LECTURE C.2

Recall that our aim in this C-series of lectures is to construct derived Galois deformation rings. Just as classical Galois deformation rings arise as the representing objects of certain formal moduli problems, and are constructed by verifying representability criteria, derived Galois deformation rings arise as the representing objects of certain derived formal moduli problems (enhancing the classical formal moduli problems), and again are constructed by verifying representability criteria. In this lecture, our aim is to understand what a derived formal moduli problem is and the representability criteria for such.

§1. Artin local simplicial commutative rings

Classically, a formal moduli problem over a field k is a functor $\mathrm{Art}_k^0 \to \mathrm{Set}$, where Art_k^0 denotes the category of Artin local commutative rings with residue field identified with k. The main point of the derived enhancement is to enlarge our category of test objects.

Definition 1.1. Let A be a simplicial commutative ring. We say that A is *Artin local* if $\pi_0(A)$ is an Artin local ring and $\pi_*(A)$ is a finitely generated $\pi_0(A)$-module (in other words, $\pi_i(A)$ is a finitely generated $\pi_0(A)$-module for each i and vanishes for $i \gg 0$).

If A is Artin local, the *residue field* of A is defined to be the residue field of $\pi_0(A)$; this is a field k uniquely characterized by the data of a map of simplicial commutative rings $\epsilon : A \to k$ inducing a surjection of commutative rings $\pi_0(A) \to \pi_0(k) \simeq k$.

For k a field, we let Art_k denote the full subcategory of $\mathrm{SCR}_{/k}$ consisting of those maps of simplicial commutative rings $\epsilon : A \to k$ where A is Artin local and ϵ exhibits k as the residue field of A.

Example 1.2. Any Artin local commutative ring can be regarded as an Artin local simplicial commutative ring. For any field k, this determines a fully faithful functor $\mathrm{Art}_k^0 \hookrightarrow \mathrm{Art}_k$.

Example 1.3. Let k be a field and let M be a simplicial k-module. Then we have the square-zero extension $k \oplus M \in \mathrm{SCR}_{/k}$, obtained by forming ordinary square-zero extensions at each simplicial level, and with zeroth homotopy group $\pi_0(k \oplus M)$ given by the ordinary square-zero extension $k \oplus \pi_0(M)$. If $\pi_*(M)$ is a finite-dimensional k-vector space, then $k \oplus M$ is Artin local, hence lies in Art_k.

There is an important special case of this construction. Let $n \geq 0$, and let S^n denote the simplicial n-sphere $\Delta^n/\partial\Delta^n$. We can form the free simplicial k-module $k[S^n]$, and then quotient out the basepoint to obtain a simplicial k-module that we will denote by $k[n]$ (note that $\pi_*(k[n])$ is k concentrated in degree n). This determines a square-zero Artin local simplicial commutative ring $k \oplus k[n] \in \mathrm{Art}_k$. When $n = 0$, this recovers the usual dual numbers $k[\epsilon]/(\epsilon^2)$.

In summary, just like an ordinary Artin local commutative ring, we think of an Artin local simplicial commutative ring as a finite infinitesimal/nilpotent thickening of its residue field, but now possibly with new kinds of nilpotents, namely elements in the higher homotopy groups. We will return to say more about the structure of Artin local simplicial commutative rings in §4.

§2. Derived formal moduli problems

Definition 2.1. For k a field, a *derived formal moduli problem over k* is a homotopy invariant functor $\mathrm{Art}_k \to \mathrm{sSet}$.

Example 2.2. Let k be a field and let R be a cofibrant simplicial commutative ring equipped with a map $\epsilon : R \to k$. Then $\mathcal{F}_R(A) := \mathrm{SCR}_{/k}(R, A)$ defines a derived formal moduli problem over k (encoding the formal completion of $\mathrm{Spec}(R)$ at the point ϵ).

More generally, suppose that $R = \{R_j\}_{j \in J}$ is a pro-object in $\mathrm{SCR}_{/k}$, i.e. a cofiltered diagram $J \to \mathrm{SCR}_{/k}$, with each R_j cofibrant. Then $\mathcal{F}_R(A) := \mathrm{colim}_{j \in J}\, \mathrm{SCR}_{/k}(R_j, A)$ defines a derived formal moduli problem over

k. We say that a derived formal moduli problem is *pro-representable* if it is naturally weakly equivalent to one of this form where each R_j in fact lies in $\mathrm{Art}_k \subseteq \mathrm{SCR}_{/k}$.

