CHROMATIC ABERRATIONS OF GEOMETRIC SATAKE I: THE REGULAR LOCUS

S. K. DEVALAPURKAR

It is an established practice to take old theorems about ordinary homology, and generalise them so as to obtain theorems about generalised homology theories.

J. F. Adams, [Ada69]

Abstract. Let G be a connected and simply-connected semisimple group over $\mathbf{C}$, and let T be a maximal torus. The derived geometric Satake equivalence of Bezrukavnikov-Finkelberg gives an $\mathbf{E}_3$-monoidal equivalence between the ∞-category $\mathrm{DMod}_{\mathfrak{L}^+G}(\mathrm{Gr}_G)$ and $\mathrm{QCoh}(\check{\mathfrak{g}}[2]/\check{G})$, where $\check{G}$ is the Langlands dual group over $\mathbf{C}$. This localizes to an equivalence between a full subcategory of $\mathrm{Mod}_{\mathrm{H}^*_{\mathfrak{L}^+G}(\mathrm{Gr}_G;\mathbf{C})}$ and $\mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}[2]/\check{G})$, which can be thought of as a version of the geometric Satake equivalence "over the regular locus". In this article, we study the story when $\mathrm{Mod}_{\mathrm{H}^*_{\mathfrak{L}^+G}(\mathrm{Gr}_G;\mathbf{C})}$ is replaced by the ∞-category of modules over the cohomology $C^*_T(\mathrm{Gr}_G(\mathbf{C});A)$, where A is a complex-oriented even-periodic $\mathbf{E}_\infty$-ring equipped with an oriented group scheme $\mathbf{G}$. We show that upon rationalization, $\mathrm{Mod}_{C^*_T(\mathrm{Gr}_G(\mathbf{C});A)}$ can be described in terms of the spectral geometry of various Langlands-dual stacks associated to A and $\mathbf{G}$. For example, we show that a full subcategory of $\mathbf{Q}\otimes\mathrm{Mod}_{C^*_{\check{G}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})}$ is equivalent to the 2-periodification of the ∞-category of quasicoherent sheaves on $\check{G}^{\mathrm{reg}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}$.

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1. Introduction

Let G be a simply-connected semisimple algebraic group or a torus over $\mathbf{C}$. Many deep results in geometric representation theory are concerned with describing the "topological"/A-side category of D-modules on algebraic (ind-)schemes associated to G (such as the flag variety, the nilpotent cone, the affine Grassmannian, the affine flag variety, etc.) in terms of representation-theoretic/algebro-geometric B-side data associated to $\check{G}$, the Langlands dual. By the Riemann-Hilbert equivalence, the A-side category of D-modules on X may be interpreted instead as categories of constructible sheaves of $\mathbf{C}$-vector spaces on $X(\mathbf{C})$. The goal of this manuscript is to study analogues of some of these equivalences when we instead consider the category of constructible sheaves of A-module spectra on $X(\mathbf{C})$, where A is a complex-oriented even-periodic $\mathbf{E}_\infty$-ring (such as topological K-theory KU, or an elliptic cohomology theory).

1.1. Summary of content.

In this article, we take first technical steps into a chromatic homotopy-theoretic analogue of the derived geometric Satake equivalence[1]. Let B be a Borel subgroup of G. Let $\mathcal{K}$ denote $\mathbf{C}(\!(t)\!)$, and let $\mathcal{O}$ denote $\mathbf{C}[\![t]\!]$. The affine Grassmannian Gr_G is defined as the sheafification of the functor of points $\mathrm{CAlg}_{\mathbf{C}} \ni R \mapsto G(R \otimes_{\mathbf{C}} \mathcal{K})/G(R \otimes_{\mathbf{C}} \mathcal{O})$, and it has the property that $\mathrm{Gr}_G(\mathbf{C})$ is homotopy equivalent to $\Omega G_c \simeq \Omega^2 BG_c$, where G_c is a maximal compact subgroup of $G(\mathbf{C})$. (Note that G_c is homotopy equivalent to $G(\mathbf{C})$, so for most of this article, the distinction between them will be irrelevant.) The classical geometric Satake equivalence says:

Theorem 1.1.1 (Classical geometric Satake, [MV07])**.** *The abelian category* $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G; \mathbf{Q})$ *of* $G(\mathcal{O})$*-equivariant perverse sheaves on* Gr_G *is equivalent to* $\mathrm{Rep}(\check{G}_{\mathbf{Q}})$*, where* $\check{G}_{\mathbf{Q}}$ *is the Langlands dual group over* $\mathbf{Q}$*.*

In [BF08], Bezrukavnikov-Finkelberg proved a *derived* analogue of the geometric Satake equivalence:

Theorem 1.1.2 (Derived geometric Satake, [BF08])**.** *There is an equivalence* $\mathrm{DMod}_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}_{\mathbf{C}}[2]/\check{G}_{\mathbf{C}})$ *of* $\mathbf{C}$*-linear* ∞*-categories, where* $\check{\mathfrak{g}}_{\mathbf{C}}[2]$ *is the derived* $\mathbf{C}$*-scheme* $\mathrm{Spec}\,\mathrm{Sym}_{\mathbf{C}}(\check{\mathfrak{g}}^*_{\mathbf{C}}[-2])$*.*

Remark 1.1.3. There is a Betti analogue of the Bezrukavnikov-Finkelberg equivalence, where $\mathrm{DMod}_{G(\mathcal{O})}(\mathrm{Gr}_G)$ is replaced by $\mathrm{Shv}_{G(\mathcal{O})}(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C}) \simeq \mathrm{Shv}_{G_c}(\Omega G_c; \mathbf{C})$.

Our goal in this article (partly inspired by Adams' quote above, and partly inspired by the discussion in [Tel14] and Appendix A) is to begin exploring the analogous story when $\mathbf{C}$ is replaced by a generalized cohomology theory. Specifically, we will replace $\mathbf{C}$ with an even-periodic $\mathbf{E}_\infty$-ring equipped with specific additional data.

Remark 1.1.4. Part of the reason the derived contributions are vital to generalizing the geometric Satake equivalence is that when one considers sheaves with coefficients in a 2-periodic $\mathbf{E}_\infty$-ring (or any $\mathbf{E}_\infty$-ring with nonzero homotopy in positive degrees), contributions from higher cohomology are circulated to degree 0. For instance, the result of Bezrukavnikov-Finkelberg implies that $\mathrm{Shv}_{G(\mathcal{O})}(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}_{\mathbf{C}}[2]/\check{G}_{\mathbf{C}}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$ where $|\beta| = 2$; but this is in turn equivalent to $\mathrm{QCoh}(\check{\mathfrak{g}}_{\mathbf{C}}/\check{G}_{\mathbf{C}}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$, which is *not* the 2-periodification of $\mathrm{Rep}(\check{G}_{\mathbf{C}})$.

Our goal will be to study a result of Arkhipov-Bezrukavnikov-Ginzburg (ABG) from [ABG04], which is closely related to the geometric Satake equivalence. Namely, let $I = G(\mathcal{O}) \times_G B$ denote the Iwahori subgroup of $G(\mathcal{O})$. Then:

Theorem 1.1.5 (Arkhipov-Bezrukavnikov-Ginzburg)**.** *There is an equivalence* $\mathrm{DMod}_I(\mathrm{Gr}_G) \simeq \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})$*, where* $\widetilde{\check{\mathcal{N}}}$ *is the Springer resolution. This is in turn equivalent to* $\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}_{\mathbf{C}}[2]/\check{G}_{\mathbf{C}})$ *by Koszul duality, where* $\widetilde{\check{\mathfrak{g}}}_{\mathbf{C}}[2] = \check{G} \times^{\check{B}} \check{\mathfrak{b}}[2]$ *is a shifted analogue of the Grothendieck-Springer resolution.*

Remark 1.1.6. Again, there is a Betti analogue of the ABG equivalence, where $\mathrm{DMod}_I(\mathrm{Gr}_G)$ is replaced by $\mathrm{Shv}_I(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C}) \simeq \mathrm{Shv}_{T_c}(\Omega G_c; \mathbf{C})$. Upon 2-periodification, we therefore see that $\mathrm{Shv}_I(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}_{\mathbf{C}}/\check{G}_{\mathbf{C}}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$.

[1] We warn the reader that up to Section 3, all statements rely on results which are present in the literature. In Section 2.1, we review some constructions from [Lur09a]; some, but not all, of this theory has been written down in [Lur18a, Lur18b, Lur19].

Taking global sections defines a functor $\mathrm{Shv}_I(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C}[\beta^{\pm 1}]) \to \mathrm{Mod}_{\mathrm{H}^*_I(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})} \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$. A first step towards reproving the main result of [ABG04] using techniques similar to those of [BF08] is to describe $\mathrm{Mod}_{\mathrm{H}^*_I(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})}$ in Langlands dual terms. Note that pullback along the inclusion of a point into $\mathrm{Gr}_G(\mathbf{C})$ defines a symmetric monoidal functor $\mathrm{Shv}_I(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C}[\beta^{\pm 1}]) \to \mathrm{Shv}_I(*; \mathbf{C}[\beta^{\pm 1}])$, and there is an equivalence $\mathrm{Loc}_{T_c}(G_c; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{End}_{\mathrm{Shv}_{T_c}(\Omega G_c;\mathbf{C}[\beta^{\pm 1}])}(\mathrm{Shv}_{T_c}(*; \mathbf{C}[\beta^{\pm 1}]))$. Using the ABG theorem, one can prove an equivalence

$$\mathrm{Loc}_{T_c}(G_c; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{t}} \times_{\tilde{\check{\mathfrak{g}}}/\check{G}} \check{\mathfrak{t}}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}], \tag{1}$$

where the map $\check{\mathfrak{t}} \to \tilde{\check{\mathfrak{g}}}/\check{G}$ is given by the Kostant slice.

Our first result in this article is a generalization of (1). Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A, and let $\mathbf{G}$ be an oriented group scheme in the sense of [Lur18b]. If T is a torus and X is a (finite) T-space, one can define an A-linear ∞-category $\mathrm{Loc}_T(X; A)$ of "genuine T-equivariant Mod_A-valued local systems on X"; see Section 2.2.

Let $\mathcal{M}_T$ denote the Hom-stack $\mathrm{Hom}(\mathbb{X}^\bullet(T), \mathbf{G})$. Let $\mathfrak{D}(\mathbf{G})$ denote the group scheme $\mathrm{Hom}(\mathbf{G}, B\mathbf{G}_m)$ (this is a slight variant of the construction studied in [Mou21]). If X is a scheme, let $\mathcal{L}_{\mathbf{G}}X$ denote the "$\mathbf{G}$-loop space" $\mathrm{Map}(\mathfrak{D}(\mathbf{G}), X)$, so that if $X = BG$, then $\mathcal{L}_{\mathbf{G}}BG = \mathrm{Bun}_G(\mathfrak{D}(\mathbf{G}))$. Then, one of our main results states (see Section 4.3 for a proof):

Theorem 1.1.7. *Suppose that G is a connected and simply-connected semisimple algebraic group or a torus over $\mathbf{C}$, and let T act on G by conjugation. Let G_c denote the maximal compact subgroup of $G(\mathbf{C})$, and fix a principal nilpotent element of $\check{\mathfrak{n}}$. Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A, and let $\mathbf{G}$ be an oriented group scheme in the sense of* [Lur18b] *which is dualizable (in the sense that $\mathbf{G} \xrightarrow{\sim} \mathfrak{D}(\mathfrak{D}(\mathbf{G}))$) and licit in the sense of Definition 4.3.7.*

Then there is a "Kostant slice" $\kappa : \mathcal{M}_{\check{T}} \hookrightarrow \mathcal{L}_{\mathbf{G}}B\check{B}$, and if $\mathcal{L}^0_{\mathbf{G}}B\check{B}$ denotes the connected component of $\mathcal{L}_{\mathbf{G}}B\check{B}$ containing the Kostant slice, there is an equivalence of $A_{\mathbf{Q}}$-linear ∞-categories:

$$\mathrm{Loc}_{T_c}(G_c; A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}((\mathcal{M}_{\check{T}})_{\mathbf{Q}} \times_{\mathcal{L}^0_{\mathbf{G}}B\check{B}_{A_{\mathbf{Q}}}} (\mathcal{M}_{\check{T}})_{\mathbf{Q}}).$$

Remark 1.1.8. Since the Betti analogue of the Bezrukavnikov-Finkelberg equivalence is concerned with $\mathrm{Shv}_{G_c}(\Omega G_c; \mathbf{C})$, one can view Theorem 1.1.7 as a "cobar construction" of an analogue of the geometric Satake equivalence. Moreover, motivated by [GPS18, Theorem 1.1], one can heuristically interpret Theorem 1.1.7 as describing a version of mirror symmetry for the wrapped Fukaya category of the symplectic orbifold $T^*(G_c/_{\mathrm{ad}}T_c)$, albeit with coefficients in the complex-oriented even-periodic $\mathbf{E}_\infty$-ring A.

Remark 1.1.9. The reason that the left-hand side of Theorem 1.1.7 is not merely $\mathrm{Loc}_{T_c}(G_c; \mathbf{Q}) \otimes_{\mathbf{Q}} A_{\mathbf{Q}}$ (which could then be described by (1)) is that the rationalization of equivariant A-(co)homology is essentially never isomorphic to equivariant $A \otimes \mathbf{Q}$-(co)homology. This is the key driving force behind Theorem 1.1.7. For example, if X is a finite CW-complex equipped with an action of a group H, then $\mathrm{KU}^*(X) \otimes \mathbf{Q} \cong \mathrm{H}^*(X; \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}]$, but $\mathrm{KU}^*_H(X) \otimes \mathbf{Q}$ is generally not isomorphic to $\mathrm{H}^*_H(X; \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}]$. Indeed, they already differ if X is a point and H is a finite group: in this case, $\mathrm{KU}^*_H(X) \otimes \mathbf{Q}$ is the rationalization of the representation ring of H, while $\mathrm{H}^*_H(X; \mathbf{Q}) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}] \cong \mathbf{Q}[\beta^{\pm 1}]$.

Theorem 1.1.7 implies the following instantiation of Langlands duality:

Theorem (See Theorem 4.4.7 for a precise statement)**.** *Under the above assumptions, there is an equivalence of $A_{\mathbf{Q}}$-linear ∞-categories between (a slight variant of)* $\mathrm{Coh}_{C^*_T(\mathrm{Gr}_G(\mathbf{C});A)} \otimes \mathbf{Q}$ *and an explicit full subcategory of* $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}}B\check{B}_{A_{\mathbf{Q}}})$ *related to a $\mathbf{G}$-analogue of the Kostant slice.*

We view the above result as a first step towards describing $\mathrm{Shv}_{T_c}(\Omega G_c; A) \otimes \mathbf{Q}$ in a manner analogous to [ABG04]. We will complete this description in a sequel to this article (Part II), and further use the above result to revisit (the 2-periodification of) the ABG equivalence in future work.

For instance:

(a) When $A = \mathbf{Q}[\beta^{\pm 1}]$, Theorem 4.4.7 describes an equivalence between a full subcategory of the 2-periodification of $\mathrm{Coh}_{C^*_T(\mathrm{Gr}_G(\mathbf{C});\mathbf{Q})}$ and the 2-periodification of $\mathrm{QCoh}(\tilde{\check{\mathfrak{g}}}^{\mathrm{reg}}/\check{G})$. This is a rather formal consequence of the following observation proved in Proposition 4.1.7:

Observation 1.1.10. There is a "Kostant section" $\kappa : \check{\mathfrak{t}} \to \widetilde{\check{\mathfrak{g}}}/\check{G}$ and a Cartesian square

$$\begin{tikzcd} (T^*\check{T})^{\mathrm{bl}} \arrow[r] \arrow[d] & \mathfrak{t} \cong \check{\mathfrak{t}} \arrow[d, "\kappa"] \\ \check{\mathfrak{t}} \arrow[r, "\kappa"'] & \widetilde{\check{\mathfrak{g}}}/\check{G}, \end{tikzcd}$$

where $(T^*\check{T})^{\mathrm{bl}}$ is a particular affine blowup of $T^*\check{T} \cong \check{T} \times \mathfrak{t}$.

This can be viewed as an analogue of [Ngo10, Proposition 2.2.1] and [BFM05, Proposition 2.8], and it can be used to reprove [YZ11, Theorem 6.1]. There is an isomorphism $\widetilde{\check{\mathfrak{g}}}/\check{G} \cong \check{\mathfrak{b}}/\check{B}$, and in characteristic zero, this can be identified with $\mathrm{Bun}^0_{\check{B}}(B\mathbf{G}_a)$, viewed as a closed substack of the shifted tangent bundle of $B\check{B}$. Moreover, there is an isomorphism $\mathrm{Spec}\,\mathrm{H}^T_*(\mathrm{Gr}_G(\mathbf{C}); \mathbf{Q}) \cong (T^*\check{T})^{\mathrm{bl}}$, and $(T^*\check{T})^{\mathrm{bl}}$ admits a W-action (via the W-action on $\check{T}$ and $T^*_{\{1\}}\check{T} \cong \mathfrak{t}$) such that $(T^*\check{T})^{\mathrm{bl}}/\!\!/W$ is isomorphic to the group scheme of regular centralizers in $\check{\mathfrak{g}}$.

In this case, Theorem 1.1.7 says that if T acts on G by conjugation, then there is an equivalence

$$\mathrm{Loc}_{T_c}(G_c; \mathbf{Q}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{t}}_{\mathbf{Q}} \times_{\widetilde{\check{\mathfrak{g}}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}_{\mathbf{Q}}) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}].$$

Similarly, if G acts on itself by conjugation, one obtains an equivalence

$$\mathrm{Loc}_{G_c}(G_c; \mathbf{Q}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{t}}_{\mathbf{Q}}/\!\!/W \times_{\check{\mathfrak{g}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}_{\mathbf{Q}}/\!\!/W) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}].$$

These equivalences can be de-periodified (Remark 4.3.14). Motivated by [GPS18, Theorem 1.1], these equivalences suggest viewing $\check{\mathfrak{t}} \times_{\widetilde{\check{\mathfrak{g}}}/\check{G}} \check{\mathfrak{t}}$ (resp. $\check{\mathfrak{t}}/\!\!/W \times_{\check{\mathfrak{g}}/\check{G}} \check{\mathfrak{t}}/\!\!/W$) as a (derived) mirror to the symplectic orbifold $T^*(G_c/_{\mathrm{ad}}T_c)$ (resp. $T^*(G_c/_{\mathrm{ad}}G_c)$). Concretely, these results show that if f is a regular nilpotent element of $\check{\mathfrak{g}}$ and $Z_f(\check{B})$ is its centralizer in $\check{B}$, then $Z_f(\check{B})$ is a mirror to $G(\mathbf{C}) = T^*(G_c)$ viewed as a symplectic manifold. These results are not new, and can easily be deduced from the work of Bezrukavnikov-Finkelberg [BF08] and Yun-Zhu [YZ11].

Remark 1.1.11. Upon adding loop rotation equivariance, there is an equivalence between a full subcategory of $\mathrm{LMod}_{C^*_{T\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})}$ and a particular localization of the universal category $\check{\mathcal{O}}^{\mathrm{univ}} = U_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{N},(\check{T},w)}$ from [KS20, Section 2.4]; see Theorem 4.1.11 for a more precise statement.

See Example A.3 for an explicit description of $C^{G\times S^1_{\mathrm{rot}}}_*(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ when $G = \mathrm{SL}_2$. We remark that from the homotopical perspective, the action of S^1 by loop rotation on $\mathrm{Gr}_G(\mathbf{C})$ arises by viewing $\mathrm{Gr}_G(\mathbf{C}) \simeq \Omega^\lambda BG(\mathbf{C})$, where λ is the 2-dimensional rotation representation of S^1; in other words, $\mathrm{Gr}_G(\mathbf{C})$ admits the structure of a framed $\mathbf{E}_2$-algebra, and the action of S^1 is via change-of-framing.

(b) When $A = \mathrm{KU}$, Theorem 4.4.7 describes an equivalence between a full subcategory of $\mathrm{Coh}_{C^*_T(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})} \otimes \mathbf{Q}$ and the 2-periodification of $\mathrm{QCoh}(\widetilde{\check{G}}^{\mathrm{reg}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}})$, where $\widetilde{\check{G}}^{\mathrm{reg}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}$ is the regular locus in the stacky quotient of the multiplicative Grothendieck-Springer resolution $\widetilde{\check{G}}_{\mathbf{Q}} = \check{G}_{\mathbf{Q}} \times_{\check{B}_{\mathbf{Q}}} \check{B}_{\mathbf{Q}}$. As above, this is a rather formal consequence of the following observation, which is a *multiplicative* analogue of [Ngo10, Proposition 2.2.1] and [BFM05, Proposition 2.8]:

Observation 1.1.12. There is a "Kostant section" $\kappa : \check{T} \to \widetilde{\check{G}}/\check{G}$ and a Cartesian square

$$\begin{tikzcd} (\check{T} \times T)^{\mathrm{bl}} \arrow[r] \arrow[d] & \check{T} \arrow[d, "\kappa"] \\ \check{T} \arrow[r, "\kappa"'] & \widetilde{\check{G}}/\check{G} \simeq \mathrm{Bun}^0_{\check{B}}(S^1), \end{tikzcd}$$

where $(\check{T} \times T)^{\mathrm{bl}}$ is a particular affine blowup of $\check{T} \times T$. Moreover, there is an isomorphism $\mathrm{Spec}\, C^T_*(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q}$ and a 2-periodification of $(\check{T} \times T)^{\mathrm{bl}}$.

There is also a W-action on $(\check{T}\times T)^{\mathrm{bl}}$ (by the W-action on T and $\check{T}$) such that $(\check{T}\times T)^{\mathrm{bl}}/\!\!/W$ is isomorphic to the group scheme of regular centralizers in $\check{G}$.

In this case, Theorem 1.1.7 says that if T acts on G by conjugation, then there is an equivalence

$$\mathrm{Loc}_{T_c}(G_c;\mathrm{KU})\otimes\mathbf{Q}\simeq\mathrm{QCoh}(\check{T}_{\mathbf{Q}}\times_{\tilde{\check{G}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}}\check{T}_{\mathbf{Q}})\otimes_{\mathbf{Q}}\mathbf{Q}[\beta^{\pm1}].$$

Similarly, if G acts on itself by conjugation, one obtains an equivalence

$$\mathrm{Loc}_{G_c}(G_c;\mathrm{KU})\otimes\mathbf{Q}\simeq\mathrm{QCoh}(\check{T}_{\mathbf{Q}}/\!\!/W\times_{\check{G}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}}\check{T}_{\mathbf{Q}}/\!\!/W)\otimes_{\mathbf{Q}}\mathbf{Q}[\beta^{\pm1}].$$

If $\{f\}$ is a regular unipotent element of $\check{G}_{\mathbf{Q}}$ (determined by the image of the origin in $\check{T}_{\mathbf{Q}}/\!\!/W$ under the multiplicative Kostant slice), then the preceding equivalence in turn implies an equivalence

$$\mathrm{Loc}(G_c;\mathrm{KU})\otimes\mathbf{Q}\simeq\mathrm{QCoh}(\{f\}\times_{\check{G}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}}\check{T}_{\mathbf{Q}}/\!\!/W)\otimes_{\mathbf{Q}}\mathbf{Q}[\beta^{\pm1}].$$

However, $\{f\}\times_{\check{G}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}}\check{T}_{\mathbf{Q}}/\!\!/W\cong\{f\}\times_{\check{B}_{\mathbf{Q}}/\check{B}_{\mathbf{Q}}}\check{T}_{\mathbf{Q}}$ is precisely the centralizer $Z^{\mu}_f(\check{B}_{\mathbf{Q}})$ of $f\in\check{G}_{\mathbf{Q}}$. Therefore, $Z^{\mu}_f(\check{B}_{\mathbf{Q}})$ can be viewed as a KU-theoretic mirror to $G(\mathbf{C})=T^*(G_c)$ viewed as a symplectic manifold. The main input into these results are not new, and can be deduced from the work of Bezrukavnikov-Finkelberg-Mirkovic [BFM05].

Remark 1.1.13. We expect (see Conjecture 4.2.9 for a more precise statement) that upon adding loop rotation equivariance, there is an equivalence between a full subcategory of $\mathrm{LMod}_{C^*_{T\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})}\otimes\mathbf{Q}$ and a 2-periodification of a particular localization of the quantum universal category $\check{\mathcal{O}}^{\mathrm{univ}}_q$ from [KS20, Section 2.4]. Using the calculations in this article, we reduce this expected equivalence to proving an analogue of [Gin18, Theorem 8.1.2] for the quantum group and the multiplicative nil-Hecke algebra.

We also expect (see Conjecture 4.2.12) that there is an equivalence between a full subcategory of $\mathrm{LMod}_{C^*_{T\times\mu_{p,\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})[\frac{1}{q-1}]}$ and a 2-periodification of a particular localization of $\check{\mathcal{O}}^{\mathrm{univ}}_{\zeta_p}$, i.e., the quantum universal category $\check{\mathcal{O}}$ at a primitive pth root of unity.

The reader is referred to Example A.5 for an explicit description of $C^{G\times S^1_{\mathrm{rot}}}_*(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})\otimes\mathbf{Q}$ when $G=\mathrm{SL}_2$.

(c) Suppose A is a complex-oriented even-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}$ is an oriented elliptic curve over A (in the sense of [Lur18b]). Let E be the underlying classical scheme of $\mathbf{G}$ over the classical ring $\pi_0(A)$, so that E is an elliptic curve, and let $E^\vee$ be the dual elliptic curve. The Cartesian squares from (a) and (b) above can be generalized to this setting (see Theorem 4.3.10). For simplicity, let us explain this in the case $G=\mathrm{SL}_2$, i.e., $\check{G}=\mathrm{PGL}_2$.

Observation 1.1.14. Then, there is a "Kostant section" $\kappa: E=\mathrm{Pic}^0(E^\vee)\to\mathrm{Bun}^0_{\check{B}}(E^\vee)$ which sends a line bundle $\mathcal{L}$ to the trivial extension $\mathcal{O}_{E^\vee}\subseteq\mathcal{O}_{E^\vee}\oplus\mathcal{L}$ if $\mathcal{L}\not\cong\mathcal{O}_{E^\vee}$, and to the Atiyah extension $\mathcal{O}_{E^\vee}\subseteq\mathcal{F}_2$ if $\mathcal{L}$ is trivial. (Recall that $\mathcal{F}_2$ is the rank 2 bundle on $E^\vee$ defined by a generator of $\mathrm{Ext}^1(\mathcal{O}_{E^\vee},\mathcal{O}_{E^\vee})\cong\mathrm{H}^1(E^\vee)$.) Note that by construction, the $\check{G}$-bundle underlying $\kappa(\mathcal{L})$ is semistable of degree 0. Moreover, Theorem 4.3.10 says that there is a Cartesian square

$$\begin{tikzcd}
(\mathbf{G}_m\times E)^{\mathrm{bl}} \arrow[r] \arrow[d] & E \arrow[d, "\kappa"] \\
E \arrow[r, "\kappa"'] & \mathrm{Bun}^0_{\check{B}}(E^\vee),
\end{tikzcd}$$

where $(\mathbf{G}_m\times E)^{\mathrm{bl}}$ is a particular affine blowup of $\mathbf{G}_m\times E$. [2]

Notice that $\mathbf{G}_m\times E$ admits an action of $W=\mathbf{Z}/2$, via inversion on $\mathbf{G}_m$ and E; this extends to an action of $\mathbf{Z}/2$ on $(\mathbf{G}_m\times E)^{\mathrm{bl}}$, and the above diagram suggests viewing $(\mathbf{G}_m\times E)^{\mathrm{bl}}/\!\!/(\mathbf{Z}/2)$ as an *elliptic* analogue of the group scheme of regular centralizers.

[2] The desired affine blowup $(\mathbf{G}_m\times E)^{\mathrm{bl}}$ is obtained by blowing up $\mathbf{G}_m\times E$ at the diagonal wall cut out by the zero sections of $\mathbf{G}_m$ and E, and deleting the proper preimage of the zero section of E; see also [BFM05, Lemma 4.1].

Furthermore, there is an isomorphism $\Gamma((\mathbf{G}_m\times E)^{\mathrm{bl}}; \mathcal{O}_{(\mathbf{G}_m\times E)^{\mathrm{bl}}})\otimes_{\mathbf{Q}}\pi_*(A_{\mathbf{Q}}) \cong C_*^T(\mathrm{Gr}_G(\mathbf{C}); A)\otimes \mathbf{Q}$ between a base-change of the coherent cohomology of $(\mathbf{G}_m \times E)^{\mathrm{bl}}$ and the rationalization of the T-equivariant A-homology of $\mathrm{Gr}_G(\mathbf{C})$. Using this, one can show (see Theorem 4.4.7) that there is an equivalence between a variant of $\mathrm{Coh}_{C_T^*(\mathrm{Gr}_G(\mathbf{C});A)}\otimes\mathbf{Q}$ and an explicit full subcategory of $\mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(E^\vee)) \otimes_{\mathbf{Q}} \pi_*(A_{\mathbf{Q}})$.

In this case, Theorem 1.1.7 says that if T acts on G by conjugation, then there is an equivalence

$$\mathrm{Loc}_{T_c}(G_c; A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(E \times_{\mathrm{Bun}^0_{\check{B}}(E^\vee)} E) \otimes_{\pi_0 A} A_{\mathbf{Q}}.$$

If $\{\mathcal{O}_{E^\vee} \subseteq \mathcal{F}_2\} \in \mathrm{Bun}^0_{\check{B}}(E^\vee)$ denotes the Atiyah bundle, then the preceding equivalence in turn implies an equivalence

$$\mathrm{Loc}(G_c; A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\{\mathcal{O}_{E^\vee} \subseteq \mathcal{F}_2\} \times_{\mathrm{Bun}^0_{\check{B}}(E^\vee)} E) \otimes_{\pi_0 A} A_{\mathbf{Q}}.$$

One can view $Z_f^E(\check{B}_{A_{\mathbf{Q}}}) := (\{\mathcal{O}_{E^\vee} \subseteq \mathcal{F}_2\} \times_{\mathrm{Bun}^0_{\check{B}}(E^\vee)} E) \otimes_{\pi_0 A} A_{\mathbf{Q}}$ as the "centralizer in $\check{B}$ of the regular 'elli-potent' element $\{\mathcal{O}_{E^\vee} \subseteq \mathcal{F}_2\} \in \mathrm{Bun}^0_{\check{B}}(E^\vee)$". Therefore, $Z_f^E(\check{B}_{A_{\mathbf{Q}}})$ can be viewed as an A-theoretic mirror to $G(\mathbf{C}) = T^*(G_c)$ viewed as a symplectic manifold.

Remark 1.1.15. One might hope that these results hold without rationalization, but we do not know how to prove such a statement. In the case of KU, for instance, the key obstruction is that we do not know whether the 2-periodification of $\widetilde{\check{G}}_{\mathbf{Q}}^{\mathrm{reg}}/\check{G}_{\mathbf{Q}}$ can be lifted to a flat stack $(\widetilde{\check{G}}^{\mathrm{reg}}/\check{G})_{\mathrm{KU}}$ over KU. If it did, then the arguments below would go through to prove a KU-linear equivalence of the form $\mathrm{Mod}(C_T^*(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU})) \simeq \mathrm{QCoh}((\widetilde{\check{G}}^{\mathrm{reg}}/\check{G})_{\mathrm{KU}})$. When A is specialized to motivic cohomology $\mathrm{H}\mathbf{Z}_{\mathrm{mot}}$, Theorem 4.4.7 recovers a version of the motivic analogue of the geometric Satake equivalence (studied in [RS21b, RS21a]) over the regular locus.

Remark 1.1.16. In a sequel to this article (Part II), we will use Theorem 4.4.7 to show that there is an equivalence between a motivic variant of $\mathrm{Shv}_T(\mathrm{Gr}_G(\mathbf{C}); A)\otimes\mathbf{Q}$ and a variant of $\mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}_{A_{\mathbf{Q}}}}(\mathfrak{D}(\mathbf{G})_{\mathbf{Q}}))$, as well as a motivic variant of $\mathrm{Shv}_G(\mathrm{Gr}_G(\mathbf{C}); A) \otimes \mathbf{Q}$ and a variant of $\mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss},0}_{\check{G}_{A_{\mathbf{Q}}}}(\mathfrak{D}(\mathbf{G})_{\mathbf{Q}}))$, where $\mathrm{Bun}^{\mathrm{ss},0}_{\check{G}_{A_{\mathbf{Q}}}}$ denotes the moduli *stack* of semistable $\check{G}_{A_{\mathbf{Q}}}$-bundles of degree 0. For instance, when $A = \mathrm{KU}$, this specializes to an equivalence between a motivic variant of $\mathrm{Shv}_G(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q}$ and a variant of $\mathrm{QCoh}(\check{G}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}]$. (A conjecture of this form was proposed in [CK18], although I do not understand the definition of the "constructible side" in *loc. cit.*)

One important hurdle is that the $\mathbf{E}_\infty$-$\mathbf{Q}$-algebra of functions on the *derived* stacks $\check{G}/\check{G}$ and $\check{G}^{\mathrm{reg}}/\check{G}$ (temporarily omitting the subscript $\mathbf{Q}$ for simplicity) do not agree, because the algebraic Hartogs lemma fails to hold derivedly[3]. This difficulty is often avoided when studying constructible sheaves with coefficients in $\mathbf{C}$ by cleverly using tilting sheaves, etc; see, e.g., [Ric17, Sections 5.5 and 5.6]. Unfortunately, these techniques are often unavailable when studying constructible sheaves with more exotic coefficients, so we are forced to find a workaround. In Part II, this will be done using the techniques of [Dev22b].

In Appendix A, we discuss some motivation for this article stemming from the Coulomb branches of 3d $\mathcal{N} = 4$, 4d $\mathcal{N} = 2$, and 5d $\mathcal{N} = 1$ pure gauge theories. We also give explicit generators and relations for the Coulomb branches of 3d $\mathcal{N} = 4$ and 4d $\mathcal{N} = 2$ pure gauge theories with gauge group SL_2 (i.e., $C_G^*(\mathrm{Gr}_G(\mathbf{C}); \mathbf{Q})$ and $C_G^*(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU})$ with $G = \mathrm{SL}_2$). The 4-dimensional case is a q-analogue of the quantization of the Atiyah-Hitchin manifold from [BDG17, Equation 5.51].

Finally, in Appendix C, we give examples of $\mathfrak{D}(\mathbf{G})$ for several examples of group schemes $\mathbf{G}$. For many $\mathbf{G}$, one obtains a Fourier-Mukai equivalence between $\mathrm{QCoh}(\mathbf{G})$ and $\mathrm{QCoh}(\mathfrak{D}(\mathbf{G}))$. This simultaneously recovers the classical Fourier-Mukai equivalence when $\mathbf{G}$ is an abelian variety, the monodromy equivalence $\mathrm{QCoh}(\mathbf{G}_m) \simeq \mathrm{Loc}(S^1)$ when $\mathbf{G} = \mathbf{G}_m$, the Koszul duality equivalence $\mathrm{QCoh}(\hat{\mathbf{G}}_a) \simeq \mathrm{QCoh}(B\mathbf{G}_a)$ over an algebraically closed field of characteristic zero when $\mathbf{G} = \hat{\mathbf{G}}_a$, the Mellin transform $\mathrm{DMod}(\mathbf{G}_m) \simeq \mathrm{QCoh}(\mathbf{A}^1/\mathbf{Z})$ over an algebraically closed field of characteristic zero when $\mathbf{G} = \mathbf{G}_{m,\mathrm{dR}}$, and the Fourier-Deligne transform $\mathrm{DMod}(\mathbf{G}_a) \simeq \mathrm{DMod}(\mathbf{G}_a)$. When applied to the diffracted Hodge

[3]For instance, if k is a commutative ring, then $\Gamma(\mathbf{A}^2 - \{0\}; \mathcal{O}_{\mathbf{A}^2-\{0\}})$ has π_0 given by $k[x, y]$, but π_{-1} is given by the local cohomology $k[x, y]/(x^\infty, y^\infty)$.

stack of $\mathbf{G}_a$ (in the sense of [BL22]), one also obtains a Fourier transform for Hodge-Tate crystals on $\mathbf{G}_a$.

We will use the following notation throughout; furthermore, the reader should keep in mind that *everything* in this article will be derived, unless explicitly mentioned otherwise.

Notation 1.1.17. Let G be a connected (often simply-connected) semisimple group over $\mathbf{C}$ (or a torus). Fix a maximal torus $T \subseteq B$ contained in a Borel subgroup of G. Let $U = [B, B]$ denote the unipotent radical of B, so that $B/U \cong T$. Let Φ be the set of roots of G, Φ^+ the set of positive roots, and Δ a set of simple roots. Let W be the Weyl group; if $w \in W$, let $\dot{w} \in N_G(T)$ denote a lift of w to the normalizer of T in G. Let $\mathbb{X}^*$ denote the lattice of characters of T, $\mathbb{X}_*$ the lattice of cocharacters of T, Λ the weight lattice, and Λ^+ the lattice of dominant weights.

