



Moduli of local shtukas and Harris’s conjecture

David Hansen

We prove, under a certain assumption of “Hodge–Newton reducibility”, a strong form of a conjecture of Harris on the cohomology of moduli spaces of mixed-characteristic local shtukas for GL_n . Our strategy is roughly based on a previous strategy developed by Mantovan in the setting of p -divisible groups, but the arguments are completely different. In particular, we reinterpret and generalize the Hodge–Newton filtration of a p -divisible group in terms of modified vector bundles on the Fargues–Fontaine curve. We also compute the dualizing complex and compactly supported étale cohomology of any positive Banach–Colmez space over any base; this should be of independent interest.

1. Introduction	749
2. Preliminaries	764
3. Canonical filtrations on an admissible modification	777
4. Applications to moduli of shtukas	781
Acknowledgements	798
References	798

1. Introduction

Since their dramatic appearance in Harris and Taylor’s proof [2001] of the local Langlands conjecture for GL_n , moduli spaces of p -divisible groups have played a central role in the Langlands program. Until recently, the most general available family of spaces of this type was essentially the “Rapoport–Zink spaces” defined and studied in [Rapoport and Zink 1996]. In his 2014 course at Berkeley [Scholze and Weinstein 2020], Scholze vastly generalized these ideas by constructing moduli spaces of *mixed-characteristic local shtukas*. Like Shimura varieties, these spaces are constructed axiomatically from simple group-theoretic input data. In general they are not rigid analytic varieties, and must be interpreted as *diamonds*; this is related to the fact that the Hodge type of the p -adic Hodge structures they parametrize is not necessarily minuscule.

MSC2020: 11S37, 14G22, 14G45.

Keywords: local shtukas, Harris’s conjecture, perfectoid spaces, diamonds.

In this paper, we study a conjecture of Harris on the cohomology of these spaces, using the language and tools developed in [Scholze and Weinstein 2020; Scholze 2017]. Roughly speaking, Harris’s conjecture says that when the underlying “local shtuka datum” is not basic, the (class in the Grothendieck group of the) cohomology of the space is parabolically induced. In the original setting of Rapoport–Zink spaces, Mantovan [2008] proved many cases of Harris’s conjecture in a beautiful paper. Mantovan’s wonderful idea is that under a certain assumption of “Hodge–Newton reducibility”, the spaces themselves are parabolically induced.

Our goal here, broadly stated, is to extend Mantovan’s strategy to the more general spaces considered in [Scholze and Weinstein 2020]. However, out of necessity, the ingredients and details of our arguments are completely different from those in [Mantovan 2008]. In particular, one of our main observations is that the structures observed by Mantovan are entirely accounted for by the actions of certain group diamonds. We also reinterpret and generalize the Hodge–Newton filtration of a p -divisible group considered by Mantovan and Viehmann [2010], which plays a key role in [Mantovan 2008], in the language of modified vector bundles on the Fargues–Fontaine curve. Finally, we carry out a difficult calculation in the étale cohomology of diamonds, which should have interesting applications beyond those given here.

1A. Local shtukas and their moduli. In this section we define moduli spaces of mixed-characteristic local shtukas with infinite level structure, summarizing some material from [Scholze and Weinstein 2020].

Fix a finite extension E/\mathbb{Q}_p with uniformizer π and residue field \mathbb{F}_q . Set $\Gamma = \text{Gal}(\bar{E}/E)$ and $\check{E} = \bar{E}^{\text{unr}}$, and let $\sigma \in \text{Aut}(\check{E}/E)$ be the natural q -Frobenius. Choose a reductive group \mathbf{G}/E . For simplicity in this introduction, we assume that \mathbf{G} is split over E , so we may choose a Borel subgroup $\mathbf{B} \subset \mathbf{G}$ and a split maximal torus $\mathbf{T} \subset \mathbf{B}$ both defined over E . In the remainder of this article, we always denote algebraic groups over E in boldface, and we denote their E -points in standard font, so $G = \mathbf{G}(E)$, $B = \mathbf{B}(E)$, etc.

For the purposes of this introduction, a *local shtuka datum* is a triple (\mathbf{G}, μ, b) , where $\mu \in X_*(\mathbf{T})_{\text{dom}}$ is a \mathbf{B} -dominant cocharacter and $b \in \mathbf{G}(\check{E})$ is an element whose σ -conjugacy class $[b] \in B(\mathbf{G})$ lies in the Kottwitz set $B(\mathbf{G}, \mu^{-1})$.¹ Let \mathbf{J}_b denote the σ -centralizer of b , so this is the algebraic group over E whose functor of points sends any E -algebra R to the group

$$\mathbf{J}_b(R) = \{g \in \mathbf{G}(R \otimes_E \check{E}) \mid g = b(\text{id} \otimes \sigma)(g)b^{-1}\}.$$

For any perfectoid space S over \check{E} , the datum of b gives rise to a \mathbf{G} -bundle \mathcal{E}_{b,S^\flat} on the relative Fargues–Fontaine curve $\mathcal{X}_{S^\flat} = \mathcal{X}_{S^\flat,E}$, functorially in S , whose

¹Again, we do not take a maximally general setup here.

isomorphism class depends only on the σ -conjugacy class $[b]$. We then make the following definition:

Definition 1.1. The *moduli space of local shtukas* (with one leg, and with infinite level structure) associated with the datum (G, μ, b) is the functor

$$\mathrm{Sht}_{G, \mu, b} : \mathrm{Perfd}_{/\check{E}} \rightarrow \mathrm{Sets}$$

on perfectoid spaces over \check{E} sending $S \in \mathrm{Perfd}_{/\check{E}}$ to the set of isomorphism classes of triples (\mathcal{F}, u, α) , where \mathcal{F} is a G -bundle over \mathcal{X}_{S^\flat} ,

$$u : \mathcal{F}|_{\mathcal{X}_{S^\flat} \setminus S} \xrightarrow{\sim} \mathcal{E}_{b, S^\flat}|_{\mathcal{X}_{S^\flat} \setminus S}$$

is an isomorphism which extends to a type- μ meromorphic modification of \mathcal{E}_{b, S^\flat} along the natural closed immersion $S \subset \mathcal{X}_{S^\flat}$, and $\alpha : \mathcal{E}_{\mathrm{triv}, S^\flat} \xrightarrow{\sim} \mathcal{F}$ is a G -bundle isomorphism trivializing \mathcal{F} .

We note that, strictly speaking, it is more natural to consider slightly larger moduli spaces parametrizing shtukas whose meromorphic type is bounded by μ rather than being given exactly by μ , which we denote by $\mathrm{Sht}_{G, \leq \mu, b}$. Indeed, this is what is done in [Scholze and Weinstein 2020]. From the point of view of the present paper, however, it is more natural to fix the meromorphic type exactly. When μ is minuscule, there is no difference.

The space $\mathrm{Sht}_{G, \mu, b}$ is something like an infinite-level Rapoport–Zink space, and the following heuristic might be helpful in parsing the definition of $\mathrm{Sht}_{G, \mu, b}$:

- the data of \mathcal{E}_{b, S^\flat} is “like” the p -divisible group $H = H_b \times_{\overline{\mathbb{F}}_q} S^\flat$ over S^\flat , where $H_b/\overline{\mathbb{F}}_q$ is the (nonexistent) “ π -divisible \mathcal{O}_E -module with G -structure” whose \mathcal{O}_E -Dieudonné module is determined by b ;
- the data of \mathcal{F} and u is “like” a quasideformation \tilde{H} of H to S ;
- the data of α is “like” a trivialization of the rational Tate module $V_p \tilde{H}$.

In some cases when μ is minuscule, this heuristic can be made into literal truth, but in general $\mathrm{Sht}_{G, \mu, b}$ is unrelated to p -divisible groups.

In any case, the functor $\mathrm{Sht}_{G, \mu, b}$ is a very rich object. First of all, it's obviously fibered over $\mathrm{Spa} \check{E}$ as a functor, and it's not hard to check that $\mathrm{Sht}_{G, \mu, b}$ defines a sheaf on the big pro-étale site $\mathrm{Perfd}_{/\check{E}}^{\mathrm{proet}}$. By the equivalence of sites

$$\mathrm{Perfd}_{/\check{E}}^{\mathrm{proet}} \cong \mathrm{Perf}_{/\mathrm{Spd} \check{E}}^{\mathrm{proet}},$$

where $\mathrm{Perf} \subset \mathrm{Perfd}$ denotes the category of characteristic p perfectoid spaces, we can and do regard $\mathrm{Sht}_{G, \mu, b}$ as a pro-étale sheaf on Perf fibered over the diamond $\mathrm{Spd} \check{E}$.

Next, we observe that there are two natural commuting group actions on $\mathrm{Sht}_{G, \mu, b}$: on the one hand, the group $G \cong \mathrm{Aut}(\mathcal{E}_{\mathrm{triv}})$ acts on $\mathrm{Sht}_{G, \mu, b}$ via the right action

sending α to $\alpha \circ g$; on the other hand, $J_b < \text{Aut}(\mathcal{E}_b)$ acts via the left action sending u to $j \circ u$. Here, following our conventions, J_b is the group of E -points in the σ -centralizer J_b .

There are also two period maps out of $\text{Sht}_{G,\mu,b}$, the Grothendieck–Messing period map,

$$\pi_{\text{GM}} : \text{Sht}_{G,\mu,b} \rightarrow \text{Gr}_{G,\mu},$$

and the Hodge–Tate period map,

$$\pi_{\text{HT}} : \text{Sht}_{G,\mu,b} \rightarrow \text{Gr}_{G,\mu^{-1}}.$$

Here $\text{Gr}_{G,\mu}$ is an open Schubert cell in the \mathbb{B}_{dR} -affine Grassmannian $\text{Gr}_G/\text{Spd } \check{E}$, whose definition we will recall later. By the results of [Scholze and Weinstein 2020], $\text{Gr}_{G,\mu}$ is a locally spatial diamond over $\text{Spd } \check{E}$. The map π_{GM} forgets α , while the map π_{HT} interprets $\alpha^{-1} \circ u^{-1}$ as giving a (nontrivialized) type- μ^{-1} modification of $\mathcal{E}_{\text{triv}}$ along S . There are natural actions of J_b and G on $\text{Gr}_{G,\mu}$ and $\text{Gr}_{G,\mu^{-1}}$, respectively, such that π_{GM} and π_{HT} are $G \times J_b$ -equivariant for (respectively) the trivial actions of G and J_b on their targets.

The starting point for the investigations in this paper is the following result.

Theorem 1.2 (Scholze, via Caraiani–Scholze, Fargues–Fontaine, Kedlaya–Liu, Scholze–Weinstein). *Let the notation and assumptions be as above. Then:*

- (i) *The image of the period morphism π_{GM} is a nonempty open and partially proper subdiamond $\text{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$ of the diamond $\text{Gr}_{G,\mu}$, stable under the action of J_b .*
- (ii) *The induced morphism*

$$\pi_{\text{GM}} : \text{Sht}_{G,\mu,b} \rightarrow \text{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$$

is representable and pro-étale, and it makes $\text{Sht}_{G,\mu,b}$ into a pro-étale G -torsor over $\text{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$. In particular, $\text{Sht}_{G,\mu,b}$ is a locally spatial diamond over $\text{Spd } \check{E}$.

- (iii) *For any open compact subgroup $K \subset G$, the quotient $\text{Sht}_{G,\mu,b}/\underline{K}$ parametrizing shtukas with K -level structure is a locally spatial diamond étale over $\text{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$.*
- (iv) *When μ is minuscule, the diamonds $\text{Gr}_{G,\mu}$, $\text{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$ and $\text{Sht}_{G,\mu,b}/\underline{K}$ are in the essential image of the functor $(-)^{\diamond}$ from smooth rigid analytic spaces over $\text{Spa } \check{E}$. In particular, the rigid space \mathcal{M}_K over $\text{Spa } \check{E}$ such that*

$$\mathcal{M}_K^{\diamond} \cong \text{Sht}_{G,\mu,b}/\underline{K}$$

*is the **local Shimura variety with K -level structure** associated with the datum (G, μ, b) sought by Rapoport and Viehmann [2014].*

- (v) When μ is minuscule and $G = \mathrm{GL}_n$, there is a natural isomorphism $\mathrm{Sht}_{G,\mu,b} \cong \mathcal{M}_{H_b,\infty}^\diamond$ compatible with all structures, where $\mathcal{M}_{H_b,\infty}$ is a certain infinite-level Rapoport–Zink space.

In the case where $G = \mathrm{GL}_n/E$, we give a detailed proof of parts (i)–(iv) of this theorem in [Section 2C](#) below. Let us emphasize that we don't claim any real originality here: the existence of (some version of) the space $\mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$ together with its universal \mathbb{Q}_p -local system was announced over eight years ago by Kedlaya and Liu, and the possibility of constructing this space by some version of the argument we give was one of the primary motivations for the writing of [\[Kedlaya and Liu 2015\]](#). In the case of minuscule μ , Rapoport and Viehmann [\[2014\]](#) formulated (prior to the invention of diamonds and the proof of [Theorem 1.2](#)) a very precise qualitative description of the spaces $\mathrm{Sht}_{G,\mu,b}$. As far as we can tell, though, the idea that spaces like $\mathrm{Sht}_{G,\mu,b}$ might exist in some reasonable geometric category for an *arbitrary* Hodge cocharacter μ is due entirely to Scholze. (We also note that an analogous theorem holds for the spaces $\mathrm{Sht}_{G,\leq\mu,b}$, and this is a much more subtle result; indeed, the fact that $\mathrm{Sht}_{G,\leq\mu,b}$ is a locally spatial diamond is one of the main theorems of [\[Scholze and Weinstein 2020\]](#).)

1B. Cohomology of moduli of local shtukas. One of the primary motivations for studying spaces like $\mathrm{Sht}_{G,\mu,b}$ is that their $\overline{\mathbb{Q}}_\ell$ -cohomology is widely expected to realize instances of the local Langlands correspondence for G and the local Jacquet–Langlands correspondence between G and J_b . In the particular case when b is basic, the group J_b is an inner form of G , and a precise conjecture was formulated by Kottwitz. Very roughly, Kottwitz's conjecture predicts that for $\varphi : W_E \rightarrow {}^L G(\overline{\mathbb{Q}}_\ell)$ a discrete L -parameter and π (resp. ρ) an irreducible smooth representation of G (resp. J_b) in the discrete L -packet associated with φ , the virtual W_E -representation²

$$\sum_{i \geq 0} (-1)^i \mathrm{Hom}_{G \times J_b} \left(H_c^i \left(\mathrm{Sht}_{G,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} \widehat{E}, \overline{\mathbb{Q}}_\ell \right), \pi \boxtimes \rho \right)$$

coincides with some number of copies of $r_\mu \circ \varphi$, where r_μ is the algebraic representation of ${}^L G$ with highest weight μ , and that this number can be read off from expected properties of the local Langlands correspondence. In particular, Kottwitz's conjecture implies that for b basic, *every* supercuspidal representation of G occurs in the geometric étale cohomology of $\mathrm{Sht}_{G,\mu,b}$.

On the other hand, Harris [\[2001\]](#) conjectured that for b *not* basic, *no* supercuspidal representation of G contributes to the Euler characteristic of $H_c^*(\mathrm{Sht}_{G,\mu,b}, \overline{\mathbb{Q}}_\ell)$. This follows from a more quantitative statement, which we now describe. To formulate Harris's conjecture, let $M_{[b^{-1}]}$ be the standard Levi subgroup centralizing the B -dominant Newton cocharacter $\nu_{[b^{-1}]} \in X_*(T)_{\mathbb{Q},\mathrm{dom}}$. After possibly replacing

²See [Section 4C](#) for a precise discussion of the cohomology groups $H_c^*(-, \overline{\mathbb{Q}}_\ell)$ considered here.

b by a σ -conjugate, we can and do assume that $b \in M_{[b^{-1}]}(\check{E})$ and that $v_{b^{-1}}$ is $M_{[b^{-1}]}(\check{E})$ -conjugate to $v_{[b^{-1}]}$; if these properties hold, we say b is *well-chosen*.³ For any standard Levi subgroup M containing $M_{[b^{-1}]}$, consider the finite set of cocharacters

$$\mathcal{W}_{\mu,b}(M) = \{\lambda \in X_*(T)_{M\text{-dom}} \cap W \cdot \mu \mid [b^{-1}] \in B(M, \lambda)\},$$

where W denotes the absolute Weyl group of G . For each $\lambda \in \mathcal{W}_{\mu,b}(M)$, the tuple (M, λ, b) defines a local shtuka datum. In the setting of p -divisible groups (i.e., for minuscule μ), Harris conjectured a formula expressing (roughly) the ρ -part of the virtual representation

$$\sum_{i \geq 0} (-1)^i H_c^i(\text{Sht}_{G,\mu,b} \times_{\text{Spd } \check{E}} \text{Spd } \widehat{\check{E}}, \overline{\mathbb{Q}}_\ell)$$

in terms of the ρ -part of the virtual representation

$$\sum_{\lambda \in \mathcal{W}_{\mu,b}(M)} \sum_{i \geq 0} (-1)^i \text{Ind}_P^G(H_c^i(\text{Sht}_{M,\lambda,b} \times_{\text{Spd } \check{E}} \text{Spd } \widehat{\check{E}}, \overline{\mathbb{Q}}_\ell)).$$

Here M is the Levi factor of the standard parabolic P , and ρ denotes any irreducible smooth representation of J_b . We note that Harris's original formulation of his conjecture was slightly wrong for nonsplit G , and the corrected formulation given above (and its generalization beyond the quasisplit case) is due to Viehmann, see [Rapoport and Viehmann 2014, Conjecture 8.4] for the most general statement. We follow Rapoport in calling this general statement the *Harris–Viehmann conjecture*.

Let us note right away that it's a little delicate to extend Harris's conjecture beyond the case of minuscule μ , since there are cases where the set $\mathcal{W}_{\mu,b}(M)$ as defined above is empty (although there is a natural way to modify the definition of $\mathcal{W}_{\mu,b}(M)$ which fixes this problem). However, it still seems reasonable to expect that for any $\lambda \in \mathcal{W}_{\mu,b}(M)$, the cohomology of $\text{Sht}_{M,\lambda,b}$ contributes to the cohomology of $\text{Sht}_{G,\mu,b}$ in some way. Note in particular that if $\mu \in \mathcal{W}_{\mu,b}(M)$, we at least have a natural map of diamonds

$$\text{Sht}_{M,\mu,b} \rightarrow \text{Sht}_{G,\mu,b},$$

and one can ask how close this map, or some parabolic induction of it, comes to describing the total cohomology of $\text{Sht}_{G,\mu,b}$. One of the main results of this article is that when $G = \text{GL}_n$ and the datum (G, μ, b) is *Hodge–Newton reducible* in the sense defined below, there is a canonical Levi subgroup $M \subsetneq G$ such that this map completely accounts for the cohomology of its target.

³The presence of inverses here (and elsewhere) is related to the negation of slopes which occurs when passing from b to the associated bundle \mathcal{E}_b : the “slope cocharacter” of \mathcal{E}_b is given by $v_{b^{-1}}$.

1C. A canonical retraction of period domains. We now set up the notation and terminology necessary to state our results precisely. For the remainder of the paper, we restrict our attention to the case $\mathbf{G} = \mathrm{GL}_n/E$. Let \mathbf{B} be the upper-triangular Borel, and choose a \mathbf{B} -dominant diagonal cocharacter μ with exponents $(k_1 \geq \cdots \geq k_n) \in \mathbb{Z}^n$. As usual, we conflate μ with the ordered tuple of the k_i , and we conflate \mathbf{G} -bundles on any $\check{\mathcal{X}}_{S^\flat} = \check{\mathcal{X}}_{S^\flat, E}$ with rank n vector bundles.

Fix an element $b \in \mathrm{GL}_n(\check{E})$ with $[b] \in B(\mathbf{G}, \mu^{-1})$, and let \mathcal{E}_{b, S^\flat} denote the associated rank n vector bundle on the relative Fargues–Fontaine curve \mathcal{X}_{S^\flat} for any $S \in \mathrm{Perfd}/\check{E}$ as before. Any map $S \rightarrow T$ induces a canonical map $\mathcal{X}_{S^\flat} \rightarrow \mathcal{X}_{T^\flat}$ such that the pullback of \mathcal{E}_{b, T^\flat} identifies naturally with \mathcal{E}_{b, S^\flat} . In particular, we sometimes denote the bundle agnostically by \mathcal{E}_b .⁴ For simplicity we assume that $k_n \geq 0$,⁵ so an S -point $f : S \rightarrow \mathrm{Gr}_{\mathrm{GL}_n, \mu}$ corresponds to a modification of vector bundles

$$u : \mathcal{F}|_{\mathcal{X}_{S^\flat} \setminus S} \xrightarrow{\sim} \mathcal{E}_{b, S^\flat}|_{\mathcal{X}_{S^\flat} \setminus S}$$

which is *effective*, i.e., a modification for which u extends to an injection $u : \mathcal{F} \hookrightarrow \mathcal{E}_{b, S^\flat}$ of finite locally free $\mathcal{O}_{\mathcal{X}}$ -modules. By definition, such a modification lies in the *admissible locus* $\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$ defined earlier if and only if \mathcal{F} is pointwise-semistable of slope zero at all points of $|S|$; we'll refer to an S -point $(\mathcal{F}, u) \in \mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}(S)$ as an *admissible (type- μ) modification of \mathcal{E}_b along S* .

Let $0 = \mathcal{E}_b^0 \subsetneq \mathcal{E}_b^1 \subsetneq \mathcal{E}_b^2 \subsetneq \cdots \subsetneq \mathcal{E}_b^s = \mathcal{E}_b$ denote the slope filtration of \mathcal{E}_b , where s denotes the number of distinct slopes of \mathcal{E}_b . Each $\mathcal{E}_b^i/\mathcal{E}_b^{i-1}$ is a semistable vector bundle, with strictly decreasing slopes as a function of i . The condition $[b] \in B(\mathbf{G}, \mu^{-1})$ unwinds in this setting to the usual relation between Newton and Hodge polygons, or equivalently to the condition that the inequality

$$\deg(\mathcal{E}_b^i) \leq \sum_{1 \leq j \leq \mathrm{rank}(\mathcal{E}_b^i)} k_j$$

holds for every $1 \leq i \leq s$, with equality for $i = s$. Let $\mathcal{I} \subseteq \{1, \dots, s\}$ denote the ordered set of integers for which this inequality is an *equality*; since $s \in \mathcal{I}$ always, $|\mathcal{I}| \geq 1$.

Definition 1.3. The datum (\mathbf{G}, μ, b) is *Hodge–Newton reducible (HN-reducible)* if $|\mathcal{I}| \geq 2$. We say \mathcal{E}_b^i is *HN-reducing* if $i \in \mathcal{I} \setminus \{s\}$.

