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COHOMOLOGY OF ARITHMETIC GROUPS – FIELDS MEDAL LECTURE

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Abstract

The topology of "arithmetic manifolds", such as the space of lattices in $\mathbf{R}^n$ up to rotation, encodes subtle features of the arithmetic of algebraic varieties. In some cases, this can be explained because the arithmetic manifold itself carries the structure of an algebraic variety.

I will talk about some of the phenomena one encounters in the other, "non-algebraic," cases.

Contents

1 Introduction

I have tried to write this paper, an expanded version of my ICM talk, for a broad audience. The style is very informal: I have suppressed many technicalities, at times I have been imprecise and used idiosyncratic terminology, and the final sections are rather speculative. I apologize for any confusion this causes. I will try in future to write down a more detailed account of the novel parts.

Informally speaking, an *arithmetic group* is obtained by taking the integral points of a matrix group. For example, $\mathrm{SL}_n(\mathbf{Z})$ is the group of $n \times n$ integer matrices with determinant 1. We will be interested in the cohomology of such groups. More geometrically: writing Y_n for the space of lattices in $\mathbf{R}^n$ up to rotation, we have

$$H^*(\mathrm{SL}_n(\mathbf{Z}), \mathbf{Q}) \simeq H^*(Y_n, \mathbf{Q}),$$

see §2 for details. The space Y_n is what one might call the "arithmetic manifold"[1] associated to $\mathrm{SL}_n(\mathbf{Z})$; every arithmetic group Γ has such an arithmetic manifold Y_Γ.

Now, for some arithmetic groups, for example $\Gamma = \mathrm{Sp}_{2g}(\mathbf{Z})$, the associated arithmetic manifolds Y_Γ live in the world of algebraic geometry: they can be identified with the complex points of an algebraic variety. We will call these cases the "Hermitian cases," because the universal cover of Y_Γ is a Hermitian symmetric domain. Over the past 50 years these situations have been deeply studied; this examination has motivated many fundamental ideas in the theory of automorphic forms, and the field remains extremely active.

Based on this study and other evidence, Langlands and others proposed that there is an association

(1) cohomology of arithmetic groups $\rightsquigarrow$ arithmetic local systems

This association is part of Langlands' reciprocity conjectures. The phrase "arithmetic local system" is meant to capture several related concepts (Galois representations and motives) in a way that is not too threatening. The reader should think of it as a local system on the spectrum Spec $\mathbb{Z}$ of the integers although it is not simple to give a precise definition. See §5 for details.

1.1 Outline of the paper. Non-Hermitian cases – e.g. $\mathrm{SL}_n(\mathbf{Z})$ when $n > 2$ – have been (relatively speaking) neglected. However, despite their absence of direct algebraic structure, they have other new features that are not seen in the Hermitian case. I want to give some flavour of these features in this paper.

(i) In the first part of the paper (sections §2 - §4) we will try to describe some of the structures that exist on the cohomology of arithmetic groups, but emphasizing the phenomena (torsion, derived Hecke operators) that are only prominent in the non-Hermitian case.

(ii) In the middle part, §5 and §6, we try to explain a little bit about Langlands' reciprocity conjecture (1).

(iii) In the final part of the paper, §7 – §9, we examine how various features on one side of (1) manifest themselves on the other side. The discussion of §8 and §9 is quite speculative.

[1] This is not a standard term. It might be more usually described as "the locally symmetric space associated to Γ."

For example, as we will discuss in §8, there is a natural notion of complexity (or "height") on both sides of (1), and we propose that these notions should match up with each other. Most of our discussion is not specific to the non-Hermitian case, but is by far the most interesting there.

In an ideal world, one would understand (1) by exhibiting a machine that takes input from the left-hand side and produces output on the right-hand side. In this world, the kind of exercise we do in (iii) is pointless. After all, we could simply stick an invariant of one side of (1) into our machine, wind the crank, and see what it produces on the other side.

But for the moment, *in the non-Hermitian case*, the mechanism for (1) remains a complete mystery (despite substantial progress recently in verifying some of its implications). Perhaps the conjectures formulated in §7 – §9 might, in fact, teach us something about this still unknown mechanism.

1.2 Acknowledgements. In some form or the other, the cohomology of arithmetic groups has been my primary object of study for the past few years.

The ideas presented here draw specifically on my joint work Bergeron and Venkatesh [2013] and Bergeron, Şengün, and Venkatesh [2016]with Haluk Sengun and Nicolas Bergeron, and also Prasanna and Venkatesh [2016] with Kartik Prasanna. My overall picture of the situation has been fundamentally shaped by my work Calegari and Venkatesh [2012] with Frank Calegari, and by Calegari's many patient explanations about basic properties of Galois representations. I learned a lot from discussions with Soren Galatius, and these discussions led to the paper Galatius and Venkatesh [2018]. My understanding of derived Hecke algebra has been much advanced through my collaboration Harris and Venkatesh [2018] and Darmon, Harris, Rotger, and Venkatesh [n.d.] with Michael Harris and with Henri Darmon, Harris, and Victor Rotger. Finally, Aravind Asok and Nicolas Bergeron gave me helpful feedback on this text.

I thank all of them, and all the other mathematicians, past and present, who discovered and developed this particular intellectual paradise. (I'm sorry that this brief text can't do justice to their contributions.)

2 Arithmetic groups and their cohomology

Warning: what we call "arithmetic groups," for short, would be usually called "congruence arithmetic group."

This introduction is partly borrowed from my similar introduction to Venkatesh [2017].

2.1 Arithmetic groups. An *arithmetic group* is a group such as

$$\mathrm{SL}_n(\mathbf{Z}), \mathrm{Sp}_{2n}(\mathbf{Z}), \mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}]), \cdots$$

obtained, roughly speaking, by taking the "$\mathbb{Z}$-points" of a classical group of matrices.

Now $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}])$ looks different to the others at first, but it can be presented similarly: one can regard 2×2 matrices over $\mathbb{Z}[i]$ as 4×4 matrices over $\mathbb{Z}$:

$$\begin{pmatrix} a+bi & c+di \\ e+fi & g+hi \end{pmatrix} \mapsto \begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ e & f & g & h \\ -f & e & -h & g \end{pmatrix}$$

A precise definition, adequate for our purposes, is given as follows: A "congruence arithmetic group" (which we shall abbreviate simply to "arithmetic group" for the purpose of this document) is a group Γ obtained by taking a semisimple $\mathbf{Q}$-group $\mathbf{G} \subset \mathrm{SL}_N$, an integer M, and taking

$$\Gamma = \{g \in \mathbf{G}(\mathbf{Q}) : g \text{ has integral entries, } g \equiv \mathrm{Id}_N \text{ modulo } M\}$$

or, more generally, allowing any Γ that contains such a subgroup with finite index. Each such group Γ is contained in an ambient Lie group, namely the real points of $\mathbf{G}$:

$$\Gamma \leqslant G = \mathbf{G}(\mathbf{R}).$$

In fact, Γ is a "lattice" in G: this just means that the quotient G/Γ has finite volume.

In the examples above, the ambient Lie groups are as follows:

$$\mathrm{SL}_n(\mathbf{Z}) \leqslant \mathrm{SL}_n(\mathbf{R}), \mathrm{Sp}_{2g}(\mathbf{Z}) \leqslant \mathrm{Sp}_{2g}(\mathbf{R}), \mathrm{SL}_2(\mathbb{Z}[\sqrt{-1}]) \leqslant \mathrm{SL}_2(\mathbf{C}).$$

2.2 Symmetric spaces. Each such group Γ acts on a canonically associated Riemannian manifold S (the "symmetric space" for the ambient Lie group G). In general S is, as a manifold, the quotient of G by a maximal compact subgroup $K \subset G$; it is known that all such K are conjugate inside G. It's easy to verify that G preserves a Riemannian metric on S.

For example, if $\Gamma = \mathrm{SL}_2(\mathbf{Z})$, we have $G = \mathrm{SL}_2(\mathbf{R})$ and can take $K = \mathrm{SO}_2$; the associated geometry $S = G/K$ can be identified with the Poincaré upper-half plane

$$S = \{z \in \mathbf{C} : \mathrm{Im}(z) > 0\}$$

and the action of G is by fractional linear transformations; it preserves the standard hyperbolic metric $|dz|^2/\mathrm{Im}(z)^2$.

If $\Gamma = \mathrm{SL}_2(\mathbb{Z}[i])$ we have $G = \mathrm{SL}_2(\mathbf{C})$ and can take $K = \mathrm{SU}_2$; the associated geometry S can be identified with the three-dimensional hyperbolic space $\mathbb{H}^3$.

Finally, for $\Gamma = \mathrm{SL}_n(\mathbf{Z})$ the situation is less familiar: $G = \mathrm{SL}_n(\mathbf{R})$, we can take $K = \mathrm{SO}_n$, and the geometry S can be identified with the space of positive definite, symmetric, real-valued $n \times n$ matrices A with $\det(A) = 1$. It is an enjoyable exercise to determine the invariant notion of distance in this case.

2.3 Motivation via reduction theory. Although not important for our later purposes, we now describe, by way of motivation, a problem that naturally leads to considering the cohomology of arithmetic groups.

For $\Gamma = \mathrm{SL}_n(\mathbf{Z})$, it is a classical problem – the reduction theory of quadratic forms – to explicitly describe a fundamental domain for Γ acting on S. The case of $n = 2$ was very well understood by Gauss, and Minkowski wrote down explicit fundamental domains for $n \le 6$; for example, for a 3×3 positive definite, symmetric matrix

$$S = \begin{pmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{pmatrix},$$

one can define a fundamental domain:

$$S_{11} \le S_{22} \le S_{33}, 0 \le S_{12} \le S_{11}/2, 0 \le S_{23} \le S_{22}/2, |S|_{13} \le S_{11}/2, \text{ eight more.}$$

Such examples suggest that the complexity of such a fundamental domain (e.g. the number of inequalities needed to define it) increases very rapidly as one increases n.

Must a fundamental domain be complicated? One reason that would force it to be so is if the quotient

$$Y_\Gamma = S/\Gamma = \text{“arithmetic manifold” associated to } \Gamma$$

is topologically complicated, since one can make a model for Y_Γ by gluing sides in the fundamental domain.[2] This leads us naturally to the study of the Betti numbers, and thus the cohomology, of Y_Γ.

