p-adic Hodge theory: an introduction

Jared Weinstein

Boston University

March 29, 2022

Hodge theory: the big questions

- What sorts of *linear algebra objects* (cohomology theories) can be attached to manifolds, varieties over $\mathbf{C}$, varieties over $\overline{\mathbf{F}}_p$, varieties over $\mathbf{C}_p$, etc.?
- How do these cohomology theories interact? (Periods, comparison isomorphisms, Hodge structures...)
- How should we define Shimura varieties and their analogues (Rapoport-Zink spaces, local Shimura varieties, local shtuka spaces...) in terms of moduli of Hodge structures?
- How can Shimura varieties and their analogues help us with the Langlands program?

4 steps to p-adic Hodge theory

This talk builds up p-adic Hodge theory in four steps:

1. The complex picture,
2. The picture over a perfect field k of char. p,
3. The picture over $\mathbf{C}_p$,
4. The picture over a perfectoid space.

The ultimate goal is to understand the p-adic analogues of Shimura varieties, known as *local shtuka spaces*.

The complex picture: Hodge structures and Shimura varieties

A smooth manifold X has singular cohomology $H^i_{\mathrm{sing}}(X, \mathbf{Z})$ and de Rham cohomology $H^i_{\mathrm{dR}}(X)$, and these can be identified over $\mathbf{R}$ by integration.

But when X is also a projective variety over $\mathbf{C}$ (or just Kähler), then $V = H^i_{\mathrm{sing}}(X, \mathbf{Z})$ admits a Hodge structure (=Hodge decomposition of $V \otimes \mathbf{C}$), which we can describe with a homomorphism of real groups $\mu \colon \mathbf{C}^\times \to \mathrm{GL}(V)$.

Can generalize from $\mathrm{GL}(V)$ to G, a reductive group over $\mathbf{Q}$. Occasionally the conjugacy class of $\mu \colon C^\times \to G$ is a Hermitian symmetric domain, in which case we get a tower of Shimura varieties $\mathrm{Sh}(G, \mu)$.

The complex picture: Elliptic curves

If $X = \mathbf{R}^2/\mathbf{Z}^2$, then $H^1_{\text{sing}}(X, \mathbf{Z}) \cong \mathbf{Z}^{\oplus 2}$.

A complex structure on X turns it into an elliptic curve, which has a unique-up-to-scalar $\omega \in H^{1,0}(X) \subset H^1_{\text{dR}}(X)$. The μ describing this Hodge structure is conjugate to $z \mapsto \text{diag}(z, \bar{z})$.

Let $\gamma_1, \gamma_2 \in H_1(X, \mathbf{Z})$ be a basis; then the ratio $(\int_{\gamma_1} \omega : \int_{\gamma_2} \omega)$ determines a point of $\mathcal{H} = \mathbf{P}^1(\mathbf{C}) \backslash \mathbf{P}^1(\mathbf{R})$.

Conversely, a point $z \in \mathcal{H}$ determines an elliptic curve $\mathbf{C}/[z, 1]$.

The complex picture: elliptic curves

The following are in bijection:

1. Complex structures on $\mathbf{R}^2/\mathbf{Z}^2$,
2. Elliptic curves $E/\mathbf{C}$ with $\mathbf{Z}^2 \stackrel{\sim}{\to} H_1(E, \mathbf{Z})$,
3. Hodge structures on $\mathbf{Z}^2$ of type μ,
4. Points of $\mathcal{H}$.

The group $\mathrm{GL}_2(\mathbf{Z})$ acts on everything, for instance in (2) by changing the basis.

For each congruence subgroup $\Gamma \subset \mathrm{GL}_2(\mathbf{Z})$, get Shimura variety $\mathrm{Sh}(\mathrm{GL}_2, \mu)_\Gamma = \mathcal{H}/\Gamma$, a modular curve.

