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Some Elementary Theorems about Algebraic Cycles on Abelian Varieties

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Abstract. The structure of the group of 0-cycles modulo rational equivalence on an n-dimensional abelian variety A over an algebraically closed field k is studied. This group forms an augmented $\mathbb{Z}$-algebra under Pontryagin product, with augmentation given by the degree map. The $(n+1)$-st power of the augmentation ideal $I(=0$-cycles of degree $0)$ is shown to be zero, while for suitable k (e.g. $k=$complex numbers) the n-th power of I is non-zero. As corollary, every 0-cycle of degree 0 is shown to be rationally equivalent to a sum of intersections of divisors. Partial results, analogous to the isogeny between A and $\mathrm{Pic}^0 A$, are proved relating quotients I^{*r}/I^{*r+1} to cycles of codimension r on A.

§ 0. Introduction

The purpose of this paper is to apply the calculus of algebraic cycles on abelian varieties, as developed by Weil [6] and Lang [2] in their study of divisors, to cycles of codimension greater than one. Let X be a smooth projective variety over an algebraically closed field k, and let $r \geqq 0$ be an integer. The Chow group of codimension r cycles on X, $CH^r(X)$, is defined by

$$CH^r(X)=Z^r(X)/B^r(X)$$

where $Z^r(X)$ is the free abelian group generated by the irreducible codimension r subvarieties of X, and $B^r \subset Z^r$ is the subgroup generated by cycles $\Gamma(\infty)-\Gamma(0)$ for $\Gamma \subset \mathbb{P}^1 \times X$ a codimension r subvariety of the product. We also define the Chow group of cycles dimension r, $CH_r(X)=CH^{n-r}(X)$. We write

$$CH^*(X)=\bigoplus_{r\geqq 0} CH_r(X), \qquad CH_*(X)=\bigoplus_{r\geqq 0} CH_r(X),$$

for the graded objects. $CH^*(X)$ is contravariant functorial and $CH_*(X)$ is covariant functorial for maps of varieties.

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$CH^*(X)$ forms a graded ring under intersection of cycles. When $X=A$ is an abelian variety, $CH_*(A)$ also has a ring structure (Pontryagin product [2]) first exploited by Lang, and defined as follows: let $\mu\colon A\times A\to A$ be the group law. Given cycles $\gamma\in CH_r(A)$, $\tau\in CH_s(A)$, we have $\gamma\times\tau\in CH_{r+s}(A\times A)$. The Pontryagin product $\gamma*\tau$ is defined by

$$\gamma*\tau=\mu_*(\gamma\times\tau)\in CH_{r+s}(A).$$

Note that the zero cycles $CH_0(A)\subset CH_*(A)$ form a subring. In fact, $CH_0(A)$ is an augmented $\mathbb{Z}$-algebra, with augmentation given by the *degree map*

$$\deg\colon CH_0(A)\to\mathbb{Z}.$$

Let $I=I(A)\subset CH_0(A)$ denote the Kernel of deg. Then I is an ideal for Pontryagin multiplication, and we have an exact sequence

$$0\to I\to CH_0(A)\xrightarrow{\deg}\mathbb{Z}\to 0.$$

For $a\in A$ a k-point, $(a)\in CH_0(A)$ will denote the corresponding cycle. Thus the origin $o\in A$ corresponds to the identity element $(o)=1\in CH_0(A)$, and I is generated by cycles $(a)-(o)$, for $a\in A$. The Pontryagin powers of I will be denoted I^{*r}. For example, I^{*2} is generated by cycles

$$((a)-(o))*((b)-(o))=(a+b)-(a)-(b)+(o)$$

There is a natural map $I\to A$ given by $(a)-(o)\mapsto a$, and one gets an exact sequence

$$0\to I^{*2}\to I\to A\to 0.$$

The identitication $A=I/I^{*2}$ will be frequently used.

(0.1) **Theorem.** *Let A be an abelian variety of dimension n over an algebraically closed field k. Then $I^{*n+1}=0$.*

When A is an elliptic curve and $a,b\in A$, (0.1) amounts to the well-known fact that there exists a rational function f on A with

$$(f)_0=(a)+(b)\quad\text{and}\quad(f)_\infty=(a+b)+(o).$$

When A is an abelian surface ($n=2$), we find for any 3 points $a,b,c\in A$ the relation

$$\begin{aligned}0&=((a)-(o))*((b)-(o))*((c)-(o))\\&=(a+b+c)-(a+b)-(a+c)-(b+c)+(a)+(b)+(c)-(o).\end{aligned}$$

Bowing to custom, we will often write $\operatorname{Pic}(A)$ in place of $CH^1(A)$. $\operatorname{Pic}^0(A)\subset\operatorname{Pic}(A)$ will denote the subgroup of divisors algebraically equivalent to zero. $\operatorname{Pic}^0(A)$ is known to be (the group of closed points of) an abelian variety, and if $D\in\operatorname{Pic}(A)$ is an ample divisor, the map

$$\Phi_D\colon A\to\operatorname{Pic}^0(A);\qquad\Phi_D(a)=D_a-D=D*((a)-(o))$$

is well defined, and is an isogeny of abelian varieties. Thus $\operatorname{Pic}^0(A)=I*CH^1(A)$. As a consequence of (0.1), we find that the intersection map

$$\operatorname{Pic}(A)^{\otimes n-1}\otimes\operatorname{Pic}^0(A)\to I$$

is surjective, i.e. every o-cycle of degree o is rationally equivalent to a sum of intersections of divisors.