The cohomology of Y_Γ as a topological space coincides with the group cohomology of Γ as a discrete group:

$$H^*(\Gamma, \mathbf{C}) \simeq H^*(Y_\Gamma, \mathbf{C}).$$

Thus we can study the topology of Y_Γ without explicit recourse to the geometry of S, if so desired.

Two technical warnings:

- What we call “arithmetic manifold” would be, in more standard terminology, the “locally symmetric space” associated to Γ.
- It often happens that it is convenient and important to group together Y_Γ for several very closely related arithmetic groups Γ. This process is achieved in number theory by considering adèle groups, that is to say, by replacing the role of $\mathbb{Z} \subset \mathbb{R}$ by $\mathbb{Q} \subset \mathbf{A}$. For simplicity we suppress this in this document, but the statements in the later section should be understood with the enlarged Y_Γ. In the case when $\mathbf{G}$ is simply connected, there is no difference.

3 The defect δ of an arithmetic group

Let Γ be an arithmetic group. As we have discussed (§2) it lies as a lattice $\Gamma \leqslant G$ inside some ambient Lie group G. The behavior of its cohomology depends very strongly on

[2] If Γ is torsion-free, S/Γ has the structure of manifold; even if not, it still has the structure of an orbifold. We will, however, continue to use the world “manifold” to describe S/Γ, even though it may have finite quotient singularities.

G. For example, numerical computations suggest that $H^*(\Gamma, \mathbf{Q})$ is "typically" much larger when $G = \mathrm{SL}_2$ than when $G = \mathrm{SL}_3$. There is an important numerical invariant that accounts for this discrepancy.

Let $K \subset G$ be a maximal compact subgroup, and let

$$\delta = \mathrm{rank}(G) - \mathrm{rank}(K).$$

For example, when $G = \mathrm{SL}_n(\mathbf{R})$, we have $K = \mathrm{SO}_n$, and

$$\delta = \mathrm{rank}(\mathrm{SL}_n(\mathbf{R})) - \mathrm{rank}(\mathrm{SO}_n) = (n-1) - [n/2] = \lceil \frac{n-1}{2} \rceil,$$

so that $\delta = 0$ for SL_2 but $\delta = 1$ for SL_3. We call δ the *defect*.

Now, all Hermitian cases have $\delta = 0$. By contrast, in this paper, many of the statements and situations we consider will primarily be of interest in the case when $\delta > 0$.

The invariant δ controls much of the behavior of the cohomology (for reasons we cannot get into here). We will discuss just one example, namely, it controls growth of cohomology as Γ shrinks. There are different ways of quantifying Γ "shrinking." The weakest possible notion is asking that the volume of G/Γ get larger. For ease of statement we impose simplifying conditions (see Abert, Bergeron, Biringer, Gelander, Nikolov, Raimbault, and Samet [2017] for discussion of the general case).

We consider a sequence

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_N \supset \ldots$$

of congruence arithmetic subgroups of $\Gamma_1 = \Gamma$, such that the common intersection of all Γ_N is the trivial subgroup. We suppose also that S/Γ is compact.

Theorem 1. *(de George and Wallach, de George and Wallach [1978]; see also Lück Lück [1994]): The limit*

$$\lim_{N\to\infty} \frac{\dim H^j(\Gamma_N, \mathbf{C})}{\mathrm{vol}(S/\Gamma_N)}$$

exists. It equals zero unless $\delta = 0$ and $j = \frac{1}{2}\dim(S)$, in which case it is a nonzero quantity depending only on S (and volume normalization).

Thus $\delta = 0$ is distinguished by the rapid growth of Betti numbers. The following counterpart, suggesting that $\delta = 1$ is distinguished by rapid growth of torsion, was proposed in Bergeron and Venkatesh [2013]. This paper also contains significant evidence for the conjecture (e.g., replacing $\mathbb{Z}$ by other Γ-modules) that we don't describe.

Conjecture 1. *(Bergeron–Venkatesh, Bergeron and Venkatesh [ibid.]): The limit*

$$\lim_{N\to\infty} \frac{\log \# H^j(\Gamma_N, \mathbf{Z})_{\mathrm{tors}}}{\mathrm{vol}(S/\Gamma_N)}$$

exists. It equals zero unless $\delta = 1$ and $j = \frac{1}{2}\dim(S) + 1$, in which case it is a nonzero quantity depending only on S (and volume normalization).

The picture suggested by these statements is as follows:

- when the defect δ vanishes, lattices Γ tend to have a lot of cohomology in characteristic zero; this is concentrated in the"middle degree" $\frac{\dim(S)}{2}$.
- When the defect δ equals 1, lattices Γ tend to have a lot of torsion in their cohomology; this is concentrated in the degree immediately above middle.
- When the defect is larger than 1, both torsion and characteristic zero cohomology are scarcer.[3]

As we have mentioned, until fairly recently, there was relatively little study of cases with $\delta > 0$ (but it was certainly appreciated that there are interesting new phenomena there – see, for example, Mennicke and Grunewald [1980] and Ash and McConnell [1992]). Our emphasis in this paper is on this case of positive defect.

4 Operators

The defining feature of the cohomology of arithmetic groups is the existence of a large number of extra symmetries, i.e. extra operations

$$T : H^j(\Gamma) \longrightarrow H^j(\Gamma). \tag{2}$$

In a slightly different setting, the existence of such "Hecke" operators goes back at least to Mordell Mordell [1917].

When $\delta > 0$ there also exist operators that shift cohomological degree, i.e. maps like this:

$$H^j(\Gamma) \to H^{j+1}(\Gamma).$$

These were introduced in my recent paper Venkatesh [2016], where they are called derived Hecke operators. However, in a purely algebraic context – i.e. separate from the application to arithmetic groups – the derived Hecke algebra was studied previously by Schneider Schneider [2018].

We now describe these in a little more detail.

4.1 Hecke operators. Operations as in (2) comes from "almost automorphisms" of the group.

Namely, call an "almost automorphism" of Γ a triple $(\Gamma_1, \Gamma_2, \varphi)$ of two finite index subgroups Γ_1 and Γ_2, and an isomorphism $\varphi : \Gamma_2 \to \Gamma_1$ between them. Although not quite an automorphism of Γ itself, this data nonetheless induces an endomorphism of cohomology, as follows:

$$H^*(\Gamma) \to H^*(\Gamma_1) \xrightarrow{\varphi^*} H^*(\Gamma_2) \dashrightarrow H^*(\Gamma). \tag{3}$$

[3]These results should not be interpreted as saying that there is very little cohomology for $\delta > 1$. An heuristic proposed in Bergeron and Venkatesh [2013], phrased informally, states that if we choose Γ "at random," the chance that $H^*(\Gamma, \mathbb{Z}/p\mathbf{Z})$ is nonzero is about $p^{-\delta}$. This has to be applied with great care, but should give the rough picture: if we fix p, there is a positive, though small, probability that one sees at least a mod p cohomology class. Characteristic zero cohomology is rarer, but it is again reasonable to think if one chooses K different choices of Γ, a positive power $K^?$ will have interesting cohomology classes in $H^*(\Gamma, \mathbf{C})$.

The first two maps here come from functoriality. The last map is the transfer: a finite index inclusion of groups also induces a map on cohomology in the "wrong" direction.

Arithmetic groups, such as $\mathrm{SL}_n(\mathbf{Z})$, admit a rich supply of almost-automorphisms. Namely, if $\alpha \in \mathrm{SL}_n(\mathbf{Q})$, then conjugation by α induces

$$\Gamma_1 := \Gamma \cap \alpha\Gamma\alpha^{-1} \simeq \alpha^{-1}\Gamma\alpha \cap \Gamma =: \Gamma_2.$$

The resulting endomorphisms of homology or cohomology are called Hecke operators.

What is surprising, and not obvious, is that these operators commute, at least if we mildly restrict α. For example, if $\Gamma \leqslant \mathrm{SL}_n(\mathbf{Z})$, there exists an integer

$$N = N(\Gamma) \tag{4}$$

such that this is true if we restrict to α whose denominators are relatively prime to N. In what follows we suppose that we always restrict to Hecke operators arising from such α. We will call any choice of such an integer N the *level* of Γ; we can harmlessly replace N by any multiple of itself. A similar notion of level may be defined for any arithmetic group.

The resulting commutative algebra of Hecke operators can be used to decompose cohomology. For example, if K is a field, we may try to decompose $H^*(\Gamma, K)$ into common eigenspaces (or generalized eigenspaces) of the Hecke operators. An element of such a common eigenspace is often called a *Hecke eigenclass.* The corresponding "eigenvalue" is a homomorphism to K, from the algebra generated by all Hecke operators.

This commutative algebra generated by all Hecke operators is usually called the Hecke algebra $\mathbb{T}$ (for Γ). For simplicity we will always regard $\mathbb{T}$ as what we get by considering endomorphisms of integral cohomology; thus, $\mathbb{T}$ is a commutative subalgebra

$$\mathbb{T} \leqslant \text{endomorphisms of } H^*(\Gamma, \mathbf{Z}). \tag{5}$$

(To handle torsion and other issues it is better to make a more careful definition, see Khare and J. A. Thorne [2017, §6.5].)

A key idea is that:

> much of the information in $H^*(\Gamma)$ is actually encoded by the ring-theoretic structure of $\mathbb{T}$.

For example, rather than deal directly with the generalized eigenspace decomposition of $H^*(\Gamma, K)$, we will prefer to directly deal with homomorphisms $\pi : \mathbb{T} \to K$. It is helpful to think of such a π as indexing a Hecke eigenclass. Indeed, for such π, there exists $\alpha \in H^*(\Gamma, \mathbf{Z}) \otimes K$ which is scaled by $\mathbb{T}$ according to the character π:

$$T \cdot \alpha = \pi(T)\alpha$$

for all $T \in \mathbb{T}$.

4.2 Derived Hecke operators, when $\delta > 0$. When $\delta > 0$, one often observes in a closer study of $H^*(\Gamma, \mathbf{Q})$ that many phenomena are not confined to a single cohomological degree: if they happen in one degree, they happen in a range of several contiguous degrees. For an attempted introduction to this phenomenon, see Venkatesh [2017].

