Geometric metaplectic parameters

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Geometric metaplectic parameters

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Abstract

The objective of this work is to understand metaplectic parameters of the geometric Langlands program. In the usual Langlands program, metaplectic parameters are given by Brylinski–Deligne data, i.e., central extensions of the reductive group G by the second algebraic K-theory group $\mathbf{K}_2$. They admit a combinatorial description [11], making it possible to define metaplectic Langlands dual data [69].

In the geometric setting, we first show that Brylinski–Deligne data are equivalent to factorization line bundles on the affine Grassmannian (joint work with J. Tao [62]). Guided by the principle that factorization "twisting agents" should serve as geometric metaplectic parameters [32], we explain the notion of factorization gerbes in various sheaf-theoretic contexts. We prove that in the étale and analytic contexts, they admit combinatorial descriptions and are related to factorization line bundles via the first Chern class map.

The second half of this work addresses the de Rham context, where the sheaf theory is that of algebraic $\mathcal{D}$-modules. The naïve notion of gerbes in this context does not behave as expected, and we suggest the modified notion of factorization "tame gerbes" to serve as metaplectic parameters. We also relate factorization gerbes to factorization twistings which play the role of quantum parameters of the geometric Langlands program and discuss a natural compactification of the latter.

Contents

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The theorem relating $\mathbf{K}_2$ to factorization line bundles is proved jointly with James Tao. The impact of his ideas on my subsequent work can hardly be overestimated.

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Introduction

The metaplectic group.

The double cover of the special orthogonal group, known as the spin group, is an important object of study owing to its applications to quantum mechanics. Historically, the metaplectic group Mp_{2n} emerged as its counterpart for the real symplectic group $\mathrm{Sp}_{2n}(\mathbb{R})$. Since the fundamental group of $\mathrm{Sp}_{2n}(\mathbb{R})$ is infinite cyclic, the existence of a double cover is guaranteed by abstract reasons. However, unlike the spin group, Mp_{2n} is not the real points of a linear algebraic group. Therefore its generalization to fields other than $\mathbb{R}$ is not immediate.

The first systematic construction of metaplectic groups over an arbitrary local field was achieved by A. Weil [68] in the 1960s. He realized Mp_{2n} as a group of unitary operators on an infinite-dimensional Hilbert space. To summarize Weil's construction, let us be given a local field k of characteristic $\neq 2$ and an n-dimensional k-vector space V equipped with a quadratic form q. To these data, we may attach the Heisenberg group $\mathrm{Heis}_{(V,q)}$, which is an extension:

$$1 \to \mathrm{U}(1) \to \mathrm{Heis}_{(V,q)} \to V \times V^* \to 1.$$

The Heisenberg group acts canonically on the Hilbert space of functions $L^2(V)$. In fact, $L^2(V)$ is an irreducible unitary representation of $\mathrm{Heis}_{(V,q)}$ on which $\mathrm{U}(1)$ acts identically. Let B denote the automorphism group of $\mathrm{Heis}_{(V,q)}$. Then $L^2(V)$ defines a projective unitary representation of B, which in turn determines a central extension:

$$1 \to \mathrm{U}(1) \to \widetilde{B} \to B \to 1.$$

Finally, the group $\mathrm{Sp}(V)$, acting as transformations on $V \times V^*$ preserving its canonical symplectic form, embeds in B, so we obtain an extension of $\mathrm{Sp}(V)$ by $\mathrm{U}(1)$ and Weil's theorem shows that this extension is indeed induced from $\mathbb{Z}/2\mathbb{Z}$. We refer the curious reader to *loc.cit.* for details, or Gelbart [38, §2] for a review.

A remarkable feature of Weil's construction is that it generalizes to the adèles $\mathbb{A}_{\mathbf{F}}$ of a global field $\mathbf{F}$. We recall that $\mathbb{A}_{\mathbf{F}}$ is the restricted topological product of the local fields $\mathbf{F}_\nu$ for every place ν of $\mathbf{F}$. The metaplectic groups attached to each $\mathbf{F}_\nu$ combine into a double cover of the topological group $\mathrm{Sp}_{2n}(\mathbb{A}_{\mathbf{F}})$, equipped with a splitting over $\mathrm{Sp}_{2n}(\mathbf{F})$. Among other things, this construction suggests that the study of automorphic forms attached to the symplectic group, i.e., functions on $\mathrm{Sp}_{2n}(\mathbb{A}_{\mathbf{F}})/\mathrm{Sp}_{2n}(\mathbf{F})$, might find a natural generalization to the metaplectic group.

For the time being, we focus on the simplest case $V = \mathbf{F}_\nu$, so the symplectic group $\mathrm{Sp}(V)$ is identified $\mathrm{SL}_2(\mathbf{F}_\nu)$. We thus obtain a topological central extension:

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu) \to \mathrm{SL}_2(\mathbf{F}_\nu) \to 1. \tag{0.1}$$

Its global version is summarized in the following diagram:

$$\begin{tikzcd}
 & & & \mathrm{SL}_2(\mathbf{F}) \arrow[dl] \arrow[d] & \\
1 \arrow[r] & \mathbb{Z}/2\mathbb{Z} \arrow[r] & \widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbf{F}}) \arrow[r] & \mathrm{SL}_2(\mathbb{A}_{\mathbf{F}}) \arrow[r] & 0.
\end{tikzcd}$$

The covering group $\widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu)$ is called the *metaplectic cover* of $\mathrm{SL}_2(\mathbf{F}_\nu)$ and its global version $\widetilde{\mathrm{SL}}_2(\mathbb{A}_{\mathbf{F}})$ is likewise named.

Enter $\mathbf{K}_2$.

Let us now look at the metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu)$ from the angle of group cohomology. Recall that given a topological group H and a trivial H-module A, central extensions of H by A are parametrized by topological 2-cocycles on H, with values in A. It is thus natural to describe (0.1) via an explicit 2-cocycle on $\mathrm{SL}_2(\mathbf{F}_\nu)$.

The answer has to do with the quadratic Hilbert symbol on $\mathbf{F}_\nu$:

$$(-,-) : \mathbf{F}_\nu^\times \underset{\mathbb{Z}}{\otimes} \mathbf{F}_\nu^\times \to \mathbb{Z}/2\mathbb{Z}.$$

It is defined by $(x,y) := 1$ if and only if x is a norm in $\mathbf{F}_\nu(\sqrt{y})$. Following T. Kubota [44], we define a function $\chi : \mathrm{SL}_2(\mathbf{F}_\nu) \to \mathbf{F}_\nu^\times$ by the formula:

$$\chi \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} c & \text{if } c \neq 0 \\ d & \text{otherwise.} \end{cases}$$

Kubota's theorem shows that the metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu)$ identifies with the central extension defined by the explicit 2-cocycle:

$$s_1, s_2 \rightsquigarrow (\chi(s_1), \chi(s_2)) + (-\chi(s_1)^{-1}\chi(s_2), \chi(s_1 s_2)).$$

The construction using the Heisenberg group can be seen as defining a particular infinite-dimensional representation of $\widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu)$, called the *Weil representation*.

This description of $\widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu)$ puts metaplectic covers in close contact with the second algebraic K-theory group. Indeed, for a field k, the abelian group $\mathbf{K}_2(k)$ is isomorphic to the quotient:

$$\mathbf{K}_2(k) \xrightarrow{\sim} k^\times \underset{\mathbb{Z}}{\otimes} k^\times / \langle u \otimes v \text{ for } u + v = 1 \rangle.$$

As such, $\mathbf{K}_2(k)$ is the universal recipient of symbols on k. According to H. Matsumoto [51], $\mathrm{SL}_2(k)$ admits a universal central extension whose kernel identifies with $\mathbf{K}_2(k)$. The metaplectic cover $\widetilde{\mathrm{SL}}_2(\mathbf{F}_\nu)$ is none other than the push-out of the universal central extension of $\mathrm{SL}_2(\mathbf{F}_\nu)$ along the Hilbert symbol map $\mathbf{K}_2(\mathbf{F}_\nu) \to \mathbb{Z}/2\mathbb{Z}$ associated to the local field $\mathbf{F}_\nu$.

The interpretation of metaplectic coverings via algebraic K-theory has shaped several subsequent developments on the subject. One remarkable work along this line is that of J.-L. Brylinski and P. Deligne [11] in the early 2000s. They considered the sheafified $\mathbf{K}_2$ on the big Zariski site of a field k and posed the following question: *given a reductive group G over k, what are all central extensions:*

$$1 \to \mathbf{K}_2 \to E \to G \to 1 \tag{0.2}$$

as big Zariski sheaves? The significance of this question lies in the fact that for $k = \mathbf{F}_\nu$ (or $\mathbf{F}$), central extensions by $\mathbf{K}_2$ gives rise to topological covers of $G(\mathbf{F}_\nu)$ (and $G(\mathbb{A}_{\mathbf{F}})$) by a very general procedure, and a large class of interesting topological covers arise this way. One the other hand, the datum (0.2) is of algebraic nature, making its study more amenable to methods of algebraic geometry. Let us refer to central extensions (0.2) as "Brylinski–Deligne data."

The Brylinski–Deligne theorem.

The main result of [11] is a description of the *groupoid* of central extensions (0.2) as an essentially combinatorial gadget. It is valid over any base scheme S which is regular of finite type over a field k. Let us explain in more details the combinatorial groupoid which appears as the parameter space of central extensions by $\mathbf{K}_2$.

(1) For a split torus T, the groupoid consists of pairs $(q, \mathcal{L}^{(\lambda)})$, where q is an integer-valued quadratic form on the co-character lattice Λ_T, and $\mathcal{L}^{(\lambda)}$ is a Λ_T-indexed family of line bundles which depend multiplicatively on λ, but are only commutative up to a "twist" by the bilinear form κ attached to q. We denote this groupoid by $\Theta^+(\Lambda_T; \mathbf{Pic})$.

(2) For G simple and simply connected (i.e., the Chevalley group attached to an irreducible root system), the combinatorial gadget is a copy of $\mathbb{Z}$, with a generator corresponding to the universal central extension, as exemplified above with $G = \mathrm{SL}_2$.

(3) For any reductive group[1] G, the definition is a combination of the cases (1) and (2). More precisely, it consists of triples $(q, \mathcal{L}^{(\lambda)}, \varepsilon)$, where $(q, \mathcal{L}^{(\lambda)})$ is an object of $\Theta^+(\Lambda_T; \mathbf{Pic})$, for T a maximal torus, and ε is an isomorphism between its restriction to the co-root lattice inside Λ_T and the data coming from applying classification (2) to the simply connected form of G.

[1]Assumed split for simplicity.

Let us denote the classification groupoid by $\Theta_G^+(\Lambda_T; \mathbf{Pic})$ to emphasize the role of G. Thus, the main result of Brylinski–Deligne can be summarized as an equivalence of categories:

$$\Psi_{\mathrm{BD}} : \mathbf{CExt}(G, \mathbf{K}_2) \xrightarrow{\sim} \Theta_G^+(\Lambda_T; \mathbf{Pic}). \tag{0.3}$$

Their theorem implies that metaplectic covers, understood as a topological cover coming from (0.2), are parametrized by combinatorics. It has thus become customary to view the equivalent groupoids in (0.3) as "metaplectic parameters."

A coincidence?

Also in the early 2000s, an analogous groupoid appeared in Beilinson and Drinfeld's study of chiral algebras [8, §3.10] as the classification data for "factorization" $\mathbb{Z}/2\mathbb{Z}$-graded line bundles on the space of Λ-colored divisors on an algebraic curve X. They called this groupoid Θ-*data*. In fact, when the $\mathbb{Z}/2\mathbb{Z}$-grading is trivial, their Θ-data $\Theta(\Lambda_T; \mathbf{Pic})$ are equivalent to the groupoid $\Theta^+(\Lambda_T; \mathbf{Pic})$ introduced above for the base $S = X$. The equivalence shifts each $\mathcal{L}^{(\lambda)}$ by the $q(\lambda)$-th power of the canonical sheaf ω_X. (This was the reason for our inclusion of the superscript $+$ in the latter.)

One could say that the simultaneous appearance of the Θ-data groupoid in these two different contexts motivated the present work. Namely, our goal will be to understand Brylinski–Deligne data through the lens of algebraic geometry, particularly the geometry related to factorization structures. Inspired by Beilinson–Drinfeld [8], we will call the combinatorial gadget $\Theta_G^+(\Lambda_T; \mathbf{Pic})$ attached to G *enhanced* Θ-*data* (with an ω-shift).

The affine Grassmannian.

Let us now introduce the main player on the algebraic geometry side: the affine Grassmannian Gr_G. We will fix an algebraically closed ground field k. The global field $\mathbf{F}$ will be taken as the field of fractions of a connected, smooth algebraic curve X over k.

For concreteness, we again specialize to $G = \mathrm{SL}_2$. Suppose ν is the place corresponding to a closed point $x \in X$. Upon choosing a uniformizer, the local field $\mathbf{F}_\nu$ is identified with formal Laurent series $k(\!(t)\!)$, and its ring of integers is the formal power series $k[\![t]\!]$. The affine Grassmannian Gr_G parametrizes free $k[\![t]\!]$-submodules $L \subset k(\!(t)\!)^{\oplus 2}$, called *lattices*, with trivialized determinant. Suppose we fix an integer $n \geq 0$. We may consider those lattices situated between two standard lattices:

$$t^n k[\![t]\!]^{\oplus 2} \subset L \subset t^{-n} k[\![t]\!]^{\oplus 2}.$$

By taking quotient, they correspond to t-invariant subspaces in the finite-dimensional vector space $t^{-n}k[\![t]\!]^{\oplus 2}/t^n k[\![t]\!]^{\oplus 2}$ together with a condition on the determinant. In other words, these lattices form a closed subvariety of a usual Grassmannian. When we increase n, they exhaust all lattices inside $k(\!(t)\!)^{\oplus 2}$ and we obtain a presentation of Gr_G as an infinite union of subvarieties of the usual Grassmannian.

The situation for an arbitrary reductive group G is analogous. The resulting object Gr_G is what one may call an *ind-scheme*, formally defined as a presheaf on the category of schemes which is a filtered colimit of representables. The affine Grassmannian has received considerable attention in recent decades, thank to its prominent role in geometric representation theory.

Beilinson and Drinfeld [8] have observed a remarkable structure of the affine Grassmannian, which involves varying the point $x \in X$. As the affine Grassmannian attached to distinct points are (non-canonically) isomorphic, they were able to formalize a merging behavior as those points collide. For example, there is an ind-scheme Gr_{G,X^2} over the product X^2, with fibers:

$$\mathrm{Gr}_{G,X^2}\Big|_{(x,x)} \xrightarrow{\sim} \mathrm{Gr}_{G,x},$$

for all $x \in X$, whereas for $x \neq y$ in X, there is an isomorphism:

$$\mathrm{Gr}_{G,X^2}\big|_{(x,y)} \xrightarrow{\sim} \mathrm{Gr}_{G,x} \times \mathrm{Gr}_{G,y}\,.$$

This structure is called *factorization* of the affine Grassmannian.

More formally, one views the affine Grassmannian as a prestack $\mathrm{Gr}_{G,\mathrm{Ran}}$ over the "Ran space" of X, which is a parameter space of finite sets of points in X. Then the factorization structure is specified by an isomorphism over the disjoint locus in $\mathrm{Ran} \times \mathrm{Ran}$, which satisfies a cocycle condition over the triple product.

It is also sensible to speak of various geometric constructions (e.g., line bundles) over the affine Grassmannian as *factorizable*, or having *factorization*, when they are equipped with compatibility data with the factorization structure of $\mathrm{Gr}_{G,\mathrm{Ran}}$. We remark that for a torus T, the affine Grassmannian is a close relative of the space of Λ_T-colored divisors on X, whose factorization line bundles have been classified by Beilinson and Drinfeld.

$\mathbf{K}_2$ versus factorization.

In 2018, D. Gaitsgory and S. Lysenko [32] proposed a conjecture which asserts that Brylinski–Deligne data attached to any reductive group G are equivalent to factorization line bundles on $\mathrm{Gr}_{G,\mathrm{Ran}}$. The significance of their conjecture is that it gives a purely geometric interpretation of metaplectic parameters when the base is an algebraic curve.

Concretely, Gaitsgory [31] defined a canonical functor betwen the two groupoids:

$$\Xi_{\mathbf{Pic}} : \mathbf{CExt}(G, \mathbf{K}_2) \to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

It is a variant of the residue map on $\mathbf{K}_2$, although the actual construction involved some technical challenges because $\mathrm{Gr}_{G,\mathrm{Ran}}$ is not smooth. The existence of $\Xi_{\mathbf{Pic}}$ also hinges on the condition that $\mathrm{char}(k)$ cannot divide an integer N_G which depends on

G. Our first result, proved jointly with J. Tao, affirms the conjecture of Gaitsgory–Lysenko.

Theorem A. *Assume* $\mathrm{char}(k) \nmid N_G$. *Then* $\Xi_{\mathbf{Pic}}$ *is an equivalence.*

The proof of this theorem will be explained in Chapter 2. It relies on prior work of G. Faltings [24] and X. Zhu [73] [74]. In fact, we will build a functor $\Psi_{\mathbf{Pic}}$ from the groupoid of factorization line bundles to that of enhanced Θ-data:

$$\Psi_{\mathbf{Pic}} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \Theta_G(\Lambda_T; \mathbf{Pic}).$$

Then we use a geometric argument to show that $\Psi_{\mathbf{Pic}}$ is fully faithful. These results are not conditional on $\mathrm{char}(k)$. On the other hand, we prove that when Gaitsgory's functor $\Xi_{\mathbf{Pic}}$ exists, the Brylinski–Deligne classification functor Ψ_{BD} factors through $\Psi_{\mathbf{Pic}}$:

Theorem B. *Assume* $\mathrm{char}(k) \nmid N_G$. *The following diagram is canonically commutative:*

(0.4)
$$\begin{tikzcd} \mathbf{CExt}(G, \mathbf{K}_2) \arrow[r, "\Xi_{\mathbf{Pic}}"] \arrow[d, "\Psi_{\mathrm{BD}}", "\cong"'] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{Pic}}"] \\ \Theta_G^+(\Lambda_T; \mathbf{Pic}) \arrow[r, "\omega\text{-shift}", "\sim"'] & \Theta_G(\Lambda_T; \mathbf{Pic}) \end{tikzcd}$$

Combining the two results, we conclude that both $\Xi_{\mathbf{Pic}}$ and $\Psi_{\mathbf{Pic}}$ are equivalences.

In particular, the geometrically defined functor $\Psi_{\mathbf{Pic}}$ gives a classification of factorization line bundles by enhanced Θ-data. A further corollary of our theorem is that such line bundles canonically descend to the moduli stack of G-bundles Bun_G on X, when X is proper. This descent behavior is expected since the fibers of $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Bun}_G$ are, in a certain sense, contractible (see [28]).

From functions to sheaves.

Our next goal is to understand metaplectic parameters in the presence of a sheaf theory. Before stating our results, let us take a digression and explain how these considerations are motivated by the geometric study of automorphic forms, i.e., replacing functions on the adèlic group $G(\mathbb{A}_{\mathbf{F}})$ by sheaves.

To begin with, the metaplectic cover $\widetilde{G}$ of $G(\mathbb{A}_{\mathbf{F}})$ provides a natural generalization of automorphic forms, where one considers functions on $\widetilde{G}/G(\mathbf{F})$ equivariant against a central character. The study of automorphic forms has, since the 1970s, been guided by a host of far-reaching conjectures known as the Langlands program. The ultimate goal of these conjectures is to relate automorphic forms to representations of $\mathrm{Gal}(\overline{\mathbf{F}}/\mathbf{F})$ valued in a different group ${}^{\mathrm{L}}G$ defined in terms of the root data of G. Using the Brylinski–Deligne theorem, M. Weissman [69] was able to define the L-group of metaplectic coverings, thereby casting metaplectic groups under the scope of the Langlands program.

In the geometric set-up where $k = \mathbb{F}_q$ and $\mathbf{F}$ is the function field of a curve, automorphic forms can be accessed via constructible $\overline{\mathbb{Q}}_\ell$-sheaves by taking the trace of Frobenius. Reformulating the problem in sheaf-theoretic language allows one to manipulate the *six opérations* of $\overline{\mathbb{Q}}_\ell$-sheaves and has proved to be immensely fruitful. This approach underlies the work of V. Drinfeld [17], L. Lafforgue [45], and V. Lafforgue [46].

Going one step further, one may pose Langlands-type questions purely in terms of sheaves and thus remove their dependence on the particular ground field such as $\mathbb{F}_q$. The subject which emerged from these considerations is called the *geometric Langlands program*. It has been developed in the work of Beilinson–Drinfeld [7] and a categorical reformulation has recently been proposed by Arinkin–Gaitsgory [1]. The objects of central interest in the geometric theory are sheaves on Bun_G, a categorification of automorphic functions.

In order to study automorphic sheaves, one must first fix a sheaf-theoretic context. There are a few of them which have received considerable attention:

(1) The ground field $k = \overline{\mathbb{F}}_q$, and we consider constructible $\overline{\mathbb{Q}}_\ell$-sheaves;

(2) The ground field $k = \mathbb{C}$, and we consider constructible analytic sheaves;

(3) The ground field k is algebraically closed with $\mathrm{char}(k) = 0$, and as sheaf theory we consider algebraic $\mathcal{D}$-modules.

Within each of these contexts, one can ask for a metaplectic generalization. The agent which plays the role of line bundles will now be a *gerbe* of the sheaf theory. In context (1), this will be an étale A-gerbe for A a suitable torsion abelian group, and in context (2), this will be an analytic $\mathbb{C}^\times$-gerbe. There is a well-defined notion of sheaves twisted by a gerbe, categorifying the notion of functions twisted by a line bundle, i.e., sections.

Realizations.

In view of the relationship between Brylinski–Deligne data and factorization line bundles, Gaitsgory–Lysenko [32] proposed that the geometric (i.e., sheaf-theoretic) analogue of a metaplectic parameter should be a factorization gerbe on $\mathrm{Gr}_{G,\mathrm{Ran}}$. One expects factorization gerbes to also admit a combinatorial description, which should furthermore allow one to define geometric metaplectic dual data, in analogy with Weissman's L-group.

Our next result is the sought-for combinatorial classification. Since we will need the theorem in several sheaf-theoretic contexts, we follow a unified approach in the proof. More precisely, we axiomatize a *theory of gerbes* to be a sheaf $\mathbf{G}$ on the category of finite type schemes, valued in strictly commutative Picard 2-groupoids, which is further equipped with a divisor class map ("first Chern class"):

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} A(-1) \to \mathbf{G}, \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a.$$

Here, $A(-1)$ is a certain abelian group attached to $\mathbf{G}$, to be thought of as a Tate twist of its coefficient group. Thus, the category of interest is that of factorization sections $\mathbf{G}^{\text{fact}}(\text{Gr}_{G,\text{Ran}})$, which forms a (strictly commutative) Picard 2-groupoid.

Parellel to the case of line bundles, each theory of gerbes $(\mathbf{G}, c_1)$ comes with its own enhanced Θ-data $\Theta_G(\Lambda_T; \mathbf{G})$, now a Picard 2-groupoid. Our result is that when $(\mathbf{G}, c_1)$ satisfies a list of properties, there is a canonical equivalence between $\mathbf{G}^{\text{fact}}(\text{Gr}_{G,\text{Ran}})$ and $\Theta_G(\Lambda_T; \mathbf{G})$. The properties include: relative purity, $\mathbb{A}^1$-homotopy invariance, a weak form of proper base change, and descent with respect to a sufficiently strong topology t which allows for resolutions of singularities. We call such a theory of gerbes a *motivic t-theory of gerbes*. Let us now state our theorem more formally.

Theorem C. *Suppose $(\mathbf{G}, c_1)$ is a motivic t-theory of gerbes. Then there is an equivalence $\Psi_{\mathbf{G}}$ between factorization gerbes (i.e., sections of $\mathbf{G}$) on $\text{Gr}_{G,\text{Ran}}$ and $\Theta_G(\Lambda_T; \mathbf{G})$.*

Furthermore, the following diagram is canonically commutative:

$$\begin{tikzcd} \mathbf{Pic}^{\text{fact}}(\text{Gr}_{G,\text{Ran}}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}"] & \mathbf{G}^{\text{fact}}(\text{Gr}_{G,\text{Ran}}) \arrow[d, "\Psi_{\mathbf{G}}", "\cong"'] \\ \Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathbf{G}) \end{tikzcd}$$

The structure of the proof of Theorem C is the same as the classification of factorization line bundles. However, certain arguments become simpler because cohomology theories are less sensitive to geometric properties than K-theory. Notably, we obtain the essential surjectivity of $\Psi_{\mathbf{G}}$ unconditionally.

Theorem C is a rather crucial technical input for us, as we will apply it to four motivic theories of gerbes: étale gerbes valued in a prime-to-char(k) torsion abelian group, analytic $\mathbb{C}^\times$-gerbes[2], and two theories of gerbes ("additive" and "tame") appearing in the de Rham context. Chapter 3 is devoted to the proof of Theorem C, as well as a discussion pertaining to the étale and analytic contexts. The de Rham context, which possesses the richest theory, requires a chapter of its own.

In the étale context, it appears that central extensions by $\mathbf{K}_2$ have a cohomological counterpart, an idea proposed by Gaitsgory [31]. Namely, it is the groupoid of central extensions of G by $\mathrm{B}^2_{\text{ét}}\, A(2)$. By definition, this datum is equivalent to that of a group homomorphism $X \times G \to \mathrm{B}^3_{\text{ét}}\, A(2)$, and by delooping, a pointed map $X \times \mathrm{B}_{\text{ét}}\, G \to \mathrm{B}^4_{\text{ét}}\, A(2)$. Let us denote this groupoid by $\mathbf{CExt}(G, \mathrm{B}^2_{\text{ét}}\, A(2))$.

In [31, §6.3], the following functor is defined. (It can be seen as a variant of Soulé's regulator on algebraic K-theory):

$$\mathbf{R}_{\text{ét}} : \mathbf{CExt}(G, \mathbf{K}_2) \to \mathbf{CExt}(G, \mathrm{B}^2_{\text{ét}}\, A(2)).$$

Furthermore, Gaitsgory conjectured the canonical commutivity of the following diagram:

$$\begin{tikzcd}
\mathbf{CExt}(G, \mathbf{K}_2) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "\mathbf{R}_{\text{ét}}(-1)"] \arrow[d, "\Xi_{\mathbf{Pic}}"] & \mathbf{CExt}(G, \mathrm{B}^2_{\text{ét}}\, A(1)) \arrow[d, "\Xi_{\text{ét}}"] \\
\mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] & \mathbf{Ge}^{\text{fact}}_{\text{ét}}(\mathrm{Gr}_{G,\mathrm{Ran}}).
\end{tikzcd}$$

Here, $\Xi_{\text{ét}}$ is a "construction functor" of factorization étale gerbes, defined using the trace map of étale cohomology. We give an indication of how one may deduce this conjecture from our work. In the analytic context, we expect a similar picture to exist.

The de Rham context.

[2]In this context, a false classification result with a bogus proof [54] has been in circulation.

The next chapter is devoted to understanding metapletic parameters in the de Rham context, i.e., we take the ground field k to be algebraically closed of characteristic zero, and the sheaf theory to be $\mathcal{D}$-modules. Using the interpretation of $\mathcal{D}$-modules as crystals of quasi-coherent sheaves, one arrives at the notion of a *gerbe* for the metaplectic theory as a $\mathbb{G}_m$-gerbe on the de Rham prestack. In other words, the theory of de Rham gerbes $\mathbf{Ge}_{\mathrm{dR}}$ sends a scheme Y to the Picard 2-groupoid $\mathrm{Maps}(Y_{\mathrm{dR}}, \mathrm{B}^2\,\mathbb{G}_m)$.

It turns out that the theory of de Rham gerbes is *not* motivic. In fact, we do not have adequate methods to fully understand factorization de Rham gerbes on $\mathrm{Gr}_{G,\mathrm{Ran}}$. However, we shall show later that a tweak of the definition of $\mathbf{Ge}_{\mathrm{dR}}$ makes everything work out smoothly.

To start with, we relate $\mathbf{Ge}_{\mathrm{dR}}$ to a different kind of objects, called "twistings," which are k-linear and hence easier to understand. There is a canonical functor from twistings to $\mathbf{Ge}_{\mathrm{dR}}$, whose fiber identifies with $\mathbf{Pic}$. (Thank to Theorem A, we have a good understanding of factorization line bundles.) Then we study the behavior of factorization twistings on $\mathrm{Gr}_{G,\mathrm{Ran}}$ and give the following answers concerning their parametrization:

(1) When G is semisimple and simply connected, factorization twistings are classified by enhanced Θ-data, which reduce to invariant symmetric bilinear forms on $\mathfrak{t}$.

(2) For a torus, we provide evidences that "extra" factorization twistings exist. A combinatorial classification thus seems unlikely, or at least more complicated.

(3) Slightly tweaking the definitions, we are able to construct a good theory of metaplectic parameters in the de Rham context. It is given by factorization *tame gerbes* and *tame twistings*, both of which are classified by enhanced Θ-data.

Before proceeding further, let us point out that twistings are important objects in their own right. By definition, a twisting on Y is a $\mathbb{G}_m$-gerbe on Y_{dR} whose pullback to Y is equipped with a trivialization. This latter datum equips the category of twisted $\mathcal{D}$-modules with a forgetful functor to $\mathrm{QCoh}(Y)$, whose existence is chiefly responsible for the application of Lie theory to the geometric Langlands program. The metaplectic theory produced by twistings is more commonly called the *quantum* Langlands program.

Factorization twistings on $\mathrm{Gr}_{G,\mathrm{Ran}}$ come with a natural analogue of Brylinski–Deligne data. This is the groupoid of central extensions of the Lie-$*$ algebra $\mathfrak{g}_{\mathcal{D}} := \mathfrak{g} \otimes \mathcal{D}_X$ by ω_X, subject to an integrability condition. These Lie-$*$ algebras were first studied by Beilinson–Drinfeld [8]. Following ideas of Gaitsgory, we produce a "construction functor" of factorization twistings:

$$\Xi_{\mathbf{Tw}} : \mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X) \to \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

which passes through factorization central extensions of the loop Lie algebra $\mathrm{L}_{\mathrm{Ran}}\mathfrak{g}$, and also multiplicative factorization twistings on the loop group $\mathcal{L}_{\mathrm{Ran}}G$.

On the other hand, the groupoid of enhanced Θ-data for twistings is easy to describe. It consists of pairs (κ, E), where κ is a W-invariant symmetric bilinear form on $\mathfrak{t}$, and E is an extension of $\mathfrak{z} \otimes \mathcal{O}_X$ by ω_X as $\mathcal{O}_X$-modules. The Kac–Moody Lie-$*$ algebra construction defines a functor:

$$\Theta_G(\Lambda_T; \mathbf{Tw}) \to \mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X), \quad (\kappa, E) \rightsquigarrow \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}. \tag{0.5}$$

We now make precise the affirmative part of our parametrization.

Theorem D. *Suppose G is semisimple and simply connected. Then both $\Xi_{\mathbf{Tw}}$ and (0.5) are equivalences.*

The strategy of the proof is as follows. We apply Theorem C to the motivic theory of *additive* de Rham gerbes, i.e., with value group $\mathbb{G}_a$, to obtain a functor from $\mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$ to bilinear forms. We check that the application of this functor to $\Xi_{\mathbf{Tw}}(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})$ recovers κ. In the case of a semisimple, simply connected group G, we prove "by hand" that this functor, as well as the Kac–Moody construction (0.5), are equivalences.

For more general G, we are not able to produce a classification functor $\Psi_{\mathbf{Tw}}$ from factorization twistings to $\Theta_G(\Lambda_T; \mathbf{Tw})$. Furthermore, for tori, there exist Lie-$*$ central extensions which are not Kac–Moody, i.e., (0.5) is not essentially surjective. These Lie-$*$ algebras give rise to extra multiplicative factorization twistings on the loop group. We suspect that their descent to $\mathrm{Gr}_{T,\mathrm{Ran}}$ remain non-isomorphic to the usual ones. At present, we cannot affirm this due to lack of a sensitive invariant for factorization twistings.[3]

Tameness.

The remaining part of Chapter 4 serves to construct an alternative theory of gerbes and twistings which are better suited as metaplectic, respectively quantum, parameters. We call them *tame gerbes* and *tame twistings*.

Before giving a formal definition, let us mention some features of these objects.

(1) A usual twisting on a *smooth* scheme X is a torsor for the complex $\Omega_X^1 \to \Omega_X^{2,\mathrm{cl}}$, whereas a tame twisting is a torsor for the subsheaf $\mathring{\Omega}_X^1$ of Ω_X^1 whose sections over U consists of differentials of logarithmic growth along a good compactification $\overline{U}$ of U. In particular, the process of inducing twistings from line bundles factors through tame twistings by the map $d\log : \mathcal{O}_X^\times \to \mathring{\Omega}_X^1$.

[3]However, one expects any "correct" notion of metaplectic parameters to be canonically multiplicative over the loop group. Thus one sees already that factorization twistings are inadequate as metaplectic parameters.

(2) In contrast to usual twistings or gerbes, the category of $\mathcal{D}$-modules twisted by a tame twisting (or tame gerbe) has a natural notion of *regularity* generalizing the usual notion of regular $\mathcal{D}$-modules.

(3) When $k = \mathbb{C}$, tame gerbes on X form a full subcategory of $\mathbb{C}^\times$-gerbes on the analytification X^{an}.

These properties suggest that tame gerbes/twistings naturally arise when we consider twisted $\mathcal{D}$-modules in conjunction with complex constructible sheaves—this is, indeed, something one does for the purpose of the geometric Langlands program (c.f. [48]). We emphasize that tameness is not a condition, but an additional piece of structure.

Formally, we define tame gerbes $\mathring{\mathbf{Ge}}$ as the éh-sheafification of the classifying 2-stack of the stack of regular local systems, where "regular" is understood in the sense of $\mathcal{D}$-modules. There is a canonical map from $\mathring{\mathbf{Ge}}$ to the derived **éh**-sheafification[4] of $\mathrm{B}^2\,\mathbb{G}_m$ and we let the stack of tame twistings $\mathring{\mathbf{Tw}}$ be the fiber of this map. Analogous to the usual setting, we have a fiber sequence relating line bundles, tame twistings, and tame gerbes.

$$\mathbf{Pic} \to \mathring{\mathbf{Tw}} \to \mathring{\mathbf{Ge}}. \tag{0.6}$$

The crucial observation is that $\mathring{\mathbf{Ge}}$ is a motivic theory of gerbes, so Theorem C allows us to classify factorization tame gerbes on $\mathrm{Gr}_{G,\mathrm{Ran}}$ by enhanced Θ-data.

On the other hand, even though $\mathring{\mathbf{Tw}}$ is not motivic, it satisfies relative purity, in contrast to the usual twistings. This makes it possible to define a classification functor $\Psi_{\mathring{\mathbf{Tw}}}$ for factorization tame twistings. Then, making use of the fiber sequence (0.6), we trap factorization tame twistings between objects that have already been classified. Modulo some technicalities, this allows us to obtain the following result.

[4] We use bold characters to emphasize topologies defined on derived schemes.

Theorem E. *There is a canonical equivalence* $\Psi_{\mathring{\mathbf{Tw}}}$ *between factorization tame twistings on* $\mathrm{Gr}_{G,\mathrm{Ran}}$ *and* $\Theta_G(\Lambda_T;\mathring{\mathbf{Tw}})$.

Furthermore, the following diagram is canonically commutative:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}", "\cong"'] & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathring{\mathbf{Tw}}}", "\cong"'] \\ \Theta_G(\Lambda_T;\mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] & \Theta_G(\Lambda_T;\mathring{\mathbf{Tw}}) \end{tikzcd}$$

We will also interpret $\Theta_G(\Lambda_T;\mathring{\mathbf{Tw}})$ as the groupoid of pairs $(\kappa, \mathring{E})$, where κ is a W-invariant symmetric bilinear form on $\mathfrak{t}$ as before, and $\mathring{E}$ is an extension of $\mathfrak{z}$ by $\mathring{\Omega}^1_X$ as Zariski sheaves of k-vector spaces. A calculation of the cohomology of $\mathring{\Omega}^1_X$ then shows that the canonical map:

$$\Theta_G(\Lambda_T;\mathring{\mathbf{Tw}}) \to \Theta_G(\Lambda_T;\mathbf{Tw})$$

is an equivalence if and only if X is proper. In particular, this implies that over a proper curve, the factorization twistings attached to usual quantum parameters (κ, E) have a canonical tame structure. Combined with Theorem D, this also shows that for a semisimple, simply connected group, factorization tame twistings agree with the usual ones.

It is tempting to ask for a "construction functor" $\Xi_{\mathring{\mathbf{Tw}}}$ for factorization tame twistings. Ideally, it should start from some variant of the Lie-$*$ central extension $\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X)$ and land in factorization tame twistings. In fact, an appropriate way of bounding the order of poles in the Lie-$*$ bracket might do the trick. This groupoid is also likely to receive a functor from $\mathbf{CExt}(G, \mathbf{K}_2)$.[5] We shall save these questions for future investigations.

[5]What can this be except "taking the derivative"?

Let us put everything together in the easiest example of SL_2 (or any simple, simply connected group.) The groupoids on the left are all classified by a "number," which are related to each other as indicated by the diagram on the right.

$$
\begin{tikzcd}
 & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d] & \\
 & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[dl] \arrow[dr] & \\
\mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) & & \mathring{\mathbf{Ge}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[dr] & \\
 & & & \mathbf{Ge}^{\mathrm{fact}}_{\mathrm{an}}(\mathrm{Gr}_{G,\mathrm{Ran}})
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z} \arrow[d] & \\
 & \mathbb{C} \arrow[dl, "\mathrm{id}"'] \arrow[dr] & \\
\mathbb{C} & & \mathbb{C}/\mathbb{Z} \arrow[dr, "\exp(2\pi i-)"] & \\
 & & & \mathbb{C}^{\times}.
\end{tikzcd}
$$

Compactification of quantum parameters.

In the final chapter, we address the compactification of the groupoid of quantum parameters $\Theta_G(\Lambda_T; \mathbf{Tw})$. Namely, we will view $\Theta_G(\Lambda_T; \mathbf{Tw})$ as the k-points of an algebraic stack Par_G, and then add "points at ∞" to Par_G. The results of this chapter depend very lightly on the previous ones.

The main observation is that the construction of factorization twistings from points of Par_G generalize, more or less verbatim, to the compactification $\overline{\mathrm{Par}}_G$. These extra points produce what we call *pre-twistings*. They are a natural generalization of twistings and share most of the same features—in particular, each pre-twisting has a module category, equipped with a forgetful functor to quasi-coherent sheaves (or more appropriately, ind-coherent sheaves).

It turns out that the pre-twistings "at ∞" are very relevant to the geometric Langlands program. Namely, they formalize the degeneration of various DG categories naturally appearing in the program. The quintessential example is the degeneration:

$$\mathcal{D}\text{-Mod}^{\kappa}(\mathrm{Bun}_G) \rightsquigarrow \mathrm{QCoh}(\mathrm{LocSys}_G), \quad \text{as } \kappa \to \infty,$$

which realizes the spectral side of the global geometric Langlands program as a limit of the automorphic side.

In the case of a simple group G, an explicitly $\mathbb{P}^1$-family realizing this degeneration has been constructed by Stoyanovsky [59], although the idea likely dates back further. Away from the case of a simple group, LocSys_G is in general a derived algebraic stack and Stoyanovsky's construction does not carry over. Furthermore, the space of quantum parameters $\overline{\mathrm{Par}}_G$ involves the additional term E and we want it to be naturally included in the construction.

We provide this construction in Chapter 5 and show that it has the expected behavior.

Theorem F. *To every S-point (κ, E) of $\overline{\mathrm{Par}}_G$, there is an S-family of pre-twistings $\mathcal{T}_{\mathrm{glob}}^{(\kappa,E)}$ on Bun_G and an S-family of factorization pre-twistings $\mathcal{T}_{\mathrm{loc}}^{(\kappa,E)}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ with the following properties:*

(1) The pre-twisting $\mathcal{T}_{\mathrm{loc}}^{(\kappa,E)}$ identifies with the pullback of $\mathcal{T}_{\mathrm{glob}}^{(\kappa,E)}$ along the projection $S \times \mathrm{Gr}_{G,\mathrm{Ran}} \to S \times \mathrm{Bun}_G$;

(2) The category of $\mathcal{T}_{\mathrm{glob}}^{(\kappa,E)}$-modules on Bun_G is equivalent to $\mathrm{QCoh}(\mathrm{LocSys}_G)$ at the k-point $(\infty, 0) \in \overline{\mathrm{Par}}_G$.

Part (2) of the theorem means that the pre-twistings $\mathcal{T}_{\mathrm{glob}}^{(\kappa,E)}$ realize the degeneration of twisted $\mathcal{D}$-modules on Bun_G to $\mathrm{QCoh}(\mathrm{LocSys}_G)$.

In fact, one can play the same game with the pre-twisting $\mathcal{T}_{\mathrm{loc}}^{(\kappa,E)}$ and the various DG categories appearing in the local geometric Langlands program to understand their degeneration behavior as $\kappa \to \infty$. Examples include the Kazhdan–Lusztig category, the Whittaker category, the semi-infinite category, etc. In fact, our construction has an algorithmic flavor, so one really can "input" any relevant DG category and obtain its "output" as limit at $\kappa \to \infty$.

CHAPTER 1

Preliminaries

This chapter contains a list of topics which are used throughout the text.

We start with a review of the Brylinski–Deligne classification of central extensions by $\mathbf{K}_2$, as it forms the basis of our work. Then we recall the affine Grassmannian Gr_G. We summarize the geometric properties of Gr_G both as an ind-scheme (over a point in the base curve) and as a factorization prestack over the Ran space. In the final section, we collect some facts about higher algebra which will play a major role in the last two chapters.

As all the results in this chapter are known, we shall seldom give proofs. Instead, we put emphasis on making explicit the various constructions ("which functor is it?") that will be used in subsequent chapters.

1. Central extensions by $\mathbf{K}_2$

In this section, we survey the work of Brylinski–Deligne [11]. These results will be applied in Chapter 2, so we choose to present them in a way better suited for our purpose. In particular, we shall work in less generality than *loc.cit.*.

It is important to note that the combinatorial gadget which shows up in their classification, to be denoted by $\Theta_G^+(\Lambda_T;\mathbf{Pic})$, is the prototype of what we call "enhanced Θ-data." We will explore this kind of gadgets in greater generality in Chapters 3 and 4.

1.1. $\mathbf{K}_2$.

1.1.1. For any Noetherian ring R of finite dimension, we let $\mathbf{K}_{\geq 0}(R)$ denote the connective K-theory spectrum associated to (perfect complexes over) R. Then $\mathbf{K}_{\geq 0}(R)$

has an $\mathbb{E}_\infty$-structure [22, §VI]. For a key manipulation in §7 of Chapter 2, we will need to know that $\mathbf{K}_{\geq 0}$ satisfies Zariski descent. This fact is established by the celebrated theorem of Thomason–Trobaugh [63]. Denote by $\mathrm{K}_n(R)$ the homotopy group $\pi_n\mathbf{K}_{\geq 0}(R)$ and by $\mathbf{K}_n$ the truncation $\tau_{\geq n}\tau_{\leq n}\mathbf{K}_{\geq 0}$ as Zariski sheaves. Thus $\mathbf{K}_n$ is the Zariski sheafification of K_n.

1.1.2. For the purpose of this section, however, we will not need K-groups beyond degree 2. The study of these "lower" K-groups can be done in purely algebraic terms, and we refer the reader to [67] for their constructions and basic properties. The lower K-groups of a field $\mathbf{F}$ can be described explicitly as follows.

$$\mathrm{K}_0(\mathbf{F}) \cong \mathbb{Z}, \quad \mathrm{K}_1(\mathbf{F}) \cong \mathbf{F}^\times, \quad \mathrm{K}_2(\mathbf{F}) \cong \mathbf{F}^\times \otimes \mathbf{F}^\times / \langle u \otimes v \mid u + v = 1 \rangle.$$

In fact, the Zariski sheaf $\mathbf{K}_0$ is the constant sheaf $\underline{\mathbb{Z}}$ and $\mathbf{K}_1$ is isomorphic to the multiplicative group $\mathbb{G}_m$. The $\mathbb{E}_\infty$-structure on $\mathbf{K}_{\geq 0}$ induces a graded commutative product:

$$\{-,-\} : \mathbf{K}_1 \underset{\mathbb{Z}}{\otimes} \mathbf{K}_1 \to \mathbf{K}_2. \tag{1.1}$$

For a field $\mathbf{F}$, the pairing (1.1) is the tautological one sending f, g to $f \otimes g$.

1.1.3. To study the Zariski sheaf $\mathbf{K}_n$, we need an important tool—the *Gersten resolution* of D. Quillen. For a *regular* scheme S which is of finite type over a field k, Quillen's result asserts that the following complex of Zariski sheaves on S is acyclic.

$$\mathbf{K}_{n,S} \to \bigoplus_{x \in S^{(0)}} \mathbf{K}_n(x) \to \bigoplus_{x \in S^{(1)}} \mathbf{K}_{n-1}(x) \to \cdots \to \bigoplus_{x \in S^{(n)}} \mathbb{Z}.$$

Here, $S^{(d)}$ denote the set of codimension–d points in S. The Gersten resolution often allows to reduce questions regarding $\mathbf{K}_{n,S}$ to the case of fields. It enters, for example, in C. Sherman's proof of the following result.

1.1.4. **Lemma.** *Let S be a regular scheme of finite type over a field k. Then:*

(1) Pulling back along $p : \mathbb{A}^1_S \to S$ defines an isomorphism:

$$\mathbf{K}_n \xrightarrow{\sim} \mathrm{R}\, p_* \mathbf{K}_n;$$

(2) Pulling back along $p : \mathbb{G}_{m,S} \to S$ together with the pairing $\{t, p^*(-)\}$ with the canonical section $t \in \Gamma(\mathbb{G}_{m,S}, \mathbf{K}_1)$ defines an isomorphism:

$$\mathbf{K}_n \oplus \mathbf{K}_{n-1} \xrightarrow{\sim} \mathrm{R}\, p_* \mathbf{K}_n.$$

Proof. See [56]. □

1.2. **Brylinski–Deligne data.**

1.2.1. We now fix a ground field k and $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ denote the category of *separated*[6] schemes of finite type over k. Fix $S \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ which is furthermore regular. We shall be concerned with Zariski sheaves on the category $\mathbf{Sch}^{\mathrm{ft}}_{/S}$ of separated schemes of finite type over S, which we refer to as *big* Zariski sheaves over S. One such example is $\mathbf{K}_n$ for any $n \geq 0$.

1.2.2. Let G be a split reductive group over k.[7] We may regard G as a big Zariski sheaf over S. The *Brylinski-Deligne data* refer to the Picard groupoid of central extensions of big Zariski sheaves of abelian groups over S:

$$0 \to \mathbf{K}_2 \to \mathbf{E} \to G \to 1. \tag{1.2}$$

We denote this groupoid by $\mathbf{CExt}(G, \mathbf{K}_2)$, with the base scheme S being tacitly understood. It is straightforward to see that (1.2) is equivalent to a multiplicative $\mathbf{K}_2$-torsor over G, or yet equivalently, a *group* morphism $G \to \mathrm{B}_{\mathrm{Zar}}\, \mathbf{K}_2$ where $\mathrm{B}_{\mathrm{Zar}}\, \mathbf{K}_2$ is the classifying stack of $\mathbf{K}_2$ in the Zariski topology. The latter point of view makes

[6]As a matter of convention, all our schemes are assumed seperated unless otherwise noted.
[7]The setting of [11] covers any reductive group over S, but we shall not need it.

(1.2) much more amenable to study, as one may apply the Gersten resolution of $\mathbf{K}_{2,G_S}$.

1.2.3. The main result of [11] is a classification theorem which expresses $\mathbf{CExt}(G, \mathbf{K}_2)$ by a more concrete groupoid, which we shall presently define. This groupoid, and hence the classification theorem, depends on the choice of a split maximal torus $T \subset G$.

1.2.4. We first discuss the case of a split torus T. Define $\Theta^+(\Lambda_T; \mathbf{Pic})$ to be the category of pairs $(q, \mathcal{L}^{(\lambda)})$ where $q \in \mathcal{Q}(\Lambda_T; \mathbb{Z})$ is an integral-valued quadratic form on the cocharacter lattice Λ_T. Denote by κ its associated bilinear form:

$$\kappa(\lambda, \mu) := q(\lambda + \mu) - q(\lambda) - q(\mu).$$

Note that this association defines a bijection between quadratic forms and symmetric bilinear forms κ for which $\kappa(\lambda, \lambda) \in 2\mathbb{Z}$ for all $\lambda \in \Lambda_T$. The data $\mathcal{L}^{(\lambda)}$ are a Λ_T-indexed family of line bundles on S, equipped with multiplicativity isomorphisms:

$$c_{\lambda,\mu} : \mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda+\mu)}.$$

The data $c_{\lambda,\mu}$ ought to satisfy the associativity condition for any triple λ, μ, ν. Furthermore, each pair λ, μ is required to satisfy a κ-*twisted* commutativity condition, i.e.,

$$c_{\lambda,\mu}(a \otimes b) = (-1)^{\kappa(\lambda,\mu)} \cdot c_{\mu,\lambda}(b \otimes a).$$

1.2.5. A more succient way to package the data of the line bundles $\mathcal{L}^{(\lambda)}$ is as a central extension of (small) Zariski sheaves on S:

$$0 \to \mathcal{O}_S^{\times} \to \mathcal{E} \to \Lambda_T \to 0, \tag{1.3}$$

whose commutator pairing $[-,-] : \Lambda_T \underset{\mathbb{Z}}{\otimes} \Lambda_T \to \mathcal{O}_S^{\times}$ is given by $[\lambda, \mu] = (-1)^{\kappa(\lambda,\mu)}$. Indeed, $\mathcal{L}^{(\lambda)}$ corresponds to the pre-image of $\mathcal{E} \to \Lambda_T$ in (1.3), which is an $\mathcal{O}_S^{\times}$-torsor.

1.2.6. We note that $\Theta^+(\Lambda_T; \mathbf{Pic})$ has the structure of a Picard groupoid. In fact, it is a *strict* Picard groupoid, in the sense that the commutativity constraint $\beta_{c,c'}$ is the identity on $c \otimes c$ for $c = c'$.

