Fundamental group-scheme of some curve-connected varieties and associated ones

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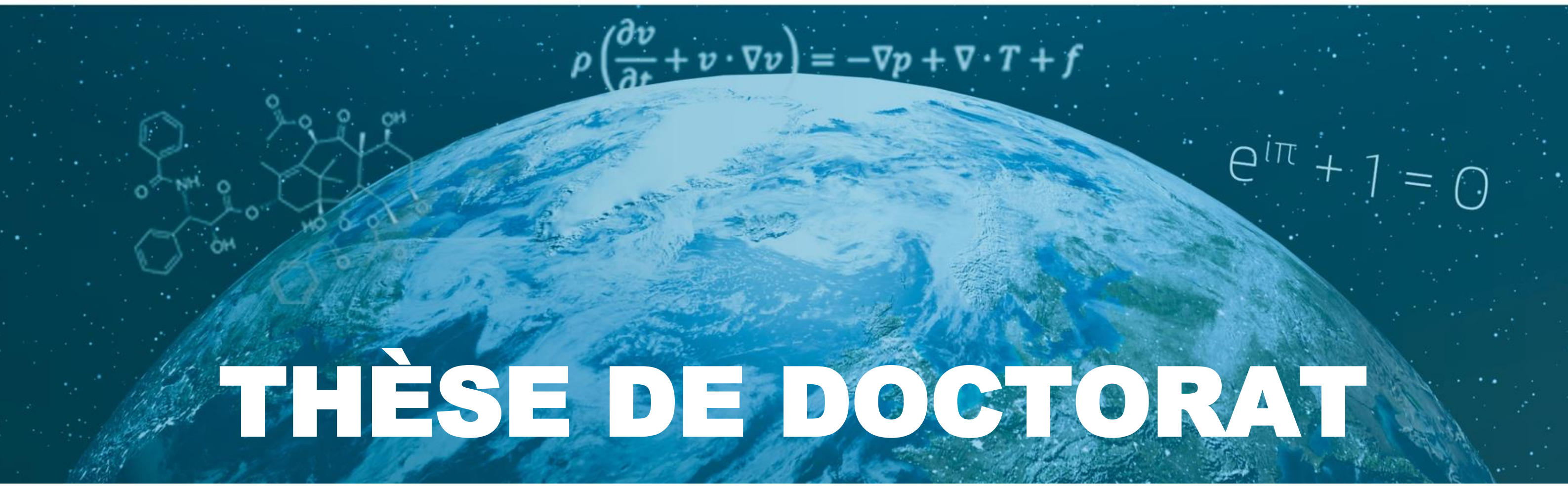

THÈSE DE DOCTORAT

Schéma en groupes fondamental de quelques variétés connexes par courbes et associées

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FUNDAMENTAL GROUP-SCHEME OF SOME CURVE-CONNECTED VARIETIES AND ASSOCIATED ONES

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ABSTRACT

Titre: Schéma en groupes fondamental de quelques variétés connexes par courbes et associées

Resumé

Dans ce travail de thèse on étudie le schéma en groupes fondamental des variétés connexes par courbes ou qui sont associées à ces variétés. Les variétés connexes par courbes sont la généralisation des variétés rationnellement connexes, dont la définition a été conçu par J. Kollár. Ces notions sont les plus proches en géométrie algébrique à la notion de connexité par arcs en topologie, car sur un corps algébriquement clos (non dénombrable), par deux points très généraux d'une variété connexe par courbes (par chaînes resp.) il existe une courbe (chaîne de courbes resp.) avec un morphisme vers la variété dont l'image contient les deux points-ci considérés. En dépendant du type de courbes qu'on considère, on a les notions de g-connexité (par chaînes resp.) où on considère exclusivement des courbes (chaînes de courbes resp.) où chaque composante irréductible est une courbe lisse et projective de genre g, et la notion de la C-connexité pour une courbe fixe C où par deux points très généraux on peut faire passer l'image d'un morphisme depuis la courbe C.

En utilisant des résultats classiques et récents de la théorie des schémas en groupes fondamentaux, qui classifient des torseurs sous l'action d'un schéma en groupes affine, notamment le schéma en groupes fondamental de Nori et le S-schéma en groupes fondamental, on essaie à décrire le schéma en groupes fondamental de Nori des certains types des variétés connexes par courbes, dont le cas rationnellement connexe est déjà connu, et ceux des certaines variétés associées.

Pour obtenir ces résultats, on utilise tous les aspects qui interviennent dans la théorie du schéma en groupes fondamental: les schémas en groupes affines, les catégories tannakiennes des fibrés vectoriels sur des variétés propres et la théorie des torseurs affines. En plus, on construit des nouveaux schémas en groupes fondamentaux associés aux catégories tannakiennes des fibrés pour des variétés où tout pair de points peut être connecté par des chaînes de courbes appartenant à des familles arbitraires de courbes, ce qui généralise une construction récente de I. Biswas, P.H. Hai et J.P. Dos Santos et qui pourrait fournir un nouveau cadre pour l'étude des schémas en groupes fondamentaux des variétés connexes par courbes.

Plus spécifiquement, on propose deux approches différentes pour décrire ces schémas en groupes fondamentaux, appliquer le nouveau cadre des schémas en groupes fondamentaux décrit dans le paragraphe précédent aux variétés g-connexes, et utiliser la fibration rationnellement connexe maximale et décrire le comportement du schéma en groupes fondamental sur cette fibration. Inspiré par la deuxième approche, on décrit le schéma en groupes fondamental des fibrations sur des variétés abéliennes avec fibres rationnellement connexes, inspiré par la description des variétés elliptiquement connexes en caractéristique zéro par F. Gounelas. Ces variétés ne sont pas nécessairement elliptiquement connexes en caractéristique positive, mais la description de ses schémas en groupes fondamentaux est possible avec la suite exacte d'homotopie.

Mot clés: Géométrie algébrique, Schéma en groupes fondamental, Catégories tannakiennes, Courbe algébrique, Variétés connexes par courbes.

Title: Fundamental group-scheme of some curve-connected varieties and associated ones

Abstract

In this thesis work we study the fundamental group-scheme of curve-connected varieties or associated to them. Curve-connected varieties are the generalization of rationally connected varieties, whose definition was conceived by J. Kollár. These notions are the closest ones in algebraic geometry, to the notion of arc connectedness in topology, because over an algebraically closed field (uncountable), over any pair of two very general points in a curve-connected variety (resp. chain-connected), there exists a curve (resp. chain of curves) with a morphism to the variety whose image contains the two points mentioned before. Depending on the type of curves we consider, we have the notions of g-connectedness (resp. chain g-connectedness) where we consider exclusively curves (resp. chains of curves) with irreducible components are smooth and projective curves of genus g, and the notion of C-connectedness for a fixed curve C where over any two very general points, we can contain them in the image of a morphism from C to the variety.
Using classical and recent results from the theory of fundamental group-schemes, which classifies torsors under the action of an affine group-scheme, notably Nori fundamental group-scheme and the S-fundamental group-scheme, we try to describe the Nori fundamental group-scheme of certain types of curve-connected varieties, for which the rationally connected case is known, and some associated varieties. To obtain these results, we use all the aspects that play a role in the theory of the fundamental group-scheme: affine group-schemes, tannakian categories of vector bundles over proper varieties, and the theory of affine torsors. Moreover, we build new fundamental group-schemes associated to tannakian categories of vector bundles over

varieties where we can join any pair of points by a chain of curves belonging to arbitrary families of curves, generalizing a recent construction of I.Biswas, P.H. Hai and J.P. Dos Santos which could provide a new framework for the study of fundamental group-schemes of curve-connected varieties.
More specifically, we propose two different approaches to understand these fundamental group-schemes, apply the new framework for fundamental group-schemes described in the paragraph above for g-connected varieties and to utilize the maximal rationally connected fibration and describe the behaviour of the fundamental group over it. Inspired by the second approach, we describe the fundamental group-scheme of fibrations over elliptic curves with rationally connected fibers, inspired by the description of elliptically connected varieties in characteristic zero made by F. Gounelas. These varieties are not necessarily elliptically connected in positive characteristic, but the description of their fundamental group-schemes is possible with the homotopy exact sequence.

Keywords: Algebraic geometry, Fundamental group-scheme, Tannakian categories, Algebraic curve, Curve-connected varieties.

Dedicado a mi Gola mami, mi Papi y mi Linditi.

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aunque hayamos hecho una “prueba de convivencia” en Costa Rica. Después de todos estos años, este departamento no puede estar más hermoso: lleno de cosas de Harry Potter, tus peluches, tus repisas, tus decoraciones y tus flores. Eres capaz de ir a cualquier esquina del mundo conmigo y hacerla un hogar, sin dudar, y tomando muchos más riesgos que otras personas. De verdad, no creo que hubiese podido terminar la tesis sin ti, para animarme, para decirme que todo va a estar bien, para hacerme entender que estaba haciendo mal y asi mejorarlo, para iluminar mi día con tus demostraciones de ternura y amor, para pedirme que te “mirara”, entre muchas otras cosas que han hecho estos años los mejores contigo. Espero que atesores estos años tanto como yo, estas “vacaciones de la vida” como tu los llamaste alguna vez. No necesitaremos vacaciones si siempre vivimos en la primavera, con las flores de nuestro amor y las que vayas a plantar en el jardín. Te amo, contigo por siempre y por donde nos lleve la vida, juntos.

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ACRONYMS

EF Essentially finite

FGS Fundamental group-scheme

NR Nori-reduced

RC Rationally connected

RCC Rationally chain connectec

SRC Separably rationally connected

SRCC Separably rationally chain connected

1 INTRODUCTION

In [52] and [53], M.V. Nori developed the fundamental group-scheme[1] or Nori fundamental group-scheme to avoid confusion with other fundamental group-schemes, a pro-finite affine group-scheme $\pi^N(X,x)$ that classifies pointed finite torsors in terms of group-schemes by associating G-torsors with morphisms $\pi^N(X,x) \to G$ when G is a finite group-scheme, over reduced and connected schemes over a field after fixing rational point.
Not only that, but following the spirit of the étale fundamental group-scheme of Grothendieck, it is also a "group-scheme of automorphisms for the fiber functor", under additional hypotheses for the base scheme. By group-scheme automorphism we mean that the fundamental group-scheme is the group-scheme associated to a neutral tannakian category, and tannakian categories always come with a "fiber functor" that defines a group-scheme in terms of automorphisms of this functor. Thus, we see that the FGS sits in the confluence of three mathematical objects: torsors, group-schemes and neutral tannakian categories.
The tannakian category we are considering in this case, is the category of "essentially finite vector bundles". The building blocks of essentially finite vector bundles are finite bundles, which are vector bundles that satisfy polynomial identities $p(\mathcal{E}) \cong q(\mathcal{E})$ where $p(x)$ and $q(x)$ are two different polynomials with non-negative integer coefficients, and sums and powers are interpreted as direct sums and tensor product powers respectively. Essentially finite bundles are quotients of sub-bundles of finite bundles, or kernels between vector bundle morphisms between two finite bundles in more recent formulations of the theory. Here we find a glaring issue: how are we so sure that essentially finite bundles are indeed vector bundles? In the category of vector bundles over a scheme X, with morphism of $\mathcal{O}_X$-modules as morphisms, it is not always true that kernels and cokernels of morphisms are again vector bundles, they are certainly quasi-coherent, but we need additional information to obtain vector bundles.
Nori introduced a supplementary type of vector bundle, that includes finite bundles, and that serves as an abelian category environment for finite and essentially finite bundles. These vector bundles are now called "Nori-semistable bundles", and their definition hinges on morphisms of curves over our base scheme, and the theory of vector bundles over curves. A vector bundle $\mathcal{E}$ is Nori-semistable if for any morphism $f : C \to X$ where C is a smooth projective connected curve over k with f birational onto its image the pull-back $f^*(\mathcal{E})$ is a

1 FGS for short.

semi-stable vector bundle of slope zero, which is the original definition of Nori. Thus, category of Nori-semistable vector bundles over a proper reduced and connected scheme is abelian, and as Nori showed, finite bundles are Nori-semistable, so essentially finite bundles effectively exist as vector bundles.

More than 40 years later, many developments have strengthened the theory and have led to some variants of fundamental group-schemes coming from tannakian categories of vector bundles, like the category of F-trivial bundles and the F-fundamental group-scheme [48], the S-fundamental group-scheme [42, 43], the F-divided fundamental group-scheme [19] among others. More notably, N. Borne and A. Vistoli conceived in [13] the fundamental gerbe of a fibered category, a generalization of the fundamental group-scheme that allows for example, to parameterize *all* finite torsors, pointed or not, over a scheme or fibered category or stack, by extending the objects it parameterizes to finite gerbes.

In the cases of the S-fundamental group-scheme and the fundamental gerbe, two different approaches to consider essentially finite bundles are given:

- For the S-fundamental group-scheme over a proper scheme, the "birational onto its image" has been dropped and non-constant morphisms $f : C \to X$ from a curve are considered, with the same original definition of demanding the pull-back along f to be semi-stable of slope zero, and thanks to a more developed theory of vector bundles over curves, the category of Nori-semistable in this case is also neutral tannakian and it contains the category of essentially finite bundles, so we obtain another fundamental group-scheme $\pi^S(X, x)$ for which the Nori fundamental group-scheme is a quotient
$$\pi^S(X, x) \to \pi^N(X, x),$$
and we can see this as the category of essentially finite bundles is more constrained than the category of Nori-semistable bundles so its automorphism group-scheme is "smaller".

- In the case of the fundamental gerbe, we consider fibered categories X that are "pseudo-proper", meaning that they are quasi-compact over a field and for any vector bundle $\mathcal{E}$ over X the vector space of global sections $H^0(X, \mathcal{E})$ is finitely dimensional. This allows to conceive essentially finite bundles as kernels between morphisms of finite bundles, without the need of Nori-semistable bundles. And the resulting category is tannakian if and only if the base is "inflexible" which is essentially a condition equivalent to X having a fundamental gerbe that classifies finite gerbes over X.

We see then, that outside the fundamental gerbe, curves over schemes are essential in constructing the tannakian part of the theory of the fundamental group-scheme, but by needing to add the proper hypothesis each time. This is because a result of Ramanujan[2] that shows that over proper schemes any two points can be joined by the image of a morphism $f : C \to X$ from a smooth projective and connected curve, and this is the property needed to show that the category of Nori-semistable bundles is abelian, and as mentioned before, this is key when ensuring essentially finite bundles form a neutral tannakian category.

This thesis has two objectives, related to the discussion above: Extend the formulation of the Nori fundamental group-scheme, generalizing a recent formulation by I. Biswas, P.H. Hai and J.P. Dos Santos [11, §7] that centers on developing the S-fundamental group-scheme and the Nori fundamental group-scheme for schemes where we suppose that two points can be joined by curves, specifically a finite chain of them, without supposing that the base scheme is proper. And secondly, to propose a road map to understand and characterize the Nori fundamental group-schemes of schemes connected by curves, which generalize the notion of "rationally connected varieties", the closest notion in algebraic geometry to path-connected topological spaces. We intend to do this by using a wide array of results and elements of the theory of fundamental group-schemes and their generalizations available to this day, and by showing generalizations and constructions from these results to do so. Hoping they could be of use in the future to widen the scope of what is possible to do with the Nori fundamental group-scheme, and to describe the FGS of more examples and families of schemes.

1.1 MOTIVATION AND RESULTS

As mentioned in the introduction, we have two objectives:

(i) To generalize the construction of the FGS for schemes that are "connected by chains of curves".

(ii) Propose a road map to understand and describe the FGS of varieties connected by curves, which are a series of generalizations of the notion of rationally connected varieties, along with some results in this direction. Using either the generalization of part (i), or by utilizing results and element of the already established theory of the FGS.

About point (i), I. Biswas, P.H. Hai and J.P. Dos Santos introduced in [11, §7] the concept of *CPC-schemes* (Definition 3.3.60): schemes X over a field k, such that any pair of points $x, y \in X$ are contained in the image $\bigcup_{i=1}^{n} \mathrm{Im}(\gamma_i)$ coming from a *proper chain of curves*, a collection of morphisms $\{\gamma_i : C_i \to X\}_{i=1}^{n}$ where C_i is a curve, whose joint image

2 See Lemma 3.3.25.

is connected, proper schemes are naturally CPC, so by not excluding this case, we are generalizing the theory of the S-fundamental group-scheme of Langer over proper varieties.
For CPC-schemes X with a given rational point $x \in X(k)$, the category $NSS(X)$ of Nori-semistable bundles over X is tannakian and thus we can define essentially finite bundles over it, giving us a full inclusion $EF(X) \hookrightarrow NSS(X)$ that induces a quotient morphism of associated fundamental group-schemes $\pi^S(X,x) \to \pi^N(X,x)$ by tannakian correspondence (Corollary 2.4.137). Our generalization, consist on limiting the curves that could appear in the proper chains of curves passing through any pair of points: If $\mathscr{C}$ is a non-empty family of projective connected curves, not necessarily smooth, we say that a scheme is $\mathscr{C}$ if any two points can be joined by a proper chain of curves $\{\gamma_i : C_i \to X\}_{i=1}^n$ in which $C_i \in \mathscr{C}$ for all $i = 1..n$. In this case, we consider as our model for Nori-semistable bundles vector bundles $\mathcal{E}$ such that $f^*(\mathcal{E})$ is semi-stable of slope zero, where $f : \hat{C} \to X$ is a non-constant morphism from a curve $\hat{C}$ that is the normalization, thus smooth, of a curve belonging to $\mathscr{C}$. These vector bundles are called *$\mathscr{C}$-Nori-semistable*, see Definition 3.3.63.
The category of $\mathscr{C}$-Nori-semistable bundles is thus neutral tannakian (Proposition 3.3.65) and its corresponding FGS is called the *(S,$\mathscr{C}$)-fundamental group-scheme* $\pi^S_{\mathscr{C}}(X,x)$. Classical Nori-semistable bundles are simply $\mathscr{C}$-Nori-semistable over the family of all curves, so they are less constrained and we thus obtain a natural quotient morphism

$$\pi^S_{\mathscr{C}}(X,x) \to \pi^S(X,x)$$

for any non-empty family $\mathscr{C}$ of projective connected curves. Moreover, a similar quotient is obtained for any inclusion of families $\mathscr{C}' \subset \mathscr{C}$, and we can also consider vector bundles $\mathcal{E}$ for which the pull-back over a morphism $f : \hat{C} \to X$ from a normalized curve of family $\mathscr{C}$ is essentially finite over $\hat{C}$, also forms a tannakian category $\text{PB-EF}_{\mathscr{C}}(X)$, with a corresponding fundamental group-scheme $\pi^{PB}_{\mathscr{C}}(X,x)$ (Proposition 3.3.76).
Finally, the category of essentially finite vector bundles, with its corresponding fundamental group-scheme $\pi^N(X,x)$ is the category of *essentially finite objects* (Definition 3.3.69) for all the categories mentioned above, and this implies by Proposition 3.3.71 that $\pi^N(X,x)$ contains all the finite and pro-finite quotients of the other fundamental group-schemes, so we say that $\pi^N(X,x)$ is the *maximal pro-finite quotient* of the other FGS.
We finish this part by showing a diagram comparing all the funda-

mental group-schemes describe and their relationships: for an inclusion of families $\mathscr{C}' \subset \mathscr{C}$ we have

$$\begin{tikzcd}
\pi^S_{\mathscr{C}'}(X,x) \arrow[r] \arrow[d] & \pi^{PB}_{\mathscr{C}'}(X,x) \arrow[d] \arrow[dr] & \\
\pi^S_{\mathscr{C}}(X,x) \arrow[r] \arrow[d] & \pi^{PB}_{\mathscr{C}}(X,x) \arrow[r] \arrow[d] & \pi^N(X,x) \\
\pi^S(X,x) \arrow[r] & \pi^{PB}(X,x) \arrow[ur] &
\end{tikzcd}$$

where all the morphisms are natural with respect to morphisms $X \to Y$ over schemes that are both $\mathscr{C}$-CPC and $\mathscr{C}'$-CPC, all morphisms in the diagram above are quotients, i.e., faithfully flat morphism of group-schemes. This is all detailed in Subsection 3.3.2.
Now we can pass to part (ii). We consider generalizations of the concept of rationally connected varieties. These generalizations were coined by F. Gounelas in [29] and there are of two types:

1. g-*connected varieties or varieties connected by genus* $g \geqslant 0$ *curves*, where there exists a variety Y and a family $\mathscr{C} \to Y$ of smooth proper connected curves, with a morphism $u : \mathscr{C} \to X$, making $u^{(2)} : \mathscr{C} \times_Y \mathscr{C} \to X \times_k X$ dominant. Notable cases here are *rationally connected varieties* ($g = 0$) and *elliptically connected varieites* ($g = 1$).

2. C-connected varieties where C is a fixed proper connected curve, where there exists a variety Y and a constant family $C \times_k Y \to Y$ with a morphism $u : C \times_k Y \to X$, such that the induced map $u^{(2)} : C \times_k C \times_k Y \to X \times_k X$ is dominant.

If $u^{(2)}$ is smooth at the generic point in any of the previous definitions, we call the corresponding varieties *separably* g-*connected varieties* and *separably* C-*connected varieties* respectively.
If k is an uncountable algebraically closed field, a g-connected variety X over k is the same as a variety for which any pair very general (Definition 4.2.8) points of X, there exists a smooth connected proper curve C' of genus g, with a morphism $f : C' \to X$ whose image contains the points. And the C-connected case is analogous by replacing C' for C and requiring that any pair of very general of points are connected by C and C alone.
Very general is not enough for the $\mathscr{C}$-CPC property described in the previous point, so we cannot use smooth curves of genus g or a single curve to define a corresponding FGS of the type $\pi^S_{\mathscr{C}}(X,x)$. The only exception being rationally chain connected varieties, which are $\mathbb{P}^1_k$-CPC, i.e., for any pair of points there is a finite chain of proper smooth rational curves, essentially $\mathbb{P}^1_k$, connecting them, see [39, IV.3 Prop. 3.6].
To remedy this in the case of g-connected varieties, we extend the

class of curves we consider to stable curves of arithmetic genus g (Definition 4.3.4). By a result of Araujo and Kollár [5, Theorem 50] families stable curves of arithmetic genus g over a projective scheme X, with n marked points, posses a projective coarse moduli space $\overline{M}_{g,0}(X)$, and using the fact that this moduli space is projective, we show that projective g-connected varieties over an uncountable algebraically closed field k are $\underline{\mathrm{Curv}}_k(g)$-CPC where $\underline{\mathrm{Curv}}_k(g)$ is the family of projective and irreducible curves of arithmetic genus $\leqslant g$, see Proposition 4.3.14 and Corollary 4.3.15, so we can use chains of curves of genus $\leqslant g$ to connect any pair of points of a projective g-connected variety.

This allows us to define the g-*fundamental group-scheme* $\pi^S_g(X,x)$ for projective g-connected varieties (Definition 4.3.16) for any $g \geqslant 0$. It is not known if we can do the same for C-connected varieties, as it is not clear that C-connected varieties are C-CPC, if it were the case, we can also consider a special fundamental group-scheme that we called the C-fundamental group-scheme $\pi^S_C(X,x)$ (Definition 4.3.21).

Now we will outline a road map to understand the FGS of curve connected varieties. First we will outline two important cases where the FGS is fully described, the first example involves rationally connected varieties: normal rationally connected proper varieties have a finite fundamental group-scheme [2] and smooth proper separably rationally connected varieties have a trivial one [9]. We a use result of I. Biswas and J.P. Dos Santos [10] that directly show that $\pi^S_0(X,x)$ is trivial, see Remark 4.3.24. We believe (Problem 4.3.25) that we can use $\pi^S_0(X,x)$ to provide another proof for the description of the Nori FGS in the rationally chain connected case, we would need to show that $\pi^S_0(X,x)$ is a finite group-scheme.

The second important example is the FGS of abelian varieties. If S is an abelian variety, then we have

$$\pi^N_1(S,0) = \lim_{\substack{\leftarrow \\ n}} S[n]$$

where $S[n]$ is the kernel of the n-th multiplication morphism $m_n : S \to S$ which is an isogeny, and a finite torsor over the neutral element $0 \in S(k)$.

To understand the FGS of C-connected varieties and g-connected varieties, let X be a normal and projective variery, we will use the *MRC sequence*, which is a sequence of rational maps between normal varieties

$$X := R^0(X) \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X) \dashrightarrow \cdots$$

and each rational map in the chain is a *maximal rationally connected fibration or MRC fibration*, it is a rational morphism $X \dashrightarrow Z$ such that for an open dense subset $U \subset X$ we have a proper morphism $f : U \to Z$ such that $f_*(\mathcal{O}_U) = \mathcal{O}_Z$ and the fibers of f are rationally chain connected, and it is maximal in the sense if $\phi' : X \dashrightarrow Y$ is

another rational map with the properties just described, there exists a rational map $g : Y \dashrightarrow Z$ such that $\phi = g \circ \phi'$, making Z unique up to birational equivalence, see Definitions 4.4.1 and 4.4.5.
We can show that the dimension of the members in the MRC sequence is decreasing and the sequence is eventually stationary, i.e. $R^{j+1}(X) \cong R^j(X)$ for $j \geqslant N$ where N is the least integer with this property, and thus we will call the variety $R^N(X)$ is the *end of the MRC sequence*. Moreover, if X is either g-connected or C-connected all the $R^i(X)$ will be too, so we can try to study the FGS of g-connected and C-connected varieties of lower dimensions, given by the possible ends of the sequence, and the morphisms $\pi^N(R^i(X)) \to \pi^N(R^{i+1}(X))$, we conjecture that these morphisms have finite kernel (Conjecture 4.4.18).
There are two cases where we can describe the end of the sequence:

- If X is C-connected, $R^N(X)$ is either a C-connected surface, a non-rational curve C' with a surjective morphism $C \to C'$, or a normal proper rationally connected variety, but without control of its dimension. See Remark 4.4.20
- If X is elliptically connected, then $R^N(X)$ is either an elliptic curve or a normal proper rationally connected variety of arbitrary dimension. See Remark 5.1.5.

And in this thesis, specifically in Chapter 5, we will study the FGS in the case the end is a curve in both scenarios. If $R^N(X)$ is a curve C', then $g : R^{N+1}(X) \to C'$ is a proper faithfully flat globally defined morphism with rationally connected fibers.
Thus, we can study a more general morphism: $f : X \to S$ is proper faithfully flat morphism between a proper variety X and a smooth projective connected curve, with geometrically connected and geometrically reduced geometric fibers, that have a finite FGS as the in the case of the MRC sequence, the FGS of the rationally connected fibers is finite.
To finish this section, we will mention the results obtained in Chapter 5: for a general curve S we get the following partial result for the induced morphism $\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s)$ between their FGS.

Proposition (Proposition 5.2.19)**.** *Let k be an algebraically closed field, X a proper variety over k and let S be a smooth and proper curve over k, with a proper faithfully flat morphism $f : X \to S$ between them. We will assume that all geometric fibers are reduced, connected and possess a finite fundamental group-scheme.*
Then, $\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s)$ is faithfully flat and for any pure *Nori-reduced finite G-torsor $t : T \to X$ over X, the pull-back $T_{\bar\eta} \to X_{\bar\eta}$ to the geometric generic fiber of f is Nori-reduced, where $\bar\eta$ is the geometric generic point of S.*

In the statement above, pure torsors (Definition 5.2.1) are torsors that are not the pull-back of a torsor over the base S, and its non-trivial quotients are not pull-backs either. They are related to the kernel of $\pi^N(f)$, see Remark 5.2.2.
In the case S is an elliptic curve we obtain a complete description, aided by the homotopy exact sequence (Theorem 5.5.2) applied to the fundamental group-scheme:

Theorem (Theorem 5.5.1)**.** *Let k be an uncountable algebraically closed field, let X be a proper variety over k and let S be an elliptic curve over k. If $f : X \to S$ is a proper faithfully flat morphism, such that all geometric fibers are reduced, connected and possess a finite fundamental group-scheme*[3]*. Then, there exist rational points $x \in X(k)$ and $s \in S(k)$ such that $f(x) = s$ and the following sequence of group-schemes is exact:*

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1.$$

For a more detailed explanation of the proof of this results, see Section 5.1.2.

1.2 ORGANIZATION OF THE MANUSCRIPT

This thesis is divided in two parts: the preliminaries and the FGS of curve connected varieties. Starting with the preliminaries:

- In Chapter 2 we will outline the main aspects that are the base for the theory of the fundamental group-schemes that we will consider in later chapters. It is divided in three main sections, one devoted to each aspect:
  - Section 2.2 presents the basics of the theory of group-schemes over fields, with an emphasis on the existence of quotients of group-schemes and the isomorphism theorem for group-schemes, affine group-schemes and pro-finite group-schemes.
  - Section 2.3 deals with the basic theory of torsors over schemes, G-equivariant sheaves and their relationship with torsors, and additional constructions like contracted products and projective limits.
  - Finally, Section 2.4, delves into the theory of neutral tannakian categories, the equivalence between neutral tannakian categories and affine group-schemes, called tannakian correspondences. Which is obtained by studying the category of finitely dimensional representations of group-schemes, that also is covered in this subsection.
- Chapter 3 introduces the different notions of fundamental group-schemes that will be used in later chapters. It is divided as follows:

3 For example, if X is normal and the fibers are rationally connected

- Section 3.2 presents the non-tannakian part of the Nori fundamental group-schemes, where $\pi^N(X,x)$ serves as the group-scheme that classifies finite pointed torsors over schemes.
- In Section 3.3, tannakian fundamental group-schemes associated with neutral tannakian categories of vector bundles are shown. This section is divided in two subsections:
  * In Subsection 3.3.1 we conceive the tannakian categories of Nori-semistable and essentially finite vector bundles over schemes. We define and state the main properties of the associated group-schemes in Subsubsection 3.3.1.1 and in Subsubsection 3.3.1.2 we show how to unify the non-tannakian and tannakian approaches to the construction on fundamental group-schemes. Showing in particular that the non-tannakian Nori fundamental group-scheme and the fundamental group-scheme associated with essentially finite bundles is the same.
  * In Subsection 3.3.2 the theory of tannakian fundamental group-schemes for schemes connected by chains of proper curves is presented, along with basic results and comparisons between the different FGS obtained.
- In the last section, Section 3.4, we will list all the more advanced results for the FGS that we will use in the second part of thesis, notably in Chapter 5. Descriptions for the FGS of rationally connected varieties and abelian varieties are given in Subsection 3.4.2.

In the second part we have:

- Chapter 4 we define our main notions of varieties connected by curves, we construct adapted fundamental group-schemes over them, using the approach of Subsection 3.3.2. It is divided in:
  - In Section 4.2 we define and list the main properties of curve-connected varieties, borrowing from Gounelas' article [29, §3].
  - Section 4.3 shows how to construct the g-fundamental group-scheme for g-connected varieties using stable curves, their moduli, and the FGS of $\mathscr{C}$-CPC schemes. We give some basic properties of these fundamental group-schemes in Subsection 4.3.3.
  - The final section, Section 4.4, we define the MRC fibrationa and we discuss its in Subsection 4.4.1. We follow with Subsection 4.4.2 where we state properties and the general behavior of the FGS along the MRC sequence and we state

possible approaches to understand the FGS of C-connected varieties.

- Chapter 5 makes a full description of a particular type of morphism arising from the MRC sequence of elliptically connected varieties. In Section 5.1 we introduce the problem by explaining the MRC sequence of elliptically connected varieties and possible approaches to understand their FGS in Subsection 5.1.1 as motivation, followed by a summary of the main results and Sections 5.2 through 5.5 in Subsection 5.1.2.

1.3 PREREQUISITES

We will assume the reader is familiar with algebraic geometry at a medium level, basic category theory and abstract algebra. A list of concepts that will be assumed as understood are:

- Category theory: functor, Hom-set, natural transformation, equivalence of categories and (co)limits like projective and direct limits.
- Algebraic geometry: Reduced, connected and irreducible schemes over fields and their geometrical counterparts.
  Noetherian and locally noetherian schemes.
  Affine, finite, proper, projective, (faithfully) flat, normal and smooth morphisms/schemes.
  Varieties, basic cohomology (mostly H^0).
  Generic point/fiber, Geometric point/fiber.
  Quasi-coherent sheaf, locally free sheaf, vector bundle.
- Abstract algebra: Groups, rings, vector spaces, R-modules, ideals, sub-modules, tensor product (of modules), k-algebras.

Besides these assumptions, this thesis intends to be as self-contained as possible, detailing most of the results in Chapter 2 so they can be consulted within this document, references are often given when necessary in later chapters.
Experts in the field may skip Chapter 2 completely, and even Section 3.2 and Subsection 3.3.1 to our recommended start from Subsection 3.3.2 onwards.
We finish by giving a list of general references for various introductory and advanced topics:

- Abstract algebra: Atiyah & Mc. Donald [8], Dummit & Foote [20].
- Algebraic geometry: All the series "Éléments de Géométrie Algébrique (EGA)", Görtz & Wedhorn [27], Harthshorne [36], Liu [45] and Stacks Project [63].

- Category theory: Mac Lane [46].
- Group-schemes: Demazure & Gabriel [18], Milne [49] and Waterhouse [68].
- Tannakian categories: Deligne & Milne [17] and Saavedra-Rivano [59].

1.4 GENERAL CONVENTIONS

1. All k-algebras we will consider will be commutative with unit $(1 \neq 0)$.
2. By "variety over a field k", we will mean an integral (thus reduced and irreducible) separated scheme of finite type over $\mathrm{Spec}(k)$.
3. We will use the words "curve" and "surface" when referring to varieties of dimension 1 and 2 respectively.
4. Sometimes we will use the adjective "finite" instead of "finitely dimensional" for k-vector spaces.
5. Tied to the last point, we will denote the category of k-vector space as $\mathrm{Vect}(k)$ while we will denote the category of finitely dimensional k-vector spaces as Vectf_k.
6. By vector bundle over a scheme X we will mean a locally free $\mathcal{O}_X$-module of finite rank. We will identify these bundles with finite and flat morphisms $f : Y \to X$ locally of finite presentation where $f_*(\mathcal{O}_Y)$ is a locally free $\mathcal{O}_X$-module of finite rank. If X is locally noetherian, finite and flat suffice to ensure $f_*(\mathcal{O}_Y)$ is locally free. See [63, Tag 02K9].
7. The last point is a particular case of the anti-equivalence between affine morphisms $f : Y \to X$ and quasi-coherent $\mathcal{O}_X$-algebras, where $Y \cong \underline{\mathrm{Spec}}_X(f_*(\mathcal{O}_Y))$ ([63, Tag 01S5]). For $\mathcal{O}_X$-modules, an additional construction is needed, see [31, §1.7]. As we mentioned in the last point, vector bundles correspond under this equivalence to finite and flat morphisms locally of finite presentation.

Part I

PRELIMINARIES

2 THE THREE ASPECTS OF THE FGS

2.1 INTRODUCTION

As the central object of this work, it is reasonable to ask: *What is the fundamental group-scheme?*.
In short, it is a generalization of the étale fundamental group, developed by Grothendieck in [35, Exp. V] and [34, Exp. IX & X], which is itself a generalization of the fundamental group of a path-connected topological space. Of course, we cannot generalize the topological fundamental group with paths, as the topology of schemes is far from being path-connected. Instead, the approach Grothendieck used is the approach of "coverings": if X is a "nice" path-connected topological space, we can define topological coverings $p : Y \to X$ and consider its automorphism group $\mathrm{Aut}(Y|X)$ of homeomorphisms $\phi : Y \to Y$ such that $p = p \circ \phi$, this group acts naturally on the pre-image $p^{-1}(x)$ of a point $x \in X$, which is a discrete topological space. Moreover, we have a *universal cover* $u : \hat{X} \to X$ that dominates all covers of X, i.e., for any cover $p : Y \to X$ there is a unique continuous morphism $\hat{X} \to Y$ which is itself a cover of Y and that factors through u. Finally, we can recover the fundamental group of X as $\mathrm{Aut}(\hat{X}|X)$, and automorphisms groups $\mathrm{Aut}(Y|X)$ for intermediate covers are quotients of $\pi_1(X)$.
Now, for the étale fundamental group, let X be a connected scheme over a field k, and let $\bar{x} : \mathrm{Spec}(\Omega) \to X$ ($\bar{\Omega} = \Omega$) be a geometric point of X. For the étale fundamental group, the covers are *étale coverings*: finite and faithfully flat morphism such that the fiber $Y_{\bar{x}}$ is a finite disjoint union of copies of $\mathrm{Spec}(\Omega)$ for any geometric point over x. Here we also have a group of automorphisms $\mathrm{Aut}(Y|X)$ that acts of $Y_{\bar{x}}$, and thus when fixing a geometric point we get a functor $\mathrm{Fib}_{\bar{x}} : \acute{\mathrm{E}}\mathrm{t}(X) \to \mathrm{Set}$, called the *fiber functor*, that assigns a finite étale cover $Y \to X$ its fiber Y_x. Among étale covers of X there are those where the action over the geometric fiber is transitive, such covers are called *Galois covers* and any cover $Y \to X$ can be dominated (in the sense of the last paragraph) by a Galois cover $Z \to X$, such that the morphism $Z \to Y$ is a Galois cover and $\mathrm{Aut}(Y|X)$ is a quotient of $\mathrm{Aut}(Z|X)$. In this context, the étale fundamental group $\pi^{\text{ét}}(X, \bar{x})$ is the group $\mathrm{Aut}(\mathrm{Fib}_{\bar{x}})$ of automorphisms of the fiber functor, making $\mathrm{Fib}_{\bar{x}}$ a functor valued in sets with a continuous $\pi^{\text{ét}}(X, \bar{x})$ action, such that Galois covers correspond to finite quotients of $\pi^{\text{ét}}(X, \bar{x})$. Alternatively, we can define non-finite étale covers, so that we would have a universal one $\hat{X} \to X$ with $\pi^{\text{ét}}(X, \bar{x}) = \mathrm{Aut}(\hat{X}|X)$ and Galois covers of X correspond to finite quotients of this group. We will not go too deep into details, but we

will refer to [64] when necessary in later chapters, the reader can also consult [51].
Having described the theory of the étale fundamental group, we can give a more complete answer to the question from the beginning of this section. Firstly, the fundamental group-scheme of a scheme X over a field k is a group-scheme, which is a scheme that behaves like a group. As such, we need to define group-schemes and also how they act on schemes, once that is done we can pass to describe coverings, called *pointed torsors*, and then we can either conceive the fundamental group-scheme as a universal covering or some kind of "automorphism group-scheme" associated to a "fiber functor". The "automorphism group-scheme" part holds in fact for any group-scheme, and we can consider it in its full generality.
Finally, these two conceptions for the fundamental group-scheme must also coincide in the case both apply to our base scheme X.
In short, we need the following:

Ingedients needed for the FGS

(a) Group-schemes, that will be explained in Section 2.2.

(b) Coverings, that will be discussed in Section 2.3.

(c) Express group-schemes as automorphism group-schemes associated to a certain functor. This will be detailed in Section 2.4.

With these ingredients, the "universal covering" approach utilizes points (a) and (b), while the "automorphism group-scheme" one utilizes points (a) and (c). They also can be shown to coincide if both approaches apply over the base scheme. This will be all explained in Chapter 3.
The purpose of this chapter is to establish the basic concepts needed for the FGS, we have divided these in three aspects, group-schemes, torsors and tannakian categories, which are the contents of the next three sections, following the order of the list just mentioned. Finally, we must point out that tannakian categories are key in the construction of the many fundamental group-schemes we will define in Section 3.3, making it the most important section in this chapter.

2.2 GROUP-SCHEMES

In this section we will state the main concepts and properties of group-schemes that we will need for later chapters, will borrow most concepts and results from [49] and [68].

2.2.1 *Basic Definitions*

Let S be a scheme, the definition of group-scheme is the following:

Definition 2.2.1. *A* group-scheme *is a scheme* G *over* S *with morphisms* $m : G \times_S G \to G$, $e : S \to G$ *and* $i : G \to G$, *called multiplication, unit point and inverse resp., that satisfy the following commutative diagrams:*

Group-schemes

$$\begin{tikzcd} G \times_S G \times_S G \arrow[r, "(m,\mathrm{id}_G)"] \arrow[d, "(\mathrm{id}_G,m)"'] & G \times_S G \arrow[d, "m"] \\ G \times_S G \arrow[r, "m"'] & G \end{tikzcd}$$

Associativity axiom

$$\begin{tikzcd} S \times_S G \cong G \arrow[r, "(e,\mathrm{id}_G)"] \arrow[dr, "\mathrm{id}_G"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} G \times_S S \cong G \arrow[r, "(\mathrm{id}_G,e)"] \arrow[dr, "\mathrm{id}_G"'] & G \times_S G \arrow[d, "m"] \\ & G \end{tikzcd}$$

Unit element axioms

$$\begin{tikzcd} G \arrow[r, "(i,\mathrm{id}_G)"] \arrow[d] & G \times_S G \arrow[d, "m"] \\ S \arrow[r, "e"'] & G \end{tikzcd} \qquad \begin{tikzcd} G \arrow[r, "(\mathrm{id}_G,i)"] \arrow[d] & G \times_S G \arrow[d, "m"] \\ S \arrow[r, "e"'] & G \end{tikzcd}$$

Inverse axioms

The diagrams we just showed are the "diagrammatic" versions of group axioms, which generalize the set-theoretic versions, and are used to define *group-objects* in general categories with analogous diagrams. Group-schemes then, are group-objects in the category Sch_S of schemes over S. To differentiate group-schemes from classical groups, we will call the latter *abstract groups* from now on.
To define morphisms of group-schemes, we need a morphism of schemes that preserves the structure of the group-schemes involved:

Definition 2.2.2. *A* morphism of group-schemes *is a morphism of* S*-schemes* $\phi : G \to H$ *between two group-schemes, such that* $\phi \circ m_G = m_H \circ (\phi, \phi)$, *meaning that* ϕ *commutes with the multiplication morphisms of* G *and* H, *where* m_G *and* m_H *are the multiplication morphisms of* G *and* H *respectively, and* $(\phi, \phi) : G \times_S G \to H \times_S H$ *is the morphism that "applies* ϕ *on each coordinate" in the fibered product.*

Morphisms of group-schemes

Remark 2.2.3. *As in the case of abstract groups, the reader can easily verify that if* $\phi : G \to H$ *is a morphisms of group-schemes, we have that* $\phi \circ e_G = e_H$ *and* $\phi \circ i_G = i_H$, *where the sub-index denotes if the unit point or inverse morphism belong to* G *or* H, *meaning that* ϕ *also preserves the unit point and inverse morphisms.*

Remark 2.2.4. *In this thesis, we will mostly work with group-schemes over* $\mathrm{Spec}(k)$ *where* k *is a field. Also, most of the group-scheme we will consider are affine group-schemes, i.e., affine* k*-schemes that are also group-schemes over* k.

Before introducing some examples, we must introduce an equivalent definition of group-schemes over k.

Functor of points

Definition 2.2.5. *Let Alg_k be the category of finitely generated k-algebras. For a scheme X over k, we define its* functor of points *as the covariant functor $\widetilde{X} : Alg_k \to Set$ given by*

$$R \mapsto Hom_{Sch_k}(\mathrm{Spec}(R), X).$$

If $\widetilde{X}(R)$ is a group for any k-algebras R and for any morphism of k-algebras $R \to S$, the morphism $\widetilde{X}(R) \to \widetilde{X}(R)$ is a morphism of groups, we say that $\widetilde{X}$ is a group-valued *functor of points.*

Convention 2.2.6. *Without too deep into technicalities, as we are going to cite many results utilizing functors of points from [49] we will follow the convention of considering a small category of k-algebras: let Alg_k^0 be the small category of k-algebras of the form $k[T_1, \cdots, T_n]/I$ where I is an ideal and $\{T_i\}_{i \in I}$ is a countable set of symbols. The full inclusion of categories $Alg_k^0 \to Alg_k$ is an equivalence of categories, and thus we will identify algebras of finite type with objects of Alg_k^0 using this equivalence so that all functors of points considered from here on will be defined over Alg_k^0.*

To better understand the properties of functors of points for schemes of finite type over k, we need the most essential "Yoneda's lemma" (see [46, III §2]):

Yoneda's lemma

Lemma 2.2.7. *Let $\mathcal{C}$ be a category and $F : \mathcal{C} \to Set$ a functor. Then, for any object c of $\mathcal{C}$, we have that $Nat(Hom(c, \cdot), F) \cong F(c)$, where "Nat" denotes the set of natural transformations between two functors. In particular, all natural transformations $Hom(c, \cdot) \to Hom(d, \cdot)$ come from a unique morphism $f : d \to c$ so that it is defined by composition with f.*

Remark 2.2.8. *As a consequence of Yoneda's lemma, for two schemes X, Y over k, the only possible natural transformations $\widetilde{X} \to \widetilde{Y}$ are the natural ones, i.e., those coming from a morphism of schemes $f : X \to Y$ over k. We will denote the natural transformation induced by f by $\widetilde{f} : \widetilde{X} \to \widetilde{Y}$.*

This remark shows that we have a "functor of points" functor $\widetilde{(\cdot)}$: $\mathrm{Sch}_k \to \mathrm{Fun}(\mathrm{Alg}_k^0, \mathrm{Set})$ where the latter category is the category of functors from Alg_k^0 to Set, whose morphisms are natural transformations. Yoneda's lemma shows that this functor is fully faithful, i.e, it induces bijections between corresponding Hom sets under the functor $\widetilde{(\cdot)}$. Moreover, we have:

Proposition 2.2.9. *The functor $\widetilde{(\cdot)}$: $Sch_k \to Fun(Alg_k^0, Set)$ induces an equivalence of categories between Sch_k and a full sub-category of the category $Fun(Alg_k^0, Set)$ of functors between Alg_k^0 and Set. In particular, schemes of finite type over k are completely determined by their functors of points.*

Proof. This is a consequence of [21, Prop. VI-2], plus the fact that a scheme of finite type over k is covered by affine open sets, corresponding to k-algebras that are finitely generated over k. □

A natural question arises from this characterization, how we identify affine schemes of finite type? The following definition is the answer:

Definition 2.2.10. *A covariant functor* $F : \mathcal{C} \to$ *Set is* representable *if it is of the form* $Hom_{\mathcal{C}}(c, \cdot)$ *where* c *is an object of* c.

Applying this definition to functors from Alg_k^0, we have that a representable functor is of the form $\mathrm{Hom}_{\mathrm{Alg}_k^0}(A, \cdot)$ where A is a finitely generated k-algebra, but from the properties of ring spectra we have that $\mathrm{Hom}_{\mathrm{Alg}_k^0}(A, \cdot) = \mathrm{Hom}_{\mathrm{Sch}_k}(\cdot, \mathrm{Spec}(A))$ showing that representable functors of points correspond to affine schemes of finite type over k. Now we can come back to group-schemes and their examples, as we can characterize now their functors of points using Yoneda's lemma and Proposition 2.2.9:

Proposition 2.2.11. *A scheme* G *of finite type over* k *is a group-scheme if and only if* $\widetilde{G}$ *is group-valued (Definition 2.2.5).*

If $G = \mathrm{Spec}(A)$ is an affine group-scheme, which kind of properties should A satisfy that would make the functor of points $R \mapsto \mathrm{Hom}_{\mathrm{Alg}_k^0}(A, R)$ group-valued? The short answer is that they are the "opposite" of those outlined in Definition 2.2.1, as the functor

$$\mathrm{Spec}(\cdot) : A \mapsto \mathrm{Spec}(A)$$

is contravariant. The precise definition is:

Definition 2.2.12. *A* Hopf algebra *is an algebra* A *with three* k*-algebra morphisms* $\Delta : A \to A \otimes_k A$, $\epsilon : A \to k$ *and* $S : A \to A$, *called* comultiplication, counit and antipode (or coinverse) *resp., that satisfy the following commutative diagrams:*

Hopf algebras

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes_k A \arrow[d, "(\Delta, \mathrm{id}_A)"] \\ A \otimes_k A \arrow[r, "(\mathrm{id}_A, \Delta)"'] & A \otimes_k A \otimes_k A \end{tikzcd}$$

Coassociativity axiom

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes_k A \arrow[d, "(\epsilon, \mathrm{id}_A)"] \\ & k \otimes_k A \cong A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[dr, "\mathrm{id}_A"'] & A \otimes_k A \arrow[d, "(\mathrm{id}_A, \epsilon)"] \\ & A \otimes_k k \cong A \end{tikzcd}$$

Counit axioms

$$\begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & A \otimes_k A \arrow[d, "\mathrm{mult}(S, \mathrm{id}_A)"] \\ k \arrow[r] & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "\Delta"] \arrow[d, "\epsilon"'] & A \otimes_k A \arrow[d, "\mathrm{mult}(\mathrm{id}_A, S)"] \\ k \arrow[r] & A \end{tikzcd}$$

Antipode axioms

where $\mathrm{mult} : A \otimes_k A \to A$ *is the natural multiplication on the tensor product given by* $a \otimes b \mapsto ab$.

As we mentioned before, using the fact that the functor $A \mapsto \mathrm{Spec}(A)$ is contravariant and using Yoneda's lemma, we obtain the following:

Proposition 2.2.13. *An affine scheme $G = \mathrm{Spec}(A)$ of finite type over k is if and only if A is a Hopf algebra that is finitely generated over k.*

Remark 2.2.14. *If we consider general affine group-schemes over k, they are still the spectra of Hopf algebras but not necessarily finitely generated. This comes from the fact that Proposition 2.2.9 is valid for general schemes over k, if we define functors of points over general k-algebras, not only finitely generated ones.*

Hopf algebras serve as a big source of examples of group-schemes, we will show some now:

Example 2.2.15. *Let us show some examples of group-schemes of finite type over k:*

General linear group-scheme

1. *Let V be a k-vector space, for any finitely generated k-algebra, we can define the group-valued functor*

$$R \mapsto \mathit{Aut}_{R-\mathrm{mod}}(V \otimes_k R)$$

of R-module automorphisms of the tensor product $V \otimes_k R$. This functor is represented by an affine group-scheme, known as the General linear group-scheme over V, *that we will denote as $GL(V)$. This group-scheme will play an essential role in Subsection 2.4.2.*
If V is finitely generated, it is not hard to see that $GL(V)$ is of finite type: as a Hopf algebra, after fixing a base of V, we can identify it with k^n and in that case we will denote $GL(k^n)$ as $GL_n(k)$, in that case we have that $GL_n(k)(R) = \mathit{Hom}_{\mathit{Alg}^0_k}(A, R)$ where

$$A = k[x_{11}, x_{12}, \cdots, x_{nn}, 1/\det]$$

where each variable x_{ij} can be thought as a coordinate of a $n \times n$ matrix, and $\det$ denotes the general formula for the determinant, which is polynomial on the coordinates of the matrix. The Hopf algebra co-operations of A come from matrix operations: Δ comes the general formula of matrix multiplication, defined over the base as

$$\Delta(x_{ij}) = \sum_{k=1}^{n} x_{kj} \otimes x_{ik},$$

ϵ comes from the coordinates of the identity matrix $\epsilon(x_{ij}) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta, and the antipode S comes from the general formula for the inverse of a $n \times n$ matrix M:

$$M^{-1} = \frac{1}{\det(M)} \mathit{adj}(M)^T.$$

The interested reader can write down $S(x_{ij})$ following the formula above and verify the Hopf algebra axioms in this case.

2. *A group-scheme* G *is* commutative *if the multiplication morphism satisfies the identity* $m \circ t = m$*, where* $t : G \times_k G \to G \times_k G$ *is the morphism corresponding to the transposition of coordinates in the fibered product.* (Commutative group-schemes)
If $G = \mathrm{Spec}(A)$ *is affine, this implies that* $t \circ \Delta = \Delta$ *where* $t : A \otimes_k A$ *is the morphism that acts on elemental tensors as* $t(a \otimes b) = b \otimes a$*. In short,* G *is commutative and affine if the formula for* Δ *is symmetrical with respect to the transposition of tensors. For example,* $\Delta(x) = x \otimes x$ *is symmetrical while* $\Delta(x) = x \otimes 1$ *is not.*
The next two examples are commutative group-schemes, along with the last one, which justifies the name choice of "commutative" group-schemes over "abelian" group-schemes.

3. *The* additive group-scheme $\mathbb{G}_a$ *is the group-scheme representing the functor* $R \mapsto (R, +)$ *that forgets the multiplicative structure over a* k*-algebra* R*, resulting in the additive abelian group* $(R, +)$*.* (Additive and Multiplicative group-schemes)
As an affine scheme, it is $\mathbb{A}^1 = \mathrm{Spec}(k[x])$ *with Hopf algebra structure given by:* $\Delta(x) = x \otimes 1 + 1 \otimes x$*,* $\epsilon(x) = 0$ *and* $S(x) = -x$*.*
A related example is the multiplicative group-scheme $\mathbb{G}_m$*, that represents the group-valued functor* $R \mapsto R^*$ *of units of* R*. Schematically this group-scheme is* $\mathbb{G}_m = \mathbb{A}^1 \backslash \{0\} = \mathrm{Spec}(k[x, x^{-1}])$ *with Hopf algebra operations given by:* $\Delta(x) = x \otimes x$*,* $\epsilon(x) = 1$ *and* $S(x) = x^{-1}$*.*
We easily see that both of these group-schemes are commutative, but they are diametrically opposite to each other: any morphism of group-schemes $\mathbb{G}_m \to \mathbb{G}_a$ *or* $\mathbb{G}_a \to \mathbb{G}_m$ *is trivial, i.e, they factor through the unit morphism.*

4. *Let* k *be a field of positive characteristic* p*, we consider the group-schemes* α_p *and* μ_p *given by* (Two non-reduced group-schemes)

$$\alpha_p = \mathrm{Spec}(k[x]/x^p), \quad \mu_p = \mathrm{Spec}(k[x]/(x^p - 1))$$

with the same Hopf algebra operations as $\mathbb{G}_a$ *and* $\mathbb{G}_m$ *respectively. These group-schemes represent the functors* $R \mapsto \{r \in R : r^p = 0\}$ *and* $R \mapsto \{r \in R : r^p = 1\}$*. We easily see that the Hopf algebras associated to these group-schemes are not reduced, as either* x *or* $x - 1$ *are nilpotent. Moreover, these group-schemes have only one rational point, corresponding to the unit point, and they are also connected. We will characterize these kind of group-schemes in Subsection 2.2.3.*

5. *The theory of group-schemes extends the theory of abstract groups. To consider an abstract group as a group-scheme, let* Γ *be a finite abstract group, we define the* constant group-scheme *as the group-scheme* $\mathrm{Spec}(k^\Gamma) = \mathrm{Spec}(\prod_{g \in \Gamma} k_g)$ *of ord*(Γ) *copies of* k*. After choosing a base* $(e_g)_{g \in \Gamma}$ *where* $e_g \in k_g \backslash \{0\}$*, the Hopf algebra structure is given by:* $\Delta(e_g) = \sum_{hk=g} e_h \otimes e_k$*,* $\epsilon(e_g) = \delta_{ug}$ *where* $u \in \Gamma$ *is the unit and* $S(e_g) = e_{g^{-1}}$*.* (Constant group-schemes)
We will denote the constant group-scheme simply as Γ *or as* $(\Gamma)_k$ *if*

confusion could arise. This group-scheme is highly disconnected, because as a scheme, it is a disjoint union of copies of $\mathrm{Spec}(k)$.

Abelian varieties

6. *An* abelian variety *is a proper variety over* k *that is also a connected commutative group-scheme. It can be shown that these varieties are projective and smooth over* k *[49, pp. 8.37 & 8.45] , abelian varieties of dimension one are simply elliptic curves.*
These group-scheme represent examples of anti-affine *group-schemes (see [49, §8e]), which are group-schemes* G *of finite type over* k *that satisfy* $H^0(G, \mathcal{O}_G) = k$ *like in the case of abelian varieties, the property of being smooth over* k *is shared by all anti-affine group-schemes. They can be considered as the opposite of affine group-schemes which satisfy* $H^0(G, \mathcal{O}_G) = A$ *if* $G = \mathrm{Spec}(A)$.

We continue with subgroup-schemes:

Subgroup-schemes

Definition 2.2.16. *Let* G *be a group-scheme over* k*, a* subgroup-scheme *of finite type is a group-scheme* H *that is a sub-scheme of* G*, such that the inclusion morphism* $i : H \to G$ *is a morphism of group-schemes.*

Remark 2.2.17. *If* $H \subset G$ *is a subgroup-scheme, then for any* k*-algebra* R *of finite type we have a natural inclusion of groups* $\widetilde{H}(R) \subset \widetilde{G}(R)$*, this means that the functor of points* $\widetilde{H}$ *is a* sub-functor *of* $\widetilde{G}$*, we will denote this relationship as* $\widetilde{H} \subset \widetilde{G}$*. In short, the functor of points of a subgroup-scheme is a group-valued sub-functor of the functor of points of the larger group-scheme, but one must be careful as not all group-valued sub-functors of* $\widetilde{G}$ *correspond to subgroup-schemes of* G.

Using this characterization as sub-functors, we can define normal subgroup-schemes:

Definition 2.2.18. *A subgroup-scheme* $N \subset G$ *is* normal *if for any* k*-algebra* R *of finite type, we have that* $\widetilde{N}(R)$ *is a normal subgroup of* $\widetilde{G}(R)$*. If* $N \subset G$ *is normal we will denote it as* $N \trianglelefteq G$ *and likewise for normal abstract subgroups.*

In terms of schematic properties, subgroup-schemes satisfy the following:

Proposition 2.2.19 (Prop. 1.41 [49])**.** *Subgroup-schemes of finite type of a group-scheme of finite type over* k *are closed sub-schemes, i.e, the inclusion* $i : H \to G$ *is a closed immersion.*

In the case of affine group-schemes we can characterize their subgroup-schemes with the following:

Hopf Ideals

Definition 2.2.20. *Let* A *be a Hopf algebra over a field* k*. An ideal* $I \subset A$ *is a* Hopf ideal *if it satisfies the following properties:*

(i) $\Delta(I) \subset I \otimes A + I \otimes A$.

(ii) $S(I) \subset I$.

(iii) $\epsilon(I) = 0$.

We can easily see that with these conditions, the quotient A/I *has a natural structure of Hopf algebra inherited from* A*, that makes the projection morphism* $\pi : A \to A/I$ *a morphism of Hopf algebras, i.e., a morphism of* k*-algebras that preserves the co-operations* Δ*,* ϵ *and* S*.*

Using Proposition 2.2.19 and the fact that closed sub-schemes of affine schemes are given by quotients, we conclude the following corollary:

Corollary 2.2.21. *Let* $G = \mathrm{Spec}(A)$ *be an affine group-scheme of finite type over* k*, and let* $H \subset G$ *be a subgroup-scheme. Then,* $H = \mathrm{Spec}(A/I)$ *is an affine group-scheme, where* I *is a Hopf ideal of* A*.*

Remark 2.2.22. *(a) Corollary 2.2.21 holds indeed for any affine group-scheme, so we can drop "of finite type over* k*" for* G *in its statement. We won't prove this directly, but the curious reader can utilize Proposition 2.2.94 and Proposition 2.2.97 together to sketch a proof.*

(b) A Hopf ideal corresponding to a normal subgroup-scheme of an affine group-scheme $G = \mathrm{Spec}(A)$ *satisfies additional conditions, we leave the reader to figure those out.*

One of the most important normal subgroup-schemes is the kernel of a morphism, which is defined as:

Definition 2.2.23. *Let* $\phi : G \to H$ *be a morphism of group-schemes. The* kernel *of* ϕ *is the group-scheme defined as* $\ker(\phi) = G \times_H \mathrm{Spec}(k)$*, which is the fiber over the unit point* $e_H : \mathrm{Spec}(k) \to H$*.*

Kernel of a morphism of group-schemes

We easily see that $\ker(\phi)(R) = \ker(G(R) \to H(R))$ *for any finitely generated* k*-algebra, and thus* $\ker(\phi) \lhd G$*.*

Remark 2.2.24. *Let* $\phi : G \to H$ *be a morphism of affine group-schemes, corresponding to a morphism of Hopf algebras* $f : B \to A$*. The kernel of the counit morphism* $I_\epsilon = \ker(\epsilon)$*, which is a maximal ideal of* A*, is called the* augmentation ideal *of* A*. From the properties of Hopf ideals (Definition 2.2.20(iii)), we see that all Hopf ideals are contained in the augmentation ideal.*
In this case, the kernel of ϕ *corresponds to the quotient* $A/I_{\epsilon'}A \cong A \otimes_B k$ *where* ϵ' *is the counit morphism of* B *and thus* $I_{\epsilon'}$ *is the augmentation ideal of* B*.*

We have just shown subgroup-schemes, it would be fitting then, that we show quotients, but before this we need another very important concept:

Definition 2.2.25. *Let* G *be a group-scheme of finite type over* k *and let* X *be a scheme of finite type over* k*. An* action *of* G *over* X*, or a* G-action *over*

Actions by group-schemes

X, is a morphism of schemes $\mu : G \times_k X \to X$ that satisfies the following commutative diagrams:

Group-scheme action axioms

$$\begin{tikzcd} G \times_k G \times_k X \arrow[r, "(\mathrm{id}_G,\mu)"] \arrow[d, "(m,\mathrm{id}_X)"'] & G \times_k X \arrow[d, "\mu"] \\ G \times_k X \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Spec}(k) \times_k X \arrow[r, "(e,\mathrm{id}_X)"] \arrow[dr, "\cong"'] & G \otimes_k X \arrow[d, "\mu"] \\ & X \end{tikzcd} \cdot$$

Remark 2.2.26. *We have, strictly speaking, left actions of group-schemes. We could define right actions either directly, that we will leave as an exercise, or by diagrammatically expressing the following trick for abstract groups: If an abstract group G acts on the left a over set X, we can define an associated right action $X \times G \to X$ by setting $x \cdot g := g^{-1} \cdot x$. This kind of trick can be used both ways to pass from left to right actions and vice versa, thus obtaining a bijection between left and right and allowing us to compose right and left actions if needed.*

In terms of schematic properties for actions, we have:

Lemma 2.2.27. *Let X be a k-scheme of finite type with an action $\mu : G \times_k X \to X$. Then, μ is always faithfully flat, and if G is smooth, affine, proper, or finite, then μ is smooth, affine, proper, or finite as well.*

Proof. This comes from the following commutative diagram:

$$\begin{tikzcd} G \times_k X \arrow[r] \arrow[d, "\mu"'] & G \times_k X \arrow[d, "p_2"] \\ X \arrow[r, "\mathrm{id}_X"'] & X \end{tikzcd}$$

Where the upper horizontal morphism is (id_G, μ).
As both horizontal arrows are isomorphisms, because the top one has the morphism $(\mathrm{id}_G, \mu(i, \mathrm{id}_X))$ as inverse, where $i : G \to G$ is the inverse morphism of G (Definition 2.2.1), we see that μ has the same properties as the projection $p_2 : G \times_k X \to X$ over the second coordinate, which concludes the proof. □

In terms of functors of points, we have:

Actions by group-valued functors

Definition 2.2.28. *Let $F : Alg_k^0 \to Set$ be a functor and let $G : Alg_k^0 \to Grp$ be a group-valued functor. An* action *of G over F is a natural transformation $\mu : G \times F \to F$ such that for any k-algebra R the morphism $\mu(R) : G(R) \times F(R) \to F(R)$ is an action of $G(R)$ over $F(R)$. The fact that μ is a natural transformation implies that for any morphism $R \to S$ of k-algebras, we have the following commutative diagram*

$$\begin{tikzcd} G(R) \times F(R) \arrow[r, "\mu(R)"] \arrow[d] & F(S) \arrow[d] \\ G(S) \times F(S) \arrow[r, "\mu(S)"'] & F(S) \end{tikzcd}$$

.

By Yoneda's lemma, a schematic action $\mu : G \times_k X \to X$ yields an action of functors $\widetilde{\mu} : \widetilde{G} \times \widetilde{X} \to \widetilde{X}$ by taking functors of points. The approach of functors of points permits to define further concepts and properties around actions:

Definition 2.2.29. *Let X be a k-scheme of finite type with an action $\mu : G \times_k X \to X$.*

(a) The action is transitive *if the morphism $(p_2, \mu) : G \times_k X \to X \times_k X$ is faithfully flat where $p_2 : G \times_k X \to X$ is the projection onto the second coordinate.*

(b) The action is free *if (p_2, μ) is a monomorphism, i.e., for any k-algebra R the induced morphism $\widetilde{G}(R) \times \widetilde{X}(R) \to \widetilde{X}(R) \times \widetilde{X}(R)$ is injective.*

Definition 2.2.30. *Let X and X' be two schemes of finite type over k with G-actions μ and μ' respectively. A morphism $f : X \to X'$ is* equivariant *or a* G-morphism *if at the level of functors of points we have that $\widetilde{f}(R)\left(\widetilde{\mu(R)}(g, x)\right) = \widetilde{\mu}'(R)(g, \widetilde{F}(R)(x))$ for any $g \in \widetilde{G}(R)$ and $x \in \widetilde{X}(R)$.*

2.2.2 *Quotients of Group-schemes and the Isomorphism Theorem*

Now we will introduce quotients of group-schemes, mostly following [49, Ch. 5 & Appendix B]. In algebraic geometry the concept of a "quotient scheme" is a delicate one, but we must first define what we will mean by quotient in this context:

Definition 2.2.31. *Let $\mathcal{C}$ be a category that has products and fibered products. For objects X_0, X_1, Y of $\mathcal{C}$, forming the following diagram:*

$$\begin{tikzcd} X_1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & X_0 \arrow[r, "u"] & Y. \end{tikzcd}$$

We say that Y is the cokernel *in $\mathcal{C}$ of the pair (u_0, u_1) if the following conditions hold:*

- $u \circ u_0 = u \circ u_1$.
- *The morphism u is universal in the sense that if $v : X_0 \to Z$ is another morphism such that $v \circ u_0 = v \circ u_1$, there exists a unique arrow $f : Y \to Z$ such that the following diagram is commutative:*

$$\begin{tikzcd} X_1 \arrow[r, shift left, "u_0"] \arrow[r, shift right, "u_1"'] & X_0 \arrow[r, "u"] \arrow[dr, "v"'] & Y \arrow[d, dashed, "f"] \\ & & Z \end{tikzcd},$$

or in other words, for any object T of $\mathcal{C}$, the following induced diagram of Hom-sets

$$\begin{tikzcd} Hom_{\mathcal{C}}(Y, T) \arrow[r] & Hom_{\mathcal{C}}(X_0, T) \arrow[r, shift left] \arrow[r, shift right] & Hom_{\mathcal{C}}(X_1, T) \end{tikzcd}$$

is an equalizer (c.f. [46, p. 70]), i.e,

$$Hom_{\mathcal{C}}(Y,T) = \{\phi : X_0 \to T : \phi \circ u_0 = \phi \circ u_1\}.$$

Remark 2.2.32. *From its properties, we see that the cokernel of a pair of morphisms is unique up to isomorphism.*

Example 2.2.33. *Let us consider the problem of the existence of the cokernel of the diagram* $X_1 \overset{u_0}{\underset{u_1}{\rightrightarrows}} X_0$ *in the category RingSp of ringed spaces, we remark that this category contains the category of schemes, as any scheme is a ringed space with extra conditions.*
In this category, the cokernel of (u_0, u_1) *exists: we take as* Y *the space where we identify elements* X_0 *when* $u_0(x) = u_1(x)$ *for some* $x \in X_1$ *and we give* Y *the quotient topology.*
In terms of the structural sheaf, from the quotient morphism $u : X_0 \to Y$*, we have the following diagram of shaves over* Y*:*

$$u_*(\mathcal{O}_{X_0}) \overset{v_0}{\underset{v_1}{\rightrightarrows}} u_*\left((u_0)_*(\mathcal{O}_{X_1})\right) = u_*\left((u_1)_*(\mathcal{O}_{X_1})\right)$$

where the morphisms v_0 *and* v_1 *are the direct images to* Y *of the sheaf-level part of the morphisms* u_0 *and* u_1 *respectively. With this, we see that we can define* $\mathcal{O}_Y$ *as the sub-sheaf of* $u_*(\mathcal{O}_{X_0})$ *formed by the sections* $s \in u_*(\mathcal{O}_{X_0})(U)$*, where* U *is an open set of* Y*, such that* $v_0(s) = v_1(s)$*, making the diagram*

$$\mathcal{O}_Y \longrightarrow u_*(\mathcal{O}_{X_0}) \overset{v_0}{\underset{v_1}{\rightrightarrows}} u_*\left((u_0)_*(\mathcal{O}_{X_1})\right) = u_*\left((u_1)_*(\mathcal{O}_{X_1})\right)$$

an equalizer.

The main diagram that we will consider in this part involves group-scheme actions, so we will introduce a relevant concept:

Definition 2.2.34. *Let* X *be a scheme of finite type over a field* k*, with an action* $\mu : G \times_k X \to X$ *from a group-scheme* G *of finite type over* k*. A morphism* $f : X \to Y$ *is* G-invariant *if* $f \circ \mu = f \circ p_2$ *where* $p_2 : G \times_k X \to X$ *is the projection over the second coordinate.*
At the level of functors of points, $f : X \to Y$ *is* G*-invariant if*

$$\widetilde{f}(R)(\widetilde{\mu}(R)(g,x)) = \widetilde{f}(R)(x)$$

for any k*-algebra* R*,* $g \in \widetilde{G}(R)$ *and* $x \in \widetilde{X}(R)$*, or in other words, if* x_1, x_2 *are on the same orbit under the action of* $\widetilde{G}(R)$*, then* $\widetilde{f}(R)(x_1) = \widetilde{f}(R)(x_2)$*.*

Example 2.2.35. *Let* X *be a scheme of finite type over a field* k*, with an action* $\mu : G \times_k X \to X$ *from a group-scheme* G *of finite type over* k*. For the diagram* $G \times_k X \overset{\mu}{\underset{p_2}{\rightrightarrows}} X$*, if the cokernel of the pair* (μ, p_2) *exists as a scheme over* k*, we will denote it as* X/G *and call it (by abuse of language)* *the* quotient of X by the action μ (or by G if the action is clear). *By definition, the morphism* $\pi : X \to X/G$ *is* G*-invariant, and by the universal property of the cokernel, for any* G*-invariant morphism* $f : X \to Z$ *there is a unique morphism* $h : X/G \to Z$ *such that* $f = h \circ \pi$.

The most important quotients by actions of group-schemes, coming from this terminology are the following:

Definition 2.2.36. *Let* G *be a group-scheme of finite type over* k*, a quotient of a group-scheme* G/H *where* H *is a subgroup-scheme of* G *is the quotient by the restricted multiplication action* $\mu_H : H \times_k G \to G$ *when it exists as a scheme over* k.

Remark 2.2.37. *We do not need for* H *to be normal to wonder if* G/H *exists, but if* G/H *exists regardless of the subgroup-scheme chosen,* G/H *should be a group-scheme when* H *is normal. We will show this later in Proposition 2.2.55.*

Not all quotient by actions might exist as schemes, but all restricted multiplication actions share the property of being free (Definition 2.2.29(b)), this brings a particular case of cokernel diagram:

Definition 2.2.38. *A pair of morphisms* $u_0, u_1 : X_1 \to X_0$ *in a category* $\mathcal{C}$ *is an* equivalence relation *if for any object* T *of* $\mathcal{C}$*, the induced morphism of Hom-sets* $Hom_{\mathcal{C}}(T, X_1) \overset{(\widetilde{u_0}, \widetilde{u_1})}{\to} Hom_{\mathcal{C}}(T, X_0) \times Hom_{\mathcal{C}}(T, X_0)$ *is a bijection between* $Hom_{\mathcal{C}}(T, X_1)$ *and the graph of an equivalence relation on* $Hom_{\mathcal{C}}(T, X_0)$.
More explicitly, if $x, x' \in Hom_{\mathcal{C}}(T, X_0)$ *are in the same equivalence class, then there exists* $y \in Hom_{\mathcal{C}}(T, X_1)$ *such that* $u_0(y) = x$ *and* $u_1(y) = x'$.

Definition 2.2.39. *Let* $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ *be an equivalence relation. A morphism* $u : X_0 \to Y$ *is an* effective epimorphism *if* u *is the cokernel of* (p_1, p_2) *where* p_1 *and* p_2 *are the projection morphisms from the fibered product* $X_0 \times_Y X_0 \underset{p_2}{\overset{p_1}{\rightrightarrows}} Y$.
We say that the cokernel u *of an equivalence relation* (u_1, u_2) *(or by abuse of language Y) is a* quotient *if the morphism* $X_1 \overset{(u_0, u_1)}{\to} X_0 \times_Y X_0$ *is an isomorphism, implying that* u *is an effective epimorphism.*

Now we will state the existence results for quotients that will allow us to consider quotients of group-schemes over a field k, we must remark that in the reference [49, Appendix B] we gave at the beginning of this subsection, the problem of quotients is considered for schemes over the spectra of noetherian rings, which requires more properties to show that quotients of equivalence relations exist. The general idea is to incrementally weaken the hypotheses of the morphisms in the diagram $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ until we have established that quotients exist for all group-schemes of finite type.

The starting case is when X_1 and X_0 are affine, and u_0 is a *locally free* morphism, i.e., flat and finite over the base scheme for which the quotient $u : X_0 \to Y$ exists and is a locally free morphism [49, Theorem B.18], followed by the case where X_0 and X_1 are of finite type over the base, u_0 is locally free and for any $x \in X_0$ the set $u_0(u_1^{-1}(x))$ representing the equivalence class of x, is contained in an affine sub-scheme of X_0. In this case, the quotient also exists and it is locally free like the former case [49, Theorem B.26]. Applying this to the case of an action of a finite group-scheme over a scheme X of finite type over k, we obtain the following, with the statement taken mostly from [50, §12 Theorem 1], that includes the case of a non-free actions:

Quotients by actions of finite group-schemes

Theorem 2.2.40. *Let X be a scheme of finite type over k, with an action $\mu : G \times_k X \to X$ of a finite group-scheme G.*
For the diagram $G \times_k X \underset{p_2}{\overset{\mu}{\rightrightarrows}} X$, let us suppose that for any point $x \in X$, the set $\mu(p_2^{-1}(x))$ [1] *is contained in an affine sub-scheme of X.*
Then, the cokernel $\pi : X \to Y$ of the pair (μ, p_2) exists and is a finite morphism over k. If we denote it as $\pi : X \to X/G$, it satisfies the following properties:

(i) Topologically X/G is a quotient space with the quotient topology.

(ii) $\pi : X \to X/G$ is G-invariant (Definition 2.2.34) and any G-invariant morphism $f : X \to Z$ factors through π, as explained in Example 2.2.35.

(iii) If $\mathcal{O}_X^G \subset \pi_*(\mathcal{O}_X)$ denotes the sheaf of sections $s \in \pi_*(\mathcal{O}_X)(U) = \mathcal{O}_X(\pi^{-1}(U))$ such that $\mu^*(s) = p_2^*(s)$, called G-invariant sections, *then we have an isomorphism $\mathcal{O}_X^G \cong \mathcal{O}_{X/G}$ of sheaves over X/G.*

If the action is moreover free (Definition 2.2.29(b)), and $n = \dim_k(A)$ where $G = \mathrm{Spec}(A)$, then $\pi : X \to X/G$ is surjective and locally free (flat and finite) of rank n, i.e, $\pi_*(\mathcal{O}_X)$ is a locally free sheaf over X/G of rank n, and it is also a quotient according to Definition 2.2.39.

Applying this to group-schemes we obtain:

Quotients by of finite subgroup-schemes

Corollary 2.2.41. *Let G be a group-scheme of finite type over k, and let $H \subset G$ be a finite subgroup-scheme. Then, the quotient G/H exists and $\pi : G \to G/H$ is finite and faithfully flat. We also have an isomorphism $H \times_k G \cong G \times_{G/H} G$.*

Now we will state the general theorem for quotients, followed by the sketch of the proof for the existence of quotients for group-schemes of finite type. In synthesis, under the right hypotheses, a cokernel is defined over an open sub-scheme, and thus we need to show that for group-schemes the cokernel is globally defined.

1 The reader can compare such set with the orbits of an abstract group action over a set.

Definition 2.2.42. *Let* $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ *be a equivalence relation of schemes, i.e., we have an equivalence relation at the level of functors of points*

$$\widetilde{X_1}(R) \underset{\widetilde{u_1}}{\overset{\widetilde{u_0}}{\rightrightarrows}} \widetilde{X_0}(R)$$

for any k*-algebra* R *(Definition 2.2.38). A sub-functor* $F \subset \widetilde{X_0}$ *is* saturated *if the functor* $\widetilde{u_1}(\widetilde{u_0}^{-1}(F))$ *is a sub-functor of* F *itself, meaning that* $F(R)$ *is a union of equivalency classes of the form* $\widetilde{u_1}(R)\left((\widetilde{u_0}(R))^{-1}(x)\right)$ *for* $x \in \widetilde{X}_0(R)$.
If $U \subset X_0$ *is a sub-scheme, we will say that* U *is* saturated *if its functor of points* $\widetilde{U} \subset \widetilde{X_0}$ *is saturated.*
Saturated sub-functors (sub-schemes) allows us to restrict equivalence relations by considering the diagram $F_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} F$ *where* $F_1 = u_0^{-1}(F) = u_1^{-1}(F)$.

Now we can state the general cokernel theorem:

Theorem 2.2.43. *Let* $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ *be a equivalence relation of schemes, with* u_0 *flat and* X_0 *of finite type. Then, there exists a dense open sub-scheme* W *of* X *for which the restricted equivalence relation has a quotient.*

Quotients by flat equivalence relations over schemes of finite type

Remark 2.2.44. *The statement of this theorem is [49, Theorem B.35] minus the hypothesis of* X_0 *being quasi-projective stated there, we will sketch the proof of this result, but the proof of the theorem without the quasi-projective assumption can be found in [6, Exp. V Théorème 8.1].*

Sketch of the proof. There are two things to prove: the existence of W and the fact that we have a quotient over it.
For the quotient part, in [49, Theorem B.32] is shown that an equivalence relation $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ that has a *quasi-section* possesses a quotient $u : X_0 \to Y$ which is surjective and u is open (resp. flat, universally closed) if u_0 is. A quasi-section is a sub-scheme $Y_0 \subset X_0$ such that the restriction of u_1 to $u_0^{-1}(Y_0)$ is surjective and finite locally free over X, and for any $x \in Y_0$ the set $u_1(u_0^{-1}(x)) \cap Y_0$ is contained in an open affine sub-scheme of Y_0. These two conditions together imply that Y_0 intersects every equivalence class $u_0(u_1^{-1}(x))$ $(x \in X_0)$ in a finite set, and that this intersection is contained in an open affine sub-scheme of Y_0. This allows us to induce an equivalence relation $Y_1 \underset{v_1}{\overset{v_0}{\rightrightarrows}} Y_0$ where $Y_1 = u_0^{-1}(Y_0) \cap u_1^{-1}(Y_0)$ for which v_0 is finite and locally free, thus giving us a quotient by [49, Theorem B.26] and the rest of the proof of [49, Theorem B.32] gives a quotient in the presence of a quasi-section.
For the existence of W, it can be shown that for any closed point $z \in X_0$, there is a closed sub-scheme $Z \subset X_0$ containing z such that

the restriction of u_1 to $u_0^{-1}(Z)$ is flat at the points of $u_1^{-1}(z)$ and $Z \cap u_1(u_0^{-1}(z))$ is finite and non-empty. The latter property is easy to obtain constructively with a decreasing sequence of closed sub-schemes, and we can get the former property from this construction. If we take the restriction $u_1' : u_0^{-1}(Z) \to Z$, its fiber over z is finite by construction, and if we consider the open sub-scheme U of $u_0^{-1}(Z)$ where u_1' is flat and quasi-finite, there is an open sub-scheme W_z of $u_1'(U)$ which is the largest such that u_1' is flat and finite.

It can be shown that W_z contains the generic point of all irreducible components of X_0 containing z, it is saturated and $\left(u_1'\right)^{-1}(W_z) = \left(u_0'\right)^{-1}(U)$, and then by construction U is a quasi-section for the restricted equivalence relation on W_z, showing the existence of a quotient over W_z.

Finally, if the sub-scheme $u_0^{-1}(X_0 \backslash Z)$ is empty, W_z is the dense open sub-scheme we we looking for, and thus we are done. If $u_0^{-1}(X_0 \backslash Z)$ is not empty, it is saturated and we can find another closed point $z' \in X_0$ and an open sub-scheme $W_{z'}$ containing z' and all the generic points of the irreducible components that contain z' for which the restricted quotient exists. It is easy to show that $W_z \cap W_{z'} = \emptyset$ for different closed points and thus after taking a finite union of open sets of the form W_z until we have covered all the (finitely many) irreducible components of X_0 we will obtain the desired dense open sub-scheme with a quotient defined over it. □

Applying this to actions of group-schemes, we obtain:

Quotients by free actions of group-schemes of finite type

Corollary 2.2.45. *Let X be a scheme of finite type over k, with an action $\mu : G \times_k X \to X$ of a finite group-scheme G. Then, there exists an open dense sub-scheme $U \subset X$, that is saturated for the equivalence relation $G \times_k X \underset{p_2}{\overset{\mu}{\rightrightarrows}} X$ and for which the quotient U/G by the restricted action of G exists. The quotient morphism $\pi : U \to U/G$ is faithfully flat, and we have an isomorphism $G \times_k U \cong U \times_{U/G} U$.*

Proof. This is a direct consequence of Theorem 2.2.43, plus the fact that the action morphism $\mu : G \times_k X \to X$ is faithfully flat (Lemma 2.2.27). □

Quotients by subgroup-schemes, general case

Corollary 2.2.46 (Theorem B.37 [49]). *Let G be a group-scheme of finite type over k, and let $H \subset G$ be a subgroup-scheme. Then, the quotient G/H exists and $\pi : G \to G/H$ is faithfully flat. We also have an isomorphism $H \times_k G \cong G \times_{G/H} G$.*

Sketch of the proof. There are 3 steps involved. The first step is to show that the quotient is globally defined after taking a finite extension of k. If $k' \supset k$ is a finite extension of k, then we consider the open sub-scheme $U[k']$ defined as the union of all open sub-schemes of $G_{k'}$ that are closed under the action of $H_{k'}$ for which the quotient exists. From

its maximal property, we see that $U[k']$ is stable under the action of the group $G(k')$ and thus $U[k] \subset G$ is dense from Corollary 2.2.45. As such, $U[k]$ must contain a closed point that we can assume to be k-rational after considering a finite extension of k, thus $U[k] \supset G(k)$ and the same is true for any finite extension of k, thus $U[k] = G$ by [49, Lemma B.36].
The second part is to show that if we have a quotient $\pi : G \to G/H$, then any finite set of points of G/H is contained in an affine open sub-scheme whose proof we will omit here, and thirdly, if K is a finite extension of k for which the quotient G_K/H_K exists, then the quotient G/K over k exists as the quotient of the pull-back ([49, Definition B.4]) of the equivalence relation $\mathrm{Spec}(K \otimes_k K) \underset{p_2}{\overset{p_1}{\rightrightarrows}} \mathrm{Spec}(K)$ by the morphism $G_K/H_K \to \mathrm{Spec}(K)$, using the second step of the proof and the existence of a quotient for an equivalence relation with a finite locally free morphism for which the equivalency classes are contained in an open affine sub-scheme [49, Theorem B.32] for the pulled-back relation. □

Remark 2.2.47. *The latter proof assumes G is quasi-projective. As we said before, this property can be removed to obtain the same result. Otherwise, for a proof of the fact that group-schemes of finite type over a field are quasi-projective, see [49, B.38 & 8.43].*

Now that we have established the existence of quotients of group-schemes, we will proceed to their functional aspects and the isomorphism theorem. We start with a remark:

Remark 2.2.48. *Let G be a group-scheme of finite type over k, and let $H \subset G$ be a subgroup-scheme. We can consider two quotients from the action $\mu : H \times_k G \to G$, the schematic one $\pi : G \to G/H$ and the induced quotient of functors of points, $\widetilde{H} \times \widetilde{G} \underset{\widetilde{p_2}}{\overset{\widetilde{\mu}}{\rightrightarrows}} \widetilde{G} \xrightarrow{u} Q$. It is not hard to see that $Q(R) = G(R)/H(R)$ for any k-algebra R, in particular, Q is group-valued if $H \lhd G$, and its universal property is that for any morphism of functors $\widetilde{G} \to F$ such that for any k-algebra R the function $\widetilde{G}(R) \to F(R)$ is constant on the right cosets of $\widetilde{G}(R)$ by $\widetilde{H}(R)$, there exist a unique morphism of functors $Q \to F$ that factors $\widetilde{G} \to F$.*
This means that we have a natural transformation $Q \to \widetilde{G/H}$, and we will see shortly that this is a special kind of sub-functor.