1.2. **Relationship to other work.** The technical basis for our work (which will described in a future article) on chromatic analogues of the geometric Satake relies on working with a motivic variant of the spherical Hecke category. As mentioned above, the idea of proving a motivic analogue of the geometric Satake equivalence has also appeared in [RS21b, RS21a]. These results work with coefficients in $\mathrm{H}\mathbf{Z}_{\mathrm{mot}}$, and only at the level of *abelian* 1-categories; at this level, one cannot see the derived contributions which feature prominently in our approach to the geometric Satake equivalence. It would be interesting to study the relationship of the work in this article to [YZ21].

Connection to Ben-Zvi–Sakellaridis–Venkatesh. Our work suggests an extension of the exciting program of Ben-Zvi–Sakellaridis–Venkatesh (see [Sak21, BSV21] for an overview). Let us briefly review our understanding of their program, and how our work conjecturally relates to it. Suppose G is a semisimple algebraic group or a torus over $\mathbf{C}$. For a large class of smooth Hamiltonian G-varieties M of the form T^*X with G acting on X, the program of Ben-Zvi–Sakellaridis–Venkatesh (with previous input by Gaitsgory-Nadler [GN10]) conjectures that there is a dual Hamiltonian $\check{G}$-variety $\check{M}$ such that there is an equivalence between the A-side category $\mathrm{Shv}(X(\mathcal{K})/G(\mathcal{O}))$ and the B-side category $\mathrm{QCoh}^{\mathrm{sh}}(\check{M}/\check{G})$. Moreover, this equivalence should satisfy certain compatibility criteria. Here, $\mathrm{QCoh}^{\mathrm{sh}}(\check{M}/\check{G})$ denotes the ∞-category of quasicoherent sheaves on a shearing of $\check{M}$.

Let us give some examples:

(a) For the pair $(M = T^*G, G \times G)$, the A-side category is $\mathrm{Shv}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))$, i.e., the spherical Hecke category. By the derived geometric Satake equivalence, we may identify this with $\mathrm{QCoh}(\check{\mathfrak{g}}[2]/\check{G}) \simeq \mathrm{QCoh}(\check{G}\backslash T^*[2](\check{G})/\check{G})$; the dual pair is therefore $(T^*\check{G}, \check{G} \times \check{G})$.

(b) For the pair $(M = T^*(G/N), G\times T)$, the A-side category is $\mathrm{Shv}(T(\mathcal{O})\backslash \mathfrak{L}(G/N)/G(\mathcal{O}))$. Since the $\mathbf{C}$-points of N is contractible, we may interpret the A-side category as $\mathrm{Shv}(T\backslash LG(\mathbf{C})/G(\mathbf{C})) \simeq \mathrm{Shv}_T(\mathrm{Gr}_G)$. (Technically, one should interpret $\mathrm{Shv}(T(\mathcal{O})\backslash \mathfrak{L}(G/N)/G(\mathcal{O}))$ as the ∞-category $\mathrm{Shv}(\mathrm{Fl}_G^{\infty/2}/G(\mathcal{O}))$, where $\mathrm{Fl}_G^{\infty/2}$ is the semi-infinite flag variety $G(\mathcal{K})/N(\mathcal{K})\cdot T(\mathcal{O})$. However, in [Ras14, Corollary 17.2.3], it was shown that $\mathrm{Shv}(\mathrm{Fl}_G^{\infty/2}/G(\mathcal{O}))$ is equivalent to $\mathrm{Shv}(\mathrm{Fl}/G(\mathcal{O})) \simeq \mathrm{Shv}(\mathrm{Gr}_G/I)$.) The work of Arkhipov-Bezrukavnikov-Ginzburg identifies this with a shearing of $\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}/\check{G})$. However, $\widetilde{\check{\mathfrak{g}}}/\check{G} \cong \check{T}\backslash\check{\mathfrak{b}}/\check{N}$ is isomorphic to $\check{T}\backslash T^*(\check{G}/\check{N})/\check{G}$, so that the dual pair is $(M = T^*(\check{G}/\check{N}), \check{G} \times \check{T})$.

(c) Let $\psi : N \to \mathbf{G}_a$ be a nondegenerate character, such as the composite

$$N \to N/[N, N] \cong \prod_{\alpha\in\Delta} \mathbf{G}_a^\alpha \xrightarrow{\sum_\alpha} \mathbf{G}_a.$$

Composition with the map $N(\mathcal{K}) \to N$ given by the residue defines a nondegenerate character of $N(\mathcal{K})$, which we will also denote by ψ. Let $T^*(G/_\psi N)$ denote $(\psi+\mathfrak{b}^*)\times_N G \cong (\psi+\mathfrak{b})\times_N G$. For the pair $(M = T^*(G/_\psi N), G)$, the A-side category is $\mathrm{Shv}((G/_\psi N)(\mathcal{K})/G(\mathcal{O}))$, which we may interpret as the Whittaker category $\mathrm{Shv}(\mathrm{Gr}_G)^{(N(\mathcal{K}),\psi)}$. By the geometric Casselman-Shalika equivalence of [FGKV98, FGV01], we may identify this with $\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(B\check{G})$; the dual pair is therefore $(T^*{*}, \check{G})$.

We summarize these examples in a table, a much-expanded version of which appears at [Wan22]:

An interesting observation (which, as far as we know, is not part of the work of Ben-Zvi–Sakellaridis–Venkatesh) is that one can generalize the above discussion further by adding in loop-rotation equivariance. Let us explain this in the above examples:

A-side pair	$(T^*G, G\times G)$	$(T^*(G/N), G\times T)$	$(T^*(G/_\psi N), G)$
B-side pair	$(T^*\check{G}, \check{G}\times\check{G})$	$(T^*(\check{G}/\check{N}), \check{G}\times\check{T})$	$(T^*\ast, \check{G})$

(a) For the pair $(M = T^*G, G\times G)$, adding in loop-rotation equivariance to the A-side category produces $\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O}))$, i.e., the spherical Hecke category. By the quantized derived geometric Satake equivalence of [BF08], there is an equivalence

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(G(\mathcal{O})\backslash G(\mathcal{K})/G(\mathcal{O})) \simeq \mathrm{DMod}_\hbar(\check{G})^{\check{G}\times\check{G},\mathrm{wk}}.$$

Note that the right-hand side is the quantization of the dual pair $(T^*\check{G}, \check{G}\times\check{G})$.

(b) For the pair $(M = T^*(G/N), G\times T)$, adding in loop-rotation equivariance to the A-side category produces $\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G/I)$. In forthcoming work, we quantize the results of Arkhipov-Bezrukavnikov-Ginzburg to identify

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G/I) \simeq \mathrm{DMod}_\hbar(\check{G}/\check{N})^{\check{G}\times\check{T},\mathrm{wk}} \simeq U_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{N},(\check{T},\mathrm{wk})}.$$

Note that the right-hand side is the quantization of the dual pair $(M = T^*(\check{G}/\check{N}), \check{G}\times\check{T})$. Using [Ras14, Corollary 17.2.3], we may view the left-hand side as $\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(T(\mathcal{O})\backslash\mathfrak{L}(G/N)/G(\mathcal{O}))$, so that the above equivalence can be stated more symmetrically as

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(T(\mathcal{O})\backslash\mathfrak{L}(G/N)/G(\mathcal{O})) \simeq \mathrm{DMod}_\hbar(\check{G}/\check{N})^{\check{G}\times\check{T},\mathrm{wk}}.$$

In fact, we will also quantize the equivalence of [Bez16]: we show that there is an equivalence

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Fl}_G/I) \simeq \mathrm{DMod}_\hbar(\check{N}\backslash\check{G}/\check{N})^{\check{T}\times\check{T},\mathrm{wk}}.$$

Note that we may heuristically view the left-hand side as $\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(T(\mathcal{O})\backslash\mathfrak{L}(N\backslash G/N)/T(\mathcal{O}))$, so that the above equivalence can be stated more symmetrically as

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(T(\mathcal{O})\backslash\mathfrak{L}(N\backslash G/N)/T(\mathcal{O})) \simeq \mathrm{DMod}_\hbar(\check{N}\backslash\check{G}/\check{N})^{\check{T}\times\check{T},\mathrm{wk}}.$$

(c) For the pair $(M = T^*(G/_\psi N), G)$, adding in loop-rotation equivariance to the A-side category produces $\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G)^{(N(\mathcal{K}),\psi)}$. However, this can be identified with $\mathrm{Shv}(\mathrm{Gr}_G)^{(N(\mathcal{K}),\psi)}\otimes \mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(\ast)$, so that the geometric Casselman-Shalika equivalence of [FGKV98, FGV01] identifies this with $\mathrm{Rep}(\check{G})\otimes\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(\ast)$. Again, this is the quantization of the dual pair $(T^*\ast, \check{G})$.

Therefore, for a pair (T^*X, G) with dual pair $(T^*\check{X}, \check{G})$, we have equivalences

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(X(\mathcal{K})/G(\mathcal{O})) \simeq \mathrm{DMod}_\hbar(\check{X})^{\check{G},\mathrm{wk}}.$$

Assume our ring k of coefficients (in Shv) is a $\mathbf{Q}$-algebra. Then Koszul duality allows us to identify $\mathrm{DMod}_\hbar(\check{X})$ with the ∞-category $\mathrm{IndCoh}(\mathrm{Map}(B\mathbf{G}_a, \check{X}))^{B\mathbf{G}_a}$ of $B\mathbf{G}_a$-equivariant ind-coherent sheaves on the "de Rham" loop space $\mathrm{Map}(B\mathbf{G}_a, \check{X})$. (If our ring k of coefficients is not a $\mathbf{Q}$-algebra, we must replace $B\mathbf{G}_a$ by $B\mathbf{W}[F]$, where $\mathbf{W}[F]$ is the kernel of Frobenius on the Witt ring scheme.) Therefore, the above equivalence can be restated as

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(X(\mathcal{K})/G(\mathcal{O})) \simeq \mathrm{IndCoh}(\mathrm{Map}(B\mathbf{G}_a, \check{X}))^{B\mathbf{G}_a\times\check{G}}. \tag{2}$$

In fact, as we will explain in a future article [Dev22a], one can refine the statement of the local geometric Langlands conjecture (which predicts an equivalence $\mathbf{L} : \mathrm{DMod}(G(\mathcal{K}))\text{-mod} \xrightarrow{\sim} 2\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{Loc}_{\check{G}}(\mathring{D}))$ of $(\infty,2)$-categories) to keep track of loop rotation, and more generally the canonical $\mathrm{Aut}(\mathring{D})$-equivariance on both sides. As stated, the equivalence $\mathbf{L}$ is in fact *not* even equivariant for the action of loop rotation on $\mathring{D}$.

The work in this article suggests a generalization of the conjectural duality of Ben-Zvi–Sakellaridis–Venkatesh to the case of "$\mathbf{G}_{A_\mathbf{Q}}$-Hamiltonian G-varieties", where $\mathbf{G}$ is an oriented 1-dimensional group scheme over an even-periodic $\mathbf{E}_\infty$-ring A. Motivated by [Saf16], one can define "$\mathbf{G}_{A_\mathbf{Q}}$-Hamiltonian G-varieties" to be stacks equipped with a Lagrangian morphism to $\mathrm{Map}(\mathfrak{D}(\mathbf{G}_{A_\mathbf{Q}}), BG_{A_\mathbf{Q}})$. Since we do not have a precise statement of the generalization of the Ben-Zvi–Sakellaridis–Venkatesh conjecture, we illustrate this suggestion through an example. Assume that G is a connected and simply connected semisimple algebraic group or a torus over $\mathbf{C}$. We restrict to the case when $\mathbf{G} = \mathbf{G}_m$ (so $A = \mathrm{KU}$), so that the examples deal with more classical objects (for instance, using [Saf16], $\mathbf{G}_m$-Hamiltonian G-varieties are precisely quasi-Hamiltonian G-varieties in the sense of [AMM98]).

Using Theorem 4.4.7, we will show in future work (in a manner analogous to Arkhipov-Bezrukavnikov-Ginzburg [ABG04]) that there is an equivalence between a motivic variant of $\mathrm{Shv}_T(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q}$ and a variant of $\mathrm{QCoh}(\widetilde{\check{G}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}}) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}]$. Note that we may view $\widetilde{\check{G}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \cong \check{B}^-_{\mathbf{Q}}/\check{B}^-_{\mathbf{Q}}$ as the two-sided quotient $\check{T}_{\mathbf{Q}} \backslash (\check{B}^-_{\mathbf{Q}} \times_{\check{N}_{\mathbf{Q}}} \check{G}_{\mathbf{Q}})/\check{G}_{\mathbf{Q}}$. Moreover, $B^- \times_N G$ can be viewed as a multiplicative analogue of $T^*(G/N) \cong \mathfrak{b}^* \times_N G \cong \mathfrak{b}^- \times_N G$; to this extent, we will write $T^*_{\mathbf{G}_m}(G/N)$ to denote $B^- \times_N G$. Then, following (b) above, the preceding discussion suggests that the quasi-Hamiltonian pairs $(T^*_{\mathbf{G}_m}(G/N), G \times T)$ and $(T^*_{\mathbf{G}_m}(\check{G}/\check{N}), \check{G} \times \check{T})$ are dual to each other.

The above discussion suggests a generalization of the Ben-Zvi–Sakellaridis–Venkatesh conjecture (which we state as a slogan, for lack of precision):

Slogan 1.2.1. Let G be a connected and simply connected semisimple algebraic group over $\mathbf{C}$, and let X be a variety over $\mathbf{C}$ with a G-action. Let $\mathbf{G}$ be an oriented 1-dimensional group scheme over an even-periodic $\mathbf{E}_\infty$-ring A. Let $\check{G}_{A_{\mathbf{Q}}}$ denote $\check{G}_{\mathbf{Q}} \otimes_{\mathbf{Q}} A_{\mathbf{Q}}$. Then there is an $A_{\mathbf{Q}}$-scheme $\check{X}_{A_{\mathbf{Q}}}$ with an action of $\check{G}_{A_{\mathbf{Q}}}$, such that there is an equivalence between a motivic variant of $\mathrm{Shv}_{G(\mathcal{O})}(X(\mathcal{K}); A) \otimes \mathbf{Q}$ and a variant of $\mathrm{IndCoh}(\mathrm{Map}(\mathfrak{D}(\mathbf{G})_{\mathbf{Q}}, \check{X}_{A_{\mathbf{Q}}}))^{\check{G}_{A_{\mathbf{Q}}}}$. Similarly, there is an equivalence between a motivic variant of $\mathrm{Shv}_{G(\mathcal{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}}(X(\mathcal{K}); A) \otimes \mathbf{Q}$ and a variant of $\mathrm{IndCoh}(\mathrm{Map}(\mathfrak{D}(\mathbf{G})_{\mathbf{Q}}, \check{X}_{A_{\mathbf{Q}}})/\mathfrak{D}(\mathbf{G})_{\mathbf{Q}})^{\check{G}_{A_{\mathbf{Q}}}}$.

Even more informally, the rationalization of the ∞-category of A-valued $G(\mathcal{O})$-constructible sheaves on $X(\mathcal{K})$ is equivalent to the ∞-category of $\check{G}_{A_{\mathbf{Q}}}$-equivariant quasicoherent sheaves on a "$\mathbf{G}$-Hamiltonian $\check{G}_{A_{\mathbf{Q}}}$-space" over $A_{\mathbf{Q}}$.

Example 1.2.2. For instance:

- If $A = \mathbf{Q}[\beta^{\pm 1}]$ and $\mathbf{G} = \hat{\mathbf{G}}_a$, then Slogan 1.2.1 recovers an equivalence of the form (2). More generally, the ∞-category of $\mathbf{Q}$-valued $G(\mathcal{O})$-constructible sheaves on $X(\mathcal{K})$ should be equivalent to the ∞-category of $\check{G}_{\mathbf{Q}}$-equivariant quasicoherent sheaves on a *Hamiltonian $\check{G}_{\mathbf{Q}}$-space* over $\mathbf{Q}$. This is effectively the program of Ben-Zvi–Sakellaridis–Venkatesh (see [Sak21, BSV21]).
- If $A = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, Slogan 1.2.1 says that there is an equivalence between a motivic variant of $\mathrm{Shv}_{G(\mathcal{O})}(X(\mathcal{K}); \mathrm{KU}) \otimes \mathbf{Q}$ and a variant of $\mathrm{IndCoh}(\mathrm{Map}(S^1, \check{X}))^{\check{G}}$. We will use Theorem 4.4.7 to prove such a statement in a sequel to this article. With loop-rotation equivariance, Slogan 1.2.1 says that there is an equivalence between a motivic variant of $\mathrm{Shv}_{G(\mathcal{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}}(X(\mathcal{K}); \mathrm{KU}) \otimes \mathbf{Q}$ and a variant of $\mathrm{IndCoh}(\mathrm{Map}(S^1, \check{X})/S^1)^{\check{G}}$. In particular, we expect that the ∞-category of S^1-equivariant ind-coherent sheaves on $\check{G} \backslash \mathrm{Map}(S^1, \check{G})/\check{G}$ is closely related to the Harish-Chandra category for $U_q(\check{\mathfrak{g}})$.

  More generally, the rationalization of the ∞-category of KU-valued $G(\mathcal{O})$-constructible sheaves on $X(\mathcal{K})$ should be equivalent to the 2-periodification of the ∞-category of $\check{G}_{\mathbf{Q}}$-equivariant quasicoherent sheaves on a *quasi-Hamiltonian $\check{G}_{\mathbf{Q}}$-space* (in the sense of [AMM98]) over $\mathbf{Q}$.
- If A is a complex-oriented even-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}$ is an oriented elliptic curve over A (in the sense of [Lur18b]), let E be the underlying classical scheme of $\mathbf{G}$ over the classical ring $\pi_0(A)$ (so that E is an elliptic curve), and let $E^\vee$ be the dual elliptic curve. Then Slogan 1.2.1 says that there is an equivalence between a motivic variant of $\mathrm{Shv}_{G(\mathcal{O})}(X(\mathcal{K}); A) \otimes \mathbf{Q}$ and a variant of $\mathrm{IndCoh}(\mathrm{Map}(E^\vee, \check{X}))^{\check{G}}$. We will use Theorem 4.4.7 to prove such a statement in a sequel to this article.

  With loop-rotation equivariance, the above slogan says that there is an equivalence between a motivic variant of $\mathrm{Shv}_{G(\mathcal{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}}(X(\mathcal{K}); A) \otimes \mathbf{Q}$ and a variant of $\mathrm{IndCoh}(\mathrm{Map}(E^\vee, \check{X})/E^\vee)^{\check{G}}$, where $E^\vee$ acts on $\mathrm{Map}(E^\vee, \check{X})$ by translation on the elliptic curve. It would be very interesting to relate $\mathrm{IndCoh}(\mathrm{Map}(E^\vee, \check{G})/E^\vee)^{\check{G} \times \check{G}}$ to some version of elliptic quantum groups à la [Fel95].

  More generally, the rationalization of the ∞-category of A-valued $G(\mathcal{O})$-constructible sheaves on $X(\mathcal{K})$ should be equivalent to the ∞-category of $\check{G}_{A_{\mathbf{Q}}}$-equivariant quasicoherent sheaves on an "elliptic $\check{G}_{A_{\mathbf{Q}}}$-space" over $A_{\mathbf{Q}}$.

Making the above slogan precise and exploring its consequences seems like an exciting avenue of study in both homotopy theory and geometric representation theory. In forthcoming work, we will expand on the above program further, and provide additional evidence for Slogan 1.2.1 by studying the case of toric varieties.

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2. Homotopy theory background

2.1. **Review of generalized equivariant cohomology.** We review the construction of generalized equivariant cohomology from [Lur09a], in a form suitable for our applications. This review will necessarily be brief, since a detailed exposition may be found in *loc. cit.*.

Setup 2.1.1. Fix an $\mathbf{E}_\infty$-ring A and a commutative A-group $\mathbf{G}$, so $\mathbf{G}$ defines a functor $\mathrm{CAlg}_A \to \mathrm{Mod}_{\mathbf{Z},\geq 0}$ which is representable by a *flat* A-algebra. We will write $\mathbf{G}_0$ to denote the resulting commutative group scheme over $\pi_0 A$.

Remark 2.1.2. The equivalence $\Omega^\infty : \mathrm{Sp}_{\geq 0} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{S}_*)$ extends to an equivalence between $\mathrm{Mod}_{\mathbf{Z},\geq 0}$ and topological abelian groups. More precisely, by the Dold-Kan correspondence and the Schwede-Shipley theorem, there are equivalences of categories

$$\mathrm{Mod}_{\mathbf{Z}}^{\geq 0} \simeq \mathrm{Ch}_{\geq 0}(\mathbf{Z}) \simeq \mathrm{Fun}(\mathbf{\Delta}^{op}, \mathrm{Ab}) = s\mathrm{Ab}.$$

The image of $\mathrm{Mod}_{\mathbf{Z}}^{\geq 0}$ under the equivalence $\Omega^\infty : \mathrm{Sp}_{\geq 0} \xrightarrow{\sim} \mathrm{CAlg}(\mathcal{S}_*)$ can be characterized as follows. Let us model grouplike infinite loop spaces X as functors $X : \mathrm{Fin}_* \to \mathcal{S}$ such that $\pi_0 \mathrm{Map}_{\mathcal{S}}(Y, X)$ is an abelian group for all spaces Y (i.e., X is grouplike) and such that the map $X([n]) \to X([1])^n$ is an equivalence. Such an object should be in the image of $\mathrm{Mod}_{\mathbf{Z}}^{\geq 0}$ iff it is "strictly commutative". One way to make this precise is as follows. Let Lattice denote the full subcategory of the category of abelian groups spanned by the groups $\mathbf{Z}^n$ with $n \geq 0$, so there is a functor $\mathrm{Fin}_* \to \mathrm{Lattice}$. Then an infinite loop space is in the image of $\mathrm{Mod}_{\mathbf{Z}}^{\geq 0}$ if and only if the functor $\mathrm{Fin}_* \to \mathcal{S}$ classifying it factors through a finite-product-preserving functor $\mathrm{Lattice} \to \mathcal{S}$. In other words, $\mathrm{Mod}_{\mathbf{Z}}^{\geq 0}$ is equivalent to the full subcategory spanned by the grouplike objects in the category $\mathrm{Fun}^\pi(\mathrm{Lattice}, \mathcal{S})$. This is a very strong condition to impose on an infinite loop space: it forces the infinite loop space to decompose as a product of Eilenberg-Maclane spaces. For example, $\mathbf{C}P^\infty$ admits such a factorization, but BU (with either the additive or multiplicative infinite loop space structure) does not.

Definition 2.1.3. A *preorientation of* $\mathbf{G}$ is a pointed map $S^2 \to \Omega^\infty \mathbf{G}(A)$ of spaces, i.e., a map $\Sigma^2 \mathbf{Z} \to \mathbf{G}(A)$ of $\mathbf{Z}$-modules (by adjunction). This induces a map $\mathbf{C}P^\infty = \Omega^\infty \Sigma^2 \mathbf{Z} \to \Omega^\infty \mathbf{G}(A)$ of topological abelian groups, and hence a map $\mathrm{Spf}\, A^{\mathbf{C}P^\infty} \to \mathbf{G}$ of $\mathbf{E}_\infty$-A-group schemes. (Note that $\mathrm{Spf}\, A^{\mathbf{C}P^\infty}$ need not admit the structure of a commutative A-group scheme: for instance, $A^{\mathbf{C}P^\infty}$ need not be flat over A.)

Definition 2.1.4. Given a preorientation $S^2 \to \Omega^\infty \mathbf{G}(A)$, we obtain a map $\mathcal{O}_{\mathbf{G}} \to C^*(S^2; A)$ of $\mathbf{E}_\infty$-A-algebras. On π_0, this induces a map $\pi_0 \mathcal{O}_{\mathbf{G}} = \mathcal{O}_{\mathbf{G}_0} \to \pi_0 C^*(S^2; A)$. However, the target can be identified with the trivial square-zero extension $\pi_0 A \oplus \pi_{-2} A$, so that the preorientation defines a derivation $\mathcal{O}_{\mathbf{G}_0} \to \pi_{-2} A$. This defines a map $\beta : \omega = \Omega^1_{\mathbf{G}_0/\pi_0 A} \to \pi_{-2} A$. The preorientation is called an *orientation* if $\mathbf{G}_0$ is smooth of relative dimension 1 over $\pi_0 A$, and the composite

$$\pi_n(A) \otimes_{\pi_0 A} \omega \to \pi_n(A) \otimes_{\pi_0 A} \pi_{-2} A \xrightarrow{\beta} \pi_{n-2} A$$

is an isomorphism for each $n \in \mathbf{Z}$. This forces A to be 2-periodic (but does not force its homotopy to be concentrated in even degrees).

Warning 2.1.5. As discussed in [Lur09a, Section 3.2], the universal $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra over which the additive group scheme $\mathbf{G}_a$ admits an orientation is given by $\mathbf{Z}[\mathbf{C}P^\infty][\frac{1}{\beta}] = \mathbf{Q}[\beta^{\pm 1}]$. Therefore, we are allowed to let $\mathbf{G} = \mathbf{G}_a$ in the story below only when A is a 2-periodic $\mathbf{E}_\infty$-$\mathbf{Q}$-*algebra*. (If A is not an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra, one cannot in general define $\mathbf{G}_a = \mathrm{Spec}\, A[t]$ as a commutative A-group: the coproduct $A[t] \to A[x, y]$ will in general not be a map of $\mathbf{E}_\infty$-A-algebras.)

We can now review the definition of T-equivariant A-cohomology when T is a torus.

Construction 2.1.6. Fix an $\mathbf{E}_\infty$-ring A as above and a commutative A-group $\mathbf{G}$. Given a diagonalizable group scheme T (over $\mathbf{C}$), define an A-scheme $\mathcal{M}_T$ by the mapping stack $\mathrm{Hom}(\mathbb{X}^*, \mathbf{G})$. We will be particularly interested in the case when T is a torus. Let $\mathcal{T}$ be the full subcategory of $\mathcal{S}$ spanned by those spaces which are homotopy equivalent to $BT(\mathbf{C})$ with T being a diagonalizable group scheme. By arguing as in [Lur19, Theorem 3.5.5], a preorientation of $\mathbf{G}$ is equivalent to the data of a functor

$\mathcal{M} : \mathcal{T} \to \mathrm{Aff}_A$ along with compatible equivalences $\mathcal{M}(BT) \simeq \mathcal{M}_T$. The $\mathbf{E}_\infty$-A-algebra $\mathcal{O}_{\mathcal{M}_T}$ is the T-equivariant A-cochains of a point, and will occasionally be denoted by A_T.

We can now sketch the construction of the T-equivariant A-cochains of more general T-spaces; see [Lur09a, Theorem 3.2]. Let T be a torus over $\mathbf{C}$ for the remainder of this discussion, and let $\mathbf{G}$ be an *oriented* commutative A-group. Let $\mathcal{S}(T)$ denote the ∞-category of finite T-spaces, i.e., the smallest subcategory of $\mathrm{Fun}(BT, \mathcal{S})$ which contains the quotients T/T' for closed subgroups $T' \subseteq T$, and which is closed under finite colimits. There is a functor $\mathcal{F}_T : \mathcal{S}(T)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_T)$ which is uniquely characterized by the requirement that it preserve finite limits and sends $T/T' \mapsto q_* \mathcal{O}_{\mathcal{M}_{T'}}$. Here, $q : \mathcal{M}_{T'} \to \mathcal{M}_T$ is the canonical map induced by the inclusion $T' \subseteq T$. If $X \in \mathcal{S}(T)$, then the *T-equivariant A-cochains of X* is the global sections $\Gamma(\mathcal{M}_T; \mathcal{F}_T(X))$; we will denote it by $C^*_T(X; A)$.

Remark 2.1.7. Note that since $\mathcal{M}_T$ is an affine group scheme, we may identify $\mathrm{QCoh}(\mathcal{M}_T)$ with $\mathrm{Mod}(A_T)$. We will therefore also denote the functor $\mathcal{F}_T : \mathcal{S}(T)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_T)$ by $C^*_T(-; A) : \mathcal{S}(T)^{\mathrm{op}} \to \mathrm{Mod}(A_T)$.

Definition 2.1.8. If $X \in \mathcal{S}(T)$, then the *T-equivariant A-chains of X* is the quasicoherent sheaf on $\mathcal{M}_T$ given by the $\mathcal{O}_{\mathcal{M}_T}$-linear dual $\mathcal{F}_T(X)^\vee$. We will denote its global sections by $C^T_*(X; A)$. Note that $C^T_*(*; A) \simeq A_T$, so this definition and notation does *not* agree with the usual notion of equivariant chains/homology.

Warning 2.1.9. Let A be an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra, and let $\mathbf{G} = \mathbf{G}_a$; then Warning 2.1.5 says that A must be an $\mathbf{E}_\infty$-$\mathbf{Q}[\beta^{\pm 1}]$-algebra. Suppose for simplicity that $T = \mathbf{G}_m$; then $\pi_* C_*(BT; A)$ may therefore be identified with the divided power algebra $\Gamma_{\pi_*(A)}(\hbar^\vee)$ with $|\hbar^\vee| = 2$. Since A is rational, this may further be identified with the polynomial ring $\pi_*(A)[\hbar^\vee]$. Unfortunately, many sources in the representation theory literature seem to identify this with $\pi_*(A_T)$, albeit with the reversed grading. Although this identification is technically correct, it is rather abusive: there is no canonical way to identify A_T with $C_*(BT; A)$ when A is an $\mathbf{E}_\infty$-$\mathbf{Q}[\beta^{\pm 1}]$-algebra. We will therefore refrain from making this identification, since it is not valid for more general $\mathbf{E}_\infty$-rings A.

Notation 2.1.10. Let $\lambda : T \to \mathbf{G}_m$ be a character, and let $T_\lambda = \ker(\lambda)$. Then the map $q : \mathcal{M}_{T_\lambda} \to \mathcal{M}_T$ is a closed immersion, and we will denote the ideal in $\mathcal{O}_{\mathcal{M}_T}$ defined by this closed immersion by $\mathcal{I}_\lambda$. Equivalently, let V_λ denote the T-representation obtained by the projection $T \to T_\lambda$. Then $\mathcal{I}_\lambda$ is given by the line bundle $\mathcal{F}_T(S^{V_\lambda})$.

It is trickier to extend the definition of equivariant cochains to nonabelian groups, but a construction is sketched in [Lur09a, Section 3.5].

Construction 2.1.11. Let G be a reductive group scheme over $\mathbf{C}$. Let $\mathcal{S}(G)$ denote the smallest subcategory of $\mathrm{Fun}(BG, \mathcal{S})$ which contains the quotients G/T' for closed *commutative* subgroups $T' \subseteq G$, and which is closed under finite colimits. Then there is a functor $C^*_G(-; A) : \mathcal{S}(G)^{\mathrm{op}} \to \mathrm{Mod}(A)$ which is uniquely characterized by the requirement that it preserve finite limits and sends $G/T' \mapsto A_{T'}$. According to [Lur09a, End of Section 3.5], when G is connected, we there is a flat A-scheme $\mathcal{M}_G$ and a functor $\mathcal{F}_G : \mathcal{S}(G)^{\mathrm{op}} \to \mathrm{QCoh}(\mathcal{M}_G)$, such that composition with the forgetful functor $\mathrm{QCoh}(\mathcal{M}_G) \to \mathrm{Mod}(A)$ is the functor $C^*_G(-; A)$. If $X \in \mathcal{S}(G)$, we will write $\mathcal{F}_G(X)^\vee$ to denote the linear dual of $\mathcal{F}_G(X)$ in $\mathrm{QCoh}(\mathcal{M}_G)$, and refer to it as the *G-equivariant A-chains* on X.

Remark 2.1.12. Let X be a space with a G-action. If X can be written as the filtered colimit of a diagram $\{X_i\}$ of subspaces, each of which are in $\mathcal{S}(G)$, then we write $C^*_G(X; A)$ to denote $\varprojlim_i C^*_G(X_i; A)$. Similarly for $\mathcal{F}_G(X)$.

Example 2.1.13. Let G be a connected reductive group, and let T be a maximal torus in G. The flag variety G/T is a G-space whose stabilizers are commutative, and therefore $G/T \in \mathcal{S}(G)$. Therefore, $C^*_G(G/T; A) = A_T$. For the remainder of this text, we will make the following *assumption*: after inverting $|W|$, there is a (homotopy-coherent) W-action on A_T by maps of $\mathbf{E}_\infty$-A-algebras, and $A_G := C^*_G(*; A)$ is equivalent to A_T^{hW} as an $\mathbf{E}_\infty$-A-algebra.

2.2. **Categories of equivariant local systems.** Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A and an oriented A-group scheme $\mathbf{G}$. Let T be a torus over $\mathbf{C}$; in this section, we will not distinguish between T (viewed as an algebraic group), $T(\mathbf{C})$ (viewed as a complex manifold), and the maximal compact

T_c (viewed as a compact manifold). Let $X \in \mathcal{S}(T)$ be a finite T-space. The following categorifies the T-equivariant A-cochains $C^*_T(X;A)$.

Construction 2.2.1. Let $\mathrm{Loc}_T(*;A)$ denote the ∞-category $\mathrm{QCoh}(\mathcal{M}_T)$. Let $T' \subseteq T$ be a closed subgroup, so that there is an associated morphism $q : \mathcal{M}_{T'} \to \mathcal{M}_T$. This defines a symmetric monoidal functor $\mathrm{QCoh}(\mathcal{M}_T) \to \mathrm{QCoh}(\mathcal{M}_{T'})$, which equips $\mathrm{QCoh}(\mathcal{M}_{T'})$ with the structure of a $\mathrm{QCoh}(\mathcal{M}_T)$-module. Let $\mathcal{L}\mathrm{oc}_T(-;A) : \mathcal{S}(T)^{\mathrm{op}} \to \mathrm{CAlg}(\mathrm{ShvCat}(\mathcal{M}_T))$ be the functor uniquely characterized by the requirement that it preserve finite limits and send $T/T' \mapsto \mathrm{QCoh}(\mathcal{M}_{T'})$. If $X \in \mathcal{S}(T)$, then the ∞-category $\mathrm{Loc}_T(X;A)$ of *T-equivariant local systems of A-modules on X* is defined to be the global sections of the quasicoherent stack $\mathcal{L}\mathrm{oc}_T(X;A)$ on $\mathcal{M}_T$. If $f : X \to Y$ is a map in $\mathcal{S}(T)$, the associated symmetric monoidal functor $f^* : \mathrm{Loc}_T(Y;A) \to \mathrm{Loc}_T(X;A)$ (induced by taking global sections of the morphism $f^* : \mathcal{L}\mathrm{oc}_T(Y;A) \to \mathcal{L}\mathrm{oc}_T(X;A)$ of $\mathbf{E}_\infty$-algebras in quasicoherent stacks over $\mathcal{M}_T$) will be called the *pullback*. One can show that $\mathrm{Loc}_T(X;A)$ is a presentable stable ∞-category, and that f^* preserves small colimits (so it has a right adjoint f_*, which will be called *pushforward*).

Example 2.2.2. If $T = \{1\}$, then $\mathrm{Loc}_T(X;A)$ is equivalent to the ∞-category $\mathrm{Loc}(X;A) := \mathrm{Fun}(X, \mathrm{Mod}_A)$ of local systems on X.

Remark 2.2.3. Let X be a finite T-space. The *constant local system* $\underline{A}_T$ is defined to be the image of $\mathcal{O}_{\mathcal{M}_T}$ under the symmetric monoidal functor $\mathrm{Loc}_T(*;A) \simeq \mathrm{QCoh}(\mathcal{M}_T) \to \mathrm{Loc}_T(X;A)$ induced by pullback along $f : X \to *$. Observe that if $\underline{A}_T$ denotes the constant local system, then $\mathrm{End}_{\mathrm{Loc}_T(X;A)}(\underline{A}_T) \simeq C^*_T(X;A)$. Indeed, $\mathrm{End}_{\mathrm{Loc}_T(X;A)}(\underline{A}_T) \simeq \Gamma(\mathcal{M}_T; f_* f^* \mathcal{O}_{\mathcal{M}_T})$, but it is easy to see that $f_* f^* \mathcal{O}_{\mathcal{M}_T} = \mathcal{F}_T(X) \in \mathrm{QCoh}(\mathcal{M}_T)$. The desired claim then follows from Construction 2.1.6.

Remark 2.2.4. If T were a *finite* diagonalizable group scheme (such as μ_n), the desired category $\mathrm{Loc}_T(X;A)$ is closely related to the ∞-category of *$\mathbf{G}$-tempered local systems* on the orbispace $X /\!\!/ T$, as described in [Lur19]. Our understanding is that Lurie is planning to describe an extension of the work in [Lur19] and its connections to equivariant homotopy theory in a future article. We warn the reader that Construction 2.2.1 is somewhat *ad hoc*; so the resulting category of equivariant local systems may or may not agree with the output of forthcoming work of Lurie.

