Annals of Mathematics, 127 (1988), 629–646

# Tamagawa numbers

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Let $F$ be a number field, let $G$ be a connected reductive group over $F$, and let $\tau(G)$ denote the Tamagawa number of $G$. This invariant of $G$ has been studied by Demazure, Lai, Langlands, Mars, Ono, Tamagawa and Weil, and classically by Siegel and earlier authors. Ono's appendix to [36] provides an overview of their work.

Weil conjectured that $\tau(G) = 1$ for all simply connected semisimple groups $G$, and the authors mentioned above proved this conjecture except for certain exceptional $G$. Jacquet-Langlands [16] proposed a two-step method for proving Weil's conjecture. The first step was to generalize to the quasi-split case the method used by Langlands [25] for split groups. This was done by Lai [24]. The second step was to use the trace formula to show that $\tau(G) = \tau(G_0)$, where $G_0$ denotes a quasi-split inner form of $G$. In order to illustrate the method, Jacquet-Langlands carried it out with $G_0 = \mathrm{SL}_2$ (actually they worked with $\mathrm{GL}_2$).

In this paper we will see that Jacquet-Langlands's second step can be carried out in general, modulo the Hasse principle for $H^1$ of simply connected groups. Since Kneser and Harder [23], [12] have proved the Hasse principle for groups without $E_8$ factors, we get the following:

Theorem. *Assume that $G$ is a simply connected semisimple group having no $E_8$ factors. Then $\tau(G) = 1$.*

Ono [29] determined the Tamagawa number for tori and also showed how Tamagawa numbers behave under isogenies. Ono's results can be reformulated and slightly generalized. For this see Section 5 of [19]; in the notation of that article we have

$$\tau(G) = \left|\pi_0\left(Z(\hat{G})^{\Gamma}\right)\right| \cdot \left|\ker^1\left(F, Z(\hat{G})\right)\right|^{-1}$$

for any connected reductive group $G$ over $F$ such that $G_{sc}$ has no $E_8$ factors.

In order to prove the theorem we need a simple form of the trace formula (see §5). This is due to Arthur and requires all of his work on the trace formula.

*Partially supported by NSF Grant DMS-8601121.

We also need to use certain functions $f_{\mathrm{EP}}$ on $p$-adic groups, which I call Euler-Poincaré functions. The orbital integrals and character values $\operatorname{tr}\pi(f_{\mathrm{EP}})$ of this function are particularly simple, making it very convenient to use in the trace formula. The functions $f_{\mathrm{EP}}$ are studied in Section 2. They are closely related to Serre's Euler-Poincaré measures [31], and in order to show that Euler-Poincaré functions on inner twists have matching stable orbital integrals for semisimple elements we need to determine the behavior of Euler-Poincaré measures under inner twisting. We do this in Section 1, answering a question of Casselman [7]. In Section 3 we give two simple applications of this result; these are not needed for the proof of the main theorem but will be useful elsewhere [27].

The Hasse principle is needed in the proof that $\tau(G) = \tau(G_0)$ because the proof relies on the partial stabilization of the trace formula given in [20], which makes use of all of the basic facts about the Galois cohomology of reductive groups over local and global fields. In fact [20] uses the Hasse principle twice. The first time is in Lemma 6.3. The second time, on page 394 of [20], it is used implicitly in the reference to Lemma 4.3.1 of [19].

I would like to thank W. Casselman, G. Prasad and R. Steinberg for suggesting improvements in my original manuscript and allowing me to incorporate them into this paper. Theorem 2′ is due to Casselman; it uses his determination for any $p$-adic group of the irreducible admissible representations having non-trivial continuous cohomology. Prasad and Steinberg realized that Steinberg's formula for Poincaré series in the twisted affine case gives a proof of the $p$-adic part of Theorem 1 that avoids the case-by-case considerations of my original manuscript.

We use the following notation: $F$ denotes a local or global field of characteristic 0 and $\bar{F}$ an algebraic closure of $F$. If $F$ is $p$-adic we denote by $\mathfrak{o}$, $k$, $q$ the valuation ring of $F$, the residue field of $\mathfrak{o}$, and the cardinality of $k$ respectively.

## 1. Euler-Poincaré measures and inner twists

Let $F$ be a local field of characteristic 0 and let $G$ be a connected reductive group over $F$. For any invariant measure $\mu$ on $G(F)$ and any discrete cocompact subgroup $\Gamma$ of $G(F)$ we write $\mu(\Gamma \backslash G(F))$ for the volume of $\Gamma \backslash G(F)$ with respect to the invariant measure on $\Gamma \backslash G(F)$ obtained from $\mu$. Serre [31] has shown that there exists an invariant measure $\mu_G$ on $G(F)$ having the following property: for every discrete, torsion-free, cocompact subgroup $\Gamma$ of $G(F)$ the Euler-Poincaré characteristic of $H^{\bullet}(\Gamma, \mathbf{Q})$ is equal to $\mu_G(\Gamma \backslash G(F))$. Borel and Harder [4] have shown that $G(F)$ always has discrete, torsion-free, cocompact subgroups; therefore $\mu_G$ is uniquely determined by the property above. Serre calls $\mu_G$ the Euler-Poincaré measure of $G$.

The Euler-Poincaré measure can be negative or zero. It is non-zero if and only if $G$ has an anisotropic maximal $F$-torus (in the $p$-adic case this happens if and only if the connected center of $G$ is anisotropic [22]). In this case its sign is $(-1)^{q(G)}$ where $q(G)$ is the $F$-rank of $G_{\text{der}}$ in the $p$-adic case and half the dimension of the symmetric space attached to $G_{\text{der}}$ in the real case (see Propositions 23, 28 of [31]).

Now suppose that we are given an inner twist $H$ of $G$ and invariant measures $\mu$ and $\mu'$ on $G(F)$ and $H(F)$ respectively. We need to recall the standard method of comparing $\mu, \mu'$. Choose an inner twisting $\psi \colon H \to G$ over $\bar{F}$ and a non-zero invariant differential form $\omega$ on $G$ over $F$ of top degree; note that $\psi^*\omega$ is defined over $F$. We say that $\mu, \mu'$ are *compatible* if there exists $c \in \mathbf{R}$ such that $\mu = c|\omega|$ and $\mu' = c|\psi^*\omega|$. The compatibility of $\mu, \mu'$ is independent of the choice of $\psi$, $\omega$.

