QUOTIENTS OF ALGEBRAIC GROUP ACTIONS

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1. Introduction

The main goal of this paper is to investigate whether an algebraic group action on a variety has a nice algebraic quotient. A classical book in this respect is "Geometric Invariant Theory" of Mumford (see [15]). The major part of the book only concerns reductive groups. More recently some work has been done to do similar things for general algebraic groups (see [8], [5], [6], [7] and [4]).

We introduce the following notation: k will be an algebraically closed field, X is a variety over k (a separated integral scheme of finite type over k), G will denote a group-variety, $\sigma : G \times X \to X$ is an algebraic group action, Y is a scheme over k (possibly non-separated) and $\pi : X \to Y$ is a morphism of schemes over k.

2. Examples of quotients

There exist several definitions of a quotient. We will mention three of them, namely the "categorical quotient", the "geometric quotient" and a "locally trivial quotient". For the notion of "good quotient" see [18]. From a categorical point of view, there is a natural way to define a quotient, and this leads to the notion of a categorical quotient. In general categorical quotients behave in a pathological way and don't follow our geometric intuition. For example, one thing we certainly would like to have is that the quotient space parametrizes all the orbits. Essentially, this is the definition of a geometric quotient.

Definition 1 *Categorical quotient (see [15, Definition 0.5])*

The morphism $\pi : X \to Y$ is called a categorical quotient if π is constant on orbits, and is universal with respect to this property: if a morphism $\psi : X \to Z$ of schemes over k is also constant on orbits, then there exist a unique morphism $\varphi : Y \to Z$ such that $\psi = \varphi\pi$. In categorical language it just means that

$$\begin{tikzcd} G \times X \arrow[r, "\sigma"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

is a push-out diagram (in the category of schemes over k).

Definition 2 *Geometric quotient (see [15, Definition 0.6])*

The map $\pi : X \to Y$ is a geometric quotient if

1. π is surjective and the image of $\Psi = (\sigma, p_2) : G \times X \to X \times X$ is equal to $X \times_Y X$. In other words, the geometric fibres of π are precisely the orbits of geometric points.
2. π is submersive (Y has the quotient topology): every $U \subseteq Y$ is open if and only if $\pi^{-1}(U)$ is open.
3. $\mathcal{O}_Y = \pi_\star \mathcal{O}_X^G$, this means $\mathcal{O}_Y(U) = \mathcal{O}_X(\pi^{-1}(U))^G$ for every $U \subseteq Y$ open.

Remark 3 *If Y is normal, the the two last properties follow from the first one (see [1, p. 174])*

A geometric quotient is always a categorical quotient.

Example 4 *Scalar multiplication on a vector space*

Let $X = k^n$, G is the multiplicative group $\mathcal{G}_m$ ($k^\star$ with group the usual multiplication as group operation). The group action $\sigma : k^\star \times k^n \to k^n$ is defined as

$$\sigma(\lambda, v) = \lambda v$$

We will prove that the categorical quotient has only 1 point. Let $Y = \mathrm{Spec}(k)$, and $\pi : X \to Y$ be the only morphism defined over k. Suppose that $\psi : X \to Z$ is a morphism which is constant on orbits. If $p \in k^n$ then $\psi^{-1}(\psi(p))$ is closed and contains the orbit of p, so it contains 0. This proves that $\psi(p) = \psi(0)$, so we can conclude that $\psi(X) = \{\psi(0)\}$. If we define $\varphi : \mathrm{Spec}(k) \to Z$ the morphism which maps the only point of $\mathrm{Spec}(k)$ to $\psi(0)$, then φ is the unique map satisfying $\psi = \varphi\pi$. Of course, π is not at all a geometric quotient.

