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# On the calculation of local terms in the Lefschetz–Verdier trace formula and its application to a conjecture of Deligne

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## Introduction

The Lefschetz trace formula for the Frobenius morphism (a theorem of A. Grothendieck; see [12] or [9, Rapport 3.2]) plays a fundamental role in the study of the Galois representations on the étale cohomology of algebraic varieties. One important application is a proof of the rationality of the zeta function of a variety over a finite field (conjectured by A. Weil [18] and first proved by B. Dwork [10]). It also has consequences for compatible systems of $\ell$-adic representations coming from cohomology (see P. Deligne [6, 1.6]). To obtain finer information one would like to split up the total cohomology using algebraic cycles and to describe the Galois representations on the individual factors. The process of splitting can be described in terms of correspondences; hence one wants to use a Lefschetz trace formula for the twist of a correspondence by Frobenius. In fact, for the applications mentioned above, it suffices to have such a trace formula for the twist by all sufficiently large powers of a fixed Frobenius. In that range Deligne has conjectured a Lefschetz trace formula of an explicit, particularly simple form. In the present article we prove Deligne's conjecture modulo resolution of singularities. Our method is a natural generalization of L. Illusie's proof in the one-dimensional case (see [13, IIIb, §§1-4]).

A significant part of the method works for correspondences over an arbitrary, algebraically closed field $k$. A correspondence is a diagram

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X$$

of compactifiable separated schemes of finite type over $k$. Fix a prime $\ell$ different from the characteristic of $k$, a finite field extension $\Lambda$ of $\mathbf{Q}_\ell$, and a constructible $\Lambda$-sheaf $M$ on $X$. (More generally $\Lambda$ can be any commutative ring with identity, which is either finite and annihilated by some power of $\ell$, or a finite extension of $\mathbf{Z}_\ell$ or $\mathbf{Q}_\ell$. Moreover $M$ can be any object of the cat-

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egory $D^b_{ctf}(X, \Lambda)$; see subsection (1.1).) A cohomological correspondence can be defined as a homomorphism

$$u : b_{2!}b_1^*M \to M.$$

(Here $b_{2!}$ denotes the derived direct image with compact support.) Assume that $b_1$ is proper. Then such data determine an endomorphism $u_!$ of the cohomology with compact support of $X$ with coefficients in $M$, whence a well-defined global term $\operatorname{tr}(u_!) \in \Lambda$ (see subsection (1.3)). Viewing $b = (b_1, b_2)$ as a morphism $B \to X \times X$, we define the scheme of fixed points $\operatorname{Fix} b$ as the inverse image of the diagonal $b^{-1}(\operatorname{diag}(X))$.

When $X$ is proper over $k$, one disposes of a general Lefschetz–Verdier trace formula (see [13, III, Cor. 4.7], also (2.2.1) below), which equates the global term with the sum of abstractly defined "local" terms for all connected components of $\operatorname{Fix} b$. In general one has neither an explicit formula for these local terms, nor the analogous trace formula for nonproper $X$. When $X$ is not proper, one can nevertheless extend the correspondence $b$ by zero to a compactification of $X$. This leaves the global term unchanged so that the problem of dealing with nonproper $X$ reduces to calculating the local terms at the boundary for the extended correspondence. In particular one may ask: When do these local terms vanish? (See (2.3).)

For the local terms associated to fixed points "at finite distance" the problem can be phrased as follows: Let $\beta$ be an isolated fixed point of $b$ and assume that $b_2$ is quasifinite in a neighborhood of $\beta$. Then the homomorphism $u$ determines an endomorphism $u_\beta$ of a stalk of $M$, whose trace can be viewed as a "naïve" local term at $\beta$ (see (1.4)). The question becomes: When do the naïve and the "true" local terms coincide? (See (2.4).)

Thus in both cases, at finite distance and at the boundary, one has a specific guess for a local term. In the present article we derive a sufficient condition for this guess to be correct. First the equality of local terms at finite distance is reduced to the vanishing of a true local term for another correspondence. Then both cases can be treated in a unified fashion. Our sufficient condition (dubbed "in good position"; see (4.4.3)) describes an aspect of the geometric behavior of the correspondence near a stratum. It is related to the condition of "general position" in [13, IIIb, 1.2] (see (5.2)). Unfortunately we need a resolution of singularities in order to formulate, not to mention to prove the sufficiency of, this geometric condition.

In the context of Deligne's conjecture one assumes that $k$ is the algebraic closure of a finite field with $q$ elements and that $X$ and $M$ are already defined over this subfield. This means that $X$ is endowed with a Frobenius endomorphism $\Phi_q$ and that $M$ comes with an isomorphism $M \cong \Phi_q^*M$. For any nonnegative integer $n$ one can then consider the twisted correspondence, defined by $b_1^{(n)} = \Phi_q^n \circ b_1$ and $b_2^{(n)} = b_2$ and where $u^{(n)}$ is the composite

$$b_{2!}^{(n)} b_1^{(n)*} M = b_{2!} b_1^* \Phi_q^{n*} M \cong b_{2!} b_1^* M \xrightarrow{u} M.$$

Assume that $b_2$ is quasifinite. One easily shows that all fixed points of $b^{(n)}$ are isolated whenever $q^n$ is greater than the degree of $b_2$. Deligne's conjecture asserts that the naïve trace formula

$$\operatorname{tr}\left(u_!^{(n)}\right) = \sum_{\beta \in \operatorname{Fix} b^{(n)}} \operatorname{tr}\left(u_\beta^{(n)}\right)$$

holds for all sufficiently large $n$ (see (7.1.3)). (There does not seem to be a general conjecture on an explicit lower bound. For instance, certain easy examples show that the lower bound cannot depend only on $b_2$ or its degree, as in the one-dimensional case.)

The following instances of this conjecture are known. For the Frobenius correspondence itself (i.e., $B = X$ and $b_1 = b_2 = \mathrm{id}$) it has been proved by Grothendieck ([12] or [9, Rapport, 3.2]). By Deligne and Lusztig's proof of 3.3 in [8], this result extends to the case of an automorphism of finite order (i.e., $B = X$, $b_2 = \mathrm{id}$, and $b_1$ is an automorphism of finite order). In the case $\dim(X) = 1$, the conjecture is a corollary of Illusie (see [13, IIIb, 1.2], also [1]). The case of a smooth surface and a lisse sheaf with finite monodromy has been covered by Th. Zink [19]. Special higher-dimensional Shimura varieties have been considered respectively by J.-L. Brylinski and J.-P. Labesse [5] and R. E. Kottwitz and M. Rapoport [15]. Finally, in the smooth case of arbitrary dimension, the conjecture has been proved independently by E. Shpiz [17], who approached the problem from a slightly different angle, but used the same basic principle of blowups as in this article. For consequences of Deligne's conjecture for moduli spaces over finite fields see Y. Z. Flicker [11].

We prove this conjecture modulo resolution of singularities in Theorem 7.2.2 and Corollary 7.2.3. More precisely we prove that, in the presence of suitable blowups, the geometric condition mentioned above holds when $n$ is sufficiently large. In order to formulate an unconditional theorem we take pains to state precisely which blowups we need in Data 7.2.1. Although a little awkward, this makes the theorem applicable. In important applications, such as that to Hecke correspondences on Shimura varieties, there is reasonable hope that the necessary construction can be performed directly. Also, under hypotheses like smoothness, etc., the assumptions become much simpler to state.

The structure of our article is as follows: Section 1 contains basic definitions, in particular those of the global term and the naïve local term. In Sections 2 through 6 we study correspondences over an arbitrary, algebraically closed field. While this constitutes an important part of our (conditional) proof of Deligne's conjecture, it may be skipped entirely by a reader who is interested only in understanding the results.

In Section 2 we discuss the relation with the general Lefschetz–Verdier trace formula in the proper case, in particular how the questions posed above reduce to the annihilation of a certain "true" local term. A stronger assertion is that a certain morphism in the derived category vanishes (see (3.1)). In the remainder of Section 3 we give instances for the functorial behavior of this morphism. In Sections 4 through 6 we deduce this vanishing from certain geometric conditions. Section 4 treats the fundamental case: that of a smooth variety $X$, a divisor with normal crossings $D$, and a constant sheaf. The method here is a direct generalization of parts of [13, IIIb, §§2–4]; a central role is played by a certain blowup of $X \times X$, which is an isomorphism outside $D \times D$ (see (4.3)). In Section 5 we use Galois coverings to extend the result to arbitrary lisse sheaves. In (5.2) we try to clarify the nature of the geometric condition. In Section 6, by splitting up correspondences along a certain stratification and using blowups, we extend the result to our most general case. In technical details all these sections rely heavily on both [2] and [13, III].

In Section 7 we specialize the results of the earlier sections to the situation conjectured by Deligne. We state the conjecture and our theorem (see subsections (7.1) and (7.2)); the remainder of the section finishes the proof of Theorem 7.2.2. Section 8 discusses the application to Galois representations arising from étale cohomology, in particular the result on independence of $\ell$ (see Theorem 8.4.2). Finally the Appendix lists some well-known conjectures on the resolution of singularities that play a role in this article.

## 1. The general setting

(1.1) *Notation.* Throughout this article, except in Section 8, $S$ denotes the spectrum of an algebraically closed field $k$. We consider only compactifiable separated schemes of finite type over $S$. Also, only morphisms over $S$ are considered, and absolute attributes like properness are to be understood relative to $S$.

We fix a prime $\ell$, which is invertible over $S$, and a commutative ring with identity $\Lambda$, which is either finite and annihilated by some power of $\ell$, or a finite extension of $\mathbf{Z}_\ell$ or $\mathbf{Q}_\ell$. We shall always (with a single exception, for a technical reason, in the proof of Proposition 4.4.2) work in the category $D^b_{ctf}(\ ,\Lambda)$ for the étale topology (for finite $\Lambda$, see [9, Rapport 4.6]; in the other cases, [7, 1.1.2-3]). To a morphism $f : X \to Y$ there are associated the usual functors between $D^b_{ctf}(X, \Lambda)$ and $D^b_{ctf}(Y, \Lambda)$: the inverse image functor $f^*$, the direct image functor $f_*$, the direct image functor with compact support $f_!$

(defined in [2, XVII, 5.1.9]), and the extraordinary inverse image functor $f^!$ (defined in [2, XVIII, 3.1.5–7]). The dualizing complex on $X$ will be denoted by $K_X$, the functor $\mathrm{R}\mathcal{H}om(\ ,K_X)$ by $D$.

When $i : Y \hookrightarrow X$ is a subscheme, the *restriction* to $Y$ of a complex $M \in \mathcal{O}b(D^b_{ctf}(X, \Lambda))$ is defined as the ordinary inverse image $M|_Y = i^*M$. If $u : M_1 \to M_2$ is a morphism of two such complexes on $X$, we say that $u$ vanishes on $Y$ if and only if the morphism $i^*u : i^*M_1 \to i^*M_2$ is zero. Given two morphisms $X_i \to X$ and complexes $M_i$ in $D^b_{ctf}(X_i, \Lambda)$ we write, as in [13, III, 1.5], $M_1 \otimes^{\mathbf{L}}_X M_2$ for the complex $\mathrm{pr}_1^* M_1 \otimes^{\mathbf{L}} \mathrm{pr}_2^* M_2$ on $X_1 \times_X X_2$. By abuse of notation we shall write $\Lambda$ for the constant sheaf with stalk $\Lambda$.

We shall repeatedly use various base-change morphisms. For the general definition we refer to [2, XVII, 2.1.3; see also XII, §4]. For the definition of those involving the extraordinary inverse image functor see [2, XVIII, 3.1.12.3, 3.1.13.2 and 3.1.14.2]. For Künneth formulas see [13, III, §1] and [2, XVII, 5.4.3].

(1.2) *Correspondences.* Let $X$ be a scheme over $S$. A (*geometric*) *correspondence* is a morphism $b : B \to X \times X$ (heeding the conventions of (1.1)). We denote the projection to the $i^{\text{th}}$ factor by $\mathrm{pr}_i : X \times X \to X$ and put $b_i = \mathrm{pr}_i \circ b$ so that giving $b$ is equivalent to giving the diagram

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X.$$

Let $M$ be an object of $D^b_{ctf}(X, \Lambda)$. A *cohomological correspondence* on $M$ with support in $b$ is a morphism

$$u : b_1^* M \to b_2^! M.$$

(1.3) *The global term.* In order to define the global term we require that

(1.3.1) $\qquad b_1$ be *proper.*

Then $u$ induces an endomorphism $u_!$ of $R\Gamma_c(X, M)$, defined as the composite

$$\begin{array}{ccc} R\Gamma_c(X,M) & & R\Gamma_c(X,M) \\ \text{adj} \downarrow & & \uparrow \text{adj} \\ R\Gamma_c(X, b_{1*}b_1^*M) \cong R\Gamma_c(B, b_1^*M) & \xrightarrow{u} & R\Gamma_c(B, b_2^!M) \cong R\Gamma_c(X, b_{2!}b_2^!M), \end{array}$$

where adj denotes the respective adjunction morphisms. Then $R\Gamma_c(X, M)$ can be represented by a perfect complex of $\Lambda$-modules and the endomorphism $u_!$ possesses a well-defined trace $\mathrm{tr}(u_!) \in \Lambda$ (see [9, Rapport, §4]). This is the *global term* in the desired Lefschetz–Verdier trace formula.

(1.4) *Local terms.* Consider a cohomological correspondence as in subsection (1.2). Let $A = X \hookrightarrow X \times X$ be the diagonal embedding and put

$\operatorname{Fix} b = B \times_{(X \times X)} A$. This is the *fixed-point locus* of the correspondence $b$. One would like to define a *local term* for every connected component $\beta \subset \operatorname{Fix} b$, i.e., a "number" in $\Lambda$, which depends only on the (étale) local behavior of $b$ and $u$ near $\beta$, such that under suitable conditions one can prove a trace formula comparing these local terms with the global term defined above. When $\beta$ is proper, there is a very abstract definition ([13, III, §4]) that leads to a general trace formula (see (2.2)), but in many cases it is not amenable to calculation. On the other hand, sometimes one can guess a simple formula yielding what might be called a *naïve local term*. In that case, one should try to find sufficient and practicable conditions for the naïve local term and the "true" local term to be equal. This is more or less what we are going to do in this article.

The true local term, denoted by $\mathrm{LT}_\beta(u) \in \Lambda$, is defined whenever $b$ and $\beta$ are *proper*. We shall recall its definition in subsection (2.1). Observe that $b$ is proper if assumption (1.3.1) is in force, since $b_1 = \mathrm{pr}_1 \circ b$ and $\mathrm{pr}_1$ is separated. Note also that when $\beta$ is an *isolated fixed point*, i.e., when $\dim(\beta) = 0$, it is automatically proper. Thus in important cases $\mathrm{LT}_\beta(u)$ is defined.

(1.5) *The naïve local term.* We are still considering the situation of the correspondences in (1.2). Let $\beta \in \operatorname{Fix} b$ be an isolated fixed point, put $b_1(\beta) = b_2(\beta) = x \in X$ and assume that $b_2$ is quasifinite over a neighborhood of $x$. Consider the morphisms between stalks

$$M_x \xrightarrow{\sim} (b_1^* M)_\beta \xrightarrow{u} (b_2^! M)_\beta \hookrightarrow (b_{2!} b_2^! M)_x \xrightarrow{\mathrm{adj}} M_x. \tag{1.5.1}$$

Here the isomorphism on the left is the obvious one, while the third arrow is the embedding of $(b_2^! M)_\beta$ as a direct summand of $(b_{2!} b_2^! M)_x \cong R\Gamma_c(b_2^{-1}(x), b_2^! M)$ and adj denotes the adjunction morphism. Let $u_\beta : M_x \to M_x$ denote the composite morphism in (1.5.1). As in subsection (1.3) it has a well-defined trace $\operatorname{tr}(u_\beta) \in \Lambda$; this is the *naïve local term*.

This definition is a generalization of that in [13, IIIb, 1.1.1]. It should be noted that we shall eventually make special geometric assumptions, without which the naïve local term defined here is, in general, not equal to the true one. When these assumptions are not satisfied, it is still sometimes possible to calculate the true local term, e.g., under smoothness conditions by [13, III, 4.3, 4.12], or in the general one-dimensional case, by [1]. Similar remarks apply to the following case, which plays an essential role in this article: Let $\beta \in \operatorname{Fix} b$ be a connected component of arbitrary dimension, but assume that the restriction of $b_1^* M$ to $\beta$ is zero. In this case we can think of the naïve local term as zero.