On the basis of this, it is reasonable to expect the existence of operators that can shift degree in cohomology by 1. In order to make them, one wants to cup with some classes (like the Lefschetz operators of algebraic geometry). Unfortunately, in most cases of interest, $H^1(\Gamma, \mathbf{Q}) = 0$, so one cannot imitate the idea of Lefschetz operators too directly.

The key observation is that:

> There is (for any fixed m) a large supply of subgroups $\Gamma_1 \leqslant \Gamma$ with $H^1(\Gamma, \mathbb{Z}/m\mathbf{Z})$ nonvanishing.

In fact, we can even find such a Γ_1 which arises as in (3). Given this, and a class $\beta \in H^1(\Gamma_1, \mathbb{Z}/m\mathbf{Z})$, we can "insert" cupping with β into the sequence (3), where we now take cohomology with $\mathbb{Z}/m\mathbb{Z}$ coefficients:

$$H^*(\Gamma) \longrightarrow H^*(\Gamma_1) \stackrel{\cup\beta}{\to} H^*(\Gamma_1) \xrightarrow{\varphi^*} H^*(\Gamma_2) \longrightarrow H^*(\Gamma).$$

Call this composite $T_{\varphi,\beta}$. Finally, one can find a sequence of φ, β and m, and patch the resulting operators $T_{\varphi,\beta}$ together into a characteristic zero limit. This gives rise to an algebra

$$\widetilde{\mathbb{T}} \leqslant \text{endomorphisms of } H^*(\Gamma, \mathbb{Q}_p),$$

which is called in Venkatesh [2016] the derived Hecke algebra. The real content of Venkatesh [ibid.] is providing evidence that $\widetilde{\mathbb{T}}$ is bigger than $\mathbb{T}$, i.e. really does contain interesting degree-shifting operators.

This derived Hecke algebra is a feature of life when $\delta > 0$. It explains why many phenomena occur in several cohomological degrees. However, it has the following failing, compared to the usual Hecke algebra: because it is constructed from operations that exist only for torsion coefficients, it does not produce endomorphisms of *rational* or *integral* cohomology. We return to this point in §7.

5 Local systems in topology and arithmetic

Let K be a field. Let Γ be an arithmetic group, and N the level of Γ, as in (4). In practice, the prime divisors of N are primes where something complicated happens. In this document we will ignore them, but really one needs to instead study them rather carefully to obtain a good understanding of the situation.

As we will see later, Langlands' reciprocity conjecture predicts that there is a mapping

(6)

$$\text{elements of a certain basis for } H^*(\Gamma, K) \rightsquigarrow \begin{array}{c}\dim(G)\text{-dimensional "local system"} \\ \text{of } K\text{-vector spaces on } \mathrm{Spec}\mathbb{Z}[1/N]\end{array}$$

But we need to make sense of the right hand side; note that Spec $\mathbb{Z}[1/N]$ is the prime spectrum of $\mathbb{Z}[1/N]$ – i.e., it has a point for every prime ideal of $\mathbb{Z}$, excepting divisors of N – and is thus far from a nice topological space like a manifold. It is easy to make a good definition for the right hand side in some cases – for example, for K finite. But for $K = \mathbf{Q}$, say, it is quite difficult to make a definition that is large enough to accommodate (6). So we digress to discuss this issue, which is part of the reason why the Langlands correspondence is so difficult to formulate correctly.

To try to make our discussion a bit more comprehensible we use the paradigm of "arithmetic topology," namely, the analogy

$$\text{Spec } \mathbb{Z}[1/N] \leftrightarrow \text{(open, nonorientable) 3-manifold.}$$

I do not have a strong philosophical commitment to this viewpoint, but it certainly makes exposition easier.

Accordingly, we first review local systems over a manifold, and then try to describe local systems on $\mathbb{Z}[1/N]$ in a parallel way. Whatever the exact definition, in what follows, we will use the phrase *arithmetic local system* to mean a local system of K-vector spaces on Spec $\mathbb{Z}[1/N]$.

5.1 Local systems in topology. For a manifold M, a local system of K-vector spaces M is a locally constant sheaf of K-modules on M. In plainer language: it is an assignment $m \rightsquigarrow L_m$ of a K-vector space to each point $m \in M$, together with compatible identifications $L_m \simeq L_{m'}$ for nearby points $m \approx m'$.

Assuming M connected, the local system L is entirely specified by its monodromy representation

$$\pi_1(M, m_0) \longrightarrow \mathrm{Aut}_K(L_{m_0})$$

for any basepoint $m_0 \in M$. Conversely, any such representation gives a local system on M, with fiber L_{m_0} at m_0.

Such local systems have the following important properties:

(i) A fiber bundle $E \to M$ and a choice of integer $j \geq 0$ gives rise to a local system: to $m \in M$ we assign the cohomology $H^j(E_m; K)$,

(ii) We can take cohomology with coefficients in a local system. These groups $H^*(M, L)$ are K-vector spaces. (If L arose from the construction of (i), then these K-vector spaces are related to the cohomology of E.)

5.2 Arithmetic local systems. How does this discussion adapt to the space Spec $\mathbb{Z}[1/N]$? The answer depends on the nature of the coefficient field K.

5.2.1 Finite or p-adic fields. If K is finite, there is a reasonable way of adapting this discussion to the space Spec $\mathbb{Z}[1/N]$, simply by using the étale topology. Namely, a local system of K-vector spaces should be a locally constant sheaf of K-vector spaces for the étale topology. In order that this behave well, we must always require that the characteristic of K divide N; we can arrange this by increasing N.

The situation is very similar if K is a p-adic field, i.e. a finite extension of the p-adic numbers $\mathbb{Q}_p$. Again, if we are using this notion, we will always require that p divide N.

In both cases, then, a "local system with K-coefficients" is specified by giving a homomorphism

$$\pi_1^{\text{et}}(\text{Spec } \mathbb{Z}[1/N]) \longrightarrow \text{GL}_n(K). \tag{7}$$

In these settings, one obtains analogues of (i) and (ii) just from formal properties of the étale topology.

In number theory one usually formulates things in a different (equivalent) way. Namely, the étale π_1 of $\mathbb{Z}[1/N]$ is a quotient of the absolute Galois group of $\overline{\mathbb{Q}}/\mathbf{Q}$, and thus a local system of K-vector spaces is specified by giving a representation

$$\text{Gal}(\overline{\mathbb{Q}}/\mathbf{Q}) \longrightarrow \text{GL}_n(K). \tag{8}$$

This is the perspective that is often adopted in number theory: local systems are encoded by means of such "Galois representations".

5.2.2 Number fields. What if K is $\mathbf{Q}$, or a finite extension of $\mathbf{Q}$? The reader without familiarity with the theory of motives should skip this section and just take on faith that:

- one can make a working definition of "local system with K coefficients" for K a finite extension of $\mathbf{Q}$, but
- it is much more subtle than the case of K finite, being closely related to the theory of motives and motivic cohomology.

Suppose, then, that $K = \mathbf{Q}$. Of course we can still try to use (7) or (8) as a definition, but the resulting class of objects is far too small. There are simply not enough homomorphisms from the profinite group π_1^{et} to the group $\text{GL}_n(\mathbf{Q})$ to accommodate the left-hand side of (6). The problem is that $\text{GL}_n(\mathbf{Q})$ has too few finite subgroups.

We can make a definition that captures more natural phenomena by replacing étale sheaves with their motivic analogues. Informally, this corresponds to trying to turn point (i) of §5.1 into an actual definition, declaring that all local systems arise from "factors" of this construction.

Recent advances Ayoub [2007a,b] and Cisinski and Déglise [2009] permit one to construct, at least, a derived category of motivic sheaves over $\mathbb{Z}[1/N]$, and it may well be possible to make a good working definition from this theory. Historically, however, the theory of motivic sheaves was not available, so number theory resorted to various cruder substitutes. Since my own familiarity with motivic sheaves is minimal, we will use such a crude substitute in this document, with the hope that the sophisticated reader can construct a better version.

Working definition:

- For K a number field, we declare for this document that a "local system over $\mathbb{Z}[1/N]$ with coefficients in K" means a Chow motive M over $\mathbf{Q}$ and with coefficients in K, pure of some fixed weight w, and whose associated p-adic Galois representation has good reduction outside Np for every prime p.

- For a local system arising from M as in (i), we declare the "cohomology with coefficients in the local system"

$$H^i(\mathbb{Z}[1/N], M) \tag{9}$$

  to be the $\mathbf{Q}$-vector space obtained by applying the construction of Scholl [2000], informally, "classes in the motivic cohomology $H^{i+w}(M, \mathbf{Q})$ of M that extend to a projective, flat, regular model over $\mathbb{Z}$."

At the very least, as best as we can see from automorphic forms, such a provisional definition does a pretty good job at producing a large enough class of local systems for the needs of the Langlands program (cf. conjectures in Clozel [2016]). But the given definition certainly has some rather undesirable properties – simple results like finite dimensionality of cohomology are in most cases unknown. (See e.g. Prasanna and Venkatesh [2016, §2.1.8, 2.19] for some other problems posed by this definition that are relevant to our later analysis.) For our purposes it is in any case just a working definition.

To summarize, here are our various definitions of "arithmetic local systems of K-vector spaces":

K	local system of K vector spaces on $\mathbb{Z}[1/N]$
$\mathbf{F}_p$	étale sheaf of n-dimensional $\mathbf{F}_p$-vector spaces on $\mathbb{Z}[1/N]$
$\mathbf{Q}_p$	étale sheaf of n-dimensional $\mathbb{Q}_p$-vector spaces on $\mathbb{Z}[1/N]$
$\mathbf{Q}$	n-dimensional Chow motive + extra properties
$\mathbf{C}$	I have no idea

We added $K = \mathbf{C}$ in this table to draw attention to its plight. The Langlands program (which is much broader than the cohomology of arithmetic groups) suggests that there should be a notion of arithmetic local system with $\mathbf{C}$ coefficients, substantially larger than what one gets by extending from $\mathbf{Q}$ or any number field. However, at present, there is no suggestion how to define it (cf. Arthur [2002]). This seems to me a fundamental problem of number theory.

