Proposition 3). We must show that the map $x \to x^{-1}$ on $\bar{U}$ is k-regular. For this purpose we may assume $\mathfrak{R}$ is the ring of polynomials in some variables over $W_n(k)$ modulo the ideal of polynomials in which all monomials are of degree greater than some fixed integer; for it is easy to see that every ring under consideration is the homomorphic image of such a ring, and we may pass to the quotient. In this polynomial ring, the units are those polynomials whose constant term is a unit in $W_n(k)$, and the formula for the inverse involves only the inverse of the constant term in the denominator. Thus we are reduced to the case $\mathfrak{R} = W_n(k)$. The statement is obvious if $n = 0$. For $n > 0$ it is easy to check from the recursive formulas for multiplication of Witt vectors (the inverse of $(x_0, \cdots, x_n)$ has coordinates which are rational functions of the x's having only powers of x_0 in the denominators). To show the dimension of $\bar{\mathfrak{R}}$ equals its length, we proceed by induction on the length. For length 1, $\bar{R} = \Omega$ and the dimension is 1. Assuming dimension equals length for rings of length $n - 1$, we choose a minimal ideal $\mathfrak{a}$ in $\bar{\mathfrak{R}}$. Applying the inductive hypothesis to $\bar{\mathfrak{R}}/\mathfrak{a}$ and using the additivity of both dimension and length, we are reduced to proving the variety $\mathfrak{a}$ has dimension 1. Now $\mathfrak{a}$ is generated by an element a and is annihilated by $\bar{\mathfrak{m}}$, so that the mapping $x \to xa$ induces a bijective regular homomorphism of $S^1 = \bar{\mathfrak{R}}/\bar{\mathfrak{m}}$ onto $\mathfrak{a}$. For (6), we remark that the submodule spanned by the identity element of R is a direct summand [15, § 2, exercise 4]. The maximal structure then insures that the bijective mapping $x \to x \cdot 1$ of W_n onto this subvariety is biregular. (7) follows from (6).

2. Local ringed spaces

In this and the next four sections we use Grothendieck's language of schemata. We begin with some conventions on notation and terminology.

Recall that a *ringed space* is a pair (X, O_X) consisting of a topological space X and a sheaf O_X of rings on X. A *morphism* from such a pair to another $(X', O_{X'})$ is a continuous map $f\colon X \to X'$ together with a homomorphism f' from the sheaf $O_{X'}$ to the direct image of O_X by f. We make the abusive notation of writing X for the pair (X, O_X) and f for the pair (f, f'). When we wish to refer to the base space only, we will denote it by $b(X)$, and the structure sheaf will be denoted O_X or $O(X)$. The fibre over any point x in the base space will be denoted O_x or $O(x)$, and the ring of global sections by $\Gamma(X)$. The morphism f induces a ring homomorphism $\Gamma(X') \to \Gamma(X)$ which we denote $\Gamma(f)$; Γ is then a contravariant functor to the category of rings. The ring of sections over an open subspace U will be denoted $\Gamma(U)$ or $\Gamma(U, O_X)$.

We introduce the category of *local ringed spaces*: these are ringed spaces in which all the fibres are required to be local rings (commutative rings with only one maximal ideal). The morphisms in this category are by definition those morphisms f of ringed spaces satisfying the following additional condition: for any $x \in b(X)$, the homomorphism of fibres $O_{f(x)} \to O_x$ induced by f is a *local homomorphism*, i.e., the pre-image of the maximal ideal in O_x is the maximal ideal in $O_{f(x)}$. We will often restrict our attention to local ringed spaces over a fixed local ringed space S. These are local ringed spaces X together with a given "structural" morphism $X \to S$. The S-morphisms are then those morphisms which commute with the structural morphisms.

Recall that to any commutative ring A we associate a local ringed space Spec A (spectrum of A) as follows: $b(\text{Spec } A)$ is the set of prime ideals in A with the Zariski topology. $O(\text{Spec } A)$ is the sheaf whose fibre over the prime ideal x in the base space is the ring of fractions A_x of A with respect to the semigroup of elements not in x. A local ringed space isomorphic to Spec A for some A is called an *affine schema*, and a local ringed space which is locally an affine schema is called a *pre-schema*. The pre-schemata form a subcategory of the category of local ringed spaces. In this subcategory, finite products exist. A *schema* is then a pre-schema X such that the diagonal of $X \times X$ is closed.

If X, X' are two local ringed spaces, we denote by $X(X')$ the set of all morphisms $X' \to X$. This is a bifunctor to the category of sets, covariant in X and contravariant in X'. If f: $X' \to X''$ is a morphism, the induced map $X(X'') \to X(X')$ will be denoted $X(f)$. If we deal only with spaces over S and S-morphisms we denote the corresponding set by $X(X')_S$, although if S is understood from the context, we will often omit this subscript. And if $X' = \text{Spec } A$, we will write simply $X(A)$ instead of $X(\text{Spec } A)$.

For later use, we must generalize to local ringed spaces certain facts known for pre-schemata.

PROPOSITION 1. *Let $X = \text{Spec } A$ be an affine schema over $S = \text{Spec } B$, and let Y be any local ringed space over S. Then $g \to \Gamma(g)$ is a bijection of the set of all S-morphisms g: $Y \to X$ with the set of all B-homo-*

morphisms from A to $\Gamma(Y)$.

PROOF. Let $C = \Gamma(Y)$. Grothendieck [5, III, 1, 5] has shown that there is a canonical morphism $h\colon Y \to Y'$, where $Y' = \operatorname{Spec} C$. Given a B-homomorphism $\varphi\colon A \to C$, there is a unique S-morphism $f\colon Y' \to X$ such that $\Gamma(f) = \varphi$, and then $\Gamma(fh) = \varphi$ since $\Gamma(h) =$ identity. Conversely, given $g\colon Y \to X$, we can show that $\varphi = \Gamma(g)$ completely determines g. For if $y \in Y$ and $x = g(y)$, the commutative diagram

$$\begin{array}{ccc} C & \xleftarrow{\varphi} & A \\ \downarrow & & \downarrow \\ O_y & \longleftarrow & A_x \end{array}$$

shows that the prime ideal x in A is the pre-image of the maximal ideal in the fibre O_y under the homomorphism $A \xrightarrow{\varphi} C \to O_y$, and that the homomorphism $A_x \to O_y$, is uniquely determined from φ by the universal property of the ring of fractions A_x.

PROPOSITION 2. *If X_1, X_2 are pre-schemata over a pre-schema S, $(X_1 \times X_2, \pi_1, \pi_2)$ the product of these S-pre-schemata, Y a local ringed space over S, and $f_i\colon Y \to X_i$ S-morphisms, $i = 1, 2$, then there is a unique S-morphism $f\colon X \to X_1 \times X_2$ such that $\pi_i f = f_i$, $i = 1, 2$.*

The meaning of this proposition is that the universal property of the product of pre-schemata applies to the larger category of local ringed spaces. If X_1, X_2, S are affine with rings B_1, B_2, A, then $X_1 \times X_2 = \operatorname{Spec} B_1 \otimes_A B_2$, and there is a unique A-homomorphism $\varphi\colon B_1 \otimes_A B_2 \to \Gamma(Y)$ such that $\varphi\Gamma(\pi_i) = \Gamma(f_i)$, $i = 1, 2$. By Proposition 1, we can write $\varphi = \Gamma(f)$, $f\colon Y \to X_1 \times X_2$. In the general case where X_1, X_2, S need not be affine, the arguments of Grothendieck [5, I, 3.2] hold without change, and will not be repeated.

