

RESEARCH STATEMENT: OVERVIEW

My research revolves around foundational aspects of pure mathematics and machine learning. In mathematics, I concentrate on the geometric Langlands program. In machine learning, I explore associative memory models through empirical, algebraic, and topological lenses. The first three pages of this document provide a summary, followed by detailed discussions of my work in both mathematics and machine learning.

P-ADIC GEOMETRY AND LANGLANDS PROGRAM

In the geometric Langlands program, my graduate work

has focused on extensions in the mixed characteristic setting, [ILZ24], this is joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn Univeristy). Our work applied the framework of Zhu's perfect geometry [Zhu17] to prove the Casselman-Shalika formula [NP01] for mixed characteristics. The Casselman-Shalika formula computes the "fourier coefficients" of automorphic forms, fundamental to modern works of geometric Langlands [FR22]. Moving forward, I will continue my exploration in two directions:

- (1) **Relative aspects of Langlands**, joint with Yuta Takaya (University of Tokyo), we aim to explore relative aspects of the Langlands program on the Fargues-Fontaine curve, [FS24], from recent conjectures [BSV], particularly the relationship between period sheaves and L -sheaves as [FW24].
- (2) **Metaplectic aspects of Langlands**, joint with Toan Pham (Johns Hopkins University), we intend to give a geometric metaplectic Casselman-Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+24].

GEOMETRY OF ASSOCIATIVE MEMORY NETWORKS

In machine learning, I am interested in the foundations of modern associative memory networks [KH16]. These networks bridge biological realism, computational efficiency, and interpretable network design.

- (1) **Polytopal Decomposition of Memory Networks:** In joint work with Chris Hillar (Redwood Research, Berkeley), we study the polytopal decomposition of the weight spaces of memory networks and its relation to network scaling, drawing parallels with tropical geometry [ZNL18]. Using Manin and Marcolli's recent formalism [MM24], which leverages *summing functors* and *Gamma spaces*, we explore the homotopy type of memory networks – a deeper invariant than homology¹. This approach aims to uncover the relationship between memory capacity and homotopical invariants.
- (2) **Associative Memory Models Beyond Storage Capacity:** In ongoing work with Muhan Gao (Johns Hopkins University), we empirically analyze modern energy-based memory networks for language modeling and classification tasks. Our focus is on the regime where stored memories exceed theoretical capacity [KH16, Equations (5) and (6)]. This study explores the limitations of synthetic memory networks and the relation between generalization and memory capacity.

¹For an overview of homology in topological data analysis, see [Cur16].

REFERENCES

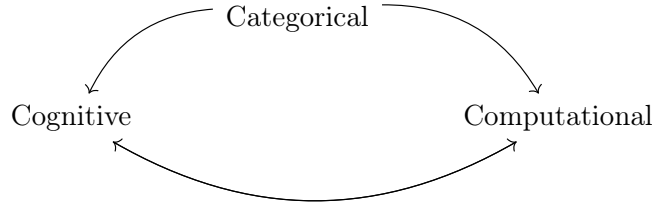
- [Bru+24] Brubaker, Ben, Buciumas, Valentin, Bump, Daniel, and Gustafsson, Henrik P. A. *Meta-plectic Iwahori Whittaker functions and supersymmetric lattice models*. 2024. arXiv: [2012.15778](https://arxiv.org/abs/2012.15778) [math.RT]. URL: <https://arxiv.org/abs/2012.15778> (cit. on p. 1).
- [BSV] Ben-Zvi, David, Sakellaridis, Yiannis, and Venkatesh, Akshay. “Relative Langlands duality”. In: () (cit. on p. 1).
- [Cur16] Curto, Carina. *What can topology tell us about the neural code?* 2016. arXiv: [1605.01905](https://arxiv.org/abs/1605.01905) [q-bio.NC]. URL: <https://arxiv.org/abs/1605.01905> (cit. on p. 1).
- [FR22] Faergeman, Joakim and Raskin, Sam. *Non-vanishing of geometric Whittaker coefficients for reductive groups*. 2022. arXiv: [2207.02955](https://arxiv.org/abs/2207.02955) [math.RT]. URL: <https://arxiv.org/abs/2207.02955> (cit. on p. 1).
- [FS24] Fargues, Laurent and Scholze, Peter. “Geometrization of the local Langlands correspondence”. In: *arXiv e-prints*, arXiv:2102.13459 (Feb. 2024), arXiv:2102.13459. arXiv: [2102.13459](https://arxiv.org/abs/2102.13459) [math.RT] (cit. on p. 1).
- [FW24] Feng, Tony and Wang, Jonathan. *Geometric Langlands duality for periods*. 2024. arXiv: [2402.00180](https://arxiv.org/abs/2402.00180) [math.NT]. URL: <https://arxiv.org/abs/2402.00180> (cit. on p. 1).
- [GL22] Gaitsgory, D. and Lysenko, S. *Parameters and duality for the metaplectic geometric Langlands theory*. 2022. arXiv: [1608.00284](https://arxiv.org/abs/1608.00284) [math.AG]. URL: <https://arxiv.org/abs/1608.00284> (cit. on p. 1).
- [ILZ24] Iyengar, Ashwin, Lin, Milton, and Zou, Konrad. *Geometric Casselman-Shalika in mixed characteristic*. 2024. arXiv: [2408.07953](https://arxiv.org/abs/2408.07953) [math.AG]. URL: <https://arxiv.org/abs/2408.07953> (cit. on p. 1).
- [KH16] Krotov, Dmitry and Hopfield, John J. *Dense Associative Memory for Pattern Recognition*. 2016. arXiv: [1606.01164](https://arxiv.org/abs/1606.01164) [cs.NE]. URL: <https://arxiv.org/abs/1606.01164> (cit. on p. 1).
- [McN16] McNamara, Peter J. “The metaplectic Casselman-Shalika formula”. In: *Trans. Amer. Math. Soc.* 368.4 (2016), pp. 2913–2937. ISSN: 0002-9947,1088-6850. URL: <https://doi.org/10.1090/tran/6597> (cit. on p. 1).
- [MM24] Manin, Yuri and Marcolli, Matilde. “Homotopy Theoretic and Categorical Models of Neural Information Networks”. In: *Compositionality* Volume 6 (2024) (Sept. 2024). ISSN: 2631-4444. URL: <http://dx.doi.org/10.46298/compositionality-6-4> (cit. on p. 1).
- [NP01] Ngô, B. C. and Polo, P. “Résolutions de Demazure affines et formule de Casselman-Shalika géométrique”. In: *J. Algebraic Geom.* 10.3 (2001), pp. 515–547. ISSN: 1056-3911,1534-7486 (cit. on p. 1).
- [Zhu17] Zhu, Xinwen. “Affine Grassmannians and the geometric Satake in mixed characteristic”. In: *Ann. of Math. (2)* 185.2 (2017), pp. 403–492. ISSN: 0003-486X,1939-8980. URL: <https://doi.org/10.4007/annals.2017.185.2.2> (cit. on p. 1).
- [ZNL18] Zhang, Liwen, Naitzat, Gregory, and Lim, Lek-Heng. *Tropical Geometry of Deep Neural Networks*. 2018. arXiv: [1805.07091](https://arxiv.org/abs/1805.07091) [cs.LG]. URL: <https://arxiv.org/abs/1805.07091> (cit. on p. 1).

