Derived Algebraic Geometry VIII: Quasi-Coherent Sheaves and Tannaka Duality Theorems

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Introduction

A standard technique in mathematics is to study an algebraic object A by considering actions of A on objects which are better understood. For instance, we might try to analyze a group G by studying the category $\mathrm{Rep}(G)$ of linear representations of G, or a commutative ring R by studying the category of R-modules. Using the language of algebraic geometry, we can see these two examples as not merely analogous but as instances of the same general paradigm:

(a) If G is an algebraic group defined over a field k (for example, a finite group), then the category $\mathrm{Rep}(G)$ of (algebraic) representations of G is equivalent to the category of quasi-coherent sheaves on the classifying stack BG.

(b) If R is a commutative ring, then the category of R-modules is equivalent to the category of quasi-coherent sheaves on the affine scheme $\operatorname{Spec} R$.

More generally, we can associate to every algebraic stack X the abelian category $\mathrm{QC}(X)$ of quasi-coherent sheaves on X. Understanding this category is often key to understanding the geometry of X. In fact, we can often recover X from the category $\mathrm{QC}(X)$. For example, the classical theory of Tannaka duality (see, for example, [7]) guarantees that we can recover an affine algebraic group G (over field k) as the automorphism group of the forgetful functor $\mathrm{Rep}(G) \to \mathrm{Vect}_k$, regarded as a symmetric monoidal functor (here Vect_k denotes the category of vector spaces over k).

Our aim in this paper is to adapt the theory of quasi-coherent sheaves to the setting of spectral algebraic geometry. In §2, we will associate to every spectral Deligne-Mumford stack $\mathfrak{X}$ a stable ∞-category $\mathrm{QCoh}(\mathfrak{X})$, which we call the *$\infty$-category of quasi-coherent sheaves on* $\mathfrak{X}$. This stable ∞-category is equipped with a t-structure, and its heart $\mathrm{QCoh}(\mathfrak{X})^\heartsuit$ can be identified with the abelian category of quasi-coherent sheaves on the ordinary Deligne-Mumford stack underlying $\mathfrak{X}$. In the special case where $\mathfrak{X}$ arises from an ordinary Deligne-Mumford stack, we can recover $\mathrm{QCoh}(\mathfrak{X})$ as the derived ∞-category of its heart (at least at the level of bounded objects). In general this need not be true: indeed, the stable ∞-category $\mathrm{QCoh}(\mathfrak{X})$ generally does not arise as the derived ∞-category of *any* abelian category. The discrepancy between $\mathrm{QCoh}(\mathfrak{X})$ and the derived ∞-category of its heart can be regarded as measuring the extent to which $\mathfrak{X}$ is a "derived" algebro-geometric object, and is therefore a central concern in the theory of spectral algebraic geometry.

The main theme of this paper can be summarized as follows: in good cases, we can recover a algebro-geometric object $\mathfrak{X}$ from the stable ∞-category $\mathrm{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on $\mathfrak{X}$. More precisely, specifying a map of geometric objects $f : \mathfrak{Y} \to \mathfrak{X}$ should be equivalent to giving a (symmetric monoidal) functor $f^* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ satisfying some natural conditions. In §3 we will prove a general result of this type assuming that $\mathfrak{X}$ is quasi-compact and the diagonal map $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is affine (see Theorem 3.4.2). In fact, we do not even need to assume that $\mathfrak{X}$ is Deligne-Mumford: our result holds for a more general class of *geometric stacks* which are only assumed to admit a faithfullly flat affine covering, rather than étale affine covering (see Definition 3.4.1). For a related result in the setting of ordinary algebraic geometry, we refer the reader to [39].

In [63], Toën introduces the notion of an *affine stack* over a field k. Roughly speaking, a (higher) stack X is affine if it can be recovered from its cochain complex $C^*(X; \mathcal{O}_X)$, suitably interpreted as some kind of generalized commutative ring. In §4, we will give an exposition of Toën's theory in the special case where the field k has characteristic zero. Our main result is that affine stacks (over a field of characteristic zero) are governed by a Tannaka duality principle: that is, an affine stack X can be recovered from the symmetric monoidal ∞-category of quasi-coherent sheaves on X (Corollary 4.6.19).

Remark 0.0.1. In the body of this paper, we will use the term *coaffine stack* in place of Toën's usage *affine stack*. This is to avoid a conflict with our own terminology (we are principally interested in studying spectral Deligne-Mumford stacks, which may also be affine).

Let k be a field and let $\mathcal{C}$ be a k-linear symmetric monoidal abelian category. The category $\mathcal{C}$ is said to be *Tannakian* if, after passing to some extension field k' of k, we can identify $\mathcal{C}$ with the category of finite-dimensional representations of an affine group scheme. When the field k has characteristic zero, Deligne has given necessary and sufficient conditions for $\mathcal{C}$ to be Tannakian (see [7]; we give a review of the statement as Theorem 5.0.1). In §5 we will prove an ∞-categorical analogue of Deligne's result. More precisely, we will introduce the notion of a *generalized algebraic gerbe* (over a field k of characteristic zero). We show that a generalized algebraic gerbe X can be functorially recovered from the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X. Moreover, we show that a symmetric monoidal ∞-category $\mathcal{C}$ has the form $\mathrm{QCoh}(X)$ (for some generalized algebraic gerbe X) if and only if $\mathcal{C}$ is *locally dimensional* (see Definition 5.6.4): that is, if and only if $\mathcal{C}$ satisfies an ∞-categorical generalization of Deligne's criterion (Theorem 5.6.19).

Throughout this paper, we will make extensive use of the theory of spectral algebraic geometry introduced in [43]. In §1 we collect some general facts about spectral Deligne-Mumford stacks which are not treated in [43], but play a role in this paper (and in future papers in this series).

Remark 0.0.2. For other treatments of "derived" Tannaka duality, we refer the reader to the work of Fukuyama-Iwanari ([18]) and Wallbridge ([74]).

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Notation and Terminology

This paper will make extensive use of the theory of ∞-categories, as developed in [40]. We will also need the theory of structured ring spectra, which is presented from an ∞-categorical point of view in [41]. Finally, we will make use of the theory of geometries developed in [42], and earlier paper in this series. For convenience, we will adopt the following reference conventions:

(T) We will indicate references to [40] using the letter T.

(A) We will indicate references to [41] using the letter A.

(V) We will indicate references to [42] using the Roman numeral V.

(VII) We will indicate references to [43] using the Roman numeral VII.

For example, Theorem T.6.1.0.6 refers to Theorem 6.1.0.6 of [40].

We will assume general familiarity with the notations of [40] and [41]. We let Sp denote the ∞-category of spectra. We regard Sp as endowed with the smash product monoidal structure. If M and N are spectra, we will denote their smash product by $M \otimes N$. We let CAlg denote the ∞-category CAlg(Sp) of commutative algebra objects of Sp; we will refer to the objects of CAlg as $\mathbb{E}_\infty$*-rings*. We let Mod denote the ∞-category Mod(Sp) of commutative module objects of Sp: that is, the objects of Mod are pairs (A, M), where A is an $\mathbb{E}_\infty$-ring and M is an A-module object of Sp. In this case, we will say that M is an *A-module spectrum* or simply that M is an *A-module*. For each $\mathbb{E}_\infty$-ring A, we let $\mathrm{Mod}_A = \mathrm{Mod} \times_{\mathrm{CAlg}} \{A\}$ denote the ∞-category $\mathrm{Mod}_A(\mathrm{Sp})$ of A-module spectra.

We say that an $\mathbb{E}_\infty$-ring A is *connective* if the homotopy groups $\pi_i A$ vanish for $i < 0$, and that A is *discrete* if the homotopy groups $\pi_i A$ vanish for $i \neq 0$. We let $\mathrm{CAlg}^{\mathrm{cn}}$ and CAlg^0 denote the full subcategory of CAlg spanned by the connective and discrete $\mathbb{E}_\infty$-rings, respectively. The construction $A \to \pi_0 A$ determines an equivalence between CAlg^0 and the (nerve of the) ordinary category of commutative rings. We will generally abuse terminology by using this equivalence to identify commutative rings with their images in $\mathrm{CAlg}^0 \subseteq \mathrm{CAlg}$.

We will assume that the reader is familiar the theory of spectral Deligne-Mumford stacks introduced in [43]. We will identify a nonconnective spectral Deligne-Mumford stack $\mathfrak{X}$ with a pair $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. The collection of all nonconnective spectral Deligne-Mumford stacks is organized into an ∞-category which we will denote by $\mathrm{Stk}^{\mathrm{nc}}$. We let Stk denote the full subcategory of $\mathrm{Stk}^{\mathrm{nc}}$ spanned by the spectral Deligne-Mumford stacks.

In this paper, we will need to consider several different avatars of the spectrum of a commutative ring (or $\mathbb{E}_\infty$-ring):

- If R is a commutative ring, we let $\mathrm{Spec}^{\mathrm{Z}} R$ denote collection of prime ideals $\mathfrak{p} \subseteq R$. We regard $\mathrm{Spec}^{\mathrm{Z}} R$ as endowed with the Zariski topology: it has a basis of open sets given by $U_x = \{\mathfrak{p} \in \mathrm{Spec}^{\mathrm{Z}} R : f \notin \mathfrak{p}\}$, where x ranges over the elements of R. We will sometimes refer to $\mathrm{Spec}^{\mathrm{Z}} R$ as the *Zariski spectrum* of the commutative ring R.

  If R is an $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}^{\mathrm{Z}} R$ denote the Zariski spectrum $\mathrm{Spec}^{\mathrm{Z}}(\pi_0 R)$ of the commutative ring $\pi_0 R$.

- If R is a commutative ring, we let $\mathrm{Spec}^{\mathrm{c}} R$ denote the affine scheme associated to R, in the sense of classical scheme theory. More precisely, $\mathrm{Spec}^{\mathrm{c}} R$ is the ringed space $(\mathrm{Spec}^{\mathrm{Z}} R, \mathcal{O})$, where $\mathcal{O}$ is the sheaf of commutative rings on $\mathrm{Spec}^{\mathrm{Z}} R$ given by $\mathcal{O}(U_x) = R[\frac{1}{x}]$ (the superscript c is intended to indicate the word "classical").

- If R is an $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}\, R$ denote the affine nonconnective Deligne-Mumford stack associated to R. More precisely, we can identify $\mathrm{Spec}\, R$ with the spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ denotes the ∞-topos of functors $\mathrm{CAlg}_R^{\text{ét}} \to \mathcal{S}$ which are sheaves with respect to the étale topology and $\mathcal{O}$ is the sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$ given by the forgetful functor $\mathrm{CAlg}_R^{\text{ét}} \to \mathrm{CAlg}$. Here $\mathrm{CAlg}_R^{\text{ét}}$ denotes the ∞-category of étale R-algebras.

- If R is a connective $\mathbb{E}_\infty$-ring, we let $\mathrm{Spec}^{\mathrm{f}} R$ denote the functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ given by the formula $(\mathrm{Spec}^{\mathrm{f}} R)(A) = \mathrm{Map}_{\mathrm{CAlg}}(R, A)$. We will refer to $\mathrm{Spec}^{\mathrm{f}} R$ as the *functor corepresented by* R (the superscript f is meant to emphasize that we are thinking of the spectrum of R as a functor). Equivalently, $\mathrm{Spec}^{\mathrm{f}} R$ is the functor represented by the affine spectral Deligne-Mumford stack $\mathrm{Spec}\, R$ (that is, we have a canonical homotopy equivalence $(\mathrm{Spec}^{\mathrm{f}} R)(A) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, A, \mathrm{Spec}\, R)$.

We say that a nonconnective spectral Deligne-Mumford stack $\mathfrak{X}$ is *affine* if it has the form $\mathrm{Spec}\, R$, for some $\mathbb{E}_\infty$-ring R. More generally, we say that an object $U \in \mathcal{X}$ is *affine* if the nonconnective spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is affine.

In this paper, we will generally be interested in geometric objects X which are more general than spectral Deligne-Mumford stacks. We will generally treat such objects using the "functor-of-points" philosophy: that is, we will think of X as a functor from some avatar of the category of commutative rings (usually the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings) to the ∞-category $\mathcal{S}$ of spaces. We will generally be interested in functors which are sheaves with respect to an appropriate Grothendieck topology on the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$. With this in mind, we let $\mathcal{S}\mathrm{hv}_{\mathrm{fpqc}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the flat topology introduced in §VII.5. For technical reasons (stemming from the fact that $\mathrm{CAlg}^{\mathrm{cn}}$ is not small), it is often more convenient to work in a slightly larger ∞-category: we let $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves with respect to the flat topology; here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small.

1 Generalities on Spectral Deligne-Mumford Stacks

Our goal in this section is to collect some general facts about spectral Deligne-Mumford stacks which will be needed in the later sections of this paper (and in later papers in this series). We begin in §1.1 by studying *pointts* of spectral Deligne-Mumford stacks. Recall that a spectral Deligne-Mumford stack $\mathfrak{X}$ is a pair $(\mathcal{X}, \mathcal{O})$, where $\mathcal{X}$ is an ∞-topos and $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We define a *point* η of $\mathfrak{X}$ to be a point of the ∞-topos $\mathcal{X}$: that is, a geometric morphism $\eta^* : \mathcal{X} \to \mathcal{S}$. If $\mathfrak{X}$ is the spectrum of a separably closed field k, then the ∞-topos $\mathcal{X}$ is equivalent to $\mathcal{S}$ so that $\mathfrak{X}$ has a unique point (up to equivalence). More generally, any map $\phi : \operatorname{Spec} k \to \mathfrak{X}$ where k is a separably closed field determines a point of $\mathfrak{X}$. One of our main results (Proposition 1.1.15) asserts that every point of $\mathfrak{X}$ arises in this way. Moreover, if we demand further that ϕ be *minimal* (see Definition 1.1.10), then the field k is uniquely determined up to (noncanonical) isomorphism.

Recall that a map of spectrally ringed ∞-topoi $\phi : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is said to be *étale* if the underlying geometric morphism $\phi_* : \mathcal{X} \to \mathcal{Y}$ is an étale morphism of ∞-topoi (that is, it induces an equivalence $\mathcal{X} \simeq \mathcal{Y}_{/U}$ for some object $U \in \mathcal{Y}$) and the map of structure sheaves $\phi^* \mathcal{O}_\mathcal{Y} \to \mathcal{O}_\mathcal{X}$ is an equivalence. If $f : A \to B$ is an étale morphism of $\mathbb{E}_\infty$-rings, then the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is an étale morphism of nonconnective spectral Deligne-Mumford stacks. In §1.2, we will prove the converse: every étale morphism between affine nonconnective spectral Deligne-Mumford stacks arises from an étale morphism of $\mathbb{E}_\infty$-rings (Theorem 1.2.1). Using this result, we give a convenient algebraic criterion for an arbitrary map of spectral Deligne-Mumford stacks $\phi : \mathfrak{X} \to \mathfrak{Y}$ to be étale (Corollary 1.2.2).

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack, and let $n \geq 0$ be an integer. We say that $\mathfrak{X}$ is *n-localic* if the ∞-topos $\mathcal{X}$ is n-localic: that is, if $\mathcal{X}$ is equivalent to an ∞-category of the form $\operatorname{Shv}(\mathcal{C})$, where $\mathcal{C}$ is an n-category equipped with a Grothendieck topology. In §1.3, we will give another characterization of the class of n-localic spectral Deligne-Mumford stacks: if $n \geq 1$, then $\mathfrak{X}$ is n-localic if and only if the mapping space $\operatorname{Map}_{\operatorname{Stk}}(\operatorname{Spec} R, \mathfrak{X})$ is n-truncated for every commutative ring R (Proposition 1.3.3).

Recall that a scheme $(X, \mathcal{O}_X)$ is said to be *quasi-compact* if the underlying topological space X is quasi-compact, and to be *quasi-separated* if the intersection $U \cap V$ is quasi-compact for every pair of quasi-compact open subsets $U, V \subseteq X$. In §1.4, we will study analogous conditions in the setting of spectral Deligne-Mumford stacks. We say that a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ is *n-quasi-compact* if the underlying ∞-topos $\mathcal{X}$ is n-coherent, in the sense of Definition VII.3.1. In the case $n = 0$, this is equivalent to the condition that $\mathcal{X}$ be quasi-compact; when $n = 1$, it is analogous to the requirement of being quasi-compact and quasi-separated.

Our discussion in this section barely scratches the surface of the theory of spectral Deligne-Mumford stacks. There are a host other other conditions on a spectral Deligne-Mumford stack $\mathfrak{X}$ (or, more generally, on a morphism $\phi : \mathfrak{X} \to \mathfrak{Y}$) which are play a role in spectral algebraic geometry. In §1.5, we sketch a general paradigm for discussing properties which are *local*: that is, which can be tested after passing to a covering of $\mathfrak{X}$. We give a few examples in §1.5; others will be introduced as the need arises.

1.1 Points of Spectral Deligne-Mumford Stacks

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a spectral Deligne-Mumford stack. It follows from Theorem VII.4.1 that the hypercompletion $\mathcal{X}^\wedge$ has enough points. Our goal in this section is to obtain an explicit description of the points of $\mathcal{X}$. We begin by introducing some terminology.

Definition 1.1.1. Let A be an $\mathbb{E}_\infty$-ring. We will say that A is *strictly Henselian* if the commutative ring $\pi_0 A$ is strictly Henselian, in the sense of Definition VII.7.17.

Remark 1.1.2. An $\mathbb{E}_\infty$-ring A is strictly Henselian if and only if it is strictly Henselian when regarded as a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathcal{S}$, in the sense of Definition VII.8.3.

Proposition 1.1.3. *Let $A \to B$ be a map of $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(1) *The $\mathbb{E}_\infty$-ring B is strictly Henselian.*

(2) *The functor $\mathcal{G}^{\mathrm{nSp}}_{\acute{e}t}(A) \to \mathcal{S}$ determined by B is a $\mathcal{G}^{\mathrm{nSp}}_{\acute{e}t}(A)$-structure on $\mathcal{S}$.*

(3) *Let A' be a compact A-algebra, and suppose we are given a finite collection of étale maps $\{A' \to A'_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat. Then any A-algebra map $A' \to B$ factors (up to homotopy) through A'_α for some index α.*

(4) *Let R be an arbitrary $\mathbb{E}_\infty$-ring and suppose we are given a a finite collection of étale maps $\{R \to R_\alpha\}$ such that $R \to \prod_\alpha R_\alpha$ is faithfully flat. Then any map $R \to B$ factors (up to homotopy) through R_α for some index α.*

(5) *Let $\{B \to B_\alpha\}$ be a finite collection of étale maps such that the induced map $B \to \prod_\alpha B_\alpha$ is faithfully flat. Then there exists an index α and an B-algebra map $B_\alpha \to B$.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) is a special case of Proposition VII.8.13, and the equivalence (2) $\Leftrightarrow$ (3) is tautological. The implication (4) $\Rightarrow$ (3) is obvious, and the converse follows from Proposition VII.8.17. The implication (5) $\Rightarrow$ (4) follows by taking $R = A'$, and implication (4) $\Rightarrow$ (3) by replacing the maps $\{R \to R_\alpha\}$ by the induced map $\{A' \to R_\alpha \otimes_R A'\}$. □

We will need a few elementary facts about strictly Henselian $\mathbb{E}_\infty$-rings.

Proposition 1.1.4. *Let R be a Henselian commutative ring and let $I \subseteq R$ be a proper ideal. Then the quotient R/I is also a Henselian local ring.*

Proof. Let $\mathfrak{m}$ denote the maximal ideal of R. Suppose we are given an étale map $R/I \to R'_0$ and an R/I-algebra map $\phi : R'_0 \to R/\mathfrak{m}$. Using the structure theory of étale maps (see Proposition VII.8.10), we deduce the existence of an étale R-algebra R' such that $R'/IR' \simeq R'_0$. Since R is Henselian, the composite map $R' \to R'/IR' \simeq R'_0 \to R/\mathfrak{m}$ lifts to a map $R' \to R$. Reducing modulo I, we obtain a map $R'_0 \to R/I$ lifting ϕ. □

Corollary 1.1.5. *Let $f : A \to A'$ be a map of $\mathbb{E}_\infty$-rings which induces a surjective ring homomorphism $\pi_0 A \to \pi_0 A'$. If A is strictly Henselian and $A' \neq 0$, then A' is strictly Henselian.*

Proof. This follows from Proposition 1.1.4, since the local rings $\pi_0 A$ and $\pi_0 A'$ have the same residue field. □

Fix an $\mathbb{E}_\infty$-ring A. Let $\mathrm{CAlg}_A^{\text{ét}}$ denote the full subcategory of CAlg_A spanned by the $\mathbb{E}_\infty$-algebras which are étale over A. Using the construction described in §V.2.2, we see that the underlying topos of $\mathrm{Spec}(A)$ can be identified with $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})$, where we regard $(\mathrm{CAlg}_A^{\text{ét}})^{op}$ as endowed with the étale topology. By definition, a *point* of $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})$ is a geometric morphism $f^* : \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op}) \to \mathcal{S}$. Composition with the Yoneda embedding $(\mathrm{CAlg}_A^{\text{ét}})^{op} \hookrightarrow \mathcal{P}((\mathrm{CAlg}_A^{\text{ét}})^{op})$ induces an equivalence between the ∞-category of points of $\mathcal{P}((\mathrm{CAlg}_A^{\text{ét}})^{op})$ with the full subcategory $\mathrm{Ind}(\mathrm{CAlg}_A^{\text{ét}}) \subseteq \mathrm{Fun}((\mathrm{CAlg}_A^{\text{ét}})^{op}, \mathcal{S}) = \mathcal{P}(\mathrm{CAlg}_A^{\text{ét}})$. We will say that an A-algebra B is Ind-*étale* if it is a filtered colimit of étale A-algebras. Since every étale A-algebra is a compact object of CAlg_A (Corollary A.7.5.4.4), we can identify $\mathrm{Ind}(\mathrm{CAlg}_A^{\text{ét}})$ with a full subcategory $\mathrm{CAlg}_A^{\mathrm{Ind}-\text{ét}} \subseteq \mathrm{CAlg}_A$ spanned by the Ind-étale A-algebras.

Proposition 1.1.6. *Let A be an $\mathbb{E}_\infty$-ring, let B be an* Ind*-étale A-algebra, and let $\eta_* : \mathcal{S} \subseteq \mathcal{P}((\mathrm{CAlg}_A^{\acute{e}t})^{op})$ be the geometric morphism determined by B. The following conditions are equivalent:*

(1) *The geometric morphism η_* factors through* $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\acute{e}t})^{op})$.

(2) *The $\mathbb{E}_\infty$-ring B is strictly Henselian.*

Proof. Using Proposition T.6.2.3.20, we see that (1) is equivalent to the following condition:

($*$) Let A' be an étale A-algebra, and suppose we are given a finite collection of étale maps $\{A' \to A'_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat. Then any A-algebra map $A' \to B$ factors (up to homotopy) through A'_α for some index α.

The implication (2) $\Rightarrow$ ($*$) follows immediately from Proposition 1.1.3. Conversely, suppose that ($*$) is satisfied. We will prove that B is strictly Henselian by verifying condition (5) of Proposition 1.1.3. Suppose we are given a finite collection of étale morphisms $\{B \to B_\alpha\}$ which induce a faithfully flat map $\theta : B \to \prod_\alpha B_\alpha$; we wish to show that there is an index α and a map of B-algebras $B_\alpha \to B$.

Write B as a filtered colimit of étale A-algebras $B(\beta)$. Using the structure theorem for étale morphisms (Proposition VII.8.10), we can choose an index β and étale morphisms $\{B(\beta) \to B^{(}\beta)_\alpha\}$ such that $B_\alpha \simeq B \otimes_{B(\beta)} B(\beta)_\alpha$. The image of the induced map

$$\coprod_\alpha \operatorname{Spec}^{\mathrm{Z}} B(\beta)_\alpha \to \operatorname{Spec}^{\mathrm{Z}} B(\beta)$$

is a quasi-compact open subset $U \subseteq \operatorname{Spec}^{\mathrm{Z}} B(\beta)$ (Proposition VII.0.2)), corresponding to a radical ideal $I \subseteq B(\beta)$. Since θ is faithfully flat, the image of I generates the unit ideal in $\pi_0 B$. Changing our index β, we may suppose that I is the unit ideal, so that the map $B(\beta) \to \prod_\alpha B(\beta)_\alpha$ is faithfully flat. It follows from $(*)$ that there exists an index α and a map of $B(\beta)$-algebras $B(\beta)_\alpha \to B$, which determines a map of B-algebras $B_\alpha \to B$. □

Proposition 1.1.6 yields the following description for points of an *affine* spectral Deligne-Mumford stacks:

Corollary 1.1.7. *Let A be an $\mathbb{E}_\infty$-ring, and let $\mathcal{C}$ be the full subcategory of* $\operatorname{Fun}(\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\acute{e}t})^{op}), \mathcal{S})$ *spanned by those functors which are left exact and preserve small colimits. Then composition with the Yoneda embedding* $(\mathrm{CAlg}_A^{\acute{e}t})^{op} \to \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\acute{e}t})^{op})$ *induces an equivalence of $\mathcal{C}$ with the full subcategory of* $\operatorname{Ind}(\mathrm{CAlg}_A^{\acute{e}t}) \simeq \mathrm{CAlg}_A^{\mathrm{Ind}-\acute{e}t} \subseteq \mathrm{CAlg}_A$ *spanned by those A-algebras which are strictly Henselian and* Ind*-étale over A.*

Remark 1.1.8. Let A be an $\mathbb{E}_\infty$-ring and let $(\mathcal{X}, \mathcal{O}) = \operatorname{Spec}(A)$ be the corresponding spectral Deligne-Mumford stack. Let $f^* : \mathcal{X} \to \mathcal{S}\mathrm{hv}(*) = \mathcal{S}$ be a point of $\mathcal{X}$, which corresponds under the equivalence of Corollary 1.1.7 to a strictly Henselian A-algebra $A' \simeq \varinjlim A'_\alpha$, where each A'_α is an étale A-algebra. Let $\mathcal{C}$ be an arbitrary compactly generated ∞-category, and let $\mathcal{F} \in \mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathcal{C}}((\mathrm{CAlg}_A^{\acute{e}t})^{op})$. Unwinding the definitions, we obtain a canonical equivalence $f^* \mathcal{F} \simeq \varinjlim \mathcal{F}(A'_\alpha)$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(*) \simeq \mathcal{C}$. In particular, A' can be identified with the stalk of the structure sheaf $f^* \mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(*) \simeq \mathrm{CAlg}$.

Remark 1.1.9. Let A be an $\mathbb{E}_\infty$-ring and let $(\mathcal{X}, \mathcal{O}) = \operatorname{Spec}(A)$. Let B be a strictly Henselian $\mathbb{E}_\infty$-ring; let us identify B with the corresponding object of $\mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(*)$, so that $(\mathcal{S}\mathrm{hv}(*), B)$ is an object of $\mathrm{RingTop}_{\acute{e}t}$. We then have a canonical homotopy equivalence

$$\operatorname{Map}_{\mathrm{RingTop}_{\acute{e}t}}((\mathcal{S}\mathrm{hv}(*), B), (\mathcal{X}, \mathcal{O})) \to \operatorname{Map}_{\mathrm{CAlg}}(A, B).$$

Under the equivalence of Corollary 1.1.7, this assertion translates as follows: every map of $\mathbb{E}_\infty$-rings $f : A \to B$ factors uniquely as a composition

$$A \xrightarrow{f'} A' \xrightarrow{f''} B,$$

where A' is strictly Henselian and Ind-étale over A, and the map f'' is local. We will refer to A' as the *Henselization of* A along the map f.

It is often convenient to describe points of spectral Deligne-Mumford stacks by Henselizing the spectra of separably closed fields.

Definition 1.1.10. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. A *geometric point* of $\mathfrak{X}$ is a morphism of spectral Deligne-Mumford stacks $\eta : \mathfrak{X}_0 \to \mathfrak{X}$, where $\mathfrak{X}_0$ is the spectrum of a separably closed field k. We will say that a geometric point $\eta : \mathfrak{X}_0 \to \mathfrak{X}$ is *minimal* if it factors as a composition

$$\mathfrak{X}_0 \xrightarrow{\eta'} \operatorname{Spec} A \xrightarrow{\eta''} \mathfrak{X}$$

where η'' is étale and η' induces a map of commutative rings $\phi : \pi_0 A \to k$ having kernel $\mathfrak{p} \subseteq \pi_0 A$, which exhibits k as a separable closure of the residue field $\kappa(\mathfrak{p})$.

For each object $\mathfrak{X} \in \mathrm{Stk}$, we let $\mathrm{GPt}(\mathfrak{X})$ denote the full subcategory of $\mathrm{Stk}_{/\mathfrak{X}}$ spanned by the minimal geometric points $\eta : \mathfrak{X}_0 \to \mathfrak{X}$.

Remark 1.1.11. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\eta : \operatorname{Spec} k \to \mathfrak{X}$ be a minimal geometric point of $\mathfrak{X}$. For *every* factorization

$$\operatorname{Spec} k \xrightarrow{\eta'} \operatorname{Spec} A \xrightarrow{\eta''} \mathfrak{X}$$

of η where η'' is étale, the map η' exhibits k as a separable closure of the residue field $\kappa(\mathfrak{p})$ of some prime ideal $\mathfrak{p} \subseteq \pi_0 A$.

Remark 1.1.12. Let $\phi : \mathfrak{U} \to \mathfrak{X}$ be an étale map of spectral Deligne-Mumford stacks, and suppose we are given a geometric point $\eta : \operatorname{Spec} k \to \mathfrak{U}$. Then η is a minimal geometric point of $\mathfrak{U}$ if and only if $\phi \circ \eta$ is a minimal geometric point of $\mathfrak{X}$.

Remark 1.1.13. Suppose we are given a commutative diagram of fields

$$\begin{tikzcd} & k \arrow[dl, "\phi"'] \arrow[dr, "\psi"] & \\ k' \arrow[rr, "\theta"] & & k''. \end{tikzcd}$$

If ϕ and ψ exhibit k' and k'' as separable closures of k, then θ is an isomorphism. It follows that if $\mathfrak{X}$ is a spectral Deligne-Mumford stack, then every morphism between minimal geometric points of $\mathfrak{X}$ is an equivalence: that is, the ∞-category $\mathrm{GPt}(\mathfrak{X})$ is a Kan complex.

Let $\eta : \mathfrak{X}_0 \to \mathfrak{X}$ be a geometric point of a connective spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Then $\mathfrak{X}_0$ is the spectrum of a separably closed field, so that underlying ∞-topos of $\mathfrak{X}_0$ is canonically equivalent to $\mathcal{S}$. Consequently, the pullback functor η^* can be viewed as a geometric morphism $\mathcal{X} \to \mathcal{S}$.

Remark 1.1.14. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne stack. Let $U \in \mathcal{X}$ be an object of let $\mathfrak{U} = (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$. The étale map of spectral Deligne-Mumford stacks $\phi : \mathfrak{U} \to \mathfrak{X}$ induces a map of Kan complexes $\theta : \mathrm{GPt}(\mathfrak{U}) \to \mathrm{GPt}(\mathfrak{X})$. Using Remarks 1.1.12 and V.2.3.4, we deduce that the homotopy fiber of θ over a point $\eta \in \mathrm{GPt}(\mathfrak{X})$ can be identified with the space $\eta^*(U) \in \mathcal{S}$.

Proposition 1.1.15. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The construction $(\eta : \mathfrak{X}_0 \to \mathfrak{X}) \mapsto (\eta^* \in \mathrm{Fun}(\mathcal{X}, \mathcal{S}))$ determines an equivalence from the ∞-category $\mathrm{GPt}(\mathfrak{X})$ of minimal geometric points of $\mathfrak{X}$ to the subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{S})$ whose objects are functors which preserve small colimits and finite limits and whose morphisms are equivalences.*

Remark 1.1.16. In other words, the ∞-category of minimal geometric points of a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ can be identified with the underlying Kan complex of the ∞-category of points of the the ∞-topos $\mathcal{X}$ (that is, geometric morphisms $\eta^* : \mathcal{X} \to \mathcal{S}$).

Proof. For every object $U \in \mathcal{X}$, let $\mathfrak{X}_{/U}$ denote the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U)$. Let $\mathrm{Pt}(U)$ denote the subcategory of $\mathrm{Fun}(\mathcal{X}_{/U}, \mathcal{S})$ whose objects are functors which preserve small colimits and finite limits, and whose morphisms are equivalences. Let $\mathbf{1}$ denote the final object of $\mathcal{X}$. The proof of Lemma V.2.3.11 shows that we can write $\mathbf{1}$ as a colimit $\varinjlim U_\alpha$, where each $\mathfrak{X}_{/U_\alpha}$ is an affine spectral Deligne-Mumford stack. We have a commutative diagram

$$\begin{tikzcd} \varinjlim \mathrm{GPt}(\mathfrak{X}_{/U_\alpha}) \arrow[r] \arrow[d] & \varinjlim \mathrm{Pt}(U_\alpha) \arrow[d] \\ \mathrm{GPt}(\mathfrak{X}) \arrow[r] & \mathrm{Pt}(\mathbf{1}). \end{tikzcd}$$

Using Remark 1.1.14 and Proposition T.6.3.5.5, we deduce that the vertical maps are equivalences. Consequently, to prove that the map $\mathrm{GPt}(\mathfrak{X}) \to \mathrm{Pt}(\mathbf{1})$ is a homotopy equivalence, it suffices to show that each of the maps $\mathrm{GPt}(\mathfrak{X}_{/U_\alpha}) \to \mathrm{Pt}(U_\alpha)$ is a homotopy equivalence. We may therefore replace $\mathfrak{X}$ by $\mathfrak{X}_{/U_\alpha}$ and thereby reduce to the case where $\mathfrak{X} = \operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A.

Let $\mathfrak{X}_0 = \mathrm{Spec}(\pi_0 A)$. The underlying ∞-topoi of $\mathfrak{X}$ and $\mathfrak{X}_0$ are the same, and the canonical map

$$\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, k, \mathfrak{X}_0) \to \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, k, \mathfrak{X})$$

is a homotopy equivalence for every discrete $\mathbb{E}_\infty$-ring k. We may therefore replace A by $\pi_0 A$ and thereby reduce to the case where A is a discrete $\mathbb{E}_\infty$-ring.

Let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{S})$ spanned by those functors which preserve small colimits and finite limits. Using Corollary 1.1.7, we can identify $\mathrm{Fun}^*(\mathcal{X}, \mathcal{S})$ with the full subcategory of $\mathcal{C} \subseteq \mathrm{CAlg}_A$ whose objects are A-algebras B which are strictly Henselian and can be written as a filtered colimit of étale A-algebras. We can identify $\mathrm{GPt}(\mathfrak{X})$ with the nerve of the groupoid consisting of those A-algebras k which are separable closures of some residue field of A. We will denote the functor $\theta : \mathrm{GPt}(\mathfrak{X}) \to \mathcal{C}$ by $k \mapsto A_k$. This construction can be characterized by the following universal property: for every object $k \in \mathrm{GPt}(\mathfrak{X})$ and every étale A-algebra B, we have a canonical bijection

$$\mathrm{Hom}_A(B, A_k) \simeq \mathrm{Hom}_k(B \otimes_A k, k) \simeq \mathrm{Hom}_A(B, k).$$

Here $\mathrm{Hom}_R(R', R'')$ denotes the set of R-algebra maps from R' to R''. Note that this bijection extends naturally to the case where B is a filtered colimit of étale A-algebras.

For every object $B \in \mathcal{C}$, let $\mathfrak{p}_B$ denote the inverse image in A of the maximal ideal of B. Any morphism $B \to B'$ in $\mathcal{C}$ determines an inclusion of prime ideals $\mathfrak{p}_{B'} \subseteq \mathfrak{p}_B$. We let $\mathcal{C}_0$ denote the subcategory of $\mathcal{C}$ consisting of those morphisms $B \to B'$ for which $\mathfrak{p}_B = \mathfrak{p}_{B'}$. Since $\mathcal{C}^0$ contains all equivalences in $\mathcal{C}$, the map θ factors through $\mathcal{C}_0$. We will show that θ induces an equivalence $\mathrm{GPt}(\mathfrak{X}) \to \mathcal{C}_0$. From this, it will follow that $\mathcal{C}_0$ is a Kan complex, hence that $\mathcal{C}_0$ is the largest Kan complex contained in $\mathcal{C}$ and therefore that θ exhibits $\mathrm{GPt}(\mathfrak{X})$ as equivalent to the largest Kan complex contained in $\mathcal{C}$.

Let $k \in \mathrm{GPt}(\mathfrak{X})$, and let $\mathfrak{p} = \mathfrak{p}_{A_k}$. Note that $\mathrm{Hom}_A(A[u^{-1}], A_k)$ is empty if and only if $u \in \mathfrak{p}$. It follows that $\mathfrak{p}$ is the kernel of the map $A \to k$, so that k is a separable closure of the residue field $\kappa(\mathfrak{p})$ of A. Since A_k is a filtered colimit of étale A-algebras, the quotient $(A_k/\mathfrak{p}A_k) \simeq A_k \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of finite étale algebras over $\kappa(\mathfrak{p})$. Since A_k is strictly Henselian, the quotient $A_k/\mathfrak{p}A_k$ is also strictly Henselian and therefore a separable closure of the residue field $\kappa(\mathfrak{p})$. Let k' be another separable closure of $\kappa(\mathfrak{p})$. The canonical map

$$v : \mathrm{Hom}_A(k, k') \simeq \mathrm{Map}_{\mathrm{GPt}(\mathfrak{X})}(k, k') \to \mathrm{Map}_{\mathcal{C}}(A_k, A_{k'}) \simeq \mathrm{Hom}_A(A_k, A_{k'}) \simeq \mathrm{Hom}_A(A_k, k')$$

is given by composition with a map $v_0 : A_k/\mathfrak{p}A_k \to k$. Here v_0 is a $\kappa(\mathfrak{p})$-algebra map between separable closures of $\kappa(\mathfrak{p})$, and therefore an isomorphism. It follows that v is bijective, which proves that $\theta : \mathrm{GPt}(\mathfrak{X}) \to \mathcal{C}_0$ is fully faithful.

It remains to prove that θ is essentially surjective. Let $B \in \mathcal{C}$ and let $\mathfrak{p} = \mathfrak{p}_B$. Then $k = B/\mathfrak{p}B \simeq B \otimes_A \kappa(\mathfrak{p})$ is a filtered colimit of étale $\kappa(\mathfrak{p})$-algebras. Since k is strictly Henselian, we deduce that k is a separable closure of $\kappa(\mathfrak{p})$. In particular, $\mathfrak{p}B$ is the maximal ideal of B, and we can identify k with an object of $\mathrm{GPt}(\mathfrak{X})$. For every étale A-algebra B', we have a canonical map

$$\mathrm{Hom}_A(B', B) \to \mathrm{Hom}_A(B', B/\mathfrak{p}B) = \mathrm{Hom}_A(B', k) \simeq \mathrm{Hom}_A(B', A_k).$$

Since B is Henselian, this map is bijective. Since B and A_k can both be obtained as a filtered colimit of étale A-algebras, we conclude that $B \simeq A_k$. □

Remark 1.1.17. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, and suppose we are given a morphism $\eta : \mathrm{Spec}\, k \to \mathfrak{X}$, where k is a separably closed field. Then η factors as a composition

$$\mathrm{Spec}\, k \xrightarrow{\eta'} \mathfrak{U} \xrightarrow{\eta''} \mathfrak{X},$$

where $\mathfrak{U}$ is affine and η'' is étale. Write $\mathfrak{U} = \mathrm{Spec}\, A$, so that η' determines a map of $\mathbb{E}_\infty$-rings $A \to k$. The image of the map of commutative rings $\pi_0 A \to k$ generates a subfield of $k' \subseteq k$. Let $k_0 \subseteq k$ denote the separable closure of k' in k. Then η factors as a composition

$$\mathrm{Spec}\, k \to \mathrm{Spec}\, k_0 \xrightarrow{\eta_0} \mathfrak{X},$$

where η_0 is a minimal geometric point of $\mathfrak{X}$ and the inclusion $k_0 \subseteq k$ is an extension of separably closed fields.

Definition 1.1.18. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We will say that $\mathfrak{X}$ is *empty* if $\mathcal{X}$ is a contractible Kan complex (that is, if $\mathcal{X}$ is equivalent to the ∞-category of sheaves on the empty topological space).

Lemma 1.1.19. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *The spectral Deligne-Mumford stack $\mathfrak{X}$ is not empty.*

(2) *There exists a nonzero connective $\mathbb{E}_\infty$-ring A and an étale map $\operatorname{Spec} A \to \mathfrak{X}$.*

(3) *There exists a minimal geometric point $\operatorname{Spec} k \to \mathfrak{X}$.*

Proof. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (1) are obvious. We prove that (1) $\Rightarrow$ (2). Assume that (1) is satisfied. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so that there exists an object of $\mathcal{X}$ which is not initial. It follows that there exists an affine object $U \in \mathcal{X}$ which is not initial. Then $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} |U)$ is equivalent to $\operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A. We therefore have an étale map $\operatorname{Spec} A \to \mathfrak{X}$. Since U is not an initial object of $\mathcal{X}$, A is nonzero; this proves (2). $\square$

Proposition 1.1.20. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(1) *Let $\mathcal{X}$ and $\mathcal{Y}$ denote the underlying ∞-topoi of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively. Then every geometric morphism of ∞-topoi $\eta_* : \mathcal{S} \to \mathcal{Y}$ factors through the geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$.*

(2) *For every field k and every map $\eta : \operatorname{Spec} k \to \mathfrak{Y}$, there exists a field extension K of k such that the composite map $\operatorname{Spec} K \to \operatorname{Spec} k \to \mathfrak{Y}$ factors through f.*

(3) *For every field k and every map $\eta : \operatorname{Spec} k \to \mathfrak{Y}$, the fiber product $\operatorname{Spec} k \times_{\mathfrak{Y}} \mathfrak{X}$ is nonempty.*

Proof. The implication (2) $\Rightarrow$ (3) is obvious, and the implication (3) $\Rightarrow$ (1) follows from Lemma 1.1.19. We will show that (1) $\Rightarrow$ (2). Let $\eta : \operatorname{Spec} k \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks. We wish to show that, after enlarging k if necessary, the map η factors through $\mathfrak{X}$. Without loss of generality, we may assume that k is separably closed. Note that η determines a geometric morphism of ∞-topoi $\eta_* : \mathcal{S} \to \mathcal{Y}$. Using condition (1), we deduce that η_* factors as a composition

$$\mathcal{S} \xrightarrow{\eta'_*} \mathcal{X} \xrightarrow{f_*} \mathcal{Y}.$$

According to Proposition 1.1.15, the geometric morphism η'_* is determined by a minimal geometric point $\eta' : \operatorname{Spec} k' \to \mathfrak{X}$. Using Remark 1.1.17, we see that η and $f \circ \eta'$ admit factorizations

$$\operatorname{Spec} k \to \operatorname{Spec} k_0 \xrightarrow{\eta_0} \mathfrak{Y}$$

$$\operatorname{Spec} k' \to \operatorname{Spec} k'_0 \xrightarrow{\eta'_0} \mathfrak{Y},$$

where k_0 and k'_0 are separably closed subfields of k and k', respectively, and η_0 and η'_0 are minimal geometric points of $\mathfrak{Y}$. By construction, the pushforward functors $(\eta_0)_*, (\eta'_0)_* : \mathcal{S} \to \mathcal{Y}$ are homotopic. It follows from Proposition 1.1.15 that there is an isomorphism of fields $k_0 \simeq k'_0$ such that the diagram

$$\begin{tikzcd} \operatorname{Spec} k_0 \arrow[rr] \arrow[dr, "\eta_0"'] & & \operatorname{Spec} k'_0 \arrow[dl, "\eta'_0"] \\ & \mathfrak{Y} & \end{tikzcd}$$

commutes up to homotopy. Let K be any residue field of the tensor product $k \otimes_{k_0} k'$. Then the composite map $\operatorname{Spec} K \to \operatorname{Spec} k \xrightarrow{\eta} \mathfrak{Y}$ factors through $f \circ \eta$, and therefore lifts to a map $\operatorname{Spec} K \to \mathfrak{X}$. $\square$

Definition 1.1.21. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *surjective* if it satisfies the equivalent conditions of Proposition 1.1.20.

Remark 1.1.22. Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y}. \end{tikzcd}$$

If f is surjective, then f' is surjective.

Remark 1.1.23. Suppose we are given maps of spectral Deligne-Mumford stacks

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z},$$

where f is surjective. Then g is surjective if and only if $g \circ f$ is surjective.

Remark 1.1.24. Let $f : \mathfrak{X} \to \mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be an étale morphism of spectral Deligne-Mumford stacks, so that $\mathfrak{X} \simeq (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} | U)$ for some object $U \in \mathfrak{Y}$. The following conditions are equivalent:

(i) The object $U \in \mathcal{Y}$ is 0-connective.

(ii) The map f is surjective.

This follows immediately from Theorem VII.4.1 (since the ∞-topos $\mathcal{Y}$ is locally coherent).

1.2 Étale Morphisms

Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of nonconnective spectral Deligne-Mumford stacks. We say that f is *étale* if it is étale when regarded as a morphism of spectrally ringed ∞-topoi. In other words, we say that f is étale if the underlying geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is an étale morphism of ∞-topoi and $f^* \mathcal{O}_{\mathcal{Y}} \to \mathcal{O}_{\mathcal{X}}$ is an equivalence of sheaves of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Our goal in this section is to relate the theory of étale morphisms in Stk to the theory of étale morphisms in CAlg. Our main result can be stated as follows:

Theorem 1.2.1. *Let $\phi : A \to B$ be a map of $\mathbb{E}_\infty$-rings. Then ϕ is étale if and only if the induced map* $\operatorname{Spec} B \to \operatorname{Spec} A$ *is an étale map of nonconnective spectral Deligne-Mumford stacks.*

Corollary 1.2.2. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map between nonconnective spectral Deligne-Mumford stacks. The following conditions are equivalent:*

(i) *The map f is étale.*

(ii) *For every commutative diagram*

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

in which the horizontal maps are étale, the underlying map of $\mathbb{E}_\infty$-rings $A \to B$ is étale.

Proof. The implication $(ii) \Rightarrow (i)$ is obvious, and the converse follows from Theorem 1.2.1. □

The proof of Theorem 1.2.1 will occupy our attention throughout this section. We begin by treating the case where A is a field (regarded as a discrete $\mathbb{E}_\infty$-ring).

Lemma 1.2.3. *Let k be a field. Suppose we are given an étale map of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \operatorname{Spec} k$. If $\mathfrak{X}$ is affine, then $\mathfrak{X} \simeq \operatorname{Spec} k'$ where k' is an étale k-algebra.*

Proof. Write $\operatorname{Spec} k = (\mathcal{X}, \mathcal{O})$, so that $\mathfrak{X} \simeq (\mathcal{X}_{/U}, \mathcal{O}|U)$ for some object $U \in \mathcal{X}$. Write $\mathfrak{X} = \operatorname{Spec} A$ for some $\mathbb{E}_\infty$-algebra A over k. Choose a separable closure $\overline{k}$ of k, so that the category of étale sheaves of sets on $\mathrm{CAlg}_k^{\text{ét}}$ can be identified with the category of sets equipped with a continuous action of the Galois group $G = \mathrm{Gal}(\overline{k}/k)$. Note that A is connective. For every discrete commutative ring R, the mapping space $\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X})$ is discrete. It follows that U is a discrete object of $\mathcal{X}$, which we can identify with a set equipped with a continuous action of G. Decomposing this set into G-orbits, we deduce that $\mathfrak{X}$ is a coproduct of spectral Deligne-Mumford stacks of the form $\operatorname{Spec} k_\alpha$, where each k_α is a finite separable extension of k. Since $\mathfrak{X}$ is quasi-compact, this coproduct is finite, from which it follows that $A \simeq \prod_\alpha k_\alpha$ is an étale k-algebra. □

Remark 1.2.4. Suppose that we are given a map of connective $\mathbb{E}_\infty$-rings $\phi : A \to B$ such that the induced map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ is étale. For every point $x \in \operatorname{Spec}^{\mathrm{Z}} A$, let $\kappa(x)$ denote the residue field of $\pi_0 A$ at the point x, and let $B_x = B \otimes_A \kappa(x)$. Then $\operatorname{Spec} B_x \to \operatorname{Spec} \kappa(x)$ is a pullback of f and therefore étale. Using Lemma 1.2.3, we deduce that B_x is a finite étale κ_x-algebra. In particular, we can identify B_x with a finite-dimensional vector space over κ_x. We will denote the dimension of this vector space by $e_\phi(x)$.

Lemma 1.2.5. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings, and suppose that the induced map $\operatorname{Spec} B \to \operatorname{Spec} A$ is étale. Then the function $e_\phi : \operatorname{Spec}^{\mathrm{Z}} A \to \mathbf{Z}$ of Remark 1.2.4 is bounded above.*

Proof. We may assume without loss of generality that A is discrete. Since ϕ is étale and $\operatorname{Spec} B$ is quasi-compact, we can choose a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} C \arrow[rr, "g"] \arrow[dr] & & \operatorname{Spec} B \arrow[dl] \\
& \operatorname{Spec} A &
\end{tikzcd}$$

where g is an étale surjection and the induced map $\psi : A \to C$ exhibits C as an étale A-algebra. Using Theorem VII.7.11, we can choose a finite A-algebra C' and an A-algebra homomorphism $C' \to C$ which induces an open immersion of schemes. Choose a surjection of A-modules $A^n \to C'$ for some integer n. For each $x \in \operatorname{Spec}^{\mathrm{Z}} A$, we have

$$e_\phi(x) \leq e_\psi(x) \leq \dim_{\kappa(x)} \mathrm{Tor}_0^A(C', \kappa(x)) \leq n.$$

□

Lemma 1.2.6. *Let $\mathcal{X}$ be an n-localic ∞-topos. Then any closed subtopos of $\mathcal{X}$ is also n-localic.*

Proof. Any closed subtopos of $\mathcal{X}$ is a topological localization of $\mathcal{X}$ (Proposition T.7.3.2.4) and therefore also n-localic (Proposition T.6.4.5.9). □

Proposition 1.2.7. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings which induces a surjection $\pi_0 A \to \pi_0 B$. Then the corresponding map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ induces a closed immersion of underlying ∞-topoi.*

Proof. Using the results of §V.2.2, we can identify $\mathcal{X}_A$ with $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op}) \subseteq \mathrm{Fun}(\mathrm{CAlg}_A^{\text{ét}}, \mathcal{S})$ and $\mathcal{X}_B$ with $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op}) \subseteq \mathrm{Fun}(\mathrm{CAlg}_B^{\text{ét}}, \mathcal{S})$. Under these identifications, f_* is given by composition with the functor

$$\mathrm{CAlg}_A^{\text{ét}} \to \mathrm{CAlg}_B^{\text{ét}}$$

$$A' \mapsto A' \otimes_A B.$$

Let $U \in \mathrm{CAlg}_A^{\text{ét}} \to \mathcal{S}$ be the functor described by the formula

$$U(A') = \begin{cases} \Delta^0 & \text{if } A' \otimes_A B \simeq 0 \\ \emptyset & \text{otherwise.} \end{cases}$$

Then U is a (-1)-truncated object of $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})$, and the pushforward functor f_* carries the ∞-category $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op})$ into $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})/U$. We will complete the proof by showing that the adjoint functors

$$\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})/U \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op}$$

are mutually inverse equivalences of ∞-categories. Note that $\mathrm{CAlg}_A^{\text{ét}}$ and $\mathrm{CAlg}_B^{\text{ét}}$ are equivalent to the nerves of categories, by Theorem A.7.5.4.2. It follows that $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})$ and $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op})$ are 1-localic ∞-topoi. Using Lemma 1.2.6, we conclude that $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})/U$ is also a 1-localic ∞-topos. Consequently, it will suffice to show that the adjoint functors (f^*, f_*) induce mutually inverse equivalences when restricted to 1-truncated objects of $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})/U$ and $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op})$.

We begin by showing that the functor f_* is conservative. Let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a morphism in $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op})$ such that $f_*(\alpha)$ is an equivalence. We will show that α induces an equivalence $\mathcal{F}(B') \to \mathcal{F}'(B')$ for every étale B-algebra B'. Using Proposition VII.8.10, we deduce the existence of a pushout diagram

$$\begin{tikzcd}
A\{x_1, \ldots, x_m\} \arrow[r, "g"] \arrow[d] & B \arrow[d] \\
A\{y_1, \ldots, y_m\}[\Delta^{-1}] \arrow[r] & B',
\end{tikzcd}$$

where the left vertical map carries each x_i to some polynomial $f_i(y_1, \ldots, y_m) \in \pi_0 A\{y_1, \ldots, y_m\}$ and $\Delta \in \pi_0 A\{y_1, \ldots, y_m\}$ denotes the determinant of the Jacobian matrix $[\frac{\partial f_i}{\partial y_j}]_{1 \leq i,j \leq m}$. Since the map $A \to B$ is surjective on π_0, the map g factors through A. Set $A' = A \otimes_{A\{x_1,\ldots,x_m\}} A\{y_1, \ldots, y_m\}[\Delta^{-1}]$. Then A' is an étale A-algebra. Since $f_*(\alpha)$ is an equivalence, we deduce that

$$\mathcal{F}(B') \simeq (f_* \mathcal{F})(A') \to (f_* \mathcal{F}')(A') \simeq \mathcal{F}'(B')$$

is an equivalence, as desired.

To complete the proof, it will suffice to show that if $\mathcal{F}$ is a 1-truncated object of $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})/U$, then the unit map $u : \mathcal{F} \to f_* f^* \mathcal{F}$ is an equivalence. Since both $\mathcal{F}$ and $f_* f^* \mathcal{F}$ are 1-truncated, they are hypercomplete: it will therefore suffice to show that the map u is ∞-connective. According to Theorem VII.4.1, the ∞-topos $\mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})^\wedge$ has enough points. It will therefore suffice to show that for every geometric morphism $\eta^* : \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op}) \to \mathcal{S}$, the map $\eta^*(u)$ is a homotopy equivalence.

According to Corollary 1.1.7, the geometric morphism η^* corresponds to a strictly Henselian A-algebra A' which can be written as a filtered colimit $\varinjlim A'_\alpha$, where each A'_α is étale over A. More precisely, the functor η^* is given by the formula $\eta^* \mathcal{G} \simeq \varinjlim \mathcal{G}(A'_\alpha)$. There are two cases to consider:

(1) Suppose that $A' \otimes_A B \simeq 0$. Then $1 = 0$ in $\pi_0(A' \otimes_A B) \simeq \varinjlim \pi_0(A'_\alpha \otimes_A B)$, so that $A'_\alpha \otimes_A B \simeq 0$ for some α. Reindexing our diagram, we may suppose that $A'_\alpha \otimes_A B \simeq 0$ for all α. Thus $\eta^* \mathcal{G} \simeq \varinjlim \mathcal{G}(A'_\alpha)$ is contractible whenever $\mathcal{G} \in \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_A^{\text{ét}})^{op})/U$. In particular, $\eta^*(u)$ is a map between contractible spaces and therefore a homotopy equivalence.

(2) The tensor product $A' \otimes_A B \neq 0$. Note that $A' \otimes_A B \simeq \varinjlim A'_\alpha \otimes_A B$ is a filtered colimit of étale B-algebras. The map $A' \to A' \otimes_A B$ induces a surjection $\pi_0 A' \to \pi_0(A' \otimes_A B)$. It follows from Proposition 1.1.4 that $A' \otimes_A B$ is strictly Henselian, and therefore determines a map $\eta'^* : \mathcal{S}\mathrm{hv}((\mathrm{CAlg}_B^{\text{ét}})^{op}) \to \mathcal{S}$. Moreover, the composite map $(\mathcal{S}\mathrm{hv}(*), A' \otimes_A B) \to (\mathcal{S}\mathrm{hv}(*), A') \to (\mathcal{X}_A, \mathcal{O}_A)$ determines the same point of $\mathcal{X}_A$; it follows that $\eta^* \simeq \eta'^* \circ f^*$. We therefore have a chain of equivalences

$$\begin{aligned}
\eta^*(f_* f^* \mathcal{F}) &\simeq \varinjlim (f_* f^* \mathcal{F})(A'_\alpha) \\
&\simeq \varinjlim (f^* \mathcal{F})(A'_\alpha \otimes_A B) \\
&\simeq \eta'^* f^* \mathcal{F} \\
&\simeq \eta^* \mathcal{F}.
\end{aligned}$$

whose composition is a homotopy inverse to $\eta^*(u)$.

$\square$

Proposition 1.2.8. *Let $\mathcal{X}$ be an ∞-topos containing a (-1)-truncated object U, and let $i_* : \mathcal{X}/U \to \mathcal{X}$ be the corresponding closed immersion of ∞-topoi. The following conditions are equivalent:*

(1) *The geometric morphism i_* is étale.*

(2) *The (-1)-truncated object U is complemented: that is, there exists an object $U' \in \mathcal{X}$ such that the coproduct $U \amalg U'$ is a final object of $\mathcal{X}$.*

Proof. The implication (2) $\Rightarrow$ (1) is clear: if $U \amalg U'$ is a final object of $\mathcal{X}$, then the construction $X \mapsto U \amalg X$ determines an equivalence of ∞-categories $\mathcal{X}_{/U'} \to \mathcal{X}/U$. Conversely, suppose that (2) is satisfied. Then the pullback functor $i^* : \mathcal{X} \to \mathcal{X}/U$ admits a left adjoint $i_!$. Let $\mathbf{1}$ denote a final object of $\mathcal{X}$ and let $V = i_! i^* \mathbf{1}$. Then

$$U \times V \simeq U \times i_! i^* \mathbf{1} \simeq i_!(i^* U \times i^* \mathbf{1}) \simeq i_! i^*(U).$$

Since $i^*(U)$ is an initial object of $\mathcal{X}/U$, the object $U \times V \simeq i_! i^* U$ is an initial object of $\mathcal{X}$. Let $U' = \tau_{\leq -1} V$, so that $U \times U'$ is an initial object of $\mathcal{X}$. It follows that $U \amalg U'$ is also a (-1)-truncated object of $\mathcal{X}$. The identity map $\mathrm{id} : i_! i^* \mathbf{1} \to V$ induces a map $i^* \mathbf{1} \to i^* V$ in $\mathcal{X}/U$, which determines a map

$$\mathbf{1} \simeq i_* i^* \mathbf{1} \to i_* i^* V \simeq U \amalg_{U \times V} V \simeq U \amalg V.$$

Composing with the projection map $V \to \tau_{\leq -1} V$, we obtain a map $\mathbf{1} \to U \amalg U'$, so that $U \amalg U'$ is a (-1)-truncated, 0-connective object of $\mathcal{X}$ and therefore a final object of $\mathcal{X}$. $\square$

Definition 1.2.9. Let $f_* : \mathcal{X} \to \mathcal{Y}$ be a geometric morphism of ∞-topoi. We will say that f_* is a *clopen immersion* if it satisfies the equivalent conditions of Proposition 1.2.8: that is, if it is both étale and a closed immersion. We will say that a map of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a *clopen immersion* if it is étale and the underlying geometric morphism $f_* : \mathcal{X} \to \mathcal{Y}$ is a clopen immersion of ∞-topoi.

We will need a few standard results from commutative algebra.

Lemma 1.2.10. *Suppose we are given a pushout diagram of commutative rings*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"] & A' \arrow[d, "\phi'"] \\ B \arrow[r] & B', \end{tikzcd}$$

where ψ is faithfully flat. If B' is finitely generated over A', then B is finitely generated over A.

Proof. If B' is finitely generated over A', then there exists a finitely generated A-subalgebra $B_0 \subseteq B$ such that $A' \otimes_A B_0 \to B'$ is surjective. It follows that $A' \otimes_A (B/B_0) \simeq 0$, so the faithful flatness of A' over A implies that $B/B_0 \simeq 0$. We conclude that $B \simeq B_0$ is finitely generated. $\square$

Lemma 1.2.11. *Suppose we are given a pushout diagram of commutative rings*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"] & A' \arrow[d, "\phi'"] \\ B \arrow[r] & B', \end{tikzcd}$$

where ψ is faithfully flat. If B' is of finite presentation over A', then B is of finite presentation over A.

Proof. Lemma 1.2.10 implies that we can write B as a quotient $A[x_1, \ldots, x_n]/I$, for some ideal I. Then B' can be identified with the quotient of $A'[x_1, \ldots, x_n]$ by the ideal $IA'[x_1, \ldots, x_n]$. Since B' is of finite presentation over A', we conclude that $IA'[x_1, \ldots, x_n]$ is finitely generated. We may therefore choose a finitely generated subideal $I_0 \subseteq I$ such that $I_0 A'[x_1, \ldots, x_n] = IA'[x_1, \ldots, x_n]$. Since $A'[x_1, \ldots, x_n]$ is faithfully flat over $A[x_1, \ldots, x_n]$, we conclude that $I = I_0$ is finitely generated, so that B is of finite presentation over A. $\square$

Lemma 1.2.12. *Let $\phi : A \to B$ be a map of connective $\mathbb{E}_\infty$-rings. Suppose there exists a faithfully flat morphism $A \to A'$ which induces an étale morphism $\phi' : A' \to B'$, where $B" = A' \otimes_A B$. Then ϕ is étale.*

Proof. It follows from Lemma 1.2.11 that $\pi_0 B$ is finitely presented as a commutative algebra over $\pi_0 A$. It will therefore suffice to show that the relative cotangent complex $L_{B/A}$ is trivial (Lemma VII.8.9). Since A' is faithfully flat over A, it will suffice to show that

$$A' \otimes_A L_{B/A} \simeq B' \otimes_B L_{B/A} \simeq L_{B'/A'}$$

vanishes, which follows from Corollary A.7.5.4.5 (since ϕ' is étale). $\square$

Proof of Theorem 1.2.1. Let $\phi : A \to B$ be a map of $\mathbb{E}_\infty$-rings which induces an étale map $f : \operatorname{Spec} B \to \operatorname{Spec} A$ of nonconnective spectral Deligne-Mumford stacks. We wish to prove that ϕ is étale. We first treat the case where A is discrete.

Let $e_\phi : \operatorname{Spec}^{\mathrm{Z}} A \to \mathbf{Z}$ be defined as in Remark 1.2.4. Lemma 1.2.5 guarantees the existence of an integer $n \geq 0$ such that $e_\phi(x) \leq n$ for every point $x \in \operatorname{Spec}^{\mathrm{Z}} A$. We proceed by induction on n. If $n = 0$, then $B \simeq 0$ and there is nothing to prove. Let us therefore assume that $n > 0$.

Let $U \subseteq \operatorname{Spec}^{\mathrm{Z}} A$ be the open subset corresponding to the image of ϕ. Since $\operatorname{Spec} B$ is quasi-compact, U is quasi-compact. We can therefore write U as a union of finitely many open sets of the form $\operatorname{Spec}^{\mathrm{Z}} A[\frac{1}{a_i}]$. Note that the elements $\phi(a_i)$ generate the unit ideal in $\pi_0 B$. According to Lemma 1.2.12, it will suffice to show that each of the localizations $B[\frac{1}{\phi(a_i)}]$ is étale over $A[\frac{1}{a_i}]$. We may therefore replace A by $A[\frac{1}{a_i}]$ and thereby reduce to the case where the map f is surjective.

Since f is an étale surjection, there exists a faithfully flat étale map $A \to C$ such that the induced map $\operatorname{Spec} C \to \operatorname{Spec} A$ factors through f. Using Lemma 1.2.12, we are reduced to proving that the map $C \to C \otimes_A B$ is étale. We may therefore replace A by C and thereby reduce to the case where f admits a section $s : \operatorname{Spec} A \to \operatorname{Spec} B$. Note that the underlying map of $\mathbb{E}_\infty$-rings $B \to A$ induces a surjection $\pi_0 B \to \pi_0 A$ (since it admits a right inverse). Proposition 1.2.7 implies that s induces a closed immersion at the level of ∞-topoi. Since s is étale, we deduce that s is a clopen immersion: that is, $\operatorname{Spec} B$ decomposes as a disjoint union $\mathfrak{X} \coprod \operatorname{Spec} A$ (see Proposition 1.2.8). It follows that $B \simeq A \times B'$, where $\mathfrak{X} = \operatorname{Spec} B'$. Let ϕ' denote the restriction of ϕ to $\mathfrak{X}$, and define $e_{\phi'}$ as in Remark 1.2.4. For each $x \in \operatorname{Spec}^{\mathrm{Z}} A$, we have $e_{\phi'}(x) = e_\phi(x) - 1 \leq n - 1$. It follows from the inductive hypothesis that B' is étale over A, so that B is étale over A as desired. This completes the proof of Theorem 1.2.1 in the case where A is discrete.

We now treat the general case. Write $\operatorname{Spec} A = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\operatorname{Spec} B = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. Then $(\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$ is étale over $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}})$. It follows from the first part of the proof that $\pi_0 B$ is étale over $\pi_0 A$. Using Theorem A.7.5.0.6, we can choose an étale A-algebra B' and an isomorphism of $\pi_0 A$-algebras $\alpha_0 : \pi_0 B' \simeq \pi_0 B$. Theorem A.7.5.4.2 implies that α_0 can be lifted (in an essentially unique way) to a map of A-algebras $\alpha : B' \to B$. Since $\operatorname{Spec} B$ and $\operatorname{Spec} B'$ are both étale over $\operatorname{Spec} A$, we conclude that α induces an étale map $g : \operatorname{Spec} B \to \operatorname{Spec} B'$. Since α_0 is an isomorphism of commutative rings, the map g induces an equivalence at the level of underlying ∞-topoi, and is therefore an equivalence. It follows that $B \simeq B'$ is étale over A, as desired. $\square$

We close this section with a useful result concerning étale morphisms between $\mathbb{E}_\infty$-rings.

Proposition 1.2.13. *Suppose we are given maps of $\mathbb{E}_\infty$-rings $A \xrightarrow{f} B \xrightarrow{g} C$, where g is étale and faithfully flat. Then f is étale if and only if $g \circ f$ is étale.*

Proof. The "only if" direction is obvious. For the converse, assume that $g \circ f$ is étale. Lemma 1.5.13 implies that f is flat. It will therefore suffice to show that the commutative ring $\pi_0 B$ is étale over $\pi_0 A$. Replacing A, B, and C by their connective covers, we can reduce to the case where A, B, and C are connective. We have a fiber sequence

$$C \otimes_B L_{B/A} \to L_{C/A} \to L_{C/B}.$$

Since C is étale over both A and B, we have $L_{C/A} \simeq L_{C/B} \simeq 0$. It follows that $C \otimes_B L_{B/A} \simeq 0$. Since C is faithfully flat over B, this implies that $L_{B/A} \simeq 0$ (Remark VII.5.3). To complete the proof that B is étale over A, it will suffice to show that $\pi_0 B$ is finitely presented as a commutative algebra over $\pi_0 A$ (Lemma VII.8.9). We first prove that $\pi_0 B$ is finitely generated over $\pi_0 A$. Since g is étale, the commutative algebra $\pi_0 C$ is finitely presented over $\pi_0 B$. We may therefore choose a finitely generated $\pi_0 A$-subalgebra $R \subseteq \pi_0 B$ and an étale morphism $R \to R'$ such that $\pi_0 C \simeq (\pi_0 B) \otimes_R R'$. Since $\pi_0 C$ is finitely generated over $\pi_0 A$, we may assume (after enlarging R if necessary) that the map $R' \to \pi_0 C$ is surjective. Since $\pi_0 C$ is faithfully flat over $\pi_0 B$, we conclude that the inclusion $R \hookrightarrow \pi_0 B$ is surjective, so that $\pi_0 B = R$ is finitely generated over $\pi_0 A$. Choose a surjection $S \to \pi_0 B$, where S is finitely presented over $\pi_0 A$. Let I denote the kernel of ϕ; we wish to show that I is a finitely generated ideal. Using the structure theory of étale morphisms (Proposition VII.8.10, we can choose an étale morphism $S \to S'$ and an isomorphism $\pi_0 C \simeq \pi_0 B \otimes_S S'$. Replacing S by the quotient S/J for some finitely generated ideal $J \subseteq I$, we can assume that S' is faithfully flat over S. It follows that the canonical map $S" \to \pi_0 C$ is surjection with kernel $S' \otimes_S I$. Since $\pi_0 C$ is finitely presented over $\pi_0 A$, the ideal $S' \otimes_S I$ is finitely generated as a module over S'. Because S' is faithfully flat over S, the ideal I is finitely generated as an S-module, as desired. □

1.3 Localic Spectral Deligne-Mumford Stacks

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We say that $\mathfrak{X}$ is *n-localic* if the ∞-topos $\mathcal{X}$ is n-localic. In this section, we will give a reformulation of this condition (Proposition 1.3.3) in terms of the functor represented by $\mathfrak{X}$ (at least when $n \geq 1$).

Definition 1.3.1. Let $n \geq 0$. A *spectral Deligne-Mumford n-stack* is a spectral Deligne-Mumford stack $\mathfrak{X}$ with the following property: for every discrete commutative ring R, the mapping space

$$\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$$

is n-truncated. A *spectral algebraic space* is a spectral Deligne-Mumford 0-stack.

Remark 1.3.2. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The condition that $\mathfrak{X}$ be a spectral algebraic space depends only on the underlying 0-truncated spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0\, \mathcal{O}_{\mathcal{X}})$.

Our main result is:

Proposition 1.3.3. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack, and let $n \geq 1$ be an integer. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{X}$ is n-localic.*

(2) *For every commutative ring R, the mapping space* $\mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X})$ *is n-truncated. That is, $\mathfrak{X}$ is a spectral Deligne-Mumford n-stack.*

The proof will require a few general observations about n-localic ∞-topoi.

Lemma 1.3.4. *Let $\mathcal{X}$ be an ∞-topos, let $n \geq 0$ be an integer, and let U be an n-truncated object of $\mathcal{X}$. If $\mathcal{X}$ is n-localic, then $\mathcal{X}_{/U}$ is n-localic.*

Proof. If $\mathcal{X}$ is n-localic, then we can write $\mathcal{X}$ as a topological localization of $\mathcal{P}(\mathcal{C})$, for some small n-category $\mathcal{C}$ (see the proof of Proposition T.6.4.5.7). Let us identify $\mathcal{X}$ with the corresponding subcategory of $\mathcal{P}(\mathcal{C})$. Then $\mathcal{X}_{/U}$ is a topological localization of $\mathcal{P}(\mathcal{C})_{/U}$. According to Proposition T.6.4.5.9, it will suffice to show that the ∞-topos $\mathcal{P}(\mathcal{C})_{/U}$ is n-localic. The presheaf $U : \mathcal{C}^{op} \to \mathcal{S}$ classifies a right fibration of ∞-categories $\theta : \widetilde{\mathcal{C}} \to \mathcal{C}$. Since U is n-truncated, the fibers of θ are n-truncated Kan complexes, so that $\widetilde{\mathcal{C}}$ is also an n-category. We complete the proof by observing that there is a canonical equivalence of ∞-categories $\mathcal{P}(\mathcal{C})_{/U} \simeq \mathcal{P}(\widetilde{\mathcal{C}})$. □

Lemma 1.3.5. *Let $\mathcal{X}$ and $\mathcal{Y}$ be ∞-topoi, and suppose that $\mathcal{Y}$ is n-localic for some $n \geq 0$. Let $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{X}, \mathcal{Y})$ spanned by the geometric morphisms $f^* : \mathcal{X} \to \mathcal{Y}$. Then $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ is equivalent to an n-category.*

Proof. Let $\tau_{\leq n}\mathcal{X}$ and $\tau_{\leq n}\mathcal{Y}$ be the underlying n-topoi of $\mathcal{X}$ and $\mathcal{Y}$. Since $\mathcal{Y}$ is n-localic, we can identify $\mathrm{Fun}^*(\mathcal{X}, \mathcal{Y})$ with the full subcategory of $\mathrm{Fun}(\tau_{\leq n-1}\mathcal{X}, \tau_{\leq n-1}\mathcal{Y})$ spanned by those functors which preserve small colimits and finite limits. The desired result now follows from the observation that $\tau_{\leq n-1}\mathcal{Y}$ is equivalent to an n-category. □

Lemma 1.3.6. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be spectral Deligne-Mumford stacks. Assume that $\mathcal{O}_{\mathcal{Y}}$ is n-truncated, and that $\mathcal{X}$ is n-localic. Then the mapping space $\mathrm{Map}_{\mathrm{Stk}}(\mathfrak{Y}, \mathfrak{X})$ is n-truncated.*

Proof. There is an evident forgetful functor $\theta : \mathrm{Map}_{\mathrm{Stk}}(\mathfrak{Y}, \mathfrak{X}) \to \mathrm{Map}_{{}^{\mathrm{L}}\mathcal{T}\mathrm{op}}(\mathcal{X}, \mathcal{Y})$, where the codomain of θ is n-truncated by Lemma 1.3.5. It will therefore suffice to show that the fiber of θ over every geometric morphism $f^* : \mathcal{X} \to \mathcal{Y}$ is n-truncated. Unwinding the definitions, we see that this fiber can be identified with a summand of the mapping space $\mathrm{Map}_{\mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{Y})}(f^* \mathcal{O}_{\mathcal{X}}, \mathcal{O}_{\mathcal{Y}})$, which is n-truncated by virtue of our assumption that $\mathcal{O}_{\mathcal{Y}}$ is n-truncated. □

Remark 1.3.7. Using exactly same argument, we can deduce the analogous result for spectral schemes. In particular, every connective 0-localic spectral scheme represents a functor which carries discrete $\mathbb{E}_\infty$-rings to discrete spaces.

Proof of Proposition 1.3.3. The implication (1) ⇒ (2) follows from Lemma 1.3.6. Assume now that (2) is satisfied. Replacing $\mathcal{O}_{\mathcal{X}}$ by $\pi_0 \mathcal{O}_{\mathcal{X}}$, we may assume that $\mathcal{O}_{\mathcal{X}}$ is discrete. It follows from Theorem VII.8.42 that there exists a 1-localic spectral Deligne-Mumford stack $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and a 2-connective object $U \in \mathcal{Y}$ such that $\mathfrak{X} \simeq (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} |U)$. To prove that $\mathfrak{X}$ is 1-localic, it will suffice to show that the object U is n-truncated (Lemma 1.3.4). Let $\mathcal{Y}_0$ be the full subcategory of $\mathcal{Y}$ spanned by those objects $Y \in \mathcal{Y}$ such that $\mathrm{Map}_{\mathcal{Y}}(Y, U)$ is n-truncated. We wish to show that $\mathcal{Y}_0 = \mathcal{Y}$. Since $\mathcal{Y}_0$ is closed under small colimits in $\mathcal{Y}$, it will suffice to show that $\mathcal{Y}_0$ contains every object Y for which $(\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}} |Y) \simeq \operatorname{Spec} R$ is affine (note that in this case, R is automatically discrete). We now observe that $\mathrm{Map}_{\mathcal{Y}}(Y, U)$ can be identified with the homotopy fiber of the forgetful map

$$\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y}).$$

Here $\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X})$ is n-truncated by assumption (2), and $\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y})$ is 1-truncated by Lemma 1.3.6, so that the homotopy fiber is also n-truncated. □

Corollary 1.3.8. *Let $\mathfrak{X}$ be a spectral algebraic space. Then $\mathfrak{X}$ is 1-localic.*

Remark 1.3.9. Suppose we are given a pullback diagram of connective spectral Deligne-Mumford stacks

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y}. \end{tikzcd}$$

Assume that $\mathfrak{X}$ and $\mathfrak{Y}'$ are spectral Deligne-Mumford n-stacks, and that $\mathfrak{Y}$ is a spectral Deligne-Mumford $(n+1)$-stack. Then $\mathfrak{X}'$ is a spectral Deligne-Mumford n-stack. In particular, if $\mathfrak{Y}'$ and $\mathfrak{Y}$ are affine and $\mathfrak{X}$ is a spectral Deligne-Mumford n-stack, then $\mathfrak{X}'$ is a spectral Deligne-Mumford n-stack.

1.4 Quasi-Compactness of Spectral Deligne-Mumford Stacks

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral scheme. We will say that $\mathfrak{X}$ is *quasi-compact* if the ∞-topos $\mathcal{X}$ is quasi-compact. Our goal in this section is to study quasi-compactness and related finiteness conditions in the setting of spectral algebraic geometry.

Definition 1.4.1. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral scheme or a nonconnective spectral Deligne-Mumford stack, and let $n \geq 0$ be an integer. We will say that $\mathfrak{X}$ is *n-quasi-compact* if the ∞-topos $\mathcal{X}$ is n-coherent (see §VII.3). We will say that $\mathfrak{X}$ is *quasi-compact* if it is 0-quasi-compact. We will say that $\mathfrak{X}$ is *∞-quasi-compact* if it is n-quasi-compact for every integer n.

Proposition 1.4.2. *Let A be an $\mathbb{E}_\infty$-ring. Then the nonconnective spectral scheme* $\mathrm{Spec}^{\mathrm{Zar}}(A)$ *and the nonconnective spectral Deligne-Mumford stack* $\mathrm{Spec}^{\acute{e}t}(A)$ *are ∞-quasi-compact.*

Proof. Since the Grothendieck topologies on $\mathcal{G}^{\mathrm{nSp}}_{\acute{\mathrm{e}}\mathrm{t}}$ and $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ are finitary, this follows from Proposition VII.3.19 together with the construction of spectra described in §V.2.2. □

Corollary 1.4.3. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral scheme or a nonconnective spectral Deligne-Mumford stack. Then the ∞-topos $\mathcal{X}$ is locally coherent.*

Proposition 1.4.4. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of nonconnective spectral Deligne-Mumford stacks (or nonconnective spectral schemes). Let $n \geq 0$ be an integer. The following conditions are equivalent:*

(1) *For every n-coherent object $U \in \mathcal{Y}$, the pullback f^*U is n-coherent.*

(2) *For every affine object $U \in \mathcal{Y}$, the pullback f^*U is an n-coherent object of $\mathcal{X}$.*

(3) *There exists a full subcategory $\mathcal{Y}_0 \subseteq \mathcal{Y}$ with the following properties:*

  (a) *Each object $U \in \mathcal{Y}_0$ is n-coherent.*

  (b) *For each $U \in \mathcal{Y}_0$, the pullback $f^*(U)$ is n-coherent.*

  (c) *For each object $Y \in \mathcal{Y}$, there exists an effective epimorphism $\coprod Y_i \to Y$, where each $Y_i \in \mathcal{Y}_0$.*

Moreover, if $n > 0$, then these conditions imply:

(4) *For every relatively $(n-1)$-coherent morphism $u : U \to Y$ in $\mathcal{Y}$, the pullback $f^*(u)$ is a relatively $(n-1)$-coherent morphism in $\mathcal{X}$.*

Proof. We proceed by induction on n. The implication (1) ⇒ (2) is immediately from Proposition 1.4.2. To see that (2) ⇒ (3), we take $\mathcal{Y}_0$ to be the collection of all objects $U \in \mathcal{Y}$ such that $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} | U)$ is affine. We next show that (3) ⇒ (4) if $n > 0$. Let $u : U \to Y$ be an $(n-1)$-coherent morphism in $\mathcal{Y}$; we wish to show that $f^*(u)$ is a relatively $(n-1)$-coherent morphism in $\mathcal{X}$. Choose an effective epimorphism $\coprod_{i \in I} Y_i \to Y$, where each $Y_i \in \mathcal{Y}$. Using Corollary VII.3.11, we are reduced to proving that the induced map $f^*(U \times_Y \coprod_{i \in I} Y_i) \to f^*(\coprod_{i \in I} Y_i)$ is relatively $(n-1)$-coherent. We may therefore replace Y by some Y_i and thereby reduce to the case where Y is n-coherent. Then U is $(n-1)$-coherent. Using (2) together with the inductive hypothesis, we deduce that f^*Y is n-coherent and that f^*U is $(n-1)$-coherent, so that $f^*(u)$ is relatively $(n-1)$-coherent as desired.

We now prove that (3) implies (1). Fix an n-coherent object $U \in \mathcal{Y}$; we wish to prove that $f^*(U)$ is an n-coherent object of $\mathcal{X}$. Choose an effective epimorphism $\coprod_{i \in I} U_i \to U$ where each $U_i \in \mathcal{Y}_0$. Since U is quasi-compact, we may assume without loss of generality that I is finite. Using (2) and Remark VII.3.5, we conclude that $\coprod f^*(U_i)$ is an n-coherent object of $\mathcal{X}$. Moreover, the map $\coprod f^*(U_i) \to f^*(U)$ is an effective epimorphism which is $(n-1)$-coherent if $n > 0$ (by virtue of (4)). Using Proposition VII.3.9 we conclude that f^*U is n-coherent as desired. □

Definition 1.4.5. Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a morphism of nonconnective spectral schemes or nonconnective spectral Deligne-Mumford stacks. We will say that f is *n-quasi-compact* if it satisfies the equivalent conditions (1) and (2) of Proposition 1.4.4. We will say that f is *quasi-compact* if it is 0-quasi-compact, and *∞-quasi-compact* if it is n-quasi-compact for every integer $n \geq 0$.

Example 1.4.6. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map between affine nonconnective spectral Deligne-Mumford stacks (or affine nonconnective spectral schemes). Then f is ∞-quasi-compact; this follows immediately from Proposition 1.4.2.

Remark 1.4.7. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks and let $0 \leq n \leq \infty$. The following conditions are equivalent:

(1) The map f is n-coherent.

(2) For every étale map $\operatorname{Spec} A \to \mathfrak{Y}$, the fiber product $\operatorname{Spec} A \times_{\mathfrak{Y}} \mathfrak{X}$ is n-coherent.

Proposition 1.4.8. *Let $\mathfrak{X}$ be a quasi-compact nonconnective spectral Deligne-Mumford stack, and let $n > 0$. The following conditions on $\mathfrak{X}$ are equivalent:*

(1) *For every pair of maps $\operatorname{Spec} A \to \mathfrak{X} \leftarrow \operatorname{Spec} B$, the fiber product $\operatorname{Spec} A \times_{\mathfrak{X}} \operatorname{Spec} B$ is $(n-1)$-coherent.*

(2) *Every map $f : \operatorname{Spec} A \to \mathfrak{X}$ is $(n-1)$-coherent.*

(3) *For every pair of maps $\operatorname{Spec} A \to \mathfrak{X} \xleftarrow{u} \operatorname{Spec} B$ where u is étale, the fiber product $\operatorname{Spec} A \times_{\mathfrak{X}} \operatorname{Spec} B$ is $(n-1)$-coherent.*

(4) *Every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$ is $(n-1)$-coherent.*

(5) *For every pair of étale maps $\operatorname{Spec} A \to \mathfrak{X} \leftarrow \operatorname{Spec} B$, the fiber product $\operatorname{Spec} A \times_{\mathfrak{X}} \operatorname{Spec} B$ is $(n-1)$-coherent.*

(6) *The nonconnective spectral Deligne-Mumford stack $\mathfrak{X}$ is n-coherent.*

Proof. The equivalences (1) $\Leftrightarrow$ (2) and (3) $\Leftrightarrow$ (4) are obvious. The equivalences (2) $\Leftrightarrow$ (3) and (4) $\Leftrightarrow$ (5) follow from Remark 1.4.7, and the equivalence (5) $\Leftrightarrow$ (6) follows from Corollary VII.3.10. $\square$

Proposition 1.4.9. *Suppose we are given a pullback diagram σ :*

$$\begin{tikzcd}
(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \arrow[r, "f'"] \arrow[d, "g'"] & (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'}) \arrow[d, "g"] \\
(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r, "f"] & (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})
\end{tikzcd}$$

of nonconnective spectral schemes or nonconnective spectral Deligne-Mumford stacks. If f is n-quasi-compact, then f' is n-quasi-compact.

Proof. Let $\mathcal{Y}'_0$ be the full subcategory of $\mathcal{Y}'$ spanned by those objects $Y' \in \mathcal{Y}'$ with the following properties:

(i) The pair $(\mathcal{Y}'_{/Y'}, \mathcal{O}_{\mathcal{Y}'} | Y')$ is affine.

(ii) There exists an object $Y \in \mathcal{Y}$ and a map $Y' \to g^* Y$, where $(\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}} | Y)$ is affine.

This subcategory satisfies requirements (a), (b), and (c) of Proposition 1.4.4; it will therefore suffice to show that $f'^* Y$ is an n-coherent object of $\mathcal{X}'$.

Replacing σ by the diagram

$$\begin{tikzcd}
(\mathcal{X}'_{/f'^*Y'}, \mathcal{O}_{\mathcal{X}'} | f'^*Y') \arrow[r, "f'"] \arrow[d] & (\mathcal{Y}'_{/Y'}, \mathcal{O}_{\mathcal{Y}'} | Y') \arrow[d, "g"] \\
(\mathcal{X}_{/f^*Y}, \mathcal{O}_{\mathcal{X}} | f^*Y) \arrow[r, "f"] & (\mathcal{Y}_{/Y}, \mathcal{O}_{\mathcal{Y}} | Y),
\end{tikzcd}$$

we can reduce to the case where $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and $(\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$ are affine. Since f is n-quasi-compact, the ∞-topos $\mathcal{X}$ is n-coherent; we wish to prove that $\mathcal{X}'$ is n-coherent. To prove this, it suffices to show that the map g' is n-quasi-compact. This assertion is local on $\mathcal{X}$; we may therefore assume that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine. Since σ is a pullback diagram, we conclude that $(\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$ is affine and the desired result follows from Example 1.4.6. □

Corollary 1.4.10. *Let $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a map of nonconnective spectral schemes or nonconnective spectral Deligne-Mumford stacks. Assume that $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is affine. Then f is n-quasi-compact if and only if $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is n-quasi-compact.*

Proof. The "only if" direction is obvious (and requires only that $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be n-quasi-compact). Conversely, suppose that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is affine. Let $U \in \mathcal{Y}$ be such that $(\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} | U)$ is affine. We wish to prove that f^*U is an n-coherent object of $\mathcal{X}$. We have a pullback diagram

$$\begin{tikzcd}
(\mathcal{X}_{/f^*U}, \mathcal{O}_{\mathcal{X}} | f^*U) \arrow[r] \arrow[d, "g"] & (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} | U) \arrow[d, "g'"] \\
(\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \arrow[r] & (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}).
\end{tikzcd}$$

The map g' is n-quasi-compact by Example 1.4.6, so that Proposition 1.4.9 guarantees that g is n-quasi-compact. Since the final object $\mathbf{1} \in \mathcal{X}$ is n-coherent, we conclude that $g^*\mathbf{1} \in \mathcal{X}_{/f^*U}$ is n-coherent: that is, f^*U is an n-coherent object of $\mathcal{X}$. □

Proposition 1.4.11. *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks (or nonconnective spectral schemes)*

$$\begin{tikzcd}
\mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\
\mathfrak{Y}' \arrow[r, "g"] & \mathfrak{Y},
\end{tikzcd}$$

and let $0 \leq n \leq \infty$. If f is n-quasi-compact, then f' is n-quasi-compact. The converse holds if g is an étale surjection.

Proof. The first assertion follows from Proposition 1.4.9. We will prove the second assertion for nonconnective spectral Deligne-Mumford stacks (the case of nonconnective spectral schemes is similar). We may assume without loss of generality that $\mathfrak{Y} = \operatorname{Spec} A$ is affine. Choose an étale map $\operatorname{Spec} A' \to \mathfrak{Y}'$ such that the composite map $\operatorname{Spec} A' \to \mathfrak{Y}' \to \mathfrak{Y}$ is an étale surjection. Replacing $\mathfrak{Y}'$ by $\operatorname{Spec} A'$, we can reduce to the case where $\mathfrak{Y}'$ is also affine. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{X}' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Then $\mathcal{X}'$ is n-coherent, and we wish to prove that $\mathcal{X}$ is n-coherent (Corollary 1.4.10). We claim more generally that if $X \in \mathcal{X}$ is an object such that f^*X is m-coherent, then X is m-coherent. The proof proceeds by induction on m. When $m = 0$, it suffices to observe that the pullback functor $f^* : \mathcal{X}_{/X} \to \mathcal{X}'_{/f^*X}$ is an étale surjection of ∞-topoi. Assume that $m > 0$. According to Corollary VII.3.10, it will suffice to show that if we are given affine objects $U, V \in \mathcal{X}_{/X}$, then the fiber product $U \times_X V$ is $(m-1)$-coherent. By the inductive hypothesis, it suffices to show that $f^*(U \times_X V) \simeq f^*U \times_{f^*X} f^*V$ is $(m-1)$-coherent. This follows from the m-coherence of f^*X, since f^*U and f^*V are affine. □

We now discuss the relationship of Definition 1.4.1 with classical scheme theory.

Lemma 1.4.12. *Let X be a topological space. The following conditions are equivalent:*

(1) *The ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is coherent.*

(2) *The ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is 1-coherent.*

(3) *The topological space X is coherent: that is, the collection of quasi-compact open subsets $U \subseteq X$ is closed under finite intersections and forms a basis for the topology of X (in particular, X is quasi-compact).*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. We prove that (2) $\Rightarrow$ (3). For each $U \subseteq X$, let $\chi_U \in \mathcal{S}\mathrm{hv}(X)$ be the sheaf given by the formula

$$\chi_U(V) = \begin{cases} \Delta^0 & \text{if } V \subseteq U \\ \emptyset & \text{otherwise.} \end{cases}$$

We note that χ_U is a quasi-compact object of $\mathcal{X}$ if and only if U is quasi-compact as a topological space. If $\mathcal{S}\mathrm{hv}(X)$ is 1-coherent, then the collection of quasi-compact objects of $\mathcal{S}\mathrm{hv}(X)$ are closed under products. Since the construction $U \mapsto \chi_U$ carries finite intersections to finite products, we conclude that the collection of quasi-compact open subsets of X is closed under finite intersections. We claim that the quasi-compact open subsets form a basis for the topology of X. To prove this, choose an arbitrary open subset $U \subseteq X$. Since $\mathcal{S}\mathrm{hv}(X)$ is 1-coherent, there exists an effective epimorphism $\theta : \coprod \mathcal{F}_i \rightarrow \chi_U$, where each $\mathcal{F}_i$ is a quasi-compact object of $\mathcal{S}\mathrm{hv}(X)$. For each index i, we have $\tau_{\leq -1} \mathcal{F}_i \simeq \chi_{U_i}$ for some open set $U_i \subseteq X$. It follows that θ induces an effective epimorphism $\coprod_i \chi_{U_i} \rightarrow \chi_U$, so that $U = \bigcup U_i$. We claim that each U_i is quasi-compact: equivalently, each of the sheaves χ_{U_i} is a quasi-compact object of $\mathcal{S}\mathrm{hv}(X)$. This follows from the observation that we have effective epimorphisms $\mathcal{F}_i \rightarrow \chi_{U_i}$.

We now complete the proof by showing that (3) implies (1). Assume that X is a coherent topological space. Let $\mathcal{C} \subseteq \mathcal{S}\mathrm{hv}(X)$ be the full subcategory spanned by objects of the form χ_U, where U is a quasi-compact open subset of X. Since the quasi-compact open subsets of X form a basis for the topology of X, the ∞-category $\mathcal{C}$ generates $\mathcal{S}\mathrm{hv}(X)$ under small colimits. It will therefore suffice to show that $\mathcal{C}$ consists of coherent objects of $\mathcal{S}\mathrm{hv}(X)$. We prove by induction on n that the objects of $\mathcal{C}$ are n-coherent. The case $n = 0$ is clear. Assume that the objects of $\mathcal{C}$ are known to be n-coherent for $n \geq 0$. We wish to prove that if $U \subseteq X$ is a quasi-compact open subset, then χ_U is $(n+1)$-coherent. According to Corollary VII.3.10, it will suffice to show that for every pair of objects $\chi_V, \chi_{V'} \in \mathcal{C}$, every fiber product $\chi_V \times_{\chi_U} \chi_{V'}$ is n-coherent. Unwinding the definitions, this is equivalent to the statement that $V \cap V'$ is quasi-compact, which follows from our assumption that the quasi-compact open subsets of X are closed under finite intersections. $\square$

Proposition 1.4.13. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a 0-localic spectral scheme, so that $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(X)$ for some topological space X, and $(X, \pi_0 \mathcal{O}_{\mathcal{X}})$ is a scheme (see Proposition VII.2.37). Then:*

(1) *The spectral scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is 0-quasi-compact if and only if the scheme $(X, \pi_0 \mathcal{O}_{\mathcal{X}})$ is quasi-compact.*

(2) *If $1 \leq n \leq \infty$, then $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is n-quasi-compact if and only if the scheme $(X, \pi_0 \mathcal{O}_{\mathcal{X}})$ is quasi-compact and quasi-separated.*

Proof. Assertion (1) follows from the observation that a topological space X is quasi-compact if and only if the ∞-topos $\mathcal{S}\mathrm{hv}(X)$ is quasi-compact. Assertion (2) follows from Lemma 1.4.12. $\square$

Proposition 1.4.14. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral scheme and let $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathrm{Spec}^{\acute{e}t}_{\mathrm{Zar}}(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be the associated spectral Deligne-Mumford stack. Let $0 \leq n \leq \infty$. The following conditions are equivalent:*

(1) *The spectral scheme $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is n-quasi-compact.*

(2) *The spectral Deligne-Mumford stack $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is n-quasi-compact.*

Proof. We proceed by induction on n; when $n = 0$, the desired result follows from Lemma VII.9.7 (see Remark VII.3.2).

Assume first that $n > 0$ and that $\mathcal{Y}$ is n-coherent; we wish to prove that $\mathcal{X}$ is n-coherent. The inductive hypothesis shows that an object $U \in \mathcal{X}$ is $(n-1)$-coherent if and only if f^*U is an $(n-1)$-coherent object of $\mathcal{Y}$. Since f^* commutes with finite products and the collection of $(n-1)$-coherent objects of $\mathcal{Y}$ is closed under finite products, we conclude that the collection of $(n-1)$-coherent objects of $\mathcal{X}$ is closed under finite products. Since $\mathcal{X}$ is locally coherent, it follows immediately that $\mathcal{X}$ is n-coherent.

We next prove that if $n > 0$, then f^* carries relatively $(n-1)$-coherent morphisms in $\mathcal{X}$ to relatively $(n-1)$-coherent morphisms in $\mathcal{Y}$. Suppose that $u : U \to X$ is a relatively $(n-1)$-coherent morphism in $\mathcal{X}$. We wish to prove that $f^*(u)$ is a relatively $(n-1)$-coherent morphism in $\mathcal{Y}$. Choose an effective epimorphism $\coprod X_i \to X$, where each $(\mathcal{X}_{/X_i}, \mathcal{O}_{\mathcal{X}} | X_i)$ is affine. According to Corollary VII.3.11, it will suffice to show that the induced map

$$f^*(U \times_X \coprod X_i) \to f^*(\coprod X_i)$$

is relatively n-coherent. We may therefore replace X by some X_i and thereby reduce to the case where $(\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}} | X)$ is affine. Since u is relatively $(n-1)$-coherent, it follows that U is $(n-1)$-coherent. By the inductive hypothesis, f^*U is an $(n-1)$-coherent object of $\mathcal{Y}$. Since $(\mathcal{Y}_{/f^*X}, \mathcal{O}_{\mathcal{Y}} | f^*X)$ is affine, f^*X is an n-coherent object of $\mathcal{Y}$ (Proposition 1.4.2), so the map $f^*U \to f^*X$ is relatively $(n-1)$-coherent as desired.

We now claim that if $n > 0$ and $\mathcal{X}$ is n-coherent, then $\mathcal{Y}$ is n-coherent. We note that $\mathcal{X}$ is quasi-compact, so there exists an effective epimorphism $u : \coprod_{i \in I} U_i \to \mathbf{1}$ in $\mathcal{X}$, where each $(\mathcal{X}_{/U_i}, \mathcal{O}_{\mathcal{X}} | U_i)$ is affine and the index set I is finite; here $\mathbf{1}$ denotes the final object of $\mathcal{X}$. Proposition 1.4.2 implies that $U = \coprod U_i$ is $(n-1)$-coherent, so that the map u is relatively $(n-1)$-coherent. The above argument shows that $f^*(u)$ is a relatively $(n-1)$-coherent effective epimorphism. Since $f^*(U) \simeq \coprod_{i \in I} f^*U_i$ is n-coherent (using Proposition 1.4.2 again), we can invoke Proposition VII.3.9 to conclude that $f^*\mathbf{1}$ is an n-coherent object of $\mathcal{Y}$: that is, the ∞-topos $\mathcal{Y}$ is n-coherent. □

To simplify the notation, we will now confine our attention to the quasi-compactness of spectral Deligne-Mumford stacks. The results below have analogues in the setting of spectral schemes, which can be proven either directly (using the same arguments) or by invoking Proposition 1.4.14.

Proposition 1.4.15. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks and let $0 \leq n \leq \infty$. Then:*

(1) *If $\mathfrak{Y}$ is n-quasi-compact and f is n-quasi-compact, then $\mathfrak{X}$ is n-quasi-compact.*

(2) *If $\mathfrak{X}$ is n-quasi-compact and $\mathfrak{Y}$ is $(n+1)$-quasi-compact, then f is n-quasi-compact.*

Proof. We proceed by induction on n. We begin with assertion (1). Assume that f and $\mathfrak{Y}$ are n-quasi-compact; we wish to prove that $\mathfrak{X}$ is n-quasi-compact. Choose an étale surjection $\operatorname{Spec} R \to \mathfrak{Y}$. Then the fiber product $\mathfrak{X}' = \operatorname{Spec} R \times_{\mathfrak{Y}} \mathfrak{X}$ is n-coherent. We have an étale surjection $\mathfrak{X}' \to \mathfrak{X}$, so that $\mathfrak{X}$ is quasi-compact. This completes the proof when $n = 0$. Assume now that $n > 0$. In view of Proposition 1.4.8, it will suffice to show that every map $\operatorname{Spec} A \to \mathfrak{X}$ is $(n-1)$-quasi-compact. Using Proposition 1.4.11, we are reduced to showing that the induced map

$$u : \operatorname{Spec} R \times_{\mathfrak{Y}} \operatorname{Spec} A \to \mathfrak{X}'$$

is $(n-1)$-quasi-compact. Since $\mathfrak{X}'$ is n-quasi-compact and $\operatorname{Spec} R \times_{\mathfrak{Y}} \operatorname{Spec} A$ is $(n-1)$-quasi-compact (using Proposition 1.4.8 and the n-quasi-compact of $\mathfrak{Y}$), the $(n-1)$-quasi-compactness of u follows from the inductive hypothesis.

We now prove (2). Assume that $\mathfrak{X}$ is n-quasi-compact and that $\mathfrak{Y}$ is $(n+1)$-quasi-compact. We wish to show that for every map $\operatorname{Spec} R \to \mathfrak{Y}$, the fiber product $\mathfrak{X}' = \operatorname{Spec} R \times_{\mathfrak{Y}} \mathfrak{X}$ is n-quasi-compact. By (1), it will suffice to show that the projection map $\mathfrak{X}' \to \mathfrak{X}$ is n-quasi-compact. This follows from Proposition 1.4.11, since the map $\operatorname{Spec} R \to \mathfrak{Y}$ is n-quasi-compact by Corollary 1.4.6. □

Corollary 1.4.16. *Suppose we are given maps $f : \mathfrak{X} \to \mathfrak{Y}$ and $g : \mathfrak{Y} \to \mathfrak{Z}$ of nonconnective spectral Deligne-Mumford stacks. Then:*

(i) *If f and g are n-quasi-compact, then $g \circ f$ is n-quasi-compact.*

(ii) *If g is $(n+1)$-quasi-compact and $g \circ f$ is n-quasi-compact, then f is n-quasi-compact.*

Corollary 1.4.17. *Let $\mathfrak{Z}$ be a quasi-compact nonconnective spectral Deligne-Mumford stack and let $n \geq 0$. Then $\mathfrak{Z}$ is $(n+1)$-quasi-compact if and only if the following condition is satisfied: for every pair of maps $\mathfrak{X} \to \mathfrak{Z} \leftarrow \mathfrak{Y}$ where $\mathfrak{X}$ and $\mathfrak{Y}$ are n-quasi-compact, the fiber product $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is n-quasi-compact.*

Proof. The "if" direction follows immediately from Proposition 1.4.8 (take $\mathfrak{X}$ and $\mathfrak{Y}$ to be affine). Conversely, suppose that $\mathfrak{Z}$ is $(n+1)$-quasi-compact. If $\mathfrak{X}$ is n-quasi-compact, then the map $\mathfrak{X} \to \mathfrak{Z}$ is n-quasi-compact. It follows from Proposition 1.4.11 that the projection map $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y} \to \mathfrak{Y}$ is n-quasi-compact. Since $\mathfrak{Y}$ is also n-quasi-compact, Proposition 1.4.15 implies that $\mathfrak{X} \times_{\mathfrak{Z}} \mathfrak{Y}$ is n-quasi-compact. □

Corollary 1.4.18. *The collection of ∞-quasi-compact nonconnective spectral Deligne-Mumford stacks is closed under the formation of fiber products.*

1.5 Local Properties of Spectral Deligne-Mumford Stacks

In this section, we study properties of spectral Deligne-Mumford stacks $\mathfrak{X}$ which are of a *local* nature on $\mathfrak{X}$, in that they can be tested after passing to any étale covering of $\mathfrak{X}$.

Definition 1.5.1. Let $f : \mathfrak{X} \to \mathfrak{Y} = (\mathcal{Y}, \mathcal{O})$ be an étale morphism of spectral Deligne-Mumford stacks. We will say that a collection of étale morphisms $\{f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y}\}$ is *jointly surjective* if the induced map $\coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{Y}$ is surjective (see Definition 1.1.21).

Definition 1.5.2. Let P be a property of nonconnective spectral Deligne-Mumford stacks. We will say that P is *local for the étale topology* if the following conditions hold:

(i) For every étale morphism of nonconnective spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$, if $\mathfrak{Y}$ has the property P, then $\mathfrak{X}$ also has the property P.

(ii) Given a jointly surjective collection of étale maps $\{\mathfrak{X}_\alpha \to \mathfrak{Y}\}$, if each $\mathfrak{X}_\alpha$ has the property P, then $\mathfrak{Y}$ has the property P.

Remark 1.5.3. Let P be a property of nonconnective spectral Deligne-Mumford stacks which is local for the étale topology. A nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ has the property P if and only if, for every affine $U \in \mathcal{X}$, the affine nonconnective spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X} | U)$ has the property P. Consequently, P is determined by the full subcategory $\mathrm{CAlg}(P) \subseteq \mathrm{CAlg}$ spanned by those $\mathbb{E}_\infty$-rings A such that $\operatorname{Spec} A$ has the property P. Using the fact that P satisfies étale descent, we deduce that $\mathrm{CAlg}(P)$ has the following properties:

(i) If $f : A \to A'$ is an étale morphism of $\mathbb{E}_\infty$-rings and $A \in \mathrm{CAlg}(P)$, then $A' \in \mathrm{CAlg}(P)$.

(ii) Given a finite collection of étale maps $\{A \to A_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat, if each $A_\alpha \in \mathrm{CAlg}(P)$, then $A \in \mathrm{CAlg}(P)$.

Conversely, given a full subcategory $\mathrm{CAlg}(P) \subseteq \mathrm{CAlg}$ satisfying conditions (i) and (ii), we obtain a property P of nonconnective spectral Deligne-Mumford stacks as follows: a pair $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ has the property P if and only if, whenever we have an equivalence $(\mathcal{X}_{/U}, \mathcal{O}_\mathcal{X} | U) \simeq \operatorname{Spec} A$, the $\mathbb{E}_\infty$-ring A belongs to $\mathrm{CAlg}(P)$.

Recall that an $\mathbb{E}_\infty$-ring A is said to be *Noetherian* if A is connective, $\pi_0 A$ is a Noetherian commutative ring, and $\pi_n A$ is a finitely generated module over $\pi_0 A$ for every integer n.

Definition 1.5.4. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We will say that $\mathfrak{X}$ is *locally Noetherian* if, whenever $U \in \mathcal{X}$ is affine so that $(\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}} | U) \simeq \operatorname{Spec} A$, the $\mathbb{E}_\infty$-ring A is Noetherian.

Proposition 1.5.5. *The property of being a locally Noetherian spectral Deligne-Mumford stack is local for the étale topology.*

Lemma 1.5.6. *Let $f : A \to B$ be a faithfully flat map of $\mathbb{E}_\infty$-rings. If B is Noetherian, then A is Noetherian.*

Proof. We first show that $\pi_0 A$ is a Noetherian commutative ring. We claim that the collection of ideals in $\pi_0 A$ satisfies the ascending chain condition. To prove this, it will suffice to show that the construction $I \mapsto I\pi_0 B$ determines an injection from the partially ordered set of ideals of $\pi_0 A$ to the partially ordered set of ideals of $\pi_0 B$. Since $\pi_0 B$ is flat over A, the map $I \otimes_{\pi_0 A} \pi_0 B$ is an injection with image $I\pi_0 B$. Given a pair of ideals $I, J \subseteq \pi_0 A$ we have an exact sequence of $\pi_0 A$-modules

$$0 \to I \cap J \to I \oplus J \to I + J \to 0.$$

This sequence remains exact after tensoring with $\pi_0 B$, so that $(I \cap J)\pi_0 B = I\pi_0 B \cap J\pi_0 B$. It follows that if $I\pi_0 B = J\pi_0 B$, then the inclusion $(I \cap J)\pi_0 B \hookrightarrow I\pi_0 B$ is bijective, so that $I/(I \cap J) \otimes_{\pi_0 A} \pi_0 B \simeq 0$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, this implies that $I/(I \cap J) = 0$, so that $I \subseteq J$. A similar argument shows that $J \subseteq I$, so that $I = J$. This completes the proof that $\pi_0 A$ is a Noetherian commutative ring.

Since f is faithfully flat, we have $\pi_n B \simeq \pi_n A \otimes_{\pi_0 A} \pi_0 B$. Since $\pi_n B \simeq 0$ for $n < 0$, the faithful flatness of $\pi_0 B$ over $\pi_0 A$ implies that $\pi_n A \simeq 0$. This proves that A is connective. To complete the proof, we must show that each $\pi_n A$ is finitely generated as a module over $\pi_0 A$. Since $\pi_n B$ is finitely generated as a $\pi_0 B$-module, we can choose a finitely generated submodule $M \subseteq \pi_n A$ such that the map

$$M \otimes_{\pi_0 A} \pi_0 B \to \pi_n A \otimes_{\pi_0 A} \pi_0 B \simeq \pi_n B$$

is surjective. The cokernel of this map is given by $(\pi_n A)/M \otimes_{\pi_0 A} \pi_n B$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we deduce that $(\pi_n A)/M \simeq 0$, so that $\pi_n A \simeq M$ is finitely generated as a module over $\pi_0 A$. □

Proof of Proposition 1.5.5. In view of Remark 1.5.3, it will suffice to prove the following assertions:

(i) If $f : A \to A'$ is an étale morphism of $\mathbb{E}_\infty$-rings and A is Noetherian, then A' is also Noetherian.

(ii) Given a finite collection of étale maps $\{A \to A_\alpha\}$ such that $A \to \prod_\alpha A_\alpha$ is faithfully flat, if each A_α is Noetherian, then A is Noetherian.

Assertion (i) is obvious, and assertion (ii) follows immediately from Lemma 1.5.6. □

We now turn our attention to properties of morphisms.

Definition 1.5.7. Let P be a property of morphisms between nonconnective spectral Deligne-Mumford stacks. We will say that P is *local on the source with respect to the étale topology.* if the following conditions hold:

(i) For every composable pair of morphisms

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z},$$

if f is étale and g has the property P, then $g \circ f$ has the property P.

(ii) Given a jointly surjective collection of étale maps $\{f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y}\}$ and a morphism $g : \mathfrak{Y} \to \mathfrak{Z}$, if each of the composite maps $g \circ f_\alpha$ has the property P, then g has the property P.

Example 1.5.8. Let P be the property of being an étale morphism between nonconnective spectral Deligne-Mumford stacks. Then P is local on the source with respect to the étale topology.

Definition 1.5.9. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks. We will say that f is *flat* if the following condition is satisfied:

($*$) For every commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

in which the horizontal maps are étale, the underlying map of $\mathbb{E}_\infty$-rings $A \to B$ is flat.

