Chen-Wei (Milton) Lin

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Research Interests

Mathematics: Relative Geometric Langlands, Representation Theory of Metaplectic Groups, p-Adic Geometry, Higher Category Theory, K-theory.

Education

Ph.D. of Mathematics, Johns Hopkins University, 2019-2025 (Expected)
Supervisor: David Gepner
Thesis title: Geometric and Categorical Aspects of the Langlands Program.

Masters of Mathematics, University of Oxford, 2018-2019
Dissertation Topic: Index of Operators and KK-theory. Supervisor: Dr. Andre Henriques.
Fourth year examinations, ranked 4th in cohort
Best dissertation award in Mathematics Department

BA Mathematics, University of Oxford, 2015-2018
Supervisors: Prof. Glenys Luke, Prof. Tom Sanders.
Preliminary Examinations, ranked Top 10 of approx. 200 students.
Third Year Examinations, ranked Top 10 of approx. 150 students.

Awards and honors

Gibbs Dissertation Prize for Mathematics
Awarded by the Oxford Mathematical Institute.
Best Masters of Mathematics dissertation.

Alison Sheppard Prize for Mathematics
Awarded by St Hugh's College, Oxford.
Third year mathematician with highest first class in College.

St Hugh's College Scholarship Award
Awarded by St Hugh's College, Oxford, annually.
First Class Honors in each year.

Invited Talks

Technical University of Darmstadt, Feburary 7th, 2025.
University of Minnesota Student Number Theory Seminar, November 19th, 2024.
Johns Hopkins University, Topology seminar, September 12th, 2024.

Seminar Talks

2024
Topology E-theory seminar, JHU, on *Gross-Hopkins Period Map*.

Number theory learning seminar, JHU, motivic periods, two talks on *Chen's Theorem.*

2023
Topology Seminar, JHU, on *Dieudonné modules, following Lurie and Hopkins.*
Topics in representation theory seminar, JHU, on *Uniformization of G-bundles.*
Topological Quantum Field Theory learning seminar, JHU, on *Classical Field Theory and σ-models.*
Topics in representation theory seminar, JHU, on *Affine Grassmanian.*
Prismatic cohomology Seminar organizer, with Naruki Masuda and David Gepner.

2022
Heegner points study group, JHU, on *Selmer structures and duality.*
Derived deformation theory seminar, JHU, three talks on *Calegari-Geraghty Method in Modularity Lifting.*
Jacquet Langlands Correspondence student seminar, JHU, four talks.

2021
eCHT Hermitian K-theory, on *Poincaré Categories.*
Category theory seminar, on *Differential Cohomology and Cohesive Topoi.*
Derived deformation theory seminar, JHU, on formal moduli problems.
Seminar on Stack of Langlands Parameter, joint with U Chicago, on *Representation Stacks.*
Non-archimedean study group, on *Formal schemes and Rigid Generic Fiber.*

2020
DaFra Seminar on Condensed mathematics, a talk on *Solid Abelian Groups.*
Étale homotopy study group, Kings College London, a talk on *Étale Homotopy Obstruction.*
Topological Hochschild Homology Seminar, UIC, two talks on *Construction of THH.*
Spectral Algebraic Geometry Seminar, UIC, two talks on *Spectrally Ringed ∞-Topoi.*
eCHT Kan Fall Seminar, two talks on chapter 1 of *A Survey of Elliptic Cohomology, J. Lurie.*
Number Theory Seminar, Uni. of Melbourne, two talks on *Contragredient representations.*
Oberseminar, Uni. of Regensburg, a talk on *The p-complete Frobenius.*

2019
Masters presentation, University of Oxford. On *The Atiyah Singer-Index Theorem.*
Reading Group, University of Oxford. On *Model Categories, Dwyer and Saplinski.*

Professional service

All roles listed below were conducted at Johns Hopkins University.

Graduate Mentorship
(2023-2025) Yashi Jain, serving as a secondary advisor. Primary advisor: David Savitt.

Undergraduate Mentorship
Spring 2024: Viggy Vanchinathan, mentored a DRP project on understanding addition with transformers.
Fall 2023: Spencer Huang, Dev Lalwani, mentored a DRP project on mechanistic interpretability.
Spring 2023: Orisis Zheng, mentored a DRP project on Zariski's lemma in Algebraic Geometry.
Fall 2022: Nick Lombardi: mentored a project on an introduction to the Langlands program.

