G-TORSORS OVER A DEDEKIND SCHEME

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Abstract. We prove the equivalence of three "points of view" on the notion of a G-torsor when the base scheme is a Dedekind scheme, generalizing known results when the base is a field. The two main tools that we generalize are Chevalley's theorem on semi-invariants (cf. [1, II.5.1]) and a Tannakian description of G-torsors given by Nori and Saavedra (cf. [10, Sec. 2] and [13, II.4.2]). As an application, we show that the fibered category of G-torsors on a regular proper curve over a field k is an Artin stack locally of finite presentation over k.

1. Introduction

Let us first fix some notation. We fix a Dedekind scheme X (the base scheme). That is, X is a scheme that has a finite affine open cover by the spectra of Dedekind domains. Unless stated otherwise, any unadorned product is assumed to be over X, and for two X-schemes Y and T we often write $Y_T = Y \times T = Y \times_X T$. If Y is a scheme over X, we use the "functor of points notation" and write $y \in Y$ to denote a morphism $y : T \to Y$ of schemes over X. In the same spirit, if V is a locally free $\mathscr{O}_X$-module of finite rank, we denote also by V the functor $V : T \mapsto V \otimes \mathscr{O}_T$, for T an X-scheme. This functor is represented by $\mathrm{Spec}\,(\mathrm{Sym}\, V^*)$, where $V^* = \mathscr{H}om_{\mathscr{O}_X}(V, \mathscr{O}_X)$ denotes the dual of V. For any Y, if M is an $\mathscr{O}_Y$-module and $N \subset M$ is an $\mathscr{O}_Y$-submodule, we say N is *locally split (in M)* if N is Zariski-locally on Y a direct summand of M.

We fix G a *flat algebraic group* over X, by which we mean a flat, affine group scheme of finite type over X. By a *representation of* G, we mean a finite rank, locally free $\mathscr{O}_X$-module V with a linear G-action (for details, the reader is referred to §3 below). If Y is an X-scheme, a *G_Y-torsor* is a scheme P faithfully flat and affine over Y, provided with a right G_Y-action such that the following two conditions hold:

(i) The map $P \to Y$ is G_Y-invariant.
(ii) The natural map
$$P \times_Y G_Y \to P \times_Y P;\ (p, g) \mapsto (p, pg)$$
is an isomorphism.

It follows from faithfully flat descent ([6, 2.7.1]) that a G_Y-torsor is also finitely presented over Y, since G is finitely presented over X. A map $P \to P'$ of G_Y-torsors is a G_Y-equivariant map of Y-schemes. A *trivial G_Y-torsor* is a G_Y-torsor $P \to Y$ that is isomorphic as a G_Y-torsor to the projection map $Y \times G \to Y$. Given this terminology, condition (ii) is equivalent to:

(ii′) The map $P \to Y$ admits a section fppf-locally on Y.

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Let X_{Zar} denote the small Zariski site on X, that is, the category whose objects are open subsets $U \subset X$ and whose morphisms are inclusions. Denote by $\mathbf{Rep}\, G$ the fibered category over X_{Zar} where for an object U in X_{Zar}, $\mathbf{Rep}\, G(U) = \mathrm{Rep}_U\, G$ is the category of representations of G_U on locally free $\mathscr{O}_U$-modules of finite rank. For a scheme Y over X, let $\mathbf{Bun}_Y$ denote the fibered category over X_{Zar} where for an object U in X_{Zar}, $\mathbf{Bun}_Y(U) = \mathrm{Bun}_{Y_U}$ is the category of all finite rank vector bundles on Y_U. Both $\mathbf{Rep}\, G$ and $\mathbf{Bun}_Y$ are tensor categories (as described in §4), and by a *tensor functor* $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ we mean a functor of fibered categories respecting the tensor structure.

Let V be a representation of G, $\{X_1, \ldots, X_r\}$ the (nonempty) connected components of X and $\mathbf{i} = (i_1, \ldots, i_r)$ a sequence of natural numbers. We denote by $\bigwedge^{\mathbf{i}} V$ the vector bundle such that $\bigwedge^{\mathbf{i}} V|X_k = \bigwedge^{i_k} V|X_k$, for $k = 1, \ldots, r$. We denote by $t(V)$ some finite iteration of the operations $\otimes$, $\bigwedge^{\mathbf{i}}$, Sym^j, $\oplus$, and $(\cdot)^*$. We call such an iteration a *tensorial construction*. We remark that a tensor functor always respects the operations $\otimes$, $\oplus$ and $(\cdot)^*$, but need not respect $\bigwedge^{\mathbf{i}}$ or Sym^j. However, it is a consequence of Theorem 4.8 that if Y is faithfully flat over X, and $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ is a tensor functor that is exact and faithful on the fibers over X_{Zar}, then F respects all tensorial constructions.

If V is a vector bundle on X, and $L \subset V$ is a locally split line bundle, we denote by $\underline{\mathrm{Aut}}\,(V, L)$ the representable functor whose T-points are automorphisms f of $V \otimes \mathscr{O}_T$ such that $f(L \otimes \mathscr{O}_T) = L \otimes \mathscr{O}_T$. We now state our main theorems.

Theorem 1.1. *Let G be a flat algebraic group over a Dedekind scheme X. There is a representation V of G, a tensorial construction $t(V)$, and a locally split line bundle $L \subset t(V)$, such that $G \xrightarrow{\sim} \underline{\mathrm{Aut}}\,(V, L)$.*

Proof. This is Theorem 3.5. □

Theorem 1.2. *Let G and X be as above. Let Y be a scheme faithfully flat over X. There is a natural equivalence that is functorial in Y of the following groupoids:*

(i) the groupoid of G_Y-torsors;
(ii) the groupoid of tensor functors $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ that on each fiber over X_{Zar} are faithful and exact.

Proof. This is Theorem 4.8 (see also Remark 4.9 for an explanation of notation.) □

We can immediately state a corollary to Theorem 1.1, for which we make the following defintion. Let V be a vector bundle on X, $t(V)$ a tensorial construction and $L \subset t(V)$ a line bundle. For an X-scheme Y, we define a *Y-twist of* (V, L), to be a pair $(\mathscr{E}, \mathscr{L})$ consisting of a locally free sheaf $\mathscr{E}$ on Y provided with a locally split line bundle $\mathscr{L} \subset t(\mathscr{E})$ that is fppf-locally isomorphic as a pair to (V, L). That is, there is an fppf cover $Y' \to Y$ and an isomorphism $f : \mathscr{E}_{Y'} \xrightarrow{\sim} V_{Y'}$ such that $f(\mathscr{L}_{Y'}) = L_{Y'}$. An isomorphism of Y-twists $f : (\mathscr{E}, \mathscr{L}) \to (\mathscr{E}', \mathscr{L}')$ is an isomorphism of vector bundles $f : \mathscr{E} \to \mathscr{E}'$ such that $f(\mathscr{L}) = \mathscr{L}'$.