These definitions all have the analogue of properties (i) and (ii) for manifolds; in particular, a local system ρ of K-vector spaces gives rise to a K-vector space $H^i(\mathbb{Z}[1/N], \rho)$ (which we will call $H^i(\rho)$ for short, that is, the cohomology with coefficients in the local system ρ), given by étale cohomology in the first two cases, and as described in (9) when K is a number field. Also a local system ρ of $\mathbf{Q}$-vector spaces gives, in a natural way, a local system $\rho \otimes \mathbb{Q}_p$ of $\mathbb{Q}_p$-vector spaces.

For each of the field K above there is a canonical one-dimensional local system called $K(1)$, the "Tate twist." This local system plays a distinguished role. When K is p-adic or finite, for example, $K(1)$ may be regarded (in the metaphor of arithmetic topology) as the orientation sheaf of $\mathbb{Z}[1/N]$.

5.3 Trace of monodromy. We will want to extract certain numerical invariants from local systems. For a rank m local system L of K-vector spaces on a manifold M, a map

$$\gamma : S^1 \longrightarrow M$$

defines a conjugacy class of $m \times m$ matrices, the monodromy around the loop; in particular, by taking the trace, we get a well defined invariant in K. Thus:

$$\text{trace of monodromy around } \gamma \in K.$$

There is a natural class of "loops" on Spec $\mathbb{Z}[1/N]$. In algebraic geometry, the spectrum of $\mathbf{F}_\ell$ looks like a circle; its étale fundamental group is a (pro-)cyclic group with a distinguished generator. The inclusion

$$\text{Spec } \mathbf{F}_\ell \hookrightarrow \text{Spec } \mathbb{Z}[1/N]$$

behaves similarly to the inclusion $S^1 \to M$ of a circle into a manifold. If ρ is a local system of K-vector spaces, and ℓ is a prime number not dividing N we may then similarly define

$$\text{trace of monodromy around Spec } \mathbf{F}_\ell \in K. \tag{10}$$

(Strictly, we can only do this for K finite or p-adic using the étale topology, and for K a number field we must make a choice of p and tensor with $\mathbb{Q}_p$; it is expected, although not proved in general, that the result is in K and independent of choice, see Illusie [2006]. We will suppress this issue, which causes no problem for our purposes.)

5.4 Boundary conditions. We warn that with our definitions, the association $\rho \rightsquigarrow H^i(\rho)$ does *not* commute with passings from $\mathbf{Q}$ to $\mathbb{Q}_p$. There is a map

$$H^i(\rho) \otimes \mathbb{Q}_p \longrightarrow H^i(\rho \otimes \mathbb{Q}_p), \tag{11}$$

but it need not be an isomorphism.

There are multiple reasons for this failure. One serious reason is that étale cohomology of $\mathbb{Z}[1/N]$ has a good duality theory, and motivic cohomology does not seem to.

Another reason has to do with boundary conditions. The space Spec $\mathbb{Z}[1/N]$ fails to be compact, in the arithmetic topology paradigm, because it is missing points related to the prime divisors of N, and it is also reasonable to try to further compactify it by adjoining Spec $\mathbb{R}$. Correspondingly it admits different types of cohomology according to what conditions we impose at the boundary. For example, we can consider an analog of "compactly supported cohomology," or, if we fix a subspace $L \subset H^i(\mathbb{Q}_p, \rho)$, we may consider the subspace

$$H^i_L(\rho) = \{\alpha \in H^i(\rho) : \text{image of } \alpha \text{ in } H^i(\mathbb{Q}_p, \rho) \text{ lies in } L.\},$$

or perhaps carry out a chain-level version of this condition. One can think of this as akin to considering classes in the cohomology of a 3-manifold M which lie in some specified subspace of $H^i(\partial M)$.

By imposing suitable boundary conditions on the definition of $H^i(\rho)$, for ρ a p-adic local system, we can (at least conjecturally) force (11) to be an isomorphism again for $i = 0, 1$. How to get these conditions right is a very subtle manner, addressed in generality by Bloch and Kato in their famous paper Bloch and Kato [1990].

5.5 Determinants of cohomology (topology) . (This section §5.5 is motivational. Both it and and §5.6 are only used in the final section, §9).

Local systems on a manifold can be assigned, in some cases, an interesting numerical invariant: Reidemeister torsion. We describe the algebra of it. In discussing this we will introduce some space-saving notation: if V is a vector space over the field k, we write $[V]$ for the determinant of V, i.e., the top exterior power of V. We write $[V][W]$ instead of $[V] \otimes [W]$, $\frac{[V]}{[W]}$ instead of $[V] \otimes [W]^*$, and so on. For a graded vector space V^i we define $[V^*] = \prod_i [V^i]^{(-1)^i}$. These are all k-lines, i.e. one-dimensional k vector spaces. [4]

Suppose, in what follows, that M is a compact odd-dimensional manifold. and that L is a local system of K-vector spaces on M. One can form the determinant of cohomology

$$[H^*(M, L)] = \prod_i [H^i(M, L)]^{(-1)^i},$$

which is a K-line.

Now fix a smooth triangulation T of M. This identifies the above determinant with the determinant of a certain chain complex, the complex of cochains for T valued in L:

$$[H^*(M, L)] \simeq \prod_i [C^i(L)]^{(-1)^i} \tag{12}$$

For L trivial there is a preferred basis for the right hand space, at least up to sign, coming from the basis of C^i arising from the dual basis of cells; and even for L nontrivial one can obtain a preferred basis after fixing some extra data (an "Euler structure," for this and independence of choice of triangulation see Turaev [1989]). Thus, for a smooth manifold (+ a little extra data) we obtain a preferred trivialization of the determinant of cohomology of any local system.

This algebraic data gives rise to a numerical invariant. Suppose, for example, that L is acyclic, i.e. $H^j(M, L) = 0$ for every j, then both sides of the above isomorphism (12) have distinguished bases. However, these bases do not match up under (12); they differ by an element of K^*, well defined at least up to sign. We will ignore the sign ambiguity in what follows. This element of K^* is the Reidemeister torsion:

$$L \text{ local system} \rightsquigarrow \mathrm{RT}(M, L) \in K \tag{13}$$

(let us agree this invariant is zero if L is not acyclic). If, for example, $K = \mathbf{Q}$ and L has an integral structure $L_{\mathbb{Z}}$, this invariant computes an alternating product of the sizes of $H^i(M, L_{\mathbb{Z}})$.

5.6 Determinant of cohomology (arithmetic). (This section §5.6 is only used in the final section, §9).

The theory of L-functions in number theory, in favorable circumstances, attaches an invariant valued in $(K \otimes \mathbf{C})$ to a local system of K-vector spaces ρ:

$$\rho \rightsquigarrow L(\rho) \in (K \otimes \mathbf{C}). \tag{14}$$

[4]Since we do not need to be careful about signs that occur in this theory, and so we will not attempt to think of $[V]$ as a graded k-line, just as a k-line.

(Read this only if you want details: according to our definition, a local system of K-vector spaces is a Chow motive M over $\mathbf{Q}$ with coefficients in K, and we define $L(\rho)$ so that its image under an embedding $\sigma : K \hookrightarrow \mathbb{C}$ gives the value at $s = 0$ of the L-function $L(s, M^\sigma)$, see Deligne [1979b, §2]. The phrase "in favorable circumstances" above refers to the fact that this L-function is not known in general to extend holomorphically to a region containing $s = 0$.)

Now, with respect to the analogy between Spec $\mathbb{Z}[1/N]$ and a 3-manifold, the Reidemeister torsion has strong formal analogies with L-functions in number theory. (Unfortunately, I do not know how to attribute this analogy). Besides various straightforward formal analogies, there is an interesting parallel between Fried's formula Fried [1986] and the product formula for an L-function.

By analogy with the situation with manifolds, then, one might hope that there is an extra algebraic structure at play: a trivialization of cohomology determinant lines. This viewpoint has been proposed in papers of Fontaine and Perrin-Riou [1994] and by Kato [1993]. For example Kato [ibid., §3.2] formulates a precise version of the following

> Conjecture: If ρ is a local system over Spec $\mathbb{Z}[1/N]$ with finite or profinite coefficients, then there is a distinguished trivialization of the determinant line $[H_c^*(\rho)]$ (see below for definition), characterized in terms of L-functions.

This conjecture, an algebraic incarnation of the theory of L-functions, seems to be difficult to approach directly, at least in any generality.

As we have mentioned in §5.4 one needs to be careful, when discussing cohomology of arithmetic local systems, to get the boundary conditions right. This is certainly the case in the above conjecture: it uses not usual cohomology but a modification H_c^*. Here Kato [1993] and Fontaine and Perrin-Riou [1994] use different normalizations and we have followed Fontaine and Perrin-Riou [1994]. For detailed formulations of the following discussion see Fontaine and Perrin-Riou [ibid., §4] and also the survey article Flach [2004].

- For ρ a local system of $\mathbf{F}_p$ or $\mathbb{Q}_p$-vector spaces, we define $[H_c^*(\rho)]$ to be $[H^*(\rho)]$ modified by a boundary term:

  $$[H_c^*(\rho)] = [H^*(\rho)] \otimes [\partial\rho]^{-1}$$

  where (see Fontaine and Perrin-Riou [1994, §4.1]) $[\partial\rho]$ is the K-line defined by

  $$[\partial\rho] = \prod_{v \in V} \text{determinant of étale cohomology of } \rho \text{ restricted to Spec } \mathbb{Q}_v$$

  where V is the set of primes dividing N, together with ∞; we regard $\mathbb{Q}_\infty = \mathbb{R}$.

- If ρ is a $\mathbf{Q}$-local system, there is an analogous but more *ad hoc* definition of $[H_c^*(\rho)]$.

  First of all, we define $H^j(\mathbb{Z}[1/N], \rho)$ by means of motivic cohomology as in (9) for $j \leq 1$, and extend this definition to degrees 2 and 3 by forcing it to satisfy an analogous duality to étale cohomology.