The picture over k: crystalline cohomology

Let k be a perfect field of characteristic p. Let W be its ring of Witt vectors. The Frobenius $\mathrm{Fr}_p \in \mathrm{Aut}\, k$ induces $\sigma \in \mathrm{Aut}\, W$.

Let X/k be smooth and proper. Can form its crystalline cohomology

$$H^i_{\mathrm{crys}}(X/W) := H^i_{\mathrm{dR}}(\tilde{X}/W),$$

where $\tilde{X}/W$ is a smooth proper lift of k.

Loosely in analogy with H^i_{dR} of a real manifold. No Hodge filtration on $H^i_{\mathrm{crys}}(X/W)$ yet.

But there is one new bit of structure: The relative Frobenius $X \to X^{(p)} = X \times_{k,\mathrm{Fr}_p} k$ induces a σ-linear endomorphism F of $H^i_{\mathrm{crys}}(X/W)$.

The picture over k: H^1 and Dieudonné modules

A *Dieudonné module* is a finite free W-module D together with σ-linear and σ^{-1}-linear endomorphisms F, V satisfying $FV = p$.

Dieudonné-Manin classification ($k = \overline{k}$): each D decomposes into irreducibles D_λ with "slope" $\lambda \in \mathbf{Q} \cap [0,1]$. Here $\lambda = p$-adic valuation of eigenvalue of F.

Fontaine: There's an anti-equivalence $\mathcal{G} \mapsto D(\mathcal{G})$ between p-divisible groups over k and Dieudonné modules.

Also if A/k is an ab. var. then $H^1_{\text{crys}}(A/W) \cong D(A[p^\infty])$.

Examples: $D(\mathbf{Q}_p/\mathbf{Z}_p) = D_0$, $D(\mu_{p^\infty}) = D_1$,
$H^1_{\text{crys}}(E/W) \cong D_0 \oplus D_1$ (ordinary), $H^1_{\text{crys}}(E/W) \cong D_{1/2}$ (supersingular).

The picture over k: isocrystals and G-structure

Let $K_0 = W[1/p]$. An *isocrystal* is a fin. dim. K_0-vector space with σ-linear automorphism F.

Isom. classes of isocrystals correspond to elements of $\mathrm{GL}_n(K_0)$ up to σ-conjugacy: $b \sim x^\sigma b x^{-1}$.

For $\lambda = m/n$, the isocrystal D_λ corresponds to

$$b = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ p^m & & & \cdots & 0 \end{pmatrix} \in \mathrm{GL}_n(K_0).$$

For a reductive $G/\mathbf{Q}_p$, Kottwitz introduced a notion of "G-isocrystal". Isom. classes of these $\cong B(G) := G(K_0)$ up to σ-conjugacy.

The picture over C: moduli of p-divisible groups

Let $C/\mathbf{Q}_p$ be a complete algebraically closed field with ring of integers $\mathcal{O}_C$ and residue field k.

How to classify p-divisible groups $\mathcal{G}/\mathcal{O}_C$? Here are 3 invariants of $\mathcal{G}$:

1. The Tate module $T = T_p\mathcal{G}$, a free $\mathbf{Z}_p$-module of rank $n = \mathrm{height}(\mathcal{G})$.
2. The Lie algebra $\mathrm{Lie}\,\mathcal{G}$, a C-vector space of dimension $d = \dim(\mathcal{G})$,
3. The special fiber $\mathcal{G}_k$, which corresponds to a Dieudonné module, in turn corresponding to $b \in B(\mathrm{GL}_n)$.

How are these all related?

The picture over C: moduli of p-divisible groups

Let $\mathcal{G}$ be a p-divisible group over $\mathcal{O}_C$.

Theorem (Fargues, the Hodge-Tate exact sequence)

There is a natural short exact sequence

$$0 \to \operatorname{Lie} \mathcal{G} \otimes C(1) \to T_p\mathcal{G} \otimes_{\mathbf{Z}_p} C \to (\operatorname{Lie} \mathcal{G}^*)^* \otimes C \to 0$$

Letting $T = T_p\mathcal{G}$ and $W = \operatorname{Lie} G \otimes C(1)$, we have a pair (T, W) with $\operatorname{rank}_{\mathbf{Z}_p} T = n$, $\dim_C W = d$ and $W \subset T \otimes C$.