Fall 2024
Graduate Algebra, Teaching Assistant.
Introduction to Proofs, Teaching Assistant.

Spring 2024

Directed Reading Program, Co-organizer.

Fall 2023
SOUL Course: Interpretability in AI, Lecturer.
Honors Single Variable Calculus, Lecturer.
Directed Reading Program, Organizer and Mentor.

Spring 2023
Calculus III, Head Teaching Assistant.
Directed Reading Program, Co-organizer and Mentor.

Fall 2022
Calculus II, Teaching Assistant.
Directed Reading Program, Co-organizer and Mentor.

AI Projects

The following are some small personal projects during my free time, reflecting my interests in, 1. Social Impact and Ethical AI, 2. Multilingual Natural Language Processing, focusing on improving low-resource language performance, and 3. AI and Mathematics, exploring the role of language models in mathematical research, education.

Spoken MASSIVE joint with Chutong Meng (George Mason University), this project studies synthetic data in the context of spoken language understanding (SLU). We created the first multilingual SLU dataset. The dataset was synthesized from MASSIVE, and SLU models were trained and evaluated.

Emotion Fine Tuning This is an experimental project, with Prof. Levine (Cornell University), exploring the role of emotion in AI behavior. The first step is to quantify if emotional feedback can be as useful as RLHF in training data.

Mitigating Social Biases in Language Models with Adversarial Debate This project is joint work with Cole Molloy (Johns Hopkins University) and Lois Wang (Johns Hopkins University). We explored how in-context adversarial debates between language models can be structured to mitigate social biases in pretrained language models.

Some Thoughts on AI and Mathematical Research, in 2023 Written with Sina Hazratpour (Postdoc at Johns Hopkins University), this article surveys the impact of large language models in theorem-proving community.

Skills

Programming Languages: Python, R, MATLAB
AI Frameworks: TensorFlow, PyTorch
Tools: Jupyter, Git, LaTeX
Languages: Mandarin (native), English (fluent), Ukrainian (elementary proficiency)

RESEARCH STATEMENT

MILTON LIN

Introduction

My areas of interest in mathematics are:

(1) The **Langlands correspondence**, which is now a huge web of conjectures, from special values of L functions to conformal field theory. My particular interest is in various incarnations of the **Casselman–Shalika formula**, A, and **relative aspects of the ℓ-adic geometric local Langlands correspondence**, C.

(2) **Stable homotopy theory**, through the lens of **categorical deformation theory**, F.

(3) **Motivic phenomena**, where I hope to explore the motivic version of Casselman–Shalika formula, D, and a stacky approach to periods, E.

Majority of the research presented here originates from my study of the Casselman–Shalika formula in the mixed characteristic setting, as outlined in Section 1. These areas of research are interconnected, as shown in the following diagram.

$$
\begin{tikzcd}
\begin{array}{c}\text{Section 2 C}\\ \text{Relative Langlands on}\\ \text{Fargues–Fontaine curve}\end{array} & & \begin{array}{c}\text{Section 4 E}\\ \text{Stacky approach}\\ \text{to periods}\end{array} \arrow[ll, dashed] \\
\begin{array}{c}\textbf{ℓ-adic geometric}\\ \textbf{local Langlands}\end{array} \arrow[u] \arrow[d] \arrow[r] & \begin{array}{c}\text{Section 1 A,}\\ \text{Geometriization of}\\ \text{metaplectic/quasi-split}\\ \text{Casselman–Shalika}\end{array} \arrow[d, dashed] & \begin{array}{c}\textbf{"Motivic"}\\ \textbf{Phenomena}\end{array} \arrow[u] \arrow[d] \\
\begin{array}{c}\text{Section 1 B}\\ \text{Perfect geometry of}\\ \text{affine Schubert Varieties}\end{array} & \begin{array}{c}\text{Section 5 F}\\ \text{Categorical deformation}\\ \text{of representation category}\end{array} & \begin{array}{c}\text{Section 3 D}\\ \text{Motivic}\\ \text{Casselman–Shalika}\end{array} \arrow[l, dashed] \\
 & \textbf{Categorical deformation theory} \arrow[u] &
\end{tikzcd}
$$

The priority of research is listed in the following order[1],

A=C=F>D>E> B.