Having done this, we again define $[H_c^*(\rho)]$ to be $[H^*(\rho)]$ modified by a boundary term:

$$[H_c^*(\rho)] = [H^*(\rho)] \otimes [\partial\rho]^{-1} \tag{15}$$

where $[\partial\rho]$ is now the $\mathbf{Q}$-vector space defined by

$$[\partial\rho] = [H_B^+] \otimes [H_{\mathrm{dR}}/F^0 H_{\mathrm{dR}}]^{-1}. \tag{16}$$

Here, since ρ corresponds to a Chow motive, one can speak of its Betti cohomology H_B and its de Rham cohomology H_{dR}, both taken with $\mathbf{Q}$-coefficients; H_B^+ is the fixed part under complex conjugation. For details see Fontaine and Perrin-Riou [1994, p. 4.4.1]; this $[\partial\rho]$ is an algebraic incarnation of the Deligne period introduced in Deligne [1979b].

With the above definitions, the two definitions given above are conjecturally compatible with one another: under certain conjectures about comparison between motivic and étale cohomology, there is a canonical isomorphism

$$[H_c^*(\rho)] \simeq [H_c^*(\rho \otimes \mathbb{Q}_p)] \tag{17}$$

if ρ is an arithmetic local system with $\mathbf{Q}$ coefficients. See again Fontaine and Perrin-Riou [1994, §3.2] for discussion of the precise conjectures under which (17) is valid.

6 Langlands' vision

Having introduced a working definition of an "arithmetic local system," i.e. a local system of K-modules on $\mathbb{Z}[1/N]$, for various classes of fields K, we are in a position to formulate the special case of Langlands' reciprocity conjectures[5] that applies to the cohomology of arithmetic groups.

We formulate the conjectures in a non-traditional way. It is also imprecise, because we do not explicate the compatibility condition with Hecke operators. For a better formulation of this conjecture, in standard language, see Clozel [2016].

6.1 The reciprocity conjecture.

6.1 The reciprocity conjecture. The basic content is that *cohomology of arithmetic groups provide arithmetic local systems.*

To be a little more specific we have to use the Hecke algebra: as in §4.1, $H^*(\Gamma, \mathbf{Z})$ admits a commutative algebra $\mathbb{T}$ of endomorphisms.

Langlands reciprocity proposes that[6], attached to this situation, there should exist an arithmetic local system of locally free $\mathbb{T}$-modules on $\mathbb{Z}[1/N]$, with rank equal to

[5]The word "reciprocity" here, deriving from quadratic reciprocity, is traditional but confusing. Namely quadratic reciprocity permits us to understand how the number of solutions to $x^2 = N$ modulo p varies with the prime p. When unwound, the conjecture below yields similar (if far more complicated) statements, with $x^2 = N$ replaced by a more complicated system of equations.

[6]Note that the usual formulation of this conjecture involves the dual group to G, but we are describing a simplified version using the adjoint representation of that dual group –see §6.2 below.

$\dim(G)$, which comes along with a very explicit formula for the monodromy around each loop $\operatorname{Spec} \mathbf{F}_\ell \hookrightarrow \operatorname{Spec} \mathbb{Z}[1/N]$ – see (10).

Unfortunately, we cannot give a precise formulation of this version, because we have not defined local systems of $\mathbb{T}$-modules for rings as general as $\mathbb{T}$. So to make a more precise statement, we use homomorphisms $\pi : \mathbb{T} \to K$, where K is one of the fields discussed earlier. Also, to be careful, we must allow a finite extension of K.

Conjecture 2. *(Langlands' reciprocity, simplified version):*

Let K be a finite or p-adic or number field. Let $\mathbb{T}$ be the commutative subalgebra of endomorphisms of $H^*(\Gamma, \mathbf{Z})$ defined in §4.1.

For each character $\pi : \mathbb{T} \to K$, there exists an arithmetic local system ρ_π of dimension equal to $(\dim G)$, with coefficients in a finite extension of K, and such that

$$\textit{monodromy trace of } \rho_\pi \textit{ around } \operatorname{Spec} \mathbf{F}_\ell = \pi(T_\ell), \tag{18}$$

where $T_\ell \in \mathbb{T}$ is described by a completely explicit recipe.

This perhaps seems underwhelming or incomprehensible at first sight. The important features for us are, firstly, that there are $(\dim G)$-dimensional arithmetic local systems that can be extracted from the cohomology of Γ; and, secondly, that they control via (18) how Hecke operators act on that cohomology.

For simplicity in the rest of the text we suppose that this arithmetic local system in fact has K coefficients (as should be the case in most cases), rather than a finite extension of K. Recalling our prior discussion after (5) we can then informally think of the conjecture as yielding an association

(19)

Hecke eigenclasses in $H^*(\Gamma, K) \rightsquigarrow$ arithmetic local systems ρ of K-vector spaces.

6.2 Translation to usual terminology. As we now explain for readers familiar with the usual formulations, our "arithmetic local system" is, in effect, the adjoint Galois representation attached to π. To fix ideas take $K = \overline{\mathbb{Q}_p}$ and assume that the semisimple $\mathbf{Q}$-group $\mathbf{G}$ associated to Γ is simply connected. It has an L-group, ${}^L\widehat{G}$. Usual formulations propose that to π is attached a Galois representation

$$\rho : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbf{Q}) \longrightarrow {}^L\widehat{G}(K).$$

Write $d = \dim(\mathbf{G}) = \dim(G)$, so that one has an adjoint representation ${}^L\widehat{G} \to \mathrm{GL}_d$; composing ρ with this we get

$$\mathrm{Ad} \circ \rho : \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_d(K).$$

This is the local system proposed in Conjecture 2. Of course, we are throwing away information by only considering the adjoint of ρ, but this extra information isn't needed for our current purposes.

6.3 The role of the defect. Recall from §3 that attached to Γ there is a numerical invariant, the "defect" δ, that controls many aspects of the cohomology of Γ.

There is an enormous amount of evidence for Conjecture 2 in cases of vanishing defect. See for example Shin [2011]. But until rather recently there were very few *general statements* supporting it in the case of positive defect – although progress in the broader Langlands program nonetheless provided a great deal of evidence for it. The difficulty lies in the fact that the space Y_Γ never has the structure of complex algebraic variety when the defect is positive.

Recent work has improved the situation, particularly with torsion coefficients. Works of Harris, Lan, Taylor, and J. Thorne [2016] and Scholze [2015] have provided many examples where one can construct the association $\pi \rightsquigarrow \rho_\pi$ for K a finite or p-adic field (under some technical conditions, which we omit). The importance of this work demands more than a paragraph, but for lack of space I cannot go further into it, beyond saying that:

- the techniques involves *reducing* to a case with vanishing defect. Indeed, some of the deepest parts of Harris, Lan, Taylor, and J. Thorne [2016] and Scholze [2015] are new theorems in the case $\delta = 0$.
- the techniques do not seem, in their current form, to really give anything when K is a number field. They are essentially based on the extra flexibility of torsion coefficients.

There is a more fundamental reason, beyond technique alone, that cases with $\delta > 0$ are more complicated. The deformation theory of the arithmetic local systems that appear in Langlands' conjecture are controlled by δ. (Here, to make sense of deformation theory, we must work with finite or profinite coefficients, using the étale topology.) But when $\delta > 0$, the deformation theory is always obstructed, and to fully capture it, we need to consider deformations over differential graded rings or a similar concept.

For this reason, when we study cases of positive defect, one needs some form of derived geometry to accurately describe the situation. Because of the technical nature of this we will not discuss it here. See Galatius and Venkatesh [2018], which is partly a reformulation in more intrinsic terms of ideas in the fundamental paper Calegari and Geraghty [2018a] of Calegari and Geraghty.

7 Rational structures and motivic cohomology

Our task in this and the remaining sections is to examine how various features (on one or the other side) interact with Langlands' reciprocity, (19):

Hecke eigenclasses in $H^*(\Gamma, K) \rightsquigarrow$ arithmetic local systems ρ of K-vector spaces.

We have already described, in §4.2, the existence of operators on $H^*(\Gamma, K)$ that shift cohomological degree. However, these operators exist only with torsion coefficients, or (by passing to a limit) p-adic coefficients. The construction does not work with rational coefficients. It is thus reasonable to ask to:

> Construct a family of endomorphisms of $H^*(\Gamma, \mathbf{Q})$ which, upon extension of scalars to $\mathbb{Q}_p$, recovers the derived Hecke algebra of §4.2.

The answer to this seems to be quite subtle, and at present exists only at a conjectural level. It involves cohomology with coefficients in the arithmetic local system ρ; that is why we could not formulate it without first formulating the reciprocity conjecture.

7.1 Rough version. We begin with an informal version before we get to details. Recall the first, informal formulation of Langlands reciprocity from §6.1: there is an arithmetic local system of locally free $\mathbb{T}$-modules associated to $H^*(\Gamma, \mathbf{Q})$. The cohomology of $\mathbb{Z}[1/N]$ with coefficients in that local system gives a $\mathbb{T}$-module Λ. Then

> Rough conjecture: The $\mathbb{T}$-dual of Λ, i.e. $\mathrm{Hom}_{\mathbb{T}}(\Lambda, \mathbb{T})$, acts on $H^*(\Gamma, \mathbf{Q})$, compatibly with $\mathbb{T}$, and increasing cohomological degree by 1.

As before we do not really have the language to formulate this properly, so we localize everything by means of a character $\pi : \mathbb{T} \to \mathbf{Q}$. In order to make statements that are reasonably precise we shall need to impose some genericity conditions on π, and we turn to this in §7.2. The reader can probably skip this and refer back to it as needed.

7.2 Standing assumptions for the remainder of this document. In §7, 8, 9, we will work with a Hecke eigenclass

$$\alpha \in H^*(\Gamma, \mathbf{Q}), \text{ with eigenvalue given by } \pi : \mathbb{T} \to \mathbb{Q}.$$

As mentioned earlier, it is most convenient to express everything in terms of π, and we will do so, but the reader can think of everything as being indexed by α if they prefer. (We may also replace $\mathbf{Q}$ by a finite extension of $\mathbf{Q}$, but then some of the later statements must be altered.)

We impose some genericity conditions on π, referring to Prasanna and Venkatesh [2016] for details. Namely, as in Prasanna and Venkatesh [ibid., §1.1] we assume that π is *cuspidal* and *tempered*. What we mean by this is that every automorphic representation for $\mathbf{G}$ that has Hecke eigenvalues given by π is cuspidal and tempered.[7]

We will also suppose that Y_Γ is a genuine manifold, as opposed to an orbifold, to avoid some mild issues with torsion.