## 2. The relevance of the annihilation of a certain local term

(2.1) *The true local term.* Let us first review the relevant part of [13, III, §4]. Let $X$ be a scheme over $S$. Consider two *proper* morphisms $a : A \to X \times X$, $b : B \to X \times X$, and define $C := A \times_{(X \times X)} B$. The definition will make sense for arbitrary $A$, but *we shall always assume that $A$ is the diagonal in $X \times X$, except where the contrary is explicitly mentioned* (this will be only in (3.3) and (5.4)). We have $C = \operatorname{Fix} b$, as in subsection (1.4). We identify $\operatorname{Fix} b$ and the diagonal $A$ with their images in $B$ and $X \times X$, respectively. We are going to deal with the following morphisms:

$$
\begin{tikzcd}[column sep=small]
 & C \arrow[dl, "b'"'] \arrow[dd, "c"] \arrow[dr, "a'"] & \\
A \arrow[dr, "a"'] & & B \arrow[dl, "b"] \\
 & X \times X \arrow[dl, "\mathrm{pr}_1"'] \arrow[dr, "\mathrm{pr}_2"] & \\
X \arrow[dr] & & X \arrow[dl] \\
 & S &
\end{tikzcd}
\tag{2.1.1}
$$

As in subsection (1.2) we write $a_i = \mathrm{pr}_i \circ a$ and $b_i = \mathrm{pr}_i \circ b$. Let $M$ be an object of $D^b_{ctf}(X, \Lambda)$ and abbreviate

$$P := M \otimes^{\mathbf{L}}_S DM \quad \text{and} \quad Q := DM \otimes^{\mathbf{L}}_S M.$$

The evaluation morphisms ([13, III, 2.1.1 together with 2.2.2])

$$\mathrm{pr}_i^* DM \otimes^{\mathbf{L}} \mathrm{pr}_i^* M = \mathrm{pr}_i^* \,\mathrm{R}\mathcal{H}\mathrm{om}(M, K_X) \otimes^{\mathbf{L}} \mathrm{pr}_i^* M \to \mathrm{pr}_i^* K_X$$

induce a morphism

$$
\begin{aligned}
P \otimes^{\mathbf{L}} Q &\stackrel{\mathrm{def}}{=} (\mathrm{pr}_1^* M \otimes^{\mathbf{L}} \mathrm{pr}_2^* DM) \otimes^{\mathbf{L}} (\mathrm{pr}_1^* DM \otimes^{\mathbf{L}} \mathrm{pr}_2^* M) \\
&\to \mathrm{pr}_1^* K_X \otimes^{\mathbf{L}} \mathrm{pr}_2^* K_X \\
&= K_{X \times X}.
\end{aligned}
\tag{2.1.2}
$$

By [13, III, 4.2.1] there is a canonical morphism

$$a^! P \otimes^{\mathbf{L}}_{X \times X} b^! Q \to c^!(P \otimes^{\mathbf{L}} Q), \tag{2.1.3}$$

which one can, following [13, IIIb, §2, p.149], write as the composite

$$\begin{aligned} a^! P \otimes^{\mathbf{L}}_{X\times X} b^! Q &\overset{\text{def}}{=} b'^* a^! P \otimes^{\mathbf{L}} a'^* b^! Q \\ &\xrightarrow{\text{(base change)}\otimes\text{id}} a'^! b^* P \otimes^{\mathbf{L}} a'^* b^! Q \\ &\xrightarrow{\text{projection}} a'^! (b^* P \otimes^{\mathbf{L}} b^! Q) \\ &\xrightarrow{\text{projection}} a'^! b^! (P \otimes^{\mathbf{L}} Q) \\ &= c^! (P \otimes^{\mathbf{L}} Q); \end{aligned} \tag{2.1.4}$$

here the morphisms called "projection" are derived by adjunction from the isomorphism [2, XVII, 5.2.9]. Applying $c_!$, we have a Künneth isomorphism ([2, XVII, 5.4.2.2])

$$c_!(a^! P \otimes^{\mathbf{L}}_{X\times X} b^! Q) \cong a_! a^! P \otimes^{\mathbf{L}} b_! b^! Q. \tag{2.1.5}$$

Combining (2.1.2), (2.1.3) and (2.1.5) yields a morphism

$$a_! a^! P \otimes^{\mathbf{L}} b_! b^! Q \to c_! c^! K_{X\times X} \cong c_! K_C. \tag{2.1.6}$$

Since $a$ and $b$ are proper, so is $c$. Applying $H^0(X\times X, -)$ to (2.1.6), we obtain a pairing

$$H^0(A, a^! P) \otimes H^0(B, b^! Q) \to H^0(C, K_C). \tag{2.1.7}$$

By [13, III, 3.1.1 and 3.2.1] we have canonical isomorphisms

$$b^! Q \overset{\text{def}}{=} b^! (DM \otimes^{\mathbf{L}}_S M) \cong b^! \,\mathrm{R}\mathcal{H}\mathrm{om}(\mathrm{pr}_1^* M, \mathrm{pr}_2^! M) \cong \mathrm{R}\mathcal{H}\mathrm{om}(b_1^* M, b_2^! M),$$

and similarly,

$$a^! P \cong \mathrm{R}\mathcal{H}\mathrm{om}(a_2^* M, a_1^! M).$$

Plugging these into (2.1.7) yields a pairing

$$\mathrm{Hom}(a_2^* M, a_1^! M) \otimes \mathrm{Hom}(b_1^* M, b_2^! M) \to H^0(C, K_C). \tag{2.1.8}$$

Finally observe that $H^0(C, K_C)$ is the direct sum of $H^0(\beta, K_\beta)$ for all connected components $\beta$ of $C$. Fix one such $\beta$, which is *proper*, and consider

$$H^0(C, K_C) \xrightarrow{\mathrm{proj}_\beta} H^0(\beta, K_\beta) \xrightarrow{\mathrm{adj}} \Lambda.$$

The composite with (2.1.8) is a pairing

$$\mathrm{Hom}(a_2^* M, a_1^! M) \otimes \mathrm{Hom}(b_1^* M, b_2^! M) \to \Lambda. \tag{2.1.9}$$

Now recall that $A$ is the diagonal in $X \times X$; so the morphisms $a_i$ are just the identity on $X$ (in particular, proper), and we can consider the identity morphism

$$a_2^* M \cong M \xrightarrow{\mathrm{id}} M \cong a_1^! M.$$

*Definition* 2.1.10. The true local term $\mathrm{LT}_\beta(u) \in \Lambda$ is the image of $\mathrm{id} \otimes u$ under the pairing (2.1.9).

(2.2) *The general Lefschetz–Verdier trace formula.* Consider a cohomological correspondence, as in subsection (1.2). If we assume that $X$ and $B$ are proper, then so is every connected component $\beta \subset \operatorname{Fix} b$, and all local and global terms are defined. In this case there is a general trace formula.

THEOREM 2.2.1. *If $X$ and $B$ are proper, then*

$$\operatorname{tr}(u_!) = \sum_{\beta \in \pi_0(\operatorname{Fix} b)} \mathrm{LT}_\beta(u).$$

(See [13, III, 4.7]. That the global term in [loc. cit.] coincides with that given in (1.3) follows from [13, III, 3.6.2].)

(2.3) *Extension by zero.* When $X$ is not proper, but $b_1$ is proper, then the correspondence can be extended by zero to a compactification, as follows: Consider a commutative diagram

$$\begin{array}{ccccc} X & \xleftarrow{b_1} & B & \xrightarrow{b_2} & X \\ j \downarrow & & j_B \downarrow & & \downarrow j \\ \bar{X} & \xleftarrow{\bar{b}_1} & \bar{B} & \xrightarrow{\bar{b}_2} & \bar{X}, \end{array}$$

where the vertical arrows are open embeddings and the bottom line is proper over $S$. Since $b_1$ is proper, after replacing $\bar{B}$ by the Zariski closure of $B$, we may, and do, assume that the left-hand square is cartesian. Applying proper base change shows that the composite morphism

$$\bar{u} : \bar{b}_1^* j_! M \cong j_{B!} b_1^* M \xrightarrow{u} j_{B!} b_2^! M \xrightarrow{\text{base change}} \bar{b}_2^! j_! M$$

is a cohomological correspondence on $j_! M$ with support in $\bar{b}$. For this extended correspondence, the general Lefschetz–Verdier trace formula (2.2.1) applies. Moreover we have the next lemma:

LEMMA 2.3.1. *The global terms are equal:* $\operatorname{tr}(u_!) = \operatorname{tr}(\bar{u}_!)$.

*Proof.* By the definition of the base-change morphism, the following diagram commutes:

$$\begin{array}{ccc} \bar{b}_{2!} j_{B!} b_2^! & \xrightarrow{\text{base change}} & \bar{b}_{2!} \bar{b}_2^! j_! \\ \wr\| & & \downarrow \text{adj} \\ j_! b_{2!} b_2^! & \xrightarrow{\text{adj}} & j_!. \end{array}$$

The definitions of $u_!$ and $\bar{u}_!$ thus yield a commutative diagram

$$\begin{array}{ccc} R\Gamma_c(X,M) & \xrightarrow{u_!} & R\Gamma_c(X,M) \\ \wr\| & & \wr\| \\ R\Gamma_c(\bar{X},j_!M) & \xrightarrow{\bar{u}_!} & R\Gamma_c(\bar{X},j_!M), \end{array}$$

which proves the assertion. □

For any connected component $\beta \subset \operatorname{Fix} b$ which is proper, the local terms of $u$ and $\bar{u}$ are equal, since they depend only on the local behavior near $\beta$. Thus if $\operatorname{Fix} b$ is proper, the defect in the trace formula for $u$ is measured by the local terms of $\bar{u}$ at all $\beta \subset \operatorname{Fix} \bar{b} \setminus \operatorname{Fix} b$. Changing notation, we are led to the general problem of studying the local term $\mathrm{LT}_\beta(u)$ when the restriction of $b_1^*M$ to $\beta$ is zero. The same question occurs in how to determine local terms at finite distance, which we shall discuss presently.

(2.4) *Local terms at finite distance.* Consider the situation described in subsection (2.1). Let $\beta \in \operatorname{Fix} b$ be an isolated fixed point, put $b_1(\beta) = b_2(\beta) = x \in X$ and assume that $b_2$ is quasifinite over a neighborhood of $x$. After removing from $X$ the finitely many closed points $b_1(b_2^{-1}(x)) \setminus \{x\}$, which leaves the local term unchanged, we have

(2.4.1) $$b_2^{-1}(x) \subset b_1^{-1}(x).$$

Consider the open embedding $j : U = X \setminus \{x\} \hookrightarrow X$. By (2.4.1), a commutative diagram exists

(2.4.2) $$\begin{array}{ccccc} U & \xleftarrow{b_{1j}} & b_1^{-1}(U) & \xrightarrow{b_{2j}} & U \\ {\scriptstyle j}\big\downarrow & \square & {\scriptstyle j_B}\big\downarrow & & \big\downarrow{\scriptstyle j} \\ X & \xleftarrow{b_1} & B & \xrightarrow{b_2} & X \end{array}$$

whose left-hand side is cartesian, as indicated. The composite morphism

$$u_j : b_1^* j_! j^! M \cong j_{B!} b_{1j}^* j^! M \cong j_{B!} j_B^! b_1^* M \xrightarrow{u} j_{B!} j_B^! b_2^! M \cong j_{B!} b_{2j}^! j^! M \to b_2^! j_! j^! M,$$

where the unmarked arrows and isomorphisms are trivial or come from a base change, is a cohomological correspondence on $j_!j^!M$ with support in $b$. It is, in fact, the extension by zero of the pullback of $u$ to $b_1^{-1}(U)$. The following proposition measures the defect in the naïve local term $\mathrm{tr}(u_\beta)$ in terms of the true local term for $u_j$. Thus the problem of calculating the true local term for $u$ boils down to the same question posed in subsection (2.3), namely whether another local term vanishes.

PROPOSITION 2.4.3. *In the above situation,*

$$\mathrm{LT}_\beta(u) = \mathrm{tr}(u_\beta) + \mathrm{LT}_\beta(u_j).$$

*Proof.* Consider the filtered complex $(M, F)$, where $F^{-1}M = 0$, $F^0M = j_!j^!M$ and $F^1M = M$. The correspondence $u$ extends uniquely to a correspondence in the filtered derived category whose graded part in degree 0 is $u_j$. The graded part in degree 1 is a correspondence $u_i$ on the complex $i_*i^*M$, which can be described as follows: There is a commutative diagram, analogous to (2.4.2),

$$\begin{tikzcd}
\{x\} \arrow[d, hook, "i"'] & b_2^{-1}(x) \arrow[l, "b_{1i}"'] \arrow[r, "b_{2i}"] \arrow[d, hook, "i_B"'] \arrow[dr, phantom, "\square"] & \{x\} \arrow[d, hook, "i"] \\
X & B \arrow[l, "b_1"'] \arrow[r, "b_2"] & X
\end{tikzcd}$$

whose right-hand side is cartesian, as indicated. We can define the pullback of $u$ as the composite

$$i^*u\colon\ b_{1i}^*i^*M \cong i_B^*b_1^*M \xrightarrow{u} i_B^*b_2^!M \xrightarrow{\text{base change}} b_{2i}^!i^*M.$$

It is easy to verify that $u_i$ can be given as the direct image of $i^*u$ under $i$, i.e., that $u_i$ is the composite morphism

$$b_1^*i_*i^*M \xrightarrow{\text{base change}} i_{B*}b_{1i}^*i^*M \xrightarrow{i^*u} i_{B*}b_{2i}^!i^*M \cong b_2^!i_*i^*M,$$

where the isomorphism comes from the transpose of a proper base change. Now by [13, III, 4.13] the local term is additive:

$$\mathrm{LT}_\beta(u) = \mathrm{LT}_\beta(u_j) + \mathrm{LT}_\beta(u_i).$$

Moreover, as a special case of [13, III, 4.5], we have

$$\mathrm{LT}_\beta(u_i) = \mathrm{LT}_\beta(i^*u).$$

Finally observe that the functors $i^*$ and $i_B^*$ mean taking the stalk at $x$ and $\beta$, respectively, so that the morphism $i^*u$ corresponds to the endomorphism $u_\beta : M_x \to M_x$ defined in subsection (1.5). Since the geometry of the correspondence $i^*u$ is trivial, it follows that

$$\mathrm{LT}_\beta(i^*u) = \mathrm{tr}(u_\beta),$$

and we are done. □

## 3. Functorial behavior of a morphism

(3.1) *Vanishing of a morphism.* From now on, until Section 6, we are concerned with the problem that occurred in subsections (2.3)–(2.4), namely that of studying the local term at a connected component $\beta \subset \mathrm{Fix}\, b$, where the restriction of $b_1^*M$ to $\beta$ is zero. Our aim is to find sufficient and practicable conditions which imply that $\mathrm{LT}_\beta(u) = 0$.

Consider the situation described in subsection (2.1). Recall that in (2.1.4) we made use of the base-change morphism

$$\varphi(b, M) : b'^* a^! P \to a'^! b^* P, \tag{3.1.1}$$

where $P = M \otimes^{\mathbf{L}}_S DM$. If this morphism vanishes on $\beta$, and if $\beta$ is proper, then the pairing (2.1.3) is zero; hence $\mathrm{LT}_\beta(u) = 0$, as desired. We shall, therefore, try to find out where $\varphi(b, M)$ vanishes.

Observe that the vanishing of $\varphi(b, M)$ is a little stronger than the annihilation of the local term. For instance, suppose that $\beta$ is a proper, connected component of $\mathrm{Fix}\, b$ and that $\varphi(b, M)$ vanishes on a closed subset $\gamma \subset \beta$ such that $b_1^* M|_\gamma = 0$. Then this vanishing could perhaps be used to define—or even to calculate—a local term for the nonproper fixed-point locus $\beta \smallsetminus \gamma$.