Graphically, this is equivalent to requiring that aside from touching at their end-points, the Newton and Hodge polygons also touch at some interior break-point of the Newton polygon.

⁴Whereby we're really regarding it as an $\overline{\mathbb{F}}_q$ -point of the stack Bun_n .

⁵This is no restriction on our results, since it can always be achieved by a suitable “central twisting” of the datum (b, μ) which leaves all spaces in question essentially unchanged.

Remark. When μ is minuscule, it's easy to check that at most two interior break-points of the Newton polygon touch the Hodge polygon. For nonminuscule μ , however, the number of such points can be arbitrarily large.

Let $\mathcal{I} = \{i_1 < \cdots < i_k = s\} \subset \{1, \dots, s\}$ be the ordered set of indices in the slope filtration as defined above, and let $\{d_1, \dots, d_k\} \in \mathbb{N}^k$ denote the ordered set

$$\{\text{rank}(\mathcal{E}_b^{i_1}), \text{rank}(\mathcal{E}_b^{i_2}/\mathcal{E}_b^{i_1}), \dots, \text{rank}(\mathcal{E}_b^{i_k}/\mathcal{E}_b^{i_{k-1}})\}.$$

Consider the standard Levi

$$M = \begin{pmatrix} \text{GL}_{d_1} & & & \\ & \text{GL}_{d_2} & & \\ & & \ddots & \\ & & & \text{GL}_{d_k} \end{pmatrix} \subset G,$$

and let $P = MU$ be the associated standard parabolic. We shall refer to M and P as the *Hodge–Newton Levi* (resp. *Hodge–Newton parabolic*) associated with the datum (G, μ, b) . Note that (G, μ, b) is Hodge–Newton reducible if and only if $M \subsetneq G$. After possibly replacing b by a σ -conjugate, we can and do assume that b is well-chosen, so in particular we have an inclusion

$$b \in M_{[b^{-1}]}(\check{E}) \subset M(\check{E}).$$

Writing b_m for the projection of b into the m -th block of M , we then get a decomposition $\mathcal{E}_b \cong \bigoplus_{1 \leq i \leq k} \mathcal{E}_{b_i}$, or equivalently a canonical reduction of \mathcal{E}_b to an M -bundle, such that the induced P -bundle structure on \mathcal{E}_b is a coarsening of the slope filtration.

Writing μ_m for the projection of μ into the m -th block of M , one easily checks that we have a product decomposition

$$\text{Gr}_{M, \mu} \cong \text{Gr}_{\text{GL}_{d_1}, \mu_1} \times_{\text{Spd } \check{E}} \cdots \times_{\text{Spd } \check{E}} \text{Gr}_{\text{GL}_{d_k}, \mu_k}$$

of diamonds over $\text{Spd } \check{E}$. One also checks that (M, μ, b) defines a local shtuka datum, which is naturally decomposed into a direct product of local shtuka data:

$$(M, \mu, b) \cong \prod_{m=1}^k (\text{GL}_{d_m}, \mu_m, b_m).$$

This product decomposition induces canonical compatible isomorphisms

$$\text{Gr}_{M, \mu}^{\mathcal{E}_b\text{-adm}} \cong \text{Gr}_{\text{GL}_{d_1}, \mu_1}^{\mathcal{E}_{b_1}\text{-adm}} \times_{\text{Spd } \check{E}} \cdots \times_{\text{Spd } \check{E}} \text{Gr}_{\text{GL}_{d_k}, \mu_k}^{\mathcal{E}_{b_k}\text{-adm}},$$

and

$$\text{Sht}_{M, \mu, b} = \text{Sht}_{\text{GL}_{d_1}, \mu_1, b_1} \times_{\text{Spd } \check{E}} \cdots \times_{\text{Spd } \check{E}} \text{Sht}_{\text{GL}_{d_k}, \mu_k, b_k}.$$

There is also a compatible and canonical J_b -equivariant inclusion

$$i : \mathrm{Gr}_{M,\mu} \hookrightarrow \mathrm{Gr}_{G,\mu}$$

induced by the decomposition $\mathcal{E}_b \cong \bigoplus_{1 \leq i \leq k} \mathcal{E}_{b_i}$ and thereby sending $\mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}}$ into $\mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$, and this inclusion fits into a diagram

$$\begin{array}{ccc} \mathrm{Sht}_{M,\mu,b} & \xrightarrow{i_\infty} & \mathrm{Sht}_{G,\mu,b} \\ \pi_{\mathrm{GM}} \downarrow & & \downarrow \pi_{\mathrm{GM}} \\ \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} & \xrightarrow{i} & \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}} \end{array}$$

equivariant for all obvious group actions.

Next, we observe that the left and right columns of this diagram admit canonical actions of certain groups objects $\mathcal{J}_{b,\check{E}}^M$ and $\mathcal{J}_{b,\check{E}}$, respectively, extending the action of J_b : here $\mathcal{J}_{b,\check{E}}$ and $\mathcal{J}_{b,\check{E}}^M$ are the functors on $\mathrm{Perf}/\mathrm{Spd}\check{E}$ sending any S/\check{E} to the group of bundle automorphisms $\mathrm{Aut}(\mathcal{E}_{b,S^\flat})$, resp. to the subgroup of automorphisms which respect the canonical M -bundle structure described above. Again, an element j acts by sending a pair (\mathcal{F}, u) to $(\mathcal{F}, j \circ u)$. It's not hard to see that $\mathcal{J}_{b,\check{E}}$ canonically decomposes as the semidirect product $\mathcal{J}_{b,\check{E}}^M \ltimes \mathcal{J}_{b,\check{E}}^U$, where $\mathcal{J}_{b,\check{E}}^U$ is the subgroup of elements $j \in \mathrm{Aut}(\mathcal{E}_b)$ such that $j - \mathrm{id}$ carries each $\mathcal{E}_b^{i_m}$ into $\mathcal{E}_b^{i_m-1}$, and that $J_b < \mathcal{J}_b^M$ compatibly with all group actions. The functor $\mathcal{J}_{b,\check{E}}$ and its decorated variants are examples of *group diamonds* over $\mathrm{Spd}\check{E}$. Our main observation, roughly speaking, is that in the HN-reducible setting these group diamonds are “large enough” to account for the difference between the period domains and shtuka spaces associated with M and those associated with G .

Our first precise result along these lines is as follows.

Theorem 1.4. *Maintain the notation and assumptions as above. Then:*

- (i) *The inclusion $i : \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} \hookrightarrow \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$ admits a canonical $\mathcal{J}_{b,\check{E}}^M$ -equivariant retraction*

$$r : \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}} \rightarrow \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}}.$$

In other words, any admissible type- μ modification (\mathcal{F}, u) of \mathcal{E}_{b,S^\flat} along S admits a canonical reduction to a collection of admissible type- μ_m modifications (\mathcal{F}_m, u_m) of the bundles $\mathcal{E}_{b_m,S^\flat}$ along S , for all $1 \leq m \leq k = |\mathcal{I}|$.

- (ii) *The natural action map*

$$\mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} \times_{\mathrm{Spd}\check{E}} \mathcal{J}_{b,\check{E}}^U \rightarrow \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$$

induced by i is surjective and pro-étale.

This result seems to be new even in the setting of p -divisible groups.

Let us illustrate this theorem in the simple case where $E = \mathbb{Q}_p$, $G = \mathrm{GL}_2$, $\mu = (1, 0)$, and $b = \mathrm{diag}(p^{-1}, 1)$. Then M is the diagonal maximal torus and

$$\mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} = \mathrm{Gr}_{M,\mu} = \mathrm{Spd} \check{\mathbb{Q}}_p$$

is a single point. We also have a natural isomorphism $\mathrm{Gr}_{G,\mu} \cong P_{\check{\mathbb{Q}}_p}^{1,\diamond}$ which, by an old result of Dwork, induces an isomorphism

$$\mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}} \cong \mathbb{A}_{\check{\mathbb{Q}}_p}^{1,\diamond}$$

(see [Rapoport and Zink 1996]). Then i is just the inclusion of $\mathrm{Spd} \check{\mathbb{Q}}_p$ at the origin, and r is the structure map to $\mathrm{Spd} \check{\mathbb{Q}}_p$. More exotically, we find that

$$\mathcal{J}_{b,\check{\mathbb{Q}}_p}^U \cong \mathbb{B}_{\mathrm{crys}}^{+,\varphi=p} \cong (\mathrm{Spf} \check{\mathbb{Z}}_p \llbracket T^{1/p^\infty} \rrbracket)_\eta^{\mathrm{ad}}$$

is representable by an open preperfectoid ball over $\mathrm{Spa} \check{\mathbb{Q}}_p$, and the action of this on $\mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$ is given as follows: for any $S = \mathrm{Spa}(A, A^+) \in \mathrm{Perf}/\check{\mathbb{Q}}_p$, an element $j \in \mathbb{B}_{\mathrm{crys}}^{+,\varphi=p}(A)$ acts by sending an element

$$a \in A = \mathrm{Hom}_{/\mathrm{Spd} \check{\mathbb{Q}}_p}(\mathrm{Spd}(A, A^+), \mathbb{A}_{\check{\mathbb{Q}}_p}^{1,\diamond})$$

to the element $a + \theta(j)$. Finally, the map in part (ii) is just the usual surjection $\theta : \mathbb{B}_{\mathrm{crys}}^{+,\varphi=p} \twoheadrightarrow \mathbb{A}^{1,\diamond}$. Thus, even in this simple case, $\mathcal{J}_{b,\check{\mathbb{Q}}_p}^U$ is not a classical object: it's a group object in preperfectoid spaces. This may explain why the actions of $\mathcal{J}_{b,\check{\mathbb{Q}}_p}$ and $\mathcal{J}_{b,\check{\mathbb{Q}}_p}^U$ at the level of period domains haven't been much exploited.

1D. The idea behind the canonical retraction. We'd like to explain in detail the construction of the retraction r from Theorem 1.4(i) at the level of S -points in the case where $|S|$ is a single point, i.e., when $S = \mathrm{Spa}(K, \mathcal{O}_K)$ for some perfectoid field K/\check{E} . Choose such an S , and let $\mathcal{X} = \mathcal{X}_{K^\flat,E}$ be the associated Fargues–Fontaine curve. This is a locally Noetherian quasicompact adic curve, and we have a natural closed immersion $i : S \rightarrow \mathcal{X}$.

Let

$$0 \rightarrow \mathcal{F} \xrightarrow{u} \mathcal{E} \rightarrow \mathcal{Q} \rightarrow 0$$

be a short exact sequence of coherent sheaves on \mathcal{X} , where \mathcal{F} and \mathcal{E} are rank n vector bundles and \mathcal{Q} is supported at the distinguished point $x(\infty) := i(|S|) \in |\mathcal{X}|$. The stalk $\mathcal{Q} = \mathcal{Q}_{x(\infty)}$ is then a finite torsion module over the discrete valuation ring

$$\widehat{\mathcal{O}}_{\mathcal{X},x(\infty)} \cong \mathbb{B}_{\mathrm{dR}}^+(K).$$

With $\mu = (k_1 \geq \dots \geq k_n) \in \mathbb{Z}^n$ as before, we say (\mathcal{F}, u) is a *type- μ modification of \mathcal{E} along S* if there is an isomorphism

$$\mathcal{Q} \simeq \bigoplus_{1 \leq i \leq n} \mathbb{B}_{\mathrm{dR}}^+(K)/\xi^{k_i}$$

(here ξ denotes any uniformizer of $\mathbb{B}_{\mathrm{dR}}^+(K)$).

Theorem 1.5. *With the notation and assumptions as above, let $\mathcal{E}^+ \subseteq \mathcal{E}$ be any saturated subbundle, i.e., any subbundle such that $\mathcal{E}/\mathcal{E}^+$ is also a vector bundle; set $\mathcal{F}^+ = \mathcal{F} \cap \mathcal{E}^+$. Then if \mathcal{F} is semistable of slope zero, we have the inequality*

$$\deg(\mathcal{E}^+) \leq \sum_{1 \leq j \leq \text{rank}(\mathcal{E}^+)} k_j.$$

If \mathcal{F} is semistable of slope zero and equality holds in the previous inequality, then \mathcal{F}^+ is also semistable of slope zero, and

$$\mathcal{E}^+/\mathcal{F}^+ \simeq \bigoplus_{1 \leq i \leq \text{rank}(\mathcal{E}^+)} \mathbb{B}_{\text{dR}}^+(K)/\xi^{k_i}$$

is a direct summand of Q .

In the setting of Theorem 1.4, we apply this result with $\mathcal{E} = \mathcal{E}_{b, S^\flat}$ and with (\mathcal{F}, u) corresponding to a (K, \mathcal{O}_K) -point of $\text{Gr}_{G, \mu}^{\mathcal{E}_{b, S^\flat}\text{-adm}}$. Then for any $1 \leq i \leq s$ such that $\mathcal{E}^i = \mathcal{E}_{b, S^\flat}^i$ is HN-reducing, the equality $\deg(\mathcal{E}^i) = \sum_{1 \leq j \leq \text{rank}(\mathcal{E}^i)} k_j$ holds by assumption, so by Theorem 1.5 the bundle $\mathcal{F}^i = \mathcal{F} \cap \mathcal{E}^i$ is semistable of slope zero, and the module

$$Q^i := \text{im}(\mathcal{E}^i \rightarrow Q) = \mathcal{E}^i/\mathcal{F}^i \subseteq Q$$

is described by the isomorphism

$$Q^i \simeq \bigoplus_{1 \leq j \leq \text{rank}(\mathcal{E}^i)} \mathbb{B}_{\text{dR}}^+(K)/\xi^{k_j}.$$

Moreover, Q^i is a direct summand of Q . Forming these objects for all $i \in \mathcal{I}$, we get a canonical flag

$$0 \subsetneq \mathcal{F}^{i_1} \subsetneq \dots \subsetneq \mathcal{F}^{i_k} = \mathcal{F}$$

of vector subbundles of \mathcal{F} with each step semistable of slope zero and with successive graded pieces of ranks d_1, \dots, d_k ; we call this flag the *Hodge–Newton filtration* of \mathcal{F} . The successive quotients

$$\mathcal{F}_m = \mathcal{F}^{i_m}/\mathcal{F}^{i_{m-1}}$$

are all semistable of slope zero as well, and they sit in natural short exact sequences

$$0 \rightarrow \mathcal{F}_m \xrightarrow{u_m} \mathcal{E}^{i_m}/\mathcal{E}^{i_{m-1}} \rightarrow Q^{i_m}/Q^{i_{m-1}} \rightarrow 0.$$

But now an easy induction on m shows that

$$Q^{i_m}/Q^{i_{m-1}} \simeq \bigoplus_{\text{rank}(\mathcal{E}^{i_{m-1}}) < j \leq \text{rank}(\mathcal{E}^{i_m})} \mathbb{B}_{\text{dR}}^+(K)/\xi^{k_j},$$

so we conclude that each pair (\mathcal{F}_m, u_m) is canonically an admissible type- μ_m modification of $\mathcal{E}^{i_m}/\mathcal{E}^{i_{m-1}} \cong \mathcal{E}_{b_m}$ along S ! Therefore we get a canonical map

$$r : \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}(K, \mathcal{O}_K) \rightarrow \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}}(K, \mathcal{O}_K), \quad (\mathcal{F}, u) \mapsto \prod_{1 \leq m \leq k} (\mathcal{F}_m, u_m)$$

on (K, \mathcal{O}_K) -points, and this is clearly a retraction of the inclusion

$$\mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}}(K, \mathcal{O}_K) \subset \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}(K, \mathcal{O}_K).$$

Let us remark here that the Hodge–Newton filtration

$$0 \subsetneq \mathcal{F}^{i_1} \subsetneq \cdots \subsetneq \mathcal{F}^{i_k} = \mathcal{F}$$

defined above is our analogue of the Hodge–Newton filtration of a p -divisible group [Mantovan and Viehmann 2010], and the role it plays in this paper is analogous with the role of the Hodge–Newton filtration in [Mantovan 2008]. In fact, it’s not hard to reprove the main results in [Mantovan and Viehmann 2010] by combining Theorem 1.5 with some of the ideas in [Scholze and Weinstein 2013], but we won’t pursue this here.

Happily, the argument for Theorem 1.5 is short and direct, requiring only the basic properties of modifications and slopes together with a piece of elementary commutative algebra. However, in order to deduce Theorem 1.4 in full, we need a *relative* version of Theorem 1.5 treating the situation where $S = \mathrm{Spa}(A, A^+)$ is an arbitrary affinoid perfectoid space over \check{E} . This is harder, for at least two good reasons:

- (1) The relative curve \mathcal{X}_{S^\flat} is not locally Noetherian in general.
- (2) In the relative setting there’s no *a priori* reason for $\mathcal{F} \cap \mathcal{E}^+$ to even be a vector bundle.

Our strategy for the relative version of Theorem 1.5, which is stated and proved as Theorem 3.1 below, is to reduce to the pointwise result above by way of some careful commutative algebra over the relative Fontaine ring $\mathbb{B}_{\mathrm{dR}}^+(A)$. The arguments here are somewhat technical, and rely crucially on various results from Kedlaya and Liu’s foundational volumes [2015; 2019] on relative p -adic Hodge theory.

1E. Adding infinite level structure, and cohomological consequences. Returning to the setting of Section 1C, we now want to equivariantly lift the structures exhibited in Theorem 1.4 to similar structures on moduli of shtukas with infinite level structure. Following Mantovan, we do this by defining an intermediate space $\mathrm{Sht}_{P,\mu,b}$ of P -shtukas. The precise definition is as follows: for any perfectoid space S over \check{E} , the S -points of $\mathrm{Sht}_{P,\mu,b}$ consist of isomorphism classes of triples

$(\mathcal{F}, u, \alpha^P)$, where (\mathcal{F}, u) is an S -point of $\mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$ and $\alpha^P : \mathcal{O}_{\mathcal{X}_{S^b}}^n \xrightarrow{\sim} \mathcal{F}$ is a trivialization matching the flag

$$0 \subsetneq \mathcal{O}_{\mathcal{X}_{S^b}}^{d_1} \subsetneq \mathcal{O}_{\mathcal{X}_{S^b}}^{d_1+d_2} \subsetneq \cdots \subsetneq \mathcal{O}_{\mathcal{X}_{S^b}}^n$$

with the Hodge–Newton flag

$$0 \subsetneq \mathcal{F}^{i_1} \subsetneq \mathcal{F}^{i_2} \subsetneq \cdots \subsetneq \mathcal{F}$$

constructed in [Section 1D](#). In particular, we have inclusions of subfunctors

$$\mathrm{Sht}_{M,\mu,b} \stackrel{(1)}{\subset} \mathrm{Sht}_{P,\mu,b} \stackrel{(2)}{\subset} \mathrm{Sht}_{G,\mu,b},$$

and there is a natural action of P on $\mathrm{Sht}_{P,\mu,b}$ compatible with the M - and G -actions on $\mathrm{Sht}_{M,\mu,b}$ and $\mathrm{Sht}_{G,\mu,b}$. There is also a natural action of $\mathcal{J}_{b,\check{E}} = \mathcal{J}_{b,\check{E}}^{M_\check{E}} \ltimes \mathcal{J}_{b,\check{E}}^U$ on $\mathrm{Sht}_{P,\mu,b}$ making the inclusions (1) and (2) $\mathcal{J}_{b,\check{E}}^{M_\check{E}}$ -equivariant and $\mathcal{J}_{b,\check{E}}$ -equivariant, respectively.

The next two theorems give a precise meaning to the expectation that $\mathrm{Sht}_{P,\mu,b}$ should mediate between $\mathrm{Sht}_{G,\mu,b}$ and $\mathrm{Sht}_{M,\mu,b}$.

Theorem 1.6. *The inclusion $\mathrm{Sht}_{P,\mu,b} \subset \mathrm{Sht}_{G,\mu,b}$ induces a canonical equivariant identification*

$$\mathrm{Sht}_{G,\mu,b} \cong \mathrm{Sht}_{P,\mu,b} \times^P \underline{G}.$$

In particular, the period map

$$\pi_{\mathrm{GM}} : \mathrm{Sht}_{P,\mu,b} \rightarrow \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$$

is a pro-étale P -torsor, and there is a canonical G -equivariant isomorphism

$$H_c^*(\mathrm{Sht}_{G,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} C, \mathbb{Z}/\ell^n) \cong \mathrm{ind}_P^G(H_c^*(\mathrm{Sht}_{P,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} C, \mathbb{Z}/\ell^n))$$

of smooth G -representations preserving degrees and compatible with all additional structures; here ind_P^G denotes unnormalized smooth induction, and C/\check{E} is any complete algebraically closed field. A similar formula holds for $\overline{\mathbb{Q}}_\ell$ -coefficients.

This theorem is a completely straightforward consequence of the results we've proved so far. On the other hand, the following theorem is not obvious.

Theorem 1.7. *The natural action map*

$$a_\infty : \mathrm{Sht}_{M,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathcal{J}_{b,\check{E}}^U \rightarrow \mathrm{Sht}_{P,\mu,b}$$

is an isomorphism of diamonds. In particular, the retraction r lifts canonically to a retraction r_∞ of the natural inclusion $\mathrm{Sht}_{M,\mu,b} \subset \mathrm{Sht}_{P,\mu,b}$, with fibers given by

canonically trivial \mathcal{J}_b^U -torsors. More precisely, the diagram

$$\begin{array}{ccc}
 & \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} \times_{\mathrm{Spd} \check{E}} \mathcal{J}_{b,\check{E}}^U & \\
 \nearrow & & \searrow a \\
 \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} & \xleftarrow{r} & \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}} \\
 \xleftarrow{i} & & \xrightarrow{i}
 \end{array}$$

has a canonical equivariant lifting to a diagram

$$\begin{array}{ccc}
 & \mathrm{Sht}_{M,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathcal{J}_{b,\check{E}}^U & \\
 \nearrow & & \searrow a_\infty \\
 \mathrm{Sht}_{M,\mu,b} & \xleftarrow{r_\infty} & \mathrm{Sht}_{P,\mu,b} \\
 \xleftarrow{i_\infty} & & \xrightarrow{i_\infty}
 \end{array}$$

with the map r_∞ defined as a_∞^{-1} followed by the natural projection

$$\mathrm{Sht}_{M,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathcal{J}_{b,\check{E}}^U \rightarrow \mathrm{Sht}_{M,\mu,b}.$$

The extremely simple structure of the map a_∞ came as a surprise to us.⁶ Note that by the first claim of this theorem, the product $\mathrm{Sht}_{M,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathcal{J}_{b,\check{E}}^U$ inherits a canonical P -action; we caution the reader that although the action of $M \subset P$ is indeed the obvious one, given by its natural action on the first factor, the full P -action mixes both factors in a way which seems a little tricky to describe directly. In particular, it seems hard to see the simple structures in this theorem at any *finite* level; they only reveal themselves at infinite level.