Let $H^*(\Gamma, \mathbf{Q})_\pi$ be the π-eigenspace on cohomology: those classes which transform, under the Hecke algebra, according to π:

$$H^*(\Gamma, \mathbf{Q})_\pi = \{\alpha' \in H^*(\Gamma, \mathbf{Q}) : T\alpha' = \pi(T)\alpha' \text{ for all } T \in \mathbb{T}.\}. \tag{20}$$

We can make the same definition with coefficients in $\mathbb{R}$ or $\mathbb{C}$. Our assumptions on π imply that this is the same as the generalized π eigenspace, and in particular is a direct summand of cohomology.

Hodge theory – that is to say, representing cohomology classes for Y_Γ by means of harmonic differential forms on Y_Γ – allows one to compute various features of this. For example, we know the range of dimensions in which $H^*(\Gamma, \mathbf{Q})_\pi$ is supported: it turns out that

$$H^j(\Gamma, \mathbf{Q})_\pi \neq 0 \iff j \in [q, \dim(Y_\Gamma) - q], \tag{21}$$

[7] This setting is really the "generic" one. It is still desirable to generalize beyond it; then one must, at least, take account of the Arthur SL_2 parameter.

where q is chosen so that $\dim(Y_\Gamma) - 2q = \delta$, with δ the defect of §3. One can also compute the dimensions of these groups: they turn out to be a constant multiple of $\binom{\delta}{(j-q)}$.

We now supply a more precise version of the discussion of §7.1, by specializing to the situation described in §7.2. Associated to $\pi : \mathbb{T} \to \mathbf{Q}$ there is, as in §6.1, a $\dim(G)$-dimensional arithmetic local system ρ_π with $\mathbf{Q}$ coefficients. We may twist ρ_π by $\mathbb{Q}(1)$ (see page 278) and then, as for any local system, we can form the cohomology with coefficients in it:

$$\Lambda := H^1 \text{ with coefficients in } \rho_\pi \otimes \mathbb{Q}(1)$$

which is a $\mathbf{Q}$-vector space. Conjecturally (see Prasanna and Venkatesh [2016] for discussion of which conjectures are involved!) $\dim_{\mathbb{Q}} \Lambda = \delta$.

The key conjecture is that the dual vector space Λ^* acts on $H^*(\Gamma, \mathbf{Q})_\pi$, shifting degrees by 1, and when we extend scalars to $\mathbb{Q}_p$ this recovers the derived Hecke action:

Conjecture 3. *(An action of a motivic cohomology group on the cohomology of an arithmetic group:) There is a (graded, free) action of $\wedge^* \Lambda^*$ on $H^*(\Gamma, \mathbf{Q})_\pi$, specified by either of the following properties:*

- *the corresponding action of $\wedge^*(\Lambda \otimes \mathbf{C})^*$ is as described in Prasanna and Venkatesh [ibid.] (explicitly described in terms of operations on differential forms).*
- *the corresponding action of $\wedge^*(\Lambda \otimes \mathbb{Q}_p)^*$ is as described in Venkatesh [2016] (explicitly described in terms of the derived Hecke algebra).*

This conjecture makes predictions that are numerically testable, and these have been tested in different ways in Harris and Venkatesh [2018], Darmon, Harris, Rotger, and Venkatesh [n.d.], and Prasanna and Venkatesh [2016].

The conjecture says that there are are extra, degree-shifting, operations on $H^*(\Gamma, \mathbf{Q})$, indexed by the dual of $H^1(\rho_\pi \otimes \mathbb{Q}(1))$. Now, according to our definitions, $H^1(\rho_\pi \otimes \mathbb{Q}(1))$ is really a certain motivic cohomology group; our statement above says that there are *extra operations on $H^*(\Gamma, \mathbf{Q})$ indexed by classes in the motivic cohomology of certain algebraic varieties.*

At present I don't know of any framework, even heuristic, into which this phenomenon should fit. One hopes that there is a way to formulate Langlands' reciprocity conjecture in such a way that this should "obviously" be true, but I don't know how to do this at the moment.

8 Heights on cohomology and heights of motives

This section is speculative. As with §7, our goal is to match up various invariants on either side of the correspondence (see §6):

$$\text{Hecke eigenclasses in } H^*(\Gamma, K) \rightsquigarrow \text{arithmetic local systems } \rho \text{ of } K\text{-vector spaces.}$$

In this section, we explain how there are natural notions of "complexity" on either side, which we will call "height" in line with arithmetic tradition, and propose that they should match up. Although we use the same word in both cases, the nature of the definitions is quite different on either side.

Up to the exact formulation, our proposal is well-known in the case of the classical modular curve. The main point is to suggest that it *always* works. This becomes much more surprising in cases when $\delta > 0$.

As in §7.2, starting with an eigenvector for Hecke operators on $H^*(\Gamma, \mathbf{Q})$ we get a character $\pi : \mathbb{T} \to \mathbf{Q}$ of the Hecke algebra. We impose the same assumptions on this π as in §7.2. We want to assign to π two notions of height:

- An automorphic height
$$\text{height}(\pi) \in \mathbb{R}_{>0},$$
which measures, roughly speaking, how complicated it is to exhibit cycles on Y_Γ representing elements of $H^*(\Gamma, \mathbf{Q})_\pi \simeq H^*(Y_\Gamma, \mathbf{Q})_\pi$.

- An arithmetic height
$$\text{height}(\rho_\pi) \in \mathbb{R}_{>0}$$
for the associated arithmetic local system ρ_π.

  Recalling that ρ arises, in effect, from a Chow motive. Although we did not define this concept, a motive is an object constructed out of a $\mathbf{Q}$-variety V. Then height(ρ) is intended to give a measure of how large the coefficients of a defining equation for V must be. Of course, a motive is not the same as a variety, but fortunately Kato [2018] has proposed a way to assign a height to a motive, rather than just to a variety, and we will make use of his idea.

After introducing these two different notions of height, we discuss the conjectured relationship between them in §8.4.

8.1 The height of π: motivation. We will give some motivation for the definition that follows in §8.2.

In number theory, if X is a (preferably projective) variety over $\mathbb{Z}$ and x an integral point of X, the "height of x" measures the size of the coordinates of x. Of course, this requires some choices to make sense of "size of coordinates." A nice way to do it is to pick

(i) a line bundle $\mathfrak{L}$ on X

(ii) for each complex point x a norm $|\cdot|_x$ on the complex line $\mathfrak{L}_x$, varying smoothly.[8]

Given these data, we get a height function:

$$\text{height} : \text{integral points of } X \longrightarrow \mathbf{R}_{>0}$$

as follows: pulling back $\mathfrak{L}$ to an integral point gives a free, rank one, $\mathbb{Z}$-module; choose a generator s and compute $|s|^{-1}$. (This definition is hard to wrap your mind around

[8]For us, a norm on the one-dimensional complex vector space W is just a function $W \to \mathbf{R}_{\geq 0}$, not identically zero, with $|zw| = |z||w|$ $(z \in \mathbf{C}, w \in W)$.

when you first see it, but it is short and has good formal properties. It can be motivated by analogy with computing the degree of a line bundle on a curve.)

Now, because the Hecke algebra $\mathbb{T}$ is finite over $\mathbb{Z}$, our homomorphism π actually has image in the integers,

$$\pi : \mathbb{T} \longrightarrow \mathbb{Z},$$

so we can think of π as an integral point of Spec $\mathbb{T}$. This scheme has only finitely many complex points, so it is not geometrically very interesting, but as a scheme over $\mathbb{Z}$ it can be quite complicated. Anyway, according to our discussion above, to define a "height" in this setting, we should specify a line bundle over Spec $\mathbb{T}$, and a hermitian metric on this line bundle.

Now $\mathbb{T}$ does come with a certain class of modules, namely, the cohomology $H^*(\Gamma, \mathbf{Z})$. Let us suppose that, for some r, $H^r(\Gamma, \mathbf{Z})$ is actually *free* over $\mathbb{T}$; thus it gives rise to a vector bundle over Spec $\mathbb{T}$. We can always make a line bundle out of a vector bundle (take its determinant). Moreover, identifying cohomology classes for Γ with harmonic differential forms on Y_Γ gives rise to a norm on this line bundle. This extra data gives rise to a height function on Spec $\mathbb{T}$, as desired.

In practice, it is hard to check the freeness result mentioned above, but one can just write down a formula inspired by this reasoning, and it makes sense even without freeness. This is what we do next.

8.2 Height of π: definition. As in §7.2 let q be the minimum degree for which $H^q(\Gamma, \mathbf{Q})_\pi \neq 0$. The identification

(22) $\quad H^q(\Gamma, \mathbf{R})_\pi \simeq$ harmonic q-forms on Y_Γ, transforming according to π

together with the Riemannian structure on Y_Γ, endows $H^q(\Gamma, \mathbf{R})_\pi$ with the structure of a finite-dimensional inner product space (since we can take inner product of two harmonic forms, using the Riemannian metric).

Put

$$H^q(\Gamma, \mathbf{Z})_\pi := \text{integral classes inside } H^q(\Gamma, \mathbf{R})_\pi.$$

i.e. the intersection of $H^q(\Gamma, \mathbf{R})_\pi$ with the image of $H^q(\Gamma, \mathbf{Z}) \to H^q(\Gamma, \mathbf{R})$, or, said differently, classes in $H^q(\Gamma, \mathbf{R})_\pi$ all of whose periods over cycles are integral. Now define height(π) in such a way that

$$\begin{aligned} \text{height}(\pi)^{h_q} &= \text{volume}\left(H^q(Y, \mathbf{Z})_\pi\right)^2, \\ &= (\det\langle e_i, e_j\rangle) \end{aligned}$$

where $e_1, \ldots, e_{h_q}$ is a basis for the free $\mathbb{Z}$ module $H^q(\Gamma, \mathbf{Z})_\pi$. For example:

- When $h_q = 1$, we can choose a harmonic differential q-form ω generating the one-dimensional real vector space $H^q(\Gamma, \mathbf{R})_\pi$. It has a lattice of periods: the collection of integrals of ω over q-cycles is of the form $\mathbb{Z}.m$ for some $m \in \mathbb{R}$. Scale ω so that $m = \pm 1$. Then the height of π is given by $\langle \omega, \omega \rangle$.