Lemma 1.5.10. *Let $\phi : A \to B$ be an étale morphism of $\mathbb{E}_\infty$-rings, and let M be a B-module spectrum. If M is flat over A, then it is flat over B.*

Proof. If M is flat over A, then the tensor product $B \otimes_A M \simeq (B \otimes_A B) \otimes_B M$ is flat over B. Since ϕ is étale, B is a retract of $B \otimes_A B$, so that $M \simeq B \otimes_B M$ is a retract of $(B \otimes_A B) \otimes_B M$ and therefore also flat over B. ☐

Proposition 1.5.11. *Let $\phi : A \to B$ be a map of $\mathbb{E}_\infty$-rings. The following conditions are equivalent:*

(1) *The map ϕ is flat.*

(2) *The map ϕ induces a flat morphism $\operatorname{Spec} B \to \operatorname{Spec} A$ of nonconnective spectral Deligne-Mumford stacks.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious. Conversely, suppose that (1) is satisfied. Suppose we are given a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B' \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A' \arrow[r] & \operatorname{Spec} A \end{tikzcd}$$

where the horizontal maps are étale ; we wish to prove that B' is flat over A'. Using Theorem 1.2.1, we deduce that the map of $\mathbb{E}_\infty$-rings $B \to B'$ is étale. It follows that B' is flat over A. Since A' is étale over A (Theorem 1.2.1), the desired result follows from Lemma 1.5.10. ☐

Proposition 1.5.12. *The property of being a flat morphism (between nonconnective spectral Deligne-Mumford stacks) is local on the source with respect to the étale topology.*

Lemma 1.5.13. *Let $f : A \to B$ and $g : B \to C$ be maps of $\mathbb{E}_\infty$-rings. Then:*

(1) *If f and g are flat, then $g \circ f$ is flat.*

(2) *If $g \circ f$ is flat and g is faithfully flat, then f is flat.*

Proof. We first prove (1). The map $\pi_0 A \to \pi_0 C$ is a composition of two flat maps between commutative rings, and therefore a flat map. It therefore suffices to show that for each integer n, the map of abelian groups $\operatorname{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 C) \to \pi_n C$ is an isomorphism. We can factor this map as a composition

$$\begin{aligned} \operatorname{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 C) &\simeq \operatorname{Tor}_0^{\pi_0 B}(\operatorname{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B), \pi_0 C) \\ &\xrightarrow{\phi} \operatorname{Tor}_0^{\pi_0 B}(\pi_n B, \pi_0 C) \\ &\xrightarrow{\psi} \pi_n C. \end{aligned}$$

We conclude by observing that ϕ is an isomorphism because f is assumed to be flat, and the map ψ is an isomorphism because g is assumed to be flat.

We now prove (2). We first claim that f induces a flat map of commutative rings $\pi_0 A \to \pi_0 B$. To prove this, choose a monomorphism $M \to N$ of (discrete) $\pi_0 A$-modules, and let K be the kernel of the induced map $\mathrm{Tor}_0^{\pi_0 A}(M, \pi_0 B) \to \mathrm{Tor}_0^{\pi_0 A}(N, \pi_0 B)$; we wish to prove that $K = 0$. Since $\pi_0 B \to \pi_0 C$ is faithfully flat, it suffices to show that $\mathrm{Tor}_0^{\pi_0 C}(K, \pi_0 B)$ is zero. Using the flatness of $\pi_0 B \to \pi_0 C$, we can identify $\mathrm{Tor}_0^{\pi_0 C}(K, \pi_0 B)$ with the kernel of the map

$$\mathrm{Tor}_0^{\pi_0 B}(\mathrm{Tor}_0^{\pi_0 A}(M, \pi_0 B), \pi_0 C) \simeq \mathrm{Tor}_0^{\pi_0 A}(M, \pi_0 C) \to \mathrm{Tor}_0^{\pi_0 A}(N, \pi_0 C) \simeq \mathrm{Tor}_0^{\pi_0 B}(\mathrm{Tor}_0^{\pi_0 A}(N, \pi_0 B), \pi_0 C).$$

This map is a monomorhism, since $g \circ f$ is assumed to be flat.

To complete the proof that f is flat, we must show that for each integer n, the map $\mathrm{Tor}_0^{\pi_0 A}(\pi_n A, \pi_0 B) \to \pi_n B$ is an isomorphism. Since g is faithfully flat, we reduce to proving that the map ϕ above is an isomorphism. By a two-out-of-three argument, we are reduced to proving that the maps $\psi \circ \phi$ and ψ are isomorphisms. This follows from our assumption that g and $g \circ f$ are flat. $\square$

Proof of Proposition 1.5.12. Condition (i) of Definition 1.5.7 follows immediately from the definition. To prove (ii), suppose we are given a morphism $g : \mathfrak{Y} \to \mathfrak{Z}$ and a jointly surjective collection of étale maps $f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Z}$ such that each composition $g \circ f_\alpha$ is flat. We wish to show that g is flat. Choose a commutative diagram

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r] \arrow[d] & \mathfrak{Y} \arrow[d, "g"] \\ \operatorname{Spec} A \arrow[r] & \mathfrak{Z} \end{tikzcd}$$

where the horizontal maps are étale. We wish to show that B is flat over A. Since the maps f_α are jointly surjective, we can choose a finite collection of étale maps $\{B \to B_\beta\}$ such that $B \to \prod_\beta B_\beta$ is faithfully flat, and each of the induced maps $\operatorname{Spec} B_\beta \to \operatorname{Spec} B \to \mathfrak{Y}$ factors through some $\mathfrak{X}_\alpha$. Since $g \circ f_\alpha$ is assumed to be flat, we deduce that B_β is flat as an A-module. It follows that $\prod_\beta B_\beta$ is flat as an A-module. Using Lemma 1.5.13, we deduce that B is flat over A. $\square$

Remark 1.5.14. Let k be a commutative ring, regarded as a discrete $\mathbb{E}_\infty$-ring. Let $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a nonconnective spectral Deligne-Mumford stack over k. We say that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is *flat over* k if the map $(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to \operatorname{Spec} k$ is flat, in the sense of Definition 1.5.9. In this case, the structure sheaf $\mathcal{O}_\mathcal{X}$ is automatically discrete. Suppose that $\mathcal{X}$ is 1-localic and $\mathcal{O}_\mathcal{X}$ is discrete, so we may identify $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ with an ordinary Deligne-Mumford stack $\mathfrak{X}$ over k (Proposition VII.8.36). Then $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is flat over k if and only if $\mathfrak{X}$ is flat over k (in the sense of classical algebraic geometry). Consequently, Proposition VII.8.36 yields an equivalence between the ∞-category of 1-localic spectral Deligne-Mumford stacks which are flat over k and the ∞-category of ordinary Deligne-Mumford stacks which are flat over k.

Remark 1.5.15. Suppose we are given a pullback diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

of nonconnective spectral Deligne-Mumford stacks. If f is flat, then f' is flat.

Remark 1.5.16. Suppose we are given morphisms of nonconnective spectral Deligne-Mumford stacks

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z}.$$

If f and g are flat, then the composition $g \circ f$ is flat. To prove this, suppose we are given a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} C \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\
\operatorname{Spec} A \arrow[r] & \mathfrak{Z}
\end{tikzcd}$$

where the horizontal maps are étale ; we wish to show that C is flat over A. This assertion is local on $\operatorname{Spec} C$ with respect to the étale topology (Proposition 1.5.12), so we may assume that the map $\operatorname{Spec} C \to \operatorname{Spec} A \times_{\mathfrak{Z}} \mathfrak{Y}$ factors as a composition $\operatorname{Spec} C \to \operatorname{Spec} B \xrightarrow{u} \operatorname{Spec} A \times_{\mathfrak{Z}} \mathfrak{Y}$ where u is étale. Since f is flat, C is flat over B. Because g is flat, B is flat over A. It follows from Lemma 1.5.13 that C is flat over A.

Remark 1.5.17. Let k' be an $\mathbb{E}_\infty$-ring and let $k = \tau_{\geq 0} k'$ be its connective cover. The proof of Proposition VII.2.33 shows that if A is an $\mathbb{E}_\infty$-algebra over k' with $\operatorname{Spec} A = (\mathcal{X}, \mathcal{O}_\mathcal{X})$, then $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ is a spectral Deligne-Mumford stack which can be identified with the spectrum of $\tau_{\geq 0 A}$. In particular, if $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is flat over k', then $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ is flat over k. By reduction to the affine case, we obtain the more general global assertion: if $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a flat over k', then $(\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ is flat over k.

Notation 1.5.18. For every $\mathbb{E}_\infty$-ring A, we let $\mathrm{Stk}^{\mathrm{nc}}_A = \mathrm{Stk}^{\mathrm{nc}}_{/ \operatorname{Spec} A}$ denote the ∞-category of nonconnective spectral Deligne-Mumford stacks $\mathfrak{X}$ equipped with a map $\theta : \mathfrak{X} \to \operatorname{Spec} A$. We let $\mathrm{Stk}^\flat_A$ denote the full subcategory of $\mathrm{Stk}^{\mathrm{nc}}_A$ spanned by those objects for which the map θ is flat.

Proposition 1.5.19. *Let $f : A \to B$ be a morphism of $\mathbb{E}_\infty$-rings, and suppose that f induces an isomorphism $\pi_n A \to \pi_n B$ for $n \geq 0$. Then the pullback functor*

$$\mathfrak{X} \mapsto \mathfrak{X} \times_{\operatorname{Spec} A} \operatorname{Spec} B$$

induces an equivalence of ∞-categories $f^* : \mathrm{Stk}^\flat_B \to \mathrm{Stk}^\flat_A$.

Proof. It follows from Remark 1.5.15 that if $\mathfrak{X}$ is flat over A, then $f^* \mathfrak{X}$ is flat over B. Let $\overline{A}$ denote a connective cover of A (which is also a connective cover of B, since $\pi_n A \simeq \pi_n B$ for $n \geq 0$). We have a commutative diagram of pullback functors

$$\begin{tikzcd}
\mathrm{Stk}^\flat_B \arrow[rr] \arrow[dr] & & \mathrm{Stk}^\flat_A \arrow[dl] \\
& \mathrm{Stk}^\flat_{\overline{A}}. &
\end{tikzcd}$$

It will therefore suffice to prove that the vertical functors are equivalences of ∞-categories. We may therefore reduce to the case where f exhibits A as a connective cover of B. In this case, the functor f^* has a right adjoint G, given informally by the formula $G(\mathcal{X}, \mathcal{O}_\mathcal{X}) = (\mathcal{X}, \tau_{\geq 0} \mathcal{O}_\mathcal{X})$ (this functor preserves flatness by Remark 1.5.17). Consequently, it suffices to show that the unit and counit transformations

$$FG(\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{X}, \mathcal{O}_\mathcal{X}) \qquad (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \to GF(\mathcal{X}, \mathcal{O}_\mathcal{X})$$

are equivalences whenever $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is a nonconnective spectral Deligne-Mumford stack which is flat over B or $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is a spectral Deligne-Mumford stack which is flat over A. These assertions are local on $\mathcal{X}$ and $\mathcal{Y}$; we may therefore reduce to the affine case, where the desired result follows from Proposition A.7.2.2.24. □

We now introduce an analogue of Definition 1.5.2 for the flat topology.

Definition 1.5.20. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is a *flat covering* if the following conditions are satisfied:

(1) The map f is flat.

(2) For every quasi-compact open substack $\mathfrak{V} \hookrightarrow \mathfrak{Y}$, there exists a quasi-compact open substack $\mathfrak{U} \hookrightarrow \mathfrak{V}$ such that f induces a surjection $\mathfrak{U} \to \mathfrak{V}$.

Example 1.5.21. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an étale surjection. Then f is a flat covering. Condition (1) of Definition 1.5.20 is obvious. To prove (2), we first replace $\mathfrak{Y}$ by $\mathfrak{V}$ and thereby reduce to the case where $\mathfrak{Y}$ is quasi-compact. Choose an étale surjection $\operatorname{Spec} A \to \mathfrak{Y}$. Write $\operatorname{Spec} A \times_{\mathfrak{Y}} \mathfrak{X}$ as $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. For every affine object $X \in \mathcal{X}$, we can write $(\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}} | X) \simeq \operatorname{Spec} B$ for some étale A-algebra B, so that the map $\operatorname{Spec}^{\mathrm{Z}} B \to \operatorname{Spec}^{\mathrm{Z}} A$ has image given by some open subset $U_X \subseteq \operatorname{Spec}^{\mathrm{Z}} A$ (Proposition VII.0.2). Since f is surjective, the open sets U_X cover $\operatorname{Spec}^{\mathrm{Z}} A$. Since $\operatorname{Spec}^{\mathrm{Z}} A$ is quasi-compact, this open cover has a finite subcover. Taking the disjoint union of the corresponding objects of $\mathcal{X}$, we obtain an affine object $X \in \mathcal{X}$ such that the induced map $(\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}} | X) \to \operatorname{Spec} A$ is surjective. Since X is quasi-compact, the image of $(\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}} | X)$ in $\mathfrak{X}$ is a quasi-compact open substack $\mathfrak{U} \subseteq \mathfrak{X}$ having the desired properties.

Definition 1.5.22. Let P be a property of nonconnective spectral Deligne-Mumford stacks. We will say that P is *local for the fpqc topology* if the following conditions hold:

(i) For every flat morphism of nonconnective spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$, if $\mathfrak{Y}$ has the property P, then $\mathfrak{X}$ also has the property P.

(ii) Given a collection of flat morphisms $\{\mathfrak{X}_\alpha \to \mathfrak{Y}\}$, if each $\mathfrak{X}_\alpha$ has the property P and the induced map $\coprod \mathfrak{X}_\alpha \to \mathfrak{Y}$ is a flat covering, then $\mathfrak{Y}$ has the property P.

Proposition 1.5.23. *Let P be a property of nonconnective spectral Deligne-Mumford stacks. Then P is local for the fpqc topology if and only if the following conditions are satisfied:*

(1) *The property P is local for the étale topology (Definition 1.5.22).*

(2) *If $f : A \to B$ is a flat morphism of $\mathbb{E}_\infty$-rings such that* $\operatorname{Spec} A$ *has the property P, then* $\operatorname{Spec} B$ *has the property P. The converse holds provided that f is faithfully flat.*

Proof. If P is local for the fpqc topology, then condition (2) is obvious and condition (1) follows from Example 1.5.21. Conversely, suppose that conditions (1) and (2) are satisfied. We first verify condition (i) of Definition 1.5.22. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a flat morphism of spectral Deligne-Mumford stacks, and assume that $\mathfrak{Y}$ has the property P. We wish to show that $\mathfrak{X}$ has the property P. By virtue of assumption (1), this condition is local with respect to the étale topology on $\mathfrak{X}$. We may therefore assume that $\mathfrak{X} \simeq \operatorname{Spec} B$ and that the map f factors as a composition

$$\mathfrak{X} \to \operatorname{Spec} A \xrightarrow{f''} \mathfrak{Y}$$

where f'' is étale. Then $\operatorname{Spec} A$ has the property P. Since f is flat, B is flat over A. It then follows from (2) that $\mathfrak{X} \simeq \operatorname{Spec} B$ has the property P, as desired.

We now verify condition (ii). Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a flat covering and suppose that $\mathfrak{X}$ has the property P; we wish to show that $\mathfrak{Y}$ has the property P. The assertion is local with respect to the étale topology on $\mathfrak{Y}$, so we may suppose that $\mathfrak{Y} \simeq \operatorname{Spec} A$ is affine. In particular, $\mathfrak{Y}$ is quasi-compact. Replacing $\mathfrak{X}$ by an open substack if necessary (and using (1)), we can reduce to the case where $\mathfrak{X}$ is quasi-compact. We can then choose an étale surjection $\operatorname{Spec} B \to \mathfrak{X}$. Then $\operatorname{Spec} B$ has the property P (by (1)) and B is faithfully flat over A, so that $\operatorname{Spec} A$ has the property P by (2). $\square$

Example 1.5.24. Let P be the property of being a spectral Deligne-Mumford stack, so that a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ has the property P if and only if the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is connective. Then P is local with respect to the fpqc topology.

Example 1.5.25. Let P be the property of being an n-truncated spectral Deligne-Mumford stack, so that $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ has the property P if and only if the structure sheaf $\mathcal{O}_{\mathcal{X}}$ is connective and n-truncated. Then P is local with respect to the fpqc topology.

Definition 1.5.26. Let P be a property of morphisms between nonconnective spectral Deligne-Mumford stacks. We will say that P is *local on the source with respect to the fpqc topology* if the following conditions hold:

(i) For every composable pair of morphisms

$$\mathfrak{X} \xrightarrow{f} \mathfrak{Y} \xrightarrow{g} \mathfrak{Z},$$

if f is flat and g has the property P, then $g \circ f$ has the property P.

(ii) Given a collection of flat morphisms $\{f_\alpha : \mathfrak{X}_\alpha \to \mathfrak{Y}\}$ which determine a flat covering $\coprod_\alpha \mathfrak{X}_\alpha \to \mathfrak{Y}$ and a morphism $g : \mathfrak{Y} \to \mathfrak{Z}$, if each of the composite maps $g \circ f_\alpha$ has the property P, then g has the property P.

Proposition 1.5.23 has the following analogue for properties of morphisms between nonconnective spectral Deligne-Mumford stacks, which is proven in the same way:

Proposition 1.5.27. *Let P be a property of morphisms between nonconnective spectral Deligne-Mumford stacks. Then P is local on the source for the fpqc topology if and only if the following conditions are satisfied:*

(1) *The property P is local on the source for the étale topology (Definition 1.5.22).*

(2) *Suppose we are given a pair of maps*

$$\operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A \xrightarrow{g} \mathfrak{Z}$$

such that B is flat over A. If g has the property P, then $g \circ f$ has the property P. The converse holds if B is faithfully flat over A.

Example 1.5.28. The property of being a flat morphism is local with respect to the fpqc topology. In view of Proposition 1.5.12, it will suffice to show that the property of flatness satisfies condition (2) of Proposition 1.5.27. The first assertion is obvious (since the collection of flat morphisms is closed under composition, by Remark 1.5.16). Conversely, suppose that B is faithfully flat over A and that we are given a map $g : \operatorname{Spec} A \to \mathfrak{Z}$ such that the composite map $\operatorname{Spec} B \to \operatorname{Spec} A \to \mathfrak{Z}$ is flat. We wish to show that g is flat. This follows immediately from the definitions, together with Lemma 1.5.13.

2 Quasi-Coherent Sheaves

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. In this section, we will introduce the notion of a *quasi-coherent sheaf* on $\mathfrak{X}$. The collection of quasi-coherent sheaves on $\mathfrak{X}$ is naturally organized into a ∞-category, which we will denote by $\mathrm{QCoh}(\mathfrak{X})$. The ∞-category $\mathrm{QCoh}(\mathfrak{X})$ is our principal object of study in this paper.

We begin in a more general setting. Let $(\mathcal{X}, \mathcal{O})$ be an arbitrary spectrally ringed ∞-topos. In §2.1, we will introduce a stable ∞-category $\mathrm{Mod}_{\mathcal{O}}$, whose objects are $\mathcal{O}$-module objects in the ∞-category $\mathrm{Stab}(\mathcal{X}) \simeq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ of sheaves of spectra on $\mathcal{X}$. Our main result (Proposition 2.1.8) asserts that if $\mathcal{O}$ is discrete and $\mathcal{X}$ is 1-localic, then $\mathrm{Mod}_{\mathcal{O}}$ contains $\mathcal{D}^+(\mathcal{A})$ as a full subcategory, where $\mathcal{A}$ is the abelian category of discrete sheaves of $\mathcal{O}$-modules on $\mathcal{X}$.

Suppose now that $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack. In §2.3, we define a full subcategory $\mathrm{QCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$, which we call the *$\infty$-category of quasi-coherent sheaves on* $\mathfrak{X}$. It is uniquely characterized by the following pair of assertions:

(a) Let $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{X}$. Then the condition that $\mathcal{F}$ be quasi-coherent is of a local nature on $\mathcal{X}$. In particular, $\mathcal{F}$ is quasi-coherent if and only if, for every affine $U \in \mathcal{X}$, the restriction $\mathcal{F}|U$ is a quasi-coherent sheaf on the spectral Deligne-Mumford stack $(\mathcal{X}_{/U}, \mathcal{O}|U)$ (Remark 2.3.7).

(b) Suppose that $\mathfrak{X} = \operatorname{Spec} A$ is affine. Then the global sections functor $\Gamma : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_A$ admits a fully faithful left adjoint, whose essential image is the full subcategory $\mathrm{QCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ (Proposition 2.3.11).

The class of quasi-coherent sheaves can be given an alternative characterization using the language of geometries developed in [42]. We will describe the relevant geometries in §2.2. This material is of a somewhat technical nature (and is not really needed in the rest of the paper); the reader should feel free to skip over it.

Recall that a scheme X is said to be *quasi-affine* if X is quasi-compact and there exists an open immersion of X into an affine scheme. This notion has an obvious generalization to the setting of spectral algebraic geometry: we say that a spectral Deligne-Mumford stack $\mathfrak{X}$ is *quasi-affine* it is equivalent to a quasi-compact open substack of an affine spectral Deligne-Mumford stack. In §2.4, we will study the class of quasi-affine spectral Deligne-Mumford stacks. Our main results are that a quasi-affine spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ can be recovered from the $\mathbb{E}_\infty$-ring of global sections $A = \Gamma(\mathcal{X}; \mathcal{O})$ (Proposition 2.4.9), and that the global sections functor induces an equivalence of stable ∞-categories $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_A$ (Proposition 2.4.4).

Let $f : (\mathcal{X}, \mathcal{O}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ be a map of spectrally ringed ∞-topoi. Then f determines a pair of adjoint functors

$$\mathrm{Mod}_{\mathcal{O}_\mathcal{Y}} \underset{f_*}{\overset{f^*}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}_\mathcal{X}} .$$

If $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ and $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ are spectral Deligne-Mumford stacks, then the pullback functor f^* carries quasi-coherent sheaves on $\mathfrak{Y}$ to quasi-coherent sheaves on $\mathfrak{X}$. In §2.5, we will study conditions on f which guarantee that the pushforward f_* also preserves quasi-coherence. In particular, we will show that f_* preserves coherence when the morphism f is quasi-affine (Corollary 2.5.15). We will deduce this from a more general statement (Proposition 2.5.12) which we will later use to prove analogous results in the setting of quasi-compact, quasi-separated spectral algebraic spaces.

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. In §2.6, we will study properties of the quasi-coherent sheaf $\mathcal{F}$ which can be tested locally on $\mathfrak{X}$. We are particularly interested in finiteness conditions on $\mathcal{F}$. For example, when $\mathfrak{X}$ is locally Noetherian (Definition 1.5.4), we will show that there is a well-behaved full subcategory $\mathrm{Coh}(\mathfrak{X}) \subseteq \mathrm{QCoh}(\mathfrak{X})$, which we call the *$\infty$-category of coherent sheaves on* $\mathfrak{X}$ (Definition 2.6.20).

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. Then $\mathfrak{X}$ determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, given informally by $X(R) = \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X})$. In §V.2.4, we saw that $\mathfrak{X}$ is determined by the functor X, up to canonical equivalence. In §2.7, we will explain how to reconstruct the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ directly from the functor X. More precisely, we associate to any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ an ∞-category $\mathrm{QCoh}(X)$; roughly speaking, an object $\mathcal{F} \in \mathrm{QCoh}(X)$ is a rule which associates to each point $\eta \in X(R)$ an R-module $\eta^* \mathcal{F}$, depending functorially on R. Our main result (Proposition 2.7.18) asserts that if X is represented by a spectral Deligne-Mumford stack $\mathfrak{X}$, then the ∞-categories $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(\mathfrak{X})$ are equivalent to one another. Consequently, we may view the construction $X \mapsto \mathrm{QCoh}(X)$ as a reasonable extrapolation of the theory of quasi-coherent sheaves to the setting of non-representable functors.

2.1 Sheaves on a Spectrally Ringed ∞-Topos

Let X be a topological space and let $\mathcal{O}$ be a sheaf of commutative rings on X. A *sheaf of $\mathcal{O}$-modules* is a sheaf of abelian groups $\mathcal{F}$ on X such that $\mathcal{F}(U)$ is equipped with the structure of a module over the commutative ring $\mathcal{O}(U)$ for every open subset $U \subseteq X$, which depends functorially on U. Our goal in this section is to introduce an ∞-categorical analogue of the theory of sheaves of modules. We will replace the topological space X with an arbitrary ∞-topos $\mathcal{X}$, and $\mathcal{O}$ by an arbitrary sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

Definition 2.1.1. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O} \in \mathrm{Shv}_{\mathrm{CAlg}}(\mathcal{X})$ be a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Recall that $\mathcal{O}$ can be identified with a commutative algebra object of the symmetric monoidal ∞-category $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$ of sheaves of spectra on $\mathcal{X}$ (see §VII.9.25). We let $\mathrm{Mod}_{\mathcal{O}}$ denote the ∞-category $\mathrm{Mod}_{\mathcal{O}}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}))$ of $\mathcal{O}$-module objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$. Then $\mathrm{Mod}_{\mathcal{O}}$ can be regarded as a symmetric monoidal ∞-category with respect to the relative tensor product $\otimes_{\mathcal{O}}$ (see §A.3.4.4). We will refer to the objects of $\mathrm{Mod}_{\mathcal{O}}$ as *sheaves of $\mathcal{O}$-modules on* $\mathcal{X}$, or sometimes just as *$\mathcal{O}$-modules.*

Warning 2.1.2. Let X be a topological space and let $\mathcal{O}$ be a sheaf of commutative rings on X. Then we can identify $\mathcal{O}$ with a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathrm{Shv}(X)$. In this case, Definition 2.1.1 does not recover the classical theory of sheaves of $\mathcal{O}$-modules on X, because we allow ourselves to consider sheaves of spectra rather than sheaves of abelian groups. In a moment, we will prove that the ∞-category $\mathrm{Mod}_{\mathcal{O}}$ is stable and equipped with a natural t-structure (Proposition 2.1.3). The classical theory of sheaves of $\mathcal{O}$-modules can be recovered as the abelian category $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$. Moreover, the ∞-category $\mathrm{Mod}_{\mathcal{O}}$ is closely related to the derived ∞-category of the abelian category $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ (see Proposition 2.1.8).

We now summarize some of the formal properties of Definition 2.1.1.

Proposition 2.1.3. *Let $\mathcal{X}$ be an ∞-topos and $\mathcal{O}$ a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then:*

(1) *The ∞-category $\mathrm{Mod}_{\mathcal{O}}$ is stable.*

(2) *The ∞-category $\mathrm{Mod}_{\mathcal{O}}$ is presentable and the tensor product $\otimes_{\mathcal{O}} : \mathrm{Mod}_{\mathcal{O}} \times \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}}$ preserves small colimits separately in each variable.*

(3) *The forgetful functor $\theta : \mathrm{Mod}_{\mathcal{O}} \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is conservative and preserves small limits and colimits.*

Suppose that $\mathcal{O}$ is connective. Then:

(4) *The ∞-category $\mathrm{Mod}_{\mathcal{O}}$ admits a t-structure, defined by $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0} = \theta^{-1}\,\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ and $(\mathrm{Mod}_{\mathcal{O}})_{\leq 0} = \theta^{-1}\,\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$.*

(5) *The t-structure on $\mathrm{Mod}_{\mathcal{O}}$ is accessible: that is, the ∞-category $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ is presentable (see Proposition A.1.4.5.13).*

(6) *The t-structure on $\mathrm{Mod}_{\mathcal{O}}$ is compatible with the symmetric monoidal structure on $\mathrm{Mod}_{\mathcal{O}}$. In other words, $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ contains the unit object of $\mathrm{Mod}_{\mathcal{O}}$ and is stable under tensor product.*

(7) *The t-structure on $\mathrm{Mod}_{\mathcal{O}}$ is compatible with filtered colimits: in other words, $(\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$ is stable under filtered colimits in $\mathrm{Mod}_{\mathcal{O}}$.*

(8) *The t-structure on $\mathrm{Mod}_{\mathcal{O}}$ is right complete.*

Proof. Assertion (1) follows from Proposition A.7.1.1.4, assertion (2) follows from Theorem A.3.4.4.2, and assertion (3) follows from Corollaries A.3.4.3.2 and A.3.4.4.6. Suppose that $\mathcal{O}$ is connective. We observe that $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ can be identified with the ∞-category of $\mathcal{O}$-module objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$, so that (5) follows from Theorem A.3.4.4.2. Assume (4) for the moment. To prove (6), we note that the unit object of $\mathrm{Mod}_{\mathcal{O}}$ has image $\mathcal{O} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$, and that for every pair of objects $\mathcal{F}, \mathcal{G} \in \mathrm{Mod}_{\mathcal{O}}$, the image in $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ of the relative tensor product $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ can be identified with the geometric realization of a simplicial object whose entires are tensor products $\mathcal{F} \otimes \mathcal{O} \otimes \cdots \otimes \mathcal{O} \otimes \mathcal{G}$. Since $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{O}$ are connective, the above tensor product is connective (Proposition VII.1.16); because $\mathrm{Mod}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$ is closed under colimits we conclude that $\mathcal{F} \otimes_{\mathcal{O}} \mathcal{G}$ is connective. Assertion (7) follows from (3) together Proposition VII.1.7, and assertion (8) follows from (7) and Proposition A.1.2.1.19.

It remains to prove (4). Since the full subcategory $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0} \subseteq \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})$ is closed under small colimits and extensions, we conclude that $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ is closed under small colimits and extensions in $\mathrm{Mod}_{\mathcal{O}}$ (since the forgetful functor θ preserves small colimits, by (3)). Using Proposition A.1.4.5.11, we deduce the existence of an accessible t-structure $((\mathrm{Mod}_{\mathcal{O}})_{\geq 0}, \mathrm{Mod}'_{\mathcal{O}})$ on $\mathrm{Mod}_{\mathcal{O}}$. To complete the proof, it will suffice to show that $\mathrm{Mod}'_{\mathcal{O}} = (\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$. Suppose first that $\mathcal{F} \in \mathrm{Mod}'_{\mathcal{O}}$. Then the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}(\mathcal{G}[1], \mathcal{F})$ is contractible for every object $\mathcal{G} \in (\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$. In particular, for every connective sheaf of spectra $\mathcal{M} \in \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$, the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}((\mathcal{M} \otimes \mathcal{O})[1], \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})}(\mathcal{M}[1], \theta(\mathcal{F}))$ vanishes, so that $\theta(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$ and therefore $\mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$.

Conversely, suppose that $\mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$. We wish to prove that $\mathcal{F} \in \mathrm{Mod}'_{\mathcal{O}}$. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Mod}_{\mathcal{O}}$ spanned by those objects $\mathcal{G} \in \mathrm{Mod}_{\mathcal{O}}$ for which the mapping space $\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}(\mathcal{G}[1], \mathcal{F})$ is contractible. We wish to prove that $\mathcal{C}$ contains $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$. Condition (3) shows that θ induces a functor $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0} \rightarrow \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\geq 0}$which is conservative and preserves small colimits; moreover, this functor has a left adjoint F, given informally by the formula $F(\mathcal{M}) \simeq \mathcal{O} \otimes \mathcal{M}$. Using Proposition A.6.2.2.11, we conclude that $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ is generated under the formation of geometric realizations by the essential image of F. Since $\mathcal{C}$ is stable under colimits, it will suffice to show that $\mathcal{C}$ contains the essential image of F. Unwinding the definitions, we are reduced to proving that the mapping space

$$\mathrm{Map}_{\mathrm{Mod}_{\mathcal{O}}}(F(\mathcal{M})[1], \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})}(\mathcal{M}[1], \theta(\mathcal{F}))$$

is contractible for every connective sheaf of spectra $\mathcal{M}$ on $\mathcal{X}$, which is equivalent to our assumption that $\theta(\mathcal{F}) \in \mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})_{\leq 0}$. □

Remark 2.1.4. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. We will say a sheaf of $\mathcal{O}$-modules $\mathcal{M}$ is *connective* if it is connective as an object of $\mathrm{Shv}_{\mathrm{Sp}}(\mathcal{X})$: that is, if it belongs to $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$. We will often denote the ∞-category $(\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ by $\mathrm{Mod}_{\mathcal{O}}^{\mathrm{cn}}$.

Remark 2.1.5. Let $\mathcal{X}$ be an ∞-topos and let $\mathcal{O}$ be a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then we can identify $\pi_0 \mathcal{O}$ with a commutative ring object in the underlying topos of $\mathcal{X}$. Unwinding the definitions, we can identify the heart $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ of the t-structure of Proposition 2.1.3 with (the nerve of) the ordinary category of $(\pi_0 \mathcal{O})$-module objects in the underlying topos of $\mathcal{X}$.

Warning 2.1.6. The t-structure of Proposition 2.1.3 is generally not left-complete.

Notation 2.1.7. Let $(\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos, and suppose we are given objects $\mathcal{F}, \mathcal{F}' \in \mathrm{Mod}_{\mathcal{O}}$. For every integer n, we let $\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{F}, \mathcal{F}')$ denote the abelian group $\mathrm{Ext}^n_{\mathrm{Mod}_{\mathcal{O}}}(\mathcal{F}, \mathcal{F}')$ of homotopy classes of maps from $\mathcal{F}$ to $\mathcal{F}'[n]$ in $\mathrm{Mod}_{\mathcal{O}}$.

Let $\mathcal{X}$ be an ∞-topos and $\mathcal{O}$ a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, so that the ∞-category $\mathrm{Mod}_{\mathcal{O}}$. In what follows, we will abuse notation by identifying the heart $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ (which is a full subcategory of $\mathrm{Mod}_{\mathcal{O}}$) with its homotopy category (which is an abelian category). According to Remark A.1.3.4.23, the inclusion $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit} \subseteq \mathrm{Mod}_{\mathcal{O}}$ admits an essentially unique extension to a t-exact functor $\mathcal{D}^+(\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}) \rightarrow \mathrm{Mod}_{\mathcal{O}}$, where $\mathcal{D}^+(\mathrm{Mod}_{\mathcal{O}}^{\heartsuit})$ denotes the derived ∞-category of $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ (see §A.1.3.1).

Proposition 2.1.8. *Let $\mathcal{X}$ be a 1-localic ∞-topos and let $\mathcal{O}$ be a discrete sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then the functor*

$$F : \mathcal{D}^+(\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}) \rightarrow \mathrm{Mod}_{\mathcal{O}}$$

defined above is fully faithful, and its essential image is the full subcategory $\bigcup_n (\mathrm{Mod}_{\mathcal{O}})_{\leq n}$ of left-bounded objects of $\mathrm{Mod}_{\mathcal{O}}$.

The proof of Proposition 2.1.8 will require a brief digression. Let $\mathcal{X}$ be an ∞-topos and $\mathcal{O}$ a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$. Then for every object $X \in \mathcal{X}$, we let $\mathcal{O}\,|X$ denote the composition of $\mathcal{O}$ with the forgetful functor $\pi : \mathcal{X}_{/X} \rightarrow \mathcal{X}$, so that $\mathcal{O}\,|X$ is a sheaf of $\mathbb{E}_\infty$-rings on the ∞-topos $\mathcal{X}_{/X}$. Composition with π determines a pullback functor $\mathrm{Mod}_{\mathcal{O}} \rightarrow \mathrm{Mod}_{\mathcal{O}\,|X}$, which we will denote by π^*. The functor π^* preserves small limits and colimits, and therefore admits a left adjoint $\pi_! : \mathrm{Mod}_{\mathcal{O}\,|X} \rightarrow \mathrm{Mod}_{\mathcal{O}}$ (Corollary T.5.5.2.9).

Lemma 2.1.9. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{O}$ a connective sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and X a discrete object of $\mathcal{X}$. Then the functor $\pi_! : \mathrm{Mod}_{\mathcal{O}\,|X} \rightarrow \mathrm{Mod}_{\mathcal{O}}$ is t-exact (with respect to the t-structures introduced in Proposition 2.1.3).*

Proof. The functor $\pi_!$ is obviously right t-exact (since it is the left adjoint of the t-exact pullback functor $\pi^* : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}\,|X}$). It will therefore suffice to show that $\pi_!$ is left t-exact: that is, that $\pi_!$ carries $(\mathrm{Mod}_{\mathcal{O}\,|X})_{\leq 0}$ to $(\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$.

Without loss of generality, we may assume that $\mathcal{X}$ is an accessible left-exact localization of a presheaf ∞-category $\mathcal{P}(\mathcal{C})$, for some small ∞-category $\mathcal{C}$; we will identify X with the corresponding discrete $\mathcal{P}(\mathcal{C})$. Then $\mathcal{O}$ can be obtained as the pullback of a connective sheaf of $\mathbb{E}_\infty$-rings $\mathcal{O}' \in \mathcal{S}\mathrm{hv}_{\mathrm{CAlg}}(\mathcal{P}(\mathcal{C})) \simeq \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{CAlg})$. We have a homotopy commutative diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Mod}_{\mathcal{O}'\,|X} \arrow[r, "\pi'_!"] \arrow[d, "g^*"] & \mathrm{Mod}_{\mathcal{O}'} \arrow[d, "f^*"] \\ \mathrm{Mod}_{\mathcal{O}\,|X} \arrow[r] & \mathrm{Mod}_{\mathcal{O}}, \end{tikzcd}$$

where the vertical maps are given by pullback along the geometric morphisms

$$f_* : \mathcal{X} \subseteq \mathcal{P}(\mathcal{C}) \qquad g_* : \mathcal{X}_{/X} \subseteq \mathcal{P}(\mathcal{C})_{/X}$$

(and are therefore t-exact). For any object $\mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}\,|X})_{\leq 0}$, there exists an object $\mathcal{F}' \in (\mathrm{Mod}_{\mathcal{O}'\,|X})_{\leq 0}$ such that $\mathcal{F} \simeq g^* \mathcal{F}'$: for example, we can take $\mathcal{F}'$ to be the pushforward $g_* \mathcal{F}$. Since the functor f^* is t-exact, to prove that $\pi_! \mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$, it will suffice to show that $\pi'_! \mathcal{F}' \in (\mathrm{Mod}_{\mathcal{O}'})_{\leq 0}$. In other words, we wish to show that for every object $C \in \mathcal{C}$, the $\mathcal{O}'(C)$-module spectrum $(\pi'_! \mathcal{F}')(C)$ belongs to $\mathrm{Sp}_{\leq 0}$. Since X is discrete, we may assume without loss of generality that X is a $\mathcal{S}\mathrm{et}$-valued functor on $\mathcal{C}^{op}$. Note that $(\pi'_! \mathcal{F}')(C)$ can be written as a coproduct

$$\coprod_{\eta \in X(C)} \mathcal{F}'(C_\eta)$$

where $C_\eta \in \mathcal{P}(\mathcal{C})_{/X}$ denotes map $j(C) \to X$ representing $\eta \in X(C)$, where $j : \mathcal{C} \to \mathcal{P}(\mathcal{C})$ is the Yoneda embedding. Since $\mathcal{F}' \in (\mathrm{Mod}_{\mathcal{O}'\,|X})_{\leq 0}$, each of the spectra $\mathcal{F}'(C_\eta) \in (\mathrm{Sp})_{\leq 0}$, so that $(\pi_! \mathcal{F}')(C) \in \mathrm{Sp}_{\leq 0}$ as desired. ☐

We will need an amplification of Remark T.7.2.2.17:

Lemma 2.1.10. *Let $\mathcal{C}$ be a small category which admits finite limits and is equipped with a Grothendieck topology. Let $\mathcal{O}$ be a sheaf of commutative rings on $\mathcal{C}$, and let $\mathcal{F}$ be an injective object of the abelian category of sheaves of discrete $\mathcal{O}$-modules on $\mathcal{C}$. Then $\mathrm{H}^n(\mathcal{S}\mathrm{hv}(\mathcal{C}); \mathcal{F}) \simeq 0$ for $n > 0$ (where the cohomology groups $\mathrm{H}^n(\mathcal{S}\mathrm{hv}(\mathcal{C}); \mathcal{F})$ are defined as in § T.7.2.2); here we identify $\mathcal{F}$ with the underlying sheaf of abelian groups on $\mathcal{C}$.*

Proof. We can identify the heart $\mathrm{Sp}^{\heartsuit}$ of the ∞-category of spectra with the nerve of the ordinary category of abelian groups. By means of this identification, we can identify $\mathcal{F}$ with a functor $\mathcal{F}' : \mathrm{N}(\mathcal{C})^{op} \to \mathrm{Sp}$ which takes values in $\mathrm{Sp}^{\heartsuit}$. Using the injectivity of $\mathcal{F}$, we will prove the following:

$(*)$ The functor $\mathcal{F}'$ is a Sp-valued sheaf on $\mathrm{N}(\mathcal{C})$.

Assuming $(*)$ for the moment, let us complete the proof. Assertion $(*)$ implies that for every integer $n > 0$, the composite functor

$$\mathrm{N}(\mathcal{C})^{op} \xrightarrow{\mathcal{F}'} \mathrm{Sp} \xrightarrow{\Omega^{\infty - n}} \mathcal{S}_*$$

is a $\mathcal{S}$-valued sheaf $\mathcal{F}[n]$ on $\mathrm{N}(\mathcal{C})$, which we may identify with a pointed object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$. We note that $\mathcal{F}[n]$ is n-connective (since its value on any object $C \in \mathcal{C}$ is n-connective) and that $\Omega^n \mathcal{F}[n]$ can be identified with $\mathcal{F}$, so that $\mathcal{F}[n]$ is an Eilenberg-MacLane object $K(\mathcal{F}, n)$ of the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$. In particular, the cohomology group $\mathrm{H}^n(\mathcal{S}\mathrm{hv}(\mathcal{C}); \mathcal{F})$ can be identified with $\pi_0 \Gamma(\mathcal{S}\mathrm{hv}(\mathcal{X}); \mathcal{F}[n]) \simeq \pi_0 \mathcal{F}[n](\mathbf{1}_{\mathcal{C}}) \simeq *$, where $\mathbf{1}_{\mathcal{C}}$ denotes the final object of $\mathcal{C}$.

It remains to prove $(*)$. Fix an object $C \in \mathcal{C}$ and a covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$; we wish to prove that the natural map

$$\mathcal{F}'(C) \to \varprojlim_{C' \in \mathcal{C}^{(0)}_{/C}} \mathcal{F}'(C')$$

is an equivalence of spectra. Replacing $\mathcal{C}$ by $\mathcal{C}_{/C}$, we may assume that C is the final object of $\mathbf{1}_{\mathcal{C}} \in \mathcal{C}$ (Lemma 2.1.10 implies that the forgetful functor from sheaves on $\mathcal{C}$ to sheaves on $\mathcal{C}_{/C}$ has an exact left adjoint, and therefore carries injective objects to injective objects). We may therefore identify $\mathcal{C}^{(0)}_{/C}$ with a sieve $\mathcal{C}^{(0)} \subseteq \mathcal{C}$. Fix a collection of objects $\{C_\alpha \in \mathcal{C}^{(0)}\}_{\alpha \in B}$ which generate the sieve $\mathcal{C}^{(0)}$. Let $\mathbf{\Delta}_B$ denote the category whose objects are pairs $([n], c)$ where $[n]$ is an object of $\mathbf{\Delta}$ and $c : [n] \to B$ is a map of sets, and whose morphisms are commutative diagrams

$$\begin{tikzcd} {[n]} \arrow[rr, "f"] \arrow[dr, "c"'] & & {[n']} \arrow[dl, "c"] \\ & B & \end{tikzcd}$$

where f is a morphism in $\mathbf{\Delta}$. There is an evident functor $\tau : \mathbf{\Delta}_B^{op} \to \mathcal{C}^{(0)}$ which carries a pair $([n], c)$ to the product $\prod_{0 \leq i \leq n} C_{c(i)}$. We claim that the induced map $\mathrm{N}(\mathbf{\Delta}_B^{op}) \to \mathrm{N}(\mathcal{C}^{(0)})$ is left cofinal. In view of Theorem T.4.1.3.1, it will suffice to show that for every object $D \in \mathcal{C}^{(0)}$, the fiber product $\mathbf{\Delta}_B^{op} \times_{\mathcal{C}^{(0)}} (\mathcal{C}^{(0)})_{D/}$ has weakly contractible nerve. Unwinding the definitions, the fiber product category can be identified with $\mathbf{\Delta}_{B'}^{op}$, where $B' = \coprod_{\alpha \in B} \mathrm{Hom}_{\mathcal{C}}(D, C_\alpha)$. This is the category of simplices of the simplicial set $\mathrm{N}(B')$, where we view B' as a groupoid in which there is a unique isomorphism between each pair of objects. Since the objects C_α generate the sieve $\mathcal{C}^{(0)}$, the set B' is nonempty, so that $\mathrm{N}(B')$ is a contractible Kan complex.

Since τ is left cofinal, it is sufficient to show that the canonical map

$$\mathcal{F}'(\mathbf{1}_{\mathcal{C}}) \to \varprojlim(\mathcal{F}' \circ \tau)$$

is an equivalence of spectra. Let $X^\bullet : \mathrm{N}(\mathbf{\Delta}) \to \mathrm{Sp}$ be a right Kan extension of $\mathcal{F}' \circ \tau$ along the forgetful functor $\mathrm{N}(\mathbf{\Delta}_B) \to \mathrm{N}(\mathbf{\Delta})$, so that $\varprojlim(\mathcal{F}' \circ \tau)$ is equivalent to the limit of the cosimplicial spectrum $X^\bullet$. More informally, $X^\bullet$ can be described by the formula

$$X^n \simeq \prod_{c:[n] \to B} \mathcal{F}'(C_{c(0)} \times \cdots \times C_{c(n)}).$$

In particular, each X^n belongs to the heart $\mathrm{Sp}^\heartsuit$, and can be identified with the abelian group

$$\pi_0 X^n \simeq \prod_{c:[n] \to B} \mathcal{F}(C_{c(0)} \times \cdots \times C_{c(n)}).$$

According to Example A.1.2.4.8, the homotopy groups of $\varprojlim(\mathcal{F}' \circ \tau)$ are given by the cohomology groups of the cochain complex

$$\pi_0 X^0 \to \pi_0 X^1 \to \pi_0 X^2 \to \cdots$$

To complete the proof, we must show that the augmented cochain complex

$$0 \to \mathcal{F}(\mathbf{1}_{\mathcal{C}}) \to \pi_0 X^0 \to \pi_0 X^1 \to \cdots$$

is acyclic.

Let $\mathcal{A}$ denote the abelian category of sheaves of (discrete) $\mathcal{O}$-modules on $\mathcal{C}$. For each object $D \in \mathcal{C}$, the evaluation functor $\mathcal{G} \mapsto \mathcal{G}(D)$ is corepresented by an object $e(D) \in \mathcal{A}$. Let $A_\bullet$ be the augmented simplicial object of $\mathcal{A}$ given by the formula $A_n = \bigoplus_{c:[n]\to B} e(C_{c(0)} \times \cdots \times C_{c(n)})$ Using this notation, we can rewrite the above chain complex as

$$0 \to \mathrm{Hom}_{\mathcal{A}}(A_{-1}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{A}}(A_0, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{A}}(A_1, \mathcal{F}) \to \cdots$$

We wish to prove that this chain complex of abelian groups is exact. Since $\mathcal{F}$ is an injective object of $\mathcal{A}$, it suffices to show that the chain complex

$$\cdots \to A_2 \to A_1 \to A_0 \to A_{-1} \to 0$$

is exact in $\mathcal{A}$. Since exactness can tested locally, we are free to replace $\mathcal{C}$ by $\mathcal{C}_{/C_\alpha}$ for some $\alpha \in B$, and thereby reduce to the case where there exists a map $s : \mathbf{1}_{\mathcal{C}} \to C_\alpha$. In this case, the above chain complex is split exact, since we can use s to endow $A_\bullet$ with the structure of a split augmented simplicial object of $\mathcal{A}$. □

Proof of Proposition 2.1.8. In view of Proposition A.1.3.2.6, it will suffice to prove the following: if $\mathcal{F}$ and $\mathcal{G}$ are objects of the heart $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ and $\mathcal{G}$ is injective, then there exists an epimorphism $\mathcal{F}' \to \mathcal{F}$ in $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ such that the abelian groups $\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{F}', \mathcal{G})$ vanish for $n > 0$.

Since $\mathcal{X}$ is 1-localic, we can assume without loss of generality that $\mathcal{X} \simeq \mathcal{S}\mathrm{hv}(\mathrm{N}(\mathcal{C}))$, where $\mathcal{C}$ is a small category which admits finite limits and which is equipped with a Grothendieck topology. For every object $C \in \mathcal{C}$, let $X_C \in \mathcal{S}\mathrm{hv}(\mathcal{C})$ denote the sheafification of the presheaf represented by C and $\mathcal{O}_C$ the restriction $\mathcal{O}\,|X_C$. We have a pair of adjoint functors

$$\mathrm{Mod}_{\mathcal{O}_C} \underset{\pi(C)^*}{\overset{\pi(C)_!}{\rightleftarrows}} \mathrm{Mod}_{\mathcal{O}}.$$

Since $\mathcal{C}$ is the nerve of a category, X_C is a discrete object of $\mathcal{S}\mathrm{hv}(\mathcal{C})$ so that Lemma 2.1.9 guarantees that $\pi(C)_!$ is t-exact.

We can identify a section $\eta \in \mathcal{F}(C)$ with a map $\mathcal{O}_C \to \pi(C)^* \mathcal{F}$ of $\mathcal{O}_C$-modules. In particular, η induces a map $u(\eta) : \pi(C)_! \mathcal{O}_C \to \mathcal{F}$. Let $\mathcal{F}' = \bigoplus_{C \in \mathcal{C}, \eta \in \mathcal{F}(C)} \pi(C)_! \mathcal{O}_C$. The maps $u(\eta)$ amalgamate to a morphism $u : \mathcal{F}' \to \mathcal{F}$ of discrete $\mathcal{O}$-modules. We note that u is an epimorphism (by construction, it determines an epimorphism after evaluation at any object $C \in \mathcal{C}$).

To complete the proof, it will suffice to show that the groups $\mathrm{Ext}^n_{\mathcal{O}}(\mathcal{F}', \mathcal{G})$ vanish for $n > 0$. These groups can be written as a product

$$\prod_{C,\eta} \mathrm{Ext}^n_{\mathcal{O}}(\eta(C)_! \mathcal{O}_C, \mathcal{G}) \simeq \prod_{C,\eta} \mathrm{Ext}^n_{\mathcal{O}_C}(\mathcal{O}_C, \pi(C)^* \mathcal{G}).$$

It therefore suffices to show that, for each $C \in \mathcal{C}$ and each $n > 0$, the groups $\mathrm{Ext}^n_{\mathcal{O}_C}(\mathcal{O}_C, \pi(C)^* \mathcal{G}) \simeq \mathrm{H}^n(\mathcal{X}; \pi(C)^* \mathcal{G})$ vanish. This follows from Lemma 2.1.10 (note that $\pi(C)^*$ is the right adjoint to a t-exact functor, and therefore carries injective objects of $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$ to injective objects of $\mathrm{Mod}_{\mathcal{O}_C}^{\heartsuit}$). □

Remark 2.1.11. Let $(\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos. Then the construction $(U \in \mathcal{X}) \mapsto \mathrm{Mod}_{\mathcal{O}\,|U}$ determines a functor from $\mathcal{X}^{op}$ into the ∞-category $\widehat{\mathcal{C}\mathrm{at}}_\infty$ of (not necessarily small) ∞-categories. Moreover, this functor preserves small limits.

To see this, consider the coCartesian fibration $p : \mathrm{Fun}(\Delta^1, \mathcal{X}) \to \mathrm{Fun}(\{1\}, \mathcal{X}) \simeq \mathcal{X}$ given by evaluation at $\{1\} \subseteq \Delta^1$. This coCartesian fibration is classified by a functor $\chi : \mathcal{X} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$, which assigns to each object $U \in \mathcal{X}$ the ∞-topos $\mathcal{X}^{/U}$. We claim that this functor preserves small colimits. To prove this, it suffices to show that the opposite functor $\chi : \mathcal{X}^{op} \to \mathcal{P}\mathrm{r}^{\mathrm{L}\,op} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}}$ preserves small limits; this functor classifies p as a *Cartesian* fibration, and is a limit diagram by virtue of Theorems T.6.1.3.9 and T.5.5.3.18 together Proposition T.5.5.3.13. For any presentable ∞-category $\mathcal{C}$, we obtain a new functor given by the composition

$$\mathcal{X} \xrightarrow{\chi} \mathcal{P}\mathrm{r}^{\mathrm{L}} \xrightarrow{\otimes \mathcal{C}} \mathcal{P}\mathrm{r}^{\mathrm{L}},$$

which assigns to each object $U \in \mathcal{X}$ the ∞-category $\mathcal{S}\mathrm{hv}_{\mathcal{C}}(\mathcal{X}^{/U})$ (see Remark V.1.1.5). The same reasoning yields a limit-preserving functor $\mathcal{X}^{op} \to \mathcal{P}\mathrm{r}^{\mathrm{L}^{op}} \simeq \mathcal{P}\mathrm{r}^{\mathrm{R}}$ which, by virtue of Theorem T.5.5.3.18, gives a limit-preserving functor $\chi_{\mathcal{C}} : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$.

The evident forgetful functor Mod $\to$ CAlg determines a natural transformation of functors $\chi_{\mathrm{Mod}} \to \chi_{\mathrm{CAlg}}$ from $\mathcal{X}^{op}$ to $\widehat{\mathcal{C}\mathrm{at}}_{\infty}$. Every sheaf $\mathcal{O}$ of $\mathbb{E}_{\infty}$-rings on $\mathcal{X}$ determines a natural transformation $* \to \chi_{\mathrm{CAlg}}$, where $*$ denotes the constant functor $\mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$ taking the value Δ^0. Forming a pullback diagram

$$\begin{tikzcd} \chi_{\mathcal{O}} \arrow[r] \arrow[d] & \chi_{\mathrm{Mod}} \arrow[d] \\ * \arrow[r] & \chi_{\mathrm{CAlg}}, \end{tikzcd}$$

we obtain a new limit-preserving functor $\chi_{\mathcal{O}} : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_{\infty}$. Unwinding the definitions, we see that $\chi_{\mathcal{O}}$ assigns to each object $U \in \mathcal{X}$ the ∞-category $\mathrm{Mod}_{\mathcal{O}\,|U}$, and to every morphism $f : U \to V$ in $\mathcal{X}$ the associated pullback functor $f^* : \mathrm{Mod}_{\mathcal{O}\,|V} \to \mathrm{Mod}_{\mathcal{O}\,|U}$. Since χ_{CAlg} and χ_{Mod} preserve small limits, so does $\chi_{\mathcal{O}}$.

2.2 Module Geometries

Let $(\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos. In §2.1, we studied the ∞-category $\mathrm{Mod}_{\mathcal{O}}$ of sheaves of $\mathcal{O}$-module spectra on $\mathcal{X}$. In this section, we will study the dependence of this ∞-category on the pair $(\mathcal{X}, \mathcal{O})$. We begin with a simple observation.

Remark 2.2.1. Let Mod = Mod(Sp) denote the ∞-category of pairs (A, M), where A is an $\mathbb{E}_{\infty}$-ring and M is an A-module spectrum. There are evident forgetful functors

$$\mathrm{Sp} \xleftarrow{p} \mathrm{Mod} \xrightarrow{q} \mathrm{CAlg},$$

given informally by the formulas $p(A, M) = M$ and $q(A, M) = A$. Let K be a small simplicial set. Since the ∞-category of spectra admits small limits, Theorem A.3.4.3.1 implies that every commutative diagram

$$\begin{tikzcd} K \arrow[r, "f_0"] \arrow[d] & \mathrm{Mod} \arrow[d, "q"] \\ K^{\triangleleft} \arrow[r, "g"] \arrow[ur, dashed, "f"] & \mathrm{CAlg} \end{tikzcd}$$

admits an extension as indicated, where f is a q-limit diagram. Moreover, this extension f is characterized by the requirement that $p \circ f$ be a limit diagram in the ∞-category Sp. In particular, if we choose g to be a limit of the diagram $q \circ f_0$ (which exists by virtue of Corollary A.3.2.2.5), then f is a limit diagram in Mod (Proposition T.4.3.1.5). Using the uniqueness properties of limits, we deduce:

(∗) The ∞-category Mod admits small limits. Moreover, a diagram $f : K^{\triangleleft} \to \mathrm{Mod}$ is a limit diagram if and only if $p \circ f$ is a limit diagram in Sp and $q \circ f$ is a limit diagram in CAlg. In particular, the functors p and q preserve small limits.

Let $\mathcal{X}$ be an ∞-topos. Assertion (∗) implies that the canonical isomorphism of simplicial sets

$$\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Mod}(\mathrm{Sp})) \simeq \mathrm{Mod}(\mathrm{Fun}(\mathcal{X}^{op}, \mathrm{Sp}))$$

restricts to an isomorphism of full subcategories

$$\mathcal{S}\mathrm{hv}_{\mathrm{Mod}}(\mathcal{X}) \simeq \mathrm{Mod}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X})).$$

In other words, we may identify a Mod-valued sheaf on $\mathcal{X}$ with a pair $(\mathcal{O}, \mathcal{F})$, where $\mathcal{O}$ is a sheaf of $\mathbb{E}_{\infty}$-rings on $\mathcal{X}$ and $\mathcal{F}$ is a $\mathcal{O}$-module. To exploit this observation, we need a few simple facts about compact objects of the ∞-category Mod.

Proposition 2.2.2. *The ∞-category* Mod *is compactly generated. Moreover, an object* $(A, M) \in$ Mod *is compact if and only if the following pair of conditions is satisfied:*

(1) *The $\mathbb{E}_\infty$-ring A is a compact object of* CAlg.

(2) *The A-module M is perfect (that is, M is a compact object of* Mod_A*; see §A.7.2.5).*

The proof of Proposition 2.2.2 requires the following preliminary.

Lemma 2.2.3. *The ∞-category* Mod *admits small colimits. The forgetful functor* $q : \mathrm{Mod} \to \mathrm{CAlg}$ *preserves small colimits, and the forgetful functor* $p : \mathrm{Mod} \to \mathrm{Sp}$ *preserves sifted colimits.*

Proof. For every $\mathbb{E}_\infty$-ring A, the ∞-category Mod_A admits small colimits; moreover, every map of $\mathbb{E}_\infty$-rings $A \to B$ induces a functor $\mathrm{Mod}_A \to \mathrm{Mod}_B$ which preserves small colimits. It follows that every commutative diagram

$$\begin{tikzcd} K \arrow[r, "f_0"] \arrow[d] & \mathrm{Mod} \arrow[d, "q"] \\ K^{\triangleright} \arrow[r, "g"] \arrow[ur, dashed, "f"] & \mathrm{CAlg} \end{tikzcd}$$

admits an extension as indicated, where f is a q-colimit diagram. Applying this to the situation where g is a colimit of $q \circ f_0$, we deduce that f_0 admits a colimit f (Proposition T.4.3.1.5) such that $g = q \circ f$ is a colimit diagram in CAlg. It follows that Mod admits small colimits and that the forgetful functor $q : \mathrm{Mod} \to \mathrm{CAlg}$ preserves small colimits. To prove the last claim, we invoke Proposition A.4.4.1.4 to deduce that Mod fits into a homotopy pullback diagram

$$\begin{tikzcd} \mathrm{Mod} \arrow[r, "p'"] \arrow[d] & \mathrm{LMod}(\mathrm{Sp}) \arrow[d, "f"] \\ \mathrm{CAlg}(\mathrm{Sp}) \arrow[r, "p'_0"] & \mathrm{Alg}(\mathrm{Sp}). \end{tikzcd}$$

Using Proposition A.3.2.3.1, we deduce that p'_0 and f preserve sifted colimits, so that p' preserves sifted colimits. The functor p factors as a composition

$$\mathrm{Mod} = \mathrm{Mod}(\mathrm{Sp}) \xrightarrow{p'} \mathrm{LMod}(\mathrm{Sp}) \xrightarrow{p''} \mathrm{Sp},$$

where p'' preserves sifted colimits by Proposition A.3.2.3.1; it follows that p preserves sifted colimits as well. $\square$

Proof of Proposition 2.2.2. Let $\mathrm{Sp} \xleftarrow{p} \mathrm{Mod} \xrightarrow{q} \mathrm{CAlg}$ be the forgetful functors of Remark 2.2.1. The functors p admits a left adjoint, given informally by the formula $X \mapsto (S, X)$; here S denotes the sphere spectrum. Since p preserves filtered colimits, this left adjoint carries compact objects of Sp to compact objects of Mod; in particular, we deduce that the object $(S, S[n]) \in \mathrm{Mod}$ is compact for every integer n. Similarly, q has a left adjoint, given informally by the formula $A \mapsto (A, 0)$. Since q preserves small colimits, we deduce that the object $(A, 0) \in \mathrm{Mod}$ is compact whenever A is a compact $\mathbb{E}_\infty$-ring.

Fix a compact $\mathbb{E}_\infty$-ring A, and let $\mathcal{C} \subseteq \mathrm{Mod}_A$ denote the full subcategory spanned by those A-modules M such that (A, M) is a compact object of Mod. The functor $(A, M) \mapsto (A, M[-1])$ is a self-equivalence of Mod, and therefore carries compact objects to compact objects. It follows that $\mathcal{C}$ is closed under translation in Mod_A. Using Propositions T.4.3.1.10, we conclude that every pushout diagram in Mod_A is a q-pushout diagram in Mod and therefore a pushout diagram in Mod. Since the collection of compact objects in Mod is stable under pushouts, we deduce that $\mathcal{C}$ is stable under pushouts in Mod. It follows that $\mathcal{C}$ is a stable subcategory of Mod_A, which is obviously stable under retracts. Since the collection of compact objects in Mod is stable under coproducts, we conclude that $(S, S) \coprod (A, 0) \simeq (A, A)$ is compact, so that $A \in \mathcal{C}$.

It follows that $\mathcal{C}$ contains all perfect A-modules (see Definition A.7.2.5.1). This proves that every object $(A, M) \in \mathrm{Mod}$ satisfying conditions (1) and (2) is compact.

Conversely, suppose that $(A, M) \in \mathrm{Mod}$ is compact. The functor q admits a right adjoint (also given by $A \mapsto (A, 0)$) which preserves filtered colimits, so that q carries compact objects of Mod to compact objects of CAlg; it follows that $A \in \mathrm{CAlg}$ is compact. We claim that M is a perfect A-module. To prove this, consider a filtered diagram $\{N_\alpha\}$ of A-modules with colimit N; we wish to prove that the canonical map $\theta : \varinjlim \mathrm{Map}_{\mathrm{Mod}_A}(M, N_\alpha) \to \mathrm{Map}_{\mathrm{Mod}_A}(M, N)$ is a homotopy equivalence. To prove this, we observe that θ fits into a map of homotopy fiber sequences

$$\begin{tikzcd}
\varinjlim \mathrm{Map}_{\mathrm{Mod}_A}(M, N_\alpha) \arrow[r] \arrow[d, "\theta"] & \varinjlim \mathrm{Map}_{\mathrm{Mod}}((A, M), (A, N_\alpha)) \arrow[r] \arrow[d, "\theta'"] & \varinjlim \mathrm{Map}_{\mathrm{CAlg}}(A, A) \arrow[d, "\theta''"] \\
\mathrm{Map}_{\mathrm{Mod}_A}(M, N) \arrow[r] & \mathrm{Map}_{\mathrm{Mod}}((A, M), (A, N)) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}}(A, A).
\end{tikzcd}$$

The map θ'' is obviously a homotopy equivalence, and the map θ' is a homotopy equivalence by virtue of our assumption that $(A, M) \in \mathrm{Mod}$ is compact (since $(A, N) \simeq \varinjlim (A, N_\alpha)$ in Mod, by virtue of Lemma 2.2.3).

Now let $\mathcal{D}$ denote the full subcategory of Mod spanned by objects satisfying conditions (1) and (2); note that $\mathcal{D}$ is essentially small and admits finite colimits. The inclusion $\mathcal{D} \to \mathrm{Mod}$ induces a functor $F : \mathrm{Ind}(\mathcal{D}) \to \mathrm{Mod}$. The functor F is fully faithful by Proposition T.5.3.5.11; we wish to show that F is an equivalence. Note that F preserves small colimits (Proposition T.5.5.1.9). Since $\mathrm{Ind}(\mathcal{D})$ is presentable, we deduce from Corollary T.5.5.2.9 (and Remark T.5.5.2.10) that F admits a right adjoint G. To prove that F is fully faithful it suffices to show that G is conservative. Let $\alpha : (A, M) \to (B, N)$ be a morphism in Mod such that $G(\alpha)$ is an equivalence. In particular, we deduce that

$$\mathrm{Map}_{\mathrm{Mod}}((C, 0), (A, M)) \simeq \mathrm{Map}_{\mathrm{Alg}}(C, A) \to \mathrm{Map}_{\mathrm{Alg}}(C, B) \simeq \mathrm{Map}_{\mathrm{Mod}}((C, 0), (B, N))$$

is an equivalence for every compact object $C \in \mathrm{CAlg}$; since CAlg is compactly generated, it follows that α induces an equivalence $A \simeq B$. Similarly, we have

$$\mathrm{Map}_{\mathrm{Mod}}((S, S[n]), (A, M)) \simeq \Omega^{\infty+n} M \to \Omega^{\infty+n} N \simeq \mathrm{Map}_{\mathrm{Mod}}((S, S[n]), (B, N))$$

for every integer n, so that α induces an equivalence of spectra $M \simeq N$. It follows that α is an equivalence, as desired. $\square$

Let $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$ denote full subcategory of Mod^{op} spanned by the compact objects of Mod: that is, spanned by objects of the form (A, M), where A is a compact $\mathbb{E}_\infty$-ring and M is a perfect A-module. We will regard $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$ as a discrete geometry (see Definition V.1.2.5).

Notation 2.2.4. We let $\mathrm{RingTop}^M$ denote the ∞-category ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}})^{op}$ whose objects are ∞-topoi $\mathcal{X}$ equipped with a left-exact functor $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}} \to \mathcal{X}$. We will identify the objects of $\mathrm{RingTop}^M$ with triples $(\mathcal{X}, \mathcal{O}, \mathcal{F})$, where $\mathcal{X}$ is an ∞-topos, $\mathcal{O}$ is a sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, and $\mathcal{F}$ is a sheaf of $\mathcal{O}$-module spectra on $\mathcal{X}$ (see Remark V.1.1.6).

According to Lemma 2.2.3, the forgetful functor $q : \mathrm{Mod} \to \mathrm{CAlg}$ preserves colimits, and therefore induces a transformation of (discrete) geometries $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}} \to \mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}$, where $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}$ is the geometry of Remark VII.1.26. Composition with this transformation determines a forgetful functor

$$\theta : \mathrm{RingTop}^M \to \mathrm{RingTop},$$

which carries an object $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \in \mathrm{RingTop}^M$ to its underlying spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O})$. For every object $(\mathcal{X}, \mathcal{O}) \in \mathrm{RingTop}$, the fiber $\theta^{-1}\{(\mathcal{X}, \mathcal{O})\}$ can be identified with the ∞-category $\mathrm{Mod}_{\mathcal{O}}$ of sheaves of $\mathcal{O}$-module spectra studied in §2.1.

Notation 2.2.5. We let $\mathrm{RingTop}^M_{\mathrm{Zar}}$ and $\mathrm{RingTop}^M_{\text{ét}}$ denote the fiber products

$$\mathrm{RingTop}^M \times_{\mathrm{RingTop}} \mathrm{RingTop}_{\mathrm{Zar}} \qquad \mathrm{RingTop}^M \times_{\mathrm{RingTop}} \mathrm{RingTop}_{\text{ét}}\,.$$

In particular, the objects of $\mathrm{RingTop}^M_{\mathrm{Zar}}$ ($\mathrm{RingTop}^M_{\text{ét}}$) are triples $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \in \mathrm{RingTop}^M$ such that $\mathcal{O}$ is a local (strictly Henselian) sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$.

The ∞-categories $\mathrm{RingTop}^M_{\mathrm{Zar}}$ and $\mathrm{RingTop}^M_{\text{ét}}$ can be obtained by introducing suitable modifications of the geometry $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$, involving the Zariski and étale topologies on the ∞-category of $\mathbb{E}_\infty$-algebras, respectively. We begin by describing a general paradigm for introducing such modifications.

Proposition 2.2.6. *Let $\mathcal{G}$ be a geometry, and let $q : \mathcal{G}' \to \mathcal{G}$ be a Cartesian fibration of ∞-categories. Assume that:*

(1) *For every object $X \in \mathcal{G}$, the fiber $\mathcal{G}'_X$ is idempotent complete and admits finite limits.*

(2) *For every morphism $f : X \to Y$ in $\mathcal{G}$, the pullback functor $f^* : \mathcal{G}'_Y \to \mathcal{G}'_X$ preserves finite limits.*

Let us say that a morphism $f : X \to Y$ in $\mathcal{G}'$ is admissible if f is q-Cartesian and $q(f)$ is an admissible morphism in $\mathcal{G}$. Let us say that a collection of admissible morphisms $\{f_\alpha : U_\alpha \to X\}$ in $\mathcal{G}'$ is a covering if the images $\{q(f_\alpha) : q(U_\alpha) \to q(X)\}$ is an admissible covering in $\mathcal{G}$. Then the collection of admissible morphisms and admissible coverings exhibit $\mathcal{G}'$ as a geometry.

Proof. We first show that $\mathcal{G}'$ admits finite limits. Using conditions (1) and (2) together with Corollary T.4.3.1.11, we deduce that for every finite simplicial set K and every diagram of simplicial sets

$$\begin{tikzcd} K \arrow[r, "f_0"] \arrow[d] & \mathcal{G}' \arrow[d, "q"] \\ K^{\triangleleft} \arrow[r, "g"] \arrow[ur, dashed, "f"] & \mathcal{G} \end{tikzcd}$$

there exists a dotted arrow as indicated such that f is a q-limit. Taking g to be a limit of $q \circ f_0$ and applying Proposition T.4.3.1.5, we deduce that f is a limit diagram such that $g = q \circ f$ is also a limit diagram. It follows that $\mathcal{G}'$ admits finite limits and that q preserves finite limits.

We next show that $\mathcal{G}'$ is idempotent-complete. Let Idem be the ∞-category defined in §T.4.4.5 and let $f_0 : \mathrm{Idem} \to \mathcal{G}'$ be a diagram; we wish to prove that f admits a colimit. Since $\mathcal{G}'$ is idempotent-complete, we deduce that $q \circ f_0$ admits a colimit g. In view of Proposition T.4.3.1.5, it will suffice to show that the diagram

$$\begin{tikzcd} \mathrm{Idem} \arrow[r, "f_0"] \arrow[d] & \mathcal{G}' \arrow[d, "q"] \\ \mathrm{Idem}^{\triangleleft} \arrow[r, "g"] \arrow[ur, dashed, "f"] & \mathcal{G} \end{tikzcd}$$

admits an extension as indicated, where f is a q-colimit diagram. The existence of f follows from Corollary T.4.3.1.11, since each fiber $\mathcal{G}'_X$ of q is idempotent complete and therefore admits Idem-indexed colimits (note that every functor automatically preserves Idem-indexed colimits; see §T.4.4.5).

It is clear that the collection of admissible morphisms in $\mathcal{G}'$ is stable under composition. We now claim that the collection of admissible morphisms satisfies conditions (i), (ii), and (iii) of Definition V.1.2.1:

(i) Given a pullback diagram σ :

$$\begin{tikzcd} U' \arrow[r] \arrow[d, "f'"] & U \arrow[d, "f"] \\ X' \arrow[r] & X \end{tikzcd}$$

in $\mathcal{G}'$, if f is admissible, then f' is admissible. The only nontrivial point is to verify that f' is q-Cartesian. Note that the construction of limits above shows that σ is a q-limit diagram. Since the diagram $U \to X \leftarrow X'$ is a q-right Kan extension of the restricted diagram $X \leftarrow X'$, we deduce from Lemma T.4.3.2.7 that the triangle

$$\begin{tikzcd} U' \arrow[d, "f'"] \arrow[dr] & \\ X' \arrow[r] & X \end{tikzcd}$$

is a q-limit diagram. Since the inclusion $\{1\} \subseteq \Delta^1$ is left cofinal, we conclude that the diagram $U' \xrightarrow{f'} X'$ is a q-limit, so that f' is q-coCartesian as desired.

(*ii*) Suppose given a commutative triangle

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{G}$, where g and h are admissible. Then f is admissible. Since $q(g)$ and $q(h)$ are admissible morphisms in $\mathcal{G}$, we deduce that $q(f)$ is admissible. It remains only to show that f is q-Cartesian, which follows from Proposition T.2.4.1.7.

(*iii*) The collection of admissible morphisms in $\mathcal{G}'$ is stable under retracts; this follows immediately from the definition.

We conclude by showing that the collection of admissible coverings determines a Grothendieck topology on $\mathcal{G}'$. More precisely, we will say that a sieve $\mathcal{G}'^{(0)}_{/X} \subseteq \mathcal{G}'_{/X}$ on an object $X \in \mathcal{G}'$ is *covering* if it contains an admissible covering. The only nontrivial point is to verify that the collection of covering sieves is stable under pullback, which follows easily from (*i*). □

In the situation of Proposition 2.2.6, the Cartesian fibration $q : \mathcal{G}' \to \mathcal{G}$ is a transformation of geometries. Let $\mathcal{G}'_0 \subseteq \mathcal{G}'$ be the full subcategory spanned by those objects $X \in \mathcal{G}'$ such that X is a final object of the fiber $\mathcal{G}'_{q(X)}$. Then q restricts to a trivial Kan fibration $\mathcal{G}'_0 \to \mathcal{G}$, which admits a section s. The functor s is right adjoint to q, and therefore preserves finite limits. It carries all morphisms in $\mathcal{G}$ to q-Cartesian morphisms in $\mathcal{G}'$, and in particular carries admissible morphisms in $\mathcal{G}$ to admissible morphisms in $\mathcal{G}'$. It follows that s is also a transformation of geometries from $\mathcal{G}$ to $\mathcal{G}'$.

Proposition 2.2.7. *Let $q : \mathcal{G}' \to \mathcal{G}$ be as in Proposition 2.2.6, and let $s : \mathcal{G} \to \mathcal{G}'$ be the transformation of geometries described above. Let $\mathcal{X}$ be an ∞-topos. Then:*

(1) *A left-exact functor $\mathcal{O} : \mathcal{G}' \to \mathcal{X}$ is a $\mathcal{G}'$-structure on $\mathcal{X}$ if and only if $\mathcal{O} \circ s$ is a $\mathcal{G}$-structure on $\mathcal{X}$.*

(2) *Let $\alpha : \mathcal{O} \to \mathcal{O}'$ be a natural transformation between $\mathcal{G}'$-structures $\mathcal{O}, \mathcal{O}' : \mathcal{G}' \to \mathcal{X}$. Then α is local if and only if the induced map $\mathcal{O} \circ s \to \mathcal{O}' \circ s$ is a local transformation of $\mathcal{G}$-structures.*

Proof. We first prove (1). Let $X \in \mathcal{G}'$ and let $\{U_\alpha \to X\}$ be an admissible covering of X. We wish to prove that the map $\coprod \mathcal{O}(U_\alpha) \to \mathcal{O}(X)$ is an effective epimorphism in the ∞-topos $\mathcal{X}$. We note that the collection of maps $\{q(U_\alpha) \to q(X)\}$ is an admissible covering of $q(X) \in \mathcal{G}$. Since $\mathcal{O} \circ s$ is a $\mathcal{G}$-structure on $\mathcal{X}$, we deduce that the maps $\coprod \mathcal{O}((s \circ q)(U_\alpha)) \to \mathcal{O}((s \circ q)(X))$ is an effective epimorphism. We have an evident commutative diagram

$$\begin{tikzcd} \coprod \mathcal{O}(U_\alpha) \arrow[r] \arrow[d] & \mathcal{O}(X) \arrow[d] \\ \coprod \mathcal{O}((s \circ q)(U_\alpha)) \arrow[r] & \mathcal{O}((s \circ q)(X)). \end{tikzcd}$$

To complete the proof of (1), it suffices to show that this square is a pullback. Since colimits in $\mathcal{X}$ are universal and the functor $\mathcal{O}$ is left exact, it will suffice to show that for each index α, the diagram σ :

$$\begin{tikzcd} U_\alpha \arrow[r] \arrow[d] & X \arrow[d] \\ (s\circ q)(U_\alpha) \arrow[r] & (s\circ q)(X) \end{tikzcd}$$

is a pullback square in $\mathcal{G}'$. We observe that $q(\sigma)$ is a diagram in which the vertical maps are equivalences, and is therefore a pullback square in $\mathcal{G}$. In view of Proposition T.4.3.1.5, it will suffice to show that σ is a q-pullback diagram. Since $(s\circ q)(U_\alpha)$ and $(s\circ q)(X)$ are q-initial objects of $\mathcal{G}'$, we can use Lemma T.4.3.2.7 to reduce to proving that the morphism $U_\alpha \to X$ is q-Cartesian, which follows from our assumption of admissibility.

To prove (2), let $f : U \to X$ be an admissible morphism in $\mathcal{G}'$. Let $U' = (s\circ q)(U)$ and $X' = (s\circ q)(X)$, and consider the commutative diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}'(U) \arrow[r] \arrow[d] & \mathcal{O}'(U') \arrow[d] \\ \mathcal{O}(X) \arrow[r] & \mathcal{O}'(X) \arrow[r] & \mathcal{O}'(X') \end{tikzcd}$$

in $\mathcal{X}$. We wish to show that the left square is a pullback. The first part of the proof shows that the right square is a pullback, so it suffices to show that the outer rectangle is a pullback. To prove this, we consider the diagram

$$\begin{tikzcd} \mathcal{O}(U) \arrow[r] \arrow[d] & \mathcal{O}(U') \arrow[r] \arrow[d] & \mathcal{O}'(U') \arrow[d] \\ \mathcal{O}(X) \arrow[r] & \mathcal{O}(X') \arrow[r] & \mathcal{O}'(X') \end{tikzcd}$$

The left square is a pullback by the first part of the proof, and the right square is a pullback because α induces a local map of $\mathcal{G}$-structures $\mathcal{O}\circ s \to \mathcal{O}'\circ s$. It follows that the outer rectangle is a pullback diagram as desired. $\square$

Example 2.2.8. The Cartesian fibration $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}} \to \mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}$ satisfies the hypotheses of Proposition 2.2.7. It follows that for any geometry with underlying ∞-category $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{disc}}$, we obtain a new geometry with underlying ∞-category $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$. In particular, the geometries $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{nSp}}_{\mathrm{\acute{e}t}}$ of §VII.2 and §VII.8 determine a pair of geometries $\mathcal{G}^{\mathrm{nM}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{nM}}_{\mathrm{\acute{e}t}}$. Proposition 2.2.7 yields an isomorphism of ∞-categories

$${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nM}}_{\mathrm{Zar}})^{op} \simeq \mathrm{RingTop}^{M}_{\mathrm{Zar}} \subseteq \mathrm{RingTop}^{M}$$

$${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nM}}_{\mathrm{\acute{e}t}})^{op} \simeq \mathrm{RingTop}^{M}_{\mathrm{\acute{e}t}} \subseteq \mathrm{RingTop}^{M}.$$

Example 2.2.9. Let $\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}}$ denote the full subcategory of $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$ spanned by those objects (A, M), where A is a connective compact $\mathbb{E}_\infty$-ring and M is a connective perfect A-module. We observe that $\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}}$ is closed under retracts and finite limits in $\mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$. Using Proposition T.5.3.5.11, we deduce that the inclusion $(\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}})^{op} \subseteq \mathrm{Mod}$ extends to a fully faithful embedding $F : \mathrm{Ind}((\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}})^{op}) \to \mathrm{Mod}$. We claim that the essential image of F is precisely the full subcategory $\mathrm{Mod}^{\mathrm{cn}} = \mathrm{Mod}(\mathrm{Sp}^{\mathrm{cn}}) \subseteq \mathrm{Mod}$ spanned by those pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is a connective A-module. Since this subcategory contains $(\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}})^{op}$ and is stable under colimits, the containment $F(\mathrm{Ind}((\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}})^{op})) \subseteq \mathrm{Mod}^{\mathrm{cn}}$ is clear. To prove the reverse inclusion, we note that F preserves small colimits (Proposition T.5.5.1.9) and therefore admits a right adjoint G (Corollary T.5.5.2.9 and Remark T.5.5.2.10). It therefore suffices to show that $G|\,\mathrm{Mod}^{\mathrm{cn}}$ is conservative. In other words, we must show that if $\alpha : (A, M) \to (B, N)$ is a morphism in $\mathrm{Mod}_{\geq 0}$ which induces a homotopy equivalence $\mathrm{Map}_{\mathrm{Mod}^{\mathrm{cn}}}(X, (A, M)) \to \mathrm{Map}_{\mathrm{Mod}^{\mathrm{cn}}}(X, (B, N))$ for every object $X \in \mathcal{G}^{\mathrm{nM}}_{\mathrm{disc}}$, then α is an equivalence. This is clear; taking $X = (S\{x\}, 0)$ we deduce that the map $\Omega^\infty A \to \Omega^\infty B$ is a homotopy equivalence, so that α induces an equivalence from A to B (since A and B are connective); similarly, taking $X = (S, S)$ we deduce that $\Omega^\infty M \to \Omega^\infty N$ is an equivalence so that $M \simeq N$.

There is an evident Cartesian fibration $\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}} \to \mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}$ which satisfies the hypotheses of Proposition 2.2.7. Consequently, every structure of geometry on the ∞-category $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{disc}}$ determines a structure of geometry on $\mathcal{G}^{\mathrm{M}}_{\mathrm{disc}}$. In particular, the geometries $\mathcal{G}^{\mathrm{Sp}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{Sp}}_{\text{ét}}$ defined in §VII.2 and §VII.8 determine geometries $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{M}}_{\text{ét}}$. Using Proposition 2.2.7 and Remark 2.2.1, we can identify ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}})^{op}$ and ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{M}}_{\text{ét}})^{op}$ with the full subcategories of $\mathrm{RingTop}^{M}_{\mathrm{Zar}}$ and $\mathrm{RingTop}^{M}_{\text{ét}}$ spanned by those triples $(\mathcal{X}, \mathcal{O}, \mathcal{F})$, where $\mathcal{O}$ and $\mathcal{F}$ are connective as sheaves of spectra on $\mathcal{X}$.

We conclude this section by showing that the geometries $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ and $\mathcal{G}^{\mathrm{M}}_{\text{ét}}$ can be described as geometric envelopes. First, we treat the analogous question for the discrete topology.

Proposition 2.2.10. *The ∞-category $\mathrm{Mod}_{\geq 0}$ of Example 2.2.9 is generated by compact projective objects. Moreover, an object $(A, M) \in \mathrm{Mod}^{\mathrm{cn}}$ is compact and projective if and only if A is a compact projective object of the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ of connective $\mathbb{E}_\infty$-rings, and M is a finitely generated projective A-module (that is, a retract of an A-module of the form A^n; see Proposition A.7.2.5.20).*

Proof. We have an evident forgetful functor $q^{\mathrm{cn}} : \mathrm{Mod}^{\mathrm{cn}} \to \mathrm{Alg}^{\mathrm{cn}}$, which is a restriction of the forgetful functor $q : \mathrm{Mod} \to \mathrm{CAlg}$ of Remark 2.2.1 and therefore preserves small colimits (Lemma 2.2.3). It follows that the left adjoint $A \mapsto (A, 0)$ of q^{cn} carries compact projective objects of $\mathrm{CAlg}^{\mathrm{cn}}$ to compact projective objects of $\mathrm{Mod}^{\mathrm{cn}}$. Similarly, the forgetful functor $p^{\mathrm{cn}} : \mathrm{Mod}^{\mathrm{cn}} \to \mathrm{Sp}^{\mathrm{cn}}$ preserves sifted colimits (Lemma 2.2.3) so its left adjoint $M \mapsto (S, M)$ preserves compact projective objects (here S denotes the sphere spectrum): that is, the objects $(S, S^n) \in \mathrm{Mod}^{\mathrm{cn}}$ are compact and projective. Since compact projective objects are stable under finite coproducts, we deduce that if $A \in \mathrm{CAlg}^{\mathrm{cn}}$ is compact and projective, then the coproduct $(A, 0) \coprod (S, S^n) \simeq (A, A^n)$ is compact and projective in $\mathrm{Mod}^{\mathrm{cn}}$. If M is a finitely generated projective A-module, then it is a retract of A^n for some integer n, so that (A, M) is also a compact projective object of $\mathrm{Mod}^{\mathrm{cn}}$.

Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Mod}^{\mathrm{cn}}$ spanned by objects of the form (A, A^m), where A is a free algebra of the form $S\{x_1, \ldots, x_n\}$. Then $\mathcal{C}$ admits small coproducts and consists of compact projective objects of $\mathrm{Mod}^{\mathrm{cn}}$, so Proposition T.5.5.8.22 shows that the inclusion $\mathcal{C} \subseteq \mathrm{Mod}^{\mathrm{cn}}$ extends to a fully faithful embedding $F : \mathcal{P}_\Sigma(\mathcal{C}) \to \mathrm{Mod}^{\mathrm{cn}}$. Since $\mathcal{C}$ contains (S, S) and $(S\{x\}, 0)$, the argument of Example 2.2.9 shows that F is essentially surjective, so that $\mathrm{Mod}^{\mathrm{cn}} \simeq \mathcal{P}_\Sigma(\mathcal{C})$ is generated by compact projective objects. Moreover, this shows that an object $(A, M) \in \mathrm{Mod}^{\mathrm{cn}}$ is compact and projective if and only if it is a retract of an object of (P, P^m) in $\mathcal{C}$. In particular, A is a retract of $P \simeq S\{x_1, \ldots, x_n\}$, so that A is a compact projective object of $\mathrm{CAlg}_{\geq 0}$. Then M is a retract of $P^m \otimes_P A \simeq A^m$ and is therefore a finitely generated projective A-module. □

Notation 2.2.11. We let $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$ denote the full subcategory of $(\mathrm{Mod}^{\mathrm{cn}})^{op}$ spanned by objects of the form (A, A^m) where A has the form $S\{x_1, \ldots, x_n\}[\frac{1}{y}]$ for some $y \in \pi_0 S\{x_1, \ldots, x_n\} \simeq \mathbf{Z}[x_1, \ldots, x_n]$, We will say that a morphism in $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$ is admissible if it is admissible as a morphism of $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$, and that a collection of admissible morphisms $\{U_\alpha \to X\}$ in $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$ is a covering if it is a covering in $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$.

Similarly, we let $\mathcal{T}^{\mathrm{M}}_{\text{ét}}$ denote the full subcategory of $(\mathrm{Mod}^{\mathrm{cn}})^{op}$ denote the full subcategory spanned by objects of the form (A, A^m) for which there exists an étale morphism of $\mathbb{E}_\infty$-rings $S\{x_1, \ldots, x_n\} \to A$. We will say that a morphism in $\mathcal{T}^{\mathrm{M}}_{\text{ét}}$ is admissible if it is admissible as a morphism of $\mathcal{G}^{\mathrm{M}}_{\text{ét}}$, and that a collection of admissible morphisms $\{U_\alpha \to X\}$ in $\mathcal{T}^{\mathrm{M}}_{\text{ét}}$ is a covering if it is a covering in $\mathcal{G}^{\mathrm{M}}_{\text{ét}}$.

Proposition 2.2.12. (*a*) *The collection of admissible morphisms and admissible coverings determines the structure of a pregeometry on $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$, and the inclusion $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}} \to \mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ exhibits $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ as a geometric envelope of $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$.*

(*b*) *The collection of admissible morphisms and admissible coverings determines the structure of a pregeometry on $\mathcal{T}^{\mathrm{M}}_{\acute{e}t}$, and the inclusion $\mathcal{T}^{\mathrm{M}}_{\acute{e}t} \to \mathcal{G}^{\mathrm{M}}_{\acute{e}t}$ exhibits $\mathcal{G}^{\mathrm{M}}_{\acute{e}t}$ as a geometric envelope of $\mathcal{T}^{\mathrm{M}}_{\acute{e}t}$.*

Proof. Let $\mathcal{T}_0$ denote the full subcategory of $(\mathrm{Mod}^{\mathrm{cn}})^{op}$ spanned by objects of the form (A, A^m), where A is a free algebra of the form $S\{x_1, \ldots, x_n\}$. To prove (a) and (b), it will suffice to show that the inclusions $\mathcal{T}_0 \subset \mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ and $\mathcal{T}_0 \subset \mathcal{G}^{\mathrm{M}}_{\text{ét}}$ satisfy conditions (1) through (6) of Proposition V.3.4.5. Conditions (1) and (2) are obvious and (3) follows from Remark V.3.4.6 and the proof of Proposition 2.2.10. To prove (4) in case (a), we note that every admissible morphism $(A[\frac{1}{a}], M \otimes_A A[\frac{1}{a}]) \to (A, M)$ in $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ is the pullback of an admissible morphism $(S\{x\}[x^{-1}], 0) \to (S\{x\}, 0)$ under the map $(A, M) \to (S\{x\}, 0)$ which is determined up to homotopy by the requirement that it carries $x \in \pi_0 S\{x\}$ to $a \in \pi_0 A$. In case (b), we use similar reasoning together with Proposition VII.8.10.

To prove (5) in case (a), let $\mathcal{G}$ denote the ∞-category $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ endowed with the Grothendieck topology generated by the collection of admissible coverings in $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$; we wish to prove that every admissible covering $\{f_i : (A[\frac{1}{a_i}], M_i) \to (A, M)\}_{1 \leq i \leq n}$ in $\mathcal{G}^{\mathrm{M}}_{\mathrm{Zar}}$ generates a covering sieve in $\mathcal{G}$. Since the morphisms f_i are pullbacks of the maps $(A[\frac{1}{a_i}], 0) \to (A, 0)$, we may assume without loss of generality that $M \simeq 0$ so that each $M_i \simeq 0$. The proof now proceeds exactly as in the proof of Proposition VII.2.20. We have an equation of the form

$$a_1 b_1 + \ldots + a_n b_n = 1$$

in the commutative ring $\pi_0 A$. Let $B = S\{x_1, \ldots, x_n, y_1, \ldots, y_n\}[\frac{1}{x_1 y_1 + \ldots + x_n y_n}]$. There is a morphism $\phi : B \to A$ carrying each $x_i \in \pi_0 B$ to $a_i \in \pi_0 A$, and each $y_i \in \pi_0 B$ to $b_i \in \pi_0 A$ (in fact, ϕ is uniquely determined up to homotopy). Each map f_i fits into a pullback diagram

$$\begin{tikzcd}
(A[\frac{1}{a_i}], 0) \arrow[r, "f_i"] \arrow[d] & (A, 0) \arrow[d] \\
(B[\frac{1}{x_i}], 0) \arrow[r, "g_i"] & (B, 0)
\end{tikzcd}$$

in $\mathcal{G}$; it now suffices to observe that the maps $\{g_i : \operatorname{Spec} B[\frac{1}{x_i}] \to \operatorname{Spec} B\}$ determine an admissible covering of $\operatorname{Spec} B$ in $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}(k)$. The proof in case (b) proceeds in much the same way, using the proof of Proposition VII.8.17.

It remains to verify condition (6). We will give the proof in case (a). Let $\mathcal{C}$ be an idempotent-complete ∞-category which admits finite limits and let $\alpha : f \to f'$ be a natural transformation between admissible functors $f, f' : \mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}} \to \mathcal{C}$ such that α induces an equivalence $f|\mathcal{T}_0 \simeq f'|\mathcal{T}_0$. We wish to prove that α is an equivalence. Fix an object (A, A^m) in $\mathcal{T}_0$; we wish to prove that α induces an equivalence $f(A, A^m) \to f'(A, A^m)$. If $A \simeq S\{x_1, \ldots, x_n\}$ is a free algebra, then $(A, A^m) \in \mathcal{T}_0$ and the result is obvious. Since $(A, A^m) \simeq (A, 0) \times (S, S^m)$ and the functors f and f' preserve finite products, we are reduced to proving the result in the case $m = 0$. In other words, we wish to prove that α induces an equivalence between admissible functors $f \circ s, f' \circ s : \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}} \to \mathcal{C}$, where $s : \mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}} \to \mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$ denotes the functor given informally by $s(\operatorname{Spec} A) = (A, 0)$. We are now reduced to verifying that condition (6) holds for the pregeometry $\mathcal{T}^{\mathrm{Sp}}_{\mathrm{Zar}}$, which is verified in the proof of Proposition VII.2.20. The proof in case (b) is the same (except that we reduce instead to the proof of Proposition VII.8.28). $\square$

Remark 2.2.13. Using Propositions V.3.2.5 and V.3.2.8, we see that it is possible to introduce a number of variations on the pregeometries $\mathcal{T}^{\mathrm{M}}_{\mathrm{Zar}}$ and $\mathcal{T}^{\mathrm{M}}_{\text{ét}}$ of Notation 2.2.11 without changing the underlying scheme-theory. For example, we can replace $\mathcal{T}^{\mathrm{M}}_{\text{ét}}$ by the larger ∞-category of pair (A, M), where A is a smooth S-algebra (in the sense of Definition VII.8.24) and M is a finitely generated projective A-module.

2.3 Quasi-Coherent Sheaves

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectrally ringed ∞-topos. In §2.1, we defined a stable ∞-category $\mathrm{Mod}_{\mathcal{O}}$ whose objects are sheaves of $\mathcal{O}$-module spectra on $\mathcal{X}$. In this section, we specialize to the case where $(\mathcal{X}, \mathcal{O})$ is a spectral

Deligne-Mumford stack. In this case, we will define a full subcategory $\mathrm{QCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$, whose objects we will refer to as *quasi-coherent sheaves* on $\mathfrak{X}$.

Remark 2.3.1. The theory of quasi-coherent sheaves developed here is closely related to the theory of quasi-coherent sheaves in classical algebraic geometry. Let $\mathfrak{X}$ be a Deligne-Mumford stack and let $\mathcal{O}_{\mathfrak{X}}$ be its structure sheaf (regarded as a sheaf of commutative rings). We can associate to $\mathfrak{X}$ an abelian category $\mathcal{A}_{\mathfrak{X}}$, whose objects are étale sheaves of (discrete) $\mathcal{O}_{\mathfrak{X}}$-modules. Let $\mathcal{A}_{\mathfrak{X}}^{\mathrm{qc}}$ denote the full subcategory of $\mathcal{A}_{\mathfrak{X}}$ spanned by the quasi-coherent sheaves (in the sense of classical algebraic geometry).

If $(\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack, then the ∞-category $\mathrm{QCoh}(\mathcal{X})$ defined below is a stable ∞-category equipped with a t-structure, and its heart $\mathrm{QCoh}(\mathcal{X})^{\heartsuit}$ is a equivalent (as an abelian category) to $\mathcal{A}_{\mathfrak{X}}^{\mathrm{qc}}$, where $\mathfrak{X}$ is the underlying Deligne-Mumford stack of $(\mathcal{X}, \mathcal{O})$: see Remark 2.3.20 below.

If $\mathcal{X}$ is a 1-localic ∞-topos and $\mathcal{O}$ is a discrete sheaf of $\mathbb{E}_\infty$-rings on $\mathcal{X}$, then Proposition 2.1.8 furnishes a fully faithful embedding of stable ∞-categories $\mathcal{D}^+(\mathcal{A}_{\mathfrak{X}}) \to \mathrm{Mod}_{\mathcal{O}}$, which induces an equivalence onto the full subcategory $\bigcup_{n \in \mathbf{Z}} (\mathrm{Mod}_{\mathcal{O}})_{\leq n} \subseteq \mathrm{Mod}_{\mathcal{O}}$. Under this equivalence, $\bigcup_{n \in \mathbf{Z}} \mathrm{QCoh}(\mathcal{X})_{\leq n}$ corresponds to the full subcategory $\mathcal{D}^+_{\mathrm{qc}}(\mathcal{A}_{\mathfrak{X}}) \subseteq \mathcal{D}^+(\mathcal{A}_{\mathfrak{X}})$ spanned by those cochain complexes whose cohomologies are quasi-coherent (that is, belong to the subcategory $\mathcal{A}_{\mathfrak{X}}^{\mathrm{qc}} \subseteq \mathcal{A}_{\mathfrak{X}}$): see Proposition 2.3.21. In many cases, one can show that this subcategory is equivalent to the derived ∞-category $\mathcal{D}^+(\mathcal{A}_{\mathfrak{X}}^{\mathrm{qc}})$.

Remark 2.3.2. In this section, we will devote our attention to the study of quasi-coherent sheaves on (possibly nonconnective) spectral Deligne-Mumford stacks. There is an entirely parallel theory of quasi-coherent sheaves on spectral schemes. All of the results of this section carry over to the setting of spectral schemes, with essentially the same proofs.

We now proceed to define the ∞-category of quasi-coherent sheaves on a (possibly nonconnective) spectral Deligne-Mumford stack. Let $p : \mathrm{RingTop}^M \to {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$ be the forgetful functor which associates to each triple $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ the underlying ∞-topos $\mathcal{X}$. We can identify the ∞-category Mod with the inverse image $p^{-1}\{\mathcal{S}\}$ of the final object $\mathcal{S} \in {}^{\mathrm{R}}\mathcal{T}\mathrm{op}$. Since p is a coCartesian fibration, the inclusion $\mathrm{Mod} \to \mathrm{RingTop}^M$ admits a left adjoint $\Gamma : \mathrm{RingTop}^M \to \mathrm{Mod}$. Concretely, the functor Γ assigns to each triple $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ the pair $(\mathcal{O}(\mathbf{1}), \mathcal{F}(\mathbf{1})) \in \mathrm{Mod}$, where $\mathbf{1}$ denotes a final object of $\mathcal{X}$ (so that $\mathcal{F}(\mathbf{1})$ is a module spectrum over the $\mathbb{E}_\infty$-ring $\mathcal{O}(\mathbf{1})$). By restriction, Γ determines a functor

$$\Gamma_{\text{ét}} : \mathrm{RingTop}^M_{\text{ét}} \to \mathrm{Mod}\,.$$

Applying the constructions of §V.2.2 in the case of the geometry $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$ of Example 2.2.8, we obtain the following result:

Proposition 2.3.3. *The functor* $\Gamma_{\text{ét}} : \mathrm{RingTop}^M_{\text{ét}} \to \mathrm{Mod}$ *admits a right adjoint* $\mathrm{Spec}^{\text{ét}}_{\mathrm{M}} : \mathrm{Mod} \to \mathrm{RingTop}^M_{\text{ét}}$*.*

There is an evident forgetful functor $q : \mathcal{G}^{\mathrm{nM}}_{\text{ét}} \to \mathcal{G}^{\mathrm{nSp}}_{\text{ét}}$. Composition with q induces a functor ${}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nSp}}_{\text{ét}}) \to {}^{\mathrm{L}}\mathcal{T}\mathrm{op}(\mathcal{G}^{\mathrm{nM}}_{\text{ét}})$, which is given informally by the formula $(\mathcal{X}, \mathcal{O}) \mapsto (\mathcal{X}, \mathcal{O}, 0)$. Unwinding the definitions, we see that the relative spectrum functor associated to q is given by the forgetful functor $\mathrm{RingTop}^M_{\text{ét}} \to \mathrm{RingTop}_{\text{ét}}$. From this observation, we immediately deduce the following results:

Proposition 2.3.4. *Let A be an $\mathbb{E}_\infty$-ring, and let M be an A-module. Then the object* $\mathrm{Spec}^{\text{ét}}_{\mathrm{M}}(A, M) \in \mathrm{RingTop}^M_{\text{ét}}$ *can be identified with a triple $(\mathcal{X}, \mathcal{O}, \mathcal{F})$, where the spectrally ringed ∞-topos $(\mathcal{X}, \mathcal{O})$ is given by* $\mathrm{Spec}\, A \in \mathrm{RingTop}_{\text{ét}}$*.*

Proposition 2.3.5. *Let* $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \in \mathrm{RingTop}^M_{\text{ét}}$ *be a $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme. Then $(\mathcal{X}, \mathcal{O})$ is a nonconnective spectral Deligne-Mumford stack.*

Definition 2.3.6. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{X}$. We will say that $\mathcal{F}$ is *quasi-coherent* if the triple $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ is a $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme. We let $\mathrm{QCoh}(\mathfrak{X})$ denote the full subcategory of $\mathrm{Mod}_{\mathcal{O}}$ spanned by the quasi-coherent sheaves of $\mathcal{O}$-modules on $\mathcal{X}$.

In other words, a sheaf $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}}$ is a quasi-coherent sheaf on a nonconnective spectral Deligne-Mumford stack $(\mathcal{X}, \mathcal{O})$ if $\mathcal{X}$ admits a covering by objects U_α such that each of the triples $(\mathcal{X}_{/U_\alpha}, \mathcal{O}\,|U_\alpha, \mathcal{F}\,|U_\alpha)$ has the form $\mathrm{Spec}^{\text{ét}}_{\mathrm{M}}(A_\alpha, M_\alpha)$, for some $\mathbb{E}_\infty$-ring A_α and some A_α-module spectrum M_α.

Remark 2.3.7. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack, and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{X}$. The condition that $\mathcal{F}$ be quasi-coherent is local on $\mathcal{X}$. In other words, for every morphism $U \to V$ in $\mathcal{X}$, if $\mathcal{F}\,|V$ is a quasi-coherent sheaf on $(\mathcal{X}_{/V}, \mathcal{O}\,|V)$, then $\mathcal{F}\,|U$ is a quasi-coherent sheaf on $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$. Conversely, if we are given an effective epimorphism $\coprod_\alpha U_\alpha \to V$ and each restriction $\mathcal{F}\,|U_\alpha$ is a quasi-coherent sheaf on $(\mathcal{X}_{/U_\alpha}, \mathcal{O}\,|U_\alpha)$, then $\mathcal{F}\,|V$ is a quasi-coherent sheaf on $(\mathcal{X}_{/V}, \mathcal{O}\,|V)$. This follows immediately from Proposition V.2.3.10.

Our next goal is to describe quasi-coherent sheaves over affine spectral Deligne-Mumford stacks. We begin with a few general remarks.

Lemma 2.3.8. *Let A be an $\mathbb{E}_\infty$-ring, let M be an A-module, and let $(\mathcal{X}, \mathcal{O}, \mathcal{F}) = \mathrm{Spec}^{\acute{e}t}_{\mathrm{M}}(A, M)$. Let $U \in \mathcal{X}$ be an object such that $(\mathcal{X}_{/U}, \mathcal{O}\,|U) \simeq \mathrm{Spec}\, B$ is affine. Then we have a canonical equivalence $M \otimes_A B \to \mathcal{F}(U)$.*

Proof. Let $\mathcal{C}$ denote the full subcategory of $\mathrm{Mod}_{(A,M)/}$ spanned by those objects (B, N) such that the map $A \to B$ is étale and the map $M \otimes_A B \to N$ is an equivalence. Using the construction of §V.2.2, we can identify $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}(\mathcal{C}^{op})$ and $\mathcal{F}$ with the sheafification of the presheaf of spectra $(B, N) \mapsto N$ on $\mathcal{C}^{op}$. Using Corollary VII.6.14 (and Proposition VII.5.7) we deduce that this presheaf is already a sheaf. In other words, for every étale A-algebra B, we have $\mathcal{F}(U) \simeq M \otimes_A B$, where $U \in \mathcal{X}$ is such that $\mathrm{Spec}\, B \simeq (\mathcal{X}_{/U}, \mathcal{O}\,|U)$. Invoking Theorem 1.2.1, we deduce that this condition holds *whenever* $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is affine. $\square$

Lemma 2.3.9. *Let $(\mathcal{X}, \mathcal{O}) \simeq \mathrm{Spec}\, A$ be an affine nonconnective spectral Deligne-Mumford stack. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}$-module and let $M = \Gamma(\mathcal{X}; \mathcal{F})$ be the global sections of $\mathcal{F}$, regarded as an $A \simeq \Gamma(\mathcal{X}; \mathcal{O})$-module. Then the canonical map $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \to \mathrm{Spec}^{\acute{e}t}_{\mathrm{M}}(A, M)$ is an equivalence (in the ∞-category $\mathrm{RingTop}^{M}_{\acute{e}t}$).*

Remark 2.3.10. We can regard Lemma 2.3.9 as a converse to Proposition 2.3.4: it implies that a $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme is affine if and only if the underlying nonconnective spectral Deligne-Mumford stack is affine.

Proof. Let $\mathcal{C}$ be the opposite of the full subcategory of CAlg_A spanned by the étale A-algebras, so that we can identify $\mathcal{X}$ with the ∞-topos $\mathcal{S}\mathrm{hv}(\mathcal{C})$ (see §V.2.2). Since $\mathcal{F}$ is quasi-coherent, there exists a collection of objects $U_\alpha \in \mathcal{X}$ which cover the final object such that each $(\mathcal{X}_{/U_\alpha}, \mathcal{O}\,|U_\alpha, \mathcal{F}\,|U_\alpha)$ is an affine $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme. In particular, $(\mathcal{X}_{/U_\alpha}, \mathcal{O}\,|U_\alpha)$ is an affine nonconnective spectral Deligne-Mumford stack (Proposition 2.3.4) and therefore has the form $\mathrm{Spec}\, B_\alpha$ for some étale A-algebra $B_\alpha \in \mathcal{C}$ (Theorem 1.2.1). Without loss of generality, we may assume that the set of indices α is finite. Let $B = \prod B_\alpha$ so that $\mathrm{Spec}\, B \simeq (\mathcal{X}_{/U}, \mathcal{O}\,|U)$ for $U = \coprod_\alpha U_\alpha$; we observe that $(\mathcal{X}_{/U}, \mathcal{O}\,|U, \mathcal{F}\,|U)$ is an affine $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme of the form $\mathrm{Spec}^{\text{ét}}_{\mathrm{M}}(B, N)$ for some B-module N.

Let us abuse notation by interpreting the pair $(\mathcal{O}, \mathcal{F})$ as a functor $\mathcal{C}^{op} \to \mathrm{Mod}$. Using Lemma 2.3.8, we deduce that the canonical map $\mathcal{F}(R) \otimes_R R' \to \mathcal{F}(R')$ is an equivalence whenever $R \to R'$ is a morphism in $\mathcal{C}$ such that the étale map $A \to R$ factors through B. Let $B^\bullet$ be the Čech nerve of the faithfully flat morphism $A \to B$. Since $\mathcal{F}$ is a sheaf, we have $M = \mathcal{F}(A) = \varprojlim \mathcal{F}(B^\bullet)$. The proof of Theorem VII.6.12 shows that the canonical map $M \otimes_A B \to \mathcal{F}(B)$ is an equivalence, so that $M \otimes_A R \to \mathcal{F}(R)$ is an equivalence for any étale map $A \to R$ which factors through B. Let $\mathrm{Spec}^{\text{ét}}_{\mathrm{M}}(A, M) \simeq (\mathcal{X}, \mathcal{O}, \mathcal{F}')$, so that the map $M \to \mathcal{F}(A)$ induces a morphism of sheaves of $\mathcal{O}$-modules $\mathcal{F}' \to \mathcal{F}$. Using Lemma 2.3.8, we deduce that α induces an equivalence $\mathcal{F}'(R) \to \mathcal{F}(R)$ whenever $A \to R$ is an étale map which factors through B. Since $\mathcal{F}'$ and $\mathcal{F}$ are sheaves, they are determined by their restriction to this covering sieve on $\mathcal{C}$ so that α is an equivalence as desired. $\square$

Proposition 2.3.11. *Let $(\mathcal{X}, \mathcal{O}) \simeq \mathrm{Spec}\, A$ be an affine nonconnective spectral Deligne-Mumford stack. Then the global sections functor $\Gamma : \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_A$ admits a fully faithful left adjoint, whose essential image is precisely the collection of quasi-coherent $\mathcal{O}$-modules.*

Proof. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}}$ be the functor described informally by the formula $\mathrm{Spec}^{\text{ét}}_M(A, M) \simeq (\mathcal{X}, \mathcal{O}, F(M))$. Unwinding the definitions, we deduce immediately that F is a left adjoint to Γ. It is clear that F carries A-modules to quasi-coherent $\mathcal{O}$-modules. Conversely, Lemma 2.3.9 implies that every quasi-coherent $\mathcal{O}$-module belongs to the essential image of F. To prove that F is fully faithful, it suffices to show that for every A-module M, the unit map $M \to \Gamma(\mathcal{X}, F(M))$ is an equivalence, which is a special case of Lemma 2.3.8. □

Proposition 2.3.12. *Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{X}$. The following conditions are equivalent:*

(1) *The sheaf $\mathcal{F}$ is quasi-coherent.*

(2) *Let $f : U \to V$ be a morphism in $\mathcal{X}$ such that $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ and $(\mathcal{X}_{/V}, \mathcal{O}\,|V)$ are affine. Then the canonical map $\mathcal{F}(V) \otimes_{\mathcal{O}(V)} \mathcal{O}(U) \to \mathcal{F}(U)$ is an equivalence.*

Proof. Assume first that (1) is satisfied. To prove (2), we are free to replace $\mathcal{X}$ by $\mathcal{X}_{/V}$ and thereby reduce to the case where $(\mathcal{X}, \mathcal{O})$ is affine. It follows that $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ is an affine $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme (see Remark 2.3.10), so that assertion (2) follows from Lemma 2.3.8.