The following diagram summarizes the situation so far in a manner which we hope is suggestive:

$$\begin{array}{ccccc}
 & & \mathrm{Sht}_{P,\mu,b} & & \\
 & \nearrow r_\infty & & \searrow \mathrm{Ind}_P^G(-) & \\
 \mathrm{Sht}_{M,\mu,b} & \xrightarrow{i_\infty} & & & \mathrm{Sht}_{G,\mu,b} \\
 \downarrow \pi_{\mathrm{GM}} & & & & \downarrow \pi_{\mathrm{GM}} \\
 \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} & \xleftarrow{r} & & & \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}} \\
 & \xleftarrow{i} & & &
 \end{array}$$

⁶It turns out there is a heuristic explanation for this structure based on comparing the Hodge–Tate period maps out of $\mathrm{Sht}_{G,\mu,b}$ and $\mathrm{Sht}_{M,\mu,b}$, but we only discovered this heuristic after the fact.

Finally, combining the preceding analysis with a calculation of the geometric étale cohomology of $\mathcal{J}_{b,\check{E}}^U$ and some consequences of a cohomological formalism for diamonds recently developed by Scholze [2017], we deduce our main cohomological result.⁷

Theorem 1.8. *There are canonical G -equivariant isomorphisms*

$$H_c^i(\mathrm{Sht}_{G,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} C, \mathbb{Z}/\ell^n) \cong \mathrm{ind}_P^G(H_c^{i-2d}(\mathrm{Sht}_{M,\mu,b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} C, \mathbb{Z}/\ell^n)(-d))$$

for all $i \geq 0$ compatible with all additional structures, where

$$d = \dim \mathrm{Sht}_{G,\mu,b} - \dim \mathrm{Sht}_{M,\mu,b};$$

in particular, if $C = \widehat{\check{E}}$, these isomorphisms are compatible with the natural W_E -actions on both sides. Here again ind_P^G denotes unnormalized smooth induction, and C/\check{E} is any complete algebraically closed field.

Let us sketch the proof of this result. Since Theorem 1.6 describes the cohomology of $\mathrm{Sht}_{G,\mu,b}$ as the smooth induction of the cohomology of $\mathrm{Sht}_{P,\mu,b}$, it's enough to relate the cohomologies of $\mathrm{Sht}_{P,\mu,b}$ and $\mathrm{Sht}_{M,\mu,b}$. The idea now is that from the viewpoint of ℓ -adic cohomology, $\mathcal{J}_{b,\check{E}}^U$ is “contractible” in a certain precise sense, so the map a_∞ from Theorem 1.7 should induce an isomorphism between the cohomologies of $\mathrm{Sht}_{M,\mu,b}$ and $\mathrm{Sht}_{P,\mu,b}$, at least up to a twist and a shift in degree. For a precise statement, see Theorem 4.13.

The main technical point here is the calculation of the compactly supported étale cohomology $R\Gamma_c(\mathcal{J}_{b,C}^U, \mathbb{Z}/\ell^n)$. Forgetting any possible Galois actions, it is not so hard to show that this is the expected shift of the constant sheaf \mathbb{Z}/ℓ^n . However, when $C = \widehat{\check{E}}$ we would also like a precise description of the W_E -action, and although the answer is easy to guess, proving it turns out to be *much* more subtle. This calculation occupies most of Section 4C, and exploits the full power of the six-functor formalism for diamonds developed in [Scholze 2017]. The essential point here is Proposition 4.8, which should have numerous other applications.

One can also prove a similar result with $\overline{\mathbb{Q}}_\ell$ -coefficients.

1F. Other results. At the time we first posted these results in preprint form, in the summer of 2016, we planned to treat the case of general groups in a sequel paper written jointly with Jared Weinstein. However, we then learned that Gaisin and Imai had also been working along similar lines, and very shortly afterwards they were able to treat the case of general groups, by combining our results on canonical filtrations with Tannakian methods in the expected way [Gaisin and Imai 2016].

⁷See Section 4C for a discussion of the results from [Scholze 2017] which we need.

Let us also note that Scholze has announced a proof of the full Harris–Viehmann conjecture (and presumably its nonminuscule analogue as well), using ideas drawn from geometric Langlands. As far as we can tell, the results of the present paper (together with their generalizations in [Gaisin and Imai 2016]) are neither strictly weaker nor stronger than Scholze’s results: our methods only apply in the Hodge–Newton reducible case, but when they do apply they yield very precise information. In particular, rather than studying an alternating sum of cohomology groups as in the Harris–Viehmann conjecture, we prove (as Mantovan did before us) that in the Hodge–Newton reducible case, the *individual* geometric étale cohomology groups of $\mathrm{Sht}_{G,\mu,b}$ are all parabolically induced.

2. Preliminaries

2A. Notation, terminology, and assumptions. This paper freely uses the language of diamonds as developed in [Scholze 2017; Scholze and Weinstein 2020]; in Section 4C we’ll also use the full power of the six-functor formalism developed in [Scholze 2017].

If X is any analytic adic space over $\mathrm{Spa}\,\mathbb{Z}_p$ with associated diamond X^\diamond , there is a natural equivalence of categories $\mathrm{Perfd}_X \cong \mathrm{Perf}_{X^\diamond}$, and we will use this without any particular comment.

If R is a DVR with uniformizer π , and M is a nonzero finite torsion R -module, then $M \simeq \bigoplus_{1 \leq i \leq n} R/\pi^{k_i}$ for some uniquely determined ordered sequence of positive integers $\mu(M) = (k_1 \geq \cdots \geq k_n)$ (where of course $n = \dim_{R/\pi} M/\pi M$). We will slightly abusively refer to the ordered sequence of the k_i as the *elementary divisors* of M (as an R -module). For convenience we extend k_i to a function of all $i \geq 1$ by setting $k_{n+1} = k_{n+2} = \cdots = 0$.

Throughout the remainder of the paper, we shall only explicitly consider the case $E = \mathbb{Q}_p$ of the objects and results described in the introduction. From the point of view of proofs, there is no danger here: the only real change necessary to explicitly treat general E in the proofs which follow is to make the following systematic replacements:

- replace all appearances of $\check{\mathbb{Q}}_p$ with \check{E} ;
- replace all appearances of the Witt vectors $W(-)$ with the functor

$$W_{\mathcal{O}_E}(-) := W(-) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$$

of \mathcal{O}_E -Witt vectors;

- replace all appearances of the Witt vector Frobenius with the natural q -Frobenius $\varphi_q = \varphi^f \otimes 1$ on $W_{\mathcal{O}_E}(-)$.

2B. Some module theory over $\mathbb{B}_{\mathrm{dR}}^+$. In this section we do some module theory over the relative Fontaine ring $\mathbb{B}_{\mathrm{dR}}^+$. The relevance of the material here will become clear in the next subsection (in particular see Remark 2.8 and Theorem 2.9).

Fix a perfectoid Tate ring A/\mathbb{Q}_p , so we have the usual period ring $\mathbb{B}_{\mathrm{dR}}^+(A)$, defined as the $\ker \theta$ -adic completion of $W(A^{\mathrm{ob}})[1/p]$, where $\theta : W(A^{\mathrm{ob}})[1/p] \rightarrow A$ is the usual surjection of p -adic Hodge theory. Recall that the kernel of θ is principal and generated by a non-zero-divisor ξ which is unique up to multiplication by a unit. We fix a choice of ξ in what follows; nothing will depend on this choice.

Definition 2.1. A $\mathbb{B}_{\mathrm{dR}}^+(A)$ -module M is *finite projective virtually* (fpv) over A if M can be resolved by a short exact sequence

$$0 \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0,$$

where P_0 and P_1 are finite projective $\mathbb{B}_{\mathrm{dR}}^+(A)$ -modules, and some finite power of ξ kills M .

In other words, M is fpv over A if M is ξ -torsion and 1-fpd as a $\mathbb{B}_{\mathrm{dR}}^+(A)$ -module in the sense of [Kedlaya and Liu 2019, §1.1]. Note the placement of the word “virtually”: M is typically not an A -module, since $\mathbb{B}_{\mathrm{dR}}^+(A)$ is not an A -algebra. Observe that any finite direct sum $\bigoplus_i \mathbb{B}_{\mathrm{dR}}^+(A)/\xi^{n_i}$ is fpv over A , and so is any finite projective A -module regarded as a $\mathbb{B}_{\mathrm{dR}}^+(A)$ -module via θ . Observe also that if $A = K$ is a perfectoid field, then $\mathbb{B}_{\mathrm{dR}}^+(K)$ is a DVR, in which case modules fpv over K coincide with finite torsion $\mathbb{B}_{\mathrm{dR}}^+(K)$ -modules.

We regard modules fpv over A as a full subcategory of the category of $\mathbb{B}_{\mathrm{dR}}^+(A)$ -modules. By [Kedlaya and Liu 2019, Lemma 1.1.5(d)], the property of being fpv over A is stable under formation of extensions. We record some further properties as a lemma.

Lemma 2.2. *Let*

$$0 \rightarrow M_1 \rightarrow M_2 \rightarrow M_3 \rightarrow 0$$

be an exact sequence of $\mathbb{B}_{\mathrm{dR}}^+(A)$ -modules.

- (i) *If M_2 is finite projective and M_3 is fpv over A , then M_1 is finite projective. In fact, if M_2 is finite projective and M_3 is ξ -torsion, then M_1 is finite projective if and only if M_3 is fpv over A .*
- (ii) *If M is fpv over A , then so are the submodules $\xi^n M$ and $M[\xi^n]$ for any n .*
- (iii) *If $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is an exact sequence of $\mathbb{B}_{\mathrm{dR}}^+(A)$ -modules such that N and L are both fpv, then M is fpv.*
- (iv) *If $0 \rightarrow M \rightarrow N \rightarrow L \rightarrow 0$ is an exact sequence of $\mathbb{B}_{\mathrm{dR}}^+(A)$ -modules such that M and L are both fpv, then N is fpv.*

In part (ii) here (and elsewhere in what follows), $\xi^n M \subset M$ is shorthand for $\mathrm{im}(M \xrightarrow{\xi^n} M)$.

Proof. Parts (i), (iii) and (iv) are easy, by repeated application of [Kedlaya and Liu 2019, Lemma 1.1.5]. For part (ii), we note that $M/\xi^n M$ is 2-fpd by [Kedlaya and Liu 2019, Remark 1.1.3], so then considering the sequence

$$0 \rightarrow \xi^n M \rightarrow M \rightarrow M/\xi^n M \rightarrow 0$$

[Kedlaya and Liu 2019, Lemma 1.1.5(f)] shows that $\xi^n M$ is 1-fpd, and hence fpv. But then looking at the sequence

$$0 \rightarrow M[\xi^n] \rightarrow M \rightarrow \xi^n M \rightarrow 0,$$

part (iii) implies that $M[\xi^n]$ is fpv. \square

Proposition 2.3. *If M is a $\mathbb{B}_{\text{dR}}^+(A)$ -module which is fpv over A , and $N \subseteq M$ is a direct summand of M , then N is fpv over A .*

Proof. Let $e(M) < \infty$ be the smallest positive integer e such that ξ^e kills M . We prove the claim by induction on $e(M)$. When $e(M) = 1$, the result is clear: in this case, M is a finite projective A -module, and N is a direct summand thereof, so also finite projective over A . In general, we have a commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & N[\xi] & \longrightarrow & N & \longrightarrow & \xi N \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & M[\xi] & \longrightarrow & M & \longrightarrow & \xi M \longrightarrow 0 \end{array}$$

where the vertical arrows identify the upper row as a direct summand of the lower row, in the evident sense. But $e(M[\xi]) = 1$ and $e(\xi M) = e(M) - 1$, so $N[\xi]$ and ξN are fpv over A by the induction hypothesis. Since the property of being fpv over A is stable under forming extensions, we get the result. \square

The next result gives a pointwise criterion for a $\mathbb{B}_{\text{dR}}^+(A)$ -module to be fpv over A ; this criterion plays a key role in our proof of Theorem 1.4.

Proposition 2.4. *Let N be a nonzero $\mathbb{B}_{\text{dR}}^+(A)$ -module which is finitely generated and ξ -torsion. If the elementary divisors of $N_x = N \otimes_{\mathbb{B}_{\text{dR}}^+(A)} \mathbb{B}_{\text{dR}}^+(K_x)$ as a $\mathbb{B}_{\text{dR}}^+(K_x)$ -module are locally constant (i.e., continuous) as functions of $x \in \text{Spa}(A, A^\circ)$, then N is fpv over A .*

Before proving this, we need a little lemma.

Lemma 2.5. *Let M be a finite projective A -module, viewed as a $\mathbb{B}_{\text{dR}}^+(A)$ -module via θ . Then*

$$\text{Tor}_1^{\mathbb{B}_{\text{dR}}^+(A)}(M, \mathbb{B}_{\text{dR}}^+(K_x)) = 0$$

for any $x \in \text{Spa}(A, A^\circ)$.

Proof. We easily reduce to the case $M = A$. Applying $-\otimes_{\mathbb{B}_{\text{dR}}^+(A)} \mathbb{B}_{\text{dR}}^+(K_x)$ to the resolution

$$0 \rightarrow \mathbb{B}_{\text{dR}}^+(A) \xrightarrow{\xi} \mathbb{B}_{\text{dR}}^+(A) \xrightarrow{\theta} A \rightarrow 0,$$

the result then follows from the fact that ξ is a non-zero-divisor in $\mathbb{B}_{\text{dR}}^+(A)$ and in $\mathbb{B}_{\text{dR}}^+(K_x)$. \square

Proof of Proposition 2.4. Let $k_{1,x} \geq k_{2,x} \geq \dots$ be the elementary divisors of N_x as a $\mathbb{B}_{\text{dR}}^+(K_x)$ -module, so by hypothesis the function $x \mapsto k_{i,x}$ is locally constant. Note that $e(N) = \sup_x k_{1,x}$.

We first show that $N/\xi N$ is a finite projective A -module. To see this, note that our assumptions imply the rank of

$$N_x/\xi N_x = (N/\xi N) \otimes_A K_x$$

as a K_x -vector space is locally constant, since this rank is simply the number of i for which $k_{i,x} > 0$. Since $N/\xi N$ is a finitely generated A -module and A is a uniform Banach ring, [Kedlaya and Liu 2015, Proposition 2.8.4] now implies that $N/\xi N$ is a finite projective A -module.

We now argue by induction on $e(N)$. If $e(N) = 1$, then $N = N/\xi N$ is a finite projective A -module by the argument of the previous paragraph, so N is fpv over A . If $e(N) > 1$, consider the short exact sequence

$$0 \rightarrow \xi N \rightarrow N \rightarrow N/\xi N \rightarrow 0.$$

Since $N/\xi N$ is a finite projective A -module, we see by the previous lemma that this sequence remains exact after applying $-\otimes_{\mathbb{B}_{\text{dR}}^+(A)} \mathbb{B}_{\text{dR}}^+(K_x)$, so in particular the natural map

$$(\xi N)_x = (\xi N) \otimes_{\mathbb{B}_{\text{dR}}^+(A)} \mathbb{B}_{\text{dR}}^+(K_x) \rightarrow \xi N_x$$

is an isomorphism for any $x \in \text{Spa}(A, A^\circ)$. Since the elementary divisors of ξN_x are given by the locally constant functions $\max(k_{i,x} - 1, 0)$, this implies that the elementary divisors of $(\xi N)_x$ are locally constant; since moreover $e(\xi N) = e(N) - 1$, the induction hypothesis now shows that ξN is fpv over A . Looking again at the sequence

$$0 \rightarrow \xi N \rightarrow N \rightarrow N/\xi N \rightarrow 0$$

and using the fact that the property of being fpv over A is stable under forming extensions, we deduce that N is fpv over A . \square

2C. Vector bundles and modifications on relative curves. Throughout this section, let S denote a perfectoid space over \mathbb{Q}_p , with tilt S^\flat . Unless explicitly stated otherwise, we assume S is affinoid perfectoid (so S^\flat is as well), in which case

we write $S = \mathrm{Spa}(A, A^+)$ and $S^\flat = \mathrm{Spa}(R, R^+)$; we choose this notation for compatibility with [Kedlaya and Liu 2015, §8].

We first summarize some material from [Kedlaya and Liu 2015, §§8.7–8.9]. For any affinoid perfectoid $S = \mathrm{Spa}(A, A^+)$ as above, let $\mathcal{X} = \mathcal{X}_{S^\flat}$ denote the adic Fargues–Fontaine curve over S^\flat . This is defined as the quotient $\mathcal{Y}/\varphi^\mathbb{Z}$, where

$$\mathcal{Y} = \mathcal{Y}_{S^\flat} \subset \mathrm{Spa} W(R^+)$$

is the adic space

$$\mathcal{Y}_{S^\flat} = \mathrm{Spa} W(R^+) \setminus \{x \mid |p[\varpi]|_x = 0\}$$

and φ is the natural (properly discontinuous) automorphism of \mathcal{Y} induced by the Witt vector Frobenius. (Here $\varpi \in R^+$ is any pseudouniformizer for R .) There is a canonical Zariski-closed embedding $i : S \rightarrow \mathcal{X}_{S^\flat}$ of adic spaces over \mathbb{Q}_p which realizes S as a relative Cartier divisor inside \mathcal{X}_{S^\flat} . Writing $\pi : \mathcal{Y}_{S^\flat} \rightarrow \mathcal{X}_{S^\flat}$ for the canonical projection, i lifts canonically along π to a Zariski-closed embedding $\tilde{i} : S \rightarrow \mathcal{Y}_{S^\flat}$, with this latter embedding coming (at the level of rings) from the usual theta map

$$\theta : W(R^+) \twoheadrightarrow A^+.$$

Let $\mathcal{O}(1)$ be the canonical ample line bundle on \mathcal{X} , and define the graded ring

$$P_R = \bigoplus_{i \geq 0} H^0(\mathcal{X}, \mathcal{O}(i)).$$

Then $X = X_{S^\flat} = \mathrm{Proj}(P_R)$ is the schematic Fargues–Fontaine curve associated with S^\flat . Set $Z = \mathrm{Spec}(A)$, so we have a canonical closed immersion $Z \rightarrow X$ such that the completion of X along Z is canonically identified with $\tilde{Z} := \mathrm{Spec} \mathbb{B}_{\mathrm{dR}}^+(A)$. Furthermore, the subscheme $X \setminus Z$ of X is affine; we define

$$\mathbb{B}_e(A) = H^0(X \setminus Z, \mathcal{O}_X)$$

to be its coordinate ring. These objects all fit together into a canonical commutative diagram of locally ringed spaces

$$\begin{array}{ccccc} & & & \mathcal{Y} & \\ & & & \downarrow \pi & \uparrow \tilde{i} \\ X \setminus Z \coprod \tilde{Z} & \xrightarrow{(f_e, f_{\mathrm{dR}}^+)} & X & \xleftarrow{f^{\mathrm{an}}} & \mathcal{X} \\ & \uparrow i_{\mathrm{sch}} & \uparrow i & & \\ & Z & \xleftarrow{m} & S & \end{array}$$

over $\mathrm{Spec} \mathbb{Q}_p$, covariantly functorial in morphisms

$$f : S = \mathrm{Spa}(A, A^+) \rightarrow T = \mathrm{Spa}(B, B^+)$$

of affinoid perfectoid spaces over \mathbb{Q}_p . One easily checks that if $f : S \rightarrow T$ is an open immersion, then so is the induced map $\mathcal{Y}_{S^\flat} \rightarrow \mathcal{Y}_{T^\flat}$; combining this with an easy gluing argument shows that \mathcal{X} and \mathcal{Y} , and the rightmost column of the above diagram, exist for arbitrary (i.e., possibly not affinoid) perfectoid spaces over \mathbb{Q}_p .

Theorem 2.6 (Kedlaya–Liu). *Let $S = \mathrm{Spa}(A, A^+)$ be any affinoid perfectoid space over \mathbb{Q}_p . Then with the setup as above:*

- (i) *Pullback along the morphism f^{an} induces an equivalence of exact tensor categories from vector bundles on X to vector bundles on \mathcal{X} .*
- (ii) *Pulling back along the pair of morphisms (f_e, f_{dR}^+) and then passing to global sections induces an equivalence of exact tensor categories from vector bundles on X to B -pairs over A .*
- (iii) *Pullback along the morphism π induces an equivalence of exact tensor categories from vector bundles on \mathcal{X} to φ -equivariant vector bundles on \mathcal{Y} .*

Proof. Parts (i) and (ii) follow immediately by combining Theorems 8.7.7 and 8.9.6 of [Kedlaya and Liu 2015], and part (iii) is trivial. \square

In this context, a B -pair over A is a pair $M = (M_e, M_{\mathrm{dR}}^+)$ where M_e is a finite projective $\mathbb{B}_e(A)$ -module and M_{dR}^+ is a finite projective $\mathbb{B}_{\mathrm{dR}}^+(A)$ -lattice inside the finite projective $\mathbb{B}_{\mathrm{dR}}(A)$ -module

$$M_{\mathrm{dR}} = M_e \otimes_{\mathbb{B}_e(A)} \mathbb{B}_{\mathrm{dR}}(A).$$

If \mathcal{E} is a vector bundle on \mathcal{X} (or on X), we write $M(\mathcal{E}) = (M_e(\mathcal{E}), M_{\mathrm{dR}}^+(\mathcal{E}))$ for the associated B -pair; we denote the inverse functor from B -pairs to vector bundles by $M \mapsto \mathcal{V}(M)$.

We remark that by the functoriality of the assignment $S \rightsquigarrow \mathcal{X}_{S^\flat}$, any point $x \in \mathrm{Spa}(A, A^+)$ gives rise to a morphism

$$s_x : \mathcal{X}_{\mathrm{Spa}(K_x, K_x^+)^{\flat}} \rightarrow \mathcal{X}_{S^\flat}.$$

If \mathcal{E} is a vector bundle on \mathcal{X}_{S^\flat} , we abbreviate the pullback $s_x^* \mathcal{E}$ on $\mathcal{X}_{\mathrm{Spa}(K_x, K_x^+)^{\flat}}$ by \mathcal{E}_x . Note that \mathcal{E}_x corresponds to the B -pair over K_x given by

$$(M_e(\mathcal{E}) \otimes_{\mathbb{B}_e(A)} \mathbb{B}_e(K_x), M_{\mathrm{dR}}^+(\mathcal{E}) \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x)).$$

Definition 2.7. Let S be any perfectoid space over \mathbb{Q}_p . An *effective modification along S* is a triple $(\mathcal{E}, \mathcal{F}, u)$ where \mathcal{E} and \mathcal{F} are vector bundles on \mathcal{X}_{S^\flat} , and $u : \mathcal{F} \rightarrow \mathcal{E}$ is an injective map of $\mathcal{O}_{\mathcal{X}}$ -modules such that $\mathcal{E}/u(\mathcal{F})$ is killed (locally on \mathcal{X}) by a finite power of the ideal sheaf cutting out S in \mathcal{X}_{S^\flat} . When \mathcal{E} is given, we also speak of (\mathcal{F}, u) as being an *effective modification of \mathcal{E} along S* .