Corollary 1.3. *Let G and X be as above. Fix a pair (V, L) as in Theorem 1.1 so that $G \xrightarrow{\sim} \underline{\mathrm{Aut}}\,(V, L)$. For any scheme Y over X, there is a natural equivalence that is functorial in Y of the following groupoids:*

(i) the groupoid of G_Y torsors;
(ii) the groupoid of Y-twists of (V, L).

Proof. This is a standard construction. Given a G_Y-torsor P and a representation W of G, we can form the associated vector bundle

$$P \times^G W := P \times W/(pg, w) \sim (p, g^{-1}w).$$

Note that this construction respects tensorial constructions (see the proof of Lemma 4.1 for details).

Let a G_Y-torsor P be given. Define $\mathscr{E} = P \times^G V$ and $\mathscr{L} = P \times^G L$. Then it is straightforward to check that $(\mathscr{E}, \mathscr{L})$ is a Y-twist of (V, L)

For a quasi-inverse, given $(\mathscr{E}, \mathscr{L})$, we get a G_Y-torsor by considering the associated "frame bundle" $P = \underline{\mathrm{Isom}}\left((V_Y, L_Y), (\mathscr{E}, \mathscr{L})\right)$. □

Remark 1.4. Combining the equivalences stated in Theorem 1.2 and Corollary 1.3, we get an equivalence from the groupoid of functors as in Theorem 1.2 and the groupoid of Y-twists of (V, L). This has a simple description. Namely, it is given by $F \mapsto (F(V), F(L))$.

To see this, given a functor $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$, the equivalence in Theorem 1.2 assigns to F the G-torsor $F(G)$ (the notation is explained in Remark 4.9). Corollary 1.3 then assigns to $F(G)$ the pair $(F(G) \times^G V, F(G) \times^G L)$. There is a map $F(G) \times^G V \to F(V)$ induced by applying F to the G-map $G \times V_0 \to V$ (where V_0 is V provided with the trivial G-action). That this gives a well-defined isomorphism $(F(G) \times^G V, F(G) \times^G L) \xrightarrow{\sim} (F(V), F(L))$ is shown in the proof of Theorem 4.8.

As we mentioned in the abstract, Theorems 1.1 and 1.2 were known when the base is a field. Furthermore, the idea of confining oneself to locally free, finite rank representations of G (rather than all quasicoherent sheaves with G-action) over Dedekind schemes is already present in Saavedra's book on Tannakian categories [13]. Nonetheless, the equivalence in Theorem 1.2 is only proven there when the base is a field (cf. [13, II.4.2.2]).

Finally, we remark that the formalism involving fibered categories over the Zariski site on X used in Theorem 1.2 is not necessary when X is affine. In that case, one need only consider exact, faithful tensor functors $F : \mathrm{Rep}\, G \to \mathrm{Bun}_X$.

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2. Application to the moduli of G-torsors

Before proceeding with the proof of Theorem 1.1, we give an application to the stack of G-torsors over a curve. By an *Artin stack*, we mean an algebraic stack as defined in [7, 4.1]. In particular, we assume that an Artin stack has a separated and quasicompact diagonal. For this section only, let k be a field, and assume that X is a connected, regular, proper curve over k. In particular, X is a Dedekind scheme. We also assume for this section that G *has connected generic fibre*. Finally, for this section only we use the convention that for k-schemes Y and T, $Y_T = Y \times_{\mathrm{Spec}\, k} T$.

Let $G\mathrm{Tor}_X$ denote the fibered category that assigns to a k-scheme T the groupoid of G_{X_T}-torsors. The goal of this section is to prove the following theorem. We are grateful to Brian Conrad for pointing out this application of Theorem 1.1.

Theorem 2.1. *The fibered category $G\mathrm{Tor}_X$ is an Artin stack, locally of finite presentation over k.*

We recall the following definition from [12, 3.3.3], a key input into the proof of the theorem, although the reader can take the statements of the subsequent theorem and lemmas as a black box. Let S be a scheme and T a scheme locally of finite presentation over S. We define the *relative associated primes* of T over S, denoted $\mathrm{Ass}\,(T/S)$, by

$$\mathrm{Ass}\,(T/S) = \bigcup_{s\in S} \mathrm{Ass}\,(T_s).$$

For a point $s \in S$, denote by $(\widetilde{S}, \widetilde{s})$ the henselization of the pair (S, s), and let $\widetilde{T} = T \times_S \widetilde{S}$. We say that T is *pure along* T_s if for each element $\widetilde{t} \in \mathrm{Ass}\,(\widetilde{T}/\widetilde{S})$, the closure of $\widetilde{t}$ in $\widetilde{T}$ meets $\widetilde{T}_{\widetilde{s}}$. We say that T is *S-pure* (or that the map $T \to S$ is *pure*) if it is pure along T_s for each $s \in S$.

A simple example of a map that is not pure is given by $S = \mathrm{Spec}\, R$ for R a complete DVR, $T = \mathrm{Spec}\, K$ where K is the fraction field of R and $T \to S$ the natural inclusion. Then T_s is in fact empty for s the closed point of S.

The reason why we introduce this notion of purity is that pure maps have "flattening stratifications." More precisely, we have the following theorem.

Theorem 2.2. *Suppose that $T \to S$ is pure. Then there is a monomorphism $Z \hookrightarrow S$ that is locally of finite presentation such that for any S-scheme S', $T \times_S S' \to S'$ is flat if and only if $S' \to S$ factors through Z.*

Proof. This is [12, I.4.3.1]. □

Lemma 2.3. *With G and X as above, G is X-pure.*

Proof. Let $\xi \in X$ be the generic point of X. By assumption G_ξ is connected, so it is in fact geometrically irreducible by [3, $\mathrm{VI_A}$ 2.4]. By [6, 2.3.7], since G is flat over X, and X is irreducible, the image of G_ξ in G is dense. In particular, since G_ξ is irreducible so is G. Since X has ξ as its unique associated prime, $\mathrm{Ass}\, G = \mathrm{Ass}\, G_\xi$ by the X-flatness of G (see [6, 3.3.1], which describes associated primes along fibers). Let $\eta \in G$ be its generic point. We claim that $\mathrm{Ass}\, G_\xi = \{\eta\}$. Suppose on the contrary that $Z \subset G_\xi$ is an embedded component. In particular $\dim G_\xi > 0$. Denote by $\overline{\xi}$ an algebraic closure of ξ. Then $Z_{\overline{\xi}} \subset G_{\overline{\xi}}$ is a union of finitely many embedded components. Furthermore, for each closed point $g \in G_{\overline{\xi}}$, $gZ_{\overline{\xi}}$ is also a union of finitely many embedded components. Since $G_{\overline{\xi}}$ is irreducible, there must be infinitely many distinct closed sets amongst the pairwise disjoint $\{gZ_{\overline{\xi}}\}_{g\in G}$. But, this is a contradiction since $G_{\overline{\xi}}$ is of finite type over $\overline{\xi}$ hence has only finitely many associated primes.

