STACKY APPROACH TO MOTIVIC PERIODS

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Contents

Notations and conventions 1
1. Introduction 2
1.1. Periods 2
1.2. Goal 3
1.3. The de Rham unipotent homotopy groups 3
1.4. Further directions 4
2. Recollection on Chen's theorem 5
2.1. Differential forms on loop space 5
2.2. Relation to Malcev–Lie algebra 5
3. Recollection on Stacks Approach 5
3.1. Stacks approach to unipotent group scheme 5
3.2. Homotopy and hypercomplete sheaves 6
3.3. Representability results of Toën 7
3.4. Nori's unipotent scheme 7
4. Betti Analytic Stack 7
References 9

Notations and conventions

- **Any sections with *to be written/added* are only sketched.**
- $\mathcal{S}$ denotes the ∞-category of anima. For $\mathcal{C}$ an ∞-category, and $X, Y \in \mathcal{C}$ we let $\mathrm{Map}_{\mathcal{C}}(X, Y) \in \mathcal{S}$ denote the mapping space.

Fix base R, a discrete commutative ring. We consider the following homotopy rings:

- $\mathrm{CAlg} := \mathrm{CAlg}(\mathrm{Sp})$, the ∞-category of $\mathbb{E}_\infty$ algebra. This has two full subcategories, the *coconnective* and *connective* algebras.

$$\begin{tikzcd}
\mathrm{CAlg}^{\mathrm{ccn}} := \mathrm{CAlg}_{\leq 0} \arrow[r, hook, shift right] \arrow[d, shift left] & \mathrm{CAlg} \arrow[l, shift right, "\tau_{\leq 0}"'] \arrow[r, shift right, "\tau_{\geq 0}"'] \arrow[d] & \mathrm{CAlg}_{\geq 0} := \mathrm{CAlg}^{\mathrm{cn}} \arrow[l, hook', shift right] \arrow[d, shift left] \\
\mathrm{Mod}_{\leq 0} \arrow[u, shift left, "\mathrm{Sym}^{\mathrm{co}}"] \arrow[r, hook, shift right] & \mathrm{Mod} \arrow[l, shift right, "\tau_{\leq 0}"'] \arrow[r, shift right, "\tau_{\geq 0}"'] & \mathrm{Mod}_{\geq 0} \arrow[l, hook', shift right] \arrow[u, shift left, "\mathrm{Sym}"]
\end{tikzcd}$$

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- SCR the ∞-category of simplicial commutative rings. This is the sifted completion of the category of polynomial algebra $\mathrm{Poly}_{\mathbb{Z}}$. Dually, we have coSCR, the ∞-category of co-simplicial rings. Importantly, there is are Dold Kan and co dual Dold-Kan inducing
$$\theta : \mathrm{SCR} \to \mathrm{CAlg}^{\mathrm{cn}}$$
$$\mathrm{co}\theta : \mathrm{coSCR} \to \mathrm{CAlg}^{\mathrm{ccn}}$$
these are equivalences when we consider relative over a base field k of characteristic zero.

- All three categories have the ordinary category of discrete rings, $\mathrm{CAlg}^{\heartsuit}$ embedding to it. We let $\mathrm{Aff}_R^{\heartsuit} := (\mathrm{CAlg}_R^{\mathrm{op}})^{\heartsuit} \hookrightarrow \mathrm{Aff}_R$ be the ordinary category of affine schemes over R.

- $\mathrm{CAlg}_R^{\mathrm{aug}} := (\mathrm{CAlg}_R^{\mathrm{cn}})_{/R}$, be the ∞-category of augmented R-algebra.

- $\mathrm{Stk}_R := \mathrm{Shv}_R := \mathrm{Shv}_{\mathcal{S}}(\mathrm{CAlg}_R^{\heartsuit}, \tau) \hookrightarrow \mathrm{PStk}_R$ denotes the ∞- category of *stacks*[1] and prestacks over R. Unless stated otherwise, τ is the fpqc-topology. Let the category of *pointed* stacks be denoted as $(\mathrm{Stk}_R)_* := (\mathrm{Shv}_R)_*$.

- $\mathrm{dStk}_R := \mathrm{Shv}_{\mathcal{S}}(\mathrm{Aff}_R, \tau)$, the category of *derived stacks*.

Remark 0.1. One can formulate a similar theory for Cdga, the ∞-category of commutative differential graded algebra (we use cohomological grading, as per convention here).

If $R \in \mathrm{Cdga}^{\heartsuit}$, there are equivalences $\mathrm{Cdga}_R \simeq \mathrm{CAlg}_R$, with the ∞-category of $\mathbb{E}_\infty$-algebras over R. We will freely interchange between the variations in this case. Cdga_R is not as useful outside of characteristic zero, as there does not exist model categories.