RESEARCH STATEMENT IN FOUNDATIONS OF MACHINE LEARNING

MILTON LIN

PERSONAL EXPERIENCES

My research in the geometric Langlands program has centered on bridging discrete structures (number theory) with continuous spaces (topology) through *algebraic formalism*. Building on this, I aim to explore how **algebraic and categorical models** can reveal the **qualitative dynamics** of modern language models and neural circuits. This involves examining the extent to which **biological realism** can be mirrored in computational frameworks.



The overarching goal is to unify insights from *cognitive science*, *category theory*, and *computation* to understand the *limiting behaviors* of complex networks, both artificial and biological. Specifically, my research will offer new perspectives on *scaling laws* in large language models [Kap+20], [SMK23] and propose biologically grounded designs for neural networks. In Section 1, I introduce *associative memory networks* like *Hopfield Networks* and *Simplicial Hopfield Networks*. Our research studies the weight space decomposition of these models and evaluates their scaling behavior. In Section 2, I highlight how my background in algebra and geometry positions me to contribute to this interdisciplinary research.¹

1. ASSOCIATIVE MEMORY NETWORKS

Associative Memory Networks, or Hopfield Networks, are well-known models where memories are stored and recalled based on Lyapunov energy functions that govern the network’s dynamics [Hop84]. The local minima in the energy landscape correspond to stored memories. Recent models, such as Dense Associative Memories, improve memory capacity exponentially to $2^{d/2}$ by modifying the energy dynamics [KH16; Dem+17]. These have been connected to attention mechanisms in transformers [Ram+21], though their full relations remain largely unexplored.

Research Goal: Scaling Properties of Associative Memory and Modern Models. The two key research areas are, joint with Chris Hillar (Redwood Research), Tenzin Chan (Algebraic) and Muhan Gao (Johns Hopkins University)

1. *Polytopal Decomposition of Weight Spaces in Toy Models:* We will extend the polytopal weight space decomposition, as present in literature on threshold linear networks, [CLM20], [CGM23], to higher order memory networks, such as *simplicial Hopfield networks* inspired by setwise connection [BF23] or *dense associative memories*. This connects to recent approaches using spline theory to

¹I am indebted to my discussions with Prof. Daniel Khashabi and Prof. Leyla Isik.

understand neural networks, [Bb18], [Bla+22]. We will study how the decomposition changes as the *network size increases*.

2. *Energy Transformer Experiments*: We will evaluate the scaling properties of the *Energy Transformer* [Hoo+23], a model based on dense associative memory principles, hypothesizing across three dimensions: - i) Parameters: increasing parameters increases number of attractors (local minima). - ii) Data: scaling data size will refine attractors, resulting in more meaningful minima. - iii) Compute: increasing compute (depth or iterations) will lead to faster convergence to low-energy retrievals. We will then investigate how

- (i) *data augmentation* during training affects memory storage and retrieval efficiency, as in prior work on transformers [AL24].
- (ii) one can leverage interpretable aspects of the energy transformers to explore various representation hypotheses, such as the *platonic representation hypothesis*, [Huh+24].

The end goal is to provide both empirical and theoretical comparison with modern networks [Hoo+24]; some works which studied scaling properties of memory networks include [ND21], [Niu+24], and [CDB24].

2. CATEGORICAL MODELS AND HOMOTOPY THEORY

Categorical approaches have gained momentum as a systematic framework for studying network structures [Gav+24]. This has been particularly successful in the field of *geometric deep learning* [Bro+21], where abstract mathematical structures help describe complex neural networks. We propose to explore *Hopfield networks* using a recent formalism by Manin et al. [MM24], which uses *summing functors* and *Gamma spaces* to model the allocation of resources in neural networks. These concepts will allow us to understand how the complexity of memory networks scales as network size increases. The formalism allows us to study a *homotopy type* - a mathematical construct at a deeper level than *homology*². Homotopy captures invariants of network up to continuous deformations. Previous studies have shown that stimulus space can be reconstructed up to homotopy [Man15].

Research Goal: Homotopical Complexity Under Scaling. This research will investigate how the **homotopical complexity** of memory networks evolves as their size increases. Specifically, we will examine how *memory capacity* correlates with *homotopical invariants* like *Betti numbers* (which measure the number of independent cycles in a space) and *simplicial complexes* (which provide a higher-dimensional generalization of networks). Burns and Fukai have already done early work in this direction [BF23], but much remains to be explored.

3. EXPECTED IMPACT:

These research will highlight

- the limitations of synthetic memory networks, particularly in the case when the number of parameters exceed the storage capacities. This sheds light in regarding these networks as proxies for explaining biological networks, see also [KH21].
- the possibilities of creating hybrid models that respect biological constraints while maintaining the computational power of synthetic networks.
- lens through which to study the behavior of neural networks algebraically.

²which is commonly used in topological data analysis (TDA). For a short survey of topology and neural code, see [Cur16].