Let X, Y be local ringed spaces over a local ringed space S. For any open $U \subset X$, let $F(U)$ be the set of all S-morphisms $U \to Y$, where U is considered as a local ringed space over S with the structure induced by X. If $V \subset U$ is another open, define $\rho_{V,U}\colon F(U) \to F(V)$ to be ordinary restriction of morphisms. It is immediately verified that these data satisfy the axioms for a sheaf on $b(X)$, called *the sheaf of germs of S-morphisms from X to Y*. F is just a sheaf of sets; later, with special choice of Y, we will see that F becomes a sheaf of rings on $b(X)$. If F' is the sheaf of germs of S-morphisms into Y'; and $g\colon Y \to Y'$, a given S-morphism, then composing g with each element of $F(U)$ defines a mapping $F(U) \to F'(U)$ compatible with restriction, hence a sheaf homomorphism $F \to F'$. Hence we have a covariant functor from local ringed spaces over S to sheaves on $b(X)$.

3. Ring pre-schemata

We now consider the schematic analogues of algebraic groups and algebraic rings. Following the general program of Grothendieck, we define a *group pre-schema* X over S by requiring that $X(Y)_S$ be given a structure of group for every S-pre-schema Y, such that for any S-morphism $g\colon Y \to Y'$, the induced mapping $X(g)\colon X(Y')_S \to X(Y)_S$ is a homomorphism of groups. Similarly, replacing the word "group" with "ring" we have the notion of *ring pre-schema* over S.

PROPOSITION 1. *If X is a group pre-schema over S, then there exist unique S-morphisms $\sigma\colon X \times X \to X$ and $\tau\colon X \to X$ such that for every S-pre-schema Y and $f, g \in X(Y)_S$, we have*

$$fg = \sigma\circ(f, g)$$
$$f^{-1} = \tau\circ f\,.$$

Moreover, if $e\colon S \to X$ is the neutral element of the group $X(S)_S$, and $\psi\colon Y \to S$ the structural morphism of Y, then $e\circ\psi$ is the neutral element of $X(Y)_S$.

PROOF. Let π_1, π_2, be the projections of $X \times X$ on X, and put $\sigma = \pi_1\pi_2$ (product in the group $X(X \times X)_S$). Then the first formula follows from the diagram

$$\begin{tikzcd}
 & Y \arrow[dl, "f"'] \arrow[d, "{(f,g)}"] \arrow[dr, "g"] & \\
X & X \times X \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & X
\end{tikzcd}$$

and the fact that (f, g) induces a homomorphism $X(X \times X)_S \to X(Y)_S$. Similarly, let i be the identity morphism of X, and put $\tau = i^{-1}$ (inverse of i in the group $X(X)_S$). The second formula follows from the fact that any $f \in X(Y)_S$ induces a homomorphism $X(X)_S \to X(Y)_S$. Uniqueness of σ is obtained by taking $f = \pi_1$, $g = \pi_2$, for $(\pi_1, \pi_2) =$ identity. Uniqueness of τ is obtained by taking $f = i$. Finally, the homomorphism $X(S)_S \to X(Y)_S$ induced by ψ takes the neutral element e into the neutral element $e\circ\psi$.

For a ring pre-schema R over S, there are two S-morphisms $R \times R \to R$ corresponding to the two laws of composition.

PROPOSITION 2. *Let R be a ring pre-schema over S. Then there exists* *a group pre-schema U over S and a monomorphism j: $U \to R$ such that for every S-pre-schema Y, $j(Y)$ is an isomorphism of $U(Y)_S$ with the group of units in the ring $R(Y)_S$.*

PROOF. Let e be the neutral element for the multiplication in $R(S)_S$, ψ: $R \to S$ the structural morphism of R. Since $\psi \circ e$ is the identity morphism of S, e is an immersion of S in R [5, I, 5.3.13], i.e., an isomorphism of S with a subpre-schema E of R. If σ: $R \times R \to R$ is the morphism corresponding to the multiplicative law of composition, then $U = \sigma^{-1}(E)$ is a subpre-schema of $R \times R$. Let h: $U \to R \times R$ be the injection morphism, π: $R \times R \to R$ one of the projections, $j = \pi \circ h$.

For any S-pre-schema Y, if we identify $(R \times R)(Y)_S = R(Y)_S \times R(Y)_S$, then π induces the ordinary projection of the product of sets, σ the multiplication in $R(Y)_S$. The image of $U(Y)_S$ under $h(Y)$ is the set of pairs (x, y) such that $xy = 1$. Thus $j(Y)$ is a bijection of $U(Y)_S$ on the group of units in $R(Y)_S$, and we can transport the group structure to $U(Y)_S$ so that U becomes a group pre-schema. Since $j(Y)$: $U(Y)_S \to R(Y)_S$ is an injection for every Y, j is by definition a monomorphism.

Let R be a ring pre-schema over S. If X is any S-pre-schema, the sheaf of germs of S-morphisms from X to R is now a sheaf of rings on $b(X)$. We form a ringed space $G_R(X)$ having $b(X)$ as base space but with the sheaf of germs of S-morphisms from X to R instead of the structural sheaf of X. We denote by $R(x)$ the fibre of this sheaf over any point $x \in b(X)$. Then G_R is a covariant functor from pre-schemata over S to ringed spaces over $G_R(S)$.

Let R, R' be ring pre-schemata over S. An S-morphism f: $R \to R'$ will be called a *homomorphism* if for every S-pre-schema Y, the induced mapping $f(Y)$: $R(Y)_S \to R'(Y)_S$ is a homomorphism of rings. (Similarly for group pre-schemata.) Then the ring pre-schemata over S and homomorphisms form a category; and for X fixed, $G_R(X)$ is a contravariant functor from this category to the category of ringed spaces over $G_R(S)$.

Given a homomorphism f: $R \to R'$, we can define its *kernel*: The neutral element $e \in R'(S)_S$ is an isomorphism of S with a subpre-schema E of R'. Then we can form the subpre-schema $f^{-1}(E)$ of R, and $f^{-1}(E)(Y)$ is the kernel of $f(Y)$ for every S-pre-schema Y. If R' is separated over S, the kernel of f is a closed subpre-schema of R, since e is a closed immersion.

