METAPLECTIC GROUP SCHEMES

YIFEI ZHAO

ABSTRACT. Given a reductive group scheme G, we give a linear algebraic description of reduced étale 4-cocycles on its classifying stack B(G). These cocycles form a 2-groupoid, which we interpret as parameters of metaplectic covers of G. We use our linear algebraic description to define the Langlands dual of a metaplectic cover.

CONTENTS

INTRODUCTION

The purpose of this paper is to develop an algebraic theory of metaplectic groups which shares structural features with reductive group schemes. For an integer $N \geq 1$ and a $\mathbb{Z}[\frac{1}{N}]$-scheme S, our notion of a "metaplectic group scheme" is a pair (G, μ), where $G \to S$ is a reductive group scheme and μ is an object of étale cohomological nature, which we call an "N-fold metaplectic cover" of G. We shall prove:

(1) Metaplectic group schemes are classified by linear algebraic data, extending the root data classification of reductive group schemes.
(2) When we fix a coefficient ring E and an injective character $\zeta : \mu_N \to E^\times$, a metaplectic group scheme (G, μ) has canonically defined dual data.

(3) Metaplectic group schemes define topological covering groups over the spectrum of a local or global field. They include all covering groups which can be obtained from central extensions of G by $\mathbf{K}_2$.

It is an old idea that metaplectic groups should arise from objects of algebraic geometry. While reductive groups are points of reductive group schemes, their metaplectic covers appear to have cohomological origin. Using étale cohomology, Deligne ([Del96]) gave a uniform construction of metaplectic covers of semisimple, simply connected groups. Shortly after, Brylinski and Deligne ([BD01]) found a different construction using central extensions by $\mathbf{K}_2$, and by a *tour de force*, classified them for all reductive group schemes.

The K-theoretic parametrization of metaplectic covers has seen tremendous applications. For example, it forms the base of the emerging metaplectic Langlands program ([Wei18], [GG18]). Its étale cohomology sibling, we believe, deserves an equal amount of attention: the étale theory needs no regularity assumption on the base, thanks to excellent base change properties, and as we shall see, the K-theoretic constructions of metaplectic groups and their L-groups both factor through the étale theory. Given this context, the present paper can be seen as an extension of [Del96] to the case of reductive group schemes. It is a necessary step for the application of the étale theory to Langlands duality of metaplectic groups.

0.1. Defining μ

0.1.1. Fix an integer $N \geq 1$ and a $\mathbb{Z}[\frac{1}{N}]$-scheme S. Let $G \to S$ be a reductive group scheme. Our notion of an N-*fold metaplectic cover* of G is a morphism of pointed étale stacks:

$$\mu : B(G) \to B^{(4)}(\mu_N^{\otimes 2}). \tag{0.1}$$

Here, $B(G)$ denotes the classifying stack of G, a familiar object in algebraic geometry. The target $B^{(4)}(\mu_N^{\otimes 2})$ stands for the fourth iterated classifying stack of $\mu_N^{\otimes 2}$. Making sense of it requires n-categories for $n \geq 4$. Instead of exhibiting 4-morphisms, we import Lurie's theory of ∞-categories ([Lur09], [Lur17]) to systematically handle such objects.

0.1.2. The ∞-groupoid of N-fold metaplectic covers of G turns out to be 2-truncated, with nth homotopy group given by the reduced cohomology group $\tilde{H}_{\text{ét}}^{4-n}(BG; \mu_N^{\otimes 2})$. Moreover, it is a highly structured ∞-groupoid—a connective $\mathbb{Z}/N$-module spectrum—allowing us to manipulate metaplectic covers as if they were objects of linear algebra.

0.1.3. When the base is a nonarchimedean local field F such that $\mu_N(F)$ has cardinality N, various notions of metaplectic covers are related as in the following commutative diagram:

$$\begin{Bmatrix} \text{central extensions of} \\ \text{G by } \mathbf{K}_2 \end{Bmatrix} \xrightarrow{\ \Phi\ } \begin{Bmatrix} \text{topological covers of} \\ \text{G(F) by } \mu_N(F) \end{Bmatrix} \tag{0.2}$$

with $\mathbf{R}_{\text{ét}}$ down to $\begin{Bmatrix} \text{pointed morphisms} \\ B(G) \to B^{(4)}(\mu_N^{\otimes 2}) \end{Bmatrix} \xrightarrow{\ \Phi_{\text{ét}}\ }$

Here, Φ is the functor of Brylinski–Deligne [BD01, §10] and $\mathbf{R}_{\text{ét}}$ is defined by Gaitsgory ([Gai20, §6.3]). The functor $\Phi_{\text{ét}}$ is known classically ([Del96, §2]), but we shall sketch a construction nevertheless: a pointed morphism $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$ is equivalent to an $\mathbb{E}_1$-monoidal morphism $G \to B^{(3)}(\mu_N^{\otimes 2})$. Evaluating at $\text{Spec}(F)$ and using $H_{\text{ét}}^3(F; \mu_N^{\otimes 2}) = 0$, it

defines a multiplicative $H^2_{\text{ét}}(F; \mu_N^{\otimes 2})$-torsor on $G(F)$, i.e., a central extension:

$$0 \to H^2_{\text{ét}}(F; \mu_N^{\otimes 2}) \to \widetilde{G} \to G(F) \to 1.$$

The covering of $G(F)$ by $\mu_N(F)$ arises from local Tate duality $H^2_{\text{ét}}(F; \mu_N^{\otimes 2}) \cong \mu_N(F)$. Small modifications need to be made for F archimedean.

There is an analogous commutative diagram when F is a global field. In this case, the target category consists of topological covers of $G(\mathbb{A}_F)$ by $\mu_N(F)$, equipped with a canonical splitting over $G(F) \subset G(\mathbb{A}_F)$.

0.1.4. One possible interpretation of these commutative diagrams is that metaplectic groups naturally inhabit étale cohomology theory, although some have motivic origin. The latter ones include all the classical examples.

0.1.5. If the base scheme S is a smooth curve over an algebraically closed field, a pointed morphism $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$ defines a factorization μ_N-gerbe on the affine Grassmannian of G: a "metaplectic parameter" in the sense of Gaitsgory-Lysenko [GL18]. The 2-groupoids they form turn out to be equivalent ([Zha20][1]). From this perspective, the present paper can be viewed as a generalization of [GL18] to an arbitrary base scheme over $\mathbb{Z}[\frac{1}{N}]$.

0.2. Classification

0.2.1. When G is semisimple and simply connected, Deligne ([Del96]) determined the sheaf of pointed morphisms $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$. The assumption on G implies that this is a sheaf of discrete abelian groups. For general G, the problem is 2-categorical in an essential way. Consequently, our classification of metaplectic covers is a linear algebraic description of a sheaf of 2-categories.

0.2.2. Let us first explain how this is done in the case of a split torus T. According to [BD01, §3], central extensions of T by $\mathbf{K}_2$ are classified by pairs (Q, E) where:

(1) Q is an integral quadratic form on the character lattice Λ_T;
(2) E is a central extension of Λ_T by $\mathbb{G}_m$, with commutator $\lambda_1, \lambda_2 \mapsto (-1)^{b(\lambda_1, \lambda_2)}$, where b denotes the symmetric form associated to Q.

Roughly speaking, the classification of pointed morphisms $B(T) \to B^4(\mu_N^{\otimes 2})$ is obtained by replacing Q by a $\mathbb{Z}/N$-valued quadratic form and $\mathbb{G}_m$ by $B(\mu_N)$. Making this replacement precise turns out to be slightly elaborate.

0.2.3. Let $H^{(1)}(\Lambda_T)$ denote the abelian extension of Λ_T by $\text{Sym}^2(\Lambda_T)$ defined by the cocycle $\lambda_1, \lambda_2 \mapsto \lambda_1 \lambda_2$. Linear maps out of $H^{(1)}(\Lambda_T)$ are akin to quadratic functions without constant term: $\lambda \mapsto a\lambda^2 + b\lambda$. Restricting them to $\text{Sym}^2(\Lambda_T)$ gives the quadratic coefficients.

The relevant object for us, $H^{(2)}(\Lambda_T)$, is a categorical analogue of $H^{(1)}(\Lambda_T)$: it is a strictly commutative Picard groupoid (or equivalently, a $\mathbb{Z}$-module spectrum in cohomological degrees $[-1, 0]$). As a monoidal category, it is $\Lambda_T \times B(\wedge^2 \Lambda_T)$, but the commutativity constraint $\lambda_1 + \lambda_2 \cong \lambda_2 + \lambda_1$ is *not* the identity, but multiplication by $\lambda_1 \wedge \lambda_2$. In any event, $H^{(2)}(\Lambda_T)$ is an extension of strictly commutative Picard groupoids:

$$B(\wedge^2 \Lambda_T) \to H^{(2)}(\Lambda_T) \to \Lambda_T. \tag{0.3}$$

We think of linear morphisms out of $H^{(2)}(\Lambda_T)$ as "quadratic functions of the second kind." Key example: any morphism $H^{(2)}(\Lambda_T) \to B(\mathbb{G}_m)$ can first be restricted to $B(\wedge^2 \Lambda_T)$, yielding

---

[1]The proof of Theorem 5.5 in *loc. cit.* applies to divisible coefficient groups. The statement remains valid for coefficients in μ_N and we will supply a proof in a subsequent paper.

an alternating form $A : \wedge^2(\Lambda_T) \to \mathbb{G}_m$ (its "quadratic coefficient"). The monoidal section $\Lambda_T \to H^{(2)}(\Lambda_T)$ determines a monoidal morphism $\Lambda_T \to B(\mathbb{G}_m)$: this is precisely a central extension of Λ_T by $\mathbb{G}_m$ whose commutator equals A.

The second Brylinski–Deligne datum E of §0.2.2 can thus be interpreted as follows: it is a morphism $H^{(2)}(\Lambda_T) \to B(\mathbb{G}_m)$ of strictly commutative Picard groupoids, with prescribed "quadratic coefficient" $\lambda_1, \lambda_2 \mapsto (-1)^{b(\lambda_1,\lambda_2)}$.

0.2.4. We can now state our description of the sheaf $\mathrm{Maps}_*(BT, B^{(4)}\mu_N^{\otimes 2})$ of metaplectic covers of T. It is given by a Cartesian product of sheaves:

$$\begin{array}{ccc}
\mathrm{Maps}_*(BT, B^{(4)}\mu_N^{\otimes 2}) & \longrightarrow & \mathrm{Quad}(\Lambda_T; \mathbb{Z}/N) \\
\downarrow & & \downarrow \\
\mathrm{Maps}_{\mathbb{Z}}(H^{(2)}(\Lambda_T), B^{(2)}\mu_N) & \to & \mathrm{Maps}_{\mathbb{Z}}(\wedge^2(\Lambda_T), B\mu_N)
\end{array} \qquad (0.4)$$

Hence a metaplectic cover of T is classified by a triple (Q, F, h) where:
  (1) Q is a $\mathbb{Z}/N$-valued quadratic form on Λ_T;
  (2) $F : H^{(2)}(\Lambda_T) \to B^{(2)}(\mu_N)$ is a morphism of $\mathbb{Z}$-module spectra; and
  (3) h is an isomorphism between its "quadratic coefficient" with a map $\wedge^2(\Lambda_T) \to B(\mu_N)$ determined by Q.

We give the sheaf of such triples a name: $\vartheta^{(2)}(\Lambda_T)$, the sheaf of *étale ϑ-data*. The author has made a previous attempt to define them in rather cumbersome terms ([Zha20, §5.3]).

The label (2) indicates that there is also $\vartheta^{(1)}(\Lambda_T)$, defined in terms of $H^{(1)}(\Lambda_T)$. This sheaf classifies pointed morphisms $T \to B^{(2)}(\mu_N^{\otimes 2})$. These two sheaves are related by Koszul duality. This relationship is hardly surprising since the cohomology of T (an alternating algebra) is Koszul dual to the cohomology of $B(T)$ (a symmetric algebra).

0.2.5. The classification of metaplectic covers of a split reductive group G is formulated in terms of étale ϑ-data for its universal Cartan T and the classification for its simply connected form G_{sc}. We say that a $\mathbb{Z}/N$-valued quadratic form Q on Λ_T is *strict* if its symmetric bilinear form b satisfies:

$$b(\alpha, \lambda) = \langle \check{\alpha}, \lambda \rangle Q(\alpha) \quad \text{(for all } \alpha \in \Delta \text{ and } \lambda \in \Lambda_T).$$

Metaplectic covers of G_{sc} are classified by strict quadratic forms on $\Lambda_{T_{sc}}$ alone. The latter are in turn isomorphic to a sum of copies of $\mathbb{Z}/N$ indexed by simple factors of G_{sc}.

If we choose a Borel subgroup $B \subset G$, then any metaplectic cover of G defines one for the universal Cartan T. Indeed, we restrict it along $B(B) \to B(G)$ and observe that it canonically descends to $B(T)$. In particular, strict quadratic forms on $\Lambda_{T_{sc}}$ define metaplectic covers of T_{sc}, and in turn étale ϑ-data on $\Lambda_{T_{sc}}$.

Theorem A. *Suppose that $N \geq 1$ and S is an $\mathbb{Z}[\frac{1}{N}]$-scheme. Let $G \to S$ be a split reductive group scheme with a chosen Borel $B \subset G$. Then $\mathrm{Maps}_*(BG, B^4\mu_N^{\otimes 2})$ is canonically equivalent to the sheaf of quadruples (Q, F, h, φ), where:*
  (1) $(Q, F, h) \in \vartheta^{(2)}(\Lambda_T)$ and Q is strict;
  (2) φ is an isomorphism between the restriction of (Q, F, h) to $\vartheta^{(2)}(\Lambda_{T_{sc}})$ and the étale ϑ-datum associated to the restriction of Q to $\Lambda_{T_{sc}}$.

0.2.6. The formulation of Theorem A is in complete parallel with [BD01, Theorem 7.2], except that for later purposes, we prefer to choose a Borel subgroup rather than a maximal

torus. In the main text, we work with slightly weaker assumptions but Theorem A will only be applied when G splits. The equivalence of Theorem A *depends* on the choice of B (or in an alternative formulation, the choice of a maximal torus). This is one aspect in which it differs from the root data classification of split reductive group schemes.

Let us also mention that Theorem A implies the analogous statement for singular cochains on a compact Lie group. If we put aside the relationship to metaplectic groups, it naturally belongs to a chapter on categorical structures of linear algebraic groups.

0.3. The Langlands dual

0.3.1. We now turn to the second main topic of the paper: the definition of the Langlands dual, or L-group, of a metaplectic group scheme. Let us share the bad news first: it will not be a group but rather a tensor functor. In what follows, we shall attempt to justify why we think this is appropriate to the deceived reader.

The good news, however, is that it is canonically attached to (G, μ) without any splitness assumption on G. The only auxiliary data one needs are a coefficient ring E and an injective character $\chi : \mu_N \to E^\times$.

0.3.2. Let us first focus on the case where G splits. If a Borel subgroup $B \subset G$ is chosen, then any metaplectic cover μ has classification datum (Q, F, h, φ). We shall use it to construct a pair (H, F_μ), where H is a pinned reductive group scheme over $\mathrm{Spec}(E)$ and:

$$F_\mu : \check{Z}_H \to B^{(2)}(E^\times) \tag{0.5}$$

is an $\mathbb{E}_\infty$-monoidal morphism out of the character group of its center Z_H.

0.3.3. The reductive group H is well known as the quantum Frobenius quotient. It depends on Q alone. Inside the cocharacter lattice Λ_T of the universal Cartan, we find the kernel $\Lambda_T^\sharp$ of its symmetric form b. For a simple coroot $\alpha \in \Delta$, its $\mathrm{ord}(Q(\alpha))$-multiple belongs to $\Lambda_T^\sharp$, so we find a system $\Delta^\sharp = \{\mathrm{ord}(Q(\alpha)) \mid \alpha \in \Delta\}$ of simple coroots, generating the coroot lattice $\Lambda_T^{\sharp,r}$. Doing the dual construction for $\check{\Lambda}_T$, we obtain a based root datum. It defines a pinned reductive group H whose character lattice is $\Lambda_T^\sharp$, cocharacter lattice is $\check{\Lambda}_T^\sharp$, etc.

0.3.4. The construction of F_μ uses the remaining data (F, h, φ). Recall that $F : H^{(2)}(\Lambda_T) \to B^{(2)}(\mu_N)$ is a $\mathbb{Z}$-linear morphism whose "quadratic coefficient" is determined by Q via h. Moreover, F defines an $\mathbb{E}_1$-monoidal morphism $F_1 : \Lambda_T \to B^{(2)}(\mu_N)$ by restriction along the $\mathbb{E}_1$-monoidal section of (0.3). We then make the following series of moves:

   (1) the restriction of F_1 to $\Lambda_T^\sharp$ lifts to an $\mathbb{E}_\infty$-morphism, by $b = 0$ and h;
   (2) the restriction of F_1 to $\Lambda_T^{\sharp,r}$ is trivialized, by $Q = 0$ and φ;
   (3) from (1) and (2), F_1 factors through an $\mathbb{E}_\infty$-morphism $\bar{F}_1 : \Lambda_T^\sharp / \Lambda_T^{\sharp,r} \to B^{(2)}(\mu_N)$.

The source $\Lambda_T^\sharp / \Lambda_T^{\sharp,r} \cong \check{Z}_H$ and the target can be composed with χ. This yields (0.5).

0.3.5. Let us summarize pictorially the two sublattices in Λ_T and the special behavior of F_1 on them owing to the vanishing of b, respectively Q:

$$
\begin{array}{ccccc}
\Lambda_T^{\sharp,r} & \longrightarrow & \Lambda_T^\sharp & \longrightarrow & \Lambda_T \\
& {\scriptstyle *\cong}\searrow & \ \downarrow{\scriptstyle \mathbb{E}_\infty} & \swarrow{\scriptstyle \mathbb{E}_1} & \\
& & B^{(2)}(\mu_N) & &
\end{array}
$$

The main result we prove about this construction is as follows.

Theorem B. *The construction of* (H, F_μ) *is canonically independent of* B.

0.3.6. This statement is not obvious because the classification of metaplectic covers (Theorem A) depends on the choice of the Borel subgroup $B \subset G$. It is only after the extraction of (H, F_μ) that the independence is restored. We prove Theorem B via a quantitative analysis of the dependence on B in the classification of metaplectic covers.

0.3.7. Theorem B gives us a functor from the 2-groupoid of metaplectic group schemes (G, μ) where G splits to the 2-groupoid of pairs (H, F) where H is a pinned reductive group scheme over $\mathrm{Spec}(E)$ and $F : \check{Z}_H \to B^{(2)}(E^\times)$ is an $\mathbb{E}_\infty$-morphism.

The data H, F conjoin to spawn a third object: an étale stack of tensor (i.e., symmetric monoidal E-linear) categories. General paradigm: given a stack of tensor categories $\mathcal{C}$ compatibly graded by an abelian group Γ, any $\mathbb{E}_\infty$-morphism $F : \Gamma \to B^{(2)}(E^\times)$ can be used to form a twisted stack of tensor categories $\mathcal{C}_F$. We apply this paradigm to the stack $\mathrm{Rep}_{H,S}$ of H-representations on E-local systems over S, graded by Z_H-weights. The result is a stack of tensor categories $(\mathrm{Rep}_{H,S})_F$, equipped with a "fiber functor":

$$(\mathrm{Rep}_{H,S})_F \to (\mathrm{Rep}_{Z_H,S})_F. \tag{0.6}$$

The target is isomorphic to a direct sum of twisted stacks of E-local systems indexed by $\check{Z}_H$. It has a nontrivial symmetric monoidal structure captured by the $\check{Z}_H$-grading. It is also allowed to vary along the base scheme S, which explains the sheaf-theoretic formulation.

0.3.8. We view (0.6) as the Langlands dual (or more precisely, the L-group) of (G, μ). Generalizing this construction of the Langlands dual to all metaplectic group schemes is straightforward, because stacks of tensor categories are of local nature. Indeed, we work étale locally with split G, extract the pair (H, F_μ), then the fiber functor (0.6), and finally glue them into a fiber functor between stacks of tensor categories over S.

0.3.9. In a forthcoming paper, we shall give another construction of the Langlands dual fiber functor when S is a smooth curve X over an algebraically closed field (with suitable E). It asserts that when a ϑ-characteristic on X (i.e., square root of the canonical bundle) is chosen, the tensor category of "μ-twisted" spherical sheaves on the affine Grassmannian of G is canonically equivalent to global sections of $(\mathrm{Rep}_{H,X})_{F_\mu}$, compatibly with natural fiber functors. This equivalence is one justification for our definition of the Langlands dual.

0.3.10. In existing literature, two notions of the metaplectic Langlands dual have been proposed. Weissman ([Wei18]) constructed the first one using the K-theoretic parametrization of metaplectic covers. It is an honest L-group in the spirit of Langlands. Our construction is closely related to Weissman's "second twist", but differs from it in one crucial aspect: in the representation category of any group, invertible objects form a strictly commutative subcategory, but this is not the case for our symmetric monoidal categories.

The second construction is due to Gaitsgory and Lysenko ([GL18]) and is particular to the function field context. Their construction agrees with ours once a ϑ-characteristic is chosen. We are tempted to think that the group-theoretic input of the metaplectic geometric Langlands program is a metaplectic group scheme (G, μ), while its geometric input is a *spin curve* $(X, \omega_X^{1/2})$. These two aspects are decoupled in our definition of the Langlands dual, the advantage being that it requires little assumption on the base.

This perspective also brings us to Weissman's "first twist" (the meta-Galois group), which the author views as an arithmetic incarnation of the ϑ-characteristic and prefers to treat it independently of the group-theoretic input.

0.4. Guide

0.4.1. In §1, we recall some notions from Higher Algebra which allows us to handle algebraic structures on 2-categories in an efficient manner. (Defining a symmetric monoidal 2-category is already a rather imposing task.) In §2, we explain how to produce topological covering groups from our notion of metaplectic covers.

0.4.2. Sections §3, 4, 5 are the technical core of the paper: they prove the classification Theorem A. We begin in §3 by constructing an étale analogue of the equality $\{a, a\} = \{a, -1\}$ in the second algebraic K-group. In §4, we define étale ϑ-data. The key result is Lemma 4.2.6, which describes the sheaf of étale ϑ-data both as a pullback and as a pushout. In §5, we classify metaplectic covers first for tori (Proposition 5.1.2) and then for reductive group schemes with a chosen Borel (Theorem 5.3.3).

0.4.3. In §6, we use the classification theorem to define the Langlands dual of a metaplectic group scheme in the split case. We prove the independence of the Borel (Theorem 6.2.2) using an earlier calculation of equivariance structures (Proposition 5.4.7). Then we invoke the twisting formalism of Appendix A to construct the Langlands dual fiber functor for all metaplectic group schemes.

0.4.4. In §7, we explain the relationship between metaplectic group schemes and central extensions by $\mathbf{K}_2$. We conclude with some remarks on the differences between our Langlands dual and the existing definitions due to Weissman [Wei18] and Gaitsgory–Lysenko [GL18].

0.4.5. Strictly speaking, this paper contains only constructions and no theorem. However, labeling them as constructions throughout the paper makes rather awkward reading. Hence, whenever we assert that "there is a canonical something," we mean that the proof will construct it, although a precise description is not needed in the sequel.

0.4.6. When it comes to ∞-categories, there is some displacement between the language we use and the mathematical formalism it invokes. To build a functor between ∞-categories, one ought to perform operations purely on ∞-categories. Saying what the functor does on objects achieves nothing.

However, to elucidate the ideas, we will often take certain objects as fixed and perform operations on them, with the tacit understanding that these operations are functorial. Of course, when we make an additional choice in the construction, we always verify that the choice does not matter.

0.5. Acknowledgements

1. General topology

The notion of metaplectic group schemes we develop is 2-categorical in an essential way. The 2-isomorphisms we shall encounter fall into two kinds: ones which exist for *general* reasons and ones which exist for *particular* reasons. While the bulk of the paper deals with the latter, we also need an effective language to handle the former. This language is provided by Lurie's theory of Higher Algebra.

1.1. Structured spaces

1.1.1. Let us begin with Grothendieck's dictionary between chain complexes in cohomological degrees $[-1, 0]$ and small, strictly commutative Picard groupoids ([AGV+06, Exposé XVIII, §1.4]). Recall: a symmetric monoidal groupoid $\mathcal{A}$ is a *Picard groupoid* if the monoidal product $- \otimes a : \mathcal{A} \to \mathcal{A}$ is an equivalence for all $a \in \mathcal{A}$. It is *strictly commutative* if the commutativity constraint $a_1 \otimes a_2 \cong a_2 \otimes a_1$ is the identity map whenever $a_1 = a_2$.

To a chain complex $K^{-1} \xrightarrow{d} K^0$, we attach a small, strictly commutative Picard groupoid whose objects are elements $a \in K^0$ and there is an isomorphism $a_1 \xrightarrow{\sim} a_2$ for each $f \in K^{-1}$ with $df = a_2 - a_1$. This association defines an equivalence of categories between chain complexes in cohomological degrees $[-1, 0]$ and small, strictly commutative Picard groupoid ([AGV+06, Exposé XVIII, Proposition 1.4.15]).

The central objects of this paper—metaplectic covers—form the 2-categorical version of a small, strictly commutative Picard groupoid. Homotopical algebra provides us with the tools to concisely express the data defining them.

1.1.2. Let Spc (resp. Spc$_*$) denote the ∞-category of (resp. pointed) spaces. The stable ∞-category of spectra Sptr is the stabilization of Spc$_*$ (c.f. [Lur17, Proposition 1.4.2.24]). It is equipped with a canonical functor $\Omega^\infty : \mathrm{Sptr} \to \mathrm{Spc}_*$ and a t-structure such that Ω^∞ factors through the ∞-category Sptr$^{\leq 0}$ of connective spectra ([Lur17, Proposition 1.4.3.4]). The resulting functor Sptr$^{\leq 0} \to \mathrm{Spc}_*$, still denoted by Ω^∞, has left adjoint Σ^∞_+.

1.1.3. Let us put the functor Ω^∞ in a different context. For an integer $0 \leq n \leq \infty$, write $\mathbb{E}_n(\mathrm{Spc})$ for the ∞-category of $\mathbb{E}_n$-monoid objects in Spc. (See [Lur17, §5.1] for the definition of the $\mathbb{E}_n$-operad.) Equivalently, this is the ∞-category of $\mathbb{E}_n$-algebras in Spc with respect to the Cartesian symmetric monoidal structure ([Lur17, Proposition 2.4.2.5]). We refer to an object of $\mathbb{E}_n(\mathrm{Spc})$ simply as an $\mathbb{E}_n$-*space*. The ∞-category $\mathbb{E}_0(\mathrm{Spc})$ is equivalent to Spc$_*$.

For $n \geq 1$, an $\mathbb{E}_n$-space $\mathcal{A}$ is grouplike if and only if $\pi_0(\mathcal{A})$ is a group with respect to the induced monoid structure ([Lur17, Definition 5.2.6.2, Example 5.2.6.4]). We denote the ∞-category of grouplike $\mathbb{E}_n$-spaces by $\mathbb{E}_n^{\mathrm{grp}}(\mathrm{Spc})$. A version of May's recognition theorem ([Lur17, Remark 5.2.6.26]) states that there is a canonical equivalence of ∞-categories:

$$\mathrm{Sptr}^{\leq 0} \cong \mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc}). \tag{1.1}$$

Under (1.1), the forgetful functor $\mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc}) \to \mathbb{E}_0(\mathrm{Spc})$ corresponds to Ω^∞. This allows us to view connective spectra as spaces equipped with an "algebraic" structure.

1.1.4. Every commutative ring R may be viewed as an $\mathbb{E}_\infty$-algebra of Sptr with respect to the smash product ([Lur17, Remark 7.1.0.3]). The ∞-categorical version of the Schwede–Shipley theorem ([Lur17, Theorem 7.1.2.13]) compares the ∞-derived category D(R) and the ∞-category Mod$_\mathrm{R}$ of R-module spectra. It states that there is a canonical equivalence of symmetric monoidal ∞-categories:

$$\mathrm{D(R)} \cong \mathrm{Mod_R}. \tag{1.2}$$

Under (1.2), the t-structure on R-module spectra corresponds to the natural t-structure on $D(R)$. In particular, the ∞-category of connective R-module spectra $\mathrm{Mod}_R^{\leq 0}$ is equivalent to that of nonpositively graded complexes of R-modules $D^{\leq 0}(R)$.

This last equivalence can be viewed as a generalization of Grothendieck's dictionary, where the role of small, strictly commutative Picard groupoids is played by connective $\mathbb{Z}$-module spectra. The equivalence in §1.1.1 is the special case for $R = \mathbb{Z}$, when both sides are restricted to 1-coconnective objects.

1.1.5. The following chain of forgetful functors relates all "structured spaces" which we shall consider in this paper:

$$\mathrm{Mod}_R^{\leq 0} \to \mathrm{Mod}_{\mathbb{Z}}^{\leq 0} \to \mathrm{Sptr}^{\leq 0}$$
$$\cong \mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc}) \to \mathbb{E}_1^{\mathrm{grp}}(\mathrm{Spc}) \to \mathrm{Spc}_* \to \mathrm{Spc}. \tag{1.3}$$

Lemma 1.1.6. *All functors in* (1.3) *are conservative. They preserve sifted colimits and arbitrary limits.*

Proof. The first two functors preserve limits by [Lur17, Proposition 4.6.2.17]. Since the $\mathbb{E}_\infty$-operad is coherent, we may apply [Lur17, Corollary 3.4.4.6] to conclude that they preserve arbitrary colimits and deduce from [Lur17, Corollary 3.4.3.3] that they are conservative.