REFERENCES

- [AL24] Allen-Zhu, Zeyuan and Li, Yuanzhi. *Physics of Language Models: Part 3.1, Knowledge Storage and Extraction*. 2024. arXiv: [2309.14316](https://arxiv.org/abs/2309.14316) [cs.CL]. URL: <https://arxiv.org/abs/2309.14316> (cit. on p. 2).
- [Bb18] Balestrieri, Randall and baraniuk, richard. “A Spline Theory of Deep Learning”. In: *Proceedings of the 35th International Conference on Machine Learning*. Ed. by Jennifer Dy and Andreas Krause. Vol. 80. Proceedings of Machine Learning Research. PMLR, Oct. 2018, pp. 374–383. URL: <https://proceedings.mlr.press/v80/balestrieri18b.html> (cit. on p. 2).
- [BF23] Burns, Thomas F and Fukai, Tomoki. “Simplicial Hopfield networks”. In: *The Eleventh International Conference on Learning Representations*. 2023. URL: https://openreview.net/forum?id=_QLsH8gatwx (cit. on pp. 1, 2).
- [Bla+22] Black, Sid, Sharkey, Lee, Grinsztajn, Leo, Winsor, Eric, Braun, Dan, Merizian, Jacob, Parker, Kip, Guevara, Carlos Ramón, Millidge, Beren, Alfour, Gabriel, and Leahy, Connor. *Interpreting Neural Networks through the Polytope Lens*. 2022. arXiv: [2211.12312](https://arxiv.org/abs/2211.12312) [cs.LG]. URL: <https://arxiv.org/abs/2211.12312> (cit. on p. 2).
- [Bro+21] Bronstein, Michael M., Bruna, Joan, Cohen, Taco, and Veličković, Petar. *Geometric Deep Learning: Grids, Groups, Graphs, Geodesics, and Gauges*. 2021. arXiv: [2104.13478](https://arxiv.org/abs/2104.13478) [cs.LG]. URL: <https://arxiv.org/abs/2104.13478> (cit. on p. 2).
- [CDB24] Cabannes, Vivien, Dohmatob, Elvis, and Bietti, Alberto. *Scaling Laws for Associative Memories*. 2024. arXiv: [2310.02984](https://arxiv.org/abs/2310.02984) [stat.ML]. URL: <https://arxiv.org/abs/2310.02984> (cit. on p. 2).
- [CGM23] Curto, Carina, Geneson, Jesse, and Morrison, Katherine. *Stable fixed points of combinatorial threshold-linear networks*. 2023. arXiv: [1909.02947](https://arxiv.org/abs/1909.02947) [q-bio.NC]. URL: <https://arxiv.org/abs/1909.02947> (cit. on p. 1).
- [CLM20] Curto, Carina, Langdon, Christopher, and Morrison, Katherine. *Combinatorial Geometry of Threshold-Linear Networks*. 2020. arXiv: [2008.01032](https://arxiv.org/abs/2008.01032) [math.CO]. URL: <https://arxiv.org/abs/2008.01032> (cit. on p. 1).
- [Cur16] Curto, Carina. *What can topology tell us about the neural code?* 2016. arXiv: [1605.01905](https://arxiv.org/abs/1605.01905) [q-bio.NC]. URL: <https://arxiv.org/abs/1605.01905> (cit. on p. 2).
- [Dem+17] Demircigil, Mete, Heusel, Judith, Löwe, Matthias, Uppang, Sven, and Vermet, Franck. “On a Model of Associative Memory with Huge Storage Capacity”. In: *Journal of Statistical Physics* 168.2 (May 2017), pp. 288–299. ISSN: 1572-9613. URL: <http://dx.doi.org/10.1007/s10955-017-1806-y> (cit. on p. 1).
- [Gav+24] Gavranović, Bruno, Lessard, Paul, Dudzik, Andrew, Glehn, Tamara von, Araújo, João G. M., and Veličković, Petar. *Position: Categorical Deep Learning is an Algebraic Theory of All Architectures*. 2024. arXiv: [2402.15332](https://arxiv.org/abs/2402.15332) [cs.LG]. URL: <https://arxiv.org/abs/2402.15332> (cit. on p. 2).
- [Hoo+23] Hoover, Benjamin, Liang, Yuchen, Pham, Bao, Panda, Rameswar, Strobelt, Hendrik, Chau, Duen Horng, Zaki, Mohammed J., and Krotov, Dmitry. *Energy Transformer*. 2023. arXiv: [2302.07253](https://arxiv.org/abs/2302.07253) [cs.LG]. URL: <https://arxiv.org/abs/2302.07253> (cit. on p. 2).
- [Hoo+24] Hoover, Benjamin, Strobelt, Hendrik, Krotov, Dmitry, Hoffman, Judy, Kira, Zsolt, and Chau, Duen Horng. *Memory in Plain Sight: Surveying the Uncanny Resemblances of Associative Memories and Diffusion Models*. 2024. arXiv: [2309.16750](https://arxiv.org/abs/2309.16750) [cs.LG]. URL: <https://arxiv.org/abs/2309.16750> (cit. on p. 2).
- [Hop84] Hopfield, John J. “Neurons with graded response have collective computational properties like those of two-state neurons.” In: *Proceedings of the national academy of sciences* 81.10 (1984), pp. 3088–3092 (cit. on p. 1).

- [Huh+24] Huh, Minyoung, Cheung, Brian, Wang, Tongzhou, and Isola, Phillip. *The Platonic Representation Hypothesis*. 2024. arXiv: [2405.07987](https://arxiv.org/abs/2405.07987) [cs.LG]. URL: <https://arxiv.org/abs/2405.07987> (cit. on p. 2).
- [Kap+20] Kaplan, Jared, McCandlish, Sam, Henighan, Tom, Brown, Tom B., Chess, Benjamin, Child, Rewon, Gray, Scott, Radford, Alec, Wu, Jeffrey, and Amodei, Dario. *Scaling Laws for Neural Language Models*. 2020. arXiv: [2001.08361](https://arxiv.org/abs/2001.08361) [cs.LG]. URL: <https://arxiv.org/abs/2001.08361> (cit. on p. 1).
- [KH16] Krotov, Dmitry and Hopfield, John J. *Dense Associative Memory for Pattern Recognition*. 2016. arXiv: [1606.01164](https://arxiv.org/abs/1606.01164) [cs.NE]. URL: <https://arxiv.org/abs/1606.01164> (cit. on p. 1).
- [KH21] Krotov, Dmitry and Hopfield, John. *Large Associative Memory Problem in Neurobiology and Machine Learning*. 2021. arXiv: [2008.06996](https://arxiv.org/abs/2008.06996) [q-bio.NC]. URL: <https://arxiv.org/abs/2008.06996> (cit. on p. 2).
- [Man15] Manin, Yuri I. *Neural codes and homotopy types: mathematical models of place field recognition*. 2015. arXiv: [1501.00897](https://arxiv.org/abs/1501.00897) [math.HO]. URL: <https://arxiv.org/abs/1501.00897> (cit. on p. 2).
- [MM24] Manin, Yuri and Marcolli, Matilde. “Homotopy Theoretic and Categorical Models of Neural Information Networks”. In: *Compositionality* Volume 6 (2024) (Sept. 2024). ISSN: 2631-4444. URL: <http://dx.doi.org/10.46298/compositionality-6-4> (cit. on p. 2).
- [ND21] Nichol, Alex and Dhariwal, Prafulla. *Improved Denoising Diffusion Probabilistic Models*. 2021. arXiv: [2102.09672](https://arxiv.org/abs/2102.09672) [cs.LG]. URL: <https://arxiv.org/abs/2102.09672> (cit. on p. 2).
- [Niu+24] Niu, Xueyan, Bai, Bo, Deng, Lei, and Han, Wei. *Beyond Scaling Laws: Understanding Transformer Performance with Associative Memory*. 2024. arXiv: [2405.08707](https://arxiv.org/abs/2405.08707) [cs.LG]. URL: <https://arxiv.org/abs/2405.08707> (cit. on p. 2).
- [Ram+21] Ramsauer, Hubert, Schäfl, Bernhard, Lehner, Johannes, Seidl, Philipp, Widrich, Michael, Gruber, Lukas, Holzleitner, Markus, Adler, Thomas, Kreil, David, Kopp, Michael K, Klambauer, Günter, Brandstetter, Johannes, and Hochreiter, Sepp. “Hopfield Networks is All You Need”. In: *International Conference on Learning Representations*. 2021. URL: <https://openreview.net/forum?id=tL89RnzIiCd> (cit. on p. 1).
- [SMK23] Schaeffer, Rylan, Miranda, Brando, and Koyejo, Sanmi. *Are Emergent Abilities of Large Language Models a Mirage?* 2023. arXiv: [2304.15004](https://arxiv.org/abs/2304.15004) [cs.AI]. URL: <https://arxiv.org/abs/2304.15004> (cit. on p. 1).