An effective modification along S is *admissible* if \mathcal{F}_x is semistable of slope zero for all points $x \in S$.

Regarding this last piece of terminology, recall that when $S = \mathrm{Spa}(K, \mathcal{O}_K)$ is a point, Fargues and Fontaine [2018] constructed a canonical slope filtration on any bundle over \mathcal{X}_{S^\flat} ; we say a bundle is semistable if its slope filtration has a unique nonzero step. As usual, we'll use the terms “semistable of slope zero” and “étale” interchangeably.

Effective modifications along S form an exact tensor category $\mathcal{E}ff_{/S}$ in an obvious manner, and any morphism $f : S \rightarrow T$ of perfectoid spaces over \mathbb{Q}_p induces a pullback functor $f^* : \mathcal{E}ff_{/T} \rightarrow \mathcal{E}ff_{/S}$.

Remark 2.8. If \mathcal{E} is a vector bundle on \mathcal{X} with associated B-pair (M_e, M_{dR}^+) , and $N \subseteq M_{\mathrm{dR}}^+$ is any $\mathbb{B}_{\mathrm{dR}}^+(A)$ submodule such that $N[1/\xi] = M_{\mathrm{dR}}^+[1/\xi] = M_{\mathrm{dR}}$, then the following are equivalent:

- (1) N is a finite projective $\mathbb{B}_{\mathrm{dR}}^+(A)$ -module.
- (2) M_{dR}^+/N is fpv over A .
- (3) The pair (M_e, N) is in the essential image of $M(-)$ (so $\mathcal{V}(M_e, N) \rightarrow \mathcal{E}$ is an effective modification of \mathcal{E} along S).

Indeed, (1) and (2) are equivalent by Lemma 2.2(i), and (1) and (3) are equivalent by Theorem 2.6. This explains the appearance of the fpv condition in the following theorem.

Theorem 2.9. *Let $S = \mathrm{Spa}(A, A^+)$ be an affinoid perfectoid space over \mathbb{Q}_p , and let \mathcal{E} be a vector bundle on \mathcal{X}_{S^\flat} . Then we have a functorial identification between the set of isomorphism classes of effective modifications of \mathcal{E} along S , and the set of $\mathbb{B}_{\mathrm{dR}}^+(A)$ -submodules $N \subseteq M_{\mathrm{dR}}^+(\mathcal{E})$ such that $M_{\mathrm{dR}}^+(\mathcal{E})/N$ is fpv over A .*

Proof. The functor in one direction sends (\mathcal{F}, u) to

$$M_{\mathrm{dR}}^+(u) \circ M_{\mathrm{dR}}^+(\mathcal{F}) \subseteq M_{\mathrm{dR}}^+(\mathcal{E}).$$

For the functor in the other direction, set $M' = (M_e(\mathcal{E}), N)$. This is a B-pair, and by construction there is a natural injection of B-pairs $\iota : M' \rightarrow M(\mathcal{E})$; set $\mathcal{F} = \mathcal{V}(M')$ and $u = \mathcal{V}(\iota)$. Since $\mathcal{V}(-)$ is an equivalence of exact tensor categories, $u : \mathcal{F} \rightarrow \mathcal{E}$ is injective. Moreover, since ι is an isomorphism on the \mathbb{B}_e -terms of the B-pairs $M(\mathcal{E})$ and M' , u is an isomorphism away from the closed immersion $S \subset \mathcal{X}$. The remaining verifications are then an easy unwinding, taking Remark 2.8 into account. \square

Definition 2.10. If \mathcal{E} is a vector bundle on \mathcal{X}_{S^\flat} and (\mathcal{F}, u) is an effective modification along S with associated de Rham module $N = M_{\mathrm{dR}}^+(\mathcal{F}) \subseteq M_{\mathrm{dR}}^+(\mathcal{E})$, then for any point $x \in |S|$ we define the *type* of the modification at x , denoted $\mu_x(\mathcal{F}, u)$, as the ordered sequence of elementary divisors of the finite torsion $\mathbb{B}_{\mathrm{dR}}^+(K_x)$ -module

$$(M_{\mathrm{dR}}^+(\mathcal{E})/N) \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x).$$

In this terminology, the open Schubert cell $\mathrm{Gr}_{\mathrm{GL}_n, \mu}$ can be defined as follows.

Definition 2.11. For any $n \geq 2$ and any $\mu = (k_1 \geq \cdots \geq k_n)$ with $k_n \geq 0$, $\mathrm{Gr}_{\mathrm{GL}_n, \mu}$ is the functor sending $S \in \mathrm{Perf}/\mathbb{Q}_p$ to the set of isomorphism classes of effective modifications of $\mathcal{O}_{\mathcal{X}_{S^\flat}}^n$ along S of constant type μ . Equivalently, $\mathrm{Gr}_{\mathrm{GL}_n, \mu}(\mathrm{Spa}(A, A^+))$ is the set of finite projective $\mathbb{B}_{\mathrm{dR}}^+(A)$ -submodules $N \subseteq \mathbb{B}_{\mathrm{dR}}^+(A)^n$ with ξ -torsion quotient such that

$$(\mathbb{B}_{\mathrm{dR}}^+(A)^n/N) \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x)$$

has elementary divisors given by μ for all $x \in |\mathrm{Spa}(A, A^+)|$.

By [Scholze and Weinstein 2020, Corollary 19.3.4], $\mathrm{Gr}_{\mathrm{GL}_n, \mu}$ is a locally spatial diamond.

In the remainder of this subsection, we explain the proof of Theorem 1.2(i)–(iv), in the case where $E = \mathbb{Q}_p$ and $G = \mathrm{GL}_n$. Let us fix μ as in the previous definition, together with an element $b \in \mathrm{GL}_n(\check{\mathbb{Q}}_p)$ such that $[b] \in B(\mathrm{GL}_n, \mu^{-1})$. If $S = \mathrm{Spa}(A, A^+)$ is any affinoid perfectoid space over $\check{\mathbb{Q}}_p$, then \mathcal{Y}_{S^\flat} is naturally an adic space over $\check{\mathbb{Q}}_p$ via the maps

$$\check{\mathbb{Q}}_p = W(\overline{\mathbb{F}}_p)[1/p] \rightarrow W(A^{+\flat})[1/p] \rightarrow \mathcal{O}_{\mathcal{Y}},$$

and the Frobenius φ on \mathcal{Y} is σ -semilinear. In particular, we may define a φ -equivariant rank n vector bundle on \mathcal{Y}_{S^\flat} via the formula

$$(\tilde{\mathcal{E}}_b, \varphi_{\tilde{\mathcal{E}}_b}) = (\mathcal{O}_{\mathcal{Y}} \otimes_{\check{\mathbb{Q}}_p} \check{\mathbb{Q}}_p^n, \varphi \otimes b\sigma);$$

let \mathcal{E}_{b, S^\flat} be the corresponding vector bundle on \mathcal{X}_{S^\flat} . The assignment $S \rightsquigarrow \mathcal{E}_{b, S^\flat}$ is clearly functorial in morphisms $S \rightarrow T$ of perfectoid spaces over $\check{\mathbb{Q}}_p$. Note also that if b lies in $\mathrm{GL}_n(\mathbb{Q}_{p^r})$ for some finite unramified extension $\mathbb{Q}_p \subset \mathbb{Q}_{p^r} \subset \check{\mathbb{Q}}_p$, then \mathcal{E}_{b, S^\flat} is well-defined for any S over \mathbb{Q}_{p^r} (as opposed to $\check{\mathbb{Q}}_p$).

It seems to us that “ $\mathbb{B}_{\mathrm{crys}}(A)$ ” is not well-defined for an arbitrary A ; however, if $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq A$, then it's reasonable to define this ring by the formula

$$\mathbb{B}_{\mathrm{crys}}(A) = \mathbb{B}_{\mathrm{crys}}^+(A) \otimes_{\mathbb{B}_{\mathrm{crys}}^+(\mathbb{Q}_p^{\mathrm{cyc}})} \mathbb{B}_{\mathrm{crys}}(\mathbb{Q}_p^{\mathrm{cyc}}).$$

Note that with this definition, the expected isomorphism $\mathbb{B}_e(A) \cong \mathbb{B}_{\mathrm{crys}}(A)^{\varphi=1}$ is indeed true. With this in mind, it's not hard to show that when $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq A$, the B-pair over A corresponding to $\mathcal{E}_{b, \mathrm{Spa}(A, A^+)^{\flat}}$ can be explicitly described via identifications

$$\begin{aligned} M_e(\mathcal{E}_{b, S^\flat}) &= (\mathbb{B}_{\mathrm{crys}}(A) \otimes_{\check{\mathbb{Q}}_p} \check{\mathbb{Q}}_p^n)^{\varphi \otimes b\sigma=1}, \\ M_{\mathrm{dR}}^+(\mathcal{E}_{b, S^\flat}) &= \mathbb{B}_{\mathrm{dR}}^+(A)^n. \end{aligned}$$

The key point here is the observation that the canonical map

$$(\mathbb{B}_{\mathrm{crys}}(A) \otimes_{\check{\mathbb{Q}}_p} \check{\mathbb{Q}}_p^n)^{\varphi \otimes b\sigma=1} \otimes_{\mathbb{B}_e(A)} \mathbb{B}_{\mathrm{crys}}(A) \rightarrow \mathbb{B}_{\mathrm{crys}}(A) \otimes_{\check{\mathbb{Q}}_p} \check{\mathbb{Q}}_p^n = \mathbb{B}_{\mathrm{crys}}(A)^n$$

is an isomorphism, so the scalar extension of $M_e(\mathcal{E}_{b,S^\flat})$ along $\mathbb{B}_e(A) \rightarrow \mathbb{B}_{\mathrm{dR}}(A)$ is canonically identified with $\mathbb{B}_{\mathrm{dR}}(A)^n$.

Combining this description of $M(\mathcal{E}_b)$ with [Theorem 2.9](#) and [Definition 2.11](#) (and making use of an easy pro-étale descent to get rid of the assumption $\mathbb{Q}_p^{\mathrm{cyc}} \subseteq A$), we obtain the following result.

Proposition 2.12. *For any fixed $b \in \mathrm{GL}_n(\check{\mathbb{Q}}_p)$ and any $\mu = (k_1 \geq \dots \geq k_n) \in \mathbb{Z}^n$ with $k_n \geq 0$ as before, the functor*

$$\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p} := \mathrm{Gr}_{\mathrm{GL}_n, \mu} \times_{\mathrm{Spd} \mathbb{Q}_p} \mathrm{Spd} \check{\mathbb{Q}}_p$$

is canonically identified with the functor $\mathrm{Perfd} / \check{\mathbb{Q}}_p \rightarrow \mathrm{Sets}$ sending S to the set of isomorphism classes of effective modifications (\mathcal{F}, u) of \mathcal{E}_{b,S^\flat} along S of constant type μ . In particular, the latter functor is a locally spatial diamond over $\mathrm{Spd} \check{\mathbb{Q}}_p$.

Maintaining the notation and interpretation of $\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$ provided by this proposition, let

$$\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}} \subset \mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$$

be the subfunctor defined by the following condition: an S -point

$$(\mathcal{F}, u) \in (\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p})(S)$$

factors through an S -point of $\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$ if and only if the bundle \mathcal{F}_x is semistable of slope zero at every point $x \in S$. Since all the data of n, μ, b are fixed, we'll sometimes abbreviate $\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$ to $\mathrm{Gr}^{\mathrm{adm}}$ in what follows.

Theorem 2.13. *The functor $\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$ is a locally spatial diamond over $\mathrm{Spd} \check{\mathbb{Q}}_p$, and is naturally open and partially proper as a subdiamond of $\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$.*

Proof. Let S be any perfectoid space over $\check{\mathbb{Q}}_p$, and let $f : S^\diamond \rightarrow \mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$ be any S -point of $\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$, with (\mathcal{F}, u) the associated modification of \mathcal{E}_{b,S^\flat} along S . Let $|S|^{\mathrm{adm}} \subset |S|$ be the set of points $x \in |S|$ where \mathcal{F}_x is semistable of slope zero. By Lemma 8.5.11 of [\[Kedlaya and Liu 2015\]](#), $|S|^{\mathrm{adm}} \subset |S|$ is open and partially proper, and hence corresponds to a partially proper open immersion of perfectoid spaces $S^{\mathrm{adm}} \subset S$. Putting this together with the definition of $\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$, we get a pullback square

$$\begin{array}{ccc} (S^{\mathrm{adm}})^\diamond & \xrightarrow{v} & S^\diamond \\ g \downarrow & & \downarrow f \\ \mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}} & \xrightarrow{u} & \mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p} \end{array}$$

of sheaves on $\mathrm{Perfd}^{\mathrm{proet}} / \mathrm{Spd} \check{\mathbb{Q}}_p$. Since S and f are arbitrary and v is open and partially proper, we deduce that $\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$ is an open subfunctor of $\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$, and that u is a partially proper open embedding. Finally, note that since $\mathrm{Gr}_{\mathrm{GL}_n, \mu / \check{\mathbb{Q}}_p}$ is a locally

spatial diamond, any open subfunctor $U \subset \mathrm{Gr}_{\mathrm{GL}_n, \mu/\check{\mathbb{Q}}_p}$ is a locally spatial diamond as well. \square

Now let X be any perfectoid space. By Corollary 8.7.10 of [Kedlaya and Liu 2015], the category $\mathbb{Q}_p\mathrm{Loc}(X)$ of \mathbb{Q}_p -local systems on X is functorially equivalent to the category of vector bundles \mathcal{F} on \mathcal{X}_{X^\flat} with the property that \mathcal{F}_x is étale at every point $x \in X$. If \mathcal{F} is a vector bundle with this property, let $V(\mathcal{F})$ denote the associated \mathbb{Q}_p -local system on X .

Proposition 2.14. *There is a rank n \mathbb{Q}_p -local system*

$$V^{\mathrm{univ}} \in \mathbb{Q}_p\mathrm{Loc}(\mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}})$$

characterized uniquely by the following universal property: for any perfectoid space S over $\check{\mathbb{Q}}_p$ and any S -point $f : S^\diamond \rightarrow \mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$, with (\mathcal{F}, u) the associated (pointwise-étale) modification of \mathcal{E}_{b, S^\flat} along S , there is a canonical and functorial isomorphism

$$V(\mathcal{F}) \cong f^* V^{\mathrm{univ}}$$

of \mathbb{Q}_p -local systems on S .

Proof. Choose some perfectoid space $X/\check{\mathbb{Q}}_p$ together with a surjective quasi-pro-étale map $g : X^\diamond \rightarrow \mathrm{Gr}^{\mathrm{adm}}$, and let (\mathcal{F}, u) be the associated “universal” modification of \mathcal{E}_{b, X^\flat} along X . Since \mathcal{F} is pointwise-étale by definition, we can form the associated rank n \mathbb{Q}_p -local system $V(\mathcal{F})$ on X . Now consider the pullback diagram

$$\begin{array}{ccc} X^\diamond \times_{\mathrm{Gr}^{\mathrm{adm}}} X^\diamond & \xrightarrow{\mathrm{pr}_2} & X^\diamond \\ \downarrow \mathrm{pr}_1 & & \downarrow g \\ X^\diamond & \xrightarrow{g} & \mathrm{Gr}^{\mathrm{adm}} \end{array}$$

of diamonds over $\check{\mathbb{Q}}_p$; note that $X^\diamond \times_{\mathrm{Gr}^{\mathrm{adm}}} X^\diamond$ is representable. By the definition of $\mathrm{Gr}^{\mathrm{adm}}$, it's easy to see that there is a canonical isomorphism $\mathrm{pr}_1^* \mathcal{F} \cong \mathrm{pr}_2^* \mathcal{F}$ which satisfies the usual cocycle condition, so we get a descent datum for \mathcal{F} relative to the quasi-pro-étale cover g . By the functoriality of $V(-)$, this induces a descent datum for $V(\mathcal{F})$ relative to g . Since \mathbb{Q}_p -local systems on diamonds satisfy effective descent with respect to quasi-pro-étale covers, this descent datum (unlike the one for \mathcal{F}) is effective, and we define V^{univ} as the associated descent of $V(\mathcal{F})$. The uniqueness and the claimed properties of V^{univ} are then an easy verification. \square

At this point, we're almost ready to construct $\mathrm{Sht}_{\mathrm{GL}_n, \mu, b}$. Before doing so, we quickly prove the following result. Given any diamond \mathcal{D} together with a rank n \mathbb{Q}_p -local system V on \mathcal{D} , let

$$\mathrm{Triv}_{V/\mathcal{D}} : \mathrm{Perf}_{/\mathcal{D}} \rightarrow \mathrm{Sets}$$

be the functor on perfectoid spaces over \mathcal{D} sending any $f : T \rightarrow \mathcal{D}$ to the set

$$\mathrm{Isom}_{\mathbb{Q}_p\mathrm{Loc}(T)}(\underline{\mathbb{Q}}_p^n, f^*V)$$

of trivializations of f^*V .

Proposition 2.15. *The natural map $\mathrm{Triv}_{V/\mathcal{D}} \rightarrow \mathcal{D}$ is surjective and pro-étale (so in particular, $\mathrm{Triv}_{V/\mathcal{D}}$ is a diamond), and the natural $\mathrm{GL}_n(\mathbb{Q}_p)$ -action on $\mathrm{Triv}_{V/\mathcal{D}}$ makes it into a pro-étale $\mathrm{GL}_n(\mathbb{Q}_p)$ -torsor over \mathcal{D} . If $K \subset \mathrm{GL}_n(\mathbb{Q}_p)$ is any open compact subgroup, $\mathrm{Triv}_{V/\mathcal{D}}/K \rightarrow \mathcal{D}$ is separated and étale.*

Proof. Note that if X is any adic space and V is any \mathbb{Q}_p -local system on X , then V admits a \mathbb{Z}_p -lattice locally in the analytic topology on X ; more precisely, we can find a covering of X by open affinoids U_i together with \mathbb{Z}_p -local systems $L_i \subset V|_{U_i}$ such that $V|_{U_i} \simeq L_i \otimes_{\mathbb{Z}_p} \underline{\mathbb{Q}}_p$. This follows immediately from Remark 8.4.5 and Corollary 8.4.7 in [Kedlaya and Liu 2015].

Now, let T be any perfectoid space equipped with a rank n \mathbb{Q}_p -local system V . By construction, $\mathrm{Triv}_{V/T} \rightarrow T$ is a $\mathrm{GL}_n(\mathbb{Q}_p)$ -pretorsor, so it suffices to prove that $\mathrm{Triv}_{V/T}$ is representable and that $\mathrm{Triv}_{V/T} \rightarrow T$ is a pro-étale cover. These claims are local on T , so we can assume that T is affinoid and that V admits a \mathbb{Z}_p -lattice $L_0 \subset V$. As in [Kedlaya and Liu 2015, Remark 1.4.7], let $L_m(L_0) \rightarrow T$ be the functor parametrizing \mathbb{Z}_p -local systems $L \subset L_0 \otimes_{\mathbb{Z}_p} \underline{\mathbb{Q}}_p = V$ such that $p^m L_0 \subseteq L \subseteq p^{-m} L_0$. This functor is representable by an affinoid perfectoid space and the map $L_m(L_0) \rightarrow T$ is finite étale and surjective (since it has a section). Moreover, the natural map $L_m(L_0) \rightarrow L_{m'}(L_0)$ is an open and closed immersion for any $m \leq m'$. Let

$$Y = \mathcal{L}at_{V/T} = \mathrm{colim}_{m \rightarrow \infty} L_m(L_0) \xrightarrow{h} T$$

be the functor parametrizing \mathbb{Z}_p -lattices in V ; by our observations so far, Y is a countable disjoint union of affinoid perfectoid spaces each finite étale over T , and the map h is surjective, so in particular $Y \rightarrow T$ is an étale cover. Over Y we have a universal \mathbb{Z}_p -lattice $L^{\mathrm{univ}} \subset h^*V$. Let

$$\mathrm{Triv}_{L^{\mathrm{univ}}/Y} : \mathrm{Perf}_Y \rightarrow \mathrm{Sets}, \quad \{g : W \rightarrow Y\} \mapsto \mathrm{Isom}_{\mathbb{Z}_p\mathrm{Loc}(W)}(\underline{\mathbb{Q}}_p^n, g^*L^{\mathrm{univ}})$$

be the functor parametrizing trivializations of L^{univ} , so there is a natural equivariant isomorphism $\mathrm{Triv}_{V/T} \cong \mathrm{Triv}_{L^{\mathrm{univ}}/Y}$ given by sending an X -point $\beta : \underline{\mathbb{Q}}_p^n \xrightarrow{\sim} f^*V$ of $\mathrm{Triv}_{V/T}$ (lying over a given $f : X \rightarrow T$) to the lattice $\beta(\underline{\mathbb{Z}}_p^n) \subset f^*V$ together with its evident trivialization.

It thus suffices to prove that for any \mathbb{Z}_p -local system L on any perfectoid space Y , the functor $\mathrm{Triv}_{L/Y}$ is representable and $\mathrm{Triv}_{L/Y} \rightarrow Y$ is pro-(finite étale surjective). Set $L/p^j = L \otimes_{\mathbb{Z}_p} \underline{\mathbb{Z}}/p^j$, so L/p^j is a sheaf of \mathbb{Z}/p^j -modules on Y_{proet} which is

locally free of rank n , and $\mathbf{L} \cong \lim_{\leftarrow j} \mathbf{L}/p^j$. Let

$$\mathcal{Triv}_{(\mathbf{L}/p^j)/Y} : \mathrm{Perf}_Y \rightarrow \mathrm{Sets}, \quad \{f : T \rightarrow Y\} \mapsto \mathrm{Isom}_{\mathbb{Z}/p^j\mathrm{Loc}(T)}((\mathbb{Z}/p^j)^n, f^*\mathbf{L}/p^j)$$

be the evident functor with its natural action of $\mathrm{GL}_n(\mathbb{Z}/p^j)$. Since by definition

$$\mathcal{Triv}_{\mathbf{L}/Y} \cong \lim_{\leftarrow j} \mathcal{Triv}_{(\mathbf{L}/p^j)/Y},$$

it suffices to show that each $\mathcal{Triv}_{(\mathbf{L}/p^j)/Y}$ is representable by a finite étale $\mathrm{GL}_n(\mathbb{Z}/p^j)$ -torsor over Y . This is trivial if \mathbf{L}/p^j is generated by global sections; since \mathbf{L}_0/p^j is generated by global sections étale-locally on Y , there is some étale cover $\tilde{Y} \rightarrow Y$ such that $\mathcal{Triv}_{(\mathbf{L}/p^j)/Y} \rightarrow Y$ pulls back to a surjective finite étale map along $\tilde{Y} \rightarrow Y$. We now conclude by the fact that finite étale maps of perfectoid spaces satisfy effective descent with respect to étale covers, see [Weinstein 2017, Lemma 4.2.4].