1.2.7. Let us now define a functor from Brylinski–Deligne data for T to the Picard groupoid $\Theta^+(\Lambda_T; \mathbf{Pic})$. Given a central extension $\mathbf{E}$ of T by $\mathbf{K}_2$, we have a commutator pairing $[-,-] : T \underset{\mathbb{Z}}{\otimes} T \to \mathbf{K}_2$. For any pair $\lambda, \mu \in \Lambda_T$, we obtain a morphism:

$$\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m \to \mathbf{K}_2, \quad z_1, z_2 \rightsquigarrow [z_1^\lambda, z_2^\mu].$$

Since S is smooth, such a morphism must be an integral multiple of (1.1) (c.f. [10]). The integer $\kappa(\lambda, \mu)$ is then set to be this multiple. By [11, Proposition 3.13], κ defines a quadratic form q. To obtain the central extension (1.3), we pull back $\mathbf{E}$ along the projection $p : \mathbb{G}_{m,S} \to S$ and then pushforward. The vanishing of Lemma 1.1.4(2) gives a central extension:

$$0 \to p_*\mathbf{K}_2 \to p_*\mathbf{E} \to p_*T \to 1. \tag{1.4}$$

Note that each $\lambda \in \Lambda_T$ defines a section of T over $\mathbb{G}_{m,S}$. These give rise to a map of sheaves $\Lambda_T \to p_*T$ which respects the group structure. Thus we obtain (1.3) by pulling back (1.4) along the map $\Lambda_T \to p_*T$ and inducing along $p_*\mathbf{K}_2 \to \mathbf{K}_1$.

1.2.8. **Lemma.** *The above procedure defines a canonical equivalence of Picard groupoids:*

$$\Psi_{\mathrm{BD},T} : \mathbf{CExt}(T, \mathbf{K}_2) \xrightarrow{\sim} \Theta^+(\Lambda_T; \mathbf{Pic}).$$

Proof. This is [11, Theorem 3.16]. $\square$

1.2.9. Let us now turn to the case of a split semisimple, simply connected group G_{sc}, with a fixed split maximal torus T_{sc}. We let W denote the Weyl group attached to $(G_{\mathrm{sc}}, T_{\mathrm{sc}})$. It acts on T_{sc} and thus on its co-character lattice $\Lambda_{T_{\mathrm{sc}}}$.

Given any Brylinski–Deligne datum $\mathbf{E}$ of G_{sc}, pulling back along $T_{\mathrm{sc}} \to G_{\mathrm{sc}}$ and applying $\Psi_{\mathrm{BD},T_{\mathrm{sc}}}$, we obtain an object of $\Theta^+(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic})$. In particular, we may take its underlying quadratic form q, and the result will be a functor from $\mathbf{CExt}(G_{\mathrm{sc}}, \mathbf{K}_2)$ to $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})$.

1.2.10. **Lemma.** *The above functor defines a canonical equivalence of Picard groupoids:*

$$\mathbf{Q}_{\mathrm{BD},G_{\mathrm{sc}}} : \mathbf{CExt}(G_{\mathrm{sc}}, \mathbf{K}_2) \xrightarrow{\sim} \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W.$$

Proof. This is [11, Theorem 4.7]. In fact, *loc.cit.* also shows that $\mathbf{Q}_{\mathrm{BD},G_{\mathrm{sc}}}$ factors through an equivalence between *pointed* morphisms $G_{\mathrm{sc}} \to \mathrm{B}_{\mathrm{Zar}}\, \mathbf{K}_2$ and $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W$. $\square$

In particuar, the following composition defines a functor from $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W$ to $\Theta^+(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic})$.

(1.5)
$$\begin{tikzcd} \mathbf{CExt}(G_{\mathrm{sc}}, \mathbf{K}_2) \arrow[d] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W \arrow[l, "\mathbf{Q}^{-1}_{\mathrm{BD},G_{\mathrm{sc}}}"', "\sim"] \\ \Theta^+(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic}). & \end{tikzcd}$$

1.2.11. Finally, we consider a split reductive group G with a fixed split maximal torus T. Let G_{der} be its derived subgroup and $\widetilde{G}_{\mathrm{der}}$ the universal cover. The preimage of T defines maximal split tori $T_{\mathrm{der}} \subset G_{\mathrm{der}}$ and $\widetilde{T}_{\mathrm{der}} \subset \widetilde{G}_{\mathrm{der}}$. The group $\widetilde{G}_{\mathrm{der}}$ is split semisimple and simply connected, so Lemma 1.2.10 applies to it.

We define $\Theta^+_G(\Lambda_T; \mathbf{Pic})$ to be category of triples $(q, \mathcal{L}^{(\lambda)}, \varepsilon)$, where $(q, \mathcal{L}^{(\lambda)})$ is an object of $\Theta^+(\Lambda_T; \mathbf{Pic})$, and ε is an isomorphism between the following objects of $\Theta^+(\Lambda_{\widetilde{T}_{\mathrm{der}}}; \mathbf{Pic})$:

(1) The restriction of $(q, \mathcal{L}^{(\lambda)})$ to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$;
(2) The object associated via (1.5) to the restriction of q to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$.

1.2.12. Then we have a functor:

$$\Psi_{\mathrm{BD},G} : \mathbf{CExt}(G, \mathbf{K}_2) \to \Theta^+_G(\Lambda_T; \mathbf{Pic}),$$

obtained as follows. The datum $(q, \mathcal{L}^{(\lambda)})$ arises from pulling back along $T \to G$ and applying $\Psi_{\mathrm{BD},T}$. The isomorphism ε comes from the canonicity of the classification for tori, as we restrict a Brylinski–Deligne datum along the two circuits of the commutative diagram:

$$\begin{tikzcd} \widetilde{T}_{\mathrm{der}} \arrow[r] \arrow[d] & \widetilde{G}_{\mathrm{der}} \arrow[d] \\ T \arrow[r] & G \end{tikzcd}$$

1.2.13. **Theorem.** *The functor $\Psi_{\mathrm{BD},G}$ is an equivalence of Picard groupoids.*

Proof. This is [11, Theorem 6.2]. □

2. Pointwise geometry of Gr_G

In this section, we begin studying the geometry of the affine Grassmannian. We fix an algebraically closed ground field k. Let H be a linear algebraic group over k.

In this text, we use the term *ind-scheme* to refer to prestacks which can be represented as a filtered colimit $\underset{\alpha\in A}{\mathrm{colim}}\, X_\alpha$ of (separated) schemes X_α such that:

(1) the transition maps $X_\alpha \to X_\beta$ are closed immersions;

(2) the index category has cardinality bounded by $\aleph_0$.

Thus, ind-schemes for us are what one may call "seperated $\aleph_0$-strict ind-schemes." We denote the category they form by $\mathbf{IndSch}_{/k}$. Let $\mathbf{IndSch}^{\mathrm{ft}}_{/k}$ be its full subcategory of ind-schemes of ind-finite type, i.e., in a presentation one can take each $X_\alpha \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$.[8]

2.1. First properties.

[8] Any property P of schemes which is inherited by closed subschemes makes sense for ind-schemes.

2.1.1. Consider the local field of formal Laurent series $k(\!(t)\!)$. It contains the formal power series $k[\![t]\!]$ as the ring of integers. The affine Grassmannian Gr_H gives an ind-scheme structure on the quotient of $H(k(\!(t)\!))$ by $H(k[\![t]\!])$. More precisely, Gr_H is the prestack which associates to a k-algebra R the set of pairs $(\mathcal{P}_H, \alpha)$, where $\mathcal{P}_H$ is a H-torsor on $\mathrm{Spec}(R(\!(t)\!))$ and α is a trivialization of its restriction to $\mathrm{Spec}(R[\![t]\!])$.

2.1.2. **Lemma.** *The prestack* Gr_H *is representable by an ind-scheme of ind-finite type. Furthermore, when* H *is reductive,* Gr_H *is ind-projective.*

Proof. This is [74, Theorem 1.2.2]. □

2.1.3. We also define the *loop group* as the prestack $H(\!(t)\!)$ whose group of R-points is given by $\mathrm{Maps}(\mathrm{Spec}(R(\!(t)\!)), H)$. It contains two distinguished sub-group prestacks, the *arc group* $H[\![t]\!]$ whose R-points are $\mathrm{Maps}(\mathrm{Spec}(R[\![t]\!]), H)$ and the *opposite arc group* whose R-points are $\mathrm{Maps}(\mathrm{Spec}(R[t^{-1}]), H)$. The following Lemma summarizes their basic properties.

2.1.4. **Lemma.** *The following statements hold.*

(1) $H(\!(t)\!)$ *is representable by a group ind-scheme;*

(2) $H[\![t]\!]$ *is representable by an affine group scheme;*

(3) $H[t^{-1}]$ *is representable by an group ind-scheme of ind-finite type;*

(4) The natural map $H(\!(t)\!) \to \mathrm{Gr}_H$ *is an étale* $H[\![t]\!]$*-torsor.*

Proof. The representability of $H(\!(t)\!)$ and $H[\![t]\!]$ is [74, Proposition 1.3.2]. Then Proposition 1.3.6 (c.f. Remark 1.3.8) of *loc.cit.* shows that $H(\!(t)\!)$ is an étale $H[\![t]\!]$-torsor over Gr_H. The representability of $H[t^{-1}]$ is [74, Lemma 4.1.4]. □

2.2. **Schubert strata.**

2.2.1. We now assume that G is a *semisimple* and *simply connected* algebraic group. The finer properties of Gr_G can be studied via its Schubert stratification. We fix a maximal torus $T \subset G$ and a Borel subgroup B containing T. Let W denote the Weyl group of the pair (G, T).

2.2.2. Each co-character $\lambda \in \Lambda_T$ determines a k-point t^λ of $T(\!(t)\!)$ and thus of $G(\!(t)\!)$ and Gr_G, which will still by denoted by t^λ. The $G[\![t]\!]$-orbits on Gr_G are in bijection with the set of dominant co-characters Λ_T^+, where $\lambda \in \Lambda_T^+$ corresponds to $\mathrm{Gr}_G^\lambda := G[\![t]\!] \cdot t^\lambda$. Furthermore, the closure relationship of $G[\![t]\!]$-orbits is given by:

$$\mathrm{Gr}_G^{\lambda_1} \subset \overline{\mathrm{Gr}}_G^{\lambda_2} \iff \lambda_2 - \lambda_1 \in \Lambda_T^+.$$

We shall denote by $\mathrm{Gr}_G^{\leq\lambda}$ the closure of Gr_G^λ. It has dimension $\langle \lambda, 2\check{\rho} \rangle$ where $2\check{\rho}$ is the sum of the positive roots of G.

2.2.3. We will also need the *affine flag variety* Fl_G, which as a prestack as R-points given by an R-point $(\mathcal{P}_G, \alpha)$ of Gr_G together with a reduction of $\mathcal{P}_G$ to B over the "closed point" $\mathrm{Spec}(R) \subset \mathrm{Spec}(R[\![t]\!])$. Then the canonical map:

$$\pi : \mathrm{Fl}_G \to \mathrm{Gr}_G \tag{2.1}$$

realizes Fl_G as an étale locally trivial fiber bundle with typical fiber G/B. In particular, Fl_G is also representable by an ind-scheme of ind-finite type. Writing $I \subset G[\![t]\!]$ for the subgroup defined as the fiber of B along the projection $G[\![t]\!] \to G$, we see that Fl_G is also the étale quotient of $G(\!(t)\!)$ by I.

2.2.4. Let W^{aff} denote the *affine Weyl group* $W \ltimes \Lambda_T$ formed using the tautological W-action on Λ_T. Thus we have a short exact sequence of groups:

$$0 \to \Lambda_T \to W^{\mathrm{aff}} \to W \to 1,$$

together with a splitting $W \to W^{\mathrm{aff}}$. The affine Weyl group is a Coxeter group on the generators s_i, which are simple reflections corresponding to the affine simple roots $\Delta^{\mathrm{aff}} = \{\alpha_0\} \sqcup \Delta$.

2.2.5. For an element $\tilde{w} = w\lambda \in W^{\mathrm{aff}}$, we write $t^{\tilde{w}}$ for the element $w \cdot t^{\lambda}$ in Fl_G. It is well defined since T commutes with t^{λ} and belongs to I as a subgroup of $G[\![t]\!]$. The I-orbits in Fl_G are in bijection with W^{aff}, with $\tilde{w}$ passing to the orbit $\mathrm{Fl}_G^{\tilde{w}} := I \cdot t^{\tilde{w}}$. Their closure relationship is given by:

$$\mathrm{Fl}_G^{\tilde{w}_1} \subset \overline{\mathrm{Fl}}_G^{\tilde{w}_2} \iff \tilde{w}_1 \preceq \tilde{w}_2 \text{ in the Bruhat ordering.}$$

This Bruhat ordering is the one associated to the Coxeter group W^{aff}. As before, we let $\mathrm{Fl}_G^{\leq \tilde{w}}$ denote the closure of $\mathrm{Fl}_G^{\tilde{w}}$. It has dimension equal to the length of $\tilde{w}$.

2.2.6. For each reduced expression of $\tilde{w} = s_1 \cdots s_r$, one can associate an *affine Bott–Samelson resolution* of the Schubert cell $\mathrm{Fl}_G^{\leq \tilde{w}}$, defined as the product $\mathcal{P}_{s_1} \overset{I}{\times} \cdots \overset{I}{\times} \mathcal{P}_{s_r}/I$, for each $\mathcal{P}_{s_i}$ being the parahoric subgroup associated to s_i (c.f. [24, §3]):

$$\pi_{s_1 \cdots s_r} : \widetilde{\mathrm{Fl}}_{G, s_1 \cdots s_r}^{\leq \tilde{w}} \to \mathrm{Fl}_G^{\leq \tilde{w}}.$$

Its properties are analogous to the usual Bott–Samelson resolution of the Schubert cells in G/B.

2.2.7. **Lemma.** *Suppose G is semisimple and simply connected.*

(1) The scheme $\widetilde{\mathrm{Fl}}_{G, s_1 \cdots s_r}^{\leq \tilde{w}}$ is a successive $\mathbb{P}^1$-bundle, with each factor correponding to an embedding $\mathcal{P}_{s_i}/I \cong \mathbb{P}^1 \hookrightarrow \widetilde{\mathrm{Fl}}_{G, s_1 \cdots s_r}^{\leq \tilde{w}}$. In particular, $\widetilde{\mathrm{Fl}}_{G, s_1 \cdots s_r}^{\leq \tilde{w}}$ is smooth and proper.

(2) Pulling back defines an isomorphism:

$$\mathcal{O}_{\mathrm{Fl}_G^{\leq \tilde{w}}} \xrightarrow{\sim} \mathrm{R}(\pi_{s_1 \cdots s_r})_* \mathcal{O}_{\widetilde{\mathrm{Fl}}_{G, s_1 \cdots s_r}^{\leq \tilde{w}}}.$$

Proof. The first statement is immediate. The second statement is [24, Lemma 9]. □

2.2.8. In fact, the affine Bott–Samelson resolution generalizes the usual one in the following precise sense. There is a closed immersion $G/B \to \mathrm{Fl}_G$ which is also the fiber of (2.1) at the unit k-point. Let $\mathrm{X}^{\leq w}$ denote the closure of the usual Schubert cell BwB/B corresponding to $w \in W$. The inclusion $W \to W^{\mathrm{aff}}$ respects the Bruhat ordering and the length function. Thus we obtain a commutative diagram:

$$\begin{tikzcd}
\widetilde{\mathrm{X}}^{\leq w}_{s_1\cdots s_r} \arrow[r] \arrow[d, "\cong"] & \mathrm{X}^{\leq w} \arrow[r, hook] \arrow[d, "\cong"] & G/B \arrow[d] \\
\widetilde{\mathrm{Fl}}^{\leq w}_{G,s_1\cdots s_r} \arrow[r] & \mathrm{Fl}^{\leq w}_G \arrow[r, hook] & \mathrm{Fl}_G
\end{tikzcd}$$

where $\widetilde{\mathrm{X}}^{\leq w}_{s_1\cdots s_r}$ is the Bott–Samelson resolution associated to a reduced expression $w = s_1 \cdots s_r$.

2.2.9. The Schubert strata of Gr_G and Fl_G can be compared as follows. Fixing $\lambda \in \Lambda^+_T$, since every element in $G(\!(t)\!)$ of the form $w_1 t^\lambda w_2$ projects to the same $G[\![t]\!]$-orbit Gr^λ_G, the preimage of Gr^λ_G along π (2.1) contains $\mathrm{Fl}^{\tilde{w}}_G$ for all $\tilde{w} \in W\lambda W$. By the theory of Tits systems, the double coset $W\lambda W \subset W^{\mathrm{aff}}$ contains a *unique* longest element, to be denoted by $\tilde{w}_\lambda$. Then we have a fiber product diagram:

$$\begin{tikzcd}
\mathrm{Fl}^{\leq \tilde{w}_\lambda}_G \arrow[r] \arrow[d, "\pi^{\leq\lambda}"] & \mathrm{Fl}_G \arrow[d, "\pi"] \\
\mathrm{Gr}^{\leq\lambda}_G \arrow[r] & \mathrm{Gr}_G
\end{tikzcd} \tag{2.2}$$

Indeed, this is because the canonical map from $\mathrm{Fl}^{\leq \tilde{w}_\lambda}_G$ to $\mathrm{Gr}^{\leq\lambda}_G \underset{\mathrm{Gr}_G}{\times} \mathrm{Fl}_G$ is a dominant closed immersion between reduced schemes, hence an isomorphism. We now deduce an affine analogue of the Borel–Weil–Bott theorem from this description.

2.2.10. **Lemma.** *Suppose G is semisimple, simply connected, and $\lambda \in \Lambda^+_T$. Then $\mathrm{R}\Gamma(\mathrm{Gr}^{\leq\lambda}_G, \mathcal{O})$ is canonically isomorphic to k.*

In particular, we have $\mathrm{H}^i(\mathrm{Gr}^{\leq\lambda}_G, \mathcal{O}) = 0$ for all $i \geq 1$.

Proof. Let $\tilde{w}_\lambda$ be the longest element in the double coset $W\lambda W$ of W^{aff}. The fiber product diagram (2.2) and Lemma 2.2.7(2) supply isomorphisms:

$$\mathrm{R}\Gamma(\mathrm{Gr}_G^{\leq\lambda}, \mathcal{O}) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathrm{Fl}_G^{\leq\tilde{w}_\lambda}, \mathcal{O}) \xrightarrow{\sim} \mathrm{R}\Gamma(\widetilde{\mathrm{Fl}}_{G,s_1\cdots s_r}^{\leq\tilde{w}_\lambda}, \mathcal{O}),$$

for any reduced expression $\tilde{w}_\lambda = s_1 \cdots s_r$. We win because $\widetilde{\mathrm{Fl}}_{G,s_1\cdots s_r}^{\leq\tilde{w}_\lambda}$ is a successive $\mathbb{P}^1$-bundle, according to Lemma 2.2.7(1). $\square$

Let us also record the following result of G. Faltings.

2.2.11. **Lemma.** *Suppose G is semisimple, simply connected, and $\lambda \in \Lambda_T^+$. Then $\mathrm{Gr}_G^{\leq\lambda}$ is normal. Furthermore, the ind-scheme $\operatorname{colim}_{\lambda\in\Lambda_T^+} \mathrm{Gr}_G^{\leq\lambda}$ canonically identifies with Gr_G.*

Proof. This is [24, Theorem 8]. $\square$

2.3. **The Picard group.**

2.3.1. Assume that G is *simple* and simply connected and fix $T \subset B \subset G$ as above. We now review Faltings' classification of line bundles on Gr_G.

2.3.2. Let $\mathring{I}^- \subset G[t^{-1}]$ denote the fiber of N^- along the projection $G[t^{-1}] \to G$. According to [24, §3], the $\mathring{I}^-$-orbits in Fl_G are again in bijection with W^{aff}, and we denote by $\mathrm{Fl}_{G,\tilde{w}}$ the $\mathring{I}^-$-orbit containing $t^{\tilde{w}}$. By contrast, the $\mathring{I}^-$-orbit $\mathrm{Fl}_{G,\tilde{w}}$ are of finite co-dimension $l(\tilde{w})$, and their closure relation is the *opposite* Bruhat ordering:

$$\mathrm{Fl}_{G,\tilde{w}_1} \subset \overline{\mathrm{Fl}}_{G,\tilde{w}_2} \iff \tilde{w}_2 \preceq \tilde{w}_1 \text{ in the Bruhat ordering.}$$

In particular, the *big cell* $\mathrm{Fl}_{G,e}$ is open dense inside Fl_G.

2.3.3. Fix a finite subset $A \subset W^{\mathrm{aff}}$ closed under descendance, i.e., $\tilde{w} \in A$ and $\tilde{w}' \preceq \tilde{w}$ implies that $\tilde{w}' \in A$. Write Ω_A for the union of opens:

$$\Omega_A := \bigcup_{\tilde{w}\in A} \tilde{w} \cdot \mathrm{Fl}_{G,e}.$$

Then Ω_A is again $\mathring{I}^-$-stable, and contains precisely those $\mathrm{Fl}_{G,\tilde{w}}$ for $\tilde{w} \in A$. On the other hand, for $n \geq 1$, we write $\mathring{I}^-(n)$ for the subgroup of $\mathring{I}^-$ of elements lying in T modulo t^{-n}. For sufficiently large integer n relative to A, the quotient $\Omega_A/\mathring{I}^-(n)$ is a *smooth* scheme (Lemma 6 of *loc.cit.*), and furthermore, the $\mathring{I}^-$-orbits in Ω_A are preimages of affine spaces:

$$\begin{tikzcd} \mathrm{Fl}_{G,\tilde{w}} \arrow[r, hook] \arrow[d, "\mathring{I}^-(n)"] & \Omega_A \arrow[d, "\mathring{I}^-(n)"] \\ \mathbb{A}^d \arrow[r, hook] & \Omega_A/\mathring{I}^-(n). \end{tikzcd}$$

Suppose A contains all the simple reflections. Then the $\mathring{I}^-$-orbits in Ω_A of codimension-1 are precisely Fl_{G,s_i} for s_i the simple reflection associated to $\alpha_i \in \Delta^{\mathrm{aff}}$. To each $\alpha_i \in \Delta^{\mathrm{aff}}$ we may attach the corresponding effective Cartier divisor on $\Omega_A/\mathring{I}^-(n)$ and pull back the corresponding line bundle to Ω_A. This procedure defines a map:

$$\mathrm{Maps}(\Delta^{\mathrm{aff}}, \mathbb{Z}) \to \mathrm{Pic}(\Omega_A), \tag{2.3}$$

where $\mathrm{Pic}(\Omega_A)$ is the Picard group of Ω_A.

2.3.4. **Theorem.** *For A containing all simple reflections, the map* (2.3) *is bijective. In particular, we have a bijection:*

$$\mathrm{Maps}(\Delta^{\mathrm{aff}}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Fl}_G).$$

Furthermore, the element $\mathcal{L}_f \in \mathrm{Pic}(\mathrm{Fl}_G)$ corresponding to $f : \Delta^{\mathrm{aff}} \to \mathbb{Z}$ has the property that its pullback along $\mathbb{P}^1 = \mathcal{P}_i/I \to \mathrm{Fl}_G$ has degree $f(\alpha_i)$.

Proof. This is a combination of Theorem 7 and the first part of Corollary 12 of [24]. $\square$

We use Theorem 2.3.4 to determine the Picard group of Gr_G. Although this is also accomplished in Faltings' work, we find it helpful to expand on the details as they will be used later in a different context. The classification relies on a convenient Lemma describing descent of fiberwise trivial line bundles.

2.3.5. **Lemma.** *Suppose $p : X \to Y$ is a proper, flat morphism in $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ such that its fiber X_y at every k-point $y \in Y$ satisfies $k \xrightarrow{\sim} \mathrm{H}^0(X_y, \mathcal{O})$ and $\mathrm{H}^1(X_y, \mathcal{O}) = 0$. Then for any line bundle $\mathcal{L}$ over X, the following are equivalent:*

(1) $\mathcal{L}$ is trivial along the fibers of p at k-points;

(2) $p_*\mathcal{L}$ is a line bundle over Y and the canonical map $p^*p_*\mathcal{L} \to \mathcal{L}$ is an isomorphism.

Proof. The implication (2) $\implies$ (1) is obvious, so we only need to prove the converse. This is an application of the "cohomology and base change" theorem ([39, §7.7]). Indeed, $\mathcal{L}$ is flat over Y and the fiberwise triviality shows that for all $y \in Y$:

$$\mathrm{H}^1(X_y, \mathcal{L}\big|_{X_y}) \xrightarrow{\sim} \mathrm{H}^1(X_y, \mathcal{O}) = 0.$$

Therefore the base change map $f_*\mathcal{L} \underset{\mathcal{O}_Y}{\otimes} k(y) \to \mathrm{H}^0(X_y, \mathcal{O})$ is an isomorphism. Since $f_*\mathcal{L}$ is coherent, this implies that $f_*\mathcal{L}$ is a line bundle. The morphism $p^*p_*\mathcal{L} \to \mathcal{L}$ is an isomorphism since it is so at every k-point $x \in X$. $\square$

Let us also note a criterion for product decomposition of line bundles.

2.3.6. **Lemma.** *Suppose $X, Y \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ are connected, and furthermore X is integral, projective, with $\mathrm{H}^1(X, \mathcal{O}) = 0$. Then external product defines an isomorphism:*

$$\boxtimes : \mathrm{Pic}(X) \times \mathrm{Pic}(Y) \xrightarrow{\sim} \mathrm{Pic}(X \times Y).$$

Proof. The hypothesis on X implies that the Picard scheme Pic_X is representably by a scheme locally of finite type with vanishing tangent spaces, i.e., a discrete scheme. Since Y is connected, any morphism $Y \to \mathrm{Pic}_X$ is constant. By definition, this means that up to a line bundle from Y, every line bundle on $X \times Y$ is equivalent to a pullback from X. $\square$

2.3.7. Since G/B is connected with $\mathrm{H}^1(G/B, \mathcal{O}) = 0$, we conclude from the Lemma that:

$$\pi^* : \mathrm{Pic}(\mathrm{Gr}_G) \to \mathrm{Pic}(\mathrm{Fl}_G)$$

is injective and its image identifies with fiberwise trivial line bundles. On the other hand, Lemma 2.3.6 together with the fact that Fl_G is an étale fiber bundle over the connected scheme Gr_G implies that $\mathcal{L} \in \mathrm{Pic}(\mathrm{Fl}_G)$ is fiberwise trivial as long as it is trivial on just one fiber.

2.3.8. We consider the unit fiber $G/B \to \mathrm{Fl}_G$. The description in Theorem 2.3.4 implies that $\mathcal{L}_f \in \mathrm{Pic}(\mathrm{Fl}_G)$ is trivial on G/B if and only if $f(\alpha_i) = 0$ for all $\alpha_i \in \Delta$. Therefore, we obtain an isomorphism $\mathrm{Pic}(\mathrm{Gr}_G) \cong \mathbb{Z}$ making the following diagram commute.

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d, "\cong"] & \mathrm{Maps}(\Delta^{\mathrm{aff}}, \mathbb{Z}) \arrow[d, "\cong"] \\
\mathrm{Pic}(\mathrm{Gr}_G) \arrow[r] & \mathrm{Pic}(\mathrm{Fl}_G)
\end{tikzcd}$$

The top arrow sends $\mathbf{1}$ to the characteristic function on the affine simple root α_0. We denote by min the image of $\mathbf{1} \in \mathbb{Z}$ in $\mathrm{Pic}(\mathrm{Gr}_G)$, regarded as a line bundle trivialized at the unit $e \in \mathrm{Gr}_G$. In fact, min descends to an ample line bundle to Bun_G for any proper, smooth curve X.

2.3.9. **Theorem.** *Suppose G is semisimple and simply connected, with $\mathbf{S}$ the set of its simple factors. Then we have a canonical isomorphism:*

$$\mathrm{Maps}(\mathbf{S}, \mathbb{Z}) \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Gr}_G), \quad \mathbf{1}_s \rightsquigarrow \mathrm{min}_s.$$

Proof. For G simple, this has been established above. The general case follows from the product decomposition (2.3.6) together with the H^1-vanishing of affine Schubert cells (Lemma 2.2.10). □

3. Factorization

We continue to fix $k = \bar{k}$. Suppose X is a smooth curve over k. Following Beilinson–Drinfeld [8], the affine Grassmannian Gr_H can be defined relative to the Ran space of X, and as such, it possesses an important piece of structure called "factorization." In this section, we explain this structure and some basic geometric properties associated to the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{H,\mathrm{Ran}}$.

3.1. **Factorization prestacks.**

3.1.1. Let Ran denote the prestack on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ whose S-points are finite sets of maps $x^{(i)} : S \to X$. Write $\mathbf{fSet}^{\mathrm{surj}}$ for the category of finite nonempty sets I together with surjective maps $I \twoheadrightarrow J$. The the canonical map $\operatorname*{colim}_{I \in \mathbf{fSet}^{\mathrm{surj}}} X^I \to \mathrm{Ran}$ is an equivalence of prestacks.

3.1.2. For $n \geq 1$, we let $\mathrm{Ran}^{\times n}_{\mathrm{disj}}$ denote the open sub-prestack of $\mathrm{Ran}^{\times n}$ consisting of points $\{x^{(i)}\}_{i \in I_k, 1 \leq k \leq n}$ such that $x^{(i)}$ and $x^{(j)}$ are disjoint as long as i, j belong to I_k and $I_{k'}$ for $k \neq k'$. There is a morphism of "disjoint union":

$$\sqcup_{(n)} : \mathrm{Ran}^{\times n}_{\mathrm{disj}} \to \mathrm{Ran}.$$

3.1.3. We shall only be concerned with prestacks valued in *sets*. Let us recall that a *factorization prestack* over X is a prestack $\mathcal{Y}$ over Ran equipped with the additional data, called a *factorization isomorphism* over $\mathrm{Ran}^{\times 2}_{\mathrm{disj}}$:

$$f_{(2)} : \sqcup^*_{(2)} \mathcal{Y} \xrightarrow{\sim} (\mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}}.$$

The isomorphism $f_{(2)}$ is required to satisfy a coherence condition over $\mathrm{Ran}^{\times 3}_{\mathrm{disj}}$ expressing that the three ways on can form an isomorphism $\sqcup^*_{(3)} \mathcal{Y} \xrightarrow{\sim} (\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}}$ out of $f_{(2)}$ are identical. A convenient way to express this is as follows. We assume to be given:

$$f_{(3)} : \sqcup^*_{(3)} \mathcal{Y} \xrightarrow{\sim} (\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}},$$

such that for each surjection $\varphi : \{1,2,3\} \to \{1,2\}$, the map $\sqcup_\varphi : \mathrm{Ran}^{\times 3}_{\mathrm{disj}} \to \mathrm{Ran}^{\times 2}_{\mathrm{disj}}$ of taking unions along each of φ makes the following diagram commute.

$$\begin{tikzcd}
\sqcup^*_{(3)} \mathcal{Y} \arrow[r, "f_{(3)}", "\sim"'] \arrow[d, "\cong"'] & (\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}} \\
\sqcup^*_\varphi \sqcup^*_{(2)} \mathcal{Y} \arrow[r, "\sqcup^*_\varphi f_{(2)}", "\sim"'] & \sqcup^*_\varphi (\mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}} \arrow[u, "\cong"', "f_{(2),\varphi}"]
\end{tikzcd}$$

Here, $f_{(2),\varphi}$ means applying $f_{(2)}$ on the factor corresponding to the element of $\{1,2\}$ with two preimages. Clearly, $f_{(3)}$ is not an additional piece of structure.

3.1.4. Suppose $\mathcal{Y}$ is a factorization prestack over X. Then a *factorization line bundle* $\mathcal{L} \in \mathbf{Pic}^{\mathrm{fact}}(\mathcal{Y})$ is a section $\mathcal{L} \in \mathbf{Pic}(\mathcal{Y})$ equipped with *factorization isomorphisms*:

$$\sqcup^*_{(n)} \mathcal{L} \xrightarrow{\sim} \mathcal{L}^{\boxtimes n} \text{ in } \mathbf{Pic}(\sqcup^*_{(n)} \mathcal{Y} \xrightarrow{\sim} (\mathcal{Y}^{\times n})_{\mathrm{disj}}), \tag{3.1}$$

for $n = 2, 3$. Furthermore, for each surjection $\varphi : \{1,2,3\} \to \{1,2\}$, the following diagram is commutative:

$$\begin{tikzcd}
\sqcup^*_{(3)} \mathcal{L} \arrow[r, "\sim"] \arrow[d, "\cong"'] & \mathcal{L} \boxtimes \mathcal{L} \boxtimes \mathcal{L} \\
\sqcup^*_\varphi \sqcup^*_{(2)} \mathcal{L} \arrow[r, "\sim"] & \sqcup^*_\varphi (\mathcal{L} \boxtimes \mathcal{L}) \arrow[u, "\cong"]
\end{tikzcd}
\quad \text{in} \quad \mathbf{Pic}\left(
\begin{tikzcd}
\sqcup^*_{(3)} \mathcal{Y} \arrow[r, "\sim"] \arrow[d, "\cong"'] & (\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}} \\
\sqcup^*_\varphi \sqcup^*_{(2)} \mathcal{Y} \arrow[r, "\sim"] & \sqcup^*_\varphi (\mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}} \arrow[u, "\cong"]
\end{tikzcd}
\right).$$

As above, the factorization isomorphism (3.1) for $n = 3$ is not an additional piece of structure.

3.2. **The Beilinson–Drinfeld Grassmannian.**

3.2.1. We shall now introduce the main example of a factorization prestack: the *Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_{H,\mathrm{Ran}}$ associated to X and a linear algebraic group H. It is defined as the prestack over Ran whose fiber at an S-point $x^{(i)} : S \to X$ is the set of pairs $(\mathcal{P}_H, \alpha)$ where $\mathcal{P}_H$ is an étale H-torsor over $S \times X$ and α is a trivialization of $\mathcal{P}_H$ on the complement of the graphs:

$$\alpha : \mathcal{P}_H \xrightarrow{\sim} \mathcal{P}^0_H\big|_{S\times X\setminus \bigcup_{i\in I}\Gamma_{x^{(i)}}}.$$

The Beauville–Laszlo lemma shows that $\mathrm{Gr}_{H,\mathrm{Ran}}$ has the structure of a factorization prestack over X (c.f. [74]).

3.2.2. Fixing a k-point $x \in X$, we can consider the base change $\mathrm{Gr}_{H,x} := \mathrm{Gr}_{H,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \{x\}$. Suppose we furthermore fix a uniformizer t of the completed local ring $\widehat{\mathcal{O}}_{X,x}$, we obtain isomorphism between $\mathrm{Gr}_{H,x}$ and the "pointwise" affine Grassmannian Gr_H of §2.

3.2.3. In fact, all the basic properties of Gr_H generalize to the relative setting. Namely, the projection:

$$\pi : \mathrm{Gr}_{H,\mathrm{Ran}} \to \mathrm{Ran} \tag{3.2}$$

is ind-schematic and ind-finite type, i.e., for every $S \in \mathrm{Ran}$ with $S \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, the fiber product $\mathrm{Gr}_{H,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S$ is representable by an ind-scheme of ind-finite type. When H is reductive, π is furthermore ind-proper [74, Theorem 3.1.3]. For a finite set I, we will denote by Gr_{H,X^I} the fiber product $\mathrm{Gr}_{H,X^I} := \mathrm{Gr}_{H,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} X^I$. The morphism (3.2) admits a *unit* section, defined by sending $x^{(i)}$ to the trivial H-torsor $\mathcal{P}^0_H$ equipped with the tautological trivialization:

$$e : \mathrm{Ran} \to \mathrm{Gr}_{H,\mathrm{Ran}}\,.$$

3.2.4. We also have the factorization versions of the loop and arc groups. Namely, let $\mathcal{L}^+_{\mathrm{Ran}}H$ denote the prestack over Ran whose fiber at an S-point $x^{(i)} : S \to X$ is a map from the formal completion D_{x^I} of the graph $\bigcup_{i\in I}\Gamma_{x^{(i)}} \subset S \times X$ to H. The prestack $\mathcal{L}_{\mathrm{Ran}}H$ has as fiber maps from $\mathring{D}_{x^I} := D_{x^I}\setminus \bigcup_{i\in I}\Gamma_{x^{(i)}}$ to H. Then $\mathcal{L}^+_{\mathrm{Ran}}H$ is a relative affine group scheme over Ran, $\mathcal{L}_{\mathrm{Ran}}H$ is a relative group ind-scheme over

Ran, and $\mathrm{Gr}_{H,\mathrm{Ran}}$ identifies with the étale quotient $\mathcal{L}_{\mathrm{Ran}}H/\mathcal{L}^+_{\mathrm{Ran}}H$. We denote the base change of $\mathcal{L}^+_{\mathrm{Ran}}H$ and $\mathcal{L}_{\mathrm{Ran}}H$ to X^I by $\mathcal{L}^+_{X^I}H$, respectively $\mathcal{L}_{X^I}H$.

3.2.5. *Connection along* Ran. Another important piece of structure of $\mathrm{Gr}_{H,\mathrm{Ran}}$ is its canonical connection along Ran. A convenient way to formulate it is as follows. For any prestack $\mathcal{Y}$, we denote by $\mathcal{Y}_{\mathrm{dR}}$ the *de Rham prestack* associated to $\mathcal{Y}$. For any test affine scheme S, we have:

$$\mathrm{Maps}(S, \mathcal{Y}_{\mathrm{dR}}) := \mathrm{Maps}(S_{\mathrm{red}}, \mathcal{Y}).$$

We can define a prestack $\mathrm{Gr}_{H,\mathrm{Ran}_{\mathrm{dR}}}$ by associating to $x^I : S \to \mathrm{Ran}_{\mathrm{dR}}$ the set of H-torsors on $S \times X$ with trivialization on $S \times X \backslash \bigcup_{i\in I} \Gamma_{x^{(i)}}$—it is well defined since the latter open subscheme depends only on the underlying closed subset of $\Gamma_{x^{(i)}}$. Therefore, we have a Cartesian diagram:

$$\begin{tikzcd}
\mathrm{Gr}_{H,\mathrm{Ran}} \arrow[r] \arrow[d] & \mathrm{Gr}_{H,\mathrm{Ran}_{\mathrm{dR}}} \arrow[d] \\
\mathrm{Ran} \arrow[r] & \mathrm{Ran}_{\mathrm{dR}}
\end{tikzcd}$$

Note that the projection $\mathrm{Gr}_{H,\mathrm{Ran}_{\mathrm{dR}}} \to \mathrm{Ran}_{\mathrm{dR}}$ is still ind-schematic of ind-finite type. This is because any test affine scheme $S \to \mathrm{Ran}_{\mathrm{dR}}$ lifts (non-canonically) to Ran by smoothness of X.

The same argument applies to the loop, respectively arc groups. We obtain group prestacks $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}H$ and $\mathcal{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}H$ over $\mathrm{Ran}_{\mathrm{dR}}$ whose base changes to Ran recover the usual prestacks.

3.3. **Degeneration of Schubert strata.**

3.3.1. Let us consider now a semisimple, simply connected group G with fixed maximal torus and Borel subgroup $T \subset B \subset G$. The Schubert strata of §2.2 can be generalized as follows. Fix a finite set I and an I-tuple λ^I of elements of Λ_T. Then

$\mathrm{Gr}_{T,X^I} \to X^I$ has a section:

$$t^{\lambda^I} : X^I \to \mathrm{Gr}_{T,X^I}, \tag{3.3}$$

which lifts an S-point $x^{(i)}$ of X^I to the T-bundles $\bigotimes_{i\in I} \mathcal{O}(\lambda^{(i)}\Gamma_{x^{(i)}})$ equipped with its canonical trivialization away from $\bigcup_{i\in I} \Gamma_{x^{(i)}}$. Therefore, we also obtain a section of Gr_{G,X^I}, continued to be denoted by t^{λ^I}. We caution the reader that, contrary to the pointwise situation, t^{λ^I} does not lift to the loop group over X^I.

3.3.2. Suppose each $\lambda^{(i)}$ belongs to Λ_T^+. Then we denote the orbit closure of t^{λ^I} under the $\mathcal{L}_{X^I}G$-action by $\mathrm{Gr}_G^{\le\lambda^I}$. In other words, it is the schematic image of the morphism $\mathrm{act}_{t^{\lambda^I}} : \mathcal{L}_{X^I}G \to \mathrm{Gr}_{G,X^I}$. We call $\mathrm{Gr}_G^{\le\lambda^I}$ the (closed) Schubert stratum corresponding to λ^I. For $|I| = 1$ and $x \in X$, one sees immediately that the base change $\mathrm{Gr}_G^{\le\lambda} \underset{X}{\times} \{x\}$ identifies with the Schubert stratum of §2.2, upon the choice of a uniformizer of $\widehat{\mathcal{O}}_{X,x}$.

3.3.3. **Lemma.** *There holds:*

1. *The projection $\pi^{\le\lambda^I} : \mathrm{Gr}_G^{\le\lambda^I} \to X^I$ is flat;*
2. *For every surjection $\varphi : I \twoheadrightarrow J$, writing $\lambda^{(j)} = \sum_{i\in\varphi^{-1}(j)} \lambda^{(i)}$, the following canonical map is an isomorphism:*

$$\mathrm{Gr}_G^{\le\lambda^J} \xrightarrow{\sim} \mathrm{Gr}_G^{\le\lambda^I}\big|_{\Delta_{I\twoheadrightarrow J}}.$$

Proof. The problem is étale local on X, so we may assume $X = \mathbb{A}^1$ and both statements are immediate for $|I| = 1$. Let $\mathcal{L}$ be the ample generator of the Picard group of $\mathrm{Bun}_G(\mathbb{P}^1)$. The preimage of $\mathcal{L}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ has a factorization structure [74, Proposition 3.1.16], and is ample relative to Ran. Furthermore, for each $\lambda, \mu \in \Lambda_T^+$ and $d \ge 1$, we have a canonical isomorphism [73, Theorem 1.2.2]:

$$\mathrm{H}^0(\mathrm{Gr}_{G,x}^{\le\lambda+\mu}, \mathcal{L}^{\otimes d}) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Gr}_{G,x}^{\le\lambda}, \mathcal{L}^{\otimes d}) \otimes \mathrm{H}^0(\mathrm{Gr}_{G,x}^{\le\mu}, \mathcal{L}^{\otimes d}). \tag{3.4}$$

In light of (3.4) and the criterion of flatness by the Hilbert polynomial, we see that the statement (2) $\implies$ (1). The statement (2) for $|I| = 2$ is [74, Proposition 3.1.14]. We omit the proof for $|I| \geq 3$ which is similar. $\square$

4. Some derived techniques

In this section, we collect some basic notions and techniques related to ∞-categories and derived algebraic geometry. In Chapters 2 and 3, our use of higher category theory is very localized, so the reader can safely skip this section for now.

The main parts where higher categories and derived algebraic geometry play an essential role concern the de Rham context, in Chapters 4 and 5, where more foundational materials will be supplied as we go.

4.1. **Glossary.**

4.1.1. We use *quasi-categories* as the model of ∞-categories, as is done in Lurie [49]. Let $\mathbf{Spc}$ denote the ∞-category of ∞-groupoids, otherwise referred to as *spaces*. It is the localization of Kan complexes with respect to the usual weak equivalences. Therefore a map $\mathbf{S}_1 \to \mathbf{S}_2$ of spaces is an equivalence if it induces isomorphisms on π_i for all $i \geq 0$.

4.1.2. The higher categorical replacement for an abelian category is a *stable ∞-category*. It is defined as an ∞-category $\mathbf{C}$ with a zero object, all finite limits and finite colimits, and a square is Cartesian if and only if it is co-Cartesian [50, §1]. An important example of a stable ∞-category is that of spectra $\mathbf{Sptr}$. It is the stabilization of the ∞-category of spaces, and is related to the latter by a pair of adjoint functors:

$$\mathbf{Spc} \underset{\Omega^\infty}{\overset{\Sigma^\infty}{\rightleftarrows}} \mathbf{Sptr}.$$

4.1.3. There is a well-defined notion of symmetric monoidal ∞-categories. Furthermore, in any symmetric monoidal ∞-category $\mathbf{C}$, there is a notion of *commutative* (or $\mathbb{E}_\infty$-) algebras. They form an ∞-category denoted by $\mathrm{ComAlg}(\mathbf{C})$. Taking $\mathbf{A} \in \mathrm{ComAlg}(\mathbf{C})$, one may consider the $\mathbf{A}$-module objects in $\mathbf{C}$, which form a stable ∞-category $\mathbf{A}$-Mod.

4.1.4. In the presence of a field k, we let $\mathbf{Vect}_{/k}$ be the derived ∞-category of (unbounded) chain complexes of k-vector spaces. Then $\mathbf{Vect}_{/k}$ has the structure of a symmetric monoidal stable ∞-category. In particular, one may consider $\mathbf{Vect}_{/k}$-module objects in stable, co-complete ∞-categories. Such an object is also called a *DG category*. Informally, a DG category is a stable ∞-category $\mathbf{C}$ such that for a pair of objects $c_1, c_2 \in \mathbf{C}$, there is a chain complex $\mathrm{Hom}(c_1, c_2)$ of k-vector spaces.

4.1.5. We use the ∞-category of simplicial commutative rings as model for derived algebraic geometry. Namely, the ∞-category of *affine derived schemes* $\mathbf{DSch}^{\mathrm{aff}}$ is defined to be its opposite category, and all geometric objects will be presheaves on $\mathbf{DSch}^{\mathrm{aff}}$ valued in $\mathbf{Spc}$, otherwise called *derived prestacks*. These include derived schemes, derived algebraic spaces, derived *algebraic stacks*, etc. (see [35, Chapter II]). As is our convention with classical schemes, derived schemes are assumed separated.

4.1.6. *Coconnectivity.* We call a derived prestack $\mathcal{Y}$ *classical* if it is the left Kan extension of its restriction to $\mathbf{Sch}^{\mathrm{aff}}$. Namely, the following canonical map is an equivalence:

$$\operatorname*{colim}_{\substack{S \to T \\ T \in \mathbf{Sch}^{\mathrm{aff}}}} \mathrm{Maps}(T, \mathcal{Y}) \to \mathrm{Maps}(S, \mathcal{Y}).$$

The classical prestacks form a full subcategory of derived prestacks which is *not* closed under fiber products. In fact, they are related by a pair of adjoint functors:

$$\mathbf{PStk}^{\mathrm{cl}} \underset{}{\overset{\mathrm{LKE}}{\rightleftarrows}} \mathbf{PStk} \tag{4.1}$$

with the left adjoint being fully faithful ([35, II, §1.3.6]). More generally, one considers *n-coconnective* prestacks for each $n \geq 0$. They are the left Kan extensions of prestacks on n-coconnective affine schemes, i.e., $S \in \mathbf{DSch}^{\mathrm{aff}}$ with $\pi_i \mathcal{O}_S = 0$ for $i > n$. Let us denote the subcategory in $\mathbf{DSch}^{\mathrm{aff}}$ of n-coconnective affine schemes by ${}^{\leq n}\mathbf{DSch}^{\mathrm{aff}}$.

4.1.7. *Laft prestacks.* We also recall the definition of being "almost of finite type" over a ground field k. An affine derived scheme S is *almost of finite type* if $\pi_0 \mathcal{O}_S$ is of finite type over k, and each $\pi_i \mathcal{O}_S$ is a finitely genereated $\pi_0 \mathcal{O}_S$-module. A prestack $\mathcal{Y}$ is *locally almost of finite type* if:

(1) $\mathcal{Y}$ is convergent, i.e., for any $S \in \mathbf{DSch}^{\mathrm{aff}}$, the canonical map $\mathcal{Y}(S) \to \lim_n \mathcal{Y}(\tau^{\leq n} S)$ is an equivalence.
(2) the restriction of $\mathcal{Y}$ to each ${}^{\leq n}\mathbf{DSch}^{\mathrm{aff}}$ is locally of finite type, i.e., it takes co-filtered limits to colimits.

We denote by $\mathbf{DSch}^{\mathrm{aff,\ aft}}_{/k}$ the ∞-category of derived affine schemes almost of finite type, and by $\mathbf{PStk}^{\mathrm{laft}}_{/k}$ the ∞-category of derived prestacks locally of finite type ("laft"). When $\mathrm{char}(k) = 0$, there is an established theory of ind-coherent sheaves on laft prestacks. We will review it in Chapter 4 where it is needed.

4.1.8. The ∞-category $\mathbf{DSch}^{\mathrm{aff,\ aft}}_{/k}$ contains the full subcategory $\mathbf{DSch}^{\mathrm{aff,ft}}_{/k}$ of *finite type* affine schemes, defined to satisfy the additional assumption of being n-coconnective for some $n \geq 0$. Analogously, a derived scheme X is of finite type if it is quasi-compact and is covered by finite type affine schemes. They form a full subcategory $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ of $\mathbf{DSch}_{/k}$.

4.2. **Derived h-descent.**

4.2.1. An important aspect of derived algebraic geometry is that topologies of "resolution of singularity" type become subcanonical. Let us explain what this means.

4.2.2. Recall that on the category of classical separated schemes of finite type $\mathbf{Sch}^{\mathrm{ft}}_{/k}$, one can define the h-*topology* as the topology generated by "universal topological submersions." More precisely, a morphism $f : Y \to X$ is a *topological submersion* if the induced map on underlying topological spaces is a quotient map, and is a *universal topological submersion* if it remains a topological submersion after any base change. Thus, a proper surjection is an h-cover. More precisely, one can show that a (classical) presheaf $\mathcal{F}$ is an h-sheaf if it has descent with respect to Nisnevich covers and proper surjections.

4.2.3. The *derived* $\mathbf{h}$-*topology*[9] is defined analogously, where we take as covers morphisms which are topological submersions on the underlying classical schemes after any base change. We note, however, that given an h-cover of classical schemes $f : \widetilde{X} \to X$, its Čech complex formed as derived schemes is *not* in general the same as the one formed as classical schemes. In particular, the restriction of a derived $\mathbf{h}$-sheaf to the category of classical schemes is not in general an h-sheaf.

Our use of the derived $\mathbf{h}$-topology largely owes to the following result of Halpern-Leistner–Preygel [40].

4.2.4. **Lemma.** *The* ∞*-prestack* Perf *satisfies derived* $\mathbf{h}$*-descent.*

Proof. This is [40, Theorem 3.3.1]. □

When we work over a ground field k with $\mathrm{char}(k) = 0$, the result also follows from the derived $\mathbf{h}$-descent of ind-coherent sheaves, established by Gaitsgory [27].

4.2.5. In particular, the standard Tannakian reconstruction argument shows that every quasi-compact derived algebraic stack with affine diagonal satisfies $\mathbf{h}$-descent. This includes the particular cases of $\mathrm{B}_{\text{ét}}H$ for a linear algebraic group H.

[9]The bold character is meant to emphasize topologies defined on derived schemes.

Moreover, $\mathrm{B}^n_{\mathrm{Zar}}\,\mathbb{G}_a$ is an **h**-sheaf for any $n \geq 1$ because:

$$\mathrm{Maps}(X, \mathrm{B}^n_{\mathrm{Zar}}\,\mathbb{G}_a) \xrightarrow{\sim} \tau^{\leq 0}\mathrm{Hom}_{\mathrm{Perf}(X)}(\mathcal{O}_X, \mathcal{O}_X[n]),$$

and we conclude again by Lemma 4.2.4.

4.3. **The "derived" affine Grassmannian.**

4.3.1. Fix a ground field k and a linear algebraic group H. The affine Grassmannian $\mathrm{Gr}_{H,\mathrm{Ran}}$ can be defined directly as a derived prestack over Ran. Namely, its fiber at an S-point of Ran given by maps $x^{(i)} : S \to X$, for $S \in \mathbf{DSch}^{\mathrm{aff}}_{/k}$, is the fiber of the following map of spaces.

$$\mathrm{Maps}(S \times X, \mathrm{B}_{\text{ét}}\, H) \to \mathrm{Maps}(S \times X \backslash \bigcup_{i \in I} \Gamma_{x^{(i)}}, \mathrm{B}_{\text{ét}}\, H).$$

The following Lemma eliminates potential ambiguity.