Remark 2.3. A derived formal moduli problem over a field k has an underlying classical formal moduli problem. Namely, given a functor $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSet}$, we can define a functor $\mathcal{F}^0 : \mathrm{Art}_k^0 \to \mathrm{Set}$ via the composition

$$\mathrm{Art}_k^0 \hookrightarrow \mathrm{Art}_k \xrightarrow{\mathcal{F}} \mathrm{sSet} \xrightarrow{\pi_0} \mathrm{Set}.$$

For example, if $\mathcal{F}$ is pro-represented (in the sense defined just above) by $\{R_j\}_{j\in J}$, then $\mathcal{F}^0$ is pro-represented (in the classical sense) by $\{\pi_0(R_j)\}_{j\in J}$. Note that in the pro-representable case, the value of $\mathcal{F}$ on a classical Artin local ring is already homotopy discrete, i.e. $\pi_i(\mathcal{F}(A)) \simeq 0$ for $i > 0$ and $A \in \mathrm{Art}_k^0$ (so that the final π_0 in the above composition is essentially redundant).

§3. The representability theorem

In the derived setting, we have the following analogue of Schlessinger's criteria for pro-representability:

Theorem 3.1. [Lurie] Suppose that $\Omega^1_{k/\mathbb{Z}} \simeq 0$. Then a derived formal moduli problem $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSet}$ is pro-representable if and only if the following three conditions hold:

(a) $\mathcal{F}(k)$ is weakly contractible;

(b) given a homotopy cartesian diagram in Art_k

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

with $\pi_0(B) \to \pi_0(D)$ and $\pi_0(C) \to \pi_0(D)$ surjective, the induced diagram of simplicial sets

$$\begin{tikzcd} \mathcal{F}(A) \arrow[r] \arrow[d] & \mathcal{F}(B) \arrow[d] \\ \mathcal{F}(C) \arrow[r] & \mathcal{F}(D) \end{tikzcd}$$

is also homotopy cartesian.

(c) $\mathcal{F}(k \oplus k[0])$ is homotopy discrete (i.e. its higher homotopy groups vanish).

Remark 3.2. Let us briefly discuss the notion of homotopy cartesian diagram invoked in Theorem 3.1. Up to weak equivalence, simplicial sets can be replaced by Kan complexes (even functorially). For $Y \to X \leftarrow Z$ a diagram of Kan complexes, the homotopy pullback $Y \times^h_X Z$ is defined to be the simplicial set $(Y \times Z) \times_{(X\times X)} X^{\Delta^1}$, so that a map of simplicial sets $S \to Y \times^h_X Z$ consists of maps $S \to Y$ and $S \to Z$ together with a homotopy of the two composites $S \to X$. In particular, a commutative diagram

$$\begin{tikzcd} S \arrow[r] \arrow[d] & Z \arrow[d] \\ Y \arrow[r] & X \end{tikzcd}$$

determines a map $S \to Y \times^h_X Z$, and we say that the diagram is homotopy cartesian if this map is a weak equivalence. We note also the following points:

(a) A weak equivalence of diagrams $Y \to X \leftarrow Z$ induces a weak equivalence of homotopy pullbacks (which is not true in general for usual pullbacks).

(b) If $Z = \Delta^0$, the homotopy pullback $Y \times^h_X Z$ is a homotopy fiber F of $Y \to X$ and we have a long exact sequence

$$\cdots \to \pi_1(X) \to \pi_0(F) \to \pi_0(Y) \to \pi_0(Z).$$

This shows in particular that if X and Y are homotopy discrete, then F is too. One can also use this to analyze more general homotopy pullbacks, using the fact that the homotopy fibers of $Y \times^h_X Z \to Z$ are weakly equivalent to the homotopy fibers of $Y \to X$. For example, one can check by this method that if X, Y, Z are all homotopy discrete, then $Y \times^h_X Z$ is too.

(c) Any simplicial abelian group, in particular any simplicial commutative ring, is a Kan complex, and the homotopy pullback of a diagram of such retains the same algebraic structure. For $Y \to X \leftarrow Z$ a diagram of simplicial abelian groups, there is a long exact sequence

$$\cdots \to \pi_1(X) \to \pi_0(Y \times^h_X Z) \to \pi_0(Y) \oplus \pi_0(Z) \to \pi_0(X).$$

In other words, the sequence $Y \times^h_X Z \to Y \oplus Z \to X$ is (up to a connectivity issue) a distinguished triangle on the other side of the Dold–Kan correspondence. This shows that for $B \to D \leftarrow C$ as in (b), the homotopy pullback $B \times^h_D C$ in is in fact an Artin local simplicial commutative ring. For example, for $n \geq 1$, there is an equivalence of simplicial k-modules $0 \times^h_{k[n]} 0 \simeq k[n-1]$, and therefore an equivalence of simplicial commutative rings $k \times^h_{k \oplus k[n]} k \simeq k \oplus k[n-1]$.

§4. Structure of Artin local simplicial commutative rings

One ingredient in the proof of the classical Schlessinger theorem is the fact that Artin local commutative rings can be analyzed inductively: for A an Artin local commutative ring with residue field k, the quotient map $A \to k$ can be factored in Art^0_k as a finite sequence of maps $A = A_m \to A_{m-1} \to \cdots \to A_0 = k$ where each $A_i \to A_{i-1}$ is a square-zero extension by k. This has the following generalization, which is used to prove Theorem 3.1:

Proposition 4.1. Let A be an Artin local simplicial commutative ring with residue field k. Then there is a sequence of maps $A \to A_m \to A_{m-1} \to \cdots \to A_0 \to k$, where the first and last are weak equivalences, together with weak equivalences from A_i to the homotopy pullback of a diagram $A_{i-1} \to k \oplus k[n_i] \leftarrow k$ for some $n_i \geq 1$.