RESEARCH STATEMENT IN PURE MATHEMATICS

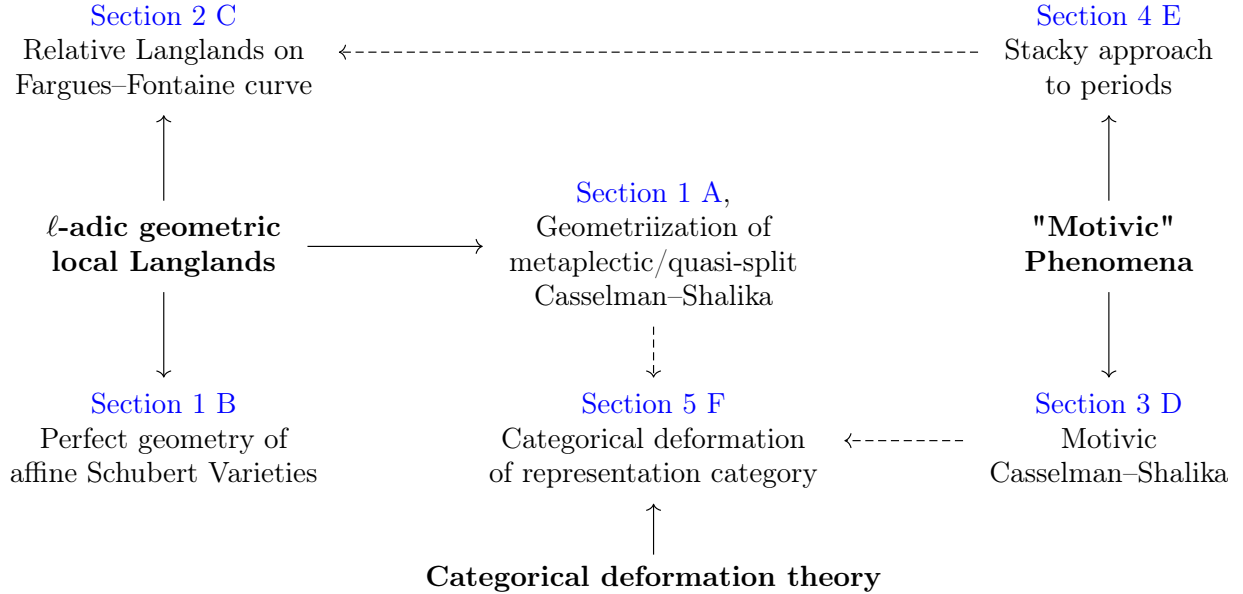
MILTON LIN

INTRODUCTION

My areas of interest in mathematics are:

- (1) The **Langlands correspondence**, which is now a huge web of conjectures, from special values of L functions to conformal field theory. My particular interest is in various incarnations of the **Casselman–Shalika formula**, [A](#), and **relative aspects of the ℓ -adic geometric local Langlands correspondence**, [C](#).
- (2) **Stable homotopy theory**, through the lens of **categorical deformation theory**, [F](#).
- (3) **Motivic phenomena**, where I hope to explore the motivic version of Casselman–Shalika formula, [D](#), and a stacky approach to periods, [E](#).

Majority of the research presented here originates from my study of the Casselman–Shalika formula in the mixed characteristic setting, as outlined in [Section 1](#). These areas of research are interconnected, as shown in the following diagram.



The priority of research is listed in the following order¹,

$$\mathbf{A=C=F>D>E>B.}$$

Date: November 19, 2024.

¹The alphabet links to the goal rather than the section.

Notations. Theorems stated have full proof written by either me or joint with collaborators, *unless* it is annotated with: **conjecture** – no proofs have been written down but is believed to hold, or **in progress** – where we have partial progress. We use freely the language of higher categories and higher algebra, [Lur09], [Lur18].

1. MIXED CHARACTERISTIC GEOMETRY AND THE CASSELMAN–SHALIKA FORMULA

Let G be a connected reductive group over a nonarchimedean local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld [AG15], [Gai14]. For a fixed choice of Whittaker datum, [FS24, I.10.2] conjectured a $\mathrm{IndPerf}(\mathrm{LS}_{\hat{G}})$ -linear equivalence

$$\mathbb{L}_G : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \simeq \mathrm{IndCoh}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}})$$

which can be thought of as a generalization of the classical local Langlands. One fundamental aspect of the program is to understand the *Whittaker Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \rightarrow D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, as described *op.cit.*, this corresponds to finding the Fourier coefficients of automorphic functions:

Example 1.1. Let $G = \mathrm{PGL}_2$ be the projective linear group over \mathbb{Q} . A modular function, f , has an adelic formulation, \tilde{f} on $G(\mathbb{A}_{\mathbb{Q}})$. Up to a *normalizing factor*, the Fourier coefficients coincide with the Whittaker coefficients (Jacquet period),

$$a_m \sim \int_{N(\mathbb{Q}) \backslash N(\mathbb{A}_{\mathbb{Q}})} \tilde{f}(n\alpha_m)\psi(-n) \, dn \quad \text{for } m \geq 1$$

where $\alpha_m \in T(\mathbb{A}_{\mathbb{Q}}^{\mathrm{fin}})$ is m considered as a finite idèle and ψ is a standard character on $N(\mathbb{A}_{\mathbb{Q}}) = \mathbb{A}_{\mathbb{Q}}$, where N is the radical of standard Borel. For more details, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which we aim to replicate in the mixed characteristic setting. In joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn University) [ILZ24], we proved a variation of this problem: the *geometric Casselman-Shalika formula* over the Witt vector affine Grassmannian Gr_G , analogous to the equi-characteristic geometrization carried out in [NP01].

Theorem 1.2. [ILZ24] *The geometric Casselman-Shalika formula holds over the Witt vector Grassmannian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori-Satake equivalence, [Bez+19] without using nearby cycles.

Theorem 1.3 (I.-Lin-Z., in progress). *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.*²

As of writing, D. Hansen, L. Hamman, and L. Mann have announced significant advancements to the global version of the formula. Building on all these progresses, I have developed a strong interest in exploring the geometrization of the Casselman-Shalika formula for covering groups [GGW18].

²In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

Research Goal A. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula over equal-characteristic local fields. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+20; Bru+24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

Lastly, we would like to highlight that in our recent work [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\leq\mu}$, for minuscule and quasi-minuscule μ . In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal B. Geometry of general perfect Schubert variety *Prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhat–Tits building of G , we have an associated group scheme \mathcal{G} over \mathcal{O} .*

Conjecture 1.4. The smooth locus of $\mathrm{Gr}_{\mathcal{G},\leq s}$ is $\mathrm{Gr}_{\mathcal{G},s}$ in perfect geometry, in the sense of [Zhu17].