For the final statement, we may assume that K is contained in $\mathrm{GL}_n(\mathbb{Z}_p)$, so the map $\mathcal{Triv}_{V/T}/\underline{K} \rightarrow T$ factors over a map

$$q : \mathcal{Triv}_{V/T}/\underline{K} \rightarrow \mathcal{Triv}_{V/T}/\underline{\mathrm{GL}_n(\mathbb{Z}_p)} \cong \mathcal{Lat}_{V/T}.$$

It then suffices to check that the map q is finite étale, which again follows from effective descent of finite étale maps with respect to étale covers. \square

Proposition 2.16. *There is a natural $\mathrm{GL}_n(\mathbb{Q}_p)$ -equivariant isomorphism*

$$\mathrm{Sht}_{\mathrm{GL}_n, \mu, b} \cong \mathcal{Triv}_{V^{\mathrm{univ}}/\mathrm{Gr}^{\mathrm{adm}}},$$

of functors over $\mathrm{Gr}^{\mathrm{adm}} = \mathrm{Gr}_{\mathrm{GL}_n, \mu}^{\mathcal{E}_b\text{-adm}}$.

Proof. Given an S -point (\mathcal{F}, u, α) of $\mathrm{Sht}_{\mathrm{GL}_n, \mu, b}$ lying over an S -point $f : S^\diamond \rightarrow \mathrm{Gr}^{\mathrm{adm}}$, we simply apply $V(-)$ to the isomorphism

$$\alpha : \mathcal{O}_{\mathcal{X}_{S^\diamond}}^n \xrightarrow{\sim} \mathcal{F}$$

of pointwise-étale vector bundles on \mathcal{X}_{S^\diamond} . Since $V(\mathcal{O}_{\mathcal{X}_{S^\diamond}}) = \underline{\mathbb{Q}}_p \in \mathbb{Q}_p\mathrm{Loc}(S)$, this gives an isomorphism

$$V(\alpha) : \underline{\mathbb{Q}}_p^n \xrightarrow{\sim} V(\mathcal{F}) \cong f^*V^{\mathrm{univ}},$$

and thus an S -point of $\mathcal{Triv}_{V^{\mathrm{univ}}/\mathrm{Gr}^{\mathrm{adm}}}$. The map in the other direction is similar, taking into account the universal property of V^{univ} . \square

Putting together this result with the remarks on $\mathcal{Triv}_{V/\mathcal{D}}$ directly preceding it and the conclusions of Theorem 2.13, we immediately deduce parts (i)–(iii) of Theorem 1.2. Part (iv) follows from the fact that for minuscule μ , the Bialynicki-Birula map $\mathrm{Gr}_{G, \mu} \rightarrow \mathcal{FL}_{G, \mu}^\diamond$ is an isomorphism [Scholze and Weinstein 2020, Proposition 19.4.2], so in particular $\mathrm{Gr}_{G, \mu}$ is the diamond of a smooth rigid space over $\check{\mathbb{Q}}_p$. For any smooth rigid space Y , the functor $(-)^{\diamond}$ on rigid spaces over Y

induces an equivalence $Y_{\text{ét}} \cong Y_{\text{ét}}^{\diamond}$ [Scholze 2017, Lemma 15.6], so combining this with part (iii) shows that $\text{Sht}_{G,\mu,b}/\underline{K}$ is in the essential image of this functor, as desired.

2D. Section and automorphism functors of a bundle.

Definition 2.17. Choose any $n \geq 1$ and $b \in \text{GL}_n(\check{\mathbb{Q}}_p)$, with \mathcal{E}_b the associated bundle. We define functors $\mathcal{H}^0(\mathcal{E}_b)$ and \mathcal{J}_b as follows:

- (1) $\mathcal{H}^0(\mathcal{E}_b) : \text{Perf}_{/\text{Spd } \overline{\mathbb{F}}_p} \rightarrow \text{Sets}$ is the functor sending any perfectoid space $S/\overline{\mathbb{F}}_p$ to the set $H^0(\mathcal{X}_S, \mathcal{E}_{b,S})$
- (2) $\mathcal{J}_b : \text{Perf}_{/\text{Spd } \overline{\mathbb{F}}_p} \rightarrow \text{Sets}$ is the functor sending any perfectoid space $S/\overline{\mathbb{F}}_p$ to the group $\text{Aut}(\mathcal{E}_{b,S})$.

Note that \mathcal{J}_b is a subfunctor of $\mathcal{H}^0(\mathcal{E}_b^{\vee} \otimes \mathcal{E}_b)$. Again, if $b \in \text{GL}_n(\mathbb{Q}_{p^r})$ for some $r \geq 1$, then $\mathcal{H}^0(\mathcal{E}_b)$ and \mathcal{J}_b are more naturally defined as functors on $\text{Perf}_{/\text{Spd } \mathbb{F}_{p^r}}$, and the results which follow all descend to this smaller base; we will not spell this out, although we will take this perspective in some of Section 4C.

Proposition 2.18. *The functors $\mathcal{H}^0(\mathcal{E}_b)$ and \mathcal{J}_b are pro-étale sheaves on $\text{Perf}_{/\text{Spd } \overline{\mathbb{F}}_p}$.*

Sketch. Any \mathcal{X}_S is preperfectoid; vector bundles and morphisms of vector bundles on preperfectoid spaces can be glued pro-étale-locally; and if $S \rightarrow T$ is pro-étale then so is the map $\mathcal{X}_S \rightarrow \mathcal{X}_T$. \square

Proposition 2.19. *For any $b \in \text{GL}_n(\check{\mathbb{Q}}_p)$, the structure maps $\mathcal{H}^0(\mathcal{E}_b) \rightarrow \text{Spd } \overline{\mathbb{F}}_p$ and $\mathcal{J}_b \rightarrow \text{Spd } \overline{\mathbb{F}}_p$ are representable in locally spatial diamonds. In particular, the base changes $\mathcal{H}^0(\mathcal{E}_b)_{\check{\mathbb{Q}}_p} = \mathcal{H}^0(\mathcal{E}_b) \times_{\text{Spd } \overline{\mathbb{F}}_p} \text{Spd } \check{\mathbb{Q}}_p$ and $\mathcal{J}_{b,\check{\mathbb{Q}}_p} = \mathcal{J}_b \times_{\text{Spd } \overline{\mathbb{F}}_p} \text{Spd } \check{\mathbb{Q}}_p$ are locally spatial diamonds.*

Proof. Arguing as in [Birkbeck et al. 2020, Propositions 3.3.6–3.3.7], one checks that \mathcal{J}_b is an open subfunctor of $\mathcal{H}^0(\mathcal{E}_b^{\vee} \otimes \mathcal{E}_b)$. It then suffices to show that the map $\mathcal{H}^0(\mathcal{E}_b) \rightarrow \text{Spd } \overline{\mathbb{F}}_p$ is representable in locally spatial diamonds. This follows from Proposition 4.7 below. \square

We note that for any $S = \text{Spa}(A, A^+)$ over $\check{\mathbb{Q}}_p$, the S -points of $\mathcal{J}_{b,\check{\mathbb{Q}}_p}$ are just the group

$$\{g \in \text{GL}_n(\mathbb{B}_{\text{crys}}^+(A)) \mid g = b\varphi(g)b^{-1}\},$$

but we will not need this.

We now put ourselves in the situation of Section 1C. As in the discussion there, let d_1, \dots, d_k be positive integers, and let $b_i \in \text{GL}_{d_i}(\check{\mathbb{Q}}_p)$ be some elements with the property that the slopes of \mathcal{E}_{b_i} are strictly greater than the slopes of $\mathcal{E}_{b_{i+1}}$ for every $1 \leq i < k$. Set $n = d_1 + \dots + d_k$, and let $\mathbf{M} \cong \prod \text{GL}_{d_i} \subset \text{GL}_n$ be the associated standard Levi. Let \mathbf{U} be the unipotent radical of the standard parabolic \mathbf{P} associated with \mathbf{M} . Let $b \in \mathbf{M}(\check{\mathbb{Q}}_p) \subset \text{GL}_n(\check{\mathbb{Q}}_p)$ be the element defined by the

totality of the b_i in the obvious way, so $\mathcal{E}_{b,S} \cong \bigoplus_{1 \leq i \leq k} \mathcal{E}_{b_i,S}$ functorially for any $S \in \text{Perf}_{/\text{Spd } \mathbb{F}_p}$.

Proposition 2.20. *In this situation, the group sheaf \mathcal{J}_b decomposes canonically into the semidirect product $\mathcal{J}_b^{\mathbf{M}} \ltimes_{\text{Spd } \mathbb{F}_p} \mathcal{J}_b^U$, where*

$$\mathcal{J}_b^{\mathbf{M}} = \mathcal{J}_{b_1} \times_{\text{Spd } \mathbb{F}_p} \cdots \times_{\text{Spd } \mathbb{F}_p} \mathcal{J}_{b_k} =: \prod_{1 \leq i \leq k} /_{\text{Spd } \mathbb{F}_p} \mathcal{J}_{b_i}$$

is the group of \mathbf{M} -bundle automorphisms of \mathcal{E}_b and where

$$\mathcal{J}_b^U = \prod_{1 \leq i < j \leq k} /_{\text{Spd } \mathbb{F}_p} \mathcal{H}^0(\mathcal{E}_{b_j}^\vee \otimes \mathcal{E}_{b_i})$$

is the kernel of the natural map $\mathcal{J}_b \rightarrow \mathcal{J}_b^{\mathbf{M}}$. Via base change, we obtain an analogous decomposition $\mathcal{J}_{b,\check{\mathbb{Q}}_p} \cong \mathcal{J}_{b,\check{\mathbb{Q}}_p}^{\mathbf{M}} \ltimes_{\text{Spd } \check{\mathbb{Q}}_p} \mathcal{J}_{b,\check{\mathbb{Q}}_p}^U$.

Proof. Since any automorphism of \mathcal{E}_b preserves the slope filtration, this is clear. \square

3. Canonical filtrations on an admissible modification

The main result in this section is the following theorem.

Theorem 3.1. *Let S be a perfectoid space over \mathbb{Q}_p , and let $(\mathcal{E}, \mathcal{F}, u)$ be an admissible effective modification along S of constant type $\mu = (k_1 \geq k_2 \geq \cdots)$. Let $\mathcal{E}^+ \subseteq \mathcal{E}$ be a saturated subbundle with the property that for every point $x \in |S|$, we have an equality*

$$\deg(\mathcal{E}_x^+) = \sum_{1 \leq i \leq \text{rank}(\mathcal{E}_x^+)} k_i.$$

Then the sheaf $\mathcal{F}^+ = \mathcal{F} \cap \mathcal{E}^+$ defines a subvector bundle of \mathcal{F} , and the bundle \mathcal{F}^+ is pointwise semistable of slope zero.

We remind the reader that our strategy is to first give a proof in the special case where $S = \text{Spa}(K, \mathcal{O}_K)$ is a single point, i.e., we first prove (a slightly more general version of) [Theorem 1.5](#). We then bootstrap from this situation to the case of a general S . These two steps are realized in [Sections 3A](#) and [3B](#), respectively.

3A. The case of a point. The following lemma plays a key role in our argument.

Lemma 3.2. *Let R be a DVR with uniformizer π , and let M be a finite torsion R -module, so $M \simeq \bigoplus_{1 \leq i \leq n} R/\pi^{k_i}$ with $\mu(M) = (k_1 \geq \cdots \geq k_n)$ the elementary divisors of M . Let $N \subseteq M$ be an R -submodule generated by j elements. Then $\ell(N) \leq k_1 + \cdots + k_j$, and if equality holds then N is a direct summand.*

Here and throughout, ℓ denotes R -module length (where R will always be a specific DVR clear from the context).

Proof. For the first claim, it clearly suffices to show the complementary inequality $\ell(M/N) \geq \sum_{j < i \leq n} k_i$. For this we use Fitting ideals. Recall that for any finite torsion module Q over R with elementary divisors k_i , we have an equality $\text{Fitt}_j(Q) = (\pi^{\sum_{j < i} k_i})$ for any $j \geq 0$; in particular, $\text{Fitt}(Q) = \text{Fitt}_0(Q) = (\pi^{\ell(Q)})$, and $\text{Fitt}_m(Q) = R$ if Q is generated by $\leq m$ elements. Returning to the situation at hand, we have an inclusion

$$\text{Fitt}_j(N)\text{Fitt}(M/N) \subseteq \text{Fitt}_j(M) = (\pi^{\sum_{j < i \leq n} k_i})$$

(this is a special case of [Lang 2002, Proposition XIII.10.7]). But $\text{Fitt}_j(N) = R$ since N is generated by j elements, so we get

$$(\pi^{\ell(M/N)}) = \text{Fitt}(M/N) \subseteq \text{Fitt}_j(M) = (\pi^{\sum_{j < i \leq n} k_i}),$$

and this immediately implies the desired inequality.

For the second claim, we argue by induction on j ; the case $j = 1$ is easy and left to the reader. For the induction step, choose a projection $f : M \rightarrow R/\pi^{k_1}$ onto a maximal-length cyclic direct summand of M , so $\ker f \simeq \bigoplus_{2 \leq i \leq n} R/\pi^{k_i}$. Let n_1, \dots, n_j be a set of elements generating N . After rearranging the n_i , we may assume that $f(N) = f(C)$ where $C = Rn_1 \subseteq N$, i.e., that $f(N)$ is generated by $f(n_1)$. After then possibly replacing n_i by $n_i - r_i n_1$ for all $2 \leq i \leq j$, we may assume that $\ker f$ contains the submodule N' generated by n_2, \dots, n_j . Note that we have inequalities $\ell(C) \leq k_1$ and $\ell(N') \leq k_2 + \dots + k_j$, the former because π^{k_1} kills M and the latter by applying the first half of the lemma to $N' \subseteq \ker f$. By assumption we have $\ell(N) = k_1 + \dots + k_j$, so now the chain of inequalities

$$\ell(N) = \ell(N' + C) \leq \ell(N') + \ell(C) \leq k_1 + \dots + k_j = \ell(N)$$

forces the equalities $\ell(N') = k_2 + \dots + k_j$ and $\ell(C) = k_1$. Since N' and C are generated by $j - 1$ elements and 1 element, respectively, they are both direct summands of M by the induction hypothesis. Finally, the above chain of inequalities also forces the equality $\ell(N' + C) = \ell(N') + \ell(C)$, which implies that $N' \cap C = 0$ inside M , so $N \cong N' \oplus C \subseteq M$ is a direct summand of M . \square

Theorem 3.3. *Let K be any perfectoid field over \mathbb{Q}_p , and set $S = \text{Spa}(K, \mathcal{O}_K)$. Let \mathcal{E} be a rank n vector bundle on $\mathcal{X}_{\mathbb{G}}$, and let $(\mathcal{E}, \mathcal{F}, u)$ be an effective modification along S of type $(k_1 \geq \dots \geq k_n)$. Let $\mathcal{E}^+ \subseteq \mathcal{E}$ be any saturated subbundle, and set $\mathcal{F}^+ = \mathcal{F} \cap \mathcal{E}^+$, so \mathcal{F}^+ is a saturated subbundle of \mathcal{F} .*

(i) *If \mathcal{F} is semistable of slope zero, we have the inequality*

$$\deg(\mathcal{E}^+) \leq \sum_{1 \leq j \leq \text{rank}(\mathcal{E}^+)} k_j.$$

(ii) If \mathcal{F} is semistable of slope zero and equality holds in the inequality of part (i), then \mathcal{F}^+ is also semistable of slope zero, and furthermore

$$\mathcal{E}^+/\mathcal{F}^+ \simeq \bigoplus_{1 \leq i \leq \text{rank}(\mathcal{E}^+)} \mathbb{B}_{\text{dR}}^+(K)/\xi^{k_i}$$

as submodules of

$$Q = \mathcal{E}/\mathcal{F} \simeq \bigoplus_{1 \leq i \leq n} \mathbb{B}_{\text{dR}}^+(K)/\xi^{k_i},$$

so in particular $\mathcal{E}^+/\mathcal{F}^+$ is a direct summand of Q .

Proof. Let Q^+ denote the image of the stalk $\mathcal{E}_{x(\infty)}^+$ in Q (here, as before, $x(\infty) = i(S) \in \mathcal{X}_{S^0}$). It's easy to see the equality

$$\deg(\mathcal{E}^+) = \deg(\mathcal{F}^+) + \ell(Q^+),$$

where ℓ denotes length as a $\mathbb{B}_{\text{dR}}^+(K)$ -module. Since \mathcal{F} is semistable of slope zero, \mathcal{F}^+ must have degree ≤ 0 , so dropping $\deg(\mathcal{F}^+)$ from this equality gives $\deg(\mathcal{E}^+) \leq \ell(Q^+)$. If r denotes the rank of \mathcal{E}^+ , clearly $\mathcal{E}_{x(\infty)}^+$ and then also Q^+ are generated by r elements, so [Lemma 3.2](#) implies the inequality $\ell(Q^+) \leq \sum_{1 \leq i \leq r} k_i$. Combining these inequalities gives

$$\deg(\mathcal{E}^+) \leq \ell(Q^+) \leq \sum_{1 \leq i \leq r} k_i,$$

so the first part of the theorem follows.

For the second part, we argue as follows. Putting together the equality $\deg(\mathcal{E}^+) = \deg(\mathcal{F}^+) + \ell(Q^+)$ with the inequality $\ell(Q^+) \leq \sum_{1 \leq i \leq r} k_i$, we get

$$\deg(\mathcal{E}^+) \leq \deg(\mathcal{F}^+) + \sum_{1 \leq i \leq r} k_i,$$

so if $\deg(\mathcal{E}^+) = \sum_{1 \leq i \leq r} k_i$ then \mathcal{F}^+ has degree ≥ 0 . But \mathcal{F} is semistable of slope zero, so \mathcal{F}^+ has degree ≤ 0 . Therefore \mathcal{F}^+ has degree zero. But then \mathcal{F}^+ must be semistable of degree zero, since otherwise it would have a positive-degree subbundle as a step in its slope filtration, contradicting the semistability of \mathcal{F} . Finally, since $\deg(\mathcal{F}^+) = 0$ we get an equality

$$\ell(Q^+) = \deg(\mathcal{E}^+) = \sum_{1 \leq i \leq r} k_i,$$

so then [Lemma 3.2](#) immediately shows that Q^+ is a direct summand of Q , and the maximality of its length relative to its number of generators then forces it to have the claimed shape. \square

3B. The general case.

Proof of Theorem 3.1. We argue at the level of B-pairs over A . Precisely, set $Q = M_{\text{dR}}^+(\mathcal{E})/M_{\text{dR}}^+(\mathcal{F})$; this is a $\mathbb{B}_{\text{dR}}^+(A)$ -module which is fpv over A by Theorem 2.9. Consider the $\mathbb{B}_{\text{dR}}^+(A)$ -submodule

$$Q^+ = \text{im}(M_{\text{dR}}^+(\mathcal{E}^+) \rightarrow Q)$$

of Q ; this is finitely generated and ξ -torsion. We are going to prove that Q^+ is fpv over A . Granted this, Lemma 2.2 implies that

$$N = \ker(M_{\text{dR}}^+(\mathcal{E}^+) \rightarrow Q^+) = M_{\text{dR}}^+(\mathcal{E}^+) \cap M_{\text{dR}}^+(\mathcal{F})$$

is a finite projective $\mathbb{B}_{\text{dR}}^+(A)$ -module. Then $(M_e(\mathcal{E}^+), N)$ defines a B-pair, and we obtain \mathcal{F}^+ as the associated vector bundle.

To show that Q^+ is fpv over A , we first note that it sits in a short exact sequence of finitely generated $\mathbb{B}_{\text{dR}}^+(A)$ -modules

$$0 \rightarrow Q^+ \rightarrow Q \rightarrow Q_- \rightarrow 0,$$

where

$$Q_- = M_{\text{dR}}^+(\mathcal{E}) / (M_{\text{dR}}^+(\mathcal{F}) + M_{\text{dR}}^+(\mathcal{E}^+)) = \text{coker}(M_{\text{dR}}^+(\mathcal{F}) \oplus M_{\text{dR}}^+(\mathcal{E}^+) \rightarrow M_{\text{dR}}^+(\mathcal{E})).$$

Since Q is fpv over A , we see from Lemma 2.2(iii) that to prove Q^+ is fpv over A , it suffices to show that Q_- is fpv over A . We're going to check that Q_- is fpv by applying the pointwise criterion from Proposition 2.4.

Note that unlike Q^+ (at least a priori), Q and Q_- interact well with specializing to arbitrary points $x \in |S|$. In particular, for any $x \in |S|$ we have a commutative diagram of $\mathbb{B}_{\text{dR}}^+(K_x)$ -modules

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_{\text{dR}}^+(\mathcal{E}_x^+) \cap M_{\text{dR}}^+(\mathcal{F}_x) & \longrightarrow & M_{\text{dR}}^+(\mathcal{E}_x^+) & \longrightarrow & T_x \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & M_{\text{dR}}^+(\mathcal{F}_x) & \longrightarrow & M_{\text{dR}}^+(\mathcal{E}_x) & \longrightarrow & Q \otimes_{\mathbb{B}_{\text{dR}}^+(A)} \mathbb{B}_{\text{dR}}^+(K_x) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 0 & \longrightarrow & S_x & \longrightarrow & M_{\text{dR}}^+(\mathcal{E}_x)/M_{\text{dR}}^+(\mathcal{E}_x^+) & \longrightarrow & Q_- \otimes_{\mathbb{B}_{\text{dR}}^+(A)} \mathbb{B}_{\text{dR}}^+(K_x) \longrightarrow 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & & 0 & & 0
 \end{array}$$

with exact rows and columns and with everything in the first two columns finitely generated and free. (Here T_x and S_x are defined by the commutativity of this diagram.) By hypothesis, the elementary divisors of $Q \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x)$ are independent of x and are simply given by the k_i in the statement of the theorem. By our assumptions, it's easy to see that the bundles \mathcal{E}_x , \mathcal{F}_x and \mathcal{E}_x^+ satisfy the hypotheses of [Theorem 3.3\(ii\)](#) for any $x \in |S|$. Therefore, applying [Theorem 3.3\(ii\)](#), we deduce that T_x is a *direct summand* of $Q \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x)$ with elementary divisors $k_1 \geq \dots \geq k_{\mathrm{rank}(\mathcal{E}_x^+)}$ for any point $x \in |S|$. This immediately implies that $Q_- \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x)$ has elementary divisors

$$k_{\mathrm{rank}(\mathcal{E}_x^+)+1} \geq k_{\mathrm{rank}(\mathcal{E}_x^+)+2} \geq \dots$$

for any $x \in |S|$. In particular, since $\mathrm{rank}(\mathcal{E}_x^+)$ is locally constant, the elementary divisors of $Q_- \otimes_{\mathbb{B}_{\mathrm{dR}}^+(A)} \mathbb{B}_{\mathrm{dR}}^+(K_x)$ are locally constant as functions of $x \in |S|$. Thus [Proposition 2.4](#) applies, and so Q_- is fpv over A . This completes the proof. \square

Proof of Theorem 1.4(i). This is immediate upon combining [Theorem 3.1](#) with the discussion in [Section 1D](#). \square

4. Applications to moduli of shtukas

4A. Surjectivity of the retraction. Fix all notation and assumptions as in the lead-up to [Theorem 1.4](#). In this section we prove the surjectivity portion of [Theorem 1.4\(ii\)](#), which we restate for the convenience of the reader.