4.3.2. **Lemma.** *The above definition of* $\mathrm{Gr}_{H,\mathrm{Ran}}$ *yields a classical prestack.*

Proof. Since $\mathrm{Gr}_{H,\mathrm{Ran}}$ is the colimit of Gr_{H,X^I} over $I \in \mathbf{fSet}^{\mathrm{surj}}$ and the inclusion (4.1) preserves colimits, it suffices to show that each Gr_{H,X^I} is classical. The follows from the observation that Gr_{H,X^I} is derived formally smooth ([34, Proposition 9.3.2]) and its restriction to $\mathbf{Sch}^{\mathrm{aff}}_{/k}$ yields an ind-scheme locally of finite type, so [34, Theorem 9.1.6] applies. □

4.3.3. In other words, the Lemma guarantees the classical prestack $\mathrm{Gr}_{H,\mathrm{Ran}}$ of §3, when viewed as a derived prestack by left Kan extension, has the moduli interpretation given above. As a consequence, we see that if $G' := G \underset{H}{\times} H'$ is a fiber product of linear algebraic groups, then the Cartesian diagram of classical prestacks:

$$\begin{tikzcd} \mathrm{Gr}_{G',\mathrm{Ran}} \arrow[r] \arrow[d] & \mathrm{Gr}_{G,\mathrm{Ran}} \arrow[d] \\ \mathrm{Gr}_{H',\mathrm{Ran}} \arrow[r] & \mathrm{Gr}_{H,\mathrm{Ran}} \end{tikzcd}$$

is also derived Cartesian, i.e., a Cartesian diagram in $\mathbf{PStk}_{/k}$. This observation was used in Gaitsgory's construction of the functor $\Xi_{\mathbf{Pic}}$ (to be explained in the next chapter), and we will also apply it in a reduction step in the proof of Chapter 2, Theorem 8.1.2.

CHAPTER 2

Integral metaplectic parameters

In this chapter, we provide a first geometrization of Brylinski–Deligne data via factorization line bundles on the Beilinson–Drinfeld Grassmannian. These can be regarded as integral metaplectic parameters of the Langlands program. More precisely, we will show that the Brylinski–Deligne classification functor $\Psi_{\mathrm{BD},G}$ factors through $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$, and under a mild hypothesis (always satisfied when $\mathrm{char}(k) = 0$, for example), we have equivalences:

$$\mathbf{CExt}(G, \mathbf{K}_2) \cong \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \cong \Theta^+(\Lambda_T; \mathbf{Pic}).$$

Throughout this chapter, we fix a ground field $k = \bar{k}$. The assumption on algebraic closedness can be weakened: all the results we shall prove hold more generally for a perfect field k by Galois descent.

The results of this chapter are proved jointly with J. Tao [62].

5. Tori and Θ-data

In this section, we classify factorization line bundles on the Beilinson–Drinfeld Grassmannian associated to a torus T. We will define a Picard groupoid $\Theta(\Lambda_T; \mathbf{Pic})$, a slight variant of $\Theta^+(\Lambda_T; \mathbf{Pic})$ introduced in §1.2, and the main result is an equivalence of Picard groupoids:

$$\Psi_{\mathbf{Pic},T} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \xrightarrow{\sim} \Theta(\Lambda_T; \mathbf{Pic}). \tag{5.1}$$

In the course of the proof, we will introduce various versions of $\mathrm{Gr}_{T,\mathrm{Ran}}$ and show that factorization line bundles on them are all equivalent to one another.

5.1. **Variants of** $\mathrm{Gr}_{T,\mathrm{Ran}}$.

5.1.1. Suppose T is a torus over k. We let Λ_T denote its co-character lattice. The objects we will introduce can be organized in the following commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}_{T,\mathrm{comb}} \arrow[r] & \mathrm{Gr}_{T,\mathrm{Ran}} \arrow[r] \arrow[d] & \mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T \arrow[d] \\ & \mathrm{Gr}_{T,\mathrm{lax}} \arrow[r] & \mathrm{Gr}_{T,\mathrm{rat}} \end{tikzcd} \tag{5.2}$$

5.1.2. *The combinatorial variant* $\mathrm{Gr}_{T,\mathrm{comb}}$. Consider an index category whose objects are pairs (I, λ^I), where I is a finite set, and λ^I is an I-family of elements in Λ_T (its element corresponding to $i \in I$ is denoted by $\lambda^{(i)}$). A morphism $(I, \lambda^I) \to (J, \lambda^J)$ in this category consists of a *surjective* map $\varphi : I \twoheadrightarrow J$ such that $\lambda^{(j)} = \sum_{i\in\varphi^{-1}(j)} \lambda^{(i)}$ for all $j \in J$. We set:

$$\mathrm{Gr}_{T,\mathrm{comb}} := \operatorname*{colim}_{(I,\lambda^I)} X^I.$$

Then $\mathrm{Gr}_{T,\mathrm{comb}}$ has the structure of a factorization prestack over Ran. Furthermore, we have a canonical map $\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_{T,\mathrm{Ran}}$ sending the copy of X^I corresponding to (I, λ^I) to $\mathrm{Gr}_{T,\mathrm{Ran}}$ along the map t^{λ^I} (3.3). We will write $X^{\lambda^I} = X^{(\lambda^{(1)},\cdots,\lambda^{(|I|)})}$ for the corresponding closed subscheme of $\mathrm{Gr}_{T,\mathrm{Ran}}$.

5.1.3. *The lax variant* $\mathrm{Gr}_{T,\mathrm{lax}}$. Recall that a *lax prestack* is a presheaf on the category of affine k-schemes valued in categories. Let $\mathrm{Ran}_{\mathrm{lax}}$ denote the lax prestack whose category of S-points consists of finite sets of maps $x^{(i)} : S \to X$ and there is a morphism $x^I \to x^J$ whenever $x^I \subset x^J$. We refer the reader to [29, §2] for an introduction to lax prestacks.

We let $\mathrm{Gr}_{T,\mathrm{lax}}$ denote the lax prestack whose value at S is the category whose objects are triples $(x^I, \mathcal{P}_T, \alpha)$ as in $\mathrm{Gr}_T(S)$, but there is a morphism:

$$(x^I, \mathcal{P}_T, \alpha) \to (x^J, \mathcal{P}'_T, \alpha'),$$

whenever $x^I \subset x^J$, $\mathcal{P}_T \xrightarrow{\sim} \mathcal{P}'_T$, and the trivialization α restricts to α' over the complement of $\bigcup_{j\in J}\Gamma_{x^{(j)}}$. Such a morphism is non-invertible when $x^I \subset x^J$ is a proper inclusion.

$\mathrm{Gr}_{T,\mathrm{lax}}$ has the structure of a factorization lax prestack over the lax version of the Ran space $\mathrm{Ran}(X)_{\mathrm{lax}}$, analogously to §3. Furthermore, we have a canonical map $\mathrm{Gr}_{T,\mathrm{Ran}} \to \mathrm{Gr}_{T,\mathrm{lax}}$ sending $(x^I, \mathcal{P}_T, \alpha)$ to the very same object. In fact, this map identifies $\mathrm{Gr}_{T,\mathrm{Ran}}(S)$ as the maximal sub-groupoid of $\mathrm{Gr}_{T,\mathrm{lax}}(S)$.

5.1.4. *The rational variant* $\mathrm{Gr}_{T,\mathrm{rat}}$. We define $\mathrm{Gr}_{T,\mathrm{rat}}$ as a prestack whose value at S is the groupoid of T-bundles $\mathcal{P}_T$ over $S\times X$ equipped with a *rational trivialization*, i.e., for some open $U \subset S\times X$ which is schematically dense after arbitrary base change $S' \to S$, the T-bundle $\mathcal{P}_T$ admits a trivialization over U; we regard two rational trivializations as equivalent if they agree on the overlaps.

Even though $\mathrm{Gr}_{T,\mathrm{rat}}$ does not live over any version of the Ran space, one can still make sense of factorization line bundles (or any other gadget) over $\mathrm{Gr}_{T,\mathrm{rat}}$. Namely, it is a line bundle $\mathcal{L}$ over $\mathrm{Gr}_{T,\mathrm{rat}}$ together with isomorphisms:

$$c_{\mathcal{P}_T^{(1)},\mathcal{P}_T^{(2)}} : \mathcal{L}\big|_{\mathcal{P}_T} \xrightarrow{\sim} \mathcal{L}\big|_{\mathcal{P}_T^{(1)}} \otimes \mathcal{L}\big|_{\mathcal{P}_T^{(2)}},$$

whenever $\mathcal{P}_T^{(1)}$ (resp. $\mathcal{P}_T^{(2)}$) admits a trivialization over $U^{(1)}$ (resp. $U^{(2)}$) such that the complements of $U^{(1)}$ and $U^{(2)}$ are disjoint, and $\mathcal{P}_T$ is the gluing of $\mathcal{P}_T^{(1)}\big|_{U^{(2)}}$ and $\mathcal{P}_T^{(2)}\big|_{U^{(1)}}$ along $U^{(1)}\cap U^{(2)}$, where they are both trivialized. The isomorphisms $c_{\mathcal{P}_T^{(1)},\mathcal{P}_T^{(2)}}$ are required to satisfy the obvious compatibility conditions in the presence of three T-bundles.

5.1.5. **Remark.** The objects $\mathrm{Gr}_{T,\mathrm{lax}}$ and $\mathrm{Gr}_{T,\mathrm{rat}}$ have analogues for a general group G, but we will not use them.

5.1.6. *The divisorial variant* $\mathrm{Div}(X)\underset{\mathbb{Z}}{\otimes}\Lambda_T$. Recall the prestack $\mathrm{Div}(X)$ whose value at S is the abelian group of Cartier divisors of $S\times X$ relative to S. We take $\mathrm{Div}(X)\underset{\mathbb{Z}}{\otimes}\Lambda_T$

as its extension of scalars to Λ_T. There is a morphism $\mathrm{Div}(X) \to \mathrm{Gr}_{\mathbb{G}_m,\mathrm{rat}}$ defined by associating to a Cartier divisor D the line bundle $\mathcal{O}_{S\times X}(D)$. It extends to a morphism $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T \to \mathrm{Gr}_{T,\mathrm{rat}}$.

As in the previous case, we make sense of factorization line bundles over $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T$ as follows. It is a line bundle $\mathcal{L}$ together with isomorphisms:

$$c_{D_1,D_2} : \mathcal{L}\big|_{D_1+D_2} \xrightarrow{\sim} \mathcal{L}\big|_{D_1} \otimes \mathcal{L}\big|_{D_2},$$

whenever the support of D_1 and D_2 are disjoint. The isomorphisms c_{D_1,D_2} are required to satisfy the obvious compatibility conditions for three divisors.

5.2. **Classification by Θ-data.**

5.2.1. We define a variant of the groupoid $\Theta^+(\Lambda_T; \mathbf{Pic})$ introduced in Chapter 1, §1.2. Denote by $\Theta(\Lambda_T; \mathbf{Pic})$ the groupoid of pairs $(q, \mathcal{L}^{(\lambda)})$ where:

(1) $q \in \mathcal{Q}(\Lambda_T, \mathbb{Z})$ is an integral valued quadratic form on Λ_T; we use κ to denote its symmetric bilinear form, defined by the formula:

$$\kappa(\lambda, \mu) := q(\lambda + \mu) - q(\lambda) - q(\mu);$$

(2) $\mathcal{L}^{(\lambda)}$ is a system of line bundles on X parametrized by $\lambda \in \Lambda_T$, equipped with isomorphisms $c_{\lambda,\mu}$:

$$c_{\lambda,\mu} : \mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda+\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)}, \tag{5.3}$$

which are associative, and satisfy the κ-twisted commutativity condition, i.e.

$$c_{\lambda,\mu}(a \otimes b) = (-1)^{\kappa(\lambda,\mu)} \cdot c_{\mu,\lambda}(b \otimes a). \tag{5.4}$$

Clearly, there is an isomorphism, called "ω-shift":

$$\Theta(\Lambda_T; \mathbf{Pic}) \xrightarrow{\sim} \Theta^+(\Lambda_T; \mathbf{Pic}), \quad (q, \mathcal{L}^{(\lambda)}) \rightsquigarrow (q, \mathcal{L}^{(\lambda)} \otimes \omega_X^{q(\lambda)}).$$

We first classify the factorization line bundles on $\mathrm{Gr}_{T,\mathrm{comb}}$.

5.2.2. **Lemma.** *There is a canonical equivalence of Picard groupoids:*

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \xrightarrow{\sim} \Theta(\Lambda_T; \mathbf{Pic}).$$

Proof. Given a factorization line bundle over $\mathrm{Gr}_{T,\mathrm{comb}}$, we denote its pullback along the inclusion $X \to \mathrm{Gr}_{T,\mathrm{comb}}$ corresponding to $(\{1\}, \lambda)$ by $\mathcal{L}^{(\lambda)}$, and its pullback along $X^2 \to \mathrm{Gr}_{T,\mathrm{comb}}$ corresponding to $(\{1,2\}, (\lambda, \mu))$ by $\mathcal{L}^{(\lambda,\mu)}$. The factorization isomorphism shows that there is an isomorphism $\mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)}\big|_{x^2-\Delta} \xrightarrow{\sim} \mathcal{L}^{(\lambda,\mu)}$. It extends to an isomorphism

$$\mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)} \xrightarrow{\sim} \mathcal{L}^{(\lambda,\mu)} \otimes \mathcal{O}_{X^2}(-\kappa(\lambda,\mu)\Delta), \tag{5.5}$$

for some uniquely determined integer $\kappa(\lambda, \mu)$; its dependency on λ, μ is bilinear, by considering $\mathcal{L}^{(\lambda,\mu,\nu)}$ for a triple $(\{1,2,3\}, (\lambda,\mu,\nu))$, using the compatibility between factorization isomorphism and composition. Since $\mathcal{L}^{(\lambda,\mu)}$ restricts to $\mathcal{L}^{(\lambda+\mu)}$ along $\Delta \hookrightarrow X^2$, the isomorphism (5.5) restricts to a system of isomorphisms $c_{\lambda,\mu}$ as in (5.3).

Next, because the factorization isomorphisms are Σ_2-invariant, so are the isomorphisms (5.5). In other words, we have a commutative diagram:

$$\begin{array}{ccc} \mathcal{L}^{(\lambda)} \boxtimes \mathcal{L}^{(\mu)} & \xrightarrow{\sim} & \mathcal{L}^{(\lambda,\mu)} \otimes \mathcal{O}_{X^2}(-\kappa(\lambda,\mu)\Delta) \\ \downarrow{\cong} & & \downarrow{\cong} \\ \sigma^*(\mathcal{L}^{(\mu)} \boxtimes \mathcal{L}^{(\lambda)}) & \xrightarrow{\sim} & \sigma^*\mathcal{L}^{(\mu,\lambda)} \otimes \sigma^*\mathcal{O}_{X^2}(-\kappa(\mu,\lambda)\Delta), \end{array} \tag{5.6}$$

where σ is the isomorphism $X^{(\lambda,\mu)} \xrightarrow{\sim} X^{(\mu,\lambda)}$. One deduces from this fact that κ is also symmetric. Restricting (5.6) to the diagonal, we obtain a commutative diagram:

$$\begin{tikzcd}
\mathcal{L}^{(\lambda)} \otimes \mathcal{L}^{(\mu)} \arrow[r, "c_{\lambda,\mu}"] \arrow[d, "\cong"] & \mathcal{L}^{(\lambda+\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)} \arrow[d, "(-1)^{\kappa(\lambda,\mu)}"] \\
\mathcal{L}^{(\mu)} \otimes \mathcal{L}^{(\lambda)} \arrow[r, "c_{\mu,\lambda}"] & \mathcal{L}^{(\mu+\lambda)} \otimes \omega_X^{\kappa(\mu,\lambda)}
\end{tikzcd}$$

where the multiplication by $(-1)^{\kappa(\lambda,\mu)}$ appears because the isomorphism $\mathcal{O}_{X^2}(-\Delta)\big|_\Delta \xrightarrow{\sim} \omega_X$ is only Σ_2-invariant *up to a sign.* This commutative diagram expresses the identity (5.4). Finally taking $\lambda = \mu$, we see that $(-1)^{\kappa(\lambda,\lambda)} = 1$, so $\kappa(\lambda,\lambda) = 2q(\lambda)$ for an integral quadratic form q on Λ_T.

The above procedure defines the functor $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \to \Theta(\Lambda_T; \mathbf{Pic})$. Checking that it is an equivalence is straightforward. $\square$

5.2.3. We can now state the main result of this section. By pulling back along the morphisms of (5.2), we obtain a diagram of Picard groupoids, where the leftmost equivalence comes from Lemma 5.2.2:

(5.7)

$$\begin{tikzcd}
\Theta(\Lambda_T; \mathbf{Pic}) & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \arrow[l, "\sim"'] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[l] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T) \arrow[l, "(a)"'] \\
 & & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{lax}}) \arrow[u, "(c)"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{rat}}) \arrow[l, "(b)"] \arrow[u]
\end{tikzcd}$$

5.2.4. **Theorem.** *All morphisms in* (5.7) *are equivalences.*

Proof. We shall deduce from existing literature how each of the labeled maps is an equivalence. First, by [8, §3.10.7, Proposition], the composition of the top row defines an equivalence:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T) \xrightarrow{\sim} \theta(\Lambda_T).$$

This shows that the map (a) has a left inverse.

By [5, Proposition 5.2.2], the map $\mathrm{Gr}_{T,\mathrm{lax}} \to \mathrm{Gr}_{T,\mathrm{rat}}$ induces an equivalence after fppf sheafification. Hence pulling back defines an equivalence

$$\mathbf{Pic}(\mathrm{Gr}_{T,\mathrm{rat}}) \xrightarrow{\sim} \mathbf{Pic}(\mathrm{Gr}_{T,\mathrm{lax}}).$$

One immediately checks that the additional data defining factorization structures on both are also equivalent. Hence (b) is an equivalence.

By [74, Theorem 4.3.9(2)], pulling back along $\mathrm{Gr}_T \to \mathrm{Gr}_{T,\mathrm{rat}}$ defines an equivalence on *rigidified* line bundles. On the other hand, every factorization line bundle on Gr_T pulls back to one along the unit section $\mathrm{Ran}(X) \to \mathrm{Gr}_T$, which is canonically trivial by Lemma 5.2.2 (applied to the trivial group). Thus a factorization line bundle on Gr_T descends to a line bundle on $\mathrm{Gr}_{T,\mathrm{rat}}$, and the result has a canonical factorization structure as well, so we have an equivalence

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{rat}}) \xrightarrow{\sim} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}).$$

This shows that (c) is an equivalence. The undecorated maps in (5.7) are now equivalences by the 2-out-of-3 property. $\square$

5.2.5. **Remark.** The cited result [74, Theorem 4.3.9(2)] contains a gap in the proof. Namely, it relies on the unproved fact that line bundles on $S \underset{k}{\times} \mathrm{Ran}$ canonically descend to S. The latter follows instead from Tao's general contractibility theorem [61].

5.2.6. **Remark.** When X is proper, [12, Theorem 2.3.3] shows that the map $\mathrm{Div}(X) \underset{\mathbb{Z}}{\otimes} \Lambda_T \to \mathrm{Gr}_{T,\mathrm{rat}}$ is an isomorphism of prestacks, which immediately implies that factorization line bundles on them are equivalent.

5.2.7. **Remark.** We have the following equivalence for any smooth, fiberwise connected, affine group scheme $\mathbf{G}$ over X:

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\mathbf{G},\mathrm{rat}}) \xrightarrow{\sim} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\mathbf{G},\mathrm{lax}}) \xrightarrow{\sim} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{\mathbf{G}}).$$

6. The classification functor $\Psi_{\mathbf{Pic}}$

In this section, we first classify factorization line bundles on $\mathrm{Gr}_{G,\mathrm{Ran}}$ for G a semisimple, simply connected group. Then we use this result to build the classification functor $\Psi_{\mathbf{Pic}}$ for any reductive group.

6.1. **Quadratic form and determinant.**

6.1.1. We shall first note some constructions valid for any reductive group G over k. We fix a maximal torus and a Borel subgroup $T \subset B \subset G$. Pulling back along the closed immersion $\mathrm{Gr}_{T,\mathrm{Ran}} \hookrightarrow \mathrm{Gr}_{G,\mathrm{Ran}}$ and appealing to the classification of $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}})$ by Θ-data (Theorem 5.2.4), we obtain a functor:

$$\mathbf{Q}_{\mathbf{Pic},G} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathcal{Q}(\Lambda_T;\mathbb{Z}). \tag{6.1}$$

Namely, we can attach an integral quadratic form to any factorization line bundle on $\mathrm{Gr}_{G,\mathrm{Ran}}$.

6.1.2. Let us also recall the standard construction of determinant line bundles, following [32]. Let $\mathbf{S}$ denote the set of simple factors of $\widetilde{G}_{\mathrm{der}}$. Then for each $s \in \mathbf{S}$, the corresponding Lie algebra $\mathfrak{g}_s$ can be regarded as a G-representation. Consequently, we may define a line bundle $\det_{\mathfrak{g}_s}$ over $\mathrm{Gr}_{G,\mathrm{Ran}}$ by specifying its fiber at an S-point $(x^{(i)}, \mathcal{P}_G, \alpha)$ to be the relative determinant of the vector bundles $(\mathfrak{g}_s)_{\mathcal{P}_G}$ and $(\mathfrak{g}_s)_{\mathcal{P}_G^0}$, identified outisde $\bigcup_{i\in I}\Gamma_{x^{(i)}}$. Then $\det_{\mathfrak{g}_s}$ has the canonical structure of a factorization line bundle over $\mathrm{Gr}_{G,\mathrm{Ran}}$ (c.f. [32, §5.2]). Thus we have a map:

$$\det : \mathrm{Maps}(\mathbf{S}, \mathbb{Z}) \to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}), \quad \mathbf{1}_s \rightsquigarrow \det{}_{\mathfrak{g}_s}. \tag{6.2}$$

6.1.3. **Lemma.** *For each $s \in \mathbf{S}$, the quadratic form $\mathbf{Q}_{\mathbf{Pic},G}(\det_{\mathfrak{g}_s})$ is the Killing form associated to s, defined by the formula:*

$$q_{\det,s}(\lambda) := \frac{1}{2} \sum_{\check{\alpha} \in \Phi_s} \langle \check{\alpha}, \lambda \rangle^2.$$

Proof. Indeed, the pullback of $\det{}_{\mathfrak{g}_s}$ to $\mathrm{Gr}_{T,\mathrm{Ran}}$ is the determinant line bundle associated to the T-representation $\mathfrak{n}_s \oplus \mathfrak{n}_s^-$ which has weights Φ_s. The Lemma then follows from unwinding the definitions. □

6.2. **Semisimple, simply connected groups.**

6.2.1. Let us now further assume $G = G_{\mathrm{sc}}$ to be semisimple and simply connected. We continue to fix the data $T_{\mathrm{sc}} \subset B_{\mathrm{sc}} \subset G_{\mathrm{sc}}$. Let W be the Weyl group of $(G_{\mathrm{sc}}, T_{\mathrm{sc}})$.

6.2.2. An important tool in analyzing line bundles over $\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}$ is an exact sequence relating line bundles over X^I with those defined away from the diagonal. Let $\mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I}$ denote the (small) étale sheaf on X^I which associates to $S \to X^I$ the abelian group of line bundles on $\mathrm{Gr}_{G_{\mathrm{sc}},X^I} \underset{X^I}{\times} S$ trivialized over the unit section e.

6.2.3. Let B denote the abelian group $\mathrm{Maps}(\mathbf{S}, \mathbb{Z})$ and $\underline{B}_X$ its étale sheafification over X. Since B is equivalent to rigidified line bundles on $\mathrm{Gr}_{G_{\mathrm{sc}},x}$ for any k-point $x \in X$, we have a map:

$$\mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I} \to \boxtimes_{i \in I} \underline{B}_X$$

defined by restriction away from all diagonals and applying the product decomposition (Lemma 2.3.6 and 2.2.10).

6.2.4. **Lemma.** *The above procedure defines an exact sequence:*

$$0 \to \mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I} \to \boxtimes_{i\in I}\underline{B}_X \to \bigoplus_{\substack{I\twoheadrightarrow J\\ |J|=|I|-1}} (\Delta_{I\twoheadrightarrow J})_* \boxtimes_{j\in J} \underline{B}_X, \tag{6.3}$$

where the second map is given by taking difference along each diagonal $\Delta_{I\twoheadrightarrow J}$.

Proof. For G_{sc} *simple* and simply connected, this is [74, Lemma 3.4.3] and is valid over a Schubert stratum of $\mathrm{Gr}_{G_{\mathrm{sc}},X^I}$. The semisimple case follows from the product decomposition. $\square$

6.2.5. **Theorem.** *The functor* $\mathbf{Q}_{\mathbf{Pic},G_{\mathrm{sc}}}$ *defines an equivalence:*

$$\Psi_{\mathbf{Pic},G_{\mathrm{sc}}} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \xrightarrow{\sim} \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, \mathbb{Z})^W.$$

Proof. Since factorization line bundles on Ran are canonically trivial, restriction to $x \in X$ factors through line bundles rigidified at the unit section. We have a composition:

$$\begin{aligned}\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) &\xrightarrow{(a)} \mathbf{Pic}^e(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}})\\ &\xrightarrow{(b)} \mathbf{Pic}^e(\mathrm{Gr}_{G_{\mathrm{sc}},x}) \xrightarrow{\sim} \mathrm{Maps}(\mathbf{S}, \mathbb{Z}),\end{aligned}$$

where the last isomorphism is due to Theorem 2.3.9. On the other hand, the exact sequence (6.3) shows that arrows (a) and (b) are also equivalences.

Finally, we need to prove that the above equivalence makes the following diagram commute:

$$\begin{tikzcd} \mathrm{Maps}(\mathbf{S},\mathbb{Z}) \arrow[r, "\sim"] \arrow[d, "(c)"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \arrow[d, "\mathbf{Q}_{G_{\mathrm{sc}}}"] \\ \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}},\mathbb{Z})^W \arrow[r, hook] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}},\mathbb{Z}) \end{tikzcd}$$

Here, the vertical map (c) sends $\mathbf{1}_s$ to the minimal quadratic form $q_{\min,s}$ for the factor s, i.e., its value at any short coroot in Φ_s is 1 and vanishes on all components other than s. To check the commutativity, we start with $2\check{h}_s \cdot \mathbf{1}_s$ where $\check{h}_s$ is the dual Coxeter number of the simple group G_s. The vertical map sends it to $q_{\mathrm{det},s} = 2\check{h}_s \cdot q_{\mathrm{min},s}$. On the other hand, we have an identification $\det_{\mathfrak{g}_s} \cong \mathrm{min}_s^{\otimes 2\check{h}_s}$ in $\mathbf{Pic}(\mathrm{Gr}_{G_{\mathrm{sc}},x})$ and the quadratic form attached to $\det_{\mathfrak{g}_s}$ is also $q_{\mathrm{det},s}$, by Lemma 6.1.3. $\square$

6.2.6. In particular, one may use Theorem 6.2.5 to produce a functor:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \arrow[d] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, \mathbb{Z})^W \arrow[l, "\Psi^{-1}_{\mathbf{Pic},G_{\mathrm{sc}}}"'] \\ \Theta(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic}) & \end{tikzcd} \tag{6.4}$$

6.3. Construction of $\Psi_{\mathbf{Pic},G}$.

6.3.1. We now return to the general case of a reductive group G. Let us define the groupoid of *enhanced* Θ*-data* for line bundles. Indeed, let $\Theta_G(\Lambda_T; \mathbf{Pic})$ to be category of triples $(q, \mathcal{L}^{(\lambda)}, \varepsilon)$, where $(q, \mathcal{L}^{(\lambda)})$ is an object of $\Theta(\Lambda_T; \mathbf{Pic})$, and ε is an isomorphism between the following objects of $\Theta(\Lambda_{\widetilde{T}_{\mathrm{der}}}; \mathbf{Pic})$:

(1) The restriction of $(q, \mathcal{L}^{(\lambda)})$ to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$;
(2) The object associated *via* (6.4) to the restriction of q to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$.

Then we have a functor:

$$\Psi_{\mathbf{Pic},G} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \Theta_G(\Lambda_T; \mathbf{Pic}),$$

obtained as follows. The datum $(q, \mathcal{L}^{(\lambda)})$ arises from pulling back along $\mathrm{Gr}_{T,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ and applying $\Psi_{\mathbf{Pic},T}$. The isomorphism ε comes from the canonicity of the classification for tori, as we restrict a factorization line bundle along the two circuits of the commutative diagram:

$$\begin{tikzcd} \mathrm{Gr}_{\widetilde{T}_{\mathrm{der}},\mathrm{Ran}} \arrow[r] \arrow[d] & \mathrm{Gr}_{\widetilde{G}_{\mathrm{der}},\mathrm{Ran}} \arrow[d] \\ \mathrm{Gr}_{T,\mathrm{Ran}} \arrow[r] & \mathrm{Gr}_{G,\mathrm{Ran}} \end{tikzcd}$$

6.3.2. In the next section, we shall compare the map (6.4) with its counterpart (1.5) for K-theory. It will then allows us to compare $\Theta_G(\Lambda_T; \mathbf{Pic})$ with the object $\Theta_G^+(\Lambda_T; \mathbf{Pic})$ defined in §1.2.

7. Interaction with Brylinski–Deligne data

In this section, we recall a functor $\Xi_{\mathbf{Pic},G}$ defined by D. Gaitsgory [31] which goes from Brylinski–Deligne data to factorization line bundles on $\mathrm{Gr}_{G,\mathrm{Ran}}$, subject to a condition on G. Our goal is to show that "ω-shift" induces an isomorphism:

$$\Theta_G(\Lambda_T; \mathbf{Pic}) \xrightarrow{\sim} \Theta_G^+(\Lambda_T; \mathbf{Pic}),$$

and furthermore, when Gaitsgory's functor is defined, the following diagram is canonically commutative:

$$\begin{tikzcd} \mathbf{CExt}(G, \mathbf{K}_2) \arrow[r, "\Xi_{\mathbf{Pic},G}"] \arrow[d, "\Psi_{\mathrm{BD},G}"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{Pic},G}"] \\ \Theta_G^+(\Lambda_T; \mathbf{Pic}) \arrow[r, "\omega\text{-shift}"] & \Theta_G(\Lambda_T; \mathbf{Pic}) \end{tikzcd}$$

7.1. The construction functor $\Xi_{\mathbf{Pic},G}$.

7.1.1. Let G be a reductive group over k with maximal torus T. For any finite-dimensional representation $G \to G_V := \mathrm{GL}(V)$, one can consider the induced map on quadratic forms:

$$\mathcal{Q}(\Lambda_{T_V}; \mathbb{Z})^{W_V} \to \mathcal{Q}(\Lambda_T; \mathbb{Z})^W.$$

Let $\mathcal{Q}_1 \subset \mathcal{Q}(\Lambda_T;\mathbb{Z})^W$ be the subgroup generated by their images as we vary V. It is easy to see that the inclusion of $\mathcal{Q}_1$ becomes bijective after tensoring with $\mathbb{Q}$, so it is a subgroup of finite index. We define N_G as the smallest integer which annihilates the quotient.

7.1.2. The main result of [31] is that when $\mathrm{char}(k) \nmid N_G$, there is a functor of Picard groupoids:

$$\Xi_{\mathbf{Pic},G} : \mathbf{CExt}(G, \mathbf{K}_2) \to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

where $\mathbf{CExt}(G, \mathbf{K}_2)$ is the Picard groupoid of Brylinski–Deligne data of §1.2. Even though the definition of $\Xi_{\mathbf{Pic},G}$ is somewhat involved, it is easy to describe the image line bundle over a *regular* test scheme $S \to \mathrm{Gr}_{G,\mathrm{Ran}}$.

7.1.3. Indeed, let S be a regular affine scheme over k and $\pi : \mathfrak{X} \to S$ be a smooth relative curve with connected fibers; although we shall apply the construction to $\mathfrak{X} = S \times X$, this generality is needed for a later manipulation. Suppose we have a finite set x^I of sections $x^{(i)} : S \to \mathfrak{X}$. Let Γ_{x^I} denote the *scheme-theoretic* union of their images, and $U_{x^I} := \mathfrak{X} - \Gamma_{x^I}$ be its complement.

7.1.4. *Taking residue.* We will first construct a functor between Picard groupoids, referred to hereafter as *taking residue* along π:

$$\mathrm{Res}_\pi : \left\{ \begin{matrix} \text{Zariski } \mathbf{K}_2\text{-gerbes } \mathcal{G} \text{ on } \mathfrak{X} \text{ with} \\ \text{trivialization } \gamma \text{ over } U_{x^I} \end{matrix} \right\} \to \mathbf{Pic}(S). \tag{7.1}$$

Indeed, the datum $(\mathcal{G}, \gamma)$ is equivalent to a section of $\iota^!\mathbf{K}_2[2]$ over $\mathfrak{X}$, where $\iota : \Gamma_{x^I} \hookrightarrow \mathfrak{X}$ is the closed immersion. On the other hand, the Gersten resolution of $\mathbf{K}_2$ on $\mathfrak{X}$ shows that $\iota^!\mathbf{K}_2[2]$ is quasi-isomorphic to the following complex concentrated in degrees $[-1, 0]$:

$$\bigoplus_{i \in I} (\iota_{\eta^{(i)}})_* K_1(\eta) \to \bigoplus_{\substack{\mathrm{codim}(\nu)=1 \\ \text{in } \Gamma_{x^I}}} (\iota_\nu)_* \mathbb{Z}, \tag{7.2}$$

where $\iota_{\eta^{(i)}}$ (resp. ι_ν) denotes the inclusion of the generic point of the ith section (resp. codimension-one point ν of Γ_{x^I}). On the other hand, $\mathbf{K}_1[1]$ over S is quasi-isomorphic to:

$$(\iota_\eta)_* K_1(\eta) \to \bigoplus_{\substack{\mathrm{codim}(\nu)=1 \\ \text{in } S}} (\iota_\nu)_* \mathbb{Z}.$$

Thus the image of (7.2) under π maps to $\mathbf{K}_1[1]$ via summation. Hence a section of $\iota^! \mathbf{K}_2[2]$ over $\mathfrak{X}$ gives rise to a section of $\mathbf{K}_1[1] \cong \mathcal{O}_S^\times[1]$, i.e., a line bundle on S.

7.1.5. Given a Brylinski–Deligne datum $\mathbf{E}$ and a map $S \to \mathrm{Gr}_G$ with S a regular affine scheme over k, we shall produce a line bundle $\mathcal{L}$ over S. Indeed, suppose the map is specified by the triple $(x^I, \mathcal{P}_G, \alpha)$. The Drinfeld–Simpson theorem [21] shows that $\mathcal{P}_G$ is Zariski locally trivial over an étale cover of S. Thus by étale descent of line bundles, we may assume $\mathcal{P}_G$ to be Zariski locally trivial on $S \times X$.

7.1.6. Thus, we obtain a Zariski $\mathbf{K}_2$-gerbe $\mathcal{G}$ over $S \times X$ by inducing along the map $\mathrm{B}_{\mathrm{Zar}}\, G \to \mathrm{B}^2_{\mathrm{Zar}}\, \mathbf{K}_2$. Concretely, $\mathcal{G}$ classifies an $\mathbf{E}$-torsor $\mathcal{P}_{\mathbf{E}}$ equipped with an identification of its induced G-torsor $(\mathcal{P}_{\mathbf{E}})_G \xrightarrow{\sim} \mathcal{P}_G$. The trivialization α of $\mathcal{P}_G$ gives rise to a trivialization γ of $\mathcal{G}$ over U_{x^I}. Then $(\mathcal{G}, \gamma)$ produces a line bundle on S by taking residue (7.1) along $\pi : S \times X \to S$. This construction yields the composition:

$$\begin{aligned} \mathbf{CExt}(G, \mathbf{K}_2) &\xrightarrow{\Xi_{\mathbf{Pic},G}} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \\ &\xrightarrow{(x^I, \mathcal{P}_G, \alpha)^*} \mathbf{Pic}(S). \end{aligned}$$

7.1.7. For general S, Gaitsgory defines an analogue of the residue construction (7.1) when we replace $\mathbf{K}_2$ by $\mathbf{K}_{\geq 2}$. More precisely, write $\iota : \Gamma_{x^I} \to S \times X$ and $\pi : \Gamma_{x^I} \to S$ for the obvious maps. Then have a morphism, where the sheaves are valued in $\mathbb{E}_\infty$-spaces ([31, §3.2.2]):

$$\pi_* \iota^! \mathbf{K}_{\geq 2} \to \mathbf{Pic}. \tag{7.3}$$

This is possible by virtue of the existence of the direct image functor from perfect complexes on $S \times X$ with set-theoretic support on Γ_{x^I} to perfect complexes on S.

7.1.8. The essential difficulty, however, is that a Brylinski–Deligne datum may not lift:

$$\begin{tikzcd} & \mathrm{B}^2_{\mathrm{Zar}}\,\mathbf{K}_{\geq 2} \arrow[d] \\ X \times \mathrm{B}_{\mathrm{Zar}}\,G \arrow[r] \arrow[ur, dotted] & \mathrm{B}^2_{\mathrm{Zar}}\,\mathbf{K}_2 \end{tikzcd}$$

Roughly speaking, this problem is overcome by proving that for $G = \mathrm{GL}_n$, a lift of the "absolute" Brylinski–Deligne datum $\mathrm{B}_{\mathrm{Zar}}\,G \to \mathrm{B}^2_{\mathrm{Zar}}\,\mathbf{K}_2$ always exists. Therefore for general G, the N_G-multiple of an absolute Brylinski–Deligne datum lifts. Showing that a factorization structure on the N_G-power of a line bundle $\mathcal{L}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ induces a factorization structure on $\mathcal{L}$ then requires $\mathrm{char}(k) \nmid N_G$.

7.1.9. We note another fact about the functor $\Xi_{\mathbf{Pic},G}$. Recall that a *multiplicative* factorization line bundle $\mathcal{L}$ on $\mathcal{L}_{\mathrm{Ran}}G$ means the additional datum of an isomorphism:

$$\mathrm{mult}^*\,\mathcal{L} \xrightarrow{\sim} \mathcal{L} \boxtimes \mathcal{L} \tag{7.4}$$

over $\mathcal{L}_{\mathrm{Ran}}G \underset{\mathrm{Ran}}{\times} \mathcal{L}_{\mathrm{Ran}}G$ that satisfies the cocycle condition on the triple product, and is compatible with the factorization isomorphisms on both sides.

We let $\mathbf{Pic}^{\mathrm{fact},\times}(\mathcal{L}_{\mathrm{Ran}}G)$ (resp. $\mathbf{Pic}^{\mathrm{fact},\times}_{/\mathcal{L}^+_{\mathrm{Ran}}G}(\mathcal{L}_{\mathrm{Ran}}G)$) denote the Picard groupoid of multiplicative factorization line bundles on $\mathcal{L}_{\mathrm{Ran}}G$ (resp. equipped with a trivialization *as such* over $\mathcal{L}^+_{\mathrm{Ran}}G$). Clearly, there is a functor by étale descent:

$$\mathbf{Pic}^{\mathrm{fact},\times}_{/\mathcal{L}^+_{\mathrm{Ran}}G}(\mathcal{L}_{\mathrm{Ran}}G) \to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{fact}}). \tag{7.5}$$

The fact we will need pertains only to a torus T. Note that $N_T = 1$, so the functor $\Xi_{\mathbf{Pic},T}$ exists unconditionally.

7.1.10. **Lemma.** *The functor* $\Xi_{\mathbf{Pic},T}$ *canonically factors through* (7.5)*:*

$$\Xi_{\mathbf{Pic},T} : \mathbf{CExt}(T, \mathbf{K}_2) \xrightarrow{\Xi^{\times}_{\mathbf{Pic},T}} \mathbf{Pic}^{\mathrm{fact},\times}_{/\mathcal{L}^{+}_{\mathrm{Ran}}T}(\mathcal{L}_{\mathrm{Ran}}T)$$
$$\to \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}).$$

Proof. Interpret $\mathbf{E}$ as a pointed morphism $e : X \times \mathrm{B}_{\mathrm{Zar}}\, T \to \mathrm{B}^2_{\mathrm{Zar}}\, \mathbf{K}_2$. Then e lifts non-canonically to some $\tilde{e} : X \times \mathrm{B}_{\mathrm{Zar}}\, T \to \mathbf{K}_{\geq 2}$ ([31, §5.3.1]). Hence the data $(x^I, \mathcal{P}_T, \alpha)$ of an S-point of $\mathrm{Gr}_{T,\mathrm{Ran}}$ (where we may again assume $\mathcal{P}_T$ to be Zariski-locally trivial) give us a section of $\mathbf{K}_{\geq 2}$ over $S \times X$ with support on Γ_{x^I}. The line bundle $\mathcal{L}_{\tilde{e}}\big|_S$ is then constructed using the map (7.3). For two lifts $\tilde{e}$ and $\tilde{e}'$, we need to produce a canonical isomorphism $\mathcal{L}_{\tilde{e}} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}$. This is done as follows:

(1) for S the spectrum of an Artinian k-algebra, (7.3) factors through $\pi_*\iota^!\mathbf{K}_2$, so we obtain a *canonical* isomorphism $\mathcal{L}_{\tilde{e}}\big|_S \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}\big|_S$;
(2) there exists an isomorphism $\mathcal{L}_{\tilde{e}} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}$ which restricts to the one in (1) for any S the spectrum of an Artinian k-algebra (§5.3.4-6 of *loc.cit.*), which then must be uniquely determined.

We now claim that $\mathcal{L}_{\tilde{e}}\big|_{\mathcal{L}_{\mathrm{Ran}}T}$ acquires a canonical multiplicative structure. Indeed, $\tilde{e}$ induces a morphism $X \times T \to \mathrm{B}_{\mathrm{Zar}}\, \mathbf{K}_{\geq 2}$. Given S-points t, t' of $\mathcal{L}_{\mathrm{Ran}}T$ over the same point x^I in Ran, we may view them both as maps from $\mathring{D}_{x^I}$ to $X \times T$. There is a canonical homotopy between $\tilde{e}(t) + \tilde{e}(t')$ and $\tilde{e}(tt')$ as maps from $\mathring{D}_{x^I}$ to $\mathrm{B}_{\mathrm{Zar}}\, \mathbf{K}_{\geq 2}$. Under the map $\mathrm{B}_{\mathrm{Zar}}\, \mathbf{K}_{\geq 2}\big|_{\mathring{D}_{x^I}} \to \iota^!\mathbf{K}_{\geq 2}$ of sheaves over D_{x^I}, we obtain a canonical homotopy between the corresponding sections of $\iota^!\mathbf{K}_{\geq 2}$. It gives rise to the desired multiplicative structure $\mathcal{L}_{\tilde{e}}\big|_t \otimes \mathcal{L}_{\tilde{e}}\big|_{t'} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}}\big|_{tt'}$ under (7.3).

It remains to check that for two lifts $\tilde{e}$ and $\tilde{e}'$, the isomorphism $\mathcal{L}_{\tilde{e}} \xrightarrow{\sim} \mathcal{L}_{\tilde{e}'}$ is compatible with the multiplicative structures on both sides. This amounts to checking that the following diagram of line bundles over $\mathcal{L}_{\mathrm{Ran}}T \underset{\mathrm{Ran}}{\times} \mathcal{L}_{\mathrm{Ran}}T$ commutes:

$$\begin{tikzcd} \mathrm{mult}^* \mathcal{L}_{\tilde{e}} \arrow[r] \arrow[d] & \mathcal{L}_{\tilde{e}} \boxtimes \mathcal{L}_{\tilde{e}} \arrow[d] \\ \mathrm{mult}^* \mathcal{L}_{\tilde{e}'} \arrow[r] & \mathcal{L}_{\tilde{e}'} \boxtimes \mathcal{L}_{\tilde{e}'}. \end{tikzcd}$$

It suffices to test the commutativity over S the spectrum of an Artinian k-algebra. Note again that for such S, (7.3) factors through $\pi_*\iota^!\mathbf{K}_2$, so the construction of the multiplicative structure is defined indepentently of any lift. Therefore, we have equipped $\mathcal{L}\big|_{\mathcal{L}_{\mathrm{Ran}}T}$ with a canonical multiplicative structure. $\square$

7.1.11. Consider a k-point $x \in X$ and denote by $\mathbf{F}_x$ the field of fractions of $\mathbf{O}_x := \widehat{\mathcal{O}}_{X,x}$. From the description of $\Xi_{\mathbf{Pic},T}(\mathbf{E})$ on regular test schemes, we see that $\Xi^{\times}_{\mathbf{Pic},T}(\mathbf{E})\big|_t$ for $t \in T(\mathbf{F}_x)$ agrees with the $k^{\times}$-torsor induced from the central extension $\mathbf{E}(\mathbf{F}_x)$ along the residue map $\mathbf{K}_2(\mathbf{F}_x) \to k^{\times}$. Furthermore, this identification respects the multiplicative structures. In other words, we have a push-out of abstract groups:

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{K}_2(\mathbf{F}_x) \arrow[r] \arrow[d] & \mathbf{E}(\mathbf{F}_x) \arrow[r] \arrow[d] & T(\mathbf{F}_x) \arrow[r] & 0 \\ & k^{\times} \arrow[r] & \Xi^{\times}_{\mathbf{Pic},T}(\mathbf{E})\big|_x(k) & & \end{tikzcd} \tag{7.6}$$

The same pointwise description remains valid for a k'-point where k' is any field extension of k, not necessarily algebraic.

7.2. **Compatibility diagram for tori.**

7.2.1. Fix a torus T. Recall the classification functors $\Psi_{\mathrm{BD},T}$ (§1.2) and $\Psi_{\mathbf{Pic},T}$ (§5.2), as well as the ω-shift. Our current goal is to prove the following compatibility statement.

7.2.2. **Proposition.** *The following diagram is canonically commutative:*

$$\begin{tikzcd} \mathbf{CExt}(T, \mathbf{K}_2) \arrow[r, "\Xi_{\mathbf{Pic},T}"] \arrow[d, "\Psi_{\mathrm{BD},T}"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{Pic},T}"] \\ \Theta^+(\Lambda_T; \mathbf{Pic}) \arrow[r, "\omega\text{-shift}"] & \Theta(\Lambda_T; \mathbf{Pic}) \end{tikzcd}$$

The proof will occupy the rest of this subsection.

7.2.3. *Notations.* Fix an object $\mathbf{E}$ of $\mathbf{CExt}(T, \mathbf{K}_2)$. We denote its image in $\Theta^+(\Lambda_T; \mathbf{Pic})$ under $\Psi_{\mathrm{BD},T}$ by $(q, \mathcal{L}_+^{(\lambda)})$, with multiplicative structure $c^+_{\lambda,\mu}$, and its image under $\Xi_{\mathbf{Pic},T}$ by $\mathcal{L}$. The image of $\mathcal{L}$ in $\Theta(\Lambda_T; \mathbf{Pic})$ will be denoted by $(q', \mathcal{L}^{(\lambda)})$, with multiplicative structure $c_{\lambda,\mu}$. We need to establish the following statements.

(1) $q = q'$;

(2) there is a canonical system of isomorphisms:

$$\mathcal{L}_+^{(\lambda)} \xrightarrow{\sim} \mathcal{L}^{(\lambda)} \otimes \omega_X^{q(\lambda)} \tag{7.7}$$

which respects $c^+_{\lambda,\mu}$ and $c_{\lambda,\mu}$.

7.2.4. *Quadratic form.* First, we show $q = q'$ by checking that their bilinear forms κ and κ' agree. Fixing a closed point $x \in X$ and any co-character $\mu \in \Lambda_T$, we will show that $\kappa(-, \mu)$ and $\kappa'(-, \mu)$ define the same character $T(k') \to \mathbb{G}_m(k')$ for every field extension $k \subset k'$. This will imply that $\kappa = \kappa'$. Indeed, for every $\lambda \in \Lambda_T$, suppose $z \rightsquigarrow z^{\kappa(\lambda,\mu)}$ and $z \rightsquigarrow z^{\kappa'(\lambda,\mu)}$ define the same map $\mathbb{G}_m(k') \to \mathbb{G}_m(k')$ for all field extension $k \subset k'$. By suitably choosing k', we can ensure that $(k')^\times$ contains an element of infinite order. Thus $\kappa(\lambda, \mu)$ agrees with $\kappa'(\lambda, \mu)$.

We now further fix a uniformizer t of the completed local ring $t \in \widehat{\mathcal{O}}_{X,x}$, so we regard t^μ as an element of $T(\mathbf{F}_x)$. Evaluating the central extension $\mathbf{E}$ at $\mathbf{F}_x$, and pushing out along the residue map $\mathbf{K}_2(\mathbf{F}_x) \to k^\times$, we obtain central extension:

$$0 \to k^\times \to \mathbf{E}' \to T(\mathbf{F}_x) \to 0.$$

The conjugation action of $T(\widehat{\mathcal{O}}_{X,x})$ on the fiber of $\mathbf{E}(\mathbf{F}_x) \to T(\mathbf{F}_x)$ at t^μ induces a map:

$$T(\widehat{\mathcal{O}}_{X,x}) \to k^\times. \tag{7.8}$$

We will calculate this map (and its variant for a field extension $k \subset k'$) in two ways.

Step 1. We first show that the map (7.8) is given by the composition:

$$T(\widehat{\mathcal{O}}_{X,x}) \xrightarrow{\mathrm{ev}} T(k) \xrightarrow{\kappa(-,\mu)} k^\times.$$

Indeed, recall (§1.2) that the composition

$$\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m \xrightarrow{\lambda\otimes\mu} T \underset{\mathbb{Z}}{\otimes} T \xrightarrow{[-,-]} \mathbf{K}_2$$

is the $\kappa(\lambda,\mu)$-multiple of the product map on K-theory. Thus the map:

$$\mathbb{G}_m(\mathbf{F}_x) \underset{\mathbb{Z}}{\otimes} \mathbb{G}_m(\mathbf{F}_x) \xrightarrow{\lambda\otimes\mu} T(\mathbf{F}_x) \underset{\mathbb{Z}}{\otimes} T(\mathbf{F}_x) \xrightarrow{[-,-]} \mathbf{K}_2(\mathbf{F}_x) \xrightarrow{\mathrm{res}} k^\times$$

is the $\kappa(\lambda,\mu)$-multiple of the Contou-Carrère symbol $\{f,g\} := (f^{\mathrm{ord}(g)}/g^{\mathrm{ord}(f)})(0)$. Hence the conjugation action of $f \in \mathbb{G}_m(\widehat{\mathcal{O}}_{X,x})$ on $\mathbf{E}'$ through the character λ is given by $e' \rightsquigarrow \{f,t\}^{\kappa(\lambda,\mu)}e'$. Note that $\{f,t\} = f(0)$, as required.

For a field extension $k \subset k'$, the above computation is valid without modification.