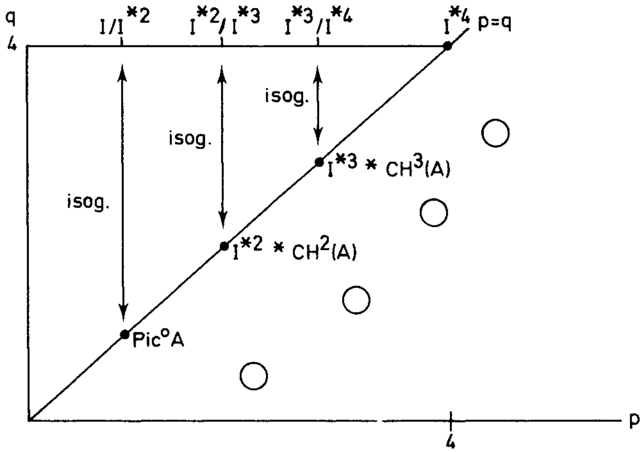
It seems natural to study the successive quotients I^{*r}/I^{*r+1} for $0 \leqq r \leqq n$.

(0.2) **Conjecture.** *Let D be an ample divisor class on A, and let $r \geqq 0$ be an integer. Write $D^r = D \cdot \cdots \cdot D$ (r times). Then $I^{*r+1} * CH^r(A) = 0$ and the map*

$$\Phi_{D^r}: I^{*r}/I^{*r+1} \to I^{*r} * CH^r(A); \qquad \Phi_{D^r}(t) = t * D^r$$

is an isogeny, i.e. Φ_{D^r} is surjective and there exists a map $v: I^{*r} * CH^r(A) \to I^{*r}/I^{*r+1}$ defined by a correspondence such that $v \circ \Phi_{D^r}$ = multiplication by N for some non-zero integer N.

In §4, we verify the conjecture for $r=1$, $n-2$, $n-1$, and n (the case $r=0$ can also be included if we agree to call any non-zero map $\mathbb{Z} \to \mathbb{Z}$ an isogeny). Suppose for example $n=4$ and associate to an integer point (p, q) the group $I^{*p} * CH^q(A)/I^{*p+1} * CH^q(A)$. We get



The paper is organised as follows: in §1 we prove some elementary Lemmas. §2 contains the proof of (0.1) and some corollaries. In §3 we use a technique involving differentials which goes back to Severi, but which was first really exploited by Mumford [3] and then developed further by Roitman [4, 5], in order to show for certain fields k (e.g. k = complex numbers) (0.1) is sharp, $I^{*n} \neq 0$. We give also a result of Swan that $I^{*2} = 0$ when $k = \overline{\mathbb{F}}_p$, the algebraic closure of a finite field. §4 is devoted to partial results related to (0.2).

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§ 1. Some Lemmas

It is convenient to group together some lemmas which will be used repeatedly in the sequel. Throughout, A will denote an abelian variety of dimension n over an algebraically closed field k, $I \subset CH_0(A)$ will be the group of 0-cycles of degree 0, and notations like $\gamma * \tau$, C^{*r}, I^{*r} will indicate Pontryagin product. $\mathrm{Pic}^0(A) \subset CH^1(A)$ will denote the group of divisors algebraically equivalent to zero. One knows ([2], p. 100) $\mathrm{Pic}^0(A) = I * CH^1(A)$.

(1.1) **Lemma.**(i) *Let $\gamma, \gamma' \in CH^*(A)$, and let $D \in \mathrm{Pic}^0(A)$. Then one has the formula in $CH^*(A)$:*

$$(\gamma * \gamma') \cdot D = (\gamma \cdot D) * \gamma' + (\gamma' \cdot D) * \gamma,$$

i.e. intersection with D is a derivation for the Pontryagin ring structure.

(ii) *Let $C \subset A$ be a curve, $M \leqq N$ integers, $D_1, \ldots, D_M \in \mathrm{Pic}^0(A)$. Let*

$$s_i = D_i\, C \in CH_0(A),$$

and let $\gamma \in CH^p(A)$. Then we have a congruence in $CH^{p-N+M}(A)$

$$(\gamma * C^{*N}) \cdot D_1, \ldots, D_M \equiv \frac{N!}{(N-M)!}\, \gamma * C^{*N-M} * s_1 * \cdots * s_M$$

$$\bmod \left[\sum_{i=1}^{\min(P,M)} (\gamma \cdot \underbrace{\mathrm{Pic}^0(A) \ldots \mathrm{Pic}^0(A)}_{i \text{ times}}) * C^{*N-M+i} * I^{*M-i} \right]$$

Proof. (ii) follows easily from (i). The assertion in (i) is linear in γ and γ' so we may assume γ and γ' are the classes of irreducible subvarieties X, X'. We have a diagram

$$\begin{tikzcd} X \times X' \arrow[rr, hook, "i"] \arrow[dr, "\pi"'] & & A \times A \arrow[dl, "\mu"] \\ & A & \end{tikzcd}$$

where i = product of the inclusions $X \hookrightarrow A$, $X' \hookrightarrow A$, and μ = addition, $\pi = \mu \circ i$. By definition, one has

$$(\gamma * \gamma') \cdot D = \pi_*(X \times X') \cdot D = \pi_*\, \pi^* D = \pi_*\, i^*\, \mu^* D,$$
$$(\gamma \cdot D) * \gamma' = \pi_*\, i^*(D \times A),$$
$$\gamma * (\gamma' \cdot D) = \pi_*\, i^*(A \times D).$$

Since D is algebraically equivalent to 0, one knows ([2], p. 90) $\mu^* D = D \times A + A \times D$, so the lemma is immediate. Q.E.D.