2. RELATIVE LANGLANDS ON THE FARGUES FONTAINE CURVE

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on \mathcal{A} -side (automorphic) and L -sheaves on \mathcal{B} -side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin–Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues–Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with (\hat{G}, \hat{X}) of hyper-spherical varieties. Let $\Lambda = \bar{\mathbb{Q}}_l$. We considered the *Iwasawa–Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \rightarrow \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \rightarrow \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_! \Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \pi_* \omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. Bun_G has a Hardar–Narasimhan stratification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G)$. In our case, $G = \mathbb{G}_m$, $\mathrm{Bun}_{\mathbb{G}_m}$ is stratified by $\mathrm{Bun}_{\mathbb{G}_m}^n$ for $n \in \mathbb{Z} = B(T)$. Interesting phenomena occurs for $n > 0$, and the study of period sheaves reduces to the study of Bun_G^X restricted to Bun_G^n . This corresponds to the Abel–Jacobi map previously studied by Fargues [Far20] and Hansen [Han21]. On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles, [FW24, Ch. 6] recently developed by Le-Bras et al.

Conjecture 2.1. [Lin-T., in progress] Under the geometric local Langlands correspondence, \mathbb{L}_G , (appropriately normalized) \mathcal{P}_X is sent to³ $\mathcal{L}_{\hat{X}}$.

Research Goal C. Relative Langlands on the Fargues Fontaine curve. *Complete Conjecture 2.1 as a first step and then the Hecke case, which classically corresponds to Hecke’s integral representation of standard L -function for GL_2 . Lastly, one can ask whether on the \mathcal{B} -side, the same constructions of [BSV, Ch. 11] works for the p -adic (Emerton–Gee) L -parameter stacks, which potentially give new interpretations to p -adic L -functions.*

³One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on X which we do not discuss.

3. MOTIVIC PHENOMENA

In this section S denotes a base scheme satisfying the Beilinson-Soulé vanishing conjecture, such as $\text{Spec } \mathbb{Z}$, finite field. R denotes a regular coherent coefficient ring. I discuss one side project I have done, and one area of research, [D](#), which extends my joint work [\[ILZ24\]](#).

In Grothendieck's quest towards the Weil conjectures, he reduced to problems of motives. Beilinson proposed an extension of the notion of pure motives to mixed motives, which was realized through the works of Voevodsky, Levine for a field, and extended by Ayoub and Cisikinki-Dégliše, [\[CD19\]](#). For a finite type S -scheme $X \in \text{Sch}_S^{\text{ft}}$, we can construct the derived categories of motives over X , $\text{DM}(X, R)$. If X is smooth then its Ext-groups,

$$\pi_{-m} \text{Map}_{\text{DM}(X, R)}(1_X, 1_X(n)) \simeq \text{CH}^n(X, 2n - m)_R$$

are Bloch's higher Chow groups. Motivic categories and the Chow groups are difficult to explicitly work with due to the lack of motivic t -structures. One method is due to the work of Beauville [\[Bea83\]](#), using Fourier transform. In my joint work, [\[Has+24\]](#) we extended the work of Beauville,

Theorem 3.1 (Lin et al.). *Let $X \rightarrow Y$ be an abelian scheme, whose base Y is smooth and quasi projective over a field. There is an explicit N , such that one obtains a Beauville decomposition*

$$\text{CH}^i(X)_{\mathbb{Z}[1/N]} \simeq \bigoplus_s \text{CH}_{(s)}^i(X)_{\mathbb{Z}[1/N]}$$

where $\text{CH}_{(s)}^i(X)_R := \{x \in \text{CH}^i(X)_R : [n]_X^* x = n^{2i-s} x \quad \forall n \in \mathbb{Z}\}$ and $[n]$ is the multiplication of an abelian scheme.

This extends to a \mathfrak{sl}_2 action, which we discussed in *op.cit*; and if S were an algebraically closed field, this implies various structural results. The key ingredient was using G. Pappas' version of integral Grothendieck-Riemann-Roch, [\[Pap07\]](#).

Returning to motivic t -structures, it was shown by Levine that t -structures exist on a nice subcategory $\text{DTM}(X) \hookrightarrow \text{DM}(X)$ of *mixed Tate motives* for nice schemes X . This was extended to schemes with cellular Whitney-Tate stratification by Soergel and Wendt, and to prestacks in [\[RS20\]](#).

Research Goal D. Motivic Whittaker categories. *Define a Whittaker category in motivic setting, obtaining a similar equivalence at [\[FGV01\]](#). To begin, we can prove the same statement in [\[NP01\]](#) in the category of mixed Tate motives, in which all the relevant objects are well defined. The difficulty, however, is that the original argument requires characteristic 0 coefficients, whilst in positive characteristic, there are problems of extensions.*

4. STACKY APPROACHES AND PERIODS

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeault cohomology – admit a factorization to the category of stacks over some affine scheme $\text{Spec } R$,

$$\begin{aligned} \text{Sch}_{\mathbb{Z}}^{\text{sep, ft}} &\rightarrow \text{Stk}_R \rightarrow D(R) \\ X &\mapsto X^? \mapsto \Gamma(X^?, \mathcal{O}_{X^?}) \end{aligned}$$

For instance, the *de Rham stack* X^{dR} over \mathbb{Q} , has points given by $X^{\text{dR}}(A) := X(A_{\text{red}})$ for any \mathbb{Q} -algebra A (cf. [\[GR14\]](#)). This is often referred to as a *stacky approach* [\[Dri22\]](#) or *transmutation* [\[Bha23\]](#), which allows one to use six functor formalism and geometric techniques. I explored this concept in basic constructions of F. Brown's work on *motivic periods*, [\[Bro14\]](#). If X were a smooth variety over \mathbb{Q} , the matrix coefficient from Grothendieck's comparison theorem

$$H_{\text{dR}}^*(X) \otimes_{\mathbb{Q}} \mathbb{C} \xrightarrow{\sim} H_{\text{Betti}}^*(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

with respect to the \mathbb{Q} structure of de-Rham and Betti cohomology (of $X(\mathbb{C})$) are *periods associated to X* . These periods along with their enhancements through Hodge structures, has a natural action of "Galois group"⁴ which should govern the arithmetic structure of periods.