Now suppose that $G$ has an anisotropic maximal $F$-torus $T$, so that the Euler-Poincaré measure $\mu_G$ is non-zero. We write $\mathfrak{D}(T, G)$ for the finite set

$$\ker\left[H^1(F, T) \to H^1(F, G)\right].$$

As before let $H$ be an inner form of $G$. Then it is well-known that $T$ transfers to $H$ (see §10 of [20] for example), and thus we can also consider the finite set $\mathfrak{D}(T, H)$.

THEOREM 1. *The invariant measure* $(-1)^{q(G)}|\mathfrak{D}(T, G)|^{-1}\mu_G$ *on* $G(F)$ *is compatible with the invariant measure* $(-1)^{q(H)}|\mathfrak{D}(T, H)|^{-1}\mu_H$ *on* $H(F)$.

In the $p$-adic case the vanishing of $H^1$ for simply connected groups [22] implies that $\mathfrak{D}(T, G)$ and $\mathfrak{D}(T, H)$ are both equal to

$$\operatorname{im}\left[H^1(F, T_{sc}) \to H^1(F, T)\right],$$

and therefore the theorem just says that $(-1)^{q(G)}\mu_G$ and $(-1)^{q(H)}\mu_H$ are compatible.

In the real case, $\mathfrak{D}(T, G)$ and $\mathfrak{D}(T, H)$ need not have the same cardinality, as one sees from Shelstad's [32] description of $\mathfrak{D}(T, G)$ for anisotropic $T$:

$$\mathfrak{D}(T, G) = \Omega(G(\mathbf{R}), T(\mathbf{R})) \setminus \Omega(G(\mathbf{C}), T(\mathbf{C})).$$

Here $\Omega$ stands for Weyl group.

Now we prove the theorem. For real groups it is enough to consider the case in which $H$ is the compact form of $G$ (an inner form of $G$ since we have assumed that $G$ has a compact maximal torus). Then the theorem is implicit in Serre's proof [31, Prop. 23] that the sign of $\mu_G$ is $(-1)^{q(G)}$. Let $K$ be a maximal compact subgroup of $G(\mathbf{R})$ containing $T(\mathbf{R})$ and let $dk$ be Haar measure on $K$ giving $K$ measure 1. Then $H$ can be obtained from $G$ by twisting by the Cartan involution of $G$ that fixes $K$ pointwise. This twisting does not change $K$, so that we have

$$T(\mathbf{R}) \subset K \subset H(\mathbf{R}).$$

Serre's proof shows that the invariant measure $(-1)^{q(G)}\mu_G$ is compatible with the invariant measure $dh$ on $H(\mathbf{R})$ such that $dh/dk$ is the Gauss-Bonnet measure on $H(\mathbf{R})/K$. But Hopf and Samelson [15] showed that the Euler-Poincaré characteristic of $H(\mathbf{R})/K$ is equal to the cardinality of

$$\Omega(K, T(\mathbf{R})) \setminus \Omega(H(\mathbf{R}), T(\mathbf{R})).$$

Since $\Omega(K, T(\mathbf{R})) = \Omega(G(\mathbf{R}), T(\mathbf{R}))$ and $\Omega(H(\mathbf{R}), T(\mathbf{R})) = \Omega(G(\mathbf{C}), T(\mathbf{C}))$, we now see that $(-1)^{q(G)}|\mathfrak{D}(T, G)|^{-1}\mu_G$ is compatible with the invariant measure on $H(\mathbf{R})$ that gives $H(\mathbf{R})$ measure 1, which is in fact the Euler–Poincaré measure of $H$ because $H(\mathbf{R})$ is compact. Since the factors $(-1)^{q(H)}$, $|\mathfrak{D}(T, H)|$ are both 1, this proves the theorem in the real case.

Next we consider the $p$-adic case. First of all there is a routine reduction to the case in which $G$ is semisimple and simply connected; we will assume that $G$ is of this type for the rest of the section.

In order to check compatibility under inner twists we need a volume form on $G$ (a non-zero invariant differential form on $G$ of top degree). We get one by considering the extension of $G$ to a smooth group scheme over $\mathfrak{o}$ determined by an Iwahori subgroup $B$ of $G(F)$. For this $\mathfrak{o}$-structure on $G$ we have $G(\mathfrak{o}) = B$. Let $\omega$ be a volume form on $G$ over $\mathfrak{o}$ whose restriction to the special fiber $G_k$ is non-zero. Any two such forms $\omega$ differ by a unit and therefore the Haar measure $|\omega|$ is independent of the choice of $\omega$. It is also independent of the choice of $B$. We use $\nu_G$ to denote this Haar measure.

Let $F_1$ be a maximal unramified extension of $F$, and let $\mathfrak{o}_1$ be its valuation ring. We will write $\sigma$ for the Frobenius element of $\mathrm{Gal}(F_1/F)$. Since $G$ is residually split [35, 1.10], the group $G(\mathfrak{o}_1)$ is an Iwahori subgroup of $G(F_1)$ and the $\mathfrak{o}_1$-structure on the $F_1$-group $G$ is the one determined by this Iwahori subgroup. Since $H^1(F_1, G_{\mathrm{ad}})$ is trivial by a theorem of Steinberg [33], we can find an inner twisting $\psi$: $G \to H$ defined over $F_1$. After composing $\psi$ with an inner automorphism of $G$ over $F_1$ we can assume that $\psi$ is defined over $\mathfrak{o}_1$, where $H$ has the $\mathfrak{o}$-structure coming from some Iwahori subgroup of $H(F)$. It is now clear that $\nu_G$ and $\nu_H$ are compatible.

Later we will need a little more information about the relationship between $H$ and $G$. Choose a maximal $F$-split torus $S$ in $G$ whose apartment contains the chamber corresponding to the Iwahori subgroup $B$. Bruhat-Tits [6] have shown that there exists an $F$-torus $S_1$ in $G$, containing $S$, such that $S_1$ is a maximal $F_1$-split torus in $G$ over $F_1$. Choose $S_H$, $S_{H,1}$ of the same sort in $H$. Then by composing $\psi$ with an inner automorphism of $G$ over $F_1$, we may assume that $\psi$ carries $S_1$ into $S_{H,1}$. Then $\psi^{-1} \circ \sigma(\psi)$ is an inner automorphism of $G$ over $F_1$ which normalizes $S_1$.