The simplest and most important example of the above lemma is the case $\gamma = (o) \in CH_0(A)$ and $M = N$. We get

$$C^{*N} \cdot D_1 \cdot \cdots \cdot D_N = N!\, s_1 * \cdots * s_N.$$

(1.2) **Lemma.** a) *Let τ and γ be cycles on A, and let $a \in A$. Write τ_a, γ_a in place of $\tau * (a)$, $\gamma * (a)$. We have*

$$\begin{aligned}(\tau \cdot \gamma) * ((a) - (o)) &= (\tau_a - \tau)\, \gamma + \tau_a \cdot (\gamma_a - \gamma) \\ &= (\tau_a - \tau) \cdot \gamma + \tau(\gamma_a - \gamma) + (\tau_a - \tau) \cdot (\gamma_a - \gamma).\end{aligned}$$

b) *Suppose we have* $\gamma * I^{*s} = \tau * I^{*t} = 0$ *for given integers* s, t. *Then* $(\gamma \cdot \tau) * I^{*s+t-1} = 0$. *In particular, if* $D_1, \ldots, D_s$ *are divisors on* A, *then* $(D_1 \cdot \cdots \cdot D_s) * I^{*s+1} = 0$.

c) *Let* D *be a divisor on* A, $s \geqq t$ *integers, and* $a_1, \ldots, a_t \in A$ *points. Then*

$$(D^s) * ((a_1) - (o)) * \cdots * ((a_t) - (o)) = \frac{s!}{(s-t)!} (D_{a_1} - D) \cdot \cdots \cdot (D_{a_t} - D) \cdot D^{s-t} + T,$$

where T *is a sum of terms of the form*

$$(D_{b_1} - D) \cdot \cdots \cdot (D_{b_u} - D) \cdot D^{s-u}$$

for $t < u \leqq s$ *and* $b_i \in A$. (*In particular,* $T = 0$ *if* $s = t$.)

Proof. a) Straight computation, using the fact that $(\gamma \cdot \tau)_a = \gamma_a \cdot \tau_a$.

b) Induction on $s+t$ together with the above. Note if either s or $t = 0$, $\gamma \cdot \tau = 0$. Also the theorem of the square implies $D * I^2 = 0$ for a divisor D.

c) The formula is easy when $s = 1$, and if follows without difficulty from a) when $t = 1$ (s arbitrary):

$$D^s * ((a_1) - (o)) = s(D_{a_1} - D) D^{s-1} + T_1.$$

Now prodeed by induction on s, noting that $D_{a_1} - D$ is translation invariant:

$$\begin{aligned} &D^s * ((a_1) - (o)) * \cdots * ((a_t) - (o)) \\ &\quad = s(D_{a_1} - D)[D^{s-1} * ((a_2) - (o)) * \cdots * ((a_t) - (o))] \\ &\qquad + T_1 * ((a_2) - (o) * \cdots * ((a_t) - (o)) \\ &\quad = s(D_{a_1} - D) \left[\frac{(s-1)!}{(s-t)!} (D_{a_2} - D) \cdot \cdots \cdot (D_{a_t} - D) \cdot D^{s-t} + T_2 \right] \\ &\qquad + T_1 * ((a_2) - (o)) * \cdots * ((a_t) - (o)). \end{aligned}$$

Using the inductive hypothesis,

$$T = s(D_{a_1} - D) \cdot T_2 + T_1 * ((a_2) - (o)) * \cdots * ((a_t) - (o))$$

has the desired form for the remainder. Q.E.D.

(1.3) **Lemma.** *Let* X *be a smooth variety proper over an algebraically closed field* k. *Let* $s \geqq 0$ *be an integer, and let* $CH^s(X)_{\text{alg}} \subset CH^s(X)$ *denote the subgroup of cycles algebraically equivalent to* 0. *Then* $CH^s(X)_{\text{alg}}$ *is a divisible group.*

Proof. It is known (*op. cit.* p. 60) that $CH^s(X)_{\text{alg}}$ is generated by the images of divisors on curves via correspondences, i.e. there exists a collection of smooth curves C_i with jacobians $J(C_i)$, and a surjection of groups.

$$\coprod J(C_i) \to CH^s(A)_{\text{alg}}.$$

Divisibility on the right now follows from divisibility of $J(C_i)$. Q.E.D.

As a consequence of (1.3), we see immediately that the groups I^{*r} are divisible for all $r \geqq 1$.

(1.4) **Lemma.** *For any s, the intersection map*

$$CH^s(A)_{\mathrm{alg}} \times CH_s(A)_{\mathrm{alg}} \to CH_0(A), \qquad (\gamma, \tau) \mapsto \gamma \cdot \tau$$

has image contained in I^{*2}.

Proof. The image is clearly contained in I, so we can compose to get $CH^s(A)_{\mathrm{alg}} \times CH_s(A)_{\mathrm{alg}} \to I \to I/I^{*2} \cong A$. If C (resp. C') is a smooth curve parametrizing a family of cycles of codimension s (resp. dimension s) on A, we obtain a map of varieties

$$J(C) \times J(C') \to A$$

which is trivial on $\{0\} \times J(C')$ and $J(C) \times \{0\}$. By rigidity ([2], p. 23) if follows that this map is zero. Arguing as in (1.3), we conclude that the intersection map

$$CH^s(A)_{\mathrm{alg}} \times CH_s(A)_{\mathrm{alg}} \to A = I/I^{*2}$$

is zero, i.e. that the intersection of algebraically equivalent to zero cycles of complementary dimensions lies in I^{*2}. Q.E.D.

§ 2. Boundedness Results for 0-Cycles

As before, A will be an abelian variety of dimension n over an algebraically closed field k, and $I \subset CH_0(A)$ will be the group of 0-cycles of degree zero modulo rational equivalence. I^{*s} will denote the Pontryagin product of I with itself s times. In this section we will verify that $I^{*n+1} = 0$ (Theorem (0.1)). As corollaries we obtain results about representing 0-cycles on A as sums of complete intersections, about the structure of 0-cycles on the Kummer variety $A/\{\pm 1\}$, and about the behavior of $CH_0(A)$ under endomorphisms of A.