1. Introduction

Let $X \in \mathrm{Sch}_{\mathbb{Q}}^{\mathrm{sm,proj}}$, and $X^\circ : X \backslash D$, where D is a divisor with normal crossing.

Example 1.1. $X^\circ = \mathbb{P}^1 \backslash \{0, 1, \infty\}$, studied in [Del89].

1.1. Periods.
Periods are classically integrals of rational differential forms:
$$\log(2) = \int_{1 \le z \le 2} \frac{dz}{z}, \quad \zeta(2) = \int_{0 \le t_1 \le t_2 \le 1} \frac{dt_1}{1 - t_1} \frac{dt_2}{t_2}$$
More generally, they are the matrix coefficient from Grothendieck's comparison theorem
$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^*_{\mathrm{Betti}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$
with respect to the $\mathbb{Q}$ structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to* X. These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"[2] which should govern the arithmetic structure of periods.

[1] we simply refer sheaves as stacks, which is not the convention. Often these require some *geometric context*, see [Toë06], [Lur11a].

[2] For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

1.2. **Goal.** We will study the *pro unipotent homotopy groups*

$$\pi_1^{U,?}(X^\circ, x)$$

in various realization $? \in \{\text{ét}, \text{Betti}, \text{dR}, \text{cris}\}$. We discuss the de Rham version in Section 1.3.

1.3. **The de Rham unipotent homotopy groups.** In this section k would be a field of characteristic 0. What should be $\pi_1^{U,\text{dR}}$? We propose a definition in Definition 1.2. In Proposition 1.1 we prove its equivalence to a classical definition. For a stack X one can associate its unipotent homotopy type $\mathbf{U}(X)$ see Section 3.1.

Definition 1.1. Denote the *de Rham complex functor*

$$\begin{tikzcd} \mathrm{CAlg}_k^{\heartsuit,\mathrm{sm}} \arrow[r, "\mathrm{dR}"] \arrow[d] & \mathrm{CAlg}_k^{\mathrm{ccn}} \\ \mathrm{CAlg}_k^{\mathrm{cn}} \arrow[ur] & \end{tikzcd}$$

where on smooth discrete algebras,

$$\mathrm{dR}(A) = \Omega^*_{A/k}$$

is the algebraic de Rham complex. This is then left Kan extended to $\mathrm{CAlg}_k^{\mathrm{cn}}$.

Lemma 1.1. *$A \in Poly_k \hookrightarrow CAlg_k^{\heartsuit}$, a finitely generated polynomial algebra over k. then*

$$(\operatorname{Spec} A)^{dR} \simeq \operatorname{Spec} dRA$$

where

$$(-)^{dR} : \mathrm{Shv}_k \to \mathrm{Shv}_k$$

is the associated endo fucntor of de Rham stack functor in Example 3.1, and Spec *is the Yoneda embedding, see Proposition 3.1.*

Proof. This is [Mon22, Lem 2.0.5] combined with [Mon22, Thm 2.0.1]. □

Definition 1.2. For a pointed cohomologically connected scheme $X \in (\mathrm{Sch}_k)_*$, we let

$$\pi_1^{u,\mathrm{dR}}(X) := \pi_1(\mathbf{U}(X^{\mathrm{dR}}), *)$$

be its unipotent de Rham fundamental group scheme.

Proposition 1.1. *If X° is $\left(Aff_{\mathbb{Q}}^{\heartsuit,ft}\right)_*$, a finite type pointed, connected, affine scheme over $\mathbb{Q}$, then*

$$\pi_1^{u,dR}(X^\circ) \simeq \operatorname{Spec} H^0(B(dR(X^\circ)))$$

in $Grp(\mathrm{Sch}_{\mathbb{Q}})$, where the right hand side is the Bar complex construction definition of Haine [Hai87] the right-hand object being what is classically used to define the unipotent de Rham homotopy group, [Bro14].