Theorem (Scholze-W., 2012)

The functor $\mathcal{G} \mapsto (T, W)$ is an equivalence of categories.

Question: given (T, W), how to read off $\mathcal{G}_k$?

The picture over C: p-divisible groups with $(n, d) = (2, 1)$

As an example, consider $\mathcal{G}/\mathcal{O}_C$ with height 2 and dimension 1. Choose $\mathbf{Z}_p^2 \cong T$. Such $\mathcal{G}$ are in bijection with $W \in \mathbf{P}^1(C)$. Some will have $D(\mathcal{G}_k) = D_{1/2}$ (basic case) and the rest will have $D(\mathcal{G}_k) = D_0 \oplus D_1$.

In fact $\mathcal{G}_k$ is basic if and only if $W \in \mathcal{H} = \mathbf{P}^1(C) \backslash \mathbf{P}^1(\mathbf{Q}_p)$ (Drinfeld's upper half-plane).

Example of a local Shimura variety: Let $\mathcal{M}^{\mathrm{Dr}}$ classify triples $(\mathcal{G}, \alpha, \iota)$, where $\mathcal{G}/\mathcal{O}_C$ is a p-divisible group, $\alpha \colon \mathbf{Z}_p^{\oplus 2} \xrightarrow{\sim} T_p\mathcal{G}$, and $\iota \colon D_{1/2}[1/p] \cong D(\mathcal{G}_k)[1/p]$ is an isomorphism of isocrystals.

$\mathcal{M}^{\mathrm{Dr}}$ is the *Drinfeld tower*, it is a pro-étale torsor over $\mathcal{H}$ with group $D^\times$, where $D = \operatorname{End} D_{1/2}[1/p]$ is the quaternion algebra over $\mathbf{Q}_p$. The map $\mathcal{M}^{\mathrm{Dr}} \to \mathcal{H}$ is equivariant for the action of $\mathrm{GL}_2(\mathbf{Q}_p)$.

How to determine $D(\mathcal{G}_k)$ from (T, W) in general?

To answer, we need rings larger than C. Construction (Fontaine):

$$\mathcal{O}_{C^\flat} := \varprojlim_{\mathrm{Fr}_p} \mathcal{O}_C/p,$$

a perfect valuation ring in char p, with pseudo-uniformizer $p^\flat = (p, p^{1/p}, p^{1/p^2}, \dots)$. The projection $\mathcal{O}_{C^\flat} \to \mathcal{O}_C/p$ induces a surjection

$$\theta\colon W(\mathcal{O}_{C^\flat}) \to \mathcal{O}_C.$$

Let $A_{\text{inf}} = W(\mathcal{O}_{C^\flat})$, a 2-dimensional local ring with endomorphism φ induced by Fr_p.

The picture over C: $A_{\inf}$ and related rings

We have $\mathcal{O}_C$, its tilt $\mathcal{O}_{C^\flat}$, and $A_{\inf} = W(\mathcal{O}_{C^\flat})$. This 2-d ring has 3 obvious 1-d quotients:

$x_{C^\flat}$ $A_{\inf} \to \mathcal{O}_{C^\flat}$, kernel gen'd by p. Complete local ring $= W(C^\flat)$.

x_{K_0} $A_{\inf} \to W(k)$, kernel gen'd by $[x]$ for $x \in$ max. ideal of $\mathcal{O}_{C^\flat}$.