Example 5 *Projective space as a quotient space*

We will restrict the $\mathcal{G}_m$ action as defined in example 4 to $k^n \setminus \{0\}$. So let $X = k^n \setminus \{0\}$, $G = \mathcal{G}_m$, and $\sigma : k^\star \times k^n \to k^n$ is defined as

$$\sigma(\lambda, v) = \lambda v$$

If we define $Y = \mathbb{P}^{n-1}$ and define $\pi : X \to Y$ by

$$(x_1, x_2, \ldots, x_n) \in k^n \setminus \{0\} \to (x_1 : x_2 : \ldots : x_n) \in \mathbb{P}^{n-1}$$

then π is a geometric quotient, because the geometric fibres of π are exactely the G-orbits, and $\mathbb{P}^{n-1}$ is normal (see Remark 3).

These two examples show that a G-stable open subset might have a geometric quotient, if the variety itself doesn't have any. This is a general fact:

Theorem 6 *(Rosenlicht) If X is any G-variety, then there exists an open G-stable subset U of X which has a geometric G-quotient (see [21] and [17]).*

Definition 7 *Locally trivial quotient*

A geometric quotient $\pi : X \to Y$ is called locally trivial in the Zariski topology if there is an open cover $\{U_i\}_{i \in I}$ of Y, such that for all i the variety $\pi^{-1}(U_i)$ is G-isomorphic to $G \times U_i$.

A geometric quotient $\pi : X \to Y$ is called locally trivial in the étale topology if there exists a surjective étale morphism of schemes $U \to Y$ such that $X \times_Y U$ is G-isomorphic to $G \times U$. In this case we call X a principal G-bundle over Y.

The reader should observe that locally trivial in the Zariski topology implies locally trivial in the étale topology (take U the disjunct union of all U_i).

Example 8 *Projective space*

Look again at example 5. Define open subsets $U_i \subseteq \mathbb{P}^{n-1}$ by

$$U_i = \{(x_1 : x_2 : \ldots : x_n) \in \mathbb{P}^{n-1} \mid x_i \neq 0\}$$

Of course the U_i cover $\mathbb{P}^{n-1}$ and

$$\pi^{-1}(U_i) = \{(x_1, x_2, \ldots, x_n) \in k^n \mid x_i \neq 0\} \cong \mathcal{G}_m \times U_i$$

Where the isomorphism is given by

$$(x_1, x_2, \ldots, x_n) \in \pi^{-1}(U_i) \to (x_i, (x_1 : x_2 : \ldots : x_n)) \in \mathcal{G}_m \times U_i$$

This proves that π is locally trivial in the Zariski topology.

Example 9 *Quotients with respect to subgroups*
Suppose H is a closed subgroup of G. Now H acts on G on the right:

$$h.g = gh^{-1} \text{ for all } h \in H,\, g \in G$$

It is known that this action has a geometric quotient which we will denote by G/H (see [20, p. 133] or [1, p. 181]). Any variety on which G acts transitively is called a homogeneous space. Such a variety is isomorphic to G/H for some closed subgroup H. The quotient map $G \to G/H$ is locally trivial in the étale topology.

Let Z be an H-variety. Now H acts on $G \times Z$ because it acts on G and Z as defined before. This action always has a geometric quotient which we will denote by $G \star^H Z$. The object $G \star^H Z$ is called the associated fibre bundle of the action $H \times Z \to Z$. The quotient map $G \times Z \to G \star^H Z$ is locally trivial in the étale topology. We will list now some nice properties (without proof) of $G \star^H Z$ (see [19] and [2, III §4]):

Let G act on $G \times Z$ as left multiplication on G (action on Z is trivial). This action commutes with the action of H defined before, and therefore the G-action induces an action on $G \star^H Z$. There is a 1-1 correspondence between H-orbits of Z and G-orbits of $G \star^H Z$. For any scheme we have: Y is a geometric H-quotient of Z if and only if Y is a geometric G-quotient of $G \star^H Z$. The natural map $G \times Z \to G/H$ is constant on H-orbits, so it factors through a morphism $G \star^H Z \to G/H$ which is G-equivariant. If V is any G-variety and $f : V \to G/H$ is G-equivariant, then V is G-isomorphic to $G \star^H f^{-1}(eH)$. A fibre bundle $G \star^H Z \to G/H$ is trivial if and only if the action of H on Z can be extended to a G-action.