Proof. Let $A := \mathrm{dR}(X^\circ)$. Consider the homotopy sheaf Definition 3.3,

$$\big(\pi_1 \operatorname{Spec} A : R \mapsto \pi_1(\mathrm{Map}_{\mathrm{CAlg}_k}(A, R), *)\big) \in \mathrm{Shv}_{\mathrm{Set}}(\mathrm{CAlg}_k^{\heartsuit}, \mathrm{fpqc})$$

By Toën's representability Theorem 3.1 and hypercompleteness of affine stacks Proposition 3.2 this sheaf is representable by pro-unipotent group scheme, $\pi_1^u(\operatorname{Spec} A, *) \in \mathrm{Grp}(\mathrm{Aff}_k)$. By [Ols16, Ch.6] we have

$$\pi_1(\mathbf{U}(\operatorname{Spec} A), *) \simeq \pi_1^u \operatorname{Spec} A \simeq \operatorname{Spec} H^0 BA$$

where first equivalence is by Definition 3.2. Lastly,

$$\operatorname{Spec} A \simeq (X^\circ)^{\mathrm{dR}}$$

by Lemma 1.1. □

Remark 1.1. We would hope the proof generalize to schemes with log structures. A definition of de Rham homotopy of scheme with log structures can be found in [Shi00].

Conjecture 1.1. *(1) There exists X^{an} which is the (analytic) Betti stack of X, see Section 4, such that the unipotent Betti homotopy group $\pi_1^{u,Betti}(X(\mathbb{C}), x)$ as defined in [Bro17] is isomorphic to $\pi_1(\boldsymbol{U}(X^{an}))$.*

(2) A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,dR}(X, x) \simeq \pi_1^{u,Betti}(X(\mathbb{C}), x) \otimes \mathbb{C}$$

Remark 1.2. this is a little different to the comparison theorem as suggested [Toë06, Ch. 3.5]. In this case X° is only smooth, but *not projective.*

1.4. **Further directions.** By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [3] We collect a few examples below that suggests avenues with a view towards p-adic cohomology theories, such as $X^{\Delta}, X^{\mathrm{crys}}$ and X^{dR}. (prismatic, crsytalline and de Rham stack, respectively). We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].

In the examples below, let V be a complete discrete valuation ring with a perfect residue field of characteristic $p > 0$ and fraction field K, $K_0 := \mathrm{Frac} W(k) \hookrightarrow K$. $X \in \mathrm{Sch}_V^{\mathrm{sm,prop}}$.

Example 1.2. $\pi_1^{u,\mathrm{crys}}$ has a Tannakian description as given in Shiho's [Shi00]. Part of the strategy is formal: for one Tannakian category when can consider the *nilpotent* part. In *op.cit. Ch.5* one constructs a unipotent crystalline de-Rham comparison map,

$$\pi_1^{u,\mathrm{crys}}(X_V^\circ, x) \otimes_{K_0} K \simeq \pi_1^{u,\mathrm{dR}}(X_K^\circ, x)$$

which has been shown in the case of cohomologies by Berthelot and Ogus.

[3] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

2. Recollection on Chen's theorem

Morgan showed that the homotopy Lie algebra of a smooth complex algebraic variety has a mixed Hodge structure by using Sullivan's minimal models. Haine [Hai87] generalized this result to arbitrary complex variety, the key result was using Chen's theorem, Theorem 2.1. We begin by discussing an interpretation of Chen's theorem, Section 2.1.

2.1. **Differential forms on loop space.**

To be written. References: [Che73]. Our goal is to briefly review the proof of the following theorem.

Theorem 2.1. *Let $x, y \in X^\circ(\mathbb{C})$. For all integer $N \geq 0$,*

$$\mathcal{O}(\pi_{1,N}^{uni,dR}(BdR(X^\circ)) \otimes \mathbb{C} \simeq \mathcal{O}(\pi_{1,N}^{Betti}(X^\circ, x, y)) \otimes \mathbb{C} \tag{1}$$

where

$$\mathcal{O}(\pi_{1,N}^{uni,dR}(BdR(X^\circ)) := L_N B(dR(X^\circ))$$

L_N being the length filtration on the bar complex. taking colimit along N, we induce

$$\mathcal{O}(\pi_1^{uni,dR}(BdR(X^\circ)) \otimes \mathbb{C} \simeq \mathcal{O}(\pi_1^{uni,Betti}(X^\circ, x, y)) \otimes \mathbb{C}$$

The proof follows by using a combinatorial presentation of relative cohomology.

2.2. **Relation to Malcev–Lie algebra.**

3. Recollection on Stacks Approach

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\begin{gathered}\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R) \\ X \mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?})\end{gathered}$$

Example 3.1. The *de Rham stack* X^{dR} over $\mathbb{Q}$, has points given by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A (cf. [GR14]).

This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques.

3.1. **Stacks approach to unipotent group scheme.**

We recall the work of [MR23]. Let $(X, x) \in (\mathrm{Sch}_k)_*$ such that it is cohomologically connected [4] A classical homotopical invariant for schemes is the étale fundamental group introduced by Grothendieck. Nori, upgraded this definition to that of a *group scheme*, $\pi_1^N(X, x)$, which is constructed using Tannakian methods. One can associate a unipotent homotopy type $\mathbf{U}(X)$, which recovers Nori's unipotent homotopy group scheme, Definition 3.2,

$$\pi_1(\mathbf{U}(X), x) \xrightarrow{\simeq} \pi_1^{U,N}(X, x) \tag{2}$$

This is proved in [MR23, §3.1.].