For the functors on the bottom row, conservativity is a consequence of [Lur17, Lemma 3.2.2.6]. For any integer $0 \leq n \leq \infty$, the functor $\mathbb{E}_n(\mathrm{Spc}) \to \mathrm{Spc}$ preserves arbitrary limits ([Lur17, Corollary 3.2.2.4]) and *sifted* colimits ([Lur17, Proposition 3.2.3.1]). Together with conservativity, this implies the same for the functors:

$$\mathbb{E}_\infty(\mathrm{Spc}) \to \mathbb{E}_1(\mathrm{Spc}) \to \mathrm{Spc}_* \to \mathrm{Spc}.$$

Finally, the full subcategory of grouplike objects in $\mathbb{E}_n(\mathrm{Spc})$ ($n \geq 1$) is closed under arbitrary limits (by definition) and arbitrary colimits ([Lur17, Remark 5.2.6.9]). $\square$

1.2. Sheaves

1.2.1. Let $\mathcal{C}$ be a site. Denote by $\mathrm{PShv}(\mathcal{C})$ the ∞-category of presheaves of spaces on $\mathcal{C}$. It contains the full subcategory $\mathrm{Shv}(\mathcal{C}) \subset \mathrm{PShv}(\mathcal{C})$ of sheaves of spaces, characterized by the property that for any covering sieve $\mathcal{S} \subset \mathrm{Hom}_{\mathcal{C}}(-, c)$, the canonical map:

$$\mathcal{F}(c) \to \lim_{(f : c_1 \to c) \in \mathcal{S}} F(c_1)$$

is an equivalence. The ∞-category $\mathrm{Shv}(\mathcal{C})$ is an ∞-topos in the sense of [Lur09, §6].

1.2.2. For each ∞-category in (1.3), we may consider the ∞-category of (pre)sheaves valued in it. Since the functors connecting them preserve limits (Lemma 1.1.6), the corresponding functors on presheaves preserve the full subcategories of sheaves.

Lemma 1.1.6 also implies that sheaves valued in $\mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Spc})$ (resp. $\mathbb{E}_1^{\mathrm{grp}}(\mathrm{Spc})$ or Spc_*) are canonically equivalent to grouplike $\mathbb{E}_\infty$-monoids (resp. grouplike $\mathbb{E}_1$-monoids or pointed objects) in $\mathrm{Shv}(\mathcal{C})$. In particular, (1.3) gives rise to a chain of ∞-categories:

$$\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_R^{\leq 0}) \to \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0}) \to \mathrm{Shv}(\mathcal{C}, \mathrm{Sptr}^{\leq 0})$$
$$\cong \mathbb{E}_\infty^{\mathrm{grp}}(\mathrm{Shv}(\mathcal{C})) \to \mathbb{E}_1^{\mathrm{grp}}(\mathrm{Shv}(\mathcal{C})) \to \mathrm{Shv}_*(\mathcal{C}) \to \mathrm{Shv}(\mathcal{C}).$$
$$\tag{1.4}$$

The functors in (1.4) are conservative and limit-preserving. We could add to (1.4) a limit-preserving (but evidently not conservative) functor $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_R) \to \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_R^{\leq 0})$ defined by $\tau^{\leq 0}$ on the underlying presheaves.

1.2.3. Let $\mathcal{A}$ be a symmetric monoidal ∞-category with arbitrary limits and colimits and such that the unit $\mathbf{1}$ is both final and initial. Then for $n \geq 0$, the ∞-category of $\mathbb{E}_n$-algebras of $\mathcal{A}$ and the ∞-category of $\mathbb{E}_n$-algebras of $\mathcal{A}^{\mathrm{op}}$ (i.e. $\mathbb{E}_n$-coalgebras of $\mathcal{A}$) are related by a pair of adjoint functors ([Lur17, Remark 5.2.3.6]):

$$\mathrm{Bar}^{(n)} : \mathbb{E}_n(\mathcal{A}) \rightleftarrows \mathbb{E}_n(\mathcal{A}^{\mathrm{op}})^{\mathrm{op}} : \mathrm{Cobar}^{(n)}.$$

We shall apply this construction to $\mathcal{A} = \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathrm{R}}^{\leq 0})$, $\mathrm{Shv}(\mathcal{C}, \mathrm{Sptr}^{\leq 0})$, $\mathrm{Shv}_*(\mathcal{C})$, equipped with the *Cartesian* symmetric monoidal structure.

In the first two cases, the Cartesian symmetric monoidal structure coincides with the co-Cartesian one, so the forgetful functor $\mathbb{E}_n(\mathcal{A}) \to \mathcal{A}$ is an equivalence ([Lur17, Proposition 2.4.3.9]). Under this equivalence, $\mathrm{Bar}^{(n)}$ corresponds to n-fold suspension $[n]$ ([Lur17, Example 5.2.2.4]). Since the functor $\Omega^\infty : \mathrm{Shv}(\mathcal{C}, \mathrm{Sptr}^{\leq 0}) \to \mathrm{Shv}_*(\mathcal{C})$ is symmetric monoidal, the following diagram is commutative:

$$
\begin{array}{ccc}
\mathrm{Shv}(\mathcal{C}, \mathrm{Sptr}^{\leq 0}) & \xrightarrow{\ [n]\ } & \mathrm{Shv}(\mathcal{C}, \mathrm{Sptr}^{\leq 0}) \\
{\scriptstyle \mathbb{E}_n(\Omega^\infty)}\downarrow & & \downarrow{\scriptstyle \Omega^\infty} \\
\mathbb{E}_n^{\mathrm{grp}}(\mathrm{Shv}(\mathcal{C})) & \xrightarrow{\ \mathrm{B}^{(n)}\ } & \mathrm{Shv}_*(\mathcal{C})
\end{array}
\qquad (1.5)
$$

Here, the functor $\mathrm{B}^{(n)}$ is obtained from $\mathrm{Bar}^{(n)}$ by composing with the functor forgetting the $\mathbb{E}_n$-coalgebra structure. We have an analogous commutative diagram when we replace $\mathrm{Sptr}^{\leq 0}$ in the top row by $\mathrm{Mod}_{\mathrm{R}}^{\leq 0}$.

1.2.4. Given $\mathcal{F} \in \mathrm{Shv}(\mathcal{C})$ and integer $n \geq 0$, we let $\pi_n \mathcal{F}$ denote the sheafification of the presheaf $c \mapsto \pi_n(\mathcal{F}(c))$. Write $\mathrm{Shv}(\mathcal{C})_{\geq n}$ for the full subcategory of $\mathrm{Shv}(\mathcal{C})$ consisting of objects $\mathcal{F}$ with $\pi_k \mathcal{F} = 0$ for all $0 \leq k < n$. Namely, this is the ∞-category of n-connective objects of $\mathrm{Shv}(\mathcal{C})$. We also use the notation $\mathrm{Shv}_*(\mathcal{C})_{\geq n}$ for the pointed version.

Because $\mathrm{Shv}(\mathcal{C})$ is an ∞-topos, $\mathrm{B}^{(n)}$ in (1.5) defines an equivalence of ∞-categories onto $\mathrm{Shv}_*(\mathcal{C})_{\geq n}$ ([Lur17, Theorem 5.2.6.15]).

1.2.5. Let us mention a concrete description of $\mathrm{B} := \mathrm{B}^{(1)}$. Every $\mathbb{E}_1$-monoid in $\mathrm{Shv}(\mathcal{C})$ defines a simplicial object $\mathcal{F}^{[n]}$ ($[n] \in \Delta^{\mathrm{op}}$) of $\mathrm{Shv}(\mathcal{C})$ with $\mathcal{F}^{[0]} \cong \mathrm{pt}$. The functor B sends it to the geometric realization $\mathrm{colim}_{[n]} \mathcal{F}^{[n]}$, pointed by $\mathcal{F}^{[0]}$ ([Lur17, Example 5.2.6.13]).

In particular, given a sheaf of groups H on $\mathcal{C}$, $\mathrm{B}(\mathrm{H})$ is the classifying stack of H in the classical sense. For a sheaf of abelian groups (resp. R-modules) A on $\mathcal{C}$ and $n \geq 1$, we view $\mathrm{B}^{(n)}(\mathrm{A})$ as the n-fold classifying stack of A. The commutative diagram (1.5) tells us that $\mathrm{B}^{(n)}(\mathrm{A})$ is canonically identified with the image of $\mathrm{A}[n]$ under Ω^∞. In particular, $\mathrm{B}^{(n)}(\mathrm{A})$ inherits the structure of a sheaf valued in $\mathrm{Sptr}^{\leq 0}$ (resp. $\mathrm{Mod}_{\mathrm{R}}^{\leq 0}$).

1.2.6. For later purposes, we shall describe the mapping space $\mathrm{Maps}_{\mathbb{E}_1}(\mathrm{H}, \mathrm{B}^{(n+1)}(\mathrm{A}))$ (where $n \geq 1$) for H a sheaf of groups and A a sheaf of abelian groups. Taking fibers at $* \to \mathrm{B}^{(n+1)}(\mathrm{A})$ defines a functor:

$$\mathrm{Maps}_{\mathbb{E}_1}(\mathrm{H}, \mathrm{B}^{(n+1)}(\mathrm{A})) \to \mathbb{E}_1(\mathrm{Shv}(\mathcal{C}))_{\pi_0 = \mathrm{H}, \pi_n = \mathrm{A}}. \qquad (1.6)$$

The target denotes the ∞-category of objects $\mathcal{G} \in \mathbb{E}_1(\mathrm{Shv}(\mathcal{C}))$ equipped with isomorphisms $\pi_0 \mathcal{G} \cong \mathrm{H}$, $\pi_n \mathcal{G} \cong \mathrm{A}$, and $\pi_k \mathcal{G} = 0$ for all $k \neq 0, n$. Such $\mathcal{G}$ is automatically grouplike.

Lemma 1.2.7. *For each $n \geq 1$, the functor* (1.6) *is an equivalence.*

Proof. The functor B defines an equivalence of mapping spaces (§1.2.4):

$$\mathrm{Maps}_{\mathbb{E}_1}(\mathrm{H}, \mathrm{B}^{(n+1)}(\mathrm{A})) \cong \mathrm{Maps}_*(\mathrm{B}(\mathrm{H}), \mathrm{B}^{(n+2)}(\mathrm{A})).$$

Using Postnikov truncation ([Lur09, §5.5.6]), the latter space is equivalent to the space of objects $\mathcal{G}_1 \in \mathrm{Shv}_*(\mathcal{G})$ equipped with isomorphisms $\pi_1(\mathcal{G}_1) \cong \mathrm{H}$, $\pi_{n+1}(\mathcal{G}_1) \cong \mathrm{A}$, and $\pi_k(\mathcal{G}_1) = 0$ for all $k \neq 1, n+1$. Applying the inverse of B yields the desired equivalence. $\square$

Remark 1.2.8. (1) For $n = 0$, the space of $\mathbb{E}_1$-monoidal morphisms $\mathrm{H} \to \mathrm{B}(\mathrm{A})$ is equivalent to the groupoid of central extensions of H by A. By analogy, one may interpret Lemma 1.2.7 as saying that for $n \geq 1$, any $\mathbb{E}_1$-monoidal extension of H by $\mathrm{B}^{(n)}\mathrm{A}$ is automatically "central" (although we do not define this term).

(2) For $n \geq 1$ and H abelian, one may consider $\mathrm{Maps}_{\mathbb{E}_k}(\mathrm{H}, \mathrm{B}^{(n+1)}(\mathrm{A}))$ for any $0 \leq k \leq \infty$. This space is equivalent to $\mathbb{E}_k$-objects of $\mathrm{Shv}(\mathcal{C})$ equipped with isomorphisms $\pi_0\mathcal{G} \cong \mathrm{H}$, $\pi_n\mathcal{G} \cong \mathrm{A}$, and $\pi_k\mathcal{G} = 0$ for all $k \neq 0, n$. Indeed, for finite k one argues as in Lemma 1.2.7. For $k = \infty$, one argues instead in $\mathrm{Sptr}^{\leq 0}$.

1.3. Complexes

1.3.1. In light of (1.2), an object of the ∞-category $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})$ can be viewed as a sheaf of complexes of R-modules. In the classical context, we are more accustomed to complexes of sheaves of R-modules. Let us explain how these two points of views are related.

We continue to fix a site $\mathcal{C}$ and a commutative ring R. The construction of $\pi_n\mathcal{F}$ for an object $\mathcal{F} \in \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})$ defines a t-structure on $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})$. Its heart is identified with the category of sheaves of R-modules on $\mathcal{C}$. According to [Lur18, Corollary 2.1.2.4], we have a fully faithful, t-exact functor:

$$\mathrm{D}^+(\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})^\heartsuit) \to \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R}). \tag{1.7}$$

The essential image of (1.7) is the full subcategory $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})^{>-\infty}$ of left-bounded objects.

1.3.2. Let $\mathrm{ob} : \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R}) \to \mathrm{PShv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})$ denote the forgetful functor. Since ob is a right adjoint, it is left t-exact. We denote by $\mathrm{R}(\mathrm{ob}^\heartsuit)$ the right derived functor of its truncation. Explicitly, given a left-bounded complex $\mathcal{F}$ of sheaves of R-modules, $\mathrm{R}(\mathrm{ob}^\heartsuit)(\mathcal{F})$ is the presheaf whose value at $c \in \mathcal{C}$ is the complex of R-modules $\mathrm{R}\Gamma(c, \mathcal{F})$.

The following commutative diagram arises from the analogous diagram for the sheafification functor by passing to the right adjoint:

$$
\begin{array}{ccc}
\mathrm{D}^+(\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})^\heartsuit) & \overset{\cong}{\longrightarrow} & \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})^{>-\infty} \\
\downarrow{\scriptstyle \mathrm{R}(\mathrm{ob}^\heartsuit)} & & \downarrow{\scriptstyle \mathrm{ob}} \\
\mathrm{D}^+(\mathrm{PShv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})^\heartsuit) & \overset{\cong}{\longrightarrow} & \mathrm{PShv}(\mathcal{C}, \mathrm{Mod}_\mathrm{R})^{>-\infty}
\end{array}
\tag{1.8}
$$

It implies that the image of $\mathcal{F}$ under (1.7) has an explicit description: its value at $c \in \mathcal{C}$ is the complex of R-modules $\mathrm{R}\Gamma(c, \mathcal{F})$.

Lemma 1.3.3. *Suppose that* A *is a sheaf of* R-*modules on* $\mathcal{C}$. *For integers* $0 \leq k \leq n$ *and* $c \in \mathcal{C}$, *there is a canonical isomorphism of* R-*modules:*

$$\pi_k\Gamma(c, \mathrm{B}^{(n)}(\mathrm{A})) \cong \mathrm{H}^{n-k}(c, \mathrm{A}).$$

Proof. This follows from (1.8) and the discussion in §1.2.5. $\square$

2. Metaplectic covers

We introduce the central notion of this paper.

Definition 2.0.1. Let $N \geq 1$ denote an integer and S be a $\mathbb{Z}[\frac{1}{N}]$-scheme. Suppose that $G \to S$ is a reductive group scheme. Define the ∞-groupoid of N-*fold metaplectic covers* of G to be the mapping space of pointed sheaves $\mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_N^{\otimes 2})$ on the étale site of affine S-schemes. It admits the structure of a connective $\mathbb{Z}/N$-module spectrum (see §1.2.5).

We call a pair (G, μ), where G is as above and μ is an N-fold metaplectic cover of G, an N-*fold metaplectic group scheme.* Clearly, N-fold metaplectic group schemes form an ∞-groupoid fibered over the groupoid of reductive group schemes over S. In what follows, we always fix N and omit the adjective "N-fold."

In the remainder of this section, we explain the relationship between our notion of metaplectic covers and the classical notion of topological coverings of reductive groups over a local or global field. This relationship is due to Deligne ([Del96, §2, §5-6]). All we do here is reinterpreting the constructions of *op.cit.* in the language of §1, which allows us to avoid direct contact with resolutions.

2.1. Local fields

2.1.1. Suppose that $F \neq \mathbb{R}$ is a local field such that $\mu_N(F)$ has cardinality N. Let (G, μ) be a metaplectic group scheme over $\mathrm{Spec}(F)$. To these data, we shall attach a central extension of topological groups:
$$1 \to \mu_N(F) \to \widetilde{G}_\mu \to G(F) \to 1. \tag{2.1}$$
Here, $\mu_N(F)$ is discrete and $G(F)$ is equipped with the topology induced from F.

In the case $F = \mathbb{R}$, an analogous construction applies to metaplectic covers with "trivial signature", a notion which we will introduce in §2.1.4 in an *ad hoc* manner.

2.1.2. Let $F \neq \mathbb{R}$ and (G, μ) be as above. The pointed map μ is equivalent to an $\mathbb{E}_1$-monoidal morphism $G \to \mathrm{B}^{(3)}\mu_N^{\otimes 2}$ of étale sheaves over $\mathrm{Spec}(F)$ (§1.2.4). Evaluation at $\mathrm{Spec}(F)$ yields a morphism of $\mathbb{E}_1$-monoids in Spc:
$$G(F) \to \Gamma(F; \mathrm{B}^{(3)}\mu_N^{\otimes 2}). \tag{2.2}$$
By Lemma 1.3.3, the right-hand-side has $\mathrm{H}^3(F; \mu_N^{\otimes 2})$ as its group of connected components, which vanishes ([Ser65, II, 4.3]). This implies that $\Gamma(F; \mathrm{B}^{(3)}\mu_N^{\otimes 2})$ is identified with the Bar construction of $\Gamma(F; \mathrm{B}^{(2)}\mu_N^{\otimes 2})$, viewed as an object of $\mathbb{E}_1^{\mathrm{grp}}(\mathrm{Spc})$. In particular, (2.2) defines a morphism of $\mathbb{E}_1$-monoids:
$$G(F) \to \mathrm{B}(\mathrm{H}^2(F; \mu_N^{\otimes 2})), \tag{2.3}$$
by projecting $\Gamma(F; \mathrm{B}^{(2)}\mu_N^{\otimes 2})$ onto its group of connected components. The morphism (2.3) is equivalent to a central extension of $G(F)$ by $\mathrm{H}^2(F; \mu_N^{\otimes 2})$ as abstract groups. For $F \neq \mathbb{C}$, we invoke the local Tate duality isomorphism ([Ser65, II, 5.2]):
$$\mathrm{H}^2(F; \mu_N^{\otimes 2}) \cong \mu_N(F) \tag{2.4}$$
to obtain the central extension (2.1) as abstract groups. For $F = \mathbb{C}$, we use the trivial map instead of (2.4) so the resulting central extension is canonically split.

2.1.3. It remains to put a topology on $\widetilde{G}_\mu$ so that (2.1) is a central extension of topological groups. As explained in [Del96, 2.9-2.10], this may be achieved by constructing a system of local sections $\widetilde{G}_\mu \to G(F)$ which are compatible with the group structure. The essential point

is that an étale morphism $X \to Y$ of finite type F-schemes defines a local homeomorphism $X(F) \to Y(F)$. The morphism $G \to B^{(3)}\mu_N^{\otimes 2}$ defines a fiber sequence of 2-stacks:

$$B^{(2)}\mu_N^{\otimes 2} \to E_\mu \to G,$$

where the second morphism is surjective in the étale topology. The desired system of local sections follows.

2.1.4. For $F = \mathbb{R}$, the group $H^3(\mathbb{R}; \mu_N^{\otimes 2})$ does not vanish. We shall say that μ has *trivial signature* if the map (2.2) lands in the neutral connected component. The above construction carries through for metaplectic covers of trivial signature.

In §5.5, we shall define the "signature" of a metaplectic cover μ over $\mathrm{Spec}(\mathbb{R})$. It turns out to only concern the restriction of μ to the maximal split torus of G, where it can be determined explicitly. For now, we mention that if G is semisimple and simply connected, then every metaplectic cover μ has trivial signature. More generally, this holds for all μ coming from algebraic K-theory (see §7.2.4).

2.1.5. What we have said in §2.1.2-2.1.4 for an individual metaplectic cover μ can be made functorial. Denote by $\mathrm{Maps}_*^\circ(BG, B^{(4)}\mu_N^{\otimes 2})$ the full subgroupoid of $\mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2})$ consisting of objects of trivial signature if $F = \mathbb{R}$ (and the entire groupoid otherwise). Then we have a composition of functors of ∞-groupoids:

$$\Phi_{\text{ét}} : \mathrm{Maps}_*^\circ(BG, B^{(4)}\mu_N^{\otimes 2}) \xrightarrow{\sim} \mathrm{Maps}_{\mathbb{E}_1}^\circ(G, B^{(3)}\mu_N^{\otimes 2})$$

$$\to \left\{ \begin{matrix} \text{Topological central extensions} \\ \text{of } G(F) \text{ by } \mu_N(F) \end{matrix} \right\}$$

The first functor is the equivalence of §1.2.4, its target being characterized by the trivial signature property. These ∞-groupoids are equipped with the structures of $\mathbb{Z}/N$-module spectra (see §1.2.5) and the functors preserve them. The last ∞-groupoid is 1-truncated and its $\mathbb{Z}/N$-module structure is defined by the Baer sum.

2.2. Global fields

2.2.1. Suppose that F is a global field such that $\mu_N(F)$ has cardinality N. Denote by $\mathbb{A}_F$ the topological ring of adèles of F. Let (G, μ) be a metaplectic group scheme over $\mathrm{Spec}(F)$ whose restriction to every real place is of trivial signature (see §2.1.4). We shall attach to (G, μ) a central extension of topological groups:

$$1 \to \mu_N(F) \to \widetilde{G}_\mu \to G(\mathbb{A}_F) \to 1, \tag{2.5}$$

which is furthermore equipped with a canonical splitting over $G(F) \subset G(\mathbb{A}_F)$.

2.2.2. Let $O_F \subset F$ denote the ring of integers. Consider the filtered category of triples (U, G_U, γ) where $U \subset \mathrm{Spec}(O_F)$ is an affine open subscheme, $G_U \to U$ is a reductive group scheme, and α is an isomorphism $G \cong (G_U) \times_U \mathrm{Spec}(F)$ of group schemes over $\mathrm{Spec}(F)$. Functoriality of the mapping space $\mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2})$ with respect to the base scheme defines a functor of connective $\mathbb{Z}/N$-module spectra:

$$\operatorname*{colim}_{(U, G_U, \gamma)} \mathrm{Maps}_*(B(G_U), B^{(4)}\mu_N^{\otimes 2}) \to \mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2}). \tag{2.6}$$

Lemma 2.2.3. *The functor* (2.6) *is an equivalence.*

Proof. The existence of an integral model implies that G is identified with $\lim_{(U, G_U, \gamma)} G_U$, where the limit is taken in the category of affine schemes. The assertion of the Lemma, when BG is replaced by G^n (for $[n] \in \Delta^{\mathrm{op}}$), follows from the local finite presentation of the stack of étale local systems ([Sta18, 0GL2]). Since $B^{(4)} \mu_N^{\otimes 2}$ is 4-truncated, the mapping space $\mathrm{Maps}_*(BG, B^{(4)} \mu_N^{\otimes 2})$ is computed by a cosimplicial limit over a finite subcategory of Δ^{op}, so we conclude by the commutation of filtered colimits with finite limits. $\qquad\square$

2.2.4. In light of Lemma 2.2.3, it suffices to functorially construct (2.5) out of the data U, G_U, μ_U as in §2.2.2, where μ_U is a metaplectic cover of G_U whose restriction to each real place has trivial signature. To ease the notation, we denote G_U (resp. μ_U) simply by G (resp. μ) until the end of the construction in §2.2.7.

For any open subscheme $V \subset U$, the construction in §2.1 defines a topological central extension of $G(F_x)$ by $\mu_N(F)$ for all $x \notin V$. Their product defines a central extension of $\prod_{x \notin V} G(F_x)$ by $\bigoplus_{x \notin V} \mu_N(F)$. Inducing along the summation map $\bigoplus_{x \notin V} \mu_N(F) \to \mu_N(F)$ and taking product with $\prod_{x \in V} G(O_x)$, we obtain a topological central extension:

$$1 \to \mu_N(F) \to \widetilde{G}_{\mu, V} \to \prod_{x \notin V} G(F_x) \times \prod_{x \in V} G(O_x) \to 1. \tag{2.7}$$

Taking the colimit of (2.7) as V shrinks, we obtain (2.5).

2.2.5. To construct the splitting over $G(F) \subset G(\mathbb{A}_F)$, we shall again work with (2.7) and construct a canonical splitting over $G(V)$, when V is sufficiently small. This uses a standard calculation of étale cohomology of rings of integers.

Lemma 2.2.6. *Suppose that* $V \subset \mathrm{Spec}(O_F)$ *is an open subscheme such that there is at least one nonarchimedean place outside* V *and* N *is invertible on* V. *Then:*

(1) restriction along real places defines an isomorphism:

$$H^3(V; \mu_N^{\otimes 2}) \cong \begin{cases} 0 & N \neq 2; \\ \bigoplus_{x \text{ real}} H^3(F_x; \mu_N^{\otimes 2}) & N = 2. \end{cases}$$

(2) summation of local Tate duality maps (2.4) defines an exact sequence:

$$H^2(V; \mu_N^{\otimes 2}) \to \bigoplus_{x \notin V} H^2(F_x; \mu_N^{\otimes 2}) \to \mu_N(F) \to 1.$$

Proof. Using the existence of nonarchimedean places outside V, the canonical short exact sequence of global class field theory ([Tat67, §10.2]) yields the short exact sequence:

$$0 \to H^2(V; \mathbb{G}_m) \to \bigoplus_{x \notin V} H^2(F_x; \mathbb{G}_m) \to \mathbb{Q}/\mathbb{Z} \to 0, \tag{2.8}$$

as well as the vanishing of $H^3(V; \mathbb{G}_m)$. The second map in (2.8) is the summation of Hasse invariants. Since N is invertible on V, we obtain the assertions on N-torsion coefficients using the Kummer short exact sequence. $\qquad\square$

2.2.7. We return to the context of §2.2.4 and assume furthermore that V satisfies the hypotheses of Lemma 2.2.6. Viewing μ as an $\mathbb{E}_1$-monoidal morphism $G \to B^3 \mu_N^{\otimes 2}$ over V, evaluation at V and taking connected components yield a group homomorphism:

$$G(V) \to H^3(V; \mu_N^{\otimes 2}). \tag{2.9}$$

Claim: (2.9) vanishes. By Lemma 2.2.6(1), (2.9) vanishes for trivial reasons when $N \neq 2$, or else it is completely described by its image in $H^3(F_x; \mu_N^{\otimes 2})$ for real places x. Its composition

to $\mathrm{H}^3(\mathrm{F}_x; \mu_\mathrm{N}^{\otimes 2})$ factors through $\mathrm{G}(\mathrm{V}) \to \mathrm{G}(\mathrm{F}_x)$ and the local map (2.2). Since μ has trivial signature at each F_x, one again concludes that (2.9) vanishes.

The vanishing of (2.9) allows us to define a central extension $\widetilde{\mathrm{G}}^\mathrm{V}_\mu$ of $\mathrm{G}(\mathrm{V})$ by $\mathrm{H}^2(\mathrm{V}; \mu_\mathrm{N}^{\otimes 2})$. The central extensions $\widetilde{\mathrm{G}}^\mathrm{V}_\mu$ and $\widetilde{\mathrm{G}}_{\mu,\mathrm{V}}$ (2.7) are mediated through the product of local central extensions, as displayed in the commutative diagram:

$$
\begin{array}{ccccc}
\mathrm{H}^2(\mathrm{V}; \mu_\mathrm{N}^{\otimes 2}) & \longrightarrow & \widetilde{\mathrm{G}}^\mathrm{V}_\mu & \longrightarrow & \mathrm{G}(\mathrm{V}) \\
\downarrow & & \downarrow & & \downarrow \\
\bigoplus_{x \notin \mathrm{V}} \mathrm{H}^2(\mathrm{F}_x; \mu_\mathrm{N}^{\otimes 2}) \to \prod_{x \notin \mathrm{V}} \widetilde{\mathrm{G}}_{\mu,x} \times \prod_{x \in \mathrm{V}} \mathrm{G}(\mathrm{O}_x) & \to & \prod_{x \notin \mathrm{V}} \mathrm{G}(\mathrm{F}_x) \times \prod_{x \in \mathrm{V}} \mathrm{G}(\mathrm{O}_x) \\
\downarrow & & \downarrow & & \downarrow \cong \\
\mu_\mathrm{N}(\mathrm{F}) & \longrightarrow & \widetilde{\mathrm{G}}_{\mu,\mathrm{V}} & \longrightarrow & \prod_{x \notin \mathrm{V}} \mathrm{G}(\mathrm{F}_x) \times \prod_{x \in \mathrm{V}} \mathrm{G}(\mathrm{O}_x)
\end{array}
$$

It follows from Lemma 2.2.6(2) that the first vertical composition vanishes. Hence $\widetilde{\mathrm{G}}_{\mu,\mathrm{V}}$ admits a canonical splitting over $\mathrm{G}(\mathrm{V}) \subset \prod_{x \notin \mathrm{V}} \mathrm{G}(\mathrm{F}_x) \times \prod_{x \in \mathrm{V}} \mathrm{G}(\mathrm{O}_x)$. The colimit of these splittings as V shrinks defines the splitting of $\widetilde{\mathrm{G}}_\mu$ over $\mathrm{G}(\mathrm{F}) \subset \mathrm{G}(\mathbb{A}_\mathrm{F})$.

2.2.8. We return to the context of §2.2.1 for a summary. Write $\mathrm{Maps}^\circ_*(\mathrm{B}\mathrm{G}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2})$ for the full subgroupoid of $\mathrm{Maps}_*(\mathrm{B}\mathrm{G}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2})$ consisting of objects with trivial signature at each real place of F. (It is the entire groupoid if F is totally imaginary or a function field.) As in the local case, our construction defines a functor of ∞-groupoids, compatible with the $\mathbb{Z}/\mathrm{N}$-module spectra structure:

$$
\Phi_{\text{ét}} : \mathrm{Maps}^\circ_*(\mathrm{B}\mathrm{G}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2}) \to \left\{
\begin{array}{c}
\text{Topological central extensions} \\
\text{of } \mathrm{G}(\mathbb{A}_\mathrm{F}) \text{ by } \mu_\mathrm{N}(\mathrm{F}) \text{ equipped with} \\
\text{a splitting over } \mathrm{G}(\mathrm{F}) \subset \mathrm{G}(\mathbb{A}_\mathrm{F})
\end{array}
\right\}.
$$

The target of this functor is 1-truncated. Its structure as a $\mathbb{Z}/\mathrm{N}$-module spectrum is given explicitly by Baer sum.

$$
3. \ \{a, a\} \cong \{a, -1\}
$$

Until the end of §5, our goal is to give an explicit description of the $\mathbb{Z}/\mathrm{N}$-module spectrum of metaplectic covers. This section fulfills the first step, which is of general nature. Namely, we exhibit a canonical isomorphism between the self-cup product of the Kummer μ_N-torsor Ψ on $\mathbb{G}_m$, viewed as an étale 2-cocycle, and its Yoneda product with the μ_N-torsor η of Nth roots of (-1) (Proposition 3.1.7).