The construction of $G \star^H Z$ is an important tool in the invariant theory, and we will use it later on.

Example 10 *Locally trivial in the étale topology doesn't imply locally trivial in the Zariski topology*
Let $\mathcal{G}_m$ be the multiplicative group and μ_2 the multiplicative group of order 2. The quotient map $\mathcal{G}_m \to \mathcal{G}_m/\mu_2$ is locally trivial in the étale topology, because

$$\mathcal{G}_m \times_{\mathcal{G}_m/\mu_2} \mathcal{G}_m \cong \mathcal{G}_m \times \mu_2$$

It is clear that this quotient map is not locally trivial in the Zariski topology.

3. Properties of actions

Of course it would be nice to be able to predict whether a geometric quotient exists, or whether this quotient is locally trivial. First, one has to assume some nescessary conditions like separatedness or freeness. The main question of this section is: In which cases are these conditions also sufficient?

Definition 11 *Proper, separated and free actions (see [15, Definition 0.8])*
Define $\Psi = (\sigma, p_2) : G \times X \to X \times X$.

1. The action σ is called proper if Ψ is a proper morphism.
2. An action $\sigma : G \times X \to X$ is called separated if the image of Ψ is closed and it is called locally separated if the image is locally closed.
3. The action σ is called free if if ψ is a closed immersion.

The following implications are clear:

$$\text{free} \Rightarrow \text{proper} \Rightarrow \text{separated} \Rightarrow \text{locally separated}$$

Theorem 12 *If σ has a geometric quotient $\pi : X \to Y$ then the action is locally separated. If Y is a separated scheme then the action is separated.*

If σ has a geometric quotient $\pi : X \to Y$, then the image of Ψ is equal to the subscheme $X \times_Y X$ of $X \times X$ which is locally closed, so σ is locally separated. If Y is a seperated scheme then $X \times_Y X$ is a closed subscheme of $X \times X$, so the action σ is separated.

Theorem 13 *If σ has a locally trivial geometric quotient $\pi : X \to Y$ (in the étale topology) then the action is free.*

For some surjective étale map $U \to Y$ we have $U \times_Y X \cong G \times U$. All isotropy groups of the G-action on $G \times U$ are trivial, therefore all isotropy groups of the action σ are trivial. So $\Psi : G \times X \to X \times X$ is injective. It is not hard to show that this map is also injective on tangent spaces. Since the image of Ψ is a closed subscheme because of theorem 12 we can conclude that Ψ is a closed immersion.

We will study now the converses of these statements:

Question 14 *Does every separated action have a geometric quotient?*

Question 15 *Does every free action have a locally trivial quotient?*

In general, the answer to both questions is NO! Nagata and Hironaka both gave an example of a proper reductive group action on a quasi-projective variety without a geometric quotient (see [15, IV §3],[16] and [10]). The case that G is reductive and X is affine will be treated below. If G is any algebraic group acting on an affine variety, then again the answer to questions 14 and 15 is negative. In [3] give Finston and Deveney an example of a free $\mathcal{G}_a$-action on k^5 which has no geometric quotient. Another counterexample will be given in section 4.

Definition 16 *An algebraic group is called reductive if every connected solvable normal subgroup is trivial (see [20, 6.14] or [1, 11.21])*

We now will assume that G is a reductive group and X is an affine variety. We denote the ring of regular functions on X by $\mathcal{O}(X)$. It is a well-known fact (finiteness theorem, see [21]) that the ring of G-invariant functions $\mathcal{O}(X)^G$ is finitely generated over k. We define the variety X/G as $\mathrm{Spec}(\mathcal{O}(X)^G)$. The inclusion $\mathcal{O}(X)^G \to \mathcal{O}(X)$ defines a dominant morphism $\pi : X \to X/G$.