Let $V = X^*(S_1) \otimes \mathbf{C}$ and let $W$ be the Weyl group of $S_1$ in $G$ over $F_1$. Then $\mathrm{Gal}(F_1/F)$ acts on $W$ and the semidirect product of $\mathrm{Gal}(F_1/F)$ and $W$ acts linearly on $V$. Using $\psi$, we can identify $V$ with the corresponding space for the group $H$. Then the action of $\sigma$ on $V$ for $H$ and the action of $\sigma$ on $V$ for $G$ differ by the element of $W$ obtained from $\psi^{-1} \circ \sigma(\psi)$.

In order to show that $(-1)^{q(G)}\mu_G, (-1)^{q(H)}\mu_H$ are compatible, we must show that the number $c_G$ defined by

$$c_G = (-1)^{q(G)}\mu_G(B)/\nu_G(B)$$

is invariant under inner twisting.

It is easy to evaluate $\nu_G(B)$. First of all we have

$$\nu_G(B) = |G(k)|q^{-\dim(G)}$$

(see [36] for example). The special fiber $G_k$ is a connected solvable linear algebraic group over $k$ (see [35], remembering that $G$ is simply connected). The torus $S_1$ extends uniquely to a torus over $\mathfrak{o}$, and Bruhat-Tits [6, Prop. 4.6.4(i)] have shown that its special fiber is a maximal torus in $G_k$. The unipotent radical of $G_k$ just contributes a power of $q$ to $|G(k)|$. Therefore

$$\begin{aligned} \nu_G(B) &= |S_1(k)|q^{-\dim(S_1)} \\ &= \det\left(1 - q^{-1}\sigma; V\right). \end{aligned}$$

Next we need to find a suitable expression for $\mu_G(B)$. Here we follow a suggestion of Prasad and Steinberg, who noticed that the results in Section 3 of [34] are just what is needed. This makes unnecessary the use of case-by-case considerations. Serre proved [31, Thm. 6] that

$$(-1)^{q(G)}\mu_G(B) = W\left(\mathbf{q}^{-1}\right)^{-1},$$

where $W(\mathbf{q}^{-1})$ is the Poincaré series of the affine Weyl group of $G(F)$ with the numbers $q_s^{-1}$ of [35, 3.3.1] substituted for the indeterminates $t_s$.

Steinberg evaluated this Poincaré series in [34]. He worked with the affine Weyl group $W'$ of $S_1(F_1)$ in $G(F_1)$. Then $\sigma$ acts on $W'$ with fixed point set the affine Weyl group of $S(F)$ in $G(F)$. To see this, one compares Tits's description [35, 1.10.1] of the affine roots of $G$ with Steinberg's description [34, Cor. 1.33] of the analogous set for the group of fixed points $(W')^\sigma$ of $\sigma$. In order to see that Steinberg's version of the Poincaré series agrees with Serre's we need to check that for any simple reflection $s$ in $(W')^{\sigma}$ the number $q_s$ is equal to $q^{N(s)}$, where $N(s)$ is the length of $s$ in $W'$. Let $v$ be the vertex corresponding to $s$ in the relative local Dynkin diagram of $G$. From 3.3.1 and 1.11 of [35] we have $q_s = q^{d(v)}$, where $d(v)$ is the number of positive absolute roots in the relative rank one $k$-group $G_v$ corresponding to $v$. From 1.30 and 1.32 of [34] we see that $s$ determines an orbit $\pi$ of $\sigma$ in the set of simple reflections of $W'$ and that $s$ is the longest element in the subgroup $W'_{\pi}$ of $W'$ generated by the elements in $\pi$. Since $W'_{\pi}$ is the absolute Weyl group of $G_v$, we see that $d(v) = N(s)$, as desired.

Using Steinberg's formula [34, Thm. 3.10] for $W(\mathbf{q}^{-1})$, we get

$$c_G = \prod_j \left(1 - \varepsilon_j q^{1-d(j)}\right) \Big/ \left(1 - \varepsilon_j q^{-d(j)}\right),$$

where $\varepsilon_j$, $d(j)$ are as follows. Let $R = \oplus_{k=0}^{\infty} R_k$ be the graded $\mathbf{C}$-algebra of $W$-invariants in the symmetric algebra $\mathrm{Sym}^*(V) = \oplus_{k=0}^{\infty}\mathrm{Sym}^k(V)$, let $R_+$ be the maximal ideal $\oplus_{k=1}^{\infty} R_k$, and let $V_R$ be the graded vector space $R_+/(R_+)^2$. A theorem of Chevalley states that $R$ is (non-canonically) isomorphic to the symmetric algebra $\mathrm{Sym}^*(V_R)$. Choose a basis $\{v_j\}$ for $V_R$ consisting of homogeneous eigenvectors for $\sigma$. Then $\varepsilon_j$ is the eigenvalue of $\sigma$ on $v_j$ and $d(j)$ is the degree of $v_j$. Note that the invariance of $c_G$ under inner twists is now obvious. As we saw earlier, going from $G$ to an inner form simply changes $\sigma$ by an element of $W$. Since $R$ is formed of $W$-invariants, the action of $\sigma$ on $R$ and $V_R$ is the same for the inner form as it is for $G$.

## 2. Euler-Poincaré functions

Let $F$ be a $p$-adic field and let $G$ be a connected reductive group over $F$. Assume that the connected center of $G$ is anisotropic over $F$. For any facet $\sigma$ of the building $\mathscr{B}$ of $G(F)$ we write $G(F)_{\sigma}$ for the stabilizer of $\sigma$ in $G(F)$; this stabilizer is a compact open subgroup of $G(F)$. Any element $g \in G(F)_{\sigma}$ permutes the vertices of $\sigma$, and we denote by $\mathrm{sgn}_{\sigma}(g)$ the sign of this permutation, thus obtaining a character $\mathrm{sgn}_{\sigma}$ on $G(F)_{\sigma}$ with values in $\{\pm 1\}$. We extend $\mathrm{sgn}_{\sigma}$ to a function on all of $G(F)$ by putting $\mathrm{sgn}_{\sigma}(g) = 0$ for $g \notin G(F)_{\sigma}$. Choose a Haar measure $dg$ on $G(F)$ as well as a set $\mathscr{S}$ of representatives for the orbits of $G(F)$ on the set of facets of $\mathscr{B}$. We define a function $f_{\mathrm{EP}} \in C_c^{\infty}(G(F))$ by

$$f_{\mathrm{EP}} = \sum_{\sigma \in \mathscr{S}} (-1)^{\dim(\sigma)} \mathrm{meas}\left(G(F)_{\sigma}\right)^{-1} \mathrm{sgn}_{\sigma},$$

and we call $f_{\mathrm{EP}}$ an *Euler-Poincaré function* on $G(F)$. If $dg$ is multiplied by a scalar, then $f_{\mathrm{EP}}$ is divided by that scalar. Changing the set $\mathscr{S}$ changes $f_{\mathrm{EP}}$ but not its orbital integrals.

