Research statement: mixed characteristic Casselman-Shalika formula and Whittaker categories

The current research program is conducted jointly with Ashwin Iyengar and Konrad Zou.

Let G be a split connected reductive algebraic group over the finite field $\mathbb{F}_q$. Let $\mathrm{Sph}^{\heartsuit}_{G,e} := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, e)$ be the *spherical category* of G, or the category of L^+G equivariant perverse sheaves on Gr_G with coefficients in e. For e a field, this is a *highest weight* category, with standard and costandard objects,

$$j_!(\lambda, e) := \pi_0 j_!^\lambda k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle] \text{ and } j_*(\lambda, e) := \pi_0 j_*^\lambda k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle]$$

If e is of characteristic 0, the category is semisimple, with simple objects

$$\mathcal{A}_\lambda := j_{!*}(\lambda, e)$$

and isomorphic to $\mathrm{Rep}(\widehat{G}, e)$, algebraic representations of the dual group of G with coefficients in e, [7]. The reader is welcome to skip from here to the statement of geometric Casselman-Shalika, 0.2.

0.1. The associated function from Frobenius trace.

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

defined on the set of k points of $\overline{\mathrm{Gr}^\lambda}$, can be viewed as a function of the unramified Hecke algebra [6], $\mathcal{H}_G$[1]. The constant term map

$$\mathcal{H}_G \to \mathcal{H}_T, f \mapsto f^B$$

has formula given by

$$f^B(t) := \delta^{1/2}_{B(K)}(t) \int_{N(K)} f(tu)\, du$$

The obvious basis elements $\{f_\lambda\}_{\lambda \in \Lambda_+}$ defined as indicator functions of double cosets, has a surprisingly simple formula, [8], under the constant term map

$$f^B_\lambda(t) = \int_{U(K)} A_\lambda(x\varpi^\nu)\, dx = (-1)^{2\langle \rho, \nu\rangle} q^{\langle \rho, \nu\rangle} m_\lambda(\nu)$$

0.2. The geometric Casselman-Shalika formula.

The equal characteristic *geometric* Casselman -Shalika states

Theorem 0.1. [3, 8.1.2]

$$H^i_c(S^\mu, j_{!*}(\lambda, e)\Big|_{S^\mu} \otimes_e \chi^*_\mu(\mathcal{L}_\psi)) = \begin{cases} e & \text{if } \lambda = \mu \text{ and } \langle 2\check{\rho}, \lambda\rangle = i \\ 0 & \text{otherwise.}\end{cases}$$

[1] compactly supported functions in $G(K)$ this is bi-equivariant with respect to $G(\mathcal{O})$

This is a geometrization of the classical Casselman-Shalika formula described in 0.1. The first goal of the project is therefore to give a mixed characteristic (of the geometry) version. This will make extensive use of recent of results of Fargues and Scholze, [**4**].

The project's second goal is to set up the foundations of Whittaker category in mixed characteristic, by understanding it as a left module over the spherical Hecke category. This is important in setting up geometric Langlands in the mixed characteristic setting, see 0.3.

By generalizing, suggests a fundamental property of the representation theory of reductive groups over local non-archimedean fields and allows one to import further arithmetic information.

0.3. Related works. Beyond its applications in the original paper. [**3**], the geometric CS formula in equal characteristic has been applied in recent work [**1**] to give an *Iwahori-Whittaker model* of the Satake category.

The implication of such a geometric model is twofold. Firstly, it gives a geometric description of the representation category.

$$D^b_{\mathrm{IW}}(\mathrm{Gr}_G, e) \simeq D^b(\mathrm{Rep}_e(\check{G})^{\heartsuit})$$

But further shows the derived category is *abelian*, which is much more easy to control.

Secondly, this result fits in the framework of *fundamental local equivalence* (FLE), a program initiated by D. Gaitsgory, [**5**]. The equivalence is present in [**2**, Thm. 3]. The Iwahori-Whittaker model is what the Whittaker filtration stabilizes to, see [**9**].

Bibliography

[1] Roman Bezrukavnikov, Denis Gaitsgory, Ivan Mirkovic, Simon Riche, and Laura Rider, *An Iwahori-Whittaker model for the Satake category* (2019).

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[5] Dennis Gaitsgory, *Recent progress in geometric Langlands theory* (2016).

[6] B Gross, *On the Satake isomoprhism* (1998), available at https://people.math.harvard.edu/~gross/preprints/sat.pdf.

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[9] Sam Raskin, *W-algebras and Whittaker categories* (2016).