Thus far, we have concluded that G is an irreducible scheme over X, and its generic point $\eta \in G$ is its unique associated prime. To show G is pure over a closed point $x \in X$ we may replace X by $\mathrm{Spec}\, \mathscr{O}_{X,x}$. So, we may assume that X is the spectrum of a DVR with closed point x (G is still irreducible and its generic point is its unique associated prime after this base change). Let $(\widetilde{X}, \widetilde{x})$ be the henselization of (X, x). Then $\widetilde{X}$ has its generic point at its unique associated prime. It then follows as above that $\widetilde{G} := G \times_X \widetilde{X}$ also has its generic point as its unique associated prime. Thus, $\mathrm{Ass}\,(\widetilde{G}/\widetilde{X})$ consists of the generic point of $\widetilde{G}$ together with points on $\widetilde{G}_{\widetilde{x}}$ (in fact just the generic points of the latter, but this is not needed). In particular, the closures of these points in $\widetilde{G}$ meet $\widetilde{G}_{\widetilde{x}}$. This shows that G is pure along G_x for each closed point $x \in X$, and it is straightforward to check that G is pure along G_ξ as well. Hence, G is pure over X, as claimed. □

Lemma 2.4. *Let $T \to S$ be locally of finite presentation. If $S' \to S$ is fppf, then $T \times_S S' \to S'$ is flat and pure if and only if $T \to S$ is flat and pure.*

Proof. For purity this is [12, I.3.3.7], and for flatness this is [6, 2.5.1]. □

Lemma 2.5. *Let $\mathscr{I}$ and $\mathscr{Q}$ be an Artin stacks over k, and let $f : \mathscr{I} \to X_{\mathscr{Q}}$ be representable in schemes and locally of finite presentation. The condition on $\mathscr{Q}$-schemes T that $\mathscr{I} \times_{\mathscr{Q}} T \to X_T$ is flat and pure is representable by an Artin stack locally of finite presentation over $\mathscr{Q}$.*

Proof. Let $\mathscr{Z}$ denote the fibered category over $\mathscr{Q}$ where $\mathscr{Z}(T) \subset \mathscr{Q}(T)$ is the full subcategory consisting of those objects of $\mathscr{Q}(T)$ for which $\mathscr{I} \times_{\mathscr{Q}} T \to X_T$ is flat and pure. Using Lemma 2.4, it is straightforward to verify that $\mathscr{Z}$ is a stack. We must show that the map $\mathscr{Z} \to \mathscr{Q}$ is representable and locally of finite presentation.

Let $Q \to \mathscr{Q}$ be a smooth scheme cover, and let $I = \mathscr{I} \times_{\mathscr{Q}} Q$, a smooth scheme cover of $\mathscr{I}$. It suffices to show that $Z = \mathscr{Z} \times_{\mathscr{Q}} Q$ is an algebraic space, locally of finite presentation over Q. By definition, for any k-scheme T, a map $T \to Q$ lies in $Z(T) \subset Q(T)$ if and only if $I \times_Q T \to X_T$ is flat and pure. Thus, we must represent that condition on Q-schemes. We first represent the purity condition. By [12, 3.3.8], purity is an open condition, so there is an open immersion $U' \hookrightarrow X_Q$ such that $X_T \to X_Q$ factors through U' if and only if $I \times_Q T = I \times_{X_Q} X_T$ is pure over X_T. To get an open subspace of Q representing the purity condition, we take the (closed) image of the closed complement of U' under $X_Q \to Q$ and let U be complement of that image. It then follows that $T \to Q$ factors through U if and only if $I \times_Q T$ is pure over X_T.

Thus, replacing Q by U and I by the inverse image of X_U, we may assume that $I \to X_Q$ is pure. In this case, by Theorem 2.2, there is a representable monomorphism $Z' \to X_Q$ such that $Y \to X_Q$ factors through Z' if and only if $I \times_{X_Q} Y \to Y$ is flat. We now want to represent the condition on Q-schemes T that $X_T \to X_Q$ factors through Z'. These are exactly the T-points of the restriction of scalars $\mathrm{Res}_Q^{X_Q}(Z')$. By [11, 1.5], since $X_Q \to Q$ is a proper, flat, and locally finitely presented, and $Z' \to X_Q$ is separated and locally of finite presentation, $\mathrm{Res}_Q^{X_Q}(Z')$ is an algebraic space, locally of finite presentation over Q. □

Proof of Theorem 2.1. By Theorem 1.1, we can find a representation of G on a finite rank vector bundle V, a tensorial construction $t(V)$ and a locally split line bundle $L \subset t(V)$ such that $G \xrightarrow{\sim} \underline{\mathrm{Aut}}\,(V, L)$. We now fix such a pair (V, L). Since X is connected, V has constant rank n for some $n \in \mathbb{N}$. For any X-scheme Y, the identification $G \xrightarrow{\sim} \underline{\mathrm{Aut}}\,(V, L)$ pulls back to $G_Y \xrightarrow{\sim} \underline{\mathrm{Aut}}\,(V_Y, L_Y)$. Let Bun_X^n denote the stack of rank n vector bundles over X (where n is the rank of V). That is, to each k-scheme T, $\mathrm{Bun}_X^n(T)$ is the groupoid of rank n vector bundles over $X_T = X \times_k T$. By [7, 4.6.2.1], Bun_X^n is an Artin stack, locally of finite presentation over k.