In the remainder of this section we shall prove some general facts that will be used in Sections 4 through 6. While trying to avoid unnecessary generalities, we shall often deal with the following situation: Let $b : B \to X \times X$ be a geometric correspondence and $f : Y \to X$ be a morphism. By the *pullback* of the correspondence $b$ we mean the correspondence

$$B \times_{(X \times X)} (Y \times Y) \xrightarrow{\mathrm{pr}_2} Y \times Y. \tag{3.1.2}$$

(3.2) *Splitting a correspondence along open and closed embeddings.* The additivity of local terms used in Proposition 2.4.3 has an analogue for the morphism $\varphi(b, M)$. Let us discuss this in somewhat greater generality. Consider again the situation of subsection (2.1). Let $i : Y \hookrightarrow X$ be a closed embedding and $j : U = X \smallsetminus Y \hookrightarrow X$ be the open embedding of the complement. The following two conditions are equivalent:

$$b_1^{-1}(U) \subset b_2^{-1}(U), \qquad b_2^{-1}(Y) \subset b_1^{-1}(Y). \tag{3.2.1}$$

We can interpret these conditions as saying that the closed subscheme $Y$, or the stratification $\{Y, U\}$, is *respected* by the correspondence $b$. (For most of what follows, this condition is only needed in a neighborhood of $\mathrm{Fix}\, b$ in $B$, but in our final application we can always reduce to the case where (3.2.1) holds globally.) We want to study how the morphism $\varphi(b, M)$ behaves with respect to the exact triangle

$$j_! j^! M \to M \to i_* i^* M. \tag{3.2.2}$$

PROPOSITION 3.2.3. *If* (3.2.1) *holds, there is a natural commutative diagram*

$$\begin{array}{ccc}
b'^* a^! (M \otimes^{\mathbf{L}}_S DM) & \longrightarrow & b'^* a^! (j_! j^! M \otimes^{\mathbf{L}}_S D j_! j^! M) \oplus b'^* a^! (i_* i^* M \otimes^{\mathbf{L}}_S D i_* i^* M) \\
{\scriptstyle \varphi(b,M)} \Big\downarrow & & \Big\downarrow {\scriptstyle \varphi(b, j_! j^! M) \oplus \varphi(b, i_* i^* M)} \\
a'^! b^* (M \otimes^{\mathbf{L}}_S DM) & \longleftarrow & a'^! b^* (j_! j^! M \otimes^{\mathbf{L}}_S D j_! j^! M) \oplus a'^! b^* (i_* i^* M \otimes^{\mathbf{L}}_S D i_* i^* M).
\end{array}$$

*Proof.* As in the proof of Proposition 2.4.3 we consider the filtered complex $(M, F)$, where $F^{-1}M = 0$, $F^0M = j_!j^!M$ and $F^1M = M$. Then $P = M \otimes^{\mathbf{L}}_S DM$ is a filtered complex, whose associated graded complex is zero outside the range from $-1$ to $1$, and whose graded part of degree $0$ is

$$\operatorname{gr}^0_F P = (j_!j^!M \otimes^{\mathbf{L}}_S Dj_!j^!M) \oplus (i_*i^*M \otimes^{\mathbf{L}}_S Di_*i^*M).$$

It suffices to prove that, in the commutative diagram

$$\begin{tikzcd}
b'^*a^!P \arrow[d, "\text{base change}"'] & b'^*a^!F^0P \arrow[l, "①"'] \arrow[r] \arrow[d, "\text{base change}"'] & b'^*a^!\operatorname{gr}^0_F P \arrow[d, "\text{base change}"] \\
a'^!b^*P & a'^!b^*F^0P \arrow[l] \arrow[r, "②"] & a'^!b^*\operatorname{gr}^0_F P,
\end{tikzcd}$$

the arrows ① and ② are isomorphisms. This is equivalent to

$$\begin{aligned} b'^*a^!\operatorname{gr}^1_F P &= b'^*a^!(i_*i^*M \otimes^{\mathbf{L}}_S Dj_!j^!M) = 0, \\ a'^!b^*\operatorname{gr}^{-1}_F P &= a'^!b^*(j_!j^!M \otimes^{\mathbf{L}}_S Di_*i^*M) = 0. \end{aligned} \tag{3.2.4}$$

To prove the first vanishing observe that by duality and the Künneth formulas,

$$i_*i^*M \otimes^{\mathbf{L}}_S Dj_!j^!M \cong (i \times j)_*(i^*M \otimes^{\mathbf{L}}_S Dj^!M).$$

Since $a^{-1}(Y \times U) = \emptyset$, the transpose of the proper base change implies that the composite functor $a^!(i \times j)_*$ is zero. Together this proves the first line of equation (3.2.4). The second can be treated in the same way, by assumption (3.2.1). □

(3.3) *Behavior under direct image.* Let us now study the behavior of $\varphi(b, M)$ under a direct image. Consider two diagrams, as in subsection (2.1), and some compatible proper morphisms between them. In other words, consider a commutative diagram of schemes over $S$ and proper morphisms:

$$\begin{tikzcd}
& F \arrow[dl, "e'"'] \arrow[dr, "d'"] \arrow[rrrr, "f_C"] & & & & C \arrow[dl, "b'"'] \arrow[dr, "a'"] & \\
D \arrow[dr, "d"'] \arrow[rrrr, "f_A"] & & E \arrow[dl, "e"] \arrow[rrrr, "f_B"] & & A \arrow[dr, "a"'] & & B. \arrow[dl, "b"] \\
& Y \times Y \arrow[rrrr, "f \times f"'] & & & & X \times X &
\end{tikzcd} \tag{3.3.1}$$

As in (2.1) we assume $C = A \times_{(X \times X)} B$ and $F = D \times_{(Y \times Y)} F$, but here we do not require that $A$ or $D$ be the diagonal.

PROPOSITION 3.3.2. *The following diagram of base-change morphisms commutes:*

$$\begin{tikzcd}
& b'^*a^!(f\times f)_* \arrow[r, "\varphi(b,f_*M)"] & a'^!b^*(f\times f)_* \arrow[dr, "②"] & \\
b'^*f_{A*}d^! \arrow[ur, "①"] \arrow[dr] & & & a'^!f_{B*}e^*. \\
& f_{C*}e'^*d^! \arrow[r, "f_{C*}\varphi(e,M)"] & f_{C*}d'^!e^* \arrow[ur] &
\end{tikzcd}$$

*Proof.* There are several commutative hexagons of this type, all of which result from the definition and standard properties of base-change morphisms. To prove this particular one we first transform it, using the adjointness of the functors $b'^*$ and $b'_*$. The resulting hexagon is the outer edge of the following diagram:

$$\begin{tikzcd}
a^!f_* \arrow[drr, "\text{adj}"] \arrow[drrr, "\text{adj(b.c.)}"] \arrow[ddrr, "\text{adj(b.c)}"] \arrow[ddr, "\text{adj}"] & & & \\
& & a^!b_*b^*f_* \arrow[r, phantom, "\cong"] \arrow[d, "\text{b.c.}"] & b'_*a'^!b^*f_* \arrow[d, "\text{b.c.}"] \\
& a^!f_*e_*e^* \arrow[r, phantom, "="] & a^!b_*f_{B*}e^* \arrow[r, phantom, "\cong"] & b'_*a'^!f_{B*}e^* \\
& f_{A*}d^!e_*e^* \arrow[u, "\text{b.c.}"] \arrow[r, phantom, "\cong"] & f_{A*}e'_*d'^!e^* \arrow[r, phantom, "="] & b'_*f_{C*}d'^!e^* \arrow[u, "\text{b.c.}"] \\
& & f_{A*}e'_*e'^*d^! \arrow[u, "\text{b.c.}"] \arrow[r, phantom, "="] & b'_*f_{C*}e'^*d^! \arrow[u, "\text{b.c.}"] \\
f_{A*}d^! \arrow[uuuuu, "\text{b.c.}"] \arrow[uur, "\text{adj}"] \arrow[uurr, "\text{adj(b.c)}"] \arrow[urr, "\text{adj}"] \arrow[urrr, "\text{adj(b.c)}"] & & &
\end{tikzcd}$$

Here "$=$" denotes a trivial isomorphism and "$\cong$" a proper base-change isomorphism. The abbreviation "b.c." stands for base change, "adj" denotes an adjunction map and "adj(b.c.)" means the adjoint of a base-change morphism. The inner triangles commute by the definition of the base-change morphisms involved. The commutativity of the quadrangles is trivial, and the inner hexagon commutes by the composition law for base change ([2, XII, 4.4]). Thus the whole diagram commutes, as desired. □

We are interested in cases where the upper horizontal morphism in Proposition 3.3.2 factors through the lower horizontal morphism, i.e., where morphisms ① and ② are isomorphisms. This is, for instance, the case when diagram (3.3.1) is everywhere cartesian. In fact we shall see that it suffices if the diagram is cartesian only over certain open subsets. Consider an open embedding $U \hookrightarrow X$ and put $j : V := f^{-1}(U) \hookrightarrow Y$. Consider a complex of the form $M = j_!N$. We want to look at the diagram in Proposition 3.3.2 for the complex

$$P = M \otimes^{\mathbf{L}}_S DM = j_!N \otimes^{\mathbf{L}}_S Dj_!N.$$

We shall repeatedly need the following formulas, which result from duality and the Künneth formulas ([13, III, 1.6.4 and 1.7.1])

$$\begin{aligned} j_!N \otimes^{\mathbf{L}}_S Dj_!N &\cong (j \times \mathrm{id})_!(\mathrm{id} \times j)_*(N \otimes^{\mathbf{L}}_S DN) \\ &\cong (\mathrm{id} \times j)_*(j \times \mathrm{id})_!(N \otimes^{\mathbf{L}}_S DN). \end{aligned} \tag{3.3.3}$$

Consider the diagrams

$$\begin{array}{ccccccc} d^{-1}(Y \times V) & \to & a^{-1}(X \times U) & \qquad & e^{-1}(V \times Y) & \to & b^{-1}(U \times X) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ Y \times V & \to & X \times U, & & V \times Y & \to & U \times X. \end{array} \tag{3.3.4}$$

LEMMA 3.3.5. *Assume that diagrams* (3.3.4) *are cartesian. Then morphisms* ① *and* ② *in* (3.3.2), *applied to the complex* $P = j_!N \otimes^{\mathbf{L}}_S Dj_!N$, *are isomorphisms.*

*Proof.* For ①, using the second line of formula (3.3.3), we find it suffices to prove isomorphy for

$$f_{A*}d^!(\mathrm{id} \times j)_* \xrightarrow{\text{base change}} a^!(f \times f)_*(\mathrm{id} \times j)_*.$$

Since the diagram on the left in (3.3.4) is cartesian, the transpose of proper base change implies that this is an isomorphism over $X \times U$. For the same reason one can move $(\mathrm{id} \times j)_*$ past $d^!$ and $a^!$, and the assertion follows. The same argument works for ②, except that the first line of formula (3.3.3) must be used. □

PROPOSITION 3.3.6. *If diagrams* (3.3.4) *are cartesian and the morphism* $\varphi(e, M)$ *vanishes, then the morphism* $\varphi(b, f_*M)$ *vanishes.*

*Proof.* This is a direct consequence of Proposition 3.3.2 and Lemma 3.3.5. □

We shall use this result in cases of the following three types: for closed embeddings, for finite coverings and for proper modifications at the boundary.

## 4. The smooth case with constant coefficients

(4.1) *The question.* Now we start to attack the question of where the morphism $\varphi(b, M)$ (see (3.1.1)) vanishes. Throughout this section and the next two we fix the situation and the notation given in subsection (3.2), and we assume condition (3.2.1). In addition we require that $M|_Y = 0$; in other words, that $M = j_!N$ for some complex $N$ on $U$.

We shall say that $\varphi(b, M)$ *vanishes at the boundary* if and only if it vanishes on $b^{-1}(a(Y)) \subset \operatorname{Fix} b$. We want to give sufficient and practicable conditions for this to be so, and we shall make these conditions explicit as we proceed. Modulo resolution of singularities there is essentially only one geometric condition on the correspondence $b$ near the boundary (see Definition 4.4.3).

We treat this question in three parts, each being both more general than and relying on the preceding one. In the present section we assume that $X$ is smooth, that $Y$ is a divisor with normal crossings and that every $\mathcal{H}^i(N)$ is a constant sheaf. In Section 5 we shall only require that $U$ be smooth and that every $\mathcal{H}^i(N)$ be lisse. In order to reduce this case to the first we shall make a technical assumption, which is a consequence of the resolution of singularities. Section 6 treats the most general case. There we shall need certain stratifications, which exist, for instance, when the morphism $b_2$ is *quasifinite* over $U \subset X$, and we shall need the existence of certain blowups, which follows when the resolution of embedded singularities is known.

(4.2) *Formal arguments.* Let us now explain some purely formal reduction steps, which are more or less directly derived from [13, IIIb]. We shall apply them only in the smooth case, but they work in great generality as well.

To bring the problem into a more convenient form, we can write the definition ([2, XVIII, 3.1.14]) of the base-change morphism (3.1.1) as a commutative diagram of functors

$$\begin{array}{ccc} b'^*a^! & \xrightarrow{\varphi(b,M)} & a'^!b^* \\ \text{adj}\downarrow & & \uparrow\text{adj} \\ b'^*a^!b_*b^* & \cong & b'^*b'_*a'^!b^*, \end{array}$$

where the lower isomorphism comes from the transpose of a proper base change. Since $a$ is an embedding, $\varphi(b, M)$ vanishes on $b^{-1}(a(Y))$ if the adjunction morphism

$$a_!a^!(\mathrm{id} \to b_*b^*)P \tag{4.2.1}$$

vanishes on $b(B) \cap a(Y)$.

Next we want to get rid of the functor $a^!$. The open and closed embeddings

$$X \times X \smallsetminus A \overset{\alpha}{\hookrightarrow} X \times X \overset{a}{\hookleftarrow} A$$

give rise to an exact triangle of morphisms

$$a_!a^! \to \mathrm{id} \to \alpha_*\alpha^*.$$

Thus the adjunction morphism yields a morphism of exact triangles

$$\begin{array}{ccccc} a_! a^! P & \to & P & \to & \alpha_* \alpha^* P \\ \downarrow & & \downarrow & & \downarrow \\ a_! a^! b_* b^* P & \to & b_* b^* P & \to & \alpha_* \alpha^* b_* b^* P. \end{array}$$

Since $P|_{Y\times X} = 0$ (compare (3.3.3)), this yields a commutative diagram

$$\begin{array}{ccc} \alpha_* \alpha^* P[-1]|_{Y\times X} & \xrightarrow{\sim} & a_! a^! P|_{Y\times X} \\ \downarrow & & \downarrow \\ \alpha_* \alpha^* b_* b^* P[-1]|_{Y\times X} & \longrightarrow & a_! a^! b_* b^* P|_{Y\times X}. \end{array}$$

Hence it suffices to prove that the morphism

$$\alpha_* \alpha^* (\mathrm{id} \to b_* b^*) P \tag{4.2.2}$$

vanishes on $b(B) \cap a(Y)$.

Now we translate the problem into one on a blowup of $X \times X$. Abbreviate $Z = X \times X$ and consider a proper modification $\pi : \tilde{Z} \to Z$, which is an isomorphism outside $Y \times Y$. Assumption (3.2.1) implies that $b_1^{-1}(U) = b^{-1}(U \times U)$; let $\tilde{B}$ be its closure in $B \times_Z \tilde{Z}$. (When $b$ is a closed embedding, this is just the proper transform of $B$.) We then have a commutative diagram

$$\begin{tikzcd} & \tilde{B} \smallsetminus \tilde{b}^{-1}\pi^{-1}(A) \arrow[dl, "\tilde{b}^A"'] \arrow[rr, hook, "\tilde{\alpha}_B"] & & \tilde{B} \arrow[dl, "\tilde{b}"] \arrow[dr, "\pi_B"] & \\ \tilde{Z} \smallsetminus \pi^{-1}(A) \arrow[rr, hook, "\tilde{\alpha}"] \arrow[dr] & & \tilde{Z} \arrow[dr, "\pi"] & & B \arrow[dl, "b"] \\ & Z \smallsetminus A \arrow[rr, hook, "\alpha"] & & Z & \end{tikzcd}$$

in which all oblique morphisms are proper.

LEMMA 4.2.3. *The morphism* (4.2.2) *factors through the complex*

$$\pi_* \tilde{b}_* \tilde{b}^* \tilde{\alpha}_* \tilde{\alpha}^* \pi^* P.$$

*Proof.* First consider the commutative diagram of adjunction morphisms

$$\begin{array}{ccccc} P & & \xrightarrow{\quad\quad\mathrm{adj}\quad\quad} & & b_* b^* P \\ \mathrm{adj} \downarrow & & & & \downarrow \mathrm{adj} \\ \pi_* \pi^* P & \xrightarrow{\mathrm{adj}} & \pi_* \tilde{b}_* \tilde{b}^* \pi^* P & \cong & b_* \pi_{B*} \pi_B^* b^* P. \end{array}$$

Since $\pi$ is an isomorphism over $U \times X$ and since $P$ vanishes on the complement, the left vertical arrow is an isomorphism. The same holds for the right vertical arrow. This shows that the adjunction morphism $(\mathrm{id} \to b_* b^*)P$ factors through the morphism $\pi_*(\mathrm{id} \to \tilde{b}_* \tilde{b}^*)\pi^* P$.