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the projective space minus three points over \mathbb{Q} . The period associated is studied through Chen's de Rham comparison theorem, [Che76], which relates to iterated integrals [Che73] and multiple zeta values [Bro14]. To give a "stacky" perspective of this period one uses unipotent types of stacks, which is defined in Toën, [Toë06], and recently in [MR23]. This is an endofunctor \mathbf{U} on stacks, sending a stack X to its unipotent homotopy type. My first result is:

Theorem 4.1. (*Lin*) *Unipotent de Rham fundamental group, $\pi_1^{u,dR}(X, x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{dR}))$.*

Research Goal E. A stacky approach to motivic periods. *This research aims to use geometric techniques in the study of periods. As a proof of concept, we will recover de Rham comparison theorem through the Riemann-Hilbert correspondence between the analytic Betti stack and the de Rham stack. $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, is not proper, which requires us to incorporate log structures. I expect to prove:*

Conjecture 4.2. There exists X^{Betti} such that the unipotent Betti group $\pi_1^{u,\text{Betti}}(X(\mathbb{C}), x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{\text{Betti}}))$. A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,dR}(X, x) \simeq \pi_1^{u,\text{Betti}}(X(\mathbb{C}), x) \otimes \mathbb{C}$$

By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure.⁵ We hope that such work can spark new techniques and new phenomena, such as those used in p -adic integration theory, [Vol01].

5. DEFORMATION THEORY AND THE SPHERE SPECTRUM \mathbb{S}

Let \mathcal{S} denote the ∞ -category of ∞ -groupoids/anima. The *stabilization* of \mathcal{S} is \mathbf{Sp} , the ∞ -category of spectra. This is the natural category to study cohomological invariants. Within \mathbf{Sp} , lies the universal cohomology theory, \mathbb{S} , the *sphere spectrum*. By Chevalley's works, connected reductive groups over \mathbb{C} have a canonical split \mathbb{Z} -form $G_{\mathbb{Z}}$, see [Con15]. One can analogously ask: *is there a \mathbb{S} -form for algebraic groups?* A first approximation is the existence of an algebraic category $\text{Rep}_{\mathbb{S}}(G_{\mathbb{S}})$, which *deforms* to $\text{Rep}_{\mathbb{Z}}(G_{\mathbb{Z}})$. To study, we begin with *formal deformation of categories*, which we briefly recall.

Let \mathcal{C} be a symmetric monoidal category. There is a natural hierarchy of commutativity fitting in the diagram

$$\cdots \rightarrow \mathbb{E}_n(\mathcal{C}) \rightarrow \mathbb{E}_{n-1}(\mathcal{C}) \rightarrow \cdots \rightarrow \mathbb{E}_1(\mathcal{C})$$

where $\text{CAlg}(\mathcal{C}) := \mathbb{E}_{\infty}(\mathcal{C}) := \varprojlim \mathbb{E}_n(\mathcal{C})$ of symmetric algebra objects can be identified with the limit. One can formalize the notion of \mathbb{E}_n -algebra objects via *disk operads*, or *little cubes operads*.

Example 5.1. Let $\mathcal{C} = (\mathcal{S}, \times)$. Let $Y \in \mathcal{S}_*$ is a pointed ∞ -groupoid, its k -fold based loop spaces, $\Omega_*^k Y$ is a classical example of \mathbb{E}_k algebra object in (\mathcal{S}, \times) .

Let $R \in \mathbb{E}_n(\mathbf{Sp})$ be an \mathbb{E}_n ring, and consider LMod_R , the derived category of left R -modules, as an \mathbb{E}_1 object in Pr^{st} , the category of presentable stable categories. This defines $\text{RMod}_{\text{LMod}_R}(\text{Pr}^{\text{st}})$, the category of presentable stable (right) R -linear categories, [Lur18, Appendix D]. Set $\text{Pr}_R^{\text{st}, \text{cg}}$, as

⁴For instance, in the approach of Deligne, he defined a *systems of realizations* [Del89]

⁵This is important in Brown's approach, where he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

the full subcategory spanned by those whose underlying category is compactly generated.⁶ For G a connected reductive group over a field k , $D^b(\text{Rep}_k^{\text{fd}}(G))$, the bounded derived category of finite dimensional algebraic representations with k coefficients lies in $\text{Pr}_R^{\text{st}, \text{cg}}$.

From now on, $k = \mathbb{C}$. Let $\text{Art}_k^{(n)}$ denote the category of \mathbb{E}_n artinian ring spectrum, for $n \geq 0$ over k . We refer to [Lur18, Ch. 15] for definition. This extends the classical definition of Artinian local ring, in particular, $R \in \text{Art}_k^{(n)}$ admits an augmentation map $\epsilon : R \rightarrow k$. One defines the \mathbb{E}_{n+2} -formal moduli problem,

$$\begin{aligned} \text{CatDef}^{(n)}(\mathcal{C}) : \text{Art}_k^{(n+2)} &\rightarrow \hat{\mathcal{S}} \\ R &\mapsto |\{\mathcal{C}\} \times_{\text{Pr}_R^{\text{st}, \text{cg}}} \text{Pr}_k^{\text{st}, \text{cg}}| \end{aligned}$$

where $|\cdot|$ is the underlying Kan complex of the ∞ -category. An object consists of: a \mathcal{C}_B right stable R -linear category, and an equivalence $\mathcal{C}_B \otimes_{\text{LMod}_B} \text{LMod}_k \simeq \mathcal{C}$. Our \mathbb{E}_4 -moduli problem is when $n = 2$ and $\mathcal{C} = D^b(\text{Rep}_k^{\text{fd}}(G))$. The geometric Casselman-Shalika [FGV01], which is the \mathbb{E}_2 -algebra equivalence of \mathcal{C} with the *Whittaker sheaves* on the *affine Grassmanian* $\text{Gr}_{\hat{G}}$, describes this moduli problem. Consider moduli of functor of \mathbb{G}_m -gerbes over X

$$\text{Ge}_{\mathbb{G}_m}(X) : R \mapsto \text{Map}_{\mathbb{E}_2(\mathcal{S})}(X, B^2 R^\times) \quad R \in \text{Art}_k^{(4)}$$

where $R^\times \subset \Omega^\infty R$ are the invertible elements of the underlying space of R ⁷ and B^2 is the second deloop. It was stated without proof in [Lur10]

Theorem 5.2 (Lurie). *There is an equivalence of formal \mathbb{E}_4 -moduli problems*

$$\widehat{\text{Ge}_{\mathbb{G}_m}}(\text{Gr}_{\hat{G}}) \xrightarrow{\simeq} \text{CatDef}^{(2)}(\text{Rep}_k^{\text{fd}} G)$$

where $\widehat{}$ is the formal completion of the moduli functor at a base point.

Research Goal F. Categorical deformations of the representation category We will first document carefully Lurie's theorem, [Theorem 5.2](#). Then, we will explore deformations of representation of Lusztig's small quantum group, as suggested in *op.cit.* Remark 10.12, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.

⁶The compact generation is only a smallness condition for our version.