Let $\mathscr{E}^{\mathrm{univ}}$ denote the universal rank n vector bundle on $X \times \mathrm{Bun}_X^n$. Let $\mathscr{Q}$ denote the relative quot scheme over Bun_X^n classifying all rank one, locally split subbundles of $t(\mathscr{E}^{\mathrm{univ}})$ (where t is the same tensorial construction as that defining G). That is, for a scheme T over Bun_X^n, $\mathscr{Q}(T)$ is the groupoid of locally split line bundles $\mathscr{L}_{X_T} \subset t(\mathscr{E}^{\mathrm{univ}})_{X_T} = t(\mathscr{E}^{\mathrm{univ}}_{X_T})$ on X_T. Since X is projective over k, it follows from [5, no. 221 Theorem 3.1] that $\mathscr{Q} \to \mathrm{Bun}_X^n$ is representable and locally of finite presentation. Let $\mathscr{L}^{\mathrm{univ}} \subset t(\mathscr{E}^{\mathrm{univ}}_{X_\mathscr{Q}})$ denote the universal line bundle on $X_\mathscr{Q}$. Finally, let $\mathscr{I}$ over $X_\mathscr{Q}$ denote the fibered category, where for an $X_\mathscr{Q}$-scheme T, $\mathscr{I}(T) = \mathrm{Isom}\left((V_T, L_T), (\mathscr{E}^{\mathrm{univ}}_T, \mathscr{L}^{\mathrm{univ}}_T)\right)$. Then $\mathscr{I} \to X_\mathscr{Q}$ is representable in schemes, affine and of finite presentation.

By Lemma 2.5, there is an Artin stack $\mathscr{Z}$ locally of finite presentation over $\mathscr{Q}$ representing the condition on $\mathscr{Q}$-schemes T that $\mathscr{I} \times_\mathscr{Q} T$ is flat and pure over X_T. In particular, $\mathscr{I} \times_\mathscr{Q} \mathscr{Z}$ is flat over $X_\mathscr{Z}$. Let $\mathscr{U}' \subset X_\mathscr{Q}$ denote its open image. Let $\mathscr{U} \subset \mathscr{Q}$ denote the complement of the closed image of the complement of $\mathscr{U}'$ under the projection $X_\mathscr{Q} \to \mathscr{Q}$. Thus, $\mathscr{U}$ represents the condition on $\mathscr{Q}$-schemes T that $\mathscr{I} \times_\mathscr{Q} T$ is flat, surjective (hence fppf since $\mathscr{I} \to X_\mathscr{Q}$ is finitely presented) and pure over X_T. Furthermore, we still have that $\mathscr{U}$ is locally of finite presentation over $\mathscr{Q}$. We now show that $\mathscr{U}$ is naturally isomorphic to $G\mathrm{Tor}_X$. By Corollary 1.3, $G\mathrm{Tor}_X$ is isomorphic to the fibered category that assigns to a k-scheme T the groupoid of X_T-twists of (V, L). It suffices to show that $\mathscr{U}$ is naturally isomorphic to this latter fibered category.

Let T be a $\mathscr{Q}$-scheme and denote by $f : X_T \to X_\mathscr{Q}$ the corresponding map. For ease, we write $f^*\mathscr{I}$ for the pullback of $\mathscr{I}$ along f. The map $f : X_T \to X_\mathscr{Q}$ gives rise to a pair $(f^*\mathscr{E}^{\mathrm{univ}}_{X_\mathscr{Q}}, f^*\mathscr{L}^{\mathrm{univ}})$. We claim that $(f^*\mathscr{E}^{\mathrm{univ}}_{X_\mathscr{Q}}, f^*\mathscr{L}^{\mathrm{univ}})$ is an X_T-twist of (V, L) if and only if T factors through $\mathscr{U}$. First assume that $T \to \mathscr{Q}$ factors through $\mathscr{U}$. In particular, $f^*\mathscr{I} \to X_T$ is fppf. Note that the canonical projection $f^*\mathscr{I} \to \mathscr{I}$ gives an isomorphism $(\mathscr{E}^{\mathrm{univ}}_{f^*\mathscr{I}}, \mathscr{L}^{\mathrm{univ}}_{f^*\mathscr{I}}) \cong (V_{f^*\mathscr{I}}, L_{f^*\mathscr{I}})$. Thus, $f^*\mathscr{I} \to X_T$ gives the desired fppf cover. Conversely, if $(f^*\mathscr{E}^{\mathrm{univ}}_{X_\mathscr{Q}}, f^*\mathscr{L}^{\mathrm{univ}})$ is an X_T-twist of (V, L), then $f^*\mathscr{I}$ is a G_{X_T}-torsor (cf. the proof of Corollary 1.3), and so fppf over X_T. Furthermore, since G is X-pure by Lemma 2.3, it follows by Lemma 2.4 the G_{X_T}-torsor $f^*\mathscr{I}$ is X_T-pure. Thus, T factors through $\mathscr{U}$. We conclude that $\mathscr{U}$ is naturally isomorphic to the desired fibered category, which completes the proof. □

3. Algebraic groups over Dedekind schemes

With notation as in the introduction, let G be a flat algebraic group scheme over X. This means that G is a flat affine group scheme of finite type over X. Let $f : G \to X$ denote the structure map. We will abuse notation and denote the $\mathscr{O}_X$-bialgebra $f_*(\mathscr{O}_G)$ simply by $\mathscr{O}_G$. Let $\Delta : \mathscr{O}_G \to \mathscr{O}_G \otimes \mathscr{O}_G$ denote the comultiplication map and $\varepsilon : \mathscr{O}_G \to \mathscr{O}_X$ the counit. As above, if $W \subset V$ is Zariski-locally on X a direct summand as an $\mathscr{O}_X$-module, we will call the inclusion *locally split*. If W and V are (compatibly) $\mathscr{O}_G$-comodules, that the inclusion $W \subset V$ is locally split does not imply in general that $W \subset V$ is locally a direct summand as an $\mathscr{O}_G$-comodule. Finally, recall that $GL(V)$ is an algebraic group scheme that is represented by $\mathrm{Spec}\,(\mathrm{Sym}\,(V \otimes V^*)[1/\det])$. Our presentation of this section follows [16, Chap. 3] and [1, Chap. 5], generalized to our current situation.

Lemma 3.1. *Let V be an X-flat quasicoherent $\mathscr{O}_G$-comodule. Then V is the direct limit of $\mathscr{O}_G$-comodules that are locally free $\mathscr{O}_X$-modules of finite rank.*

Proof. For X affine, this is the Corollary to Proposition 1.2 in [14]. We quickly sketch the proof in the general case as the details are the same as in *ibid.* Since X is noetherian, by [4, 9.4.9] any quasicoherent sheaf is the direct limit of its coherent subsheaves. Since a coherent $\mathscr{O}_X$-submodule of V is locally free, it suffices to show that for any coherent submodule $W \subset V$, W is contained in a coherent $\mathscr{O}_G$-subcomodule of V. Let $\rho : V \to V \otimes \mathscr{O}_G$ denote the comodule map. Since $\rho(W)$ is coherent, there is a coherent submodule $W' \subset V$ such that $\rho(W) \subset W' \otimes \mathscr{O}_G$. Define a quasicoherent $\mathscr{O}_X$-module $E = \rho^{-1}(W' \otimes \mathscr{O}_G)$. By working over open affines in X, one can show that $E \subset W'$, so it is coherent, and E is an $\mathscr{O}_G$-comodule (cf. [14, Section 1.5]). $\square$