Next there is a trivial isomorphism of functors $\alpha_* \alpha^* \pi_* \cong \pi_* \tilde{\alpha}_* \tilde{\alpha}^*$. On the other hand, from the definition of the base-change morphism $\tilde{b}^* \tilde{\alpha}_* \to \tilde{\alpha}_{B*} \tilde{b}^{A*}$, one can derive the commutativity of the following diagram of functors:

$$\begin{array}{ccccc}
\tilde{\alpha}_*\tilde{\alpha}^* & \xrightarrow{\text{adj}} & \tilde{\alpha}_*\tilde{\alpha}^*\tilde{b}_*\tilde{b}^* & \overset{\text{(trivial)}}{\cong} & \tilde{\alpha}_*\tilde{b}^A_*\tilde{b}^{A*}\tilde{\alpha}^* \\
\text{adj}\downarrow & & & & \wr\| \\
\tilde{b}_*\tilde{b}^*\tilde{\alpha}_*\tilde{\alpha}^* & & \xrightarrow{\qquad\text{base change}\qquad} & & \tilde{b}_*\tilde{\alpha}_{B*}\tilde{b}^{A*}\tilde{\alpha}^*.
\end{array}$$

This implies the assertion. □

By the lemma it suffices to prove that the complex in Lemma 4.2.3 vanishes on $b(B) \cap a(Y)$. Using proper base change for the functors $\pi_*$ and $\tilde{b}_*$, we have proved the following result:

PROPOSITION 4.2.4. *The morphism $\varphi(b, M)$ vanishes on $b^{-1}(a(Y))$ if the complex*

$$\tilde{\alpha}_*\tilde{\alpha}^*\pi^*P$$

*vanishes on $\tilde{b}(\tilde{B}) \cap \pi^{-1}(a(Y))$.*

This reduction is useful, because the complex in Proposition 4.2.4 does not depend at all on the correspondence. One can therefore proceed as follows: First try to find a suitable blowup $\tilde{Z}$ such that the complex in Proposition 4.2.4 vanishes at sufficiently many points. Then $\varphi(b, M)$ will vanish on $b^{-1}(a(Y))$ whenever $\tilde{b}(\tilde{B}) \cap \pi^{-1}(a(Y))$ is contained in this "good" set of points. This will be a purely geometric condition. In effect this allows one to separate the geometry of the correspondence from the geometry of $\bar{X}$.

(4.3) *Construction of a blowup.* Now we shall construct a blowup for which the above program works. We consider only the special situation described in subsection (4.1): Assume that $X$ is smooth and that $Y = D \subset X$ is a divisor with normal crossings. Let $D_1, \ldots, D_m$ be the pairwise-distinct, irreducible components of $D$. For simplicity assume that every $D_i$ is smooth, i.e., that $D$ has no self-intersections. This extra condition is always satisfied after replacing $X$ by a blowup or else by an étale covering and the correspondence by the associated pullback.

As in subsection (4.2) we abbreviate $Z = X \times X$. For every $1 \le i \le m$ let $\mathcal{J}_i$ be the ideal sheaf on $Z$ of the subscheme $D_i \times D_i$. Let $\mathcal{J}$ be the product of all these. We define $\tilde{Z}$ as the blowup of $Z$ in $\mathcal{J}$. Abbreviate $D^1 = D \times X$ and $D^2 = X \times D$, and let $\tilde{D}^i \subset \tilde{Z}$ denote the proper transform of $D^i$ ($i = 1, 2$). This makes sense, since $\pi$ is an isomorphism at all generic points of $D^i$.

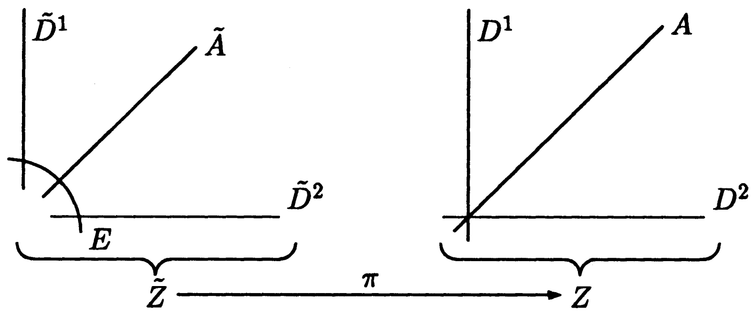
To visualize this modification, it suffices to look at a neighborhood of the diagonal. Locally for the étale topology, $U = X \setminus D \hookrightarrow X$ is isomorphic to the open embedding $\mathbf{G}_m^d \hookrightarrow \mathbf{A}^d$, so we only have to consider this case. First let us describe the situation in detail for $d = 1$.

In this case $Z = \mathbf{A}^1 \times \mathbf{A}^1 = \mathbf{A}^2$, and $\tilde{Z}$ is just the blowup in the origin; let us denote it by $\tilde{\mathbf{A}}^2$. The boundary divisors $D^1, D^2$ are coordinate axes. The standard, open, affine covering of $\tilde{Z}$ is

$$\tilde{\mathbf{A}}^2 \smallsetminus \tilde{D}^1 = \operatorname{Spec} k[x,v],$$
$$\tilde{\mathbf{A}}^2 \smallsetminus \tilde{D}^2 = \operatorname{Spec} k[u,y],$$

where $u = x/y$ and $v = y/x$. Moreover $\tilde{D}^1 \subset \operatorname{Spec} k[u,y]$ is given by $u = 0$, and $\tilde{D}^2 \subset \operatorname{Spec} k[x,v]$ by $v = 0$. The exceptional divisor $E = \pi^{-1}(\{(0,0)\})$ has the equation $y = 0$, $x = 0$ in the respective charts. The proper transform $\tilde{A}$ of the diagonal $A$ is given by $u = 1$ and $v = 1$, respectively. Since the tangent directions of $D^1$, $D^2$ and $A$ at the origin are pairwise distinct, their proper transforms are pairwise disjoint. The following sketch visualizes this:

(4.3.1)




For arbitrary $d$, everything decomposes into a product corresponding to the $d$ coordinates on $\mathbf{A}^d$. More precisely, under the isomorphism

$$\begin{array}{ccc} \mathbf{A}^d \times \mathbf{A}^d & \xrightarrow{\sim} & (\mathbf{A}^2)^d, \\ ((x_1,\ldots,x_d),(y_1,\ldots,y_d)) & \mapsto & ((x_1,y_1),\ldots,(x_d,y_d)), \end{array}$$

$\tilde{Z}$ becomes isomorphic to $(\tilde{\mathbf{A}}^2)^d$. The above-mentioned facts generalize: for instance,

$$\tilde{Z} \smallsetminus \tilde{D}^1 = \operatorname{Spec} k[x_1,\ldots,x_d, y_1/x_1,\ldots,y_d/x_d], \tag{4.3.2}$$

and the irreducible components of the divisor $\tilde{D}^1$ are the inverse images under the $i^{\text{th}}$ projection of the proper transform of $\{0\} \times \mathbf{A}^1$ in $\tilde{\mathbf{A}}^2$, for all $1 \le i \le d$.

(4.4) *Vanishing of a complex.* With this blowup we can indeed carry out the program outlined above.

PROPOSITION 4.4.1. *The restriction of the complex*

$$\tilde{\alpha}_*\tilde{\alpha}^*\pi^*(j \times \mathrm{id})_!(\mathrm{id} \times j)_*\Lambda$$

*to* $\tilde{D}^1 \cap \pi^{-1}(a(Y))$ *vanishes.*

*Proof.* Since the assertion is local in the étale topology on $Z$, it suffices to consider the coordinate-hyperplane case described in subsection (4.3). Now everything decomposes into a product:

$$\begin{aligned} U &= \mathbf{G}_m^d, \\ X &= (\mathbf{A}^1)^d, \\ Z &= (\mathbf{A}^2)^d, \\ \tilde{Z} &= (\tilde{\mathbf{A}}^2)^d, \\ A &= (A_0)^d, \end{aligned}$$

where $A_0$ denotes the diagonal in $\mathbf{A}^2$. By the Künneth formulas ([13, III, 1.6.1, 1.6.4, and 1.7.1]) the complex in question is quasi-isomorphic to an external tensor product of analogous complexes for each of these factors. Every irreducible component of $\tilde{D}^1$ is also a product. By the Künneth decomposition it suffices to prove the vanishing on the $i^{\text{th}}$ factor. In other words we have reduced the assertion to the case $d = 1$. The rest can be viewed as a special case of [13, IIIb, §4].

By the explicit description in subsection (4.3), and by the same notation, it suffices to calculate inside the neighborhood $V = \tilde{\mathbf{A}}^2 \smallsetminus (\tilde{D}^2 \cup \tilde{A})$ of $\tilde{D}^1$. The complex in question is defined in terms of the morphisms

$$\begin{tikzcd}
& & \tilde{\mathbf{A}}^2 \smallsetminus \pi^{-1}(A) \arrow[d, hook, "\tilde{\alpha}"] \\
\mathbf{A}^2 \smallsetminus (D^1 \cup D^2) \arrow[r, hook, "\mathrm{id}\times j"] & \mathbf{A}^2 \smallsetminus D^1 \arrow[r, hook, "\tilde{j}"] \arrow[dr, "j\times\mathrm{id}"'] & \tilde{\mathbf{A}}^2 \arrow[d, "\pi"] \\
& & \mathbf{A}^2.
\end{tikzcd}$$

Since trivially $\pi^* \circ (j \times \mathrm{id})_! \cong \tilde{j}_!$, we only have to deal with the upper part of this diagram. Intersecting everything with $V$ (and keeping the old notation for the morphisms $\tilde{j}$ and $\tilde{\alpha}$) yields

$$\begin{tikzcd}
& & V \smallsetminus E \arrow[d, hook, "\tilde{\alpha}"] \\
V \smallsetminus (\tilde{D}^1 \cup E) \arrow[r, "\mathrm{id}"] & V \smallsetminus (\tilde{D}^1 \cup E) \arrow[r, hook, "\tilde{j}"] & V.
\end{tikzcd}$$

Thus it suffices to prove the vanishing of $\tilde{\alpha}_* \tilde{\alpha}^* \tilde{j}_! \Lambda$ on $\tilde{D}^1$. Changing the notation for our morphisms, we can consider the diagram

$$\begin{tikzcd}
V \smallsetminus (\tilde{D}^1 \cup E) \arrow[r, hook, "j_1'"] \arrow[d, hook, "j_2'"'] & V \smallsetminus E \arrow[d, hook, "j_2"] \\
V \smallsetminus \tilde{D}^1 \arrow[r, hook, "j_1"] & V.
\end{tikzcd}$$

The complex in question is

$$j_{2*}j_2^*(j_2 \circ j_1')_!\Lambda \cong j_{2*}j_{1!}'\Lambda,$$

and it remains to show that this is isomorphic to $j_{1!}j_{2*}'\Lambda$. This follows from another way to apply the Künneth formulas: namely, by embedding $V$ into the affine plane such that $\tilde{D}^1$ and $E$ correspond to the coordinate axes. Indeed, by (4.3), we can write

$$V = \tilde{\mathbf{A}}^2 \smallsetminus (\tilde{D}^2 \cup \tilde{A}) = \operatorname{Spec} k\left[u, \frac{1}{u-1}, y\right]$$

with $u = x/y$, and in this neighborhood, $\tilde{D}^1$ and $E$ are given by the equations $u = 0$ and $y = 0$, respectively. Thus we can write everything as a product with respect to the coordinates $u$, $y$, as desired. □

Coming back to (4.1), we find that Proposition 4.4.1 implies another result:

PROPOSITION 4.4.2. *Assume that $M = j_!N$ and that every $\mathcal{H}^i(N)$ is a constant sheaf on $U$. Then the complex $\tilde{\alpha}_*\tilde{\alpha}^*\pi^*P$ vanishes on $\tilde{D}^1 \cap \pi^{-1}(a(Y))$.*

*Proof.* Since the assertion is local for the Zariski topology, we may assume that $X$, and hence $U$, are equidimensional. We must apply the functor

$$F:\ K \mapsto (\tilde{\alpha}_*\tilde{\alpha}^*\pi^*(j \times \mathrm{id})_!(\mathrm{id} \times j)_*K)\big|_{\tilde{D}^1 \cap \pi^{-1}(a(Y))}$$

to the complex $N \otimes_S^{\mathbf{L}} DN$. First observe that every $\mathcal{H}^i(N \otimes_S^{\mathbf{L}} DN)$ is a constant sheaf. Indeed, since $U$ is smooth and equidimensional over an algebraically closed field, the dual of the constant sheaf $\Lambda$ on $U$ is isomorphic to the constant sheaf $\Lambda$ placed in cohomological degree $-2\dim(U)$. This implies that every $\mathcal{H}^i(DN)$ is constant, and the same follows for $N \otimes_S^{\mathbf{L}} DN$, as desired.

We want to use dévissage to prove that $FK = 0$ for suitable complexes $K$. Since, in general, truncation does not preserve the category $D^b_{ctf}(\ ,\Lambda)$, we must work in the larger category $D^b_c(\ ,\Lambda)$. It suffices to prove that $FK = 0$ for all $K \in \mathcal{O}\mathrm{b}(D^b_c(Z, \Lambda))$ such that all $\mathcal{H}^i(K)$ are constant sheaves. Using truncation (which is now permitted) and a shift in cohomological degree reduces to the case of a constant constructible sheaf of $\Lambda$-modules. Using a filtration by constant subsheaves further reduces to the case where the stalk is a cyclic $\Lambda$-module, say, $\Lambda/I$ for an ideal $I$. Now we just apply Proposition 4.4.1 with $\Lambda/I$ in place of $\Lambda$, and thus we are done. □

Now Propositions 4.2.4 and 4.4.2 suggest the following definition:

*Definition* 4.4.3. We say that $b$ is *in good position with respect to $D$* if and only if

$$\tilde{b}(\tilde{B}) \cap \pi^{-1}(a(Y)) \subset \tilde{D}^1.$$

PROPOSITION 4.4.4. *Consider the situation of subsection* (4.1) *where $X$ is smooth, $Y$ is a union of smooth divisors with at most normal crossings and every $\mathcal{H}^i(N)$ is a constant sheaf. If $b$ is in good position with respect to $D$, then the morphism $\varphi(b, M)$ vanishes on the boundary.*

## 5. Reduction to constant coefficients

(5.1) *The result.* In this section we shall consider the second special case of subsection (4.1): $U$ is smooth and every $\mathcal{H}^i(N)$ is lisse. We make no assumption on $X$, but in order to reduce this case to the former we assume that we are given an explicit "nice" covering that annihilates the wild ramification of each $\mathcal{H}^i(N)$. Precisely we consider the following data:

DATA 5.1.1. *These consist of a cartesian diagram*

$$\begin{array}{ccc} U^\natural & \hookrightarrow & X^\natural \\ f_U \downarrow & \square & \downarrow f \\ U & \hookrightarrow & X \end{array}$$

*such that*

(a) *$f$ is proper;*

(b) *$f_U$ is a finite, étale, Galois covering with Galois group $G$;*

(c) *$X^\natural$ is smooth and $D^\natural := X^\natural \setminus U^\natural = f^{-1}(Y)$ is a union of smooth divisors with, at most, normal crossings;*

(d) *every $f_U^*\mathcal{H}^i(N)$ is, at most, tamely ramified along $D^\natural$.*

The existence of such data follows from a resolution of singularities:

LEMMA 5.1.2. *Given $U \hookrightarrow X$ and $N$ as above, the existence of such data is implied by Conjecture* (A.1) *for the field $k$ (see Appendix).*

*Proof.* Without loss of generality we may assume that $U$ is irreducible. Choose a finite, étale, Galois covering $U^\natural \to U$, which trivializes the non-pro-$\ell$ part of the global monodromy of every $\mathcal{H}^i(N)$. Then part (d) of Data 5.1.1 follows from the other conditions. To construct $X^\natural$ apply Conjecture A.1 to the normalization of $X$ in the function field of $U^\natural$. $\square$

Let $b^\natural : B^\natural \to X^\natural \times X^\natural$ be the pullback (cf. (3.1.2)) of the correspondence $b$. With $A^\natural$ equal to the diagonal in $X^\natural \times X^\natural$, we are precisely in the situation considered in the preceding section. In particular

$$\tilde{b}^\natural(\tilde{B}^\natural),\ \tilde{D}^{\natural 1} \subset \tilde{Z}^\natural \xrightarrow{\pi^\natural} Z^\natural = X^\natural \times X^\natural$$

are defined as in subsections (4.2) and (4.3). The aim of this section is to prove the following result:

PROPOSITION 5.1.3. *In the situation above, if $b^\natural$ is in good position with respect to $D^\natural$, then the morphism $\varphi(b, M)$ vanishes on the boundary.*

The proof will be given in subsection (5.4), but first we want to discuss the notion of good position in more detail.