⁷is the union of the connected components of invertibles in the $\pi_0 R$ of the 0th space of R and is equivalent to the n th loop space of some space, $R^\times \simeq \Omega^n Z$, $n \geq 4$, hence the deloop $B^2 R^\times \in \mathbb{E}_2(\mathcal{S})$

REFERENCES

- [AG15] Arinkin, D. and Gaitsgory, D. “Singular support of coherent sheaves and the geometric Langlands conjecture”. In: *Selecta Math. (N.S.)* 21.1 (2015), pp. 1–199. ISSN: 1022-1824,1420-9020. URL: <https://doi.org/10.1007/s00029-014-0167-5> (cit. on p. 2).
- [BBM21] Bezrukavnikov, Roman, Braverman, Alexander, and Mirkovic, Ivan. *Some results about the geometric Whittaker model*. 2021. arXiv: [math/0210250](https://arxiv.org/abs/math/0210250) [[math.AG](#)] (cit. on p. 2).
- [Bea83] Beauville, A. “Quelques remarques sur la transformation de Fourier dans l’anneau de Chow d’une variété abélienne”. In: *Algebraic geometry (Tokyo/Kyoto, 1982)*. Vol. 1016. Lecture Notes in Math. Springer, Berlin, 1983, pp. 238–260. ISBN: 3-540-12685-6. URL: <https://doi.org/10.1007/BFb0099965> (cit. on p. 4).
- [Bez+19] Bezrukavnikov, Roman, Gaitsgory, Dennis, Mirković, Ivan, Riche, Simon, and Rider, Laura. “An Iwahori-Whittaker model for the Satake category”. In: *J. Éc. polytech. Math.* 6 (2019), pp. 707–735. ISSN: 2429-7100,2270-518X. URL: <https://doi.org/10.5802/jep.104> (cit. on p. 2).
- [Bha23] Bhatt, Bhargav. *Prismatic F -gauges*. 2023 (cit. on pp. 4, 5).
- [Bro14] Brown, Francis. *Motivic periods and the projective line minus three points*. 2014. arXiv: [1407.5165](https://arxiv.org/abs/1407.5165) [[math.NT](#)]. URL: <https://arxiv.org/abs/1407.5165> (cit. on pp. 4, 5).
- [Bro17] Brown, Francis. “Notes on motivic periods”. In: *Commun. Number Theory Phys.* 11.3 (2017), pp. 557–655. ISSN: 1931-4523,1931-4531. URL: <https://doi.org/10.4310/CNTP.2017.v11.n3.a2> (cit. on p. 5).
- [Bru+20] Brubaker, Ben, Buciumas, Valentin, Bump, Daniel, and Gustafsson, Henrik P. A. *Vertex operators, solvable lattice models and metaplectic Whittaker functions*. 2020. arXiv: [1806.07776](https://arxiv.org/abs/1806.07776) [[math.RT](#)]. URL: <https://arxiv.org/abs/1806.07776> (cit. on p. 3).
- [Bru+24] Brubaker, Ben, Buciumas, Valentin, Bump, Daniel, and Gustafsson, Henrik P. A. *Metaplectic Iwahori Whittaker functions and supersymmetric lattice models*. 2024. arXiv: [2012.15778](https://arxiv.org/abs/2012.15778) [[math.RT](#)]. URL: <https://arxiv.org/abs/2012.15778> (cit. on p. 3).
- [BSV] Ben-Zvi, David, Sakellaridis, Yiannis, and Venkatesh, Akshay. “Relative Langlands duality”. In: () (cit. on p. 3).
- [CD19] Cisinski, Denis-Charles and Déglise, Frédéric. *Triangulated categories of mixed motives*. Springer Monographs in Mathematics. Springer, Cham, [2019] ©2019, pp. xlii+406. ISBN: 978-3-030-33241-9; 978-3-030-33242-6. URL: <https://doi.org/10.1007/978-3-030-33242-6> (cit. on p. 4).
- [Che73] Chen, Kuo-tsai. “Iterated integrals of differential forms and loop space homology”. In: *Ann. of Math. (2)* 97 (1973), pp. 217–246. ISSN: 0003-486X. URL: <https://doi.org/10.2307/1970846> (cit. on p. 5).
- [Che76] Chen, Kuo Tsai. “Reduced bar constructions on de Rham complexes”. In: *Algebra, topology, and category theory (a collection of papers in honor of Samuel Eilenberg)*. Academic Press, New York-London, 1976, pp. 19–32 (cit. on p. 5).
- [Con15] Conrad, Brian. “Non-split reductive groups over \mathbf{Z} ”. In: *Autours des schémas en groupes. Vol. II*. Vol. 46. Panor. Synthèses. Soc. Math. France, Paris, 2015, pp. 193–253. ISBN: 978-2-85629-819-0. URL: <https://doi.org/10.1017/CB09781316092439> (cit. on p. 5).
- [Del89] Deligne, Par P. “Le groupe fondamental de la droite projective moins trois points”. In: *Galois Groups over \mathbb{Q} Proceedings of a Workshop Held March 23–27, 1987*. Springer, 1989, pp. 79–297 (cit. on p. 5).
- [Dri22] Drinfeld, Vladimir. *A stacky approach to crystals*. 2022. arXiv: [1810.11853](https://arxiv.org/abs/1810.11853) [[math.AG](#)]. URL: <https://arxiv.org/abs/1810.11853> (cit. on p. 4).
- [Far20] Fargues, Laurent. “Simple connexité des fibres d’une application d’Abel-Jacobi et corps de classes local”. In: *Ann. Sci. Éc. Norm. Supér. (4)* 53.1 (2020), pp. 89–124. ISSN: 0012-9593,1873-2151. URL: <https://doi.org/10.24033/asens.2418> (cit. on p. 3).