As an example of a ring pre-schema, let $\Omega = \operatorname{Spec} \mathbf{Z}[T]$, where $\mathbf{Z}[T]$ is the polynomial ring in one variable T over the integers. For any local ringed space Y, Proposition 1, § 1, gives us the bijection $g \to \Gamma(g)$ of $\Omega(Y)$ with $\operatorname{Hom}(\mathbf{Z}[T], \Gamma(Y)) = \Gamma(Y)$. If f: $Y \to Y'$ is a morphism, we have the commutative diagram

$$\begin{array}{ccc} \Omega(Y) & \longleftarrow & \Omega(Y') \\ \downarrow & & \downarrow \\ \Gamma(Y) & \underset{\Gamma(f)}{\longleftarrow} & \Gamma(Y') \end{array} .$$

This shows that if we transport the ring structure of $\Gamma(Y)$ to $\Omega(Y)$ by our bijection, Ω becomes a commutative ring schema and the functors $Y \to \Omega(Y)$ and $Y \to \Gamma(Y)$ become canonically isomorphic.

For the relative theory over S, we can extend the base to $\Omega_S = \Omega \times S$. Let π: $\Omega^S \to \Omega$ be the projection. For every local ringed space Y over S with structural morphism ψ, $\pi(Y)$: $\Omega^S(Y)_S \to \Omega(Y)$ is a bijection whose inverse is $g \to (g, \psi)$. This bijection is functorial in Y, so that once again Ω^S is a ring schema over S and $\Omega^S(Y)$ is isomorphic to $\Gamma(Y)$. We call Ω^S *the affine line over* S. (More generally, the product of Ω^S with itself n times will be called *affine n-space* over S.) Passing to the sheaf of germs of morphisms, we obtain at once

PROPOSITION 3. *If Y is any local ringed space over S, and L is the affine line over S, then the ringed space $G_L(Y)$ is canonically functorially isomorphic to Y.*

It follows in particular that $G_L(Y)$ is a local ringed space, not just a ringed space. In general, given a commutative ring pre-schema R over S such that the functor G_R takes its values in the category of local ringed spaces, we call R a *local ring pre-schema* over S. We define the category of local ring pre-schemata by the condition that for Y fixed, the functor $G_R(Y)$ in the variable R takes its values in the category of local ringed spaces. In the special case $Y = \operatorname{Spec} A$, where A is a local ring, these requirements imply the following: $R(A)_S$ is a local ring (consider the fibre over the unique closed point of Y). Any homomorphism $R \to R'$, R' another local ring pre-schema, (resp. $A \to A'$, A' another local ring) induces a local homomorphism $R(A)_S \to R'(A)_S$ (resp. $R(A)_S \to R(A')_S$).

If R is a local ring pre-schema, then there is a canonical morphism $G_R(S) \to \operatorname{Spec} R(S)_S$ (because $R(S)_S = \Gamma(G_R(S))$). Then $G_R(Y)$ can also be regarded as a local ringed space over $R(S)_S$.

4. Realizations

Throughout this section, we place ourselves in the following situation: We are given a commutative ring k as ground ring and a local ring pre-schema R over k. We denote by Ω the affine line over k. R is assumed to satisfy the following

HYPOTHESIS I. *The pre-schema R is k-isomorphic to affine n-space* *over k for some n.*

It follows that R is actually a schema over k. We emphasize that the isomorphism in our hypothesis is not an isomorphism of ring pre-schemata—for any k-pre-schema Y, the induced mapping $R(Y)_k \to \Omega(Y)^n$ is a bijection of sets and not an isomorphism of rings. We will indicate later exactly where this hypothesis is used.

The results of § 1 show how we can arrive at such a situation. Let k be a perfect ground field. Let $\mathfrak{R}$ be an Artin local ring with residue field k, endowed with a structure of k-algebra if the characteristic is 0. Now we use the fact that for any field k, the category of algebraic spaces over k is equivalent to the category of reduced schemata of finite type over k. We can therefore translate the results of § 1 into the language of schemata and associate to $\mathfrak{R}$ a ring schema R over k satisfying Hypothesis I such that $R(k)_k = \mathfrak{R}$. We must show that R is a local ring schema.

The results of § 1 insure that there is an open sub-schema U of R such that $U(Y)_k$ is the group of units in $R(Y)_k$ for every k-pre-schema Y. We also have a "maximal ideal" sub-schema M of R, kernel of the homomorphism ψ: $R \to \Omega$ of § 1, whose base space is complementary to $b(U)$. Given a k-pre-schema Y and $y \in b(Y)$. If f is a unit in the fibre $R(y)$ over y of the sheaf $O(G_R(Y))$, then $f(y) \in b(U)$. Conversely, if the latter condition holds, the fact that U is open insures the existence of an open neighborhood V of y and $f' \in R(V)$ representing f such that $f'(V) \subset U$; f' is then a unit in $R(V)$, hence f is a unit in $R(y)$. But those $f \in R(y)$ such that $f(y) \in b(M)$ form an ideal in $R(y)$, so that $R(y)$ is indeed a local ring. Thus $G_R(Y)$ is a local ringed space. If g: $Y' \to Y$ is a k-morphism and $g(y') = y$, then for $f' \in R(V)$, $(f' \circ g)(y') \in b(U)$ if and only if $f'(y) \in b(U)$, so that $R(g)$ is a local homomorphism.

We now use the simpler notation GY for $G_R(Y)$.

Definition. Let X be a pre-schema over the ring $R(k)$. A pair $(FX, \lambda X)$ consisting of pre-schema FX over k and an $R(k)$-morphism λX of the local ringed space GFX into X is called *a realization of X over k* if the following universal property holds: For every pre-schema Y over k and $R(k)$-morphism g of the local ringed space GY into X, there is a unique k-morphism g^*: $Y \to FX$ giving the commutative diagram

$$\begin{tikzcd} & Y & \\ GY \arrow[ur, "g"] \arrow[rr, "G(g^*)"'] & & GFX \arrow[ul, "\lambda X"'] \end{tikzcd} .$$

In other words, we require that the mapping $g^* \to (\lambda X) \circ G(g^*)$ should be a bijection of $(FX)(Y)$ with $X(GY)$. (We are dropping subscripts for simplicity, e.g., we should really write $(FX)(Y)_k$.)

The main application we have in mind is the situation of § 1, where k is a field and $R(k)$ is an Artin local ring. If k' is any extension field of k, then $G(\text{Spec } k')$ has a single point as base space and the ring $R(k')$ as fibre over that point, i.e., $G(\text{Spec } k') = \text{Spec } R(k')$. (This is still true if we allow k' to be any Artin local algebra over k, not necessarily a field.) We then obtain a bijection of $FX(k')$ with $X(R(k'))$. Thus the rational points of X in the ring $R(k')$ are "realized" as the rational points in the field k' of a pre-schema over k. This explains our terminology.

We are treating the case where k is any commutative ring because there is no extra work required for this generality. However, this raises some questions. We no longer have $G(\text{Spec } k) = \text{Spec } R(k)$, only a canonical morphism $G(\text{Spec } k) \to \text{Spec } R(k)$. Should we take X to be a pre-schema over the local ringed space $G(\text{Spec } k)$ instead of over $R(k)$? Can Spec k be replaced by an arbitrary ground pre-schema? Future applications may determine the answers to these questions, so we proceed now without further comment.

The universal property yields at once the following uniqueness statement.

PROPOSITION 1. *If $(FX, \lambda X)$, $(F'X, \lambda' X)$ are realizations of X over k, then there is a unique k-isomorphism g^* of FX with $F'X$ such that* $(\lambda' X)\circ G(g^*) = \lambda X$.