[4] $H^0(X, k) \simeq k$.

Proposition 3.1. *We have an adjunction [Toë06, Cor. 2.2.4]*

$$\begin{tikzcd} CAlg_k^{\heartsuit} \arrow[d, hook] \arrow[dr, hook] & \\ (CAlg_k^{ccn})^{op} \arrow[r, shift left, hook, "\mathrm{Spec}"] & PShv(CAlg_k^{\heartsuit}) \arrow[l, shift left, "\boldsymbol{U}"] \end{tikzcd}$$

The right adjoint is alternatively denoted as $\Gamma(-, \mathcal{O})$, the *global sections* functor.

Definition 3.1. An object in the essential image of Spec in Proposition 3.1 is an *affine stack*, and the right adjoint $\mathbf{U}$ is called the *affinization*.

From the results of [Toë06], discussed in Section 3.3, we can define the homotopy groups:

Definition 3.2. Let $X \in (\mathrm{Sch}_k)_*$ which is cohomologically connected. Define

$$\pi_i^u(X) := \pi_i(\mathbf{U}(X), *) \in \mathrm{Grp}(\mathrm{Aff}_k)$$

as the *unipotent homotopy groups of* X, where $\mathbf{U}$ is as defined in Proposition 3.1.

Remark 3.1. The unipotent type can be defined for $(\mathrm{Stk}_R)_*$. But they are not necessarily representable, see [Mon22, p. 5].

Proposition 3.2. Spec *factors through* $\mathrm{Shv}_k^\wedge$.

Proof. By faithfully flat descent, Spec factors through Shv_k. For hypercompleteness see [Lur11b, Appendix D]. □

Example 3.2. $K(\mathbb{G}_a, i) := \mathrm{Spec}\,\mathrm{Sym}_k^{\mathrm{co}}\, k[-i]$ for $i > 0$ are affine stacks.

Example 3.3. Zero truncated quasi-affine stacks are *not* affine.

3.2. **Homotopy and hypercomplete sheaves.** Let $X \in \mathrm{Stk}_k$, $R \in \mathrm{CAlg}^{\heartsuit}$ in this section. In this paper, we would only be considering hypercomplete sheaves.

Definition 3.3. Let $n \geq 0$, then

$$\pi_n(X, *) \in \mathrm{Shv}_{\mathrm{Set}}(\mathrm{CAlg}_R^{\heartsuit}, \mathrm{fpqc})$$

is the sheafification of the presheaf

$$A \mapsto \pi_n(X(A), *)$$

We will be interested in hypercomplete sheaves, see [CM21] for a discusscusion in the prestable setting.

Definition 3.4. A morphism $f : X \to Y$ in an ∞-topos $\mathfrak{X}$ is ∞-*connective* if

(1) it is an effective epimorphism.

(2) $\pi_k f = *$ for $k \geq 0$.

Definition 3.5. $X \in \mathfrak{X}$ is *hypercomplete* iff it is local to ∞-connective morphism. We denote the hypercomplete objects as $\mathfrak{X}^\wedge$, fitting into an adjunction

$$\mathfrak{X}^\wedge \underset{\hookrightarrow}{\overset{\longleftarrow}{\rightleftarrows}} \mathfrak{X}$$

Hypercompleteness can also be characterized by hypercoverings.

Example 3.4. Let $(\mathcal{C}, \tau)$ be an ∞-stie, [Lur09, Ch.6]. Let $\mathcal{D}$ be an $(n+1, 1)$ category for $n \geq 0$, [Lur09, 2.3.4]. Then $F \in \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, D)$ satisfies descent for coverings iff it satisfies descent for hypercovering. In particular, this is useful when $(\mathcal{C}, \tau)$ is an ordinary category as the representables factors through Set $\hookrightarrow \mathcal{S}$.

3.3. **Representability results of Toën.**

Theorem 3.1. *[Toë06, Thm. 2.4.1, 2.4.5] Let $X \in (\mathrm{Shv}_k^\wedge)_*$, such that $\pi_0 X \simeq *$, then X is an affine stack iff $\pi_i(X, *)$ is representable by an affine group scheme $\pi_i^u X$ for all $i > 0$.*

Remark 3.2. [Toë06, Thm. 2.4.], if $H^0(B) \simeq k$, for $B \in (\mathrm{CAlg}_k^{\mathrm{ccn}})_*$, then $\mathrm{Spec}\, B$ is pointed connected.