Step 2. We now calculate the map (7.8) alternatively as follows. Recall the canonical multiplicative structure on $\mathcal{L}\big|_{\mathcal{L}_{\mathrm{Ran}}T}$ from Lemma 7.1.10. It induces a *strong* $\mathcal{L}^+_{\mathrm{Ran}}T$-equivariance structure on $\mathcal{L}$ (c.f. [32, §7.3.4]) with respect to the trivial left $\mathcal{L}^+_{\mathrm{Ran}}T$-action. In particular over X^2, the twisted product $\mathcal{L}\widetilde{\boxtimes}\mathcal{L}$ on the convolution Grassmannian $\widetilde{\mathrm{Gr}}_{T,X^2}$ is identified with the pullback of $\mathcal{L}\big|_{\mathrm{Gr}_{T,X^2}}$ along the action map $\widetilde{\mathrm{Gr}}_{T,X^2} \to \mathrm{Gr}_{T,X^2}$, in a way that is compatible with the factorization structure of $\mathcal{L}$.

Furthermore, its value at $t^\mu \in \mathrm{Gr}_{T,x}$ is given by the conjugation action (7.8), according to the push-out diagram (7.6). Consequently, the map (7.8) is given by

$$T(\widehat{\mathcal{O}}_{X,x}) \xrightarrow{\mathrm{ev}} T(k) \xrightarrow{\kappa'(-,\mu)} k^\times.$$

Indeed, this is because for a factorization line bundle $\mathcal{L}$ on $\mathrm{Gr}_{T,\mathrm{Ran}}$ with associated bilinear form κ', every strong $\mathcal{L}^+_{\mathrm{Ran}}T$-equivariance structure acts on $t^\mu \in \mathrm{Gr}_{T,x}$ through the composition $\mathcal{L}^+_x T \xrightarrow{\mathrm{ev}} T \xrightarrow{\kappa'(-,\mu)} \mathbb{G}_m$ (c.f. [32, §7.4]).

Again for a field extension $k \subset k'$, the above computation holds without modification. This finishes the proof that $\kappa = \kappa'$.

7.2.5. *Isomorphisms of line bundles.* We now construct the isomorphisms (7.7). The strategy is to first canonically identify $\mathcal{L}^{(\lambda)}$ with the twist of $\mathcal{L}^{(\lambda)}_+$ by some power of the tangent sheaf $\mathcal{T}_X$, and then determine this power.

Step 1. Consider the diagonal embedding $\Delta : X \hookrightarrow X \times X$. Define $\mathcal{G}^{(\lambda)}$ as the Zariski $\mathbf{K}_2$-gerbe on $X \times X$ classifying a $\mathrm{pr}_2^* \mathbf{E}$-torsor $\mathcal{P}_{\mathbf{E}}$, together with an isomorphism $(\mathcal{P}_{\mathbf{E}})_T \xrightarrow{\sim} \mathcal{O}(\lambda\Delta)$. Then $\mathcal{G}^{(\lambda)}$ comes equipped with a trivialization γ over $X \times X - \Delta$. The line bundle $\mathcal{L}^{(\lambda)}$ arises from $(\mathcal{G}^{(\lambda)}, \gamma)$ by taking residue along pr_1 (c.f. §7.1).

Let $X \times \mathbb{A}^1 \hookrightarrow \mathfrak{X} \to \mathbb{A}^1$ be the deformation of the diagonal embedding to the normal cone, constructed as the blow-up of $X \times X \times \mathbb{A}^1$ along the diagonally embedded subscheme $X \times \{0\}$, where we then remove the strict transform of $X \times X \times \{0\}$. It has the following features:

(1) $X \times \{t\} \hookrightarrow \mathfrak{X}\big|_t$ identifies with $X \hookrightarrow X \times X$ for $t \neq 0$;

(2) $X \times \{0\} \hookrightarrow \mathfrak{X}\big|_0$ identifies with the embedding of X as the zero section inside the total space of the tangent sheaf T_X.

(3) there is a canonical map $\mathfrak{X} \xrightarrow{\mathrm{pr}_1, \mathrm{pr}_2} X \times X$ which is identity for $t \neq 0$, and the canonical projection $T_X \xrightarrow{p,p} X \times X$ at $t = 0$.

Consider $\mathfrak{Z} := X \times \mathbb{A}^1$ as a divisor inside $\mathfrak{X}$. We define $\widetilde{\mathcal{G}}^{(\lambda)}$ as the $\mathbf{K}_2$-gerbe classifying a $\mathrm{pr}_2^* \mathbf{E}$-torsor $\widetilde{\mathcal{P}}_{\mathbf{E}}$ over $\mathfrak{X}$, together with an isomorphism $(\widetilde{\mathcal{P}}_{\mathbf{E}})_T \xrightarrow{\sim} \mathcal{O}(\lambda\mathfrak{Z})$. Note that $\widetilde{\mathcal{G}}^{(\lambda)}$ is equipped with a trivialization over $\mathfrak{X} - \mathfrak{Z}$, so we may take residue along pr_1 to obtain a line bundle $\widetilde{\mathcal{L}}^{(\lambda)}$ over $X \times \mathbb{A}^1$.

Tautologically, $\widetilde{\mathcal{L}}^{(\lambda)}\big|_{X\times\{t\}}$ identifies with $\mathcal{L}^{(\lambda)}$ for $t \neq 0$. On the other hand, every line bundle on $X \times \mathbb{A}^1$ canonically identifies with the pullback of a line bundle from X. Thus, we obtain an isomorphism $\widetilde{\mathcal{L}}^{(\lambda)}\big|_{X\times\{t\}} \xrightarrow{\sim} \widetilde{\mathcal{L}}^{(\lambda)}\big|_{X\times\{0\}}$. This shows that $\mathcal{L}^{(\lambda)}$ arises from the residue of $(\mathcal{G}^{(\lambda)}_{T_X}, \gamma_{T_X})$ along $p : T_X \to X$, where:

(1) $\mathcal{G}^{(\lambda)}_{T_X}$ is the Zariski $\mathbf{K}_2$-gerbe on T_X classifying a $p^*\mathbf{E}$-torsor $\mathcal{P}_{\mathbf{E}}$, together with an isomorphism $(\mathcal{P}_{\mathbf{E}})_T \xrightarrow{\sim} \mathcal{O}(\lambda\{0\})$, where $\{0\}$ denotes the zero section $X \hookrightarrow T_X$; and

(2) γ_{T_X} is the tautological trivialization of $\mathcal{G}^{(\lambda)}_{T_X}$ over $T_X - \{0\}$.

Step 2. In the above description, suppose we replaced $p : T_X \to X$ by the trivial line bundle $\mathbb{A}^1_X \to X$, then the line bundle arising from taking residue of the analogously defined pair $(\mathcal{G}^{(\lambda)}_{\mathbb{A}^1_X}, \gamma_{\mathbb{A}^1_X})$ would identify with $\mathcal{L}^{(\lambda)}_+$. Indeed, this follows from comparing the "taking residue" construction with that of §1.2.

We now explain an alternative way to arrive at $\mathcal{L}^{(\lambda)}$ via twisting the line bundle $\mathbb{A}^1_X \to X$ in the above construction. Consider the $\mathbb{G}_m$-action on $\mathbb{A}^1_X$ by scaling. The pair $(\mathcal{G}^{(\lambda)}_{\mathbb{A}^1_X}, \gamma_{\mathbb{A}^1_X})$ admits a $\mathbb{G}_m$-equivariance structure. Hence $L^{(\lambda)}_+$ (the total space of $\mathcal{L}^{(\lambda)}_+$) is equipped with a fiberwise $\mathbb{G}_m$-action. Since $\mathcal{G}^{(\lambda)}_{T_X}$ identifies with the twisted product $\mathcal{G}^0 \widetilde{\boxtimes} \mathcal{G}^{(\lambda)}_{\mathbb{A}^1_X}$ on the total space $T_X^{\times} \overset{\mathbb{G}_m}{\times} \mathbb{A}^1_X$ (where $\mathcal{G}^0$ denotes the trivial gerbe), we find $L^{(\lambda)} \xrightarrow{\sim} T_X^{\times} \overset{\mathbb{G}_m}{\times} L^{(\lambda)}_+$. In other words, suppose the fiberwise $\mathbb{G}_m$-action on $L^{(\lambda)}_+$ is given by some character $q_1(\lambda) \in \mathbb{Z}$, then there is a canonical isomorphism:

$$\mathcal{L}^{(\lambda)} \xrightarrow{\sim} \mathcal{T}_X^{q_1(\lambda)} \otimes \mathcal{L}^{(\lambda)}_+. \tag{7.9}$$

Step 3. We now determine the integer $q_1(\lambda)$ for all λ. It suffices to calculate the character of the $\mathbb{G}_m$-action at a closed point $x \in X$. The k-points of the line $L_+^{(\lambda)}\big|_{x\in X}$ admit a simple description, as follows (c.f. §1.2). Evaluating $\mathbf{E}$ at $\mathbb{G}_{m,x} := \mathrm{Spec}(k[t, t^{-1}])$, we obtain an exact sequence:

$$0 \to \mathbf{K}_2(k[t, t^{-1}]) \to \mathbf{E}(k[t, t^{-1}]) \to T(k_x[t, t^{-1}]) \to 0, \tag{7.10}$$

and consequently a $\mathbf{K}_2(k[t, t^{-1}])$-torsor $\mathbf{E}(z)$ at every point $z \in T(k[t, t^{-1}])$. The line $L_+^{(\lambda)}\big|_{x\in X}(k)$ is the $k^\times$-torsor induced from $\mathbf{E}(t^\lambda)$ along the residue map $\mathbf{K}_2(k[t, t^{-1}]) \to k^\times$.

To unburden the notation, we again use $L_+^{(\lambda)}$ to denote this line; the $\mathbb{G}_m(k)$-action on it also admits a simple description. Take $a \in \mathbb{G}_m(k)$, the action by $a^{q_1(\lambda)}$:

$$\cdot\, a^{q_1(\lambda)} : L_+^{(\lambda)} \xrightarrow{\sim} L_+^{(\lambda)} \tag{7.11}$$

is given as follows.

(1) Consider the scaling map $k[t, t^{-1}] \to k[t, t^{-1}]$, $t \rightsquigarrow t \cdot a$. It induces a group automorphism $\mathbf{E}(k[t, t^{-1}]) \xrightarrow{a_*} \mathbf{E}(k[t, t^{-1}])$, covering the analogously defined automorphism on $T(k[t, t^{-1}])$. In particular, we obtain a map $a_* : \mathbf{E}(t^\lambda) \to \mathbf{E}(t^\lambda a^\lambda)$ (*incompatible* with the $\mathbf{K}_2(k[t, t^{-1}])$-torsor structures.)

After inducing to $k^\times$-torsors, we obtain a map *compatible* with the $k^\times$-torsor structures:

$$a_* : L_+^{(\lambda)} \to L_+(t^\lambda a^\lambda) := \mathbf{E}(t^\lambda a^\lambda)_{k^\times},$$

since $a_* : \mathbf{K}_2(k[t, t^{-1}]) \to \mathbf{K}_2(k[t, t^{-1}])$ induces the identity on $k^\times$.

(2) On the other hand, every element in $T(k[t])$ admits a lift to $\mathbf{E}(k[t])$, up to an element from $\mathbf{K}_2(k[t])$ (as follows from $\mathrm{R}^1\, p_*\mathbf{K}_2 = 0$, see Lemma 1.1.4) Hence we have another map $\mathbf{E}(t^\lambda) \to \mathbf{E}(t^\lambda a^\lambda)$, defined as right-multiplying by *any* lift of $a^\lambda \in T(k[t])$.

Inducing along $\mathbf{K}_2(k[t,t^{-1}]) \to k^\times$, we again obtain a map of $k^\times$-torsors:

$$R_{a^\lambda} : L_+^{(\lambda)} \to L_+(t^\lambda a^\lambda).$$

Note that this map is independent of the choice of the lift.

(3) The automorphism (7.11) identifies with the composition $R_{a^\lambda}^{-1} \circ a_*$.

Step 4. We shall now deduce two identities:

$$q_1(2\lambda) - \kappa(\lambda,\lambda) = 2 \cdot q_1(\lambda) \tag{7.12}$$

$$4 \cdot q_1(\lambda) = q_1(2\lambda) \tag{7.13}$$

The combination of these identities will show that $q_1(\lambda) = \frac{1}{2}\kappa(\lambda,\lambda) = q(\lambda)$. Then the desired isomorphism follows from (7.9).

Proof of (7.12). This follows from the mutiplicative structure on $\mathbf{E}(k[t,t^{-1}])$. Indeed, consider the following commutative diagrams:

$$\begin{tikzcd} L_+^{(2\lambda)} \arrow[r, "a_*"] \arrow[d, "\cong"] & L_+(t^{2\lambda}a^{2\lambda}) \arrow[d, "\cong"] \\ L_+^{(\lambda)} \otimes L_+^{(\lambda)} \arrow[r, "a_*\otimes a_*"] & L_+(t^\lambda a^\lambda) \otimes L_+(t^\lambda a^\lambda) \end{tikzcd} \qquad \begin{tikzcd} L_+^{(2\lambda)} \arrow[r, "a^{\kappa(\lambda,\lambda)}\cdot R_{a^{2\lambda}}"] \arrow[d, "\cong"] & L_+(t^{2\lambda}a^{2\lambda}) \arrow[d, "\cong"] \\ L_+^{(\lambda)} \otimes L_+^{(\lambda)} \arrow[r, "R_{a^\lambda}\otimes R_{a^\lambda}"] & L_+(t^\lambda a^\lambda) \otimes L_+(t^\lambda a^\lambda) \end{tikzcd}$$

where vertical arrows witness the multiplicativity of $\mathcal{L}_+^{(\lambda)}$. The first diagram commutes because a_* defines a group homomorphism on $\mathbf{E}(k[t,t^{-1}])$. The second diagram commutes (note the factor $a^{\kappa(\lambda,\lambda)}$) because it calculates the commutator $[a^\lambda, t^\lambda] \in \mathbf{K}_2(k[t,t^{-1}])$, whose residue identifies with $a^{\kappa(\lambda,\lambda)}$.

Now, tracing through the horizontal arrows gives rise to the identity $a^{q_1(2\lambda)-\kappa(\lambda,\lambda)} = a^{2\cdot q_1(\lambda)}$ in $k^\times$. Since the same calculation is valid for any field extension $k \subset k'$, we obtain (7.12). $\square$

Proof of (7.13). This follows from the functoriality of $\mathbf{E}(k[t,t^{-1}])$ with respect to the double covering map $\mathrm{sq}(t) = t^2$ on $k[t,t^{-1}]$. Note that $\mathrm{sq}_* : \mathbf{E}(k[t,t^{-1}]) \to \mathbf{E}(k[t,t^{-1}])$ induces a *quadratic* map of $k^\times$-torsors[10]:

$$\mathrm{sq}_* : L_+^{(\lambda)} \to L_+^{(2\lambda)}.$$

On the other hand, we have the following commutative diagrams:

$$\begin{tikzcd}
L_+^{(\lambda)} \arrow[r, "(a^2)_*"] \arrow[d, "\mathrm{sq}_*"] & L_+(t^\lambda a^{2\lambda}) \arrow[d, "\mathrm{sq}_*"] \\
L_+^{(2\lambda)} \arrow[r, "a_*"] & L_+(t^{2\lambda} a^{2\lambda})
\end{tikzcd}
\qquad
\begin{tikzcd}
L_+^{(\lambda)} \arrow[r, "R_{a^{2\lambda}}"] \arrow[d, "\mathrm{sq}_*"] & L_+(t^\lambda a^{2\lambda}) \arrow[d, "\mathrm{sq}_*"] \\
L_+^{(2\lambda)} \arrow[r, "R_{a^{2\lambda}}"] & L_+(t^{2\lambda} a^{2\lambda})
\end{tikzcd}$$

The first diagram commutes tautologically. The second diagram commutes because $a^{2\lambda}$ belongs to the subgroup $T(k) \hookrightarrow T(k[t,t^{-1}])$, and we may first lift $a^{2\lambda}$ to $\mathbf{E}(k)$ so that its image in $\mathbf{E}(k[t,t^{-1}])$ is fixed by the automorphism sq_*.

Tracing through the horizontal maps and using the quadraticity of vertical maps, we find $a^{4\cdot q_1(\lambda)} = a^{q_1(2\lambda)}$ in $k^\times$. Again because the same calculation is valid for any field extension $k \subset k'$, we obtain (7.13). $\square$

$\square$(Proposition 7.2.2)

Compatibility diagram in general.

7.2.6. We now use the compatibility statement for tori to compare the Picard groupoids of enhanced Θ-data $\Theta_G(\Lambda_T; \mathbf{Pic})$ and $\Theta_G^+(\Lambda_T; \mathbf{Pic})$. The goal is to show that they are isomorphic upon an "ω-shift."

7.2.7. **Lemma.** *There is a canonical equivalence of Picard groupoids:*

$$\Theta_G(\Lambda_T; \mathbf{Pic}) \xrightarrow{\sim} \Theta_G^+(\Lambda_T; \mathbf{Pic}), \quad (q, \mathcal{L}^{(\lambda)}) \rightsquigarrow (q, \mathcal{L}^{(\lambda)} \otimes \omega_X^{q(\lambda)}).$$

[10]i.e., the $k^\times$-action on the two lines intertwines $k^\times \to k^\times$, $a \rightsquigarrow a^2$.

Proof. Unwinding the definitions, we reduce to prove the following statement. For $G = G_{\mathrm{sc}}$ a semisimple, simply connected group, the two functors (1.5), (6.4) from $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, \mathbb{Z})^W$ to $\Theta^+(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic})$, respectively $\Theta(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic})$ are identified by an ω-shift. We fit these two functors in the following diagram:

$$\begin{tikzcd}
\Theta^+(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic}) \arrow[d, "\omega\text{-shift}", "\cong"'] & \mathbf{CExt}(T_{\mathrm{sc}}, \mathbf{K}_2) \arrow[l, "\Psi_{\mathrm{BD},T_{\mathrm{sc}}}"', "\sim"] \arrow[d, "\Xi_{\mathbf{Pic},T_{\mathrm{sc}}}"] & \mathbf{CExt}(G_{\mathrm{sc}}, \mathbf{K}_2) \arrow[l] \arrow[d, dotted, "\Xi_{\mathbf{Pic},G_{\mathrm{sc}}}"] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, \mathbb{Z})^W \arrow[l, "\Psi^{-1}_{\mathrm{BD},G_{\mathrm{sc}}}"'] \arrow[d, "\mathrm{id}", "\cong"'] \\
\Theta(\Lambda_{T_{\mathrm{sc}}}; \mathbf{Pic}) & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_{\mathrm{sc}},\mathrm{Ran}}) \arrow[l, "\Psi_{\mathbf{Pic},T_{\mathrm{sc}}}"', "\sim"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \arrow[l] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, \mathbb{Z})^W \arrow[l, "\Psi^{-1}_{\mathbf{Pic},G_{\mathrm{sc}}}"']
\end{tikzcd}$$

By Proposition 7.2.2, the left square is commutative. On the other hand, $\Xi_{\mathbf{Pic},G_{\mathrm{sc}}}$ is only well defined after pulling back to a regular test scheme $S \to \mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}$, where it renders the middle and right squares commutative. The solid diagram is thus commutative, because $\Psi_{\mathbf{Pic},T_{\mathrm{sc}}}$ factors through an equivalence with $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_{\mathrm{sc}},\mathrm{comb}})$ and two objects there are isomorphic if they are so compatibly on regular test schemes. $\square$

We can *a fortiori* extend the definition of $\Xi_{\mathbf{Pic},G}$ to all semisimple, simply connected groups, in a way that is consistent with Gaitsgory's functor.

Finally, we prove the general compatibility statement.

7.2.8. **Theorem.** *Let G be a reductive group with maximal torus T. Suppose either $\mathrm{char}(k) \nmid N_G$ or G is semisimple and simply connected. Then the following diagram is commutative:*

$$\begin{tikzcd}
\mathbf{CExt}(G, \mathbf{K}_2) \arrow[r, "\Xi_{\mathbf{Pic},G}"] \arrow[d, "\Psi_{\mathrm{BD},G}", "\cong"'] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{Pic},G}"] \\
\Theta^+_G(\Lambda_T; \mathbf{Pic}) \arrow[r, "\omega\text{-shift}", "\sim"'] & \Theta_G(\Lambda_T; \mathbf{Pic})
\end{tikzcd}$$

The hypothesis on G guarantees that $\Xi_{\mathbf{Pic},G}$ is well-defined functorially in G.

Proof. Given a Brylinski–Deligne datum $\mathbf{E}$, we have to construct an isomorphism φ between two objects of $\Theta(\Lambda_T; \mathbf{Pic})$, corresponding to the two circuits of the diagram, and check that φ respects the two isomorphisms $\varepsilon, \varepsilon'$ between objects of $\Theta(\Lambda_{\widetilde{T}_{\mathrm{der}}}; \mathbf{Pic})$. The isomorphism φ comes from the commutativity datum of Proposition 7.2.2, and the required compatibility follows from its functoriality with respect to the map of tori $\widetilde{T}_{\mathrm{der}} \to T$. $\square$

8. Proof of equivalence

In this section, we state and prove the classification of factorization line bundles on $\mathrm{Gr}_{G,\mathrm{Ran}}$ by enhanced Θ-data. We use a geometric argument to show that $\Psi_{\mathbf{Pic},G}$ is always fully faithful. Then we use Brylinski–Deligne data to produce enough factorization line bundles.

8.1. **The main theorem.**

8.1.1. We now state our main theorem concerning the behavior of the classification functor $\Psi_{\mathbf{Pic},G}$ for factorization line bundles, introduced in §6.3.

8.1.2. **Theorem.** *Let G be a reductive group with maximal torus T.*

(1) The functor $\Psi_{\mathbf{Pic},G}$ is fully faithful;

(2) Suppose either $\mathrm{char}(k) \nmid N_G$ or G is semisimple and simply connected. Then $\Psi_{\mathbf{Pic},G}$ is an equivalence.

A few preliminary remarks are in order. First, the statement (1) $\implies$ (2) by Theorem 7.2.8. Second, we have already proved statement (1) in the special cases where $G = T$ is a torus (§5.2) or $G = G_{\mathrm{sc}}$ is semisimple and simply connected (§6.2).

8.1.3. **Remark.** We expect both Theorem 7.2.8 and Theorem 8.1.2 to hold in greater generality, i.e., without the restriction $\mathrm{char}(k) \nmid N_G$ or that G is constant along X. However, our techniques do not apply in these contexts. Recently, J. Tao has suggested an approach of proving that $\Psi_{\mathbf{Pic},G}$ is essentially surjective without appealing to Brylinski–Deligne data.

8.2. **Reduction to $\pi_1 G_{\mathrm{der}} = 0$.**

8.2.1. We shall reduce the proof of Theorem 8.1.2(1) to the case where G_{der} is simply connected. Let t denote a topology on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ which is stronger than the étale topology and such that every $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ is t-locally smooth.

8.2.2. **Lemma.** *Suppose $G' \to G$ is a map of reductive groups whose kernel is a torus. Then the morphism $\mathrm{Gr}_{G',\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ is surjective in the t-topology.*

Proof. One takes an S-point of Gr_G represented by $(x^{(i)}, \mathcal{P}_G, \alpha)$. By the Drinfeld–Simpson theorem, we may assume that $\mathcal{P}_G$ is Zariski-locally trivial after an étale cover of S. A reduction of the datum $(\mathcal{P}_G, \alpha)$ to the structure group G' is thus equivalent to the trivialization of a section of $i^! T[2]$ in the Zariski topology of $S \times X$, where i denotes the closed immersion:

$$\bigcup_{i\in I} \Gamma_{x^{(i)}} \xrightarrow{i} S \times X \xleftarrow{j} U_{\{x^{(i)}\}}.$$

We shall show that over a t-cover $\widetilde{S} \to S$ with $\widetilde{S}$ smooth, every section of $i^! T[2]$ admits a trivialization. To prove this statement, one reduces to $T = \mathbb{G}_m$. The canonical triangle $i^! \mathbb{G}_m \to \mathbb{G}_m \to \mathrm{R}\, j_* \mathbb{G}_m$ induces a long exact sequence:

$$\mathrm{Pic}(\widetilde{S} \times X) \to \mathrm{Pic}(U_{\{x^{(i)}\}}) \to \mathrm{H}^2(\widetilde{S} \times X; i^! \mathbb{G}_m) \to 0.$$

The map on Picard groups is surjective by smoothness of $\widetilde{S}$. Thus $\mathrm{H}^2(\widetilde{S} \times X; i^! \mathbb{G}_m) = 0$. $\square$

8.2.3. Recall that a *z-extension* of G is a short exact sequence of reductive groups:

$$1 \to T_2 \to G' \to G \to 1.$$

where the derived subgroup $G'_{\mathrm{der}} \subset G'$ is simply connected. Its existence is assured by the combinatorics of root data (c.f. [53, Proposition 3.1]). We fix a z-extension of G and let T_1 be the quotient torus G'/G'_{der}. Then the quotient of lattices $\Lambda_{T_1}/\Lambda_{T_2}$ identifies with $\pi_1 G$.

8.2.4. It is easy to see that T_2 is central in G'. Thus the Čech nerve of $G' \to G$ is in fact a simplicial system of group schemes $G' \times T_2^\bullet$. We shall consider the corresponding augmented simplicial system of affine Grassmannians.

$$\cdots \mathrel{\substack{\longrightarrow\\[-1ex]\longrightarrow\\[-1ex]\longrightarrow}} \mathrm{Gr}_{G'\times T_2,\mathrm{Ran}} \rightrightarrows \mathrm{Gr}_{G',\mathrm{Ran}} \longrightarrow \mathrm{Gr}_{G,\mathrm{Ran}}. \tag{8.1}$$

8.2.5. **Lemma.** *The pullback map defines an equivalence of Picard groupoids:*

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \lim_{\Delta^{\mathrm{op}}} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G'\times T_2^\bullet,\mathrm{Ran}}).$$

Proof. We use the fact that $\mathbf{Pic}$ satisfies *derived* $\mathbf{h}$-descent (Chapter 1, Lemma 4.2.4). On the other hand, the simplicial system (8.1) is the Čech nerve associated to the map of derived prestacks $\mathrm{Gr}_{G',\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$, since the formation of affine Grassmannian commutes with fiber product of groups (see Chapter 1, §4.3). Finally, Lemma 8.2.2 shows that $\mathrm{Gr}_{G',\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ is surjective in $\mathbf{h}$-topology. Hence $\mathbf{Pic}(\mathrm{Gr}_{G,\mathrm{Ran}})$ is equivalent to $\lim_{\Delta^{\mathrm{op}}} \mathbf{Pic}(\mathrm{Gr}_{G'\times T_2^\bullet,\mathrm{Ran}})$. Applying the same argument for the morphism

$$(\mathrm{Gr}_{G',\mathrm{Ran}} \times \mathrm{Gr}_{G',\mathrm{Ran}})_{\mathrm{disj}} \to (\mathrm{Gr}_{G,\mathrm{Ran}} \times \mathrm{Gr}_{G,\mathrm{Ran}})_{\mathrm{disj}},$$

we upgrade this equivalence to one between factorization line bundles. $\square$

8.2.6. On the other hand, a direct calculation shows $\Theta_G(\Lambda_T; \mathbf{Pic})$ is canonically equivalent to the totalizaiton of enhanced Θ-data for $G' \times T_2^\bullet$ and the following diagram commutes:

$$\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\sim"] \arrow[d, "\Psi_{\mathbf{Pic},G}"] & \lim_{\Delta^{\mathrm{op}}} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G'\times T_2^\bullet,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{Pic},G'\times T_2^\bullet}"] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \arrow[r, "\sim"] & \lim_{\Delta^{\mathrm{op}}} \Theta_{G'\times T_2^\bullet}(\Lambda_{T'\times T_2^\bullet}, \mathrm{Ran})
\end{tikzcd}$$

Thus we reduce the problem for G to that for $G' \times T_2^{\times n}$, with $n \geq 0$. The latter groups have simply connected derived subgroups.

8.3. **Finishing the proof.**

8.3.1. We now suppose G has simply connected derived subgroup. Write T_1 for the quotient torus G/G_{der}. Then Λ_{T_1} identifies with $\pi_1 G$. It remains to prove that the classification functor:

$$\Psi_{\mathbf{Pic},G} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \Theta_G(\Lambda_T; \mathbf{Pic})$$

is fully faithful.

8.3.2. Recall the functor $\mathbf{Q}_{\mathbf{Pic},G}$ (6.1) and we let $\mathbf{Pic}_0^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$ denote its fiber. Then we have a map between fiber sequences of Picard groupoids.

$$\begin{tikzcd}
\mathbf{Pic}_0^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r] \arrow[d, "(a)"] & \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\mathbf{Q}_{\mathbf{Pic},G}"] \arrow[d, "\Psi_{\mathbf{Pic},G}"] & \mathfrak{Q}(\Lambda_T; \mathbb{Z}) \arrow[d, "\mathrm{id}", "\cong"'] \\
\mathrm{Hom}(\Lambda_{T_1}, \mathbf{Pic}(X)) \arrow[r] & \Theta_G(\Lambda_T; \mathbf{Pic}) \arrow[r] & \mathfrak{Q}(\Lambda_T; \mathbb{Z})
\end{tikzcd}$$

Therefore, it suffices to show that the functor (a) is an equivalence. On the other hand, (a) has a section, defined by the composition:

$$\begin{aligned}\mathrm{Hom}(\Lambda_{T_1}, \mathbf{Pic}(X)) &\xrightarrow{\sim} \mathbf{Pic}_0^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}}) \\ &\to \mathbf{Pic}_0^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}).\end{aligned}$$

Here, the first isomorphism comes from the classification for torus (Theorem 5.2.4) and the second one is pullback along the projection:

$$\pi_1 : \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{T_1,\mathrm{Ran}}.$$

We shall deduce the seoncd map being an equivalence from the following more general result.

8.3.3. **Proposition.** *Pulling back along* π_1 *defines an equivalence between*

(1) $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}})$, *and*

(2) the full subgroupoid

$$\mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_{G,\mathrm{Ran}}) \subset \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$$

which consists of objects $\mathcal{L}$ *such that* $\mathbf{Q}_{\mathbf{Pic},G}(\mathcal{L})$ *vanishes on* $\Lambda_{T_{\mathrm{der}}}$.

Indeed, having Proposition 8.3.3 at our disposal, we find that $\mathbf{Pic}^{\mathrm{fact}}_0(\mathrm{Gr}_{G,\mathrm{Ran}})$ is equivalent to the full subgroupoid of $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}})$ where the associated quadratic form on Λ_{T_1} vanishes when pulled back along the surjection $\Lambda_T \twoheadrightarrow \Lambda_{T_1}$, i.e., it vanishes.

The proof of Proposition 8.3.3 is geometric in nature. We first note a Lemma.

8.3.4. **Lemma.** *The map* π_1 *realizes* $\mathrm{Gr}_{G,\mathrm{Ran}}$ *as an étale locally trivial* $\mathrm{Gr}_{G_{\mathrm{der}},\mathrm{Ran}}$*-bundle over* $\mathrm{Gr}_{T_1,\mathrm{Ran}}$*, i.e., for every affine scheme* $S \to \mathrm{Gr}_{T_1,\mathrm{Ran}}$*, there is an étale cover* $\widetilde{S} \to S$ *and an isomorphism:*

$$\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Gr}_{T_1,\mathrm{Ran}}}{\times} \widetilde{S} \xrightarrow{\sim} \mathrm{Gr}_{G_{\mathrm{der}},\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \widetilde{S}.$$

Proof. We first show that $G \to T_1$ has a non-canonical splitting. Indeed, the maximal torus $T \subset G$ surjects onto T_1, so it suffices to show that the kernel $T \cap G_{\mathrm{der}}$ is connected. The latter follows since $T \cap G_{\mathrm{der}}$ is a maximal torus of G_{der}.

Given an S-point $S \xrightarrow{\gamma} \mathrm{Gr}_{T_1,\mathrm{Ran}}$, we denote by $S \xrightarrow{\gamma_0} \mathrm{Gr}_{T_1,\mathrm{Ran}}$ the "neutral point" corresponding to γ, i.e., the composite:

$$S \xrightarrow{\gamma} \mathrm{Gr}_{T_1,\mathrm{Ran}} \xrightarrow{\pi} \mathrm{Ran} \hookrightarrow \mathrm{Gr}_{T_1,\mathrm{Ran}}.$$

Since $\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Gr}_{T_1,\mathrm{Ran}},\gamma_0}{\times} S$ identifies with $\mathrm{Gr}_{\widetilde{G}_{\mathrm{der}},\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S$, it suffices to produce an isomorphism:

$$\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Gr}_{T_1,\mathrm{Ran}},\gamma}{\times} \widetilde{S} \xrightarrow{\sim} \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Gr}_{T_1,\mathrm{Ran}},\gamma_0}{\times} \widetilde{S} \tag{8.2}$$

after passing to some étale cover $\widetilde{S} \to S$.

We choose $\widetilde{S} \to S$ such that the elements $\gamma, \gamma_0 \in \mathrm{Maps}_{/\mathrm{Ran}}(\widetilde{S}, \mathrm{Gr}_{T_1,\mathrm{Ran}})$ differ by the action of some $\alpha \in \mathrm{Maps}_{/\mathrm{Ran}}(\widetilde{S}, \mathcal{L}_{\mathrm{Ran}}T_1)$—this is possible, for example, by lifting $S \to \mathrm{Gr}_{T_1,\mathrm{Ran}}$ to $\widetilde{S} \to \mathcal{L}_{\mathrm{Ran}}T_1$. The above discussion shows that we have a splitting of the canonical projection $\mathcal{L}_{\mathrm{Ran}}G \to \mathcal{L}_{\mathrm{Ran}}T_1$. Hence α can be lifted to an element $\widetilde{\alpha} \in \mathrm{Maps}_{/\mathrm{Ran}}(\widetilde{S}, \mathcal{L}_{\mathrm{Ran}}G)$. The equivariance property of π_1 shows that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \widetilde{S} \arrow[r, "\mathrm{act}_{\widetilde{\alpha}}"] \arrow[d] & \mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \widetilde{S} \arrow[d] \\
\mathrm{Gr}_{T_1,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \widetilde{S} \arrow[r, "\mathrm{act}_{\alpha}"] & \mathrm{Gr}_{T_1,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \widetilde{S}
\end{tikzcd}$$

Since act_α transforms the section $\gamma : \widetilde{S} \to \mathrm{Gr}_{T_1,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \widetilde{S}$ to γ_0, we obtain the required isomorphism (8.2) as $\mathrm{act}_{\widetilde{\alpha}} \underset{\mathrm{act}_\alpha}{\times} \mathrm{id}_{\widetilde{S}}$. □

We will also need an elementary fact about étale sheaves.

8.3.5. **Lemma.** *Let Y be a connected, Noetherian scheme and $\mathcal{F}$ be an étale sheaf on Y. Suppose furthermore that $\mathcal{F}$ is étale locally isomorphic to a subsheaf of a constant sheaf. Then a section $s \in \mathrm{H}^0(Y, \mathcal{F})$ vanishes if and only if it does so over some étale open $V \to Y$.*

Proof. One can pick finitely many étale maps $V_i \to Y$ $(i \in I)$ so that:

(1) each V_i is connected;

(2) $\mathcal{F}\big|_{V_i}$ is isomorphic to a subsheaf of a constant sheaf;

(3) the images U_i of V_i collectively cover Y.

We induct on the cardinality of I over all connected, Noetherian schemes admitting such a cover; the base case $I = \emptyset$ is trivial. The image U of $V \to Y$ must intersect some U_i. The condition (b) implies that the restriction $s_i \in \mathrm{H}^0(U_i, \mathcal{F})$ vanishes. Now, let $\overset{\circ}{Y} := \bigcup_{j \neq i} U_i$. It is *not* necessarily connected. However, the fact that Y is connected shows that U_i intersects every connected component of $\overset{\circ}{Y}$. We apply the induction hypothesis to each connected component of $\overset{\circ}{Y}$ to conclude that s vanishes.

□

Proof of Proposition 8.3.3. A temporary piece of notation: let $\mathbf{Pic}^{\mathrm{fact}}_{\natural}(\mathrm{Gr}_{G,\mathrm{Ran}})$ denote the full subgroupoid of $\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$ consisting of objects which are trivial along all fibers of π_1 at k-points. Thus we can factor the canonical functor (1) $\Longrightarrow$ (2) as follows.

$$\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}}) \xrightarrow{(a)} \mathbf{Pic}^{\mathrm{fact}}_{\natural}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{(b)} \mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

To prove that (a) is an equivalence, we note a generalization of Lemma 2.3.5 to ind-schemes. Suppose $p : \mathcal{X} \to Y$ is ind-schematic morphism with target $Y \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, represented by morphisms $p_i : X_i \to Y$ of schemes satisfying the hypothesis of Lemma 2.3.5. Then $p^* : \mathbf{Pic}(Y) \to \mathbf{Pic}(\mathcal{X})$ has a partially defined right adjoint:

$$p_*\mathcal{L} := \lim_i (p_i)_*\mathcal{L}_i, \quad \text{while representing } \mathcal{L} \text{ by the inverse system } \mathcal{L}_i \in \mathbf{Pic}(X_i)$$

which is well defined on the full subcategory of $\mathbf{Pic}(\mathcal{X})$ consisting of line bundles which are trivial along the fibers of p, and we furthermore have an isomorphism $p^*p_*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$. For any line bundle $\mathcal{M}$ from the base Y, it is also clear that $\mathcal{M} \xrightarrow{\sim} p_*p^*\mathcal{M}$. Hence p^* defines an equivalence from $\mathbf{Pic}(Y)$ to the full subcategory of $\mathbf{Pic}(\mathcal{X})$ consisting of fiberwise trivial line bundles.

The above discussion, together with Lemma 8.3.4 and properties of the Schubert strata (Lemma 2.2.10 and Lemma 3.3.3), shows that $\mathfrak{p}^*$ defines an equivalence between $\mathbf{Pic}(\mathrm{Gr}_{T_1,\mathrm{Ran}})$ and fiberwise trivial objects in $\mathbf{Pic}(\mathrm{Gr}_{G,\mathrm{Ran}})$. To see that this upgrades to an equivalence of factorization line bundles, we simply note that the ind-schematic morphism

$$\mathrm{Gr}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{T_1,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{T_1,\mathrm{Ran}}$$

satisfies the same hypothesis after base change to a scheme. This finishes the proof that (a) is an equivalence.

To show that (b) is an equivalence, we let $\mathbf{P}^{\lambda^I}$ denote the presheaf of relative Picard group of $\mathrm{Gr}_{G,X^I} \underset{\mathrm{Gr}_{T_1,X^I}}{\times} X^{\lambda^I}$ over X^{λ^I}, i.e., it associates to every étale map $S \to X^{\lambda^I}$ the cofiber of the Picard groupoids:

$$\mathbf{Pic}(S) \to \mathbf{Pic}(\mathrm{Gr}_{G,X^n} \underset{\mathrm{Gr}_{T_1,X^n}}{\times} S).$$

Since $G \to T_1$ has a non-caninical splitting (c.f. the proof of Lemma 8.3.4), $\mathbf{P}^{\lambda^I}$ is an étale sheaf valued in abelian groups. Given an object $\mathcal{L}$ of $\mathbf{Pic}^{\mathrm{fact}}_{q_{\mathrm{der}}=0}(\mathrm{Gr}_{G,\mathrm{Ran}})$, it defines a global section l^{λ^I} of $\mathbf{P}^{\lambda^I}$ for every n-tuple λ^I. The goal is to show that all l^{λ^I} vanish.

On the other hand, Lemma 8.3.4 shows that the sheaf $\mathbf{P}^{\lambda^I}$ is étale locally isomorphic to $\mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{der}}}/X^I}$ under the identification $X^{\lambda^I} \xrightarrow{\sim} X^I$. By the description (6.3) of the latter, we see that $\mathbf{P}^{\lambda^I}$ is étale locally a subsheaf of $\boxtimes_{i\in I}\underline{B}_X$.

Our proof that each $l^{(\lambda^I)}$ vanishes now proceeds as follows:

Step 1: $l^{(0)} = 0$. Indeed, since line bundles on $\mathrm{Gr}_{G_{\mathrm{der}},X}$ are classified by the quadratic form q_{der}, we see that $\mathcal{L}$ is trivialized when pulled back along $\mathrm{Gr}_{G_{\mathrm{der}},X} \to \mathrm{Gr}_{G,X}$. On the other hand, $\mathrm{Gr}_{G_{\mathrm{der}},X}$ appears as the fiber of π_1 along the unit section $X \hookrightarrow \mathrm{Gr}_{T_1,X}$. Hence $l^{(0)} = 0$.

Step 2: $l^{(\lambda)} = 0$ for all $\lambda \in \Lambda_{T_1}$. Consider the section $l^{(\lambda,-\lambda)}$ of $\mathbf{P}^{(\lambda,-\lambda)}$. It is represented by some line bundle $\mathcal{L}^{(\lambda,-\lambda)}$ over $\mathrm{Gr}_{G,X^2} \underset{\mathrm{Gr}_{T_1,X^2}}{\times} X^{(\lambda,-\lambda)}$. We know from Step 1 that the restriction of $\mathcal{L}^{(\lambda,-\lambda)}$ to the diagonal comes from the base $X^{(0)} \hookrightarrow X^{(\lambda,-\lambda)}$. Hence, over an étale neighborhood of $X^{(0)}$, the section $l^{(\lambda,-\lambda)}$ has to vanish by the identification of $\mathbf{P}^{(\lambda,-\lambda)}$ with $\mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{der}}}/X^2}$. We then apply Lemma 8.3.5 to conclude that $l^{(\lambda,-\lambda)}$ vanishes.

Now, under the identification of $\mathbf{P}^{(\lambda,-\lambda)}$ with $\mathbf{P}^{(\lambda)} \boxtimes \mathbf{P}^{(-\lambda)}$ away from the diagonal, the section $l^{(\lambda,-\lambda)}$ passes to $l^{(\lambda)} \boxtimes l^{(-\lambda)}$. The fact that $l^{(\lambda,-\lambda)} = 0$ now implies that $l^{(\lambda)}$ and $l^{(-\lambda)}$ both vanish.

Step 3: $l^{\lambda^I} = 0$ for all I-tuple λ^I. When $|I| \geq 2$, we may use the factorization property of l^{λ^I} to see that l^{λ^I} vanishes away from the union of the diagonals in X^{λ^I}. Hence by Lemma 8.3.5 again we have $l^{\lambda^I} = 0$.

This finishes the proof that (b) is an equivalence. $\square$

$\square$(Theorem 8.1.2)

CHAPTER 3

Sheaf theories and realizations

This chapter is devoted to a further geometrization of Brylinski–Deligne data in the presence of a sheaf theory.

For example, in the Langlands program for the function field of a curve, one studies ℓ-adic sheaves on Bun_G and the function-theoretic aspects arise from taking the trace of Frobenius. The metaplectic data of Chapter 2 can be used to form a twisted category of sheaves, upon fixing an element of $A(-1)$. Here, A is a torsion subgroup of $\overline{\mathbb{Q}}_\ell^\times$ whose elements have orders indivisible by $\mathrm{char}(k)$. However, a more direct approach is possible via "factorization gerbes" on the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$.

The main result of this chapter, reproduced from [70], is to prove a classification theorem of these gadgets analogous to factorization line bundles and to explain the relationship between the two theorems. These results have been anticipated by Gaitsgory–Lysenko [32] in the ℓ-adic context. However, for later applications, we work independently of sheaf-theoretic contexts.

We continue to assume $k = \bar{k}$ throughout the chapter.

9. Motivic theory of gerbes

The goal of this section is to characterize a sufficiently topological notion of gerbes by axioms. We shall call such a notion a "motivic theory of gerbes."

9.1. **Sheaf-theoretic context.**

9.1.1. The geometrization of the Langlands program proceeds by replacing functions by sheaves. In order to do so, we must first fix a "sheaf-theoretic context." The basic paradigm is as follows. We fix an abelian group E, regarded as a "coefficient group." Then a sheaf-theoretic context is a contravariant assignment of an E-linear ∞-category $\mathrm{Shv}(X)$ to every scheme X over k, which is furthermore lax symmetric monoidal. The latter means that there is a canonically defined functor:

$$\mathrm{Shv}(X) \otimes \mathrm{Shv}(Y) \to \mathrm{Shv}(X \times Y),$$

which may not be an equivalence. Nevertheless, this datum does equip each $\mathrm{Shv}(X)$ with the structure of a symmetric monoidal ∞-category.

9.1.2. In order to have a well functioning theory, we often want Shv to have a six-functor formalism. Instead of specifying the axioms, we note two quintessential example of Shv:

(1) When E is a torsion abelian group such that the order of its elements are not divisible by $\mathrm{char}(k)$, we let $\mathrm{Shv}(X)$ be the ∞-category of constructible complexes of étale E-sheaves.
(2) When $k = \mathbb{C}$ and $E = \mathbb{C}$ as well, we let $\mathrm{Shv}(X)$ be the ∞-category of constructible complexes of $\mathbb{C}$-sheaves on the analytification X^{an}.

In each sheaf-theoretic context, there is a notion of a "gerbe" $\mathcal{G}$ that can be used to form a twisted category of sheaves $\mathrm{Shv}_{\mathcal{G}}(X)$. The metaplectic generalization of the geometric Langlands program amounts to the passage from usual sheaves to twisted sheaves.

9.1.3. Let us note how gerbes naturally arise from a sheaf theory. We single out a full subfunctor $\mathbf{L}$ of Shv valued in an *abelian* category of invertible objects, so $\mathbf{L}$ takes values in abelian groups. Then we take a suitable classifying 2-stack $\mathbf{G} := \mathrm{B}\,\mathbf{L}$ and call its sections "gerbes." Since $\mathbf{L}$ acts as automorphisms on Shv, every gerbe $\mathcal{G}$ on X allows to form a twisted category $\mathrm{Shv}_{\mathcal{G}}(X)$. In the setting of analytic spaces, we may let $\mathbf{L}$ be the stack of rank-1 local systems, i.e., analytic $\mathbb{C}^\times$-torsors, and our gerbe theory $\mathbf{G}$ will be analytic $\mathbb{C}^\times$-gerbes.

In the example (1), the natural choice of $\mathbf{L}$ would be étale $E^\times$-torsors and the theory of gerbes $\mathbf{G}$ would be étale $E^\times$-gerbes. Such gerbes, however, are poorly behaved because $E^\times$ can have torsion divisible by $\mathrm{char}(k)$. Thus a better choice would be étale A-gerbes for $A \subset E^\times$ a subgroup without such elements.

9.1.4. Since there does not seem to be a uniform way of extracting a theory of gerbes from a sheaf-theoretic context, we find it more convenient to formulate axiomatically a "motivic theory of gerbes," which is sufficiently well behaved for the purpose of the Langlands program. Then we shall interpret geometric metaplectic data as factorization gerbes on the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$. We will apply these results to the two sheaf-theoretic contexts mentioned above, and explain their relationship to Brylinski–Deligne data.

Besides these two examples, there is also the context of $\mathcal{D}$-modules when $\mathrm{char}(k) = 0$. This context does not have all six functors unless we restrict to the holonomic subcategory. However, it turns out to be especially rich, and involves several non-equivalent notions of gerbes. We will discuss them thoroughly in Chapter 4.

9.2. Picard n-groupoids.

9.2.1. In this text, we refer to commutative group objects of $\mathbf{Spc}$ as *Picard groupoids*. More precisely, Picard groupoids $\mathbf{A}$ form the full subcategory of $\mathbb{E}_\infty$-spaces characterized by the property of being *grouplike*, i.e., $\pi_0\mathbf{A}$ is a group under the commutative multiplication. A Picard groupoid $\mathbf{A} \in \mathrm{ComGrp}(\mathbf{Spc})$ with $\pi_i\mathbf{A} = 0$ for $i > 1$ is thus a Picard groupoid in the classical sense (c.f. [2, Exposé XVIII]).

9.2.2. The ∞-category $\mathrm{ComGrp}(\mathbf{Spc})$ is also equivalent to that of connective spectra:

$$\mathrm{ComGrp}(\mathbf{Spc}) \xrightarrow{\sim} \mathbf{Sptr}_{\geq 0}.$$

We note that the forgetful functor from ComGrp(**Spc**) to **Spc**, which passes to Ω^∞ on the level of spectra, preserves limits and filtered colimits.

9.2.3. We will also need to consider the more restricted notion of *strict Picard groupoids*. These are the $\mathrm{H}\,\mathbb{Z}$-module objects in ComGrp(**Spc**). The Dold–Kan correspondence identifies the following ∞-categories:

(1) Nonpositively graded cochain complexes of abelian groups $\mathbb{Z}\text{-Mod}^{\leq 0}$;

(2) $\mathrm{H}\,\mathbb{Z}$-module objects in ComGrp(**Spc**).

Under this correspondence, the H^{-i} of a cochain complex identifies with π_i of the $\mathrm{H}\mathbb{Z}$-module, for all $i \geq 0$. We will denote this ∞-category by $\mathrm{ComGrp}^{\mathrm{st}}(\mathbf{Spc})$, often passing without mention the Dold–Kan correspondence.

9.2.4. **Remark.** For every $\mathbf{A} \in \mathrm{ComGrp}^{\mathrm{st}}(\mathbf{Spc})$, the commutativity constraint $\mathbf{A} \otimes \mathbf{A} \xrightarrow{\sim} \mathbf{A} \otimes \mathbf{A}$ is homotopy equivalent to $\mathrm{id}_{\mathbf{A}\otimes\mathbf{A}}$. For $\mathbf{A} \in \mathrm{ComGrp}(\mathbf{Spc})$ with $\pi_i\mathbf{A} = 0$ for $i > 1$, being strict is a condition but this will no longer be the case in general.

We shall call a (resp. strict) Picard groupoid $\mathbf{A}$ with $\pi_i\mathbf{A} = 0$ for $i > n$ a (resp. *strict*) *Picard n-groupoid*. One of the main objects we shall be concerned with—gerbes—form a strict Picard 2-groupoid.

9.2.5. Let us note the sheaf-theoretic analogue of the above discussion. For $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, there is a functor from the ∞-category of complexes of étale sheaves of abelian groups on X to the ∞-category of $\mathrm{ComGrp}^{\mathrm{st}}(\mathbf{Spc})$-valued étale sheaves:

$$\mathcal{F}^\bullet \rightsquigarrow \mathbf{F}, \quad \mathbf{F}(U) := \mathrm{DK}(\tau^{\leq 0}\,\mathrm{R}\,\Gamma(U, \mathcal{F}^\bullet)). \tag{9.1}$$

Here $\tau^{\leq 0}$ denotes cohomological trunction and DK is the Dold–Kan correspondence. The fact that $\mathbf{F}$ is again a sheaf follows from the preservation of limits under $\tau^{\leq 0}$ and DK. We say that the étale sheaf of strict Picard groupoids $\mathbf{F}$ is *represented by* the complex $\mathcal{F}^\bullet$.