Definition 2.2.49. *Let $F : Alg_k^0 \to Set$ be a functor. A sub-functor $D \subset F$ is* fat *if for any k-algebra R and for any element $x \in F(R)$, there exists a faithfully flat R-algebra $R \to R'$ such that the image $x' \in F(R')$ of x lies in $D(R')$.*

Example 2.2.50. *The functor $R \mapsto \{r^n : r \in R^*\}$ is a fat sub-functor of $R \mapsto R^*$ as any unit r of R becomes a n-th power in $R' = R[T]/(T^n - r)$ which is a faithfully flat R-algebra.*

Fat sub-functors appear naturally between functors of points coming from faithfully flat morphisms of schemes:

Lemma 2.2.51 (Prop. 5.7 [49])**.** *Let $f : X \to Y$ be a faithfully flat morphism between schemes of finite type over a field k. Then, the functor $R \mapsto \widetilde{f}(\widetilde{X}(R))$ is a fat sub-functor of $\widetilde{Y}$.*

Corollary 2.2.52. *Let G be a group-scheme of finite type over k, and let $H \subset G$ be a subgroup-scheme. Then, the functor $R \mapsto \widetilde{G}(R)/\widetilde{H}(R)$ is a fat sub-functor of $\widetilde{G/H}$.*

Proof. Let us denote by Q the functor $R \mapsto \widetilde{G}(R)/\widetilde{H}(R)$. If $\pi : G \to G/H$ is the canonical morphism for the quotient group-scheme, we have the following commutative diagram of functors

$$\begin{tikzcd} \widetilde{H} \times \widetilde{G} \arrow[r, shift left, "\widetilde{\mu}"] \arrow[r, shift right, "\widetilde{p_2}"'] & \widetilde{G} \arrow[r, "u"] \arrow[dr, "\widetilde{\pi}"'] & Q \arrow[d, "q"] \\ & & \widetilde{G/H} \end{tikzcd}$$

coming from the universal property of Q (Remark 2.2.48). From Lemma 2.2.51, $R \mapsto \widetilde{\pi}(\widetilde{G}(R))$ is a fat sub-functor of $\widetilde{G/H}$. We also see from this diagram that q is a sub-functor as for any k-algebra R, different elements of $Q(R)$ correspond to disjoint cosets of $\widetilde{G}(R)$, and from the isomorphism of schemes $H \times_k G \overset{(\mu,p_2)}{\cong} G \times_{G/H} G$, we have at the level of functors of points that different cosets of $\widetilde{G}(R)$ have different images in $\widetilde{G/H}(R)$ under $\widetilde{\pi}(R)$.
For the fat property, if $x \in \widetilde{G/H}(R)$, as $\widetilde{\pi}$ is fat there exists a faithfully flat R-algebra $R \to R'$ such that the image x' of x in $\widetilde{G/H}(R')$ belongs to $\widetilde{\pi}(R')(\widetilde{G}(R'))$, if $y' \in \widetilde{G}(R')$ is a pre-image of x', then $z' = u(R')(x')$ is an element of $\widetilde{Q}(R')$ such that $q(R')(z') = x'$, showing that Q is fat. □

Now we will show that if $H \subset G$ is normal, then G/H is a group-scheme.

Proposition 2.2.53 (5.10 & 5.11 [49])**.** *Let X and Y be schemes of finite type over k. If D is a fat sub-functor of $\widetilde{X}$, then any morphism of functors $\phi : D \to \widetilde{Y}$ extends to a unique morphism $\widetilde{X} \to \widetilde{Y}$, or equivalently by Yoneda's lemma, a morphism of schemes $X \to Y$.*
In particular, if X and X' are two schemes of finite type over k and we take fat sub-functors D and D' of $\widetilde{X}$ and $\widetilde{X}'$ respectively. Then, any isomorphism $D \cong D'$ extends to an isomorphism $X \cong X'$.

Corollary 2.2.54. *Let X be a scheme of finite type over k and let D be a fat sub-functor of $\widetilde{X}$. If D is group-valued, then its group structure extends to a group-scheme structure over X.*

Proof. If D is $\widetilde{\text{group}}$-valued, we have morphisms of functors $m : D \times D \to D$, $e : \mathrm{Spec}(k) \to D$ and $\mathrm{inv} : D \to D$ that satisfy the commutative diagrams of Definition 2.2.1.
As we can easily see that $D \times D$ is a fat sub-functor of $\widetilde{X} \times \widetilde{X}$, we can uniquely extend the morphisms m, e and inv to $\widetilde{X}$ using Proposition 2.2.53 in a way that the commutative diagrams needed to give $\widetilde{X}$ a group structure hold as well. Thus, we conclude that the functor $\widetilde{X}$ is group-valued and thus X is group-scheme by Yoneda's lemma. □

Applying this to quotients of group-schemes, we obtain the following:

Proposition 2.2.55. *Let* G *be a group-scheme of finite type over* k*, and let* $H \subset G$ *be a subgroup-scheme. Then* G/H *is a group-scheme if and only if* $H \lhd G$

Proof. If H is normal, then the functor $R \mapsto \widetilde{G}(R)/\widetilde{H}(R)$ is group-valued, thus, from Corollary 2.2.52 and Corollary 2.2.54, we see that G/H is a group-scheme.
For the opposite sense, if G/H is a group-scheme, then let K be the kernel of $\pi : G \to G/H$, we will show that $K = H$. From the diagram of group-valued functors

$$\widetilde{G} \xrightarrow{\widetilde{\pi}} \widetilde{G}/\widetilde{H} \xrightarrow{q} \widetilde{G/H}$$

we see that $\ker(\widetilde{\pi}) = \widetilde{H} \subset \ker(q \circ \widetilde{\pi}) = \widetilde{K}$. On the other hand, as q is a fat sub-functor, it is in particular injective for any k-algebra R, and thus $\ker(q)(R)$ is trivial, meaning that the elements of $\ker(q \circ \widetilde{\pi})(R)$ that map to the identity of $\widetilde{G/H}(R)$ map to the identity element of $\widetilde{G}(R)/\widetilde{H}(R)$ via $\widetilde{\pi}$. Thus, we conclude that $\widetilde{H} = \widetilde{K}$ and then H is a normal subgroup-scheme of G. □

We know from the isomorphism theorem of abstract groups that surjective morphisms of groups are the same as quotients by its kernel. In the case of group-schemes, what constitutes a "surjective morphism"? The answer is the following:

Definition 2.2.56. *Let* $\phi : G \to Q$ *be a morphism of group-schemes over a field* k*. We say that* ϕ *is a* quotient *or a* quotient morphism *if* ϕ *is faithfully flat.*

Quotient morphisms of group-schemes

We will now show that "quotient" is an appropriate name, as quotient morphisms are indeed the projection morphism to a quotient by a subgroup-scheme, and we will show later after the isomorphism theorem that surjective is not enough to get a quotient in Remark 2.2.69.
The first step to understand quotient morphisms is a corollary of Proposition 2.2.55:

Corollary 2.2.57. *Any normal subgroup-scheme* N *of a group-scheme* G *of finite type over* k *is the kernel of a quotient morphism.*

Proof. The projection $\pi : G \to G/N$ is a quotient morphism as it is faithfully flat by Corollary 2.2.46 and thus it is a quotient, and the proof of Proposition 2.2.55 shows that N is the kernel of this morphism. □

Now we will show that quotient morphisms are tied with their kernels:

Proposition 2.2.58. *Let* $\phi : G \to Q$ *be a quotient morphism of group-schemes of finite type over* k *and let* N *be its kernel. Then, if* $f : G \to H$ *is a morphism of group-schemes such that* $\ker(f)$ *contains* N*, then there exists a unique morphism of group-schemes* $g : Q \to H$ *such that the following diagram is commutative:*

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[dr, "f"'] & Q \arrow[d, "g"] \\ & H \end{tikzcd} .$$

Proof. Let F be the functor $R \mapsto \widetilde{G}(R)/\widetilde{N}(R)$, as the kernel of f contains N, we easily we have a factorization $\widetilde{G} \to F \xrightarrow{h} \widetilde{H}$ of $\widetilde{f}$.
On the other hand, as $R \mapsto \widetilde{\phi}(R)(\widetilde{G}(R))$ is a fat sub-functor of $\widetilde{Q}$ and we also have a similar factorization $\widetilde{G} \to F \to \widetilde{Q}$ of $\widetilde{\phi}$ from the isomorphism theorem for abstract groups, and it is not difficult to see that this factorization makes F a fat sub-functor of $\widetilde{Q}$. Thus, the morphism $h : F \to \widetilde{H}$ extends uniquely to a morphism of functors $\widetilde{g} : \widetilde{Q} \to \widetilde{H}$, that gives us the morphism g we were looking for. □

Corollary 2.2.59. *Let* $\phi : G \to Q$ *be a quotient morphism between group-schemes of finite type over* k*. If* N *is the kernel of* ϕ*, then* $Q \cong G/N$ *and* ϕ *becomes the projection morphism* $\pi : G \to G/N$ *under this isomorphism.*

Proof. In this case we have morphisms $Q \to G/N$ and $G/N \to Q$ coming from Proposition 2.2.58. We clearly see that both of their possible compositions are the identity, thus these group-schemes are isomorphic and then the conclusion follows. □

Corollary 2.2.60. *For any group-scheme* G *of finite type over* k*, there is a one-to-one correspondence between quotient morphisms* $\phi : G \to Q$ *and normal subgroup-schemes of* G*.*

One essential part of the isomorphism theorem involves morphisms of group-schemes that have trivial kernel, in full generality these are:

Definition 2.2.61. *Let* $f : X \to Y$ *be a morphism of objects in a category* $\mathcal{C}$*. We call this morphism a* monomorphism *if for any object* Z *and any pair of morphisms* $Z \underset{h}{\overset{g}{\rightrightarrows}} X$ *such that* $f \circ g = f \circ h$*, then* $g = h$*.*

For group-schemes we have the following characterization:

Proposition 2.2.62 (Prop. 5.31 [49]). *Let $\phi : G \to H$ be a morphism of group-schemes of finite type over k. Then, the following statements are equivalent:*

(a) For any k-algebra R, the map $\widetilde{\phi}(R) : \widetilde{G}(R) \to \widetilde{H}(R)$ is injective.

(b) $\ker(\phi)$ is trivial.

(c) ϕ is a monomorphism in the category of group-schemes of finite type over k.

(d) ϕ is a monomorphism in the category of schemes of finite type over k.

For abstract groups, the concepts of monomorphism and subgroup are interchangeable, for group-schemes we have an analogous statement:

Proposition 2.2.63. *A morphism of group-schemes of finite type over k is a monomorphism if and only if it is a closed immersion.*

Proof. Let $\phi : G \to H$ be a morphism of group-schemes, we recall that the kernel of ϕ is the fiber over the unit point of H (Definition 2.2.23). If ϕ is a closed immersion, then the fiber over the unit point of H must be a single point, the unit point of G, thus the kernel is trivial.
If $\phi : G \to H$ is a monomorphism, then we have a quotient of G corresponding to equivalence relation coming from the diagram $H \times_k G \overset{\mu}{\underset{p_2}{\rightrightarrows}} G$ by Corollary 2.2.46, where the morphism $\mu : H \times_k G \to H$ comes from the functor $(h, g) \mapsto hg$ defined for any k-algebra where $g \in \widetilde{G}(R)$ and $h \in \widetilde{H}(R)$ considered as an element of $\widetilde{G}(R)$ using the monomorphism ϕ and the property in Proposition 2.2.62(a).
If $\pi : G \to Q$ is this quotient, then it is not hard to see that the fiber of Q over the unit point of G is H, as the unit point of G is closed, we see that the image of ϕ is closed in G and thus ϕ is a closed immersion. □

The last result needed for the isomorphism theorem is the following:

Proposition 2.2.64. *A morphism of group-schemes that is a monomorphism and a quotient morphism at the same time is an isomorphism.*

Proof. See [49, Prop. 5.33]. □

Now we can state the isomorphism theorem:

The isomorphism theorem for group-schemes

Theorem 2.2.65. *Let $\phi : G \to H$ be a morphism of group-schemes of finite type over k. Then, this morphism uniquely factors as*

$$G \xrightarrow{q} I \xrightarrow{i} H$$

where I is a group-scheme, q is a quotient morphism (in particular faithfully flat) and i is a closed immersion.

Proof. Let N be the kernel ϕ. Then $q : G \to G/N$ is a quotient morphism and q factors through ϕ from Proposition 2.2.58, so we have a factorization $\phi = i \circ q$ where $i : G/N \to H$ is a monomorphism as it has trivial kernel. The uniqueness is clear from this, so the conclusion follows. □

Definition 2.2.66. *Let Let $\phi : G \to H$ be a morphism of group-schemes of finite type over k. The group-scheme I that appears in the middle of the factorization in Theorem 2.2.65 is called the* image *of ϕ. We will denote it as Im(ϕ) and it is the smallest subgroup-scheme of H trough which ϕ factors.*

With the definition of an image, we have an alternative diagram for Theorem 2.2.65 for a morphism $\phi : G \to H$:

$$\begin{tikzcd} G \arrow[r, "\phi"] \arrow[d, "q"'] & H \\ G/N \arrow[r, "\cong"'] & \mathrm{Im}(\phi) \arrow[u, "i"'] \end{tikzcd}$$

where N is the kernel of ϕ. With the isomorphism theorem, we can completely characterize quotient morphisms of group-schemes and the functor of points of the image subgroup-scheme:

Proposition 2.2.67 (Prop. 5.43 [49]). *Let $\phi : G \to Q$ be a morphism of group-schemes of finite type over k. The following statements are equivalent:*

(a) ϕ is a quotient morphism.

(b) The functor $R \mapsto \phi(\widetilde{G}(R))$ is a fat sub-functor of $\widetilde{Q}$.

(c) The induced morphism of sheaves $\mathcal{O}_Q \to \phi_*(\mathcal{O}_G)$ determined by ϕ is injective.

Corollary 2.2.68. *Let $\phi : G \to H$ be a morphism of group-schemes of finite type over k. If I is the image of ϕ, then the morphism $q : G \to I$ is a quotient morphism and for any k-algebra R, $\widetilde{I}(R)$ consists of all the elements of $x \in \widetilde{H}(R)$ whose image in $\widetilde{H}(R')$ lies in $\widetilde{\phi}(R')(G(R'))$ for a faithfully flat R-algebra $R \to R'$.*

Remark 2.2.69. *If we would have known from the beginning that the image of group-scheme morphism always existed, which was not clear until now, a reasonable replacement for "surjective morphisms of abstract groups" would be "a morphism $\phi : G \to H$ such that Im$(\phi) = H$". The isomorphism theorem shows that this is the case if and only if ϕ is a quotient morphism and thus this is the correct notion of "surjective".*
On the other hand, surjective morphisms of schemes, i.e., morphisms of schemes that induce a surjective morphisms of underlying topological spaces, are not enough as a replacement: If k is a field of positive characteristic, the unit morphism $e : \mathrm{Spec}(k) \to \mu_p$ is clearly surjective as the underlying space

of μ_p *is just a point but it is not a quotient morphism as it is not faithfully flat.*
In any case, if $\phi : G \to H$ *is surjective and* H *is reduced, then it is faithfully flat ([49, Prop. 1.70]). Thus, the morphism of the last paragraph serves as a counterexample when* H *is not reduced.*

With images of morphisms of group-schemes, we can define exact sequences:

Definition 2.2.70. *A sequence of morphisms of group-schemes of finite type over* k

$$G \xrightarrow{f} H \xrightarrow{g} K$$

is exact *if* $Im(f) = \ker(g)$.
We will often use short exact sequences *which are sequences of morphisms of group-schemes*

$$1 \to G \xrightarrow{f} H \xrightarrow{g} K \to 1$$

where the "1" in sequence denotes the trivial group-scheme, that are exact whenever possible, meaning from left to right that $f : G \to H$ *is a closed immersion,* $Im(f) = \ker(g)$*, and* $g : H \to K$ *is a quotient morphism. In the future we will consider sequences where either the "1" on the left or the "1" on the right is omitted in the sequence above. Those sequences will then be exact if we have the same properties for the sequence above minus the closed immersion one or the quotient morphism one depending of the omitted "1" in the sequence.*

In the case of affine group-schemes, the isomorphism theorem is very easy to show from the following result:

Proposition 2.2.71 (Theorem §14.1 [68])**.** *Any inclusion* $B \subset A$ *of Hopf algebras over a field* k *is faithfully flat.*

Remark 2.2.72. *From this proposition, if* $\phi : \mathrm{Spec}(A) \to \mathrm{Spec}(B)$ *is a morphism of affine group-schemes, by applying the isomorphism theorem for algebras for the induced morphism* $\varphi : B \to A$ *and verifying that its kernel is a Hopf ideal, we have a factorization* $B \xrightarrow{f} C \xrightarrow{g} A$ *with* C *a Hopf algebra,* f *surjective and* g *injective, thus faithfully flat by Proposition 2.2.71. This immediately gives us the isomorphism theorem after taking spectra on this factorization.*
Moreover, the isomorphism theorem holds for any affine group-scheme over a field k*, not necessarily of finite type, and we can consider* $\mathrm{Spec}(C)$ *as a quotient scheme as long as they exist, we will characterize quotients of finite affine group-schemes later.*

Now we will present a slightly different definition of the quotient by a subgroup-scheme, the existence of this alternative notion is also assured by Corollary 2.2.46, but it adds aspects the we will need in later chapters, more specifically in Subsection 5.2.3.
From now on, we will suppose that if H is a subgroup-scheme of a group-scheme G of finite type over k, the multiplication action of H

over G is on the right. This does not change any of the results we have stated, but at the level of functors of points the functor $R \mapsto \widetilde{G}(R)/\widetilde{H}(R)$ is given by left cosets if H acts on the right rather than right cosets if H acts on the left.

Alternative version of the quotient by a subgroup-scheme

Definition 2.2.73. *Let G be a group-scheme of finite type over k and let H be a subgroup-scheme. An* M-quotient *of G by H is a scheme X of finite type over k equipped with an action $\mu : G \times_k X \to X$, and a point $o \in \widetilde{X}(k)$ such that for any k-algebra R the following properties hold:*

(a) The non-empty fibers of the map $g \mapsto \widetilde{\mu}(R)(g, o)$ from $\widetilde{G}(R)$ to $\widetilde{X}(R)$ are the (left) cosets of $\widetilde{H}(R)$ in $\widetilde{G}(R)$.

(b) Any element of $x \in \widetilde{X}(R)$ lies in $\widetilde{G}(R')$ for a faithfully flat R-algebra $R \to R'$ under the map mentioned before.

If X is a quotient of G by H under this definition, we will denote it as a triple (X, μ, o).

Remark 2.2.74. *Part (b) of the definition for the M-quotient is the same as demanding that the morphism $\mu_o : G \to X$ determined by the map of part (a) makes the functor $\widetilde{G}/\widetilde{H}$ a fat sub-functor of $\widetilde{X}$. Equivalently, we can demand that the action of G on X contains at the level of functors of points the multiplication action of $\widetilde{G}$ on the left cosets of $\widetilde{G}/\widetilde{H}$ given by $g \cdot g'\widetilde{H}(R) = gg'\widetilde{H}(R)$ for any k-algebra R and $g, g \in \widetilde{G}(R)$, and that any element in the image of the action $\widetilde{\mu}(R)$ is of the form mentioned above over a faithfully flat R-algebra $R \to R'$.*
From the last paragraph, and by the existence of the quotient of group-schemes in the sense of Definition 2.2.36, stated in Corollary 2.2.46, together with Corollary 2.2.52 we see that notions of quotient and M-quotient coincide for group-schemes of finite type over a field, in particular any result about quotients so far holds for the M-quotient and vice versa. Moreover, the notion of an M-quotient adds a left action from G over a quotient of group-schemes that we have not considered before.

The universal properties that can be derived directly from the definition of the M-quotient are:

Proposition 2.2.75. *Let (X, μ, o) be an M-quotient of a group-scheme G of finite type by a subgroup-scheme H. Then the following statements hold:*

(a) Let $\phi : G \to X'$ be a morphism of schemes of finite type over k. If for any k-algebra R the map $\widetilde{\phi}(R)$ is constant on the cosets of $\widetilde{H}(R)$ in $\widetilde{G}(R)$, then there exists a unique morphism $X \to X'$ so that ϕ factors trough $\mu_o : G \to X$.

$$\begin{tikzcd} G \arrow[r, "\mu_o"] \arrow[dr, "\phi"'] & X \arrow[d, dashed] \\ & X' \end{tikzcd} .$$

(b) *If X' is another scheme with an action μ' of G, and $o' \in X'(k)$ is a point fixed by H, then there is a unique G-equivariant morphism $X \to X'$ making the following diagram commutative:*

$$\begin{tikzcd} G \arrow[r, "\mu_o"] \arrow[dr, "\mu'_0"'] & X . \arrow[d, dashed] \\ & X' \end{tikzcd}$$

Proof. Part (b) is a direct consequence of (a), and part (a) can be easily deduced from the fact that $\widetilde{G}/\widetilde{H}$ is a fat sub-functor of $\widetilde{X}$. □

Now we will introduce another notion that rends the question of the existence of a quotient of group-schemes easier. The answer for this question is not easier in this context, but the presentation is, in the opinion of the author, more straightforward.

Definition 2.2.76. *Let $F : Alg_k^0 \to Set$ be a functor. We say that F is a* flat sheaf *if it satisfies the following properties:*

(a) *For any finite family of finitely generated k-algebras $R_1, R_2, \cdots, R_n$ we have an isomorphism*

$$F(R_1 \times \cdots \times R_n) \cong F(R_1) \times \cdots \times F(R_n).$$

In this case we say that F is local*.*

(b) *For any k-algebra R and faithfully flat R-algebra $R \to R'$, the diagram of sets*

$$F(R) \longrightarrow F(R') \rightrightarrows F(R' \otimes_R R')$$

is an equalizer, where the parallel arrows are induced by the k-algebra morphisms $R \to R' \otimes_R R'$ given by $r \mapsto r \otimes 1$ and $r \mapsto 1 \otimes r$ respectively. This property is called the descent property*.*

A morphism of flat sheaves *is simply a natural transformation.*

Remark 2.2.77. *The notion of a flat sheaf is simply the notion of a fppf sheaf, i.e., a sheaf over the category of schemes with the fppf topology. For an introduction to these concepts, namely, Grothendieck topologies, sheaves, topoi, etc. the interested reader can consult A. Vistoli's notes, see [67].*

Example 2.2.78. *A representable functor $F = Hom_{Alg_k^0}(A, \cdot)$ is a flat sheaf as the diagram*

$$R \longrightarrow R' \rightrightarrows R' \otimes_R R'$$

is an equalizer for any faithfully flat R-algebra $R \to R'$.
Moreover, if X is a scheme of finite type over k, its functor of points $\widetilde{X}$ is a flat sheaf by [49, Lemma 5.9]. If a flat sheaf F is isomorphic to the functor of points of a scheme X of finite type over k, we will say that F is represented by a scheme*.*

Fat sub-functors of flat sheaves allows us to extend morphisms to sheaves, extending Proposition 2.2.53 for flat sheaves:

Lemma 2.2.79. *Let D be a fat sub-functor of a flat sheaf S. Then any morphism of functors $D \to S'$ to a flat sheaves extends uniquely to a morphism of sheaves $S \to S'$.*

Proof. See the proof of [49, Lemma 5.10]. □

Like in the case of sheaves over topological spaces, we can associate to any functor a flat sheaf, in the following sense:

Definition 2.2.80. *Let $F : Alg_k^0 \to Set$ be a functor. A pair (aF, α) is called the* sheaf associated to F, *or* sheafification of F *if aF is a flat sheaf and the morphism of functors $\alpha : F \to aF$ is universal for all flat sheaves in the sense that for any sheaf S and a morphism $\beta : F \to S$, there exists a unique morphism of sheaves $aF \to S$ such that the following diagram commutative:*

$$\begin{tikzcd} F \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & aF \arrow[d, dashed, "\exists!"] \\ & S \end{tikzcd} .$$

If F possesses a sheafification, it is unique up to isomorphism.

Proposition 2.2.81 (Prop. 5.68 [49]). *Any functor $F : Alg_k^0 \to Set$ possesses a sheafification (aF, α).*

Remark 2.2.82. *Let R be a k-algebra, a family $(R_i)_{i\in I}$ of R-algebras is* faithfully flat *if the morphism $R \to \prod_{i\in I} R_i$ is a faithfully flat morphism of k-algebras.*
Let $D \subset S$ be a sub-functor of a flat sheaf S. The pair $(S, D \hookrightarrow S)$ coming from the inclusion is a sheafification of D if for any k-algebra R and $x \in S(R)$, there exists a finite faithfully flat family of R-algebras $(R_i)_{i\in I}$, such that the image x_i of x in $S(R_i)$ lies in $D(R_i)$ for all $i \in I$. Thus, the fat property is a strong case of this property, for which we can select faithfully flat families composed of a single R-algebra.
If moreover D is local (Definition 2.2.76 (a)) and separated, *i.e., if for any faithfully flat family of R-algebras $(R_i)_{i\in I}$, the induced morphism $F(R) \to \prod_{i\in I} F(R_i)$ is injective, the we can explicitly calculate its associated sheaf as*

$$(aF)(R) = \varinjlim Eq\left(F(R') \rightrightarrows F(R' \otimes_R R') \right)$$

where R is a k-algebra, $Eq(\cdots)$ denotes the equalizer of an equalizer diagram, and the direct limit is taken over all faithfully flat R-algebras $R \to R'$.

Sheafifications are important to develop a theory of group-valued functors that are flat sheaves:

Definition 2.2.83. *A flat sheaf $G : Alg_k^0 \to Set$ is a* (flat) group-sheaf *if it is a group-valued functor (Definition 2.2.5).*

Emulating the proof of Corollary 2.2.54 we obtain:

Lemma 2.2.84. *Let* D *be a group-valued functor. If* G *is the sheafification of* D*, then* G *is a group-sheaf.*

Definition-Remark 2.2.85. *Lemma 2.2.84 allows us to transfer abstract groups concepts to group-sheaves concepts:*

(a) *A* morphism of group-sheaves *is a morphism of group-sheaves* $\phi : G \to H$ *such that for any* k*-algebra* R *the morphism* $\phi(R) : G(R) \to H(R)$ *is a morphism of abstract groups.*

(b) *A* subgroup-sheaf *is a group-valued sub-sheaf* $H \subset G$. H *is normal if* $H(R)$ *is normal in* $G(R)$ *for any* k*-algebra* R.

(c) *The* kernel *of a morphism of group-sheaves* $\phi : G \to H$ *is the sheaf* $R \mapsto \ker(\phi(R))$.

(d) *The* image *of a morphism of group-sheaves* $\phi : G \to H$ *is the sheaf associated to the group-valued functor* $R \mapsto \phi(R)(G(R)) \subset H(R)$.

(e) *The* quotient *of the group-sheaf* G *by a subgroup-sheaf* H *is the sheafification of the functor* $R \mapsto G(R)/H(R)$ *which is a group-sheaf if* H *is normal.*

It is not hard to see that all the results we have established for group-schemes, for example the Isomorphism Theorem (Theorem 2.2.65), are valid for group-sheaves. One can in fact argue that to establish these results it suffices to take all the needed results for abstract groups and change them mutatis mutandis *to obtain the corresponding results for group-sheaves using the translations given above.*

From Corollary 2.2.46 and using either the definition of quotient of Definition 2.2.39 or Definition 2.2.73, we can bridge the definition of quotient group-sheaf with quotient of group-schemes:

Proposition 2.2.86. *Let* G *be a group-scheme of finite type over* k*, and let* $H \subset G$ *be a subgroup-scheme. Then, the quotient sheaf* Q *associated to the functor* $\widetilde{G}/\widetilde{H}$ *is representable by the scheme* G/H *(see Example 2.2.78).*

Quotients of group-schemes are quotient group-sheaves

Remark 2.2.87. *The construction of a quotient sheaf is arguably easier conceptually than the construction of the quotient scheme, modulo the knowledge of the fppf topology and fppf sheaves. For group-schemes over* k *that are not of finite type, the more reasonable criteria for quotients is essentially the statement of Proposition 2.2.86, demanding that the quotient scheme* G/H *exists if and only if the flat sheaf associated to* $\widetilde{G}/\widetilde{H}$ *is represented by a (unique) scheme.*

To finish this section, we will characterize quotients of affine group-schemes, not necessarily of finite type over a field:

Example 2.2.88. *Let* $G = \mathrm{Spec}(A)$ *be an affine group-scheme, and let* $H = \mathrm{Spec}(B)$ *be a subgroup-scheme, then we have* $B = A/I$ *where* I *is a Hopf ideal of* A *according to Corollary 2.2.21. We would like to show that*

Quotients of affine group-schemes

the quotient G/H *exists if* H *is normal, that it is affine, and explicitly show its corresponding Hopf algebra.*
Let $\Delta : A \to A \otimes_k A$ *be the comultiplication morphism of* A *(Definition 2.2.12), associated to the multiplication morphism* $m : G \times_k G \to G$. *The restricted multiplication from* H *(on the right),* $\mu_H : G \times_k H \to G$ *corresponds to* Δ *modulo* $A \otimes A/I$ *that we will denote as* $\tau_H : A \to A \otimes_k A/I$. *As* Δ *is also a morphism of* k*-algebras we see that* τ_H *is a morphism of* k*-algebras as well, it also satisfies the dualized versions of the commutative diagrams we outlined for actions (Definition 2.2.25), this will appear again when we will study representations in Subsection 2.4.2.*
Let us suppose that H *is normal, to obtain a quotient group-scheme* G/H *we can consider the dualized diagram associated to the diagram* [2]

$$G \times_k H \underset{p_1}{\overset{\mu_H}{\rightrightarrows}} G$$

which becomes

$$A \underset{i_1}{\overset{\tau_H}{\rightrightarrows}} A \otimes_k A/I$$

where i_1 *is the morphism* $a \mapsto a \otimes 1(\mod A \otimes_k A/I)$. *If a quotient* Q *exists for the schematic diagram, we would like the projection morphism* $G \to Q$ *to be a quotient morphism 2.2.56, and thus faithfully flat. As we mentioned before, any inclusion of Hopf algebras is faithfully flat (Proposition 2.2.71) so it suffices to show a Hopf sub-algebra* $C \subset A$ *such that the diagram*

$$C \longrightarrow A \underset{i_1}{\overset{\tau_H}{\rightrightarrows}} A \otimes_k A/I$$

is commutative. This Hopf sub-algebra exists, and its called the algebra of H-invariants, *defined as* $A^H = \{a \in A : \tau_H(a) = a \otimes 1(\mod A \otimes_k A/I)\}$. *This is in fact a Hopf sub-algebra of* A *and if* $\phi : G \to Q$ *is a quotient morphism of affine group-schemes, then* $Q \cong \mathrm{Spec}(A^N)$ *where* $N = \ker(\phi)$ *[68, Lemma p.124] and any quotient of* G *by a normal subgroup-scheme is of the form* $\mathrm{Spec}(A^H)$ *[68, Theorem p.123], giving a one-to-one correspondence analogous to Corollary 2.2.60.*
In general, for any subgroup-scheme H *we can still consider* $A^H \subset A$ *the* H*-invariant elements under the action of* τ_H*, which is a sub-algebra of* A *but not a Hopf sub-algebra unless* H *is normal. This sub-algebra has additional structure, related to* G*, and by proxy* A *(see Example 2.4.85(5)).*

2.2.3 *More Properties*

The subject of group-schemes of finite type over a field is an enormously ample one, in this subsection, we will state all the more specific properties and results that will be used in later chapters.
We will start with finite group-schemes:

2 The reader should notice the subtle change in the projection, as we are considering right actions.

Finite group-schemes

Definition 2.2.89. *A group-scheme G over a field k is* finite *if its structural morphism $G \to \mathrm{Spec}(k)$ is finite. Thus, $G = \mathrm{Spec}(A)$ is affine with A a Hopf algebra that is a finitely generated k-vector space. If $X = \mathrm{Spec}(A)$ is a finite scheme over k, the* order *of X is the integer $\mathit{ord}(X) = \dim_k(A)$.*

Orders of finite group-schemes behave as orders of abstract groups:

Proposition 2.2.90. *Let G a finite group-scheme over k and let $H \subset G$ be a subgroup-scheme, thus H is finite. Then, we have the following identity*

$$\mathit{ord}(G) = \mathit{ord}(H) \cdot \mathit{ord}(G/H).$$

Proof. From the part of Theorem 2.2.40 about free actions from finite group-schemes, the projection $\pi : G \to G/H$ is a locally free morphism of rank $\mathrm{ord}(H)$, meaning that $\pi_*(\mathcal{O}_G)$ is a locally free sheaf of rank $\mathrm{ord}(H)$, as G/H is finite of order $\mathrm{ord}(G/H)$ over k, the conclusion follows from the identity $\dim_k(\Gamma(G, \mathcal{O}_G)) = \dim_k(\Gamma(G/H, \pi_*(\mathcal{O}_G)))$. □

Having defined finite group-schemes, we can consider pro-finite group-schemes, which are one of the most important group-schemes we will work on later:

Pro-finite and pro-algebraic group-schemes

Definition 2.2.91. *Let I be a totally ordered set and let $(G_i)_{i\in I}$ be a projective system of group-schemes, where all the morphisms in the system are morphisms of group-schemes. If the projective limit of this system $G = \varprojlim_{i\in I} G_i$ exists, it is easy to see that it is a group-scheme by the universal properties of projective limits.*
If all the group-schemes G_i are finite, we will say that G is pro-finite*, and if all the G_i are affine and of finite type over k, we will say that G is* pro-algebraic*.*

Notation 2.2.92. *If $(G_i)_{i\in I}$ is a projective system of group-schemes, for any pair $i \leqslant j$, we will denote the corresponding transition morphism of group-schemes in the system as $\phi_{ij} : G_j \to G_i$. Moreover, if the projective limit G exists, the structural morphism to the i-ith object of the system will be denoted as $\pi_i : G \to G_i$.*

Remark 2.2.93. *In both the pro-finite and pro-algebraic case, the projective limit $G = \varprojlim_{i\in I} G_i$ exists as the transition arrows $\phi_{ij} : G_j \to G_i$ ($i \leqslant j$) are affine morphisms over k by [33, Prop. 8.2.3].*
Moreover, as all $G_i = \mathrm{Spec}(A_i)$ are affine where A_i is a Hopf algebra, these algebras form an direct system $(A_i)_{i\in I}$ with transition arrows $f_{ij} : A_i \to A_j$ corresponding to ϕ_{ij} such that its direct limit $A = \varinjlim_{i\in I} A_i$ exists, it is a Hopf algebra, and $G = \mathrm{Spec}(A)$. In particular, G is affine.

From now on, any projective limit of group-schemes we will consider will be either pro-finite or pro-algebraic. As affine group-schemes over k, we can consider morphisms between them, in particular quotient morphisms (Definition 2.2.56) and monomorphisms (Definition

2.2.61). For group-schemes that are not of finite type, it is not always true that monomorphisms are closed immersions, however, for pro-finite and pro-algebraic group-schemes this is true:

Proposition 2.2.94. *Let $\phi : G \to H$ a morphism of pro-finite (resp. pro-algebraic) group-schemes. Then ϕ is a monomorphism if and only if it is a closed immersion.*

Proof. As a monomorphism of group-schemes has trivial kernel by Proposition 2.2.62 and the isomorphism theorem holds for arbitrary affine group-schemes (Remark 2.2.72), so the conclusion follows. □

Now we will state a special property of pro-algebraic and pro-finite group-schemes:

Proposition 2.2.95. *Let $G = \lim_{\leftarrow i \in I} G_i$ be a pro-algebraic group-scheme. Then, any morphism $G \to H$ to an affine group-scheme of finite type over k factors trough $\pi_i : G \to G_i$ for some $i \in I$.*
If G is pro-finite, the same holds for H finite over k.

Proof. We have $H = \mathrm{Spec}(B)$ where B is a finitely generated Hopf algebra, and the morphism $G \to H$ then becomes $f : B \to A$ where $G = \mathrm{Spec}(A)$ is a direct limit of Hopf algebras finitely generated over k. If $(A_i)_{i \in I}$ is the direct system defining A with $G_i = \mathrm{Spec}(A_i)$, we need to factor f as $f = \psi_i \circ f_i$ where $\psi_i : A_i \to A$ is the morphism associated to π_i and $f_i : B \to A_i$.
As B is noetherian and finitely generated over k, it is finitely presented as

$$B \cong k[x_1, x_2, \cdots, x_n]/(p_1, \cdots, p_m)$$

for some integers n, m where the p_j are polynomials on the x_l for all $j = 1 \cdots m$.
If for some A_i the morphism f_i is defined, this Hopf algebra at least must contain the images $f(x_l)$ ($l = 1..n$) of the generators of B, and $f_i(p_j) = 0$ for all $j = 1..m$. As the system $(A_i)_{i \in I}$ is directed we can easily find such an algebra, let it be A_N ($N \in I$), then we have a k-algebra morphism $f_N : B \to A_N$ that might not be a morphism of Hopf algebras. If it was, we would have the identities $f_N(S(x_l)) = S(f_N(x_l))$ and $(f_N, f_N)(\Delta(x_l)) = \Delta(f_N(x_l))$, if we denote $\Delta(x_l) = \sum_{p=1}^{s_l} a_{pl} \otimes b_{pl}$, we see , as $f : B \to A$ is already a morphism of Hopf algebras, that the elements $f(a_{pl})$ and $f(b_{pl})$ are present in A by using the identity $(f, f)(\Delta(x_l)) = \Delta(f(x_l))$, so we can take a Hopf algebra A_M with a larger index $M \gg N$ in the direct system so that the elements $f(x_l)$, $f(a_{pl})$, $f(b_{pl})$ and $f(S(x_l))$ for $l = 1..n$ and $p = 1..s_l$ belong to A_M. In that case, we have a morphism $f_M : B \to A_M$ of Hopf algebras and thus we are done in the pro-algebraic case.
For the pro-finite case, B is a finitely generated k-vector space and if $(x_i)_{i=1}^n$ is a base then we can easily obtain the antipodes $S(x_l)$ and the comultiplications $\Delta(x_l)$ by obtaining the right coefficients over the

base, this makes the problem of finding a Hopf algebra in the directed system easier, as we only need that this Hopf algebra A_M contains $f(x_l)$ for $i = 1..n$, as A_M is also a finitely generated k-vector space in this case the conclusion follows as well. $\square$

Corollary 2.2.96. *Let* $G = \varprojlim_{i \in I} G_i$ *be a pro-algebraic group-scheme. The natural transformation defined for any affine group-scheme* H *of finite type over* k *as*

$$\varinjlim_{i \in I} \mathit{Hom}(G_i, H) \to \mathit{Hom}(G, H)$$

is an isomorphism of functors.

Proof. Proposition 2.2.95 gives an inverse to the natural transformation given in the statement. $\square$

Pro-algebraic group-schemes are pivotal to understand arbitrary affine group-schemes over a field:

Proposition 2.2.97. *Let* G *be an affine group-scheme over* k*, then it is pro-algebraic and we can chose a directed family* $(G_i)_{i \in I}$ *where each* G_i *is of finite type over* k *and a quotient of* G*.*

Proof. Let $G = \mathrm{Spec}(A)$, the proof the statement of the proposition is equivalent to show that A is a directed limit of Hopf sub-algebras finitely generated over k, whose proof can be found in [68, §3.3][3] and it comes from the fact that any finite set of elements of A is contained in a Hopf sub-algebra, finitely generated over k which proof follows the idea for the proof of Proposition 2.2.95. $\square$

Corollary 2.2.98. *Any affine (or equivalently pro-algebraic) group-scheme over* k *is the direct limit of its quotients of finite type.*

Proof. Let G be an affine group-scheme over k, and let $(G_j)_{j \in J}$ be the family of all quotients of G of finite type over k, which is directed as for any pair of quotients $\pi_l : G \to H_l$ $(l = 1, 2)$ there is always a third quotient $\pi_3 : G \to H_3$ corresponding to the kernel of π_3 being $\ker(\pi_3) = \ker(\pi_1 \cap \pi_2)$. From Proposition 2.2.97 $G = \varprojlim_{i \in I} G_i$, with $I \subset J$ a directed subset and any G_i is a quotient of G of finite type.
If we denote $Q = \varprojlim_{i \in I} G_i$ there is a natural morphism $G \to Q$ as any member of the limit forming Q is quotient of G, and $Q \to G$ coming from the inclusions $I \subset J$. We can easily see that $G \to Q$ is faithfully flat by looking at the Hopf algebras, and thus the composition $G \to Q \to G$ is the identity, making $G \to Q$ a monomorphism as well, giving us an isomorphism by Proposition 2.2.64. $\square$

These results for pro-algebraic group-schemes imply their analogues for pro-finite group-schemes

3 Alternatively, this is a consequence of Corollary 2.4.87 that is derived from the same reference.

Corollary 2.2.99. *Let G be a pro-finite group-scheme. Then any quotient of G of finite type over k is finite.*

Proof. If we write $G = \mathrm{Spec}(A)$, a quotient of finite type of G corresponds to a Hopf sub-algebra $B \subset A$ finitely generated over k. If $\{b_1, b_2, \cdots, b_n\}$ generate B, we can find a finite group-scheme $H = \mathrm{Spec}(C)$ belonging to the projective system defining G such that the image of C on A contains the $b_i \in B$ $(i = 1..n)$. But the image of C has finite dimension as a vector space and it contains B, thus B is finite. □

Corollary 2.2.100. *Let G be an pro-finite group-scheme over k, then we can chose a directed family $(G_i)_{i \in I}$ where each G_i is finite and a quotient of G.*

Proof. The proof of Corollary 2.2.99 implies this. □

Corollary 2.2.101. *Any pro-finite group-scheme over k is the direct limit of its finite quotients.*

Proof. The proof of this statement is similar to the proof of Corollary 2.2.98, using Corollary 2.2.100. □

The final property that we will state for pro-finite group-schemes is the following:

Proposition 2.2.102. *A pro-finite group-scheme that is of finite type over k is finite.*

Proof. Let us suppose that $G = \mathrm{Spec}(A)$ is of finite type but not finite, then its Hopf algebra contains a finite set of generators $\{a_1, a_2, \cdots, a_n\}$ that should be contained in a Hopf sub-algebra, finitely generated over k, corresponding to a quotient of G. But from Corollary 2.2.101 this Hopf sub-algebra is a finitely dimensional k-vector space, thus A is as well. □

We will finish this subsection by characterizing étale group-schemes and finite group-schemes in characteristic 0:

Definition-Proposition 2.2.103 (Prop. 1.5.6 [64])**.** *The following statements are equivalent for a k-algebra A finitely generated as a k-vector space:*

- $A \cong \prod_{i=1}^{n} k_i$ *where k_i is a finite separable extension of k for any* $i = 1..n$.
- $A \otimes_k \bar{k} \cong \bar{k}^n$ *where $\bar{k}$ is the algebraic closure of k.*
- $A \otimes_k \bar{k}$ *is reduced or equivalently* $\mathrm{Spec}(A)$ *is geometrically reduced.*

An algebra satisfying any of the equivalent conditions is called an étale *algebra.*

Étale algebras are intimately tied to the Galois pro-finite abstract group $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ where k^{sep} is the separable closure of k. We recall that pro-finite abstract groups are topological groups with the so called "Krull topology" (see [64, p. 12]), finite quotients of $\mathrm{Gal}(k^{\mathrm{sep}}/k)$ are in bijection with the Galois groups of finite Galois extensions of k.
The correspondence we have just mentioned, can be extended fully to étale algebras:

Theorem 2.2.104 (Thm. 1.5.4 [64])**.** *Let k be a field, the functor $A \mapsto \mathrm{Hom}_k(A, k^{sep})$ (the latter set is finite) is an equivalence of categories, from the category of étale k-algebras to the category of finite sets with a continuous left action from the Galois group $\mathrm{Gal}(k^{sep}/k)$[4].*
Under this equivalence, finite separable extensions of k correspond to sets with a transitive action and finite Galois extensions of k correspond to the action of $\mathrm{Gal}(k^{sep}/k)$ over its finite quotients.

Galois correspondence for étale algebras

Now we will define étale group-schemes:

Definition 2.2.105. *A finite scheme X over k is* étale *if its corresponding k-algebra is étale as in Definition-Proposition 2.2.103. A finite group-scheme is* étale *if its corresponding Hopf algebra is étale.*

Étale group-schemes

Remark 2.2.106. *Let $G = \mathrm{Spec}(A)$ an étale group-scheme. Then, we see that $\mathrm{Hom}_k(A, k^{sep}) = \widetilde{G}(k^{sep})$ is an abstract group and the action of $\mathrm{Gal}(k^{sep}/k)$ over $\widetilde{G}(k^{sep})$ has the following special property: if $\gamma \in \mathrm{Gal}(k^{sep}/k)$ the map $g \mapsto \gamma \cdot g$ for $g \in \widetilde{G}(k^{sep})$ is a morphism of abstract groups, in this case we say that $\mathrm{Gal}(k^{sep}/k)$* acts on $\widetilde{G}(k^{\mathrm{sep}})$ by abstract group morphisms*.*
The inverse is true from the correspondence of Theorem 2.2.104: the functor $G \mapsto \widetilde{G}(k^{sep})$ establishes an equivalence between the category of étale group-schemes and discrete finite abstract groups with a continuous action of $\mathrm{Gal}(k^{sep}/k)$ by abstract group morphisms.

Example 2.2.107. *If Γ is a finite abstract group, it is easy to see that the associated constant group-scheme $(\Gamma)_k$ (Example 2.2.15 (5)) is étale as its associated Hopf algebra is reduced. Constant group-schemes correspond to finite abstract groups with a trivial action from $\widetilde{G}(k^{sep})$ under the correspondence of Theorem 2.2.104.*

Étale group-schemes fully characterize finite group-schemes over fields of characteristic zero, this is because of the following lemma:

Lemma 2.2.108 (Prop. 5.1.31 [64])**.** *A finite scheme $X = \mathrm{Spec}(A)$ over k is smooth over k if and only if A is étale.*

and this strong result:

Proposition 2.2.109 (Cartier)**.** *Let k be a field of characteristic zero. Then any affine group-scheme of finite type over k is smooth. In particular, all finite group-schemes over k are étale.*

4 In this case all these finite sets have the discrete topology.

Proof. See [49, Theorem 3.23], the part about finite group-schemes comes directly from Lemma 2.2.108. □

2.3 TORSORS

2.3.1 Definition and basic Properties

Now we will study torsors over k-schemes, where k is a field. We start with the main definition:

Torsors

Definition 2.3.1. *Let X be a scheme over k and let G be a group-scheme over k. A scheme T with a faithfully flat morphism $t : T \to X$ is a* G-torsor *if T possesses a right action $\mu_T : T \times_X G_X \to T$[5] where $G_X = G \times_k X$, such that $t : T \to X$ is G_X-invariant (Definition 2.2.34) and the morphism*

$$T \times_X G_X \cong T \times_k G \overset{(\mathrm{id}_T, \mu_T)}{\to} T \times_X T$$

is an isomorphism.
If $x \in \widetilde{X}(k)$ is a fixed rational point, we say that a G-torsor $t : T \to X$ is pointed *if there exists a rational point $y \in T(k)$ such that $t(y) = x$.*

Remark 2.3.2. *Let z be a point of X. If $t : T \to X$ is a G-torsor, it is not hard to see that the fiber T_z has an action from $G_{\kappa(z)}$ that is free and transitive (Definition 2.2.29). In particular, if $x \in \widetilde{X}(k)$ is a rational point, and $t : T \to X$ is pointed, we see that T_x can be identified with G after selecting a k-point in T over x via t.*

From the definition of a torsor, we can deduce the following reasonable property:

Base change of torsors

Lemma 2.3.3. *Let X be a scheme over k and let G be a group-scheme over k. If $t : T \to X$ is a G-torsor and $Z \to X$ is a morphism, then the base change $t_Z : T \times_X Z \to Z$ is a G-torsor as well.*

Proof. It is not hard to see that t_Z is G_Z-invariant and faithfully flat. Moreover, if we denote $T_Z = T \times_X Z$, the isomorphism $T \times_k G \cong T \times_X T$ becomes $T_Z \times_k G \cong T_Z \times_Z T_Z$ as both sides of the second isomorphism are just the base change to Z of the isomorphism coming from the fact that $t : T \to X$ is a torsor. □

Now we will introduce some examples of torsors:

Example 2.3.4. *Let G be a group-scheme of finite type over k.*

Trivial torsor

1. *The* trivial torsor *over a scheme X over k is the torsor $G_X = X \times_k G \to X$ coming from the first projection, where G_X acts on $X \times_k G$ by the natural multiplication action $m_X : G_X \times_X G_X \to G_X$ which is the*

5 The action here is simply Definition 2.2.25 but over the base scheme X instead of Spec(k).

base change of the multiplication m *of* G*. As* G_X *is a group-scheme, it is easy to see that the morphism*

$$G_X \times_X G_X \overset{(\mathrm{id}_{G_X}, m_X)}{\to} G_X \times_X G_X$$

is an isomorphism, and thus we have a torsor. We also note that

$$(X \times_k G) \times_X G_X \cong X \times_k G \times_k G = G_X \times_k G.$$

2. *If* G *and* Q *are group-scheme of finite type over* k*, then any quotient morphism* $\phi : G \to Q$ *with kernel* $N \subset G$ *is an* N*-torsor as* $Q \cong G/N$ *(Corollary 2.2.59) and we have an isomorphism* $G \times_k N \cong G \times_{G/N} G$ *(Corollary 2.2.46). In general, if* $p : X \to Y$ *a is a quotient by an equivalence relation* $X \times_k G \overset{p_1}{\underset{\mu}{\rightrightarrows}} X$ *(Definition 2.2.39), given by a free and transitive group-scheme action is a* G*-torsor.*

Quotients of group-schemes

There are a few examples that we will show later, as we need the to state an important property of torsors, but first we have to define the following:

Definition 2.3.5. *A morphism* $f : X \to Y$ *is* fpqc[6] *if it is faithfully flat and for any point* $x \in X$*, there exists a neighborhood* U *containing* x *such that* $f(U) \subset Y$ *is open and the restriction* $f|_U : U \to f(U)$ *is a* quasi-compact *morphism, i.e., the inverse image of any affine open subset of* $f(U)$ *is a quasi-compact topological space of* U*. A family of morphisms* $\{f_i : U_i \to X\}_{i \in I}$ *is an* fpqc cover *if the induced morphism from the disjoint union* $f : \coprod_{i \in I} U_i \to X$ *is fpqc.*

Remark 2.3.6. *Fpqc covers form what is known as a* Grothendieck topology*, we will not define it here, but we have mentioned another topology, the fppf topology in Remark 2.2.77. We will refer to [67] when we will need some results from the theory of Grothendieck topologies and descent. There are many equivalent ways to define the second part of an fpqc morphism, see [67, Prop. 2.33] for the equivalencies.*

The main properties of fpqc morphisms are the following:

Proposition 2.3.7 (Prop. 2.35 [67])**.** *Let* $f : X \to Y$ *and* $g : Y \to Z$ *be morphisms of schemes.*

(a) *If both* f *and* g *are fpqc, then* $g \circ f$ *is fpqc.*

(b) *If there is an open covering* $\{V_j\}_{j \in J}$ *of* Y *such that all the restrictions* $f|_{f^{-1}(V_j)} : f^{-1}(V_j) \to V_j$ *are fpqc, then* f *is fpqc.*

(c) *If* f *is open and faithfully flat, then* f *is fpqc.*

(d) *If* $f : X \to Y$ *is fpqc and* $h : U \to Y$ *is a morphism, then the base change morphism* $X \times_Y U \to U$ *is fpqc.*

6 Fpqc is an abbreviature of “fidèlement plat et quasi-compact”.

(e) *If* $f : X \to Y$ *is fpqc and* $U \subset Y$ *is a subset, then* U *is open if and only if* $f^{-1}(U)$ *is open in* X.

Fpqc covers also come with a handy property, known as descent:

Fpqc descent

Proposition 2.3.8 (Proposition 2.7.1 [32])**.** *Let* $g : Y \to X$ *be a morphism of schemes and let* $\{f_i : U_i \to X\}_{i \in I}$ *be an fpqc cover. If for all* $i \in I$ *the base change morphism* $g_i : U_i \times_X Y \to U_i$ *satisfies one of the following properties:*

(a) *Separated.*

(b) *Of finite type.*

(c) *Quasi-compact.*

(d) *Proper.*

(e) *Affine.*

(f) *Finite.*

(g) *Flat.*

(h) *Smooth.*

(i) *Étale.*

(j) *A closed immersion.*

(k) *Surjective.*

(l) *Isomorphism.*

then so does $g : Y \to X$.

The relationship between torsors and fqpc covers is the following:

Proposition 2.3.9. *Let* X *be a scheme over* k, G *a group-scheme over* k, *and let* $t : T \to X$ *be a scheme over* X *with an action of* G_X *over* T *such that* t *is* G_X*-invariant. Then* $t : T \to X$ *is a* G*-torsor if and only if there exists an fpqc cover* $\{f_i : U_i \to X\}_{i \in I}$ *of* X $U_i \times_X T \to U_i$ *are trivial torsors.*

Proof. If $t : T \to X$ is a G-torsor, then $\{t : T \to X\}$ is an fpqc cover and the isomorphism $T \times_X G_X \cong T \times_X T$ implies that $T \times_T T \to T$ is a trivial torsor.
For the other implication, let $\{f_i : U_i \to X\}_{i \in I}$ be a fpqc cover of X such that the base changes $U_i \times_X T \to U_i$ are trivial torsors. This means that for any $i \in I$ we have an isomorphism $\psi_i : T \times_X U_i \to U_i \times_X G_X \cong U_i \times_k G$.
Now, as all base changes of $t : T \to X$ become $p_1 : U_i \times_k G \to U_i$ and the latter morphism is faithfully flat for all $i \in I$, by fpqc descent (Proposition 2.3.8(g) & (k)), we have that t is faithfully flat. Moreover, the morphism $T \times_k G \overset{(id_T, \mu_T)}{\to} T \times_X T$ becomes an isomorphism after

taking the base change over $f_i : U_i \to X$ for any $i \in I$ as $T \times_X U_i$ is a torsor over U_i, thus the same result of fpqc descent implies that $T \times_k G \overset{(id_T, \mu_T)}{\to} T \times_X T$ is an isomorphism, thus we have a G-torsor. $\square$

This result means that a G-torsor is *locally trivial* in the fpqc topology, where by local means over the components of an fpqc cover $\{f_i : U_i \to X\}_{i \in I}$ as one would to in topology by replacing the morphisms f_i by the inclusions by open subsets of a topological space. This property is comparable to a topological cover, that can be thought as a morphism of topological spaces that is locally an homeomorphism.
Also, using this property we can give more examples:

Example 2.3.10. *We will show two examples of torsors coming from the locally trivial property:*

1. *Let $L \supset k$ be a finite Galois extension of fields and let $Gal(L/k)$ be its Galois group. If we consider this group as a constant group-scheme, we have an obvious action $\mu_L : \mathrm{Spec}(L) \times_k Gal(L/k) \to L$ and an invariant faithfully flat morphism $t : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$.* (Galois extensions)
   This arrow is a torsor: by the primitive element theorem, $L = k(\alpha)$ where α is an algebraic element over k. If $p(x) \in k[x]$ is the minimal polynomial of α, then we have $L \cong k[x]/(p(x))$ and we can easily see that
   $$L \otimes_k L = L \otimes_k k[x]/(p(x)) = L[x]/(p(x))$$
   but as $L \supset k$ is Galois, all the roots of $p(x)$ are contained in L, we have that $p(x) = \prod_{i=1}^n (x - \alpha_i)$ with $\alpha_1 = \alpha$ and thus
   $$L \otimes_k L = \prod_{i=1}^n L[x]/(x - \alpha_i) \cong \prod_{i=1}^n L$$
   and as $Gal(L/k)$ permutes the roots of $p(x)$, we have an isomorphism $\mathrm{Spec}(L) \times_k Gal(L/k) \cong \mathrm{Spec}(L) \times_k \mathrm{Spec}(L)$, implying $t : \mathrm{Spec}(L) \to \mathrm{Spec}(k)$ is a $Gal(L/k)$-torsor.

2. *Let $f : Y \to X$ is a* finite étale covering, *i.e., a finite faithfully flat morphism such that for any point $x \in X$ the fiber $Y \times_X \mathrm{Spec}(\kappa(x))$ is the spectrum of an étale $\kappa(x)$-algebra (Definition-Proposition 2.2.103).* (Étale covers)
   A finite faithfully flat $f : Y \to X$ is an étale cover if and only if there exists finite and faithfully flat morphism $U \to X$ such that the base change $Y \times_X U \to U$ is a trivial étale covering, i.e., $Y \times_X U \cong \coprod U$ is a finite disjoint union of copies of U.
   If $f : Y \to X$ is an étale cover, for a geometric point, i.e., a morphism $\bar{x} : \mathrm{Spec}(\Omega) \to X$ where Ω is an algebraically closed field, we can consider the geometric fiber $Y_{\bar{x}}$ that possesses a free action ([64, Coro. 5.3.3]) from the abstract group $\Gamma = Aut(Y|X)$ of automorphisms of f. If the action of Γ on $Y_{\bar{x}}$ is in addition transitive, we say that an étale cover is Galois. *In this case, we can chose a finite and faithfully flat*

morphism $U \to X$ *such that the base change is trivial and isomorphic to* $U \times_k \Gamma$*, thus we see that in fact* $f : Y \to X$ *is a* Γ*-torsor*[7]*. The inverse is also correct, a* G*-torsor with* G *a constant group-scheme is a Galois étale cover, with* $G(k) = Aut(Y|X)$*, as the property of being an étale cover satisfies fpqc descent by Proposition 2.3.8(i).*
In general, if G *is an étale group-scheme (Definition 2.2.105), a* G*-torsor* $t : T \to X$ *is not necessarily an étale cover, but it becomes one after taking a finite separable extension* $L \supset k$ *such that* G_L *becomes constant, in which case the base change* $T_L \to X_L$ *is an étale cover, see [64, p. 5.3.16].*

A few immediate consequences of the locally trivial property are the following:

Lemma 2.3.11. *Let* X *be a scheme over* k *and let* G *be a group-scheme over* k*. Let* $t : T \to X$ *and* $t' : T' \to X$ *be two* G*-torsors over* X*. If* $f : T \to T'$ *is an* equivariant *morphism over* X*, i.e. a morphism that satisfies* $(f, id_{G_X}) \circ \mu_T = \mu_{T'} \circ (f, id_{G_X})$[8]*, then* $T \cong T'$ *over* X*.*

Proof. It suffices to show that T is isomorphic to T' over an fpqc covering using descent. Let $\{f_i : U_i \to X\}_{i \in I}$ be an fpqc cover such that $T \times_X U_i \to U_i$ is trivial for any $i \in I$, then we have an G_{U_i}-equivariant morphism $T \times_X U_i \to T' \times_X U_i$. If $T' \times_X U_i$ would be trivial, then any G_{U_i}-equivariant morphism would be an isomorphism, but as $T' \to X$ is also a torsor, we have another fpqc cover $\{g_j : V_j \to X\}_{j \in J}$ such that $T' \times_X V_j$ is trivial.
Using the properties in Proposition 2.3.7, it is not hard to see that for any $i \in I$, the family of morphisms $\{W_{ij} = V_j \times_X U_i \to U_i\}_{j \in J}$ is an fpqc cover of U_i. By construction, over any scheme W_{ij} both base changes of T and T' are trivial, thus isomorphic over U_i by fpqc descent (Proposition 2.3.8(l)) for any $i \in I$, so the conclusion follows from applying fpqc descent again to get an isomorphism over X. □

This lemma is related to the following concept in category theory:

Definition 2.3.12. *A category* $\mathcal{C}$ *is a* groupoid *if any arrow* $f : a \to b$ *is invertible, meaning there exists another arrow* $g : b \to a$ *such that* $g \circ f = id_a$ *and* $f \circ g = id_b$*.*

Torsors over a scheme form a groupoid

Example 2.3.13. *Let* X *be a scheme. If* $\mathcal{T}_G(X)$ *is the category of* G*-torsors over* X *with* G_X*-equivariant morphism as arrows, then by Lemma 2.3.11 this category is a groupoid.*

A G-torsor $t : T \to X$ is *trivial* if it is isomorphism to the trivial G-torsor (Example 2.3.4(1)). A corollary of Lemma 2.3.11 is the following:

7 Where Γ is considered as a constant group-scheme.
8 In other words, Definition 2.2.30 but over X instead of a field.

Corollary 2.3.14. *Let* X *be a scheme over* k *and let* G *be a group-scheme over* k*. If* $t : T \to X$ *is a* G*-torsor that possesses a section, i.e. a morphism* $s : X \to T$ *such that* $t \circ s = \mathrm{id}_X$*, then* T *is trivial.*

Proof. If $t : T \to X$ has a section, then the morphism

$$X \times_k G \overset{\mu_T \circ (s, \mathrm{id}_G)}{\rightarrow} T$$

is a G-equivariant morphism from the trivial torsor, thus T is isomorphic to the trivial torsor. □

There are still morphisms between torsors that are not isomorphisms, those are:

Definition 2.3.15. *Let* X *be a scheme over* k*. If* $t : T \to X$ *and* $t' : T' \to X$ *are a* G*-torsor and a* G'*-torsor respectively, a* morphism of torsors *is a morphism* $f : T \to T'$ *over* X *together with a morphism of group-schemes* $\phi : G \to G'$*, such that* f *intertwines the actions on the respective schemes via* ϕ*, meaning that the following diagram*

Morphisms of torsors

$$\begin{tikzcd} T \times_X G_X \arrow[r, "\mu_T"] \arrow[d, "(f,\phi)"'] & T \arrow[d, "f"] \\ T' \times_X G'_X \arrow[r, "\mu_{T'}"] & T' \end{tikzcd}$$

is commutative.
If $x \in X(k)$ *is a fixed rational point and both* T *and* T' *are pointed torsors, a* morphism of pointed torsors $f : T \to T'$ *is a morphism of torsors such that if* $y \in T(k)$ *and* $y' \in T'(k)$ *are rational points over* x*, then* $f(t) = t'$*.*

For Chapter 3, we will restrict ourselves to torsors over affine group-schemes, which induce properties over their structural morphisms in the following sense:

Proposition 2.3.16. *Let* $t : T \to X$ *be a* G*-torsor. If* G *is affine, or of finite type or finite or smooth or étale over* k*, then* $t : T \to X$ *is as well.*

Proof. Let $\{f_i : U_i \to X\}_{i \in I}$ be an fpqc cover such that $T \times_X U_i \to U_i$ is trivial for any $i \in I$. If G has one of the properties mentioned in the statement over k, then the first projection $U_i \times_X G \to U_i$ has the same property as all the properties mentioned are preserved by base change, so the conclusion follows from fqpc descent (Proposition 2.3.8). □

This result justifies attaching an adjective to a torsor, according to the properties of its corresponding group-scheme. The names we will use in this thesis are:

Definition 2.3.17. *Let* X *be a scheme over* k *and let* G *be a group-scheme over* k*. If* $t : T \to X$ *is a* G*-torsor, we will call the torsor:*

(a) Finite *if* G *is finite over* k*.*

Finite torsors

Algebraic torsors (b) Algebraic *if* G *is of finite type over* k.

Affine torsors (c) Affine *if* G *is affine over* k.

Étale torsors (d) Étale *if* G *is étale (Definition 2.2.105) over* k.

Pro-finite/Pro-algebraic torsors (e) Pro-finite *if* G *is pro-finite or* Pro-algebraic *if* G *is pro-algebraic.*

in accordance to Proposition 2.3.16.

2.3.2 *G-Equivariant Sheaves*

From now on, will be work over the following setting:

Setting 2.3.18. *The scheme* X *will be of finite type over* k *and any* G*-torsor* $t : T \to X$ *will be an affine torsor over* X*, in particular* t *will be always an affine morphism by Proposition 2.3.16.*

In this subsection, we want to study the pull-back of quasi-coherent sheaves along a G-torsor $t : T \to X$. We will denote the category of quasi-coherent sheaves over X as $\mathrm{QCoh}(X)$.

Definition 2.3.19. *We define the category QCoh as the category of pairs* $(X, \mathcal{E})$ *where* X *is a scheme and* $\mathcal{E}$ *is a quasi-coherent sheaf over* X*. A morphism* $(X, \mathcal{E}) \to (Y, \mathcal{F})$ *in this category is the datum of a morphism of schemes* $f : X \to Y$ *together with a morphism of sheaves* $g : \mathcal{E} \to f^*(\mathcal{F})$*, that we will denote as a commutative diagram*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "g"] \arrow[d, mapsto] & \mathcal{F} \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \end{tikzcd}.$$

Remark 2.3.20. *There is a problem with Definition 2.3.19: categories need compositions, and if we would have to do a composition*

$$\begin{tikzcd} \mathcal{E} \arrow[r, "g"] \arrow[d, mapsto] & \mathcal{F} \arrow[r, "g'"] \arrow[d, mapsto] & \mathcal{G} \arrow[d, mapsto] \\ X \arrow[r, "f"'] & Y \arrow[r, "f'"'] & Z \end{tikzcd}$$

we have the choice to either consider a morphism $\mathcal{E} \to (f' \circ f)^*(\mathcal{G})$ *or a morphism* $\mathcal{E} \to (f')^*(f^*(\mathcal{G}))$ *as* $(f' \circ f)^*(\mathcal{G})$ *and* $(f')^*(f^*(\mathcal{G}))$ *are different sheaves. Hopefully, they are canonically isomorphic so there is a way to solve this issue.*

Pseudo-functors **Definition 2.3.21** (Definition 3.10 [67])**.** *Let* $\mathcal{C}$ *be a category. A* pseudo-functor[9] *over* $\mathcal{C}$*, denoted as* Φ*, is composed by the following layers of data:*

9 Also known as a "lax 2-functor".

(a) *For any object* U *of* $\mathcal{C}$, *a category* $\Phi(U)$.

(b) *For any morphism* $f : U \to V$, *a functor* $f^* : \Phi(V) \to \Phi(U)$.

(c) *For any object* U *of* $\mathcal{C}$, *an isomorphism of functors* $\Phi(U) \to \Phi(U)$ *denoted as*

$$\epsilon_U : (\mathrm{id}_U)^* \cong \mathrm{id}_{\Phi(U)},$$

where the functor $\mathrm{id}_{\Phi(U)} : \Phi(U) \to \Phi(U)$ *is the functor that is the identity on the objects and on the morphisms of the category* $\Phi(U)$.

(d) *For any pair of arrows* $U \xrightarrow{f} V \xrightarrow{g} W$ *between objects of* $\mathcal{C}$, *an isomorphism of functors* $\Phi(V) \to \Phi(U)$ *denoted as*

$$\alpha_{f,g} : f^* \circ g^* \cong (g \circ f)^*.$$

For Φ *to be a pseudo-functor, this data must additionally satisfy the following properties:*

(i) *For any morphism* $f : U \to V$ *and any object* η *of* $\Phi(V)$, *we have the following identity of objects in* $\Phi(U)$:

$$\alpha_{\mathrm{id}_U,f}(\eta) = \epsilon_U(f^*(\eta))$$

giving the identity of natural transformations

$$\alpha_{\mathrm{id}_U,f} = \epsilon_U(f^*) : (\mathrm{id}_U)^* \circ f^* \to f^*,$$

and also the identity

$$\alpha_{f,\mathrm{id}_U}(\eta) = f^*(\epsilon_V(\eta))$$

giving the identity of natural transformations

$$\alpha_{f,\mathrm{id}_U} = f^*(\epsilon_V) : f^* \circ (\mathrm{id}_V)^* \to f^*.$$

(ii) *For any diagram of objects and morphisms of* $\mathcal{C}$

$$U \xrightarrow{f} V \xrightarrow{g} W \xrightarrow{h} T$$

and any object $\theta \in \Phi(T)$, *we have that the following diagram*

$$\begin{tikzcd}
(f^* \circ g^* \circ h^*)(\theta) \arrow[r, "\alpha_{f,g}(h^*)(\theta)"] \arrow[d, "f^*(\alpha_{g,h})(\theta)"'] & ((g \circ f)^* \circ h^*)(\theta) \arrow[d, "\alpha_{g\circ f,h}(\theta)"] \\
(f^* \circ (h \circ g)^*)(\theta) \arrow[r, "\alpha_{f,h\circ g}(\theta)"'] & (h \circ g \circ f)^*(\theta)
\end{tikzcd}$$

is commutative.

Remark 2.3.22. *Let $\mathcal{C}$ be a category. If we have a pseudo-functor Φ over $\mathcal{C}$, we can construct a category $\mathcal{F}$ with a functor $F : \mathcal{F} \to \mathcal{C}$: The elements of $\mathcal{F}$ are pairs (U, ξ) where U is an object of $\mathcal{C}$ and ξ is an object of $\Phi(U)$, and morphisms are pairs $(f, a) : (U, \xi) \to (V, \eta)$ where $f : U \to V$ is a morphism in $\mathcal{C}$ and $a : \xi \to f^*(\eta)$ in $\Phi(U)$. The functor $F : \mathcal{F} \to \mathcal{C}$ is defined as $F(U, \xi) = U$ and $F(f, a) = f$ for an arrow $(f, a) : (U, \xi) \to (V, \eta)$.*
For $\mathcal{F}$ to be a category, we need to also define identity arrows and compositions. Starting with compositions, for two arrows $(f, a) : (U, \xi) \to (V, \eta)$ and $(g, b) : (V, \eta) \to (W, \zeta)$, their composition is the arrow $(g \circ f, b \cdot a) : (U, \xi) \to (W, \zeta)$ where $b \cdot a$ is given by the composition

$$b \cdot a = \xi \xrightarrow{a} f^*(\eta) \xrightarrow{f^*(b)} f^*(g^*(\zeta)) \xrightarrow{\alpha_{f,g}(\zeta)} (g \circ f)^*(\zeta)$$

in $\Phi(U)$, where $\alpha_{f,g}$ is the natural transformation in Definition 2.3.21(d).
For the identity, if (U, ξ) is an object in $\mathcal{F}$, we have an isomorphism $\epsilon_U(\xi) : (\mathrm{id}_U)^*(\xi) \to \xi$ (Definition 2.3.21(c)), and thus we can define the identity morphism of a pair $\mathrm{id}_{(U,\xi)} : (U, \xi) \to (U, \xi)$ as $(\mathrm{id}_U, \epsilon_U^{-1}(\xi))$.
Using the properties in Definition 2.3.21(i) and (ii), we can show that the composition in $\mathcal{F}$ is associative and that $\mathrm{id}_{(U,\xi)}$ is indeed a neutral element for the composition, see [67, §3.1.3] for more details.

To see that the category Qcoh in Definition 2.3.19 is indeed a category, it suffices to show that we are in the presence of a pseudo-functor:

Lemma 2.3.23. *Let $\mathcal{C} = Sch_S$ be the category of schemes over a base scheme S. The assignment $X \mapsto Qcoh(X)$ is a pseudo-functor with the pull-back of sheaves functor and the obvious canonical isomorphisms for compositions and the identity.*

Proof. See [67, §3.2.1]. □

As a consequence of this lemma, there is a natural functor F : Qcoh $\to$ Sch$_S$ sending a pair $(X, \mathcal{F})$ to X, following Remark 2.3.22.
Now we can define G-equivariant sheaves:

G-equivariant sheaves

Definition 2.3.24. *Let S be a fixed base scheme, and let T be a scheme over S with an action from an affine group-scheme $G \to S$ (Recall Definition 2.2.1). A* G-equivariant *sheaf over T is a quasi-coherent sheaf $\mathcal{E}$ over T, together with an action of the abstract group $\widetilde{G}(Z) = Hom_S(Z, G)$, for any Z-point $Z \to T$ of T over S, on the set $Hom_{Qcoh}((Z, \mathcal{F}), (T, \mathcal{E}))$ for any quasi-coherent sheaf $\mathcal{F}$ over Z in a commutative diagram*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\ Z \arrow[r] & T \end{tikzcd}$$

that satisfy the following two conditions:

(a) *For any morphism of objects of Qcoh*

$$\begin{tikzcd} \mathcal{G} \arrow[r] \arrow[d, mapsto] & \mathcal{F} \arrow[d, mapsto] \\ Y \arrow[r, "f"'] & Z \end{tikzcd}$$

where $f : Y \to Z$ *is a morphism of schemes over* T*, the induced map* $f^* : Hom_{Qcoh}((Z, \mathcal{F}), (T, \mathcal{E})) \to Hom_{Qcoh}((Y, \mathcal{G}), (T, \mathcal{E}))$ *intertwines the respective actions with respect to the morphism of abstract groups* $\widetilde{G}(Z) \to \widetilde{G}(Y)$.

(b) *The functor* $Hom_{Qcoh}((Z, \mathcal{F}), (T, \mathcal{E})) \to Hom_{Sch_S}(Z, T)$ *is* $\widetilde{G}(Z)$*-equivariant for every quasi-coherent sheaf* $\mathcal{F}$ *and any scheme* Z *over* T.

We will denote the category of G*-equivariant sheaves of* T *as* $Qcoh^G(T)$.

Example 2.3.25. *The structural sheaf of a scheme* T *over* S *with an action* μ *from an affine group-scheme* G *is* G*-equivariant: if* $f : Z \to T$ *is an* Z*-point of* T*, then for any quasi-coherent sheaf* $\mathcal{F}$ *over* Z*, we have a morphism in Qcoh*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, mapsto] & \mathcal{O}_T \arrow[d, mapsto] \\ Z \arrow[r, "f"'] & T \end{tikzcd}$$

where the upper arrow is a morphism $\mathcal{F} \to f^*(\mathcal{O}_T) = \mathcal{O}_Z$ *of sheaves over* Z*. This allows us to define an action (on the right) from the abstract group* $\widetilde{G}(Z)$ *for any element* $g \in \widetilde{G}(Z)$ *via the composition*

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[dr, mapsto] & \mathcal{O}_Z \arrow[r] \arrow[d, mapsto] & \mathcal{O}_T \arrow[d, mapsto] \\ & Z \arrow[r, "f \cdot g"'] & T \end{tikzcd} .$$

where $f \cdot g$ *denotes the morphism* $Z \to T$ *obtained by taking the composition* $\mu(f, g)$.

There is another construction that will be important for the tannakian aspect of the fundamental group-scheme, we will state it in Section 3.3.

Remark 2.3.26. *The classical definition of a* G*-equivariant sheaf* $\mathcal{E}$ *over* T *with an action* $\mu : T \times_S G \to T$ *is an isomorphism* $\lambda : p_1^*(\mathcal{E}) \cong \mu^*(\mathcal{E})$ *such that if* $\mu_0 : T \times_S G \times_S G \to T$ *is the natural projection,* $\mu_1 = \mu \circ p_{12} : T \times_S G \times_S G \to T$ *and* $\mu_2 = \mu \circ (\mu, id_G)$*, then we have the following commutative diagram of sheaves over* $T \times_S G \times_S G$

$$\begin{tikzcd} \mu_0^*(\mathcal{E}) \arrow[rr, "(id_T, m_G)^* \circ \lambda"] \arrow[dr, "p_{12}^* \circ \lambda"'] & & \mu_2^*(\mathcal{E}) \\ & \mu_1^*(\mathcal{E}) \arrow[ur, "(\mu, id_G)^* \circ \lambda"'] & \end{tikzcd} .$$

This definition is equivalent to Definition 2.3.24, see [67, Prop. 3.49].

The main reason to introduce G-equivariant sheaves is the following:

Pull-back of quasi-coherent sheaves over an affine torsor

Theorem 2.3.27 (Theorem 4.46 [67]). *Let G be an affine group-scheme over k, and let X be a k-scheme. If $t : T \to X$ is a G-torsor, then any pull-back of a quasi-coherent sheaf over X is G-equivariant under t, and the functor $t^* : Qcoh(X) \to Qcoh^G(T)$ is an equivalence of categories.*

This theorem applied to vector bundles yields:

Pull-back of vector bundles over an affine torsor

Corollary 2.3.28. *Let G be an affine group-scheme over k, and let X be a k-scheme. If $t : T \to X$ is a G-torsor, then the restriction of the functor functor $t^* : Qcoh(X) \to Qcoh^G(T)$ to the full sub-category $Vect(X)$ of vector bundles over X, is an equivalence of categories $t^* : Vect(X) \to Vect^G(T)$.*

Proof. This stems directly from Theorem 2.3.27 by noting that the property of being a locally free sheaf of finite rank satisfies fpqc descent [32, Prop. 2.5.2]: if $\{f_i : U_i \to X\}_{i \in I}$ is an fpqc cover of X and $\mathcal{E}$ is a quasi-coherent sheaf such that $f_i^*(\mathcal{E})$ is a locally free sheaf of finite rank $r \geqslant 1$ over U_i for any $i \in I$, then $\mathcal{E}$ is locally free of rank r over X. □

2.3.3 *Additional Constructions and Properties*

Keeping the hypotheses of Setting 2.3.18, the first thing we would like to show is that torsors are in fact quotients in the sense of Definition 2.2.39.

To this purpose we start with a property for schemes, related to fpqc covers.

Definition 2.3.29. *Let be $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ an fpqc cover of a scheme X. For another scheme Y, we will say that a collection of morphisms $\{g_i : U_i \to Y\}_{i \in I}$ is a* morphism from the cover $\mathcal{U}$ to Y, *that we will denote as an arrow $\mathcal{U} \to Y$, if for any pair of elements $i, j \in I$, the following diagram*

$$\begin{tikzcd} U_i \times_X U_j \arrow[r] \arrow[d] & U_j \arrow[d, "g_j"] \\ U_i \arrow[r, "g_i"'] & Y \end{tikzcd}$$

is commutative. We will denote the set of all morphism $\mathcal{U} \to Y$ as $Hom(\mathcal{U}, Y)$.

Remark 2.3.30. *Let S be a base scheme. For any fpqc cover $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ of a scheme X over S, there is an obvious map $Hom_{Sch_S}(X, Y) \to Hom(\mathcal{U}, Y)$ that takes a morphism $g : X \to Y$ to the morphism $\mathcal{U} \to Y$ given by the set $\{g \circ f_i : U_i \to Y\}_{i \in I}$.*

The functor $\mathrm{Hom}_{Sch_S}(X, \cdot)$ is representable, contravariant and it is a special kind of functor, but first we have to introduce some suggesting notation attached to it:

Notation 2.3.31. *Let* S *be a base scheme. If* $F : Sch_S \to Set$ *be a contravariant functor,* X *is a scheme over* S *and* $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$ *is an fpqc covering, we will denote for an element* $a \in F(X)$ *its image on* $F(U_i)$ *via* $F(f_i)$ *as* $a|_{U_i}$ *for any* $i \in I$.
Moreover, we can consider for any pair of indexes $i, j \in I$ *the image of a section* $a_k \in F(U_k)$ $(k = i, j)$ *on* $F(U_i \times_X U_j)$ *induced by the projection* $p_{ij,k} : U_i \times_X U_j \to U_k$ *as* $a_k|_{U_i \times_X U_j}$.

Definition 2.3.32. *Let* $F : Sch_S \to Set$ *be a contravariant functor. Given a scheme* X *over* S *and an fpqc covering* $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$, *we say that* F *satisfies the* sheaf property for $\mathcal{U}$ *if for any family* $(a_i)_{i \in I}$ *with* $a_i \in F(U_i)$ *such that for any pair of elements* $i, j \in I$, *we have that* $a_i|_{U_i \times_X U_j} = a_j|_{U_i \times_X U_j}$, *then there exists a unique element* $a \in F(X)$ *such that* $a_i = a|_{U_i}$ *for all* $i \in I$.
F *is a* fpqc sheaf *if* F *satisfies the sheaf property for any fpqc cover over any scheme.*

Proposition 2.3.33 (Grothendieck)**.** *Let S be a base scheme. For any scheme* X *over* S *and any fpqc cover* $\mathcal{U} = \{f_i : U_i \to X\}_{i \in I}$, *the assignment of Remark 2.3.30*

Representable functors are fpqc sheaves

$$Hom_{Sch_S}(X, Y) \to Hom(\mathcal{U}, Y)$$

for a scheme Y *defines a natural transformation* $Hom_{Sch_S}(X, \cdot) \to Hom(\mathcal{U}, \cdot)$ *which is an isomorphism of functors.*
In particular, any representable functor is an fpqc sheaf.

Proof. See [67, Theorem 2.55]. □

An easy consequence of this result that comes from applying the sheaf property (Definition 2.3.32) over a single fpqc-morphism is:

Corollary 2.3.34. *Any fpqc morphism (Definition 2.3.5)* $f : X \to Y$ *of schemes over a base scheme* S *is an effective epimorphism (Definition 2.2.39)*

This applies to torsors to obtain the following:

Proposition 2.3.35. *Let* X *be a* k*-scheme and let* G *be an affine group-scheme. If* $t : T \to X$ *is a* G*-torsor over* X, *then* $X = T/G$ *by the equivalence relation* $T \times_k G \overset{\mu_T}{\underset{p_1}{\rightrightarrows}} T$ *in the sense of Definition 2.2.39. Moreover, we have that* $\mathcal{O}_X = \mathcal{O}_T^G$ *is the sheaf of invariant sections, defined in Theorem 2.2.40(iii).*

Torsors are quotients

Proof. By the definition of torsor (Definition 2.3.1), $t : T \to X$ is G-invariant, an effective epimorphism by Corollary 2.3.34 and we have the isomorphism $T \times_k G \cong T \times_X T$ so the first part of the statement holds.
The only property missing is at the level of structural sheaves: as t is faithfully flat, we have that $\mathcal{O}_X \subset t_*(\mathcal{O}_T)$, if $\mathcal{O}_T^G \subset t_*(\mathcal{O}_T)$ is the sheaf of invariant sections, we clearly have that an inclusion $\mathcal{O}_X \subset \mathcal{O}_T^G$ so we need to show that if $s \in t_*(\mathcal{O}_T)(U) = \mathcal{O}_T(t^{-1}(U))$ is G-invariant

where $U \subset X$ is an open sub-scheme, then $s \in \mathcal{O}_X(U)$.
The fact that s is G-invariant can be expressed as the equality $(\mu_T)^*(s) = (p_1)^*(s) \in \mathcal{O}_{T\times_k G}(t^{-1}(U) \times_k G)$ but the latter scheme is isomorphic to $t^{-1}(U) \times_U t^{-1}(U)$ and we see that it is the fibered product associated to the fpqc cover $\{t^{-1}(U) \to U\}$ and the equality $(\mu_T)^*(s) = (p_1)^*(s)$ means that $s \in \mathcal{O}_T(t^{-1}(U))$ has equal restrictions to $t^{-1}(U) \times_U t^{-1}(U)$ under both canonical projections so we could use Proposition 2.3.33 to conclude that $s \in \mathcal{O}_X(U)$. The only obstacle we have, is that this proposition is stated for a representable functor, and s is a global section of a sheaf. But this is not an issue as for any scheme, we have the natural isomorphisms $\mathrm{Hom}_{\mathrm{Sch}_k}(Y, \mathbb{A}^1) \cong \Gamma(Y, \mathcal{O}_Y)$ for any scheme Y, that we can apply to all the sections considered so far, finishing the proof. □

We will finish this section by considering some constructions on torsors, that are related to morphisms of group-schemes.

Sub-torsors

Definition 2.3.36. *Let $t : T \to X$ is a G-torsor over X with G affine. A* sub-torsor *is an H-torsor $f : V \to T$ where H is a subgroup-scheme of G, with a morphism of torsors (Definition 2.3.15) that is a closed immersion $V \to T$ at the level of schemes and the inclusion morphism $H \to G$ at the level of group-schemes.*

Remark 2.3.37. *We could have defined a sub-torsor with just an* immersion[10] *$f : V \to T$ at the level of schemes, but as G is affine, any subgroup-scheme of G is closed and if $\{U_i \to X\}_{i\in I}$ is an fpqc cover that trivializes both T and V, which always exists following the proof of Lemma 2.3.11, then f becomes $U_i \times_k H \to U_i \times_k G$ over any U_i that is clearly a closed immersion, thus f is a closed immersion by fpqc descent (Proposition 2.3.8(j)).*

For pointed torsors, the notion of a sub-torsor must keep track of the rational points on each scheme, so we have

Definition 2.3.38. *Let X be a scheme over k and let $x \in X(k)$ be a rational point. If G is an affine group-scheme and $t : T \to X$ is a pointed G-torsor, a* pointed sub-torsor *is a sub-torsor $f : V \to T$ that is a pointed morphism of torsors, meaning that if $y \in T(k)$ and $v \in V(k)$ are the rational points of T and V respectively, we have that $f(v) = y$.*

Remark 2.3.39. *We see from the definition of a pointed sub-torsor, that if $y \in T(k)$ is a fixed rational point, a general sub-torsor $V \to T$ might not be a pointed sub-torsor, it is the case if the image of V on T contains the closed point y, otherwise it cannot be a pointed sub-torsor. So a priori, there are more sub-torsors than pointed sub-torsors for a given rational point of T.*

The following type of torsor is the most important type of torsor we will consider in this document:

10 An immersion of schemes is the composition of a closed immersion followed by an open immersion.

Nori-reduced torsors

Definition 2.3.40. *Let X be a scheme over k and let G be an affine group-scheme. A G-torsor $t : T \to X$ is* Nori-reduced *if T does not posses any sub-torsor apart from itself, or equivalently, any morphism of torsors $T' \to T$ over X is faithfully flat.*

We have just seen the first type of morphism between torsors that is clearly related to subgroup-schemes of an affine group-scheme G. But what about quotients or general morphisms $\phi : G \to H$? The answer is the following:

Contracted product

Definition 2.3.41. *Let X be a scheme over k with a right action $\mu_X : X \times_k G \to G$ from an affine group-scheme G. If $\phi : G \to H$ is a morphism of group-schemes, let $\mu_H : G \times_k H \to H$ be the left G-action induced by ϕ, defined as $\mu_H = m_H \circ (\phi, id_H)$.*
We define the contracted product of X by ϕ *as the quotient*[11] *of the product $X \times_k H$ by the left action of G given by $\mu_X(id_X, i_G)$ on the first coordinate where i_G is the inverse morphism of G, and μ_H on the second, if such quotient exists, we will denote it as $X \times^G H$.*

We have several remarks about this construction:

Remark 2.3.42. *Under the hypotheses of Definition 2.3.41:*

(a) *At the level of the functor of points, the action of G over the product $X \times_k H$ looks as*

$$g \cdot (x, h) = (x \cdot g^{-1}, \widetilde{\phi}(R)(g)h)$$

for $g \in \widetilde{G}(R)$, $h \in \widetilde{H}(R)$, $x \in \widetilde{X}(R)$ and $\widetilde{\phi}(R) : \widetilde{G}(R) \to \widetilde{H}(R)$ is the induced morphism coming from ϕ at the level of R-points, where R is a k-algebra not necessarily of finite type as the schemes involved are not of finite type over k in general.