Date: October 26, 2024.

[1]The alphabet links to the goal rather than the section.

Notations. Theorems stated have full proof written by either me or joint with collaborators, *unless* it is annotated with: ***conjecture*** – no proofs have been written down but is believed to hold, or ***in progress*** – where we have partial progress. We use freely the language of higher categories and higher algebra, [Lur09], [Lur18].

1. Mixed Characteristic Geometry and the Casselman–Shalika formula

Let G be a connected reductive group over a nonarchimedean local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld [AG15], [Gai14]. They constructed two actions

$$\begin{array}{ccc} & \mathrm{IndPerf}(\mathrm{LS}_{\hat{G}}) & \\ \circlearrowleft & & \circlearrowright \\ \mathrm{IndCoh}^{qc}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}}) & & D_{\mathrm{lis}}(\mathrm{Bun}_G) \end{array}$$

where $\mathrm{LS}_{\hat{G}}$, be the moduli stack of L-parameters of $\hat{G}$, and Bun_G the moduli stack of G-bundles on the Fargues–Fontaine curve. The left-hand action in the diagram is induced from the natural tensor structure, while the right-hand is the so-called *spectral action* [NY19]. For a fixed choice of Whittaker datum, [FS24, I.10.2] conjectured a $\mathrm{IndPerf}(\mathrm{LS}_{\hat{G}})$-linear equivalence

$$\mathbb{L}_G : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \simeq \mathrm{IndCoh}^{\mathrm{qc}}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}})$$

which can be thought of as a generalization of the classical local Langlands.

One fundamental aspect of the program is to understand the *Whittaker Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, this corresponds to finding the Fourier coefficients of automorphic functions.

Example 1.1. Let $G = \mathrm{PGL}_2$ be the projective linear group over $\mathbb{Q}$. A modular function, f, has an adelic formulation, $\widetilde{f}$ on $G(\mathbb{A}_\mathbb{Q})$. Up to a *normalizing factor*, the Fourier coefficients coincide with the Whittaker coefficients (Jacquet period),

$$a_m \sim \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \widetilde{f}(n\alpha_m)\psi(-n)\,dn \quad \text{for } m \geq 1$$

where $\alpha_m \in T(\mathbb{A}^{\mathrm{fin}}_\mathbb{Q})$ is m considered as a finite idèle and ψ is a standard character on $N(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_\mathbb{Q}$, where N is the radical of standard Borel. For more details, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which we aim to replicate in the mixed characteristic setting. In joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn University) [ILZ24], we proved a variation of this problem: the *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian Gr_G, analogous to the equi-characteristic geometrization carried out in [NP01].

Theorem 1.2. *[ILZ24] The geometric Casselman–Shalika formula holds over the Witt vector Grassmanian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori–Satake equivalence, [Bez+19] without using nearby cycles.

Theorem 1.3 (I.-Lin-Z., in progress)**.** *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [2]

As of writing, D. Hansen, L. Hamman, and L. Mann have announced significant advancements to the global version of the formula. Building on all these progresses, I have developed a strong interest in exploring the geometrization of the Casselman–Shalika formula for covering groups [GGW18].

Research Goal A. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula over equal-characteristic local fields. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+20; Bru+24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

Lastly, we would like to highlight that in our recent work [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\leq\mu}$, for minuscule and quasi-minuscule μ. In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal B. Geometry of general perfect Schubert variety *Prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhats-Tits building of G, we have an associated group scheme $\mathcal{G}$ over $\mathcal{O}$.*

Conjecture 1.4. The smooth locus of $\mathrm{Gr}_{\mathcal{G},\leq\mathfrak{s}}$ is $\mathrm{Gr}_{\mathcal{G},\mathfrak{s}}$ in perfect geometry, in the sense of [Zhu17].