Now suppose that (2) is satisfied. We wish to prove that $(\mathcal{X}, \mathcal{O}, \mathcal{F})$ is an $\mathcal{G}^{\mathrm{nM}}_{\text{ét}}$-scheme. The assertion is local on $\mathcal{X}$: we may therefore assume without loss of generality that $(\mathcal{X}, \mathcal{O})$ is an affine nonconnective spectral Deligne-Mumford stack $\mathrm{Spec}\, A$. Let $M = \Gamma(\mathcal{X}; \mathcal{F})$, regarded as an $A = \Gamma(\mathcal{X}; \mathcal{O})$-module. Then the identity map $M \to \Gamma(\mathcal{X}; \mathcal{F})$ induces a morphism $(\mathcal{X}, \mathcal{O}, \mathcal{F}) \to \mathrm{Spec}^{\text{ét}}_M(A, M) \simeq (\mathcal{X}, \mathcal{O}, \mathcal{F}')$ in $\mathrm{RingTop}^M_{\text{ét}}$. To complete the proof, it suffices to show that this map induces an equivalence of sheaves of spectra $\mathcal{F}' \to \mathcal{F}$. Since $\mathcal{X}$ is generated under small colimits by the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by those objects $U \in \mathcal{X}$ for which $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is affine, it will suffice to show that $\mathcal{F}'(U) \to \mathcal{F}(U)$ is an equivalence when U is affine. This follows from the observation that we have a commutative diagram

$$\begin{tikzcd} & M \otimes_A \mathcal{O}(U) \arrow[dl, "\psi"'] \arrow[dr, "\phi"] & \\ \mathcal{F}'(U) \arrow[rr] & & \mathcal{F}(U) \end{tikzcd}$$

where ϕ is an equivalence by Lemma 2.3.8 and ψ is an equivalence by assumption (2). □

Proposition 2.3.13. *Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack. Then:*

(1) *The ∞-category $\mathrm{QCoh}(\mathcal{X})$ is closed under small colimits in $\mathrm{Mod}_{\mathcal{O}}$.*

(2) *The ∞-category $\mathrm{QCoh}(\mathcal{X})$ is stable.*

(3) *The ∞-category $\mathrm{QCoh}(\mathcal{X})$ is presentable.*

Proof. We first prove (1). Suppose we are given a small diagram $\{\mathcal{F}_\alpha\}$ of quasi-coherent $\mathcal{O}$-modules, having a colimit $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}}$. We wish to prove that $\mathcal{F}$ is quasi-coherent. The assertion is local on $\mathcal{X}$: it therefore suffices to show that $\mathcal{F}\,|U \simeq \varinjlim \mathcal{F}_\alpha\,|U$ is a quasi-coherent sheaf on $\mathcal{X}_{/U}$ whenever $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is affine. Replacing $\mathcal{X}$ by $\mathcal{X}_{/U}$, we may assume that $(\mathcal{X}, \mathcal{O})$ is affine. In this case, the desired result follows from Proposition 2.3.11. Using exactly the same argument, we deduce that $\mathrm{QCoh}(\mathcal{X}, \mathcal{O})$ is closed under shifts in the stable ∞-category $\mathrm{Mod}_{\mathcal{O}}$. Assertion (2) now follows from Lemma A.1.1.3.3.

To prove (3), we let $\mathcal{X}_0 \subseteq \mathcal{X}$ denote the full subcategory spanned by those objects U for which the ∞-category $\mathrm{QCoh}(\mathcal{X}_{/U})$ is presentable. We wish to prove that $\mathcal{X}_0 = \mathcal{X}$. According to Remark 2.1.11, the construction $U \mapsto \mathrm{Mod}_{\mathcal{O}\,|U}$ defines a limit-preserving functor $\chi_{\mathcal{O}} : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. This functor is classified by a Cartesian fibration $p : \widetilde{\mathcal{X}} \to \mathcal{X}$. Let $\widetilde{\mathcal{X}}'$ denote the full subcategory of $\widetilde{\mathcal{X}}$ spanned by those objects X which correspond to quasi-coherent sheaves on $\mathcal{X}_{/p(X)}$. Remark 2.3.7 guarantees that $p|\widetilde{\mathcal{X}}'$ is also a Cartesian fibration, which is classified by another functor $\chi'_{\mathcal{O}} : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ (given informally by $U \mapsto \mathrm{QCoh}(\mathcal{X}_{/U})$). Since the condition of quasi-coherence is local (Remark 2.3.7), Proposition T.3.3.3.1 shows that $\chi'_{\mathcal{O}}$ is again a limit diagram. The functor $\chi'_{\mathcal{O}}$ evidently factors through the subcategory $\widehat{\mathcal{C}\mathrm{at}}'_\infty \subseteq \widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by those ∞-categories which admit small colimits and those functors which preserve small colimits. The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories can be identified with a full subcategory of $\widehat{\mathcal{C}\mathrm{at}}'_\infty$, so that $\mathcal{X}_0^{op} = (\chi'_{\mathcal{O}})^{-1}\,\mathcal{P}\mathrm{r}^{\mathrm{L}}$. Since $\chi'_{\mathcal{O}}$ preserves small limits, it follows from Proposition T.5.5.3.13 that $\mathcal{X}_0$ is stable under small colimits in $\mathcal{X}$. It will therefore suffice to show that $\mathcal{X}_0$ contains every object $U \in \mathcal{X}$ such that $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is affine, which follows immediately from Proposition 2.3.11. □

Proposition 2.3.14. *Let $(\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack. Then the full subcategory $\mathrm{QCoh}(\mathcal{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ contains the unit object $\mathcal{O}$ and is stable under tensor products, and therefore inherits a symmetric monoidal structure from the symmetric monoidal structure on $\mathrm{Mod}_{\mathcal{O}}$ (see Proposition A.2.2.1.1).*

Proof. The assertion is local, so we may assume that $(\mathcal{X}, \mathcal{O}) \simeq \operatorname{Spec} A$ is an affine nonconnective spectral Deligne-Mumford stack. Let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}}$ be the functor described in Proposition 2.3.11, so that the essential image of F is the full subcategory $\mathrm{QCoh}(\mathcal{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$. Then $F(A) \simeq \mathcal{O}$, so that $\mathcal{O}$ is quasi-coherent. To show that $\mathrm{QCoh}(\mathcal{X})$ is stable under tensor products, it suffices to show that $F(M) \otimes_{\mathcal{O}} F(N)$ is quasi-coherent, for every pair of A-modules $M, N \in \mathrm{Mod}_A$.

Let $\mathcal{C}$ be the full subcategory of CAlg_A^{op} spanned by the étale A-algebras, so that we can identify $\mathcal{X}$ with $\mathcal{S}\mathrm{hv}(\mathcal{C})$. For any sheaf of $\mathcal{O}$-modules $\mathcal{F}$, we can identify the pair $(\mathcal{O}, \mathcal{F})$ with a Mod-valued sheaf $\mathcal{C}^{op} \to \mathrm{Mod}$ on $\mathcal{C}$. Using Lemma 2.3.8, we see that $F(M)$ and $F(N)$ are given by the formulas

$$F(M)(B) = M \otimes_A B \qquad F(N)(B) = N \otimes_A B.$$

It follows that $F(M) \otimes_{\mathcal{O}} F(N)$ is the sheafification of the presheaf

$$B \mapsto F(M)(B) \otimes_{\mathcal{O}(B)} F(N)(B) \simeq (M \otimes_A B) \otimes_B (N \otimes_A B) \simeq (M \otimes_A N) \otimes_A B.$$

As in the proof of Lemma 2.3.8, this presheaf is already a sheaf which we will denote by $\mathcal{F}$. We have $\mathcal{F}(A) \simeq M \otimes_A N$ so the above formula shows that the canonical map $\mathcal{F}(A) \otimes_A B \to \mathcal{F}(B)$ is an equivalence for every étale A-algebra B; in other words, the counit map $F(\Gamma(\mathcal{X}; \mathcal{F})) \to \mathcal{F}$ is an equivalence, so that $\mathcal{F}$ belongs to the essential image $\mathrm{QCoh}(\mathcal{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ of the functor F. □

We now restrict our attention to the case of spectral Deligne-Mumford stacks $(\mathcal{X}, \mathcal{O})$: that is, we assume that the structure sheaf $\mathcal{O}$ is connective. In this case, the ∞-category $\mathrm{QCoh}(\mathcal{X})$ inherits a t-structure.

Lemma 2.3.15. *Let A be a connective $\mathbb{E}_\infty$-ring, let $\operatorname{Spec}^{\acute{e}t} A = (\mathcal{X}, \mathcal{O})$, and let $F : \mathrm{Mod}_A \to \mathrm{Mod}_{\mathcal{O}}$ be the fully faithful embedding of Proposition 2.3.11. Then F is t-exact.*

Proof. The functor F is left adjoint to the global sections functor $\mathcal{F} \mapsto \Gamma(\mathcal{X}; \mathcal{F})$, which is obviously left t-exact. It follows formally that F is right t-exact. To complete the proof, we will show that F is left t-exact: that is, if $M \in (\mathrm{Mod}_A)_{\leq 0}$, then $F(M) \in (\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$. Let $\mathcal{X}_0$ be the full subcategory of $\mathcal{X}$ spanned by those objects $U \in \mathcal{X}$ such that $F(M)(U) \in \mathrm{Sp}_{\leq 0}$. We wish to prove that $\mathcal{X}_0 = \mathcal{X}$. Since $\mathcal{F}$ is a sheaf and the full subcategory $\mathrm{Sp}_{\leq 0} \subseteq \mathrm{Sp}$ is stable under limits, we deduce that $\mathcal{X}_0$ is stable under colimits in $\mathcal{X}$. It will therefore suffice to show that $\mathcal{X}_0$ contains all objects $U \in \mathcal{X}$ such that $(\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is an affine spectral Deligne-Mumford stack $\operatorname{Spec}^{\text{ét}} B$. Lemma 2.3.8 gives a canonical equivalence $F(M)(U) \simeq M \otimes_A B$. The desired result now follows from Theorem A.7.2.2.15, since Theorem 1.2.1 guarantees that B is étale (and in particular flat) over A. □

Proposition 2.3.16. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. Then the full subcategory $\mathrm{QCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathcal{O}}$ is compatible with the t-structure of Proposition 2.1.3. More precisely, if $\mathcal{F} \in \mathrm{Mod}_{\mathcal{O}}$ is quasi-coherent, then the truncations $\tau_{\geq n}\mathcal{F}$ and $\tau_{\leq n}\mathcal{F}$ are quasi-coherent, for every integer n. Consequently, the full subcategories $\mathrm{QCoh}(\mathfrak{X})_{\geq 0} = \mathrm{QCoh}(\mathfrak{X}) \cap (\mathrm{Mod}_{\mathcal{O}})_{\geq 0}$ and $\mathrm{QCoh}(\mathfrak{X})_{\leq 0} = \mathrm{QCoh}(\mathfrak{X}) \cap (\mathrm{Mod}_{\mathcal{O}})_{\leq 0}$ determine a t-structure on the ∞-category $\mathrm{QCoh}(\mathfrak{X})$.*

Proof. Replacing $\mathcal{F}$ by its translates if necessary, it will suffice to show that if $\mathcal{F}$ is quasi-coherent, then $\tau_{\geq 0}\mathcal{F}$ and $\tau_{\leq -1}\mathcal{F}$ are quasi-coherent. This assertion is local on $\mathcal{X}$; we may therefore assume that $(\mathcal{X}, \mathcal{O}) \simeq \operatorname{Spec} A$ is an affine spectral Deligne-Mumford stack (where A is a connective $\mathbb{E}_\infty$-ring). Let $F : \operatorname{Mod}_A \to \operatorname{Mod}_{\mathcal{O}}$ be the functor described in Proposition 2.3.11. Since $\mathcal{F}$ is quasi-coherent, we may assume without loss of generality that $\mathcal{F} = F(M)$ for some A-module M. Since A is connective, there is a fiber sequence

$$M' \to M \to M''$$

where M' is a connective A-module and $M'' \in (\operatorname{Mod}_A)_{\leq -1}$. Applying the exact functor F, we obtain a fiber sequence

$$F(M') \to \mathcal{F} \to F(M'')$$

in $\operatorname{Mod}_{\mathcal{O}}$. Lemma 2.3.15 guarantees that $F(M') \in (\operatorname{Mod}_{\mathcal{O}})_{\geq 0}$ and $F(M'') \in (\operatorname{Mod}_{\mathcal{O}})_{\leq -1}$. We therefore obtain identifications $F(M') \simeq \tau_{\geq 0}\mathcal{F}$ and $F(M'') \simeq \tau_{\leq -1}\mathcal{F}$ which prove that $\tau_{\geq 0}\mathcal{F}$ and $\tau_{\leq -1}\mathcal{F}$ are quasi-coherent. □

Notation 2.3.17. If $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack, we will often let $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ denote the full subcategory $\operatorname{QCoh}(\mathfrak{X})_{\geq 0} \subseteq \operatorname{QCoh}(\mathfrak{X})$ defined in Proposition 2.3.16. We will say that a quasi-coherent sheaf $\mathcal{F}$ is *connective* if it belongs to $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}}$.

The basic properties of the t-structure on $\operatorname{QCoh}(\mathcal{X})$ are summarized in the following result:

Proposition 2.3.18. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack. Then:*

(1) *The t-structure on $\operatorname{QCoh}(\mathfrak{X})$ is accessible (see Definition A.1.4.5.12).*

(2) *The t-structure on $\operatorname{QCoh}(\mathfrak{X})$ is compatible with filtered colimits: that is, the full subcategory $\operatorname{QCoh}(\mathfrak{X})_{\leq 0}$ is closed under filtered colimits.*

(3) *The t-structure on $\operatorname{QCoh}(\mathfrak{X})$ is both right and left complete.*

Proof. To prove (1), it suffices to show that the full subcategory $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ is presentable (Proposition A.1.4.5.13). This follows from Proposition T.5.5.3.12, since $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}}$ can be identified with the fiber product $\operatorname{QCoh}(\mathfrak{X}) \times_{\operatorname{Mod}_{\mathcal{O}}} \operatorname{Mod}_{\mathcal{O}}^{\mathrm{cn}}$. Assertion (2) follows from Proposition 2.3.13 together with the corresponding result for $\operatorname{Mod}_{\mathcal{O}}$ (Proposition 2.1.3).

We now prove (3). Since $\operatorname{Mod}_{\mathcal{O}}$ is right-complete (Proposition 2.1.3), we deduce that $\bigcap_n \operatorname{QCoh}(\mathfrak{X})_{\leq -n} \subseteq \bigcap_n (\operatorname{Mod}_{\mathcal{O}})_{\leq -n}$ contains only zero objects. Combining this observation with (2), we deduce that $\operatorname{QCoh}(\mathfrak{X})$ is right-complete (see Proposition A.1.2.1.19).

The proof that $\operatorname{QCoh}(\mathfrak{X})$ is left-complete requires a bit more effort. Consider the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ spanned by those objects $U \in \mathcal{X}$ for which the t-structure on $\operatorname{QCoh}(\mathcal{X}_{/U})$ is left-complete. To complete the proof, it will suffice to show that $\mathcal{X}_0 = \mathcal{X}$. Using Proposition 2.3.11, Lemma 2.3.15, and Proposition A.7.1.1.13, we deduce that $\mathcal{X}_0$ contains every object $U \in \mathcal{X}$ for which $\mathfrak{X}_U = (\mathcal{X}_{/U}, \mathcal{O}\,|U)$ is affine. It will therefore suffice to show that $\mathcal{X}_0$ is closed under small colimits in $\mathcal{X}$. Since the conditions of being quasi-coherent and n-truncated are local, the proof of Proposition 2.3.13 shows that the constructions

$$U \mapsto \operatorname{QCoh}(\mathfrak{X}_U) \qquad U \mapsto \operatorname{QCoh}(\mathfrak{X}_U)_{\leq n}$$

determine limit-preserving functors $\mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. If $\{U_\alpha\}$ is a diagram in $\mathcal{X}$ having a colimit $U \in \mathcal{X}$, we have a commutative diagram

$$\begin{tikzcd}
\operatorname{QCoh}(\mathfrak{X}_U) \arrow[r] \arrow[d] & \varprojlim_\alpha \operatorname{QCoh}(\mathfrak{X}_{U_\alpha}) \arrow[d, "\theta"] \\
\varprojlim_n \operatorname{QCoh}(\mathfrak{X}_U)_{\leq n} \arrow[r] & \varprojlim_{n,\alpha} \operatorname{QCoh}(\mathfrak{X}_{U_\alpha})_{\leq n}.
\end{tikzcd}$$

where the vertical maps are equivalences. If each U_α belongs to $\mathcal{X}_0$, then the right vertical map is also an equivalence, so the left vertical map is an equivalence as well and $U \in \mathcal{X}_0$ as desired. □

Remark 2.3.19. Let $\mathcal{G}_{\text{ét}}^{\text{M}}$ be the geometry introduced in Example 2.2.9. Unwinding the definitions, we see that a $\mathcal{G}_{\text{ét}}^{\text{M}}$-scheme can be identified with a triple $(\mathcal{X}, \mathcal{O}, \mathcal{F})$, where $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ is a spectral Deligne-Mumford stack and $\mathcal{F} \in \text{QCoh}(\mathfrak{X})^{\text{cn}}$.

Remark 2.3.20. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be nonconnective spectral Deligne-Mumford stack, and let $\mathfrak{X}_0$ denote the underlying ordinary Deligne-Mumford stack (see Remark VII.8.41). If $\mathcal{F}$ is a quasi-coherent sheaf on $(\mathcal{X}, \mathcal{O})$, then each homotopy group $\pi_n \mathcal{F}$ can be identified with a quasi-coherent sheaf on $\mathfrak{X}_0$, in the sense of classical algebraic geometry. If $\mathcal{O}$ is connective, then this construction induces an equivalence from the heart $\text{QCoh}(\mathfrak{X})^{\heartsuit}$ of the ∞-category $\text{QCoh}(\mathfrak{X})$ of quasi-coherent sheaves on $(\mathcal{X}, \mathcal{O})$ to (the nerve of) the abelian category of quasi-coherent sheaves on $\mathfrak{X}_0$. For a more general assertion of this nature, see Corollary 2.5.24.

We close this section by giving a different characterization of quasi-coherence.

Proposition 2.3.21. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a nonconnective spectral Deligne-Mumford stack and let $\mathcal{F}$ be a sheaf of $\mathcal{O}$-modules on $\mathcal{X}$. Then $\mathcal{F}$ is quasi-coherent if and only if it satisfies the following conditions:*

(1) *For every integer n, the homotopy group $\pi_n \mathcal{F}$ is a quasi-coherent $\tau_{\geq 0} \mathcal{O}$-module on $\mathcal{X}$ (which we can identify with a quasi-coherent sheaf on the underlying ordinary Deligne-Mumford stack of $(\mathcal{X}, \mathcal{O})$; see Remark 2.3.20).*

(2) *The object $\Omega^\infty \mathcal{F} \in \text{Shv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$ is hypercomplete.*

Proof. Replacing $\mathcal{O}$ by its connective cover if necessary, we may assume that $\mathcal{O}$ is connective. If $\mathcal{F}$ is quasi-coherent, then Proposition 2.3.16 implies that each homotopy group $\pi_n \mathcal{F}$ is quasi-coherent as a $\mathcal{O}$-module. To prove that (2) is satisfied, it suffices to work locally on $\mathcal{X}$; we may therefore assume that $(\mathcal{X}, \mathcal{O}) \simeq \operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A. Let F be the functor of Proposition 2.3.11, so that $\mathcal{F} \simeq F(M)$ for some A-module M. Let $\mathcal{C}$ denote the full subcategory of CAlg_A^{op} spanned by the étale A-algebras, so we may identify $\mathcal{X}$ with the ∞-topos $\text{Shv}(\mathcal{C})$ and the pair $(\mathcal{O}, \mathcal{F})$ with a sheaf $\mathcal{C}^{op} \to \text{Mod}$ on $\mathcal{C}$. We note that for each $B \in \mathcal{C}$, we have

$$F(M)(B) \simeq M \otimes_A B \simeq \varprojlim \tau_{\leq n}(M \otimes_A B) \simeq \varprojlim (\tau_{\leq n} M) \otimes_A B \simeq \varprojlim F(\tau_{\leq n} M)(B).$$

It follows that $\mathcal{F} \simeq F(M) \simeq \varprojlim F(\tau_{\leq n} M)$ is a limit of truncated objects of $\text{Mod}_{\mathcal{O}}$ (Lemma 2.3.15), so that $\Omega^\infty \mathcal{F}$ is a limit of truncated objects of $\mathcal{X}$ and therefore hypercomplete.

Now suppose that $\mathcal{F} \in \text{Mod}_{\mathcal{O}}$ satisfies conditions (1) and (2). We wish to prove that $\mathcal{F}$ is quasi-coherent. Note that $\mathcal{F} \simeq \varinjlim \tau_{\geq -n} \mathcal{F}$ by Proposition 2.3.18. Since the collection of quasi-coherent sheaves is closed under colimits in $\text{Mod}_{\mathcal{O}}$, it suffices to prove that each $\tau_{\geq -n} \mathcal{F}$ is quasi-coherent. Replacing $\mathcal{F}$ by $(\tau_{\geq -n} \mathcal{F})[n]$, we may assume that $\mathcal{F}$ is connective. Since the condition of being quasi-coherent is local on $\mathcal{X}$, we may suppose that $(\mathcal{X}, \mathcal{O}) \simeq \operatorname{Spec} A$ is an affine spectral Deligne-Mumford stack, where A is a connective $\mathbb{E}_\infty$-ring; let $\mathcal{C}$ and $F : \text{Mod}_A \to \text{Mod}_{\mathcal{O}}$ be defined as above.

We now argue by induction on m that each truncation $\tau_{\leq m} \mathcal{F}$ is quasi-coherent. For $m < 0$, this is obvious. If $m \geq 0$, it follows from the existence of a fiber sequence

$$\tau_{\leq m-1} \mathcal{F} \to \tau_{\leq m} \mathcal{F} \to (\pi_m \mathcal{F})[m].$$

Using Proposition 2.3.11, we may suppose that the tower $\{\tau_{\leq m} \mathcal{F}\}$ is obtained from a tower of A-modules $\{M_m\}_{m \geq 0}$. Using Lemma 2.3.15, we deduce that for $m \leq m'$, the map $M_{m'} \to M_m$ exhibits M_m as an m-truncation $\tau_{\leq m} M_{m'}$. Since Mod_A is left complete (Proposition A.7.1.1.13), the A-module $M \simeq \varprojlim M_m$ has the property that $\tau_{\leq m} M \simeq M_m$ for every integer m. For every flat A-algebra B, we also obtain an equivalence $\tau_{\leq m}(M \otimes_A B) \simeq M_m \otimes_A B$, so that $M \otimes_A B \simeq \varprojlim (M_m \otimes_A B)$. It follows that $F(M) \simeq \varprojlim_m F(M_m) \simeq \varprojlim \tau_{\leq m} \mathcal{F}$ in the ∞-category $\text{Mod}_{\mathcal{O}}$. In particular, we obtain a map $\alpha : \mathcal{F} \to F(M)$. To prove that $\mathcal{F}$ is quasi-coherent, it will suffice to show that α is an equivalence. Since $\mathcal{F}$ and $F(M)$ are both connective, this is equivalent to the requirement that α induces an equivalence $\Omega^\infty \mathcal{F} \to \Omega^\infty F(M)$ in $\text{Shv}_{\mathcal{S}}(\mathcal{X}) \simeq \mathcal{X}$. Since $\Omega^\infty \mathcal{F}$ is hypercomplete (by (2)) and $\Omega^\infty F(M) \simeq \varprojlim \Omega^\infty F(M_m)$ is hypercomplete (since it is an inverse limit of truncated objects of $\mathcal{X}$), it will suffice to show that the map $\Omega^\infty(\alpha) : \Omega^\infty \mathcal{F} \to \Omega^\infty F(M)$ is ∞-connective. This is clear, since for every integer $m \geq 0$, the truncation $\tau_{\leq m}\Omega^\infty(\alpha)$ can be identified with the equivalence

$$\tau_{\leq m}\Omega^\infty \mathcal{F} \simeq \Omega^\infty(\tau_{\leq m} \mathcal{F}) \simeq \Omega^\infty F(M_m) \simeq \Omega^\infty F(\tau_{\leq m} M) \simeq \Omega^\infty \tau_{\leq m} F(M) \simeq \tau_{\leq m}\Omega^\infty F(M).$$

□

2.4 Quasi-Affine Spectral Deligne-Mumford Stacks

Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, let $\mathbf{1}$ denote a final object of $\mathcal{X}$, and let $A = \mathcal{O}(\mathbf{1})$ be the $\mathbb{E}_\infty$-ring of global sections of $\mathcal{O}$. Evaluation at the object $\mathbf{1}$ determines a functor $\theta : \mathrm{QCoh}(\mathfrak{X}) \subseteq \mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_A$. If $\mathfrak{X}$ is affine, then Proposition 2.3.11 implies that θ is an equivalence of ∞-categories. In this section, we will study a weaker condition on $\mathfrak{X}$ which guarantees that θ is an equivalence of ∞-categories: namely, the condition that $\mathfrak{X}$ is *quasi-affine*.

Definition 2.4.1. Let $\mathfrak{X}$ be a nonconnective spectral Deligne-Mumford stack. We say that $\mathfrak{X}$ is *quasi-affine* if $\mathfrak{X}$ is quasi-compact and there exists an open immersion $j : \mathfrak{X} \to \mathrm{Spec}^{\text{ét}} R$ for some $\mathbb{E}_\infty$-ring R.

Remark 2.4.2. Let $\mathfrak{X}$ be a quasi-affine spectral Deligne-Mumford stack. Then $\mathfrak{X}$ is schematic. In particular, $\mathfrak{X}$ is a spectral algebraic space.

Suppose that $\mathfrak{X}$ is a quasi-affine nonconnective spectral Deligne-Mumford stack. Then there exists an open immersion $j : \mathfrak{X} \to \mathfrak{X}'$, where $\mathfrak{X}'$ is affine. The following pair of results asserts that there is a canonical choice of $\mathfrak{X}'$, for which the ∞-categories $\mathrm{QCoh}(\mathfrak{X})$ and $\mathrm{QCoh}(\mathfrak{X}')$ are equivalent.

Proposition 2.4.3. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-compact nonconnective spectral Deligne-Mumford stack and let $\mathbf{1}$ denote a final object of $\mathcal{X}$. The following conditions are equivalent:*

(1) *The nonconnective spectral Deligne-Mumford stack $\mathfrak{X}$ is quasi-affine.*

(2) *The canonical map $\mathfrak{X} \to \mathrm{Spec}\, \mathcal{O}_{\mathcal{X}}(\mathbf{1})$ is an open immersion.*

Proposition 2.4.4. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-affine nonconnective spectral Deligne-Mumford stack. Let $\mathbf{1}$ denote a final object of $\mathcal{X}$ and let $A = \mathcal{O}_{\mathcal{X}}(\mathbf{1})$. Then evaluation at $\mathbf{1}$ induces an equivalence of ∞-categories $e : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_A$.*

Before giving the proofs of Propositions 2.4.3 and 2.4.4, we establish the following techinical result:

Proposition 2.4.5. *Let $\mathfrak{X}$ be a nonconnective quasi-compact spectral Deligne-Mumford stack and let $j : \mathfrak{X} \to \mathrm{Spec}^{\acute{e}t} R$ be an open immersion. Then:*

(1) *The global sections functor $\Gamma : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_R$ commutes with small colimits.*

(2) *Suppose that R is connective. Then there exists an integer n such that $j_*(\mathrm{QCoh}(\mathfrak{X})_{\geq 0}) \subseteq (\mathrm{Mod}_R)_{\geq -n}$.*

(3) *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "f'"] \arrow[d, "j'"] & \mathrm{Spec}^{\acute{e}t} R' \arrow[d, "f"] \\ \mathfrak{X} \arrow[r, "j"] & \mathrm{Spec}^{\acute{e}t} R. \end{tikzcd}$$

Then the associated diagram of ∞-categories

$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r, "j^*"] \arrow[d] & \mathrm{QCoh}(\mathfrak{X}) \arrow[d] \\ \mathrm{Mod}_{R'} \arrow[r, "j'^*"] & \mathrm{QCoh}(\mathfrak{X}') \end{tikzcd}$$

is right adjointable.

Proof. The open immersion j is determined by an open subset $U \subseteq \operatorname{Spec}^{\mathrm{Z}} R$. For every open subset $V \subseteq U$, let $\Gamma_V : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_R$ be the functor given by evaluation at V (which we can identify with a (-1)-truncated object of the underlying ∞-topos $\mathcal{X}$ of $\mathfrak{X}$). Given a pair of open sets $V', V'' \subseteq U$, we obtain a pullback diagram of functors σ:

$$\begin{tikzcd}
\Gamma_{V' \cup V''} \arrow[r] \arrow[d] & \Gamma_{V''} \arrow[d] \\
\Gamma_{V'} \arrow[r] & \Gamma_{V' \cap V''}.
\end{tikzcd}$$

To prove (1), it will suffice to show that for every quasi-compact open subset $V \subseteq U$, the functor Γ_V commutes with filtered colimits. Since V is quasi-compact, we can write V as a union $\bigcup_{1 \leq i \leq n} V_i$ where each $V_i \subseteq \operatorname{Spec}^{\mathrm{Z}} R$ is given by $\operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_i}]$ for some $x_i \in \pi_0 R$. We proceed by induction on n. If $n = 0$, then V is empty and the result is obvious. If $n > 0$, we let $V' = V_1$ and $V'' = \bigcup_{1 < i \leq n} V_i$ so that $V = V' \cup V''$. The inductive hypothesis implies that $\Gamma_{V''}$ and $\Gamma_{V' \cap V''}$ commute with filtered colimits. Using the pullback diagram σ, we are reduced to proving that $\Gamma_{V'}$ commutes with filtered colimits. This is clear, since $\Gamma_{V'}$ is given by the composition

$$\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\operatorname{Spec}^{\text{ét}} R[\frac{1}{x_1}]) \simeq \mathrm{Mod}_{R[\frac{1}{x_1}]} \to \mathrm{Mod}_R .$$

We now prove (2). Assume that R is connective. We will show that if $V \subseteq U$ is an open subset which can be written as a union $\bigcup_{1 \leq i \leq n} V_i$, where each V_i is of the form $\operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_i}]$, then Γ_V carries $\mathrm{QCoh}(\mathfrak{X})_{\geq 0}$ to $(\mathrm{Mod}_R)_{\geq 1-n}$. We proceed by induction on n. In the case $n = 0$, $V = \emptyset$ and there is nothing to prove. Assume therefore that $n > 0$ and define subsets $V', V'' \subseteq V$ as above. If $M \in \mathrm{QCoh}(\mathfrak{X})_{\geq 0}$, then the pullback diagram σ gives a fiber sequence

$$\Gamma_V(M) \to \Gamma_{V'}(M) \oplus \Gamma_{V''}(M) \to \Gamma_{V' \cap V''}(M)$$

and therefore an exact sequence of abelian groups

$$\pi_{m+1} \Gamma_{V' \cap V''}(M) \to \pi_m \Gamma_V(M) \to \pi_m \Gamma_{V'}(M) \oplus \pi_m \Gamma_{V''}(M).$$

The functor $\Gamma_{V'}$ is given by the composition

$$\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\operatorname{Spec}^{\text{ét}} R[\frac{1}{x_1}]) \simeq \mathrm{Mod}_{R[\frac{1}{x_1}]} \to \mathrm{Mod}_R .$$

and is therefore t-exact. Using the inductive hypothesis, we deduce that if $m \leq -n$, then

$$\pi_{m+1} \Gamma_{V' \cap V''}(M) \simeq \pi_m \Gamma_{V''}(M) \simeq 0,$$

from which it follows that $\pi_m \Gamma_V(M) \simeq 0$.

We now prove (3). Let $\pi : \operatorname{Spec}^{\mathrm{Z}} R' \to \operatorname{Spec}^{\mathrm{Z}} R$ be the continuous map of topological spaces induced by the map of $\mathbb{E}_\infty$-rings $R \to R'$. For every open set $V \subseteq U$, let $\Gamma_{\pi^{-1} V} : \mathrm{QCoh}(\mathfrak{X}') \to \mathrm{Mod}_{R'}$ be defined as above. Let us say that an open subset $V \subseteq U$ is *good* if the canonical map $R' \otimes_R \Gamma_V(\bullet) \to \Gamma_{\pi^{-1} V}$ is an equivalence of functors from $\mathrm{QCoh}(\mathfrak{X})$ to $\mathrm{Mod}_{R'}$. Note that if $V', V'' \subseteq U$, then the canonical map

$$\Gamma_{\pi^{-1}(V' \cup V'')} \to \Gamma_{\pi^{-1}(V')} \times_{\Gamma_{\pi^{-1}(V' \cap V'')}} \Gamma_{\pi^{-1}(V'')}$$

is an equivalence. It follows that if V', V'', and $V' \cap V''$ are good, then $V' \cup V''$ is good. We will prove that every quasi-compact open subset $V \subseteq U$ is good. Write $V = \bigcup_{1 \leq i \leq n} V_i$ as above; we proceed by induction on n. When $n = 0$, $V = \emptyset$ and there is nothing to prove. If $n > 0$, we define $V', V'' \subseteq V$ as above, so that V'' and $V' \cap V''$ are good by the inductive hypothesis. We may therefore replace V by V' and thereby reduce to the case $\mathfrak{X}_V = \operatorname{Spec}^{\text{ét}} R[\frac{1}{x_1}]$, in which case the desired result follows from Lemma VII.6.15. $\square$

Corollary 2.4.6. *Let $\mathfrak{X}$ be a quasi-compact nonconnncetive spectral Deligne-Mumford stack and let $j : \mathfrak{X} \to \operatorname{Spec}^{ét} R$ be an open immersion. Then the global sections functor $\Gamma : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Mod}_R$ is fully faithful.*

Proof. Let $M \in \operatorname{QCoh}(\mathfrak{X})$; we will show that the counit map $j^*(j_*M) \to M$ is an equivalence. The open immersion j is determined by an open subset $U \subseteq \operatorname{Spec}^{\mathrm{Z}} R$. Write U as a union $\bigcup_{1 \leq i \leq n} U_i$, where each U_i is the open subset given by $\operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_i}]$ for some $x_i \in \pi_0 R$. For $1 \leq i \leq n$, let $g_i : \mathfrak{U}_i \to \mathfrak{X}$ be the open immersion determined by the inclusion $U_i \subseteq U$. It will therefore suffice to show that each of the induced maps $\theta_i : g_i^* j^* j_* M \to g_i^* j_* M$ is an equivalence. This follows immediately from Proposition 2.4.5, since the projection map $\mathfrak{U}_i \times_{\operatorname{Spec}^{ét} R} \mathfrak{X} \to \mathfrak{U}_i$ is an equivalence. □

Proof of Proposition 2.4.3. The implication (2) ⇒ (1) is obvious. We will show that (1) ⇒ (2). Assume therefore that there exists an open immersion $j : \mathfrak{X} \to \operatorname{Spec}^{ét} R$. Set $A = \mathcal{O}_{\mathcal{X}}(\mathbf{1})$, so that j determines a map of $\mathbb{E}_\infty$-rings $\phi : R \to A$. Then $\mathfrak{X} \times_{\operatorname{Spec}^{ét} R} \operatorname{Spec}^{ét} A$ is an open substack of $\operatorname{Spec}^{ét} A$. We will complete the proof by showing that the projection map $p : \mathfrak{X} \times_{\operatorname{Spec}^{ét} R} \operatorname{Spec}^{ét} A \to \mathfrak{X}$ is an equivalence. The map j determines an open subset U of the Zariski spectrum $\operatorname{Spec}^{\mathrm{Z}} R$. Since $\mathfrak{X}$ is quasi-compact, this open subset can be written as a union $\bigcup_{1 \leq i \leq n} \operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_i}]$ for some elements $x_i \in \pi_0 R$. To show that p is an equivalence, it will suffice to show that each of the induced projection maps

$$p_i : \operatorname{Spec}^{ét} R[\frac{1}{x_i}] \times_{\operatorname{Spec}^{ét} R} \operatorname{Spec}^{ét} A \to \operatorname{Spec}^{ét} R[\frac{1}{x_i}]$$

is an equivalence.

Let $x = x_i$. We wish to prove that the map $\theta : R[\frac{1}{x}] \to R[\frac{1}{x}] \otimes_R A$ is an equivalence of $\mathbb{E}_\infty$-rings. Let $\mathcal{O}'$ denote the structure sheaf of $\operatorname{Spec} R$. For every open subset $V \subseteq \operatorname{Spec}^{\mathrm{Z}} R$, let V_0 denote the intersection of V with the open set $\operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x}]$, and let f_V denote the canonical map $R[\frac{1}{x}] \otimes_R \mathcal{O}'(V) \to \mathcal{O}'(V_0)$. We note that f_U is left inverse to θ. It will therefore suffice to show that f_U is an equivalence, which is a special case of Proposition 2.4.5. □

Proof of Proposition 2.4.4. We can identify $e : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Mod}_A$ with the pushforward functor j_* associated to the canonical map $j : \mathfrak{X} \to \operatorname{Spec} A$. Proposition 2.4.3 implies that j is an open immersion, so that j_* is fully faithful by Corollary 2.4.6. Consequently, to prove that j_* is an equivalence of ∞-categories, it will suffice to show that the unit map $u_M : M \to j_* j^* M$ is an equivalence for every A-module M. Since j_* commutes with small colimits (Proposition 2.4.5), the collection of those A-modules M for which u_M is an equivalence is closed under small colimits. It will therefore suffice to show that u_M is an equivalence in the case $M = A[n]$. We may easily reduce to the case $n = 0$, in which case the desired result follows from the definition of A. □

Warning 2.4.7. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a quasi-affine spectral Deligne-Mumford stack, let $\mathbf{1}$ be a final object of $\mathcal{X}$, and let $A = \mathcal{O}_{\mathcal{X}}(\mathbf{1})$. Then the canonical map $j : \mathfrak{X} \to \operatorname{Spec}^{ét} A$ is an open immersion (Proposition 2.4.3). However, A is generally not connective. In fact, A is connective if and only if j is an equivalence (that is, if and only if the spectral Deligne-Mumford stack $\mathfrak{X}$ is affine). The "if" direction is obvious. Conversely, suppose that A is connective. The open immersion j determines a quasi-compact open subset $U \subseteq \operatorname{Spec}^{\mathrm{Z}} A$, consisting of those prime ideals which fail to contain some finitely generated ideal $I = (x_1, \ldots, x_n) \subseteq \pi_0 R$. Let $M = (\pi_0 A)/I$, which we regard as a discrete A-module. Then $M[\frac{1}{x_i}] \simeq 0$ for $1 \leq i \leq n$, so that M is annihilated by the pullback functor $\operatorname{Mod}_A \simeq \operatorname{QCoh}(\operatorname{Spec}^{ét} A) \xrightarrow{j^*} \operatorname{QCoh}(\mathfrak{X})$. Proposition 2.4.4 implies that the pullback functor j^* is an equivalence of ∞-categories, so that $M \simeq 0$. It follows that I is the unit ideal in $\pi_0 A$, so that j is an equivalence.

In spite of Warning 2.4.7, every quasi-affine spectral Deligne-Mumford stack admits an open immersion into the spectrum of a connective $\mathbb{E}_\infty$-ring:

Proposition 2.4.8. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. The following conditions are equivalent:*

(1) *There exists a connective $\mathbb{E}_\infty$-ring R and an open immersion $j : \mathfrak{X} \to \operatorname{Spec}^{ét} R$.*

(2) *The spectral Deligne-Mumford stack $\mathfrak{X}$ is quasi-affine.*

(3) *The discrete spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X})$ is quasi-affine.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious. To prove (2) $\Rightarrow$ (3), we note that if $j : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ is an open immersion, then j induces an open immersion $(\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X}) \to \operatorname{Spec}^{\text{ét}}(\pi_0 R)$.

It remains to prove that (3) $\Rightarrow$ (1). Let $\mathbf{1}$ denote a final object of $\mathcal{X}$. For each $i \geq 0$, we let R_i denote the $\mathbb{E}_\infty$-ring $(\tau_{\leq i} \mathcal{O}_\mathcal{X})(\mathbf{1})$, and let $R = \mathcal{O}_\mathcal{X}(\mathbf{1}) \simeq \varprojlim_i R_i$. Let $\Gamma : \operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Sp}$ denote the global sections functor. Applying Proposition 2.4.5 to the quasi-affine spectral Deligne-Mumford stack $(\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X})$, we deduce that there exists an integer n such that $\Gamma(M) \in \operatorname{Sp}_{\geq -n}$ whenever M belongs to the heart of $\operatorname{QCoh}(\mathfrak{X})$. The fiber sequence

$$\pi_m \mathcal{O}_\mathcal{X}[m] \to \tau_{\leq m} \mathcal{O}_\mathcal{X} \to \tau_{\leq m-1} \mathcal{O}_\mathcal{X}$$

yields a fiber sequence of spectra

$$\Gamma(\pi_m \mathcal{O}_\mathcal{X})[m] \to R_m \to R_{m-1}$$

so that the map $\pi_i R_m \to \pi_i R_{m-1}$ is an isomorphism for $m > n + i$. It follows that each of the maps $\pi_i R \to \pi_i R_{n+i}$ is an isomorphism.

Since condition (3) is satisfied, Proposition 2.4.3 implies that the canonical map $j_0 : (\mathcal{X}, \pi_0 \mathcal{O}_\mathcal{X}) \to \operatorname{Spec}^{\text{ét}} R_0$ is an open immersion, corresponding to some quasi-compact open subset $U \subseteq \operatorname{Spec}^{\mathrm{Z}} R_0$. For each $x \in \pi_0 R_0$, let $U_x = \{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{Z}} R_0 : x \notin \mathfrak{p}\}$. We next prove the following:

$(*)$ Let x be an element of $\pi_0 R_0$ such that $U_x \subseteq U$. Then there exists an integer $m > 0$ such that x^m can be lifted to an element of $\pi_0 R$.

For every pair of integers $i \leq i'$, let $\phi_{i',i} : \pi_0 R_{i'} \to \pi_0 R_i$ be the canonical map. To prove $(*)$, we show that for each $i \geq 0$, some positive power x^m of x lies in the image of the map $\phi_{i,0} : \pi_0 R_i \to \pi_0 R_0$. Since $\pi_0 R \simeq \pi_0 R_n$, $(*)$ will follow if we prove this in the case $i = n$. We proceed by induction on i, the case $i = 0$ being trivial. Assume therefore that there exists an integer $m > 0$ such that $x^m = \phi_{i,0}(y)$ for some $y \in \pi_0 R_i$. We will prove that some positive power of y lies in the image of the map $\phi_{i+1,i}$. Using Theorem A.7.4.1.26, we deduce that $\tau_{\leq i+1} \mathcal{O}_\mathcal{X}$ is a square-zero extension of $\tau_{\leq i} \mathcal{O}_\mathcal{X}$ by the module $(\pi_{i+1} \mathcal{O}_\mathcal{X})[i+1]$. It follows that R_{i+1} is a square-zero extension of R_i by the module $\Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})[i+1]$. In particular, the image of the map $\phi_{i+1,i}$ is the kernel of a derivation $d : \pi_0 R_i \to \pi_{-i-2}\Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})$. We wish to prove that $d(y^{m'}) = 0$ for some $m' > 0$. Since d is a derivation, we have $d(y^{m'}) = m' y^{m'-1} dy$. It will therefore suffice to show that $dy \in \pi_{-i-2}\Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})$ is annihilated by some power of y. Note that $\Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})$ has the structure of a module over R_0. Moreover, Corollary 2.4.6 implies that $j_0^* \Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})$ is equivalent to $\pi_{i+1} \mathcal{O}_\mathcal{X}$, which a discrete sheaf of spectra on $\mathcal{X}$. Since $U_x \subseteq U$, we deduce that $\Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})[\frac{1}{x}]$ is discrete. Since $i + 2 \neq 0$, it follows that every element of $\pi_{-i-2}\Gamma(\pi_{i+1} \mathcal{O}_\mathcal{X})$ is annihilated by a power of x, and therefore by a power of y. This completes the proof of $(*)$.

Write U as a union of open sets $\bigcup_{1 \leq i \leq n} U_{x_i}$ for some elements $x_i \in \pi_0 R_0$. Using $(*)$, we may assume without loss of generality that each x_i is the image of some element $y_i \in \pi_0 R$. For $1 \leq i \leq n$, let V_i denote the open subset $\{\mathfrak{p} \in \operatorname{Spec}^{\mathrm{Z}} R : y_i \notin \mathfrak{p}\}$, and let $V = \bigcup_{1 \leq i \leq n} V_i$. Let $\mathfrak{V}$ denote the open substack of $\operatorname{Spec} R$ corresponding to V, and for $1 \leq i \leq n$ let $\mathfrak{V}_i$ denote the open substack of $\mathfrak{V}$ corresponding to V_i. Since V is the inverse image of U in $\operatorname{Spec}^{\mathrm{Z}} R$, the canonical map $j : \mathfrak{X} \to \operatorname{Spec}^{\text{ét}} R$ factors through $\mathfrak{V}$. We claim that j induces an equivalence $\mathfrak{X} \to \mathfrak{V}$. To prove this, it suffices to show that each of the induced maps $\mathfrak{X} \times_\mathfrak{V} \mathfrak{V}_i \to \mathfrak{V}_i$ is an equivalence. In view of Proposition 2.4.5, this is equivalent to the assertion that $\mathfrak{X} \times$ is affine. This follows from Remark VII.8.43, since the 0-truncation of $\mathfrak{X} \times_\mathfrak{V} \mathfrak{V}_i$ is given by $\operatorname{Spec} R_0[\frac{1}{x_i}]$. $\square$

Proposition 2.4.9. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be a quasi-affine nonconnective spectral Deligne-Mumford stack, let $\mathbf{1} \in \mathcal{X}$ be the final object, let $A = \mathcal{O}_\mathcal{X}(\mathbf{1})$, and let $j : \mathfrak{X} \to \operatorname{Spec}^{ét} A$ be the open immersion of Proposition 2.4.3. For every spectral Deligne-Mumford stack $\mathfrak{Y}$, the canonical map*

$$\theta : \operatorname{Map}_{\operatorname{Stk}}(\mathfrak{Y}, \mathfrak{X}) \to \operatorname{Map}_{\operatorname{Stk}}(\mathfrak{Y}, \operatorname{Spec} A)$$

is a homotopy equivalence.

Proof. The assertion is local on $\mathfrak{Y}$; we may therefore assume that $\mathfrak{Y}$ is affine, so that $\mathfrak{Y} \simeq \operatorname{Spec}^{\text{ét}} R$ for some connective $\mathbb{E}_\infty$-ring R. Since j is an open immersion, the map θ exhibits $\operatorname{Map}_{\mathrm{Stk}}(\mathfrak{Y}, \mathfrak{X})$ as a summand of $\operatorname{Map}_{\mathrm{Stk}}(\mathfrak{Y}, \operatorname{Spec}^{\text{ét}} A)$. It will therefore suffice to show that every map $f : \operatorname{Spec}^{\text{ét}} R \to \operatorname{Spec}^{\text{ét}} A$ factors through j. Form a pullback diagram

$$\begin{tikzcd}
(\mathcal{X}', \mathcal{O}_{\mathcal{X}'}) \arrow[r, "j'"] \arrow[d] & \operatorname{Spec}^{\text{ét}} R \arrow[d] \\
\mathfrak{X} \arrow[r] & \operatorname{Spec}^{\text{ét}} A
\end{tikzcd}$$

so that j' is an open immersion. Let $\mathbf{1}'$ denote a final object of $\mathcal{X}'$. Since the canonical map $A \to \mathcal{O}_{\mathcal{X}}(\mathbf{1})$ is an equivalence, Proposition 2.4.5 implies that the map $R \to \mathcal{O}_{\mathcal{X}'}(\mathbf{1}')$ is an equivalence. Since R is connective, it follows from Warning 2.4.7 that j' is an equivalence. $\square$

Let $\mathfrak{X}$ be a quasi-affine spectral Deligne-Mumford stack. Combining Proposition 2.4.9 with Theorem VII.5.14, we deduce that the functor $R \mapsto \operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X})$ is a hypercomplete sheaf with respect to the flat topology on $\mathrm{CAlg}^{\mathrm{cn}}$. In fact, we have the following stronger assertion:

Proposition 2.4.10. *Let $\mathfrak{X}$ be quasi-affine nonconnective spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$. Then X is a hypercomplete sheaf with respect to the flat topology on* CAlg.

Proof. Choose an open immersion $j : \mathfrak{X} \hookrightarrow \operatorname{Spec} A$, for some $\mathbb{E}_\infty$-ring A. It follows from Theorem VII.5.14 that the functor

$$R \mapsto \operatorname{Map}_{\mathrm{Stk}^{\mathrm{nc}}}(\operatorname{Spec} R, \operatorname{Spec} A) \simeq \operatorname{Map}_{\mathrm{CAlg}}(A, R)$$

is a hypercomplete sheaf with respect to the flat topology on CAlg. According to Lemma 3.1.20, it will suffice to show that for every map $\eta : \operatorname{Spec} R \to \operatorname{Spec} A$, the fiber product $\mathfrak{X}' = \mathfrak{X} \times_{\operatorname{Spec} A} \operatorname{Spec} R$ represents a hypercomplete sheaf with respect to the flat topology on $\mathrm{CAlg}_{R/}$. We can identify $\mathfrak{X}'$ with an open substack of $\operatorname{Spec} R$, classified by an open set $U \subseteq \operatorname{Spec}^{\mathrm{Z}} R$. Unwinding the definitions, we are reduced to showing that if $\phi : B \to B'$ is a faithfully flat morphism in CAlg_R such that the map $\operatorname{Spec}^{\mathrm{Z}} B' \to \operatorname{Spec}^{\mathrm{Z}} R$ factors through U, then $\operatorname{Spec}^{\mathrm{Z}} B \to \operatorname{Spec}^{\mathrm{Z}} R$ also factors through U. This is clear, since the map $\operatorname{Spec}^{\mathrm{Z}} B' \to \operatorname{Spec}^{\mathrm{Z}} B$ is a surjection. $\square$

In fact, we can prove an even stronger version of Proposition 2.4.10. For every $\mathbb{E}_\infty$-ring R, let $\mathrm{Stk}^{\mathrm{nc}}_R$ denote the ∞-category $\mathrm{Stk}^{\mathrm{nc}}_{/\operatorname{Spec} R}$ of nonconnective spectral Deligne-Mumford stacks $\mathfrak{X}$ equipped with a map $f : \mathfrak{X} \to \operatorname{Spec} R$. Let $\mathrm{QAff}^{\mathrm{nc}}_R$ denote the full subcategory of $\mathrm{Stk}^{\mathrm{nc}}_R$ spanned by those maps $f : \mathfrak{X} \to \operatorname{Spec} R$ where $\mathfrak{X}$ is quasi-affine. If R is connective, we let QAff_R denote the full subcategory of $\mathrm{QAff}^{\mathrm{nc}}_R$ spanned by those morphisms where $\mathfrak{X}$ is a spectral Deligne-Mumford stack.

Proposition 2.4.11. *The functor $R \mapsto \mathrm{QAff}^{\mathrm{nc}}_R$, is a hypercomplete sheaf (with values in $\widehat{\mathcal{C}\mathrm{at}}_\infty$) with respect to the flat topology on* CAlg. *The functor $R \mapsto \mathrm{QAff}_R$ is a hypercomplete sheaf with respect to the flat topology on* $\mathrm{CAlg}^{\mathrm{cn}}$.

Proof. For every $\mathbb{E}_\infty$-ring R, let $\mathrm{Aff}^{\mathrm{nc}}_R$ denote the full subcategory of $\mathrm{Stk}^{\mathrm{nc}}_R$ spanned by those morphisms $f : \mathfrak{X} \to \operatorname{Spec} R$ where $\mathfrak{X}$ is an affine nonconnective spectral Deligne-Mumford stack. We have an equivalence of ∞-categories $(\mathrm{Aff}^{\mathrm{nc}}_R)^{op} \simeq \mathrm{CAlg}^{op}_R$. Using Corollary VII.6.13, we deduce that the functor $R \mapsto \mathrm{Aff}^{\mathrm{nc}}_R$ is a hypercomplete sheaf with respect to the flat topology.

For every $\mathbb{E}_\infty$-ring R, let $Y(R)$ denote the full subcategory of $\operatorname{Fun}(\Delta^1, \mathrm{Stk}^{\mathrm{nc}}_R)$ spanned by those morphisms $f : \mathfrak{U} \to \mathfrak{X}$, where $\mathfrak{U}$ is affine and f is an open immersion. Let us regard Y as a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We claim that Y is a hypercomplete sheaf with respect to the flat topology. Evaluation at $\{1\} \subseteq \Delta^1$ determines a map $Y(R) \to \mathrm{Aff}^{\mathrm{nc}}_R$, depending functorially on R. Using Lemma 3.1.20, we are reduced to verifying the following assertion:

($*$) Let R be an $\mathbb{E}_\infty$-ring, let $f : \operatorname{Spec} A \to \operatorname{Spec} R$ be a map of affine spectral Deligne-Mumford stacks, and let $F : \mathrm{CAlg}_R \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be the functor which assigns to each R-algebra R' the ∞-category of open substacks of $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \operatorname{Spec} A$. Then F is a hypercomplete sheaf with respect to the flat topology.

This follows easily from Proposition VII.5.9.

For every $\mathbb{E}_\infty$-ring R, let $Y'(R)$ denote the full subcategory of $Y(R)$ spanned by those morphisms $f : \mathfrak{U} \to \mathfrak{X}$ where $\mathfrak{U}$ is quasi-compact. Let us regard Y' as a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We claim that Y' is a hypercomplete sheaf with respect to the flat topology. Since Y is a sheaf with respect to the flat topology, we may use Lemma 3.1.20 to reduce to proving the following concrete assertion:

($*'$) Let $f : A \to A'$ be a faithfully flat map of $\mathbb{E}_\infty$-rings, and let $U \subseteq \operatorname{Spec}^{\mathrm{Z}} A$ be an open subset. If the inverse image of U in $\operatorname{Spec}^{\mathrm{Z}} A'$ is quasi-compact, then U is quasi-compact.

This is clear, since the map $\operatorname{Spec}^{\mathrm{Z}} A' \to \operatorname{Spec}^{\mathrm{Z}} A$ is surjective.

For every $\mathbb{E}_\infty$-ring R, let $Y''(R)$ denote the full subcategory of $Y'(R)$ spanned by those morphisms $f : (\mathcal{U}, \mathcal{O}_{\mathcal{U}}) \to \operatorname{Spec} A$ which induce an equivalence of $\mathbb{E}_\infty$-rings $A \to \Gamma(\mathcal{U}; \mathcal{O}_{\mathcal{U}})$. Let us regard Y'' as a functor $\mathrm{CAlg} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. We claim that Y'' is a hypercomplete sheaf with respect to the flat topology. This follows easily from Lemma 3.1.20 and Corollary 2.5.15.

Evaluation at $\{0\} \subseteq \Delta^1$ induces a functor $\phi_R : Y''(R) \to \mathrm{QAff}^{\mathrm{nc}}_R$, depending functorially on R. Proposition 2.4.3 implies that each of these functors is an equivalence of ∞-categories. It follows that $R \mapsto \mathrm{QAff}^{\mathrm{nc}}_R$ is a hypercomplete sheaf with respect to the flat topology on CAlg. To prove that the functor $R \mapsto \mathrm{QAff}_R$ is a hypercomplete sheaf with respect to the flat topology on $\mathrm{CAlg}^{\mathrm{cn}}$, we invoke Lemma 3.1.20 to reduce to the following assertion:

($*''$) Suppose we are given a map of nonconnective spectral Deligne-Mumford stacks $\mathfrak{U} \to \operatorname{Spec} R$. Assume that R is connective and that there exists a faithfully flat morphism $R \to R'$ such that the fiber product $\mathfrak{U} \times_{\operatorname{Spec} R} \operatorname{Spec} R'$ is a spectral Deligne-Mumford stack. Then $\mathfrak{U}$ is a spectral Deligne-Mumford stack (that is, its structure sheaf is connective).

This follows immediately from Example 1.5.24. □

We conclude this section by introducing a relative version of the notion of a quasi-affine spectral Deligne-Mumford stack.

Definition 2.4.12. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks. We will say that f is *affine* if, for every map $\operatorname{Spec}^{\text{ét}} R \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec} R$ is quasi-affine. We will say that f is *quasi-affine* if, for every map $\operatorname{Spec}^{\text{ét}} R \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec}^{\text{ét}} R$ is quasi-affine.

The following assertions regarding affine and quasi-affine morphisms follow immediately from the definition:

Proposition 2.4.13. (1) *Any equivalence of nonconnective spectral Deligne-Mumford stacks is affine. Any affine morphism is quasi-affine.*

(2) *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks, and suppose that $\mathfrak{Y}$ is affine. Then f is affine (quasi-affine) if and only if $\mathfrak{X}$ is affine (quasi-affine).*

(3) *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y} . \end{tikzcd}$$

If f is affine (quasi-affine), then f' is affine (quasi-affine).

2.5 Pullbacks and Pushforwards of Quasi-Coherent Sheaves

Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a map of spectrally ringed ∞-topoi. Then f induces a symmetric monoidal functor $f^* : \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}) \to \mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})$ and a morphism $f^* \mathcal{O}_{\mathcal{X}} \to \mathcal{O}_{\mathcal{Y}}$ of commutative algebra objects of $\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{Y})$. We therefore obtain a composite functor

$$\mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}} \to \mathrm{Mod}_{f^* \mathcal{O}_{\mathcal{Y}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}.$$

We will generally abuse notation by denoting this functor also by f^*.

Proposition 2.5.1. *Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a map of nonconnective spectral Deligne-Mumford stacks. Then the pullback functor $f^* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves on $\mathcal{X}$ to quasi-coherent sheaves on $\mathcal{Y}$.*

Proof. The assertion is local on $\mathcal{X}$ and $\mathcal{Y}$. We may therefore assume that both $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $(\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ are affine, in which case the desired result follows immediately from the characterization of quasi-coherent sheaves given by Proposition 2.3.11. □

Remark 2.5.2. Proposition 2.5.1 has an obvious analogue in the setting of spectral schemes, which can be proven in the same way.

Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be as in Proposition 2.5.1. The induced functor $f^* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ is a colimit-preserving functor between presentable ∞-categories, and therefore admits a right adjoint which we will denote by f_*. In this section, we will study some conditions which guarantee that the functor f_* preserves quasi-coherence. We first need some general remarks.

Proposition 2.5.3. *Let $\mathcal{X}$ be an ∞-topos and suppose we are given a diagram σ :*

$$\begin{tikzcd} U \arrow[r, "f"] \arrow[d] & U' \arrow[d] \\ V \arrow[r, "f'"] & V', \end{tikzcd}$$

in $\mathcal{X}$. The following conditions are equivalent:

(1) *The diagram σ is both a pushout square and a pullback square, and the map f' is (-1)-truncated.*

(2) *The diagram σ is a pushout square and the map f is (-1)-truncated.*

(3) *The diagram σ is a pullback square, f' is (-1)-truncated, and if we let $i^* : \mathcal{X}_{/V'} \to \mathcal{X}_{/V'}/V$ denote the corresponding closed immersion, then i^*U' is a final object of $\mathcal{X}_{/V'}/V$.*

Proof. The equivalence of (1) and (3) is a matter of unwinding definitions, and the implication (1) ⇒ (2) is obvious. We will show that (2) ⇒ (1). Since $\mathcal{X}$ is an ∞-topos, there exists a fully faithful geometric morphism $i_* : \mathcal{X} \to \mathcal{P}(\mathcal{C})$, for some small ∞-category $\mathcal{C}$. Form a pushout diagram τ :

$$\begin{tikzcd} i_*U \arrow[r] \arrow[d] & i_*U' \arrow[d] \\ i_*V \arrow[r, "g'"] & W \end{tikzcd}$$

in $\mathcal{P}(\mathcal{C})$. Then $\sigma \simeq i^*(\tau)$. It will therefore suffice to show that τ is a pullback diagram and that g' is (-1)-truncated. In other words, we may replace $\mathcal{X}$ by $\mathcal{P}(\mathcal{C})$ and thereby reduce to the case where $\mathcal{X}$ is an ∞-category of presheaves. Working pointwise, we can reduce to the case $\mathcal{X} = \mathcal{S}$. In this case, the condition that f is (-1)-truncated guarantees that $U' \simeq U \amalg X$ for some space X, in which case $V' \simeq V \amalg X$ and the result is obvious. □

Definition 2.5.4. We will say that a diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ V \arrow[r] & V' \end{tikzcd}$$

in an ∞-topos $\mathcal{X}$ is an *excision square* if it satisfies the equivalent conditions of Proposition 2.5.3.

Definition 2.5.5. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. A *scallop decomposition* of $\mathcal{X}$ consists of a sequence of (-1)-truncated morphisms

$$U_0 \to U_1 \to \cdots \to U_n$$

in $\mathcal{X}$ satisfying the following conditions:

(a) The object $U_0 \in \mathcal{X}$ is initial and the object $U_n \in \mathcal{X}$ is final.

(b) For $1 \leq i \leq n$, there exists an excision square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \arrow[d] \\ U_{i-1} \arrow[r] & U_i \end{tikzcd}$$

where X is affine and V is quasi-compact.

In this case, we will refer to n as the *length* of the scallop decomposition.

Example 2.5.6. Let $\mathfrak{X}$ be a quasi-affine nonconnective spectral Deligne-Mumford stack. Then $\mathfrak{X}$ admits a scallop decomposition.

Remark 2.5.7. We will show later that a spectral Deligne-Mumford stack admits a scallop decomposition if and only if it is a quasi-compact, quasi-separated spectral algebraic space.

Before stating the next result, we need to introduce a bit of terminology. Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack. We say that an object $U \in \mathcal{X}$ is *semiaffine* if it is quasi-compact and there exists a (-1)-truncated map $U \to X$ in $\mathcal{X}$, where X is affine. We will say that a morphism $f : U \to V$ in $\mathcal{X}$ is *semiaffine* if the fiber product $U \times_V X$ is semiaffine, whenever $X \in \mathcal{X}$ is affine.

Proposition 2.5.8. *Let $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack which admits a scallop decomposition. Suppose that $\mathcal{C} \subseteq \mathcal{X}$ is a full subcategory satisfying the following conditions:*

(0) *The ∞-category $\mathcal{C}$ is closed under equivalence in $\mathcal{X}$.*

(1) *Initial objects of $\mathcal{X}$ belongs to $\mathcal{C}$.*

(2) *If we are given an excision square*

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ V \arrow[r] & V' \end{tikzcd}$$

of semiaffine morphisms in $\mathcal{X}$ where U' is affine and $U, V \in \mathcal{C}$, then $V' \in \mathcal{C}$.

Then $\mathcal{C}$ contains the final objects of $\mathcal{X}$.

Proof. It follows immediately from (1) and (2) that every affine object of $\mathcal{X}$ belongs to $\mathcal{C}$. We next show that if $U \in \mathcal{X}$ is semiaffine, then $U \in \mathcal{C}$. Choose a (-1)-truncated map $j : U \to X$ where X is affine, so that $(\mathcal{X}_{/X}, \mathcal{O}_{\mathcal{X}} | X) \simeq \operatorname{Spec}^{\text{ét}} R$. Then we can identify U with an open subset of the topological space $\operatorname{Spec}^{\mathrm{Z}} R$. Since U is quasi-compact, we can write U as a finite union $\bigcup_{1 \leq i \leq n} \operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_i}]$ for some elements $x_i \in \pi_0 R$. Choose n as small as possible. We proceed by induction on n. If $n = 0$, then U is an initial object of $\mathcal{X}$ and therefore $U \in \mathcal{C}$ by virtue of (1). Assume therefore that $n > 0$. Let $U_0 = \bigcup_{1 \leq i < n} \operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_i}]$, let $U_1 = \operatorname{Spec}^{\mathrm{Z}} R[\frac{1}{x_n}]$, and let $U_{01} = U_0 \cap U_1$. We identify U_0, U_1, and U_{01} with (-1)-truncated objects of $\mathcal{X}$, so that we have an excision square

$$\begin{tikzcd} U_{01} \arrow[r] \arrow[d] & U_1 \arrow[d] \\ U_0 \arrow[r] & U. \end{tikzcd}$$

Since $U_{01}, U_0 \in \mathcal{C}$ be the inductive hypothesis and U_1 is affine, we deduce that $U \in \mathcal{C}$ by (2).

Choose a scallop decomposition

$$U_0 \to U_1 \to \cdots \to U_n$$

for $\mathcal{X}$. We prove by induction on i that each U_i belongs to $\mathcal{C}$. When $i = 0$, this follows from (1). Taking $i = n$ we will obtain the result. To carry out the inductive step, suppose that $U_i \in \mathcal{C}$. Choose an excision square

$$\begin{tikzcd} V \arrow[r] \arrow[d] & X \arrow[d] \\ U_i \arrow[r] & U_{i+1} \end{tikzcd}$$

where X is affine and V is quasi-compact. The map $V \to X$ is (-1)-truncated, so that V is semiaffine and therefore $V \in \mathcal{C}$. It follows from (2) that $U_{i+1} \in \mathcal{C}$, as desired. $\square$

Corollary 2.5.9. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a nonconnective spectral Deligne-Mumford stack which admits a scallop decomposition. Suppose that $\mathcal{C} \subseteq \mathcal{X}$ is a full subcategory which contains all affine objects of $\mathcal{X}$ and is closed under pushouts. Then $\mathcal{C}$ contains the final object of $\mathcal{X}$.*

Definition 2.5.10. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of nonconnective spectral Deligne-Mumford stacks. We will say that f is *relatively scalloped* if, for every map $\operatorname{Spec}^{\text{ét}} R \to \mathfrak{Y}$, the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec}^{\text{ét}} R$ admits a scallop decomposition.

Example 2.5.11. Every quasi-affine morphism is relatively scalloped (see Example 2.5.6).

Proposition 2.5.12. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathfrak{Y}$ be a relatively scalloped map of nonconnective spectral Deligne-Mumford stacks. Then the pushforward functor $f_* : \operatorname{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \operatorname{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ carries quasi-coherent sheaves to quasi-coherent sheaves. Moreover, the induced functor $\operatorname{QCoh}(\mathfrak{X}) \to \operatorname{QCoh}(\mathfrak{Y})$ commutes with small colimits.*

Proof. The assertion is local on $\mathcal{Y}$; we may therefore assume without loss of generality that $\mathfrak{Y} = \operatorname{Spec}^{\text{ét}} R$ is affine. Assume first that $\mathfrak{X} = \operatorname{Spec}^{\text{ét}} A$ is affine. Let $\mathcal{F} \in \operatorname{QCoh}(\mathfrak{X})$ be the quasi-coherent sheaf associated to some A-module M, and let $\mathcal{F}' \in \operatorname{QCoh}(\mathfrak{Y})$ be the quasi-coherent sheaf associated to the image of M in Mod_R. The canonical map $A \otimes_R M \to M$ determines a map $f^* \mathcal{F}' \to \mathcal{F}$, which is adjoint to a map of $\mathcal{O}_{\mathcal{Y}}$-modules $\mathcal{F}' \to f_* \mathcal{F}$. We claim that this map is an equivalence. For this, we must show that $u : \mathcal{F}'(U) \to (f_* \mathcal{F})(U) \simeq \mathcal{F}(f^* U)$ is an equivalence of spectra for each $U \in \mathcal{Y}$. The collection of those objects U which satisfy this condition is stable under colimits. We may therefore assume that U is representable by an étale R-algebra R'. In this case, u can be identified with the canonical equivalence

$$M \otimes_R R' \to M \otimes_A (A \otimes_R R').$$

This proves that f_* carries quasi-coherent sheaves on $\mathfrak{X}$ to quasi-coherent sheaves on $\mathfrak{Y}$. Moreover, the restricted functor $f_* : \mathrm{Mod}_A \simeq \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y}) \simeq \mathrm{Mod}_R$ is right adjoint to the base-change functor $N \mapsto N \otimes_R A$, and can therefore be identified with the forgetful functor $\mathrm{Mod}_A \to \mathrm{Mod}_R$, which commutes with small colimits.

We now treat the general case. For each object $U \in \mathcal{X}$, let $\Gamma_U : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ denote the composite functor

$$\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}|U}(\mathcal{S}\mathrm{hv}_{\mathrm{Sp}}(\mathcal{X}_{/U})) \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}} .$$

Let us say that U is good if Γ_U restricts to a colimit-preserving functor from $\mathrm{QCoh}(\mathfrak{X})$ into $\mathrm{QCoh}(\mathfrak{Y})$. The construction $U \mapsto \Gamma_U$ carries pushout square to pullback squares. It follows that the collection of good objects of $\mathcal{X}$ is stable under finite colimits. Since every affine object of $\mathcal{X}$ is good (by the first part of the proof), Corollary 2.5.9 implies that the final object of $\mathcal{X}$ is good. □

Proposition 2.5.13. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) = \mathfrak{Y}$ be a relatively scalloped map of spectral Deligne-Mumford stacks. Assume that $\mathfrak{Y}$ is quasi-compact. Then there exists an integer n such that the pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ carries $\mathrm{QCoh}(\mathfrak{X})_{\geq 0}$ into $\mathrm{QCoh}(\mathfrak{Y})_{\geq -n}$.*

Proof. Since $\mathfrak{Y}$ is quasi-compact, we can choose an étale surjection $\mathrm{Spec}^{\text{ét}} R \to \mathfrak{Y}$ for some connective $\mathbb{E}_\infty$-ring R. Replacing $\mathfrak{Y}$ by $\mathrm{Spec}^{\text{ét}} R$, we may assume that $\mathfrak{Y}$ is affine so that $\mathfrak{X}$ admits a scallop decomposition. We define the class of *good* objects $U \in \mathcal{X}$ as in the proof of Proposition 2.5.12. For every good object $U \in \mathcal{X}$, let $\Gamma_U : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$ be defined as in the proof of Proposition 2.5.12. Let us say that U is *n-good* if $\Gamma_U(\mathrm{QCoh}(\mathfrak{X})_{\geq 0}) \subseteq \mathrm{QCoh}(\mathfrak{Y})_{\geq -n}$. Note that if we are given a pushout diagram

$$\begin{tikzcd} U \arrow[r] \arrow[d] & U' \arrow[d] \\ V \arrow[r] & V' \end{tikzcd}$$

in $\mathcal{X}$, then we have a fiber sequence of functors

$$\Gamma_{U'} \oplus \Gamma_V \to \Gamma_{V'} \to \Gamma_U[1].$$

It follows that if U' and V are n-good and U is $(n-1)$-good, then V' is also n-good. Let us say that a good object $U \in \mathcal{X}$ is *very good* if it n-good for some integer $n \geq 0$. It follows that the collection of very good objects of $\mathcal{X}$ is closed under pushouts. Any affine object of $\mathcal{X}$ is 0-good, and therefore very good. Using Corollary 2.5.9, we deduce that the final object of $\mathcal{X}$ is very good, which implies the desired result. □

The formation of pushforwards along a relatively scalloped morphism is compatible with base change:

Proposition 2.5.14. *Suppose we are given a pullback diagram of nonconnective spectral Deligne-Mumford stacks*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r, "g'"] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r, "g"] & \mathfrak{Y} \end{tikzcd}$$

where f (and therefore f') is relatively scalloped. Then the diagram of ∞-categories

$$\begin{tikzcd} \mathrm{QCoh}(\mathfrak{Y}) \arrow[r, "f^*"] \arrow[d] & \mathrm{QCoh}(\mathfrak{X}) \arrow[d] \\ \mathrm{QCoh}(\mathfrak{Y}') \arrow[r, "f'^*"] & \mathrm{QCoh}(\mathfrak{X}') \end{tikzcd}$$

is right adjointable. In other words, for every object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$, the canonical map $\lambda : g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$ is an equivalence in $\mathrm{QCoh}(\mathfrak{Y}')$.

Proof. The assertion is local on $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and $\mathfrak{Y}' = (\mathcal{Y}', \mathcal{O}_{\mathcal{Y}'})$; we may therefore assume that $\mathfrak{Y} = \operatorname{Spec} R$ and $\mathfrak{Y}' = \operatorname{Spec} R'$ are affine. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{X}' = (\mathcal{X}', \mathcal{O}_{\mathcal{X}'})$. Let $U \in \mathcal{X}$ be an object and $U' = g'^{*}U$ its pullback to $\mathfrak{X}'$. Define functors $\Gamma_U : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}}$ and $\Gamma_{U'} : \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}'}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}'}}$ as in the proof of Proposition 2.5.12. Let us say that $U \in \mathcal{X}$ is *good* if the canonical map $\lambda_U : g^{*}\Gamma_U(\mathcal{F}) \to \Gamma_{U'}(f'^{*}\mathcal{F})$ is an equivalence of $\mathcal{O}_{\mathcal{Y}'}$-modules. Since the construction $U \mapsto \lambda_U$ carries finite colimits to finite limits, the collection of good objects of $\mathcal{X}$ is closed under finite colimits. We wish to prove that the final object of $\mathcal{X}$ is good. Since $\mathcal{X}$ admits a scallop decomposition, it will suffice to show that every affine object of $\mathcal{X}$ is good (Corollary 2.5.9). We may therefore reduce to the case where $\mathcal{X}$ (and therefore also $\mathcal{X}'$) are affine, in which case the desired assertion is a special case of Lemma VII.6.15. □

Corollary 2.5.15. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a quasi-affine map of nonconnective spectral Deligne-Mumford stacks. Then the pushforward functor f_* determines a colimit-preserving functor $f_* : \mathrm{QCoh}(\mathfrak{X}) \to \mathrm{QCoh}(\mathfrak{Y})$. Moreover, for every pullback diagram*

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d] & \mathfrak{X} \arrow[d] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y}, \end{tikzcd}$$

the induced diagram

$$\begin{tikzcd} \mathrm{QCoh}(\mathfrak{Y}) \arrow[r, "f^*"] \arrow[d] & \mathrm{QCoh}(\mathfrak{X}) \arrow[d] \\ \mathrm{QCoh}(\mathfrak{Y}') \arrow[r, "f'^*"] & \mathrm{QCoh}(\mathfrak{X}') \end{tikzcd}$$

is right adjointable.

Proof. Combine Propositions 2.5.12, 2.5.14, and Example 2.5.11. □

Corollary 2.5.16. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y}$ be a quasi-affine map of spectral Deligne-Mumford stacks. Then the induced functor $\mathrm{QCoh}(\mathfrak{X}) \to \mathrm{Mod}_{f_* \mathcal{O}_{\mathcal{X}}}(\mathrm{QCoh}(\mathfrak{Y}))$ is an equivalence of ∞-categories.*

Proof. The assertion is local on $\mathfrak{Y}$. We may therefore assume that $\mathfrak{Y} = \operatorname{Spec} R$ is affine, so that $\mathfrak{X}$ is quasi-affine and the desired result follows from Proposition 2.4.4. □

If we are willing to restrict our attention to truncated quasi-coherent sheaves on spectral Deligne-Mumford stacks, then we can verify the quasi-coherence of direct images under hypotheses much weaker than those of Corollary 2.5.15.

Notation 2.5.17. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. We let $\mathrm{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$ denote the heart of the ∞-category $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$: it can be identified with (the nerve of) the abelian category of sheaves of discrete modules over $\pi_0 \mathcal{O}_{\mathcal{X}}$. We will say that an object of $\mathrm{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$ is *quasi-coherent* if it belongs to $\mathrm{QCoh}(\mathfrak{X})^{\heartsuit} = \mathrm{QCoh}(\mathfrak{X}) \cap \mathrm{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$. We will say that an object $\mathcal{F} \in \mathrm{Mod}^{\heartsuit}_{\mathcal{O}_{\mathcal{X}}}$ is *pseudo-coherent* if, for every affine object $U \in \mathcal{X}$, there exists a composition series

$$0 = \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n = \mathcal{F}\,|U$$

such that each quotient $\mathcal{F}_i / \mathcal{F}_{i-1}$ is a subobject of some quasi-coherent object $\mathcal{G}_i \in \mathrm{QCoh}(\mathfrak{U})^{\heartsuit}$, where $\mathfrak{U} = (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}\,|U)$.

We have the following coherence result:

Theorem 2.5.18. *Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be an n-quasi-compact morphism between spectral Deligne-Mumford stacks. Let $\mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}})_{\leq 0}$ be sheaf of $\mathcal{O}_{\mathcal{Y}}$-modules satisfying the following conditions:*

(a) *For $0 \leq i < n$, $\pi_{-i} \mathcal{F}$ is quasi-coherent.*

(b) *The sheaf $\pi_{-n} \mathcal{F}$ is pseudo-coherent.*

Then $f_* \mathcal{F}$ also satisfies conditions (a) and (b).

The proof of Theorem 2.5.18 will require a few preliminaries. Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a spectral Deligne-Mumford stack. Since the t-structure on $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$ restricts to a t-structure on the full subcategory $\mathrm{QCoh}(\mathfrak{X})$, we can identify $\mathrm{QCoh}(\mathfrak{X})^{\heartsuit}$ with a full subcategory of the abelian category $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit}$ which is closed under the formation of kernels, cokernels, and extensions. Our first goal is to extend these observations to pseudo-coherent sheaves.

Lemma 2.5.19. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, and suppose we are given a morphism $\alpha : \mathcal{F} \to \mathcal{G}$ in the abelian category $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$. If $\mathcal{F}$ is quasi-coherent and $\mathcal{G}$ is pseudo-coherent, then* $\ker(\alpha)$ *and* $\mathrm{im}(\alpha)$ *(formed in the abelian category $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$) are quasi-coherent.*

Proof. The assertion is local on $\mathcal{X}$; we may therefore assume that $\mathfrak{X}$ is affine so that there exists a finite filtration

$$0 = \mathcal{G}_0 \hookrightarrow \mathcal{G}_1 \hookrightarrow \cdots \hookrightarrow \mathcal{G}_n = \mathcal{G}$$

such that each quotient $\mathcal{G}_i / \mathcal{G}_{i-1}$ is a subobject of a quasi-coherent object $\mathcal{H}_i \in \mathrm{QCoh}(\mathfrak{X})^{\heartsuit}$. Let $\mathcal{K}_i$ denote the kernel of the composite map

$$\mathcal{F} \stackrel{\alpha}{\to} \mathcal{G} \to \mathcal{G} / \mathcal{G}_i .$$

For each index i, α induces a monomorphism

$$\mathcal{K}_i / \mathcal{K}_{i-1} \hookrightarrow \mathcal{G}_i / \mathcal{G}_{i-1} \to \mathcal{H}_i .$$

Thus $\mathcal{K}_{i-1}$ can be identified with the kernel of a map $\mathcal{K}_i \to \mathcal{H}_i$. Note that $\mathcal{K}_n \simeq \mathcal{F}$ is quasi-coherent. It follows by descending induction on i that each $\mathcal{K}_i$ is quasi-coherent. In particular, $\mathcal{K}_0 = \ker(\alpha)$ is quasi-coherent. Using the exact sequence

$$0 \to \ker(\alpha) \to \mathcal{F} \to \mathrm{im}(\alpha) \to 0,$$

we see that $\mathrm{im}(\alpha)$ is quasi-coherent as well. □

Lemma 2.5.20. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O})$ be a spectral Deligne-Mumford stack, and suppose we are given an exact sequence*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in the abelian category $\mathrm{Mod}_{\mathcal{O}}^{\heartsuit}$.

(a) *If $\mathcal{F}'$ and $\mathcal{F}''$ are psuedo-coherent, then $\mathcal{F}$ is pseudo-coherent.*

(b) *If $\mathcal{F}$ is pseudo-coherent, then $\mathcal{F}'$ is pseudo-coherent.*

(c) *If $\mathcal{F}'$ is quasi-coherent and $\mathcal{F}$ is pseudo-coherent, then $\mathcal{F}''$ is pseudo-coherent.*

Proof. Assertion (a) follows immediately from the definitions. We next prove (b). Without loss of generality, we may assume that $\mathfrak{X}$ is affine. Then $\mathcal{F}$ admits a finite filtration

$$0 = \mathcal{F}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n = \mathcal{F}$$

and a collection of monomorphisms $\mathcal{F}_i / \mathcal{F}_{i-1} \hookrightarrow \mathcal{G}_i$, where $\mathcal{G}_i \in \mathrm{QCoh}(\mathfrak{X})^{\heartsuit}$. Let us regard $\mathcal{F}_i$ and $\mathcal{F}'$ as subobjects of $\mathcal{F}$, and set $\mathcal{F}'_i = \mathcal{F}_i \cap \mathcal{F}'$. Then we have a filtration

$$0 = \mathcal{F}'_0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}'_n = \mathcal{F}'$$

where each quotient $\mathcal{F}'_i / \mathcal{F}'_{i-1}$ is equivalent to a subobject of $\mathcal{F}_i / \mathcal{F}_{i-1}$, and therefore to a subobject of $\mathcal{G}_i$. This proves that $\mathcal{F}'$ is pseudo-coherent.

It remains to prove (c). Again we may assume without loss of generality that $\mathfrak{X}$ is affine, so that $\mathcal{F}$ and $\mathcal{F}'$ admit composition series as indicated above. We first prove by descending induction on i that each $\mathcal{F}'_i$ is quasi-coherent. The result is obvious for $i = n$, since $\mathcal{F}'_n \simeq \mathcal{F}'$. For the inductive step, we note that $\mathcal{F}'_i$ can be described as the kernel of a map $\mathcal{F}'_{i+1} \to \mathcal{F}_{i+1} / \mathcal{F}_i \hookrightarrow \mathcal{G}_i$, and is therefore quasi-coherent. It follows that each of the quotients $\mathcal{F}'_i / \mathcal{F}'_{i-1}$ is quasi-coherent. Form a short exact sequence

$$0 \to \mathcal{F}'_i / \mathcal{F}'_{i-1} \to \mathcal{G}_i \to \mathcal{H}_i \to 0,$$

so that each $\mathcal{H}_i$ is quasi-coherent. Let $\mathcal{F}''_i$ denote the image of $\mathcal{F}_i$ in $\mathcal{F}''$. Then we have a finite filtration

$$0 = \mathcal{F}''_0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}''_n = \mathcal{F}''.$$

For each index i, the monomorphism $\mathcal{F}_i / \mathcal{F}_{i-1} \hookrightarrow \mathcal{G}_i$ induces a monomorphism $\mathcal{F}''_i / \mathcal{F}''_{i-1} \to \mathcal{H}_i$. It follows that $\mathcal{F}''$ is pseudo-coherent, as desired. □

Lemma 2.5.21. *Let $f : (\mathcal{Y}, \mathcal{O}_\mathcal{Y}) \to (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be an affine morphism between spectral Deligne-Mumford stacks. Let $\mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}_\mathcal{Y}})_{\leq 0}$ be such that $\pi_0 \mathcal{F}$ is pseudo-coherent. Then $(f_* \mathcal{F}) \in (\mathrm{Mod}_{\mathcal{O}_\mathcal{X}})_{\leq 0}$, and $\pi_0(f_* \mathcal{F})$ is pseudo-coherent.*

Proof. We first note that the pushforward functor f_* is left t-exact. Let $\mathcal{F} \in (\mathrm{Mod}_{\mathcal{O}_\mathcal{Y}})_{\leq 0}$ be such that $\pi_0 \mathcal{F}$ is pseudo-coherent; we wish to prove that $\pi_0(f_* \mathcal{F})$ is pseudo-coherent. Since f_* is left t-exact, the map $f_*(\tau_{\geq 0} \mathcal{F}) \to f_* \mathcal{F}$ induces an equivalence $\pi_0 f_*(\tau_{\geq 0} \mathcal{F}) \to \pi_0(f_* \mathcal{F})$. We may therefore replace $\mathcal{F}$ by $\tau_{\geq 0} \mathcal{F}$ and thereby reduce to the case $\mathcal{F} \in \mathrm{Mod}^{\heartsuit}_{\mathcal{O}_\mathcal{Y}}$. We may assume without loss of generality that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is affine. Since f is affine, we deduce that $(\mathcal{Y}, \mathcal{O}_\mathcal{Y})$ is affine, so that $\mathcal{F} \simeq \pi_0 \mathcal{F}$ admits a composition series $\mathcal{F} \simeq \pi_0 \mathcal{F}$ is pseudo-coherent, we can choose a composition series

$$0 = \mathcal{F}_0 \hookrightarrow \cdots \hookrightarrow \mathcal{F}_n \simeq \mathcal{F}$$

where each quotient admits a monomorphism $\mathcal{F}_i / \mathcal{F}_{i-1} \hookrightarrow \mathcal{G}_i$ foe some quasi-coherent object $\mathcal{G}_i \in \mathrm{Mod}^{\heartsuit}_{\mathcal{O}_\mathcal{Y}}$. Since f_* is left t-exact, we get an induced filtration

$$0 = \pi_0 f_* \mathcal{F}_0 \hookrightarrow \cdots \hookrightarrow \pi_0 f_* \mathcal{F}_n = \pi_0 f_* \mathcal{F}$$

where each successive quotient $(\pi_0 f_* \mathcal{F}_i)/(\pi_0 f_* \mathcal{F}_{i-1})$ admits a monomorphism

$$(\pi_0 f_* \mathcal{F}_i)/(\pi_0 f_* \mathcal{F}_{i-1}) \hookrightarrow \pi_0 f_*(\mathcal{F}_i / \mathcal{F}_{i-1}) \hookrightarrow \pi_0 f_* \mathcal{G}_i .$$

It now suffices to observe that $\pi_0 f_* \mathcal{G}_i$ is a quasi-coherent $\mathcal{O}_\mathcal{X}$-module (Corollary 2.5.15). □

Proof of Theorem 2.5.18. Without loss of generality, we may assume that $(\mathcal{X}, \mathcal{O}_\mathcal{X})$ is affine. We proceed by induction on n. Then the ∞-topos $\mathcal{Y}$ is n-coherent, and in particular quasi-compact. We may therefore choose an effective epimorphism $u : U_0 \to \mathbf{1}$ in $\mathcal{Y}$, where $\mathbf{1}$ denotes the final object and $(\mathcal{Y}_{/U_0}, \mathcal{O}_\mathcal{Y} |U_0)$ is affine. Let $U_\bullet$ denote the Čech nerve of u. For each $k \geq 0$, let $f^k : (\mathcal{Y}_{/U_k}, \mathcal{O}_\mathcal{Y} |U_k) \to (\mathcal{X}, \mathcal{O}_\mathcal{X})$ be the map induced by f, and let $\mathcal{G}^k = f^k_*(\mathcal{F} |U_k) \in (\mathrm{Mod}_{\mathcal{O}_\mathcal{X}})_{\leq 0}$. We obtain a cosimplicial object $\mathcal{G}^\bullet$ of $(\mathrm{Mod}_{\mathcal{O}_\mathcal{X}})_{\leq 0}$ whose totalization is equivalent to $f_* \mathcal{F}$. Applying Proposition A.1.2.4.6 and Variant A.1.2.4.7, we deduce the existence of a spectral sequence $\{E^{p,q}_r, d_r\}_{r \geq 1}$ in the abelian category $\mathrm{Mod}^{\heartsuit}_{\mathcal{O}_\mathcal{X}}$ with the following properties:

- (i) We have $E^{p,q}_1 \simeq \pi_{-q} \mathcal{G}^p$ for $p, q \geq 0$, and $E^{p,q}_1 \simeq 0$ otherwise.
- (ii) The differentials d_r have bidegree $(r, 1-r)$: that is, they carry $E^{p,q}_r$ into $E^{p+r,q-r+1}_r$.

(*iii*) The spectral sequence $\{E_r^{p,q}, d_r\}_{r\geq 1}$ converges to $\pi_{-p-q} f_* \mathcal{F}$ in the following sense: for every integer $k \geq 0$, there exists a finite filtration

$$0 = F^{-1}\pi_{-k}(f_* \mathcal{F}) \hookrightarrow F^0\pi_{-k}(f_* \mathcal{F}) \hookrightarrow \cdots \hookrightarrow F^k\pi_{-k}(f_* \mathcal{F}) = \pi_{-k}(f_* \mathcal{F})$$

in the abelian category $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit}$ such that each successive quotient $F^q\pi_0(f_* \mathcal{F})/F^{q-1}\pi_0(f_* \mathcal{F})$ is isomorphic to $E_r^{k-q,q}$ for $r \gg 0$.

Since $\mathcal{Y}$ is n-coherent, each of the objects $U_q \in \mathcal{Y}$ is $(n-1)$-coherent. Using the inductive hypothesis and (i), we deduce:

(*iv*) The objects $E_1^{p,q}$ are quasi-coherent for $q < n-1$ and pseudo-coherent for $q = n-1$.

When $p = 0$ we can do a bit better: since $(\mathcal{Y}_{/U_0}, \mathcal{O}_{\mathcal{Y}} | U_0)$ is affine, Lemma 2.5.21 gives:

(*v*) The objects $E_1^{0,q}$ are quasi-coherent for $q < n$ and pseudo-coherent for $q = n$.

We now prove the following statement by induction on r:

($*$) The object $E_r^{p,q}$ is psuedo-coherent if $p+q = n$, and quasi-coherent if $p+q < n$.

In the case $r = 1$, assertion $(*)$ follows from (iv), (v), and (i). In the general case, we can describe $E_r^{p,q}$ as the cohomology of a cochain complex

$$E_{r-1}^{p-r,q+r-1} \xrightarrow{\alpha} E_{r-1}^{p,q} \xrightarrow{\beta} E_{r-1}^{p+r,q-r+1}.$$

so that we have an exact sequence

$$0 \to \mathrm{im}(\alpha) \to \ker(\beta) \to E_r^{p,q} \to 0.$$

If $p+q < n$, then $E_{r-1}^{p,q}$ and $E_{r-1}^{p-r,q+r-1}$ are quasi-coherent and $E_{r-1}^{p+r,q-r+1}$ is pseudo-coherent (by the inductive hypothesis). It follows that $\mathrm{im}(\alpha)$ and $\ker(\beta)$ are quasi-coherent (Lemma 2.5.20), so that $E_r^{p,q}$ is quasi-coherent. If $p+q = n$, then the inductive hypothesis guarantees instead that $E_{r-1}^{p-r,q+r-1}$ is quasi-coherent and $E_r^{p,q}$ is pseudo-coherent. Lemma 2.5.19 then guarantees that $\mathrm{im}(\alpha)$ is quasi-coherent and Lemma 2.5.20 guarantees that $\ker(\beta)$ is pseudo-coherent, so that $E_r^{p,q}$ is pseudo-coherent by Lemma 2.5.20.

Using $(*)$ and (3), we deduce that $\pi_{-k} f_* \mathcal{F}$ admits a finite filtration by objects of $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit}$ which are quasi-coherent if $k < n$ and pseudo-coherent if $k = n$. Since the classes of quasi-coherent and pseudo-coherent objects of $\mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}^{\heartsuit}$ are stable under extensions (Lemma 2.5.20), we conclude that $\pi_{-k} f_* \mathcal{F}$ is quasi-coherent for $k < n$ and pseudo-coherent for $k = n$, as desired. □

Combining Theorem 2.5.18 with the quasi-coherence criterion of Proposition 2.3.21, we obtain:

Corollary 2.5.22. *Let $f : (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}) \to (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ be a map of spectral Deligne-Mumford stacks which is n-quasi-compact for all $n \geq 0$. Then the induced functor $f_* : \mathrm{Mod}_{\mathcal{O}_{\mathcal{Y}}} \to \mathrm{Mod}_{\mathcal{O}_{\mathcal{X}}}$ carries $\mathrm{QCoh}(\mathcal{Y})_{\leq 0}$ into $\mathrm{QCoh}(\mathcal{X})_{\leq 0}$.*

We conclude this section with a few remarks about the behavior of the pushforward functor f_* in the case where $f : \mathfrak{X} \to \mathfrak{Y}$ is a "highly connected" affine morphism.

Proposition 2.5.23. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be spectral Deligne-Mumford stacks. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an affine morphism, let $n \geq 0$ be an integer, and suppose that the fiber of the map $\mathcal{O}_{\mathcal{Y}} \to f_* \mathcal{O}_{\mathcal{X}}$ is n-connective. Then:*

(1) *The pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}_{\leq n} \to \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$ is fully faithful.*

(2) *The pushforward functor $f_* : \mathrm{QCoh}(\mathfrak{X})^{\mathrm{cn}}_{\leq n-1} \to \mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n-1}$ is an equivalence of ∞-categories.*

Proof. The assertion is local on $\mathfrak{Y}$, so we may assume without loss of generality that $\mathfrak{Y}$ is affine. Write $\mathfrak{Y} = \operatorname{Spec} A$ for some connective $\mathbb{E}_\infty$-ring A. Since f is affine, we can assume $\mathfrak{X} = \operatorname{Spec} B$ for some connective A-algebra B. Let $u : A \to B$ denote the underlying map of $\mathbb{E}_\infty$-rings, so that $\operatorname{fib}(u)$ is an n-connective spectrum. To prove (1), we must show that the forgetful functor $\phi_n : (\operatorname{Mod}_B^{\mathrm{cn}})_{\leq n} \to (\operatorname{Mod}_A^{\mathrm{cn}})_{\leq n}$ is fully faithful. We observe that ϕ_n has a left adjoint ψ_n, given by $M \mapsto \tau_{\leq n}(B \otimes_A M)$. We wish to show that the counit map $\psi_n \circ \phi_n \to \mathrm{id}$ is an equivalence. Unwinding the definitions, we must show that if $M \in (\operatorname{Mod}_B^{\mathrm{cn}})_{\leq n}$, then the canonical map $\theta : B \otimes_A M \to M$ exhibits M as an n-truncation of $B \otimes_A M$. Since M is n-truncated, this is equivalent to the requirement that $\operatorname{fib}(\theta)$ is $(n+1)$-connective. Let θ_0 be the multiplication map $B \otimes_A B \to B$, so that $\operatorname{fib}(\theta) \simeq \operatorname{fib}(\theta_0) \otimes_B M$. Since M is connective, it will suffice to show that $\operatorname{fib}(\theta_0)$ is $(n+1)$-connective. Note that θ_0 admits a section s, so we can identify $\operatorname{fib}(\theta_0)$ with the $\operatorname{cofib}(s) = B \otimes_A \operatorname{cofib}(u)$. We complete the proof of (1) by observing that $\operatorname{cofib}(u) = \operatorname{fib}(u)[1]$ is $(n+1)$-connective.

We now prove (2). Let ϕ_{n-1} and ψ_{n-1} be defined as above; we wish to show that the unit map $\mathrm{id} \to \phi_{n-1} \circ \psi_{n-1}$ is an equivalence. In other words, we wish to show that if $N \in (\operatorname{Mod}_A^{\mathrm{cn}})_{\leq n-1}$, then the canonical map $N \to B \otimes_A N$ induces an isomorphism $\pi_i N \to \pi_i(B \otimes_A N)$ for $i < n$. We have a long exact sequence

$$\pi_i(\operatorname{fib}(u) \otimes_A N) \to \pi_i N \to \pi_i(B \otimes_A N) \to \pi_{i-1}(\operatorname{fib}(u) \otimes_A N).$$

It therefore suffices to show that the homotopy groups $\pi_i(\operatorname{fib}(u) \otimes_A N)$ vanish for $i < n$. This is clear, since $\operatorname{fib}(u)$ is n-connective and N is connective. □

Corollary 2.5.24. *Let $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ and $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be spectral Deligne-Mumford stacks. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a morphism which induces an equivalence of ∞-topoi $\mathfrak{X} \simeq \mathfrak{Y}$ and an equivalence of n-truncations $\tau_{\leq n} \mathcal{O}_{\mathcal{Y}} \simeq \tau_{\leq n} f_* \mathcal{O}_{\mathcal{X}}$. Then the pushforward functor f_* induces an equivalence of ∞-categories* $\operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}}_{\leq n} \simeq \operatorname{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}$.

Proof. Let $Tr_n\, \mathfrak{X} = (\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}})$ and define $Tr_n\, \mathfrak{Y}$ similarly, so that we have a commutative diagram

$$\begin{tikzcd} Tr_n\, \mathfrak{X} \arrow[r] \arrow[d, "\phi"] & Tr_n\, \mathfrak{Y} \arrow[d, "\psi"] \\ \mathfrak{X} \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

where the upper horizontal map is an equivalence. It will therefore suffice to show that the vertical maps induce equivalences of ∞-categories

$$\phi_* : \operatorname{QCoh}(Tr_n\, \mathfrak{X})^{\mathrm{cn}}_{\leq n} \to \operatorname{QCoh}(\mathfrak{X})^{\mathrm{cn}}_{\leq n}$$

$$\psi_* : \operatorname{QCoh}(Tr_n\, \mathfrak{Y})^{\mathrm{cn}}_{\leq n} \to \operatorname{QCoh}(\mathfrak{Y})^{\mathrm{cn}}_{\leq n}.$$

Both of these assertions follow immediately from Proposition 2.5.23. □

2.6 Local Properties of Quasi-Coherent Sheaves

Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack. In §2.3, we introduced the ∞-category $\operatorname{QCoh}(\mathfrak{X})$ of *quasi-coherent* sheaves on $\mathfrak{X}$. In this section, we will study some finiteness conditions on objects of $\operatorname{QCoh}(\mathfrak{X})$. In particular, when $\mathfrak{X}$ is locally Noetherian, we will introduce a full subcategory $\operatorname{Coh}(\mathfrak{X}) \subseteq \operatorname{QCoh}(\mathfrak{X})$, which we call the *$\infty$-category of coherent sheaves* on $\mathfrak{X}$.

We begin by discussing some finiteness conditions on modules over $\mathbb{E}_1$-rings.

Definition 2.6.1. Let A be a connective $\mathbb{E}_1$-ring and let M be a left A-module. We will say that M is *perfect to order n* if, for every filtered diagram $\{N_\alpha\}$ in $(\operatorname{LMod}_A)_{\leq 0}$, the canonical map

$$\varinjlim_\alpha \operatorname{Ext}^i_A(M, N_\alpha) \to \operatorname{Ext}^i_A(M, \varinjlim N_\alpha)$$

injective when $i = n$ and bijective when $i < n$.

Remark 2.6.2. If A is a connective $\mathbb{E}_1$-ring, then a left A-module M is almost perfect (see Definition A.7.2.5.10) if and only if it is perfect to order n for every integer n.

Remark 2.6.3. In the situation of Definition 2.6.1, suppose that M is perfect to order n. When restricted to $(\mathrm{LMod}_A)_{\leq n-1}$, the functor $N \mapsto \mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ commutes with filtered colimits. In particular, $\tau_{\leq n-1} M$ is a compact object of $(\mathrm{LMod}_A)_{\leq n-1}$.

Remark 2.6.4. Let A be a connective $\mathbb{E}_1$-ring, and let M be an A-module which is perfect to order n for some integer n. Then $M \in (\mathrm{LMod}_A)_{\geq -m}$ for some m. This follows immediately from Remark 2.6.3.

Remark 2.6.5. Let A be a connective $\mathbb{E}_1$-ring, and suppose we are given a map $f : M \to M'$ such that the induced map $\pi_i M \to \pi_i M'$ is surjective when $i = n$ and bijective for $i < n$. Let $N \in (\mathrm{LMod}_A)_{\leq 0}$. Then the induced map $\mathrm{Ext}^i_A(M', N) \to \mathrm{Ext}^i_A(M, N)$ is injective for $i = n$ and bijective for $i < n$. It follows that if M is perfect to order n, so is M'.

Remark 2.6.6. Let A be a connective $\mathbb{E}_1$-ring. If M is an A-module which is perfect to order $n+1$, then $\tau_{\leq n} M$ is also perfect to order $n+1$ (this is a special case of Remark 2.6.5).

Remark 2.6.7. Let A be a connective $\mathbb{E}_1$-ring and let M be a compact object of $(\mathrm{LMod}_A)_{\leq n}$. Since LMod_A is compactly generated, we deduce that M is a retract of $\tau_{\leq n} M'$ for some compact object $M' \in \mathrm{LMod}_A$. Since M' is perfect to order $n+1$, so is M (Remark 2.6.6). Combining this observation with Remark 2.6.3, we deduce that an object of $(\mathrm{LMod}_A)_{\leq n}$ is compact in $(\mathrm{LMod}_A)_{\leq n}$ if and only if it is perfect to order $n+1$.

Remark 2.6.8. Let A be a connective $\mathbb{E}_1$-ring and suppose we are given a fiber sequence of left A-modules

$$M' \to M \to M''.$$

If M' is perfect to order n, then M is perfect to order n if and only if M'' is perfect to order n. This follows immediately from an inspection of the associated long exact sequence of Ext-groups.

Remark 2.6.9. Suppose we are given a finite collection of connective $\mathbb{E}_1$-rings $\{A_i\}_{1 \leq i \leq n}$ having product A. Let M be a left A-module, so that $M \simeq \prod M_i$ where each M_i is a left module over A_i. Then M is perfect to order n if and only if each M_i is perfect to order n.

Definition 2.6.10. Let A be a connective $\mathbb{E}_1$-ring, let M be a left A-module, and let n be an integer. We will say that M is *finitely n-presented* if M is connective, perfect to order $n+1$, and the homotopy groups $\pi_i M$ vanish for $i > n$.

Remark 2.6.11. Using Remark 2.6.7, we see that a left A-module M is finitely n-presented if and only if it is a compact object of $(\mathrm{LMod}^{\mathrm{cn}}_A)_{\leq n}$.

Our next result gives a formulation of Definition 2.6.1 which is well-adapted to making inductive arguments:

Proposition 2.6.12. *Let A be a connective $\mathbb{E}_1$-ring, let M be a connective left A-module and Then:*

(1) *The module M is perfect to order 0 if and only if $\pi_0 M$ is finitely generated as a module over $\pi_0 A$.*

(2) *Let $n > 0$ and suppose we are given a map of A-modules $\phi : A^k \to M$ which induces a surjection $\pi_0 A^k \to \pi_0 M$. Then M is perfect to order n if and only if $\mathrm{fib}(\phi)$ is perfect to order $(n-1)$.*

Proof. We first prove (1). For each $N \in (\mathrm{LMod}_A)_{\leq 0}$, we have $\mathrm{Ext}^i_A(M, N) \simeq 0$ for $i < 0$, and $\mathrm{Ext}^0_A(M, N)$ is the abelian group of $\pi_0 A$-module homomorphisms from $\pi_0 M$ into $\pi_0 N$. Consequently, M is perfect to order 0 if and only if, for every filtered diagram of discrete $\pi_0 A$-modules N_α having colimit N, the canonical map

$$\varinjlim_\alpha \mathrm{Ext}^0_{\pi_0 A}(\pi_0 M, N_\alpha) \to \mathrm{Ext}^0_{\pi_0 A}(\pi_0 M, N)$$

is injective. If $\pi_0 M$ is finitely generated as a $\pi_0 A$-module, then we can choose a surjection $(\pi_0 A)^k \to \pi_0 M$, in which case the domain and codomain of θ can be identified with subgroups of the abelian group N^k; this proves the "if" direction of (1). For the converse, suppose that M is perfect to order 0. Let $\{N_\alpha\}$ be the (filtered) diagram of all quotients of the form $(\pi_0 M)/S$, where S is a finitely generated submodule of $\pi_0 M$. Then $\varinjlim N_\alpha \simeq 0$. It follows that $\varinjlim_\alpha \mathrm{Ext}^0_{\pi_0 A}(\pi_0 M, N_\alpha) \simeq 0$, so that the canonical epimorphism $\pi_0 M \to N_\alpha$ is the zero map for some index α. This implies that $\pi_0 M$ is finitely generated.

We now prove (2). Choose a fiber sequence of connective A-modules

$$M' \to A^k \to M,$$

and suppose we are given a filtered diagram $\{N_\alpha\}$ in $(\mathrm{LMod}_A)_{\leq 0}$ having a colimit N. For every pair of object $X, Y \in \mathrm{LMod}_A$, let $\mathrm{Mor}(X,Y)$ denote the spectrum of maps from X to Y in LMod_A, so that $\mathrm{Ext}^i_A(X,Y) = \pi_{-i}\,\mathrm{Mor}(X,Y)$. Let $F(X)$ denote the fiber of the canonical map $\varinjlim_\alpha \mathrm{Mor}(X, N_\alpha) \to \mathrm{Mor}(X,N)$. Note that $F(A) \simeq 0$. We have a fiber sequence of spectra

$$F(M') \to F(A^k) \to F(M),$$

so that $F(M)$ can be identified with the suspension of $F(M')$. In particular, $\pi_i F(M) \simeq 0$ for $i \geq n$ if and only if $\pi_i F(M') \simeq 0$ for $i \geq n-1$, from which (2) follows. $\square$

Proposition 2.6.13. *Let $f : A \to B$ be a map of connective $\mathbb{E}_1$-rings and let M be a left A-module. If M is perfect to order n as an A-module, then $B \otimes_A M$ is perfect to order n as a B-module. The converse holds if B is faithfully flat (as a right module) over A.*

Proof. Assume first that M is perfect to order n as an A-module. We wish to show that $B \otimes_A M$ is perfect to order n as a B-module. Remark 2.6.4 implies that $M \in (\mathrm{LMod}_A)_{\geq -m}$ for some $m \gg 0$. Replacing M by $M[m]$ (and n by $n+m$) we may assume that M is connective and that $n \geq 0$. We proceed by induction on n. If $n = 0$, then we are reduced to proving that $\pi_0(B \otimes_A M) \simeq \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is finitely generated as a module over $\pi_0 B$, which follows immediately from the fact that $\pi_0 M$ is finitely generated over $\pi_0 A$. If $n > 0$, then we can choose a fiber sequence of connective left A-modules

$$M' \to A^k \to M.$$

Tensoring with B, we obtain a fiber sequence of connective left B-modules

$$B \otimes_A M' \to B^k \to B \otimes_A M$$

Using Proposition 2.6.12, we deduce that M' is perfect to order $(n-1)$ as an A-module. The inductive hypothesis implies that $B \otimes_A M'$ is perfect to order $(n-1)$ as a B-module. Using Proposition 2.6.12 again, we deduce that $B \otimes_A M$ is perfect to order n as a B-module.

We now prove the converse. Assume that f is faithfully flat and that $B \otimes_A M$ is perfect to order n as a B-module. Then there exists an integer m such that $\pi_i(B \otimes_A M) \simeq \mathrm{Tor}_i^{\pi_0 A}(\pi_0 B, \pi_i M)$ vanishes for $i < -m$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we deduce that $\pi_i M \simeq 0$ for $i < -m$. Replacing M by $M[m]$ and n by $n+m$, we may assume that M is connective and that $n \geq 0$. We prove that M is perfect to order n using induction on n. We first treat the case $n = 0$. We must show that $\pi_0 M$ is finitely generated as a module over $\pi_0 A$. Our assumption that $B \otimes_A M$ is perfect to order 0 guarantees that $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is finitely generated as a module over $\pi_0 B$. We may therefore choose a finitely generated submodule $M_0 \subseteq \pi_0 M$ such that the induced map $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, M_0) \to \mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, \pi_0 M)$ is surjective, so that $\mathrm{Tor}_0^{\pi_0 A}(\pi_0 B, (\pi_0 M)/M_0)) \simeq 0$. Since $\pi_0 B$ is faithfully flat over $\pi_0 A$, we deduce that $(\pi_0 M)/M_0 \simeq 0$. It follows that $\pi_0 M \simeq M_0$ is finitely generated.

Now suppose that $n > 0$. The argument above shows that $\pi_0 M$ is finitely generated, so we can choose a fiber sequence of connective left A-modules

$$M' \to A^k \to M.$$

Tensoring with B, we obtain a fiber sequence of connective left B-modules

$$B \otimes_A M' \to B^k \to B \otimes_A M.$$

Since $B \otimes_A M$ is perfect to order n, Proposition 2.6.12 implies that $B \otimes_A M'$ is perfect to order $n-1$. It follows from the inductive hypothesis that M' is perfect to order $n-1$, so that M is perfect to order n by Proposition 2.6.12. $\square$

We now introduce some terminology to place Proposition 2.6.13 in a more general context.

Definition 2.6.14. Let $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ denote the ∞-category of pairs (AM), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module spectrum. Let P be a property of objects $(A, M) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$. We will say that P is *local for the fpqc topology* if the following conditions are satisfied:

(1) Let $f : A \to B$ be a flat morphism of connective $\mathbb{E}_\infty$-rings, let M be an A-module. If (A, M) has the property P, then $(B, B \otimes_A M)$ has the property P. The converse holds if f is faithfully flat.