x_C $\theta\colon A_{\inf} \to \mathcal{O}_C$, kernel generated by

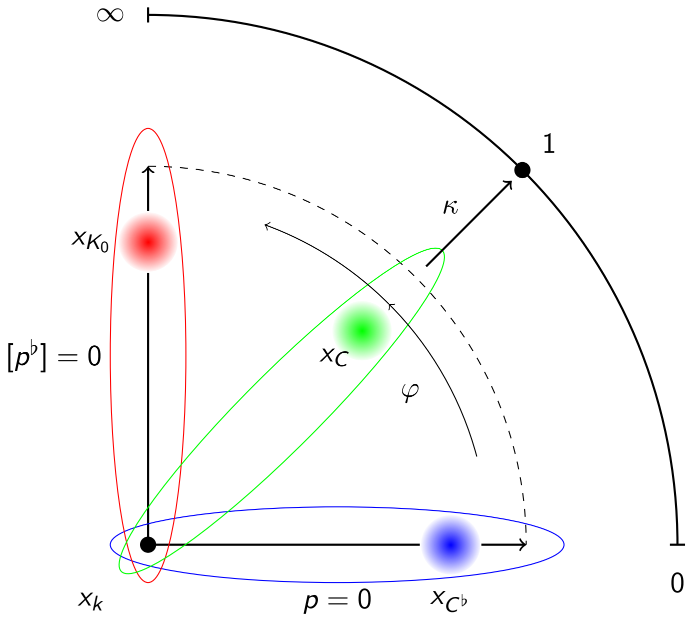
$$\xi = \frac{[\varepsilon] - 1}{[\varepsilon^{1/p}] - 1},\ \varepsilon = (1, \zeta_p, \zeta_{p^2}, \dots) \in \mathcal{O}_{C^\flat}$$

Regarding the last quotient, we will also need the completion:

$$B_{\mathrm{dR}}^+ := \varprojlim A_{\inf}[1/p]/\xi^n$$

This is a DVR with fraction field $B_{\mathrm{dR}} = B_{\mathrm{dR}}^+[1/\xi]$.

The picture over C: A_{inf} and related rings



The picture over C: The p-adic $2\pi i$ (interlude)

The map $A_{\text{inf}} \xrightarrow{\theta} \mathcal{O}_C$ has kernel $\xi = ([\varepsilon]-1)/([\varepsilon^{1/p}]-1)$, and $B_{\text{dR}}^+ = (A_{\text{inf}})\hat{}_{\xi}$.

Periods of varieties over C lie in B_{dR}. The simplest example is the element

$$t = \log[\varepsilon] = ([\varepsilon]-1) - \frac{1}{2}([\varepsilon]-1)^2 + \cdots \in B_{\text{dR}}^+$$

Then t is the period of the formal multiplicative group; note that $\text{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on t through the cyclotomic character.

The picture over C: Breuil-Kisin-Fargues modules

When we upgrade from k to C, Dieudonné modules become Breuil-Kisin-Fargues (BKF) modules.

A BKF module is a finite free A_{inf}-module M together with an isomorphism $\varphi_M\colon (\varphi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}]$.

At the special points $x_{C^\flat}, x_{K_0}, x_C \in \operatorname{Spec} A_{\mathrm{inf}}$, we get from (M, φ_M) the following data:

1. At the completion of $x_{C^\flat}$, we get a φ-module N over $W(C^\flat)$, and these are in equivalence with free finite rank $\mathbf{Z}_p$-modules T, via $N \mapsto N^{\varphi=1}$ and $T \mapsto T \otimes W(C^\flat)$.
2. At x_{K_0}, we get a φ-module over K_0, which is the same as an isocrystal.
3. At x_C, we have a B_{dR}^+-lattice $\Xi \subset T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$, measuring the failure of φ to be an isomorphism at this point.

The picture over C: Breuil-Kisin-Fargues modules

A BKF module is a finite free A_{inf}-module M together with an isomorphism $\varphi_M \colon (\varphi^* M)[\xi^{-1}] \xrightarrow{\sim} M[\xi^{-1}]$.

Theorem (Fargues)

The following categories are equivalent.