This isomorphism should be viewed as an analogue of the formula $\{a, a\} = \{a, -1\}$ in the second algebraic K-group. The étale theory expels us from the garden of an abelian category, making us search more keenly for the isomorphisms masquerading as familiar equalities. Self-cup product in the context of Selmer groups has been analyzed by Poonen–Rains [PR11], [PR12, §4]. Their method illuminated the path we take.

3.1. Self-cup product of Ψ

3.1.1. Let $\mathcal{C}$ be a site. Suppose that A is a sheaf of abelian groups on $\mathcal{C}$. Let $\mathrm{Heis}(\mathrm{A})$ denote the Heisenberg extension of $\mathrm{A} \times \mathrm{A}$ by $\mathrm{A}^{\otimes 2}$. As a sheaf of sets, $\mathrm{Heis}(\mathrm{A})$ is the product $\mathrm{A}^{\otimes 2} \times (\mathrm{A} \times \mathrm{A})$. Its group structure is specified by the cocycle $(a_1, b_1), (a_2, b_2) \mapsto b_1 \otimes a_2$ for elements $(a_1, b_1), (a_2, b_2) \in \mathrm{A} \times \mathrm{A}$.

The extension $\mathrm{Heis}(\mathrm{A})$ categorifies the cup product of degree-1 cohomology classes in the following sense. The induced map in $\mathrm{Shv}_*(\mathcal{C})$:

$$\cup : \mathrm{B}(\mathrm{A}) \times \mathrm{B}(\mathrm{A}) \cong \mathrm{B}(\mathrm{A} \times \mathrm{A}) \to \mathrm{B}^{(2)}(\mathrm{A}^{\otimes 2}) \tag{3.1}$$

defines the cup product pairing on $\mathrm{H}^1(c; \mathrm{A})$, when we evaluate it at $c \in \mathcal{C}$ and pass to the group of connected components (Lemma 1.3.3). The antisymmetry structure of (3.1) is witnessed by the 1-cochain $(a, b) \mapsto -b \otimes a$.

3.1.2. For $\mathrm{A}_1, \mathrm{A}_2 \in \mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$, the mapping space $\mathrm{Maps}_{\mathbb{Z}}(\mathrm{A}_1, \mathrm{A}_2)$ may again be viewed as an object of $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$. Its value at c is explicitly given by $\tau^{\leq 0} \mathrm{Hom}(\mathrm{A}_1(c), \mathrm{A}_2(c))$. In particular, any sheaves of abelian groups A_1, A_2 define a map in $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$:

$$\mathrm{A}_1 \to \mathrm{Maps}_{\mathbb{Z}}(\mathrm{A}_2, \mathrm{A}_1 \otimes \mathrm{A}_2) \cong \mathrm{Maps}_{\mathbb{Z}}(\mathrm{BA}_2, \mathrm{B}(\mathrm{A}_1 \otimes \mathrm{A}_2)).$$

Evaluation at any global section τ of $\mathrm{B}(\mathrm{A}_2)$ determines a morphism $\mathbf{Y}_\tau : \mathrm{A}_1 \to \mathrm{B}(\mathrm{A}_1 \otimes \mathrm{A}_2)$ in $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$. Applying B yields a morphism:

$$\mathrm{B}(\mathbf{Y}_\tau) : \mathrm{B}(\mathrm{A}_1) \to \mathrm{B}^{(2)}(\mathrm{A}_1 \otimes \mathrm{A}_2). \tag{3.2}$$

This morphism categorifies the Yoneda product with $[t] \in \mathrm{H}^1(c; \mathrm{A}_1)$ when evaluated at $c \in \mathcal{C}$. The following construction compares it with the cup product (3.1) when $\mathrm{A}_1 = \mathrm{A}_2 = \mathrm{A}$.

Lemma 3.1.3. *There is a canonical isomorphism in* $\mathrm{Maps}_*(\mathrm{BA}, \mathrm{B}^{(2)}\mathrm{A}^{\otimes 2})$:

$$\tau \cup (-) \cong \mathrm{B}(\mathbf{Y}_\tau). \tag{3.3}$$

Proof. We first exhibit $\tau \cup (-) : \mathrm{B}(\mathrm{A}) \to \mathrm{B}^{(2)}(\mathrm{A}^{\otimes 2})$ as a morphism of pointed sheaves. Since the fiber of (3.1) is identified with $\mathrm{B}(\mathrm{Heis}(\mathrm{A}))$, a pointing on $\tau \cup (-)$ amounts to a lift of $(\tau, *) \in \mathrm{B}(\mathrm{A}) \times \mathrm{B}(\mathrm{A})$ to $\mathrm{B}(\mathrm{Heis}(\mathrm{A}))$. We write $\mathrm{Heis}(\mathrm{A})$ as an extension of groups:

$$1 \to \mathrm{A}^{\otimes 2} \times \mathrm{A} \to \mathrm{Heis}(\mathrm{A}) \xrightarrow{p_2} \mathrm{A} \to 1. \tag{3.4}$$

The map p_2 is the projection of $\mathrm{Heis}(\mathrm{A})$ onto the second factor of $\mathrm{A} \times \mathrm{A}$. The required pointing is defined by the $\mathrm{Heis}(\mathrm{A})$-torsor P induced from $(*, \tau)$ along the first map of (3.4).

The 2-isomorphism (3.3) is equivalent to an $\mathbb{E}_1$-monoidal isomorphism between $\mathbf{Y}_\tau$ and the map on automorphisms of the pointing defined by $\tau \cup (-)$. These two $\mathbb{E}_1$-monoidal morphisms correspond to extensions of groups:

$$0 \to \mathrm{A}^{\otimes 2} \to \mathrm{E}_i \to \mathrm{A} \to 0 \quad (i = 1, 2),$$

where a lift of $a \in \mathrm{A}$ to E_1 is a section of $\mathbf{Y}_\tau(a)$, and a lift to E_2 is an automorphism f of the $\mathrm{Heis}(\mathrm{A})$-torsor P which gives rise to (id, a) on the induced $(\mathrm{A} \times \mathrm{A})$-torsor (which is canonically isomorphic to $(\tau, *)$).

Note that (3.4) has a semidirect product structure, with A-action on $\mathrm{A}^{\otimes 2} \times \mathrm{A}$ given by $a_1 \cdot (\lambda, a_2) = (\lambda + a_1 \otimes a_2, a_2)$. Hence an automorphism of P which gives rise to a on the p_2-induced A-torsor is equivalent to an isomorphism between the $(\mathrm{A}^{\otimes 2} \times \mathrm{A})$-torsor $(*, \tau)$ and its twist by the a-induced automorphism of $\mathrm{A}^{\otimes 2} \times \mathrm{A}$—which is $(\mathbf{Y}_\tau(a), \tau)$. This isomorphism is equivalent to a section of $\mathbf{Y}_\tau(a)$ together with an automorphism of τ. The condition that f gives identity on the p_1-induced A-torsor shows that the latter automorphism is trivial. Hence $\mathrm{E}_1 \cong \mathrm{E}_2$ as group extensions of A by $\mathrm{A}^{\otimes 2}$. $\qquad\square$

Remark 3.1.4.   (1) Lemma 3.1.3 equips the cup product (3.1) with a bilinear structure and the Yoneda product (3.2) for $\mathrm{A}_1 = \mathrm{A}_2$ with an antisymmetry structure.

(2) It is straightforward to generalize the definition of (3.2) to a morphism $B^{(n_1)}(\mathbf{Y}_\tau)$: $B^{(n_2)}(A_1) \to B^{(n_1+n_2)}(A_1 \otimes A_2)$ with $\tau \in B^{(n_2)}(A_2)$, for all $n_1, n_2 \geq 0$.

3.1.5. We restrict $\mathrm{Heis}(A)$ to the diagonal $A \subset A \times A$ and apply the symmetrization map to the kernel $A^{\otimes 2}$. The result is an extension of *abelian* groups:

$$0 \to \mathrm{Sym}^2(A) \to H^{(1)}(A) \to A \to 0. \tag{3.5}$$

It has coboundary map $\mathbf{H} : A \to B(\mathrm{Sym}(A))$ as well as the induced map:

$$B(\mathbf{H}) : B(A) \to B^{(2)}(\mathrm{Sym}^2(A)). \tag{3.6}$$

Both are morphisms in $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$ (§1.2.5). The morphism (3.6) categorifies the symmetrized self-cup product $H^1(c; A) \to H^2(c; \mathrm{Sym}^2(A))$ when evaluated at an object $c \in \mathcal{C}$. Furthermore, the 1-cochain $a \mapsto a \otimes a$ has coboundary $A \times A \to \mathrm{Sym}^2(A)$, $(a_1, a_2) \mapsto 2a_1 \otimes a_2$, which defines the square of (3.5). Hence $\mathbf{H}$ has a natural 2-torsion structure when viewed as an object of the mapping spectrum.

If A is a sheaf of R-modules for a commutative ring R, the map (3.6) is naturally compatible with the structures of R-module spectra.

3.1.6. Let $N \geq 1$ be an integer and S be a $\mathbb{Z}[\frac{1}{N}]$-scheme. The main construction of this section concerns the situation where $\mathcal{C} = S_{\text{ét}}$ is the étale site of affine S-schemes and $A = \mu_N$. We have $\mu_N^{\otimes 2} \cong \mathrm{Sym}^2(\mu_N)$, so (3.6) is a morphism in $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}/N}^{\leq 0})$:

$$B(\mathbf{H}) : B(\mu_N) \to B^{(2)}(\mu_N^{\otimes 2}), \tag{3.7}$$

equipped with the canonical 2-torsion structure induced from $\mathbf{H}$.

Let η denote the μ_N-torsor of Nth roots of (-1), viewed as a global section of $B(\mu_N)$. By §3.1.2, it defines the Yoneda pairing:

$$B(\mathbf{Y}_\eta) : B(\mu_N) \to B^{(2)}(\mu_N^{\otimes 2}), \tag{3.8}$$

equipped with the 2-torsion structure inherited from η. Finally, we let Ψ denote the Kummer μ_N-torsor, viewed as a morphism $\Psi : \mathbb{G}_m \to B(\mu_N)$ in $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$.

Proposition 3.1.7. *There is a canonical 2-isomorphism* T *in* $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$:

$$\begin{array}{ccc}
\mathbb{G}_m & \xrightarrow{\ \Psi\ } & B(\mu_N) \\
{\scriptstyle \Psi}\big\downarrow & \overset{\mathrm{T}}{\nearrow\nearrow} & \big\downarrow{\scriptstyle B(\mathbf{Y}_\eta)} \\
B(\mu_N) & \xrightarrow[B(\mathbf{H})]{} & B^{(2)}(\mu_N^{\otimes 2})
\end{array} \tag{3.9}$$

Furthermore, $2 \cdot \mathrm{T}$ *is canonically isomorphic to the identity map with respect to the 2-torsion structures of* $B(\mathbf{H})$ *and* $B(\mathbf{Y}_\eta)$.

Remark 3.1.8. Proposition 3.1.7 and Lemma 3.1.3 define isomorphisms of sections of $B^{(2)}(\mu_N^{\otimes 2})$ over $\mathbb{G}_m$ which are compatible with the 2-torsion structures:

$$\Psi \cup \Psi \cong (B\mathbf{Y}_\eta)(\Psi) \cong \eta \cup \Psi.$$

3.1.9. In what follows, we shall construct the 2-isomorphism T in three stages:

(1) we construct a 2-isomorphism comparing self-cup product valued in $\mathbb{G}_m$ and the Yoneda pairing with a 2-torsion element classifying quadratic refinements;

(2) we use the Kummer exact sequence to obtain an analogous 2-isomorphism for the self-cup product valued in μ_N—this step will show that $B(\mathbf{H})$ and $B(\mathbf{Y}_\eta)$ differ by a Bockstein term;

(3) we prove that the Bockstein term vanishes when paired with Ψ.

3.2. Pairings valued in $\mathbb{G}_m$.

3.2.1. The materials of this subsection are closely related to [PR11, Theorem 3.4]. They are most naturally formulated for the fppf site of any scheme S (i.e., no coprimality assumption). Let Γ be a finite locally free abelian group scheme over S. Since Cartier duality $\Gamma \mapsto \check{\Gamma}$ is an autoequivalence on such group schemes, we have $\mathrm{Ext}^1(\Gamma, \mathbb{G}_m) = 0$, so (3.5) defines a short exact sequence of fppf sheaves of abelian groups:

$$1 \to \check{\Gamma} \to \mathrm{Hom}(\mathrm{H}^{(1)}(\Gamma), \mathbb{G}_m) \to \mathrm{Sym}^2(\Gamma, \mathbb{G}_m) \to 1. \tag{3.10}$$

Here, $\mathrm{Sym}^2(\Gamma, \mathbb{G}_m)$ denotes the sheaf of $\mathbb{G}_m$-valued symmetric bilinear pairings on Γ.

3.2.2. Restriction along the set-theoretic section $\Gamma \subset \mathrm{H}^{(1)}(\Gamma)$ defines an isomorphism between $\mathrm{Hom}(\mathrm{H}^{(1)}(\Gamma), \mathbb{G}_m)$ and the sheaf $\check{\mathrm{H}}^{(1)}(\Gamma, \mathbb{G}_m)$ of maps $Q : \Gamma \to \mathbb{G}_m$ satisfying:

(1) $Q(1) = 1$;
(2) $Q(a_1 a_2) Q(a_1)^{-1} Q(a_2)^{-1}$ defines a symmetric bilinear form on Γ.

There is a canonical short exact sequence:

$$1 \to \check{\Gamma} \to \check{\mathrm{H}}^{(1)}(\Gamma, \mathbb{G}_m) \to \mathrm{Sym}^2(\Gamma, \mathbb{G}_m) \to 1, \tag{3.11}$$

where the surjection sends Q to the symmetric form $a_1, a_2 \mapsto Q(a_1 a_2) Q(a_1)^{-1} Q(a_2)^{-1}$. The extensions (3.10) and (3.11) are *negations* of one another via the aforementioned isomorphism $\mathrm{Hom}(\mathrm{H}^{(1)}(\Gamma), \mathbb{G}_m) \cong \check{\mathrm{H}}^{(1)}(\Gamma, \mathbb{G}_m)$.

3.2.3. We denote the coboundary morphism of (3.11) by:

$$\mathcal{Q} : \mathrm{Sym}^2(\Gamma, \mathbb{G}_m) \to B(\check{\Gamma}) \tag{3.12}$$

and view it as classifying quadratic refinements. It has a 2-torsion structure given by the canonical quadratic refinement $Q(a) = b(a \otimes a)$ of $2b$, for any $b \in \mathrm{Sym}^2(\Gamma, \mathbb{G}_m)$.

Given $b \in \mathrm{Sym}^2(\Gamma, \mathbb{G}_m)$, the Yoneda pairing with $\mathcal{Q}(b)$ defines a map $\mathbf{Y}_{\mathcal{Q}(b)} : \Gamma \to B(\mathbb{G}_m)$. Namely, it is the evaluation of the map $\Gamma \to \mathrm{Maps}(B\check{\Gamma}, B\mathbb{G}_m)$ at $\mathcal{Q}(b) \in B\check{\Gamma}$ (§3.1.2). Recall the morphism $\mathbf{H} : \Gamma \to B(\mathrm{Sym}^2(\Gamma))$ given as the coboundary of (3.5).

Lemma 3.2.4. *There is a canonical isomorphism* T_b *in* $\mathrm{Maps}_{\mathbb{Z}}(\Gamma, B\mathbb{G}_m)$:

$$\mathrm{T}_b : B(b) \circ \mathbf{H} \cong \mathbf{Y}_{\mathcal{Q}(b)}. \tag{3.13}$$

Furthermore, $2 \cdot \mathrm{T}_b$ *is the identity with repsect to the 2-torsion structures of* $\mathbf{H}$ *and* $\mathbf{Y}_{\mathcal{Q}(b)}$.

3.2.5. Lemma 3.2.4 will follow from a general construction. Given a short exact sequence of sheaves of abelian groups on any site:

$$0 \to \mathrm{A}_1 \to \mathrm{A}_2 \to \mathrm{A}_3 \to 0,$$

a sheaf of abelian groups B for which the sheaf-theoretic $\mathrm{Ext}^1(\mathrm{A}_1, B)$ vanishes, as well as sections $a \in \mathrm{A}_3$ and $b \in \mathrm{Hom}(\mathrm{A}_1, B)$, we obtain a pair of B-torsors from:

(1) the A_1-torsor $\mathrm{A}_2 \times_{\mathrm{A}_3} \{a\}$ induced along b;
(2) the $\mathrm{Hom}(\mathrm{A}_3, B)$-torsor $\mathrm{Hom}(\mathrm{A}_2, B) \times_{\mathrm{Hom}(\mathrm{A}_1, B)} \{b\}$ induced along a.

These B-torsors are canonically *inverse* to one another via the map:

$$\operatorname{Hom}(A_2, B) \times_{\operatorname{Hom}(A_1, B)} \{b\} \to (A_2 \times_{A_3} \{a\}) \times^{A_1} B, \quad \varphi \mapsto (a_2, -\varphi(a_2)), \tag{3.14}$$

where $a_2 \in A_2$ is any lift of a.

Note that the isomorphism of B-torsors defined by (3.14) is additive in $b \in B$. Namely, for $b_1, b_2 \in B$, the B-torsors corresponding to $b_1 + b_2$ in the constructions (1) and (2) are naturally identified with the sum of B-torsors corresponding to b_1 respectively b_2. Under these identifications, the isomorphism defined by (3.14) for $b_1 + b_2$ equals the sum of those isomorphisms for b_1 and b_2.

Proof of Lemma 3.2.4. We apply §3.2.5 to the short exact sequence (3.5) for $A = \Gamma$ and $B = \mathbb{G}_m$. This gives the isomorphism $T_b : B(b) \circ \mathbf{H} \cong \mathbf{Y}_{Q(b)}$.

It remains to verify that $2 \cdot T_b$ is the identity with respect to the 2-torsion structures of $\mathbf{H}$ and $\mathbf{Y}_{Q(b)}$. Since the map (3.14) is additive in b, the statement reduces to comparing the 2-torsion structures on (3.10) and (3.11). Consider $b_1 \in \operatorname{Sym}^2(\Gamma, \mathbb{G}_m)$. The lift of $2b_1$ to $\operatorname{Hom}(H^{(1)}(\Gamma), \mathbb{G}_m)$ is defined by the 1-cocycle $\Gamma \to \operatorname{Sym}^2(\Gamma)$, $a \mapsto a \otimes a$, so its restriction along the set-theoretic section $\Gamma \subset H^{(1)}(\Gamma)$ is $a \mapsto b_1(a \otimes a)$. This equals the canonical quadratic refinement of $2b_1$. $\qquad\square$

Remark 3.2.6. For a section τ of $B(\Gamma)$, the isomorphism (3.13) defines an isomorphism between sections $b(\tau \cup \tau) \cong \mathbf{Y}_{Q(b)}(\tau)$ of $B^{(2)}(\mathbb{G}_m)$. For Γ being the 2-torsion points of an abelian scheme A and b the Weil pairing induced from a self-dual morphism $A \to \hat{A}$, we recover [PR11, Theorem 3.4] on the level of cohomology.

3.2.7. Suppose that $a \in \Gamma$ with $a^N = 1$. Evaluation at a defines a homomorphism $\check{\Gamma} \to \mu_N$ as well as a homomorphism $\check{H}^{(1)}(\Gamma, \mathbb{G}_m) \to \mathbb{G}_m$, $Q \mapsto Q(a)$. Together, they define a map of short exact sequences of fppf sheaves of abelian groups:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \check{\Gamma} & \longrightarrow & \check{H}^{(1)}(\Gamma, \mathbb{G}_m) & \longrightarrow & \operatorname{Sym}^2(\Gamma, \mathbb{G}_m) & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle x \mapsto x(a)} & & \downarrow{\scriptstyle Q \mapsto Q(a)} & & \downarrow{\scriptstyle b \mapsto b(a \otimes a)^{\binom{N}{2}}} & & \\
1 & \longrightarrow & \mu_N & \longrightarrow & \mathbb{G}_m & \xrightarrow{\ \ N\ \ } & \mathbb{G}_m & \longrightarrow & 1
\end{array} \tag{3.15}$$

Since $a^N = 1$, the right square commutes by the binomial theorem:

$$Q(a)^N = b(a \otimes a)^{\binom{N}{2}} Q(a^N). \tag{3.16}$$

3.3. Pairings valued in μ_N

3.3.1. We reinstate the assumption that S is a $\mathbb{Z}[\frac{1}{N}]$-scheme. Let Γ be a finite locally free abelian group scheme over S which is N-torsion. We shall produce an isomorphism analogous to (3.13) in the mapping spectrum $\operatorname{Maps}_{\mathbb{Z}}(\Gamma, B\mu_N)$. The essential difference from the $\mathbb{G}_m$ case is that $\operatorname{Ext}^1(\Gamma, \mu_N)$ does not vanish, so the morphism induced from (3.5):

$$\operatorname{Hom}(H^{(1)}(\Gamma), \mu_N) \to \operatorname{Sym}^2(\Gamma, \mu_N) \tag{3.17}$$

is not surjective as étale sheaves. Its cofiber is canonically identified with $\operatorname{Hom}(\Gamma, B\mu_N)$.

3.3.2. The short exact sequence (3.10), combined with the Kummer exact sequence, shows that the cofiber of (3.17) is also identified with $B(\check{\Gamma}) \times \check{\Gamma}$. Let us describe the isomorphism relating these presentations of the cofiber via its two factors:

$$B(\check{\Gamma}) \times \check{\Gamma} \cong \operatorname{Hom}(\Gamma, B\mu_N).$$

(1) $\mathrm{B}(\check{\Gamma}) \to \mathrm{Hom}(\Gamma, \mathrm{B}\mu_\mathrm{N})$ sends $\tau \in \mathrm{B}(\check{\Gamma})$ to the Yoneda pairing $\mathbf{Y}_\tau$;
(2) $\check{\Gamma} \to \mathrm{Hom}(\Gamma, \mathrm{B}\mu_\mathrm{N})$ is the composition of $\check{\Gamma} \times \Gamma \to \mathbb{G}_m$ with $\Psi : \mathbb{G}_m \to \mathrm{B}\mu_\mathrm{N}$.

The second morphism admits an equivalent description in terms of the Bockstein extension μ_{N^2} of μ_N by itself, which we view via its coboundary $\mathbf{B} : \mu_\mathrm{N} \to \mathrm{B}(\mu_\mathrm{N})$. Since the pairing $\check{\Gamma} \times \Gamma \to \mathbb{G}_m$ is valued in μ_N, the second morphism sends $x \in \check{\Gamma}$ to the composition of $\mathbf{B}$ with $x : \Gamma \to \mu_\mathrm{N}$, which we call the *Bockstein pairing* with x and write it as:

$$\mathbf{B}_x : \Gamma \to \mathrm{B}(\mu_\mathrm{N}). \tag{3.18}$$

3.3.3. Denote by $\check{\mathrm{H}}^{(1)}(\Gamma, \mu_\mathrm{N})$ the étale sheaf of μ_N-valued quadratic functions on Γ, defined by the conditions of §3.2.2. The construction of *loc.cit.* realizes the canonical morphism:

$$\check{\mathrm{H}}^{(1)}(\Gamma, \mu_\mathrm{N}) \to \mathrm{Sym}^2(\Gamma, \mu_\mathrm{N}) \tag{3.19}$$

as the negation of (3.17). The induced morphism to its cofiber $\mathrm{Sym}^2(\Gamma, \mu_\mathrm{N}) \to \mathrm{B}(\check{\Gamma}) \times \check{\Gamma}$ may also be described via its two factors $\mathcal{Q}_1$, $\mathcal{Q}_2$ valued in $\mathrm{B}(\check{\Gamma})$, respectively $\check{\Gamma}$.

Lemma 3.3.4. *The morphisms $\mathcal{Q}_1$, $\mathcal{Q}_2$ admit the following descriptions:*

(1) *$\mathcal{Q}_1(b)$ is the $\check{\Gamma}$-torsor of quadratic refinements of the $\mathbb{G}_m$-valued pairing defined by b, i.e., the image of b under (3.12);*
(2) *$\mathcal{Q}_2(b)$ is the homomorphism $\Gamma \to \mathbb{G}_m$, $a \mapsto b(a \otimes a)^{\binom{\mathrm{N}}{2}}$.*

Proof. Both statements follow from combining (3.11) with the Kummer exact sequence. To obtain the formula in part (2), we note that the value of $\mathcal{Q}_2(b)$ at $a \in \Gamma$ is given by $\mathrm{Q}(a)^\mathrm{N}$ for any quadratic refinement Q of b, viewed as a $\mathbb{G}_m$-valued symmetric bilinear form. We conclude by the binomial theorem (3.16). $\square$

3.3.5. Lemma 3.2.4 determines a canonical isomorphism in $\mathrm{Maps}_\mathbb{Z}(\Gamma, \mathrm{B}(\mu_\mathrm{N}))$, which we still denote by T_b:

$$\mathrm{T}_b : \mathrm{B}(b) \circ \mathbf{H} \cong \mathbf{Y}_{\mathcal{Q}_1(b)} + \mathbf{B}_{\mathcal{Q}_2(b)}. \tag{3.20}$$

The right-hand-side denotes the sum of the Yoneda pairing with $\mathcal{Q}_1(b)$ and the Bockstein pairing with $\mathcal{Q}_2(b)$. Furthermore, $2 \cdot \mathrm{T}_b$ is the identity with respect to the 2-torsion structures on $\mathbf{H}$, $\mathbf{Y}_{\mathcal{Q}_1(b)}$, and $\mathbf{B}_{\mathcal{Q}_2(b)}$.

3.4. Construction of T

3.4.1. We shall now construct the 2-isomorphism T demanded by Proposition 3.1.7. To start, we observe that when S contains all Nth roots of unity, their product satisfies:

$$\prod_{\zeta^\mathrm{N}=1} \zeta = \begin{cases} 1 & \mathrm{N}\ \text{odd} \\ -1 & \mathrm{N}\ \text{even} \end{cases}$$

In particular, η is canonically identified with the μ_N-torsor of Nth roots of unity of $\prod_{\zeta^\mathrm{N}=1} \zeta$. For a fixed primitive root of unity ζ, this product may also be written as $\zeta^{\binom{\mathrm{N}}{2}}$.

3.4.2. Étale locally on S, we may choose a perfect pairing $b : \mu_\mathrm{N}^{\otimes 2} \cong \mu_\mathrm{N}$. Such b is equivalent to the choice of a primitive Nth root of unity ζ, related to b by $b(\zeta \otimes \zeta) = \zeta$.

From the 2-isomorphism T_b in (3.20) applied to $\Gamma = \mu_\mathrm{N}$, we obtain an isomorphism in the connective $\mathbb{Z}$-module spectrum $\mathrm{Maps}_\mathbb{Z}(\mathbb{G}_m, \mathrm{B}^{(2)}\mu_\mathrm{N})$:

$$\mathrm{T}_b : \mathrm{B}^{(2)}(b) \circ \mathrm{B}(\mathbf{H}) \circ \Psi \cong \mathrm{B}(\mathbf{Y}_{\mathcal{Q}_1(b)}) \circ \Psi + \mathrm{B}(\mathbf{B}_{\mathcal{Q}_2(b)}) \circ \Psi \tag{3.21}$$

Furthermore, $2 \cdot \mathrm{T}_b$ is canonically isomorphic to the identity with respect to the 2-torsion structures on $\mathrm{B}(\mathbf{H})$, $\mathrm{B}(\mathbf{Y}_{\mathcal{Q}_1(b)})$, and $\mathrm{B}(\mathbf{B}_{\mathcal{Q}_2(b)})$.

Lemma 3.4.3. *There are canonical isomorphisms:*

(1) $\mathrm{B}(\mathbf{Y}_{\mathcal{Q}_1(b)}) \cong \mathrm{B}^{(2)}(b) \circ \mathrm{B}(\mathbf{Y}_\eta)$ *in* $\mathrm{Maps}_{\mathbb{Z}}(\mathrm{B}\mu_{\mathrm{N}}, \mathrm{B}^{(2)}\mu_{\mathrm{N}})$;

(2) $\mathrm{B}(\mathbf{B}_{\mathcal{Q}_2(b)}) \circ \Psi \cong * $ *in* $\mathrm{Maps}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^{(2)}\mu_{\mathrm{N}})$.

Furthermore, their squares are canonically isomorphic to the identity with respect to the 2-torsion structures on $\mathrm{B}(\mathbf{Y}_{\mathcal{Q}_1(b)})$, $\mathrm{B}(\mathbf{Y}_\eta)$, *and* $\mathrm{B}(\mathbf{B}_{\mathcal{Q}_2(b)})$.