(2) Suppose we are given a finite collection of pairs $(A_i, M_i) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$, each of which has the property P. Then the product $(\prod A_i, \prod M_i) \in \mathcal{C}$ has the property P.

The following result collects some examples of local properties of module spectra:

Proposition 2.6.15. *The following conditions on an object $(A, M) \in \mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^{\mathrm{cn}}$ are local with respect to the fpqc topology:*

(1) *The condition that M is n-connective (when regarded as a spectrum), where n is a fixed integer.*

(2) *The condition that M is almost connective: that is, M is $(-n)$-connective for $n \gg 0$.*

(3) *The condition that M is n-truncated (that is, that $\pi_i M \simeq 0$ for $i > n$), where n is a fixed integer.*

(4) *The condition that M is truncated (that is, that $\pi_i M \simeq 0$ for $i \gg 0$).*

(5) *The condition that M is has* Tor*-amplitude $\leq n$, where n is a fixed integer.*

(6) *The condition that M is flat.*

(7) *The condition that M is perfect to order n over A, where n is a fixed integer.*

(8) *The condition that M is finitely n-presented over A, where $n \geq 0$ is a fixed integer.*

(9) *The condition that M is almost perfect over A.*

(10) *The condition that M is perfect over A.*

Lemma 2.6.16. *Let $f : A \to B$ be a faithfully flat morphism of connective $\mathbb{E}_\infty$-rings and let M be a left A-module. If $B \otimes_A M$ has* Tor*-amplitude $\leq n$ over B, then M has* Tor*-amplitude $\leq n$ over A (see Definition A.7.2.5.21).*

Proof. Let N be a discrete A-module; we wish to show that $M \otimes_A N$ is n-truncated. Since B is faithfully flat over A, it suffices to show that

$$B \otimes_A (M \otimes_A N) \simeq (B \otimes_A M) \otimes_B (B \otimes_A N)$$

is n-truncated. This follows from our assumption that $B \otimes_A M$ has Tor-amplitude $\leq n$, since $B \otimes_A N$ is a discrete B-module. $\square$

Proof of Proposition 2.6.15. Assertions (1) and (3) follow from Proposition A.7.2.2.13, and assertions (2) and (4) follow immediately from (1) and (3). Assertion (5) follows from Lemma 2.6.16. Assertion (6) follows from (5) and (1). Assertion (7) follows from Proposition 2.6.13 and Remark 2.6.9. Assertion (8) follows from (1), (3), and (7). Assertion (9) follows from (7). Assertion (10) follows (5), (9), and Proposition A.7.2.5.23. □

Definition 2.6.17. Let P be a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is local with respect to the fpqc topology, let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, and let $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$. We will say that $\mathcal{F}$ *has the property* P if, for every étale morphism $f : \operatorname{Spec} A \to \mathcal{F}$, the pullback $f^* \mathcal{F} \in \mathrm{QCoh}(\operatorname{Spec} A)$ corresponds to an A-module M such that (A, M) has the property P.

Let us spell out the contents of Definition 2.6.17 in the examples provided by Proposition 2.6.15. Let $\mathcal{F}$ be a quasi-coherent sheaf on a spectral Deligne-Mumford stack $\mathfrak{X}$. We say that $\mathcal{F}$ is:

(1) *n-connective* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is n-connective. This is equivalent to the requirement that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})_{\geq n}$.

(2) *almost connective* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is almost connective. If $\mathfrak{X}$ is quasi-compact, then this condition is equivalent to the requirement that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})_{\geq -n}$ for some integer n.

(3) *n-truncated* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is n-truncated. This is equivalent to the requirement that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})_{\leq n}$.

(4) *locally truncated* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is n-truncated for some integer n. If $\mathfrak{X}$ is quasi-compact, this is equivalent to the requirement that $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})_{\leq n}$ for some integer n.

(5) *of* Tor-*amplitude* $\leq n$ if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is of Tor-amplitude $\leq n$.

(6) *flat* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is flat.

(7) *perfect to order n* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is perfect to order n.

(8) *finitely n-presented* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is finitely n-presented.

(9) *almost perfect* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \mathcal{F} \in \mathrm{Mod}_A$ is almost perfect.

(10) *perfect* if, for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, the pullback $f^* \in \mathrm{Mod}_A$ is perfect.

Remark 2.6.18. In case (4), our terminology does not quite conform to the general convention of Definition 2.6.17. We use the term "locally truncated" rather than "truncated" to emphasize the fact that if $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ is a quasi-coherent sheaf whose pullback $f^* \mathcal{F}$ is truncated for every étale map $f : \operatorname{Spec} A \to \mathfrak{X}$, then $\mathcal{F}$ need not belong to $\bigcup \mathrm{QCoh}(\mathfrak{X})_{\leq n}$ unless we assume that $\mathfrak{X}$ is quasi-compact.

We have the following analogue of Proposition 1.5.7:

Proposition 2.6.19. *Let P be a property of objects of* $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ *which is local for the fpqc topology, let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, and let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. Then:*

(1) *If $f : \mathfrak{Y} \to \mathfrak{X}$ is a flat morphism and $\mathcal{F}$ has the property P, then $f^* \mathcal{F}$ has the property P. In particular, if $\mathcal{F}$ is n-connective (almost connective, n-truncated, locally truncated, of* Tor-*amplitude $\leq n$, flat, perfect to order n, finitely n-presented, almost perfect, perfect), then $f^* \mathcal{F}$ has the same property.*

(2) *If we are given a collection of flat maps $\{f_\alpha : \mathfrak{Y}_\alpha \to \mathfrak{X}\}$ which induces a flat covering $\coprod \mathfrak{Y}_\alpha \to \mathfrak{X}$, and each pullback $f_\alpha^* \mathcal{F}$ has the property P, then $\mathcal{F}$ has the property P. In particular, if each $f_\alpha^* \mathcal{F}$ is n-connective (almost connective, n-truncated, locally truncated, of* Tor*-amplitude $\leq n$, flat, perfect to order n, finitely n-presented, almost perfect, perfect), then $\mathcal{F}$ has the same property.*

Definition 2.6.20. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack and let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. We will say that $\mathcal{F}$ is *coherent* if it is almost perfect and locally truncated. We let $\mathrm{Coh}(\mathfrak{X})$ denote the full subcategory of $\mathrm{QCoh}(\mathfrak{X})$ spanned by the coherent sheaves.

Remark 2.6.21. Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, and suppose we are given a fiber sequence

$$\mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$$

in the stable ∞-category $\mathrm{QCoh}(\mathfrak{X})$. If any two of $\mathcal{F}'$, $\mathcal{F}$, and $\mathcal{F}''$ are coherent, so is the third. That is, $\mathrm{Coh}(\mathfrak{X})$ is a stable subcategory $\mathrm{QCoh}(\mathfrak{X})$. Moreover, $\mathrm{Coh}(\mathfrak{X})$ is idempotent complete.

Remark 2.6.22. Propositoin 2.6.19 implies that the property of being coherent satisfies descent for the flat topology. That is, if $f : \mathfrak{Y} \to \mathfrak{X}$ is a flat morphism of spectral Deligne-Mumford stacks, and $\mathcal{F} \in \mathrm{Coh}(\mathfrak{X})$, then $f^* \mathcal{F} \in \mathrm{Coh}(\mathfrak{Y})$. The converse holds when f is a flat covering.

We will primarily be interested in the class of coherent sheaves in the locally Noetherian setting.

Proposition 2.6.23. *Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack, and let $\mathcal{F} \in \mathrm{Coh}(\mathfrak{X})$. For every integer n, the truncations $\tau_{\geq n} \mathcal{F}$ and $\tau_{\leq n} \mathcal{F}$ belong to* $\mathrm{Coh}(\mathfrak{X})$. *In particular, the subcategories*

$$\mathrm{Coh}(\mathfrak{X})_{\leq 0} = \mathrm{QCoh}(\mathfrak{X})_{\leq 0} \cap \mathrm{Coh}(\mathfrak{X}) \qquad \mathrm{Coh}(\mathfrak{X})_{\geq 0} = \mathrm{QCoh}(\mathfrak{X})_{\geq 0} \cap \mathrm{Coh}(\mathfrak{X})$$

determine a t-structure on $\mathrm{Coh}(\mathfrak{X})$ *(see Proposition 2.3.16).*

Proof. We may assume without loss of generality that $\mathfrak{X}$ is affine, in which case the desired result follows immediately from Proposition A.7.2.5.17. ☐

Proposition 2.6.24. *Let $\mathfrak{X}$ be a locally Noetherian spectral Deligne-Mumford stack and let $\mathcal{F}$ be a quasi-coherent sheaf on $\mathfrak{X}$. If $\mathcal{F}$ is finitely n-presented for some integer n, then $\mathcal{F}$ is coherent.*

Proof. We may assume without loss of generality that $\mathfrak{X}$ is affine, of the form $\mathrm{Spec}^{\text{ét}} A$ for some connective $\mathbb{E}_\infty$-ring A. Then $\mathcal{F}$ corresponds to an A-module M. Arguing as in Remark 2.6.7, we see that M is a retract of $\tau_{\leq n} M'$, where M' is a perfect A-module. The coherence of $\mathcal{F}$ now follows from Proposition 2.6.23. ☐

2.7 Quasi-Coherent Sheaves on a Functor

In §2.3, we introduced the notion of a *quasi-coherent sheaf* on a spectral Deligne-Mumford stack $\mathfrak{X}$. Every spectral Deligne-Mumford stack $\mathfrak{X}$ determines a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, given informally by the formula

$$X(R) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, R, \mathfrak{X}).$$

In §V.2.4, we saw that $\mathfrak{X}$ can be recovered (up to canonical equivalence) from the functor X. More precisely, the construction $\mathfrak{X} \mapsto X$ determines a fully faithful embedding $h : \mathrm{Stk} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ (Theorem V.2.4.1). In particular, the ∞-category $\mathrm{QCoh}(\mathfrak{X})$ can be recovered (up to equivalence) from the functor X. Our goal in this section is to make the passage from X to $\mathrm{QCoh}(\mathfrak{X})$ explicit.

More generally, we explain how to associate to any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ an ∞-category $\mathrm{QCoh}(X)$, which we call the *∞-category of quasi-coherent sheaves* on the functor X. Roughly speaking, an object $\mathcal{F} \in \mathrm{QCoh}(X)$ can be viewed as a rule which assigns to each point $\eta \in X(R)$ an R-module $\mathcal{F}(\eta)$, which depends functorially on R in the following sense: if $\phi : R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings and η' denotes the image of η in $X(R')$, then we have a canonical equivalence $R' \otimes_R \mathcal{F}(\eta) \simeq \mathcal{F}(\eta')$. Moreover, we require that this equivalence is compatible with composition of morphisms in $\mathrm{CAlg}^{\mathrm{cn}}$, up to coherent homotopy. The first part of this section is devoted to turning the above discussion into a precise definition (Definition 2.7.8). We will then show that, in the special case where X is the functor represented by a spectral Deligne-Mumford stack $\mathfrak{X}$, there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(\mathfrak{X})$ (Proposition 2.7.18).

We begin with some general remarks. Suppose we are given a simplicial set S and a map $\psi : S \to \mathcal{C}\mathrm{at}_\infty^{op}$. Since $\mathcal{C}\mathrm{at}_\infty$ admits small limits, Theorem T.5.1.5.6 implies that ψ factors (up to homotopy) as a composition

$$S \xrightarrow{j} \mathcal{P}(S) \xrightarrow{\Psi} \mathcal{C}\mathrm{at}_\infty^{op},$$

where the functor Ψ preserves small colimits (here $j : S \to \mathcal{P}(S)$ denotes the Yoneda embedding). Abstractly, we can characterize Ψ as a left Kan extension of ψ along j. For our purposes, it will be convenient to have a more explicit description of Ψ (in terms of a Cartesian fibration $X \to S$ classified by ψ).

Lemma 2.7.1. *Let S be a simplicial set. Let $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$ denote the category of marked simplicial sets equipped with a map to S, which we regard as endowed with the Cartesian model structure (see §T.3.1). Let $F : (\mathcal{S}\mathrm{et}_\Delta)_{/S} \to (\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$ be the functor given by $X \mapsto X^\sharp$, and regard $(\mathcal{S}\mathrm{et}_\Delta)_{/S}$ as endowed with the contravariant model structure (see §T.2.1.4). Then:*

(1) *The functor F carries fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta)_{/S}$ (with respect to the contravariant model structure) to fibrant objects of $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$, and therefore induces a functor $f : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)_{/S}^o) \to \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^+)_{/S}^o)$.*

(2) *The functor f preserves small limits and colimits.*

Proof. Assertion (1) follows immediately from Proposition T.2.4.2.4. To prove (2), we observe that f fits into a homotopy commutative diagram

$$\begin{tikzcd}
\mathrm{N}(((\mathcal{S}\mathrm{et}_\Delta)^{\mathfrak{C}[S]^{op}})^{\circ}) \arrow[r, "f'"] \arrow[d] & \mathrm{N}(((\mathcal{S}\mathrm{et}_\Delta^+)^{\mathfrak{C}[S]^{op}})^{\circ}) \arrow[d] \\
\mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)_{/S}^{\circ}) \arrow[r, "f"] & \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^+)_{/S}^{\circ})
\end{tikzcd}$$

where the vertical maps are given by the unstraightening functors of §T.2.2.1 and §T.3.2.1, and therefore equivalences of ∞-categories. It therefore suffices to prove that the map f' preserves small limits. Using Proposition T.4.2.4.4, we can identify f'' with the map $\mathrm{Fun}(S, \mathcal{S}) \to \mathrm{Fun}(S, \mathcal{C}\mathrm{at}_\infty)$ induced by the inclusion $i : \mathcal{S} \to \mathcal{C}\mathrm{at}_\infty$. It therefore suffices to show that i preserves small and colimits, which follows from the observation that i admits left and right adjoints. $\square$

Construction 2.7.2. Let $q : X \to S$ be a Cartesian fibration of simplicial sets. Given another map of simplicial sets $S' \to S$, we say that a map $F \in \mathrm{Fun}_S(S', X)$ is *q-Cartesian* if it carries every edge of S' to a q-Cartesian edge of X. We let $\mathrm{Fun}_S^{\mathrm{cart}}(S', X)$ denote the full subcategory of $\mathrm{Fun}_S(S', X)$ spanned by the Cartesian maps. The construction $S' \mapsto \mathrm{Fun}_S^{\mathrm{cart}}(S', X)$ determines a functor $\Phi_0(q) : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)_{/S}^o)^{op} \to \mathcal{C}\mathrm{at}_\infty$, where we regard $(\mathcal{S}\mathrm{et}_\Delta)_{/S}$ as endowed with the contravariant model structure.

The definition above has an evident dual: if $q : X \to S$ is a coCartesian fibration of simplicial sets and we are given a map $S' \to S$, we say that a map $F \in \mathrm{Fun}_S(S', X)$ is *q-coCartesian* if it carries every edge of S' to a q-coCartesian edge of X. We let $\mathrm{Fun}_S^{\mathrm{ccart}}(S', X)$ denote the full subcategory of $\mathrm{Fun}_S(S', X)$ spanned by the q-coCartesian maps.

Remark 2.7.3. Let $q : X \to S$ be a Cartesian fibration of simplicial sets. Then the functor $\Phi_0(q)$ can be obtained by composing the functor $f : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)_{/S}^o) \to \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^+)_{/S}^o)$ of Lemma 2.7.1 with the functor $G_0 : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^+)_{/S}^o) \to (\mathcal{S}\mathrm{et}_\Delta^+)^o$ induced by the right adjoint to the left Quillen functor $K \mapsto K \times S$ from $\mathcal{S}\mathrm{et}_\Delta^+$ to $(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}$. It follows that $\Phi_0(q)$ preserves small limits.

Construction 2.7.4. Let $q : X \to S$ be a Cartesian fibration of simplicial sets. We let $\Phi(q) : \mathcal{P}(S)^{op} \to \mathcal{C}\mathrm{at}_\infty$ denote the functor Φ_0 of Construction 2.7.2 with the equivalence $\mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)^\circ_{/S}) \simeq \mathcal{P}(S)$ of Proposition T.5.1.1.1.

Lemma 2.7.5. *Let $q : X \to S$ be a Cartesian fibration of simplicial sets and let $j : S \to \mathcal{P}(S)$ denote the Yoneda embedding. Then the composition $\Phi(q) \circ j : S^{op} \to \mathcal{C}\mathrm{at}_\infty$ classifies the Cartesian fibration q.*

Proof. We begin by recalling the definition of the Yoneda embedding j. Choose a weak equivalence of simplicial categories $\phi : \mathfrak{C}[S] \to \mathcal{C}^{op}$, where $\mathcal{C}$ is fibrant (that is, the mapping space $\mathrm{Map}_{\mathcal{C}}(X, Y)$ is a Kan complex for every pair of objects $X, Y \in \mathcal{C}$). The construction $X \mapsto \mathrm{Map}_{\mathcal{C}}(X, \bullet)$ determines a simplicial functor $F : \mathcal{C}^{op} \to ((\mathcal{S}\mathrm{et}_\Delta)^{\mathcal{C}})^\circ$, which (after composing with ϕ) yields a functor $f : S \to \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ)$; here we regard $\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}}$ as endowed with the projective model structure. The Yoneda embedding j is obtained by composing f with the equivalences $\theta : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ) \to \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[S^{op}]})^\circ)$ and $\theta' : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[S^{op}]})^\circ) \to \mathrm{Fun}(S^{op}, \mathcal{S}) = \mathcal{P}(S)$. The functor $\Phi(q)$ is obtained by composing a homotopy inverse to θ', the equivalence $U : \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta^{\mathfrak{C}[S^{op}]})^\circ) \to \mathrm{N}((\mathcal{S}\mathrm{et}_\Delta)_{/S})$ induced by the unstraightening functor Un_S of §T.3.2.1, and the functor $\Phi_0(q)$ of Construction 2.7.2. It therefore suffices to show that the composition $\Phi(q) \circ j \simeq \Phi_0(q) \circ U \circ \theta \circ f = \Phi_0(q) \circ \mathrm{Un}_\phi \circ f$ classifies the Cartesian fibration q.

Without loss of generality, we may suppose that $X \simeq \mathrm{Un}_\phi^+ \chi$, where χ is a fibrant-cofibrant object of $(\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}}$ (Theorem T.3.2.0.1). Then the composition $\Phi_0(q) \circ \mathrm{Un}_\phi : \mathrm{N}(\mathcal{S}\mathrm{et}_\Delta^{\mathcal{C}})^\circ \to \mathcal{C}\mathrm{at}_\infty$ is the nerve of the simplicial functor $F \mapsto \mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^+)_{/S}}(\mathrm{Un}_\phi^+ F^\sharp, \mathrm{Un}_\phi^+ \chi)$, which is equivalent to the functor $F \mapsto \mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}}}(F^\sharp, \chi)$; here $F^\sharp : \mathcal{C} \to \mathcal{S}\mathrm{et}_\Delta^+$ denotes the functor given by $F^\sharp(C) = F(C)^\sharp$. In particular, if F is representable by an object $C \in \mathcal{C}$, the classical (simplicially enriched) version of Yoneda's lemma gives a canonical isomorphism $\mathrm{Map}_{(\mathcal{S}\mathrm{et}_\Delta^+)^{\mathcal{C}}}(F^\sharp, \chi) \simeq \chi(C)$. We conclude that $\Phi(q) \circ j \simeq \Phi_0(q) \circ \mathrm{Un}_\phi \circ f$ is adjoint to the simplicial functor $\mathfrak{C}[S] \to \mathcal{C} \xrightarrow{\chi} \mathcal{C}\mathrm{at}_\infty^\Delta$, so that $\Phi(q) \circ j$ classifies the Cartesian fibration q. □

Proposition 2.7.6. *Let $q : X \to S$ be a Cartesian fibration classified by a functor $\chi : S^{op} \to \mathcal{C}\mathrm{at}_\infty$, let $j : S \to \mathcal{P}(S)$ be the Yoneda embedding. Then $\Phi(q)$ is a right Kan extension of χ along j.*

Proof. Since $\Phi(q)$ preserves small limits (Remark 2.7.3), Lemma T.5.1.5.5 guarantees that $\Phi(q)$ is a right Kan extension of $\Phi(q) \circ j$ along j. It now suffices to observe that $\Phi(q) \circ j \simeq \chi$ by virtue of Lemma 2.7.5. □

Remark 2.7.7. There are evident dual versions of the constructions described above. For every coCartesian fibration of simplicial sets $q : X \to S$ classified by a functor $\chi : S \to \mathcal{C}\mathrm{at}_\infty$, one can define a functor $\Phi'(q) : \mathrm{Fun}(S, \mathcal{S})^{op} \simeq \mathcal{P}(S^{op})^{op} \to \mathcal{C}\mathrm{at}_\infty$, which is a right Kan extension of χ along the Yoneda embedding $j : S \to \mathcal{P}(S^{op})^{op}$.

Definition 2.7.8. Let $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ denote the ∞-category whose objects are pairs (A, M), where A is a connective $\mathbb{E}_\infty$-ring and M is an A-module spectrum. Let $q : \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \to \mathrm{CAlg}^{\mathrm{cn}}$ denote the projection onto the first factor, so that q is a coCartesian fibration. We let $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}_\infty}$ denote the functor $\Phi'(q)$ obtained by applying Remark 2.7.7 to q (after a change of universe); here $\widehat{\mathcal{S}}$ denotes the ∞-category of spaces which are not necessarily small, and $\widehat{\mathcal{C}\mathrm{at}_\infty}$ is defined similarly.

If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor, we will refer to $\mathrm{QCoh}(X) \in \widehat{\mathcal{C}\mathrm{at}_\infty}$ as the *∞-category of quasi-coherent sheaves on X*.

Remark 2.7.9. There is no real need to restrict to connective $\mathbb{E}_\infty$-rings in Definition 2.7.8. Using exactly the same procedure, we can associate to any functor $X : \mathrm{CAlg} \to \widehat{\mathcal{S}}$ an ∞-category $\mathrm{QCoh}'(F)$ of quasi-coherent sheaves on X. In some sense, this definition is strictly more general than that of Definition 2.7.8: if $X_0 : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor, then there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(X_0) \simeq \mathrm{QCoh}'(X)$, where $X : \mathrm{CAlg} \to \widehat{\mathcal{S}}$ is a left Kan extension of X_0. However, for most of our applications it will be convenient to consider functors which are defined only on the full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$ spanned by the connective $\mathbb{E}_\infty$-rings.

Remark 2.7.10. Lemma 2.7.5 implies that the composition of the Yoneda embedding

$$\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$$

with the functor $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classifies the coCartesian fibration $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \to \mathrm{CAlg}^{\mathrm{cn}}$. More informally: if $\mathrm{Spec}^{\mathrm{f}} R : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is the functor corepesented by a connective $\mathbb{E}_\infty$-ring R, then we have a canonical identification $\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R) \simeq \mathrm{Mod}_R$.

Notation 2.7.11. If $\alpha : X \to X'$ is a natural transformation between functors $X, X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, then α determines a functor $\mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$. We will denote this functor by α^*, and refer to it as the functor given by *pullback along* α.

Remark 2.7.12. By construction, if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor classifying a left fibration $\mathcal{C} \to \mathrm{CAlg}$, then the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on F can be identified with the ∞-category $\mathrm{Fun}^{\mathrm{ccart}}_{\mathrm{CAlg}}(\mathcal{C}, \mathrm{Mod})$ of Construction 2.7.2. More informally, we can think of an object $\mathcal{F} \in \mathrm{QCoh}(X)$ as a functor which assigns to every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$ (encoded by an object $\widetilde{R} \in \mathcal{C}$ lifting R) an R-module $\mathcal{F}(\eta) \in \mathrm{Mod}_R$. These modules are required to depend functorially on R in the following strong sense: if $\phi : R \to R'$ is a map of connective $\mathbb{E}_\infty$-rings and $\eta' \in X(R')$ is the image of η under ϕ (so that we have a morphism $\widetilde{f} : \widetilde{R} \to \widetilde{R}'$ in $\mathcal{C}$), then we obtain a q-coCartesian morphism $\mathcal{F}(\eta) \to \mathcal{F}(\eta')$ in Mod, corresponding to an equivalence of R'-modules $R' \otimes_R \mathcal{F}(\eta) \to \mathcal{F}(\eta')$.

Note that we can identify η with a natural transformation of functors $\mathrm{Spec}^{\mathrm{f}} R \to F$, where $\mathrm{Spec}^{\mathrm{f}} R : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is the functor corepresented by R. Then $\mathcal{F}(\eta)$ denotes the image of $\mathcal{F}$ under the composite

$$\mathrm{QCoh}(F) \xrightarrow{\eta^*} \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R) \simeq \mathrm{Mod}_R,$$

where the equivalence $\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R) \simeq \mathrm{Mod}_R$ is supplied by Remark 2.7.10. Motivated by this observation, we will sometimes denote the R-module $\mathcal{F}(\eta)$ by $\eta^* \mathcal{F}$

Definition 2.7.13. We let $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves with respect to the flat topology of Proposition VII.5.4.

Proposition 2.7.14. *Let $L : \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}}) \to \widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$ be a left adjoint to the inclusion functor. Then the functor $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ is equivalent to $\mathrm{QCoh} \circ L$, so that the restriction $\mathrm{QCoh} | (\widehat{\mathrm{Shv}}_{\mathrm{fpqc}})^{op}$ is a colimit-preserving functor $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}} \to \widehat{\mathcal{C}\mathrm{at}}_\infty^{op}$.*

Remark 2.7.15. We can formulate Proposition 2.7.14 more concretely as follows: if $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor and X' is the sheafification of F with respect to the flat topology, then the pullback map $\mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$ is an equivalence of ∞-categories.

Proof. Since the functor QCoh preserves limits (Remark 2.7.3), Proposition V.1.1.12 implies that the functor QCoh factors through L if and only if the composition of QCoh with the Yoneda embedding $\mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$ is a $\widehat{\mathcal{C}\mathrm{at}}_\infty$-valued sheaf on $\mathrm{CAlg}^{\mathrm{cn}}$. This follows from Corollary VII.6.13. $\square$

Remark 2.7.16. In fact, we can strengthen Proposition 2.7.14 slightly: the ∞-category $\mathrm{QCoh}(F)$ of quasi-coherent sheaves on a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ depends only on the *hypercompletion* of the flat sheaf $L(X)$.

We now study some formal properties of the ∞-categories of the form $\mathrm{QCoh}(F)$.

Proposition 2.7.17. (1) *For every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QCoh}(X)$ is stable and admits small colimits.*

(2) *For every natural transformation morphism $\alpha : X \to X'$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, the pullback functor $\alpha^* : \mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$ preserves small colimits.*

(3) *Suppose that $X \in \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ belongs to the smallest full subcategory of $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ which is closed under small colimits and contains the essential image of the Yoneda embedding. Then the ∞-category $\mathrm{QCoh}(X)$ is presentable.*

Proof. Let $\mathcal{C}$ denote the subcategory of $\widehat{\mathcal{C}\mathrm{at}}_\infty$ spanned by those ∞-categories which are stable and admit small colimits, and those functors which preserve small colimits. Then $\mathcal{C}$ admits limits, and the inclusion $\mathcal{C} \hookrightarrow \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves limits. Since the coCartesian fibration $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \to \mathrm{CAlg}^{\mathrm{cn}}$ is classified by a functor $\chi : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{C}$, it follows from Proposition 2.7.6 that QCoh factors through $\mathcal{C}$. This proves (1) and (2). To prove (3), we let $\mathcal{C}_0$ denote the full subcategory of $\mathcal{C}$ spanned by the presentable ∞-categories. Using Proposition T.5.5.3.13, we deduce that $\mathcal{C}_0$ is stable under small limits in $\mathcal{C}$, so that $\mathrm{QCoh}^{-1}\,\mathcal{C}_0$ is stable under small colimits in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It therefore suffices to observe that $\mathrm{QCoh}(X)$ is presentable whenever X is corepresented by a connective $\mathbb{E}_\infty$-ring R: this follows from the equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_R$ of Remark 2.7.10. □

Proposition 2.7.18. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, given informally by $X(A) = \mathrm{Map}_{\mathrm{Stk}}(\mathrm{Spec}\, A, \mathfrak{X})$. Then there is a canonical equivalence of ∞-categories $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(\mathfrak{X})$, where $\mathrm{QCoh}(\mathfrak{X})$ is defined as in §2.3.*

Proof. The forgetful functor $\mathrm{Sch}(\mathcal{G}^{\mathrm{M}}_{\mathrm{\acute{e}t}})^{op} \to \mathrm{Stk}$ is a coCartesian fibration, whose fiber over a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ can be identified with the ∞-category $\mathrm{QCoh}(\mathfrak{X})$. This coCartesian fibration is classified by a functor $\chi : \mathrm{Sch}(\mathcal{G}^{\mathrm{nSp}}_{\mathrm{\acute{e}t}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $\phi : \mathrm{Sch}(\mathcal{G}^{\mathrm{nSp}}_{\mathrm{\acute{e}t}}) \to \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ be the fully faithful embedding of Theorem V.2.4.1. We have a pullback diagram in $\widehat{\mathcal{C}\mathrm{at}}_\infty$

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod} \arrow[r] \arrow[d] & \mathrm{Sch}(\mathcal{G}^{\mathrm{M}}_{\mathrm{\acute{e}t}})^{op} \arrow[d] \\
\mathrm{CAlg}^{\mathrm{cn}} \arrow[r, "j"] & \mathrm{Stk}^{op}.
\end{tikzcd}$$

Combining this observation with Lemma 2.7.5, we deduce that χ and $\mathrm{QCoh} \circ \phi$ have equivalent restrictions to the full subcategory $\mathcal{A} \subseteq \mathrm{Stk}$ spanned by the affine spectral Deligne-Mumford stacks. Since ϕ is fully faithful, Proposition 2.7.6 implies that $\mathrm{QCoh} \circ \phi$ is a right Kan extension of its restriction to $\mathcal{A}$, so we obtain a natural transformation of functors $\alpha : \chi \to \mathrm{QCoh} \circ \phi$ (which is an equivalence when restricted to affine spectral Deligne-Mumford stacks). We will complete the proof by showing that α is an equivalence (so that for every spectral Deligne-Mumford stack $\mathfrak{X}$ representing a functor $F = \phi(\mathfrak{X})$, we have an equivalence $\mathrm{QCoh}(\mathfrak{X}) \simeq \chi(\mathfrak{X}) \to \mathrm{QCoh}(F)$).

Fix a spectral Deligne-Mumford stack $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, and let $\psi : \mathcal{X} \to \mathrm{Stk}$ be given informally by the formula $\psi(U) = (\mathcal{X}_{/U}, \mathcal{O}_{\mathcal{X}}|U)$. The natural transformation α determines a natural transformation $\alpha' : \chi \circ \psi \to \mathrm{QCoh} \circ \phi \circ \psi$ of functors $\mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$. Let $\mathcal{X}_0$ denote the full subcategory of $\mathcal{X}$ spanned by those objects U for which α' induces an equivalence

$$\mathrm{QCoh}(\mathcal{X}_{/U}) \simeq (\chi \circ \psi)(U) \to (\mathrm{QCoh} \circ \phi \circ \psi)(U).$$

Let $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{\acute{e}t}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ spanned by those functors which are sheaves with respect to the étale topology. Proposition 2.7.14 implies that the functor $\mathrm{QCoh}\,|(\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{\acute{e}t}})^{op}$ preserves small limits, and Lemma V.2.4.13 implies that the functor $\phi \circ \psi : \mathcal{X} \to \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{\acute{e}t}}$ preserves small colimits, so that $\mathrm{QCoh} \circ \phi \circ \psi : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ preserves small limits. Arguing as in the proof of Proposition 2.3.13, we deduce that $\chi \circ \psi : \mathcal{X}^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ also preserves small limits. It follows that the full subcategory $\mathcal{X}_0 \subseteq \mathcal{X}$ is stable under colimits in $\mathcal{X}$. Since $\mathcal{X}_0$ contains every affine object $U \in \mathcal{X}$, we conclude that $\mathcal{X}_0 = \mathcal{X}$. In particular, $\mathcal{X}_0$ contains a final object of $\mathcal{X}$, which proves that α induces an equivalence $\mathrm{QCoh}(\mathcal{X}) \simeq \chi(\mathcal{X}, \mathcal{O}) \to \mathrm{QCoh}(\phi(\mathcal{X}, \mathcal{O}))$ as desired. □

Many of the local properties of quasi-coherent sheaves discussed in §2.6 make sense in the context of quasi-coherent sheaves on an arbitrary functor.

Definition 2.7.19. Let P be a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$. We will say that P is *stable under base change* if, whenever a pair (A, M) has the property P and $f : A \to B$ is a map of connective $\mathbb{E}_\infty$-rings, the pair $(B, B \otimes_A M)$ also has the property P.

Proposition 2.7.20. *The following conditions on an object $(A, M) \in \mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^{\mathrm{cn}}$ are stable under base change:*

(1) *The condition that M is n-connective (when regarded as a spectrum), where n is a fixed integer.*

(2) *The condition that M is almost connective: that is, M is $(-n)$-connective for $n \gg 0$.*

(3) *The condition that M is has* Tor*-amplitude $\leq n$, where n is a fixed integer.*

(4) *The condition that M is flat.*

(5) *The condition that M is perfect to order n over A, where n is a fixed integer.*

(6) *The condition that M is almost perfect over A.*

(7) *The condition that M is perfect over A.*

Proof. Assertions (1), (2), and (7) are obvious. Assertion (3) follows from Lemma 2.6.16 and assertion (5) from Proposition 2.6.13. Assertions (4) and (6) are immediate consequences of (3) and (5). □

Definition 2.7.21. Let P be a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is invariant under base change, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We will say that an object $\mathcal{F} \in \mathrm{QCoh}(X)$ *has the property P* if, for every connective $\mathbb{E}_\infty$-ring R and every point $\eta \in X(R)$, the pair $(R, \mathcal{F}(\eta))$ has the property P, where $\mathcal{F}(\eta) \in \mathrm{Mod}_R$ is the R-module of Remark 2.7.12.

Remark 2.7.22. Let P be a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change, and let $\alpha : F \to F'$ be a natural transformation between functors $F, F' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. If $\mathcal{F} \in \mathrm{QCoh}(F')$ has the property P, then $\alpha^* \mathcal{F} \in \mathrm{QCoh}(F)$ has the property P.

Example 2.7.23. Let R be a connective $\mathbb{E}_\infty$-ring and let $\mathrm{Spec}^{\mathrm{f}} R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor corepresented by R, so that Remark 2.7.10 gives an equivalence of ∞-categories $\theta : \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R) \simeq \mathrm{Mod}_R$. If P is a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change, then an object $\mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} R)$ has the property P if and only if the pair $(R, \theta(\mathcal{F}))$ has the property P.

Remark 2.7.24. Let $\alpha : X \to X'$ be a natural transformation between functors $X, X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and suppose that α induces an equivalence after sheafification with respect to the flat topology. Proposition 2.7.14 implies that the pullback functor $\alpha^* : \mathrm{QCoh}(X') \to \mathrm{QCoh}(X)$ is an equivalence of ∞-categories. If P is a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change and $\mathcal{F} \in \mathrm{QCoh}(X')$ has the property P, then the pullback $\alpha^* \mathcal{F} \in \mathrm{QCoh}(X)$ has the property P (Remark 2.7.22). The converse holds provided that P is local with respect to the fpqc topology, in the sense of Definition 2.6.14.

Example 2.7.25. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathcal{F} \in \mathrm{QCoh}(X)$. We say that $\mathcal{F}$ is *connective* if, for every point $\eta \in X(R)$, the R-module $\mathcal{F}(\eta)$ is connective. We let $\mathrm{QCoh}(X)^{\mathrm{cn}}$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by the connective quasi-coherent sheaves on X. It is clear that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is closed under small colimits and extensions in $\mathrm{QCoh}(X)$. If F satisfies condition (3) of Proposition 2.7.17, then the proof of Proposition 2.7.17 shows that $\mathrm{QCoh}(X)^{\mathrm{cn}}$ is a presentable ∞-category. It follows from Proposition A.1.4.5.11 that the pair of subcategories $(\mathrm{QCoh}(X)^{\mathrm{cn}}, \mathrm{QCoh}(X)_{\leq 0})$ determines an accessible t-structure on $\mathrm{QCoh}(X)$, where $\mathrm{QCoh}(X)_{\leq 0}$ denotes the full subcategory of $\mathrm{QCoh}(X)$ spanned by those objects $\mathcal{G}$ for which the mapping space $\mathrm{Map}_{\mathrm{QCoh}(F)}(\mathcal{F}[1], \mathcal{G})$ is contractible for all $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$.

For a general functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ (satisfying condition (3) of Proposition 2.7.17), there is no obvious way to test that an object $\mathcal{F} \in \mathrm{QCoh}(X)$ belongs to $\mathrm{QCoh}(X)_{\leq 0}$. For example, the inclusion $\mathcal{F} \in \mathrm{QCoh}(X)$ does *not* imply that $\mathcal{F}(\eta) \in (\mathrm{Mod}_R)_{\leq 0}$ when $\eta \in X(R)$, since the property of being 0-truncated is not stable under base change. However, $\mathrm{QCoh}(X)_{\leq 0}$ does admit an explicit description in the case when X is representable by an algebraic stack $\mathfrak{X}$: in this case, one has $\mathcal{F} \in \mathrm{QCoh}(X)_{\leq 0}$ if and only if $\mathcal{F}(\eta) \in (\mathrm{Mod}_R)_{\leq 0}$ whenever η classifies a *flat* morphism $\mathrm{Spec}\, R \to \mathfrak{X}$.

Proposition 2.7.26. *Let $\mathfrak{X}$ be a spectral Deligne-Mumford stack, let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$, and let $\theta : \mathrm{QCoh}(\mathfrak{X}) \simeq \mathrm{QCoh}(F)$ be the equivalence of Proposition 2.7.18. Let P be a property of objects of $\mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}$ which is stable under base change and local for the fpqc topology. An object $\mathcal{F} \in \mathrm{QCoh}(\mathfrak{X})$ has the property P if and only if $\theta(\mathcal{F}) \in \mathrm{QCoh}(F)$ has the property P.*

Proof. The "only if" direction is obvious. For the converse, let us suppose that $\mathcal{F}$ has the property P. Let A be a connective $\mathbb{E}_\infty$-ring and let $f : \mathrm{Spec}\, A \to \mathfrak{X}$ be a map of spectral Deligne-Mumford stacks; we wish to show that the pair $(A, f^* \mathcal{F})$ has the property P. In verifying this, we are free to replace $\mathfrak{X}$ by any open substack through which f factors; we may therefore assume without loss of generality that $\mathfrak{X}$ is quasi-compact. Choose an étale surjection $u : \mathfrak{U} \to \mathfrak{X}$, where $\mathfrak{U} \simeq \mathrm{Spec}\, R$ is affine. We can then choose a faithfully flat étale map $A \to A'$ such that the composite map $\mathrm{Spec}\, A' \to \mathrm{Spec}\, A \xrightarrow{f} \mathfrak{X}$ factors through $\mathfrak{U}$. Since P is local for the fpqc topology, we may replace A by A' and thereby reduce to the case where f factors through U. Then $f^* \mathcal{F} \simeq A \otimes_R u^* \mathcal{F}$. Since P is stable under base change, we are reduced to proving that the pair $(R, u^* \mathcal{F})$ has the property P, which follows from our assumption that $\mathcal{F}$ has the property P. □

Let $\mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ denote the ∞-category of (not necessarily small) symmetric monoidal ∞-categories. We have an evident forgetful functor

$$\theta : \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \to \widehat{\mathcal{C}\mathrm{at}}_\infty,$$

which preserves limits. The functor $R \mapsto \mathrm{Mod}_R$ factors as a composition

$$\mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{U} \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty) \xrightarrow{\theta} \widehat{\mathcal{C}\mathrm{at}}_\infty,$$

where U assigns to each connective $\mathbb{E}_\infty$-ring the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\otimes}$ (see §A.4.4.3). Let $\mathrm{QCoh}^{\otimes} : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \mathrm{CAlg}(\widehat{\mathcal{C}\mathrm{at}}_\infty)$ be a right Kan extension of U along the Yoneda embedding $(\mathrm{CAlg}^{\mathrm{cn}})^{op} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$. Then the functor $\mathrm{QCoh}^{\otimes}$ assigns to each functor $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ a symmetric monoidal ∞-category $\mathrm{QCoh}(F)^{\otimes}$, whose underyling ∞-category can be identified with $\mathrm{QCoh}(F)$. We can describe the situation more informally by saying that for every functor $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the ∞-category $\mathrm{QCoh}(F)$ admits a symmetric monoidal structure. Unwinding the definitions, we see that the tensor product on $\mathrm{QCoh}(F)$ is given pointwise: that is, it is described by the formula

$$(\mathcal{F} \otimes \mathcal{F}')(\eta) \simeq \mathcal{F}(\eta) \otimes_R \mathcal{F}'(\eta)$$

for $\eta \in F(R)$. It follows that the tensor product on $\mathrm{QCoh}(F)$ preserves small colimits separately in each variable.

Notation 2.7.27. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is any functor, we let $\mathcal{O}_X$ denote the unit object of $\mathrm{QCoh}(X)$ (with respect to the symmetric monoidal structure defined above). More informally, $\mathcal{O}_X$ assigns to each point $\eta \in X(R)$ the spectrum R, regarded as a module over itself.

Proposition 2.7.28. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor, and let $\mathcal{F} \in \mathrm{QCoh}(X)$. The following conditions are equivalent:*

(1) *The quasi-coherent sheaf $\mathcal{F}$ is perfect.*

(2) *The quasi-coherent sheaf is a dualizable object of the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$.*

Proof. Using Proposition A.4.2.5.11, we can reduce to the case where X is corepresentable by a connective $\mathbb{E}_\infty$-ring R. In this case, we can identify $\mathcal{F}$ with an R-module M. We wish to show that M is a dualizable object of Mod_R if and only if M is perfect. The collection of dualizable objects of Mod_R forms a stable subcategory which is closed under retracts. Since $R \in \mathrm{Mod}_R$ is dualizable, it follows that every perfect object of Mod_R is dualizable. Conversely, suppose that M admits a dual $M^\vee$. Then the functor $N \mapsto \mathrm{Map}_{\mathrm{Mod}_R}(M, N)$ is given by $N \mapsto \Omega^\infty(M^\vee \otimes_R N)$, and therefore commutes with filtered colimits. It follows that M is a compact object of Mod_R, and therefore perfect (Proposition A.7.2.5.2). □

Note that for any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, the full subcategory $\mathrm{QCoh}(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}(X)$ contains $\mathcal{O}_X$ and is closed under tensor products, and therefore inherits the structure of a symmetric monoidal ∞-category. We close this section by describing the dualizable objects of $\mathrm{QCoh}(X)^{\mathrm{cn}}$.

Definition 2.7.29. Let A be a connective $\mathbb{E}_\infty$-ring and let M be an A-module. We will say that M is *locally free of finite rank* if there exists an integer n such that M is a direct summand of A^n.

Remark 2.7.30. Let A be a connective $\mathbb{E}_\infty$-ring and let M be an A-module which is locally free of finite rank. Then $\pi_0 M$ is a locally free $\pi_0 A$-module of finite rank (in the sense of classical commutative algebra). We can therefore choose elements $x_1, \ldots, x_n \in \pi_0 A$ which generate the unit ideal, such that each localization $(\pi_0 M)[\frac{1}{x_i}]$ is a free module over $(\pi_0 A)[\frac{1}{x_i}]$ of finite rank. It then follows from Corollary A.7.2.2.19 that $M[\frac{1}{x_i}]$ is a free module over $A[\frac{1}{x_i}]$. In other words, the condition of Definition 2.7.29 implies that M is free locally with respect to the Zariski topology on $\mathrm{Spec}^{\mathrm{Z}} A$.

We now prove a converse to Remark 2.7.30.

Proposition 2.7.31. *The property of being a locally free module of finite rank is stable under base change and local with respect to the fpqc topology.*

Proof. Let A be a connective $\mathbb{E}_\infty$-ring. According to Proposition A.7.2.5.20, an A-module M is locally free of finite rank if and only if M is flat and almost perfect. The desired assertions now follow from Propositions 2.6.15 and 2.7.20. □

If $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is a functor and $\mathcal{F} \in \mathrm{QCoh}(X)$ is a quasi-coherent sheaf on X, then we will say that $\mathcal{F}$ is *locally free of finite rank* if $\mathcal{F}(\eta) \in \mathrm{Mod}_R$ is locally free of finite rank for every point $\eta \in X(R)$ (in other words, if $\mathcal{F}$ satisfies the condition of Definition 2.7.21, where P is the property of being locally free of finite rank).

Proposition 2.7.32. *Let R be a connective $\mathbb{E}_\infty$-ring and let M be a connective R-module. The following conditions are equivalent:*

(1) *The module M is locally free of finite rank.*

(2) *The module M is a dualizable object of the symmetric monoidal ∞-category $\mathrm{Mod}_R^{\mathrm{cn}}$.*

Proof. The collection of dualizable objects of $\mathrm{Mod}_R^{\mathrm{cn}}$ is evidently closed under the formation of retracts and direct sums. Since the unit object $R \in \mathrm{Mod}_R^{\mathrm{cn}}$ is dualizable, we conclude that (1) $\Rightarrow$ (2). Conversely, suppose that M is a dualizable object of $\mathrm{Mod}_R^{\mathrm{cn}}$. Then M is a dualizable object of Mod_R, and therefore a perfect R-module (Proposition 2.7.28). Let $M^\vee$ denote the dual of M. For any discrete R-module N, we have isomorphisms $\pi_i(M \otimes_R N) \simeq \pi_i \mathrm{Map}_{\mathrm{Mod}_R}(M^\vee, N)$ for $i \geq 0$. Since $M^\vee$ is connective, we deduce that $\pi_i(M \otimes_R N)$ vanishes for $i > 0$. It follows that M is flat, so that M is locally free of finite rank by Proposition A.7.2.5.20. □

Corollary 2.7.33. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor and let $\mathcal{F} \in \mathrm{QCoh}(X)^{\mathrm{cn}}$. Then $\mathcal{F}$ is a dualizable object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$ if and only if $\mathcal{F}$ is locally free of finite rank.*

Proof. Combine Propositions A.4.2.5.11 and 2.7.32. □

3 Geometric Stacks and Tannaka Duality

In [39], we prove the following:

Theorem 3.0.1. *Let X be a quasi-compact Artin stack, and assume that the diagonal map $X \to X \times X$ is affine (for example, we can take X to be any quasi-compact separated scheme). Then for every scheme Y, the construction*

$$(f : Y \to X) \mapsto f^*$$

determines a fully faithful embedding $\theta : \mathrm{Map}(Y, X) \to \mathcal{C}$. Here $\mathrm{Map}(Y, X)$ denotes the groupoid of maps from Y to X (in the 2-category of Artin stacks) and $\mathcal{C}$ denotes the category of symmetric monoidal functors from the abelian category of quasi-coherent sheaves on X to the abelian category of quasi-coherent sheaves on Y. The essential image of θ consists of those functors F which preserve small colimits and carry flat objects to flat objects.

In this section, we will prove an analogous result in the setting of spectral algebraic geometry. In place of an stack X, we will consider an arbitrary functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Our goal is to give conditions which guarantee that X can be recovered from the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X (together with the symmetric monoidal structure on $\mathrm{QCoh}(X)$). Our main result (Theorem 3.4.2) implies that this recovery is possible whenever X is a *geometric stack*.

The bulk of this section is devoted to developing the tools necessary to formulate and prove Theorem 3.4.2. We will begin in §3.1 with a general discussion of properties of morphisms of spectral Deligne-Mumford stacks $\phi : \mathfrak{X} \to \mathfrak{Y}$ which can be tested locally on $\mathfrak{Y}$. Such a property then makes sense in the more general setting of natural transformations $f : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ (see Definition 3.1.24). In particular, we obtain a notion of *quasi-affine* morphism of functors, which we will study in detail in §3.2 (globalizing some of the ideas introduced in §2.4).

We will say that a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is a *quasi-geometric stack* if the diagonal morphism $X \to X \times X$ is quasi-affine. In §3.3, we will study the class of quasi-geometric stacks. Our main result is a prototype version of Theorem 3.0.1: if X is a quasi-geometric stack and $\mathrm{QCoh}(X)$ is presentable, then for any functor $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, the construction $(f : Y \to X) \mapsto f^*$ determines a fully faithful embedding from $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X)$ to the ∞-category $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ of symmetric monoidal functors from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$ (Proposition 3.3.11). In general, we do not know a convenient description of the essential image of the full faithful embedding

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

of Proposition 3.3.11. However, we can say more in the case where X is a geometric stack: that is, a functor which admits a faithfully flat affine covering by a corepresentable functor (see Definition 3.4.1). In §3.4 we will formulate and prove Theorem 3.4.2, which asserts that when X is geometric, a symmetric monoidal functor $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ belongs to the essential image of θ if and only if F preserves small colimits, connective objects, and flat objects. It follows that the functor X is completely determined by the ∞-category $\mathrm{QCoh}(X)$ together with its symmetric monoidal structure and t-structure.

3.1 Local Properties of Morphisms

In §1.5, we studied properties of morphisms of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$ which can be tested locally on $\mathfrak{X}$. In this section, we will study properties which can be tested locally on $\mathfrak{Y}$. We begin by introducing some terminology.

Definition 3.1.1. Let P be a property of morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ between spectral Deligne-Mumford stacks. We will say that f is *local on the target with respect to the étale topology* if the following conditions are satisfied:

(1) Suppose we are given a pullback diagram of spectral Deligne-Mumford stacks

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r, "g"] & \mathfrak{Y} \end{tikzcd}$$

where g is étale. If f has the property P, then f' also has the property P.

(2) Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks, and suppose we are given a surjective étale morphism $\coprod_\alpha \mathfrak{Y}_\alpha \to \mathfrak{Y}$. If each of the induced maps $\mathfrak{Y}_\alpha \times_{\mathfrak{Y}} \mathfrak{X} \to \mathfrak{Y}_\alpha$ has the property P, then f has the property P.

Remark 3.1.2. Let P be a property of morphisms between spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology. Then a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ has the property P if and only if, for every étale map $u : \operatorname{Spec} R \to \mathfrak{Y}$, the pullback $\operatorname{Spec} R \times_{\mathfrak{Y}} \mathfrak{X} \to \operatorname{Spec} R$ has the property P.

Remark 3.1.3. Suppose we are given a property P_0 of morphisms of spectral Deligne-Mumford stacks having the form $\mathfrak{Z} \to \operatorname{Spec} R$. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be an arbitrary morphism of spectral Deligne-Mumford stacks. We will say that f *locally has the property* P_0 if, for every étale map $\operatorname{Spec} R \to \mathfrak{Y}$, the induced map $\operatorname{Spec} R \times_{\mathfrak{Y}} \mathfrak{X} \to \operatorname{Spec} R$ has the property P_0. Suppose that P_0 satisfies the following conditions:

(i) Let $f : \mathfrak{Z} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks and $u : R \to R'$ an étale morphism of $\mathbb{E}_\infty$-rings. If f has the property P_0, then the induced map $\operatorname{Spec} R' \times_{\operatorname{Spec} R} \mathfrak{Z} \to \operatorname{Spec} R$ has the property P_0. The converse holds if u is faithfully flat.

(ii) If we are given a finite collection of morphisms $\{\mathfrak{Z}_i \to \operatorname{Spec} R_i\}$ having the property P_0, then the induced map $\coprod \mathfrak{Z}_i \to \operatorname{Spec}(\prod R_i)$ has the property P_0.

Then the condition that a morphism f locally has the property P_0 is local on the target with respect to the étale topology, in the sense of Definition 3.1.1. Moreover, a morphism $f : \mathfrak{Z} \to \operatorname{Spec} R$ satisfies this condition if and only if f has the property P_0 (this follows immediately from (i)). Combining this observation with Remark 3.1.2, we obtain a bijective correspondence between the following:

(a) Properties P of arbitrary morphisms $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks, which are local on the target with respect to the étale topology.

(b) Properties P_0 of morphisms of the form $f : \mathfrak{Z} \to \operatorname{Spec} R$ which satisfy conditions (i) and (ii).

Example 3.1.4. The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be étale is local on the target with respect to the étale topology.

Example 3.1.5. The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be an equivalence is local on the target with respect to the étale topology.

Example 3.1.6. The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be an open immersion is local on the target with respect to the étale topology. This follows from Examples 3.1.4 and 3.1.5, since f is an open immersion if and only if f is étale and the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is an equivalence.

Example 3.1.7. The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be surjective is local on the target with respect to the étale topology (see Proposition 1.1.20).

Example 3.1.8. The condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be flat is local on the target with respect to the étale topology.

Example 3.1.9. For $0 \leq n \leq \infty$, the condition that a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be n-quasi-compact is local on the target with respect to the étale topology (Proposition 1.4.11).

Example 3.1.10. For every integer $n \geq 0$, the condition that a map of spectral Deligne-Mumford stacks $f : (\mathfrak{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathfrak{Y}, \mathcal{O}_{\mathcal{Y}})$ induce an equivalence $(\mathfrak{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \to (\mathfrak{Y}, \tau_{\leq n} \mathcal{O}_{\mathcal{Y}})$ is local on the target with respect ot the étale topology.

Definition 3.1.11. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks and let $n \geq -2$ be an integer. We will say that f is a *relative Deligne-Mumford n-stack* if, for every discrete commutative ring R, the induced map

$$\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X}) \to \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{Y})$$

is n-truncated.

Example 3.1.12. When $n = -2$, a map of spectral Deligne-Mumford stacks $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ is a relative Deligne-Mumford n-stack if and only if, for every discrete commutative ring R, the map

$$\mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, (\mathcal{X}, \mathcal{O}_{\mathcal{X}})) \to \mathrm{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}}))$$

is a homotopy equivalence. This is equivalent to the requirement that f induces an equivalence $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$.

Remark 3.1.13. The condition that a map $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ be a relative Deligne-Mumford n-stack depends only on the underlying map of 0-truncated spectral Deligne-Mumford stacks $(\mathcal{X}, \pi_0 \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \pi_0 \mathcal{O}_{\mathcal{Y}})$.

Remark 3.1.14. If $n \geq 0$, then a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ is a relative Deligne-Mumford n-stack if and only if, for every discrete commutative ring R and every map $u : \operatorname{Spec} R \to \mathfrak{Y}$, the pullback $\operatorname{Spec} R \times_{\mathfrak{Y}} \mathfrak{X}$ is a spectral Deligne-Mumford n-stack (Definition 1.3.1). Using Remark 3.1.13, we see that this is equivalent to assertion that for every connective $\mathbb{E}_\infty$-ring R and every map $u : \operatorname{Spec} R \to \mathfrak{Y}$, the pullback $\operatorname{Spec} R \times_{\mathfrak{Y}} \mathfrak{X}$ is a spectral Deligne-Mumford n-stack.

Proposition 3.1.15. *Let $n \geq -2$ be an integer. The condition that a map $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be a relative spectral Deligne-Mumford n-stack is local on the target with respect to the étale topology.*

Proof. The proof proceeds by induction on n. When $n = -2$, the desired result follows from Examples 2.7.12 and 3.1.9. If $n > -2$, we observe that $f : \mathfrak{X} \to \mathfrak{Y}$ is a relative spectral Deligne-Mumford n-stack if and only if the diagonal map $\mathfrak{X} \to \mathfrak{X} \times_{\mathfrak{Y}} \mathfrak{X}$ is a relative Deligne-Mumford $(n-1)$-stack. □

Proposition 3.1.16. *The condition that a map $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be affine is local on the target with respect to the étale topology.*

Proof. Using Remark 3.1.3, we are reduced to verifying the following assertion:

- $(*)$ Let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks, and suppose there exists a faithfully flat étale morphism $R \to R^0$ such that the fiber product $\mathfrak{X}_0 = \operatorname{Spec} R^0 \times_{\operatorname{Spec} R} \mathfrak{X}$ is affine. Then $\mathfrak{X}$ is affine.

To prove $(*)$, let $R^\bullet$ be the Čech nerve of the map $R \to R^0$ (in the ∞-category CAlg^{op}). For each $n \geq 0$, the fiber product $\operatorname{Spec} R^n \times_{\operatorname{Spec} R} \mathfrak{X}$ is an affine spectral Deligne-Mumford stack, of the form $\operatorname{Spec} A^n$ for some $\mathbb{E}_\infty$-ring A^n. Let A denote the totalization of the cosimplicial $\mathbb{E}_\infty$-ring $A^\bullet$. It follows from Theorem VII.6.12 that $A^\bullet$ is the Čech nerve of the morphism $A \to A^0 \simeq R^0 \otimes_R A$. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$, so that the simplicial spectral Deligne-Mumford stack $\mathfrak{X}_\bullet$ corresponds to a simplicial object $U_\bullet$ in $\mathcal{X}$, whose geometric realization is a final object $\mathbf{1} \in \mathcal{X}$. Then we have a chain of equivalences

$$\mathcal{O}_{\mathcal{X}}(\mathbf{1}) = \mathcal{O}_{\mathcal{X}}(|\mathbf{U}_\bullet|) \simeq \varprojlim \mathcal{O}_{\mathcal{X}}(\mathbf{U}_\bullet) \simeq \varprojlim \mathbf{A}^\bullet \simeq \mathbf{A}.$$

The composite equivalence determines a map $\theta : \mathfrak{X} \to \operatorname{Spec} A$. The map θ is an equivalence, since it can be obtained as the geometric realization of an equivalence of simplicial spectral Deligne-Mumford stacks $\mathfrak{X}_\bullet \simeq \operatorname{Spec} A^\bullet$. This proves that $\mathfrak{X}$ is affine, as desired. □

Proposition 3.1.17. *The condition that a map of spectral Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$ be quasi-affine is local on the target with respect to the étale topology.*

Proof. Using Remark 3.1.3, we are reduced to proving the following:

(∗) Let $f : \mathfrak{X} \to \operatorname{Spec} R$ be a map of spectral Deligne-Mumford stacks, and suppose there exists a faithfully flat étale morphism $R \to R^0$ such that the fiber product $\mathfrak{X}_0 = \operatorname{Spec} R^0 \times_{\operatorname{Spec} R} \mathfrak{X}$ is quasi-affine. Then $\mathfrak{X}$ is quasi-affine.

Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. We first claim that the pushforward $f_* \mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf on $\operatorname{Spec} R$. This assertion can be tested locally with respect to the étale topology on $\operatorname{Spec} R$, and therefore follows from Corollary 2.5.15. We can identify $f_* \mathcal{O}_{\mathcal{X}}$ with an $\mathbb{E}_\infty$-algebra over R. Let A denote the connective cover of this $\mathbb{E}_\infty$-algebra. The map $A \to f_* \mathcal{O}_{\mathcal{X}}$ classifies a map of spectral Deligne-Mumford stacks $g : \mathfrak{X} \to \operatorname{Spec} A$. We claim that g is a quasi-compact open immersion. Since this assertion is local on the target with respect to the étale topology (Examples 3.1.6 and 3.1.9), we may replace R by R' and thereby reduce to the case where $\mathfrak{X}$ is quasi-affine. In this case, the desired result follows from Proposition 2.4.3 (see the proof of Proposition 2.4.8). □

Recall that the ∞-category Stk of spectral Deligne-Mumford stacks can be identified with a full subcategory of $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ (the identification is given by carrying a spectral Deligne-Mumford stack $\mathfrak{X}$ to the functor given informally by the formula $R \mapsto \operatorname{Map}_{\mathrm{Stk}}(\operatorname{Spec} R, \mathfrak{X})$). In §3.1, we studied local properties of morphisms between Deligne-Mumford stacks $f : \mathfrak{X} \to \mathfrak{Y}$. Many of these properties can be defined more generally for natural transformations between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Definition 3.1.18. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. We will say that f is *representable* if, for every connective $\mathbb{E}_\infty$-ring R and every natural transformation $\operatorname{Spec}^{\mathrm{f}} R \to Y$ (corresponding to a choice of point $\eta \in Y(R)$), the fiber product $X \times_Y \operatorname{Spec}^{\mathrm{f}} R$ is representable by a spectral Deligne-Mumford stack.

Proposition 3.1.19. *Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Assume that Y is representable by a spectral Deligne-Mumford stack $\mathfrak{Y}$. Then f is representable (in the sense of Definition 3.1.18) if and only if X is representable by a spectral Deligne-Mumford stack.*

The proof will require the following general observation about sheaves.

Lemma 3.1.20. *Let $\mathcal{C}$ be a small ∞-category equipped with a Grothendieck topology, and let $\alpha : \mathcal{F} \to \mathcal{F}'$ be a natural transformation of functors $\mathcal{F}, \mathcal{F}' : \mathcal{C}^{op} \to \mathcal{S}$. For every object $C \in \mathcal{C}$ and every point $\eta \in \mathcal{F}'(C)$, we define a functor $\mathcal{F}_\eta : \mathcal{C}_{/C} \to \mathcal{S}$ by taking the fiber of the induced transformation $\mathcal{F} \,|\, \mathcal{C}_{/C} \to \mathcal{F}' \,|\, \mathcal{C}_{/C}$ (over the point determined by η).*

Assume that $\mathcal{F}'$ is a sheaf on $\mathcal{C}$. Then the following conditions are equivalent:

(1) *The functor $\mathcal{F}$ is a sheaf on $\mathcal{C}$.*

(2) *For every object $C \in \mathcal{C}$ and every point $\eta \in \mathcal{F}'(C)$, the functor $\mathcal{F}_\eta$ is a sheaf on $\mathcal{C}_{/C}$ (with respect to the induced Grothendieck topology).*

Moreover, if $\mathcal{F}'$ is hypercomplete, then $\mathcal{F}$ is hypercomplete if and only if each $\mathcal{F}_\eta$ is hypercomplete.

Proof. The implication (1) ⇒ (2) is obvious, since the full subcategory $\operatorname{Shv}(\mathcal{C}_{/C}) \subseteq \operatorname{Fun}(\mathcal{C}^{op}_{/C}, \mathcal{S})$ is closed under small limits. Suppose that (2) is satisfied. Fix an object $C \in \mathcal{C}$ and a covering sieve $\mathcal{C}^{(0)}_{/C} \subseteq \mathcal{C}_{/C}$;

we wish to prove that the canonical map $\theta : \mathcal{F}(C) \to \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(0)}_{/C}$ is a homotopy equivalence. We have a commutative diagram

$$\begin{tikzcd}
\mathcal{F}(C) \arrow[r, "\theta"] \arrow[d] & \varprojlim \mathcal{F} \,|\, \mathcal{C}^{(0)}_{/C} \arrow[d] \\
\mathcal{F}'(C) \arrow[r, "\theta'"] & \varprojlim \mathcal{F}' \,|\, \mathcal{C}^{(0)}_{/C},
\end{tikzcd}$$

where the map θ' is a homotopy equivalence. Consequently, to show that θ is a homotopy equivalence, it will suffice to show that θ induces a homotopy equivalence after passing to the homotopy fiber over any point $\eta \in \mathcal{F}'(C)$; this is precisely the content of assumption (2).

Now suppose that $\mathcal{F}'$ is hypercomplete. Since the collection of hypercomplete sheaves on $\mathcal{C}_{/C}$ is closed under limits, it is easy to see that $\mathcal{F}$ is hypercomplete only if each $\mathcal{F}_\eta$ is hypercomplete. Conversely, suppose that each $\mathcal{F}_\eta$ is hypercomplete; we wish to prove that $\mathcal{F}$ is hypercomplete. Choose an ∞-connective morphism $\beta : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is hypercomplete; we wish to prove that β induces an equivalence $\beta_C : \mathcal{F}(C) \to \mathcal{G}(C)$ for each $C \in \mathcal{C}$. Since $\mathcal{F}'$ is hypercomplete, the map α factors through β; it will therefore suffice to show that β_C induces a homotopy equivalence after passing to the homotopy fiber over every point $\eta \in \mathcal{F}'(C)$. For this, it suffices to show that the induced map $\beta_\eta : \mathcal{F}_\eta \to \mathcal{G}_\eta$ is an equivalence. This is clear, since β_η is ∞-connective and both $\mathcal{F}_\eta$ and $\mathcal{G}_\eta$ are hypercomplete objects of $\mathrm{Shv}(\mathcal{C}_{/C})$. □

Proof of Proposition 3.1.19. Let $\widehat{\mathrm{Shv}}_{\text{ét}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ spanned by those functors which are sheaves with respect to the étale topology. Since the ∞-category of spectral Deligne-Mumford stacks admits fiber products, it is clear that if X is representable by a spectral Deligne-Mumford stack, then f is representable. To prove the converse, write $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$. For every object $U \in \mathcal{Y}$, let Y_U denote the functor represented by the spectral Deligne-Mumford stack $\mathfrak{Y}_U = (\mathcal{Y}_{/U}, \mathcal{O}_{\mathcal{Y}} \,|U)$, and let $X_U = X \times_Y Y_U$. Let us say that U is *good* if X_U is representable by a spectral Deligne-Mumford stack $\mathfrak{X}_U$. Assuming that f is representable by spectral Deligne-Mumford stacks, we will show that every object $U \in \mathcal{Y}$. Our assumption immediately implies that every affine object $U \in \mathcal{Y}$ is good. It will therefore suffice to show that the collection of good objects of $\mathcal{Y}$ is closed under small colimits (Lemma V.2.3.11). To this end, suppose we are given a diagram of object $\{U_\alpha\}$ in $\mathcal{Y}$ having a colimit U, and that each X_{U_α} is representable by a spectral Deligne-Mumford stack $\mathfrak{X}_{U_\alpha}$. Note that for every morphism $U_\alpha \to U_\beta$ in our diagram, the induced map $\mathfrak{X}_{U_\alpha} \to \mathfrak{X}_{U_\beta}$ is étale (since it is a pullback of the étale morphism $\mathfrak{Y}_{U_\alpha} \to \mathfrak{Y}_{U_\beta}$). It follows from Proposition V.2.3.10 that the diagram $\{\mathfrak{X}_{U_\alpha}\}$ has a colimit $\mathfrak{X}_U$ in the ∞-category Stk. Moreover, $\mathfrak{X}_U$ represents a functor $F : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which is the colimit of the diagram $\{X_{U_\alpha}\}$ in the ∞-category $\widehat{\mathrm{Shv}}_{\text{ét}}$ spanned by the étale sheaves (Lemma V.2.4.13). To prove that U is good, it will suffice to show that $F \simeq X_U$: that is, that X_U is the colimit of the diagram $\{X_{U_\alpha}\}$ in $\widehat{\mathrm{Shv}}_{\text{ét}}$. Since colimits in $\widehat{\mathrm{Shv}}_{\text{ét}}$ are universal, we are reduced to proving the following pair of assertions:

(a) The functor X is a sheaf with respect to the étale topology.

(b) The functor Y_U is a colimit of the diagram $\{Y_{U_\alpha}\}$ in $\widehat{\mathrm{Shv}}_{\text{ét}}$.

Assertion (a) follows from Lemma 3.1.20, and assertion (b) follows from Lemma V.2.4.13. □

Corollary 3.1.21. *Suppose we are given natural transformations $X \xrightarrow{f} Y \xrightarrow{g} Z$ of functors $X, Y, Z : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and assume that g is representable. Then f is representable if and only if $g \circ f$ is representable.*

Proof. Without loss of generality, we may assume that Z is corepresentable by a connective $\mathbb{E}_\infty$-ring R. Then Y is representable by a spectral Deligne-Mumford stack $\mathfrak{Y}$. The desired equivalence now follows immediately from Proposition 3.1.19. □

Definition 3.1.22. Let P be a property of morphisms of spectral Deligne-Mumford stacks. We will say that P is *stable under base change* if, for every pullback diagram of spectral Deligne-Mumford stacks

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \mathfrak{Y}' \arrow[r] & \mathfrak{Y} \end{tikzcd}$$

such that f has the property P, the morphism f' also has the property P.

Remark 3.1.23. Let P be a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology. Then P is stable under base change if and only if, for every pullback diagram

$$\begin{tikzcd} \mathfrak{X}' \arrow[r] \arrow[d, "f'"] & \mathfrak{X} \arrow[d, "f"] \\ \operatorname{Spec} R' \arrow[r] & \operatorname{Spec} R \end{tikzcd}$$

such that f has the property P, the morphism f' also has the property P.

Definition 3.1.24. Let P be a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology and stable under base change. Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. We will say that f *has the property* P if, for every connective $\mathbb{E}_\infty$-ring R and every natural transformation $\operatorname{Spec}^{\mathrm{f}} R \to Y$ (determined by a point $\eta \in Y(R)$), the fiber product $\operatorname{Spec}^{\mathrm{f}} R \times_Y X$ is representable by a spectral Deligne-Mumford stack $\mathfrak{X}_\eta$ such that the induced map $\mathfrak{X}_\eta \to \operatorname{Spec} R$ has the property P.

Remark 3.1.25. In the situation of Definition 3.1.24, the natural transformation $f : X \to Y$ has the property P if and only if the following apparently stronger condition holds: for every pullback diagram of functors

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y, \end{tikzcd}$$

if Y' is representable by a spectral Deligne-Mumford stack $\mathfrak{Y}$ (so that X' is representable by a spectral Deligne-Mumford stack $\mathfrak{X}$, by virtue of Proposition 3.1.19), the induced map $\mathfrak{X} \to \mathfrak{Y}$ has the property P.

Remark 3.1.26. Let P a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology and stable under base change. Let $\phi : \mathfrak{X} \to \mathfrak{Y}$ be a morphism of spectral Deligne-Mumford stacks, let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functors represented by $\mathfrak{X}$ and $\mathfrak{Y}$, and let $f : X \to Y$ be the natural transformation determined by ϕ. Then f has the property P if and only if ϕ has the property P.

Example 3.1.27. The following conditions on a morphism $f : \mathfrak{X} \to \mathfrak{Y}$ are stable under base change (and local on the target with respect to the étale topology, as we saw in §3.1):

(1) The condition that f is étale.

(2) The condition that f is an equivalence.

(3) The condition that f is an open immersion.

(4) The condition that f is a flat.

(5) The condition that f is a relative Deligne-Mumford n-stack, where $n \geq -2$ is some fixed integer (see Definition 3.1.11).

(6) The condition that $f : (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ induces an equivalence $(\mathcal{X}, \tau_{\leq n} \mathcal{O}_{\mathcal{X}}) \to (\mathcal{Y}, \tau_{\leq n} \mathcal{O}_{\mathcal{Y}})$, where $n \geq 0$ is some fixed integer.

(7) The condition that f is surjective.

(8) The condition that f is affine.

(9) The condition that f is quasi-affine.

Consequently, we make make sense of each of these conditions for an arbitrary representable morphism $f : X \to Y$ between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Definition 3.1.28. Let $f : X \to Y$ be a representable morphism of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that f is *faithfully flat* if it is flat and surjective (in the sense of Example 3.1.27).

Remark 3.1.29. Let P be a property of morphisms of spectral Deligne-Mumford stacks which is local on the target with respect to the étale topology and stable under base change. Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, and suppose that Y is given as the colimit of a diagram $\{Y_\alpha\}$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Then f is representable and has the property P if and only if each of the induced maps $X \times_Y Y_\alpha \to Y_\alpha$ satisfies the same conditions. The "only if" direction is obvious, and the converse follows from the observation that every map map $\mathrm{Spec}\, A \to Y$ factors through some Y_α.

3.2 Quasi-Affine Morphisms of Functors

Let $f : X \to Y$ be a representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. Our goal in this section is to show that if f is quasi-affine, then there is a close relationship between the ∞-categories $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$. We begin with some general remarks about adjunctions between symmetric monoidal ∞-categories.

Proposition 3.2.1. *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories, and suppose that the underlying functor $f : \mathcal{C} \to \mathcal{D}$ admits a right adjoint g. Then g extends to a map of ∞-operads $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$.*

Proof. Consider the diagram

$$\begin{tikzcd} \mathcal{C}^{\otimes} \arrow[rr, "F"] \arrow[dr, "p"'] & & \mathcal{D}^{\otimes} \arrow[dl] \\ & \mathrm{N}(\mathcal{F}\mathrm{in}_*). & \end{tikzcd}$$

For every object $\langle n \rangle \in \mathcal{F}\mathrm{in}_*$, the induced map $\mathcal{C}^{\otimes}_{\langle n \rangle} \to \mathcal{D}^{\otimes}_{\langle n \rangle}$ can be identified with $f^n : \mathcal{C}^n \to \mathcal{D}^n$, and therefore admits a right adjoint $g^n : \mathcal{D}^n \to \mathcal{C}^n$. Since F carries p-coCartesian morphisms to q-coCartsesian morphisms, Proposition A.7.3.2.6 guarantees the existence of a functor $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ which is a right adjoint of F relative to $\mathrm{N}(\mathcal{F}\mathrm{in}_*)$. In particular, $G|\,\mathcal{D}$ is a right adjoint to f and we may therefore assume that $G|\,\mathcal{D} = g$. To see that G is a map of ∞-operads, it suffices to observe that for every injection $\langle m \rangle^{\circ} \hookrightarrow \langle n \rangle^{\circ}$, the diagram

$$\begin{tikzcd} \mathcal{C}^n \arrow[r] \arrow[d] & \mathcal{D}^n \arrow[d] \\ \mathcal{C}^m \arrow[r] & \mathcal{D}^m \end{tikzcd}$$

is right adjointable. $\square$

Remark 3.2.2. In the situation of Proposition 3.2.1, F and G determine adjoint functors

$$\mathrm{CAlg}(\mathcal{C}) \rightleftarrows \mathrm{CAlg}(\mathcal{D}).$$

Corollary 3.2.3. *Let $F : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ be a symmetric monoidal functor between symmetric monoidal ∞-categories, and suppose that the underlying functor $f : \mathcal{C} \to \mathcal{D}$ admits a right adjoint g; let $G : \mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ be the resulting map of ∞-operads. Then:*

(1) *If $\mathbf{1}$ denotes the unit object of $\mathcal{D}$, then $A = g(\mathbf{1})$ has the structure of a commutative algebra object of $\mathcal{C}$.*

(2) *The functor G factors as a composition*

$$\mathcal{D}^{\otimes} \simeq \mathrm{Mod}_{\mathbf{1}}(\mathcal{D})^{\otimes} \to \mathrm{Mod}_{A}(\mathcal{C})^{\otimes} \to \mathcal{C}^{\otimes}.$$

Example 3.2.4. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Suppose that the pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint f_*. Then:

(1) The pushforward $f_* \mathcal{O}_X$ of the structure sheaf of X has the structure of a commutative algebra object of $\mathrm{QCoh}(Y)$.

(2) The functor f_* induces a map of ∞-operads $\mathrm{QCoh}(X)^{\otimes} \to \mathrm{Mod}_{f_* \mathcal{O}_X}(\mathrm{QCoh}(Y))^{\otimes}$.

The existence of the right adjoint f_* follows from Corollary T.5.5.2.9 if $\mathrm{QCoh}(Y)$ is presentable (for example, if X is representable by a spectral Deligne-Mumford stack), since f^* preserves small colimits.

Proposition 3.2.5. *Let $f : X \to Y$ be a quasi-affine representable morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Then the pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint f_*, which induces an equivalence of (symmetric monoidal) ∞-categories*

$$\mathrm{QCoh}(X) \to \mathrm{Mod}_{f_* \mathcal{O}_X}(\mathrm{QCoh}(Y)).$$

Proof. When Y is a corepresentable functor, the desired result follows from Corollaries 2.5.15 and 2.5.16. We now treat the general case. Write Y as the colimit of a diagram $q : S \to \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$, where each $q(s) \simeq \mathrm{Spec}\, A_s$ is affine. Since f is representable by quasi-affine spectral Deligne-Mumford stacks, each of the fiber products $X \times_Y q(s)$ is representable by a spectral Deligne-Mumford stack $\mathfrak{X}_s$. Every edge $s \to s'$ in S determines a pullback diagram

$$\begin{tikzcd} \mathfrak{X}_s \arrow[r] \arrow[d] & \mathfrak{X}_{s'} \arrow[d] \\ \mathrm{Spec}\, A_s \arrow[r] & \mathrm{Spec}\, A_{s'}. \end{tikzcd}$$

Using Corollary 2.5.15, we conclude that the diagram of functors

$$\begin{tikzcd} \mathrm{QCoh}(\mathrm{Spec}\, A_{s'}) \arrow[r] \arrow[d] & \mathrm{QCoh}(\mathfrak{X}_{s'}) \arrow[d] \\ \mathrm{QCoh}(\mathrm{Spec}\, A_s) \arrow[r] & \mathrm{QCoh}(\mathfrak{X}_s) \end{tikzcd}$$

is right adjointable. Since $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(\mathfrak{X}_s)$ and $\mathrm{QCoh}(Y) \simeq \varprojlim \mathrm{QCoh}(\mathrm{Spec}\, A_s)$, Corollary A.6.2.3.18 implies the following:

(a) The functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ admits a right adjoint.

(b) For each $s \in S$, the diagram

$$\begin{tikzcd} \mathrm{QCoh}(Y) \arrow[r, "f^*"] \arrow[d] & \mathrm{QCoh}(X) \arrow[d] \\ \mathrm{QCoh}(\operatorname{Spec} A_s) \arrow[r, "f_s^*"] & \mathrm{QCoh}(\mathfrak{X}_s) \end{tikzcd}$$

is right adjointable.

In particular, we deduce that $f_* \mathcal{O}_X$ is a quasi-coherent sheaf on Y whose restriction to each $\operatorname{Spec} A_s$ is given by $B_s = (f_s)_* \mathcal{O}_s$, where $\mathcal{O}_s$ denotes the structure sheaf of $\mathfrak{X}_s$. It follows that the functor $\mathrm{QCoh}(X) \to \mathrm{Mod}_{f_* \mathcal{O}_X}(\mathrm{QCoh}(Y))$ is given by a limit of equivalences $\mathrm{QCoh}(\mathfrak{X}_s) \to \mathrm{Mod}_{B_s}(\mathrm{QCoh}(\operatorname{Spec} A_s))$, and therefore an equivalence.

$\square$

Corollary 3.2.6. *Suppose we are given a pullback diagram*

$$\begin{tikzcd} X & Y \arrow[l, "f"'] \\ X' \arrow[u, "g"] & Y' \arrow[l, "f'"] \arrow[u, "g'"'] \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$*, where f is representable and quasi-affine. Then:*

(1) *The map f' is representable and quasi-affine.*

(2) *The diagram of ∞-categories*

$$\begin{tikzcd} \mathrm{QCoh}(X) \arrow[r, "f^*"] \arrow[d, "g^*"] & \mathrm{QCoh}(Y) \arrow[d, "g'^*"] \\ \mathrm{QCoh}(X') \arrow[r, "f'^*"] & \mathrm{QCoh}(Y') \end{tikzcd}$$

is right adjointable.

(3) *Assume that* $\mathrm{QCoh}(X)$ *and* $\mathrm{QCoh}(X')$ *are presentable. Then the diagram*

$$\begin{tikzcd} \mathrm{QCoh}(X)^{\otimes} \arrow[r] \arrow[d] & \mathrm{QCoh}(Y)^{\otimes} \arrow[d] \\ \mathrm{QCoh}(X')^{\otimes} \arrow[r] & \mathrm{QCoh}(Y')^{\otimes} \end{tikzcd}$$

is a pushout square in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})$.

Proof. Assertion (1) is obvious. If X' is affine, then (2) follows from the proof of Proposition 3.2.5. To prove (2) in general, we must show that the natural map

$$\alpha : g^* f_* \mathcal{F} \to f'_* g'^* \mathcal{F}$$

is an equivalence for every quasi-coherent sheaf $\mathcal{F}$ on Y. It will suffice to show that $h^*(\alpha)$ is an equivalence for every map $h : \operatorname{Spec} A \to X'$. Let $h' : \operatorname{Spec} A \times_{X'} Y' \to Y'$ and $f'' : \operatorname{Spec} A \times_{X'} Y' \to \operatorname{Spec} A$ be the projection maps. Since $\operatorname{Spec} A$ is affine, we deduce that the natural map $h^* f'_* \simeq f''_* h'^*$ is an equivalence; thus $h^*(\alpha)$ can be identified with the base-change map $(g \circ h)^* f_* \mathcal{F} \to f''_* (g' \circ h')^* \mathcal{F}$. We may therefore replace g by $g \circ h$ and thereby reduce to the case where X' is affine.

It remains to prove (3). Let $\mathcal{A} = f_* \mathcal{O}_Y$ and $\mathcal{A}' = f'_* \mathcal{O}_{Y'}$. Using (2), we see that the natural map $g^* \mathcal{A} \to \mathcal{A}'$ is an equivalence of commutative algebra objects of $\mathrm{QCoh}(X')$. Proposition 3.2.5 yields equivalences

$$\mathrm{QCoh}(Y) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X)) \qquad \mathrm{QCoh}(Y') \simeq \mathrm{Mod}_{\mathcal{A}'}(\mathrm{QCoh}(X')) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X')).$$

The desired result now follows from the equivalence

$$\mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X')) \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X)) \otimes_{\mathrm{QCoh}(X)} \mathrm{QCoh}(X')$$

provided by Theorem A.6.3.4.6. □

For every functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, let Aff_X denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}$ spanned by those maps $f : Y \to X$ which are representable and affine. The construction $X \mapsto \mathrm{Aff}_X$ determines a functor $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ which commutes with limits, and is therefore a right Kan extension of the composite functor

$$\mathrm{CAlg}^{\mathrm{cn}} \xrightarrow{j} \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \xrightarrow{\mathrm{Aff}} \widehat{\mathcal{C}\mathrm{at}}_\infty$$

along the Yoneda embedding $j : \mathrm{CAlg}^{\mathrm{cn}} \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$. We observe that this composite functor is given by the formula $R \mapsto (\mathrm{CAlg}^{\mathrm{cn}}_R)^{op}$. We therefore obtain an equivalence of ∞-categories $\mathrm{Aff}_X^{op} \simeq \mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$, depending functorially on X. If $\mathcal{A}$ is a commutative algebra object of $\mathrm{QCoh}(X)^{\mathrm{cn}}$, we let $\mathrm{Spec}_X \mathcal{A}$ denote the image of $\mathcal{A}$ under this equivalence; we will refer to $\mathrm{Spec}_X \mathcal{A}$ as the *spectrum of* $\mathcal{A}$ *relative to* X.

If $\mathfrak{X}$ is a spectral Deligne-Mumford stack representing a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and $\mathcal{A}$ is a connective object of $\mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{X}))$, then we can identify A with an object of $\mathrm{CAlg}(\mathrm{QCoh}(X)^{\mathrm{cn}})$. The functor $\mathrm{Spec}_X \mathcal{A}$ is representable by a spectral Deligne-Mumford stack which is affine over $\mathfrak{X}$; we will denote this spectral Deligne-Mumford stack by $\mathrm{Spec}_{\mathfrak{X}} \mathcal{A}$.

Variant 3.2.7. For any functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$, let QAff_X denote the full subcategory of

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}$$

spanned by those natural transformations $f : Y \to X$ which are representable and quasi-affine. The ∞-category QAff_X contains Aff_X as a full subcategory.

Lemma 3.2.8. *Suppose we are given natural transformations $f : Y \to X$, $g : Z \to X$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Assume that f is representable and quasi-affine, and let $\mathcal{A} = f_* \mathcal{O}_Y \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Then the canonical map*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}}(Z, Y) \to \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(Z))}(g^* \mathcal{A}, \mathcal{O}_Z)$$

is a homotopy equivalence.

Proof. When regarded as functors of Z, both the domain and codomain of θ carry colimits of functors to limits of spaces. Writing Z as a colimit of corepresentable functors, we may reduce to the case where Z is corepresentable by a connective $\mathbb{E}_\infty$-ring R. Replacing X by Z and Y by the fiber product $Y \times_X Z$, we may reduce to the case where Y is representable by a quasi-affine spectral Deligne-Mumford stack $\mathfrak{Y}$ equipped with a map $\mathfrak{Y} \to \mathrm{Spec}\, R$. In this case, the desired result is a consequence of Proposition 2.4.9. □

Proposition 3.2.9. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor, and assume that the ∞-category $\mathrm{QCoh}(X)$ is presentable. For every object $f : Y \to X$ of QAff_X, the ∞-category $\mathrm{QCoh}(Y)$ is presentable. The construction $(f : Y \to X) \mapsto \mathrm{QCoh}(Y)^{\otimes}$ determines a fully faithful embedding $\mathrm{QAff}_X \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{QCoh}(X)/}$.*

Proof. For every map $f : Y \to X$ which is representable by quasi-affine spectral Deligne-Mumford stacks, let $\mathcal{A}_Y = f_* \mathcal{O}_X \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Proposition 3.2.5 supplies an equivalence $\mathrm{QCoh}(Y) \simeq \mathrm{Mod}_{\mathcal{A}_Y}(\mathrm{QCoh}(X))$. Since $\mathrm{QCoh}(X)$ is assumed to be presentable, we deduce that $\mathrm{QCoh}(Y)$ is presentable. If we are given another map $g : Z \to X$ which is representable and quasi-affine, then Corollary A.6.3.5.18 supplies a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{QCoh}(X)/}}(\mathrm{QCoh}(Y), \mathrm{QCoh}(Z)) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(X))}(\mathcal{A}_Y, \mathcal{A}_Z) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathrm{QCoh}(Z))}(g^* \mathcal{A}_Y, \mathcal{O}_Z).$$

The desired result now follows from Lemma 3.2.8. □

Remark 3.2.10. Suppose we are given maps of functors $f : Y \to X$ and $g : Z \to X$ which are representable by quasi-affine spectral Deligne-Mumford stacks. Proposition 3.2.9 implies that if $F : \mathrm{QCoh}(Z) \to \mathrm{QCoh}(Y)$ is a $\mathrm{QCoh}(X)$-linear symmetric monoidal functor which preserves small colimits, then F is given by pullback along some map $h : Y \to Z$ such that $h \circ f \simeq g$. Moreover, if we are given two such maps $h, h' : Y \to Z$, then any $\mathrm{QCoh}(X)$-linear symmetric monoidal equivalence α between h^* and h'^* can be lifted to a homotopy between h and h'. In fact, the assumption that α is an equivalence is superfluous: any $\mathrm{QCoh}(X)$-linear symmetric monoidal functor from h^* to h'^* is automatically an equivalence (Remark A.6.3.5.8).

Proposition 3.2.9 gives in particular a fully faithful embedding

$$\mathrm{Aff}_X \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{QCoh}(X)/}.$$

Using Corollary A.6.3.5.18, we can describe the essential image of this embedding:

Proposition 3.2.11. *Let $Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor for which the ∞-category $\mathrm{QCoh}(Y)$ is presentable, and let Aff_Y be defined as above. Then the construction*

$$(f : X \to Y) \mapsto \mathrm{QCoh}(X)$$

determines a fully faithful embedding $\mathrm{Aff}_Y^{op} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{QCoh}(Y)/}$. Moreover, a symmetric monoidal functor $f^* : \mathrm{QCoh}(Y) \to \mathcal{C}$ lies in the essential image of θ if and only if the following conditions are satisfied:

- (a) *The ∞-category $\mathcal{C}$ is presentable, the tensor product on $\mathcal{C}$ preserves colimits separately in each variable.*
- (b) *The functor f^* admits a right adjoint f_*.*
- (c) *The functor f_* is conservative and preserves geometric realizations of simplicial objects.*
- (d) *For every quasi-coherent sheaf $\mathcal{F}$ on X and every object $C \in \mathcal{C}$, the canonical map $\mathcal{F} \otimes f_*(C) \to f_*(f^* \mathcal{F} \otimes C)$ is an equivalence in $\mathrm{QCoh}(Y)$.*
- (e) *The object $f_* \mathcal{O}_X \in \mathrm{QCoh}(X)$ is connective, where $\mathbf{1}$ denotes the unit object of $\mathcal{C}$.*

Proof. In view of Proposition 3.2.9, it will suffice to show that if $f^* : \mathrm{QCoh}(X) \to \mathcal{C}$ is a symmetric monoidal functor satisfying conditions (a) through (e), then $\mathcal{C}$ has the form $\mathrm{QCoh}(Y)$ for some map $f : Y \to X$ which is representable and affine. Equivalently, we must show that there is a symmetric monoidal equivalence $\mathcal{C} \simeq \mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(X))$, for some connective object $\mathcal{A} \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. The existence of $\mathcal{A}$ follows from conditions (a) through (d), together with Corollary A.6.3.5.18. In this case, the commutative algebra $\mathcal{A}$ is given by $f_* \mathbf{1}$, so that the connectivity of $\mathcal{A}$ follows from condition (e). □

Remark 3.2.12. In the situation of Proposition 3.2.11, we can replace (c) by the following apparently stronger condition:

- (c') The functor f_* is conservative and preserves small colimits.

3.3 Quasi-Geometric Stacks

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. Our goal in §3.4) is to address the following general question:

($*$) To what extent is the functor X determined by the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X?

In this section, we consider a slightly easier question:

($*'$) Let $f : X \to Y$ be a morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. To what extent is f determines by the pullback functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$?

Our main result is Proposition 3.3.11, which asserts that f can be recovered from f^* (as a symmetric monoidal functor) provided that the functor Y is *quasi-geometric* in the sense of the following definition:

Definition 3.3.1. Let $f : X \to Y$ be a natural transformation between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. We will say that f is *quasi-geometric* if the diagonal map $X \to X \times_Y X$ is representable and quasi-affine.

We will say that a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is *quasi-geometric* if the projection map $X \to *$ is quasi-geometric, where $*$ denotes a final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$.

Warning 3.3.2. The definition of a quasi-geometric morphism $f : X \to Y$ does not require that f be representable.

The class of quasi-geometric morphisms enjoys the following closure property:

Proposition 3.3.3. *Suppose we are given a commutative diagram*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$*, and assume that g is quasi-geometric. Then f is quasi-geometric if and only if h is quasi-geometric.*

To prove Proposition 3.3.3, we first need to establish the analogous assertion for the class of quasi-affine morphisms. This will require a few preliminary observations.

Lemma 3.3.4. *Let $f : \mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}}) \to \mathfrak{Y}$ be a quasi-affine morphism of spectral Deligne-Mumford stacks, and let $\mathcal{A} = \tau_{\geq 0} f_* \mathcal{O}_{\mathcal{X}}$. Then the canonical map $\mathfrak{X} \to \mathrm{Spec}_{\mathfrak{Y}} \mathcal{A}$ is a quasi-compact open immersion.*

Proof. The assertion is local on $\mathfrak{Y}$. We may therefore assume that $\mathfrak{Y}$ is affine, in which case the desired result follows from Proposition 2.4.3 (and the proof of Proposition 2.4.8). $\square$

Lemma 3.3.5. *Let $\mathfrak{Y}$ be a spectral Deligne-Mumford stack and let $\mathcal{A} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathfrak{Y})^{\mathrm{cn}})$. If $\mathfrak{Y}$ is quasi-affine, then $\mathrm{Spec}_{\mathfrak{Y}} \mathcal{A}$ is quasi-affine.*

Proof. Write $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$ and let $\mathbf{1}$ denote a final object of $\mathcal{Y}$. Let $R = \mathcal{O}_{\mathcal{Y}}(\mathbf{1})$ and $A = \mathcal{A}(\mathbf{1})$. Proposition 2.4.3 implies that the canonical map $f_0 : \mathfrak{Y} \to \mathrm{Spec}\, R$ is a quasi-compact open immersion (of nonconnective spectral Deligne-Mumford stacks). We claim that

$$\begin{tikzcd} \mathrm{Spec}_{\mathfrak{Y}} \mathcal{A} \arrow[r, "f"] \arrow[d] & \mathrm{Spec}\, A \arrow[d] \\ \mathfrak{Y} \arrow[r, "f_0"] & \mathrm{Spec}\, R \end{tikzcd}$$

is a pullback square of nonconnective spectral Deligne-Mumford stacks. To prove this, it suffices to show that the canonical map $f_0^* A \to \mathcal{A}$ is an equivalence in $\mathrm{QCoh}(\mathfrak{Y})$ (here we identify A with the corresponding quasi-coherent sheaf on $\operatorname{Spec} R$). This follows from Corollary 2.5.16. It follows that f is a quasi-compact open immersion, so that $\operatorname{Spec}_{\mathfrak{Y}} \mathcal{A}$ is quasi-affine as desired. □

Proposition 3.3.6. *Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. Suppose that $\mathfrak{Y}$ is quasi-affine and that f is quasi-affine. Then $\mathfrak{X}$ is quasi-affine.*

Proof. Write $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$. Since f is quasi-affine, the pushforward $f_* \mathcal{O}_{\mathcal{X}}$ is a quasi-coherent sheaf of $\mathbb{E}_\infty$-rings on $\mathfrak{Y}$. Let $\mathcal{A} = \tau_{\geq 0} f_* \mathcal{O}_{\mathcal{X}}$ be its connective cover, and let $\mathfrak{X}' = \operatorname{Spec}_{\mathfrak{Y}} \mathcal{A}$. Lemma 3.3.4 implies that the canonical map $\mathfrak{X} \to \mathfrak{X}'$ is a quasi-compact open immersion. It will therefore suffice to show that $\mathfrak{X}'$ is quasi-affine, which follows from Lemma 3.3.5. □

Proposition 3.3.7. *Suppose we are given a commutative diagram*

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$*, and assume that g is representable and quasi-affine. Then f is representable and quasi-affine if and only if h is representable and quasi-affine.*

Proof. Suppose first that f is representable and quasi-affine. Choose a point $\eta \in Z(R)$, classifying a natural transformation $\operatorname{Spec}^{\mathrm{f}} R \to Z$. Our hypothesis on g implies that $Y \times_Z \operatorname{Spec}^{\mathrm{f}} R$ is representable by a quasi-affine spectral Deligne-Mumford stack $\mathfrak{Y}$. Our hypothesis on f guarantees that $X \times_Z \operatorname{Spec}^{\mathrm{f}} R$ is representable by a spectral Deligne-Mumford stack $\mathfrak{X}$, and that the map $\mathfrak{X} \to \mathfrak{Y}$ is quasi-affine. Applying Proposition 3.3.6, we deduce that $\mathfrak{X}$ is quasi-affine.

Now suppose that h is representable by quasi-affine spectral Deligne-Mumford stacks; we wish to show that f has the same property. Choose a point $\eta \in Y(R)$, classifying a map $\operatorname{Spec}^{\mathrm{f}} R \to Y$. We wish to show that $X \times_Y \operatorname{Spec}^{\mathrm{f}} R$ is representable by a quasi-affine spectral Deligne-Mumford stack. Our hypotheses on g and h imply that the fiber products $X \times_Z \operatorname{Spec}^{\mathrm{f}} R$ and $Y \times_Z \operatorname{Spec}^{\mathrm{f}} R$ are representable by quasi-affine spectral Deligne-Mumford stacks $\mathfrak{X}$ and $\mathfrak{Y}$. Then $X \times_Y \operatorname{Spec}^{\mathrm{f}} R$ is representable by the fiber product $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec} R$. Since $\mathfrak{Y}$ is quasi-affine, the map $\operatorname{Spec} R \to \mathfrak{Y}$ is affine, so the projection map $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec} R \to \mathfrak{X}$ is affine. Invoking Proposition 3.3.6, we deduce that $\mathfrak{X} \times_{\mathfrak{Y}} \operatorname{Spec} R$ is quasi-affine, as desired. □

Proof of Proposition 3.3.3. The diagonal map $\delta : X \to X \times_Z X$ factors as a composition

$$X \xrightarrow{\delta'} X \times_Y X \xrightarrow{\delta''} X \times_Z X.$$

Here δ'' is a pullback of the diagonal map $Y \to Y \times_Z Y$. Our assumption on g guarantees that δ'' is representable and quasi-affine. Invoking Proposition 3.3.7, we deduce that δ' is representable and quasi-affine if and only if δ is representable and quasi-affine. □

Definition 3.3.8. Let $f : \mathfrak{X} \to \mathfrak{Y}$ be a map of spectral Deligne-Mumford stacks. We will say that f is *quasi-geometric* if it induces a quasi-geometric morphism $X \to Y$ of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. We will say that $\mathfrak{X}$ is *quasi-geometric* if the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ represented by $\mathfrak{X}$ is quasi-geometric. In other words, $\mathfrak{X}$ is quasi-geometric if the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is quasi-affine.

Remark 3.3.9. The condition that a map $f : \mathfrak{X} \to \mathfrak{Y}$ of spectral Deligne-Mumford stacks be quasi-geometric is local on the target with respect to the étale topology and stable under base change. This follows immediately from the corresponding assertion for quasi-affine morphisms (Example 3.1.27).

Proposition 3.3.10. *Let $\mathfrak{X}$ be a quasi-geometric spectral Deligne-Mumford stack, and let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be the functor represented by $\mathfrak{X}$. Then X is a hypercomplete sheaf with respect to the flat topology on* $\mathrm{CAlg}^{\mathrm{cn}}$.

Proof. Let $\widehat{\mathrm{Shv}}^{\wedge}_{\mathrm{fpqc}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors which are hypercomplete sheaves with respect to the flat topology. The inclusion $\widehat{\mathrm{Shv}}^{\wedge}_{\mathrm{fpqc}} \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}, \widehat{\mathcal{S}})$ admits a left exact left adjoint, which we will denote by L. Let $Y = LX \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. We would like to show that the unit map $X \to Y$ is an equivalence. We first claim that for every connective $\mathbb{E}_\infty$-ring R, the map $X(R) \to Y(R)$ is (-1)-truncated: that is, it exhibits $X(R)$ as a summand of $Y(R)$. To prove this, it will suffice to show that for any pair of points $x, x' \in X(R)$, the induced map $\theta : \{x\} \times_{X(R)} \{x'\} \to \{x\} \times_{Y(R)} \{x'\}$ is a homotopy equivalence. We note that x and x' determine a pair of maps from $\operatorname{Spec} R$ to $\mathfrak{X}$. Let $\mathfrak{X}'$ denote the fiber product $\operatorname{Spec} R \times_{\mathfrak{X}} \operatorname{Spec} R$ and let X' be the functor represented by $\mathfrak{X}'$. To prove that θ is a homotopy equivalence, it will suffice to show that the canonical map $\beta : X' \to \operatorname{Spec}^{\mathrm{f}} R \times_Y \operatorname{Spec}^{\mathrm{f}} R$ is an equivalence. Since the functor L is left exact, β induces an equivalence $LX' \simeq \operatorname{Spec}^{\mathrm{f}} R \times_Y \operatorname{Spec}^{\mathrm{f}} R$. It will therefore suffice to show that X' is a hypercomplete sheaf with respect to the flat topology. Since $\mathfrak{X}$ is quasi-geometric, $\mathfrak{X}'$ is quasi-affine. The desired result now follows from Proposition 2.4.10.

Note that X and Y are both sheaves with respect to the étale topology on $\mathrm{CAlg}^{\mathrm{cn}}$. To complete the proof that the unit map $X \to Y$ is an equivalence, it will suffice to show that it is an effective epimorphism with respect to the étale topology. Choose a point $\eta \in Y(R)$ for some connective $\mathbb{E}_\infty$-ring R. For every map of connective $\mathbb{E}_\infty$-rings $R \to A$, let η_A denote the image of η in $Y(A)$. We wish to prove that there exists a faithfully flat étale map $R \to R'$ such that $\eta_{R'}$ belongs to the essential image of the map $X(R') \to Y(R')$.

Since $Y = LX$, there exists finite collection of flat maps $R \to A_\alpha$ such that the induced map $R \to \prod_\alpha A_\alpha$ is faithfully flat, and each η_{A_α} belongs to the essential image of the map $X(A_\alpha) \to Y(A_\alpha)$. Let $A^0 = \prod_\alpha A_\alpha$, and let $A^\bullet$ be the cosimplicial object of $\mathrm{CAlg}^{\mathrm{cn}}$ given by the Čech nerve of the map $R \to A^0$. For every integer n, the point η_{A^n} belongs to the essential image of the fully faithful embedding $X(A^n) \to Y(A^n)$, and therefore classifies a map of spectral Deligne-Mumford stacks $\phi_n : \operatorname{Spec} A^n \to \mathfrak{X}$.

Choose a quasi-compact open substack $\mathfrak{X}' \subseteq \mathfrak{X}$ such that ϕ_0 factors through $\mathfrak{X}'$. Since $\mathfrak{X}'$ is quasi-compact, we can choose an étale surjection $\mathfrak{U} \to \mathfrak{X}'$, where $\mathfrak{U}$ is affine. For each $n \geq 0$, let $\mathfrak{V}_n = \mathfrak{U} \times_{\mathfrak{X}} \operatorname{Spec} A^n$. Since $\mathfrak{X}$ is quasi-geometric, each $\mathfrak{V}_n$ is a quasi-affine spectral Deligne-Mumford stack over A^n. Using Proposition 2.4.11, we deduce that there exists a quasi-affine spectral Deligne-Mumford stack $\mathfrak{V}$ and a map $\mathfrak{V} \to \operatorname{Spec} R$ such that $\mathfrak{V} \times_{\operatorname{Spec} R} \operatorname{Spec} A^\bullet \simeq \mathfrak{V}_\bullet$. Since $\mathfrak{U} \to \mathfrak{X}'$ is surjective, the map $\mathfrak{V}_0 \to \operatorname{Spec} A^0$ is surjective. Since A^0 is faithfully flat over R, the composite map $\mathfrak{V}_0 \to \operatorname{Spec} A^0 \to \operatorname{Spec} R$ is surjective. It follows that $\mathfrak{V} \to \operatorname{Spec} R$ is surjective. Using Lemma 1.2.12, we deduce that $\mathfrak{V} \to \operatorname{Spec} R$ is étale. Choose an étale surjection $\operatorname{Spec} R' \to \mathfrak{V}$, so that R' is a faithfully flat étale R-algebra. By construction, the point $\eta_{R'} \in Y(R')$ lifts to $X(R')$. □

We now begin to address the question raised at the beginning of this section.

Proposition 3.3.11. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be functors. Assume that X is quasi-geometric and that the ∞-category $\mathrm{QCoh}(X)$ is presentable. Then the canonical map*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

is fully faithful. Here $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$ denotes the ∞-category of symmetric monoidal functors from $\mathrm{QCoh}(X)$ to $\mathrm{QCoh}(Y)$. In particular, if we are given maps $f, f' : Y \to X$, then any $\mathrm{QCoh}(X)$-linear symmetric monoidal natural transformation $\alpha : f^* \to f'^*$ is an equivalence.

Remark 3.3.12. In the situation of Proposition 3.3.11, the hypothesis that $\mathrm{QCoh}(X)$ be presentable can be removed. We make this assumption to simplify the exposition (and because it is satisfied in all examples of interest).

Proof of Proposition 3.3.11. Let us regard X as fixed. The constructions

$$Y \mapsto \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \qquad Y \mapsto \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

carry colimits in Y to limits of ∞-categories. Writing Y as a colimit of corepresentable functors, we may reduce to the case where Y is corepresented by a connective $\mathbb{E}_\infty$-ring R. Since X is quasi-geometric, any map

$Y \to X$ is automatically representably by quasi-affine spectral Deligne-Mumford stacks. Choose a pair of points $\eta, \eta' \in X(R)$, corresponding to maps $f, f' : Y \to X$. Remark 3.2.10 implies that every $\mathrm{QCoh}(X)$-linear symmetric monoidal transformation from f^* to f'^* is an equivalence. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}}(f, f') \arrow[r] \arrow[d] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{QCoh}(X)/}}(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y)) \arrow[d] \\
\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, Y) \arrow[r] & \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}/}}(\mathrm{QCoh}(Y), \mathrm{QCoh}(Y)).
\end{tikzcd}$$

We wish to show that this diagram induces a homotopy equivalence between the homotopy fibers of the horizontal maps. To prove this, it suffices to show that the vertical maps are equivalences. This follows from Proposition 3.2.9, since the maps $Y \to X$ and $Y \to *$ are both representable and quasi-affine (here $*$ denotes the final object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$). □

3.4 Geometric Stacks

Consider a pair of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Suppose that X is quasi-geometric and that the ∞-category $\mathrm{QCoh}(X)$ is presentable. Proposition 3.3.11 implies that a natural transformation $f : Y \to X$ is determined (up to a contractible space of choices) by the symmetric monoidal functor $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$. In this section, we ask the following question:

($*$) Which symmetric monoidal functors $\mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ are given by pullback along a natural transformation of functors $Y \to X$?

To obtain a reasonable answer, we will need to strengthen our assumptions on X.

Definition 3.4.1. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We will say that X is a *geometric stack* if the following conditions are satisfied:

(1) The functor X is a sheaf with respect to the flat topology on $\mathrm{CAlg}^{\mathrm{cn}}$.

(2) There exists a map $f : X' \to X$ in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ with the following properties:

  (a) The functor X' is corepresentable by a connective $\mathbb{E}_\infty$-ring A.

  (b) The map f is representable, affine, and faithfully flat.

The main result of this section is the following:

Theorem 3.4.2 (Tannaka Duality for Geometric Stacks)**.** *Let Y be a geometric stack and $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ an arbitrary functor. Then the construction*

$$(f : X \to Y) \mapsto (f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X))$$

determines a fully faithful embedding

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X)),$$

whose essential image is the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X))$ *spanned by those symmetric monoidal functors* $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ *which preserve small colimits, connective objects, and flat objects.*

Remark 3.4.3. We will show in a moment that every geometric stack is quasi-geometric (Lemma 3.4.6). Consequently, Theorem 3.4.2 can be regarded as a strengthening of Proposition 3.3.11.

Remark 3.4.4. A close relative of Theorem 3.4.2 (under some more restrictive hypotheses) is proven in [18]).

We will give the proof of Theorem 3.4.2 at the end of this section. First, let us study some of the properties enjoyed by the class of geometric stacks.

Remark 3.4.5. Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which is representable and affine. Assume that Y is a sheaf with respect to the flat topology. Then X is a sheaf with respect to the flat topology (see Proposition 3.4.11). Moreover, if f is faithfully flat, then it is an effective epimorphism of flat sheaves.

Lemma 3.4.6. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. The following conditions are equivalent:*

(1) *Let A be a connective $\mathbb{E}_\infty$-ring and $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ a natural transformation (classified by a point $\eta \in X(A)$). Then f is representable and affine.*

(2) *For every pair of connective $\mathbb{E}_\infty$-rings A and B and every pair of morphisms $\mathrm{Spec}^{\mathrm{f}} A \to X \leftarrow \mathrm{Spec}^{\mathrm{f}} B$, the fiber product $\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} B$ is a corepresentable functor.*

(3) *The diagonal map $X \to X \times X$ is affine.*

Proof. The equivalence of (1) and (2) is clear. We next prove that (2) $\Rightarrow$ (3). Choose any morphism $\mathrm{Spec}^{\mathrm{f}} A \to X \times X$, corresponding to a pair of maps $f, g : \mathrm{Spec}^{\mathrm{f}} A \to X$. We have an equivalence

$$\mathrm{Spec}^{\mathrm{f}} A \times_{X \times X} X \simeq (\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} A) \times_{\mathrm{Spec}^{\mathrm{f}} A \times \mathrm{Spec}^{\mathrm{f}} A} \mathrm{Spec}^{\mathrm{f}} A.$$

Condition (2) guarantees that the functor $\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} A$ is corepresentable. Since the collection of corepresentable functors $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ is closed under small limits, we conclude that $\mathrm{Spec}^{\mathrm{f}} A \times_{X \times X} X$ is corepresentable.

Now assume that (3) holds; we will prove (2). Let $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ and $g : \mathrm{Spec}^{\mathrm{f}} B \to X$ be morphisms. We have an equivalence

$$\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} A \simeq \mathrm{Spec}^{\mathrm{f}} (A \otimes B) \times_{X \times X} X.$$

Since the diagonal map $X \to X \times X$ is affine, we conclude that $\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} B$ is corepresentable. □

Notation 3.4.7. We let $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$ denote the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ spanned by those functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ which are sheaves with respect to the flat topology on $\mathrm{CAlg}^{\mathrm{cn}}$.