(b) *There is a natural right action from H over the contracted product $X \times^G H$ coming from the right action of H on $X \times_k H$ by multiplication on the right over the second coordinate.*

(c) *The notion of quotient we have use in the definition of the contracted product is in fact the cokernel associated to the diagram*

$$G \times_k (X \times_k H) \underset{p_2}{\overset{\mu}{\rightrightarrows}} X \times_k H\,,$$

that we called quotient by an abuse of language in the case of an action by a group-scheme. The main case we are interested in, is when we have a G-torsor $t : T \to X$ where μ_T is free and transitive (Definition 2.2.29). From part (a), we see that in that case the action is always free, and thus $X \times^G H$ is a quotient by an equivalence relation (Definition 2.2.39) in this case.

11 In the sense of Example 2.2.35.

(d) *If we consider* G *acting on itself by multiplication on the right, it is not hard to see that the contracted product exists, and* $G \times^G H \cong H$ *as we have the commutative diagram*

$$G \times_k (G \times_k H) \underset{p_2}{\overset{\mu}{\rightrightarrows}} G \times_k H \xrightarrow{\mu_H} H$$

and we can verify the isomorphism at the level of functor of points. This fact will be important when we will consider contracted products of torsors.
Under this isomorphism, the action of H *over the contracted product described in part (b) is just the multiplication on the right.*

Now we want to prove that contracted products of torsors exist, we start with a lemma:

Lemma 2.3.43. *Let* $X_1 \underset{u_1}{\overset{u_0}{\rightrightarrows}} X_0$ *be an equivalence relation of schemes over a base scheme* S*. Let us suppose that the quotient* Y *of this relation exists, and that the projection morphism* $\pi : X_0 \to Y$ *is an fpqc cover. Then, for any morphism* $f : S' \to S$*, the base change* $Y \times_S S'$ *is the quotient of the induced equivalence relation* $X_1 \times_S S' \underset{u'_1}{\overset{u'_0}{\rightrightarrows}} X_0 \times_S S'$.

Proof. Let us denote $X'_0 = X_0 \times_S S'$, $X'_1 = X_1 \times_S S'$ and $Y' = Y \times_S S'$. The base change $\pi' : X'_0 \to Y'$ is an fpqc cover (Proposition 2.3.7(d)) and thus it is an effective epimorphism by Corollary 2.3.34. Moreover, as Y is a quotient, we have an isomorphism $X_1 \cong X_0 \times_Y X_0$ induced by u_0 and u_1 that also holds over S', and thus we have an isomorphism $X'_1 \cong X'_0 \times_{Y'} X'_0$ so the only thing we need to prove is that Y' is a cokernel.
Let Z be a scheme over S' with a morphism $g : X'_0 \to Z$ such that $g \circ u'_0 = g \circ u'_1$, then from the last paragraph, we see that $g \circ p_1 = g \circ p_2$ using the isomorphism, where p_1 and p_2 are the canonical projections of $X'_0 \times_{Y'} X'_0$. Thus, the conclusion follows using either the fact that Y' is the cokernel of the diagram $X'_0 \times_{Y'} X'_0 \underset{p_2}{\overset{p_1}{\rightrightarrows}} X'_0$ or either the sheaf property of the functor $\mathrm{Hom}_{\mathrm{Sch}_{S'}}(\cdot, Y')$ (Proposition 2.3.33). □

Corollary 2.3.44. *Let* X *be a* k*-scheme, and let* G, H *be two affine group-schemes with a morphism* $\phi : G \to H$*. Then, the contracted product* $(X \times_k G) \times^G H$ *exists and we have isomorphisms*

$$(X \times_k G) \times^G H \cong X \times_k (G \times^G H) \cong X \times_k H$$

where X *is considered with a trivial action from* G*.*

Proof. We know that $G \times^G H$ exists and that the projection $G \times_k H \to H$ is an action (Remark 2.3.42(d)), thus this morphism is faithfully flat and affine (Lemma 2.2.27), thus it is an fpqc cover and the conclusion follows from Lemma 2.3.43. □

We will apply these results using the following: Let $t : T \to X$ be a G-torsor and let $\mathcal{U} : \{U_i \to X\}_{i \in I}$ be an fpqc cover of X such that $T \times_X U_i$ is a trivial torsor over U_i for all $i \in I$. As such, we have $T \times_X U_i \cong U_i \times_k G$ and we easily see from Corollary 2.3.44 that for any morphism $\phi : G \to H$ the contracted product exists for the trivial torsor over any U_i. So, to obtain the contracted product for T over X, we need a way to "recollect" all these contracted products into a scheme over X and show that this is the quotient we are looking for. Before defining this recollection process, we need a remark:

Remark 2.3.45. *Let* X *be a scheme over a base scheme* S*, and let* $\mathcal{U} : \{U_i \to X\}_{i \in I}$ *be an fpqc cover of* X*. If* $f : Y \to X$ *is a morphism of schemes over* S*, we have the restrictions* $f_i = f|_{U_i} : Y \times_X U_i \to X$ *and if we consider the fibered products* $U_{ij} = U_i \times_X U_j$ *for* $i, j \in I$*, we have isomorphisms* $U_{ij} \cong U_{ji}$*. We also have isomorphisms for any pair* $i, j \in I$

$$\eta_{ij} : Y_{ji} = Y_j \times_{U_j} U_{ij} \cong Y_{ij} = Y_i \times_{U_i} U_{ij}.$$

as $Y_{ij} = Y \times_X U_{ij}$ *and likewise for* Y_{ji}*.*
Out of these isomorphisms, we have the identities $f_j|_{Y_{ji}} = f_i|_{Y_{ij}} \circ \eta_{ij}$ *for the restrictions of* f_i *over* U_{ij} *for any pair* $i, j \in I$*.*
We have described what happens to f *over any element of the fpqc cover, their fibered products, and we can moreover describe what happens to* Y *over the triple products* $U_{ijk} = U_i \times_X U_j \times_X U_k$ *for any triple of indexes* $i, j, k \in I$*.*
If we consider the fibered products $Y_{ijk} = Y_{ij} \times_{U_{ij}} U_{ijk} = Y_i \times_{U_i} U_{ijk}$*, we have the so called* cocycle condition

$$\eta_{ik}|_{Y_{kji}} = \eta_{ij}|_{Y_{jki}} \circ \eta_{jk}|_{Y_{kji}}.$$

The cocycle condition

In regard to the paragraph before the remark, if we consider a morphism $f : Y \to X$ to be already recollected, the identities in the remark give us the necessary definition for the data we would like to recollect:

Affine descent datums

Definition 2.3.46. *Let* X *be a scheme over a base scheme* S*, and let* $\mathcal{U} : \{U_i \to X\}_{i \in I}$ *be an fpqc cover of* X*. A family of affine morphisms* $(\{f_i\}, \{\eta_{ij}\})_{i,j \in I}$ *where* $f_i = Y_i \to U_i$ *and* $\eta_{ij} = Y_{ji} \xrightarrow{\sim} Y_{ij}$ *such that* $f_j|_{Y_{ji}} = f_i|_{Y_{ij}} \circ \eta_{ij}$ *is an* affine descent datum *on* $\mathcal{U}$ *if for any triple* $i, j, k \in I$ *the family satisfies the cocycle condition* $\eta_{ik}|_{Y_{kji}} = \eta_{ij}|_{Y_{jki}} \circ \eta_{jk}|_{Y_{kji}}$*.*
A descent datum $(\{f_i\}, \{\eta_{ij}\})_{i,j \in I}$ *over* $\mathcal{U}$ *is* effective *if there exists a scheme* Y *with an affine morphism* $f : Y \to X$ *such that* $f_i = f|_{U_i}$ *for any* $i \in I$*.*
A morphism *between two affine descent datums* $(\{f_i\}, \{\eta_{ij}\})_{i,j \in I}$ *and* $(\{g_i\}, \{\theta_{ij}\})_{i,j \in I}$ *over* $\mathcal{U}$ *with* $g_i : Z_i \to U_i$*, that we will denote as* α *:*

$(\{f_i\},\{\eta_{ij}\})_{i,j\in I} \to (\{g_i\},\{\theta_{ij}\})_{i,j\in I}$, *is a collection of morphisms* $\alpha_i : Y_i \to Z_i$ *for* $i \in I$ *such that the following diagram*

$$\begin{tikzcd} Y_i \arrow[rr, "\alpha_i"] \arrow[dr, "f_i"'] & & Z_i \arrow[dl, "g_i"] \\ & U_i & \end{tikzcd}$$

is commutative for any $i \in I$, *and the following diagram*

$$\begin{tikzcd} Y_{ji} \arrow[r, "\alpha_j|_{Y_{ji}}"] \arrow[d, "\eta_{ij}"'] & Z_{ji} \arrow[d, "\theta_{ij}"] \\ Y_{ij} \arrow[r, "\alpha_i|_{Y_{ij}}"'] & Z_{ij} \end{tikzcd}$$

is commutative for any pair $i, j \in I$.
The category of affine descent datums over $\mathcal{U}$ *will be denoted as* $Aff(\mathcal{U})$.

Effective descent for affine schemes

Theorem 2.3.47 (Theorem 4.33 [67])**.** *Let* X *be a scheme over a base scheme* S, *and let* $\mathcal{U} : \{U_i \to X\}_{i\in I}$ *be an fpqc cover of* X. *Then, any affine descent datum over* $\mathcal{U}$ *is effective.*
In other words, if $Aff(X)$ *denotes the category of schemes affine over* X *with affine morphisms, the functor* $Aff(X) \to Aff(\mathcal{U})$ *that takes a scheme affine over* $f : Y \to X$ *to the affine the descent datum it defines, described in Remark 2.3.45, is an equivalence of categories.*

Corollary 2.3.48. *Let* X *be a scheme over* k *and let* $t : T \to X$ *be a* G*-torsor over* X *with* G *an affine group-scheme. Then, for any morphism of group-schemes* $\varphi : G \to H$ *the contracted product* $T \times^G H$ *exists and it is an* H*-torsor over* X.

Proof. Let us start with the second assertion: if $T \times^G H$ exists, let us consider $T \times_k H$ and the projection $p : T \times_k H \to T$ with the left G-action described in Definition 2.3.41 and the right H-action described in Remark 2.3.42(b), we see that the composition $t \circ p$ is invariant for both G and H, as T is a torsor and p is H-invariant, thus the induced morphism from contracted product $T \times^G H \to X$ is H-invariant. Moreover, let $\mathcal{U} : \{U_i \to X\}_{i\in I}$ be an fpqc cover of X such that T becomes trivial over any U_i, the morphism $T \to T \times^G H$ over X becomes $U_i \times_k G \to U_i \times_k H$, thus the contracted product is trivialized over $\mathcal{U}$ as well and thus it is an H-torsor over X by Proposition 2.3.9.
For the existence of the contracted product, if T is trivialized by the fpqc cover $\mathcal{U}$, then for any U_i we can take the contracted product $T|_{U_i} \cong U_i \times_k G \to U_i \times_k H$ that exists by Corollary 2.3.44. Then, if we consider the affine descent datum $(\{t_i\},\{\eta_{ij}\})_{i,j\in I}$ associated to T, given by $t_i : U_i \times_k G \to U_i$ and $\eta_{ij} : U_{ji} \times_k G \xrightarrow{\sim} U_{ji} \times_k G$ for $i, j \in I$ and subject to the cocycle condition $\eta_{ik}|_{T_{kji}} = \eta_{ij}|_{T_{jki}} \circ \eta_{jk}|_{T_{kji}}$, we

see that we can take contracted products on any element of the descent datum, again by Corollary 2.3.44, thus naturally obtaining an affine descent datum $(\{u_i\},\{\theta_{ij}\})_{i,j\in I}$ with a morphism

$$\pi : (\{t_i\},\{\eta_{ij}\})_{i,j\in I} \to (\{u_i\},\{\theta_{ij}\})_{i,j\in I}.$$

Thus, by Theorem 2.3.47, there exists an scheme Y over X with a morphism $\pi : T \to Y$ such that $Y|_{U_i} = U_i \times_k H$ for any $i \in I$. Using fpqc descent (Proposition 2.3.8) we can easily see that Y fits into the commutative diagram

$$G \times_k (T \times_k H) \overset{\mu}{\underset{p_2}{\rightrightarrows}} (T \times_k H) \xrightarrow{q} Y$$

and that $G \times_k (T \times_k H) \cong (T \times_k H) \times_Y (T \times_k H)$, so to obtain a quotient according to Definition 2.2.39, we just need to show that Y is a cokernel, but this is true over any U_i, so the conclusion follows by using Proposition 2.3.33 for any scheme Z fitting into a commutative diagram similar to Y's to obtain a morphism $Y \to Z$, finishing the proof. □

Now that we have established the existence of contracted products of torsors, the main reason we introduced them is the following:

Proposition 2.3.49. *Let X be a k-scheme and let T and T' be a G-torsor and a H-torsor respectively. If $f : T \to T'$ is a morphism of torsors, for the morphism at the level of group-schemes $\phi : G \to H$ associated to f, we have $T' \cong T \times^G H$ where the contracted product at the right is taken using ϕ.*

Proof. There is a natural morphism $T \times^G H \to T'$ as T' fits in the commutative diagram

$$G \times_k (T \times_k H) \overset{\mu}{\underset{p_2}{\rightrightarrows}} (T \times_k H) \xrightarrow{g} T'$$

where $g = \mu_H \circ (f, id_H)$ where $\mu_H : T' \times_k H \to T'$ is the action morphism of T' and thus we obtain a morphism $T \times^G H \to T'$ as $T \times^G H$ is a cokernel. This morphism is also H-equivariant an thus we have a morphism of H-torsors over X, that is an isomorphism by Lemma 2.3.11. □

Using contracted products we can relate properties of morphisms between group-schemes with properties of the canonical projection to the contracted product, and thus properties of morphisms of torsors.

Remark 2.3.50. *Let $t : T \to X$ be a G-torsor with G affine and X a k-scheme. If $\phi : G \to H$ makes G a subgroup-scheme of H, we can easily see that the morphism of torsors $f : T \to T \times^G H$ is a sub-torsor (Definition 2.3.36).*
On the other extreme, if $\phi : G \to G/N$ is a quotient with kernel $N \lhd G$, by using a similar argument for the existence of the contracted product in

the proof of Corollary 2.3.48, we can show that the quotient T/N *by the restricted action of* N *over* T *exist, and thus the projection* $\pi : T \to T/N$ *makes* T *a* N*-torsor over* T/N*. Moreover,* T/N *is a* G/N*-torsor over* X *as* $T/N \cong T \times^G G/N$ *by Proposition 2.3.49 and the scheme on the right is a* G/N*-torsor.*

The last paragraph of this remark allows us to define:

Quotient torsors

Definition 2.3.51. *Let* $t : T \to X$ *be a* G*-torsor with* G *affine and* X *a* k*-scheme. If* $\phi : G \to Q$ *is a quotient morphism (Definition 2.2.56) with kernel* $N \lhd G$*.*
We will call the contracted product $T \times^G Q$ *a* quotient torsor of T*. This is equivalent to consider the quotient* T/N *by Corollary 2.2.59 as* $Q \cong G/N$*.*

Remark 2.3.50 and the isomorphism theorem for group-schemes (Theorem 2.2.65), immediately yield:

Decomposition of a morphism of torsors by the isomorphism theorem

Proposition 2.3.52. *Let* X *be a* k*-scheme and let* T *and* T' *be a* G*-torsor and a* H*-torsor respectively. If* $f : T \to T'$ *is a morphism of torsors and* $\phi : G \to H$ *is its associated morphism at the level of group-schemes, then* f *decomposes*[12] *as* $f = i \circ p$ *where* $p : T \to T/K$ *is the quotient of* T *by* $K = \ker(\phi)$ *and* $i : T/K \to T'$ *is a sub-torsor.*

Remark 2.3.53. *If we consider pointed torsors, it is not hard to see that the contracted product of a pointed torsor is pointed, so all the results we have considered so far also hold for pointed torsors.*

Now we will state the last construction of torsors that we will need for later chapters, for this purpose we will consider a scheme X of finite type over k and a G-torsor $t : T \to X$ with G finite, thus t is a finite and faithfully flat (thus locally free) morphism by Proposition 2.3.16.

Weak quotients of torsors

Definition 2.3.54. *Let* X *be a scheme of finite type over* k *and a* G*-torsor* $t : T \to X$ *with* G *finite. If* $H \subset G$ *is a subgroup-scheme that is not normal, we can consider the restricted action of* H *over* T*. By Theorem 2.2.40 the quotient by this action* T/H *exists and we will call it a* weak quotient of T.

Remark 2.3.55. *The projection morphism* $\pi : T \to T/H$ *is an* H*-torsor, and the morphism* $T/H \to X$ *is faithfully flat and locally free of degree* $ord(G)/ord(H)$*.*

2.3.4 *Projective Limits of Torsors*

Let k be any field and let X be a scheme over k. In this final subsection about torsors, we will state the basic properties of projective limits of affine torsors (Definition 2.3.17(c)) over X. As such, all group-schemes considered in this part will be affine.
First, we will introduce some notation:

12 Modulo isomorphism of torsors over X.

Notation 2.3.56. *Let* X *be a scheme over* k*. If* $\{T_i\}_{i \in I}$ *is an inverse directed system of affine torsors* $T_i \to X$ *over a partially ordered set* I*, where the transition morphisms are torsor morphisms. The limit of this system will be denoted as*

$$T := \lim_{\leftarrow\, i \in I} T_i.$$

We will also consider the associated inverse directed system of group-schemes $\{G_i\}_{i \in I}$*, being* G_i *the group-scheme associated to* T_i*.*
Finally, for the pointed case, if $x \in X(k)$*, the points* t_i *(*$i \in I$*) and* t *will denote respectively a rational point of* T_i *over* x *and a rational point of* T *over* x*, clearly* t *is the inverse limit of the directed system formed by the* t_i*.*
When needed, we may add an index 0 *to the set* I *such that* $T_0 := X$ *and* $t_0 := x$*.*

The first question one should ask, is if general projective limits of schemes exists. If all schemes over X are affine, this is indeed the case, and if more hypotheses hold for X, we can obtain additional properties:

Properties of affine projective limits

Lemma 2.3.57. *Let* X *be a scheme (not necessarily over a field) and let* $\{Y_i\}_{i \in I}$ *and* $\{Z_i\}_{i \in I}$ *be two inverse directed system of affine schemes over* X *with the same indexes. We will denote as* Y *and* Z *the respective projective limits if these exists as schemes. Then:*

(a) Y *and* Z *exist as schemes over* X*.*

(b) The formation of projective limits commutes with base change.

(c) If X *is quasi-compact and quasi-separated, for any scheme* $V \to Y$ *of finite presentation*[13] *over* Y*, there exists an index* $i \in I$ *and a scheme* V_i *of finite presentation over* Y_i*, such that the following diagram is cartesian:*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & Y \arrow[d] \\ V_i \arrow[r] & Y_i \end{tikzcd} .$$

(d) If X *is quasi-compact and quasi-separated, and if every scheme* Y_i *is quasi-compact and quasi-separated over* X *and every* Z_i *is locally of finite presentation over* X *for all* $i \in I$*, then morphisms* $f : Y \to Z$ *over* X *are in bijective correspondence with directed inverse systems of morphisms* $f_i : Y_i \to Z_i$ *over* X *for* $i \in I$*.*

(e) If X *is quasi-compact and quasi-separated, and if both* Y_i *and* Z_i *are of finite presentation over* X *for every* $i \in I$*, then for any morphism* $f : Y \to Z$ *over* X*, there exists an index* $i \in I$ *such that the base change of the morphism* $f_i : Y_i \to Z_i$ *to* Z *is* f*.*

13 See [63, Tag 01TO] for a definition and basic properties of morphisms of finite presentation.

We recall that X is *quasi-separated* over S if the diagonal morphism $\Delta_{X/S} : X \to X \times_S X$ is quasi-compact (Definition 2.3.5).

Proof. The proof for all parts of this lemma, can be found in [33].
The proof of part (a) is in Proposition 8.2.3 of the reference above while part (b) stems from Proposition 8.2.5.
Finally, parts (c) and (d) correspond to parts (ii) and (i) of Théorème 8.8.2 respectively and part (e) is a direct consequence of the mentioned theorem. □

Together with the existence of projective limits, we need to relate certain properties of morphisms between projective limits to the corresponding property for morphisms between schemes from the respective inverse directed systems.

Descent property for affine projective limits

Lemma 2.3.58 (Théorème 8.10.5 [33])**.** *Keeping the notation of Lemma 2.3.57. Let us suppose that X is quasi-compact and that for any $i \in I$ the schemes Y_i and Z_i are of finite presentation over X. If we have a given morphism $f_j : Y_j \to Z_j$ for some $j \in I$, with its respective base change $f : Y \to Z$. Then, for the following list of properties for morphisms:*

(a) Closed immersion.

(b) Separated.

(c) Affine.

(d) Isomorphism.

(e) Surjective.

(f) Finite.

(g) Proper.

we have that f has one of the properties listed above if and only if there exists an index $k \geqslant j$ such that $f_k : Y_k \to Z_k$, the base change of f_j over Z_k, possesses the same property. In this case all morphisms $f_l : Y_l \to Z_l$ share the same property for all $l \geqslant k$.
Moreover, if X is also quasi-separated, we have that f possess one property from above if and only if there exists an index $i \in I$ such that f_i shares the same property.

The last part of the lemma's statement uses Lemma 2.3.57(e).

Remark 2.3.59. *Let X be a quasi-compact scheme over k. Then, any algebraic torsor $t : T \to X$ is of finite presentation: this uses the fact that morphisms of finite presentation are preserved by base change ([63, Tag 01TS]) and we can apply fpqc descent with them ([32, Prop. 2.7.1(vi)]).*
As any algebraic group-scheme G over a field is of finite presentation because $\mathrm{Spec}(k)$ *is noetherian and being of finite presentation coincides with being*

of finite type in this case (see [63, Tag 01TX]), we have that the isomorphism $T \times_X G \cong T \times_X T$ *is equivalent to say that the commutative square*

$$\begin{tikzcd} T \times_X G \arrow[r, "p_1"] \arrow[d, "\mu_T"'] & T \arrow[d, "t"] \\ T \arrow[r, "t"'] & X \end{tikzcd}$$

is cartesian, and thus, as $T \times_X G$ *is of finite presentation by base change, we conclude that* $t : T \to X$ *is of finite presentation by fpqc descent.*

Now we can establish the existence of projective limits of algebraic torsors (Definition 2.3.17(b)):

Existence of projective limits of algebraic torsors

Proposition 2.3.60. *Let* X *be a quasi-compact scheme over* k*. If* $\{T_i\}_{i \in I}$ *is a inverse directed system of algebraic torsors over* X *indexed by a partially ordered set* I*. Then, the projective limit*

$$T = \varprojlim_{i \in I} T_i$$

exists as a scheme over X*, and moreover it is a* G*-torsor over* X *where* G *is the projective limit to the associated inverse directed system of group-schemes* $\{G_i\}_{i \in I}$*. Moreover, if* $x \in X(k)$ *is a rational point of* X *and all the torsors in the system are pointed over* X*, then* $T \to X$ *is pointed too.*

Proof. By Lemma 2.3.57(a) the projective limit $t : T \to X$ exists, so we now need to proof that it is a G-torsor, recall that G is a group-scheme by Remark 2.2.93.
Firstly, G_X acts on T by using Lemma 2.3.57(b) and "passing to the limit" the action of each $(G_i)_X$ over T_i. The same principle shows that the morphism $t : T \to X$ is G-invariant (Definition 2.2.34).
The isomorphisms $T_i \times_k G_i \cong T_i \times_X T_i$ induce the isomorphism $T \times_k G \cong T \times_X T$ over the corresponding limits using Lemma 2.3.58(d) as the torsors T_i are of finite presentation over X (Remark 2.3.59).
To show that T is faithfully flat over X, we can use the fact that projective limits of flat schemes are flat (see [30, p. 6.1.2]) and that $T \to X$ is surjective using Lemma 2.3.58(e).
Finally, the last part for pointed torsors can be easily verified. □

Now that we have established the existence of projective limits for algebraic torsors, we need to characterize torsor morphisms between them, in the case X is quasi-separated we have some nice descent properties, that we will use in Chapter 5:

Descent of finite torsors over projective limits of algebraic torsors

Proposition 2.3.61. *Let* $T := \varprojlim T_i$ *be a projective limit of algebraic torsors over a quasi-compact and quasi-separated scheme* X *over a field* k*. Let* $V, W \to T$ *be two finite torsors over* T*, where* G *and* H *are their corresponding group-schemes respectively. Then:*

(a) *There exist an index* $i \in I$ *and a finite* G*-torsor* $V_i \to T_i$ *such that the following diagram is cartesian*

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V_i . \arrow[d] \\ T \arrow[r] & T_i \end{tikzcd}$$

In addition, if V *is Nori-reduced, then so is* V_i.

(b) *If* $\phi : V \to W$ *is a morphism of finite torsors over* T*, there exists an index* $i \in I$ *such that* V_i *and* W_i *are finite torsors over* T_i*, with a morphism of torsors* $\phi_i : V_i \to W_i$ *over* T_i *such that* ϕ *is the pull-back of* ϕ_i *over* T*. In that case,* ϕ *is a sub-torsor (Definition 2.3.36), a quotient of* V*, or an isomorphism if and only if for an index* $j \geqslant i$ *the pull-back of* ϕ_i *to* T_j *is of the same type.*

If the torsors T*,* V *and* W *are pointed, all morphisms above are of pointed torsors.*

Proof. For part (a), first we should note that T is separated and quasi-compact over k, as all the torsors T_i are so we can apply Lemma 2.3.58(b).

Now, V is of finite presentation over T, and thus we can apply Lemma 2.3.57(c) so there exists an index $i \in I$ and a scheme V_i over T_i such that $V \cong V_i \times_{T_i} T$, moreover by taking a larger index if necessary the morphism $V_i \to T_i$ is finite (Lemma 2.3.58(f)).

V_i is not necessarily a torsor over T_i, but the isomorphism $V \times_T G_T \cong V \times_T V$ implies that there exists a large enough index $j \geqslant i$ such that we have a finite group-scheme G_j and an isomorphism $V_j \times_{T_j} G_j \cong V_j \times_{T_j} V_j$ by applying Lemma 2.3.58(d), making $V_j \to T_j$ a G_j-torsor, G_j comes from applying Lemma 2.3.57(c) to the underlying morphism between the group-schemes associated to V and T. It is easy to see that V_j is Nori-reduced if V is.

For part (b), we first chose two indexes i and j such that V descends to V_i over T_i and W so does to W_j over T_j. We can assume $i \leqslant j$ as we can take another larger index to both i and j, in that case we can take $V_j = V_i \times_{T_i} T_j$. Both schemes satisfy the hypotheses of Lemma 2.3.57(d) and thus we have a bijection between T-morphisms between V and W and the directed limit of the sets $\mathrm{Hom}_{T_l}(V_l, W_l)$ where $l \geqslant j$ with $V_l = V_k \times_{T_j} T_l$ and $W_l = W_k \times_{T_j} T_l$. This means that ϕ can be seen as a directed system of morphisms $\phi_l : V_l \to W_l$ and by picking a possibly larger index we can assume that it is compatible with the actions of the respective group-schemes, as we can apply Lemma 2.3.57(d) again to get a bijection for $\mathrm{Hom}_T(G_T, H_T)$ and $\mathrm{Hom}_T(G_T \times_T V, H_T \times_T W)$ and its corresponding directed system of

Hom-sets for each respective Hom-set. This bijection allows us to conclude that the commutative diagram

$$\begin{tikzcd} G_T \times V \arrow[r] \arrow[d] & H_T \times W \arrow[d] \\ V \arrow[r] & W \end{tikzcd}$$

that compatibilizes the actions descends to a diagram that makes the actions compatible over a certain index, larger that j, which means that we have a morphism of torsors over T_j. The final part of the statement in part (b) comes the decomposition for torsor morphisms by the isomorphism theorem of group-schemes (Proposition 2.3.52) and from Lemma 2.3.58 applied to isomorphisms and closed immersions we have the desired result for sub-torsors and isomorphisms. In the case of quotients, the same lemma tells us that $V \to T$ is surjective if and only if $V_j \to T_j$ is surjective for an index $j \in J$ large enough and the flatness is induced in both ways by descent (Proposition 2.3.8(g)) and base change respectively.
The pointed part is trivial, by applying Lemma 2.3.57(c). □

Among all the possible limits of torsors we can construct, we are mostly interested in limits where all torsors in the inverse directed system are Nori-reduced (Definition 2.3.40), Proposition 2.3.61 gives us a handy corollary that we will use in subsequent sectios, notably in Sections 3.3 and 5.3:

Corollary 2.3.62. *Let X be a quasi-compact and quasi-separated scheme over k. If $\{T_i\}_{i\in I}$ is a inverse directed system of algebraic torsors over X indexed by a partially ordered set I. If all the torsors T_i are Nori-reduced, then $T = \varprojlim_{i\in I} T_i$ is Nori-reduced as well.*

Proof. Let us suppose that T has a non-trivial and proper sub-torsor $V \hookrightarrow T$, then by using Lemma 2.3.58(a) and Proposition 2.3.61 we can find an index $i \in I$ such that V descends to a closed immersion of torsors $V_i \to T_i$, but as T_i is Nori-reduced, V_i is either trivial or isomorphic to T_i, which implies by base change that V is of the same type, contradicting our initial assumption. □

2.4 TANNAKIAN CATEGORIES

Let G be a compact topological abstract group. In [65] and [41], T. Tannaka and M.G. Krein showed independently that G can be recovered from its category of finitely dimensional complex representations $\mathrm{Rep}_{\mathbb{C}}(G)$ in a functorial bijective way, and that there is a criterion in order to show that a category of finitely dimensional complex vector spaces is the category of representations of a compact topological abstract group. This duality principle between groups and representations is known as the *Tannaka-Krein duality*.

Currently, there are many different flavors of this duality, and in this part we will establish the duality associated to affine group-schemes over a field and *neutral tannakian categories*. We will borrow most results from [17], [59] and [64].

2.4.1 *Definition of Tannakian Categories*

In order to define tannakian categories, we first need several concepts from category theory, that we will outline them in this section. We will start with abelian categories in Subsubsection 2.4.1.1, tensor categories in Subsubsection 2.4.1.2 and we will finish with neutral tannakian categories in Subsubsection 2.4.1.3. For this section, when considering objects in a category $\mathcal{C}$, we will often use the notation $x \in \mathrm{Obj}(\mathcal{C})$.

2.4.1.1 *Abelian Categories*

We will now define abelian categories, following [46, Ch. VIII]. The first step is to define zero objects and arrows, kernels and cokernels.

Zero object and zero arrow

Definition 2.4.1. *Let $\mathcal{C}$ be a category. An object $z \in Obj(C)$ which is initial and final at the same time is called a* null (or zero) object of $\mathcal{C}$.
If $\mathcal{C}$ has a null object z, then for any $a, b \in Obj(\mathcal{C})$, we can consider the unique canonical arrows $a \to z$ and $z \to b$ and their composition

$$0 = 0_a^b : a \to z \to b$$

that we will call the zero morphism (or zero arrow).

Remark 2.4.2. *The composition of any arrow with a zero arrow is a zero arrow, the zero object in a category $\mathcal{C}$ is unique up to isomorphism and the zero arrow is independent of the choice of a zero object.*

Example 2.4.3. *In the category Grp of abstract groups, the trivial group $\{1\}$ is the zero object and for any pair of abstract groups G, H, the composition $G \to \{1\} \to H$ is the zero arrow between G and H.*
Analogously, in the category Vect(k) of vector spaces over a field k, the zero vector space $\{0\}$ is the zero object and the composition $V \to \{0\} \to W$ where V and W are vector spaces is the zero arrow between them.

Kernel and cokernel in a category

Definition 2.4.4. *Let $\mathcal{C}$ be a category with a null object, if $f : a \to b$ is a morphism, the* kernel of f *is the equalizer* $s \xrightarrow{k} a \underset{0}{\overset{f}{\rightrightarrows}} b$.

For an arrow $g : a \to b$, the cokernel of f *is the co-equalizer* $a \underset{0}{\overset{g}{\rightrightarrows}} b \xrightarrow[c]{} t$.
.

Remark 2.4.5. *Let $\mathcal{C}$ be a category with a null object. If the kernel $k : s \to a$ of an arrow $f : a \to b$ exists, it is a monomorphism (Definition 2.2.61) in $\mathcal{C}$, meaning that if* $d \underset{h}{\overset{g}{\rightrightarrows}} s$ *are two arrows such that $k \circ g = k \circ h$, then*

$g = h$. *This means that* k *can be canceled on the left.*
As such kernels can be considered as sub-objects, like in the case of kernel of morphisms of vector spaces, modules, abstract groups, etc. Or the kernel of a morphism of group-schemes (Definition 2.2.23) in the category of group-schemes over a field k *with the trivial group-scheme* $\mathrm{Spec}(k)$ *as the zero object, which are closed subgroup-schemes and thus sub-objects.*

There are some easy identities that can be shown for kernels and cokernels:

Proposition 2.4.6 (p. 189 [46]). *Let* $\mathcal{C}$ *be a category with zero object, zero arrows, and such that any arrow between objects of* $\mathcal{C}$ *has a kernel and a cokernel. Then, for any arrow* u *of* $\mathcal{C}$*, we have the following identities:*

(a) $\ker(\mathit{coker}(\ker(u))) = \ker(u)$.

(b) $\mathit{coker}(\ker(\mathit{coker}(u))) = \mathit{coker}(u)$.

Moreover, the following properties hold:

1. *An arrow* g *is a kernel in and only if* $g = \ker(\mathit{coker}(g))$.

2. *Any arrow* $f : a \to b$ *has a unique factorization,* $f = m \circ q$ *where* $m = \ker(\mathit{coker}(f))$.

Definition 2.4.7. *Let* $f : a \to b$ *be a morphism of objects in a category* $\mathcal{C}$. f *is an* epimorphism *if for any object* c *and any pair of morphisms* $b \underset{h}{\overset{g}{\rightrightarrows}} c$ *such that* $g \circ f = h \circ f$*, then* $g = h$.

The factorization given in Proposition 2.4.6 has special properties:

Lemma 2.4.8 (Lemma 1 p.189 [46]). *Keeping the hypotheses of Proposition 2.4.6, if* $f : a \to b$ *is a morphism of* $\mathcal{C}$ *and* $f = m' \circ q'$ *is a factorization where* m' *is a kernel, then in the following commutative diagram there exists a unique diagonal morphism* t

$$\begin{tikzcd} a \arrow[r, "q"] \arrow[d, "q'"'] \arrow[dr, phantom, "\exists ! t" near start] & c \arrow[d, "m"] \arrow[dl, dashed] \\ d \arrow[r, "m'"'] & b \end{tikzcd}$$

such that $m = m' \circ t$ *and* $q = t \circ q'$*. Moreover, if* $\mathcal{C}$ *has equalizers and any monomorphism is a kernel, then* q *is an epimorphism.*

Now we will state the properties of Hom-sets in abelian categories. The essential properties of Hom-sets in abelian categories come from the categories of modules and abelian abstract groups.

AB categories

Definition 2.4.9. *A category $\mathcal{A}$ is an* Ab category *if any pair of objects $a, b \in Obj(\mathcal{A})$, the set $Hom_{\mathcal{A}}(a, b)$ has the structure of an abelian abstract group*[14]*, such that for any other object $c \in Obj(\mathcal{A})$ the composition map*

$$\begin{array}{rcl} Hom_{\mathcal{A}}(a,b) \times Hom_{\mathcal{A}}(b,c) & \to & Hom_{\mathcal{A}}(a,c) \\ (f,g) & \mapsto & g \circ f \end{array}$$

is a biadditive morphism. In this case, the unit element of the abelian abstract group $Hom_{\mathcal{A}}(a, b)$ is a morphism $0 : a \to b$, called the zero arrow, *which does not necessarily coincide with the null arrow of Definition 2.4.1 if for example the category $\mathcal{A}$ does not have a zero object. It is clear that the composition of zero arrows is a zero arrow.*

Remark 2.4.10. *As abelian abstract groups are $\mathbb{Z}$-modules, we have that for any object a of an AB category $\mathcal{A}$ there is a canonical arrow $\mathbb{Z} \to Hom_{\mathcal{A}}(a, a)$ such that the image of $1 \in \mathbb{Z}$ is id_a.*
The biadditive property means that for any triple of objects $a, b, c \in Obj(\mathcal{A})$ and morphisms $f, f' \in Hom_{\mathcal{A}}(a, b)$ and $g, g' \in Hom_{\mathcal{A}}(b, c)$, we have that

$$(g + g') \circ (f + f') = g \circ f + g \circ f' + g' \circ f + g' \circ f'.$$

We can also express this property by considering $Hom_{\mathcal{A}}(b, c) \otimes_{\mathbb{Z}} Hom_{\mathcal{A}}(a, b)$, writing the composition as

$$\begin{array}{rcl} Hom_{\mathcal{A}}(b,c) \otimes_{\mathbb{Z}} Hom_{\mathcal{A}}(a,b) & \to & Hom_{\mathcal{A}}(a,c) \\ g \otimes f & \mapsto & g \circ f \end{array}$$

and demanding that it is $\mathbb{Z}$-linear.

Functors between AB categories should preserve the abelian abstract group structures on Hom-sets, these morphisms are called:

Additive functors

Definition 2.4.11. *Let $\mathcal{A}$ and $\mathcal{B}$ be two AB categories. A functor $T : \mathcal{A} \to \mathcal{B}$ is* additive *if for any pair of objects $a, a' \in Obj(\mathcal{A})$, the map $Hom_{\mathcal{A}}(a, a') \to Hom_{\mathcal{B}}(T(a), T(a'))$ induced by T is a morphism of abelian abstract groups.*

Remark 2.4.12. *Keeping the notations of Definition 2.4.11. We can consider additive contravariant functors for which the morphism of abelian abstract groups is $Hom_{\mathcal{A}}(a, a') \to Hom_{\mathcal{B}}(T(a'), T(a))$.*
It is clear that the composition of additive functors is additive.

To bridge AB categories with categories having a zero object, we need the following proposition:

Proposition 2.4.13 (§VIII.2 Proposition 1 [46]). *Let z be an object of an AB category $\mathcal{A}$. Then, the following assertions are equivalent:*

(a) z is an initial object.

(b) z is a final object.

14 We will always use the additive notation for abelian abstract groups of morphisms.

(c) $\mathrm{id}_z = 0 \in Hom_{\mathcal{A}}(z, z)$.

(d) *The abelian abstract group* $Hom_{\mathcal{A}}(z, z)$ *is the zero abelian abstract group.*

In particular any initial (or final) object in an AB category is a zero object (Definition 2.4.1).

One feature of the category of abelian abstract groups or modules, is that the direct product coincides with the coproduct, the direct sum $\oplus$ being this special construction in these two categories.

Biproducts

Definition 2.4.14. *Let $\mathcal{A}$ be an AB category. A* biproduct diagram *is the following type of diagram in $\mathcal{A}$*

$$\begin{tikzcd} a \arrow[r, shift right, "i_1"'] & c \arrow[l, shift right, "p_1"'] \arrow[r, shift left, "p_2"] & b \arrow[l, shift left, "i_2"] \end{tikzcd}$$

where $p_1 \circ i_1 = \mathrm{id}_a$, $p_2 \circ i_2 = \mathrm{id}_b$ and $i_1 \circ p_1 + i_2 \circ p_2 = \mathrm{id}_c$.
In a biproduct diagram, the object c is called the biproduct of a and b.

The feature we mentioned before for abelian abstract groups and modules can be expressed abstractly as:

Theorem 2.4.15 (§VIII.2 Theorem 2 [46])**.** *Let $\mathcal{A}$ be an AB category and let a, b be two objects of $\mathcal{A}$. Then, the product of a and b exists if and only if their coproduct exists, if and only if they possess a biproduct. More specifically, given a biproduct diagram as in Definition 2.4.14, the object c with the projections p_1 and p_2 is the product of a and b while the same c with the inclusions i_1 and i_2 is the coproduct of a and b.*

We can now consider categories on which all the properties considered so far hold, we will define them as:

Additive categories

Definition 2.4.16. *An* additive category *is an AB category $\mathcal{A}$ that has a null object, where the biproduct of any pair of objects of $\mathcal{A}$ exists.*

Proposition 2.4.17 (§VIII.2 Prop. 3 [46])**.** *Let $\mathcal{A}$ be an additive category, and let $f, f' : a \to b$ be two morphisms of $\mathcal{A}$. Then, we have the identity*

$$f + f' = \check{\delta}^b \circ (f \oplus f') \circ \delta_a$$

where $\delta_a : a \to a \times a$ is the natural diagonal morphism of the product, and $\check{\delta}^b : b \coprod b \to b$ is the co-diagonal morphism of the coproduct.

Using this proposition, we can characterize additive functors between additive categories in terms of biproducts.

Proposition 2.4.18 (§VIII.2 Prop. 4 [46])**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be two AB categories that have all biproducts, and let $T : \mathcal{A} \to \mathcal{B}$ be a functor. Then, T is additive if and only if the image via T of any biproduct diagram of objects of $\mathcal{A}$ is the biproduct diagram in $\mathcal{B}$ of their images.*

Now we are ready to define abelian categories:

Abelian categories

Definition 2.4.19. *An AB category (Definition 2.4.9) $\mathcal{A}$ is an* abelian category *if it satisfies the following conditions:*

(a) $\mathcal{A}$ has a null object (Definition 2.4.1).

(b) The biproduct (Definition 2.4.14) of any pair of objects of $\mathcal{A}$ exists.

(c) Every morphism between objects of $\mathcal{A}$ has a kernel and a cokernel (Definition 2.4.4).

(d) Every monomorphism is a kernel and every epimorphism is a cokernel.

Remark 2.4.20. *Let $\mathcal{A}$ be an abelian category. Conditions (a) and (b) in Definition 2.4.19 imply that $\mathcal{A}$ is an additive category (Definition 2.4.16). By Proposition 2.4.13 we can replace "null object" with initial of final object in condition (a) and by Theorem 2.4.15 we can replace "biproduct" by product or coproduct in condition (b).*
Condition (d) is powerful: a morphism $f : a \to b$ that is a monomorphism and an epimorphism at the same time is an isomorphism as in this case $f = \ker(g)$ where $g : b \to c$ is another morphism with domain b, which implies that $g \circ f = 0 = 0 \circ f$, but as f is an epimorphism we can "cancel f on the right" and thus $g = 0$, as the kernel the zero morphism from b is id_b we conclude that f is an isomorphism.

Example 2.4.21. *Now we will give some examples of abelian categories:*

Abelian abstract groups

1. *The category Ab of abelian abstract groups is an abelian category. We should point out that if we consider the category of all abstract groups Grp, this category is not abelian as not all morphisms of abstract groups have cokernels, they do if the image is normal on the codomain which is not always the case.*

Modules

2. *Let R be a commutative ring with unit, then the category $R-$Mod of R-modules is an abelian category.*

Vector spaces

3. *Let k be a field. The category Vect(k) of vector spaces over k is an abelian category in which we can easily see that Hom-sets are also k-vector spaces, this property is important and we will define it later.*

Commutative group-schemes

4. *If k is a field, the category of commutative group-schemes (Example 2.2.15(2)) of finite type over k is an abelian category, see [49, Theorem 5.62].*

Applying Lemma 2.4.8 to abelian categories, we obtain the abstract version of the isomorphism theorem that we can find on any of the examples in Example 2.4.21.

Proposition 2.4.22 (§VIII.3 Prop. 1 [46])**.** *Let $\mathcal{A}$ an abelian category. Then any arrow $f : a \to b$ can be factored as $f = m \circ e$ where m is a monomorphism and e is an epimorphism and we moreover have $m = \ker(coker(f))$ y $e = coker(\ker(f))$.*

If $f' = m' \circ e' : c \to d$ is the factorization of another arrow of $\mathcal{A}$, and we have a commutative diagram

$$\begin{tikzcd} a \arrow[r, "f"] \arrow[d, "g"'] & b \arrow[d, "h"] \\ c \arrow[r, "f'"'] & d \end{tikzcd} .$$

If q and q' are the codomains of e and e' respectively, there exist a unique arrow $k : q \to q'$ such that the following diagram is commutative:

$$\begin{tikzcd} a \arrow[r, "e"] \arrow[rr, bend left, "f"] \arrow[d, "g"'] & q \arrow[r, "m"] \arrow[d, dashed, "k"'] & b \arrow[d, "h"] \\ c \arrow[r, "e'"'] \arrow[rr, bend right, "f'"'] & q' \arrow[r, "m'"'] & d \end{tikzcd} .$$

Definition 2.4.23. *Let $\mathcal{A}$ an abelian category, and let $f : a \to b$ be an arrow. If $f = m \circ e$ is the factorization of f given in Proposition 2.4.22, we call m the* image of f *$im(f)$ and e the* coimage of f *$coim(f)$. By the aforementioned proposition, the image and the coimage are unique up to isomorphism.*

Remark 2.4.24. *For any arrow $f : a \to b$ in an abelian category $\mathcal{A}$, we have that its image is a "sub-object" and its coimage is a "quotient" which are in short, equivalency classes of monomorphisms and epimorphisms respectively, for the specific definitions of these see [46, §V.7].*
For the usual abelian categories, those mentioned in Example 2.4.21, sub-objects and quotients are well understood along with the existence of the factorization of Proposition 2.4.22.

On abelian categories we can naturally define exactness:

Definition 2.4.25. *Let $\mathcal{A}$ an abelian category. We say that a pair of composable arrows*

Exactness in abelian categories

$$a \xrightarrow{f} b \xrightarrow{g} c$$

is exact at b *o that we have* exactness at b *if $im(f) \equiv \ker(g)$ where the equivalency is between sub-objects of b, or equivalently, that $coker(f) \equiv coim(g)$ as an equivalency of quotients of b.*

Remark 2.4.26. *Let $a \xrightarrow{f} b \xrightarrow{g} c$ be a diagram in an abelian category $\mathcal{A}$. We note that $im(f) \leqq ker(g)$, meaning that the monomorphism $im(f)$ factors through $\ker(g)$, if and only if $g \circ f = 0$ while $im(f) \geqq \ker(g)$, meaning that $\ker(g)$ factors through $im(f)$, if and only if any morphism k such that $g \circ k = 0$ factors as $k = m \circ k'$ where m is the monomorphism in the factorization $f = m \circ e$ of Proposition 2.4.22.*
The last paragraph applied to the usual abelian categories (abelian abstract groups, modules, etc) implies that we have exactness at b if and only if $g \circ f = 0$ and any element of b that maps to 0 via g lies in the image of f.

Short exact sequences

Definition 2.4.27. *Let $\mathcal{A}$ an abelian category. The diagram*

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0,$$

where 0 denotes the null object of $\mathcal{A}$, is a short exact sequences *if we have exactness at a, b and c.*

Remark 2.4.28. *Keeping the notation of Definition 2.4.27, the fact that the sequence*

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$$

is exact means that f is a monomorphism, g is an epimorphism, $f = \ker(g)$ and $g = coker(f)$.
Related to this, if we just demand the equality $h = coker(f)$, this is the same as requiring that in the diagram

$$a \xrightarrow{f} b \xrightarrow{h} c \to 0$$

we have exactness at b and c. We will call this situation a right exact sequence.
Analogously, if we have that $k = \ker(f)$, this is equivalent to say that the sequence

$$0 \to a \xrightarrow{k} b \xrightarrow{f} c$$

is exact at a and b, and we will call this a left exact sequence.
The factorization $f = m \circ e$ of Proposition 2.4.22 comes with two short exact sequences, appearing in the upper horizontal part and the right vertical part of the following commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & k \arrow[r, "\ker(f)"] & a \arrow[r, "coim(f)"] \arrow[dr, "f"'] & i \arrow[d, "im(f)"] \\
& & & b \arrow[d, "coker(f)"] \\
& & & q \arrow[d] \\
& & & 0
\end{tikzcd} .$$

The last concept associated to abelian categories we need, are the functors that preserve exact sequences, these are:

Exact functors

Definition 2.4.29. *Let $\mathcal{A}$ and $\mathcal{B}$ be two abelian categories. A functor $T : \mathcal{A} \to \mathcal{B}$ is* exact *if it is additive and it preserves kernels and cokernels.*
T is left exact *if it is additive and it preserves kernels, and it is* right exact *if it is additive and it preserves cokernels.*

Remark 2.4.30. *An equivalent definition for exact functors, is that of an additive functor that preserves short exact sequences, meaning that if*

$$0 \to a \xrightarrow{f} b \xrightarrow{g} c \to 0$$

is an exact sequence of objects of $\mathcal{A}$*, then the sequence*

$$0 \to T(a) \overset{T(f)}{\to} T(b) \overset{T(g)}{\to} T(c) \to 0$$

in $\mathcal{B}$ *is exact*[15] *if* T *is covariant, or the sequence*

$$0 \to T(c) \overset{T(g)}{\to} T(b) \overset{T(f)}{\to} T(a) \to 0$$

is exact if T *is contravariant.*
In the same vein, a left exact functor is the same as an additive functor that preserves left exact sequences while a right exact functor is an additive one that preserves right exact sequences.

2.4.1.2 *Tensor Categories*

The next step to define tannakian categories, is to define tensor categories. We will do so, following Deligne-Milne [17, Ch. 1].
The definition of a tensor category is charged with very complex axioms, so we will define these first before defining tensor categories.

Definition 2.4.31. *Let* $\mathcal{C}$ *be a category, and let* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ *be a functor, for which we will use the notation* $(X, Y) \mapsto X \otimes Y$.

(a) *An* associativity constraint *for the pair* $(\mathcal{C}, \otimes)$ *is a natural isomorphism of functors defined over* $\mathcal{C} \times \mathcal{C} \times \mathcal{C}$ *for any triple* $X, Y, X \in Obj(\mathcal{C})$ *as*

Associativity constraint

$$\phi_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

that satisfies the so called pentagon axiom *for any quadruple of objects,* $X, Y, Z, T \in Obj(\mathcal{C})$

$$\begin{tikzcd}
& X \otimes (Y \otimes (Z \otimes T)) \arrow[dl, "id_X \otimes \phi_{Y,Z,T}"'] \arrow[dr, "\phi_{X,Y,Z\otimes T}"] & \\
X \otimes ((Y \otimes Z) \otimes T) \arrow[d, "\phi_{X,Y\otimes Z,T}"'] & & (X \otimes Y) \otimes (Z \otimes T) \arrow[d, "\phi_{X\otimes Y,Z,T}"] \\
(X \otimes (Y \otimes Z)) \otimes T \arrow[rr, "\phi_{X,Y,Z} \otimes id_T"'] & & ((X \otimes Y) \otimes Z) \otimes T
\end{tikzcd}.$$

(b) *A* commutativity constraint *for the pair* $(\mathcal{C}, \otimes)$ *is a natural isomorphism of functors defined over* $\mathcal{C} \times \mathcal{C}$ *for any pair* $X, Y \in Obj(\mathcal{C})$ *as*

Commutativity constraint

$$\psi_{X,Y} : X \otimes Y \to Y \otimes X$$

such that the composition $\psi_{Y,X} \circ \psi_{X,Y} : X \otimes Y \to X \otimes Y$ *is the identity of* $X \otimes Y$.

15 As T is additive, $T(0) = 0$.

Compatibility between associativity and commutativity constraints

(c) *We say that an associativity constraint ϕ and a commutativity constraint ψ for the pair $(\mathcal{C}, \otimes)$ are* compatible, *if the following diagram, called the* hexagon diagram, *is commutative:*

$$\begin{tikzcd}
& X \otimes (Y \otimes Z) \arrow[r, "\phi_{X,Y,Z}"] \arrow[dl, "\mathrm{id}_X \otimes \psi_{Y,Z}"'] & (X \otimes Y) \otimes Z \arrow[dr, "\psi_{X \otimes Y, Z}"] & \\
X \otimes (Z \otimes Y) \arrow[dr, "\phi_{X,Z,Y}"'] & & & Z \otimes (X \otimes Y) \arrow[dl, "\phi_{Z,X,Y}"] \\
& (X \otimes Z) \otimes Y \arrow[r, "\psi_{X,Z} \otimes \mathrm{id}_Y"'] & (Z \otimes X) \otimes Y &
\end{tikzcd} .$$

Identity object for the tensor product

(d) *A pair (U, u) where $U \in Obj(\mathcal{C})$ and $u : U \to U \otimes U$ is an isomorphism is a* identity object for $\otimes$ *if the functor $U \otimes (\cdot) : \mathcal{C} \to \mathcal{C}$ defined as $X \mapsto U \otimes X$ is an equivalence of categories.*

Remark 2.4.32. *All of these outlined conditions can be traced back to Saavedra-Rivano's book "Catégories Tannakiennes" [59]. The associativity and commutativity constraints can be found and §I.1 1.1 and §I.1 1.2 respectively, in §I.2 2.1 the compatibility between the constraints is stated, and finally the identity object appears in §I.1 1.3.*

All of the conditions of Definition 2.4.31 put together form what we will mean by tensor categories:

Tensor categories

Definition 2.4.33. *Let $\mathcal{C}$ be a category with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. A quadruple $(\mathcal{C}, \otimes, \phi, \psi)$ is a* tensor (or monoidal) category *if ϕ and ψ are compatible associativity and commutative constraints respectively, and $\mathcal{C}$ possesses an identity object (U, u) for $\otimes$.*
If $\mathcal{C}$ is a tensor category, we will call $\otimes$ the tensor product of $\mathcal{C}$.

Notation 2.4.34. *We will often omit the constrains when referring to a tensor category, meaning that we will either refer a tensor category by the pair $(\mathcal{C}, \otimes)$ or by simply $\mathcal{C}$ if the product $\otimes$ is clear from the context.*

Example 2.4.35. *For any commutative ring R with unit, the category of R-modules $R - Mod$ with the tensor product $\otimes_R$ is a tensor category, if U is a free R-module of rank one, the pair (U, u) is the identity object of this category where $u : U \to U \otimes U$ is isomorphism sending the only basis element $u_0 \in U$ to $u_0 \otimes u_0$, all identity objects of $R - Mod$ are of this type. We will use the category of R-modules as our model for tensor categories.*
If we would have considered $R - Mod$ with the tensor product $(x, y) \mapsto -x \otimes_R y$, this functor does not satisfy the pentagon diagram (Definition 2.4.31(a)), even though the commutativity constraint is satisfied, so we do not have a tensor category in this case.

In Definition 2.4.31(d) we have defined a seemingly left identity object, and in Example 2.4.35 we have many identity objects in the category of R-modules, though they are all isomorphic. The following propositions gives further properties for the identity object in a tensor category to clarify these points:

Proposition 2.4.36 (Prop. 1.3 [17]). *Let $(\mathcal{C}, \otimes, \phi, \psi)$ be a tensor category and let (U, u) be an identity object. Then:*

(a) For any $X \in Obj(\mathcal{C})$ there exists a natural isomorphism $l_X : X \to U \otimes X$ such that $l_U = u$ and both the diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, equal] \arrow[d, "l_X \otimes id_Y"'] & X \otimes Y \arrow[d, "l_{X \otimes Y}"] \\ (U \otimes X) \otimes Y \arrow[r, "\phi_{U,X,Y}"'] & U \otimes (X \otimes Y) \end{tikzcd}$$

and the diagram

$$\begin{tikzcd} X \otimes Y \arrow[r, "l_X \otimes id_Y"] \arrow[d, "id_X \otimes l_Y"'] & (U \otimes X) \otimes Y \arrow[d, "\psi_{U,X} \otimes id_Y"] \\ X \otimes (U \otimes Y) \arrow[r, "\phi_{X,U,Y}"'] & (X \otimes U) \otimes Y \end{tikzcd}$$

are commutative.

(b) If (U', u') is another identity object, there exists a unique isomorphism $a : U \to U'$ making the following diagram

$$\begin{tikzcd} U \arrow[r, "u"] \arrow[d, "a"'] & U \otimes U \arrow[d, "a \otimes a"] \\ U' \arrow[r, "u'"'] & U' \otimes U' \end{tikzcd}$$

commutative.

Remark 2.4.37. *If (U, u) is an identity object in a tensor category $(\mathcal{C}, \otimes)$, then we have a functorial isomorphism $r_X : X \to X \otimes U$ with analogous properties of those stated in Proposition 2.4.36(a). This isomorphism is defined as $r_X = \psi_{U,X} \circ l_X$, showing that the identity object is bilateral and unique up to isomorphism. As such, we will from now on denote this object as the pair $(\mathbb{1}, e)$.*

We must point out that the notion of tensor category given here is not standard. See [17, Remark 1.4] for other names given to Definition 2.4.33.

Remark 2.4.38. *Let $(\mathcal{C}, \otimes)$ be a tensor category. By the associativity constraint there is an essentially unique way to consider the tensor product of several objects of $\mathcal{C}$, meaning that we have a functor*

Extension of the tensor product over several objects

$$\bigotimes_{i \in I} : \mathcal{C}^I \to \mathcal{C}$$

for any finite set I *representing the product of* $|I|$ *elements of* $\mathcal{C}$ *so that any permutation in the ordering of the elements appearing in the tensor product indexed by* I *yields isomorphic objects of* $\mathcal{C}$*. See [17, Proposition 1.5] for more details.*

Invertible elements

Definition 2.4.39. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. An object* $L \in Obj(\mathcal{C})$ *is* invertible *if the functor* $C \to C$ *given by* $X \mapsto L \otimes X$ *is an equivalence of categories.*
If this is the case, there exists an object L' *such that* $L \otimes L' = \mathbb{1}$*.*
It is worth noting that the mere existence of L' *with the property listed above is enough to get that* L *is invertible, so this is an equivalent way to define invertible objects.*
A pair (L', δ) *is an* inverse of L *if* $L' \in Obj(\mathcal{C})$*, and* $\delta : L \otimes L' \to \mathbb{1}$ *is an isomorphism, we will denote the inverse of* L *as* L^{-1}*.*

Remark 2.4.40. *The definition of inverse element is symmetrical, meaning that if* (L', δ) *is an inverse of* L*, then* (L, δ) *is an inverse of* L^{-1}*.*
The inverse of any object L *is unique up to isomorphism: if* (L_1, δ_1) *and* (L_2, δ_2) *are inverses of* L*, then there exists a unique isomorphism* $\alpha : L_1 \to L_2$ *such that* $\delta_2 \circ (\mathrm{id}_L \otimes \alpha) : L \otimes L_1 \to L \otimes L_2 \to \mathbb{1}$ *equals* δ_1*.*

Example 2.4.41. *In the category of* R*-modules, free modules of rank 1 are invertible see [59, p. I 0.2.2.2].*
If X *is a scheme over* k*, we see that the category Vect*(X) *of vector bundles over* X *is tensorial by identifying vector bundles with locally free sheaves over* $\mathcal{O}_X$ *of finite rank. In Vect*(X) *line bundles, i.e., locally free sheaves of rank 1 are invertible.*

Another feature of categories like $R-\mathrm{mod}$ and $\mathrm{Vect}(X)$ is the existence of duals, so we would like to define a notion of dual on tensor categories. We will start with the following:

Internal Hom

Definition 2.4.42. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. If the functor* $T \mapsto Hom_{\mathcal{C}}(T \otimes X, Y)$ *is representable, we will denote its representative by* $\underline{Hom}(X, Y)$ *and call it the* internal Hom-set of X and Y. *Also, we will consider* $ev_{X,Y} : \underline{Hom}(X, Y) \otimes X \to Y$ *the morphism corresponding to* $\mathrm{id}_{\underline{Hom}(X,Y)}$ *under the first functor mentioned, known as the* evaluation morphism of X over Y.

Remark 2.4.43. *Immediately from the definition, if* $g : T \otimes X \to Y$ *is a morphism, there exists a unique morphism* $f : T \to \underline{Hom}(X, Y)$ *such that* $ev_{X,Y} \circ (f \otimes \mathrm{id}_X) = g$*.*
In other words, we have the commutative diagrams:

$$\begin{tikzcd} T \arrow[d, dashed, "f"'] \\ \underline{Hom}(X,Y) \end{tikzcd} \qquad \begin{tikzcd} T \otimes X \arrow[d, dashed, "f \otimes \mathrm{id}_X"'] \arrow[dr, "g"] & \\ \underline{Hom}(X,Y) \otimes X \arrow[r, "ev_{X,Y}"'] & Y \end{tikzcd} .$$

Example 2.4.44. *In the category* $R-Mod$*, the internal Hom of two objects* $\underline{Hom}(X,Y)$ *always exists, and it is simply* $Hom_{R-Mod}(X,Y)$ *with its natural structure of* R*-module. This is indeed an internal Hom as for any triple of* R*-modules* T,X,Y *we have*

$$Hom_{R-Mod}(T, Hom_{R-Mod}(X,Y)) \cong Hom_{R-Mod}(T \otimes_R X, Y).$$

Moreover, in this case the evaluation morphism $ev_{X,Y}$ *is simply given by* $f \otimes x \mapsto f(x)$.

We will now list several properties of internal Hom's:

Remark 2.4.45. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category and let us suppose that any pair of objects of* $\mathcal{C}$ *possess an internal Hom-set.*

(a) *In this case we have a composition morphism for internal Hom-sets: if* $X,Y,Z \in Obj(\mathcal{C})$ *then we have a morphism*

$$\underline{Hom}(X,Y) \otimes \underline{Hom}(Y,Z) \to \underline{Hom}(X,Z)$$

corresponding to the composition

$$(\underline{Hom}(X,Y) \otimes \underline{Hom}(Y,Z)) \otimes X \xrightarrow{ev} \underline{Hom}(Y,Z) \otimes Y \xrightarrow{ev} Z$$

which is a morphism of $Hom_{\mathcal{C}}(\underline{Hom}(X,Y) \otimes \underline{Hom}(Y,Z) \otimes X, Z)$*, and this set is isomorphic to* $Hom_{\mathcal{C}}(\underline{Hom}(X,Y) \otimes \underline{Hom}(Y,Z), \underline{Hom}(X,Z))$ *by the definition of internal Hom (Definition 2.4.42), from which we obtain the desired composition morphism of internal Hom's.*

(b) *Along with the isomorphism* $Hom_{\mathcal{C}}(Z, \underline{Hom}(X,Y)) \cong Hom_{\mathcal{C}}(Z \otimes X, Y)$ *for any triple of objects* $X,Y,Z \in Obj(\mathcal{C})$*, we also have the isomorphism*

$$\begin{aligned} Hom_{\mathcal{C}}(T, \underline{Hom}(Z, \underline{Hom}(X,Y))) &\cong Hom_{\mathcal{C}}(T \otimes Z, \underline{Hom}(X,Y)) \\ &\cong Hom_{\mathcal{C}}(T \otimes Z \otimes X, Y) \\ &\cong Hom_{\mathcal{C}}(T, \underline{Hom}(Z \otimes X, Y)) \end{aligned} .$$

(c) *If we apply the identity that characterizes the internal Hom-set to the identity element of* $\mathcal{C}$*, we obtain:*

$$Hom_{\mathcal{C}}(\mathbb{1}, \underline{Hom}(X,Y)) \cong Hom_{\mathcal{C}}(\mathbb{1} \otimes X, Y) \cong Hom_{\mathcal{C}}(X,Y)$$

.

Internal Hom-sets allows us to define a familiar concept in the category of R-modules or k-vector spaces in general:

Definition 2.4.46. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. For any object* $X \in Obj(\mathcal{C})$*, the* dual *of* X *is the object* $X^{\vee} = \underline{Hom}(X, \mathbb{1})$ *when it exists.*
The evaluation morphism (Definition 2.4.42) associated to a dual will be denoted as $ev_X : X^{\vee} \otimes X \to \mathbb{1}$.

Duals in tensor categories

Remark 2.4.47. *Keeping the notation of the definition, as the representative of a functor, we have for any* $T \in \mathcal{C}$ *the natural isomorphism* $Hom_{\mathcal{C}}(T, X^{\vee}) \cong Hom_{\mathcal{C}}(T \otimes X, \mathbb{1})$.
The assignment $X \mapsto X^{\vee}$ *can be extended to a contravariant functor* $(\cdot)^{\vee} : \mathcal{C} \to \mathcal{C}$ *by assigning to a morphism* $f : X \to Y$*, the morphism of duals* $f^{\vee} : Y^{\vee} \to X^{\vee}$ *defined as the only possible one that makes the following diagram*

$$\begin{tikzcd} Y^{\vee} \otimes X \arrow[r, "f^{\vee} \otimes id_X"] \arrow[d, "id_{Y^{\vee}} \otimes f"'] & X^{\vee} \otimes X \arrow[d, "ev_X"] \\ Y^{\vee} \otimes Y \arrow[r, "ev_Y"'] & \mathbb{1} \end{tikzcd}$$

commutative.

Example 2.4.48. *Coming back to the example of the category of* R*-modules, we know that in this category duals exists, they are defined as* $X^{\vee} = Hom(X, R)$*, and in this case the morphism* $f^{\vee}$ *of Remark 2.4.47 is defined by the equation* $\langle f^{\vee}(y), x\rangle_X = \langle y, f(x)\rangle_Y$ *where* $\langle\cdot,\cdot\rangle_X$ *and* $\langle\cdot,\cdot\rangle_Y$ *are alternative ways to denote* ev_X *and* ev_Y *respectively.*

Let V be a vector space of finite dimension over a field k, we know that the double dual $V^{\vee\vee}$ is isomorphic to V itself, this behavior in general is called:

Reflexive object in a tensor category

Definition 2.4.49. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category and* $X \in Obj(\mathcal{C})$*. Let* $i_X : X \to X^{\vee\vee}$ *be the morphism coming from the composition* $ev_X \circ \psi_{X,X^{\vee}} : X \otimes X^{\vee} \to \mathbb{1}$*. We will say that* X *is* reflexive *if* i_X *is an isomorphism.*

Remark 2.4.50. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. If an object* X *has an inverse (Definition 2.4.39)* (X^{-1}, δ)*, then the morphism* $X^{-1} \to X^{\vee}$ *induced by* $\delta : X \otimes X^{-1} \to \mathbb{1}$ *is an isomorphism, and thus we see that* X *is reflexive.*

Finite products of internal Hom-sets

Remark 2.4.51. *We have the following property for products of internal Hom-sets of finite products: if* $(\mathcal{C}, \otimes)$ *is a tensor category, and we have two finite families* $(X_i)_{i\in I}$ *and* $(Y_i)_{i\in I}$ *of objects of* $\mathcal{C}$*, then we have a morphism:*

$$\bigotimes_{i\in I} \underline{Hom}(X_i, Y_i) \to \underline{Hom}\left(\bigotimes_{i\in I} X_i, \bigotimes_{i\in I} Y_i\right)$$

corresponding to

$$\bigotimes_{i\in I} \underline{Hom}(X_i, Y_i) \otimes \bigotimes_{i\in I} X_i \overset{\cong}{\to} \bigotimes_{i\in I} (\underline{Hom}(X_i, Y_i) \otimes X_i) \overset{\otimes_{i\in I} ev}{\to} \bigotimes_{i\in I} Y_i$$

using the correspondence of Definition 2.4.42.
This morphism applied to duals yields a morphism

$$\bigotimes_{i\in I} X_i^{\vee} \to \left(\bigotimes_{i\in I} X_i\right)^{\vee}$$

and as a particular case of this, we obtain the morphism

$$X^\vee \otimes Y \to \underline{Hom}(X, Y)$$

by taking $I = \{1, 2\}$, $X_1 = X$, $X_2 = \mathbb{1} = Y_1$ *and* $Y_2 = Y$.

We will now define tensor categories that have internal Hom-sets for any pair of elements, and such that any object is reflexive, resembling the behavior of the category of k-vector spaces in terms of duals:

Definition 2.4.52. *Let* $(\mathcal{C}, \otimes)$ *be a tensor category. We say that the category* $\mathcal{C}$ *is* rigid *if it satisfies the following conditions:*

Rigid tensor categories

(a) For any pair of objects $X, Y \in Obj(\mathcal{C})$, *their internal Hom-set* $\underline{Hom}(X, Y)$ *(Definition 2.4.42) exists.*

(b) For any quadruple X_1, X_2, Y_1, Y_2 *of objects of* $\mathcal{C}$, *the morphism between the products of internal Homs, mentioned in Remark 2.4.51,*

$$\underline{Hom}(X_1, Y_1) \otimes \underline{Hom}(X_2, Y_2) \to \underline{Hom}\,(X_1 \otimes X_2, Y_1 \otimes Y_2)$$

is an isomorphism.

(c) Any object of $\mathcal{C}$ *is reflexive (Definition 2.4.49).*

Remark 2.4.53. *Let* $(\mathcal{C}, \otimes)$ *be a rigid tensor category.*

(a) Condition (b) of Definition 2.4.52 implies by induction that for any finite family $(X_i)_{i \in I}$ *of objects of* $\mathcal{C}$, *the morphism of Remark 2.4.51*

$$\bigotimes_{i \in I} \underline{Hom}(X_i, Y_i) \to \underline{Hom}\left(\bigotimes_{i \in I} X_i, \bigotimes_{i \in I} Y_i\right)$$

is an isomorphism.

(b) The functor $X \mapsto X^\vee$ *is an anti-equivalency of categories as it is contravariant, this is because this functor composed twice is isomorphic to the identity as any object of* $\mathcal{C}$ *is reflexive.*
Moreover, we have an isomorphism $\underline{Hom}(X, Y) \cong \underline{Hom}(Y^\vee, X^\vee)$ *coming from the composition*

$$\underline{Hom}(X, Y) \cong X^\vee \otimes Y \cong X^\vee \otimes Y^{\vee\vee} \cong Y^{\vee\vee} \otimes X^\vee \cong \underline{Hom}(Y^\vee, X^\vee).$$

(c) As any object X *of* $\mathcal{C}$ *is reflexive, we have a morphism*

$$\underline{Hom}(X, X) \cong X^\vee \otimes X \stackrel{ev_X}{\to} \mathbb{1}.$$

Composing this isomorphism with the functor $Hom(\mathbb{1}, \cdot)$ *we obtain a morphism* $End_{\mathcal{C}}(X) \to End_{\mathcal{C}}(\mathbb{1})$ *using Remark 2.4.45(c).*

Trace and rank in rigid tensor categories

Definition 2.4.54. *Let $(\mathcal{C}, \otimes)$ be a rigid tensor category and let X be an object of $\mathcal{C}$. The morphism $Tr_X : End_{\mathcal{C}}(X) \to End_{\mathcal{C}}(\mathbb{1})$ of Remark 2.4.53(c) is called the* trace morphism of X.
The rank of X *is* $rank(X) = Tr_X(\mathrm{id}_X) \in End_{\mathcal{C}}(\mathbb{1})$.

Lemma 2.4.55 (I 5.1.4 [59]). *Let $(\mathcal{C}, \otimes)$ be a rigid tensor category and let X, X' be objects of $\mathcal{C}$. Then, for any pair of endomorphisms $f \in End_{\mathcal{C}}(X)$ and $f' \in End_{\mathcal{C}}(X')$ we have the identities*

$$\begin{aligned} Tr_{X \otimes X'}(f \otimes f') &= Tr_X(f) \circ Tr_{X'}(f') \\ Tr_{\mathbb{1}}(f) &= f \end{aligned}$$

and

$$\begin{aligned} rank(X \otimes X') &= rank(X) \circ rank(X') \\ rank(\mathbb{1}) &= \mathrm{id}_{\mathbb{1}}. \end{aligned}$$

Example 2.4.56. *Let k be a field. The category of k-vector spaces of finite dimension $Vectf_k$ is rigid as for any pair of vector spaces V and W, their Hom-set $Hom_{Vectf_k}(V, W)$ has a natural structure of k-vector spaces and thus it is their internal Hom-set, we also have duals that are simply given $V^{\vee} = Hom_{Vectf_k}(V, k)$ as k is the identity element for the tensor product here.*
On the other hand, for a ring R the category $R-Modf$ of finitely generated modules is not rigid, even if we have internal Hom-sets (Example 2.4.44) as not any finitely generated module has a dual.
In $Vectf_k$, the trace of a vector space $Tr_X : End_{Vectf_k}(V) \to k$ becomes the well known trace of matrices after choosing a base of V.

Now we will consider the functors and natural transformations between tensor categories, they must preserve tensor category structures and thus the definition is the following:

Tensor functors

Definition 2.4.57. *Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be tensor categories. A* tensor functor between $\mathcal{C}$ and $\mathcal{C}'$ *is a pair (F, c) consisting in a functor $F : C \to C'$ along with an isomorphism of functors given for $X, Y \in Obj(\mathcal{C})$ as $c_{X,Y} : F(X) \otimes' F(Y) \to F(X \otimes Y)$ that satisfy the following properties:*

(a) For any triple of objects X, Y, Z of $\mathcal{C}$, the following diagram is commutative

$$\begin{tikzcd} F(X) \otimes' (F(Y) \otimes' F(Z)) \arrow[r, "\mathrm{id}_{F(X)} \otimes' c_{Y,Z}"] \arrow[d, "\phi'_{F(X),F(Y),F(Z)}"'] & F(X) \otimes' (F(Y \otimes Z)) \arrow[r, "c_{X,Y\otimes Z}"] & F(X \otimes (Y \otimes Z)) \arrow[d, "F(\phi_{X,Y,Z})"] \\ (F(X) \otimes' F(Y)) \otimes' F(Z) \arrow[r, "c_{X,Y} \otimes' \mathrm{id}_{F(Z)}"'] & F(X \otimes Y) \otimes' F(Z) \arrow[r, "c_{X\otimes Y, Z}"'] & F((X \otimes Y) \otimes Z) \end{tikzcd} .$$

Where ϕ and ϕ' are the isomorphisms coming from the associativity constrains (Definition 2.4.31(a)) of $\mathcal{C}$ and $\mathcal{C}'$ respectively.

(b) For any pair of objects X, Y *of* $\mathcal{C}$*, the following diagram is commutative*

$$\begin{tikzcd} F(X) \otimes' F(Y) \arrow[r, "c_{X,Y}"] \arrow[d, "\psi'_{F(X),F(Y)}"'] & F(X \otimes Y) \arrow[d, "F(\psi_{X,Y})"] \\ F(Y) \otimes' F(X) \arrow[r, "c_{Y,X}"'] & F(Y \otimes X) \end{tikzcd} .$$

Where ψ *and* ψ' *are the isomorphisms coming from the commutativity constrains (Definition 2.4.31(b)) of* $\mathcal{C}$ *and* $\mathcal{C}'$ *respectively.*

(c) If (U, u) *is an identity object of* $\mathcal{C}$ *(Definition 2.4.31(d)), then the pair* $(F(U), F(u))$ *is an identity object of* $\mathcal{C}'$*.*

We can make some initial remarks about these functors:

Remark 2.4.58. *Let* (F, c) *be a tensor functor between two tensor categories* $(\mathcal{C}, \otimes)$ *and* $(\mathcal{C}', \otimes')$*.*

(a) For any finite family $\{X_i\}_{i \in I}$ *of objects of* $\mathcal{C}$*, we have a well defined isomorphism*

$$c : \bigotimes_{i \in I} F(X_i) \to F\left(\bigotimes_{i \in I} X_i\right)$$

for the extensions of the tensor product over the family I *(Remark 2.4.38), that can be seen as an extension of* c *in Definition 2.4.33 for several objects instead of two.*
In particular, F *maps invertible objects (Definition 2.4.39) of* $\mathcal{C}$ *to invertible objects of* $\mathcal{C}'$*.*

(b) If the internal Hom-set $\underline{Hom}(X, Y)$ *of two objects* X, Y *of* $\mathcal{C}$ *exists, then the morphism*

$$F(ev_{X,Y}) : F(\underline{Hom}(X, Y)) \otimes' F(X) \to F(Y)$$

induces a morphism $F_{X,Y} : F(\underline{Hom}(X, Y)) \to \underline{Hom}(F(X), F(Y))$*. In particular, if the dual of an object* X *of* $\mathcal{C}$ *exists, then* F *induces a morphism* $F_X : F(X^\vee) \to F(X)^\vee$*.*

For rigid tensor categories, the morphisms in part (b) of this remark are isomorphisms:

Lemma 2.4.59 (Prop. 1.9 [17])**.** *Let* $(\mathcal{C}, \otimes)$ *and* $(\mathcal{C}', \otimes')$ *be two rigid tensor categories, and let* (F, c) *be a tensor functor. Then for any pair of objects* X, Y *of* $\mathcal{C}$*, the morphism discussed in Remark 2.4.58(b)*

$$F_{X,Y} : F(\underline{Hom}(X, Y)) \to \underline{Hom}(F(X), F(Y))$$

is an isomorphism.

Remark 2.4.60. *Let* (F, c) *be a tensor functor between two rigid tensor categories* $(\mathcal{C}, \otimes)$ *and* $(\mathcal{C}', \otimes')$*. Then,* F *induces a natural morphism* $F_{End} : End_{\mathcal{C}}(\mathbb{1}) \to End_{\mathcal{C}'}(\mathbb{1}')$ *which in turn implies the following identities for traces and ranks:*

$$\begin{aligned} Tr_{F(X)}(F_{End}(f)) &= F_{End}(Tr_X(f)) \\ Rank(F(X)) &= F_{End}(Rank(X)). \end{aligned}$$

Tensor equivalences

Definition 2.4.61. *Let* (F, c) *be a tensor functor between two tensor categories* $(\mathcal{C}, \otimes)$ *and* $(\mathcal{C}', \otimes')$*. We say that* (F, c) *is a* tensor equivalence *if* F *is an equivalence of categories.*

We can define equivalences of categories in terms of natural transformations, and we would like a similar version of this while preserving the tensor category structure in the categories involved for a tensor equivalence. So now we will define these special natural transformations:

Morphisms of tensor functors

Definition 2.4.62. *Let* (F, c) *and* (G, d) *be two tensor functors from the tensor category* $(\mathcal{C}, \otimes)$ *to the tensor category* $(\mathcal{C}', \otimes')$*. A* morphism of tensor functors *is a natural transformation* $\lambda : F \to G$ *such that for any finite family* $\{X_i\}_{i \in I}$ *of objects of* $\mathcal{C}$*, the diagram*

$$\begin{tikzcd} \bigotimes_{i\in I} F(X_i) \arrow[r, "c"] \arrow[d, "\otimes_{i\in I}\lambda_{X_i}"'] & F\left(\bigotimes_{i\in I} X_i\right) \arrow[d, "\lambda_{\otimes_{i\in I} X_i}"] \\ \bigotimes_{i\in I} G(X_i) \arrow[r, "d"'] & G\left(\bigotimes_{i\in I} X_i\right) \end{tikzcd}$$

is commutative, along with the commutative diagram

$$\begin{tikzcd} \mathbb{1} \arrow[r, "\cong"] \arrow[d, equal] & F(\mathbb{1}) \arrow[d, "\lambda_{\mathbb{1}}"] \\ \mathbb{1} \arrow[r, "\cong"] & G(\mathbb{1}) \end{tikzcd}$$

which gives in particular an isomorphism between $F(\mathbb{1})$ *and* $G(\mathbb{1})$ *as they are both identity objects.*
We will denote by $Hom^{\otimes}(F, G)$ *the set*[16] *of morphisms of tensor functors between* F *and* G*.*
An isomorphism of tensor functors *is a natural isomorphism that is a morphism of tensor functors.*

Now we can characterize tensor equivalences using morphisms of tensor functors:

Proposition 2.4.63 (Ch. I §4.4 [59])**.** *Let* $(F, c) : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ *be a tensor equivalence between two tensor categories. Then, there exists a tensor functor* $(F', c') : (\mathcal{C}', \otimes') \to (\mathcal{C}, \otimes)$ *such that the compositions* $F \circ F'$ *and* $F' \circ F$ *are isomorphic as tensor functors to* $\mathrm{id}_{\mathcal{C}'}$ *and* $\mathrm{id}_{\mathcal{C}}$ *respectively.*

16 Most likely the class.

Morphisms of tensor functors between rigid tensor categories share the same property G-torsors, with G a fixed group-scheme, over schemes do (See 2.3.11):

Proposition 2.4.64. *Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be two rigid tensor categories, and let (F, c) and (G, d) be two tensor functors from $\mathcal{C}$ to $\mathcal{C}'$. Then, any morphisms $\lambda : F \to G$ of tensor functors is an isomorphism.*

Groupoid property for morphisms of tensor functors between rigid tensor categories

Proof. By Lemma 2.4.59, for any object X of $\mathcal{C}$ we have an isomorphism $F(X)^\vee \cong F(X^\vee)$ and the same holds for G. This allows us to define a morphism $\mu : G \to F$ of tensor functors that makes the diagram

$$\begin{tikzcd} F(X^\vee) \arrow[r, "\lambda_{X^\vee}"] \arrow[d, "\cong"'] & G(X^\vee) \arrow[d, "\cong"'] \\ F(X)^\vee \arrow[r, "(\mu_X)^\vee"'] & G(X)^\vee \end{tikzcd}$$

commutative using the isomorphism of Remark 2.4.53(b).
This morphism of tensor functors is an inverse of λ in the sense of Proposition 2.4.63, the proof of this fact is more subtle than it looks, for more details see the proof made by Todd Trimble[17] that arose from a relevant question posed in mathoverflow[18]. □

Example 2.4.65. *Let k be a field, for any k-algebra R there is a canonical tensor functor ϕ_R : $Vectf_k \to R-Modf$ given by $V \mapsto V \otimes_k R$. If $(\mathcal{C}, \otimes)$ is a tensor category and $(F, c), (G, d) : \mathcal{C} \to Vectf_k$ are two tensor functors, then we can define the functor $\underline{Hom}^\otimes(F, G) : Alg_k \to Set$ given by*

$$\underline{Hom}^\otimes(F, G)(R) = Hom^\otimes(\phi_R \circ F, \phi_R \circ G).$$

This example is important, as it will appear in Subsubsections 2.4.2.2 and 2.4.3.1 when we will talk about recovering an affine group-scheme from its finite representations (Definition 2.4.78).

The last concept of tensor categories we need is tensor sub-categories:

Definition 2.4.66. *Let $\mathcal{C}'$ be a strictly full sub-category of a tensor category $(\mathcal{C}, \otimes)$. We say that $\mathcal{C}'$ is a* tensor sub-category *it $\mathcal{C}'$ is closed under the formation of finite tensor products of elements of $\mathcal{C}'$ or equivalently, if $\mathcal{C}'$ contains $\mathbb{1}$ and $X \otimes Y$ whenever $X, Y \in Obj(\mathcal{C}')$.*
If $(\mathcal{C}, \otimes)$ is moreover rigid, a rigid tensor sub-category *is a tensor sub-category $\mathcal{C}'$ such that $X^\vee \in Obj(\mathcal{C}')$ when $X \in Obj(\mathcal{C}')$.*

Tensor sub-categories

Remark 2.4.67. *If $\mathcal{C}'$ is a tensor sub-category of a tensor category $(\mathcal{C}, \otimes)$. Then, the pair $(\mathcal{C}, \otimes)$ is a tensor category, where $\otimes$ in this case denotes the restriction of the tensor product over $\mathcal{C}$ to $\mathcal{C}'$, which is well defined following the definition of tensor sub-categories.*
We easily see that the inclusion $i : (\mathcal{C}', \otimes) \to (\mathcal{C}, \otimes)$ is a tensor functor.

17 https://ncatlab.org/toddtrimble/published/Morphisms+between+tensor+functors
18 https://mathoverflow.net/questions/116104/functors-on-rigid-tensor-categories

2.4.1.3 *Neutral Tannakian Categories*

Now we will combine abelian categories and tensor categories in order to define neutral tannakian categories towards the end. We will always suppose that functors between additive categories (Definition 2.4.16) are additive (Definition 2.4.11).