9.2.6. **Lemma.** *Under the functor* $\mathcal{F}^\bullet \rightsquigarrow \mathbf{F}$ (9.1)*, there holds:*

(1) For any $x \in X$*, we have an isomorphism of stalks* $\mathbf{F}_x \xrightarrow{\sim} \mathrm{DK}(\tau^{\leq 0}\mathcal{F}^\bullet_x)$*;*

(2) Suppose $f : X \to Y$ *is a morphism in* $\mathbf{Sch}^{\mathrm{ft}}_{/k}$*. Then the pushforward* $f_*\mathbf{F}$ *identifies with the* $\mathrm{ComGrp}^{\mathrm{st}}(\mathbf{Spc})$*-valued sheaf associated to* $\mathrm{R}\, f_*\mathcal{F}^\bullet$*.*

Proof. Part (a) follows from the identification of $\mathcal{F}^\bullet_x$ with $\operatorname*{colim}_U \mathrm{R}\,\Gamma(U, \mathcal{F}^\bullet)$, where U ranges over étale neighborhoods of x, and the commutation of $\tau^{\leq 0}$ with filtered colimits. Part (b) follows from the fact that for every étale $V \to Y$, the complex $\mathrm{R}\,\Gamma(V, \mathrm{R}\, f_*\mathcal{F}^\bullet)$ identifies with $\mathrm{R}\,\Gamma(V \underset{Y}{\times} X, \mathcal{F}^\bullet)$. □

9.3. Motivic t-theory of gerbes.

9.3.1. Let $\mathbf{G}$ be an étale stack on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ valued in *strict* Picard 2-groupoids (c.f. §9.2). We write $A(-1)$ for the fiber of the restriction map $\mathbf{G}(\mathbb{A}^1) \to \mathbf{G}(\mathbb{A}^1 \backslash \{\mathbf{0}\})$ and think of it as a "Tate twist" of some coefficient group A (although we do not define A). Note that *a priori* $A(-1)$ is a strict Picard 2-groupoid as opposed to an abelian group. We define a *theory of gerbes* to be such $\mathbf{G}$, equipped with a map of stacks of strict Picard groupoids:

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} A(-1) \to \mathbf{G}, \quad (\mathcal{L}, \lambda) \rightsquigarrow \mathcal{L}^\lambda, \tag{9.2}$$

which we shall call a *divisor class map* ("first Chern class"). We will often refer to $\mathbf{G}$ as a theory of gerbes, the datum of (9.2) being tacitly included. For $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, the notation $\mathbf{G}_X$ means the restriction of $\mathbf{G}$ to the small étale site of X.

9.3.2. Let us fix a topology t on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ which is finer than the étale topology and such that every $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ is t-locally smooth. Examples of t include the éh-topology when $\mathrm{char}(k) = 0$ and the h-topology in the general case. We call a theory of gerbes $\mathbf{G}$ a *t-theory of gerbes* if $\mathbf{G}$ furthermore satisfies t-descent.

9.3.3. Here is a list of properties that we shall consider for a theory of gerbes $\mathbf{G}$.

(RP1) $A(-1)$ is discrete, and for any $X \in \mathbf{Sm}_{/k}$ and $i : Z \hookrightarrow X$ a smooth divisor, the map of étale stacks induced from the divisor class map is an equivalence:

$$\underline{A}(-1) \xrightarrow{\sim} \mathrm{Fib}(\mathbf{G}_X \to j_*\mathbf{G}_{X\setminus Z}), \quad a \rightsquigarrow \mathcal{O}_X(Z)^a.$$

(RP2) For any $X \in \mathbf{Sm}_{/k}$ and $i : Z \hookrightarrow X$ a closed subscheme of pure codimension ≥ 2, the morphism is an equivalence:

$$\mathbf{G}_X \xrightarrow{\sim} j_*\mathbf{G}_{X\setminus Z}.$$

(A) For any $X \in \mathbf{Sm}_{/k}$, the pullback morphism is an equivalence:

$$\mathbf{G}(X) \xrightarrow{\sim} \mathbf{G}(X \times \mathbb{A}^1).$$

(B) For any proper morphism $p : Y \to X$ in $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ and every k-point $x \in X$, the étale stalk $(p_*\mathbf{G}_Y)_x$ maps fully faithfully to the fiber $\mathbf{G}(Y \underset{X}{\times} \{x\})$.

The names of these properties are *relative purity in codimension* 1 (RP1), *relative purity in codimension* ≥ 2 (RP2), $\mathbb{A}^1$*-invariance* (A), and *weak proper base change* (B). We call a t-theory of gerbes $\mathbf{G}$ satisfying all the above properties a *motivic* t*-theory of gerbes.*

We note that although property (B) refers only to k-points, the assumption $k = \bar{k}$ guarantees that we have enough of them.

9.3.4. **Lemma.** *Let* $\mathbf{F}$ *be an étale sheaf on* $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ *valued in strict Picard groupoids. If the stalk* $\mathbf{F}_x = 0$ *for all* k*-points* $x \in X$*. Then* $\mathbf{F} = 0$.

Thus a morphism $\mathbf{F} \to \mathbf{G}$ is an isomorphism if and only if its stalks at all k-points are.

Proof. It suffices to show $\pi_i\mathbf{F}$, the sheafification of $U \rightsquigarrow \mathbf{F}(U)$, vanishes. Since $(\pi_i\mathbf{F})_{\bar{\eta}} = \pi_i(\mathbf{F}_{\bar{\eta}})$ for every geometric point $\bar{\eta} \to X$, the problem reduces to the case where $\mathbf{F}$ is valued in abelian groups. The problem then reduces to the fact that the étale neighborhood of any geometric point $\bar{\eta} \to \eta \in X$ contains a k-point in the closure of η. $\square$

9.4. **Immediate consequences.**

9.4.1. We now note some quick implications of the aforementioned properties of a theory of gerbes. First, we observe that relative purity in codimension 1 can be generalized to the situation of multiple divisors.

9.4.2. **Lemma.** *Let* $\mathbf{G}$ *be a theory of gerbes satisfying* (RP1). *Then for any* $X \in \mathbf{Sm}_{/k}$ *together with a closed immersion* $i : Z \hookrightarrow X$ *where* Z *is a finite union of smooth divisors* $i_\alpha : Z_\alpha \hookrightarrow X$. *Then the following map is an equivalence:*

$$\bigoplus_\alpha (i_\alpha)_* \underline{A}(-1) \xrightarrow{\sim} \mathrm{Fib}(\mathbf{G}_X \to j_*\mathbf{G}_{X\backslash Z}), \quad (a_\alpha) \rightsquigarrow \bigotimes_\alpha \mathcal{O}_X(Z_\alpha)^a$$

The conclusion is, of course, trivial if $\mathbf{G}$ also satisfies (RP2).

Proof. For notational simplicity, we only prove the case $Z = Z_1 \cup Z_2$. Factor the open immersion $j : X\backslash Z \hookrightarrow X$ as such:

$$X\backslash Z \xrightarrow{j_2} X\backslash Z_1 \xrightarrow{j_1} X,$$

where the complement of j_2 is the locally closed subscheme $\mathring{Z}_2 := Z_2\backslash Z_1$. Applying relative purity to the open immersion j_2, we obtain a fiber sequence:

$$(i_{\mathring{Z}_2})_* \underline{A}(-1) \to \mathbf{G}_{X\backslash Z_1} \to (j_2)_*\mathbf{G}_{X\backslash Z}.$$

Applying $(j_1)_*$ to this fiber sequence. Using the fact that $\underline{A}(-1)$ is a constant sheaf so its pushforward under $j_1 \circ i_{\mathring{Z}_2}$ identifies with $(i_2)_*\underline{A}(-1)$, we find a fiber sequence:

$$(i_2)_*\underline{A}(-1) \to (j_1)_*\mathbf{G}_{X\setminus Z_1} \to j_*\mathbf{G}_{X\setminus Z}. \tag{9.3}$$

On the other hand, relative purity applied to the open immersion j_1 yields:

$$(i_1)_*\underline{A}(-1) \to \mathbf{G}_X \to (j_1)_*\mathbf{G}_{X\setminus Z_1}. \tag{9.4}$$

Combining (9.3) and (9.4), we see that the fiber of $\mathbf{G}_X \to j_*\mathbf{G}_{X\setminus Z}$ is an extension of $(i_2)_*\underline{A}(-1)$ by $(i_1)_*\underline{A}(-1)$. The symmetry of the situation implies that this extension canonically splits. $\square$

9.4.3. We now explain that property (A) can be enhanced in the presence of t-descent. Namely, $\mathbf{G}$ is trivial on "$\mathbb{A}^1$-contractible" ind-schemes of ind-finite type. Note that by our convention[11], $X \in \mathbf{IndSch}^{\mathrm{ft}}_{/k}$ has the property that $X \to X \times X$ is a schematic closed immersion.

Given $X \in \mathbf{IndSch}^{\mathrm{ft}}_{/k}$ equipped with a $\mathbb{G}_m$-action, the action is called *contracting* if it extends to an action of the multiplicative monoid $\mathbb{A}^1$. Such an extension is unique if it exists. Indeed, given two action maps $\mathbb{A}^1 \times X \underset{\mathrm{act}_2}{\overset{\mathrm{act}_1}{\rightrightarrows}} X$, the locus on which they agree maps to $\mathbb{A}^1 \times X$ via a schematic closed immersion. Therefore, if the locus contains $\mathbb{G}_m \times X$, it is all of $\mathbb{A}^1 \times X$.

Let $X^0 \hookrightarrow X$ be the fixed-point locus of a contracting $\mathbb{G}_m$-action. Then X^0 is again an ind-scheme of ind-finite type. We have a commutative diagram:

$$\begin{tikzcd}
\{\mathbf{0}\} \times X \arrow[r, "q"] \arrow[d] & X^0 \arrow[d, "i"] \\
\mathbb{A}^1 \times X \arrow[r, "\mathrm{act}"] & X
\end{tikzcd}$$

[11]Namely, all schemes are assumed separated unless stated otherwise.

Furthermore, the composition $X^0 \xrightarrow{i} \{\mathbf{0}\} \times X \xrightarrow{q} X^0$ is the identity map. This is because $\mathbb{G}_m$ acts trivially on X^0, so it extends uniquely to the trivial $\mathbb{A}^1$-action.

9.4.4. **Lemma.** *Suppose* $\mathbf{G}$ *is a motivic* t*-theory of gerbes satisfying* (A).

(1) For any $X \in \mathbf{IndSch}^{\mathrm{ft}}_{/k}$*, the pullback morphism is an equivalence:*

$$\mathbf{G}(X) \xrightarrow{\sim} \mathbf{G}(X \times \mathbb{A}^1).$$

(2) Suppose $X \in \mathbf{IndSch}^{\mathrm{ft}}_{/k}$ *is equipped with a contracting* $\mathbb{G}_m$*-action. Then restriction to the fixed-point locus is an equivalence:*

$$i^* : \mathbf{G}(X) \xrightarrow{\sim} \mathbf{G}(X^0).$$

Proof. For part (1), we first prove the result for $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$. Indeed, take a t-hypercovering of X consisting of smooth schemes $\widetilde{X}^{\bullet}$, the pullback $\widetilde{X}^{\bullet} \times \mathbb{A}^1$ is a t-hypercovering of $X \times \mathbb{A}^1$, so we win by t-descent. For the general case, we represent X by $\operatorname{colim}_{\nu} X^{(\nu)}$ with $X^{(\nu)} \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$. Then $X \times \mathbb{A}^1$ agrees with $\operatorname{colim}_{\nu} (X^{(\nu)} \times \mathbb{A}^1)$, so the result follows from the schematic case.

For part (2), we note that $\mathbb{A}^1$-invariance gives a canonical isomorphism of functors:

$$\mathrm{pr}^* \xrightarrow{\sim} \mathrm{act}^* : \mathbf{G}(X) \to \mathbf{G}(\mathbb{A}^1 \times X).$$

Composing with the pullback to $\{\mathbf{0}\} \times X$, we find that the identity functor on $\mathbf{G}(X)$ is equivalent to $q^* \circ i^*$. On the other hand, $i^* \circ q^*$ is the identity functor on $\mathbf{G}(X^0)$ as observed above, so the result follows. $\square$

9.4.5. We now show that property (B) implies a Künneth type formula when some rigidity is assumed of one of the factors. For any $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, write $\mathbf{G}(X/\mathrm{pt})$ as the cofiber of $\mathbf{G}(\mathrm{pt}) \to \mathbf{G}(X)$ calculated in the ∞-category of strict Picard groupoids. Any choice of a k-point $x \in X$ identifies $\mathbf{G}(X/\mathrm{pt})$ with the fiber $\mathbf{G}(X; x)$ of x^* :

$\mathbf{G}(X) \to \mathbf{G}(\mathrm{pt})$, i.e., gerbes rigidified at x. In particular, $\mathbf{G}(X/\mathrm{pt})$ is still a 2-groupoid.

9.4.6. **Lemma.** *Let* $\mathbf{G}$ *be a theory of gerbes satisfying* (B). *Let* $X_1, X_2 \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ *be connected schemes and furthermore suppose:*

(1) X_1 *is proper, and*

(2) $\mathbf{G}(X_1/\mathrm{pt})$ *is discrete.*

Then the external product defines an equivalence:

$$\boxtimes : \mathbf{G}(X_1/\mathrm{pt}) \times \mathbf{G}(X_2/\mathrm{pt}) \xrightarrow{\sim} \mathbf{G}(X_1 \times X_2/\mathrm{pt}). \tag{9.5}$$

Proof. We let $\underline{\mathbf{G}(X_1)}$ be the étale sheafification of the constant presheaf with value $\mathbf{G}(X_1)$ on X_2 (and similarly for $\underline{\mathbf{G}(\mathrm{pt})}$). Let $p : X_1 \times X_2 \to X_2$ denote the projection map. External product defines a morphism:

$$\boxtimes : \underline{\mathbf{G}(X_1)} \bigsqcup_{\underline{\mathbf{G}(\mathrm{pt})}} \mathbf{G}_{X_2} \to p_* \mathbf{G}_{X_1 \times X_2}. \tag{9.6}$$

Here, the push-out is calculated in the ∞-category of étale sheaves valued in strict Picard groupoids. We claim that (9.6) is an equivalence. It suffices to check that the stalks at every k-point $x_2 \in X_2$ agree (Lemma 9.3.4). We first note that $\mathbf{G}(\mathrm{pt}) \to \mathbf{G}_{X_2,x_2}$ is an equivalence since the restriction $\mathbf{G}_{X_2,x_2} \to \mathbf{G}(x_2)$ is fully faithful (Property (B)). Thus the composition:

$$\mathbf{G}(X_1) \bigsqcup_{\mathbf{G}(\mathrm{pt})} \mathbf{G}_{X_2,x_2} \to (p_* \mathbf{G}_{X_1 \times X_2})_{x_2} \to \mathbf{G}(X_1 \times \{x_2\})$$

is an equivalence. Since the second map is fully faithful (Property (B)), the first map is an equivalence. This proves that (9.6) is indeed an equivalence.

To prove that (9.5) is an equivalence, we can fix points $x_1 \in X_1$ and $x_2 \in X_2$ and instead prove that the external product is an equivalence for rigidified gerbes:

$$\boxtimes : \mathbf{G}(X_1; x_1) \times \mathbf{G}(X_2; x_2) \to \mathbf{G}(X_1 \times X_2; (x_1, x_2)). \tag{9.7}$$

The splitting of $\mathbf{G}(X_1)$ as the bi-product $\mathbf{G}(X_1; x_1) \times \mathbf{G}(\mathrm{pt})$ implies that $\underline{\mathbf{G}(X_1)} \underset{\underline{\mathbf{G}(\mathrm{pt})}}{\sqcup} \mathbf{G}_{X_2}$ is isomorphic to $\underline{\mathbf{G}(X_1; x_1)} \times \mathbf{G}_{X_2}$. Since $\mathbf{G}(X_1; x_1)$ is discrete and X_2 is connected, the global section of (9.6) yields an equivalence:

$$\mathbf{G}(X_1; x_1) \times \mathbf{G}(X_2) \xrightarrow{\sim} \mathbf{G}(X_1 \times X_2).$$

Adding the rigidification at x_2, respectively (x_1, x_2), implies the equivalence (9.7). $\square$

10. Classification of factorization gerbes

In this section, we define factorization gerbes on the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$ and a strict Picard 2-groupoid of "gerbe-theoretic" enhanced Θ-data attached to any theory of gerbes $\mathbf{G}$.

The definition is entirely parallel to the situation for line bundles in Chapter §2. The only difference is that we do not use any classification result for $\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$ in the case of a semisimple, simply connected group. Rather, we appeal to the classification for line bundles (Theorem 6.2.5 of Chapter §2) and the functoriality provided by the divisor class map. We choose to do so because it is convenient to have a notion of enhanced Θ-data even when the theory of gerbes is not motivic.

10.1. **Factorization gerbes.**

10.1.1. Let us recall the notion of a factorization prestack from Chapter 1, §3. We want to replicate the definition of factorization line bundles for defining factorization gerbes. Since the latter takes values in strict Picard 2-groupoids, we need to specify homotopy coherence data for one level higher.

10.1.2. Suppose we are given a presheaf $\mathbf{F}$ on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ valued in strict Picard 2-groupoids. We extend $\mathbf{F}$ to prestacks by the process of right Kan extension:

$$\mathbf{F}(\mathcal{Y}) = \lim_{\substack{X \to \mathcal{Y} \\ X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}}} \mathbf{F}(X).$$

10.1.3. Suppose $\mathcal{Y}$ is a factorization prestack over X. Then a *factorization section* $\mathcal{S} \in \mathbf{F}^{\mathrm{fact}}(\mathcal{Y})$ is a section $\mathcal{S} \in \mathbf{F}(\mathcal{Y})$ equipped with *factorization isomorphisms*:

$$\sqcup^*_{(n)} \mathcal{S} \xrightarrow{\sim} \mathcal{S}^{\boxtimes n} \text{ in } \mathbf{F}(\sqcup^*_{(n)} \mathcal{Y} \xrightarrow{\sim} (\mathcal{Y}^{\times n})_{\mathrm{disj}}),$$

for $n = 2, 3$. Furthermore, for each surjection $\varphi : \{1, 2, 3\} \to \{1, 2\}$, we are supplied a 2-isomorphism witnessing the commutativity of the following diagram:

$$\begin{tikzcd} \sqcup^*_{(3)} \mathcal{S} \arrow[r, "\sim"] \arrow[d, "\cong"'] & \mathcal{S} \boxtimes \mathcal{S} \boxtimes \mathcal{S} \arrow[dl, Rightarrow] \\ \sqcup^*_{\varphi} \sqcup^*_{(2)} \mathcal{S} \arrow[r, "\sim"] & \sqcup^*_{\varphi}(\mathcal{S} \boxtimes \mathcal{S}) \arrow[u, "\cong"'] \end{tikzcd} \quad \text{in} \quad \mathbf{F}\left(\begin{tikzcd} \sqcup^*_{(3)} \mathcal{Y} \arrow[r, "\sim"] \arrow[d, "\cong"'] & (\mathcal{Y} \times \mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}} \\ \sqcup^*_{\varphi} \sqcup^*_{(2)} \mathcal{Y} \arrow[r, "\sim"] & \sqcup^*_{\varphi}(\mathcal{Y} \times \mathcal{Y})_{\mathrm{disj}} \arrow[u, "\cong"'] \end{tikzcd} \right).$$

These 2-isomorphisms are required to satisfy a coherence condition over $\mathrm{Ran}^{\times 4}_{\mathrm{disj}}$ which we shall not specify.

Thus $\mathbf{F}^{\mathrm{fact}}(\mathcal{Y})$ naturally forms a strict Picard 2-groupoid, and the forgetful map $\mathbf{F}^{\mathrm{fact}}(\mathcal{Y}) \to \mathbf{F}(\mathcal{Y})$ is a morphism of such. In the particular case where $\mathbf{G}$ is a theory of gerbes, we call sections of $\mathbf{G}^{\mathrm{fact}}(\mathcal{Y})$ *factorization gerbes* on $\mathcal{Y}$.

10.2. Enhanced Θ-data.

10.2.1. We now introduce the enhanced Θ-data for a theory of gerbes $\mathbf{G}$. Suppose we are given the following data:

(1) a smooth, connected algebraic curve X;
(2) a reductive group G over k with maximal torus $T \subset G$;

(3) a theory of gerbes $\mathbf{G}$ such that $A(-1)$ is a *divisible* abelian group (in particular, discrete).

Then we shall attach a strict Picard 2-groupoid $\Theta_G(\Lambda_T; \mathbf{G})$ called *enhanced* Θ-*data* for $\mathbf{G}$. It will consists of triples $(q, \mathcal{G}^{(\lambda)}, \varepsilon)$ to be specified below.

10.2.2. *Restricted quadratic form.* Let W denote the Weyl group of (G, T). It acts on the cocharacter lattice Λ_T. Let $\mathcal{Q}(\Lambda_T; A(-1))^W$ denote the abelian group of W-invariant $A(-1)$-valued quadratic forms on Λ_T. Any such quadratic form gives rise to a W-invariant bilinear form κ defined by:

$$\kappa(\lambda, \mu) := q(\lambda + \mu) - q(\lambda) - q(\mu).$$

In particular, $\kappa(\lambda, \lambda) = 2q(\lambda)$.

Following Gaitsgory–Lysenko [32], we specify a subgroup

$$\mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}} \subset \mathcal{Q}(\Lambda_T; A(-1))^W,$$

called *restricted* quadratic forms, by the property that $q \in \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$ if:

$$\kappa(\alpha, \lambda) = \langle \check{\alpha}, \lambda \rangle q(\alpha), \quad \text{for all } \alpha \in \check{\Phi}, \lambda \in \Lambda_T, \tag{10.1}$$

where $\check{\alpha}$ denotes the root associated to α. We note that there always holds $2\kappa(\alpha, \lambda) = 2\langle \check{\alpha}, \lambda \rangle q(\alpha)$; indeed, this is because $\kappa(\alpha, \lambda) = \kappa(-\alpha, s_\alpha(\lambda))$ by W-invariance, where $s_\alpha(\lambda) = \lambda - \langle \check{\alpha}, \lambda \rangle \alpha$. Analogously, if each co-root is twice a co-character (e.g. $G = \mathrm{GL}_2$, PGL_2), then (10.1) always holds. Let $\Lambda_T^r \subset \Lambda_T$ denote the co-root lattice and $\pi_1 G := \Lambda_T / \Lambda_T^r$ be the algebraic fundamental group of G.

We note an elementary fact. Recall the Killing form $q_{\mathrm{det},s}(\lambda) := \frac{1}{2} \sum_{\alpha \in \Phi_s} \langle \check{\alpha}, \lambda \rangle^2$ which has appeared in Chapter 2, §6.1.

10.2.3. **Lemma.** *Suppose* $q \in \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$. *Then there is a (non-canonical) decomposition* $q = q_1 + q_2$ *where:*

(1) q_1 is an $A(-1)$-linear sum of Killing forms $q_{\det,s}$, attached to each irreducible component Φ_s ($s \in \mathbf{S}$) of the coroot system of (G,T);

(2) q_2 descends to a quadratic form on $\pi_1 G$.

Proof. For each $s \in \mathbf{S}$, let α_s be a short coroot of Φ_s. Since $A(-1)$ is divisible, there exists some $b_s \in A(-1)$ such that $q(\alpha_s) = b_s q_{\det,s}(\alpha_s)$. We set $q_1 := \sum_{s\in\mathbf{S}} b_s q_{\det,s}$ and $q_2 := q - q_1$. Thus q_2 still belongs to $\mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$. The identity (10.1) implies that the Λ_T^r lies in the kernel of the bilinear form attached to q_2, so it descends to a quadratic form on $\pi_1 G$. $\square$

Consider the injective map:

$$\mathcal{Q}(\Lambda;\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1) \hookrightarrow \mathcal{Q}(\Lambda; A(-1))^W_{\mathrm{restr}}. \tag{10.2}$$

10.2.4. **Lemma.** *Suppose G_{der} is simply connected. Then* (10.2) *is bijective.*

Proof. The hypothesis shows that $\pi_1 G$ is torsion-free. Hence every $A(-1)$-valued quadratic form on $\pi_1 G$ lives in $\mathcal{Q}(\pi_1 G;\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1)$. $\square$

10.2.5. Θ-*data.* We temporarily relax the condition: $A(-1)$ is only assumed discrete in this paragraph.

Given a lattice Λ, we let $\Theta(\Lambda;\mathbf{G})$ denote the strict Picard 2-groupoid consisting of a quadratic form $q \in \mathcal{Q}(\Lambda; A(-1))$, and a Λ-indexed system of gerbes $\mathcal{G}^{(\lambda)} \in \mathbf{G}(X)$ equipped with multiplicative structures:

$$c_{\lambda,\mu} : \mathcal{G}^{(\lambda)} \otimes \mathcal{G}^{(\mu)} \xrightarrow{\sim} \mathcal{G}^{(\lambda+\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)}, \tag{10.3}$$

together with associativity *constraint* and κ-twisted commutativity *constraint*, i.e., a homotopy $h_{\lambda,\mu}$ witnessing the commutative diagram:

$$\begin{tikzcd} \mathcal{G}^{(\lambda)} \otimes \mathcal{G}^{(\mu)} \arrow[r, "c_{\lambda,\mu}"] \arrow[d] & \mathcal{G}^{(\lambda+\mu)} \otimes \omega_X^{\kappa(\lambda,\mu)} \arrow[d, "(-1)^{\kappa(\lambda,\mu)}"] \arrow[dl, Rightarrow, "h_{\lambda,\mu}"'] \\ \mathcal{G}^{(\mu)} \otimes \mathcal{G}^{(\lambda)} \arrow[r, "c_{\mu,\lambda}"] & \mathcal{G}^{(\mu+\lambda)} \otimes \omega_X^{\kappa(\mu,\lambda)} \end{tikzcd}$$

satisfying the usual coherence conditions for every triple $\mathcal{G}^{(\lambda)}$, $\mathcal{G}^{(\mu)}$, $\mathcal{G}^{(\nu)}$, as well as an additional condition expressing that *strictness* ought to be respected. Namely, for $\lambda = \mu$, as the automorphism $(-1)^{\kappa(\lambda,\lambda)} \xrightarrow{\sim} (-1)^{2q(\lambda)}$ is canonically trivialized, we require that $h_{\lambda,\lambda}$ be the identity 2-homotopy. The strict Picard 2-groupoid $\Theta(\Lambda; \mathbf{G})$ is called Θ-*data for* $\mathbf{G}$.

10.2.6. *Enhanced* Θ*-data.* We reinstall the assumption that $A(-1)$ be divisible.

For G semisimple, simply connected, the morphism (6.4) of Chapter 2, coupled with the divisor class map for $\mathbf{G}$, gives rise to a morphism:

$$\mathcal{Q}(\Lambda_T; \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1) \to \Theta(\Lambda_T; \mathbf{G}), \quad (q, a) \rightsquigarrow (q, (\mathcal{L}^{(\lambda)})^a). \tag{10.4}$$

For a reductive group G, define the *enhanced* Θ*-data* $\Theta_G(\Lambda_T; \mathbf{G})$ *for* $\mathbf{G}$ as the strict Picard 2-groupoid of triples $(q, \mathcal{G}^{(\lambda)}, \varepsilon)$ where:

(1) $q \in \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$ is a *restricted* quadratic form in the sense above, whose bilinear form is denoted κ;

(2) $\mathcal{G}^{(\lambda)}$ is a Λ_T-indexed system in $\mathbf{G}(X)$, equipped with multiplicative structure (10.3), associativity constraint, and κ-twisted commutativity constraint, making $(q, \mathcal{G}^{(\lambda)})$ an object of $\Theta(\Lambda_T; \mathbf{G})$;

(3) ε is an isomorphism between the restriction of $\mathcal{G}^{(\lambda)}$ to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$ and the system of gerbes $\mathcal{G}_q^{(\lambda)}$ attahced to the restriction of q to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$ via (10.4), compatible with the associativity and κ-twisted commutativity constraints.

Therefore, we have a fiber sequence of strict Picard 2-groupoids:

$$\mathbf{Hom}(\pi_1 G, \mathbf{G}(X)) \to \Theta_G(\Lambda_T; \mathbf{G}) \to \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}, \tag{10.5}$$

where $\mathbf{Hom}(\pi_1 G, \mathbf{G}(X))$ denotes the groupoid of morphisms $\pi_1 G \to \mathbf{G}(X)$ in the category of *strict* Picard 2-groupoids.

10.2.7. **Remark.** By the proof of Lemma 7.2.7, the above morphism (10.4) can alternatively be defined using K-theory and the Brylinski–Deligne classification for G semisimple, simply connected.

10.2.8. *ω-shift.* We note a variant in the definition of enhanced Θ-data where we incorporate shifts by a power of ω_X. This is entirely analogous to the situation in Chapter 2.

Define the *ω-shifted* enhanced Θ-data $\Theta^+_G(\Lambda_T; \mathbf{G})$ for $\mathbf{G}$ to be the strict Picard 2-groupoid of triples $(q, \mathcal{G}^{(\lambda)}, \varepsilon)$ where:

(1) $q \in \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$ is as before;

(2) $\mathcal{G}^{(\lambda)}$ is a Λ_T-indexed system in $\mathbf{G}(X)$, equipped with multiplicative structures:

$$c^+_{\lambda,\mu} : \mathcal{G}^{(\lambda)} \otimes \mathcal{G}^{(\mu)} \xrightarrow{\sim} \mathcal{G}^{(\lambda+\mu)},$$

together with associativity constraint and κ-twisted commutativity constraint:

$$\begin{tikzcd}
\mathcal{G}^{(\lambda)} \otimes \mathcal{G}^{(\mu)} \arrow[r, "c^+_{\lambda,\mu}"] \arrow[d] & \mathcal{G}^{(\lambda+\mu)} \arrow[d, "(-1)^{\kappa(\lambda,\mu)}"] \arrow[dl, Rightarrow, "h^+_{\lambda,\mu}"'] \\
\mathcal{G}^{(\mu)} \otimes \mathcal{G}^{(\lambda)} \arrow[r, "c^+_{\mu,\lambda}"] & \mathcal{G}^{(\mu+\lambda)}
\end{tikzcd}$$

satisfying coherence conditions for every triple $\mathcal{G}^{(\lambda)}$, $\mathcal{G}^{(\mu)}$, $\mathcal{G}^{(\nu)}$ and respects strictness.

(3) ε is an isomorphism between the restriction of $\mathcal{G}^{(\lambda)}$ to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$ and the system of gerbes $\mathcal{G}_q^{(\lambda)} \otimes \omega_X^{q(\lambda)}$, where $\mathcal{G}_q^{(\lambda)}$ is the system attahced to the restriction of q to $\Lambda_{\widetilde{T}_{\mathrm{der}}}$ via (10.4), compatible with the associativity and κ-twisted commutativity constraints.

Clearly, there is an equivalence between the two kinds of enhanced Θ-data:

$$\Theta_G(\Lambda_T; \mathbf{G}) \xrightarrow{\sim} \Theta_G^+(\Lambda_T; \mathbf{G}), \quad (q, \mathcal{G}^{(\lambda)}, \varepsilon) \rightsquigarrow (q, \mathcal{G}^{(\lambda)} \otimes \omega_X^{q(\lambda)}, \varepsilon).$$

10.3. **Classification: statement.**

10.3.1. We can now state the main classification theorem for factorization gerbes. Informally, it states that motivic factorization gerbes are classified by their enhanced Θ-data, in a way compatible with the classification of factorization line bundles (Chapter 2, Theorem 8.1.2).

We fix a topology t on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ satisfying the properties in §9.3. The following Theorem applies to any reductive group G over k with maximal torus $T \subset G$, and any connected, smooth algebraic curve X.

10.3.2. **Theorem.** *Let $\mathbf{G}$ be a motivic t-theory of gerbes whose coefficient $A(-1)$ is a divisible abelian group. Then there is a canonical equivalence between strict Picard 2-groupoids:*

$$\Psi_{\mathbf{G}} : \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \Theta_G(\Lambda_T; \mathbf{G}),$$

which makes the following diagram commute:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}"] & \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{G}}", "\cong"'] \\ \Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathbf{G}) \end{tikzcd} \tag{10.6}$$

We call $\Psi_{\mathbf{G}}$ the *classification functor* for factorization gerbes on $\mathrm{Gr}_{G,\mathrm{Ran}}$. As before, we sometimes denote it by $\Psi_{\mathbf{G},G}$ to emphasize the role of the reductive group G. We recall that $\Psi_{\mathbf{Pic}}$ is also an equivalence when $\mathrm{char}(k) \nmid N_G$, and when G is semisimple and simply connected.

11. Proof of the classification theorem

This entire section is devoted to the proof of Theorem 10.3.2. The proof uses the same strategy as that of Chapter 2, Theorem 8.1.2 and many of the same techniques. However, thank to the following two major differences, the proof is in fact simpler.

(1) The sheaf $\mathbf{G}$ satisfies t-descent, so the proof for tori becomes considerably simpler and no derived technique is needed in the general case.

(2) The coefficient group $A(-1)$ is divisible, so the proof of essential surjectivity of $\Psi_{\mathbf{G}}$ is nearly tautological. In particular, this allows us to obtain the result without any restriction on $\mathrm{char}(k)$.

The only additional complication is that $A(-1)$ can have 2-torsion elements, hence the appearance of restricted quadratic forms.

Throughout this section, we fix a topology t on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ stronger than the étale topology and such that every object of $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ is t-locally smooth. Let X be a smooth curve, G a reductive group, and $\mathbf{G}$ be a motivic t-theory of gerbes whose coefficient group $A(-1)$ is divisible.

11.1. **Tori.**

11.1.1. We first recall the combinatorial affine Grassmannian $\mathrm{Gr}_{T,\mathrm{comb}}$ of §5.

11.1.2. **Lemma.** *There is a canonical equivalence of strict Picard* 2*-groupoids:*

$$\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \xrightarrow{\sim} \Theta(\Lambda_T; \mathbf{G}).$$

Proof. One argues as in Chapter 2, Lemma 5.2.2. $\square$

We define $\Psi_{\mathbf{G},T}$ as the composition:

$$\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \to \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \\ \xrightarrow{\sim} \Theta(\Lambda_T; \mathbf{G}).$$

11.1.3. **Lemma.** *The following canonical map is an isomorphism after* t*-sheafification:*

$$\mathrm{Gr}_{T,\mathrm{comb}} \to \mathrm{Gr}_{T,\mathrm{Ran}}.$$

Proof. The map is clearly a monomorphism of prestacks.[12] It suffices to check that it is surjective in the t-topology, and we reduce immediately to the case $T = \mathbb{G}_m$. Consider any S-point $(x^{(i)}, \mathcal{L}, \alpha)$ of $\mathrm{Gr}_{\mathbb{G}_m,\mathrm{Ran}}$. It belongs to $\mathrm{Gr}_{\mathbb{G}_m,\mathrm{comb}}$ if and only if L is isomorphic to $\mathcal{O}(\sum_i \lambda_i \Gamma_{x^{(i)}})$ for some $\lambda_i \in \mathbb{Z}$, and α identifies with its canonical trivialization. This is indeed the case after passing to any τ-cover $\widetilde{S} \to S$ with $\widetilde{S}$ smooth. $\square$

11.1.4. **Proposition.** *The functor* $\Psi_{\mathbf{G},T}$ *is an equivalence of strict Picard* 2*-groupoids.*

Proof. Since $\mathbf{G}$ satisfies t-descent, the Lemma implies that we have an isomorphism:

$$\mathbf{G}(\mathrm{Gr}_{T,\mathrm{Ran}}) \xrightarrow{\sim} \mathbf{G}(\mathrm{Gr}_{T,\mathrm{comb}}). \tag{11.1}$$

Moreover, the map $\mathrm{Gr}_{T,\mathrm{Ran}}^{\times n} \to \mathrm{Gr}_{T,\mathrm{comb}}^{\times n}$ is an isomorphism after t-sheafification for all $n \geq 1$. Therefore the isomorphism (11.1) lifts to one between factorization sections of $\mathbf{G}$. $\square$

11.2. **Semisimple, simply connected groups.**

[12]Here, it is crucial that we work with *classical* prestacks.

11.2.1. For any reductive group G with a fixed maximal torus T, we consider the composition:

$$\begin{aligned}\mathbf{Q}_{\mathbf{G},G} : \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to & \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \\ & \xrightarrow{\sim} \Theta(\Lambda_T; \mathbf{G}) \to \mathcal{Q}(\Lambda_T; A(-1))\end{aligned}$$

Thus $\mathbf{Q}_{\mathbf{G},G}$ associates a quadratic form to any factorization gerbe. This functor will be the basis of the classification of factorization gerbes for semisimple, simply connected groups.

11.2.2. Let G_{sc} be a semisimple, simply connected group with maximal torus T_{sc}. We let $\mathbf{S}$ denote the set of its simple factors. We recall the following equivalence of Chapter 2, Theorem 6.2.5:

$$\Psi_{\mathbf{Pic},G_{\mathrm{sc}}} : \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \xrightarrow{\sim} \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W. \tag{11.2}$$

In fact, $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W$ canonically identifies with $\mathrm{Maps}(\mathbf{S}, \mathbb{Z})$. For each $s \in \mathbf{S}$, the mapping which sends s to 1 and all other elements to zero passes to the *minimal* quadratic form $q_{\mathrm{min},s}$ on $\Lambda_{T_{\mathrm{sc}}}$ which has $q_{\mathrm{min},s}(\alpha_s) = 1$ for α_s a short coroot in Φ_s and vanishes on components associated to other simple factors. Under (11.2), this passes to the minimal line bundle min_s which has a canonical factorization structure.

11.2.3. The inverse of (11.2) paired with the divisor class map defines a functor:

$$\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1) \to \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}). \tag{11.3}$$

By construction, the composition of (11.3) with $\mathbf{Q}_{\mathbf{G},G}$ is the forgetful map from $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1)$ to $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; A(-1))$.

11.2.4. Fix a point $x \in X$. By the classification for tori, we see that every factorization gerbe on $\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}$ is canonically trivialized when pulled back to the unit section. Thus the functor of restriction to x factors through the category of gerbes rigidified at the unit point:

$$\operatorname{Res}_x : \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \to \mathbf{G}^e(\mathrm{Gr}_{G_{\mathrm{sc}},x}). \tag{11.4}$$

11.2.5. **Lemma.** *The composition of* (11.3) *with* Res_x *is an equivalence:*

$$\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}};\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1) \xrightarrow{\sim} \mathbf{G}^e(\mathrm{Gr}_{G_{\mathrm{sc}},x}).$$

In particular, $\mathbf{G}^e(\mathrm{Gr}_{G_{\mathrm{sc}},x})$ is discrete.

Proof. By the product decomposition (Lemma 9.4.6), we reduce to the case $\mathbf{S} = \{1\}$, i.e., G_{sc} is simple and simply connected. We shall denote it simply by G. Choose a uniformizer t of $\widehat{\mathcal{O}}_{X,x}$ and identify $\mathrm{Gr}_{G,x}$ with the étale quotient $G(\!(t)\!)/G[\![t]\!]$. Recall that the projection $\mathrm{Fl}_G \to \mathrm{Gr}_{G,x}$ is an étale-locally trivial fiber bundle with typical fiber G/B (Chapter §1).

We first observe that $\mathbf{G}^e(G/B)$ is canonically isomorphic to $\mathrm{Maps}(\Delta, A(-1))$ for Δ the set of simple roots. Indeed, a gerbe rigidified at the unit point e of the big Bruhat cell N^-e must be trivialized over N^-e (Property (A)). The complement of N^-e is an effective Cartier divisor whose irreducible components are labeled by Δ. An appliction of Properties (RP1) and (RP2) shows that $\mathbf{G}^e(G/B) \xrightarrow{\sim} \mathrm{Maps}(\Delta, A(-1))$.

Therefore, the product decomposition and étale descent shows that the pullback:

$$\mathbf{G}^e(\mathrm{Gr}_{G,x}) \to \mathbf{G}^e(\mathrm{Fl}_G)$$

is fully faithful, and its image identifies with *fiberwise* trivial objects. Since $\mathrm{Gr}_{G,x}$ is connected, the condition on fiberwise triviality is equivalent to triviality along the unit fiber $G/B \hookrightarrow \mathrm{Fl}_G$.

Next, we classify gerbes on Fl_G using the geometric description in Chapter 1, §2.3. Namely, we let $A \subset W^{\mathrm{aff}}$ be a subset closed under descendance. Take $n \geq 1$

sufficiently large with respect to A, and we have a Cartesian diagram for any $\tilde{w} \in A$:

$$\begin{tikzcd} \mathrm{Fl}_{G,\tilde{w}} \arrow[r, hook] \arrow[d] & \Omega_A \arrow[d] \\ \mathbb{A}^d \arrow[r, hook] & \Omega_A/\mathring{I}^-(n) \end{tikzcd}$$

Furthermore, $\mathrm{Fl}_{G,\tilde{w}}$ of codimension $l(\tilde{w})$.

We now make the observation that $\mathring{I}^-(n)$ has a contracting $\mathbb{G}_m$-action by scaling t. Indeed, $G[t^{-1}]$ already admits a contracting $\mathbb{G}_m$-action which preserves $\mathring{I}^-(n)$. The fixed-point locus in $G[t^{-1}]$ is the subgroup G and we have $\mathring{I}^-(n) \cap G = \{1\}$. By Lemma 9.4.4, $\mathbf{G}(\Omega_A)$ identifies with $\mathbf{G}(\Omega_A \times \mathring{I}^-(n)^\bullet)$, so étale descent implies an equivalence:

$$\mathbf{G}(\Omega_A/\mathring{I}^-(n)) \xrightarrow{\sim} \mathbf{G}(\Omega_A).$$

On the other hand, for A sufficiently large, the complement of the big cell $\mathrm{Fl}_{G,1}/\mathring{I}^-(n)$ in $\Omega_A/\mathring{I}^-(n)$ is the union of effective Cartier divisors corresponding to the set of simple *affine* roots $\Delta^{\mathrm{aff}} = \Delta \sqcup \{\alpha_0\}$. Thus an argument as for the usual flag variety implies that $\mathbf{G}^e(\Omega_A/\mathring{I}^-(n)) \xrightarrow{\sim} \mathrm{Maps}(\Delta^{\mathrm{aff}}, A(-1))$. Summarizing, we have:

$$\begin{aligned} \mathbf{G}^e(\mathrm{Gr}_{G,x}) \hookrightarrow \mathbf{G}^e(\mathrm{Fl}_G) \xrightarrow{\sim} \mathbf{G}^e(\Omega_A) & \\ \xrightarrow{\sim} \mathbf{G}^e(\Omega_A/\mathring{I}^-(n)) \xrightarrow{\sim} \mathrm{Maps}(\Delta^{\mathrm{aff}}, A(-1)). \end{aligned}$$

It remains to observe that the restriction $\mathbf{G}^e(\mathrm{Fl}_G) \to \mathbf{G}^e(G/B)$ to the unit fiber passes to the restriction of functions $\mathrm{Maps}(\Delta^{\mathrm{aff}}, A(-1)) \to \mathrm{Maps}(\Delta, A(-1))$, and furthermore, the gerbe $\mathcal{G} \in \mathbf{G}^e(\mathrm{Gr}_G)$ corresponding to the function with value $a \in A(-1)$ at α_0 is precisely the ath power of the minimal line bundle on $\mathrm{Gr}_{G,x}$. $\square$

11.2.6. We now analyze the process of restriction to $x \in X$. Let A' denote the abelian group $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1) \cong \mathrm{Maps}(\mathbf{S}, A(-1))$. Write $\mathbf{G}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^n}$ for the (small) étale sheaf on X^n whose value at $S \to X^n$ is the strict Picard 2-groupoid of gerbes on $\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} S$ trivialized at the unit section. For $n = 1$, the functor (11.3) defines a morphism of étale sheaves on X:

$$\underline{A}'_X \to \mathbf{G}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X}. \tag{11.5}$$

By Lemma 11.2.5 and Property (B), the stalks of (11.5) at any k-point $x \in X$ are mutual retracts. Hence (11.5) is an isomorphism. Now, the divisor class map and the exact sequence of Chapter 2, Lemma 6.2.4 induces a morphism:

(11.6)

$$\begin{tikzcd} 0 \arrow[r] & \mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I} \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r] \arrow[d, "(a)"] & \boxtimes_{i\in I}\underline{A}'_X \arrow[r, "(b)"] \arrow[d, "\cong"] & \bigoplus_{\substack{I\twoheadrightarrow J \\ |J|=|I|-1}} (\Delta_{I\twoheadrightarrow J})_* \boxtimes_{j\in J} \underline{A}'_X \arrow[d, "\cong"] \\ & \mathbf{G}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I} \arrow[r] & \boxtimes_{i\in I}\underline{A}'_X \arrow[r] & \bigoplus_{\substack{I\twoheadrightarrow J \\ |J|=|I|-1}} (\Delta_{I\twoheadrightarrow J})_* \boxtimes_{j\in J} \underline{A}'_X \end{tikzcd}$$

Here, the morphism $\mathbf{G}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I} \to \boxtimes_{i\in I}\underline{A}'_X$ is defined by restriction away from all diagonals using (11.5). By checking on stalks using Property (B), we see that (a) is also an equivalence. This implies that $\mathbf{G}^e_{\mathrm{Gr}_{G_{\mathrm{sc}}}/X^I}$ identifies with kernel of the map (b).

Since (b) is defined by taking difference along each diagonal, we see that restriction to $x \in X$ defines an equivalence:

$$\mathbf{G}^e(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \xrightarrow{\sim} \mathbf{G}^e(\mathrm{Gr}_{G_{\mathrm{sc}},x}).$$

Tautologically, the functor Res_x (11.4) factors through the above equivalence.

11.2.7. **Lemma.** *The functor* Res_x *is fully faithful.*

Proof. It remains to prove that the forgetful functor:

$$\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \to \mathbf{G}^e(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}})$$

is fully faithful. Since rigidified gerbes on $(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}})^{\times 2}_{\mathrm{disj}}$ are classified by the discrete abelian group $A' \times A'$, a factorization structure is unique if it exists. $\square$

We now finish the classification for G_{sc}.

11.2.8. **Lemma.** *Let G_{sc} be a semisimple, simply connected group. Then $\mathbf{Q}_{\mathbf{G},G}$ has image in $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}, A(-1))^W_{\mathrm{restr}}$ and defines an equivalence:*

$$\Psi_{\mathbf{G},G_{\mathrm{sc}}} : \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}}}) \xrightarrow{\sim} \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; A(-1))^W_{\mathrm{restr}}.$$

Proof. Recall that $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; A(-1))^W_{\mathrm{restr}}$ identifies with $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; \mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} A(-1)$ (Lemma 10.2.4). We have seen that there is a factoring of its embedding inside $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; A(-1))$ as follows.

$$\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; A(-1))^W_{\mathrm{restr}} \to \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}}}) \xrightarrow{\mathcal{Q}_{\mathbf{G},G}} \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}}; A(-1)).$$

The first functor is an equivalence by combining Lemma 11.2.5 and Lemma 11.2.7. $\square$

11.3. Construction of $\Psi_{\mathbf{G}}$.

11.3.1. We start with a mild generalization of the classification result for semisimple, simply connected groups. Let G be a reductive group whose derived subgroup G_{der} is simply connected. Denote by T_1 the quotient torus G/G_{der}. We know by Chapter 2, Lemma 8.3.4 that the projection $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{T_1,\mathrm{Ran}}$ is an étale fiber bundle with typical fiber $\mathrm{Gr}_{G_{\mathrm{der}},\mathrm{Ran}}$. In other words, to every S-point of $\mathrm{Gr}_{T_1,\mathrm{Ran}}$ one can associate an étale cover $\widetilde{S} \to S$ and an isomorphism:

$$\widetilde{S} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{G_{\mathrm{der}},\mathrm{Ran}} \xrightarrow{\sim} \widetilde{S} \underset{\mathrm{Gr}_{T_1,\mathrm{Ran}}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}\,. \tag{11.7}$$

We will now identify the fiber of $\mathcal{Q}_{\mathbf{G},G}$ when G_{der} is simply connected.

11.3.2. **Lemma.** *Suppose G_{der} is simply connected. Then pulling back along $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{T_1,\mathrm{Ran}}$ defines a fiber sequence of strict Picard 2-groupoids:*

$$\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}}) \to \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathcal{Q}(\Lambda_{T_{\mathrm{der}}}; A(-1)).$$

Proof. Let $\mathbf{G}_{\mathrm{Gr}_G/\mathrm{Gr}_{T_1}}$ denote the étale sheafification of the presheaf on Gr_{T_1}:

$$S \rightsquigarrow \mathrm{Cofib}(\mathbf{G}(S) \to \mathbf{G}(S \underset{\mathrm{Gr}_{T_1,\mathrm{Ran}}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}})).$$

Let $\mathbf{Pic}_{\mathrm{Gr}_G/\mathrm{Gr}_{T_1}}$ be the analogously defined étale sheaf where we replace $\mathbf{G}$ by $\mathbf{Pic}$. We claim that the divisor class map $\mathbf{Pic}_{\mathrm{Gr}_G/\mathrm{Gr}_{T_1}} \underset{\mathbb{Z}}{\otimes} A(-1) \to \mathbf{G}_{\mathrm{Gr}_G/\mathrm{Gr}_{T_1}}$ is an isomorphism. Indeed, it suffices to show the map on presheaves is an étale local equivalence. Take any S-point of Gr_{T_1}, an étale cover $\widetilde{S}$ together with an isomorphism (11.7) reduces the claim to identifying the cofibers of the horizontal maps:

$$\begin{array}{ccc} \mathbf{Pic}(\widetilde{S}) \underset{\mathbb{Z}}{\otimes} A(-1) & \longrightarrow & \mathbf{Pic}(\widetilde{S} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{G_{\mathrm{der}}}) \underset{\mathbb{Z}}{\otimes} A(-1) \\ \downarrow & & \downarrow \\ \mathbf{G}(\widetilde{S}) & \longrightarrow & \mathbf{G}(\widetilde{S} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{G_{\mathrm{der}}}) \end{array}$$

This in turn follows from the identification $\mathbf{Pic}^e_{\mathrm{Gr}_{G_{\mathrm{der}}}/X^I} \underset{\mathbb{Z}}{\otimes} A(-1) \xrightarrow{\sim} \mathbf{G}^e_{\mathrm{Gr}_{G_{\mathrm{der}}}/X^I}$ of (11.6).

In particular, $\mathbf{G}_{\mathrm{Gr}_G/\mathrm{Gr}_{T_1}}$ is a subsheaf of a constant étale sheaf of abelian groups, so one may apply the argument of [62, §3.4.3]. Namely, starting with a section g of $\mathbf{G}_{\mathrm{Gr}_G/\mathrm{Gr}_{T_1}}$ over $\mathrm{Gr}_{T_1,\mathrm{Ran}}$, the hypothesis shows that g vanishes over the unit section. Then factorization of $\mathrm{Gr}_{T_1,\mathrm{Ran}}$ allows one to bootstrap to the vanishing of g. □

Let us control the type of quadratic forms that can arise from factorization gerbes. We remove the assumption on G_{der} and instead consider any reductive group G.