3.4. **Nori's unipotent scheme.** To be written.

4. Betti Analytic Stack

To be written. Such stacks was discussed in [KpT08], [PY16]. The name *Betti analytic stack* can be misleading. This is stack X^{an} is designed so that

$$\pi_1(X^{\mathrm{an}}, *) \simeq \pi_1(|X(\mathbb{C})|, *)$$

where $X(\mathbb{C})$ is given the analytic topology. [5]

Remark 4.1. We do not yet have a correct definition of what X^{an} should be. There are - as of now - two ways to construct. One follows [PY16], which we write X_{loc}, the other from [Sch22, Ch.1], which we write X_{Betti}. At the level of cohomology:

$$R\Gamma(X_{\mathrm{loc}}, \mathcal{O}_{X_{\mathrm{loc}}}) \simeq R\Gamma(X_{\mathrm{Betti}}, \mathcal{O}_{X_{\mathrm{Betti}}}) \simeq R\Gamma(|X(\mathbb{C})|, \mathbb{C})$$

However, we still need to compute π_1.

We first recall the construction of X_{loc}. The natural map $\pi : \mathrm{Aff}_{\mathbb{C}} \to *$, induces a geometric morphism

$$\mathrm{Stk}_{\mathbb{C}} \underset{\pi_*}{\overset{\pi^*}{\rightleftarrows}} \mathcal{S}$$

Definition 4.1. $(\mathrm{Stn}_{\mathbb{C}}, \tau_{\mathrm{an}})$, denote the category of Stein complex analytic spaces with the analytic topology: this consists coverings $\{U_i \to X\}_{i \in I}$, where $U_i \hookrightarrow_{\mathrm{open}} X$ are open immersions, and $\coprod_{i \in I} U_i \to X$ is a surjection. [6] Let $\mathrm{AnStk}_{\mathbb{C}} := \mathrm{Shv}_{\mathcal{S}}(\mathrm{Stn}_{\mathbb{C}}, \tau_{\text{ét}})$.

[5] The reason for this (bad) choice is also not to be confused with recent works, [SC23].

[6] One can also consider with respect to the $\tau_{\text{ét}}$ étale topology, i.e. $U_i \to X$

Proposition 4.1. *There is an analytification functor*

$$(-)^{an} : \left(Aff_{\mathbb{C}}^{dfp}, \tau_{\acute{e}t}\right) \to (Stn_{\mathbb{C}}, \tau_{an})$$

Proof. See [Lur11a], [Por18]. □

In particular there is a well defined functor

$$\mathrm{Stk}_{\mathbb{C}} \to \mathcal{S}$$
$$X \mapsto |X(\mathbb{C})|$$

sending a stack to its underlying analytic topology.

Definition 4.2. we let $X_{\mathrm{loc}} := \pi^*(|X(\mathbb{C})|) \in \mathrm{Stk}_{\mathbb{C}}$ be the *analytic stack of local systems.*

Lemma 4.1. *(1)* $*_{loc} \simeq \operatorname{Spec}\mathbb{C}$.

(2) For $\operatorname{Spec} A \in Stk_{\mathbb{C}}$, $QCoh(X_{loc} \times \operatorname{Spec} A) \simeq Fun(|X(\mathbb{C})|, Mod_A)$.

Proof. (1) π^* is left exact, so it preserves the terminal object.

(2) This is by induction. Write $|X(\mathbb{C})|$ as the colimit of a tower cells,

$$|X(\mathbb{C})| \simeq \mathrm{colim}_{n \in \mathbb{N}} X_n$$

Then use that $\pi^*(-)$, $\mathrm{QCoh}(-)$, and $\mathrm{Fun}(-, \mathrm{Mod}_A)$ commutes with colimits in their variables

□

(2) justifies why we call this the stack of local systems. $\mathrm{Fun}(|X(\mathbb{C})|, \mathrm{Mod}_A)$ identify with the locally constant sheaves on $\mathrm{Op}(|X(\mathbb{C})|)$, the site of open subsets of $X(\mathbb{C})$. This implies that $\mathcal{O}_{X^{\mathrm{an}}}$ corresponds to the constant sheaf. In particular $\pi_*\mathcal{O}_{X^{\mathrm{an}}} \simeq R\Gamma(|X(\mathbb{C})|, \mathbb{C})$, where $\pi_* : \mathrm{QCoh}(X_{\mathrm{loc}}) \to \mathrm{QCoh}(*) \simeq \mathrm{Mod}_{\mathbb{C}}$.

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