Proof. Since μ_{N} plays two roles in the proof, we denote it also by Γ. From (3.15), we obtain a commutative diagram for any $a \in \Gamma$:

$$
\begin{array}{ccc}
\mathrm{Sym}^2(\Gamma, \mathbb{G}_m) & \xrightarrow{\mathcal{Q}} & \mathrm{B}(\check{\Gamma}) \\
\downarrow{\scriptstyle b \mapsto b(a \otimes a)^{\binom{\mathrm{N}}{2}}} & & \downarrow{\scriptstyle x \mapsto x(a)} \\
\mathbb{G}_m & \xrightarrow{\Psi} & \mathrm{B}(\mu_{\mathrm{N}})
\end{array}
$$

Let us view b as a section of $\mathrm{Sym}^2(\Gamma, \mathbb{G}_m)$, so its image under the upper-right circuit yields $\mathbf{Y}_{\mathcal{Q}_1(b)}(a)$ (Lemma 3.3.4(1)). Claim: its image under the lower-left circuit yields $\mathrm{B}(b) \circ \mathbf{Y}_\eta(a)$. Indeed, the identity $b(\zeta \otimes \zeta)^{\binom{\mathrm{N}}{2}} = \zeta^{\binom{\mathrm{N}}{2}}$ implies that, writing $a = \zeta^k$, the element $b(a \otimes a)^{\binom{\mathrm{N}}{2}}$ is the kth multiple of the 2-torsion element $\zeta^{\binom{\mathrm{N}}{2}} \in \mathbb{G}_m$. Thus its induced μ_{N}-torsor is the kth multiple of η (§3.4.1). This defines the isomorphism $\mathbf{Y}_{\mathcal{Q}_1(b)} \cong \mathrm{B}(b) \circ \mathbf{Y}_\eta$ in $\mathrm{Maps}_{\mathbb{Z}}(\Gamma, \mathrm{B}\mu_{\mathrm{N}})$ compatible with the 2-torsion structures. Its image under B gives (1).

The trivialization of $\mathrm{B}(\mathbf{B}_{\mathcal{Q}_2(b)}) \circ \Psi$ is constructed as follows. By construction, it is the composition in the lower row of the diagram:

$$
\begin{array}{ccccccc}
\mathbb{G}_m & \xrightarrow{\Psi'} & \mathrm{B}(\mu_{\mathrm{N}^2}) & \xrightarrow{\binom{\mathrm{N}}{2}} & \mathrm{B}(\mu_{\mathrm{N}^2}) & \longrightarrow & 1 \\
\downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow & & \downarrow \\
\mathbb{G}_m & \xrightarrow{\Psi} & \mathrm{B}(\mu_{\mathrm{N}}) & \xrightarrow{\mathcal{Q}_2(b)} & \mathrm{B}(\mu_{\mathrm{N}}) & \xrightarrow{\mathrm{B}(\mathbf{B})} & \mathrm{B}^{(2)}(\mu_{\mathrm{N}})
\end{array}
\tag{3.22}
$$

Here, the two middle vertical arrows are the Nth power map $\mu_{\mathrm{N}^2} \to \mu_{\mathrm{N}}$. The morphism Ψ' is the coboundary of the Kummer μ_{N^2}-torsor. To see that the middle diagram commutes, note that by Lemma 3.3.4(2), $\mathcal{Q}_2(b)$ sends ζ to $b(\zeta \otimes \zeta)^{\binom{\mathrm{N}}{2}} = \zeta^{\binom{\mathrm{N}}{2}}$, so any $a = \zeta^k \in \mu_{\mathrm{N}}$ is mapped to $a^{\binom{\mathrm{N}}{2}}$. The upper composition of (3.22) then defines a trivialization:

$$
\mathrm{R}_b : \mathrm{B}(\mathbf{B}_{\mathcal{Q}_2(b)}) \circ \Psi \cong *.
$$

The isomorphism $2 \cdot \mathrm{R}_b \cong *$ is an additional piece of structure: $2 \cdot \mathrm{R}_b$ and $2\mathcal{Q}_2(b) = 1$ define *a priori* distinct trivializations of $\mathrm{B}(\mathbf{B}_{2\mathcal{Q}_2(b)}) \circ \Psi$. To compare them, we note that $2 \cdot \mathrm{R}_b$ is the trivialization defined by replacing $\binom{\mathrm{N}}{2}$ in the upper row of (3.22) by $2 \cdot \binom{\mathrm{N}}{2}$. The 2-isomorphism of these trivializations comes from the commutativity of the middle and left squares with $\binom{\mathrm{N}}{2}$ replaced by $2 \cdot \binom{\mathrm{N}}{2}$ and $\mathcal{Q}_2(b)$ replaced by 1. $\qquad\square$

3.4.4. Combining (3.21) with the isomorphisms supplied by Lemma 3.4.3, we arrive at an isomorphism in $\mathrm{Maps}_{\mathbb{Z}}(\mathbb{G}_m, \mathrm{B}^{(2)}\mu_{\mathrm{N}})$:

$$
\mathrm{T}_b : \mathrm{B}^{(2)}(b) \circ \mathrm{B}(\mathbf{H}) \circ \Psi \cong \mathrm{B}^{(2)}(b) \circ \mathrm{B}(\mathbf{Y}_\eta) \circ \Psi.
\tag{3.23}
$$

Then $b^{-1}\mathrm{T}_b := \mathrm{B}^{(2)}(b^{-1}) \circ \mathrm{T}_b$ defines an isomorphism:

$$
b^{-1}\mathrm{T}_b : \mathrm{B}(\mathbf{H}) \circ \Psi \cong \mathrm{B}(\mathbf{Y}_\eta) \circ \Psi,
$$

i.e., it renders diagram (3.9) commutative. The statement on compatibility with the 2-torsion structures follows from the analogous statement for T_b. To conclude the construction of T, it remains to address the independence of $b^{-1}\mathrm{T}_b$ on the perfect pairing b.

Lemma 3.4.5. *The isomorphism $b^{-1}\mathrm{T}_b$ is canonically independent of b.*

Proof. For two perfect pairings $b_1, b_2 : \mu_\mathrm{N}^{\otimes 2} \cong \mu_\mathrm{N}$, we need to supply a 2-isomorphism $\mathrm{R}_{1,2}$ rendering the following diagram in $\mathrm{Maps}_\mathbb{Z}(\mathbb{G}_m, \mathrm{B}^{(2)}(\mu_\mathrm{N}))$ commutative:

$$
\begin{array}{ccc}
\mathrm{B}(\mathbf{H}) \circ \Psi & \xrightarrow{b_1^{-1}\mathrm{T}_{b_1}} & \mathrm{B}(\mathbf{Y}_\eta) \circ \Psi \\
\Big\downarrow{\cong} & \overset{\mathrm{R}_{1,2}}{\diagup\diagup} & \Big\downarrow{\cong} \\
\mathrm{B}(\mathbf{H}) \circ \Psi & \xrightarrow[b_2^{-1}\mathrm{T}_{b_2}]{} & \mathrm{B}(\mathbf{Y}_\eta) \circ \Psi
\end{array}
$$

In addition, for perfect pairings b_1, b_2, b_3, the equality $\mathrm{R}_{2,3}\mathrm{R}_{1,2} = \mathrm{R}_{1,3}$ must hold.

We construct $\mathrm{R}_{1,2}$ using the additive dependence of T_b on b, observed in §3.2.5 and carried over throughout the construction. Indeed, $b_2 = k \cdot b_1$ for a unique $k \in (\mathbb{Z}/\mathrm{N})^\times$. The 2-isomorphism T_{b_2} is canonically identified with $k \cdot \mathrm{T}_{b_1}$ when $\mathrm{B}^{(2)}(b_2) \circ \mathrm{B}(\mathbf{H}) \circ \Psi$ (resp. $\mathrm{B}^{(2)}(b_2) \circ \mathrm{B}(\mathbf{Y}_\eta) \circ \Psi$) is identified with the kth multiple of $\mathrm{B}^{(2)}(b_1) \circ \mathrm{B}(\mathbf{H}) \circ \Psi$ (resp. $\mathrm{B}^{(2)}(b_1) \circ \mathrm{B}(\mathbf{Y}_\eta) \circ \Psi$). The existence of $\mathrm{R}_{1,2}$ thus follows. $\qquad\square$

4. Theta-data of étale cochains

The goal of this section is to make a construction in linear algebra. It takes as input a sheaf of finite, locally free $\mathbb{Z}$-modules Λ and produces a sheaf of connective $\mathbb{Z}/\mathrm{N}$-module spectrum. In fact, we make two such constructions, to be denoted by $\vartheta^{(1)}(\Lambda)$ and $\vartheta^{(2)}(\Lambda)$ (§4.2). Their purposes will become clear in §5.1.

The last subsection §4.3 studies the $\mathbb{E}_\infty$-structure on a piece of data involved in $\vartheta^{(2)}(\Lambda)$. It will be used in §6 when we define the metaplectic dual.

4.1. Linear algebra

4.1.1. Let $\check\Lambda$ be a finite free $\mathbb{Z}$-module. The permutation group Σ_2 acts on $\check\Lambda \otimes \check\Lambda$ by exchanging its factors. Derived Σ_2-coinvariants $(\check\Lambda \otimes \check\Lambda)_{\Sigma_2}$ are computed by the complex:

$$
[\cdots \xrightarrow{\mathrm{Ant}} \check\Lambda \otimes \check\Lambda \xrightarrow{\mathrm{Sym}} \check\Lambda \otimes \check\Lambda \xrightarrow{\mathrm{Ant}} \check\Lambda \otimes \check\Lambda] \in \mathrm{Mod}_\mathbb{Z}^{\leq 0}. \tag{4.1}
$$

Here, Sym (resp. Ant) denotes the (anti-)symmetrizer sending $x_1 \otimes x_2$ to $x_1 \otimes x_2 + x_2 \otimes x_1$ (resp. $x_1 \otimes x_2 - x_2 \otimes x_1$). In H^0, we recover $\mathrm{Sym}^2(\check\Lambda)$. In H^{-1}, we find $\check\Lambda/2$ according to a commutative diagram of four short exact sequences:

$$
\begin{array}{ccc}
\mathrm{Sym}^2(\check\Lambda) & \xrightarrow{\cong} & \mathrm{Sym}^2(\check\Lambda) \\
\downarrow & & \downarrow \\
\Gamma^2(\check\Lambda) & \longrightarrow \check\Lambda \otimes \check\Lambda \longrightarrow & \wedge^2(\check\Lambda) \\
\downarrow & \downarrow & \downarrow{\cong} \\
\check\Lambda/2 & \longrightarrow \mathrm{Ant}^2(\check\Lambda) \longrightarrow & \wedge^2(\check\Lambda)
\end{array} \tag{4.2}
$$

Next, $\mathrm{H}^{-2} = 0$ since the sequence below is exact:

$$
0 \to \wedge^2(\check\Lambda) \to \check\Lambda \otimes \check\Lambda \to \mathrm{Sym}^2(\check\Lambda) \to 0. \tag{4.3}
$$

4.1.2. Let Λ denote the $\mathbb{Z}$-module dual to $\check{\Lambda}$. Items in (4.2) admit descriptions as integral "forms" on Λ. Bilinear forms are classified by $\check{\Lambda} \otimes \check{\Lambda}$ in the obvious way, $\wedge^2(\check{\Lambda})$ consists of alternating forms via the inclusion in (4.3), and $\Gamma^2(\check{\Lambda})$ consists of symmetric bilinear forms via the inclusion in (4.2).

Next, $\mathrm{Sym}^2(\check{\Lambda})$ consists of quadratic forms Q whose value at $\lambda \in \Lambda$ is given by $c(\lambda, \lambda)$ for any lift $c \in \check{\Lambda} \otimes \check{\Lambda}$ along (4.3). Finally, $\mathrm{Ant}^2(\check{\Lambda})$ consists of pairs (A, f) where A is an alternating form on Λ and $f : \Lambda \to \mathbb{Z}/2$ is a function satisfying:

$$f(\lambda_1 + \lambda_2) - f(\lambda_1) - f(\lambda_2) = A(\lambda_1, \lambda_2) \mod 2.$$

The maps $\mathrm{Sym}^2(\check{\Lambda}) \to \Gamma^2(\check{\Lambda})$, respectively $\mathrm{Ant}^2(\check{\Lambda}) \to \wedge^2(\check{\Lambda})$ in (4.2) encode the following operations: going from a quadratic form Q to its symmetric bilinear form b, or going from a pair (A, f) to its alternating form A.

Remark 4.1.3. We may also consider the Σ_2 action on $\check{\Lambda} \otimes \check{\Lambda}$ twisted by the sign character. The derived coinvariants are computed by a complex similar to (4.1), but which starts with Sym as the differential d^{-1}. In particular, we find $\mathrm{Ant}^2(\check{\Lambda})$ in H^0. Then we have $\mathrm{H}^{-1} = 0$, $\mathrm{H}^{-2} \cong \check{\Lambda}/2$, and the rest of the H^{-n} alternate between them.

The vanishing of H^{-1} has a practical consequence: given $A \in \mathrm{Mod}_{\mathbb{Z}}^{\geq -1}$, the mapping space of Σ_2-equivariant objects $\mathrm{Maps}_{\Sigma_2}(\check{\Lambda} \otimes \check{\Lambda}, A)$, *a priori* equivalent to $\mathrm{Maps}_{\mathbb{Z}}((\check{\Lambda} \otimes \check{\Lambda})_{\Sigma_2}, A)$ for the derived Σ_2-coinvariants, is actually identified with $\mathrm{Maps}_{\mathbb{Z}}(\mathrm{Ant}^2(\check{\Lambda}), A)$.

4.1.4. We shall also meet two kinds of "quadratic functions" on Λ, one of which has made an appearance in §3.2.2. Denote by $\mathrm{H}^{(1)}(\Lambda)$ the symmetrized Heisenberg extension of Λ by $\mathrm{Sym}^2(\Lambda)$ (see (3.5)). By restriction along the set-theoretic section $\Lambda \subset \mathrm{H}^{(1)}(\Lambda)$, elements of the dual $\check{\mathrm{H}}^{(1)}(\Lambda)$ are functions $Q : \Lambda \to \mathbb{Z}$ satisfying:

(1) $Q(0) = 0$;
(2) $Q(\lambda_1 + \lambda_2) - Q(\lambda_1) - Q(\lambda_2)$ defines a symmetric bilinear form on Λ.

Associating the symmetric bilinear form (2) to Q defines a homomorphism $\check{\mathrm{H}}^{(1)}(\Lambda) \to \Gamma^2(\check{\Lambda})$ which is *negative* to the dual of the natural inclusion $\mathrm{Sym}^2(\Lambda) \to \mathrm{H}^{(1)}(\Lambda)$.

4.1.5. Denote by $\mathrm{H}^{(2)}(\Lambda)$ the chain complex $[\Lambda \otimes \Lambda \to \mathrm{H}^{(1)}(\Lambda)]$ in cohomological degrees $[-1, 0]$. The differential is given by the projection $\Lambda \otimes \Lambda \to \mathrm{Sym}^2(\Lambda)$ followed by the natural inclusion. The complex $\mathrm{H}^{(2)}(\Lambda)$ fits into a canonical triangle:

$$\wedge^2(\Lambda)[1] \to \mathrm{H}^{(2)}(\Lambda) \to \Lambda. \tag{4.4}$$

This is the alternatrized "second Heisenberg extension." Its dual $\check{\mathrm{H}}^{(2)}(\Lambda)$ is a complex in cohomological degrees $[0, 1]$.

Remark 4.1.6. (1) The strictly commutative Picard groupoid corresponding to $\mathrm{H}^{(2)}(\Lambda)$ (§1.1.1) is equivalent to the following one. It is $\Lambda \times B(\wedge^2(\Lambda))$ as a monoidal groupoid but the commutativity constraint is multiplication by $\lambda_1 \wedge \lambda_2$ for objects $\lambda_1, \lambda_2 \in \Lambda$. In particular, the projection $\mathrm{H}^{(2)}(\Lambda) \to \Lambda$ admits an $\mathbb{E}_1$-monoidal section.

(2) The strictly commutative Picard groupoid corresponding to the shifted dual:

$$\check{\mathrm{H}}^{(2)}(\Lambda)[1] \cong \mathrm{Hom}(\mathrm{H}^{(2)}(\Lambda), \mathbb{Z}[1]) \in \mathrm{Mod}_{\mathbb{Z}}^{[-1,0]}$$

may also be described explicitly. Restriction along (4.4) defines an alternating form $b : \wedge^2(\Lambda) \to \mathbb{Z}$. By restriction along the $\mathbb{E}_1$-monoidal section $\Lambda \to \mathrm{H}^{(2)}(\Lambda)$, objects of $\mathrm{Hom}(\mathrm{H}^{(2)}(\Lambda), \mathbb{Z}[1])$ are equivalent to such b together with a central extension of Λ

by $\mathbb{Z}$ with commutator b. Maps out of $\mathrm{H}^{(2)}(\Lambda)$ are the "quadratic functions of the second kind."

(3) The central extension (4.4) plays a similar role as (3.5) for a sheaf of abelian groups A on a site $\mathcal{C}$. Namely, the analogously defined extension $\mathrm{H}^{(2)}(\mathrm{A})$ of A by $\wedge^2(\mathrm{A})[1]$ gives rise to a coboundary map $\mathrm{B}^{(2)}(\mathrm{A}) \to \mathrm{B}^{(4)}(\wedge^2 \mathrm{A})$. It categorifies the alternatrized self-cup product of a cohomology class in $\mathrm{H}^2(c; \mathrm{A})$ for any $c \in \mathcal{C}$.

4.1.7. The $\mathbb{Z}$-module $\check{\Lambda}$ has the natural coalgebra structure with respect to the Cartesian symmetric monoidal structure on $\mathrm{Mod}_{\mathbb{Z}}$. It is defined by the cosimplicial object $[n] \mapsto \check{\Lambda}^{\oplus n}$, whose coface maps are insertions and whose degeneracy maps are projections. We display the maps between $\check{\Lambda}$ and $\check{\Lambda}^{\oplus 2}$ and leave the rest as an exercise in pattern recognition:

$$d_i^2(x) = \begin{cases} (0, x) & i = 0 \\ (x, x) & i = 1 \\ (x, 0) & i = 2 \end{cases} \text{ and } s_i^1(x_0, x_1) = \begin{cases} x_1 & i = 0 \\ x_0 & i = 1 \end{cases}$$

Functoriality of the constructions $\check{\Lambda} \mapsto \mathrm{F}(\check{\Lambda}) \coloneqq \check{\Lambda} \otimes \check{\Lambda}$, $\Gamma^2(\check{\Lambda})$, $\wedge^2(\check{\Lambda})$, $\mathrm{Sym}^2(\check{\Lambda})$, $\mathrm{Ant}^2(\check{\Lambda})$, $\check{\mathrm{H}}^{(1)}(\Lambda)$, $\check{\mathrm{H}}^{(2)}(\Lambda)$ determine cosimplicial objects $[n] \mapsto \mathrm{F}(\check{\Lambda}^{\oplus n})$ of $\mathrm{Mod}_{\mathbb{Z}}$.

Lemma 4.1.8. *There are canonical isomorphisms in* $\mathrm{Mod}_{\mathbb{Z}}$:

(1) $\lim_{[n]}(\check{\Lambda}^{\oplus n}) \cong \check{\Lambda}[-1]$;

(2) $\lim_{[n]}(\check{\Lambda}^{\oplus n} \otimes \check{\Lambda}^{\oplus n}) \cong \check{\Lambda} \otimes \check{\Lambda}[-2]$;

(3) $\lim_{[n]} \mathrm{F}_1(\check{\Lambda}^{\oplus n}) \cong \mathrm{F}_2(\check{\Lambda})[-2]$ *for all functors notated* $\mathrm{F}_1 \Rightarrow \mathrm{F}_2$ *below:*

$$\Gamma^2 \Rightarrow \wedge^2 \Rightarrow \mathrm{Sym}^2 \Rightarrow \mathrm{Ant}^2;$$

(4) $\lim_{[n]}(\check{\mathrm{H}}^{(1)}(\Lambda^{\oplus n})) \cong \check{\mathrm{H}}^{(2)}(\Lambda)[-1]$.

Proof. The cosimplicial limit of $\mathrm{F}(\check{\Lambda}^{\oplus n})$ is equivalent to the chain complex:

$$[0 \xrightarrow{d} \mathrm{F}(\check{\Lambda}) \xrightarrow{d} \mathrm{F}(\check{\Lambda}^{\oplus 2}) \xrightarrow{d} \cdots] \in \mathrm{Mod}_{\mathbb{Z}}^{\geq 0} \tag{4.5}$$

where d denotes the alternating sum of the face maps. Inclusion of the subcomplex $\bigcap \ker(s_i)$ of nondegenerate cochains is a homotopy equivalence. This yields (1) and (2).

For (3) with $\mathrm{F} = \Gamma^2$, $\wedge^2$, or Sym^2, the essential observation is that nondegenerate n-cochains vanish for $n \geq 3$ and nondegenerate 2-cochains are equivalent to $\check{\Lambda} \otimes \check{\Lambda}$. The differential it receives from nondegenerate 1-cochains $d : \mathrm{F}(\check{\Lambda}) \to \check{\Lambda} \otimes \check{\Lambda}$ is the natural inclusion, so we conclude by the short exact sequences in (4.2) and (4.3) (c.f. [BD01, §3.9]).

For (4), we identify elements of $\check{\mathrm{H}}^{(1)}(\Lambda^{\oplus n})$ with quadratic functions $\mathrm{Q}^{(n)} : \Lambda^{\oplus n} \to \mathbb{Z}$ (see §4.1.4). Thus nondegenerate cochains for $n \geq 3$ vanish and are isomorphic to $\check{\Lambda} \otimes \check{\Lambda}$ for $n = 2$: those $\mathrm{Q}^{(2)}$ which vanish on $\Lambda \oplus 0$ and $0 \oplus \Lambda$ are uniquely determined by the pairing between $\Lambda \oplus 0$ and $0 \oplus \Lambda$, which can be arbitrary. The nondegenerate cochains form the complex:

$$[\check{\mathrm{H}}^{(1)}(\Lambda) \xrightarrow{d} \check{\Lambda} \otimes \check{\Lambda}] \in \mathrm{Mod}_{\mathbb{Z}}^{[1,2]},$$

where d takes $\mathrm{Q}^{(1)}$ to the bilinear form $\lambda_1, \lambda_2 \mapsto \mathrm{Q}^{(1)}(\lambda_1) + \mathrm{Q}^{(1)}(\lambda_2) - \mathrm{Q}^{(1)}(\lambda_1 + \lambda_2)$ on Λ. Taking the negative sign in §4.1.4 into account, this is precisely $\check{\mathrm{H}}^{(2)}(\Lambda)[-1]$. $\qquad\square$

4.1.9. Finally, we mention that the constructions of this subsection naturally generalize to a sheaf-theoretic setting. Namely, for a site $\mathcal{C}$ and a sheaf $\check{\Lambda}$ of finite locally free $\mathbb{Z}$-modules, the constructions $\check{\Lambda} \otimes \check{\Lambda}$, $\Gamma^2(\check{\Lambda})$, $\wedge^2(\check{\Lambda})$, $\mathrm{Sym}^2(\check{\Lambda})$, $\mathrm{Ant}^2(\check{\Lambda})$, $\check{\mathrm{H}}^{(1)}(\Lambda)$, $\check{\mathrm{H}}^{(2)}(\Lambda)$ make sense as objects of $\mathrm{Shv}(\mathcal{C}, \mathrm{Mod}_{\mathbb{Z}})$ and are related by (4.2), (4.3), and Lemma 4.1.8.

4.2. Étale ϑ-data

4.2.1. Let S be a $\mathbb{Z}[\frac{1}{N}]$-scheme. Consider the étale site $S_{\text{ét}}$ of affine S-schemes. Recall that η denotes the μ_N-torsor of Nth roots of (-1). The 2-torsion structure on η allows us to view it as a morphism in $\text{Shv}(S_{\text{ét}}, \text{Mod}_{\mathbb{Z}})$:

$$\eta : \mathbb{Z}/2 \to \mu_N[1]. \tag{4.6}$$

For an étale sheaf of finite locally free $\mathbb{Z}$-module Λ (with dual $\check{\Lambda}$), we denote by $\vartheta^{(1)}(\Lambda)$ the pushout of the following diagram in $\text{Shv}(S_{\text{ét}}, \text{Mod}_{\mathbb{Z}/N})$:

$$\check{\Lambda} \otimes \mu_N[1] \xleftarrow{\check{\Lambda} \otimes \eta} \Gamma^2(\check{\Lambda})/N \longrightarrow (\check{\Lambda} \otimes \check{\Lambda})/N.$$

The right arrow is the natural inclusion. The left arrow is equivalently described, via adjunction, by a morphism $\Gamma^2(\check{\Lambda}) \to \check{\Lambda} \otimes \mu_N[1]$ in $\text{Shv}(S_{\text{ét}}, \text{Mod}_{\mathbb{Z}})$. As such, it is the composition of the projection $\Gamma^2(\check{\Lambda}) \to \check{\Lambda}/2$ with $\check{\Lambda} \otimes \eta$.

4.2.2. We shall write $\vartheta^{(1)}(\Lambda)$ as a pullback in $\text{Shv}(S_{\text{ét}}, \text{Mod}_{\mathbb{Z}/N})$. The canonical projection $(\check{\Lambda} \otimes \check{\Lambda})/N \to \wedge^2(\check{\Lambda})/N$ vanishes on $\Gamma^2(\check{\Lambda})/N$, so it determines a map:

$$\vartheta^{(1)}(\Lambda) \to \wedge^2(\check{\Lambda})/N. \tag{4.7}$$

There is also a natural map defined by the push-out description of $\vartheta^{(1)}(\Lambda)$:

$$\vartheta^{(1)}(\Lambda) \to \check{H}^{(1)}(\Lambda) \otimes \mu_N[1], \tag{4.8}$$

   (1) $\check{\Lambda} \otimes \check{\Lambda} \to \check{H}^{(1)}(\Lambda) \otimes \mu_N[1]$ is the composition of $\check{\Lambda} \otimes \check{\Lambda} \to \check{H}^{(1)}(\Lambda)/2$, sending a bilinear form c to the $\mathbb{Z}/2$-valued quadratic function $\lambda \mapsto c(\lambda, \lambda) \mod 2$, with $\check{H}^{(1)}(\Lambda) \otimes \eta$;
   (2) $\check{\Lambda} \otimes \mu_N[1] \to \check{H}^{(1)}(\Lambda) \otimes \mu_N[1]$ is the natural inclusion tensored with $\mu_N[1]$;
   (3) when restricted to $\Gamma^2(\check{\Lambda})$, the first map factors through $\check{\Lambda}/2 \subset \check{H}^{(1)}(\Lambda)/2$, followed by $\check{\Lambda} \otimes \eta$.

4.2.3. The morphisms (4.7) and (4.8) are naturally identified after mapping both to $\Gamma^2(\check{\Lambda}) \otimes \mu_N[1]$. Let us explain how. For $\check{H}^{(1)}(\Lambda) \otimes \mu_N[1]$, the map to $\Gamma^2(\check{\Lambda}) \otimes \mu_N[1]$ is the natural projection tensored with $\mu_N[1]$. For $\wedge^2(\check{\Lambda})/N$, the morphism $\wedge^2(\check{\Lambda}) \to \Gamma^2(\check{\Lambda})/2$, sending an alternating form c to the symmetric form $c \mod 2$, may be composed with $\Gamma^2(\check{\Lambda}) \otimes \eta$ to yield a map to $\Gamma^2(\check{\Lambda}) \otimes \mu_N[1]$.

The desired identification concerns the pre-compositions with (1) and (2), with a compatibility condition against (3). For (2), both compositions to $\Gamma^2(\check{\Lambda}) \otimes \mu_N[1]$ are canonically trivialized. For (1), the $\mathbb{Z}/2$-valued quadratic form $\lambda \mapsto c(\lambda, \lambda) \mod 2$ has symmetric form defined by the image of c in $\wedge^2(\check{\Lambda})$.

Lemma 4.2.4. *In the following commutative diagram in* $\text{Shv}(S_{\text{ét}}, \text{Mod}_{\mathbb{Z}/N})$:

$$
\begin{array}{ccc}
\Gamma^2(\check{\Lambda})/N & \longrightarrow & (\check{\Lambda} \otimes \check{\Lambda})/N \\
\downarrow{\scriptstyle \check{\Lambda} \otimes \eta} & & \downarrow \\
\check{\Lambda} \otimes \mu_N[1] \longrightarrow \vartheta^{(1)}(\Lambda) & \longrightarrow & \wedge^2(\check{\Lambda})/N \\
\downarrow & & \downarrow{\scriptstyle \Gamma^2(\check{\Lambda}) \otimes \eta} \\
\check{H}^{(1)}(\Lambda) \otimes \mu_N[1] & \longrightarrow & \Gamma^2(\check{\Lambda}) \otimes \mu_N[1]
\end{array}
\tag{4.9}
$$

   (1) *both squares are Cartesian (and co-Cartesian);*

(2) the middle horizontal sequence is canonically a triangle.

Proof. The top square is Cartesian by definition. For (2), we observe that the top morphism $\Gamma^2(\check{\Lambda})/N \to (\check{\Lambda} \otimes \check{\Lambda})/N$ has cofiber $\wedge^2(\check{\Lambda})/N$, identified with the rightmost term of the middle sequence along the given maps. For the bottom square, we observe that the bottom morphism $\check{H}^{(1)}(\Lambda) \otimes \mu_N[1] \to \Gamma^2(\check{\Lambda}) \otimes \mu_N[1]$ has fiber $\check{\Lambda} \otimes \mu_N[1]$, identified with the leftmost term of the middle sequence along the given maps. $\qquad\square$

4.2.5. Consider the cosimplicial system $[n] \mapsto \check{\Lambda}^{\oplus n}$ of §4.1.7. Define:

$$\vartheta^{(2)}(\Lambda) := \lim_{[n]} \vartheta^{(1)}(\Lambda^{\oplus n})[2],$$

where the limit is taken in $\mathrm{Shv}(S_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}/N})$. To describe $\vartheta^{(2)}(\Lambda)$ in explicit terms, we need to introduce two additional canonical morphisms defined by η:

(1) $\wedge^2(\check{\Lambda})/N \to \check{\Lambda} \otimes \mu_N[2]$ defined by the coboundary map $\wedge^2(\check{\Lambda}) \to \check{\Lambda}/2[1]$ (see $\mathrm{Ant}^2(\check{\Lambda})$ in (4.2)) followed by $\check{\Lambda} \otimes \eta$;

(2) $\mathrm{Sym}^2(\check{\Lambda})/N \to \wedge^2(\check{\Lambda}) \otimes \mu_N[1]$ defined by $\mathrm{Sym}^2(\check{\Lambda}) \to \wedge^2(\check{\Lambda})/2$, sending a quadratic form Q to the alternating form $\lambda_1, \lambda_2 \mapsto b(\lambda_1, \lambda_2) \mod 2$, followed by $\wedge^2(\check{\Lambda}) \otimes \eta$.