- In Bergeron, Şengün, and Venkatesh [2016] it is shown that, when Y_Γ is a 3-manifold, height(π) is related to the minimal Thurston norm of an element of $H_2(Y_\Gamma, \mathbb{Q})$ that transforms according to π. In that case, then, this can truly be seen as a measure of the minimum geometric complexity of a cycle whose eigenvalue is given by π.

8.3 Kato's height. We are now going to define the arithmetic height of an arithmetic local system ρ of $\mathbf{Q}$-vector spaces, or, rather, we are going to summarize Kato's definition. See Kato [2018, 2017] and Koshikawa [2015]. Any errors here are due to to me, of course.

Stepping back, one would like to define the *height* of an algebraic variety V over $\mathbf{Q}$ as a measure of the amount of information needed to describe V. One way of doing this is to measure the largest coefficient in a system of defining equations. Since V can be presented in many ways, one takes the minimum over all such presentations. This is however a rather inconvenient definition. Faltings found (see Faltings [1983]) a much more elegant definition for abelian varieties, and Kato found a generalization to all motives. We briefly sketch some of the key ideas:

Think of V, for a moment, as a point on some moduli space $\mathcal{V}$ of varieties, which we suppose is a projective $\mathbf{Q}$-variety. According to the discussion of §8.1, to attach a height to V, we need a line bundle $\mathcal{L}$ on this moduli space, normed at each complex point, and an extension of $(\mathcal{V}, \mathcal{L})$ from $\mathbf{Q}$ to $\mathbb{Z}$. Faltings observed that, in the case of abelian varieties of dimension g,

- we can take $\mathcal{L}$ to be the line bundle that attaches to an abelian variety A the line $H^0(A, \Omega^g)$ of algebraic volume forms, the squared norm being given by integrating $\Omega \wedge \overline{\Omega}$ over $A(\mathbf{C})$, for $\Omega \in H^0(A, \Omega^g)$, and
- One can get by without extending $(\mathcal{V}, \mathcal{L})$ to $\mathbb{Z}$. Rather, it is enough to specify an integral structure on $H^0(A, \Omega^g)$ for each abelian variety over $\mathbf{Q}$, and this can be done using the miracle of the Néron model, which provides for each such abelian variety a "best" model over $\mathbb{Z}$.

Said slightly differently, up to constants,

$$\text{Faltings height of } A = \left|\langle \Omega, \overline{\Omega} \rangle\right|^{-1/2} \tag{23}$$

where $\Omega \in H^0(A, \Omega^g) \subset H^g(A, \mathbf{C})$ is normalized up to sign using the Néron model, and the pairing is the duality pairing on $H^g(A, \mathbf{C})$.

In this way, the resulting definition makes no mention of the moduli space $\mathcal{V}$; admittedly, to prove that the resulting height has reasonable properties requires some analysis on $\mathcal{V}$ (see Faltings [ibid.]).

Kato adapts this definition to a general motive; for simplicity let us talk only about general (projective, smooth) $\mathbf{Q}$-varieties V. In the discussion above, we replace $H^0(A, \Omega^g)$ by Griffiths' "canonical line" constructed out of various pieces of the Hodge filtration on $H^*(V)$ (cf. Griffiths [1970, (7.13)]), and one uses elementary properties of Hodge structures to make a norm on it. Now there is no "best" model for V over $\mathbb{Z}$;

but by using integral p-adic Hodge theory one can still make a good candidate for an integral structure on the Hodge filtration, and this method is so flexible that it works for motives too.

Having given this rough motivation, we now explain the version of it that is relevant to our situation.

Recall that the "arithmetic local system" ρ_π is, by definition, a Chow motive M over $\mathbf{Q}$ (with some extra properties). In fact, although we did not specify it, this Chow motive is actually self-dual, and we suppose it comes equipped with a preferred pairing (for example, associated to the Killing form, see Prasanna and Venkatesh [2016, §4] for discussion). It has a de Rham realization $H_{\mathrm{dR}}(M)$, which is a filtered $\mathbf{Q}$-vector space by means of the Hodge filtration, and inherits a $\mathbf{Q}$-valued pairing. Let $F^1 H_{\mathrm{dR}}(M)$ be the degree 1 step of this filtration. Then $F^1 H_{\mathrm{dR}}(M)$ is equipped with a Hermitian form, arising from pairing ω and $\overline{\omega}$. Correspondingly, the determinant of $F^1 H_{\mathrm{dR}}(M)$ also inherits a norm, given by

$$\|\omega_1 \wedge \cdots \wedge \omega_k\|^2 = \left|\det\langle \omega_i, \overline{\omega_j}\rangle\right|.$$

where $\langle -, - \rangle$ denotes the duality on $H_{\mathrm{dR}}(M)$. (This is related to Kato's $H_{\clubsuit}$, in the case when M is of the form $N \otimes N^*$.)

Now suppose that the $\mathbf{Q}$-vector space $H_{\mathrm{dR}}(M)$ comes with an integral lattice. Kato and Koshikawa describe a specific way to produce such a lattice, which we will mention below; call a lattice that arises by their construction *admissible.* Writing $\omega_1, \ldots, \omega_k$ for a basis for the integral lattice induced on $F^1 H_{\mathrm{dR}}(M)$, we may now define

$$\begin{aligned} \text{arithmetic height of } \pi &= \|\omega_1 \wedge \cdots \wedge \omega_k\|^{-1} \\ &= \left|\det\langle \omega_i, \overline{\omega_j}\rangle\right|^{-1/2.} \end{aligned}$$

compare with (23)!

This is not a real definition, because it depends on the choice of an admissible integral lattice. However, we will permit ourselves to use it with this ambiguity, bearing in mind:

(i) Kato and Koshikawa describe (in slightly different ways) how to construct the integral lattice in $H_{\mathrm{dR}}(M)$ starting from a Galois-invariant lattice in the Betti cohomology $H_B(M)$ and using p-adic Hodge theory. An integral lattice in $H_{\mathrm{dR}}(M)$ thus constructed will be called *admissible*.

While there can be infinitely many possible Galois-stable lattices in $H_B(M)$, and correspondingly infinitely many possible admissible integral lattices in $H_{\mathrm{dR}}(M)$, the height does not change too much. In particular as one changes the integral lattice, only finitely many possibilities for the height occur, at least in Koshikawa's definition: Koshikawa [2015, Theorem 9.8].

(ii) For many primes p, in our situation, there is a unique admissible lattice at p (at least up to rescaling, which doesn't affect height). Namely, if π is not Eisenstein at p in the rather strong sense that the p-adic local system $\rho_\pi \otimes \mathbb{Q}_p$ remains irreducible modulo p, then there will be a unique admissible lattice at p, up to rescaling.

8.4 Automorphic versus arithmetic height. I would (tentatively) conjecture that there is an identity

$$\frac{\text{automorphic height}}{\text{arithmetic height}} \overset{?}{=} \text{fudge factor} \cdot L(\rho_\pi \otimes \mathbb{Q}(1)). \tag{24}$$

valid for all Γ, π.

In more conventional notation, $L(\rho_\pi \otimes \mathbb{Q}(1))$ is the value of the adjoint L-function at the edge of its critical strip. From the analytic point of view L-functions at the edge of their critical strip behave in a very mild fashion – not too big, not too small – and so we should think of (24) as saying that the sizes of automorphic and arithmetic heights are comparable.

Recall that, in our current state of affairs, the arithmetic height requires a choice of admissible lattice – see discussion in §8.3 – so we interpret (24) as asserting that there is a choice of admissible lattice for which the above equality is valid.

I apologize for the existence of the "fudge factor" – I have not yet worked out enough examples in the detail needed to make a precise formulation, and without some limitations on this fudge factor, the equation (24) is contentless. But it seems reasonable to expect that this factor is a product of purely local contributions:

$$c_\infty \prod_{p|N} c_p \prod_{p \text{ Eis}} c_p :$$

- c_∞ is a constant depending only on the ambient group G and the choice of invariant metric on S;
- c_p depends only on the local behavior of Γ and π at primes p dividing the level of Γ.
- The final factor tracks contributions arising from "Eisenstein" primes p. These are primes at which there are multiple choices for admissible lattice up to scaling. Optimistically, this term could be eliminated by making the correct choice of admissible lattice.

8.5 Discussion. Despite the common use of the word height, such a relationship

$$\text{heights of automorphic forms } \pi \longleftrightarrow \text{height of local systems (or motives) } \rho_\pi$$

seems to me surprising. To see why, let us interpret both sides as the height of an integral point on a variety, in the sense of §8.1.

- π defines a $\mathbb{Z}$-point of the spectrum of $\mathbb{T}$, and the left hand side measures the height of this point.
- If ρ_π arises from an algebraic variety V over $\mathbf{Q}$, we should think of the right-hand side as measuring the height of V considered as a point of some moduli space $\mathfrak{V}$ of algebraic varieties.

Thus, both sides have to do with heights on some algebraic varieties – but the two varieties have nothing to do with one another! $\mathcal{V}$ is often some positive-dimensional moduli space; Spec $\mathbb{T}$ is (over $\mathbb{C}$) zero-dimensional. Moreover, Spec $\mathbb{T}$ is morally speaking determined by just the "topological" structure of Spec $\mathbb{Z}$, whereas $\mathcal{V}$ really uses its algebraic structure.

Also, different aspects are clear on the two sides. On the automorphic side, it is easy – given Γ – to give a (terrible) *a priori* upper bound on the height of π, whereas on the arithmetic side this is a very subtle question related to effective bounds for Diophantine equations. In the reverse direction, if we start from a given variety V (explicitly presented in terms of some polynomial equations) it is clear that the height is bounded by a polynomial in the coefficients of those equations, but that bound is not at all clear on the automorphic side! See Frey [1989] and Murty and Pasten [2013] for examples where this was exploited.