Lemma 3.4.8. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a sheaf with respect to the flat topology which is equipped with a natural transformation $X \to \mathrm{Spec}^{\mathrm{f}} A$, for some connective $\mathbb{E}_\infty$-ring A. Suppose that there exists a faithfully flat map $A \to A^0$ such that $X \times_{\mathrm{Spec}^{\mathrm{f}} A} \mathrm{Spec}^{\mathrm{f}} A^0$ is corepresentable. Then X is corepresentable.*

Proof. Let $A^\bullet$ be the Čech nerve of the map $A \to A^0$ (in the ∞-category CAlg^{op}). For each $n \geq 0$, the fiber product $X \times_{\mathrm{Spec}^{\mathrm{f}} A} \mathrm{Spec}^{\mathrm{f}} A^n$ is corepresentable, hence of the form $\mathrm{Spec}^{\mathrm{f}} B^n$ for some connective $\mathbb{E}_\infty$-ring B^n. Let $B = \varprojlim B^\bullet$. It follows from Theorem VII.6.12 that $B^\bullet$ is the Čech nerve of the faithfully flat morphism $B \to A^0 \otimes_A B \simeq B^0$. We therefore obtain an equivalence

$$X \simeq X \times_{\mathrm{Spec}^{\mathrm{f}} A} |\mathrm{Spec}^{\mathrm{f}} A^\bullet| \simeq |X \times_{\mathrm{Spec}^{\mathrm{f}} A} \mathrm{Spec}^{\mathrm{f}} A^\bullet| \simeq |\mathrm{Spec}^{\mathrm{f}} B^\bullet| \simeq \mathrm{Spec}^{\mathrm{f}} B$$

in the ∞-category $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$. □

Lemma 3.4.9. *Suppose we are given a pushout diagram of $\mathbb{E}_\infty$-rings*

$$\begin{tikzcd} A \arrow[r, "\psi"] \arrow[d, "\phi"] & A' \arrow[d, "\phi'"] \\ B \arrow[r] & B', \end{tikzcd}$$

where ψ is faithfully flat. If B' is flat over A', then B is flat over A.

Proof. We first suppose that A is connective. Since ψ and ϕ' are flat, we deduce that A' and B' are also connective. The faithful flatness of $B \to B'$ implies that B is also connective. To show that B is flat over A, it will suffice to show that for every discrete A-module M, the tensor product $B \otimes_A M$ is also discrete. (Theorem A.7.2.2.15). Since $B \to B'$ is faitfhully flat, this is equivalent to the assertion that $B' \otimes_B (B \otimes_A M) \simeq B' \otimes_{A'} (A' \otimes_A M)$ is discrete, which follows from the flatness of ϕ' and ψ.

We now treat the general case. Passing to connective covers, we deduce that the map $\pi_0 A \to \pi_0 B$ is flat. It will therefore suffice to show that for every integer n, the canonical map

$$\theta : \pi_n A \otimes_{\pi_0 A} \pi_0 B \to \pi_n B$$

is an isomorphism. Since $\pi_0 B'$ is faithfully flat over $\pi_0 B$, it will suffice to show that θ induces an isomorphism

$$\pi_n A \otimes_{\pi_0 A} \pi_0 B' \to \pi_n B \otimes_{\pi_0 B} \pi_0 B' \simeq \pi_n B'.$$

This follows from the flatness of the composite map $\phi' \circ \psi$. $\square$

Lemma 3.4.10. *Suppose given a pullback diagram*

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

in the ∞-category $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$. Suppose further that g is an effective epimorphism. If f' is representable and affine, then so is f. In this case, if f' is flat (faithfully flat), then so is f.

Proof. Suppose we are given a connective $\mathbb{E}_\infty$-ring A and a map $\mathrm{Spec}^{\mathrm{f}} A \to Y$; we wish to show that the fiber product $\mathrm{Spec}^{\mathrm{f}} A \times_Y X$ is corepresentable. Replacing X by $\mathrm{Spec}^{\mathrm{f}} A \times_Y X$ and Y' by $\mathrm{Spec}^{\mathrm{f}} A \times_Y Y'$, we can assume that $Y = \mathrm{Spec}^{\mathrm{f}} A$ is corepresentable. Since the map $Y' \to Y$ is an effective epimorphism of flat sheaves, there exists a faithfully flat map $A \to A'$ such that the induced map $\mathrm{Spec}^{\mathrm{f}} A' \to Y$ factors through Y'. Since f' is representable by affine spectral Deligne-Mumford stacks, we deduce that the fiber product $\mathrm{Spec}^{\mathrm{f}} A' \times_Y X \simeq \mathrm{Spec}^{\mathrm{f}} A' \times_{Y'} X'$ is corepresentable. It follows from Lemma 3.4.8 that $X \simeq \mathrm{Spec}^{\mathrm{f}} B$ is corepresentable. To complete the proof, it suffices to show that B is flat (faithfully flat) over A if and only if $B \otimes_A A'$ is flat (faithfully flat) over A'. This follows from Lemma 3.4.9. $\square$

Proposition 3.4.11. *Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a geometric stack. Then the diagonal map $X \to X \times X$ is representable and affine.*

Proof. Choose a connective $\mathbb{E}_\infty$-ring A and a map $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ which is representable by spectral Deligne-Mumford stacks, affine, and faithfully flat. Consider the diagram of flat sheaves

$$\begin{tikzcd} \mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} A \arrow[r] \arrow[d] & \mathrm{Spec}^{\mathrm{f}} A \times \mathrm{Spec}^{\mathrm{f}} A \arrow[d, "f \times f"] \\ X \arrow[r] & X \times X \end{tikzcd}$$

in $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$. Since $f \times f$ is an effective epimorphism (Remark 3.4.5) and $\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} A$ is corepresentable, Lemma 3.4.10 implies that the diagonal map $X \to X \times X$ is representable by affine spectral Deligne-Mumford stacks. $\square$

Corollary 3.4.12. *Let X be a geometric stack, and let A be a connective $\mathbb{E}_\infty$-ring. Then every natural transformation $\mathrm{Spec}^{\mathrm{f}} A \to X$ is representable and affine.*

Corollary 3.4.13. *Suppose we are given a pullback diagram*

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & X \arrow[d] \\ Y' \arrow[r] & Y \end{tikzcd}$$

in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$*. If* X*,* Y*, and* Y' *are geometric stacks, then* X' *is a geometric stack.*

Proof. It is clear that X' is a sheaf with respect to the flat topology. Since X and Y' are geometric, we can choose connective $\mathbb{E}_\infty$-rings and faithfully flat $f : \mathrm{Spec}^{\mathrm{f}} A \to X$, $g : \mathrm{Spec}^{\mathrm{f}} B \to Y'$. It follows that the induced map $\mathrm{Spec}^{\mathrm{f}} A \times_Y \mathrm{Spec}^{\mathrm{f}} B \to X \times_Y Y' = X'$ is representable, affine, and faithfully flat. Since Y is geometric, Proposition 3.4.11 implies that $\mathrm{Spec}^{\mathrm{f}} A \times_Y \mathrm{Spec}^{\mathrm{f}} B$ is a corepresntable functor. It follows that X' is geometric, as desired. □

Corollary 3.4.14. *Let* $\mathfrak{X}$ *be a spectral Deligne-Mumford stack, and let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ *be the functor represented by* $\mathfrak{X}$*. The following conditions are equivalent:*

(1) *The functor* X *is a geometric stack.*

(2) *The spectral Deligne-Mumford stack* $\mathfrak{X}$ *is quasi-compact, and the diagonal map* $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ *is affine.*

Proof. Suppose first that (1) is satisfied. Then there exists a faithfully flat affine morphism $\mathrm{Spec}\, A \to \mathfrak{X}$, which proves that $\mathfrak{X}$ is quasi-compact. It follows from Proposition 3.4.11 that the diagonal $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is affine.

Conversely, suppose that $\mathfrak{X}$ satisfies condition (2). Since $\mathfrak{X}$ is quasi-compact, we can choose a surjective étale morphism $\mathrm{Spec}\, A \to \mathfrak{X}$. Because the diagonal of $\mathfrak{X}$ is affine, the induced map $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ is representable by affine spectral Deligne-Mumford stacks. It is clear that f is faithfully flat (in fact, it is even étale). To complete the proof, it will suffice to show that X is a sheaf with respect to the flat topology. This follows from Proposition 3.3.10, since $\mathfrak{X}$ is quasi-geometric. □

Definition 3.4.15. We will say that a spectral Deligne-Mumford stack $\mathfrak{X}$ is *geometric* if it satisfies the equivalent conditions of Corollary 3.4.14.

Lemma 3.4.16. *Let* P *be a property of pairs* $(A, M) \in \mathrm{CAlg}^{\mathrm{cn}} \times_{\mathrm{CAlg}} \mathrm{Mod}(\mathrm{Sp})$ *which is stable under base change and local with respect to the fpqc topology. Let* X *be a geometric stack and choose a faithfully flat morphism* $f : \mathrm{Spec}^{\mathrm{f}} A \to X$*. Let* $\mathcal{F} \in \mathrm{QCoh}(X)$ *and set* $M = f^* \mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} A) \simeq \mathrm{Mod}_A$*. Then* $\mathcal{F}$ *has the property* P *if and only if the pair* (A, M) *has the property* P*. In particular,* $\mathcal{F}$ *is connective if and only if* M *is connective.*

Proof. The "only if" assertion is obvious. Conversely, suppose that the pair (A, M) has the property P. Let B be a connective $\mathbb{E}_\infty$-ring, $g : \mathrm{Spec}^{\mathrm{f}} B \to X$ be an arbitrary morphism, and $N = g^* \mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} B) \simeq \mathrm{Mod}_B$. We wish to show that the pair (B, N) has the property P. We have $\mathrm{Spec}^{\mathrm{f}} A \times_X \mathrm{Spec}^{\mathrm{f}} B = \mathrm{Spec}^{\mathrm{f}} B'$ for some connective $\mathbb{E}_\infty$-ring B' which is faithfully flat over B. Since P is local with respect to the fpqc topology, we may replace g by the induced map $\mathrm{Spec}^{\mathrm{f}} B' \to X$, and thereby reduce to the case where g factors through f. In this case, the desired result follows immediately from our assumption that P is stable under base change. □

Proposition 3.4.17. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *be a geometric stack. Then:*

(1) *The* ∞*-category* $\mathrm{QCoh}(X)$ *is presentable.*

(2) *There exists an accessible t-structure* $(\mathrm{QCoh}(X)_{\geq 0}, \mathrm{QCoh}(X)_{\leq 0})$ *on* $\mathrm{QCoh}(X)$*, where* $\mathrm{QCoh}(X)_{\geq 0}$ *is the full subcategory of* $\mathrm{QCoh}(X)$ *spanned by the connective objects.*

(3) *Choose a connective $\mathbb{E}_\infty$-ring A and a faithfully flat map $f : \mathrm{Spec}^{\mathrm{f}} A \to X$. Then a quasi-coherent sheaf $\mathcal{F}$ on X belongs to $\mathrm{QCoh}(X)_{\leq 0}$ if and only if $f^* \mathcal{F} \in (\mathrm{Mod}_A)_{\leq 0}$.*

(4) *The t-structure on $\mathrm{QCoh}(X)$ is compatible with filtered colimits: that is, $\mathrm{QCoh}(X)_{\leq 0}$ is closed under filtered colimits.*

(5) *The t-structure on $\mathrm{QCoh}(X)$ is both right and left complete.*

Proof. Let $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ be as in (3). Let $X_0 = \mathrm{Spec}^{\mathrm{f}} A$, and let $X_\bullet$ be the Čech nerve of f, formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ of flat sheaves. It follows from Remark 3.4.5 that X can be identified with the geometric realization of $X_\bullet$. It follows that $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$. For $n \geq 0$, the functor $X_n \simeq \mathrm{Spec}^{\mathrm{f}} A^n$ for some connective $\mathbb{E}_\infty$-ring A_n, so that $\mathrm{QCoh}(X_n) \simeq \mathrm{Mod}_{A^n}$ is presentable. It follows from Proposition T.5.5.3.13 that $\mathrm{QCoh}(X)$ is presentable.

Since $\mathrm{QCoh}(X)_{\geq 0}$ can be described as the fiber product $\mathrm{QCoh}(X) \times_{\mathrm{Mod}_A} (\mathrm{Mod}_A)_{\geq 0}$, it is presentable (Proposition T.5.5.3.12). Since $\mathrm{QCoh}(X)_{\geq 0}$ is clearly stable under colimits and extensions, assertion (2) follows from Proposition A.1.4.5.11.

Write $X_1 = \mathrm{Spec}^{\mathrm{f}} B$ for some connective $\mathbb{E}_\infty$-ring B, which we can regard as an A-algebra in two different ways. Consider the cosimplicial ∞-category given by $\mathrm{QCoh}(X_\bullet)$. Using Corollary 3.2.6, we conclude that every map $[m] \to [n]$ in $\mathbf{\Delta}$ induces a right adjointable diagram

$$\begin{tikzcd} \mathrm{QCoh}(X_m) \arrow[r] \arrow[d] & \mathrm{QCoh}(X_{m+1}) \arrow[d] \\ \mathrm{QCoh}(X_n) \arrow[r] & \mathrm{QCoh}(X_{n+1}). \end{tikzcd}$$

It follows from Theorem A.6.2.4.2 that $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$ can be identified with the ∞-category of algebras over the comonad on $\mathrm{QCoh}(X_0) \simeq \mathrm{Mod}_A$ given by tensor product with B (regarded as an A-bimodule). Since B is flat over A, this functor is t-exact. Assertions (3) and (5) now follow from Proposition VII.6.20. Assertion (4) is clear from the definitions. □

Remark 3.4.18. Let $X : \mathrm{CAlg} \to \widehat{\mathcal{S}}$ be a geometric stack. It is clear from the definitions that the t-structure on $\mathrm{QCoh}(X)$ is compatible with the symmetric monoidal structure on $\mathrm{QCoh}(X)$: that is, the collection of connective objects contains the structure sheaf $\mathcal{O}_X$ and is stable under tensor products.

Remark 3.4.19. Let $f : X \to Y$ be a representable affine morphism between geometric stacks, and let $\mathcal{F}$ be a quasi-coherent sheaf on X. Then:

(a) The sheaf $\mathcal{F}$ belongs to $\mathrm{QCoh}(X)_{\geq 0}$ if and only if $f_* \mathcal{F}$ belongs to $\mathrm{QCoh}(Y)_{\geq 0}$.

(b) The sheaf $\mathcal{F}$ belongs to $\mathrm{QCoh}(X)_{\leq 0}$ if and only if $f_* \mathcal{F}$ belongs to $\mathrm{QCoh}(Y)_{\leq 0}$.

We will prove (a); the proof of (b) is similar. Choose a connective $\mathbb{E}_\infty$-ring A and a faithfully flat affine morphism $g : \mathrm{Spec}^{\mathrm{f}} A \to Y$. Then $\mathrm{Spec}^{\mathrm{f}} A \times_Y X \simeq \mathrm{Spec}^{\mathrm{f}} B$ for some connective A-algebra B; let $g' : \mathrm{Spec}^{\mathrm{f}} B \to X$ denote the projection map. Note that $\mathcal{F} \in \mathrm{QCoh}(X)_{\geq 0}$ if and only if $M = g'^* \mathcal{F}$ is a connective B-module, and $f_* \mathcal{F}$ is connective if and only if $N = g^* f_* \mathcal{F}$ is a connective A-module. The desired result now follows by observing that N can be identified with the underlying A-module of M (Corollary 3.2.6).

Remark 3.4.20. Let X be a geometric stack, and let $\mathcal{F} \in \mathrm{QCoh}(X)$. The following conditions are equivalent:

(1) The quasi-coherent sheaf $\mathcal{F}$ is flat.

(2) The operation of tensor product with $\mathcal{F}$ determines a t-exact functor from $\mathrm{QCoh}(X)$ to itself.

The implication (1) $\Rightarrow$ (2) is obvious. Conversely, suppose that (2) is satisfied. Since $\mathcal{O}_X \in \mathrm{QCoh}(X)_{\geq 0}$, the t-exactness of tensor product with $\mathcal{F}$ implies that $\mathcal{F} \in \mathrm{QCoh}(X)_{\geq 0}$. Choose a connective $\mathbb{E}_\infty$-ring A and a map $f : \mathrm{Spec}^{\mathrm{f}} A \to X$. We wish to prove that $f^* \mathcal{F} \in \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} A) \simeq \mathrm{Mod}_A$ is flat. Let M be a discrete A-module, which we can identify with an object of $\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} A)$. Then $M \otimes f^* \mathcal{F}$ is a connective A-module, and we wish to show that it is discrete. Equivalently, we wish to show that the groups

$$\mathrm{Ext}^i_{\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}} A)}(\mathcal{O}_{\mathrm{Spec}^{\mathrm{f}} A}, M \otimes f^* \mathcal{F}) \simeq \mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{O}_X, f_*(M \otimes f^* \mathcal{F})) \simeq \mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{O}_X, (f_* M) \otimes \mathcal{F})$$

are trivial for $i < 0$. Since $\mathcal{O}_X$ is connective, it will suffice to show that $(f_* M) \otimes \mathcal{F} \in \mathrm{QCoh}(X)_{\leq 0}$. This follows from condition (2), since $f_* M \in \mathrm{QCoh}(X)_{\leq 0}$.

Proposition 3.4.21. *Let X be a geometric stack. Assume that there exists a connective $\mathbb{E}_\infty$-ring A and a t-exact equivalence of symmetric monoidal ∞-categories $\mathrm{QCoh}(X)^{\otimes} \simeq \mathrm{Mod}_A$. Then $X \simeq \mathrm{Spec}^{\mathrm{f}} A$.*

Proof. Choose a connective $\mathbb{E}_\infty$-ring A^0 and a faithfully flat morphism $f : \mathrm{Spec}^{\mathrm{f}} A^0 \to X$. Let $X_0 = \mathrm{Spec}^{\mathrm{f}} A^0$ and let $X_\bullet$ denote the Čech nerve of f. Since f is an effective epimorphism in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ (Remark 3.4.5), we can extend $X_\bullet$ to a colimit diagram $\overline{X}_\bullet : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ with $\overline{X}_{-1} = X$. For $n \geq 0$, we have $X_n \simeq \mathrm{Spec}^{\mathrm{f}} A^n$ for some connective $\mathbb{E}_\infty$-ring A^n, so that $\mathrm{QCoh}(X_n) \simeq \mathrm{Mod}_{A^n}$. Since the construction $R \mapsto \mathrm{Mod}_R$ is a fully faithful embedding of $\mathrm{CAlg}(\mathrm{Sp})$ into $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{Sp}\,/}$ (Corollary A.6.3.5.18), we can extend $A^\bullet$ to an augmented cosimplicial $\mathbb{E}_\infty$-ring $\overline{A}^\bullet$, satisfying $\mathrm{QCoh}(\overline{X}_\bullet) \simeq \mathrm{Mod}_{\overline{A}^\bullet}$. Then $\overline{A}^{-1} \simeq A$. To complete the proof, it will suffice to show that $\mathrm{Spec}^{\mathrm{f}} \overline{A}^\bullet$ is the Čech nerve of a faithfully flat map $\mathrm{Spec}^{\mathrm{f}} A^0 \to \mathrm{Spec}^{\mathrm{f}} A$: this will guarantee the existence of equivalences

$$\mathrm{Spec}^{\mathrm{f}} A \simeq |\,\mathrm{Spec}^{\mathrm{f}} A^\bullet| \simeq |X_\bullet| \simeq X$$

in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$.

Since f is faithfully flat, the functor $f_* f^*$ determines a t-exact functor from $\mathrm{QCoh}(X)$ to itself. Under the equivalence $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_A$, this functor is given by tensor product with A^0. It follows that the map $u : A \to A^0$ is faithfully flat. We will complete the proof by showing that $\overline{A}^\bullet$ is the Čech nerve of u (in the ∞-category CAlg^{op}). It suffices to show that $\mathrm{Mod}_{\overline{A}^\bullet} \simeq \mathrm{QCoh}(\overline{X}_\bullet)$ is the Čech nerve of the morphism $f^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_0)$ in the ∞-category $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})^{op}$, which follows immediately from Corollary 3.2.6. □

Remark 3.4.22. Let $f : X \to Y$ be a representable affine morphism between functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Assume that Y is a geometric stack. Then X is a geometric stack. Indeed, Remark 3.4.5 shows that X is a sheaf with respect to the flat topology. Choose a faithfully flat morphism $\mathrm{Spec}^{\mathrm{f}} A \to Y$. The induced map $\mathrm{Spec}^{\mathrm{f}} A \times_Y X \to X$ is also representable, affine, and faithfully flat. Since f is affine, the fiber product $\mathrm{Spec}^{\mathrm{f}} A \times_Y X$ is corepresentable, so that X is geometric.

Corollary 3.4.23. *Let $f : X \to Y$ be a natural transformation of functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. Assume that Y is a geometric stack. Then X is corepresentable if and only if the following conditions are satisfied:*

(1) *The map f is representable and affine.*

(2) *There exists a symmetric monoidal equivalence $\theta : \mathrm{QCoh}(X)^{\otimes} \simeq \mathrm{Mod}_A^{\otimes}$, for some connective $\mathbb{E}_\infty$-ring A.*

(3) *The composition $\theta \circ f^*$ is right t-exact.*

Remark 3.4.24. In the situation of Corollary 3.4.23, Proposition 3.2.11 guarantees that $\mathbb{E}_\infty$-ring A is determined up to equivalence. In particular, condition (3) is independent of the choice of θ.

Proof. The necessity of conditions (2) and (3) is clear, and the necessity of (1) follows from Proposition 3.4.11. Conversely, suppose that (1), (2), and (3) are satisfied. Remark 3.4.22 guarantees that X is geometric. Let $\theta : \mathrm{QCoh}(X)^{\otimes} \simeq \mathrm{Mod}_A^{\otimes}$ be as in (2). According to Proposition 3.4.21, it will suffice to show θ is t-exact.

In other words, we must show that θ restricts to an equivalence from $\mathrm{QCoh}(X)_{\geq 0}$ to $(\mathrm{Mod}_A)_{\geq 0}$. Since θ is a symmetric monoidal functor, it carries $\mathcal{O}_X \in \mathrm{QCoh}(X)_{\geq 0}$ to A, which generates $(\mathrm{Mod}_A)_{\geq 0}$ under small colimits. It therefore suffices to show that $\theta(\mathrm{QCoh}(X)_{\geq 0}) \subseteq (\mathrm{Mod}_A)_{\geq 0}$. Using Proposition A.6.2.2.11 and Remark 3.4.19, we see that every object of $\mathrm{QCoh}(X)_{\geq 0}$ can be written as a geometric realization of objects belonging to $f^* \mathrm{QCoh}(Y)_{\geq 0}$. It therefore suffices to show that $\theta(f^* \mathrm{QCoh}(Y)_{\geq 0}) \subseteq (\mathrm{Mod}_A)_{\geq 0}$, which is a restatement of assumption (3). □

Corollary 3.4.25. *Let X be a geometric stack and let $\mathcal{C}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/X}$ spanned by those maps $f : Y \to X$ for which Y is corepresentable. The construction*

$$(f : Y \to X) \to \mathrm{QCoh}(Y)$$

determines a fully faithful embedding $\theta : \mathcal{C}^{op} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})_{\mathrm{QCoh}(X)/}$. Moreover, a symmetric monoidal functor $f^* : \mathrm{QCoh}(X) \to \mathcal{C}$ belongs to the essential image of θ if and only if the following conditions are satisfied:

- (a) *There exists a symmetric monoidal equivalence $\mathcal{C} \simeq \mathrm{Mod}_A$ for some connective $\mathbb{E}_\infty$-ring A.*
- (b) *The functor f^* admits a right adjoint f_*.*
- (c) *The functor f_* is conservative and preserves geometric realizations of simplicial objects.*
- (d) *For every quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ and every object $C \in \mathcal{C}$, the canonical map $\mathcal{F} \otimes f_*(C) \to f_*(f^* \mathcal{F} \otimes C)$ is an equivalence in $\mathrm{QCoh}(Y)$.*
- (e) *The functor $f^* : \mathcal{C} \to \mathrm{Mod}_A$ is right t-exact.*

Proof. Combine Proposition 3.2.11 with Corollary 3.4.23. □

Our next goal is to find a slightly different characterization of the essential image of the functor θ appearing in Corollary 3.4.25.

Proposition 3.4.26. *Let X be a geometric stack, let A be a connective $\mathbb{E}_\infty$-ring, and let $f^* : \mathrm{QCoh}(X)^{\otimes} \to \mathrm{Mod}_A^{\otimes}$ be a symmetric monoidal functor. The following conditions are equivalent:*

1. *The functor f^* is equivalent to the pullback functor associated to a map of geometric stacks $\mathrm{Spec}^{\mathrm{f}} A \to X$ (uniquely determined up to equivalence, by Corollary 3.4.25).*
2. *The functor f^* satisfies conditions (b) through (e) of Corollary 3.4.25.*
3. *The functor f^* is right t-exact, preserves small colimits, and carries flat objects of $\mathrm{QCoh}(X)$ to flat A-modules.*

Proof. The equivalence (1) ⇔ (2) is Corollary 3.4.25, and the implication (1) ⇒ (3) is obvious. We will show that (3) ⇒ (2). Assume that f^* is right t-exact, preserves small colimits, and preserves flatness. We wish to prove that f^* satisfies conditions (b) through (e) of Corollary 3.4.25. The existence of a right adjont f_* to f^* follows from Corollary T.5.5.2.9 (since f^* preserves small colimits and $\mathrm{QCoh}(X)$ is presentable by Proposition 3.4.17). We next prove that f_* is conservative. Suppose we are given a map $\alpha : M \to M'$ of A-modules such that $f_*(\alpha)$ is an equivalence. We wish to prove that α is an equivalence: that is, that α induces an isomorphism $\pi_n M \to \pi_n M'$ for every integer n. Replacing M by $M[-n]$ and M' by $M'[-n]$, we can assume that $n = 0$. Then α induces the isomorphism

$$\begin{aligned}
\pi_0 M &\simeq \mathrm{Hom}_{\mathrm{hMod}_A}(f^* \mathcal{O}_X, M) \\
&\simeq \mathrm{Hom}_{\mathrm{hQCoh}(X)}(\mathcal{O}_X, f_* M) \\
&\simeq \mathrm{Hom}_{\mathrm{hQCoh}(X)}(\mathcal{O}_X, f_* M') \\
&\simeq \mathrm{Hom}_{\mathrm{hMod}_A}(A, M') \\
&\simeq \pi_0 M'.
\end{aligned}$$

We next show that f_* preserves geometric realizations of simplicial objects. In fact, we prove that f_* preserves all small colimits. Choose a faithfully flat morphism $g : X_0 \to X$, where X_0 is corepresented by a connective $\mathbb{E}_\infty$-ring B. Let $\mathcal{B} = g_* \mathcal{O}_{X_0} \in \mathrm{CAlg}(\mathrm{QCoh}(X))$. Since g is faithfully flat, $\mathcal{B}$ is a faithfully flat commutative algebra object of $\mathrm{QCoh}(X)$, so both $\mathcal{B}$ and the cofiber of the unit map $\mathcal{O}_X \to \mathcal{B}$ are flat (Lemma VII.6.22). Since f^* preserves flatness, we deduce that $A^0 = f^* \mathcal{B}$ and the cofiber of the unit map $A \to A^0$ are flat A-modules, so that A^0 is faithfully flat over A (Lemma VII.6.22). Let $\mathcal{B}^\bullet$ denote the Čech nerve of the unit map $\mathcal{O}_X \to \mathcal{B}$ (in the ∞-category $\mathrm{CAlg}(\mathrm{QCoh}(X))^{op}$), and $A^\bullet = f^* \mathcal{B}^\bullet$. Let $X_\bullet$ be the Čech nerve of g, so that we have an equivalence of cosimplicial ∞-categories $\mathrm{QCoh}(X_\bullet) \simeq \mathrm{Mod}_{\mathcal{B}^\bullet}(\mathrm{QCoh}(X))$. Note that we have $X_\bullet \simeq \mathrm{Spec}^{\mathrm{f}} B^\bullet$, for some cosimplicial connective $\mathbb{E}_\infty$-ring $B^\bullet$.

The functor f^* determines a natural transformation of cosimplicial ∞-categories

$$\mathrm{Mod}_{B^\bullet} \simeq \mathrm{QCoh}(X_\bullet) \simeq \mathrm{Mod}_{\mathcal{B}^\bullet}(\mathrm{QCoh}(X)) \to \mathrm{Mod}_{A^\bullet}(\mathrm{Mod}_A) \simeq \mathrm{Mod}_{A^\bullet}.$$

Using Lemma VII.6.15, we deduce that for every map $[m] \to [n]$ in $\mathbf{\Delta}$, the diagram of ∞-categories

$$\begin{array}{ccc} \mathrm{QCoh}(X_m) & \xrightarrow{f(m)^*} & \mathrm{Mod}_{A^m} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X_n) & \xrightarrow{f(n)^*} & \mathrm{Mod}_{A^n} \end{array}$$

is right adjointable; moreover, the right adjoint to $f(n)^* : \mathrm{QCoh}(X_n) \to \mathrm{Mod}_{A^n}$ can be identified with the forgetful functor $f(n)_* : \mathrm{Mod}_{A^n} \to \mathrm{Mod}_{B^n}$ induced by a map of $\mathbb{E}_\infty$-rings $B^n \to A^n$. In particular, $f(n)_*$ commutes with small colimits.

Since $X \simeq |X_\bullet|$ in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, we get a canonical equivalence $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$. Theorem VII.6.27 guarantees that $\mathrm{Mod}_A \simeq \varprojlim \mathrm{Mod}_{A^\bullet}$, so that f^* can be identified with the limit of the functors $f(n)^*$. It follows from Corollary A.6.2.3.18 that each of the diagrams

$$\begin{array}{ccc} \mathrm{QCoh}(X) & \xrightarrow{f^*} & \mathrm{Mod}_{A} \\ \downarrow & & \downarrow \\ \mathrm{QCoh}(X_n) & \xrightarrow{f(n)^*} & \mathrm{Mod}_{A^n} \end{array}$$

is right adjointable.

To prove that $f_* : \mathrm{Mod}_A \to \mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$ preserves small colimits, it suffices to show that each of the composite functors $\mathrm{Mod}_A \to \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_n)$ preserves small colimits. By the above, this is equivalent to the assertion that the composite functor

$$\mathrm{Mod}_A \to \mathrm{Mod}_{A^n} \xrightarrow{f(n)_*} \mathrm{QCoh}(X_n) \simeq \mathrm{Mod}_{B^n}$$

preserves small colimits, which is clear (since $f(n)_*$ preserves small colimits).

It remains to verify that the functor $f_* : \mathrm{Mod}_A \to \mathrm{QCoh}(X)$ satisfies condition (d) of Corollary 3.4.25. Fix an A-module M and a quasi-coherent sheaf $\mathcal{F}$ on X; we wish to show that the canonical map $\alpha : \mathcal{F} \otimes f_* M \to f_*(f^* \mathcal{F} \otimes M)$ is an equivalence in $\mathrm{QCoh}(X)$. Since the pullback functor $g^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(X_0) \simeq \mathrm{Mod}_B$ is conservative, it will suffice to show that $g^*(\alpha) : g^* \mathcal{F} \otimes g^* f_* M \to g^* f_*(f^* \mathcal{F} \otimes M)$ is an equivalence of B-modules. This map fits into a commutative diagram

$$\begin{array}{ccc} g^* \mathcal{F} \otimes g^* f_* M & \longrightarrow & g^* f_*(f^* \mathcal{F} \otimes M) \\ \downarrow & & \downarrow \\ g^* \mathcal{F} \otimes f(0)_*(A^0 \otimes M) & \longrightarrow & f(0)_*(A^0 \otimes f^* \mathcal{F} \otimes M) \end{array}$$

where the vertical maps are equivalences. We may therefore replace X by $\mathrm{Spec}^{\mathrm{f}} B$ and $\mathcal{F}$ by $g^* \mathcal{F}$, in which case the assertion is obvious. $\square$

We are now ready to give the proof of our main result.

Proof of Theorem 3.4.2. Let Y be a fixed geometric stack, and let $F : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathcal{C}\mathrm{at}_\infty}$ be the functor which assigns to each $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X))$ spanned by those functors which preserve small colimits, connective objects, and flat objects. Note that if $X \in \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ is the colimit of a system of a diagram functors $\{X_\alpha\}$, then $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\alpha)$. Moreover, an object of $\mathrm{QCoh}(X)$ is flat if and only if its image in each $\mathrm{QCoh}(X_\alpha)$ is flat and connective if and only if its image in each $\mathrm{QCoh}(X_\alpha)$ is connective. It follows that the functor F preserves limits. There is an evident natural transformation

$$\theta_X : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y) \to F(X),$$

We wish to prove that θ_X is an equivalence of ∞-categories. We note that the collection of functors $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ for which this statement holds is stable under colimits. It will therefore suffice to treat the case where X is corepresented by a connective $\mathbb{E}_\infty$-ring A.

Note that $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y)$ can be identified with the homotopy fiber

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/Y} \times_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})} \{X\}.$$

Using Proposition 3.2.11, we can identify this with the homotopy fiber

$$\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/Y} \times_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}})} \{\mathrm{Mod}_A\}.$$

Using Corollary 3.4.25 and Proposition 3.4.26, we see that θ induces an equivalence from this fiber product to the largest Kan complex contained in $F(X)$. We now invoke Remark 3.2.10 to conclude that $F(X)$ is already a Kan complex, so that θ_X is an equivalence. □

Corollary 3.4.27. *Let X be a geometric stack. Then X is a hypercomplete object of $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$.*

Proof. Let A be a connective $\mathbb{E}_\infty$-ring and let $A^\bullet$ be a flat hypercovering of A; we wish to prove that the canonical map $\phi : X(A) \to \varprojlim X(A^\bullet)$ is a homotopy equivalence. We have a commutative diagram

$$\begin{tikzcd}
X(A) \arrow[r] \arrow[d] & \varprojlim X(A^\bullet) \arrow[d] \\
\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_A) \arrow[r] & \varprojlim \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_{A^\bullet})
\end{tikzcd}$$

where the vertical maps are fully faithful (Theorem 3.4.2). Using Theorem VII.6.12, we conclude that $\mathrm{Mod}_A \simeq \varprojlim \mathrm{Mod}_{A^\bullet}$, so that the lower horizontal map is an equivalence of ∞-categories. It follows that ϕ is fully faithful. To complete the proof, it suffices to show that ϕ is surjective on connected components. Using Theorem 3.4.2, this amounts to the following assertion:

- $(*)$ Let $f^* : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ be a symmetric monoidal functor. If each of the composite functors $\mathrm{QCoh}(X) \xrightarrow{f^*} \mathrm{Mod}_A \to \mathrm{Mod}_{A^n}$ preserves small colimits, flat objects, and connective objects, then f^* has the same properties.

Suppose that f^* is as in $(*)$. Since the base-change functor $\mathrm{Mod}_A \to \mathrm{Mod}_{A^0}$ is conservative and preserves small colimits, we conclude that f^* preserves small colimits. The remaining assertions follow from Proposition 2.6.15. □

Corollary 3.4.28. *Let X be a geometric stack. Then, for every connective $\mathbb{E}_\infty$-ring A, the space $X(A)$ is essentially small.*

Proof. Proposition 3.4.17 guarantees that $\mathrm{QCoh}(X)$ is a presentable ∞-category. Consequently, if κ is a sufficiently large regular cardinal, then $\mathrm{QCoh}(X) \simeq \mathrm{Ind}_\kappa(\mathrm{QCoh}(X)_\kappa)$, where $\mathrm{QCoh}(X)_\kappa$ denotes the full subcategory of $\mathrm{QCoh}(X)$ spanned by the κ-compact objects. Enlarging κ if necessary, we may assume that the full subcategory $\mathrm{QCoh}(X)_\kappa$ contains the unit object $\mathcal{O}_X$ and is closed under tensor products. It follows that $\mathrm{QCoh}(X)_\kappa$ inherits the structure of a symmetric monoidal ∞-category. Proposition A.6.3.1.10 guarantees that $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)_\kappa^{\otimes}, \mathrm{Mod}_A^{\otimes})$ can be identified with the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\otimes}, \mathrm{Mod}_A^{\otimes})$ spanned by those symmetric monoidal functors which preserves κ-filtered colimits. Combining this with Theorem 3.4.2, we obtain a fully faithful embedding $\theta : X(A) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)_\kappa^{\otimes}, \mathrm{Mod}_A^{\otimes})$. Let $(\mathrm{Mod}_A)_\kappa$ denote the full subcategory of Mod_A spanned by the κ-compact objects. To complete the proof, it will suffice to show that θ factors through the (essentially small) subcategory $\mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)_\kappa^{\otimes}, (\mathrm{Mod}_A)_\kappa^{\otimes}) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)_\kappa^{\otimes}, \mathrm{Mod}_A^{\otimes})$. In other words, we must show that if $f : \mathrm{Spec}^{\mathrm{f}} A \to X$ is any map, then the associated pullback functor f^* preserves κ-compact objects. This is equivalent to the assertion that the right adjoint f_* preserves κ-filtered colimits. But f_* preserves all colimits, since f is affine (Proposition 3.4.11). $\square$

The requirement that a symmetric monoidal functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ preserve flat objects is sometimes technically inconvenient. When X is corepresentable, it can be avoided:

Corollary 3.4.29. *Let Y be a geometric stack, $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ a corepresentable functor, and $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ a symmetric monoidal functor. The following conditions are equivalent:*

(1) *The functor f^* belongs to the essential image of the fully faithful embedding $\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X))$ of Theorem 3.4.2.*

(2) *The functor f^* preserves small colimits, connective objects, and flat objects.*

(3) *The functor f^* admits a right adjoint $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which is left t-exact. Moreover, for every pair of objects $\mathcal{F} \in \mathrm{QCoh}(Y)$, $\mathcal{F}' \in \mathrm{QCoh}(X)$, the canonical map $\mathcal{F} \otimes f_* \mathcal{F}' \to f_*(f^* \mathcal{F} \otimes \mathcal{F}')$ is an equivalence.*

Proof. The equivalence (1) $\Leftrightarrow$ (2) follows from Theorem 3.4.2. We next prove that (1) $\Rightarrow$ (3). Suppose that f^* is given by the pullback functor associated to a natural transformation $f : X \to Y$. Since X is corepresentable and Y is geometric, the morphism f is representable and affine. It follows that $\mathcal{A} = f_* \mathcal{O}_X$ is a connective object of $\mathrm{CAlg}(\mathrm{QCoh}(Y))$, and we can identify f_* with the forgetful functor $\mathrm{Mod}_{\mathcal{A}}(\mathrm{QCoh}(Y)) \to \mathrm{QCoh}(Y)$, for which assertion (3) is evident.

We complete the proof by showing that (3) $\Rightarrow$ (2). The existence of a right adjoint f_* implies that f^* preserves small colimits, and the right t-exactness of f^* is equivalent to the left t-exactness of f_*. Let $\mathcal{F} \in \mathrm{QCoh}(Y)$ be flat; we wish to prove that $f^* \mathcal{F} \in \mathrm{QCoh}(X)$ is flat. Since $\mathrm{QCoh}(X) \simeq \mathrm{Mod}_A$ for some connective $\mathbb{E}_\infty$-ring A, it will suffice to show that for every discrete object $\mathcal{F}' \in \mathrm{QCoh}(X)$, the tensor product $(f^* \mathcal{F}) \otimes \mathcal{F}' \in \mathrm{QCoh}(X)$ is discrete. It is clear that $(f^* \mathcal{F}) \otimes \mathcal{F}'$ is connective; it will therefore suffice to show that $(f^* \mathcal{F}) \otimes \mathcal{F}' \in \mathrm{QCoh}(X)_{\leq 0}$. Equivalently, we wish to show that the abelian groups $\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{O}_X, (f^* \mathcal{F}) \otimes \mathcal{F}'))$ are trivial for $i < 0$. We have

$$\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{O}_X, (f^* \mathcal{F}) \otimes \mathcal{F}') \simeq \mathrm{Ext}^i_{\mathrm{QCoh}(Y)}(\mathcal{O}_Y, f_*((f^* \mathcal{F}) \otimes \mathcal{F}')) \simeq \mathrm{Ext}^i_{\mathrm{QCoh}(Y)}(\mathcal{O}_Y, \mathcal{F} \otimes f_* \mathcal{F}').$$

Since $\mathcal{O}_Y$ is connective, it is sufficient to show $\mathcal{F} \otimes f_* \mathcal{F}'$ belongs to $\mathrm{QCoh}(Y)_{\leq 0}$. Using the flatness of $\mathcal{F}$, we are reduced to proving that $f_* \mathcal{F}' \in \mathrm{QCoh}(Y)_{\leq 0}$. This follows from the left t-exactness of f_*. $\square$

Remark 3.4.30. Let X be a geometric stack. Corollary 3.4.29 implies that for every connective $\mathbb{E}_\infty$-ring A, we can describe the space $X(A)$ in terms of the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$, together with its t-structure. If we are willing to restrict our attention to the case where A is a *discrete* $\mathbb{E}_\infty$-ring, then we can do better: $X(A)$ can be identified with a full subcategory of the *ordinary* category of symmetric monoidal functors between the abelian categories $\mathrm{QCoh}(X)^{\heartsuit}$ and $\mathrm{Mod}_A^{\heartsuit}$. See Theorem 3.0.1 for a more precise statement, and [39] for a proof.

4 Coaffine Stacks

Let X be an affine scheme, and let $R = \Gamma(X; \mathcal{O}_X)$ be the commutative ring $\mathcal{O}_X(X)$ of global sections of $\mathcal{O}_X$. Then:

(a) We can recover X as the spectrum $\operatorname{Spec}^c R$. More precisely, for any scheme Y, the canonical map $\operatorname{Hom}(Y, X) \to \operatorname{Hom}(R, \Gamma(Y; \mathcal{O}_Y))$ is bijective; here the Hom-set on the left is computed in the category of schemes, while the Hom-set on the right is computed in the category of commutative rings.

(b) The global sections functor $\mathcal{F} \mapsto \Gamma(X; \mathcal{F})$ determines an exact functor from the abelian category of quasi-coherent sheaves on X to the category of abelian groups. This construction underlies an equivalence between the category of quasi-coherent sheaves on X and the category of R-modules.

Assertion (a) is essentially the definition of an affine scheme. Under some mild additional hypotheses, one can show that (b) is also equivalent to the requirement that X be affine.

In the setting of spectral algebraic geometry, there is a large class of geometric objects which are not affine but nonetheless exhibit the sort of "affine behavior" embodied by statements (a) and (b). For example, suppose that $\mathfrak{X} = (\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is a quasi-affine spectral Deligne-Mumford stack, and let $R = \Gamma(\mathcal{X}; \mathcal{O}_{\mathcal{X}})$ be the $\mathbb{E}_\infty$-ring of global sections of $\mathcal{O}$. For every spectral Deligne-Mumford stack $\mathfrak{Y} = (\mathcal{Y}, \mathcal{O}_{\mathcal{Y}})$, the canonical map $\operatorname{Map}_{\mathrm{Stk}}(\mathfrak{Y}, \mathfrak{X}) \to \operatorname{Map}_{\mathrm{CAlg}}(R, \Gamma(\mathcal{Y}; \mathcal{O}_{\mathcal{Y}}))$ is a homotopy equivalence (Proposition 2.4.9), and the global sections functor $\mathcal{F} \mapsto \Gamma(\mathfrak{X}; \mathcal{F})$ induces an equivalence of ∞-categories $\operatorname{QCoh}(\mathfrak{X}) \to \operatorname{Mod}_R$ (Proposition 2.4.4).

Let us now describe another example of the same phenomenon. Let k be a field of characteristic zero, and let CAlg_k^0 denote the ∞-category of discrete $\mathbb{E}_\infty$-algebras over k (so that CAlg_k^0 is equivalent to the nerve of the ordinary category of commutative k-algebras). We let $B\mathbb{G}_a$ denote the classifying stack of the additive group over k. We will identify $B\mathbb{G}_a$ with a functor $\mathrm{CAlg}_k^0 \to \mathcal{S}$, given by the formula $R \mapsto BR$, where BR denotes the classifying space of the additive group of R. Let A denote the $\mathbb{E}_\infty$-ring of global sections of the structure sheaf of $\mathbb{G}_{\mathfrak{D}}$: that is, the limit

$$\varprojlim_{R \in \mathrm{CAlg}_k^0, \eta \in B\mathbb{G}_a(R)} R$$

formed in the ∞-category CAlg_k of $\mathbb{E}_\infty$-algebras over k. Unwinding the definitions (and performing a calculation), we obtain isomorphisms

$$\pi_i A \simeq \mathrm{H}^{-i}(\mathbb{G}_a; k) \simeq \begin{cases} k & \text{if } i \in \{0, -1\} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that A can be identified with the free $\mathbb{E}_\infty$-algebra $\operatorname{Sym}^* k[-1]$ on one generator (in homological degree -1). Consequently, for any discrete k-algebra R, we obtain a homotopy equivalence

$$\operatorname{Map}_{\mathrm{CAlg}_k}(A, R) \simeq \operatorname{Map}_{\mathrm{Mod}_k}(k[-1], R) \simeq BR \simeq B\mathbb{G}_a(R).$$

In other words, the classifying stack $B\mathbb{G}_a$ satisfies a version of condition (a) above, and can be recovered from its structure sheaf. One can also prove an analogue of (b) (thereby describing the derived category of representations of the additive group $\mathbb{G}_a$): see Proposition 4.5.2.

In this section, we will study a large class of geometric objects which exhibit behavior similar to that of the classifying stack $B\mathbb{G}_a$, which we call *coaffine stacks*. Roughly speaking, a coaffine stack over a field k of characteristic zero is a functor $X : \mathrm{CAlg}_k^0 \to \mathcal{S}$ such that each of the spaces $X(R)$ is connected, and each of the functors $R \mapsto \pi_i X(R)$ (defined relative to a choice of base point $\eta \in X(k)$) is representable by a prounipotent group scheme over k.

Remark 4.0.1. Our actual definition of coaffine stack will be slightly different: we will consider functors defined on the larger ∞-category $\mathrm{CAlg}_k^{\mathrm{cn}}$ of all connective $\mathbb{E}_\infty$-algebras over k. However, the distinction is ultimately irrelevant: every coaffine stack is determined by its value on discrete k-algebras: see Proposition 4.4.6.

Our main goal in this section is to prove that coaffine stacks satisfy analogues of the statements (a) and (b) given above. More precisely, every coaffine stack X is given by the formula $X(R) = \mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ for a canonically determined object $A \in \mathrm{CAlg}_k$. Moreover, we can explicitly describe those objects $A \in \mathrm{CAlg}_k$ which arise in this manner: they are precisely those algebras for which the unit map $k \to A$ exhibits k as a connective cover of A. The collection of $\mathbb{E}_\infty$-algebras over k satisfying this condition span a full subcategory $\mathrm{CAlg}_k^{\mathrm{cc}} \subseteq \mathrm{CAlg}_k$. We will study this ∞-category in detail in §4.1 and §4.3. To carry out our analysis, we will need some general facts about ∞-categories which are generated by noncompact projective objects, which we explain in §4.2.

In §4.4, we will *define* a coaffine stack to be a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which is corepresented by a coconnective $\mathbb{E}_\infty$-algebra A. We will show that A is uniquely determined (Theorem 4.4.1), and show that the class of coaffine stacks can be characterized in the manner described above (Proposition 4.4.8).

Our next goal is to prove an analogue of (b) in the setting of coaffine stacks. In §4.5, we will study the ∞-category of quasi-coherent sheaves $\mathrm{QCoh}(X)$ on a coaffine stack X. We will see that $\mathrm{QCoh}(X)$ admits a t-structure, whose heart is equivalent to the category of algebraic representations of $\pi_1 X$, regarded as a prounipotent group scheme over the field k. Moreover, we will show that there is a global sections functor $\mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which is fully faithful when restricted to the subcategory $\bigcup_n \mathrm{QCoh}(X)_{\leq n}$ (Proposition 4.5.2). Moreover, we can recover $\mathrm{QCoh}(X)$ as the completion of the stable ∞-category Mod_A with respect to suitable t-structure.

Our final objective in this section is to prove that coaffine stacks satisfy a version of Tannaka duality. More precisely, suppose that we are given functors $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, where X arises from a coaffine stack over a field of characteristic zero. We will prove that there is a fully faithful embedding

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)),$$

whose essential image is the collection of symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ which preserve small colimits and connective objects. We will prove this in §4.6 (see Corollary 4.6.19) as an application of some general remarks about completions of stable ∞-categories.

Remark 4.0.2. The notion of a coaffine stack was introduced by Toën, who uses the term *affine stack*. We refer the reader to [63] for an exposition of Toën's theory of affine stacks (over a field of arbitrary characteristic), which contains many of the results that we treat in this section.

4.1 Coconnective Algebras

Recall that an $\mathbb{E}_\infty$-ring A is said to be *connective* if it is connective when regarded as a spectrum: that is, if the homotopy groups $\pi_i A$ vanish for $i < 0$. The collection of all connective $\mathbb{E}_\infty$-rings spans a full subcategory $\mathrm{CAlg}^{\mathrm{cn}} \subseteq \mathrm{CAlg}$, which is closed under small colimits in CAlg.

In this section, we will study $\mathbb{E}_\infty$-ring A which satisfy the following alternative connectivity condition:

$(*)$ The homotopy groups $\pi_i A$ vanish for $i > 0$.

Though condition $(*)$ is formally analogous to the requirement that A is connective, it behaves quite differently in practice. For example, if A is an arbitrary $\mathbb{E}_\infty$-ring, then the spectrum $\tau_{\geq 0} A$ inherits the structure of a (connective) $\mathbb{E}_\infty$-ring, which is universal among connective $\mathbb{E}_\infty$-rings B equipped with a map $B \to A$. There is no analogous procedure for "killing" the higher homotopy groups of an $\mathbb{E}_\infty$-ring: the spectrum $\tau_{\leq 0} A$ generally does not admit a ring structure. Consequently, the full subcategory $\mathcal{C} \subseteq \mathrm{CAlg}$ spanned by those $\mathbb{E}_\infty$-ring satisfying $(*)$ is poorly behaved. To obtain any kind of reasonable theory, it is necessary to introduce a stronger version of condition $(*)$.

Definition 4.1.1. Let k be a field and let A be an $\mathbb{E}_0$-algebra over k: that is, A is a k-module spectrum equipped with a unit map $u : k \to A$. Let $n \geq 1$ be an integer. We will say that A is *n-coconnective* if the following conditions are equivalent:

(1) The unit map u induces an isomorphism of abelian groups $k \to \pi_0 A$.

(2) The homotopy groups $\pi_i A$ vanish for $i > 0$ and $-n < i < 0$.

If A is an $\mathbb{E}_m$-algebra over k for $0 \leq m \leq \infty$, then we will say that A is *n-coconnective* if it is n-coconnective when regarded as an $\mathbb{E}_0$-algebra. We will say that A is *coconnective* if it is 1-coconnective: that is, if the unit map $u : k \to A$ exhibits k as a connective cover of A.

We let $\mathrm{CAlg}_k^{\mathrm{cc}}$ denote the full subcategory of CAlg_k spanned by the coconnective $\mathbb{E}_\infty$-algebras over k.

The remainder of this section is devoted to establishing some of the formal properties of the ∞-category $\mathrm{CAlg}_k^{\mathrm{cc}}$ in the case where k is a field of characteristic zero. For example, we will show that every coconnective $\mathbb{E}_\infty$-algebra A over k can be built out of coconnective free algebas (Proposition 4.1.4), and use this to prove that the ∞-category $\mathrm{CAlg}_k^{\mathrm{cc}}$ is compactly generated (Proposition 4.1.8).

Notation 4.1.2. Let k be a field. We let $\mathrm{Sym}^* : \mathrm{Mod}_k \to \mathrm{CAlg}_k$ denote a left adjoint to the forgetful functor. For every k-module spectrum V, the underlying k-module of $\mathrm{Sym}^*(V)$ is given by $\bigoplus_{n \geq 0} \mathrm{Sym}^n(V)$, where $\mathrm{Sym}^n(V)$ denotes the coinvariants for the natural action of the symmetric group Σ_n on $V^{\otimes n}$ (see §A.3.1.3). We let $\mathrm{Sym}^{>0}(V)$ denote the coproduct $\bigoplus_{n>0} \mathrm{Sym}^n(V)$; it can be identified with the fiber of the canonical map of $\mathbb{E}_\infty$-algebras $\mathrm{Sym}^*(V) \to k$ (induced by the zero map $V \to k$).

Lemma 4.1.3. *Let k be a field of characteristic zero, let A be a coconnective $\mathbb{E}_\infty$-algebra over k, and V a k-module spectrum. Let $m \geq 1$ be an integer such that $\pi_n V \simeq 0$ for $n \geq -m$. Suppose we are given a pushout diagram*

$$\begin{tikzcd} \mathrm{Sym}^* V \arrow[r] \arrow[d] & A \arrow[d] \\ k \arrow[r] & A' \end{tikzcd}$$

of $\mathbb{E}_\infty$-algebras over k.

(*a*) *The map $\pi_n A \to \pi_n A'$ is an isomorphism for $n > -m$. In particular, A' is coconnective.*

(*b*) *There is an exact sequence of vector spaces*

$$0 \to \pi_{-m} A \to \pi_{-m} A' \to \pi_{-m-1} V \to \pi_{-m-1} A \to \pi_{-m-1} A'$$

(*c*) *For every object $R \in \mathrm{CAlg}_k^{\mathrm{cn}}$, the induced map $\theta : \mathrm{Map}_{\mathrm{CAlg}_k}(A', R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ is m-connective.*

Proof. We can write A' as a relative tensor product $A \otimes_{\mathrm{Sym}^* V} k$, which can be computed as the geometric realization of a simplicial object $A_\bullet$ in Mod_k with $A_n = A \otimes_k (\mathrm{Sym}^* V)^{\otimes n}$. Consequently, we obtain a spectral sequence $\{E_r^{p,q}, d_r\}$ converging to the homotopy groups $\pi_{p+q} A'$, with $E_1^{p,*} \simeq \pi_*(A \otimes_k (\mathrm{Sym}^{>0} V)^{\otimes p})$. Since k has characteristic zero, we can identify $\pi_*(\mathrm{Sym}^d V)$ with a quotient of $(\pi_* V)^{\otimes d}$; it follows that the homotopy group $\pi_i(\mathrm{Sym}^{>0} V)$ vanish for $i \geq -m$. Consequently, if $E_1^{p,q}$ does not vanish, we must have $q \leq p(-m-1)$, so that $p + q \leq pm$. It follows that the edge map $\pi_n A = E_1^{0,n} \to \pi_n A'$ is an isomorphism for $n > -m$, and that we have an exact sequence of low degree terms

$$0 \to \pi_{-m} A \to \pi_{-m} A' \to E_1^{1,-m-1} \to \pi_{-m-1} A \to \pi_{-m-1} A'$$

with $E_1^{1,-m-1} \simeq \pi_{-m-1} V$. This proves (*a*) and (*b*). Assertion (*c*) follows from the observation that θ is a pullback of the map

$$\theta' : * \simeq \mathrm{Map}_{\mathrm{CAlg}_k}(k, R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}^*(V), R) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(V, R).$$

Writing V as a coproduct of k-modules of the form $k[-n]$ for $n > m$, we see that $\mathrm{Map}_{\mathrm{Mod}_k}(V, R)$ is a product of spaces of the form $\Omega^\infty R[n]$ where $n > m$. Since R is connective, each of these spaces is $(m+1)$-connective. It follows that θ is m-connective. □

Proposition 4.1.4. *Let k be a field of characteristic zero, and let $\phi : A \to B$ be a map of coconnective $\mathbb{E}_\infty$-algebras over k. Let $m \geq 1$ be an integer, and assume that the map $\pi_n A \to \pi_n B$ is a bijective for $n > -m$ and injective for $n = -m$. Then there exists a diagram*

$$A = A(0) \to A(1) \to A(2) \to \cdots$$

in $\mathrm{CAlg}_k^{\mathrm{cc}}$ with the following properties:

(a) For each $i \geq 0$, we have a pushout diagram

$$\begin{tikzcd} \mathrm{Sym}^* V \arrow[r] \arrow[d] & A(i) \arrow[d] \\ k \arrow[r] & A(i+1), \end{tikzcd}$$

where V is a k-module spectrum such that $\pi_i V \simeq 0$ for $i \geq -m$.

(b) Each of the maps $A(i) \to A(i+1)$ induces a bijection $\pi_n A(i) \to \pi_n A(i+1)$ for $n > -m$, and an injection $\pi_{-m} A(i) \to \pi_{-m} A(i+1)$. In particular, each $A(i)$ is coconnective.

(c) The colimit $\varinjlim A(i)$ is equivalent to B (as an object of CAlg_A).

Proof. We construct a compatible sequence of maps $\phi_i : A(i) \to B$ satisfying the following condition:

$(*)$ The map $\pi_n A(i) \to \pi_n B$ is bijective for $n > -m$ and injective for $n = -m$.

The construction goes by recursion on i, beginning with the case $i = 0$ where we set $\phi_0 = \phi$. Assume that ϕ_i has been constructed, and form a pullback diagram

$$\begin{tikzcd} V \arrow[r] \arrow[d] & A(i) \arrow[d, "\phi_i"] \\ 0 \arrow[r] & B \end{tikzcd}$$

in the ∞-category Mod_k. We then obtain a diagram

$$\begin{tikzcd} \mathrm{Sym}^* V \arrow[r] \arrow[d] & A(i) \arrow[d, "\phi_i"] \\ k \arrow[r] & B \end{tikzcd}$$

in CAlg_k, which is classified by a map $\phi_{i+1} : A(i+1) \to B$ with $A(i+1) = A(i) \otimes_{\mathrm{Sym}^* V} k$. Since ϕ_i satisfies condition $(*)$, the homotopy groups $\pi_n V$ vanish for $n \geq -m$. It follows from Lemma 4.1.3 that the map $\pi_n A(i) \to \pi_n A(i+1)$ is bijective for $n > -m$, so that $\pi_n A(i+1) \simeq \pi_n B$. Moreover, we have a map of exact sequences

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & \pi_{-m} A(i) \arrow[r] \arrow[d] & \pi_{-m} A(i+1) \arrow[r] \arrow[d] & \pi_{-m-1} V \arrow[r] \arrow[d] & \pi_{-m-1} A \arrow[d] \\ 0 \arrow[r] & \pi_{-m} A \arrow[r] & \pi_{-m} B \arrow[r] & \pi_{-m-1} V \arrow[r] & \pi_{-m-1} A \end{tikzcd}$$

which proves that the map $\pi_{-m} A(i+1) \to \pi_{-m} B$ is an isomorphism.

To complete the proof, it will suffice to show that the canonical map $\phi : \varinjlim A(i) \to B$ is an equivalence of $\mathbb{E}_\infty$-algebras over k. It will suffice to show that ϕ is an equivalence of k-modules. In the ∞-category Mod_k, we have a larger diagram

$$A(0) \to A(0)/\operatorname{fib}(\phi_0) \to A(1) \to A(1)/\operatorname{fib}(\phi_1) \to \cdots.$$

It will therefore suffice to show that the canonical map $\varinjlim A(i)/\operatorname{fib}(\phi_i) \to B$ is an equivalence in Mod_k. This is clear, since the diagram $\{A(i)/\operatorname{fib}(\phi_i)\}$ is equivalent to the constant diagram taking the value B. □

Remark 4.1.5. The diagram $A = A(0) \to A(1) \to A(2) \to \cdots$ constructed in the proof of Proposition 4.1.4 depends functorially on the initial map $\phi : A \to B$.

Corollary 4.1.6. *Let $m \geq 1$, and let B be an m-coconnective $\mathbb{E}_\infty$-algebra over a field k of characteristic zero. Then we can write B as the colimit of a sequence $k = A(0) \to A(1) \to \cdots$ in $\mathrm{CAlg}_k^{\mathrm{cc}}$ with the following property: for each i, we have an equivalence $A(i+1) \simeq A(i) \otimes_{\mathrm{Sym}^* V} k$ for some k-module spectrum V such that $\pi_i V \simeq 0$ for $i \geq -m$.*

Proof. Apply Lemma 4.1.4 in the special case $A = k$. □

Corollary 4.1.7. *Let $m \geq 1$, let A be an m-coconnective $\mathbb{E}_\infty$-algebra over a field k of characteristic zero, and let $R \in \mathrm{CAlg}_k^{\mathrm{cn}}$. Then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ is m-connective.*

Proof. Write A as the colimit of a sequence

$$k = A(0) \to A(1) \to A(2) \to \cdots$$

as in Corollary 4.1.6. It follows that $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ is the limit of the tower of spaces

$$\cdots \to \mathrm{Map}_{\mathrm{CAlg}_k}(A(2), R) \xrightarrow{\theta(2)} \mathrm{Map}_{\mathrm{CAlg}_k}(A(1), R) \xrightarrow{\theta(1)} \mathrm{Map}_{\mathrm{CAlg}_k}(A(0), R) \simeq *$$

To prove that this limit is m-connective, it will suffice to show that each of the maps $\theta(i)$ is m-connective. This follows from assertion (c) of Lemma 4.1.3. □

Proposition 4.1.8. *Let k be a field of characteristic zero and let $\mathcal{C}$ denote the full subcategory of $\mathrm{CAlg}_k^{\mathrm{cc}}$ spanned by those coconnective $\mathbb{E}_\infty$-algebras over k which are compact objects of CAlg_k. Then the inclusion $\mathcal{C} \subseteq \mathrm{CAlg}_k^{\mathrm{cc}}$ induces an equivalence of ∞-categories $\mathrm{Ind}(\mathcal{C}) \simeq \mathrm{CAlg}_k^{\mathrm{cc}}$.*

Proof. Note that the ∞-category $\mathcal{C}$ is essentially small. Proposition T.5.3.5.10 implies that the inclusion $\mathcal{C} \subseteq \mathrm{CAlg}_k^{\mathrm{cc}}$ is homotopic to a composition

$$\mathcal{C} \xrightarrow{j} \mathrm{Ind}(\mathcal{C}) \xrightarrow{F} \mathrm{CAlg}_k^{\mathrm{cc}}$$

where F is a functor which preserves filtered colimits (determined up to contractible choice). Proposition T.5.3.5.11 implies that F is fully faithful; consequently, F is an equivalence of $\mathrm{Ind}(\mathcal{C})$ onto the smallest full subcategory $\mathcal{C}' \subseteq \mathrm{CAlg}_k^{\mathrm{cc}}$ which contains $\mathcal{C}$ and is stable under filtered colimits. We wish to prove that $\mathcal{C}' = \mathrm{CAlg}_k^{\mathrm{cc}}$. Fix a coconnective k-algebra A; we wish to prove that $A \in \mathcal{C}'$. Choose a sequence of coconnective k-algebras

$$k = A(0) \to A(1) \to A(2) \to \cdots$$

with colimit A as in Proposition 4.1.4. Since $\mathcal{C}'$ is stable under filtered colimits, it will suffice to show that each $A(i) \in \mathcal{C}'$. The proof now proceeds by induction on i, the case $i = 0$ being obvious. Assume that $A(i) \in \mathcal{C}$ and write $A(i+1) \simeq A(i) \otimes_{\mathrm{Sym}^* V} k$, where $V \in (\mathrm{Mod}_k)_{\leq -2}$; we wish to prove that $A(i+1) \in \mathcal{C}'$. Write V as a filtered colimit $\varinjlim V_\alpha$, where each $V_\alpha \in (\mathrm{Mod}_k)_{\leq -2}$ is a perfect k-module. Then $A(i+1) \simeq \varinjlim_\alpha A(i) \otimes_{\mathrm{Sym}^* V_\alpha} k$; it will therefore suffice to prove that each $A(i) \otimes_{\mathrm{Sym}^* V_\alpha} k$ belongs to $\mathcal{C}'$. Write $A(i)$ as a filtered colimit $\varinjlim_\beta A(i)_\beta$, where each $A(i)_\beta$ belongs to $\mathcal{C}$. Since $\mathrm{Sym}^* V_\alpha$ is compact, the map $\mathrm{Sym}^* V_\alpha \to A(i)$ factors through some $A(i)_\beta$. It follows that after reindexing the colimit, we can assume that the map $\mathrm{Sym}^* V_\alpha \to A(i)$ is a filtered colimit of maps $\mathrm{Sym}^* V_\alpha \to A(i)_\beta$. Then we have $A(i) \otimes_{\mathrm{Sym}^* V_\alpha} k \simeq \varinjlim_\beta A(i)_\beta \otimes_{\mathrm{Sym}^* V_\alpha} k$. It will therefore suffice to show that each tensor product $A(i)_\beta \otimes_{\mathrm{Sym}^* V_\alpha} k$ belongs to $\mathcal{C}'$. This is clear, since $A(i)_\beta \otimes_{\mathrm{Sym}^* V_\alpha} k$ is compact by construction and coconnective by Lemma 4.1.3. □

We now discuss some linearized versions of the above results.

Proposition 4.1.9. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field k and let M be a left A-module. Then there exists a sequence of A-modules*

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

with the following properties:

(a) *For each $n \geq 0$, there exists a k-module spectrum V such that $\pi_i V(n) \simeq 0$ for $i \geq 0$ and a cofiber sequence of left A-modules*
$$A \otimes_k V(n) \to M(n) \to M(n+1).$$

(b) *There exists a map $\theta : \varinjlim M(n) \to M$ which induces an isomorphism $\pi_m \varinjlim M(n) \to \pi_m M$ for $m \leq 0$.*

Remark 4.1.10. In the situation of Proposition 4.1.9, it follows easily by induction on n that $\pi_m M(n) \simeq 0$ for $m > 0$. It follows that $\pi_m \varinjlim M(n) \simeq 0$ for $m > 0$, so that the map $\theta : \varinjlim M(n) \to M$ is an equivalence if and only if $\pi_m M \simeq 0$ for $m < 0$.

Proof. Let M' be the underlying k-module spectrum of M, and let $V(1) = \tau_{\leq 0} M'$. Since k is a field, the canonical map $M' \to V(1)$ admits a section s. Set $M(1) = A \otimes_k V(1)$, so that s determines a map of left A-modules $M(1) \to M$. By construction, the map $\pi_m M(1) \to \pi_m M$ is surjective for $m \leq 0$ and bijective for $m = 0$. We construct $M(n) \in (\mathrm{LMod}_A)_{/M}$ for $n > 1$ by induction on n. Assume that we have already constructed $M(n-1) \in (\mathrm{LMod}_A)_{/M}$, and that the map $e_m : \pi_m M(n-1) \to \pi_m M$ is bijective for $m = 0$ and surjective for $m < 0$. Let W_m denote the kernel of e_m (as a vector space over k), and let $V(n) = \bigoplus_{m<0} W_m[m]$ (as a k-module spectrum). We have an evident map of k-modules $V(n) \to \mathrm{fib}(M(n-1) \to M)$, hence a map of left A-modules $f : (A \otimes_k V(n)) \to \mathrm{fib}(M(n-1) \to M)$. Let $M(n)$ denote the cofiber of the map $A \otimes_k V(n) \to M(n-1)$, so that f determines a map $M(n-1) \to M(n)$ in $(\mathrm{LMod}_A)_{/M}$ and we have a cofiber sequence
$$A \otimes_k V(n) \to M(n-1) \to M(n).$$
For each $m \in \mathbf{Z}$, let $e'_m : \pi_m M(n) \to \pi_m M$ be the evident map. It is clear that e'_m is surjective for $m < 0$ (since e_m factors through e'_m). We claim that e'_m is bijective when $m = 0$. To prove this, it suffices to show that the evident map $\pi_0 M(n-1) \to \pi_0 M(n)$ is bijective. We have a long exact sequence

$$\pi_1(A \otimes_k V(n)) \to \pi_0 M(n-1) \xrightarrow{e'_n} \pi_0 M(n) \to \pi_0(A \otimes_k V(n)) \xrightarrow{\phi} \pi_{-1} M(n-1).$$

Since A is coconnective and $\pi_m V(n) \simeq 0$ for $m > 0$, the vector space $\pi_1(A \otimes_k V(n))$ is trivial and $\pi_0(A \otimes_k V(n)) \simeq \pi_0 V(n) \simeq \ker(\pi_{-1} M(n-1) \to \pi_{-1} M)$. It follows that ϕ is injective so that e'_0 is an isomorphism.

It remains to prove that the map $\theta : \varinjlim M(n) \to M$ induces an isomorphism on π_m for $m \leq 0$. It is clear that the map $\vec{e}_m : \pi_m \varinjlim M(n) \to \pi_m M$ is surjective for $m \leq 0$. If η belongs to the kernel of $\vec{e}_m$, then η can be represented by an element of $\pi_m M(n-1)$ belonging to the kernel of e_m for some $n \gg 0$. By construction, the image of this class in $\pi_m M(n)$ vanishes, so that $\eta = 0$. $\square$

Corollary 4.1.11. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field k, let M be a left A-module, and let N be a right A-module. Suppose that $\pi_i M \simeq \pi_i N \simeq 0$ for $i > 0$. Then $\pi_i(N \otimes_A M) \simeq 0$ for $i > 0$. Moreover, the map $(\pi_0 N) \otimes_k (\pi_0 M) \to \pi_0(N \otimes_k M)$ is injective.*

Proof. Let $\{M(n)\}_{n \geq 0}$ be as in the proof of Proposition 4.1.9, so that $M \simeq \varinjlim M(n)$ by Remark 4.1.10. Then $\pi_i(N \otimes_A M) \simeq \varinjlim \pi_i(N \otimes_A M(n))$, and we have $M(1) \simeq A \otimes_k M$ so that

$$\pi_0(N \otimes_A M(1)) \simeq \pi_0(N \otimes_k M) \simeq (\pi_0 N) \otimes_k (\pi_0 M).$$

It will therefore suffice to show that $\pi_i(N \otimes_A M(1)) \simeq 0$ for each $i > 0$ and that the maps $\pi_0(N \otimes_A M(1)) \to \pi_0(N \otimes_A M(n))$ are injective, for which we use induction on n. When $n = 1$, the result is obvious. Otherwise, we have a cofiber sequence
$$A \otimes_k V(n) \to M(n-1) \to M(n)$$

where $V(n) \in (\mathrm{Mod}_k)_{\leq -1}$, whence a cofiber sequence of spectra

$$N \otimes_k V(n) \to N \otimes_A M(n-1) \to N \otimes_A M(n).$$

The desired result now follows from the inductive hypothesis, since $\pi_i(N \otimes_k V(n)) \simeq 0$ for $i \geq 0$. □

Corollary 4.1.12. *Let k be a field and let $\phi : A \to B$ be a map of coconnective $\mathbb{E}_1$-algebras over k. Let M be a left A-module such that $\pi_i M \simeq 0$ for $i > 0$. Then the homotopy groups $\pi_i(B \otimes_A M)$ vanish for $i > 0$, and the map $\pi_0 M \to \pi_0(B \otimes_A M)$ is injective.*

Corollary 4.1.13. *Let k be a field and let $\phi : A \to B$ be a map of coconnective $\mathbb{E}_1$-algebras over k. Let M be a left A-module such that $\pi_i M \simeq 0$ for $i > 0$. If $B \otimes_A M \simeq 0$, then $M \simeq 0$.*

Proposition 4.1.14. *Let A be a coconnective $\mathbb{E}_1$-algebra over a field k. Let M and N be left A-modules. Assume that $\pi_m M \simeq 0$ for $m > 0$ and that $\pi_m N \simeq 0$ for $m \leq 0$. Then any map $f : M \to N$ is nullhomotopic.*

Proof. Let $\{M(n)\}_{n \geq 0}$ be as in the proof of Proposition 4.1.9, so that $M \simeq \varinjlim M(n)$ by Remark 4.1.10. We may therefore identify $\mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ with the homotopy limit of the tower $\{\mathrm{Map}_{\mathrm{LMod}_A}(M(n), N)\}_{n \geq 0}$. To prove that $\mathrm{Map}_{\mathrm{LMod}_A}(M, N)$ is connected, it will suffice to show that each $\mathrm{Map}_{\mathrm{LMod}_A}(M(n), N)$ is connected, and that each map

$$\pi_1 \mathrm{Map}_{\mathrm{LMod}_A}(M(n), N) \to \pi_1 \mathrm{Map}_{\mathrm{LMod}_A}(M(n-1), N)$$

is surjective. We proceed by induction on n. Using the cofiber sequence

$$A \otimes_k V(n) \to M(n-1) \to M(n),$$

we obtain a fiber sequence of spaces

$$\mathrm{Map}_{\mathrm{LMod}_A}(M(n), N) \to \mathrm{Map}_{\mathrm{LMod}_A}(M(n-1), N) \to \mathrm{Map}_{\mathrm{LMod}_k}(V(n), N).$$

It will therefore suffice to show that $\pi_1 \mathrm{Map}_{\mathrm{LMod}_k}(V(n), N) \simeq 0$. Since k is a field, this follows immediately from the our assumptions that $\pi_m V(n) \simeq 0$ for $m \geq 0$ and $\pi_m N \simeq 0$ for $m \leq 0$. □

We now study the cotangent complexes of coconnective $\mathbb{E}_\infty$-algebras.

Proposition 4.1.15. *Let k be a field of characteristic zero, let $f : A \to B$ be a map of coconnective $\mathbb{E}_\infty$-algebras over k, and let $m \geq 1$ be an integer. Assume that the induced map $\pi_i A \to \pi_i B$ is bijective for $i > -m$ and injective when $i = -m$. Then:*

(1) *Let $L_{B/A}$ denote the relative cotangent complex of B over A. Then the homotopy groups $\pi_i L_{B/A}$ vanish for $i > -m$.*

(2) *Let B/A denote the cofiber of f (in the ∞-category of A-modules). Then the universal derivation $d : B \to B \oplus L_{B/A}$ induces an injection $\pi_{-m}(B/A) \to \pi_{-m} L_{B/A}$.*

Proof. Let $A = A(0) \to A(1) \to A(2) \to \cdots$ be the sequence constructed in the proof of Proposition 4.1.4. Note that $L_{B/A}$ can be identified with the colimit $\varinjlim(B \otimes_{A(i)} L_{A(i)/A})$. We have $A(1) = A \otimes_{\mathrm{Sym}^*(B/A[-1])} k$ so that $B \otimes_{A(1)} L_{A(1)/A}$ can be identified with $B \otimes_A (B/A)$. Moreover, the map $B/A \to L_{B/A}$ described in (2) is given by the composition

$$B/A \to B \otimes_A (B/A) \simeq B \otimes_{A(1)} L_{A(1)/A} \to L_{B/A}.$$

Proposition 4.1.9 guarantees that $\pi_i(B \otimes_A B/A)$ vanishes for $i > -m$ and that the map $\pi_{-m}(B/A) \to \pi_{-m}(B \otimes_A B/A)$ is injective. Consequently, it will suffice to prove the following for each $n \geq 1$:

($*$) The map $\pi_i(B \otimes_{A(n)} L_{A(n)/A}) \to \pi_i(B \otimes_{A(n+1)} L_{A(n+1)/A})$ is bijective for $i > -m$ and injective when $i = -m$.

In view of the fiber sequence

$$B \otimes_{A(n)} L_{A(n)/A} \to B \otimes_{A(n+1)} L_{A(n+1)/A} \to B \otimes_{A(n+1)} L_{A(n+1)/A(n)},$$

condition $(*)$ is equivalent to the requirement that the groups $\pi_i(B \otimes_{A(n+1)} L_{A(n+1)/A(n)})$ vanish for $i > -m$.

By assumption, we have a pushout diagram

$$\begin{tikzcd}
\mathrm{Sym}^* V \arrow[r] \arrow[d] & k \arrow[d] \\
A(n) \arrow[r] & A(n+1)
\end{tikzcd}$$

where $\pi_i V \simeq 0$ for $i \geq m$. This pushout diagram gives an equivalence $L_{A(n+1)/A(n)} \simeq A(n+1) \otimes_k L_{k/\mathrm{Sym}^* V} \simeq A(n+1) \otimes_k V[1]$, so that $\pi_i(B \otimes_{A(n+1)} L_{A(n+1)/A(n)}) \simeq \pi_i(B \otimes_k V[1])$ vanishes for $i > -m$ as desired. $\square$

Corollary 4.1.16. *Let k be a field of characteristic zero and let A be a coconnective $\mathbb{E}_\infty$-algebra over k. Then the homotopy groups $\pi_i L_{A/k}$ vanish for $i \geq 0$.*

Proposition 4.1.17. *Let k be a field of characteristic zero, let $f : A \to B$ be a morphism of coconnective $\mathbb{E}_\infty$-algebras over k, and choose a map of k-algebras $\eta : B \to k$ (Corollary 4.1.7). Then f is an equivalence if and only if, for each $i \leq -1$, the induced map $\phi_i : \pi_i(k \otimes_A L_{A/k}) \to \pi_i(k \otimes_B L_{B/k})$ is an isomorphism of vector spaces over k.*

Proof. The "only if" direction is clear. Conversely, suppose that each of the maps ϕ_i is an isomorphism. Let A/k denote the cofiber of the unit map $k \to A$, and let B/k and B/A be defined similarly. We have a commutative diagram

$$\begin{tikzcd}
\pi_{-1}(A/k) \arrow[r] \arrow[d] & \pi_{-1}(B/k) \arrow[d] \\
\pi_{-1} L_{A/k} \arrow[r] \arrow[d] & \pi_{-1} L_{B/k} \arrow[d] \\
\pi_{-1}(k \otimes_A L_{A/k}) \arrow[r, "\phi_{-1}"] & \pi_{-1}(k \otimes_B L_{B/k}).
\end{tikzcd}$$

Using Corollary 4.1.12 and Proposition 4.1.15, we deduce that the vertical maps are injections. Since ϕ_{-1} is an isomorphism, we conclude that the map $\pi_{-1} A/k \to \pi_{-1} B/k$ is injective so that $\pi_i B/A \simeq 0$ for $i \geq 0$. Assume that f is not an equivalence, and let $m \geq 1$ be the smallest integer such that $\pi_{-m} B/A$ is nonzero. Using Proposition 4.1.15, we deduce that the map $\pi_{-m} B/A \to \pi_{-m} L_{B/A}$ is injective, and that $\pi_i L_{B/A} \simeq 0$ for $i > -m$. It follows from Corollary 4.1.12 that $\pi_i(k \otimes_B L_{B/A}) \simeq 0$ for $i > -m$ and that $\pi_{-m}(k \otimes_B L_{B/A})$ is nonzero. Using the exact sequence

$$\pi_{-m}(k \otimes_A L_{A/k}) \xrightarrow{\phi_{-m}} \pi_{-m}(k \otimes_B L_{B/k}) \to \pi_{-m}(k \otimes_B L_{B/A}) \to \pi_{-m-1}(k \otimes_A L_{A/k}) \xrightarrow{\phi_{-m-1}} \pi_{-m-1}(k \otimes_B L_{B/k}),$$

we obtain a contradiction with our assumption that ϕ_{-m} and ϕ_{-m-1} are isomorphisms. $\square$

4.2 Digression: Strong Projectivity

Let R be an associative ring and let M be a left R-module. Recall that a *projective resolution* of M is an exact sequence of left R-modules

$$\cdots \to P_2 \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective. Using the Dold-Kan correspondence (§A.1.2.3), we can identify the chain complex P_* with a simplicial left R-module $P_\bullet$. The exactness of the sequence

$$P_1 \to P_0 \to M \to 0$$

implies that M can be described as the geometric realization $|P_\bullet|$ in the abelian category of left R-modules. The exactness in higher degrees leads to a stronger assertion: we can identify M with the geometric realization $|P_\bullet|$ in the ∞-category LMod_R of left R-module spectra. This a special case of a much more general assertion: if R is any connective $\mathbb{E}_1$-ring and M is a connective left module over R, then we can describe M as the geometric realization of a simplicial object $P_\bullet$ of LMod_R, where each P_n is projective. We can describe the situation informally by saying that the ∞-category $\mathrm{LMod}_R^{\mathrm{cn}}$ is *generated by projective objects*. Our goal in this section is to make a general study of this phenomenon.

Let $\mathcal{C}$ be an arbitrary ∞-category which admits small colimits. Recall that an object $P \in \mathcal{C}$ is said to be *projective* it the functor $C \mapsto \mathrm{Map}_{\mathcal{C}}(P, C)$ preserves geometric realizations of simplicial objects. The collection of projective objects of $\mathcal{C}$ comprise a full subcategory $\mathcal{C}^{\mathrm{proj}} \subseteq \mathcal{C}$. We would like to show that, in good cases, we can functorially recover the ∞-category $\mathcal{C}$ from the subcategory $\mathcal{C}^{\mathrm{proj}}$. Note that if $P_\bullet$ and $Q_\bullet$ are simplicial objects of $\mathcal{C}^{\mathrm{proj}}$, then we can describe the mapping space

$$\mathrm{Map}_{\mathcal{C}}(|P_\bullet|, |Q_\bullet|) \simeq \varprojlim_{[m]} \mathrm{Map}_{\mathcal{C}}(P_m, |Q_\bullet|) \simeq \varprojlim_{[m]} |\mathrm{Map}_{\mathcal{C}^{\mathrm{proj}}}(P_m, Q_n)|$$

entirely in terms of the mapping spaces in $\mathcal{C}^{\mathrm{proj}}$. The main question, then, is to decide when an arbitrary object $C \in \mathcal{C}$ can be obtained as a geometric realization $|P_\bullet|$, where each P_n is projective. In this case, we have a map $u : P_0 \to C$ which is *surjective* in the following sense: for every projective object $Q \in \mathcal{C}$, every map $Q \to C$ factors through u up to homotopy. We might attempt to construct P_0 by taking a large coproduct of projective objects of $\mathcal{C}$ which map to C. However, we encounter the following technical obstacle: though the collection of projective objects of $\mathcal{C}$ is always stable under finite coproducts (Remark T.5.5.8.19), it is generally not stable under *infinite* coproducts (for example, a discrete space is not a projective object of $\mathcal{S}$ if it has infinitely many path components). To address this point, we introduce a stronger notion of projectivity.

Definition 4.2.1. Let $\mathcal{C}$ be an ∞-category which admits geometric realizations. We will say that an object $P \in \mathcal{C}$ is *strongly projective* if P corepresents a functor $e : \mathcal{C} \to \mathcal{S}$ with the following property:

($*$) For every simplicial object $X_\bullet$ in $\mathcal{C}$, the simplicial space $e(X_\bullet)$ is a hypercovering of $e(|X_\bullet|)$.

Remark 4.2.2. Every strongly projective object of an ∞-category $\mathcal{C}$ is projective.

Remark 4.2.3. Let $\mathcal{C}$ be the full subcategory of $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+^{op}), \mathcal{S})$ spanned by those augmented simplicial spaces $X_\bullet$ whose underlying simplicial space is a hypercovering of X_{-1}. Then $\mathcal{C}$ is stable under products in $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+^{op}), \mathcal{S})$: this follows from the observation that the collection of effective epimorphisms in $\mathcal{S}$ is stable under products.

Proposition 4.2.4. *Let $\mathcal{C}$ be an ∞-category which admits geometric realizations. Then the collection of strongly projective objects of $\mathcal{C}$ is stable under all coproducts which exist in $\mathcal{C}$.*

Proof. This is an immediate consequence of Remark 4.2.3. □

The following result guarantees that ∞-categories of an algebraic flavor have a good supply of strongly projective objects.

Proposition 4.2.5. *Let $\mathcal{C}$ be an ∞-category which admits geometric realizations, and let P be a cogroup object of $\mathcal{C}$ (that is, a group object of the opposite ∞-category $\mathcal{C}^{op}$). Then P is projective if and only if it is strongly projective.*

The proof of Proposition 4.2.5 will require some preliminaries.

Proposition 4.2.6. *Let $\mathcal{X}$ be an ∞-topos, $\mathcal{G}\mathrm{rp}(\mathcal{X})$ the ∞-category of group objects of $\mathcal{X}$ (which we regard as a full subcategory of the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{X})$ of simplicial objects of $\mathcal{X}$), and K a sifted simplicial* *set. Suppose we are given a pullback diagram*

$$\begin{tikzcd} W \arrow[r] \arrow[d] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

in the ∞*-category* $\mathrm{Fun}(K^{\triangleright}, \mathcal{G}\mathrm{rp}(\mathcal{X}))$ *satisfying the following conditions:*

(i) *The functors* X, Y, *and* Z *are colimit diagrams.*

(ii) *For every vertex* v *of* K, *the map* $Y(k) \to Z(k)$ *induces an effective epimorphism in* $\mathcal{X}$.

Then W *is a colimit diagram in* $\mathcal{G}\mathrm{rp}(\mathcal{X})$.

Proof. We observe that condition (ii) is also satisfied when v is the cone point of $K^{\triangleleft}$, since the collection of effective epimorphisms in $\mathcal{X}$ is stable under colimits.

Since $\mathcal{X}$ is an ∞-topos, the formation of geometric realizations determines an equivalence of ∞-categories from $\mathcal{G}\mathrm{rp}(\mathcal{X})$ to the ∞-category $\mathcal{X}_*^{\geq 1}$ of pointed connected objects of $\mathcal{X}$. Applying this equivalence, we have a commutative diagram σ :

$$\begin{tikzcd} W' \arrow[r] \arrow[d] & X' \arrow[d] \\ Y' \arrow[r] & Z' \end{tikzcd}$$

of functors from $K^{\triangleleft}$ to $\mathcal{X}_*$. Since the forgetful functor $\mathcal{G}\mathrm{rp}(\mathcal{X}) \to \mathcal{X}$ is conservative and preserves sifted colimits, we deduce that X', Y', and Z' are colimit diagrams, and we wish to prove that W' is a colimit diagram. Using Lemma A.5.3.6.17, we are reduced to proving that σ is a pullback square. The diagram σ is evidently a pullback square in $\mathrm{Fun}(K^{\triangleleft}, \mathcal{X}_*^{\geq 1})$, so it will suffice to show that the fiber product $X' \times_{Z'} Y'$ (formed in the larger ∞-category $\mathrm{Fun}(K^{\triangleleft}, \mathcal{X}_*)$) belongs to $\mathrm{Fun}(K^{\triangleleft}, \mathcal{X}_*^{\geq 1})$. In other words, we wish to show that for every vertex $v \in K^{\triangleleft}$, the fiber product $X'(v) \times_{Z'(v)} Y'(v)$ is a connected object of $\mathcal{X}$. Since the map $Y(v) \to Z(v)$ is an effective epimorphism, we deduce that its delooping $Y'(v) \to Z'(v)$ is 1-connective. It follows that the projection map $X'(v) \times_{Z'(v)} Y'(v) \to X'(v)$ is 1-connective. The desired result now follows from the observation that $X'(v)$ is connected. ☐

Corollary 4.2.7. *Let* $\mathcal{X}$ *be an* ∞*-topos, let* $X_\bullet$ *be a simplicial object in the* ∞*-category* $\mathcal{G}\mathrm{rp}(\mathcal{X})$. *Then* $X_\bullet$ *is a hypercovering of its geometric realization* $|X_\bullet|$.

Proof. Without loss of generality, we may suppose that $\mathcal{X}$ is the essential image of a left exact localization functor $L : \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{C})$, for some small ∞-category $\mathcal{C}$. We may assume without loss of generality that $X_\bullet \simeq LY_\bullet$, for some simplicial object $Y_\bullet$ of $\mathcal{G}\mathrm{rp}(\mathcal{P}(\mathcal{C}))$ (for example, we can take $Y_\bullet = X_\bullet$). Since $L : \mathcal{P}(\mathcal{C}) \to \mathcal{X}$ preserves colimits, we have an equivalence $L|Y_\bullet| \simeq |X_\bullet|$. Since L preserves hypercoverings, it will suffice to show that $Y_\bullet$ is a hypercovering of $|Y_\bullet|$. For this, we need only show that $Y_\bullet(C)$ is a hypercovering of $|Y_\bullet(C)|$ in $\mathcal{S}$, for each object $C \in \mathcal{C}$. In other words, we may assume that $\mathcal{X}$ is the ∞-topos $\mathcal{S}$ of spaces.

Let $\mathbf{A}$ denote the category of simplicial groups, regarded as a simplicial model category; we then have a canonical equivalence of ∞-categories $\mathrm{N}(\mathbf{A}^{o}) \to \mathcal{G}\mathrm{rp}(\mathcal{S})$ (see Remark A.5.1.3.13). Let

$$\overline{X}_\bullet \in \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+^{op}), \mathcal{G}\mathrm{rp}(\mathcal{S}))$$

be a colimit of $X_\bullet$. Using Proposition T.4.2.4.4, we may assume that $\overline{X}_\bullet$ is image of an augmented simplicial object $G : \mathbf{\Delta}_+^{op} \to \mathbf{A}$. We will identify G with a simplicial object in the category $\mathbf{A}_{/G([-1])}$. For every simplicial set K, let $G(K) \in \mathbf{A}$ denote the limit $\varprojlim_{\sigma \in \mathrm{Hom}_{\mathrm{Set}_\Delta}(\Delta^n, K)} G([n])$, computed in the category $\mathbf{A}_{/G([-1])}$. Without loss of generality, we may assume that G is Reedy fibrant. Then the map from X_n to the matching object $M_n(X_\bullet)$ (computed in the ∞-category $\mathcal{S}_{/|X_\bullet|}$) can be identified with the map θ :

$G(\Delta^n) \to G(\partial\,\Delta^n)$. Consequently, to prove that $X_\bullet$ is a hypercovering $|X_\bullet|$, it will suffice to show that the map $\pi_0 G(\Delta^n) \to \pi_0 G(\partial\,\Delta^n)$ is surjective. Since θ is a Kan fibration (by virtue of our assumption that G is Reedy fibrant), this is equivalent to the requirement that $\theta : G(\Delta^n)_0 \to G(\partial\,\Delta^n)_0$ is a surjection of groups.

Given an inclusion of simplicial sets $A \subseteq B$, we let $G(B, A)$ denote the fiber of the restriction map $G(B) \to G(A)$. Since the map $\overline{X}_0 \to \overline{X}_{-1}$ is an effective epimorphism, the fibration $G(\Delta^0) \to G(\emptyset)$ is surjective on connected components and therefore induces a surjection $G(\Delta^0)_0 \to G(\emptyset)_0$. Every nonempty simplicial set K contains Δ^0 as a retract, so that the map $G(K)_0 \to G(\emptyset)_0$ is likewise surjective (this is evidently true also if $K = \emptyset$). We have a commutative diagram

$$\begin{tikzcd}
G(\Delta^n, \emptyset)_0 \arrow[r] \arrow[d, "\theta'"] & G(\Delta^n)_0 \arrow[r] \arrow[d, "\theta"] & G(\emptyset)_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
G(\partial\,\Delta^n, \emptyset)_0 \arrow[r] & G(\partial\,\Delta^n)_0 \arrow[r] & G(\emptyset)_0 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Consequently, to prove that θ is surjective, it will suffice to show that θ' is surjective.

Let $H_\bullet$ denote the simplicial group given by the formula $H_n = G(\Delta^n, \emptyset)_0$. Then $H_\bullet$ is automatically fibrant. Consequently, the map $G(\Delta^n, \emptyset)_0 \to G(\Lambda^n_0, \emptyset)_0$ is surjective. We have a commutative diagram

$$\begin{tikzcd}
G(\Delta^n, \Lambda^n_0)_0 \arrow[r] \arrow[d, "\theta''"] & G(\Delta^n, \emptyset)_0 \arrow[r] \arrow[d, "\theta'"] & G(\Lambda^n_0, \emptyset)_0 \arrow[r] \arrow[d] & 0 \arrow[d] \\
G(\partial\,\Delta^n, \Lambda^n_0)_0 \arrow[r] & G(\partial\,\Delta^n, \emptyset) \arrow[r] & G(\Lambda^n_0, \emptyset)_0 \arrow[r] & 0
\end{tikzcd}$$

with exact rows. Consequently, to prove that θ' is surjective, it will suffice to show that the map

$$\theta'' : G(\Delta^n, \Lambda^n_0)_0 \to G(\partial\,\Delta^n, \Lambda^n_0)_0 \simeq G(\Delta^{\{1,\ldots,n\}}, \partial\,\Delta^{\{1,\ldots,n\}})_0$$

is surjective. To complete the proof, we will verify the following:

- $(*)$ Let $G : \mathbf{\Delta}_+^{op} \to \mathbf{A}$ be an augmented simplicial object of the category $\mathbf{A}$ of simplicial groups. Assume that G is Reedy fibrant and is a homotopy colimit diagram in $\mathbf{A}$. Then the map $\theta'' : G(\Delta^n, \Lambda^n_0)_0 \to G(\Delta^{\{1,\ldots,n\}}, \partial\,\Delta^{\{1,\ldots,n\}})_0$ is surjective.

We will prove $(*)$ by induction on n. The case $n = 0$ is obvious, since the group $G(\Delta^{\{1,\ldots,n\}}, \partial\,\Delta^{\{1,\ldots,n\}})_0 \simeq G(\partial\,\Delta^n, \Lambda^n_0)_0$ is trivial. To handle the inductive step, let TG denote the augmented simplicial group given by the formula $TG([m]) = G([m] \star [0]) = G([m+1])$, and form a pullback diagram (in the category of augmented simplicial objects of $\mathbf{A}$)

$$\begin{tikzcd}
G' \arrow[r] \arrow[d] & * \arrow[d] \\
TG \arrow[r] & G.
\end{tikzcd}$$

Since each of the face maps $TG([m]) \simeq G([m+1]) \to G([m])$ is a fibration, the above diagram is a homotopy pullback square. Note that TG is a split augmented simplicial object of $\mathbf{A}$, and therefore automatically a homotopy colimit diagram. For $n \geq 0$, the face map $TG([m]) \to G([m])$ admits a section, and therefore determines an effective epimorphism in $\mathcal{S}$. Invoking Proposition 4.2.6, we deduce that G' is a homotopy colimit diagram in $\mathbf{A}$. We have a commutative diagram

$$\begin{tikzcd}
G(\Delta^n, \Lambda^n_0) \arrow[r] \arrow[d] & G(\Delta^{\{1,\ldots,n\}}, \partial\,\Delta^{\{1,\ldots,n\}}) \arrow[d] \\
G'(\Delta^{n-1}, \Lambda^{n-1}_0) \arrow[r] & G'(\Delta^{\{1,\ldots,n-1\}}, \partial\,\Delta^{\{1,\ldots,n-1\}})
\end{tikzcd}$$

in which the vertical maps are isomorphisms of simplicial groups. The inductive hypothesis guarantees that $G'(\Delta^{n-1}, \Lambda^{n-1}_0)_0 \to G'(\Delta^{\{1,\ldots,n-1\}}, \partial\,\Delta^{\{1,\ldots,n-1\}})_0$ is surjective. This implies that the map θ'' : $G(\Delta^n, \Lambda^n_0)_0 \to G(\Delta^{\{1,\ldots,n\}}, \partial\,\Delta^{\{1,\ldots,n\}})_0$ is also surjective as required. □

Corollary 4.2.8. *Let* $\mathcal{G}\mathrm{rp}(\mathcal{S})$ *denote the* ∞*-category of group objects of* $\mathcal{S}$*. Let* $F : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{G}\mathrm{rp}(\mathcal{S})) \to \mathcal{S}$ *denote the composition of the forgetful functor* $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{G}\mathrm{rp}(\mathcal{S})) \to \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{S})$ *with the geometric realization functor* $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{S}) \to \mathcal{S}$*. Then* F *commutes with small products.*

Proof. It suffices to show that the collection of augmented simplicial objects of $\mathcal{G}\mathrm{rp}(\mathcal{S})$ which determine colimit diagrams in $\mathcal{S}$ is stable under products. This follows immediately from Corollary 4.2.7 together with Remark 4.2.3. □

Proof of Proposition 4.2.5. It is obvious that if $P \in \mathcal{C}$ is strongly projective, then P is projective. For the converse, we observe that because P is a cogroup object, the functor $e : \mathcal{C} \to \mathcal{S}$ corepresented by P can be lifted to a functor $\overline{e} : \mathcal{C} \to \mathcal{G}\mathrm{rp}(\mathcal{S})$. It follows from Corollary 4.2.7 that e carries every simplicial object $X_\bullet$ of $\mathcal{C}$ to a hypercovering of $|e(X_\bullet)|$. If e is projective, then this geometric realization can be identified with $e|X_\bullet|$. □

Let $\mathcal{C}$ be a presentable ∞-category, and assume that there exists a set of of compact projective generators for $\mathcal{C}$ (Definition T.5.5.8.23). Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory of $\mathcal{C}$ spanned by the compact projective objects. According to Proposition T.5.5.8.25, the ∞-category $\mathcal{C}$ is equivalent to $\mathcal{P}_\Sigma(\mathcal{C}_0)$, the full subcategory of $\mathrm{Fun}(\mathcal{C}_0^{op}, \mathcal{S})$ spanned by those functors which preserve finite products. Our goal for the remainder of this section is to establish a similar reconstruction result in case where the projective generators which are not assumed to be compact.

Definition 4.2.9. Let $\mathcal{C}$ be an ∞-category. We will say that $\mathcal{C}$ is a *socle* if the following conditions are satisfied:

(1) The ∞-category $\mathcal{C}$ is locally small.

(2) The ∞-category $\mathcal{C}$ admits small coproducts.

(3) There exists an essentially small full subcategory $\mathcal{C}_0 \subseteq \mathcal{C}$ with the following property: every object $C \in \mathcal{C}$ is a retract of a coproduct $\coprod_\alpha C_\alpha$, where each $C_\alpha \in \mathcal{C}_0$.

(4) Every object $C \in \mathcal{C}$ has the structure of a cogroup (that is, we can regard C as a group object of $\mathcal{C}^{op}$).

If $\mathcal{C}$ is a socle, we let $\mathcal{P}_\sigma(\mathcal{C})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ spanned by those functors which preserve small products.

Example 4.2.10. Let $\mathcal{C}$ be a pointed ∞-category which admits finite colimits. Then for every object $X \in \mathcal{C}$, the suspension ΣX is a cogroup object of $\mathcal{C}$. In particular, if the suspension functor $\Sigma : \mathcal{C} \to \mathcal{C}$ is essentially surjective (for example, if $\mathcal{C}$ is stable), then $\mathcal{C}$ satisfies condition (4) of Definition 4.2.9.

Proposition 4.2.11. *Let* $\mathcal{C}$ *be a socle. Then:*

(1) *The full subcategory* $\mathcal{P}_\sigma(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ *contains the essential image of the Yoneda embedding* j : $\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ *and is stable under small limits in* $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$*.*

(2) *The full subcategory* $\mathcal{P}_\sigma(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ *is closed under the formation of geometric realizations of simplicial objects.*

(3) *When regarded as a functor from* $\mathcal{C}$ *to* $\mathcal{P}_\sigma(\mathcal{C})$*, the Yoneda embedding* j *preserves small coproducts.*

(4) *For every object* $X \in \mathcal{P}_\sigma(\mathcal{C})$*, there exists an augmented simplicial object* $X_\bullet \in \mathcal{P}_\sigma(\mathcal{C})$ *with the following properties:*

(a) *For $n \geq 0$, the object X_n belongs to the essential image of j.*

(b) *The augmented simplicial object $X_\bullet$ is a hypercovering; that is, for every $C \in \mathcal{C}$, the augmented simplicial space $X_\bullet(C)$ is a hypercovering. In particular, $X_\bullet$ is a colimit diagram in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$.*

(c) *We have an equivalence $X_{-1} \simeq X$.*

(5) *The $\mathcal{P}_\sigma(\mathcal{C})$ is the smallest full subcategory of $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ which contains the essential image of the Yoneda embedding and is stable under geometric realizations of simplicial objects.*

(6) *Let $\mathcal{D}$ be any ∞-category which admits geometric realizations, and let $\mathrm{Fun}_\sigma(\mathcal{P}_\sigma(\mathcal{C}), \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{P}_\sigma(\mathcal{C}), \mathcal{D})$ spanned by those functors which preserve geometric realization of simplicial objects. Then composition with the Yoneda embedding induces an equivalence of ∞-categories $\mathrm{Fun}_\sigma(\mathcal{P}_\sigma(\mathcal{C}), \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}, \mathcal{D})$.*

Proof. Assertion (1), (3) and the implication (4) $\Rightarrow$ (5) are obvious. Assertion (4) is a special case of Proposition A.7.2.1.4, and the implication (5) $\Rightarrow$ (6) follows from Remark T.5.3.5.9. We next prove (2). Let $X_\bullet$ be a simplicial object of $\mathcal{P}_\sigma(\mathcal{C})$, and let X denote the geometric realization $|X_\bullet|$ formed in the ∞-category $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$. We wish to prove that $X \in \mathcal{P}_\sigma(\mathcal{C})$. In other words, we wish to show that if $\{C_\alpha\}$ is a small collection of objects of $\mathcal{C}$ having a coproduct $C \in \mathcal{C}$, then the natural map $X(C) \to \prod_\alpha X(C_\alpha)$ is an equivalence. We must show that the map $\phi : |X_\bullet(C)| \to \prod_\alpha |X_\bullet(C_\alpha)|$ is an equivalence. Since each X_n belongs to $\mathcal{P}_\sigma(\mathcal{C})$, we can identify ϕ with the natural map

$$|\prod_\alpha X_\bullet(C_\alpha)| \to \prod_\alpha |X_\bullet(C_\alpha)|.$$

Since each C_α is a cogroup object of $\mathcal{C}$ and each X_n carries finite coproducts to finite products, we deduce that each $X_\bullet(C_\alpha)$ can be identified with a simplicial object in the ∞-category $\mathcal{G}\mathrm{rp}(\mathcal{S})$ of group objects of $\mathcal{S}$. The desired result now follows from Corollary 4.2.8. □

Corollary 4.2.12. *Let $\mathcal{C}$ be a socle and let $X \in \mathcal{P}_\sigma(\mathcal{C})$. Then X is a projective object of $\mathcal{P}_\sigma(\mathcal{C})$ if and only if X is a retract of $j(C)$ for some $C \in \mathcal{C}$; here $j : \mathcal{C} \to \mathcal{P}_\sigma(\mathcal{C}) \subseteq \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$ denotes the Yoneda embedding.*

Proof. Since $\mathcal{P}_\sigma(\mathcal{C})$ is stable under geometric realizations in $\mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, we deduce immediately that $j(C) \in \mathcal{P}_\sigma(\mathcal{C})$ is projective for each $C \in \mathcal{C}$, and consequently any retract of $j(C)$ is projective. Conversely, suppose that $X \in \mathcal{P}_\sigma(\mathcal{C})$ is projective, and write X as the geometric realization of a simplicial object $X_\bullet$ as in the proof of Proposition 4.2.11. Since X is projective, the identity map $X \simeq |X_\bullet|$ lies in the image of the map $\mathrm{Map}_{\mathcal{P}_\sigma(\mathcal{C})}(X, X_0) \to \mathrm{Map}_{\mathcal{P}_\sigma(\mathcal{C})}(X, |X_\bullet|)$. It follows that X is a retract of $X_0 \in j(\mathcal{C})$. □

Corollary 4.2.13. *Let $\mathcal{C}$ be a socle. Then the ∞-category $\mathcal{P}_\sigma(\mathcal{C})$ is locally small.*

Proof. Let $\mathcal{D} \subseteq \mathcal{P}_\sigma(\mathcal{C})$ be the full subcategory spanned by those objects X for which the mapping space $\mathrm{Map}_{\mathcal{P}_\Sigma(\mathcal{C})}(X, Y)$ is essentially small for every $Y \in \mathcal{P}_\sigma(\mathcal{C})$. We wish to prove that $\mathcal{D} = \mathcal{P}_\sigma(\mathcal{C})$. In view of Proposition 4.2.11, this follows from the following pair of observations:

(a) The full subcategory $\mathcal{D}$ contains the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{S})$, since $\mathrm{Map}_{\mathcal{P}_\sigma(\mathcal{C})}(j(C), Y) \simeq Y(C)$.

(b) The full subcategory $\mathcal{D}$ is stable under geometric realizations of simplicial objects, since the collection of essentially small spaces is stable under the formation of totalizations of cosimplicial objects.

□

Corollary 4.2.14. *Let $\mathcal{C}$ be a socle. Then the ∞-category $\mathcal{P}_\sigma(\mathcal{C})$ admits small colimits. Moreover, if $\mathcal{D}$ is another ∞-category which admits small colimits and $F : \mathcal{P}_\sigma(\mathcal{C}) \to \mathcal{D}$ is a functor, then F preserves small colimits if and only if the following conditions are satisfied:*

(1) *The functor F preserves geometric realizations of simplicial objects.*

(2) *The composition $\mathcal{C} \xrightarrow{j} \mathcal{P}_\sigma(\mathcal{C}) \xrightarrow{F} \mathcal{D}$ preserves small coproducts.*

Proof. The necessity of conditions (1) and (2) is clear. Conversely, suppose that $F : \mathcal{P}_\sigma(\mathcal{C}) \to \mathcal{D}$ is a functor satisfying (1) and (2) (for example, the identity functor); we will show that $\mathcal{P}_\sigma(\mathcal{C})$ admits small colimits and that those colimits are preserved by F. We first show that $\mathcal{P}_\sigma(\mathcal{C})$ admits small coproducts (and that these coproducts are preserved by F). Let $\{X^\alpha\}_{\alpha \in A}$ be a small collection of objects in $\mathcal{P}_\sigma(\mathcal{C})$. For each index α, we use Proposition 4.2.11 to choose a simplicial object $X^\alpha_\bullet$ with the following properties:

(a) Each X^α_n has the form $j(C^\alpha_n)$ for some $C^\alpha_n \in \mathcal{C}$.

(b) The geometric realization of $X^\alpha_\bullet$ is equivalent to X^α.

For each $n \geq 0$, the objects X^α_n admit a coproduct $X_n = j(\coprod_\alpha C^\alpha_n)$. It follows that the simplicial objects $\{X^\alpha_\bullet\}_{\alpha \in A}$ admit a coproduct $X_\bullet$ having a colimit $|X_\bullet|$ which can be identified with a coproduct $\coprod_\alpha X^\alpha$. We note that

$$\begin{aligned} F(|X_\bullet|) &\simeq |F(X_\bullet)| \\ &\simeq |\coprod_\alpha F(X^\alpha_\bullet)| \\ &\simeq \coprod_\alpha |F(X^\alpha_\bullet)| \\ &\simeq \coprod_\alpha F(|X^\alpha_\bullet|) \\ &\simeq \coprod_\alpha F(X^\alpha) \end{aligned}$$

so that F preserves small coproducts.

To complete the proof, it will suffice to show that $\mathcal{P}_\sigma(\mathcal{C})$ admits coequalizers which are preserved by the functor F (Propositions T.4.4.3.2 and T.4.4.3.3). Choose a coequalizer diagram

$$Y' \rightrightarrows Y,$$

which we identify with a functor $G : \mathrm{N}(\mathbf{\Delta}^s_{\leq 1})^{op} \to \mathcal{P}_\sigma(\mathcal{C})$. Since $\mathcal{P}_\sigma(\mathcal{C})$ admits finite coproducts, there exists a functor $X_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{P}_\sigma(\mathcal{C})$ which is a left Kan extension of G along the inclusion $i : \mathrm{N}(\mathbf{\Delta}^s_{\leq 1})^{op} \hookrightarrow \mathrm{N}(\mathbf{\Delta})^{op}$; more concretely, $X_\bullet$ is the simplicial object of $\mathcal{P}_\sigma(\mathcal{C})$ given informally by the formula $X_n = Y \coprod Y' \coprod \cdots \coprod Y'$. Then the derived coequalizer can be identified with the geometric realization $|X_\bullet|$. Similarly, since F preserves coproducts, the functor $F \circ X_\bullet$ is a left Kan extension of $F \circ G$ along i, so we can identify $F(|X_\bullet|) \simeq |F(X_\bullet)|$ with the coequalizer of the diagram

$$F(Y') \rightrightarrows F(Y).$$

□

We close this section by characterizing those ∞-categories which are of the form $\mathcal{P}_\sigma(\mathcal{C})$, where $\mathcal{C}$ is a socle.

Proposition 4.2.15. *Let $\mathcal{C}$ be a socle, let $\mathcal{D}$ be an ∞-category which admits geometric realizations of simplicial objects, let $F : \mathcal{P}_\sigma(\mathcal{C}) \to \mathcal{D}$ be a functor which preserves geometric realizations of simplicial objects, and let f denote the composition $\mathcal{C} \xrightarrow{j} \mathcal{P}_\sigma(\mathcal{C}) \xrightarrow{F} \mathcal{D}$ (so that F is determined by f up to equivalence, by Proposition 4.2.11). Assume that:*

(i) *The functor f is fully faithful.*

(ii) *For every object $C \in \mathcal{C}$, the image $f(C) \in \mathcal{D}$ is projective.*

Then F is fully faithful. Moreover, F is an equivalence if and only if the following additional conditions are satisfied:

(iii) *Let $\alpha : D \to D'$ in $\mathcal{D}$ be a morphism such that for every object $C \in \mathcal{C}$, composition with α induces a homotopy equivalence $\mathrm{Map}_{\mathcal{D}}(f(C), D) \to \mathrm{Map}_{\mathcal{D}}(f(C), D')$. Then α is an equivalence.*

(iv) *The functor f preserves small coproducts.*

Proof. Let $\mathcal{X}$ denote the full subcategory of $\mathcal{P}_\sigma(\mathcal{C})$ spanned by those objects X such that, for every object $Y \in \mathcal{P}_\sigma(\mathcal{C})$, the map $\mathrm{Map}_{\mathcal{P}_\sigma(\mathcal{C})}(X, Y) \to \mathrm{Map}_{\mathcal{D}}(F(X), F(Y))$ is a homotopy equivalence. We wish to prove that $\mathcal{X} = \mathcal{P}_\sigma(\mathcal{C})$. Since F preserves geometric realizations of simplicial objects, the full subcategory $\mathcal{X}$ is stable under geometric realizations in $\mathcal{P}_\sigma(\mathcal{C})$. It will therefore suffice to show that $\mathcal{X}$ contains the essential image of j.