Theorem 4.1. *The natural action map*

$$\mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}} \times_{\mathrm{Spd} \, \check{\mathbb{Q}}_p} \mathcal{J}_{b,\check{\mathbb{Q}}_p}^U \rightarrow \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$$

is pro-étale-locally surjective.

Proof. We argue as follows. Pick any point $f : S = \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Gr}_{G,\mu}^{\mathcal{E}_b\text{-adm}}$, with (\mathcal{F}, u) the corresponding admissible type- μ modification of \mathcal{E}_{b,S^\flat} along S . Let $0 \subsetneq \mathcal{F}^{i_1} \subsetneq \dots \subsetneq \mathcal{F}^{i_k} = \mathcal{F}$ be the Hodge–Newton flag inside \mathcal{F} (where $k = |\mathcal{I}|$ as before). Then applying the canonical retraction, i.e., looking at the point $r \circ f : \mathrm{Spa}(A, A^+) \rightarrow \mathrm{Gr}_{M,\mu}^{\mathcal{E}_b\text{-adm}}$, we get a collection $(\mathcal{F}_m, u_m)_{1 \leq m \leq k}$ of admissible type- μ_m modifications of the summands $\mathcal{E}_{b_m,S^\flat}$. The point $i \circ r \circ f$ then corresponds to viewing $(\bigoplus_{1 \leq m \leq k} \mathcal{F}_m, \bigoplus_{1 \leq m \leq k} u_m)$ as a type- μ modification of $\mathcal{E}_{b,S^\flat} = \bigoplus_{1 \leq m \leq k} \mathcal{E}_{b_m,S^\flat}$. We're going to (pro-étale-locally on S) find an element $j \in \mathcal{J}_{b,\check{\mathbb{Q}}_p}^U(S)$ which transports the point $i \circ r \circ f$ to the point f .

Now, the fact that f and $i \circ r \circ f$ have the same retraction translates into the following fact: After choosing compatible isomorphisms

$$\iota_m : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \dots \oplus \mathcal{F}_m \simeq \mathcal{F}^{i_m}$$

(which we can do pro-étale-locally on S), the compatible-in- m maps

$$v_m : u|_{\mathcal{F}^{i_m}} \circ \iota_m : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_m \rightarrow \mathcal{E}^{i_m} \cong \bigoplus_{1 \leq i \leq m} \mathcal{E}_{b_i}$$

and

$$\eta_m : u_1 \oplus \cdots \oplus u_m : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_m \rightarrow \mathcal{E}^{i_m} \cong \mathcal{E}^{i_m} \cong \bigoplus_{1 \leq i \leq m} \mathcal{E}_{b_i}$$

coincide after projection along $\mathcal{E}^{i_m} \twoheadrightarrow \mathcal{E}_{b_m}$. We are going to show that each $v_m \circ \eta_m^{-1}$, which is initially only a *meromorphic* endomorphism of \mathcal{E}^{i_m} , actually defines a global section of $(\mathcal{E}^{i_m})^\vee \otimes \mathcal{E}^{i_m}$ such that $v_m \circ \eta_m^{-1} - 1$ defines a section of the subbundle $(\mathcal{E}^{i_m})^\vee \otimes \mathcal{E}^{i_{m-1}}$. To do this, note that by an easy induction, each map $\eta_m - v_m : \mathcal{F}^{i_m} \rightarrow \mathcal{E}^{i_{m-1}}$ has zeros of order $\geq k_{d_1+\dots+d_{m-1}}$ along $S \subset \mathcal{X}_{S^\flat}$. On the other hand, $u_m^{-1} : \mathcal{E}_{b_m} \rightarrow \mathcal{F}_m$ has poles of order $\leq k_{d_1+\dots+d_{m-1}+1}$ along S .⁸ Now, formally, we have the identity

$$\begin{aligned} v_m \circ \eta_m^{-1} &= v_{m-1} \circ \eta_{m-1}^{-1} + v_m \circ u_m^{-1} \\ &= v_{m-1} \circ \eta_{m-1}^{-1} + (v_m - \eta_m + \eta_m) \circ u_m^{-1} \\ &= v_{m-1} \circ \eta_{m-1}^{-1} + (v_m - \eta_m) \circ u_m^{-1} + \eta_m \circ u_m^{-1}. \end{aligned}$$

But $(v_m - \eta_m) \circ u_m^{-1} : \mathcal{E}_{b_m} \rightarrow \mathcal{E}^{i_{m-1}}$ is well-defined by our previous remarks on zeros and poles, and $\eta_m \circ u_m^{-1} : \mathcal{E}_{b_m} \rightarrow \mathcal{E}^{i_m}$ is just the canonical inclusion as a direct summand. Thus we get the desired properties of $v_m \circ \eta_m^{-1}$ by induction, noting that $v_1 \circ \eta_1^{-1} = \text{id}$. But this analysis shows that the section

$$j = v_k \circ \eta_k^{-1} \in H^0(\mathcal{X}_{S^\flat}, \mathcal{E}_b^\vee \otimes \mathcal{E}_b)$$

defines an element of $\mathcal{J}_{b, \mathbb{Q}_p}^U(S)$, and by construction it transports

$$\left(\bigoplus_{1 \leq m \leq k} \mathcal{F}_m, \bigoplus_{1 \leq m \leq k} u_m \right) \quad \text{to} \quad (\mathcal{F}, u) \simeq \left(\bigoplus_{1 \leq m \leq k} \mathcal{F}_m, u \circ \iota_k \right),$$

so we're done. \square

4B. The retraction at infinite level.

Proof of Theorem 1.7. We construct a two-sided inverse to a_∞ . Let $S \in \text{Perfd}/\text{Spa } \mathbb{Q}_p$ and $(\mathcal{F}, u, \alpha) \in \text{Sht}_{P, \mu, b}(S)$ be given. We need to construct a point

$$\prod_{1 \leq m \leq k} (\mathcal{G}_m, v_m, \beta_m) \in \text{Sht}_{M, \mu, b}(S)$$

and an element $j \in \mathcal{J}_{b, \mathbb{Q}_p}^U(S)$. The first is easier to find: applying the retraction on period domains to $(\mathcal{F}, u) \in \text{Gr}_{G, \mu}^{\mathcal{E}_b\text{-adm}}(S)$ gives a point

$$(\text{gr}(\mathcal{F}), \text{gr}(u)) = \prod_{1 \leq m \leq k} (\mathcal{F}_m, u_m) \in \text{Gr}_{M, \mu}^{\mathcal{E}_b\text{-adm}}(S).$$

⁸It's easy to make these statements about poles and zeros precise; the point is that the ideal sheaf cutting out $S \subset \mathcal{X}_{S^\flat}$ is locally principal and generated by a non-zero-divisor.

Now by the definition of $\text{Sht}_{P,\mu,b}$, it's easy to see check that “ $\text{gr}(\alpha)$ ” gives a well-defined sequence of trivializations $\alpha_m : \mathcal{O}_{\mathcal{X}_{S^b}}^{d_m} \xrightarrow{\sim} \mathcal{F}_m$, and this gives a point

$$(\text{gr}(\mathcal{F}), \text{gr}(u), \text{gr}(\alpha)) = \prod_{1 \leq m \leq k} (\mathcal{F}_m, u_m, \alpha_m) \in \text{Sht}_{M,\mu,b}(S)$$

as desired.

To construct j , recall from the proof of [Theorem 4.1](#) that after making any choices of compatible isomorphisms $\iota_m : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_m \simeq \mathcal{F}^{i_m}$ ($1 \leq m \leq k$), the two maps

$$\nu_k : u \circ \iota_k : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k \rightarrow \mathcal{E}_b \cong \bigoplus_{1 \leq i \leq k} \mathcal{E}_{b_i}$$

and

$$\eta_k : u_1 \oplus \cdots \oplus u_k : \mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_k \rightarrow \mathcal{E}_b \cong \bigoplus_{1 \leq i \leq k} \mathcal{E}_{b_i}$$

have the property that $\nu_k \circ \eta_k^{-1}$ defines an element of \mathcal{J}_b^U . Now we simply observe that at infinite level, there is a canonical choice for the ι_m , as indicated by the diagram

$$\mathcal{F}_1 \oplus \mathcal{F}_2 \oplus \cdots \oplus \mathcal{F}_m \xleftarrow[\alpha_1 \oplus \cdots \oplus \alpha_m]{\sim} \mathcal{O}_{\mathcal{X}}^{d_1 + \cdots + d_m} \xrightarrow[\alpha | (\mathcal{O}_{\mathcal{X}}^{d_1 + \cdots + d_m} \oplus 0)]{\sim} \mathcal{F}^{i_m}.$$

In other words, we take

$$\iota_k = \alpha \circ (\alpha_1^{-1} \oplus \cdots \oplus \alpha_m^{-1}) = \alpha \circ \text{gr}(\alpha)^{-1},$$

and then

$$\nu_k \circ \eta_k^{-1} = u \circ \iota_k \circ \text{gr}(u)^{-1} = u \circ \alpha \circ \text{gr}(\alpha)^{-1} \circ \text{gr}(u)^{-1} \in \mathcal{J}_b^U(S)$$

is the unique element such that

$$j \cdot (\text{gr}(\mathcal{F}), \text{gr}(u), \text{gr}(\alpha)) = (\mathcal{F}, u, \alpha),$$

as desired. □

4C. Consequences for cohomology. In order to apply our geometric results to calculations of cohomology, we need some well-behaved cohomological formalism for diamonds. Such a theory was recently developed by Scholze [\[2017\]](#). One of the key outcomes of this formalism is that many of the results in Huber's book [\[1996\]](#) have natural generalizations to morphisms of well-behaved diamonds. In this subsection, we freely use various notations, terminology and constructions from [\[Scholze 2017\]](#), including the notions of locally spatial diamonds, partially proper morphisms, and ℓ -cohomologically smooth morphisms. For brevity, we introduce the following conventions:

- A morphism $f : X \rightarrow Y$ of small v-stacks is *good* if (in the terminology of [\[Scholze 2017\]](#)) it is compactifiable, representable in locally spatial diamonds, and of finite $\dim.\text{trg}$.

- A morphism $f : X \rightarrow Y$ of small v-stacks is *smooth* if (in the terminology of [Scholze 2017]) it is ℓ -cohomologically smooth for all $\ell \neq p$.

Note that by convention, any smooth map is good, and any good map is representable in locally spatial diamonds.

Let Λ be a coefficient ring killed by some integer n prime to p . For any small v-stack X , let $D_{\text{ét}}(X, \Lambda)$ denote the triangulated category defined in [Scholze 2017]. When X is a locally spatial diamond this agrees with (the left completion of) the honest derived category $D(X_{\text{ét}}, \Lambda)$, but in general the latter is not the correct object. The categories $D_{\text{ét}}(X, \Lambda)$ support the expected “four functors” f^* , Rf_* , $- \otimes_{\Lambda}^L -$, $R\mathcal{H}om_{\Lambda}(-, -)$, and these satisfy all expected compatibilities and adjunctions. Less obviously, Scholze also proved that if $f : X \rightarrow Y$ is a good morphism of small v-stacks, there is a well-behaved derived direct image functor with proper supports

$$Rf_! : D_{\text{ét}}(X, \Lambda) \rightarrow D_{\text{ét}}(Y, \Lambda)$$

with all expected properties. In particular, $Rf_!$ satisfies the proper base change theorem and the projection formula, and admits a well-behaved right adjoint $Rf^!$. When X and Y are the diamonds associated with analytic adic spaces over $\text{Spa } \mathbb{Z}_p$, these constructions all agree with the constructions in Huber’s book.

In what follows we adopt the following convention: if X is a locally spatial diamond which comes with an evident structure map $f : X \rightarrow \text{Spd } C$ for some complete algebraically closed field C , and moreover f is good, then we set

$$R\Gamma_c(X, F) = R\Gamma(\text{Spd } C, Rf_! F)$$

for any $F \in D_{\text{ét}}(X, \Lambda)$. Our main goal in this section is the calculation of the compactly supported étale cohomology groups $R\Gamma_c(\mathcal{J}_{b,C}, \mathbb{Z}/n\mathbb{Z})$, where $\mathcal{J}_{b,C}$ denotes the base change of $\mathcal{J}_{b, \check{\mathbb{Q}}_p}$ along any complete algebraically closed extension $\check{\mathbb{Q}}_p \rightarrow C$. Forgetting any possible Galois actions, it is not so hard to show that this is the expected shift of $\mathbb{Z}/n\mathbb{Z}$. However, when $C = \widehat{\mathbb{Q}}_p$ we would also like a precise description of the $W_{\mathbb{Q}_p}$ -action, and this is significantly harder. The key point is Proposition 4.8.

Proposition 4.2 (Künneth formula). *Let*

$$\begin{array}{ccc} X \times_S Y & \xrightarrow{q} & Y \\ p \downarrow & \searrow h & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

be a cartesian diagram of small v-stacks such that f and g are good. Then for any $F \in D_{\text{ét}}(X, \Lambda)$ and $G \in D_{\text{ét}}(Y, \Lambda)$, there is a natural isomorphism

$$Rh_!(p^* F \otimes_{\Lambda}^L q^* G) \cong Rf_! F \otimes_{\Lambda}^L Rg_! G.$$

In particular, if $S = \mathrm{Spd} \mathbb{C}$ is a geometric point, then

$$R\Gamma_c(X \times_S Y, p^* F \otimes_{\Lambda}^L q^* G) \cong R\Gamma_c(X, F) \otimes_{\Lambda}^L R\Gamma_c(Y, G).$$

As usual, this follows formally from the proper base change theorem [Scholze 2017, Proposition 22.19] and the projection formula [loc. cit., Proposition 22.23].

Proposition 4.3. *Let*

$$\begin{array}{ccc} W & \xrightarrow{f'} & Y \\ g' \downarrow & \searrow h & \downarrow g \\ X & \xrightarrow{f} & S \end{array}$$

be a cartesian diagram of small v -stacks, where f and g are good. Suppose moreover that one of f or g is smooth. Then there is a natural isomorphism

$$Rh^! \Lambda \cong g'^* Rf^! \Lambda \otimes_{\Lambda}^L f'^* Rg^! \Lambda.$$

Proof. By symmetry, we can assume g is smooth, so g' is as well. Then

$$\begin{aligned} Rh^! \Lambda &\cong Rg'^! Rf^! \Lambda \\ &\cong g'^* Rf^! \Lambda \otimes_{\Lambda}^L Rg'^! \Lambda \\ &\cong g'^* Rf^! \Lambda \otimes_{\Lambda}^L Rg'^! f^* \Lambda \\ &\cong g'^* Rf^! \Lambda \otimes_{\Lambda}^L f'^* Rg^! \Lambda. \end{aligned}$$

Here the first and third lines are trivial; the second line follows from [Scholze 2017, Proposition 23.12(i)], and the fourth line follows from [Scholze 2017, Proposition 23.12(iii)]. \square

We'll also need the following results.

Proposition 4.4. *Let $f : X \rightarrow Y$ be any proper map of spatial diamonds, and let Λ be any coefficient ring killed by some integer prime to p . Then the functor Rf_* carries $D_{\text{ét}}^{[a,b]}(X, \Lambda)$ into $D_{\text{ét}}^{[a,b+2\dim.\mathrm{trg} f]}(Y, \Lambda)$.*

Proof. This is implicit in the proof of [Scholze 2017, Theorem 22.5]. \square

Proposition 4.5. *Let X be a spatial diamond, and let Λ be a coefficient ring killed by some integer prime to p . Suppose that for some complete algebraically closed field C/\mathbb{F}_p there is a map $f : X \rightarrow \mathrm{Spd}(C, C^\circ)$ with $\dim.\mathrm{trg} f < \infty$, or more generally that there exists an integer N such that $H^i(X_{\text{ét}}, \mathcal{F}) = 0$ for all $i > N$ and all sheaves of Λ -modules \mathcal{F} on $X_{\text{ét}}$. Then $D(X_{\text{ét}}, \Lambda)$ is left-complete and compactly generated. In particular, there is a natural equivalence $D_{\text{ét}}(X, \Lambda) \cong D(X_{\text{ét}}, \Lambda)$.*

Note that if such a map $f : X \rightarrow \mathrm{Spd}(C, C^\circ)$ exists, then $H^i(X_{\text{ét}}, \mathcal{F}) = 0$ for all $i > 2\dim.\mathrm{trg} f$ and all \mathcal{F} , which justifies the wording in the second sentence of the proposition. To see this, note that by [Scholze 2017, Proposition 21.11], $H^i(X_{\text{ét}}, \mathcal{F}) = 0$ for all \mathcal{F} and all $i > \dim|X| + \sup_{\ell \neq p} \sup_x \mathrm{cd}_\ell x$, where x runs over the maximal points of X . Then

$$\dim|X| = \dim f \leq \dim.\mathrm{trg} f$$

by a straightforward computation, and

$$\mathrm{cd}_\ell x \leq \dim.\mathrm{trg} f$$

for all maximal points and all $\ell \neq p$ by [Scholze 2017, Proposition 21.16].

Proof. Let $j : U \rightarrow X$ be any quasicompact separated étale map. By [Scholze 2017, Remark 21.14], $j_{\text{ét}*}$ is an exact functor, so $H^i(U_{\text{ét}}, \mathcal{F}) = H^i(X_{\text{ét}}, j_{\text{ét}*}\mathcal{F})$ for any abelian étale sheaf \mathcal{F} on U . Combining this with the hypotheses of the theorem, we deduce the existence of an integer N such that for any quasicompact separated étale map $U \rightarrow X$ and any sheaf \mathcal{F} of Λ -modules on $U_{\text{ét}}$, the group $H^i(U_{\text{ét}}, \mathcal{F})$ vanishes for all $i > N$. Granted this, left-completeness of $D(X_{\text{ét}}, \Lambda)$ follows by arguing as in [Stacks 2005–, Tag 0719]. The equivalence $D_{\text{ét}}(X, \Lambda) \cong D(X_{\text{ét}}, \Lambda)$ then follows from [Scholze 2017, Proposition 14.15].

For compact generation, we leave it to the reader to check that varying over all quasicompact separated étale maps $j : U \rightarrow X$ and all $n \in \mathbb{Z}$, the objects $j_!\Lambda[n]$ give a generating set of compact objects in $D(X_{\text{ét}}, \Lambda)$. \square

Proposition 4.6. *Let S be a small v -sheaf, and let $\tilde{f} : \tilde{X} \rightarrow S$ be a smooth map of small v -sheaves. Suppose that \tilde{X} is equipped with a free \underline{K} -action for some pro- p group K , lying over the trivial action on S . Set $X = \tilde{X}/\underline{K}$, and let $f : X \rightarrow S$ be the natural map. Then:*

- (i) *The natural map $q : \tilde{X} \rightarrow X$ is a \underline{K} -torsor, and the map $f : X \rightarrow S$ is smooth.*
- (ii) *Any choice of a Λ -valued Haar measure on K determines an isomorphism*

$$Rf^!\Lambda \simeq (q_* R\tilde{f}^!\Lambda)^K.$$

In particular, if $S = \mathrm{Spd} k$ for some nonarchimedean field k of residue characteristic p , and \tilde{X} arises from a connected smooth adic space $Y/\mathrm{Spa} k$, then any choice of a Haar measure on K gives rise to an isomorphism

$$Rf^!\Lambda \simeq \Lambda(d)[2d]$$

with $d = \dim Y$.

Note that up to a shift, $q_* R\tilde{f}^!\Lambda$ is concentrated in one degree, i.e., is a sheaf, so the group action in part (ii) makes sense with the “naive” interpretation.

Proof. The key tool here is [Scholze 2017, Proposition 24.2]. Part (i) follows directly from this result together with [Scholze 2017, Lemma 10.13]. For part (ii), recall from [Scholze 2017, Proposition 24.2] that any choice of a Haar measure on K determines a natural isomorphism of functors

$$q^* Rf^! \cong Rq^! Rf^! = R\tilde{f}^!$$

(of course there is a canonical choice of Haar measure, giving K total volume 1; however, in a later argument we'll actually need the freedom of choosing a different Haar measure). In particular, if we set $\omega_X = Rf^! \Lambda$ and $\omega_{\tilde{X}} = R\tilde{f}^! \Lambda$, then such a choice determines an isomorphism $q^* \omega_X \cong Rq^! \omega_X = \omega_{\tilde{X}}$. On the other hand, it is easy to see that for any shift of any abelian étale sheaf \mathcal{F} on X , the natural map

$$\mathcal{F} \rightarrow (q_* q^* \mathcal{F})^K$$

is an isomorphism.⁹ Putting these observations together, we compute that

$$\omega_X \cong (q_* q^* \omega_X)^K \cong (q_* \omega_{\tilde{X}})^K,$$

which proves the first half of (ii). Finally, if \tilde{X} arises from a connected smooth $Y/\mathrm{Spa} k$ as in the proposition, then $\omega_{\tilde{X}} \cong \omega_Y \cong \Lambda(d)[2d]$ by the results in [Berkovich 1993; Huber 1996], so then $(q_* \omega_{\tilde{X}})^K \simeq \Lambda(d)[2d]$, as desired. \square

We now put ourselves in the following situation. Let $b \in \mathrm{GL}_n(\mathbb{Q}_p)$ be any element, so the bundle $\mathcal{E}_{b,S}$ is well-defined on \mathcal{X}_S for any perfectoid space S/\mathbb{F}_p , functorially in S . We may therefore consider the functor $\mathcal{H}^0(\mathcal{E}_b)$ sending any $S \in \mathrm{Perf}$ to $H^0(\mathcal{X}_S, \mathcal{E}_{b,S})$, with its structure map $f_b : \mathcal{H}^0(\mathcal{E}_b) \rightarrow \mathrm{Spd} \mathbb{F}_p$.¹⁰ For C/\mathbb{F}_p any complete algebraically closed extension, write $f_{b,C} : \mathcal{H}^0(\mathcal{E}_b)_C \rightarrow \mathrm{Spd} C$ for the base change of the situation along $\mathbb{F}_p \rightarrow C$.