Theorem 17 *(see [21]) The map $\pi : X \to X/G$ has the following properties:*

1. *π is a categorical quotient*
2. *π is surjective and X/G has the quotient topology*
3. *if $Z \subseteq X$ is G-stable and closed, then $\pi(Z)$ is closed*
4. *if $Z_1, Z_2 \subseteq X$ are both G stable and disjunct, then $\pi(Z_1)$ and $\pi(Z_2)$ are also disjunct.*

In this case the answer to question 14 is affirmative. If σ is a separated action, then all orbits are closed. The map π separates all orbits because of property 4. So conditions 1 and 2 of a geometric quotient are satisfied. The third condition is not too hard to prove.

A consequence of Luna's slice theorem is that the answer to question 15 is also yes (see [14] and [19]).

4. A counterexample

Not every separated action on an affine variety has a geometric quotient, and not every free action has a locally trivial quotient. Finston and Deveney give an example in [3] of a free $\mathcal{G}_a$-action on k^5 which doesn't have a geometric quotient. In this section we will give another example of a free action without geometric quotient. We will use only geometric reasoning, no computer calculation is needed.

Example 18 *A free action with no geometric quotient*

Denote the $n \times h$ matrices with entries in k by $\mathrm{M}_{n,h}$ (we will abbreviate $\mathrm{M}_{n,n}$ by M_n). Look at the following morphism

$$\mathrm{M}_{n,h} \times \mathrm{M}_{h,n} \xrightarrow{\pi} \mathrm{M}_n$$

The morphism π is given by $\pi(A, B) = AB$. This morphism is also studied in [11, II, §4], [12] and [13]. Let J be the $n \times n$ matrix

$$\begin{pmatrix} 0 & -1 & 0 & \dots & 0 \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

and define $V_{n,h} = \pi^{-1}(J)$. The group SL_2 acts on the set of homogeneous polynomials of the degree $n-1$ by substitution:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} .p(x,y) = p(ax+cy, bx+dy)$$

(this is in fact the action of SL_2 on $S^{n-1}(k^2)$, where SL_2 acts as usual on k^2). Let $\phi(g)$ be the matrix of $g \in \mathrm{SL}_2$ on the basis $x^{n-1}, x^{n-2}y, \ldots, y^{n-1}$. The group SL_n acts on $\mathrm{M}_{n,h} \times \mathrm{M}_{h,n}$ and M_n as follows:

$$\begin{array}{rcll} g(A,B) & = & (\phi(g)A, B\phi(g)^t) & g \in \mathrm{SL}_n,\ (A,B) \in \mathrm{M}_{n,h} \times \mathrm{M}_{h,n} \\ gC & = & \phi(g)C\phi(g)^t & g \in \mathrm{SL}_n,\ C \in \mathrm{M}_n \end{array}$$

The map π is an SL_2-equivariant morphism. We will view the additive group $\mathcal{G}_a$ as a subgroup of SL_2 by identifying them with all the matrices of the form

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

where $t \in k$. Observe that $V_{n,h}$ is $\mathcal{G}_a$-stable, because J is invariant under $\mathcal{G}_a$.

The action of $\mathcal{G}_a$ on $V_{n,h}$ is free: define again $\Psi : \mathcal{G}_a \times V_{n,h} \to V_{n,h} \times V_{n,h}$ by

$$\Psi(g,(A,B)) = ((\phi(g)A, B\phi(g)^t), (A,B))$$

and define $s : V_{n,h} \times V_{n,h} \to \mathcal{G}_a \times V_{n,h}$ by

$$s((C,D),(A,B)) = (u,(A,B))$$

where u is the entry of CB with index $(1,1)$. If $(A,B) \in V_{n,h}$ and $t \in \mathcal{G}_a$, then

$$s(\Psi(t,(A,B))) = s((\phi(t)A, B\phi(t)^t),(A,B)) =$$

$$((\phi(t)AB)_{1,1},(A,B)) = ((\phi(t)J)_{1,1},(A,B)) = (t,(A,B))$$

So $s \circ \Psi = \mathrm{id}$ and this proves that Ψ is a closed immersion.