Abelian tensor categories

Definition 2.4.68. *An* additive (resp. abelian) tensor category *is a tensor category* $(\mathcal{C}, \otimes)$ *such that* $\mathcal{C}$ *is an additive category (resp. abelian category*[19]*) and the tensor product* $\otimes$ *is a biadditive functor*[20]*.*

Now we will introduced some concepts exclusive to these hybrid categories:

Definition 2.4.69. *Let* $(\mathcal{C}, \otimes)$ *be an abelian tensor category, we will denote as* $\oplus$ *the biproduct in* $\mathcal{C}$ *(Definition 2.4.14).*

Sub-quotients

(a) A sub-quotient *of an object* $X \in Obj(\mathcal{C})$ *is a cokernel (quotient)* V/V' *where* $V' \subset V \subset X$ *is a chain of sub-objects of* X*.*

Tensor generating families

(b) A family $(X_i)_{i \in I}$ *of objects of* $\mathcal{C}$ *is a* tensor generating family *if every object of* $\mathcal{C}$ *is isomorphic to a sub-quotient of the element* $P(X_i)$ *where* $P(T_i) \in \mathbb{Z}_{\geqslant 0}[x][T_i]_{i \in I}$ *is a polynomial with non-negative integer coefficients with variables indexed by* I*, and the object* $P(X_i)$ *is the object of* $\mathcal{C}$ *resulting from changing each sum between monomials for* $\oplus$*, and changing any monomial* $nT_{i_1}^{a_1} \cdots T_{i_k}^{a_k}$ *of* P *by*

$$(X_{i_1}^{\otimes a_1} \otimes \cdots \otimes X_{i_k}^{\otimes a_k})^{\oplus n}$$

for $n, k, a_i \geqslant 0$ *where* $(\cdot)^{\oplus a}$ *is the biproduct of* $a \geqslant 0$ *copies of an object,* $(\cdot)^{\otimes a}$ *is the tensor product of* a *copies of an object, with the conventions* $(\cdot)^{\oplus 0} = 0_{\mathcal{C}}$ *where* $0_{\mathcal{C}}$ *is the zero object of* $\mathcal{C}$*, and* $(\cdot)^{\otimes 0} = \mathbb{1}$*.*

Remark 2.4.70. *Let* $(\mathcal{C}, \otimes)$ *be an abelian tensor category. If we denote* $R = End_{\mathcal{C}}(\mathbb{1})$*, we have that* R *is a ring and for any object* X *of* $\mathcal{C}$*,* R *acts on* X *via the isomorphism* $l_X : X \xrightarrow{\cong} \mathbb{1} \otimes X$ *(Proposition 2.4.36(a)). We see that this action commutes with any endomorphism, so in particular* R *is a commutative ring.*
If $(\mathbb{1}', e')$ *is another unit, by Proposition 2.4.36(b) we have an isomorphism* $a : (\mathbb{1}, e) \to (\mathbb{1}', e')$ *so that* $R \cong End_{\mathcal{C}}(\mathbb{1}')$*.*

Definition 2.4.71. *Let* $(\mathcal{C}, \otimes)$ *be an additive tensor category. If* R *is a commutative ring with unit, we say that* $\mathcal{C}$ *is* R-linear *if for any pair of objects* X, Y *of* $\mathcal{C}$*, the set* $Hom_{\mathcal{C}}(X, Y)$ *has a structure of* R*-module such that the composition is* R*-bilinear*[21] *and the tensor product is a* R*-bilinear functor so it preserves the* R*-modules structure on Hom-sets.*

19 Definition 2.4.19
20 Like in the case of the composition in AB categories, see Remark 2.4.10.
21 Which is the same property stated in Remark 2.4.10 over R instead of $\mathbb{Z}$.

By Remark 2.4.70, any abelian tensor category is R-linear where $R = \mathrm{End}(\mathbb{1})$.

Remark 2.4.72. *If $(\mathcal{C}, \otimes)$ is an abelian rigid tensor category, and $R = End_{\mathcal{C}}(\mathbb{1})$, then the trace morphism $Tr_X : End_{\mathcal{C}}(X) \to R$ is R-linear.*

Now we will state some properties of abelian tensor categories:

Proposition 2.4.73 (Prop. 1.16 [17])**.** *Let $(\mathcal{C}, \otimes)$ be an rigid tensor category. If $\mathcal{C}$ is moreover abelian, then $\otimes$ is a bilinear additive functor, that commutes with direct and projective limits in both variables, in particular it is exact (Definition 2.4.29) on each variable.*

Proposition 2.4.74 (Prop. 1.17 [17])**.** *Let $(\mathcal{C}, \otimes)$ be an abelian rigid tensor category. If U is a sub-object of $\mathbb{1}$, then we have a decomposition $\mathbb{1} = U \oplus U^{\perp}$ where $U^{\perp} = \ker(\mathbb{1} \to U^{\vee})$. In particular, if $End_{\mathcal{C}}(\mathbb{1})$ is a field then $\mathbb{1}$ is simple, i.e. it does not possess any sub-object different from the zero object and itself.*

Remark 2.4.75. *This proposition shows that there is a bijection between sub-objects of $\mathbb{1}$ and idempotents elements in the ring $End_{\mathcal{C}}(\mathbb{1})$.*
If e is an idempotent endomorphism of $\mathbb{1}$, then we can decompose $\mathcal{C}$ as $\mathcal{C} = \mathcal{C}' \times \mathcal{C}''$ where $\mathcal{C}'$ and $\mathcal{C}''$ are tensor categories, and any object $\mathcal{C}$ is in $\mathcal{C}'$ when e acts like the identity on it, and in $\mathcal{C}''$ when $1-e$ acts like the identity. This decomposition is comparable of the decomposition of affine schemes in the presence of idempotents (see [36, II Exc. 2.19]).

Proposition 2.4.76 (Prop. 1.20 [17])**.** *Let $(\mathcal{C}, \otimes)$ and $(\mathcal{C}', \otimes')$ be two abelian rigid tensor categories. If $\mathbb{1}$ and $\mathbb{1}'$ are identity objects of $\mathcal{C}$ and $\mathcal{C}'$ respectively. If $End_{\mathcal{C}}(\mathbb{1})$ is a field and $\mathbb{1}' \neq 0$, then any exact tensor functor $F : (\mathcal{C}, \otimes) \to (\mathcal{C}', \otimes')$ is faithful.*

We finish this subsection with the definition of neutral tannakian categories:

Neutral tannakian categories

Definition 2.4.77. *Let k be a field and let $\mathcal{C}$ be a k-linear abelian rigid tensor category. We say that $\mathcal{C}$ is a* neutral tannakian category over k *if $End_{\mathcal{C}}(\mathbb{1}) = k$ and $\mathcal{C}$ possesses an exact faithful tensor functor $\omega : \mathcal{C} \to Vectf_k$. The functor ω is called the* fiber functor of $\mathcal{C}$.

In Subsection 2.4.2 we will define and study the model neutral tannakian category, the category of representations of affine group-schemes, followed by Subsection 2.4.3 where we show that any neutral tannakian category over k is equivalent to a category of representations of an affine group-scheme over k.

2.4.2 *Representations of Group-schemes*

In this part we will study representations of group-schemes. All group-schemes considered in this part will be affine. The main focus of this

subsection is to study the category of all finitely dimensional representations of an affine group-scheme, where the final goal is to show that this category is neutral tannakian. This will be the content of Subsubsection 2.4.2.1
Secondly, we will see in Subsubsection 2.4.2.2 how an affine group-scheme can be recovered form its category of representations in a functorial way.

2.4.2.1 *Categories of comodules and representations*

Representation of affine group-schemes

Definition 2.4.78. *Let* k *be a field and let* G *be a group-scheme over* k*. A* representation of G (over V) *is a morphism* $r : G \to GL(V)$ *where* V *is* k*-vector space and* $GL(V)$ *is the general linear group-scheme, defined in Example 2.2.15 (1).*
Let $r : G \to GL(V)$ *and* $s : G \to GL(W)$ *be two representations of* G*, a* morphism of representations, *is a morphism of group-schemes* $\phi : GL(V) \to GL(W)$*, coming from a* k*-linear morphism* $V \to W$*, such that* $s = \phi \circ r$*.*
A representation $u : G \to GL(V)$ *where* V *is a finitely dimensional* k*-vector space will be called a* finite or finitely dimensional *representation. The category of finite representations of a group-scheme* G *will be denoted as* $Rep_k(G)$*.*

Remark 2.4.79. *Let* k *be a field and let* G *be a group-scheme over* k*. A representation* $r : G \to GL(V)$ *of* G *equates to a functorial left action (Definition 2.2.28)*

$$\widetilde{G}(R) \times (V \otimes_k R) \to (V \otimes_k R)$$

by R*-linear morphisms, for any* k*-algebra* R*.*

Faithful representations of group-schemes

Definition 2.4.80. *Let* k *be a field and let* G *be a group-scheme over* k*. If a representation* $r : G \to GL(V)$ *is a monomorphism of group-schemes (Proposition 2.2.62), then we will say that the representation* r *is* faithful.

Remark 2.4.81. *If a group-scheme* G *possesses a finite faithful representation, we can easily conclude that* G *is of finite type as the representation morphism* $r : G \to GL(V)$ *is a closed immersion and* $GL(V)$ *is of finite type over* k *(see Example 2.2.15(1)). We will see later that the converse assertion is also true, i.e., any group-scheme of finite type over a field has a faithful finite representation (see Corollary 2.4.88).*

If $G = \mathrm{Spec}(A)$ is an affine group-scheme, we can express representation $r : G \to \mathrm{GL}(V)$ of G in terms of the underlying Hopf algebra A and V. In fact, we do not even need all the axioms of a Hopf algebra to do so, so we will state the minimal properties we need to define this expression.

Co-algebras and comodules

Definition 2.4.82. *A* co-algebra over k *is* k*-vector space* C *with two* k*-linear morphisms* $\Delta : C \to C \otimes C$ *and* $\epsilon : C \to k$ *that satisfy the coassociativity and counit axioms stated in Definition 2.2.12.*
If C *is a co-algebra over* k *and* V *is a* k*-vector space, a* comodule over C *is a*

k-linear morphism $\rho : V \to V \otimes_k C$ that satisfies the following commutative diagrams

$$\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[dr, "\mathrm{id}_V"'] & V \otimes_k C \arrow[d, "(\mathrm{id}_V, \epsilon)"] \\
 & V \otimes_k k \cong V
\end{tikzcd}$$

and

$$\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[d, "\rho"'] & V \otimes_k C \arrow[d, "(\rho, \mathrm{id}_C)"] \\
V \otimes_k C \arrow[r, "(\mathrm{id}_V, \Delta)"'] & V \otimes_k C \otimes_k C
\end{tikzcd} .$$

Let $\rho : V \to V \otimes_k C$ and $\sigma : W \to W \otimes_k C$ be two comodules over C, a morphism of comodules over C *is a k-linear morphism of vector spaces $\phi : V \to W$ such that the following diagram is commutative:*

$$\begin{tikzcd}
V \arrow[r, "\rho"] \arrow[d, "\phi"'] & V \otimes_k C \arrow[d, "(\phi, \mathrm{id}_C)"] \\
W \arrow[r, "\sigma"'] & W \otimes_k C
\end{tikzcd} .$$

If V is a finitely dimensional k-vector space, we will call the corresponding comodule a finite or finitely dimensional *comodule. The category of all finite comodules of a given co-algebra C will be denoted as $Comod_k(C)$.*

Remark 2.4.83. *If C is a co-algebra over k, then $\Delta : C \to C \otimes_k C$ makes C a comodule over itself, the axioms we demanded for Δ and ϵ in Definition 2.4.82 imply this. In particular, if A is a Hopf algebra A is a comodule over itself by via its comultiplication morphism.*

Now we can express representation of group-schemes with comodules:

Proposition 2.4.84. *For any affine group-scheme $G = \mathrm{Spec}(A)$ over k, there is a bijective and natural correspondence between representations $r : G \to GL(V)$ of G and comodules $\rho : V \to V \otimes_k A$ over A.*

Correspondence between representations and comodules

Proof. We will only mention here how to go from a comodule to a representation and vice versa, for more details see [68, Theorem §3.2]. If $r : G \to GL(V)$ is a representation of G, as $\widetilde{G}$ and $\widetilde{GL(V)}$ are representable functors of k-algebras, we can show using Yoneda's lemma (Lemma 2.2.7) that the natural transformation $\widetilde{r}$, induced by r, is determined by the image of $\mathrm{id}_G \in \widetilde{G}(A)$ under $\widetilde{r}$. This image is a member of $\widetilde{GL(V)}(A) = \mathrm{Aut}_{A-mod}(V \otimes_k A)$ so it corresponds to an automorphism $V \otimes_k A \to V \otimes_k A$ that can be shown to be determined by its restriction

$$V \cong V \otimes_k k \subset V \otimes_k A \to V \otimes_k A$$

giving us a comodule over A.

On the other hand, if V is a comodule over A, the morphism ρ :

$V \to V \otimes_k A$ determines a representation of G over V, that at the level of functor of points, it is given for any k-algebra R and any $g : A \to R \in \widetilde{G}(R)$, as an automorphism $g_V : V \otimes_k R \to V \otimes_k R$ whose restriction to V is

$$V \xrightarrow{\rho} V \otimes_k A \xrightarrow{(id_V, g)} V \otimes_k R\,.$$

□

Now as we have established this bijection, we can show some examples of comodules/representations:

Example 2.4.85. *Let $G = \mathrm{Spec}(A)$ be an affine group-scheme.*

The regular representation

1. *As we mentioned in Remark 2.4.83, A is a comodule over itself via the comultiplication morphism $\Delta : A \to A \otimes_k A$. We will call this representation the* regular representation of A (or G). *We clearly see that G is a finite group-scheme if and only its regular representation is finite.*

Actions of group-schemes and comodules

2. *Let $X = \mathrm{Spec}(B)$ be an affine scheme of finite type over k, with a given right action $\mu : X \times_k G \to X$. In this case, this action is the same as considering a morphism of k-algebras $\rho : B \to B \otimes_k A$ and we can easily see that by looking at the definition of an action (Definition 2.2.25) and taking their corresponding axioms at the level of k-algebras, that B is a comodule over A. If X is finite over k, then the corresponding comodule is finite.*

Sub-comodules and sub-representations

3. *If $\rho : V \to V \otimes C$ is a comodule over a co-algebra C, then a* sub-comodule *of ρ is a sub-space $W \subset V$ such that $\rho(W) \subset W \otimes_k C$, so the restricted morphism $\rho|_W$ makes W a comodule over C.*
In the same vein, for a representation $r : G \to GL(V)$ of a group-scheme, a sub-representation *is a representation $s : G \to GL(W)$ over a sub-space $W \subset V$ such that $f(W \otimes R) \subset W \otimes R$ for any k-algebra R and any automorphism $f : V \otimes R \to V \otimes R$ so we have a well defined restriction morphism $(\cdot)|_W : GL(V) \to GL(W)$ with $s = r|_W$. Clearly, by the correspondence of Proposition 2.4.84 $W \subset V$ is sub-comodule if and only if the corresponding representation over W is a sub-representation.*

Fixed sub-comodule

4. *Let C be a co-algebra and let $\rho : V \to V \otimes C$ be a comodule over C, we define the* fixed sub-comodule *as the sub-comodule*

$$V^C = \{v \in V : \rho(v) = v \otimes 1\}.$$

When A is the Hopf algebra associated to an affine group-scheme G we will denote this sub-comodule as V^G. In this case this sub-comodule can be equivalently defined as

$$V^G = \{v \in V : g \cdot v_R = v_R \textit{ for all } k\textit{-algebras } R \textit{ and } g \in \widetilde{G}(R)\}$$

where $v_R = v \otimes 1 \in V \otimes_k R$.

5. *In Example 2.2.88 we characterized quotients of affine group-schemes, now we will relate this with representations of affine group-schemes. Let* $H = \mathrm{Spec}(B)$ *be a subgroup-scheme, then we have* $B = A/I$ *where* I *is a Hopf ideal of* A *(Corollary 2.2.21). The action by multiplication* $G \times_k H \to G$ *yields a comodule* $\tau_H : A \to A \otimes_k A/I$ *over* A/I.
Let us consider the quotient $Q = G/H$, *as we mentioned in Example 2.2.88,* $Q = \mathrm{Spec}(A^H)$ *is the spectrum of the fixed sub-comodule that we defined in the last example (which is also a* k*-algebra) of* A *by* τ_H*, i.e,*
$$A^H = \{a \in A : \tau_H(a) = a \otimes 1 \ (mod\ A \otimes_k A/I)\}$$
Moreover, this quotient defines a representation of G *from left action of* G *over* Q *(recall Definition 2.2.73), that we will denote by* $\mu_H : G \to GL(A^H)$*. We will consider again this representation in Subsection 5.2.3.*

Representation associated to a quotient of group-schemes

Coming back to the content of Proposition 2.4.84, it shows that for an affine group-scheme $G = \mathrm{Spec}(A)$, the categories $\mathrm{Rep}_k(G)$ and $\mathrm{Comod}_k(A)$ are equivalent, but how equivalent are they?
It is not as clear at first glance that both these categories are tensorial, and the equivalence between representations and comodules might not come from a tensor functor, if we suppose $\mathrm{Comod}_k(A)$ and $\mathrm{Rep}_k(G)$ are tensorial in the first place, that we have not established yet.
To obtain that $\mathrm{Rep}_k(G)$ is a neutral tannakian category over k, we will study the properties of $\mathrm{Comod}_k(C)$ starting from C as a co-algebra up to a Hopf algebra, focusing on what we will add along the way as we add more structure to the base co-algebra. In the following, each time the reader sees a result about comodules, they should do the mental exercise of figuring the corresponding property for representations, aided by the equivalence of categories in Proposition 2.4.84.
Before that, we will list some basic properties of comodules:

Lemma 2.4.86 (Prop. 2.3 [17], §3.3 [68])**.** *Let* C *be a co-algebra and let* V *be a comodule over* C*. Then every finite subset* $\{v_1, v_2, \cdots, v_n\}$ *of* V *is contained in a finite sub-comodule of* V*.*

Tying this with and Proposition 2.4.84, we obtain:

Corollary 2.4.87 (Corollary 2.4 [17])**.** *Let* $G = \mathrm{Spec}(A)$ *be an affine group-schemes, then any representation of* G *is a directed union of its finite sub-representations.*

Corollary 2.4.88. *An affine group-scheme* $G = \mathrm{Spec}(A)$ *is of finite type over* k *if and only if it has a faithful finite representation.*

Proof. The "only if" part was already established in Remark 2.4.81.
For the "if" part, let us suppose that G is of finite type over k. We can write the regular representation $r : G \to GL(A)$ of G as a directed limit of finite sub-representations $G \to GL(V_i)$ for some indexes i

belonging to an eventually infinite set I by Corollary 2.4.87. It is not hard to see that the regular representation is faithful, so we have

$$\{1\} = \ker(r) = \cap_{i \in I} (\ker(G \to GL(V_i)))$$

which is an intersection of closed sub-schemes of G. As G is noetherian, there should exists an index $i_0 \in I$ such that the kernel of the corresponding representation $G \to GL(V_{i_0})$ is trivial, giving us the desired faithful finite representation. □

Now we will pivot to the study of the categories of finite comodules and representations. As we would like to obtain that $\mathrm{Rep}_k(G)$ (and thus $\mathrm{Comod}_k(A)$) is a neutral tannakian categories, we need fiber functors to the category Vectf_k of finitely-dimensional k-vector spaces, according to Definition 2.4.77, so we will start by defining these functors:

Forgetful functor of comodules and representations

Definition 2.4.89. *Let C be a co-algebra over k. The* forgetful functor of $\mathrm{Comod}_k(C)$ *is the functor* $\omega_C : \mathit{Comod}_k(C) \to \mathit{Vectf}_k$ *that takes a finite comodule V over C and considers it solely as a k-vector space detached of its comodule structure.*
Let G be an affine group-scheme. The forgetful functor of $\mathrm{Rep}_k(G)$ *is the functor* $\omega_G : \mathit{Rep}_k(G) \to \mathit{Vectf}_k$ *that takes a finite representation* $r : G \to GL(V)$ *and sends it to the finite vector space V.*

Remark 2.4.90. *Let* $G = \mathrm{Spec}(A)$ *be an affine group-scheme. The equivalence between finite comodules over A and finite representations over G in Proposition 2.4.84 commutes with the forgetful functors of these categories. This means that, if we denote as* $F_C^R : \mathit{Comod}_k(A) \to \mathit{Rep}_k(G)$ *is the functor that takes a comodule and assigns it to its corresponding representation, as we did in the proof of Proposition 2.4.84, then* $F_C^R(\omega_A) = \omega_G$ *and the same is valid if we take the inverse functor* $F_R^C : \mathit{Rep}_k(G) \to \mathit{Comod}_k(A)$*, so we have* $F_R^C(\omega_G) = \omega_C$.

The first easy characterization of the category of comodules is:

Lemma 2.4.91. *Let C be a co-algebra over k. Then, the category* $\mathit{Comod}_k(C)$ *is additive (Definition 2.4.16). Moreover, the forgetful functors is additive too (Definition 2.4.11).*

Proof. The zero vector space clearly equips both $\mathrm{Comod}_k(C)$ with a zero object.
For the AB category structures, as any morphism of comodules rests on morphism between k-vector spaces, Hom-sets of comodules have natural structures of k-vector spaces as any sum or scalar multiplication of morphisms between comodules is again of the same type. Also, it is not hard to see that compositions induce k-linear morphisms between Hom-sets, so $\mathrm{Comod}_k(C)$ is an AB category (Definition 2.4.9), and moreover, its Hom-sets are well-behaved k-vector spaces. This discussion also implies that both forgetful functors are additive, and moreover, they induce k-linear morphism between Hom-sets

of comodules and those of finite k-vector spaces.
Finally, if $\rho : V \to V \otimes_k C$ and $\sigma : W \to W \otimes_k C$ are two finite comodules, then the morphism

$$\rho \oplus \sigma : V \oplus W \to (V \oplus W) \otimes_k C \cong (V \otimes_k C) \oplus (W \otimes_k C)$$

shows that $V \oplus W$ is a well-defined comodule over C and thus the category $\mathrm{Comod}_k(C)$ possesses biproducts, so it is additive. □

Remark 2.4.92. *Let C be a co-algebra over k, as $\mathrm{Comod}_k(C)$ is not necessarily tensorial, we cannot say that this category is k-linear in the sense of Definition 2.4.71. We just have a better version of Remark 2.4.10 as we have added scalar multiplication from k on the Hom-sets that behaves well with respect to composition. We will call kind of category call a* k-linear additive AB category*, if the category in question is additive or abelian, we will add the adjective k-linear as well.*
Functors $F : \mathcal{C} \to \mathcal{D}$ between k-linear AB, additive or abelian categories that induce k-linear morphisms between Hom-sets will be called k-linear functors*.*

Lemma 2.4.91 leads to:

Proposition 2.4.93. *Let C be a co-algebra over k. Then, the category $\mathrm{Comod}_k(C)$ is k-linear and abelian (Definition 2.4.19).*
Moreover, ω_C is exact (Definition 2.4.29), k-linear and faithful.

Finite comodules form an abelian category

Proof. As morphisms of comodules have morphisms of vector spaces included, it suffices to show that any morphism of finite comodules has a kernel and a cokernel to conclude that $\mathrm{Comod}_k(C)$ is an abelian category. In fact, if V, W are finite comodules over C, and $f : V \to W$ is a morphism of comodules between them, we will show that this morphism has a kernel and a cokernel, such that $\omega_C(\ker(f)) = \ker(\omega_C(f))$ and $\omega_C(\mathrm{coker}(f)) = \mathrm{coker}(\omega_C(f))$.
We will start with the kernel. Let us denote the structural comodule morphism of V and W as $\rho : V \to V \otimes_k C$ and $\sigma : W \to W \otimes_k C$. Now, let $K = \ker(f) \subset V$ be the kernel of f as a morphism of vector spaces, if we show that K is a sub-comodule of V (Example 2.4.85(3)) we can conclude that K is a kernel as a comodule as well. For this we just need to show that $\rho(K) \subset K \otimes_k C$. Indeed, if we take $k \in K$ and we denote $\rho(k) = \sum_i v_i \otimes c_i$ with $v_i \in V$ and $c_i \in C$ for all i, we will suppose that $\rho(k)$ does not possess any zero pure tensor in it, meaning that $c_i \neq 0$ for all i. As f is a morphism of comodules, we that

$$\sigma(\underbrace{f(k)}_{0}) = 0 = (f, \mathrm{id}_C)(\rho(k)) = \sum_i f(v_i) \otimes c_i$$

so for all i, $f(v_i) = 0$ and thus $v_i \in K$ so K is effectively a sub-comodule of V.
For the cokernel, if $q : W \to Q$ is the cokernel of f as a morphism of

vector spaces, we simply need to show that Q has a comodule structure over C such that q becomes a morphism of comodules. As Q is a quotient by $\mathrm{im}(f)$ and it is not hard to see that the image of f is a sub-comodule of W, we will show in general that if $U \subset W$ is a sub-comodule, then W/U has a natural structure of comodule over C such that the projection $\pi : W \to W/U$ is a morphism of comodules. Let us define $\tau : W/U \to W/U \otimes_k C$ as $\tau(\bar{w}) = (\pi, \mathrm{id}_C)(\sigma(w))$ for $w \in W$, if τ is well-defined, W/U is a comodule over C and π becomes a morphism of comodules. Let $w, w' \in W$ be two elements such that $w = w' + u$ with $u \in U$ so that $\bar{w} = \bar{w'}$. Then we have that

$$\sigma(w) = \sigma(w') + \underbrace{\sigma(u)}_{\in U \otimes_k C}$$

thus after taking (π, id_C) we obtain that $(\pi, \mathrm{id}_C)(\sigma(w) = (\pi, \mathrm{id}_C)(\sigma(w')$ and thus W/U is indeed a comodule over C. This proves the first part of the statement. For the second part, if $f : V \to W$ is the zero morphism, then it is in particular the zero morphism of vector spaces, so ω_C is faithful. ω_C is also exact as sub-comodules are sub-spaces of course, and quotients of comodules are quotients of vector spaces, as we showed in the last paragraph. □

Quotients of comodules

Remark-Definition 2.4.94. *The second to last paragraph of the proof above shows that if C is a co-algebra over a field k, V is a comodule over C and $U \subset V$ is a sub-comodule, then the* quotient comodule *exists and it is simply the vector space quotient V/U with a natural comodule morphism $\tau : W/U \to W/U \otimes_k C$ coming from the quotient morphism $\pi : V \to V/U$ at the level of vector spaces.*

At the beginning of this section, we said that tannakian categories allows us to recover an affine group-scheme from its finite representation, but what about its underlying Hopf algebra?
The short answer is that the Hopf algebra associated to a group-scheme can be recovered from its category of finite comodules, which makes sense if the reader takes the correspondence of Proposition 2.4.84 into consideration. In fact, this holds already for co-algebras, but we need to introduce some notation first:

Notation 2.4.95. *Let C be a co-algebra over k. If V is a fixed k-vector space, we will denote the functor $W \mapsto \omega_C(W) \otimes_k V$, where W is a finite comodule over C, as $\omega_C \otimes V : Comod_k(C) \to Vect(k)$.*

Proposition 2.4.96. *Let C be a co-algebra over k. For any k-vector space V, there is a natural isomorphism*

$$Hom_{Vect(k)}(C, V) \cong Hom(\omega_C, \omega_C \otimes V)$$

where the Hom-set on the right hand side denotes the set of all natural transformations between ω_C and $\omega_C \otimes V$, viewed as functors from the category $Comod_C$ to $Vect(k)$. In particular, the functor

$$V \mapsto Hom(\omega_C, \omega_C \otimes V)$$

is representable.

Proof. We will describe two natural transformations

$$\Psi : \mathrm{Hom}_{\mathrm{Vect}(k)}(C, \cdot) \to \mathrm{Hom}(\omega_C, \omega_C \otimes (\cdot))$$

and

$$\Xi : \mathrm{Hom}(\omega_C, \omega_C \otimes (\cdot)) \to \mathrm{Hom}_{\mathrm{Vect}(k)}(C, \cdot),$$

that will establish the isomorphism of functors in the statement. For the details showing that these are natural transformations, and that the two compositions $\Psi \circ \Xi$ and $\Xi \circ \Psi$ are the corresponding identity natural transformation, see [60, Lemma 2.2.1] or [64, Prop. 6.2.1].
We will start with Ψ: there is a canonical natural transformation (of functors over the category of arbitrary k-vector spaces)

$$\Pi : \omega_C \to \omega_C \otimes C$$

induced by the comodule morphism $\rho : W \to W \otimes_k C$ for any finite comodule W over C, and if $\phi : C \to V$ is any morphism of k-vector spaces, we have a natural transformation $(\mathrm{id}, \phi) : \omega_C \otimes C \to \omega_C \otimes V$ given by the morphism

$$\omega_C(W) \otimes_k C \xrightarrow{(\mathrm{id}_{\omega_C(W)}, \phi)} \omega_C(W) \otimes_k V$$

for any finite comodule W. With this, we have that $\Psi = (\mathrm{id}, \phi) \circ \Pi$.
To define Ξ, let V be a fixed k-vector space and let $\varphi : \omega_C \to \omega_C \otimes V$ be a natural transformation. We need to provide a morphism $C \to V$, let us suppose for a moment that C is finitely dimensional, so we have a morphism of k-vector spaces $\varphi(C) : \omega_C(C) \to \omega_C(C) \otimes_k V$ as C is a comodule over itself. With this, we can easily define a morphism $\Xi(V) : C \to V$ by taking $\Xi(V) = (\epsilon, \mathrm{id}_V) \circ \varphi(C) : C \to V$ where ϵ is the counit morphism of C.
In general, C is not finitely dimensional so we cannot do this directly. Instead, for any element $c \in C$, by Lemma 2.4.86 there exists a finite sub-comodule U containing c, thus we have a morphism $\varphi(U) : \omega_C(U) \to \omega_C(U) \otimes_k V$ and we can try do define $\Xi(V)$ over c as $((\epsilon|_U, \mathrm{id}_V) \circ \varphi(U))(c)$. This is well-defined as it is not hard to verify that the value of this composition is independent of the choice of the sub-comodule that contains c. □

Corollary 2.4.97. *Let C be a co-algebra k. C is determined up to a unique isomorphism by the category Comod_C and the functor ω_C.*

Recovering a co-algebra from its comodules

Proof. By Proposition 2.4.96 and Yoneda's lemma (Lemma 2.2.7), C as a vector space is determined by a unique isomorphism. To recover the comultiplication $\Delta : C \to C \otimes_k C$, we use the isomorphism of sets

$$\mathrm{Hom}_{\mathrm{Vect}(k)}(C, C \otimes_k C) \cong \mathrm{Hom}(\omega_C, \omega_C \otimes C \otimes_k C)$$

from Proposition 2.4.96 and we identify Δ with the natural transformation $(\Pi, \mathrm{id}_C) \circ \Pi : \omega_C \to \omega_C \otimes_k C$ where Π is the natural transformation that used to describe the functor Ψ in the proof of Proposition 2.4.96.
Finally, to recover $\epsilon : C \to k$, we use the natural isomorphism of functors $\omega_C \cong \omega_C \otimes k$. □

This Corollary fully describes the category of finite comodules over a co-algebra C. Now we will tackle the tensor structure in Comod_C.
Let us consider two finite comodules V, W over C, a tensor product of these comodules should give $\omega_C(V) \otimes \omega_C(W)$ after taking the forgetful functor, as ω_C must be a tensor functor (Definition 2.4.57) when Comod_C is a tensor category (Definition 2.4.33).
If $\rho : V \to V \otimes_k C$ and $\sigma : W \to W \otimes_k C$ are the structural morphisms of V and W respectively, we can take

$$(\rho, \sigma) : V \otimes_k W \to (V \otimes_k C) \otimes_k (W \otimes_k C) \cong (V \otimes_k W) \otimes (C \otimes_k C)$$

so, if we would have a "multiplication morphism" $m : C \otimes_k C \to C$ we could get a comodule structure on $V \otimes_k W$ by taking

$$(\mathrm{id}_{V \otimes_k W}, m) \circ (\rho, \sigma) : V \otimes_k W \to V \otimes_k W \otimes C$$

but according to Definition 2.4.82 we need additional conditions that should be imposed on this multiplication so we get comodule structure over the tensor product:

Compatible multiplication and tensor product of comodules

Definition 2.4.98. *Let C be a co-algebra over a field k. A morphism $m : C \otimes_k C \to C$ is a* compatible multiplication *if it commutes with both Δ and ϵ.*
If C is a co-algebra with a compatible multiplication, then the tensor product of two comodules V, W is a comodule with the structural morphism

$$\tau = (\mathrm{id}_{V \otimes_k W}, m) \circ (\rho, \sigma) : V \otimes_k W \to V \otimes_k W \otimes C$$

where ρ and σ are the structural morphisms of V and W respectively. We will denote this comodule as $V \otimes_m W$.

Remark 2.4.99. *If C is a co-algebra with a compatible multiplication $m : C \otimes_k C \to C$, then for two comodules V, W over C, we clearly have that $\omega_C(V \otimes_m W) = \omega_C(V) \otimes_k \omega_C(W)$. We have not supposed that m is associative or commutative, this is intentional as will show now the effect of this properties have on the structure of Comod_C.*
The multiplication m is compatible if and only if m is a morphism of co-algebras, i.e, it is a morphism that commutes with the respective comultiplications and counits, where the co-algebra structure of $C \otimes_k C$ is simply given by (Δ, Δ) and (ϵ, ϵ).

We want to establish an analog result to Proposition 2.4.96 for the tensor product of comodules, for this we need to add some notation, in the same vein as what we outlined in Notation 2.4.95:

Notation 2.4.100. *Let* C *be a co-algebra over* k *with a compatible multiplication* m*. If* V *is a fixed* k*-vector space, we will denote the functor* $\omega_C \otimes \omega_C$ *as the functor*

$$\omega_C \otimes \omega_C : Comod_k(C) \times Comod_k(C) \to Vect(k)$$

that takes a pair of comodules (V, W) *over* C *and send it to*

$$\omega_C \otimes \omega_C(V, W) = \omega_C(V \otimes_m W) = \omega_C(V) \otimes_k \omega_C(W).$$

We will also consider the functor $\omega_C \otimes \omega_C \otimes V$*, with the same domain and codomain as before, defined as*

$$(V, W) \mapsto \omega_C(V) \otimes_k \omega_C(W) \otimes_k V.$$

As $C \otimes_k C$ is a co-algebra if C has a compatible multiplication, we can consider the category $\mathrm{Comod}_{C \otimes_k C}$, and using an analogous proof to the proof of Proposition 2.4.96 we get:

Proposition 2.4.101. *Let* C *be a co-algebra over a field* k *with compatible multiplication* m*. Then, we have for any* k*-vector space* V *a natural isomorphism*

$$Hom_{Vect(k)}(C \otimes_k C, V) \cong Hom(\omega_C \otimes \omega_C, \omega_C \otimes \omega_C \otimes V).$$

Remark 2.4.102. *Applying Proposition 2.4.101 with with* C *we obtain a bijection*

$$Hom_{Vect(k)}(C \otimes_k C, C) \cong Hom(\omega_C \otimes \omega_C, \omega_C \otimes \omega_C \otimes C)$$

that allows us to recover the compatible multiplication m *by taking two finite comodules* V *and* W *over* C *and taking the natural transformation given by the composition*

$$\begin{array}{rcl} \omega_C(V) \otimes_k \omega_C(W) \xrightarrow{\sim} \omega_C(V \otimes_m W) & \to & \omega_C(V \otimes_m W) \otimes_k C \\ & \xrightarrow{\sim} & \omega_C(V) \otimes_k \omega_C(W) \otimes_k C \end{array}$$

where the second arrow in the composition comes from the comodule structure of $V \otimes_m W$*.*

Using this idea, we can show the following:

Properties of the tensor of comodules v/s properties of the multiplication

Corollary 2.4.103. *Let* C *be a co-algebra over a field* k *with compatible multiplication* m*. Then,*

(a) m *is commutative if and only if the tensor product on* Comod_C *defined by it satisfies the commutativity constraint (Definition 2.4.31(b)), if and only if for any two finite comodules* V, W *over* C*, the natural isomorphism*

$$\omega_C(V) \otimes_k \omega_C(W) \cong \omega_C(W) \otimes_k \omega_C(V)$$

coming from the commutativity constraint satisfied in the category $Vectf_k$ *comes from an isomorphism*

$$V \otimes_m W \cong W \otimes_m V$$

of comodules over C *via* ω_C*.*

(b) m *is associative if and only if the tensor product on* Comod$_C$ *defined by it satisfies the commutativity constraint (Definition 2.4.31(a)), if and only if for any three finite comodules* V, W, U *over* C*, the natural isomorphism*

$$(\omega_C(V) \otimes_k \omega_C(W)) \otimes_k \omega_C(U) \cong \omega_C(V) \otimes_k (\omega_C(W) \otimes_k \omega_C(U))$$

coming from the associativity constraint satisfied in the category Vectf$_k$ *comes from an isomorphism*

$$(V \otimes_m W) \otimes_m U \cong V \otimes_m (W \otimes_m U)$$

of comodules over C *via* ω_C.

Moreover, if m *is both commutative and associative, then the tensor product in* Comod$_C$ *has compatible constraints, in the sense of Definition 2.4.31(c).*

Proof. Let us start with the commutative property. m is commutative if and only if $m = m \circ t$ where $t : C \otimes_k C \to C \otimes_k C$ is the morphism defined as $t(a \otimes b) = b \otimes a$ for $a, b \in C$.
We can then consider for two comodules V, W over C the two tensor products $V \otimes_m W$ and $V \otimes_{m \circ t} W$ and both m and $m \circ m$ correspond to two natural morphisms of k-vector spaces

$$\omega_C(V \otimes_m W) \to \omega_C(V \otimes_m W) \otimes_k C$$

and

$$\omega_C(V \otimes_{m \circ t} W) \to \omega_C(V \otimes_{m \circ t} W) \otimes_k C$$

respectively by Remark 2.4.102. Under this correspondence, $m = m \circ t$ if and only if the morphisms of k-vector spaces outlined above are the same, but by these morphisms are determined by the comodule structures on $V \otimes_m W$ and $V \otimes_{m \circ t} W$ so these comodules are isomorphic if and only if we have the equality of natural transformations, if and only if the multiplication of C is commutative.
The proofs for the associativity constraint and the compatibility if m is both associative and commutative are analogous. □

If a co-algebra B is also a k-algebra, it would not only have an associative and commutative compatible multiplication, but also a unit by the conventions outlined in Section 1.4, so we also need to outline the effect that having a compatible unit $e \in B$, that can be identified with the image of $1 \in k$ under a morphism $e : k \to B$, has on the category of comodules. We will start by outlining what we mean by "compatible" in the case of a unit:

Compatible unit for comodules with multiplication

Definition 2.4.104. *Let* C *be a co-algebra over a field* k *with compatible multiplication* m*. A morphism of* k*-vector spaces* $e : k \to A$ *that commutes with* m*, is a* compatible unit *if* e *is a morphism of co-algebras where we consider* k *as a co-algebra with the comultiplication* $1 \mapsto 1 \otimes 1$ *and the identity as a counit.*

Proposition 2.4.105. *Let C be a co-algebra over a field k with compatible multiplication m. Then C possesses a compatible unit $e : k \to C$ if and only if the morphism e makes k a comodule over C such that the tensor product of comodules defined by m satisfies the identity object axiom (Definition 2.4.31(d) but on the right) in the category Comod_k, if and only if for any finite comodule V over C, the natural isomorphism*

Effect of a compatible unit over the category of comodules

$$\omega_C(V) \otimes_k k \cong \omega_C(V)$$

comes from an isomorphism

$$V \otimes_m k \cong V$$

of comodules over C.

Proof. For any finite comodule V over C, with structural morphism $\rho : V \to V \otimes_k C$. The composition of morphisms of k-vector spaces

$$\omega_C(V) \cong \omega_C(V) \otimes_k k \overset{(\omega_C(\rho),e)}{\to} \omega_C(V) \otimes_k A \otimes_k A \overset{(\mathrm{id},m)}{\to} \omega_C(V) \otimes_k A$$

defines a natural transformation $\omega_C \to \omega_C \otimes A$ that corresponds by Proposition 2.4.96 to the morphism $A \to A$ given by $a \mapsto m(a, e)$.
If the right identity axiom for the unit holds, then the natural transformation above must be the same as the one given by the comodule structures of finite comodules

$$\omega_C(V) \overset{\omega_C(\rho)}{\longrightarrow} \omega_C(V) \otimes_k A.$$

This should happen if and only if $V \otimes_m k \cong V$ as comodules over C so that they yield the canonical isomorphism $\omega_C(V) \cong \omega_C(V) \otimes_k k$ after taking ω_C. □

The algebraic structure that is a comodule with multiplication and unit, which encompasses all of the compatibility properties mentioned in Definitions 2.4.98 and 2.4.104 is the following:

Definition 2.4.106. *A* bi-algebra *over a field k is an algebra B that also has a comodule structure, so that both the multiplication and unit morphisms of B are compatible with it.*

Bi-algebras

Putting together Corollary 2.4.103 and Proposition 2.4.105 and applying them to the category of finite comodules over a bi-algebra, we obtain:

Proposition 2.4.107. *Let B be a bi-algebra over k. Then, the category Comod_B is a tensor category (Definition 2.4.33) where the tensor product of comodules is given by the multiplication of B, according to Definition 2.4.98. Moreover, the forgetful functor ω_B is a tensor functor (Definition 2.4.57).*
Additionally, the converse is also true, i.e., if C is a co-algebra such that Comod_C is a tensor category and ω_C is a tensor functor, then C is a bi-algebra.

Category of comodules over a bi-algebra

Remark 2.4.108. *Let B be a bi-algebra over k. It can easily be seen that $End_{Comod_B}(\mathbb{1}) = k$ where $\mathbb{1}$ is simply k viewed as a comodule over B as described in Proposition 2.4.105.*
As $Comod_B$ is legitimately a tensor category, Definition 2.4.71 holds in this case and as $Comod_B$ it is also abelian, we conclude from Remarks 2.4.70 and 2.4.92 that $Comod_B$ is a k-linear abelian tensor category.

If B is a bi-algebra over k. The difference between B and a Hopf algebra A is that possesses an antipode morphism $S : A \to A$ (Definition 2.2.12) which is a morphism of k-algebras. On the category of comodules' side, the difference between $Comod_B$ and a tannakian category is that the latter is rigid. So, if the category of finite comodules over a Hopf algebra $Comod_A$ was tannakian, it would be so because the antipode of A makes this category rigid. This in indeed the case as the antipode allows us to provide a comodule structure on duals of vector spaces:

Comodule structure for duals of comodules over a Hopf algebra

Definition 2.4.109. *Let A be a Hopf algebra over k, and let $\rho : V \to V \otimes_k A$ be a finite comodule over A. If we consider the dual vector space $V^\vee$ (Definition 2.4.46), it has a comodule structure over A whose structural morphism $\rho^\vee : V^\vee \to V^\vee \otimes_k A \cong Hom_{Vect(k)}(V, A)$ is given for any element $\phi \in V^\vee \cong Hom_{Vect(k)}(V, k)$ as the composition*

$$\rho^\vee(\phi) : V \xrightarrow{\rho} V \otimes_k A \xrightarrow{(\phi, S)} k \otimes_k A \cong A \,.$$

Lemma 2.4.110. *Let A be a Hopf algebra over k, and let $\rho : V \to V \otimes_k A$ be a finite comodule over A. The morphism $\rho^\vee$ defined in Definition 2.4.109 effectively makes $V^\vee$ a structure of comodule over A. Moreover, this comodule structure makes the evaluation morphism*

$$ev_V : V^\vee \otimes_m V \to k$$

in Definition 2.4.46 a morphism of comodules over A where $\otimes_m$ denotes the comodules structure over the tensor product $V^\vee \otimes_k V$ with respect to the multiplication m of A (Definition 2.4.98).

Proof. See [64, Lemma 6.2.6]. □

As we showed in Remark 2.4.102, Corollary 2.4.103 and Proposition 2.4.105, if the category of comodules has certain properties, we can recover the operations or co-operations along with some of their properties in the corresponding co-algebra or bi-algebra. This behavior extends to duals in the category of comodules:

Recovering the antipode from the existence of duals of comodules

Proposition 2.4.111. *Let A be a bi-algebra over k. If for any finite comodule V over A, the dual vector space $V^\vee$ has a comodule structure $\rho^\vee : V^\vee \to V^\vee \otimes_k A$ that makes the evaluation morphism*

$$ev_V : V^\vee \otimes_m V \to k$$

a morphism of comodules over A. Then, A has an antipode $S : A \to A$ so that A is a Hopf algebra.

Proof. We will only show how to obtain the antipode $S : A \to A$ from a natural transformation $\omega_A \to \omega_A \otimes A$ using Proposition 2.4.96. For details on how the morphism S conceived using this correspondence is effectively an antipode, see [66, Theorem].

To define the natural transformation corresponding to S, let V be a finite comodule over A. Firstly, we should note that we have the equality $\omega_A(V^\vee) = \omega_A(V)^\vee$ and using it, we can define a natural morphism $\omega_A(V) \to \omega_A(V) \otimes_k A$ using the following composition (we will omit sub-indexes on identity morphisms):

$$\begin{array}{rcl} \omega_A(V) & \overset{(\delta_{\omega_A(V)},\mathrm{id})}{\to} & \omega_A(V) \otimes_k \omega_A(V)^\vee \otimes_k \omega_A(V) \\ & = & \omega_A(V) \otimes_k \omega_A(V^\vee) \otimes_k \omega_A(V) \\ & \overset{(\mathrm{id},\rho^\vee,\mathrm{id})}{\to} & \omega_A(V) \otimes_k \omega_A(V^\vee) \otimes_k A \otimes_k \omega_A(V) \\ & \cong & \omega_A(V) \otimes_k \omega_A(V^\vee) \otimes_k \omega_A(V) \otimes_k A \\ & = & \omega_A(V) \otimes_k \omega_A(V)^\vee \otimes_k \omega_A(V) \otimes_k A \\ & \overset{(\mathrm{id},\epsilon_{\omega_A(V)})}{\to} & \omega_A(V) \otimes_k A \end{array}$$

where for any finitely dimensional k-vector space W, the morphism

$$\delta_W : k \to W \otimes_k W^\vee$$

known as the *co-evaluation morphism* of W coming from the isomorphism $\varphi : \mathrm{End}_{\mathrm{Vectf}_k}(W^\vee) \cong W \otimes_k W^\vee$ coming from the last morphism in Remark 2.4.51, which is an isomorphism as W has finite dimension. This isomorphism induces a morphism $\delta_W : k \to W \otimes_k W^\vee$ defined as the only morphism that sends $1 \in k$ to $\varphi(\mathrm{id}_{W^\vee})$. $\square$

Let A be a Hopf algebra over k. The equality $\omega_A(V^\vee) = \omega_A(V)^\vee$ for any finite comodule over A shows that Comod_A is a rigid (Definition 2.4.52), so we have finally arrived to the desired characterization:

Theorem 2.4.112. *Let A be a Hopf algebra over k. Then, the category Comod_A of finite comodules over A, together with the forgetful functor $\omega_A : \mathrm{Comod}_k(A) \to \mathrm{Vectf}_k$ is a neutral tannakian category over k (Definition 2.4.77).*

Finite comodules over a Hopf algebra form a neutral tannakian category

Moreover, if $G = \mathrm{Spec}(A)$ is an affine group-scheme, using the correspondence between comodules and representations in Proposition 2.4.84, we easily see that any additional structure (tensor, abelian, rigid, etc) on the category Comod_A transfers to $\mathrm{Rep}_k(G)$ and as the correspondence commutes with the respective forgetful functors (Remark 2.4.90), we obtain:

Theorem 2.4.113. *Let G be a an affine group-scheme over k. Then, the category $\mathrm{Rep}_k(G)$ of finite representations of G, together with the forgetful functor $\omega_G : \mathrm{Rep}_k(G) \to \mathrm{Vectf}_k$ is a neutral tannakian category over k (Definition 2.4.77).*

Finite representations of an affine group-scheme form a neutral tannakian category

To finish this part, we will resume how the additional operations given to a co-algebra C give additional structure to the category of finite comodules up to a neutral tannakian category in the form of a table:

Structure of C	Properties of Comodf_A	References
C is a co-algebra	- The category is abelian and ω_C is exact and faithful. - The category is also k-linear and ω_C is a k-linear functor. - Both C and ω_G can be recovered from the category.	- Proposition 2.4.93 - Remark 2.4.92 - Corollary 2.4.97
C has a compatible multiplication m	- Comodules can be tensored $- \otimes_m -$. - m can be recovered from the category.	- Definition 2.4.98 - Remark 2.4.102
m is commutative	$- \otimes_m -$ satisfies the commutativity constraint	Corollary 2.4.103(a)
m is associative	$- \otimes_m -$ satisfies the associativity constraint.	Corollary 2.4.103(b)
C has a compatible unit e	$- \otimes_m -$ satisfies the unit object axiom with k as identity element.	Proposition 2.4.105
C is a bi-algebra	- The category is tensorial. - ω_C is a tensor functor. - $\mathrm{End}(k) = k$ and the category is k-linear.	- Proposition 2.4.107 - Remark 2.4.108
C is a bi-algebra with an antipode S	- Any dual of a comodule over C has a comodule structure. - S can be recovered from the category.	- Definition 2.4.109 - Proposition 2.4.111
C is a Hopf algebra	The category together with ω_C is neutral tannakian.	Theorem 2.4.112

Table 1: Properties of the category of comodules with respect to the operations in a co-algebra (with references).

2.4.2.2 *Recovering a group-scheme from its category of representations*

Now that we have established that the category of finite representations of an affine group-scheme G over a field k is neutral tannakian. We will now show how to recover G from the category $\mathrm{Rep}_k(G)$.
While, the last Subsubsection was mostly focused on comodules, now we will shift attention to representations[22].
To start, we will apply Example 2.4.65 to $\omega_G : \mathrm{Rep}_k(G) \to \mathrm{Vectf}_k$, the forgetful functor of $\mathrm{Rep}_k(G)$. Meaning that we will consider the functor

$$\underline{\mathrm{End}}^{\otimes}(\omega_G) = \underline{\mathrm{Hom}}^{\otimes}(\omega_G, \omega_G) : \mathrm{Alg}_k \to \mathrm{Set}$$

that for any k-algebra R, it considers the set of tensor morphisms (Definition 2.4.62), between $\phi_R \circ \omega_G$ and itself, where

$$\phi_R : \mathrm{Vectf}_k \to R - \mathrm{Modf}$$

is the functor that takes any k-vector space W and tensors it with R.
The composition $\phi_R \circ \omega_G$ is simply the functor $V \mapsto \omega_G(V) \otimes_k R$ where V is the finite k-vector space corresponding to a representation $r : G \to GL(V)$.
As the category $\mathrm{Rep}_k(G)$ is rigid, we have:

Remark 2.4.114. *The functor $\underline{End}^{\otimes}(\omega_G)$ is group-valued (Definition 2.2.5) as $Rep_k(G)$ is a rigid tensor category, for any k-algebra R the essential image of $\phi_R \circ \omega_G$ is composed of finite and free R-modules so the full subcategory of $R - Modf$ they conform is rigid by [24, Exercise 2.10.16.], and thus any morphism of tensor functors in $\underline{End}^{\otimes}(\omega_G)(R)$ is an isomorphism (Proposition 2.4.64).*

Notation 2.4.115. *Let G be an affine group-scheme. For any k-algebra R, we will denote the composition $\phi_R \circ \omega_G$ from the beginning of this subsubsection as $\omega_G \otimes R$. Notice the similarity with the notation in Notation 2.4.95.*

From Remark 2.4.114 the set $\underline{\mathrm{End}}^{\otimes}(\omega_G)$ defined above this remark is composed exclusively of automorphisms of the tensor functor $\omega_G \otimes R$, so we will define:

Definition 2.4.116. *Let G be an affine group-scheme. The group-valued functor*

$$\underline{Aut}^{\otimes}(\omega_G) : Alg_k \to Grp$$

is called the automorphism functor of the forgetful functor associated to G. *If this functor is representable, it is represented by a group-scheme that we will call the* automorphism group-scheme of the forgetful functor associated to G.

Automorphism functor of the forgetful functor of representations

This functor is key for what we want to achieve:

22 Though the reader should always have the correspondence of Proposition 2.4.84 in mind for any result we state here.

Recovering an affine group-scheme from its finite representations

Proposition 2.4.117. *Let* G *be an affine group-scheme. Then, we have an isomorphism of functors* $\widetilde{G} \cong \underline{Aut}^{\otimes}(\omega_G)$.

Proof. Let us consider

$$\underline{\mathrm{End}}(\omega) : \mathrm{Alg}_k \to \mathrm{Set}$$

the functor that maps any k-algebra R to the set of natural transformations, that are not necessarily tensor morphisms, of $\omega_G \otimes R$ to itself. We obviously have an inclusion $\underline{\mathrm{Aut}}^{\otimes}(\omega_G)(R) \subset \underline{\mathrm{End}}(\omega)(R)$ for any k-algebra R.
The desired isomorphism of functors come from the more general natural isomorphisms

$$\underline{\mathrm{End}}(\omega)(R) \cong \mathrm{Hom}_{R-\mathrm{Mod}}(A \otimes_k R, R) \cong \mathrm{Hom}_{\mathrm{Vect}(k)}(A, R)$$

for any k-algebra R, where A is the Hopf algebra associated to G. This implies the desired isomorphism $\widetilde{G} = \mathrm{Hom}_{\mathrm{Alg}_k}(A, \cdot) \cong \underline{\mathrm{Aut}}^{\otimes}(\omega_G)$ if we show that under the isomorphism above $\underline{\mathrm{Aut}}^{\otimes}(\omega_G)(R) \subset \underline{\mathrm{End}}(\omega)(R)$ corresponds to the subset of $\mathrm{Hom}_{\mathrm{Vect}(k)}(A, R)$ composed of all k-algebra morphisms between A and R.
To establish the isomorphisms, let R be a k-algebra. Firstly we have that $\mathrm{Hom}_{R-\mathrm{Mod}}(A \otimes_k R, R) \cong \mathrm{Hom}_{\mathrm{Vect}(k)}(A, R)$ as any morphism of R-modules $f : A \otimes_k R \to R$ becomes a morphism of k-vector spaces $A \to R$ by composing f with the morphism of k-vector spaces $i : A \to A \otimes_k R$ defined as $i(a) = a \otimes 1$ for $a \in A$, and any morphism of k-vector spaces $g : A \to R$ induces a morphism of R-modules $f : A \otimes_k R \to R$ by the formula $f(a \otimes r) = m(g(a) \otimes r)$ where m is the multiplication morphism of R.
If we consider R solely as a vector space, we have an isomorphism $\mathrm{Hom}_{\mathrm{Vect}(k)}(A, R) \cong \mathrm{Hom}(\omega_G, \omega_G \otimes R)$ by Proposition 2.4.96[23] where we are considering the natural transformations between the functors ω_G and $\omega_G \otimes R$ as functors of vector spaces by forgetting the k-algebra structure of R. But as for any finite comodule V over A, $\omega_G(V) \otimes_k R$ is a k-algebra, we have by using the isomorphism in the previous paragraph that

$$\mathrm{Hom}_k(\omega_G, \omega_G \otimes R) \cong \mathrm{Hom}_R(\omega_G \otimes R, \omega_G \otimes R) = \underline{\mathrm{End}}(\omega)(R)$$

where the Hom-set with subscript k means "natural transformation of functors of vector spaces" while the Hom-set with subscript R means "natural transformation of functors of R-modules" so we have indeed the natural isomorphism of functors

$$\underline{\mathrm{End}}(\omega) \cong \mathrm{Hom}_{\mathrm{Vect}(k)}(A, \cdot)$$

mentioned in the beginning of the proof.
To finish the proof, we just need to show that for any k-algebra R,

23 We are identifying ω_G with ω_A by Proposition 2.4.84.

any isomorphism of tensor functors $\Phi(R) : \omega_G \otimes R \to \omega_G \otimes R$ corresponds bijectively to a k-algebra morphism $\phi : A \to R$. The fact Φ_R that is a tensor functor implies that the following diagram of natural transformations is commutative

$$\begin{tikzcd}
\omega_G((\cdot)\otimes(\cdot))\otimes R \arrow[r, "\Phi_R"] \arrow[d] & \omega_G((\cdot)\otimes(\cdot))\otimes R \arrow[d] \\
\omega_G(\cdot)\otimes\omega_G(\cdot)\otimes R \arrow[r, "(\Phi_R,\Phi_R)"'] & \omega_G(\cdot)\otimes\omega_G(\cdot)\otimes R
\end{tikzcd}$$

but using Proposition 2.4.96, Proposition 2.4.101 and the isomorphism from the beginning, we can conclude that this is equivalent to have the following commutative diagram

$$\begin{tikzcd}
A \otimes_k A \arrow[r, "(\phi,\phi)"] \arrow[d, "m_A"'] & R \otimes_k R \arrow[d, "m_R"] \\
A \arrow[r, "\phi"'] & R
\end{tikzcd}$$

where $\phi : A \to V$ is the morphism of k-vector spaces associated to Φ_R, which is precisely the condition we need to impose to ϕ in order to show that it is a morphism of k-algebras. $\square$

This result will allows us to prove the tannakian correspondence in Subsubsection 2.4.3.1, and relate properties of G to properties $\mathrm{Rep}_k(G)$, along with properties of morphisms of affine group-schemes in Subsubsection 2.4.3.2.
For both these purposes, we also need to relate morphisms of group-schemes and the tensor functors these morphisms actually induce between their categories of representations.

Definition 2.4.118. *Let G, G' be affine group-schemes over k. If $\phi : G \to G'$ is a morphism of group-schemes, any finite representation $r : G' \to GL(V)$ induces a representation of G by taking the composition $r \circ \phi : G \to GL(V)$.*
This defines a functor $\phi^* : Rep_k(G') \to Rep_k(G)$ called the pull-back functor (of representations) induced by ϕ.

Pull-back of representations induced by morphisms of group-schemes

Remark 2.4.119. *Let G, G' be affine group-schemes over k and let $\phi : G \to G'$ be a morphism of group-schemes. The pull-back functor ϕ^* is a tensor functor that clearly preserves the respective forgetful functors, i.e., $\omega_G \circ \phi^* = \omega'_G$.*

A corollary of Proposition 2.4.117 that relates to pull-back functors is:

Corollary 2.4.120. *Let G, G' be affine group-schemes over k. There is a bijective correspondence between morphisms of group-schemes $\phi : G \to G'$ and tensor functors $F : Rep_k(G') \to Rep_k(G)$ that satisfy $\omega_G \circ F = \omega'_G$.*

Proof. It suffices to show that the assignment $\phi \mapsto \phi^*$ has an inverse. Let $F : \mathrm{Rep}_k(G') \to \mathrm{Rep}_k(G)$ be a functor as in the statement, then any tensor automorphism of ω_G induces an automorphism of $\omega_{G'}$ via F, thus the same is true for $\omega_G \otimes R$ over $\omega_{G'} \otimes R$, so F induces a morphism of group-valued functors

$$\underline{\mathrm{Aut}}^{\otimes}(\omega_G) \to \underline{\mathrm{Aut}}^{\otimes}(\omega_{G'})$$

that becomes a morphism of group-schemes $G \to G'$ by Proposition 2.4.117 and Yoneda's lemma, which provides an inverse to the assignment at the beginning of the proof. □

2.4.3 *Tannakian Correspondence*

Now we have almost all the tools to show that any neutral tannakian category over a field k is equivalent to the category of representations of a group-scheme. We can actually show that we have a correspondence between k-linear abelian categories with an exact and faithful functor to the category of finitely dimensional k-vector spaces and the category of finite comodules over co-algebra, in such a way that if the corresponding co-algebra was a Hopf algebra the category must be neutral tannakian and vice versa, this will be the content of Subsubsection 2.4.3.1. We will follow the approach of Serre in [61, §2.5] with statements borrowed from [49, Sections 9d & 9e].
After this, in Subsubsection 2.4.3.2 we will combine the contents of Subsubsection 2.4.2.2 together with tannakian correspondence to show how some properties of an affine group-scheme are reflected on its category of representations.

2.4.3.1 *Proof of tannakian correspondence*

Let C be a co-algebra over a field k. We know from Proposition 2.4.93 and Remark 2.4.92, that the category Comod_C of finite comodules over C is k-linear abelian. Moreover, the forgetful functor ω_C is exact and faithful. The opposite statement is true, and it is the main result of this section:

Correspondence between k-linear abelian categories with a fiber functor with comodule categories

Theorem 2.4.121. *Let $\mathcal{C}$ be a k-linear abelian category with a k-linear exact and faithful functor $\omega : \mathcal{C} \to \mathit{Vectf}_k$. Then, there exists a co-algebra C such that $\mathcal{C}$ is equivalent to the category Comod_C with the forgetful functor ω_C.*

A functor $\omega : \mathcal{C} \to \mathrm{Vectf}_k$ like the one described in the statement of the theorem will be called a *fiber functor*, extending with Definition 2.4.77 outside neutral tannakian categories.

Remark 2.4.122. *If $\mathcal{C}$ is a k-linear abelian category with a fiber functor ω. We will always suppose that ω is in addition a tensor functor (Definition 2.4.57) when $\mathcal{C}$ is also a tensor category.*

We will proof Theorem 2.4.121 starting with an easier case of the correspondence of Theorem 2.4.121, but first we need a definition:

Simple objects, composition series and length

Definition 2.4.123. *Let $\mathcal{A}$ be an abelian category. A* simple *object in an abelian category is an object S such that any monomorphism (or sub-object) $i : S' \to S$ is either the zero morphism or an isomorphism. In other words, simple objects do not have sub-objects different from the zero object and themselves.*
Let X be an object of an abelian category, a composition series of X, *is a sequence of sub-objects*

$$X = P^0 \supset P^1 \supset P^2 \supset \cdots$$

such that for any $i \geqslant 0$ the quotient P^i/P^{i+1} is a simple object , if such a sequence exists.
X has finite length *if it has a composition series with a finite amount of members.*

and some remarks:

Remark 2.4.124. *Let $\mathcal{C}$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$.*

(a) Any morphism u of $\mathcal{C}$ such that $\omega(u)$ is the zero morphism (Definition 2.4.1) of $Vectf_k$ is necessarily the zero morphism as ω is faithful.
As ω is also exact and using Definition 2.4.19(d), a morphism ν of $\mathcal{C}$ is a monomorphism or an epimorphism or an isomorphism if and only if $\omega(\nu)$ has the corresponding property.
Another consequence of the faithfulness of ω is that for any pair of objects X, Y of $\mathcal{C}$, we have an inclusion of vector spaces $Hom_{\mathcal{C}}(X, Y) \subset Hom_{Vectf_k}(\omega(X), \omega(Y))$, in particular $Hom_{\mathcal{C}}(X, Y)$ always have finite dimension.

(b) The faithfulness and exactness of ω imply that the lattice of sub-objects of any object X of $\mathcal{C}$ is injectively mapped to the sub-object lattice of $\omega(X)$, where the sub-object lattice is defined by the relation $f \leqslant f'$, where $f : Y \to X$ and $f' : Y' \to X$ are sub-objects of X, if and only if there exists a unique morphism $i : Y \to Y'$ such that $f = f' \circ i$.
This implies that any object of X has finite length as certainly $\omega(X)$ must have finite length as it is a finitely dimensional k-vector space, and ω preserves simple objects.
We should also note that the analogous lattice of quotients of X is also injectively mapped to the quotient lattice of $\omega(X)$.

We also need to introduce a construction that we will be widely use in the context of tannakian categories:

Generated sub-category by sub-quotients

Definition 2.4.125. *Let $\mathcal{A}$ be an abelian category. If $\mathcal{S}$ is a set of sub-objects of X, we will denote as $\langle \mathcal{S} \rangle$ the full sub-category of $\mathcal{A}$ consisting of sub-quotients (Definition 2.4.69(a)) of finite direct sums of objects of $\mathcal{S}$, that we will call the* sub-category of sub-quotients generated by $\mathcal{S}$. *If $\mathcal{A}$ equals*

$\langle \mathcal{S} \rangle$ we will say that $\mathcal{C}$ is generated by sub-quotients of $\mathcal{S}$.
In the special case we are considering a single object X, we will denote its sub-category of sub-quotients generated by it by $\langle X \rangle$.

Now we can introduce a key simpler case of Theorem 2.4.121:

Correspondence for categories generated by a single element

Proposition 2.4.126. *Let $\mathcal{C}$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$. If $\mathcal{C} = \langle X \rangle$ is generated by sub-quotients of a single object X. Then, there exists a co-algebra C_X such that $\mathcal{C}$ is equivalent to the category $Comod_{C_X}$ with the forgetful functor ω_{C_X}.*

Definition 2.4.127. *Let $\mathcal{C}$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$. Let X be an object of $\mathcal{C}$, if $S \subset \omega(X)$ is a sub-object, the minimal element*[24] *of the lattice of sub-objects $Y \subset X$ such that $S \subset \omega(Y)$ will be called the* sub-object of X generated by S, *where we are using the same ordering for sub-objects as the one described in Remark 2.4.124(b) .*
A sub-object $Y \subset X$ generated by a single element $y \in \omega(X)$ is called a monogenic object.

Remark 2.4.128. *Let $\mathcal{C}$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$ and let X be an object of $\mathcal{C}$. Let $Y \subset X$ be a monogenic sub-object generated by $y \in \omega(X)$, Y being monogenic is equivalent to the condition*

$$Y' \subset Y \text{ and } y \in \omega(Y') \implies Y = Y'$$

for any sub-object Y' of X.

Now let us suppose that our starting abelian k-linear category is $\mathcal{C} = \langle X \rangle$ generated by sub-quotients by a single element. The proof of Proposition 2.4.126 lies on three lemmas, that we will present without proof, the first being:

Lemma 2.4.129 (Lemma 9.33 [49]). *Let $\mathcal{C} = \langle X \rangle$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$. Let X be an object of $\mathcal{C}$. Then, for any monogenic sub-object $Y \subset X$ we have*

$$\dim_k(\omega(Y)) \leqslant \dim_k(\omega(X))^2.$$

This lemma implies that the dimensions of monogenic sub-objects of the generator by sub-quotients are always bounded. If such a dimension is maximal for a certain monogenic object, we have

Lemma 2.4.130 (Lemma 9.34 [49]). *Let $\mathcal{C} = \langle X \rangle$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$ and let P be a monogenic sub-object of X such that $\dim_k(\omega(Y))$ is the largest possible. Then, if the object $p \in \omega(X)$ generates P, we have:*

(a) For any object Z of $\mathcal{C}$ and any element $z \in \omega(Z)$, there exists a unique morphism $f : P \to Z$ such that $\omega(f)(p) = x$.

(b) The functor $Hom_{\mathcal{C}}(P, \cdot)$ is exact and faithful.

---

24 That exists by Zorn's lemma.

An object P of an abelian category for which the always left exact functor $\mathrm{Hom}_{\mathcal{C}}(P, \cdot)$ is exact is called *projective*, if this functor is moreover faithful P is called a *projective generator*. Thus, the object P described in Lemma 2.4.130 is a projective generator of $\mathcal{C}$.
If we consider the set of endomorphisms of P, $\mathrm{End}(P)$[25], it has a natural structure of an associative k-algebra with unit that is not necessarily commutative. As P is a projective generator, we have

Lemma 2.4.131 (Lemma 9.35 [49]). *Let $\mathcal{C} = \langle X \rangle$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$ and let P be a projective generator of $\mathcal{C}$. Then, $\mathcal{C}$ is equivalent to the category of finite right $\mathrm{End}(P)$-modules in such a way ω becomes the forgetful functor of the latter category.*

To finish the proof Proposition 2.4.126, we need the following remark:

Remark 2.4.132. *Let C be a co-algebra over k. If we consider C as a k-vector space, we can consider its dual $C^\vee$ and in fact as the dual acts as a covariant functor (Remark 2.4.47) the co-operations Δ and ϵ become a multiplication and a unit morphisms for $C^\vee$ so this dual is an associative k-algebra with unit that is not necessarily commutative. If $C^\vee$ is commutative, we will say that C is* co-commutative.

Proof of Proposition 2.4.126. Let P be a projective generator of $\mathcal{C} = \langle X \rangle$ which exists thanks to Lemmas 2.4.129 and 2.4.130.
By Lemma 2.4.131 $\mathcal{C}$ is equivalent to the category of right $\mathrm{End}(P)$-modules that carries ω to the forgetful functor of the latter, but we can easily see that a right module over $\mathrm{End}(P)$ becomes a comodule over $\mathrm{End}(P)^\vee$. This establishes an equivalence of categories between finite right $\mathrm{End}(P)$-modules and finite comodules over $\mathrm{End}(P)^\vee$, under this equivalence the forgetful functor of one becomes the forgetful functor of the other, thus we have

$$\mathcal{C} \cong \mathrm{Comod}_{\mathrm{End}(P)^\vee}$$

as desired. ☐

Remark 2.4.133. *Let $\mathcal{C} = \langle X \rangle$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to Vectf_k$. If P is a projective generator of $\mathcal{C}$, Lemma 2.4.130(a) is equivalent to say that P represents the fiber functor ω. This allows us to canonically associate a co-algebra to an abelian category of the form $\langle X \rangle$ by using Yoneda's lemma, as we can easily see that we have an isomorphism of associative k-algebras*[26] *with unit*

Canonical comodule associated to abelian categories generated by a single element

$$End(P) \cong End(\omega)$$

25 We have omitted the sub-script $\mathcal{C}$ for convenience.
26 Not necessarily commutative.

where the set on the right is the set of natural transformations from ω *to itself. Thus, by Proposition 2.4.126 we have a canonical equivalence of categories*

$$\mathcal{C} \cong \mathit{Comod}_{\mathit{End}(\omega)^\vee}$$

that carries ω *to* $\omega_{\mathit{End}(\omega)^\vee}$ *which is also independent of the choice of a projective generator of* $\mathcal{C}$.

Notation 2.4.134. *Following Remark 2.4.133, if* $\mathcal{C} = \langle X \rangle$ *is a* k*-linear abelian category with a fiber functor* $\omega : \mathcal{C} \to \mathit{Vectf}_k$*, it is canonically isomorphic to the category* $\mathit{Comod}_{\mathit{End}(\omega)^\vee}$*, and thus, in this case we will denote the algebra* $\mathit{End}(\omega)$ *and its dual* $\mathit{End}(\omega)^\vee$ *as* A_X *and* C_X *respectively, so we have*

$$\mathcal{C} = \langle X \rangle \cong \mathit{Comod}_{C_X}.$$

If X *is an object of a* k*-linear abelian category* $\mathcal{D}$ *with fiber functor* ω*, not necessarily generated by a single element by sub-quotients, we will also use* A_X *and* C_X *to denote*

$$A_X = \mathit{End}\left(\omega|_{\langle X \rangle}\right) \text{ and } C_X = A_X^\vee.$$

Remark 2.4.135. *Let* $\mathcal{C}$ *be a* k*-linear abelian category with a fiber functor* $\omega : \mathcal{C} \to \mathit{Vectf}_k$*. If* X *is an object of* $\mathcal{C}$*, we can consider the full sub-category* $\langle X \rangle$ *within* $\mathcal{C}$*. We know that* $\langle X \rangle \cong \mathit{Comod}_{C_X}$ *by Proposition 2.4.126. Moreover* A_X *acts on the* k*-vector space* $\left(\omega|_{\langle X \rangle}\right)(Y)$ *for any object* Y *in* $\langle X \rangle$ *so* $\left(\omega|_{\langle X \rangle}\right)(Y)$ *is an* A_X*-module, or equivalently, a comodule over* C_X*. In short, the functor* $Y \mapsto \left(\omega|_{\langle X \rangle}\right)(Y)$ *establishes the equivalence of categories between* $\langle X \rangle$ *and* Comod_{C_X} *of Proposition 2.4.126.*

Now we are ready to proof the main result:

Proof of Theorem 2.4.121. Let $\mathcal{C}$ be a k-linear abelian category with a fiber functor $\omega : \mathcal{C} \to \mathrm{Vectf}_k$. For any object X of $\mathcal{C}$, we will denote as $[X]$ its isomorphism class in the skeletal category associated to $\mathcal{C}$.
We can define a partial ordering on these classes by setting

$$[X] \leqslant [Y] \iff \langle X \rangle \text{ is a full sub-category of } \langle Y \rangle\,.$$

The set of these classes with this partial ordering is directed as for any two classes $[X], [Y]$ we have that $[X], [Y] \leqslant [X \oplus Y]$. Moreover, for any pair of classes with $[X] \leqslant [Y]$ we have a restriction morphism $A_Y \to A_X$ coming from the full inclusion $\langle X \rangle \subset \langle Y \rangle$.
Thus, we have a projective system of k-algebras $\{A_X\}_{[X]}$ and a direct system of co-algebras $\{C_X\}_{[X]}$, so we can consider the directed limit

$$C(\omega) = \varinjlim_{[X]} C_X$$

of these comodules[27]. By construction $C(\omega)$ acts on any k-vector space $\omega(Z)$ where Z is an object of $\mathcal{C}$ so $\omega(Z)$ is a comodule over $C(\omega)$, thus, in the same vein as Remark 2.4.135, the functor $X \mapsto \omega(X)$ establishes an equivalence between $\mathcal{C}$ and $\mathrm{Comod}_{C(\omega)}$, carrying ω to $\omega_{C(\omega)}$, as desired. □

Now we are almost ready to establish the tannakian correspondence.

Remark 2.4.136. *Let $\mathcal{C}$ be a neutral tannakian category over k, with fiber functor ω (Definition 2.4.77). Following Example 2.4.65 and the first paragraphs of Subsubsection 2.4.2.2, we have for any k-algebra R a functor*

$$X \mapsto \omega(X) \otimes_k R$$

that we will denote as $\omega \otimes R$ following Notation 2.4.115. We can consider the functor $\underline{End}^{\otimes}(\omega) : \mathcal{C} \to R - Modf$ that assigns to each k-algebra R the set of endomorphisms of $\omega \otimes R$. As $\mathcal{C}$ is rigid, Remark 2.4.114 also holds for $\mathcal{C}$ so $\underline{End}^{\otimes}(\omega)$ is actually composed of tensor automorphisms and it is group-valued. So we will denote it as $\underline{Aut}^{\otimes}(\omega)$.

The correspondence we have established for comodules implies tannakian correspondence by the contents of Subsubsection 2.4.2.1:

Tannakian correspondence

Corollary 2.4.137. *Let $\mathcal{C}$ be a neutral tannakian category over k, with fiber functor ω. Then, the functor $\underline{Aut}^{\otimes}(\omega)$ is represented by an affine group-scheme G over k, such that the functor $X \mapsto \omega(X)$ establishes an equivalence between $\mathcal{C}$ and $Rep_k(G)$ that carries ω to the forgetful functor ω_G.*

Fundamental group-scheme of a tannakian category

Definition 2.4.138. *Let $\mathcal{C}$ be a neutral tannakian category over k, with fiber functor ω. The group-scheme G that makes $\mathcal{C}$ equivalent to $Rep_k(G)$ is called the* fundamental group-scheme of $\mathcal{C}$.

Proof of Corollary 2.4.137. We know by Theorem 2.4.121 and its proof, that $\mathcal{C}$ is equivalent to the category of comodules $\mathrm{Comod}_{C(\omega)}$ via the forgetful functor ω with

$$C(\omega) = \varinjlim_{[X]} C_X$$

where the limit above is taken over the set of all isomorphism classes of objects of $\mathcal{C}$, and

$$C_X = \mathrm{End}\left(\omega|_{\langle X \rangle}\right)^{\vee}.$$

Now $\mathcal{C}$ is not only an abelian category, but rather an abelian tensor category, so if $\mathrm{Comod}_{C(\omega)}$ is a tensor category, and the equivalence between $\mathcal{C}$ and $\mathrm{Comod}_{C(\omega)}$ is a tensorial functor, we would

27 Technically, we need that $\mathcal{C}$ is *locally small* so $C(\omega)$ exists. $\mathcal{C}$ is locally small if its Hom-sets are effectively sets and not proper classes. We will suppose that this is the case and move on, but it should be pointed out that this limit might not exists without this assumption.

obtain that $C(\omega)$ is in fact a Hopf algebra as its category of comodules would be tensorial (Proposition 2.4.107) and rigid (Proposition 2.4.111). This implies the corollary as $\mathcal{C}$ is then equivalent to $\mathrm{Rep}_k(G)$ where $G = \mathrm{Spec}(C(\omega))$ using the correspondence between finite representations of group-schemes and finite comodules over Hopf algebras (Proposition 2.4.84), and thus $\underline{\mathrm{Aut}}^{\otimes}(\omega) \cong \widetilde{G}$ by Proposition 2.4.117.
To show that $\mathrm{Comod}_{C(\omega)}$ and $\omega : \mathcal{C} \to \mathrm{Comod}_{C(\omega)}$ are tensorial, we will use a correspondence between morphisms of co-algebras and functors between their categories of comodules, in a similar fashion to Corollary 2.4.120: there is a bijective correspondence between morphisms of co-algebras $f : C \to C'$ and functors $F : \mathrm{Comod}_C \to \mathrm{Comod}_{C'}$ such that $\omega_C = \omega_{C'} \circ F$, see [49, Lemma 9.43] for a proof of this fact.
Now, as $\mathcal{C}$ is tensorial, the tensor functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ becomes a morphism of co-algebras $m : C(\omega) \otimes_k C(\omega) \to C(\omega)$ as we can easily see that $\mathcal{C} \times \mathcal{C}$ is equivalent to $\mathrm{Comod}_{C(\omega)\otimes_k C(\omega)}$. In particular C possesses a compatible multiplication (Definition 2.4.98) and the tensor category axioms of $\mathcal{C}$, listed in Definition 2.4.31, translate into diagrams involving $C(\omega)$ and its multiplication that make it commutative and associative, together with the existence of a compatible unit $e : k \to C(\omega)$ (Definition 2.4.104). Finally, by Corollary 2.4.103 and Proposition 2.4.105 we conclude that $\mathrm{Comod}_{C(\omega)}$ is a tensor category and this easily implies that the equivalence $\omega : \mathcal{C} \to \mathrm{Comod}_{C(\omega)}$ is tensorial, finishing the proof. $\square$

Remark 2.4.139. *It should be noted that the correspondence for k-liner abelian categories of Theorem 2.4.121 we have just proven is highly dependent of the fiber functor: the same k-linear abelian category $\mathcal{C}$ with two different fiber functors ω and ω' would yield a priori two different co-algebras C and C', so that $\mathcal{C} \cong \mathrm{Comod}_C$ or $\mathcal{C} \cong \mathrm{Comod}_{C'}$ depending on which fiber functor we are considering, this is a consequence of the characterization of the co-algebras in the proof Theorem 2.4.121, as we have*

$$C = \varinjlim_{[X]} \mathit{End}\left(\omega|_{\langle X\rangle}\right)^{\vee} \quad \textit{and} \quad C' = \varinjlim_{[X]} \mathit{End}\left(\omega'|_{\langle X\rangle}\right)^{\vee}$$

where the limit is taken over all the isomorphism classes of objects of $\mathcal{C}$. Clearly this extends to neutral tannakian categories over k: a priori, different choices of fiber functors yield different associated fundamental group-schemes.

2.4.3.2 *Properties of group-schemes v/s properties of categories of representations*

We finish this chapter by showing some results about how properties of affine group-schemes are reflect in the respective categories of finite representations. The categorical properties we will outline here

hold as well for arbitrary neutral tannakian categories over a field k by tannakian correspondence.
Also, from this point on, we will use "representation over a group-scheme" and "comodule over a Hopf algebra" interchangeably by the correspondence of Proposition 2.4.84.
Let $G = \mathrm{Spec}(A)$ be an affine group-scheme over a field k, we will denote as $\rho_G : A \to A \otimes_k A$ be its regular representation 2.4.85(1). We will start with the case when G is finite:

Free comodule

Definition 2.4.140. *Let $G = \mathrm{Spec}(A)$ be an affine group-scheme over a field k and let V be a finitely dimensional k-vector space. The comodule*

$$(\mathrm{id}_V, \Delta) : V \otimes_k A \to V \otimes_k A \otimes_k A$$

over A is called the free comodule over V.

Remark 2.4.141. *Let $G = \mathrm{Spec}(A)$ be an affine group-scheme over a field k and let V be a finitely dimensional k-vector space. The choice of a base of V characterizes the free comodule over V as a direct sum of n copies of ρ_G where $n = \dim_k(V)$, we will denote this direct sum as $\rho_G^{\oplus n}$.*

Lemma 2.4.142. *Let $G = \mathrm{Spec}(A)$ be an affine group-scheme over a field k and let V be a non-zero finite comodule over A. Then V is a sub-comodule (Example 2.4.85(3)) of $\rho_G^{\oplus n}$ for some integer $n \geqslant 1$.*

Proof. Let us consider $\omega_A(V) \otimes_k V$, the free comodule over $\omega_A(V)$. The fact that V is a comodule over A is equivalent to saying that the structural morphism $\rho : V \to V \otimes_k A$ is a morphism of comodules if we consider the target vector space as the free comodule over V. Moreover, ρ is injective as $(\mathrm{id}_V, \epsilon) \circ \rho$ equals id_V, implying the result as V is then a sub-comodule of $\rho_G^{\oplus \dim_k(V)}$. □

This result fully characterizes $\mathrm{Rep}_k(G)$ if G is finite.

Category of representations of a finite group-scheme

Corollary 2.4.143. *Let $G = \mathrm{Spec}(A)$ be an affine group-scheme over a field k. Then, G is finite if and only if $\mathrm{Rep}_k(G) = \langle V \rangle$ for some finite representation V of G.*

Proof. If G is finite, ρ_G is a finite representation of G and thus $\mathrm{Rep}_k(G) = \langle \rho_G \rangle$ by Lemma 2.4.142.
On the other hand, if $\mathrm{Rep}_k(G) = \langle V \rangle$ for some finite representation V of G, by Remark 2.4.133 we have that the Hopf algebra associated to G is the dual of the finitely dimensional k-vector space $\mathrm{End}_{\mathrm{Comod}_A}(P) \subset \mathrm{End}_{\mathrm{Vectf}_k}(P)$, where P is a projective generator of Comod_A, thus A is finitely dimensional. □

Now we will characterize the category of representations of $G = \mathrm{Spec}(A)$ group-scheme of finite type over k. Recall by Corollary 2.4.88 that G possesses a faithful finite representation $\rho : V \to V \otimes_k A$.

Category of representations of a group-scheme of finite type

Proposition 2.4.144. *Let* $G = \mathrm{Spec}(A)$ *be an affine group-scheme over a field* k*. Then,* G *is of finite type over* k *if and only if there exists a finite representation* W *such that* $\mathit{Rep}_k(G)$ *is tensor generated by* $\{W, W^\vee\}$ *(Definition 2.4.69(b)), or equivalently, any representation of* G *can be obtained from* W *by taking tensor products, direct sums, duals and sub-quotients.*

Proof. If G is of finite type over k, then it has a faithful finite representation $\rho : V \to V \otimes_k A$ and it can be shown that $\{V, V^\vee\}$ is a tensor generating family for $\mathrm{Rep}_k(G)$, see [49, Thm. 4.14].
If $\mathrm{Rep}_k(G)$ is tensor generated by $\{W, W^\vee\}$ where W is a finite representation of G, let us suppose that G is of finite type and that W is not faithful, then its underlying representation morphism $G \to GL(W)$ has a kernel $N \subset G$, the representation corresponding to $W^\vee$ would have a kernel containing N too and so will any representation of the form $W^{\oplus n} \otimes \left(W^\vee\right)^{\oplus m}$ for $m, n \geqslant 0$ and any finite sum of these, thus $\mathrm{Rep}_k(G)$ would not contain any faithful representation, contradicting the fact that G must have a faithful representation. □

We would like to characterize the category of representations for pro-finite and pro-algebraic group-schemes as well. For this purpose we could use Corollaries 2.2.100 and 2.2.98 respectively, but this demands the characterization of a quotient morphism (Definition 2.2.56) of group-schemes $q : G \to H$ in terms of the induced functor

$$q^* : \mathrm{Rep}_k(H) \to \mathrm{Rep}_k(G)$$

of Definition 2.4.118 under the correspondence of Corollary 2.4.120. Moreover, this characterization is useful on itself, so we will consider this characterization and the corresponding one for monomorphisms of group-schemes, that are closed immersions by Propositions 2.2.63 and 2.2.94.