Lemma 3.2. *There is a representation V of G such that the map $G \to GL(V)$ is a closed embedding.*

Proof. Consider the regular representation $\Delta : \mathscr{O}_G \to \mathscr{O}_G \otimes \mathscr{O}_G$. By Lemma 3.1, there is a locally free, finite rank $\mathscr{O}_G$-subcomodule $V \subset \mathscr{O}_G$ that locally contains a finite system of $\mathscr{O}_X$-algebra generators of $\mathscr{O}_G$. By restricting Δ to V, we have an $\mathscr{O}_G$-comodule $\rho : V \to V \otimes \mathscr{O}_G$. To check the corresponding map $G \to GL(V)$ is a closed embedding, we may assume that $X = \operatorname{Spec} R$, where R is a DVR. In this case, $V \cong R^n$, and $\mathscr{O}_{GL(V)} \cong R[x_{11}, \ldots, x_{nn}][1/\det]$. The verification that $\mathscr{O}_{GL(V)} \to \mathscr{O}_G$ is surjective is then identical to the proof in [16, 3.4].

Namely, if we choose a basis $\{v_1, \ldots, v_n\}$ of V and write $\rho(v_i) = \sum v_j \otimes a_{ij}$, then the map $\mathscr{O}_{GL(V)} \to \mathscr{O}_G$ is given by $x_{ij} \mapsto a_{ij}$. Since $v_j = (\varepsilon \otimes 1)\Delta(v_j) = \sum \varepsilon(v_i) a_{ij}$, the image of the map $\mathscr{O}_{GL(V)} \to \mathscr{O}_G$ contains V, hence is surjective since V contains the algebra generators of $\mathscr{O}_G$. $\square$

Let $\{X_1, \ldots, X_r\}$ denote the set of (nonempty) connected components of X. Let $\mathscr{K}_{X_i}$ denote the stalk of $\mathscr{O}_{X_i}$ at the generic point of X_i, and write $\mathscr{K}_X = \prod \mathscr{K}_{X_i}$. If M is a locally free of finite rank $\mathscr{O}_X$-module, and $N' \subset M$ is a coherent submodule, we call $N = (N' \otimes \mathscr{K}_X) \cap M \subset M \otimes \mathscr{K}_X$ the *saturation of* N' *in* M. (The point is that N' may not be a subbundle of M.)

Lemma 3.3. *Let W be a representation of G, $U' \subset W$ a subrepresentation, and let U denote the saturation of U' in W. Then, U is a subrepresentation of W that is locally split as an $\mathscr{O}_X$-module.*

Proof. Since X is Dedekind, it is straightforward to check that U is locally split in W (say, by looking at stalks and using the elementary divisors theorem). It remains to show that U is G-stable. Let $\rho : W \to W \otimes \mathscr{O}_G$ denote the comodule map. We wish to show that $\rho(U) \subset U \otimes \mathscr{O}_G$. This can be checked Zariski-locally on X, so can assume that $X = \operatorname{Spec} A$ is a Dedekind domain, and U/W is free. To show that the image of U in $W \otimes \mathscr{O}_G$ is contained in $U \otimes \mathscr{O}_G$, we must show the image of any element in $W \otimes \mathscr{O}_G$ goes to zero in $(U/W) \otimes \mathscr{O}_G$. Since this latter A-module is flat, we can check that the image is zero on the generic point of $\operatorname{Spec} A$. But, over the generic point $U = U'$, so the result follows from the G-stability of U'. $\square$

Lemma 3.4. *Let W be a finite rank vector bundle on X, and suppose $U \subseteq W$ is a locally split subbundle. Let $\mathbf{d} = (d_1, \ldots, d_r)$ be the sequence of ranks of U on each nonempty connected component of X. Define $L = \bigwedge^{\mathbf{d}} U \subset \bigwedge^{\mathbf{d}} W$. Let $g \in GL(W)$. Then*

$$gL = L \iff gU = U.$$

Proof. The statement is local on X, so we suppose that $X = \operatorname{Spec} A$ for a Dedekind domain A, and that $U \subset W$ is a rank d direct summand. The direction $\Longleftarrow$ is immediate by functoriality, so we assume now that $gL = L$. First, note that for any A-algebra B,

$$U \otimes B = \{\omega \in W \otimes B \mid \omega \wedge (L \otimes B) = 0\}.$$

If $g \in GL(W \otimes B)$ and $u \in U \otimes B$, then

$$gu \wedge (L \otimes B) = g(u \wedge g^{-1}(L \otimes B)) = g(u \wedge L \otimes B) = 0.$$

It follows from the previous remark that $gu \in U \otimes B$, as desired. $\square$

Theorem 3.5. *There is a representation V of G, a tensorial construction $t(V)$, and a locally split line bundle $L \subset t(V)$ such that*

$$G = \{g \in GL(V) \mid gL = L\}.$$

Proof. By Lemma 3.2, we can fix a representation V of G such that $G \to GL(V)$ is a closed embedding. We must now construct $t(V)$ and $L \subset t(V)$. We can write

$$\mathscr{O}_{GL(V)} = \varinjlim_i \left(\bigoplus_{m \geq 0} \operatorname{Sym}^m (V \otimes V^*) \cdot \det{}^{-i} \right). \tag{3.1}$$

Identifying G as a closed subgroup of $GL(V)$, G is defined by a coherent sheaf of ideals $\mathscr{I} \subset \mathscr{O}_{GL(V)}$. Note that since G is flat over X, $\mathscr{I}$ is saturated in $\mathscr{O}_{GL(V)}$. Choose a finite open affine cover $\{X_i\}$ of X. On each X_i, $\mathscr{I}|X_i$ is finitely generated in $\mathscr{O}_{GL(V)}|X_i$ as an $\mathscr{O}_{X_i}$-module. Hence, by taking integers M and N sufficiently large, we can ensure that the module generators of $\mathscr{I}$ on each X_i are contained in

$$t'(V) = \bigoplus_{m=0}^{M} \operatorname{Sym}^m (V \otimes V^*) \cdot \det{}^{-N}.$$

Let $U' = \mathscr{I} \cap t'(V)$. Let $G' = \{g \in GL(V) \mid gU' = U'\}$. We claim that $G = G'$. First, note that

$$G = \{g \in GL(V) \mid g\mathscr{I} = \mathscr{I}\}.$$

In particular, $G \subseteq G'$. On the other hand, if $g \in G'(B)$, then by definition the induced map $(1 \otimes g) \circ \Delta : U' \to \mathscr{O}_{GL(V)} \otimes B$ factors through $U' \otimes B$. However, since $(1 \otimes g) \circ \Delta$ is an $\mathscr{O}_X$-algebra map, it follows that $\mathscr{I} \to \mathscr{O}_{GL(V)} \otimes B$ factors through $\mathscr{I} \otimes B$. That is, $G' \subseteq G$, thus $G = G'$.