(2.1) **Lemma.** *Suppose* $I^{*N} = 0$ *for some* $N \gg 0$. *Then* $I^{*n+1} = 0$.

Proof. Let $C \overset{i}{\hookrightarrow} A$ be a smooth curve generating A such that the composition

$$\mathrm{Pic}^0(A) \xrightarrow{i^*} \mathrm{Pic}^0(C) = J(C) \xrightarrow{i_*} A$$

is an isogeny (e.g. take C to be the intersection of $n-1$ general hyperplane sections of A). We can view $i_* i^*$ as a map $\mathrm{Pic}^0(A) \to I$ whose image generates $\bmod I^{*2}$.

We will assume $I^{*N} = 0$ for some $N > n+1$, and verify in the case that $I^{*N-1} = 0$. Notice that $I^{*N-1} = I^{*N-1}/I^{*N}$ is generated by the image of $\bigotimes_{N-1} I/I^{*2}$, and hence by cycles of the form

$$(D_1 \cdot C) * \cdots * (D_{N-1} \cdot C), \qquad D_i \in \mathrm{Pic}^0(A),$$

so it suffices to show these are zero. Also $\mathrm{Pic}^0(A)$ is a divisible group and the above expression is linear in each D_i, so it will suffice to show

$$(N-1)!\,(D_1 \cdot C) * \cdots * (D_{N-1} \cdot C) = 0.$$

But by (1.1), this expression is equal to

$$C^{*N-1} \cdot D_1 \cdot \cdots \cdot D_{N-1}.$$

Since $N-1>n$, we get zero by reason of dimension. Q.E.D.

Remark. Without assumptions, the proof gives $I^{*N}=I^{*N+1}$ for $N>n$.

(2.2) **Lemma.** *Suppose $A=J(C)$ is the jacobian of a curve of genus n. Then $I^{*n+1}=0$.*

Proof. It suffices by (1.3) to show $(n+1)!\, I^{*n+1}=0$. Fix an embedding $C \xrightarrow{i} A$ such that $o \in i(C)$, and let $S \subset I$ denote the *set* of cycles of the form $(c)-(o)$ for $c \in C \subset A$. Note that elements of S together with their translates under A generate I. Indeed, since C generates A, we can write any $a \in A$ as $a=c_1+\cdots+c_n$ with $c_j \in C \subset A$. We get

$$(a)-(o)=T_{c_1+\cdots+c_{n-1}}((c_n)-(o))+T_{c_1+\cdots+c_{n-2}}((c_{n-1})-(o))+\cdots+((c_1)-(o))$$

where T_x denotes translation by x. Since a relation

$$\tau_1 * \cdots * \tau_{N+1}=0, \qquad \tau_i \in I$$

implies a relation for any $x \in A$

$$(T_x \tau_1) * \tau_2 * \cdots * \tau_{n+1}=0,$$

our problem reduces to verifying

$$(n+1)!\,((c_1)-(o)) * \cdots * ((c_{n+1})-(o))=0, \qquad c_j \in C.$$

Let Θ be a theta divisor on $A=J(C)$. We know $(c_j)-(o)=i_* \, i^*(\Theta_{c_i}-\Theta)$ ([6], p. 76). Applying (1.1) with $D_i=\Theta_{c_i}-\Theta$, we must show

$$C^{*n+1} \cdot (\Theta_{c_1}-\Theta) \cdot \cdots \cdot (\Theta_{c_{n+1}}-\Theta)=0.$$

This is clear, again by reason of dimension. Q.E.D.

(0.1) *Proof of Theorem.* Let $C \subset A$ be a smooth curve (A is now any abelian variety) of genus $N \geqq n$ such that the induced map $J(C) \to A$ is surjective. We get a surjection on cycle groups $I(J(C)) \twoheadrightarrow I(A)$ which is compatible with Pontryagin product. By (2.2), $I(J(C))^{*N+1}=0$, hence $I(A)^{*N+1}=0$ also. By (2.1) this implies $I(A)^{*n+1}=0$. Q.E.D.

(2.3) **Corollary.** *Any zero cycle of degree* 0 *on A is rationally equivalent to a sum of intersections of divisors. More precisely, the intersection map*

$$\mathrm{Pic}^0(A) \otimes CH^1(A)^{\otimes n-1} \to I$$

is surjective.

Proof. Let D be an ample divisor on A. Applying (1.2) (c), it suffices to show that I is generated by cycles of the form

$$D^n * ((a_1)-(o)) * \cdots * ((a_t)-(o)).$$

Note $D^n \equiv (\deg D^n) \cdot (o) \bmod I$ so

$$D^n * \tau \equiv (\deg D^n) \cdot \tau \bmod I^{*t+1}, \qquad \tau \in I^{*t}.$$

Since I is divisible (1.3), cycles of the desired sort generate I/I^{*N} for any N (use successive approximation). Since $I^{*n+1} = 0$, they generate I. Q.E.D.

(2.4) **Corollary.** *Let A be an abelian surface, m an integer, and let $m\delta\colon A \to A$ denote multiplication by m. Let $T(A) \subset I$ be the kernel of the Albanese map $I \to A$. Then the functorial maps $(m\delta)^*$, $(m\delta)_*\colon T(A) \to T(A)$ are both multiplication by m^2.*

Proof. Notice $T(A) = I^{*2} = I^{*2}/I^{*3}$ is naturally a quotient of $A \otimes A = (I/I^{*2}) \otimes (I/I^{*2})$ under Pontryagin product. We get a commutative diagram

$$\begin{array}{ccc} A \otimes A & \xrightarrow{m\delta \otimes m\delta} & A \otimes A \\ {\scriptstyle \text{P. prod}} \downarrow & & \downarrow {\scriptstyle \text{P. prod}} \\ T(A) & \xrightarrow{(m\delta)_*} & T(A) \end{array}$$

so $(m\delta)_* =$ multiplication by m^2.