11.3.3. **Lemma.** *The image of $\mathcal{Q}_{\mathbf{G},G}$ is contained in $\mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$.*

Proof. Let $\mathcal{G} \in \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$ and $q := \mathcal{Q}_{\mathbf{G},G}(\mathcal{G})$. We need to establish the following identities for each simple co-root α_i and co-character $\lambda \in \Lambda_T$.

(1) $q(s_{\alpha_i}(\lambda)) = q(\lambda)$;

(2) $\kappa(\alpha_i, \lambda) = \langle \check{\alpha}_i, \lambda \rangle q(\alpha_i)$.

Consider the parabolic subgroup $P \subset G$ generated by T and α_i. The quotient of P by its nilradical N_P is a reductive group M of semisimple rank 1. We have the following maps:

$$\begin{tikzcd} & \mathrm{Gr}_{P,\mathrm{Ran}} \arrow[dl, "\mathfrak{p}"'] \arrow[dr, "\mathfrak{q}"] & \\ \mathrm{Gr}_{G,\mathrm{Ran}} & & \mathrm{Gr}_{M,\mathrm{Ran}} . \end{tikzcd}$$

We observe that $\mathfrak{q}$ is an étale fiber bundle with typical fiber $\mathrm{Gr}_{N_P,\mathrm{Ran}}$. Indeed, this is because $P \to M$ admits a splitting, so one may argue as in Chapter 2, Lemma 8.3.4.

On the other hand, there is a contracting $\mathbb{G}_m$-action on $\mathrm{Gr}_{N_P,\mathrm{Ran}}$ given by the co-root α_i whose fixed point locus is the unit section. By Lemma 9.4.4 and étale descent, we see that $\mathfrak{p}^*\mathcal{G}$ canonically identifies with $\mathfrak{q}^*\mathcal{G}_M$ for some $\mathcal{G}_M \in \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{M,\mathrm{Ran}})$. Regarding α_i as a co-root of M, we reduce the problem to reductive groups of semisimple rank 1, with unique simple co-root α. Such a group G must be the direct product of a torus T_1 with $G_1 = \mathrm{SL}_2$, GL_2, or PGL_2.

To verify (1), we exhibit two paths $\gamma_1, \gamma_2 : \mathbb{A}^1 \to G$ such that:

$$\gamma_1(0) = e, \quad \gamma_1(1) = \gamma_2(1), \quad \gamma_2(0) = \tilde{s}_\alpha.$$

where $\tilde{s}_\alpha$ a lift of $s_\alpha \in W$ to G. For instance, we may set γ_1, γ_2 to be identity on the factor T_1 and be given by the following matrices for the G_1 factor:

$$\gamma_1(t) = \begin{pmatrix} 1 & 2t \\ 0 & 1 \end{pmatrix}, \quad \gamma_2(t) = \begin{pmatrix} t & t+1 \\ t-1 & t \end{pmatrix}.$$

As G acts on itself by inner automorphisms, we have action morphisms $\mathbb{A}^1 \times \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ defined by γ_1 and γ_2. Pulling back $\mathcal{G}$ produces two factorization gerbes $\mathcal{G}_{\gamma_1}$, $\mathcal{G}_{\gamma_2}$ on $\mathbb{A}^1 \times \mathrm{Gr}_{G,\mathrm{Ran}}$. Thus $\mathbb{A}^1$-invariance (Lemma 9.4.4) gives isomorphisms:

$$\mathcal{G} \xrightarrow{\sim} \gamma_1(1)^*\mathcal{G} \xrightarrow{\sim} \gamma_2(1)^*\mathcal{G} \xrightarrow{\sim} \tilde{s}_\alpha^*\mathcal{G}.$$

This proves identity (1).

For identity (2), we only need to consider the case $G = T_1 \times \mathrm{SL}_2$ as the other two cases are vacuous (c.f. §10.2). We claim that external product defines an equivalence:

$$\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}}) \times \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{\mathrm{SL}_2,\mathrm{Ran}}) \xrightarrow{\sim} \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

Indeed, given $\mathcal{G} \in \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_G)$, pulling back along $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{\mathrm{SL}_2,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$ and taking the quotient, we obtain a gerbe $\mathcal{G}_1 \in \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_G)$ whose associated quadratic form vanishes on $\Lambda_{T_{\mathrm{der}}}$. Since SL_2 is simply connected, Lemma 11.3.2 applies and we see that $\mathcal{G}_1$ is pulled back from $\mathrm{Gr}_{T_1,\mathrm{Ran}}$. Having the product decomposition, the desired identity follows from the classification for semisimple, simply connected groups (Lemma 11.2.8). □

11.3.4. We now combine the above ingredients to build the classification functor:

$$\Psi_{\mathbf{G}} : \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \Theta_G(\Lambda_T; \mathbf{G}).$$

Indeed, given $\mathcal{G} \in \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$, the procedure of §11.2 produces a Θ-datum $(q, \mathcal{G}^{(\lambda)}) \in \Theta(\Lambda_T; \mathbf{G})$. Lemma 11.3.3 shows that q indeed lies in $\mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}$.

It remains to produce the isomorphism ε. Indeed, the restriction of $\mathcal{G}$ to $\mathrm{Gr}_{\widetilde{G}_{\mathrm{der}},\mathrm{Ran}}$ is the factorization gerbe classified by $q\big|_{\Lambda_{\widetilde{T}_{\mathrm{der}}}}$ via Lemma 11.2.8. Thus we obtain an isomorphism ε of Θ-data for the lattice $\Lambda_{\widetilde{T}_{\mathrm{der}}}$ by functoriality of pullback along the following diagram.

$$\begin{tikzcd}
\mathrm{Gr}_{\widetilde{T}_{\mathrm{der}},\mathrm{Ran}} \arrow[r] \arrow[d] & \mathrm{Gr}_{\widetilde{G}_{\mathrm{der}},\mathrm{Ran}} \arrow[d] \\
\mathrm{Gr}_{T,\mathrm{Ran}} \arrow[r] & \mathrm{Gr}_{G,\mathrm{Ran}}
\end{tikzcd}$$

11.4. $\Psi_{\mathbf{G}}$ is an equivalence.

11.4.1. Our final goal is to prove that the classification functor $\Psi_{\mathbf{G}}$, constructed in the previous subsection, is an equivalence of categories. As in Chapter 2, §8.2, we first want to reduce to the case where the derived subgroup is simply connected.

11.4.2. We fix a z-extension G' of G and let T_1 be the quotient torus G'/G'_{der}. Then the quotient of lattices $\Lambda_{T_1}/\Lambda_{T_2}$ identifies with $\pi_1 G$. By Chapter 2, Lemma 8.2.2, we see that the morphism:

$$\mathrm{Gr}_{G',\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$$

is surjective in the t-topology. Since $\mathbf{G}$ satisfies t-descent, we have a commutative diagram of strict Picard 2-groupoids, where the horizontal maps are equivalences:

$$\begin{tikzcd}
\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\sim"] \arrow[d, "\Psi_{\mathbf{G},G}"] & \lim_{\Delta^{\mathrm{op}}} \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G'\times T_2^{\bullet},\mathrm{Ran}}) \arrow[d, "\Psi_{\mathbf{G},G'\times T_2^{\bullet}}"] \\
\Theta_G(\Lambda_T;\mathbf{G}) \arrow[r, "\sim"] & \lim_{\Delta^{\mathrm{op}}} \Theta_{G'\times T_2^{\bullet}}(\Lambda_{T'\times T_2^{\bullet}}, \mathrm{Ran})
\end{tikzcd}$$

Therefore, in proving that $\Psi_{\mathbf{G},G}$ is an equivalence, we may assume:

—*the derived subgroup G_{der} is simply connected.*

Under this assumption, we can write $T_1 = G/G_{\mathrm{der}}$ and Λ_{T_1} is isomorphic to $\pi_1 G$.

11.4.3. **Lemma.** *Suppose G_{der} is simply connected. Then $\Psi_{\mathbf{G},G}$ is an equivalence.*

Fully faithfulness. Since $\Psi_{\mathbf{G},G}$ is a morphism of strict Picard 2-groupoids, it suffices to show that $\Psi_{\mathbf{G}}$ has contractible fiber at $\mathbf{0} \in \Theta_G(\Lambda_T;\mathbf{G})$. Let $(\mathcal{G};\alpha)$ be an object of the fiber, so $\mathcal{G} \in \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$ and α is a trivialization of its image $(q, \mathcal{G}^{(\lambda)}, \varepsilon) \in \Theta_G(\Lambda_T;\mathbf{G})$. Since $q = 0$, Lemma 11.3.2 implies that $\mathcal{G}$ descends to a factorization gerbe $\mathcal{G}_1$ over $\mathrm{Gr}_{T_1,\mathrm{Ran}}$.

By the classification for tori (§11.1), we see that $\mathcal{G}_1$ corresponds to an object in $\Theta(\Lambda_T;\mathbf{G})$ with vanishing quadratic form, i.e., an object of $\mathbf{Hom}(\Lambda_{T_1}, \mathbf{G}(X))$. In particular, the datum of the trivialization α is equivalent to a trivialization of $\mathcal{G}_1$. $\square$

Essential surjectivity. We have a morphism between fiber sequences of strict Picard 2-groupoids, where the top fiber sequence comes from Lemma 11.3.2 and the classification for tori.

$$\begin{tikzcd}
\mathbf{Hom}(\Lambda_{T_1}, \mathbf{G}(X)) \arrow[r] \arrow[d, "\cong"] & \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\alpha"] \arrow[d, "\Psi_{\mathbf{G}}"] & \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}} \arrow[d, "\cong"] \\
\mathbf{Hom}(\Lambda_{T_1}, \mathbf{G}(X)) \arrow[r] & \Theta_G(\Lambda_T; \mathbf{G}) \arrow[r] & \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}
\end{tikzcd}$$

By the 4-lemma, it is enough to show that α is surjective. We note that the determinant line bundle construction of Chapter 2, §6.1 gives a section:

$$\begin{tikzcd}
 & \bigoplus_{s \in \mathbf{S}} A(-1) \arrow[dl, "\det"'] \arrow[d, "\mathrm{Kil}"] \\
\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\alpha"] & \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}
\end{tikzcd}$$

Thus, by Lemma 10.2.3, it remains to consider quadratic forms pulled back from $\mathcal{Q}(\Lambda_{T_1}; A(-1))$. However, each such form q lifts to some Θ-datum $(q, \mathcal{G}^{(\lambda)}) \in \Theta(\Lambda_{T_1}; \mathbf{G})$ after choosing a square root $\frac{1}{2}q$. Indeed, such choice is possible because Λ_{T_1} is free and $A(-1)$ is divisible. We are thus done by the section ν:

$$\begin{tikzcd}
 & \Theta(\Lambda_{T_1}; \mathbf{G}) \arrow[dl, "\nu"'] \arrow[d] \\
\mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\alpha"] & \mathcal{Q}(\Lambda_T; A(-1))^W_{\mathrm{restr}}
\end{tikzcd}$$

constructed by composing the equivalence $\Psi^{-1}_{\mathbf{G},T_1} : \Theta(\Lambda_{T_1}; \mathbf{G}) \xrightarrow{\sim} \mathbf{G}^{\mathrm{fact}}(\mathrm{Gr}_{T_1,\mathrm{Ran}})$ with the pullback along $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{T_1,\mathrm{Ran}}$. $\square$

$\square$(Theorem 10.3.2)

12. The étale context

In this section, we specialize to the context of constructible étale sheaves. We will introduce the étale theory of gerbes valued in a suitable torision abelian group.

12.1. $\mathbf{Ge}_{\text{ét}}$.

12.1.1. We fix a torsion abelian group A the order of whose elements are indivisible by $p := \text{char}(k)$. We shall describe a motivic h-theory of gerbes with coefficients in A. In practice, this gerbe theory can be used to twist the DG category of constructible étale $\mathbb{Z}/n\mathbb{Z}$-sheaves and A will be a subgroup of $(\mathbb{Z}/n\mathbb{Z})^\times$ (well chosen so that A has no p-torsion). In the context of metaplectic Langlands program, this gerbe theory has been considered by Gaitsgory–Lysenko [32].

On the other hand, if one is interested in the DG category of constructible étale $\mathbb{F}_p$-sheaves, such as the context of Cass [13], then taking $A = \mathbb{F}_p^\times$ suffices.

12.1.2. We define the sheaf $\mathbf{Ge}_{\text{ét}}$ of strict Picard 2-groupoids on $\mathbf{Sch}^{\text{ft}}_{/k}$ by:

$$\mathbf{Ge}_{\text{ét}}(X) := \text{Maps}(X, \mathrm{B}^2_{\text{ét}}\, A).$$

Thus the fiber of $\mathbf{Ge}_{\text{ét}}(\mathbb{A}^1) \to \mathbf{Ge}_{\text{ét}}(\mathbb{A}^1 \backslash \{\mathbf{0}\})$ identifies with the usual Tate twist:

$$A(-1) \xrightarrow{\sim} \operatorname*{colim}_{n|n'} \text{Hom}(\mu_n(k), A),$$

where for $n \mid n'$, the transition map $\mu_{n'}(k) \to \mu_n(k)$ is given by raising to (n'/n)th power. As A has no p-torsion, we may take n to be indivisible by p in this colimit. Since $k = \bar{k}$, the map $\underline{\mu_n(k)} \to \mu_n$ is an isomorphism of étale sheaves on $\mathbf{Sch}^{\text{ft}}_{/k}$. Therefore $A(-1)$ is also the colimit of Hom-groups of étale sheaves $\operatorname*{colim}_{n|n'} \text{Hom}(\mu_n, \underline{A})$.

12.1.3. The divisor class map:

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} A(-1) \to \mathbf{Ge}_{\text{ét}}, \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a$$

can be constructed as follows (c.f. [32, §1.4]). The Kummer exact sequence gives rise to a map $\theta_n : \mathbf{Pic} \to \mathrm{B}^2_{\text{ét}}\,\mu_n$ for each n indivisible by p, such that for $n \mid n'$ the following diagram commutes:

$$\begin{tikzcd} & \mathrm{B}^2_{\text{ét}}\,\mu_{n'} \arrow[dd, "(-)^{n'/n}"] \\ \mathbf{Pic} \arrow[ur, "\theta_{n'}"] \arrow[dr, "\theta_n"'] & \\ & \mathrm{B}^2_{\text{ét}}\,\mu_n \end{tikzcd}$$

Therefore, a pair $(\mathcal{L}, a)$ gives rise to a section of $\mathrm{B}^2_{\text{ét}}\,A$, to be denoted by $\mathcal{L}^a$.

12.1.4. **Lemma.** *The sheaf* $\mathbf{Ge}_{\text{ét}}$ *and* c_1 *define a motivic* h*-theory of gerbes.*

Proof. The properties (RP1), (RP2), (A), and (B) are all standard facts of étale cohomology. Finally, we claim that $\mathbf{Ge}_{\text{ét}}$ satisfies h-descent. Indeed, by a theorem of Suslin–Voevodsky [60], $\underline{A}$ is a sheaf in the h-topology and one has canonical isomorphisms:

$$\mathrm{H}^i_{\text{ét}}(X; A) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{h}}(X; A), \quad \text{for all } i \geq 0.$$

In particular, this shows that étale A-gerbes agree with A-gerbes in the h-topology.

$\square$

12.1.5. **Corollary.** *There is a canonical equivalence of strict Picard* 2*-groupoids:*

$$\Psi_{\text{ét}} : \mathbf{Ge}^{\text{fact}}_{\text{ét}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \Theta_G(\Lambda_T; \mathbf{Ge}_{\text{ét}}),$$

which makes the following diagram commute:

$$
\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}"] & \mathbf{Ge}_{\text{ét}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\text{ét}}", "\cong"'] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathbf{Ge}_{\text{ét}})
\end{tikzcd}
\tag{12.1}
$$

Proof. This follows from Theorem 10.3.2 and Lemma 12.1.4. □

12.2. **Construction functor.**

12.2.1. We now introduce the étale gerbe-theoretic analogue of the Brylinski–Deligne data. It will admit a canonical functor $\Xi_{\text{ét}}$ to $\mathbf{Ge}_{\text{ét}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}})$, and we will discuss its compatibility with the construction functor $\Xi_{\mathbf{Pic}}$. For convenience, we will assume $A = \mathbb{Z}/\ell\mathbb{Z}$ with $\mathrm{char}(k) \nmid \ell$.

12.2.2. Let us consider the Picard groupoid of central extensions of G by $\mathrm{B}^2_{\text{ét}}\, A(2)$ as big étale sheaves over X. By definition, such an extension is a multiplicative morphism $X \times G \to \mathrm{B}^3_{\text{ét}}\, A(2)$, or equivalently, a pointed morphism $X \times \mathrm{B}_{\text{ét}}\, G \to \mathrm{B}^4_{\text{ét}}\, A(2)$. Thus we have an equivalence:

$$\mathbf{CExt}(G, \mathrm{B}^2_{\text{ét}}\, A(2)) \xrightarrow{\sim} \mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(2)).$$

12.2.3. Following [32], we shall construct a canonical functor:

$$\Xi_{\text{ét}} : \mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(1)) \to \mathbf{Ge}_{\text{ét}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

Indeed, given a central extension, regarded as a map $\mathbf{E}$ to $\mathrm{B}^4_{\text{ét}}\, A(1)$ with pointing γ, as well as a point $S \to \mathrm{Gr}_{G,\mathrm{Ran}}$ represented by $(x^I, \mathcal{P}_G, \alpha)$, we obtain two canonically commutative squares:

$$\begin{tikzcd}
U_{x^I} \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] \arrow[dl, Rightarrow, "\alpha"'] & \mathrm{pt} \arrow[d] \arrow[dl, Rightarrow, "\gamma"'] \\
X \times S \arrow[r, "\mathcal{P}_G"] & X \times \mathrm{B}_{\text{ét}}\, G \arrow[r, "\mathbf{E}"] & \mathrm{B}^4_{\text{ét}}\, A(1),
\end{tikzcd}$$

where U_{x^I} denotes the complement of the union of graphs $\iota : \Gamma_{x^I} \hookrightarrow S \times X$. In other words, we obtain a section of the complex of étale sheaves $\iota^! \underline{A}(1)[4]$. Let $\pi : \Gamma_{x^I} \to S$ denote the finite projection. The canonical map:

$$\pi_* \iota^! \underline{A}(1)[4] \to \pi_* \underline{A}_{\Gamma_{x^I}}[2] \xrightarrow{\mathrm{Tr}} \underline{A}_S[2]$$

then allows to produce an étale A-gerbe over S. Since this construction is functorial in S, we obtain the functor $\Xi_{\text{ét}}$.

12.2.4. On the other hand, there is a functor from the Brylinski–Deligne data of Chapter 1, constructed by Gaitsgory [31, §6]:

$$\mathbf{CExt}(G, \mathbf{K}_2) \to \mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(2)). \tag{12.2}$$

It is defined by first showing that the canonical map from motivic cohomology to K-theory gives rise to an equivalence:

$$\tau_{\leq 2}\mathrm{Maps}_{X/}(X \times \mathrm{B}_{\mathrm{Zar}}\, G, \mathrm{B}^4_{\mathrm{Zar}}\, \mathbb{Z}_{\mathrm{mot}}(2)) \xrightarrow{\sim} \mathbf{CExt}(G, \mathbf{K}_2).$$

Therefore, we obtain the desired map (12.2) as the composition

$$\begin{aligned}
\tau_{\leq 2}\mathrm{Maps}_{X/}(X \times \mathrm{B}_{\mathrm{Zar}}\, G, \mathrm{B}^4_{\mathrm{Zar}}\, \mathbb{Z}_{\mathrm{mot}}(2)) &\to \tau_{\leq 2}\mathrm{Maps}_{X/}(X \times \mathrm{B}_{\mathrm{Zar}}\, G, \mathrm{B}^4_{\mathrm{Zar}}\, \mathbb{Z}_{\mathrm{mot}}/\ell(2)) \\
&\to \tau_{\leq 2}\mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(2)),
\end{aligned}$$

where in the second map, we used Suslin's rigidity to identify the étale sheafification of $\mathbb{Z}_{\mathrm{mot}}/\ell(2)$ with $A(2)$. For the last groupoid, the truncation $\tau_{\leq 2}$ is immaterial because it has vanishing homotopy groups in degrees $i > 2$ ([32, Appendix]). The functor (12.2) is intimately related to the Soulé regulator of algebraic K-theory.

12.2.5. Combining with a Tate twist, we obtain a functor:

$$\mathbf{R}_{\text{ét}} : \mathbf{CExt}(G, \mathbf{K}_2) \underset{\mathbb{Z}}{\otimes} A(-1) \to \mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(1)).$$

12.2.6. We now mention a conjectural compatibility statement between the construction functors, proposed by D. Gaitsgory [31]. We work under the assumption that $\mathrm{char}(k) \nmid N_G$ or that G is semisimple and simply connected, so that $\Xi_{\mathbf{Pic}}$ is well-defined and is in fact an equivalence (Chapter 2, Theorem 8.1.2). The conjecture asserts that the following diagram is canonically commutative, with vertical arrows being equivalences.

$$\begin{tikzcd}
\mathbf{CExt}(G, \mathbf{K}_2) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "\mathbf{R}_{\text{ét}}"] \arrow[d, "\Xi_{\mathbf{Pic}}", "\cong"'] & \mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(1)) \arrow[d, "\Xi_{\text{ét}}", "\cong"'] \\
\mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "c_1"] & \mathbf{Ge}^{\text{fact}}_{\text{ét}}(\mathrm{Gr}_{G,\mathrm{Ran}})
\end{tikzcd}$$

The interest of the conjecture lies in that it gives a geometric interpretation of the realization functor $\mathbf{R}_{\text{ét}}$ as the first Chern class map.

Indication of proof. We compose this diagram with (12.1), so it suffices to prove that the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{CExt}(G, \mathbf{K}_2) \underset{\mathbb{Z}}{\otimes} A(-1) \arrow[r, "\mathbf{R}_{\text{ét}}"] \arrow[d, "\Psi_{\mathbf{Pic}} \circ \Xi_{\mathbf{Pic}}"] & \mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(1)) \arrow[d, "\Psi_{\text{ét}} \circ \Xi_{\text{ét}}"] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathbf{Ge}_{\text{ét}})
\end{tikzcd}$$

By the compatibility Theorem 7.2.8 of Chapter 2, the functor $\Psi_{\mathbf{Pic}} \circ \Xi_{\mathbf{Pic}}$ identifies with the classification functor Ψ_{BD} for Brylinski–Deligne data, up to an ω-shift.

On the other hand, the calculation of homotopy groups of $\mathrm{Maps}_{X/}(X \times \mathrm{B}_{\text{ét}}\, G, \mathrm{B}^4_{\text{ét}}\, A(1))$ ([32, Appendix]) allows us also to interpret $\Psi_{\text{ét}} \circ \Xi_{\text{ét}}$ as an explicit equivalence. This is the cohomological version of the aforementioned theorem, and its proof is presumably not harder. Hence we have removed factorization structures from the problem.

Finally, by the definition of the Θ-data groupoid, it suffices to verify commutativity for a torus, where all functors have explicit descriptions, so it should be straightforward. $\square$

13. The analytic context

In this section, we specialize to the context of constructible analytic sheaves. The results are parellel to the previous section; however, we have not thought through the possibility of a "construction functor" Ξ_{an}, so this discussion is absent.

We take $\mathbb{C}$ as the ground field.

13.1. $\mathbf{Ge}_{\text{an}}$.

13.1.1. Let $\mathbf{Ge}_{\text{an}}$ denote the presheaf of strict Picard 2-groupoids on $\mathbf{Sch}^{\text{ft}}_{/k}$ which associates $\mathbb{C}^\times$-gerbes over the analytification:

$$\mathbf{Ge}_{\text{an}}(X) := \mathbf{Ge}_{\mathbb{C}^\times}(X^{\text{an}}).$$

Equivalently, $\mathbf{Ge}_{\text{an}}(X)$ is the space of maps from the homotopy type of X^{an} to the Eilenberg–MacLane space $K(2; \mathbb{C}^\times)$. Its coefficient group $A(-1)$ identifies with $\mathbb{C}^\times$. Given any h-cover $\widetilde{X} \to X$, the geometric realization of the Čech complex of $\widetilde{X}^{\text{an}} \to X^{\text{an}}$ gives an equivalence of homotopy types ([9, Proposition 3.21]). In particular, $\mathbf{Ge}_{\text{an}}$ is an h-sheaf on $\mathbf{Sch}^{\text{ft}}_{/k}$.

13.1.2. We build a divisor class map:

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \to \mathbf{Ge}_{\mathrm{an}}, \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a, \tag{13.1}$$

as follows. Let us first construct a map, where $\mathbf{Loc}_{\mathrm{an}}$ stands for the stack of analytic $\mathbb{C}^\times$-local systems, i.e., maps to $K(1; \mathbb{C}^\times)$:

$$\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \to \mathbf{Loc}_{\mathrm{an}}, \quad (f, a) \rightsquigarrow f^a. \tag{13.2}$$

Since $\mathbf{Loc}_{\mathrm{an}}$ satisfies h-descent, it suffices to construct the map (13.2) compatibly over smooth schemes. We know that $\mathcal{O}^\times_{Y^{\mathrm{an}}} \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times$ is quasi-isomorphic to the complex $\mathcal{O}^\times_{Y^{\mathrm{an}}} \to \mathcal{O}^\times_{Y^{\mathrm{an}}} \underset{\mathbb{Z}}{\otimes} \mathbb{C}$ placed in degrees $[-1, 0]$, where we use $\exp(2\pi i -) : \mathbb{C} \to \mathbb{C}^\times$ for the quotient map. On the other hand, $\mathbf{Loc}_{\mathrm{an}} \cong \mathbb{C}^\times[1]$ is quasi-isomorphic to the complex $\mathcal{O}^\times_{Y^{\mathrm{an}}} \xrightarrow{d\log} \Omega^{1,\mathrm{cl}}_{Y^{\mathrm{an}}}$ placed in degrees $[-1, 0]$, so the desired map arises from the commutative diagram:

$$\begin{tikzcd}
\mathcal{O}^\times_{Y^{\mathrm{an}}} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"] & \mathcal{O}^\times_{Y^{\mathrm{an}}} \underset{\mathbb{Z}}{\otimes} \mathbb{C} \arrow[d] & f \otimes \lambda \arrow[d, mapsto] \\
\mathcal{O}^\times_{Y^{\mathrm{an}}} \arrow[r, "d\log"] & \Omega^{1,\mathrm{cl}}_{Y^{\mathrm{an}}} & \lambda d\log f
\end{tikzcd}$$

The construction is obviously functorial in Y. The desired map (13.1) then arises from delooping.

13.1.3. **Lemma.** *The sheaf* $\mathbf{Ge}_{\mathrm{an}}$ *and* c_1 *define a motivic* h-*theory of gerbes.*

Proof. The h-descent has already been noted above. The properties (RP1), (RP2), and (A) are standard facts. To verify the weak proper base change property (B), we shall show that the restriction map:

$$\operatorname*{colim}_U \mathrm{H}^i(Y^{\mathrm{an}} \underset{X^{\mathrm{an}}}{\times} U^{\mathrm{an}}; \mathbb{C}^\times) \to \mathrm{H}^i(Y^{\mathrm{an}} \underset{X^{\mathrm{an}}}{\times} \{x\}; \mathbb{C}^\times), \quad i \geq 0, \tag{13.3}$$

where U ranges over étale neighborhoods of $x \in X$, is in fact an isomorphism.

Note that there is an exact sequence of abelian groups:

$$0 \to \mathbb{C}^\times_{\mathrm{tors}} \to \mathbb{C}^\times \to \mathbb{C}/\mathbb{Q} \to 0,$$

where $\mathbb{C}^\times_{\mathrm{tors}}$ denotes the torsion subgroup of $\mathbb{C}^\times$. By Artin's comparison theorem, the map (13.3) is an isomorphism for coefficients in $\mathbb{C}^\times_{\mathrm{tors}}$ and $\mathbb{Q}_\ell$ for any prime ℓ. The same statement must also be true for coefficients in $\mathbb{Q}$ as the operation $- \underset{\mathbb{Q}}{\otimes} \mathbb{Q}_\ell$ is conservative. Thus it remains true for $\mathbb{C}/\mathbb{Q}$ as it is a direct sum of copies of $\mathbb{Q}$. This implies the result for coefficients in $\mathbb{C}^\times$. $\square$

13.1.4. **Corollary.** *There is a canonical equivalence of strict Picard* 2*-groupoids:*

$$\Psi_{\mathrm{an}} : \mathbf{Ge}^{\mathrm{fact}}_{\mathrm{an}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \Theta_G(\Lambda_T; \mathbf{Ge}_{\mathrm{an}}),$$

which makes the following diagram commute:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}", "\cong"'] & \mathbf{Ge}^{\mathrm{fact}}_{\mathrm{an}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathrm{an}}", "\cong"'] \\ \Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathbf{Ge}_{\mathrm{an}}) \end{tikzcd} \tag{13.4}$$

Proof. This follows from Theorem 10.3.2 together with Lemma 13.1.3. $\square$

CHAPTER 4

The de Rham context

For now on, we shall focus on the de Rham context. Hence we set ground field $k = \bar{k}$ with $\mathrm{char}(k) = 0$, and take our sheaf theory to be $\mathcal{D}$-modules. The main feature of the de Rham context, which underlies much of the richness of the theory, is that there is a forgetful functor from $\mathcal{D}$-modules to quasi-coherent (or ind-coherent) sheaves.

In this chapter, we introduce several "twisting agents" for $\mathcal{D}$-modules which allow to construct a twisted category of sheaves equipped with a forgetful functor. In successive generality, they are "tame twistings", "twistings", and "pre-twistings", related by functors:

$$\{\text{tame twistings}\} \to \{\text{twistings}\} \hookrightarrow \{\text{pre-twistings}\}$$

The notion of twistings is introduced by Gaitsgory–Rozenblyum [33]. By constrast, a tame twisting is a twisting with an additional structure which allows to define regularity for its twisted $\mathcal{D}$-modules. Pre-twistings are a generalization of twistings which appear in the compactification of quantum parameters for the geometric Langlands program—we leave more details about them for Chapter 5.

Concerning factorization tame twistings on $\mathrm{Gr}_{G,\mathrm{Ran}}$, we are able to construct a classification functor $\Psi_{\mathbf{Tw}^\circ}$ and prove that it is an equivalence, using the results of Chapters 2 and 3. For twistings, we only have a construction functor $\Xi_{\mathbf{Tw}}$, which is not known to be an equivalence in general, although we shall prove that it is so for semisimple, simply connected groups.

The material on pre-twistings has been documented in [72] and the results concerning tame twistings will also appear in the forthcoming work [70].

14. Pre-twistings and twistings

In this section, we explain the notion of pre-twistings and specialize it to twistings. All the results here have been proved for twistings in [33]. Their generalization to pre-twistings contains no surprise. The main difference is that instead of the de Rham prestack, we work with an arbitrary formal moduli problem, which involves some additional (but rather light) technical baggage.

14.1. **Crystals.**

14.1.1. Let us first recall the DG category of crystals as a way to formalize $\mathcal{D}$-modules in the setting of derived algebraic geometry [33].

14.1.2. Under the $\mathrm{char}(k) = 0$ assumption, there is a well behaved theory of ind-coherent sheaves on dervied schemes locally almost of finite type (see Chapter 1, §4). This is a functor out of the $(\infty, 2)$-category of correspondences on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$. This formalism makes precise the following functorialities: IndCoh attaches to each $X \in \mathbf{DSch}^{\mathrm{ft}}_{/k}$ a k-linear ∞-category $\mathrm{IndCoh}(X)$ together with two functors attached to each morphism $f : Y \to X$:

1. a functor $f^! : \mathrm{IndCoh}(X) \to \mathrm{IndCoh}(Y)$;
2. a functor $f^{\mathrm{IndCoh}}_* : \mathrm{IndCoh}(Y) \to \mathrm{IndCoh}(X)$; and
3. for each Cartesian product $Y' = Y \underset{X}{\times} X'$, a canonical commutativity of the diagram:

$$\begin{tikzcd} \mathrm{IndCoh}(Y') \arrow[d, "(f')^{\mathrm{IndCoh}}_*"] & \mathrm{IndCoh}(Y) \arrow[l, "(\alpha')^!"'] \arrow[d, "f^{\mathrm{IndCoh}}_*"] \\ \mathrm{IndCoh}(X') & \mathrm{IndCoh}(X) \arrow[l, "\alpha^!"'] \end{tikzcd}$$

14.1.3. Furthermore, $\mathrm{IndCoh}(X)$ receives a fully faithful functor Υ_X from $\mathrm{QCoh}(X)$ which intertwines f^* on $\mathrm{QCoh}(X)$ and $f^!$ on $\mathrm{IndCoh}(X)$. The functor Υ_X is defined as follows. Let $\omega_X \in \mathrm{Coh}(X)$ be the dualizing complex. Then the action of $\mathrm{Perf}(X)$ on $\mathrm{Coh}(X)$ ind-extends to a functor $\Upsilon_X : \mathrm{QCoh}(X) \to \mathrm{IndCoh}(X)$.

For any prestack $\mathcal{Y}$, we let $\mathcal{Y}_{\mathrm{dR}}$ denote its *de Rham prestack*, whose S-points are defined as $\mathrm{Maps}(X_{\mathrm{red}}, \mathcal{Y})$. The formation of de Rham prestacks commutes with all limits and colimits.

14.1.4. By definition, the presheaf Crys on $\mathbf{DSch}^{\mathrm{aff}}_{/k}$ associates to an affine derived scheme X the stable ∞-category $\mathrm{QCoh}(X_{\mathrm{dR}})$. When X is locally of finite type, the functor $\Upsilon_{X_{\mathrm{dR}}}$ identifies $\mathrm{QCoh}(X_{\mathrm{dR}})$ with $\mathrm{IndCoh}(X_{\mathrm{dR}})$. Therefore, pulling back along $X \to X_{\mathrm{dR}}$ defines two forgetful functors related by Υ_X:

$$\begin{tikzcd} & \mathrm{Crys}(X) \arrow[dl, "\mathrm{oblv}^l"'] \arrow[dr, "\mathrm{oblv}^r"] & \\ \mathrm{QCoh}(X) \arrow[rr, "\Upsilon_X"] & & \mathrm{IndCoh}(X) \end{tikzcd}$$

These can be viewed as realizations of a crystal as a left, versus right $\mathcal{D}$-module.

We will now restrict our consideration to $\mathbf{DSch}^{\mathrm{ft}}_{/k}$. Recall the derived $\mathbf{h}$-topology introduced in Chapter 1, §4.

14.1.5. **Lemma.** *The presheaf* Crys *satisfies derived* $\mathbf{h}$*-descent.*

Proof. This folllows from the derived $\mathbf{h}$-descent of IndCoh, see [33, Proposition 3.2.2]. $\square$

14.2. **Pre-twistings.**

14.2.1. In order to introduce pre-twistings, we have to consider more general prestacks which behave "like X_{dR}." They are formalized by the notion of formal moduli problems. Let $\mathbf{PStk}^{\text{laft-def}}_{/k}$ denote the full subcategory of $\mathbf{PStk}_{/k}$ consisting of prestacks $\mathcal{Y}$ which are:

(1) locally almost of finite type; and

(2) admit deformation theory.

Informally, the second condition means that the tangent complex $\mathbb{T}_{\mathcal{Y}/k}$ is a well-defined object of $\mathrm{IndCoh}(\mathcal{Y})$ and has the expected properties. We refer the reader to [36, Chapter I] for the precise definitions.

14.2.2. We let $\mathbf{FMod}_{/k}$ be the ∞-category of morphisms $\alpha : \mathcal{Y} \to \mathcal{Y}^\flat$ in $\mathbf{PStk}_{/k}^{\text{laft-def}}$ such that α is a nil-isomorphism, i.e., it induces an equivalence on every reduced test scheme. Objects of $\mathbf{FMod}_{/k}$ are called *formal moduli problems.*

There is an obvious forgetful functor:

$$\mathbf{FMod}_{/k} \to \mathbf{PStk}_{/k}^{\text{laft-def}}, \quad (\alpha : \mathcal{Y} \to \mathcal{Y}^\flat) \rightsquigarrow \mathcal{Y},$$

whose fiber at $\mathcal{Y}$ is denoted by $\mathbf{FMod}_{/k}(\mathcal{Y})$. Its objects are called formal moduli problems *under* $\mathcal{Y}$ (or "pointed by $\mathcal{Y}$"). One recovers the classical notion of formal moduli problems as those pointed by $\mathrm{Spec}(k)$.

14.2.3. The classical correspondence between formal moduli problems and formal Lie algebras also admits a "multi-point" generalization. To wit, we let $\mathbf{FGpd}_{/k}$ denote the ∞-category of formal groupoids, i.e., groupoid objects in $\mathbf{PStk}_{/k}^{\text{laft-def}}$ all of whose morphisms are nil-isomorphisms. Let $\mathbf{FGpd}_{/k}(\mathcal{Y})$ denote the fiber of the forgetful functor:

$$\mathbf{FGpd}_{/k} \to \mathbf{PStk}_{/k}^{\text{laft-def}}, \quad \mathcal{Y}^\bullet \rightsquigarrow \mathcal{Y}^{[0]}.$$

The Čech nerve construction defines a functor:

$$\Omega^f : \mathbf{FMod}_{/k} \to \mathbf{FGpd}_{/k}. \tag{14.1}$$

14.2.4. **Lemma.** *The functor Ω^f is an equivalence of ∞-categories.*

Proof. This is [36, Chapter V, Theorem 2.3.2]. □

14.2.5. We shall denote the inverse of (14.1) by B^f. We emphasize that the functor B^f is *not* the delooping functor of prestacks; the latter fails to preserve $\mathbf{PStk}_{/k}^{\text{laft-def}}$ in general. The equivalence (14.1) induces an equivalence for every $\mathcal{Y} \in \mathbf{PStk}_{/k}^{\text{laft-def}}$:

$$\Omega_{\mathcal{Y}}^f : \mathbf{FMod}_{/k}(\mathcal{Y}) \xrightarrow{\sim} \mathbf{FGpd}_{/k}(\mathcal{Y}) : \mathrm{B}_{\mathcal{Y}}^f .$$

Given $\mathcal{Y} \in \mathbf{PStk}_{/k}^{\text{laft-def}}$, the ∞-category $\mathbf{FMod}_{/k}(\mathcal{Y})$ has $\mathcal{Y}_{\mathrm{dR}}$ as the terminal object. On the level of formal groupoids, $\mathcal{Y}_{\mathrm{dR}}$ passes to the infinitesimal groupoid, whose n-simplex is the completion of $\mathcal{Y}^{\times n}$ along its main diagonal. By unwinding the definitions, one sees that the prestack colimit of this groupoid identifies with $\mathcal{Y}_{\mathrm{dR}}$ if and only if $\mathcal{Y}$ is formally smooth.

14.2.6. *Pre-twistings.* We define the ∞-category of *pre-twistings* on $\mathcal{Y}$ as an object $\alpha : \mathcal{Y} \to \mathcal{Y}^\flat$ of $\mathbf{FMod}_{/k}(\mathcal{Y})$ together with an object of the k-linear groupoid:

$$\mathcal{T} \in \mathrm{Fib}(\mathrm{Maps}(\mathcal{Y}^\flat, \mathrm{B}^2_{\mathrm{Zar}}\,\mathbb{G}_a) \xrightarrow{\alpha^*} \mathrm{Maps}(\mathcal{Y}, \mathrm{B}^2_{\mathrm{Zar}}\,\mathbb{G}_a)). \tag{14.2}$$

We refer to $\mathcal{Y}^\flat$ as the *base* of the pre-twisting $(\alpha : \mathcal{Y} \to \mathcal{Y}^\flat, \mathcal{T})$. In particular, the ∞-category $\mathbf{PTw}_{/k}(\mathcal{Y}/\mathcal{Y}^\flat)$ of pre-twistings based at $\mathcal{Y}^\flat$ has a k-linear structure.

The ∞-category of pre-twistings is functorial in $\mathcal{Y}$, i.e., it upgrades to a presheaf $\mathbf{PTw}_{/k}$ on $\mathbf{PStk}_{/k}^{\text{laft-def}}$ which associates to every morphism $f : \mathcal{Y} \to \mathcal{X}$ the following functor:

$$f^* : \mathbf{PTw}_{/k}(\mathcal{X}) \to \mathbf{PTw}_{/k}(\mathcal{Y}), \quad (\alpha : \mathcal{Y} \to \mathcal{Y}^\flat, \mathcal{T}) \rightsquigarrow (f^*\alpha : \mathcal{X} \to \mathcal{Y}^\flat \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{X}_{\mathrm{dR}}, (f^\flat)^*\mathcal{T}),$$

where $f^\flat$ denotes the induced functor $\mathcal{Y}^\flat \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{X}_{\mathrm{dR}} \to \mathcal{Y}^\flat$.

14.2.7. *Changing the topology.* Recall that $\mathrm{B}^2_{\mathrm{Zar}}\,\mathbb{G}_a$ satisfies derived $\mathbf{h}$-descent (Chapter 1, Lemma 4.2.4). Thus in (14.2), we may replace the Zariski topology with any topology on $\mathbf{DSch}_{/k}^{\mathrm{ft}}$ weaker than $\mathbf{h}$ without changing the resulting groupoid (e.g., étale, fppf). In fact, one can also replace $\mathrm{B}^2_{\mathrm{Zar}}\,\mathbb{G}_a$ by the prestack delooping $\mathrm{B}^2\,\mathbb{G}_a$.

This is because for an *affine* derived scheme S, the groupoid $\tau^{\leq 0}\mathrm{R}\Gamma_{\mathrm{Zar}}(S, \mathcal{O}_S[2])$ has vanishing π_0, so the canonical map:

$$\mathrm{Maps}(S, \mathrm{B}^2\,\mathbb{G}_a) \to \mathrm{Maps}(S, \mathrm{B}^2_{\mathrm{Zar}}\,\mathbb{G}_a)$$

is an equivalence.

14.2.8. *Changing the structure group.* We discuss another kind of flexibility of the defintion—that of changing the structure group. Let A be a prestack which takes values in commutative groups. Denote by $A_{\widehat{e}}$ its formal completion at identity, i.e., the prestack fiber product $\mathrm{pt} \underset{A_{\mathrm{dR}}}{\times} A$. Then $A_{\widehat{e}}$ is also valued in commutative groups. Fix a topology $\mathbf{t}$ on $\mathbf{DSch}^{\mathrm{aff}}_{/k}$.

14.2.9. **Lemma.** *Let $\alpha : \mathcal{Y} \to \mathcal{Y}^\flat$ be a formal moduli problem. Then induction along $A_{\widehat{e}} \to A$ defines an equivalence:*

$$\begin{aligned}&\mathrm{Fib}(\mathrm{Maps}(\mathcal{Y}^\flat, \mathrm{B}^2_{\mathbf{t}}\, A_{\widehat{e}}) \to \mathrm{Maps}(\mathcal{Y}, \mathrm{B}^2_{\mathbf{t}}\, A_{\widehat{e}}))\\ &\qquad\qquad \xrightarrow{\sim} \mathrm{Fib}(\mathrm{Maps}(\mathcal{Y}^\flat, \mathrm{B}^2_{\mathbf{t}}\, A) \to \mathrm{Maps}(\mathcal{Y}, \mathrm{B}^2_{\mathbf{t}}\, A)).\end{aligned}$$

Proof. We shall construct the inverse functor by canonically trivializing the induced section of $\mathrm{Maps}(\mathcal{Y}^\flat, \mathrm{B}^2_{\mathbf{t}}\, A_{\mathrm{dR}})$, i.e., an A_{dR}-gerbe on $\mathcal{Y}^\flat$, attached to any object $\mathcal{T}$ of the second groupoid. Indeed, let $\widehat{\mathcal{Y}}^\flat_A$ be the total space of some A-gerbe on $\mathcal{Y}^\flat$, locally trivial in the $\mathbf{t}$-topology. Then $(\widehat{\mathcal{Y}}^\flat)_{\mathrm{dR}} \to \mathcal{Y}_{\mathrm{dR}}$ is an A_{dR}-gerbe locally trivial in the $\mathbf{t}$-topology. On the other hand, suppose $\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}}}$ is the A_{dR}-gerbe induced from $\widehat{\mathcal{Y}}^\flat_A$. We obtain a canonical morphism of A_{dR}-gerbes on $\mathcal{Y}^\flat$, both locally trivial in the $\mathbf{t}$-topology:

$$\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}}} \to (\widehat{\mathcal{Y}}^\flat_A)_{\mathrm{dR}} \underset{\mathcal{Y}_{\mathrm{dR}}}{\times} \mathcal{Y}^\flat. \tag{14.3}$$

Thus (14.3) is an isomorphism of prestacks.

Consequently, a section of the A_{dR}-gerbe $\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}}}$ amounts to filling in the dotted arrow

$$\begin{tikzcd}
\widehat{\mathcal{Y}}^\flat_{A_{\mathrm{dR}}} \arrow[r] \arrow[d, "A_{\mathrm{dR}}"] & (\widehat{\mathcal{Y}}^\flat_A)_{\mathrm{dR}} \arrow[d, "A_{\mathrm{dR}}"] \\
\mathcal{Y}^\flat \arrow[r] \arrow[ur, dotted] & \mathcal{Y}_{\mathrm{dR}}
\end{tikzcd}$$

making the lower-right triangle commute. On the other hand, the structure of $\mathcal{T}$ supplies a section $\mathcal{Y} \to \widehat{\mathcal{Y}}^\flat_A$ over $\mathcal{Y}^\flat$. Hence we obtain a map $\mathcal{Y}^\flat \to \mathcal{Y}_{\mathrm{dR}} \to (\widehat{\mathcal{Y}}^\flat_A)_{\mathrm{dR}}$ over $\mathcal{Y}_{\mathrm{dR}}$. □

14.2.10. Summarizing, the groupoid (14.2) remains unchanged for any structure group among

$$\mathbb{G}_a, \widehat{\mathbb{G}}_a(\text{formal completion at } e), \mathbb{G}_m, \widehat{\mathbb{G}}_m,$$

as well as any topology $\mathbf{t}$ weaker than the $\mathbf{h}$-topology. In particular, for a fixed formal moduli problem $\alpha : \mathcal{Y} \to \mathcal{Y}^\flat$, we have a functor:

$$\mathbf{Pic}(\mathcal{Y}) \underset{\mathbb{Z}}{\otimes} k \to \mathbf{PTw}_{/k}(\mathcal{Y}/\mathcal{Y}^\flat),$$

which sends $\mathcal{L}$ to the trivial Zariski $\mathbb{G}_m$-gerbe on $\mathcal{Y}^\flat$ whose pullback to $\mathcal{Y}$ is equipped with the trivialization determined by $\mathcal{L}$. One then extends the map by k-linearity.

14.3. **Classical counterparts.**

14.3.1. Let us note what pre-twistings look like on a classical scheme $Y \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$. We shall show that they are generalizations of central extensions of Lie algebroids by the structure sheaf.

14.3.2. Recall that a *Lie algebroid* on Y is an $\mathcal{O}_Y$-module $\mathcal{L}$ together with an k-linear Lie bracket $[-,-]$ and an $\mathcal{O}_Y$-module map $\sigma : \mathcal{L} \to \mathcal{T}_Y$ such that the following properties are satisfied:

(1) $[l_1, f \cdot l_2] = \sigma(l_1)(f) \cdot l_2 + f[l_1, l_2]$;

(2) σ sends $[-,-]$ to the canonical Lie bracket on $\mathcal{T}_Y$.

We denote the category of Lie algebroids on Y by $\mathbf{Lie}_{/k}(Y)$.

14.3.3. Given a formal moduli problem $\alpha : Y \to Y^\flat$ such that the tangent complex $\mathbb{T}_{Y/Y^\flat}$ belongs to $\Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$, one may canonically attach a Lie algebroid as the tangent algebroid of its corresponding formal groupoid (c.f. (14.1)). This procedure defines an equivalence between Lie algebroids on Y and formal moduli problems under Y satisfying the above condition ([36, Chapter VIII]), which we shall call *classical*:

$$\mathbf{FMod}_{/k}(Y)^{\mathrm{cl}} \xrightarrow{\sim} \mathbf{Lie}_{/k}(Y), \quad (\alpha : Y \to Y^\flat) \rightsquigarrow \Upsilon_Y^{-1}(\mathbb{T}_{Y/Y^\flat}).$$

In fact, the underlying $\mathcal{O}_Y$-module of a classical formal moduli problem is precisely $\Upsilon_Y^{-1}(\mathbb{T}_{Y/Y^\flat})$.

14.3.4. We now fix a classical formal moduli problem $Y^\flat$ on Y. Given a formal moduli problem $\widehat{Y}^\flat \to Y^\flat$ such that $\mathbb{T}_{Y/\widehat{Y}^\flat} \in \Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$, one can assign a Lie algebroid $\widehat{\mathcal{L}}$ equipped with a map $\widehat{\mathcal{L}} \to \mathcal{L}$. Furthermore, a morphism $\widehat{Y}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{Y}^\flat$ in $\mathbf{FMod}_{/k}(Y)$ induces a map

$$\widehat{\mathcal{L}} \oplus \mathcal{O}_Y \to \widehat{\mathcal{L}}, \quad (l, f) \rightsquigarrow l + f\mathbf{1} \tag{14.4}$$

where $\mathbf{1}$ is the image of $(0, f)$ in $\widehat{\mathcal{L}}$. If the morphism $\widehat{Y}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{Y}^\flat$ realizes $\widehat{Y}^\flat$ as a $\widehat{\mathbb{G}}_m$-gerbe over $Y^\flat$, then we see that $\mathcal{O}_Y \to \widehat{\mathcal{L}}$, $f \rightsquigarrow f\mathbf{1}$ is the kernel of the canonical map $\widehat{\mathcal{L}} \to \mathcal{L}$. The fact that (14.4) preserves Lie bracket then implies $\mathcal{O}_Y$ is central inside $\widehat{\mathcal{L}}$. In other words, the map $\widehat{\mathcal{L}} \to \mathcal{L}$ is a central extension of classical Lie algebroids by $\mathcal{O}_Y$. We denote this k-linear category by $\mathbf{CExt}_{/k}(\mathcal{L}, \mathcal{O}_Y)$.

14.3.5. Now, given a pre-twisting based at $Y^\flat$, interpreted using structure group $\widehat{\mathbb{G}}_m$ and the trivial topology, we claim that the corresponding formal moduli problem $\widehat{Y}^\flat$ (the total space of the $\widehat{\mathbb{G}}_m$-gerbe) satisfies the property that $\mathbb{T}_{Y/\widehat{Y}^\flat}$ lies in

$\Upsilon_Y(\mathrm{QCoh}(Y)^{\heartsuit})$. Indeed, we have a canonical triangle in $\mathrm{IndCoh}(Y)$:

$$\omega_Y \cong \mathbb{T}_{\widehat{Y}^\flat/Y^\flat}\big|_Y \to \mathbb{T}_{Y/\widehat{Y}^\flat} \to \mathbb{T}_{Y/Y^\flat}$$

and the outer terms lie in the essential image of $\mathrm{QCoh}(Y)^{\heartsuit}$. Hence we have defined a functor:

$$\mathbf{PTw}_{/k}(Y/Y^\flat) \to \mathbf{CExt}_{/k}(\mathcal{L}, \mathcal{O}_Y). \tag{14.5}$$

14.3.6. **Proposition.** *The functor* (14.5) *is an equivalence of categories.*

In particular, $\mathbf{PTw}_{/k}(Y/Y^\flat)$ is an ordinary category.