2. Relative langlands on the Fargues Fontaine curve

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and L-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with $(\hat{G}, \hat{X})$ of hyperspherical varieties. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_!\Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \pi_*\omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. Bun_G has a Hardar-Narasimhan straification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G)$. In our case, $G = \mathbb{G}_m$, $\mathrm{Bun}_{\mathbb{G}_m}$ is stratified by $\mathrm{Bun}_{\mathbb{G}_m}^n$ for $n \in \mathbb{Z} = B(T)$. The following theorem explicitly computes the period sheaf.

Proposition 2.1 (Lin-T.)**.** *Let $n > 0$. Let $\mathcal{BC}(n)$ be the Banach-Colmez space of the line bundle $\mathcal{O}(n)$. The relative stack $\mathrm{Bun}_G^{X,n} := \pi^{-1}(\mathrm{Bun}_G^n)$ is a $\mathcal{BC}(n)$-torsor over Bun_G^n. $\mathcal{P}_X^n := \mathcal{P}_X|_{\mathrm{Bun}_G^n} \simeq \Lambda[-2n]$.*

On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles (cf. [FW24, Ch. 6]).

[2] In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

Conjecture 2.2. [Lin-T., in progress] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[3] $\mathcal{L}_{\hat{X}}$.

Research Goal C. Relative Langlands on the Fargues Fontaine curve. *Complete Conjecture 2.2 as a first step and then the Hecke case, which classically corresponds to Hecke's integral representation of standard L-function for GL_2. Lastly, one can ask whether on the $\mathcal{B}$-side, the same constructions of [BSV, Ch. 11] works for the p-adic (Emerton-Gee) L-parameter stacks, which potentially give new interpretations to p-adic L-functions.*

3. Motivic Phenomena

In this section S denotes a base scheme satisfying the Beilinson-Soulé vanishing conjecture, such as $\operatorname{Spec}\mathbb{Z}$, finite field. R denotes a regular coherent coefficient ring. I discuss one side project I have done, and one area of research, D, which extends my joint work [ILZ24].

In Grothendieck's quest towards the Weil conjectures, he reduced to problems of motives. Beilinson proposed an extension of the notion of pure motives to mixed motives, which was realized through the works of Voevodsky, Levine for a field, and extended by Ayoub and Cisikinki-Déglise, [CD19]. For a finite type S-scheme $X \in \mathrm{Sch}_S^{\mathrm{ft}}$, we can construct the derived categories of motives over X, $\mathrm{DM}(X, R)$. If X is smooth then its Ext-groups,

$$\pi_{-m} \operatorname{Map}_{\mathrm{DM}(X,R)}(1_X, 1_X(n)) \simeq \mathrm{CH}^n(X, 2n-m)_R$$

are Bloch's higher Chow groups. Motivic categories and the Chow groups are difficult to explicitly work with due to the lack of motivic t-structures. One method is due to the work of Beauville [Bea83], using Fourier transform. In my joint work, [Has+24] we extended the work of Beauville,

Theorem 3.1 (Lin et al.)**.** *Let $X \to Y$ be an abelian scheme, whose base Y is smooth and quasi projective over a field. There is an explicit N, such that one obtains a Beauville decomposition*

$$CH^i(X)_{\mathbb{Z}[1/N]} \simeq \bigoplus_s CH^i_{(s)}(X)_{\mathbb{Z}[1/N]}$$

where $CH^i_{(s)}(X)_R := \left\{x \in CH^i(X)_R \,:\, [n]_X^* x = n^{2i-s}x \quad \forall n \in \mathbb{Z}\right\}$ *and* $[n]$ *is the multiplication of an abelian scheme.*

This extends to a $\mathfrak{sl}_2$ action, which we discussed in *op.cit*; and if S were an algebraically closed field, this implies various structural results. The key ingredient was using G. Pappas' version of integral Grothendieck–Riemann–Roch.

Returning to motivic t-structures, it was shown by Levine that t-structures exist on a nice subcategory $\mathrm{DTM}(X) \hookrightarrow \mathrm{DM}(X)$ of *mixed Tate movies* for nice schemes X. This was extended to schemes with cellular Whitney-Tate stratification by Soergel and Wendt, and to prestacks in [RS20].

Research Goal D. Motivic Whittaker categories. *Define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we can prove the same statement in [NP01] in the category of mixed Tate motives, in which all the relevant objects are well defined. The difficulty, however, is that the original argument requires characteristic* 0 *coefficients, whilst in positive characteristic, there are problems of extensions.*

[3] One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on X which we do not discuss.