If Y is a *connected* space, the ∞-category $\mathrm{Loc}(Y;A) = \mathrm{Fun}(Y, \mathrm{Mod}_A)$ of local systems on Y is equivalent by Koszul duality to $\mathrm{LMod}_{C_*(\Omega Y;A)}$. This property of local systems is very useful, since it allows one to study of local systems using (derived) algebra. A similar property is true for $\mathrm{Loc}_T(X;A)$:

Proposition 2.2.5. *Let X be a connected finite T-space. Then there is an equivalence* $\mathrm{Loc}_T(X;A) \simeq \mathrm{LMod}_{\mathcal{F}_T(\Omega X)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$.

Proof. Let $s : * \to X$ denote the inclusion of a point. We claim that $s^* : \mathrm{Loc}_T(X;A) \to \mathrm{QCoh}(\mathcal{M}_T)$ admits a left adjoint $s_!$. Indeed, the statement for general X follows formally from the case of $X = T/T'$ for some closed subgroup $T' \subseteq T$ (so s is the inclusion of the trivial coset). In this case, s^* is the functor $\mathrm{QCoh}(\mathcal{M}_{T'}) \to \mathrm{QCoh}(\mathcal{M}_T)$ given by pushforward along the associated morphism $q : \mathcal{M}_{T'} \to \mathcal{M}_T$, so it has a left adjoint $s_!$ given by q^*. Note that s^* also has a right adjoint; in particular, it preserves small limits and colimits. Observe now that $s_! \mathcal{O}_{\mathcal{M}_T}$ is a compact generator of $\mathrm{Loc}_T(X;A)$: indeed, suppose $\mathcal{F} \in \mathrm{Loc}_T(X;A)$ such that $\mathrm{Map}_{\mathrm{Loc}_T(X;A)}(s_! \mathcal{O}_{\mathcal{M}_T}, \mathcal{F})) \simeq 0$ as an object of $\mathrm{QCoh}(\mathcal{M}_T)$. Because $s^* \mathcal{F} \simeq \mathrm{Map}_{\mathrm{Loc}_T(X;A)}(s_! \mathcal{O}_{\mathcal{M}_T}, \mathcal{F}))$ in $\mathrm{QCoh}(\mathcal{M}_T)$, we see that $s^* \mathcal{F} \simeq 0$. Using the connectivity of X, we see that $\mathcal{F}$ itself must be zero, which implies that $s_! \mathcal{O}_{\mathcal{M}_T}$ is a compact generator of $\mathrm{Loc}_T(X;A)$. It follows from the Barr-Beck-Lurie theorem ([Lur16, Theorem 4.7.3.5]) that $\mathrm{Loc}_T(X;A)$ is equivalent to the ∞-category of left $\mathrm{End}_{\mathrm{Loc}_T(X;A)}(s_! \mathcal{O}_{\mathcal{M}_T})$-modules in $\mathrm{QCoh}(\mathcal{M}_T)$. But $\mathrm{End}_{\mathrm{Loc}_T(X;A)}(s_! \mathcal{O}_{\mathcal{M}_T}) \simeq s^* s_! \mathcal{O}_{\mathcal{M}_T}$, which identifies with $\mathcal{F}_T(\Omega X)^\vee$. □

2.3. GKM and complex periodic $\mathbf{E}_\infty$-rings.

We review the main result of [HHH05], which proves a generalization of a result of Goresky-Kottwitz-MacPherson to generalized cohomology theories.

Setup 2.3.1. Let A be a complex-oriented even-periodic $\mathbf{E}_\infty$-ring, and let $\mathbf{G}$ be an oriented commutative A-group. Fix a torus T over $\mathbf{C}$ (which we will often identify with a maximal compact subgroup of $T(\mathbf{C})$, thereby equipping T-representations with positive definite Hermitian forms). We will consider (ind-finite; see Remark 2.1.12) T-spaces X such that the following assumptions hold.

(a) X admits a T-invariant stratification $\bigcup_{w\in W} X_x$ with only *even-dimensional* cells, with only finitely many in each dimension.
(b) The T-action on each cell $X_w = \mathbf{A}^{\ell(w)}$ is via a linear action, whose weights are pairwise relatively prime.
(c) For each weight λ of the T-action on $X_w = \mathbf{A}^{\ell(w)}$, the closure of $\mathbf{C}_\lambda \subseteq X_w$ is a sphere S^λ such that 0 and ∞ are fixed points of the T-action.

Definition 2.3.2. The *GKM graph* Γ asssociated to an X as in Setup 2.3.1 is defined as follows. The vertices are the (isolated) fixed points of the T-action, and there is an edge $x \to y$ labeled by a character λ if $x = 0$ and $y = \infty$ in the closure S^λ of $D(\lambda) \subseteq D(\mathbf{A}^{2\ell(w)})$. Let V denote the set of vertices of Γ, and E the set of edges.

Theorem 2.3.3 ([HHH05, Theorem 3.1]). *In Setup 2.3.1, the map $\mathcal{F}_T(X) \to \mathrm{Map}(V, \mathcal{O}_{\mathcal{M}_T}) \simeq \mathcal{F}_T(X^T)$ induces an injection on homotopy sheaves, and there is an equalizer diagram*

$$\mathcal{F}_T(X) \to \mathrm{Map}(V, \mathcal{O}_{\mathcal{M}_T}) \rightrightarrows \prod_{\alpha\in E} \mathcal{O}_{\mathcal{M}_{T_\alpha}}.$$

Here, the two maps are induced by the inclusion of the source and target of $\alpha : x \to y$.

Proof sketch. The argument is exactly as in [HHH05, Theorem 3.1] (where the spaces denoted F_i are points, corresponding to the origin in $\mathbf{A}^{\ell(w)}$), so we only give a sketch. We will work locally on $\mathbf{G}$. In this case, we need to show that the map $\mathcal{F}_T(X) \to \mathrm{Map}(V, \mathcal{O}_{\mathcal{M}_T}) \simeq \mathcal{F}_T(X^T)$ is injective, and there is an equalizer diagram

$$\mathcal{F}_T(X) \to \mathcal{F}_T(X^T) \rightrightarrows \prod_{\alpha\in E} \mathcal{F}_{T_\alpha}.$$

For the injectivity claim, we first claim that $\mathcal{F}_T(X)^{tT} \simeq \mathcal{F}_T(X^T)^{tT}$. (This is a version of Atiyah-Bott localization.) Since X is generated by finite colimits from T-orbits T/T', it suffices to prove this claim when X is of that form. Then $\mathcal{F}_T(T/T') \simeq \mathcal{F}_{T'}(*) = q_*\mathcal{O}_{\mathcal{M}_{T'}}$; this has zero Tate construction if $T' \neq T$. On the other hand, $X^T = \emptyset$ if $T' \neq T$, so $\mathcal{F}_T(X^T)^{tT} = 0$ as desired. If $T' = T$, then $X^T = *$, so that both sides are simply A^{tT}.

Note that $\mathcal{F}_T(X^T)^{tT} \simeq \mathcal{F}_T(X^T) \otimes_A A^{tT}$. Since $\mathcal{F}_T(X)^{tT} \simeq \mathcal{F}_T(X) \otimes_{\mathcal{O}_{\mathcal{M}_T}} A^{tT}$ is a localization, it suffices to prove that the map $\mathcal{F}_T(X) \to \mathcal{F}_T(X)^{tT}$ induces an injection on homotopy. For this, it suffices to prove that $\mathcal{F}_T(X)$ is a free $\mathcal{O}_{\mathcal{M}_T}$-module. This is a consequence of the assumptions on X.

To prove the statement about the equalizer diagram, the key case is when $X = S^W$ for a T-representation W; the general case is obtained by induction on the stratification of X. Let $\lambda_1, \cdots, \lambda_n$ be the weights of W, so that $X = \bigotimes_{i=1}^n S^{\lambda_i}$. Therefore, X is the quotient of $\prod_{i=1}^n S^{\lambda_i}$ by its $(2n-2)$-skeleton. Using this observation, it is not difficult to reduce to the case when $W = \lambda$ is a character of T.

In this case, $X = S^\lambda$ has T-fixed points given by $\{0, \infty\}$. There is a cofiber sequence $S(\lambda) \to * \to S^\lambda$, which induces a pushout square

$$\begin{tikzcd} S(\lambda)_+ \arrow[r] \arrow[d] & S^0 = \{\infty\}_+ \arrow[d] \\ S^0 = \{0\}_+ \arrow[r] & S^\lambda_+. \end{tikzcd}$$

Therefore, we get an equalizer diagram

$$\mathcal{F}_T(S^\lambda) \to \mathcal{O}_{\mathcal{M}_T} \rightrightarrows \mathcal{F}_T(S(\lambda)).$$

However, if $T_\lambda = \ker(\lambda : T \to \mathbf{G}_m)$, then $S(\lambda) \simeq T/T_\lambda$, so that $\mathcal{F}_T(S(\lambda)) \simeq q_*\mathcal{O}_{\mathcal{M}_{T_\lambda}}$. It follows that $\mathcal{F}_T(S^\lambda)$ is the fiber of the map $\mathcal{O}_{\mathcal{M}_T} \oplus \mathcal{O}_{\mathcal{M}_T} \to q_*\mathcal{O}_{\mathcal{M}_{T_\lambda}}$ given by the following composite:

$$\mathcal{O}_{\mathcal{M}_T} \oplus \mathcal{O}_{\mathcal{M}_T} \xrightarrow{(x,y)\mapsto x-y} \mathcal{O}_{\mathcal{M}_T} \to q_*\mathcal{O}_{\mathcal{M}_{T_\lambda}}.$$

However, the map $\mathcal{O}_{\mathcal{M}_T} \to q_*\mathcal{O}_{\mathcal{M}_{T_\lambda}}$ is precisely given by quotienting by the ideal $\mathcal{I}_\lambda$ (by Notation 2.1.10). Therefore, $\mathcal{F}_T(S^\lambda)$ is described by the claimed equalizer diagram. ∎

Remark 2.3.4. Informally, the image on homotopy sheaves of the map $\mathcal{F}_T(X) \to \mathrm{Map}(V, \mathcal{O}_{\mathcal{M}_T}) \simeq \mathcal{F}_T(X^T)$ consists of those $f \in \pi_* \mathcal{O}_{\mathcal{M}_T}^V$ such that $f(x) \equiv f(y) \pmod{\mathcal{I}_\alpha}$ for every edge $\alpha : x \to y$ in Γ. Here, $\mathcal{I}_\alpha$ is as in Notation 2.1.10.

2.4. **Affine blowups.**

Recollection 2.4.1. Let R be an $\mathbf{E}_n$-ring, and let $y \in \pi_{2*}R$. Then one can form the localization $R[\frac{1}{y}]$: this is an $\mathbf{E}_n$-algebra under R such that the $\mathbf{E}_n$-map $R \to R[\frac{1}{y}]$ induces the localization map on $\pi_*(R)$ at the level of homotopy groups. The universal property of $R[\frac{1}{y}]$ is the following: for any $\mathbf{E}_n$-algebra A, the map $R \to R[\frac{1}{y}]$ induces a fully faithful map $\mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_n}}(R[\frac{1}{y}], A) \to \mathrm{Map}_{\mathrm{Alg}_{\mathbf{E}_n}}(R, A)$, whose essential image consists of those $\mathbf{E}_n$-maps $f : R \to A$ such that $f(y) \in \pi_*(A)$ is invertible.

We will discuss a slight generalization of this notion: given $x \in \pi_{2*}R$ of the same degree as y, we will construct an $\mathbf{E}_n$-algebra $R[\frac{x}{y}]$ under R such that the $\mathbf{E}_n$-map $R \to R[\frac{x}{y}]$ induces the eponymous map on $\pi_*(R)$ at the level of homotopy groups.

Construction 2.4.2. Let $\mathcal{C}$ be a presentably symmetric monoidal stable ∞-category. Let $X \in \mathrm{Mod}_{\mathbf{1}[z]}(\mathcal{C})$, where $\mathbf{1}[t] = S^0[z] \otimes \mathbf{1}$ with $|z| = 0$. Therefore, X is equipped with an endomorphism $z : X \to X$. There is a symmetric monoidal functor $\mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C}) \to \mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C}^{\mathrm{gr}})$ which sends $(X, z : X \to X)$ to the graded object $\bigoplus_{n \in \mathbf{Z}} X(n)$, and where t acts by the map $z : X(1) \to X$. There is an equivalence $\mathrm{Rees} : \mathcal{C}^{\mathrm{fil}} \xrightarrow{\sim} \mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C}^{\mathrm{gr}})$ of symmetric monoidal ∞-categories, so we obtain a symmetric monoidal functor $\mathrm{F}_t^\star : \mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C}) \to \mathcal{C}^{\mathrm{fil}}$, which sends $(X, z : X \to X)$ to the filtered object $\cdots \xrightarrow{z} X(1) \xrightarrow{z} X(0) \to \cdots$. We will only focus on the functor $\mathrm{F}_t^{\star \geq 0} : \mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C}) \to \mathcal{C}^{\mathrm{fil}}$ obtained by composing F with the restriction along $\mathbf{N} \to \mathbf{Z}$ (i.e., restrict to the nonnegative part of the filtered object $\mathrm{F}_t^\star$).

Remark 2.4.3. Let $\mathbf{1}[\sigma_d]$ denote $S^0[\sigma_d] \otimes \mathbf{1}$ with $|\sigma_d| = 2d$. By shearing, we obtain an $\mathbf{E}_n$-monoidal functor $\mathrm{Mod}_{\mathbf{1}[\sigma_d]}(\mathcal{C}) \to \mathcal{C}^{\mathrm{fil}}$, which sends $(X, \sigma_d : \Sigma^{2d} X \to X)$ to the filtered object $\cdots \to \Sigma^{2d} X(1) \xrightarrow{\sigma_d} X(0) \xrightarrow{\sigma_n} \cdots$.

Lemma 2.4.4. *Let $R \in \mathrm{Alg}_{\mathbf{E}_n}(\mathcal{C})$, and fix a map $x : \mathbf{1} \to R$, so that x extends to an $\mathbf{E}_1$-map $\mathrm{Free}_{\mathbf{E}_1}(\mathbf{1}) = \mathbf{1}[t] \to R$ with $|t| = 0$. If x lifts to an $\mathbf{E}_n$-map $\mathbf{1}[t] \to R$, then the resulting object $(R, x) \in \mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C})$ defines an $\mathbf{E}_{n-1}$-algebra in $\mathrm{Mod}_{\mathbf{1}[t]}(\mathcal{C})$, i.e., $\mathrm{F}_x^\star R$ admits the structure of an $\mathbf{E}_{n-1}$-algebra in $\mathcal{C}^{\mathrm{fil}}$.*

It is not always possible to extend the $\mathbf{E}_1$-map $\mathbf{1}[t] \to R$ to a map of $\mathbf{E}_n$-algebras as in Lemma 2.4.4, but this does happen in some cases. See Appendix B for some negative results.

Example 2.4.5. Let R be a *simplicial commutative* $\mathbf{Z}$-algebra. Then any class $x \in \pi_0(R)$ extends to an $\mathbf{E}_\infty$-map $\mathbf{Z}[t] \to R$ sending $t \mapsto x$, so that $\mathrm{F}_x^\star R \in \mathrm{CAlg}(\mathrm{Mod}_{\mathbf{Z}}^{\mathrm{fil}})$. (This is not necessarily true if R is merely an $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra.)

Example 2.4.6. Let R be an $\mathbf{E}_2$-ring whose homotopy is concentrated in even degrees. Then any class $x \in \pi_0(R)$ extends to an $\mathbf{E}_2$-map $S^0[t] \to R$ sending $t \mapsto x$, so that $\mathrm{F}_x^\star R \in \mathrm{Alg}(\mathrm{Sp}^{\mathrm{fil}})$. Indeed, since $\mathrm{Bar}^{[2]}(\mathbf{N}) \simeq \mathbf{C}P^\infty$ admits a cell structure with one cell in each even dimension, $S^0[t]$ admits an $\mathbf{E}_2$-cell structure with one cell in each even dimension. Since R is assumed to have homotopy only in even degrees, the obstructions to extending $x : S^0 \to R$ along the $\mathbf{E}_2$-skeleta of $S^0[t]$ all vanish. (See [Dev22b, Lemma 3.2.8] for a related statement.)

We can now define affine blowups.

Construction 2.4.7. Let R be an $\mathbf{E}_n$-algebra. If $I = (x_1, \cdots, x_j)$ is a finitely generated ideal in $\pi_0(R)$ such that each $\mathrm{F}_{x_i}^\star R$ admits the structure of an $\mathbf{E}_n$-algebra in $\mathrm{Sp}^{\mathrm{fil}}$, then define $\mathrm{F}_I^\star R$ to be the $\mathbf{E}_n$-algebra $\bigotimes_{i=1}^j \mathrm{F}_{x_i}^\star R$ in $\mathrm{Sp}^{\Lambda\text{-fil}}$ with $\Lambda = \mathbf{Z}^j$. Define the *blowup algebra* $\mathrm{Bl}_I(R)$ to be the $\mathbf{E}_n$-algebra in $\mathrm{Mod}_{S^0[t_1, \cdots, t_j]}(\mathrm{Sp}^{\mathrm{gr}})$ given by $\mathrm{Rees}(\mathrm{F}_I^\star R)$.

Let $a \in I \subseteq \pi_0(R)$ be an element, and regard a as defining a class $\widetilde{a} : S^0(1) \to \mathrm{Bl}_I(R)$ in degree 0 and weight 1. The *affine blowup algebra* $R[\frac{I}{a}]$ is the $\mathbf{E}_n$-algebra in $\mathrm{Mod}_{S^0[t_1, \cdots, t_j]}(\mathrm{Sp}^{\mathrm{gr}})$ given by $\mathrm{Bl}_I(R)[\frac{1}{\widetilde{a}}]$. We will often also write $R[\frac{I}{a}]$ for the underlying $\mathbf{E}_n$-algebra.

Example 2.4.8. Let A be an $\mathbf{E}_n$-algebra, and let $R = A[t_1, \cdots, t_j]$. Let I denote the ideal $(t_1, \cdots, t_j)$, so that $\mathrm{Bl}_I R$ admits the structure of an $\mathbf{E}_n$-algebra in $\mathrm{Mod}_{S^0[x_1,\cdots,x_j]}(\mathrm{Sp}^{\mathrm{gr}})$. In other words, $\mathrm{Bl}_I(R) \simeq A[t_1, \cdots, t_j, x_1, \cdots, x_j]/(x_i t_j = t_i x_j)$ where the t_j live in weight 1. Let $a = t_1$; then

$$R[\tfrac{I}{t_1}] \simeq R[y_1, \cdots, y_j]/(t_1 y_j = t_j) \simeq R[\tfrac{t_2}{t_1}, \cdots, \tfrac{t_j}{t_1}].$$

Example 2.4.9. Let R be an $\mathbf{E}_2$-ring, let $I = (x_1, \cdots, x_j) \subseteq \pi_0(R)$, and let $a \in I$. Then Example 2.4.5 and Example 2.4.6 imply that $R[\frac{I}{a}]$ is a well-defined:

(a) $\mathbf{E}_n$-algebra when R is an $\mathbf{E}_{n+1}$-ring such that the $\mathbf{E}_1$-map $x_1, \cdots, x_j : S^0[t_1, \cdots, t_j] \to R$ extends to an $\mathbf{E}_{n+1}$-map.
(b) $\mathbf{E}_\infty$-$\mathbf{Z}$-algebra when R is a simplicial commutative $\mathbf{Z}$-algebra;
(c) $\mathbf{E}_1$-algebra when $\pi_*(R)$ is concentrated in even degrees.

If $I = (x, y)$ is generated by two elements and $a = x$, then we write $R[\frac{y}{x}]$ to denote $R[\frac{I}{a}]$.

Proposition 2.4.10. *Let R be an $\mathbf{E}_2$-ring, and assume that $R[\frac{y}{x}]$ is an $\mathbf{E}_1$-algebra. If $A \in \mathrm{Alg}(\mathrm{Sp})$, then the functor $\mathrm{Map}_{\mathrm{Alg}(\mathrm{Sp})}(R[\frac{y}{x}], A) \to \mathrm{Map}_{\mathrm{Alg}(\mathrm{Sp})}(R, A)$ is fully faithful, with essential image consisting of those $\mathbf{E}_1$-maps $f : R \to A$ such that $f(y) \equiv 0 \pmod{f(x)}$ in $\pi_0(A)$.*

Example 2.4.11. Let A be an $\mathbf{E}_2$-ring, M be an A-module, and let S be a space. If $x \in \pi_0 A$ and $y \in \pi_0 A[S]$, then an A-linear map $f : A[S][\frac{y-1}{x}] \to M$ yields an A-linear map $f : A[S] \to M$ (i.e., a map $f : S \to \Omega^\infty M$ of spaces) such that $f(yz) \equiv f(z) \pmod{x}$ for all $z \in \pi_0 A[S]$. If A is a classical commutative ring, M is a flat A-module, and S is a discrete set, then this determines $A[S][\frac{y-1}{x}]$ as an A-module.

3. Equivariant cohomology of the affine Grassmannian

3.1. **Cohomology of Kac-Moody flag varieties.** Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A and an oriented commutative A-group $\mathbf{G}$.

Observation 3.1.1. Let $\mathcal{G}$ be a Kac-Moody group, and let $\mathcal{P} \subseteq \mathcal{G}$ be a parabolic subgroup associated to a subset $J \subseteq \Delta$ of simple roots. Let $T = T_\mathcal{G}/Z(\mathcal{G})$ denote the torus of $\mathcal{G}/Z(\mathcal{G})$, and let W be the Weyl group associated to $\mathcal{G}$. Let $W_\mathcal{P}$ denote the subgroup of W generated by s_{α_j} for $\alpha_j \in J$, and let W^J denote the set of minimal-length coset representatives in $W_\mathcal{G}/W_\mathcal{P}$.

Then $(\mathcal{G}/\mathcal{P})^T \cong W^\mathcal{P}$, and the Schubert decomposition $\mathcal{G}/\mathcal{P} = \coprod_{w\in W^\mathcal{P}} \mathcal{B}\dot{w}\mathcal{P}/\mathcal{P}$ is a T-invariant stratification, where $\overline{w} = \dot{w}\mathcal{P}/\mathcal{P}$ is the unique T-fixed point in the cell $\mathcal{B}\dot{w}\mathcal{P}/\mathcal{P}$. We claim that $\mathcal{G}/\mathcal{P}$ satisfies the hypotheses of Setup 2.3.1. Clearly, condition (a) is satisfied. For condition (b), observe that the tangent space to $\mathcal{B}\overline{w}$ at $\overline{w}$ is

$$T_{\overline{w}}\mathcal{B}\dot{w}\mathcal{P}/\mathcal{P} = \mathfrak{b}/(\mathfrak{b} \cap w \cdot \mathfrak{p}) = \bigoplus_{\alpha\in\Phi^+ - w\Phi^+(\mathfrak{p})} \mathfrak{g}_\alpha,$$

where each $\mathfrak{g}_\alpha$ is 1-dimensional. The weights are therefore all distinct, so condition (b) in Setup 2.3.1 is satisfied. For condition (c), let $\alpha \in \Phi^+ - w\Phi^+(\mathfrak{p})$, and let $i_\alpha : \mathrm{SL}_2 \to \mathcal{G}$ denote the associated subgroup. The closure of $\mathcal{B}_\alpha\overline{w}$ is $\mathrm{SL}_2\overline{w} = \mathbf{P}^1$, where the point at 0 corresponds to $\overline{w}$, and the point at ∞ corresponds to $\overline{s_\alpha w}$. Then the GKM graph Γ of $\mathcal{G}/\mathcal{P}$ has vertices $W^\mathcal{P}$ and edges $w \to s_\alpha w$ labeled by $s_\alpha \in W_\mathcal{G}$.

Warning 3.1.2. In the following, the reader should replace the symbol "$\mathcal{F}_T(\mathcal{G}/\mathcal{P})$" by $\mathcal{F}_T(X_{\leq w})$ where $X_{\leq w}$ is a Schubert cell in $\mathcal{G}/\mathcal{P}$. In this case, $X_{\leq w}$ is a finite CW-complex, so that $\mathcal{F}_T(X_{\leq w})$ is a *perfect* $\mathcal{O}_{\mathcal{M}_T}$-module. This implies that the T-equivariant *homology* $\mathcal{F}_T(X_{\leq w})^\vee$ is the $\mathcal{O}_{\mathcal{M}_T}$-linear dual of $\mathcal{F}_T(X_{\leq w})$; note that this is not true of $\mathcal{F}_T(\mathcal{G}/\mathcal{P})$ when the Kac-Moody group is not of finite type. (In general, homology is a predual of cohomology, but the linear dual of cohomology does not recover homology in the non-finite case.) We *define* $\mathcal{F}_T(\mathcal{G}/\mathcal{P})^\vee$ as the inverse limit of $\mathcal{F}_T(X_{\leq w})^\vee$. We trust the reader to make the appropriate modifications below (which we have not done to avoid an overbearance of notation), so that the calculation of the T-equivariant homology $\mathcal{F}_T(\mathrm{Gr}_G)^\vee$ in Theorem 3.2.10 by taking the linear dual of $\mathcal{F}_T(\mathrm{Gr}_G)$ is indeed valid.

Since $\mathcal{G}/\mathcal{P}$ satisfies the hypotheses of Setup 2.3.1 by Observation 3.1.1, we may apply Theorem 2.3.3 to calculate $\mathcal{F}_T(\mathcal{G}/\mathcal{P})$. See [LSS10] for a related discussion.

Theorem 3.1.3. *There is an equalizer diagram*

$$\mathcal{F}_T(\mathcal{G}/\mathcal{P}) \to \mathrm{Map}(W^\mathcal{P}, \mathcal{O}_{\mathcal{M}_T}) \rightrightarrows \prod_{\alpha: w\to s_\alpha w} \mathcal{O}_{\mathcal{M}_{T_\alpha}},$$

where the two maps are given by restriction and applying s_α to $W^\mathcal{P}$. Therefore, $\pi_0\mathcal{F}_T(\mathcal{G}/\mathcal{P})$ is the sub-$\pi_0\mathcal{O}_{\mathcal{M}_T}$-algebra of $\mathrm{Map}(W^\mathcal{P}, \pi_0\mathcal{O}_{\mathcal{M}_T})$ consisting of those maps $f : W^\mathcal{P} \to \pi_0\mathcal{O}_{\mathcal{M}_T}$ such that

$$f(s_\alpha w) \equiv f(w) \pmod{\mathcal{I}_\alpha} \text{ for all } w \in W^\mathcal{P}, \alpha \in \Phi. \tag{3}$$

Motivated by Theorem 3.1.3, we may define an algebraic generalization of $\pi_0\mathcal{F}_T(\mathcal{G}/\mathcal{P})$ as follows.

Construction 3.1.4. Let (W, S) be a Coxeter system, and let $V = \mathbf{R}^S$ denote the associated geometric representation. For $s \in S$, let α_s denote the associated vector, let $\Phi = \{w(\alpha_s) | s \in S, w \in W\}$ be the set of roots, and let $\Phi^+ \subseteq \Phi$ denote the set of positive roots. Let $\Lambda = \mathbf{Z}\Phi \subseteq V$ denote the associated root lattice. Fix a smooth 1-dimensional affine group scheme $\mathbf{G}_0$ over a commutative ring R, and let $\mathbf{G}_{0,T} = \mathrm{Hom}(\Lambda^\vee, \mathbf{G}_0)$. Given a character λ, let c_λ denote a function which cuts out the closed subscheme $\mathbf{G}_{\ker(\lambda)} \hookrightarrow \mathbf{G}_{0,T}$. Define $\mathbf{K}$ to be the sub-$\mathcal{O}_{\mathbf{G}_{0,T}}$-algebra of $\mathrm{Map}(W, \mathcal{O}_{\mathbf{G}_{0,T}})$ consisting of those maps $f : W \to \mathcal{O}_{\mathbf{G}_{0,T}}$ satisfying (3), i.e., such that $f(s_\alpha w) \equiv f(w) \pmod{c_\alpha}$ for $\alpha \in \Phi$ and $w \in W$.

Remark 3.1.5. Note that if λ is a character, then the function c_λ on $\mathcal{M}_T$ is given by the T-equivariant Thom class of the representation of T given by $\lambda : T \to \mathbf{G}_m^{\mathrm{rot}}$. Note that c_λ generates $\mathcal{I}_\lambda$.

Lemma 3.1.6. *Let $s_\alpha \in W$, and let $T_\alpha = \ker(\alpha) \subseteq T$. Then we have the following commutative diagram of R-schemes (where the non-vertical arrows are closed immersions):*

$$\begin{tikzcd} \mathbf{G}_{0,T_\alpha} \arrow[r, "q"] \arrow[dr, "q"'] & \mathbf{G}_{0,T} \arrow[d, "s_\alpha"] \\ & \mathbf{G}_{0,T}; \end{tikzcd}$$

informally, $s_\alpha \equiv 1 \pmod{\mathcal{I}_\alpha}$.

Proof. This follows from the fact that the character lattice of T_α is the quotient of $\mathbb{X}^*(T)$ by the rank 1 sublattice generated by α; therefore, if $\chi \in \mathbb{X}^*(T)$, then $s_\alpha \chi|_{T_\alpha} = \chi|_{T_\alpha}$. $\square$

Theorem 3.1.3 and Example 2.4.11 implies the following:

Corollary 3.1.7. *Suppose $\mathbf{G}$ is affine. Then there is an equivalence $\mathcal{F}_T(\mathcal{G}/\mathcal{P})^\vee \simeq \mathcal{O}_{\mathcal{M}_T}[W^{\mathcal{P}}, \frac{s_\alpha - 1}{c_\alpha}, \alpha \in \Phi]$ of $\mathcal{O}_{\mathcal{M}_T}$-modules.*

Recall that if $w \in W$, then $\mathrm{inv}(w) \subseteq \Phi^+$ denotes the set of positive roots α such that $s_\alpha w < w$. The following is then the analogue of [LSS10, Lemma 2.3, Lemma 2.5, Proposition 2.6].

Proposition 3.1.8. *Suppose that $\mathbf{G}$ is affine. In Construction 3.1.4, $\mathbf{K}$ is a free $\mathcal{O}_{\mathbf{G}_{0,T}}$-module spanned by functions $\psi_w : W \to \mathcal{O}_{\mathbf{G}_{0,T}}$ for $w \in W$, where ψ_w is uniquely characterized by the property that it satisfies* (3) *and the following two properties:*

$$\psi_w(v) = 0 \text{ if } v < w,$$
$$\psi_w(w) = \prod_{\alpha \in \mathrm{inv}(w)} c_\alpha.$$

Proof. The two stated conditions define ψ_w on the interval $[1, w] \subseteq W$. We will now define an extension of ψ_w to the whole of W. We will in fact prove the following more general claim by induction on $\ell(w)$:

($*$) Let $w \in W$, and let $[1, w]^\circ = [1, w] - \{w\}$. Then any function $\psi : [1, w]^\circ \to \mathcal{O}_{\mathbf{G}_{0,T}}$ satisfying (3) extends to a function $[1, w] \to \mathcal{O}_{\mathbf{G}_{0,T}}$ satisfying (3).

To see this, write $w = s_{i_1} \cdots s_{i_n}$, let $\alpha = \alpha_{i_1}$, and let $w' = s_\alpha w$ (so that $w' < w$). Consider the restriction of ψ to $[1, w']^\circ$, so that ψ itself is an extension to $[1, w']$. Define $\psi' : [1, w']^\circ \to \mathcal{O}_{\mathbf{G}_{0,T}}$ by the formula $\psi'(v) = s_\alpha \psi(s_\alpha v)$. Then ψ' also satisfies (3): indeed, if β is another root, then $\psi'(s_\beta v) \equiv \psi'(v) \pmod{\mathcal{I}_\beta}$ if and only if $\psi(s_\alpha s_\beta v) \equiv \psi(s_\alpha v) \pmod{s_\alpha \mathcal{I}_\beta}$. However, $s_\alpha \mathcal{I}_\beta = \mathcal{I}_{s_\alpha(\beta)}$, while $s_\alpha s_\beta = s_{s_\alpha(\beta)} s_\alpha$. The claim therefore follows from the assumption that ψ satisfies (3).

Since $w' < w$, the inductive hypothesis says that ψ' extends to a function $\psi' : [1, w'] \to \mathcal{O}_{\mathbf{G}_{0,T}}$ which satisfies (3). If $v \in [1, w']^\circ$, then

$$\psi(v) - \psi'(v) = \psi(v) - s_\alpha \psi(s_\alpha v) \equiv (1 - s_\alpha)\psi(v) \pmod{\mathcal{I}_\alpha}.$$

By Lemma 3.1.6, we see that $\psi(v) - \psi'(v) \equiv 0 \pmod{\mathcal{I}_\alpha}$, so we may define a function $p_v \in \mathcal{O}_{\mathbf{G}_{0,T}}$ by the formula $\frac{\psi(v) - \psi'(v)}{c_\alpha}$. If $\beta \in \Phi^+$ is such that $s_\beta w' < w'$, then:

$$\begin{aligned} \psi(w') - \psi'(w') &\equiv \psi(s_\beta w') - \psi'(s_\beta w') \pmod{\mathcal{I}_\beta} \\ &= c_\alpha p_{s_\beta w'} \pmod{\mathcal{I}_\beta}. \end{aligned}$$

In particular, there is a function $p_{w'} \in \mathcal{O}_{\mathbf{G}_{0,T}}$ such that

$$\psi(w') - \psi'(w') \equiv c_\alpha p_{w'} \pmod{\mathcal{I}_\beta}$$

for all $\beta \in \Phi^+$ such that $s_\beta w' < w'$, i.e., $\beta \in \mathrm{inv}(w')$. In particular,

$$\psi(w') - \psi'(w') \equiv c_\alpha p_{w'} \pmod{\prod_{\beta \in \mathrm{inv}(w')} \mathcal{I}_\beta}. \tag{4}$$

Note that $s_\alpha \mathrm{inv}(w')$ is the set of $\beta \in \Phi^+ - \{\alpha\}$ such that $s_\beta w' < w'$. Define

$$\psi(w) = s_\alpha \psi'(w') + x \prod_{\beta \in s_\alpha \mathrm{inv}(w')} c_\beta$$

for some x that we will determine in a moment. We check that ψ satisfies (3). Let $\alpha' \in \Phi^+$ be such that $s_{\alpha'} w < w$. Then:

(a) If $\alpha' = \alpha$, then

$$\begin{aligned}\psi(w) - \psi(s_\alpha w) &= s_\alpha \psi'(w') - \psi(w') + x \prod_{\beta \in s_\alpha \mathrm{inv}(w')} c_\beta \\ &\equiv s_\alpha(\psi'(w') - \psi(w')) + x \prod_{\beta \in s_\alpha \mathrm{inv}(w')} c_\beta \pmod{\mathcal{I}_\alpha}\end{aligned}$$

However, (4) implies that

$$s_\alpha(\psi(w') - \psi'(w')) \equiv c_{-\alpha} s_\alpha(p_{w'}) \pmod{\prod_{\beta \in s_\alpha \mathrm{inv}(w')} \mathcal{I}_\beta}$$

Therefore, taking x to be the negative of the residue of $s_\alpha(\psi(w') - \psi'(w')) - c_{-\alpha} s_\alpha(p_{w'})$ modulo $\prod_{\beta \in s_\alpha \mathrm{inv}(w')} \mathcal{I}_\beta$, we see that

$$\psi(w) - \psi(s_\alpha w) \equiv c_{-\alpha} s_\alpha(p_{w'}) \equiv 0 \pmod{\mathcal{I}_\alpha},$$

as desired.

(b) If $\alpha' \neq \alpha$, then $\alpha' \in s_\alpha \mathrm{inv}(w')$. Then, we have

$$\begin{aligned}\psi'(w') &\equiv \psi'(s_{s_\alpha(\alpha')} w') \pmod{\mathcal{I}_{s_\alpha(\alpha')}} \\ &= s_\alpha \psi(s_\alpha s_{s_\alpha(\alpha')} s_\alpha w) \pmod{\mathcal{I}_{s_\alpha(\alpha')}} \\ &= s_\alpha \psi(s_{\alpha'} w) \pmod{\mathcal{I}_{s_\alpha(\alpha')}}.\end{aligned}$$

In particular, $s_\alpha \psi'(w') \equiv \psi(s_{\alpha'} w) \pmod{\mathcal{I}_{\alpha'}}$. But this implies that

$$\begin{aligned}\psi(w) - \psi(s_{\alpha'} w) &\equiv s_\alpha \psi'(w') - \psi(s_{\alpha'} w) \pmod{\mathcal{I}_{\alpha'}} \\ &\equiv 0 \pmod{\mathcal{I}_{\alpha'}},\end{aligned}$$

as desired.

This finishes the proof of $(*)$.