Proof. We explicitly construct the functor inverse to (14.5). Namely, given a central extension $\widehat{\mathcal{L}}$ of $\mathcal{L}$, we equip its corresponding formal moduli problem $\widehat{Y}^\flat$ with the structure of a $\widehat{\mathbb{G}}_m$-gerbe over $\mathcal{Y}^\flat$. As before, the action map $\widehat{Y}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{Y}^\flat$ arises from the morphism of classical Lie algebroids over Y:

$$\widehat{\mathcal{L}} \oplus \mathcal{O}_Y \to \widehat{\mathcal{L}}, \quad (l, f) \rightsquigarrow l + f\mathbf{1}.$$

The morphism induced by action and projection $\widehat{Y}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m \to \widehat{Y}^\flat \underset{Y^\flat}{\times} \widehat{Y}^\flat$ is an isomorphism since the same holds for the corresponding map of classical Lie algebroids:

$$\widehat{\mathcal{L}} \oplus \mathcal{O}_Y \to \widehat{\mathcal{L}} \underset{\mathcal{L}}{\times} \widehat{\mathcal{L}}, \quad (l, f) \rightsquigarrow (l + f\mathbf{1}, l).$$

It remains to show that $\widehat{Y}^\flat \to Y^\flat$ admits a section over any affine derived scheme T mapping to $\mathcal{Y}^\flat$. We shall deduce the existence of this section from the following claim: *the morphism $\widehat{Y}^\flat \to Y^\flat$ is formally smooth.*

To prove the claim, let T be any affine derived scheme with a morphism $\widehat{y} : T \to \widehat{Y}^\flat$. By the criterion of formal smoothness [36, Chapter I, §7.3], we ought to show

$\mathrm{Maps}(\mathcal{T}^*_{\widehat{Y}^\flat/Y^\flat}|_{\widehat{y}}, \mathcal{F}) \in \mathbf{Vect}^{\leq 0}$ for all $\mathcal{F} \in \mathrm{QCoh}(T)^\heartsuit$. The Cartesian square:

$$\begin{tikzcd} T \arrow[r, "(\widehat{y},\widehat{y})"] & \widehat{Y}^\flat \underset{Y^\flat}{\times} \widehat{Y}^\flat \arrow[r] \arrow[d] & \widehat{Y}^\flat \arrow[d] \\ & \widehat{Y}^\flat \arrow[r] & Y^\flat \end{tikzcd}$$

together with the isomorphism above gives a chain of isomorphisms:

$$\mathcal{T}^*_{\widehat{Y}^\flat/Y^\flat}|_{\widehat{y}} \xrightarrow{\sim} \mathcal{T}^*_{\widehat{Y}^\flat \underset{Y^\flat}{\times} \widehat{Y}^\flat/\widehat{Y}^\flat}|_{(\widehat{y},\widehat{y})} \xrightarrow{\sim} \mathcal{T}^*_{\widehat{Y}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m/\widehat{Y^\flat}}|_{(\widehat{y},1)} \xrightarrow{\sim} \mathcal{O}_T[-1].$$

One deduces from this the required degree estimate.

Using the claim, we will construct a section of $\widehat{Y}^\flat \to Y^\flat$ over $T \to Y^\flat$ as follows. First consider the fiber product $T \underset{Y^\flat}{\times} Y$, which is equipped with a nil-isomorphism to T. We obtain a solid commutative diagram:

$$\begin{tikzcd} T^{\mathrm{red}} \arrow[r] \arrow[dr] & T \underset{Y^\flat}{\times} Y \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & \widehat{Y}^\flat \arrow[dl] \\ & T \arrow[r] \arrow[urr, dotted] & Y^\flat & \end{tikzcd}$$

Formal smoothness now implies the existence of the dotted arrow. □

14.4. **Pre-twisted crystals.**

14.4.1. We now explain how a pre-twisting $(\alpha : \mathcal{Y} \to \mathcal{Y}^\flat, \mathcal{T})$ allows to form a DG category $\mathrm{Crys}^r_{\mathcal{T}}(\mathcal{Y})$ equipped with a forgetful functor:

$$\mathrm{oblv}_{\mathcal{T}} : \mathrm{Crys}^r_{\mathcal{T}}(\mathcal{Y}) \to \mathrm{IndCoh}(\mathcal{Y}). \tag{14.6}$$

We use the interpretation of $\mathcal{T}$ by structure group $\mathbb{G}_m$ and a $\mathbf{t}$-topology for some $\mathbf{t}$ weaker than $\mathbf{h}$. The crucial fact is that IndCoh satisfies $\mathbf{t}$-descent.

14.4.2. Denote by $\mathcal{T}^\flat$ the corresponding $\mathbb{G}_m$-gerbe on $\mathcal{Y}^\flat$. Let $\mathbf{DSch}^{\mathrm{aff},\mathcal{T}\text{-split}}_{/\mathcal{Y}^\flat}$ denote the ∞-category consisting of an object $S \in \mathbf{DSch}^{\mathrm{aff}}_{/\mathcal{Y}^\flat}$ and a trivialization γ of $\mathcal{T}^\flat\big|_S$. Since $\mathcal{T}^\flat$ is Zariski locally trivial, the morphism:

$$\mathbf{DSch}^{\mathrm{aff},\mathcal{T}\text{-split}}_{/\mathcal{Y}^\flat} \to \mathbf{DSch}^{\mathrm{aff}}_{/\mathcal{Y}^\flat}$$

is a basis for the Zariski site $\mathbf{DSch}^{\mathrm{aff}}_{/\mathcal{Y}^\flat}$. By $\mathbf{t}$-descent of IndCoh, we have

$$\mathrm{IndCoh}(\mathcal{Y}^\flat) \xrightarrow{\sim} \lim_{S \in \mathbf{DSch}^{\mathrm{aff},\mathcal{T}\text{-split}}_{/\mathcal{Y}^\flat}} \mathrm{IndCoh}(S).$$

To form the twisted category $\mathrm{Crys}^r_{\mathcal{T}}(\mathcal{Y})$, we take a limit over the same index category, but adjust the transition functor corresponding to $(S,\gamma) \to (S',\gamma')$ by:

$$- \otimes (\gamma'\big|_S \cdot \gamma^{-1}) : \mathrm{IndCoh}(S') \to \mathrm{IndCoh}(S),$$

where $\gamma'\big|_S \cdot \gamma^{-1}$ is regarded as a section of $\mathrm{B}_{\mathrm{Zar}}\,\mathbb{G}_m$ over S.

It follows that if $\mathcal{T}^\flat$ is equipped with a global trivialization, then $\mathrm{Crys}^r_{\mathcal{T}}(\mathcal{Y})$ is canonically equivalent to $\mathrm{IndCoh}(\mathcal{Y}^\flat)$. Applying this construction to $\mathcal{Y}$ and the trivialized $\mathbb{G}_m$-gerbe $\mathcal{T}^\flat\big|_{\mathcal{Y}}$, we obtain the functor (14.6).

14.4.3. In fact, choosing $\mathbf{t}$ to be weaker than fpqc and using QCoh instead of IndCoh, we would obtain a DG category $\mathrm{Crys}^l_{\mathcal{T}}(\mathcal{Y})$ equipped with a forgetful functor to $\mathrm{QCoh}(\mathcal{Y})$. In general, the DG categories $\mathrm{Crys}^l_{\mathcal{T}}(\mathcal{Y})$ and $\mathrm{Crys}^r_{\mathcal{T}}(\mathcal{Y})$ are not equivalent.

We note a monadicity property related to the functor (14.6).

14.4.4. **Lemma.** *The functor* $\mathrm{oblv}_{\mathcal{T}}$ *admits a left adjoint* $\mathrm{ind}_{\mathcal{T}}$*, and the pair* $(\mathrm{ind}_{\mathcal{T}}, \mathrm{oblv}_{\mathcal{T}})$ *is monadic.*

Proof. Let $\widehat{\mathcal{Y}}^\flat$ be the total space of the $\widehat{\mathbb{G}}_m$-gerbe on $\mathcal{Y}^\flat$ corresponding to $\mathcal{T}$, and $\widehat{\mathcal{Y}}$ be its pullback to $\mathcal{Y}$, which is equipped with a canonical trivialization. The functor

$\mathrm{oblv}_{\mathcal{T}}$ identifies with the totalization of the !-pullback functors:

$$(\alpha^{(n)})^! : \mathrm{IndCoh}(\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) \to \mathrm{IndCoh}(\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}),$$

where $\alpha^{(n)} : \widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n} \to \widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}$ is the canonical map.

Each $(\alpha^{(n)})^!$ admits a left adjoint $\alpha^{(n)}_{*,\mathrm{IndCoh}}$. Furthermore, the diagram induced from an arbitrary face map:

$$\begin{tikzcd}
\mathrm{IndCoh}(\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) \arrow[r] \arrow[d, "\alpha^{(n)}_{*,\mathrm{IndCoh}}"] & \mathrm{IndCoh}(\widehat{\mathcal{Y}} \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times m}) \arrow[d, "\alpha^{(m)}_{*,\mathrm{IndCoh}}"] \\
\mathrm{IndCoh}(\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times n}) \arrow[r] & \mathrm{IndCoh}(\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}_m^{\times m})
\end{tikzcd}$$

which *a priori* commutes up to a natural transformation, actually commutes. Hence $\mathrm{oblv}_{\mathcal{T}}$ admits a left adjoint $\mathrm{ind}_{\mathcal{T}} := \mathrm{Tot}(\alpha^{(n)}_{*,\mathrm{IndCoh}})$. We now observe:

(1) $\mathrm{oblv}_{\mathcal{T}}$ is conservative; this is because it identifies with the composition $(\alpha^{(0)})^! \circ \mathrm{ev}^0$, where both functors are conservative.

(2) $\mathrm{oblv}_{\mathcal{T}}$ preserves colimits; this is obvious as pullback on IndCoh does.

The result then follows from the Barr–Beck–Lurie theorem. □

14.5. **Twistings.**

14.5.1. For any $\mathcal{Y} \in \mathbf{PStk}^{\text{laft-def}}_{/k}$, we define $\mathbf{Tw}_{/k}(\mathcal{Y})$ to be the full subcategory of pre-twistings $\mathbf{PTw}_{/k}(\mathcal{Y})$ consisting of objects $(\mathcal{Y}^\flat, \mathcal{T})$ for which the canonical map $\mathcal{Y}^\flat \to \mathcal{Y}_{\mathrm{dR}}$ is an isomorphism. It is clear that our definition is consistent with that of Gaitsgory–Rozenblyum [33, §6].

Besides the properties already mentioned above, we note a strong descent property.

14.5.2. **Lemma.** *The presheaf* $\mathbf{Tw}_{/k}$ *on* $\mathbf{DSch}_{/k}$ *satisfies* $\mathbf{h}$*-descent.*

Proof. We have already noted that $\mathrm{B}^2_{\mathrm{Zar}}\mathbb{G}_a$ satisfies **h**-descent. The result follows from the fact that for an **h**-covering $f : Y \to X$, the induced map $f_{\mathrm{dR}} : Y_{\mathrm{dR}} \to X_{\mathrm{dR}}$ is again an **h**-covering. □

14.5.3. **Remark.** All the results in this section generalize to the setting where the base scheme is $S \in \mathbf{Sch}_{/k}$. Thus we have an inclusion of $\mathcal{O}_S$-linear stacks $\mathbf{Tw}_{/S} \subset \mathbf{PTw}_{/S}$. The only modification needed in setting up the theory is that for an object $\mathcal{Y} \in \mathbf{PStk}^{\text{laft-def}}_{/S}$, the de Rham prestack should be replaced by $\mathcal{Y}_{\mathrm{dR}} \underset{S_{\mathrm{dR}}}{\times} S$.

15. Tameness I: the éh-topology

From this section until §17, we will set up the theory of tame twistings. These gadgets are defined to allow for a notion of "regular" twisted crystals. It turns out that in order to have a robust theory of tame twistings, additional techniques must be introduced.

The purpose of the present section is thus to gather some preliminaries on the éh-topology, a weaker variant of the h-topology.

15.1. **Classical** éh**-topology.**

15.1.1. Recall the h-topology on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ whose coverings are generated by universal topological epimorphisms (Chapter 1). By de Jong's alteration, every scheme $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ is h-locally smooth. For a technical reason in the definition of tame twistings, we shall instead use the éh-topology on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ introduced by Geisser [37]. It is generated by étale coverings and abstract blow-up squares.

The following diagram summarizes its relationship to several other topologies on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$, where $\preceq$ denotes the "coarser than" relation.

$$\begin{array}{ccccc} \mathrm{cdh} & \preceq & \text{éh} & \preceq & \mathrm{h} \\ \curlyvee\!\mid & & \curlyvee\!\mid & & \\ \mathrm{Nis} & \preceq & \text{ét} & & \end{array}$$

In fact, the éh topology bears the same relationship to the étale topology as the cdh topology (c.f. Voevodsky [64]) does to the Nisnevich topology.

15.1.2. Let us recall the definition of éh. A Cartesian square in $\mathbf{Sch}^{\mathrm{ft}}_{/k}$:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ Z \arrow[r, "i"] & X \end{tikzcd} \tag{15.1}$$

is an *abstract blow-up square* if i is a closed immersion, p is a proper morphism and induces an isomorphism $Y \backslash E \xrightarrow{\sim} X \backslash Z$. Let t_0 denote the coarsest topology on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ including the empty sieve of $\emptyset$ and the sieve generated by $\{p, i\}$ for every abstract blow-up square (15.1) as coverings.

Abstract blow-up squares are obviously stable under pullback and given an abstract blow-up square (15.1), the induced square:

$$\begin{tikzcd} E \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_p"] \\ E \underset{Z}{\times} E \arrow[r, "{(i,i)}"] & Y \underset{X}{\times} Y \end{tikzcd}$$

is again an abstract blow-up square [66, Lemma 2.14]. Thus the conditions of [3, Theorem 3.2.5] are satisfied and one sees that a $\mathbf{Spc}$-valued presheaf $\mathcal{F}$ on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ is a t_0-sheaf if and only if $\mathcal{F}(\emptyset)$ is contractible and for every abstract blow-up square

(15.1), the induced square is homotopy Cartesian:

$$\begin{tikzcd} \mathcal{F}(E) & \mathcal{F}(Y) \arrow[l] \\ \mathcal{F}(Z) \arrow[u] & \mathcal{F}(X) \arrow[l] \arrow[u] \end{tikzcd}$$

The éh-topology on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ is defined as the coarsest topology contaning the étale topology and t_0. Under the $\mathrm{char}(k) = 0$ assumption of this Chapter, we know that every $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ is éh-locally smooth by Hironaka's resolution of singularities.

15.1.3. We note that the étale covering sieves together with t_0 define a *quasi-topology* on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$, i.e., if S is a covering sieve on X, then for every morphism $f : Y \to X$, the pullback f^*S is again a covering sieve. The presheaves on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ satisfying descent with respect to this quasi-topology are precisely étale sheaves which turn every abstract blow-up square into a homotopy Cartesian square. According to [41, Corollary C.2], this condition precisely characterizes the éh-sheaves in $\mathrm{PSh}(\mathbf{Sch}^{\mathrm{ft}}_{/k})$.

The following Lemma describes a "normal form" of éh covers of a smooth scheme.

15.1.4. **Lemma.** *Let $X \in \mathbf{Sm}_{/k}$. Every* éh*-cover of X has a refinement of the form $\{U_i \to X' \to X\}$ where $\{U_i \to X'\}$ is an étale cover and $X' \to X$ is a composition of blow-ups along smooth centers.*

Proof. This is [37, Corollary 2.6]. $\square$

15.2. Lemmas of Geisser and Friedlander–Voevodsky.

15.2.1. We note two results comparing cohomology groups calculated in éh-versus-étale topologies. These results apply to sheaves valued in *abelian groups*, so we temporarily assume the convention that presheaves are valued in sets instead of higher groupoids.

15.2.2. Let us consider the inclusing of sites:

$$\rho : \mathbf{Sm}_{/k} \to \mathbf{Sch}^{\mathrm{ft}}_{/k}.$$

The éh-topology on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ induces an éh-*topology on* $\mathbf{Sm}_{/k}$ in the sense of [2, Exposé III, §3.1], i.e., it is the finest topology for which presheaf restriction along ρ takes sheaves to sheaves. Furthermore, since every $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ is éh-locally smooth, restriction defines an equivalence $\mathrm{Shv}_{\text{éh}}(\mathbf{Sch}^{\mathrm{ft}}_{/k}) \xrightarrow{\sim} \mathrm{Shv}_{\text{éh}}(\mathbf{Sm}_{/k})$ (Théorème 4.1 of *loc. cit.*).

15.2.3. On the other hand, the functor ρ is (topologically) co-continuous, so the following diagram commutes [2, Exposé III, Proposition 2.3(2)] (or the left adjoint of [58, 00XK]):

$$\begin{tikzcd}
\mathrm{PSh}(\mathbf{Sch}^{\mathrm{ft}}_{/k}) \arrow[r, "\rho^p"] \arrow[d, "L"] & \mathrm{PSh}(\mathbf{Sm}_{/k}) \arrow[d, "L"] \\
\mathrm{Shv}_{\text{éh}}(\mathbf{Sch}^{\mathrm{ft}}_{/k}) \arrow[r, "\sim"] & \mathrm{Shv}_{\text{éh}}(\mathbf{Sm}_{/k})
\end{tikzcd}$$

In particular, the functor of left Kan extension along ρ followed by éh-sheafification[13] identifies with éh-sheafification within the presheaf category on $\mathbf{Sm}_{/k}$:

$$L : \mathrm{PSh}(\mathbf{Sm}_{/k}) \to \mathrm{Shv}_{\text{éh}}(\mathbf{Sm}_{/k}).$$

Analogously, starting with an étale sheaf on $\mathbf{Sm}_{/k}$ (or any topology weaker than éh), left Kan extension along ρ followed by éh-sheafification identifies with the functor:

$$L : \mathrm{Shv}_{\text{ét}}(\mathbf{Sm}_{/k}) \to \mathrm{Shv}_{\text{éh}}(\mathbf{Sm}_{/k}), \tag{15.2}$$

which is, in particular, exact.

Let $\mathbb{G}_{m,\text{éh}}$ be the éh-sheaf on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ associated to $\mathbb{G}_m$. The following Lemma is a special case of a theorem of Geisser [37].

[13]This composition is denoted by ρ_d^* in [37] (for $d = \infty$) and by $\mathcal{F} \rightsquigarrow \mathcal{F}_{\mathrm{cdh}}$ in [25] for its cdh version.

15.2.4. **Lemma.** *Suppose $X \in \mathbf{Sm}_{/k}$. Then the canonical map is an isomorphism for all $i \geq 0$:*

$$\mathrm{H}^i_{\text{ét}}(X; \mathbb{G}_m) \xrightarrow{\sim} \mathrm{H}^i_{\text{éh}}(X; \mathbb{G}_{m,\text{éh}}).$$

Proof. Geisser [37, Theorem 4.3] proves the comparison result for all motivic complexes $\mathbb{Z}(n)$. On the other hand, $\mathbb{G}_{m,\text{éh}}[-1]$ is quasi-isomorphic to $\mathbb{Z}(1)$ as a complex of éh-sheaves on $\mathbf{Sch}^{\text{ft}}_{/k}$, as follows from the analogous fact for complexes in $\mathrm{Shv}_{\text{ét}}(\mathbf{Sm}_{/k})$ and the exactness of (15.2) ([37, Lemma 4.1]). $\square$

15.2.5. We now turn to a comparison result due to Friedlander–Voevodsky. Let $\mathbf{Sm}^{\text{Cor}}_{/k}$ denote the category whose objects are the same as $\mathbf{Sm}_{/k}$, but a morphism $X \dashrightarrow Y$ is given by a k-linear combination of algebraic cycles $W \subset X \times Y$ which are finite over X. The graph construction gives a functor $\mathbf{Sm}_{/k} \to \mathbf{Sm}^{\text{Cor}}_{/k}$, and a presheaf of abelian groups on $\mathbf{Sm}_{/k}$ has a *transfer structure* if it comes equipped with an extension to $\mathbf{Sm}^{\text{Cor}}_{/k}$. On the other hand, a presheaf $\mathcal{F}$ on $\mathbf{Sm}_{/k}$ is said to be $\mathbb{A}^1$-invariant, if the canonical map:

$$\mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$$

is an isomorphism for all $X \in \mathbf{Sm}_{/k}$.

The following Lemma is the étale version of [25, Theorem 5.5(1)], whereas *loc.cit.* compares Nisnevich and cdh cohomology of an $\mathbb{A}^1$-invariant presheaf with transfer. Since the proofs are nearly identical, we only indicate the modifications needed.

15.2.6. **Lemma.** *Let $\mathcal{F}$ be an $\mathbb{A}^1$-invariant* éh-*sheaf with transfer on $\mathbf{Sm}_{/k}$ valued in $\mathbb{Q}$-vector spaces. Then for $X \in \mathbf{Sm}_{/k}$, the following canonical map is an isomorphism for all $i \geq 0$:*

$$\mathrm{H}^i_{\text{ét}}(X; \mathcal{F}) \xrightarrow{\sim} \mathrm{H}^i_{\text{éh}}(X; \mathcal{F}).$$

The assumption on rational coefficients guarantees that the forgetful functor:

$$\text{oblv} : \mathrm{Shv}_{\text{ét}}(\mathbf{Sm}_{/k}; \mathbb{Q}) \to \mathrm{Shv}_{\mathrm{Nis}}(\mathbf{Sm}_{/k}; \mathbb{Q}) \tag{15.3}$$

is exact, c.f. [65, Proposition 5.27].

Proof. Arguing as in [25, Theorem 5.5(1)], the Lemma reduces to the following statement: given an *étale* sheaf $\mathcal{F}_1$ of abelian groups on $\mathbf{Sm}_{/k}$ such that the éh sheafification $(\mathcal{F}_1)_{\text{éh}} = 0$, then for any $\mathbb{A}^1$-invariant pretheory $\mathcal{G}$ satisfying étale descent, one has:

$$\mathrm{Ext}^i(\mathcal{F}_1, \mathcal{G}) = 0, \quad \text{for all } i \geq 0. \tag{15.4}$$

Analogous to [25, Lemma 5.4], the proof consists of two steps:

(1) Establish (15.4) for $\mathcal{F}_1 = \mathrm{Coker}(\mathbb{Z}_{\text{ét}}(U') \to \mathbb{Z}_{\text{ét}}(U))$, where $U' \to U$ is a composition of n blow-ups with smooth centers. An induction argument reduces to $n = 1$, where the result follows from the Nisnevich version [25, Lemma 5.3] together with the exactness of (15.3).

(2) Reduction to case (1). Indeed, since $\mathcal{F}_1$ is already an étale sheaf. Lemma 15.1.4 shows that to each section $a \in \mathcal{F}_1(U)$, one can find a sequence of blow-ups with smooth centers $p : U' \to U$ such that $p^*a = 0$. Thus the same argument as in [25, Lemma 5.4] applies. ☐

15.3. **Derived éh-topology.**

15.3.1. We introduce a variant of the éh-topology for derived schemes, based on the modified version of abstract blow-up square introduced by Halpern-Leistner–Preygel [40]. We call a homotopy Cartesian square of derived prestacks:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & Y \arrow[d, "p"] \\ \mathcal{Z} \arrow[r, "i"] & X \end{tikzcd} \tag{15.5}$$

a *derived* abstract blow-up square if $X, Y \in \mathbf{DSch}^{\mathrm{ft}}_{/k}$, i is the formal completion along a closed subset in the topological space $|X|$, and p is proper and induces an isomorphism $Y\backslash\mathcal{E} \xrightarrow{\sim} X\backslash\mathcal{Z}$. We note that $\mathcal{Z}$ and $\mathcal{E}$ are thus derived ind-schemes ([40, Proposition 2.1.2]).

Let $\mathbf{t}_0$ denote the coarsest topology on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ such that the empty sieve covers $\emptyset$ and for every derived abstract blow-up square (15.5), the sieve generated by $\{p, i\}$ is a covering sieve of X.

15.3.2. To give an alternative description, let $\mathbf{S}$ denote the set of morphisms from the geometric realization $|\check{\mathrm{C}}(\mathfrak{U})| \to X$ in $\mathrm{PSh}(\mathbf{DSch}^{\mathrm{ft}}_{/k})$, where $\check{\mathrm{C}}(\mathfrak{U})$ is the Čech nerve associated to $\mathfrak{U} = \{p, i\}$ for any derived abstract blow-up square. Then $\mathbf{F} \in \mathrm{PSh}(\mathbf{DSch}^{\mathrm{ft}}_{/k})$ is a $\mathbf{t}_0$-sheaf if and only if it is $\mathbf{S}$-local. Indeed, the presheaf $|\check{\mathrm{C}}(\mathfrak{U})|$ is equivalent to the sieve generated by $\mathfrak{U}$, so the result again follows from [41, Corollary C.2].

We note that derived abstract blow-up squares verify the (∞-categorical version of the) conditions of [3, Theorem 3.2.5]. More precisely:

(1) every derived abstract blow-up square is homotopy Cartesian;

(2) derived abstract blow-up squares are stable under base change in $\mathbf{DSch}^{\mathrm{ft}}_{/k}$;

(3) for every (15.5), i is a monomorphism of presheaves;

(4) given (15.5), the induced square is still a derived abstract blow-up:

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & Y \arrow[d, "\Delta_p"] \\ \mathcal{E} \underset{\mathcal{Z}}{\times} \mathcal{E} \arrow[r, "{(i,i)}"] & Y \underset{X}{\times} Y \end{tikzcd}$$

Thus we have the following analogus of [3, Theorem 3.2.5].

15.3.3. **Lemma.** *Let* $\mathbf{F}$ *be a presheaf on* $\mathbf{DSch}^{\mathrm{ft}}_{/k}$*. Then it is a* $\mathbf{t}_0$*-sheaf if and only if* $\mathbf{F}(\emptyset)$ *is contractible and for every derived abstract blow-up square* (15.5)*, the induced* *square is homotopy Cartesian:*

$$
(15.6) \qquad \begin{tikzcd} \operatorname{Hom}(\mathcal{E}, \mathbf{F}) & \mathbf{F}(Y) \arrow[l] \\ \operatorname{Hom}(\mathcal{Z}, \mathbf{F}) \arrow[u] & \mathbf{F}(X) \arrow[l] \arrow[u] \end{tikzcd}
$$

Proof. The proof of *loc.cit.* applies verbatim. $\square$

We remark that Condition (c) would fail if $\mathcal{Z}$ was a closed subscheme of X instead of a formal completion.

15.3.4. *Derived* **éh**-*topology.* We define **éh** to be the coarsest topology on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ containing the étale topology, the topology generated by surjective closed immersions, and $\mathbf{t}_0$. Thus, a **Spc**-valued presheaf $\mathbf{F}$ on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ is an **éh**-sheaf if and only if it satisfies:

(1) $\mathbf{F}$ is an étale sheaf;
(2) $\mathbf{F}$ satisfies descent along surjective closed immersions;
(3) $\mathbf{F}$ turns every derived abstract blow-up square into a homotopy Cartesian square.

Given a derived abstract blow-up square (15.5), the sieve generated by $\{p, i\}$ can be refined by a proper surjective cover (for instance, taking any closed subscheme Z of X with the same underlying set as $\mathcal{Z}$, we obtain a proper surjection $Z \sqcup Y \to X$). Therefore **éh** is coarser than the derived **h**-topology. We obtain relations analogous to the classical situation:

$$\text{étale} \preceq \mathbf{éh} \preceq \mathbf{h}.$$

However, we caution the reader that the restriction of an **éh**-sheaf to the full subcategory $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ is not necessarily an éh-sheaf in the classical sense.

We record some facts which will be used later.

15.3.5. **Lemma.** *Let $\mathcal{F}$ (resp. $\mathbf{F}$) be an* éh-*sheaf on* $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ *(resp.* **éh**-*sheaf on* $\mathbf{DSch}^{\mathrm{ft}}_{/k}$*).*

(1) *The tautological extension of* $\mathcal{F}$ *to* $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ *is an* **éh***-sheaf:*

$$(\mathbf{DSch}^{\mathrm{ft}}_{/k})^{\mathrm{op}} \to \mathbf{Spc}, \quad X \rightsquigarrow \mathcal{F}(\pi_0 X)$$

(2) *If* $\mathbf{F}$ *is nil-invariant, then its restriction to* $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ *is an* éh*-sheaf.*

In particular, given a nil-invariant presheaf $\mathbf{F}$ on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$, satisfying **éh** descent is equivalent to its restriction to $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ satisfying éh descent.

Proof. The étale descent is clear is both statements. To prove (1), we note that $\mathcal{F}$ is nil-invariant so its extension has descent along surjective closed immersions. Let us now be given a derived abstract blow-up square (15.5) where $\mathcal{Z}$ is the formal completion of $Z \subset |X|$. We represent $\mathcal{Z}$ as a filtered colimit of Z_α, where each Z_α is a closed subscheme of X with underlying set Z. Then $\mathcal{E}$ identifies with $\operatorname{colim}_\alpha E_\alpha$ for $E_\alpha := Z_\alpha \underset{X}{\times} Y$. The square (15.6) is equivalent to:

$$\begin{tikzcd}
\lim_\alpha \mathcal{F}(\pi_0 E_\alpha) & \mathcal{F}(\pi_0 Y) \arrow[l] \\
\lim_\alpha \mathcal{F}(\pi_0 Z_\alpha) \arrow[u] & \mathcal{F}(\pi_0 X) \arrow[l] \arrow[u]
\end{tikzcd}$$

which is a limit of homotopy Cartesian diagrams. To prove (2), let us be given an abstract blow-up square (15.1). Let $\mathcal{Z}$ (resp. $\mathcal{E}$) be the completion of Z inside X (resp. E inside Y). Then we obtain a derived abstract blow-up square, so the following square is homotopy Cartesian:

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{E}, \mathbf{F}) & \mathbf{F}(Y) \arrow[l] \\
\mathrm{Hom}(\mathcal{Z}, \mathbf{F}) \arrow[u] & \mathbf{F}(X) \arrow[l] \arrow[u]
\end{tikzcd}$$

Since $\mathbf{F}$ is nil-invariant, the left vertical map identifies with $\mathbf{F}(Z) \to \mathbf{F}(E)$. $\square$

15.3.6. **Lemma.** *Suppose* $\mathbf{F}$ *is an n-truncated presheaf on* $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ *for some* $n \geq 0$*, i.e.* $\pi_i\mathbf{F}(X) = 0$ *for all* $i > n$ *and* $X \in \mathbf{DSch}^{\mathrm{ft}}_{/k}$*. Then* $\mathbf{F}_{\text{\bf{éh}}}$ *is nil-invariant.*

Proof. Any **éh**-hypersheaf is nil-invariant since the constant simplicial system X_{red} is an **éh**-hypercover of $X \in \mathbf{DSch}^{\mathrm{ft}}_{/k}$. The n-truncation hypothesis implies that the **éh**-sheafification and hypersheafification agree. $\square$

We now let $\mathrm{PSh}^{\mathrm{nil},\leq n}(\mathbf{DSch}^{\mathrm{ft}}_{/k})$ denote the ∞-category of nil-invariant, n-truncated presheaves on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$. Combining Lemma 15.3.5 and Lemma 15.3.6, we have commutative diagrams:

$$\begin{tikzcd} \mathrm{PSh}^{\mathrm{nil},\leq n}(\mathbf{DSch}^{\mathrm{ft}}_{/k}) \arrow[r, "\sim"] \arrow[d, shift right, "L"'] & \mathrm{PSh}^{\mathrm{nil},\leq n}(\mathbf{Sch}^{\mathrm{ft}}_{/k}) \arrow[d, shift right, "L"'] \\ \mathrm{Shv}^{\mathrm{nil},\leq n}_{\text{\bf{éh}}}(\mathbf{DSch}^{\mathrm{ft}}_{/k}) \arrow[r, "\sim"] \arrow[u, shift right] & \mathrm{Shv}^{\mathrm{nil},\leq n}_{\text{éh}}(\mathbf{Sch}^{\mathrm{ft}}_{/k}) \arrow[u, shift right] \end{tikzcd} \tag{15.7}$$

In other words, for n-truncated nil-invariant presheaves, the **éh** and éh-topologies give rise to the same sheaf theory with the same functorialities.

16. Tameness II: differential forms of moderate growth

In this section, we introduce another ingredient needed for the study of tame twistings—the sheaf of differential forms of moderate growth $\mathring{\Omega}^p$ $(p \geq 0)$. Its definition is based on some Hodge-theoretic input, and its most important properties for us are the h-descent and a Gersten resolution.

Although technically, we will only need the case for $p = 1$, we find it helpful to develop the techniques in greater generality.

16.1. **Point of departure.**

16.1.1. An effective Cartier divisor D in a smooth scheme X is said to be of *normal crossing* if, étale locally on X, D is defined by the vanising of $x_1 \cdots x_k$ $(k \leq n)$ where $x_1, \cdots, x_n$ is a system of coordinates on X. Although globally, D may not be a union of smooth divisors, the normalization $\nu : \widetilde{D} \to D$ always produces a smooth $\widetilde{D}$. In the situation of a normal crossing divisor with complement $\mathring{X}$:

$$\mathring{X} \xrightarrow{j} X \xleftarrow{i} D,$$

one may define a locally free $\mathcal{O}_X$-module $\Omega_X^p(\log D)$ for each $p \geq 0$. We refer the reader to [16, §II.3] for its basic properties.

16.1.2. Let $X \in \mathbf{Sm}_{/k}$. A *good compactification* of X is an open immersion $X \hookrightarrow \overline{X}$, where $\overline{X}$ is proper, smooth, and $D := \overline{X} \backslash X$ is a normal crossing divisor. Hironaka's desingularization shows that a good compactification always exists. The complex $\Omega^\bullet_{\overline{X}}(\log D)$ equipped with the Hodge filtration (i.e., stupid truncation) yields a spectral sequence:

$$ {}_F\mathrm{E}_1^{p,q} = \mathrm{H}^q(\overline{X}; \Omega^p_{\overline{X}}(\log D)) \implies \mathbb{H}^{p+q}(\overline{X}; \Omega^\bullet_{\overline{X}}(\log D)), \tag{16.1}$$

which degenerates at E_1 ([15, Corollaire 3.2.13(ii)]). Since $\mathbb{H}^p(\overline{X}; \Omega^\bullet_{\overline{X}}(\log D))$ and the Hodge filtration it carries are canonically independent of the good compactification ([15, Théroème 3.2.5(ii)]), so must be its pth graded piece

$$ {}_F\,\mathrm{Gr}^p\,\mathbb{H}^p(\overline{X}; \Omega^\bullet_{\overline{X}}(\log D)) \xrightarrow{\sim} \mathrm{H}^0(\overline{X}; \Omega^p_{\overline{X}}(\log D)).$$

16.1.3. We are thus led to the following definition. For $p \geq 0$, define $\mathring{\Omega}^p$ as the subpresheaf of Ω^p on $\mathbf{Sm}_{/k}$, consisting of those differential forms $\omega \in \Omega^p(X)$ which extend to $\mathrm{H}^0(\overline{X}; \Omega^p_{\overline{X}}(\log D))$ for a good compactification $X \hookrightarrow \overline{X}$. The above observation implies that $\mathring{\Omega}^p$ is a well-defined presheaf on $\mathbf{Sm}_{/k}$. We extend $\mathring{\Omega}^p$ to $\mathbf{Sch}^{\mathrm{ft}}_{/k}$

by the procedure of right Kan extension:

$$\mathring{\Omega}^p(X) := \lim_{\substack{Y\to X\\ Y\in \mathbf{Sm}_{/k}}} \mathring{\Omega}^p(Y).$$

Let us note some quick consequences of the definition:

16.1.4. **Lemma.** *The presheaves $\mathring{\Omega}^p$ $(p \geq 0)$ satisfy:*

(1) $\mathring{\Omega}^0$ is canonically the constant sheaf $\underline{k}$;

(2) $\mathring{\Omega}^p$ takes values in finite-dimensional k-vector spaces;

(3) For $X \in \mathbf{Sm}_{/k}$, the subspace $\mathring{\Omega}^p(X) \subset \Omega^p(X)$ belongs to closed p-forms;

(4) $\mathring{\Omega}^p$ is a sheaf in the Zariski topology on $\mathbf{Sm}_{/k}$.

Proof. (1) is immediate. (2) follows from the smooth case by taking a smooth hypercover. (3) is a consequence of the degeneration of (16.1) at E_1 ([15, Corollaire 3.2.14]). For (4), it is clear that $\mathring{\Omega}^p$ is a separated presheaf. To check gluing, we cover $X \in \mathbf{Sm}_{/k}$ by opens U and V, the Mayer-Vietoris sequence on de Rham cohomology:

$$\mathbb{H}^p(X) \to \mathbb{H}^p(U) \oplus \mathbb{H}^p(V) \to \mathbb{H}^p(U \cap V)$$

is exact and strictly compatible with the Hodge filtration ([15, Théorème 1.2.10(iii)]), so it remains exact after applying ${}_F\mathrm{Gr}^p$ ([15, Proposition 1.1.11(ii)]). $\square$

16.2. h-descent.

16.2.1. In this section, we shall prove:

16.2.2. **Proposition.** *For all $p \geq 0$, the presheaf $\mathring{\Omega}^p$ on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ satisfies* h*-descent.*

Instead of giving a direct argument, we compare $\mathring{\Omega}^p$ to the h-sheafification Ω^p_h of the usual differential p-forms, studied by Huber–Jörder [42]. Their theorem is that

Ω^p_{h} identifies with the right Kan extension of Ω^p from $\mathbf{Sm}_{/k}$:

$$\Omega^p_{\mathrm{h}}(X) \xrightarrow{\sim} \lim_{\substack{Y \to X \\ Y \in \mathbf{Sm}_{/k}}} \Omega^p(Y).$$

This implies that $\mathring{\Omega}^p$ can be regarded as a subpresheaf of Ω^p_{h}, characterized by the property that a section $\omega \in \Omega^p_{\mathrm{h}}(X)$ belongs to $\mathring{\Omega}^p(X)$ if and only if its pullback to any smooth scheme $Y \to X$ belongs to $\mathring{\Omega}^p(Y)$.

Therefore, in order to prove Proposition 16.2.2, we only need to show that for $\pi : \widetilde{X} \to X$ an h-covering in $\mathbf{Sch}^{\mathrm{ft}}_{/k}$, if $\omega \in \Omega^p_{\mathrm{h}}(X)$ has the property that $\pi^*\omega$ belongs to $\mathring{\Omega}^p(\widetilde{X})$, then $\omega \in \mathring{\Omega}^p(X)$. By mapping a smooth scheme Y to X and considering a further smooth h-cover of $Y \underset{X}{\times} \widetilde{X}$, we may assume that $\widetilde{X}$ and X are both smooth. Fitting $\widetilde{X} \to X$ into a map between good compactifications, the Proposition follows from the Lemma below.

16.2.3. **Lemma.** *Suppose there is a commutative diagram in* $\mathbf{Sm}_{/k}$*:*

$$\begin{tikzcd} \mathring{Y} \arrow[r, hook] \arrow[d] & Y \arrow[d, "\pi"] \\ \mathring{X} \arrow[r, hook] & X \end{tikzcd}$$

where $\mathring{X} \hookrightarrow X$ (resp. $\mathring{Y} \hookrightarrow Y$) is an open immersion whose boundary is a normal crossing divisor D (resp. E). Assume furthermore that π is a proper surjection. Then given any $\omega \in \Omega^p(\mathring{X})$, it extends to $\Omega^p_X(\log D)$ if and only if $\pi^*\omega$ extends to $\Omega^p_Y(\log E)$.

Proof. The "only if" direction is clear as $\pi^{-1}D$ is set-theoretically contained in E. Let us argue the converse. The question is étale local on X. Since $\Omega^p_X(\log D)$ is locally free, it suffices to show that ω extends to $\Omega^p_X(\log D)$ away from codimension ≥ 2. Thus we will choose coordinates $x_1, \cdots, x_n \in \mathcal{O}_X$ such that D is defined by $x_1 = 0$ and Ω^1_X is free on $dx_1, \cdots, dx_n$.

We will also replace Y by its formal neighborhood around some $y \in Y$ contained in the smooth locus of an irreducible component E_1 of E which dominates D. Since the normalization $\widetilde{E}_1 \to D$ is a proper surjection and $\widetilde{E}_1$ is connected and smooth, we see that $\Omega^p(D) \to \Omega^p(E_1)$ is injective. In other words, we shall assume:

(1) $Y = \mathrm{Spec}(k[\![y_1, \cdots, y_m]\!])$, E_1 is defined by $y_1 = 0$, and $\mathring{Y} = Y \backslash E_1$ is the preimage of $\mathring{X}$;

(2) The map $\Omega^p(D) \to \Omega^p(E_1)$ is injective.

Thus $\pi^* x_1 = u y_1^e$ for some $e \geq 1$ and $u \in \mathcal{O}_Y^\times$. Hensel's lemma finds an eth root of u, so after an automorphism on Y fixing E_1, we may further assume:

(3) $\pi^* x_1 = y_1^e$.

Let us now consider a meromorphic form $\omega \in \Omega^p(X)[x_1^{-1}]$ such that $\pi^*\omega \in \Omega^p(Y)[y_1^{-1}]$ is logarithmic along E_1. Write

$$\omega = \omega_1 + \omega_2 \wedge \frac{dx_1}{x_1},$$

where $\omega_1, \omega_2 \in \Omega^p(X)[x_1^{-1}]$ do not feature dx_1. In what follows we assume ω_1, ω_2 are both nonzero (the case where either is zero being similar but simpler). Write $\omega_1 = x_1^{d_1} \tilde{\omega}_1$ and $\omega_2 = x_1^{d_2} \tilde{\omega}_2$ where $\tilde{\omega}_1$ and $\tilde{\omega}_2$ are holomorphic and not divisible by x_1. Then:

$$\begin{aligned} \pi^*\omega &= (y_1^e)^{d_1} \pi^* \tilde{\omega}_1 + (y_1^e)^{d_2} \pi^* \tilde{\omega}_2 \wedge e \frac{dy_1}{y_1} \\ &= (y_1^e)^{d_1} (\eta_1^{(1)} + \eta_1^{(2)} \wedge dy_1) + (y_1^e)^{d_2} \eta_2^{(1)} \wedge e \frac{dy_1}{y_1} \end{aligned} \tag{16.2}$$

where $\pi^* \tilde{\omega}_1 = \eta_1^{(1)} + \eta_1^{(2)} \wedge dy_1$ is its decomposition into parts where $\eta_1^{(1)}$, $\eta_1^{(2)}$ do *not* feature dy_1 (and analogously for $\pi^*\omega_2$). Then assumption (2) implies that $\pi^*\tilde{\omega}_1, \pi^*\tilde{\omega}_2$ are *nonzero* after pulling back to E_1. Thus $\eta_1^{(1)}$ and $\eta_2^{(1)}$ are *not* divisible by y_1. Now, analyzing the part of the expression (16.2) *not* featuring dy_1, we see that $d_1 \geq 0$.

Hence the first term is holomorphic, so the second term is necessarily logarithmic along y_1. Since $\eta_2^{(1)}$ is not divisible by y_1, we see that $d_2 \geq 0$ as well. $\square$

$\square$(Proposition 16.2.2)

16.2.4. A particular consequence of the h-descent of $\mathring{\Omega}^p$ is a canonical transfer structure on the restriction of $\mathring{\Omega}^p$ to $\mathbf{Sm}_{/k}$. We recall the category of correspondences $\mathbf{Sm}_{/k}^{\mathrm{Cor}}$ mentioned in §15. According to J. Scholbach [55, Lemma 2.1], the representable presheaf $\mathbb{Z}_{\mathrm{tr}}(X)$ on $\mathbf{Sm}_{/k}^{\mathrm{Cor}}$ for any $X \in \mathbf{Sm}_{/k}$ has the property that its h-sheafification identifies with that of $\mathbb{Z}(X)$ on $\mathbf{Sm}_{/k}$:

$$\mathbb{Z}_{\mathrm{h}}(X) \xrightarrow{\sim} (\mathbb{Z}_{\mathrm{tr}}(X)|_{\mathbf{Sm}_{/k}})_{\mathrm{h}}.$$

Consequently, for any h-sheaf of abelian groups $\mathcal{F}$ on $\mathbf{Sm}_{/k}$ there is an isomorphism:

$$\mathcal{F}(X) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{PSh}(\mathbf{Sm}_{/k})}(\mathbb{Z}_{\mathrm{tr}}(X), \mathcal{F}),$$

so $\mathcal{F}$ acquires a canonical transfer structure.

16.2.5. **Lemma.** *The restriction of $\mathring{\Omega}^p$ ($p \geq 0$) to $\mathbf{Sm}_{/k}$ is an $\mathbb{A}^1$-invariant sheaf with a canonical transfer structure.*

Proof. The $\mathbb{A}^1$-invariance is a direct consequence of the identification of $\mathring{\Omega}^p(X)$ with the pth graded piece of $\mathbb{H}^p(\overline{X}, \Omega^\bullet_{\overline{X}}(\log D))$ with respect to the Hodge filtration. The canonical transfer structure has just been noted above. $\square$

By construction, the transfer structure on $\mathring{\Omega}^p$ is compatible with that of Ω^p. For an explicit formula of the latter, we refer the reader to the trace construction of Lecomte–Wach [47]. In particular, the morphism $d\log : \mathbb{G}_m \to \mathring{\Omega}^1$ commutes with transfer.

16.3. **Cohomological properties.**

16.3.1. Suppose $\mathcal{F}$ is a presheaf on $\mathbf{Sm}_{/k}$ valued in abelian groups. Following Voevodsky [65, §3.1], we define $\mathcal{F}_{-1}$ to be the presheaf:

$$\mathcal{F}_{-1} : X \rightsquigarrow \mathrm{Coker}(\mathcal{F}(X \times \mathbb{A}^1) \to \mathcal{F}(X \times (\mathbb{A}^1 \backslash \mathbf{0}))).$$

The presheaf $\mathcal{F}_{-n}$ is then defined iteratively.

16.3.2. **Lemma.** *There holds:*

(1) The sheaf $(\mathring{\Omega}^0)_{-1}$ is identically zero;

(2) For any $p \geq 1$, there is a canonical isomorphism $(\mathring{\Omega}^p)_{-1} \xrightarrow{\sim} \mathring{\Omega}^{p-1}$.

Proof. Part (1) is tautological. Part (2) follows either from the Hodge-theoretic interpretation of $\mathring{\Omega}^p$ or a direct calculation making use of the product formula for logarithmic forms [16, §II, Proposition 3.2(iii)]. □

16.3.3. For notational convenience, we extend $\mathring{\Omega}^p$ to smooth local schemes (i.e., localizations of smooth schemes at a point) by the formula:

$$\mathring{\Omega}^p(\eta) := \operatorname*{colim}_{U_\alpha} \mathring{\Omega}^p(U_\alpha),$$

where U_α is a cofiltered limit presentation of η with each U_α smooth, affine and each $U_\alpha \to U_\beta$ an open immersion. The following Theorem summarizes the cohomological properties of $\mathring{\Omega}^p$:

16.3.4. **Theorem.** *Let $p \geq 0$ and τ be one of the following Grothendieck topologies on $\mathbf{Sm}_{/k}$: Zariski, Nisnevich, étale,* cdh, éh, qfh, h. *There holds:*

(1) For all $n \geq 0$, the presheaf $X \rightsquigarrow \mathrm{H}^n_\tau(X; \mathring{\Omega}^p)$ on $\mathbf{Sm}_{/k}$ is an $\mathbb{A}^1$-invariant presheaf with transfer, and is canonically independent of the choice of τ;

(2) For $X \in \mathbf{Sm}_{/k}$, the Zariski sheaf $\mathring{\Omega}^p_X$ is quasi-isomorphic to the following complex concentrated in degrees $[0,p]$:

$$\bigoplus_{x \in X^{(0)}} (i_x)_* \mathring{\Omega}^p(x) \to \bigoplus_{x \in X^{(1)}} (i_x)_* \mathring{\Omega}^{p-1}(x) \to \cdots \to \bigoplus_{x \in X^{(p)}} (i_x)_* k.$$

Here, $X^{(n)}$ denotes the set of codimension-n points of X.

Proof. Statement (1) is valid for any $\mathbb{A}^1$-invariant h-sheaf of $\mathbb{Q}$-vector spaces, by Scholbach [55, Theorem 2.11]; the only choice of τ not covered in *loc.cit.* is the éh-topology, which follows from Lemma 15.2.6. For statement (2), Mazza–Voevodsky–Weibel [52, Theorem 24.11] shows that an $\mathbb{A}^1$-invariant pretheory $\mathcal{F}$ satisfying Zariski descent admits a Gersten resolution with terms given by $\bigoplus_{x \in X^{(n)}} (i_x)_* F_{-n}(x)$. We are done by the calculation of $(\mathring{\Omega}^p)_{-n}$ in Lemma 16.3.2. □

16.3.5. **Remark.** A. Beilinson has kindly pointed out that the Gersten resolution in (2) also follows directly from applying ${}_F\mathrm{Gr}^p$ to the Gersten resolution of algebraic de Rham cohomology obtained from the Bloch–Ogus theorem.

16.3.6. **Example.** We calculate the cohomology of $\mathring{\Omega}^1$ on a smooth curve X. Since the cohomology groups will be independent of the chosen Grothendieck topology (Theorem 16.3.4(1)), we may as well calculate them in the Zariski topology using the Gersten resolution (Theorem 16.3.4(2)). The answer is as follows:

(1) if X is affine, then $\mathrm{H}^1(X; \mathring{\Omega}^1) = 0$;

(2) if X is proper, then the canonical map $\mathrm{R}\,\Gamma(X; \mathring{\Omega}^1) \to \mathrm{R}\,\Gamma(X; \Omega^1)$ is an isomorphism.

Indeed, the affine case amounts to the problem of contructing ω with prescribed poles and follows from $\mathrm{H}^1(\overline{X}; \Omega^1(E)) = 0$ for the boundary divisor $E := \overline{X} \backslash X$ in a smooth completion $\overline{X}$.

For the proper case, the nontrivial part is cohomology in degree 1. We reduce to X connected (with generic point η) and remove one closed point $\mathring{X} := X\backslash x$. The sum-of-residue formula and the vanishing of $\mathrm{H}^1(\mathring{X}, \mathring{\Omega}^1)$ shows that the cokernel of d is indeed identified with k:

$$\begin{tikzcd}
 & & \mathring{\Omega}(\eta) \arrow[d, "d"'] \arrow[dr] & & \\
0 \arrow[r] & k \arrow[r] & \bigoplus_{x\in X^{(1)}} k \arrow[r] & \bigoplus_{x\in \mathring{X}^{(1)}} k \arrow[r] & 0.
\end{tikzcd}$$

16.3.7. *Tangential remarks.* We conclude this section with some remarks concerning the interaction between $\mathring{\Omega}^p$ and algebraic cycles. These facts will not play a role in this text.