Fix an object $C \in \mathcal{C}$, and let $\mathcal{Y} \subseteq \mathcal{P}_\sigma(\mathcal{C})$ be the full subcategory spanned by those objects Y such that the map $Y(C) \simeq \mathrm{Map}_{\mathcal{P}_\sigma(\mathcal{C})}(j(C), Y) \to \mathrm{Map}_{\mathcal{D}}(f(C), F(Y))$ is an equivalence. Since $j(C)$ is a projective object of $\mathcal{P}_\sigma(\mathcal{C})$ (Corollary 4.2.12), $f(C)$ is a projective object of $\mathcal{D}$ (by assumption (ii)), and F preserves geometric realizations of simplicial objects, we conclude that $\mathcal{Y}$ is stable under geometric realizations of simplicial objects in $\mathcal{P}_\sigma(\mathcal{C})$. It will therefore suffice to show that $\mathcal{Y}$ contains the essential image of j. That is, we wish to show that if $C' \in \mathcal{C}$, then the canonical map $\mathrm{Map}_{\mathcal{P}_\sigma(\mathcal{C})}(j(C), j(C')) \to \mathrm{Map}_{\mathcal{D}}(f(C), f(C'))$ is a homotopy equivalence. This follows from assumption (i), since the Yoneda embedding $j : \mathcal{C} \to \mathcal{P}_\sigma(\mathcal{C})$ is fully faithful.

This completes the proof that F is fully faithful. Note that if F is an equivalence of ∞-categories, then condition (iii) is obvious and (iv) follows from Proposition 4.2.11. Conversely, suppose that (iii) and (iv) are satisfied. Using (iv), we deduce the existence of a functor $G : \mathcal{D} \to \mathcal{P}_\sigma(\mathcal{C})$, given informally by the formula $G(D)(C) = \mathrm{Map}_{\mathcal{D}}(f(C), D)$. This functor is right adjoint to F, so that F is essentially surjective if and only if G is conservative: this is equivalent to condition (iii). □

4.3 Elementary Coconnective Algebras

Let k be a field of characteristic zero, and let $\mathrm{CAlg}_k^{\mathrm{cc}}$ denote the ∞-category of coconnective $\mathbb{E}_\infty$-algebras over k. Our goal in this section is to show that that the ∞-category $(\mathrm{CAlg}_k^{\mathrm{cc}})^{op}$ has a good supply of (noncompact) projective objects, which we will call *elementary* coconnective k-algebras. We begin with a concrete description of these k-algebras.

Proposition 4.3.1. *Let k be a field of characteristic zero, and let A be an $\mathbb{E}_\infty$-algebra over k. The following conditions are equivalent:*

(1) *The k-algebra A is coconnective and $\pi_i A \simeq 0$ for $i \leq -2$.*

(2) *There exists a k-vector space V such that A is equivalent to the square-zero extension $k \oplus V[-1]$.*

Definition 4.3.2. We will say that a k-algebra A is *elementary* if it satisfies the equivalent conditions of Proposition 4.3.1.

Remark 4.3.3. Proposition 4.3.1 asserts that if a coconnective k-algebra A satisfies $\pi_i A \simeq 0$ for $i \leq -2$, then A is *formal*: that is, it is determined up to equivalence by its homotopy groups $\pi_* A$.

Proof of Proposition 4.3.1. The implication (2) $\Rightarrow$ (1) is clear. Suppose that A satisfies (1), and let $V = \pi_{-1} A$. We will prove that there exists a map of k-algebras $\phi : A \to k \oplus V[-1]$ such that the induced map

$$V \simeq \pi_{-1} A \xrightarrow{\phi} \pi_{-1}(k \oplus V[-1]) \simeq V$$

is the identity; it will follow immediately that ϕ is an equivalence, so that $A \simeq k \oplus V[-1]$. We begin by choosing a map $\phi_0 : A \to k$ (such a map exists and is unique up to equivalence, by Corollary 4.1.7). Lifting

$\phi_0 : A \to k$ to a map $\phi : A \to k \oplus V[-1]$ is equivalent to giving a map of k-module spectra $k \otimes_A L_{A/k} \to V[-1]$. The collection of homotopy classes of such maps is given by the k-vector space $\mathrm{Hom}_k(\pi_{-1}(k \otimes_A L_{A/k}, V)$. To complete the proof, we must show that there exists a k-linear map $\psi : \pi_{-1}(k \otimes_A L_{A/k}) \to V$ such that the composition

$$V \simeq \pi_{-1} A \xrightarrow{\psi''} \pi_{-1} L_{A/k} \xrightarrow{\psi'} \pi_{-1}(k \otimes_A L_{A/k}) \xrightarrow{\psi} V$$

is the identity map. In other words, we wish to construct a splitting of the linear map $\psi' \circ \psi''$. Since k is a field, the existence of such a splitting is equivalent to the assertion that $\psi' \circ \psi''$ is injective. It follows from Proposition 4.1.15 that ψ'' is injective, and that the homotopy groups $\pi_i L_{A/k}$ vanish for $i > -1$. Using Corollary 4.1.12, we conclude that ψ' is also injective. $\square$

Our next goal is to show that if $A = k \oplus V[-1]$ is elementary, then A is a projective object of $(\mathrm{CAlg}^{\mathrm{cc}}_{k/\,/k})^{op}$. To prove this, we will need to study the formation of totalizations of cosimplicial objects in CAlg_k. We begin with a few general observations.

Remark 4.3.4. Let $\mathcal{C}$ be an ∞-category which admits small colimits and let $X_\bullet : \mathrm{N}(\mathbf{\Delta})^{op} \to \mathcal{C}$ be a simplicial object of $\mathcal{C}$. Since the inclusion $\mathrm{N}(\mathbf{\Delta}_s)^{op} \hookrightarrow \mathrm{N}(\mathbf{\Delta})^{op}$ is left cofinal (Lemma T.6.5.3.7), we can identify the geometric realization $X_\bullet$ with the colimit $\varinjlim X_\bullet | \mathrm{N}(\mathbf{\Delta}_s)^{op}$. The category $\mathbf{\Delta}_s$ admits a filtration by full subcategories $\mathbf{\Delta}_{s,\leq n}$ (spanned by the objects $[m]$ for $m \leq n$), so (using the results of §T.4.2.3) we can identify $|X_\bullet|$ with the colimit

$$\varinjlim_n (\varinjlim X_\bullet | \mathrm{N}(\mathbf{\Delta}_{s,\leq n})^{op}).$$

Fix an integer $n \geq 0$, and let $i : \{[n]\} \hookrightarrow \mathbf{\Delta}_{s,\leq n}$ and $j : \mathbf{\Delta}_{s,\leq n-1} \hookrightarrow \mathbf{\Delta}_{s,\leq n}$ denote the inclusions. These inclusions determine restriction functors

$$\mathcal{C} \simeq \mathrm{Fun}(\{[n]\}, \mathcal{C}) \xleftarrow{i^*} \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{s,\leq n})^{op}, \mathcal{C}) \xrightarrow{j^*} \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{s,\leq n-1})^{op}, \mathcal{C}).$$

These restriction functors admit left adjoints $i_!$ and $j_!$ (given by left Kan extension along i and j, respectively). For any functor $F : \mathrm{N}(\mathbf{\Delta}_{s,\leq n})^{op} \to \mathcal{C}$, we obtain a diagram

$$\begin{tikzcd}
j_! j^* i_! i^* F \arrow[r] \arrow[d] & i_! i^* F \arrow[d] \\
j_! j^* F \arrow[r] & F.
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}^{op}_{s,\leq n}), \mathcal{C})$. In fact, this diagram is a pushout square: the vertical maps are equivalences when evaluated on $[n]$, and the horizontal maps are equivalences when evaluated on $[m]$ for $m < n$. Passing to colimits, we get a pushout square

$$\begin{tikzcd}
\varinjlim j^* i_! F([n]) \arrow[r] \arrow[d] & F([n]) \arrow[d] \\
\varinjlim F | \mathrm{N}(\mathbf{\Delta}^{op}_{s,\leq n-1}) \arrow[r] & \varinjlim F.
\end{tikzcd}$$

To understand the term in the upper left, we factor the map i as a composition

$$\{[n]\} \xrightarrow{i'} (\mathbf{\Delta}_{s,\leq n})_{/[n]} \xrightarrow{i''} \mathbf{\Delta}_{s,\leq n}.$$

Since $[n]$ is a final object of $(\mathbf{\Delta}_{s,\leq n})_{/[n]}$, the functor $i'_!$ carries the object $F([n]) \in \mathcal{C}$ to the constant functor F' taking the value $F([n])$. Let $\mathcal{D}$ denote the fiber product $(\mathbf{\Delta}_{s,\leq n})_{/[n]} \times_{\mathbf{\Delta}_{s,\leq n}} \mathbf{\Delta}_{s,\leq n-1}$ and let $p : \mathcal{D} \to \mathbf{\Delta}_{s,\leq n-1}$ denote the projection. It is easy to see that the natural map $p_!(F' | \mathcal{D}) \to j^* i''_! F'$ is an equivalence, so that $j^* i_! F \simeq j^* i''_! F'$ can be identified with $p_!(F' | \mathcal{D})$. Note that the category $\mathcal{D}$ can be identified with the category of simplices of the simplicial set $\partial\,\Delta^n$, so that $\varinjlim p_!(F'|\,\mathcal{D}) \simeq \varinjlim(F'|\,\mathcal{D}) \simeq F([n]) \otimes \partial\,\Delta^n$, where we regard $\mathcal{C}$ as tensored over spaces as explained §T.4.4.4.

Applying the above discussion in the case where $F = X_\bullet|\,\mathrm{N}(\mathbf{\Delta}_{s,\leq n})^{op}$, we deduce the existence of a pushout square

$$\begin{tikzcd} X_n \otimes \partial\,\Delta^n \arrow[r] \arrow[d] & X_n \arrow[d] \\ \varinjlim(X|\,\mathrm{N}(\mathbf{\Delta}^{op}_{s,\leq n-1})) \arrow[r] & \varinjlim(X|\,\mathrm{N}(\mathbf{\Delta}^{op}_{s,\leq n})). \end{tikzcd}$$

Proposition 4.3.5. *Let $\mathcal{C}$ be an ∞-category which is equivalent to an n-category for some $n \geq 0$, and assume that $\mathcal{C}$ admits finite colimits. Then:*

(1) *The ∞-category $\mathcal{C}$ admits geometric realizations of simplicial objects.*

(2) *For every simplicial object $X_\bullet$ of $\mathcal{C}$, the geometric realization of $X_\bullet$ is a colimit of the diagram $X_\bullet|\,\mathrm{N}(\mathbf{\Delta}^{op}_{s,\leq n+1})$.*

Proof. Let $X_\bullet$ be a simplicial object of $\mathcal{C}$. For each $m \geq 0$ let $Y(m)$ denote the colimit of the diagram $X_\bullet|\,\mathrm{N}(\mathbf{\Delta}^{op}_{s,\leq m})$ in $\mathcal{C}$. Using Lemma T.6.5.3.7 and the results of §T.4.2.3, we obtain a sequence of maps

$$Y(0) \to Y(1) \to Y(2) \to \cdots$$

whose colimit, if it exists, is a geometric realization of $X_\bullet$. To complete the proof, it will suffice to show that the map $Y(m) \to Y(m+1)$ is an equivalence for $m \geq n+1$. Applying Remark 4.3.4 in the ∞-category $\mathrm{Ind}(\mathcal{C})$, we obtain a pushout diagram

$$\begin{tikzcd} X_{m+1} \otimes \partial\,\Delta^{m+1} \arrow[r] \arrow[d] & X_{m+1} \arrow[d] \\ Y(m) \arrow[r] & Y(m+1). \end{tikzcd}$$

It will therefore suffice to show that the upper horizontal map is an equivalence for $m \geq n+1$, which follows from our assumption that $\mathcal{C}$ is equivalent to an n-category. ☐

Notation 4.3.6. Let $\mathcal{C}$ be an ∞-category which admits small limits. It follows that the diagonal embedding $\mathcal{C} \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathcal{C})$ admits a right adjoint, which carries each cosimplicial object $X^\bullet$ of $\mathcal{C}$ to its limit $\varprojlim_{[n]\in\mathbf{\Delta}} X^n$. We will denote this functor by $\mathrm{Tot} : \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathcal{C}) \to \mathcal{C}$, and refer to it as the *totalization functor.*

Corollary 4.3.7. *Let $\mathrm{Sp}_{\leq 0}$ denote the full subcategory of Sp spanned by those spectra X such that $\pi_i X \simeq 0$ for $i > 0$. Then the totalization functor $\mathrm{Tot} : \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta})^{op}, \mathrm{Sp}_{\leq 0}) \to \mathrm{Sp}_{\leq 0}$ commutes with filtered colimits.*

Proof. It suffices to show that for every integer n, the $(\Omega^{\infty-n} \circ \mathrm{Tot}) : \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathrm{Sp}_{\leq 0}) \to \mathcal{S}$ preserves filtered colimits. Proposition 4.3.5 implies that $\Omega^{\infty-n} \circ \mathrm{Tot} = \varprojlim \circ \Omega^{\infty-n}$ is equivalent to the composition

$$\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}, \mathrm{Sp}_{\leq 0})) \xrightarrow{\Omega^{\infty-n}} \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}), \mathcal{S}) \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_{s,\leq n+1}), \mathcal{S})) \xrightarrow{\varprojlim} \mathcal{S}\,.$$

The last functor preserves filtered colimits by Proposition T.5.3.3.3 (since $\mathrm{N}(\mathbf{\Delta}_{s,\leq n+1})$ has only finitely many nondegenerate simplices). ☐

Lemma 4.3.8. *Let k be a field, let $V^\bullet$ and $W^\bullet$ be cosimplicial objects of Mod_k, and assume that $\pi_n V^\bullet \simeq \pi_n W^\bullet \simeq 0$ for $n > 0$. Then the canonical map*

$$\theta : (\mathrm{Tot}\, V^\bullet) \otimes_k (\mathrm{Tot}\, W^\bullet) \to \mathrm{Tot}(V^\bullet \otimes W^\bullet)$$

is an equivalence.

Proof. For every integer m, let $V(m)^\bullet = \tau_{\geq -m} V^\bullet$, so that $V^\bullet \simeq \varinjlim V(m)^\bullet$. It follows from Corollary 4.3.7 that $\operatorname{Tot} V^\bullet \simeq \varinjlim_m \operatorname{Tot} V(m)^\bullet$ and $\operatorname{Tot}(V^\bullet \otimes W^\bullet) \simeq \varinjlim_m \operatorname{Tot}(V(m)^\bullet \otimes W^\bullet)$. It will therefore suffice to show that the map

$$\theta_m : (\operatorname{Tot} V(m)^\bullet) \otimes_k (\operatorname{Tot} W^\bullet) \to \operatorname{Tot}(V(m)^\bullet \otimes W^\bullet)$$

is an equivalence for every integer m. We proceed by induction on m, the case $m < 0$ being trivial. For the inductive step, we have a map of exact triangles

$$\begin{tikzcd}
(\operatorname{Tot} V(m-1)^\bullet) \otimes_k (\operatorname{Tot} W^\bullet) \arrow[r] \arrow[d, "\theta_{m-1}"] & (\operatorname{Tot} V(m)^\bullet) \otimes_k (\operatorname{Tot} W^\bullet) \arrow[r] \arrow[d, "\theta_m"] & (\operatorname{Tot} V(m)^\bullet / V(m-1)^\bullet) \otimes_k \operatorname{Tot} W^\bullet \arrow[d, "\psi"] \\
\operatorname{Tot}(V(m-1)^\bullet \otimes W^\bullet) \arrow[r] & \operatorname{Tot}(V(m)^\bullet \otimes W^\bullet) \arrow[r] & \operatorname{Tot}(V(m)^\bullet / V(m-1)^\bullet \otimes W^\bullet).
\end{tikzcd}$$

By the inductive hypothesis, we may assume that θ_{m-1} is an equivalence; we are therefore reduced to proving that ψ is an equivalence. Replacing $V^\bullet$ by $V(m)^\bullet / V(m-1)^\bullet[m]$, we are reduced to the situation where $V^\bullet$ is a cosimplicial *discrete* k-module. Similarly, we may assume that $W^\bullet$ is a cosimplicial discrete k-module. In this case, we can identify $V^\bullet$ and $W^\bullet$ with cosimplicial vector spaces over k. One can show that the map θ induces on homotopy the classical Alexander-Whitney isomorphism

$$\mathrm{H}^*(V^\bullet) \otimes_k \mathrm{H}^*(W^\bullet) \to \mathrm{H}^*(V^\bullet \otimes W^\bullet).$$

However, one can also argue more indirectly as follows. Let $\mathcal{A}$ denote the abelian category of cochain complexes of vector spaces

$$0 \to U^0 \to U^1 \to \cdots$$

By the Dold-Kan correspondence, there is an equivalence of categories of $\mathcal{A}$ with the category of cosimplicial k-vector spaces, which assigns to each cosimplicial vector space $V^\bullet$ its normalized cochain complex $0 \to N(V)^0 \to N(V)^1 \to \cdots$. Let $\mathcal{A}_0$ denote the full subcategory of $\mathcal{A}$ spanned by those objects which correspond to cosimplicial vector spaces $V^\bullet$ such that the map $\theta : (\operatorname{Tot} V^\bullet) \otimes_k (\operatorname{Tot} W^\bullet) \to \operatorname{Tot}(V^\bullet \otimes W^\bullet)$ is an equivalence, for *every* cosimplicial k-vector space $W^\bullet$. We wish to prove that $\mathcal{A}_0 = \mathcal{A}$. We make the following observations:

- (a) The subcategory $\mathcal{A}_0 \subseteq \mathcal{A}$ is stable under filtered colimits and finite direct sums, and therefore under arbitrary direct sums.
- (b) Given an exact sequence $0 \to U'^\bullet \to U^\bullet \to U''^\bullet \to 0$ in $\mathcal{A}$, if $U^\bullet$ and $U''^\bullet$ belong to $\mathcal{A}_0$, then $U'^\bullet$ belongs to $\mathcal{A}_0$.
- (c) Any acyclic complex belongs to $\mathcal{A}_0$ (in this case, the domain and codomain of θ are both zero objects).
- (d) Since k is a field, every object of $\mathcal{A}$ can be written as a direct sum of indecomposable objects $\{C(n), D(n)\}$, where $C(n)$ is the cochain complex $\cdots \to 0 \to k \to 0 \to \cdots$ with k appearing in cohomological degree n, and $D(n)$ is the cochain complex $\cdots \to 0 \to k \xrightarrow{\mathrm{id}} k \to 0 \to \cdots$ with k appearing in cohomological degreees n and $n+1$.

By virtue of (a) and (d), it will suffice to show that for each n, the chain complexes $C(n)$ and $D(n)$ belong to $\mathcal{A}_0$. The proof proceeds by induction on n. Since $D(n)$ is acyclic, it belongs to $\mathcal{A}_0$ by (c). The exact sequence $0 \to C(n+1) \to D(n) \to C(n) \to 0$ shows that $C(n) \in \mathcal{A}_0$ implies $C(n+1) \in \mathcal{A}_0$. We are therefore reduced to proving that $C(0) \in \mathcal{A}_0$. This is clear: the corresponding cosimplicial k-vector space $V^\bullet$ is the constant cosimplicial vector space k, and the map θ corresponds to the composition

$$(\operatorname{Tot} V^\bullet) \otimes_k (\operatorname{Tot} W^\bullet) \simeq k \otimes_k \operatorname{Tot} W^\bullet \simeq \operatorname{Tot} W^\bullet \simeq \operatorname{Tot}(V^\bullet \otimes_k W^\bullet).$$

$\square$

Lemma 4.3.9. *Let k be a field of characteristic zero and let $V^\bullet$ be a cosimplicial object of* Mod_k *such that $\pi_n V^\bullet \simeq 0$ for $n > 0$. Then the natural map $\theta : \mathrm{Sym}^m(\mathrm{Tot}\, V^\bullet) \to \mathrm{Tot}\,\mathrm{Sym}^m V^\bullet$ is an equivalence for every integer m.*

Proof. If W is a k-module spectrum, let $\Gamma^m W$ denote the invariants for the action of the symmetric group Σ_m on $W^{\otimes m}$. We have a commutative diagram

$$\begin{tikzcd}
\Gamma^m(\mathrm{Tot}\, V^\bullet) \arrow[r] \arrow[d] & (\mathrm{Tot}\, V^\bullet)^{\otimes m} \arrow[r] \arrow[d, "\theta'"] & \mathrm{Sym}^m(\mathrm{Tot}\, V^\bullet) \arrow[d, "\theta"] \\
\mathrm{Tot}(\Gamma^m V^\bullet) \arrow[r] & \mathrm{Tot}(V^\bullet)^{\otimes m} \arrow[r] & \mathrm{Tot}(\mathrm{Sym}^m V^\bullet).
\end{tikzcd}$$

Lemma 4.3.8 shows that θ' is an equivalence. Since k is a field of characteristic zero, the horizontal composite maps are equivalences. It follows that the above diagram exhibits θ as a retract of θ', so that θ is also an equivalence. $\square$

Lemma 4.3.10. *Let k be a field, let $V^\bullet$ be a cosimplicial object of* Mod_k *satisfying $\pi_i V^\bullet \simeq 0$ for $i > 0$, let $W \in \mathrm{Mod}_k^{\mathrm{cn}}$, and let $F : \mathrm{Mod}_k^{op} \to \mathcal{S}$ denote the functor represented by W. Then $F(V^\bullet)$ is a hypercovering of $F(\mathrm{Tot}\, V^\bullet)$.*

Proof. The functor F factors as a composition

$$\mathrm{Mod}_k^{op} \xrightarrow{F'} \mathrm{Mod}_k \xrightarrow{\Omega^\infty} \mathcal{S},$$

where F' is the internal Hom-functor $V \mapsto \mathrm{Mor}_{\mathrm{Mod}_k}(V, W)$ with respect to the tensor structure on Mod_k (that is, $F'(V)$ is universal among objects of Mod_k which admit a map $V \otimes_k F'(V) \to W$). Choose an extension of $V^\bullet$ to a limit diagram $\overline{V}^\bullet : \mathrm{N}(\boldsymbol{\Delta}_+) \to \mathrm{Mod}_k$. Since Ω^∞ factors through the ∞-category $\mathcal{G}\mathrm{rp}(\mathcal{S})$ of group objects of $\mathcal{S}$, we see from Corollary 4.2.7 that it suffices to show that $F(\overline{V}^\bullet)$ is a colimit diagram in $\mathcal{S}$.

Since k is a field, any object $V \in (\mathrm{Mod}_k)_{\leq k}$ can be written as a coproduct of objects of the form $k[-n]$ for $n \geq 0$. It follows that $F'(V)$ can be identified with a product of k-modules of the form $W[n]$, $n \geq 0$, so that $F'(V) \in \mathrm{Mod}_k^{\geq 0}$. In particular, $F'(\overline{V}^\bullet)$ is an augmented cosimplicial object of $\mathrm{Mod}_k^{\mathrm{cn}}$. Since the forgetful functor $\Omega^\infty : \mathrm{Mod}_k^{\mathrm{cn}} \to \mathcal{S}$ preserves sifted colimits (Proposition A.7.2.2.7), it will suffice to show that $F'(\overline{V}^\bullet)$ is a colimit diagram in $\mathrm{Mod}_k^{\mathrm{cn}}$. Because $\overline{V}^\bullet$ is a colimit diagram in Mod_k^{op}, this is a consequence of the following assertion:

- $(*)$ The functor $F'_0 = F' | (\mathrm{Mod}_k)^{op}_{\leq 0} : (\mathrm{Mod}_k)^{op}_{\leq 0} \to \mathrm{Mod}_k^{\mathrm{cn}}$ preserves geometric realizations of simplicial objects.

The ∞-category $\mathrm{Mod}_k^{\mathrm{cn}}$ can be written as the limit of the tower of ∞-categories $\{\mathrm{Mod}_k^{\mathrm{cn}} \cap (\mathrm{Mod}_k)_{\leq 0}\}_{m \geq 0}$. Consequently, $(*)$ is equivalent to:

- $(*')$ For each $m \geq 0$, the composite functor $\tau_{\leq m} \circ F'_0 : (\mathrm{Mod}_k)^{op}_{\leq 0} \to \mathrm{Mod}_k^{\mathrm{cn}} \cap (\mathrm{Mod}_k)_{\leq m}$ preserves geometric realizations of simplicial objects.

Since the intersection $\mathcal{C} = \mathrm{Mod}_k^{\mathrm{cn}} \cap (\mathrm{Mod}_k)_{\leq m}$ is equivalent to an $(m+1)$-category, Proposition 4.3.5 implies that the formation of geometric realizations in $\mathcal{C}$ can be written as a finite colimit. Consequently, it will suffice to prove:

- $(*'')$ For each $m \geq 0$, the composite functor $\tau_{\leq m} \circ F'_0 : (\mathrm{Mod}_k)^{op}_{\leq 0} \to \mathrm{Mod}_k^{\mathrm{cn}} \cap (\mathrm{Mod}_k)_{\leq m}$ preserves finite colimits.

Since the truncation functor $\tau_{\leq m}$ preserves finite colimits, we are reduced to proving that $F'_0 : (\mathrm{Mod}_k)^{op}_{\leq 0} \to \mathrm{Mod}_k^{\mathrm{cn}}$ preserves finite colimits. In fact, we claim that $F' : \mathrm{Mod}_k^{op} \to \mathrm{Mod}_k$ preserves finite colimits. This follows from the stability of Mod_k, since F' preserves small limits (see Proposition A.1.1.4.1). $\square$

Proposition 4.3.11. *Let k be a field of characteristic zero, let $A^\bullet$ be a cosimplicial object of* CAlg_k *and* $A = \operatorname{Tot} A^\bullet$ *its totalization. Assume that each $A^\bullet$ is coconnective. Then:*

(1) *The k-algebra A is coconnective.*

(2) *Let $R \in \mathrm{Alg}_k$ be such that the homotopy groups $\pi_i R$ vanish for $i \leq -2$. Then the simplicial space* $\mathrm{Map}_{\mathrm{CAlg}_k}(A^\bullet, R)$ *is a hypercovering of* $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$.

(3) *The canonical map* $|\mathrm{Map}_{\mathrm{CAlg}_k}(A^\bullet, R)| \to \mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ *is a homotopy equivalence.*

Proof. To prove (1), we note that the cofiber of the unit map $k \to A$ can be identified with the totalization of a cosimplicial object of $(\mathrm{Mod}_k)_{\leq -1}$ (given by $[n] \mapsto \mathrm{cofib}(k \to A^n)$). It follows that $\mathrm{cofib}(k \to A) \in (\mathrm{Mod}_k)_{\leq -1}$, so that A is coconnective.

The implication (2) $\Rightarrow$ (3) follows from Theorem VII.4.20. To prove (2), we use the construction of Proposition 4.1.4 (see Remark 4.1.5) to choose a sequence of cosimplicial coconnective k-algebras

$$A(0)^\bullet \to A(1)^\bullet \to A(2)^\bullet \to \cdots$$

with the following properties:

(a) The cosimplicial algebra $A(0)^\bullet$ is constant, taking the value $k \in \mathrm{CAlg}_k$.

(b) For each $i \geq 0$, there is a cosimplicial k-module $V^\bullet$ such that $\pi_n V^m \simeq 0$ for $n \geq -1$, and an equivalence

$$A(i+1)^\bullet \simeq A(i)^\bullet \otimes_{\mathrm{Sym}^* V^\bullet} k$$

of cosimplicial objects of CAlg_k.

(c) There is an equivalence $A^\bullet \simeq \varinjlim A(i)^\bullet$ of cosimplicial objects of CAlg_k.

For each i, choose an extension of $A(i)^\bullet$ to a limit diagram $\overline{A(i)}^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_k$, and let $\overline{A}^\bullet = \varinjlim \overline{A(i)}^\bullet$. We claim that $\overline{A}^\bullet$ is also a limit diagram in CAlg_k (so that $A \simeq \overline{A}^{-1}$). To prove this, it suffices to show that for every integer $n \geq 0$, the augmented cosimplicial space $Z^\bullet = \Omega^{\infty - n} \overline{A}^\bullet$ is a limit diagram. Consider the diagram

$$Z^{-1} \xrightarrow{\phi} \operatorname{Tot}(Z^\bullet | \mathrm{N}(\mathbf{\Delta})) \xrightarrow{\phi'} \operatorname{Tot}(Z^\bullet | \mathrm{N}(\mathbf{\Delta}_{s, \leq n+1})).$$

Since each A^p is coconnective, each of the spaces Z^p is n-truncated so that the map ϕ' is an equivalence by Proposition 4.3.5. It will therefore suffice to show that the composite map $\phi' \phi$ is an equivalence. Because filtered colimits in $\mathcal{S}$ commute with finite limits (Proposition T.5.3.3.3), we can write $\phi' \circ \phi$ as the colimit of a sequence of compositions

$$Z(i)^{-1} \xrightarrow{\phi_i} \operatorname{Tot}(Z(i)^\bullet | \mathrm{N}(\mathbf{\Delta})) \xrightarrow{\phi'_i} \operatorname{Tot}(Z(i)^\bullet | \mathrm{N}(\mathbf{\Delta}_{s, \leq n+1})),$$

where each $Z(i)^\bullet$ is given by $\Omega^{\infty - n} \overline{A(i)}^\bullet$. It will therefore suffice to show that each composition $\phi'_i \circ \phi_i$ is an equivalence. This is clear: ϕ_i is an equivalence by assumption, and ϕ'_i is an equivalence by Proposition 4.3.5.

Let us now fix $i \geq 0$ and study the passage from $A(i)^\bullet$ to $A(i+1)^\bullet$. Let $V^\bullet$ be as in (b), and extend $V^\bullet$ to a limit diagram $\overline{V}^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{Mod}_k$. We have an evident commutative square of augmented cosimplicial k-module spectra

$$\begin{tikzcd}
\overline{V}^\bullet \arrow[r] \arrow[d] & \overline{A(i)}^\bullet \arrow[d] \\
0 \arrow[r] & \overline{A(i+1)}^\bullet,
\end{tikzcd}$$

which induces a map $\theta : \overline{A(i)}^\bullet \otimes_{\mathrm{Sym}^* \overline{V}^\bullet} k \to \overline{A(i+1)}^\bullet$. We claim that θ is an equivalence: that is, the augmented cosimplicial object $\overline{A(i)}^\bullet \otimes_{\mathrm{Sym}^* \overline{V}^\bullet} k$ is a limit diagram in CAlg_k. To prove this, we recall that the relative tensor product $B \otimes_{\mathrm{Sym}^* W} k$ can be computed as the geometric realization of a simplicial object $\mathrm{Bar}_{\mathrm{Sym}^* W}(B, k)_\bullet$ (see §A.4.3.5). In particular, $B \otimes_{\mathrm{Sym}^* W} k$ can be written as the colimit of a sequence $\{B \otimes^n_{\mathrm{Sym}^* W} k\}_{n \geq 0}$, where $B \otimes^n_{\mathrm{Sym}^* W} k$ is given by the colimit of the restriction of $\mathrm{Bar}_{\mathrm{Sym}^* W}(B, k)_\bullet$ to $\mathrm{N}(\mathbf{\Delta}^{op}_{s, \leq n})$. In view of Corollary 4.3.7, it will suffice to prove the following pair of assertions:

- (i) For each $n \geq 0$, the augmented cosimplicial object $\overline{A(i)}^\bullet \otimes^n_{\mathrm{Sym}^* \overline{V}^\bullet} k$ is a limit diagram in Mod_k.
- (ii) The homotopy groups $\pi_m \overline{A(i)}^\bullet \otimes^n_{\mathrm{Sym}^* \overline{V}^\bullet} k$ vanish for $m \geq 0$.

Using Remark 4.3.4 and the stability of the ∞-category Mod_k, we are reduced to proving:

- (i') For each $n \geq 0$, the augmented cosimplicial object $\mathrm{Bar}_{\mathrm{Sym}^* \overline{V}^\bullet}(\overline{A(i)}^\bullet, k)_n \simeq \overline{A(i)}^\bullet \otimes_k (\mathrm{Sym}^* \overline{V}^\bullet)^{\otimes n}$ is a limit diagram in Mod_k.
- (ii') The homotopy groups $\pi_m(\overline{A(i)}^\bullet \otimes_k (\mathrm{Sym}^* \overline{V}^\bullet)^{\otimes n})$ vanish for $m > -n$.

Assertion (i') follows from Corollary 4.3.7, Lemma 4.3.8, and Lemma 4.3.9 (since $\overline{A(i)}^\bullet$ and $\overline{V}^\bullet$ are limit diagrams by assumption). Assertion (ii') follows from our assumption that the homotopy groups $\pi_m \overline{V}$ vanish for $m \geq -1$.

For every augmented cosimplicial k-algebra $B^\bullet$ and every integer $n \geq 0$, let $M_n(B^\bullet)$ denote the nth matching object of the cosimplicial space $\mathrm{Map}_{\mathrm{CAlg}_k}(B^\bullet, R)$. We wish to prove that the canonical map $\psi : \mathrm{Map}_{\mathrm{CAlg}_k}(A^n, R) \to M_n(\overline{A}^\bullet)$ is surjective on connected components. Note that ψ can be written as the homotopy limit of a tower of maps $\{\psi_i : \mathrm{Map}_{\mathrm{CAlg}_k}(A(i)^n, R) \to M_n(\overline{A(i)}^\bullet)\}$; it will therefore suffice to show that each ψ_i is surjective on connected components. The proof proceeds by induction on i, the case $i = 0$ being trivial. To carry out the inductive step, we consider the map of homotopy fiber sequences

$$\begin{tikzcd}
\mathrm{Map}_{\mathrm{CAlg}_k}(A(i+1)^n, R) \arrow[r] \arrow[d, "\psi_{i+1}"] & \mathrm{Map}_{\mathrm{CAlg}_k}(A(i)^n, R) \arrow[r] \arrow[d, "\psi_i"] & \mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}^* V^n, R) \arrow[d, "\psi'"] \\
M_n(\overline{A(i+1)}^\bullet) \arrow[r] & M_n(\overline{A(i)}^\bullet) \arrow[r] & M_n(\mathrm{Sym}^* \overline{V}^\bullet)
\end{tikzcd}$$

The inductive hypothesis guarantees that ψ_i has nonempty homotopy fibers. Consequently, to prove that the homotopy fibers of ψ_{i+1} are nonempty, it will suffice to prove that the homotopy fibers of ψ' are connected.

Let $X_\bullet : \mathrm{N}(\mathbf{\Delta}_+)^{op} \to \mathcal{S}$ be the augmented simplicial space given informally by

$$X_\bullet \simeq \mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}^* \overline{V}^\bullet, R) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(\overline{V}^\bullet, R).$$

Note that the spaces $X_\bullet$ are naturally pointed. The assertion that the map ψ' has connected homotopy fibers (for every integer n) is equivalent to the assertion that the augmented cosimplicial spaces $X_\bullet$ and $\Omega X_\bullet \simeq \mathrm{Map}_{\mathrm{Mod}_k}(\overline{V}^\bullet[1], R)$ are hypercoverings, which is a special case of Lemma 4.3.10. □

Before stating the next result, let us introduce a bit of notation. Every object $A \in \mathrm{CAlg}^{\mathrm{cc}}_k$ admits an *augmentation* $\epsilon : A \to k$, which is unique up to homotopy (Corollary 4.1.7). We let $\mathrm{CAlg}_{k//k} = (\mathrm{CAlg}_k)_{/k}$ denote the ∞-category of *augmented* $\mathbb{E}_\infty$-algebras over k, and $\mathrm{CAlg}^{\mathrm{cc}}_{k//k}$ the full subcategory of $\mathrm{CAlg}_{k//k}$ spanned by the coconnective augmented $\mathbb{E}_\infty$-algebras over k.

Corollary 4.3.12. *Let k be a field of characteristic zero and let $A \in \mathrm{CAlg}_k$ be elementary. Then:*

1. *The k-algebra A is a projective object of $(\mathrm{CAlg}^{\mathrm{cc}}_k)^{op}$.*
2. *Choose an augmentation $\epsilon : A \to k$. Then ϵ exhibits A as a projective object of $(\mathrm{CAlg}^{\mathrm{cc}}_{k//k})^{op}$.*

Proof. Assertion (1) follows immediately from Proposition 4.3.11. To prove (2), let $B^\bullet$ be a cosimplicial object of $\mathrm{CAlg}^{\mathrm{cc}}_{k//k}$ and extend $B^\bullet$ to a limit diagram $\overline{B}^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}^{\mathrm{cc}}_{k//k}$. We wish to show that $F(\overline{B}^\bullet)$ is a colimit diagram in $\mathcal{S}$, where $F : (\mathrm{CAlg}^{\mathrm{cc}}_{k//k})^{op} \to \mathcal{S}$ denotes the functor corepresented by A. We have a pullback diagram of functors

$$\begin{tikzcd} F \arrow[r] \arrow[d] & * \arrow[d] \\ F' \arrow[r] & F'', \end{tikzcd}$$

where F' is given by the formula $F'(B) = \mathrm{Map}_{\mathrm{CAlg}_k}(B, A)$ and F'' is given by $F''(B) = \mathrm{Map}_{\mathrm{CAlg}_k}(B, k)$. We can therefore identify $F \circ \overline{B}^\bullet$ with the fiber of the map of augmented simplicial spaces $F'(\overline{B}^\bullet) \to F''(\overline{B}^\bullet)$. Proposition 4.3.11 implies that $F'(\overline{B}^\bullet)$ and $F''(\overline{B}^\bullet)$ are colimit diagrams in $\mathcal{S}$, and Corollary 4.1.7 guarantees that each $F''(\overline{B}^n)$ is connected. It follows from Lemma A.5.3.6.17 that $F(\overline{B}^\bullet)$ is also a colimit diagram in $\mathcal{S}$. □

Proposition 4.3.13. *Let k be a field of characteristic zero, $\overline{\mathcal{E}}$ denote the opposite of the ∞-category $\mathrm{CAlg}^{\mathrm{cc}}_{k//k}$ of coconnective k-algebras, and $\mathcal{E} \subseteq \overline{\mathcal{E}}$ the full subcategory spanned by the elementary coconnective k-algebras. Then:*

(1) *The ∞-category $\mathcal{E}$ is a socle (Definition 4.2.9).*

(2) *The inclusion $\mathcal{E} \subseteq \overline{\mathcal{E}}$ is homotopic to a composition*

$$\mathcal{E} \xrightarrow{j} \mathcal{P}_\sigma(\mathcal{E}) \xrightarrow{F} \overline{\mathcal{E}},$$

where $\mathcal{P}_\sigma(\mathcal{E}) \subseteq \mathrm{Fun}(\mathcal{E}^{op}, \mathcal{S})$ (see Definition 4.2.9), j denotes the Yoneda embedding, and the functor F preserves geometric realization of simplicial objects. Moreover, the functor F is unique up to homotopy.

(3) *The functor F is an equivalence of ∞-categories.*

More informally: the ∞-category $(\mathrm{CAlg}^{\mathrm{cc}}_{k//k})^{op}$ is freely generated by elementary k-algebras under the formation of geometric realizations of simplicial objects.

Proof. It is clear that $\mathcal{E}$ is locally small. The construction $V \mapsto k \oplus V[-1]$ carries products of k-vector spaces to coproducts in the ∞-category $\mathcal{E}$. It follows immediately that $\mathcal{E}$ admits small coproducts. Since every vector space V is a group object in the category of k-vector spaces, we conclude that every elementary coconnective k-algebra has the structure of a cogroup object of $\mathcal{E}$. To complete the proof that $\mathcal{E}$ is a socle, it suffices to show that there exists a full subcategory $\mathcal{E}_0 \subseteq \mathcal{E}$ which is essentially small, such that every elementary k-algebra is equivalent to a retract of a coproduct of objects of $\mathcal{E}_0$. To prove this, it suffices to construct a k-vector space V_0 such that every k-vector space V is a retract of a product of copies of V_0. In fact, we can take $V_0 = k$: for any vector space V, a choice of basis $\{v_i\}_{i \in I}$, which determines an injection of vector spaces

$$V \simeq \coprod_{i \in I} k \hookrightarrow \prod_{i \in I} k$$

which is automatically split. This proves (1).

Assertion (2) follows from Proposition 4.2.11, since $\overline{\mathcal{E}}$ admits geometric realizations of simplicial objects. It remains to prove (3). We will verify that F satisfies the hypotheses of Proposition 4.2.15:

(*i*) The composite $\mathcal{E} \xrightarrow{j} \mathcal{P}_\sigma(\mathcal{E}) \xrightarrow{F} \overline{\mathcal{E}}$ is fully faithful. This is clear, since $F \circ j$ is homotopic to the inclusion $\mathcal{E} \subseteq \overline{\mathcal{E}}$.

(*ii*) The essential image of $F \circ j$ consists of projective objects of $\overline{\mathcal{E}}$. This follows from Corollary 4.3.12.

(*iii*) Let $\alpha : A \to B$ be a morphism of augmented coconnective k-algebras with the property that for every k-vector space V, composition with α induces a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}_{k//k}}(B, k \oplus V[-1]) \to \mathrm{Map}_{\mathrm{CAlg}_{k//k}}(A, k \oplus V[-1]).$$

We wish to prove that α is an equivalence. In view of Proposition 4.1.17, it will suffice to show that α induces an isomorphism of k-vector spaces $\phi : \pi_i(k \otimes_A L_{A/k}) \to \pi_i(k \otimes_B L_{B/k})$ for $i \leq -1$. This is clear, since the induced map of dual spaces can be identified with the map

$$\pi_{-1-i} \mathrm{Map}_{\mathrm{CAlg}_{k//k}}(B, k \oplus k[-1]) \to \pi_{-1-i} \mathrm{Map}_{\mathrm{CAlg}_{k//k}}(A, k \oplus k[-1]).$$

(*iv*) The functor f preserves small coproducts: this is clear from the construction.

□

Corollary 4.3.14. *Let k be a field of characteristic zero and let A be a connective $\mathbb{E}_\infty$-algebra over k. Then:*

(1) *Let $\epsilon : A \to k$ be a morphism in CAlg_k, and identify A with the corresponding object of $\mathrm{CAlg}^{\mathrm{cc}}_{k//k}$. Then there exists a cosimplicial object $A^\bullet$ of $\mathrm{CAlg}^{\mathrm{cc}}_{k//k}$ where each A^i is elementary, and an equivalence $A \simeq \mathrm{Tot}\, A^\bullet$ in the ∞-category $\mathrm{CAlg}^{\mathrm{cc}}_{k//k}$.*

(2) *There exists a cosimplicial object $A^\bullet$ of $\mathrm{CAlg}^{\mathrm{cc}}_k$ where each A^i is elementary, such that $A \simeq \mathrm{Tot}\, A^\bullet$.*

Proof. Assertion (1) follows from Propositions 4.3.13 and 4.2.11. Assertion (2) follows from (1) and Corollary 4.1.7. □

The following property of elementary algebras will be needed in §4.4:

Proposition 4.3.15. *Let k be a field of characteristic zero, let $A \in \mathrm{CAlg}_k$ be elementary, and let $\eta : A \to k$ be a morphism in CAlg_k. Let $A^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_k$ be a Čech nerve of η in the ∞-category CAlg_k^{op} (so that A^n is the nth tensor power of k over A; in particular $A^{-1} \simeq A$ and $A^0 \simeq k$). Then:*

(1) *For $n \geq 0$, the k-algebra A^n is discrete.*

(2) *The augmented cosimplicial object $A^\bullet$ is a limit diagram in CAlg_k.*

Lemma 4.3.16. *Let k be a field, let A be a coconnective $\mathbb{E}_1$-algebra over k, and let $\eta : A \to k$ be a map of $\mathbb{E}_1$-algebras over k, which induces a forgetful functor $\theta : \mathrm{Mod}_k \to \mathrm{LMod}_A$. Let M be a left A-module such that $\pi_i M \simeq 0$ for $i \neq 0$. Then M belongs to the essential image of θ.*

Proof. Let M' denote the image of M in Mod_k. We will prove that there exists a map $\alpha : M \to \theta(M')$ which induces the identity map

$$\pi_0 M \xrightarrow{\alpha} \pi_0 \theta(M') \simeq \pi_0(M') \simeq \pi_0 M.$$

Such a map is automatically an equivalence and will therefore exhibit M as belonging to the essential image of θ.

Supplying the map α is equivalent to giving a map of k-module spectra $k \otimes_A M \to M'$. The collection of homotopy classes of such maps can be identified with the k-vector space $\mathrm{Hom}_k(\pi_0(k \otimes_A M), \pi_0 M')$. Consequently, constructing a map α is equivalent to splitting the map of k-vector spaces $\phi : \pi_0 M \to \pi_0(k \otimes_A M)$. Such a splitting exists, since ϕ is injective by Corollary 4.1.12. □

Proof of Proposition 4.3.15. Let $M = k$, regarded as an A-module via the map η, and let

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

be the sequence of A-modules constructed in the proof of Proposition 4.1.9, so that $M(n+1)$ is the cofiber of a map $A \otimes_k V(n) \to M(n)$ where $V(n) = \mathrm{fib}(M(n) \to M)$. We prove the following by simultaneous induction on n:

(a_n) The homotopy groups $\pi_i M(n)$ vanish for $i \notin \{0, -1\}$, and the map $\pi_0 M(n) \to \pi_0 M \simeq k$ is injective.

(b_n) The k-module spectrum $V(n)[1]$ is discrete; that is, $\pi_i V(n) \simeq 0$ for $i \neq -1$.

Assertion (a_0) is clear, and the implication (a_n) $\Rightarrow$ (b_n) follows from the long exact sequence

$$\pi_{i+1} M \to \pi_i V(n) \to \pi_i M(n) \to \pi_i M.$$

We claim that (a_n) and (b_n) imply (a_{n+1}). To prove this, we consider the exact sequence

$$\pi_i M(n) \to \pi_i M(n+1) \to \pi_{i-1}(A \otimes_k V(n)).$$

If $i \notin \{0, 1\}$, the left side vanishes by (a_n), and the right side vanishes by (b_n) together with our assumption that A is elementary. The injectivity of the map $\pi_0 M(n) \to \pi_0 M$ was established in the proof of Proposition 4.1.9.

We now prove (1). We wish to show that A^m is discrete for $m \geq 0$. For $m = 0$ this is clear; we establish the general case using induction on m. We have $A^m \simeq A^1 \otimes_k A^{m-1}$. The collection of discrete k-module spectra is stable under tensor products, it will suffice to show that A^1 is discrete. We have $A^1 \simeq k \otimes_A k \simeq \varinjlim k \otimes_A M(n)$; it therefore suffices to show that each $k \otimes_A M(n)$ is discrete. We now proceed by induction on n, the case $n = 0$ being obvious. If $n > 0$, then we have an exact sequence

$$\pi_i(k \otimes_A M(n)) \to \pi_i(k \otimes_A M(n+1)) \to \pi_{i-1} V(n).$$

If $i \neq 0$, then the outer terms vanish (by the inductive hypothesis and (b_n)) so that $\pi_i(k \otimes_A M(n+1)) \simeq 0$ as desired.

We now prove (2). Let $F : \mathrm{Mod}_A \to \mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+), \mathrm{Mod}_k)$ denote the functor given by $M \mapsto M \otimes_A A^\bullet$. We wish to show that $F(A)$ is a limit diagram in Sp. The map η gives a cofiber sequence of A-modules $A \to k \to M$, whence a cofiber sequence of diagrams $F(A) \to F(k) \to F(M)$. Consequently, to prove that $F(A)$ is a limit diagram, it will suffice to show that $F(k)$ and $F(M)$ are limit diagrams. Since A is elementary, we have $\pi_i M \simeq 0$ for $i \neq 0$. It follows from Lemma 4.3.16 that M belongs to the essential image of the functor $\theta : \mathrm{Mod}_k \to \mathrm{Mod}_A$ determined by η. To complete the proof, we will establish the following:

($*$) Let V be a k-module spectrum. Then $F(\theta(V))$ admits the structure of a split augmented cosimplicial object of Mod_k (see Definition A.6.2.1.2), and therefore a limit diagram in Mod_k (Lemma T.6.1.3.16).

To prove ($*$), we note that $F(\theta(V))$ can be identified with the composite functor

$$\mathrm{N}(\mathbf{\Delta}_+) \overset{F(k)}{\to} \mathrm{Mod}_k \overset{\otimes_k V}{\to} \mathrm{Mod}_k .$$

It therefore suffices to show that $F(k)$ admits the structure of a split augmented cosimplicial object of Mod_k. This follows from the observation that $F(k)$ can be identified with the cosimplicial object $[n] \mapsto A^{n+1}$. $\square$

Remark 4.3.17. In the proof of Proposition 4.3.15, it is possible to be much more precise. Let $A = k \oplus W[-1]$ for some k-vector space W. An induction on n shows that each of the vector spaces $V(n)$ appearing in the proof can be identified with the nth tensor power $W^{\otimes n}$; we therefore obtain a filtration of $A^1 = k \otimes_A k$ by subspaces $F^n A^1 \simeq k \otimes_A M(n)$ such that the successive quotients $F^n A^1 / F^{n-1} A^1$ are isomorphic to $W^{\otimes n}$. Note that, for every discrete k-algebra R, the set $\mathrm{Hom}(A^1, R) = \pi_0 \mathrm{Map}_{\mathrm{CAlg}_k}(A^1, R) \simeq \pi_1 \mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ has a natural group structure: thus A^1 can be regarded as a commutative Hopf algebra over k. This Hopf algebra can be identified with the tensor algebra $T(W) = \bigoplus_n W^{\otimes n}$, equipped with the classical *shuffle product*. If W is a vector space of finite dimension m, then $A^1 \simeq T(W)$ is the algebra of functions on the prounipotent completion of a free group on m generators.

4.4 Coaffine Stacks

Let k be a connective $\mathbb{E}_\infty$-ring, and let $A \in \mathrm{CAlg}_k$. We let $\mathrm{cSpec}\, A : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ denote the functor corepresented by A, given by $R \mapsto \mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$. The construction $A \mapsto \mathrm{cSpec}\, A$ can be regarded as a restricted Yoneda embedding. It not fully faithful in general. For example, if the ring $\pi_* A$ contains invertible elements of nonzero degree, then then the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R)$ is empty for every nonzero connective k-algebra R. However, there are many circumstances in which one can recover the object $A \in \mathrm{CAlg}_k$ from the functor $\mathrm{cSpec}\, A$. For example, if A is connective, then Yoneda's lemma (Proposition T.5.1.3.1) implies that A is determined by $\mathrm{cSpec}\, A$, up to a contractible space of choices. In this section, we will study a case which is in some sense at the opposite extreme: that in which A is a coconnective $\mathbb{E}_\infty$-algebra over a field k of characteristic zero. The starting point for our investigation is the following observation:

Theorem 4.4.1. *Let k be a field of characteristic zero. Then the functor $\mathrm{cSpec} : \mathrm{CAlg}_k^{op} \to \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$ described above induces a fully faithful embedding*

$$(\mathrm{CAlg}_k^{\mathrm{cc}})^{op} \to \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S}).$$

Proof. Let $\widehat{\mathcal{S}}$ denote the ∞-category of spaces which are not necessarily small, and let $\widehat{\mathrm{CAlg}}_k$ denote the ∞-category of $\mathbb{E}_\infty$-algebras over k which are not necessarily small. Theorem T.5.1.5.6 implies that there exists a limit-preserving functor $\Gamma : \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op} \to \widehat{\mathrm{CAlg}}_k$ such that the composition of Γ with the Yoneda embedding $j' : (\mathrm{CAlg}_k^{\mathrm{cn}}) \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})^{op}$ is homotopic to the the inclusion $\mathrm{CAlg}_k^{\mathrm{cn}} \hookrightarrow \widehat{\mathrm{CAlg}}_k$; moreover, Γ is determined uniquely up to homotopy equivalence. For every object $X \in \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})$ and every object $A \in \mathrm{Alg}_k^{\mathrm{cc}}$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, \mathrm{cSpec}\, A) \simeq \mathrm{Map}_{\widehat{\mathrm{CAlg}}_k}(A, \Gamma(X)).$$

To prove that cSpec is fully faithful, we must show that for every pair of coconnective k-algebras A and B, the natural map

$$\mathrm{Map}_{\mathrm{CAlg}_k}(A, B) \to \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})}(\mathrm{cSpec}\, B, \mathrm{cSpec}\, A) \simeq \mathrm{Map}_{\widehat{\mathrm{CAlg}}_k}(A, \Gamma(\mathrm{cSpec}\, B))$$

is a homotopy equivalence. To prove this, it suffices to show that the unit map $\psi_B : B \to \Gamma(\mathrm{cSpec}\, B)$ is an equivalence in $\widehat{\mathrm{CAlg}}_k$.

According to Corollary 4.3.14, we can write B as the limit of a cosimplicial k-algebra $B^\bullet$, where each B^n is elementary. Proposition 4.3.11 implies that $\mathrm{cSpec}\, B$ can be identified with the colimit of the simplicial object $\mathrm{cSpec}\, B^\bullet$ in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Since Γ preserves all limits, we deduce that $\Gamma(\mathrm{cSpec}\, B) \simeq \mathrm{Tot}\, \Gamma(\mathrm{cSpec}\, B^\bullet)$. Consequently, to prove that ψ_B is an equivalence, it will suffice to show that ψ_{B^n} is an equivalence for each $n \geq 0$. Replacing B by B^n, we may suppose that B is elementary.

Using Corollary 4.1.7, we can choose a map of k-algebras $\eta : B \to k$. Let $C^\bullet : \mathrm{N}(\boldsymbol{\Delta}_+) \to \mathrm{CAlg}_k$ be the Čech nerve of η in the ∞-category CAlg_k^{op}: that is, let $C^\bullet$ be the cosimplicial k-algebra such that C^n is the $(n+1)$st tensor power of k over B. Let $X_\bullet$ be the augmented simplicial object of $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})$ given by $X_\bullet(R) = \mathrm{Map}_{\mathrm{CAlg}_k}(C^\bullet, R)$. There is an evident unit map $u : C^\bullet \to \Gamma \circ X_\bullet$. For $n \geq 0$, Proposition 4.3.15 implies that C^n is connective, so that X_n lies in the essential of j' and so u induces an equivalence $C^n \to \Gamma X_n$ by construction. We wish to prove that u induces an equivalence $B \simeq C^{-1} \to \Gamma X_{-1} \simeq \Gamma(\mathrm{cSpec}\, B)$. Since $C^\bullet$ is a limit diagram (Proposition 4.3.15), it will suffice to show that $\Gamma \circ X_\bullet$ is a limit diagram in $\widehat{\mathrm{CAlg}}_{k/}$. Since Γ preserves limits by construction, we are reduced to proving that $X_\bullet$ is a colimit diagram in $\mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Note that $X_\bullet$ can be identified with the Čech nerve of the map $q : \mathrm{cSpec}\, k \to \mathrm{cSpec}\, B$ determined by η. It is therefore a colimit diagram if and only if q is an effective epimorphism (see §T.6.2.3): that is, if and only if the map

$$* \simeq (\mathrm{cSpec}\, k)(R) \xrightarrow{q} (\mathrm{cSpec}\, B)(R)$$

is surjective on connected components, for every connective k-algebra R. This is equivalent to the connectivity of the mapping space $\mathrm{Map}_{\mathrm{CAlg}_k}(B, R)$, which follows from Corollary 4.1.7. $\square$

Definition 4.4.2. Let k be a field of characteristic zero. We will say that a functor $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ is a *coaffine stack* if has the form $\mathrm{cSpec}\, A$, where A is a coconnective $\mathbb{E}_\infty$-algebra over k.

Remark 4.4.3. If $X : \mathrm{Fun}(\mathrm{CAlg}_k^{\mathrm{cn}}, \mathcal{S})$ is a coaffine stack, then X is automatically a hypercomplete sheaf with respect to the flat topology (Theorem VII.5.14).

The remainder of this section is devoted to studying the class of coaffine stacks over a field k of characteristic zero. We begin by showing that a coaffine stack X is determined by its restriction to *discrete* k-algebras (Proposition 4.4.6).

Proposition 4.4.4. *Let k be a field of characteristic zero and let A be a coconnective $\mathbb{E}_\infty$-algebra over k. Then there exists an augmented cosimplicial object $A^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_k$ with the following properties:*

(1) *We have equivalences $A^{-1} \simeq A$ and $A^0 \simeq k$.*

(2) *For $n \geq 0$, the k-algebra A^n is discrete.*

(3) *The augmented cosimplicial object $A^\bullet$ is a limit diagram.*

(4) *For every connective k-algebra R, the augmented simplicial space $\mathrm{Map}_{\mathrm{CAlg}_k}(A^\bullet, R)$ is a colimit diagram.*

(5) *For every injective map $[m] \to [n]$ in $\mathbf{\Delta}$, the induced map of discrete k-algebras $A^m \to A^m$ is flat.*

Proof. Corollary 4.1.7 implies that there exists an augmentation $\eta : A \to k$. Using Corollary 4.3.14, we can choose a limit diagram $\overline{B}^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_{k//k}$ such that composite map $B^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_{k//k} \to \mathrm{CAlg}_k$ has the following properties:

(a) There is an equivalence of k-algebras $B^{-1} \simeq A$.

(b) For $n \geq 0$, the k-algebra B^n is elementary.

We can regard $\overline{B}^\bullet$ as giving a map of augmented cosimplicial k-algebras $f : B^\bullet \to k^\bullet$, where $k^\bullet$ denotes the constant augmented cosimplicial k-algebra taking the value k. Let $B^{\bullet,\bullet} : \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+) \to \mathrm{CAlg}_k$ be a Čech nerve of f (in the ∞-category $\mathrm{Fun}(\mathrm{N}(\mathbf{\Delta}_+), \mathrm{CAlg}_k)^{op}$, and let $A^\bullet$ be the composition

$$\mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{N}(\mathbf{\Delta}_+, \mathbf{\Delta}_+) \xrightarrow{B^{\bullet,\bullet}} \mathrm{CAlg}_k$$

where the left map is the diagonal embedding. We claim that $A^\bullet$ has the desired properties.

Condition (1) is clear from the construction, and condition (2) follows from Proposition 4.3.15. We now prove (3). Since $\mathrm{N}(\mathbf{\Delta})^{op}$ is sifted (Lemma T.5.5.8.4), it will suffice to show that the restriction $B^{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta})^{\triangleleft}$ is a limit diagram. For each integer $n \geq 0$, the augmented cosimplicial algebra $B^{n,\bullet}$ is a Čech nerve of the map $B^n \to k$ (in CAlg_k^{op}), and therefore a limit diagram (Proposition 4.3.15). It follows that $B^{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)$ is a right Kan extension of $B^{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta})$. Using Lemma T.4.3.2.7, we can reduce to the problem of showing that $B^{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)^{\triangleleft}$ is a limit diagram. Since the inclusion $\mathrm{N}(\mathbf{\Delta} \times \{[-1]\}) \hookrightarrow \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)$ is right cofinal, it will suffice to show that $B^{\bullet,-1} \simeq B^\bullet$ is a limit diagram, which follows by construction.

The proof of (4) is similar. Fix a connective k-algebra R, and let $X_{\bullet,\bullet} : \mathrm{N}(\mathbf{\Delta}_+ \times \mathbf{\Delta}_+)^{op} \to \mathcal{S}$ be defined by the formula $X_{\bullet,\bullet} = \mathrm{Map}_{\mathrm{CAlg}_k}(B^{\bullet,\bullet}, R)$. We wish to prove that augmented simplicial space $[n] \mapsto X_{n,n}$ is a colimit diagram in $\mathcal{S}$. Since $\mathrm{N}(\mathbf{\Delta})^{op}$ is sifted, it will suffice to show that $X_{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta}^{op} \times \mathbf{\Delta}^{op})^{\triangleright}$ is a colimit diagram in $\mathcal{S}$. Each of the augmented simplicial spaces $X_{n,\bullet}$ is the Čech nerve of the map $\phi : \Delta^0 \simeq \mathrm{Map}_{\mathrm{CAlg}_k}(k, R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(B^n, R)$. Since $\mathrm{Map}_{\mathrm{CAlg}_k}(B^n, R)$ is connected (Corollary 4.1.7), ϕ is an effective epimorphism so that $X_{n,\bullet}$ is a colimit diagram. It follows that $X_{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)^{op}$ is a left Kan extension of $X_{\bullet,\bullet} | \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta})^{op}$. Using Lemma T.4.3.2.7 again, we are reduced to proving that $X_{\bullet,\bullet} | (\mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)^{op})^{\triangleright}$ is a colimit diagram. Since the inclusion $\mathrm{N}(\mathbf{\Delta} \times \{[-1]\}) \hookrightarrow \mathrm{N}(\mathbf{\Delta} \times \mathbf{\Delta}_+)$ is right cofinal, it will suffice to show that the augmented simplicial space $X_{\bullet,-1}$ is a colimit diagram. This follows from Proposition 4.3.11.

We now prove (5). Let $\alpha : [m] \to [n]$ be an injection in $\mathbf{\Delta}$; we wish to prove that the induced map $\gamma : A^m \to A^n$ is a flat map of discrete commutative k-algebras. Factoring α as a composition $[m] \hookrightarrow [m+1] \hookrightarrow \cdots \hookrightarrow [n]$, we can reduce to the case where $m = n-1$ and the image of α is $\{0, 1, \ldots, i-1, i+1, \ldots, n\} \subseteq [n]$. Note that there exists a map $\beta : [n] \to [m]$ in $\mathbf{\Delta}$ such that $\beta \circ \alpha$ is the identity. Let $H^m = B^{m,1} \simeq k \otimes_{B^m} k$ and let $H^n = B^{n,1} \simeq k \otimes_{B^n} k$. Then H^m and H^n are commutative Hopf algebras, and α induces a map $f : H^m \to H^n$. Since f is an injective map of Hopf algebras (it has a left inverse given by β), it is automatically flat. The map γ factors as a composition

$$A^m \simeq (H^m)^{\otimes m} \xrightarrow{f^{\otimes m}} (H^n)^{\otimes m} \xrightarrow{\theta} (H^n)^{\otimes n} \simeq A^n.$$

We observe that θ is a tensor product of finitely many copies of the identity map id_{H^n} with either the unit map $k \to H^n$ (if $i \in \{0, n\}$) or the comultiplication $\Delta : H^n \to H^n \otimes_k H^n$ (if $0 < i < n$); in either case, it is clear that θ is flat. It follows that the composition $\theta \circ f^{\otimes n}$ is flat, which completes the proof of (5). $\square$

Notation 4.4.5. Let k be a field. We let CAlg^0_k denote the full subcategory of CAlg_k spanned by the discrete $\mathbb{E}_\infty$-algebras over k (so that CAlg^0_k is equivalent to the nerve of the ordinary category of commutative k-algebras; see Remark A.7.1.0.3).

Proposition 4.4.6. *Let k be a field of characteristic zero. If $X : \mathrm{CAlg}^{\mathrm{cn}}_k \to \mathcal{S}$ is a coaffine stack, then X is a left Kan extension of the restriction $X | \mathrm{CAlg}^0_k$.*

Proof. Let $\mathcal{C} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})$ be the full subcategory spanned by those functors X which are left Kan extensions of $X | \mathrm{CAlg}^0_k$. Then $\mathcal{C}$ is stable under small colimits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})$; we wish to show that $\mathcal{C}$ contains $\mathrm{cSpec}\, A$ for every coconnective k-algebra A. Let $A^\bullet : \mathrm{N}(\mathbf{\Delta}_+) \to \mathrm{CAlg}_k$ be as in Proposition 4.4.4. We wish to show that $\mathrm{cSpec}\, A^{-1}$ belongs to $\mathcal{C}$. Proposition 4.4.4 implies that $\mathrm{cSpec}\, A^{-1}$ is the colimit of the functors $\mathrm{cSpec}\, A^n$ (taken over $[n] \in \mathbf{\Delta}^{op}$). It will therefore suffice to show that each $\mathrm{cSpec}\, A^n$ belongs to $\mathcal{C}$. This is clear, since each A^n is discrete. $\square$

Corollary 4.4.7. *Let k be a field of characteristic zero. Then the composite functor*

$$(\mathrm{CAlg}^{\mathrm{cc}}_k)^{op} \subseteq \mathrm{CAlg}^{op}_k \xrightarrow{\mathrm{cSpec}} \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})$$

is fully faithful.

Proof. Combine Theorem 4.4.1, Proposition 4.4.6, and Proposition T.4.3.2.15. Alternatively, we can repeat the proof of Theorem 4.4.1, replacing $\mathrm{CAlg}^{\mathrm{cn}}_k$ by CAlg^0_k everywhere. $\square$

Our next goal is to obtain a characterization of those functors $X : \mathrm{CAlg}^{\mathrm{cn}}_k \to \mathcal{S}$ which are coaffine stacks. According to Proposition 4.4.6, X must be a left Kan extension of its restriction to the subcategory $\mathrm{CAlg}^0_k \subseteq \mathrm{CAlg}^{\mathrm{cn}}_k$. We may therefore formulate the problem as follows: which functors $\mathrm{CAlg}^0_k \to \mathcal{S}$ are corepresented by coconnective $\mathbb{E}_\infty$-algebras over k? Our next result provides an answer:

Proposition 4.4.8. *Let k be a field of characteristic zero and let $X : \mathrm{CAlg}^0_k \to \mathcal{S}$ be a functor. Then X is the restriction of a coaffine stack $\overline{X} : \mathrm{CAlg}^{\mathrm{cn}}_k \to \mathcal{S}$ (which is uniquely determined up to equivalence, by Proposition 4.4.6) if and only if the following conditions are satisfied:*

(1) *For every discrete k-algebra R, the space $X(R)$ is connected.*

It follows from (1) that there exists a point $\eta \in X(k)$. For every k-algebra R, we will identify η with its image in the space $X(R)$.

(2) *For every integer $i > 0$, the functor $R \mapsto \pi_i(X(R), \eta)$ is representable by a prounipotent group scheme over k.*

Remark 4.4.9. In the situation of Proposition 4.4.8, the functors $R \mapsto \pi_i(X(R), \eta)$ depend on the choice of point $\eta \in X(k)$. However, since $X(k)$ is connected, any other choice of point $\eta' \in X(k)$ yields an isomorphic functor $R \mapsto \pi_i(X(R), \eta')$. The isomorphism $\pi_i(X(R), \eta) \simeq \pi_i(X(R), \eta')$ is not canonically determined: it depends on a choice of path from η to η' in $X(k)$ (which is generally not simply connected). Nevertheless, the condition that $R \mapsto \pi_i(X(R), \eta)$ be representable by a prounipotent group scheme does depend on the choice of point η.

Remark 4.4.10. If W and W' are vector spaces over the field k, we let $\mathrm{Hom}_k(W, W')$ denote the set of linear maps from W to W'. For every vector space W, the scheme $\mathrm{Spec}^{\mathrm{c}} \mathrm{Sym}^* W$ is a prounipotent group scheme over k, representing the functor $R \mapsto \mathrm{Hom}_k(W, R)$. Since k has characteristic zero, this construction determines a fully faithful embedding of the category of k-vector spaces into the category of prounipotent group schemes over k. The essential image of this functor is the category of *commutative* prounipotent group schemes.

Proof of Proposition 4.4.8. Suppose first that X is the restriction of a coaffine stack $\overline{X}$, and write $\overline{X} = \mathrm{cSpec}\, A$ for some coconnective $\mathbb{E}_\infty$-algebra A over k. Assertion (1) follows from Corollary 4.1.7. Fix $\eta \in X(k)$, and for $i > 0$ let U_i denote the functor from the category of commutative rings to the category of groups given by the formula $U_i(R) = \pi_i(X(R), \eta)$. We wish to prove that each U_i is representable by a prounipotent group scheme over k.

Choose a sequence

$$k = A(0) \to A(1) \to A(2) \to \cdots$$

satisfying the conditions of Lemma 4.1.4. For each $j \geq 0$, let $\overline{X}(j) = \mathrm{cSpec}\, A(j)$ and let $X(j) = \overline{X}(j)| \mathrm{CAlg}^0_k$. For every commutative ring R, we have a canonical map $X(k) \to X(R) \to X(j)(R)$; we will identify η with its image under this map. For $i > 0$ and $j \geq 0$, we let $U(j)_i$ be the functor from commutative rings to groups given by the formula $U(j)_i(R) = \pi_i(X_j(R), \eta)$. We first prove:

(2′) Each of the functors $U(j)_i$ is representable by a prounipotent group scheme over k.

The proof proceeds by induction on j. If $j = 0$, the result is obvious. Assume therefore that $j > 0$ and that each $U(j-1)_i$ is representable by a prounipotent group scheme over k. By assumption, we have a cofiber sequence

$$\mathrm{Sym}^* V \to A(j-1) \to A(j)$$

in $\mathrm{CAlg}_{k//k}$, where $\pi_n V \simeq 0$ for $n > -2$. It follows that for $i > 0$, we have a canonical long exact sequence

$$U(j-1)_{i+1}(R) \xrightarrow{\theta} \mathrm{Hom}_k(\pi_{-i-1}V, R) \to U(j)_i(R) \to U(j-1)_i(R) \xrightarrow{\theta'} \mathrm{Hom}_k(\pi_{-i}V, R).$$

By the inductive hypothesis, we may assume that $U(j-1)_{i+1}$ is representable by a commutative prounipotent group scheme over k: that is, there exists a k-vector space W and a functorial isomorphism $U(j-1)_{i+1}(R) \simeq \mathrm{Hom}_k(W, R)$. The natural transformation θ is induced by a map of vector spaces $\lambda : \pi_{-i-1}V \to W$, and $\mathrm{coker}(\theta)$ is the functor $R \mapsto \mathrm{Hom}_k(\ker(\lambda), R)$.

Suppose now that $i = 1$. We have an exact sequence (depending functorially on R)

$$0 \to \mathrm{Hom}_k(\ker(\lambda), R) \to U(j)_1(R) \xrightarrow{p_R} U(j-1)_1(R) \to 0.$$

By the inductive hypothesis, the functor $U(j-1)_1$ is corepresentable by a commutative ring A. Since p_A is surjective, we can lift the canonical element $U(j-1)_1(A)$ to a class $\epsilon \in U(j)_1(A)$. This lifting determines a natural transformation of set-valued functors $U(j-1)_1 \to U(j)_1$ which is a section of p. Using the group structure on $U(j)_1$, we get a canonical isomorphism of set-valued functors

$$U(j)_1 \simeq U(j-1)_1 \times \mathrm{Hom}_k(\ker(\lambda), \bullet)$$

so that $U(j)_1$ is corepresentable by the commutative k-algebra $A \otimes_k \mathrm{Sym}^* \ker(\lambda)$. It follows that $U(j)_1$ is representable by an affine group scheme. This affine group scheme is an extension of the prounipotent group scheme $U(j-1)_1$ by the commutative group scheme given by the $\operatorname{Spec}^c \operatorname{Sym}^* \ker(\lambda)$, and therefore prounipotent.

Now suppose that $i > 1$. The inductive hypothesis implies that $U(j-1)_i$ is representable by a commutative prounipotent group scheme over k, and therefore has the form $R \mapsto \operatorname{Hom}_k(W', R)$ for some k-vector space W'. The map θ' is induced by a map of k-vector spaces $\lambda' : \pi_{-i} V \to W'$. It follows that the fiber of θ can be identified with the functor $R \mapsto \operatorname{Hom}_k(\operatorname{coker}(\lambda'), R)$. We therefore have a functorial exact sequence of abelian groups

$$0 \to \operatorname{Hom}_k(\ker(\lambda), R) \to U(j)_i(R) \to \operatorname{Hom}_k(\operatorname{coker}(\lambda'), R) \to 0.$$

Arguing as above, we conclude that $U(j)_i$ is representable by an affine group scheme over k. This group scheme is an extension of $\operatorname{Spec}^c \operatorname{Sym}^* \operatorname{coker}(\lambda')$ by the $\operatorname{Spec}^c \operatorname{Sym}^* \ker(\lambda)$, and is therefore prounipotent. This completes the proof of $(2')$.

We now prove that each U_i is representable by a prounipotent scheme over k. Since the collection of prounipotent group schemes is closed under limits, it will suffice to show that each of the maps $U_i \to \varprojlim_j U(j)_i$ is an equivalence. Since $A \simeq \varinjlim A(j)$, we have an equivalence $X \simeq \varprojlim X(j)$ giving rise to Milnor short exact sequences

$$0 \to \varprojlim{}^1 U(j)_{i+1}(R) \to U_i(R) \to \varprojlim U(j)_i(R) \to 0.$$

Each $U(j)_{i+1}$ is representable by a commutative prounipotent group scheme, and therefore has the form $R \mapsto \operatorname{Hom}_k(W(j), R)$ for some vector space $W(j)$. The vector spaces $W(j)$ form a direct system $0 = W(0) \to W(1) \to W(2) \to \cdots$. To complete the proof of (2), it will suffice to show that for every commutative k-algebra R, the group $\varprojlim^1 \operatorname{Hom}_k(W(j), R)$ vanishes. To see this, we note that $\varprojlim^1 \operatorname{Hom}_k(W(j), R)$ can be identified with

$$\pi_0 \varprojlim \operatorname{Map}_{\operatorname{Mod}_k}(W(j), R[1]) \simeq \pi_0 \operatorname{Map}_{\operatorname{Mod}_k}(\varinjlim W(j), R[1]) = \operatorname{Ext}^1_k(\varinjlim W(j), R),$$

which vanishes since $\varinjlim W(j)$ is automatically a projective k-module (because k is a field).

Now suppose that conditions (1) and (2) are satisfied. We wish to prove that X is the restriction of a coaffine stack $\overline{X}$. For each $n \geq 0$, let $\tau_{\leq n} X$ denote the n-truncation of X, defined by the formula

$$(\tau_{\leq n} X)(R) = \tau_{\leq n} X(R).$$

We will prove:

- $(*)$ For each $n \geq 0$, the functor $\tau_{\leq n} X$ is the restriction of a coaffine stack $\operatorname{cSpec} A_n$, for some coconnective k-algebra A_n.

Assuming $(*)$, we deduce from Theorem 4.4.1 and Proposition 4.4.6 that the tower

$$\cdots \to \tau_{\leq 2} X \to \tau_{\leq 1} X \to \tau_{\leq 0} X$$

arises from an essentially unique diagram of coconnective k-algebras

$$A_0 \to A_1 \to A_2 \to \cdots.$$

We conclude by observing that $X \simeq \varprojlim \tau_{\leq n} X$ is corepresentable by the coconnective k-algebra $A \simeq \varinjlim A_i$.

It remains to prove $(*)$. We work by induction on n. If $n = 0$, then condition (1) guarantees that $(\tau_{\leq 0} X)(R)$ is contractible for every commutative ring R, so that $\tau_{\leq 0} X$ is corepresentable by $A_0 = k$. Assume therefore that $n > 0$. Choose a point $\eta \in X(k)$ and for each i, let U_i denote the functor $R \mapsto \pi_i(X(R), \eta)$. Assumption (2) guarantees that U_n is representable by a prounipotent group scheme which is acted on by the prounipotent group scheme U_1. It follows that U_n admits a filtration by U_1-invariant closed subgroups

$$\cdots \subseteq F^2 U_n \subseteq F^1 U_n \to F^0 U_n = U_n$$

with the following properties:

(a) The canonical map $U_n \to \varprojlim U_n/F^pU_n$ is an isomorphism.

(b) For each $p \geq 0$, the group scheme U_1 acts trivially on the quotient $F^pU_n/F^{p+1}U_n$; in particular, $F^pU_n/F^{p+1}U_n$ is a commutative prounipotent group scheme, representing the functor $R \mapsto \mathrm{Hom}_k(W_p, R)$ for some k-vector space W_p.

For each $p \geq 0$, we can choose a partial truncation $X \to X_p$ in $\mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})$ which is characterized up to equivalence by the following requirements:

(i) For $m < n$, the map $\pi_m(X(R); \eta) \to \pi_m(X_p(R); \eta)$ is bijective.

(ii) For $m = n$, the map $\pi_m(X(R); \eta) \to \pi_m(X_p(R), \eta)$ is a surjection with kernel $F^pU_n(R)$.

(iii) For $m > n$, the homotopy group $\pi_m(X_p(R), \eta)$ vanishes.

These truncations form an inverse system

$$\cdots \to X_2 \to X_1 \to X_0 = \tau_{\leq n-1}X$$

with $\varprojlim X_p \simeq \tau_{\leq n}X$. We will prove:

$(*')$ For each $p \geq 0$, the functor X_p is the restriction of a coaffine stack $\mathrm{cSpec}\, B_n$, for some coconnective k-algebra B_n.

Assuming $(*')$, we get a direct system $B_0 \to B_1 \to B_2 \to \cdots$ whose colimit is a coconnective k-algebra A_n which corepresents the functor $\tau_{\leq n}X$. It will therefore suffice to prove $(*')$. For this, we use induction on p. If $p = 0$, we observe that $X_0 = \tau_{\leq n-1}X$ is corepresented by the coconnective k-algebra A_{n-1}. Assume therefore that $p > 0$, and that X_{p-1} is corepresented by some connective k-algebra B_n. We note that $\theta : X_p \to X_{p-1}$ is an n-gerbe banded by the functor $R \mapsto \mathrm{Hom}_k(W_p, R)$ (see §T.7.2.2). It follows that θ fits into a fiber sequence

$$\begin{tikzcd} X_p \arrow[r] \arrow[d] & * \arrow[d] \\ X_{p-1} \arrow[r, "\phi"] & Y, \end{tikzcd}$$

where Y is the functor $R \mapsto K(\mathrm{Hom}_k(W_p, R), n+1)$ corepresented by $\mathrm{Sym}^* W[-n-1]$. Using Theorem 4.4.1 and Proposition 4.4.6, we conclude that the map ϕ is classified by a map of coconnective k-algebras $\mathrm{Sym}^* W[-n-1] \to B_{p-1}$. Lemma 4.1.3 implies that the pushout $B_p = B_{p-1} \otimes_{\mathrm{Sym}^* W[-n-1]} k$ is a coconnective k-algebra; by construction this k-algebra represents the functor X_p. $\square$

Notation 4.4.11. Let k be a field of characteristic zero and $X : \mathrm{CAlg}^{\mathrm{cn}}_k \to \mathcal{S}$ a coaffine stack. For every integer i, we let $\pi_i X$ denote the prounipotent group scheme over k representing the functor $R \mapsto \pi_i(X(R), \eta)$, for some chosen point $\eta \in X(k)$. This notation is abusive: $\pi_i X$ depends not only on X but also on the point η (however, the isomorphism class of $\pi_i X$ does not depend on η).

We now study the connectivity properties of coaffine stacks.

Proposition 4.4.12. *Let A be a coconnective $\mathbb{E}_\infty$-algebra over a field k of characteristic zero, and let $X = \mathrm{cSpec}\, A$ be the corresponding coafffine stack. For each integer $n > 0$, the following conditions are equivalent:*

(1) *For every connective k-algebra R, the space $X(R)$ is n-connective.*

(2) *For every discrete k-algebra R, the space $X(R)$ is n-connective.*

(3) *For $0 < i < n$, the prounipotent group scheme $\pi_i X$ is trivial.*

(4) *If $\eta : A \to k$ is a map of k-algebras, then the homotopy groups $\pi_i(k \otimes_A L_{A/k})$ vanish for $i > -n$.*

(5) *The homotopy groups $\pi_i L_{A/k}$ vanish for $i > -n$.*

(6) *For $0 < i < n$, the homotopy group $\pi_{-i} A$ vanishes.*

Definition 4.4.13. We will say that a coaffine stack X is *n-connective* if it satisfies the equivalent conditions of Proposition 4.4.12.

Remark 4.4.14. Let k be a field of characteristic zero, $X = \mathrm{cSpec}\, A$ a coaffine stack over k, and choose a base point $\eta \in X(k)$. For every integer $n \geq 1$, let $\Omega_n(X)$ denote the k-vector space of left invariant 1-forms on the prounipotent group scheme $\pi_n X$. For every k-vector space V, we can identify the set of k-linear maps $\mathrm{Hom}_k(\Omega_n(X), V)$ with the kernel of the group homomorphism $(\pi_n X)(k \oplus V) \to (\pi_n X)(k)$. It follows that $\Omega_n(X)$ can be identified with the vector space $\pi_{-n}(k \otimes_A L_{A/k})$.

Proof of Proposition 4.4.12. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious. The implication (3) $\Rightarrow$ (4) follows from Remark 4.4.14, the implication (4) $\Rightarrow$ (5) follows from Corollaries 4.1.16 and 4.1.12, and the implication (5) $\Rightarrow$ (6) follows from Proposition 4.1.15. We will prove that (6) $\Rightarrow$ (1). Applying Proposition 4.1.4 to the map $k \to A$, we deduce the existence of a sequence of algebra maps

$$k = A(0) \to A(1) \to A(2) \to \cdots$$

with colimit A, where each $A(i+1)$ can be written as a tensor product $A(i) \otimes_{\mathrm{Sym}^* V(i)} k$ with $\pi_j V(i) \simeq 0$ for $j \geq -n$. Let R be a connective k-algebra; we wish to prove that $\mathrm{Map}_{\mathrm{CAlg}_k}(A, R) \simeq \varprojlim \mathrm{Map}_{\mathrm{CAlg}_k}(A(i), R)$ is n-connective. It will therefore suffice to show that each of the restriction maps

$$\mathrm{Map}_{\mathrm{CAlg}_k}(A(i+1), R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(A(i), R)$$

has n-connective homotopy fibers. These can be identified with the homotopy fibers of the restriction map

$$* = \mathrm{Map}_{\mathrm{CAlg}_k}(k, R) \to \mathrm{Map}_{\mathrm{CAlg}_k}(\mathrm{Sym}^* V(i), R) \simeq \mathrm{Map}_{\mathrm{Mod}_k}(V(i), R);$$

we are therefore reduced to proving that $\mathrm{Map}_{\mathrm{Mod}_k}(V(i), R)$ is $(n+1)$-connective. This follows from our assumption that R is connective and $\pi_j V(i) \simeq 0$ for $j \geq -n$. $\square$

Proposition 4.4.15. *Let k be a field of characteristic zero, $A \in \mathrm{CAlg}_k^{\mathrm{cc}}$ a coconnective $\mathbb{E}_\infty$-algebra over k, and $X = \mathrm{cSpec}\, A$ the corresponding coaffine stack. The following conditions are equivalent:*

(1) *There exists a finite sequence of k-algebra morphisms*

$$k = A(0) \to A(1) \to \cdots \to A(n) = A$$

such that, for $0 \leq j < n$, there is a pushout diagram of k-algebras

$$\begin{tikzcd} \mathrm{Sym}^* k[a_j] \arrow[r] \arrow[d] & k \arrow[d] \\ A(j) \arrow[r] & A(j+1) \end{tikzcd}$$

for some integer $a_j \leq -2$.

(2) *The k-algebra A is a compact object of CAlg_k.*

(3) *The relative cotangent complex $L_{A/k}$ is a perfect A-module.*

(4) *For some point $\eta : A \to k$, the tensor product $k \otimes_A L_{A/k}$ is a perfect k-module: that is, the homotopy groups $\pi_i(k \otimes_A L_{A/k})$ are finite-dimensional for all i, and vanish for all but finitely many i.*

(5) *Each of the prounipotent group schemes $\pi_i X$ has finite type, and $\pi_i X$ is trivial for $i \gg 0$.*

Proof. The implication (1) $\Rightarrow$ (2) is obvious (since each $\mathrm{Sym}^* k[a_j]$ is a compact object of CAlg_k). To see that (2) $\Rightarrow$ (3), we note the square-zero extension functor $(M \in \mathrm{Mod}_A) \mapsto (A \oplus M \in \mathrm{CAlg}_{k//A})$ commutes with filtered colimits. Consequently, if $A \in \mathrm{CAlg}_k$ is compact, then the functor

$$M \mapsto \mathrm{Map}_{\mathrm{Mod}_A}(L_{A/k}, M) \simeq \mathrm{Map}_{\mathrm{CAlg}_{k//A}}(A, A \oplus M)$$

commutes with filtered colimits: that is, $L_{A/k}$ is a compact A-module and therefore perfect. The implication (3) $\Rightarrow$ (4) is obvious. Note that the homotopy groups $\pi_i(k \otimes_A L_{A/k})$ vanish for $i \geq 0$ by Corollaries 4.1.16 and 4.1.12, so that the equivalence of (4) and (5) follows from Remark 4.4.14.

Suppose that condition (5) is satisfied, and choose a base point $\eta \in X(k)$. We will prove (1) using induction on the integer $n = \sum_i \dim(\pi_i X)$. If $n = 0$, then $A \simeq k$ and there is nothing to prove. Otherwise, let m denote the largest integer such that the prounipotent group scheme $U = \pi_m X$ is nontrivial. Let $U_0 \subseteq U$ be a closed subgroupscheme which is invariant under the action of $\pi_1 X$ on U such that U_0 is isomorphic to the additive group $\mathbb{G}_a$. Choose a map $X \to X'$ in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_k^0, \mathcal{S})$ with the following property: for every discrete k-algebra R, the map $\pi_i X(R) \to \pi_i X'(R)$ is an isomorphism for $i \neq m$ and fits into a short exact sequence

$$0 \to U_0(R) \to \pi_i X(R) \to \pi_i X'(R) \to 0$$

when $i = m$. Proposition 4.4.8 implies that X' is a coaffine stack, and the inductive hypothesis implies the existence of a finite sequence

$$k = A(0) \to \cdots \to A(n-1)$$

with $X' \simeq \mathrm{cSpec}\, A(n-1)$. Since $X \to X'$ is an m-gerbe banded by U_0, we have a pullback diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & X' \arrow[d] \\ * \arrow[r] & K(U_0, m+1). \end{tikzcd}$$

Corollary 4.4.7 guarantees that this pullback diagram is the image of a diagram of coconnective k-algebras σ :

$$\begin{tikzcd} A & A(n-1) \arrow[l] \\ k \arrow[u] & \mathrm{Sym}^* k[-m-1], \arrow[l] \arrow[u] \end{tikzcd}$$

which is a pushout square in $\mathrm{CAlg}_k^{\mathrm{cc}}$. It follows from Lemma 4.1.3 that $A \simeq A(n-1) \otimes_{\mathrm{Sym}^* k[-m-1]} k$ (that is, the square σ is also a pushout diagram in CAlg_k), which completes the proof of (1). $\square$

From Remark 4.4.14 we immediately deduce the following:

Proposition 4.4.16. *Let k be a field of characteristic zero, $A \in \mathrm{CAlg}_k^{\mathrm{cc}}$, and $X = \mathrm{cSpec}\, A$ the corresponding coaffine stack. Choose a base point $\eta \in X(k)$, corresponding to a map $A \to k$. The following conditions are equivalent:*

(1) *For each $n \geq 1$, the prounipotent group scheme $\pi_i X$ has finite type.*

(2) *The vector spaces $\pi_i(k \otimes_A L_{A/k})$ are finite dimensional for every integer i.*

Definition 4.4.17. We will say that a coaffine stack X is of *finite type* if it satisfies the equivalent conditions of Proposition 4.4.16.

Proposition 4.4.18. *Let k be a field of characteristic zero, $A \in \mathrm{CAlg}_k^{\mathrm{cc}}$, and $X = \mathrm{cSpec}\, A$ the corresponding coaffine stack. Then:*

(1) *If X is of finite type, then each homotopy group $\pi_i A$ is a finite dimensional vector space over k.*

(2) *Suppose that each homotopy group $\pi_i A$ is finite dimensional and that $\pi_{-1} A \simeq 0$. Then X is of finite type.*

Remark 4.4.19. Assertion (2) of Proposition 4.4.18 is not true if we do not require that $\pi_{-1} A \simeq 0$ has finite type. For example, it fails if A is an elementary algebra $k \oplus V[-1]$ when V has dimension ≥ 2; see Remark 4.3.17.

Proof. We first prove (1). Assume that X is of finite type. Each truncation $\tau_{\leq n} X$ is corepresentable by a coconnective k-algebra A_n, with $A \simeq \varinjlim A_n$. It will suffice to prove the following:

(a) Each A_n is of finite type.

(b) For $n > 0$, the map $A_{n-1} \to A_n$ induces an isomorphism $\pi_{-i} A_{n-1} \to \pi_{-i} A_n$ for $0 < i < n$.

The proof proceeds by induction on n, the case $n = 0$ being trivial. Let $U_n = \pi_n X$. Since U_n has finite type, it admits a *finite* U_1-invariant filtration

$$0 = F^m U_n \subseteq \cdots F^{m-1} U_n \subseteq \cdots \subseteq F^1 U_n \subseteq F^0 U_n = 0$$

Let $X_p \in \mathrm{Fun}(\mathrm{CAlg}_k^0, \mathcal{S})$ and B_p be as in the proof of Proposition 4.4.12, so that $B_m \simeq A_n$ and $B_0 \simeq A_{n-1}$. Assertions (a) and (b) are consequences of the following more refined assertions:

(a') Each B_p is of finite type.

(b') For $p > 0$, the map $B_{p-1} \to B_p$ induces an isomorphism $\pi_{-i} B_{p-1} \to \pi_{-i} B_p$ for $0 < i < n$.

We prove (a') and (b') by induction on p, the case $p = 0$ being evident. Assume that $p > 0$, so that $F^{p-1} U_n / F^p U_n$ is a commutative unipotent group scheme representing the functor $R \mapsto \mathrm{Hom}_k(W, R)$ for some finite dimensional k-vector space W. Let $V = W[-n-1] \in \mathrm{Mod}_k$, so we can write $B_p \simeq B_{p-1} \otimes_{\mathrm{Sym}^* V} k$ as in the proof of Proposition 4.4.12. We have a spectral sequence $\{E_r^{a,b}, d_r\}$ converging to the homotopy groups $\pi_{a+b} B_p$, with $E_1^{a,*} \simeq (\pi_* B_{p-1}) \otimes_k (\mathrm{Sym}^{>0} \pi_* V)^{\otimes a}$. Using the inductive hypothesis, we deduce that each $E_1^{a,b}$ is a finite-dimensional vector space over k, and that $E_1^{a,b}$ vanishes unless $0 \leq a \leq -(n+1)b$, and $b \leq 0$. In particular, for a fixed integer $i \geq 0$, there are only finitely many nonzero groups $E_1^{a,b}$ with $a + b = -i$ (since we must then have $0 \leq -b \leq \frac{i}{n}$). Since $\pi_{-i} B_p$ admits a finite filtration by subquotients of the groups $E_1^{a,b}$, we conclude that each $\pi_{-i} B_p$ is finite dimensional: that is, B_p is of finite type. This proves (a). To prove (b), we note that if $i < n$, the vanishing $E_1^{-i-b,b} \simeq E_1^{1-i-b,b}$ for $b \neq 0$ gives isomorphisms $\pi_{-i} B_p \simeq E_\infty^{-i,0} \simeq E_1^{-i,0} \simeq \pi_{-i} B_{p-1}$. This completes the proof of (1).

We now prove (2). We will prove by induction on n that the unipotent group scheme $U_n = \pi_n X$ has finite type, the case $n = 1$ being trivial. Let $A_0 \to A_1 \to \cdots$ be defined as above. Arguing as above, we deduce that the maps $\pi_{-n} A_m \to \pi_{-n} A_{m+1}$ are isomorphisms for $m \geq n$. The inductive hypothesis implies that $\tau_{\leq n-1} X$ is of finite type, so that A_{n-1} is of finite type by assertion (1). The commutative unipotent group scheme U_n represents the functor $R \mapsto \mathrm{Hom}_k(W, R)$ for some vector space W over k; we wish to prove that W is finite dimensional. Let $V = W[-n-1] \in \mathrm{Mod}_k$, so that $A_n \simeq A_{n-1} \otimes_{\mathrm{Sym}^* V} k$ and therefore the homotopy groups $\pi_{a+b} A_n$ are computed by the spectral sequence $\{E_r^{a,b}, d_r\}$ above. This spectral sequence gives rise to an exact sequence of vector spaces

$$\pi_{-n} A_{n-1} \to \pi_{-n} A_n \to W \to \pi_{-n-1} A_{n-1}.$$

Since A_{n-1} has finite type, we deduce that W is finite dimensional if and only if $\pi_{-n} A_n \simeq \varinjlim \pi_{-n} A_m \simeq \pi_{-n} A$ is finite dimensional. $\square$

4.5 Quasi-Coherent Sheaves on Coaffine Stacks

Let k be a field of characteristic zero and let $X : \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathcal{S}$ be a coaffine stack, so that $X = \mathrm{cSpec}\, A$ for some coconnective $\mathbb{E}_\infty$-algebra A over k. In this section, we will study the ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X, and study its relationship to the ∞-category Mod_A of A-module spectra. Our first step is to define $\mathrm{QCoh}(X)$.

Notation 4.5.1. Let X be a coaffine stack over a field k of characteristic zero. Then X classifies a left fibration of ∞-categories $q : \mathcal{C} \to \mathrm{CAlg}_k^{\mathrm{cn}}$. The composite map $\mathcal{C} \xrightarrow{q} \mathrm{CAlg}_k^{\mathrm{cn}} \to \mathrm{CAlg}^{\mathrm{cn}}$ is also a left fibration, classified by a functor $X^+ : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$. The functor X^+ is a left Kan extension of X along the forgetful functor $\mathrm{CAlg}_k^{\mathrm{cn}} \to \mathrm{CAlg}_k$. We let $\mathrm{QCoh}(X)$ denote the symmetric monoidal ∞-category $\mathrm{QCoh}(X^+)$ defined in §2.7. More concretely, if we write $X = \mathrm{cSpec}\, A$, where A is a coconnective $\mathbb{E}_\infty$-algebra over k, then $\mathrm{QCoh}(X)$ is given by the inverse limit $\varprojlim_{\phi : A \to R} \mathrm{Mod}_R$, where the limit is taken over all maps of $\mathbb{E}_\infty$-rings $\phi : A \to R$ for which the codomain R is connective. In particular, we have a symmetric monoidal pullback functor $\mathrm{Mod}_A \to \mathrm{QCoh}(X)$.

We can now formulate the main result of this section.

Proposition 4.5.2. *Let X be a coaffine stack over a field k of characteristic zero, and fix a point $\eta \in X(k)$. For every object $M \in \mathrm{QCoh}(X)$, we let $M(\eta)$ denote the image of M in Mod_k. Then:*

(1) *The stable ∞-category $\mathrm{QCoh}(X)$ is presentable.*

(2) *Let $\mathrm{QCoh}(X)_{\geq 0} \subseteq \mathrm{QCoh}(X)$ be the full subcategory spanned by those objects M for which $\pi_i M(\eta) \simeq 0$ for $i < 0$, and let $\mathrm{QCoh}(X)_{\leq 0}$ denote the full subcategory of $\mathrm{QCoh}(X)$ spanned by those objects M for which $\pi_i M(\eta) \simeq 0$ for $i > 0$. Then $(\mathrm{QCoh}(X)_{\geq 0}, \mathrm{QCoh}(X)_{\leq 0})$ is an accessible t-structure on $\mathrm{QCoh}(X)$.*

(3) *The t-structure of (2) is compatible with filtered colimits: that is, the full subcategory $\mathrm{QCoh}(X)_{\leq 0}$ is closed under filtered colimits in $\mathrm{QCoh}(X)$.*

(4) *The t-structure of (2) is both right and left complete.*

(5) *The heart of $\mathrm{QCoh}(X)$ can be identified with the abelian category of algebraic representations of the prounipotent group scheme $\pi_1 X$.*

(6) *Write $X = \mathrm{cSpec}\, A$ for some coconnective k-algebra A. The pullback functor $F : \mathrm{Mod}_A \to \mathrm{QCoh}(X)$ is t-exact (with respect to the t-structure on Mod_A described in Proposition 4.5.4 below).*

(7) *The functor F induces an equivalence of ∞-categories $(\mathrm{Mod}_A)_{\leq 0} \to \mathrm{QCoh}(X)_{\leq 0}$.*

Remark 4.5.3. Let X be a coaffine stack over a field k of characteristic zero. Since the ∞-categories $(\mathrm{Mod}_k)_{\leq 0}$ and $(\mathrm{Mod}_k)_{\geq 0}$ both contain the unit object of Mod_k and are stable under tensor products, we conclude that $\mathrm{QCoh}(X)_{\geq 0}$ and $\mathrm{QCoh}(X)_{\leq 0}$ contain the unit object of $\mathrm{QCoh}(X)$ and are stable under tensor products.

The proof of Proposition 4.5.2 will be given at the end of this section. First, we need to describe the t-structure on the ∞-category Mod_A appearing in assertion (6).

Proposition 4.5.4. *Let k be a field of characteristic zero and A a coconnective $\mathbb{E}_\infty$-algebra over k. We let $\mathrm{Mod}_A^{\leq 0}$ denote the full subcategory of Mod_A spanned by those A-module spectra M for which the homotopy groups $\pi_i M$ vanish for $i > 0$. Then:*

(1) *The full subcategory $\mathrm{Mod}_A^{\leq 0}$ determines an accessible t-structure $(\mathrm{Mod}_A^{\geq 0}, \mathrm{Mod}_A^{\leq 0})$ on the stable ∞-category Mod_A.*

(2) *Let $M \in \mathrm{Mod}_A$. The following conditions are equivalent:*

(a) *The module M belongs to $\mathrm{Mod}_A^{\geq 0}$.*

(b) *For every map $A \to R$, where R is a connective k-algebra, the tensor product $R \otimes_A M$ is a connective R-module.*

(c) *There exists a map of k-algebras $A \to R$, where R is connective and nonzero, such that $R \otimes_A M$ is a connective R-module.*

(3) *The full subcategory $\mathrm{Mod}_A^{\leq 0} \subseteq \mathrm{Mod}_A$ is stable under filtered colimits.*

(4) *The t-structure $(\mathrm{Mod}_A^{\geq 0}, \mathrm{Mod}_A^{\leq 0})$ is right complete.*

(5) *The t-structure on Mod_A is compatible with the symmetric monoidal structure on Mod_A. That is, the full subcategory $\mathrm{Mod}_A^{\geq 0} \subseteq \mathrm{Mod}_A$ contains the unit object $A \in \mathrm{Mod}_A$ and is stable under tensor products.*

(6) *The t-structure on Mod_A is compatible with the induced symmetric monoidal structure on Mod_A^{op}. In other words, the full subcategory $\mathrm{Mod}_A^{\leq 0}$ contains the unit object $A \in \mathrm{Mod}_A$ and is stable under tensor products.*

Proof. Let $\mathrm{Mod}_A^{\geq 0}$ be the smallest full subcategory of Mod_A is contains A and is closed under extensions and small colimits. It follows from Proposition A.1.4.5.11 that there exists an accessible t-structure $(\mathrm{Mod}_A^{\geq 0}, \mathrm{Mod}_A^{\leq 0})$ on Mod_A, where $\mathrm{Mod}_A^{\leq 0}$ is the full subcategory of Mod_A spanned by those objects M for which the groups $\pi_i M \simeq \mathrm{Ext}_A^{-i}(A, M)$ vanish for $i > 0$. This proves (1).

We now prove (2). It is clear from the definition of $\mathrm{Mod}_A^{\geq 0}$ that if $\phi : A \to R$ is a map of k-algebras where R is connective, then $R \otimes_A M$ is connective for each $M \in \mathrm{Mod}_A^{\geq 0}$. This shows that $(a) \Rightarrow (b)$, and the implication $(b) \Rightarrow (c)$ is clear (since there exists a map $A \to k$ by virtue of Corollary 4.1.7). We now prove that $(c) \Rightarrow (a)$. Let $M \in \mathrm{Mod}_A$ be such that $R \otimes_A M$ is connective for some nonzero connective A-algebra R. Replacing R by $\tau_{\leq 0} R$, we may assume that R is discrete. Using Corollary 4.1.7, we may assume that the map $A \to R$ factors as a composition $A \to k \to R$. Then $R \otimes_A M \simeq R \otimes_k (k \otimes_A M)$ is connective. Since R is faithfully flat over k, we deduce that $k \otimes_A M$ is connective: that is, we may reduce to the case where $R = k$. Choose a fiber sequence

$$M' \to M \to M''$$

with $M' \in \mathrm{Mod}_A^{\geq 0}$, and $M'' \in \mathrm{Mod}_A^{\leq -1}$. The implication $(a) \Rightarrow (b)$ shows that $k \otimes_A M'$ is connective; it follows that $k \otimes_A M''$ is connective. Using Corollary 4.1.12, we deduce that $k \otimes_A M'' \in \mathrm{Mod}_k^{\leq -1}$, so that $k \otimes_A M'' \simeq 0$. Using Corollary 4.1.12, we deduce by descending induction on i that $\pi_i M'' \simeq \pi_i(k \otimes_A M'') \simeq 0$, so that $M'' \simeq 0$ and $M \in \mathrm{Mod}_A^{\geq 0}$ as desired.

Assertion (3) is obvious. Assertion (4) follows from (3), Proposition A.1.2.1.19, and the observation that $\bigcap_n \mathrm{Mod}_A^{\leq n}$ consists only of zero objects of Mod_A. We now prove (5). It is clear that $A \in \mathrm{Mod}_A^{\geq 0}$. To show that $\mathrm{Mod}_A^{\geq 0}$ is closed under tensor products, let $\mathcal{C}$ denote the full subcategory of Mod_A spanned by those objects M for which the functor $N \mapsto M \otimes_A N$ carries $\mathrm{Mod}_A^{\geq 0}$ into itself. It follows immediately that $\mathcal{C}$ is stable under colimits and extensions in Mod_A. Since $A \in \mathcal{C}$, we conclude that $\mathrm{Mod}_A^{\geq 0} \subseteq \mathcal{C}$ as desired.

We now prove (6). It is clear that $A \in \mathrm{Mod}_A^{\leq 0}$. Suppose that $M, N \in \mathrm{Mod}_A^{\leq 0}$; we wish to prove that $M \otimes_A N \in \mathrm{Mod}_A^{\leq 0}$. Let

$$0 = M(0) \to M(1) \to M(2) \to \cdots$$

be as in the proof of Corollary 4.1.12, so that $M \simeq \varinjlim M(i)$. By virtue of (3), it will suffice to show that each tensor product $M(i) \otimes_A N$ belongs to $\mathrm{Mod}_A^{\leq 0}$. We prove this by induction on i, using the exact triangle

$$M(i) \otimes_A N \to M(i+1) \otimes_A N \to (K_0(i) \otimes_k A)[1] \otimes_A N \to (M(i) \otimes_A N)[1]$$

Since $\mathrm{Mod}_A^{\leq 0}$ is stable under extensions, we are reduced to proving that $(K_0(i) \otimes_k A)[1] \otimes_A N \simeq K_0(i)[1] \otimes_k N \in \mathrm{Mod}_A^{\leq 0}$. Since the homotopy groups $\pi_n K_0(i)$ vanish for $n \geq 0$, we deduce that $K_0(i)[1]$ is equivalent to a coproduct of k-modules of the form $k[m]$ where $m \leq 0$. It follows that $K_0(i)[1] \otimes_k N$ is equivalent to a coproduct of A-modules of the form $N[m]$, where $m \geq 0$, and therefore belongs to $\mathrm{Mod}_A^{\leq 0}$ as desired. $\square$

Remark 4.5.5. Let A be a coconnective $\mathbb{E}_\infty$-algebra over a field k of characteristic zero. The t-structure of Proposition 4.5.4 is not left complete in general. For example, let $A = \mathrm{Sym}^*(k[-2])$ denote the free k-algebra $k[x]$ on one generator in cohomological degree 2. Then the A-module $k[x, x^{-1}]$ is a nonzero object of the intersection $\mathrm{Mod}_A^{\geq\infty} = \bigcap_n \mathrm{Mod}_A^{\geq n}$.

Remark 4.5.6. Let X be a coaffine stack over a field k of characteristic zero, and write $X = \mathrm{cSpec}\, A$ where A is a coconnective $\mathbb{E}_\infty$-algebra over k. Proposition 4.5.2 implies that the ∞-category $\mathrm{QCoh}(X)$ can be identified with the left completion of the ∞-category Mod_A (with respect to the t-structure of Proposition 4.5.4).

Proof of Proposition 4.5.2. Write $X = \mathrm{cSpec}\, A$ for a coconnective $\mathbb{E}_\infty$-algebra A over k. According to Proposition 4.4.4 there exists a cosimplicial k-algebra $A^\bullet$ such that each A^n is discrete, $A \simeq \varprojlim A^\bullet$, and X is a colimit (in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}_k^{\geq 0}, \mathcal{S})$) of a simplicial object $X_\bullet$ given by the formula $X_\bullet(R) = \mathrm{Map}_{\mathrm{CAlg}_k}(A^\bullet, R)$. Moreover, we may assume that the map $A^m \to A^n$ is flat when $[m] \to [n]$ is injective. It follows that $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet) \simeq \varprojlim \mathrm{Mod}_{A^\bullet}$ is presentable. Form a pullback diagram

$$\begin{tikzcd} \mathcal{M} \arrow[r] \arrow[d, "q"] & \mathrm{Mod} \arrow[d] \\ \mathrm{N}(\mathbf{\Delta}) \arrow[r, "A^\bullet"] & \mathrm{CAlg}. \end{tikzcd}$$

Then $q : \mathcal{M} \to \mathrm{N}(\mathbf{\Delta})$ is a presentable fibration (Definition T.5.5.3.2), and Proposition T.3.3.3.1 allows us to identify $\mathrm{QCoh}(X)$ with the full subcategory of $\mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})}(\mathrm{N}(\mathbf{\Delta}), \mathcal{M})$ spanned by the coCartesian sections of q. Assertion (1) now follows from Proposition T.5.5.3.17.

Let $\mathcal{M}_{\geq 0}$ denote the full subcategory of $\mathcal{M}$ spanned by those pairs $([n], M)$ where M is a connective A^n-module. Let $\mathcal{C} = \mathrm{Fun}_{\mathrm{N}(\mathbf{\Delta})}(\mathrm{N}(\mathbf{\Delta}), \mathcal{M})$ be the ∞-category of all sections of q, and let $\mathcal{C}_{\geq 0}$ be the full subcategory spanned by those sections which factor through $\mathcal{M}_{\geq 0} \subseteq \mathcal{M}$. For every map $[m] \to [n]$ in $\mathbf{\Delta}$, the relative tensor product functor $M \mapsto A^n \otimes_{A^m} M$ carries connective A^m-modules to connective A^n-modules. It follows that $\mathcal{C}_{\geq 0}$ is a colocalization of $\mathcal{C}$. Moreover, a map $M^\bullet \to N^\bullet$ in $\mathcal{C}$ exhibits $M^\bullet$ as a $\mathcal{C}_{\geq 0}$-localization of $N^\bullet$ if and only if, for every integer n, the induced map of A^n-modules $M^n \to N^n$ exhibits M^n as a connective cover of N^n. Let $\mathrm{QCoh}(X)'_{\geq 0} \subseteq \mathrm{QCoh}(X)$ denote the inverse image of $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$. Then $\mathrm{QCoh}(X)'_{\geq 0}$ can be identified with the ∞-category of coCartesian sections of $q|\,\mathcal{M}_{\geq 0}$, and is therefore presentable (Proposition T.5.5.3.17). We note that $\mathrm{QCoh}(X)'_{\geq 0}$ is closed under colimits and extensions in $\mathrm{QCoh}(X)$, and therefore determines an accessible t-structure $(\mathrm{QCoh}(X)'_{\geq 0}, \mathrm{QCoh}(X)'_{\leq 0})$ on $\mathrm{QCoh}(X)$ (Proposition A.1.4.5.13). Since every object $[n] \in \mathbf{\Delta}$ admits a map $[0] \to [n]$, a coCartesian section $M^\bullet$ of q belongs to $\mathcal{C}_{\geq 0}$ if and only M^0 is a connective module over $A^0 \simeq k$. It follows that $\mathrm{QCoh}(X)'_{\geq 0} = \mathrm{QCoh}(X)_{\geq 0}$ is the full subcategory of $\mathrm{QCoh}(X)$ spanned by those objects M for which $M(\eta) \in \mathrm{Mod}_k$ is connective.