4. Stacky approaches and periods

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R)$$
$$X \mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?})$$

For instance, the *de Rham stack* X^{dR} over $\mathbb{Q}$, has points given by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A (cf. [GR14]). This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques. I explored this concept in basic constructions of F. Brown's work on *motivic periods*, [Bro14]. If X were a smooth variety over $\mathbb{Q}$, the matrix coefficient from Grothendieck's comparison theorem

$$H^*_{\mathrm{dR}}(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\simeq} H^*_{\mathrm{Betti}}(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

with respect to the $\mathbb{Q}$ structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to X*. These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"[4] which should govern the arithmetic structure of periods.

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the projective space minus three points over $\mathbb{Q}$. The period associated is studied through Chen's de Rham comparison theorem, [Che76], which relates to iterated integrals [Che73] and multiple zeta values [Bro14]. To give a "stacky" perspective of this period one uses unipotent types of stacks, which is defined in Toën, [Toë06], and recently in [MR23]. This is an endofunctor $\mathbf{U}$ on stacks, sending a stack X to its unipotent homotopy type. My first result is:

Theorem 4.1. *(Lin) Unipotent de Rham fundamental group, $\pi_1^{u,dR}(X, x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{dR}))$.*

Research Goal E. A stacky approach to motivic periods. *This research aims to use geometric techniques in the study of periods. As a proof of concept, we will recover de Rham comparison theorem through the Riemann-Hilbert correspondence between the analytic Betti stack and the de Rham stack. $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, is not proper, which requires us to incorporate log structures. I expect to prove:*

Conjecture 4.2. There exists X^{Betti} such that the unipotent Betti group $\pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}), x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{\mathrm{Betti}}))$. A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,\mathrm{dR}}(X, x) \simeq \pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}), x) \otimes \mathbb{C}$$

By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [5] *We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].*

5. Deformation theory and the sphere spectrum $\mathbb{S}$

Let $\mathcal{S}$ denote the ∞-category of ∞-groupoids/anima. The *stabilization* of $\mathcal{S}$ is Sp, the ∞-category of spectra. This is the natural category to study cohomological invariants. Within Sp, lies the universal cohomology theory, $\mathbb{S}$, the *sphere spectrum* By Chevalley's works, connected reductive groups over $\mathbb{C}$ have a canonical split $\mathbb{Z}$-form $G_{\mathbb{Z}}$, see [Con15]. One can analogously ask: *is there a $\mathbb{S}$-form for algebraic groups*? A first approximation is the existence of an algebraic category $\mathrm{Rep}_{\mathbb{S}}(G_{\mathbb{S}})$, which

[4] For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

[5] This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

deforms to $\mathrm{Rep}_{\mathbb{Z}}(G_{\mathbb{Z}})$. To study, we begin with *formal deformation of categories*, which we briefly recall.

Let $\mathcal{C}$ be a symmetric monoidal category. There is a natural hierarchy of commutativity fitting in the diagram

$$\cdots \to \mathbb{E}_n(\mathcal{C}) \to \mathbb{E}_{n-1}(\mathcal{C}) \to \cdots \to \mathbb{E}_1(\mathcal{C})$$

where $\mathrm{CAlg}(\mathcal{C}) := \mathbb{E}_\infty(\mathcal{C}) := \varprojlim \mathbb{E}_n(\mathcal{C})$ of symmetric algebra objects can be identified with the limit. One can formalize the notion of $\mathbb{E}_n$-algebra objects via *disk operads*, or *little cubes operads*.

Example 5.1. Let $\mathcal{C} = (\mathcal{S}, \times)$. Let $Y \in \mathcal{S}_*$ is a pointed ∞-groupoid, its *k-fold based loop spaces*, $\Omega^k_* Y$ is a classical example of $\mathbb{E}_k$ algebra object in $(\mathcal{S}, \times)$.

