1 More Set Theory, power set construction

Week 2: will miss one class due to Labor day.
Reading: [3, Ch.3, Ch.4.1-2], [2, 2].

Learning Objectives

In last lectures, we

- Defined $\mathbb{N}$ axiomatically, **??**.
- Used induction to prove properties of operations as $+$ and $\times$ on $\mathbb{N}$.

In the next two lectures

- Discuss the remaining axioms of set theory. We begin by discussing new notions: *subsets*, 1.1, and *ordered pairs*, 1.2. We end with the construction of the power set, 1.7.
- Discuss *equivalence relation*, 2, which constructs the integers and the rationals

1.1 Subcollections

Definition 1.1. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1,2\} \subset \{1,2,3\}$.

1.2 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 1.2. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1,2\} = \{2,1\}$

In ordered pairs

- $(1,2) \neq (2,1)$

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

1.3 Remaining axioms of set theory

Week 2

In this section we continue from previous lecture and discuss more axioms from set theory. We complete the *Zermelo-Fraenkel axioms of set theory*, due to Ernest Zermelo and Abraham Fraenkel.

Axiom 1.3. Axiom of pairwise union. Given any two sets A, B there exists a set $A \cup B$ whose elements which belong to either A or B or both.

Often we would require a stronger version.

Axiom 1.4. Axiom of union. Let A be a set of sets. Then there exists a set

$$\bigcup A$$

whose objects are precisely the elements of the set.

Example

Let

- $A = \{\{1,2\},\{1\}\}$
- $A = \{\{1,2,3\},\{9\}\}$

Discussion

Using the axioms, can we get from $\{1,3,4\}$ to $\{2,4,5\}$?

We will now state the power set axiom for completeness but revisit again.

Axiom 1.5. Axiom of power set. Let X, Y be sets. Then there exists a set Y^X consists of all functions $f : X \to Y$,

We will review definition of function later, **??**.

Axiom 1.6. Axiom of replacement. For all $x \in A$, and y any object, suppose there is a statement $P(x, y)$ pertaining to x and y. There is a set

$$\{y \,:\, P(x, y) \text{ is true for some } x \in A\}$$

This can intuitively be thought of as the set

$$\{y \,:\, y = f(x) \text{ some } x \in A\}$$

Proposition 1.7. The collection

$$\{Y \,:\, Y \text{ is a subset } X\}$$

is a set.

Proof. We have $\{0, 1\}^X$ is a set, 1.5. For $Y \subset X$, $f \in \{0, 1\}^X$, let $P(Y, f)$ be the property that

$$Y = f^{-1}(1) := \{x \in X \,:\, f(x) = 1\}$$

by axiom of replacement, 1.6, we obtain our desired collection. $\square$

2 Equivalence Relation

Week ?

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [1, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 2.1. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$
$$a \sim a$$
- *Transitive*. For all $a, b, c \in A$,
$$a \sim b, b \sim c \Rightarrow a \sim c$$
- *Symmetric*. For all $a, b \in A$,
$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Definition 2.2. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as

$$[x] := \{y \in A : x \sim y\}$$

Definition 2.3. Quotient set. Given a relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

There is a relation between equivalence and partition of sets.

Definition 2.4. A *partition* of a set X is a collection ???

2.1 Integers

What are the integers? What is *subtraction* or the *negative* numbers.

Discussion

- What is the difference between "$4-6$" and "$3-5$"?
- If we suppose subtraction is well-defined, how do we define addition?

The *integers* is an *expression*[1] of non-negative integers, (a, b), thought of as $a - b$. Where two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 2.5. The *integers* is the set

$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

2.2 Rational nunbers

In a similar manner

Definition 2.6. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote a pair (a, b) by a/b.

Again, we need the notion of addition, multiplication, and negation.