(5.2) *Nature of this geometric condition.* Consider the situation described in subsection (4.4), i.e., $X$ is smooth, and $Y = D \subset X$ is a union of smooth divisors with, at most, normal crossings. We want to reformulate condition (4.4.3). Consider the diagram

$$\begin{array}{ccccccc} b^{-1}(U \times U) & \to & U \times U & \hookrightarrow & \tilde{Z} \setminus \tilde{D}^1 & \subset & \tilde{Z} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \pi \\ B & \xrightarrow{b} & X \times X & = & Z & = & Z. \end{array}$$

By abuse of notation we view $U \times U$ as a subscheme of $\tilde{Z}$.

LEMMA 5.2.1. *The following conditions are equivalent:*
(a) *$b$ is in good position with respect to $D$.*
(b) *The closure of $b(B) \cap (U \times U)$ in $\tilde{Z}$ meets $\pi^{-1}(a(Y))$ at most in $\tilde{D}^1$.*

*Proof.* The definition of $\tilde{B}$ in subsection (4.2) implies that $\tilde{b}(\tilde{B})$ is the closure of $b(B) \cap (U \times U)$ in $\tilde{Z}$. This implies the desired equivalence. □

*Remarks.* In the special case $\dim(X) = 1$, consider a point $x \in D$ such that $z = (x, x) \in b(B)$. Our notion of good position means that $b(B)$ meets the point $z$ tangentially to the divisor $D^1 = \{x\} \times X$. This condition is stronger than that of *general position* in [13, IIIb, 1.2]. For our application to Deligne's conjecture, however, this makes no difference.

To understand the nature of condition (4.4.3) a little better it is worthwhile to compare our situation with that of the Lefschetz trace formula in singular cohomology. There the behavior of $\mathrm{LT}_\beta(u)$ seems to be governed by the principle: *contracting* fixed points contribute what one expects naïvely, *expanding* fixed points give a formula in terms of the dual correspondence, and other fixed points are more complicated. To clarify the terminology consider the case where the correspondence is the transpose of the graph of a map, i.e., where $B = \{(f(x), x) \mid x \in X\} \subset X \times X$ for a morphism $f$. Then we call the correspondence contracting, resp. expanding, at a fixed point if and only if $f$ has the same property in the literal sense (with respect to some chosen metric). This point of view has been developed by J. Bewersdorff in [4] and in as yet unpublished work by M. Goresky and R. MacPherson.

Our geometric condition can be viewed as an algebraic analog of the above *contracting* hypothesis. In fact, in the one-dimensional case, when $b$

is the transpose of the graph of a map $f$, then condition (4.4.3) means that the derivative of $f$ must vanish at the fixed point. In the higher-dimensional case there is not such an easy description, but perhaps the condition can be interpreted as saying that the correspondence be contracting at least in one suitable direction.

(5.3) *Invariance of "good position".* Now we shall prove that the property of good position is stable under finite coverings and blowups at the boundary. For this we come back to the situation of subsection (5.1), still assuming that $X$ is smooth and that $Y = D \subset X$ is a union of smooth divisors with, at most, normal crossings.

LEMMA 5.3.1. *Over a neighborhood of the diagonal in $Z = X \times X$ there exists a unique morphism $\tilde{Z}^\natural \smallsetminus \tilde{D}^{\natural 1} \to \tilde{Z} \smallsetminus \tilde{D}^1$ inducing $f \times f$ on $U^\natural \times U^\natural$.*

*Proof.* The uniqueness follows from the density of $U \times U$ and the separation hypotheses. In particular the assertion is local with respect to the Zariski topology; so we may assume that $X = \operatorname{Spec} A$ for a regular integral domain $A$ such that $D$ is given by the equation $t_1 \cdots t_r = 0$ for independent parameters $t_\nu \in A$ $(1 \leq \nu \leq r)$. Then $Z = \operatorname{Spec} A \otimes_k A$, and by equation (4.3.2),

$$\tilde{Z} \smallsetminus \tilde{D}^1 = \operatorname{Spec} A \otimes_k A \left[ \left. \frac{1 \otimes t_\nu}{t_\nu \otimes 1} \right|_{1 \leq \nu \leq r} \right].$$

After localizing on $X^\natural$ in the same way, write $\varphi : A \to A^\natural$ for the ring homomorphism dual to $f$. Let $t^\natural_\mu \in A^\natural$ $(1 \leq \mu \leq s)$ be independent parameters defining $D^\natural$. The problem is to extend $\varphi \otimes \varphi : A \otimes_k A \to A^\natural \otimes_k A^\natural$ to a homomorphism

$$A \otimes_k A \left[ \left. \frac{1 \otimes t_\nu}{t_\nu \otimes 1} \right|_{1 \leq \nu \leq r} \right] \to A^\natural \otimes_k A^\natural \left[ \left. \frac{1 \otimes t^\natural_\mu}{t^\natural_\mu \otimes 1} \right|_{1 \leq \mu \leq s} \right].$$

Observe that, since $f^{-1}(D) = D^\natural$ and $A^\natural$ is a unique factorization domain, we have

$$\varphi(t_\nu) = u \cdot \prod_{\mu=1}^{s} (t^\natural_\mu)^{\alpha_{\mu,\nu}}$$

for a unit $u \in (A^\natural)^\times$ and some nonnegative integers $\alpha_{\mu,\nu}$. Thus the definition

$$(\varphi \otimes \varphi) \left( \frac{1 \otimes t_\nu}{t_\nu \otimes 1} \right) := \frac{1 \otimes \varphi(u)}{\varphi(u) \otimes 1} \cdot \prod_{\mu=1}^{s} \left( \frac{1 \otimes t^\natural_\mu}{t^\natural_\mu \otimes 1} \right)^{\alpha_{\mu,\nu}}$$

yields the desired extension. $\square$

PROPOSITION 5.3.2. *If $b$ is in good position with respect to $D$, then $b^\natural$ is in good position relative to $D^\natural$.*

*Proof.* The closure of $b^\natural(B^\natural) \cap (U^\natural \times U^\natural)$ in $\tilde{Z}^\natural \setminus \tilde{D}^{\natural 1}$ maps, under the morphism constructed in Lemma 5.3.1, to the closure of $b(B) \cap (U \times U)$ in $\tilde{Z} \setminus \tilde{D}^1$. By assumption and Lemma 5.2.1, the latter does not meet $\pi^{-1}(a(Y))$. Thus the former does not meet $\pi^{\natural -1}(a^\natural(Y^\natural))$; hence by Lemma 5.2.1 as applied to $Z^\natural$, etc., we are done. □

(5.4) *Proof of Proposition* 5.1.3. We shall first need a lemma.

LEMMA 5.4.1. *It suffices to prove the assertion universally under the assumption that every $\mathcal{H}^i(N)$ becomes constant over $U^\natural$.*

*Proof.* First, by limit arguments, we may reduce the assertion to the case where $\Lambda$ is a finite ring. We want to replace $U^\natural$ by a larger covering, which trivializes every $\mathcal{H}^i(N)$ and which possesses a smooth toroidal compactification. Such a covering can be constructed as follows: Without loss of generality we may assume that $X$ and $X^\natural$ are connected. Let $U^{\natural\natural} \to U^\natural$ be the unique, smallest, finite, étale, Galois covering that trivializes every $\mathcal{H}^i(N)$ and let $X^{\natural\natural}$ be the normalization of $X^\natural$ in the function field of $U^{\natural\natural}$. By assumption, $X^{\natural\natural} \to X^\natural$ is at most tamely ramified over $D^\natural$. It follows that $X^{\natural\natural}$ is again smooth and that the complement $D^{\natural\natural} = X^{\natural\natural} \setminus U^{\natural\natural}$ is again a divisor with normal crossings. After replacing $X^{\natural\natural}$ by a blowup, if necessary, we find that $D^{\natural\natural}$ has no self-intersections. Making the same definitions as in subsection (5.1) for $X^{\natural\natural}$ in place of $X^\natural$, we know that Proposition 5.3.2 implies that $b^{\natural\natural}$ is in good position. Observe by the uniqueness of $U^{\natural\natural}$ that the finite étale covering $U^{\natural\natural} \to U$ is again Galois. We may therefore replace $X^\natural$ by $X^{\natural\natural}$, after which we may assume that every $\mathcal{H}^i(N)$ becomes *constant* over $U^\natural$. □

For the next step denote the finite covering $U^\natural \to U$ by $f_U$.

LEMMA 5.4.2. *It suffices to prove the assertion for $f_{U*}f_U^*N$ in place of $N$, with $N$ as in Lemma* 5.4.1.

*Proof.* Since $f_U$ is an étale Galois covering with the Galois group $G$, there is a natural action of $G$ on $f_{U*}f_U^*N$, which makes this into an object of $D^b_{ctf}(U, \Lambda[G])$. Moreover we have a canonical isomorphism $N \cong (f_{U*}f_U^*N) \otimes^{\mathbf{L}}_{\Lambda[G]} \Lambda$, where $G$ acts trivially on $\Lambda$. Since the derived functors $a^!$, $b^*$, etc. commute with the tensor product $(\;) \otimes^{\mathbf{L}}_{\Lambda[G]} \Lambda$, the reduction follows. □

The remainder of the proof consists of applying Propositions 4.4.4 and 3.3.6. Since the inverse image of the diagonal $A$ decomposes into the diagonal $A^\natural$ and other components, we will break up the problem into several steps.

Let $X^\flat$ be the normalization of $X$ in the function field of $U^\natural$. We then have a cartesian diagram

$$\begin{tikzcd}
U^\natural \arrow[r, hook, "j^\natural"] \arrow[d, equal] & X^\natural \arrow[d] \\
U^\natural \arrow[r, hook, "j^\flat"] \arrow[d] & X^\flat \arrow[d] \\
U \arrow[r, hook, "j"] & X.
\end{tikzcd}$$

Define $Z^\flat = X^\flat \times X^\flat$ as usual and let $A^\flat$ be the diagonal in $Z^\flat$. Let $A^{\flat\flat}$ be the disjoint union of $|G|$ copies of $A^\flat$ and let $a^{\flat\flat} : A^{\flat\flat} \to Z^\flat$ be the morphism that is given by $(g \times \mathrm{id}) \circ a^\flat$ on the component associated to $g \in G$. Defining $b^\flat : B^\flat \to Z^\flat$ as a pullback, we obtain a commutative diagram

$$\begin{tikzcd}
A^\natural \arrow[r, hook, "a^\natural"] \arrow[d] & Z^\natural \arrow[d] & B^\natural \arrow[l, "b^\natural"'] \arrow[d] \\
A^\flat \arrow[r, hook, "a^\flat"] \arrow[d, hook] & Z^\flat \arrow[d, equal] & B^\flat \arrow[l, "b^\flat"'] \arrow[d, equal] \\
A^{\flat\flat} \arrow[r, "a^{\flat\flat}"] \arrow[d] & Z^\flat \arrow[d] & B^\flat \arrow[l, "b^\flat"'] \arrow[d] \\
A \arrow[r, hook, "a"] & Z & B \arrow[l, "b"']
\end{tikzcd} \tag{5.4.3}$$

whose right half is cartesian, but *not* its left half. Away from the boundary, however, the upper and lower squares on the left are cartesian.

Let $N$ be as in Lemma 5.4.1. We shall look at the different rows of diagram (5.4.3), beginning at the top. By Proposition 4.4.4 the morphism $\varphi(b^\natural, j^\natural_! f^*_U N)$ vanishes at the boundary. Next, by Proposition 3.3.6 applied to the upper two rows of (5.4.3), the same follows for the morphism $\varphi(b^\flat, j^\flat_! f^*_U N)$. Looking at the third row, we have a compatible action of $G \times G$ on everything, so it suffices to prove vanishing on representatives for all orbits. These representatives are provided by $A^\flat \subset A^{\flat\flat}$, i.e., by the second row. Thus the vanishing at the boundary also holds for the third row. (Here one must keep in mind that $A^{\flat\flat}$ is *not* the diagonal.) Finally we can apply Proposition 3.3.6 again to the last two rows of (5.4.3), which implies that the morphism $\varphi(b, j_! f_{U*} f^*_U M)$ vanishes at the boundary. By Lemma 5.4.2 this finishes the proof of Proposition 5.1.3. □

## 6. Reduction via a stratification

(6.1) *Construction of a stratification.* Now let us consider the general case of subsection (4.1). In order to reduce this to the smooth case treated in the preceding section we must split up our correspondence along a stratification. Precisely we want to have the following data:

DATA 6.1.1. *These consist of a finite filtration by reduced closed subschemes*

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_m = Y$$

*with the property that for all* $0 \leq i < m$,

(a) $S_i := Y_i \smallsetminus Y_{i+1}$ *is smooth over* $S$;

(b) *every* $\mathcal{H}^j(N)|_{S_i}$ *is lisse; and*

(c) $S_i \cap \overline{b_1(b_2^{-1}(S_i)) \smallsetminus Y_i} = \emptyset$.

It would be nice to have a filtration satisfying (a) and (b) and which is respected by the correspondence in a neighborhood of $\operatorname{Fix} b$, in the sense of assumption (3.2.1). In that case we could split up the correspondence successively along this stratification, without any additional work. In general one cannot find such a nice filtration, and we shall see below that the weaker condition (c) is a suitable replacement of condition (3.2.1).

PROPOSITION 6.1.2. *Assume that the morphism* $b_2$ *is quasifinite over* $U \subset X$. *Then there exists a filtration, as in Data* 6.1.1.

*Proof.* Fix a filtration by reduced closed subschemes

$$X = X_0 \supset X_1 \supset \cdots \supset X_n = Y$$

such that every $\mathcal{H}^\nu(N)|_{X_j \smallsetminus X_{j+1}}$ is lisse. The desired filtration $\{Y_i\}$ will be a refinement, i.e., every $X_j$ will be equal to some $Y_i$. Then every $S_i$ will be contained in some $X_j \smallsetminus X_{j+1}$, and condition (b) will hold.

We shall construct the $Y_i$ by induction over $i$. Consider $0 \leq j \leq n$ and suppose that we have already constructed the reduced closed subschemes $Y_0 \supset Y_1 \supset \cdots \supset Y_i$ such that $X_j \supset Y_i \supsetneq X_{j+1}$ and that properties (a), (b) and (c) hold for all $0 \leq i' < i$. The induction step consists of constructing $Y_i \supsetneq Y_{i+1} \supset X_{j+1}$ such that (a) and (c) hold. Assuming that this can be done, we obtain a decreasing sequence of closed subschemes. Since $X$ is noetherian, this sequence must stop. But the only way it can stop is when $Y_i = X_n = Y$, in which case we are done.

In order to perform the induction step, put

$$Z := X_{j+1} \cup \left(Y_i \cap \overline{b_1\left(b_2^{-1}(Y_i \smallsetminus X_{j+1})\right) \smallsetminus Y_i}\right).$$

We claim:

$$Y_i \supsetneq Z \supset X_{j+1}. \tag{6.1.3}$$

Indeed, by definition, we have $Y_i \supset Z \supset X_{j+1}$ so that it suffices to show that $Y_i$ is not contained in $Z$. Let $d = \dim(Y_i \smallsetminus X_{j+1})$. Since $b_2$ is quasifinite over

$X \smallsetminus X_{j+1}$, the dimension of $b_2^{-1}(Y_i \smallsetminus X_{j+1})$ is at most $d$. Thus each stratum in some stratification of the constructible set

$$b_1(b_2^{-1}(Y_i \smallsetminus X_{j+1})) \smallsetminus Y_i,$$

by locally closed subsets, has dimension at most $d$. What is added to this set when the closure is taken therefore has dimension at most $d-1$. Since $Y_i \smallsetminus X_{j+1}$ has dimension $d$, it cannot be contained in this closure; whence $Y_i \not\subset Z$, as desired.