Let U be the saturation of U' in $t'(V)$. By Lemma 3.3, U is G-stable and locally split in $t'(V)$. It follows that $G \subseteq \{g \in GL(V) \mid gU = U\}$. Conversely, to check that $\{g \in GL(V) \mid gU = U\} \subseteq G$, it suffices to check on an affine cover of X. Then one can see that $\{g \in GL(V) \mid gU = U\} \subseteq G$ exactly as in the proof of Lemma 3.3. Thus, $G = \{g \in GL(V) \mid gU = U\}$. Let $\mathbf{d} = (d_1, \ldots, d_n)$ be the sequence of ranks of U on each nonempty connected component of X. Define $t(V) = \bigwedge^{\mathbf{d}} t'(V)$ and $L = \bigwedge^{\mathbf{d}} U \subset t(V)$. By Lemma 3.4, we have that $G = \{g \in GL(V) \mid gL = L\}$, as claimed. $\square$

4. Tannakian viewpoint

We recall the notation from the introduction. As usual, G denotes a flat algebraic group over a Dedekind scheme X. In this section, we fix a faithfully flat X-scheme Y. Recall that unadorned products are fiber products over X and for an X-scheme T, $Y_T = Y \times T = Y \times_X T$. For each open subscheme $U \subset X$, let $\mathscr{O}_U = \mathscr{O}_X|U$. We write $\operatorname{Rep}_U G$ for the category of representations of G_U on finite rank, locally free $\mathscr{O}_U$-modules. Then, $\operatorname{Rep}_U G$ is an $\mathscr{O}_U$-linear, rigid tensor category. Here, rigid means that $\operatorname{Rep}_U G$ has internal homs. Of course, unless $\mathscr{O}_U$ is a field, this will not be an abelian category. Denote by X_{Zar} the small Zariski site on X. Denote by $\mathbf{Rep}\, G$ the fibered over X_{Zar} where for an object U in X_{Zar}, $\mathbf{Rep}\, G(U) = \operatorname{Rep}_U G$. Then $\mathbf{Rep}\, G$ is a *(fibered) tensor category* in the following sense:

(i) There is a monoidal structure
$$\mathbf{Rep}\, G \times_{X_{\mathrm{Zar}}} \mathbf{Rep}\, G \to \mathbf{Rep}\, G$$
(along with associativity and commutativity constraints) that over each U in X_{Zar} induces the usual tensor structure on $\operatorname{Rep}_U G$.

(ii) There is an object $1_X \in \operatorname{Rep}_X G$ that pulls back to the unit object in $\operatorname{Rep}_U G$ for each U in X_{Zar}.

(iii) For each $U' \subset U$, the pullback map $\operatorname{Rep}_U G \to \operatorname{Rep}_{U'} G$ is a tensor functor.

Let $\mathbf{Bun}_Y$ denote the fibered category over X_{Zar} where for an object U in X_{Zar}, $\mathbf{Bun}_Y(U) = \operatorname{Bun}_{Y_U}$ is the category of all finite rank vector bundles on Y_U (not to be confused with Bun_Y^n in §2). Then $\mathbf{Bun}_Y$ is a tensor category each of whose fibers over X_{Zar} is $\mathscr{O}_{Y_U}$-linear and rigid. By a *(fibered) tensor functor* $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$, we mean a functor of fibered categories over X_{Zar} that induces a tensor functor (in the usual sense) on each fiber. In particular, F must respect unit objects on each fiber.

Let $P \to Y$ be a G_Y-torsor. Then, for each object U in X_{Zar}, P_U is a G_{Y_U}-torsor. We define a functor $F_P : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ as follows. For an object U in X_{Zar}, and V in $\operatorname{Rep}_U G$,
$$F_P : V \mapsto P_U \times^{G_{Y_U}} (V \times_U Y_U) = P_U \times (V \times_U Y_U)/((p,v) \sim (pg, g^{-1}v)).$$
Concretely, we are pushing out P along the map $G \to GL(V)$ to associate to the G_{Y_U}-torsor P a $GL(V_U)$-torsor, that is, a vector bundle on Y_U. It is clear F_P respects pullback maps, so it is a functor of fibered categories. When no confusion will arise, we will write $F_P(V) = P \times^G V$ for notational ease.

Lemma 4.1. *The functor F_P is a tensor functor that on each fiber over X_{Zar} is faithful and exact.*

Proof. It is clear that F_P is a functor of fibered categories over X_{Zar}, so we must show it is an exact, faithful tensor functor on each fiber. Fix an object U in X_{Zar}. Since G acts transitively on P, it is straightforward to check from the definition that $F_P(\mathscr{O}_U) = \mathscr{O}_{Y_U}$, where $\mathscr{O}_U$ has the trivial G_U-action. Thus, F_P respects unit objects. For V and W in $\operatorname{Rep}_U G$, there is a natural map
$$P \times^G (V \otimes W) \to (P \times^G V) \otimes (P \times^G W);\ (p, v \otimes w) \mapsto (p,v) \otimes (p,w), \tag{4.1}$$
which is straightforward to check is well defined. It suffices to check that (4.1) is an isomorphism fppf-locally on U, so we may assume that $P_U = G_{Y_U} \times_U Y_U$ is the trivial G_{Y_U}-torsor. Under the identification, $P \times^G V = V_{Y_U}$, (4.1) becomes the identity map. Hence, F_P is a tensor functor.

To show that F_P is exact, we must show that if $0 \to V' \to V \to V'' \to 0$ is exact, then so is $0 \to P \times^G V' \to P \times^G V \to P \times^G V'' \to 0$. Again, we can check that this sequence is exact fppf-locally on Y_U, so we can assume that $P_U = G_{Y_U} \times_U Y_U$. We can then identify the latter exact sequence with $0 \to V'_{Y_U} \to V_{Y_U} \to V''_{Y_U} \to 0$, which is exact since Y is flat over X. Finally, to show that F_P is faithful, we assume that $F_P(V) = 0$. Passing to an fppf-cover of Y_U, this implies that $V_{Y_U} = 0$. Hence $V = 0$ since Y is faithfully flat over X. □

Remark 4.2. The proof above that F_P respects tensor products generalizes easily to show that in fact F_P respects any tensorial construction.