THEOREM 2. *The orbital integrals of $f_{\mathrm{EP}}$ are* 0 *except for elliptic semisimple orbits in $G(F)$. Let $\gamma$ be an elliptic semisimple element of $G(F)$, let $I$ be its connected centralizer, and let $di$ be the Euler-Poincaré measure on $I(F)$ (note that $di \neq 0$). Then*

$$\int_{I(F)\backslash G(F)} f_{\mathrm{EP}}(g^{-1}\gamma g)\, dg/di = 1.$$

COROLLARY (Rogawski). *When Euler-Poincaré measures are used on connected centralizers of elliptic semisimple elements of $G(F)$, the Shalika germ for the identity element of $G(F)$ is identically* 1 *on every elliptic maximal torus of $G$.*

We get this from the theorem by considering the germ expansion for the orbital integrals of $f_{\mathrm{EP}}$ about the identity element of $G(F)$. The result is due to Rogawski [30], and the proof here, including that of the part of Theorem 2 that concerns non-semisimple $\gamma$, is just a variant of his proof. The advantage of using the Euler-Poincaré function is that it has such simple orbital integrals.

Now we prove the theorem. First consider the case of a semisimple element $\gamma \in G(F)$. We write $\mathscr{F}$ for the set of facets of $\mathscr{B}$, $\mathscr{F}(\gamma)$ for the set of fixed points of $\gamma$ in $\mathscr{F}$, and $\mathscr{B}(\gamma)$ for the set of fixed points of $\gamma$ in $\mathscr{B}$. For any $\tau \in \mathscr{F}(\gamma)$, the intersection of $\tau$ and $\mathscr{B}(\gamma)$ is non-empty; we will denote it by $\tau(\gamma)$. In this way we may regard $\mathscr{B}(\gamma)$ as a polysimplicial complex whose set of facets is $\{\tau(\gamma)\}_{\tau \in \mathscr{F}(\gamma)}$. It is easy to verify that

$$\operatorname{sgn}_\tau(\gamma) = (-1)^{\dim(\tau) - \dim(\tau(\gamma))}.$$

A standard calculation (see the first section of [21] for example) shows that the orbital integral of $f_{\mathrm{EP}}$ over the orbit of $\gamma$ is equal to

$$\sum_{\sigma \in \mathscr{S}} (-1)^{\dim(\sigma)} \sum_{\tau} \operatorname{meas}(I(F)_\tau)^{-1} \operatorname{sgn}_\tau(\gamma),$$

where $\tau$ runs through a set of representatives for the orbits of $I(F)$ on $(G(F) \cdot \sigma) \cap \mathscr{F}(\gamma)$, and where $I(F)_\tau$ denotes the stabilizer of $\tau$ in $I(F)$. Thus our orbital integral is equal to

$$\sum_{\rho} (-1)^{\dim(\rho)} \operatorname{meas}(I(F)_\rho)^{-1},$$

where $\rho$ runs over a set of representatives for the orbits of $I(F)$ on the set of facets of $\mathscr{B}(\gamma)$. If $\mathscr{B}(\gamma)$ is empty, the orbital integral is 0. This can happen only if $\gamma$ is non-elliptic. If $\mathscr{B}(\gamma)$ is non-empty, then it is contractible (see Lemma 7.2 of [17]) and satisfies all the conditions of [31, 3.3] relative to the group $I$. Note that Serre's condition v) is a consequence of the semisimplicity of $\gamma$. Proposition

24 of [31] says that

$$\left[\sum_{\rho}(-1)^{\dim(\rho)}\operatorname{meas}\left(I(F)_{\rho}\right)^{-1}\right] di$$

is the Euler-Poincaré measure on $I(F)$. If $\gamma$ is non-elliptic, then the Euler-Poincaré measure on $I(F)$ is 0, which implies that our orbital integral is also 0. If $\gamma$ is elliptic, then we are taking $di$ to be the Euler-Poincaré measure on $I(F)$, and since this measure is non-zero, our orbital integral must be equal to 1.

At this point we have proved the theorem for semisimple elements. It remains to show that the orbital integrals of $f_{\mathrm{EP}}$ are 0 for non-semisimple elements. Let $\gamma$ be a semisimple element of $G(F)$, and let $I$ be its connected centralizer. We consider the Shalika germ expansion for $f_{\mathrm{EP}}$ about $\gamma$:

$$O_t(f_{\mathrm{EP}}) = \sum_{u \in U} \Gamma_u(t) O_{\gamma u}(f_{\mathrm{EP}}),$$

valid for all regular semisimple $t$ in $G(F)$ that lie in $I(F)$ and are sufficiently close to $\gamma$. This expression requires some explanation. We are using $O_g$ to denote orbital integral with respect to $g$. The set $U$ is a set of representatives for the conjugacy classes of unipotent elements in $I(F)$, and for $u \in U$ we write $\Gamma_u$ for the Shalika germ associated to $u$.

We are going to use some of Harish-Chandra's results in [14]. For simplicity of exposition we will temporarily adopt his normalizations of the Haar measures used to form orbital integrals. However we will not use the discriminant factor that he always used to normalize regular semisimple orbital integrals. Look first at the left side of the germ expansion. We have $O_t(f_{\mathrm{EP}}) = 1$ if $t$ is elliptic and $O_t(f_{\mathrm{EP}}) = 0$ if $t$ is not elliptic. Theorem 15 and Corollary 2 of Lemma 20 of [14] tell us that there exists a real number $c$ such that $O_t(f_{\mathrm{EP}}) = c\Gamma_1(t)$. Therefore

$$\sum_{u \in U} c_u \Gamma_u = 0,$$

where $c_u = O_{\gamma u}(f_{\mathrm{EP}})$ if $u \neq 1$ and $c_1 = O_{\gamma}(f_{\mathrm{EP}}) - c$. Lemma 24 of [14] implies that $O_{\gamma u}(f_{\mathrm{EP}}) = 0$ for all $u \neq 1$, which is what we needed to prove.

The next theorem is due to Casselman, who kindly allowed me to include it here. It is a natural complement to Theorem 2. Let $(\pi, V)$ be an irreducible admissible representation of $G(F)$. We denote by $H_e^i(G(F), V)$ the continuous cohomology groups of [5, X.5.1]; they are finite dimensional complex vector spaces and are trivial except for finitely many values of $i$. As usual in the $p$-adic case we write $q(G)$ for the $F$-rank of $G_{\mathrm{der}}$.