Definition 2.4.145. *Let* $F : \mathcal{C} \to \mathcal{D}$ *be a functor between two categories* $\mathcal{C}$ *and* $\mathcal{D}$*. We say that the essential image of* F *is* closed by sub-objects *if for any object* X *of* $\mathcal{D}$ *and any sub-object* $Y \subset F(X)$ *there exists a sub-object* $Z \subset X$ *such that* $F(Z) \cong Y$*.*

Pull-back functor of representations for quotient morphisms

Proposition 2.4.146. *Let* G, H *be two affine group-schemes over* k*, and let* $\phi : G \to H$ *be a morphism of group-schemes over* k*. Then,* ϕ *is a quotient morphism if and only if the induced pull-back functor of representations*

$$\phi^* : \mathit{Rep}_k(H) \to \mathit{Rep}_k(G)$$

is fully faithful and the essential image of ϕ^* *is closed by sub-objects.*

Proof. If ϕ is a quotient morphisms, by the universal property of quotient morphisms 2.2.58, we can conceive $\mathrm{Rep}_k(H)$ a the full subcategory of $\mathrm{Rep}_k(G)$ of representations $G \to GL(V)$ that factor through H, thus the functor ϕ^* clearly satisfies the properties in the statement. Now let us suppose that ϕ^* is fully faithful and its essential image is closed by sub-objects, then the former property implies that ϕ^* induces an equivalence between $\mathrm{Rep}_k(H)$ and a full sub-category of $\mathrm{Rep}_k(G)$, moreover, as the essential image of ϕ^* is closed by sub-objects, for any object X of $\mathrm{Rep}_k(H)$, the sub-category $\langle X \rangle$ generated by X by sub-quotients is equivalent to $\langle \phi^*(X) \rangle$. If we write $G = \mathrm{Spec}(A)$ and $H = \mathrm{Spec}(B)$, by Tannakian correspondence (Corollary 2.4.137) we have

$$A = \varinjlim_{[X]} \mathrm{End}\left(\omega_G|_{\langle X \rangle}\right)^{\vee} \text{ and } B = \varinjlim_{[Y]} \mathrm{End}\left(\omega_H|_{\langle Y \rangle}\right)^{\vee}$$

where the brackets $[\cdot]$ denote classes of isomorphisms of objects in the respective categories.
Finally, as ϕ^* preserves fiber functors, we have

$$B = \varinjlim_{[Y]} \mathrm{End}\left(\omega_H|_{\langle Y \rangle}\right)^{\vee} = \varinjlim_{[Y]} \mathrm{End}\left(\omega_G|_{\langle \phi^*(Y) \rangle}\right)^{\vee} \subset A$$

and thus we have an inclusion of Hopf algebras $B \subset A$ that is faithfully flat by Proposition 2.2.71, finishing the proof. □

Remark 2.4.147. *In an article by I. Biswas, P.H. Hai and J.P. Dos Santos, Proposition 2.4.146 has been improved: in short, a morphism $\phi : G \to H$ of affine group-schemes over k is faithfully flat if and only if ϕ^* is fully faithful and ϕ^* satisfies one second condition from the list below, that replaces the condition that the essential image of ϕ^* is closed by sub-objects. These conditions, which are equivalent to one another by taking duals, are:*

(a) *For any representation V of H and a sub-representation $L \subset \phi^*(V)$ of G with $\dim_k(L) = 1$, then L is also a representation of H.*

(b) *For any representation V of H and a quotient representation $\phi^*(V) \to L$ of G with $\dim_k(L) = 1$, then L is a representation of G and the quotient morphism $\phi^*(V) \to L$ is a morphism of representations over G.*

Moreover, if H is pro-finite, then ϕ^* begin faithfully flat is enough to establish the equivalence. See [11, Lemma 2.1].

Proposition 2.4.148. *Let G, H be two affine group-schemes over k, and let $\phi : G \to H$ be a morphism of group-schemes over k. Then, ϕ is a closed immersion morphism if and only if any element of $\mathrm{Rep}_k(G)$ is isomorphic to a sub-quotient of an element of the form $\phi^*(X)$ for some object X of $\mathrm{Rep}_k(H)$.*

Pull-back functor of representations for closed immersions

Proof. Let $\mathcal{C}$ be the full sub-category of $\mathrm{Rep}_k(G)$ generated by objects of the form $\phi^*(X)$, where X is an object of $\mathrm{Rep}_k(H)$, by sub-quotients. Thus, the condition of the statement is equivalent to say that $\mathrm{Rep}_k(G)$ is equivalent to $\mathcal{C}$.
We have a factorization of ϕ^*

$$\mathrm{Rep}_k(H) \to \mathcal{C} \to \mathrm{Rep}_k(G)$$

and as $\mathcal{C}$ has a fiber functor $\omega_{\mathcal{C}} = \omega_G|_{\mathcal{C}}$ and ϕ^* preserves it, if we write $G = \mathrm{Spec}(A)$, $H = \mathrm{Spec}(B)$, and we denote by C the co-algebra corresponding to $\mathcal{C}$ by the correspondence of Theorem 2.4.121 and using [49, Lemma 9.43], the factorization above becomes a composition of morphisms of co-algebras

$$B \to C \to A.$$

The same argument at the end of the proof of Proposition 2.4.146 shows that the morphism $C \to A$ is injective. On the other hand, the other morphism $B \to C$ is surjective as for any object X of $\mathrm{Rep}_k(H)$ the morphism

$$\mathrm{End}\left(\omega_{\mathcal{C}}|_{\langle\phi^*(X)\rangle}\right) \to \mathrm{End}\left(\omega_H|_{\langle X\rangle}\right)$$

is injective, and thus by taking duals and direct limits we conclude that $B \to C$ is surjective.
This allows us to conclude the proof, because if ϕ is a closed immersion, then the morphism $B \to A$ is surjective and thus $A \cong C$, and if $\mathrm{Rep}_k(G)$ is equivalent to $\mathcal{C}$, then $C \cong A$ and thus the morphism $B \to A$ is surjective. □

Using these results, we conclude the chapter by characterizing the category of representations of pro-finite and pro-algebraic group-schemes:

Category of representations of pro-finite(algebraic) group-schemes

Proposition 2.4.149. *Let G be an affine group-scheme over k. Then $\mathrm{Rep}_k(G)$ is equivalent to the direct limit of full sub-categories of the form $\mathrm{Rep}_k(G_i)$ with G_i is finite for any i if G is pro-finite or of finite type over k for any i if G is pro-algebraic. In particular, any object X of $\mathrm{Rep}_k(G)$ is contained in full sub-category of the form $\mathrm{Rep}_k(G_i)$.*

Proof. This result is simply equivalent to Corollaries 2.2.100 and 2.2.98 at the level of categories of representations for pro-finite group-schemes and pro-algebraic group-schemes respectively, by the characterization of Proposition 2.4.146 for quotient morphism of group-schemes, where the directed system of sub-categories here is given by the full inclusions of categories induced by these quotients. □

Tannakian sub-category generated by an object

Remark 2.4.150. *Let G be an affine group-scheme, for any object X of $\mathrm{Rep}_k(G)$. Proposition 2.4.149 ensures the existence of a "smallest"[28] tannakian full sub-category of $\mathrm{Rep}_k(G)$ that contains X. We will denote this category as $\langle X\rangle^{\otimes}$ and we will called the* full tannakian sub-category generated by X.
For an arbitrary object, this sub-category is a priori larger than $\langle X\rangle$, and if we have the equality $\langle X\rangle = \langle X\rangle^{\otimes}$, this is equivalent to say that group-scheme associated to $\langle X\rangle^{\otimes}$ is finite by Corollary 2.4.143. In particular, if G is pro-finite, all full tannakian sub-category generated by single objects satisfy this property.

28 In the sense of full inclusions of categories.

If G *is a general affine group-scheme, it is pro-algebraic (Proposition 2.2.97), and thus any tannakian full sub-category* $\langle \mathsf{X} \rangle^{\otimes}$ *is tensor generated by* $\{\mathsf{Y}, \mathsf{Y}^{\vee}\}$ *where* Y *is some finite representation of* G*, but one must note that* Y *might not be faithful as a representation of* G*, but it will as a representation of the group-scheme associated to* $\langle \mathsf{X} \rangle^{\otimes}$*.*

3 THE FUNDAMENTAL GROUP-SCHEME

3.1 INTRODUCTION

Now that we have established the three aspects of the fundamental group-scheme in Chapter 2, we will put them together in this chapter to conceive various fundamental group-schemes.
As we mentioned in Section 2.1, there are two approaches to conceive fundamental group-schemes, as a "universal cover" and as an "automorphism group-scheme".
We will start in Section 3.2 with the classical theory of the FGS of Nori, contained in [53, Ch. II] that considers the fundamental group-scheme $\pi^N(X,x)$ of a scheme over a field k with a prescribed rational point $x \in X(k)$ as a universal covering, where the coverings we will consider are pointed (pro-)finite torsors (Definition 2.3.17(a) & (e)) over X. Here we will show sufficient conditions for X that imply the existence of the FGS, along with some basic results.
Then, in Section 3.3 we will consider fundamental group-schemes as automorphism group-schemes, obtained by considering tannakian categories of vector bundles over proper reduced and connected k-schemes. Starting in Subsection 3.3.1, we will obtain two fundamental group-schemes, a tannakian way to ultimately conceive the fundamental group-scheme of Section 3.2, and another fundamental group-scheme $\pi^S(X,x)$, called the S-fundamental group-scheme, first devised by I. Biswas, A.J. Parameswaran and S. Subramanian for proper smooth curves in [12] and later generalized by A. Langer for proper reduced and connected schemes over k in [42, 43]. We will also show how to bridge the theory of fundamental group-schemes as universal covers with the theory of tannakian fundamental group-schemes, enriching both theories.
For the second half of Section 3.3, we will show in Subsection 3.3.2 a generalization of a new construction for fundamental group-schemes over "varieties connected by proper chains" or CPC-varieties by I. Biswas, P.H. Hai and J.P. Dos Santos in [11, §7], that allows us to replace the proper condition needed for the existence of the fundamental group-schemes of Subsection 3.3.1. This new construction enriches the theory tannakian fundamental group-schemes of vector bundles over k-schemes, and puts the Nori fundamental group-scheme and the S-fundamental group-scheme in a larger network of fundamental group-schemes that could be used in future applications. This is the theoretical base for the new fundamental group-schemes that we will consider in Section 4.3.

Finally, in Section 3.4 we will state more advanced results and properties, mostly for the Nori fundamental group-scheme, in Subsection 3.4.1 that we will use throughout this manuscript in later chapters, along with examples of descriptions of fundamental group-schemes in Subsection 3.4.2 for certain types of schemes that we will also use later, specially in Chapter 5.
All torsors considered in this chapter will be affine (Definition 2.3.17(c)) and thus we will omit this adjective when referring to torsors, the base field k will be of positive characteristic $p > 0$ unless stated otherwise.

3.2 NON-TANNAKIAN DEFINITION

3.2.1 *Definition and existence*

We will start by defining the category of torsors that we will use in this subsection.

Category of (pro-)finite pointed torsors

Definition 3.2.1. *Let X be a scheme over k with a rational point $x \in X(k)$. We will denote as $Ftors_{X,x}$ the* category of finite pointed (over x) torsors over X*(Definition 2.3.17(a)) with morphisms of pointed torsors (Definition 2.3.15) as morphisms.*
We will also consider the category $PFTors_{X,x}$ of category of pro-finite pointed (over x) torsors over X*(Definition 2.3.15(e)) with the same kind of morphisms.*
There is a canonical full inclusion of categories $Ftors_{X,x} \to PFTors_{X,x}$.

Now we will state the main definition of this section:

Possessing a FGS

Definition 3.2.2. *Let X be a scheme over k with a rational point $x \in X(k)$. We say that X* possesses a FGS *if there exists a unique pro-finite group-scheme $\pi_1^N(X,x)$ over k and a unique*[1] *pointed $\pi_1^N(X,x)$-torsor, denoted as $\hat{X} \to X$ and called the* universal torsor of X.
The torsor $\hat{X}$ is universal in the sense that there exists a unique morphism of torsors $\hat{X} \to T$ for any pointed (pro-)finite torsor $T \to X$.
Equivalently, X possesses a FGS if for any (pro-)finite group-scheme G, there exists a natural bijection of sets

$$\{t : T \to X : T \text{ is a pointed } G\text{-torsor over } X\} \tilde{\to} \left\{\text{Morphisms of group-schemes } \pi_1^N(X,x) \to G \text{ over } k\right\}.$$

The group-scheme $\pi^N(X,x)$ is called the (Nori) fundamental group-scheme.

We will write "Nori" before fundamental group-scheme if necessary to avoid confusion with other fundamental group-schemes.

Remark 3.2.3. *If X is a scheme over k with a rational point $x \in X(k)$ that possesses a FGS. The bijection of Definition 3.2.2 can be stated as an isomorphism of functors*

$$\mathcal{T}_{X,x}(\cdot) \cong Hom_{PFGrp\text{-}Sch_k}(\pi^N(X,x), \cdot)$$

1 Up to an unique isomorphism.

where PFGrp-Sch$_k$ denotes the category of pro-finite group-schemes over k*, and* $\mathcal{T}_{X,x}(\cdot)$ *denotes the functor that assigns to any pro-finite group-scheme the set* $\mathcal{T}_{X,x}(G)$ *of pointed* G*-torsors over* X*.*
This bijection is given explicitly as the following: for any pointed G*-torsor* $t : T \to X$*, the unique morphism from the universal torsor* $\hat{X} \to T$ *induces a morphism of group-schemes* $\pi^N(X,x) \to G$ *by taking fibers over* x *(Remark 2.3.2). The inverse of this assignment, takes a morphism* $\pi^N(X,x) \to G$ *and associates to it the contracted product* $\hat{X} \times^{\pi^N(X,x)} G$ *(Definition 2.3.41) that is a pointed* G*-torsor over* X *(Corollary 2.3.48). The fact that both assignments are inverses of each other is supported on Proposition 2.3.49.*
Finally, the fact that this bijection is natural means that for any morphism $\phi : G \to H$ *of pro-finite group-schemes, we have a commutative diagram*[2]

$$\begin{tikzcd} \mathcal{T}_{X,x}(G) \arrow[r] \arrow[d] & Hom(\pi^N(X,x), G) \arrow[d] \\ \mathcal{T}_{X,x}(H) \arrow[r] & Hom(\pi^N(X,x), H) \end{tikzcd}$$

where the horizontal arrows are isomorphisms, and the left vertical is the map $\mathcal{T}_{X,x}(G) \to \mathcal{T}_{X,x}(H)$ *that associates any pointed* G*-torsor* T *over* X *to the contracted product* $T \times^G H$ *that is a pointed* H*-torsor over* X*.*

To effectively assess if a scheme X possesses a FGS, we will do it by introducing a property of the category $\mathrm{Ftors}_{X,x}$ whose presence is equivalent to possessing a FGS:

Definition 3.2.4. *Let* X *be a scheme over* k *with a rational point* $x \in X(k)$*. We say that the category Ftors*$_{X,x}$ *of finite pointed torsos over* X *is* closed by fibered products *if for any pair of morphism of pointed torsors over* X $f_i : T_i \to T$ *(*$i = 1, 2$*) with fixed target, the fibered product* $T_1 \times_T T_2$*, which is a priori just a scheme over* T*, is also a finite pointed torsor over* X*, thus this torsor belongs to Ftors*$_{X,x}$*.*

Remark 3.2.5. *The reader must note that the fibered product* $T_1 \times_T T_2$ *in Definition 3.2.4 is different to the fibered product* $T_1 \times_X T_2$ *which is in fact a pointed torsor over the group-scheme* $G_1 \times_k G_2$ *where* G_i *is the group-scheme associated to* T_i *for* $i = 1, 2$*.*
The scheme $T_1 \times_T T_2$ *has naturally an action from the group-scheme* $G_1 \times_G G_2$ *where* G *is the group-scheme associated to* T*. This implies that if* $T_1 \times_T T_2$ *is a torsor over* X*, it is a* $G_1 \times_G G_2$*-torsor. Moreover, there is a morphism* $T_1 \times_T T_2 \to T_1 \times_X T_2$ *that commutes with the respective group-scheme actions, coming from the natural morphism* $G_1 \times_G G_2 \to G_1 \times_k G_2$*.*
Finally, if both the torsors T_i *are pointed for* $i = 1, 2$ *and* T *is pointed as well, it is not hard to see that if* $T_1 \times_T T_2$ *has morphism over* X*, it would have a rational point over* x *even if it is not a torsor.*

Using the fact that $T_1 \times_X T_2$ is a torsor, we can conclude that the action of $G_1 \times_G G_2$ over $T_1 \times_T T_2$ is free, so we could apply Theorem

2 From now on we will omit the subscript PFGrp-Sch$_k$ for convenience.

2.2.40 to obtain a quotient (Definition 2.2.39) $T_1 \times_T T_2 \to Y$ that we would like to be X, as $T_1 \times_T T_2$ is clearly a torsor over Y. The complete general property of this fibered product is:

Lemma 3.2.6. *Keeping the notation of Definition 3.2.4 and Remark 3.2.5, the scheme $T_1 \times_T T_2$ is a pointed $G_1 \times_G G_2$-torsor over a closed sub-scheme $Y \subset X$ that contains the rational point $x \in X(k)$.*

Proof. See [53, Ch. II Lemma 1] or [14, Prop. 4.4]. □

Now we will show how the fact that fibered products of torsors are torsors implies the existence of the fundamental group-scheme:

Criteria for possessing a FGS

Proposition 3.2.7. *Let X be a quasi-compact scheme over k with a rational point $x \in X(k)$. Then, X possesses a FGS if and only if the category $Ftors_{X,x}$ is closed by fibered products.*

Proof. If X possesses a FGS, the for any pair $f_i : T_i \to T$ $(i = 1, 2)$ of morphisms of finite torsors over X, then we have a morphism $\hat{X} \to T_1 \times_T T_2$ naturally, as for any $i = 1, 2$, we have morphisms $\phi_i : \hat{X} \to T_i$ that come from the universal property of $\hat{X}$ and

$$f_1 \circ \phi_1 = f_2 \circ \phi_2 : \hat{X} \to T$$

is the unique morphism from $\hat{X}$ to T. Thus, we have a commutative diagram

$$\begin{tikzcd} \hat{X} \arrow[rr] \arrow[dr] & & T_1 \times_T T_2 \arrow[dl] \\ & X & \end{tikzcd}$$

but the morphism $\hat{X} \to X$ is faithfully flat while $T_1 \times_T T_2 \to X$ factors through a closed sub-scheme Y of X by Lemma 3.2.6, thus $Y = X$ and thus $Ftors_{X,x}$ is closed by fibered products.
If the category $Ftors_{X,x}$ is closed by finite products, its torsors $\{T_i\}_{i \in I}$ and associated group-schemes $\{G_i\}_{i \in I}$ form inverse directed systems of schemes over X and $\mathrm{Spec}(k)$ respectively, the system of torsors is projective by the fact that $Ftors_{X,x}$ is closed by fibered products, while $\{G_i\}_{i \in I}$ is always an inverse directed system, and its projective limits exists as a pro-finite group-scheme $G = \varprojlim_{i \in I} G_i$ over k (Remark 2.2.93), and thus we can form the projective limit given by these systems to obtain a universal torsor $\hat{X} = \varprojlim_{i \in I} T_i$ using Proposition 2.3.60. □

We see that if X is also quasi-separated, and X possesses a FGS, then $\hat{X}$ is a Nori-reduced pointed $\pi^N(X, x)$-torsor by Corollary 2.3.62. Now we will state the main result of this part, due to Nori, that establishes sufficient conditions for a scheme over k to possess a FGS.

Proposition 3.2.8 (Proposition 2 [53]). *Let X be a quasi-compact scheme over k with a rational point $x \in X(k)$. If X is reduced and connected, it possesses a FGS.*

Reduced and connected schemes possess a FGS

Proof. By Proposition 3.2.7, it suffices to show that $\mathrm{Ftors}_{X,x}$ is closed by fibered products. Let $f_i : T_i \to T$ ($i = 1, 2$) be two morphism of finite pointed torsors over X, we will need to show that $T_1 \times_T T_2$ is an object of $\mathrm{Ftors}_{X,x}$ as well.
By Lemma 3.2.6, $T_1 \times_T T_2$ is a pointed torsor over a closed sub-scheme $Y \subset X$ with $x \in Y(k)$. Let G be the finite group-scheme associated to T, let p_i ($i = 1, 2$) be the compositions $T_1 \times_X T_2 \to T_i \xrightarrow{f_i} T$, the fact that the action of G over T is free and transitive (Definition 2.2.29), there exists a morphism $z : T_1 \times_X T_2 \to G$ such that $p_1 = \mu_T(p_2, z)$ where μ_T is the action morphism of G over T. With this notation, we can see that $T_1 \times_T T_2$ equals $z^{-1}(e)$ where $e : \mathrm{Spec}(k) \to G$ is the unit morphism of G, in particular it is a closed sub-scheme of $T_1 \times_X T_2$.
Now let G^0 the connected component of the unit of G, which is a subgroup-scheme of G as it is the kernel of the morphism $\pi_0 : G \to \pi_0(G)$ where $\pi_0(G)$ is the scheme representing the connected components of G, which is an étale group-scheme over k, corresponding to the largest étale (Definition-Proposition 2.2.103) sub-algebra of A, if $G = \mathrm{Spec}(A)$.
Because G is finite, G^0 is clopen[3] sub-scheme of G, thus $z^{-1}(G^0)$ is a clopen sub-scheme as well, and if $t : T_1 \times_X T_2 \to X$ is the structural morphism of X, as it is finite thus proper, and faithfully flat thus an open immersion [32, Théorème 2.4.6], we have that $t(z^{-1}(G^0)) \subset X$ is clopen as well, thus $t(z^{-1}(G^0)) = X$ as X is connected.
Finally, if we check the respective reduced sub-schemes for the inequality above, we will obtain X on the right hand side as X is reduced, while we will obtain Y on left hand side as $(G^0)_{\mathrm{red}} = \mathrm{Spec}(k)$ is the unit point of G by [49, p. 2.17], thus $z^{-1}\left((G^0)_{\mathrm{red}}\right) = T_1 \times_T T_2$ in terms of their underlying sets, taking t over this equality implies that $Y = X$ as sets and thus $Y = X$ as schemes, finishing the proof. □

Remark 3.2.9. *The hypotheses of Proposition 3.2.8 is just a sufficient condition for possessing a FGS. If we consider for instance non-reduced quasi-compact schemes over k, they could still possess a FGS, see Proposition 3.4.15 in Subsection 3.4.1 for an important example.*
In general, non-reduced schemes over Dedekind schemes with a rational point have a universal torsor, eventually a proper class of distinct universal torsors. This class is closed by fibered products in analogous sense to that of Definition 3.2.4. See [3] for more details.

Now we will show the behavior of the FGS over morphisms of schemes.

3 Closed and open.

Induced morphism of FGS

Definition 3.2.10. *Let X and Y be two schemes with rational points $x \in X(k)$ and $y \in Y(k)$. If $f : X \to Y$ is a morphism that $f(x) = y$ we will say that f is* compatible with the respective rational points. *In this case, the pull-back to X of the universal torsor $\hat{Y} \to Y$ of Y is a pointed $\pi^N(Y, y)$-torsor over X. Thus, we have an unique morphism $\hat{X} \to \hat{Y} \times_Y X$ of torsors over X inducing a morphism*

$$\pi^N(f) : \pi^N(X, x) \to \pi^N(Y, y)$$

of group-schemes over X by taking fibers over x. The morphism $\pi^N(f)$ is called the induced morphism of fundamental group-schemes induced by f.

Notation 3.2.11. *In case we have not specified rational points over two schemesX and Y, we will say that a morphism $f : X \to Y$* induces a morphism of FGS for compatible rational points *if we have an induced morphism $\pi^N(f) : \pi^N(X, x) \to \pi^N(Y, y)$ for some rational points $x \in X(k)$ and $y \in Y(k)$ that are compatible under f, provided such rational points exist.*

Remark 3.2.12. *Let X and Y be two schemes over k, and let us suppose that both these schemes possess a FGS. If a morphism $f : X \to Y$ induces a morphism $\pi^N(f) : \pi^N(X, x) \to \pi^N(Y, y)$ for compatible rational points, then using the bijection of Definition 3.2.2, we see that for any pointed G-torsor $T \to Y$ that corresponds to a morphism of group-schemes $\pi^N(Y, y) \to G$, its pull-back $T \times_Y X$ to X is a pointed G-torsor over X (Lemma 2.3.3) and it corresponds to the composition*

$$\pi^N(X, x) \xrightarrow{\pi^N(f)} \pi^N(X, x) \longrightarrow G\ .$$

This will be important for the next subsection.

We finish this subsection with a remark:

Remark 3.2.13. *In Proposition 3.2.7 we do not really need to exclusively consider rational points, in fact we can take S-points $x : S \to X$ where S is a non-empty scheme over k, thus we can have a FGS with S-points $\pi^N(X, x; S)$.*
A particular case of interest, is the case when we take geometric points $\bar{x}$: Spec$(\Omega) \to X$ where Ω is an algebraically closed field. The FGS obtained with this type of points can be called the arithmetic FGS, *that we can denote as $\pi^N(X, \bar{x})$. The name arithmetic comes in part by the fact that if k is any field, $\pi^N(\mathrm{Spec}(k), x)$ is the trivial group-scheme while $\pi^N(X, \bar{x})$ is generally not trivial if k is not algebraically closed and its pointed torsor are related to extensions of k, separable and purely inseparable among them, alluding to what some mathematicians call "arithmetic".*
On the other hand, $\pi^N(X, x)$ is "geometric" as many results and examples of fundamental group-schemes are exclusive for schemes over algebraically closed fields. We will see results of this kind throughout this manuscript. See [70] for more details on the arithmetic FGS.

3.2.2 *Nori-reduced torsors revisited and induced morphisms of FGS*

Let X be a scheme over k and let G be an affine group-scheme. Recall that a G-torsor $t : T \to X$ is Nori-reduced if it does not possess proper sub-torsors (Definition 2.3.40). If X possesses a FGS, Nori-reduced pointed torsors are a key element of the theory of the FGS of X. In this subsection we will show these torsors allow to characterize induced morphisms $\pi^N(f) : \pi^N(X, x) \to \pi^N(Y, y)$ of FGS by morphisms $f : X \to Y$ for compatible rational points when they are faithfully flat or closed immersions, among other properties, like we did for tannakian categories and morphisms of group-schemes in Subsubsection 2.4.3.2.

The main property of pointed Nori-reduced torsors for schemes possessing a FGS can be easily deduced from their definition and the bijection of Definition 3.2.2, it is the following:

Main characterization of Nori-reduced torsors

Lemma 3.2.14. *Let X be a scheme over k with a rational point $x \in X(k)$ that possesses a FGS. Then a pointed G-torsor $t : T \to X$ is Nori-reduced if and only if its corresponding arrow $\pi^N(X, x) \to G$ is faithfully flat.*

Remark 3.2.15. *Let X be a scheme over k with a rational point $x \in X(k)$ that possesses a FGS. If $t : T \to X$ is a pointed Nori-reduced torsor, by Remark 2.3.50, the unique morphism $\hat{X} \to T$ from the universal torsor of X is a K-torsor with $K = \ker(\pi^N(X, x) \to G)$.*
Let $T' \to X$ be a pointed H-torsor over X, if we apply the isomorphism theorem for group-schemes (Theorem 2.2.65) to the associated morphism $\pi^N(X, x) \to H$ of group-schemes, we have that the image of this morphism $H' \subset H$ corresponds to a sub-torsor $V \subset T'$ of T' by the bijection of Definition 3.2.2, and V is a Nori-reduced H'-torsor over X. This is a reformulation of Proposition 2.3.52 to the morphism $\hat{X} \to T'$ showing that for a scheme that possesses a FGS, this proposition can be trivially deduced from the isomorphism theorem of group-schemes.

Canonical Nori-reduced sub-torsor

Definition 3.2.16. *Let X be a scheme over k with a rational point $x \in X(k)$ that possesses a FGS. If $t : T \to X$ is a pointed G-torsor over X. Then, the Nori-reduced pointed sub-torsor $V \subset X$ corresponding to the image of $\pi^N(X, x) \to G$ is called the* canonical Nori-reduced sub-torsor of T.

Remark 3.2.17. *Let X be a scheme over k with a rational point $x \in X(k)$ that possesses a FGS. If $T \to X$ is a pointed torsor over X, we can define a partial order over the set of pointed sub-torsors of T by defining $W \leqslant W'$ for two sub-torsors $W, W' \subset T$ if and only if the closed immersion $W \hookrightarrow T$ factors through the closed immersion of $W' \hookrightarrow T$.*
If V is the canonical Nori-reduced sub-torsor of T, then V is the minimal element under this partial ordering, i.e., V is the smallest sub-torsor of T.

Nori-reduced torsors are the most important for describing the FGS of a scheme over k. The following proposition justifies this:

Proposition 3.2.18. *Let X be a scheme over k with a rational point $x \in X(k)$ that possesses a FGS. Then, $\pi^N(X, x)$ is the projective limit of all its quotients that are finite and Nori-reduced.*

Proof. This is essentially the torsor version of Corollary 2.2.101, that pro-finite group-schemes are the limit of its finite quotients. The only non-trivial part is to show that the family of all pointed Nori-reduced torsors over X is directed.
As the category $\mathrm{Ftors}_{X,x}$ is closed by fibered products by Proposition 3.2.7, if $f_i : T_i \to T$ $(i = 1, 2)$ are two morphisms of Nori-reduced torsors, then $T_1 \times_T T_2$ is a finite pointed torsor over X but it might not be Nori-reduced, so we need a Nori-reduced alternative, which is simply the canonical Nori-reduced sub-torsor of $T_1 \times_T T_2$, finishing the proof. □

This proposition allows us to characterize morphisms of FGS's $\pi^N(f) : \pi^N(X, x) \to \pi^N(Y, y)$ induced morphisms $f : X \to Y$ for compatible rational points:

Properties of morphisms between FGS's in terms of Nori-reduced torsors

Proposition 3.2.19. *Let X and Y be two schemes with rational points $x \in X(k)$ and $y \in Y(k)$ and let $f : X \to Y$ be a morphism that is compatible with the respective rational points. If both X and Y possess a FGS, and we consider the induced morphism $\pi^N(f) : \pi^N(X, x) \to \pi^N(Y, y)$ then we have:*

(a) $\pi^N(f)$ has trivial image if and only if the pull-back to X of any finite Nori-reduced torsor $T \to Y$ is a trivial torsor (Example 2.3.4(1)) over X.

(b) $\pi^N(f)$ is faithfully flat if and only if the pull-back of any finite Nori-reduced torsor over Y is Nori-reduced over X.

(c) $\pi^N(f)$ is a closed immersion if and only if any finite Nori-reduced torsor over X is a sub-torsor of the pull-back of a finite Nori-reduced torsor over Y.

Proof. Let $t : T \to X$ be a finite Nori-reduced pointed G-torsor over X and let $u : U \to Y$ be a finite Nori-reduced pointed H-torsor over Y, for some finite group-schemes G and H. Then:

Proof of (a): If $\pi^N(f)$ factors through the trivial group-scheme, then for any faithfully flat arrow $\pi^N(Y, y) \to H$ the composition $\pi^N(X, x) \to \pi^N(Y, y) \to H$ has trivial image, and thus the corresponding pull-back torsor is trivial by the bijection of Definition 3.2.2. On the other hand, if any pull-back of a Nori-reduced torsor $u : U \to Y$ is trivial over X, then $U \times_Y X \cong X \times_k H$ and thus if I is a set that indexes Nori-reduced pointed torsors over Y, where

$U_i \to Y$ for $i \in I$ is a Nori-reduced H_i-torsor, using Proposition 3.2.18 we have

$$\begin{aligned}\hat{Y} \times_Y X &= \left(\lim_{\leftarrow\, i\in I} U_i\right) \times_Y X \\ &\cong \lim_{\leftarrow\, i\in I} (X \times_Y H_i) \\ &= X \times_Y \left(\lim_{\leftarrow\, i\in I} H_i\right) \\ &= X \times_Y \pi^N(Y, y)\end{aligned}$$

that implies that $\pi^N(f)$ is trivial (Definition 3.2.10).

Proof of (b): If $\pi^N(f)$ is faithfully flat, as the morphism $\pi^N(Y, y) \to H$ is faithfully flat as U is Nori-reduced (Lemma 3.2.14), then we conclude that the composition $\pi^N(X, x) \to \pi^N(Y, y) \to H$ is also faithfully flat. On the other hand, (b) is equivalent to say that for any individual Nori-reduced H-torsor $U \to Y$ the composition $\pi^N(X, x) \to \pi^N(Y, y) \to H$ individually, then by passing to the limit over all arrows $\pi^N(X, x) \to H$ for Nori-reduced torsors over Y, we will obtain that $\pi^N(X, x) \to \pi^N(Y, y)$ is faithfully flat by looking at the respective Hopf algebra morphisms, that are all injective.

Proof of (c): If $\pi^N(f)$ is a closed immersion, for any Nori-reduced G-torsor $T \to X$ let K_G be $K_G = \ker(\pi^N(X, x) \to G)$, then we can consider the subgroup-scheme $\pi^N(K_G) \subset \pi^N(Y, y)$ and N_G the smallest normal subgroup-scheme of $\pi^N(Y, y)$, that exists by Zorn's lemma. Thus, we have a commutative diagram

$$\begin{tikzcd} \pi^N(X, x) \arrow[r, "\pi^N(f)"] \arrow[d] & \pi^N(Y, y) \arrow[d] \\ G \arrow[r] & H = \pi^N(Y, y)/N_G \end{tikzcd}$$

where both vertical morphisms are faithfully flat, and both horizontal morphisms are closed immersions. As $\pi^N(f)$ is a closed immersion and G is not trivial, we cannot have that $N_G = \pi^N(Y, y)$ as H would be trivial in that case, and thus H is not trivial. Finally, if H is finite, we are done, otherwise H is a profinite quotient of $\pi^N(Y, y)$ and there exists a finite quotient $H \to Q$ such that $G \to Q$ is a closed immersion by [33, Théorème 8.10.5(iv)].

For the opposite implication, let us suppose that any Nori-reduced G torsor $T \to X$ is a sub-torsor of the pull-back of a Nori-reduced H-torsor $V \to Y$, in this case we have a commutative diagram

$$\begin{tikzcd} \pi^N(X,x) \arrow[r, "\pi^N(f)"] \arrow[d] & \pi^N(Y,y) \arrow[d] \\ G \arrow[r] & H \end{tikzcd}$$

where both vertical morphisms are faithfully flat, and the lower horizontal morphism is a closed immersion. If $K = \ker(\pi^N(f)$ we can see by following the diagram that the composition

$$K \hookrightarrow \pi^N(X,x) \to G$$

has trivial image. As this holds for any finite quotient of $\pi^N(X,x)$, by passing to the limit we see that the closed immersion $K \hookrightarrow \pi^N(X,x)$ has trivial image, implying that K is trivial.

□

3.2.3 *Comparison with the étale fundamental group*

In this subsection we will compare the fundamental group-scheme with the étale fundamental group. This will also show that over fields of characteristic zero the non-tannakian FGS is redundant as it is essentially the étale fundamental group.

Let X be a scheme over a field k, recall (Example 2.3.10(2)) that a finite and faithfully flat morphism $Y \to X$ is a finite étale covering if for any point $x \in X$ the fiber $Y \times_X \mathrm{Spec}(\kappa(x))$ is the spectrum of an étale $\kappa(x)$-algebra (Definition-Proposition 2.2.103). For any geometric point $\bar{x} : \mathrm{Spec}(\Omega) \to X$, the abstract group $\Gamma = \mathrm{Aut}(Y|X)$ of automorphisms of f acts freely over $Y_{\bar{x}}$, and if this action is moreover transitive, we say that the étale cover $Y \to X$ is Galois.

Category of finite étale covers and fiber functor

Definition 3.2.20. *Let X be a scheme over a field k, and let $\bar{x} : \mathrm{Spec}(\Omega) \to X$ be a fixed geometric point of X. We will denote the* category of finite étale covers of X *as Fét$_X$. Objects in this category are finite étale cover over X, and morphisms are simply scheme morphism over X.*
Moreover, we will consider the functor Fib$_{\bar{x}}$: Fét$_X$ $\to$ Set that assigns to every étale cover $Y \to X$ the underlying set of its geometric fiber $Y_{\bar{x}}$. This functor is known as the fiber functor over $\bar{x}$*.*

With this, we can define the étale fundamental group:

Étale fundamental group

Definition 3.2.21. *Let X be a scheme over a field k, and let $\bar{x} : \mathrm{Spec}(\Omega) \to X$ be a fixed geometric point of X. The* étale fundamental group *is the group $\pi_1(X,\bar{x})$ of natural automorphism of Fib$_{\bar{x}}$.*

Remark 3.2.22. *As an automorphism group for the functor Fib$_{\bar{x}}$, the étale fundamental group $\pi_1(X,\bar{x})$ acts on each geometric fiber $Y_{\bar{x}}$, so the fiber functor can be consider as a functor*

$$\mathit{Fib}_{\bar{x}} : \mathit{F\acute{e}t}_X \to \pi_1(X,\bar{x}) - \mathit{Set}$$

where $\pi_1(X,\bar{x}) - Set$ *denotes the category of sets with a right action from* $\pi_1(X,\bar{x})$.

The main theorem, underlying the basic properties of the étale fundamental group is:

Main properties of the étale fundamental group

Theorem 3.2.23 (§V.7 [35])**.** *Let* X *be a connected scheme over a field* k*, and let* $\bar{x} : \mathrm{Spec}(\Omega) \to X$ *be a fixed geometric point of* X*. If* $\pi_1(X,\bar{x})$ *is the étale fundamental group of* X *with respect to* $\bar{x}$*, then:*

(a) The action of $\pi_1(X,\bar{x})$ *on every geometric fiber is continuous*[4] *and the fiber functor* $Fib_{\bar{x}}$ *establishes an equivalence between* $Fét_X$ *and the category of finite sets with continuous right action of* $\pi_1(X,\bar{x})$.

(b) Under the equivalence of the point above, connected étale coverings of X *correspond to sets with transitive actions of* $\pi_1(X,\bar{x})$*. Moreover, connected Galois covers correspond to finite quotients of* $\pi_1(X,\bar{x})$*, i.e., for any Galois cover* $Y \to X$*,* $Aut(Y|X)$ *is a finite quotient of the étale fundamental group.*

(c) $\pi_1(X,\bar{x})$ *is a pro-finite abstract group. It is the projective limit of all the automorphism groups of connected Galois étale covers of* X.

For a proof of this theorem, the reader can consult [64, Thm. 5.4.2 & Coro. 5.4.8].
This result allows us to to relate $\pi_1(X,\bar{x})$ with the fundamental group-scheme $\pi^N(X,x)$ when the base field k is algebraically closed, using the next definition:

Maximal pro-$\mathcal{P}$ quotient

Definition 3.2.24. *Let* G *be an affine group-scheme over* k*, and let* $\mathcal{P}$ *a property of group-schemes, for example, finite or étale. We will suppose the following supplementary property for set of quotients of* G *that satisfy property* $\mathcal{P}$*: if* Q*,* Q_1 *and* Q_2 *are quotients of* G *that satisfy the property* $\mathcal{P}$*, then for any pair of faithfully flat morphisms* $q_i : Q_i \to Q$ *(*$i = 1, 2$*) that commute with the quotients morphisms from* G*, the fibered product* $Q_1 \times_Q Q_2$ *also has property* $\mathcal{P}$.
Then, the projective limits of all these quotients is a quotient group-scheme $G \to G^{\mathcal{P}}$ *that we will call the* maximal pro-$\mathcal{P}$ quotient of G*. If the property* $\mathcal{P}$ *is "finite" the resulting quotient will be called the* maximal pro-finite quotient of G *that we will denote as* $G^{pro\text{-}F}$ *and if* $\mathcal{P}$ *"étale" the corresponding quotient will be called the* maximal pro-étale quotient of G *that we will denote as* $G^{ét}$.

Remark 3.2.25. *Let* G *be an affine group-scheme over* k*, and let* $\mathcal{P}$ *a property of group-schemes, let us suppose that property* $\mathcal{P}$ *is always inherited by subgroup-schemes. If* G *has a pro-*$\mathcal{P}$ *quotient* $G^{\mathcal{P}}$*, it has the following universal property, using the Isomorphism Theorem for group-schemes (Theorem 2.2.65): For any morphism of group-schemes over* k*,* $\phi : G \to H$

4 Here the finite sets corresponding to the geometric fibers are considered with the discrete topology.

where H *is a group-scheme that satisfies property* $\mathcal{P}$*, then there exists a unique morphism* $G^{\mathcal{P}} \to H$ *that factors through* ϕ*. If* $\mathcal{P}$ *does not get inherited by all subgroup-schemes, then the universal property above only holds for faithfully flat morphisms.*

Comparison between the étale fundamental group and the FGS

Proposition 3.2.26. *Let* X *be a reduced and connected scheme over an algebraically closed field* k*, with a rational point* $x \in X(k)$*. Then, the étale fundamental group-scheme* $\pi_1(X, x)$ *is isomorphic to* $\left(\pi^N(X,x)^{\acute{e}t}\right)(k)$*, the set of* k *rational points of the maximal pro-étale quotient of the FGS.*

Proof. By the Galois correspondences for étale algebras (Theorem 2.2.104) and Example 2.2.107, if G is a finite étale group-scheme over an algebraically closed field, it is necessarily the constant group-scheme associated to a finite abstract group Γ. Constant group-schemes over a field k are fully determined by their $\bar{k}$-points, so as k is algebraically closed, we have that

$$\left(\pi^N(X,x)^{\acute{e}t}\right)(k) = \lim_{\substack{\leftarrow \\ i \in I}} G_i(k) \cong \lim_{\substack{\leftarrow \\ i \in I}} \Gamma_i$$

where I indexes the set of étale quotients of $\pi^N(X, x)$ and Γ_i is the abstract group associated to G_i so it suffices to show that the projective limit on the right is $\pi_1(X, x)$. But as noted in Example 2.3.10(2), Nori-reduced G_i-torsors over constant group-schemes are the same as Galois étale covers with automorphism group $G_i(k) = \Gamma_i$, moreover they are connected, thus the desired isomorphism follows from Theorem 3.2.23. □

FGS is characteristic zero

Remark 3.2.27. *If* k *is an algebraically closed field of characteristic zero and* X *is a scheme over* k *that possesses a FGS, then* $\pi^N(X, x) = \pi^N(X,x)^{\acute{e}t}$ *as any finite group-scheme over* k *is étale (Proposition 2.2.109), so there is essentially no difference between the FGS and the étale fundamental group. If* k *is not algebraically closed,* $\pi^N(X, x)$ *is a priori "larger" than* $\pi_1(X, x)$ *as Nori-reduced torsors over non-constant étale group-schemes are not Galois étale covers, rather, they are geometrically so. See the last paragraph of Example 2.3.10(2).*

Remark 3.2.28. *If now* k *has positive characteristic, then* $\pi^N(X, x)$ *is a priori larger than* $\pi^N(X,x)^{\acute{e}t}$ *as not all finite group-schemes over* k *are reduced, for example the group-scheme*

$$\alpha_p = \operatorname{Spec}\left(k[x]/(x^p)\right)$$

that represents the group-valued functor of k*-algebras* $R \mapsto \{r \in R : r^p = 0\}$ *is highly non-reduced by looking at its underlying Hopf algebra. This Hopf algebra is local, and group-schemes of these type are called* local or infinitesimal group-schemes *(See [18, II §4.7] for more details). Thus, the eventual presence of torsors over local group-schemes implies that* $\pi^N(X, x)$ *is larger than* $\pi^N(X,x)^{\acute{e}t}$ *if* X *possesses such torsors. Moreover, if* $Y \to X$ *is an étale cover of a reduced scheme,* Y *is reduced as well ([64, Prop. 5.2.12])* *but this might not longer true in positive characteristic for arbitrary finite torsors over* X*, Nori-reduced or not.*

3.3 TANNAKIAN FUNDAMENTAL GROUP-SCHEMES

3.3.1 *Tannakian Nori and S FGS's*

Now we will construct two fundamental group-schemes out of neutral tannakian categories over a field k of vector bundles over a scheme X over k. There are two possible ways to construct these fundamental group-schemes, one is the construction for proper schemes that combines Nori [53, Ch. I] and Langer [42] approaches by considering "finite bundles" over X and "semi-stable bundles" over proper smooth curves. This will be the content of Subsubsection 3.3.1.1
The other approach is for "pseudo-proper" schemes, due to N. Borne and A. Vistoli in [13] that extends beyond schemes and simplifies the tannakian formulation of the Nori fundamental group-scheme.
We will focus more on the former approach in this subsection, but we will briefly mention the key points and results for the pseudo-proper approach at the end, see Remark 3.3.59.
We will also show how these tannakian approaches are related to the fundamental group-scheme obtained out of pointed finite torsors in Section 3.2, unifying both approaches in Subsubsection 3.3.1.2.
Throughout this subsection, X will be a proper (thus of finite type), reduced and connected scheme over a field k. In particular, X is noetherian and $\Gamma(X, \mathcal{O}_X) = k$. Thus, vector bundles $\mathcal{E}$ over X correspond to schemes $f : Y \to X$ finite and flat over X with $\mathcal{E} \cong f_*(\mathcal{O}_Y)$ (Section 1.4(6)).

3.3.1.1 *Nori-semistable and essentially finite vector bundles*

We will start with some details about vector bundles, that we will cover now.

Definition 3.3.1. *Let* X *be a proper, reduced and connected scheme over a field* k*. If* $\mathcal{F}$ *and* $\mathcal{G}$ *are two vector bundles, that we will identify as finite and flat morphisms* $f : Y \to X$ *and* $g : Z \to X$ *with* $f_*(\mathcal{O}_Y) = \mathcal{F}$ *and* $g_*(\mathcal{O}_Z) = \mathcal{G}$*.* *Morphisms of vector bundles*
A morphism of vector bundles $h : \mathcal{F} \to \mathcal{G}$ *is a morphism of schemes* $h : Z \to Y$ *over* X *such that* $g = f \circ h$ *and for every geometric point* $\bar{x} : \mathrm{Spec}(\Omega) \to X$*, the morphism* $Z_{\bar{x}} \to Y_{\bar{x}}$ *of fibers over* $\bar{x}$ *is a morphism of* Ω*-vector spaces of constant rank that is independent of the geometric point* $\bar{x}$*.*

Remark 3.3.2. *In practice, a morphism of vector bundles at the* $\mathcal{O}_X$*-module level is a morphism of* $\mathcal{O}_X$*-modules* $h : \mathcal{F} \to \mathcal{G}$ *of constant rank over any geometric point over* X*, i.e., the morphism of fibers over geometric points* $\mathcal{F}_{\bar{x}} \to \mathcal{G}_{\bar{x}}$ *has a constant rank that is independent of the given geometric*

point, this is sustained by two facts: the first fact is that over a reduced scheme a sheaf is locally free if and only if it has constant rank on all stalks, the constancy of rank does not imply local freeness over non-reduced schemes, thus the second fact [44, Prop. 1.7.2] ensures the kernel and the image are vector bundles, but we could have used another version for the definition of vector bundle morphisms from the literature with the same result.
We see then, that the conditions we impose on vector bundle morphisms are stronger than the conditions required of a simple morphism of $\mathcal{O}_X$-modules, as a morphism of vector bundles ensures the kernel and the image sheaves are indeed vector bundles, which is not true in general if we only consider morphisms of $\mathcal{O}_X$-modules.

Now we will introduce sub-bundles:

Sub-bundles

Definition 3.3.3. *Let X be a proper, reduced and connected scheme over a field k and let $\mathcal{F}$ be a vector bundle over X. A quasi-coherent sub-sheaf $\mathcal{E} \subset \mathcal{F}$ is a* sub-bundle *if the inclusion morphism is a morphism of vector bundles.*

Remark 3.3.4. *Keeping the notation of Definition 3.3.3, as the image of an inclusion of bundles $i : \mathcal{E} \hookrightarrow \mathcal{F}$ is a vector bundle, $\mathcal{E}$ has a* complement vector sub-bundle, *i.e., there exists a sub-bundle $\mathcal{E}' \subset \mathcal{F}$ such that $\mathcal{F} = \mathcal{E} \oplus \mathcal{E}'$.*

We finish the discussion of vector bundles with quotient bundles:

Quotient bundle

Definition 3.3.5. *Let X be a proper, reduced and connected scheme over a field k, let $\mathcal{F}$ be a vector bundle over X and let $\mathcal{E}$ be sub-bundle of $\mathcal{F}$. The* quotient bundle *is the quotient sheaf $\mathcal{F}/\mathcal{E}$ that is a vector bundle by Remark 3.3.4.*

Now we will define the first main type of vector bundle we will work with:

Finite bundles

Definition 3.3.6. *Let $\mathcal{F}$ be a vector bundle over X. We say that $\mathcal{F}$ is* finite *if there exist two different polynomials $f, g \in \mathbb{Z}_{\geqslant 0}[x]$ such that $f(\mathcal{F}) \cong g(\mathcal{F})$, following the notation of Definition 2.4.69(b). We will denote the full sub-category of Qcoh(X) whose objects are finite bundles as Fin(X).*

This definition is due to Weil, so the name "Weil finite" is used too.

Finite torsors yield finite vector bundles

Example 3.3.7. *Let $t : T \to X$ be a finite G-torsor over X. As t is finite and faithfully flat, $V_T = t_*(\mathcal{O}_T)$ is a locally free sheaf, moreover, the isomorphism $T \times_X G_X \cong T \times_X T$ corresponds to the isomorphism of vector bundles $V^{\oplus n} \cong V \otimes V = V^{\otimes 2}$, where $n = ord(G)$ is the order of G (Definition 2.2.89). Thus, we see that V_T is a finite bundle by using the polynomials $p(x) = nx$ and $q(x) = x^2$.*

We would like to study the category Fin(X) as a full sub-category of the category Vect(X) of vector bundles of finite rank which we will consider as a full sub-category of Qcoh(X), the category of quasi-coherent sheaves over X, thus we are considering vector bundles with

morphisms of $\mathcal{O}_X$-modules instead of vector bundle morphisms (Definition 3.3.1). It is not hard to see that $\mathrm{Qcoh}(X)$ is k-linear abelian, tensorial but not rigid, while $\mathrm{Vect}(X)$ is a k-linear rigid tensor category that is additive but not abelian, as kernels and cokernels of morphisms of $\mathcal{O}_X$-modules between vector bundles might not be vector bundles (Remark 3.3.2).
To better study $\mathrm{Fin}(X)$, we need an equivalent definition for finite bundles, so we will introduce the following concepts:

Krull-Schmidt categories

Definition 3.3.8. *Let $\mathcal{C}$ be a k-linear*[5] *additive category.*

(a) An object I of $\mathcal{C}$ is indecomposable *if I cannot be written as a direct sum of two objects, i.e., if $I = I_1 \oplus I_2$ then either I_1 or I_2 is the zero object.*

(b) Let X be an object of $\mathcal{C}$. An endomorphism $e \in End_{\mathcal{C}}(X)$ is idempotent *if $e \circ e = e$. We say that an idempotent $e \in End_{\mathcal{C}}(X)$ of $\mathcal{C}$* splits *if there exists an object Y of $\mathcal{C}$ and morphisms $f : X \to Y$ and $g : Y \to X$ such that $g \circ f = e$ and $f \circ g = id_Y$.*

(c) The category $\mathcal{C}$ is a Krull-Schmidt category *if for any object X of $\mathcal{C}$ idempotent endomorphisms $e \in End_{\mathcal{C}}(X)$ split.*

Decompositions in Krull-Schmidt categories

Lemma 3.3.9. *Let $\mathcal{C}$ be a k-linear additive category. If $\mathcal{C}$ is Krull-Schmidt and Hom-sets are finitely dimensional over k, then every object X of $\mathcal{C}$ has a finite decomposition in indecomposable objects*

$$X \cong E_1 \oplus E_2 \cdots E_n$$

where E_i is indecomposable for any $i = 1..n$. This decomposition is unique in the following sense: if

$$X \cong E_1 \oplus E_2 \cdots E_n \cong F_1 \oplus F_2 \cdots F_m$$

are two decompositions of X, then $m = n$ and $F_i = E_{\sigma(i)}$ for some permutation $\sigma \in S_n$.

Over proper schemes, Hom-sets of vector bundles are finitely dimensional, so we have:

Proposition 3.3.10 (Atiyah [7])**.** *Let X be a proper reduced and connected scheme over a field k. Then, the category $Vect(X)$ is Krull-Schmidt category, in particular any vector bundle can be decomposed into a finite direct sum of indecomposable vector bundles.*

With this we can characterize finite vector bundles further:

Proposition 3.3.11. *Let X be a proper reduced and connected scheme over a field k. Let $\mathcal{E}$ be a vector bundle over X, and let $I(\mathcal{E})$ denote the set of*

5 In the sense of Remark 2.4.92.

isomorphism classes of indecomposable vector bundles appearing in the decomposition of $\mathcal{E}$*. Then,* $\mathcal{E}$ *is finite if and only if the set*

$$S(\mathcal{E}) = \bigcup_{n \geqslant 1} I(\mathcal{E}^{\otimes n})$$

is finite.

Proof. See [64, Prop. 6.7.4]. □

Finite line bundles

Example 3.3.12. *Let* X *be a proper reduced and connected scheme over a field* k *and let* $\mathcal{L}$ *be a* line bundle*, i.e., a vector bundle of rank 1.*
In this case, $\mathcal{L}$ *and its tensor powers are all line bundles, and thus indecomposable. So we have*

$$S(\mathcal{E}) = \{\mathcal{L}^{\otimes n} : n \geqslant 1\},$$

and this allows us to conclude that $\mathcal{L}$ *is finite if and only if it is a* torsion line bundle*, i.e.,* $\mathcal{L}^{\otimes m} \cong \mathcal{O}_X$ *for some integer* $m \geqslant 1$*.*

Using this proposition and by carefully working with the sets $S(\mathcal{E})$, we have:

Corollary 3.3.13. *Let* X *be a proper reduced and connected scheme over a field* k*. The category* $Fin(X)$ *is stable under tensor products, duals, direct sums and direct summands, thus it is a* k*-linear rigid additive Krull-Schmidt tensor category.*

Sadly, we have not arrived at an abelian category as we have the same issue the category of vector bundles has with kernels and cokernel. We need to find an abelian, or better still, a neutral tannakian category of vector bundles over X that contains $\mathrm{Fin}(X)$.
In any case, we need a fiber functor for a neutral tannakian category, so we will define it now and then we will define the larger "good" class of bundles that we will use.

Definition 3.3.14. *Let* X *be a proper reduced and connected scheme over a field* k*. If* $x \in X(k)$ *is a rational point, we will consider the functor* ω_x *that assigns to each vector bundle* V*, associated to a morphism* $f : \underline{\mathrm{Spec}}(V) \to X$*, its fiber over* $x : \mathrm{Spec}(k) \to X$*, that we will denote as* $V_x = \underline{\mathrm{Spec}}(V) \times_X \mathrm{Spec}(k)$*.*
It is not hard to see that V_x *is a finitely dimensional vector space, so we have a functor* $\omega_x : Vect(X) \to Vectf_k$*.*

Remark 3.3.15. *Keeping the notations of Definition 3.3.14,* ω_x *is a* k*-linear additive tensor functor, moreover, it is fully faithful as* $\omega_x(V) = 0$ *is the zero vector space if and only if* V *is the zero vector bundle.*
If $\mathcal{C}$ *is a full sub-category of* $Vect(X)$ *that is also abelian and such that any morphism between elements of* $\mathcal{C}$ *is a vector bundle morphism, it is not hard to see that* $\omega_x|_{\mathcal{C}}$ *is also exact.*

Let C be a proper (thus projective) smooth curve over k, we will consider a special family of vector bundles over curves to obtain an abelian category of vector bundles over proper reduced and connected schemes X that contains $\mathrm{Fin}(X)$.
But first, we need to introduce some concepts for vector bundles over curves:

Definition 3.3.16. *Let C be a proper smooth curve over k.*

(a) Let $\mathcal{E}$ be a vector bundle over C of rank $r \geqslant 1$, the determinant of $\mathcal{E}$ *is the the r-th exterior power of $\mathcal{E}$, $\det(\mathcal{E}) = \bigwedge^r \mathcal{E}$. It is a line bundle over C. By [36, Ch. II 6.11 & 6.13], this line bundle corresponds to a divisor $D_{\mathcal{E}} = \sum_{i=1}^{s} n_i c_i$ ($n_i \in \mathbb{Z}$, $c_i \in C$ is a closed point for any i) whose degree $deg(D_{\mathcal{E}}) = \sum_{i=1}^{s} n_i \in \mathbb{Z}$ is called the* degree of $\mathcal{E}$, *we will denote it as $deg(\mathcal{E})$.* *Degree*

(b) Let $\mathcal{E}$ be a vector bundle over C of rank $r \geqslant 1$, the slope of $\mathcal{E}$ *is the ratio* *Slope*

$$\mu(\mathcal{E}) = \frac{deg(\mathcal{E})}{r} \in \mathbb{Q}.$$

(c) Let $\mathcal{E}$ be a vector bundle over C of rank $r \geqslant 1$, we say that $\mathcal{E}$ is semi-stable *if for every non-zero sub-bundle $\mathcal{E}' \subset \mathcal{E}$ we have* *Semi-stable bundles*

$$\mu(\mathcal{E}') \leqslant \mu(\mathcal{E})$$

if the inequality above is strict for every sub-bundle, we say that $\mathcal{E}$ is stable.

Remark 3.3.17. *Let C be a proper smooth curve over k. We have several remarks related to Definition 3.3.16:*

(a) For any pair of vector bundles $\mathcal{E}$ and $\mathcal{F}$, we have that

$$\mu\left(\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{F}\right) = \mu(\mathcal{E}) + \mu(\mathcal{F}).$$

(b) If $\mathcal{E}$ is a vector bundle. An equivalent definition of semi-stability for $\mathcal{E}$ is that

$$\mu(\mathcal{E}) \leqslant \mu(\mathcal{F})$$

for any non-zero quotient sheaf $\mathcal{E} \to \mathcal{F}$.

(c) Let

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \mathcal{E}_2 \to 0$$

be a short exact sequence of vector bundles over C with vector bundle morphisms. If both $\mathcal{E}_1$ and $\mathcal{E}_2$ are semi-stable with the same slope μ, then $\mathcal{E}$ is also semi-stable with slope μ, see [62, Corollaire 7].

(d) The property of being semi-stable is geometric, meaning that a vector bundle $\mathcal{E}$ over C is semi-stable if and only if the pull-back of this bundle over C_L is semi-stable for any extension L of k. We will use this mostly to affirm that semi-stable bundles remain so over $C_{\bar{k}}$ where $\bar{k}$ is the algebraic closure of k. If we consider stable bundles, this property is no longer true in general, see [38, pp. 1.3.8 & 1.3.9].

The last point has in fact a stronger version:

Proposition 3.3.18 (Proposition 8 [62]). *Let C be a proper smooth curve over k and let $f : \mathcal{E} \to \mathcal{F}$ be a morphism of $\mathcal{O}_C$-modules between two vector bundles $\mathcal{E}$ and $\mathcal{F}$ with $\mu(\mathcal{E}) = \mu(\mathcal{F}) = \mu$.*
Then, f is a morphism of vector bundles and the bundles $\ker(f)$ *and coker*(f) *are semi-stable of slope μ.*

This allows us to characterize the following category:

Category of semi-stable bundles of constant slope

Definition 3.3.19. *Let C be a proper smooth curve over k. For a fixed rational number μ, we will denote the* category of semi-stable bundles of slope μ over C *as* $SS_C(\mu)$.

Due to Proposition 3.3.18, we can characterize the category $SS_C(\mu)$ as a corollary:

Corollary 3.3.20. *Let C be a proper smooth curve over k and let μ be a fixed rational number. Then, the category $SS_C(\mu)$ is a k-linear abelian category.*

What about the tensor product? By Remark 3.3.17(a) the only possible case where the category $SS_C(\mu)$ could be tensorial is when $\mu = 0$, but the tensor product of semi-stable bundles of slope zero is not necessarily semi-stable, we will show a little bit later a sufficient condition for two semi-stable vector bundles of slope zero to have a semi-stable tensor product.
We have introduced semi-stable bundles for two reasons, that the categories $SS_C(\mu)$ are all k-linear abelian, and:

Finite bundles are semi-stable of slope zero

Lemma 3.3.21. *Let C be a proper smooth curve over k. If $\mathcal{E}$ is a finite vector bundle over C, then it is semi-stable of slope zero.*

Proof. See [64, Prop. 6.7.8]. □

With this we can define a better category containing $\mathrm{Fin}(X)$ for a proper reduced and connected scheme X over k.

Nori-semistable bundles

Definition 3.3.22. *Let X be a proper reduced and connected scheme over a field k. A vector bundle $\mathcal{E}$ over X is* Nori-semistable *if for any non-constant morphism $f : C \to X$ where C is a proper smooth curve over k the pull-back bundle $f^*(\mathcal{E})$ is semi-stable of slope zero. We will denote the category of Nori-semistable bundles as NSS(X) and we will consider it as a full sub-category Qcoh(X), thus morphisms in NSS(X) are a priori of $\mathcal{O}_X$-modules.*

Remark 3.3.23. *This definition of Nori-semistable bundles is not standard, as there are other versions in the literature. For instance, Nori defined Nori-semistable bundles as bundles $\mathcal{E}$ over X such that $f^*(\mathcal{E})$ is semi-stable of slope zero for any morphism $f : C \to X$ that is birational onto its image ([53, Definition p.81]), this hinders the possibility of a tensor structure over NSS(X). A similar definition, is that $f^*(\mathcal{E})$ is semi-stable of slope zero for any normalization $f : \widetilde{C} \to C \to X$ for any integral closed sub-scheme $C \subset X$*

of dimension 1 ([64, p.253]), this definition equally hinders a possible tensor category structure in NSS(X).
The definition given here permits a richer structure for the category NSS(X) *that allows to define fundamental group-schemes different from* $\pi^N(X,x)$, *that we will define shortly and in Subsection 3.3.2.*

Regardless of the different definitions in the literature, they always ensure the following property for the category of Nori-semistable bundles:

Proposition 3.3.24. *Let* X *be a proper reduced and connected scheme over a field* k. *Then, the category NSS*(X) *is* k*-linear and abelian. Moreover, any morphism of* $\mathcal{O}_X$*-modules between Nori-semistable bundles is a morphism of vector bundles.*

NSS(X) *is* k*-linear abelian*

For the proof of this proposition, we need the following lemma:

Lemma 3.3.25. *Let* X *be a proper reduced and connected scheme over a field* k. *The for any pair of closed points* $x, y \in X$ *there exists a finite set of proper smooth curves* $\{C_i\}_{i=1}^n$ *with morphisms* $\gamma_i : C_i \to X$ *such that* $\bigcup_{i=1}^n Im(\gamma_i)$ *is connected and contains these points.*

The finite set of curves described above is called a *chain of proper curves*, see Definition 3.3.60.

Proof. If X is projective, a sketch of the proof can be found in [45, Exc. 8.1.5].
If X is proper, we can use Chow's lemma ([31, Théorème 5.6.1]) to get a surjective birational morphism $\pi : X' \to X$ where X' is a projective scheme. It can be shown that X' is a reduced and connected as well, so any pair of points in X' can be joined by a chain of proper curves by the last paragraph. Thus, as π is surjective, we obtain the same property for any pair of points of X. $\square$

Proof of Proposition 3.3.24. By Remark 3.3.2, if any morphism $f : \mathcal{E} \to \mathcal{F}$ of $\mathcal{O}_X$-modules between Nori-semistable bundles is a morphism of vector bundles, then the kernel and cokernel of these morphisms are Nori-semistable bundles as well. In effect, for any non-constant morphism $g : C \to X$ from a smooth and proper curve, we have $g^*(\ker(f)) = \ker(g^*(f))$ and $g^*(\operatorname{coker}(f)) = \operatorname{coker}(g^*(f))$, and as $g^*(\mathcal{E})$ and $g^*(\mathcal{F})$ belong to the abelian category $SS_C(0)$ (Corollary 3.3.20), we conclude that $\ker(f)$ and $\operatorname{coker}(f)$ are Nori-semistable if $\mathcal{E}$ and $\mathcal{F}$ are Nori-semistable and f is a morphism of vector bundles.
Now let us show that $f : \mathcal{E} \to \mathcal{F}$ is a morphisms of vector bundles. Let $\bar{x}, \bar{y}$ be any pair of geometric points of X such that their images over X correspond to two closed points x, y respectively. By Lemma 3.3.25, X contains a curve C that contains x and y, so if we normalize C we obtain a non-constant morphism $g : \widetilde{C} \to X$. Thus, the pull-back morphism $g^*(f) : g^*(\mathcal{E}) \to g^*(\mathcal{E})$ is a morphism of vector bundles of slope 0, thus the rank of f is constant over all geometric points along

the image of $C \subset X$, in particular the rank is constant over $\bar{x}$ and $\bar{y}$. We can then conclude that the rank of f is constant over the fibers over all closed points of X, thus it is everywhere constant as the rank is an upper semi-continuous function over X (see [36, II Example 12.7.2]), this concludes the proof. □

Remark 3.3.26. *Let X be a proper reduced and connected scheme over a field k. If $\mathcal{E}$ is a finite bundle over X, then for any non-constant morphism $f : C \to X$ from a smooth and proper curve $f^*(\mathcal{E})$ is finite as the pull-back commutes with tensor products and direct sums. By Lemma 3.3.21 we conclude that $\mathcal{E}$ is Nori-semistable. Thus, we have a full inclusion of categories Fin$(X) \subset$ NSS(X).*
Another consequence of Proposition 3.3.24, is that any morphism between finite vector bundles is a vector bundle morphism.

We have then, that the category of Nori-semistable bundles is a good k-linear abelian category of vector bundles of X that contains all finite bundles. As we have established a good "abelian master category"for finite bundles, we can consider another important type of vector bundles:

Essentially finite bundles

Definition 3.3.27. *Let X be a proper reduced and connected scheme over a field k. A vector bundle $\mathcal{E}$ is* essentially finite *if it is a sub-quotient (Definition 2.4.69(a)) of a finite bundle.*
We will denote the category of essentially finite bundles over X as EF(X).

Remark 3.3.28. *By definition, EF(X) is the category of sub-quotients generated by the category of finite bundles over X (Definition 2.4.125) as direct sums of finite bundles are finite (Corollary 3.3.13). Thus, it is the "smallest" k-linear abelian full sub-category of NSS(X) that contains Fin(X).*
Thus, we have full inclusions of categories Fin$(X) \subset$ EF$(X) \subset$ NSS(X) and all morphisms in these categories are vector bundle morphisms.

Now we will study the tensor category structure of the categories NSS(X) and EF(X). For the category of Nori-semistable bundles, it is not hard to see that the tensor product $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ of two Nori-semistable bundles stays Nori-semistable if the tensor product $f^*(\mathcal{E}) \otimes_{\mathcal{O}_C} f^*(\mathcal{F})$ is semi-stable of slope 0 for any non-constant morphism $f : C \to X$ where C is a smooth and proper curve. But as we mentioned before, in general the tensor product of two semi-stable bundles over curves is not semi-stable.
A method to obtain such examples, is to consider the pull-backs of vector bundles over curves under relative the Frobenius morphism and its iterations, see [26] and [58, p. 4.1]. And in fact, a sufficient condition to ensure tensor products of semi-stable bundles is related to these pull-backs, but first we will fix notation for Frobenius morphisms:

Absolute and relative Frobenius morphisms

Definition 3.3.29. *Let k be a field of positive characteristic $p > 0$. Over k we have the* Frobenius morphism $F : k \to k$ *given by $a \mapsto a^p$. We can apply this to any k-algebra as well and thus we have the Frobenius morphism $F : A \to A$ on A as well. This morphism is not k-linear as it applies p-th powers on the elements of k, but we can consider the tensor product $A^{(p)} = A \otimes_{k,F} k$ that corresponds to the diagram*

$$\begin{tikzcd} k \arrow[r] \arrow[d, "F"'] & A \arrow[d] \\ k \arrow[r] & A \otimes_{k,F} k \end{tikzcd}$$

and the morphism $A^{(p)} \to A$ given by $a \otimes c \mapsto ca^p$. This morphism is k-linear and thus its corresponding morphism of schemes $F_{\mathrm{Spec}(A)} : \mathrm{Spec}(A) \to \mathrm{Spec}(A^{(p)}) := \mathrm{Spec}(A)^{(p)}$ is known as the relative Frobenius morphism. *The* absolute Frobenius morphism *of $\mathrm{Spec}(A)$ is the morphism of schemes $F : \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ coming from the Frobenius morphism of A, it factors as $\mathrm{Spec}(A) \overset{F_{\mathrm{Spec}(A)}}{\to} \mathrm{Spec}(A)^{(p)} \to \mathrm{Spec}(A)$.*
For a general scheme X over k, the absolute Frobenius morphism *of X is the morphism $\sigma_X : X \to X$ that is the identity on the underlying topological spaces but such that for any open subset $U \subset X$, the morphism induced by σ_X at the level of structural sheaves $\mathcal{O}_X(U) \to \mathcal{O}_X(U)$ is the absolute Frobenius morphism. The* relative Frobenius morphism *of X is the morphism $F_X : X \to X^{(p)}$ where $X^{(p)}$ is the fibered product $X \times_k k$ corresponding to the cartesian diagram*

$$\begin{tikzcd} X^{(p)} \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}(k) \arrow[r, "\sigma_{\mathrm{Spec}(k)}"'] & k \end{tikzcd}$$

and thus the relative Frobenius morphism is the morphism $X \to X^{(p)}$ appearing in the factorization $X \to X^{(p)} \to X$ of σ_X. F_X is a morphism of k-schemes, while σ_X is not. For any integer $n \geqslant 1$ the n-th iterate of the relative Frobenius morphism is denoted as $F_X^n : X \to X^{(p^n)}$ coming from the n-th iterate if the Frobenius morphism of k. It is essentially what we have defined in previous paragraphs by replacing the Frobenius morphism with $a \mapsto a^{p^n}$ over k and k-algebras.

Remark 3.3.30. *We will state the main properties of the relative Frobenius morphism without proof:*

(a) *The assignment $X \mapsto X^{(p)}$ is functorial, meaning that for any morphism $f : X \to Y$ of schemes over k we have a unique morphism $f^{(p)} : X^{(p)} \to Y^{(p)}$ making the following diagram commutative*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "F_X"'] & Y \arrow[d, "F_Y"] \\ X^{(p)} \arrow[r, "f^{(p)}"'] & Y^{(p)} \end{tikzcd} .$$

(b) *The formation of* $X^{(p)}$ *commutes with products, meaning that there is a canonical isomorphism* $(X \times_k Y)^{(p)} \cong X^{(p)} \times_k Y^{(p)}$ *such that* $F_{X \times_k Y}$ *is* $F_X \times F_Y$ *composed with this isomorphism.*

(c) *The relative Frobenius morphism* $F_X : X \to X^{(p)}$ *is compatible with extensions of the base field, meaning that for any extension* L *of* k*,* $(X_L)^{(p)}$ *is canonically isomorphic to* $(X^{(p)})_L$ *so that* $F_{(X_L)^{(p)}}$ *becomes* $(F_X)_L$*, the base change of* F_X *to* L*, under this isomorphism.*

(d) *For any* k*-algebra* R*, we denote* ${}_fR$ *the algebra* R *but with the structural morphism from* k *given by the composition* $k \xrightarrow{F} k \xrightarrow{i} R$ *where* i *is the original inclusion of* k *on* R*. With this, the functor of points of* $X^{(p)}$ *is* $R \mapsto \widetilde{X}({}_fR)$*.*

(e) *For any scheme* X *over* k*,* F_X *induces a homeomorphism of underlying topological spaces. If* X *is of finite type over* k*, then* F_X *is also finite [27, Exc. 4.17 & 12.5].*

Now we can state a condition for bundles over curves, which is stronger than semi-stability:

Strongly semi-stable bundles

Definition 3.3.31. *Let* C *be a proper smooth curve over* k*. A semi-stable bundle* $\mathcal{E}$ *over* C *is* strongly semi-stable *if for any* $n \geqslant 1$*, the pull-back* $(F_C^n)^*(\mathcal{E})$ *of* $\mathcal{E}$ *under the relative Frobenius morphism of* C *is semi-stable.*

The following result, due to Ramanan and Ramanathan, is key to establish the tensor category structure over NSS(X):

Proposition 3.3.32 (Theorem 3.23 [58])**.** *Let* C *be a proper smooth curve over* k*. If* $\mathcal{E}$ *and* $\mathcal{F}$ *are strongly semi-stable bundles over* C*, then the tensor product* $\mathcal{E} \otimes_{\mathcal{O}_C} \mathcal{F}$ *is semi-stable as well.*

Now we can fully characterize the categories NSS(X) and EF(X).

Corollary 3.3.33. *Let* X *be a proper reduced and connected scheme over a field* k*. Then, the categories NSS*(X) *and EF*(X) *are* k*-linear rigid abelian tensor categories. In particular, if* $x \in X(k)$ *is a rational point, then these categories together with the restricted fiber functor* ω_x *over* x *(Definition 3.3.14) are neutral tannakian categories over* k*.*

We will avoid denoting the restrictions of the functor ω_x to NSS(X) or EF(X) as $\omega_x|_{\mathrm{NSS}(X)}$ or $\omega_x|_{\mathrm{EF}(X)}$, we will simply denote the fiber functor over these categories as ω_x if no confusion should arise.

Proof. If the categories NSS(X) and EF(X) are effectively k-linear rigid abelian tensor categories, then they are neutral tannakian after choosing a rational point $x \in X(k)$ as we can easily see that $\mathrm{End}_{\mathrm{Vect}(X)}(\mathcal{O}_X) = k$ as $\Gamma(X, \mathcal{O}_X) = k$, and by using Remark 3.3.15 as morphisms of vector bundles in these categories are vector bundle morphisms (Proposition 3.3.24).
As we already know the categories NSS(X) and EF(X) are k-linear

abelian, it suffices to show that these categories are stable under tensor products and duals.
Let us start with NSS(X). If $\mathcal{E}$ and $\mathcal{F}$ are Nori-semistable bundles over X, then for any non-constant morphism $f : C \to X$ from a smooth and proper curve we have that $f^*(\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}) = f^*(\mathcal{E}) \otimes_{\mathcal{O}_C} f^*(\mathcal{F})$ is a vector bundle of slope 0, so by Proposition 3.3.32 it suffices to show that $f^*(\mathcal{E})$ and $f^*(\mathcal{F})$ are strongly semi-stable of slope 0, but this is true as we can pre-compose f with the relative Frobenius morphism F_C or any of its iterations to obtain that $\left(F_C^n\right)^* (f^*(\mathcal{E}))$ is semi-stable of slope 0 for any $n \geqslant 1$ as $\mathcal{E}$ is Nori-semistable and the same holds for $f^*(\mathcal{F})$.
Finally, as duals commute with pull-backs we just need to show that the dual of a semi-stable vector bundle of slope 0 is of the same type. The slope part is easy to see, and the semi-stability is a consequence of Remark 3.3.17(b), thus NSS(X) has the desired properties.
For essentially finite bundles, if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two sub-quotients of the finite bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ respectively, then $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$ is a sub-quotient of $\mathcal{E}_1 \otimes_{\mathcal{O}_X} \mathcal{E}_2$ and by Corollary 3.3.13 we conclude that the tensor product $\mathcal{F}_1 \otimes_{\mathcal{O}_X} \mathcal{F}_2$ is essentially finite. The same corollary applied to duals shows that the dual of an essentially finite vector bundle is essentially finite. $\square$

As we have just obtained two tannakian categories, by tannakian correspondence (Corollary 2.4.137) we can define two new fundamental group-schemes:

S and EF fundamental group-schemes

Definition 3.3.34. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. The fundamental group-schemes (Definition 2.4.138) associated to the tannakian categories NSS(X) and EF(X) will be denotes as $\pi^S(X, x)$ and $\pi^{EF}(X, x)$ respectively and we will call them the* S-fundamental group-scheme *and the* essentially finite fundamental group-scheme (or EF-fundamental group-scheme).

Remark 3.3.35. *Let X, Y be proper reduced and connected schemes over a field k with a rational points $x \in X(k)$ and $y \in Y(k)$.*

(a) The essentially finite FGS is none other that the fundamental group-scheme $\pi^N(X, x)$ of Section 3.2, but for now we will use the different notation for it as we have not established why $\pi^{EF}(X, x)$ is the same as the Nori FGS.

(b) The full inclusion EF$(X) \subset$ NSS(X) is clearly closed by sub-objects (Definition 2.4.145) and thus by Corollary 2.4.146 we have a natural faithfully flat morphism $\pi^S(X, x) \to \pi^N(X, x)$.

(c) Also, it is not hard to see that if $f : Y \to X$ is a morphism of schemes then for any Nori-semistable (resp. essentially finite) bundle $\mathcal{E}$, the pull-back $f^*(\mathcal{E})$ is Nori-semistable (resp. essentially finite) as well. Thus we have a tensor functor $f^* :$ NSS$(X) \to$ NSS(Y) (resp.

$f^* : EF(X) \to EF(Y)$) *that for compatible rational points* $f(y) = x$ *induces morphisms of fundamental group-schemes* $\pi^S(f) : \pi^S(Y, y) \to \pi^S(X, x)$ *(resp.* $\pi^{EF}(f) : \pi^{EF}(Y, y) \to \pi^{EF}(X, x)$*).*

We will give more details about properties (b) and (c) over a more general framework for tannakian fundamental group-schemes of vector bundles in Subsection 3.3.2.

To show that $\pi^N(X, x)$ and $\pi^{EF}(X, x)$ coincide, we need to show how we can associate representations of affine group-schemes with vector bundles over schemes. We will show how to do this later in Subsubsection 3.3.1.2, but we will state two properties of the fundamental group-schemes we have just defined.
First, we will start by characterizing $\pi^S(C, c)$ when C is a proper and smooth curve over k. For this, we need a general lemma about morphisms between curves $f : D \to C$. Recall that any non-constant morphism between proper and smooth curves is finite and surjective [36, II Prop. 6.8], moreover these morphisms are determined by the finite extension of rational function fields $K(D) \supset K(C)$. In positive characteristic, finite field extensions are a mix of two flavors: separable and purely inseparable, this influences morphisms between curves in the following way:

Lemma 3.3.36. *Let* $f : D \to C$ *be a finite and surjective morphism between two smooth and proper curves over* k*. Then,* f *decomposes as*

$$\begin{tikzcd} D \arrow[r, "F_D^n"'] \arrow[rr, bend left, "f"] & D^{(p^n)} \arrow[r, "g"'] & C \end{tikzcd}$$

where F_D *is the relative Frobenius morphism of* D*, and* g *is a* separable *morphism between smooth and proper curves, i.e., the field extension* $K(D^{(p^n)}) \supset K(C)$ *is separable.*

Proof. See [36, pp. III 2.5 & 2.5.4] or [63, Tag 0CD2]. □

This allows us to fully characterize Nori-semistable bundles over curves:

Nori-semistable bundles over curves

Proposition 3.3.37. *Let* C *be a proper and smooth curve over* k*. Then, the category* $NSS(C)$ *coincides with the category of strongly semi-stable bundles of slope 0 over* C*.*

Proof. By taking the identity morphism of C and all its Frobenius iterates, we see that any bundle V of $NSS(C)$ is strongly semi-stable of slope zero.
To conclude both categories are the same, we just need to show that any strongly semi-stable bundle of degree zero is Nori-semistable: Let $f : D \to C$ be a non-constant morphism from another proper smooth and irreducible curve and let $\mathcal{E}$ be strongly semi-stable bundle of

degree 0. Then, by Lemma 3.3.36 there exists an integer $n \geqslant 0$ such that f factors as

$$\begin{tikzcd} D \arrow[r, "F_D^n"'] \arrow[rr, bend left, "f"] & D^{(p^n)} \arrow[r, "g"'] & C \end{tikzcd}$$

where F_r^n is the n-ith iterated relative Frobenius morphism over D and $g : D \to C$ is separable. By [25, Lemma 1.1] $g^*(\mathcal{E})$ is a semi-stable bundle over $D^{(p^n)}$. Moreover, by Remark 3.3.30(a), we see that $g^*(\mathcal{E})$ is strongly semi-stable over $D^{(p^n)}$ if and only if $\mathcal{E}$ is strongly semi-stable over C, as we have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "F_E"'] & C \arrow[d, "F_C"] \\ E^{(p)} \arrow[r, "g^{(p)}"'] & C^{(p)} \end{tikzcd}$$

where $E = D^{(p^n)}$ and $f^{(p)}$ is separable as f is separable by tracing the underlying rational function fields in the diagram above.
Thus, we see that $f^*(\mathcal{E}) = \left(F_D^n\right)^* (g^*(\mathcal{E}))$ is semi-stable as we wanted. □

The second property we will show now, is that $\pi^{EF}(X,x)$ is pro-finite, like $\pi^N(X,x)$ is:

Proposition 3.3.38. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. Then, $\pi^{EF}(X,x)$ is pro-finite.*

Proof. Let $\mathcal{E}$ be an essentially finite vector bundle over X. By definition, $\mathcal{E}$ belongs to a full-subcategory of the form $\langle \mathcal{F} \rangle \subset EF(X)$ where $\mathcal{F}$ is a finite vector bundle. We will show that for any finite bundle, we have $\left\langle \widetilde{\mathcal{F}} \right\rangle = \langle \mathcal{F} \rangle^{\otimes}$ (Remark 2.4.150) for a certain essentially finite vector bundle that depends on $\mathcal{F}$, in particular this sub-category is tannakian and it corresponds to a finite quotient $\pi^{EF}(X,x) \to G$ by Proposition 2.4.143, this implies $\pi^{EF}(X,x)$ is pro-finite.
For a finite vector bundle $\mathcal{F}$, as the bundle $\mathcal{F} \oplus \mathcal{F}^\vee$ is finite by Corollary 3.3.13, we have that the direct sum

$$\widetilde{\mathcal{F}} = \bigoplus_{\mathcal{E}_i \in S(\mathcal{F} \oplus \mathcal{F}^\vee)} \mathcal{E}_i$$

has a finite amount of factors by Proposition 3.3.11 and thus it is a finite bundle again by Corollary 3.3.13. Now if we consider the full-subcategory $\left\langle \widetilde{\mathcal{F}} \right\rangle$, by construction, this category contains any tensor product of the form $\mathcal{F}^{\otimes i} \otimes \left(\mathcal{F}^\vee\right)^{\otimes j}$ for $i, j \geqslant 0$ by putting the appropriate indecomposable sub-bundles appearing in $S(\mathcal{F} \oplus \mathcal{F}^\vee)$ and thus we can easily conclude that $\left\langle \widetilde{\mathcal{F}} \right\rangle = \langle \mathcal{F} \rangle^{\otimes}$ as the category on the left is closed by tensor products and duals, finishing the proof. □

3.3.1.2 *Unifying the non-tannakian and the tannakian approaches for FGS's*

Now we will show how to relate $\pi^N(X,x)$ with $\pi^{EF}(X,x)$: let G be an affine group-scheme and let $t : T \to X$ be a G-torsor, if we recall Theorem 2.3.27 and its particular version for vector bundles (Corollary 2.3.28), the pull-back of any quasi-coherent sheaf (resp. vector bundle) over X by t is a G-equivariant sheaf (resp. vector bundle) in such a way we obtain an equivalence of categories, the reader may go back to the definition of G-equivariant sheaves (Definition 2.3.24) before continuing.
If we write the equivalence of the last paragraph as $t^* : \mathrm{Qcoh}(X) \to \mathrm{Qcoh}^G(T)$ where $\mathrm{Qcoh}^G(T)$ denotes the category of G-equivariant sheaves over T, we could try the following to inject representations of G into the mix: let $r : G \to GL(V)$ be a finite representation of G, if we could show a way to obtain a G-equivariant sheaf out of V, we could eventually obtain a functor $F_G : \mathrm{Rep}_k(G) \to \mathrm{Qcoh}^G(T)$ whose composition with the inverse of t^* would yield a functor $F_G(T) : \mathrm{Rep}_k(G) \to \mathrm{Qcoh}(X)$ that is determined by the torsor T. And conversely, if we consider an affine group-scheme G and a functor $F_G : \mathrm{Rep}_k(G) \to \mathrm{Qcoh}(X)$ that satisfies certain properties, could we get a G-torsor T over X out of it in such a way F_G is isomorphic to $F_G(T)$ as functors? The answer to this question is yes, and we will outline how to obtain torsors out of functors from $\mathrm{Rep}_k(G)$ to $\mathrm{Qcoh}(X)$, but first we will explicitly define the functor $F_G(T)$ for any given G-torsor T and for this purpose we will expand Example 2.3.25:

The G-equivariant vector bundle associated to a representation

Example 3.3.39. *Let S be a base scheme and let T be a scheme over k with with an action μ from an affine group-scheme G over k. We established in Example 2.3.25 that the structural sheaf $\mathcal{O}_T$ is G-equivariant, we will use this to show that for any finite representation V of G the free vector bundle $V \otimes_k \mathcal{O}_T$ is G-equivariant.*
Let $f : Z \to T$ be a scheme over T and let $\mathcal{F}$ be a quasi-coherent sheaf. We need to show that we have an action of $\widetilde{G}(Z) = Hom_S(Z,G)$ over $Hom_{Qcoh}((Z,\mathcal{F}),(T, V\otimes_k \mathcal{O}_T))$. By Definition 2.3.19, an element of this Hom-set is a commutative diagram

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[d, dash] & V\otimes_k \mathcal{O}_T \arrow[d, dash] \\ Z \arrow[r, "f"'] & T \end{tikzcd}$$

where the upper arrow corresponds to a morphism of quasi-coherent sheaves $\mathcal{F} \to f^*(V\otimes_k \mathcal{O}_T) = V\otimes_k \mathcal{O}_Z$. For convenience we will expand this diagram as

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\phi"] \arrow[drr, dash] & V\otimes_k \mathcal{O}_Z \arrow[r, "id"] \arrow[dr, dash] & V\otimes_k \mathcal{O}_Z \arrow[r, "f^*"] \arrow[d, dash] & V\otimes_k \mathcal{O}_T \arrow[d, dash] \\ & & Z \arrow[r, "f"'] & T \end{tikzcd} .$$

Over the sheaf $V \otimes_k \mathcal{O}_Z$, any element $g \in \widetilde{G}(Z)$ acts $V \otimes_k \mathcal{O}_Z$ as follows: for any open sub-set $U \subset Z$, the action of g over V induces a morphism $V \otimes_k \mathcal{O}_Z(U) \to V \otimes_k \mathcal{O}_Z(U)$ which gives us a morphism of $\mathcal{O}_Z$-modules $\rho_g : V \otimes_k \mathcal{O}_Z \to V \otimes_k \mathcal{O}_Z$, now we combine this action with the action of $\widetilde{G}(Z)$ coming from the fact that $\mathcal{O}_T$ is G-equivariant to obtain an action over $V \otimes_k \mathcal{O}_Z$ corresponding to the composition

$$\begin{tikzcd}
\mathcal{F} \arrow[r, "\phi"] \arrow[rrd] & V \otimes_k \mathcal{O}_Z \arrow[r, "\rho_g"] \arrow[rd] & V \otimes_k \mathcal{O}_Z \arrow[r, "(f\cdot g)^*"] \arrow[d] & V \otimes_k \mathcal{O}_T \arrow[d] \\
 & & Z \arrow[r, "f\cdot g"'] & T
\end{tikzcd} .$$

In simpler terms, g roughly acts on a section $v \otimes_k z$ of $V \otimes_k \mathcal{O}_Z$ on the right as

$$(v \otimes z) \cdot g = (g^{-1} \cdot v) \otimes (z \cdot g)^{6}.$$

In particular, if $V = k$ we will obtain the G-equivariant structure of $\mathcal{O}_T$ of Example 2.3.25.
Finally, it should be remarked that we can consider non-finite representations of G in this construction, and in that case this yields quasi-coherent G-equivariant $\mathcal{O}_T$-modules.