From now on we will take $n = 3$, $h = 6$ and $V = V_{3,6}$. In this case we have $h \geq 2n$, so π is a flat, normal morphism according to [12, Teorema 4.2.]. The kernel of ϕ is $\{\pm I\}$, so the action of SL_2 induces a faithful action of $\mathrm{PSL}_2 = \mathrm{SL}_2/\{\pm I\}$ on $\mathrm{M}_{3,6} \times \mathrm{M}_{6,3}$, M_3 and V. From now on we will identify $\mathcal{G}_a$ with all elements

$$\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \text{ modulo } \{\pm I\}$$

Let S be the PSL_2-orbit of J. So

$$S = \{ \begin{pmatrix} 0 & p^2 & pq \\ -p^2 & 0 & q^2 \\ -pq & -q^2 & 0 \end{pmatrix} \in \mathrm{M}_3 \mid (p,q) \in k^2 \setminus \{0\}\}$$

The set S is a locally closed subset of M_3 and is isomorphic to $\mathrm{PSL}_2/\mathcal{G}_a$ because $\mathcal{G}_a$ is the isotropy group of J. Let $U = \pi^{-1}(S)$. The restriction $\pi : U \to S$ is PSL_2-equivariant, and S is a homogeneous space, so U is PSL_2-isomorphic to $\pi^{-1}(J) \star^{\mathcal{G}_a} \mathrm{PSL}_2 = V \star^{\mathcal{G}_a} \mathrm{PSL}_2$. The closure $\overline{U}$ is normal because $\overline{U} \subseteq \pi^{-1}(\overline{S})$ and $\overline{S}$ is isomorphic to a quadric which is normal. Furthermore $\overline{U} - U \subseteq \pi^{-1}(\overline{S} - S) = \pi^{-1}(0)$. Because π is equidimensional we see that $\overline{U} - U$ has codimension ≥ 2 in $\overline{U}$. So we get $\mathcal{O}(U) = \mathcal{O}(\overline{U})$ and $\overline{U} \cong \mathrm{Spec}(\mathcal{O}(U))$.

Suppose that V has a geometric quotient V/G_a. Then $V \star^{\mathcal{G}_a} \mathrm{PSL}_2 \to V/G_a$ is a geometric quotient with respect to PSL_2. Taking global sections we get a ringhomomorphism $\mathcal{O}(V/G_a) = \mathcal{O}(V)^{G_a} \to \mathcal{O}(U) = \mathcal{O}(\overline{U})$, which corresponds with a morphism $\mathrm{Spec}(\mathcal{O}(U)) = \overline{U} \to \mathrm{Spec}(\mathcal{O}(V)^{\mathcal{G}_a})$. This yields a commutative diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & \overline{U} \arrow[d] \\ V/\mathcal{G}_a \arrow[r] & \mathrm{Spec}(\mathcal{O}(V)^{\mathcal{G}_a}) \end{tikzcd}$$

Using a similar computation as in [9, II, Example 6.5.2] one can show that V is factorial. From [4, Proposition 2.5.2.] now follows that
$V/\mathcal{G}_a \to \mathrm{Spec}(\mathcal{O}(V)^{\mathcal{G}_a})$ is an open immersion. Let p_α be the point

$$\left(\begin{pmatrix} 1 & 0 & \alpha & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right) \in V$$

where $\alpha \in k$. All p_α have different PSL_2-orbits. But the point $q \in \overline{U}$ defined by

$$q = \left(\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right)$$

lies in the closure of the orbit of p_α for every α. All p_α must map via $U \to \overline{U} \to \mathrm{Spec}(\mathcal{O}(V)^{\mathcal{G}_a}) = \overline{U} \to V/\mathcal{G}_a \to \mathrm{Spec}(\mathcal{O}(V)^{\mathcal{G}_a})$ to the same point. But $V/\mathcal{G}_a \to \mathrm{Spec}(\mathcal{O}(V)^{\mathcal{G}_a})$ is an open immersion, therefore all p_α have the same image in $V/\mathcal{G}_a$. This contradicts with $V/\mathcal{G}_a$ being a geometric quotient of U with respect to PSL_2. Therefore the assumption that a geometric quotient of V by $\mathcal{G}_a$ exists is false.

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