THEOREM 2′ (Casselman). (a) *There is an equality*

$$\operatorname{tr}\pi(f_{\mathrm{EP}}) = \sum_i (-1)^i \dim H_e^i(G(F), V).$$

*Moreover* $(-1)^{q(G)} f_{\mathrm{EP}}$ *is a pseudo-coefficient for the Steinberg representation of* $G(F)$.

(b) *Assume that* $G$ *is simple and* $\pi$ *is unitary. Then* $\operatorname{tr}\pi(f_{\mathrm{EP}}) = 0$ *except in the following two cases: the trace of* $f_{\mathrm{EP}}$ *on the trivial representation of* $G(F)$ *is* 1, *and the trace of* $f_{\mathrm{EP}}$ *on the Steinberg representation of* $G(F)$ *is* $(-1)^{q(G)}$.

The theorem follows immediately from results of Casselman [8] and Casselman-Wigner [9] on continuous cohomology in the $p$-adic case. These results were reproved and slightly generalized by Borel-Wallach [5]. Everything we need here can be found in Chapters X and XI of [5]. First of all it is obvious from the definition of $f_{\mathrm{EP}}$ that

$$\operatorname{tr}\pi(f_{\mathrm{EP}}) = \sum_{\sigma\in\mathscr{S}} (-1)^{\dim(\sigma)} \dim V_\sigma,$$

where $V_\sigma$ denotes the biggest subspace of $V$ on which $G(F)_\sigma$ acts by the character $\operatorname{sgn}_\sigma$. On the other hand, by [5, X.2.4, X.5.1] the continuous cohomology of $V$ is the cohomology of a certain complex which Borel-Wallach denote by $C^*(Y; V)^G$, whose $i$-th term is isomorphic to $\oplus_\sigma V_\sigma$, where the sum is taken over those $\sigma \in \mathscr{S}$ for which $\dim(\sigma) = i$. This gives the first statement of (a). The second statement of (a) follows from [5, XI.3.8], which says that the Steinberg representation of $G(F)$ is the only tempered representation of $G(F)$ having non-zero continuous cohomology.

Statement (b) follows from [5, XI.3.9], which says that if $G$ is simple and $\pi$ is unitary, then $H_e^i(G(F), V)$ is 0 unless $i = 0$ and $V$ is the trivial representation, or $i = q(G)$ and $V$ is the Steinberg representation, in which cases $H_e^i(G(F), V)$ is one-dimensional.

## 3. Stable orbital integrals for semisimple elements

The results in this section are not needed for the proof of the main theorem. Let $F$ be a $p$-adic field and let $G$ be a connected reductive group over $F$. Let $\gamma$ be a semisimple element of $G(F)$ and let $I = (G_\gamma)^0$. Choose Haar measures $dg, di$ on $G(F)$, $I(F)$ respectively, and let $O_\gamma$ be the distribution on $G(F)$ given by

$$O_\gamma(f) = \int_{I(F)\backslash G(F)} f(g^{-1}\gamma g)\, dg/di$$

for $f \in C_c^\infty(G(F))$. For any stable conjugate $\gamma' \in G(F)$ of $\gamma$ the group $I' =$

$(G_{\gamma'})^0$ is an inner twist of $I$, and we denote by $di'$ the Haar measure on $I'(F)$ that is compatible with $di$. We use $dg, di'$ to form $O_{\gamma'}$, and we define a distribution $SO_\gamma$ by

$$SO_\gamma = \sum_{\gamma'} e(I')a(\gamma')O_{\gamma'},$$

where $\gamma'$ runs over a set of representatives for the conjugacy classes in the stable conjugacy class of $\gamma$. The number $a(\gamma')$ is the cardinality of the finite set

$$\ker\left[H^1(F, I') \to H^1(F, G_{\gamma'})\right];$$

it is 1 if $G_{\gamma'}$ is connected, which is always the case if $G_{\mathrm{der}}$ is simply connected. The number $e(I')$ is the sign attached to $I'$ in [18]. In the notation of Section 1 it is $(-1)^{q(I_0)-q(I')}$, where $I_0$ denotes a quasi-split inner form of $I$.

We will now see how to express $SO_\gamma$ for arbitrary semisimple $\gamma$ in terms of stable orbital integrals for regular semisimple $t$. We will need Rogawski's result on germs (see the corollary to Theorem 2) plus Theorem 1. Let $Z$ denote the split component of the center of $G$. Fix a Haar measure $dz$ on $Z(F)$. If $\gamma$ is elliptic, then the connected center of $I/Z$ is anisotropic, and the Euler–Poincaré measure $\mu_{I/Z}$ on $I/Z$ is non-zero. Since $Z$ is split, we have

$$(I/Z)(F) = I(F)/Z(F),$$

and hence the Haar measure $(-1)^{q(I)}\mu_{I/Z}$ on $(I/Z)(F)$ and the measure $dz$ on $Z(F)$ determine a Haar measure $di$ on $I(F)$. In the same way $(-1)^{q(G)}\mu_{G/Z}$ and $dz$ determine a Haar measure $dg$ on $G(F)$. We use $dg, di$ to form $O_\gamma$, $SO_\gamma$ for such $\gamma$; note that $O_\gamma$, $SO_\gamma$ are independent of the choice of $dz$. Of course we have used Theorem 1 in order to know that for stably conjugate $\gamma, \gamma'$ the measures $di, di'$ are compatible.

Let $\gamma$ be an elliptic semisimple element of $G(F)$ and let $T$ be an elliptic maximal torus of $G$ containing $\gamma$. Let $J$ be a set of representatives for the $G(F)$-conjugacy classes of embeddings $j$: $T \to G$, stably conjugate to the inclusion $T \to G$. Of course $J$ can be identified with $\mathfrak{D}(T, G)$. Let $f \in C_c^\infty(G(F))$. For each $j \in J$ we get a Shalika germ expansion for the orbital integrals $O_{j(t)}(f)$ around $j(\gamma)$, valid for $t \in T(F)_{\mathrm{reg}}$ sufficiently near $\gamma$:

$$O_{j(t)}(f) = \sum_{u \in U_j} \Gamma_u(I_j, j(t))O_{j(\gamma)u}(f).$$

Here $I_j$ denotes the connected centralizer of $j(\gamma)$ in $G$, and we have chosen a set $U_j$ of representatives for the conjugacy classes of unipotent elements in $I_j(F)$. We are using $\Gamma_u$ to denote the Shalika germ associated to $u$. With our normalization of measures, Rogawski's result on germs says that $\Gamma_1(I_j, j(t))$ is identically equal to $(-1)^{q(I_j)}$.