To finish the proof of the proposition, note that the two conditions on ψ_w specify it on $[1, w]$, and hence on the subset of W consisting of elements of length $\ell(w)$. By $(*)$, we may inductively extend ψ_w to the subset of W consisting of elements of length $\geq \ell(w)$, and hence to all of W. It remains to show that any $\psi \in \mathrm{Map}(W, \mathcal{O}_{\mathbf{G}_{0,T}})$ satisfying (3) can be written as a $\mathcal{O}_{\mathbf{G}_{0,T}}$-linear combination of the ψ_w; see the second half of [LSS10, Proposition 2.6] for the following argument.

Let $\mathrm{Supp}(\psi)$ denote the subset of $w \in W$ such that $f(\psi) \neq 0$. Let $v \in \mathrm{Supp}(\psi)$ be minimal. If $\alpha \in \mathrm{inv}(v)$ (so $s_\alpha v < v$), then $\psi(v) \equiv \psi(s_\alpha v) = 0 \pmod{\mathcal{I}_\alpha}$. This implies that $\psi(v) \equiv 0 \pmod{\psi_v(v)}$. Define $\psi' : W \to \pi_0 \mathcal{O}_{\mathbf{G}_{0,T}}$ by $\psi'(w) = \psi(w) - \frac{\psi(v)}{\psi_v(v)} \psi_v(w)$; then ψ' satisfies (3) (since ψ and ψ_v do). By construction, $v \notin \mathrm{Supp}(\psi')$, and $\mathrm{Supp}(\psi') - \mathrm{Supp}(\psi)$ consists of elements which are strictly larger than v. Therefore, we may repeat this argument for ψ', and induct; this yields the desired result. $\square$

3.2. **The affine Grassmannian.**

Setup 3.2.1. Fix notation as in Notation 1.1.17, and assume that G is semisimple. Then we have an associated affine root datum: the affine simple roots are $\Delta_{\mathrm{aff}} = \Delta \cup \{0\}$, and the affine weight lattice is given by $\mathbf{Z}K \oplus \bigoplus_{\alpha_i \in \Delta_{\mathrm{aff}}} \mathbf{Z}\alpha_i$. (In particular, we denote the affine root by α_0.) Thus the associated Kac-Moody algebra is $\hat{\mathfrak{g}} = \mathfrak{g}(\!(t)\!) \oplus \mathbf{C}\alpha_0 \oplus \mathbf{C}K$, where K is the central class, and α_0 is the scaling factor. Let $\mathcal{G}$ denote the associated Kac-Moody group, and let $W^{\mathrm{aff}} = \Lambda^\vee \rtimes W$ denote associated Weyl group. If $\lambda^\vee \in \Lambda^\vee$, we write $t_{\lambda^\vee}$ to denote the associated element of W^{aff}. If $\alpha + n\alpha_0$ is an affine root and $x \in \mathfrak{t}$, then

$$s_{\alpha + n\alpha_0}(x) = x - (\langle x, \alpha \rangle + n)\alpha^\vee = s_\alpha(x) + n\alpha^\vee.$$

Let $\mathcal{B}$ denote the Iwahori subgroup, and T_{aff} the maximal torus of $\mathcal{G}$. Then $\mathcal{G}/\mathcal{B}$ is the affine flag variety Fl_G; similarly, Gr_G is the Kac-Moody flag variety associated to the subset $\Delta \subseteq \Delta_{\mathrm{aff}}$. Up to keeping track of the central torus, we may view $\mathcal{G}$ as $G(\!(t)\!)$, and $\mathcal{B}$ as the Iwahori I. Thus $T = T^{\mathrm{aff}} \cap G$ is the maximal torus of G. Let $\widetilde{T}$ denote the extended torus $T \times \mathbf{G}_m^{\mathrm{rot}}$ (where $\mathbf{G}_m^{\mathrm{rot}}$ is the loop rotation torus); we may identify its Lie algebra $\widetilde{\mathfrak{t}}$ with $\mathfrak{t} \oplus \mathbf{C}\alpha_0$.

Remark 3.2.2. Let $\alpha \in \Phi$ and $n \in \mathbf{Z}$. Then $n\alpha_0$ is the $\mathbf{G}_m^{\mathrm{rot}}$-representation of weight n. Write $[n](\hbar_A)$ to denote the local section of $\pi_0\mathcal{O}_{\mathcal{M}_{\mathbf{G}_m^{\mathrm{rot}}}} \cong \pi_0\mathcal{O}_{\mathbf{G}}$ given by the Thom class of the line bundle $\mathcal{F}_{\mathbf{G}_m^{\mathrm{rot}}}(S^{n\alpha_0})$ on $\mathcal{M}_{\mathbf{G}_m^{\mathrm{rot}}} \simeq \mathbf{G}$, so that it is $\hbar_A +_{\mathbf{G}} \cdots +_{\mathbf{G}} \hbar_A$. Note that $\alpha + n\alpha_0$ defines an ideal sheaf $\mathcal{I}_{\alpha+n\alpha_0} \subseteq \pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}} = \pi_0\mathcal{O}_{\mathcal{M}_T} \otimes_{\pi_0 A} \pi_0\mathcal{O}_{\mathbf{G}}$.

Theorem 3.1.3 gives an explicit description of $\pi_0\mathcal{F}_{T^{\mathrm{aff}}}(\mathrm{Fl}_G)$ and $\pi_0\mathcal{F}_{T^{\mathrm{aff}}}(\mathrm{Gr}_G)$. Using that

$$(\mathrm{Fl}_G)^T = (\mathrm{Fl}_G)^{\widetilde{T}} = W^{\mathrm{aff}}$$
$$(\mathrm{Gr}_G)^T = (\mathrm{Gr}_G)^{\widetilde{T}} = W^{\mathrm{aff}}/W \cong \Lambda^\vee,$$

this further immediately specializes to the following explicit description of $\pi_0\mathcal{F}_{\widetilde{T}}(\mathrm{Fl}_G)$ and $\pi_0\mathcal{F}_{\widetilde{T}}(\mathrm{Gr}_G)$:

Corollary 3.2.3. *The following statements are true:*

(a) *We may identify* $\pi_0\mathcal{F}_{\widetilde{T}}(\mathrm{Fl}_G) \cong \pi_0\mathcal{F}_{\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Fl}_G/I)$ *with* $\mathbf{K}$ *from Construction 3.1.4, i.e., as the sub-*$\pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}}$*-algebra of* $\mathrm{Map}(W^{\mathrm{aff}}, \pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}})$ *consisting of those maps* $f : W^{\mathrm{aff}} \to \pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}}$ *such that*

$$f(s_{\alpha+n\alpha_0}(w)) \equiv f(w) \pmod{\mathcal{I}_{\alpha+n\alpha_0}} \tag{5}$$

for all $w \in W^{\mathrm{aff}}, \alpha \in \Phi, n \in \mathbf{Z}$.

(b) *We may identify* $\pi_0\mathcal{F}_{\widetilde{T}}(\mathrm{Gr}_G) \cong \pi_0\mathcal{F}_{\mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G/I)$ *as the sub-*$\pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}}$*-algebra of* $\mathrm{Map}(\Lambda^\vee, \pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}})$ *consisting of those maps* $f : \Lambda^\vee \to \pi_0\mathcal{O}_{\mathcal{M}_{\widetilde{T}}}$ *such that*

$$f(s_{\alpha+n\alpha_0}(\lambda)) \equiv f(\lambda) \pmod{\mathcal{I}_{\alpha+n\alpha_0}} \tag{6}$$

for all $\lambda \in \Lambda^\vee, \alpha \in \Phi, n \in \mathbf{Z}$.

Corollary 3.2.4. *The following statements are true:*

(a) *We may identify* $\pi_0\mathcal{F}_T(\mathrm{Fl}_G)$ *as the sub-*$\pi_0\mathcal{O}_{\mathcal{M}_T}$*-algebra of* $\mathrm{Map}(W^{\mathrm{aff}}, \pi_0\mathcal{O}_{\mathcal{M}_T})$ *consisting of those maps* $f : W^{\mathrm{aff}} \to \pi_0\mathcal{O}_{\mathcal{M}_T}$ *such that*

$$f(s_{\alpha+n\alpha_0}(w)) \equiv f(w) \pmod{\mathcal{I}_\alpha} \tag{7}$$

for all $w \in W^{\mathrm{aff}}, \alpha \in \Phi, n \in \mathbf{Z}$.

(b) *We may identify* $\pi_0\mathcal{F}_T(\mathrm{Gr}_G)$ *as the sub-*$\pi_0\mathcal{O}_{\mathcal{M}_T}$*-algebra of* $\mathrm{Map}(\Lambda^\vee, \pi_0\mathcal{O}_{\mathcal{M}_T})$ *consisting of those maps* $f : \Lambda^\vee \to \pi_0\mathcal{O}_{\mathcal{M}_T}$ *such that*

$$f(s_{\alpha+n\alpha_0}(\lambda)) \equiv f(\lambda) \pmod{\mathcal{I}_\alpha} \tag{8}$$

for all $\lambda \in \Lambda^\vee, \alpha \in \Phi, n \in \mathbf{Z}$.

Observation 3.2.5. The image of $s_{\alpha+n\alpha_0}$ under the identification $W^{\mathrm{aff}}/W \cong \Lambda^\vee$ is the right coset $s_{\alpha+n\alpha_0}W$. However, $s_{\alpha+n\alpha_0}s_\alpha$ is translation by $n\alpha^\vee$. If k is a commutative ring, we may view $k[\Lambda^\vee]$ as the $\mathbf{E}_\infty$-ring of functions on $\check{T}_k$; the element $n\alpha^\vee \in \Lambda^\vee$ corresponds to the function $e^{n\alpha^\vee}$. Therefore, (8) can be restated as

$$f((e^{n\alpha^\vee} - 1)(\lambda)) \equiv 0 \pmod{\mathcal{I}_\alpha}.$$

Using Example 2.4.11, we see that if $\mathbf{G}$ is affine, then $\pi_0\mathcal{F}_T(\mathrm{Gr}_G)$ is the $\pi_0\mathcal{O}_{\mathcal{M}_T}$-linear dual of $\pi_0\mathcal{O}_{\mathcal{M}_T}[\Lambda^\vee][\frac{e^{n\alpha^\vee}-1}{c_\alpha}]$. However, note that for any $n \in \mathbf{Z}$, we may use the multiplicative formal group law to obtain $\frac{e^{n\alpha^\vee}-1}{c_\alpha}$ from $\frac{e^{\alpha^\vee}-1}{c_\alpha}$. Therefore,

$$\pi_0\mathcal{F}_T(\mathrm{Gr}_G) \cong \mathrm{Map}_{\mathrm{QCoh}(\pi_0\mathcal{M}_T)}(\pi_0\mathcal{O}_{\mathcal{M}_T}[\Lambda^\vee][\tfrac{e^{\alpha^\vee}-1}{c_\alpha}], \pi_0\mathcal{O}_{\mathcal{M}_T}).$$

Remark 3.2.6. Let $\lambda \in \Lambda^{\vee,\mathrm{pos}}$ be a dominant coweight, and let $\Lambda^{\vee,\mathrm{pos}}_{\leq\lambda}$ denote the subset of $\Lambda^{\vee,\mathrm{pos}}$ consisting of those dominant weights which are at most λ. Then we may identify

$$(\mathrm{Gr}_G^{\leq\lambda})^T = W \cdot \Lambda^{\vee,\mathrm{pos}}_{\leq\lambda} \subseteq \Lambda^\vee = (\mathrm{Gr}_G)^T,$$

which allows us to calculate that if $\mathbf{G}$ is affine, then

$$\pi_0\mathcal{F}_T(\mathrm{Gr}_G^{\leq\lambda}) \cong \mathrm{Map}_{\mathrm{QCoh}(\pi_0\mathcal{M}_T)}(\pi_0\mathcal{O}_{\mathcal{M}_T}[W \cdot \Lambda^{\vee,\mathrm{pos}}_{\leq\lambda}][\tfrac{e^{\alpha^\vee}-1}{c_\alpha}], \pi_0\mathcal{O}_{\mathcal{M}_T}).$$

In the above expression, α ranges over $\Phi \cap W \cdot \Lambda^{\vee,\mathrm{pos}}_{\leq \lambda}$; in other words, α is of the form $w\alpha_i$ with $\alpha_i \in \Delta$ such that $\alpha_i \leq \lambda$.

We can now use Corollary 3.2.4 to compute the T-equivariant homology of Gr_G, which will be a key input into using [Dev22b, Theorem 2.3.3] to prove an analogue of geometric Satake.

Lemma 3.2.7. *There is an equivalence in* $\mathrm{Alg}_{\mathbf{E}_2}(\mathrm{coCAlg}(\mathrm{QCoh}(\mathcal{M}_T)))$*:*

$$\mathcal{F}_T(\mathrm{Gr}_T(\mathbf{C}))^\vee \cong \mathcal{O}(\check{T}_A \times_{\mathrm{Spec}(A)} \mathcal{M}_T).$$

Proof. Since the action of T on $\mathrm{Gr}_T(\mathbf{C})$ is trivial, we have a canonical equivalence $\mathcal{F}_T(\mathrm{Gr}_T(\mathbf{C}))^\vee \simeq \mathrm{Gr}_T(\mathbf{C})_+ \otimes \mathcal{F}_T(*)^\vee$. By definition, $\mathcal{F}_T(*)^\vee \simeq \mathcal{O}_{\mathcal{M}_T}$. We conclude that $\mathcal{F}_T(\mathrm{Gr}_T(\mathbf{C}))^\vee$ is equivalent as an $\mathbf{E}_2$-A-algebra to $C_*(\mathrm{Gr}_T(\mathbf{C}); A) \otimes_A \mathcal{O}_{\mathcal{M}_T}$. Since $BT(\mathbf{C}) \simeq B^2\Lambda^\vee$, there is an equivalence $\mathrm{Gr}_T(\mathbf{C}) \simeq \Lambda^\vee$ of $\mathbf{E}_2$-spaces. Therefore, $C_*(\mathrm{Gr}_T(\mathbf{C}); A) \simeq A[\Lambda^\vee]$ as $\mathbf{E}_2$-A-algebras, which is $\mathcal{O}(\check{T}_A)$. This implies the desired claim. □

Question 3.2.8. Can Lemma 3.2.7 be upgraded to an equivalence of $\mathbf{E}_3$-A-*algebras*? This additional structure is crucial for a statement of the geometric Satake correspondence which is $\mathbf{E}_3$-monoidal.

Notation 3.2.9. Let $T^*_{\mathbf{G}}\check{T}_A$ denote $\check{T}_A \times_{\mathrm{Spec}(A)} \mathcal{M}_T$, and let $T^*_{\mathbf{G}}\check{T}$ denote its underlying scheme (over $\pi_0\mathcal{M}_T$). Note that if $\mathbf{G} = \mathbf{G}_a$, then $T^*_{\mathbf{G}}\check{T}$ is the cotangent bundle of $\check{T}$, while if $\mathbf{G} = \mathbf{G}_m$, then $T^*_{\mathbf{G}}\check{T} = \check{T} \times T$. If $\mathfrak{B}_{\mathbf{G}}$ denotes the blowup of $T^*_{\mathbf{G}}\check{T}$ at the closed subscheme given by $\mathcal{M}_{T_\alpha}$ and the zero set of $e^{\alpha^\vee} - 1$ for $\alpha \in \Phi$, then define $(T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}$ as the complement of the proper preimage of $\mathcal{M}_{T_\alpha}$ in $\mathfrak{B}_{\mathbf{G}}$ for $\alpha \in \Phi$.

Theorem 3.2.10. *Let G be a connected semisimple algebraic group over $\mathbf{C}$. Then there is a W-equivariant isomorphism* $\mathrm{Spec}\,\pi_0\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee \cong (T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}$ *of schemes over $\pi_0\mathcal{M}_T$. If $\mathbf{G}$ is affine, this refines to an equivalence*

$$\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee \cong \mathcal{O}(\check{T}_A \times_{\mathrm{Spec}(A)} \mathcal{M}_T)[\tfrac{e^{\alpha^\vee}-1}{c_\alpha}, \alpha \in \Phi]$$

of $\mathbf{E}_1$-$\mathcal{O}_{\mathcal{M}_T}$-algebras; note that Example 2.4.9(c) only guarantees an $\mathbf{E}_1$-algebra structure on the right-hand side.

Proof. There is an $\mathbf{E}_2$-map $\mathrm{Gr}_T(\mathbf{C}) \to \mathrm{Gr}_G(\mathbf{C})$, which induces an $\mathbf{E}_2$-map $\mathcal{F}_T(\mathrm{Gr}_T(\mathbf{C}))^\vee \to \mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee$. This is given by dualizing the map $r : \mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C})) \to \mathcal{F}_T(\mathrm{Gr}_T(\mathbf{C}))$ of $\mathbf{E}_2$-coalgebras in $\mathrm{QCoh}(\mathcal{M}_T)$. The non-$W$-equivariant claim now follows formally, since r induces an injection on π_0, and the (cocommutative) Hopf algebra structure on $\pi_0\mathcal{F}_T(\mathrm{Gr}_T(\mathbf{C}))$ is given by the dual of the equivalence of Lemma 3.2.7. Proving W-equivariance requires a bit more work, but can easily be incorporated by keeping track of the W-action throughout the above discussion; we leave details to the reader. □

Remark 3.2.11. Suppose $A = \mathrm{KU}$, so that $\mathbf{G} = \mathbf{G}_m$ and c_α is $e^\alpha - 1$. It follows from Theorem 3.2.10 that replacing T with $\check{T}$, we get an isomorphism between $\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_{\check{G}}(\mathbf{C}))^\vee$ and $\pi_0(T_A \times_{\mathrm{Spec}(A)} \check{T}_A)[\frac{e^\alpha - 1}{e^{\alpha^\vee}-1}, \alpha \in \Phi]$. Therefore, $\pi_0\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee$ and $\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_{\check{G}}(\mathbf{C}))^\vee$ are both obtained from the blowup $\mathfrak{B}_{\mathbf{G}_m}$ of $T^*_{\mathbf{G}}\check{T}$ by deleting the proper preimage of two different closed subschemes which are Langlands dual to each other. In particular, the Langlands self-duality of the blowup $\mathfrak{B}_{\mathbf{G}_m}$ swaps the affine pieces $\mathrm{Spec}\,\pi_0\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee$ and $\mathrm{Spec}\,\pi_0\mathcal{F}_{\check{T}}(\mathrm{Gr}_{\check{G}}(\mathbf{C}))^\vee$ in $\mathfrak{B}_{\mathbf{G}_m}$.

Remark 3.2.12. In the case $G = \mathrm{SL}_2$, we refer the reader to Example A.3 and Example A.5 for an explicit description of $\mathrm{H}^G_*(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$, $\mathrm{H}^{G\times S^1_{\mathrm{rot}}}_*(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$, $\mathrm{KU}^G_*(\mathrm{Gr}_G(\mathbf{C}))\otimes\mathbf{C}$, and $\mathrm{KU}^{G\times S^1_{\mathrm{rot}}}_*(\mathrm{Gr}_G(\mathbf{C}))\otimes \mathbf{C}$.

4. The coherent side

4.1. **Langlands duality over $\mathbf{Q}[\beta^{\pm 1}]$ via [BFM05].** We now turn to the coherent side of the geometric Satake equivalence. For general $\mathbf{G}$, it is not obvious what the Langlands dual algebraic stack should be; we will discuss this in Section 4.3, and focus only on $\mathbf{Q}[\beta^{\pm 1}]$ in this section.

Definition 4.1.1. Let G be a connected reductive group over $\mathbf{C}$, and fix the rest of notation as in Notation 1.1.17. Fix a principal nilpotent element $e \in \mathfrak{n}$, and let (e, f, h) be the associated $\mathfrak{sl}_2$-triple in $\mathfrak{g}$. Let $\mathfrak{g}^e$ be the centralizer (so $\mathfrak{g} = \mathfrak{g}^e \oplus [f, \mathfrak{g}]$), and let $\mathcal{S} := f + \mathfrak{g}^e \subseteq \mathfrak{g}^{\mathrm{reg}}$ be the Kostant slice. Then the composite $f + \mathfrak{g}^e \to \mathfrak{g} \to \mathfrak{g}/\!\!/G \cong \mathfrak{t}/\!\!/W$ is an isomorphism.

Let $\widetilde{\mathfrak{g}} = \mathfrak{b} \times_B G$ be the Grothendieck-Springer resolution, so that $\widetilde{\mathfrak{g}}/G \simeq \mathfrak{b}/B$. We will often work with $\widetilde{\mathfrak{g}}^*$ instead, defined as $\mathfrak{b}^* \times_B G$. There is a map $\widetilde{\chi} : \widetilde{\mathfrak{g}} \to \mathfrak{t}$ which sends a pair $(x \in \mathrm{Ad}_g(\mathfrak{b}))$ to the inverse image under the isomorphism $\mathfrak{t} \to \mathfrak{b} \to \mathfrak{b}/\mathfrak{n}$ of the image of $g^{-1}x \in \mathfrak{b}$. Let $\widetilde{\mathcal{S}}$ denote the fiber product $\mathcal{S} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$, so that $\widetilde{\mathcal{S}} \subseteq \widetilde{\mathfrak{g}}^{\mathrm{reg}} = \mathfrak{g}^{\mathrm{reg}} \times_{\mathfrak{g}} \widetilde{\mathfrak{g}}$. Then, Kostant's result on the Kostant slice implies formally that the composite $\widetilde{\mathcal{S}} \to \widetilde{\mathfrak{g}} \xrightarrow{\widetilde{\chi}} \mathfrak{t}$ is an isomorphism. We will often abusively write the inclusion of $\widetilde{\mathcal{S}}$ as a map $\kappa : \mathfrak{t} \to \widetilde{\mathfrak{g}}$.

In fact, we will only care about the composite $\mathfrak{t} \to \widetilde{\mathfrak{g}} \to \widetilde{\mathfrak{g}}/G$ below, so we will also denote it by κ. If we identify $\widetilde{\mathfrak{g}}/G \cong \mathfrak{b}/B$, then the map κ admits a simple description: it is the composite $\mathfrak{t} \to \mathfrak{b} \to \mathfrak{b}/B$ which sends $x \mapsto f + x$.

Notation 4.1.2. If R is an $\mathbf{E}_\infty$-ring and V is an R-module, let $V[n]$ denote the affine scheme $\mathrm{Spec}\,\mathrm{Sym}_R(V^*[-n])$; note that since $R \otimes_{\mathrm{Sym}_R(V^*[-n])} R = \mathrm{Sym}_R(V^*[-(n-1)])$, we have $\{0\} \times_{V[n]} \{0\} = V[n-1]$. We warn the reader that this is generally *different* from the usual notation employed in representation theory when R is not a $\mathbf{Q}$-algebra. Indeed, if $R = k$ is a field, $V = k$, and $n = 1$, then $\mathcal{O}_{V[1]}$ is $\mathrm{Sym}_k(k[-1])$; this is the global sections of the structure sheaf of $B\mathbf{G}_a$. By [Jan03, Section 4.27], this is an exterior algebra on a class in degree -1 if and only if k is a field of characteristic 0.

Fix a nondegenerate invariant bilinear form on $\mathfrak{g}$, to identify $\mathfrak{g}$ with $\mathfrak{g}^*$. The first main result of this section is the following:

Theorem 4.1.3. *Let G be a connected and simply-connected semisimple algebraic group over $\mathbf{C}$. Let A be an $\mathbf{E}_\infty$-$\mathbf{Q}[\beta^{\pm 1}]$-algebra, and let $\mathbf{G} = \mathbf{G}_a$ (so $\mathcal{M}_T$ is the affine space $\mathfrak{t}[2]$ over A). View $\check{\mathfrak{t}}^*$, $\check{\mathfrak{n}}$, $\check{\mathfrak{g}}$, and $\check{B}$ as schemes over $\mathbf{Q}$. Then $\mathrm{QCoh}(\check{\mathfrak{t}}^*)$ admits the structure of a module over $\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})$, where the fiber product is (always) derived, such that there is an equivalence*

$$\mathrm{End}_{\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})}(\mathrm{QCoh}(\check{\mathfrak{t}}^*)) \otimes_{\mathbf{Q}} A \simeq \mathrm{Mod}_{C_*^T(\mathrm{Gr}_G(\mathbf{C}); A)}.$$

Remark 4.1.4. Recall from [ABG04] that there is an Iwahori-Satake equivalence $\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G}) \simeq \mathrm{Shv}(\mathrm{Gr}_G)^I$ over $\mathbf{C}$, where the right-hand side is normalized appropriately. One should therefore regard Theorem 4.1.3 as a cobar construction ("once-looped") of the restriction of this equivalence (lifted from $\mathbf{C}$ to $\mathbf{Q}$) to the regular locus, and more optimistically as a first step towards an alternative proof. See also Example 4.4.9 for the equivalence resulting from "undoing" the cobar construction.

Remark 4.1.5. Theorem 4.1.3 (in particular, the key Proposition 4.1.7 below) can be used to reprove [YZ11, Theorem 6.1].

We now turn to the proof of Theorem 4.1.3. For the next two results, we only work on one side of Langlands duality, so we drop the "check"s for notational simplicity. Note that $(\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G} \cong (\check{\mathfrak{n}} \times_{\check{\mathfrak{g}}} \{0\})/\check{B}$; it will be more convenient to work with the latter description.

Lemma 4.1.6. *There is a Koszul duality equivalence $\mathrm{QCoh}(\widetilde{\mathfrak{g}}^*/G) \simeq \mathrm{IndCoh}((\mathfrak{n} \times_{\mathfrak{g}} \{0\})/B)$.*

Proof. Observe that $\mathfrak{n} \times_{\mathfrak{g}} \{0\} \cong \mathrm{Spec}\,\mathrm{Sym}(\mathfrak{n}^* \times_{\mathfrak{g}^*} 0)$. Since $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{b}^-$, we see that $\mathfrak{n}^* \times_{\mathfrak{g}^*} 0$ is equivalent as a B-equivariant module to $\mathfrak{b}[-1]$. (Here, we use the invariant bilinear form on $\mathfrak{g}$.) It follows that $\mathfrak{n} \times_{\mathfrak{g}} \{0\} \cong \mathrm{Spec}\,\mathrm{Sym}(\mathfrak{b}[-1])$, so that $\mathrm{IndCoh}((\mathfrak{n} \times_{\mathfrak{g}} \{0\})/B) \simeq \mathrm{IndCoh}(\mathrm{Sym}(\mathfrak{b}[-1]))^B$. By Koszul duality, $\mathrm{IndCoh}(\mathrm{Sym}(\mathfrak{b}^*[-1]))^B \simeq \mathrm{QCoh}(\mathfrak{b}^*/B)$; this implies the claim, since $\mathfrak{b}^*/B \simeq \widetilde{\mathfrak{g}}^*/G$. □

We will give two proofs of the following fact.

Proposition 4.1.7 (Variant of [BFM05, Proposition 2.8])**.** *Work over a field k of characteristic* 0*, and view* $\mathrm{QCoh}(\mathfrak{t}^*)$ *as a* $\mathrm{QCoh}(\widetilde{\mathfrak{g}}^*/G)$*-module via the Kostant slice* $\kappa : \mathfrak{t}^* \to \widetilde{\mathfrak{g}}^*$*. Then there is an equivalence* $\mathrm{End}_{\mathrm{QCoh}(\widetilde{\mathfrak{g}}^*/G)}(\mathrm{QCoh}(\mathfrak{t}^*)) \simeq \mathrm{QCoh}((T^*T)^{\mathrm{bl}})$.

First proof of Proposition 4.1.7. We may identify $\mathrm{End}_{\mathrm{QCoh}(\widetilde{\mathfrak{g}}^*/G)}(\mathrm{QCoh}(\mathfrak{t}^*))$ with $\mathrm{QCoh}(\mathfrak{t}^* \times_{\widetilde{\mathfrak{g}}^*/G} \mathfrak{t}^*)$. We will show, in fact, that there is a Cartesian square

(9)
$$\begin{tikzcd} (T^*T)^{\mathrm{bl}} \arrow[r] \arrow[d] & \mathfrak{t}^* \arrow[d, "\kappa"] \\ \mathfrak{t}^* \arrow[r, "\kappa"'] & \widetilde{\mathfrak{g}}^*/G \simeq \mathfrak{b}^*/B. \end{tikzcd}$$

This is an analogue of [Ngo10, Proposition 2.2.1] and [BFM05, Proposition 2.8]. (Note that since $\mathfrak{t}^* \to \widetilde{\mathfrak{g}}^*$ lands in the open locus $\widetilde{\mathfrak{g}}^{*,\mathrm{reg}}$, it does not matter whether we intersect $\mathfrak{t}^*$ with itself in $\widetilde{\mathfrak{g}}^*/G$ or in $\widetilde{\mathfrak{g}}^{*,\mathrm{reg}}/G$; indeed, the intersection $\widetilde{\mathfrak{g}}^{*,\mathrm{reg}} \times_{\widetilde{\mathfrak{g}}^*} \widetilde{\mathfrak{g}}^{*,\mathrm{reg}}$ is just $\widetilde{\mathfrak{g}}^{*,\mathrm{reg}}$.) In what follows, it will be convenient to use the chosen nondegenerate invariant bilinear form on $\mathfrak{g}$ to identify $\mathfrak{b}^*$ with the opposite Borel $\mathfrak{b}^-$ and N with its opposite unipotent, and then to flip the role of $\mathfrak{b}$ and $\mathfrak{b}^-$, etc.

Recall that the Kostant slice $\mathcal{S} \subseteq \mathfrak{g}$ is transverse to the regular G-orbits, and intersects each orbit exactly once; this implies that the image of the map $\kappa : \mathfrak{t} \to \widetilde{\mathfrak{g}}$ is transverse to the regular G-orbits on $\widetilde{\mathfrak{g}}$, and intersects each orbit exactly once. In particular, if C denotes the locally closed subvariety of $\widetilde{\mathfrak{g}} \times G$ consisting of pairs (x, g) with $x \in \widetilde{\mathfrak{g}}^{\mathrm{reg}}$ and $\mathrm{Ad}_g(x) = x$, then $C /\!\!/ G = \mathfrak{t} \times_{\widetilde{\mathfrak{g}}/G} \mathfrak{t}$ (so we may assume without loss of generality that $x \in \mathfrak{t}$). To compute $C /\!\!/ G$, one can reduce to the case when G has semisimple rank 1 by the argument of [BFM05, Section 4.3]. To work out this case, we will assume $G = \mathrm{SL}_2$ and $G = \mathrm{PGL}_2$.

There are "two" ways to compute in these cases. First, we present the argument which is essentially present in [BFM05]; for this, we will assume $G = \mathrm{SL}_2$. The Grothendieck-Springer resolution $\widetilde{\mathfrak{g}}$ is the total space of $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ over $\mathbf{P}^1$; we will think of a point in $\widetilde{\mathfrak{g}}$ as a pair $(x \in \mathfrak{sl}_2, \ell \subseteq \mathbf{C}^2)$ such that x preserves ℓ. The Kostant slice $\kappa : \mathfrak{t} \cong \mathbf{A}^1 \to \widetilde{\mathfrak{g}}$ is the map sending $\lambda \in \mathbf{A}^1$ to the pair (x, ℓ) with $x = \left(\begin{smallmatrix} 0 & \lambda^2 \\ 1 & 0 \end{smallmatrix}\right)$ and $\ell = [\lambda : 1]$. Indeed, this is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{A}^1 \cong \mathfrak{t} \arrow[r, "\kappa"] \arrow[d, "\lambda \mapsto \lambda^2"'] & \widetilde{\mathfrak{sl}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \mathfrak{t}/\!\!/W \arrow[r, "\kappa", "\lambda \mapsto \left(\begin{smallmatrix} 0 & \lambda \\ 1 & 0 \end{smallmatrix}\right)"'] & \mathfrak{sl}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathfrak{g}}$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, we compute that

$$\mathrm{Ad}_g\left(\begin{smallmatrix} 0 & \lambda^2 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} bd - ac\lambda^2 & (a\lambda)^2 - b^2 \\ d^2 - (c\lambda)^2 & ac\lambda^2 - bd \end{smallmatrix}\right), \ g \cdot [\lambda : 1] = [a\lambda + b : c\lambda + d].$$

From this, we see that $\mathrm{Ad}_g(x) = x$ if and only if $a = d$ and $b = c\lambda^2$, in which case g also fixes $[\lambda : 1]$. In other words, $g = \left(\begin{smallmatrix} a & c\lambda^2 \\ c & a \end{smallmatrix}\right)$ with $a, c \in k$; in order for $\det(g) = 1$, we need $a^2 - c^2\lambda^2 = 1$. When $\lambda \neq 0$, both x and g are diagonalized by the matrix $\frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -\lambda^{-1} & -\lambda^{-1} \end{smallmatrix}\right) \in \mathrm{SL}_2$: the diagonalization of x is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and the diagonalization of g is $\left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix}\right)$ where $2a = t + t^{-1}$ and $2\lambda c = t - t^{-1}$. Since we have $\det(g) = a^2 - (c\lambda)^2 = 1$, this shows that if k is not of characteristic 2, then $\mathfrak{t} \times_{\widetilde{\mathfrak{sl}}_2/\mathrm{SL}_2} \mathfrak{t} \cong \mathrm{Spec}\, k[\lambda, t^{\pm 1}, \frac{t - t^{-1}}{\lambda}]$.

The second way to reach this calculation (still with $G = \mathrm{SL}_2$) is to use the fact that $\kappa : \mathfrak{t} \to \widetilde{\mathfrak{g}}/G$ can be identified with the composite $\mathfrak{t} \to \mathfrak{b} \to \mathfrak{b}/B$ sending $x \mapsto f + x$. Then, $\mathfrak{t} \times_{\mathfrak{b}/B} \mathfrak{t}$ is isomorphic to the subvariety of $\mathfrak{t} \times B$ consisting of pairs (x, g) with $x \in \mathfrak{t}$ and $\mathrm{Ad}_g(x + f) = x + f$. If $g = \left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right) \in B$, then

$$\mathrm{Ad}_g\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ 2a^{-1}bx + a^{-2} & -x \end{smallmatrix}\right).$$

Therefore, $\mathrm{Ad}_g(x + f) = x + f$ if and only if

$$2a^{-1}bx + a^{-2} = 1,$$

which forces $b = \frac{a - a^{-1}}{2x}$. This implies that $\mathfrak{t} \times_{\mathfrak{b}/B} \mathfrak{t}$ is isomorphic to $\mathrm{Spec}\, k[x, a^{\pm 1}, \frac{a - a^{-1}}{x}]$, as desired.

We will now do the calculation with $G = \mathrm{PGL}_2$ via the second method. Again, $\mathfrak{t} \times_{\mathfrak{b}/B} \mathfrak{t}$ is isomorphic to the subvariety of $\mathfrak{t} \times B$ consisting of pairs (x, g) with $x \in \mathfrak{t}$ (identified with the matrix $\left(\begin{smallmatrix} x & 0 \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{gl}_2$) and $\mathrm{Ad}_g(x + f) = x + f$. If $g = \left(\begin{smallmatrix} a & 0 \\ b & 1 \end{smallmatrix}\right) \in B$, then

$$\mathrm{Ad}_g\left(\begin{smallmatrix} x & 0 \\ 1 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} x & 0 \\ (bx+1)a^{-1} & 0 \end{smallmatrix}\right).$$

Therefore, $\mathrm{Ad}_g(x + f) = x + f$ if and only if

$$(bx + 1)a^{-1} = 1,$$

which forces $b = \frac{a-1}{x}$. This implies that $\mathfrak{t} \times_{\mathfrak{b}/B} \mathfrak{t}$ is isomorphic to $\mathrm{Spec}\, k[x, a^{\pm 1}, \frac{1-a}{x}]$, as desired. □

Second proof of Proposition 4.1.7. As in the first proof of Proposition 4.1.7, it will be convenient to use the chosen nondegenerate invariant bilinear form on $\mathfrak{g}$ to identify $\mathfrak{b}^*$ with the opposite Borel $\mathfrak{b}^-$ and N with its opposite unipotent, and then to flip the role of $\mathfrak{b}$ and $\mathfrak{b}^-$, etc. We will prove the following variant of Proposition 4.1.7, which in turn implies the desired result: view $\mathrm{QCoh}(\mathfrak{t}^*/\!\!/W)$ as a $\mathrm{QCoh}(\mathfrak{g}^*/G)$-module via the Kostant slice. Then there is an equivalence $\mathrm{End}_{\mathrm{QCoh}(\mathfrak{g}^*/G)}(\mathrm{QCoh}(\mathfrak{t}^*/\!\!/W)) \simeq \mathrm{QCoh}((T^*T)^{\mathrm{bl}}/\!\!/W)$.