Remark 3.3.40. *Keeping the notations of Example 3.3.39, if the scheme T with an action of a group-scheme G is in fact a G-torsor $t : T \to X$, then for any representation V of G, not necessarily finite, and any scheme $f : Z \to T$ over T the action of $\widetilde{G}(Z)$ over $Hom_{Qcoh}((Z, \mathcal{F}), (T, V \otimes_k \mathcal{O}_T))$ is free.*

Functor of G-invariant sheaves

Definition 3.3.41. *Let X a scheme over k, let G be an affine group-scheme over k and let $t : T \to X$ be a G-torsor. For a quasi-coherent sheaf $\mathcal{E}$ over T, the* sheaf of G-invariants *is the sheaf $\mathcal{E}^G \subset t_*(\mathcal{E})$ of sections $s \in t_*(\mathcal{E})(U) = \mathcal{E}(t^{-1}(U))$ such that $\mu_T^*(s) = p_1^*(s)$ where $\mu_T : T \times_k G \to T$ denotes the action morphism of T and p_1 is the projection of $T \times_k G$ over the first coordinate.*
The functor $F_{T,G} : Rep_k(G) \to Qcoh(X)$ defined as

$$F_{T,G}(V) = (V \otimes_k \mathcal{O}_T)^G$$

for a finite representation V of G, will be called the functor of G-invariants induced by T.

Remark 3.3.42. *Keeping the notation of Definition 3.3.41, the functor $F_{T,G}$ is well-defined as $X = T/G$ by Proposition 2.3.35 which implies that for any quasi-coherent $\mathcal{O}_T$-module $\mathcal{E}$, $\mathcal{E}^G$ is a clearly a quasi-coherent $\mathcal{O}_T^G = \mathcal{O}_X$-module, this also extends to vector bundles, and thus we can consider $F_{T,G}$ as a functor of vector bundles by restricting this functor to $Vect(T)$.*
This functor provides an inverse to the functor $t^* : Qcoh(X) \to Qcoh^G(T)$ that establishes the equivalences of Theorem 2.3.27 and Corollary 2.3.28 as

6 Here the action of $\widetilde{G}(Z)$ is on the left, recall Remark 2.4.79.

clearly $t^*(\mathcal{E}^G)$ *is* G*-equivariant if we use the equivalent definition of* G*-equivariant sheaf stated in Remark 2.3.26.*
Finally, by if we consider the essential image of $F_{T,G}$ *as a full sub-category of Vect*(X) *we see that it is composed of vector bundles* $\mathcal{E}$ *over* X *that are* trivialized by T *as* $t^*(\mathcal{E}) \cong \mathcal{O}_T^{\oplus n}$ *for some integer* $n \geqslant 0$ *as any vector bundle of the form* $V \otimes_k \mathcal{O}_T$ *is trivial purely as a vector bundle.*

Example 3.3.43. *Keeping the notation of Definition 3.3.41, let us suppose* $G = \mathrm{Spec}(A)$ *is finite. In this case, we can consider the regular representation of* G *and the vector bundle of* G*-invariants associated to it* $(A \otimes_k \mathcal{O}_T)^G$*. This vector bundle is in fact* $t_*(\mathcal{O}_T)$*: let* $\{f_i : U_i \to X\}_{i \in I}$ *be an fpqc cover (Definition 2.3.5) of* X *such that the base changes* $t_i : U_i \times_X T \to U_i$ *are trivial* G*-torsors (Proposition 2.3.9). We have that* $U_i \times_X T \cong U_i \times_k G$ *for any* $i \in I$*, and the commutative diagram*

$$\begin{tikzcd} U_i \times_X T \arrow[r, "g_i"] \arrow[d, "t_i"'] & T \arrow[d, "t"] \\ U_i \arrow[r, "f_i"'] & X \end{tikzcd} .$$

We will use this diagram to compare $(A \otimes_k \mathcal{O}_T)^G$ *and* $t_*(\mathcal{O}_T)$ *with their pull-backs* $f_i^*\left((A \otimes_k \mathcal{O}_T)^G\right)$ *and* $f_i^*(t_*(\mathcal{O}_T))$ *over each* U_i*. We can easily see that*

$$f_i^*\left((A \otimes_k \mathcal{O}_T)^G\right) \cong (g_i^*(A \otimes_k \mathcal{O}_T))^G$$

using the equivalence of Theorem 2.3.27 and Remark 3.3.42, so to study the right hand side above, we need to understand how the functor $F_{U_i \times_k G, G}$ *of* G*-invariants behaves for the trivial torsor* $t_i : U_i \times_k G \to U_i$*. Let* A *be the Hopf algebra associated to* G*, as a scheme locally free over* U_i *we easily see that* $(t_i)_*(\mathcal{O}_{U_i \times_k G}) = \mathcal{O}_{U_i} \otimes_k A \cong \mathcal{O}_{U_i}^{\oplus n}$ *where* $n = \mathrm{ord}(G)$ *and for any finite representation* V *of* G*, as* G *acts over* $\mathcal{O}_{U_i} \times_k G$ *solely over the second coordinate by multiplication we have*

$$(V \otimes_k \mathcal{O}_{U_i \times_k G})^G = \mathcal{O}_{U_i} \otimes_k (V \otimes A)^G$$

where the action of G *over* $V \otimes A$ *is roughly given over a pure tensor* $v \otimes a$ *as* $(v \otimes a) \cdot g = (g^{-1} \cdot v) \otimes (a \cdot g)$*. Applying this to* $g_i^*(A \otimes_k \mathcal{O}_T) = A \otimes_k \mathcal{O}_{U_i \times_k G}$*, we obtain that*

$$(g_i^*(A \otimes_k \mathcal{O}_T))^G = \mathcal{O}_{U_i} \otimes_k (A \otimes_k A)^G$$

where the action of G *over* $A \otimes_k A$ *corresponds to the action induced by multiplication by diagonal morphism* $\Delta_G : G \to G \otimes_k G$ *on the right, which looks set-theoretically as* $(g_1, g_2) \cdot g = (g_1 \cdot g, g_2 \cdot g)$*. As* Δ_G *is a closed immersion, we have that* $\Delta_G(G)$ *is a subgroup-scheme of* $G \times_k G$ *and as in the case of abstract groups, we can easily show that* $(G \times_k G)/(\Delta_G(G)) \cong G$ *as schemes*[7]*, which implies*

$$(g_i^*(A \otimes_k \mathcal{O}_T))^G = \mathcal{O}_{U_i} \otimes_k (A \otimes_k A)^G \cong \mathcal{O}_{U_i} \otimes_k A = (t_i)_*(\mathcal{O}_{U_i \times_k G}).$$

7 It can be shown that Δ_G is normal in $G \times_k G$ if and only if G is commutative, and that the isomorphism becomes an isomorphism of group-schemes.

To finally establish that $t_*(\mathcal{O}_T) \cong (A \otimes_k \mathcal{O}_T)^G$*, we have the isomorphism* $f_i^*(t_*(\mathcal{O}_T)) \cong (t_i)_*(\mathcal{O}_{U_i \times_k G})$ *as* t *is affine using [63, Lemma 02KG]. Thus, we conclude that* $t_*(\mathcal{O}_T)$ *is isomorphic to* $(A \otimes_k \mathcal{O}_T)^G$ *as vector bundles when pull-backed over* U_i *via* f_i *for any* $i \in I$*, thus these vector bundles are isomorphic by fpqc descent (Proposition 2.3.8(l)).*

As $F_{T,G}$ is the inverse of t^* we can easily deduce some properties of this functor:

Proposition 3.3.44. *Let* X *a scheme over* k*, let* G *be an affine group-scheme over* k *and let* $t : T \to X$ *be a* G*-torsor. Then, the functor* $F_{T,G}$ *is a* k*-linear exact tensor faithful functor.*
Moreover, if G *is finite and* X *is proper reduced and connected, the essential image of* $F_{T,G}$ *lies within the category of essentially finite bundles over* X*.*

Proof. The properties of the functor $F_{T,G}$ is an easy verification, so we will only show the second part of the statement.
If $G = \mathrm{Spec}(A)$ is finite, then we can consider the regular representation A of G, we know from Example 3.3.43 that $t_*(\mathcal{O}_T) \cong F_{T,G}(A)$ and the former vector bundle is finite, as seen in Example 3.3.7. Moreover, as G is finite we have that $\mathrm{Rep}_k(G) = \langle A \rangle$ is the category of sub-quotients of direct sums of finite copies of the representation A (Corollary 2.4.143), thus the essential image of $F_{T,G}$ is a sub-category of $\langle t_*(\mathcal{O}_T) \rangle \subset \mathrm{Vect}(X)$ but as $t_*(\mathcal{O}_T)$ is finite we conclude this sub-category is fully contained in EF(X), finishing the proof. □

The functor $F_{T,G}$ shows that torsors yield neutral tannakian categories of vector bundles when X has a rational point $x \in X(k)$, but we can ask ourselves a question in the opposite sense: are k-linear exact tensor faithful functors $F : \mathcal{C} \to \mathrm{Vect}(X)$ where $\mathcal{C}$ is a neutral tannakian category over k naturally isomorphic to a functor of the form $F_{T,G}$ for some affine group-scheme G and a G-torsor T over X?
The following remark is key to answer this question:

Remark 3.3.45. *Let* X *a scheme over* k*, let* G *be an affine group-scheme over* k *and let* $t : T \to X$ *be a* G*-torsor. For any affine open sub-scheme* $i : \mathrm{Spec}(R) \to X$*, we can consider the restricted* G*-torsor*

$$t_R : T_R = \mathrm{Spec}(R) \times_X T \to \mathrm{Spec}(R).$$

It is not hard to see that $i^* \circ F_{T,G} = F_{T_R,G}$ *and the latter functor is explicitly given as* $F_{T_R,G} = ((V \otimes_k R) \otimes_R \mathcal{O}_T)^G$ *where we are identifying quasi-coherent sheaves over* $\mathrm{Spec}(R)$ *with* R*-modules via their global sections, which establishes an equivalence between quasi-coherent sheaves over* $\mathrm{Spec}(R)$ *and* R*-modules (see [36, II Coro. 5.5]).*
We can go further in this case, as we have a natural transformation of tensor functors $\omega_G \otimes R \to F_{T_R,G}$*, where* $\omega_G \otimes R$ *is the notation for the base change to* R *of the forgetful functor* ω_G *(see Notation 2.4.115). This natural transformation is an isomorphism as the essential image of* $\omega_G \otimes R$ *lies on the rigid sub-category of* $R-\mathrm{Modf}$ *of free* R*-modules by Remark 2.4.114 and*

so does $F_{T_R,G}$ *as vector bundles over* $\mathrm{Spec}(R)$ *correspond to free* R*-modules, so Proposition 2.4.64 holds in this case.*
Moreover, for any R*-algebra* S*, by recalling from Proposition 2.4.117 that the functor of points of* G *is related to its tannakian category of representations as we have* $\widetilde{G} \cong \underline{Aut}^{\otimes}(\omega_G)$*, and if we consider the functor* $\underline{Hom}^{\otimes}(\omega_G \otimes R, F_{T_R,G})$ *that assigns for any* R*-algebra* S*, the set of natural transformations*[8] *between* $\omega_G \otimes S$ *and* $F_{T_R,G} \otimes S \cong F_{T_S,G}$ *as in Example 2.4.65, where* T_S *is the base change of* T_R *to* $\mathrm{Spec}(S)$ *by the canonical morphism* $R \to S$*. The group-valued functor* $\underline{Aut}^{\otimes}(\omega_G)$ *acts of the functor* $\underline{Hom}^{\otimes}(\omega_G \otimes R, F_{T_R,G})$ *by acting on* $\omega_G \otimes S$ *for any* R*-algebra* S*, thus we have a functorial action of functors of* R*-algebras*

$$\underline{Hom}^{\otimes}(\omega_G \otimes R, F_{T_R,G}) \times \underline{Aut}^{\otimes}(\omega_G) \to \underline{Hom}^{\otimes}(\omega_G \otimes R, F_{T_R,G})$$

this action is free and transitive for each R*-algebra.*

The last part of the Remark is the key to obtain torsors out of k-linear exact tensor faithful functors $F : \mathcal{C} \to R - \mathrm{Modf}$, at least over affine schemes. We will call these functors the following:

R-valued fiber functors

Definition 3.3.46. *Let* $\mathcal{C}$ *be a neutral tannakian category over* k*. For any* k*-algebra* R*, an* R-valued fiber functor over $\mathcal{C}$ *is a* k*-linear exact tensor faithful functor* $\eta : \mathcal{C} \to R - \mathrm{Modf}$ *whose essential image lies in the full sub-category of finite and free* R*-modules.*

Remark 3.3.47. *Let* $\mathcal{C}$ *be a neutral tannakian category over* k*. If* ω *is the fiber functor of* $\mathcal{C}$*, we know by tannakian correspondence (Corollary 2.4.137), that functor* $\underline{Aut}^{\otimes}(\omega)$ *is the functor of points of an affine group-scheme* G*. If* η *is an* R*-valued fiber functor, where* R *is a* k*-algebra, we can consider the functor of natural transformations* $\underline{Hom}^{\otimes}(\omega \otimes R, \eta)$*, which is similar to the functor we considered for the functor of* G*-invariant sheaves over affine schemes in Remark 3.3.45 as for any* R*-algebra* S*,* $\underline{Hom}^{\otimes}(\omega \otimes R, \eta)(S)$ *is the set of natural transformations between* $\omega \otimes S$ *and* $\eta \otimes S$ *which are always isomorphisms by Proposition 2.4.64. Moreover, we have a functorial action*

$$\underline{Hom}^{\otimes}(\omega \otimes R, \eta) \times \underline{Aut}^{\otimes}(\omega) \to \underline{Hom}^{\otimes}(\omega \otimes R, \eta)$$

as in the case of torsors over affine schemes.
Finally, if $t : T \to \mathrm{Spec}(R)$ *is a* G*-torsor, then* $F_{T,G}$ *is clearly an* R*-valued fiber functor over* $Rep_k(G)$*.*

Now we will show the following key result, which will allows us to ultimately show that $\pi^{EF}(X, x)$ coincides with $\pi^{N}(X, x)$:

8 Which is rather $\underline{\mathrm{Isom}}^{\otimes}(\omega_G \otimes R, F_{T_R,G})$ as for any R-algebra we have isomorphisms between tensor functors, by what was mentioned in the last paragraph.

Theorem 3.3.48. *Let $\mathcal{C}$ be a neutral tannakian category over k. Then:*

Bijective correspondence between affine torsors over affine schemes and R-valued functors of finite representations

(a) For any R-valued fiber functor $\eta : \mathcal{C} \to R-Modf$, the functor $\underline{Hom}^{\otimes}(\omega \otimes R, \eta)$ is representable by a G-torsor over $\mathrm{Spec}(R)$ where G is the affine group-scheme whose functor of points is $\underline{Aut}^{\otimes}(\omega)$.

(b) The assignment $\eta \mapsto \underline{Hom}^{\otimes}(\omega \otimes R, \eta)$ establishes an equivalence of categories between the category of R-valued fiber functors over $\mathcal{C}$ and the category of G-torsors over $\mathrm{Spec}(R)$.

Remark 3.3.49. *In part (b) above, morphisms between R-valued fiber functors are simply natural transformations between them while morphisms of G-torsors are G-equivariant morphisms (Definition 2.2.30) over $\mathrm{Spec}(R)$. It must be noted that both these categories are groupoids (Definition 2.3.12), R-valued fiber functors form a groupoid by Proposition 2.4.64 while the same result for G-torsors over $\mathrm{Spec}(R)$ can be found in Lemma 2.3.11.*

The proof of this result was inspired by the proof of [17, Theorem 3.2].

Proof of Theorem 3.3.48. For part (b), if part (a) is true then the assignment $T \mapsto F_{T,G}$, where T is a G-torsor over $\mathrm{Spec}(R)$, establishes an inverse for $\eta \mapsto \underline{\mathrm{Hom}}^{\otimes}(\omega \otimes R, \eta)$ as $\mathcal{C}$ is equivalent to $\mathrm{Rep}_k(G)$ by tannakian correspondence, see the last paragraph of Remark 3.3.47.
For the proof of part (a), let X be an object of $\mathcal{C}$, recall from Lemma 2.4.130 that the k-linear abelian full sub-category $\langle X \rangle$ generated by X has a projective generator P_X, i.e, for this object the functor $\mathrm{Hom}_{\mathcal{C}}(P_X, \cdot)$ is exact and faithful, moreover $\langle X \rangle$ is equivalent to the category of finite comodules over $\mathrm{End}(P_X)^{\vee}$ (Lemma 2.4.131), and the restriction $\omega|_{\langle X \rangle}$ of the forgetful functor of $\mathcal{C}$ to becomes the forgetful functor of this category. Following Notation 2.4.134 we will call this comodule C_X and A_X its dual.
Let $\underline{\mathrm{Hom}}(\omega|_{\langle X \rangle} \otimes R, \eta|_{\langle X \rangle})$ be the functor that assigns to each R-algebra S the set

$$\underline{\mathrm{Hom}}(\omega|_{\langle X \rangle} \otimes R, \eta|_{\langle X \rangle})(S) = \mathrm{Hom}(\omega|_{\langle X \rangle} \otimes_R S, \eta|_{\langle X \rangle} \otimes_R S)$$

of natural transformations between the respective base changes. We will show that this functor is representable. For this, we should recall the natural bijections

$$\mathrm{Hom}_{\mathrm{Vect}(k)}(C_X, V) \cong \mathrm{Hom}(\omega_{C_X}, \omega_{C_X} \otimes V)$$

where ω_{C_X} is the forgetful functor of the category of comodules over C_X, and

$$\mathrm{Hom}_{R-\mathrm{Mod}}(C_X \otimes_k R, R) \cong \mathrm{Hom}_{\mathrm{Vect}(k)}(C_X, R).$$

The first bijections comes from Proposition 2.4.96 and the second is present in the proof of Proposition 2.4.117.

Now, we would like to slightly modify this using η[9]: let S be an R-algebra, thus the first bijection becomes

$$\mathrm{Hom}_{R-\mathrm{Mod}}(\eta(C_X), S) \cong \mathrm{Hom}(\omega_{C_X} \otimes_R S, \eta \otimes_R S)$$

and the second becomes

$$\mathrm{Hom}_{S-\mathrm{Mod}}(\eta(C_X) \otimes_R S, S) \cong \mathrm{Hom}_{R-\mathrm{Mod}}(\eta(C_X), S).$$

If we put these two bijections together, we will obtain for any R-algebra S that

$$\mathrm{Hom}_{R-\mathrm{Mod}}(\eta(C_X), S) \cong \mathrm{Hom}(\omega_{C_X} \otimes_R S, \eta \otimes_R S)$$

which shows that $\eta(C_X)$ represents the functor $\underline{\mathrm{Hom}}(\omega|_{\langle X\rangle} \otimes R, \eta|_{\langle X\rangle})$ as we wanted.

The proof of the first modified bijection is a modified version of the proof of the original corresponding bijection, that we will left as an exercise for the reader. About the second bijection, we will show how to get from one side of the bijection to the other, as we showed in the proof of Proposition 2.4.96, we left the verification that these functors are inverses of each other as another exercise.

Let us start with a morphism of R-modules $f : \eta(C_X) \to S$, if V is a finite comodule over C_X the structural comodule morphism $\phi : V \to V \otimes C_X$ becomes

$$\omega_{C_X}(V) \otimes_R S \to \omega_{C_X}(V) \otimes_k \omega_{C_X}(C_X) \otimes_R S$$

after applying $\omega_{C_X} \otimes_R S$ and if we apply η over the image of this morphism, we will obtain

$$\omega_{C_X}(V) \otimes_R S \to \eta(V) \otimes_R \eta(C_X) \otimes_R S$$

and thus after composing with f we will obtain a morphism

$$\omega_{C_X}(V) \otimes_R S \to \eta(V) \otimes_R S.$$

For the other direction, given a natural transformation

$$F : \omega_{C_X} \otimes_R S \to \eta \otimes_R S,$$

then we can define a morphism $\eta(C_X) \otimes_R S \to S$ by considering the counit morphism $\epsilon : C_X \to k$ and applying $\omega_{C_X} \otimes_R S$ over it to obtain a morphism

$$\omega_{C_X}(C_X) \otimes_R S \to S$$

and by using F, we will thus obtain a morphism of S-modules

$$\eta(C_X) \otimes_R S \to S$$

9 We will omit the restrictions over η in most instances from now on.

as clearly $\eta(k) = R$ that is equivalent to a morphism of R-modules $\eta(C_X) \to S$ by the second modified bijection above.
To resume, for any object X the functor $\underline{\mathrm{Hom}}^{\otimes}(\omega \otimes R, \eta)$ restricted to the abelian full sub-category $\langle X \rangle \subset \mathcal{C}$ representable. With this, if we consider

$$Q = \varinjlim_{[X]} C_X$$

where [X] denotes the isomorphism class of X we considered in the proof of Theorem 2.4.121, then we have that Q is the underlying Hopf algebra of the affine group-scheme G that established the equivalence between $\mathcal{C}$ and $\mathrm{Rep}_k(G)$, see the proof of Corollary 2.4.137. Moreover, we have for any R-algebra S that

$$\begin{aligned}
\underline{\mathrm{Hom}}^{\otimes}(\omega \otimes R, \eta)(S) &= \varprojlim_{[X]} \underline{\mathrm{Hom}}(\omega|_{\langle X \rangle} \otimes R, \eta|_{\langle X \rangle})(S) \\
&= \varprojlim_{[X]} \mathrm{Hom}(\omega|_{\langle X \rangle} \otimes_R S, \eta|_{\langle X \rangle} \otimes_R S) \\
&= \varprojlim_{[X]} \mathrm{Hom}_{R-\mathrm{Mod}}(\eta(C_X), S) \\
&= \mathrm{Hom}_{R-\mathrm{Mod}}(\eta(Q), S).
\end{aligned}$$

where $\eta(Q) = \varinjlim_{[X]} \eta(C_X)$, thus $\underline{\mathrm{Hom}}^{\otimes}(\omega \otimes R, \eta)$ is representable.
To finish the proof, we just need to show that $\eta(Q)$ is faithfully flat over R, as the fact that $\mathrm{Spec}(\eta(Q))$ is a G-torsor over $\mathrm{Spec}(R)$ will be a consequence of Remark 3.3.47.
First, for any object X of $\mathcal{C}$, the R-module $\eta(C_X)$ is finite and free, we conclude that $\eta(Q)$ is a flat R-module as flatness is preserved over direct limits and each R-module $\eta(C_X)$ is flat. Moreover, the canonical injection $\mathbb{1} \to C_X$ induces an exact sequence

$$0 \to \mathbb{1} \to C_X \to C_X/\mathbb{1} \to 0$$

which become the exact sequence

$$0 \to \underbrace{\eta(\mathbb{1})}_{=R} \to \eta(C_X) \to \eta(C_X/\mathbb{1}) \to 0$$

after applying η as this functor is exact, which shows that $\eta(C_X)$ is faithfully flat. Thus, we conclude that $\eta(Q)$ is faithfully flat as well, finishing the proof. □

Remark 3.3.50. *Let $\mathcal{C}$ be a neutral tannakian category over k. If R be a k-algebra, and $\eta : \mathcal{C} \to R-\mathit{Modf}$ is a R-valued fiber functor over $\mathcal{C}$, any rational point $f : \mathrm{Spec}(k) \to \mathrm{Spec}(R)$ defines an k-linear exact tensor faithful functor $f^* : R-\mathit{Mod} \to \mathit{Vect}(k)$.*
Let $Q \in \mathcal{C}$ the object in the proof of Theorem 3.3.48 such that $\eta(Q)$ represents the functor $\underline{\mathit{Hom}}^{\otimes}(\omega \otimes R, \eta)$. We have the composition $\eta(Q) \xrightarrow{f^*}$

$f^*(\eta(Q)) \to k$ *as* $f^*(\eta(Q))$ *is a* k*-vector space*[10], *which defines a rational point for the* G*-torsor associated to* η*, in particular this torsor is pointed.*

As a consequence of this result for affine schemes, we can deduce a general result for any scheme over k. This result is due to Nori, see [53, Ch.I Prop. 2.9].

Bijective correspondence between affine torsors and functors of finite representations over vector bundles

Corollary 3.3.51. *Let* X *a scheme over* k *and let* G *be an affine group-scheme over* k*. Then, the category* $\mathcal{T}_G(X)$ *of* G*-torsors over* X *is equivalent to the the category of* k*-linear exact tensor faithful functors* $F : Rep_k(G) \to Vect(X)$*. Under this equivalence, if* $x \in X(k)$ *is a rational point, then the torsor corresponding to a functor* $F : Rep_k(G) \to Vect(X)$ *is pointed over* x*.*

Proof. If $t : T \to G$ is a G-torsor, then the functor of G-invariant bundles $F_{T,G}$ is a k-linear exact tensor faithful functor (Proposition 3.3.44), so it suffices to show that any k-linear exact tensor faithful functor $F : \mathrm{Rep}_k(G) \to \mathrm{Vect}(X)$ corresponds to a G-torsor over X.
Let F be such a functor, and let $i : \mathrm{Spec}(R) \to X$ be an affine open sub-scheme of X, thus the $i^* \circ F$ is an R-valued fiber functor over $\mathrm{Rep}_k(G)$ as vector bundles over $\mathrm{Spec}(R)$ correspond to finite and free R-modules. Thus, this restricted functor corresponds to a G-torsor $t_R : T_R \to \mathrm{Spec}(R)$ by Theorem 3.3.48.
Let $\{U_j\}_{j\in J}$ be an open affine cover of X, if we denote as $i_j : U_i = \mathrm{Spec}(R_i) \to X$ the respective open immersion for $j \in J$, it is not hard to see that the family $\mathcal{U} = \{i_j : U_j \to X\}_{j\in J}$ is an fpqc cover of X as any i_j is flat and affine. Thus, if $t_j : T_j \to U_j$ is the G-torsor over U_j induced by F, the family of morphisms $\{t_j\}$ induces and affine descent datum over $\mathcal{U}$ (Definition 2.3.46), which is effective by Theorem 2.3.47, thus we have an affine scheme $t : T \to X$ which is also faithfully flat by fpqc descent (Proposition 2.3.8(g) & (k)). Additionally, by applying effective descent for the action morphism of G over each torsor T_j, we conclude that T is a G-torsor over X. Moreover, by construction $F_{T,G} = F$ as F restricted to each open sub-scheme U_j is $i_j^*(F_{T,G}) = F_{T_j,G}$ for all $j \in J$, finishing the proof of the equivalence.
Finally, if $x \in X(k)$ is a rational point, it will be a rational point over at least one affine open sub-scheme of X, thus any torsor corresponding to a functor k-linear exact tensor faithful functor $F : \mathrm{Rep}_k(G) \to \mathrm{Vect}(X)$ will be pointed by applying Remark 3.3.50. ☐

Having established the correspondences, we can make a few remarks:

Functorialty of the correspondence between torsors and functors

Remark 3.3.52. *Let* X *a scheme over* k*. If* $t : T \to X$ *is an affine* G*-torsor over* X*, with functor of* G*-invariants* $F_{T,G}$*.*
If $\phi : G \to H$ *is a morphism of affine group-schemes, we can consider the contracted product* $T' = T \times^G H$ *(Definition 2.3.41). Then, this torsor comes with its own functor of* H*-invariants* $F_{T',H} : Rep_k(H) \to Vect(X)$

10 Not necessarily finitely dimensional.

and in fact, Nori showed in [53, I Lemma 2.9(c)] that in this case we have a commutative diagram

$$\begin{tikzcd}
Rep_k(H) \arrow[dd, "\phi^*"'] \arrow[dr, "F_{T',H}"] & \\
& Vect(X) \\
Rep_k(G) \arrow[ur, "F_{T,G}"'] &
\end{tikzcd}$$

where $\phi^* : Rep_k(H) \to Rep_k(G)$ *is the natural tensor functor induced by* ϕ *(Definition 2.4.118). And conversely, using the correspondence of Corollary 3.3.51, we have that the diagram above corresponds to the natural morphism of torsors* $T \to T'$ *(see Corollary 2.3.48).*
This implies that for any morphism of torsors $f : T \to V$ *with an underlying morphism of group-schemes* $\phi : G \to H$*, we have a similar commutative diagram as the one above*

$$\begin{tikzcd}
Rep_k(H) \arrow[dd, "\phi^*"'] \arrow[dr, "F_{V,H}"] & \\
& Vect(X) \\
Rep_k(G) \arrow[ur, "F_{T,G}"'] &
\end{tikzcd}$$

as $V \cong T \times^G H$ *by Proposition 2.3.49.*
Conversely, by the correspondence of Corollary 3.3.51, any commutative diagram of the form

$$\begin{tikzcd}
Rep_k(H) \arrow[dd, "\phi^*"'] \arrow[dr, "F_2"] & \\
& Vect(X) \\
Rep_k(G) \arrow[ur, "F_1"'] &
\end{tikzcd}$$

where F_1 *and* F_2 *are* k*-linear exact tensor faithful functors, and* ϕ^* *is the natural functor of representations induced by a morphism of affine group-schemes* $\phi : G \to H$*, corresponds to a morphism* $T \to V$ *of torsors over* X *where* T *is the* G*-torsor associated to* F_1 *and* V *is the* H*-torsor associated to* F_2*.*
If we work with pointed torsors, then the morphisms between torsors above are morphisms of pointed torsors.

Universal torsors associated to tannakian categories of vector bundles

Remark 3.3.53. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. The full inclusions of categories $NSS(X) \subset Vect(X)$ and $EF(X) \subset Vect(X)$ correspond by Corollary 3.3.51 to two torsors*

$$\hat{X}_{EF} \to X, \quad \hat{X}_{NSS} \to X$$

which are a pointed $\pi^{EF}(X,x)$-torsor and a pointed $\pi^{S}(X,x)$-torsor respectively.

Moreover, these torsors are Nori-reduced: let us suppose we have a proper non-trivial pointed G-subtorsor $T \subset \hat{X}_{NSS}$, this corresponds at the level of group-schemes to a closed immersion $G \hookrightarrow \pi^{S}(X,x)$ which induces a commutative diagram of functors

$$\begin{tikzcd} & Rep_k(G) \arrow[r, "F_{T,G}"] & Vect(X) \\ Rep_k(\pi^S(X,x)) \cong NSS(X) \arrow[ur, "H"] \arrow[urr, "F"'] & & \end{tikzcd}$$

where F is the functor of G-invariants associated to $\hat{X}_{NSS}$. As H is a functor between neutral tannakian categories coming from a closed immersion, by Proposition 2.4.148, every finite representation V of G is a sub-quotient of a representation of the form $H(W)$ where W is a finite representation of $\pi^{S}(X,x)$, thus $F_{T,G}(V)$ must belong to $NSS(X)$ as it is clearly a sub-quotient of the Nori-semistable bundle $F(W)$ over X and thus the essential image of $F_{T,G}$ lies within $NSS(X)$, thus we have $F_{T,G} : Rep_k(G) \to NSS(X)$ and this provides a pseudo-inverse to H, thus $G = \pi^{S}(X,x)$ and the proof in the case of $\hat{X}_{EF}$ is similar by changing $NSS(X)$ for $EF(X)$ when necessary.

We must also note that the quotient morphism $\pi^{S}(X,x) \to \pi^{EF}(X,x)$ coming by the full-inclusion of categories $EF(X) \subset NSS(X)$ (Remark 3.3.35(b)) induces a faithfully flat morphism of pointed torsors $\hat{X}_{NSS} \to \hat{X}_{EF}$ over X (see Remark 3.3.52).

In general, if $\mathcal{C}$ is a full sub-category $Vect(X)$ such that morphism of objects of $\mathcal{C}$ are vector bundle morphisms and $\mathcal{C}$ is a neutral tannakian category with the fiber functor ω_x. We have a Nori-reduced pointed torsor $\hat{X}_{\mathcal{C}}$ associated to this category that we will call the universal torsor associated to the tannakian category $\mathcal{C}$ of vector bundles over X.

Remark 3.3.53 can be generalized to general neutral tannakian categories of vector bundles, and they allows us to characterize Nori-reduced torsors:

Characterization of Nori-reduced torsors

Proposition 3.3.54. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. Let $\mathcal{C}$ be a full sub-category $Vect(X)$ such that morphism of objects of $\mathcal{C}$ are vector bundle morphisms and $\mathcal{C}$ is a neutral tannakian category with the fiber functor ω_x.*

Let $t : T \to X$ be a pointed G-torsor over X with G affine over k such that its functor of G-invariant bundles has its essential image within $\mathcal{C}$. Then, the following assertions are equivalent:

(a) T *is Nori-reduced.*

(b) $F_{T,G} : Rep_k(G) \to \mathcal{C}$ *is fully faithful.*

Moreover, if G *is finite, then we can add the equivalent assertion:*

(c) $\Gamma(T, \mathcal{O}_T) = k$ *in particular* T *is geometrically connected.*

In part (c), T is geometrically connected because it is proper as X is proper, using fpqc descent (Proposition 2.3.8(d))

Proof. Before establishing the equivalencies, we should note that if $t : T \to X$ is a G-torsor with $F_{T,G} : \mathrm{Rep}_k(G) \to \mathcal{C}$, there exists a unique morphism $\hat{X}_{\mathcal{C}} \to T$ of pointed torsors over X that corresponds to $F_{T,G}$ by Corollary 2.4.120, justifying the adjective "universal" for $\hat{X}_{\mathcal{C}}$.
Now let us show (a) $\implies$ (b), if T is Nori-reduced then the morphism $\hat{X}_{\mathcal{C}} \to T$ is faithfully flat, which translates at the level of categories of representations that $F_{T,G}$ is fully faithful by Proposition 2.4.146.
For (b) $\implies$ (a), if $F_{T,G}$ is fully faithful we will show that the morphism $\hat{X}_{\mathcal{C}} \to T$ is faithfully flat. Let $H \subset G$ be the image of this morphism, if $G_{\mathcal{C}}$ is the group-scheme associated to $\mathcal{C}$, then we have an H-subtorsor $T' \subset T$ and a commutative diagram of functors

$$\begin{tikzcd} \mathrm{Rep}_k(H) \arrow[r, "F_{T',H}"] & \mathcal{C} \cong \mathrm{Rep}_k(G_{\mathcal{C}}) \\ \mathrm{Rep}_k(G) \arrow[u, "F"] \arrow[ur, "F_{T,G}"'] & \end{tikzcd} .$$

As $F_{T,G}$ is fully faithful, so is F and the essential image of $F_{T',H}$ is closed by sub-objects by Proposition 2.4.146, this allows us to show that the essential image of F is closed by sub-objects too and thus the inclusion $H \subset G$ is faithfully flat to, finishing the proof.
Now let us suppose that $G = \mathrm{Spec}(A)$ is finite, in this case by Proposition 3.3.44 the essential image of $F_{T,G}$ lies with $\mathrm{EF}(X)$ and thus the universal torsor that dominates T is $\hat{X}_{\mathrm{EF}}$, so we have a morphism $\phi : \hat{X}_{\mathrm{EF}} \to T$ of pointed torsors over X.
Starting with (c) $\implies$ (a), if $\Gamma(T, \mathcal{O}_T) = k$ and we suppose that T is not Nori-reduced, then if $H \subset G$ is the image of ϕ, then if we consider A as the regular representation of G as a representation of $\pi^{EF}(X, x)$ using the morphism $\pi^{EF}(X, x) \to G$ we have:

$$\begin{aligned} \Gamma(T, \mathcal{O}_T) &= \Gamma(X, t_*(\mathcal{O}_T)) \\ &= \Gamma(X, F_{T,G}(A)) \\ &= \Gamma(\hat{X}_{\mathrm{EF}}, A \otimes_k \mathcal{O}_{\hat{X}_{\mathrm{EF}}})^{\pi^{EF}(X,x)} \end{aligned}$$

but as $\pi^{EF}(X, x) \to G$ has image H, the fixed sub-algebra of A by the action of $\pi^{EF}(X, x)$ is the same as A^H (Example 2.2.88), and thus, as $\mathcal{O}_{\hat{X}_{\mathrm{EF}}}^{\pi^{EF}(X,x)} = \mathcal{O}_X$ we will obtain that

$$\Gamma(\hat{X}_{\mathrm{EF}}, A \otimes_k \mathcal{O}_{\hat{X}_{\mathrm{EF}}})^{\pi^{EF}(X,x)} = A^H \otimes_k \Gamma(X, \mathcal{O}_X) \cong k^{\oplus \mathrm{ord}(G/H)}$$

and this shows that $\dim_k(\Gamma(T, \mathcal{O}_T)) > 1$ if T is not Nori-reduced.
To finalize the proof, we will show that (a) $\implies$ (c): in this case the morphism of group-schemes $\pi^{EF}(X, x) \to G$ is faithfully flat, in particular we have that $A^{\pi^{EF}(X,x)} = A^G = k$ which shows that $\Gamma(T, \mathcal{O}_T) = k$ by using the the equality $\Gamma(T, \mathcal{O}_T) = \Gamma(\hat{X}_{EF}, A \otimes_k \mathcal{O}_{\hat{X}_{EF}})^{\pi^{EF}(X,x)}$ previously obtained. □

Remark 3.3.55. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. Let $\mathcal{C}$ be a full sub-category Vect(X) such that morphism of objects of $\mathcal{C}$ are vector bundle morphisms and $\mathcal{C}$ is a neutral tannakian category with the fiber functor ω_x, we will denote its corresponding fundamental group-scheme (Definition 2.4.138) as $\pi_{\mathcal{C}}(X, x)$. As X is a quasi-separated scheme[11] over k, we can alternatively say that the universal torsor $\hat{X}_{\mathcal{C}}$ is Nori-reduced by Corollary 2.3.62, as it is the projective limit of Nori-reduced torsors, which are the quotients of finite type of $\pi_{\mathcal{C}}(X, x)$ by Proposition 2.2.97. These correspond to full tannakian sub-categories of $\mathcal{C}$ (Proposition 2.4.146), and thus Nori-reduced torsors using Proposition 3.3.54.*

The fact that universal torsors associated to neutral tannakian categories are Nori-reduced implies a very strong property for morphisms between their respective fundamental group-schemes:

Morphisms between FGS's of neutral tannakian categories of vector bundles in full inclusion are always quotients

Proposition 3.3.56. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. Let $\mathcal{C}$ and $\mathcal{D}$ be neutral tannakian categories of vector bundles over X with vector bundle morphisms with the common fiber functor ω_x. We will denote as $\pi_{\mathcal{C}}(X, x)$ and $\pi_{\mathcal{D}}(X, x)$ the respective associated fundamental group-schemes.*
An exact additive tensor functor $F : \mathcal{C} \to \mathcal{D}$ is a full inclusion of categories if and only if the induced morphism between fundamental group-schemes $\pi_{\mathcal{D}}(X, x) \to \pi_{\mathcal{C}}(X, x)$ is faithfully flat.

Proof. If the morphism $\pi_{\mathcal{D}}(X, x) \to \pi_{\mathcal{C}}(X, x)$ is faithfully flat then clearly we have a full inclusion by Proposition 2.4.146.
On the other hand, the existence of a full inclusion $F : \mathcal{C} \to \mathcal{D}$ induces a morphism between fundamental group-schemes $\pi_{\mathcal{D}}(X, x) \to \pi_{\mathcal{C}}(X, x)$ and by extension, of universal torsors $\hat{F} : \hat{X}_{\mathcal{D}} \to \hat{X}_{\mathcal{C}}$. But as $\hat{X}_{\mathcal{C}}$ is Nori-reduced, $\hat{F}$ is a quotient morphisms of torsors (Definition 2.3.51), in particular the underlying morphism of fundamental group-schemes $\pi_{\mathcal{D}}(X, x) \to \pi_{\mathcal{C}}(X, x)$ is faithfully flat as desired. □

This proposition essentially states that full inclusions of neutral tannakian categories of vector bundles *always* induces faithfully flat morphisms between the associated fundamental group-schemes.
With Proposition 3.3.54, we can finally show that for a proper reduced and connected scheme X over a field k with a rational point $x \in X(k)$

11 In fact, it is separated over k

X with $x \in X(k)$, the fundamental group-schemes $\pi^N(X,x)$ are the canonically isomorphic $\pi^{EF}(X,x)$.

The essentially finite and Nori FGS's coincide

Corollary 3.3.57. *Let X be a proper reduced and connected scheme over a field k with a rational point $x \in X(k)$. Then, the Nori FGS $\pi^N(X,x)$ of X (Definition 3.2.2) is canonically isomorphic to the essentially finite FGS $\pi^{EF}(X,x)$ (Definition 3.3.34).*

Proof. As both these group-schemes are pro-finite, using 2.2.101 we will show that $\pi^N(X,x)$ and $\pi^{EF}(X,x)$ have the same finite quotients. Let us start with the quotients of $\pi^N(X,x)$: any finite quotient $\pi^N(X,x) \to G$ corresponds to a pointed Nori-reduced finite G-torsor $t : T \to X$ by Lemma 3.2.14. Thus, by the characterization of Proposition 3.3.54, the functor of G-invariants $F_{T,G} : \mathrm{Rep}_k(G) \to \mathrm{Vect}(X)$ is fully faithful, and as the essential image of this functor lies within $\mathrm{EF}(X)$ by Proposition 3.3.44. This easily implies that $F_{T,G} : \mathrm{Rep}_k(G) \to \mathrm{EF}(X) \cong \mathrm{Rep}_k(\pi^{EF}(X,x))$ is fully faithful and thus by applying Proposition 2.4.146 we obtain that G is also a quotient of $\pi^{EF}(X,x)$.
On the other hand, any finite quotient $\pi^{EF}(X,x) \to H$ corresponds to a fully faithful k-linear exact tensor functor $F : \mathrm{Rep}_k(H) \to \mathrm{EF}(X)$ that can be thought as a full inclusion of tannakian categories of vector bundles, thus this inclusions corresponds to a pointed H-torsor $\nu : V \to X$ by Corollary 3.3.51 which is Nori-reduced by applying Proposition 3.3.54 once again, thus this corresponds to a quotient $\pi^N(X,x) \to H$ and thus the pro-finite group-schemes $\pi^N(X,x)$ and $\pi^{EF}(X,x)$ are canonically isomorphic. □

Remark 3.3.58. *Keeping the hypotheses of Corollary 3.3.57, as the S-fundamental group-scheme $\pi^S(X,x)$ of X is an affine group-scheme, it is pro-algebraic (see Proposition 2.2.97) and thus any quotient $\pi^S(X,x) \to G$ of finite type over k corresponds using Proposition 3.3.54 to a pointed Nori-reduced algebraic G-torsor $t : T \to X$ (Definition 2.3.17(b)) for which the essential image of $F_{T,G}$ can be identified with a full neutral tannakian subcategory of NSS(X). Moreover, T is an algebraic quotient of the universal torsor $\hat{X}_{NSS}$ associated to $\pi^S(X,x)$ (Remark 3.3.53). It must be noted that $\pi^S(X,x)$ DOES NOT contain in general all pointed Nori-reduced algebraic torsors over X, unlike the case of $\pi^N(X,x)$ that contains all finite pointed Nori-reduced torsors. $\pi^S(X,x)$ just contains a certain directed family of algebraic Nori-reduced torsors that contains at least all the finite ones, and for instance, we will show "larger" fundamental group-schemes in Subsection 3.3.2.*
Also, Corollary 3.3.57 shows that the canonical quotient morphism $\pi^S(X,x) \to \pi^N(X,x)$ mentioned in Remark 3.3.35(b) makes $\pi^N(X,x)$ the maximal pro-finite quotient (Definition 3.2.24) of $\pi^S(X,x)$ as in the description of the last paragraph, finite quotients of $\pi^S(X,x)$ correspond to finite Nori-reduced quotients of $\hat{X}_{NSS}$, which are in turn quotients of the universal torsor $\hat{X} = \hat{X}_{EF}$.

We finish this subsection with some commentary on the pseudo-proper approach for fundamental group-schemes:

The pseudo-proper approach

Remark 3.3.59. *Let X be a quasi-compact (see Definition 2.3.5) scheme over k. We say that X is* pseudo-proper *if for any vector bundle $\mathcal{E}$ over X we have $\Gamma(X, \mathcal{E}) = H^0(X, E)$ is a finitely dimensional k-vector space. Proper schemes over k are clearly pseudo-proper, but as far as the author knows there are no known examples of quasi-compact schemes over k that are pseudo-proper but not proper. An extreme example of the difference proper does to global sections of vector bundles is the projective line $\mathbb{P}^1_k$ which is clearly proper thus pseudo-proper, while $\mathbb{A}^1_k \subset \mathbb{P}^1_k$ is not pseudo-proper as $H^0(\mathbb{A}^1_k, \mathcal{O}_{\mathbb{A}^1_k}) = k[x]$ which is not finitely dimensional over k.*
"Pseudo-properness" can be seen as a technical condition to generalize the theory of the Nori-fundamental group-scheme beyond schemes, not only to classify pointed finite torsors, but also non-pointed ones. In the case of pointed torsors, Definition 3.2.2 shows that schemes suffice, under the right hypotheses[12], but if we fix an affine group-scheme G and we consider the category of all G-torsors over k-schemes, we cannot longer use schemes to get an "universal torsor". Non-pointed G-torsors schemes over k-schemes form a gerbe *which is a special kind of fibered category (see [59, III §2]).*
In [13] N. Borne and A. Vistoli generalized the theory of the Nori fundamental group-schemes to fibered categories, using what is called the "fundamental gerbe" that which is a gerbe $\Pi_{X/k}$ associated to a fibered category over k that classifies all (pro-)finite gerbes over X, in particular the gerbe of (pro-)finite G-torsors over k-schemes. In the terminology of the fundamental gerbe, a fibered category X is inflexible *if X essentially possesses a fundamental gerbe, akin to our Definition 3.2.2. A sufficient condition if we consider object "of finite type over k" is being geometrically connected and geometrically reduced (see [13, Prop. 5.5]), which is comparable to the hypotheses of Proposition 3.2.8.*
There is also a theory of tannakian categories associated to gerbes [17, Ch. 3], and for pseudo-proper inflexible fibered categories the fundamental gerbe is the "tannakian fundamental gerbe" associated to the tannakian category essentially finite bundles, see [13, §7]. In the same section, an equivalent and simplified definition of essentially finite bundles (Definition 3.3.27) can be found: they are the kernels of vector bundle morphisms between finite vector bundles. See [14, Ch. 6] for an exposition on the tannakian theory of the Nori fundamental group-scheme for pseudo-proper geometrically connected and geometrically reduced schemes over k that offers an alternative approach to this subsection. The S-fundamental group-scheme can also be generalized to a gerbe, see [1, §4].

12 The hypotheses present in Proposition 3.2.8 serve as an example.

3.3.2 *New tannakian FGS's for schemes connected by chains of proper curves*

There is a third alternative to conceive tannakian categories of vector bundles over non-proper k-schemes: Instead of considering a technical condition as pseudo-properness, we can impose that any two closed points can be joined by smooth and proper curves, which is key to show that the category of Nori-semistable bundles is abelian and thus neutral tannakian, see the proof of Proposition 3.3.24. And with a larger neutral tannakian category of Nori-semistable bundles, we can easily define essentially finite bundles.
This approach was developed recently by I. Biswas, P.H. Hai and J.P. Dos Santos in [11, §7]. We will outline this approach in this subsection, and generalize it further to allow restricted families of curves when joining two points, which is the needed theoretical base to associate to varieties that are proper and connected by curves of genus g, new fundamental group-schemes that exploit the latter curve related property, we will do this in Section 4.3.
We will obtain once again the S-fundamental group-scheme the and Nori fundamental group-scheme that we conceived for proper schemes in the last subsection, many more fundamental group-schemes and certain properties that we haven't considered so far in order to show them in full generality, like what happens with to a FGS when we change the base rational point, among others.
Throughout this subsection, X will be a reduced and connected scheme of finite type over a perfect field k.

Chains of proper curves and CPC schemes

Definition 3.3.60. *Let X be a k-scheme of finite type over k*[13]*. A* chain of proper curves *on X is a finite family of morphisms $\{\gamma_i : C_i \to X\}_{i=1}^n$ from a finite set of proper and irreducible curves to X, such that the closed subset $\bigcup_{i=1}^n Im(\gamma_i)$ is connected. We will also refer to the set $\bigcup_{i=1}^n Im(\gamma_i)$ as a chain of proper curves.*
We say that X is connected by proper chains*, or* CPC *for short, if any two points of X can be joined by a chain of proper curves. If $\mathscr{C}$ is a non empty family of proper and irreducible curves*[14]*, we say that X is* $\mathscr{C}$-CPC *if any pair of points of X can be joined by a chain of proper curves $\{\gamma_i : C_i \to X\}_{i=1}^n$ with $C_i \in \mathscr{C}$ for all $i = 1..n$.*

We can outline some basic properties of CPC-schemes

Remark 3.3.61. *If X is proper over k then it is CPC by Lemma 3.3.25.*
The property of being CPC can be inherited: if $f : Y \to X$ is a surjective k-morphism and Y is CPC, then X inherits the CPC property, and the same holds for $\mathscr{C}$-CPC schemes.
If X is projective and f is an open embedding with Y connected and big in

13 Here k can be any field, even of characteristic zero.
14 We will not require that curves are smooth when considering families of curves in general.

X, in which case we have that Y is CPC, see [11, pp. 7.2 & 7.3]. Outside the latter example, it is not known if there are other non-proper CPC k-schemes, but for any non-empty family $\mathscr{C}$ of proper and connected curves, the property of being $\mathscr{C}$-CPC is more restrictive than being proper and CPC.

Convention 3.3.62. *From now on, we will focus on $\mathscr{C}$-CPC schemes. Clearly a CPC scheme is a $\mathscr{C}$-CPC scheme where $\mathscr{C} = Curv_k$ is the family of all proper and irreducible curves over k, and thus it is enough to restrict ourselves to $\mathscr{C}$-CPC schemes. In any case, we will mention to the relevant concept for CPC schemes each time we will define a new concept.*
For the CPC case, we will omit the symbol $\mathscr{C}$ when defining objects for $\mathscr{C}$-CPC schemes.

Recall the definition of semi-stale bundle and slope Definition 3.3.16. The first generalization we can make, is to Nori-semistable bundles:

$\mathscr{C}$-Nori-semistable bundles

Definition 3.3.63. *Let $\mathscr{C}$ be a non-empty family of irreducible and proper curves and let X be a $\mathscr{C}$-CPC scheme of finite type over k. We will say that a vector bundle $\mathcal{E}$ over X is* $\mathscr{C}$-Nori-semistable *if the pull-back of $\mathcal{E}$ over any non-constant morphism $g : \hat{C} \to X$, where $\hat{C}$ is the normalization of a curve of the family $\mathscr{C}$, is semi-stable of slope 0.*
In the case $\mathscr{C} = Curv_k$ we will simply say that a vector bundle $\mathcal{E}$ over X is Nori-semistable *and the definition above becomes Definition 3.3.22.*
We will denote the category of $\mathscr{C}$-Nori-semistable bundles as $NSS_{\mathscr{C}}(X)$ and the category of Nori-semistable bundles over X as $NSS(X)$.

For the moment, morphisms between objects in the categories $\mathrm{NSS}_{\mathscr{C}}(X)$ and $\mathrm{NSS}(X)$ are morphisms of $\mathcal{O}_X$-modules, but we will show later that these are vector bundle morphisms like in the proper case (Proposition 3.3.24).

Inclusions of curve families induce full inclusions of NSS categories

Remark 3.3.64. *Let X be a reduced and connected scheme of finite type over k, and let $\mathscr{C}$ and $\mathscr{C}'$ be two non-empty families of proper and irreducible curves.*
If X is both $\mathscr{C}$-CPC and $\mathscr{C}'$-CPC, and all curves of $\mathscr{C}$ belong to $\mathscr{C}'$, then we have a full inclusion of categories $NSS_{\mathscr{C}'}(X) \hookrightarrow NSS_{\mathscr{C}}(X)$. In particular if X is CPC, as any family is contained in $Curv_k$, we always have a full inclusion $NSS(X) \hookrightarrow NSS_{\mathscr{C}}(X)$ for any family $\mathscr{C}$.

Now let us suppose we have a rational point $x \in X(k)$, we can then consider the fiber functor over x $\omega_x : \mathrm{Vect}(X) \to \mathrm{Vectf}_k$ (Definition 3.3.14). We will not use bars to denote restrictions.
The following result is the $\mathscr{C}$-CPC version of Proposition 3.3.24 and Corollary 3.3.33.

Proposition 3.3.65. *Let X be a reduced and connected scheme of finite type over k that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves.*
Then, if X has a rational point $x \in X(k)$, morphism of $\mathscr{C}$-Nori-semistable

bundles are vector bundle morphisms and the category $NSS_{\mathscr{C}}(X)$ is a neutral tannakian category over k with fiber functor ω_x.

Proof. We will start by showing that $\mathrm{NSS}_{\mathscr{C}}(X)$ is abelian and that morphisms of $\mathscr{C}$-Nori-semistable bundles are vector bundle morphisms (Definition 3.3.1): let $\phi : \mathcal{E} \to \mathcal{F}$ be a morphism between two $\mathscr{C}$-Nori-semistable bundles, we need to prove the rank of ϕ over any geometric point of X is constant. This also shows that $\ker(\phi)$ and $\mathrm{coker}(\phi)$ are $\mathscr{C}$-Nori-semistable bundles, see Remark 3.3.2.
Let $\{\gamma_i : C_i \to X\}_{i=1}^n$ be a chain of proper curves with $C_i \in \mathscr{C}$, we can always consider their normalizations, and thus we can suppose that the curves C_i are normalizations of curves belonging to $\mathscr{C}$.
As $SS_{C_i}(\mathcal{O})$ is a k-linear abelian category where all morphisms are vector bundle morphisms (Corollary 3.3.20), for any $i = 1..n$, the rank of the morphism $\gamma_i^*(\phi) : \gamma_i^*(\mathcal{E}) \to \gamma_i^*(\mathcal{F})$ is constant and thus the rank of ϕ is constant along the points of $\bigcup_{i=1}^n \mathrm{Im}(\gamma_i)$ and the rest of the proof is similar to the proof of Proposition 3.3.24.
The fact that $\mathrm{NSS}_{\mathscr{C}}(X)$ is abelian and morphisms are vector bundle morphisms also shows that $\omega_x : \mathrm{NSS}_{\mathscr{C}}(X) \to \mathrm{Vectf}_k$ is faithful, as for any $\mathscr{C}$-Nori-semistable bundle $\mathcal{E}$ we have that

$$H^0(X, \mathcal{E}) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E})$$

is a subspace of $\mathrm{Hom}_{\mathrm{Vectf}_k}(k, \omega_x(\mathcal{E}))$ thus it is finitely dimensional, and in particular we obtain that $H^0(X, \mathcal{O}_X) = k$ which is a required property of a neutral tannakian category. ω_x is also exact by Remark 3.3.15
For the tensor structure, we need to prove that for two $\mathscr{C}$-Nori-semistable bundles $\mathcal{E}, \mathcal{F}$ and $\gamma : C \to X$ where C is the normalization of a curve belonging to $\mathscr{C}$, the tensor product $\gamma^*(\mathcal{E}) \otimes_{\mathcal{O}_C} \gamma^*(\mathcal{F})$ belongs to $SS_C(\mathcal{O})$ which is not a tensor category in general. In this case we have indeed that the tensor product belongs to $SS_C(\mathcal{O})$, as we can easily see that $\gamma^*(\mathcal{E})$ and $\gamma^*(\mathcal{F})$ are strongly semi-stable bundles (Definition 3.3.31), and thus we can use Proposition 3.3.32.
Finally, we remark that $\mathrm{NSS}_{\mathscr{C}}(X)$ is closed by duals and those are reflexive: if $\mathcal{E}$ is a $\mathscr{C}$-Nori-semistable bundle and $\gamma : C \to X$ is a non-constant morphism from a smooth, irreducible and projective curve that is the normalization of a curve of $\mathscr{C}$, then $\gamma^*(\mathcal{E}^\vee) = (\gamma^*(\mathcal{E}))^\vee$, finishing the proof. □

Remark 3.3.66. *If X is a reduced and connected scheme of finite type over k that is $\mathscr{C}$-CPC, the proof above shows that for any $\mathscr{C}$-Nori-semistable bundle $\mathcal{E}$ its set of global sections $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{E})$ is finitely dimensional.*
Thus, we have a weaker version of pseudo-proper vector bundles for $\mathscr{C}$-CPC reduced and connected schemes of finite type over k: $H^0(X, \mathcal{E})$ is finitely dimensional for any $\mathscr{C}$-Nori-semistable bundle $\mathcal{E}$ while the pseudo-proper property holds for any locally free sheaf over X.

This will allows us to develop a theory of essentially finite bundles later, in a parallel manner to the pseudo-proper approach (Remark 3.3.59).

Proposition 3.3.65 allows us to define new fundamental group-schemes:

(S,$\mathscr{C}$)-fundamental group-scheme

Definition 3.3.67. *Let X be a reduced and connected scheme of finite type over k that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves. Let $x \in X(k)$ be a rational point, we define the* (S,$\mathscr{C}$)-fundamental group-scheme *as the fundamental group-scheme $\pi^S_{\mathscr{C}}(X,x)$ associated to $NSS_{\mathscr{C}}(X)$ with fiber functor ω_x. If $\mathscr{C} = Curv_k$, we will denote the fundamental group-scheme associated to $NSS(X)$ with fiber functor ω_x as $\pi^S(X,x)$ and we will call it the* S-fundamental group-scheme.

The S-fundamental group-scheme defined above is essentially the same as the one defined in Definition 3.3.34.

Remark 3.3.68. *Let X and Y be reduced and connected schemes of finite type over k, with rational points $x \in X(k)$ and $y \in Y(k)$. And let $\mathscr{C}$ and $\mathscr{C}'$ be two non-empty families of proper and irreducible curves. Then:*

(a) If X is both $\mathscr{C}$-CPC and $\mathscr{C}'$-CPC, and $\mathscr{C}' \subset \mathscr{C}$, the full inclusion of categories $NSS_{\mathscr{C}}(X) \hookrightarrow NSS_{\mathscr{C}'}(X)$ mentioned in Remark 3.3.64 induces a morphism of fundamental group-schemes $\pi^S_{\mathscr{C}'}(X,x) \to \pi^S_{\mathscr{C}}(X,x)$.
By Proposition 3.3.56 this morphism of group-schemes is faithfully flat.

(b) If both X and Y are $\mathscr{C}$-CPC and $\mathscr{C}'$-CPC. Let $f : Y \to X$ is a morphism of k-schemes compatible with the respective rational points, as both schemes have (S,$\mathscr{C}'$)-fundamental group-schemes and a (S,$\mathscr{C}$)-fundamental group-scheme, the pull-back of vector bundles induces morphisms

$$\pi^S_{\mathscr{C}}(f) : \pi^S_{\mathscr{C}}(Y,y) \to \pi^S_{\mathscr{C}}(X,x) \quad \textit{and} \quad \pi^S_{\mathscr{C}'}(f) : \pi^S_{\mathscr{C}'}(Y,y) \to \pi^S_{\mathscr{C}'}(X,x).$$

Both morphism come from the pull-back functor

$$f^* : NSS_{\mathscr{C}}(X) \to NSS_{\mathscr{C}}(Y) \quad \textit{and} \quad f^* : NSS_{\mathscr{C}'}(X) \to NSS_{\mathscr{C}'}(Y)$$

that is a tensor functor that preserves the respective fiber functors.
The pull-back f^* commutes with the respective inclusions of categories, and thus we have a commutative diagram with faithfully flat vertical morphisms

$$\begin{tikzcd} \pi^S_{\mathscr{C}'}(Y,y) \arrow[r, "\pi^S_{\mathscr{C}'}(f)"] \arrow[d] & \pi^S_{\mathscr{C}'}(X,x) \arrow[d] \\ \pi^S_{\mathscr{C}}(Y,y) \arrow[r, "\pi^S_{\mathscr{C}}(f)"'] & \pi^S_{\mathscr{C}}(X,x) \end{tikzcd}$$

coming from the corresponding commutative diagram at the level of tannakian categories of vector bundles which shows that the morphism $\pi^S_{\mathscr{C}'}(X,x) \to \pi^S_{\mathscr{C}}(X,x)$ mentioned in point (a) is natural.

(c) *Applying this to Nori-semistable bundles, we have a faithfully flat morphism of fundamental group-schemes* $\pi^{S}_{\mathscr{C}}(X, x) \to \pi^{S}(X, x)$ *such that for any morphism of* k*-schemes* $f : Y \to X$ *compatible with the respective rational points the diagram*

$$\begin{tikzcd} \pi^{S}_{\mathscr{C}'}(Y, y) \arrow[r, "\pi^{S}_{\mathscr{C}}(f)"] \arrow[d] & \pi^{S}_{\mathscr{C}'}(X, x) \arrow[d] \\ \pi^{S}(Y, y) \arrow[r, "\pi^{S}(f)"'] & \pi^{S}(X, x) \end{tikzcd}$$

is commutative with faithfully flat columns, thus we have a natural transformation.

Now we will introduce essentially finite bundles, for this purpose we will generalize the concept of essentially finite bundles to suitable categories that resemble $\mathrm{Vect}(X)$: Let $\mathcal{T}$ be a small k-linear abelian tensor category with finitely dimensional Hom-sets. The isomorphism classes of objects of $\mathcal{T}$ form a set with a natural structure of a semi-ring with the direct product as the addition (which lacks an additive inverse) and the tensor product as the multiplication.
Following Nori in [53, I §2.3], we can consider the ring $K(\mathcal{T})$ which is the monoid of fractions for the addition of the semi-ring we defined before, we must remark it is not the Grothendieck ring as we are not expressing short exact sequences as sums.
Let us suppose that $\mathcal{T}$ is an Krull-Schmidt category (Definition 3.3.8(c)), in this case the Lemma 3.3.9 holds for $\mathcal{T}$: any object of $\mathcal{T}$ can be decomposed as a direct sum of indecomposable sub-objects that it is unique modulo permutation of the direct summands. In this case, the set of isomorphism classes of indecomposable objects is an additive base of $K(\mathcal{T})$.
Over $\mathcal{T}$, integer polynomial evaluations of any object makes sense (see Definition 2.4.69(b)), so we can define:

Finite and essentially finite objects

Definition 3.3.69. *Let* $\mathcal{T}$ *be a small* k*-linear abelian tensor category with finitely dimensional Hom-sets that is also a Krull-Schimdt category.*

(a) *An object* V *of* $\mathcal{T}$ *is* finite *if there exist two different polynomials* $p(x), q(x) \in \mathbb{Z}_{\geqslant 0}$ *such that* $p(V) \cong q(V)$.

(b) *An object* W *of* $\mathcal{T}$ *is* essentially finite *if* W *is a sub-quotient (see Definition 2.4.69(a)) of a finite object. The full subcategory of* $\mathcal{T}$ *which consists of the essentially finite objects will be denoted as* $EF(\mathcal{T})$.

Remark 3.3.70. *Let* $\mathcal{T}$ *be a small* k*-linear abelian tensor category with finitely dimensional Hom-sets that is also a Krull-Schmidt category. Finite objects can be characterized as being integral over the minimal prime ring of* $K(\mathcal{T})$*, in particular by applying known results for integral elements and integral ring extensions [47, Theorem 9.1], we can see that tensor products,*

duals, direct sums and direct summands of finite objects are finite, as it was the case for finite bundles (Lemma 3.3.13). Moreover, finite objects of Vect(X) *as defined in Definition 3.3.69 coincide with the previous definition of finite bundles (Definition 3.3.6).*
Finally, as an alternative equivalent definition for essentially finite object, we can say that W *is essentially finite if and only if it is the kernel of a morphism* $\phi : V \to V'$ *between finite objects of* $\mathcal{T}$*, following the pseudo-proper approach (Remark 3.3.59).*

Before redefining essentially finite bundles for $\mathscr{C}$-CPC schemes, we will state some general properties of this construction when applied to neutral tannakian categories. If G is an affine group-scheme, it is not hard to see that $\mathrm{Rep}_k(G)$ is clearly a Krull-Schmidt category, thus we can study its essentially finite elements:

Essentially finite elements in neutral tannakian categories

Proposition 3.3.71 (Prop. 7.12 [11]). *Let* G *be an affine group-scheme over* k*, we will denote as* EF_G *the category* $EF(\mathrm{Rep}_k(G))$*. Then, we have:*

(a) The category $EF(\mathcal{G})$ *is closed under sub-objects, quotients, tensor products and duals.*

(b) G *is pro-finite if and only if* $EF_G = \mathrm{Rep}_k(G)$*.*

(c) Let $\omega_G^{EF} : EF_G \to \mathrm{Vect}(k)$ *be the restriction of the forgetful functor* ω_G *to* EF_G*. Then* EF_G *is a neutral tannakian category with* ω_G^{EF} *as the fiber functor. If* G^{EF} *is its associated group-scheme, it is pro-finite.*

(d) Moreover, if $\phi : G \to G^{EF}$ *is the morphism of group-schemes associated to the full inclusion functor* $EF_G \to \mathrm{Rep}_k(G)$*, then* ϕ *is faithfully flat and it makes* G^{EF} *the maximal pro-finite quotient (Definition 3.2.24) of* G*.*

Proof. Parts (a) and (c) are easy verifications, similar to the verifications present in Corollary 3.3.33.
For part (b), let us start with the case when G is finite. In this case, if we write $G = \mathrm{Spec}(A)$ and we consider A as the regular representation of G, then the multiplication morphism $m : G \times_k G \to G$ of G is clearly a G-torsor, thus applying Example 3.3.7 we see that $A \otimes_k A \cong A^{\oplus n}$ where $n = \mathrm{ord}(G)$ in particular A is finite, thus we clearly have $EF_G = \mathrm{Rep}_k(G)$ by applying Lemma 2.4.142. If G is now pro-finite, as any quotient of G is finite, then any object of $\mathrm{Rep}_k(G)$ lies $\mathrm{Rep}_k(Q)$ where $G \to Q$ is a finite quotient of G, thus we obtain the same conclusion as in the finite case.
Now let us suppose that G is just affine and $EF_G = \mathrm{Rep}_k(G)$, it suffices to show that any quotient $\phi : G \to Q$, that we can suppose to be of finite type over k by Proposition 2.2.97 and Corollary 2.2.98. In this case $\mathrm{Rep}_k(Q)$ is tensor generated by V and $V^\vee$ (Definition 2.4.69) by Proposition 2.4.144, where V is a faithful representation of Q (Corollary 2.4.88). Now ϕ induces a tensor functor $\phi^* : \mathrm{Rep}_k(Q) \to \mathrm{Rep}_k(G)$ and if we consider $\phi^*(V)$, as this element is

essentially finite, we have that $\phi^*(V)$ is a sub-quotient of a finite object W of $\mathrm{Rep}_k(G)$ and it is not hard to see that the same holds for $\phi^*(V^\vee)$, in particular $\phi^*(V)$ and $\phi^*(V^\vee)$ belong to $\langle W\rangle^\otimes$ the neutral tannakian category generated by W. Thus, as V and $V^\vee$ are tensor generators and the essential image of ϕ^* is closed by sub-objects (Proposition 2.4.146), then we conclude that $\mathrm{Rep}_k(Q) \subset \langle W\rangle^\otimes$ and as the larger neutral tannakian category corresponds to a finite group-scheme by Corollary 2.4.143, we conclude that H is finite as we wanted.
Finally, for part (d), it is clear that $\mathrm{EF}(\mathrm{EF}_G) = \mathrm{EF}_G$ thus G^{EF} is profinite by part (b) and for any finite quotient H of G, as $\mathrm{Rep}_k(H)$ is the full sub-category of $\mathrm{Rep}_k(G)$ composed of sub-quotients of a finite representation, we easily conclude that we have a finer full inclusion $\mathrm{Rep}_k(H) \subset \mathrm{EF}_G$ which effectively shows (d). □

Now we can define essentially finite bundles by applying Definition 3.3.69 to the categories of $\mathscr{C}$-Nori-semistable bundles of Definition 3.3.63:

Essentially finite bundles for $\mathscr{C}$-CPC schemes

Definition 3.3.72. *Let X be a reduced and connected scheme of finite type over k that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves. A vector bundle $\mathcal{E}$ belonging to $EF(NSS_{\mathscr{C}}(X))$ will be called a $\mathscr{C}$-*essentially finite *bundle. In the particular case of $NSS(X)$, a bundle of $EF(NSS(X))$ is called an* essentially finite *bundle.*
We will also denote the categories $EF(NSS(X))$ and $EF(NSS_{\mathscr{C}}(X))$ as $EF(X)$ and $EF_{\mathscr{C}}(X)$ respectively.

It is clear that finite and essentially finite bundles of $\mathrm{NSS}(X)$ as defined using Definition 3.3.69 are just the same bundles that we defined in Definitions 3.3.6 and 3.3.27 in the approach where X is proper.
Moreover, the definition of $\mathscr{C}$-essentially finite bundles is superfluous:

Lemma 3.3.73. *Let X be a reduced and connected scheme of finite type over k that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves. Then, $EF_{\mathscr{C}}(X) = EF(X)$ and there is just one single category of essentially finite bundles.*

Proof. This stems from the fact that a finite $\mathscr{C}$-Nori-semistable bundle $\mathcal{E}$ is simply finite as in Definition 3.3.6, and in particular $\mathcal{E}$ is always Nori-semistable by Lemma 3.3.21, thus the natural full inclusion $\mathrm{EF}(X) \subset \mathrm{EF}_{\mathscr{C}}(X)$ that comes from the inclusion in Remark 3.3.64 is an equality. □

EF Fundamental group-scheme for $\mathscr{C}$-CPC schemes

Definition 3.3.74. *Let X be a reduced and connected scheme of finite type over k with a rational point $x \in X(k)$ that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves. We will call the group-scheme associated to the tannakian category $EF(X)$ together with the functor ω_x the* essentially finite fundamental group-scheme *of X and we will denote it by $\pi^N(X, x)$.*

Recall that by Remark 3.3.53 and Corollary 3.3.51 that the fundamental group-schemes $\pi^S_{\mathscr{C}}(X,x)$, $\pi^S(X,x)$ and $\pi^N(X,x)$ have universal torsors associated to them, and the group-scheme quotients of these FGS's correspond to pointed Nori-reduced torsors over X. Using this together with Proposition 3.3.71, we have:

Lemma 3.3.75. *Let X be a reduced and connected scheme of finite type over k with a rational point $x \in X(k)$ that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves. Then, $\pi^N(X,x)$ is pro-finite and it is the maximal pro-finite quotient of both $\pi^S_{\mathscr{C}}(X,x)$ and $\pi^S(X,x)$, moreover it coincides with the Nori fundamental group-scheme of Definition 3.2.2*

Proof. The first assertion of the statement is a direct consequence of Proposition 3.3.71(c) & (d) while the second one is simply the content of Corollary 3.3.57. □

Now recall that for a smooth and proper curve C over k, by Proposition 3.3.37 the category $NSS(C)$ coincides with the tannakian category of strongly semi-stable bundles over C (Definition 3.3.31), this allows us to consider a new kind of FGS:

The tannakian category of "EF when pulled-back" bundles

Proposition 3.3.76. *Let X be a reduced and connected scheme of finite type over k with a rational point $x \in X(k)$ that is $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves.*
Let PB-EF$_{\mathscr{C}}(X)$ be the full sub-category of vector bundles $\mathcal{E}$ of NSS$_{\mathscr{C}}(X)$ such that for any curve C, that is the normalization of a curve of the family $\mathscr{C}$, together with a non-constant morphism $f : C \to X$ the pull-back bundle $f^*(\mathcal{E})$ belongs to EF(C), then PB-EF$_{\mathscr{C}}(X)$ with ω_x is a neutral tannakian category over k.
If $\pi^{PB}_{\mathscr{C}}(X,x)$ is the affine group-scheme associated with PB-EF$_{\mathscr{C}}(X)$, we have a full inclusion of categories EF$(X) \to$ PB-EF$_{\mathscr{C}}(X)$ inducing a faithfully flat morphism of group-schemes $\pi^{PB}_{\mathscr{C}}(X,x) \to \pi^N(X,x)$. Moreover, we have EF$(X) =$ EF (PB-EF$_{\mathscr{C}}(X)$).

Proof. It is not hard to see PB-EF$_{\mathscr{C}}(X)$ is closed under direct sums, tensor products, duals, kernels and cokernels, which shows these categories are tannakian with ω_x. The rest of the properties outlined in the statement are easily verified. □

Now we have defined all of the types of fundamental group-schemes, so we will add this remark to recapitulate their relationships:

Relationships between all the fundamental group-schemes

Remark 3.3.77. *Let X be a reduced and connected scheme of finite type over k, with a rational point $x \in X(k)$. And let $\mathscr{C}$ and $\mathscr{C}'$ be two non-empty families of proper and irreducible curves.*
We note that the category PB-EF$_{\mathscr{C}}(X)$ is a priori "larger" than the category EF$_{\mathscr{C}}(X)$. For the specific case of $\mathscr{C} = Curv_k$, we will drop the subscript $\mathscr{C}$ and thus we have the tannakian category PB-EF(X) whose associated group-scheme will be denoted as $\pi^{PB}(X,x)$.
Now let us suppose that X is both $\mathscr{C}$-CPC and $\mathscr{C}'$-CPC, for any inclusion of

families of curves $\mathscr{C}' \subset \mathscr{C}$, we have the following commutative diagram of group-schemes where all arrows are faithfully flat:

$$\begin{tikzcd}
\pi^S_{\mathscr{C}'}(X,x) \arrow[r] \arrow[d] & \pi^{PB}_{\mathscr{C}'}(X,x) \arrow[d] \arrow[dr] & \\
\pi^S_{\mathscr{C}}(X,x) \arrow[r] \arrow[d] & \pi^{PB}_{\mathscr{C}}(X,x) \arrow[r] \arrow[d] & \pi^N(X,x) \\
\pi^S(X,x) \arrow[r] & \pi^{PB}(X,x) \arrow[ur] &
\end{tikzcd}$$

These arrows come from the associated full inclusions of their respective tannakian categories of bundles over X, and they are all faithfully flat by Proposition 3.3.56.
If Y is another k-scheme with the same properties as X that has a rational point $y \in Y(k)$. For any $f : Y \to X$ morphism of k-schemes compatible with the respective rational points we have induces morphisms between fundamental group-schemes

$$\begin{aligned}
\pi^S_{\mathscr{C}}(f) : \quad \pi^S_{\mathscr{C}}(Y,y) &\to \pi^S_{\mathscr{C}}(X,x) \\
\pi^S_{\mathscr{C}'}(f) : \quad \pi^S_{\mathscr{C}'}(Y,y) &\to \pi^S_{\mathscr{C}'}(X,x) \\
\pi^S(f) : \quad \pi^S(Y,y) &\to \pi^S(X,x) \\
\pi^{PB}_{\mathscr{C}}(f) : \quad \pi^{PB}_{\mathscr{C}}(Y,y) &\to \pi^{PB}_{\mathscr{C}}(X,x) \\
\pi^{PB}_{\mathscr{C}'}(f) : \quad \pi^{PB}_{\mathscr{C}'}(Y,y) &\to \pi^{PB}_{\mathscr{C}'}(X,x) \\
\pi^{PB}(f) : \quad \pi^{PB}(Y,y) &\to \pi^{PB}(X,x) \\
\pi^N(f) : \quad \pi^N(Y,y) &\to \pi^N(X,x)
\end{aligned}$$

that commute with all the respective morphisms in the analogue diagram above for the fundamental group-schemes of Y and those over X, showing that all these fundamental group-schemes are functorial for morphisms compatible with the respective rational points, and that the faithfully flat morphisms between them over a fixed scheme are natural.

To finish this subsection, we will outline what happens to a tannakian fundamental group-scheme of vector bundles changes when we change rational points. For this purpose, we first need the following definition:

Definition 3.3.78. *Let G be a group-scheme of finite type over a field k.*

(a) *If $g \in G(k)$, then g induces a group-scheme automorphism $G \to G$ of G given for $x \in \widetilde{G}(R)$ with R a k-algebra, as $x \mapsto g_R x g_R^{-1}$ where g_R is the image of g via the canonical morphism $k \to R$. We will denote this automorphism of G as $inn(g)$.* *Inner automorphism induced by a k-point*

(b) *An* inner automorphism of G *is an automorphism $f : G \to G$ that after changing the base field to $\bar{k}$, the algebraic closure of k, becomes* *Inner automorphisms* *inn*$(\bar{g})$ *for some element* $\bar{g} \in G(\bar{k})$. *In other words, internal automorphisms become inner automorphisms by conjugation over the algebraic closure of* k.

Inner forms

(c) *An* inner form *of* G *is another group-scheme* G' *of finite type over* k, *such that over* $\bar{k}$ *the base change* $G'_{\bar{k}}$ *is isomorphic to* $G_{\bar{k}}$ *via an inner automorphism inn*$(\bar{g})$ *of* $G_{\bar{k}}$ *for some* $\bar{g} \in G(\bar{k})$.

Remark 3.3.79. *Let* G *be a group-scheme of finite type over a field* k. *As in the case of abstract groups, we can define the* center of G $Z \subset G$ *which functions similarly to its abstract group counterpart, see [49, p. 34]. Internal automorphisms of* G *are determined by* $Z(\bar{k})$ *where* Z *is the center of* G, *thus, if* G *is commutative any inner form is is equal to* G *itself [49, p. 17.63].*

Change of base point for tannakian categories of vector bundles

Proposition 3.3.80. *Let* X *be a reduced and connected scheme of finite type over a field* k. *If* $x, y \in X(k)$ *are rational points of* X *and* G *is an affine group-scheme associated to a tannakian sub-category* $\mathcal{D}$ *of Vect*(X) *with fiber functor* ω_x. *Then* $\mathcal{D}$ *with* ω_y *is also a tannakian category, and if* G' *is its associated group-scheme, then* G' *is an inner form of* G. *In particular, if* k *is algebraically closed, these group-schemes are isomorphic and differ by an inner automorphism of* G *or they coincide if* G *is commutative.*

Proof. As G is affine, it is pro-algebraic by Proposition 2.2.97. Let G_i be a quotient of finite type of G, it corresponds to a full neutral tannakian sub-category of $\mathcal{D}$ and also to a pointed (over x) Nori-reduced G_i-torsor by applying Corollary 3.3.51.
Nori showed in [53, p. 88] that if $T_i \to X$ is the pointed (over x) Nori-reduced G_i-torsor, we can construct a G'_i-torsor $T'_i \to X$ pointed over y where G'_i is an inner form of G_i (see also [22, §2.1]). This induces an equivalence of tannakian categories between $\mathcal{D}_i$ with ω_x and $\mathcal{D}_i$ with ω_y. Then, passing to the projective limit yields the analogous equivalence for $\mathcal{D}$ with the two different fiber functors, concluding the proof by considering the associated group-schemes. □

Remark 3.3.81. *If* k *is algebraically closed, by Proposition 3.3.80, we will sometimes omit rational points when writing fundamental group-schemes as different rational points yield isomorphism fundamental group-schemes. So for example, write fundamental group-schemes as* $\pi^N(X)$ *or* $\pi^S(X)$ *or* $\pi^N_{\mathscr{C}}(X)$, *etc.*

3.4 ADVANCED PROPERTIES AND RESULTS

In this final section we will list many miscellaneous results, most of them for the Nori fundamental group-scheme with some related to the S-fundamental group-scheme, that we will use later, specially in Chapter 5.
We will state them here without proof, with the corresponding citation, in Subsection 3.4.1. In Subsection 3.4.2 we will also outline some

results that describe the specific properties of the FGS for certain types of schemes, the main two types of schemes for which we will describe their FGS are rationally connected varieties and abelian varieties.

3.4.1 *Properties and results*

We will divide this subsection in several subsubsections to group the results thematically.
To ease navigation, we will list the subsubsections here in sequential order:

- Subsubsection 3.4.1.1: "Induced morphisms of FGS's for open immersions and birational invariance".
- Subsubsection 3.4.1.2: "Global sections of essentially finite bundles and Grauert's theorem for finite torsors".
- Subsubsection 3.4.1.3: "Faithfully flat induced morphisms of FGS's and isomorphisms of FGS's".
- Subsubsection 3.4.1.4: "Towers of torsors and FGS of Nori-reduced torsors".

3.4.1.1 *Induced morphisms of FGS's for open immersions and birational invariance*

Let us start our list of results with open immersions:

Induced morphism of FGS over open immersions

Proposition 3.4.1 (p. 90 [53]). *Let X be a normal, connected, reduced and of finite type over a field k. Let $i : U \to X$ be an open immersion, then for rational points $u \in U(k)$ and $x \in X(k)$ that are compatible under i, the induced morphisms between fundamental group-schemes*

$$\pi^N(i) : \pi^N(U, u) \to \pi^N(X, x)$$

is faithfully flat.
Moreover, if i is dominant and U is big, *i.e, we have codim*[15]$(X - U) \geqslant 2$ *, then $\pi^N(i)$ is an isomorphism.*

Now we will outline how the Nori and S-fundamental group-schemes behave under birational equivalence:

Birational invariance for the Nori and S FGS's

Proposition 3.4.2. *Let X and Y be smooth and projective varieties over an algebraically closed field k. If X and Y are birationally equivalent, for compatible rational points $x \in X(k)$ and $y \in Y(k)$ we have a isomorphisms of fundamental group-schemes*

$$\begin{aligned} \pi^N(X, x) &\cong \pi^N(Y, y) \\ \pi^S(X, x) &\cong \pi^S(Y, y). \end{aligned}$$

15 See [36, Definition p.86] for a definition of codimension.

Proof. The statement for the Nori fundamental group-scheme can be found in [53, Ch. II Prop. 8] while the birational invariance for the S-fundamental group-scheme can be found in [37]. □

3.4.1.2 *Global sections of essentially finite bundles and Grauert's theorem for finite torsors*

Now we will state some results about global sections of essentially finite bundles, so we could use the semi-continuity theorem and Grauert's theorem for vector bundles associated to pointed finite torsors over proper and flat morphisms $f : X \to Y$ compatible with the respective rational points in Subsubsection 3.4.1.3. For this, we need to study global sections of essentially finite bundles. Let us start with a general result for global sections of essentially finite bundles:

Lemma 3.4.3 (Lemma 2.2 [69])**.** *Let X be a reduced and connected scheme over a field k, with a rational point $x \in X(k)$, that is either proper or $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves.*
If $\mathcal{E}$ is an essentially finite bundle over X, then the natural morphism

$$\Gamma(X, \mathcal{E}) \otimes_k \mathcal{O}_X \to \mathcal{E}$$

is an embedding that makes $\Gamma(X, \mathcal{E}) \otimes \mathcal{O}_X$ the maximal trivial sub-bundle of $\mathcal{E}$.

From this lemma, we can deduce the following as a corollary:

Corollary 3.4.4. *Under the hypotheses of Lemma 3.4.3. If $\mathcal{E}$ is an essentially finite bundle over X of rank $r \geqslant 1$, then*

$$1 \leqslant \dim_k(H^0(X, \mathcal{E})) \leqslant r.$$

Moreover, $\mathcal{E}$ is trivial if and only if $\dim_k(H^0(X, \mathcal{E})) = r$.