Thus, F_P is a tensor functor that on each fiber over X_{Zar} is faithful and exact. We now prove that the converse is true. Let $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ be a tensor functor that on each fiber over X_{Zar} is faithful and exact. We show that there is a natural equivalence $F \xrightarrow{\sim} F_P$ for a uniquely defined G_Y-torsor P. We closely follow the elegant presentation in [10, Sec. 2], generalizing to our current situation. The main idea to define P is to apply F to the regular representation of G. Of course, this is not a finite rank representation, so we must first suitably extend F.

We denote by $\mathbf{Rep}'\, G$ the fibered category over X_{Zar}, where for each U in X_{Zar}, $\mathbf{Rep}'\, G(U) = \mathrm{Rep}'_U\, G$ is the category of flat quasicoherent $\mathscr{O}_U$-modules that are also $\mathscr{O}_{G_U}$-comodules. Denote by $\mathbf{QCoh}_Y$ the fibered category over X_{Zar} where for each U in X_{Zar}, $\mathbf{QCoh}_Y(U) = \mathrm{QCoh}_{Y_U}$ is the category of quasicoherent $\mathscr{O}_{Y_U}$-modules.

Since we will be working over open subschemes of X, we will need the following slight generalization of Lemma 3.1.

Lemma 4.3. *Let $U \subset X$ be an open subscheme and let V be an object of $\mathrm{Rep}'_U\, G$. Then, V is the direct limit of its subobjects in $\mathrm{Rep}_U\, G$.*

Proof. The proof is identical to that of Lemma 3.1. One need only note that it is still the case that any coherent $\mathscr{O}_U$-submodule of V is locally free. □

Lemma 4.4. *The functor F extends uniquely to a tensor functor $F : \mathbf{Rep}'\, G \to \mathbf{QCoh}_Y$ such that:*

(i) On each fiber over X_{Zar}, F is exact and faithful.
(ii) The extended F respects direct limits.
(iii) The $\mathscr{O}_Y$-module $F(\mathscr{O}_G)$ is faithfully flat.

Proof. Fix an object U in X_{Zar}. To extend F, let V be a flat, quasicoherent $\mathscr{O}_U$-module, and define

$$F(V) = \varinjlim_{W \subset V} F(W),$$

where the colimit is over all coherent $\mathscr{O}_G$-subcomodules $W \subset V$. By Lemma 4.3, this is a direct limit. Since filtered colimits are exact and commute with tensor product, $F(V)$ is flat, and the extended functor is a tensor functor that is exact. This establishes (i)

Next, we show that the extended F respects colimits. Suppose $W = \varinjlim_\alpha W_\alpha$, and write $W_\alpha = \varinjlim_\beta W_{\alpha\beta}$, where each $W_{\alpha\beta}$ is a finite rank $\mathscr{O}_G$-comodule. Since colimits can be iterated by [9, IX.8], we have $W = \varinjlim_{\alpha,\beta} W_{\alpha,\beta}$. It follows that

$$F(W) = \varinjlim_{\alpha,\beta} F(W_{\alpha,\beta}) = \varinjlim_{\alpha} \varinjlim_{\beta} F(W_{\alpha\beta}) = \varinjlim_{\alpha} F(W_\alpha),$$

hence F respects colimits, which establishes (ii).

It remains to show that $F(\mathscr{O}_G)$ is faithfully flat. By [6, 2.2.1], $F(\mathscr{O}_G)$ is faithfully flat over Y if and only if the functor $M \mapsto F(\mathscr{O}_G) \otimes_{U'} M$ is an exact and faithful functor on $\mathrm{QCoh}_{U'}$ for all $U' \subset Y$ open. Since $F(\mathscr{O}_G)$ is flat, $M \mapsto F(\mathscr{O}_G) \otimes_{U'} M$ is exact. It remains to show that for any $M \neq 0$, $F(\mathscr{O}_G) \otimes_{U'} M$ is nonzero. Since $\mathscr{O}_G$ has $\mathscr{O}_X$ as a direct summand, and F is exact, $F(\mathscr{O}_G)$ contains $F(\mathscr{O}_X) = \mathscr{O}_Y$ as a direct summand. In particular, $F(\mathscr{O}_G) \otimes_{U'} M = M \oplus M'$ (for some M') is nonzero, which completes the proof. □

Lemma 4.5. *The functor F naturally induces a functor from the fibered category over X_{Zar} of U-schemes with G_U-action that are flat and affine over U to the fibered category over X_{Zar} of schemes flat and affine over Y_U. The resulting functor, which we again denote by F, respects products and has the property that if T_0 has a trivial G_U-action then $F(T_0) = Y_U \times_U T_0$.*

Proof. Fix an object U in X_{Zar}. Let T be a scheme flat and affine over U with G_U-action. Then (the pushforward of) $\mathscr{O}_T$ is an $\mathscr{O}_U$-algebra and $\mathscr{O}_{G_U}$-comodule. Furthermore, the multiplication map $\mathscr{O}_T \otimes \mathscr{O}_T \to \mathscr{O}_T$ is an $\mathscr{O}_{G_U}$-comodule map. Thus, since F is a tensor functor, $F(\mathscr{O}_T)$ is naturally an $\mathscr{O}_{Y_U}$-algebra and flat by Lemma 4.4. We can therefore define

$$F(T) = \operatorname{Spec} F(\mathscr{O}_T),$$

a scheme that is flat and affine over Y_U. Since F is a tensor functor, it is clear that it respects products.

To verify the last claim, we identify the full subcategory of trivial representations in $\mathrm{Rep}_U\, G$ with the category of finite rank vector bundles on U. For each affine open $U' \subset Y_U$, we will give a natural isomorphism

$$F(V)|U' \xrightarrow{\sim} \mathscr{O}_{U'} \otimes_{\mathscr{O}_U} V$$

and it will be clear from the construction that these isomorphisms will agree on overlaps. Thus, we may assume that $Y = \operatorname{Spec} B$ is affine.

Furthermore, it suffices to prove the result for some affine cover of X so we may assume that $X = \operatorname{Spec} A$ is affine. Since F is a tensor functor, $F(A) = B$. Thus, for any vector bundle V the composition

$$V \xrightarrow{\sim} \mathrm{Hom}_A(A, V) \xrightarrow{F} \mathrm{Hom}_B(B, F(V))$$

gives rise to a natural map of B-modules $\psi : V \otimes_A B \to F(V)$ by adjunction. Furthermore, ψ is an isomorphism for $V = A^n$. Since any vector bundle is a direct summand of a free module, it follows that ψ is an isomorphism for all V. □

Remark 4.6. The above lemma establishes the aim of this section in the case that G is the trivial group. When X is not affine, the use of fibered categories is crucial to establish this result.