It follows from (1.2) (*c*) (taking $s = t = 2$), that the intersection map $\mathrm{Pic}^0(A) \otimes \mathrm{Pic}^0(A) \to T(A)$ is surjective. Since intersection is compatible with pullback, we get another commutative diagram

$$\begin{array}{ccc} \mathrm{Pic}^0(A) \otimes \mathrm{Pic}^0(A) & \xrightarrow{(m\delta)^* \otimes (m\delta)^*} & \mathrm{Pic}^0(A) \otimes \mathrm{Pic}^0(A) \\ {\scriptstyle \text{Intersect. p.}} \downarrow & & \downarrow {\scriptstyle \text{Intersect. p.}} \\ T(A) & \xrightarrow{(m\delta)^*} & T(A) \end{array}$$

so $(m\delta)^*$ is multiplication by m^2 on $T(A)$ also. Q.E.D.

For any surface X, we define $T(X) = \mathrm{Ker}\,(CH_0(X)_{\deg o} \to \mathrm{Alb}\,(X))$.

(2.5) **Corollary.** *Let A be an abelian surface, and let $X = A/\{\pm 1\}$ (with singularities resolved) be the Kummer surface of A. Let $\pi\colon A \to X$ be the rational map. There are induced maps*

$$T(A) \xrightarrow{\pi_*} T(X) \xrightarrow{\pi^*} T(A)$$

and the compositions $\pi_* \pi^*$ and $\pi^* \pi_*$ are both multiplication by 2. In particular, $T(X)$ is isogenous to $T(A)$.

Proof. π_* and π^* can be defined via the correspondence given by the graph of π in $A \times X$. A zero cycle γ on X (resp. on A) can be moved off the finite number of exceptional curves (resp. points of order two), after which $\pi^* \gamma$ (resp. $\pi_* \gamma$) can be computed via the set-theoretic inverse (resp. direct) image. It follows easily that $\pi_* \pi^* =$ multiplication by 2 on $T(X)$. A similar argument gives $\pi^* \pi_*(\gamma) = \gamma + (-\delta)_* \gamma$ for $\gamma \in T(A)$. By (2.4), $(-\delta)_* \gamma = (-1)^2 \cdot \gamma = \gamma$ so $\pi^* \pi_*(\gamma) = 2\gamma$. Q.E.D.

(2.6) *Remarks.* i) The reader can easily provide analogs of (2.4) and (2.5) for abelian varieties of dimension > 2.

ii) Roitman has recently proved that groups such as $T(X)$ and I^{*s}, $r \geqq 2$, are torsion free. This gives $T(x) \cong T(A)$ in (2.5), and enables one to sharpen various results in §4 as well. I have chosen not to incorporate these statements in the text, first because *I* am uncertain of the precise scope of Roitman's results and second because one hopes eventually to make the *groups* $T(X)$, I^{*r}, etc... into *functors.* In this richer setting one does not expect isomorphisms $T(X) \cong T(A)$.

§ 3. Some Unboundedness Results for 0-Cycles

For certain fields k, the assertion $I^{*n+1}=0$ is best possible. For others it is not.

(3.1) **Theorem.** *Let A be an abelian variety of dimension n over an algebraically closed field k, and let $I \subset CH_0(A)$ be the group of 0-cycles of degree 0 modulo rational equivalence.*

a) *Assume k is of characteristic 0 and is uncountable. Then $I^{*n} \neq 0$.*

b) *(Swan). Assume $k=\overline{\mathbb{F}_p}$ is the algebraic closure of a finite field. Then $I^{*2}=0$.*

Proof of b). We have seen that elements of I come via correspondences from jacobians of curves (cf. the proof of (1.3)). When $k=\overline{\mathbb{F}_p}$, this implies that I is torsion as well as divisible. Since I^{*2} is a quotient of $I \otimes I=0$, it follows that $I^{*2}=0$.

The proof of a) uses some techniques of Mumford [3] and Roitman [4], [5], involving differentials. The general statements (e.g. [5], §3) and proofs of these results are somewhat delicate because one must work with differentials on singular varieties (namely symmetric products of varieties of dimension >1). In our case, however, we can get by with a crude version involving ordinary products. The proof is immediate from Roitman's results, and is omitted.

(3.2) **Lemma.** *Let X be a non-singular projective variety over a field k which is uncountable, algebraically closed, and of characteristic 0. Let $N>0$ be an integer, and let $\gamma: X^N \times X^N \to CH_0(X)_{\deg 0}$ denote the map*

$$\gamma(x_1, \ldots, x_N; y_1, \ldots, y_N)=\Sigma(x_i)-\Sigma(y_i).$$

Let Z be a non-singular variety, and suppose given a morphism

$$f=(f_1, f_2): Z \to X^N \times X^N$$

such that the composition $\gamma \circ f: Z \to CH_0(X)$ is the zero map. Let $\omega \in \Gamma(X, \Omega^q_{X/k})$ be a q-form on X for some $q \geqq 1$. Define a differential $\tilde{\omega} \in \Gamma(X^N, \Omega^q)$ by $\tilde{\omega}= p_1^* \omega+\cdots+p_N^* \omega$, where p_i: $X^N \to X$ denotes projection on the i-th factor. Then $f_1^* \tilde{\omega}=f_2^* \tilde{\omega}$ on Z.