In (4.10) below, they are labeled by terms involving η, whereas the top and bottom morphisms arise from natural maps $\wedge^2(\check{\Lambda}) \to \check{\Lambda} \otimes \check{\Lambda}$ and $\check{H}^{(2)}(\Lambda)[1] \to \wedge^2(\check{\Lambda})$ (see (4.4)).

Lemma 4.2.6. *There is a commutative diagram in* $\mathrm{Shv}(S_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}/N})$:

$$
\begin{array}{ccc}
\wedge^2(\check{\Lambda})/N & \longrightarrow & (\check{\Lambda} \otimes \check{\Lambda})/N \\
\downarrow{\scriptstyle \check{\Lambda} \otimes \eta} & & \downarrow \\
\check{\Lambda} \otimes \mu_N[2] & \longrightarrow \vartheta^{(2)}(\Lambda) \longrightarrow & \mathrm{Sym}^2(\check{\Lambda})/N \\
& \downarrow & \downarrow{\scriptstyle \wedge^2(\check{\Lambda}) \otimes \eta} \\
\check{H}^{(2)}(\Lambda) \otimes \mu_N[2] & \longrightarrow & \wedge^2(\check{\Lambda}) \otimes \mu_N[1]
\end{array}
\qquad (4.10)
$$

with the following properties:

(1) both squares are Cartesian (and co-Cartesian);

(2) the middle horizontal sequence is canonically a triangle.

Proof. Each term in (4.9) is functorially assigned to $\check{\Lambda}$. Taking the cosimplicial limit over $[n] \mapsto \check{\Lambda}^{\oplus n}$ and applying Lemma 4.1.8, we obtain (4.10) with the required properties in (1) and the structure in (2). $\qquad\square$

4.2.7. Let us view $\vartheta^{(2)}(\Lambda)$ as a Spc-valued sheaf (§1.2.2) and see what its sections consist of, using the bottom Cartesian square in (4.10). First, $\mathrm{Sym}^2(\check{\Lambda})/N$ consists of $\mathbb{Z}/N$-valued quadratic forms on Λ. The underlying Spc-valued sheaf of $\wedge^2(\check{\Lambda}) \otimes \mu_N[1]$ (resp. $\check{H}^{(2)}(\Lambda) \otimes \mu_N[2]$) is the sheaf of $\mathbb{Z}$-linear maps $\wedge^2(\Lambda) \to B\mu_N$ (resp. $H^{(2)}(\Lambda) \to B^{(2)}\mu_N$). Thus a section of $\vartheta^{(2)}(\Lambda)$ is a triple (Q, F, h) where:

(1) Q is a $\mathbb{Z}/N$-valued quadratic form on Λ;

(2) $F : H^{(2)}(\Lambda) \to B^{(2)}\mu_N$ is a $\mathbb{Z}$-linear morphism;

(3) h is an isomorphism between the map $\wedge^2(\Lambda) \to B\mu_N$ induced from Q and the restriction of F along $B(\wedge^2\Lambda) \to H^{(2)}(\Lambda)$ (see (4.4)).

Note that the restriction of F along the $\mathbb{E}_1$-monoidal section $\Lambda \to H^{(2)}(\Lambda)$ (Remark 4.1.6) defines an $\mathbb{E}_1$-monoidal morphism (a "quadratic function of the second kind"):

$$F_1 : \Lambda \to B^{(2)}(\mu_N). \tag{4.11}$$

4.3. $\mathbb{E}_\infty$-structure on F_1

4.3.1. We remain in the context of §4.2.1. Fix a sheaf of finite locally free $\mathbb{Z}$-modules Λ. Let (Q, F, h) be a section of $\vartheta^{(2)}(\Lambda)$ and F_1 the corresponding $\mathbb{E}_1$-monoidal morphism (4.11). From the middle triangle in (4.10), we see that if $Q = 0$, then F_1 is canonically a $\mathbb{Z}$-linear morphism. This subsection addresses what it takes for F_1 to have an $\mathbb{E}_\infty$-structure.

4.3.2. Denote by $E_\infty(\Lambda)$ the sheaf of symmetric monoidal groupoids whose underlying sheaf of monoidal groupoids is $\Lambda \times B(\text{Ant}^2(\Lambda))$ but the commutativity constraint is multiplication by the image of $\lambda_1 \otimes \lambda_2$ in $\text{Ant}^2(\Lambda)$, for objects $\lambda_1, \lambda_2 \in \Lambda$. By Remark 4.1.6(1), there is an $\mathbb{E}_\infty$-monoidal morphism $E_\infty(\Lambda) \to H^{(2)}(\Lambda)$.

Suppose that A is a sheaf of abelian groups and $G : E_\infty(\Lambda) \to B^{(2)}(A)$ is an $\mathbb{E}_\infty$-monoidal morphism. Restriction along the $\mathbb{E}_1$-monoidal section $\Lambda \to E_\infty(\Lambda)$ defines an $\mathbb{E}_1$-monoidal morphism:

$$G_1 : \Lambda \to B^{(2)}(A). \tag{4.12}$$

Denote by b the restriction of G to $B(\text{Ant}^2(\Lambda))$, viewed as a $\mathbb{Z}$-linear morphism $\text{Ant}^2(\Lambda) \to B(A)$. (Since the target is 1-truncated, $\mathbb{Z}$-linearity is equivalent to being $\mathbb{E}_\infty$-monoidal.)

Lemma 4.3.3. *The groupoid of null-homotopies of b is canonically equivalent to the groupoid of lifts of G_1 to an $\mathbb{E}_\infty$-monoidal morphism.*

Proof. The fiber $G_1^\dagger$ of (4.12) is a sheaf of monoidal groupoids. By Lemma 1.2.7 and Remark 1.2.8, an $\mathbb{E}_\infty$-structure on G_1 is equivalent to a lift of the monoidal structure on $G_1^\dagger$ to a symmetric monoidal structure. On the other hand, the groupoid of $\mathbb{Z}$-linear morphisms $\text{Ant}^2(\Lambda) \to B(A)$ is equivalent to the groupoid of $\mathbb{Z}$-linear morphisms $\Lambda \otimes \Lambda \to B(A)$ which are Σ_2-equivariant against the sign character (Remark 4.1.3).

For $\lambda \in \Lambda$, write $G(\lambda)$ for the fiber of $G_1^\dagger \to \Lambda$ at λ. Let $a_1 \in G(\lambda_1)$ and $a_2 \in G(\lambda_2)$. The $\mathbb{E}_\infty$-structure on G supplies an isomorphism of sections of $B(A)$:

$$\text{Maps}_{G_1^\dagger}(a_1 \otimes a_2, a_2 \otimes a_1) \cong b(\lambda_1, \lambda_2), \tag{4.13}$$

which is functorial in a_1, a_2 and is subject to the following conditions:

(1) for $a_i \in G(\lambda_i)$ $(i = 1, 2, 3)$, composition of isomorphisms in $G_1^\dagger$:

$$\text{Maps}_{G_1^\dagger}(a_1 \otimes a_2, a_2 \otimes a_1) \otimes a_3 + a_2 \otimes \text{Maps}_{G_1^\dagger}(a_1 \otimes a_3, a_3 \otimes a_1)$$
$$\cong \text{Maps}_{G_1^\dagger}(a_1 \otimes (a_2 \otimes a_3), (a_2 \otimes a_3) \otimes a_1)$$

passes to $b(\lambda_1, \lambda_2) + b(\lambda_1, \lambda_3) \cong b(\lambda_1, \lambda_2 + \lambda_3)$;

(2) for $a_i \in G(\lambda_i)$ $(i = 1, 2)$, inversion of isomorphisms in $G_1^\dagger$:

$$\text{Maps}_{G_1^\dagger}(a_1 \otimes a_2, a_2 \otimes a_1) \cong \text{Maps}_{G_1^\dagger}(a_2 \otimes a_1, a_1 \otimes a_2)$$

passes to $b(\lambda_1, \lambda_2) \cong b(\lambda_2, \lambda_1)$.

By (4.13), a null-homotopy of $b(\lambda_1, \lambda_2)$ for all $\lambda_1, \lambda_2 \in \Lambda$ is equivalent to a functorial choice of isomorphisms $a_1 \otimes a_2 \cong a_2 \otimes a_1$ for all $a_1, a_2 \in G_1^\dagger$. The condition that the null-homotopy of b be $\mathbb{Z}$-linear in the second variable is equivalent to the hexagon axiom. The condition that the null-homotopy of b be Σ_2-equivariant is equivalent to the inverse axiom. $\square$

Proposition 4.3.4. *Let* (Q, F, h) *be a section of* $\vartheta^{(2)}(\Lambda)$ *with corresponding* $\mathbb{E}_1$-*monoidal morphism* $F_1 : \Lambda \to B^{(2)}(\mu_N)$. *If the symmetric form* b *of* Q *vanishes, then* F_1 *has a canonical* $\mathbb{E}_\infty$-*monoidal structure.*

Proof. If $b = 0$, then h supplies a null-homotopy of the restriction of F to $\mathrm{Ant}^2(\Lambda)$ along the composition $\mathrm{Ant}^2(\Lambda) \to \wedge^2(\Lambda) \to H^{(2)}(\Lambda)$. This null-homotopy defines an $\mathbb{E}_\infty$-structure on F_1 by Lemma 4.3.3. $\qquad\square$

4.3.5. Supppose that Γ is a sheaf of abelian groups. The sheaf of $\mathbb{E}_\infty$-monoidal morphisms $\Gamma \to B^{(2)}(\mu_N)$ fits into a fiber sequence:

$$\mathrm{Maps}_{\mathbb{Z}}(\Gamma, B^{(2)}\mu_N) \to \mathrm{Maps}_{\mathbb{E}_\infty}(\Gamma, B^{(2)}\mu_N) \xrightarrow{\mathrm{inv}} \mathrm{Maps}_{\mathbb{Z}}(\Gamma/2, \mu_N). \tag{4.14}$$

The second morphism inv is defined as follows: regard an $\mathbb{E}_\infty$-morphism $G : \Gamma \to B^{(2)}(\mu_N)$ as a morphism of sheaves of symmetric monoidal categories $G^\dagger \to \Gamma$ with fiber $B(\mu_N)$, then to any $a \in G^\dagger$ we associate the commutativity constraint of $a \otimes a$, which determines a $\mathbb{Z}$-linear map $\Gamma/2 \to \mu_N$.

Lemma 4.3.6. *The morphism* inv *in* (4.14) *is a surjection of étale sheaves.*

Proof. Let us make an étale local identification $\mu_N \cong \mathbb{Z}/N$. There is nothing to prove if N is odd. If N is even, $\mathbb{Z}/N$ contains a unique nonzero 2-torsion element ϵ. We lift a homomorphism $f : \Gamma/2 \to \mathbb{Z}/N$ to a symmetric monoidal category whose underlying monoidal category is $\Gamma \times B(\mathbb{Z}/N)$ but whose commutativity constraint for $\lambda_1, \lambda_2 \in \Gamma$ is the multiplication by ϵ if both $f(\lambda_1) = \epsilon$ and $f(\lambda_2) = \epsilon$, and the identity otherwise. $\qquad\square$

4.3.7. We record a consequence of Lemma 4.3.6. Suppose that $\vartheta = (Q, F, h)$ is a section of $\vartheta^{(2)}(\Lambda)$ with corresponding $\mathbb{E}_1$-monoidal morphism $F_1 : \Lambda \to B^{(2)}(\mu_N)$. Assume $b = 0$, so F_1 upgrades to an $\mathbb{E}_\infty$-monoidal morphism (Proposition 4.3.4).

Suppose that $\Lambda_0 \subset \Lambda$ is a subsheaf of $\mathbb{Z}$-modules and φ is a trivialization of the restriction of ϑ to Λ_0. We claim that F_1 canonically factors through an $\mathbb{E}_\infty$-morphism:

$$\bar{F}_1 : \Lambda/\Lambda_0 \to B^{(2)}(\mu_N).$$

Indeed, the cofiber sequence (4.14) implies that $\mathrm{Maps}_{\mathbb{E}_\infty}(\Lambda/\Lambda_0, B^{(2)}\mu_N)$ is canonically identified with the fiber of the restriction map $\mathrm{Maps}_{\mathbb{E}_\infty}(\Lambda, B^{(2)}\mu_N) \to \mathrm{Maps}_{\mathbb{E}_\infty}(\Lambda_0, B^{(2)}\mu_N)$.

Remark 4.3.8. There is an $\mathbb{E}_2$-version of Lemma 4.3.3. Define $E_2(\Lambda)$ to be the sheaf of braided monoidal groupoids whose underlying sheaf of monoidal groupoids is $\Lambda \times B(\Lambda \otimes \Lambda)$ but the braiding is multiplication by $\lambda_1 \otimes \lambda_2$, for objects $\lambda_1, \lambda_2 \in \Lambda$. As before, an $\mathbb{E}_2$-monoidal morphism $G : E_2(\Lambda) \to B^{(2)}(A)$ restricts to an $\mathbb{E}_1$-monoidal morphism $G_1 : \Lambda \to B^{(2)}(A)$.

The corresponding assertion is that an $\mathbb{E}_2$-structure on G_1 is equivalent to a trivialization of the restriction of G to $B(\Lambda \otimes \Lambda)$, viewed as a $\mathbb{Z}$-linear morphism $b : \Lambda \otimes \Lambda \to B(A)$. Change in the proof: replace "Σ_2-equivariance" of b by "linearity in the first variable."

5. Classification

In this section, we classify metaplectic covers of a reductive group scheme G equipped with a Borel subgroup scheme. Namely, we describe the étale sheaf of connective $\mathbb{Z}/N$-module spectrum $\mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2})$ in terms of linear algebra. We first prove the classification result for a torus (Proposition 5.1.2), taking substantial input from §3 and §4. The subsection 5.2 which follows computes the étale cohomology of BG in degrees ≤ 4 over a separably closed

ground field. This calculation allows us to describe $\mathrm{Maps}_*(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})$, where G_{sc} is the simply connected cover of $\mathrm{G}_{\mathrm{der}}$, by a discrete spectrum. The full classification theorem (Theorem 5.3.3) is formulated in terms of the results for G_{sc} and T.

The remainder of this section is devoted to certain technical aspects: §5.4 addresses equivariance structures and §5.5 addresses particular features of the base $\mathrm{Spec}(\mathbb{R})$.

5.1. Torus

5.1.1. Let $\mathrm{N} \geq 1$ be an integer and S be a $\mathbb{Z}[\frac{1}{\mathrm{N}}]$-scheme. Suppose that $\mathrm{T} \to \mathrm{S}$ is a torus. We work with the étale site $\mathrm{S}_{\text{ét}}$ of affine S-schemes. Let Λ_{T} denote the sheaf of cocharacters of T. Its dual $\check{\Lambda}_{\mathrm{T}}$ is the sheaf of characters. For any two pointed sheaves $\mathrm{A}_1, \mathrm{A}_2$ over S, we view $\mathrm{Maps}_*(\mathrm{A}_1, \mathrm{A}_2)$ again as a pointed sheaf over S. Let us recall the sheaves $\vartheta^{(1)}(\Lambda_{\mathrm{T}})$, $\vartheta^{(2)}(\Lambda_{\mathrm{T}})$ introduced in §4.2.

Proposition 5.1.2. *There are canonical isomorphisms in* $\mathrm{Shv}(\mathrm{S}_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}/\mathrm{N}}^{\leq 0})$:

$$\vartheta^{(1)}(\Lambda_{\mathrm{T}}) \cong \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2}), \tag{5.1}$$

$$\vartheta^{(2)}(\Lambda_{\mathrm{T}}) \cong \mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2}). \tag{5.2}$$

5.1.3. Let us first construct the canonical map from $\vartheta^{(1)}(\Lambda_{\mathrm{T}})$ to $\mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2})$. This will make use of Proposition 3.1.7. By the definition of $\vartheta^{(1)}(\Lambda_{\mathrm{T}})$ as a pushout, it suffices to construct a commutative diagram:

$$\begin{array}{ccc}
\Gamma^2(\check{\Lambda}_{\mathrm{T}})/\mathrm{N} & \longrightarrow & (\check{\Lambda}_{\mathrm{T}} \otimes \check{\Lambda}_{\mathrm{T}})/\mathrm{N} \\
{\scriptstyle \check{\Lambda}_{\mathrm{T}} \otimes \eta} \downarrow & & \downarrow {\scriptstyle (1)} \\
\check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}(\mu_{\mathrm{N}}) & \overset{(2)}{\to} & \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2})
\end{array} \tag{5.3}$$

The cup product of pullbacks of the Kummer torsor Ψ (see §3.1) determines a morphism in $\mathrm{Shv}(\mathrm{S}_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}}^{\leq 0})$, which gives (1) by adjunction:

$$\check{\Lambda}_{\mathrm{T}} \otimes \check{\Lambda}_{\mathrm{T}} \to \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2}), \quad x_1 \otimes x_2 \mapsto x_1^*\Psi \cup x_2^*\Psi. \tag{5.4}$$

The morphism (2) is the Yoneda product with the pullback of Ψ:

$$\check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}(\mu_{\mathrm{N}}) \to \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2}), \quad x \otimes \tau \mapsto (\mathrm{B}\mathbf{Y}_\tau)(x^*\Psi). \tag{5.5}$$

5.1.4. A $\mathbb{Z}/\mathrm{N}$-linear isomorphism between their restrictions to $\Gamma^2(\check{\Lambda}_{\mathrm{T}})/\mathrm{N}$ is equivalent to a $\mathbb{Z}$-linear isomorphism between their restrictions to $\Gamma^2(\check{\Lambda}_{\mathrm{T}})$. Note that the canonical antisymmetry of the cup product factors (5.4) through a morphism $\mathrm{Ant}^2(\check{\Lambda}_{\mathrm{T}}) \to \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2})$. (Here, we have invoked Remark 4.1.3 using that the target is 1-truncated.) Its restriction to $\check{\Lambda}_{\mathrm{T}}/2$ (see (4.2)) is induced from:

$$\check{\Lambda}_{\mathrm{T}} \to \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2}), \quad x \mapsto x^*(\Psi \cup \Psi), \tag{5.6}$$

using the 2-torsion structure of $\Psi \cup \Psi$. Namely, it is defined by the composition of $\check{\Lambda}_{\mathrm{T}} \otimes \mathrm{T} \to \mathbb{G}_m$ with $\Psi \cup \Psi$, viewed as a 2-torsion element of $\mathrm{Maps}_*(\mathbb{G}_m, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2})$. On the other hand, the restriction of (5.5) to $\check{\Lambda}_{\mathrm{T}}/2$ along the morphism $\check{\Lambda} \otimes \eta$ is defined by:

$$\check{\Lambda}_{\mathrm{T}} \to \mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}(\mu_{\mathrm{N}}^{(2)})), \quad x \mapsto (\mathrm{B}\mathbf{Y}_\eta)(x^*\Psi), \tag{5.7}$$

using the 2-torsion structure of η. Under the isomorphism $(\mathrm{B}\mathbf{Y}_\eta)(x^*\Psi) \cong x^*(\mathrm{B}\mathbf{Y}_\eta)(\Psi)$, this map is defined by the composition of $\check{\Lambda}_{\mathrm{T}} \otimes \mathrm{T} \to \mathbb{G}_m$ with $(\mathrm{B}\mathbf{Y}_\eta)(\Psi)$, viewed as a 2-torsion

element of $\mathrm{Maps}_*(\mathbb{G}_m, \mathrm{B}^{(2)}\mu_\mathrm{N}^{\otimes 2})$. The desired identification between (5.6) and (5.7) comes from $\Psi \cup \Psi \cong (\mathrm{B}\mathbf{Y}_\eta)(\Psi)$ (Proposition 3.1.7).

Proof of Proposition 5.1.2. For the isomorphism (5.1), it suffices to prove that (5.3) is a pushout. This assertion is equivalent to obtaining an isomorphism on the cofibers along the horizontal morphisms. Write $p : \mathrm{T} \to \mathrm{S}$ for the projection. The question reduces to one concerning sheaves of abelian groups $\pi_n\mathrm{Maps}_*(\mathrm{T}, \mathrm{B}^{(2)}\mu_\mathrm{N}^{\otimes 2}) \cong \mathrm{R}^n p_*(\mathrm{T}, \mu_\mathrm{N}^{\otimes 2})$ (for $n = 1, 2$) using Lemma 1.3.3. Since the étale cohomology of T commutes with arbitrary base change ([Del96, Rappel 1.5.1]), we pass to S being the spectrum of a separably closed field and conclude by the isomorphisms:

$$\check{\Lambda}_\mathrm{T} \otimes \mu_\mathrm{N} \cong \mathrm{H}^1(\mathrm{T}; \mu_\mathrm{N}^{\otimes 2});$$

$$(\wedge^2 \check{\Lambda}_\mathrm{T})/\mathrm{N} \cong \mathrm{H}^2(\mathrm{T}; \mu_\mathrm{N}^{\otimes 2}),$$

defined by the Kummer class $[\Psi]$. To construct (5.2), it suffices to construct an isomorphism in $\mathrm{Shv}(\mathrm{S}_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}/\mathrm{N}}^{\leq 0})$:

$$\mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2}) \cong \lim_{[n]} \mathrm{B}^{(2)}\mathrm{Maps}_*(\mathrm{T}^{\times n}, \mathrm{B}^{(2)}\mu_\mathrm{N}^{\otimes 2}),$$

where $[n] \mapsto \mathrm{T}^{\times n}$ is the simplicial system defining T as a group scheme. We have:

$$\mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2}) \cong \lim_{[n]} \mathrm{Maps}_*(\mathrm{T}^{\times n}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2})$$

$$\cong \lim_{[n]} \mathrm{B}(\mathrm{Maps}_*(\mathrm{T}^{\times n}, \mathrm{B}^{(3)}\mu_\mathrm{N}^{\otimes 2})), \tag{5.8}$$

where the second isomorphism holds because the zeroth simplex agrees with the pointing. On the other hand, $\mathrm{Maps}_*(\mathrm{T}^{\times n}, \mathrm{B}^{(3)}\mu_\mathrm{N}^{\otimes 2})$ has sheaf of connected components:

$$\mathrm{R}^3 p_*(\mathrm{T}^n; \mu_\mathrm{N}^{\otimes 2}) \cong \wedge^3(\check{\Lambda}_\mathrm{T}^{\oplus n}) \otimes \mu_\mathrm{N}^{\otimes -1}. \tag{5.9}$$

As a sheaf of $\mathbb{Z}/\mathrm{N}$-module spectrum, we have $\lim_{[n]}(\wedge^3(\check{\Lambda}_\mathrm{T}^{\oplus n})) \cong \mathrm{Sym}^3(\check{\Lambda}_\mathrm{T})[-3]$. Thus the Bar construction of (5.9), viewed as sheaves of connective spectrum under $\tau^{\leq 0}$, has vanishing limit. The last limit in (5.8) is thus identified with $\lim_{[n]} \mathrm{B}^{(2)}\mathrm{Maps}_*(\mathrm{T}^{\times n}, \mathrm{B}^{(2)}\mu_\mathrm{N}^{\otimes 2})$. □

5.1.5. It follows from Lemma 4.2.6 and Proposition 5.1.2 that $\mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2})$ may be described by a pushout diagram in $\mathrm{Shv}(\mathrm{S}_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}/\mathrm{N}}^{\leq 0})$:

$$
\begin{array}{ccc}
\wedge^2(\check{\Lambda}_\mathrm{T})/\mathrm{N} & \longrightarrow & (\check{\Lambda}_\mathrm{T} \otimes \check{\Lambda}_\mathrm{T})/\mathrm{N} \\
\Big\downarrow{\scriptstyle \check{\Lambda}_\mathrm{T} \otimes \eta} & & \Big\downarrow{\scriptstyle (1)} \\
\check{\Lambda}_\mathrm{T} \otimes \mathrm{B}^2(\mu_\mathrm{N}) & \overset{(2)}{\to} & \mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2})
\end{array}
$$

The morphisms (1) and (2) are described as follows. The Kummer torsor Ψ defines a section $\mathrm{B}\Psi : \mathrm{B}\mathbb{G}_m \to \mathrm{B}^2\mu_\mathrm{N}$. Then (1) arises, by adjunction, from the cup product:

$$\check{\Lambda}_\mathrm{T} \otimes \check{\Lambda}_\mathrm{T} \to \mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2}), \quad x_1 \otimes x_2 \mapsto x_1^*(\mathrm{B}\Psi) \cup x_2^*(\mathrm{B}\Psi),$$

whereas (2) arises from the Yoneda product:

$$\check{\Lambda}_\mathrm{T} \otimes \mathrm{B}^2(\mu_\mathrm{N}) \to \mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2}), \quad x \otimes \tau \mapsto (\mathrm{B}^{(2)}\mathbf{Y}_\tau)(x^*(\mathrm{B}\Psi)).$$

The canonical triangle in (4.10) corresponds to the description of $\mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_\mathrm{N}^{\otimes 2})$ as an extension of its $\pi_0 \cong \mathrm{Sym}^2(\check{\Lambda}_\mathrm{T})/\mathrm{N}$ by its $\pi_2 \cong \check{\Lambda}_\mathrm{T} \otimes \mu_\mathrm{N}$ (up to a shift).

5.2. Cohomology of $B(G)$

5.2.1. Suppose that $(\Lambda, \check{\Lambda}, \Phi, \check{\Phi}, \Delta, \check{\Delta})$ is a based reduced root datum ([ABD$^+$66, XXI]). Write W for its Weyl group. Let A be an abelian group. Denote by $\mathrm{Quad}(\Lambda; A)$ the abelian group $\mathrm{Sym}^2(\check{\Lambda}) \otimes A$, viewed as A-valued quadratic forms on Λ. Namely, $c \otimes a$ defines the quadratic form $Q : \lambda \mapsto a \cdot \bar{c}(\lambda, \lambda)$, where $\bar{c}$ is an arbitrary lift of $c \in \mathrm{Sym}^2(\check{\Lambda})$ to $\check{\Lambda} \otimes \check{\Lambda}$. For a quadratic form Q, we always write b for its associated symmetric form. Define:

$$\mathrm{Quad}(\Lambda; A)_{\mathrm{st}} \subset \mathrm{Quad}(\Lambda; A)$$

as the subgroup of quadratic forms Q satisfying:

$$b(\alpha, \lambda) = \langle \check{\alpha}, \lambda \rangle Q(\alpha), \quad (\alpha \in \Delta, \lambda \in \Lambda). \tag{5.10}$$

We call elements of $\mathrm{Quad}(\Lambda; A)_{\mathrm{st}}$ *strict* quadratic forms.

Remark 5.2.2. (1) For general A, (5.10) is strictly stronger than W-invariance.
  (2) Note that any W-invariant Q verifies (5.10) after multiplying both sides by 2, so $\mathrm{Quad}(\Lambda; A)_{\mathrm{st}}$ equals $\mathrm{Quad}(\Lambda; A)^{\mathrm{W}}$ when A has no 2-torsion. This applies, in particular, to $A = \mathbb{Z}$. For general A, the condition (5.10) is strictly weaker than admitting a lift to $\mathrm{Quad}(\Lambda; \mathbb{Z})^{\mathrm{W}}$.
  (3) The equality (5.10) implies the same equality for all $\alpha \in \Phi$. Indeed, (5.10) implies W-invariance and every coroot is carried to a simple one by an element of W.

Lemma 5.2.3. *Suppose that Λ equals the sublattice generated by Φ. Then the canonical map below is bijective:*

$$\mathrm{Quad}(\Lambda; \mathbb{Z})^{\mathrm{W}} \otimes A \xrightarrow{\cong} \mathrm{Quad}(\Lambda; A)_{\mathrm{st}}.$$

Proof. The hypothesis implies that $(\Lambda, \check{\Lambda}, \Phi, \check{\Phi}, \Delta, \check{\Delta})$ decomposes under the W-action into a direct sum of based irreducible reduced root data ([ABD$^+$66, XXI, Corollaire 7.1.6]). The condition (5.10) implies that distinct summands are orthogonal under b. Hence the problem reduces to a single summand. In this case, we have an isomorphism $\mathrm{Quad}(\Lambda; A)_{\mathrm{st}} \cong A$ given by evaluating Q at a short coroot. We conclude by the equality $\mathrm{Quad}(\Lambda; \mathbb{Z})^{\mathrm{W}} = \mathrm{Quad}(\Lambda; \mathbb{Z})_{\mathrm{st}}$ as subgroups of $\mathrm{Quad}(\Lambda; \mathbb{Z})$. $\square$

Remark 5.2.4. The assertion of Lemma 5.2.3 remains valid if the quotient of Λ by the sublattice generated by Φ is torsion-free. The proof is slightly more complicated and we will not use this fact.

5.2.5. Let k be a separably closed field. Suppose that G is a reductive group scheme over k. Fix a Borel subgroup $B \subset G$ with unipotent radical U. The quotient torus $T := B/U$ is canonically independent of B. Since U is affine, the projection $B(B) \to B(T)$ induces an isomorphism on étale cohomology groups.

Let G_{sc} denote the simply connected cover of the derived subgroup $G_{\mathrm{der}} \subset G$. It has an induced Borel $B_{\mathrm{sc}} \to B$ with quotient torus T_{sc}. The cocharacter lattice $\Lambda_{T_{\mathrm{sc}}}$ equals the sublattice of Λ_T generated by the coroots Φ. Write $\pi_1(G) := \Lambda_T / \Lambda_{T_{\mathrm{sc}}}$.