Proposition 4.7. *Let $b \in \mathrm{GL}_n(\mathbb{Q}_p)$ be any element such that the slopes of \mathcal{E}_b are all positive, and let f_b and $f_{b,C}$ be as above.*

- (i) *The structure map f_b is partially proper, representable in locally spatial diamonds and smooth.*
- (ii) *For any complete algebraically closed extension C/\mathbb{F}_p , there is an isomorphism $Rf_{b,C}^! \Lambda \simeq \Lambda[2d]$, where d is the degree of \mathcal{E}_b .*
- (iii) *For any complete algebraically closed extension C/\mathbb{F}_p , the natural map*

$$\Lambda \rightarrow Rf_{b,C*} \Lambda$$

is an isomorphism.

⁹Using the identification $q_* q^* \mathcal{F} = \mathcal{F} \otimes_{\Lambda} q_* \Lambda$, one reduces to the special case where $\mathcal{F} = \Lambda$, which is trivial.

¹⁰This is a slight change from the notation in Section 2D, so, e.g., the functor defined in Definition 2.17(1) is the base change of what we are presently denoting by $\mathcal{H}^0(\mathcal{E}_b)$ along the map $\mathrm{Spd} \mathbb{F}_p \rightarrow \mathrm{Spd} \mathbb{F}_p$.

Proof. First, we note that it suffices to prove (i) after base change to $S = \mathrm{Spd} C$ for C/\mathbb{F}_p any complete algebraically closed field C . To see this, note that $\mathrm{Spd} C \rightarrow \mathrm{Spd} \mathbb{F}_p$ is surjective as a map of v -sheaves, so [Scholze 2017, Proposition 13.4(v)] shows that if $f_{b,C}$ is separated and representable in locally spatial diamonds, then so is f_b . Moreover, ℓ -cohomological smoothness is a v -local property [Scholze 2017, Proposition 23.15], and being separated is v -local as well [Scholze 2017, Proposition 10.11(ii)]. Finally, observe that f_b automatically satisfies the valuative portion of the definition of partial properness [Scholze 2017, Definition 18.4], using the fact that

$$H^0(\mathcal{X}_{\mathrm{Spa}(R, R^+)}, \mathcal{E}) = H^0(\mathcal{X}_{\mathrm{Spa}(R, R^\circ)}, \mathcal{E})$$

for any characteristic p perfectoid Tate–Huber pair (R, R^+) and any vector bundle \mathcal{E} on $\mathcal{X}_{\mathrm{Spa}(R, R^+)}$.

Next, observe that

$$\mathcal{H}^0(\mathcal{E}_{b_1} \oplus \mathcal{E}_{b_2}) \cong \mathcal{H}^0(\mathcal{E}_{b_1}) \times_{\mathrm{Spd} \mathbb{F}_p} \mathcal{H}^0(\mathcal{E}_{b_2}),$$

and that $\mathcal{H}^0(\mathcal{E}_b) \simeq \mathcal{H}^0(\mathcal{E}_{b'})$ if b and b' are σ -conjugate. With these isomorphisms in mind, an easy inductive argument together with Proposition 4.3 shows that for (i) and (ii) it suffices to treat the case where

$$b = \begin{pmatrix} & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \\ p^{-d} & & & & & \end{pmatrix} \in \mathrm{GL}_h(\mathbb{Q}_p)$$

for some coprime h, d , in which case we write $\mathcal{E}_b = \mathcal{O}(d/h)$ as usual.

If $d \leq h$ then $\mathcal{H}^0(\mathcal{O}(d/h))_C$ coincides with (the base change to C of) the universal cover of an isoclinic p -divisible group $\mathcal{G}/\overline{\mathbb{F}}_p$ of height h and dimension d , so by the calculations in [Fargues and Fontaine 2018; Scholze and Weinstein 2013], $\mathcal{H}^0(\mathcal{O}(d/h))_C$ is representable by the perfectoid space

$$X = \mathrm{Spf}(\mathcal{O}_C \llbracket T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty} \rrbracket)_\eta^{\mathrm{ad}}.$$

In this case partial properness is clear, smoothness of $f_{b,C}$ follows from [Scholze 2017, Proposition 24.4], and part (ii) follows from classical calculations of Berkovich and Huber (and in fact the isomorphism in part (ii) is canonical in this case, being realized by a suitable trace map, see [Berkovich 1993; Huber 1996]).

We may thus suppose that $d > h$. We may choose a short exact sequence of vector bundles

$$0 \rightarrow \mathcal{O}^m \rightarrow \mathcal{O}(1)^d \rightarrow \mathcal{O}(d/h) \rightarrow 0$$

on $\mathcal{X}_{\mathrm{Spa} C}$, which induces a short exact sequence of \mathbb{Q}_p -vector diamonds

$$0 \rightarrow \mathcal{H}^0(\mathcal{O}^m)_C \simeq \underline{\mathbb{Q}_p^m} \rightarrow \mathcal{H}^0(\mathcal{O}(1)^d)_C \rightarrow \mathcal{H}^0(\mathcal{O}(d/h))_C \rightarrow 0$$

over S . As in [Fargues and Fontaine 2018; Scholze and Weinstein 2013], the diamond $\mathcal{H}^0(\mathcal{O}(1)^d)_C$ is naturally representable by the perfectoid space

$$\tilde{B}^d = (\mathrm{Spa} \mathcal{O}_C \llbracket T_1^{1/p^\infty}, \dots, T_d^{1/p^\infty} \rrbracket)_\eta,$$

and moreover this isomorphism identifies multiplication by p on $\mathcal{H}^0(\mathcal{O}(1)^d)$ with the relative Frobenius $\varphi : T_i \mapsto T_i^p$. Choose a quasicompact open subgroup $U_0 \subset \tilde{B}^d \simeq \mathcal{H}^0(\mathcal{O}(1)^d)_C$, and set $A_0 = \mathbb{Q}_p^m \cap U_0(C)$, where the intersection is taken inside the C -points of $\mathcal{H}^0(\mathcal{O}(1)^d)_C$. Then $A_0 \simeq \mathbb{Z}_p^m$, and the \mathbb{Q}_p^m -action on $\mathcal{H}^0(\mathcal{O}(1)^d)_C$ restricts to a compatible A_0 -action on U_0 . Writing $U_n = p^{-n} U_0$ and $A_n = p^{-n} A_0$ for their preimages under multiplication by p^n , we get a rising union of quasicompact open subgroups $U_0 \subset U_1 \subset U_2 \subset \dots$ covering $\mathcal{H}^0(\mathcal{O}(1)^d)_C$, each of which is the perfection of a closed rigid analytic disk over C , with compatible free actions of the group sheaves $A_0 \subset A_1 \subset \dots$. In particular, the U_n are ℓ -cohomologically smooth over S , and writing $\tilde{f}_n : U_n \rightarrow S$ for the structure map, we have natural isomorphisms $R\tilde{f}_n^! \Lambda \cong \Lambda[2d]$ compatible with varying n . By the previous proposition, we deduce that the quotient diamonds $V_n = U_n/A_n \subset \mathcal{H}^0(\mathcal{O}(d/h))_C$ are spatial and ℓ -cohomologically smooth over S , and that $Rf_n^! \Lambda \simeq \Lambda[2d]$ for any n , where $f_n : V_n \rightarrow S$ denotes the structure map. A careful examination of the proof of the previous proposition shows that these isomorphisms may be chosen compatibly with varying n ,¹¹ in which case they glue into an isomorphism $Rf_{b,C}^! \Lambda \simeq \Lambda[2d]$.

For part (iii) of the proposition, we return to the case of general b . Let s be the number of distinct slopes of \mathcal{E}_b . On $\mathcal{X}_{\mathrm{Spa} C}$, choose an isomorphism $\mathcal{E}_b \simeq \bigoplus_{1 \leq i \leq s} \mathcal{E}_i$ where each \mathcal{E}_i is semistable. As in the previous paragraph, we may write each $\mathcal{H}^0(\mathcal{E}_i)$ as a rising union $\bigcup U_{n,i}/A_{n,i}$, where $U_{n,i}$ is the perfection of a closed rigid disk and $A_{n,i} \simeq \mathbb{Z}_p^{m_i}$. (If \mathcal{E}_i has slope ≤ 1 , then one chooses the $A_{n,i}$ to be trivial.) Taking the product of these presentations, we get an analogous presentation of $\mathcal{H}^0(\mathcal{E}_b)_C$ as a rising union

$$\mathcal{H}^0(\mathcal{E}_b)_C \simeq \bigcup_n U_n/A_n$$

where $U_n = U_{n,1} \times_S \dots \times_S U_{n,s}$ and $A_n = \prod A_{n,i}$. Let

$$\tilde{f}_n : U_n \rightarrow S \quad \text{and} \quad f_n : V_n = U_n/A_n \rightarrow S$$

¹¹Recall that the natural isomorphisms defined in Proposition 24.2 of [Scholze 2017] depend on a choice of Haar measure on the group K ; in the present application, the key point is to fix the Haar measure on A_0 with $\mu(A_0) = 1$ and then choose the Haar measures on the A_n compatibly with the inclusions $A_0 \subset A_n$, so in particular the total measure of A_n is p^{mn} .

be the structure maps. We then have a natural isomorphism

$$Rf_{b,C*}\Lambda \cong R\lim_{\leftarrow} Rf_{n*}\Lambda,$$

see Lemma 3.9.2 of [Huber 1996] and its proof. Using the Cartan–Leray spectral sequence for the Galois cover $U_n \rightarrow V_n$, we compute that

$$\begin{aligned} Rf_{n*}\Lambda &\cong R\Gamma(A_n, R\tilde{f}_{n*}\Lambda) \\ &\cong R\Gamma(A_n, \Lambda) \\ &\cong \Lambda. \end{aligned}$$

Under these isomorphisms, it is trivial to check that the transition maps in the inverse system of the $Rf_{n*}\Lambda$ are given by the identity map, so $R\lim_{\leftarrow} Rf_{n*}\Lambda \cong \Lambda$, as desired. \square

Proposition 4.8. *As in the previous proposition, let $b \in \mathrm{GL}_n(\mathbb{Q}_p)$ be any element such that the slopes of \mathcal{E}_b are all positive, and consider the functor $\mathcal{H}^0(\mathcal{E}_b)$ with its structure map $f_b : \mathcal{H}^0(\mathcal{E}_b) \rightarrow \mathrm{Spd} \mathbb{F}_p$. Set $d = \deg \mathcal{E}_b$. Then:*

- (i) *The natural adjunction map $Rf_{b!}Rf_b^!\Lambda \rightarrow \Lambda$ is an isomorphism.*
- (ii) *There is a natural isomorphism $Rf_b^!\Lambda \cong \Lambda(d)[2d]$.*
- (iii) *There is a natural isomorphism $Rf_{b!}\Lambda \cong \Lambda(-d)[-2d]$.*

In particular, if C is any algebraically closed field, then

$$R\Gamma_c(\mathcal{H}^0(\mathcal{E}_b)_C, \Lambda) \simeq \Lambda(-d)[-2d].$$

Proof. The final point follows from (iii) by proper base change. Next, note that (iii) follows quickly from (i) and (ii). Indeed, combining (i) and (ii) with the projection formula gives a chain of natural isomorphisms

$$\Lambda(d)[2d] \otimes_{\Lambda}^L Rf_{b!}\Lambda \cong Rf_{b!}(\Lambda(d)[2d]) \cong Rf_{b!}Rf_b^!\Lambda \cong \Lambda.$$

The result then follows by tensoring both sides with $\Lambda(-d)[-2d]$.

Let C/\mathbb{F}_p be any complete algebraically closed extension, and let

$$f_{b,C} : \mathcal{H}^0(\mathcal{E}_b)_C \rightarrow \mathrm{Spd} C$$

be the base change along $\mathbb{F}_p \rightarrow C$ as before. For (i), note that by a combination of smooth and proper base change, it suffices to prove that for some arbitrary choice of C , the natural adjunction $Rf_{b,C!}Rf_{b,C}^!\Lambda \rightarrow \Lambda$ is an isomorphism in $D_{\text{ét}}(\mathrm{Spd} C, \Lambda) \cong D(\Lambda)$. Next, observe that there are natural isomorphisms

$$\begin{aligned} Rf_{b,C!}Rf_{b,C}^!\Lambda &\cong Rf_{b,C!}Rf_{b,C}^!(\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}/n\mathbb{Z}}^L \Lambda) \\ &\cong Rf_{b,C!}(Rf_{b,C}^!\mathbb{Z}/n\mathbb{Z} \otimes_{\mathbb{Z}/n\mathbb{Z}}^L \Lambda) \\ &\cong (Rf_{b,C!}Rf_{b,C}^!\mathbb{Z}/n\mathbb{Z}) \otimes_{\mathbb{Z}/n\mathbb{Z}}^L \Lambda, \end{aligned}$$

where the second line follows from [Scholze 2017, Theorem 1.10(i)] and the third line follows from the projection formula. This reduces us further to the case where $\Lambda = \mathbb{Z}/n\mathbb{Z}$.

Let K denote the cone of the map

$$Rf_{b,C}! Rf_{b,C}^! \Lambda \rightarrow \Lambda,$$

and let $K^\vee = R\mathcal{H}om_\Lambda(K, \Lambda)$ be its dual. Dualizing the distinguished triangle

$$Rf_{b,C}! Rf_{b,C}^! \Lambda \rightarrow \Lambda \rightarrow K,$$

we see that $K^\vee[1]$ is isomorphic to the cone of the dualized map

$$\Lambda \rightarrow R\mathcal{H}om_\Lambda(Rf_{b,C}! Rf_{b,C}^! \Lambda, \Lambda).$$

Next, observe that there are canonical isomorphisms

$$R\mathcal{H}om_\Lambda(Rf_{b,C}! Rf_{b,C}^! \Lambda, \Lambda) \cong Rf_{b,C*} R\mathcal{H}om_\Lambda(Rf_{b,C}^! \Lambda, Rf_{b,C}^! \Lambda) \cong Rf_{b,C*} \Lambda,$$

where the first isomorphism follows from Verdier duality [Scholze 2017, Theorem 1.8(iv)] and the second follows from biduality [Scholze 2017, Theorem 1.12]. Moreover, the resulting map $\Lambda \rightarrow Rf_{b,C*} \Lambda$ coincides with the obvious adjunction map, which is an isomorphism by Proposition 4.8. Therefore K^\vee is zero. But

$$R\mathcal{H}om_\Lambda(-, \Lambda) : D(\Lambda) \rightarrow D(\Lambda)$$

is conservative (using that $\Lambda = \mathbb{Z}/n\mathbb{Z}$), so then K is zero as well.

For (ii), we again reduce via Proposition 4.3 to the case where

$$b = \begin{pmatrix} & & 1 & & \\ & & & 1 & \\ & & & & \ddots \\ & & & & & 1 \\ p^{-d} & & & & & \end{pmatrix} \in \mathrm{GL}_h(\mathbb{Q}_p)$$

for some coprime $h, d \geq 1$, so $\mathcal{E}_b = \mathcal{O}(d/h)$. In what follows, we only give full details in the case where $h = 1$; the general case is very similar.

In the subsequent arguments, the following conventions are convenient. For any small v-sheaf X , we say that X is absolutely good (resp. absolutely smooth) if the canonical map $\pi_X : X \rightarrow \mathrm{Spd} \mathbb{F}_p$ is good (resp. smooth). If X is absolutely good, we write $\omega_X = R\pi_X^! \Lambda$. If X is connected and absolutely smooth, we write $\delta(X)$ for the unique integer such that ω_X is concentrated in degree $-\delta(X)$. For example, $\mathrm{Spd} \mathbb{Q}_p$ is connected and absolutely smooth, and $\delta(\mathrm{Spd} \mathbb{Q}_p) = 2$.

For brevity, set $X_d = \mathcal{H}^0(\mathcal{O}(d))$ for any $d \geq 1$, with structure map $f_d : X_d \rightarrow \mathrm{Spd} \mathbb{F}_p$. We are trying to produce a reasonably canonical isomorphism $\omega_{X_d} \cong \Lambda(d)[2d]$.

When $d = 1$, such an isomorphism follows from work of Berkovich and Huber, as we'll explain below. The general case is somewhat tricky. Before giving the argument, we introduce a plethora of spaces related to X_d which we'll need.

Let $Y_d \subset X_d$ denote the open subfunctor with S -points parametrizing sections of $\mathcal{O}(d)$ which don't vanish identically on any fiber of $|\mathcal{X}_S| \rightarrow |S|$, with structure map $g_d : Y_d \rightarrow \mathrm{Spd} \mathbb{F}_p$. There is a continuous \mathbb{Q}_p^\times -action on X_d given by scaling sections, which restricts to a free action on Y_d , and we set $\mathrm{Div}^d = Y_d / \mathbb{Q}_p^\times$. By [Fargues 2020, Proposition 2.18], the natural map $q : (\mathrm{Div}^1)^d \rightarrow \mathrm{Div}^d$ induces an isomorphism $(\mathrm{Div}^1)^d / S_d \cong \mathrm{Div}^d$. Let U_d be the complement of all the partial diagonals in $(\mathrm{Div}^1)^d$, or equivalently the maximal open subfunctor of $(\mathrm{Div}^1)^d$ on which the S_d -action restricts to a free action. Then $V_d = U_d / S_d$ is naturally an open subfunctor of Div^d , and the map q restricts to a finite étale S_d -cover $q : U_d \rightarrow V_d$. Finally, let $W_d \subset Y_d$ be the preimage of V_d under the natural map $Y_d \rightarrow \mathrm{Div}^d$. Note that all of these functors are partially proper and absolutely good.

We now argue in a series of steps.

Step 1. *There is a natural isomorphism $\omega_{\mathrm{Div}^1} \cong \Lambda(1)[2]$.*

Proof. Note that X_1 identifies with the functor on Perf sending any $\mathrm{Spa}(R, R^+)$ to R° . In particular, it is an open subfunctor of the functor \mathbb{B} sending any $\mathrm{Spa}(R, R^+)$ to R^+ . By [Scholze 2017, Theorem 24.1], there is a canonical natural isomorphism $\omega_{\mathbb{B}} \cong \Lambda(1)[2]$, which then restricts to give a natural isomorphism $\omega_{X_1} \cong \Lambda(1)[2]$. Since Y_1 is open in X_1 , we also get a natural isomorphism $\omega_{Y_1} \cong \Lambda(1)[2]$. Let $Y_1 \rightarrow \mathrm{Div}^1$ be the natural \mathbb{Q}_p^\times -torsor, which can be factored canonically as

$$Y_1 \xrightarrow{\gamma} T \xrightarrow{\gamma'} \mathrm{Div}^1$$

where $T = Y_1 / (1 + p\mathbb{Z}_p)$; here we use the usual canonical decomposition $\mathbb{Q}_p^\times = p^\mathbb{Z} \times \mathbb{F}_p^\times \times (1 + p\mathbb{Z}_p)$. Note that γ (resp. γ') is a $1 + p\mathbb{Z}_p$ -torsor (resp. a $\mathbb{Z} \times \mathbb{F}_p^\times$ -torsor); in particular, γ is proper and pro-étale, while γ' is separated and étale. Applying Proposition 4.6(ii) (with the canonical choice of Haar measure giving $1 + p\mathbb{Z}_p$ volume one), the previous identification of ω_{Y_1} descends to an isomorphism $\omega_T \cong \Lambda(1)[2]$. Next, setting $G = \mathbb{Z} \times \mathbb{F}_p^\times$, we observe that since $R\gamma'^! = \gamma'^*$, there is a natural adjunction map $\gamma'_! \gamma'^* \rightarrow \mathrm{id}$ which induces a functorial isomorphism

$$(\gamma'_! \gamma'^* \mathcal{F})_G \cong \mathcal{F} \tag{*}$$

for any (shifted) étale sheaf of Λ -modules \mathcal{F} , where the subscript on the left-hand side denotes the coinvariants for the natural G -action (proof follows). Since

$$\gamma'^* \omega_{\mathrm{Div}^1} \cong \omega_T \cong \Lambda(1)[2] = \gamma'^* \Lambda(1)[2]$$

compatibly with the G -actions, this induces an isomorphism $\omega_{\mathrm{Div}^1} \cong \Lambda(1)[2]$, as desired. \square

Proof of ().* By the projection formula, $\gamma'_! \gamma'^* \mathcal{F} \cong \mathcal{F} \otimes \gamma'_! \Lambda$, so we immediately reduce to the case $\mathcal{F} = \Lambda$. Taking stalks at some geometric point $\bar{x} \rightarrow \text{Div}^1$, one then concludes by observing that $(\gamma'_! \Lambda)_{\bar{x}}$ identifies with the G -module of locally constant compactly supported functions $f : |T \times_{\text{Div}^1} \bar{x}| \rightarrow \Lambda$. Since $T \times_{\text{Div}^1} \bar{x}$ is just a countably infinite number of copies of \bar{x} permuted simply transitively by G , the coinvariants of $(\gamma'_! \Lambda)_{\bar{x}}$ identify with Λ via the map taking a compactly supported function on $|T \times_{\text{Div}^1} \bar{x}|$ to the sum of its values. Finally, one checks that the latter map coincides with the coinvariants of the adjunction map in question. \square

Step 2. *There is a natural isomorphism $\omega_{(\text{Div}^1)^d} \cong \Lambda(d)[2d]$.*

Proof. This follows from [Step 1](#) by repeated applications of [Proposition 4.3](#). \square

Step 3. *There is a natural isomorphism $\omega_{V_d} \cong \Lambda(d)[2d]$.*

Proof. Since U_d is open in $(\text{Div}^1)^d$, restricting the isomorphism exhibited in [Step 2](#) gives a natural isomorphism $\omega_{U_d} \cong \Lambda(d)[2d]$. Next, since $q : U_d \rightarrow V_d$ is a finite étale and Galois S_d -cover, the natural map

$$\mathcal{F} \rightarrow (q_* q^* \mathcal{F})^{S_d}$$

is an isomorphism for any (shifted) étale sheaf of Λ -modules \mathcal{F} on V_d . Since $q^* = Rq^!$, we get natural identifications

$$q^* \omega_{V_d} \cong \omega_{U_d} \cong \Lambda(d)[2d] \cong q^* \Lambda(d)[2d]$$

compatible with the S_d -actions, so applying $(q_* -)^{S_d}$ gives the desired result. \square

Step 4. *There is a natural isomorphism $\omega_{W_d} \cong \Lambda(d)[2d]$.*

Proof. Since the map $W_d \rightarrow V_d$ is a \mathbb{Q}_p^\times -torsor, we may factor it as $W_d \xrightarrow{\beta} W'_d \xrightarrow{\beta'} V_d$ analogously to the argument in [Step 1](#), where β is a $\mathbb{Z} \times \mathbb{F}_p^\times$ -torsor and β' is a $1+p\mathbb{Z}_p$ -torsor. Then

$$\omega_{W'_d} \cong R\beta'^! \omega_{V_d} \cong \beta'^* \omega_{V_d} \cong \beta'^* \Lambda(d)[2d] \cong \Lambda(d)[2d],$$

where the first isomorphism is immediate, the second follows from [Proposition 4.6](#), and the third follows from [Step 3](#).