Now take the sum of the germ expansions for $j$ ranging through $J$. We get

$$SO_t(f) = (-1)^{q(I_0)} \sum_{j \in J} e(I_j) O_{j(\gamma)}(f) + A(t)$$

for all $t \in T(F)_{\text{reg}}$ sufficiently near $\gamma$, where $A(t)$ is a linear combination of germs $\Gamma_u(I_j, j(t))$ for $u \neq 1$. Using Sections 1, 4, 10 of [20], one can check that

$$\sum_{j \in J} e(I_j) O_{j(\gamma)} = |\mathfrak{D}(T, I_0)| SO_\gamma.$$

Therefore

$$SO_t(f) = (-1)^{q(I_0)} |\mathfrak{D}(T, I_0)| SO_\gamma(f) + A(t).$$

Using the exponential map, we can think of the Shalika germs as functions on the Lie algebra of $T$. Then they are homogeneous [14, Thm. 14(1)]: for all $\alpha \in F^\times$ and $H \in \text{Lie}(T(F))_{\text{reg}}$ one has

$$\Gamma_u(I, \alpha^2 H) = |\alpha|^{d(u)} \Gamma_u(I, H),$$

where $d(u) = -\dim(G/G_u)$. Note that $\Gamma_1$ is the only Shalika germ which is homogeneous of degree 0. Therefore the germ of $SO_t(f)$ around $\gamma$ determines $SO_\gamma(f)$; more precisely,

$$(-1)^{q(I_0)} |\mathfrak{D}(T, I_0)| SO_\gamma(f)$$

is the homogeneous part of degree 0 of the germ of $SO_t(f)$ around $\gamma$. From this we get two propositions.

Proposition 1. *Let $\gamma$ be any semisimple element of $G(F)$. Then $SO_\gamma$ is a stable distribution.*

We need to show that $SO_\gamma(f) = 0$ for $f \in C_c^\infty(G(F))$ such that $SO_t(f) = 0$ for every regular semisimple $t$. Let $S$ be the split component of the center of $(G_\gamma)^0$ and let $M$ be the centralizer of $S$ in $G$. Let $T$ be an elliptic maximal torus of $M$. Using that

$$\ker\left[H^1(F, M) \to H^1(F, G)\right]$$

is trivial, we see that

$$\mathfrak{D}(T, M) = \mathfrak{D}(T, G).$$

The usual descent argument gives us a function $f^M \in C_c^\infty(M(F))$ such that

$$SO_\gamma(f) = c(\gamma) SO_\gamma(f^M),$$
$$SO_t(f) = c(t) SO_t(f^M) \qquad (t \in T(F)_{\text{reg}})$$

for non-zero real numbers $c(\gamma)$, $c(t)$. We have

$$SO_t(f^M) = c(t)^{-1} SO_t(f) = 0$$

for all $t \in T(F)_{\mathrm{reg}}$. Applying our previous discussion to the elliptic element $\gamma$ of $M(F)$, we see that

$$SO_\gamma(f) = c(\gamma) SO_\gamma(f^M) = 0.$$

PROPOSITION 2. *Let $H$ be a quasi-split inner form of $G$. Let $f \in C_c^\infty(G(F))$ and $f^H \in C_c^\infty(H(F))$, and suppose that $f$, $f^H$ have matching stable orbital integrals for all regular semisimple orbits. Then they have matching stable orbital integrals for all semisimple orbits.*

Let $\gamma_H$ be a semisimple element of $H(F)$. We are asserting that $SO_{\gamma_H}(f^H) = 0$ unless $\gamma_H$ comes from some $\gamma \in G(F)$, in which case $SO_{\gamma_H}(f^H) = SO_\gamma(f)$, provided that compatible measures are used on the two sides. Our hypothesis is that these statements are true for regular semisimple $\gamma_H$.

As in the proof of Proposition 1 a descent argument reduces us to the case in which $\gamma_H$ is elliptic. In this case $\gamma_H$ automatically comes from some $\gamma \in G(F)$ (see [20, §10]). The equality

$$SO_{\gamma_H}(f^H) = SO_\gamma(f)$$

follows from the discussion preceding Proposition 1, since the numbers $q(I_0)$, $|\mathfrak{D}(T, I_0)|$ are the same for $\gamma_H$, $\gamma$.

## 4. Tamagawa numbers

Let $F$ be a number field and let $G$ be a simply connected semisimple group over $F$. Assume that $G$ has no $E_8$ factors. Our goal in this section is to prove the following theorem.

THEOREM 3. *The Tamagawa number $\tau(G)$ of $G$ is* 1.

We prove this by induction on the dimension of $G$. We need to use Arthur's simple form of the trace formula (see §5). For this we consider a function $f \in C_c^\infty(G(\mathbf{A}))$ of the form $\prod f_v$ for $f_v \in C_c^\infty(G(F_v))$. Moreover we assume that there are two distinct finite places $v_1, v_2$ of $F$ such that $f_{v_1}$ is a coefficient of a supercuspidal representation of $G(F_{v_1})$ and $f_{v_2}$ is an Euler-Poincaré function. Then the trace formula for $f$ reduces to

$$\sum_\gamma \tau(G_\gamma) O_\gamma(f) = \sum_\pi m(\pi) \operatorname{tr} \pi(f).$$

The sum on the right is taken over cuspidal automorphic representations $\pi$ of

$G(\mathbf{A})$, and $m(\pi)$ denotes the multiplicity of $\pi$ in $L^2_{\text{cusp}}(G(F)\backslash G(\mathbf{A}))$. The sum on the left is taken over a set of representatives $\gamma$ for the elliptic semisimple conjugacy classes in $G(F)$. We are writing $O_\gamma(f)$ for the orbital integral

$$\int_{G_\gamma(\mathbf{A})\backslash G(\mathbf{A})} f(g^{-1}\gamma g)\, dg/dt,$$

normalized by using the canonical measures on $G(\mathbf{A}), G_\gamma(\mathbf{A})$ (the ones used to define the Tamagawa numbers of $G, G_\gamma$).