To apply the lemma, let N denote the number of (unordered) subsets $I=\{i_1, \ldots, i_r\} \subset \{1, \ldots, n\}$ such that r is even (we include the empty set). Write $r=\# I$, and, for $\alpha=(\alpha_1, \ldots, \alpha_n) \in A^n$, write $\alpha_I=\alpha_{i_1}+\cdots+\alpha_{i_r} \in A$ ($\alpha_\emptyset=0$). Define

$$f_1(\alpha)=\prod_{\# I \text{ even}} \alpha_I \in A^N$$

$$f_2(\alpha)=\prod_{\# I \text{ odd}} \alpha_I \in A^N$$

and let $f=(f_1, f_2)\colon A^n \to A^N \times A^N$. The point is that the composition $\gamma \circ f$: $A^n \to CH_0(A)_{\deg 0}$ is $\gamma \circ f(\alpha_1, \ldots, \alpha_n) = ((\alpha_1)-(o)) * \cdots * ((\alpha_n)-(o))$. In particular, assuming $I^{*n}=0$, we get γ is the zero map.

On the other hand, let $\omega_1, \ldots, \omega_n$ be a basis for $\Gamma(A, \Omega_A^1)$, so $\omega = \omega_1 \wedge \cdots \wedge \omega_n \neq 0$. I claim $f_1^* \tilde{\omega} - f_2^* \tilde{\omega} \neq 0$ on A^n. It suffices to prove:

(3.3) **Lemma.** *Let S_n denote the group of permutations on $\{1, \ldots, n\}$. Then*

$$f_1^* \tilde{\omega} - f_2^* \tilde{\omega} = \pm \sum_{\sigma \in S_n} p_1^* \omega_{\sigma(1)} \wedge \cdots \wedge p_n^* \omega_{\sigma(n)}.$$

Proof. For $J \subset \{1, \ldots, n\}$, let $\rho_J\colon A^n \to A$ be defined by $\rho_J(\alpha_1, \ldots, \alpha_n) = \sum_{i \in J} \alpha_i$. Then

$$f_1^* \tilde{\omega} - f_2^* \tilde{\omega} = \sum_J (-1)^{\# J} \rho_J^* \omega.$$

We have

$$\rho_J^* \omega_j = \sum_{i \in J} p_i^* \omega_j$$

$$\rho_J^* \omega = \Big(\sum_{i \in J} p_i^* \omega_1\Big) \wedge \cdots \wedge \Big(\sum_{i \in J} p_i^* \omega_n\Big).$$

Let $(i_1, \ldots, i_n)$ be an ordered n-triple of integers with $1 \leqq i_j \leqq n$. We need to evaluate the coefficient of $p_{i_1}^* \omega_1 \wedge \cdots \wedge p_{i_n}^* \omega_n$ in $f_1^* \tilde{\omega} - f_2^* \tilde{\omega}$.

Case 1. $\{1, \ldots, n\} = \{i_1, \ldots, i_n\}$. In this case, the term in question occurs only in $\rho_{\{1, \ldots, n\}}^* \omega$, and the coefficient is $(-1)^n$.

Case 2. $\{i_1, \ldots, i_n\} \subsetneqq \{1, \ldots, n\}$. For an integer r, define

$$\Phi_{\{i_1, \ldots, i_n\}}(r) = \# \{J \subset \{1, \ldots, n\} \mid \# J = r \text{ and } \{i_1, \ldots, i_n\} \subset J\}.$$

The desired coefficient is

$$\sum_{s=m}^{n} (-1)^r \Phi(r), \qquad m = \# \{i_1, \ldots, i_n\} < n.$$

Since $\Phi(r) = \binom{n-m}{r-m}$, we get

$$\sum_{r=m}^{n} (-1)^r \Phi(r) = \sum_{r=m}^{n} (-1)^r \binom{n-m}{r-m} = (-1)^m \sum_{r=0}^{n-m} (-1)^r \binom{n-m}{r}$$
$$= (-1)^m (1-1)^{n-m} = 0. \quad \text{Q.E.D.}$$

§ 4. Extensions to Cycles of Dimension >0

The symbols A, I and n will have the same meaning as before.

(4.1) **Theorem.** *We have $I^{*r+1} * CH^r(A) = (0)$ in the following cases:*

$r = 0, 1, n-2, n-1, n.$

Proof. The case $r=0$ is easy, and $r=1$ is a consequence of the theorem of the square. $r=n$ is a consequence of § 2, so it remains to consider the cases $r=n-1$, $r=n-2$.

(4.2) **Lemma.** *Let $\gamma \in CH^{n-1}(A)$. Then $\gamma * I^{*n}=0$. If, moreover, γ is algebraically equivalent to zero, then $\gamma * I^{*n-1}=0$.*

Proof of Lemma. Note $I^{*n}=I^{*n}/I^{*n+1}$, so for the first assertion it suffices to show $n!\gamma * s_1 * \cdots * s_n=0$ with s_i running through a system of generators of I/I^{*2}. Let D be an ample divisor class on A, and take $C=D^{n-1}\in CH^{n-1}(A)$. The map

$$\mathrm{Pic}^0(A) \xrightarrow{\cdot C} I/I^{*2}$$

is surjective, so we may take $s_i=E_i\cdot C$ for $E_i \in \mathrm{Pic}^0(A)$.

The cycle $\gamma * C^{*n}$ is zero for reasons of dimension, so by (1.1)

$$0=(\gamma * C^{*n})\cdot E_1 \cdot \cdots \cdot E_n = n!\gamma * s_1 * \cdots * s_n + \tau$$

where $\tau \in (\gamma \cdot \mathrm{Pic}^0 A) * C * I^{*n-1}$. Since $\gamma \cdot \mathrm{Pic}^0 A \subseteq I$, we get $\tau \in C * I^{*n} = D^{n-1} * I^{*n}$. It follows that $\tau=0$ as desired.