A few words about the origin of the conjecture: The motivation to study height(π) arose from studying the growth of torsion cohomology, i.e. Conjecture 1. It turns out that Conjecture 1 is intimately connected to getting good upper bounds on height(π), see my paper Bergeron, Şengün, and Venkatesh [2016] with N. Bergeron and H. Sengun. This paper along with Calegari and Venkatesh [2012] gave some evidence that the conjecture might be true for hyperbolic 3-manifolds. That it might be true more generally was suggested by a striking coincidence between some of the expressions that appear in my work Prasanna and Venkatesh [2016] with Kartik Prasanna, and some of the formulas in Kato's paper Kato [1993].

9 Special cycles and L-functions

This section is speculative. We continue to examine how various features that appear on one side of

$$\text{Hecke eigenclasses in } H^*(\Gamma, K) \rightsquigarrow \text{arithmetic local systems } \rho \text{ of } K\text{-vector spaces.}$$

(see §6) manifest themselves on the other side.

Recall that we have discussed, in §5.6, how – by analogy with the case of manifolds – we might like to produce trivializations of cohomology determinants for arithmetic local systems. In this section we explain how we can use special cycles on Y_Γ to produce such trivializations – at least for certain very special *virtual* arithmetic local systems.

The ideas here are in some sense a reworking of computations contained in my joint work Prasanna and Venkatesh [2016] with Kartik Prasanna. The main point of carrying out this reworking is that everything is set up in a way that it could also work with torsion coefficients (although we do not examine this here).

9.1 Cohomology determinants. We continue with the setting of §7.2, and the discussion following it.

Recall that we have defined a **Q**-vector space Λ there:

$$\Lambda = H^1(\rho_\pi \otimes \mathbb{Q}(1)).$$

Let q and q^* be (respectively) the minimum and maximum dimensions where $H^*(\Gamma, \mathbf{Q})_\pi$ is nonzero. Conjecture 3 implies that the highest exterior power $[\Lambda^*]$ of the $\mathbf{Q}$-dual Λ^* indexes an isomorphism between them. In symbols, $H^q(\Gamma, \mathbf{Q})_\pi \otimes [\Lambda^*] \cong H^{q^*}(\Gamma, \mathbf{Q})_\pi$ and now, taking determinants, we get

$$[H^q(\Gamma, \mathbf{Q})_\pi] \cdot [\Lambda^*]^{h_q} \simeq [H^{q^*}(\Gamma, \mathbf{Q})_\pi],$$

where $h^q = \dim H^q(\Gamma, \mathbf{Q})_\pi$. Taking account of Poincaré duality (our assumptions in §7.2 mean that the π-eigenspace is self-dual) we arrive at an isomorphism:

$$[\Lambda]^{h_q} \simeq [H^q(\Gamma, \mathbf{Q})_\pi]^2. \tag{25}$$

In the situations we are in, the cohomology with coefficients in the arithmetic local system $\rho_\pi \otimes \mathbb{Q}(1)$ vanishes in degrees other than 1. Therefore $[\Lambda]$ is identified with $[H^*(\rho_\pi \otimes \mathbb{Q}(1))]^{-1}$ (the negation because the contribution comes from cohomological degree 1). Thus (25) can be reformulated as an identification

$$[H^*(\rho_\pi \otimes \mathbb{Q}(1))]^{-h_q} \simeq [H^q(\Gamma, \mathbf{Q})_\pi]^2. \tag{26}$$

To emphasize, (26) says:

> the determinant of $H^q(\Gamma, \mathbf{Q})_\pi$ can be (up to a certain power) canonically identified with the determinant of cohomology of an arithmetic local system.

This conclusion is just a restatement of a small piece of Conjecture 3, and is pretty much useless taken in isolation (although it is very much related to (24), relating arithmetic and automorphic heights).

The next step is to use certain distinguished classes in $H^q(\Gamma, \mathbf{Q})$ to produce classes in the cohomology determinant which are meaningful from the point of view of L-functions (see §5.6).

We can only do this in some very special cases. The details are a real mess, so we give only a brief sketch below.

9.2 Special cycles. The spaces Y_Γ often admit natural submanifolds: an inclusion of arithmetic groups, such as $\mathrm{SL}_{n-1}(\mathbf{Z}) \to \mathrm{SL}_n(\mathbf{Z})$, induces an inclusion $Z \hookrightarrow Y$ of associated arithmetic manifolds. These "special cycles" play an important role in the theory of automorphic forms and are related to L-functions in a way sketched below.

In Prasanna and Venkatesh [ibid., §6.2] we examine four cases that fit into this paradigm. Assume for simplicity that we are in a case with $h_q = 1$ (in the other cases the discussion is slightly more complicated, see Prasanna and Venkatesh [ibid.]). In the cases examined in Prasanna and Venkatesh [ibid.], Y_Γ admits a sub-arithmetic manifold Z of dimension q (as in (21)), and one can either prove, or expects Gan, Gross, and Prasad [2012] the validity of, a result of the following nature:

$$\langle [Z]_\pi, [Z]_\pi \rangle \sim \frac{L(\rho')}{L(\rho_\pi \otimes \mathbb{Q}(1))}. \tag{27}$$

Here:

- $[Z]_\pi$ is the projection of the fundamental class of Z to $H^*(Y, \mathbf{Q})_\pi$;
- $\langle -, - \rangle$ denotes the inner product on cohomology induced by an identification with harmonic forms, as in (22);
- ρ' is another arithmetic local system, functorially associated with ρ_π (in terms of the discussion of §6.2, it is obtained from yet another representation of ${}^L\hat{G}$).
- L denotes L-function, as in (14).

The notation $\sim$ means that the equality holds up to certain factors of π and algebraic numbers, at least for suitable choice of metric.

We will now indicate how to use the cycle Z to construct a

$$\text{distinguished basepoint for the } \mathbf{Q}\text{-line } \frac{\det H_c^*(\rho')}{\det H_c^*(\rho_\pi \otimes \mathbb{Q}(1))},$$

in the case when $L(\rho') \neq 0$. This might be seen as an algebraic incarnation of formula (27). As we have mentioned, the work Fontaine and Perrin-Riou [1994] and Kato [1993] proposes that there should be such a basepoint for $[H_c^*(\rho')]$ and $[H_c^*(\rho_\pi \otimes \mathbb{Q}(1))]$ individually, and presumably the basepoint we construct should be closely related to the ratio of these basepoints.

Recall that $[H_c^*]$ is not the determinant of cohomology, but rather a modification of it (see §5.6 especially (15)) by a "boundary" term that we have called $[\partial\rho]$. The key point is that there is an identification

$$[\partial(\rho_\pi \otimes \mathbb{Q}(1))] \sim [\partial\rho'] \tag{28}$$

(the $\sim$ means that there is an identification up to twisting by factors related to cyclotomic motives, which we will ignore here). The equality (28) is a puzzling piece of numerology, which is verified implicitly by the computations in Prasanna and Venkatesh [2016].

Accordingly,

$$\frac{[H_c^*(\rho')]}{[H_c^*(\rho_\pi \otimes \mathbb{Q}(1)]} = \frac{[H^*(\rho')]}{[H^*(\rho_\pi \otimes \mathbb{Q}(1))]} \left(\frac{[\partial\rho']}{[\partial(\rho_\pi \otimes \mathbb{Q}(1))]} \right)^{-1}.$$

where $=$ really means there is a canonical identification. But we can now perceive a distinguished element on the right-hand side, up to sign and possibly after extending scalar from $\mathbf{Q}$ to a quadratic extension:

- Using the identification (26), in the case $h_q = 1$, the fundamental class $[Z]_\pi$ give rise to an element $z_\pi \in [H^*(\rho_\pi \otimes \mathbb{Q}(1))]^{-2}$.

  We may take the square root of this element to get a class (defined up to sign) in $[H^*(\rho_\pi \otimes \mathbb{Q}(1))] \otimes_{\mathbb{Q}} E$ for some quadratic extension $E/\mathbf{Q}$, well defined up to sign. (Hopefully we can take $E = \mathbf{Q}$).
- The second fraction on the right is trivialized using (28).
- Conjecturally, the fact that $Z_\pi \neq 0$ implies that $H^*(\rho')$ is vanishing. Thus there is an identification $[H^*(\rho')] \simeq \mathbf{Q}$.

In this way (up to possibly extending scalars to a quadratic field) we have constructed[9] a class in the left-hand side, i.e. a class in

> determinant of cohomology $[H_c^*]$ for the *virtual* representation $\rho' - \rho_\pi \otimes \mathbb{Q}(1)$.

I believe this construction is interesting since it gives some (slightly) more explicit understanding, in some special cases, of the trivializations of cohomology determinants predicted by Fontaine and Perrin-Riou [1994] and Kato [1993]. To emphasize, the key non-formal point in the construction is that the "boundary term" from (16) *vanishes* for this virtual representation – at least up to simple factors related to Tate twists.

To me, the most interesting direction is this: in the whole discussion above we could replace $\mathbf{Q}$ coefficients by $\mathbf{F}_q$ coefficients. The case of $\mathbf{F}_q$ coefficients is particularly interesting because it cannot be accessed through the theory of L-functions.

10 Conclusion

We have been discussing

$$\text{Hecke eigenclasses in } H^*(\Gamma, K) \rightsquigarrow \text{arithmetic local systems } \rho \text{ of } K\text{-vector spaces.}$$

It important to note that the arrow is not surjective. When $K = \mathbf{Q}$, say, one sees only a small fraction of all the possible arithmetic local systems this way. Said more plainly, only very particular algebraic varieties are related to the cohomology of arithmetic groups.

As a result, while the conjectures that we have presented are important in the context of the above correspondence, they are of relatively small significance in the greater scope of arithmetic algebraic geometry. For example, nothing in this document has any relevance whatsoever to a smooth projective curve C over $\mathbf{Q}$ of genus 3.

The Langlands program, however, is larger than the study of $H^*(\Gamma, K)$ and it does not forget about C. It relates C to an eigenfunction of the Hecke operators, but no longer in cohomology – rather, to an eigenfunction inside the space of all L^2-functions on a suitable Y_Γ. This eigenfunction and its ambient L^2-space do not possess the rich algebraic structures of cohomology.

Although it seems far-fetched at the moment, any extension of our discussion to this more transcendental setting would be of great importance.

In the unlikely event of a reader having gotten this far, I thank them for their patience!

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[9]This construction as stated is conditional on various conjectures, but it should be possible to make unconditional shadows of it.

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