If X admits a realization over k, we will henceforth automatically denote it by $(FX, \lambda X)$.

PROPOSITION 2. *If X, X' are pre-schemata over $R(k)$ having realizations over k, and $g\colon X \to X'$ is any $R(k)$-morphism, then there is a unique k-morphism $F(g)\colon FX \to FX'$ yielding the commutative diagram*

$$\begin{tikzcd} X \arrow[r, "g"] & X' \\ GFX \arrow[u, "\lambda X"] \arrow[r, "GF(g)"'] & GFX' \arrow[u, "\lambda X'"'] \end{tikzcd}.$$

PROOF. Apply the universal property of $(FX', \lambda X')$ to $g\circ(\lambda X)$.

The following corollaries are immediate consequences of the uniqueness (the same notation).

COROLLARY 1. *If $h\colon X' \to X''$ and X'' has a realization over k, then $F(hg) = F(h)F(g)$. If g is the identity of X, $F(g)$ is the identity of FX.*

COROLLARY 2. *If $h\colon GY \to X$, then $(gh)^* = F(g)h^*$.*

COROLLARY 3. *If k is a field and k' any Artin local algebra over k, then we have the commutative diagram*

$$\begin{array}{ccc} X(R(k')) & \xrightarrow{g(R(k'))} & X'(R(k')) \\ \downarrow & & \downarrow \\ FX(k') & \xrightarrow[F(g)(k')]{} & FX'(k') \end{array}$$

where the vertical arrows are the bijections induced by the realizations.

It follows from Corollary 1 that if, to each pre-schema X in a subcategory of pre-schemata over $R(k)$, we can associate a realization $(FX, \lambda X)$ over k, then F is a covariant functor from this category to the category of pre-schemata over k, and λ is a morphism of functors from the composite functor GF to the identity functor.

Let S^n be affine n-space over $R(k)$, so that $\Gamma(S^n) = R(k)\,[T_1, \cdots, T_n]$, the polynomial ring in n variables over the ring $R(k)$. Let R^n be the product of R with itself n times. In the ring $R(R^n)$ of all k-morphisms $R^n \to R$, we single out the projections $\pi_1, \cdots, \pi_n$, and define an $R(k)$-homomorphism φ: $\Gamma(S^n) \to R(R^n)$ by $\varphi(T_i) = \pi_i$, $i = 1, \cdots, n$. If $n = 0$, we put $S^0 = \operatorname{Spec} R(k)$, $R^0 = \operatorname{Spec} k$, $\varphi =$ identity. According to Proposition 1, § 2, there is a unique $R(k)$-morphism λ_n of GR^n into S^n such that $\Gamma(\lambda_n) = \varphi$.

PROPOSITION 3. *For all $n \geqq 0$, (R^n, λ_n) as defined above is a realization of S^n over k.*

PROOF. Given g: $GY \to S^n$. Then $\Gamma(g)(T_i)$, $i = 1, \cdots, n$ are n morphisms $Y \to R$, so by the universal property of the product, there is a unique morphism g^*: $X \to R^n$ such that $\pi_i \circ g^* = \Gamma(g)(T_i)$ for all i. Since $R(g^*)\circ\Gamma(\lambda_n) = \Gamma(g)$, the statement is true for $n > 0$. If $n = 0$, $\Gamma(g)$ can only be the canonical homomorphism $R(k) \to R(Y)$, and g^* must be the structural morphism of Y over k.

In the next result, we use our hypothesis on R.

PROPOSITION 4. *Let X be an affine schema over $R(k)$ having a realization which is an affine schema over k. Let X' be a closed sub-schema of X, j: $X' \to X$ the injection morphism. Then there is a unique closed sub-schema FX' of FX and $R(k)$-morphism $\lambda X'$: $GFX' \to X'$ such that $(FX', \lambda X')$ is a realization of X' over k and the diagram*

$$\begin{array}{ccc} X' & \xrightarrow{j} & X \\ {\scriptstyle \lambda X'}\uparrow & & \uparrow{\scriptstyle \lambda X} \\ GFX' & \xrightarrow[G(j')]{} & GFX \end{array}$$

is commutative, where j': $FX' \to FX$ is the injection morphism.

PROOF. We choose the isomorphism $R \to \Omega^n$ of our hypothesis so that under the induced bijection $R(k) \to k^n$, zero has coordinates $(0, \cdots, 0)$. Then the same property holds for the induced bijection $R(Y) \to \Omega(Y)^n$ for any k-pre-schema Y. Consider the homomorphism $\Gamma(\lambda X)$: $\Gamma(X) \to R(FX)$. If $\mathfrak{a}$ is the ideal in $\Gamma(X)$ defining X', this homomorphism carries $\mathfrak{a}$ into a certain subset $\mathfrak{a}'$ of $R(FX)$. To each $f \in \mathfrak{a}'$ corresponds by our bijection n elements $f_1, \cdots, f_n \in \Gamma(FX)$, provided we pass from $\Omega(FX)$ to $\Gamma(FX)$ by the canonical isomorphism (Proposition 3, § 3). Let $\mathfrak{b}$ be the ideal generated in $\Gamma(FX)$ by these $f_1, \cdots, f_n$ as f runs over the set $\mathfrak{a}'$, and let FX' be the closed sub-schema of FX defined by $\mathfrak{b}$. Then the composite homomorphism $\Gamma(X) \to R(FX) \to R(FX')$ vanishes on $\mathfrak{a}$ by our choice of coordinates for zero, hence there is a unique homomorphism $\Gamma(X') \to R(FX')$ giving the commutative diagram

$$\begin{tikzcd} \Gamma(X') \arrow[d] & \Gamma(X) \arrow[l] \arrow[d] \\ R(FX') & R(FX) \arrow[l] \end{tikzcd} .$$

This homomorphism determines $\lambda X'$: $GFX' \to X'$ according to Proposition 1, § 2.

Given any Y and g: $GY \to X'$, there is a unique morphism g^{**}: $Y \to FX$ such that $(\lambda X)\circ G(g^{**}) = j\circ g$. The homomorphism $R(g^{**})$: $R(FX) \to R(Y)$ vanishes on $\mathfrak{a}'$, so that $\Gamma(g^{**})$ vanishes on $\mathfrak{b}$. We then obtain by passage to the quotient a homomorphism $\Gamma(FX') \to \Gamma(Y)$, and this homomorphism has the form $\Gamma(g^*)$ for some morphism g^*: $Y \to FX'$ since FX' is affine. This g^* satisfies the universal property.

If $(F'X', \lambda'X')$ were another such realization, j'': $F'X' \to FX$ the injection morphism, then the universal property yields an isomorphism h: $F'X' \to FX'$ such that $(\lambda X')\circ G(h) = \lambda' X'$. But uniqueness implies $j'' = j'\circ h$, so that $F'X' \leqq FX'$ in Grothendieck's notation. The situation being symmetric, we must have h = identity.