Remark 4.2. If we just think additively, forming a homotopy pullback $A' \to k \oplus k[n] \leftarrow k$ is equivalent to forming an extension of A' by $k[n-1]$.

The proof of Proposition 4.1 involves an iterative procedure. Let us just illustrate this procedure with the following special case:

Lemma 4.3. Let A, A' be Artin local commutative rings with residue field k, and let $A \to A'$ be a square-zero extension by k. Then there is a weak equivalence from A to the homotopy pullback of a diagram $A'' \to k \oplus k[1] \leftarrow k$ where A'' is weakly equivalent to A'.

Proof. Choose a factorization $A \hookrightarrow A'' \to A'$ where the first map is a cofibration and the second map is a weak equivalence, and consider the simplicial tensor product $A'' \otimes_A k$: this models $A' \otimes^L_A k$, with homotopy groups $\mathrm{Tor}^A_*(A', k)$, the first two which are both k. Now, given a simplicial commutative ring R, there is a canonical truncated simplicial commutative ring $\tau_{\leq 1}(R)$ equipped with a map $R \to \tau_{\leq 1}(R)$ inducing an isomorphism on π_0 and π_1 and with $\pi_i(\tau_{\leq 1}(R)) \simeq 0$ for $i \geq 2$. Thus $\tau_{\leq 1}(A'' \otimes_A k)$ has homotopy groups isomorphic to those of $k \oplus k[1]$. Then we have a commutative diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & k \arrow[d] \\ A'' \arrow[r] & \tau_{\leq 1}(A'' \otimes_A k) \end{tikzcd}$$

which is homotopy cartesian. One can show that there is in fact a weak equivalence $\tau_{\leq 1}(A'' \otimes_A k) \xrightarrow{\sim} k \oplus k[1]$ commuting with the maps from k, which combined with the above finishes the proof. □

Remark 4.4. Let $\mathcal{F} : \mathrm{Art}_k \to \mathrm{sSet}$ be a derived formal moduli problem and let $A \to A'$ be as in Lemma 4.3. Supposing that $\mathcal{F}$ satisfies the first two conditions of Theorem 3.1, $\mathcal{F}(A)$ is weakly equivalent to the homotopy fiber of a map $\mathcal{F}(A') \to \mathcal{F}(k \oplus k[1])$. This gives a long exact sequence

$$\cdots \to \pi_0(\mathcal{F}(A)) \to \pi_0(\mathcal{F}(A')) \to \pi_0(\mathcal{F}(k \oplus k[1])),$$

so that given a class $x \in \mathcal{F}^0(A') = \pi_0(\mathcal{F}(A'))$, there is an "obstruction class" in $\pi_0(\mathcal{F}(k \oplus k[1]))$ that vanishes if and only if x lifts to a class in $\mathcal{F}^0(A) = \pi_0(\mathcal{F}(A))$. This kind of statement may feel familiar from classical deformation theory, and indeed those statements can be explained from this perspective.

§5. The proof of the representability theorem

Let us now explain the inductive procedure used to prove Theorem 3.1. What Proposition 4.1 shows is that it suffices to find a pro-system $R = \{R_j\}_{j\in J}$ of cofibrant objects in Art_k and $x \in \mathcal{F}(R) = \lim_{j\in J} \mathcal{F}(R_j)$ such that the corresponding natural transformation $\mathcal{F}_R \to \mathcal{F}$ induces equivalences $\mathcal{F}_R(k \oplus k[n]) \to \mathcal{F}(k \oplus k[n])$ for each $n \geq 1$ (this point may be explained more thoroughly in Lecture C.3). We produce this pro-object using the following inductive procedure.

Suppose given $S \in \mathrm{Art}_k$ cofibrant with $y \in \mathcal{F}(S)$, and for some $n \geq 1$ a point in the homotopy fiber of the map

$$\mathrm{SCR}_{/k}(S, k \oplus k[n]) = \mathcal{F}_S(k \oplus k[n]) \xrightarrow{y_*} \mathcal{F}(k \oplus k[n]),$$

determining some map $S \to k \oplus k[n]$. Then let S' be (a cofibrant replacement of) the homotopy pullback of $S \to k \oplus k[n] \leftarrow k$. By construction, the point y can be lifted to a point $y' \in \mathcal{F}(S')$, and the map

$$\mathcal{F}_{S'}(k \oplus k[n]) \xrightarrow{y'_*} \mathcal{F}(k \oplus k[n])$$

is one step closer to being an equivalence.