By formula (6.1.3) we may choose $Y_i \supsetneqq Y_{i+1} \supset Z$ arbitrary such that $Y_i \smallsetminus Y_{i+1}$ is smooth. It remains to verify condition (c). But by construction,

$$(Y_i \smallsetminus Y_{i+1}) \cap \overline{b_1\big(b_2^{-1}(Y_i \smallsetminus Y_{i+1})\big) \smallsetminus Y_i} \subset Z \smallsetminus Y_{i+1} = \emptyset,$$

and we are done. □

(6.2) *Separating strata in blowups.* The merit of condition (c) in Data 6.1.1 is that when blowing up $X$ in the locus over which the $i^{\text{th}}$ filtration step is not respected, if certain strata separate in this blowup, then we find that the pullback of the correspondence respects the $i^{\text{th}}$ filtration step where necessary. The splitting can then be carried out on this blowup. To be precise we need the following data:

DATA 6.2.1. *These consist of a sequence of proper modifications*

$$X^{\langle 0\rangle} \to X^{\langle 1\rangle} \to \ldots \to X^{\langle m\rangle} = X$$

*such that, for all* $0 \leq i < m$,

(a) *the morphism* $\pi^{\langle i\rangle} : X^{\langle i\rangle} \to X$ *is an isomorphism outside* $Y_{i+1}$;

(b) *the closures of* $S_i$ *and* $b_1(b_2^{-1}(S_i)) \smallsetminus Y_i$ *in* $X^{\langle i\rangle}$ *are disjoint.*

The existence of such data follows from the resolution of singularities:

LEMMA 6.2.2. *Given Data* 6.1.1, *the existence of Data* 6.2.1 *is implied by Conjecture* A.3 *for the field* $k$ *(see Appendix).*

*Proof.* Use downward induction on $i$ and condition (c) of Data 6.1.1. □

Now we fix Data 6.2.1.

LEMMA 6.2.3. *For every* $0 \leq i < m$ *there exists a covering of* $X^{\langle i\rangle}$ *by finitely many open subsets* $U_\nu^{\langle i\rangle}$ *such that, for all* $\nu$, *the pullback of* $b$ *to* $U_\nu^{\langle i\rangle}$ *respects the closed subscheme* $(\pi^{\langle i\rangle})^{-1}(Y_i) \cap U_\nu^{\langle i\rangle}$.

*Proof.* We proceed by downward induction on $i$, beginning with the case where $i = m-1$. Since $b_2^{-1}(Y) \subset b_1^{-1}(Y)$ by assumption (3.2.1), the same holds for the pullback of $b$ to $X^{\langle m-1\rangle}$ and the closed subset $(\pi^{\langle m-1\rangle})^{-1}(Y_m)$.

The set of points that violate condition (3.2.1) for $(\pi^{\langle m-1\rangle})^{-1}(Y_{m-1})$ therefore lies completely over $U \times U$, and we can calculate on $X$. The offending set is

$$\begin{aligned} Z := & \, b_2^{-1}(Y_{m-1}) \smallsetminus b_1^{-1}(Y_{m-1}) \\ = & \, b_2^{-1}(S_{m-1}) \smallsetminus b_1^{-1}(Y_{m-1}), \end{aligned}$$

which implies

$$\begin{aligned} & b_1(Z) = b_1\big(b_2^{-1}(S_{m-1})\big) \smallsetminus Y_{m-1}, \\ & b_2(Z) \subset S_{m-1}. \end{aligned}$$

By assumption (b) of Data 6.2.1 the closures of these sets (more precisely, of their inverse images) in $X^{\langle m-1\rangle}$ are disjoint. Defining

$$U_\nu^{\langle m-1\rangle} := X^{\langle m-1\rangle} \smallsetminus \overline{(\pi^{\langle m-1\rangle})^{-1}(b_\nu(Z))}$$

for $\nu = 1, 2$ shows that the assertion follows for $i = m-1$.

For arbitrary $0 \leq i \leq m-2$ it suffices to look at what happens over any fixed open subset $U_\nu^{\langle i+1\rangle} \subset X^{\langle i+1\rangle}$, over which the $(i+1)^{\mathrm{st}}$ filtration step is respected. After replacing $X$ by $U_\nu^{\langle i+1\rangle}$, the correspondence by its pullback, $m$ by $i+1$, and $Y_{i'} \subset X$ by $U_\nu^{\langle i+1\rangle} \cap (\pi^{\langle i+1\rangle})^{-1}(Y_{i'})$ for $0 \leq i' \leq i+1$, we are precisely in the case above, where $i$ equals the new $m-1$. By induction the assertion follows. □

(6.3) *The result.* Given Data 6.1.1 and 6.2.1, we can now reduce to the smooth case. Let us first introduce some notation: Identify $S_i$ with its inverse image under $\pi^{\langle i\rangle}$ and denote its closure in $X^{\langle i\rangle}$ by $\bar{S}_i$. Let $b_{i,\nu}$ denote the pullback of the correspondence $b$ to $\bar{S}_i \cap U_\nu^{\langle i\rangle} \hookrightarrow X^{\langle i\rangle}$. By Lemma 6.2.3, condition (3.2.1) holds for each $b_{i,\nu}$ and the open embedding $S_i \cap U_\nu^{\langle i\rangle} \hookrightarrow \bar{S}_i \cap U_\nu^{\langle i\rangle}$.

PROPOSITION 6.3.1. *If the morphism $\varphi(b_{i,\nu}, M|_{\bar{S}_i})$ vanishes at the boundary for all $0 \leq i < m$ and all $\nu$, then the morphism $\varphi(b, M)$ also vanishes at the boundary.*

*Proof.* We use induction over $m$. First, since $\pi^{\langle m-1\rangle}$ is an isomorphism outside $Y$, by Proposition 3.3.6 it suffices to prove the vanishing for the pullback to $X^{\langle m-1\rangle}$. We may also concentrate on a single $U_\nu^{\langle m-1\rangle}$. Without loss of generality we may replace $\bar{X}$ by $U_\nu^{\langle m-1\rangle}$ and everything else by its pullback. Then the correspondence respects the open and closed embeddings

$$\bar{X} \smallsetminus Y_{m-1} \xrightarrow{j_{m-1}} \bar{X} \xleftarrow{i_{m-1}} Y_{m-1}.$$

Thus by Proposition 3.2.3 we may replace $M$ by either of the complexes $j_{m-1,!}j_{m-1}^{!}M$ and $i_{m-1,*}i_{m-1}^{*}M$. For the former complex, the vanishing follows from the induction hypothesis. On the other hand, the latter complex vanishes outside $S_{m-1}$. We may apply Proposition 3.3.6 to the closed embedding $\bar{S}_{m-1} \hookrightarrow \bar{X}$ and the complex $M|_{\bar{S}_{m-1}}$. Now the desired vanishing follows from the assumption in Proposition 6.3.1. □

## 7. Deligne's conjecture

(7.1) *Statement of the conjecture.* Now we shall specialize everything to correspondences over a finite field. To be precise, let $k$ be the algebraic closure of a finite field $\mathbf{F}_q$ with $q$ elements. Let $X$ be a scheme over $k$ (with the conventions of (1.1)) together with a model over $\mathbf{F}_q$, i.e., with the corresponding Frobenius morphism $\Phi_q : X \to X$. Let $M$ be an object of $D^b_{ctf}(X, \Lambda)$ endowed with an isomorphism $M \cong \Phi_q^* M$ (e.g., a complex of Weil sheaves over $\mathbf{F}_q$ in the sense of [7, 1.1.10]). Consider a geometric correspondence

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X$$

and a cohomological correspondence on $M$ with support in $b$:

$$u : b_1^* M \to b_2^! M.$$

We assume that

(7.1.1) $\qquad b_1$ is proper, and $b_2$ is quasifinite.

For any nonnegative integer $n$ we consider the "twisted" morphism

$$b^{(n)} := (\Phi_q^n \times \mathrm{id}) \circ b = (\Phi_q^n \circ b_1, b_2)$$

and the "twisted" cohomological correspondence

$$u^{(n)} : (\Phi_q^n \circ b_1)^* M = b_1^* \Phi_q^{n*} M \cong b_1^* M \xrightarrow{u} b_2^! M.$$

LEMMA 7.1.2. *If $b_2$ is everywhere of degree less than $q^n$, then* $\dim(\mathrm{Fix}\, b^{(n)}) = 0$.

*Proof* (reproduced from [9, 2.3]). Suppose that $\mathrm{Fix}\, b^{(n)}$ contains an irreducible curve $C$. The degree of $C$ over its image under $b_2$ is at most the degree of $b_2$. On the other hand, the degree of $C$ over its image under $\Phi_q^n \circ b_1$ is at least $q^n$. Hence the maps $b_2$ and $\Phi_q^n \circ b_1$ cannot agree on $C$, a contradiction. □

Since $b_1^{(n)}$ is proper, the global term $\mathrm{tr}(u_!^{(n)})$ is defined (see (1.3)). For $n$ sufficiently large, the correspondence $b^{(n)}$ has at most finitely many isolated fixed points, and the "naïve" local terms $\mathrm{tr}(u_\beta^{(n)})$ are defined (see (1.5)).

CONJECTURE 7.1.3 (Deligne). *There exists an integer $m_0$ such that, whenever $q^n > m_0$,*

$$\mathrm{tr}\left(u_!^{(n)}\right) = \sum_{\beta \in \mathrm{Fix}\, b^{(n)}} \mathrm{tr}\left(u_\beta^{(n)}\right).$$

For previous results on this conjecture see the Introduction.

(7.2) *The main result.* In its most general formulation, our proof of Conjecture 7.1.3 depends on rather elaborate resolution hypotheses. These follow from general conjectures on the resolution of singularities, which, unfortunately, are not known to date. On the other hand, in applications such as that to Shimura varieties, one may be able to perform the necessary constructions directly. Therefore, and out of the desire to formulate an unconditional theorem, we take the time to state these hypotheses in detail. We consider the statement of (7.1) and are interested in the following data:

DATA 7.2.1. *These consist of an open embedding $j : X \hookrightarrow \bar{X}$, defined over $\mathbf{F}_q$, where $\bar{X}$ is proper; a finite filtration by reduced closed subschemes*

$$X = Y_0 \supset Y_1 \supset \cdots \supset Y_m = \emptyset,$$

*defined over $\mathbf{F}_q$, such that for all $0 \leq i < m$:*

(a) *$S_i := Y_i \smallsetminus Y_{i+1}$ is smooth over $S$;*

(b) *every $\mathcal{H}^j(M)|_{S_i}$ is lisse;*

*a sequence of proper modifications*

$$X^{\langle 0\rangle} \to X^{\langle 1\rangle} \to \ldots \to X^{\langle m\rangle} = \bar{X},$$

*defined over $\mathbf{F}_q$, such that for all $0 \leq i < m$:*

(c) *$X^{\langle i\rangle} \to \bar{X}$ is an isomorphism over $X \smallsetminus Y_{i+1}$,*

(d) *the closures of $S_i$ and $b_1(b_2^{-1}(S_i)) \smallsetminus Y_i$ in $X^{\langle i\rangle}$ are disjoint.*

*For every $0 \leq i < m$ let $\bar{S}_i$ be the closure of $S_i$ in $X^{\langle i\rangle}$. The data comprise furthermore a cartesian diagram*

$$\begin{array}{ccc} S_i^\natural & \hookrightarrow & \bar{S}_i^\natural \\ f_i \downarrow & \square & \downarrow \bar{f}_i \\ S_i & \hookrightarrow & \bar{S}_i \end{array}$$

*such that*

(e) *$\bar{f}_i$ is proper;*

(f) *$f_i$ is a finite, étale, Galois covering;*

(g) *$\bar{S}_i^\natural$ is smooth and $D_i^\natural := \bar{S}_i^\natural \smallsetminus S_i^\natural$ is a union of smooth divisors with, at most, normal crossings;*

(h) *every $f_i^*(\mathcal{H}^j(M)|_{S_i})$ is, at most, tamely ramified along $D_i^\natural$.*

In practice (and in particular, in the intended application to Hecke correspondences on Shimura varieties) one will often have a filtration satisfying (a) and (b) and which is respected by the correspondence $b$; i.e., such that $b_2^{-1}(Y_i) \subset b_1^{-1}(Y_i)$ for all $0 \leq i < m$. In that case one can do without successive blowups; in other words one can take all $X^{\langle i\rangle} = \bar{X}$, and conditions (c) and

(d) become void. We leave it to the reader to formulate other special cases of Data 7.2.1, e.g., under smoothness hypotheses. Our main result is the following theorem:

THEOREM 7.2.2. *If Data* 7.2.1 *exist, then Deligne's Conjecture* 7.1.3 *is true. Moreover,* $m_0$ *depends at most on* $X$, $\Phi_q$, $B$, $b$, $\bar{X}$ *and Data* 7.2.1.

The proof will be given in subsection (7.4).

COROLLARY 7.2.3. *Deligne's Conjecture* 7.1.3 *is implied by the resolution of singularities in the form of Conjectures* A.1–A.3 *for the field* $\mathbf{F}_q$ *(see Appendix).*

*Proof.* We have to construct Data 7.2.1. Choose any compactification $X \hookrightarrow \bar{X}$. By Proposition 6.1.2 and Lemma 6.2.2 we can find a filtration and a sequence of proper modifications that satisfy conditions (a)–(d). Strictly speaking we know these are defined only over the algebraic closure $k$, but the proofs of Proposition 6.1.2 and Lemma 6.2.2 work equally well over $\mathbf{F}_q$. The rest follows from Lemma 5.1.2. □

(7.3) *Geometric consequences of twisting.* For the proof of Theorem 7.2.2 we need to study how the property of good position behaves under a Frobenius twist. We must deal with two different situations: one of local terms at the boundary and the other of local terms at finite distance. In either case we consider the statement of (7.1), assuming that $X$ is smooth and $b$ is proper, but we do *not* require assumption (7.1.1).

In the first case let us consider an open subscheme $j : U \hookrightarrow X$, whose complement $D = X \setminus U$ is a union of smooth divisors with, at most, normal crossings. We assume that $U$, $D$, $B$ and $b$ are defined over $\mathbf{F}_q$ and that condition (3.2.1) holds, i.e., that $b_2^{-1}(D) \subset b_1^{-1}(D)$. The same condition then also holds for all $b^{(n)}$.

PROPOSITION 7.3.1. *There exists an integer* $m_0$ *such that the twisted correspondence* $b^{(n)}$ *is in good position with respect to* $D$ *whenever* $q^n > m_0$.

*Proof.* By Lemma 5.2.1 we must prove that the closure of $b^{(n)}(B) \cap (U \times U)$ in $\tilde{Z}$ meets $\pi^{-1}(a(Y))$, at most, in $\tilde{D}^1$. In other words we have to show that $b^{(n)}(B) \cap (U \times U)$ is closed in $\tilde{Z} \setminus \tilde{D}^1$, at least, over a neighborhood of the diagonal in $Z = X \times X$.

The assertion is Zariski local on $X$. We want to use local coordinates, as in the proof of Lemma 5.3.1, but the irreducible components of $D$ are not necessarily individually defined over $\mathbf{F}_q$. Nevertheless $X$ possesses a finite covering by $\Phi_q$-stable open affines, which trivialize the ideal sheaf of every irreducible component of $D$. Indeed recall that $X = X_0 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q$ and consider a closed point $x_0$ of $X_0$. Since every invertible sheaf on a semilocal scheme is trivial, the ideal sheaves in question are trivial on some open neighborhood of the finite subscheme $x_0 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q$. Intersecting the different Galois conjugates yields a $\Phi_q$-stable neighborhood with the same property. The contention now follows from the quasicompactness of $X$.