Proposition 5.2.6. *The subgroup $B \subset G$ induces isomorphisms:*

$$H^i(BG; \mu_N^{\otimes 2}) \cong \begin{cases} 0 & i = 1; \\ \mathrm{Hom}(\pi_1(G), \mu_N) & i = 2; \\ \mathrm{Ext}^1(\pi_1(G), \mu_N) & i = 3; \\ \mathrm{Quad}(\Lambda_T; \mathbb{Z}/N)_{\mathrm{st}} & i = 4. \end{cases} \tag{5.11}$$

5.2.7. The isomorphisms in (5.11) are defined by pulling back along $B(B) \subset B(G)$ and using the description of cohomology groups of $B(T)$. This assertion has a clear meaning for $i = 2, 4$, whereas for $i = 3$ we shall construct an exact sequence:

$$0 \to H^2(BG; \mu_N^{\otimes 2}) \to H^2(BT; \mu_N^{\otimes 2}) \to H^2(BT_{sc}; \mu_N^{\otimes 2}) \to H^3(BG; \mu_N^{\otimes 2}) \to 0, \tag{5.12}$$

with middle arrow given by the restriction along $BT_{sc} \to BT$.

Proof of Proposition 5.2.6. (This is supposedly well-known.) Consider the morphism $f : B(B) \to B(G)$. It is a fiber bundle locally trivial in the étale topology with typical fiber G/B. The direct images $R^i f_*(\mu_N^{\otimes 2})$ $(i \geq 0)$ are constant sheaves with fiber $H^i(G/B; \mu_N^{\otimes 2})$. The Bruhat decomposition $G/B \cong \bigsqcup_{w \in W} X^w$ yields isomorphisms:

$$H^2(G/B; \mu_N^{\otimes 2}) \cong \bigoplus_{w \in W^{(1)}} \mu_N \cdot [X^w]; \tag{5.13}$$

$$H^4(G/B; \mu_N^{\otimes 2}) \cong \bigoplus_{w \in W^{(2)}} \mathbb{Z}/N \cdot [X^w]. \tag{5.14}$$

Here, $W^{(n)}$ denotes the subset of length-n elements of W and $[X^w]$ denotes the cohomology class represented by the subscheme X^w. The triangle:

$$\mu_N^{\otimes 2} \to Rf_*(\mu_N^{\otimes 2}) \to \tau^{\geq 1} Rf_*(\mu_N^{\otimes 2}) \tag{5.15}$$

relates cohomology groups of $B(G)$, $B(B)$, and G/B. The immediate conclusions from (5.15) are $H^1(BG; \mu_N^{\otimes 2}) = 0$ and the existence of an exact sequence:

$$0 \to H^2(BG; \mu_N^{\otimes 2}) \to H^2(BT; \mu_N^{\otimes 2}) \to H^2(G/B; \mu_N^{\otimes 2}) \to H^3(BG; \mu_N^{\otimes 2}) \to 0. \tag{5.16}$$

Under the isomorphisms:

$$H^2(BT_{sc}; \mu_N^{\otimes 2}) \xrightarrow{\sim} H^2(G_{sc}/B_{sc}; \mu_N^{\otimes 2}) \xleftarrow{\sim} H^2(G/B; \mu_N^{\otimes 2}),$$

(5.16) becomes (5.12) and completes our descriptions of $H^i(BG; \mu_N^{\otimes 2})$ with $1 \leq i \leq 3$.

To calculate $H^4(BG; \mu_N^{\otimes 2})$, we write $\tau^{\geq 1} Rf_*(\mu_N^{\otimes 2})$ as an extension of $\tau^{\geq 4} Rf_*(\mu_N^{\otimes 2})$ by $R^2 f_*(\mu_N^{\otimes 2})[-2]$. From (5.13), (5.14), (5.15), we find a commutative diagram:

$$
\begin{array}{c}
0 \\
\downarrow \\
\bigoplus_{w \in W^{(1)}} \mathrm{Hom}(\pi_1 G, \mathbb{Z}/N) \cdot [X^w] \\
\downarrow \\
0 \to H^4(BG; \mu_N^{\otimes 2}) \to H^4(BT; \mu_N^{\otimes 2}) \longrightarrow H^4(BG; \tau^{\geq 1} Rf_*(\mu_N^{\otimes 2})) \\
\downarrow \\
\bigoplus_{w \in W^{(2)}} \mathbb{Z}/N \cdot [X^w]
\end{array}
\tag{5.17}
$$

Let us determine the composition $H^4(B(T); \mu_N^{\otimes 2}) \to \bigoplus_{w \in W^{(2)}} \mathbb{Z}/N \cdot [X^w]$ by viewing elements in the source as quadratic forms. Consider $x_1 \otimes x_2 \in \check{\Lambda}_T \otimes \check{\Lambda}_T$. Its image in $H^4(BT; \mu_N^{\otimes 2})$ is the cup product $x_1^*[\Psi] \cup x_2^*[\Psi]$. For $x \in \check{\Lambda}_T$, the class in $H^2(G/B; \mu_N)$ defined by $x^*[\Psi]$ is $\sum_{\alpha \in \Delta} x(\alpha) \cdot [X^{s_\alpha}]$ where Δ is the set of simple coroots determined by B. Hence the class in $H^4(G/B; \mu_N^{\otimes 2})$ determined by $x_1^*[\Psi] \cup x_2^*[\Psi]$ is:

$$\sum_{\alpha, \beta \in \Delta} x_1(\alpha) x_2(\beta) \cdot [X^{s_\alpha}] \cup [X^{s_\beta}]. \tag{5.18}$$

Chevalley's formula for the Chow ring of G/B ([Che94, Proposition 10]) indicates how to calculate $[X^{s_\alpha}] \cup [X^{s_\beta}]$. Namely, it is given by $\sum_{\beta \le w \in W^{(2)}} c(w, \alpha) \cdot [X^w]$ where the sum is over $w \in W^{(2)}$ dominating β in the Bruhat order and $c(w, \alpha)$ is the coefficient in α of the unique positive coroot γ for which $w = s_\beta s_\gamma$. Summing over α first, (5.18) is equal to:

$$\sum_{\alpha, \beta \le w \in W^{(2)}} x_1(\alpha) x_2(\beta) c(w, \alpha) \cdot [X^w] = \sum_{\beta \le w \in W^{(2)}} x_1(\gamma) x_2(\beta) \cdot [X^w], \qquad (5.19)$$

where γ depends on β and w in the manner specified above.

The Coxeter presentation of W shows that each $w \in W^{(2)}$ is uniquely written as a product $s_{\alpha_1} s_{\alpha_2}$ for distinct simple coroots α_1, α_2, the only relationship between them being $s_{\alpha_1} s_{\alpha_2} = s_{\alpha_2} s_{\alpha_1}$ when α_1 and α_2 are orthogonal. Hence (5.19) contains precisely two terms for a given component $w = s_{\alpha_1} s_{\alpha_2}$, coming from $(\beta, \gamma) = (\alpha_1, \alpha_2)$ and $(\beta, \gamma) = (\alpha_2, s_{\alpha_2}(\alpha_1))$. The coefficient of (5.19) in front of $[X^w]$ is thus the sum of $x_1(\alpha_2) x_2(\alpha_1)$ and $x_1(s_{\alpha_2}(\alpha_1)) x_2(\alpha_2)$.

Replacing $x_1 \otimes x_2$ by a general pairing $c \in \check{\Lambda}_T \otimes \check{\Lambda}_T$ whose corresponding quadratic form (resp. symmetric form) is Q (resp. b), we conclude that its image in $\bigoplus_{w \in W^{(2)}} \mathbb{Z}/N \cdot [X^w]$ has the following coefficient in front of $[X^w]$ (writing $w = s_{\alpha_1} s_{\alpha_2}$):

$$c(\alpha_2, \alpha_1) + c(s_{\alpha_2}(\alpha_1), \alpha_2) = c(\alpha_2, \alpha_1) + c(\alpha_1, \alpha_2) - \langle \check{\alpha}_2, \alpha_1 \rangle \cdot c(\alpha_2, \alpha_2)$$
$$= b(\alpha_2, \alpha_1) - \langle \check{\alpha}_2, \alpha_1 \rangle Q(\alpha_2).$$

In particular, quadratic forms $Q \in H^4(BT; \mu_N^{\otimes 2})$ verifying $b(\alpha_2, \alpha_1) = \langle \check{\alpha}_2, \alpha_1 \rangle Q(\alpha_2)$ for any simple coroots α_1, α_2 map to $\bigoplus_{w \in W^{(1)}} \mathrm{Hom}(\pi_1(G), \mathbb{Z}/N) \cdot [X^w]$ in (5.17). The target may be identified with $\mathbb{Z}/N$-valued pairings between $\Lambda_{T_{sc}}$ and $\pi_1(G)$. Under this identification, Q maps to the pairing induced from $\alpha \otimes \lambda \mapsto b(\lambda, \alpha) - \langle \check{\alpha}, \lambda \rangle Q(\alpha)$ (for $\alpha \in \Delta$) by a similar analysis. This implies that the image of $H^4(BG; \mu_N^{\otimes 2})$ in $H^4(BT; \mu_N^{\otimes 2})$ consists of quadratic forms satisfying (5.10). $\qquad\square$

5.3. Classification

5.3.1. Let N be an integer and S be a $\mathbb{Z}[\frac{1}{N}]$-scheme. Suppose that $G \to S$ is a reductive group scheme equipped with a Borel subgroup scheme $B \subset G$ whose quotient torus is denoted by T.[2] Write Λ_T (resp. $\check{\Lambda}_T$) for its sheaf of cocharacters (resp. characters). Let G_{sc} denote the simply connected cover of the derived subgroup $G_{der} \subset G$, equipped with induced Borel subgroup scheme B_{sc} and quotient torus T_{sc}. The subsheaf of *strict* quadratic forms:

$$\mathrm{Quad}(\Lambda_T; \mathbb{Z}/N)_{st} \subset \mathrm{Quad}(\Lambda_T; \mathbb{Z}/N) \qquad (5.20)$$

is defined by the formula (5.10) étale locally on S where G splits.

5.3.2. Denote by $\vartheta^{(2)}(\Lambda_T)_{st} \subset \vartheta^{(2)}(\Lambda_T)$ the full subsheaf with sections (Q, F, h) where Q belongs to $\mathrm{Quad}(\Lambda_T; \mathbb{Z}/N)_{st}$. It is a sheaf of connective $\mathbb{Z}/N$-module spectra. By Proposition 5.2.6, restricting any pointed morphism $\mu : B(G) \to B^{(4)} \mu_N^{\otimes 2}$ along $B(B) \to B(G)$ defines a section whose associated quadratic form Q lies in $\mathrm{Quad}(\Lambda_T; \mathbb{Z}/N)_{st}$. This implies that the isomorphism (5.2) induces a functor:

$$R_B : \mathrm{Maps}_*(BG, B^{(4)} \mu_N^{\otimes 2}) \to \vartheta^{(2)}(\Lambda_T)_{st}. \qquad (5.21)$$

---

[2]If S is affine, the projection $B \to T$ splits noncanonically ([ABD+66, XXII, Corollaire 5.9.7]), but we shall not choose such a splitting.

Since $\pi_1(G_{sc}) = 0$, Proposition 5.2.6 implies that $\mathrm{Maps}_*(BG_{sc}, B^{(4)}\mu_N^{\otimes 2})$ is isomorphic to its π_0 which is $\mathrm{Quad}(\Lambda_{T_{sc}}; \mathbb{Z}/N)_{st}$. Hence $R_{B_{sc}}$ may also be viewed as a functor:

$$R_{B_{sc}} : \mathrm{Quad}(\Lambda_{T_{sc}}; \mathbb{Z}/N)_{st} \to \vartheta^{(2)}(\Lambda_{T_{sc}})_{st}. \tag{5.22}$$

Theorem 5.3.3. *There is a canonical Cartesian square in* $\mathrm{Shv}(S_{\text{ét}}, \mathrm{Mod}_{\mathbb{Z}/N}^{\leq 0})$:

$$
\begin{array}{ccc}
\mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2}) & \xrightarrow{\ R_B\ } & \vartheta^{(2)}(\Lambda_T)_{st} \\
\downarrow & & \downarrow \\
\mathrm{Quad}(\Lambda_{T_{sc}}; \mathbb{Z}/N)_{st} & \xrightarrow{\ R_{B_{sc}}\ } & \vartheta^{(2)}(\Lambda_{T_{sc}})_{st}
\end{array}
\tag{5.23}
$$

Proof. The two circuits are isomorphic by the functoriality of R_B with respect to the map of pairs $(G_{sc}, B_{sc}) \to (G, B)$. To show that (5.23) is Cartesian, we calculate the induced morphisms on π_i on the fibers $\mathcal{F}$, $\mathcal{F}_{sc}$ of the horizontal morphisms. Since étale cohomology of BG commutes with arbitrary base change ([Del96, Rappel 1.5.1]), the problem reduces to S being the spectrum of a separably closed field.

Note that both horizontal maps in (5.23) induce bijections on π_0. On the other hand, $\vartheta^{(2)}(\Lambda_T)$ and $\vartheta^{(2)}(\Lambda_{T_{sc}})$ have vanishing π_1. Hence, the relevant homotopy sheaves are π_1 and π_2 of the fibers, which are related by the map of exact sequences:

$$
\begin{array}{ccccccccc}
0 & \to & \mathrm{Hom}(\pi_1(G), \mu_N) & \longrightarrow & \mathrm{Hom}(\Lambda_T, \mu_N) & \longrightarrow & \pi_1(\mathcal{F}) & \longrightarrow & \mathrm{Ext}^1(\pi_1(G), \mu_N) & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & & \longrightarrow & & \mathrm{Hom}(\Lambda_{T_{sc}}, \mu_N) & \to & \pi_1(\mathcal{F}_{sc}) & & \longrightarrow & & 0
\end{array}
$$

In particular, both $\pi_2(\mathcal{F})$ and $\pi_2(\mathcal{F}_{sc})$ vanish. To conclude, we only need to observe that the morphism on π_1 induced from $\mathcal{F} \to \mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2})$ is identified with the canonical map $H^2(BT_{sc}; \mu_N^{\otimes 2}) \to H^3(BG; \mu_N^{\otimes 2})$ of (5.12). $\qquad\square$

5.3.4. It follows from Theorem 5.3.3 that a section of $\mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2})$, viewed as a Spc-valued sheaf on $S_{\text{ét}}$, is given by a pair (ϑ, φ) where:

(1) ϑ is a section of $\vartheta^{(2)}(\Lambda_T)$ whose quadratic form Q is strict;
(2) φ is an isomorphism between $R_{B_{sc}}(Q_{sc})$ and ϑ_{sc}; here, Q_{sc} (resp. ϑ_{sc}) denotes the restriction of Q (resp. ϑ) to $\Lambda_{T_{sc}}$.

The form Q being given, the datum ϑ is equivalent to pair (F, h) where $F : H^{(2)}(\Lambda_T) \to B^{(2)}\mu_N$ is a $\mathbb{Z}$-linear morphism and h is an isomorphism between its restriction to $B(\wedge^2(\Lambda_T))$ and the map defined by Q (see §4.2.7). Then φ may be viewed as an isomorphism between the restriction of (F, h) to $\Lambda_{T_{sc}}$ and the pair (F_{sc}, h_{sc}) determined by Q_{sc}. In other words, we may also view a section of $\mathrm{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2})$ as a quadruple (Q, F, h, φ).

5.4. Equivariance and commutation

5.4.1. We remain in the context of §5.3.1. The horizontal morphisms exhibited in (5.23) depend on the choice of B in the following sense: for two Borels B_1, B_2, there is no *canonical* isomorphism between R_{B_1} and R_{B_2}. However, an isomorphism exists if we are given a section $g \in G$ which conjugates B_1 into B_2. In this subsection, we study this isomorphism in the special case $B_1 = B_2 = B$, which reduces to a calculation of Deligne ([Del96, §3-4]).

5.4.2. Fix a section of $\mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})$. It is equivalently described by an $\mathbb{E}_1$-monoidal morphism, which we now call $\mu : \mathrm{G} \to \mathrm{B}^{(3)}\mu_{\mathrm{N}}^{\otimes 2}$ (see §1.2.4). Any grouplike $\mathbb{E}_1$-monoid $\mathcal{A}$ acts on itself by inner automorphisms. If the $\mathbb{E}_1$-structure on $\mathcal{A}$ lifts to an $\mathbb{E}_\infty$-structure, then this action is canonically trivialized.

Since $\mathrm{B}^{(3)}\mu_{\mathrm{N}}^{\otimes 2}$ is $\mathbb{E}_\infty$-monoidal, μ admits a G-equivariance structure with respect to the inner G-action on G and the trivial G-action on $\mathrm{B}^{(3)}\mu_{\mathrm{N}}^{\otimes 2}$. If we view μ as a section of $\mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})$, this means that it has a canonical G-equivariance structure with respect to the inner G-action on BG.

5.4.3. Inner automorphisms by sections of $\mathrm{B} \subset \mathrm{G}$ define automorphisms of the pair (G, B). Since R_{B} (5.21) is functorially assigned to (G, B), the image $\mathrm{R}_{\mathrm{B}}(\mu)$ admits a B-equivariance structure with respect to the *trivial* B-action on $\vartheta^{(2)}(\Lambda_{\mathrm{T}})_{\mathrm{st}}$. (Inner automorphisms by B induce the trivial action on the quotient T.) Consequently, this B-equivariance structure is described by a pointed morphism:

$$\mathrm{int}(\mu) : \mathrm{B} \to \check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}\mu_{\mathrm{N}}, \qquad\qquad (5.24)$$

where the target arises as the sheaf of automorphisms of $* \in \vartheta^{(2)}(\Lambda_{\mathrm{T}})_{\mathrm{st}}$.

5.4.4. The projection $\mathrm{B} \to \mathrm{T}$ defines an equivalence of groupoids between:
   (1) pointed morphisms $\mathrm{B} \to \check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}\mu_{\mathrm{N}}$, and
   (2) pointed morphisms $\mathrm{T} \to \check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}\mu_{\mathrm{N}}$.
Indeed, this follows from the triviality of the étale cohomology of the unipotent radical $\mathrm{U} \subset \mathrm{B}$. In turn, both are equivalent to:
   (3) pairings $\Lambda_{\mathrm{T}} \otimes \Lambda_{\mathrm{T}} \to \mathbb{Z}/\mathrm{N}$
by the calculation of étale cohomology of T in degrees ≤ 1 (Proposition 5.1.2). In particular, (5.24) is equivalent to a pairing $\Lambda_{\mathrm{T}} \otimes \Lambda_{\mathrm{T}} \to \mathbb{Z}/\mathrm{N}$, which we continue to denote by $\mathrm{int}(\mu)$.

Remark 5.4.5. In the special case $\mathrm{G} = \mathrm{T}$, the pairing $\mathrm{int}(\mu)$ for $\mu \in \mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})$ is the commutator of the extension of T by $\mathrm{B}^{(2)}(\mu_{\mathrm{N}}^{\otimes 2})$ corresponding to μ. Namely, since T acts trivially on both T and $\mathrm{B}^{(2)}(\mu_{\mathrm{N}}^{\otimes 2})$, the T-equivariance structure on the extension amounts to a pointed morphism:

$$\mathrm{T} \to \mathrm{Maps}_{\mathbb{E}_1}(\mathrm{T}, \mathrm{B}^{(2)}\mu_{\mathrm{N}}^{\otimes 2}) \cong \check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}\mu_{\mathrm{N}},$$

where the second isomorphism follows from the calculation of étale cohomology of $\mathrm{B}(\mathrm{T})$ in degrees ≤ 3. This morphism equals $\mathrm{int}(\mu)$.

5.4.6. We shall also address the situation with G_{sc}. Denote by G_{ad} the adjoint quotient of G and let $\mathrm{B} \to \mathrm{B}_{\mathrm{ad}}$ be the induced map of Borel subgroup schemes. The conjugation action of G_{ad} on G preserves $\mathrm{G}_{\mathrm{der}}$. By functoriality of the simply connected cover, there is an induced G_{ad}-action on G_{sc}, which induces an action of $\mathrm{B}_{\mathrm{ad}} \subset \mathrm{G}_{\mathrm{ad}}$.

Since B_{ad} acts trivially on $\mathrm{Quad}(\Lambda_{\mathrm{T}_{\mathrm{sc}}}, \mathbb{Z}/\mathrm{N})_{\mathrm{st}}$, the B-equivariance on any given section μ_{sc} of $\mathrm{Maps}_*(\mathrm{BG}_{\mathrm{sc}}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})$ lifts to a B_{ad}-equivariance. Functoriality of the construction of $\mathrm{R}_{\mathrm{B}_{\mathrm{sc}}}$ (5.22) implies that the image $\mathrm{R}_{\mathrm{B}_{\mathrm{sc}}}(\mu_{\mathrm{sc}})$ admits a B_{ad}-equivariance structure with respect to the trivial B_{ad}-action on $\vartheta^{(2)}(\Lambda_{\mathrm{T}_{\mathrm{ad}}})_{\mathrm{st}}$. This structure is described by a pointed morphism:

$$\mathrm{int}(\mu_{\mathrm{sc}}) : \mathrm{B}_{\mathrm{ad}} \to \check{\Lambda}_{\mathrm{T}_{\mathrm{sc}}} \otimes \mathrm{B}\mu_{\mathrm{N}}, \qquad\qquad (5.25)$$

which is equivalent to a pairing $\Lambda_{\mathrm{T}_{\mathrm{sc}}} \otimes \Lambda_{\mathrm{T}_{\mathrm{ad}}} \to \mathbb{Z}/\mathrm{N}$.

Proposition 5.4.7. *Let* (G, B) *be as in §5.3.1.*

(1) *Suppose that a pointed morphism* $\mu : B(G) \to B^{(4)}(\mu_N^{\otimes 2})$ *has quadratic form* Q *whose symmetric form is* b, *then* (5.24) *corresponds to the pairing:*

$$(-b) : \Lambda_T \otimes \Lambda_T \to \mathbb{Z}/N.$$

(2) *Suppose that a pointed morphism* $\mu_{sc} : B(G_{sc}) \to B^{(4)}(\mu_N^{\otimes 2})$ *has quadratic form* Q_{sc}, *then* (5.25) *corresponds to the unique extension* $\Lambda_{T_{sc}} \otimes \Lambda_{T_{ad}} \to \mathbb{Z}/N$ *of:*

$$(\alpha, \lambda) \mapsto -\langle \check{\alpha}, \lambda \rangle Q_{sc}(\alpha) \quad (\alpha \in \Delta, \lambda \in \Lambda_{T_{ad}}).$$

Proof. It suffices to verify the equalities when S is the spectrum of a separably closed field. In particular, we may choose a splitting $T \to B$ and consider T (resp. T_{ad})-equivariance structures instead. Statement (1) then reduces to the case $G = T$, where it is calculated in [Del96, Proposition 3.2]. Statement (2) is [Del96, Proposition 4.9]. $\square$

Remark 5.4.8. If μ_{sc} is the image of μ in $\mathrm{Maps}_*(BG_{sc}, B^{(4)}\mu_N^{\otimes 2})$, then $\mathrm{int}(\mu)$ and $\mathrm{int}(\mu_{sc})$ must be compatible over $\Lambda_{T_{sc}} \otimes \Lambda_T$. This is indeed observed by the equality (5.10).

5.5. The case of $\mathrm{Spec}(\mathbb{R})$

5.5.1. In this subsection, we specialize to $S = \mathrm{Spec}(\mathbb{R})$ and $N = 2$. Let us begin with $G = \mathbb{G}_m$. The triangle of Lemma 4.2.6 exhibits $\vartheta^{(2)}(\Lambda_{\mathbb{G}_m})$ as an extension of $\mathrm{Quad}(\Lambda_{\mathbb{G}_m}; \mathbb{Z}/2)$ by $B^2(\mu_2)$. This triangle admits a canonical splitting:

$$\mathrm{Quad}(\Lambda_{\mathbb{G}_m}; \mathbb{Z}/2) \cong (\check{\Lambda}_{\mathbb{G}_m} \otimes \check{\Lambda}_{\mathbb{G}_m})/2 \to \vartheta^{(2)}(\Lambda_{\mathbb{G}_m}), \tag{5.26}$$

where the second map belongs to the top square in (4.10). Thus we obtain a map:

$$\vartheta^{(2)}(\Lambda_{\mathbb{G}_m}) \to B^2(\mu_2). \tag{5.27}$$

On the other hand, every pointed morphism $\mu : B(\mathbb{G}_m) \to B^{(4)}(\mu_2^{\otimes 2})$ defines an $\mathbb{E}_1$-monoidal morphism $\mathbb{G}_m \to B^{(3)}(\mu_2^{\otimes 2})$, hence a map $\mathbb{G}_m(\mathbb{R}) \to H^3(\mathbb{R}; \mu_2^{\otimes 2})$. Its value at $(-1) \in \mathbb{G}_m(\mathbb{R})$ defines an element $\mathrm{sgn}(\mu) \in H^3(\mathbb{R}; \mu_2^{\otimes 2})$.

Lemma 5.5.2. *The element* $\mathrm{sgn}(\mu)$ *equals the image of* μ *under* (5.27) *followed by:*

$$\Gamma(\mathbb{R}; B^2\mu_2) \to H^2(\mathbb{R}; \mu_2) \cong H^3(\mathbb{R}; \mu_2^{\otimes 2}). \tag{5.28}$$

(The second isomorphism is unique as both are identified with $\mathbb{Z}/2$.)

Proof. We first argue that the image μ of any $Q \in \mathrm{Quad}(\Lambda_{\mathbb{G}_m}; \mathbb{Z}/2)$ along the splitting (5.26) satisfies $\mathrm{sgn}(\mu) = 0$. Indeed, the associated $\mathbb{E}_1$-monoidal morphism $\mathbb{G}_m \to B^{(3)}(\mu_2^{\otimes 2})$ is trivial after forgetting the $\mathbb{E}_1$-monoid structure, but the element $\mathrm{sgn}(\mu)$ is independent of the $\mathbb{E}_1$-monoid structure.

It remains to prove that for a pointed morphism $\mu : B(\mathbb{G}_m) \to B^{(4)}(\mu_2^{\otimes 2})$ coming from $\tau \in B^2(\mu_2)$, the element $\mathrm{sgn}(\mu)$ is the image of τ under (5.28). Indeed, the $\mathbb{E}_1$-monoidal morphism $\mathbb{G}_m \to B^{(3)}(\mu_2^{\otimes 2})$ is the $\mathbb{Z}$-linear morphism defined as the Yoneda product $B(\mathbf{Y}_\tau)(\Psi)$ (see §5.1.5). Evaluation at $(-1) \in \mathbb{G}_m(\mathbb{R})$ corresponds to pulling back along $(-1) : \mathrm{Spec}(\mathbb{R}) \to \mathbb{G}_m$. By replacing Yoneda product with τ by $[\tau] \cup (-)$, we obtain an equality of cohomology classes:

$$\mathrm{sgn}(\mu) = [\tau] \cup (-1)^*[\Psi] \in H^3(\mathbb{R}; \mu_2^{\otimes 2}).$$

The cohomology ring of $\mathrm{Spec}(\mathbb{R})$ is isomorphic to $\mathbb{F}_2[x]$ for a degree-1 generator x. Under this isomorphism, $(-1)^*[\Psi]$ equals x. Cup product with $(-1)^*[\Psi]$ thus agrees with the isomorphism in (5.28). $\square$

5.5.3. Let G be a reductive group scheme over $\mathbb{R}$ together with a maximal torus T. Write $T_0 \subset T$ for its maximal split subtorus. Given a pointed morphism $\mu : BG \to B^{(4)}(\mu_2^{\otimes 2})$, we obtain a homomorphism:

$$\Lambda_{T_0} \to \mathbb{Z}/2, \quad \lambda \mapsto \mathrm{sgn}(\lambda^* \mu). \tag{5.29}$$

Here, we implicitly identify $H^3(\mathbb{R}; \mu_2^{\otimes 2})$ with $\mathbb{Z}/2$. The notation $\lambda^* \mu$ denotes the restriction of μ along $\lambda : \mathbb{G}_m \to T_0$. Lemma 5.5.2 indicates how to calculate $\mathrm{sgn}(\lambda^* \mu)$.

On the other hand, μ defines an $\mathbb{E}_1$-monoidal morphism $G \to B^{(3)}(\mu_2^{\otimes 2})$ and consequently a map of groups:

$$G(\mathbb{R}) \to H^3(\mathbb{R}; \mu_2^{\otimes 2}). \tag{5.30}$$

Lemma 5.5.4. *The map* (5.30) *is identically zero if and only if* (5.29) *vanishes.*

Proof. The "if" direction requires a proof. By [Del96, Lemme 2.10], the homomorphism (5.30) is continuous with respect to the topology on $G(\mathbb{R})$ induced from $\mathbb{R}$. Thus it factors through $\pi_0(G(\mathbb{R}))$. According to Borel–Tits [BT65, Théorème 14.4], the map $\pi_0(T_0(\mathbb{R})) \to \pi_0(G(\mathbb{R}))$ is surjective. Hence, it suffices to show that the induced map of abelian groups:

$$\pi_0(T_0(\mathbb{R})) \to H^3(\mathrm{Spec}(\mathbb{R}); \mu_2^{\otimes 2})$$

vanishes identically. Now, $\pi_0(T_0(\mathbb{R}))$ is a direct sum of copies of $\mathbb{Z}/2$, and the hypothesis shows that its restriction to each copy of $\mathbb{Z}/2$ vanishes. $\square$

6. Metaplectic Langlands dual

In this section, we define the Langlands dual of a metaplectic group scheme (G, μ). If G has a Borel subgroup scheme $B \subset G$, the datum μ is classified in Theorem 5.3.3. When G splits, we construct a pair (H, F_μ) where H is a pinned reductive group scheme over the coefficient ring E and F_μ is an $\mathbb{E}_\infty$-morphism from the character group of its center Z_H to $B^{(2)}(E^\times)$. The key fact we prove is that (H, F_μ) is canonically independent of B (Theorem 6.2.2), in spite of the dependence of the classifying data on B.

The independence theorem allows a functorial association $(G, \mu) \mapsto (H, F_\mu)$ whenever G is split. In §6.3, we explain how to replace the target by a fiber functor between étale sheaves of E-linear symmetric monoidal categories. This viewpoint gives the "Langlands dual fiber functor" of (G, μ) without any splitness assumption on G.

6.1. The dual pair (H, F_μ)

6.1.1. Let $N \geq 1$ be an integer and S be a $\mathbb{Z}[\frac{1}{N}]$-scheme. We shall fix another ring E and an injective character $\chi : \mu_N \to E^\times$, where $E^\times$ is viewed as a constant étale sheaf over S.