Choose an inner twisting $\psi: G_0 \to G$ with $G_0$ quasi-split over $F$. Choose a set $E_0^*$ of representatives for the non-central elliptic semisimple stable conjugacy classes in $G_0(F)$, and let $Z$ denote the center of $G$. Then from (9.6.5) of [20] we see that the left side of the trace formula can be rewritten as the sum of $A$ and $B$, where

$$A = \sum_{z \in Z(F)} \tau(G) f(z),$$

$$B = \sum_{\gamma_0 \in E_0^*} \tau_1(G) \sum_{\gamma} \sum_{\kappa} \langle \text{obs}(\gamma), \kappa\rangle e(\gamma) O_\gamma(f),$$

where $\gamma$ now runs over a set of representatives for the $G(\mathbf{A})$-conjugacy classes in $G(\mathbf{A})$ contained in the $G(\overline{\mathbf{A}})$-conjugacy class of $\psi(\gamma_0)$, and $\kappa$ runs over $\mathfrak{K}(I_0/F)$, where $I_0$ denotes the centralizer of $\gamma_0$ in $G_0$. For the definition of $\mathfrak{K}(I_0/F)$ and $\text{obs}(\gamma)$ see 4.6 and 6.1 of [20]. The number $\tau_1(G)$ is the relative Tamagawa number $\tau(G)/\tau(G_{\text{sc}})$, which is 1 since we are assuming that $G$ is simply connected. The number $e(\gamma)$ is the product over all places of $F$ of the signs $e(I_v)$ attached in [18] to $I_v$, the centralizer in $G$ of the $v$-component of $\gamma$.

The finiteness results in [20] show that the triple sum in $B$ has only finitely many non-zero terms, so that we may interchange the order of the sums over $\kappa$ and $\gamma$. Suppose that $\kappa \neq 1$. We will now show that

$$\sum_{\gamma} \langle \text{obs}(\gamma), \kappa\rangle e(\gamma) O_\gamma(f) = 0.$$

This is trivially true if the sum is empty; thus we may as well assume that there exists $\delta \in G(\mathbf{A})$ in the $G(\overline{\mathbf{A}})$-conjugacy class of $\psi(\gamma_0)$. Then the expression we are considering can be rewritten as a product of $\kappa$-orbital integrals (see §5 of [20]):

$$\langle \text{obs}(\delta), \kappa\rangle \prod_v O^{\kappa(v)}_{\delta(v)}(f_v).$$

Here $\kappa(v)$ denotes the image of $\kappa$ under

$$\mathfrak{K}(I_0/F) \to \mathfrak{K}(I_0/F_v),$$

and $\delta(v)$ denotes the $v$-component of $\delta$. Since $f_{v_2}$ is an Euler-Poincaré function, the $\kappa$-orbital integral of $f_{v_2}$ is 0 unless $\delta(v_2)$ is elliptic in $G(F_{v_2})$. If $\delta(v_2)$ is elliptic in $G(F_{v_2})$, then

$$\mathfrak{K}(I_0/F) \to \mathfrak{K}(I_0/F_{v_2})$$

is injective, so that $\kappa(v_2) \neq 1$. Then Theorems 1 and 2 imply that the $\kappa$-orbital integral of $f_{v_2}$ is 0.

We now know that

$$B = \sum_{\gamma} SO_{\gamma}(f),$$

where $\gamma$ runs over a set of representatives for the stable conjugacy classes in $G(\mathbf{A})$ that come from a non-central elliptic semisimple element of $G_0(F)$, and $SO_{\gamma}(f)$ denotes the stable orbital integral of $f$ for $\gamma$ (see [20]). This whole discussion can also be applied to any function $f_0$ on $G_0(\mathbf{A})$ of the same kind we are using on $G(\mathbf{A})$.

At this point we need to be more specific about which functions $f$, $f_0$ we will use. Let $S$ be a finite set of places of $F$ such that for all $v$ outside $S$ there exists an $F_v$-isomorphism

$$\psi_v\colon G_0 \to G$$

differing from $\psi$ by an inner automorphism of $G$ over $\overline{F}_v$. We may assume that $S$ contains at least one finite place of $F$. At each finite place in $S$ we take the local components of $f$ and $f_0$ to be Euler-Poincaré functions. At each infinite place $v$ in $S$ we take for $f_v$ any $K$-finite function in $C_c^{\infty}(G(F_v))$ such that $f_v(1) \neq 0$, and we take for $f_{0,v}$ any $K$-finite function in $C_c^{\infty}(G_0(F_v))$ whose stable orbital integrals match those of $f_v$ for all regular semisimple orbits. See [11], [32] for the existence of $f_{0,v}$. Shelstad has observed that Section 37 of [13] implies that $f_v$ and $f_{0,v}$ have matching stable orbital integrals for all semisimple orbits [27].

At all places outside $S$ we take functions $f_v$, $f_{0,v}$ that correspond under $\psi_v$ and satisfy the following additional requirements. We choose a finite place $v_1$ outside $S$ and take $f_{v_1}$ to be a coefficient of a supercuspidal representation of $G(F_{v_1})$; we can (and do) assume that $f_{v_1}(1) \neq 0$. We set aside for later use a finite place $u$ outside $S$ at which $G$ is unramified. We require that $f_v(1) \neq 0$ for all $v$ outside $S$ except the place $u$. By shrinking the support of $f_v$ at some place outside $S$ that we have not yet used, we may also assume that $f(z) = 0$ for all $z \in Z(F)$ except $z = 1$.

For $f$, $f_0$ as above we have

$$\tau(G)f(1) + \sum_{\gamma} SO_{\gamma}(f) = \sum_{\pi} m(\pi)\operatorname{tr}\pi(f),$$

$$\tau(G_0)f_0(1) + \sum_{\gamma_0} SO_{\gamma_0}(f_0) = \sum_{\pi_0} m(\pi_0)\operatorname{tr}\pi_0(f_0).$$

We are going to subtract the first formula from the second. Since $f$, $f_0$ have matching stable orbital integrals for semisimple orbits, a term $SO_{\gamma_0}(f_0)$ in the second formula will be cancelled by the corresponding term in the first formula unless $\gamma_0$ does not come from $G(\mathbf{A})$. But in this case $SO_{\gamma_0}(f_0) = 0$, and hence the difference of the two left-hand sides is simply

$$[\tau(G) - \tau(G_0)]\, f(1).$$

Now we use the finite place $u$ that we set aside. We choose a hyperspecial maximal compact subgroup of $G(F_u)$ and form the corresponding Hecke algebra $\mathscr{H}$. We fix all local components of $f$ except the $u$-component, which we let vary through $\mathscr{H}$. Then the difference of the left-hand sides is a non-zero constant times

$$[\tau(G) - \tau(G_0)]\, f_u(1),$$

while the difference of the right-hand sides is an absolutely convergent series of the form

$$\sum_{i=1}^{\infty} c_i \operatorname{tr} \pi_i(f_u),$$

where the $\pi_i$ are unramified unitary representations of $G(F_u)$. The distribution $f_u \mapsto f_u(1)$ is given in terms of $\operatorname{tr} \pi(f_u)$ for unramified tempered $\pi$ by integration against the Plancherel measure, which is continuous in an obvious sense [28]. The argument used by Langlands on pp. 210–11 of [26] shows that

$$[\tau(G) - \tau(G_0)]\, f_u(1) = 0$$

for all $f_u \in \mathscr{H}$. Therefore $\tau(G)$ is equal to $\tau(G_0)$. Since Lai [24] has shown that $\tau(G_0) = 1$, we have now proved the theorem.