Now without loss of generality we may assume that $X$ is affine, say, $X = \operatorname{Spec} A$, and that the irreducible component $D_\nu$ of $D$ is given by the equation $t_\nu = 0$ for some $t_\nu \in A$. We use the same notation as in the proof of Lemma 5.3.1. Denote by $\Psi_q$ the endomorphism of $A$ that is dual to the Frobenius morphism. Since the total divisor $D$ is defined over $\mathbf{F}_q$ and given by the equation $t := \prod_{\nu=1}^r t_\nu = 0$, we have $\Psi_q(t) = at^q$ for some unit $a \in A$. The relevant morphisms of schemes are represented by the following ring homomorphisms:

$$\begin{array}{ccc}
R^{\circ\circ} & := & A \otimes_k A \left[\dfrac{1}{t \otimes 1}, \dfrac{1}{1 \otimes t}\right] \\
\cup & & \cup \\
R^{\circ} & := & A \otimes_k A \left[\dfrac{1}{t \otimes 1}\right] \\
\cup & & \cup \\
\tilde{R} & := & A \otimes_k A \left[\left.\dfrac{1 \otimes t_\nu}{t_\nu \otimes 1}\right|_{1 \le \nu \le r}\right] \\
\cup & & \cup \\
R & := & A \otimes_k A
\end{array}
\qquad
\begin{tikzcd}
U \times U \arrow[d, hook] & & \\
U \times X \arrow[ddd] & = & \tilde{Z} \smallsetminus \pi^{-1}(D^1) \arrow[d, hook] \\
 & & \tilde{Z} \smallsetminus \tilde{D}^1 \arrow[dd] \\
 & & \\
X \times X & = & Z.
\end{tikzcd}$$

Let $I^{\circ\circ} \subset R^{\circ\circ}$ be the ideal of $b(B) \cap (U \times U)$. The closure in $U \times X$ is given by the ideal $I^{\circ\circ} \cap R^{\circ}$. By assumption, $b(B) \cap (U \times U) = b(B) \cap (U \times X)$, which is closed in $U \times X$. Thus the inclusion induces an isomorphism

$$R^{\circ}/I^{\circ\circ} \cap R^{\circ} \xrightarrow{\sim} R^{\circ\circ}/I^{\circ\circ}. \tag{7.3.2}$$

The closure of $b^{(n)}(B) \cap (U \times U) = (\Phi_q^n \times \mathrm{id})(b(B) \cap (U \times U))$ in $\tilde{Z} \smallsetminus \tilde{D}^1$ is given by the ideal

$$(\Psi_q^n \otimes \mathrm{id})^{-1}(I^{\circ\circ}) \cap \tilde{R}.$$

Thus we must prove that, for all sufficiently large $n$, the inclusion induces an isomorphism

$$\tilde{R}/(\Psi_q^n \otimes \mathrm{id})^{-1}(I^{\circ\circ}) \cap \tilde{R} \xrightarrow{\sim} R^{\circ\circ}/(\Psi_q^n \otimes \mathrm{id})^{-1}(I^{\circ\circ}). \tag{7.3.3}$$

Observe that $R^{\circ\circ} = R^{\circ}[(1 \otimes t)^{-1}] = \tilde{R}[(1 \otimes t)^{-1}]$. Thus equality (7.3.2) means that there exists $u \in R^{\circ}$ such that $1 - u \cdot (1 \otimes t) \in I^{\circ\circ}$. For the same reason, equality (7.3.3) is equivalent to the existence of an element $w \in \tilde{R}$ such that $(\Psi_q^n \otimes \mathrm{id})(1 - w \cdot (1 \otimes t)) \in I^{\circ\circ}$.

We shall now prove this assertion for all sufficiently large $n$. Since $R^\circ = R[(t\otimes 1)^{-1}]$, we can write $u = v\cdot(t\otimes 1)^{-e}$ for some element $v\in R$ and some nonnegative integer $e$. Put

$$w = \frac{(\mathrm{id}\otimes\Psi_q^n)(v)\cdot(1\otimes t)^{q^n-1}}{(1\otimes a)\cdot(t\otimes 1)^e}.$$

This is an element of $\tilde{R}$ whenever $q^n-1\geq e$. Since $\Psi_q(t)=t^q$, it follows that

$$\begin{aligned}(\Psi_q^n\otimes\mathrm{id})(1-w\cdot(1\otimes t)) &= (\Psi_q^n\otimes\mathrm{id})\left(1-\frac{(\mathrm{id}\otimes\Psi_q^n)(v)\cdot(1\otimes t)^{q^n-1}}{(1\otimes a)\cdot(t\otimes 1)^e}\cdot(1\otimes t)\right)\\ &= (\Psi_q^n\otimes\Psi_q^n)\left(1-\frac{v\cdot(1\otimes t)}{(t\otimes 1)^e}\right)\\ &= (\Psi_q\otimes\Psi_q)^n(1-u\cdot(1\otimes t)) \in I^{\circ\circ},\end{aligned}$$

because $I^{\circ\circ}$ is, as assumed, stable under Frobenius. This is the desired assertion; the lower bound is $m_0=e$. □

In general we do not know much about the lower bound $m_0$. In dealing with local terms at finite distance, however, we need a more explicit lower bound, because the fixed points change under twist. It turns out that, here, the lower bound in Lemma 7.1.2 is enough. So assume that $b_2$ is quasifinite of degree less than $q^n$ and let $\beta$ be an (isolated) fixed point of $b^{(n)}$. Its image in $X$ is a closed point $x$, and the correspondence $b^{(n)}$ respects the closed subscheme $\{x\}$ over a neighborhood of $x$ (compare (2.4)). Recall that $X$ is smooth. Let $X'\to X$ be the blowup at $x$ and regard the exceptional divisor $D$ as the boundary. Consider the pullback of the geometric correspondence $b^{(n)}$.

PROPOSITION 7.3.4. *When $q^n>\deg b_2$, then for all $\beta$ as above the pullback of $b^{(n)}$ via $X'\to X$ is in good position with respect to $D$.*

*Proof.* First let $\beta$ and $x$ be arbitrary closed points of $B$ and $X$, respectively, and let $\mathbf{m}(\beta)$, $\mathbf{m}(x)$ be the associated (maximal) ideal sheaves. Clearly we have for the stalks:

(a) $\left(\Phi_q^*\mathbf{m}(\Phi_q(x))\right)_x\subset\mathbf{m}(x)_x^q$,

(b) $\left(b_1^*\mathbf{m}(b_1(\beta))\right)_\beta\subset\mathbf{m}(\beta)_\beta$,

(c) $\mathbf{m}(\beta)_\beta^d\subset\left(b_2^*\mathbf{m}(b_2(\beta))\right)_\beta$,

where $d<q^n$ is the degree of $b_2$. Applying (a), (b), (c) in this order yields

$$\left(b_1^{(n)*}\mathbf{m}(b_1^{(n)}(\beta))\right)_\beta\subset\left(b_1^*\mathbf{m}(b_1(\beta))\right)_\beta^{q^n}\subset\mathbf{m}(\beta)_\beta^{q^n}\subset\mathbf{m}(\beta)_\beta\cdot\left(b_2^*\mathbf{m}(b_2(\beta))\right)_\beta.$$

Let us come back to the original situation, where $\beta$ is a fixed point that maps to $x\in X$. After replacing $b$ by $b^{(n)}$, we know that

$$\left(b_1^*\mathbf{m}(x)\right)_\beta\subset\mathbf{m}(\beta)_\beta\cdot\left(b_2^*\mathbf{m}(x)\right)_\beta. \tag{7.3.5}$$

and we only have to show that the pullback of $b$ via $X' \to X$ is in good position with respect to $D$. Since $b$ is still a finite morphism, there exists a positive integer $e$ (for instance, $e = d$) such that

$$\mathbf{m}(\beta)^e_\beta \subset \big(b^*\mathbf{m}((x,x))\big)_\beta.$$

Plugging this into formula (7.3.5), we obtain

$$\big(b_1^*\mathbf{m}(x)\big)^e_\beta \subset \big(b^*\mathbf{m}((x,x))\big)_\beta \cdot \big(b_2^*\mathbf{m}(x)\big)^e_\beta. \tag{7.3.6}$$

The assertion is Zariski local at $x$, so we may replace $X$ by its localization at $x$. Then $X = \operatorname{Spec} A_0$ for a regular local ring, whose maximal ideal $\mathbf{m}_0$ corresponds to $x$. Let $I_0 \subset R_0 := A_0 \otimes_k A_0$ be the ideal of $b(B)$. Condition (7.3.6) is equivalent to

$$(\mathbf{m}_0 \otimes A_0)^e \subset (\mathbf{m}_0 \otimes A_0 + A_0 \otimes \mathbf{m}_0) \cdot (A_0 \otimes \mathbf{m}_0)^e + I_0. \tag{7.3.7}$$

The blowup $X' \to X$ is covered by open affines associated to $A := A_0[t^{-1} \cdot \mathbf{m}_0]$ for some $t \in \mathbf{m}_0 \smallsetminus \mathbf{m}_0^2$. Observe that $\mathbf{m}_0 \cdot A$ is the principal ideal generated by $t$. Abbreviating $R := A \otimes_k A$, as in the proof of Proposition 7.3.1, we find that relation (7.3.7) implies

$$(t \otimes 1)^e \in ((t \otimes 1)R + (1 \otimes t)R) \cdot (1 \otimes t)^e + I_0. \tag{7.3.8}$$

Moreover $\tilde{Z} \smallsetminus \tilde{D}^1$ is the spectrum of the ring $\tilde{R} = R[(1 \otimes t) \cdot (t \otimes 1)^{-1}]$. In this ring $1 \otimes t$ is a multiple of $t \otimes 1$, so that (7.3.8) implies

$$(t \otimes 1)^e \in (t \otimes 1)^e \cdot (1 \otimes t) \cdot \tilde{R} + I_0;$$

i.e.,

$$1 \in (1 \otimes t) \cdot \tilde{R} + \frac{1}{(t \otimes 1)^e} \cdot I_0.$$

As in the proof of Proposition 7.3.1, this implies the condition in Definition 4.4.3. $\square$

(7.4) *Proof of Theorem* 7.2.2. Most of the work has been done above; however it remains to put everything together. As in subsection (2.3), choose a commutative diagram

$$\begin{tikzcd} X \arrow[d, hook, "j"'] & B \arrow[l, "b_1"'] \arrow[r, "b_2"] \arrow[d, hook] \arrow[dl, phantom, "\square"] & X \arrow[d, hook, "j"] \\ \bar{X} & \bar{B} \arrow[l, "\bar{b}_1"'] \arrow[r, "\bar{b}_2"] & \bar{X}, \end{tikzcd}$$

whose left-hand side is cartesian, and where $\bar{B}$ is proper. (Such $\bar{B}$ can be constructed easily: assumption (7.1.1) implies that $b$ is finite, so just define $\bar{B}$ as the relative spectrum of a suitable coherent $\mathcal{O}_{\bar{X}\times\bar{X}}$-subalgebra of $(j \times j)_* b_* \mathcal{O}_B$.) We want to apply the general Lefschetz–Verdier trace formula in the proper case of Theorem 2.2.1 to the extension by zero $\bar{u}^{(n)}$. Assume that $q^n$ is greater than the degree of $b_2$ so that Lemma 7.1.2 applies. As explained in subsection (2.3) we are left with the problem of calculating all "true" local terms $\mathrm{LT}_\beta(\bar{u}^{(n)})$, when $n$ is sufficiently large. For fixed points at finite distance we want to prove that the true local term equals the naïve local term (compare (2.4)). For fixed-point components at the boundary we want to prove that the true local term is zero.

It suffices to prove these assertions when $B$, $b$, $\bar{B}$ and all $\bar{f}_i : \bar{S}_i^\natural \to \bar{S}_i$ are defined over $\mathbf{F}_q$. In fact this is an easy application of the restriction of scalars à la Grothendieck. Suppose that $B$, $b$ and $\bar{B}$ come from schemes and morphisms $B_1$, etc., over $\mathbf{F}_{q^m}$. Viewing $B_1$ as a scheme over $\mathbf{F}_q$ instead of $\mathbf{F}_{q^m}$ shows that the composite morphism

$$B_1 \xrightarrow{b} X_0 \times_{\mathbf{F}_q} \mathbf{F}_{q^m} \xrightarrow{\mathrm{pr}_1} X_0$$

is defined over $\mathbf{F}_q$. By base extension to $\bar{\mathbf{F}}_q$ we obtain a (geometric) correspondence on $X$, extending our original correspondence. Since the $S_i$, $Y_i$ and $X^{\langle i\rangle}$ are defined over $\mathbf{F}_q$, this extended correspondence also satisfies the conditions of Data 7.2.1. As $B = B_1 \times_{\mathbf{F}_{q^m}} \bar{\mathbf{F}}_q$ is open and closed in $B_1 \times_{\mathbf{F}_q} \bar{\mathbf{F}}_q$, we can extend $u$ in any way we like, and it suffices to prove the assertion for the extended correspondence. Similarly, when $\bar{f}_i : \bar{S}_i^\natural \to \bar{S}_i$ is defined only over $\mathbf{F}_{q^m}$, we simply construct a larger covering by the Grothendieck restriction and, again, all properties are preserved.

After these preliminary reductions, consider the "boundary" $Y := \bar{X} \smallsetminus X$ and, for every $0 \leq i < m$, put $\bar{Y}_i := Y_i \cup Y$. These subschemes form a filtration of $\bar{X}$; together with the blowups $X^{\langle i\rangle}$ they satisfy the conditions of Data 6.1.1 and 6.2.1, with $X$ of loc. cit. replaced by $\bar{X}$. We shall split up the extended correspondence $\bar{b}$ along this stratification. Fix finite open coverings $\{U_\nu^{\langle i\rangle}\}$ of $X^{\langle i\rangle}$, as given by Lemma 6.2.3. Since everything is defined over $\mathbf{F}_q$, the proof of the lemma actually produces open subsets that are themselves defined over $\mathbf{F}_q$. Therefore we may, and do, assume that the $U_\nu^{\langle i\rangle}$ are defined over $\mathbf{F}_q$.

As in subsection (6.3) let $b_{i,\nu}$ denote the pullback of the correspondence $\bar{b}$ to $\bar{S}_i \cap U_\nu^{\langle i\rangle} \hookrightarrow X^{\langle i\rangle}$. Then condition (3.2.1) holds for $b_{i,\nu}$ and the open embedding $S_i \cap U_\nu^{\langle i\rangle} \hookrightarrow \bar{S}_i \cap U_\nu^{\langle i\rangle}$. The same follows for the pullback $b_{i,\nu}^\natural$ of $b_{i,\nu}$ under $\bar{f}_i : \bar{S}_i^\natural \to \bar{S}_i$. Applying Proposition 7.3.1 to this situation, we find that $b_{i,\nu}^{(n)}$ is in good position whenever $n$ is sufficiently large. We can even achieve this for all pairs $(i, \nu)$ at the same time, since there are only finitely many of them. Applying Propositions 5.1.3 and 6.3.1 shows that the morphism $\varphi(\bar{b}^{(n)}, j_! M)$ vanishes at the boundary $Y \subset \bar{X}$. As explained in subsection (3.1) this implies that $\mathrm{LT}_\beta(\bar{u}^{(n)}) = 0$ for all $\beta \in \pi_0(\mathrm{Fix}\, \bar{b}^{(n)} \smallsetminus \mathrm{Fix}\, b^{(n)})$. This finishes the case of fixed points at the boundary.

For fixed points at finite distance the matter is a little more complicated. By Lemma 7.1.2 we have to deal only with isolated fixed points. We may fix $(i,\nu)$ and concentrate on fixed points that lie over $S_i \cap U_\nu^{\langle i\rangle}$. Suppose that $n$ exceeds the bound given by Proposition 7.3.4 for the pullback of $b_{i,\nu}$ to $S_i \cap U_\nu^{\langle i\rangle}$. Let $\beta$ be a fixed point of $b^{(n)}$ that maps to a point $x \in S_i \cap U_\nu^{\langle i\rangle}$. Consider the open embedding $j_x : X \smallsetminus \{x\} \hookrightarrow X$. In subsection (2.4) we defined a cohomological correspondence $u_j^{(n)}$ on $j_{x!}j_x^! M$ with support in $b^{(n)}$, at least over a neighborhood of $x$. By Proposition 2.4.3 the difference between the true local term and the naïve local term at $\beta$ is equal to $\mathrm{LT}_\beta(u_j^{(n)})$.

To prove that this local term vanishes we now proceed as we did for local terms at the boundary. We shall show that the morphism $\varphi(b^{(n)}, j_{x!}j_x^! M)$ vanishes at $\beta$. Since we are only interested in a neighborhood of $x$, we may replace $X$ and $\bar{X}$ by the open subscheme $U_\nu^{\langle i\rangle} \smallsetminus \pi^{\langle i\rangle -1}(\bar{Y}_{i+1})$ and everything else by its pullback. After this reduction we must deal with the filtration $\{Y_{i'}\}$ and with the blowups $\{X^{\langle i'\rangle}\}$ only in the range $0 \le i' \le i$. The filtration and the proper modifications

$$X = Y_0 \supset \ Y_1 \ \supset \ldots \supset \ Y_i \ \supset \{x\},$$
$$X^{\langle 0\rangle} \to X^{\langle 1\rangle} \to \ldots \to X^{\langle i\rangle} = X^{\langle i\rangle},$$

satisfy the assumptions of Data 6.1.1 and 6.2.1. We want to apply Proposition 6.3.1 to this situation, with the "boundary" $\{x\}$. Recall that above we have chosen $n$ such that the morphism $\varphi(b_{i',\nu'}^{(n)}, -)$ (for the pullback of $M$ to $S_{i'}$, extended by zero to $\bar{S}_{i'}$) vanishes at the boundary $\bar{S}_{i'} \smallsetminus S_{i'}$ for every $0 \le i' < i$. Thus by Proposition 6.3.1 it remains to prove that the morphism $\varphi(b_{i,\nu}^{(n)}, (j_{x!}j_x^! M)|_{S_i})$ vanishes at $\beta$. To show this let $S_i'$ be the blowup of $S_i$ at $x$ and $D$ be the exceptional divisor. Since $n$ was chosen larger than the bound given by Proposition 7.3.4, the pullback of $b^{(n)}$ to $S_i'$ is in good position. The desired vanishing now follows from Proposition 5.1.3, using the identical covering of $S_i$, $\bar{S}_i$. The theorem is proved. □

## 8. Application to $\ell$-adic representations

(8.1) *Decomposing cohomology under correspondences.* In this section we want to discuss consequences of Deligne's conjecture for $\ell$-adic representations associated to algebraic varieties. We fix a number field $K$ (of finite degree over $\mathbf{Q}$) and an algebraic closure $\bar{K}$. Fix a scheme $X$ of finite type over $K$ (with the conventions of (1.1)), and another number field $E$. For every finite prime $\lambda$ of $E$ we have a continuous representation of $\mathrm{Gal}(\bar{K}/K)$ on the graded $E_\lambda$-vector space

$$V_\lambda^\bullet := H_c^\bullet(X_{\bar{K}}, E_\lambda).$$

We want to decompose this graded representation using correspondences. (It would be interesting to consider the cohomology of more general complexes. For similar results one should probably require that the complex and the cohomological correspondences be "of geometric origin" in a suitable sense, perhaps in the spirit of [3, 6.2.4].)