Let $G \to S$ be a reductive group scheme and μ be a pointed morphism $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$. The notations G_{sc}, T, and T_{sc} are as in §5.3.1. They are functorially attached to G.

6.1.2. Let us first assume that G splits. For a choice of a Borel subgroup scheme $B \subset G$, we shall construct a pair (H, F_μ) where H is a pinned reductive group scheme over $\mathrm{Spec}(E)$ and F_μ is an $\mathbb{E}_\infty$-morphism $F_\mu : \check{Z}_H \to B^{(2)}(E^\times)$, where Z_H denotes the center of H and $\check{Z}_H$ the abelian group of its characters.

6.1.3. The construction of (H, F_μ) relies on the classification Theorem 5.3.3. Indeed, using $B \subset G$, the datum μ defines a quadruple (Q, F, h, φ) where:

(1) Q is a strict quadratic form on Λ_T valued in $\mathbb{Z}/N$;
(2) $F : H^{(2)}(\Lambda_T) \to B^{(2)} \mu_N$ is a $\mathbb{Z}$-linear morphism;

(3) h is an isomorphism between the restriction of F to $B(\wedge^2 \Lambda_T)$ and the canonical map
$\wedge^2(\Lambda_T) \to B\mu_N$ defined by Q; and

(4) φ is an isomorphism between the restriction of (F, h) to $\Lambda_{T_{sc}}$ and the pair (F_{sc}, h_{sc})
defined by Q_{sc} (see §5.3.4).

As usual, b denotes the symmetric form associated to Q.

6.1.4. The construction of $H \to \operatorname{Spec}(E)$ is well known ([Lus93, 2.2.5]). Let $\Lambda_T^\sharp \subset \Lambda_T$ be the
kernel of b. Define $\Phi^\sharp \subset \Lambda_T^\sharp$ to be the subset $\{\operatorname{ord}(Q(\alpha))\alpha \mid \alpha \in \Phi\}$, where $\operatorname{ord}(Q(\alpha)) \in \mathbb{N}$
denotes the order of $Q(\alpha)$. Let $\check{\Lambda}_T^\sharp$ be the dual of $\Lambda_T^\sharp$. Since Q is strict, $\operatorname{ord}(Q(\alpha))^{-1}\check{\alpha}$ takes
integral values on $\Lambda_T^\sharp$. Hence one has a well-defined subset $\check{\Phi}^\sharp = \{\operatorname{ord}(Q(\alpha))^{-1}\check{\alpha} \mid \check{\alpha} \in \check{\Phi}\}$ of
$\check{\Lambda}_T^\sharp$. The bijection $\Phi \cong \check{\Phi}$ induces a bijection $\Phi^\sharp \cong \check{\Phi}^\sharp$. The systems of simple coroots and
roots $\Delta^\sharp, \check{\Delta}^\sharp$ are determined by the corresponding elements of $\Delta, \check{\Delta}$.

One verifies that $(\Lambda_T^\sharp, \check{\Lambda}_T^\sharp, \Phi^\sharp, \check{\Phi}^\sharp, \Delta^\sharp, \check{\Delta}^\sharp)$ defines a based reduced root datum. Let $G^\sharp$ be
the corresponding pinned reductive group over S. Define H to be the Langlands dual of $G^\sharp$,
viewed as a pinned reductive group over $\operatorname{Spec}(E)$. (This means that $\Lambda_T^\sharp$ is the *character*
lattice of the maximal torus of H.)

6.1.5. The construction of F_μ uses the remaining data (F, h, φ). Let F_1 be the $\mathbb{E}_1$-monoidal
morphism $\Lambda_T \to B^{(2)}\mu_N$ associated to F (see (4.11)). By Proposition 4.3.4, the isomorphism
h lifts its restriction to $\Lambda_T^\sharp$ to an $\mathbb{E}_\infty$-monoidal morphism:

$$(F_1)^\sharp : \Lambda_T^\sharp \to B^{(2)}(\mu_N). \tag{6.1}$$

Write $\Lambda_T^{\sharp;r}$ for the sublattice of $\Lambda_T^\sharp$ generated by $\Phi^\sharp$, so $\check{Z}_H$ is canonically identified with
$\Lambda_T^\sharp/\Lambda_T^{\sharp;r}$. The quadratic form Q_{sc} vanishes on $\Lambda_T^{\sharp;r}$, so the isomorphism φ trivializes the
restriction of (F, h) to $\Lambda_T^{\sharp;r}$. According to §4.3.7, the $\mathbb{E}_\infty$-morphism (6.1) factors through an
$\mathbb{E}_\infty$-morphism $\check{Z}_H \to B^{(2)}(\mu_N)$. Composing with the character $\chi : \mu_N \to E^\times$, we obtain the
$\mathbb{E}_\infty$-morphism:

$$F_\mu : \check{Z}_H \to B^{(2)}(E^\times). \tag{6.2}$$

6.2. Independence of B

6.2.1. We remain in the context of §6.1.1 and assume that G *splits*. For a pair (G, B) and
a pointed morphism $\mu : B(G) \to B^{(4)}(\mu_N^{\otimes 2})$, we view the corresponding pair (H, F_μ) as an
object of the 2-groupoid of pairs consisting of a pinned reductive group scheme over $\operatorname{Spec}(E)$
together with an $\mathbb{E}_\infty$-morphism from the character group of its center to $B^{(2)}(E^\times)$.

Theorem 6.2.2. *The pair* (H, F_μ) *is canonically independent of* B.

6.2.3. Let us first observe that the functor defined by R_B (5.21) composed with the extraction of quadratic forms from $\vartheta^{(2)}(\Lambda_T)$:

$$\operatorname{Maps}_*(BG, B^{(4)}\mu_N^{\otimes 2}) \to \operatorname{Quad}(\Lambda_T; \mathbb{Z}/N)_{st} \tag{6.3}$$

is independent of B. (This is a condition.)

Indeed, étale locally on S, two choices of Borel subgroup schemes are conjugate by some
$g \in G$. The inner automorphism int_g may be viewed as a morphism of pairs $(G, B) \to$
(G, gBg^{-1}). Functoriality of the construction $\mu \mapsto R_B(\mu)$ with respect to (G, B) supplies
an isomorphism $R_{gBg^{-1}}(\mu) \cong R(\operatorname{int}_g^*\mu)$. On the other hand, the natural G-equivariance of μ
(§5.4.2) gives $\operatorname{int}_g^*\mu \cong \mu$. Their combination is an isomorphism in $\vartheta^{(2)}(\Lambda_T)_{st}$:

$$R_{gBg^{-1}}(\mu) \cong R_B(\mu). \tag{6.4}$$

The existence of (6.4) implies that their quadratic forms are equal.

6.2.4. Fix a strict quadratic form Q on Λ_{T}. The sublattice $\Lambda_{\mathrm{T}}^{\sharp}$ is defined as in §6.1.4. We let $\mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}}$ denote the subsheaf of $\mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})$ whose associated quadratic form equals Q. This is a Spc-valued sheaf on $\mathrm{S}_{\text{ét}}$. The next Lemma uses the calculation of B-equivariance in Proposition 5.4.7.

Lemma 6.2.5. *The composition of* R_{B} *with the restriction* $\vartheta^{(2)}(\Lambda_{\mathrm{T}}) \to \vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp})$:

$$\mathrm{R}_{\mathrm{B}}^{\sharp} : \mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}} \to \vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp}) \tag{6.5}$$

is canonically independent of B.

Proof. Take $\mu \in \mathrm{Maps}_*(\mathrm{BG}, \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}}$. The mapping $\mathrm{B}' \mapsto \mathrm{R}_{\mathrm{B}'}(\mu)$ may be viewed as a morphism of Spc-valued sheaves $\mathrm{G}/\mathrm{B} \to \vartheta^{(2)}(\Lambda_{\mathrm{T}})$. The Lemma demands us to show that its composition with the restriction map $\vartheta^{(2)}(\Lambda_{\mathrm{T}}) \to \vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp})$ is canonically constant. The restriction along $\mathrm{G} \to \mathrm{G}/\mathrm{B}$ defines the morphism:

$$\mathrm{G} \to \vartheta^{(2)}(\Lambda_{\mathrm{T}}), \quad g \mapsto \mathrm{R}_{g\mathrm{B}g^{-1}}(\mu), \tag{6.6}$$

equipped with B-equivariance structure. The isomorphisms (6.4) (for $g \in \mathrm{G}$) exhibit an isomorphism between (6.6) and the constant sheaf with value $\mathrm{R}_{\mathrm{B}}(\mu)$.

The compatibility between this constancy-homotopy with the B-equivariance structure on (6.6) is only valid *after* post-composition to $\vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp})$. To see this, we first restrict (6.6) to $\mathrm{B} \subset \mathrm{G}$. For $g \in \mathrm{B}$, the above isomorphism $\mathrm{R}_{g\mathrm{B}g^{-1}}(\mu) \cong \mathrm{R}_{\mathrm{B}}(\mu)$ is an automorphism of $\mathrm{R}_{\mathrm{B}}(\mu)$, and its dependence in g is described by a pointed morphism:

$$\mathrm{B} \to \check{\Lambda}_{\mathrm{T}} \otimes \mathrm{B}(\mu_{\mathrm{N}}). \tag{6.7}$$

By Proposition 5.4.7(1), this morphism is classified by the (negation of the) symmetric form b. In particular, the composition of (6.7) with the restriction to $\check{\Lambda}_{\mathrm{T}}^{\sharp} \otimes \mathrm{B}(\mu_{\mathrm{N}})$ is canonically trivialized. The case for $g \in \mathrm{G}$ is analogous: the mapping from B to automorphisms of $\mathrm{R}_{g\mathrm{B}g^{-1}}(\mu)$ induced from the B-equivariance structure agrees with (6.7). Consequently, the morphism $\mathrm{G}/\mathrm{B} \to \vartheta^{(2)}(\Lambda_{\mathrm{T}})$ is canonically constant after post-composition to $\vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp})$. $\quad\square$

Proof of Theorem 6.2.2. Let us denote by $\mathrm{R}^{\sharp}$ the functor (6.5), which we may now view as canonically attached to G and Q. In particular, the morphism $\mathrm{G}_{\mathrm{sc}} \to \mathrm{G}$ induces a commutative diagram:

$$\begin{array}{ccc}
\mathrm{Maps}_*(\mathrm{BG}; \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}} & \xrightarrow{\mathrm{R}^{\sharp}} & \vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp}) \\
\downarrow & & \downarrow \\
\mathrm{Maps}_*(\mathrm{BG}_{\mathrm{sc}}; \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}_{\mathrm{sc}}} & \xrightarrow{\mathrm{R}_{\mathrm{sc}}^{\sharp}} & \vartheta^{(2)}(\Lambda_{\mathrm{T}_{\mathrm{sc}}}^{\sharp})
\end{array} \tag{6.8}$$

Note that $\mathrm{Maps}_*(\mathrm{BG}_{\mathrm{sc}}; \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}_{\mathrm{sc}}}$ is the constant sheaf with value $\{\mathrm{Q}_{\mathrm{sc}}\}$. The compositions of both circuits of (6.8) with the further restriction to $\vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp,r})$ define a morphism:

$$\mathrm{Maps}_*(\mathrm{BG}; \mathrm{B}^{(4)}\mu_{\mathrm{N}}^{\otimes 2})_{\mathrm{Q}} \to \vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp}) \times_{\vartheta^{(2)}(\Lambda_{\mathrm{T}}^{\sharp,r})} \{0\}. \tag{6.9}$$

The pair $(\mathrm{H}, \mathrm{F}_{\mu})$ depends only on the image of μ along (6.9), which proves the assertion. $\quad\square$

Remark 6.2.6. For split G, a pointed map $\mu : B(G) \to B^{(4)}(\mu_N^{\otimes 2})$ defines a pair (H, F_μ). The sheaf $\check{Z}_H$ (isomorphic to $\Lambda_T^\sharp / \Lambda_T^{\sharp, r}$) is a locally constant étale sheaf of abelian groups.

Without splitness assumption on G, we obtain from μ a locally constant étale sheaf $\check{Z}$ by gluing the sheaves $\check{Z}_H$, locally attached (G, μ) where G splits. Functoriality of the construction of F_μ then defines an $\mathbb{E}_\infty$-monoidal functor $F_\mu : \check{Z} \to B^{(2)}(E^\times)$.

6.3. Langlands dual as fiber functor

6.3.1. Let us start with some generalities: S is any scheme and E is any ring. The term *tensor category* refers to a symmetric monoidal E-linear additive category, and *tensor functor* refers to a symmetric monoidal E-linear additive functor between them. Denote by $\mathrm{Mod}_E^\square$ the tensor category of finite projective E-modules.

6.3.2. Consider the stack of tensor categories Lis_E over $S_{\text{ét}}$ whose value at $S_1 \to S$ is the category of lisse (=locally constant constructible) sheaves of finite projective E-modules.

Given an affine group scheme H over E, an H-*representation* in $\mathrm{Lis}_E(S_1)$ is an object $L \in \mathrm{Lis}_E(S_1)$ equipped with functorial automorphisms:

$$(L \otimes_E R \to L \otimes_E R) \in \mathrm{Lis}_R$$

for any E-algebra R and $h \in H(R)$. Write $\mathrm{Rep}_H(S_1)$ for the category of H-representations in $\mathrm{Lis}_E(S_1)$. The association $S_1 \mapsto \mathrm{Rep}_H(S_1)$ is itself a stack of tensor categories over S, to be denoted by $\mathrm{Rep}_{H,S}$.

Remark 6.3.3. One may pass to $\mathrm{Ind}(\mathrm{Lis}_E(S_1))$ for a more economical definition. The category $\mathrm{Ind}(\mathrm{Lis}_E(S_1))$ is tensored over $\mathrm{Ind}(\mathrm{Mod}_E^\square)$. Suppose that $\mathcal{O}_H$ belongs to $\mathrm{Ind}(\mathrm{Mod}_E^\square)$. Then the datum of an H-representation on L is equivalent to a morphism $L \to L \otimes \mathcal{O}_H$ in $\mathrm{Ind}(\mathrm{Lis}_E(S_1))$ satisfying the axioms of a coaction.

6.3.4. Let Γ be a finitely generated abelian group and $\check{\Gamma}$ be its Cartier dual group scheme over $\mathrm{Spec}(E)$. Then there is a canonical equivalence:

$$\mathrm{Rep}_{\check{\Gamma}}(S_1) \cong \bigoplus_{\lambda \in \Gamma} \mathrm{Lis}_E(S_1).$$

(The copy of $\mathrm{Lis}_E(S_1)$ corresponding to $\lambda \in \Gamma$ has $\check{\Gamma}$-action of weight λ.) This is a classical fact if S_1 is the spectrum of a separably closed field ([ABD+66, I, Proposition 4.7.3]). The proof is unchanged in our setting: $\mathcal{O}_{\check{\Gamma}} \cong E[\Gamma]$ and $L \in \mathrm{Rep}_{\check{\Gamma}}(S_1)$ decomposes according to the image of the coaction map $L \to L \otimes E[\Gamma] \cong \bigoplus_{\lambda \in \Gamma} L$.

6.3.5. Let H be a split reductive group scheme over $\mathrm{Spec}(E)$ with center Z_H. Then we have a direct sum decomposition via restriction to the Z_H-action:

$$\mathrm{Rep}_H(S_1) \cong \bigoplus_{\lambda \in \check{Z}_H} \mathrm{Rep}_H^\lambda(S_1).$$

Indeed, this is because H-action fixes Z_H-weights. By varying S_1, we obtain a direct sum decomposition of the stack $\mathrm{Rep}_{H,S}$.

6.3.6. The constant sheaf of abelian groups $E^\times$ acts multiplicatively on $\mathrm{id}_{\mathrm{Rep}_{H,S}}$ (§A.1.1). In particular, given an $\mathbb{E}_\infty$-monoidal morphism $F : \check{Z}_H \to B^{(2)}(E^\times)$, we obtain another stack of tensor categories $(\mathrm{Rep}_{H,S})_F$ by the twisting construction of §A.2.4. It decomposes as $\bigoplus_{\lambda \in \check{Z}_H} (\mathrm{Rep}_{H,S}^\lambda)_{F(\lambda)}$. Functoriality with respect to $Z_H \to H$ yields a tensor functor:

$$(\mathrm{Rep}_{H,S})_F \to \bigoplus_{\lambda \in \check{Z}_H} (\mathrm{Lis}_E)_{F(\lambda)}. \tag{6.10}$$

It is conservative, faithful, and preserves short exact sequences (a "fiber functor").

6.3.7. Let N, S, E, χ be as in §6.1.1. By Theorem 6.2.2, there is a functor of 2-groupoids:

$$\left\{\begin{array}{c}\text{metaplectic group schemes } (G,\mu) \\ \text{over S such that G splits}\end{array}\right\} \to \left\{\begin{array}{c}\text{pinned reductive group schemes } H \to \mathrm{Spec}(E) \\ \text{with } \mathbb{E}_\infty\text{-morphism } F : \check{Z}_H \to B^{(2)}(E^\times) \text{ over S}\end{array}\right\},$$

sending (G,μ) to (H, F_μ). On the other hand, the target 2-groupoid admits a functor to the 2-groupoid of tensor functors between stacks of tensor categories, sending (H, F) to (6.10). Stacks of tensor categories are étale local objects, so we obtain a functor of 2-groupoids by working over the covering sieve of S where G splits:

$$\left\{\begin{array}{c}\text{metaplectic group schemes} \\ (G,\mu) \text{ over S}\end{array}\right\} \to \left\{\begin{array}{c}\text{tensor functors of étale stacks} \\ \text{of tensor categories over S}\end{array}\right\}. \tag{6.11}$$

Remark 6.3.8. We view (6.11) as a combinatorial definition of the metaplectic Langlands dual group (or rather the L-group). In the tensor category of representations of a group, invertible objects form a strictly commutative symmetric monoidal subcategory. However, tensor categories arising from our construction generally do not have this property, so we will not attempt to define the Langlands dual as a group.

7. Central extensions by $\mathbf{K}_2$

In this section, we explain the relationship between our notion of metaplectic group schemes (G,μ) and central extensions of G by $\mathbf{K}_2$. These central extensions are classified by Brylinski–Deligne [BD01] when the base scheme S is regular of finite type over a field. (The classification makes crucial use of Gersten's resolution of $\mathbf{K}_2$.)

For S smooth, there is a functor $\mathbf{R}_{\text{ét}}$ from central extensions of G by $\mathbf{K}_2$ to pointed morphisms $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$, defined by Gaitsgory ([Gai20]). We explain its construction in §7.1. In §7.2, we summarize the relationship between our classification and the one of Brylinski and Deligne. The final §7.3 addresses the differences betwen our definition of the Langlands dual and the ones of Weissman [Wei18] and Gaitsgory–Lysenko [GL18].

7.1. Étale realization

7.1.1. Suppose that S is a smooth scheme over a field k and N is invertible in k. Write $\mathbf{K}_2$ for the Zariski sheafification of the second algebraic K-group $\mathrm{Spec}(R) \mapsto K_2(R)$. Let $G \to S$ be a reductive group scheme.

When we say a *central extension of* G *by* $\mathbf{K}_2$, we mean a central extension as sheaves on the Zariski site S_{Zar} of smooth S-schemes (as opposed to $S_{\text{ét}}$). In this section, we write $B(G)$ for the Bar construction in the ∞-category of Spc-valued presheaves on whichever category of S-schemes. Analogous constructions with respect to a topology will be decorated by its name, e.g. $B^{(2)}_{\mathrm{Zar}}(\mathbf{K}_2)$ or $B^{(4)}_{\text{ét}}(\mu_N^{\otimes 2})$.

7.1.2. A central extension of G by $\mathbf{K}_2$ is equivalent to an $\mathbb{E}_1$-monoidal morphism $G \to B_{\mathrm{Zar}}(\mathbf{K}_2)$ ([BD01, §1.2]), which is in turn equivalent to a pointed morphism $B(G) \to B^{(2)}_{\mathrm{Zar}}(\mathbf{K}_2)$ (§1.2.4). The presheaf $\mathrm{Maps}_*(BG, B^{(2)}_{\mathrm{Zar}}\mathbf{K}_2)$ of them satisfies étale descent by works of Colliot-Thélène and Suslin (see [BD01, §2]).

7.1.3. We shall construct a functor of étale sheaves:

$$\mathbf{R}_{\text{ét}} : \mathrm{Maps}_*(BG, B^{(2)}_{\mathrm{Zar}}\mathbf{K}_2) \to \mathrm{Maps}_*(BG, B^{(4)}_{\text{ét}}\mu_N^{\otimes 2}). \tag{7.1}$$

Since the étale site is invariant under universal homeomorphisms, we reduce to the case of perfect k by taking a direct limit perfection. Since the target is étale local, we reduce to the case where G splits. In this case, the definition of $\mathbf{R}_{\text{ét}}$ is due to Gaitsgory ([Gai20]), making use of Voevodsky's motivic complex $\mathbb{Z}_{\text{mot}}(2)$ (see [MVW06]).

7.1.4. The complex $\mathbb{Z}_{\text{mot}}(2)$ intervenes through a morphism of complexes on S_{Zar}:

$$\mathbb{Z}_{\text{mot}}(2)[2] \to \mathbf{K}_2. \tag{7.2}$$

Let S_1 be a smooth S-scheme. Restricted to the small Zariski site of S_1, both complexes in (7.2) admit coniveau filtrations $F^{\geq n}(\mathbb{Z}_{\text{mot}}(2)[2])$ and $F^{\geq n}\mathbf{K}_2$ (for $0 \leq n \leq 2$) whose associated graded pieces are given by:

$$\text{Gr}_F^n(\mathbb{Z}_{\text{mot}}(2)[2]) \cong \bigoplus_{x \in S_1^{(n)}} \mathbb{Z}_{\text{mot}}(2-n)_x[2-n],$$

$$\text{Gr}_F^n(\mathbf{K}_2) \cong \bigoplus_{x \in S_1^{(n)}} \mathbf{K}_{2-n,x}.$$

Here $S_1^{(n)}$ denotes the set of codimension-n points of S_1. In degrees ≤ 2, the K-groups agree with Milnor K-groups of fields. The comparison between motivic cohomology and Milnor K-theory yields canonical morphisms:

$$\mathbb{Z}_{\text{mot}}(2-n)_x[2-n] \to \mathbf{K}_{2-n,x} \quad (0 \leq n \leq 2). \tag{7.3}$$

For $n = 1, 2$, they are in fact isomorphisms. The compatibility of (7.3) with residue maps then yields the morphism of complexes (7.2).

Lemma 7.1.5. *Suppose that k is a perfect field and S is a smooth k-scheme. For a split reductive group scheme $G \to S$, the morphism (7.2) induces an isomorphism:*

$$\text{R}\Gamma_*(G; \mathbb{Z}_{\text{mot}}(2)[2]) \cong \text{R}\Gamma_*(G; \mathbf{K}_2).$$

Here, $\text{R}\Gamma_*(G; -)$ denotes the fiber of $e^* : \text{R}\Gamma(G; -) \to \text{R}\Gamma(S; -)$.

Proof. We observe that at a smooth S-scheme S_1, the fiber of (7.2) is the truncated complex $\bigoplus_{x \in S_1^{(0)}} \tau^{\leq -1}(\mathbb{Z}_{\text{mot}}(2)_x[2])$. Indeed, this is because (7.3) is an isomorphism for $n = 1, 2$, and for $n = 0$, the complex $\mathbb{Z}_{\text{mot}}(2)_x[2]$ is concentrated in cohomological degrees ≤ 0, with its H^0 identified with $\mathbf{K}_{2,x}$ ([MVW06, Theorem 5.1]). In particular, we find a canonical triangle of complexes of $\mathbb{Z}$-modules:

$$\bigoplus_{x \in S_1^{(0)}} \tau^{\leq -1}(\mathbb{Z}_{\text{mot}}(2)_x[2]) \to \text{R}\Gamma(S_1; \mathbb{Z}_{\text{mot}}(2)[2]) \to \text{R}\Gamma(S_1; \mathbf{K}_2). \tag{7.4}$$

To prove the assertion, we may assume that S has a unique generic point η_S and write η_G for the generic point of G. The statement is equivalent to showing that the morphism given by pulling back along $G \to S$ on the truncated complexes is an isomorphism:

$$\tau^{\leq -1}(\mathbb{Z}_{\text{mot}}(2)_{\eta_S}[2]) \cong \tau^{\leq -1}(\mathbb{Z}_{\text{mot}}(2)_{\eta_G}[2]). \tag{7.5}$$

Since G is split reductive, it is birational to an affine space $\mathbb{A}_S^k \to S$. We apply the canonical triangle (7.4) to $S_1 = \mathbb{A}_S^k \to S$. Since $\text{R}\Gamma(S_1; \mathbf{K}_2) \in \text{Mod}_{\mathbb{Z}}^{\geq 0}$, it suffices to show that $\text{R}\Gamma(S; \mathbb{Z}_{\text{mot}}(2)[2]) \to \text{R}\Gamma(\mathbb{A}_S^k; \mathbb{Z}_{\text{mot}}(2)[2])$ induces an isomorphism on H^n for all $n \leq -1$. This follows from the $\mathbb{A}^1$-invariance of motivic cohomology. $\square$

7.1.6. Suppose that $N \geq 1$ is invertible in k. Then $(\mathbb{Z}/N)_{\mathrm{mot}}(2) := \mathbb{Z}_{\mathrm{mot}}(2) \otimes \mathbb{Z}/N$ is a sheaf on the étale site of smooth S-schemes ([MVW06, Corollary 6.4]), canonically identified with $\mu_N^{\otimes 2}$ ([MVW06, Theorem 10.3]). It follows from Lemma 7.1.5 that we have a canonical map of complexes of $\mathbb{Z}$-modules $\mathrm{R}\Gamma_*(\mathrm{G}; \mathbf{K}_2) \to \mathrm{R}\Gamma_*(\mathrm{G}; \mu_N^{\otimes 2}[2])$.

We shift this morphism by [2], apply truncation $\tau^{\leq 0}$, and view the results as mapping spectra (§1.3):

$$\mathrm{Maps}_*(\mathrm{G}; \mathrm{B}_{\mathrm{Zar}}^{(2)} \mathbf{K}_2) \to \mathrm{Maps}_*(\mathrm{G}; \mathrm{B}_{\text{ét}}^{(4)}(\mu_N^{\otimes 2})). \tag{7.6}$$

Finally, replacing G by the simplicial system of split reductive groups $[n] \mapsto \mathrm{G}^{[n]}$ and taking the cosimplicial limit of (7.6), we obtain the desired functor $\mathbf{R}_{\text{ét}}$.

7.2. Compatibilities

7.2.1. We remain in the context of §7.1.1. Let us explain how the classification of pointed morphisms $\mathrm{B}(\mathrm{G}) \to \mathrm{B}^{(4)}(\mu_N^{\otimes 2})$ is related to Brylinski and Deligne's classification of central extension of G by $\mathbf{K}_2$ ([BD01, Theorem 7.2]). For a sheaf of finite locally free $\mathbb{Z}$-modules Λ, we write $\vartheta^{(2)}(\Lambda; \mathbb{Z})$ for the sheaf of pairs (Q, F) where:

(1) Q is an integral quadratic form on Λ;
(2) $\mathrm{F} : \mathrm{H}^{(2)}(\Lambda) \to \mathrm{B}(\mathbb{G}_m)$ is a $\mathbb{Z}$-linear morphism whose restriction to $\mathrm{B}(\wedge^2(\Lambda))$ is given by the map $\wedge^2(\Lambda) \to \mathbb{G}_m$, $\lambda_1 \wedge \lambda_2 \mapsto (-1)^{b(\lambda_1, \lambda_2)}$.

The notation $\mathrm{H}^{(2)}(\Lambda)$ is introduced in §4.1.5. As usual, b denotes the symmetric form associated to Q.

Restriction of F along the $\mathbb{E}_1$-monoidal section $\Lambda \to \mathrm{H}^{(2)}(\Lambda)$ determines an $\mathbb{E}_1$-monoidal morphism $\mathrm{F}_1 : \Lambda \to \mathrm{B}(\mathbb{G}_m)$, or equivalently, a central extension of Λ by $\mathbb{G}_m$. When Q is given, the condition in (2) says that its commutator pairing is $\lambda_1 \wedge \lambda_2 \mapsto (-1)^{b(\lambda_1, \lambda_2)}$. Conversely, any such central extension determines the $\mathbb{Z}$-linear morphism F.

7.2.2. Let $\mathrm{T} \to \mathrm{S}$ be a torus whose sheaf of cocharacters is Λ_T. The equivalence (5.2) is compatible with [BD01, Theorem 3.16] in the form of a commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}_{\mathrm{Zar}}^{(2)} \mathbf{K}_2) & \xrightarrow{\;\cong\;} & \vartheta^{(2)}(\Lambda_\mathrm{T}; \mathbb{Z}) \\
\downarrow{\scriptstyle \mathbf{R}_{\text{ét}}} & & \downarrow{\scriptstyle \Psi_*} \\
\mathrm{Maps}_*(\mathrm{BT}, \mathrm{B}_{\text{ét}}^{(4)} \mu_N^{\otimes 2}) & \xrightarrow{\;\cong\;} & \vartheta^{(2)}(\Lambda_\mathrm{T})
\end{array}
$$

The right vertical arrow sends (Q, F) to the triple $(\bar{\mathrm{Q}}, \bar{\mathrm{F}}, h)$ where $\bar{\mathrm{Q}}$ is the reduction of Q mod N, $\bar{\mathrm{F}} : \mathrm{H}^{(2)}(\Lambda_\mathrm{T}) \to \mathrm{B}^{(2)}(\mu_N)$ is the composition of F with the Kummer map $\Psi : \mathbb{G}_m \to \mathrm{B}(\mu_N)$, and h is the isomorphism coming from the equality in §7.2.1(2). We denote it by Ψ_* to indicate that it is essentially "pushing forward along Ψ."