1. *Pairs (T, Ξ), where T is a finite free $\mathbf{Z}_p$-module and $\Xi \subset T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}$ is a B_{dR}^+-lattice.*
2. *BKF modules.*

When we restrict to those Ξ for which this condition holds:

$$T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}^+ \subset \Xi \subset \xi^{-1}(T \otimes_{\mathbf{Z}_p} B_{\mathrm{dR}}^+),$$

so that Ξ corresponds to a C-subspace of $T \otimes_{\mathbf{Z}_p} C(-1)$, the category becomes equivalent to the category of p-divisible groups over $\mathcal{O}_C$.

The picture over C: some moduli spaces of shtukas

By the theory of elementary divisors, $\mathrm{GL}_n(B_{\mathrm{dR}}^+)$-orbits of lattices $\Xi \subset B_{\mathrm{dR}}^{\oplus n}$ are in bijection with tuples $k_1 \geq \cdots \geq k_n$, which are in turn in bijection with conjugacy classes of cocharacters $\mu\colon \mathbf{G}_m \to \mathrm{GL}_n$, via $\mu(t) = \mathrm{diag}(t^{k_1}, \ldots, t^{k_n})$.

Now let μ be a cocharacter and $b \in B(\mathrm{GL}_n)$. We can now define the moduli space of shtukas $\mathrm{Sht}(\mathrm{GL}_n, b, \mu)$, at least on the level of C-points. They are:

BKF modules where the Ξ is of type μ, and the isocrystal is $D(b)$.

There's a period morphism $\mathrm{Sht}(\mathrm{GL}_n, b, \mu) \to \mathrm{Gr}_{\mathrm{GL}_n,\mu}$ recording the Ξ; here $\mathrm{Gr}_{\mathrm{GL}_n,\mu}$ is the B_{dR}^+-affine Grassmannian.

The picture over C: contact with integral p-adic Hodge theory

Looking beyond p-divisible groups, we have the following "master comparison theorem":

Theorem (Bhatt-Morrow-Scholze)

Let $X/\mathcal{O}_C$ be smooth and proper. There is a perfect complex of A_{inf}-modules $R\Gamma_{A_{\mathrm{inf}}}(X)$, equipped with a φ-linear map $R\Gamma_{A_{\mathrm{inf}}}(X)$ inducing a quasi-isomorphism

$$R\Gamma_{A_{\mathrm{inf}}}(X)[1/\xi] \xrightarrow{\sim} R\Gamma_{A_{\mathrm{inf}}}[1/\varphi(\xi)].$$

It specializes to the following cohomology theories:

1. *At x_{K_0}, $R\Gamma_{A_{\mathrm{inf}}}(X) \otimes_{A_{\mathrm{inf}}} W \cong R\Gamma_{\mathrm{crys}}(X_k/W)$.*
2. *At x_C, $R\Gamma_{A_{\mathrm{inf}}}(X) \otimes_{A_{\mathrm{inf}}} \mathcal{O}_C \cong R\Gamma_{\mathrm{dR}}(X/\mathcal{O}_C)$.*
3. *Near $x_{C^\flat}$, $R\Gamma_{A_{\mathrm{inf}}}(X) \cong R\Gamma_{\text{ét}}(X, \mathbf{Z}_p) \otimes A_{\mathrm{inf}}$ as φ-modules.*

The picture over a perfectoid space

To actually define a moduli space like $\mathrm{Sht}(G, b, \mu)$ or $\mathrm{Gr}_{G,\mu}$, need to be able to work in families, not just over C.

When μ is minuscule, $\mathrm{Gr}_{G,\mu}$ is a flag variety, and the finite layers of $\mathrm{Sht}(G, b, \mu)$, being étale over $\mathrm{Gr}_{G,\mu}$ (via Gross-Hopkins period map), are rigid-analytic spaces (Scholze). These are the *local Shimura varieties*. They encompass all Rapoport-Zink spaces.

But in general, $\mathrm{Gr}_{G,\mu}$ is not a rigid-analytic variety. What is it?