COROLLARY 1. *Every affine schema of finite type over $R(k)$ has a realization which is an affine schema of finite type over k.*

PROOF. Every such schema is isomorphic to a closed sub-schema of some S^n. Apply Proposition 3.

COROLLARY 2. *Let X, X' be affine schemata of finite type over $R(k)$, g: $X' \to X$ a closed immersion. Then $F(g)$: $FX' \to FX$ is a closed immersion.*

PROOF. Let gX' be the closed sub-schema of X isomorphic to X' under

g. Taking $F(gX')$ to be the closed sub-schema of FX in Proposition 4, the functoriality of F implies that $F(g)$ is an isomorphism of FX' with $F(gX')$.

PROPOSITION 5. *Let X be any pre-schema over $R(k)$ having a realization over k. Let X' be any open subpre-schema of X, and let FX' be the open subpre-schema of FX having $(\lambda X)^{-1}(b(X'))$ as base space. Then there is a unique $R(k)$-morphism $\lambda X'$ providing the commutative diagram*

$$\begin{array}{ccc} X' & \xrightarrow{j} & X \\ {\scriptstyle \lambda X'}\uparrow & & \uparrow{\scriptstyle \lambda X} \\ GFX' & \xrightarrow[G(j')]{} & GFX \end{array}$$

(where j and j' are the injection morphisms), and $(FX', \lambda X')$ is a realization of X' over k. These two properties characterize FX' among subpre-schemata of FX.

PROOF. Take $\lambda X'$ to be the restriction of λX to the open subspace $(\lambda X)^{-1}(b(X'))$. If Y, g, g^{**} are as in the proof of Proposition 4, it is clear that g^{**} maps into FX', so can be factored into $g^{**} = j' \circ g^*$, $g^*\colon Y \to FX'$. Uniqueness is shown as in Proposition 4.

COROLLARY 1. *If X, X' are pre-schemata over $R(k)$ having realizations over k, and $g\colon X' \to X$ is an open immersion, then $F(g)\colon FX' \to FX$ is an open immersion.*

PROOF. The argument is the same as for closed immersions.

COROLLARY 2. *Every sub-schema of an affine schema of finite type over $R(k)$ admits a realization over k.*

PROOF. Such a schema is isomorphic to an open sub-schema of an affine schema of finite type. Apply Proposition 5 and Corollary 1 to Proposition 4.

COROLLARY 3. *Let X be a pre-schema over $R(k)$ admitting a realization over k. For any open subpre-schema X' of X, let FX' be the open subpre-schema of FX defined in Proposition 5. If F is considered as a function from the lattice of opens in X to the lattice of opens in FX, then F is a lattice homomorphism.*

PROOF. Inverse images by λX preserve all lattice properties.

An examination of the proof of Proposition 4 shows that we cannot make the same statement about closed subpre-schemata. If X' is closed in X as in Proposition 4, then $b(FX')$ is contained in but does not coincide with the pre-image of $b(X')$ by λX. In fact, $b(FX')$ may even be empty though X' is not.

PROPOSITION 6. *Let S be a pre-schema over $R(k)$, $X_1\, X_2$ pre-schemata over S. Assume X_1, X_2, S have realizations over k. Let $Z = FX_1 \times_{FS} FX_2$, with projections ρ_i, $i = 1, 2$. Then there is a unique $R(k)$-morphism $\zeta: GZ \to X_1 \times_S X_2$ such that $\pi_i\zeta = (\lambda X_i)\circ G(\rho_i)$, $i = 1, 2$ (where π_1, π_2 are the projections of $X_1 \times_S X_2$), and (Z, ζ) is a realization of $X \times_S X'$ over k. If $\big(F(X_1 \times_S X_2), \lambda(X_1 \times_S X_2)\big)$ is any realization of $X_1 \times_S X_2$ over k, there is a unique isomorphism g of $F(X_1 \times_S X_2)$ with Z such that $\lambda(X_1 \times_S X_2) = \zeta\circ G(g)$, $\rho_i g = F(\pi_i)$, $i = 1, 2$.*

PROOF. The existence and uniqueness of ζ follow from Proposition 2, § 2. The rest of the argument is purely a formal manipulation of the universal properties of products and realizations.

The following corollaries can be deduced at once from the fact that F is a functor (Corollary 1 to Proposition 2). All pre-schemata mentioned are assumed to have realizations over k.

COROLLARY 1. *If g_i: $X \to X_i$ are S-morphisms, $i = 1, 2$, then $F\big((g_1, g_2)_S\big) = \big(F(g_1), F(g_2)\big)_{F(S)}$.*

COROLLARY 2. *If Γ_g: $X_1 \to X_1 \times_S X_2$ is the graph of an S-morphism g: $X_1 \to X_2$, then $F(\Gamma_g)$ is the graph of $F(g)$.*

COROLLARY 3. *Given g_i: $X_i' \to X_i$, $i = 1, 2$, S-morphisms, we have $F(g_1 \times_S g_2) = F(g_1) \times_{F(S)} F(g_2)$.*

As special cases of the compatibility of F with fibre products, we have

(1) compatibility of F with products (take $S =$ Spec $R(k)$),

(2) compatibility of F with pre-images (given a morphism g: $X_1 \to S$, with X_2 a subpre-schema of S), and

(3) compatibility of F with intersection (or greatest lower bound) of subpre-schemata (X_1, X_2 are subpre-schemata of S).

COROLLARY 4. *If X is a group pre-schema over $R(k)$ having a realization over k, then FX is a group pre-schema over $R(k)$, and for every k-pre-schema Y, the bijection $(GY)(X) \to (FX)(Y)$ induced by λX is an isomorphism of groups.*

This follows at once from the functoriality of F with respect to products.

The argument in the next proposition is purely formal and belongs in the general theory of local categories.

PROPOSITION 7. *Every pre-schema of finite type over $R(k)$ has a realization over k.*

PROOF. Let X be the given pre-schema, and let $(X_i)_{i\in I}$ be a covering of X by affine opens. Let t_i: $X_i \to X$ be the injection morphism, X_{ij} the sub-schema $X_i \cap X_j$ of X considered as a sub-schema of X_i, t_{ji}: $X_{ij} \to X_{ji}$ the isomorphism obtained by restricting $t_j^{-1}t_i$. These isomorphisms satisfy the compatibility condition $t_{ji}t_{ih} = t_{jh}$ in $X_{hij} = X_{hi} \cap X_{hj}$.

By Corollary 2 to Proposition 5, there exist realizations of X_i, and by Proposition 5 we can choose realizations of X_{ij}, X_{ijh} which are open subpre-schemata of FX_i. Applying the functor F, we obtain isomorphisms $F(t_{ji})$: $FX_{ij} \to FX_{ji}$ satisfying the compatibility conditions $F(t_{ji})F(t_{ih}) = F(t_{jh})$ on FX_{hij}. We can therefore apply the standard construction lemma for piecing together ringed spaces (cf., [11, no. 4], or [1, exposé 2, Proposition 4]) and obtain a ringed space FX over k, an open covering $(Z_i)_{i\in I}$ of FX, and for each i an isomorphism s_i of Z_i with FX_i such that $s_i(Z_i \cap Z_j) = FX_{ij}$ and $F(t_{ij}) = s_i s_j{}^{-1}$. The isomorphisms s_i show that FX is a pre-schema over k, since each FX_i is an affine schema over k (Corollary 1 to Proposition 4).