If we restrict $\bar{\mathrm{F}}$ along the $\mathbb{E}_1$-monoidal section $\Lambda_\mathrm{T} \to \mathrm{H}^{(2)}(\Lambda_\mathrm{T})$, we obtain an $\mathbb{E}_1$-monoidal morphism $\bar{\mathrm{F}}_1 : \Lambda_\mathrm{T} \to \mathrm{B}^{(2)}(\mu_N)$ which agrees with the composition of F_1 with Ψ.

7.2.3. We slightly modify the formulation of Theorem 5.3.3 for a comparison with [BD01, Theorem 7.2]. Namely, instead of choosing a Borel subgroup scheme, we assume that a maximal torus $\mathrm{T} \subset \mathrm{G}$ is fixed. Then we obtain a Cartesian diagram, whose horizontal

morphisms are restrictions along $B(T) \to B(G)$ (resp. $B(T_{sc}) \to B(G_{sc})$):

$$
\begin{array}{ccc}
\mathrm{Maps}_*(BG, B_{\text{ét}}^{(4)} \mu_N^{\otimes 2}) & \xrightarrow{\ R_T\ } & \vartheta^{(2)}(\Lambda_T)_{st} \\
\downarrow & & \downarrow \\
\mathrm{Quad}(\Lambda_{T_{sc}}; \mathbb{Z}/N)_{st} & \xrightarrow{\ R_{T_{sc}}\ } & \vartheta^{(2)}(\Lambda_{T_{sc}})_{st}
\end{array}
\tag{7.7}
$$

Likewise, [BD01, Theorem 7.2] asserts the existence of a canonical Cartesian diagram:

$$
\begin{array}{ccc}
\mathrm{Maps}_*(BG, B_{\text{Zar}}^{(2)} \mathbf{K}_2) & \xrightarrow{\ R_T\ } & \vartheta^{(2)}(\Lambda_T; \mathbb{Z})_{st} \\
\downarrow & & \downarrow \\
\mathrm{Quad}(\Lambda_{T_{sc}}; \mathbb{Z})_{st} & \xrightarrow{\ R_{T_{sc}}\ } & \vartheta^{(2)}(\Lambda_{T_{sc}}; \mathbb{Z})_{st}
\end{array}
\tag{7.8}
$$

Here, strict quadratic forms are precisely the W-invariant ones (Remark 5.2.2(2)). The label "st" in the right column means that the quadratic form Q of the pair (Q, F) is strict. The Cartesian diagrams (7.7) and (7.8) are related by $\mathbf{R}_{\text{ét}}$ and reduction mod N on the left column, and Ψ_* on the right column.

7.2.4. Let F be a local field such that $\mu_N(F)$ has cardinality N. Brylinski–Deligne [BD01, §10] constructs a functor Φ from $\mathrm{Maps}_*(BG, B_{\text{Zar}}^{(2)} \mathbf{K}_2)$ to the category of topological $\mu_N(F)$-covers of $G(F)$. It is related to the functor $\Phi_{\text{ét}}$ (§2.1.5) via the following diagram:

$$
\begin{array}{ccc}
\mathrm{Maps}_*(BG, B_{\text{Zar}}^{(2)} \mathbf{K}_2) & & \\
\downarrow{\scriptstyle \mathbf{R}_{\text{ét}}} & \xrightarrow{\ \Phi\ } & \left\{ \begin{array}{c} \text{topological covers of} \\ G(F) \text{ by } \mu_N(F) \end{array} \right\}. \\
\mathrm{Maps}_*^\circ(BG, B_{\text{ét}}^{(4)} \mu_N^{\otimes 2}) & \xrightarrow{\ \Phi_{\text{ét}}\ } &
\end{array}
\tag{7.9}
$$

Part of the assertion is that if $F = \mathbb{R}$ and $N = 2$, then $\mathbf{R}_{\text{ét}}$ has essential image contained in the subgroupoid of objects μ which induce the trivial map $G(\mathbb{R}) \to H_{\text{ét}}^3(\mathbb{R}; \mu_N^{\otimes 2})$. By definition of $\mathbf{R}_{\text{ét}}$, this map factors through $G(\mathbb{R}) \to H_{\text{Zar}}^3(\mathbb{R}; \mathbb{Z}_{\text{mot}}(2))$ which vanishes since $\mathbb{Z}_{\text{mot}}(2)$ is concentrated in cohomological degrees ≤ 2.

The commutativity of (7.9) follows from the comparison of the Galois symbol with the map from motivic cohomology to Milnor K-theory: the canonical map $H_{\text{Zar}}^2(F; \mathbb{Z}_{\text{mot}}(2)) \to H_{\text{ét}}^2(F; \mu_N^{\otimes 2})$ followed by local Tate duality (or the unique isomorphism onto $\mu_N(F)$ if $F = \mathbb{R}$ and $N = 2$) equals its map to $\mathbf{K}_2(F)$ ([MVW06, §5]) followed by the Nth Hilbert symbol.

7.2.5. Let F be a global field such that $\mu_N(F)$ has cardinality N. We have an analogous commutative diagram:

$$
\begin{array}{ccc}
\mathrm{Maps}_*(BG, B_{\text{Zar}}^{(2)} \mathbf{K}_2) & & \\
\downarrow{\scriptstyle \mathbf{R}_{\text{ét}}} & \xrightarrow{\ \Phi\ } & \left\{ \begin{array}{c} \text{topological covers of} \\ G(\mathbb{A}_F) \text{ by } \mu_N(F) \text{ equipped with} \\ \text{a splitting over } G(F) \end{array} \right\}, \\
\mathrm{Maps}_*^\circ(BG, B_{\text{ét}}^{(4)} \mu_N^{\otimes 2}) & \xrightarrow{\ \Phi_{\text{ét}}\ } &
\end{array}
\tag{7.10}
$$

thanks to the commutativity of (7.9) for each place of F.

7.3. Incompatibilities

7.3.1. Our construction of the metaplectic dual fiber functor is closely related to Weissman's second twist ([Wei18, §3]) and Gaitsgory and Lysenko's definition of the metaplectic dual data ([GL18, §6]). Let us address their differences when G → S is a split reductive group. The assertions in §7.3.5-§7.3.7 require proofs which will appear in a subsequent paper.

7.3.2. We begin with a cautionary remark on ϑ-data. Let (Q, F) be a section of $\vartheta^{(2)}(\Lambda_T; \mathbb{Z})$ (§7.2.1). Write $F_1 : \Lambda_T \to B(\mathbb{G}_m)$ for the $\mathbb{E}_1$-monoidal morphism and b the symmetric form of Q. Let $(\bar{Q}, \bar{F}, h)$ denote the image of (Q, F) in $\vartheta^{(2)}(\Lambda_T)$ along Ψ_*.

If b happens to take even values, then $F_1 : \Lambda_T \to B(\mathbb{G}_m)$ is $\mathbb{Z}$-linear. (The corresponding central extension is abelian.) In this case, one may compose it with Ψ to obtain a $\mathbb{Z}$-linear morphism $\bar{F}'_1 : \Lambda_T \to B^{(2)}(\mu_N)$. On the other hand, if the mod N reduction $\bar{b} = 0$, then $\bar{F}_1$ lifts to an $\mathbb{E}_\infty$-monoidal morphism (Proposition 4.3.4). Caution: $\bar{F}_1$ and $\bar{F}'_1$ agree as $\mathbb{E}_1$-monoidal morphisms but generally *disagree* as $\mathbb{E}_\infty$-monoidal morphisms.

7.3.3. Suppose that we are given a central extension of G by $\mathbf{K}_2$. Using the Cartesian diagram (7.8), we view its classification datum as a triple (Q, F, φ) where (Q, F) belongs to $\vartheta^{(2)}(\Lambda_T; \mathbb{Z})_{st}$ and φ is an isomorphism between the restriction of F to $\Lambda_{T_{sc}}$ and the morphism defined by Q_{sc} (the restriction of Q to $\Lambda_{T_{sc}}$). We write $(\bar{Q}, \bar{F}, h, \bar{\varphi})$ for the quadruple classifying the induced pointed morphism $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$. Let E, χ be as in §6.1.1 and write $(H, \bar{F}_\mu)$ for its Langlands dual defined in §6.1.

7.3.4. If N is odd, we assume that b takes even values on the sublattice $\Lambda_T^\sharp \subset \Lambda_T$ defined in §6.1 (c.f. [Wei18, Assumption 3.1]). The construction of §7.3.2 yields a $\mathbb{Z}$-linear morphism $\bar{F}'_1 : \Lambda_T^\sharp \to B^{(2)}(\mu_N)$ canonically trivialized on $\Lambda_T^{\sharp,r}$. Then $\bar{F}'_1$ defines a $\mathbb{Z}$-linear morphism:

$$\bar{F}'_\mu : \check{Z}_H \to B^{(2)}(E^\times).$$

(Equivalently, it is a $Z_H(E)$-gerbe on $S_{\text{ét}}$.) In Weissman's definition of the L-group, the "second twist" amounts to replacing $\text{Rep}_{H,S}$ by the twisted category $(\text{Rep}_{H,S})_{\bar{F}'_\mu}$.

7.3.5. When S is the spectrum of a field F (with fixed algebraic closure $\bar{F}$) and E = $\mathbb{C}$, global sections of $(\text{Rep}_{H,\text{Spec}(F)})_{\bar{F}'_\mu}$ over $\text{Spec}(F)$ form the category of representations of an extension of $\text{Gal}(\bar{F}/F)$ by $H(\mathbb{C})$, algebraic on the $H(\mathbb{C})$-factor.

This category is *different* from $(\text{Rep}_{H,\text{Spec}(F)})_{\bar{F}_\mu}$ appearing in our definition of the Langlands dual, which uses $\bar{F}_\mu$ instead of $\bar{F}'_\mu$. By the discussion in §7.3.2, they agree as monoidal categories but generally disagree as symmetric monoidal categories. In particular, the difference disappears on K_0.

Remark 7.3.6. We do not include Weissman's first twist (the "meta-Galois group") in our definition of the Langlands dual. Its construction is particular to local and global fields and has no analogue over a general base scheme.

We *do* think that the first twist is an arithmetic manifestation of the ϑ-characteristic: for a local function field, a splitting of the meta-Galois group is equivalent to the choice of a ϑ-characteristic.

7.3.7. Let us now turn to the context of Gaitsgory–Lysenko ([GL18]). The base S = X is a smooth curve over an algebraically closed field. The coefficient ring E is also a field. Any pointed morphism $B(G) \to B^{(4)}(\mu_N^{\otimes 2})$ defines a factorization gerbe on the affine Grassmannian ([GL18, §3.1]). The definition of metaplectic dual data in *op.cit.* is then obtained as a

triple $(H, \mathcal{G}_Z, \epsilon)$, leading to a symmetric monoidal category $(\mathrm{Rep}_H)_{\mathcal{G}_Z}^\epsilon$. Let us forget about factorization structure for the moment.

This symmetric monoidal category is *different* from global sections of our $(\mathrm{Rep}_{H,X})_{F_\mu}$ (§6.3). More precisely, there is a canonical $\mathbb{Z}$-linear morphism $\omega_X^Q : \check{Z}_H \to B^{(2)}(E^\times)$ sending λ to the gerbe of ϑ-characteristics induced along $Q(\lambda)$, viewed as a map $\{\pm 1\} \to E^\times$. Hence F_μ defines an $\mathbb{E}_\infty$-monoidal morphism $F_\mu^+ \coloneqq F_\mu \otimes \omega_X^Q$. The symmetric monoidal category $(\mathrm{Rep}_H)_{\mathcal{G}_Z}^\epsilon$ of *op.cit.* is equivalent to global sections of $(\mathrm{Rep}_{H,X})_{F_\mu^+}$. In particular, the difference disappears when a ϑ-characteristic is chosen.

APPENDIX A. TWISTING CONSTRUCTION

Given a sheaf of groups A, an A-torsor P, and a sheaf of sets X equipped with an A-action, we obtain another sheaf of sets $X_P \coloneqq P \times^A X$: the twist of X by P. When A is abelian, two A-torsors P_1, P_2 define a third one $P_1 \otimes P_2$. When X has a monoid structure, we obtain a multiplication map:

$$X_{P_1} \times X_{P_2} \to X_{P_1 \otimes P_2}, \tag{A.1}$$

if the equality $(a_1 a_2) \cdot (x_1 x_2) = (a_1 \cdot x_1)(a_2 \cdot x_2)$ holds for all $a_1, a_2 \in A$ and $x_1, x_2 \in X$.

Let us involve another piece of structure: X is now a sheaf of E-algebras equipped with a grading $X = \bigoplus_{\lambda \in \Gamma} X_\lambda$ by some abelian group Γ, such that $1 \in X_0$ and $x_1 x_2$ has grading $\lambda_1 + \lambda_2$ if x_1, x_2 have gradings λ_1, λ_2. Then any multiplicative A-torsor P on Γ defines a new sheaf of E-algebras $X_P \coloneqq \bigoplus_{\lambda \in \Gamma} (X_\lambda)_{P(\lambda)}$ with multiplicative rule (A.1). Here, $P(\lambda)$ denotes the A-torsor $P \times_\Gamma \{\lambda\}$.

The goal of this section is to explain an analogous construction where X is replaced by a Γ-graded stack of tensor categories $\mathcal{C}$. We explain the meaning of an A-action on $\mathrm{id}_\mathcal{C}$ and how to form a twisted stack of tensor categories $\mathcal{C}_F$ for an $\mathbb{E}_\infty$-functor $F : \Gamma \to B^{(2)}(A)$.

Remark A.0.1. There are analogous constructions when $\mathcal{C}$ is monoidal (resp. braided monoidal) and $F : \Gamma \to B^{(2)}(A)$ is $\mathbb{E}_1$-monoidal (resp. $\mathbb{E}_2$-monoidal). The result $\mathcal{C}_F$ is then a sheaf of monoidal (resp. braided monoidal) categories.

A.1. Actions

A.1.1. Suppose that A is an abelian group. Let $\mathcal{C}$ be a category. We say that A *acts on* $\mathrm{id}_\mathcal{C}$ if there is a group homomorphism $A \to \mathrm{Aut}(\mathrm{id}_\mathcal{C})$. Here, $\mathrm{id}_\mathcal{C}$ denotes the identity functor viewed as an object of the category of endofunctors of $\mathcal{C}$. Concretely, an A-action on $\mathrm{id}_\mathcal{C}$ means that to each $a \in A$ and $c \in \mathcal{C}$, there is an isomorphism $a_c : c \cong c$. They satisfy:

(1) 1_c is the identity for all $c \in \mathcal{C}$;
(2) $(a_1 a_2)_c = (a_1)_c \circ (a_2)_c$ for all $a_1, a_2 \in A$ and $c \in \mathcal{C}$;
(3) $a_{c_2} \circ f = f \circ a_{c_1}$ for all $a \in A$ and $f : c_1 \to c_2$ in $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a symmetric monoidal category. We say that an A-action on $\mathrm{id}_\mathcal{C}$ is *multiplicative* if the following condition is satisfied:

(4) $(a_1)_{c_1} \otimes (a_2)_{c_2} = (a_1 a_2)_{c_1 \otimes c_2}$ for all $a_1, a_2 \in A$ and $c_1, c_2 \in \mathcal{C}$.

A.1.2. Denote by $B(A)$ the groupoid with a single object $*$ and $\mathrm{Aut}(*) = A$ (i.e. the Bar construction in Spc). It has a symmetric monoidal structure defined by the group structure of A. The notion of an A-action on $\mathrm{id}_\mathcal{C}$ is really a description of a $B(A)$-action on $\mathcal{C}$, in the following sense: it defines a groupoid object $[n] \mapsto \mathcal{C}^{[n]}$ in the 2-category of categories

covering the groupoid object $[n] \mapsto B(A)^{\times[n]} := B(A)^{\times n}$, together with an isomorphism $\mathcal{C}^{[0]} \cong \mathcal{C}$, such that the following diagram is Cartesian for both $i = 0, 1$:

$$
\begin{array}{ccc}
\mathcal{C}^{[1]} & \xrightarrow{\ \delta^i\ } & \mathcal{C}^{[0]} \\
\downarrow & & \downarrow \\
B(A) & \longrightarrow & *
\end{array}
$$

The groupoid object $[n] \mapsto \mathcal{C}^{[n]}$ is explicitly constructed by $[n] \mapsto B(A)^{\times n} \times \mathcal{C}$. One of the boundary maps, say δ^0, passes to projection onto $\mathcal{C}$. The other one δ^1 is given by:

$$
\mathrm{act} : B(A) \times \mathcal{C} \to \mathcal{C}, \quad (a, f) \mapsto af := a_{c_2} \circ f = f \circ a_{c_1}. \tag{A.2}
$$

(The formula in (A.2) describes what act does to morphisms $a \in A$ and $f : c_1 \to c_2$.) The higher boundary maps are compositions of actions and projections. The degeneracy maps are insertions along $* \in B(A)$. Conditions (1) and (2) of §A.1.1 ensure that the simplicial object is well defined. Taking geometric realization of the morphism $\mathcal{C}^{[n]} \to B(A)^{[n]}$, we obtain a functor of 2-categories (see §1.2.5):

$$
\mathcal{C}^{[-1]} \to B^{(2)}(A). \tag{A.3}
$$

A.1.3. If $\mathcal{C}$ is a symmetric monoidal category and the A-action on $\mathcal{C}$ is multiplicative, then (A.2) is itself a functor of symmetric monoidal categories. The isomorphism between $\mathrm{act}(*, c_1) \otimes \mathrm{act}(*, c_2)$ and $\mathrm{act}(*, c_1 \otimes c_2)$ is the obvious one. However, it demands a commutative diagram:

$$
\begin{array}{ccc}
c_1 \otimes d_1 & \xrightarrow{\ \cong\ } & c_1 \otimes d_1 \\
{\scriptstyle (a_1 f) \otimes (a_2 g)}\downarrow & & \downarrow{\scriptstyle (a_1 a_2) \otimes (fg)} \\
c_2 \otimes d_2 & \xrightarrow{\ \cong\ } & c_2 \otimes d_2
\end{array}
$$

This follows from conditions (3) and (4) of §A.1.1. It follows that $[n] \to \mathcal{C}^{[n]}$ is a simplicial object in the 2-category of symmetric monoidal categories. Furthermore, the morphism $\mathcal{C}^{[n]} \to B(A)^{[n]}$ is a morphism of such. It follows that (A.3) lifts to a functor of $\mathbb{E}_\infty$-monoidal 2-categories. (We have used the fact that forgetting the $\mathbb{E}_\infty$-structure commutes with sifted colimits, see the proof of Lemma 1.1.6.)

A.1.4. The above constructions carry sheaf-theoretic meaning. Fix a site $\mathcal{X}$ and let A be a sheaf of abelian groups, $\mathcal{C}$ be a stack of categories. Then an A-action on $\mathrm{id}_{\mathcal{C}}$ defines a morphism of simplicial stacks of categories $\mathcal{C}^{[n]} \to B(A)^{[n]}$. Here, $B(A)$ denotes the Bar construction of A in the ∞-category of Spc-valued sheaves. By taking the geometric realization, we obtain a morphism of sheaves of 2-categories:

$$
\mathcal{C}^{[-1]} \to B^{(2)}(A). \tag{A.4}
$$

If $\mathcal{C}$ carries a symmetric monoidal structure and the A-action is multiplicative, then (A.4) lifts to a morphism of sheaves of $\mathbb{E}_\infty$-monoidal 2-categories.

A.2. How $(\mathcal{C}, F)$ defines $\mathcal{C}_F$

A.2.1. We continue to fix a site $\mathcal{X}$ and let A be a sheaf of abelian groups, $\mathcal{C}$ be a stack of categories. Suppose that A acts on $\mathrm{id}_{\mathcal{C}}$. Given any section F of $B^{(2)}(A)$ over x, the fiber product of (A.4) with $F : x \to B^{(2)}(A)$ defines a stack of categories over x. We denote it by $\mathcal{C}_F$ and view it as the "F-twist of $\mathcal{C}$."

Note that for any $x_1 \to x$ such that the pullback F_{x_1} of F is trivialized, i.e., factors as $x_1 \to * \to B^{(2)}(A)$, the pullback of $\mathcal{C}_F$ to x_1 is isomorphic to the pullback of $\mathcal{C}$.

A.2.2. If E is a ring and $\mathcal{C}$ is an E-linear category, the same structure is inherited by $\mathcal{C}_F$. Indeed, we may let $\mathcal{S} \subset \operatorname{Hom}(-, x)$ be the covering sieve consisting of morphisms $x_1 \to x$ such that F_{x_1} is trivial. Then we have:

$$\mathcal{C}_F(x) \cong \lim_{(x_1 \to x) \in \mathcal{S}} \mathcal{C}_F(x_1).$$

On the right-hand-side, $\mathcal{C}_F(x_1)$ has an E-linear structure by choosing *any* trivialization of F_{x_1} and transport the E-linear structure from $\mathcal{C}(x_1)$. Two distinct trivializations of F_{x_1} differ by a section of $B(A)$, which locally defines an automorphism of $\mathcal{C}(x_1)$ by the action map (A.2). Since this action map is E-linear on the Hom-sets, the category $\mathcal{C}_F(x_1)$ acquires an E-linear structure independently of the trivialization of F_{x_1}. The same structure then passes to $\mathcal{C}_F(x)$.

A.2.3. Let us now suppose that $\mathcal{C}$ is a stack of symmetric monoidal E-linear additive categories, together with a decomposition $\mathcal{C} = \bigoplus_{\lambda \in \Gamma} \mathcal{C}_\lambda$ for an abelian group Γ such that:

   (1) $\mathbf{1}_\mathcal{C} \in \mathcal{C}_0$;
   (2) $c_1 \otimes c_2 \in \mathcal{C}_{\lambda_1 + \lambda_2}$ if $c_1 \in \mathcal{C}_{\lambda_1}$ and $c_2 \in \mathcal{C}_{\lambda_2}$.

These data may be packaged differently: let $\mathcal{C}^\sqcup$ denote the stack of categories over Γ whose fiber at $\lambda \in \Gamma$ is $\mathcal{C}_\lambda$. Then $\mathcal{C}^\sqcup$ admits a symmetric monoidal structure such that $\mathcal{C}^\sqcup \to \Gamma$ is a symmetric monoidal functor. (Here, Γ is viewed as a discrete category whose symmetric monoidal structure comes from the group operations.)

Conversely, given a symmetric monoidal functor $\mathcal{D} \to \Gamma$ where $\mathcal{D}$ is a stack of symmetric monoidal E-linear categories whose *fibers* over Γ are additive, we obtain a stack of symmetric monoidal E-linear additive categories $\mathcal{D}^\oplus := \bigoplus_{\lambda \in \Gamma} \mathcal{D}_\lambda$ by taking direct sum of the fibers.

A.2.4. Let $\mathcal{C}$ be as in §A.2.3. Suppose that A acts multiplicatively on $\operatorname{id}_\mathcal{C}$. Then it induces a multiplicative action on $\operatorname{id}_{\mathcal{C}^\sqcup}$. It also acts trivially on id_Γ and the functor $\mathcal{C}^\sqcup \to \Gamma$ is tautologically compatible with the actions. The construction of (A.4) is functorial in $\mathcal{C}$. In particular, the symmetric monoidal functor $\mathcal{C}^\sqcup \to \Gamma$ yields an $\mathbb{E}_\infty$-monoidal functor:

$$\mathcal{C}^{\sqcup, [-1]} \to \Gamma \times B^{(2)}(A). \tag{A.5}$$

Suppose that $F : \Gamma \to B^{(2)}(A)$ is an $\mathbb{E}_\infty$-monoidal morphism. Taking fiber product of (A.5) with $(\operatorname{id}_\Gamma, F)$ yields an $\mathbb{E}_\infty$-monoidal functor $\mathcal{C}_F^\sqcup \to \Gamma$.

Finally, we apply the construction of §A.2.3 to obtain $\mathcal{C}_F := (\mathcal{C}_F^\sqcup)^\oplus$, which is a stack of symmetric monoidal E-linear additive categories equipped with a compatible Γ-grading. This is the "F-twist" of $\mathcal{C}$.

Remark A.2.5. Let us give an informal account of $\mathcal{C}_F$. It admits a Γ-grading:

$$\mathcal{C}_F \cong \bigoplus_{\lambda \in \Gamma} (\mathcal{C}_\lambda)_{F(\lambda)},$$

where $(\mathcal{C}_\lambda)_{F(\lambda)}$ is the $F(\lambda)$-twist of $\mathcal{C}_\lambda$ in the sense of §A.2.1. The monoidal operation on $\mathcal{C}_F$ is given as follows: for $\lambda_1, \lambda_2 \in \Gamma$, we have $(\mathcal{C}_{\lambda_1})_{F(\lambda_1)} \times (\mathcal{C}_{\lambda_2})_{F(\lambda_2)} \to (\mathcal{C}_{\lambda_1 + \lambda_2})_{F(\lambda_1 + \lambda_2)}$ coming from the monoidal operation on $\mathcal{C}$ and the monoidal structure on F. The symmetric monoidal structure is likewise induced from those of $\mathcal{C}$ and F.

References

[ABD⁺66] Michael Artin, Jean-Etienne Bertin, Michel Demazure, Alexander Grothendieck, Pierre Gabriel, Michel Raynaud, and Jean-Pierre Serre, *Schémas en groupes. Séminaire de géométrie algébrique du Bois Marie 1962–64 (SGA 3)*, Séminaire de Géométrie Algébrique de l'Institut des Hautes Études Scientifiques, Institut des Hautes Études Scientifiques, Paris, 1963/1966.

[AGV⁺06] Michael Artin, Pierre Grothendieck, Jean-Louis Verdier, Nicolas Bourbaki, Pierre Deligne, and Bernard Saint-Donat, *Théorie des topos et cohomologie etale des schémas. Séminaire de géométrie algébrique du Bois-Marie 1963-1964 (SGA 4): Tome 3*, vol. 305, Springer, 2006.

[BD01] Jean-Luc Brylinski and Pierre Deligne, *Central extensions of reductive groups by* $\mathbf{K}_2$, Publ. Math. Inst. Hautes Études Sci. (2001), no. 94, 5–85. MR 1896177

[BD04] Alexander Beilinson and Vladimir Drinfeld, *Chiral algebras*, American Mathematical Society Colloquium Publications, vol. 51, American Mathematical Society, Providence, RI, 2004. MR 2058353

[BT65] Armand Borel and Jacques Tits, *Groupes réductifs*, Inst. Hautes Études Sci. Publ. Math. (1965), no. 27, 55–150. MR 207712

[Che94] C. Chevalley, *Sur les décompositions cellulaires des espaces* G/B, Algebraic groups and their generalizations: classical methods (University Park, PA, 1991), Proc. Sympos. Pure Math., vol. 56, Amer. Math. Soc., Providence, RI, 1994, With a foreword by Armand Borel, pp. 1–23. MR 1278698

[Del96] P. Deligne, *Extensions centrales de groupes algébriques simplement connexes et cohomologie galoisienne*, Inst. Hautes Études Sci. Publ. Math. (1996), no. 84, 35–89 (1997). MR 1441006

[Gai20] Dennis Gaitsgory, *Parameterization of factorizable line bundles by* K-*theory and motivic cohomology*, Selecta Math. (N.S.) **26** (2020), no. 3, Paper No. 44, 50. MR 4117995

[GG18] Wee Teck Gan and Fan Gao, *The Langlands-Weissman program for Brylinski-Deligne extensions*, L-groups and the Langlands program for covering groups, no. 398, Société Mathématique de France, Paris, 2018, L-groups and the Langlands program for covering groups, pp. 187–275. MR 3802419

[GL18] D. Gaitsgory and S. Lysenko, *Parameters and duality for the metaplectic geometric Langlands theory*, Selecta Math. (N.S.) **24** (2018), no. 1, 227–301. MR 3769731

[Lur09] Jacob Lurie, *Higher topos theory*, Annals of Mathematics Studies, vol. 170, Princeton University Press, Princeton, NJ, 2009. MR 2522659

[Lur17] ———, *Higher algebra*, https://people.math.harvard.edu/~lurie/papers/HA.pdf, 2017, preprint.

[Lur18] ———, *Spectral algebraic geometry*, https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf, February 2018, preprint.

[Lus93] George Lusztig, *Introduction to quantum groups*, Progress in Mathematics, vol. 110, Birkhäuser Boston, Inc., Boston, MA, 1993. MR 1227098

[MVW06] Carlo Mazza, Vladimir Voevodsky, and Charles Weibel, *Lecture notes on motivic cohomology*, Clay Mathematics Monographs, vol. 2, American Mathematical Society, Providence, RI; Clay Mathematics Institute, Cambridge, MA, 2006. MR 2242284

[PR11] Bjorn Poonen and Eric Rains, *Self cup products and the theta characteristic torsor*, Math. Res. Lett. **18** (2011), no. 6, 1305–1318. MR 2915483

[PR12] ———, *Random maximal isotropic subspaces and Selmer groups*, J. Amer. Math. Soc. **25** (2012), no. 1, 245–269. MR 2833483

[Ser65] Jean-Pierre Serre, *Cohomologie Galoisienne*, Lecture Notes in Mathematics, No. 5, Springer-Verlag, Berlin-New York, 1965, With a contribution by Jean-Louis Verdier, Troisième édition, 1965. MR 0201444

[Sta18] The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu, 2018.

[Tat67] J. T. Tate, *Global class field theory*, Algebraic Number Theory (Proc. Instructional Conf., Brighton, 1965), Thompson, Washington, D.C., 1967, pp. 162–203. MR 0220697

[TZ21] James Tao and Yifei Zhao, *Central extensions by* $\mathbf{K}_2$ *and factorization line bundles*, Math. Ann. **381** (2021), no. 1-2, 769–805. MR 4322626

[Wei18] Martin H. Weissman, L-*groups and parameters for covering groups*, Astérisque, no. 398, Société Mathématique de France, 2018, L-groups and the Langlands program for covering groups, pp. 33–186. MR 3802418

[Zha20]   Yifei Zhao, *Tame twistings and Θ-data*, `https://arxiv.org/abs/2004.09671`, 2020, arXiv preprint.

Email address: `yifei.zhao@uni-muenster.de`