Let χ be a nondegenerate character on $\mathfrak{n}^-$. The N^--action on G via conjugation induces a Hamiltonian N^--action on T^*G; let $N^-{}_\chi\backslash(T^*G)/_\chi N^-$ denote the bi-Whittaker reduction of T^*G with respect to this N^--action at the character $\chi \in \mathfrak{n}^{-,*}$. Then $(T^*T)^{\mathrm{bl}}/\!\!/W \cong N^-{}_\chi\backslash(T^*G)/_\chi N^-$; see [Tel14, Theorem 6.3], for instance. There is a Morita equivalence between $\mathrm{QCoh}(\mathfrak{g}^*/G)$ and $\mathrm{QCoh}(T^*G)$ (equipped with the convolution monoidal structure); under this equivalence, the $\mathrm{QCoh}(\mathfrak{g}^*/G)$-module $\mathrm{QCoh}(\mathfrak{g}^*/_\chi N^-)$ is sent to the $\mathrm{QCoh}(T^*G)$-module $\mathrm{QCoh}((T^*G)/_\chi N^-)$. We conclude the series of equivalences:

$$\begin{aligned}
\mathrm{QCoh}((T^*T)^{\mathrm{bl}}/\!\!/W) &\simeq \mathrm{QCoh}(N^-{}_\chi\backslash(T^*G)/_\chi N^-) \\
&\simeq \mathrm{End}_{\mathrm{QCoh}(T^*G)}(\mathrm{QCoh}((T^*G)/_\chi N^-)) \\
&\simeq \mathrm{End}_{\mathrm{QCoh}(\mathfrak{g}^*/G)}(\mathrm{QCoh}(\mathfrak{g}^*/_\chi N^-)).
\end{aligned}$$

However, Kostant's theorem identifies $\mathfrak{g}^*/_\chi N^-$ with $\mathfrak{t}^*/\!\!/W$, which finishes the proof. □

Proof of Theorem 4.1.3. By Theorem 3.2.10, we have $C_*^T(\mathrm{Gr}_G(\mathbf{C}); A) = \mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee \cong \mathcal{O}_{(T^*T)^{\mathrm{bl}}}$ as $\mathbf{E}_1$-algebras in $\mathrm{QCoh}(\mathfrak{t}_A)$. It follows that $\mathrm{Mod}_{C_*^T(\mathrm{Gr}_G(\mathbf{C}); A)} \simeq \mathrm{QCoh}((T^*\check{T})^{\mathrm{bl}}_A)$, although only as $\mathbf{E}_0$-monoidal categories (i.e., an exact equivalence of presentable stable ∞-categories). Since $\mathrm{QCoh}((T^*\check{T})^{\mathrm{bl}}) = \mathrm{QCoh}((T^*\check{T})^{\mathrm{bl}}) \otimes_{\mathbf{Q}} A$, and $\mathrm{End}_{\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})}(\mathrm{QCoh}(\check{\mathfrak{t}}^*)) \simeq \mathrm{QCoh}((T^*\check{T})^{\mathrm{bl}})$ by Lemma 4.1.6 and Proposition 4.1.7, we conclude the desired result. □

Remark 4.1.8. So far, we have not emphasized the role of Whittaker reduction in the above story (except for the second proof of Proposition 4.1.7). However, we take a moment to describe this briefly, since it is conceptually useful. Recall that a theorem of Kostant's gives an isomorphism $(f + \mathfrak{b})/N \cong \mathcal{S} = f + \mathfrak{g}^e$. In terms of Whittaker reduction, this says that $\mathcal{S} \cong \mathfrak{g}/_\chi N^-$. Since Proposition 4.1.7 is concerned with $\widetilde{\mathfrak{g}}$ instead of $\mathfrak{g}$, we need a slight variant of this statement. Namely, recall the map $\pi : \widetilde{\mathfrak{g}} \to \mathfrak{g}$, let $\mu : \mathfrak{g} \to \mathfrak{n}$ be the moment map for the adjoint N-action on $\mathfrak{g}$, and let $\widetilde{\mu}$ denote the composite $\widetilde{\mathfrak{g}} \to \mathfrak{g} \to \mathfrak{n}$. Then $\mu^{-1}(f)$ is the variety $f + \mathfrak{b}$, so that $\widetilde{\mu}^{-1}(f)$ is the subscheme of $\widetilde{\mathfrak{g}}$ spanned by those pairs $(\mathfrak{b}', y \in \mathfrak{b}' \cap (f + \mathfrak{b}))$. Kostant's result implies that there is an isomorphism $\widetilde{\mu}^{-1}(f)/N^- \xrightarrow{\sim} \mathfrak{t}$. Note that this map is easily seen to be surjective: an inverse map is given by sending $x \in \mathfrak{t}$ to the pair $(\mathfrak{b}^-, f + x \in \mathfrak{b}^-)$. Injectivity is equivalent to the claim that the space of pairs $(\mathfrak{b}', y \in \mathfrak{n}' \cap (f + \mathfrak{b}))$ is an N-torsor, which is harder to prove. In some sense, Whittaker reduction is a key aspect of the the Langlands-dual side of Theorem 4.1.3: it is needed to even define the action of $\mathrm{QCoh}(\widetilde{\mathfrak{g}}^*/G)$ on $\mathrm{QCoh}(\mathfrak{t}^*)$.

Remark 4.1.9. In [BFM05], the following analogue of (9) is established (over $\mathbf{C}$, but this does not affect the statement): there is a Cartesian square

$$\begin{tikzcd}
(T^*\check{T})^{\mathrm{bl}}/\!\!/W \arrow[r] \arrow[d] & \mathfrak{t}/\!\!/W \arrow[d, "\kappa"] \\
\mathfrak{t}/\!\!/W \arrow[r, "\kappa"'] & \check{\mathfrak{g}}/\check{G},
\end{tikzcd} \tag{10}$$

where the top-left corner can be identified with $\operatorname{Spec} C_*^G(\mathrm{Gr}_G(\mathbf{C});\mathbf{Q})$. We can take the fiber product of (9) with itself over (10) to obtain a Cartesian square

$$\begin{tikzcd} (T^*\check{T})^{\mathrm{bl}} \times_{(T^*\check{T})^{\mathrm{bl}}/\!\!/W} (T^*\check{T})^{\mathrm{bl}} \arrow[r] \arrow[d] & \mathfrak{t} \times_{\mathfrak{t}/\!\!/W} \mathfrak{t} \arrow[d, "\kappa"] \\ \mathfrak{t} \times_{\mathfrak{t}/\!\!/W} \mathfrak{t} \arrow[r, "\kappa"'] & (\widetilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\check{G}. \end{tikzcd} \tag{11}$$

Using Theorem 4.1.3 and the above discussion, one can use (11) to show that $\mathrm{End}_{\mathrm{QCoh}((\widetilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\check{G})}(\mathrm{QCoh}(\mathfrak{t}\times_{\mathfrak{t}/\!\!/W} \mathfrak{t})) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}]$ can be identified with $\mathrm{LMod}_{C_*^T(\mathrm{Fl}_G(\mathbf{C});\mathbf{Q}[\beta^{\pm1}])}$. This can be viewed as a "once-looped" version of Bezrukavnikov's equivalence from [Bez16].

One can quantize Theorem 4.1.3 as follows.

Definition 4.1.10. Following [KS20], define the (Langlands dual) *universal category* $\check{\mathcal{O}}_\hbar^{\mathrm{univ}}$ to be $\mathrm{DMod}^\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},w)} \simeq U_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{N},(\check{T},w)}$. The ∞-category $\check{\mathcal{O}}_\hbar^{\mathrm{univ}}$ is a quantization of $\mathrm{QCoh}(\check{\mathfrak{b}}^-/\check{B}^-)$, since there are isomorphisms

$$\check{\mathfrak{b}}^-/\check{B}^- \cong \widetilde{\check{\mathfrak{g}}}/\check{G} \cong \check{T}\backslash T^*(\check{G}/\check{N})/\check{G}.$$

Theorem 4.1.11. *Let A be an $\mathbf{E}_\infty$-$\mathbf{C}[\beta^{\pm1}]$-algebra, and let G be a connected and simply-connected semisimple algebraic group or a torus over $\mathbf{C}$. Then there is a Kostant functor $\check{\mathcal{O}}_\hbar^{\mathrm{univ}} \to \mathrm{QCoh}(\check{\mathfrak{t}}^* \times \mathbf{A}^1_\hbar)$ and a left $A[\![\hbar]\!]$-linear equivalence*

$$\mathrm{LMod}_{C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});A)} \simeq \mathrm{End}_{\check{\mathcal{O}}_\hbar^{\mathrm{univ}}}(\mathrm{QCoh}(\check{\mathfrak{t}}^* \times \mathbf{A}^1_\hbar)) \otimes_{\mathbf{C}[\![\hbar]\!]} \mathbf{C}[\![\hbar]\!][\beta^{\pm1}].$$

In fact, the periodification is unnecessary.

Proof sketch; compare to the second proof of Proposition 4.1.7. We will assume $A = \mathbf{C}[\beta^{\pm1}]$, so that $C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});A)$ is a 2-periodification of $C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})$. Let $\mathcal{H}(\widetilde{\check{\mathfrak{t}}}^*, W^{\mathrm{aff}})$ be the nil-Hecke algebra associated to $\widetilde{\check{\mathfrak{t}}}^* \cong \check{\mathfrak{t}}^* \oplus \mathbf{C}\alpha_0$, and let $e = \frac{1}{\#W}\sum_{w\in W} w \in \mathbf{Q}[W]$ be the symmetrizer idempotent. Using Corollary 3.2.4, one can then show that $\mathrm{H}_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})$ is isomorphic to $\mathcal{H}(\widetilde{\check{\mathfrak{t}}}^*, W^{\mathrm{aff}})e$, where the loop rotation parameter $\hbar$ corresponds to the affine root α_0; see [KK90]. The evenness of $C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});A)$ lets us identify $C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});A) \simeq \mathrm{H}_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});A)$ as $\mathbf{E}_1$-A-algebras, so that one can similarly identify $\mathrm{LMod}_{C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});A)}$ with $\mathrm{LMod}_{\mathcal{H}(\widetilde{\check{\mathfrak{t}}}^*, W^{\mathrm{aff}})e}$.

We now construct the Kostant functor $\kappa_\hbar : \check{\mathcal{O}}_\hbar^{\mathrm{univ}} \to \mathrm{QCoh}(\check{\mathfrak{t}}^* \times \mathbf{A}^1_\hbar)$. Since $\check{\mathcal{O}}_\hbar^{\mathrm{univ}} = \mathrm{DMod}^\hbar(\check{G}/\check{N})^{(\check{G}\times\check{T},w)}$, it suffices to construct an isomorphism $\Gamma(\check{G}/\check{N};\mathcal{D}_{\check{G}/\check{N}})^{\check{G}\times\check{T}} \cong U(\check{\mathfrak{t}}) \cong \mathcal{O}_{\check{\mathfrak{t}}^*}$, where $\mathcal{D}_{\check{G}/\check{N}}$ denotes the sheaf of differential operators on $\check{G}/\check{N}$. This is essentially part of the Beilinson-Bernstein theorem; for example, [Mil, Theorem 2.6.5] shows that there is an $\check{G}$-equivariant isomorphism $\Gamma(\check{G}/\check{N};\mathcal{D}^\hbar_{\check{G}/\check{N}})^{\check{T}} \cong U_\hbar(\check{\mathfrak{g}}) \otimes_{Z(\check{\mathfrak{g}})} U(\check{\mathfrak{t}})$, so the claim follows from the fact that $U_\hbar(\check{\mathfrak{g}})^{\check{G}} \cong Z(\check{\mathfrak{g}}) \otimes_{\mathbf{C}} \mathbf{C}[\![\hbar]\!]$ and that $U(\check{\mathfrak{t}})$ is a flat $Z(\check{\mathfrak{g}})$-module.

To finish, we need to show that $\mathrm{LMod}_{\mathcal{H}(\widetilde{\check{\mathfrak{t}}}^*, W^{\mathrm{aff}})e} \simeq \mathrm{End}_{\check{\mathcal{O}}_\hbar^{\mathrm{univ}}}(\mathrm{QCoh}(\check{\mathfrak{t}}^* \times \mathbf{A}^1_\hbar))$. We will denote $\mathcal{D}_{\check{G}} \otimes_{Z(\check{\mathfrak{g}})} U(\check{\mathfrak{t}})$ by $\widetilde{\mathcal{D}_{\check{G}}}$, so that $\widetilde{\mathcal{D}_{\check{G}}}$ is a quantization of $\widetilde{\check{\mathfrak{g}}} \times \check{G}$. Kostant's isomorphism $U(\check{\mathfrak{g}})/_\chi \check{N}^- \cong Z(\check{\mathfrak{g}})$ from [Kos78] implies that

$$\mathrm{End}_{\check{\mathcal{O}}_\hbar^{\mathrm{univ}}}(\mathrm{QCoh}(\check{\mathfrak{t}}^* \times \mathbf{A}^1_\hbar)) \simeq \mathrm{LMod}_{\check{N}^-{}_\chi\backslash \widetilde{\mathcal{D}^\hbar_{\check{G}}}/_\chi \check{N}^-}.$$

The desired claim now follows from the observation that we have

$$\check{N}^-{}_\chi\backslash \widetilde{\mathcal{D}_{\check{G}}}/_\chi \check{N}^- \cong \mathrm{Sym}(\check{\mathfrak{t}}) \otimes_{Z(\check{\mathfrak{g}})} (\check{N}^-{}_\chi\backslash \mathcal{D}_{\check{G}}/_\chi \check{N}^-) \cong \mathcal{H}(\widetilde{\check{\mathfrak{t}}}^*, W^{\mathrm{aff}})e,$$

where the final isomorphism is given by [Gin18, Theorem 8.1.2]. □

Remark 4.1.12. In fact, one can quantize the result of [ABG04]: namely, there is an equivalence

$$\mathrm{DMod}_{I\rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G) \simeq \check{\mathcal{O}}_\hbar^{\mathrm{univ}}. \tag{12}$$

We do not have a reference for this fact when G lives over $\mathbf{C}$, but if G lives over $\overline{\mathbf{F}_p}$ and DMod is replaced with $\overline{\mathbf{Q}_\ell}$-adic sheaves, then (12) can be deduced from [Dod11, Theorem 84] and the parabolic-Whittaker duality for the affine Grassmannian from [BY13]. Just as with Theorem 4.1.3, Theorem 4.1.11 may be regarded as a "once-looped" version of (12). One can similarly show that there is an equivalence

$$\mathrm{DMod}_{I\rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Fl}_G) \simeq \mathrm{DMod}_\hbar(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},\mathrm{wk})}, \tag{13}$$

which quantizes Bezrukavnikov's equivalence from [Bez16]. (Note that $\check{T}\backslash T^*(\check{N}\backslash\check{G}/\check{N})/\check{T}$ is isomorphic to $(\widetilde{\check{\mathfrak{g}}} \times_{\check{\mathfrak{g}}} \widetilde{\check{\mathfrak{g}}})/\check{G}$.) There is a "once-looped" version of (13), stating that there is a Kostant functor $\mathrm{DMod}_\hbar(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},\mathrm{wk})} \to \mathrm{QCoh}((\check{\mathfrak{t}}^* \times_{\check{\mathfrak{t}}^*/\!\!/W} \check{\mathfrak{t}}^*) \times \mathbf{A}^1_\hbar)$ and a left $\mathbf{Q}[\![\hbar]\!][\beta^{\pm 1}]$-linear equivalence

$$\mathrm{LMod}_{C_*^{\check{T}}(\mathrm{Fl}_G;\mathbf{Q})} \simeq \mathrm{End}_{\mathrm{DMod}_\hbar(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},\mathrm{wk})}}(\mathrm{QCoh}((\check{\mathfrak{t}}^* \times_{\check{\mathfrak{t}}^*/\!\!/W} \check{\mathfrak{t}}^*) \times \mathbf{A}^1_\hbar)) \otimes_{\mathbf{Q}[\![\hbar]\!]} \mathbf{Q}[\![\hbar]\!][\beta^{\pm 1}].$$

Remark 4.1.13. If G is a connected and simply-connected semisimple algebraic group or a torus over $\mathbf{C}$, let $\mathrm{HC}_\hbar(\check{G})$ denote the ∞-category $U_\hbar(\check{\mathfrak{g}})\text{-mod}^{\check{G},w}$. Then $\Gamma(\check{G};\mathcal{D}_{\check{G}})^{\check{G}\times\check{G}} \cong U(\check{\mathfrak{g}})^{\check{G}} \cong \mathrm{Sym}(\check{\mathfrak{t}})^W$. An argument very similar to Theorem 4.1.11 proves that there is a Kostant functor $\mathrm{HC}_\hbar(\check{G}) \to \mathrm{QCoh}(\check{\mathfrak{t}}^*/\!\!/W \times \mathbf{A}^1_\hbar)$ and a left $A[\![\hbar]\!]$-linear equivalence

$$\mathrm{LMod}_{C_*^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});A)} \simeq \mathrm{End}_{\mathrm{HC}_\hbar(\check{G})}(\mathrm{QCoh}(\check{\mathfrak{t}}^*/\!\!/W \times \mathbf{A}^1_\hbar)) \otimes_{\mathbf{C}[\![\hbar]\!]} \mathbf{C}[\![\hbar]\!][\beta^{\pm 1}]. \tag{14}$$

The periodification is in fact unnecessary. This is closely related to [Gin18], [Lon18], and [Gan22a, Theorem 1.4]; these articles provide a monoidal "Fourier transform" equivalence $\mathrm{DMod}(\check{N}^-{}_\chi\backslash\check{G}/_\chi\check{N}^-) \simeq \mathrm{IndCoh}(\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}})$. (See [Gan22b] for a definition of $\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}}$.) Note that combined with the preceding discussion, we obtain an equivalence

$$\mathrm{IndCoh}(\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}}) \simeq \mathrm{End}_{\mathrm{HC}(\check{G})}(\mathrm{QCoh}(\check{\mathfrak{t}}^*/\!\!/W)). \tag{15}$$

In the same way, we will show that there are equivalences

$$\mathrm{IndCoh}(\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}}) \simeq \mathrm{End}_{\mathrm{DMod}(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},w)}}(\mathrm{QCoh}(\check{\mathfrak{t}}^*)), \tag{16}$$

$$\mathrm{IndCoh}(\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}}) \simeq \mathrm{Fun}^L_{\check{\mathfrak{g}}\text{-mod}^{\check{N}}}(\mathrm{QCoh}(\check{\mathfrak{t}}^*/\Lambda^\vee), \mathrm{QCoh}(\check{\mathfrak{t}}^*/\!\!/W)). \tag{17}$$

We expect that the techniques of [BGO20] and Section 4.4 can be used to show that the equivalences (15) and (17) imply the equivalences conjectured in [Gan22a, Remark 6.22]. Let us now prove (16):

$$\begin{aligned}
\mathrm{IndCoh}(\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}}) &\simeq \mathrm{DMod}(\check{N}^-{}_\chi\backslash\check{G}/_\chi\check{N}^-) \\
&\simeq \mathrm{End}_{\mathrm{DMod}(\check{G})}(\mathrm{DMod}(\check{G}/_\chi\check{N}^-)) \\
&\simeq \mathrm{End}_{\mathrm{DMod}(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},w)}}(\mathrm{DMod}(\check{N}\backslash\check{G}/_\chi\check{N}^-)^{\check{T},w}) \\
&\simeq \mathrm{End}_{\mathrm{DMod}(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},w)}}(\mathrm{DMod}(\check{T})^{\check{T},w}) \\
&\simeq \mathrm{End}_{\mathrm{DMod}(\check{N}\backslash\check{G}/\check{N})^{(\check{T}\times\check{T},w)}}(\mathrm{QCoh}(\check{\mathfrak{t}}^*)).
\end{aligned}$$

The third equivalence above uses [BGO20, Corollary 1.2], and the fourth equivalence above is the well-known fact that restriction to the big cell in $\check{G}$ defines an equivalence $\mathrm{DMod}(\check{N}\backslash\check{G}/_\chi\check{N}^-) \xrightarrow{\sim} \mathrm{DMod}(\check{N}\backslash\check{B}/_\chi\check{N}^-) \simeq \mathrm{DMod}(\check{T})$; see [Gan22a, Proposition 1.8], for instance. The proof of (17) is similar:

$$\begin{aligned}
\mathrm{IndCoh}(\check{\mathfrak{t}}/\!\!/W^{\mathrm{aff}}) &\simeq \mathrm{End}_{\mathrm{DMod}(\check{G})}(\mathrm{DMod}(\check{G}/_\chi\check{N}^-)) \\
&\simeq \mathrm{Fun}^L_{\mathrm{DMod}(\check{G})^{\check{N},(\check{G},w)}}(\mathrm{DMod}(\check{G}/_\chi\check{N}^-)^{\check{N}}, \mathrm{DMod}(\check{G}/_\chi\check{N}^-)^{\check{G},w}) \\
&\simeq \mathrm{Fun}^L_{\check{\mathfrak{g}}\text{-mod}^{\check{N}}}(\mathrm{DMod}(\check{T}), \check{\mathfrak{g}}\text{-mod}^{(\check{N}^-,\chi)}) \\
&\simeq \mathrm{Fun}^L_{\check{\mathfrak{g}}\text{-mod}^{\check{N}}}(\mathrm{QCoh}(\check{\mathfrak{t}}^*/\Lambda^\vee), \mathrm{QCoh}(\check{\mathfrak{t}}^*/\!\!/W)).
\end{aligned}$$

The penultimate line uses two observations. The first is that the Mellin transform gives an equivalence $\mathrm{QCoh}(\check{\mathfrak{t}}^*/\Lambda^\vee) \simeq \mathrm{DMod}(\check{T})$, which can be identified with $\mathrm{DMod}(\check{N}\backslash\check{G}/_\chi\check{N}^-)$. The second observation is that there is an equivalence $\check{\mathfrak{g}}\text{-mod}^{(\check{N}^-,\chi)} \simeq \mathrm{QCoh}(\check{\mathfrak{t}}^*/\!\!/W)$, given by the Skryabin equivalence (see the appendix of [Pre02]).

Remark 4.1.14. Since $\check{\mathfrak{g}}/\check{G} = \mathrm{Map}(B\mathbf{G}_a, B\check{G})$, the canonical orientation of $B\mathbf{G}_a$ defined a 1-shifted symplectic structure on $\check{\mathfrak{g}}/\check{G}$ via [PTVV13, Theorem 2.5]. The quasi-classical limit (i.e., $\hbar \to 0$) of the quantized equivalence (14) gives the following strengthening of Theorem 4.1.3. The Kostant slice $\check{\mathfrak{t}}/\!\!/W \to \check{\mathfrak{g}}/\check{G}$ is a Lagrangian morphism by [Saf20, Proposition 4.18], so that the self-intersection $\check{\mathfrak{t}}/\!\!/W \times_{\check{\mathfrak{g}}/\check{G}} \check{\mathfrak{t}}/\!\!/W$ admits the structure of a symplectic stack (using [PTVV13, Theorem 2.9]). Since this fiber product is isomorphic to $(T^*\check{T})^{\mathrm{bl}}/\!\!/W$ by (10), we obtain a Poisson bracket on $\mathcal{O}_{(T^*\check{T})^{\mathrm{bl}}/\!\!/W} \cong \mathrm{H}_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$. This structure can be seen topologically: using one of the main results of [Kla18], $C_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ can be identified with the $\mathbf{E}_3$-center of $C_*(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$. This defines a 2-shifted Poisson bracket on $\mathrm{H}_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$, which can be identified after 2-periodification with the Poisson bracket on $\mathcal{O}_{(T^*\check{T})^{\mathrm{bl}}/\!\!/W}$.

4.2. **Rationalized Langlands duality over** KU **via** [BFM05]**.** Let us now discuss the K-theoretic analogue of Theorem 4.1.3.

Definition 4.2.1. Let G be a simply-connected semisimple algebraic group or a torus. Given $w \in W$, let $N_w = N \cap w^{-1} N^- w$, so that $N_w = \prod_{\alpha \in \Phi_w} U_\alpha$, where Φ_w is the set of roots made negative by w. Let $w = \prod_{\alpha \in \Delta} s_\alpha \in W$ be a Coxeter element, and let $\dot{w}$ be a lift of w to $N_G(T)$. Define the Steinberg slice $\Sigma = \dot{w} N_w \subseteq G$. Then [Ste65] proved/stated that the composite $\Sigma \to G \to G/\!\!/G \cong T/\!\!/W$ is an isomorphism. Let $\widetilde{G} = B \times_B G$ be the multiplicative Grothendieck-Springer resolution, so that $\widetilde{G}/G = B/B$. There is a map $\widetilde{G} \to T$ sending a pair $x \in gBg^{-1}$ to $x \pmod{g[B,B]g^{-1}}$. Let $\widetilde{\Sigma}$ denote the fiber product $\Sigma \times_G \widetilde{G}$, so that the composite $\widetilde{\Sigma} \to \widetilde{G} \to T$ is an isomorphism. We will denote the inclusion of $\widetilde{\Sigma}$ by $\sigma : T \to \widetilde{G}$.

Theorem 4.2.2. *Let G be a connected and simply-connected semisimple algebraic group or a torus over $\mathbf{C}$. Let A be an $\mathbf{E}_\infty$-KU-algebra, and let $\mathbf{G} = \mathbf{G}_m$ (so $\mathcal{M}_T$ is the torus T over A). View $\widetilde{\check{G}}$ as a scheme over $\mathbf{Q}$. If $\mathrm{QCoh}(\check{T})$ is viewed as a module over $\mathrm{QCoh}(\widetilde{\check{G}}/\check{G})$ via κ^*, then there is an equivalence*

$$\mathrm{End}_{\mathrm{QCoh}(\widetilde{\check{G}}/\check{G})}(\mathrm{QCoh}(\check{T})) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}] \simeq \mathrm{Mod}_{C_*^T(\mathrm{Gr}_G(\mathbf{C}); A)} \otimes_{\mathrm{KU}} \mathrm{KU}_{\mathbf{Q}}.$$

Proof. We will assume without loss of generality that $A = \mathrm{KU}$, and identify $\mathrm{KU}_{\mathbf{Q}} \simeq \mathbf{Q}[\beta^{\pm 1}]$ as $\mathbf{E}_\infty$-$\mathbf{Q}$-algebras. By Theorem 3.2.10, there is an equivalence $C_*^T(\mathrm{Gr}_G(\mathbf{C}); A) = \mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee \simeq \mathcal{O}_{T^*_{\mathbf{G}_m} \check{T}_A}$ as $\mathbf{E}_1$-algebras in $\mathrm{QCoh}(T_A)$. It follows that $\mathrm{Mod}_{C_*^T(\mathrm{Gr}_G(\mathbf{C}); A)} \simeq \mathrm{QCoh}((T^*_{\mathbf{G}_m} \check{T}_A)^{\mathrm{bl}})$, although only as $\mathbf{E}_0$-monoidal categories (i.e., an exact equivalence of presentable stable ∞-categories). There is an equivalence $(T^*_{\mathbf{G}_m} \check{T}_A)^{\mathrm{bl}} \otimes_{\mathrm{KU}} \mathrm{KU}_{\mathbf{Q}} \simeq (T^*_{\mathbf{G}_m} \check{T}_{\mathbf{Q}})^{\mathrm{bl}} \otimes_{\mathbf{Q}} \mathrm{KU}_{\mathbf{Q}}$. Note that since G is assumed to be simply-connected, $\check{G}$ is of adjoint type. It therefore suffices to show that over a field k of characteristic zero, there is an equivalence $\mathrm{End}_{\mathrm{QCoh}(\widetilde{\check{G}}/\check{G})}(\mathrm{QCoh}(\check{T})) \simeq \mathrm{QCoh}((T^*_{\mathbf{G}_m} \check{T})^{\mathrm{bl}})$.

As in Proposition 4.1.7, there is an equivalence $\mathrm{End}_{\mathrm{QCoh}(\widetilde{\check{G}}/\check{G})}(\mathrm{QCoh}(\check{T})) \simeq \mathrm{QCoh}(\check{T} \times_{\widetilde{\check{G}}/\check{G}} \check{T})$, so it suffices to establish the existence of a Cartesian square

$$\begin{tikzcd} (T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}} \arrow[r] \arrow[d] & \check{T} \arrow[d, "\sigma"] \\ \check{T} \arrow[r, "\sigma"'] & \widetilde{\check{G}}/\check{G}. \end{tikzcd} \tag{18}$$

Again, one can reduce to the case when $\check{G}$ has semisimple rank 1 by the argument of [BFM05, Section 4.3]. Since we are assuming that $\check{G}$ is of adjoint type, we need to consider $\check{G} = \mathrm{PGL}_2$. However, most of the following argument works for general connected $\check{G}$, so we will illustrate the calculation when $\check{G} = \mathrm{SL}_2$. (We will describe an alternative simpler calculation in the case $\check{G} = \mathrm{PGL}_2$ later.) View a point in $\widetilde{\check{G}}$ as a pair $(x \in \mathrm{SL}_2, \ell \subseteq \mathbf{C}^2)$ such that x preserves ℓ. The Steinberg slice $\sigma : \check{T} \cong \mathbf{G}_m \to \widetilde{\mathrm{SL}_2}$ is the map sending $\lambda \in \mathbf{G}_m$ to the pair (x, ℓ) with

$$x = \left(\begin{smallmatrix} \lambda + \lambda^{-1} & -1 \\ 1 & 0 \end{smallmatrix}\right), \ \ell = [\lambda : 1].$$

Note that this indeed a well-defined point in $\widetilde{\mathrm{SL}}_2$, since one can check that x preserves ℓ. This calculation of $\sigma(\lambda)$ is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}_m \cong \check{T} \arrow[r, "\sigma"] \arrow[d, "\lambda\mapsto\lambda+\lambda^{-1}"'] & \widetilde{\mathrm{SL}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \check{T}/\!\!/W \arrow[r, "\sigma", "\lambda\mapsto\left(\begin{smallmatrix}\lambda & -1\\ 1 & 0\end{smallmatrix}\right)"'] & \mathrm{SL}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathrm{SL}}_2$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, one can directly compute that g commutes with $\left(\begin{smallmatrix} \lambda+\lambda^{-1} & -1 \\ 1 & 0 \end{smallmatrix}\right)$ if and only if $a = c(\lambda+\lambda^{-1}) + d$ and $b = -c$. Therefore, $g = \left(\begin{smallmatrix} c(\lambda+\lambda^{-1})+d & -c \\ c & d \end{smallmatrix}\right)$ for $c, d \in k$. In order for $\det(g) = 1$, we need

$$c^2 + d^2 + cd(\lambda + \lambda^{-1}) = 1.$$

As long as $\lambda \neq \pm 1$, both x and g can be simultaneously diagonalized by $\left(\begin{smallmatrix} \lambda & \lambda^{-1} \\ 1 & 1 \end{smallmatrix}\right)$: the diagonalization of x is $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right)$, and the diagonalization of g is $\left(\begin{smallmatrix} c\lambda+d & 0 \\ 0 & c\lambda^{-1}+d \end{smallmatrix}\right)$. If $t = c\lambda + d$, then $c\lambda^{-1} + d = t^{-1}$ by the above determinant relation. We also have that $a = t - \frac{\lambda(t-t^{-1})}{\lambda-\lambda^{-1}}$ and $c = \frac{t-t^{-1}}{\lambda-\lambda^{-1}}$. This shows that $\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 1}, \frac{t-t^{-1}}{\lambda-\lambda^{-1}}]$ (even if if k is of characteristic 2). $\square$

Remark 4.2.3. In the above argument for Theorem 4.2.2, we assumed $\check{G} = \mathrm{SL}_2$. The resulting intersection $\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m$ is *not* isomorphic to $(T^*_{\mathbf{G}_m}\mathbf{G}_m)^{\mathrm{bl}}$. Indeed, $(T^*_{\mathbf{G}_m}\mathbf{G}_m)^{\mathrm{bl}} = \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 1}, \frac{t-1}{\lambda-1}]$, while we computed that $\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong \operatorname{Spec} k[\lambda^{\pm 1}, t^{\pm 1}, \frac{t^2-1}{\lambda^2-1}]$. This minor modification is why we assume that $\check{G}$ is of adjoint type in Theorem 4.2.2; for example, we have $\mathbf{G}_m \times_{\widetilde{\mathrm{PGL}}_2/\mathrm{PGL}_2} \mathbf{G}_m \cong (T^*_{\mathbf{G}_m}\mathbf{G}_m)^{\mathrm{bl}}$. See the calculation in the second proof of Theorem 4.2.2 below.

An alternative argument for the Cartesian square (18) can be given using the multiplicative Kostant slice.

Definition 4.2.4. Let $e \in \mathfrak{n}$ be a principal nilpotent element. Then the map $\mathbf{G}_a \to G$ corresponding to e factors through the map $\mathbf{G}_a = B \to \mathrm{SL}_2$; we will denote the image of the standard generator $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right) \in B^-$ under the map $\mathrm{SL}_2 \to G$ by $f \in G$. Let $Z_G(e)^\circ$ be the connected component of the identity in the centralizer of e in G. Define the *multiplicative Kostant slice* $\mathcal{S}_\mu$ by $Z_G(e)^\circ \cdot f \subseteq G$. Since G is assumed to be simply-connected, the composite $\mathcal{S}_\mu \to G \to G/\!\!/G \cong T/\!\!/W$ is an isomorphism. We will often denote the inclusion of the Kostant slice by $\kappa : T/\!\!/W \to G$. Let $\widetilde{\mathcal{S}}_\mu$ denote the fiber product $\widetilde{\mathcal{S}}_\mu \times_G \widetilde{G}$, so that the composite $\widetilde{\mathcal{S}}_\mu \to \widetilde{G} \to T$ is an isomorphism; we will denote the inclusion of $\widetilde{\mathcal{S}}_\mu$ as a map $\kappa : \widetilde{\mathcal{S}}_\mu \cong T \to \widetilde{G}$.

As with the additive Kostant slice, we will only care about the composite $T \to \widetilde{G} \to \widetilde{G}/G$ below, so we will also denote it by κ. If we identify $\widetilde{G}/G \cong B/B$, then the map κ admits a simple description: it is the composite $T \to B \to B/B$ which sends $x \mapsto xf$.

Alternative argument for the Cartesian square (18). We will give an alternative argument of the Cartesianness of (18), where the map $\check{T} \to \widetilde{\check{G}}/\check{G}$ is chosen to be the multiplicative Kostant slice instead of the Steinberg slice. Again, we only review the calculation for $\check{G} = \mathrm{SL}_2$; this was done in [BFM05]. For convenience, we will drop the "check"s. As before, there are "two" ways to compute in the case $G = \mathrm{SL}_2$. First, we present the argument which is essentially present in [BFM05]. If $\lambda \in \mathbf{G}_m$, we denote $\lambda + \lambda^{-1} \in \mathbf{A}^1$ by $f(\lambda)$. The Kostant slice $\kappa : \check{T} \cong \mathbf{G}_m \to \widetilde{\mathrm{SL}}_2$ is the map sending $\lambda \in \mathbf{G}_m$ to the pair (x, ℓ) with

$$x = \left(\begin{smallmatrix} f(\lambda)-1 & f(\lambda)-2 \\ 1 & 1 \end{smallmatrix}\right), \ \ell = [\lambda - 1 : 1].$$

Note that this indeed a well-defined point in $\widetilde{\mathrm{SL}}_2$, since one can check that x preserves ℓ: the key point is the conic relation

$$2\lambda = f(\lambda) - \sqrt{f(\lambda)^2 - 4}.$$

Indeed, this calculation of $\kappa(\lambda)$ is essentially immediate from the requirement that the following diagram commutes:

$$\begin{tikzcd} \mathbf{G}_m \cong \check{T} \arrow[r, "\kappa"] \arrow[d, "\lambda\mapsto f(\lambda)"'] & \widetilde{\mathrm{SL}}_2 \arrow[d] \\ \mathbf{A}^1 \cong \check{T}/\!\!/W \arrow[r, "\kappa", "\lambda\mapsto\left(\begin{smallmatrix}\lambda-1 & \lambda-2\\ 1 & 1\end{smallmatrix}\right)"'] & \mathrm{SL}_2. \end{tikzcd}$$

Moreover, the SL_2-action on $\widetilde{\mathrm{SL}}_2$ sends $g \in \mathrm{SL}_2$ and (x, ℓ) to $(\mathrm{Ad}_g(x), g\ell)$. If $g = \left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$, we directly compute that $\mathrm{Ad}_g(x) = x$ if and only if $b = c(f(\lambda)-2)$ and $a - d = (f(\lambda)-2)c$, in which case g also preserves ℓ. Therefore, $g = \left(\begin{smallmatrix}(f(\lambda)-2)c+d & (f(\lambda)-2)c\\ c & d\end{smallmatrix}\right)$ for $c, d \in k$. In order for $\det(g) = 1$, we need

$$d^2 + c(f(\lambda)-2)(d-c) = 1.$$

Both x and g can be simultaneously diagonalized (if $f(\lambda) \neq \pm 2$); note that $\lambda + \lambda^{-1}$ is an eigenvalue of x. If t is an eigenvalue of g, then we have $c = \frac{t-t^{-1}}{\lambda-\lambda^{-1}}$ and $d = \frac{t^2\lambda+1}{t(\lambda+1)}$. When k is not of characteristic 2, this shows that $\mathbf{G}_m \times_{\widetilde{\mathrm{SL}}_2/\mathrm{SL}_2} \mathbf{G}_m \cong k[\lambda^{\pm 1}, t^{\pm 1}, \frac{t-t^{-1}}{\lambda-\lambda^{-1}}]$, as desired.

