1 The remaining axioms of set theory and the power set construction

Week 2: will miss one class due to Labor day.
Reading: [2, Ch.3.1-4], [1, 2].

Learning Objectives

In last lectures, we

- Defined $\mathbb{N}$ axiomatically using the Peano axioms.
- Used induction to prove properties of operations as $+$ and $\times$ on $\mathbb{N}$.

In the next two lectures

- Discuss the remaining axioms of set theory. We begin by discussing new notions: *subsets*, 1.1, We end with the construction of the power set.
- Discuss *equivalence relation*, **??**, and *ordered pairs*, **??**. which constructs the integers and the rationals

1.1 Subcollections

Definition 1.1. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1,2\} \subset \{1,2,3\}$.

1.2 Remaining axioms of set theory

Week 2
In this section we continue from previous lecture and discuss remaining axioms from what is known as the *Zermelo-Fraenkel (ZF) axioms of set theory*, due to Ernest Zermelo and Abraham Fraenkel.

Axiom 1.2. Singleton set axiom. If a is an object. There is a set $\{a\}$ consists of just one element.

Axiom 1.3. Axiom of pairwise union. Given any two sets A, B there exists a set $A \cup B$ whose elements which belong to either A or B or both.

Often we would require a stronger version.

Axiom 1.4. Axiom of union. Let A be a set of sets. Then there exists a set

$$\bigcup A$$

whose objects are precisely the elements of the set.

Example

Let

- $A = \{\{1, 2\}, \{1\}\}$
- $A = \{\{1, 2, 3\}, \{9\}\}$

Discussion

Using the axioms, can we get from $\{1, 3, 4\}$ to $\{2, 4, 5\}$?

We will now state the power set axiom for completeness but revisit again.

Axiom 1.5. Axiom of power set. Let X, Y be sets. Then there exists a set Y^X consists of all functions $f : X \to Y$.

We will review definition of function later, 1.7.

Axiom 1.6. Axiom of replacement. For all $x \in A$, and y any object, suppose there is a statement $P(x, y)$ pertaining to x and y. There is a set

$$\{y : P(x, y) \text{ is true for some } x \in A\}$$

This can intuitively be thought of as the set

$$\{y : y = f(x) \text{ some } x \in A\}$$

1.3 Function

Discussion

How would you intuitively define a function?

Definition 1.7. Let X, Y be two sets. Let

$$P(x, y)$$

be a *property* pertaining to $x \in X$ and $y \in Y$, such that for all $x \in X$, there *exists a unique* $y \in Y$ such that $P(x, y)$ is true. A *function associated to* P is an object

$$f_P : X \to Y$$

such that for each $x \in X$ assigns an output $f(x) \in Y$, to be the unique object such that $P(x, f(x))$ is true.

- X is called the *domain*
- Y is called the *codomain*.

Definition 1.8. The *image*

Discussion

When is what kind of properties P does not satisfy the condition of being function?

- "$y^2 = x$".
- "$y = x^2$".

Homework for week 2

Due: Week 3, Wednesday. All questions in 1.4, Boolean algebra is compulsory. Select 3 other questions to be graded.

Reading: We refer to the axioms of set theory we have discussed thus far collectively as the ZF axioms. *The only axiom we did not discuss is the axiom of replacement,* [2, 3.5]. *This will be left as required reading for certain problems.*

Problems

1. Let A, B, C be sets.
   (a) Prove set inclusion, def. 1.1, is reflexive and transitive. $A \subseteq B, B \subseteq C$ then $A \subseteq C$.
   (b) Prove that the union operation $\cup$ on sets 1.3, is associative and commutative:
   $$A \cup (B \cup C) = (A \cup B) \cup C$$
   $$A \cup B = B \cup A$$
2. (**) Let I be a set and that for all $\alpha \in I$, I have a set A_α. Read about the axiom of replacement, see [2, Axiom 3.5] or 1.6.
   - Prove that under the ZF axioms, one can form the union of the collection:
   $$\bigcup_{\alpha \in I} A_\alpha = \bigcup \{A_\alpha : \alpha \in I\}$$
   - Give a one line explanation briefly describing why axiom of union 1.4 is not sufficient.
3. (*) Let A, B, C, D be sets. This exercise shows that we can actually construct *ordered pairs* using the ZF axioms. Another definition is discussed in the notes, **??**. Prove
   - We can construct the following set
   $$\langle A, B \rangle := \{A, \{A, B\}\}$$
   from the axioms of set theory.
   - Prove $\langle A, B \rangle = \langle C, D \rangle$ if and only if $A = B, C = D$.

4. (***) Show that the collection

$$\{Y : Y \text{ is a subset } X\}$$

is a set using the ZF axioms. You will need to use the axiom of replacement.

1.4 Boolean algebras

This section is compulsory. Boolean algebras form the foundation of probability theory.

Definition 1.9. Let Ω be a set. A *Boolean algebra* in Ω is a set $\mathcal{E}$ of subsets of Ω (equivalently, $\mathcal{E} \subseteq 2^{\Omega}$) satisfying

1. $\emptyset \in \mathcal{E}$
2. closed under unions and intersections.

$$E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$$

$$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$$

3. closed under complements.

A *σ-algebra in* Ω is a Boolean algebra in Ω such that it satisfies

4. Countable[1] closure. If $A_i \in \mathcal{E}$ for $i \in \mathbb{N}$, then $\bigcup A_i \in \mathcal{E}$.

Problems

1. Prove that $\mathcal{E} := \{\emptyset, \Omega\}$ is a σ-algebra.
2. Prove that $2^{\Omega} := \{E : E \subset \Omega\}$ is a σ-algebra.
3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Hints for problems

3. There are 3 cases. What happens $A = \emptyset$ or $A = \Omega$? Now consider the case $A \neq \emptyset$ and $A \neq \Omega$.

[1] A set X is countable if it is in bijection with $\mathbb{N}$. We will explore this word in further detail in the future.

References

[1] Jonathan Pila, *B1.2 set theory*.

[2] Terence Tao, *Analysis I, 4th edition*, 2022.