Lemma 3.4.3 also allows us to show:

Global sections of vector bundles associated to finite torsors

Lemma 3.4.5. *Let X be a reduced and connected scheme over a field k, with a rational point $x \in X(k)$, that is either proper or $\mathscr{C}$-CPC, where $\mathscr{C}$ is a non-empty family of irreducible and proper curves. If $t : T \to X$ is a finite pointed torsor, corresponding to the morphism $\pi^N(X, x) \to G$ and $H \subset G$ is the image of this morphism, then*

$$\dim_k \left(H^0(X, t_*(\mathcal{O}_T))\right) = \mathit{ord}(G/H).$$

Proof. By Corollary 3.4.4 and Proposition 3.3.54, we can suppose that T is neither trivial nor Nori-reduced, as the statement is already true in those cases, so we will suppose that $H \subset G$ is a non-trivial proper subgroup-scheme of G.
Let $F_{T,G} : \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(\pi_1^N(X, x)) \cong \mathrm{EF}(X)$ be the functor of G-invariants (Definition 3.3.41) associated to T. We know that $t_*(\mathcal{O}_T)$ is the image of ρ_G via this functor (Example 3.3.43). Now, if $G =$

$\mathrm{Spec}(A)$ where A is a Hopf algebra over k, we will write $H = \mathrm{Spec}(A/I)$ where I is a Hopf ideal. The quotient morphism $A \to A/I$ induces a comodule structure over A, $\tau_H : A \to A \otimes A/I$ so A is a representation of H, see Example 2.4.85(5). Thus, we see that in the factorization $\mathrm{Rep}_k(G) \to \mathrm{Rep}_k(H) \to \mathrm{Rep}_k(\pi_1^N(X,x))$ the representation $\tau_H \in \mathrm{Rep}_k(H)$ maps to $t_*(\mathcal{O}_T)$ as well.
Now let $U \subset t_*(\mathcal{O}_T)$ be the maximal trivial sub-bundle, as $\pi_1^N(X,x) \to H$ is faithfully flat, using Proposition 2.4.146, we can find a trivial subrepresentation of τ_H whose image in $\mathrm{Rep}_k(\pi_1^N(X,x))$ is isomorphic to U. We can easily see that this the largest sub-representation of τ_H with a trivial action of H and thus it corresponds to A^H (Example 2.4.85(4)), finishing the proof. □

Remark 3.4.6. *If H is a normal subgroup-scheme of G, then the quotient T/H is a trivial G/H-torsor and thus, if $q : T/H \to X$ is the canonical morphism form the quotient T/H of T by H, we have that $q_*(\mathcal{O}_{T/H}) \cong \mathcal{O}_X^{\oplus ord(G/H)}$ and this bundle is the maximal trivial sub-bundle of $t_*(\mathcal{O}_T)$*

Now let X and Y be two connected and reduced schemes of finite type over k, both with rational points $x \in X(k)$ and $y \in Y(k)$. If $f : X \to Y$ is a proper and flat morphism compatible with the respective rational points, and $t : T \to X$ is a finite pointed G-torsor over X, then the semi-continuity theorem holds [50, Corollary p.50] for the vector bundle $t_*(\mathcal{O}_X)$, and by applying Grauert's theorem in Corollary 2 p.50 in loc.cit. we obtain:

Grauert's theorem for finite torsors

Proposition 3.4.7. *Let X and Y be two connected and reduced schemes of finite type over k, both with rational points $x \in X(k)$ and $y \in Y(k)$. If $f : X \to Y$ is a proper and flat morphism compatible with the respective rational points, and $t : T \to X$ is a finite pointed G-torsor over X such that for any point $y \in Y$ the pull-back of T to the fiber X_y is a trivial torsor.*
Then, there exists a pointed finite G-torsor $t' : T' \to Y$ such that T is isomorphic to the pull-back $T'_X = T' \times_Y X$.
Moreover, if we replace $t_*(\mathcal{O}_T)$ by an essentially finite bundle $\mathcal{E}$ over X with the same property as above, then $f^*(f_*(\mathcal{E})) \in EF(X)$.

Proof. This proof is due to Nori, see the proof of [53, Ch.II Prop 9]. In this case by applying Grauert's theorem we have that $\mathcal{F} = f_*(t_*(\mathcal{O}_T))$ is locally free and $f^*(\mathcal{F}) \cong t_*(\mathcal{O}_T)$. If $t' : T' \to Y$ is the finite flat morphism with $(t')_*(\mathcal{O}_{T'}) = \mathcal{F}$, then we have that the composition $T \to T' \xrightarrow{t'} Y$ is the Stein factorization (see [36, II Coro. 11.5] or [63, Tag 03H0][16].) of the composition $T \xrightarrow{t} X \xrightarrow{f} Y$.
Finally, by applying the Stein factorization to the action morphism $\mu_T : T \times_X G \to T$ and the isomorphism $T \times_X G \cong T \times_X T$ and the fact that the Stein factorization commutes with base change, we conclude

16 https://stacks.math.columbia.edu/tag/03H0

that $t' : T' \to Y$ is a pointed G-torsor whose pull-back to X is isomorphic to T.
In the case of an essentially finite bundle, Grauert's theorem shows that $f^*(f_*(\mathcal{E})) \cong \mathcal{E}$ as we have a cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "g"] \arrow[d, "u"'] & Z' \arrow[d, "u'"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

with finite and flat vertical arrows and $u_*(\mathcal{O}_Z) = \mathcal{E}$ and $(u')_*(\mathcal{O}_{Z'}) = f_*(\mathcal{E})$, so [36, II Prop. 9.3] applies as f is flat. □

Remark 3.4.8. *In Chapter 5 we will use Proposition 3.4.7 in a specific setting: Let X and Y be two proper varieties over k, both with rational points $x \in X(k)$ and $y \in Y(k)$.*
Let $f : X \to Y$ is a proper and flat morphism with reduced and connected geometric fibers, compatible with the respective rational points, we will denote the generic point of Y as $\eta : \mathrm{Spec}(L) \to Y$ and the geometric generic fiber of X as $X_{\bar{\eta}}$ where $\bar{\eta}$ denotes the geometric generic point of Y. If $t : T \to X$ a finite pointed G-torsor such that $T_{\bar{\eta}} \to X_{\bar{\eta}}$, its pull-back to $X_{\bar{\eta}}$, is a trivial torsor, then T is the pull-back of a G-torsor over Y.
This is a direct consequence of Grauert's theorem for finite torsors as in this case, if r is the rank of $t_*(\mathcal{O}_T)$, then by Corollary 3.4.4 we have that

$$\dim_k \left(H^0\left(X_{\bar{\eta}}, t_*(\mathcal{O}_T)|_{X_{\bar{\eta}}}\right)\right) = r$$

thus the same equality holds for the generic fiber X_η, and by semi-continuity we can easily deduce that for any point $y \in Y$, we will also have dimension r for the global sections of $t_*(\mathcal{O}_T)|_{X_y}$ and thus we conclude that Proposition 3.4.7 applies in this case. For an essentially finite vector bundle $\mathcal{E}$ with trivial restriction to $X_{\bar{\eta}}$, Proposition 3.4.7 shows that $f^*(f_*(\mathcal{E})) \in EF(X)$.

3.4.1.3 *Faithfully flat induced morphisms of FGS's and isomorphisms of FGS's*

Now we will show a sufficient condition for an induced morphism of fundamental group-schemes to be faithfully flat, in the proper case:

Sufficient condition for a faithfully flat morphism of FGS's

Proposition 3.4.9. *Let X and Y be proper, reduced and connected schemes over a field k, both with rational points $x \in X(k)$ and $y \in Y(k)$. If $f : X \to Y$ is a faithfully flat morphisms with $f_*(\mathcal{O}_X) = \mathcal{O}_Y$[17] that is compatible with the respective rational points, then the induced morphisms of fundamental group-schemes*

$$\begin{aligned} \pi^S(f) : \pi^S(X,x) &\to \pi^S(Y,y) \\ \pi^N(f) : \pi^N(X,x) &\to \pi^N(Y,y). \end{aligned}$$

are both faithfully flat.

17 This is equivalent to assume that the fibers of f are geometrically connected.

Proof. For the S-fundamental group-scheme, this is [42, Lemma 8.1]. Thus, the conclusion for the Nori fundamental group-schemes derives from the commutative diagram

$$\begin{tikzcd} \pi^S(X,x) \arrow[r, "\pi^S(f)"] \arrow[d] & \pi^S(Y,y) \arrow[d] \\ \pi^N(X,x) \arrow[r, "\pi^N(f)"'] & \pi^N(Y,y) \end{tikzcd}$$

with faithfully flat vertical arrows and upper horizontal arrow, that comes from combining Remark 3.3.35(b) & (c). ☐

Remark 3.4.10. *An older version of Proposition 3.4.9 that has stronger hypotheses and only holds for the Nori fundamental group-scheme can be found in the first corollary in [53, p. 90].*
In the proof of [42, Lemma 8.1], Langer used the projection formula ([36, II Excercise 5.1]) to show that the pull-back functor $f^* : NSS(Y) \to NSS(X)$ is fully faithful, thus using the stronger version of Proposition 2.4.146 present in Remark 2.4.147, we obtain a more general version of Proposition 3.4.9 for the Nori fundamental group-scheme as only the hypothesis $f_*(\mathcal{O}_X) = \mathcal{O}_Y$ is necessary to obtain a faithfully flat morphism

$$\pi^N(f) : \pi^N(X,x) \to \pi^N(Y,y)$$

as these group-schemes are pro-finite, and this result holds in both the proper and $\mathscr{C}$-CPC approaches.

Now we will consider cases when we can obtain an isomorphism of fundamental group-schemes:

Sufficient condition for an isomorphism of FGS's

Proposition 3.4.11. *Let X and Y be proper, reduced and connected schemes over a field k, both with rational points $x \in X(k)$ and $y \in Y(k)$. Let $f : X \to Y$ be a proper and flat morphism with reduced and connected geometric fibers, that is compatible with the respective rational points. If either one of the two following conditions hold:*

(a) The geometric fibers of f have trivial fundamental group-schemes.

(b) Any Nori-reduced finite pointed G-torsor $t : T \to X$ becomes a trivial torsor when pulled-back to the geometric generic fiber.

then, the induced morphism of Nori fundamental group-schemes

$$\pi^N(f) : \pi^N(X,x) \to \pi^N(Y,y)$$

is an isomorphism.

Proof. The hypotheses imply that $\pi^N(f)$ is faithfully flat by Remark 3.4.10. So we just need to show that this morphism is a closed immersion, and for both of the assumptions, this is an application of

Proposition 3.4.7 and Proposition 3.2.19(c).
Let $t : T \to X$ be a Nori-reduced finite pointed G-torsor over X. If we assume (a), we directly have that global sections of the restriction of $t_*(\mathcal{O}_T)$ to all fibers of f have the same constant dimension by Corollary 3.4.4 and thus Grauert's theorem applies, and we would obtain the result if we assume (b) as stated in Remark 3.4.8. □

3.4.1.4 *Towers of torsors and FGS of Nori-reduced torsors*

In this final subsubsection, we will outline two results from the theory of the Nori fundamental gerbe in the article [1] by M. Antei, I. Biswas, M. Emsalem, F. Tonini and L. Zhang, in their simpler versions for schemes and the fundamental group-scheme.
We will start by introducing towers of torsors:

Towers of torsors, envelopes and closures

Definition 3.4.12 (Definition 3.8 [1]). *Let X be a scheme over a field k with $x \in X(k)$, and let $Z \to Y$ and $Y \to X$ be finite pointed torsors*[18]*. A* tower of finite pointed torsors *is simply a diagram $Z \to Y \to X$.*
If G and H are the finite group-schemes associated to $Z \to Y$ and $Y \to X$ respectively, we say that a finite pointed K-torsor $U \to X$ is an envelope *of or that U* envelops *the tower $Z \to Y \to X$ if we have morphisms of group-schemes $\alpha : K \to G$ and $\beta : \ker(\alpha) \to H$, and a morphism $U \to Z$ making the following diagram commutative*

$$\begin{tikzcd} & U \arrow[dl] \arrow[d] \arrow[dr] & \\ Z \arrow[r] & Y \arrow[r] & X \end{tikzcd}$$

so that we have a morphism of torsors $U \to Y$ over X that intertwines the respective group-scheme actions via α, and a morphism of schemes $U \to Z$ over W that intertwines the corresponding actions via β[19] *and the marked rational point of U is mapped to the respective marked rational points of the schemes in the tower. An envelope is* Nori-reduced *if $U \to X$ is Nori-reduced.*
Finally, if an envelope U is minimal in the sense that every other envelope U' possesses a canonical arrow $U' \to U$ that is a morphism of torsors over X, we say that U is the closure *of the tower $Z \to Y \to X$. Closures, when they exist, are unique up to isomorphism.*

Now we can state the second result:

Existence of closures for towers of torsors

Theorem 3.4.13 (Theorem III [1]). *Let X be a proper, reduced and connected scheme over a field k, with a rational point $x \in X(k)$. Then, any tower of torsors over X possesses a Nori-reduced closure.*
Moreover, if $Z \to Y \to X$ is a tower of torsors over X and $U \to X$ is its closure, the morphism $U \to Z$ is faithfully flat if and only if both members

18 This means that we have rational points $y \in Y(k)$ and $z \in Z(k)$ and z maps to y while y maps to x.
19 A priori, this morphisms is not a morphisms of torsors.

of the tower are Nori-reduced over its respective bases, and in that case, U *is a pointed Nori-reduced torsor over* Z *and* Y*.*

Remark 3.4.14. *Under the hypotheses and notations of Theorem 3.4.13, and* $f : Z \to X$ *denotes the composition of the morphism in the tower of torsors* $Z \to Y \to X$*, we have that* $f_*(\mathcal{O}_Z)$ *is an essentially finite bundle over* X*.*

We also have a very strong result for finite Nori-reduced pointed torsors:

Proposition 3.4.15 (Corollary I [1]). *Let* X *be a proper, reduced and connected scheme over a field* k*, with a rational point* $x \in X(k)$*. Then, a pointed and finite* G*-torsor* $T \to X$ *possesses a FGS (Definition 3.2.2) if and only if it is Nori-reduced, and in that case, for* $t \in T(k)$ *over* x*, we have an exact sequence of Nori fundamental group-schemes*

Nori-reduced finite torsors possess a FGS

$$1 \to \pi_1^N(T, t) \to \pi_1^N(X, x) \to G \to 1.$$

Remark 3.4.16. *Using Theorem 3.4.13, we can see that the universal torsor of a Nori-reduced torsor* $T \to X$ *is* $\hat{X}$*, the universal torsor of* X*.*

3.4.2 *Examples of descriptions of fundamental group-schemes*

In this subsection, we will describe the Nori fundamental group-scheme (and sometimes the S-FGS) for certain type of schemes. We will use some results, namely the FGS of abelian varieties and the FGS of rationally connected varieties, in Chapter 5.
We will start with projective spaces:

Proposition 3.4.17. *Let* k *be an algebraically closed field and* $n \geqslant 1$*, then for any rational point* x *of* $\mathbb{P}^n$*, both fundamental group-schemes* $\pi^N(\mathbb{P}^n, x)$ *and* $\pi^S(\mathbb{P}^n, x)$ *are trivial.*

FGS of projective spaces

Proof. By [42, Prop. 8.2], $\pi^S(\mathbb{P}^n, x)$ is trivial, thus the result for the Nori FGS follows. □

A variety X over k is *rational* if it is birationally equivalent to a projective space, thus by Proposition 3.4.2 we obtain:

Corollary 3.4.18 (Ch. II Prop. 8 [53]). *Let* X *be a proper and normal rational variety over an algebraically closed field* k*. Then, for* $x \in X(k)$ *we have that* $\pi^N(X, x)$ *and* $\pi^S(X, x)$ *are trivial.*

FGS of proper rational varieties

Now we will define rationally connected varieties and describe their FGS, we will go deeper into this concept and its generalizations in Chapter 4.

Definition 3.4.19 (Definition 3.2 §IV.3 [39]). *Let* X *be a variety over* k*. We say for that* X *is* rationally connected (resp. rationally chain connected), *if there exist a proper and flat family of curves* $\mathscr{C} \to Y$ *where* Y *is a variety over* k*, whose geometric fibers are proper smooth irreducible rational curves*

Rationally connected varieties

(resp. proper connected curves with smooth irreducible components that are rational curves), such that there exists a morphism $u : \mathscr{C} \to X$, *making*

$$u^{(2)} : \mathscr{C} \times_Y \mathscr{C} \to X \times_k X$$

a dominant morphism.
Moreover, if X *is rationally connected (resp. rationally chain connected) and* $u^{(2)}$ *is smooth at the generic point, we say that* X *is* separably rationally connected (resp. rationally chain connected).

We will sometimes abbreviate "rationally connected", "rationally chain connected", "separably rationally connected" and "separably rationally chain connected" as RC, RCC, SRC and SRCC respectively.

Remark 3.4.20. *If* $char(k) = 0$*, by generic smoothness, a rationally connected (resp. rationally chain connected) variety is also separably rationally connected (resp. separably rationally chain connected), so these notions coincide. In positive characteristic, there are examples of rationally connected varieties that are not separably rationally connected, see [39, Ch. V 5.19].*

In terms of the FGS, we have the following result:

Proposition 3.4.21. *Let* X *be a proper and normal variety over an algebraically closed field* k*, with a rational point* $x \in X(k)$*. If* X *is rationally chain connected, then* $\pi_1^N(X, x)$ *is finite.*
If in addition, X *is separably rationally connected and smooth, then* $\pi_1^N(X, x)$ *is trivial.*

Proof. For the first assertion, see [2]. The second one can be found in [9]. □

Recall that a variety S over k is an abelian variety if S is a projective, smooth and connected commutative group-scheme over k (Example 2.2.15(6)).

n-th power morphisms of group-schemes

Definition 3.4.22. *Let* G *be a group-scheme of finite type over* k *and* $n \geqslant 1$*, then the* n-nth power morphism of G *is the morphisms* $m_n : G \to G$ *that is represented at the level of functors of points as the* n*-th power morphism of abstract groups*

$$\begin{array}{ccc} \widetilde{G}(R) & \to & \widetilde{G}(R) \\ g & \mapsto & g^n \end{array}$$

for any k*-algebra* R*.*

FGS of an abelian variety

Proposition 3.4.23 ([54])**.** *Let* S *be an abelian variety over a perfect field* k*. For* $n \geqslant 1$*, let* $S[n]$ *be the kernel of the* n*-th power morphism* $m_n : S \to S$*, which is a Nori-reduced* $S[n]$*-torsor. Then*

$$\pi_1^N(S, 0) = \varprojlim_n S[n].$$

Remark 3.4.24. *In [54], Nori showed that if $t : T \to S$ is a pointed G-torsor over 0, then there exists a unique integer $N \geqslant 1$ an a morphism of pointed torsors $p : S \to T$ such that $t \circ p = m_N$, and thus we have a morphism of group schemes $S[N] \to G$ that commutes with the respective morphism from $\pi^N(S, 0)$.*
At the time Nori published this result, the Künneth formula for the FGS of products of schemes was not yet established for the FGS, but he conjectured the limit formula for $\pi_1^N(S, 0)$ if the result was true. Later, Mehta and Subramanian showed the Künneth formula in [48].

Proposition 3.4.23 allows to fully describe pointed Nori-reduced finite torsors over abelian varieties:

Nori-reduced torsors over abelian varieties

Corollary 3.4.25. *Let S be an abelian variety over a perfect field k. If $t : T \to S$ is a Nori-reduced finite torsor over S, pointed over 0, then T is an abelian variety. When k is algebraically closed, and T a Nori-reduced finite torsor over S, but is pointed over $s \in S(k)$ different from 0, then T is a smooth projective variety over k.*

Proof. If $t : T \to S$ is pointed over S, then by Remark 3.4.24, there exists a unique integer $N \geqslant 1$ an a morphism of pointed torsors $p : S \to T$ such that $t \circ p = m_N$. If G is the group-scheme associated to t, then G is quotient of $S[N]$ and thus by using Proposition 2.3.49 we have $T \cong S \times^{S[N]} G$, and we can easily see that $T \cong S/K$ where K is the kernel of the quotient morphism $S[N] \to G$, and by applying [49, Prop. 1.62(b)] we conclude that T is an abelian variety as finite morphisms are projective.
If k is algebraically closed and T is pointed over s, by post-composing t by a translation, we see that T is isomorphic to a Nori-reduced torsor T' that is be pointed over 0, which concludes the proof. □

Remark 3.4.26. *The fact that pointed Nori-reduced finite torsors over abelian varieties are abelian varieties themselves, or at least smooth projective varieties is rare. In general, we cannot expect much regularity from Nori-reduced pointed torsors, besides the fact that they are geometrically connected.*
For example, in [23, Remark 2.3 2)] an example of a non-reduced Nori-reduced torsor is shown.

Part II

FUNDAMENTAL GROUP-SCHEME OF CURVE-CONNECTED VARIETIES

4 CURVE-CONNECTED VARIETIES AND THEIR FGS

4.1 INTRODUCTION

The concept of curve-connected varieties started from the introduction of rationally connected varieties (Definition 3.4.19). Independently, F. Campana in [15] and J. Kollár, Y. Miyaoka and S. Mori in [40] introduced the concept of rationally connected varieties motivated by the study of Fano varieties of high dimension. One can think of the concept of rationally connected as a rough generalization of uniruled varieties: over uniruled varieties for any general point we can pass a rational curve over it, while for rationally connected for any two very general points we can pass a rational curve through them, at least if the base field is algebraically closed. We can also think of rationally connected varieties as an algebro-geometric analogue of "path connectedness' which is specially evident for complex varieties.
Nowadays, rationally connected varieties are an important part in the study of higher-dimensional varieties, and the purpose of this chapter is to introduce this concept and generalizations where we use curves of higher genus or a fixed curve instead of just rational curves, along with new fundamental group-schemes adapted to these varieties.
We will start in Section 4.2, where we will introduce the generalized notions of curve-connected varieties, introduced by F. Gounelas in [29] along with some basic results for these varieties. Then, in Section 4.3 we will show that g-connected varieties are $\mathscr{C}$-CPC (Definition 3.3.60) varieties, where the family $\mathscr{C}$ depends on the genus of the curves, and thus we can attach new fundamental group-schemes to these varieties following the CPC approach of Subsection 3.3.2.
Finally, in Section 4.4 we will apply the construction of the "maximal rationally connected fibration" to g-connected varieties and their fundamental group-schemes, specially in the case of elliptically connected varieties, which serves as a motivation for the results in Chapter 5.
Most results from the theory of rationally connected varieties and g-connected varieties will be stated without proof, with the respective reference for the interested reader.

4.2 VARIETIES CONNECTED BY CURVES

In this section we will state the main notions of curve-connectedness that we will work on in this chapter. These notions were defined by F. Gounelas in [29] as a generalization of the notion of rationally connected varieties. In this section k will be a field of arbitrary characteristic.

g-connected varieties

Definition 4.2.1 (Definition 3.1 [29]). *Let X be a variety over k. We say for that X is* connected by curves of genus g (resp. chain connected by curves of genus g) *for some $g \geqslant 0$, if there exists a proper and flat family of curves $\mathscr{C} \to Y$ where Y is a variety, whose geometric fibers are proper irreducible curves of genus g (resp. proper connected schemes of dimension 1 whose irreducible components are smooth curves of genus g), such that there exists a morphism $u : \mathscr{C} \to X$, making $u^{(2)} : \mathscr{C} \times_Y \mathscr{C} \to X \times_k X$ dominant.*
Moreover, if X is connected (resp. chain connected) by curves of genus g and $u^{(2)}$ is smooth at the generic point, we say that X is separably connected by curves of genus g (resp. chain connected by curves of genus g).
We will often use the shorter names (separable) g-connected, (separable) g-chain connected.

Remark 4.2.2. *Definition 4.2.1 is a generalization of the notion of (separable) rationally (chain) connected varieties (Definition 3.4.19) which is of course the latter definition with $g = 0$.*
In the case $g = 1$, we will call 1-connected varieties elliptically connected varieties.

There is another notion of curve-connected varieties that we will consider, in which we will use a single curve instead of many curves of a certain genus:

C-connected varieties

Definition 4.2.3 (Definition 3.3 [29]). *Let X be a variety over k. Let C be a proper curve, we say that X is* C-connected *if there exist a variety Y and a morphism $u : C \times_k Y \to X$, such that the induced map $u^{(2)} : C \times_k C \times_k Y \to X \times_k X$ is dominant.*
Moreover, $u^{(2)}$ is smooth at the generic point, we say that X is separably C-connected.

Clearly if X is a C-connected variety, then X is g-connected where g is the genus of C.
The theory of curve-connected varieties diverges significantly depending on the characteristic:

Remark 4.2.4. *If char$(k) = 0$, by generic smoothness (see [27, Exc. 10.40(a)]), all the notions of curve-connectedness we have defined are automatically separably connected, thus these terms coincide. In positive characteristic, the "separably" condition is stronger. For example, in [39, Ch. V 5.19] Kollár showed examples of rationally connected varieties that are not separably rationally connected.*

Now we will outline some basic properties of g-connected varieties, let us start with:

Lemma 4.2.5. *Let X be a smooth and projective variety over k.*

(a) There exists $g \geqslant 0$ such that X is g-connected.

(b) *If* X *is* g*-connected, then it is also* g'*-connected for any* $g' \geqslant 2g-1$.

Proof. The proof of part (a) uses Bertini's theorem ([36, II Thm. 8.18]) and can be found in [28, Lemma 2.20], while the proof of part (b) can be found in [29, Lemma 3.2]. □

Another property for g-connected varieties, that will become very handy in Section 4.4, is the following:

Lemma 4.2.6 (Lemma 3.4 (1) [29]). *Let* C *be a smooth, irreducible and projective curve and let* X *be a variety over* k*. Let us suppose we have a rational dominant map* $X \dashrightarrow Y$ *to a proper variety* Y*. If* X *is connected by genus* g *curves (resp.* C*-connected), then* Y *is connected by genus* g *curves (resp.* C*-connected) as well.*

Induced curve-connectedness over rational dominant maps

Remark 4.2.7. *The last two results hold for projective varieties. However, Lemma 4.2.6 allows us to transfer results about projective curve-connected varieties to proper curve-connected varieties by using Chow's lemma: There exists a projective variety* $\overline{X}$ *with a proper birational morphism* $\overline{X} \to X$*, see [27, Theorem 13.100]. So, we can safely assume that a result of curve-connectedness that holds for projective varieties hold for proper ones as well.*

If we look back at Definition 4.2.1, it is not clear how we can pass a curve over two points as we stated in the introduction. In fact, this property is geometrical, i.e., it holds when k is algebraically closed. To fully describe g-connected varieties over algebraically closed field, we first need:

Definition 4.2.8. *Let* X *be a variety over* k*. A point* $x \in X$ *is* very general *if belongs to a complement* $U = X \backslash Z$ *where* Z *is a countable union of proper closed sub-varieties of* X*.*

Proposition 4.2.9 (Lemma 3.4 (4) & (5) [29]). *Let* k *be an uncountable algebraically closed field, let* X *be a variety and let* C *be a smooth, irreducible and projective curve. Then:*

g-connected varieties over algebraically closed fields

(a) X *is* C*-connected, if and only if for any pair* x_1, x_2 *of very general closed points of* X*, there exists a morphism* $C \to X$ *passing through them.*

(b) *If* k *is an uncountable algebraically closed field, then* X *is* g*-connected if and only if for two very general points of* X *there exist a smooth irreducible curve of genus* g *with a morphism to* X *that contains these points.*

Remark 4.2.10. *In Proposition 4.2.9 above, the hypothesis of* k *being uncountable is necessary as a variety* X *might not have very general points at all if* k *is countable or finite.*

Sadly, this result is not enough to show that g-connected varieties over uncountable algebraically closed fields are $\mathscr{C}$-CPC as we would need a smooth proper curve over <u>any pair</u> of two closed points. We will address this in the next section, but we will finish this section with a result that serves as a starting point in the study of C-connected varieties, and their FGS.

C-connected varieties arising from generically SRC fibrations

Proposition 4.2.11 (Proposition 3.5 [29]). *Let X be a projective and smooth variety over an algebraically closed field k, and let $f : X \to C$ be a flat morphism to a smooth and projective curve whose geometric generic fiber is separably rationally connected. Then X is C-connected.*

4.3 FUNDAMENTAL GROUP-SCHEMES FOR G-CONNECTED VARIETIES

4.3.1 *Introduction and motivation*

Let k be an uncountable algebraically closed field, and let X be a g-connected variety. If $\mathscr{C} \to Y$ is the family of curves, parameterized by Y, the fact that we can join two very general points of X by a smooth proper curve of genus g stems from the fact that the morphism $u^{(2)}$: $\mathscr{C} \times_Y \mathscr{C} \to X \times_k X$ in Definition 4.2.1 is dominant, if $u^{(2)}$ would be surjective instead, we would be able to connect any pair of points of X using a curve in the family $\mathcal{C}$, so X would be $\mathscr{C}$-CPC and thus would have a (S,$\mathscr{C}$)-fundamental group-scheme $\pi^S_{\mathcal{C}}(X, x)$ for some rational point $x \in X(k)$ (Definition 3.3.67).
A sufficient condition for $u^{(2)}$ to be surjective is that $\mathscr{C}$ is proper, as the morphism $\mathscr{C} \to Y$ is clearly proper and then, we would obtain that $u^{(2)}$ is a proper dominant morphism.
There is an obstacle for the family $\mathscr{C}$ to be proper: $\mathscr{C}$ is an irreducible component of a moduli space of curves over X, namely, of curves that pass through pair of points of X, and if we restrict ourselves to only smooth proper curves of genus g, the corresponding moduli space is not proper or projective. But as in the case of Remark 4.2.7 we can try to "compactify the moduli", so the newly obtain moduli space of curves would be proper or projective, so that the closure $\overline{\mathscr{C}} \to \overline{Y}$ of $\mathscr{C}$ would be a new proper or projective family of curve so we would effectively obtain a proper dominant morphism $\overline{u}^{(2)} : \overline{\mathscr{C}} \times_{\overline{Y}} \overline{\mathscr{C}} \to X \times_k X$.
For this purpose, we need to extend the types of curves we will allow to form families with. These curves are called *stable curves*, and we will show that we can effectively use them to connect any pair of points of X, so X becomes $\overline{\mathscr{C}}$-CPC for a larger but reasonable family of curves, that will allows us to define the desired fundamental group-schemes for g-connected varieties.

4.3.2 *Stable curves and their moduli*

Let us start by defining more precisely what we mean by families of curves:

Definition 4.3.1. *Let k be a field and let X be a k-scheme. For a finite morphism $Q \to \mathrm{Spec}(k)$ a* family of curves over X with marking Q *is a triple $(\mathscr{C}, f, \rho)$ where:*

Curve families

(a) *We have a flat and proper morphism $\mathscr{C} \to Y$ over a k-scheme Y whose geometric fibers are proper connected schemes of dimension 1 whose irreducible components are smooth curves.*

(b) *The morphism $f : \mathscr{C} \to X \times_k Y$ is a morphism of schemes over Y.*

(c) *We have an embedding $\rho : Q \times_k Y \to \mathscr{C}$ into the smooth locus[1] of $\mathscr{C}$ with disjoint images.*

Remark 4.3.2. *For such a family the arithmetic genus ([36, III Exc. 5.3]) $g \geqslant 0$ and degree of any geometric fiber is constant by [36, III Coro. 9.10] as proper schemes of dimension $\leqslant 1$ are projective.*
If X is projective with a fixed ample bundle H, we can consider the degree of the pull-back over a geometric fiber, that we will denote as $d = deg_{\mathscr{C}}(f^*(H)) \geqslant 0$, which is constant in the family.

Before defining stable curves, it must be remarked that they are not smooth in general, but their singularities are always "nodal" so we need to define what we mean by it:

Definition 4.3.3. *Let X be a scheme locally of finite type over k and of dimension 1.*

Nodal singularities

(a) *If k is algebraically closed, we say that a point $x \in X$ is a* node *if we consider m_x the maximal ideal of the stalk $\mathcal{O}_{X,x}$, and the m_x-adic completion $\widehat{\mathcal{O}_{X,x}}$ of $\mathcal{O}_{X,x}$ (see [36, p. 193]), then we have that*

$$\widehat{\mathcal{O}_{X,x}} \cong k[[x,y]]/(xy)$$

where the double bracket denotes the k-algebra of formal power series.

(b) *For general k, a point $x \in X$ is a* node *if there exists a node $\bar{x} \in X_{\bar{k}}$ that maps onto x via the base change morphism $X_{\bar{x}} \to X$.*

(c) *X has* at worst nodal singularities *if any point $x \in X$ is either a node or smooth over k.*

(d) *A proper connected scheme of dimension 1 whose irreducible components are curves will be called a* nodal curve.

Now we can define stable curves:

1 The set of points of $\mathscr{C}$ that are smooth over $\mathrm{Spec}(k)$.

Stable curve families

Definition 4.3.4. *Let k be a field and let X be a k-scheme. A family $(\mathscr{C}, f, \rho)$ of curves over X with marking Q is* stable *if in addition to the conditions of Definition 4.3.1 we have:*

(d) All the geometric fibers in the family $\mathscr{C} \to Y$ over an k-scheme Y are nodal curves.

(e) For every rational point $\operatorname{Spec}(k) \to Y$, the curve C_k has at most a finite amount of automorphisms such that, if $f_k : C_k \to X$ is the restriction of f to C_k, we have that $f_k = f_k \circ \rho$ and they fix the closed points of $\rho(Q_k)$ where Q_k is the respective restriction of Q.

Geometric stable curves

We will call families of stable curves of arithmetic genus g with $\operatorname{Spec}(L) = Y$ where L is an algebraically closed extension of k a geometric stable curve of arithmetic genus g over L.

Remark 4.3.5. *Definitions 4.3.1 and 4.3.4 are particular cases of [5, Definition 49] where there are no prescribed base points over X, that we do not need for this thesis, and the base scheme is $\operatorname{Spec}(k)$.*

Definition 4.3.6. *Let X be a scheme over k, and let $(\mathscr{C}, f, \rho)$ and $(\mathscr{C}', f', \rho')$ be two family, stable or otherwise, of curves over X with marking Q, parameterized by the same k-scheme Y.*

Isomorphisms of curve families

(a) An isomorphism of families *between $(\mathscr{C}, f, \rho)$ and $(\mathscr{C}', f', \rho')$, is an isomorphism $\phi : \mathscr{C} \to \mathscr{C}'$ over Y such that that $f' \circ \phi = f$ and $\phi \circ \rho = \rho'$.*

Pull-back of curve families

(b) If $g : Z \to Y$ is a morphism of schemes over k, the pull-back *of $(\mathscr{C}, f, \rho)$* under g *is the family $(\mathscr{C}_Z, f_Z, \rho)$ that results of base changing the morphism $f : \mathscr{C} \to X \times_k Y$ in Definition 4.3.1(b) via g. The pull-back of a stable family is stable.*

With this, we can define functors over schemes that we will attach moduli spaces to:

Functors of curve families

Definition 4.3.7. *Let k be a field, $g \geqslant 0$ and let $Q \to \operatorname{Spec}(k)$ be a finite morphism with $n = |Q|$, the* functor of curve families of genus g with marking Q *is the functor $\mathfrak{F}_{g,d} : Sch_k \to Set$ that associates to each scheme X over k the set of isomorphism classes of families of curves over X of arithmetic genus g and marking Q and for any morphism $g : Z \to X$ of schemes over k, $\mathfrak{F}_{g,d}(g)$ maps a family $(\mathscr{C}, f, \rho)$ to its pull-back under g (Definition 4.3.6(b)).*
The analogous functor that associates to each scheme the set of isomorphism classes of stable families of curves over X of arithmetic genus g and marking Q will be denoted as $\overline{\mathfrak{F}}_{g,d} : Sch_k \to Set$, and called the functor of stable curve families of genus g with marking Q.

Now we can define what we will mean by moduli space:

Definition 4.3.8. *Let* k *be a field, and let* $F : Sch_k \to Set$ *be a functor. An algebraic space*[2] M *over* k*, with a natural transformation* $\tau_M : F \to Hom(\cdot, M)$[3] *that satisfies the following:*

Coarse moduli spaces

(a) *For any algebraically closed field* L, $\tau_M(\mathrm{Spec}(L))$ *is an isomorphism.*

(b) *For any scheme* N *over* k*, and natural transformation* $\tau_N : F \to Hom(\cdot, N)$*, there exists a unique morphism* $f : M \to N$ *such that* $\tau_N = \tau_f \circ \tau_M$ *where*

$$\tau_f : Hom(\cdot, M) \to Hom(\cdot, N)$$

is the natural transformation induced by composition with f.

And thus we can start with the coarse moduli of smooth curves:

Definition 4.3.9. *Let* X *be a scheme over* k*. For* $g \geqslant 0$ *and a finite morphism* $Q \to \mathrm{Spec}(k)$ *with* $n = |Q|$*. The* coarse moduli of smooth curves of genus g over X with n marked points *is the coarse moduli space* $M_{g,n}(X)$ *for the functor* $\mathfrak{F}_{g,d} : Sch_k \to Set$ *in Definition 4.3.7. The disjoint union for of the moduli spaces* $M_{g,n}(X)$ *for* $n \geqslant 0$ *we will denoted as* $M_g(X)$ *and is known as the* coarse moduli of smooth curves of genus g over S, *it is quasi-projective.*

Coarse moduli of smooth curves of genus g

Remark 4.3.10. *If* X *is projective with a fixed ample bundle* H*, we can consider the degree* $d = deg_{\mathscr{C}}(f^*(H)) \geqslant 0$*, which is constant in curve families (Remark 4.3.2), and its respective coarse moduli space will be denoted as* $M_g(X, d)$*, this space can be further divided as* $M_g(X, d) = \sqcup_{n \geqslant 0} M_{g,n}(X, d)$ *where each moduli space* $M_{g,n}(X, d)$ *classifies the functor of isomorphism classes of families of smooth and irreducible projective curves of degree* d *over* X *with marking* Q.

The compactification of these moduli spaces, associated to the functor of stable curve families of genus g with marking Q, exists and we have:

Theorem 4.3.11 (Theorem 50 [5])**.** *Let* X *be a projective scheme over* k *with an ample bundle* H*. For a finite morphism* $Q \to \mathrm{Spec}(k)$ *with* $|Q| = n$ *and integers* $g, d \geqslant 0$ *there exists a separated algebraic space* $\overline{M}_{g,n}(X, d)$ *of finite type over* k *that is a coarse moduli space for the functor* $\overline{\mathfrak{F}}_{g,d} : Sch_k \to Set$ *of stable curve families of genus* g *with marking* Q *(Definition 4.3.7), such that* $deg_{\mathscr{C}}(f^*(H)) = d$*. Moreover,* $\overline{M}_{g,n}(X, d)$ *is projective over* k.

Coarse moduli space of stable curves of arithmetic genus g

2 See [57, Definition 5.1.10] for a definition of algebraic space, in short, it is a sheaf in the big étale site of Sch_k whose diagonal is representable and is covered by an étale morphism from a scheme.

3 In the big étale site schemes are representable functors, thus the Hom-set here denotes the set of étale sheaf morphism.

"Inclusion" of the coarse moduli space of smooth curves into the stable moduli space

Remark 4.3.12. *Let X be a projective scheme over k with an ample bundle H. It can be shown that for any $g \geqslant 0$ the coarse moduli space $M_{g,n}(X, d)$ is a scheme over k, and thus the obvious natural transformation $\iota : \mathfrak{F}_{g,d} \to \overline{\mathfrak{F}}_{g,d}$ that comes from the inclusion of classes families smooth curves of genus g over X with n marked points in a class of stable curve families, induces a morphism $i_{g,n,d} : M_{g,n}(X, d) \to \overline{M}_{g,n}(X, d)$ that makes the following diagram commutative of functors and natural transformations:*

$$\begin{tikzcd} \mathfrak{F}_{g,d} \arrow[r, "\tau_{M_{g,n}(X,d)}"] \arrow[d, "\iota"'] & \mathit{Hom}(\cdot, M_{g,n}(X, d)) \arrow[d, "\tau_{i_{g,n,d}}"] \\ \overline{\mathfrak{F}}_{g,d} \arrow[r, "\tau_{\overline{M}_{g,n}(X,d)}"'] & \mathit{Hom}(\cdot, \overline{M}_{g,n}(X, d)) \end{tikzcd}$$

4.3.3 *New fundamental group-schemes for curve-connected varieties*

Now we are ready to define new fundamental group-schemes to g-connected proper varieties. We start with the following observation:

Remark 4.3.13. *Let k be a field, and let X be a g-connected variety.*
If $\mathscr{C} \to Y$ is the family of smooth proper curves of genus g that makes X g-connected, by looking at Definition 4.2.1 we see that in fact the family is a family of curves over X without marked points, meaning that it is related to the coarse moduli space $M_{g,0}(X, d)$ for a certain integer $d \geqslant 0$.
Moreover, the projection $p_2 : \mathscr{C} \times_Y \mathscr{C} \to \mathscr{C}$ defines a family smooth proper curves of genus g now parameterized by $\mathscr{C}$ that as a section, the diagonal morphism $\Delta_{\mathscr{C}}$ of $\mathscr{C}$ over Y, thus we have a family with one marked point, associated to $M_{g,1}(X, d)$. In particular, as these are coarse moduli spaces, these families correspond to points over them, such that the following diagram is commutative:

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] & M_{g,1}(X, d) \arrow[d] \\ Y \arrow[r] & M_{g,0}(X, d) \end{tikzcd}$$

where the morphism $M_{g,1}(X, d) \to M_{g,0}(X, d)$ arises from the natural transformation that forgets the marking of a family of smooth proper curves of genus g, using Definition 4.3.8(b).
Moreover, there is a natural morphism $ev_X : M_{g,1}(X, d) \to X$ as for any scheme Z over k, a family of smooth curves of genus g and degree parameterized by Z defines a morphism $Z \to X$ using the section provided by the marking Q, in particular we have a natural transformation

$$\mathit{Hom}(\cdot, M_{g,1}(X, d)) \to \widetilde{X}$$

that corresponds to the morphism above, so we can complete the commutative diagram as follows:

$$\begin{tikzcd} \mathscr{C} \arrow[r] \arrow[d] \arrow[rr, bend left, "u"] & M_{g,1}(X,d) \arrow[r, "ev_X"'] \arrow[d] & X \\ Y \arrow[r] & M_{g,0}(X,d) & \end{tikzcd}$$

Thus, this diagram induces a dominant morphism

$$M_{g,1}(X,d) \times_{M_{g,0}(X,d)} M_{g,1}(X,d) \to X \times_k X$$

that gives us curves that pass through any pair of very general points if k *is algebraically closed by applying Definition 4.3.8(a). We can replace* $M_{g,1}(X,d)$ *and* $M_{g,1}(X,d)$ *by* $M_{g,1}(X)$ *and* $M_{g,0}(X)$ *respectively in the last morphism.*

Proposition 4.3.14. *Let* k *be an uncountable algebraically closed field, and let* X *be a projective variety that is* g*-connected. Then, for any pair of points* x, y *of* X*, there exist a geometric stable connected curve of arithmetic genus* g *over* k *(Definition 4.3.4) and a morphism* $f : C \to X$ *whose image contains* x *and* y*.*

Points in projective g-connected varieties can be connected by stable curves of genus g

Proof. By Lemma 4.2.6 we can assume X is projective.
As X is g-connected, applying Proposition 4.2.9 and Remark 4.3.13, we have a dominant morphism $M_{g,1}(X) \times_{M_{g,0}(X)} M_{g,1}(X) \to X \times_k X$. If $\mathcal{C}$ is a family of smooth projective and irreducible curves of genus g that makes X g-connected, it induces a morphism $\mathcal{C} \to M_{g,1}(X,d)$ for some degree $d \geqslant 0$. Thus, by Remark 4.3.12, we have another dominant morphism $\overline{M}_{g,1}(X,d) \times_{\overline{M}_{g,0}(X,d)} \overline{M}_{g,1}(X,d) \to X \times_k X$, but as $\overline{M}_{g,1}(X,d)$ is projective, this morphism is surjective and by looking at the k-points, by Definition 4.3.8(a) we obtain geometric stable curves of arithmetic genus g over k with morphisms over X that pass through any pair of closed points. □

Corollary 4.3.15. *Let* k *be an algebraically closed field and let* X *be a projective variety that is* g*-connected (*$g \geqslant 0$*). Then* X *is* $\underline{Curv}_k(g)$*-CPC where* $\underline{Curv}_k(g)$ *is the family of projective and irreducible curves of arithmetic genus* $\leqslant g$*.*

Projective curve connected varieties are $\mathscr{C}$*-CPC*

Proof. Let $C \to X$ be a geometric stable curve of arithmetic genus g over k passing through two points of X, then any irreducible component of C must have arithmetic genus $\leqslant g$ and thus the conclusion follows. □

Thus, we can define an (S,$\mathscr{C}$)-fundamental group-scheme for projective varieties. From this point on, k will be an uncountable algebraically closed field of positive characteristic.

The g-fundamental group-scheme

Definition 4.3.16. *Let X be a projective g-connected variety ($g \geqslant 0$) over k, with a rational point $x \in x(k)$. The* g-fundamental group-scheme *of X is the $(S, \mathcal{C})$-fundamental group-scheme of Definition 3.3.67 associated to the family $\underline{Curv}_k(g)$. We will denote it as $\pi_g^S(X, x)$. We can also define the* g-PB-fundamental group-scheme *as the fundamental group-scheme of Proposition 3.3.76 applied to the family of curves $\underline{Curv}_k(g)$, we will denote it as $\pi_g^{PB}(X, x)$.*

Remark 4.3.17. *Let X be a projective g-connected variety ($g \geqslant 0$) over k, with a rational point $x \in x(k)$. To fix some notation, recall that $\pi_g^S(X, x)$ is the group-scheme associated to the tannakian category $NSS_g(X)$ (we will use this notation instead of $NSS_{\underline{Curv}_k(g)}(X)$) of vector bundles over X whose pull-backs along morphisms from smooth, irreducible and projective curves of genus $\leqslant g$ are strongly semi-stable of degree 0. In the case of $\pi_g^{PB}(X, x)$, it is associated to the category $PB\text{-}EF_g(X)$ (as before, we will use this instead of $PB\text{-}EF_{\underline{Curv}_k(g)}(X)$) of vector bundles over X whose pull-backs along morphisms from smooth, irreducible and projective curves of genus $\leqslant g$ are essentially finite.*

Remark 4.3.18. *If X is a g-connected projective variety over k, for any genus $g' \geqslant g$ for which X is also g'-connected, we have a natural morphism of fundamental group-schemes*

$$\pi_g^S(X, x) \to \pi_{g'}^S(X, x)$$

coming from the full inclusion of categories $NSS_{g'}(X) \subset NSS_g(X)$ (see Remark 3.3.64). The same applies to $\pi_g^{PB}(X, x)$ and $\pi_{g'}^{PB}(X, x)$.
As $NSS(X)$ is fully included in $NSS_g(X)$ for any $g \geqslant 0$, we have a composition $\pi_g^S(X, x) \to \pi^S(X, x) \to \pi^N(X, x)$ for any g that is faithfully flat[4], and likewise for the g-PB-fundamental group-scheme.

If we apply the properties in Lemma 4.2.5 to the g-fundamental group-scheme, we will obtain:

Lemma 4.3.19. *Let X be a smooth and projective variety over k, with a rational point $x \in x(k)$.*

(a) There exists $g \geqslant 0$ such that X possesses a g-fundamental group-scheme.

(b) If X is g-connected, for any $g' \geqslant 2g - 1$, we have a morphism of fundamental group-schemes $\pi_g^S(X, x) \to \pi_{g'}^S(X, x)$.

We finish this sub-section by studying the 0-fundamental group-scheme. To start, in the case of rationally connected variety, we do not need to use Proposition 4.3.14:

4 The first morphism in this composition might not be faithfully flat though, see Remark 3.3.68(a).

Lemma 4.3.20 (IV.3 Corollary 3.5.1 [39]). *Let X be a proper rationally chain connected over an algebraically closed field k. Then, for any pair of points of X there exists a curve $C \subset X$ containing them, where all irreducible components of C are rational.*

Over algebraically closed fields, the only smooth proper rational curves is $\mathbb{P}^1_k$ so we will introduce new notation in the case a scheme is connected by chains of proper curves, with the same irreducible component.

Definition 4.3.21. *Let C be a proper curve over k and let X be a scheme of finite type over k, with a rational point. If we consider the singleton family $\mathscr{C} = \{C\}$, we will simply denote a $\mathscr{C}$-CPC scheme X as C-CPC.*
In this case we will denote the corresponding fundamental group-schemes as $\pi^S_C(X, x)$ and $\pi^{PB}_C(X, x)$ and we will call them the C-fundamental group-scheme *and the* C-PB-fundamental group-scheme *respectively.*

The C-fundamental group-scheme

Lemma 4.3.20 yields:

Corollary 4.3.22. *Let X be a proper rationally chain connected variety over an algebraically closed field k, not necessarily uncountable.*
Then, X is $\mathbb{P}^1_k$-CPC. Moreover, if X is rationally connected and k is uncountable, for any pair of very general points x_1, x_2 of X, there exists a morphism $\mathbb{P}^1_k \to X$ from a single rational smooth curve whose image contains x_1 and x_2.

Proper rationally chain connected varieties are $\mathbb{P}^1_k$-CPC

Proof. By taking normalizations of the rational curves over points of X, we easily see that X is $\mathbb{P}^1_k$-CPC. The same can be applied to the rationally connected case, using [39, IV.3 Prop. 3.6] to conclude the second part. □

Thus, for proper rationally chain connected varieties over k, the fundamental group-schemes $\pi^S_{\mathbb{P}^1_k}(X, x)$ and $\pi^S_0(X, x)$ coincide. We will stick with the latter notation.
Using the fact that the FGS of $\mathbb{P}^1_k$ is trivial (Proposition 3.4.17), this can be easily deduce the following:

Lemma 4.3.23. *Let X be a projective (proper) rationally chain connected variety over k. For any rational point $x \in X(k)$ the fundamental group-schemes $\pi^S_0(X, x)$, $\pi^{PB}_0(X, x)$ and $\pi^N(X, x)$ coincide.*

Remark 4.3.24. *Recall Proposition 3.4.21. In [2] M. Antei and I. Biswas showed that a normal rationally chain connected variety has finite FGS, and so do $\pi^S_0(X, x)$ and $\pi^{PB}_0(X, x)$ from Lemma 4.3.23.*
For separably rationally connected varieties, I. Biswas in [9] showed that $\pi^N(X)$ is trivial for a smooth separably rationally connected variety.
Another take on the latter result, is that with the same hypotheses, I. Biswas and J.P. Dos Santos showed in [10] that any bundle over X that becomes trivial under any non-constant morphism $\mathbb{P}^1 \to X$ is trivial. This implies that $\pi^S_0(X, x)$ is trivial, and thus $\pi^N(X, x)$ is as well from the natural faithfully flat morphism $\pi^S_0(X, x) \to \pi^N(X, x)$.

Finally, we state a problem associated to these FGS, that could provide a new approach to describe the FGS of rationally connected varieties, independent from the proof of Proposition 3.4.21 by M. Antei and I. Biswas in [2]:

0-FGS of RCC varieties

Problem 4.3.25. *Let X be a projective (proper) rationally chain connected variety over k. Show independently that $\pi_0^S(X, x)$ is a finite group-scheme.*

4.4 THE MAXIMAL RATIONALLY CONNECTED FIBRATION

We finish this chapter with the maximal rationally connected fibration, or MRC fibration. We will define the MRC fibration a state some properties of it in Subsection 4.4.1, followed by the study of this fibration for C-connected (Definition 4.2.3) proper normal varieties in Subsection 4.4.2. Finally, we discuss possible approaches to understand the FGS for these varieties, see Remark 4.4.20.

4.4.1 *Definition and properties*

Let k be a field. We start by defining the MRC fibration:

Rationally connected fibration and the MRC fibration

Definition 4.4.1 (IV.5 5.1 [39]). *Let X be a variety over k a* rationally connected fibration or RC fibration *is a rational morphism $X \dashrightarrow Z$ such that for an open dense subset $U \subset X$ we have a proper morphism $f : U \to Z$ such that $f_*(\mathcal{O}_U) = \mathcal{O}_Z$ and the fibers of f are rationally chain connected. A rationally connected fibration $\phi : X \dashrightarrow Z$ is* maximal or MRC *if for any other rationally connected fibration $\phi' : X \dashrightarrow Y$, there exists a rational map $g : Y \dashrightarrow Z$ such that $\phi = g \circ \phi'$, making Z unique up to birational equivalence.*

Remark 4.4.2. *Let X be a variety over k. Let us suppose that X has an MRC fibration $X \dashrightarrow Z$, it is not hard to see that if Z is a point, then X is rationally chain connected.*

A variety might not have a MRC fibration or even a RC fibration in general, but with sufficient hypotheses, we have:

Existence of the MRC fibration

Proposition 4.4.3 (IV.5 5.2 [39]). *Let X be a normal and proper variety, then X has an MRC fibration $f : X \dashrightarrow Z$. Moreover, if k is algebraically closed and uncountable, it is characterized by the following property: For any very general point $z \in Z$, if $C \subset X$ is a rational curve over X that intersects the fiber X_z, then $C \subset X_z$.*

The main idea that shows the existence of the MRC fibration, is that we can define an equivalence relation (Definition 2.2.38) over X: two points x_1 of x_2 are related if there exists a chain of rational proper smooth curves that connect them. Then, by a result that generalizes Theorem 2.2.43 we obtain the MRC fibration, see [39, §IV.4 4.10 & 4.17].

We will make some adjustments on MRC fibrations, so we can iterate them:

Remark 4.4.4. *If X is normal an proper, we will denote the MRC fibration as $X \dashrightarrow R^1(X)$. Kollár warned that the open subset $U \subset X$ where the MRC fibration is defined is not unique, so we will assume that we have an open subset $V \subset R^1(X)$ with a morphism $f : U \to V$ satisfying the properties of a rationally connected fibration in Definition 4.4.1. After getting rid of the bad locus we will always assume that $R^1(X)$ is normal and proper after normalizing V, taking its Nagata compactification $\bar{V}$ (see [16]), and then taking the normalization $W \to \bar{V}$ which is a finite and birational morphism.*

Thus, under the assumptions of Remark 4.4.4 we can iterate MRC fibrations using Lemma 4.4.3, to obtain:

Definition 4.4.5. *Let X be a normal and proper variety over k. The sequence of rational maps*

The MRC sequence

$$X := R^0(X) \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X) \dashrightarrow \cdots$$

such that $R^{i+1}(X) = R^1(R^i(X))$ is an MRC fibration over $R^i(X)$ for $i \geqslant 0$ will be called the MRC sequence of X.
This sequence is eventually stationary, we will call the least member for which the sequence stabilizes afterwards the end of the MRC fibration.

4.4.2 *Towards a description of the FGS of C-connected varieties*

Now we will show an approach to study and possibly describe the fundamental group-scheme of C-connected varieties, where C is a fixed smooth and proper curve. We will suppose that k is an uncountable algebraically closed field for this subsection.
We will start with a neat feature of the MRC sequence, which is a direct consequence of Lemma 4.2.6:

Lemma 4.4.6. *Let C be a smooth, irreducible and projective curve over k, and let X be a normal proper variety over k. If X is g-connected (resp. C-connected), then for any $i \geqslant 1$ and $g \geqslant 1$, $R^i(X)$ is g-connected (resp. C-connected) as well.*

Curve-connectedness is preserved along the MRC sequence

If X is a C-connected normal proper variety, then it could happen that $R^i(X) = X$ for all $i \geqslant 1$, so we would not get so much from Lemma 4.4.6.
Thus, we need a way to ensure that the dimension of $R^i(X)$ always decreases, up to a certain point for C-connected varieties. One sufficient condition for this, is the following:

Definition 4.4.7. *Let X be a variety over k with $\dim(X) = m$. We say that X is* uniruled *if there exists a variety Y of dimension $m-1$ together with a dominant morphism $\mathbb{P}^1 \times_k Y \to X$.*

Uniruled varieties fulfill our needs, because:

Remark 4.4.8. *If X is a normal proper and uniruled variety over k and we consider its MRC fibration $X \dashrightarrow R^1(X)$, by the property that characterizes the MRC over an algebraically closed field (Proposition 4.4.3), we see that $\dim(R^1(X)) < \dim(X)$ and thus the dimensions of two consecutive varieties $R^i(X) \dashrightarrow R^{i+1}(X)$ in the MRC sequence must decrease if $R^i(X)$ is uniruled.*

Are C-connected varieties uniruled then? The answer is the following:

Proposition 4.4.9 (Prop. 3.6 [29])**.** *Let X be a C-connected variety of dimension at least 3. Then X is uniruled.*

Applying this to the MRC sequence of a normal and proper C-connected variety we obtain the following:

MRC sequence of C-connected varieties

Proposition 4.4.10 (Prop. 3.7 [29])**.** *Let X be a normal and proper C-connected variety over k, with C a smooth and proper curve. Then, all members of the MRC sequence of X are C-connected and the sequence ends in either a surface, a curve or a point.*

This means that there exist $n \geqslant 0$ such that $R^n(X)$ is a surface, a curve or a point, and after that point the MRC sequence stabilizes.

Proof. The fact that all members are C-connected comes from Lemma 4.2.6.

For the end of the sequence part, if $\dim(X) < 3$ there is nothing to prove. If $\dim(X) \geqslant 3$ then X is uniruled by Proposition 4.4.9 and thus $\dim(R^1(X)) < \dim(X)$ as mentioned in Remark 4.4.8. By Lemma 4.2.6 we have that $R^i(X)$ is C-connected for every $i \geqslant 1$, and thus $\dim(R^{i+1}(X)) < \dim(R^i(X))$ as long as $\dim(R^i(X)) \geqslant 3$.

As the dimension strictly decreases, there will be a point when $R^n(X)$ has dimension smaller than 3, thus the MRC fibration will either stabilize after $R^n(X)$ or further in the sequence, with an end of dimension smaller than 3, finishing the proof. □

If we briefly touch the case of characteristic zero, we have a completely different behavior in this case:

MRC fibrations in characteristic zero

Lemma 4.4.11. *Let X be a normal proper variety over an algebraically closed field of characteristic zero. Then, $R^1(X)$ is not uniruled.*

Proof. This is a consequence of Proposition 4.2.11 applied to $\mathbb{P}^1_k$: Any flat morphism $f : X \to \mathbb{P}^k_1$ whose geometric generic fiber is separably rationally connected is $\mathbb{P}^1_k$-connected, thus X rationally connected. In characteristic zero, "rationally connected" and "separably rationally connected" are the same concept (Remark 3.4.20).

In short we have: if X is a normal property over k, with a flat morphism $f : X \to \mathbb{P}^k_1$ whose geometric generic fiber is rationally connected, then X rationally connected. So we conclude that the MRC fibration is not uniruled in this case by [39, §IV.5 Prop. 5.7]. □

This lemma combined with Proposition 4.4.10 yields:

MRC fibration of a C-connected variety in characteristic zero

Corollary 4.4.12. *Let X be a normal and proper C-connected variety over an algebraically closed field of characteristic zero, with C a smooth and proper curve. Then, $R^1(X)$ is either a point, a curve or a C-connected surface.*
In the case, $R^1(X)$ is a curve C', the MRC fibration of X is a globally defined morphism $f : X \to C'$.

The last part of the corollary comes from [55, Remark 4].
Now we will begin the study of the FGS of a C-connected normal proper variety over k, with a rational point $x \in X(k)$ and k of positive characteristic.
The easiest case to describe, is the one inspired by the MRC fibration of a C-connected variety in characteristic zero, and is a corollary of Proposition 4.2.11.

Corollary 4.4.13. *Let X be a projective and smooth variety over an algebraically closed field k, and let $f : X \to C$ be a flat morphism to a smooth and projective curve whose geometric generic fiber is separably rationally connected, so X is C-connected. Then, the morphism $f : X \to C$ induces an isomorphism of fundamental group-schemes $\pi^N(X) \cong \pi^N(C)$ for compatible rational points.*

Proof. In this case, as k is perfect the geometric generic fiber of f is smooth and irreducible using [33, Théorème 12.1.6] and [27, Exc. 6.20], in particular it has a trivial FGS by Proposition 3.4.21, so the conclusion follows from Proposition 3.4.11. □

At the time of writing the description of the FGS for C-connected varieties remains an open problem, but we will state here the best guess the author has so far about it:

Definition 4.4.14. *A morphism of affine group-schemes $\phi : G \to H$ over a field k has* finite index *if for any faithfully flat morphism $\rho : H \to Q$ to a group-scheme Q, such that the composition $\rho \circ \phi : G \to Q$ has finite image, we have that Q is finite.*

Conjecture: FGS of C-connected varieties

Conjecture 4.4.15. *Let X be a projective (proper) normal and C-connected variety over k, where C is a smooth projective curve. Let $x \in X(k)$ be a rational point, then there exists a non-constant morphism $f : C \to X$ passing trough x, such that the image of the induced morphism between fundamental group-schemes $\pi^N(f) : \pi^N(C, c) \to \pi^N(X, x)$ has finite index where $c \in C(k)$ is a rational point over x.*

Corollary 4.4.13 shows that, for the strong hypotheses of Proposition 4.2.11 the conjecture is true, and Proposition 3.4.21 shows that the conjecture is true for rationally connected (i.e. $\mathbb{P}^1_k$-connected) varieties.
Before discussing some approaches to solve Conjecture 4.4.15, the following results detail the general behavior of the FGS along a MRC sequence.

Lemma 4.4.16. *Let X be a normal and proper variety over k. Then if $X \dashrightarrow R^1(X)$ is the MRC fibration of X, defined over an open dense subset $U \subset X$ as $f : U \to V$ where V is an open subset of $R^1(X)$. Then for $x \in X(k)$ and $z \in V(k)$ such that $f(x) = z$, the induced morphism $\pi^N(f) : \pi^N(U,x) \to \pi^N(V,z)$ is faithfully flat and so is the composition $\pi^N(U,x) \to \pi^N(R^1(X),z)$.*

This lemma is easily deduced from [53, II Prop. 6] and Proposition 3.4.1.

Remark 4.4.17. *Keeping the hypotheses of Lemma 4.4.16, as X is normal the open immersion $U \hookrightarrow X$ induces a faithfully flat morphism $\pi^N(U,x) \to \pi^N(X,x)$ for $x \in X(k)$ lying in U, we have the following diagram of FGSs where all arrows are faithfully flat:*

$$\begin{tikzcd} & \pi^N(X,x) & \\ \pi^N(U,x) \arrow[ur] \arrow[r, "\pi^N(f)"'] & \pi^N(V,z) \arrow[r] & \pi^N(R^1(X),z) \end{tikzcd} .$$

For MRC fibrations we have the following conjecture:

Conjecture: FGS along MRC fibrations

Conjecture 4.4.18. *Let X be a normal and proper variety over k, with a rational point $x \in X(k)$. If X is either C-connected, where C is a smooth projective curve, or g-connected for $g \geqslant 1$, and $X \dashrightarrow R^1(X)$ is the MRC fibration of X, defined over an open dense subset $U \subset X$ as $f : U \to R^1(X)$. Then, for $x \in X(k)$ and $z \in R^1(X)(k)$ such that $f(x) = z$, the induced morphism $\pi^N(f) : \pi^N(U,x) \to \pi^N(R^1(X),z)$ has finite kernel.*

Remark 4.4.19. *The conjecture might not be true, as there are examples of varieties X for which the MRC sequence ends in a point, yet the étale fundamental group of X is infinite, see [29, §8]. The example in question is not curve-connected as demanded in Conjecture 4.4.18.*

We finish this chapter by outlining two possible approaches to describe the FGS of C-connected varieties.

Two possible approaches to describe the FGS of C-connected varieties

Remark 4.4.20. *Let X be a normal and proper C-connected variety over an algebraically closed field k, with C a smooth and proper curve. If $x \in X(k)$, to describe $\pi^N(X,x)$ we could:*

1. *Study C-CPC varieties and the C-fundamental group-scheme $\pi^C_S(X,x)$ (Definition 4.3.21), and show that C-connected varieties are C-CPC, which is an open problem as we lack a projective/proper moduli space to introduce in the dominant morphism*

$$u^{(2)} : C \times_k C \times_k Y \to X \times_k X$$

in Definition 4.2.3, where Y is a variety. It can be shown that Y can be taken to be a closed sub-variety of the Hom-scheme $\mathrm{Hom}_k(C,X)$ ([29, Lemma 3.4 (2)]), but this scheme is quasi-projective.

2. *Use the MRC sequence: for simplicity, we will assume all the rational maps in the MRC are globally defined morphisms, so we have a sequence of morphisms*

$$X := R^0(X) \to R^1(X) \to \cdots \to R^n(X)$$

where $R^n(X)$ *is the end of the sequence. Assuming Conjecture 4.4.18, the problem of describing* $\pi^N(X, x)$ *may amount to study the morphisms of fundamental group-schemes* $\pi^N(R^i(X)) \to \pi^N(R^{i+1}(X))$ *and show that if Conjecture 4.4.15 is true for* $R^{i+1}(X)$*, then it is also true for* $R^i(X)$*, so we can inductively show that* $\pi^N(X, x)$ *satisfies the conjecture. Thus, from Proposition 4.4.10 we need to study three base cases:*

   (a) *If* $R^n(X)$ *is a point, then* $R^{n-1}(X)$ *is rationally chain connected of arbitrary dimension and* C*-connected, so this case could be the "wildcard" of the three as it is the most general.*

   (b) *If* $R^n(X)$ *is a surface, then we need to study* C*-connected surfaces. An interesting example of this type, suggested by F. Gounelas, is the where* S *is an abelian* C*-connected surface, as in this case it can be shown that* S *is* C*-CPC and this extends to abelian varieties of dimension* > 2*.*

   (c) *If* $R^n(X)$ *is a non-rational smooth proper curve* C'*, as this variety is* C *connected, it comes with a prescribed morphism* $f : C \to C'$ *and a commutative diagram*

$$\begin{tikzcd} & R^{n-1}(X) \arrow[dd, "h"] \\ C \arrow[ur, "g"] \arrow[dr, "f"'] & \\ & C' \end{tikzcd}$$

   where $h : R^{n-1}(X) \to C'$ *is a proper faithfully flat morphism with rationally chain connected, geometrically normal and geometrically irreducible geometric fibers, by [33, Théorème 12.1.6] and [27, Exc. 6.20]. The key here, would be describing*

$$\pi^N(h) : \pi^N(R^{n-1}(X)) \to \pi^N(C'),$$

   which is at least faithfully flat by Lemma 4.4.16, and

$$\pi^N(f) : \pi^N(C) \to \pi^N(C').$$

   We conjecture $\pi^N(f)$ *has finite index.*

5 FUNDAMENTAL GROUP-SCHEME OF SOME RATIONALLY CONNECTED FIBRATIONS

5.1 INTRODUCTION

In this chapter, we will apply the theory of the FGS, explained in Chapters 2 and 3 to describe a the fundamental group-scheme of proper varieties fibered over curves, with rationally connected fibers.
We will provide a complete description when the base curve is an elliptical curve, and a partial one for arbitrary curves.
Let k be a field, and let $f : X \to S$ be a faithfully flat proper morphism over k between a proper normal variety and a smooth proper curve, with geometrically connected fibers, we will call such a morphism a *fibration over a curve*.
If k is algebraically closed of positive characteristic, and motivated by the problem of describing the FGS of curve-connected varieties, for instance, C-connected varieties (Definition 4.2.3), and as discussed in Remark 4.4.20(2), it is necessary to study the FGS of "rationally connected fibrations" over curves, where by rationally connected fibration we mean a fibration over a curve with rationally (chain) connected fibers.
In this case, the geometric fibers of f are also normal and geometrically reduced by [33, Théorème 12.1.6] and [27, Exc. 6.20], in particular any fiber has a finite FGS by Proposition 3.4.18! A particular fiber of interest is the geometric generic fiber $X_{\bar{\eta}}$ where $\bar{\eta} : \mathrm{Spec}(L) \to S$ is the geometric generic point of S which has finite FGS in this case, and the study of the relation between torsors over X and their pull-backs to $X_{\bar{\eta}}$ will be key through this chapter.
Another curve-connectedness property, that motivated the main result of this chapter, are elliptically connected varieties, that leads to the study of rationally connected fibrations $f : X \to S$ where S is an elliptic curve, see Remark 5.1.5 below. The FGS of abelian varieties is well known (Proposition 3.4.23), and Nori-reduced torsors over abelian varieties have strong properties (Corollary 3.4.25) that allows us to have more tools at our disposal to describe the morphism of fundamental group-schemes $\pi^N(f) : \pi^N(X) \to \pi^N(S)$ for which we know the fundamental group-schemes of both the base and the fibers.
Before introducing the main result, let us start with by describing some properties of elliptically connected varieties.

5.1.1 *Motivation: Understanding the FGS of elliptically connected varieties*

Let k be an uncountable algebraically closed field of positive characteristic, in this subsection we will study elliptically connected varieties over k (see Remark 4.2.2), and the MRC sequences (Definition 4.4.5) of these, in a similar fashion to what we did in Subsection 4.4.2 for C-connected varieties.
At the end, we will mention two approaches we can undertake in order to describe the FGS of these varieties, serving as motivation for the main result in the next subsection.
In the case of C-connected varieties, in Proposition 4.4.10 we fully described the MRC sequence using the fact that C-connected varieties of dimension greater or equal than 3 are uniruled (Definition 4.4.7). We have in fact a stronger description for elliptically connected varieties:

MRC sequence of elliptically connected varieties

Proposition 5.1.1. *Let X be a smooth and proper elliptically connected variety over k. Then, all members of the MRC sequence of X*

$$X := R^0(X) \dashrightarrow R^1(X) \dashrightarrow \cdots \dashrightarrow R^n(X) \dashrightarrow \cdots$$

are elliptically connected and the sequence ends in either a curve or a point.

The proof of this proposition is similar to the proof of Proposition 4.4.10, but the ending of the MRC sequence in this case is more restricted thanks to:

Proposition 5.1.2 (Prop. 6.1 [29])**.** *Let X a smooth and proper elliptically connected variety over k of dimension at least 2. Then X is uniruled.*

With this, we can fully describe elliptically connected varieties when k has characteristic zero, recall that in characteristic zero, for the MRC fibration $X \dashrightarrow R^1(X)$ the variety $R^1(X)$ cannot be uniruled (Lemma 4.4.11), and combining this with Proposition 5.1.1, we obtain:

Elliptically connected varieties in characteristic zero

Proposition 5.1.3 (Theorem 6.2 [29])**.** *Let X be a smooth and projective variety over an algebraically closed field of characteristic 0. Then X is elliptically connected if and only if it is either rationally connected or a rationally connected fibration over an elliptic curve.*

In the case of a rationally fibration $f : X \to S$, the geometric generic fiber of f is separably rationally connected, but in characteristic zero this is a given (Remark 3.4.20). But if we carry these hypotheses to the positive characteristic case we obtain:

Proposition 5.1.4. *Let X be a projective and smooth variety over an algebraically closed field k, and let $f : X \to S$ be a flat morphism to an elliptic curve whose geometric generic fiber is separably rationally connected, so X is S-connected[1]. Then, the morphism $f : X \to S$ induces an isomorphism of fundamental group-schemes $\pi^N(X) \cong \pi^N(S)$ for compatible rational points.*

1 And elliptically connected too

which is a particular case of Corollary 4.4.13.

For general smooth proper elliptically connected varieties, we can outline two approaches to understand their FGS.

Two possible approaches to describe the FGS of elliptically connected varieties

Remark 5.1.5. *Let X a smooth and projective proper connected variety over k. If $x \in X(k)$, we propose two approaches to describe $\pi^N(X, x)$:*

1. *We showed in Proposition 4.3.15 that if X is projective, it is $\underline{Curv}_k(1)$-CPC where $\underline{Curv}_k(1)$ is the family of projective and irreducible curves of arithmetic genus $\leqslant 1$.*
   Thus, X possesses an associated $(S, \mathcal{C})$-fundamental group-scheme of Definition 3.3.67, called the 1-fundamental group-scheme, that will denote as $\pi_1^S(X, x)$, see Definition 4.3.16, and we can try to study the faithfully flat natural morphism
   $$\pi_1^S(X, x) \to \pi^N(X, x).$$
   One possible approach here, is to study in dept the categories NSS(E) and EF(E) when E is an elliptic curve. Descriptions of vector bundles over elliptic curves in positive characteristic are well known, see [56].

2. *Use the MRC sequence: for simplicity, we will assume all the rational maps in the MRC are globally defined morphisms, so we have a sequence of morphisms*
   $$X := R^0(X) \to R^1(X) \to \cdots \to R^n(X)$$
   where $R^n(X)$ is the end of the sequence. Here make the same analysis we did in Remark 4.4.20(2).
   Thus, we need study two base cases, determined by Proposition 5.1.1, which are:
   (a) *If $R^n(X)$ is a point, then $R^{n-1}(X)$ is rationally chain connected of arbitrary dimension, and we are in the same situation as in 4.4.20(2a).*
   (b) *If $R^n(X)$ is a non-rational smooth proper curve S that is elliptically connected, and then it comes with a prescribed morphism $f : C \to S$ where C is an elliptic curve. Thus, we must have that S is an elliptic curve as well, and we have a commutative diagram*
   $$\begin{tikzcd} & R^{n-1}(X) \arrow[dd, "h"] \\ C \arrow[ur, "g"] \arrow[dr, "f"'] & \\ & S \end{tikzcd}$$
   where $h : R^{n-1}(X) \to S$ is proper a rationally connected fibration over an elliptic curve. So we are in a similar situation to Proposition 5.1.4 except the geometric generic fiber $X_{\bar{\eta}}$ is just

rationally connected.
It is not hard to show that

$$\pi^N(f) : \pi^N(C) \to \pi^N(S)$$

has finite index (Definition 4.4.14) by using Remark 3.4.24 and Proposition 3.4.15.
So we need to describe the morphism of fundamental group-schemes

$$\pi^N(h) : \pi^N(R^{n-1}(X)) \to \pi^N(S)$$

which is the main result of this chapter (see Theorem 5.5.10), this morphism is a priori faithfully flat by Lemma 4.4.16.

5.1.2 *Main results and summary*

Motivated by Remark 5.1.5(2b), we would like to describe for a rationally connected proper fibration over an elliptic curve $f : X \to S$, the induced morphism $\pi^N(f) : \pi^N(X, x) \to \pi^N(S, s)$ between fundamental group-schemes for compatible rational points $x \in X(k)$ and $s \in S(k)$.
In this case, we know the FGS of both the base (an elliptic curve) and the FGS of fibers, that are normal proper and rationally connected, so we could use the "homotopy exact sequence" for the FGS (Theorem 5.5.2), that allows us to relate all of these fundamental group-schemes. L.Zhang showed in [69] that this exact sequence holds if one of a set of equivalent conditions hold. For a proper rationally connected fibration over an elliptic curve $f : X \to S$ we can indeed show that this exact sequence hold, and in fact we can prove this in a more general situation:

Main theorem

Theorem (Theorem 5.5.1)**.** *Let k be an uncountable algebraically closed field, let X be a proper variety over k and let S be an elliptic curve over k.*
If $f : X \to S$ is a proper faithfully flat morphism, such that all geometric fibers are reduced, connected and possess a finite fundamental group-scheme[2].
Then, there exist rational points $x \in X(k)$ and $s \in S(k)$ such that $f(x) = s$ and the following sequence of group-schemes is exact:

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1.$$

First, we need to remark that if this exact sequence holds, as $\pi_1^N(X_s, x)$ is finite, then the kernel $\ker(\pi^N(f))$ of $\pi^N(f)$ must be finite too. In fact, we will show the opposite:

Proposition (Proposition 5.4.7)**.** *Under the hypotheses of the theorem above[3], $\ker(\pi^N(f))$ is finite.*

2 For example, if X is normal and the fibers are rationally connected
3 Except the assumption that k is uncountable.

And this result implies the homotopy exact sequence. How is that so? This is because we can conceive the kernel as a FGS: there is a pro-finite pointed torsor $X^* \to X$, called the *universal pull-back torsor* (Definition 5.3.8) that possesses a FGS in the sense of Definition 3.2.2, and such that the canonical short exact sequence given by the kernel of $\pi^N(f)$ becomes (Proposition 5.3.10)

$$1 \to \underbrace{\pi_1^N(X^*, x^*)}_{\ker(\pi^N(f))} \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1$$

which is in fact, a particular case of an extension of Proposition 3.4.15 for projective limits of finite Nori-reduced torsors, called *pro-NR torsors*, see Proposition 5.3.4.
So, in order to prove that $\ker(\pi^N(f))$ is finite, we need to study finite pointed Nori-reduced torsors over X^*, so we introduce a new terminology for torsors over X that allows describe torsors over X^*, by diving torsor in 3 three disjoint categories: pure[4] torsors, pull-backs of torsors over S, and "mixed" torsors, that are a combination of the first two types (Definition 5.2.1) as Nori-reduced mixed torsors $T \to X$ can be divided as:

$$\begin{tikzcd} T \arrow[d] & \\ f^{\dagger}(T) \arrow[r, "f'"] \arrow[d] & S' \arrow[d] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

where $f^{\dagger}(T)$ is the "largest" quotient of T that is a pull-back of a torsor over S (S' in the diagram above), see Remark 5.2.2(a). T as a torsor over $f^{\dagger}(T)$ is pure with respect to f', and any finite Nori-reduced torsor over X^* descends to either a pure torsor over X or a torsor over X that is pure over the pull-back of a Nori-reduced torsor over S (Proposition 5.3.11) by applying a nice descent property satisfied by pro-NR torsors stated in Subsection 2.3.4, see Proposition 2.3.61.
Thus, to show that $\pi^N(f)$ is finite, we will show that the order of the finite group-scheme associated to any finite torsor over X^* is bounded (Lemma 5.4.6).
The main result that allows to bound these orders, which also works for for any smooth proper curve giving a partial characterization of for the morphism $\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s)$ for any proper fibration $f : X \to S$ over a smooth curve S, is:

Proposition (Proposition 5.2.19)**.** *Let* k *be an algebraically closed field,* X *a proper variety over* k *and let* S *be a smooth and proper curve over* k*, with a proper faithfully flat morphism* $f : X \to S$ *between them. We will assume that all geometric fibers are reduced, connected, so they possess a FGS. Let*

4 With respect to a given morphism.

$t : T \to X$ *be a Nori-reduced pure* G*-torsor over* X*. Then, the pull-back* $T_{\bar{\eta}} \to X_{\bar{\eta}}$ *to the geometric generic fiber of* f *is Nori-reduced, where* $\bar{\eta}$ *is the geometric generic point over* S*.*

The proof of this result is in Subsection 5.2.2, and it uses some results coming from the theory of the FGS for schemes over Dedekind schemes, see [4].
Coming back to the fibration over an elliptic curve, this result only shows that pure torsors over X have bounded orders for their associated group-schemes. But as a special feature of elliptic curves (Lemma 5.4.2), we can actually reuse the same result above for any Nori-reduced torsor $S' \to S$ over the pull-back $f' : X' \to S'$ over X, which allows us to also bound pure torsors over pull-back torsors, by comparing these torsors over the respective geometric generic fibers (of f and of f'), see Lemma 5.4.4.
Once we establish that the kernel is finite, we can tackle the homotopy exact sequence. The conditions needed for this exact sequence to hold must be satisfied for any Nori-reduced torsors, and one of them, the "base change condition", is highly dependent on a rational point $s \in S(k)$ of our choosing (see Definition 5.5.3).
So, to satisfy the conditions for this exact sequence, we can show that pull-back torsors always satisfy the conditions for the exact sequence, and that for pure or mixed Nori-reduced torsor T there exists an open dense subset $U_T \subset S$ such that T satisfies the conditions for the homotopy exact sequence (see Subsection 5.5.2). In that case, we could try to intersect all of these open subsets, for if we find a rational point on this intersection, the homotopy exact sequence would finally hold.
But the problem with this, is that the intersection of all the sets U_T might not contain any rational points at all, as there might be "too many" finite torsors over X.
Here is where the finite kernel steps in, as it is the FGS of the pro-NR torsor X^*, which is the limit over all pull-backs of Nori-reduced torsors over S, and using the fact that the universal torsor $\hat{X} \to X^*$ is finite. It descends to a finite Nori-reduced torsor $X_i^* \to X_i$ where X_i is the pull-back of a Nori-reduced torsor over S.
Depending on the torsor X_i where X^* descends to, we have:

(a) If X^* descends over X, then any pure and mixed Nori-reduced torsors is essentially a quotient of this descent, and as an elliptic curve has a countable amount of isomorphism classes of Nori-reduced finite torsors (see Proposition 3.4.23), we see that X would posses a countable amount of isomorphism classes of Nori-reduced finite torsors too, and then we can intersect all the open dense subsets U_T from previous paragraphs.
So, if k is uncountable, as this intersection is countable, we can find a very general (Definition 4.2.8) point of S for which we can establish the homotopy exact sequence, proving Theorem 5.5.1. This is Lemma 5.5.9

(b) If X^* descends over a pull-back torsor X_i, if S_i is the corresponding torsor over S with a morphism $f_i : X_i \to S_i$, then we are in the situation above and the FGS of these schemes satisfy the homotopy exact sequence

$$\pi_1^N(X_{i,s'}, x') \to \pi_1(X_i, x') \to \pi_1^N(S_i, s') \to 1$$

for some $s' \in S_i(k)$ where $X_{i,s'}$ is the corresponding fiber, and this allows us to indirectly obtain the homotopy exact sequence for $f : X \to S$.

In synthesis, in order to obtain the homotopy exact sequence, we need:

1. To divide torsors with respect to f, as pure, mixed or a pull-back. This is done in Subsection 5.2.1.

2. To show that any Nori-reduced pure torsor over X remains Nori-reduced when pulled-back over the geometric generic fiber $X_{\bar{\eta}}$ that we show in Proposition 5.2.19, using the results in Subsections 5.2.2, 5.2.3 and 5.2.4.

3. To study projective limits of Nori-reduced torsors, and generalize the short exact sequence for finite Nori-reduced torsors of Proposition 3.4.15. This is the content of Subsection 5.3.1.

4. To apply the previous point to the particular case of the family of all torsor over X that are the pull-back of a Nori-reduced torsor over S, whose limit is the universal pull-back torsor X^*, that we will do in Subsection 5.3.2.

5. To show the kernel of $\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s)$ is finite in Subsection 5.4.2 by:
   (i) Bounding the order of group-schemes acting on pure Nori-reduced torsors using point (2).
   (ii) And then, do the same for mixed torsors $T \to X$ using the nice properties of pull-backs to X of torsor over elliptic curves (Lemma 5.4.2), so we can use point (2) once again and bound the group-scheme acting on the pure part $T \to f^\dagger(T) \xrightarrow{f'} S'$ of T, where S' is a Nori-reduced torsor over S and compare the pull-backs to the geometric generic fibers over S and over S' in Subsection 5.4.1.

6. To simplify the conditions for the homotopy exact sequence, that depend on a rational point of S and must be satisfied individually for all Nori-reduced torsors over X. Finding that pull-backs always satisfy the conditions, and that mixed and pure torsors satisfy them generically, i.e., for any torsor T of the two types mentioned before, there exist a dense open subset U_T for

which the conditions are satisfied over any rational point on U_T. We show this in Subsection 5.5.2.

7. And finally, using the finiteness of the kernel in part (4) we obtain the homotopy exact sequence in Subsection 5.5.3, as we can show that one of two cases hold:
   (i) There is a countable amount of isomorphism classes of mixed and pure torsors over X so we can use part (6) to obtain a the homotopy exact sequence over a very general point of S, in the intersection of all the opens subsets U_T, for which the homotopy exact sequence is true.
   (ii) Or X has a torsor $X' \to S'$ that is the pull-back of a Nori-reduced torsor $S' \to S$, so X' is in the situation above, and thus we can indirectly show that the homotopy exact sequence is satisfied for $f : X \to S$.

5.2 PULL-BACK OF TORSORS TO THE GEOMETRIC GENERIC FIBER

5.2.1 *Pure, mixed and pull-back torsors*

Let us start by introducing a new classification for torsors with respect to a morphism of schemes:

Pull-back, pure and mixed torsors

Definition 5.2.1. *Let X and be Y two schemes over k that both possess a FGS, with rational points $x \in X(k)$ and $y \in Y(k)$ and a morphism $f : X \to Y$ that is compatible with the respective rational points.*
Let T be a G-torsor over X. We say that T is a pull-back torsor *if there exists a torsor $S' \to S$ such that its pull-back along f, $S' \times_S X \to X$ is T, trivial torsors are considered pull-back torsors.*
We say that a non-trivial torsor T is pure with respect to f *if neither T nor its non-trivial quotients (Definition 2.3.51) are the pull-back of a torsor over S via f.*
A torsor which is neither a pull-back nor a pure torsor is called a mixed torsor.