For the second method of calculation when $G = \mathrm{SL}_2$ (which works in arbitary characteristic), we use the fact that $\kappa : T \to \widetilde{G}/G$ can be identified with the composite $T \to B \to B/B$ sending $x \mapsto xf$. Then, $T \times_{B/B} T$ is isomorphic to the subvariety of $T \times B$ consisting of pairs (x, g) with $x \in T$ (identified with the matrix $\left(\begin{smallmatrix}x & 0\\ 0 & x^{-1}\end{smallmatrix}\right)$) and $\mathrm{Ad}_g(xf) = xf$. Note that xf is the matrix $\left(\begin{smallmatrix}x & 0\\ x^{-1} & x^{-1}\end{smallmatrix}\right)$. If $g = \left(\begin{smallmatrix}a & 0\\ b & a^{-1}\end{smallmatrix}\right) \in B$, then

$$\mathrm{Ad}_g\left(\begin{smallmatrix}x & 0\\ x^{-1} & x^{-1}\end{smallmatrix}\right) = \left(\begin{smallmatrix}x & 0\\ a^{-2}x^{-1}+ba^{-1}(x-x^{-1}) & x^{-1}\end{smallmatrix}\right).$$

Therefore, $\mathrm{Ad}_g(xf) = xf$ if and only if

$$a^{-2}x^{-1} + ba^{-1}(x - x^{-1}) = x^{-1},$$

which forces $b = \frac{a-a^{-1}}{x^2-1}$. This implies that $T \times_{B/B} T$ is isomorphic to $\operatorname{Spec} k[x^{\pm 1}, a^{\pm 1}, \frac{a-a^{-1}}{x^2-1}]$, as desired.

We can also run this argument in the case $G = \mathrm{PGL}_2$ (again in arbitary characteristic). Again, $T \times_{B/B} T$ is isomorphic to the subvariety of $T \times B$ consisting of pairs (x, g) with $x \in T$ (identified with the matrix $\left(\begin{smallmatrix}x & 0\\ 0 & 1\end{smallmatrix}\right)$) and $\mathrm{Ad}_g(xf) = xf$. Note that xf is the matrix $\left(\begin{smallmatrix}x & 0\\ 1 & 1\end{smallmatrix}\right)$. If $g = \left(\begin{smallmatrix}a & 0\\ b & 1\end{smallmatrix}\right) \in B$, then

$$\mathrm{Ad}_g\left(\begin{smallmatrix}x & 0\\ 1 & 1\end{smallmatrix}\right) = \left(\begin{smallmatrix}x & 0\\ ba^{-1}(x-1)+a^{-1} & x^{-1}\end{smallmatrix}\right).$$

Therefore, $\mathrm{Ad}_g(xf) = xf$ if and only if

$$ba^{-1}(x-1) + a^{-1} = 1,$$

which forces $b = \frac{a-1}{x-1}$. This implies that $T \times_{B/B} T$ is isomorphic to $\operatorname{Spec} k[x^{\pm 1}, a^{\pm 1}, \frac{a-1}{x-1}]$, as desired. $\square$

Remark 4.2.5. The above argument could have also been deduced from the first argument of Theorem 4.2.2: indeed, the Kostant section $\check{T}/\!\!/W \cong \mathbf{A}^1 \to \mathrm{SL}_2$ sending $\lambda \mapsto \left(\begin{smallmatrix}\lambda-1 & \lambda-2\\ 1 & 1\end{smallmatrix}\right)$ and the Steinberg section $\check{T}/\!\!/W \cong \mathbf{A}^1 \to \mathrm{SL}_2$ sending $\lambda \mapsto \left(\begin{smallmatrix}\lambda & -1\\ 1 & 0\end{smallmatrix}\right)$ are conjugated into each other by the matrix $\left(\begin{smallmatrix}c(\lambda-1)+d & -(c+d)\\ c & d\end{smallmatrix}\right)$ for $c, d \in k$.

Observation 4.2.6. In the second argument for the Cartesian square (18), we may replace the symbol λ by the symbol e^λ; then, $e^\lambda - 1$ is the exponential of the multiplicative formal group law. In particular, the defining equation for the line ℓ in the cases of $\mathbf{G} = \mathbf{G}_a, \mathbf{G}_m$ precisely describes the exponential for $\mathbf{G}$. This observation suggests a natural generalization of Theorem 4.1.3 and Theorem 4.2.2 for general $\mathbf{G}$, which we will discuss below.

Remark 4.2.7. In [BFM05], the following analogue of (18) is established (over $\mathbf{C}$, but this does not affect the statement): there is a Cartesian square

$$\begin{tikzcd} (T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}}/\!\!/W \arrow[r] \arrow[d] & \check{T}/\!\!/W \arrow[d, "\kappa"] \\ \check{T}/\!\!/W \arrow[r, "\kappa"'] & \check{G}/\check{G}, \end{tikzcd} \tag{19}$$

where the top-left corner can be identified with $\operatorname{Spec} C_0^{\check{G}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})\otimes_{\mathrm{KU}}\mathrm{KU}_{\mathbf{Q}}$. We can take the fiber product of (18) with itself over (19) to obtain a Cartesian square

$$\begin{tikzcd} (T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}} \times_{(T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}}/\!\!/W} (T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}} \arrow[r] \arrow[d] & \check{T}\times_{\check{T}/\!\!/W}\check{T} \arrow[d, "\kappa"] \\ \check{T}\times_{\check{T}/\!\!/W}\check{T} \arrow[r, "\kappa"'] & (\tilde{\check{G}}\times_{\check{G}}\tilde{\check{G}})/\check{G}. \end{tikzcd} \tag{20}$$

Using Theorem 4.2.2 and the above discussion, one can use (20) to show that $\mathrm{End}_{\mathrm{QCoh}((\tilde{\check{G}}\times_{\check{G}}\tilde{\check{G}})/\check{G})}(\mathrm{QCoh}(\check{T}\times_{\check{T}/\!\!/W}\check{T}))\otimes_{\mathbf{Q}}\mathbf{Q}[\beta^{\pm1}]$ can be identified with $\mathrm{LMod}_{C_*^{T}(\mathrm{Fl}_G(\mathbf{C});\mathrm{KU})}\otimes_{\mathrm{KU}}\mathrm{KU}_{\mathbf{Q}}$. This can be viewed as a "once-looped" version of a K-theoretic analogue of Bezrukavnikov's equivalence from [Bez16].

Remark 4.2.8. We expect that most of the steps of Theorem 4.1.11 can be replicated to study $\mathrm{LMod}_{C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})}\otimes\mathbf{Q}$. More precisely, let $d\in\mathbf{Z}$, and fix a symmetric bilinear form $(-,-):\Lambda\times\Lambda\to\frac{1}{d}\mathbf{Z}$ such that whose Gram matrix is the associated Cartan matrix (i.e., (α_i,α_j) is the a_{ij} entry of the associated Cartan matrix). We then have the quantum group $U_q(\mathfrak{g})$ defined over $\mathbf{Z}[q^{\pm1}]$ associated to the pairing $\Lambda\times\Lambda\to\mathbf{Z}[q^{\pm1}]$ sending $\lambda,\mu\mapsto q^{-(\lambda,\mu)}$. Following [KS20, Definition 4.24], define *quantum universal category* $\mathcal{O}_q^{\mathrm{univ}}$ as the ∞-category of $(U_q(\mathfrak{g}),U_q(\mathfrak{t}))$-bimodules whose diagonal $U_q(\mathfrak{b})$-action is integrable.

Let (W,Δ) be a crystallographic root system, let $\Lambda^\vee=\mathbf{Z}\Phi$ denote the associated root lattice, and let $T=\operatorname{Spec}\mathbf{Z}[\Lambda]$ denote the associated torus. Each $\alpha\in W$ defines an operator s_α on $\mathcal{O}_T$. Define the *multiplicative nil-Hecke algebra* $\mathcal{H}(T,W)$ as the subalgebra of $\mathrm{Frac}(\mathcal{O}_T)\rtimes\mathbf{Q}[W]$ generated by $\mathcal{O}_T$ and the operators $T_\alpha=\frac{1}{e^\alpha-1}(s_\alpha-1)$. (Also see [EW22] for a study of a multiplicative analogue of Soergel theory.) Note that there are relations

$$T_\alpha^2=T_\alpha,\ (T_\alpha T_\beta)^{m_{\alpha,\beta}}=(T_\beta T_\alpha)^{m_{\alpha,\beta}},\ x\cdot T_\alpha=T_\alpha\cdot s_\alpha(x)+T_\alpha(x),\ \alpha\in\Delta.$$

Recall that $m_{\alpha_i\alpha_j}$ is 2, 3, 4, 6, ∞ if $a_{ij}a_{ji}$ is 0, 1, 2, 3, ≥4 (respectively). See also [LSS10, Section 2.2]. Note that if $\lambda\in\Lambda$ (corresponding to the function e^λ on T), we have $T_\alpha(e^\lambda)=[\langle\alpha^\vee,\lambda\rangle]_{e^\alpha}e^\lambda$, where $[\langle\alpha^\vee,\lambda\rangle]_{e^\alpha}$ denotes the q-integer $\frac{q^{\langle\alpha^\vee,\lambda\rangle}-1}{q-1}$ with $q=e^\alpha$.

Then, we expect:

Conjecture 4.2.9. *There is a Kostant functor* $\check{\mathcal{O}}_q^{\mathrm{univ}}\to\mathrm{QCoh}(\check{T}_{\mathbf{Q}}\times\mathbf{G}_m^q)$ *(where* $\mathbf{G}_m^q=\operatorname{Spec}\mathbf{Q}[q^{\pm1}]$*) such that there is a* $\mathbf{Q}[\beta^{\pm1},q^{\pm1}]$*-linear equivalence*

$$\mathrm{LMod}_{C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})}\otimes\mathbf{Q}\simeq\mathrm{End}_{\check{\mathcal{O}}_q^{\mathrm{univ}}}(\mathrm{QCoh}(\check{T}_{\mathbf{Q}}\times\mathbf{G}_m^q))\otimes_{\mathbf{Q}[q^{\pm1}]}\mathbf{Q}[\beta^{\pm1},q^{\pm1}]. \tag{21}$$

Similarly, if $\mathrm{HC}_q(\check{G})$ *denotes the category of* [KS20, Definition 2.24], *there is a Kostant functor* $\mathrm{HC}_q(\check{G})\to\mathrm{QCoh}(\check{T}_{\mathbf{Q}}/\!\!/W\times\mathbf{G}_m^q)$ *and a* $\mathbf{Q}[\beta^{\pm1},q^{\pm1}]$*-linear equivalence*

$$\mathrm{LMod}_{C_*^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})}\otimes\mathbf{Q}\simeq\mathrm{End}_{\mathrm{HC}_q(\check{G})}(\mathrm{QCoh}(\check{T}_{\mathbf{Q}}/\!\!/W\times\mathbf{G}_m^q))\otimes_{\mathbf{Q}[q^{\pm1}]}\mathbf{Q}[\beta^{\pm1},q^{\pm1}]. \tag{22}$$

At the moment, we are only able to describe the left-hand side in terms of combinatorial data. Let $e=\frac{1}{\#W}\sum_{w\in W}w$ be the symmetrizer idempotent. Using Corollary 3.2.4 and [LSS10, Proposition 2.6] (see also Proposition 3.1.8), one can show that $C_*^{\widetilde{T}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})\otimes\mathbf{Q}$ is isomorphic to a 2-periodification of $\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e$, where the parameter $q\in\pi_0\mathrm{KU}_{\mathbf{G}_m^{\mathrm{rot}}}\cong\mathbf{Z}[q^{\pm1}]$ corresponds to the coordinate on $\mathbf{G}_m^q\subseteq\widetilde{T}$ viewed as an element of $\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e$. Similarly, $C_*^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})\otimes\mathbf{Q}$ is isomorphic to a 2-periodification of $e\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e$. The conjectural equivalence (22) then reduces to proving an (also conjectural) equivalence

$$\mathrm{End}_{\check{\mathcal{O}}_q^{\mathrm{univ}}}(\mathrm{QCoh}(\check{T}\times\mathbf{G}_m^q))\simeq\mathrm{LMod}_{\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e},$$

$$\mathrm{End}_{\mathrm{HC}_q(\check{G})}(\mathrm{QCoh}(\check{T}/\!\!/W\times\mathbf{G}_m^q))\simeq\mathrm{LMod}_{e\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e}.$$

This may be understood as a quantum analogue of [Gin18, Theorem 8.1.2]. Note that the above equivalences is now a statement which is squarely on one side of Langlands duality. In the case $G=\mathrm{SL}_2$, we described $C_*^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})\otimes\mathbf{Q}$ (and hence $e\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e$) below in Example A.5; it might be possible to use this calculation to compare with $\mathrm{End}_{\check{\mathcal{O}}_q^{\mathrm{univ}}}(\mathrm{QCoh}(\check{T}\times\mathbf{G}_m^q))$ for $\check{G}=\mathrm{PGL}_2$. Note that (for general G) just as $(T\times\check{T})^{\mathrm{bl}}$ is birational to $T\times\check{T}$, the map from the algebra of q-difference operators on $\check{T}$ to $\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})e$ is an isomorphism after a particular localization. One therefore expects $\check{\mathcal{O}}_q^{\mathrm{univ}}$ and $\mathrm{HC}_q(\check{T})$ to generically be equivalent. This is indeed true, and can be seen using [KS20, Theorem 4.33] (although the functor $\check{\mathcal{O}}_q^{\mathrm{univ}}\to\mathrm{HC}_q(\check{T})$ in *loc. cit.* is not our expected functor κ).

Remark 4.2.10. Since $\check{G}/\check{G}=\mathrm{Map}(S^1,B\check{G})$, the canonical orientation of S^1 defined a 1-shifted symplectic structure on $\check{G}/\check{G}$ via [PTVV13, Theorem 2.5]. The quasi-classical limit (i.e., $q\to 1$) of the conjectural equivalence (22) gives the following strengthening of Theorem 4.2.2. (This strengthening can be proved independently of (22).)

Observe that the Kostant slice $\check{T}/\!\!/W\to\check{G}/\check{G}$ is a Lagrangian morphism. It follows that the self-intersection $\check{T}/\!\!/W\times_{\check{G}/\check{G}}\check{T}/\!\!/W$ admits the structure of a symplectic stack by [PTVV13, Theorem 2.9]. Since this fiber product is isomorphic to $(T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}}/\!\!/W$ by (19), we obtain a Poisson bracket on $\mathcal{O}_{(T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}}/\!\!/W}\cong\pi_0 C_*^G(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})$. This structure can be seen topologically: using one of the main results of [Kla18], $C_*^G(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})$ can be identified with the $\mathbf{E}_3$-center of $C_*(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})$. This defines a 2-shifted Poisson bracket on $\pi_*C_*^G(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})$, which can be identified (using the 2-periodicity of KU) with the Poisson bracket on $\mathcal{O}_{(T^*_{\mathbf{G}_m}\check{T})^{\mathrm{bl}}/\!\!/W}$.

Remark 4.2.11. Following Conjecture 4.2.9, one can also hope for a result analous to (21) when $q\rightsquigarrow\zeta_p$ is specialized to a primitive pth root of unity. Namely, consider the ∞-category $\mathrm{LMod}_{C_*^{T\times\mu_{p,\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})}$, where $\mu_{p,\mathrm{rot}}\subseteq S^1_{\mathrm{rot}}$ acts by loop rotation. Note that $C_*^{T\times\mu_{p,\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})$ is a module over $\mathrm{KU}^{h\mathbf{Z}/p}$, and $\pi_*\mathrm{KU}^{h\mathbf{Z}/p}\cong\mathbf{Z}[\![q-1]\!][\beta^{\pm1}]/(q^p-1)$. Inverting $q-1$, we find that $C_*^{T\times\mu_{p,\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})[\frac{1}{q-1}]$ is a module over $\mathrm{KU}^{h\mathbf{Z}/p}[\frac{1}{q-1}]\simeq\mathrm{KU}^{t\mathbf{Z}/p}\simeq\mathbf{Q}(\zeta_p)[\beta^{\pm1}]$. We then expect the following (likely simpler) analogues of (21) and (22):

Conjecture 4.2.12. *There are Kostant functors* $\check{\mathcal{O}}_{\zeta_p}^{\mathrm{univ}}\to\mathrm{QCoh}(\check{T}_{\mathbf{Q}(\zeta_p)})$ *and* $\mathrm{HC}_{\zeta_p}(\check{G})\to\mathrm{QCoh}(\check{T}_{\mathbf{Q}(\zeta_p)}/\!\!/W)$ *such that there are* $\mathbf{Q}(\zeta_p)[\beta^{\pm1}]$*-linear equivalences*

$$\mathrm{LMod}_{C_*^{T\times\mu_{p,\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})[\frac{1}{q-1}]}\simeq\mathrm{End}_{\check{\mathcal{O}}_{\zeta_p}^{\mathrm{univ}}}(\mathrm{QCoh}(\check{T}_{\mathbf{Q}(\zeta_p)}))\otimes_{\mathbf{Q}}\mathbf{Q}[\beta^{\pm1}],$$
$$\mathrm{LMod}_{C_*^{G\times\mu_{p,\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})[\frac{1}{q-1}]}\simeq\mathrm{End}_{\mathrm{HC}_{\zeta_p}(\check{G})}(\mathrm{QCoh}(\check{T}_{\mathbf{Q}(\zeta_p)}/\!\!/W))\otimes_{\mathbf{Q}}\mathbf{Q}[\beta^{\pm1}].$$

Note that there is no rationalization necessary on the left-hand sides.

As with Conjecture 4.2.9, Conjecture 4.2.12 reduces to proving the (also conjectural) equivalences

$$\mathrm{End}_{\check{\mathcal{O}}_{\zeta_p}^{\mathrm{univ}}}(\mathrm{QCoh}(\check{T}_{\mathbf{Q}(\zeta_p)}))\simeq\mathrm{LMod}_{\mathcal{H}_{\zeta_p}(\widetilde{T},W^{\mathrm{aff}})e},$$
$$\mathrm{End}_{\mathrm{HC}_{\zeta_p}(\check{G})}(\mathrm{QCoh}(\check{T}_{\mathbf{Q}(\zeta_p)}/\!\!/W))\simeq\mathrm{LMod}_{e\mathcal{H}_{\zeta_p}(\widetilde{T},W^{\mathrm{aff}})e},$$

where $\mathcal{H}_{\zeta_p}(\widetilde{T},W^{\mathrm{aff}})$ denotes the algebra obtained from $\mathcal{H}(\widetilde{T},W^{\mathrm{aff}})$ by setting q (arising from the loop rotation torus in $\widetilde{T}$) to ζ_p.

4.3. G-loop spaces.

Definition 4.3.1. Let $\mathbf{G}_0$ be a commutative group scheme over a ring A_0 (even an $\mathbf{E}_\infty$-ring A_0). Let $\mathfrak{D}(\mathbf{G}_0)$ denote the stack $\mathrm{Hom}(\mathbf{G}_0,B\mathbf{G}_m)$. If X is an A_0-scheme, we will write $\mathcal{L}_{\mathbf{G}_0}X$ to denote the mapping stack $\mathrm{Map}(\mathfrak{D}(\mathbf{G}_0),X)$; this is the $\mathbf{G}_0$-*loop space* of X.

Example 4.3.2. If $\mathbf{G}_0=\mathbf{G}_m$, then $\mathfrak{D}(\mathbf{G}_0)=B\mathbf{Z}$, i.e., is S^1 viewed as a constant stack. If $\mathbf{G}_0$ is an abelian variety, then $\mathfrak{D}(\mathbf{G}_0)$ is the dual abelian variety $\mathbf{G}_0^\vee$. If $\mathbf{G}_0=\mathbf{Z}$, then $\mathfrak{D}(\mathbf{G}_0)$ is $B\mathbf{G}_m$. Let $\mathbf{W}$ denote the commutative group scheme over $\mathbf{Z}_{(p)}$ of p-typical Witt vectors, and let $\mathbf{W}_n$ denote the group scheme over $\mathbf{Z}_{(p)}$ of truncated p-typical Witt vectors of length n. Let $\mathbf{W}[F]$ denote the kernel of Frobenius on $\mathbf{W}$, and similarly for $\mathbf{W}_n[F]$. When working over $\mathbf{Z}_{(p)}$, Lemma 4.3.5 below states that if $\hat{\mathbf{G}}_a$ is the formal completion of $\mathbf{G}_a$ at the origin, then $\mathfrak{D}(\hat{\mathbf{G}}_a)\cong B\mathbf{W}[F]$. If $\mathbf{G}_0$ is a finite flat, diagonal, or constant group scheme, then $\mathfrak{D}(\mathbf{G}_0)$ can be identified with the classifying stack of the Cartier dual of $\mathbf{G}_0$. In general, there is a canonical map $\mathbf{G}_0\to\mathfrak{D}(\mathfrak{D}(\mathbf{G}_0))$, and the above examples imply that it is an isomorphism if $\mathbf{G}_0$ is a finite product of abelian varieties, classifying stacks of groups of multiplicative type, and finitely generated abelian groups. If this is the case, we say that $\mathbf{G}_0$ is *dualizable*.

Remark 4.3.3. Note that the pairing $\mathbf{G}_0 \times \mathfrak{D}(\mathbf{G}_0) \to B\mathbf{G}_m$ defines a line bundle over $\mathbf{G}_0 \times \mathfrak{D}(\mathbf{G}_0)$, which we will denote by $\mathcal{P}$ and call the *Poincaré line bundle*. If $\mathbf{G}_0$ is an abelian variety, this is the usual Poincaré line bundle over $\mathbf{G}_0 \times \mathbf{G}_0^\vee$. If $\mathbf{G}_0 = \mathbf{G}_m$, the Poincaré line bundle gives the equivalence $\mathrm{Rep}(\mathbf{Z}) \simeq \mathrm{QCoh}(\mathbf{G}_m)$ obtained by viewing $\mathbf{G}_m$ as the torus associated to the monoid $\mathbf{Z}$.

Remark 4.3.4. Arguing as in Example 4.3.2, one sees that if $\mathbf{G}_0$ is replaced by its formal completion at the zero section, then the $\mathbf{G}_0$-loop space recovers the loop space of [Mou21].

Thanks to Sasha Petrov for a discussion regarding the following (presumably well-known) result.

Lemma 4.3.5. *There is an isomorphism of group schemes over $\mathbf{Z}_{(p)}$ between $\mathbf{W}[F^n] := \ker(F^n : \mathbf{W} \to \mathbf{W})$ and the Cartier dual of the completion of $\mathbf{W}_n$.*

Proof. Let us model $\mathbf{W}$ by the big Witt vectors. Given $f(t) \in \mathbf{W}$, let $a_1, a_p, a_{p^2}, \cdots$ denote the ghost components of f, so that $td\log(f(t)) = \sum_{m\geq 0} a_{p^m} t^{p^m}$. Then $f(t) \in \mathbf{W}[F^n]$ if and only if $a_{p^m} = 0$ for $m \geq n$.

Let us first prove the claim of the proposition when $n = 1$. Then, $d\log(f(t))$ is a constant, and $f(0) = 1$; we claim that this is equivalent to the condition that f defines a homomorphism $\hat{\mathbf{G}}_a \to \mathbf{G}_m$, i.e., that $f(x+y) = f(x)f(y)$. To check this, first suppose that $f(x+y) = f(x)f(y)$. Then $\partial_x f(x+y) = f(y)f'(x)$, so that $\frac{\partial_x f(x+y)}{f(x+y)} = \frac{f'(x)}{f(x)}$ is independent of y. Taking $x = 0$, we see that $\frac{f'(x)}{f(x)} = d\log(f(x))$ is constant. The reverse direction is similar.

In the general case, note that since the Frobenius on $\mathbf{W}$ shifts the ghost components by $F : (a_1, a_p, a_{p^2}, \cdots) \mapsto (a_p, a_{p^2}, a_{p^3}, \cdots)$, the Frobenius F applied to f satisfies:

$$d\log(F^j(f)(t)) = \sum_{m=0}^{n-j} a_{p^{m+j}} t^{p^m},$$

so that there is an equality of power series

$$F^j(f)(t) = \exp\left(\sum_{m=0}^{n-j} \frac{a_{p^{m+j}}}{p^m} t^{p^m}\right).$$

Define a map $g : \hat{\mathbf{W}}_n \to \mathbf{G}_m$ on Witt components $(x_0, \cdots, x_{n-1})$ (not ghost components!) as follows:

$$g(x_0, \cdots, x_{n-1}) = \prod_{j=0}^{n-1} F^j(f)(x_j) = \exp\left(\sum_{j=0}^{n-1}\sum_{m=0}^{n-j} \frac{a_{p^{m+j}}}{p^m} x_j^{p^m}\right) = \exp\left(\sum_{m=0}^{n-1} \frac{a_{p^m}}{p^m}\left(\sum_{j=0}^{m} p^j x_j^{p^{m-j}}\right)\right).$$

The coefficient of $\frac{a_{p^m}}{p^m}$ is precisely the mth Witt polynomial, so that the function g is indeed additive on $\hat{\mathbf{W}}_n$. Moreover, the assignment $f \mapsto g$ indeed gives an isomorphism $\mathbf{W}[F^n] \xrightarrow{\sim} \mathrm{Hom}(\hat{\mathbf{W}}_n, \mathbf{G}_m)$, as one can check using the case $n = 1$ and the fact that it induces an isomorphism over $\mathbf{Q}$. $\square$

Remark 4.3.6. Lemma 4.3.5 is true over $\mathbf{F}_p$ by the theory of Dieudonné modules: the Dieudonné module of $\mathbf{W}_m[F^n]$ over $\mathbf{F}_p$ is $W(k)[F, V]/(F^n, V^m)$, while the Dieudonné module of $\mathbf{W}_n[F^m]$ over $\mathbf{F}_p$ is $W(k)[F, V]/(F^m, V^n)$.

Definition 4.3.7. Suppose $\mathbf{G}$ is 1-dimensional and connected over a ring k. Say that $\mathbf{G}$ is *licit* if the affinization of $\mathfrak{D}(\mathbf{G}) \otimes_k k_\mathbf{Q}$ is isomorphic to $B\mathbf{G}_a$. In particular, there is a canonical morphism $\mathfrak{D}(\mathbf{G}) \to B\mathbf{G}_a$ which induces an equivalence on affinizations.

Remark 4.3.8. The base change to $k_\mathbf{Q}$ is vital in Definition 4.3.7. For example, let $\mathbf{G} = \mathbf{G}_m$, so that $\mathfrak{D}(\mathbf{G}) = B\mathbf{Z}$. Over $\mathbf{F}_p$, the affinization of $B\mathbf{Z}$ is isomorphic to $B\mathbf{W}^{F=1}$, i.e., the classifying stack of the Frobenius fixed points of the p-typical Witt vectors over $\mathbf{F}_p$. Compare with Lemma 4.3.5, where we discussed the stack $B\mathbf{W}^{F=0} = B\mathbf{W}[F]$. One can interpolate between the two via the family $B\mathbf{W}^{F=\lambda}$ over $\mathbf{A}^1 = \mathrm{Spec}\, k[\lambda]$; on the Cartier dual side, this corresponds to the interpolation between $\mathbf{G}_m$ and $\mathbf{G}_a$ via the group scheme $\mathrm{Spec}\, k[t, \lambda, \frac{1}{1-t\lambda}]$. This is discussed in [MRT19]. In general, the affinization of $\mathfrak{D}(\mathbf{G})$ is an interesting p-adic object which degenerates to $B\mathbf{W}[F]$.

Assumption 4.3.9. Fix an isomorphism $\mathbb{X}^\bullet(T) \cong \mathbb{X}_\bullet(T)$ of lattices, which will be used implicitly below without further mention. (Note that we are not asking for a W-equivariant isomorphism.) This gives an isomorphism $\mathcal{M}_T \cong \mathcal{M}_{\check{T}}$ (which we will use below as an analogue of the identification between $\check{\mathfrak{t}} = \mathfrak{t}^*$ and $\mathfrak{t}$, ubiquitous in geometric representation theory). Although potentially confusing, we will see below in the proof of Theorem 4.3.10 that this identification does not run the risk of conflating different sides of Langlands duality. In future work, we will discuss an argument to prove Corollary 4.3.11 without using such an isomorphism.

Our goal is to prove the following.

Theorem 4.3.10. *Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A and an oriented commutative A-group $\mathbf{G}$, as well as a semisimple algebraic group $\check{G}$ over $\mathbf{Q}$ which is of adjoint type. Assume that $\mathbf{G}$ is dualizable and licit in the sense of Definition 4.3.7 (by assumption, it is also 1-dimensional and smooth). Given a principal nilpotent $f \in \mathfrak{n}$, there is a map $\kappa : \mathcal{M}_T \to \mathcal{L}_{\mathbf{G}} B\check{B}^-$, and if $\mathcal{L}^0_{\mathbf{G}} B\check{B}$ denotes the connected component of $\mathcal{L}_{\mathbf{G}} B\check{B}$ containing the Kostant slice, there is a Cartesian square*

$$\begin{tikzcd} (T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}} \otimes \mathbf{Q} \arrow[r] \arrow[d] & \mathcal{M}_T \otimes \mathbf{Q} \arrow[d, "\kappa"] \\ \mathcal{M}_T \otimes \mathbf{Q} \arrow[r, "\kappa"'] & \mathbf{Q} \otimes \mathcal{L}^0_{\mathbf{G}} B\check{B}^-. \end{tikzcd}$$

Combining with Theorem 3.2.10, we obtain the following:

Corollary 4.3.11. *Suppose that G is a connected and simply-connected semisimple algebraic group or a torus over $\mathbf{C}$. Assume that $\mathbf{G}$ is dualizable and licit in the sense of Definition 4.3.7. Then there is an equivalence*

$$\mathrm{End}_{\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)}(\mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}) \simeq \mathrm{Mod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee}(\mathrm{QCoh}(\mathcal{M}_T)) \otimes_A A_{\mathbf{Q}}.$$

Remark 4.3.12. Note that Theorem 4.1.11 and Conjecture 4.2.9 suggest that $\mathrm{LMod}_{\mathcal{F}_{\tilde{T}}(\mathrm{Gr}_G(\mathbf{C}))^\vee}(\mathrm{QCoh}(\mathcal{M}_{\tilde{T}})) \otimes \mathbf{Q}$ should be viewed as $\mathrm{End}_{\mathcal{C}_{\mathbf{G}}}(\mathrm{QCoh}(\mathcal{M}_{\tilde{T}}) \otimes \mathbf{Q})$ for some $A_{\mathbf{Q}}$-linear ∞-category $\mathcal{C}_{\mathbf{G}}$ which is a 1-parameter deformation of $\mathrm{QCoh}(\mathrm{Bun}_{\check{B}^-}(\mathfrak{D}(\mathbf{G})))$. The "quantization parameter" (i.e., the analogue of $\hbar$ and q) is parametrized by the group scheme $\mathbf{G}$. This putative ∞-category $\mathcal{C}_{\mathbf{G}}$ would be an analogue of the (quantum) universal category $\mathcal{O}$.

Example 4.3.13. For example, if $\mathbf{G} = \hat{\mathbf{G}}_a$, then $\mathfrak{D}(\mathbf{G}) = B\mathbf{W}[F]$ by Lemma 4.3.5, so that $\mathfrak{D}(\mathbf{G})_{\mathbf{Q}} = B\mathbf{G}_a$, and $\mathcal{L}^0_{\mathbf{G}} B\check{B}^- = \check{\mathfrak{b}}^-_{\mathbf{Q}}/\check{B}^-_{\mathbf{Q}} \cong \tilde{\check{\mathfrak{g}}}_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}$ by [MRT19, Theorem 1.2.4]. If $\mathbf{G} = \mathbf{G}_m$, then $\mathfrak{D}(\mathbf{G}) = B\mathbf{Z} = S^1$, so that $\mathcal{L}^0_{\mathbf{G}} B\check{B}^- = \mathrm{Map}(S^1_{\mathrm{KU}_{\mathbf{Q}}}, B\check{B}^-_{\mathrm{KU}_{\mathbf{Q}}})$ is isomorphic to the 2-periodification of $\check{B}^-_{\mathbf{Q}}/\check{B}^-_{\mathbf{Q}}$. If $\mathbf{G}$ is an elliptic curve E, then $\mathfrak{D}(\mathbf{G}) = E^\vee$, so that $\mathcal{L}^0_{\mathbf{G}} B\check{B}^- = \mathrm{Bun}^0_{\check{B}^-}(E^\vee)$.

We also obtain a proof of Theorem 1.1.7 (which we restate for convenience):

Corollary (Theorem 1.1.7)**.** *Suppose that G is a connected and simply-connected semisimple algebraic group or a torus over $\mathbf{C}$, and let T act on G by conjugation. Let G_c denote the maximal compact subgroup of $G(\mathbf{C})$. Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A, and let $\mathbf{G}$ be an oriented group scheme in the sense of* [Lur18b] *which is dualizable and licit in the sense of Definition 4.3.7. Then there is an equivalence of $A_{\mathbf{Q}}$-linear ∞-categories:*

$$\mathrm{Loc}_{T_c}(G_c; A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}((\mathcal{M}_{\check{T}})_{\mathbf{Q}} \times_{\mathcal{L}^0_{\mathbf{G}} B\check{B}} (\mathcal{M}_{\check{T}})_{\mathbf{Q}}).$$

Proof. Note that G_c is connected. By Proposition 2.2.5, $\mathrm{Loc}_{T_c}(G_c; A) \simeq \mathrm{LMod}_{\mathcal{F}_T(\Omega G_c)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$. Using that $\mathrm{Gr}_G(\mathbf{C}) \simeq \Omega G_c$ (as T-spaces) and combining Theorem 3.2.10 with Theorem 4.3.10, we obtain the desired result. □

Remark 4.3.14. If $A = \mathbf{Q}[\beta^{\pm 1}]$, the equivalence resulting from Theorem 1.1.7 is an equivalence of 2-periodic $\mathbf{Q}$-linear ∞-categories. However, the equivalence can be de-periodified, and one obtains an equivalence

$$\mathrm{Loc}_{T_c}(G_c; \mathbf{Q}) \simeq \mathrm{QCoh}(\check{\mathfrak{t}}[2]_{\mathbf{Q}} \times_{\tilde{\check{\mathfrak{g}}}[2]_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}[2]_{\mathbf{Q}}).$$

There is also a G_c-equivariant analogue:

$$\mathrm{Loc}_{G_c}(G_c;\mathbf{Q}) \simeq \mathrm{QCoh}(\check{\mathfrak{t}}[2]_{\mathbf{Q}}/\!\!/W \times_{\check{\mathfrak{g}}[2]_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}[2]_{\mathbf{Q}}/\!\!/W).$$

This equivalence can be de-equivariantized, to obtain an equivalence

$$\mathrm{Loc}(G_c;\mathbf{Q}) \simeq \mathrm{QCoh}(\{f\} \times_{\check{\mathfrak{g}}[2]_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}[2]_{\mathbf{Q}}/\!\!/W),$$

where $f \in \check{\mathfrak{g}}$ is the image of the origin in $\check{\mathfrak{t}}/\!\!/W$ under the Kostant slice. Note that $T^*G_c = G(\mathbf{C})$, so that the left-hand side can be interpreted as a relative of the $\mathbf{Q}$-linearization of the wrapped Fukaya category of T^*G_c by [GPS18, Theorem 1.1]. Moreover, $\{f\} \times_{\check{\mathfrak{g}}[2]_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}[2]_{\mathbf{Q}}/\!\!/W$ is isomorphic to $\{f\} \times_{\widetilde{\check{\mathfrak{g}}}[2]_{\mathbf{Q}}/\check{G}_{\mathbf{Q}}} \check{\mathfrak{t}}[2]_{\mathbf{Q}}$, which is in turn isomorphic to a shifted analogue of the centralizer $Z_f(\check{B})$ of f in $\check{B}$. In particular, this shifted analogue of $Z_f(\check{B})$ is a (derived) mirror to $G(\mathbf{C})$ viewed as a symplectic manifold.

Remark 4.3.15. The proof of Theorem 1.1.7 above uses the Koszul duality equivalence $\mathrm{Loc}_{T_c}(G_c;A) \simeq \mathrm{LMod}_{\mathcal{F}_T(\Omega G_c)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$ of Proposition 2.2.5. The category $\mathrm{LMod}_{\mathcal{F}_T(\Omega G_c)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$ (and hence the right-hand side of Theorem 1.1.7) admits a "quantization" parametrized by $\mathbf{G}$, given by $\mathrm{LMod}_{\mathcal{F}_{\widetilde{T}}(\Omega G_c)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$. For instance, if $A = \mathbf{Q}[\beta^{\pm 1}]$, the right-hand side of Theorem 1.1.7 quantizes to $\mathrm{End}_{\check{\mathcal{O}}_\hbar^{\mathrm{univ}}}(\mathrm{QCoh}(\widetilde{\mathfrak{t}}))$; and if $A = \mathrm{KU}$, the right-hand side of Theorem 1.1.7 quantizes to $\mathrm{End}_{\check{\mathcal{O}}_q^{\mathrm{univ}}}(\mathrm{QCoh}(\widetilde{T}))$. It follows that the ∞-category $\mathrm{Loc}_{T_c}(G_c;A)$ must itself admits a quantization.