We can define an $R(k)$-morphism λX: $GX \to X$ by requiring that λX coincide with $t_i \circ (\lambda X_i) \circ G(s_i)$ on Z_i. Let Y be any pre-schema over k and g: $GY \to X$ any $R(k)$-morphism. Let Y_i be the open subpre-schema of Y whose base space is $g^{-1}(X_i)$. Then composing the restriction of g with t_i^{-1} gives an $R(k)$-morphism g_i: $GY_i \to X_i$ for each i. There results a k-morphism g_i^*: $Y_i \to FX_i$ such that $(\lambda X_i) \circ G(g_i^*) = g_i$. We define a k-morphism g^*: $Y \to FX$ by requiring that the restriction of g^* to Y_i be $s_i^{-1}g_i^*$. This makes sense, i.e., g_i^* and g_j^* coincide on $Y_i \cap Y_j$, because of the uniqueness of the solution to the universal problem for the morphism $Y_i \cap Y_j \to FX_{ij}$. Then $(\lambda X) \circ G(g^*) = g$. Thus $(FX, \lambda X)$ is a realization of X over k.

PROPOSITION 8. *If X', X are pre-schemata over $R(k)$ of finite type, and g: $X' \to X$ is a closed immersion, then $F(g)$: $FX' \to FX$ is a closed immersion.*

PROOF. Let $(X_i)_{i\in I}$ be a covering of X by affine opens. Then $(FX_i)_{i\in I}$ is a covering of FX by affine opens (Corollary 3 to Proposition 5), $F(g^{-1}(X_i)) = F(g)^{-1}(FX_i)$ (Proposition 6, remark (2)), and the $F(g)^{-1}(FX_i)$ form an open covering of FX'. The restriction of $F(g)$ to $F(g^{-1}(X_i))$ is a closed immersion in FX_i (Corollary 2 to Proposition 4). Therefore $F(g)$ is a closed immersion by a result of Grothendieck [5, I, Corollary 4.2.4].

COROLLARY 1. *If g: $X' \to X$ is an immersion, so is $F(g)$: $FX' \to FX$.*

Because an arbitrary immersion can be factored into a closed immersion and an open immersion (use Corollary 1 to Proposition 5).

COROLLARY 2. *If X is separated over S then FX is separated over*

FS. In particular, FX is a schema if X is.

PROOF. By Corollary 2 to Proposition 6, F takes the diagonal morphism into the diagonal morphism.

We summarize our results.

THEOREM. *Let R be a local ring schema over the commutative ring k satisfying Hypothesis* I. *Let G be the functor induced by R from pre-schemata over k to local ringed spaces over the ring $R(k)$* (cf. § 3). *Then there is a functor F from the category of pre-schemata of finite type over $R(k)$ to the category of pre-schemata of finite type over k, and a morphism λ of the functor GF to the identity, such that*

1. $g^* \to (\lambda X)\circ G(g^*)$ *is a bijection of $Y(FX)$ with $X(GY)$ (which is an isomorphism of these bifunctors).*
2. *F preserves open, closed, and arbitrary immersions, and is compatible with the lattice operations on open subpre-schemata.*
3. *F preserves fibre products.*
4. *F takes schemata into schemata, affine schemata into affine schemata.*
5. *F takes the category of group pre-schemata over $R(k)$ into the category of group pre-schemata over k, and the bijection of property* 1 *is a group isomorphism when X is a group pre-schema.*
6. $F(S^n) = R^n$ *for all* $n > 0$ *(where S^n is the affine n-space over $R(k)$), and* $F(\mathrm{Spec}\, R(k)) = \mathrm{Spec}\, k$.
7. *If (F', λ') is another such pair having property* 1, *then there is a unique isomorphism of functors σ from F to F' such that* $\lambda'\circ G(\sigma) = \lambda$.

5. Change of local ring schema

In this section we assume given two local ring schemata R, R' over the commutative ring k, both satisfying Hypothesis I of § 4, together with a homomorphism $\varphi\colon R \to R'$. For example, we can place ourselves in the situation of § 1, take $R' = \Omega$, and take $\varphi\colon R \to \Omega$ to be the canonical homomorphism corresponding to the homomorphism of $R(k) = \mathfrak{R}$ on its residue field k. We now put subscripts R and R' to distinguish functors corresponding to these local ring schemata.

We then have the extension of base functor E_φ assigning to each pre-schema X over $R(k)$ a pre-schema $E_\varphi(X) = X^\varphi$ over $R(k)$, together with an $R(k)$-morphism $\rho_\varphi(X)\colon X^\varphi \to X$ (each pre-schema over $R'(k)$ can be regarded as a pre-schema over $R(k)$ by means of $\varphi(k)$). These have the following universal property: Given a pre-schema X' over $R'(k)$ and an $R(k)$-morphism $g\colon X' \to X$, there is a unique $R'(k)$-morphism $g^\varphi\colon X' \to X$ such that $\rho_\varphi(X)\circ g^\varphi = g$. By Proposition 2, § 2, this universal property extends to the larger category of local ringed spaces X' over $R'(k)$. In particular, we deduce

PROPOSITION 1. *Let X be a pre-schema over $R(k)$, Y a pre-schema over k. Given an $R(k)$-morphism g: $G_R Y \to X$, there is a unique $R'(k)$-morphism g^φ: $G_{R'} Y \to X^\varphi$ providing the commutative diagram*

$$\begin{tikzcd} G_{R'} Y \arrow[r, "g^\varphi"] \arrow[d, "G_\varphi(Y)"'] & X^\varphi \arrow[d, "\rho_\varphi(X)"] \\ G_R Y \arrow[r, "g"'] & X \end{tikzcd} .$$

Given a pre-schema X of finite type over $R(k)$, we can apply the functor F_R of § 4 to obtain a pre-schema $F_R X$ over k. We can also extend X to a pre-schema X^φ over $R'(k)$ and then apply the functor $F_{R'}$ to X^φ, obtaining another pre-schema $F_{R'} X^\varphi$ over k. The next result shows the existence of a connecting morphism $F_\varphi(X)$: $F_R X \to F_{R'} X^\varphi$.

PROPOSITION 2. *Let X be a pre-schema of finite type over $R(k)$. With notations as above, there is a unique k-morphism $F_\varphi(X)$: $F_R X \to F_{R'} X^\varphi$ satisfying the commutative diagram*

$$\begin{tikzcd} X & X^\varphi \arrow[l, "\rho_\varphi(X)"'] & \\ G_R F_R X \arrow[u, "\lambda_R X"] & G_{R'} F_R X \arrow[l, "G_\varphi(F_R X)"] \arrow[u, "\lambda_\varphi(X)"'] \arrow[r, "G_{R'}(F_\varphi(X))"'] & G_{R'} F_{R'} X^\varphi \arrow[ul, "\lambda_{R'} X^\varphi"'] \end{tikzcd} .$$

If W is an affine open in X, then the pre-image $F_\varphi(X)^{-1}(F_R W^\varphi)$ of the affine open $F_R W^\varphi$ in $F_{R'} X^\varphi$ is the affine open $F_R W$ in $F_R X$.