Let $R \in \mathbb{E}_n(\mathrm{Sp})$ be an $\mathbb{E}_n$ ring, and consider LMod_R, the derived category of left R-modules, as an $\mathbb{E}_1$ object in $\mathrm{Pr}^{\mathrm{st}}$, the category of presentable stable categories. This defines $\mathrm{RMod}_{\mathrm{LMod}_R}(\mathrm{Pr}^{\mathrm{st}})$, the category of presentable stable (right) R-linear categories, [Lur18, Appendix D]. Set $\mathrm{Pr}^{\mathrm{st,cg}}_R$, as the full subcategory spanned by those whose underlying category is compactly generated. [6] For G a connected reductive group over a field k, $D^b(\mathrm{Rep}^{\mathrm{fd}}_k(G))$, the bounded derived category of finite dimensional algebraic representations with k coefficients lies in $\mathrm{Pr}^{\mathrm{st,cg}}_R$.

From now on, $k = \mathbb{C}$. Let $\mathrm{Art}^{(n)}_k$ denote the category of $\mathbb{E}_n$ *artinian* ring spectrum, for $n \geq 0$ over k. We refer to [Lur18, Ch. 15] for definition. This extends the classical definition of Artinian local ring, in particular, $R \in \mathrm{Art}^{(n)}_k$ admits an augmentation map $\epsilon : R \to k$. One defines the $\mathbb{E}_{n+2}$-formal moduli problem,

$$\mathrm{Cat}\mathcal{D}\mathrm{ef}^{(n)}(\mathcal{C}) : \mathrm{Art}^{(n+2)}_k \to \hat{\mathcal{S}}$$
$$R \mapsto |\,\{\mathcal{C}\} \times_{\mathrm{Pr}^{\mathrm{st,cg}}_R} \mathrm{Pr}^{\mathrm{st,cg}}_k|$$

where $|\quad|$ is the underlying Kan complex of the ∞-category. An object consists of: a $\mathcal{C}_B$ right stable R-linear category, and an equivalence $\mathcal{C}_B \otimes_{\mathrm{LMod}_B} \mathrm{LMod}_k \simeq \mathcal{C}$. Our $\mathbb{E}_4$-moduli problem is when $n = 2$ and $\mathcal{C} = D^b(\mathrm{Rep}^{\mathrm{fd}}_k(G))$. The geometric Casselman–Shalika [FGV01], which is the $\mathbb{E}_2$-algebra equivalence of $\mathcal{C}$ with the *Whittaker sheaves* on the *affine Grassmanian* $\mathrm{Gr}_{\hat{G}}$, describes this moduli problem. Consider moduli of functor of $\mathbb{G}_m$*-gerbes over* X

$$\mathrm{Ge}_{\mathbb{G}_m}(X) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(X, B^2R^\times) \quad R \in \mathrm{Art}^{(4)}_k$$

where $R^\times \subset \Omega^\infty R$ are the invertible elements of the underlying space of R [7] and B^2 is the second deloop. It was stated without proof in [Lur10]

Theorem 5.2 (Lurie)**.** *There is an equivalence of formal $\mathbb{E}_4$-moduli problems*

$$\widehat{Ge_{\mathbb{G}_m}}(\mathrm{Gr}_{\hat{G}}) \xrightarrow{\simeq} Cat\mathcal{D}ef^{(2)}(\mathrm{Rep}^{fd}_k G)$$

where $\widehat{-}$ is the formal completion of the moduli functor at a base point.

Research Goal F. Categorical deformations for the sphere spectrum. *We will first document carefully Lurie's theorem, Theorem 5.2. Then, we will explore deformations of representation of Lusztig's small quantum group, see op.cit. Remark 10.12, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.*

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[6] The compact generation is only a smallness condition for our version.

[7] is the union of the connected components of invertibles in the $\pi_0 R$ of the 0th space of R and is equivalent to the nth loop space of some space, $R^\times \simeq \Omega^n Z$, $n \geq 4$, hence the deloop $B^2R^\times \in \mathbb{E}_2(\mathcal{S})$

References

[AG15] Arinkin, D. and Gaitsgory, D. "Singular support of coherent sheaves and the geometric Langlands conjecture". In: *Selecta Math. (N.S.)* 21.1 (2015), pp. 1–199. ISSN: 1022-1824,1420-9020. URL: https://doi.org/10.1007/s00029-014-0167-5 (cit. on p. 2).

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