Definition 2.7. Let $a/b, c/d \in \mathbb{Q}$. Then

1. Addition:
$$a/b + c/d := (ad + bc)/bd$$
2. Multiplication
$$a/b \cdot c/d := (ac)/(bd)$$
3. Negation.
$$-(a/b) := (-a)/b$$

Discussion

Is this definition well defined? What does this mean? This is hw.

[1] Rather than a pair, as an expression has multiple ways of presentation

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 2.8. $\sqrt{2}$ is not rational.

Proof. ??? $\square$

Homework for week 2

Due: Week 3, Wednesday. All questions on the section, 2.3, Boolean algebra is compulsory. Select 3 other questions to be graded.

Reading:
Working with sets requires a familiarity with definitions.

Problems

1. Let A, B, C be sets.

   (a) Prove set inclusion, def. 1.1, is reflexive and transitive. $A \subseteq B, B \subseteq C$ then $A \subseteq C$.

   (b) Prove that the union operation $\cup$ on sets 1.3, is associative and commutative:

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cup B = B \cup A$$

2. Let I be a set and that for all $i \in I$, I have a set A_i. Prove that can form the union of the collection:

$$\bigcup_{\alpha \in I} A_\alpha = \bigcup \{A_\alpha \,:\, \alpha \in I\}$$

3. Let A, B, C, D be sets. This exercise shows that we can actually construct *ordered pair* using the axioms of set theory. Prove

   - We can construct the following set

$$\langle A, B \rangle := \{A, \{A, B\}\}$$

     from the axioms of set theory.
   - Prove $\langle A, B \rangle = \langle C, D \rangle$ if and only if $A = B, C = D$

4. Show that addition, product, and negation are well-defined for rational numbers. 2.6.

5. (**) Let X be any set. Recall that a binary relation on X, is any subset $R \subset X \times X$. We define $R^{(n)}$ as follows

   - For $n = 0$,

$$R^{(0)} = \{(x, x) \,:\, x \in X\}$$

- Suppose $R^{(n)}$ has been defined.

$$R^{(n+1)} := \left\{(x,y) \in X \times X \,:\, \exists z \in X, (x,z) \in R^{(n)}, (z,y) \in R\right\}$$

- Show that $R^t := \bigcup_{n \geq 1} R^n$ defines a *smallest* transitive relation on X.
- Show that $R^{ts} := \bigcup_{n \geq 0} R^{(n)}$ is the *smallest* symmetric and transitive relation on X.

Hints for problems

3. Use axiom of replacement, and axiom of union.
4. This is an extensive use of extensionality.

2.3 Boolean algebras

This section is compulsory. Boolean algebras form the foundation of probability theory.

Definition 2.9. Let Ω be a set. A *Boolean algebra* in Ω is a set $\mathcal{E}$ of subsets of Ω (equivalently, $\mathcal{E} \subseteq 2^{\Omega}$) satisfying

1. $\emptyset \in \mathcal{E}$
2. closed under unions and intersections.
$$E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$$
$$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$$
3. closed under complements.

A *σ-algebra in* Ω is a Boolean algebra in Ω such that it satisfies

4. Countable[2] closure. If $A_i \in \mathcal{E}$ for $i \in \mathbb{N}$, then $\bigcup A_i \in \mathcal{E}$.

Problems

1. Prove that $\mathcal{E} := \{\emptyset, \Omega\}$ is a σ-algebra.
2. Prove that $2^{\Omega} := \{E : E \subset \Omega\}$ is a σ-algebra.
3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Hints for problems

3. There are 3 cases. What happens $A = \emptyset$ or $A = \Omega$? Now consider the case $A \neq \emptyset$ and $A \neq \Omega$.

[2] A set X is countable if it is in bijection with $\mathbb{N}$. We will explore this word in further detail in the future.

References

[1] Paul R. Halmos, *Naive set theory*, 1961.

[2] Jonathan Pila, *B1.2 set theory*.

[3] Terence Tao, *Analysis I, 4th edition*, 2022.