## 5. Arthur's simple trace formula

In this section we will see that the simple form of the trace formula needed in the previous section is an easy consequence of Arthur's work. Arthur himself observed that his results can be used in this way. I am indebted to Clozel for telling me which of Arthur's theorems are needed for this purpose. A simple form of the trace formula was first proved by Deligne-Kazhdan, but this version is too simple for the needs of this paper because it lacks the term $\tau(G)f(1)$.

Let $F$ be a number field and let $G$ be a semisimple group over $F$. We consider a function $f \in C_c^\infty(G(\mathbf{A}))$ of the form $\prod f_v$ with $f_v \in C_c^\infty(G(F_v))$. Moreover we assume that there is a finite place $u$ such that $f_u$ is a coefficient of a supercuspidal representation of $G(F_u)$ and another finite place $w$ such that the orbital integrals of $f_w$ are 0 except for elliptic semisimple elements of $G(F_w)$. For example $f_w$ could be an Euler-Poincaré function.

For any $f$ the trace formula takes the form

$$\sum_{\mathfrak{o}\in\mathcal{O}} J_{\mathfrak{o}}(f) = \sum_{\chi\in\mathcal{X}} J_\chi(f).$$

As Clozel shows in [10], the fact that $f_u$ is a matrix coefficient of a supercuspidal representation implies that the right side of the trace formula reduces to

$$\sum_\pi m(\pi)\operatorname{tr}\pi(f),$$

where $\pi$ runs through the cuspidal automorphic representations of $G(\mathbf{A})$ and $m(\pi)$ denotes the multiplicity of $\pi$ in $L^2_{\text{cusp}}(G(F)\backslash G(\mathbf{A}))$.

We want to show that the left side of the trace formula reduces to

$$\sum_\gamma \left|I(F)\backslash G_\gamma(F)\right|^{-1}\tau(I)O_\gamma(f),$$

where $\gamma$ runs through a set of representatives for the elliptic semisimple conjugacy classes in $G(F)$, and $I$ denotes the identity component of $G_\gamma$. We are using canonical measures to normalize $O_\gamma$, as in the last section. In view of Theorems 8.1 and 8.2 of [2], it is enough to show that $J_M(\gamma, f) = 0$ unless $M = G$ and the $w$-component of $\gamma$ is elliptic semisimple. Here we are using the notation of [2], [3]. The subscript $M$ denotes a standard Levi subgroup of $G$ over $F$, and $\gamma$ denotes an element of $M(F_S)$, where $S$ is a finite set of places of $F$, large enough that $f$ comes from $C_c^\infty(G(F_S))$. We should recall that $J_M(\gamma, f)$ depends on the choice of an admissible maximal compact subgroup $K = \prod_{v\in S}K_v$ of $G(F_S)$ (see [1]).

In order to prove this vanishing theorem for $J_M(\gamma, f)$ we start with the case in which $G_\gamma^0$ is contained in $M$. In this case we have formula (2.1*) of [3]:

$$J_M(\gamma, f) = |D(\gamma)|^{1/2}\int_{G_\gamma^0(F_S)\backslash G(F_S)} f(x^{-1}\gamma x)v_M(x)\,dx/dt.$$

The weight factor $v_M(x)$ is obtained from a $(G, M)$-family $v_P(\lambda, x)$. Write $x = gh$ with $g \in G(F_u)$ and $h \in G(F_{S'})$, where $S' = S - \{u\}$. Then $v_P(\lambda, x) = v_P(\lambda, g)v_P(\lambda, h)$. Applying Lemma 6.3 of [1] (and using the notation of that paper), we see that

$$v_M(x) = \sum_{Q\in\mathcal{F}(M)} v_M^Q(g)v_Q'(h).$$

Let $Q = M_Q \cdot N$ be the Levi decomposition of $Q$ such that $M_Q$ contains $M$. Let $g \in G(F_u)$ and write $g = mnk$ with $m \in M_Q(F_u)$, $n \in N(F_u)$, $k \in K_u$. Then it follows easily from the definition of $v_M^Q(g)$ that

$$v_M^Q(g) = v_M^Q(m).$$

Using the usual integration formula derived from the decomposition $g = mnk$, as well as the supercuspidality of $f_u$, we see that

$$\int_{G_\gamma^0(F_u)\backslash G(F_u)} f_u(g^{-1}\gamma g) v_M^Q(g)\, dg/dt = 0$$

unless $Q = G$. Since $v_G'(h)$ is identically 1, our hypothesis on $f_w$ implies that $J_M(\gamma, f) = 0$ unless the $w$-component of $\gamma$ is elliptic semisimple, in which case $M$ is automatically equal to $G$. This proves our vanishing theorem for $\gamma$ such that $G_\gamma^0 \subset M$.

The general case is a consequence of this special case. According to (6.5) of [3], in the general case we have

$$J_M(\gamma, f) = \lim_{a \to 1} \sum_{L \in \mathscr{L}(M)} r_M^L(\gamma, a) J_L(a\gamma, f),$$

with notation as in that paper. For elements $a \in A_{M,\mathrm{reg}}(F_S)$ near 1 we have $G_{a\gamma}^0 \subset M \subset L$. If $M \neq G$, then for such elements $a$ the $w$-component of $a\gamma$ is never elliptic in $G(F_w)$, and therefore every term on the right side of the formula above vanishes by what we have already proved. If $M = G$, then $J_M(\gamma, f)$ is a constant times $O_\gamma(f)$, which is 0 unless the $w$-component of $\gamma$ is elliptic semisimple. This finishes the proof.

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(Received December 2, 1986)