Suppose now that γ is algebraically equivalent to 0. To show $\gamma * I^{*n-1}=0$, note $\gamma * I^{*n}=0$ so

$$\gamma * I^{*n-1} = \gamma * (I^{*n-1}/I^{*n})$$

and we reduce as before to verifying

$$(n-1)!\gamma * (E_1 \cdot C) * \cdots * (E_{n-1}\cdot C)=0.$$

Since $\gamma * C^{*n-1}$ is algebraically equivalent to zero and of codimension 0, it is zero. Thus (by (1.1))

$$0=(\gamma * C^{*n-1})\cdot E_1 \cdot \cdots \cdot E_{n-1} = (n-1)!\gamma * (E_1 \cdot C) * \cdots * (E_{n-1}\cdot C) + \tau'$$

with $\tau' \in (\gamma \cdot \mathrm{Pic}^0 A) * C * I^{*n-2}$. By (1.4) we get $\tau' \in C * I^{*n} = 0$. Q.E.D.

To finish the proof of (4.1), we muxt show:

(4.3) **Lemma.** *Let $\delta \in CH^{n-2}(A)$. Then $I^{*n-1} * \delta = 0$.*

Proof. By downwards induction it will be enough to show that $\delta * I^{*M+1}=0$ and $M \geqq n-1$ implies $\delta * I^{*M}=0$. As in the proof of (4.2), we must show

$$M!\delta * s_1 * \cdots * s_M = 0, \quad s_i = C\cdot E_i, \; E_i \in \mathrm{Pic}^0 A.$$

The usual computation gives

$$0=(\delta * C^{*M})\cdot E_1 \cdot \cdots \cdot E_M = M!\delta * s_1 * \cdots * s_M + \tau$$

with

$$\tau \in (\delta \cdot \mathrm{Pic}^0 A) * C * I^{*M-1} + (\delta \cdot \mathrm{Pic}^0 A \cdot \mathrm{Pic}^0 A) * C^{*2} * I^{*M-2}.$$

Since $\delta \cdot \mathrm{Pic}^0 A \subset CH^{n-1}(A)$ consists of cycles algebraically equivalent to zero, and $\delta \cdot \mathrm{Pic}^0 A \cdot \mathrm{Pic}^0 A \subseteq I^{*2}$ (1.4) it will be enough to prove:

(4.4) **Sublemma.** i) $C^{*2} * I^{*n-1} = 0$

ii) *If* $\gamma \in CH^{n-1}(A)$ *is agebraically equivalent to* 0, *then* $\gamma * C * I^{*n-2} = 0$.

Proof of Sublemma. i) We know by (4.2)

$$C^{*2} * I^{*n} = C * (C * I^{*n}) = 0$$

so it suffices to verify

$$N C^{*2} * s_1 * \cdots * s_{n-1} = 0, \qquad s_i = E_i \cdot C, \ N > 0 \text{ fixed integer.}$$

The reader can check that the argument in the proof of (1.1) yields also (for some $N > 0$)

$$0 = C^{*n+1} \cdot E_1 \cdot \cdots \cdot E_{n-1} = N C^{*2} * s_1 * \cdots * s_{n-1} \cdot$$

It remains to prove (4.4)ii). Note

$$\gamma * C * I^{*n-1} = C * (\gamma * I^{*n-1}) = 0$$

by (4.3), so the usual argument reduces us to showing

$$N \gamma * C * s_1 * \cdots * s_{n-2} = 0, \qquad \text{some } N > 0. \qquad s_i = E_i C.$$

Using (1.1), we get

$$0 = (\gamma * C^{*n-1}) \cdot E_1 \cdot \cdots \cdot E_{n-2} = (n-1)! \, \gamma * C * s_1 * \cdots * s_M + \tau$$

with $\tau \in (\gamma \cdot \operatorname{Pic}^0 A) * C^{*2} * I^{*n-3}$. Since $\gamma \cdot \operatorname{Pic}^0 A \subseteq I^{*2}$, we get $\tau = 0$ by (4.4) i).

This completes the proof of (4.4), and hence of (4.3) and (4.1).

(4.5) *Definition.* Let r be an integer, and let $t \in CH^r(A)$ be a cycle such that $t * I^{*r+1} = 0$. Then the *polarization map*

$$\Phi_t \colon I^{*r}/I^{*r+1} \to I^{*r} * CH^r(A)$$

is defined by $\Phi_t(\gamma) = \gamma * t$.

When $r = 1$, this is the usual notion of a polarization on an abelian variety. When $r = n$, $n-1$, or $n-2$, Φ_t is defined for any t. For general r, Φ_t is defined when $r = D_1 \cdot \cdots \cdot D_r$ by (1.2)b). We will focus on the case $t = D^r$, where D is the class of an ample divisor on A.

(4.6) **Proposition.** *Let D be an ample divisor. The polarization map*

$$\Phi_{D_r} \colon I^{*r}/I^{*r+1} \to I^{*r} * CH^r(A)$$

identifies I^{*r}/I^{*r+1} *up to isogeny with a direct factor of* $I^{*r} * CH^r(A)$. *In other words, there exists a map* $v \colon I^{*r} * CH^r(A) \to I^{*r}/I^{*r+1}$ *which is defined by an algebraic correspondence on cycles and which satisfies* $v \circ \Phi_{D^r} =$ *multiplication by N for some integer* $N \neq 0$.