To conclude, observe that we now have a chain of isomorphisms

$$\omega_{W_d} \cong R\beta^! \omega_{W'_d} \cong \beta^* \omega_{W'_d} \cong \beta^* \Lambda(d)[2d] \cong \Lambda(d)[2d],$$

where the first and fourth are trivial, the second follows from the identification $R\beta^! = \beta^*$, and the third follows from the previous paragraph. \square

Step 5. *There is a natural isomorphism $\omega_{X_d} \cong \Lambda(d)[2d]$.*

Proof. Let $j_d : W_d \rightarrow X_d$ denote the natural open embedding, with closed complement $i_d : Z_d \rightarrow X_d$. By [Proposition 4.7](#), ω_{X_d} is a shifted local system concentrated in degree $-2d$; in other words, $\delta(X_d) = 2d$. We also note that the structure map $Z_d \rightarrow \mathrm{Spd} \mathbb{F}_p$ has $\dim.\mathrm{trg}$ equal to $d - 1$; this is easy and left to the reader. Since

$$j_d^* \omega_{X_d} = \omega_{W_d} \cong \Lambda(d)[2d] \cong j_d^* \Lambda(d)[2d]$$

by [Step 4](#), the result now follows from [Lemma 4.9](#). \square

Lemma 4.9. *Let X be a small v -sheaf which is connected and partially proper, and assume that the structure map $\pi : X \rightarrow \mathrm{Spd} \mathbb{F}_p$ is smooth, so ω_X is concentrated in degree $-\delta(X)$ for some integer $\delta(X)$. Let $i : Z \rightarrow X$ be a closed subfunctor such that $2\dim.\mathrm{trg}(Z/\mathrm{Spd} \mathbb{F}_p) + 2 \leq \delta(X)$. Set $\Lambda = \mathbb{Z}/n\mathbb{Z}$ for some n prime to p . Then the restriction functor*

$$\mathrm{LocSys}_\Lambda(X) \rightarrow \mathrm{LocSys}_\Lambda(X \setminus Z)$$

is fully faithful.

Here, for any small v -stack X , $\mathrm{LocSys}_\Lambda(X)$ denotes the category of sheaves of Λ -modules \mathcal{F} on X_v such that $\mathcal{F}|_{\tilde{X}} \simeq \Lambda^m$ for some m after pullback along some v -cover $\tilde{X} \rightarrow X$. Note that any such sheaf is in fact trivial on some finite étale cover of X : after pullback along $\tilde{X} \rightarrow X$, the functor $\mathrm{Isom}(\mathcal{F}, \Lambda^m) \rightarrow X$ parametrizing trivializations of \mathcal{F} becomes a disjoint union of finitely many copies of \tilde{X} , so $\mathrm{Isom}(\mathcal{F}, \Lambda^m) \rightarrow X$ is finite étale surjective by [\[Scholze 2017, Proposition 10.11\(iii\)\]](#).

Proof. Let $j : X \setminus Z \rightarrow X$ be the evident open embedding. For any $\mathcal{L}, \mathcal{M} \in \mathrm{LocSys}_\Lambda(X)$, we have compatible isomorphisms

$$\mathrm{Hom}_{\mathrm{LocSys}_\Lambda(X)}(\mathcal{L}, \mathcal{M}) \cong H^0(R\Gamma(X, \mathcal{L}^\vee \otimes \mathcal{M}))$$

and

$$\begin{aligned} \mathrm{Hom}_{\mathrm{LocSys}_\Lambda(X \setminus Z)}(j^* \mathcal{L}, j^* \mathcal{M}) &\cong H^0(R\Gamma(X \setminus Z, j^*(\mathcal{L}^\vee \otimes \mathcal{M}))) \\ &\cong H^0(R\Gamma(X, Rj_* j^*(\mathcal{L}^\vee \otimes \mathcal{M}))) \\ &\cong H^0(R\Gamma(X, j_* j^*(\mathcal{L}^\vee \otimes \mathcal{M}))). \end{aligned}$$

In particular, it suffices to show that for any $\mathcal{F} \in \mathrm{LocSys}_\Lambda(X)$, the natural map $\mathcal{F} \rightarrow j_* j^* \mathcal{F}$ is an isomorphism. Set $S = \mathrm{Spd} \mathbb{F}_p((t))$ and $\tilde{S} = \mathrm{Spd} \overline{\mathbb{F}_p((t))}$, so $S \rightarrow \mathrm{Spd} \mathbb{F}_p$ is a smooth v -cover and $\tilde{S} \rightarrow S$ is a profinite-étale Galois cover for some profinite group G . Let $X_S, \pi_S, \mathcal{F}_S, j_S$, etc. denote the obvious base changes of various objects to S , and likewise for \tilde{S} . It suffices to check that $\mathcal{F} \rightarrow j_* j^* \mathcal{F}$ is an isomorphism after pulling back along the v -cover $X_S \rightarrow X$. By smooth base change, the pullback of this map identifies with the natural map $\mathcal{F}_S \rightarrow j_{S*} j_S^* \mathcal{F}_S$. Writing both sides of this map as the G -invariants of their pullbacks to $X_{\tilde{S}}$, it

suffices to prove that the natural map $\mathcal{F}_{\tilde{S}} \rightarrow j_{\tilde{S}*} j_{\tilde{S}}^* \mathcal{F}_{\tilde{S}}$ is an isomorphism.¹² This map fits into the long exact sequence of cohomology sheaves associated with the habitual distinguished triangle

$$i_{\tilde{S}*} Ri_{\tilde{S}}^! \mathcal{F}_{\tilde{S}} \rightarrow \mathcal{F}_{\tilde{S}} \rightarrow Rj_{\tilde{S}*} j_{\tilde{S}}^* \mathcal{F}_{\tilde{S}}.$$

It therefore suffices to show that $Ri_{\tilde{S}}^! \mathcal{G}$ is concentrated in degrees ≥ 2 for any $\mathcal{G} \in \text{LocSys}_{\Lambda}(X_{\tilde{S}})$. Passing to a finite étale cover $U \rightarrow X_{\tilde{S}}$ if necessary, we can assume that $\mathcal{G}|_U \simeq \Lambda^{\oplus m}$ and that $R\pi_{\tilde{S}}^! \Lambda|_U \simeq \Lambda[\delta(X)]$. In particular, writing $i' : Z' \rightarrow U$ for the pullback of the closed immersion $Z_{\tilde{S}} \rightarrow X_{\tilde{S}}$ along $U \rightarrow X_{\tilde{S}}$, there are (noncanonical) isomorphisms

$$Ri_{\tilde{S}}^! \mathcal{G}|_U \simeq Ri'^! \Lambda^{\oplus m} \simeq Ri'^! (R\pi_{\tilde{S}}^! \Lambda[-\delta(X)]|_U)^{\oplus m} \simeq Rg^! \Lambda[-\delta(X)]^{\oplus m},$$

where $g : Z' \rightarrow \tilde{S}$ is the evident map. In particular, it suffices to check that $Rg^! \Lambda$ is concentrated in degrees $[-2\dim.\text{trg}(Z/\text{Spd } \mathbb{F}_p), 0]$: conditional on this concentration, we get that

$$Ri_{\tilde{S}}^! \mathcal{G}|_U \simeq Rg^! \Lambda[-\delta(X)]^{\oplus m}$$

is concentrated in degrees $[\delta(X) - 2\dim.\text{trg}(Z/\text{Spd } \mathbb{F}_p), \delta(X)]$, and

$$\delta(X) - 2\dim.\text{trg}(Z/\text{Spd } \mathbb{F}_p) \geq 2$$

by assumption.

To analyze $Rg^! \Lambda$, note that the map $g : Z' \rightarrow \tilde{S}$ is a partially proper map of locally spatial diamonds, with target a rank one geometric point, and with

$$\dim.\text{trg}(Z'/\tilde{S}) = \dim.\text{trg}(Z_{\tilde{S}}/\tilde{S}) \leq \dim.\text{trg}(Z/\text{Spd } \mathbb{F}_p) < \infty.$$

By [Proposition 4.5](#), for any open spatial subdiamond $Z'' \subset Z'$ there is a natural equivalence $D_{\text{ét}}(Z'', \Lambda) \cong D(Z''_{\text{ét}}, \Lambda)$, and in particular, we may naturally regard $Rg^! \Lambda|_{Z''}$ as an object of $D(Z''_{\text{ét}}, \Lambda)$. Making this observation, it now suffices to show that the stalk of any cohomology sheaf $\mathcal{H}^i(Rg^! \Lambda|_{Z''})$ at any geometric point $\bar{z} \rightarrow Z'' \subset Z'$ vanishes for all $i \notin [-2\dim.\text{trg}(Z/\text{Spd } \mathbb{F}_p), 0]$. Any such stalk is naturally the colimit of $H^i(R\Gamma(W, u^* Rg^! \Lambda))$ as one runs over (the cofiltered system of) all diagrams $\bar{z} \rightarrow W \xrightarrow{u} Z'$ where u is a separated étale map from a spatial diamond over which $\bar{z} \rightarrow Z'$ factors. Moreover, for any such u , the cohomology group $H^i(R\Gamma(W, u^* Rg^! \Lambda))$ is given explicitly as the Λ -linear dual of $H^{-i}(R(g \circ u)_! \Lambda)$, by an easy application of Verdier duality. It's thus enough to show that

$$R(g \circ u)_! \Lambda \in D_{\text{ét}}(\tilde{S}, \Lambda) = D(\Lambda)$$

¹²I'm very grateful to Peter Scholze for suggesting this device.

is concentrated in degrees $[0, 2\dim.\mathrm{trg}(Z_{\bar{g}}/\tilde{S})]$ for any such u . This follows from the subsequent lemma, with W , Z' , and \tilde{S} playing (respectively) the roles of the objects denoted U , X , and S below. \square

Lemma 4.10. *Let S be a spatial diamond, and let $g : X \rightarrow S$ be any good map of locally spatial diamonds, with canonical compactification $\bar{g} : \bar{X}/S \rightarrow S$. Suppose that g is partially proper, or more generally that the canonical compactification \bar{X}/S is locally spatial. Let $u : U \rightarrow X$ be any separated étale map from a spatial diamond. Then $R(g \circ u)_!$ carries $D_{\text{ét}}^{[a,b]}(U, \Lambda)$ into $D_{\text{ét}}^{[a,b+2\dim.\mathrm{trg}g]}(S, \Lambda)$.*

Proof. By [Scholze 2017, Corollary 18.8(vii)], the map u extends to a quasi-pro-étale map $\bar{u} : \bar{U}/S \rightarrow \bar{X}/S$. Since \bar{X}/S is locally spatial by assumption, we deduce that \bar{U}/S is also locally spatial by [Scholze 2017, Corollary 11.28]. Moreover, $U \rightarrow S$ is quasicompact, so $\bar{U}/S \rightarrow S$ is proper by [Scholze 2017, Corollary 18.8(vi)]. Putting these observations together, we see that \bar{U}/S is in fact a spatial diamond.

By an easy inductive argument with truncation functors, it's enough to prove that for any étale sheaf \mathcal{F} of Λ -modules on U , the complex $R(g \circ u)_! \mathcal{F}$ lies in $D_{\text{ét}}^{[0, 2\dim.\mathrm{trg}g]}(S, \Lambda)$. Let $j : U \rightarrow \bar{U}/S$ be the open embedding of U into its canonical compactification, so then

$$R(g \circ u)_! \mathcal{F} \simeq R(\bar{g} \circ \bar{u})_* j_! \mathcal{F}$$

by definition. Since

$$\bar{g} \circ \bar{u} : \bar{U}/S \rightarrow S$$

is a proper map of spatial diamonds, Proposition 4.4 shows that the functor $R^i(\bar{g} \circ \bar{u})_*$ is nonzero only in degrees

$$i \leq 2\dim.\mathrm{trg}(\bar{U}/S/S) = 2\dim.\mathrm{trg}(U/S).$$

Since $\dim.\mathrm{trg}(U/S) \leq \dim.\mathrm{trg}(X/S) = \dim.\mathrm{trg} g$, the lemma follows. \square

Incidentally, the proof of Lemma 4.9 also yields the following useful result, which seems hard to prove by purely topological considerations.

Corollary 4.11. *Let X be any small v -sheaf with a smooth partially proper map $f : X \rightarrow \mathrm{Spd} k$, where k is a field which is either a complete nonarchimedean field with residue characteristic p , or a discrete extension of \mathbb{F}_p . Assume moreover that X is equidimensional in the sense that $Rf^! \Lambda \simeq \Lambda[\delta]$ v -locally on X , for some (constant) integer δ . Then for any closed subfunctor $Z \subset X$ such that $2\dim.\mathrm{trg}(Z/\mathrm{Spd} k) + 2 \leq \delta$, the natural map*

$$\pi_0(X \setminus Z) \rightarrow \pi_0(X)$$

is a bijection.

Proof. This is immediate from the bijectivity of

$$\begin{aligned} C^0(\pi_0(X), \Lambda) &\cong \mathrm{Hom}_{\mathrm{LocSys}_\Lambda(X)}(\Lambda, \Lambda) \\ &\rightarrow \mathrm{Hom}_{\mathrm{LocSys}_\Lambda(X \setminus Z)}(\Lambda, \Lambda) \cong C^0(\pi_0(X \setminus Z), \Lambda). \quad \square \end{aligned}$$

We now return to the notation and setting of Sections 1C–1E.

Proposition 4.12. *The functor $\mathcal{J}_{b, \check{\mathbb{Q}}_p}^U$ is a locally spatial diamond, whose structure map $\mathcal{J}_{b, \check{\mathbb{Q}}_p}^U \rightarrow \mathrm{Spd} \check{\mathbb{Q}}_p$ is partially proper and smooth. Moreover, for any complete algebraically closed extension $C/\check{\mathbb{Q}}_p$, we have*

$$R\Gamma_c(\mathcal{J}_{b, C}^U, \Lambda) \simeq \Lambda[-2d](-d),$$

where $d = \langle \mu, 2\rho \rangle - \langle \mu_M, 2\rho \rangle$.

Here ρ_M denotes the half-sum of the positive roots of T occurring in the adjoint action of T on $\mathrm{Lie}(M)$.

Proof. We first show that

$$d = \sum_{1 \leq i < j \leq k} (\deg \mathcal{E}_{b_i} - \deg \mathcal{E}_{b_j}).$$

Let ρ_U denote the half-sum of positive roots of T occurring in the adjoint action on $\mathrm{Lie}(U)$, so $\rho = \rho_M + \rho_U$. It's an easy exercise from the definition of $v_{[b^{-1}]}$ to check that $\sum_{1 \leq i < j \leq k} (\deg \mathcal{E}_{b_i} - \deg \mathcal{E}_{b_j})$ coincides with $\langle v_{[b^{-1}]}, 2\rho_U \rangle$. After some mild rearranging, one finds that the final claimed equality holds if and only if $\langle v_{[b^{-1}]} - \mu, \rho_U \rangle = 0$. But direct inspection shows that ρ_U defines an element of $X^*(M)$, while on the other hand we have $[b^{-1}] \in B(M, \mu)$ by assumption, which guarantees that

$$v_{[b^{-1}]} - \mu \in X_*(T \cap M^{\mathrm{der}})_{\mathbb{Q}},$$

so $\langle v_{[b^{-1}]} - \mu, \rho_U \rangle = 0$ as desired.

By the assumptions laid out in Section 1C together with Proposition 2.20, we have a natural isomorphism

$$\mathcal{J}_{b, \check{\mathbb{Q}}_p}^U \cong \mathcal{H}^0 \left(\bigoplus_{1 \leq i < j \leq k} \mathcal{E}_{b_j}^\vee \otimes \mathcal{E}_{b_i} \right)_{\check{\mathbb{Q}}_p}$$

of diamonds over $\mathrm{Spd} \check{\mathbb{Q}}_p$. The bundle $\bigoplus_{1 \leq i < j \leq k} \mathcal{E}_{b_j}^\vee \otimes \mathcal{E}_{b_i}$ has degree d , by the previous calculation. By assumption, the slopes of \mathcal{E}_{b_i} are strictly greater than the slopes of $\mathcal{E}_{b_{i+1}}$ for all $1 \leq i < k$, so the bundle $\bigoplus_{1 \leq i < j \leq k} \mathcal{E}_{b_j}^\vee \otimes \mathcal{E}_{b_i}$ has only positive slopes. Applying Propositions 4.7 and 4.8, the results follow. \square

Theorem 4.13. *There is a natural isomorphism*

$$H_c^*(\mathrm{Sht}_{P, \mu, b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} C, \mathbb{Z}/\ell^n) \cong H_c^{*-2d}(\mathrm{Sht}_{M, \mu, b} \times_{\mathrm{Spd} \check{E}} \mathrm{Spd} C, \mathbb{Z}/\ell^n)(-d)$$

compatible with all additional structures, where $d = \langle \mu, 2\rho \rangle - \langle \mu_M, 2\rho \rangle$ and C/\check{E} denotes any complete algebraically closed extension.

Proof. This follows immediately upon combining [Theorem 1.7](#), the Künneth formula, and [Proposition 4.12](#). \square

Acknowledgements

The debt this paper owes to the works of Mantovan and Scholze should be obvious to the reader. The first written account of these ideas took the form of a letter to Jared Weinstein, and it's a pleasure to thank Jared for his incisive feedback and for some helpful discussions. I'm also grateful to Kiran Kedlaya for some helpful conversations about the material in [[Kedlaya and Liu 2015](#); [2019](#)], to Johan de Jong for some amusing discussions around [Lemma 3.2](#), to Michael Harris for his constant support and encouraging remarks, and to the anonymous referee for some helpful comments and corrections. Finally, I'm very grateful to Peter Scholze for correcting a serious mistake in the early stages of this project, for some helpful comments on various drafts of this paper, and for some very useful discussions of the material in [[Scholze 2017](#)].

References

- [Berkovich 1993] V. G. Berkovich, “Étale cohomology for non-Archimedean analytic spaces”, *Inst. Hautes Études Sci. Publ. Math.* **78** (1993), 5–161. [MR](#) [Zbl](#)
- [Birkbeck et al. 2020] C. Birkbeck, T. Feng, D. Hansen, S. Hong, Q. Li, A. Wang, and L. Ye, “Extensions of vector bundles on the Fargues–Fontaine curve”, *J. Inst. Math. Jussieu* (online publication May 2020), 1–46.
- [Fargues 2020] L. Fargues, “Simple connexité des fibres d’une application d’Abel–Jacobi et corps de classes local”, *Ann. Sci. Éc. Norm. Supér.* (4) **53**:1 (2020), 89–124. [MR](#) [Zbl](#)
- [Fargues and Fontaine 2018] L. Fargues and J.-M. Fontaine, *Courbes et fibrés vectoriels en théorie de Hodge p-adique*, *Astérisque* **406**, 2018. [MR](#) [Zbl](#)
- [Gaisin and Imai 2016] I. Gaisin and N. Imai, “Non-semi-stable loci in Hecke stacks and Fargues’ conjecture”, preprint, 2016. [arXiv](#)
- [Harris 2001] M. Harris, “Local Langlands correspondences and vanishing cycles on Shimura varieties”, pp. 407–427 in *European Congress of Mathematics, I* (Barcelona, 2000), edited by C. Casacuberta et al., *Progr. Math.* **201**, Birkhäuser, 2001. [MR](#) [Zbl](#)
- [Harris and Taylor 2001] M. Harris and R. Taylor, *The geometry and cohomology of some simple Shimura varieties*, *Annals of Mathematics Studies* **151**, Princeton University Press, 2001. [MR](#) [Zbl](#)
- [Huber 1996] R. Huber, *Étale cohomology of rigid analytic varieties and adic spaces*, *Aspects of Mathematics* **30**, Friedr. Vieweg & Sohn, 1996. [MR](#) [Zbl](#)
- [Kedlaya and Liu 2015] K. S. Kedlaya and R. Liu, *Relative p-adic Hodge theory: foundations*, *Astérisque* **371**, 2015. [MR](#) [Zbl](#)
- [Kedlaya and Liu 2019] K. S. Kedlaya and R. Liu, “Relative p-adic Hodge theory, II: Imperfect period rings”, preprint, 2019. [arXiv](#)

- [Lang 2002] S. Lang, *Algebra*, 3rd ed., Graduate Texts in Mathematics **211**, Springer, 2002. [MR](#) [Zbl](#)
- [Mantovan 2008] E. Mantovan, “On non-basic Rapoport–Zink spaces”, *Ann. Sci. Éc. Norm. Supér.* (4) **41**:5 (2008), 671–716. [MR](#) [Zbl](#)
- [Mantovan and Viehmann 2010] E. Mantovan and E. Viehmann, “On the Hodge–Newton filtration for p -divisible \mathcal{O} -modules”, *Math. Z.* **266**:1 (2010), 193–205. [MR](#) [Zbl](#)
- [Rapoport and Viehmann 2014] M. Rapoport and E. Viehmann, “Towards a theory of local Shimura varieties”, *Münster J. Math.* **7**:1 (2014), 273–326. [MR](#) [Zbl](#)
- [Rapoport and Zink 1996] M. Rapoport and T. Zink, *Period spaces for p -divisible groups*, Annals of Mathematics Studies **141**, Princeton University Press, 1996. [MR](#) [Zbl](#)
- [Scholze 2017] P. Scholze, “Étale cohomology of diamonds”, preprint, 2017. [arXiv](#)
- [Scholze and Weinstein 2013] P. Scholze and J. Weinstein, “Moduli of p -divisible groups”, *Camb. J. Math.* **1**:2 (2013), 145–237. [MR](#) [Zbl](#)
- [Scholze and Weinstein 2020] P. Scholze and J. Weinstein, *Berkeley Lectures on p -adic Geometry*, Princeton University Press, 2020.
- [Stacks 2005–] “The Stacks project”, electronic reference, 2005–, <http://stacks.math.columbia.edu>.
- [Weinstein 2017] J. Weinstein, “ $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ as a geometric fundamental group”, *Int. Math. Res. Not.* **2017**:10 (2017), 2964–2997. [MR](#) [Zbl](#)

Received 13 Jul 2020.

DAVID HANSEN:

dhansen@mpim-bonn.mpg.de

Max Planck Institute for Mathematics, Bonn, Germany