Fix an arbitrary object $M \in \mathrm{QCoh}(X)$, and identify M with a section $M^\bullet : \mathrm{N}(\mathbf{\Delta}) \to \mathcal{M}$ of q. Choose a map $\alpha : M'^\bullet \to M^\bullet$ in $\mathcal{C}$ which exhibits $M'^\bullet$ as a $\mathcal{C}_{\geq 0}$-colocalization of $M^\bullet$. We claim that $M'^\bullet$ carries every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}$ to a q-coCartesian morphism in $\mathcal{M}$. Choose a map $\beta : [0] \to [m]$; by Proposition T.2.4.1.7, it will suffice to show that $M'^\bullet$ carries β and $\alpha \circ \beta$ to q-coCartesian morphisms in $\mathcal{M}$. We may therefore reduce to the case where $m = 0$, so that $A^m \simeq k$. Unwinding the definitions, we must show that the canonical map

$$A^n \otimes_k M'^0 \simeq A^n \otimes_k \tau_{\geq 0} M^0 \to \tau_{\geq 0}(A^n \otimes_k M^0) \simeq \tau_{\geq 0} M^n \simeq M'^n$$

is an equivalence of A^n-modules. In other words, we wish to show that the map $A^n \otimes_k (\tau_{\geq 0} M^0) \to A^n \otimes_k M^0$ exhibits $A^n \otimes_k (\tau_{\geq 0} M^0)$ as a connective cover of $A^n \otimes_k M^0$; this follows immediately from our assumption that A^n is discrete. It follows that we can regard $M'^\bullet$ as determining an object $M' \in \mathrm{QCoh}(X)$, which is evidently a $\mathrm{QCoh}(X)_{\geq 0}$-colocalization of M. It follows that $M \in \mathrm{QCoh}(X)'_{\leq -1}$ if and only if $M' \simeq 0$: that is, if and only if each M^n belongs to $(\mathrm{Mod}_{A^n})_{\leq -1}$. Since $M^n \simeq A^n \otimes_k M^0$, we deduce (again using the discreteness of A^n) that $M \in \mathrm{QCoh}(X)'_{\leq -1}$ if and only if $M^0 \simeq M(\eta)$ belongs to $(\mathrm{Mod}_k)_{\leq -1}$. Replacing M by $M[1]$, we deduce that $\mathrm{QCoh}(X)'_{\leq 0} = \mathrm{QCoh}(X)_{\leq 0}$, thereby completing the proof of (2).

Assertion (3) is obvious. Note also that if $M \in \bigcap_n \mathrm{QCoh}(X)_{\leq -n}$, then $M(\eta) \in \bigcap_n (\mathrm{Mod}_k)_{\leq -n}$ so that $M(\eta) \simeq 0$ and therefore $M \simeq 0$. It follows from (3) and Proposition A.1.2.1.19 that $\mathrm{QCoh}(X)$ is right complete. To prove that $\mathrm{QCoh}(X)$ is left-complete, we let $\mathcal{M}^s = \mathcal{M} \times_{\mathrm{N}(\mathbf{\Delta})} \mathrm{N}(\mathbf{\Delta}_s)$, and let $q^s : \mathcal{M}^s \to \mathrm{N}(\mathbf{\Delta}_s)$ denote the projection map. For each integer n, let $\mathcal{M}^s_{\leq n}$ be the full subcategory of $\mathcal{M}^s$ spanned by those pairs $([m], M)$ where $M \in (\mathrm{Mod}_{A^m})_{\leq n}$. Since every injective map $[m] \to [n]$ induces a flat map $A^m \to A^n$, the map q^s restricts to a coCartesian fibration $q^s_{\leq n} : \mathcal{M}^s_{\leq n} \to \mathrm{N}(\mathbf{\Delta}_s)$. Let $\chi : \mathrm{N}(\mathbf{\Delta}_s) \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be a functor classifying the coCartesian fibration q^s, and for each integer n let $\chi_{\leq n} : \mathrm{N}(\mathbf{\Delta}_s \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ classify the coCartesian fibration $q^s_{\leq n}$. Since the inclusion $\mathrm{N}(\mathbf{\Delta}_s) \hookrightarrow \mathrm{N}(\mathbf{\Delta})$ is right cofinal (Lemma T.6.5.3.7), we have canonical equivalences $\mathrm{QCoh}(X) \simeq \varprojlim \chi$ and $\mathrm{QCoh}(X)_{\leq n} \simeq \varprojlim \chi_{\leq n}$. Consequently, $\mathrm{QCoh}(X)$ is left complete if and only if the canonical map $\varprojlim \chi \to \varprojlim_n \varprojlim \chi_{\leq n}$ is an equivalence. This follows from the observation that for each integer m, the canonical map

$$\mathrm{Mod}_{A^m} \simeq \chi([m]) \to \varprojlim_n \chi_{\leq n}([m]) \simeq \varprojlim_n (\mathrm{Mod}_{A^m})_{\leq n}$$

is an equivalence (Proposition A.7.1.1.13).

We now prove (5). Suppose first that the homotopy groups $\pi_i X$ vanish for $i > 1$. Let $\mathrm{QCoh} : \mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})^{op} \to \widehat{\mathcal{C}\mathrm{at}}_\infty$ be a left Kan extension of the functor $R \mapsto \mathrm{Mod}_R$ along the Yoneda embedding $\mathrm{CAlg}^0_k \hookrightarrow \mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})^{op}$. Let $X' = X|\,\mathrm{CAlg}^0_k$; it follows from Proposition 4.4.6 that we have a canonical equivalence $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X')$. Let $Y_\bullet$ be the simplicial object of $\mathrm{Fun}(\mathrm{CAlg}^0_k, \mathcal{S})$ given by the Čech nerve of the map $* \to X|\,\mathrm{CAlg}^0_k$ determined by η. Then each $Y_n : \mathrm{CAlg}^0_k \to \mathcal{S}$ can be identified with the $\mathcal{S}$et-valued functor on CAlg^0_k given by the formula $Y_n(R) = U(R)^n$, where $U = \pi_1 X$. Note that the heart of $\mathrm{QCoh}(X) \simeq \mathrm{QCoh}(X') \simeq \varprojlim \mathrm{QCoh}(Y_\bullet)$ can be identified with the abelian category of discrete quasi-coherent sheaves on the simplicial k-scheme $Y_\bullet$: that is, with the abelian category of algebraic representations of U.

We now treat the general case. Let $X'' : \mathrm{CAlg}^0_k \to \mathcal{S}$ be the composition of X' with the truncation functor $\tau_{\leq 1} : \mathcal{S} \to \mathcal{S}$. It follows from Proposition 4.4.8 that X'' is the restriction of a coaffine stack, and the above argument shows that heart $\mathrm{QCoh}(X'')^\heartsuit$ can be identified with the abelian category of representations of the prounipotent group scheme $U = \pi_1 X'' \simeq \pi_1 X'$. To complete the proof of (5), it will suffice to show that the truncation map $X' \to X''$ induces an equivalence of ∞-categories $\mathrm{QCoh}(X'') \to \mathrm{QCoh}(X')$. To this end, we let $\mathcal{X}' \to \mathrm{CAlg}^0_k$ be a left fibration classified by X', and $\mathcal{X}'' \to \mathrm{CAlg}^0_k$ be a left fibration classified by X''. Let $p : \mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^0_k \to \mathrm{CAlg}^0_k$ denote the projection map. We can identify $\mathrm{QCoh}(X')$ with the full subcategory of $\mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}', \mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^0_k)$ spanned by those functors which carry every morphism in $\mathcal{X}'$ to a p-coCartesian morphism in $\mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^0_k$, and similarly we can identify $\mathrm{QCoh}(X'')$ with a full subcategory of $\mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}'', \mathrm{Mod} \times_{\mathrm{CAlg}} \mathrm{CAlg}^0_k)$. Let $\mathcal{N}$ denote the full subcategory of $\mathrm{Mod} \times_{\mathrm{CAlg}} (\mathrm{CAlg}^0_k$ spanned by those pairs (R, M), where R is a discrete k-algebra and M is a discrete R-module, and let $p' = p|\,\mathcal{N}$. Unwinding the definitions, we see that the heart of $\mathrm{QCoh}(X')$ can be identified with the full subcategory of $\mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}', \mathcal{N})$ spanned by those functors which carry every morphism in $\mathcal{X}'$ to a p'-coCartesian morphism in $\mathcal{N}$, and similarly we can identify $\mathrm{QCoh}(X'')$ with a full subcategory of $\mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}'', \mathcal{N})$. Since every morphism of $\mathcal{X}''$ can be lifted to $\mathcal{X}'$, we have a pullback diagram of ∞-categories

$$\begin{tikzcd}
\mathrm{QCoh}(X'') \arrow[r] \arrow[d] & \mathrm{QCoh}(X') \arrow[d] \\
\mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}'', \mathcal{N}) \arrow[r, "\theta"] & \mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}', \mathcal{N}).
\end{tikzcd}$$

It will therefore suffice to show that θ is an equivalence. Consider the commutative diagram

$$\begin{tikzcd} \mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}'', \mathcal{N}) \arrow[r, "\theta"] \arrow[d] & \mathrm{Fun}_{\mathrm{CAlg}^0_k}(\mathcal{X}', \mathcal{N}) \arrow[d] \\ \mathrm{N}\,\mathrm{Fun}_{\mathrm{hCAlg}^0_k}(\mathrm{h}\mathcal{X}'', \mathrm{h}\mathcal{N}) \arrow[r, "\theta'"] & \mathrm{N}\,\mathrm{Fun}_{\mathrm{hCAlg}^0_k}(\mathrm{h}\mathcal{X}', \mathrm{h}\mathcal{N}). \end{tikzcd}$$

The vertical maps are categorical equivalences, because CAlg^0_k and $\mathcal{N}$ are equivalent to the nerves of their homotopy categories. Consequently, to show that θ is a categorical equivalence it suffices to show that θ' is a categorical equivalence. This follows from the observation that the map of homotopy categories $\mathrm{h}\mathcal{X}' \to \mathrm{h}\mathcal{X}''$ is an equivalence.

We now prove (6). To show that F is right t-exact, we must show that $(\mathrm{Mod}_A)_{\geq 0} \subseteq F^{-1}\,\mathrm{QCoh}(X)_{\geq 0}$. Since $F^{-1}\,\mathrm{QCoh}(X)_{\geq 0}$ is closed under colimits and extensions in Mod_A, it will suffice to prove that $A \in F^{-1}\,\mathrm{QCoh}(X)_{\geq 0}$: that is, that $F(A)(\eta) \in \mathrm{Mod}_k$ is connective. This is clear, since $F(A)(\eta) \simeq k \otimes_A A \simeq k$. The left t-exactness of the functor F is a special case of Corllary 4.1.12.

It remains to prove (7). The functor F preserves small colimits, and therefore admits a right adjoint G (Corollary T.5.5.2.9). The functor G can be described concretely as follows: if M is an object of $\mathrm{QCoh}(X)$, viewed as a section $M^\bullet : \mathrm{N}(\boldsymbol{\Delta}) \to \mathcal{M}$, then $G(M) \simeq \varinjlim M^\bullet$ (interpreted as a module over $\varinjlim A^\bullet \simeq A$). In particular, G can be obtained as the limit of a cosimplicial functor $G^\bullet$, where $G^n(M)$ is the image of M^n under the forgetful functor $\mathrm{Mod}_{A^n} \to \mathrm{Mod}_A$.

Since F is right t-exact, the functor G is left t-exact; we therefore obtain a pair of adjoint functors

$$(\mathrm{Mod}_A)_{\leq 0} \underset{g}{\overset{f}{\rightleftarrows}} \mathrm{QCoh}(X)_{\leq 0}$$

given by restricting F and G. Corollary 4.1.13 guarantees that the functor f is conservative. Consequently, to prove that f is an equivalence, it will suffice to show that the counit map $v : f \circ g \to \mathrm{id}_{\mathrm{QCoh}(X)_{\leq 0}}$ is an equivalence.

We first prove the following:

$(*)$ The functor $g : \mathrm{QCoh}(X)_{\leq 0} \to (\mathrm{Mod}_A)_{\leq 0}$ preserves filtered colimits.

Since the t-structure on Mod_A is right-complete and compatible with filtered colimits (Proposition 4.5.4), it will suffice to show that for each integer $m \geq 0$, the composite functor

$$g_m : \mathrm{QCoh}(X)_{\leq 0} \xrightarrow{g} (\mathrm{Mod}_A)_{\leq 0} \xrightarrow{\tau_{\geq -n}} (\mathrm{Mod}_A)_{\leq 0} \times_{\mathrm{Mod}_A} (\mathrm{Mod}_A)_{\geq -m}$$

preserves filtered colimits. We have $g_m \simeq \varprojlim g^\bullet_m$, where $g^\bullet_m$ is the cosimplicial functor given by $\tau_{\geq -m} \circ G^\bullet$. Because $(\mathrm{Mod}_A)_{\leq 0} \times_{\mathrm{Mod}_A} (\mathrm{Mod}_A)_{\geq -m}$ is equivalent to an $(m+1)$-category, we can identify g_m with the partial totalization $\mathrm{Tot}^{m+1}(g^\bullet_m) \simeq \tau_{\geq -m} \circ \mathrm{Tot}^{m+1} G^\bullet$ (here Tot^{m+1} denotes the limit of the diagram $g^\bullet_m | \mathrm{N}(\boldsymbol{\Delta}_{n+1,s})$; see Proposition 4.3.5). It will therefore suffice to show that the functor $\mathrm{Tot}^{m+1} G^\bullet$ preserves filtered colimits. This is clear, since the collection of functors $\mathrm{QCoh}(X)_{\leq 0} \to (\mathrm{Mod}_A)_{\leq 0}$ which preserve filtered colimits is stable under finite limits, and each G^n preserves filtered colimits.

Now suppose that $M \in \mathrm{QCoh}(X)_{\leq 0}$; we wish to prove that the unit map $v_M : (F \circ G)(M) \to M$ is an equivalence. Since $\mathrm{QCoh}(X)_{\leq 0}$ is right complete, we can write $M \simeq \varinjlim_m \tau_{\geq -m} M$. Using $(*)$, we deduce that $(F \circ G)(M) \simeq \varinjlim (F \circ G)(\tau_{\geq -m} M)$. We may therefore replace M by $\tau_{\leq -m}$ and thereby reduce to the case where $M \in \mathrm{QCoh}(X)_{\geq -m}$. The proof proceeds by induction on m, the case $m < 0$ being trivial. We have a cofiber sequence

$$\tau_{\geq 1-m} M \to M \to (\pi_{-m} M)[-m].$$

By the inductive hypothesis, we may assume that $v_{\tau_{\geq 1-m} M}$ is an equivalence; we are thereby reduced to proving that $v_{(\pi_{-m} M)[-m]}$ is an equivalence. Replacing M by $\pi_{-m} M$, we are reduced to proving that v_M

is an equivalence when M belongs to the heart $\mathrm{QCoh}(X)^{\heartsuit}$. According to (5), we may think of M as an algebraic representation of the prounipotent group scheme $U = \pi_1 X$. Then M is a filtered colimit of finite-dimensional representations of U; using $(*)$ we may reduce to the case where M is finite dimensional. We now proceed by induction on the dimension of M. If $M \simeq 0$ then there is nothing to prove. Let us assume therefore that M is a representation of positive dimension. Since U is prounipotent, M contains a nonzero U-invariant vector, given by a map of representations $k \to M$ (here we identify k with the trivial repesentation of U). We have a short exact sequence

$$0 \to k \to M \to M/k \to 0$$

in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$. By the inductive hypothesis, $v_{M/k}$ is an equivalence. Consequently, to prove that v_M is an equivalence it will suffice to show that v_k is an equivalence. We now observe that $G(k) \simeq \varprojlim G^{\bullet}(k) \simeq \varprojlim A^{\bullet} \simeq A$, and the counit map $v_k : (F \circ G)(k) \simeq F(A) \to k$ is given by the identity map from the trivial representation of U to itself. $\square$

4.6 Completed Tensor Products of Stable ∞-Categories

Let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ denote the ∞-category whose objects are presentable stable ∞-categories and whose morphisms are colimit-preserving functors. We will regard $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ as a symmetric monoidal ∞-category (see §A.6.3.1): if $\mathcal{C}$ and $\mathcal{D}$ are presentable stable ∞-categories, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal among presentable ∞-categories $\mathcal{E}$ equipped with a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserves colimits separately in each variable.

Our goal in this section is to study the behavior of t-structures on stable ∞-categories with respect to the tensor product operation described above. We can summarize our main results as follows:

- (a) If $\mathcal{C}$ and $\mathcal{D}$ are presentable stable ∞-categories equipped with accessible t-structures, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ inherits an accessible t-structure.
- (b) Let $\mathcal{C}$ and $\mathcal{D}$ are presentable stable ∞-categories equipped with right complete accessible t-structures, then the induced t-structure on $\mathcal{C} \otimes \mathcal{D}$ is right complete (Proposition 4.6.11).
- (c) If $\mathcal{C}$ and $\mathcal{D}$ are presentable stable ∞-categories which are equipped with t-structures that are both right and left complete, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ need not be left complete. However, we can instead form a *completed tensor product* $\mathcal{C} \hat{\otimes} \mathcal{D}$ (given by the left completion of $\mathcal{C} \otimes \mathcal{D}$ with respect to the t-structure of (a)). The completed tensor product operation is commutative and associative up to coherent homotopy (see Proposition 4.6.15).

As an application of these results, we will show that a coaffine stack X (over a field k of characteristic zero) can be functorially recovered from the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ of quasi-coherent sheaves on X (Corollary 4.6.19).

Notation 4.6.1. If $\mathcal{C}$ and $\mathcal{D}$ are ∞-categories which admit small colimits, we let $\mathrm{Fun}^{L}(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ spanned by those functors which preserve small colimits. We let $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ denote the ∞-category whose objects are presentable ∞-categories and whose morphisms are functors which preserve small colimits, and $\mathcal{P}\mathrm{r}^{\mathrm{R}}$ the ∞-category whose objects are presentable ∞-categories and whose morphisms are accessible functors which preserve small limits. There is a canonical equivalence of ∞-categories $\mathcal{P}\mathrm{r}^{\mathrm{L}} \simeq (\mathcal{P}\mathrm{r}^{\mathrm{R}})^{op}$.

We let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ spanned by the presentable stable ∞-categories, and regard $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ as a symmetric monoidal ∞-category, as explained in §A.6.3.1. The symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$ is encoded by a coCartesian fibration ${\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$. We let $\mathrm{h}{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}}^{\otimes}$ denote the homotopy category of ${\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}}^{\otimes}$ and $\widehat{\mathcal{S}\mathrm{et}}$ the category of sets which are not necessarily small. We define a functor $\chi : \mathrm{h}{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}}^{\otimes} \to \widehat{\mathcal{S}\mathrm{et}}$ as follows:

1. We can identify objects of $\mathrm{h}{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}}^{\otimes}$ with finite sequences $(\mathcal{C}(1), \ldots, \mathcal{C}(n))$, where each $\mathcal{C}(i)$ is a presentable stable ∞-category. To such a sequence, the functor θ assigns the set of all sequences $\{\mathcal{C}(i)_{\geq 0} \subseteq \mathcal{C}(i)\}_{1 \leq i \leq n}$, where each $\mathcal{C}(i)_{\geq 0}$ determines an accessible t-structure on $\mathcal{C}(i)$.

(2) Let $f : (\mathcal{C}(1), \ldots, \mathcal{C}(m)) \to (\mathcal{D}(1), \ldots, \mathcal{D}(n))$ be a morphism in $\mathrm{h}\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}{}^{\otimes}$, corresponding to a map $\alpha : \langle m \rangle \to \langle n \rangle$ in $\mathcal{F}\mathrm{in}_*$ together with a collection of equivalence classes of functors $F_j : \prod_{\alpha(i)=j} \mathcal{C}(i) \to \mathcal{D}(j)$ which preserve colimits separately in each variable. We then define

$$\theta(f)(\mathcal{C}(1)_{\geq 0}, \ldots, \mathcal{C}(m)_{\geq 0}) = (\mathcal{D}(1)_{\geq 0}, \ldots, \mathcal{D}(n)_{\geq 0})$$

where each $\mathcal{D}(j)_{\geq 0}$ is the smallest full subcategory of $\mathcal{D}(j)$ which is closed under colimits and extensions and contains $F_j(\prod_{\alpha(i)=j} \mathcal{C}(i)_{\geq 0})$ (it follows from Proposition A.1.4.5.11 that each $\mathcal{D}(j)_{\geq 0}$ determines an accessible t-structure on $\mathcal{D}(j)$).

It is easy to see that χ defines an $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}$-monoid object of $\mathrm{N}(\widehat{\mathcal{S}\mathrm{et}}) \subseteq \widehat{\mathcal{S}}$, which classifies a coCartesian fibration of ∞-operads $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}{}^{\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}{}^{\otimes}$. The composite map $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}{}^{\otimes} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{St}}{}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ determines a symmetric monoidal structure on the underlying ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}} = \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}{}^{\otimes} \times_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)} \{\langle 1 \rangle\}$. We will refer to $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ as the *∞-category of presentable stable ∞-categories with accessible t-structure.*

Remark 4.6.2. We will identify objects of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ with pairs $(\mathcal{C}, \mathcal{C}_{\geq 0})$, where $\mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$ determines an accessible t-structure on $\mathcal{C}$.

Example 4.6.3. The unit object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ can be identified with the pair $(\mathrm{Sp}, \mathrm{Sp}_{\geq 0})$, where Sp is the ∞-category of spectra and $\mathrm{Sp}_{\geq 0}$ the full subcategory consisting of connective spectra.

If $\mathcal{C}$ is a stable ∞-category equipped with a t-structure, then we can define right and left completions of $\mathcal{C}$ as in §A.1.2.1. Our next goal is to investigate the functoriality of these constructions. We begin with a study of the right completion.

Lemma 4.6.4. *Let $\mathcal{C}$ be a stable ∞-category equipped with a t-structure. Then $\mathcal{C}$ is right complete if and only if the following conditions are satisfied:*

(1) *The intersection $\bigcap_n \mathcal{C}_{\leq -n}$ consists only of zero objects of $\mathcal{C}$.*

(2) *Given any sequence of objects*

$$X(0) \xrightarrow{f(1)} X(1) \xrightarrow{f(2)} X(2) \to \cdots$$

in $\mathcal{C}$ with the property that $\mathrm{cofib}\, f(n) \in \mathcal{C}_{\leq -n}$ *for each n, then there exists a colimit X of the sequence. Moreover, for each $n \geq 0$ the cofiber of the map $X(n) \to X$ belongs to $\mathcal{C}_{\leq -n-1}$.*

Proof. Suppose first that $\mathcal{C}$ is right complete. Condition (1) is obvious. Let $\{X(n)\}_{n \geq 0}$ be a sequence as in (2). We now prove that (2) $\Rightarrow$ (3). For each $n \geq 0$, the sequence

$$\tau_{\geq -n} X(0) \to \tau_{\geq -n} X(1) \to \cdots$$

is eventually constant, and therefore admits a colimit $\tau_{\geq -n} X(n)$ in $\mathcal{C}_{\geq -n}$. Moreover, each of these colimits is preserved by the truncation functor $\tau_{\geq -m} : \mathcal{C}_{\geq -n} \to \mathcal{C}_{\geq -m}$. Since $\mathcal{C} \simeq \varprojlim \mathcal{C}_{\geq -n}$, it follows that the sequence $\{X(n)\}$ admits a colimit in $X \in \mathcal{C}$ such that for $n \geq 0$, $\tau_{\geq -n} X$ is a colimit of the sequence $\tau_{\geq -n} X(m)$ and therefore equivalent to $\tau_{\geq -n} X(n)$. To complete the proof that the cofiber of the map $X(n) \to X$ belongs to $\mathcal{C}_{\leq -n-1}$, it suffices to show that the map $\pi_{-n-1} X(n) \to \pi_{-n-1} X$ is a monomorphism in the abelian category $\mathcal{C}^{\heartsuit}$. Since $\tau_{\geq -n-1} X(n+1) \simeq \tau_{\geq -n-1} X$, this is equivalent to the assertion that the map $\pi_{-n-1} X(n) \to \pi_{-n-1} X(n+1)$ is a monomorphism, which follows from our assumption that $\mathrm{cofib}\, f(n+1) \in \mathcal{C}_{\leq -n-1}$.

Now suppose that (1) and (2) are satisfied. Using (2), we see that the evident functor $G : \mathcal{C} \to \varprojlim \mathcal{C}_{\geq -n}$ admits a left adjoint F, which carries a compatible sequence of objects $X(n) \in \mathcal{C}_{\geq -n}$ to the colimit $\varinjlim X(n) \in \mathcal{C}$. Moreover, the unit map $\mathrm{id} \to G \circ F$ is an equivalence, so that F is fully faithful. To complete the proof, it suffices to show that the functor G is conservative. Let $\alpha : Y \to Z$ be a morphism in $\mathcal{C}$ such that $G(\alpha)$ is an equivalence, and let $X = \ker(\alpha)$. Since $\tau_{\geq -n}(\alpha)$ is an equivalence for $n \geq 0$, we conclude that $\tau_{\geq -n} X \simeq 0$ for $n \geq 0$, so that $X \in \bigcap \mathcal{C}_{\leq -n}$. Thus $X \simeq 0$, so that α is an equivalence. □

Lemma 4.6.5. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an accessible t-structure $\mathcal{C}_{\geq 0} \subseteq \mathcal{C}$. The following statements are equivalent:*

(1) *The inclusions $\mathcal{C}_{\geq -n} \subseteq \mathcal{C}$ exhibit $\mathcal{C}$ as a colimit of the diagram*

$$\mathcal{C}_{\geq 0} \subseteq \mathcal{C}_{\geq -1} \subseteq \mathcal{C}_{\geq -2} \subseteq \cdots$$

in the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories.

(2) *The t-structure on $\mathcal{C}$ is right-complete.*

Proof. Using Corollary T.5.5.3.4, we can identify the colimit of the sequence $\{\mathcal{C}_{\geq -n}\}_{n \geq 0}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ with the limit of the tower

$$\cdots \to \mathcal{C}_{\geq -2} \stackrel{\tau_{\geq -1}}{\to} \mathcal{C}_{\geq -1} \stackrel{\tau_{\geq 0}}{\to} \mathcal{C}_{\geq 0}$$

in $\mathcal{P}\mathrm{r}^{\mathrm{R}}$. The equivalence of (1) and (2) now follows from Theorem T.5.5.3.18. □

Lemma 4.6.6. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an accessible right complete t-structure $\mathcal{C}_{\geq 0}$, and let $\mathcal{D}$ be a stable ∞-category which admits small colimits. Then the restriction functor*

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D})$$

is an equivalence of ∞-categories.

Proof. Since $\mathcal{C}$ is right complete, Lemma 4.6.5 implies that $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}) \simeq \varprojlim \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq -n}, \mathcal{D})$. It will therefore suffice to show that each of the restriction functors

$$\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq -n-1}, \mathcal{D}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq -n}, \mathcal{D})$$

is an equivalence of ∞-categories. This functor has a homotopy inverse, given by $F \mapsto \Omega_{\mathcal{D}} \circ F \circ \Sigma_{\mathcal{C}}$. □

Corollary 4.6.7. *Let $(\mathcal{C}, \mathcal{C}_{\geq 0})$ and $(\mathcal{D}, \mathcal{D}_{\geq 0})$ be objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$. If the t-structure on $\mathcal{C}$ is right complete, then the ∞-category of right t-exact functors from $\mathcal{C}$ to $\mathcal{D}$ is equivalent to the ∞-category of colimit-preserving functors from $\mathcal{C}_{\geq 0}$ to $\mathcal{D}_{\geq 0}$.*

Lemma 4.6.8. *Let $\mathcal{C}$ and $\mathcal{D}$ be presentable stable ∞-categories. Assume that $\mathcal{C}$ is equipped with a right-complete accessible t-structure $\mathcal{C}_{\geq 0}$. Then there is a canonical equivalence of ∞-categories*

$$\theta : \mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})^{op}.$$

Suppose furthermore that $\mathcal{D}$ is equipped with an accessible t-structure, so that $\mathcal{C} \otimes \mathcal{D}$ inherits an accessible t-structure. Then $(\mathcal{C} \otimes \mathcal{D})_{\leq 0} = \theta^{-1} \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op}_{\leq 0})^{op}$.

Proof. The functor θ is obtained by composing the equivalence $\mathcal{C} \otimes \mathcal{D} \simeq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}^{op})^{op}$ of Remark A.6.3.1.23 with the equivalence $\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}, \mathcal{D}^{op}) \to \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})$ of Lemma 4.6.6. Let $\boxtimes : \mathcal{C} \times \mathcal{D} \to \mathcal{C} \otimes \mathcal{D}$ be the tautological functor; unwinding the definitions, we see that for $C \in \mathcal{C}_{\geq 0}$, $D \in \mathcal{D}$, and $F \in \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})$, we have a canonical homotopy equivalence

$$\mathrm{Map}_{\mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})^{op}}(\theta(C \boxtimes D), F) \simeq \mathrm{Map}_{\mathcal{D}}(D, F(C)).$$

Now suppose that $\mathcal{D}$ is equipped with an accessible t-structure, and let $\mathcal{X} \subseteq \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})^{op}$ be the essential image of $\theta | (\mathcal{C} \otimes \mathcal{D})_{\leq 0}$. Since $(\mathcal{C} \otimes \mathcal{D})_{\geq 0}$ is generated (under colimits and extensions) by the image of $\mathcal{C}_{\geq 0} \times \mathcal{D}_{\geq 0}$, we see that an $F \in \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})^{op}$ belongs to $\mathcal{X}$ if and only if, for every pair of objects $C \in \mathcal{C}_{\geq 0}$, $D \in \mathcal{D}_{\geq 0}$, the mapping space

$$\mathrm{Map}_{\mathrm{Fun}(\mathcal{C}_{\geq 0}, \mathcal{D}^{op})^{op}}(\theta(C \boxtimes D), F) \simeq \mathrm{Map}_{\mathcal{D}}(D, F(C))$$

is discrete. Since $D \in \mathcal{D}_{\geq 0}$ is arbitrary, this is equivalent to the requirement that F takes values in the subcategory $(\mathcal{D}_{\leq 0})^{op} \subseteq \mathcal{D}^{op}$. □

Definition 4.6.9. We let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ spanned by those pairs $(\mathcal{C}, \mathcal{C}_{\geq 0})$, where $\mathcal{C}$ is a presentable stable ∞-category and $\mathcal{C}_{\geq 0}$ determines a right complete accessible t-structure on $\mathcal{C}$.

Remark 4.6.10. Let S be the collection of morphisms $f : (\mathcal{C}, \mathcal{C}_{\geq 0}) \to (\mathcal{D}, \mathcal{D}_{\geq 0})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ which induce an equivalence of ∞-categories $\mathcal{C}_{\geq 0} \to \mathcal{D}_{\geq 0}$. If $f \in S$ and $(\mathcal{B}, \mathcal{B}_{\geq 0}) \in \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$, then Corollary 4.6.7 implies that composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}((\mathcal{B}, \mathcal{B}_{\geq 0}), (\mathcal{C}, \mathcal{C}_{\geq 0})) \to \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}((\mathcal{B}, \mathcal{B}_{\geq 0}), (\mathcal{D}, \mathcal{D}_{\geq 0})).$$

We can summarize this observation informally by saying that $(\mathcal{B}, \mathcal{B}_{\geq 0})$ is an *S-colocal* object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$.

Let $(\mathcal{C}, \mathcal{C}_{\geq 0})$ be an arbitrary object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$, and let $\widehat{\mathcal{C}}$ denote the right completion of $\mathcal{C}$. Using Corollary T.5.5.3.4 and Theorem T.5.5.3.18, we can identify $\widehat{\mathcal{C}}$ with the colimit of the sequence $\mathcal{C}_{\geq -n}$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}$; in particular, we get a canonical map $f : (\widehat{\mathcal{C}}, \widehat{\mathcal{C}}_{\geq 0}) \to (\mathcal{C}, \mathcal{C}_{\geq 0})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$. It follows from Proposition A.1.2.1.17 that f induces an equivalence $\widehat{\mathcal{C}}_{\geq 0} \simeq \mathcal{C}_{\geq 0}$, so that f is a morphism in S; since $\widehat{\mathcal{C}}$ is right complete, we conclude that f exhibits $(\widehat{\mathcal{C}}, \widehat{\mathcal{C}}_{\geq 0})$ as an S-colocalization of $(\mathcal{C}, \mathcal{C}_{\geq 0})$.

We can summarize the above discussion as follows: the full subcategory $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ is a colocalization of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$. In other words, the inclusion $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ admits a right adjoint, which is given by the right completion construction described in §A.1.2.1.

Proposition 4.6.11. *The ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ contains the unit object and is stable under tensor products in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$, and therefore inherits the structure of a symmetric monoidal ∞-category (see Proposition A.2.2.1.1).*

Proof. The unit object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ can be identified with $(\mathrm{Sp}, \mathrm{Sp}_{\geq 0})$ (Example 4.6.3), and is right complete by Proposition A.1.4.3.5. Now suppose that $(\mathcal{C}, \mathcal{C}_{\geq 0})$ and $(\mathcal{D}, \mathcal{D}_{\geq 0})$ are objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$. According to Lemma 4.6.8, we can identify the tensor product $\mathcal{C} \otimes \mathcal{D}$ with $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D})$. Under this identification, each $(\mathcal{C} \otimes \mathcal{D})_{\leq -n}$ corresponds to the full subcategory $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D}_{\leq -n})$. In particular,

$$\bigcap_n (\mathcal{C} \otimes \mathcal{D})_{\leq -n} \simeq \bigcap_n \mathrm{Fun}^{\mathrm{L}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D}_{\leq -n})$$

consists only of zero objects since $\bigcap_n \mathcal{D}_{\leq -n}$ consists only of zero objects. To complete the proof, we will show that the second condition of Lemma 4.6.4 is satisfied. Suppose we are given a sequence of functors

$$F(0) \xrightarrow{\alpha(1)} F(1) \xrightarrow{\alpha(2)} \cdots$$

from $\mathcal{C}^{op}_{\geq 0}$ to $\mathcal{D}$, where $\mathrm{cofib}\,\alpha(n)$ carries $\mathcal{C}^{op}_{\geq 0}$ into $\mathcal{D}_{\leq -n}$ for $n \geq 0$. Since $\mathcal{D}$ is right complete, Lemma 4.6.4 implies that the sequence of functors $\{F(n)\}_{n \geq 0}$ has a colimit F in $\mathrm{Fun}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D})$. We claim that F preserves small limits. Since $\mathcal{D}$ is right complete, this is equivalent to the requirement that $\tau_{\geq -n} \circ F : \mathcal{C}^{op}_{\geq 0} \to \mathcal{D}_{\geq -n}$ preserves small limits for each n. Since the canonical map $\tau_{\geq -n} \circ F(n) \to \tau_{\geq -n} F$ is an equivalence, this follows from the fact that $F(n)$ preserves small limits. It follows that F can be identified with the colimit of the sequence $\{F(n)\}_{n \geq 0}$ in $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D}) \simeq \mathcal{C} \otimes \mathcal{D}$. For every object $C \in \mathcal{C}$, the cofiber of the map $F(n)(C) \to F(C)$ belongs to $\mathcal{D}_{\leq -n-1}$, so that the cofiber of the map $F(n) \to F$ can be regarded as an object of $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D}_{\leq -n-1}) \simeq (\mathcal{C} \otimes \mathcal{D})_{\leq -n-1}$. □

Definition 4.6.12. We let $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ spanned by those pairs $(\mathcal{C}, \mathcal{C}_{\geq 0})$, where $\mathcal{C}_{\geq 0}$ determines a left complete t-structure on $\mathcal{C}$ (see §A.1.2.1).

Lemma 4.6.13. *The full subcategory $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm} \subseteq \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ is a localization of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$.*

Proof. Let $\mathcal{C}$ be a presentable stable ∞-category equipped with an accessible t-structure. Let $\widehat{\mathcal{C}} \simeq \varprojlim_n \mathcal{C}_{\leq n}$ be its left completion. According to Proposition A.1.2.1.17, $\widehat{\mathcal{C}}$ is again a stable ∞-category equipped with a t-structure $\widehat{\mathcal{C}}_{\geq 0} \simeq \varprojlim_n (\mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq 0})$. Using the accessibility of the t-structure on $\mathcal{C}$ and Proposition T.5.5.3.13, we deduce that $\widehat{\mathcal{C}}$ is presentable and that $\widehat{\mathcal{C}}_{\geq 0}$ determines an accessible t-structure on $\widehat{\mathcal{C}}$. There is an evident functor $f : (\mathcal{C}, \mathcal{C}_{\geq 0}) \to (\widehat{\mathcal{C}}, \widehat{\mathcal{C}}_{\geq 0})$. Note that if $\mathcal{C}$ is right complete, then

$$\begin{aligned} \widehat{\mathcal{C}} &\simeq \varprojlim_{n} \mathcal{C}_{\leq n} \\ &\simeq \varprojlim_{n,m} \mathcal{C}_{\leq n} \cap \mathcal{C}_{\geq -m} \\ &\simeq \varprojlim_{m} \widehat{\mathcal{C}}_{\geq -m}. \end{aligned}$$

so that $\widehat{\mathcal{C}}$ is also right complete. We claim that in this case, f exhibits $(\widehat{\mathcal{C}}, \widehat{\mathcal{C}}_{\geq 0})$ as a $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$-localization of $(\mathcal{C}, \mathcal{C}_{\geq 0})$. To prove this, it suffices to show that for every presentable stable ∞-category $\mathcal{D}$ equipped with a left-complete accessible t-structure $\mathcal{D}_{\geq 0} \subseteq \mathcal{D}$, composition with f induces a homotopy equivalence

$$\mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}((\widehat{\mathcal{C}}, \widehat{\mathcal{C}}_{\geq 0}), (\mathcal{D}, \mathcal{D}_{\geq 0})) \to \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}((\mathcal{C}, \mathcal{C}_{\geq 0}), (\mathcal{D}, \mathcal{D}_{\geq 0})).$$

Let us establish a bit of notation. For each $n \geq 0$, let $\mathrm{Fun}^L_0(\mathcal{C}, \mathcal{D}_{\leq n})$ be the full subcategory of $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}_{\leq n})$ spanned by those functors which carry $\mathcal{C}_{\geq 0}$ into $\mathcal{D}_{\leq n} \cap \mathcal{D}_{\geq 0}$, and let $\mathrm{Fun}^L_0(\widehat{\mathcal{C}}, \mathcal{D}_{\leq n})$ be defined similarly. Since $\mathcal{D}$ is left complete, we can identify $\mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}((\mathcal{C}, \mathcal{C}_{\geq 0}), (\mathcal{D}, \mathcal{D}_{\geq 0}))$ with the underlying ∞-groupoid of $\varprojlim_n \mathrm{Fun}^L_0(\mathcal{C}, \mathcal{D}_{\leq n})$. Similarly, $\mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}((\widehat{\mathcal{C}}, \widehat{\mathcal{C}}_{\geq 0}), (\mathcal{D}, \mathcal{D}_{\geq 0}))$ can be identified with the ∞-category $\varprojlim_n \mathrm{Fun}^L_0(\widehat{\mathcal{C}}, \mathcal{D}_{\leq n})$. We are therefore reduced to showing that, for each $n \geq 0$, the lower horizontal map in the diagram

$$\begin{tikzcd} \mathrm{Fun}^L(\widehat{\mathcal{C}}_{\leq n}, \mathcal{D}_{\leq n}) \arrow[r] \arrow[d, "\widehat{\phi}"] & \mathrm{Fun}^L(\mathcal{C}_{\leq n}, \mathcal{D}_{\leq n}) \arrow[d, "\phi"] \\ \mathrm{Fun}^L_0(\widehat{\mathcal{C}}, \mathcal{D}_{\leq n}) \arrow[r] & \mathrm{Fun}^L_0(\mathcal{C}, \mathcal{D}_{\leq n}) \end{tikzcd}$$

is an equivalence of ∞-categories. The upper horizontal map is an equivalence of ∞-categories (since $\mathcal{C}_{\leq n} \simeq \widehat{\mathcal{C}}_{\leq n}$ by Proposition A.1.2.1.17). It therefore suffices to show that ϕ and $\widehat{\phi}$ are equivalences of ∞-categories. We will prove that ϕ is an equivalence of ∞-categories; the proof for $\widehat{\phi}$ is similar. Since the truncation functor $\tau_{\leq n} : \mathcal{C} \to \mathcal{C}_{\leq n}$ is a localization, the functor ϕ induces an equivalence of $\mathrm{Fun}^L(\mathcal{C}_{\leq n}, \mathcal{D}_{\leq n})$ with the full subcategory of $\mathrm{Fun}^L(\mathcal{C}, \mathcal{D}_{\leq n})$ spanned by those functors F with the following property:

(∗) For every map $\alpha : X \to Y$ in $\mathcal{C}$ which induces an equivalence $\tau_{\leq n} X \to \tau_{\leq n} Y$, the image $F(\alpha)$ is an equivalence in $\mathcal{D}_{\leq n}$.

We wish to prove that a colimit-preserving functor $F : \mathcal{C} \to \mathcal{D}_{\leq n}$ satisfies (∗) if and only if F belongs to $\mathrm{Fun}^L_0(\mathcal{C}, \mathcal{D}_{\leq n})$. We first prove the "only if" direction. Assume that F satisfies (∗), and let $X \in \mathcal{C}_{\geq 0}$; we wish to prove that $FX \in \mathcal{D}_{\geq 0}$. This is equivalent to the requirement that the $(n+1)$-fold suspension of FX is trivial in $\mathcal{D}_{\leq n}$. Since F commutes with colimits, this is equivalent to the requirement that $F(X[n+1]) \simeq 0$. This follows from (∗), since $\tau_{\leq n} X[n+1] \simeq 0$.

We now prove the "if" direction. Assume that $F \in \mathrm{Fun}^L_0(\mathcal{C}, \mathcal{D}_{\leq n})$, and let $\alpha : X \to Y$ be a map which induces an equivalence $\tau_{\leq n} X \simeq \tau_{\leq n} Y$. We wish to prove that $F(\alpha)$ is an equivalence. By the two-out-of-three property, it will will suffice to show that the induced maps $F(X) \to F(\tau_{\leq n} X)$ and $F(Y) \to F(\tau_{\leq n} Y)$ are equivalences. We treat the first case, the second being similar. We have a cofiber sequence

$$\tau_{\geq n+1} X \to X \to \tau_{\leq n} X.$$

Since F preserves colimits, we conclude that $F(\tau_{\leq n} X)$ can be identified with the cofiber of the map $F(\tau_{\geq n+1} X) \to F(X)$ in $\mathcal{D}_{\leq n}$. It will therefore suffice to show that $F(\tau_{\geq n+1} X) \simeq 0$. Since $\tau_{\geq n+1} X \simeq (\tau_{\geq 0} X[-n-1])[n+1]$, $F(\tau_{\geq n+1} X)$ is the $(n+1)$-fold suspension of $F(\tau_{\geq 0} X[-n-1])$ in $\mathcal{D}_{\leq n}$. Since $F \in \mathrm{Fun}^L_0(\mathcal{C}, \mathcal{D}_{\leq n})$, we conclude that $F(\tau_{\geq 0} X[-n-1]) \in \mathcal{D}_{\geq 0}$ and therefore its $(n+1)$-fold suspension vanishes in $\mathcal{D}_{\leq n}$. □

Lemma 4.6.14. *Let $L : \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$ be a left adjoint to the inclusion (given by the formation of left completion). Then L is compatible with the symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$. That is, if $(\mathcal{C}, \mathcal{C}_{\geq 0})$ is an arbitrary object of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ and $f : (\mathcal{D}, \mathcal{D}_{\geq 0}) \to (\mathcal{D}', \mathcal{D}'_{\geq 0})$ is a morphism in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+}$ such that Lf is an equivalence, then L carries the induced map $(\mathcal{C}, \mathcal{C}_{\geq 0}) \otimes (\mathcal{D}, \mathcal{D}_{\geq 0}) \to (\mathcal{C}, \mathcal{C}_{\geq 0}) \otimes (\mathcal{D}', \mathcal{D}'_{\geq 0})$ to an equivalence.*

Proof. Our assumption implies that f induces an equivalence $\mathcal{D}_{\leq 0} \to \mathcal{D}'_{\leq 0}$. Using Lemma 4.6.8, we can identify the map $(\mathcal{C} \otimes \mathcal{D})_{\leq 0} \to (\mathcal{C} \otimes \mathcal{D}')_{\leq 0}$ with the induced equivalence $\mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D}_{\leq 0}) \simeq \mathrm{Fun}^{\mathrm{R}}(\mathcal{C}^{op}_{\geq 0}, \mathcal{D}'_{\leq 0})$. □

Proposition 4.6.15. *Let $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{t}\pm}$ denote the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{t}}$ spanned by those sequences*

$$((\mathcal{C}(1), \mathcal{C}(1)_{\geq 0}), \ldots, (\mathcal{C}(n), \mathcal{C}(n)_{\geq 0}))$$

where each $\mathcal{C}(n)$ is both right and left complete. Then the projection $\mathcal{P}\mathrm{r}^{\mathrm{L}\otimes}_{\mathrm{t}\pm} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ determines a symmetric monoidal structure on $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$. Moreover, the localization functor $L : \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+} \to \mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$ of Lemma 4.6.14 is a symmetric monoidal functor.

Proof. Combine Lemma 4.6.14 with Proposition A.2.2.1.9. □

Remark 4.6.16. Let $(\mathcal{C}, \mathcal{C}_{\geq 0})$ and $(\mathcal{D}, \mathcal{D}_{\geq 0})$ be objects of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$. We will denote the tensor product of $(\mathcal{C}, \mathcal{C}_{\geq 0})$ with $(\mathcal{D}, \mathcal{D}_{\geq 0})$ in $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}$ by $(\mathcal{C} \widehat{\otimes} \mathcal{D}, (\mathcal{C} \widehat{\otimes} \mathcal{D})_{\geq 0})$. Unwinding the definitions, we see that $\mathcal{C} \widehat{\otimes} \mathcal{D}$ is given by the left completion of the usual tensor product $\mathcal{C} \otimes \mathcal{D}$ with respect to its t-structure (the usual tensor product is already right complete, by Proposition 4.6.11).

Corollary 4.6.17. *The inclusion $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm}) \subseteq \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})$ admits a left adjoint. Moreover, if $f^* : \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ is a morphism in $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})$, then f^* exhibits $\mathcal{D}^{\otimes}$ as a $\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm})$-localization of $\mathcal{C}^{\otimes}$ if and only if the underlying map of ∞-categories $\mathcal{C} \to \mathcal{D}$ exhibits $\mathcal{D}$ as a left completion of $\mathcal{C}$.*

Combining Corollary 4.6.17 with Proposition 4.5.2, we obtain:

Corollary 4.6.18. *Let k be a field of characteristic zero, let A be a coconnective $\mathbb{E}_\infty$-algebra over k, and let $X = \mathrm{cSpec}\, A$ the corresponding coaffine stack. Let $F : \mathrm{Mod}_A \to \mathrm{QCoh}(X)$ be the functor described in Proposition 4.5.2, and let $\mathcal{C} \in \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}\pm})$ be any presentable ∞-category which is right and left complete with respect to an accessible t-structure. Then composition with F induces an equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})}(\mathrm{QCoh}(X), \mathcal{C}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}+})}(\mathrm{Mod}_A, \mathcal{C}).$$

Corollary 4.6.19. *Let k be a field of characteristic zero, let A be a coconnective $\mathbb{E}_\infty$-algebra over k and $X = \mathrm{cSpec}\, A$ the corresponding coaffine stack. For every connective $\mathbb{E}_\infty$-ring B, the construction*

$$(f : A \to B) \mapsto (f^* : \mathrm{QCoh}(X)^{\otimes} \to \mathrm{Mod}_B^{\otimes})$$

determines a homotopy equivalence

$$\mathrm{Map}_{\mathrm{CAlg}}(A, B) \to \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}(\mathrm{QCoh}(X), \mathrm{Mod}_B).$$

Proof. Corollary A.6.3.5.18 implies that $\mathrm{Map}_{\mathrm{CAlg}}(A, B)$ can be identified with the full subcategory of

$$\mathrm{Fun}^{\otimes}(\mathrm{Mod}_A, \mathrm{Mod}_B)$$

spanned by those symmetric monoidal functors which preserve geometric realization of simplicial objects. Any map of $\mathbb{E}_\infty$-rings $A \to B$ induces a functor $\mathrm{Mod}_A \to \mathrm{Mod}_B$ which is right t-exact and preserve small colimits (Proposition 4.5.4), so that $\mathrm{Map}_{\mathrm{CAlg}}(A, B) \simeq \mathrm{Map}_{\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}}(\mathrm{Mod}_A, \mathrm{Mod}_B)$. The desired result now follows from Corollary 4.6.18, since Mod_B is both right and left complete. □

5 Tannaka Duality for Generalized Algebraic Gerbes

Let G be an affine group scheme over a field k, and let $\mathrm{Rep}(G)_{\mathrm{fd}}$ denote the category of finite dimensional algebraic representations of G. Tannaka duality provides a means of reconstructing the group scheme G from the category $\mathrm{Rep}(G)_{\mathrm{fd}}$. Let R be a commutative algebra over k, and let $\mathrm{Mod}_R^{\heartsuit}$ denote the abelian category of (discrete) modules over R. Then $\mathrm{Mod}_R^{\heartsuit}$ is a symmetric monoidal category, so we can consider the category $\mathrm{Fun}^{\otimes}(\mathrm{Rep}(G)_{\mathrm{fd}}, \mathrm{Mod}_R^{\heartsuit})$ of symmetric monoidal functors from $\mathrm{Rep}(G)_{\mathrm{fd}}$ into $\mathrm{Mod}_R^{\heartsuit}$. This category contains a distinguished object F_0, given by the composition

$$\mathrm{Rep}(G)_{\mathrm{fd}} \xrightarrow{F'} \mathrm{Mod}_k^{\heartsuit} \xrightarrow{\otimes R} \mathrm{Mod}_R^{\heartsuit}$$

where F' is the forgetful functor (which assigns to each representation of G its underlying vector space). Tannaka duality asserts that there is canonical isomorphism

$$G(R) \simeq \mathrm{Aut}_{\mathrm{Fun}^{\otimes}(\mathrm{Rep}(G)_{\mathrm{fd}}, \mathrm{Mod}_R^{\heartsuit})}(F_0),$$

which depend functorially on R (here $G(R)$ denotes the group of R-valued points of G). In fact, something slightly more general is true: the category of exact symmetric monoidal functors $F : \mathrm{Rep}(G)_{\mathrm{fd}} \to \mathrm{Mod}_R^{\heartsuit}$ can be identified with the groupoid of R-valued points of the classifying stack BG. In other words, giving an exact symmetric monoidal functor $F : \mathrm{Rep}(G)_{\mathrm{fd}} \to \mathrm{Mod}_R^{\heartsuit}$ is equivalent to giving a map of commutative rings $k \to R$, together with a G-torsor over $\mathrm{Spec}^{\mathrm{c}} R$.

The classifying stack BG of an affine group scheme G is an example of an *algebraic gerbe* over k. More generally, we define an *algebraic gerbe* over k to be a geometric stack X equipped with a map $X \to \mathrm{Spec}^{\mathrm{f}} k$ having the following property: there exists a field extension $k \to k'$ such that the fiber product $X \times_{\mathrm{Spec}^{\mathrm{f}} k} \mathrm{Spec}^{\mathrm{f}} k'$ is equivalent to the classifying stack of an affine group scheme over k' (for a more extensive discussion of this notion, see §5.1). To every algebraic gerbe over k, we can associate an abelian category $\mathrm{QCoh}(X)^{\mathrm{fd}}$ of locally free sheaves of finite rank X. Using Tannaka duality, one can show that the construction $X \mapsto \mathrm{QCoh}(X)^{\mathrm{fd}}$ determines a fully faithful embedding from the 2-category of algebraic gerbes to the 2-category of symmetric monoidal abelian categories. A symmetric monoidal category $\mathcal{C}$ is said to be *Tannakian* if it lies in the essential image of this embedding. In characteristic zero, Deligne has given an intrinsic characterization of the class of Tannakian categories (see [7]):

Theorem 5.0.1 (Deligne)**.** *Let $\mathcal{C}$ be an abelian category with unit object $\mathbf{1}$. Then $\mathcal{C}$ is equivalent to a category $\mathrm{QCoh}(X)^{\mathrm{fd}}$ for some algebraic gerbe X over a field of characteristic zero if and only if $\mathcal{C}$ satisfies the following conditions:*

(1) *The category $\mathcal{C}$ is abelian.*

(2) *The tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in each variable.*

(3) *The ring $\mathrm{Hom}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ is a field of characteristic zero.*

(4) *Every object $V \in \mathcal{C}$ admits a dual $V^{\vee}$.*

(5) *For every object $V \in \mathcal{C}$, let $\dim(V) \in k$ denote the composition*

$$\mathbf{1} \xrightarrow{c} V \otimes V^{\vee} \xrightarrow{e} \mathbf{1},$$

where c and e are compatible coevaluation and evaluation maps for the duality between V and $V^{\vee}$. Then $\dim(V)$ is a nonnegative integer (here we identify the ring of integers $\mathbf{Z}$ with its image in the field k).

Our goal in this section is to prove an ∞-categorical analogue of Theorem 5.0.1. In §5.2 we will introduce the notion of *generalized algebraic gerbe* over a field k of characteristic zero. Roughly speaking, a generalized gerbe over k is a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ which is an extension of an algebraic gerbe over k by a coaffine stack (for a precise definition, see Definition 5.2.1). If X is a generalized algebraic gerbe, then $\mathrm{QCoh}(X)$ is a symmetric monoidal stable ∞-category which is equipped with a natural t-structure. Our main results can be stated as follows:

- (a) If X is a generalized algebraic gerbe, then X can be recovered from the ∞-category $\mathrm{QCoh}(X)$. More precisely, for any connective $\mathbb{E}_\infty$-ring A, we have a fully faithful embedding

$$X(A) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{Mod}_A),$$

whose essential image is spanned by those symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which preserve small colimits and connective objects (Proposition 5.6.1).

- (b) Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category equipped with a t-structure. Then there exists a generalized algebraic gerbe X and a symmetric monoidal t-exact equivalence $\mathcal{C} \simeq \mathrm{QCoh}(X)$ if and only if $\mathcal{C}$ is *locally dimensional* (see Definition 5.6.4): that is, if and only $\mathcal{C}$ satisfies suitable analogues of the conditions listed in Theorem 5.0.1 (Theorem 5.6.19). In this case, X is canonically determined up to equivalence (by virtue of (a)).

We will prove both of these results in §5.6. Let us give a brief indication of the proof of (b). Suppose that $\mathcal{C}$ is a locally dimensional ∞-category. Then the heart $\mathcal{C}^{\heartsuit}$ is a compactly generated abelian category, whose compact objects form an abelian category satisfying the hypotheses of Theorem 5.0.1. Deligne's result then guarantees the existence of an algebraic gerbe Y and a symmetric monoidal equivalence of abelian categories $f : \mathrm{QCoh}(Y)^{\heartsuit} \simeq \mathcal{C}^{\heartsuit}$. The main step in the proof is to show that f extends to a symmetric monoidal functor $F : \mathrm{QCoh}(Y) \to \mathcal{C}$. The construction of F will require a number of preliminary results concerning the approximation of stable ∞-categories by abelian subcategories, which we discuss in §5.3, 5.4, and 5.5. Assuming that F has been constructed and commutes with small colimits, the adjoint functor theorem guarantees the existence of a right adjoint G to F. Then G carries the unit object of $\mathcal{C}$ to a commutative algebra object $\mathcal{A}$ of $\mathrm{QCoh}(Y)$. The commutative algebra $\mathcal{A}$ in some sense measures the failure of F to be an equivalence of ∞-categories (for example, if F is an equivalence, then $\mathcal{A}$ is the unit object of $\mathrm{QCoh}(Y)$). The fact that the functor f is an equivalence of abelian categories translates to a vanishing condition on the homotopy groups of $\mathcal{A}$, which in particular guarantees that $\mathcal{A}$ is a coconnective $\mathbb{E}_\infty$-algebra in $\mathrm{QCoh}(Y)$. Taking the spectrum of $\mathcal{A}$ relative to Y, we will obtain a functor X which is, in some sense, an extension of Y by coaffine stacks. We will then complete the proof by showing that X is a generalized algebraic gerbe and that $\mathrm{QCoh}(X)$ is equivalent to $\mathcal{C}$.

5.1 Algebraic Gerbes

Let k be a field and let G be an affine group scheme over k. Then G determines a group object in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S})$. Applying the left Kan extension functor $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}_k, \mathcal{S}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, we obtain a simplicial object of $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$, which we will denote by $G_\bullet$. More concretely, each G_n is the functor corepresented by the commutative ring of functions on the nth power of the group scheme G (formed in the category of k-schemes); in particular, G_0 is the functor represented by the field k itself. Let BG denote the geometric realization $|G_\bullet|$, formed in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ of sheaves with respect to the flat topology. Then BG is a geometric stack (in the sense of Definition 3.4.1), which we will refer to as the *classifying stack* of the group scheme G. Our goal in this section is to study geometric stacks which are, in some sense, locally equivalent to the classifying stack of an affine group scheme.

Definition 5.1.1. Let k be a field and suppose we are given a natural transformation $\alpha : X \to \mathrm{Spec}^{\mathrm{f}}\, k$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. We will say that α *exhibits X as an algebraic gerbe over k* if there exists a nonzero commutative ring R (corepresenting a functor $\mathrm{Spec}^{\mathrm{f}}\, R : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$) and a natural transformation $\beta : \mathrm{Spec}^{\mathrm{f}}\, R \to X$ satisfying the following conditions:

(a) The map β is representable, affine, and faithfully flat. In particular, the fiber product $\mathrm{Spec}^{\mathrm{f}} R \times_X \mathrm{Spec}^{\mathrm{f}} R$ is corepresentable by a commutative ring A which is flat over R.

(b) The canonical map $R \otimes_k R \to A$ (determined by the natural transformation of functors $\mathrm{Spec}^{\mathrm{f}} R \times_X \mathrm{Spec}^{\mathrm{f}} R \to \mathrm{Spec}^{\mathrm{f}} R \times_{\mathrm{Spec}^{\mathrm{f}} k} \mathrm{Spec}^{\mathrm{f}} R$) is faithfully flat.

Our definition of an algebraic gerbe X requires that X be a functor defined on the ∞-category $\mathrm{CAlg}^{\mathrm{cn}}$ consisting of all connective $\mathbb{E}_\infty$-rings. Let CAlg^0 denote the full subcategory of $\mathrm{CAlg}^{\mathrm{cn}}$ spanned by the *discrete* $\mathbb{E}_\infty$-rings, so that CAlg^0 is equivalent to the nerve of the ordinary category of commutative rings. Our first goal is to show that an algebraic gerbe X is determined by its restriction to the subcategory CAlg^0.

Definition 5.1.2. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a functor. We will say that X is *discretely determined* if the following condition is satisfied:

($*$) Let $X_0 = X|\,\mathrm{CAlg}^0$, and let $X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be a left Kan extension of X_0. Then the canonical map $X' \to X$ exhibits X as a sheafification of X' with respect to the flat topology.

Proposition 5.1.3. *Let k be a field which corepresents the functor $\mathrm{Spec}^{\mathrm{f}} k : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$, and suppose we are given a natural transformation $\alpha : X \to \mathrm{Spec}^{\mathrm{f}} k$ in $\widehat{\mathrm{Shv}_{\mathrm{fpqc}}}$. For every discrete k-algebra R, let $Y(R)$ denote the summand of $X(R)$ given by the fiber of the map $X(R) \to \mathrm{Hom}(k, R)$ (where $\mathrm{Hom}(k, R)$ denotes the set of ring homomorphisms from k into R). The following conditions are equivalent:*

(1) *The map α exhibits X as an algebraic gerbe over k (Definition 5.1.1).*

(2) *There exists a nonzero discrete commutative ring R and a map $\beta : \mathrm{Spec}^{\mathrm{f}} R \to X$. Any such map is representable, affine, faithfully flat, and induces a faithfully flat map of commutative rings $R \otimes_k R \to A$, where A corepresents the fiber product $\mathrm{Spec}^{\mathrm{f}} R \times_X \mathrm{Spec}^{\mathrm{f}} R$.*

(3) *The functor X is discretely determined, the functor $Y : \mathrm{CAlg}^0_k \to \widehat{\mathcal{S}}$ is 1-truncated and 1-connective (as a sheaf with respect to the flat topology), and there exists a nonzero discrete k-algebra R and a point $\eta \in Y(R)$ such that the sheaf of groups $\pi_1(Y, \eta)$ is representable by a flat affine group scheme over R.*

(4) *The functor X is discretely determined, the functor $Y : \mathrm{CAlg}^0_k \to \widehat{\mathcal{S}}$ is 1-truncated and 1-connective (as a sheaf with respect to the flat topology), and for every point $\eta \in Y(R)$, the sheaf of groups $\pi_1(Y, \eta)$ is representable by a flat affine group scheme over R.*

Proof. The implication (2) $\Rightarrow$ (1) is obvious. Suppose next that (1) holds, and choose a map $\beta : \mathrm{Spec}^{\mathrm{f}} R \to X$ satisfying the requirements of Definition 5.1.1. Let $X_\bullet$ be the Čech nerve of f. Since f is affine and faithfully flat, each X_n is representable by an R-algebra which is flat over R (and therefore discrete). In particular, each X_n is discretely determined. Since f is an effective epimorphism with respect to the flat topology, X can be identified with the sheafification of $|X_\bullet|$ and is therefore also discretely determined. Note that the functor Y is 1-truncated (since f determines an effective epimorphism $\mathrm{Hom}(R, \bullet) \to Y$ with discrete fibers). Choose an equivalence $\mathrm{Spec}^{\mathrm{f}} R \times_X \mathrm{Spec}^{\mathrm{f}} R \simeq \mathrm{Spec}^{\mathrm{f}} A$. We note that $\pi_0 Y$ can be identified with the coequalizer (in the category of sheaves of sets on $(\mathrm{CAlg}^0_k)^{op}$ with respect to the flat topology) of the diagram

$$\mathrm{Hom}(A, \bullet) \rightrightarrows \mathrm{Hom}(R, \bullet).$$

Since A is faithfully flat over $R \otimes_k R$, this coequalizer can be identified with the restriction of the functor $\mathrm{Spec}^{\mathrm{f}} k$: in other words, Y is 1-connective (as a sheaf in the flat topology). Note that f determines a point $\eta \in Y(R)$, and that $\pi_1(Y, \eta)$ is corepresented by the tensor product $A \otimes_{R \otimes_k R} R$ (and therefore representable by a flat affine R-scheme). This proves (3).

We now prove that (3) implies (4). Assume that X satisfies (3), so that for some point $\eta \in Y(R)$ the group $\pi_1(Y, \eta)$ is corepresented by a flat R-algebra A. Choose any other point $\eta' \in Y(R')$; we wish to show that the sheaf of groups $\pi_1(Y, \eta')$ is corepresented by a flat R'-algebra. This assertion is local with respect to the flat topology on R'. Since Y is connected (as a sheaf of spaces with respect to the flat topology), we may assume that η' is the image of η under some map of k-algebras $R \to R'$, in which case $\pi_1(Y, \eta')$ is corepresented by the R'-algebra $A \otimes_R R'$.

We now prove that (4) implies (2). Since Y is 1-connective, it is nontrivial: that is, there exists a nonzero discrete k-algebra R and a point $\eta \in Y(R)$. Let $Y_0 : \mathrm{CAlg}^0_k \to \mathcal{S}$ be the functor corepresented by R, so that η determines a map $f : Y_0 \to Y$. Since Y_0 is 1-connective, the map f is an effective epimorphism (of sheaves with respect to the flat topology). Let $Y_\bullet$ be the Čech nerve of f, so that Y is equivalent to sheafification of the geometric realization $|Y_\bullet|$. The point η determines a pair of points $\eta_0, \eta_1 \in Y(R \otimes_k R)$. Since Y is 1-connective, the collection of paths from η_0 to η_1 determines a bitorsor for the flat group schemes $\pi_1(Y, \eta_0)$ and $\pi_1(Y, \eta_1)$. It follows that Y_1 is corepresented by a faithfully flat $R \otimes_k R$-algebra A, so that each Y_n is corepresented by $A \otimes_R \cdots \otimes_R A$. Let $F : \mathrm{Fun}(\mathrm{CAlg}^0_k, \widehat{\mathcal{S}}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ be the functor of left Kan extension, so that $F(Y_n)$ is the functor corepresented by $A \otimes_R \cdots \otimes_R A$. for each $n \geq 0$. It follows that $F(Y_\bullet)$ is a groupoid object of $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$, whose geometric realization is the sheafification of $F(Y)$. Since X is discretely determined, we conclude that X is equivalent to the geometric realization $|F(Y_\bullet)|$ (formed in the ∞-category $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$). In particular, the point η induces an effective epimorphism $\beta : \mathrm{Spec}^{\mathrm{f}} R \to X$ such that the fiber product $\mathrm{Spec}^{\mathrm{f}} R \times_X \mathrm{Spec}^{\mathrm{f}} R$ is equivalent to $\mathrm{Spec}^{\mathrm{f}} A$. To complete the proof, it suffices to show that β is representable, affine, and faithfully flat. Choose any other map $\gamma : \mathrm{Spec}^{\mathrm{f}} R' \to X$; we wish to prove that the fiber product $\mathrm{Spec}^{\mathrm{f}} R' \times_X \mathrm{Spec}^{\mathrm{f}} R$ has the form $\mathrm{Spec}^{\mathrm{f}} A'$, for some faithfully flat R'-algebra A'. This assertion is local with respect to the flat topology on R'. Since β is an effective epimorphism, we may assume that γ factors through β and thereby reduce to the case $\gamma = \beta$. In this case, $A' \simeq A$ is flat over R (since A is flat over $R \otimes_k R$, which is in turn flat over R). $\square$

Remark 5.1.4. Let k be a field. Then every algebraic gerbe X over k is a geometric stack (Definition 3.4.1). It follows that the space $X(A)$ is essentially small for every connective $\mathbb{E}_\infty$-ring A (Corollary 3.4.28). Consequently, we can identify X with a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$.

Remark 5.1.5. Let k be a field and let X be an algebraic gerbe over k. Choose a nonzero commutative ring R and a natural transformation $\mathrm{Spec}^{\mathrm{f}} R \to X$ (such a morphism is automatically representable, affine, and faithfully flat; see Proposition 5.1.3). Then the Čech nerve of f is given by $\mathrm{Spec}\, R^\bullet$ for some cosimplicial k-algebra $R^\bullet$. Let $\mathcal{O}_X$ denote the unit object of $\mathrm{QCoh}(X)$. Unwinding the definitions, we see that $\mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{O}_X)$ can be identified with the limit $\varprojlim R^\bullet$ of the cosimplicial ring $R^\bullet$ (in the ordinary category of commutative rings): that is, with the equalizer of the pair of maps

$$R \underset{d^1}{\overset{d^0}{\rightrightarrows}} R^1.$$

Since R^1 is faithfully flat over $R \otimes_k R$, this equalizer can be canonically identified with k. In other words, the canonical map

$$k \simeq \mathrm{Map}_{\mathrm{Mod}_k}(k, k) \xrightarrow{f^*} \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{O}_X)$$

is a homotopy equivalence.

Remark 5.1.6. We will say that a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ is an *algebraic gerbe* if there exists a field k and a natural transformation $\alpha : X \to \mathrm{Spec}^{\mathrm{f}} k$ which exhibits X as an algebraic gerbe over k. Remark 5.1.5 shows that k can be recovered from X (up to canonical isomorphism) by the formula $k \simeq \pi_0 \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{O}_X)$. In particular, the characteristic of k depends only on X. We will say that X has *characteristic zero* if the field k has characteristic zero.

Remark 5.1.7. Let k be a field and let $\alpha : X \to \mathrm{Spec}^{\mathrm{f}} k$ be an algebraic gerbe over k. We will say that X is *neutral* if the map α admits a section s (up to homotopy). In this case, s determines a map $\mathrm{Spec}^{\mathrm{f}} k \to X$ which is representable, affine, and faithfully flat. Let $R^\bullet$ be as in Remark 5.1.5. Taking the spectrum termwise, we obtain a group object in the category of affine schemes over k, which we will denote by G. Unwinding the definitions, we can recover X as the classifying stack BG of the affine group scheme G.

Remark 5.1.8. Let X be an algebraic gerbe. Then there exists a nonzero commutative ring R such that $X(R)$ is nonempty. Let κ be the residue field of R at some prime ideal $\mathfrak{p} \subseteq R$. The map of commutative rings $R \to \kappa$ induces a map of spaces $X(R) \to X(\kappa)$, so that $X(\kappa)$ is nonempty. Choosing a point of $X(\kappa)$, we obtain a map $\beta : \mathrm{Spec}^{\mathrm{f}}\,\kappa \to X$. It follows from Proposition 5.1.3 that β is representable, affine, and faithfully flat.

Remark 5.1.9. Let k be a field and let X be an algebraic gerbe over k. If k' is an extension field of k, then the fiber product $X \times_{\mathrm{Spec}^{\mathrm{f}}\,k} \mathrm{Spec}^{\mathrm{f}}\,k'$ is an algebraic gerbe over k'.

Remark 5.1.10. If X is an algebraic gerbe over a field k, then X need not be neutral. However, it can always be neutralized after passing to some extension field of k. Namely, choose a map $\mathrm{Spec}^{\mathrm{f}}\,\kappa \to X$ as in Remark 5.1.8; then the fiber product $X \times_{\mathrm{Spec}^{\mathrm{f}}\,k} \mathrm{Spec}^{\mathrm{f}}\,\kappa$ is a neutral algebraic gerbe over κ.

Remark 5.1.11. If k is a field, then a k-module spectrum M is flat over k if and only if it is discrete. It follows that if X is an algebraic gerbe, then a quasi-coherent sheaf $\mathcal{F}$ on X is flat if and only if it belongs to the heart of the t-structure on $\mathrm{QCoh}(X)$ described in Proposition 3.4.17. If these conditions are satisfied, then $\mathcal{F}$ is faithfully flat if and only if it is nonzero.

Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack. Using Proposition 3.4.11, we see that for every connective $\mathbb{E}_\infty$-ring R corepresenting a functor $\mathrm{Spec}^{\mathrm{f}}\,R$, every natural transformation $f : \mathrm{Spec}^{\mathrm{f}}\,R \to X$ is representable by affine spectral Deligne-Mumford stacks. Moreover, f is flat (faithfully flat) if and only if the functor f_* carries the unit object $R \in \mathrm{Mod}_R \simeq \mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}}\,R)$ to a flat (faithfully flat) object of $\mathrm{QCoh}(X)$. Combining this observation with Remark 5.1.11, we obtain the following:

Lemma 5.1.12. *Let X be an algebraic gerbe and let R be a connective $\mathbb{E}_\infty$-ring. Then a map $\beta : \mathrm{Spec}^{\mathrm{f}}\,R \to X$ is flat if and only if R is discrete. It is faithfully flat if and only if R is discrete and nonzero.*

Lemma 5.1.13. *Let X be an algebraic gerbe and $\mathcal{F} \in \mathrm{QCoh}(X)^{\heartsuit}$ be a discrete quasi-coherent sheaf on X. Then $\mathcal{F}$ can be written as a filtered colimit $\varinjlim \mathcal{F}_\alpha$, where each $\mathcal{F}_\alpha$ is a subobject of $\mathcal{F}$ (in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$) and is locally free of finite rank.*

Proof. Choose a faithfully flat morphism $f : \mathrm{Spec}\,\kappa \to X$, where κ is a field. Then $f^* \mathcal{F}$ can be identified with a vector space V over κ, which is a direct limit of finite dimensional subspaces V_α. Since f_* commutes with filtered colimits, we have $f_* f^* \mathcal{F} \simeq \varinjlim f_* V_\alpha$. For each index α, let $\mathcal{F}_\alpha$ denote the inverse image of $f_* V_\alpha$ under the unit map $\mathcal{F} \to f_* f^* \mathcal{F}$ (formed in the abelian category $\mathrm{QCoh}(X)^{\heartsuit}$). Since $\mathrm{QCoh}(X)^{\heartsuit}$ is a Grothendieck abelian category (see Proposition 3.4.17), we conclude that $\mathcal{F} \simeq \varinjlim \mathcal{F}_\alpha$. Moreover, for each index α, the inclusion $f^* \mathcal{F}_\alpha \to f^* \mathcal{F} \subseteq V$ factors through V_α, so that $f^* \mathcal{F}_\alpha$ is a finite-dimensional vector space over κ and therefore $\mathcal{F}_\alpha$ is locally free of finite rank. $\square$

Proposition 5.1.14. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be functors, where Y is an algebraic gerbe. Then the evident functor*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(X, Y) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(Y), \mathrm{QCoh}(X))$$

is a fully faithful embedding. Moreover, a symmetric monoidal functor $f^* : \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ belongs to the essential image of θ if and only if f^* preserves small colimits and connective objects.

Proof. By Theorem 3.4.2, it will suffice to show that if $f^*; \mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$ is a symmetric monoidal functor which preserves small colimits and connective objects, then f^* preserves flat objects. Let $\mathcal{F}$ be a flat quasi-coherent sheaf on Y. According to Lemma 5.1.13, we can write $\mathcal{F} \simeq \varinjlim \mathcal{F}_\alpha$ where each $\mathcal{F}_\alpha \in \mathrm{QCoh}(Y)$ is locally free of finite rank. Corollary 2.7.33 guarantees that $\mathcal{F}_\alpha$ is a dualizable object of $\mathrm{QCoh}(Y)_{\geq 0}$. Since f^* is a symmetric monoidal functor which preserves connective objects, we conclude that $f^* \mathcal{F}_\alpha$ is a dualizable object of $\mathrm{QCoh}(X)_{\geq 0}$. Corollary 2.7.33 implies that $f^* \mathcal{F}_\alpha$ is locally free of finite rank. Since f^* commutes with colimits, we can use Theorem A.7.2.2.15 to conclude that $f^* \mathcal{F} \simeq \varinjlim f^* \mathcal{F}_\alpha$ is a flat quasi-coherent sheaf on X. $\square$

5.2 Generalized Algebraic Gerbes

Let k be a field. In §5.1, we introduced the notion of an algebraic gerbe over k. Every algebraic gerbe X is a geometric stack: in particular, it has the property that the homotopy groups $\pi_i(X(R), \eta)$ vanish for $i \geq 2$ (and any choice of base point η) for every commutative ring R (regarded as a discrete $\mathbb{E}_\infty$-ring). In this section, we will introduce a generalization of the notion of algebraic gerbe, where we loosen the restrictions on the higher homotopy groups of X. Roughly speaking, we will obtain this generalization by mixing the the theory of algebraic gerbes with the theory of coaffine stacks developed in §4.4.

Definition 5.2.1. Let k be a field of characteristic zero and suppose we are given a morphism $\alpha : X \rightarrow \operatorname{Spec}^{\mathrm{f}} k$ in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. We will say that α *exhibits* X *as a generalized algebraic gerbe over* k if the following conditions are satisfied:

(1) The map α factors as a composition

$$X \xrightarrow{\alpha'} Y \xrightarrow{\alpha''} \operatorname{Spec}^{\mathrm{f}} k,$$

where α'' exhibits Y as an algebraic gerbe over k (Definition 5.1.1).

(2) There exists a field κ and a pullback diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}^{\mathrm{f}} \kappa \arrow[r] & Y \end{tikzcd}$$

in $\operatorname{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, and an equivalence $X_0 \simeq \operatorname{cSpec} B$ for some 2-coconnective $\mathbb{E}_\infty$-algebra A over κ.

Remark 5.2.2. We will say that a functor $X : \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \widehat{\mathcal{S}}$ is a *generalized algebraic gerbe* if there exists a field k of characteristic zero and a map $\alpha : X \rightarrow \operatorname{Spec}^{\mathrm{f}} k$ which satisfies the conditions of Definition 5.2.1. In this case, the field k and the map α are well-defined up to canonical equivalence.

Remark 5.2.3. Let X be a generalized algebraic gerbe, and choose a diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}^{\mathrm{f}} \kappa \arrow[r, "f'"] & Y \end{tikzcd}$$

as in Definition 5.2.1. Let $X_\bullet$ be the Čech nerve of the map f. Since the map $\operatorname{Spec}^{\mathrm{f}} \kappa \rightarrow Y$ is an effective epimorphism of flat sheaves (Remark 5.1.11), f is also an effective epimorphism of flat sheaves. Consequently, X can be identified with the geometric realization of $X_\bullet$ in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$.

Lemma 5.2.4. *Let* $X : \mathrm{CAlg}^{\mathrm{cn}} \rightarrow \widehat{\mathcal{S}}$ *be a generalized gerbe. Then* X *is a hypercomplete sheaf with respect to the flat topology.*

Proof. Choose a diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}^{\mathrm{f}} \kappa \arrow[r, "f"] & Y \end{tikzcd}$$

as in Definition 5.2.1. The functor Y is a hypercomplete sheaf (Corollary 3.4.27); it will therefore suffice to show that for every connective $\mathbb{E}_\infty$-ring R and every map $\eta : \operatorname{Spec}^{\mathrm{f}} R \rightarrow Y$, the pullback $X \times_Y \operatorname{Spec}^{\mathrm{f}} R$

restricts to a hypercomplete sheaf on $\mathrm{CAlg}^{\mathrm{cn}}_R$ (Lemma 3.1.20). The assertion is local with respect to the flat topology, so we may assume that η factors through f and thereby reduce to the case $R = \kappa$: that is, we must show that X_0 is hypercomplete. Since X_0 is corepresented by a coconnective κ-algebra, this follows from Theorem VII.5.14. $\square$

Lemma 5.2.5. *Let X be a generalized algebraic gerbe. Then X is discretely determined.*

Proof. The collection of discretely determined functors is closed under small colimits in $\widehat{\mathrm{Shv}}_{\mathrm{fpqc}}$. Let $X_\bullet$ be as in Remark 5.2.3, so that $X \simeq |X_\bullet|$. It will therefore suffice to show that each X_n is discretely determined. Note that $X_n \simeq X_0 \times_{\mathrm{Spec}^{\mathrm{f}} \kappa} \mathrm{Spec}^{\mathrm{f}} R$ for some discrete κ-algebra R. According to Proposition 4.4.4, we can write X_0 as the geometric realization (in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$) of a simplicial object $\mathrm{Spec}^{\mathrm{f}} A^\bullet$, where each A^m is a discrete k-algebra. Then X_n is the colimit of functors $\mathrm{Spec}^{\mathrm{f}} A^m \times_{\mathrm{Spec}^{\mathrm{f}} \kappa} \mathrm{Spec}^{\mathrm{f}} R \simeq \mathrm{Spec}^{\mathrm{f}}(A^m \otimes_\kappa R)$, each of which is corepresented by a commutative ring and therefore discretely determined. $\square$

Lemma 5.2.6. *Let R be a commutative ring, let CAlg^0_R denote the full subcategory of CAlg_R spanned by the discrete R-algebras, and let $X : \mathrm{CAlg}^0_R \to \widehat{\mathcal{S}}$ be a functor. Fix $n \geq 1$, and assume that X is an n-truncated, 2-connective sheaf with respect to the flat topology. For $2 \leq m \leq n$, let $\pi_m X$ denote the mth homotopy sheaf of X (regarded as a sheaf of abelian groups on $(\mathrm{CAlg}^0_R)^{op}$: see §T.6.5.1; this is well-defined without a choice of base point by virtue of our assumption that X is 2-connective) and assume that that there is a functorial isomorphism $(\pi_m X)(A) \simeq \mathrm{Hom}_R(V_m, A)$ for some projective R-module V_m. Then:*

(1) *For every discrete R-algebra A, the space $X(A)$ is 2-connective.*

(2) *For every discrete R-algebra A and $2 \leq m \leq n$, the canonical map $\pi_m X(A) \to (\pi_m X)(A)$ is an isomorphism of abelian groups.*

(3) *The homotopy groups $\pi_m X(A)$ vanish for $m > n$.*

Proof. We proceed by induction on n. If $n = 1$ then $X \simeq *$ and there is nothing to prove. Assume that $n \geq 2$, and let $Y = \tau_{\leq n-1} X$ (where the truncation is carried out in the ∞-category $\mathrm{Shv}_{\widehat{\mathcal{S}}}((\mathrm{CAlg}^0_R)^{op})$ of flat sheaves). Since the presheaf $A \mapsto (\pi_n X)(A) \simeq \mathrm{Hom}_R(V_m, A)$ has vanishing higher cohomology with respect to the flat topology, we conclude that

$$\pi_m(K(\pi_n X, n+1)(A)) \simeq \begin{cases} (\pi_n X)(A) & \text{if } m = n \\ 0 & \text{if } m \neq n. \end{cases}$$

Using Theorem T.7.2.2.26, we deduce the existence of a fiber sequence

$$X \to Y \to K(\pi_n X, n+1).$$

Since $\pi_n(Y(A)) \simeq \pi_{n+1}(Y(A)) \simeq 0$ by the inductive hypothesis, the long exact sequence of homotopy groups gives isomorphisms

$$\pi_m(X(A)) \simeq \begin{cases} \pi_m(Y(A)) & \text{if } m \neq n \\ (\pi_n X)(A) & \text{if } m = n \end{cases}$$

so that the desired conclusion follows from the inductive hypothesis. $\square$

Remark 5.2.7. The proof of Lemma 5.2.6 does not require that the modules V_m be projective. However, this condition will be satisfied in the examples which are of interest to us.

Remark 5.2.8. Let R be a commutative ring. Since the category of discrete R-modules satisfies flat descent, the hypothesis of Lemma 5.2.6 is local with respect to the flat topology on CAlg^0. For example, if $X : \mathrm{CAlg}^0_R \to \widehat{\mathcal{S}}$ is a flat sheaf and the restriction $X | \mathrm{CAlg}^0_A$ satisfies the hypotheses of Lemma 5.2.6 for some faithfully flat R-algebra A, then X also satisfies the hypotheses of Lemma 5.2.6.

Lemma 5.2.9. *Let* CAlg^0 *denote the* ∞*-category of discrete* $\mathbb{E}_\infty$*-rings (so that* CAlg^0 *is equivalent oto the nerve of the ordinary category of commutative rings) and let* $X : \mathrm{CAlg}^0 \to \widehat{\mathcal{S}}$ *be a functor. For each* $n \geq 1$*, let* $\tau_{\leq n} X$ *denote the composition of* X *with the truncation functor* $\tau_{\leq n} : \widehat{\mathcal{S}} \to \widehat{\mathcal{S}}$*. If* R *is a commutative ring and* $\eta \in (\tau_{\leq 1} X)(R)$*, we let* $\pi_n(X, \eta)$ *denote the functor* $\mathrm{CAlg}^0_R \to \mathcal{A}b$ *given by the formula* $A \mapsto \pi_n(X(A) \times_{(\tau_{\leq 1} X)(A)} \{\eta_A\})$*, where* η_A *denotes the image of* η *in* $(\tau_{\leq 1} X)(A)$*. The following conditions are equivalent:*

(1) (*a*) *The functor* X *is a hypercomplete sheaf with respect to the flat topology.*

(*b*) *The sheafification of* $\tau_{\leq 1} X$ *(with respect to the flat topology) is the restriction of an algebraic gerbe* $\mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$*.*

(*c*) *There exists a field* κ *of characteristic zero and a map* $\eta : \mathrm{Spec}^{\mathrm{f}} \kappa \to \tau_{\leq 1} X$ *such that, for each* $m \geq 2$*, the sheafification of the functor* $\pi_m(X, \eta) : \mathrm{CAlg}^0_\kappa \to \mathcal{A}b$ *is given by* $A \mapsto \mathrm{Hom}_\kappa(V_m, A)$*, for some* κ*-vector space* V_m*.*

(2) (a') *The functor* X *is a sheaf with respect to the flat topology.*

(b') *The functor* $\tau_{\leq 1} X$ *extends to an algebraic gerbe* $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ *over a field of characteristic zero.*

(c') *For every discrete commutative ring* R*, every point* $\eta : \mathrm{Spec}\, R \to \tau_{\leq 1} X$*, and every* $m \geq 2$*, the functor* $\pi_m(X, \eta)$ *is given by* $A \mapsto \mathrm{Hom}_R(V_m, A)$ *for some projective* R*-module* V_m*.*

Proof. We first consider the case where X is n-truncated for $n \geq 1$. The implication $(2) \Rightarrow (1)$ is obvious. Conversely, suppose that (1) is satisfied, and let Y denote the sheafification of $\tau_{\leq 1} X$. For every map $\eta : \mathrm{Spec}^{\mathrm{f}} R \to Y$ which factors through $\mathrm{Spec}^{\mathrm{f}} \kappa$, assumption ($c$) guarantees that the fiber product $X \times_Y \mathrm{Spec}^{\mathrm{f}} R$ determines a functor satisfying the hypotheses of Lemma 5.2.6. Since these hypotheses are local with respect to the flat topology (Remark 5.2.8), the same conclusion holds for every η. It follows that the homotopy fibers of the map maps $X(R) \to Y(R)$ are 2-connective, so that the map $\tau_{\leq 1} X \to \tau_{\leq 1} Y \simeq Y$ is an equivalence. The implication $(b) \Rightarrow (b')$ is now clear, and (c') follows from Lemma 5.2.6.

We now remove the assumption that X is n-truncated. Let $X : \mathrm{CAlg}^0 \to \widehat{\mathcal{S}}$ be a flat sheaf. For $n \geq 1$, let Y_n denote the sheafification of $\tau_{\leq n} X$ with respect to the flat topology. We first show that $(1) \Rightarrow (2)$. If X satisfies (1), then each Y_n satisfies (1) and therefore (by the first part of the proof) we conclude that each Y_n satisfies (2). Let $Y = \varprojlim Y_n$. Since each Y_n satisfies (2), the canonical maps $\tau_{\leq m} Y_n \to Y_m$ are equivalences for $m \leq n$. It follows that the canonical map $Y \to Y_n$ induces an equivalence $\tau_{\leq n} Y \simeq Y_n$ for each $n \geq 1$, so that the map $X \to Y$ is ∞-connective. The sheaf X is hypercomplete by (a), and the sheaf Y is hypercomplete since it is a limit of truncated sheaves. It follows that the map $X \to Y$ is an equivalence. It therefore suffices to prove that Y satisfies (2), which is clear.

Now suppose that X satisfies (2). It is clear that X satisfies (b) and (c). Using Lemma 5.2.6, we see that each $\tau_{\leq m} X$ is a sheaf with respect to the flat topology, so that $X \simeq \varprojlim \tau_{\leq m} X$ is a limit of truncated sheaves and therefore hypercomplete. $\square$

Lemma 5.2.10. *Let* $f : X \to Y$ *be a natural transformation of functors* $X, Y : \mathrm{CAlg}^0 \to \widehat{\mathcal{S}}$ *with the following property:*

($*$) *For every discrete commutative ring* R *and every natural transformation* $\mathrm{Spec}^{\mathrm{f}} R \to Y$*, the functor* $X \times_Y \mathrm{Spec}^{\mathrm{f}} R$ *is corepresentable by a flat* R*-algebra (here we abuse notation slightly by identifying* $\mathrm{Spec}^{\mathrm{f}} R$ *with its restriction to the full subcategory* $\mathrm{CAlg}^0 \subseteq \mathrm{CAlg}^{\mathrm{cn}}$*).*

Let $F : \mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ *be the functor of left Kan extension (that is, a left adjoint to the restriction functor). For every morphism* $Y' \to Y$ *in* $\mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})$*, the diagram*

$$\begin{tikzcd} F(X \times_Y Y') \arrow[r] \arrow[d] & F(Y') \arrow[d] \\ F(X) \arrow[r] & F(Y) \end{tikzcd}$$

is a pullback square in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$.

Proof. Let $\mathcal{Y}$ be the full subcategory of $\mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y}$ spanned by those maps $Y_0 \to Y$ with the following property: for every map $Y' \to Y_0$, the diagram σ :

$$\begin{tikzcd} F(Y' \times_Y X) \arrow[r] \arrow[d] & F(Y') \arrow[d] \\ F(Y_0 \times_Y X) \arrow[r] & F(Y_0) \end{tikzcd}$$

is a pullback square. We wish to prove that $\mathcal{X} = \mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y}$. We begin with the following observation:

(a) Suppose given a map $Y_0 \to Y$. To prove that $Y_0 \in \mathcal{Y}$, it suffices to verify that σ is a pullback square whenever Y' is corepresented by a discrete ring R.

To verify (a), let us fix a map $Y_0 \to Y$ and consider the full subcategory $\mathcal{X} \subseteq \mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y_0}$ spanned by those objects Y' for which the diagram σ is a pullback square. Since F preserves colimits and colimits are universal in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})_{/Y_0}$, we conclude that $\mathcal{X}$ is stable under colimits in $\mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y_0}$. Assertion ($a$) now follows from the observation that $\mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y_0}$ is generated under colimits by objects of the form $Y' \to Y_0$, where Y' is a representable functor.

We next prove that we are given a map $Y_0 \to Y$ such that Y_0 is corepresentable, then $Y_0 \in \mathcal{Y}$. To prove this, let us suppose that Y_0 is corepresented by some discrete $\mathbb{E}_\infty$-ring R. We will verify that condition (a) is satisfied: that is, that σ is a pullback square when Y' is corepresented by a discrete $\mathbb{E}_\infty$-ring R'. Using ($*$), we deduce that $Y_0 \times_Y X$ is corepresented by a flat R-algebra A, so that $Y' \times_Y X$ is corepresented by the tensor product $A' = R' \otimes_R A$ (the assumption that R' is flat over R guarantees that A' is again discrete). Then σ can be identified with the diagram

$$\begin{tikzcd} \mathrm{Spec}^{\mathrm{f}} A' \arrow[r] \arrow[d] & \mathrm{Spec}^{\mathrm{f}} R' \arrow[d] \\ \mathrm{Spec}^{\mathrm{f}} A \arrow[r] & \mathrm{Spec}^{\mathrm{f}} R, \end{tikzcd}$$

which is evidently a pullback square in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$.

We will complete the proof by showing that $\mathcal{Y} = \mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y}$. Since $\mathcal{Y}$ contains all corepresentable functors, it will suffice to show that $\mathcal{Y}$ is stable under colimits. Fix a diagram $p : \mathcal{J} \to \mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}})_{/Y}$ which takes values in $\mathcal{Y}$ and set $Y_0 = \varinjlim(p)$; we will prove that $Y_0 \in \mathcal{Y}$. By assumption, for every morphism $I \to J$ in $\mathcal{J}$, the diagram

$$\begin{tikzcd} F(X \times_Y p(I)) \arrow[r] \arrow[d] & F(p(I)) \arrow[d] \\ F(X \times_Y p(J)) \arrow[r] & F(p(J)) \end{tikzcd}$$

is a pullback diagram in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Moreover, since F preserves small colimits, we have

$$F(Y_0) \simeq \varinjlim F(p(I)) \qquad F(X \times_Y Y_0) \simeq \varinjlim F(X \times_Y p(I)).$$

Using Theorem T.6.1.3.9, we conclude that each of the diagrams

$$\begin{tikzcd} F(X \times_Y p(I)) \arrow[r] \arrow[d] & F(p(I)) \arrow[d] \\ F(X \times_Y Y_0) \arrow[r] & F(Y_0) \end{tikzcd}$$

is a pullback square in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. We now prove that Y_0 satisfies criterion (a). Choose a map $\eta : Y' \to Y_0$, where Y' is corepresentable by some discrete ring R. Then η is classified by a point in $Y_0(R) \simeq \varinjlim p(I)(R)$, so that η factors through $p(I)$ for some object $I \in \mathcal{J}$. We have a commutative diagram

$$\begin{tikzcd}
F(X \times_Y Y') \arrow[r] \arrow[d] & F(Y') \arrow[d] \\
F(X \times_Y p(I)) \arrow[r] \arrow[d] & F(p(I)) \arrow[d] \\
F(X \times_Y Y_0) \arrow[r] & F(Y_0)
\end{tikzcd}$$

where the upper square is a pullback by virtue of our assumption that $p(I) \in \mathcal{Y}$, and the lower square is a pullback by the argument given above. It follows that the outer square is a pullback, which completes the verification that Y_0 satisfies the criterion of (a) and therefore belongs to $\mathcal{Y}$. ☐

Lemma 5.2.11. *Suppose we are given a pullback diagram σ :*

$$\begin{tikzcd}
X' \arrow[r] \arrow[d] & Y' \arrow[d] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

in the ∞-category $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. Suppose further that X, Y, and Y' are discretely determined and that f is representable, flat, and affine. Then X' is discretely determined.

Proof. Since the collection of flat sheaves is stable under limits in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$, we deduce immediately that X' is a sheaf with respect to the flat topology. Let $X'_0 = X' | \mathrm{Ring}$, and define Y'_0, X_0, and Y_0 similarly. Let $F : \mathrm{Fun}(\mathrm{CAlg}^0, \widehat{\mathcal{S}}) \to \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$ be as in Lemma 5.2.10, and let $L : \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}}) \to \widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$ be a left adjoint to the conclusion (that is, sheafification with respect to the flat topology). Applying Lemma 5.2.10 to the pullback diagram

$$\begin{tikzcd}
X'_0 \arrow[r] \arrow[d] & Y'_0 \arrow[d] \\
X_0 \arrow[r, "f"] & Y_0,
\end{tikzcd}$$

we conclude that σ':

$$\begin{tikzcd}
F(X'_0) \arrow[r] \arrow[d] & F(Y'_0) \arrow[d] \\
F(X_0) \arrow[r, "f"] & F(Y_0)
\end{tikzcd}$$

is a pullback diagram in $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. There is an evident map of diagrams $\sigma' \to \sigma$. Since Y', X, and Y are discretely determined and functor L is left exact, this map induces an equivalence $L(\sigma') \simeq \sigma$ so that X' is discretely determined as well. ☐

Proposition 5.2.12. *Let k be a field of characteristic zero,* $\mathrm{Spec}^{\mathrm{f}}\, k$ *the functor corepresented by k, and let $\alpha : X \to \mathrm{Spec}^{\mathrm{f}}\, k$ be a morphism in* $\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})$. *Let $Y : \mathrm{CAlg}^0_k \to \widehat{\mathcal{S}}$ be defined as in Proposition 5.1.3. Then α exhibits X as a generalized algebraic gerbe over k if and only if the following conditions are satisfied:*

(1) *The functor X is discretely determined.*

(2) *The functor Y is 1-connective (as a sheaf in the flat topology).*

(3) *There exists an extension field κ of k and a point $\eta \in Y(\kappa)$ with the following property: each homotopy group $\pi_i(Y, \eta)$ is representable by an affine proalgebraic group over κ, which is prounipotent if $i \geq 2$.*

Proof. Suppose first that α exhibits X as a generalized algebraic gerbe over k. Then X is discretely determined by Lemma 5.2.5. Choose a pullback diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "g'"] & X \arrow[d, "g"] \\ \operatorname{Spec}^{\mathrm{f}} \kappa \arrow[r] & Z \end{tikzcd}$$

where Z is an algebraic gerbe over k and X_0 is corepresentable by a 2-coconnective $\mathbb{E}_\infty$-algebra A over κ. We will show that $X|\operatorname{CAlg}^0$ satisfies conditions (a), (b), and (c) of Lemma 5.2.9. Assertion (a) follows from Lemma 5.2.4. To prove (b), it suffices to show that the map $g : X \to Z$ exhibits $Z|\operatorname{CAlg}^0$ as the truncation $\tau_{\leq 1}(X|\operatorname{CAlg}^0)$ in the ∞-category $\mathcal{S}\mathrm{hv}_{\widehat{\mathcal{S}}}((\operatorname{CAlg}^0)^{op})$. Since $Z|\operatorname{CAlg}^0$ is 1-truncated, it suffices to show that g induces a 2-connective map $X|\operatorname{CAlg}^0 \to Z|\operatorname{CAlg}^0$. Since the map $\operatorname{Spec}^{\mathrm{f}} \kappa|\operatorname{CAlg}^0 \to Z|\operatorname{CAlg}^0$ is an effective epimorphism of flat sheaves, this is equivalent to the assertion that the map $X_0|\operatorname{CAlg}^0 \to \operatorname{Spec}^{\mathrm{f}} \kappa|\operatorname{CAlg}^0$ is 2-connective, which follows from the observation $\pi_{-1} A \simeq 0$ (Proposition 4.4.12). Assertion (c) follows immediately from Proposition 4.4.8.

Since X_0 is a coaffine stack over κ, it admits a κ-point which determines a point $\eta \in Y(\kappa)$. Using Lemma 5.2.9, we deduce that $Z = \tau_{\leq 1} X$. It now follows from Proposition 5.1.3 that Y is 1-connective and that $\pi_1(Y; \eta)$ is represented by an affine proalgebraic group over κ. Assertion (c') of Lemma 5.2.9 implies that $\pi_i(Y; \eta)$ is represented by an affine prounipotent group over κ for $i \geq 2$.

Now suppose that X satisfies conditions (1), (2), and (3). Using Lemma 5.2.9 and Proposition 5.1.3, we deduce that $\tau_{\leq 1}(X|\operatorname{CAlg}^0) \simeq Z|\operatorname{CAlg}^0$ for some k-gerbe Z. Since X is discretely determined, the canonical map $X|\operatorname{CAlg}^0 \to Z|\operatorname{CAlg}^0$ lifts in an essentially unique fashion to a map $g : X \to Z$. Choose a field κ of characteristic zero and a point $\eta \in Z(\kappa)$, and set $X_0 = \operatorname{Spec}^{\mathrm{f}} \kappa \times_Z X$. Using Propositions 4.4.8 and 4.4.12, we conclude that there exists a 2-coconnective $\mathbb{E}_\infty$-algebra A over κ such that $X_0|\operatorname{CAlg}^0 \simeq (\operatorname{cSpec} A)|\operatorname{CAlg}^0$. Lemma 5.2.11 guarantees that X_0 is discretely determined, so that $X_0 \simeq \operatorname{cSpec} A$ by Lemma 5.2.5. □

5.3 Symmetric Monoidal Structures on Chain Complexes

Let $\mathcal{A}$ be an abelian category with enough injective objects, and let $\mathcal{D}^+(\mathcal{A})$ be its derived ∞-category (see §A.1.3.1). In this section, we will show that if $\mathcal{A}$ is equipped with a sufficiently nice symmetric monoidal structure, then $\mathcal{D}^+(\mathcal{A})$ inherits a symmetric monoidal structure (see Proposition 5.3.15). We will restrict our attention to the case where the tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact in each variable.

We begin with some general remarks.

Notation 5.3.1. Let $\mathcal{A}$ be an abelian category. We let $\operatorname{Fun}(\mathbf{\Delta}, \mathcal{A})$ denote the category of cosimplicial objects of $\mathcal{A}$. The category $\operatorname{Fun}(\mathbf{\Delta}, \mathcal{A})$ is naturally cotensored over finite simplicial sets: if K is a finite simplicial set and $A^\bullet$ is an object of $\operatorname{Fun}(\mathbf{\Delta}, \mathcal{A})$, then we set

$$(A^K)^n = \prod_{\sigma \in \operatorname{Hom}(\Delta^n, K)} A^n.$$

In particular, we can regard $\operatorname{Fun}(\mathbf{\Delta}, \mathcal{A})$ as a simplicial category, whose enrichment is characterized by the formula

$$\operatorname{Hom}_{\mathcal{S}\mathrm{et}_\Delta}(K, \operatorname{Map}_{\operatorname{Fun}(\mathbf{\Delta}, \mathcal{A})}(A, B)) = \operatorname{Hom}_{\operatorname{Fun}(\mathbf{\Delta}, \mathcal{A})}(A, B^K).$$

Remark 5.3.2. Let $\mathcal{A}$ be an abelian category and let $A^\bullet$ be a cosimplicial object of $\mathcal{A}$. We can associate to $A^\bullet$ its *unnormalized chain complex*: this is the chain complex

$$A^0 \xrightarrow{d} A^1 \xrightarrow{d} A^2 \xrightarrow{d} \cdots$$

where each differential is given by an alternating sum of the coface maps of $A^\bullet$. For each $n \geq 0$, we let $\mathrm{H}^n(A^\bullet) \in \mathcal{A}$ denote the nth cohomology of this unnormalized cochain complex. We say that a map $f : A^\bullet \to B^\bullet$ of cosimplicial objects is a *quasi-isomorphism* if, for each $n \geq 0$, the induced map $\mathrm{H}^n(A^\bullet) \to \mathrm{H}^n(B^\bullet)$ is an isomorphism in $\mathcal{A}$.

Remark 5.3.3. Let $\mathcal{A}$ be an abelian category, and let $\mathrm{Ch}(\mathcal{A})$ be the category of chain complexes

$$\cdots \to A_1 \to A_0 \to A_{-1} \to \cdots$$

with values in $\mathcal{A}$, and let $\mathrm{Ch}_{\leq 0}(\mathcal{A})$ be the full subcategory spanned by those chain complexes for which $A_n \simeq 0$ for $n > 0$. The Dold-Kan correspondence gives an equivalence of categories $N^* : \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}) \to \mathrm{Ch}_{\leq 0}(\mathcal{A})$,which assigns to each cosimplicial object $A^\bullet$ its *normalized* cochain complex (see §A.1.2.3). For each $A^\bullet$, there is a natural quasi-isomorphism of cochain complexes from $A^\bullet$ to $N^*(A^\bullet)$. In particular, a map $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism if and only if the induced map $N^*(A^\bullet) \to N^*(B^\bullet)$ is a quasi-isomorphism.

Let $\mathcal{A}$ be an abelian category with enough injective objects, and let $\mathcal{A}^{\mathrm{inj}}$ denote the full subcategory of $\mathcal{A}$ spanned by the injective objects. We can regard $\mathrm{Ch}(\mathcal{A})$ as a simplicial category via the prescription described in §A.1.3.1: that is, for every pair of chain complexes $A_\bullet$ and $B_\bullet$ and every finite simplicial set K, we have a canonical bijection

$$\mathrm{Hom}_{\mathrm{Set}_\Delta}(K, \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet, B_\bullet)) \simeq \mathrm{Map}_{\mathrm{Ch}(\mathcal{A})}(A_\bullet \otimes C_\bullet(K), B_\bullet)),$$

where $C_\bullet(K)$ denotes the normalized chain complex (of free abelian groups) associated to the simplicial set K (so that the generators of $C_n(K)$ can be identified with nondegenerate n-simplices of K). The induced simplicial structure on $\mathrm{Ch}_{\leq 0}(\mathcal{A})$ does *not* agree with N^*-pullback of the simplicial structure on $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})$ described in Construction 5.3.14. Nevertheless, the inverse of N^* determines a simplicial functor $\mathrm{Ch}_{\leq 0}(\mathcal{A}) \to \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})$ which induces an equivalence of ∞-categories

$$\mathcal{D}^+(\mathcal{A})_{\leq 0} \to \mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})),$$

where $\mathcal{D}^+(\mathcal{A})$ is the derived ∞-category of $\mathcal{A}$ (see Proposition A.1.3.1.16 and Variation A.1.3.1.19).

Remark 5.3.4. Let $\mathcal{A}$ be an abelian category with enough injective objects, and let $\mathrm{Fun}_{\mathrm{acyc}}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ denote the full subcategory of $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ spanned by those cosimplicial objects $A^\bullet$ such that $\mathrm{H}^n(A^\bullet) \simeq 0$ for $n > 0$. It follows from the above analysis (or from Proposition 5.3.12) that the functor $A^\bullet \mapsto \mathrm{H}^0(A^\bullet)$ determines an equivalence of simplicial categories $\mathrm{Fun}_{\mathrm{acyc}}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \to \mathcal{A}$ (here we regard $\mathcal{A}$ as a simplicial category with discrete mapping spaces).

Combining the above observations with Lemma A.1.3.2.10 and Proposition A.1.3.2.6, we deduce the following:

Proposition 5.3.5. *Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a right complete t-structure such that $\mathcal{C}_{\leq 0}$ is closed under filtered colimits, and choose an equivalence $f : \mathrm{N}(\mathcal{A}) \to \mathcal{C}^\heartsuit$ where $\mathcal{A}$ is an abelian category. Then $\mathcal{A}$ has enough injectives (Corollary A.1.3.4.7), and there exists an essentially unique functor $F : \mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})) \to \mathcal{C}_{\leq 0}$ with the following properties:*

(1) *The functor F is left exact.*

(2) *The restriction of F to $\mathrm{N}(\mathrm{Fun}_{\mathrm{acyc}}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$ is equivalent to the composition $\mathrm{N}(\mathrm{Fun}_{\mathrm{acyc}}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})) \simeq \mathrm{N}(\mathcal{A}) \simeq \mathcal{C}^\heartsuit \hookrightarrow \mathcal{C}$.*

Moreover, the functor F is an equivalence of ∞-categories if and only if the following condition is satisfied:

- $(*)$ *For every pair of objects $A, B \in \mathcal{A}$, if B is injective, then the abelian groups $\mathrm{Ext}^i_{\mathcal{C}}(f(A), f(B))$ are trivial for $i > 0$.*

Example 5.3.6. Let $X : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ be a geometric stack, and let $\mathcal{A} = \mathrm{hQCoh}(X)^{\heartsuit}$ be the abelian category of discrete quasi-coherent sheaves on X. Let $F : \mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})) \to \mathrm{QCoh}(X)_{\leq 0}$ be the left-exact functor of Proposition 5.3.5. Choose a connective $\mathbb{E}_\infty$-ring A and a faithfully flat morphism $\mathrm{Spec}^{\mathrm{f}} A \to X$, where $\mathrm{Spec}^{\mathrm{f}} A$ denotes the functor corepresented by A. If A is discrete, then F is an equivalence of ∞-categories. To see this, choose discrete sheaves $\mathcal{F}, \mathcal{G} \in \mathrm{QCoh}(X)^{\heartsuit}$, where $\mathcal{G}$ is an injective object of $\mathcal{A}$. We can identify $\eta^* \mathcal{G}$ with a discrete A-module M. Choose a map $M \to I$, where I is an injective A-module. Since the pullback functor $\eta^* : \mathrm{QCoh}(X)^{\heartsuit} \to \mathrm{Mod}_A^{\heartsuit}$ is exact, we conclude that $\eta_* I$ is an injective object of $\mathcal{A}$. We claim that the composite map

$$\mathcal{G} \to \eta_* \eta^* \mathcal{G} \to \eta_* I$$

is a monomorphism in $\mathcal{A}$. In other words, we claim that the map $\eta^* \mathcal{G} \to \eta^* \eta_* I$ is a monomorphism of discrete A-modules. This is clear, since the composite map $\eta^* \mathcal{G} \to \eta^* \eta_* I \to I$ is a monomorphism by construction. Because $\mathcal{G}$ is injective, it follows that the monomorphism $\mathcal{G} \to \eta_* I$ splits; consequently, to prove that the abelian groups $\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{F}, \mathcal{G})$ vanish for $i > 0$, it suffices to show that $\mathrm{Ext}^i_{\mathrm{QCoh}(X)}(\mathcal{F}, \eta_* I) \simeq \mathrm{Ext}^i_{\mathrm{Mod}_A}(\eta^* \mathcal{F}, I) \simeq 0$. Since A is discrete and I is injective, this is an immediate consequence of Proposition A.7.1.1.15.

We now review some standard results concerning the interplay between the simplicial structure on the category of cosimplicial objects of $\mathcal{A}$ and the notion of quasi-isomorphism between the corresponding chain complexes.

We begin with a simple observation. Suppose we are given an exact sequence

$$0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$$

of cosimplicial objects of an abelian category $\mathcal{A}$. There is an associated long exact sequence in cohomology

$$0 \to \mathrm{H}^0(A^\bullet) \to \mathrm{H}^0(B^\bullet) \to \mathrm{H}^0(C^\bullet) \to \mathrm{H}^1(A^\bullet) \to \cdots$$

Combining this analysis with the five lemma, we obtain the following:

Lemma 5.3.7. *Let $\mathcal{A}$ be an abelian category, and suppose we are given a commutative diagram*

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \arrow[r] \arrow[d, "f"] & B^\bullet \arrow[r] \arrow[d, "g"] & C^\bullet \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & A'^\bullet \arrow[r] & B'^\bullet \arrow[r] & C'^\bullet \arrow[r] & 0
\end{tikzcd}$$

of cosimplicial objects of $\mathcal{A}$ having exact rows. If any two of morphisms f, g, and h are quasi-isomorphisms, then so is the third.