Proof. Let $C = D^{n-1}$, and write $\cdot C \colon \operatorname{Pic}^0 A \to I/I^{*2}$ for the isogeny $E \to E \cdot C$. The following diagram is commutative (the left hand square by (1.2)c), and right by (1.1))

$$\begin{tikzcd}
\bigotimes_r (I/I^{*2}) \arrow[r, "\otimes_r \Phi_D"] \arrow[d, "\text{Pontryagin product}"] & \bigotimes_r \operatorname{Pic}^0 A \arrow[r, "\otimes_r (\cdot C)"] \arrow[d, "r!\cdot\text{intersection}"] & \bigotimes_r (I/I^{*2}) \arrow[d, "(r!)^2\ \text{Pontryagin product}"] \\
I^{*r}/I^{*r+1} \arrow[r, "\Phi_{D^r}"] & I^{*r} * CH^r(A) \arrow[r, "(\cdot C^{*r})"] & I/I^{*r+1}
\end{tikzcd}$$

The map

$$(\cdot\, C)\circ\Phi_D\colon A = I/I^{*2} \to A$$

is an isogeny, so there exists a homomorphism of abelian varieties $f\colon A\to A$ with $f\circ(\cdot\, C)\circ\Phi_D =$ multiplication by N for some $N \neq 0$. From the above diagram we conclude that the composition

$$I^{*r}/I^{*r+1} \xrightarrow{\Phi_{D^r}} I^{*r} * CH^r(A) \xrightarrow{(\cdot\, C^{*r})} I/I^{*r+1} \xrightarrow{f_*} I/I^{*r+1}$$

is multiplication by N^r. Thus $f_*\circ(\cdot\, C^{*r})$ splits Φ_{D^r} upto isogeny. Q.E.D.

(4.7) **Theorem.** *Let D be the class of an ample divisor on A. Then the maps*

$$\Phi_{D^r}\colon I^{*r}/I^{*r+1} \to I^{*r} * CH^r(A)$$

are surjective isogenies for $r = n$, $n-1$, $n-2$, and 1.

Proof. By (4.6) it suffices to show Φ_{D^r} is surjective. $\Phi_{D^n} =$ multiplication degree (D^n) which is surjective since I^{*n} is divisible. Suppose now $r = n-1$; let $C = D^{n-1}$ and let $\gamma \in CH^{n-1}(A)$. By the usual argument, it suffices to show

$$(n-1)!\,\gamma * s_1 * \cdots * s_{n-1} \in I^{*n-1} * C,$$

where $s_i = C\cdot E_i$ and $E_i \in \operatorname{Pic}^0(A)$.

By (1.1) we get

$$(\gamma * C^{*n-1})\cdot E_1\cdot \cdots \cdot E_{n-1} \equiv (n-1)!\,\gamma * s_1 * \cdots * s_{n-1} \bmod I^{*n-1} * C.$$

The left side of this congruence lies in $I^{*n-1} * C$ by (1.2)c) (take $s = t = n-1$, and choose $a_i \in A$ such that $E_i = D_{a_i} - D$), so $(n-1)!\,\gamma * s_1 * \cdots * s_{n-1} \in I^{*n-1} * C$ as desired.

It remains to consider the case $r = n-2$. Let $\delta \in CH^{n-2}(A)$. With notation as above, we must show $(n-2)!\,\delta * s_1 * \cdots * s_{n-2} \in I^{*n-2} * D^{n-2}$. Applying (1.1) and (1.2)c):

$$0 \equiv (\delta * C^{*n-2})\cdot E_1\cdot \cdots \cdot E_{n-2} \equiv (n-2)!\,\delta * s_1 * \cdots * s_{n-2} + T \bmod I^{*n-2} * D^{n-2}$$

where

$$T \in (\delta\cdot \operatorname{Pic}^0(A)) * C * I^{*n-3} + (\delta\cdot \operatorname{Pic}^0(A)\cdot \operatorname{Pic}^0(A)) * C^{*2} * I^{*n-4}.$$

Claim 1. For $N \geqq n-3$, $(\delta\cdot \operatorname{Pic}^0 A) * C * I^{*N} \subseteq D^{n-2} * I^{*N+1}$.

Indeed, for $N \geqq n+1$ the assertion follows from (0.1). Proceeding by downwards induction on N, assume the claim for $N+1$ and note that it suffices to show

$$(\delta \cdot E) * C * s_1 * \cdots * s_N \in D^{n-2} * I^{*N+1}$$

with $s_i = C \cdot E_i$, E_i and $E \in \mathrm{Pic}^0(A)$. The divisor $(\delta \cdot E) * C^{*n-2}$ lies in $\mathrm{Pic}^0(A)$, so again the results in § 1 give

$$\begin{aligned} 0 &\equiv ((\delta \cdot E) * C^{*N+1}) \cdot E_1 \cdot \cdots \cdot E_N \\ &\equiv (N+1)!\,(\delta \cdot E) * C * s_1 * \cdots * s_N \bmod (I^{*N+1} * D^{n-2} + I^{*N+1} * C^{*2}). \end{aligned}$$

It suffices to verify

Claim 2. For $N \geqq n-2$, $I^{*N} * C^{*2} \subseteq I^{*N} * D^{n-2}$.

Indeed, for $N > n-2$, $I^{*N} * C^{*2} = 0$ by (4.1). With the customary notation, we must show

$$\frac{n!}{2} C^{*2} * s_1 * \cdots * s_{n-2} \in D^{n-2} * I^{*n-2}.$$

We have by (1.1) and (1.2)c)

$$\frac{n!}{2} C^{*2} * s_1 * \cdots * s_{n-2} = C^{*n} \cdot E_1 \cdot \cdots \cdot E_{n-2} \in I^{*n-2} * D^{n-2}$$

proving claim 2.

It follows from claims 1 and 2 that the quantity T in the proof of (4.7) lies in $I^{*n-2} * D^{n-2}$. This completes the proof of (4.7).

(4.8) *Remark.* The existence of an isogeny between $\mathrm{Alb}(X)$ and $\mathrm{Pic}^0(X)$ is a general fact about algebraic varieties, and is not limited to the case $X =$ abelian variety. One can hope, in a similar vein, that a result like (4.7) is valid for any smooth projective variety X. More precisely, one can hope for a filtration $F^{\cdot} CH_0(X)$ together with isogenies for $gr^r CH_0(X)$ onto a suitable part of $CH^r(X)$.

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