PROOF. The morphism $\lambda_\varphi(X)$ is that deduced from Proposition 1. The morphism $F_\varphi(X)$ is then provided by the universal property of the realization of X^φ. The second assertion follows from the commutative diagram.

COROLLARY 1. *If $X = S^1$ (affine line over $R(k)$), so that $F_R X = R$ and $F_{R'} X^\varphi = R'$, then $F_\varphi(S^1) = \varphi$.*

COROLLARY 2. *If S is a pre-schema of finite type over $R(k)$, X_i pre-schemata of finite type over S, $i = 1, 2$, then $F_\varphi(X_1 \times_S X_2) = F_\varphi(X_1) \times_{F_\varphi(S)} F_\varphi(X_2)$. In particular if S^n is affine n-space over $R(k)$, $F_\varphi(S^n) = \varphi^n$.*

COROLLARY 3. *Given a pre-schema Y over k and an $R(k)$-morphism g: $G_R Y \to X$. Let g^*: $Y \to F_R X$ be the k-morphism g induces by the uni-*

versal property of the realization of X. Let $g^{\varphi}\colon G_{R'}Y \to X^{\varphi}$ *be as in Proposition* 1, *and* $(g^{\varphi})^{*}\colon Y \to F_{R'}X^{\varphi}$ *the k-morphism* g^{φ} *induces by the universal property of the realization of* X^{φ}. *Then* $(g^{\varphi})^{*} = F_{\varphi}(X)\circ g^{*}$.

COROLLARY 4. *Let* X_1, X_2 *be pre-schemata of finite type over* $R(k)$, $g\colon X_1 \to X_2$ *an* $R(k)$*-morphism. Then we have the commutative diagram*

$$\begin{array}{ccc} F_R X_1 & \xrightarrow{F_R(g)} & F_R X_2 \\ {\scriptstyle F_{\varphi}(X_1)}\downarrow & & \downarrow{\scriptstyle F_{\varphi}(X_2)} \\ F_{R'} X_1^{\varphi} & \xrightarrow[F_{R'}(g^{\varphi})]{} & F_{R'} X_2^{\varphi} \end{array} .$$

COROLLARY 5. *If X is a group pre-schema of finite type over* $R(k)$, *then* $F_{\varphi}(X)$ *is a homomorphism.*

These corollaries all follow from uniqueness, except Corollary 5 which follows from Corollary 2 and Corollary 4.

In the case where k is a field, k' any extension field of (or Artin local algebra over) k, Corollary 3 gives us the commutative diagram

$$\begin{array}{ccc} X(R(k')) & \longrightarrow & X^{\varphi}(R'(k')) \\ \downarrow & & \downarrow \\ (F_R X)(k') & \longrightarrow & (F_{R'} X^{\varphi})(k') \end{array} .$$

Suppose we have another local ring schema R'' over k satisfying Hypothesis I and a homomorphism $\varphi'\colon R' \to R''$.

PROPOSITION 3. *If X is a pre-schema of finite type over* $R(k)$, *then* $F_{\varphi'}(X^{\varphi})F_{\varphi}(X) = F_{\varphi'\varphi}(X)$.

We leave to the reader the task of verifying this formula, remarking that it can be accomplished by a beautiful commutative diagram relating the seventeen functors and nine morphisms of functors which we have accumulated at this point.

6. Projective limits

Let $\mathfrak{R}$ be a complete Noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k assumed perfect. Let $\psi\colon \mathfrak{R} \to k$ be the canonical homomorphism. If $\mathfrak{R}$ has the same characteristic as k, then there exists a monomorphism $\mu\colon k \to \mathfrak{R}$ such that $\psi\mu =$ identity, so that $\mathfrak{R}$ has a structure of k-algebra. If k has characteristic $p > 0$ or if k is an algebraic extension of the rational numbers, μ is uniquely determined; if k is a transcendental extension of the rational numbers, we choose one such μ so as to endow $\mathfrak{R}$ with a structure of k-algebra. If $\mathfrak{R}$ and k have different characteristics, there is a canonical injection $\mu\colon k \to \mathfrak{R}$ satisfying $\psi\mu =$

identity, $\mu(0) = 0$, and whose restriction to the multiplicative group of k is a monomorphism into the group of units of $\mathfrak{R}$. The case where $\mathfrak{R}$ has characteristic p^n was discussed in § 1, so we can assume R has characteristic 0 and k characteristic p. Then $\mu(k)$ coincides with $\mathfrak{R}^{p^\infty}$, the set of all elements of $\mathfrak{R}$ having a p^{nth} root in $\mathfrak{R}$ for all $n > 0$. The elements of $\mathfrak{R}$ of type $a = \sum_{n=0}^{\infty} \mu(a_n)p^n$ form a subring which is a complete discrete valuation ring with maximal ideal generated by p and residue field k (these power series make sense because $\mathfrak{R}$ is complete); this subring is canonically isomorphic to the ring $W_\infty(k)$ of Witt vectors of infinite length. Thus $\mathfrak{R}$ is canonically endowed with a structure of $W_\infty(k)$-algebra. (For these results, see [3].)

We are, therefore, in a more general situation than § 1, where we restricted our attention to Artin local rings (which are complete because discrete). The only change is that now we allow n to take the value ∞, and $\mathfrak{R}$ is not necessarily a module of finite type over $W_n(k)$. Our conventions of § 1 shall remain in force in this more general situation: "Homomorphism" means "$W_n(k)$-algebra homomorphism," where we assume chosen a structure of $W_0(k) = k$-algebra on $\mathfrak{R}$ if k has characteristic 0, and all extensions of k are assumed perfect.

The notion of completeness can be formulated in terms of projective limits. For each $\mathfrak{m}$-primary ideal $\mathfrak{a}$ in $\mathfrak{R}$, the residue ring $\mathfrak{R}/\mathfrak{a}$ is an Artin local ring. If $\mathfrak{a}' \subset \mathfrak{a}$ is another ideal of definition of $\mathfrak{R}$, there is a canonical epimorphism $\varphi_{\mathfrak{a},\mathfrak{a}'}$: $\mathfrak{R}/\mathfrak{a}' \to \mathfrak{R}/\mathfrak{a}$. These residue rings then form a projective system of rings and homomorphisms indexed by the family of $\mathfrak{m}$-primary ideals in $\mathfrak{R}$, and to say that $\mathfrak{R}$ is complete is equivalent to saying that the canonical monomorphism of $\mathfrak{R}$ into the projective limit of this system is an isomorphism, i.e., is surjective.

The homomorphisms $\varphi_{\mathfrak{a}}$: $\mathfrak{R} \to \mathfrak{R}/\mathfrak{a}$ in the structure of projective limit are simply the canonical homomorphisms of $\mathfrak{R}$ on its residue rings. We can also obtain $\mathfrak{R}$ as a projective limit by restricting ourselves to the cofinal system of powers $\mathfrak{m}^\nu$ of the maximal ideal $\mathfrak{m}$.