Lemma 5.3.8. *Let $\mathcal{A}$ be an abelian category, $A^\bullet$ a cosimplicial object of $\mathcal{A}$, and let $f : K \to K'$ be a weak homotopy equivalence of finite simplicial sets. Then the induced map $(A^{K'})^\bullet \to (A^K)^\bullet$ is a quasi-isomorphism.*

Proof. Fix $n \geq 0$; we will show that the map $\mathrm{H}^m((A^{K'})^\bullet) \to \mathrm{H}^n((A^K)^\bullet)$ is an isomorphism for $m < n$. Let

$$B^0 \to B^1 \to B^2 \to \cdots$$

be the normalized chain complex associated to $A^\bullet$. Let $A'^\bullet$ be the cosimplicial object corresponding to the truncated chain complex

$$0 \to B^0 \to \cdots \to B^n \to 0 \to 0 \to \cdots$$

Then we have a map $A^\bullet \to A'^\bullet$ which is an isomorphism in degrees $\leq n$, and therefore induces isomorphisms

$$\mathrm{H}^m((A^{K'})^\bullet) \to \mathrm{H}^m((A'^{K'})^\bullet) \qquad \mathrm{H}^m((A^K)^\bullet) \to \mathrm{H}^m((A'^K)^\bullet)$$

for $m < n$. We may therefore replace $A^\bullet$ by $A'^\bullet$ and thereby reduce to the case where the chain complex $B^\bullet$ is bounded. Using Lemma 5.3.7, we can reduce to the case where the chain complex $B^\bullet$ is concentrated in a single degree k, taking the value $B \in \mathcal{A}$. In this case, the map of unnormalized chain complexes $\theta : ((A^{K'})^\bullet, d) \to ((A^K)^\bullet, d)$ is obtained by tensoring B with the *dual* of the map of chain complexes of free abelian groups given by

$$\theta_0 : C_*(K \times \Delta^k, K \times \partial \Delta^k; \mathbf{Z}) \to C_*(K' \times \Delta^k, K' \times \partial \Delta^k; \mathbf{Z}).$$

If f is a weak homotopy equivalence, then θ_0 is a quasi-isomorphism and therefore (since it is map of chain complexes of free abelian groups) admits a homotopy inverse. It follows that θ admits a homotopy inverse and is therefore a quasi-isomorphism. $\square$

Lemma 5.3.9. *Let $\mathcal{A}$ be an abelian category and let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism between cosimplicial objects of $\mathcal{A}$. If K is a finite simplicial set, then the induced map $(A^K)^\bullet \to (B^K)^\bullet$ is a quasi-isomorphism.*

Proof. We work by induction on the dimension n of K and the number of nondegenerate n-simplices of K. If $K \simeq \emptyset$, the result is obvious. Otherwise, we have a pushout diagram

$$\begin{tikzcd} \partial \Delta^n \arrow[r] \arrow[d] & \Delta^n \arrow[d] \\ K_0 \arrow[r] & K, \end{tikzcd}$$

which gives rise to a map of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & (A^K)^\bullet \arrow[r] \arrow[d] & (A^{K_0})^\bullet \oplus (A^{\Delta^n})^\bullet \arrow[r] \arrow[d] & (A^{\partial \Delta^n})^\bullet \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & (B^K)^\bullet \arrow[r] & (B^{K_0})^\bullet \oplus (B^{\Delta^n})^\bullet \arrow[r] & (B^{\partial \Delta^n})^\bullet \arrow[r] & 0 \end{tikzcd}$$

Using the inductive hypothesis and Lemma 5.3.7, we are reduced to proving that the map $(A^{\Delta^n})^\bullet \to (B^{\Delta^n})^\bullet$ is a quasi-isomorphism. This follows by applying Lemma 5.3.8 to the commutative diagram

$$\begin{tikzcd} A^\bullet \arrow[r] \arrow[d] & B^\bullet \arrow[d] \\ (A^{\Delta^n})^\bullet \arrow[r] & (B^{\Delta^n})^\bullet. \end{tikzcd}$$

$\square$

Lemma 5.3.10. *Let $\mathcal{A}$ be an abelian category with enough injectives, and suppose that we are given a diagram of chain complexes*

$$\begin{tikzcd} A_\bullet \arrow[r, "h_0"] \arrow[d, "f"] & X_\bullet \arrow[d, "g"] \\ B_\bullet \arrow[r] \arrow[ur, dashed, "h"] & Y_\bullet \end{tikzcd}$$

satisfying the following conditions:

(1) *The map f is a monomorphism.*

(2) *The map g is an epimorphism.*

(3) *The kernel of g is a chain complex*

$$0 \to I_0 \to I_{-1} \to I_{-2} \to \cdots$$

where each I^n is an injective object of $\mathcal{A}$.

If either f or g is a quasi-isomorphism, then there exists a map h as indicated, rendering the diagram commmutative.

Proof. For every integer n, we let $A(n)_\bullet$ denote the truncated chain complex

$$\cdots \to A_{n+1} \to \ker(A_n \to A_{n-1}) \to 0 \to \cdots .$$

Let B'_n denote the subobject of B_n generated by the image of B_{n+1} and the image of $\ker(A_n \to A_{n-1})$; there is a natural map $B'_n \to \ker(B_n \to B_{n-1})$, which is an isomorphism when f is a quasi-isomorphism. Let $B(n)_\bullet$ denote the chain complex

$$\cdots \to B_{n+1} \to B'_n \to 0 \to \cdots ,$$

regarded as a subobject of $B_\bullet$.

We will construct h as the direct limit of a family of compatible family of chain maps $h(n) : B(n)_\bullet \to X_\bullet$, each of which fits into a commutative diagram

$$\begin{tikzcd}
A(n)_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d, "g"] \\
B(n)_\bullet \arrow[r] \arrow[ur, dashed, "h(n)"] & Y_\bullet .
\end{tikzcd}$$

If $n > 0$, then $h(n)$ is uniquely determined, since g is an isomorphism in positive degrees. We may therefore proceed by descending induction on n. Suppose that $h(n+1)$ has been defined; we wish to prove that $h(n+1)$ can be extended to a chain map $h(n)$ such that the above diagram commutes. In other words, we wish to solve the lifting problem

$$\begin{tikzcd}
B(n+1)_\bullet \coprod_{A(n+1)_\bullet} A(n)_\bullet \arrow[r] \arrow[d] & X_\bullet \arrow[d, "g"] \\
B(n)_\bullet \arrow[r] \arrow[ur, dashed] & Y_\bullet .
\end{tikzcd}$$

Note that the left vertical map is a pushout of a monomorphism of chain complexes $f' : A'_\bullet \to B'_\bullet$, where $A'_\bullet$ is the chain complex

$$\cdots \to 0 \to B'_{n+1} \coprod_{\ker(A_{n+1} \to A_n)} A_{n+1} \to \ker(A_n \to A_{n-1}) \to 0 \to \cdots$$

and $B'_\bullet$ is the chain complex

$$\cdots \to 0 \to B_{n+1} \to B'_n \to 0 \to \cdots$$

Moreover, if f is a quasi-isomorphism, then so is f'. We may therefore replace $A_\bullet$ by $A'_\bullet$ and $B_\bullet$ by $B'_\bullet$, and thereby reduce to the case where $A_\bullet$ and $B_\bullet$ are concentrated in degrees n and $n+1$, and the map $\theta : B_{n+1} \oplus A_n \to B_n$ is an epimorphism. Let $X'_\bullet = X_\bullet \times_{Y_\bullet} B_\bullet$ and let $g' : X'_\bullet \to B_\bullet$ be the projection map. Note that g is a quasi-isomorphism if and only if $\ker(g) \simeq \ker(g')$ is an acyclic chain complex, which is equivalent to the statement that g' is a quasi-isomorphism. We may therefore replace g by g' and thereby reduce to the case $B_\bullet \simeq Y_\bullet$.

Let $\widetilde{A}_{n+1} = A_{n+1} \times_{B_{n+1}} X_{n+1}$, so that we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & I_{n+1} \arrow[r] \arrow[d] & \widetilde{A}_{n+1} \arrow[r] \arrow[d] & A_{n+1} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_{n+1} \arrow[r] & X_{n+1} \arrow[r] & B_{n+1} \arrow[r] & 0 \end{tikzcd}$$

Our map $A_\bullet \to X_\bullet$ determines a map $\gamma_0 : \widetilde{A}_{n+1} \to I_{n+1}$ splitting the upper sequence. Since the map $\widetilde{A}_{n+1} \to X_{n+1}$ is a monomorphism (being a pullback of the monomorphism $A_{n+1} \to B_{n+1}$) and I_{n+1} is injective, we can extend γ_0 to a map $\gamma : X_{n+1} \to I_{n+1}$. Let $\beta : B_{n+1} \to X_{n+1}$ be the unique section of $g_{n+1} : X_{n+1} \to B_{n+1}$ such that $\mathrm{im}(\beta) \simeq \ker(\gamma)$, so that β fits into a commutative diagram

$$\begin{tikzcd} A_{n+1} \arrow[r] \arrow[d] & X_{n+1} \arrow[d] \\ B_{n+1} \arrow[r] \arrow[ur, "\beta"] & B_{n+1}. \end{tikzcd}$$

If the map f is a quasi-isomorphism, then B_n is isomorphic to the pushout $A_n \coprod_{A_{n+1}} B_{n+1}$, so that the map β extends uniquely to a map of chain complexes $h : B_\bullet \to X_\bullet$ with the desired properties. If we assume instead that g is a quasi-isomorphism, we must work a bit harder. Let K be the kernel of the map $\theta : B_{n+1} \oplus A_n \to B_n$. Since θ is surjective, we note that β extends to a map of chain complexes $h : B_\bullet \to X_\bullet$ compatible with h_0 if and only if the map $\beta \oplus h_0 : B_{n+1} \oplus A_n \to X_n$ restricts to zero on K. The composition of this map with the epimorphism $X_n \to B_n$ coincides with θ, and therefore vanishes on K; consequently, we can identify $(\beta \oplus h_0)|K$ with a map $\phi : K \to \ker(X_n \to B_n) \simeq I_n$. It is easy to see that the image of ϕ lies in the kernel of the map $I_n \to I_{n-1}$. Since g is a quasi-isomorphism, $I_\bullet$ is an acyclic complex of injectives (which is concentrated in negative homological degrees) and therefore split exact. It follows that ϕ factors as a composition $K \xrightarrow{\phi'} I_{n+1} \to I_n$. Since the map $A_n \to B_n$ is a monomorphism, the composite map

$$K \to A_n \oplus B_{n+1} \to B_{n+1}$$

is also a homomorphism. The injectivity of I_{n+1} guarantees that ϕ' extends to a map $\epsilon : B_{n+1} \to I_{n+1}$. Replacing β by $\beta - \epsilon$, we can reduce to the situation where $\phi = 0$, thereby completing the proof. □

Proposition 5.3.11. *Let $\mathcal{A}$ be an abelian category, and suppose we are given cosimplicial objects $A^\bullet$, $B^\bullet$ of $\mathcal{A}$ such that each B^n is injective. Then $\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(A^\bullet, B^\bullet)$ is a Kan complex.*

Proof. It suffices to show that if $f : K \to K'$ is a trivial cofibration finite simplicial sets, then every map $A^\bullet \to (B^K)^\bullet$ factors through $(B^{K'})^\bullet$. Since f is a weak homotopy equivalence, the map $(B^{K'})^\bullet \to (B^K)^\bullet$ is a quasi-isomorphism and the desired result follows from Lemma 5.3.10. □

Proposition 5.3.12. *Let $\mathcal{A}$ be an abelian category, and suppose we are given a quasi-isomorphism $f : A^\bullet \to B^\bullet$ between cosimplicial objects of $\mathcal{A}$. Let $C^\bullet$ be another cosimplicial object of $\mathcal{A}$ such that each C^n is injective. Then composition with f induces a homotopy equivalence of Kan complexes*

$$\theta : \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(B^\bullet, C^\bullet) \to \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(A^\bullet, C^\bullet).$$

Proof. Let $C(f)^\bullet$ denote a mapping cylinder of f, so that f factors as a composition

$$A^\bullet \xrightarrow{f'} C(f)^\bullet \xrightarrow{f''} B^\bullet.$$

where f' is a monomorphism, f' and f'' are quasi-isomorphisms, and and f'' admits a section s. It will therefore suffice to show that composition with s and f' induce homotopy equivalences

$$\mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(A^\bullet, C^\bullet) \leftarrow \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(C(f)^\bullet, C^\bullet) \rightarrow \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(B^\bullet, C^\bullet).$$

Replacing f by f' or s, we can reduce to the case where the original map f is a monomorphism. In this case, we claim that θ is a trivial Kan fibration. In other words, for every monomorphism $g : K \to K'$ of finite simplicial sets, the map $(C^{K'})^\bullet \to (C^K)^\bullet$ has the right lifting property with respect to f. This is a special case of Lemma 5.3.10. $\square$

We now study symmetric monoidal structures on ∞-categories of chain complexes. Our starting point is the following observation.

Lemma 5.3.13. *Let $\mathcal{A}$ be a symmetric monoidal abelian category, and suppose that the tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact in each variable. Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism of cosimplicial objects of $\mathcal{A}$, and let $C^\bullet$ be an arbitrary cosimplicial object of $\mathcal{A}$. Then the induced map $A^\bullet \otimes C^\bullet \to B^\bullet \otimes C^\bullet$ is a quasi-isomorphism.*

Proof. Let $\mathcal{X}$ be the collection of all cochain complexes

$$E_0 \to E_{-1} \to E_{-2} \to \cdots$$

with values in $\mathcal{A}$ which have the following property:

- $(*)$ If $C^\bullet$ is a cosimplicial object of $\mathbf{A}$ whose normalized cochain complex is isomorphic to E_* and $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism, then the induced map $A^\bullet \otimes C^\bullet \to B^\bullet \otimes C^\bullet$ is also a quasi-isomorphism.

We wish to prove that $\mathcal{X} = \mathrm{Ch}_{\leq 0}(\mathcal{A})$. Arguing as in Lemma 5.3.8, we can reduce to proving that $\mathcal{X}$ contains all bounded chain complexes in $\mathrm{Ch}_{\leq 0}(\mathcal{A})$. Using Lemma 5.3.7 (and the exactness of the tensor product on $\mathcal{A}$), we are reduced to proving that $\mathcal{X}$ contains all chain complexes consisting of an object $E \in \mathcal{A}$ concentrated in a single degree k. Replacing $A^\bullet$ and $B^\bullet$ by their tensor product with E, we can reduce to the case where E is the unit object of $\mathcal{A}$. In this case, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & A^\bullet \otimes C^\bullet \arrow[r] \arrow[d] & (A^{\Delta^k})^\bullet \arrow[r] \arrow[d] & (A^{\partial\,\Delta^k})^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & B^\bullet \otimes C^\bullet \arrow[r] & (B^{\Delta^k})^\bullet \arrow[r] & (B^{\partial\,\Delta^k})^\bullet. \arrow[r] & 0
\end{tikzcd}$$

The desired result now follows from Lemmas 5.3.7 and 5.3.9. $\square$

Construction 5.3.14. Let $\mathcal{A}$ be an abelian category equipped with a symmetric monoidal structure, and assume that the tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact in each variable. Then $\otimes$ induces a symmetric monoidal structure on the simplicial category $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})$. Let $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ denote the full simplicial subcategory of $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})$ consisting of those cosimplicial objects $A^\bullet$ such that each A^n is an injective object of $\mathcal{A}$. Then $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ inherits the structure of a simplicial colored operad (Variation A.2.1.1.3), with

$$\mathrm{Mul}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A}^{\mathrm{inj}})}(\{A_i^\bullet\}, B^\bullet) \simeq \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta},\mathcal{A})}(\bigotimes A_i^\bullet, B^\bullet).$$

We let $\mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$ denote the operadic nerve of the simplicial colored operad $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$.

In the situation of Construction 5.3.14, Proposition A.2.1.1.27 guarantees that $\mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$ is an ∞-operad. We can often do better:

Proposition 5.3.15. *Let $\mathcal{A}$ be an abelian category with enough injective objects, and suppose we are given a symmetric monoidal structure on $\mathcal{A}$ such that the tensor product is exact in each variable. Then $\mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$ is a symmetric monoidal ∞-category.*

Proof. Let $p : \mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})) \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ denote the projection map; we wish to prove that p is a coCartesian fibration. Suppose we are given an object of $\mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$, consisting of a finite sequence of objects $(A_1^\bullet, \cdots, A_m^\bullet)$ of $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$. Let $\alpha : \langle m \rangle \to \langle n \rangle$ be a morphism in $\mathcal{F}\mathrm{in}_*$. For $1 \leq j \leq n$, choose a quasi-isomorphism $\bigotimes_{\alpha(i)=j} A_i^\bullet \to B_j^\bullet$ in $\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})$, where $B_j^\bullet$ is injective (such a quasi-isomorphism exists, by virtue of our assumption that $\mathcal{A}$ has enough injectives). These choices determine a morphism $\overline{\alpha} : (A_1^\bullet, \cdots, A_m^\bullet) \to (B_1^\bullet, \ldots, B_n^\bullet)$ in $\mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$. We claim that $\overline{\alpha}$ is p-coCartesian. Using Proposition T.2.4.4.3, we can reformulate this claim as follows: for every object $(C_1^\bullet, \ldots, C_p^\bullet) \in \mathrm{N}^{\otimes}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}))$ and every morphism $\beta : \langle n \rangle \to \langle p \rangle$ in $\mathcal{F}\mathrm{in}_*$, composition with $\overline{\alpha}$ induces a homotopy equivalence

$$\prod_{1 \leq k \leq p} \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})}(\bigotimes_{\beta(j)=k} B_j^\bullet, C_k) \to \prod_{1 \leq k \leq p} \mathrm{Map}_{\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A})}(\bigotimes_{(\beta\circ\alpha)(i)=k} A_i^\bullet, C_k^\bullet).$$

Since each $C_k^\bullet$ is injective, it suffices (by Proposition 5.3.12) to show that each of the maps $\bigotimes_{(\beta\circ\alpha)(i)=k} A_i^\bullet \to \bigotimes_{\beta(j)=k} B_j^\bullet$ is a quasi-isomorphism, which follows from Lemma 5.3.13. $\square$

5.4 Approximation by Connective Objects

Let $\mathcal{C}$ be a presentable stable ∞-category equipped with a t-structure. If the t-structure on $\mathcal{C}$ is right complete, then we can recover $\mathcal{C}$ as the homotopy inverse limit of the tower

$$\cdots \to \mathcal{C}_{\geq -2} \to \mathcal{C}_{\geq -1} \to \mathcal{C}_{\geq 0}$$

This homotopy inverse limit can be described as the stabilization of $\mathcal{C}_{\geq 0}$, or equivalently as the tensor product $\mathrm{Sp} \otimes \mathcal{C}_{\geq 0}$ (formed in the symmetric monoidal ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories).

Our goal in this section is to obtain an analogous result for ∞-categories which are not assumed to be presentable. Suppose that $\mathcal{C}$ is a stable ∞-category equipped with a *right-bounded* t-structure. Our main result (Proposition 5.4.3) asserts that we can recover $\mathcal{C}$ as $\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}_{\geq 0}$, where $\mathrm{Sp}^{\mathrm{fin}}$ denotes the ∞-category of *finite* spectra and the the tensor product is formed in the setting of ∞-categories which admit finite colimits (see Notation 5.4.2).

Lemma 5.4.1. *Let $\mathcal{C}$ be a small stable ∞-category equipped with a t-structure $(\mathcal{C}_{\geq 0}, \mathcal{C}_{\leq 0})$. Then $\mathrm{Ind}(\mathcal{C})$ inherits a t-structure $(\mathrm{Ind}(\mathcal{C})_{\geq 0}, \mathrm{Ind}(\mathcal{C})_{\leq 0})$, where $\mathrm{Ind}(\mathcal{C})_{\geq 0}$ is the essential image of the fully faithful functor $\mathrm{Ind}(\mathcal{C}_{\geq 0}) \to \mathrm{Ind}(\mathcal{C})$, and $\mathrm{Ind}(\mathcal{C})_{\leq 0}$ is defined similarly. If the t-structure on $\mathcal{C}$ is right bounded, then the t-structure on $\mathrm{Ind}(\mathcal{C})$ is right complete.*

Proof. The first assertion is straightforward. To prove the second, let us assume that the t-structure on $\mathcal{C}$ is right bounded. It is clear from the construction that $\mathrm{Ind}(\mathcal{C})_{\leq 0}$ is closed under filtered colimits in $\mathrm{Ind}(\mathcal{C})$. To prove that $\mathrm{Ind}(\mathcal{C})$ is right complete, it will suffice to show that the intersection $\bigcap_n \mathrm{Ind}(\mathcal{C})_{\leq -n}$ consists only of zero objects of $\mathrm{Ind}(\mathcal{C})$ (Proposition A.1.2.1.19). To this end, let us suppose that $X \in \bigcap_n \mathrm{Ind}(\mathcal{C})_{\leq -n}$. Then $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(Y, X)$ is contractible for any $Y \in \bigcup_n \mathrm{Ind}(\mathcal{C})_{\geq -n}$, and therefore for any Y belonging to the essential image of the Yoneda embedding $j : \mathcal{C} \to \mathrm{Ind}(\mathcal{C})$. Since $\mathrm{Ind}(\mathcal{C})$ is generated under filtered colimits by the essential image of j, we conclude that $\mathrm{Map}_{\mathrm{Ind}(\mathcal{C})}(Y, X)$ is contractible for all Y and therefore X is a final object of $\mathrm{Ind}(\mathcal{C})$. $\square$

Notation 5.4.2. Let $\mathcal{K}$ denote the collection of all finite simplicial sets and let $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ denote the subcategory of $\mathcal{C}\mathrm{at}_\infty$ spanned by those ∞-categories which admit finite colimits and those functors which preserve finite colimits. We will regard $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ as a symmetric monoidal ∞-category: if $\mathcal{C}$ and $\mathcal{D}$ are objects of $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$, then the tensor product $\mathcal{C} \otimes \mathcal{D}$ is universal among ∞-categories $\mathcal{E}$ which are equipped with a functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ which preserves finite colimits separately in each variable. We will regard the construction $\mathrm{Ind} : \mathcal{C}\mathrm{at}_\infty(\mathcal{K}) \to \mathcal{P}\mathrm{r}^{\mathrm{L}}$ as a symmetric monoidal functor from $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ to the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}$ of presentable ∞-categories.

Proposition 5.4.3. *Let $\mathcal{C}$ be a stable ∞-category equipped with a right-bounded t-structure. Then there is a canonical equivalence $\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}_{\geq 0} \simeq \mathcal{C}$ (where the tensor product is formed in the ∞-category $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ of Notation 5.4.2).*

Proof. Without loss of generality, we may assume that $\mathcal{C}$ is small. We note that $\mathrm{Ind}(\mathcal{C})$ is a stable ∞-category, which inherits a right complete t-structure (Lemma 5.4.1). It follows that $\mathrm{Ind}(\mathcal{C})$ can be identified with the stabilization of the ∞-category $\mathrm{Ind}(\mathcal{C})_{\geq 0}$, which (according to Example A.6.3.1.22) is given by $\mathrm{Sp} \otimes \mathrm{Ind}(\mathcal{C})_{\geq 0} \simeq \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}_{\geq 0})$. It follows that there are fully faithful embeddings

$$\mathcal{C} \xrightarrow{j} \mathrm{Ind}(\mathcal{C}) \xleftarrow{j'} \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}_{\geq 0}.$$

We will complete the proof by showing that the essential images of j and j' are the same. Let $\boxtimes : \mathrm{Sp}^{\mathrm{fin}} \times \mathcal{C}_{\geq 0} \to \mathrm{Ind}(\mathcal{C})$ be the functor classified by j'. It is clear that the essential image of j is contained in the essential image of j': if $C \in \mathcal{C}$, then $C[n] \in \mathcal{C}_{\geq 0}$ for $n \gg 0$, so that $j(C) \simeq j'(S^{-n} \boxtimes C[n])$. We now prove the reverse inclusion. Since $\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}_{\geq 0}$ is generated under finite colimits by the essential image of $\boxtimes$ (and the functors j and j' preserve finite colimits), it suffices to show that $j'(K \boxtimes C)$ belongs to the essential image of j, for every pair $(K, C) \in \mathrm{Sp}^{\mathrm{fin}} \times \mathcal{C}_{\geq 0}$. Fixing C, we see that the collection of finite spectra K for which the conclusion holds is closed under translations and finite colimits; it therefore suffices to treat the case where K is the sphere spectrum S. In this case, $j'(K \boxtimes C) \simeq j(C)$. □

Corollary 5.4.4. *Let $\mathcal{C}$ be a stable ∞-category. Then the inclusion $\{S\} \hookrightarrow \mathrm{Sp}^{\mathrm{fin}}$ induces an equivalence $\mathcal{C} \to \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}$.*

Proof. We can regard $\mathcal{C}$ as equipped with a right-bounded t-structure by setting $\mathcal{C}_{\geq 0} = \mathcal{C}$; the desired result then follows from Proposition 5.4.3. □

Remark 5.4.5. Let $\mathcal{C}$ be an ∞-category which admits finite colimits. Then $\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}$ is stable. To see this, we note that $\mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}) \simeq \mathrm{Sp} \otimes \mathrm{Ind}(\mathcal{C})$ is equivalent to $\mathrm{Stab}(\mathrm{Ind}(\mathcal{C}))$ (Example A.6.3.1.22), and therefore stable. It therefore suffices to show that the essential image of the Yoneda embedding $j : \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C} \to \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C})$ is closed under translation and finite colimits. Closure under finite colimits is obvious. Closure under translation follows from the commutativity of the diagram

$$\begin{tikzcd}
\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C} \arrow[r] \arrow[d, "\Sigma^n \otimes \mathrm{id}"] & \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}) \arrow[d, "\Sigma^n"] \\
\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C} \arrow[r] & \mathrm{Ind}(\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}).
\end{tikzcd}$$

Proposition 5.4.6. *Let $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ be the full subcategory of $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ spanned by the stable ∞-categories. Then the inclusion $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}} \subseteq \mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ admits a left adjoint, given by $\mathcal{C} \mapsto \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}$.*

Proof. Let $L : \mathcal{C}\mathrm{at}_\infty(\mathcal{K}) \to \mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ be the functor given by $\mathcal{C} \mapsto \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}$. There is an evident natural transformation $\alpha : \mathrm{id} \to L$. Note that $\alpha \otimes \mathrm{id}$ and $\mathrm{id} \otimes \alpha$ induce equivalences $L \to L^2$ (since the natural maps $\mathrm{Sp}^{\mathrm{fin}} \to \mathrm{Sp}^{\mathrm{fin}} \otimes \mathrm{Sp}^{\mathrm{fin}}$ are equivalences by Corollary 5.4.4), so that L is a localization functor. The essential image of L is contained in $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ (by Remark 5.4.5) and contains $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ (by Corollary 5.4.4). □

Remark 5.4.7. It is clear that the localization of Proposition 5.4.6 is compatible with the symmetric monoidal structure on $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$: that is, if $f : \mathcal{C} \to \mathcal{D}$ is a functor which preserves finite colimits and induces an equivalence $\mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C} \to \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{D}$, then the induced map $\mathcal{C} \otimes \mathcal{E} \to \mathcal{D} \otimes \mathcal{E}$ has the same property, for every object $\mathcal{E} \in \mathcal{C}\mathrm{at}_\infty(\mathcal{K})$. It follows from Proposition A.2.2.1.9 that $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$ inherits the structure of a symmetric monoidal ∞-category, and that the construction $\mathcal{C} \mapsto \mathrm{Sp}^{\mathrm{fin}} \otimes \mathcal{C}$ determines a symmetric monoidal functor from $\mathcal{C}\mathrm{at}_\infty(\mathcal{K})$ to $\mathcal{C}\mathrm{at}_\infty^{\mathrm{Ex}}$.

Corollary 5.4.8. *Let $\mathcal{C}$ be a stable ∞-category equipped with a right-bounded t-structure. Suppose that $\mathcal{C}_{\geq 0}$ has the structure of a symmetric monoidal ∞-category, such that the tensor product on $\mathcal{C}_{\geq 0}$ preserves finite colimits in each variable. Then there exists a symmetric monoidal structure on $\mathcal{C}$ with the following properties:*

(1) *The tensor product on $\mathcal{C}$ is exact in each variable.*

(2) *The inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}$ lifts to a symmetric monoidal functor f.*

(3) *Let $\mathcal{D}$ be any symmetric monoidal stable ∞-category for which the tensor product is exact in each variable. Then composition with f induces a homotopy equivalence*

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))}(\mathcal{C}_{\geq 0}, \mathcal{D}).$$

Corollary 5.4.9. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category, and assume that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in each variable. Assume that $\mathcal{C}$ is equipped with a right bounded t-structure, that the unit object of $\mathcal{C}$ belongs to $\mathcal{C}_{\geq 0}$, and that $\mathcal{C}_{\geq 0}$ is stable under tensor products.*

Let $\mathcal{D}$ be a symmetric monoidal stable ∞-category, and suppose that the tensor product on $\mathcal{D}$ preserves finite colimits in each variable. Let $\mathrm{Fun}_0^{\otimes}(\mathcal{C}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ spanned by those symmetric monoidal functors which preserve finite colimits, and let $\mathrm{Fun}_0^{\otimes}(\mathcal{C}_{\geq 0}, \mathcal{D})$ be defined similarly. Then the restriction functor

$$\theta : \mathrm{Fun}_0^{\otimes}(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}_0^{\otimes}(\mathcal{C}_{\geq 0}, \mathcal{D})$$

is an equivalence of ∞-categories.

Proof. Let $\mathcal{C}'$ be the ∞-category $\mathcal{C}$, endowed with the symmetric monoidal structure described in Corollary 5.4.8. Using Corollary 5.4.8, we see that symmetric monoidal inclusion $\mathcal{C}_{\geq 0} \hookrightarrow \mathcal{C}'$ induces a symmetric monoidal equivalence $\mathcal{C} \to \mathcal{C}'$.

To complete the proof, it suffices to show that for every simplicial set K, composition with θ induces a homotopy equivalence of underlying Kan complexes

$$\mathrm{Fun}(K, \mathrm{Fun}_0^{\otimes}(\mathcal{C}, \mathcal{D}))^{\simeq} \to \mathrm{Fun}(K, \mathrm{Fun}_0^{\otimes}(\mathcal{C}_{\geq 0}, \mathcal{D}))^{\simeq}.$$

Replacing $\mathcal{D}$ by $\mathrm{Fun}(K, \mathcal{D})$, we are reduced to proving that the restriction map

$$\mathrm{Map}_{\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))}(\mathcal{C}, \mathcal{D}) \to \mathrm{Map}_{\mathrm{CAlg}(\mathcal{C}\mathrm{at}_\infty(\mathcal{K}))}(\mathcal{C}_{\geq 0}, \mathcal{D})$$

is a homotopy equivalence. This follows from Corollary 5.4.8. $\square$

5.5 Approximation by Abelian Categories

Let $\mathcal{C}$ be a stable ∞-category equipped with a right complete t-structure, and let $\mathcal{A}$ denote the heart of $\mathcal{C}$. Assume that the abelian category $\mathcal{A}$ has enough injective objects, so that the derived ∞-category $\mathcal{D}^+(\mathcal{A})$ is defined (see §A.1.3.1). Using Theorem A.1.3.2.2, we see that the inclusion $\mathrm{N}(\mathcal{A}) \hookrightarrow \mathcal{C}$ extends to an exact functor $\phi : \mathcal{D}^-(\mathcal{A}) \to \mathcal{C}$ between stable ∞-categories. Our goal in this section is to prove a symmetric monoidal analogue of this result. Let us suppose that the stable ∞-category $\mathcal{C}$ is equipped with a symmetric monoidal structure such that the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is t-exact in each variable. Then the abelian category $\mathcal{A}$ inherits a symmetric monoidal structure (and the tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact in each variable). Combining the results of §5.3 with the dual of Corollary 5.4.8, we deduce that $\mathcal{D}^-(\mathcal{A})$ inherits a symmetric monoidal structure. We will show that ϕ can be promoted to a symmetric monoidal functor (Theorem 5.5.5).

Notation 5.5.1. Let $\mathcal{A}$ be a symmetric monoidal abelian category. Assume that $\mathcal{A}$ has enough injective objects and that the tensor product functor $\otimes : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is exact in each variable. Let $\mathrm{N}^{\otimes} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ be as in Construction 5.3.14, and let $\mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \subseteq \mathrm{N}^{\otimes} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ be the full subcategory spanned by objects of the form $(A_1^\bullet, \ldots, A_n^\bullet)$, where the cohomology objects $\mathrm{H}^m(A_i^\bullet)$ vanish for $m > 0$ and $1 \leq i \leq n$. Then $\mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ is a symmetric monoidal ∞-category, and the inclusion $\mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \subseteq \mathrm{N}^{\otimes} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$ is symmetric monoidal. Moreover, the construction $A^\bullet \mapsto \mathrm{H}^0(A^\bullet)$ determines a symmetric monoidal functor $\mathrm{N}^{\otimes} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \to \mathrm{N}(\mathcal{A})^{\otimes}$, which restricts to a symmetric monoidal equivalence $\mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \to \mathrm{N}(\mathcal{A})^{\otimes}$.

The ∞-operad $\mathrm{N}^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})$ is defined as the nerve of a simplicial category $\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})^{\otimes}$, whose objects are finite sequences $(A_1^{\bullet},\cdots,A_n^{\bullet})$. Let $\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})_0^{\otimes}$ denote the discrete category underlying $\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})^{\otimes}$. We will denote the nerve of this category by $\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})$. The inclusion

$$\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})\hookrightarrow \mathrm{N}^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})$$

is a map of ∞-operads. Note that $\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})$ is generally not a symmetric monoidal ∞-category, even when the hypotheses of Proposition are satisfied.

Proposition 5.5.2. *Let $\mathcal{A}$ be an abelian category with enough injective objects, and suppose that $\mathcal{A}$ is endowed with a symmetric monoidal structure such that the tensor product on $\mathcal{A}$ is exact in each variable. Let $\mathcal{C}^{\otimes}$ be a symmetric monoidal ∞-category. Then the restriction functor*

$$\theta:\operatorname{Fun}^{\otimes}(\mathrm{N}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}}),\mathcal{C})\to \operatorname{Alg}_{\mathrm{N}_0\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})}(\mathcal{C})$$

is fully faithful. Moreover, a map of ∞-operads $f:\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})\to\mathcal{C}^{\otimes}$ belongs to the essential image of θ if and only if f satisfies the following condition:

- $(*)$ *For every morphism $\alpha:(A_1^{\bullet},\ldots,A_n^{\bullet})\to(B^{\bullet})$ in $\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})$ which induces a quasi-isomorphism $\bigotimes A_i^{\bullet}\to B^{\bullet}$, the image $f(\alpha)$ is an equivalence in $\mathcal{C}^{\otimes}$.*

Proof. The simplicial category $\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})^{\otimes}$ admits path objects. It follows from Example A.1.3.3.21 that the restriction map

$$\operatorname{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}}),\mathcal{C}^{\otimes})\to\operatorname{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}}),\mathcal{C}^{\otimes})$$

is a fully faithful embedding, whose essential image consists of those functors $f:\mathrm{N}_0^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})\to\mathcal{C}^{\otimes}$ which satisfy the following property:

- $(*')$ For every equivalence α in $\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})^{\otimes}$, the image $f(\alpha)$ is an equivalence in $\mathcal{C}^{\otimes}$.

It follows immediately that θ is fully faithful. It is clear from the proof of Proposition 5.3.15 that every functor belonging to the essential image of θ satisfies condition $(*)$. Conversely, suppose that $f\in\operatorname{Alg}_{\mathrm{N}_0\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})}(\mathcal{C})$ satisfies $(*)$. In particular, f carries quasi-isomorphisms in $\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})$ to equivalences in $\mathcal{C}$. Since f is a map of ∞-operads, we conclude that f satisfies $(*')$ and therefore extends (in an essentially unique way) to a functor $\overline{f}:\mathrm{N}^{\otimes}\operatorname{Fun}(\boldsymbol{\Delta},\mathcal{A}^{\mathrm{inj}})\to\mathcal{C}^{\otimes}$. This functor is evidently a map of ∞-operads, and condition $(*)$ guarantees that it is a symmetric monoidal functor. $\square$

Proposition 5.5.3. *Let $\mathcal{A}$ be an abelian category, $\mathcal{C}$ a stable ∞-category equipped with a right complete t-structure, and let $F:\mathrm{N}(\mathcal{A})\to\mathcal{C}_{\leq 0}$ be a functor with the following properties:*

1. *The functor F carries zero objects of $\mathcal{A}$ to zero objects of $\mathcal{C}$.*
2. *For every short exact sequence $0\to A'\to A\to A''\to 0$ in $\mathcal{A}$, the diagram*

$$\begin{tikzcd} F(A') \arrow[r] \arrow[d] & F(A) \arrow[d] \\ F(0) \arrow[r] & F(A'') \end{tikzcd}$$

is a pullback square in $\mathcal{C}$.

Then:

(a) *The ∞-category $\mathcal{C}_{\leq 0}$ admits limits of cosimplicial objects.*

(*b*) *For every quasi-isomorphism $\alpha : A^\bullet \to B^\bullet$ between cosimplicial objects of $\mathcal{A}$, the induced map of totalizations*

$$\operatorname{Tot} F(A^\bullet) \to \operatorname{Tot} F(B^\bullet)$$

is an equivalence in $\mathcal{C}$.

Proof. Assertion (*a*) follows from Lemma A.1.3.2.10. To prove (*b*), it will suffice to prove the following:

(b') Fix integers $n, k \geq 0$. Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism of cosimplicial objects of $\mathcal{A}$ satisfying $A^m \simeq 0 \simeq B^m$ for $m < k$. Then the map $\tau_{\geq -n} \operatorname{Tot} F(A^\bullet) \to \tau_{\geq -n} \operatorname{Tot} F(B^\bullet)$ is an equivalence.

Since $\mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq -n}$ is equivalent to an $(n+1)$-category, if $k > n+1$ then the argument of Proposition 4.3.5 gives equivalences

$$\tau_{\geq -n} \operatorname{Tot} F(A^\bullet) \simeq \tau_{\geq -n} \varprojlim_{[m] \in \mathbf{\Delta}_{\leq n+1,s}} F(A^m) \simeq 0$$

$$\tau_{\geq -n} \operatorname{Tot} F(B^\bullet) \simeq \tau_{\geq -n} \varprojlim_{[m] \in \mathbf{\Delta}_{\leq n+1,s}} F(B^m) \simeq 0$$

We prove (b') in general using descending induction on k. Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism with $A^m \simeq 0 \simeq B^m$ for $m < k$. Let $(X^\bullet, d)$ be the normalized cochain complex associated to $A^\bullet$, $(Y^\bullet, d')$ the normalized cochain complex associated to $B^\bullet$, and K the kernel of the map $X^k \to X^{k+1}$. Since f is a quasi-isomorphism, K is also a kernel of the map $Y^k \to Y^{k+1}$. Let $C^\bullet$ be the cosimplicial object of $\mathcal{A}$ whose unnormalized chain complex consists of the object K concentrated in (cohomological) degree k. We have a commutative diagram of short exact sequences

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C^\bullet \arrow[r] \arrow[d] & A^\bullet \arrow[r] \arrow[d] & A^\bullet/C^\bullet \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & C^\bullet \arrow[r] & B^\bullet \arrow[r] & B^\bullet/C^\bullet \arrow[r] & 0.
\end{tikzcd}$$

It follows from conditions (1) and (2) that the induced diagram

$$\begin{tikzcd}
\operatorname{Tot} F(A^\bullet) \arrow[r] \arrow[d] & \operatorname{Tot} F(A^\bullet/C^\bullet) \arrow[d] \\
\operatorname{Tot} F(B^\bullet) \arrow[r] & \operatorname{Tot} F(B^\bullet/C^\bullet)
\end{tikzcd}$$

is a pullback square. Consequently, we may replace $A^\bullet$ by $A^\bullet/C^\bullet$ and $B^\bullet$ by $B^\bullet/C^\bullet$, and thereby reduce to the case $K = 0$.

Let $A''^\bullet$ be the cosimplicial object of $\mathcal{A}$ whose normalized chain complex is given by

$$0 \to \cdots \to 0 \to \operatorname{coker}(X^k \to X^{k+1}) \to X^{k+2} \to \cdots,$$

so that we have a short exact sequence of cosimplicial objects

$$0 \to A'^\bullet \to A^\bullet \to A''^\bullet \to 0,$$

where the unnormalized chain complex of $A'^\bullet$ is isomorphic to

$$\cdots \to 0 \to X^k \xrightarrow{\mathrm{id}} X^k \to \cdots$$

Using Proposition A.1.2.4.6, we deduce that $\varprojlim F(A'^\bullet) \simeq 0$. Conditions (1) and (2) then guarantee that the map $\varprojlim F(A^\bullet) \to \varprojlim F(A''^\bullet)$ is an equivalence. The same reasoning determines a map of cosimplicial objects $B^\bullet \to B''^\bullet$ which induces an equivalence $\mathrm{Tot}\, F(B^\bullet) \to \varprojlim F(B''^\bullet)$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Tot}\, F(A^\bullet) \arrow[r] \arrow[d] & \mathrm{Tot}\, F(A''^\bullet) \arrow[d] \\
\mathrm{Tot}\, F(B^\bullet) \arrow[r] & \mathrm{Tot}\, F(B''^\bullet)
\end{tikzcd}$$

where the horizontal maps are equivalences. We may therefore replace $A^\bullet$ by $A''^\bullet$ and $B^\bullet$ by $B''^\bullet$, in which case the desired result follows from the inductive hypothesis. $\square$

Lemma 5.5.4. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category equipped with a right-complete t-structure. Assume that the tensor product on $\mathcal{C}$ is exact in each variable, carries $\mathcal{C}_{\leq 0} \times \mathcal{C}_{\leq 0}$ into $\mathcal{C}_{\leq 0}$, and that $\mathcal{C}_{\leq 0}$ contains the unit object of $\mathcal{C}_{\leq 0}$. Set*

$$\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}_{\leq 0})^{\otimes} = \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}^{\otimes}_{\leq 0}) \times_{\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathrm{N}(\mathcal{F}\mathrm{in}_*))} \mathrm{N}(\mathcal{F}\mathrm{in}_*),$$

and let $g : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}_{\leq 0}) \to \mathcal{C}_{\leq 0}$ be a right adjoint to the diagonal map (which exists by Proposition 5.5.3). Then g can be promoted to a symmetric monoidal functor $G : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}_{\leq 0})^{\otimes} \to \mathcal{C}^{\otimes}_{\leq 0}$ (which is right adjoint to the diagonal map $\mathcal{C}^{\otimes}_{\leq 0} \to \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}_{\leq 0})^{\otimes}$).

Proof. The existence of G as a map of ∞-operads follows from Proposition 3.2.1. Since $\mathrm{N}(\boldsymbol{\Delta})$ is weakly contractible, the functor G carries the unit object of $\mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}_{\leq 0})$ to the unit object of $\mathcal{C}_{\leq 0}$. To complete the proof, it suffices to show that for every pair of cosimplicial objects $A^\bullet$ and $B^\bullet$ of $\mathcal{C}_{\leq 0}$, the canonical map $\mathrm{Tot}\, A^\bullet \otimes \mathrm{Tot}\, B^\bullet \to \mathrm{Tot}(A^\bullet \otimes B^\bullet)$ is an equivalence. Since $\mathrm{N}(\boldsymbol{\Delta})^{op}$ is sifted, it suffices to show that the tensor product on $\mathcal{C}_{\leq 0}$ preserves $\mathrm{N}(\boldsymbol{\Delta})$-indexed limits separately in each variable. Because the t-structure on $\mathcal{C}$ is right complete, it suffices to prove the analogous assertion for $\mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq -n}$ for each $n \geq 0$. In this case, we can replace $\mathrm{N}(\boldsymbol{\Delta})$ by $\mathrm{N}(\boldsymbol{\Delta}_{\leq n+2})$ (Proposition 4.3.5). We are therefore reduced to proving that the tensor product on $\mathcal{C}_{\leq 0}$ preserves finite limits separately in each variable, which follows from our assumption that the tensor product on $\mathcal{C}$ is exact in each variable. $\square$

Theorem 5.5.5. *Let $p : \mathcal{C}^{\otimes} \to \mathrm{N}(\mathcal{F}\mathrm{in}_*)$ be a symmetric monoidal stable ∞-category. Assume that $\mathcal{C}$ is equipped with a right complete t-structure, such that $\mathcal{C}_{\leq 0}$ contains the unit object and is stable under tensor products. Let $G : \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta}), \mathcal{C}_{\leq 0})^{\otimes} \to \mathcal{C}^{\otimes}_{\leq 0}$ be the symmetric monoidal functor of Lemma 5.5.4.*

Let $\mathcal{A}$ be a symmetric monoidal abelian category. Assume that the tensor product on $\mathcal{A}$ is exact in each variable and that $\mathcal{A}$ has enough injective objects. Let $f : \mathrm{N}(\mathcal{A}) \to \mathcal{C}_{\leq 0}$ be a symmetric monoidal functor satisfying the hypotheses of Proposition 5.5.3. Then the composite map

$$f' : \mathrm{N}^{\otimes}_0(\mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}})) \stackrel{f}{\hookrightarrow} \mathrm{Fun}(\mathrm{N}(\boldsymbol{\Delta})^{op}, \mathcal{C}_{\leq 0})^{\otimes} \stackrel{G}{\to} \mathcal{C}^{\otimes}_{\leq 0}$$

admits an essentially unique factorization

$$\mathrm{N}^{\otimes}_0 \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}}) \to \mathrm{N}^{\otimes} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}}) \stackrel{F}{\to} \mathcal{C}^{\otimes}_{\leq 0},$$

where F is a symmetric monoidal functor. Moreover, if $\mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}}) \subseteq \mathrm{N}^{\otimes} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}})$ is defined as in Notation 5.5.1, then the restriction $F | \mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}})$ is given by the composition

$$f'' : \mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}}) \simeq \mathrm{N}(\mathcal{A})^{\otimes} \stackrel{f}{\to} \mathcal{C}^{\otimes}_{\leq 0}.$$

Proof. To prove the existence of F, it will suffice to show that f' satisfies condition $(*)$ of Proposition 5.5.2. This follows from Lemma 5.5.4 and Proposition 5.5.3. To prove the last assertion, let $\mathrm{N}^{\otimes}_{\mathrm{acyc},0} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}})$ denote the full subcategory of $\mathrm{N}^{\otimes}_0 \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}})$ spanned by those objects $(A^\bullet_1, \ldots, A^\bullet_n)$ such that $\mathrm{H}^m(A^\bullet_i) \simeq 0$ for $m > 0$. Example A.1.3.3.21 implies that the restriction functor

$$\mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}^{\otimes}_{\mathrm{acyc}} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}}), \mathcal{C}^{\otimes}_{\leq 0}) \to \mathrm{Fun}_{\mathrm{N}(\mathcal{F}\mathrm{in}_*)}(\mathrm{N}^{\otimes}_{\mathrm{acyc},0} \mathrm{Fun}(\boldsymbol{\Delta}, \mathcal{A}^{\mathrm{inj}}), \mathcal{C}^{\otimes}_{\leq 0})$$

is fully faithful. Consequently, to prove that $f'' \simeq F|\mathrm{N}^{\otimes}_{\mathrm{acyc}}\,\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$, it will suffice to show that $F|\mathrm{N}^{\otimes}_{\mathrm{acyc},0}\,\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \simeq U = f''|\mathrm{N}^{\otimes}_{\mathrm{acyc},0}\,\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$. For each $n \geq 0$, let U^n be the composite functor

$$\mathrm{N}^{\otimes}_{\mathrm{acyc},0}\,\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \to \mathrm{N}(\mathcal{A})^{\otimes} \xrightarrow{f} \mathcal{C}^{\otimes}_{\leq 0}$$

where the first map is given by evaluation at $[n] \in \mathbf{\Delta}$. The functor U is given by $U(A_1^{\bullet}, \ldots, A_n^{\bullet}) = f(\mathrm{H}^0(A_1^{\bullet}), \ldots, \mathrm{H}^0(A_n^{\bullet}))$; in particular, we have a canonical natural transformation $\alpha : U \to \mathrm{Tot}\,U^{\bullet} \simeq F|\mathrm{N}^{\otimes}_{\mathrm{acyc},0}\,\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})$. To complete the proof, it will suffice to show that α is an equivalence. Unwinding the definitions, we must show that if $A^{\bullet}$ is a cosimplicial object of $\mathcal{A}^{\mathrm{inj}}$ such that the cohomology groups $\mathrm{H}^m(A^{\bullet})$ vanish for $m > 0$, then the canonical map $f(\mathrm{H}^0(A^{\bullet})) \to \mathrm{Tot}\,f(A^{\bullet})$ is an equivalence. This is a special case of Proposition 5.5.3. $\square$

5.6 Quasi-Coherent Sheaves on Generalized Algebraic Gerbes

Let X be either a geometric stack or a coaffine stack over a field of characteristic zero. Then X can be recovered from the symmetric monoidal ∞-category $\mathrm{QCoh}(X)$ (see Theorem 3.4.2 and Corollary 4.6.19). If Y is a generalized algebraic gerbe over a field k of characteristic zero, then we can regard Y as an extension of an algebraic gerbe by a coaffine stack (see Definition 5.2.1). It is therefore natural to expect that Y can be recovered from the ∞-category $\mathrm{QCoh}(Y)$, together with its symmetric monoidal structure. In this section, we will prove that this is indeed the case. One of our main results can be stated as follows:

Proposition 5.6.1. *Let X be a generalized algebraic gerbe and let A be a connective $\mathbb{E}_{\infty}$-ring. Then the construction*

$$(\eta \in X(A)) \mapsto (\eta^* : \mathrm{QCoh}(X)^{\otimes} \to \mathrm{Mod}_A^{\otimes})$$

determines a fully faithful embedding $X(A) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\otimes}, \mathrm{Mod}_A^{\otimes})$, whose essential image is the collection of symmetric monoidal functors $F : \mathrm{QCoh}(X) \to \mathrm{Mod}_A$ which preserve small colimits and connective objects.

We will give the proof of Proposition 5.6.1 later in this section.

Corollary 5.6.2. *Let $X, Y : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be functors. Assume that X is a generalized algebraic gerbe. Then the canonical map*

$$\mathrm{Map}_{\mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \widehat{\mathcal{S}})}(Y, X) \to \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y))$$

is fully faithful, and its essential image is the collection of symmetric monoidal functors $f : \mathrm{QCoh}(X)^{\otimes} \to \mathrm{QCoh}(Y)^{\otimes}$ which preserve small colimits and carry $\mathrm{QCoh}(X)^{\mathrm{cn}}$ into $\mathrm{QCoh}(Y)^{\mathrm{cn}}$.

Let X be a generalized algebraic gerbe. We will prove below that the ∞-category $\mathrm{QCoh}(X)$ is presentable, and that the full subcategory $\mathrm{QCoh}(X)$ determines an accessible t-structure on $\mathrm{QCoh}(X)$ (Proposition 5.6.18). We may therefore view the pair $(\mathrm{QCoh}(X), \mathrm{QCoh}(X)^{\mathrm{cn}})$ as a commutative algebra object of the ∞-category $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ (see Notation 4.6.1). Combining this observation with Corollary 5.6.2, we deduce:

Corollary 5.6.3. *Let $\mathrm{Grb} \subseteq \mathrm{Fun}(\mathrm{CAlg}^{\mathrm{cn}}, \mathcal{S})$ be the full subcategory spanned by the generalized gerbes. Then the construction $X \mapsto \mathrm{QCoh}(X)^{\otimes}$ determines a fully faithful embedding $\mathrm{Grb} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$.*

In fact, we can be more precise: it is possible to give an explicit description of the essential image of the fully faithful embedding $\mathrm{Grb} \to \mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})$ (Theorem 5.6.19). For this, we need to introduce a bit of terminology.

Definition 5.6.4. A *locally dimensional ∞-category* is a symmetric monoidal stable ∞-category $\mathcal{C}^{\otimes}$ equipped with a t-structure satisfying the following additional conditions:

(1) The ∞-category $\mathcal{C}$ is presentable and the tensor product $\otimes$ preserves colimits separately in each variable.

(2) The t-structure on $\mathcal{C}$ is excellent: that is, it is an accessible t-structure which is both right complete and left complete, and the full subcategory $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ is closed under filtered colimits.

(3) The unit object $\mathbf{1} \in \mathcal{C}$ belongs to the heart of $\mathcal{C}$, and the tensor product functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is t-exact: that is, the subcategories $\mathcal{C}_{\geq 0}$ and $\mathcal{C}_{\leq 0}$ are both closed under tensor products.

(4) The unit object $\mathbf{1}$ is a compact object of $\mathcal{C}_{\leq n}$ for each $n \geq 0$.

(5) There exists an abelian subcategory $\mathcal{C}^{\heartsuit}_{\mathrm{fd}} \subseteq \mathcal{C}^{\heartsuit}$ which generates $\mathcal{C}^{\heartsuit}$ under filtered colimits, such that every object $V \in \mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ admits a dual $V^{\vee}$ in $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$.

(6) The commutative ring $k = \pi_0 \operatorname{Map}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$ is a field of characteristic zero.

(7) For every object $V \in \mathcal{C}^{\heartsuit}_{\mathrm{fd}}$, the composition

$$\mathbf{1} \xrightarrow{c} V \otimes V^{\vee} \xrightarrow{e} \mathbf{1}$$

corresponds to a nonnegative integer $\dim(V) \in \mathbf{Z} \subseteq \mathbf{Q} \hookrightarrow k$. Here c and e denote compatible coevaluation and evaluation maps for the duality between V and $V^{\vee}$.

We begin with some general observations about the class of locally dimensional ∞-categories.

Remark 5.6.5. Let $\mathcal{C}^{\otimes}$ be a locally dimensional ∞-category. Suppose that $V \in \mathcal{C}_{\geq 0}$ admits a dual $V^{\vee} \in \mathcal{C}$. For $W \in \mathcal{C}_{\geq 1}$, we have $\operatorname{Map}_{\mathcal{C}}(W, V^{\vee}) \simeq \operatorname{Map}_{\mathcal{C}}(W \otimes V, \mathbf{1}) \simeq 0$ since $W \otimes V \in \mathcal{C}_{\geq 1}$ and $\mathbf{1} \in \mathcal{C}_{\leq 0}$. It follows that $V^{\vee} \in \mathcal{C}_{\leq 0}$.

Remark 5.6.6. Let $\mathcal{C}^{\otimes}$ be a locally dimensional ∞-category. Then every dualizable object $V \in \mathcal{C}^{\heartsuit}$ is a compact object of $\mathcal{C}_{\leq n}$ for $n \geq 0$. To see this, it suffices to show that the functor $W \mapsto \operatorname{Map}_{\mathcal{C}}(V, W)$ commutes with filtered colimits (when restricted to $\mathcal{C}_{\leq n}$). Since V is dualizable, we can rewrite this functor as $W \mapsto \operatorname{Map}_{\mathcal{C}}(\mathbf{1}, V^{\vee} \otimes W)$. Since tensor product with $V^{\vee}$ carries $\mathcal{C}_{\leq n}$ to itself (Remark 5.6.5) and commutes with colimits, the desired conclusion follows from the assumption that $\mathbf{1}$ is a compact object of $\mathcal{C}_{\leq n}$.

Remark 5.6.7. Let $\mathcal{C}^{\otimes}$ be a locally dimensional ∞-category. It follows from Remark 5.6.6 and Proposition T.5.3.5.11 that the inclusion $\mathcal{C}^{\heartsuit}_{\mathrm{fd}} \subseteq \mathcal{C}^{\heartsuit}$ induces a fully faithful embedding $i : \operatorname{Ind}(\mathcal{C}^{\heartsuit}_{\mathrm{fd}}) \to \mathcal{C}^{\heartsuit}$. Since $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ generates $\mathcal{C}^{\heartsuit}$ under filtered colimits, we conclude that i is an equivalence of ∞-categories. In particular, an object of $\mathcal{C}^{\heartsuit}$ is compact if and only if it is a retract of an object of $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. Since $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ is the nerve of an abelian category, it is idempotent complete. It follows that $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ consists precisely of the compact objects of $\mathcal{C}^{\heartsuit}$.

Remark 5.6.8. Combining Remarks 5.6.6 and 5.6.7, we deduce that $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ consists precisely of those objects of $\mathcal{C}^{\heartsuit}$ which admit a dual in $\mathcal{C}$. It follows immediately that $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ contains the unit object $\mathbf{1}$ and is stable under tensor product.

Remark 5.6.9. It follows easily from the definitions that the construction $V \mapsto \dim(V)$ is multiplicative: that is, we have

$$\dim(V \otimes W) = \dim(V)\dim(W) \qquad \dim(\mathbf{1}) = 1 \qquad \dim(V) = \dim(V^{\vee}).$$

Lemma 5.6.10. *Let $\mathcal{C}^{\otimes}$ be a locally dimensional ∞-category. Then the collection of dualizable objects of $\mathcal{C}$ span a stable subcategory of $\mathcal{C}$. In particular, $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ is closed under extensions in $\mathcal{C}^{\heartsuit}$.*

Proof. For every pair of objects $X, Y \in \mathcal{C}$, let X^Y denote a classifying object for maps from Y into X. According to Lemma A.4.2.5.6, an object $Y \in \mathcal{C}$ is dualizable if and only if, for every object $X \in \mathcal{C}$, the canonical map $X \otimes \mathbf{1}^Y \to X^Y$ is an equivalence in $\mathcal{C}$. It is clear from this description that the collection of dualizable objects is a stable subcategory of $\mathcal{C}$. □

Lemma 5.6.11. *Let $\mathcal{C}^{\otimes}$ be a locally dimensional ∞-category. Suppose we are given an exact sequence*

$$0 \to V' \to V \to V'' \to 0$$

in $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. Then $\dim(V) = \dim(V') + \dim(V'')$.

Proof. Let E be the kernel of the map $V \otimes V^{\vee} \to V'' \otimes V'^{\vee}$ in the abelian category $\mathcal{C}^{\heartsuit}$, and let F be the cokernel of the map $V' \otimes V''^{\vee} \to V \otimes V^{\vee}$ in $\mathcal{C}^{\heartsuit}$. Then the coevaluation map $c : \mathbf{1} \to V \otimes V^{\vee}$ factors through E, and the evaluation map $e : V \otimes V^{\vee} \to \mathbf{1}$ factors through F. It follows that $\dim(V)$ can be identified with a composition

$$\mathbf{1} \xrightarrow{c_0} E \hookrightarrow V \otimes V^{\vee} \to F \xrightarrow{e_0} \mathbf{1}.$$

The composite map $E \hookrightarrow V \otimes V^{\vee} \to F$ factors as a composition

$$E \xrightarrow{\phi} (V' \otimes V'^{\vee}) \oplus (V'' \otimes V''^{\vee}) \xrightarrow{\psi} F$$

Here $\phi \circ c_0$ can be identified with the sum of the coevaluation maps

$$c' : \mathbf{1} \to V' \otimes V'^{\vee} \qquad c'' : \mathbf{1} \to V'' \otimes V''^{\vee}$$

and similarly $e_0 \circ \psi$ can be identified with the sum of the evaluation maps

$$e' : V' \otimes V'^{\vee} \to \mathbf{1} \qquad e'' : V'' \otimes V''^{\vee} \to \mathbf{1}$$

It follows that

$$\dim(V) = e \circ c = e_0 \circ \psi \circ \phi \circ c_0 = e' \circ c' + e'' \circ c'' = \dim(V') + \dim(V'').$$

$\square$

Lemma 5.6.12. *Let $\mathcal{C}^{\otimes}$ be a locally dimensional ∞-category and let U be a subobject of the unit object $\mathbf{1}$ in $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. Then either $U \simeq 0$ or the monomorphism $U \to \mathbf{1}$ is an equivalence.*

Proof. We have a short exact sequence

$$0 \to U \xrightarrow{i} \mathbf{1} \xrightarrow{j} V \to 0$$

in the abelian category $\mathcal{C}^{\heartsuit}$. We will show that the maps i and $j^{\vee}$ exhibit $\mathbf{1}$ as a coproduct of U and $V^{\vee}$ in $\mathcal{C}$. It will then follow that there exists a map $r : \mathbf{1} \to U$ such that $r \circ i \simeq \mathrm{id}_U$. We can regard $i \circ r$ is an element of the endomorphism ring $k = \pi_0 \operatorname{Map}_{\mathcal{C}}(\mathbf{1}, \mathbf{1})$. If $i \circ r = 0$, then $U \simeq 0$. Otherwise, since k is a field, we deduce that $i \circ r$ is invertible, so that r is a homotopy inverse to i and $U \simeq \mathbf{1}$.

It remains to show that the map $f : U \oplus V^{\vee} \to \mathbf{1}$ given by i and $j^{\vee}$ is an equivalence. Let $\mathcal{C}_0 \subseteq \mathcal{C}$ be the full subcategory spanned by those objects W for which $\mathrm{id}_W \otimes f$ is an equivalence. We wish to prove that $\mathbf{1} \in \mathcal{C}_0$. Since $\mathcal{C}_0$ is evidently a stable subcategory of $\mathcal{C}$, it will suffice to show that U and V belong to $\mathcal{C}_0$.

We begin by observing that there is an epimorphism $U \otimes \mathbf{1} \to U \otimes V$ and a monomorphism $U \otimes V \to \mathbf{1} \otimes V$ in the abelian category $\mathcal{C}^{\heartsuit}$. It follows that $U \otimes V$ can be described as the image of the composite map $U \otimes \mathbf{1} \simeq U \to V \simeq \mathbf{1} \otimes V$, which is nullhomotopic. It follows that $U \otimes V \simeq 0$. The short exact sequence

$$0 \to U \otimes U \to U \otimes \mathbf{1} \to U \otimes V \to 0$$

shows that $\mathrm{id}_U \otimes i$ is an equivalence $U \otimes U \to U$. Consequently, to prove that $\mathrm{id}_U \otimes f$ is an equivalence, it will suffice to show that $U \otimes V^{\vee} \simeq 0$. This is clear: the identity map from $U \otimes V^{\vee}$ to itself factors as a composition

$$U \otimes V^{\vee} \to U \otimes V^{\vee} \otimes V \otimes V^{\vee} \to U \otimes V^{\vee}.$$

Since $U \otimes V \simeq 0$, the middle term vanishes, so that $\mathrm{id}_{U \otimes V^\vee}$ is nullhomotopic and therefore $U \otimes V^\vee \simeq 0$. This completes the proof that $U \in \mathcal{C}_0$.

We now prove that $V \in \mathcal{C}_0$: that is, the map

$$V \otimes (U \oplus V^\vee) \simeq (V \otimes U) \oplus (V \otimes V^\vee) \to V \otimes \mathbf{1} \simeq V$$

is an equivalence. Since $V \otimes U \simeq 0$, this is equivalent to the statement that $\mathrm{id}_V \otimes j^\vee$. It now suffices to observe that the cofiber of $\mathrm{id}_V \otimes j^\vee$ can be identified with $V \otimes U^\vee \simeq (U \otimes V^\vee)^\vee \simeq 0$. □

Lemma 5.6.13. *Let $\mathcal{C}^\otimes$ be a locally dimensional ∞-category. An object $V \in \mathcal{C}^\heartsuit_{\mathrm{fd}}$ is zero if and only if $\dim(V) = 0$.*

Proof. The "if" direction is obvious. For the converse, let K denote the kernel of the coevaluation map $c : \mathbf{1} \to V \otimes V^\vee$. If V is not a zero object of $\mathcal{C}$, then c is nonzero so the monomorphism $K \to \mathbf{1}$ is not an equivalence. It follows from Lemma 5.6.12 that $K \simeq 0$: that is, we have a short exact sequence

$$0 \to \mathbf{1} \xrightarrow{c} V \otimes V^\vee \to K' \to 0$$

in $\mathcal{C}^\heartsuit_{\mathrm{fd}}$. Using Lemma 5.6.11 and Remark 5.6.9, we obtain

$$\dim(V)^2 = \dim(V \otimes V^\vee) = \dim(\mathbf{1}) + \dim(K') = 1 + \dim(K')$$

so that $\dim(V) \neq 0$. □

Lemma 5.6.14. *Let $\mathcal{C}^\otimes$ be a locally dimensional ∞-category, and let $f : U \to V$ be a monomorphism in $\mathcal{C}^\heartsuit_{\mathrm{fd}}$. Then $\dim(U) \leq \dim(V)$, with equality if and only if f is an equivalence.*

Proof. We have an exact sequence

$$0 \to U \to V \to U' \to 0.$$

Using Lemma 5.6.11, we obtain $\dim(V) = \dim(U) + \dim(U') \geq \dim(U)$. Equality holds if and only if $\dim(U') = 0$, which (according to Lemma 5.6.13) is true if and only if $U' \simeq 0$. □

Proposition 5.6.15. *Let $\mathcal{C}^\otimes$ be a locally dimensional ∞-category. Then the subcategory $\mathcal{C}^\heartsuit_{\mathrm{fd}}$ is closed under the formation of subobjects and quotient objects in $\mathcal{C}^\heartsuit$.*

Proof. Let $V \in \mathcal{C}^\heartsuit_{\mathrm{fd}}$ and let $V_0 \in \mathcal{C}^\heartsuit$ be a subobject of V. Write V_0 as the colimit of a diagram $\{U_\alpha\}_{\alpha \in A}$ in $\mathcal{C}^\heartsuit_{\mathrm{fd}}$, indexed by a filtered partially ordered set A. For each index α, let V_α denote the image of the composite map

$$U_\alpha \to V_0 \hookrightarrow V,$$

so that $V_0 \simeq \varinjlim \{V_\alpha\}$. Using Lemma 5.6.14, we see that the dimensions $\dim(V_\alpha)$ are bounded above by $\dim(V)$. Choose an index α_0 so that $\dim(V_{\alpha_0})$ is as large as possible. Replacing A by $\{\alpha : \alpha \geq \alpha_0\}$, we may assume that the subobjects V_α of V all have the same dimension, so that (by Lemma 5.6.14) the diagram $\{V_\alpha\}$ is equivalent to the constant diagram taking the value V_{α_0}. It follows that $V_0 \simeq V_{\alpha_0} \in \mathcal{C}^\heartsuit_{\mathrm{fd}}$. □

Remark 5.6.16. The proof of Proposition 5.6.15 shows that $\mathcal{C}^\heartsuit_{\mathrm{fd}}$ is a Noetherian and Artinian abelian category: that is, every object of $\mathcal{C}^\heartsuit_{\mathrm{fd}}$ has finite length. It follows that the abelian category $\mathcal{C}^\heartsuit \simeq \mathrm{Ind}(\mathcal{C}^\heartsuit_{\mathrm{fd}})$ is locally Noetherian.

Proposition 5.6.17. *Let $\mathcal{C}^\otimes$ be a locally dimensional ∞-category. Let $\mathcal{D}^\otimes$ be another symmetric monoidal stable ∞-category, which is equipped with a left complete t-structure and admits small colimits. Let $\alpha : f \to g$ be a natural transformation between symmetric monoidal functors $f, g : \mathcal{C} \to \mathcal{D}$ which are right t-exact and preserve small colimits. Then α is an equivalence.*

Proof. We wish to prove that for every object $C \in \mathcal{C}$, the map α induces an equivalence $f(C) \to g(C)$ in $\mathcal{D}$. Since $\mathcal{D}$ is left complete, it suffices to show that $\tau_{\leq n} f(C) \to \tau_{\leq n} g(C)$ is an equivalence for each C. We have a commutative diagram

$$\begin{tikzcd}
\tau_{\leq n} f(C) \arrow[r] \arrow[d] & \tau_{\leq n} g(C) \arrow[d] \\
\tau_{\leq n} f(\tau_{\leq n} C) \arrow[r] & \tau_{\leq n} g(\tau_{\leq n} C).
\end{tikzcd}$$

Since f and g are right t-exact, the vertical maps are equivalences. We may therefore replace C by $\tau_{\leq n} C$ and thereby reduce to the case where $C \in \mathcal{C}_{\leq n}$. Since f and g are exact, we can replace C by $C[-n]$ and thereby reduce to the case $n = 0$.

Since $\mathcal{C}$ is right complete, we can write $C = \varinjlim_m \tau_{\geq -m} C$. Since f and g preserve small colimits, it suffices to show that α induces an equivalence $f(\tau_{\geq -m} C) \to g(\tau_{\geq -m} C)$ for each m. We may therefore replace C by $\tau_{\geq -m} C$ and thereby reduce to the case where $C \in \mathcal{C}_{\geq -m}$. We now proceed by induction on m. If $m < 0$, then $C \simeq 0$ and the result is obvious. Otherwise, we have a map of fiber sequences

$$\begin{tikzcd}
f(\tau_{\geq 1-m} C) \arrow[r] \arrow[d] & f(C) \arrow[r] \arrow[d] & f((\pi_{-m} C)[-m]) \arrow[d] \\
g(\tau_{\geq 1-m} C) \arrow[r] & g(C) \arrow[r] & g((\pi_{-m} C)[-m]).
\end{tikzcd}$$

The left vertical map is an equivalence by the inductive hypothesis, so we are reduced to proving that the right vertical map is an equivalence. In other words, we may assume (after shifting C) that $C \in \mathcal{C}^{\heartsuit}$.

Since $\mathcal{C}$ is locally dimensional, we can write C as a filtered colimit of objects of $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. Using the fact that f and g commute with colimits, we can reduce to the case where $C \in \mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. In this case, C admits a dual $C^{\vee}$, and the map $\alpha_C : f(C) \to g(C)$ has a homotopy inverse, given by the dual of the map $\alpha_{C^{\vee}} : f(C^{\vee}) \to g(C^{\vee})$. $\square$

Proposition 5.6.18. *Let X be a generalized algebraic gerbe. Then the ∞-category $\mathrm{QCoh}(X)$ is locally dimensional. That is:*

(1) *The ∞-category $\mathrm{QCoh}(X)$ is presentable.*

(2) *The subcategory $\mathrm{QCoh}(X)^{\mathrm{cn}} \subseteq \mathrm{QCoh}(X)$ spanned by the connective objects of $\mathrm{QCoh}(X)$ determines an accessible t-structure $(\mathrm{QCoh}(X)_{\geq 0}, \mathrm{QCoh}(X)_{\leq 0})$ on $\mathrm{QCoh}(X)$.*

(3) *The t-structure on $\mathrm{QCoh}(X)$ is excellent.*

(4) *The subcategories $\mathrm{QCoh}(X)_{\leq 0}$ and $\mathrm{QCoh}(X)_{\geq 0}$ both contain the unit object and are closed under tensor products.*

(5) *The unit object $\mathcal{O}_X \in \mathrm{QCoh}(X)$ is a compact object of $\mathrm{QCoh}(X)_{\leq n}$ for each $n \geq 0$.*

(6) *If k is a field of characteristic zero, $\mathrm{Spec}^{\mathrm{f}}\, k$ is the functor corepresented by k, and $\alpha : X \to \mathrm{Spec}^{\mathrm{f}}\, k$ is a map which exhibits X as a generalized algebraic gerbe over k, then the canonical map $k \simeq \pi_0 \mathrm{Map}_{\mathrm{Mod}_k}(k,k) \to \pi_0 \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{O}_X)$ is an isomorphism.*

(7) *The full subcategory $\mathrm{QCoh}(X)^{\heartsuit}_{\mathrm{fd}}$ spanned by the dualizable objects generates $\mathrm{QCoh}(X)^{\heartsuit}$ under filtered colimits.*

(8) *For every dualizable object $V \in \mathrm{QCoh}(X)^{\heartsuit}$, the composition*

$$\mathcal{O}_X \xrightarrow{c} V \otimes V^{\vee} \xrightarrow{e} \mathcal{O}_X$$

is given by multiplication by a nonnegative integer $\dim(V)$.

Proof. We first prove (1). Choose a pullback diagram

$$\begin{tikzcd} X_0 \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}^{\mathrm{f}} \kappa \arrow[r, "f_0"] & Y \end{tikzcd}$$

where Y is an algebraic gerbe, κ is a field of characteristic zero, and X_0 is the coaffine stack associated to a 2-coconnective $\mathbb{E}_\infty$-algebra over κ. The Čech nerve of f_0 has the form $\operatorname{Spec}^{\mathrm{f}} R^\bullet$, for some cosimplicial commutative ring $R^\bullet$ with $R^0 \simeq \kappa$. Let $X_\bullet$ be the Čech nerve of the map f, so that X can be identified with the geometric realization of $X_\bullet$ in the ∞-category $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$. It follows that $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$. Using Proposition T.5.5.3.13, we are reduced to showing that each $\mathrm{QCoh}(X_n)$ is a presentable ∞-category. Regard $\mathrm{QCoh}(X_0)$ as a κ-linear ∞-category, so that the proof of Corollary 3.2.6 gives an equivalence $\mathrm{QCoh}(X_n) \simeq \mathrm{LMod}_{R^n}(\mathrm{QCoh}(X_0))$. It therefore suffices to show that $\mathrm{QCoh}(X_0)$ is presentable, which follows from Proposition 4.5.2.

We next prove (2). Our first step is to prove the following:

$(*)$ A quasi-coherent sheaf $\mathcal{F}$ on X is connective if and only if $f^* \mathcal{F} \in \mathrm{QCoh}(X_0)$ is connective.

The "only if" direction of $(*)$ is clear. Conversely, suppose that $\mathcal{F} \in \mathrm{QCoh}(X)$ is such that $f^* \mathcal{F}$ is connective. We wish to show that for every connective $\mathbb{E}_\infty$-ring A and every point $\eta \in X(A)$, the pullback $\eta^* \mathcal{F}$ is a connective A-module. Since this assertion is local with respect to the flat topology and f is an effective epimorphism with respect to the flat topology, we may assume that η lies in the image of the map $X_0(A) \to X(A)$, in which case the result is obvious.

It follows immediately from $(*)$ that $\mathrm{QCoh}(X)_{\geq 0}$ is closed under colimits and extensions in $\mathrm{QCoh}(X)$; using Proposition T.5.5.3.13 we deduce that $\mathrm{QCoh}(X)_{\geq 0} \simeq \mathrm{QCoh}(X) \times_{\mathrm{QCoh}(X_0)} \mathrm{QCoh}(X_0)_{\geq 0}$ is presentable. Assertion (2) now follows from Proposition A.1.4.5.11.

Since $\mathrm{QCoh}(X) \simeq \varprojlim \mathrm{QCoh}(X_\bullet)$, Proposition A.6.2.4.1 implies that we can recover $\mathrm{QCoh}(X)$ as the ∞-category of comodules for the comonad $f^* f_*$ on $\mathrm{QCoh}(X_0)$. Using Corollary 3.2.6, we see that this comonad is given by the formula $\mathcal{F} \mapsto R^1 \otimes_{R^0} \mathcal{F}$; in particular, it is an exact functor from $\mathrm{QCoh}(X_0)$ to itself. Using Proposition VII.6.20, we deduce that the pair $((f^*)^{-1} \mathrm{QCoh}(X_0)_{\geq 0}, (f^*)^{-1} \mathrm{QCoh}(X_0)_{\leq 0})$ is a t-structure on $\mathrm{QCoh}(X)$. Assertion $(*)$ implies that this t-structure coincides with the t-structure of (2). It follows that a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$ belongs to $\mathrm{QCoh}(X)_{\leq 0}$ if and only if $f^* \mathcal{F} \in \mathrm{QCoh}(X_0)_{\leq 0}$. Since the t-structure on $\mathrm{QCoh}(X_0)$ is right and left complete (Proposition 4.5.2), Proposition VII.6.20 implies that the t-structure on $\mathrm{QCoh}(X)$ is right and left complete. Similarly, since f^* preserves filtered colimits, Proposition 4.5.2 implies that $\mathrm{QCoh}(X)_{\leq 0}$ is closed under filtered colimits. This proves (3). Since f^* is a symmetric monoidal functor, assertion (4) follows immediately from the corresponding assertion for $\mathrm{QCoh}(X_0)$ (Remark 4.5.3).

Since X_0 is coaffine, the pullback functor $\mathrm{Mod}_\kappa^\heartsuit \to \mathrm{QCoh}(X_0)^\heartsuit$ is an equivalence. It follows that for $n \geq 0$, the pullback map $\mathrm{Mod}_{R^n}^\heartsuit \to \mathrm{QCoh}(X_n)^\heartsuit$ is an equivalence, so that the pullback functor

$$\mathrm{QCoh}(Y)^\heartsuit \simeq \varprojlim \mathrm{Mod}_{R^\bullet}^\heartsuit \to \varprojlim \mathrm{QCoh}(X_\bullet)^\heartsuit \simeq \mathrm{QCoh}(X)^\heartsuit$$

is an equivalence. Assertion (6) now follows from Remark 5.1.6, and assertion (7) from Lemma 5.1.13 and Corollary 2.7.33.

To prove (5), we must show that $\mathcal{O}_X$ is a compact object of $\mathrm{QCoh}(X)_{\leq n}$ for $n \geq 0$. For each $m \geq 0$, let $f(m)$ denote the projection map $X_m \to X$. For $\mathcal{F} \in \mathrm{QCoh}(X)$, we have $\mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{F}) \simeq \varprojlim_{[m] \in \mathbf{\Delta}} \mathrm{Map}_{\mathrm{QCoh}(X_m)}(f(m)^* \mathcal{O}_X, f(m)^* \mathcal{F})$. If $\mathcal{F} \in \mathrm{QCoh}(X)_{\leq n}$, then each of the mapping spaces

$$\mathrm{Map}_{\mathrm{QCoh}(X_m)}(f(m)^* \mathcal{O}_X, f(m)^* \mathcal{F})$$

is n-truncated, so Proposition 4.3.5 shows that we can replace $\varprojlim_{[m] \in \mathbf{\Delta}} \mathrm{Map}_{\mathrm{QCoh}(X_m)}(f(m)^* \mathcal{O}_X, f(m)^* \mathcal{F})$ by the finite limit $\varprojlim_{[m] \in \mathbf{\Delta}_{\leq n+1}} \mathrm{Map}_{\mathrm{QCoh}(X_m)}(f(m)^* \mathcal{O}_X, f(m)^* \mathcal{F})$. Let $\Gamma : \mathrm{QCoh}(X)_{\leq n} \to \mathcal{S}$ be the functor corepresented by $\mathcal{O}_X$; we wish to show that Γ commutes with filtered colimits. The above argument shows that Γ can be written as a finite limit of functors Γ_m of the form

$$\mathcal{F} \mapsto \mathrm{Map}_{\mathrm{QCoh}(X_m)}(f(m)^* \mathcal{O}_X, f(m)^* \mathcal{F}) \simeq \mathrm{Map}_{\mathrm{QCoh}(X_0)}(f^* \mathcal{O}_X, R^m \otimes_k f^* \mathcal{F}).$$

It therefore suffices to show that $f^* \mathcal{O}_X$ is a compact object of $\mathrm{QCoh}(X_0)_{\leq n}$. Write $X_0 = \mathrm{cSpec}\, A$ for some 2-coconnective $\mathbb{E}_\infty$-algebra A over κ. Using Proposition 4.5.2, we are reduced to proving that A is a compact object of $(\mathrm{Mod}_A)_{\leq n}$. This is clear, since A is a compact object of Mod_A.

We now prove (8). Let V be a dualizable object of $\mathrm{QCoh}(X)^{\heartsuit}$, and let $\dim(V)$ denote the composite map

$$\mathcal{O}_X \to V \otimes V^\vee \to \mathcal{O}_X,$$

regarded as an element of $k \simeq \pi_0 \mathrm{Map}_{\mathrm{QCoh}(X)}(\mathcal{O}_X, \mathcal{O}_X)$. We wish to show that $\dim(V)$ is a nonnegative integer. Since X_0 is a coaffine stack over κ, the projection $X_0 \to \mathrm{Spec}^{\mathrm{f}}\, \kappa$ admits a section g. The pullback $(f \circ g)^* \dim(V) \in \mathrm{Map}_{\mathrm{Mod}_\kappa}(\kappa, \kappa) \simeq \kappa$ is the image of $\dim(V)$ under the embedding $k \hookrightarrow \kappa$; it therefore suffices to prove that $(f \circ g)^* \dim(V)$ is an integer. In other words, we may replace X by $\mathrm{Spec}^{\mathrm{f}}\, \kappa$ and thereby reduce to the case where V is a finite dimensional vector space over κ. In this case, $\dim(V)$ is just the usual dimension of V. □

We now return to the result stated at the beginning of this section.

Proof of Proposition 5.6.1. Let X be a generalized algebraic gerbe, and choose a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}^{\mathrm{f}}\, \kappa \arrow[r] & Y \end{tikzcd}$$

as in Definition 5.2.1. For every connective $\mathbb{E}_\infty$-ring A, let $\widehat{X}(A) \subseteq \mathrm{Fun}^{\otimes}(\mathrm{QCoh}(X)^{\otimes}, \mathrm{Mod}_A^{\otimes})$ denote the full subcategory spanned by those symmetric monoidal functors which are right t-exact and preserve small colimits. It follows from Propositions 5.6.17 and 5.6.18 that each $\widehat{X}(A)$ is a Kan complex. We therefore regard $\widehat{X}$ as a functor $\mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. It follows from Theorem VII.6.27 that $\widehat{X}$ is a sheaf with respect to the flat topology.

Using the same reasoning, we can define functors $\widehat{X}_0, \widehat{Y}, \widehat{h}_\kappa : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$. It follows from Proposition 5.1.14 that the canonical maps

$$\mathrm{Spec}^{\mathrm{f}}\, \kappa \to \widehat{h}_\kappa \qquad Y \to \widehat{Y}$$

are equivalences. Consequently, we have a commutative diagram

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "\phi_0"] & X \arrow[d, "\phi"] \\ \widehat{X}_0 \arrow[r] \arrow[d] & \widehat{X} \arrow[d] \\ \widehat{h}_\kappa \arrow[r, "g"] & \widehat{Y} \end{tikzcd}$$

in $\widehat{\mathcal{S}\mathrm{hv}}_{\mathrm{fpqc}}$, where the lower square and the outer square are pullbacks; it follows that the upper square is a pullback as well. We wish to prove that ϕ is an equivalence. Since g is an effective epimorphism of flat sheaves, it suffices to show that ϕ_0 is an equivalence. This follows from Corollary 4.6.19. □

Our goal for the remainder of this section is to prove the following converse of Proposition 5.6.18, which can be regarded as an ∞-categorical of Deligne's characterization of Tannakian categories over fields of characteristic zero.

Theorem 5.6.19. *Let $\mathcal{C}$ be a locally dimensional ∞-category. Then there exists a field k of characteristic zero, a generalized algebraic gerbe X over k, and a t-exact symmetric monoidal equivalence $\mathcal{C} \simeq \mathrm{QCoh}(X)$.*

Corollary 5.6.20. *The construction $X \mapsto \mathrm{QCoh}(X)^{\otimes}$ induces an equivalence from the full subcategory of $\widehat{\mathrm{Shv}}_{\mathrm{CAlg}}$ spanned by the generalized algebraic gerbes to the full subcategory of $\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}}$ spanned by the locally dimensional ∞-categories.*

Proof. Combine Corollary 5.6.3 with Theorem 5.6.19. $\square$

The proof of Theorem 5.6.19 will require some preparation. We begin with an application of Corollary 5.4.9.

Lemma 5.6.21. *Let $\mathcal{C}$ be a locally dimensional ∞-category, and let $\mathcal{X} \subseteq \mathcal{C}_{\leq 0}$ be the full subcategory spanned by those objects X satisfying the following conditions:*

(∗) *The objects $\pi_n X$ belong to $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ for all n, and $\pi_n X \simeq 0$ for $n \ll 0$.*

Then the inclusion $\mathcal{X} \subseteq \mathcal{C}_{\leq 0}$ induces an equivalence of ∞-categories $\theta : \mathrm{Ind}(\mathcal{X}) \to \mathcal{C}_{\leq 0}$.

Proof. Lemma 5.6.10 implies that every object $X \in \mathcal{X}$ is dualizable, and therefore a compact object of $\mathcal{C}_{\leq n}$ for all $n \geq 0$. It follows from Proposition T.5.3.5.11 that θ is a fully faithful embedding. Note that $\mathcal{X}$ is stable under finite colimits in $\mathcal{C}_{\leq 0}$, so that θ preserves small colimits (Proposition T.5.5.1.9) and therefore admits a right adjoint G. To show that θ is an equivalence of ∞-categories, it will suffice to show that G is conservative. To this end, let $\alpha : V \to W$ be a morphism in $\mathcal{C}_{\leq 0}$ such that $G(\alpha)$ is an equivalence. We wish to prove that α is an equivalence. Let U denote the fiber of the map α, so that $G(U) \simeq 0$; we wish to prove that $U \simeq 0$. Assume otherwise. Since $\mathcal{C}$ is right complete, there exists an integer n such that $\pi_{-n} U$ is nontrivial; let us assume that n is chosen as small as possible. Since $\mathcal{C}^{\heartsuit}$ is generated by $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ under filtered colimits, there exists a nonzero map $u : X \to \pi_{-n} U$ for some $X \in \mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. Then u determines a nontrivial point of $\mathrm{Map}_{\mathcal{C}}(X[-n], U) \simeq \mathrm{Map}_{\mathrm{Ind}(\mathcal{X})}(X[-n], G(U))$, contradicting our assumption that $G(U) \simeq 0$. $\square$

Proposition 5.6.22. *Let $\mathcal{C}$ be a locally dimensional ∞-category and let X be an object of $\mathcal{C}$. The following conditions are equivalent:*

(1) *We have $\pi_n X \in \mathcal{C}^{\heartsuit}_{\mathrm{fd}}$ for all n, and $\pi_n X \simeq 0$ for all but finitely many n.*

(2) *The object X is dualizable and belongs to $\mathcal{C}_{\leq n}$ for some n.*

(3) *The object X is a compact object of $\mathcal{C}_{\leq n}$ for some n.*

Proof. The implication (1) ⇒ (2) follows from Lemma 5.6.10. The implication (2) ⇒ (3) follows from the equivalence $\mathrm{Map}_{\mathcal{C}}(X, Y) \simeq \mathrm{Map}_{\mathcal{C}}(\mathbf{1}, X^{\vee} \otimes Y)$ together with the assumption that $\mathbf{1}$ is a compact object of $\mathcal{C}_{\leq n}$ for each $n \geq 0$. We prove that (3) ⇒ (1). Without loss of generality, we may assume that $n = 0$. Lemma 5.6.21 implies that X is can be realized as a filtered colimit of objects $\{X_\alpha\}$ in $\mathcal{C}_{\leq 0}$ which satisfy condition (1). Since X is compact in $\mathcal{C}_{\leq 0}$, the identity map $X \to \varinjlim X_\alpha$ factors through some X_α. It follows that X is a retract of X_α and therefore also satisfies (1). $\square$

Definition 5.6.23. Let $\mathcal{C}$ be a locally dimensional ∞-category. We will say that an object $X \in \mathcal{C}$ is *perfect* if it satisfies the equivalent conditions of Proposition 5.6.22. We let $\mathcal{C}^{\mathrm{perf}}$ denote the full subcategory of $\mathcal{C}$ spanned by the perfect objects, and $\mathcal{C}^{\mathrm{perf}}_{\leq 0}$ the intersection $\mathcal{C}^{\mathrm{perf}} \cap \mathcal{C}_{\leq 0}$.

The next result is the main technical ingredient in our proof of Theorem 5.6.19.

Proposition 5.6.24. *Let $\mathcal{C}$ be a locally dimensional ∞-category, and let $\mathcal{D} \in \mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{t}\pm})$. Let $\mathrm{Fun}^{\otimes}_0(\mathcal{C}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}, \mathcal{D})$ spanned by those symmetric monoidal functors $f : \mathcal{C} \to \mathcal{D}$ which are right t-exact and preserve small colimits. Let $\mathrm{Fun}^{\otimes}_0(\mathcal{C}_{\leq 0}, \mathcal{D})$ denote the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}_{\leq 0}, \mathcal{D})$ spanned by those functors $f : \mathcal{C}_{\leq 0} \to \mathcal{D}$ which preserve filtered colimits, finite limits, and satisfy the following condition:*

- $(*)$ *For every object $V \in \mathcal{C}^{\heartsuit}_{\mathrm{fd}}$, $f(V) \in \mathcal{D}_{\geq 0}$.*

Then the inclusion $\mathcal{C}_{\leq 0} \subseteq \mathcal{C}$ induces an equivalence of ∞-categories $\theta : \mathrm{Fun}^{\otimes}_0(\mathcal{C}, \mathcal{D}) \to \mathrm{Fun}^{\otimes}_0(\mathcal{C}_{\leq 0}, \mathcal{D})$.

Proof. Note that $\mathcal{C}^{\mathrm{perf}}$ inherits a t-structure from $\mathcal{C}$, which in turn determines a right complete t-structure on $\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}})$ (Lemma 5.4.1). Let $\mathrm{Fun}^{\otimes}_0(\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}), \mathcal{D})$ be the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}), \mathcal{D})$ spanned by those symmetric monoidal functors which are right t-exact and preserve small colimits, and let $\mathrm{Fun}^{\otimes}_0(\mathcal{C}^{\mathrm{perf}}, \mathcal{D})$ be the full subcategory spanned by those functors which are right t-exact and preserve finite colimits. Let $\mathrm{Fun}^{\otimes}_0(\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}_{\leq 0}), \mathcal{D})$ be the full subcategory of $\mathrm{Fun}^{\otimes}(\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}_{\leq 0}), \mathcal{D})$ spanned by those functors which preserve filtered colimits, finite limits, and satisfy $(*)$, and let $\mathrm{Fun}^{\otimes}_0(\mathcal{C}^{\mathrm{perf}}_{\leq 0}, \mathcal{D})$ be the full subcategory of $\mathrm{Fun}^{\otimes}(\mathcal{C}^{\mathrm{perf}}_{\leq 0}, \mathcal{D})$ spanned by those functors which preserve finite limits and satisfy $(*)$. We have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\otimes}_0(\mathcal{C}, \mathcal{D}) \arrow[r, "\theta"] \arrow[d, "\phi"] & \mathrm{Fun}^{\otimes}_0(\mathcal{C}_{\leq 0}, \mathcal{D}) \arrow[d, "\phi'"] \\
\mathrm{Fun}^{\otimes}_0(\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}), \mathcal{D}) \arrow[r] \arrow[d, "\psi"] & \mathrm{Fun}^{\otimes}_0(\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}_{\leq 0}), \mathcal{D}) \arrow[d, "\psi'"] \\
\mathrm{Fun}^{\otimes}_0(\mathcal{C}^{\mathrm{perf}}, \mathcal{D}) \arrow[r, "\theta'"] & \mathrm{Fun}^{\otimes}_0(\mathcal{C}^{\mathrm{perf}}_{\leq 0}, \mathcal{D}).
\end{tikzcd}$$

We will prove that θ is an equivalence of ∞-categories by showing that the maps ϕ, ϕ', ψ, ψ', and θ' are equivalences of ∞-categories:

- Lemma 5.6.21 implies that the functor $\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}) \to \mathcal{C}$ exhibits $\mathcal{C}$ as a right completion of $\mathrm{Ind}(\mathcal{C}^{\mathrm{perf}})$. It follows from Corollary 4.6.17 that ϕ is an equivalence of ∞-categories.
- The statement that ϕ' is an equivalence of ∞-categories follows immediately from Lemma 5.6.21.
- It follows immediately from Corollary A.6.3.1.12 that the map ψ' is an equivalence of ∞-categories. The same argument shows that ψ is an equivalence of ∞-categories, since a functor $f : \mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}) \to \mathcal{D}$ preserves small colimits if and only if it preserves filtered colimits and the composite map $f_0 : \mathcal{C}^{\mathrm{perf}} \to \mathrm{Ind}(\mathcal{C}^{\mathrm{perf}}) \xrightarrow{f} \mathcal{D}$ preserves finite colimits (Proposition T.5.5.1.9); moreover, in this case, f is right t-exact if and only if f_0 is right t-exact.
- Corollary 5.4.9 implies that θ' is a fully faithful embedding. To see that θ' is an equivalence, it suffices to observe that if $f : \mathcal{C}^{\mathrm{perf}} \to \mathcal{D}$ is an exact functor which satisfies $(*)$, then f is right t-exact.

□

Proof of Theorem 5.6.19. It follows from Proposition 5.6.1 that the functor $X : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ (if it exists) must be given by the formula $X(R) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}r^{\mathrm{L}}_{\mathrm{t}})}(\mathcal{C}, \mathrm{Mod}_R)$. Note that this functor X is a sheaf with respect to the flat topology, by Theorem VII.6.27. To complete the proof, it will suffice to verify the following:

(1) There exists a field k of characteristic zero such that X is a generalized algebraic gerbe over k.

(2) The natural map $\mathcal{C} \to \mathrm{QCoh}(X)$ is an equivalence of ∞-categories.

The proof will require several preliminaries. Let $\mathcal{A} = \mathrm{h}\mathcal{C}^{\heartsuit}$ be the heart of $\mathcal{C}$, and let $k = \mathrm{Hom}_{\mathcal{A}}(\mathbf{1}, \mathbf{1})$ be the field of endomorphisms of the unit object of $\mathcal{A}$. Using Theorem 5.0.1, we deduce that there exists an algebraic gerbe Y over k and a symmetric monoidal equivalence $\mathrm{N}(\mathcal{A}) \simeq \mathrm{QCoh}(Y)^{\heartsuit}$. Using Theorem 5.5.5, we deduce that this equivalence can be promoted to a symmetric monoidal functor $\phi : \mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}})) \rightarrow \mathrm{QCoh}(Y)_{\leq 0}$. Proposition 5.3.5 implies that this functor is also an equivalence.

Applying Theorem 5.5.5 again, we get a symmetric monoidal functor $\mathrm{N}(\mathrm{Fun}(\mathbf{\Delta}, \mathcal{A}^{\mathrm{inj}}) \rightarrow \mathcal{C}_{\leq 0}$. Composing this functor with a homotopy inverse to ϕ, we get a left exact, symmetric monoidal functor $f : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow \mathcal{C}_{\leq 0}$. We claim that f preserves filtered colimits. Since the t-structure on $\mathcal{C}$ is right complete and compatible with filtered colimits, it suffices to show that each of the composite functors $\tau_{\geq -n} \circ \psi : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow \mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq -n}$ preserves filtered colimits. Working by induction on n, we are reduced to proving that the functors $\pi_n \circ f : \mathrm{QCoh}(Y)_{\leq 0} \rightarrow \mathcal{C}^{\heartsuit}$ preserve filtered colimits; this is clear, since $\pi_n \circ f$ is equivalent to the composition

$$\mathrm{QCoh}(Y)_{\leq 0} \xrightarrow{\pi_n} \mathrm{QCoh}(Y)^{\heartsuit} \simeq \mathrm{N}(\mathcal{A}) \simeq \mathcal{C}^{\heartsuit}.$$

Since $\mathrm{QCoh}(Y)$ is a locally dimensional ∞-category (Proposition 5.6.18), Proposition 5.6.24 implies that f can be promoted (in an essentially unique way) to a symmetric monoidal functor $F : \mathrm{QCoh}(Y) \rightarrow \mathcal{C}$ which is right t-exact and preserves small colimits. Proposition 5.6.1 gives canonical equivalences

$$Y(A) \simeq \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})}(\mathrm{QCoh}(Y), \mathrm{Mod}_R),$$

so that composition with Ψ induces a natural transformation $X \rightarrow Y$.

Since F is a colimit preserving functor between presentable ∞-categories, it admits a right adjoint G (Corollary T.5.5.2.9). It follows from Proposition 3.2.1 that G is a map of ∞-operads; in particular, G carries the unit object $\mathbf{1} \in \mathcal{C}$ to a commutative algebra object A of $\mathrm{QCoh}(Y)$. Since F is right t-exact, the functor G is left t-exact; in particular, $A \in \mathrm{QCoh}(Y)_{\leq 0}$.

Let $\mathcal{O}_Y$ denote the unit object of $\mathrm{QCoh}(Y)$. Since the functor F induces an equivalence from the heart of $\mathrm{QCoh}(Y)$ to the heart of $\mathcal{C}$, we conclude that it induces isomorphisms

$$\mathrm{Ext}^i_{\mathrm{QCoh}(Y)}(\mathcal{F}, \mathcal{O}_Y) \rightarrow \mathrm{Ext}^i_{\mathcal{C}}(F(\mathcal{F}), \mathbf{1}) \simeq \mathrm{Ext}^i_{\mathcal{C}}(\mathcal{F}, A)$$

for $i \leq 1$ and $\mathcal{F} \in \mathrm{QCoh}(Y)^{\heartsuit}$. This implies the following:

(*a*) The unit map $\mathcal{O}_Y \rightarrow A$ exhibits $\mathcal{O}_Y$ as a truncation $\tau_{\geq -1} A$. In particular, the cofiber of the map $\mathcal{O}_Y \rightarrow A$ belongs to $\mathrm{QCoh}(Y)_{\leq -2}$.

Note that F induces an equivalence from the heart of $\mathrm{QCoh}(Y)$ to the heart of $\mathcal{C}$; in particular, it carries $\mathrm{QCoh}(Y)^{\heartsuit}_{\mathrm{fd}}$ to $\mathcal{C}^{\heartsuit}_{\mathrm{fd}}$. It follows that F carries perfect objects of $\mathrm{QCoh}(Y)_{\leq 0}$ to perfect objects of $\mathcal{C}_{\leq 0}$. Since F preserves filtered colimits, Lemma 5.6.21 implies that F is left t-exact, and that $F| \mathrm{QCoh}(Y)_{\leq 0}$ carries compact objects of $\mathrm{QCoh}(Y)_{\leq 0}$ to compact objects of $\mathcal{C}_{\leq 0}$. This implies:

(*b*) The restriction $G| \mathcal{C}_{\leq 0}$ preserves filtered colimits.

For every object $M \in \mathrm{QCoh}(Y)$ and every object $V \in \mathcal{C}$, there is an evident push-pull morphism $M \otimes G(V) \rightarrow G(F(M) \otimes V)$. If M is dualizable, then this map is automatically an equivalence: for every object $N \in \mathrm{QCoh}(Y)$, the canonical map $\mathrm{Map}_{\mathrm{QCoh}(Y)}(N, M \otimes G(V)) \rightarrow \mathrm{Map}_{\mathrm{QCoh}(Y)}(N, G(F(M) \otimes V))$ is given by the composition of equivalences

$$\begin{aligned}
\mathrm{Map}_{\mathrm{QCoh}(Y)}(N, M \otimes G(V)) &\simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(M^{\vee} \otimes N, G(V)) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(F(M^{\vee} \otimes N), V) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(F(M)^{\vee} \otimes F(N), V) \\
&\simeq \mathrm{Map}_{\mathcal{C}}(F(N), F(M) \otimes V) \\
&\simeq \mathrm{Map}_{\mathrm{QCoh}(Y)}(N, G(F(M) \otimes V)).
\end{aligned}$$

Lemma 5.6.21 guarantees that every object of $\mathrm{QCoh}(Y)_{\leq 0}$ is a filtered colimit of dualizable objects. Combining this observation with (*b*), we obtain:

(c) If $M \in \mathrm{QCoh}(Y)_{\leq 0}$ and $V \in \mathcal{C}_{\leq 0}$, then the push-pull map $M \otimes G(V) \to G(F(M) \otimes V)$ is an equivalence.

The functor G factors as a composition

$$\mathrm{QCoh}(X) \stackrel{G'}{\to} \mathrm{Mod}_A(\mathrm{QCoh}(Y)) \stackrel{G''}{\to} \mathrm{QCoh}(Y).$$

Note that G' admits a left adjoint F', which is the symmetric monoidal functor given informally by the formula

$$(M \in \mathrm{Mod}_A(\mathrm{QCoh}(Y))) \mapsto (F(M) \otimes_{F(A)} \mathbf{1}).$$

Similarly, G'' admits a left adjoint F'', given by $F''(M) = A \otimes M$. Let $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\geq 0}$ be the smallest full subcategory of $\mathrm{Mod}_A(\mathrm{QCoh}(Y))$ which contains $F''(\mathrm{QCoh}(Y)_{\geq 0})$. It follows from Proposition A.1.4.5.11 that $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\geq 0}$ determines an accessible t-structure on $\mathrm{Mod}_A(\mathrm{QCoh}(Y))$. Unwinding the definitions, we see that $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0}$ is the inverse image of $\mathrm{QCoh}(Y)_{\leq 0}$ in $\mathrm{Mod}_A(\mathrm{QCoh}(Y))$. In particular, $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0}$ is closed under filtered colimits, and the intersection

$$\bigcap \mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq -n}$$

consists of zero objects of $\mathrm{Mod}_A(\mathrm{QCoh}(Y))$, so that the t-structure on $\mathrm{Mod}_A(\mathrm{QCoh}(Y))$ is right complete (Proposition A.1.2.1.19).

We next prove the following:

(d) If $M \in \mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0}$, then $F'(M) \in \mathcal{C}_{\leq 0}$ and the unit map $M \to G'F'(M)$ is an equivalence.

To prove (d), we construct a sequence of A-module objects

$$M(0) \to M(1) \to M(2) \to \cdots$$

in $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{/M}$ by induction as follows. Let $M(0) = 0$. For $n > 0$, let $C(n)$ denote the fiber of the map $M(n) \to M$, and define $M(n+1)$ to be the cofiber of the map $A \otimes C(n) \to M(n)$. Since $M \in \mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0}$, we conclude that $C(1) \simeq M[-1] \in \mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq -1}$. Note that $C(n+1)$ is given by the cofiber of the structure map $A \otimes C(n) \to C(n)$. Combining this with (a) and using induction on n, we conclude that $C(n) \in \mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq -n}$. In particular, each $M(n)$ belongs to $\mathrm{QCoh}(Y)_{\leq 0}$, and $M \simeq \varinjlim M(n)$. Using (b), we are reduced to proving that each $M(n)$ satisfies the conclusion of (d). The proof proceeds by induction on n, the case $n = 0$ being trivial. If $n > 0$, we have a cofiber sequence

$$A \otimes C(n) \to M(n-1) \to M(n).$$

Using the inductive hypothesis, we are reduced to proving that the free A-module $A \otimes C(n)$ satisfies condition (d). We note that $F'(A \otimes C(n)) \simeq F(C(n))$ belongs to $\mathcal{C}_{\leq -n}$ since F is left t-exact, and that the unit map

$$G(\mathbf{1}) \otimes C(n) \simeq A \otimes C(n) \to G(C(n)) \simeq G(\mathbf{1} \otimes C(n))$$

is an equivalence by virtue of (c).

We next claim that the functor $G| \mathcal{C}_{\leq 0}$ is conservative. Indeed, if $\alpha : V \to W$ is a morphism in $\mathcal{C}_{\leq 0}$ such that $G(\alpha)$ is an equivalence. We wish to prove that α is an equivalence. Suppose otherwise, and let $U = \mathrm{fib}(\alpha)$. Since $\mathcal{C}$ is right complete and $U \in \mathcal{C}_{\leq 0}$, the assertion that U is nonzero implies that $\pi_{-n} U \neq 0$ for some $n \geq 0$; choose n as small as possible. Since F induces an equivalence from $\mathrm{QCoh}(Y)^{\heartsuit}$ to $\mathcal{C}^{\heartsuit}$, we conclude that there exists a nontrivial map $F(M[-n]) \to U$ for some $M \in \mathrm{QCoh}(Y)^{\heartsuit}$. It follows that there is a nontrivial map $M[-n] \to G(U) \simeq \mathrm{fib}(G(\alpha))$, contradicting our assumption that α is an equivalence.

It follows that $G'| \mathcal{C}_{\leq 0}$ is also a conservative functor. Combining this observation with (d), we obtain:

(e) The functor F' induces an equivalence $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0} \to \mathcal{C}_{\leq 0}$, and therefore exhibits $\mathcal{C}$ as a left completion of $\mathrm{Mod}_A(\mathrm{QCoh}(Y))$.

Let $X' : \mathrm{CAlg}^{\mathrm{cn}} \to \widehat{\mathcal{S}}$ be the functor described by the formula

$$X'(R) = \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})}(\mathrm{Mod}_A(\mathrm{QCoh}(Y)), \mathrm{Mod}_R).$$

The symmetric monoidal functor F' induces a natural transformation $X \to X'$. Using (e) and Corollary 4.6.17, we conclude that this natural transformation is an equivalence.

Since Y is an algebraic gerbe over k, there exists a field extension κ of k corepresenting a functor $\mathrm{Spec}^{\mathrm{f}}\,\kappa : \mathrm{CAlg}^{\mathrm{cn}} \to \mathcal{S}$ and a natural transformation $g : \mathrm{Spec}^{\mathrm{f}}\,\kappa \to Y$, which is representable, affine, and faithfully flat. Let $A^0 = g^* A$, regarded as a k-algebra. It follows from (a) that the algebra A^0 is coconnective and that $\pi_{-1} A^0 \simeq 0$. Theorem A.6.3.4.6 gives an equivalence $\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}}\,\kappa) \otimes_{\mathrm{QCoh}(Y)} \mathrm{Mod}_A(\mathrm{QCoh}(Y)) \simeq \mathrm{Mod}_{A^0}$. It follows that the functor $X_0 = X \times_Y \mathrm{Spec}^{\mathrm{f}}\,\kappa \simeq X' \times_Y \mathrm{Spec}^{\mathrm{f}}\,\kappa$ can be described by the formula $X_0(R) = \mathrm{Map}_{\mathrm{CAlg}(\mathcal{P}\mathrm{r}^{\mathrm{L}}_{\mathrm{t}})}(\mathrm{Mod}_{A^0}, \mathrm{Mod}_R)$. Corollary 4.6.18 shows that X_0 can be identified with the coaffine stack determined by A^0; this completes the proof of (1).

We now prove (2). There is an evident symmetric monoidal functor $\psi : \mathcal{C} \to \mathrm{QCoh}(X)$, which is right t-exact and preserves small colimits. We claim that ψ is left t-exact. In view of (e), it will suffice to show that the composite functor $\psi \circ F'$ carries $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0}$ into $\mathrm{QCoh}(X)_{\leq 0}$. Let $g' : X_0 \to X$ be the projection map; we wish to prove that $g'^* \circ \psi \circ F'$ carries $\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0}$ into $\mathrm{QCoh}(X_0)_{\leq 0}$. This functor factors as a composition

$$\mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0} \xrightarrow{g^*} \mathrm{Mod}_A(\mathrm{QCoh}(\mathrm{Spec}^{\mathrm{f}}\,\kappa))_{\leq 0} \simeq (\mathrm{Mod}_{A^0})_{\leq 0} \to \mathrm{QCoh}(X_0),$$

so that the desired result follows from Proposition 4.5.2.

Since $\mathcal{C}$ and $\mathrm{QCoh}(X)$ are both left complete, to show that ψ is an equivalence it will suffice to show that ψ induces an equivalence $\mathcal{C}_{\leq 0} \to \mathrm{QCoh}(X)_{\leq 0}$. In view of (e), this is equivalent to the statement that ψ induces an equivalence $\theta : \mathrm{Mod}_A(\mathrm{QCoh}(Y))_{\leq 0} \to \mathrm{QCoh}(X)_{\leq 0}$.

Let $Y_\bullet$ be the Čech nerve of the map $g : \mathrm{Spec}^{\mathrm{f}}\,\kappa \to Y$, and let $X_\bullet$ denote the pullback $Y_\bullet \times_Y X$. Then θ can be identified with the totalization of a cosimplicial functor

$$\theta^\bullet : \mathrm{Mod}_A(\mathrm{QCoh}(Y_\bullet))_{\leq 0} \to \mathrm{QCoh}(X_\bullet)_{\leq 0}.$$

It therefore suffices to show that each θ^n is an equivalence. Every map $[0] \to [n]$ in Δ induces a map $Y_n \to Y_0 \simeq \mathrm{Spec}^{\mathrm{f}}\,k$ which is representable and affine, so that so that $Y_n \simeq \mathrm{Spec}^{\mathrm{f}}\,B^n$ for some commutative k-algebra B^n. We observe that θ^n can be identified with the functor

$$\mathrm{Mod}_{B^n}(\mathrm{Mod}_A(\mathrm{QCoh}(Y_0))_{\leq 0}) \to \mathrm{Mod}_{B^n}(\mathrm{QCoh}(X_0)_{\leq 0})$$

induced by θ^0. We may therefore reduce to the case $n = 0$: that is, we are reduced to proving that the functor $\mathrm{Mod}_{A^0} \to \mathrm{QCoh}(X_0)$ induces an equivalence $(\mathrm{Mod}_{A^0})_{\leq 0} \to \mathrm{QCoh}(X_0)_{\leq 0}$. This follows from Proposition 4.5.2. $\square$

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