Lemma 4.7. *Let $P = F(G)$. Then P is a G_Y-torsor naturally in F and Y.*

Proof. By Lemma 4.4(iii), P is faithfully flat over Y. Denote the group map $m : G \times G \to G$ and the identity section $e : X \to G$. Let G_0 denote the same underlying scheme as G with the trivial G-action. By Lemma 4.5, applying F to the map of G-sets $G \times G_0 \to G$ gives rise to a map $P \times_Y G_Y \to P$. Again by Lemma 4.5, applying F to the commutative diagrams of G-sets

$$\begin{tikzcd} X \times G_0 \arrow[r, "e\times 1"] \arrow[dr, equal] & G \times G_0 \arrow[d, "m"] \\ & G_0 \end{tikzcd} \qquad \begin{tikzcd} G \times G_0 \times G_0 \arrow[r, "1\times m"] \arrow[d, "m\times 1"] & G \times G_0 \arrow[d, "m"] \\ G \times G_0 \arrow[r, "m"] & G_0 \end{tikzcd}$$

establishes that the map $P \times_Y G_Y \to P$ is a right G-action.

Since the G-map

$$G \times G_0 \to G \times G;\ (g,h) \mapsto (g, gh)$$

is an isomorphism, the corresponding map induced by F, $P \times_Y G_Y \to P \times_Y P$, is an isomorphism. Thus, P is a G_Y-torsor. $\square$

Theorem 4.8. *Let Y be faithfully flat scheme over X. The functor from the category of G_Y-torsors to the category of tensor functors $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ that on each fiber over X_{Zar} are faithful and exact, given by*

$$P \mapsto [F_P : V \mapsto P_U \times^{G_{Y_U}} (V \times_U Y_U)],$$

is an equivalence of fibered categories. The quasi-inverse is given by $F \mapsto F(G)$ (see below remark).

Remark 4.9. Before we begin the proof, let us summarize the definition of $F(G)$. As in Lemma 4.4, we can define $F(\mathscr{O}_G) = \varinjlim F(V)$ where V ranges over $\mathscr{O}_X$-coherent $\mathscr{O}_G$-subcomodules of $\mathscr{O}_G$. Then, as described in the proof of Lemma 4.5, $F(\mathscr{O}_G)$ is an $\mathscr{O}_X$-algebra, so we can define $F(G) = \operatorname{Spec} F(\mathscr{O}_G)$. The G_Y-action on $F(G)$ is described in the proof of Lemma 4.7.

Proof of Theorem 4.8. We must show that the two functors are quasi-inverses. Given a G_Y-torsor P, that $F_P(G)$ is naturally isomorphic to P follows directly from the definition of F_P:

$$F_P(G) = P \times^G G = P \times G/[(p,x) \sim (pg, g^{-1}x)] \xrightarrow{\sim} P.$$

Here the last map is given by $(p,x) \mapsto px$, which respects the right action on $P \times^G G$ given by $(p,x)\cdot g = (pg, x) = (p, gx)$.

Let $F : \mathbf{Rep}\, G \to \mathbf{Bun}_Y$ be given. Let $P = F(G)$. We must show that F_P is naturally equivalent to F. For the remainder of the proof, we will make frequent use of Lemma 4.5 without explicit mention. We again use the notation that if T is some object with G-action, then T_0 is the same underlying object with the trivial G-action. Recall that the right G_Y-action on P is given by applying F to the G-map $G \times G_0 \to G$. Since F respects products, $P_U = F(G_U)$ and the right G_{Y_U}-action on P_U is given by applying F to $G_U \times_U (G_U)_0 \to G_U$. Fix an object U of X_{Zar} and let V be a representation of G_U. Applying F to $\rho : G_U \times_U V_0 \to V$ induces a map $\phi = F(\rho) : P_U \times_{Y_U} (V \times_U Y_U) \to F(V)$.

We first show that ϕ factors through the quotient map $P_U \times (V \times_U Y_U) \to P_U \times^{G_{Y_U}} (V \times_U Y_U)$. By definition, this quotient is defined to be the coequalizer of

$$P_U \times_{Y_U} G_{Y_U} \times_{Y_U} (V \times_U Y_U) \underset{\beta}{\overset{\pi_{1,3}}{\rightrightarrows}} P_U \times_{Y_U} (V \times_U Y_U),$$

where $\beta : (p,g,v) \mapsto (pg, g^{-1}v)$. Thus, it suffices to show that $\phi \circ \pi_{1,3} = \phi \circ \beta$. Denote by $\alpha : G_U \times_U (G_U)_0 \times_U V_0 \to G_U \times_U V_0$ the G_U-map $(g,h,v) \mapsto (gh, h^{-1}v)$.

Then it is immediate that the following diagram commutes.

$$\begin{tikzcd} G_U \times_U (G_U)_0 \times_U V_0 \arrow[r, "\pi_{1,3}"] \arrow[d, "\alpha"'] & G_U \times_U V_0 \arrow[d, "\rho"] \\ G_U \times_U V_0 \arrow[r, "\rho"'] & V \end{tikzcd}$$

By definition of the G-action, $\beta = F(\alpha)$. Thus, by applying F to the above diagram, we conclude that $\phi \circ \pi_{1,3} = \phi \circ \beta$. It follows that ϕ descends to a map $\phi : P_U \times^{G_{Y_U}} (V \times_U Y_U) \to F(V)$, which it remains to show is an isomorphism.

Since $P_U \to Y_U$ is faithfully flat, it suffices to show that ϕ is an isomorphism after pulling back to P_U. One checks from the definitions that we have the following sequence of isomorphisms:

$$\begin{aligned} P_U \times_{Y_U} (V \times_U Y_U) &\xrightarrow{\sim} (P_U \times_U G_{Y_U}) \times^{G_{Y_U}} (V \times_U Y_U) \\ &\xrightarrow{\sim} (P_U \times_{Y_U} P_U) \times^{G_{Y_U}} (V \times_U Y_U) \\ &\xrightarrow{\sim} P_U \times_{Y_U} (P_U \times^{G_{Y_U}} (V \times_U Y_U)). \end{aligned}$$

Thus, identifying the source of $1 \times \phi$ with the first term in the above sequence, it remains to show that the induced map $\psi : P_U \times_{Y_U} (V \times_U Y_U) \to P_U \times_{Y_U} F(V)$ is an isomorphism. Following the construction, one sees that ψ comes from applying F to the G_U-map $G_U \times_U V_0 \to G_U \times_U V$ given by $(g, v) \mapsto (g, gv)$. Since this latter map is an isomorphism, it follows that ψ is an isomorphism, whence the result follows. $\square$

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