Question 4.3.16. What is an explicit description of this 1-parameter deformation of $\mathrm{Loc}_{T_c}(G_c;A)$ over $\mathbf{G}$?

One of the reasons Question 4.3.16 seems difficult to answer is that the quantization of $\mathrm{LMod}_{\mathcal{F}_T(\Omega G_c)^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$ is via the *framed* $\mathbf{E}_2$-structure on $\Omega G_c = \Omega^2 BG_c$, which equips it with an S^1_{rot}-action. Writing $\Omega^2 BG_c$ as ΩG_c is a space-level reflection of the equivalence $\mathbf{E}_1 \otimes \mathbf{E}_1 \xrightarrow{\sim} \mathbf{E}_2$ of ∞-operads (given by the Dunn additivity theorem; see [Lur16, Theorem 5.1.2.2]). However, this equivalence involves breaking the rotation symmetry of $\mathbf{R}^2$ (by choosing a point of $\mathbf{R}P^1$), which concretely means that it is difficult to describe the S^1_{rot}-action on the $\mathbf{E}_2$-operad in terms of the tensor product $\mathbf{E}_1 \otimes \mathbf{E}_1$. Giving a description of this S^1_{rot}-action in terms of the tensor product decomposition of $\mathbf{E}_2$ would lead to an answer of Question 4.3.16.

Let us now turn to the proof of Theorem 4.3.10, which will take up the remainder of this section. The first step in proving Theorem 4.3.10 is to define the map $\kappa : \mathcal{M}_T \to \mathcal{L}^0_{\mathbf{G}} B\check{B}^-$, which in the case of $\mathbf{G} = \hat{\mathbf{G}}_a$ should agree with the classical Kostant section $\check{\mathfrak{t}}^* \to \widetilde{\check{\mathfrak{g}}}/\check{G}$. (Since we are working on one side of Langlands duality, we now drop the "check".)

Construction 4.3.17 (Kostant slice)**.** Assume that $\mathbf{G}$ is dualizable and licit, and fix a principal nilpotent $f \in \mathfrak{n}^-$. There is an isomorphism $\mathrm{Hom}(\mathfrak{D}(\mathbf{G}), BT) \simeq \mathcal{M}_T \times BT$, so that we may view a point $\lambda \in \mathcal{M}_T$ as a T-bundle $\mathcal{L}_\lambda$ on $\mathfrak{D}(\mathbf{G})$ equipped with a trivialization at the identity $e \in \mathfrak{D}(\mathbf{G})$. For each point $\lambda \in \mathcal{M}_T$, we therefore obtain a fibration $\mathcal{P}_\lambda = \mathfrak{D}(\mathbf{G}) \times_{BT} BB^-$ over $\mathfrak{D}(\mathbf{G})$ whose fibers are isomorphic to BN^-. This bundle is trivializable, since $\mathrm{H}^2(\mathfrak{D}(\mathbf{G});\mathbf{G}_a) \cong 0$, and a trivialization of $\mathcal{P}_\lambda$ is given by a map $\mathfrak{D}(\mathbf{G}) \to BN^-$. Such a map is given by composing the affinization map $\mathfrak{D}(\mathbf{G}) \to B\mathbf{G}_a$ of Definition 4.3.7 with the principal nilpotent $f : \mathbf{G}_a \to N^-$. The trivialization of this fibration defines a lifting of $\lambda : \mathfrak{D}(\mathbf{G}) \to BT$ along $BB^- \to BT$. We will denote the resulting map $\mathcal{M}_T \to \mathrm{Map}(\mathfrak{D}(\mathbf{G}), BB^-) = \mathcal{L}_{\mathbf{G}} BB^-$ by κ, and call it a *Kostant slice.*

Remark 4.3.18. Roughly, Construction 4.3.17 says that $\kappa : \mathcal{M}_T \to \mathcal{L}_{\mathbf{G}} BB^-$ is obtained by "translating" $f : \mathfrak{D}(\mathbf{G}) \to BN^-$ by $\lambda \in \mathcal{M}_T \cong \mathrm{Hom}(\mathfrak{D}(\mathbf{G}), BT)$. If $\mathbf{G} = \mathbf{G}_a$, the map $\kappa : \mathcal{M}_T \to \mathcal{L}_{\mathbf{G}} BB^-$ can be identified with the Kostant slice $\mathfrak{t} \to \mathfrak{b}^-/B^-$ (sending $x \mapsto f + x$). Similarly, if $\mathbf{G} = \mathbf{G}_m$, the map $\kappa : \mathcal{M}_T \to \mathcal{L}_{\mathbf{G}} BB^-$ can be identified with the Kostant slice $T \to B^-/B^-$ (sending $x \mapsto fx$).

If $\mathbf{G}$ is the dual of an elliptic curve E, the description of the map κ boils down to a classical observation of Atiyah's. Suppose for simplicity that $G = \mathrm{SL}_2$. Unwinding the definitions, we obtain the following explicit description of κ. Given a (degree 0) line bundle $\mathcal{L}$ over E, i.e., a point of $\mathcal{M}_T$, the map $\mathfrak{D}(\mathbf{G}) \cong E \to BB^-$ classifies the unique SL_2-bundle $\mathcal{V}$ over E with composition series $(\mathcal{L}^{-1}, \mathcal{L})$, which is split unless $\mathcal{L}^2 = \mathcal{O}_E$. In the notation of [Ati57], the bundle $\mathcal{V}$ is $\mathcal{F}_2 \otimes \mathcal{L}$ if $\mathcal{L}^2$ is trivial, and is $\mathcal{L} \oplus \mathcal{L}^{-1}$ otherwise. Note that the extension is classified by an element of $\mathrm{Ext}^1(\mathcal{L}, \mathcal{L}) \cong \mathrm{H}^1(E; \mathcal{L}^2)$.

Proof of Theorem 4.3.10. We will work over $\mathbf{Q}$, and omit it from the notation. Write X to denote the fiber product in Theorem 4.3.10, so that our goal is to identify X with $(T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}$. (The reader should keep in mind Assumption 4.3.9.) The argument of [BFM05, Section 4.3] can be used to show that $(T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}$ is flat over $\mathcal{M}_T$, and reduce to the case when $\check{G}$ has semisimple rank 1, i.e., $\check{G} = \mathrm{PGL}_2$. (Recall we assumed that $\check{G}$ is of adjoint type.) For notational convenience, we will write B instead of $\check{B}^-$, etc.; also note that since T is of rank 1, we may identify $\mathcal{M}_T \cong \mathbf{G}$.

For notational convenience, let us write $C = \mathfrak{D}(\mathbf{G})$. Unwinding the definitions, one sees that the map $\kappa : \mathbf{G} \to \mathrm{Bun}_B(C)$ sends a degree 0 line bundle $\mathcal{L}$ on C to the trivial extension $\mathcal{O}_C \subseteq \mathcal{O}_C \oplus \mathcal{L}$ if $\mathcal{L} \not\cong \mathcal{O}_C$, and to the Atiyah extension $\mathcal{O}_C \subseteq \mathcal{F}_2$ if $\mathcal{L} \cong \mathcal{O}_C$. The intersection $\mathbf{G} \times_{\mathrm{Bun}_B(C)} \mathbf{G}$ therefore consists of $\mathcal{L}, \mathcal{L}' \in \mathbf{G}$ equipped with an isomorphism $\kappa(\mathcal{L}) \cong \kappa(\mathcal{L}')$ of B-bundles over C (which in particular forces $\mathcal{L} \cong \mathcal{L}'$). By arguing as in (the elementary) [FM98, Lemma 1.14], one checks that $\mathrm{Aut}(\mathcal{F}_2) \cong B$. Therefore, the fiber of the map $\mathbf{G} \times_{\mathrm{Bun}_B(C)} \mathbf{G} \to \mathbf{G}$ over $\mathcal{L} \in \mathbf{G}$ is the torus $\mathbf{G}_m$ if $\mathcal{L} \not\cong \mathcal{O}_C$, and is the Borel B when $\mathcal{L} = \mathcal{O}_C$. Note that B is isomorphic to $\mathbf{G}_m \times \mathbf{A}^1$ as a variety, so that the generic fiber of $\mathbf{G} \times_{\mathrm{Bun}_B(C)} \mathbf{G} \to \mathbf{G}$ is $\mathbf{G}_m$, and the fiber over the zero section of $\mathbf{G}$ is $\mathbf{G}_m \times \mathbf{A}^1$. It follows that if α denotes the simple root of PGL_2, then $\mathbf{G} \times_{\mathrm{Bun}_B(C)} \mathbf{G}$ is precisely the complement of the proper preimage of the divisor $\mathcal{M}_{T_\alpha} \hookrightarrow \mathcal{M}_T$ inside the blowup of $\mathbf{G} \times \mathbf{G}_m$ at the "wall" given by the intersection of $e^{\alpha^\vee} - 1 = 0$ and $\mathcal{M}_{T_\alpha}$, i.e., the affine blowup $(T^*_{\mathbf{G}}T)^{\mathrm{bl}}$. □

4.4. **Categorical cobar constructions.**

Setup 4.4.1. Let $\mathrm{Pr}^{\mathrm{L,st}}$ be the ∞-category of compactly generated presentable ∞-categories and colimit-preserving functors which preserve compact objects. Let $\mathcal{A}$ be a presentably monoidal stable ∞-category, and let $\mathcal{M}$ be a dualizable object of $\mathrm{LMod}_{\mathcal{A}}(\mathrm{Pr}^{\mathrm{L,st}})$. Let $\mathcal{M}^\vee$ denote the $\mathcal{A}$-linear dual of $\mathcal{M}$, so that $\mathcal{M}^\vee = \mathrm{Fun}^{\mathrm{L}}_{\mathcal{A}^{\mathrm{rev}}}(\mathcal{M}, \mathcal{A})$ and $\mathcal{M}^\vee \in \mathrm{RMod}_{\mathcal{A}}(\mathrm{Pr}^{\mathrm{L,st}})$.

Let $\mathcal{C} \in \mathrm{CAlg}(\mathrm{Pr}^{\mathrm{L,st}})$, and let $\mathcal{M}$ be a dualizable object in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$. (If $\mathcal{M}$ is itself symmetric monoidal and is equipped with a $\mathcal{C}$-module structure via a symmetric monoidal functor $\mathcal{C} \to \mathcal{M}$, it is useful to view $\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})$ as a "cobar" construction.) Note that $\mathcal{M}$ is a left $\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})$-module and $\mathcal{M}^\vee$ is a right $\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})$-module. Thanks to the action of $\mathcal{C}$ on $\mathcal{M}$, there is a monoidal colimit-preserving functor $\mathcal{C} \to \mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})$. Since $\mathcal{M}$ is assumed to be dualizable in $\mathrm{LMod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$, the evaluation $\mathrm{ev} : \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{C}$ and coevaluation $\mathrm{coev} : \mathcal{C} \to \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$ maps admit right adjoints; we will denote these right adjoints by ev^{R} and $\mathrm{coev}^{\mathrm{R}}$.

There is an equivalence $\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \simeq \mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})$, under which coev can be identified with the action functor $\mathcal{C} \to \mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})$, while ev can be identified with a "trace" functor.

Definition 4.4.2. In Setup 4.4.1, we say that $\mathcal{M} \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$ is *almost proper* if $\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$ admits geometric realizations which are split by the right adjoint $\mathrm{ev}^{\mathrm{R}} : \mathcal{C} \to \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$, and those geometric realizations are preserved by ev^{R}. In particular, since $\mathcal{C}$ and $\mathcal{M}$ are assumed to be compactly generated, this condition is satisfied if ev preserves compactness; in fact, then ev^{R} preserves all small colimits, since it is right (hence left) exact.

One may view the following statement as an analogue of the fact that if X is a space and $x \in X$, then $B\Omega_x X \xrightarrow{\sim} X \times_{\pi_0(X)} \{x\}$. See also [BGO20, Section 3] for a similar result.

Proposition 4.4.3. *In Setup 4.4.1, assume that $\mathcal{M} \in \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$ is almost proper. Then there is a fully faithful colimit-preserving functor $\theta^{\mathrm{L}} : \mathcal{M}^\vee \otimes_{\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})} \mathcal{M} \to \mathcal{C}$ whose essential image is the maximal full subcategory of $\mathcal{C}$ on which $\mathrm{ev}^{\mathrm{R}} : \mathcal{C} \to \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \simeq \mathrm{End}_{\mathcal{C}}(\mathcal{M})$ is conservative.*

Proof. The relative tensor product may be computed by the geometric realization of the simplicial object $\mathcal{X}_\bullet := \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})^{\otimes_{\mathcal{C}} \bullet} \otimes_{\mathcal{C}} \mathcal{M}$. Since $\mathcal{M}$ is assumed to be dualizable, $\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M}) \simeq \mathcal{M} \otimes_{\mathcal{C}} \mathcal{M}^\vee$, so that $\mathcal{X}_\bullet \simeq (\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M})^{\otimes_{\mathcal{C}} \bullet + 1}$. Under this equivalence, the face map $d_i : \mathcal{X}_n \to \mathcal{X}_{n-1}$ is given by tensoring with the evaluation functor $\mathrm{ev} : \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{C}$ in the ith spot; similarly, the degeneracy map is given by tensoring with the coevaluation functor in the ith spot. Therefore, we may extend $\mathcal{X} : \mathbf{\Delta}^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{L,st}}$ to a coaugmented simplicial object $\mathcal{X} : \mathbf{\Delta}^{\mathrm{op}}_+ \to \mathrm{Pr}^{\mathrm{L,st}}$, by declaring $\mathcal{X}_{-1} = \mathcal{C}$, where the map $\mathcal{X}_0 = \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{C} = \mathcal{X}_{-1}$ is the evaluation map.

Let $\mathcal{Y}^\bullet$ denote the associated augmented cosimplicial object in $\widehat{\mathrm{Cat}}_\infty$ given by $(\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M})^{\otimes_{\mathcal{C}} \bullet + 1}$ for $\bullet \geq -1$, where the face and degeneracy maps are the right adjoints of the face and degeneracy maps of

$\mathcal{X}_\bullet$. By [Lur09b, Corollary 5.5.3.4], there is an equivalence

$$\mathcal{M}^\vee \otimes_{\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})} \mathcal{M} \simeq |\mathcal{X}_\bullet| \simeq \mathrm{Tot}(\mathcal{Y}^\bullet).$$

There is a functor $G : \mathcal{Y}^{-1} = \mathcal{C} \to \mathcal{Y}^0 = \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$ (given by the right adjoint to the coaugmentation on $\mathcal{X}_\bullet$), which induces a functor $\theta : \mathcal{C} \to \mathrm{Tot}(\mathcal{Y}^\bullet)$. We claim that θ admits a fully faithful left adjoint $\theta^{\mathrm{L}} : \mathcal{M}^\vee \otimes_{\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})} \mathcal{M} \to \mathcal{C}$ whose essential image is the maximal full subcategory of $\mathcal{C}$ on which $\mathrm{ev}^{\mathrm{R}} : \mathcal{C} \to \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$ is conservative. For this, we use [Lur16, Corollary 4.7.5.3], which says that it suffices to show the following:

(a) $\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$ admits geometric realizations split by the functor $G : \mathcal{C} \to \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$, and those geometric realizations are preserved by G.

(b) For every morphism $\alpha : [m] \to [n]$ in $\mathbf{\Delta}_+$, the diagram

$$\begin{tikzcd} \mathcal{Y}^m \arrow[r, "d^0"] \arrow[d, "\alpha^*"'] & \mathcal{Y}^{m+1} \arrow[d, "\alpha^*"] \\ \mathcal{Y}^n \arrow[r, "d^0"'] & \mathcal{Y}^{n+1} \end{tikzcd}$$

is left adjointable.

For (a), note that the functor G is the right adjoint ev^{R} to the evaluation map $\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \to \mathcal{C}$, which satisfies (a) by assumption on $\mathcal{M}$. For (b), let us first assume that α is the map $[-1] \to [0]$. Then, we need to show that the following diagram is left-adjointable:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "\mathrm{ev}^{\mathrm{R}}"] \arrow[d, "\mathrm{ev}^{\mathrm{R}}"'] & \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \arrow[d, "\mathrm{id}\otimes\mathrm{ev}^{\mathrm{R}}"] \\ \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \arrow[r, "\mathrm{ev}^{\mathrm{R}}\otimes\mathrm{id}"'] & (\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}) \otimes_{\mathcal{C}} (\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}); \end{tikzcd} \tag{23}$$

in other words, we need to show that the following diagram commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[d, "\mathrm{ev}^{\mathrm{R}}"] & \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} \arrow[l, "\mathrm{ev}"'] \arrow[d, "\mathrm{id}\otimes\mathrm{ev}^{\mathrm{R}}"] \\ \mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M} & (\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}) \otimes_{\mathcal{C}} (\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}). \arrow[l, "\mathrm{ev}\otimes\mathrm{id}"] \end{tikzcd}$$

But this is clear. For the case of general α, suppose α is a face map. Recall that the face maps in the cosimplicial diagram are obtained by tensoring with the right adjoint ev^{R} inserted in the appropriate position. Therefore, it suffices to show that tensoring the diagram (23) with $\mathcal{M}^\vee \otimes_{\mathcal{C}} \mathcal{M}$ over $\mathcal{C}$ preserves the property of being left adjointable. This follows from [Hai21, Corollary 3.2]. If α is a degeneracy map, then the same argument with ev^{R} replaced by $\mathrm{coev}^{\mathrm{R}}$ continues to hold. $\square$

A slightly more general argument lets us prove a less explicit statement; one may view Proposition 4.4.3 as an explicit description of the "completion" below.

Proposition 4.4.4. *In Setup 4.4.1, let $\langle\mathcal{M}\rangle$ be the smallest full subcategory of $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$ which is generated by $\mathcal{M}$ under colimits. Then the inclusion $\langle\mathcal{M}\rangle \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$ has a left adjoint, denoted $\mathcal{N} \mapsto \mathcal{N}^\wedge_{\mathcal{M}}$. Then there is an equivalence $\mathcal{M}^\vee \otimes_{\mathrm{End}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M})} \mathcal{M} \simeq \mathcal{C}^\wedge_{\mathcal{M}}$.*

Proof. Since $\mathcal{M}$ is assumed to be dualizable, the functor $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}}) \to \mathrm{LMod}_{\mathrm{End}_{\mathcal{C}}(\mathcal{M})}(\mathrm{Pr}^{\mathrm{L,st}})$ given by $\mathrm{Fun}^{\mathrm{L}}_{\mathcal{C}}(\mathcal{M}, -)$ admits a left adjoint given by tensoring with $\mathcal{M}$ over $\mathrm{End}_{\mathcal{C}}(\mathcal{M})$. The functor $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}}) \to \mathrm{LMod}_{\mathrm{End}_{\mathcal{C}}(\mathcal{M})}(\mathrm{Pr}^{\mathrm{L,st}})$ is fully faithful on $\langle\mathcal{M}\rangle$, so it only remains to show that the inclusion $\langle\mathcal{M}\rangle \to \mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$ has a left adjoint. Since $\mathrm{Pr}^{\mathrm{L,st}}$ is the ∞-category of compactly generated presentable ∞-categories, it is presentable (see, e.g., [Mat16, Corollary 2.9]. Therefore, $\mathrm{Mod}_{\mathcal{C}}(\mathrm{Pr}^{\mathrm{L,st}})$ is also presentable. Since $\langle\mathcal{M}\rangle$ is an accessible subcategory which is closed under small limits, the adjoint functor theorem [Lur09b, Corollary 5.5.2.9] gives the desired left adjoint. $\square$

We will now explore one corollary of Corollary 4.3.11 using Theorem 4.3.10.

Lemma 4.4.5. *The ∞-category* $\mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}$ *is self-dual as a module over* $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)$.

Proof. We need to show that κ^* exhibits $\mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}$ as a self-dual $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)$-module. This is a consequence of [BZFN10, Corollary 4.8]. □

Definition 4.4.6. In the setup of Corollary 4.3.11, define $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)_{\mathrm{Kost}}$ to be the completion $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)^{\wedge}_{\mathrm{QCoh}(\mathcal{M}_T)\otimes\mathbf{Q}}$ of Proposition 4.4.4. Let $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)^{\omega}_{\mathrm{Kost}}$ denote the full subcategory of $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)_{\mathrm{Kost}}$ which is spanned by those objects which map to the full subcategory $\mathrm{Perf}(\mathcal{M}_T) \otimes \mathbf{Q} \subseteq \mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}$ under the functor $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-) \xrightarrow{\kappa^*} \mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}$. When A is an $\mathbf{E}_\infty$-$\mathbf{Q}[\beta^{\pm 1}]$-algebra and $\mathbf{G} = \mathbf{G}_a$, the Koszul duality equivalence of Lemma 4.1.6 gives $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-) \simeq \mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})$; we define $\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})_{\mathrm{Kost}}$ to be the essential image of $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)_{\mathrm{Kost}}$ under this equivalence.

Theorem 4.4.7 (Chromatic ABG over the regular locus)**.** *Fix a complex-oriented even-periodic $\mathbf{E}_\infty$-ring A and an oriented commutative A-group $\mathbf{G}$ which is dualizable and licit in the sense of Definition 4.3.7. Suppose G is a connected and simply-connected semisimple algebraic group over $\mathbf{C}$. Let $\mathrm{Coh}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))}(\mathrm{QCoh}(\mathcal{M}_T))$ be the full subcategory of $\mathrm{Mod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))}(\mathrm{QCoh}(\mathcal{M}_T))$ spanned by those objects whose underlying $\mathcal{O}_{\mathcal{M}_T}$-module is perfect. Then there is an $\mathbf{E}_1$-monoidal equivalence*

$$\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)^{\omega}_{\mathrm{Kost}} \simeq \mathrm{Coh}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))}(\mathrm{QCoh}(\mathcal{M}_T)) \otimes \mathbf{Q}.$$

Proof. The equivalence of Corollary 4.3.11 fits into a commutative diagram

$$\begin{tikzcd} \mathrm{End}_{\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)}(\mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}) \arrow[r, "\sim"] \arrow[dr, "\text{action}"'] & \mathrm{Mod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee}(\mathrm{QCoh}(\mathcal{M}_T)) \otimes \mathbf{Q} \arrow[d, "\mathrm{Gr}_G(\mathbf{C}) \to *"] \\ & \mathrm{QCoh}(\mathcal{M}_T) \otimes \mathbf{Q}. \end{tikzcd}$$

We may therefore take the bar construction with respect to the above augmentations. By definition, the bar construction on the augmentation $\mathrm{Mod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee}(\mathrm{QCoh}(\mathcal{M}_T)) \to \mathrm{QCoh}(\mathcal{M}_T)$ is $\mathrm{coMod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$. Note that the $\mathcal{O}_{\mathcal{M}_T}$-linear dual of $\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee$ is $\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))$. It follows that $\mathcal{O}_{\mathcal{M}_T}$-linear duality furnishes an equivalence between $\mathrm{Coh}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))}(\mathrm{QCoh}(\mathcal{M}_T))$ and the full subcategory of $\mathrm{coMod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))^\vee}(\mathrm{QCoh}(\mathcal{M}_T))$ spanned by those comodules whose underlying object in $\mathrm{QCoh}(\mathcal{M}_T)$ is perfect. By Proposition 4.4.4, the bar construction on the diagonal augmentation in the diagram above is given by $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)_{\mathrm{Kost}}$. Matching up the finiteness conditions, we obtain the desired equivalence. □

Remark 4.4.8. Based on Theorem 4.4.7, one expects an equivalence of ∞-categories between $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)$ and $\mathrm{Shv}_T(\mathrm{Gr}_G(\mathbf{C})); A) \otimes \mathbf{Q}$. This requires a good deal of additional work, and we do not know how to prove a literal equivalence between these two categories. Nonetheless, we will use Theorem 4.4.7 along with [Dev22b, Theorem 2.3.3] (which establishes a derived and motivic version of Ginzburg's argument for the full faithfulness of cohomology) in future work to identify a version of $\mathrm{QCoh}(\mathcal{L}^0_{\mathbf{G}} B\check{B}^-)$ with a motivic analogue of $\mathrm{Shv}_T(\mathrm{Gr}_G(\mathbf{C}); \mathrm{Mod}_A)$. However, even defining the motivic side appropriately requires significant technical setup, especially since Gr_G (and hence Gr_G/I) is not of finite type; we will therefore defer further discussion to a later article.

Example 4.4.9. When $A = \mathbf{Q}[\beta^{\pm 1}]$ and $\mathbf{G} = \hat{\mathbf{G}}_a$, we have $\mathcal{L}^0_{\mathbf{G}} B\check{B}^- = \widetilde{\check{\mathfrak{g}}}/\check{G}$. Theorem 4.4.7 gives an $\mathbf{E}_1$-monoidal equivalence

$$\mathrm{IndCoh}((\widetilde{\check{\mathcal{N}}} \times_{\check{\mathfrak{g}}} \{0\})/\check{G})^{\omega}_{\mathrm{Kost}} \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}] \simeq \mathrm{Coh}_{C^*_T(\mathrm{Gr}_G(\mathbf{C}); \mathbf{Q})} \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}].$$

It seems quite reasonable to expect to split off the "unit" summand using Adams operations on $\mathbf{Q}[\beta^{\pm 1}]$, and therefore recover a variant of the Iwahori-Satake equivalence of [ABG04] over the (open) regular locus (see also Remark 4.1.4). In fact, using Example 2.4.9(b), one can upgrade this to an $\mathbf{E}_2$-monoidal equivalence, where the $\mathbf{E}_2$-monoidal structure on the right-hand side is given by the $\mathbf{E}_2$-coalgebra structure on $C^*_T(\mathrm{Gr}_G(\mathbf{C}); \mathbf{Q})$ induced from the $\mathbf{E}_2$-algebra structure on $\mathrm{Gr}_G(\mathbf{C})$.

Example 4.4.10. When $A = \mathrm{KU}$ and $\mathbf{G} = \mathbf{G}_m$, we have $\mathcal{L}^0_{\mathbf{G}} B\check{B}^- = \widetilde{\check{G}}/\check{G}$. Therefore, Theorem 4.4.7 gives an $\mathbf{E}_1$-monoidal equivalence

$$\mathrm{QCoh}(\widetilde{\check{G}}/\check{G})^\omega_{\mathrm{Kost}} \otimes_{\mathbf{Q}} \mathrm{KU}_{\mathbf{Q}} \simeq \mathrm{Coh}_{C^*_T(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU})} \otimes \mathbf{Q}.$$

Similarly, there is an $\mathbf{E}_1$-monoidal equivalence

$$\mathrm{QCoh}(\check{G}/\check{G})^\omega_{\mathrm{Kost}} \otimes_{\mathbf{Q}} \mathrm{KU}_{\mathbf{Q}} \simeq \mathrm{Coh}_{C^*_{\check{G}}(\mathrm{Gr}_G(\mathbf{C});\mathrm{KU}')} \otimes \mathbf{Q}.$$

This may be understood as a variant of a full KU-theoretic geometric Satake equivalence, over the (open) regular locus of $\check{G}$; it is presumably related to [CK18, Section 1.2].

Example 4.4.11. Suppose A is a complex-oriented even-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}$ is an oriented elliptic curve over A (in the sense of [Lur18b]). Let E be the underlying classical scheme of $\mathbf{G}$ over the classical ring $\pi_0(A)$, so that E is an elliptic curve, and let $E^\vee$ be the dual elliptic curve. Then $\mathcal{L}^0_{\mathbf{G}} B\check{B}^- = \mathrm{Bun}^0_{\check{B}^-}(E^\vee)$, and Theorem 4.4.7 gives an $\mathbf{E}_1$-monoidal equivalence

$$\mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}^-}(E^\vee))^\omega_{\mathrm{Kost}} \otimes_{\pi_0 A} A_{\mathbf{Q}} \simeq \mathrm{LMod}_{\mathcal{F}_T(\mathrm{Gr}_G(\mathbf{C}))}(\mathrm{QCoh}(\mathcal{M}_T)) \otimes \mathbf{Q}.$$

This may be understood as a variant of a full A-theoretic analogue of the ABG equivalence.

4.5. **Coefficients in the sphere spectrum?** In this brief section, we study the natural question of the analogue of Theorem 1.1.7 and Theorem 4.4.7, except with coefficients in the sphere spectrum.

Appendix A. Coulomb branches of pure supersymmetric gauge theories

In this brief appendix, we explain some motivation for the results of this article from the perspective of Coulomb branches of 4d $\mathcal{N} = 2$ and 5d $\mathcal{N} = 1$ gauge theories with a generic choice of complex structure. Our goal here is not to be precise, but instead explain some motivation for the ideas in this article.

Recollection A.1. In [BFN18, Nak16] (see also [Nak17]), it is argued that the Coulomb branch of 3d $\mathcal{N} = 4$ pure gauge theory on $\mathbf{R}^3$ can be modeled by the algebraic symplectic variety $\mathcal{M}_C := \operatorname{Spec} \mathrm{H}_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ over $\mathbf{C}$. The calculations of [BFM05] say that $\mathcal{M}_C$ is isomorphic to $(T^*\check{T})^{\mathrm{bl}}/\!\!/W$. This is in turn isomorphic by [BF08, Theorem 3] to the phase space of the Toda lattice for $\check{G}$, as well as to the moduli space of solutions of Nahm's equations on $[-1, 1]$ for a compact form of $\check{G}$ by [BFN18, Theorem A.1] with an appropriate boundary condition. The *quantized* Coulomb branch of 3d $\mathcal{N} = 4$ pure gauge theory on $\mathbf{R}^3$ is then modeled by $\mathcal{A}_\epsilon := \mathrm{H}_*^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$. In [BFM05], $\mathcal{A}_\epsilon$ was be identified with the algebra of operators of the quantized Toda lattice for $\check{G}$.

Remark A.2. The physical reason for the definition of $\mathcal{A}_\epsilon$ is the "Ω-background" (introduced in [NS09]); we refer the reader to [BBB$^+$20, Tel14] for helpful expositions on this topic. The essential idea is as follows: $C_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ admits the structure of an $\mathbf{E}_3^{\mathrm{fr}}$-algebra. In particular, the $\mathbf{E}_3$-algebra structure on $C_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ is equivariant for the action of S^1 on $C_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ via loop rotation, and the action of S^1 on $\mathbf{E}_3$ via rotation about a line $\ell \subseteq \mathbf{R}^3$. Using the fact that the fixed points of the S^1-action on $\mathbf{R}^3$ are given by the line ℓ, it is argued in [BBB$^+$20] that the homotopy fixed points of $C_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ admits the structure of an $\mathbf{E}_1$-$C^*_{S^1}(*; \mathbf{C})$-algebra. Furthermore, the associative multiplication on $C_*^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ is argued to degenerate to the 2-shifted Poisson bracket on $\mathrm{H}_*^G(\mathrm{Gr}_G(\mathbf{C}); \mathbf{C})$ obtained from the $\mathbf{E}_3$-algebra structure. The "Ω-background" is supposed to refer to the S^1-action and the compatibility with the S^1-action on the $\mathbf{E}_3$-operad.

From the mathematical perspective, the idea that S^1-actions can be viewed as a deformation quantizations has been made precise by [Pre15, Toe14], and more recently in [But20a, But20b] (at least in characteristic zero; we will study the case of positive characteristic in forthcoming work). Although often not said explicitly, the idea has been a cornerstone of Hochschild homology. (The reader can skip the following discussion, since it will not be necessary in the remainder of this section; we only include it for completeness.) Consider a smooth $\mathbf{C}$-scheme X, so that the HKR theorem gives an isomorphism $\mathrm{HH}(X/\mathbf{C}) \simeq \mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1])$. There is an isomorphism $\mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1]) \simeq \bigoplus_{n\geq 0}(\wedge^n\Omega^1_{X/\mathbf{C}})[n]$, so $\mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1])$ can be understood as a shearing of the algebra $\Omega^*_{X/\mathbf{C}} = \bigoplus_{n\geq 0}(\wedge^n\Omega^1_{X/\mathbf{C}})[-n]$ of differential forms. The HKR theorem further states that the S^1-action on $\mathrm{HH}(X/\mathbf{C})$ is a shearing of the de Rham differential on $\Omega^*_{X/\mathbf{C}}$. The Koszul dual of the algebra $\mathrm{HH}(X/\mathbf{C}) \simeq \mathrm{Sym}(\Omega^1_{X/\mathbf{C}}[1])$ is $\mathrm{Sym}(T_{X/\mathbf{C}}[-2]) \simeq \mathcal{O}_{T^*[2]X}$; in the same way, the sheaf of differential operators on X is Koszul dual to the de Rham complex of X. This can be drawn pictorially as follows:

$$\begin{tikzcd}
\mathrm{Sym}(T_{X/\mathbf{C}}[-2]) \simeq \mathcal{O}_{T^*[2]X} \arrow[r, squiggly, "\text{def. quant}"] \arrow[d, squiggly, "\text{Koszul dual}"'] & \mathcal{D}^\hbar_{X/\mathbf{C}} \arrow[d, squiggly, "\text{Koszul dual}"] \\
\mathrm{Sym}_{\mathcal{O}_X}(\Omega^1_{X/\mathbf{C}}[1]) \simeq \mathrm{HH}(X/\mathbf{C}) \arrow[r, squiggly, "S^1\text{-action}"'] & \text{shearing of } (\Omega^*_{X/k}, d_{\mathrm{dR}}).
\end{tikzcd}$$

Since the algebra $\mathcal{D}^\hbar_X$ of differential operators is a quantization of $T^*[2]X$, this drawing illustrates that the S^1-action on Hochschild homology plays the role of a Koszul dual to deformation quantization.

Example A.3. We will keep $G = \mathrm{SL}_2$ as a running example in discussing Coulomb branches (see also [SW97, Section 2]). In this case, $\mathcal{M}_C \cong \operatorname{Spec} \mathbf{C}[x, t^{\pm 1}, \frac{t-1}{x}]^{\mathbf{Z}/2} \cong \operatorname{Spec} \mathbf{C}[x^2, t + t^{-1}, \frac{t-t^{-1}}{x}]$ by Theorem 3.2.10 (and [BFM05]), where $\mathbf{Z}/2$ acts on $\mathbf{C}[x, t^{\pm 1}, \frac{t-1}{x}]$ by $x \mapsto -x$ and $t \mapsto t^{-1}$. Let us denote $\Phi = x^2$, $U = t + t^{-1}$, and $V = \frac{t-t^{-1}}{x}$. Then

$$U^2 - \Phi V^2 = (t + t^{-1})^2 - (t - t^{-1})^2 = 4,$$

so $\mathcal{M}_C$ is isomorphic to the subvariety of $\mathbf{A}^3_{\mathbf{C}}$ cut out by the equation $U^2 - \Phi V^2 = 4$. Alternatively, and perhaps more suggestively:

$$(U+2)(U-2) = \Phi V^2.$$

This is known as the *Atiyah-Hitchin manifold*, and was studied in great detail in [AH88] (see [AH88, Page 20] for the definition). In [BFN18, Theorem A.1], it was shown that the Atiyah-Hitchin manifold is isomorphic to the moduli space of solutions of Nahm's equations on $[-1,1]$ for PSU(2) with an appropriate boundary condition. Since a normal vector to the defining equation of $\mathcal{M}_C$ is $2U\partial_U - V^2\partial_\phi - 2V\Phi\partial_V$, the standard holomorphic 3-form $dU \wedge d\Phi \wedge dV$ on $\mathbf{A}^3_{\mathbf{C}}$ induces a holomorphic symplectic form $\frac{d\Phi\wedge dV}{2U}$ on $\mathcal{M}_C$. (This can also be written as $\frac{dU\wedge dV}{V^2}$ or as $\frac{d\Phi\wedge dU}{2\Phi V}$.) The associated Poisson bracket on $\mathcal{O}_{\mathcal{M}_C} \cong \mathrm{H}^G_*(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})$ agrees with the 2-shifted Poisson bracket arising from the $\mathbf{E}_3$-structure on $C^G_*(\mathrm{Gr}_G(\mathbf{C});\mathbf{C})$.