The picture over a perfectoid space:perfectoid rings

Let R be a perfectoid algebra. This means:

1. R is a topological ring,
2. R admits an open subring R_0 whose topology is generated by a single element $\varpi \in R_0 \cap R^\times$,
3. R° (power-bounded elements) is bounded in R,
4. ϖ^p divides p, and Frobenius induces an isomorphism $R^\circ/\varpi \xrightarrow{\sim} R^\circ/\varpi^p$.

Examples: $\mathbf{Q}_p^{\mathrm{cycl},\hat{}}$, $\mathbf{C}_p$, $\mathbf{F}_p((t^{1/p^\infty}))$, $K\langle T^{1/p^\infty}\rangle$ for any perfectoid field K.

The picture over a perfectoid space: tilting and B_{dR}^{+}

For a perfectoid ring R, we have define $R^{\flat\circ} = \varprojlim_{x \mapsto x^p} R/\varpi$, and then if $\varpi^\flat = (\varpi, \varpi^{1/p}, \dots, \dots) \in R^{\flat\circ}$, then $R^\flat = R^{\flat\circ}[1/\varpi^\flat]$ is a perfectoid ring of characteristic p.

There is a natural map $W(R^{\circ\flat}) \to R^\circ$, whose kernel is generated by a single element ξ. Define $B_{\mathrm{dR}}^{+}(R)$ to be the ξ-adic completion of $W(R^{\circ\flat})[1/p]$.

Thus it seems that B_{dR}^{+}, $\mathrm{Gr}_{G,\mu}$, $\mathrm{Sht}_{G,b,\mu}$, etc., should be defined as functors on perfectoid rings / perfectoid spaces.

The picture over a perfectoid space: some sheaves on Perf

Let Perf be the category of perfectoid spaces in char. p, with its pro-étale topology (akin to schemes with the étale topology). We consider the following (contravariant) sheaves on Perf:

- Let $\operatorname{Spd} \mathbf{Q}_p$ be the sheaf whose value on $S = \operatorname{Spa} R$ is the set of untilts $R^\sharp/\mathbf{Q}_p$.
- Let $B_{\mathrm{dR}}^+ \to \operatorname{Spd} \mathbf{Q}_p$ be the sheaf whose fiber over $R^\sharp$ is $B_{\mathrm{dR}}^+(R^\sharp)$.
- Let $\mathrm{Gr}_{\mathrm{GL}_n, \leq \mu} \to \operatorname{Spd} \mathbf{Q}_p$ be the sheaf whose fiber over $R^\sharp$ is the set of $B_{\mathrm{dR}}^+(R^\sharp)$-lattices Ξ in $B_{\mathrm{dR}}^{\oplus n}$ that are everywhere bounded by μ.

The picture over a perfectoid space: Diamonds

An *algebraic space* is a functor on Sch (with its étale topology) of the form X/R, where X is a scheme and $R \subset X \times X$ is an étale equivalence relation.

A *diamond* is a functor on Perf (with its pro-étale topology) of the form X/R, where X is in Perf and $R \subset X \times X$ is a pro-étale equivalence relation.

Example: $\mathrm{Spd}\,\mathbf{Q}_p = (\mathrm{Spa}\,\mathbf{Q}_p^{\mathrm{cycl},\flat})/\mathbf{Z}_p^\times$.

Theorem (Scholze)

The sheaves $\mathrm{Gr}_{\mathrm{GL}_n,\leq\mu}$ and $\mathrm{Sht}_{G,b,\mu}$ are [locally spatial] diamonds.

Sources

For Hodge theory and Shimura varieties: Milne, Introduction to Shimura Varieties.

For Dieudonné theory: Katz, Crystalline cohomology, Dieudonné modules, and Jacobi sums

For perfectoid spaces, diamonds, and "shtukas with one leg": Scholze, Weinstein, Berkeley Lectures on p-adic geometry.