Let k' be an extension field of k. If we form the extensions $(\mathfrak{R}/\mathfrak{a})^{k'}$ as in § 1 (we use k' as a superscript now) and the corresponding extensions of the homomorphisms $\varphi_{\mathfrak{a},\mathfrak{a}'}$, we obtain a new projective system of Artin local rings with residue field k'. The projective limit of this system will be denoted $\mathfrak{R}^{k'}$. The limit of the injections $\mathfrak{R}/\mathfrak{a} \to (\mathfrak{R}/\mathfrak{a})^{k'}$ (§ 1, Proposition 2) is an injection of $\mathfrak{R}$ in $\mathfrak{R}^{k'}$ by means of which we identify $\mathfrak{R}$ as a subring of $\mathfrak{R}^{k'}$.

PROPOSITION 1. *$\mathfrak{R}^{k'}$ is a complete local Noetherian ring with residue field k' and maximal ideal generated by the maximal ideal of $\mathfrak{R}$.*

PROOF. That $\mathfrak{R}^{k'}$ is complete Noetherian follows from general facts in [5, **0**, 7.2]. The limit of the maximal ideals in $(\mathfrak{R}/\mathfrak{a})^{k'}$ is a maximal ideal in $\mathfrak{R}^{k'}$ whose residue field is k'. The generators of this ideal can be taken in $\mathfrak{m}$ by a standard refinement argument [5, **0**, proof of 7.2.9]. Any element x outside this ideal has an inverse modulo $\mathfrak{a}$ for all $\mathfrak{a}$, and this projective system of inverses gives the inverse of x, so that $\mathfrak{R}^{k'}$ is local.

Note. Since each $(\mathfrak{R}/\mathfrak{a})^{k'}$ is an algebra over $W_n(k')$, so is $\mathfrak{R}^{k'}$. We therefore have a canonical homomorphism of $\mathfrak{R} \otimes_{W_n(k)} W_n(k')$ into $\mathfrak{R}^{k'}$. This homomorphism is a monomorphism, and the image is a dense subring of $\mathfrak{R}^{k'}$; if k' is a finite extension of k, the image is all of $\mathfrak{R}^{k'}$. Moreover, let $\mathfrak{S}$ be an extension ring of $\mathfrak{R}$ satisfying the conditions of Proposition 1, with a structure of $W_n(k')$-algebra inducing the given structure of $W_n(k)$-algebra on $\mathfrak{R}$ (the latter assumption is needed only when k has characteristic 0). Then there is a unique $\mathfrak{R} \otimes_{W_n(k)} W_n(k')$-isomorphism of $\mathfrak{R}^{k'}$ on $\mathfrak{S}$.

Next let X be any pre-schema over $\mathfrak{R}$. For each $\mathfrak{m}$-primary ideal $\mathfrak{a}$, we obtain a pre-schema $X_\mathfrak{a}$ over $\mathfrak{R}/\mathfrak{a}$ from X by extension of the base ring, with an $\mathfrak{R}$-morphism $\varphi_\mathfrak{a}(X)$: $X_\mathfrak{a} \to X$. If $\mathfrak{a}' \subset \mathfrak{a}$, we have an $\mathfrak{R}/\mathfrak{a}'$-morphism $\varphi_{\mathfrak{a}',\mathfrak{a}}(X)$: $X_\mathfrak{a} \to X_{\mathfrak{a}'}$ such that $\varphi_{\mathfrak{a}'}(X)\varphi_{\mathfrak{a}',\mathfrak{a}}(X) = \varphi_\mathfrak{a}(X)$. These operations are functorial in X. In particular, if k' is any extension field of k, there is a canonical map of the set $X(\mathfrak{R}^{k'})$ of points of X in $\mathfrak{R}^{k'}$ into the set $X\big((R/\mathfrak{a})^{k'}\big)$ for each $\mathfrak{a}$, and if $\mathfrak{a}' \subset \mathfrak{a}$ we have a mapping of $X_{\mathfrak{a}'}\big((\mathfrak{R}/\mathfrak{a}')^{k'}\big)$ into $X_\mathfrak{a}\big((\mathfrak{R}/\mathfrak{a})^{k'}\big)$ satisfying the commutative diagram

$$\begin{tikzcd} & X(\mathfrak{R}^{k'}) \arrow[dl] \arrow[dr] & \\ X_\mathfrak{a}\big((\mathfrak{R}/\mathfrak{a})^{k'}\big) & & X_{\mathfrak{a}'}\big((\mathfrak{R}/\mathfrak{a}')^{k'}\big) \arrow[ll] \end{tikzcd} .$$

These mappings induce a mapping $\beta(X)$ of $X(\mathfrak{R}^{k'})$ into the projective limit $\lim_\mathfrak{a} X\big((\mathfrak{R}/\mathfrak{a})^{k'}\big)$ of these sets.

PROPOSITION 2. *$\beta(X)$ is a bijection.*

PROOF. We use the fact that given a local ring A and a pre-schema X, there is a canonical one-to-one correspondence between the morphisms Spec $A \to X$ and those homomorphisms h: $O(x) \to A$ under which the preimage of the maximal ideal is the maximal ideal, x running over the base space of X. Since $\mathfrak{R}^{k'}$ is the projective limit of the local rings $(\mathfrak{R}/\mathfrak{a})^{k'}$, $\beta(X)$ gives a bijection between such homomorphisms into $\mathfrak{R}^{k'}$ and a projective system of homomorphisms into the $(\mathfrak{R}/\mathfrak{a})^{k'}$.

SUMMARY

Notation as above. Assume X is a pre-schema of finite type over $\mathfrak{R}$. Let $F_\mathfrak{a}X$ be the realization of $X_\mathfrak{a}$ over k, $\Phi_{\mathfrak{a}'\mathfrak{a}}X$: $F_\mathfrak{a}X \to F_{\mathfrak{a}'}X$ the con-

necting morphism defined in § 5 *corresponding to the canonical homomorphism* $\mathfrak{R}/\mathfrak{a}' \to \mathfrak{R}/\mathfrak{a}$. *Then the bijections* $X_{\mathfrak{a}}((\mathfrak{R}/\mathfrak{a})^{k'}) \to F_{\mathfrak{a}}X(k')$ *induced by the realizations are compatible with the connecting maps, and define a bijection of the projective limits* $\lim_{\mathfrak{a}} X_{\mathfrak{a}}((\mathfrak{R}/\mathfrak{a})^{k'}) \to \lim_{\mathfrak{a}} F_{\mathfrak{a}}X(k')$. *If we precede this with the bijection* $\beta(X)$, *we obtain a bijection* $X(R^{k'}) \to \lim_{\mathfrak{a}} F_{\mathfrak{a}}X(k')$ *which is functorial in* X. *If* X *is a group pre-schema, so is each* $F_{\mathfrak{a}}X$, *each* $\Phi_{\mathfrak{a}'\mathfrak{a}}X$ *is a homomorphism, and the bijection* $X(R^{k'}) \to \lim_{\mathfrak{a}} F_{\mathfrak{a}}X(k')$ *is an isomorphism of groups.*

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