- [FGV01] Frenkel, E., Gaitsgory, D., and Vilonen, K. “Whittaker patterns in the geometry of moduli spaces of bundles on curves”. In: *Ann. of Math. (2)* 153.3 (2001), pp. 699–748. ISSN: 0003-486X,1939-8980. URL: <https://doi.org/10.2307/2661366> (cit. on pp. 2, 4, 6).
- [FR22] Faergeman, Joakim and Raskin, Sam. *Non-vanishing of geometric Whittaker coefficients for reductive groups*. 2022. arXiv: [2207.02955](https://arxiv.org/abs/2207.02955) [math.RT]. URL: <https://arxiv.org/abs/2207.02955> (cit. on p. 2).
- [FS24] Fargues, Laurent and Scholze, Peter. “Geometrization of the local Langlands correspondence”. In: *arXiv e-prints*, arXiv:2102.13459 (Feb. 2024), arXiv:2102.13459. arXiv: [2102.13459](https://arxiv.org/abs/2102.13459) [math.RT] (cit. on p. 2).
- [FW24] Feng, Tony and Wang, Jonathan. *Geometric Langlands duality for periods*. 2024. arXiv: [2402.00180](https://arxiv.org/abs/2402.00180) [math.NT]. URL: <https://arxiv.org/abs/2402.00180> (cit. on p. 3).
- [Gai14] Gaitsgory, Dennis. *Outline of the proof of the geometric Langlands conjecture for $GL(2)$* . 2014. arXiv: [1302.2506](https://arxiv.org/abs/1302.2506) [math.AG]. URL: <https://arxiv.org/abs/1302.2506> (cit. on p. 2).
- [Gel75] Gelbart, Stephen S. *Automorphic forms on adèle groups*. Vol. No. 83. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1975, pp. x+267 (cit. on p. 2).
- [GGW18] Gan, Wee Teck, Gao, Fan, and Weissman, Martin H. “L-groups and the Langlands program for covering groups: a historical introduction”. In: 398. L-groups and the Langlands program for covering groups. 2018, pp. 1–31. ISBN: 978-2-85629-845-9 (cit. on p. 2).
- [GK20] Gurevich, Nadya and Karasiewicz, Edmund. *The Twisted Satake Transform and the Casselman-Shalika Formula for Quasi-Split Groups*. 2020. arXiv: [2012.09893](https://arxiv.org/abs/2012.09893) [math.RT]. URL: <https://arxiv.org/abs/2012.09893> (cit. on p. 3).
- [GL22] Gaitsgory, D. and Lysenko, S. *Parameters and duality for the metaplectic geometric Langlands theory*. 2022. arXiv: [1608.00284](https://arxiv.org/abs/1608.00284) [math.AG]. URL: <https://arxiv.org/abs/1608.00284> (cit. on p. 3).
- [GR14] Gaitsgory, Dennis and Rozenblyum, Nick. *Crystals and D-modules*. 2014. arXiv: [1111.2087](https://arxiv.org/abs/1111.2087) [math.AG]. URL: <https://arxiv.org/abs/1111.2087> (cit. on p. 4).
- [Hai01] Hain, Richard. *Iterated Integrals and Algebraic Cycles: Examples and Prospects*. 2001. arXiv: [math/0109204](https://arxiv.org/abs/math/0109204) [math.AG]. URL: <https://arxiv.org/abs/math/0109204> (cit. on p. 5).
- [Han21] Hansen, David. “Moduli of local shtukas and Harris’s conjecture”. In: *Tunis. J. Math.* 3.4 (2021), pp. 749–799. ISSN: 2576-7658,2576-7666. URL: <https://doi.org/10.2140/tunis.2021.3.749> (cit. on p. 3).
- [Has+24] Hasan, Junaid, Hassan, Hazem, Lin, Milton, Manivel, Marcella, McBeath, Lily, and Moonen, Ben. *Integral aspects of Fourier duality for abelian varieties*. 2024. arXiv: [2407.06184](https://arxiv.org/abs/2407.06184) [math.AG]. URL: <https://arxiv.org/abs/2407.06184> (cit. on p. 4).
- [HR20] Haines, Thomas J. and Richarz, Timo. “Smoothness of Schubert varieties in twisted affine Grassmannians”. In: *Duke Mathematical Journal* 169.17 (Nov. 2020). ISSN: 0012-7094. URL: <http://dx.doi.org/10.1215/00127094-2020-0025> (cit. on p. 3).
- [ILZ24] Iyengar, Ashwin, Lin, Milton, and Zou, Konrad. *Geometric Casselman-Shalika in mixed characteristic*. 2024. arXiv: [2408.07953](https://arxiv.org/abs/2408.07953) [math.AG]. URL: <https://arxiv.org/abs/2408.07953> (cit. on pp. 2–4).
- [Kim16] Kim, Minhyong. *Arithmetic Chern-Simons Theory I*. 2016. arXiv: [1510.05818](https://arxiv.org/abs/1510.05818) [math.NT]. URL: <https://arxiv.org/abs/1510.05818> (cit. on p. 3).
- [KW07] Kapustin, Anton and Witten, Edward. *Electric-Magnetic Duality And The Geometric Langlands Program*. 2007. arXiv: [hep-th/0604151](https://arxiv.org/abs/hep-th/0604151) [hep-th]. URL: <https://arxiv.org/abs/hep-th/0604151> (cit. on p. 3).

- [Lur09] Lurie, Jacob. *Higher topos theory*. Vol. 170. Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 2009, pp. xviii+925. ISBN: 978-0-691-14049-0; 0-691-14049-9. URL: <https://doi.org/10.1515/9781400830558> (cit. on p. 2).
- [Lur10] Lurie, Jacob. “Moduli problems for ring spectra”. In: *Proceedings of the International Congress of Mathematicians. Volume II*. Hindustan Book Agency, New Delhi, 2010, pp. 1099–1125. ISBN: 978-81-85931-08-3; 978-981-4324-32-8; 981-4324-32-9 (cit. on p. 6).
- [Lur18] Lurie, Jacob. “Spectral Algebraic Geometry”. In: (2018) (cit. on pp. 2, 5, 6).
- [McN16] McNamara, Peter J. “The metaplectic Casselman-Shalika formula”. In: *Trans. Amer. Math. Soc.* 368.4 (2016), pp. 2913–2937. ISSN: 0002-9947,1088-6850. URL: <https://doi.org/10.1090/tran/6597> (cit. on p. 3).
- [MR23] Mondal, Shubhodip and Reinecke, Emanuel. *Unipotent homotopy theory of schemes*. 2023. arXiv: [2302.10703](https://arxiv.org/abs/2302.10703) [math.AG] (cit. on p. 5).
- [NP01] Ngô, B. C. and Polo, P. “Résolutions de Demazure affines et formule de Casselman-Shalika géométrique”. In: *J. Algebraic Geom.* 10.3 (2001), pp. 515–547. ISSN: 1056-3911,1534-7486 (cit. on pp. 2, 4).
- [Pap07] Pappas, Georgios. “Integral Grothendieck–Riemann–Roch theorem”. In: *Inventiones mathematicae* 170.3 (July 2007), pp. 455–481. ISSN: 1432-1297. URL: <http://dx.doi.org/10.1007/s00222-007-0067-9> (cit. on p. 4).
- [PZ24] Pappas, Georgios and Zhou, Rong. *On the smooth locus of affine Schubert varieties*. 2024. arXiv: [2312.14827](https://arxiv.org/abs/2312.14827) [math.AG]. URL: <https://arxiv.org/abs/2312.14827> (cit. on p. 3).
- [RS20] Richarz, Timo and Scholbach, Jakob. “The intersection motive of the moduli stack of Shtuka”. In: *Forum of Mathematics, Sigma* 8 (2020). ISSN: 2050-5094 (cit. on p. 4).
- [Toë06] Toën, Bertrand. “Champs affines”. In: *Selecta Math. (N.S.)* 12.1 (2006), pp. 39–135. ISSN: 1022-1824,1420-9020. URL: <https://doi.org/10.1007/s00029-006-0019-z> (cit. on p. 5).
- [Vol01] Vologodsky, Vadim. *Hodge structure on the fundamental group and its application to p-adic integration*. 2001. arXiv: [math/0108109](https://arxiv.org/abs/math/0108109) [math.AG] (cit. on p. 5).
- [Zhu17] Zhu, Xinwen. “Affine Grassmannians and the geometric Satake in mixed characteristic”. In: *Ann. of Math. (2)* 185.2 (2017), pp. 403–492. ISSN: 0003-486X,1939-8980. URL: <https://doi.org/10.4007/annals.2017.185.2.2> (cit. on p. 3).