Consider a correspondence

$$X \xleftarrow{b_1} B \xrightarrow{b_2} X,$$

defined over $K$, where $b_1$ is proper and $b_2$ is quasifinite. Using the trace map for a quasifinite morphism (in this case, the adjoint of the morphism defined in [2, XVII, 6.2.3]), we can define a cohomological correspondence

$$u_b : b_1^* E_\lambda \cong b_2^* E_\lambda \xrightarrow{\mathrm{tr}} b_2^! E_\lambda \tag{8.1.1}$$

with support in $b$. If, moreover, we are given a *weight* $w(B^\circ) \in E$ for every connected component $B^\circ \subset B$, we multiply this morphism by $w(B^\circ)$ on each $B^\circ$ and denote the resulting cohomological correspondence by $u_{b,w}$. Clearly the endomorphism $u_{b,w!}$ of $V_\lambda^\bullet$ commutes with the action of $\mathrm{Gal}(\bar{K}/K)$.

By definition, $E$-linear combinations of endomorphisms $u_{b,w!}$ are again of this form. The same holds for composites. Indeed consider two such correspondences

$$X \xleftarrow{b_1^i} B^i \xrightarrow{b_2^i} X$$

for $i = 1, 2$. The *composite* correspondence is defined by the cartesian diagram

$$\begin{tikzcd}
 & & B \arrow[dl, "b_1^{2\prime}"'] \arrow[dr, "b_2^{1\prime}"] \arrow[ddll, bend right, "b_1"'] \arrow[ddrr, bend left, "b_2"] & & \\
 & B^1 \arrow[dl, "b_1^1"'] \arrow[dr, "b_2^1"] & & B^2 \arrow[dl, "b_1^2"'] \arrow[dr, "b_2^2"] & \\
X & & X & & X.
\end{tikzcd}$$

Again $b_1$ is proper and $b_2$ is quasifinite. Moreover the definition of the trace morphism easily implies that

$$u_{b^2!} \circ u_{b^1!} = u_{b!}, \tag{8.1.2}$$

as desired. A similar relation holds for the weighted correspondences $u_{b,w!}$.

We have just seen that the set of all endomorphisms $u_{b,w!}$ of $V_\lambda^\bullet$ is an $E$-algebra; let us call it $A$. From comparison with Betti cohomology it follows that $A$ has finite degree over $E$ and does not depend on $\lambda$ (up to canonical isomorphism). Thus for every isomorphism class of irreducible representations $\pi$ of $A$ over $E$ there exists an associated idempotent in $A$. This idempotent is a projector onto the isotypic component $V_\lambda^\bullet(\pi)$.

(8.2) *Comparison with the reduction modulo $\wp$.* Fix $u_{b,w!}$ as in subsection (8.1). Choose any extension of the morphisms $b_1, b_2$ to models of finite type over $\mathcal{O}_K$. By abuse of notation we denote this extension by $\mathcal{X} \xleftarrow{b_1} \mathcal{B} \xrightarrow{b_2} \mathcal{X}$; its generic fiber is just the original correspondence. After making $\mathcal{X}$ and $\mathcal{B}$ smaller, if necessary, we can assume that for these extended morphisms we still have $b_1$ proper and $b_2$ quasifinite, and that the irreducible components of $B$ and $\mathcal{B}$ are in bijection. Then formula (8.1.1) makes sense for this extension, and so does the weight function $w$; hence the cohomological correspondence $u_{b,w}$ also extends.

We want to compare the cohomology of the generic fiber with that of special fibers. Let $\wp$ be a finite prime of $K$, whose residue field $k_\wp$ has characteristic prime to $\lambda$. Let $\mathcal{O}_K^{(\wp)}$ be the strict henselization of $\mathcal{O}_K$ at $\wp$ into $\bar{K}$; its residue field $\bar{k}_\wp$ is an algebraic closure of $k_\wp$. Extend the identical inclusion $\mathcal{O} \subset K$ to an embedding $\mathcal{O}_K^{(\wp)} \hookrightarrow \bar{K}$. This choice determines a specialization map

$$\begin{tikzcd} H_c^\bullet(\mathcal{X}_{\bar{k}_\wp}, E_\lambda) \arrow[rr, "\mathrm{sp}"] & & H_c^\bullet(X_{\bar{K}}, E_\lambda) = V_\lambda^\bullet \\ & H_c^\bullet(\mathcal{X}_{\mathcal{O}_K^{(\wp)}}, E_\lambda) \arrow[ul, "\mathrm{pullback}", "\wr"'] \arrow[ur, "\mathrm{pullback}"'] & \end{tikzcd} \tag{8.2.1}$$

(see [9, Th. finitude, appendice 2.1], using proper base change). By the "constructibility theorem" ([9, Th. finitude 1.1]) this is an isomorphism for all but, at most, finitely many $\wp$. One can even make this independent of $\lambda$ in the sense that there exists a finite set of primes $P$ such that the map (8.2.1) is an isomorphism whenever $\wp \notin P$ and its characteristic is different from that of $\lambda$.

On the other hand we can look at the special fiber of the correspondence. The definition in subsection (8.1), applied to the special fiber at $\wp$, yields a cohomological correspondence $u_{b,w,\wp}$ with support in $b_\wp$. In fact both $u_{b,w,\wp}$ and $u_{b,w}$ can be described as the pullback of a cohomological correspondence with support in $\mathcal{B}$. Using the standard compatibilities of adjunction morphisms, etc., with pullback, one easily checks that the following diagram commutes:

$$\begin{array}{ccc} H_c^\bullet(\mathcal{X}_{\bar{k}_\wp}, E_\lambda) & \xrightarrow{u_{b,w,\wp!}} & H_c^\bullet(\mathcal{X}_{\bar{k}_\wp}, E_\lambda) \\ {\scriptstyle \mathrm{sp}}\downarrow & & \downarrow{\scriptstyle \mathrm{sp}} \\ H_c^\bullet(X_{\bar{K}}, E_\lambda) & \xrightarrow{u_{b,w!}} & H_c^\bullet(X_{\bar{K}}, E_\lambda). \end{array} \tag{8.2.2}$$

(8.3) *Application of Deligne's conjecture.* We view $V_\lambda^\bullet = H_c^\bullet(X_{\bar{K}}, E_\lambda)$ as a virtual representation of $\mathrm{Gal}(\bar{K}/K)$ by counting $V_\lambda^i$ with multiplicity $(-1)^i$. In particular the trace of a graded endomorphism $\varphi$ is defined as the alternating sum $\sum(-1)^i \operatorname{tr}(\varphi|V_\lambda^i)$ (compare (1.3)). The same convention is in force for graded subspaces of $V_\lambda^\bullet$.

Assume that $u_{b,w!}$ is a projector onto an isotypic component $V_\lambda^\bullet(\pi)$. Then for every $\sigma \in \mathrm{Gal}(\bar{K}/K)$ we have

$$\mathrm{tr}(\sigma | V_\lambda^\bullet(\pi)) = \mathrm{tr}(\sigma \circ u_{b,w!} | V_\lambda^\bullet). \tag{8.3.1}$$

This is particularly useful when the right-hand side can be calculated by a Lefschetz trace formula. Let $\wp$ be a finite prime of $K$, of characteristic different from that of $\lambda$, such that the specialization map (8.2.1) is an isomorphism (so, in particular, $V_\lambda^\bullet(\pi)$ is unramified at $\wp$). Let $\mathrm{Frob}_\wp \in \mathrm{Gal}(\bar{K}/K)$ represent the geometric Frobenius substitution at $\wp$. When $\sigma = \mathrm{Frob}_\wp^n$ for a nonnegative integer $n$, then by (8.2.2) the right-hand side of equation (8.3.1) is equal to

$$\mathrm{tr}\left(u_{b,w,\wp!}^{(n)}\right).$$

The evaluation of this trace is a case for Deligne's conjecture.

THEOREM 8.3.2. *There exist a finite set $P$ of primes of $K$ and an integer $m_0$ with the following property: For all finite primes $\wp$ of $K$ and $\lambda$ of $E$ such that $\wp \notin P$ and the residue characteristics of $\wp$ and $\lambda$ are distinct, Deligne's Conjecture* 7.1.3 *is true for the cohomological correspondence $u_{b,w,\wp}$ with coefficients in $E_\lambda$ and support in $b_\wp$ and with this lower bound $m_0$.*

*Proof.* By the resolution of singularities in characteristic zero (see Appendix) the analog of Data 7.2.1 can be constructed over $K$. Since we are dealing with constant sheaves, conditions (b) and (h) are vacuous, and we can choose $S_i^\natural = S_i$. So, in particular, we can fix one set of Data 7.2.1 that works for all $\lambda$. Choose any extension of this data to schemes of finite type over $\mathcal{O}_K$; then the desired properties hold automatically for the special fibers at all but, at most, finitely many primes $\wp$. Now Theorem 7.2.2 implies everything except that the lower bound is independent of $\wp$.

But this independence follows from a close inspection of the proof of Theorem 7.2.2. Indeed first observe that, since everything is from the start defined over $K$, there is no need for the restriction of scalars à la Grothendieck. Next the lower bound coming from Lemma 7.1.2 and Proposition 7.3.4 is equal to the degree of $b_2$ in the generic fiber, which is finite. It remains to compare the lower bound given by Proposition 7.3.1 at different closed fibers coming from the same correspondence over $\mathcal{O}_K$. After reduction to the affine case, we need for the integer $e$ in the proof of Proposition 7.3.1 to be independent of $\wp$, for almost all $\wp$. But this is clear, since $u$ comes from a function on the generic fiber. $\square$

It remains to calculate the naïve local terms. Let $\beta$ be an isolated fixed point of $b_\wp^{(n)}$, mapping to a point $x \in \mathcal{X}$. The *degree* of $b_2$ at $\beta$ is the multiplicity of $\beta$ in the cycle $b_2^{-1}(x)$. When $\beta$ lies in the reduction $B^\circ_\wp$ of an irreducible component $B^\circ \subset B$, the definition of the trace morphism implies that

$$\operatorname{tr}\left((u^{(n)}_{b,w,\wp})_\beta\right) = w(B^\circ) \cdot \deg_\beta b_2.$$

Putting everything together leads to the following: There exist a finite set $P$ of primes of $K$ and a nonnegative integer $m_0$ such that, for all $\wp \notin P$, for all $\lambda$ of characteristic different from that of $\wp$, and for all $n \geq 0$ such that $|k_\wp|^n > m_0$:

$$\operatorname{tr}\big(\operatorname{Frob}^n_\wp \,|V^\bullet_\lambda(\pi)\big) = \sum_{B^\circ \in \pi_0(B)} w(B^\circ) \cdot \sum_{\beta \in B^\circ_\wp \cap \operatorname{Fix} b^{(n)}_\wp} \deg_\beta b_2. \tag{8.3.3}$$

(8.4) *Consequences.* Two main observations about equation (8.3.3) should be made. First, the right-hand side is a number in $E$ that is independent of $\lambda$. Second, the fact that the equality may be known only for large $n$ is not really a drawback. Indeed, for every number $\alpha$ in a suitable, algebraically closed overfield, let $\mu_\alpha \in \mathbf{Z}$ be the *total multiplicity* of $\alpha$ as an eigenvalue $\alpha$ of $\operatorname{Frob}^n_\wp$ on $V^\bullet_\lambda(\pi)$. Here each occurrence in degree $i$ is counted with multiplicity $(-1)^i$. The left-hand side of (8.3.3) is then, for all $n \geq 0$, equal to

$$\sum_\alpha \mu_\alpha \cdot \alpha^n. \tag{8.4.1}$$

Since the functions $n \mapsto \alpha^n$ are linearly independent in the range $n > n_0$, it follows that the multiplicities $\mu_\alpha$ are determined even when the sum (8.4.1) is known only for all large $n$. Thus, if the right-hand side of equation (8.3.3) is known for all large $n$, then the left-hand side of (8.3.3) is known *for all* $n \in \mathbf{Z}$. In particular it is a number in $E$ that is independent of $\lambda$. The same now follows for the characteristic polynomial:

THEOREM 8.4.2 (Independence on $\ell$). *There exists a finite set $P$ of primes of $K$ such that, for all $\wp \notin P$ and all $\lambda$ of characteristic different from that of $\wp$, the "virtual characteristic polynomial"*

$$\prod_i \det\left(1 - \operatorname{Frob}_\wp \cdot T \,|V^i_\lambda(\pi)\right)^{(-1)^i}$$

*has coefficients in $E$ and is independent of $\lambda$.*

Other consequences depend on whether the right-hand side of equation (8.3.3) can be calculated. In the case of Hecke correspondences on Shimura varieties this should be possible. The same applies to Shimura varieties over function fields, but due to the lack of general theorems on the resolution of singularities, one still has to construct the necessary blowups explicitly (compare [11]).

## Appendix. Resolution of singularities

We list some well-known conjectures on the resolution of singularities that play a role in this article. Fix an arbitrary field $k$.

CONJECTURE A.1 (Resolution). *Let $X$ be a scheme of finite type over $k$ and $Z \subset X$ be a closed subscheme such that the complement $X \setminus Z$ is smooth over $k$. Then there exists a proper modification $\pi : \tilde{X} \to X$, which is an isomorphism outside $Z$, such that $\tilde{X}$ is smooth and the total transform $\pi^{-1}(Z)$ of $Z$ is a divisor with normal crossings.*

CONJECTURE A.2 (Smooth compactification). *Let $X$ be a smooth scheme of finite type over $k$. Then there exists an open embedding $X \hookrightarrow \bar{X}$, where $\bar{X}$ is smooth and proper over $k$ and the boundary $\bar{X} \setminus X$ is a divisor with normal crossings.*

CONJECTURE A.3 (Separation of closed subschemes). *Let $X$ be a scheme of finite type over $k$ and $Z_1, Z_2$ be two closed subschemes. Then there exists a proper modification $\pi : \tilde{X} \to X$, which is an isomorphism outside $Z_1 \cap Z_2$, such that the proper transforms of $Z_1, Z_2$ are disjoint.*

It is known that Conjecture A.1 implies Conjecture A.2; it suffices to apply the former to any compactification of $X$. By H. Hironaka, the following theorem holds:

THEOREM A.4. *Suppose that $k$ has characteristic zero. Then all three conjectures* A.1–A.3 *are true.*

Indeed, Conjecture A.1 follows directly from Main Theorem I of [14] (Ch. 0, §3) together with Corollary 3 of Main Theorem II of [loc. cit.] (§5). For Conjecture A.3 first observe that the assertion is local on $X$. Indeed fix a finite open covering $\{U_i\}$ of $X$. Any blowup of $U_i$ in a subscheme with support in $U_i \cap Z_1 \cap Z_2$ can be extended in some way to a blowup $\tilde{X}_i \to X$ in a subscheme with support in $Z_1 \cap Z_2$. If $\tilde{X}_i$ satisfies the desired condition over $U_i$, then the assertion holds globally with $\tilde{X}$ defined as the fiber product of all $\tilde{X}_i$ over $X$. Now we may assume that $X$ is affine. Then it can be embedded in a smooth $X'$, and it suffices to prove the assertion for $X'$. But in the smooth situation, Conjecture A.3 is a special case of Corollary 1 of the same Main Theorem II in [14].

For further results in small dimensions see [16].

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