Remark 5.2.2. *Keeping the notation in Definition 5.2.1, let us consider the induced morphism between fundamental group-schemes*

$$\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s).$$

If we assume that $\pi^N(f)$ is faithfully flat, let T be a Nori-reduced G-torsor over X, then we have the natural composition $\ker(\pi^N(f)) \to \pi_1^N(X) \to G$ whose image we will be denoted as $K \subset G$, this is a normal subgroup-scheme of G, and the relationship between this subgroup-scheme and G determines the nature of T:

(a) *If K is a proper sub-group-scheme of G, the arrow $\pi_1^N(X) \to G/K$ factors through $\pi^N(f)$ and thus it corresponds to a G/K-torsor $t' : X' \to X$ that is the pull-back of a Nori-reduced torsor $p' : S' \to S$ over the same group-scheme, this pull-back quotient is the "largest", in the sense that any other quotient of T that is a pull-back is a quotient X'. In the particular case when K is trivial, T itself is a pull-back.*
We see that X' is unique, so from now on, we will call the torsor X' the maximal pull-back quotient of T *and we will denote this torsor as $f^\dagger(T)$.*
If we denote the quotient morphisms from T as $q_\dagger : T \to f^\dagger(T)$, we see that T is a Nori-reduced K-torsor over $f^\dagger(T)$.

The maximal pull-back quotient

(b) *From the latter point, we observe that a Nori-reduced torsor is pure if and only if $K = G$.*

(c) *A mixed G-torsor $p : T \to X$ is certainly not pure with respect to f, but the torsor $q_\dagger : T \to f^\dagger(T)$ is pure with respect to the base change of f to S', $f' : f^\dagger(T) \to S'$.*

The objective of this section, is to study for a pure Nori-reduced torsors for a proper faithfully flat morphism $f : X \to S$ between a proper variety and a proper smooth curve S when we consider their pull-backs to the geometric generic fiber of X, that we will denote as $X_{\bar{\eta}}$ where $\bar{\eta}$ is the geometric generic point of S, the main result is Proposition 5.2.19.

5.2.2 *Descent of a quotient for pull-backs of pure torsors over the geometric generic fiber*

From this point on, we will work under the following setting:

Setting 5.2.3. *The field k is an algebraically closed field, X is a proper variety over k and S is a smooth and proper curve over k, with a proper faithfully flat morphism $f : X \to S$ between them. We will denote its induced morphism at the level of FGS as $\pi^N(f) : \pi_1^N(X) \to \pi_1^N(S)$.*
We will further assume that all geometric fibers are reduced, connected, so they possess a FGS (Definition 3.2.2). This includes the geometric generic fiber $X_{\bar{\eta}}$ where η is the generic point of S.

We start with a simple remark:

Remark 5.2.4. *Let $t : T \to X$ be a G-torsor where G is a finite group-scheme. Then, there exists a finite extension L of $\kappa(\eta)$ such that the base change $T_L \to X_L$ is pointed over L, and thus it corresponds to an arrow $\pi^N(X_L, x) \to G_L$ for some $x \in X_L(L)$.*

To shorten the length of the Proposition 5.2.9's statement below, we will introduce some notation:

Notation 5.2.5. *We will add to Setting 5.2.3 the following notations: If $t : T \to X$ is a finite Nori-reduced G-torsor over X, not necessarily pointed, we will denote as L the finite extension of $\kappa(\eta)$ that makes $t_L : T_L \to X_L$ a pointed torsor (Remark 5.2.4).*
Over L, we will consider $n : S' \to S$ the normalization of S over L. It is a proper Dedekind scheme ([27, p. 486]) and thus it is a smooth projective curve over k, finite and ramified over S with generic point $\eta' = \mathrm{Spec}(L)$.
Finally, we will consider the geometric generic fiber $X_{\bar\eta}$ and the pull-back to T to it $t_{\bar\eta} : T_{\bar\eta} \to X_{\bar\eta}$. If T is not the pull-back of a Nori-reduced torsor over S, then $T_{\bar\eta}$ is not trivial by Remark 3.4.8, and thus $T_{\bar\eta}$ possesses a canonical Nori-reduced sub-torsor $V \subset T_{\bar\eta}$ (Definition 3.2.16) over a non-trivial subgroup-scheme $H_{\bar\eta}$ of $G_{\bar\eta}$, corresponding to the image of the morphism $\pi^N(X_{\bar\eta}) \to G_{\bar\eta}$.

And a key lemma:

Lemma 5.2.6. *Let $S \to \mathrm{Spec}(k)$ be a smooth irreducible projective curve over k. Let G be a finite group-scheme over k and let H be a subgroup-scheme of G_S that satisfies the following conditions:*

(a) The structural morphism $s_H : H \to S$ is faithfully flat.

(b) The coherent $\mathcal{O}_S$-algebra $(s_H)_*(\mathcal{O}_H)$ satisfies
$$h^0(S, (s_H)_*(\mathcal{O}_H)) = \mathit{rank}((s_H)_*(\mathcal{O}_H)).$$

Then, if $H \to H' \to \mathrm{Spec}(k)$ is the Stein factorization of the composition $H \to S \to \mathrm{Spec}(k)$ or equivalently $H \hookrightarrow G_S \to G \to \mathrm{Spec}(k)$, then H' is a subgroup-scheme of G and $H \cong (H')_S$.

Remark 5.2.7. *For Stein factorization we mean the factorization of a proper morphism $f : X \to Y$ into $X \xrightarrow{g} Z \xrightarrow{h} Y$ where g is a proper morphism with geometrically connected fibers, $Z = \underline{\mathrm{Spec}}((g)_*(\mathcal{O}_X))$, and h is the the relative normalization of Y with in X, see [31, p. 6.3.5] or [63, Tag 035H].*
As a relative normalization, the morphism $g : X \to Z$ factors trough any factorization $Y \to W \to Y$ when the second morphism is integral, i.e. affine and universally closed, in the sense that there exists a unique morphism $Z \to W$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "g"'] \arrow[d] & W \arrow[d] \\ Z \arrow[ur, dashed, "\exists!"'] \arrow[r, "h"'] & Y \end{tikzcd}$$

is commutative (see [63, Tag 035I] for more details).
Finally, if Y is locally noetherian, then h is finite.

Proof of Lemma 5.2.6. Let us consider the cartesian diagram

$$\begin{tikzcd} G_S \arrow[r, "\sigma_S"] \arrow[d, "p_S"'] & S \arrow[d, "p"] \\ G \arrow[r, "\sigma"'] & \mathrm{Spec}(k) \end{tikzcd}$$

we will start by showing that $G_S \to G \to \mathrm{Spec}(k)$ is the Stein factorization of the arrow $p \circ \sigma_S : G_S \to \mathrm{Spec}(k)$. Let $G_S \xrightarrow{g_S} Y \xrightarrow{g} \mathrm{Spec}(k)$ be the Stein factorization, we can easily see that the factorization $G_S \to G \to \mathrm{Spec}(k)$ has the properties described before as G is finite over k, and thus we have a unique morphism $h : Y \to G$ of finite schemes over k. Moreover, we have that the fibers of $p_S : G_S \to G$ are geometrically connected as for any K-valued point $\mathrm{Spec}(K) \to G$ with K a field, the fiber of p_S over this point is isomorphism to $S \times_k \mathrm{Spec}(K)$ which geometrically connected as S is. Thus, $(p_S)_* (\mathcal{O}_{G_S}) = \mathcal{O}_G$ and with this we can easily conclude that $h_*(\mathcal{O}_Y) = \mathcal{O}_G$, implying that $Y \cong G$ as we wanted.
Now let us consider H and H' as in the statement of Lemma 5.2.6, we will denote the Stein factorization of H as $H \xrightarrow{n_S} H' \xrightarrow{n} \mathrm{Spec}(k)$. From the properties of Stein factorizations, we have a natural commutative diagram

$$\begin{tikzcd} H \arrow[r, "i"] \arrow[d, "n_S"'] & G_S \arrow[r, "\sigma_S"] \arrow[d, "p_S"'] & S \arrow[d, "p"] \\ H' \arrow[r, "j"'] & G \arrow[r, "\sigma"'] & \mathrm{Spec}(k) \end{tikzcd}$$

where i is a closed immersion.
If we denote $H'' = (H')_S$, we will start by showing that $H'' \cong H$, notice that this implies the morphism j in the diagram above is a closed immersion as the diagram becomes cartesian, and we can apply fpqc descent via the fpqc cover p_S, more specifically Proposition 2.3.8(j).
The claimed isomorphism in the last paragraph follows firstly from the morphism $\lambda : H_S \to H''$ coming from n_S and the composition $H_S \hookrightarrow G_S \to S$, this yields the following commutative diagram

$$\begin{tikzcd} H \arrow[r, "i_S"] \arrow[dr, "\lambda"'] & G_S \\ & H'' \arrow[u, "j_S"'] \end{tikzcd}$$

and as the upper horizontal arrow is a closed immersion, so is λ.
In second place, as both H and H'' are finite and faithfully flat over S, we have that they correspond to two vector bundles over S. If we denote them as $\mathcal{E}$ and $\mathcal{F}$ for H and H'' respectively, we know that $\mathcal{E}$ is a locally free $\mathcal{O}_S$-module while $\mathcal{F}$ is a free $\mathcal{O}_S$-module. If we denote their ranks as m and m' respectively, we know that $m \leqslant m'$ because of λ, so it suffices to show that $m = m'$. But from the commutative diagram

$$\begin{tikzcd} H \arrow[r, "s_H"] \arrow[d, "n_S"'] & S \arrow[d, "p"] \\ H' \arrow[r, "n"'] & \mathrm{Spec}(k) \end{tikzcd}$$

we can deduce that $p_*(\mathcal{E}) = \Gamma(H', \mathcal{O}_{H'})$, but the left hand side is a m-dimensional k-vector space by the hypothesis (b) in the statement that we have imposed on $\mathcal{E}$, while the right hand side has dimension m' so we have just have obtained the desired equality, and thus λ is an isomorphism.
The last thing we need to show now, is that H' is a subgroup-scheme of G. First we will work with the multiplication morphism of H, $m_H : H \times_S H \to H$ which commutes with the closed immersion $i : H \to G_S$. We need to show that H' possesses a multiplication morphism $m : H' \times_k H' \to H'$ that commutes with $j : H' \to G$. Let $H \times_S H \to N \to \mathrm{Spec}(k)$ be Stein factorization of $p \circ s_H \circ m_H$, we clearly have a morphism $m : N \to H'$ by the way the Stein factorization is defined, so it suffices to show that $N \cong H' \times_k H'$. By the universal property of N, we have a natural morphism $N \to H' \times_k H'$ and from the commutative diagram

$$\begin{tikzcd} & H \times_S H \arrow[dl] \arrow[dr] & \\ N \arrow[rr] & & H' \times_k H' \end{tikzcd}$$

where both diagonal morphisms are proper with geometrically connected fibers, and the horizontal morphism is finite. Thus, the morphism $N \to H' \times_k H'$ is finite with geometrically connected fibers, so it is an isomorphism as we desired.
It is not hard to see that we can do the same procedure with $H \times_S H \times_S H$ and the associativity axiom for the multiplication of H, so we conclude that the multiplication morphism of H' is associative.
Finally, as the Stein factorization of $p : S \to \mathrm{Spec}(k)$ is itself, we easily see that H' possesses a unit k-point $e' : \mathrm{Spec}(k) \to H'$ and by considering inv_H, the inverse morphism of H, and the Stein factorization of $p \circ s_H \circ \mathrm{inv}_H$, we will obtain an inverse morphism $\mathrm{inv}_{H'} : H' \to H'$. The verification that these morphisms satisfy the necessary group-scheme axioms is straightforward, so we have concluded the proof. $\square$

Remark 5.2.8. *The proof of Lemma 5.2.6 shows that for the subgroup-scheme H in the statement, that $(s_H)_*(\mathcal{O}_H)$ is a free $\mathcal{O}_S$-module.*

Now we state the main result of this subsection:

Descent of a subgroup-scheme over the geometric generic fiber

Proposition 5.2.9. *Let $f : X \to S$ be a morphism as in Setting 5.2.3, and let $t : T \to X$ be a finite Nori-reduced G-torsor over X, not necessarily pointed, that is not the pull-back of a torsor over S. Keeping the notations outlined in Notation 5.2.5, there exists a non-trivial subgroup-scheme $H' \subset G$ over k, such that the base change of H' to $\bar{\eta}$ is isomorphic to $H_{\bar{\eta}} \subset G_{\bar{\eta}}$.*

Proof. We recall that over a finite extension L of η, the generic point of S, the pull-back of T over L is pointed (Remark 5.2.4). First, we claim that $\overline{\kappa(\eta)} \supset L$ and the morphism

$$\pi^N(X_{\bar{\eta}}) \to \pi^N(X_L, x_L) \times_k \mathrm{Spec}(\kappa(\bar{\eta})),$$

that codifies the behavior of pointed torsors over X_L when pulled-back to $X_{\bar{\eta}}$, is faithfully flat as for any Nori-reduced pointed finite torsor $Y \to X_L$, if we consider the commutative diagram

$$\begin{tikzcd}
Y_{\bar{\eta}} \arrow[r] \arrow[d] & Y \arrow[d] \\
X_{\bar{\eta}} \arrow[r] \arrow[d] & X_L \arrow[d] \\
\mathrm{Spec}(\bar{L}) \arrow[r] & \mathrm{Spec}(L)
\end{tikzcd}$$

that shows that

$$\dim_{\bar{L}}(\Gamma(Y_{\bar{\eta}}, \mathcal{O}_{Y_{\bar{\eta}}})) = \dim_L(\Gamma(Y, \mathcal{O}_Y)) = 1$$

so $Y_{\bar{\eta}} \to X_{\bar{\eta}}$ is Nori-reduced by Proposition 3.3.54, and the claim is proven.

Going back to the pull-back $t_L : T_L \to X_L$ of T over L, we have that T_L cannot be trivial, otherwise $T_{\bar{\eta}}$ would be trivial and thus T satisfies the assumptions of Remark 3.4.8 and thus Grauert's theorem for finite torsors 3.4.7 applies, implying that T we would be the pull-back of a torsor over S, contradicting our assumptions for T. Thus, there exists a non-trivial subgroup-scheme $H_L \subset G_L$ so T_L possesses a canonical pointed Nori-reduced H_L-subtorsor (Definition 3.2.16) $V_L \subset T_L$. H_L corresponds then to the image of the morphism $\pi^N(X_L, x_L) \to G_L$ for some L-rational point x_L that makes T_L pointed.

As L corresponds to the generic point of S', the normalization of S over L as mentioned in Notation 5.2.5, if $T_{S'}$ denotes the base change of T over $X_{S'}$ which is a pointed $G_{S'}$-torsor, by [4, Prop. 3.1] $T_{S'}$ possesses a pointed H-subtorsor where H is the schematic closure of H_L over S', that we will denote as $V_{S'}$. We can easily see that this torsor is the canonical Nori-reduced sub-torsor of $T_{S'}$ and the base change of H to $\bar{\eta}$ is isomorphic to $H_{\bar{\eta}}$.

To conclude the proof, we could apply Lemma 5.2.6 to obtain a subgroup-scheme $H' \subset G$ over k such that $H \cong (H')_{S'}$ that will have the desired base change over $\bar{\eta}$.

To satisfy the requirements of the lemma, first we see that the structural morphism of H, $s_H : H \to S'$, is faithfully flat and thus we just need to show that $(s_H)_*(\mathcal{O}_H)$, that we will denote as $\mathcal{E}$, also satisfies part (b) of the hypotheses on the statement of Lemma 5.2.6. Thus, if n is the order of G over k, and $m \leqslant n$ is the order of H_L, which is the order of $H_{\bar{\eta}}$ and H as well, we would like to show that $h^0(S', \mathcal{E}) = m$.

Let us consider the quotient $q_L : Q_L = T_L/H_L \to X_L$, by Lemma 3.4.5 applied to $(t_L)_*(\mathcal{O}_{T_L})$, we have that $(q_L)_*(\mathcal{O}_{Q_L}) = \mathcal{O}_{X_L}^{\oplus r}$, where $r = \frac{n}{m}$ is the order of the quotient group-scheme G_L/H_L over L, see Remark 3.4.6.

If we now consider the quotient of $T_{S'}$ by H, we obtain morphism $q_{S'} : Q_{S'} \to X_{S'}$ with $(q_{S'})_*(\mathcal{O}_{Q_{S'}}) = \mathcal{O}_{X_{S'}}^{\oplus r}$ as this bundle is generically trivial. This morphism is pointed: we have morphisms $S' \to X_{S'}$ and $S' \to Q_{S'}$, such that the diagram

$$\begin{tikzcd} & S' \arrow[dl] \arrow[dr] & \\ Q_{S'} \arrow[rr, "q_{S'}"'] & & X_{S'} \end{tikzcd}$$

is commutative, and the fiber over the marked S'-point is isomorphic to $G_{S'}/H$.

Thus, this finite scheme over S' corresponds to a free $\mathcal{O}_{S'}$-module $\mathcal{F} \cong \mathcal{O}_{S'}^{\oplus r}$. This implies the desired equality $h^0(S', \mathcal{E}) = m$ as the quotient morphism $G_{S'} \to G_{S'}/H$ is an H-torsor, and the isomorphism $G_{S'} \times_{S'} H \cong G_{S'} \times_{G_{S'}/H} G_{S'}$ translates to the isomorphism of locally free sheaves over $\mathcal{O}_{S'}$

$$\mathcal{G} \otimes_{\mathcal{O}_{S'}} \mathcal{E} \cong \mathcal{G} \otimes_{\mathcal{F}} \mathcal{G}$$

where $\mathcal{G}$ is the free $\mathcal{O}_{S'}$-module of rank n corresponding to $G_{S'}$, and by developing both sides of the isomorphism we will obtain

$$\mathcal{E}^{\oplus n} \cong \mathcal{O}_{S'}^{\oplus n \cdot m}$$

and by comparing dimensions of global sections, we will obtain the desired result. □

Remark 5.2.10. *Proposition 5.2.9 could also work if we replace the smooth proper curve S with a Dedekind scheme with weaker properties.*

From this we deduce the following corollary:

Descent of a quotient over the geometric generic fiber

Corollary 5.2.11. *Let $f : X \to S$ be a morphism as in Setting 5.2.3, and let $t : T \to X$ be a pointed Nori-reduced finite G-torsor over X, that is not the pull-back of a torsor over S. Keeping the notations outlined in Notation 5.2.5, the quotient $T_{\bar{\eta}} \to T_{\bar{\eta}}/H_{\bar{\eta}}$ is the isomorphic to the pull-back of a quotient $T \to T/H'$ to $X_{\bar{\eta}}$, for a certain subgroup-scheme $H' \subset G$.*

Proof. As $H_{\bar{\eta}}$ is the image of the morphism of group-schemes $\pi^N(X_{\bar{\eta}}) \to G_{\bar{\eta}}$, by applying Proposition 5.2.9 we can find a subgroup-scheme $H' \subset G$ such that the base change of H' to $\bar{\eta}$ is isomorphic to $H_{\bar{\eta}}$.

Thus, the quotient T/H' becomes isomorphic to $T_{\bar{\eta}}/H_{\bar{\eta}}$ over $\bar{\eta}$ by Lemma 2.3.43, finishing the proof. □

5.2.3 *Core of a subgroup-scheme*

Let k be a field. In this subsection, we will generalize for finite group-scheme the following: Now we will introduce a group-theoretic notion:

Definition 5.2.12. *Let Γ be an abstract group and let $J \subset \Gamma$ be a subgroup. The* core of J *is the subgroup of Γ defined as*

Core of an abstract subgroup

$$\mathit{Core}_\Gamma(J) = \bigcap_{g \in \Gamma} gJg^{-1}.$$

We can further characterize this abstract subgroup.

Proposition 5.2.13. *Let Γ be an abstract group and let $J \subset \Gamma$ be a subgroup. Then, $\mathit{Core}_\Gamma(J)$ is the largest normal subgroup of Γ contained in J, and if we consider $n = [\Gamma : J]$ the index of J, the core is also the kernel of the morphism $\Gamma \to S_n$ associated to the action $\mu : \Gamma \times \Gamma/J \to \Gamma/J$ of left multiplication by Γ over the set of right J-cosets, where S_n is the symmetric group on n elements. In particular, the core of J is trivial if and only if the action mentioned above is faithful.*

We leave the proof of this proposition to the reader.
We are going to construct an analogous notion of core at least for finite group-schemes that satisfies the same properties of the core of abstract subgroups.

Definition 5.2.14. *Let G be a finite group-scheme over k and $H \subset G$ a subgroup-scheme. As we mentioned in Example 2.2.88, A^H is the k-algebra associated to G/H and the G-action over this scheme a comodule structure for A over A^H, or equivalently, a morphism of group-schemes $\mu_H : G \to GL(A^H)$, see Example 2.4.85(5). We define the* core of H *in G as $\mathit{Core}_G(H) := \ker(\mu_H)$.*

Core of a subgroup-scheme

It is clear from the definition that the core of a subgroup-scheme H is trivial if and only if μ_H is a faithful representation (Definition 2.4.80). We recall that as a functor, we have $GL(A^H)(R) = \mathrm{Aut}_R(A^H \otimes R)$. The main property of the core that we will use later in the section is:

Proposition 5.2.15. *Let G be a finite group-scheme over k and $H \subset G$ a subgroup-scheme. The core of H is the largest normal subgroup-scheme of G contained in H.*

Main property of the core for subgroup-schemes

Proof. Let $F : \mathrm{Alg}_k^0 \to \mathrm{Grp}$ be the functor given by

$$F(R) = \{g \in \widetilde{G}(R) : \forall \text{ R-algebra } R \to S,\ g_S \in \mathrm{Core}_{\widetilde{G}(S)}(\widetilde{H}(S))\}$$

where g_S is the image of g in $\widetilde{G}(S)$, it is clearly a sub-functor of $\widetilde{H}$ and for any k-algebra R, $F(R) \lhd \widetilde{G}(R)$. Moreover, it contains any functor of the form $\widetilde{N}$ where N is a normal subgroup-scheme of G that is contained in H from Proposition 5.2.13. We will show that F is the functor

of points of $\mathrm{Core}_G(H)$.
Let us denote the functor of points of $\mathrm{Core}_G(H)$ as $\widetilde{C}$, and let us start by showing that $\widetilde{C} \subset F$. If $g_R \in \widetilde{C}(R) \subset \widetilde{G}(R)$ we can easily see that from the fact that it induces the identity R-automorphism of $A^H \otimes_k R$, this element acts like the identity over $\widetilde{G/H}(R) = \mathrm{Hom}_{k-alg}(A^H, R)$, in particular, it acts like the identity over $\widetilde{G}(R)/\widetilde{H}(R)$ and thus $g_R \in \mathrm{Core}_{\widetilde{G}(R)}(\widetilde{H}(R))$ and likewise over any R-algebra $R \to S$, from which we conclude that $\widetilde{C} \subset F$.
On the other hand, F acts trivially over $\widetilde{G}/\widetilde{H}$ and then using the properties of $\widetilde{G/H}$ as a sheafification (Proposition 2.2.86), we see that the trivial action of F over $\widetilde{G}/\widetilde{H}$ can be lifted to a trivial action of F over $\widetilde{G/H} = \mathrm{Hom}_{k-alg}(A^H, \cdot)$, this can be viewed as a natural transformation of functors $F \to GL(A^H)$ that has a trivial image over any k-algebra. Thus, $F \subset \widetilde{C}$ finishing the proof. □

5.2.4 *Pull-back of pure torsors over the geometric generic fiber (proof)*

Now we are ready to prove the main result of the section (Proposition 5.2.19). But first, we need some lemmas, the first being:

Lemma 5.2.16. *Let $f : X \to S$ be a morphism of proper, reduced and connected schemes of finite type over k. We will further assume that the induced morphism $\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s)$ is faithfully flat for compatible rational points.*
Let $\mathcal{E}$ be an essentially finite bundle corresponding to a representation of $\pi_1^N(X, x)$ that we suppose to lay inside a full subcategory $Rep_k(G)$ for a finite group-scheme G, such that the Nori-reduced G-torsor associated to the morphism of group-schemes $\pi_1^N(X, x) \to G$ is not the pull-back of a Nori-reduced torsor over S.
Then, if $\mathcal{E}$ corresponds to a faithful representation of G, it cannot be of the form $f^*(\mathcal{F})$ for some $\mathcal{F} \in EF(S)$.

Proof. Let $\alpha : G \to GL(\mathcal{E}_x)$ be the representation morphism corresponding to $\mathcal{E}$, where $\mathcal{E}_x$ is the fiber of $\mathcal{E}$ over x. As a representation of the FGS, it can be seen as the composition $\pi_1^N(X, x) \to G \to GL(\mathcal{E}_x)$. The image of this composition is isomorphic to G.
Let us suppose that $\mathcal{E}$ is the pull-back of an essentially finite bundle $\mathcal{F} \in EF(S)$, this implies that we have a commutative diagram

$$\begin{tikzcd} \pi_1^N(X,x) \arrow[r] \arrow[d, "\pi^N(f)"'] & G \arrow[r, "\alpha"] & GL(\mathcal{E}_x) \\ \pi_1^N(S,s) \arrow[urr] & & \end{tikzcd}$$

as $\pi^N(f)$ is faithfully flat, $\pi_1^N(S, s) \to GL(\mathcal{E}_x)$ factors through G. From this, we can easily see that if $T \to X$ is the G-torsor associated to

$\pi_1^N(X, x) \to G$, it is a pull-back from a G-torsor over S, a contradiction. □

We can apply this to weak quotients (Definition 2.3.54) of pure torsors as follows:

Weak quotients of pure NR torsors are not pull-backs

Lemma 5.2.17. *Let $f : X \to S$ be a morphism of proper, reduced and connected schemes of finite type over k, where the induced morphism $\pi^N(f) : \pi_1^N(X, x) \to \pi_1^N(S, s)$ is faithfully flat for compatible rational points.*
Let $t : T \to X$ be a pure Nori-reduced G-torsor and let H be any non-trivial subgroup-scheme of G. Then, for the morphism $t' : T/H \to X$, the associated essentially finite bundle $(t')_*(\mathcal{O}_{T/H})$ is not the pull-back of any essentially finite bundle over S.

Proof. We will proceed by induction over $\text{ord}(G)$, the case of order 1 is trivial. Let us now suppose $\text{ord}(G) > 1$ and let H be a subgroup-scheme of G. If H is normal, there is nothing to prove as T is pure. Otherwise, we consider the cover $t' : T/H \to X$ corresponding in terms of essentially finite bundles to the sub-representation A^H of A where $G = \text{Spec}(A)$. Let K be the core of H (Definition 5.2.14), there are two possibilities for K:

- If K is not trivial, then we can see T/H as a quotient of T/K from Proposition 5.2.15, and thus result follows by applying the induction hypothesis.
- If K is trivial, the representation corresponding to $(t')_*(\mathcal{O}_{T/H})$ is faithful and thus it is not a pull-back from Lemma 5.2.16.

□

The last lemma we need is:

Lemma 5.2.18. *Let $f : X \to S$ be a faithfully flat morphism between proper, reduced and connected k-schemes with $f_*(\mathcal{O}_X) = \mathcal{O}_S$ and such that $f(x) = s$ for some $x \in X(k)$ and $s \in S(k)$. Let $\mathcal{E} \in \text{Vect}(S)$ be a vector bundle such that $f^*(\mathcal{E})$ is essentially finite, then $\mathcal{E}$ is essentially finite.*

Proof. By 3.4.9, the induced morphism between S-fundamental group-schemes $\pi^S(f) : \pi_1^S(X, x) \to \pi_1^S(S, s)$ is faithfully flat. This implies that the pull-back functor $f^* : \text{NSS}(S) \to \text{NSS}(X)$ is fully faithful and the essential image of this functor is closed by sub-objects (Proposition 2.4.146).
Let us start by showing that $\mathcal{E}$ is Nori-semistable. For this purpose, let $\nu : C \to S$ be a non-constant morphism from a proper and smooth curve, if we take the fibered product $C \times_S X$ this is a reduced, proper and connected scheme and we can always consider a morphism $C' \to C \times_S X$ with C' a proper and smooth curve that passes through any

pair of arbitrary points using Lemma 3.3.25. This gives us a morphism $w : C' \to X$ making the following diagram commutative

$$\begin{tikzcd} C' \arrow[r, "c"] \arrow[d, "w"'] & C \arrow[d, "v"] \\ X \arrow[r, "f"'] & S \end{tikzcd}$$

and we can moreover chose the points on the fibered product such that $c : C' \to C$ is finite, surjective and $w : C' \to X$ is non-constant.
Now, we have that $w^*(f^*(\mathcal{E})) = c^*(v^*(\mathcal{E}))$ and as $f^*(\mathcal{E})$ is essentially finite, the left hand side of the equation is semi-stable of degree 0, then so is $v^*(\mathcal{E})$. Thus, $\mathcal{E}$ is Nori-semistable.
Finally, to prove that $\mathcal{E}$ is essentially finite, we need to show that it lies in a full tannakian sub-category of NSS(S) associated to a finite group-scheme. Let $\langle f^*(\mathcal{E})\rangle^{\otimes}$ be the full subcategory of NSS(X) generated by $f^*(\mathcal{E})$, see Remark 2.4.150. As $f^*(\mathcal{E})$ is essentially finite, $\langle f^*(\mathcal{E})\rangle^{\otimes}$ is equivalent to $\mathrm{Rep}_k(H)$ where H is finite group-scheme, as it is a quotient of $\pi^N(X,x)$. We can do the same for $\mathcal{E}$ so we can consider $\langle \mathcal{E}\rangle^{\otimes} \cong \mathrm{Rep}_k(G)$ where G is a quotient of $\pi^S(X,x)$, a priori G of finite type over k.
We can clearly restrict f^* to get a tensor functor

$$f^*|_{\mathrm{Rep}_k(G)} : \mathrm{Rep}_k(G) \to \mathrm{Rep}_k(H)$$

that has the same properties of f^* over NSS(S), thus this functor induces a faithfully flat morphisms of group-schemes $\phi : H \to G$ by Corollary 2.4.120, so G is finite. □

Now to the main result:

Proposition 5.2.19. *Assuming the hypothesis and notation of Setting 5.2.3, let $t : T \to X$ be a Nori-reduced pure G-torsor over X. Then, the pull-back $T_{\bar{\eta}} \to X_{\bar{\eta}}$ to the geometric generic fiber of f is Nori-reduced, where $\bar{\eta}$ is the geometric generic point over S.*

Proof. Recall from Remark 3.4.8 that for a G-torsor $t : T \to X$, if the restriction of vector bundle $t_*(\mathcal{O}_T)$ to $X_{\bar{\eta}}$ is trivial, then T is the pull-back of a G-torsor over S by Grauert's theorem for finite torsors (Proposition 3.4.7). Also, as explained in these references, this also applies to essentially finite bundles over X: if $\mathcal{E}$ is an essentially finite bundle with trivial restriction to $X_{\bar{\eta}}$, then $\mathcal{E} \cong f^*(f_*(\mathcal{E}))$ is the pull-back of a vector bundle over S, which is essentially finite by Lemma 5.2.18.
Now, if T is pure, the pull-back torsor $T_{\bar{\eta}}$ of T corresponds to a morphism of group-schemes $\pi_1^N(X_{\bar{\eta}}) \to G_{\bar{\eta}}$. Let $H_{\bar{\eta}} \subset G_{\bar{\eta}}$ be the image of this morphism and let us suppose that it is not equal to $G_{\bar{\eta}}$: if it is trivial, we will immediately get a contradiction, by the preceding paragraph.
If $H_{\bar{\eta}}$ is a non-trivial subgroup-scheme of $G_{\bar{\eta}}$, then by Corollary 5.2.11,

there exist a subgroup-scheme $H \subset G$ such that the base change of the quotient T/H to $X_{\bar{\eta}}$ is isomorphic to $T_{\bar{\eta}}/H_{\bar{\eta}}$.
The essentially finite bundle $(t')_*(\mathcal{O}_{T/H})$ associated to $t' : T/H \to X$ has a trivial restriction to the geometric generic fiber (Remark 3.4.6), so this bundle is the pull-back of an essentially finite bundle over S as explained in the first paragraph, contradicting Lemma 5.2.17. □

FGS of weak quotients

Remark 5.2.20. *As communicated to the author by M. Emsalem, it can be shown that weak quotients of Nori-reduced torsors (Definition 2.3.54) possess fundamental group-schemes applying [1, Theorem I (1)]. This should extend the bijection between pointed G-torsors and arrows $\pi_1^N(X,x) \to G$ to weak quotients on the right and weak quotients of torsors, where weak quotients of torsors possess a FGS if and only if they are a quotient of a Nori-reduced torsor.*
With this, an alternative proof of Lemma 5.2.17 that does not require considering the core of a subgroup-scheme could be given.

5.3 PRO-NR TORSORS AND THE UNIVERSAL PULL-BACK TORSOR

5.3.1 *Pro-NR torsors*

Let k be any field, all group-schemes over k considered in this section will be affine. Let us recall Notation 2.3.56 for the convenience of the reader:

Notation 5.3.1. *Let X be a scheme over k. If $\{T_i\}_{i\in I}$ is an inverse directed system of affine torsors $T_i \to X$ over a partially ordered set I, where the transition morphisms are torsor morphisms. The limit of this system will be denoted as*

$$T := \varprojlim_{i\in I} T_i.$$

We will also consider the associated inverse directed system of group-schemes $\{G_i\}_{i\in I}$, being G_i the group-scheme associated to T_i.
Finally, for the pointed case, if $x \in X(k)$, the points t_i ($i \in I$) and t will denote respectively a rational point of T_i over x and a rational point of T over x, clearly t is the inverse limit of the directed system formed by the t_i.
When needed, we may add an index 0 to the set I such that $T_0 := X$ and $t_0 := x$.

We are interested in a particular type of projective limit of torsors:

Definition 5.3.2. *Let $\{T_i\}_{i\in I}$ be a co-filtered family of finite pointed Nori-reduced torsors over a quasi-compact scheme X over k. We will call its projective limit $T := \varprojlim T_i$ a* pro-NR torsor. *In the case we have compatible rational points over any member of the limit, according to Notation 5.3.1, we will say that this pro-NR torsor is* pointed.

The existence of this projective limit is guaranteed by Proposition 2.3.60.

We want to show that if X is of finite type, proper, reduced and connected, any pro-NR torsor possesses a FGS with the same property in terms of its FGS as in the case of Nori-reduced torsors (Proposition 3.4.15). We will extensively use Proposition 2.3.61 in order to handle finite torsors over pro-NR torsors.

Remark 5.3.3. *By Corollary 2.3.62, we see that pro-NR torsors are in fact Nori-reduced.*

The main result of this section is the following:

Pro-NR torsors possess a FGS

Proposition 5.3.4. *Let X be a reduced and proper scheme of finite type over a field k with a rational point $x \in X(k)$. Let $\{T_i\}_{i \in I}$ be a projective system of pointed Nori-reduced torsors and let T be its projective limit with $t \in T(k)$ over x.*
Then T possesses a FGS, as it has the same universal torsor as X, and we have $\pi_1^N(T,t) = \ker\{\pi_1^N(X,x) \to \varprojlim G_i\}$ where G_i is the finite group-scheme associated with to T_i.

To show this, we need an extension of Theorem 3.4.13

Closure of towers over pro-NR torsors

Lemma 5.3.5. *Let X be a proper, reduced and connected scheme of finite type over a field k with a rational point $x \in X(k)$ and let T be a pro-NR torsor with $t \in T(k)$ over x. Then, if $W \to V \to T$ is a tower of finite pointed torsors, there exists a Nori-reduced closure $U \to T$ of the tower that mimics the properties of the closure of a tower of finite torsors over X, outlined in Definition 3.4.12.*

Proof. Let us start with the existence of an envelope for the tower (Definition 3.4.12): Let $W \to V \to T$ be a tower of torsors. Applying Proposition 2.3.61(a) to V to obtain a cartesian square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V_i \arrow[d] \\ T \arrow[r] & T_i \end{tikzcd}$$

where $i \in I$ and $V_i \to T_i$ is a torsor. Applying Lemma 2.3.57(c) to the composition $W \to V \to T$ we also get the following diagram in which all possible squares are cartesian:

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W_i \arrow[d] \\ V \arrow[r] \arrow[d] & V_i \arrow[d] \\ T \arrow[r] & T_i \end{tikzcd}$$

but it is not necessarily clear if $W_i \to V_i$ is a torsor. This is indeed the case (over a possibly larger index) as we can consider the projective system $\{V_j\}_{j \geqslant i}$ with base V_i and projective limit V (see [33, p. 8.2.5]) and apply Proposition 2.3.61 for $W \to V$.

Now we can utilize Theorem 3.4.13 over the tower of finite pointed torsors $W_i \to V_i \to T_i$ as all schemes are of finite type and T_i posseses a FGS (Proposition 3.4.15), and thus we obtain a closure U_i of this tower, giving us the commutative diagram

$$\begin{tikzcd} W \arrow[r] \arrow[d] & W_i \arrow[d] & \\ V \arrow[r] \arrow[d] & V_i \arrow[d] & U_i \arrow[l] \arrow[ul] \arrow[dl] \\ T \arrow[r] & T_i & \end{tikzcd}$$

If we denote by U the pull-back of U_i to T, we obtain the following additional commutative diagram

$$\begin{tikzcd} & W \arrow[r] \arrow[d] & W_i \arrow[d] & \\ U \arrow[ur] \arrow[r] \arrow[dr] & V \arrow[r] \arrow[d] & V_i \arrow[d] & U_i \arrow[l] \arrow[ul] \arrow[dl] \\ & T \arrow[r] & T_i & \end{tikzcd}$$

where we see that the left-hand side of the diagram shows the existence of a torsor that envelopes the tower $W \to V \to T$ as U_i satisfies the properties that we require for an envelope of a tower of torsors, but U might not be a minimal torsor that envelops the tower. In any case, this process shows how to obtain an envelope for the tower $W \to V \to T$ from the closure of a tower of torsors over the finite torsor $T_i \to X$.

Now let us assume that both $W \to V$ and $V \to T$ are Nori-reduced and let us show the existence of a closure in this case. First, we notice that as U_i is the closure of a tower of Nori-reduced torsor, we have that $U_i \to T_i$ is Nori-reduced, but its pull-back $U \to T$ might not be. At least, both arrows $U \to V$ and $U \to W$ are torsors and then, if we take the canonical Nori-reduced sub-torsor (Definition 3.2.16) $\bar{U} \hookrightarrow U$, we obtain a Nori-reduced envelope that additionally is a torsor over W, so we can suppose from now on that U satisfies these properties.

The existence of a unique Nori-reduced closure in the case both torsors in the tower $W \to V \to T$ are Nori-reduced comes from applying Zorn's lemma to the (non-empty) set of isomorphism classes of Nori-reduced envelopes of this tower, i.e., we are considering the skeletal sub-category of the category of envelopes with morphisms of torsors as arrows. We will abuse notation when considering classes and individual torsors.

We consider over the classes, the partial ordering $U \leqslant U'$ iff there exists a morphism of torsors $U \to U'$ of over T. With this setting, Zorn's lemma holds as we have the following properties:

(i)) Every chain of Nori-reduced envelopes has at most a finite amount of members.

(ii) The poset of Nori-reduced envelopes is directed.

For i), if $U \leqslant U'$, let G be the group-scheme associated to $U \to T$. Then, as both torsors are Nori-reduced over T, U' is of the form U/N where N is a normal subgroup of G (Proposition 2.3.52). As there is only a finite amount of quotients of G, there is a finite amount of envelopes greater than U.
To prove ii), let U, U' be two Nori-reduced envelopes of the tower $W \to V \to T$ and let $l \in I$ be an index such that both torsors U and U' descend to torsors U_l and U'_l respectively, this is possible as I is directed. In that case, these torsors envelop the tower $W_l \to V_l \to T_l$ that descends from the tower over T and thus if Z_l is the closure of this tower, we have morphisms of torsors $U_k, U'_k \to Z$ which are quotients in this case. This implies that the pull-back Z of Z_k over T is a Nori-reduced common quotient of U and U' that envelops the tower $W \to V \to T$, showing that the poset of Nori-reduced envelopes of this tower is directed. This finished the proof when both members of the tower are Nori-reduced.
For the general case, we first note that as long as $V \to T$ is Nori-reduced, we will have a closure for the tower even if $W \to V$ is not Nori-reduced from the last paragraph, but of course, this closure might not longer be a torsor over W. This ties into the general case, because if $V \to T$ is no longer Nori-reduced, any Nori-reduced envelope U of the tower is faithfully flat over $\bar{V} \subset V$, the canonical Nori-reduced sub-torsor of V, and then, if U envelopes $W \to V \to T$, it will also envelope the tower $W \times_V \bar{V} \to \bar{V} \to T$, so that we have a commutative diagram

$$\begin{tikzcd}
 & W \times_V \bar{V} \arrow[r] \arrow[d] & W \arrow[d] \\
U \arrow[ur] \arrow[r] \arrow[drr] & \bar{V} \arrow[r] \arrow[dr] & V \arrow[d] \\
 & & T
\end{tikzcd}$$

and then we conclude that we have a closure for the initial given tower on the right side of the diagram above, finishing the proof. □

Now, we can prove the main result of this part:

Proof of Proposition 5.3.4. Let $\hat{X}$ be the universal torsor of X and let Y be a finite Nori-reduced pointed torsor over T, applying Proposition

2.3.61(a), there exist an index i and a pointed torsor $Y_i \to T_i$ making a cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & Y_i . \arrow[d] \\ T \arrow[r] & T_i \end{tikzcd}$$

As T_i is Nori-reduced, from Proposition 3.4.15, $\hat{X}$ is also the universal torsor of T_i and thus we have an arrow $\hat{X} \to Y_i$. Combining this arrow with the canonical arrow $\hat{X} \to T$ we obtain an arrow $\hat{X} \to Y \cong Y_i \times_{T_i} T$ and thus we conclude that all Nori-reduced torsors over T are a quotient of $\hat{X}$, this means that T possesses a universal pointed torsor $\hat{T}$ and we have a natural morphism of torsors over T, $\hat{X} \to \hat{T}$. To get an isomorphism, it suffices to prove that $\hat{T}$ has a no non-trivial Nori-reduced finite torsors: Let $Z \to \hat{T}$ be a finite Nori-reduced torsor over $\hat{T}$, applying Proposition 2.3.61(a) over T, which is quasi-compact and quasi-separated over k because T is affine over k, there exist a finite Nori-reduced torsor $V \to T$ and a finite Nori-reduced torsor $W \to V$ fitting into a commutative diagram where all possible squares are cartesian:

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W . \arrow[d] \\ \hat{T} \arrow[r] \arrow[dr] & V \arrow[d] \\ & T \end{tikzcd}$$

From Lemma 5.3.5, there exists a Nori-reduced closure $U \to T$ of the tower $W \to V \to T$, and thus we have a canonical arrow $\hat{T} \to U$ which composed with the arrow $U \to W$ gives a section $\hat{T} \to Z = W \times_V \hat{T}$, making Z a trivial torsor by Corollary 2.3.14 and finishing the proof. □

A particular case of the fact that the universal torsor $\hat{T}$ does not possess non-trivial finite Nori-reduced torsors coming from the proof Proposition 5.3.4 is the following:

Corollary 5.3.6. *Let X be a proper, reduced and connected scheme of finite type over a field k with a rational point $x \in X(k)$. Then, if $\hat{X}$ is the universal torsor of X, all finite Nori-reduced torsors over $\hat{X}$ are trivial.*

Proof. If $Z \to \hat{X}$ is a finite Nori-reduced torsor, we can apply Proposition 2.3.61(a) to a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & W \arrow[d] \\ \hat{X} \arrow[r] \arrow[dr] & V \arrow[d] \\ & X \end{tikzcd}$$

where $W \to V \to X$ is a tower of torsors. From here, using Theorem 3.4.13 directly, we can argue analogously as we did in the proof of Proposition 5.3.4 to obtain a section $\bar{X} \to Z$. □

Remark 5.3.7. *Keeping the notation of Proposition 5.3.4, we get the following short exact sequence for a pro-NR torsor*

$$1 \to \pi_1^N(T,t) \to \pi_1^N(X,x) \to \varprojlim G_i \to 1$$

for suitable rational points. Also, from Corollary 5.3.6, the universal torsor $\hat{X}$ of X has a trivial FGS.

5.3.2 *The universal pull-back torsor*

Now we will apply the results of the last sub-section to a particular case:

The universal pull-back torsor

Definition 5.3.8. *Let X and be S two schemes over k, with rational points $x \in X(k)$ and $s \in S(k)$ and such that both possess a FGS.*
If $f : X \to S$ is a morphism that is compatible with the respective rational points, and $\hat{S}$ denotes the universal torsor of S, we define the universal pull-back torsor *as the pointed $\pi^N(S,s)$-torsor $X^* := \hat{S} \times_S X$ over X.*

We will introduced some specific notation for this torsor:

Notation 5.3.9. *Let X and be S two schemes over k, with rational points $x \in X(k)$ and $s \in S(k)$ and such that both possess a FGS.*
If $f : X \to Y$ is a morphism that is compatible with the respective rational points, we will denote

$$\hat{S} = \varprojlim_{i \in I} S_i$$

where the limit is taken over a directed set I of indexes, so we write

$$X^* = \varprojlim_{i \in I} X_i$$

with $X_i = S_i \times_S X$. We will also add, according to Notation 5.3.1, an auxiliary index "0" such that $X_0 := X$.

FGS of the universal pull-back torsor

Proposition 5.3.10. *Let X and be S two proper, reduced and connected schemes over a field k, with rational points $x \in X(k)$ and $s \in S(k)$.*
If $f : X \to S$ is a morphism that compatible with the respective rational points, such that the induced morphism $\pi^N(f) : \pi_1^N(X,x) \to \pi_1^N(S,s)$ is faithfully flat. Then, X^* is a pro-NR torsor, and in that case, we have $\pi_1^N(X^*,x^*) = \ker(\pi^N(f))$ where $x^* \in X^*(k)$ is a rational point over x.

Proof. Using Notation 5.3.9, let $S_i \to S$ be a finite Nori-reduced torsor over S, then its pull-back $X_i := S_i \times_S X$ over X is Nori-reduced as $\pi^N(f)$ is faithfully flat. As we mentioned before, X^* is the projective limit of the torsors X_i, thus it is pro-NR and it possesses a FGS which is $\ker(\pi^N(f))$ by Proposition 5.3.4. □

Using Proposition 2.3.61, the main properties of this pro-NR torsor are the following:

Properties of the universal pull-back torsor

Proposition 5.3.11. *Under the hypotheses of Proposition 5.3.10, and using Notation 5.3.1. Let $T \to X$ be a Nori-reduced torsor, T' its pull-back to X^* and let $V \to X^*$ be a finite Nori-reduced torsor over the universal pull-back torsor. Then:*

(a) If T is pure, T' is Nori-reduced.

(b) For any index $i \in I$ for which V descends to a torsor $V_i \to X_i$ fitting into a cartesian diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & V_i \arrow[d] \\ X^* \arrow[r] & X_i \end{tikzcd}$$

we have that the torsor V_i is pure with respect to the morphism of schemes $f_i : X_i \to S_i$ where f_i is the base change of f via the morphism $S_i \to S$.

(c) If V does not descend to a torsor over X. There exist a large enough index $j \in I$ such that the descent of V over X_j, that we denote as V_j, is the quotient of a mixed torsor W over X, for which $f^\dagger(W) = X_j$.

Proof. (a) follows from Remark 5.2.2. For (b), let $f_i : X_i \to S_i$ be the base change of f that comes from $S_i \to S$, and let $\pi^N(f_i) : \pi_1^N(X_i) \to \pi_1^N(S_i)$ be the induced morphism between the corresponding FGS[5]. We will show that $\ker(\pi^N(f_i)) = \ker(\pi^N(f))$ which implies (b) also from the last remark.
Firstly, from the commutative diagram of group-schemes

$$\begin{tikzcd} \pi_1^N(X_i) \arrow[r, "\pi^N(f_i)"] \arrow[d] & \pi_1^N(S_i) \arrow[d] \\ \pi_1^N(X) \arrow[r, "\pi^N(f)"'] & \pi_1^N(S) \end{tikzcd}$$

where the vertical arrows are closed immersions, we get the inclusion $\ker(\pi^N(f_i)) \subset \ker(\pi^N(f))$. And on the other hand, from the tower of torsors

$$X^* \to X_i \to X$$

we also have a commutative diagram where all the arrows are inclusions of subgroup-schemes

$$\begin{tikzcd} \pi_1^N(X^*) = \ker(\pi^N(f)) \arrow[r] \arrow[dr] & \pi_1^N(X_i) \arrow[d] \\ & \pi_1^N(X) \end{tikzcd}$$

5 We will not write the respective rational points in this proof.

which implies, together with the diagram right above, that $\ker(\pi^N(f)) \subset \ker(\pi^N(f_i))$, finishing the proof of (b).
Lastly, to prove (c), let us take one index $i \in I$ with a descent $V_i \to X_i$ of $V \to X^*$. This gives us the tower of torsors $V_i \to X_i \to X$ and thus we can consider its closure (Definition 3.4.12) W, so we have the following commutative diagram

$$\begin{tikzcd} V_i \arrow[d] & \\ X_i \arrow[d] & W \arrow[l] \arrow[ul] \arrow[dl] \\ X & \end{tikzcd} .$$

All torsor forming the tower are Nori-reduced with respect to each corresponding base. This last fact implies that V_i is a quotient of W as a torsor over X_i.
We also see that W is a mixed torsor over X, but as such, its maximal pull-back quotient (Remark 5.2.2(a)) is not necessarily X_i. Let $W \to X_j = f^\dagger(W)$ be the maximal pull-back quotient of W, we have that $j \geqslant i$ and we can use the arrow $W \to V_i$ to get an arrow to the fibered product $W \to V_i \times_{X_i} X_j$ which is actually V_j, this arrow is a quotient of torsors, finishing the proof of (c). □

5.4 FINITENESS OF THE KERNEL

Using the results of previous sections, we are ready to show that the kernel is finite for the induced morphism between FGS in the following setting:

Setting 5.4.1. *The field k is an algebraically closed field, X is a proper variety over k and S is an elliptic curve over k, with a proper faithfully flat morphism $f : X \to S$ between them. We will denote its induced morphism at the level of FGS as $\pi^N(f) : \pi_1^N(X) \to \pi_1^N(S)$.*
We will further assume that all geometric fibers are reduced, connected and possess a finite FGS. This includes the geometric generic fiber $X_{\bar{\eta}}$ where η is the generic point of S.

From this point on, we will denote the order of a finite scheme Q over k as $|Q|$.

5.4.1 *Comparison of geometric generic pull-backs of torsors*

Going back to the assumptions of Setting 5.4.1. The fact that for our fibration $f : X \to S$, S is an elliptic curve, implies the following:

Lemma 5.4.2. *Let $f : X \to S$ be as in Setting 5.4.1. Then, let $p' : X' \to X$ be a Nori-reduced G-torsor that is the pull-back of a Nori-reduced G-torsor*

$t' : S' \to S$, and let $f' : X' \to S'$ be the base change to S' of f. Then, X' is proper, reduced and connected, S' is an elliptic curve and f' is proper faithfully flat with reduced and connected geometric fibers.

Proof. As stated in Corollary 3.4.25, S' is either an elliptic curve or at least is a smooth and projective variety over k.
Let us start by characterizing the fibers of f'. As k is algebraically closed, we have the following property ([63, Tag 055E & Tag 0576]): If

$$Y_S = \{s \in S : X_s \text{ is geometrically connected and reduced }\}$$

and $Y_{S'}$ is its counterpart for S' and X', then we have that

$$Y_{S'} = \left(t'\right)^{-1}(Y_S)\,.$$

Thus, as $Y_S = S$ in our case and t' is surjective, then we conclude that $Y_{S'} = S'$ and thus all fibers are geometrically reduced and geometrically connected. In particular they all possess a FGS.
Finally, for the properties of X', as f' is a faithfully flat morphism between schemes of finite type over a field, we conclude that the proper scheme X' is also reduced as S' and all fibers of f' are, which is a consequence of the corollary under [47, Thm. 23.9][6]. In second place, for the conectedness of X', we have that $\Gamma(X', \mathcal{O}_{X'}) = k$ by Proposition 3.3.54, so we conclude that X' is connected as it is proper too. $\square$

Remark 5.4.3. *This lemma implies that Proposition 5.2.19 holds for f': If ξ is the generic point of S', then a pure Nori-reduced torsor over X' has a Nori-reduced pull-back to the geometric generic fiber $X'_{\bar{\xi}}$.*
Now we will apply this to a special case: let $p : T \to X$ be a mixed Nori-reduced G-torsor, and let $p' : X' \to X$[7]. be its maximal pull-back quotient where $t' : S' \to S$ is the H-torsor over S whose pull-back to X is X', if $q : T \to X'$ is the quotient morphism, we have the following diagram with a cartesian square within

$$\begin{tikzcd}
T \arrow[d, "q"] \arrow[dd, bend right, "p"'] & \\
X' \arrow[r, "f'"] \arrow[d, "p'"] & S' \arrow[d, "t'"] \\
X \arrow[r, "f"'] & S
\end{tikzcd}$$

If η and ξ are the generic points of S and S' respectively, we have the following comparison result for the pull-back of T to $X'_{\bar{\xi}}$:

6 The reader can alternatively consult [27, Prop. 14.57].
7 For this subsection, we will use X' for the maximal pull-back quotient instead of $f^{\dagger}(T)$

Comparison of geometric generic pull-backs of torsors (a.k.a Lemma η-ξ)

Lemma 5.4.4. *Keeping the notations of Remark 5.4.3, the pull-back $T_{\bar{\xi}}$ of T to the geometric generic fiber $X'_{\bar{\xi}}$ is the pull-back of a Nori-reduced torsor over $X_{\bar{\eta}}$.*

Proof. First, we have a natural morphism of geometric generic points $\bar{\xi} \to \bar{\eta}$, and if we consider this morphism and the morphism $\bar{\xi} \to S'$ we then have the following diagram

$$\begin{tikzcd}
\bar{\xi} \arrow[r] \arrow[dr] & S'_{\bar{\eta}} \arrow[r] \arrow[d] & S' \arrow[d, "t'"] \\
 & \bar{\eta} \arrow[r] & S
\end{tikzcd}$$

where the square is cartesian and the triangle is commutative. As $t' : S' \to S$ is a torsor, its geometric fiber is isomorphic to $H_{\bar{\eta}}$, the base change of the k-group-scheme H to the residue field of $\bar{\eta}$. We can chose this isomorphism in such a way that the morphism $\bar{\xi} \to S'_{\bar{\eta}}$ becomes the composition $\bar{\xi} \to \bar{\eta} \xrightarrow{\epsilon_{\bar{\eta}}} H_{\bar{\eta}}$ under this isomorphism where $\epsilon_{\bar{\eta}}$ is the unit rational point of the group-scheme $H_{\bar{\eta}}$. Now if we take the pull back of this diagram over $f : X \to S$, we will obtain the following commutative diagram

$$\begin{tikzcd}
X \times_S \bar{\xi} \arrow[r] \arrow[dr] & X \times_S H_{\bar{\eta}} \arrow[r] \arrow[d] & X' \arrow[d, "p'"] \\
 & X_{\bar{\eta}} \arrow[r] & X
\end{tikzcd}$$

Notice that the product $X \times_S H_{\bar{\eta}}$ is

$$\begin{aligned}
X \times_S H_{\bar{\eta}} &= X \times_S (S' \times_S \bar{\eta}) \\
&= (X \times_S S') \times_S \bar{\eta} \\
&= X' \times_S \bar{\eta} \\
&= X'_{\bar{\eta}}.
\end{aligned}$$

In addition, If we call $X_{\bar{\xi}}$ the fibered product $X \times_S \bar{\xi}$, we see that

$$\begin{aligned}
X \times_S \bar{\xi} &= (X \times_S S') \times_{S'} \bar{\xi} \\
&= X' \times_{S'} \bar{\xi} \\
&= X'_{\bar{\xi}}
\end{aligned}$$

and then we see that $X_{\bar{\xi}}$ is the geometric generic fiber of $f' : X' \to S'$. Since X' is a pull-back torsor, we have that $X'_{\bar{\eta}}$ is a trivial torsor over $X_{\bar{\eta}}$ with a section $s_{\bar{\eta}} : X_{\bar{\eta}} \to X'_{\bar{\eta}}$ fitting into the following diagram:

$$\begin{tikzcd}
X'_{\bar{\xi}} \arrow[rr, "\theta"] \arrow[dr, "\lambda"'] & & X'_{\bar{\eta}} \arrow[dl, bend right, "p'_{\bar{\eta}}"'] \\
 & X_{\bar{\eta}} \arrow[ur, bend right, "s_{\bar{\eta}}"'] &
\end{tikzcd} \quad . \tag{1}$$

where the triangle made by θ, λ and $p'_{\bar{\eta}}$ is commutative as well as the triangle made by the first two morphisms mentioned before, but with $s_{\bar{\eta}}$ instead of $p'_{\bar{\eta}}$ as the third morphism.
Now let us consider $p : T \to X$ and take its pull-back $T_{\bar{\eta}}$ over $X_{\bar{\eta}}$ and the pull-back $T_{\bar{\xi}}$ over $X'_{\bar{\xi}}$. From Proposition 5.2.19 we know that $T_{\bar{\xi}} \to X'_{\bar{\xi}}$ is Nori-reduced as $q : T \to X'$ is pure with respect to f' and it is worth to point out that $T_{\bar{\xi}}$ over $X_{\bar{\xi}}$ is the pull-back along θ of $q_{\bar{\eta}} : T_{\bar{\eta}} \to X'_{\bar{\eta}}$, which is the pull-back to $X_{\bar{\eta}}$ of q.
Let $z : Z \to X_{\bar{\eta}}$ be the pull-back of the torsor $q_{\bar{\eta}} : T_{\bar{\eta}} \to X'_{\bar{\eta}}$ along $s_{\bar{\eta}}$. As H is a quotient of G, if we call K the normal subgroup-scheme of G such that $H = G/K$, we have that Z is K-torsor and from the last commutative triangle, we see that its pull-back along λ is $T_{\bar{\xi}}$, but this pull-back is Nori-reduced, and then so is Z, which is then the torsor over $X_{\bar{\eta}}$ we were looking for, finishing the proof. $\square$

Another consequence of the commutative diagram (1) in the proof Lemma 5.4.4 is the following:

Corollary 5.4.5. *Let $p : T \to X$ be a mixed G-torsor with maximal pull-back quotient $p' : X' \to X$, if $H = G/K$ for a certain normal subgroup-scheme K of G which is the group-scheme associated to $q' : T \to X'$. Then, the canonical Nori-reduced sub-torsor (Definition 3.2.16) of $p_{\bar{\eta}} : T_{\bar{\eta}} \to X_{\bar{\eta}}$ is a K-torsor.*

Proof. Firstly, as X' becomes trivial when taking the pull-back over $X_{\bar{\eta}}$, we have that the composition

$$\pi_1^N(X_{\bar{\eta}}) \to G \to G/K$$

is trivial, and thus the image of the first arrow is contained in K. Let $V \subset X_{\bar{\eta}}$ be the canonical Nori-reduced sub-torsor of $X_{\bar{\eta}}$ which is not Nori-reduced, as its associated group-scheme corresponds to the image of $\pi_1^N(X_{\bar{\eta}}) \to G$, we see that K is the largest possible subgroup-scheme that could be associated to V.
Moreover, let us consider the Nori-reduced K-torsor $z : Z \to X_{\bar{\eta}}$ from the last proof, we will show that it is a sub-torsor of $T_{\bar{\eta}}$ which implies that $V = Z$. We have the following cartesian diagram

$$\begin{tikzcd} Z \arrow[r, "i"] \arrow[d, "z"'] & T_{\bar{\eta}} \arrow[d, "q_{\bar{\eta}}"] \\ X_{\bar{\eta}} \arrow[r, "s_{\bar{\eta}}"'] & X'_{\bar{\eta}} \end{tikzcd}$$

and because $s_{\bar{\eta}}$ is a closed immersion, so is i, and the only thing we need to show to get that Z is a sub-torsor, is the equality $p_{\bar{\eta}} \circ i = z$.

From the cartesian diagram we see that $s_{\bar{\eta}} \circ z = q_{\bar{\eta}} \circ i$ and if we compose this equality with $p'_{\bar{\eta}}$ we obtain:

$$\begin{aligned} \underbrace{p'_{\bar{\eta}} \circ q_{\bar{\eta}}}_{=p_{\bar{\eta}}} \circ i &= \underbrace{p'_{\bar{\eta}} \circ s_{\bar{\eta}}}_{=id_{X_{\bar{\eta}}}} \circ z \\ p_{\bar{\eta}} \circ i &= z \end{aligned}$$

and effectively, Z is a sub-torsor of $X_{\bar{\eta}}$ finishing the proof. □

5.4.2 *Proof of finiteness and consequences*

Now we are ready to show that, under Setting 5.4.1, $\ker(\pi^N(f))$ is finite.

Lemma 5.4.6. *Let k be any field, and let X be a k-scheme possessing a FGS with respect to the rational point $x \in X(k)$. Let us suppose that for any pointed Nori-reduced G-torsor over X, the order of G is bounded by a fixed finite positive integer, then $\pi_1^N(X, x)$ is finite.*

Proof. Let $M \in \mathbb{N}$ be the bound for the orders of the group-schemes associated to all pointed Nori-reduced torsors over X.
Let $\mathcal{M}$ be the set of isomorphism classes of finite Nori-reduced torsors over X, this set comes with a natural partial ordering $V \leqslant V'$ iff there exists a faithfully flat morphism $V' \to V$ between representatives over X. In this case, we can use Zorn's lemma to get a maximal element of $\mathcal{M}$ that will be a finite Nori-reduced G-torsor over X with $\operatorname{ord}(G) \leqslant M$.
Let $\{V_i\}_{i \in I}$ be a chain of elements of $\mathcal{M}$. If the index set I is finite, we can index the elements of the chain as $\{V_i\}_{i=1}^n$ and we have a chain of finite group-schemes

$$G_1 \leftarrow G_2 \leftarrow \cdots \leftarrow G_n$$

with faithfully flat arrows between them, that corresponds to a chain of inclusions of Hopf-algebras

$$A_1 \hookrightarrow A_2 \hookrightarrow \cdots \hookrightarrow A_n$$

and all the Hopf-algebras on the chain have finite k-dimension, bounded by M, and thus V_n is a maximal element for the chain of torsors. We note as well that A_n is isomorphic to the directed limit of the chain of Hopf-algebras.
If I is infinite, we will get a similar chain of Hopf-algebras

$$A_0 \hookrightarrow A_1 \hookrightarrow \cdots$$

and each Hopf-algebra A_i ($i \in I$) has finite k-dimension bounded by M. As the k-dimension of the Hopf-algebras on the chain increases, this chain must be eventually stationary, i.e., $A_i \cong A_j$ for $i \leqslant j$, after

a finite amount of inclusions starting from A_{i_0}, if A_N is the least element that stabilizes the chain of Hopf-algebras, we see that the direct limit $\varinjlim A_i$ is isomorphic to A_N and thus to all A_i with $i \geqslant N$, this means that we can take V_N as the maximal element of the chain, and the dimension of A_N is bounded by M.
As we have assured the existence of finite a G-torsor $U \to X$ ($\mathrm{ord}(G) \leqslant M$), the supremum of $\mathcal{M}$, we can see that for any finite Nori-reduced torsor T over X, we have a faithfully flat arrow $U \to X$. Thus, $U \cong \hat{X}$ by Proposition 3.2.18, finishing the proof. □

Finiteness of the kernel

Proposition 5.4.7. *Let $f : X \to S$ be as in Setting 5.4.1. Then, the kernel of the induced morphism $\pi^N(f) : \pi_1^N(X) \to \pi_1^N(S)$ is finite.*

Proof. Using Lemma 5.4.6, we will show that the order of the group-scheme associated to a finite Nori-reduced torsor $V \to X^*$ is bounded by a fixed positive integer.
Keeping the notation of Notation 5.3.1, if $V \to X^*$ is a Nori-reduced torsor, there are two cases according to Proposition 5.3.11:

(a) If V descends to a finite Nori-reduced and pure torsor V_0 over X, then we see that the pull-back to $X_{\bar{\eta}}$ is Nori-reduced by Proposition 5.2.19 and thus the order of the group-scheme associated to V is bounded by $|\pi_1^N(X_{\bar{\eta}})|$, the order of $\pi_1^N(X_{\bar{\eta}})$.

(b) If V descends to a finite torsor $V_i \to X_i$, that is pure with respect to $f_i : X_i \to S_i$, by Proposition 5.3.11(c), we can suppose that there is a mixed torsor $T \to X$ such that $X_i = f^{\dagger}(T)$ is its maximal pull-back quotient and V_i is a quotient of T over X_i. From Lemma 5.4.4 and Corollary 5.4.5, the order of $T \to X_i$ is the order of the canonical Nori-reduced sub-torsor (Definition 3.2.16) of the pull-back $T_{\bar{\eta}}$ over $X_{\bar{\eta}}$, which is then again bounded by $|\pi_1^N(X_{\bar{\eta}})|$, finishing the proof.

□

Finally, we outline a strong consequence that stems from the finiteness of the kernel.

Corollary 5.4.8. *Let $f : X \to S$ be as in Setting 5.4.1, and let us suppose that $\ker(\pi^N(f))$ is finite. Then, there is a Nori-reduced torsor $X_i \to X$ that is the pull-back of a Nori-reduced torsor $S_i \to S$, such that we have an isomorphism*

$$\pi_1^N(X_i, x_i) \cong \pi_1^N(S_i, s_i) \times_k \ker(\pi^N(f))$$

for compatible rational points coming from the morphism $X_i \to S_i$.

Proof. As the kernel of $\pi^N(f)$ is finite, the torsor $\hat{X} \to X^*$ is finite. Using Proposition 2.3.61(a) as X^* is pro-NR, $\hat{X} \to X^*$ descends to a finite torsor $\hat{X}_i \to X_i$ where $X_i \to X$ is the pull-back of a Nori-reduced torsor $S_i \to S$ such that we have the following cartesian diagram

$$\begin{tikzcd} \hat{X} \arrow[r] \arrow[d] & \hat{X}_i . \arrow[d] \\ X^* \arrow[r] & X_i \end{tikzcd}$$

In particular, we have that $\hat{X} = X^* \times_{X_i} \hat{X}_i$ and this implies for $x_i \in X_i(k)$ that $\pi_1^N(X_i, x_i) = G \times \ker(\pi^N(f))$ is the product of the group-schemes corresponding to X^* and $\hat{X}_i$ over X_i. To characterize G as $\pi_1^N(S_i, s')$, we see from the commutative diagram

$$\begin{tikzcd} X^* \arrow[r] \arrow[d] & \hat{S} \arrow[d] \\ X_i \arrow[r] & S_i \end{tikzcd}$$

that $\hat{S} \to S_i$ and thus $X^* \to X_i$ is a $\pi_1^N(S_i, s_i)$-torsor for some $s_i \in S_i(k)$ compatible with x_i, finishing the proof. □

Remark 5.4.9. *In the case the torsor $\hat{X} \to X^*$ descends to a torsor over X, we will obtain that $\pi_1^N(X, x) = \pi_1^N(S, s) \times_k \ker(\pi^N(f))$ and thus we would obtain a split short exact sequence*

$$1 \to \ker(\pi^N(f)) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1.$$

5.5 PROOF OF THE MAIN THEOREM

Keeping the hypotheses and the notation outlined in Setting 5.4.1, we will restate the main theorem:

Main theorem

Theorem 5.5.1. *Let $f : X \to S$ be a fibration as in Setting 5.4.1, with k an uncountable algebraically closed field. Then, there exists a rational point $s \in S(k)$ such that the following exact sequence:*

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1$$

is exact where $x \in X(k)$ is a rational point of X over s.

5.5.1 *The homotopy exact sequence for the FGS*

Now we will state the homotopy exact sequence for the FGS, that we will apply for our setting to prove the main theorem. Afterwards, we will define one key condition in this result, the base change condition, that we will simplify according to the classification of torsors of Definition 5.2.1 in Subsection 5.5.2.

Theorem 5.5.2 ([69]). *Let $f : X \to S$ be a proper morphism, with reduced and connected geometric fibers, between two reduced and connected locally noetherian schemes over a perfect field k. We additionally suppose that S is irreducible and we take $x \in X(k)$ and $s \in S(k)$ such that $f(x) = s$. Then, the following statements are equivalent:*

The homotopy exact sequence for the FGS

1. *The sequence*

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1$$

   is exact.

2. *For any Nori-reduced G-torsor $t : T \to X$ with G finite, the vector bundle $t_*(\mathcal{O}_T)$ satisfies the base change condition at s and the image of the composition $\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to G$ is a normal subgroup-scheme of G.*

3. *For any Nori-reduced G-torsor $t : T \to X$ with G finite, the vector bundle $t_*(\mathcal{O}_T)$ satisfies the base change condition at s and there exists a Nori-reduced G'-torsor $t' : T' \to S$ with an equivariant morphism $\theta : T \to T'$ such that the induced map $(t')_* (\mathcal{O}_{T'})_s \to f_* (t_*(\mathcal{O}_T))_s$ of fibers over s coming from θ is an isomorphism.*

Moreover, if X and S are proper, then we can add an additional equivalent condition. Namely:

(4) For any Nori-reduced G-torsor $t : T \to X$ with G finite, the vector bundle $t_*(\mathcal{O}_T)$ satisfies the base change condition at s and $f_* (t_*(\mathcal{O}_T))$ is essentially finite over S.

We see that all of these conditions require a condition over the essentially finite bundle $\mathcal{E} = t_*(\mathcal{O}_T)$ for all Nori-reduced torsors $t : T \to X$ known as the "base change condition at s" for $s \in S(k)$ and another additional condition. Now we will define the base change condition:

Definition 5.5.3. *Let $f : X \to S$ be a map of schemes, and $\mathcal{F}$ a coherent sheaf of $\mathcal{O}_X$-modules. If $s : \mathrm{Spec}(\kappa(s)) \to S$ is a point, then we have the cartesian diagram:*

The base change condition

$$\begin{tikzcd} X_s \arrow[r, "i"] \arrow[d, "g"] & X \arrow[d, "f"] \\ \mathrm{Spec}(\kappa(s)) \arrow[r, "s"'] & S \end{tikzcd} .$$

We say that $\mathcal{F}$ satisfies the base change condition at s *if the canonical map*

$$s^* (f_*(\mathcal{F})) \to g_* (i^*(\mathcal{F}))$$

is surjective.

Remark 5.5.4. *Keeping the notation of Definition 5.5.3, if f is proper, S is locally noetherian, $\mathcal{F}$ is coherent and flat over S, then $\mathcal{F}$ satisfies the base change condition at s if and only if the canonical map above is an isomorphism (see [36, III Thm. 12.11]).*
The base change condition over a point is a particular case of the general base change condition about the following cartesian diagram:

$$\begin{tikzcd} X' \arrow[r, "v"] \arrow[d, "g"] & X \arrow[d, "f"] \\ S' \arrow[r, "u"'] & S \end{tikzcd} .$$

and the canonical arrow

$$u^* (f_*(\mathcal{F})) \to g_* (v^*(\mathcal{F}))$$

for a quasi-coherent sheaf $\mathcal{F}$ of X. For the condition to hold, we demand that the former arrow to be surjective.
Under certain assumptions, this condition holds for a wide family of quasi-coherent sheaves:

(a) If f is separated, of finite type and $u : S' \to S$ is a flat morphism of noetherian schemes the morphism above is an isomorphism for any quasi-coherent sheaf $\mathcal{F}$ [36, III Prop. 9.3].

(b) We also have an isomorphism if f is affine for all quasi-coherent sheaves of $\mathcal{O}_X$-modules and for any u ([63, Lemma 02KG]).

For the particular case of the base change condition, we can make a few remarks:

Remark 5.5.5. *Keeping the notation of Definition 5.5.3.*

The base change condition unpacked

(a) For our particular case, the base change condition over a point s of S means that we have to prove that the following morphism of vector spaces

$$f_*(\mathcal{F})_s \otimes_{\mathcal{O}_{S,s}} \kappa(s) \to \Gamma(X_s, \mathcal{F}|_{X_s})$$

is surjective, and thus an isomorphism.

Generic base change condition

(b) Also, as the generic point $\eta : \operatorname{Spec} L \to S$ of S is a flat morphism, the base change condition is always generically satisfied. This implies for a coherent sheaf $\mathcal{F}$ over X, that there is an open set $U_{\mathcal{F}}$, containing η, such that for all closed points $s \in U_{\mathcal{F}}$, the base change condition is also satisfied at s, see [36, III Thm. 12.11 (a)].

5.5.2 *Simplifying the base change condition*

Now let us take $f : X \to S$ as in Setting 5.4.1 and let $t : T \to X$ be a pointed Nori-reduced finite G-torsor over X, corresponding to a faithfully flat arrow $\pi_1^N(X) \to G$, we need to prove that the vector

bundle $\mathcal{E} = t_*(\mathcal{O}_T)$ satisfies the base change condition over a certain rational point of S. We can separated this problem in three, depending if T is pure, mixed or a pull-back.
The easiest case is when T is a pull-back:

Proposition 5.5.6. *Under the hypotheses of Theorem 5.5.2 with S and X proper, let $t' : X' \to X$ be a finite Nori-reduced G-torsor over X. Assume that X' is the pull-back of a G-torsor $p : S' \to S$, then it satisfies the base change condition for all rational points of S.*

Base change condition for pull-back torsors

Proof. From our hypotheses, we have the following cartesian diagram

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "t'"'] & S' \arrow[d, "p'"] \\ X \arrow[r, "f"'] & S \end{tikzcd} .$$

Here, we are in the situation of Remark 5.5.4(b) as the vertical arrows are affine, in particular we see that $f^*\left((p')_*\left(\mathcal{O}_{S'}\right)\right) \cong (t')_*\left(\mathcal{O}_{X'}\right) = \mathcal{P}$ and for $s \in S(k)$ we have that $\mathcal{P}|_{X_s}$ is trivial as the torsor X' is clearly trivial over X_s, in particular

$$\dim_k(H^0(X_s, \mathcal{P}|_{X_s})) = \operatorname{rank}(\mathcal{P}) = \operatorname{rank}(\left(p'\right)_*(\mathcal{O}_{S'}))$$

by Corollary 3.4.4 which finishes the proof as this implies an isomorphism on the equation in Remark 5.5.5(a). $\square$

Now we will study pure torsors: Now we focus our attention to pure torsors:

Proposition 5.5.7. *Under Setting 5.4.1, let $t : T \to X$ be a pure Nori-reduced torsor over X, there exists an open set $U_T \subset S$, containing the generic point of S, such that for any rational point $s \in U_T$, the pull-back of T to the fiber X_s is Nori-reduced and $\mathcal{E} = t_*(\mathcal{O}_T)$ satisfies the base change condition at s.*

Base change condition for pure torsors

Proof. In this case, there exists an open set $U \subset S$ that contains the generic point of S in which the base change condition for $\mathcal{E}$ is satisfied at all of the rational points of S contained within (Remark 5.5.5(b)).
Moreover, as the pull-back of T to the geometric generic fiber $X_{\bar{\eta}}$ is Nori-reduced by Proposition 5.2.19, we have $h^0(X_{\bar{\eta}}, \mathcal{E}|_{X_{\bar{\eta}}}) = 1$ and by the semi-continuity theorem [36, III Thm. 12.8], there exist an open set $U' \subset S$ such that $h^0(X_{\bar{\eta}}, \mathcal{E}|_{X_{\bar{\eta}}}) = 1$ for all $s \in U'$, which implies that the pull-back of T to X_s is Nori-reduced, for $s \in U'(k)$.
Thus, by taking $U_T = U \cap U'$ we finish the proof. $\square$

Finally, the only Nori-reduced torsors remaining are the mixed ones:

Base change condition for mixed torsors

Proposition 5.5.8. *Under Setting 5.4.1, let $t : T \to X$ be a mixed Nori-reduced finite G-torsor. If we write $H = G/K$ where H is the group-scheme corresponding to $f^{\dagger}(T)$, the maximal pull-back quotient of T, there exists an open set $U_T \subset S$, containing the generic point of S, such that for any $s \in U_T(k)$, the canonical Nori-reduced sub-torsor (Definition 3.2.16) of the pull-back of T to X_s is a K-torsor, and $\mathcal{E} = t_*(\mathcal{O}_T)$ satisfies the base change condition at s.*

Proof. There exists an open set $U \subset S$ where $\mathcal{E}$ satisfies the base change condition at any rational point of U (Remark 5.5.5(b)).
Moreover, as the canonical Nori-reduced sub-torsor of the pull-back $T_{\bar{\eta}}$ of T to the geometric generic fiber $X_{\bar{\eta}}$ is a K-torsor by Corollary 5.4.5, as K is the image of the morphism $\pi_1^N(X_{\bar{\eta}}) \to G$.
Applying Lemma 3.4.5, we have that $h^0(X_{\bar{\eta}}, \mathcal{E}|_{X_{\bar{\eta}}}) = r$ where $r = \operatorname{ord}(H)$, and for any $s \in S(k)$ we have that $h^0(X_s, \mathcal{E}|_{X_s}) \geqslant r$ as the image of $\pi_1^N(X_s) \to G$ corresponding to the pull-back of T to X_s has its image contained in K. Thus, by semi-continuity, there exists an open set U' where the canonical Nori-reduced sub-torsor of pull-back torsor $T_s \to X_s$ is a K-torsor for any $s \in U'(k)$.
Finally, by taking $U_T = U \cap U'$ we conclude the proof. □

5.5.3 *Proof of the exact sequence*

We are ready to finish the proof of Theorem 5.5.1.
Now let $f : X \to S$ be as in Setting 5.4.1, first we recall that, for the induced morphism $\pi^N(f) : \pi_1^N(X) \to \pi_1^N(S)$, the kernel $\ker(\pi^N(f))$ is finite. Then, from Corollary 5.4.8, we can deduce the following lemma:

A direct proof of the homotopy exact sequence

Lemma 5.5.9. *Let $f : X \to S$ be as in Setting 5.4.1. Let us suppose that $k = \bar{k}$ is an uncountable field, the torsor $\hat{X} \to X^*$ (Definition 5.3.8) is finite and that it descends to a finite torsor $\hat{X}_0$ over X according to Proposition 2.3.61(a).*
Then, there exists $s \in S(k)$ such that for compatible points, the following sequence

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1$$

is exact.

Proof. Let $t : T \to X$ be a pure Nori-reduced finite torsor, as the pull-back $T \times_X X^*$ is Nori-reduced (Proposition 5.3.11(1)), we have a faithfully flat morphism of torsors $\hat{X} \to T \times_X X^*$ over X^* and thus a faithfully flat morphism $\hat{X}_0 \to T$ over X. As both torsors are Nori-reduced, T must be a quotient of $\hat{X}_0$ by a normal subgroup-scheme of $\ker(\pi^N(f))$ and thus there is a finite amount of isomorphism classes of pure Nori-reduced torsors in this case. This allows us to consider the open set $U_P = \bigcap_{T \in \mathcal{P}} U_T$ where $\mathcal{P}$ is the finite family of isomorphism classes of pure Nori-reduced torsors over X, and U_T is the

open set defined in Proposition 5.5.7. The finiteness of $\mathcal{P}$ implies that U_P is a dense open set of S as it contains its generic point and it has rational points inside. Over any of such rational point $p \in U_P$, all pure Nori-reduced torsors satisfy the base change condition at p and their pull-backs to the fiber X_p are Nori-reduced.
If $t : T \to X$ is mixed and Nori-reduced, assuming the notation of Proposition 5.3.11, if $f^{\dagger}(T) := S_i \times_S X$ is its maximal pull-back quotient where $S_i \to S$ is Nori-reduced, by using a similar argument to the one we used for pure torsors, we see that $T \to f^{\dagger}(T)$ is a quotient torsor of the pure torsor $\hat{X}_i \to f^{\dagger}(T)$, the descent of $\hat{X} \to X^*$ over $f^{\dagger}(T)$, with respect to $f_i : f^{\dagger}(T) \to S_i$. This implies there is a finite amount of classes of isomorphic pure torsors over $f^{\dagger}(T)$. As S is an elliptic curve, it possesses a countable amount of isomorphism classes of Nori-reduced torsors (Proposition 3.4.23), and thus there is a countable amount of isomorphism classes of pull-back torsors $f^{\dagger}(T)$, so we conclude that there is a countable amount of isomorphism classes of mixed Nori-reduced torsors over X. If $\mathcal{M}$ is the family of such isomorphism classes, we see that the intersection $U_M = \bigcap_{T \in \mathcal{M}} U_T$ where U_T is the open subset of S coming from Proposition 5.5.8 is a very general (Definition 4.2.8) subset of S. As k is uncountable, we can find rational points within and thus there exists $m \in U_M(k)$ such that any mixed Nori-reduced torsor satisfies the base change condition at m and the maximal Nori-reduced sub-torsor of any pull-back of a mixed torsor over X_m is a torsor over the image $\ker(\pi^N(f)) \to \pi_1^N(X, x) \to G$ where G is the group-scheme associated to a mixed torsor T.
Finally, as pull-back torsors satisfy the base change condition at any $s \in S(k)$ by Proposition 5.5.6, by choosing a rational point $s \in U_P \cap U_M$ we have for $x \in X(k)$ over s, that the sequence

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1$$

is exact as we wanted, because we have chosen $s \in S(k)$ such that we satisfy one Zhang's equivalent conditions for the homotopy exact sequence to hold, more specifically the one stated in Theorem 5.5.2 (2). □

Proof of Theorem 5.5.1. We will obtain the homotopy exact sequence indirectly: because $\ker \pi^N(f)$ is finite, there exist a Nori-reduced torsor $S_i \to S$ and $X_i \to X$ its pull-back torsor, such that the finite torsor $\hat{X} \to X^*$ descends to a torsor $\hat{X}_i \to X_i$, pure with respect to $f_i : X_i \to S_i$.

Proof of the main theorem

These schemes satisfy the hypotheses of Lemma 5.5.9 and thus for a rational point $s' \in S_i(k)$ with have an exact sequence

$$\pi_1^N(X_{i,s'}, x') \to \pi_1(X_i, x') \to \pi_1^N(S_i, s') \to 1$$

where $X_{i,s'}$ is the fiber of f_i over s'. Let s, x be the images of the points s' and x' to X and S respectively, we see that we have a commutative diagram

$$\begin{tikzcd} X_{i,s'} \arrow[r] \arrow[d] & X_i \arrow[r, "f_i"] \arrow[d] & S_i \arrow[d] \\ X_s \arrow[r] & X \arrow[r, "f"'] & S \end{tikzcd} .$$

We can easily see that $X_{i,s'} \to X_i$ is a pointed Nori-reduced torsor and thus by taking the FGS of all the schemes involved we obtain the following commutative diagram of group-schemes

$$\begin{tikzcd} \pi_1^N(X_{i,s'}, x') \arrow[r] \arrow[d] & \pi_1^N(X_i, x') \arrow[r, "\pi^N(f_i)"] \arrow[d] & \pi_1^N(S_i, s') \arrow[d] \\ \pi_1^N(X_s, x) \arrow[r] & \pi_1^N(X, x) \arrow[r, "\pi^N(f)"'] & \pi_1^N(S, s) \end{tikzcd}$$

where all the vertical arrows are closed immersions. From the proof of Proposition 5.3.11, we see that $\ker(\pi^N(f)) = \ker(\pi^N(f_i))$ and thus we have additionally the following commutative diagram

$$\begin{tikzcd} \pi_1^N(X_{i,s'}, x') \arrow[rr] \arrow[dr] \arrow[dd] & & \pi_1^N(X_i, x') \arrow[dd] \\ & \ker(\pi^N(f)) \arrow[ur] \arrow[dr] & \\ \pi_1^N(X_s, x) \arrow[rr] \arrow[ur] & & \pi_1^N(X, x) \end{tikzcd} .$$

From the homotopy exact sequence, the arrow $\pi_1^N(X_{i,s'}, x') \to \ker(\pi^N(f))$ is faithfully flat and as the arrow $\pi_1^N(X_{i,s'}, x') \to \pi_1^N(X_s, x)$ is a closed immersion, we conclude that $\pi_1^N(X_s, x) \to \ker(\pi^N(f))$ is faithfully flat, finishing the proof. □

5.5.4 *Conclusion*

With Theorem 5.5.1, we can characterize the morphism in Remark 5.1.5(2b), serving as a first stepping stone to the understanding of the Nori fundamental group-scheme of elliptically connected varieties.

FGS of a rationally connected fibration over an elliptic curve

Theorem 5.5.10. *Let k be an uncountable algebraically closed field, and let X be a smooth projective variety over k. Assume there is a projective fibration $f : X \to S$ where S is an elliptic curve such that all geometric fibers are rationally connected. Then, there exists rational compatible points $x \in X(k)$ and $s \in S(k)$ such that the following sequence of group-schemes is exact:*

$$\pi_1^N(X_s, x) \to \pi_1^N(X, x) \to \pi_1^N(S, s) \to 1.$$

Proof. This comes directly from Theorem 5.5.1: as the geometric fibers of this morphism are rationally connected and normal by [33, Théorème 12.1.6] and [27, Exc. 6.20], their FGS are finite (Proposition 3.4.21). □

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