The quantized algebra $\mathcal{A}_\epsilon$ was described explicitly in [BF08]. Let us write $\theta = \frac{1}{x}(s-1)$, where s is the simple reflection generating the Weyl group of SL_2. Then $\mathcal{A}_\epsilon$ is generated as an algebra over $\mathbf{C}[\![\hbar]\!]$ by $\mathbf{Z}/2$-invariant polynomials in x, $t^{\pm 1}$, and θ, where x is to be viewed as $t\partial_t$. Moreover, under the isomorphism $\mathcal{A}_\epsilon/\hbar \cong \mathcal{O}_{\mathcal{M}_C}$, the class x is sent to x, and θ is sent to $\frac{t-1}{x}$. We then have the commutation relation $[x, t^{\pm 1}] = \pm\hbar t^{-1}$, induced by $[\partial_t, t] = \hbar$. This implies that $[x^2, t^{\pm 1}] = \hbar^2 t^{\pm 1} \pm 2\hbar t^{\pm 1} x$, which in turn implies that $\mathcal{A}_\epsilon$ is the quotient of the free associative $\mathbf{C}[\![\hbar]\!]$-algebra on Φ, U, and $V = \frac{1}{x}(t - t^{-1})$ subject to the relations

$$\begin{aligned}
[\Phi, V] &= 2\hbar U - \hbar^2 V, \\
[\Phi, U] &= 2\hbar\Phi V - \hbar^2 U, \\
[U, V] &= \hbar V^2, \\
U^2 - \Phi V^2 &= 4 - \hbar UV.
\end{aligned}$$

Note that the commutation relations for $[\Phi, U]$ and $[U, V]$ in [DG19, Equation B.3] have typos, but it is stated correctly in [BDG17, Equation 5.51]. The above is an explicit description of the nil-Hecke algebra $\mathcal{H}(\widetilde{\mathfrak{t}}, W^{\mathrm{aff}})e$ for SL_2. Note that the above algebra is already defined over $\mathbf{Z}[\![\hbar]\!]$, so we may then specialize $2 = 0$ (i.e., work in modular characteristic) to obtain the simpler algebra

$$\begin{aligned}
[\Phi, V] &= \hbar^2 V, \\
[\Phi, U] &= \hbar^2 U, \\
[U, V] &= \hbar V^2, \\
U^2 - \Phi V^2 &= \hbar UV.
\end{aligned}$$

Heuristic A.4. An unpublished conjecture of Gaiotto (which we learned about from Nakajima) says that the Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory over $\mathbf{R}^3 \times S^1$ with a generic choice of complex structure can be modeled by $\mathcal{M}^{4\mathrm{d}}_C := \operatorname{Spec} \mathrm{KU}^G_0(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$. Although we do not know Gaiotto's motivation for this conjecture (it is probably inspired by [SW97]), we can try to heuristically justify it as follows. Recall that $\mathrm{Gr}_G(\mathbf{C})/G$ can be viewed as $\mathrm{Bun}_G(S^2)$. It is reasonable to view $\mathrm{KU}_0(\mathrm{Bun}_G(S^2)) \otimes \mathbf{C}$ as closely related to $\mathrm{H}_*(\mathcal{L}\mathrm{Bun}_G(S^2);\mathbf{C})$, where $\mathcal{L}\mathrm{Bun}_G(S^2)$ denotes the free loop space. Since $\mathcal{L}BG \simeq B\mathcal{L}G$, we have $\mathcal{L}\mathrm{Bun}_G(S^2) \simeq \mathrm{Bun}_{\mathcal{L}G}(S^2)$, so one might view $\mathrm{H}_*(\mathcal{L}\mathrm{Bun}_G(S^2);\mathbf{C})$ as the ring of functions on the "Coulomb branch of 3d $\mathcal{N} = 4$ pure gauge theory on $\mathbf{R}^3$ with gauge group $\mathcal{L}G$". Although we do not know how to make sense of this phrase, it is often useful to view gauge theory with gauge group $\mathcal{L}G$ as "finite temperature" gauge theory with gauge group G. Recall that Wick rotation relates $(3+1)$-dimensional quantum field theory at a finite temperature T to statistical mechanics over $\mathbf{R}^3 \times S^1$ where the circle has radius $\frac{1}{2\pi T}$. This suggests that $\mathrm{H}_*(\mathcal{L}\mathrm{Bun}_G(S^2);\mathbf{C})$ (which is more precisely to be understood as $\mathrm{KU}^G_0(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$) can be viewed as the ring of functions on the "Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory on $\mathbf{R}^3 \times S^1$ with gauge group G". See [BFN18, Remark 3.14]. In [BFM05], $\operatorname{Spec} \mathrm{KU}^G_0(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$ was identified with the phase space of the relativistic Toda lattice for $\check{G}$.

One can also define a quantization of $\mathcal{M}^{4\mathrm{d}}_C$ via $\mathcal{A}^{4\mathrm{d}}_\epsilon := \mathrm{KU}^{G\times S^1_{\mathrm{rot}}}_0(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$; this can be viewed as a model for the quantized Coulomb branch of 4d $\mathcal{N} = 2$ pure gauge theory on $\mathbf{R}^3 \times S^1$. In [BFM05], $\mathcal{A}^{4\mathrm{d}}_\epsilon$ was be identified with the algebra of operators of the quantized relativistic Toda lattice for $\check{G}$.

Example A.5. When $G = \mathrm{SL}_2$, the calculations of Theorem 3.2.10 and [BFM05] tell us that $\mathcal{M}_C^{4\mathrm{d}} \cong \operatorname{Spec} \mathbf{C}[x^{\pm 1}, t^{\pm 1}, \frac{t-1}{x-1}]^{\mathbf{Z}/2} \cong \operatorname{Spec} \mathbf{C}[x + x^{-1}, t + t^{-1}, \frac{t-t^{-1}}{x-x^{-1}}]$, where $\mathbf{Z}/2$ acts on $\mathbf{C}[x^{\pm 1}, t^{\pm 1}, \frac{t-1}{x-1}]$ by $x \mapsto x^{-1}$ and $t \mapsto t^{-1}$. Let us write $\Psi = x + x^{-1}$, $W = t + t^{-1}$, and $Z = \frac{t-t^{-1}}{x-x^{-1}}$. Then, one easily verifies that $\mathcal{M}_C^{4\mathrm{d}}$ is the subvariety of $\mathbf{A}^3_{\mathbf{C}}$ cut out by the equation

$$W^2 - (\Psi^2 - 4)Z^2 = 4.$$

Alternatively, and perhaps more suggestively:

$$(W + 2)(W - 2) = (\Psi + 2)(\Psi - 2)Z^2.$$

This may be regarded as an a multiplicative analogue of the Atiyah-Hitchin manifold. It would be very interesting to understand a relationship between this manifold and the moduli space of solutions to some analogue of Nahm's equations for $\mathrm{PSU}(2)$ with an appropriate boundary condition. The complex manifold $\mathcal{M}_C^{4\mathrm{d}}$ has a holomorphic symplectic form given by $\frac{d\Psi \wedge dZ}{W}$, which can also be written as $\frac{d\Psi \wedge dW}{(\Psi^2-4)Z}$ or as $\frac{dZ \wedge dW}{\Psi Z^2}$.

We can also describe the quantized algebra $\mathcal{A}_\epsilon^{4\mathrm{d}}$ explicitly. In this case, instead of the relation $[\partial_t, t] = \hbar$ which appeared in Example A.3, we have the relation $xt = qtx$ (i.e., $xtx^{-1}t^{-1} = q$). In particular, $xt^{-1} = q^{-1}t^{-1}x$, $x^{-1}t = q^{-1}tx^{-1}$, and $x^{-1}t^{-1} = qt^{-1}x^{-1}$. It follows after some tedious calculation that $\mathcal{A}_\epsilon^{4\mathrm{d}}$ is the quotient of the free associative $\mathbf{C}[q^{\pm 1}]$-algebra on Ψ, W, and $Z = \frac{1}{x-x^{-1}}(t - t^{-1})$ subject to the relations

$$\begin{aligned}
[\Psi, W] &= (q-1)(\Psi^2 - 4)Z - \frac{(q-1)^2}{2q}((\Psi^2-4)Z + \Psi W),\\
[\Psi, Z] &= (q-1)W - \frac{(q-1)^2}{2q}(\Psi Z + W),\\
[Z, W] &= (q-1)\Psi Z^2 - \frac{(q-1)^2}{2q}(\Psi Z + W)Z,\\
W^2 - (\Psi^2-4)Z^2 &= 4 - \frac{(q-1)^2}{2q}(\Psi^2-4)Z^2 + \frac{q^2-1}{2q}\Psi W Z.
\end{aligned}$$

This algebra does not seem to have been previously recorded in the literature; it is an explicit description of the multiplicative nil-Hecke algebra $\mathcal{H}(\widetilde{T}, W^{\mathrm{aff}})e$ from Conjecture 4.2.9 for SL_2. Note that if q is specialized to -1 (i.e., a primitive square root of 1), then the above relations simplify to

$$\begin{aligned}
[\Psi, W] &= 2\Psi W,\\
[\Psi, Z] &= 2\Psi Z,\\
[Z, W] &= 2WZ,\\
W^2 &= 4 + 3(\Psi^2 - 4)Z^2.
\end{aligned}$$

This can be understood as the specialization of $\mathcal{H}(\widetilde{T}, W^{\mathrm{aff}})e$ to $q = -1$ for SL_2.

Now consider an elliptic curve $E(\mathbf{C})$ over $\mathbf{C}$. Motivated by Heuristic A.4 and [NY05], one might expect that the Coulomb branch of 5d $\mathcal{N} = 1$ pure gauge theory over $\mathbf{R}^3 \times E(\mathbf{C})$ (with some specific complex structure) can be modeled by the complexification of the G-equivariant A-homology of $\mathrm{Gr}_G(\mathbf{C})$, where A is an elliptic cohomology theory associated to a putative integral lift of E. A classical result of Tate says that there are no smooth elliptic curves over $\mathbf{Z}$, so $E(\mathbf{C})$ cannot literally lift to $\mathbf{Z}$. One can more generally simultaneously consider all possible "Coulomb branches" $\mathcal{M}_C^{5\mathrm{d}} := \operatorname{Spec} A_0^G(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$ associated to every complex-oriented even-periodic $\mathbf{E}_\infty$-ring A equipped with an oriented elliptic curve (or, equivalently, the universal example $\operatorname{Spec} \mathrm{tmf}_0^G(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$). We have described $\operatorname{Spec} A_0^T(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$ in Theorem 3.2.10, from which one can calculate $\mathcal{M}_C^{5\mathrm{d}}$. Similarly, one can even use Corollary 3.2.3 to calculate $A_0^{T \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$ and $\mathcal{A}_\epsilon^{5\mathrm{d}} := A_0^{G \times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$, but this is already incredibly complicated for $G = \mathrm{SL}_2$.

Example A.6. Let A be a complex-oriented even-periodic $\mathbf{E}_\infty$-ring equipped with an oriented elliptic curve $\widetilde{E}$, and let E denote the associated elliptic curve over $\pi_0(A) \otimes \mathbf{C}$. Let $(\mathbf{G}_m \times E)^{\mathrm{bl}}$ denote the complement of the proper preimage of the zero section of E inside the blowup of $\mathbf{G}_m \times E$ at the diagonal wall cut out by the zero sections of $\mathbf{G}_m$ and E. There is an action of $\mathbf{Z}/2$ on $(\mathbf{G}_m \times E)^{\mathrm{bl}}$, induced by the inversion on the group structures on $\mathbf{G}_m$ and E. If $G = \mathrm{SL}_2$, then Theorem 3.2.10 can be used to show that $\mathcal{M}_C^{5\mathrm{d}} = \mathrm{Spec}\, A_0^G(\mathrm{Gr}_G(\mathbf{C})) \otimes \mathbf{C}$ is isomorphic to $(\mathbf{G}_m \times E)^{\mathrm{bl}} /\!\!/ (\mathbf{Z}/2)$; this can be viewed as an elliptic analogue of the Atiyah-Hitchin manifold. We do not have a simple description for $\mathcal{A}_\epsilon^{5\mathrm{d}}$ analogous to Example A.3 and Example A.5.

It would be very interesting to give a physical interpretation to $A_0^G(\mathrm{Gr}_G(\mathbf{C}))\otimes\mathbf{C}$ and $A_0^{G\times S^1_{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C}))\otimes \mathbf{C}$ for other even-periodic $\mathbf{E}_\infty$-rings A, although we expect this to be very difficult (since most other chromatically interesting generalized cohomology theories only exist after profinite or p-adic completion, and do not admit transcendental analgues). It would also be very interesting to describe the analogue of our calculations for the schemes $\mathcal{R}_{G,\mathbf{N}}$ introduced in [BFN18]. This is not too difficult when G is a torus, but we expect it to lead to interesting geometry for nonabelian G.

Appendix B. Algebraic obstructions to mapping from polynomial rings

Let R be an $\mathbf{E}_n$-algebra for $n \geq 3$. There are often several *algebraic* obstructions to extending an $\mathbf{E}_1$-map $S^0[t] \to R$ to an $\mathbf{E}_n$-map. We include this appendix only for completeness, to complement the discussion in Section 2.4.

Proposition B.1. *Fix an odd prime p, and let R be a $K(1)$-local $\mathbf{E}_n$-ring. Then $\pi_0(R)$ admits a lift of Frobenius modulo $p^{\lfloor \frac{n+1}{2} \rfloor}$, and all maps of $K(1)$-local $\mathbf{E}_n$-rings must respect this structure.*

Proof. When $n = \infty$, Proposition B.1 was shown in [Hop14]. Let us explain the construction of this lift of Frobenius (which depends on a noncanonical choice for $n < \infty$): given a class $y : S^0 \to R$, we obtain a map $F(y) : L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)_+ \to R$ which extends y^p. There is a splitting $L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)_+ \simeq L_{K(1)}S^0 \vee L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)$, and the main observation is that if $n = 2m+1$, then there is an equivalence $L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n) \simeq L_{K(1)}S^{-1}/p^m$. (See [Dav86]; the desired equivalence can be proved by combining the observation that the canonical map $\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n) \to B\Sigma_p$ exhibits $\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)$ as the $2(p-1)m$-skeleton of $B\Sigma_p$, and that the stable transfer gives an equivalence $L_{K(1)}B\Sigma_p \simeq L_{K(1)}S^0$.) Thus, $L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)_+ \simeq L_{K(1)}(S^0 \vee S^{-1}/p^m)$.

Let $\theta : S^{-1}/p^m \to L_{K(1)}(S^0 \vee S^{-1}/p^m)$ denote the map given by $(0, -\frac{1}{(p-1)!})$, and let $\psi : S^{-1}/p^{m+1} \to L_{K(1)}(S^0 \vee S^{-1}/p^m)$ denote the map given by $(\mathrm{can}, 0)$, where can denotes the map $S^{-1}/p^{m+1} \to S^0$ given by crushing onto the top cell. Thus, ψ defines a map

$$S^{-1}/p^{m+1} \xrightarrow{\psi} L_{K(1)}(S^0 \vee S^{-1}/p^m) \simeq L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)_+ \xrightarrow{F(y)} R,$$

and hence a map $\psi(y) : S^0 \to R \otimes (S^{-1}/p^{m+1})^\vee \simeq R/p^{m+1}$ (and similarly for θ). There is a map $i : S^0 \to L_{K(1)}\mathrm{Conf}_p^{\mathrm{un}}(\mathbf{R}^n)_+$ given by taking suspension spectra of the inclusion of the basepoint; composing i with $F(y)$ produces $y^p \in \pi_0(R)$. Using the fact that the transfer is a map of degree p, one can show that we have an identity $y^p = \psi(y) - p\theta(y) \in \pi_0(R)/p^{m+1}$. Similarly, ψ defines a ring map $\pi_0(R) \to \pi_0(R)/p^{m+1}$, so it can be viewed as a lift of Frobenius on $\pi_0(R)/p^{m+1}$, as desired. $\square$

Example B.2. Let p be an odd prime. Then the same argument used to prove [Dev17, Theorem 1.3] can be adapted using Proposition B.1 to show that if R is an $\mathbf{E}_3$-ring such that $\pi_0 R$ contains a primitive pth root of unity, then $L_{K(1)}R = 0$. For instance, this implies that the spectrum $\mathrm{KU}_p[\zeta_p]$ (obtained by adjoining a primitive pth root of unity to the p-completion of KU) cannot even admit an $\mathbf{E}_3$-algebra structure. We do not know if this bound is sharp, i.e., does $\mathrm{KU}_p[\zeta_p]$ admit the structure of an $\mathbf{E}_2$-algebra?

It does admit an $\mathbf{E}_1$-algebra structure: in fact, suppose more generally that R is an $\mathbf{E}_2$-ring with no odd homotopy above degree 0. If $f(t) \in \pi_0(R)[t]$, then a simple obstruction-theoretic argument shows that $R[t]/f(t)$ admits the structure of an $\mathbf{E}_1$-R-algebra. Applying this to the pth cyclotomic polynomial viewed as an element of $\mathbf{Z}_p[q]$, we see that $\mathrm{KU}_p[\zeta_p]$ admits an $\mathbf{E}_1$-KU_p-algebra structure.

Example B.3. Proposition B.1 can be globalized: suppose p is an odd prime and X is a $\mathbf{Z}_p$-scheme equipped with a quasicoherent sheaf $\mathcal{O}_{\mathfrak{X}}$ of $K(1)$-local $\mathbf{E}_n$-rings such that $\pi_0\mathcal{O}_{\mathfrak{X}} \cong \mathcal{O}_X$ (i.e., a "lift of X to a $K(1)$-local $\mathbf{E}_n$-scheme"). Then $X \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^{\lfloor \frac{n+1}{2} \rfloor}$ admits a lift of Frobenius. For example, if $X \otimes_{\mathbf{Z}_p} \mathbf{F}_p$ is a non-ordinary abelian variety, or has Kodaira dimension ≥ 1, then $X \otimes_{\mathbf{Z}_p} \mathbf{Z}/p^2$ does not admit a lift of Frobenius. It follows that X cannot carry such a sheaf $\mathcal{O}_{\mathfrak{X}}$ of $K(1)$-local $\mathbf{E}_3$-rings. We do not know whether, in these cases, X can carry such a sheaf $\mathcal{O}_{\mathfrak{X}}$ of $K(1)$-local $\mathbf{E}_2$-*rings.* For example, suppose k is a perfect field of characteristic $p > 0$. Is there an elliptic curve E over $W(k)$ such that $E \otimes_{W(k)} k$ is supersingular, and such that there is a quasicoherent sheaf $\mathcal{O}_{\mathfrak{E}}$ of $K(1)$-local $\mathbf{E}_2$-rings on E with $\pi_0\mathcal{O}_{\mathfrak{E}} \cong \mathcal{O}_E$?

Remark B.4. If $p = 2$, then the statement of Proposition B.1 must be modified somewhat. In this case, the transfer still produces an equivalence $L_{K(1)}B\Sigma_2 = L_{K(1)}\mathbf{R}P^\infty \simeq L_{K(1)}S^0$, but the partial configuration spaces $\mathrm{Conf}_2^{\mathrm{un}}(\mathbf{R}^n) \simeq \mathbf{R}P^{n-1}$ require special cases. The vector fields on spheres problem is concerned with the question of whether the top cell of $\mathbf{R}P^k$ is unattached for odd k; since the answer (due to Adams in [Ada62]) is somewhat complicated, let us assume k is even. Then, there are $K(1)$-local equivalences

$$L_{K(1)}\mathrm{Conf}_2^{\mathrm{un}}(\mathbf{R}^{2(4n-j)+1}) = L_{K(1)}\mathbf{R}P^{2(4n-j)} = \begin{cases} L_{K(1)}S^{-1}/2^{4n} & j=0 \\ L_{K(1)}S^{-1}/2^{4n-1} & j=1. \end{cases}$$

This was proved as [DM87, Theorem 4.2]; we could not find a reference discussing the cases $j = 2, 3$, although this is certainly known to experts. Thus, we see that a $K(1)$-local $\mathbf{E}_{2(4n-j)+1}$-ring at $p = 2$ admits a lift of Frobenius modulo 2^{4n+1} if $j = 0$ (resp. modulo 2^{4n} if $j = 1$). In general, it seems reasonable to expect that a $K(1)$-local $\mathbf{E}_n$-ring at $p = 2$ admits a lift of Frobenius modulo $2^{\lfloor \frac{n+1}{2} \rfloor}$, just as with the story at odd primes.

Corollary B.5. *Let p be an odd prime, and let R be any $K(1)$-local $\mathbf{E}_3$-ring in which $p \neq 0$. Then there is no $\mathbf{E}_3$-map $S[t] \to R[x]$ sending $t \mapsto x + 1$.*

Proof. By Proposition B.1, it suffices to show that the map $f : \pi_0(S[t]) = \mathbf{Z}[t] \to \pi_0(R)[x]$ sending $t \mapsto x + 1$ does not respect the lift of Frobenius modulo $p^{\lfloor \frac{3+1}{2} \rfloor} = p^2$. In other words, we wish to show that the following diagram does not commute:

$$\begin{array}{ccc} \mathbf{Z}[t]/p^2 & \xrightarrow[f]{t\mapsto x+1} & \pi_0(R)[x]/p^2 = \pi_0(R) \otimes \mathbf{Z}[x]/p^2 \\ \big\downarrow{\scriptstyle t\mapsto t^p} & & \big\downarrow{\scriptstyle \psi\otimes(x\mapsto x^p)} \\ \mathbf{Z}[t]/p^2 & \xrightarrow[f]{t\mapsto x+1} & \pi_0(R)[x]/p^2. \end{array}$$

Consider the image of t under the two composites in the above diagram. One of the composites is given by $t \mapsto t^p \mapsto (x+1)^p \in \pi_0(R)[x]/p^2$, while the other composite is given by $t \mapsto x+1 \mapsto x^p+1 \in \pi_0(R)[x]/p^2$. Therefore, it suffices to observe that

$$(x+1)^p = x^p + 1 + \sum_{0<i<p} \binom{p}{i} x^{p-i} \neq x^p + 1$$

in $\pi_0(R)[x]/p^2$, since we assumed $p \neq 0$. □

Corollary B.6. *Let p be an odd prime, and let R be any $K(1)$-local $\mathbf{E}_3$-ring in which $p \neq 0$. Then there is no $\mathbf{E}_3$-map $S[t] \to R[x, y]$ which sends $t \mapsto x + y$ (i.e., there is no group object in $\mathbf{E}_3$-schemes over R whose underlying group scheme is $\mathbf{G}_a$).*

Proof. To prove this, note that there is an $\mathbf{E}_3$-map $R[y] \to R$ sending $y \mapsto 1$, given by applying R-chains to the crushing map $\mathbf{N} \to *$. The claim therefore follows from Corollary B.5. □

Remark B.7. Let p be an odd prime, and let R be any $\mathbf{E}_\infty$-ring (note, not merely $\mathbf{E}_3$!) in which $p \neq 0$, and such that $L_{K(n)}R \neq 0$ for some $n \geq 1$. We claim that there is no group object in $\mathbf{E}_3$-schemes over R whose underlying group scheme is $\mathbf{G}_a$, i.e., no $\mathbf{E}_3$-map $S[t] \to R[x, y]$ which sends $t \mapsto x + y$. Since the localization $R \to L_{K(1)}(R)$ is a map of $\mathbf{E}_3$-rings, it suffices to show this after replacing R by $L_{K(1)}R$. By Corollary B.6, it suffices to show that $p \neq 0$ in $\pi_0 L_{K(1)}R$.

Suppose otherwise. Then $L_{K(1)}R$ is an $\mathbf{E}_3$-ring in which $p = 0$, and therefore admits an $\mathbf{E}_2$-map $\mathbf{F}_p \to L_{K(1)}R$ by a theorem of Hopkins and Mahowald. This implies that $L_{K(1)}R = 0$, since $L_{K(1)}\mathbf{F}_p = 0$. By [Hah16], this implies that $L_{K(n)}R = 0$ for all $n \geq 1$; this contradicts our assumption on R.

Appendix C. Examples of $\mathfrak{D}(\mathbf{G})$

Let $\mathbf{G}$ be a commutative group scheme over a field k. In this section, we will give some examples of the "Cartier" dual $\mathfrak{D}(\mathbf{G}) = \mathrm{Hom}(\mathbf{G}, B\mathbf{G}_m)$ from Definition 4.3.1 further.

Recollection C.1. Suppose X is a perfect stack over an $\mathbf{E}_\infty$-ring $A \in \mathrm{CAlg}$ in the sense of [Lur17, Definition 9.4.4.1]. Then [Lur17, Corollary 9.4.3.4] says that $\mathrm{QCoh}(X)$ is self-dual as a Mod_A-module in $\mathrm{Pr}^{\mathrm{L,st}}$. If X and Y are perfect stacks, this implies that $\mathrm{Fun}_A(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \simeq \mathrm{QCoh}(X \times Y)$; see also [BZFN10, Corollary 4.10]. If $\mathcal{K} \in \mathrm{QCoh}(X \times Y)$, we will write $\mathrm{FM}_{\mathcal{K}} : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ to denote the associated functor. Explicitly, $\mathrm{FM}_{\mathcal{K}}$ is given by $\mathcal{F} \mapsto \mathrm{pr}_{2,*}(\mathrm{pr}_1^*(\mathcal{F}) \otimes \mathcal{K})$, where $\mathrm{pr}_1 : X \times Y \to X$ and $\mathrm{pr}_2 : X \times Y \to Y$ are the projections.

We now restrict to the case when our base $\mathbf{E}_\infty$-ring is an algebraically closed field k.

Construction C.2. The Poincaré line bundle $\mathcal{P}$ on $\mathbf{G} \times \mathfrak{D}(\mathbf{G})$ from Remark 4.3.3 defines a functor $\mathrm{FM}_{\mathcal{P}} : \mathrm{QCoh}(\mathbf{G}) \to \mathrm{QCoh}(\mathfrak{D}(\mathbf{G}))$, which we will denote by $\mathrm{FM}_{\mathbf{G}}$. Suppose that $\mathbf{G}$ is dualizable, i.e., the canonical map $\mathbf{G} \to \mathfrak{D}(\mathfrak{D}(\mathbf{G}))$ is an isomorphism. Then, the resulting functor $\mathrm{FM}_{\mathbf{G}} : \mathrm{QCoh}(\mathbf{G}) \to \mathrm{QCoh}(\mathfrak{D}(\mathbf{G}))$ is often an equivalence, with inverse given by $\mathrm{FM}_{\mathfrak{D}(\mathbf{G})}$. (Such a commutative group stack $\mathbf{G}$ is called "good" in [DP08, Condition A.1], and a large class of examples are given in [CZ17, Theorem A.4.6] and [DP08, Proposition A.6].) This Fourier-type transform is a simultaneous generalization of the Fourier-Mukai and Mellin transforms.

We now observe some special cases of Construction C.2.

Example C.3 (Fourier-Mukai)**.** If $\mathbf{G}$ is an abelian variety, then $\mathfrak{D}(\mathbf{G})$ is the dual abelian variety by Example 4.3.2. The functor $\mathrm{FM}_{\mathbf{G}}$ is then the classical Fourier-Mukai transform.

Example C.4 (Monodromy equivalence)**.** Let $\mathbf{G} = \mathbf{G}_m$. Then Example 4.3.2 says that $\mathfrak{D}(\mathbf{G})$ is the constant stack S^1. There is an equivalence $\mathrm{QCoh}(\mathfrak{D}(\mathbf{G})) \simeq \mathrm{Loc}(S^1; \mathrm{Mod}_k)$, and the equivalence of Construction C.2 is precisely the monodromy equivalence $\mathrm{QCoh}(\mathbf{G}_m) \simeq \mathrm{Loc}(S^1; \mathrm{Mod}_k)$. More generally, if $\mathbf{G}$ is a torus T, then the equivalence of Construction C.2 is the monodromy equivalence $\mathrm{QCoh}(T) \simeq \mathrm{Loc}(B\mathbb{X}^*(T); \mathrm{Mod}_k)$.

Example C.5 (Koszul duality)**.** Let $\mathbf{G} = \hat{\mathbf{G}}_a$. Then $\mathfrak{D}(\mathbf{G}) \cong B\mathbf{W}[F]$ by Lemma 4.3.5, so Construction C.2 gives an equivalence $\mathrm{QCoh}(\hat{\mathbf{G}}_a) \simeq \mathrm{QCoh}(B\mathbf{W}[F])$. Over a field of characteristic zero, this is an equivalence $\mathrm{QCoh}(\hat{\mathbf{G}}_a) \simeq \mathrm{QCoh}(B\mathbf{G}_a)$, which can be understood as the Koszul duality equivalence between $\mathcal{O}_{\hat{\mathbf{G}}_a} = \mathbf{Q}[\![t]\!]$ and $\mathcal{O}_{B\mathbf{G}_a} \cong \mathbf{Q}[\epsilon]/\epsilon^2$.

Note that working with $\hat{\mathbf{G}}_a$ (as opposed to $\mathbf{G}_a$) is crucial: $\mathfrak{D}(\mathbf{G}_a) \cong B\,\mathrm{Hom}(\mathbf{G}_a, \mathbf{G}_m)$. However, $\mathrm{Hom}(\mathbf{G}_a, \mathbf{G}_m)$ is not representable (its functor of points sends a $\mathbf{Q}$-algebra R to the set of nilpotent elements of R).

Example C.6 (Mellin transform)**.** Assume that k is an algebraically closed field of characteristic zero, and let $\mathbf{G} = \mathbf{G}_{m,\mathrm{dR}}$. We claim that $\mathfrak{D}(\mathbf{G}) \cong \mathbf{A}^1/\mathbf{Z}$, where $\mathbf{Z}$ acts on $\mathbf{A}^1$ by translation. To prove this, observe that the canonical map $\mathbf{G}_m \to \mathbf{G}_{m,\mathrm{dR}}$ gives a map $\mathfrak{D}(\mathbf{G}_{m,\mathrm{dR}}) \to \mathfrak{D}(\mathbf{G}_m)$. Since $\hat{\mathbf{G}}_m$ is the fiber of $\mathbf{G}_m \to \mathbf{G}_{m,\mathrm{dR}}$ over $1 \in \mathbf{G}_{m,\mathrm{dR}}$, there is a Cartesian square

$$\begin{tikzcd}
\mathfrak{D}(\mathbf{G}_{m,\mathrm{dR}}) \arrow[r] \arrow[d] & \mathfrak{D}(\mathbf{G}_m) \arrow[d] \\
\{1\} \arrow[r] & \mathfrak{D}(\hat{\mathbf{G}}_m).
\end{tikzcd}$$

Since k is a field of characteristic zero, we may identify $\hat{\mathbf{G}}_m$ with $\hat{\mathbf{G}}_a$, so that Lemma 4.3.5 identifies $\mathfrak{D}(\hat{\mathbf{G}}_a) \cong B\mathbf{G}_a$. The map $\mathfrak{D}(\mathbf{G}_m) \to \mathfrak{D}(\hat{\mathbf{G}}_m)$ identifies with the map $B\mathbf{Z} \to B\mathbf{G}_a$, which is given by the homomorphism $\mathbf{Z} \to \mathbf{G}_a$. It follows that $\mathfrak{D}(\mathbf{G}_{m,\mathrm{dR}}) \cong \mathbf{G}_a/\mathbf{Z}$, i.e., $\mathbf{A}^1/\mathbf{Z}$. Therefore, Construction C.2 gives an equivalence $\mathrm{DMod}(\mathbf{G}_m) = \mathrm{QCoh}(\mathbf{G}_{m,\mathrm{dR}}) \simeq \mathrm{QCoh}(\mathbf{A}^1/\mathbf{Z})$. More generally, if T is a torus, then $\mathfrak{D}(T_{\mathrm{dR}}) \cong \mathfrak{t}^*/\mathbb{X}^\bullet(T)$, with $\mathbb{X}^\bullet(T)$ acting by translation on $\mathfrak{t}^*$. Then Construction C.2 gives an equivalence $\mathrm{DMod}(T) = \mathrm{QCoh}(T_{\mathrm{dR}}) \simeq \mathrm{QCoh}(\mathfrak{t}^*/\mathbb{X}^\bullet(T))$, which can be identified with the Mellin transform.

Example C.7 (Rothstein-Laumon transform)**.** Assume that k is an algebraically closed field of characteristic zero, let A be an abelian variety over k, and let $\mathbf{G} = A_{\mathrm{dR}}$. If we choose an isomorphism $A_e^\wedge \cong \hat{\mathbf{G}}_a^{\dim A}$, then by arguing as in Example C.6, we compute that $\mathfrak{D}(\mathbf{G})$ is the fiber product $A^\vee \times_{B\mathbf{G}_a^{\dim A}} *$. Explicitly, if $\mathcal{A}$ denotes the Atiyah bundle on $A^\vee$ classified by $(1, \cdots, 1) \in \mathrm{H}^1(A^\vee; \mathcal{O}_{A^\vee}) \cong k^{\dim A}$, then this fiber product is isomorphic to $\mathbf{P}(\mathcal{A}) \setminus A_\infty^\vee$, where $A_\infty^\vee$ denotes the zero section of the projective bundle $\mathbf{P}(\mathcal{A})$. We may also interpret this fiber product as a twisted cotangent bundle to $A^\vee$. Further discussion in the case of an elliptic curve is in [Kat77, Appendix C]. Therefore, Construction C.2 gives an equivalence $\mathrm{DMod}(A) \simeq \mathrm{QCoh}(\mathbf{P}(\mathcal{A}) \setminus A_\infty^\vee)$. This recovers the main results of [Lau96, Rot96].

Example C.8 (Fourier-Deligne transform)**.** Let k be an algebraically closed field of arbitrary characteristic. If X is a k-scheme, let X_{dR} denote the crystalline de Rham space: this is the functor $X_{\mathrm{dR}} : \mathrm{CAlg}_k \to \mathcal{S}$ given by $X_{\mathrm{dR}}(R) = \mathrm{colim}_{(I,\gamma)} X(R/I)$, where the colimit is indexed over nilpotent ideals I in R with a PD-structure γ. Arguing similarly to [GR14], one finds that $\mathrm{QCoh}(X_{\mathrm{dR}})$ is equivalent to the ∞-category $\mathrm{DMod}(X)$ of crystalline D-modules on X. We claim that there is a canonical isomorphism $\mathfrak{D}(\mathbf{G}_{a,\mathrm{dR}}) \cong \mathbf{G}_{a,\mathrm{dR}}$. Indeed, if $\mathbf{G}_a^\sharp$ denotes the PD-completion of $\mathbf{G}_a$ at the origin, then $\mathbf{G}_a/\mathbf{G}_a^\sharp \cong \mathbf{G}_{a,\mathrm{dR}}$ (see [Dri20]). It follows that there is an extension

$$\mathbf{G}_a \to \mathbf{G}_{a,\mathrm{dR}} \to B\mathbf{G}_a^\sharp.$$

By Lemma 4.3.5 (see also [Dri20, Lemma 3.2.6]), there is an isomorphism $B\mathbf{G}_a^\sharp \cong B\mathbf{W}[F] \cong \mathfrak{D}(\mathbf{G}_a)$. It follows that $\mathfrak{D}(\mathbf{G}_{a,\mathrm{dR}})$ is isomorphic to $\mathbf{G}_{a,\mathrm{dR}}$. Therefore, Construction C.2 gives an equivalence $\mathrm{DMod}(\mathbf{G}_a) \cong \mathrm{DMod}(\mathbf{G}_a)$, which can be identified with the Fourier-Deligne transform.

Example C.9. Let $\hat{\mathbf{W}}/V(1)$ denote the pullback $\mathbf{W}/V(1) \times_{\mathbf{G}_a} \hat{\mathbf{G}}_a$, so that $\hat{\mathbf{W}}/V(1)$ is a ring scheme. We claim that there is a canonical isomorphism $\mathfrak{D}(\hat{\mathbf{W}}/V(1)) \cong \hat{\mathbf{W}}/V(1)$ of group schemes. Arguing as in Example C.8, we may reduce to showing that there is a nontrivial extension

$$B\mathbf{G}_a^\sharp \to \hat{\mathbf{W}}/V(1) \to \hat{\mathbf{G}}_a.$$

For this, observe that if R is any commutative $\mathbf{Z}_{(p)}$-algebra and $x \in W(R)$, we have $xV(1) = VF(x)$, so that $\mathbf{W}/V(1) \cong \mathbf{W}/VF\mathbf{W}$. Writing VF as the composition of F with V, we see that there is an extension

$$B\mathbf{W}[F] \to \mathbf{W}/V(1) \to \mathbf{W}/V \cong \mathbf{G}_a.$$

Base-changing along $\hat{\mathbf{G}}_a \to \mathbf{G}_a$ gives the desired extension, since $B\mathbf{W}[F] = B\mathbf{G}_a^\sharp$. Since $\mathfrak{D}(\hat{\mathbf{W}}/V(1)) \cong \hat{\mathbf{W}}/V(1)$, Construction C.2 implies that there is a symmetric monoidal equivalence $\mathrm{QCoh}(\hat{\mathbf{W}}/V(1)) \simeq \mathrm{QCoh}(\hat{\mathbf{W}}/V(1))$ which intertwines the usual tensor product of quasicoherent sheaves with the convolution tensor product arising from the addition law for $\hat{\mathbf{W}}/V(1)$.

This is closely related to the Fourier-Deligne transform of Example C.8. Indeed, by [BL22, Construction 3.8], $\mathbf{W}/V(1)$ can be identified with the "diffracted Hodge stack" $\mathbf{G}_a^{\not{D}}$ of $\mathbf{G}_a$. Therefore, we we may view the above equivalence $\mathrm{QCoh}(\hat{\mathbf{W}}/V(1)) \simeq \mathrm{QCoh}(\hat{\mathbf{W}}/V(1))$ as closely related to a Fourier-type transform for Hodge-Tate crystals on $\mathbf{G}_a$.

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1 Oxford St, Cambridge, MA 02139
Email address: `sdevalapurkar@math.harvard.edu`