Let $\mathbf{K}_p^{\mathrm{M}}$ denote the Zariski sheaf of the pth Milnor K-theory group on $\mathbf{Sm}_{/k}$. For a field F, $\mathbf{K}_p^{\mathrm{M}}(F)$ is the pth graded piece of the tensor algebra $T^{\otimes}(F^{\times})$ modulo $u \otimes v$ for $u + v = 1$. More generally, $\mathbf{K}_p^{\mathrm{M}}$ is given by a Gersten resolution. When X is furthermore projective, $\mathrm{H}^p(X, \mathbf{K}_p^{\mathrm{M}})$ identifies with the Chow group $\mathrm{CH}^p(X)$ of codimension-p cycles [57, Théorème 5]. In particular, the construction:

$$d\log : \mathbf{K}_p^{\mathrm{M}}(\eta) \to \mathring{\Omega}^p(\eta), \quad f_1 \otimes \cdots \otimes f_n \rightsquigarrow \frac{df_1}{f_1} \wedge \cdots \wedge \frac{df_n}{f_n}$$

for points η on $X \in \mathbf{Sm}_{/k}$ defines a morphism of Zariski sheaves on $\mathbf{Sm}_{/k}$:

$$d\log : \mathbf{K}_p^{\mathrm{M}} \underset{\mathbb{Z}}{\otimes} k \to \mathring{\Omega}^p. \tag{16.3}$$

We obtain the following factorization of the algebraic de Rham cycle class map:

$$\begin{tikzcd}
\mathrm{CH}^p(X) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "\sim"] \arrow[drr, "\mathrm{cl}"'] & \mathrm{H}^p(X; \mathbf{K}_p^{\mathrm{M}} \underset{\mathbb{Z}}{\otimes} k) \arrow[r, "d\log"] & \mathrm{H}^p(X; \mathring{\Omega}^p) \arrow[d, "\mathrm{can}"] \\
 & & \mathrm{H}^p(X; \Omega^p)
\end{tikzcd}$$

Indeed, its factorization through $d\log : \mathrm{H}^p(X; \mathbf{K}^{\mathrm{M}}_p \underset{\mathbb{Z}}{\otimes} k) \to \mathrm{H}^p(X; \Omega^p)$ is already observed in [23] and the further factorization through $\mathrm{H}^p(X, \mathring{\Omega}^p)$ is tautological. The Gersten resolution of $\mathring{\Omega}^p$ (Theorem 16.3.4(2)) implies that the composition $\mathrm{CH}^p(X) \underset{\mathbb{Z}}{\otimes} k \to \mathrm{H}^p(X; \mathring{\Omega}^p)$ is surjective. Thus the image of $\mathrm{H}^p(X; \mathring{\Omega}^p)$ in $\mathrm{H}^p(X; \Omega^p)$ is precisely the span of cycle classes.

17. Tameness III: definitions and properties

In this section, we gather all the ingredients and define the notion of a tame twisting. In fact, we will define two stacks: tame gerbes $\mathring{\mathbf{Ge}}$ and tame twistings $\mathring{\mathbf{Tw}}$. They are both theories of gerbes in the sense of Chapter 3. The key result we will prove is that $\mathring{\mathbf{Ge}}$ is in fact a motivic éh-theory of gerbes.

17.1. **Local systems.**

17.1.1. Let $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$. Recall the de Rham prestack X_{dR} of §14. By a *rank*–1 *local system* on X, we will mean a line bundle on X_{dR}. Denote by $\mathbf{Loc}_1$ the prestack which associates to $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$ the strict Picard (1-)groupoid of rank–1 local systems on X.

17.1.2. **Lemma.** *The prestack* $\mathbf{Loc}_1$ *satisfies* h*-descent.*

Proof. The ∞-prestack Crys which associates $\mathrm{QCoh}(X_{\mathrm{dR}})$ to $X \in \mathbf{DSch}^{\mathrm{ft}}_{/k}$ satisfies (derived) $\mathbf{h}$-descent (Lemma 14.1.5). Since Crys is nil-invariant, its restriction to $\mathbf{Sch}^{\mathrm{ft}}_{/k}$ satisfies (usual) h-descent. We observe that $\mathbf{Loc}_1(X)$ is the full subcategory of $\mathrm{Crys}(X)$ consisting of invertible objects lying in the heart of the t-structure as an object of $\mathrm{QCoh}(X)$. Thus $\mathbf{Loc}_1$ inherits h-descent from Crys. $\square$

17.1.3. **Remark.** Every object in $\mathbf{Loc}_1(X)$ can be viewed as a line bundle $\mathcal{L}$ on X equipped with an isomorphism $\mathrm{pr}_1^* \mathcal{L} \xrightarrow{\sim} \mathrm{pr}_2^* \mathcal{L}$ on the completion of the diagonal in $X \times X$, satisfying a cocycle condition [33, Proposition 3.4.3]. When X is smooth, this is equivalent to a connection $\nabla : \mathcal{L} \to \mathcal{L} \otimes \Omega^1_X$, but not in general.

17.1.4. It is clear that over $\mathbf{Sm}_{/k}$, the strict Picard groupoid $\mathbf{Loc}_1$ is represented by the complex of étale sheaves concentrated in degrees $[-1,0]$ (in the sense of Chapter 3, §9.2):

$$d\log : \mathbb{G}_m \to \Omega^{1,\mathrm{cl}}.$$

We also recall the subsheaf $\mathring{\Omega}^1_X \hookrightarrow \Omega^{1,\mathrm{cl}}_X$ of differential forms of moderate growth from §16.

17.1.5. **Lemma.** *Let $X \in \mathbf{Sm}_{/k}$, the following conditions are equivalent for any $\sigma \in \mathbf{Loc}_1(X)$.*

(1) σ belongs to the subcomplex $d\log : \mathbb{G}_m \to \mathring{\Omega}^1$;

(2) σ is regular as a $\mathcal{D}_X$-module.

Being *regular* as a $\mathcal{D}_X$-module means for any smooth curve $f : C \to X$, the pullback $f^*\sigma$ acquires a connection with at most logarithmic poles at points of $\overline{C}\backslash C$.

Proof. The implication (1) $\implies$ (2) is clear. Conversely, suppose σ is regular. To check that it belongs to the subcomplex $\mathcal{O}^\times_X \xrightarrow{d\log} \mathring{\Omega}^1_X$, it suffices to do so locally on X, so we may assume that the underlying line bundle of σ is trivial. Thus the connection 1-form is given by $d+\omega$ for some $\omega \in \Omega^{1,\mathrm{cl}}(X)$. We need to argue $\omega \in \mathring{\Omega}(X)$. Consider a good compactification $X \hookrightarrow \overline{X}$. The line bundle extends trivially to $\overline{X}$. The Lemma thus becomes the implicaion (ii) $\implies$ (iv) in [16, §II, Théorèm 4.1]. ☐

17.1.6. Let $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$. Then a local system $\sigma \in \mathbf{Loc}_1(X)$ is said to be *tame* if for all morphisms $f : Y \to X$ with Y smooth, the pullback $f^*\sigma$ satisfies the conditions of Lemma 17.1.5. We let $\mathring{\mathbf{Loc}}_1$ denote the prestack of tame rank–1 local systems on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$.

17.1.7. **Lemma.** *The prestack $\mathring{\mathbf{Loc}}_1$ satisfies h-descent.*

Proof. Since $\mathring{\mathbf{Loc}}_1$ is a full subfunctor of $\mathbf{Loc}_1$, we only need to prove the following: for an h-cover $\pi : \widetilde{X} \to X$, if $\sigma \in \mathbf{Loc}_1(X)$ has the property that $\pi^*\sigma \in \mathbf{Loc}_1(\widetilde{X})$ is tame, then so is σ. By definition, we may assume $\widetilde{X} \to X$ is a dominant morphism of smooth curves, and the result is straightforward (in fact, a very special case of Lemma 16.2.3). □

17.2. **Tame gerbes.**

17.2.1. We define $\mathring{\mathbf{Ge}}$ as the éh-sheafification of the classifying prestack of $\mathring{\mathbf{Loc}}_1$:

$$\mathring{\mathbf{Ge}} := \mathrm{B}_{\text{éh}}\, \mathring{\mathbf{Loc}}_1.$$

Informally, a section of $\mathring{\mathbf{Ge}}$ is a torsor for the sheaf $\mathring{\mathbf{Loc}}_1$ which is locally trivial in the éh-topology. For $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, we call $\mathring{\mathbf{Ge}}(X) := \mathrm{Maps}(X, \mathring{\mathbf{Ge}})$ the category of *tame gerbes* on X. It has the structure of a strict Picard 2-groupoid. Lemma 17.1.7 guarantees that the loop prestack $\mathrm{pt} \underset{\mathring{\mathbf{Ge}}}{\times} \mathrm{pt}$ identifies with $\mathring{\mathbf{Loc}}_1$.

The following result shows that tame gerbes on a smooth scheme can be defined using the weaker étale topology.

17.2.2. **Lemma.** *Suppose $X \in \mathbf{Sm}_{/k}$. Then the following canonical map is an isomorphism:*

$$\mathrm{Maps}(X, \mathrm{B}_{\text{ét}}\, \mathring{\mathbf{Loc}}_1) \xrightarrow{\sim} \mathring{\mathbf{Ge}}(X).$$

In particular, $\mathring{\mathbf{Ge}}$ is represented by the complex $d\log : \mathbb{G}_m \to \mathring{\Omega}^1$ in degrees $[-2, -1]$.

Proof. Let $\mathbf{F}_{\text{éh}}$ denote the fiber of éh-sheaves $\mathring{\mathbf{Loc}}_1 \to \mathrm{B}_{\text{éh}}\, \mathbb{G}_m$ on $\mathbf{Sm}^{\mathrm{ft}}_{/k}$. Evaluating at $X \in \mathbf{Sm}^{\mathrm{ft}}_{/k}$ produces a fiber sequence:

$$\mathbf{F}_{\text{éh}}(X) \to \mathring{\mathbf{Loc}}_1(X) \to \mathrm{Maps}(X, \mathrm{B}_{\text{éh}}\, \mathbb{G}_m).$$

The comparison Lemma 15.2.4 shows that $\mathrm{Maps}(X, \mathrm{B}_{\text{éh}}\, \mathbb{G}_m)$ identifies with $\mathrm{Maps}(X, \mathrm{B}_{\text{ét}}\, \mathbb{G}_m)$. Thus Lemma 17.1.5 implies that $\mathbf{F}_{\text{éh}}$ identifies with $\mathring{\Omega}^1$. On the other hand, $\mathring{\mathbf{Loc}}_1 \to \mathrm{B}_{\text{éh}}\, \mathbb{G}_m$ is a surjection of éh-sheaves, so $\mathring{\mathbf{Loc}}_1$ is an éh $\mathring{\Omega}^1$-torsor over $\mathrm{B}_{\text{éh}}\mathbb{G}_m$. This gives us another fiber sequence:

$$\mathring{\mathbf{Loc}}_1 \to \mathrm{B}_{\text{éh}}\,\mathbb{G}_m \to \mathrm{B}_{\text{éh}}\,\mathring{\Omega}^1.$$

Delooping and taking sections over $X \in \mathbf{Sm}_{/k}$, we obtain a fiber sequence:

$$\mathrm{Maps}(X, \mathrm{B}_{\text{éh}}\,\mathring{\mathbf{Loc}}_1) \to \mathrm{Maps}(X, \mathrm{B}^2_{\text{éh}}\,\mathbb{G}_m) \to \mathrm{Maps}(X, \mathrm{B}^2_{\text{éh}}\,\mathring{\Omega}^1).$$

Thus the result follows from the comparison Lemma 15.2.4 for $\mathbb{G}_m$ and Theorem 16.3.4(1) for $\mathring{\Omega}^1$. $\square$

17.2.3. Note that there is a morphism of sheaves of strict Picard groupoids on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$:

$$\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \to \mathring{\mathbf{Loc}}, \quad (f, a) \rightsquigarrow f^a. \tag{17.1}$$

Indeed, given $f \in \mathcal{O}_X^\times$ and $a \in k/\mathbb{Z}$, we will construct a tame local system f^a on each smooth Y mapping to X in a compatible way. This process will construct an object of $\mathring{\mathbf{Loc}}(X)$ by Lemma 17.1.7. We choose a lift $\bar{a} \in k$ of a. The local system f^a on Y is set to be

$$f^{\bar{a}} := (\mathcal{O}_Y, d + \bar{a} d\log f).$$

Indeed, another choice of the lift $\bar{a}'$ must differ from $\bar{a}$ by an integer n, and the local systems $f^{\bar{a}}$ and $f^{\bar{a}'}$ are canonically isomorphic via multiplication by $f^n \in \mathcal{O}_Y^\times$. This shows that $f^a \in \mathring{\mathbf{Loc}}(Y)$ is well-defined. It is obviously compatible with change of Y.

From (17.1), we obtain a morphism of sheaves of strict Picard 2-groupoids on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$:

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \to \mathring{\mathbf{Ge}}, \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a. \tag{17.2}$$

We call (17.2) the *divisor class map* for tame gerbes.

17.3. **Analytic comparison.**

17.3.1. When the ground field $k = \mathbb{C}$, there is a Riemann–Hilbert correspondence relating tame gerbes to analytic $\mathbb{C}^\times$-gerbes. Given a scheme $X \in \mathbf{Sch}^{\mathrm{ft}}_{/\mathbb{C}}$, we let X^{an} denote its analytification. Let $\mathbf{An}^{\mathrm{ft}}_{/\mathbb{C}}$ denote the category of separated analytic spaces of finite type over $\mathbb{C}$. We write $\mathbf{Tors}_{\mathbb{C}^\times}$ (resp. $\mathbf{Ge}_{\mathbb{C}^\times}$) for the presheaf of strict Picard 1-groupoid of analytic $\mathbb{C}^\times$-torsors (resp. 2-groupoid of $\mathbb{C}^\times$-gerbes) on $\mathbf{An}^{\mathrm{ft}}_{\mathbb{C}}$.

17.3.2. **Lemma.** *Let $X \in \mathbf{Sch}^{\mathrm{ft}}_{/\mathbb{C}}$. Then,*

(1) there is an equivalence of Picard 1-groupoids

$$\mathbf{\mathring{Loc}}_1(X) \xrightarrow{\sim} \mathbf{Tors}_{\mathbb{C}^\times}(X^{\mathrm{an}});$$

(2) there is a fully faithful functor of strict Picard 2-groupoids whose image consists of those $\mathbb{C}^\times$-gerbes trivialized over $\widetilde{X}^{\mathrm{an}} \to X^{\mathrm{an}}$ for an éh*-cover $\widetilde{X} \to X$:*

$$\mathbf{\mathring{G}e}(X) \hookrightarrow \mathbf{Ge}_{\mathbb{C}^\times}(X^{\mathrm{an}}).$$

Proof. (1) Recall that $\mathbf{\mathring{Loc}}_1$ satisfies h-descent (Lemma 17.1.7). On the other hand, given any h-cover $\widetilde{X} \to X$ in $\mathbf{Sch}^{\mathrm{ft}}_{/\mathbb{C}}$, the homotopy type of X^{an} is equivalent to the geometric realization of the Čech nerve of $\widetilde{X}^{\mathrm{an}} \to X^{\mathrm{an}}$ [9, Proposition 3.21]. This implies that $X \rightsquigarrow \mathbf{Tors}_{\mathbb{C}^\times}(X^{\mathrm{an}})$ is also an h-sheaf. Thus the problem reduces to the case of smooth X. There, $\mathbf{\mathring{Loc}}_1(X)$ is the category of invertible objects inside regular, holonomic $\mathcal{D}$-modules on X, which lie in the heart when considered as objects of $\mathrm{QCoh}(X)$.

The Riemann–Hilbert correspondence is symmetric monoidal with respect to the !-monoidal structure on the constructible derived category $\mathrm{Shv}_c(X^{\mathrm{an}})$. In particular, it preserves invertible objects. On the other hand, the invertible objects in $\mathrm{Shv}_c(X^{\mathrm{an}})$ with respect to ! and $*$-monoidal structures agree via tensoring with the dualizing complex. Thus, we see that $\mathbf{\mathring{Loc}}_1(X)$ identifies with $*$-invertible objects in $\mathrm{Shv}(X^{\mathrm{an}})$ lying in the heart. The latter category identifies with $\mathbf{Tors}_{\mathbb{C}^\times}(X^{\mathrm{an}})$.

(2) The analytification functor $\mathbf{Sch}^{\mathrm{ft}}_{/\mathbb{C}} \to \mathbf{An}^{\mathrm{ft}}_{/\mathbb{C}}$ defines a map

$$i_* : \mathrm{PSh}(\mathbf{An}^{\mathrm{ft}}_{/\mathbb{C}}) \to \mathrm{PSh}(\mathbf{Sch}^{\mathrm{ft}}_{/\mathbb{C}}).$$

By the observation above, $i_*\mathbf{Tors}_{\mathbb{C}^\times}$ and $i_*\mathbf{Ge}_{\mathbb{C}^\times}$ are h-sheaves on $\mathbf{Sch}^{\mathrm{ft}}_{/\mathbb{C}}$, so in particular are éh-sheaves. On the other hand, part (a) gives an equivalence:

$$\mathring{\mathbf{Loc}}_1 \xrightarrow{\sim} i_*\mathbf{Tors}_{\mathbb{C}^\times}.$$

By delooping, we obtain a sequence of functors:

$$\mathrm{B}_{\text{éh}}\,\mathring{\mathbf{Loc}}_1 \xrightarrow{\sim} (i_*\,\mathrm{B}\,\mathbf{Tors}_{\mathbb{C}^\times})_{\text{éh}} \hookrightarrow (i_*\,\mathrm{B}_{\mathrm{an}}\,\mathbf{Tors}_{\mathbb{C}^\times})_{\text{éh}} \xrightarrow{\sim} i_*\mathbf{Ge}_{\mathbb{C}^\times}.$$

The middle functor is fully faithful and its image consists of éh-locally trivial objects.

□

17.4. **$\mathring{\mathbf{Ge}}$ is motivic.**

17.4.1. We now establish the key property of $\mathring{\mathbf{Ge}}$. Let us recall the properties (RP1), (RP2), (A), and (B) defining a motivic theory of gerbes in Chapter 3.

17.4.2. **Proposition.** *The presheaf* $\mathring{\mathbf{Ge}}$ *and* c_1 (17.2) *define a motivic* éh-*theory of gerbes.*

Analytic proofs of (RP1), (A), and (B). We first take $k = \mathbb{C}$ and compare $\mathring{\mathbf{Ge}}$ with the theory of analytic gerbes $\mathbf{Ge}_{\mathrm{an}}$. By Lemma 17.3.2 and the construction of the divisor class map for $\mathbf{Ge}_{\mathrm{an}}$ (Chapter 3, §13), we have a commutative diagram:

$$\begin{tikzcd}
\mathbf{Pic} \underset{\mathbb{Z}}{\otimes} \mathbb{C}/\mathbb{Z} \arrow[r, "c_1"] \arrow[d, "\exp(2\pi i -)"] & \mathring{\mathbf{Ge}} \arrow[d, hook] \\
\mathbf{Pic} \underset{\mathbb{Z}}{\otimes} \mathbb{C}^\times \arrow[r, "c_1"] & \mathbf{Ge}_{\mathrm{an}}
\end{tikzcd}$$

where the right vertical arrow is fully faithful.

The properties (RP1), (A), and (B) follow immediately from the corresponding properties of $\mathbf{Ge}_{\mathrm{an}}$. To wit, (RP1) is verified because $\mathring{\mathbf{Ge}}$ is a full subfunctor of $\mathbf{Ge}_{\mathrm{an}}$. To see (A), we consider the commtutive square when $k = \mathbb{C}$:

$$\begin{tikzcd}
\mathring{\mathbf{Ge}}(X) \arrow[r] \arrow[d, hook] & \mathring{\mathbf{Ge}}(X \times \mathbb{A}^1) \arrow[d, hook] \\
\mathbf{Ge}_{\mathrm{an}}(X) \arrow[r, "\cong"] & \mathbf{Ge}_{\mathrm{an}}(X \times \mathbb{A}^1)
\end{tikzcd}$$

Thus $\mathring{\mathbf{Ge}}(X) \to \mathring{\mathbf{Ge}}(X \times \mathbb{A}^1)$ is fully faithful. It is essentially surjective since there is a retract $\mathring{\mathbf{Ge}}(X \times \mathbb{A}^1) \to \mathring{\mathbf{Ge}}(X)$ and two objects $\mathcal{G}_1, \mathcal{G}_2 \in \mathring{\mathbf{Ge}}(X \times \mathbb{A}^1)$ are identified once they are identified in $\mathbf{Ge}_{\mathrm{an}}(X \times \mathbb{A}^1)$. To prove (B), we observe that the commutative diagram below consists of fully faithful embeddings:

$$\begin{tikzcd}
(p_* \mathring{\mathbf{Ge}}_Y)_x \arrow[r] \arrow[d, hook] & \mathring{\mathbf{Ge}}(Y \underset{X}{\times} \{x\}) \arrow[d, hook] \\
(p_* \mathbf{Ge}_{\mathrm{an},Y})_x \arrow[r, "\sim"] & \mathbf{Ge}_{\mathrm{an}}(Y \underset{X}{\times} \{x\})
\end{tikzcd}$$

Therefore, the top arrow is also fully faithful. □

Below, we shall present algebraic proofs of (RP1), (RP2), and (A). They are based on the calculation of cohomology of $\mathring{\Omega}^1$ and several known facts about the Brauer group. Unfortunately, we have not found an algebraic proof of (B).

Algebraic proofs of (RP1), (RP2), and (A).

Property (RP1). Over $X \in \mathbf{Sm}_{/k}$, the étale sheaf $\mathring{\mathbf{Ge}}_X$ is represented by the complex

$$\mathcal{G}_X := \mathrm{Cofib}(\mathcal{O}_X^{\times} \to \mathring{\Omega}_X^1). \tag{17.3}$$

It suffices calculate the (derived) restriction $\tau^{\le 0} i^! \mathcal{G}$. Note that $i^!$ is a left-exact functor on étale sheaves, so (17.3) gives rise to a long exact sequence:

$$\begin{aligned} 0 \to \mathrm{H}^{-2}\, i^! \mathcal{G} \to \mathrm{H}^0\, i^! \mathcal{O}_X^\times \to \mathrm{H}^0\, i^! \mathring{\Omega}_X^1 \to \mathrm{H}^{-1}\, i^! \mathcal{G} \\ \to \mathrm{H}^1\, i^! \mathcal{O}_X^\times \xrightarrow{\beta} \mathrm{H}^1\, i^! \mathring{\Omega}_X^1 \to \mathrm{H}^0\, i^! \mathcal{G} \to \mathrm{H}^2\, i^! \mathcal{O}_X^\times. \end{aligned} \tag{17.4}$$

We make the following observations based on the tautological triangle for an étale sheaf $\mathcal{F}$:

$$i_* i^! \mathcal{F} \to \mathcal{F} \to \mathrm{R}\, j_* (\mathcal{F}\big|_{X \backslash Z}).$$

(1) $\mathrm{H}^0\, i^! \mathcal{O}_X^\times = 0$ and $\mathrm{H}^0\, i^! \mathring{\Omega}_X^1 = 0$;

(2) $\mathrm{H}^1\, i^! \mathcal{O}_X^\times \xrightarrow{\sim} \underline{\mathbb{Z}}$, since this group identifies as the cokernel of $\mathcal{O}_X^\times \to j_* \mathcal{O}_{X\backslash Z}^\times$; the analogous consideration gives $\mathrm{H}^1\, i^! \mathring{\Omega}_X^1 \xrightarrow{\sim} \underline{k}$, and the morphism β passes to the tautological inclusion $\underline{\mathbb{Z}} \to \underline{k}$.

(3) $\mathrm{H}^2\, i^! \mathcal{O}_X^\times = 0$, since this group identifies with $\mathrm{R}^1\, j_* \mathcal{O}_{X\backslash Z}^\times$, which vanishes because every line bundle on $X \backslash Z$ extends across Z.

Combining the above observations, we obtain $\mathrm{H}^{-2}\, i^! \mathcal{G} = 0$, $\mathrm{H}^{-1}\, i^! \mathcal{G} = 0$, and $\mathrm{H}^0\, i^! \mathcal{G} \xrightarrow{\sim} k/\mathbb{Z}$. It is straghtforward to see that this isomorphism agrees with (17.2).

Property (RP2). The descent property of $\mathring{\mathbf{G}}\mathbf{e}$ allows to assume X is affine. We again use the complex $\mathcal{G}_X$ (17.3), and the result reduces to the following calculations of cohomology groups:

(1) $\mathrm{H}^i_{\text{ét}}(X; \mathcal{O}_X^\times) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X \backslash Z; \mathcal{O}_X^\times)$ for $i = 0, 1, 2$. The nontrivial part is $i = 2$ which follows from purity of the Brauer group for smooth schemes over a field (see Gabber [26, §2]);

(2) $\mathrm{H}^i_{\text{ét}}(X; \mathring{\Omega}_X^1) \xrightarrow{\sim} \mathrm{H}^i_{\text{ét}}(X \backslash Z; \mathring{\Omega}_X^1)$ for $i = 0, 1, 2$. This follows from the étale-to-Zariski comparison and the Gersten resolution (Theorem 16.3.4).

Property (A). Proceeding as above, it suffices to establish $\mathbb{A}^1$-invariance of the following groups:

(1) $\mathrm{H}^i_{\text{ét}}(X; \mathcal{O}_X^\times)$ for $i = 0, 1, 2$. The case for $i = 0$ is immediate. For $i = 1$, this is the $\mathbb{A}^1$-invariance of the Picard group over a regular base. For $i = 2$, one first identifies $\mathrm{H}^2_{\text{ét}}(X; \mathcal{O}_X^\times)$ with the Brauer group using Gabber's theorem [14], and then appeals to the theorem of Auslander–Goldman [4, Proposition 7.7] (this requires $\mathrm{char}(k) = 0$.)

(2) $\mathrm{H}^i_{\text{ét}}(X; \mathring{\Omega}^1_X)$ for $i = 0, 1$. These have been established in Theorem 16.3.4. $\square$

17.5. **Tame twistings.**

17.5.1. The definition of tame twistings require us to work with the ∞-category $\mathbf{DSch}^{\mathrm{ft}}_{/k}$. We first extend $\mathring{\mathbf{Loc}}_1$ and $\mathring{\mathbf{Ge}}$ to $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ by evaluation on the underlying classical scheme. By the commutative diagram (15.7), we see that $\mathring{\mathbf{Ge}}$ is the **éh**-sheafification of $\mathrm{B}\,\mathring{\mathbf{Loc}}_1$, regarded as a presheaf on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$. Next, we consider the **éh**-sheafification $\mathrm{B}^2_{\mathbf{\acute{e}h}}\,\mathbb{G}_m$. Define $\mathring{\mathbf{Tw}}$ as the fiber:

$$\mathring{\mathbf{Tw}} := \mathrm{Fib}(\mathring{\mathbf{Ge}} \to \mathrm{B}^2_{\mathbf{\acute{e}h}}\,\mathbb{G}_m).$$

Thus $\mathring{\mathbf{Tw}}$ is an **éh**-sheaf of strict Picard groupoids on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$ whose sections are called *tame twistings*. Furthermore, since $\mathring{\mathbf{Tw}}$ identifies with $\mathrm{B}_{\mathbf{\acute{e}h}}$ applied to:

$$\mathrm{Fib}(\mathring{\mathbf{Loc}}_1 \to \mathrm{B}_{\text{ét}}\,\mathbb{G}_m) \hookrightarrow \mathrm{Fib}(\mathbf{Loc}_1 \to \mathrm{B}_{\text{ét}}\,\mathbb{G}_m),$$

which admits a k-linear structure, we see that $\mathring{\mathbf{Tw}}$ is in fact valued in $\mathrm{H}\,k$-module objects in $\mathrm{ComGrp}(\mathbf{Spc})$. Furthermore, the fiber of the canonical map $\mathring{\mathbf{Tw}} \to \mathring{\mathbf{Ge}}$ identifies with $\mathrm{B}_{\mathbf{\acute{e}h}}\,\mathbb{G}_m$, but the tautological map $\mathrm{B}_{\text{ét}}\,\mathbb{G}_m \to \mathrm{B}_{\mathbf{\acute{e}h}}\,\mathbb{G}_m$ is an equivalence by the **éh**-descent of line bundles (Chapter 1, Lemma 4.2.4). We thus obtain a fiber sequence:

$$\mathbf{Pic} \to \mathring{\mathbf{Tw}} \to \mathring{\mathbf{Ge}}. \tag{17.5}$$

17.5.2. Extension by scalar defines the *divisor class map* of tame twistings:

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} k \to \mathring{\mathbf{Tw}}, \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a. \tag{17.6}$$

This map can also be constructed in a way analogous to (17.2) by first building a map:

$$\mathbb{G}_m \underset{\mathbb{Z}}{\otimes} k \to \mathrm{Fib}(\mathring{\mathbf{Loc}}_1 \to \mathrm{B}_{\text{ét}}\,\mathbb{G}_m), \quad (f, a) \rightsquigarrow f^a$$

using the $d\log$ construction over smooth schemes. Consequently, (17.6) is compatible with the divisor class map of tame gerbes (17.2) in the sense that the following diagram canonically commutes:

$$\begin{tikzcd} & \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} k \arrow[r] \arrow[d] & \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \arrow[d] \\ \mathbf{Pic} \arrow[ur] \arrow[r] & \mathring{\mathbf{Tw}} \arrow[r] & \mathring{\mathbf{Ge}} \end{tikzcd} \tag{17.7}$$

Therefore, tame twistings form a theory of gerbes in the sense of Chapter 3, but it is *not* motivic as it fails to be nil-invariant. Nevertheless, it enjoys some features of a motivic theory of gerbes.

17.5.3. **Lemma.** *The theory of gerbes* $\mathring{\mathbf{Tw}}$ *together with* (17.6) *satisfies properties* (RP1)*,* (RP2)*, and* (A).

Proof. These are already contained in the algebraic proof of the corresponding parts of Proposition 17.4.2. □

We now give an explicit description of tame twistings over a smooth scheme.

17.5.4. **Lemma.** *Suppose* $X \in \mathbf{Sm}_{/k}$*. There is an equivalence:*

$$\mathrm{DK}(\tau^{\leq 0}\,\mathrm{R}\,\Gamma_{\text{ét}}(X, \mathring{\Omega}^1[1])) \xrightarrow{\sim} \mathring{\mathbf{Tw}}(X).$$

Proof. Write provisionally $\mathring{\mathbf{Tw}}_{\text{ét}}$ for the sheaf on $\mathbf{Sm}_{/k}$ defined by $\mathrm{Fib}(\mathrm{B}_{\text{ét}}\,\mathring{\mathbf{Loc}}_1 \to \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m)$. Then we have a canonical map $\mathring{\mathbf{Tw}}_{\text{ét}} \to \mathring{\mathbf{Tw}}$ making the following diagram commute:

$$\begin{tikzcd}
\mathbf{Pic} \arrow[r] \arrow[d, "\cong"] & \mathring{\mathbf{Tw}}_{\text{ét}} \arrow[r, "\alpha"] \arrow[d, "\gamma_1"] & \mathrm{B}_{\text{ét}}\,\mathring{\mathbf{Loc}}_1 \arrow[d, "\gamma_2"] \\
\mathbf{Pic} \arrow[r] & \mathring{\mathbf{Tw}} \arrow[r] & \mathring{\mathbf{Ge}}
\end{tikzcd}$$

The comparison Lemma 17.2.2 for tame gerbes shows that γ_2 is an equivalence. Since α is an étale local surjection, we see that γ_1 must also be an equivalence. The fact that $\mathring{\mathbf{Tw}}_{\text{ét}}$ is represented by the complex $\mathring{\Omega}^1[1]$ is a direct consequence of Lemma 17.1.5. $\square$

17.5.5. We now produce a morphism from $\mathring{\mathbf{Tw}}$ to the usual presheaf of twistings $\mathbf{Tw}_{/k}$ defined in §14. Recall that the value of $\mathbf{Tw}_{/k}$ on $X \in \mathbf{DSch}^{\text{ft}}_{/k}$ can be given equivalently as:

$$\begin{aligned}
\mathbf{Tw}_{/k}(X) :=& \mathrm{Fib}(\mathrm{Maps}(X_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m) \to \mathrm{Maps}(X, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m)) \\
&\xrightarrow{\sim} \mathrm{Fib}(\mathrm{Maps}(X_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_a) \to \mathrm{Maps}(X, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_a)).
\end{aligned}$$

Let us construct the promised morphism:

$$\mathring{\mathbf{Tw}} \to \mathbf{Tw}_{/k}. \tag{17.8}$$

We let $\mathbf{Ge}_{\mathrm{dR}}$ denote the étale stack which associates to $X \in \mathbf{DSch}^{\text{ft}}_{/k}$ the strict Picard groupoid $\mathrm{Maps}(X_{\mathrm{dR}}, \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m)$. Taking the fibers along the vertical maps in the following commutative diagram:

$$\begin{tikzcd}
\mathrm{B}_{\text{ét}}\,\mathring{\mathbf{Loc}}_1 \arrow[r] \arrow[d] & \mathrm{B}_{\text{ét}}\,\mathbf{Loc}_1 \arrow[r] & \mathbf{Ge}_{\mathrm{dR}} \arrow[d] \\
\mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m \arrow[rr, "\sim"] & & \mathrm{B}^2_{\text{ét}}\,\mathbb{G}_m
\end{tikzcd}$$

one obtains a morphism from $\mathrm{B}_{\text{ét}}\,\mathrm{Fib}(\mathbf{\mathring{L}oc}_1 \to \mathrm{B}_{\text{ét}}\,\mathbb{G}_m)$ to $\mathbf{Tw}_{/k}$. One then obtains (17.8) by noting that $\mathbf{Tw}_{/k}$ satisfies derived **éh**-descent (Lemma 14.5.2).

17.5.6. Finally, we note that tame twistings can be used to produce a twisted category of $\mathcal{D}$-modules equipped with a forgetful functor to ind-coherent sheaves. Note that any object $\mathcal{L} \in \mathbf{\mathring{L}oc}(X)$ acts as automorphism on $\mathrm{Crys}^r(X)$:

$$\mathcal{M} \rightsquigarrow \mathcal{M} \otimes \mathcal{L}, \tag{17.9}$$

and if the object in $\mathbf{Pic}(X)$ induced by $\mathcal{L}$ is trivialized, the underlying ind-coherent sheaves of $\mathcal{M}$ and $\mathcal{M} \otimes \mathcal{L}$ become canonically isomorphic.

Since both Crys^r and IndCoh are **éh**-sheaves on $\mathbf{DSch}^{\mathrm{ft}}_{/k}$, the procedure of §14.4 defines for every $\mathcal{T} \in \mathbf{\mathring{T}w}(X)$ a twisted category $\mathrm{Crys}^r_{\mathcal{T}}(X)$ equipped with a forgetful functor:

$$\mathrm{oblv} : \mathrm{Crys}^r_{\mathcal{T}}(X) \to \mathrm{IndCoh}(X).$$

This construction agrees with the usual twisted category defined by the twisting attached to $\mathcal{T}$ under the map (17.8). On the other hand, the full subcategory $\mathrm{C\mathring{r}ys}^r(X) \subset \mathrm{Crys}^r(X)$ of regular $\mathcal{D}$-modules form an **éh**-subsheaf. Since (17.9) preserves regularity (thank to tameness of $\mathcal{L}$), the same construction produces a full subcategory:

$$\mathrm{C\mathring{r}ys}^r_{\mathcal{T}}(X) \hookrightarrow \mathrm{Crys}^r_{\mathcal{T}}(X).$$

In other words, the notion of *regularity* makes sense for a crystal twisted by a tame twisting (or even a tame gerbe.)

18. Classification by quantum parameters

In this section, we prove that factorization tame twistings on the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$ are classified by enhanced Θ-data. We shall deduce this result from a combination of the classification theorem for line bundles, established in Chapter 2, and the one for tame gerbes, which falls into the paradigm of Chapter 3.

Then we simplify the k-linear groupoid $\Theta_G(\Lambda_T;\mathring{\mathbf{Tw}})$—it appears as a close variant of what is usually called "quantum parameters" of the geometric Langlands program.

18.1. The classification theorems.

18.1.1. We first note the classification theorem for tame gerbes $\mathring{\mathbf{Ge}}$.

18.1.2. **Proposition.** *There is a canonical equivalence of strict Picard 2-groupoids:*

$$\Psi_{\mathring{\mathbf{Ge}}} : \mathring{\mathbf{Ge}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \Theta_G(\Lambda_T;\mathring{\mathbf{Ge}}),$$

which makes the following diagram commute:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}", "\cong"'] & \mathring{\mathbf{Ge}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathrm{an}}", "\cong"'] \\ \Theta_G(\Lambda_T;\mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \arrow[r, "c_1"] & \Theta_G(\Lambda_T;\mathring{\mathbf{Ge}}) \end{tikzcd} \tag{18.1}$$

Proof. This combines Proposition 17.4.2 and Chapter 3, Theorem 10.3.2. □

18.1.3. We will now announce the classification theorem for tame twistings $\mathring{\mathbf{Tw}}$. Even though the statement is entirely parallel to the one above, the situation is more complicated as the theory of gerbes $\mathring{\mathbf{Tw}}$ is not *motivic* in the sense of Chapter 3.

18.1.4. **Theorem.** *There is a canonical equivalence of strict Picard 2-groupoids:*

$$\Psi_{\mathring{\mathbf{Tw}}} : \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \Theta_G(\Lambda_T;\mathring{\mathbf{Tw}}),$$

which makes the following diagram commute:

$$
\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}", "\cong"'] & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathring{\mathbf{Tw}}}", "\cong"'] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathring{\mathbf{Tw}})
\end{tikzcd}
\tag{18.2}
$$

The proof of Theorem 18.1.4 occupies the remainder of this subsection. We shall prove the theorem first for tori, then for semisimple, simply connected groups, and finally for any reductive group.

18.1.5. *Tori.* We first define $\Psi_{\mathring{\mathbf{Tw}},T}$ for a torus T by the composition:

$$
\begin{aligned}
\mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) &\to \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{comb}}) \\
&\xrightarrow{\sim} \Theta(\Lambda_T; \mathring{\mathbf{Tw}}),
\end{aligned}
$$

where the second equivalence is due to purity of $\mathring{\mathbf{Tw}}$ (Lemma 17.5.3), c.f. the proof of Chapter 2, Lemma 5.2.2. This definition makes $\Psi_{\mathring{\mathbf{Tw}},T}$ compatible with the classification functors of line bundles and tame gerbes. Thus we have a commutative diagram:

$$
\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[r] \arrow[d, "\Psi_{\mathbf{Pic},T}", "\cong"'] & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[r] \arrow[d, "\Psi_{\mathring{\mathbf{Tw}},T}"] & \mathring{\mathbf{Ge}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathring{\mathbf{Ge}},T}", "\cong"'] \\
\Theta(\Lambda_T; \mathbf{Pic}) \arrow[r] & \Theta(\Lambda_T; \mathring{\mathbf{Tw}}) \arrow[r] & \Theta(\Lambda_T; \mathring{\mathbf{G}}e)
\end{tikzcd}
$$

where the rows are fiber sequences of strict Picard 2-groupoids and $\Psi_{\mathbf{Pic},T}$ and $\Psi_{\mathring{\mathbf{Ge}},T}$ are both equivalences. In order to show that $\Psi_{\mathring{\mathbf{Tw}},T}$ is also an equivalence, it remains to prove that it is essentially surjective.

Recall that $\mathring{\mathbf{Tw}}(X)$ is equivalent to étale $\mathring{\Omega}^1_X$-torsors (Lemma 17.5.4). By the calculation of the cohomology of $\mathring{\Omega}^1_X$ (Example 16.3.6), we see that the divisor class map:

$$\mathbf{Pic}(X) \underset{\mathbb{Z}}{\otimes} k \to \mathring{\mathbf{Tw}}(X), \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a$$

is essentially surjective; indeed, this is clear for X affine, and for X proper, $\mathring{\mathbf{Tw}}(X)$ is 1-dimensional and is spanned by the image of any line bundle of nonzero degree.

Let $\mathbf{Pic}_0^{\text{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}})$ denote the full subgroupoid of $\mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}})$ consisting of factorization line bundles whose associated quadratic form vanishes. We employ the same notation for factorization twistings. From the commutative diagram:

$$\begin{tikzcd}
\mathbf{Pic}_0^{\text{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r] \arrow[d, "\cong"] & \mathring{\mathbf{Tw}}_0^{\text{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathring{\mathbf{Tw}},T}"] \\
\mathbf{Hom}(\Lambda_T, \mathbf{Pic}(X)) \underset{\mathbb{Z}}{\otimes} k \arrow[r, two heads] & \mathbf{Hom}(\Lambda_T, \mathring{\mathbf{Tw}}(X)),
\end{tikzcd}$$

we see that objects of the full subgroupoid $\mathbf{Hom}(\Lambda_T, \mathring{\mathbf{Tw}}(X))$ inside $\Theta(\Lambda_T; \mathring{\mathbf{Tw}})$ admit lifts along $\Psi_{\mathring{\mathbf{Tw}},T}$. It thus remains to show that the composition of $\Psi_{\mathring{\mathbf{Tw}},T}$ with the forgetful functor to $\mathcal{Q}(\Lambda_T; k)$ is surjective.

Now, every $q \in \mathcal{Q}(\Lambda_T; k)$ is a k-linear combination of integral forms $q_i \in \mathcal{Q}(\Lambda_T; \mathbb{Z})$. Scaling allows us to assume that each q_i is valued in $2\mathbb{Z}$. Since $\Theta(\Lambda_T; \mathbf{Pic}) \to \mathcal{Q}(\Lambda_T; \mathbb{Z})$ surjects onto even-valued forms, we find that the bottom arrow in the following diagram is surjective:

$$\begin{tikzcd}
\mathbf{Pic}^{\text{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r] \arrow[d, "\cong"] & \mathring{\mathbf{Tw}}^{\text{fact}}(\mathrm{Gr}_{T,\mathrm{Ran}}) \arrow[d, "\mathrm{oblv} \circ \Psi_{\mathring{\mathbf{Tw}},T}"] \\
\Theta(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, two heads] & \mathcal{Q}(\Lambda_T; k).
\end{tikzcd}$$

This concludes the proof that $\Psi_{\mathring{\mathbf{Tw}},T}$ is essentially surjective, hence an equivalence.

18.1.6. *Semisimple, simply connected groups.* We now turn to the case of a semisimple, simply connected group G_{sc}. We construct $\Psi_{\mathring{\mathbf{Tw}},G_{\text{sc}}}$ by first pulling back to

$\mathrm{Gr}_{T_{\mathrm{sc}},\mathrm{Ran}}$ and then pick out the quadratic form on $\Lambda_{T_{\mathrm{sc}}}$. We claim that the result q is W-invariant. Indeed, this is because any multiple $c \cdot q$ (for $c \in k^\times$) is W-invariant modulo $\mathbb{Z}$, by the compatibility with tame gerbes.

Thus, there is again a commutative diagram of fiber sequences:

$$\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \arrow[r] \arrow[d, "\Psi_{\mathbf{Pic},G_{\mathrm{sc}}}", "\cong"'] & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \arrow[r] \arrow[d, "\Psi_{\mathring{\mathbf{Tw}},G_{\mathrm{sc}}}"] & \mathring{\mathbf{Ge}}(\mathrm{Gr}_{G_{\mathrm{sc}},\mathrm{Ran}}) \arrow[d, "\Psi_{\mathring{\mathbf{Ge}},G_{\mathrm{sc}}}", "\cong"'] \\
\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}};\mathbb{Z})^W \arrow[r] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}};k)^W \arrow[r] & \mathcal{Q}(\Lambda_{T_{\mathrm{sc}}};k/\mathbb{Z})^W_{\mathrm{restr}}
\end{tikzcd}$$

We are done because $\Psi_{\mathbf{Pic},G_{\mathrm{sc}}}$ and $\Psi_{\mathring{\mathbf{Ge}},G_{\mathrm{sc}}}$ are equivalences and $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}};\mathbb{Z})^W \underset{\mathbb{Z}}{\otimes} k$ surjects (in fact, isomorphes) onto $\mathcal{Q}(\Lambda_{T_{\mathrm{sc}}};k)^W$.

18.1.7. *The general case.* As in the proof of Chapter 3, Theorem 10.3.2, the functor $\Psi_{\mathring{\mathbf{Tw}},T}$ together with the classification for semisimple, simply connected groups allows us to construct the functor

$$\Psi_{\mathring{\mathbf{Tw}}} : \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \Theta_G(\Lambda_T; \mathring{\mathbf{Tw}})$$

for any reductive group G. An analogous argument as above reduces the problem to showing that $\Psi_{\mathring{\mathbf{Tw}}}$ is essentially surjective.

Recall that every $q \in \mathcal{Q}(\Lambda_T;k)^W$ splits into the sum of $q_1 = \sum_{s\in\mathbf{S}} b_s q_{s,\mathrm{Kil}}$ and a form q_2 induced from $\pi_1 G$ (Chapter 3, Lemma 10.2.3). We first claim that the composition:

$$\mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\Psi_{\mathring{\mathbf{Tw}}}} \Theta_G(\Lambda_T; \mathring{\mathbf{Tw}}) \to \mathcal{Q}(\Lambda_T;k)^W$$

surjects onto the span of Killing forms. Indeed, this is because the determinant line bundles construction of Chapter 2, §6.1 gives a section:

$$\begin{tikzcd} \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r] & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d] \\ \bigoplus_{s\in\mathbf{S}} k \arrow[u, "\det"] \arrow[r] & \mathcal{Q}(\Lambda_T; k)^W \end{tikzcd}$$

Therefore, it remains to show that $\Psi_{\mathring{\mathbf{Tw}}}$ surjects onto the full subgroupoid of $\Theta_G(\Lambda_T; \mathring{\mathbf{Tw}})$ where the associated quadratic form descends to $\pi_1 G$. This is in turn the space of quadratic forms on the lattice of the connected component of the center Z_G°. Thus the problem reduces to showing that:

$$\mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{Z_G^\circ,\mathrm{Ran}}) \xrightarrow{\sim} \Theta(\Lambda_{Z_G}; \mathring{\mathbf{Tw}}) \tag{18.3}$$

is essentially surjective. Let $T_1 := G/G_{\mathrm{der}}$. Then $Z_G^\circ \to T_1$ is an isogeny of tori, so we have the following equivalence by the k-linear structure on tame twistings:

$$\begin{tikzcd} \Theta(\Lambda_{Z_G}; \mathring{\mathbf{Tw}}) & \mathrm{Gr}_{Z_G,\mathrm{Ran}} \arrow[dd] \arrow[dr] & \\ & & \mathrm{Gr}_{G,\mathrm{Ran}} \arrow[dl] \\ \Theta(\Lambda_{T_1}; \mathring{\mathbf{Tw}}) \arrow[uu, "\cong"] & \mathrm{Gr}_{T_1,\mathrm{Ran}} & \end{tikzcd}$$

This provides a splitting of (18.3). Hence $\Psi_{\mathring{\mathbf{Tw}}}$ is essentially surjective. $\square$(Theorem 18.1.4)

18.2. **Quantum parameters.**

18.2.1. We will now give an explicit description of the k-linear groupoid $\Theta_G(\Lambda_T; \mathring{\mathbf{Tw}})$, as "quantum parameters." Let $\mathring{\mathrm{Par}}_G$ denote the k-linear groupoid consisting of pairs $(\kappa, \mathring{E})$, where:

(1) a W-invariant bilinear form $\kappa : \mathfrak{t} \underset{k}{\otimes} \mathfrak{t} \to k$;

(2) an extension $\mathring{E}$ of $\underline{\mathfrak{z}}$ by $\mathring{\Omega}^1_X$ as Zariski sheaves of k-vector spaces on X.

We call $\mathring{\mathrm{Par}}_G$ the groupoid of *tame quantum parameters*.

By constrast, the groupoid Par_G of usual *quantum parameters* consists of pairs (κ, E) where κ is as above and

(2') E is an extension of $\mathfrak{z} \otimes \mathcal{O}_X$ by ω_X as $\mathcal{O}_X$-modules.

18.2.2. Let us first recall the equivalence by ω-shift (Chapter 3, §10.2):

$$\Theta_G(\Lambda_T; \mathring{\mathbf{Tw}}) \xrightarrow{\sim} \Theta^+_G(\Lambda_T; \mathring{\mathbf{Tw}}), \quad (q, \mathcal{T}^{(\lambda)}, \varepsilon) \rightsquigarrow (q, \mathcal{T}^{(\lambda)} \otimes \omega_X^{q(\lambda)}, \varepsilon).$$

On the other hand, $\mathring{\mathbf{Tw}}(X)$ is k-linear and $\mathfrak{z} \cong \pi_1 G \underset{\mathbb{Z}}{\otimes} k$, so we have a fiber sequence:

$$\mathbf{Hom}(\mathfrak{z}, \mathring{\mathbf{Tw}}(X)) \to \Theta^+_G(\Lambda_T; \mathring{\mathbf{Tw}}) \to \mathcal{Q}(\Lambda_T; k)^W. \tag{18.4}$$

Now, we observe that the automorphism $(-1)^{\kappa(\lambda,\mu)}$ on $\mathring{\mathbf{Tw}}(X)$ is trivial since $d\log$ annihilates all constant sections. Thus an element of $\Theta^+_G(\Lambda_T; \mathring{\mathbf{Tw}})$ consists of the data of q together with a *commutative* multiplicative system $\mathcal{T}^{(\lambda)}$, i.e., a k-linear morphism $\mathfrak{t} \to \mathring{\mathbf{Tw}}(X)$, whose restriction to $\mathfrak{t}_{\mathrm{der}}$ is determined by q. In particular, (18.4) is tautologically split. Furthermore, these summands correspond, repsectively to κ and $\mathring{E}$:

(1) $\mathcal{Q}(\Lambda_T; k)^W$ identifies with the space of W-invariant bilinear forms on $\mathfrak{t}$;

(2) $\mathbf{Hom}(\mathfrak{z}, \mathring{\mathbf{Tw}}(X))$ is the space of k-linear maps $\mathfrak{z} \to \mathrm{R}\,\Gamma_{\text{ét}}(X; \mathring{\Omega}^1[1])$ (Lemma 17.5.4). On the other hand, there is an isomorphism $\mathrm{R}\,\Gamma_{\text{ét}}(X; \mathring{\Omega}^1[1]) \xrightarrow{\sim} \mathrm{R}\,\Gamma_{\mathrm{Zar}}(X; \mathring{\Omega}^1[1])$ (Theorem 16.3.4), and we have:

$$\mathrm{R}\,\Gamma_{\mathrm{Zar}}(X; \mathring{\Omega}^1[1]) \xrightarrow{\sim} \mathrm{R}\,\mathrm{Hom}(\underline{\mathfrak{z}}, \mathring{\Omega}_X[1]),$$

by adjunction of Zariski sheaves.

This gives rise to an equivalence:

$$\Theta_G^+(\Lambda_T; \mathring{\mathbf{Tw}}) \xrightarrow{\sim} \mathring{\mathrm{Par}}_G.$$

Finally, there is a canonical functor from $\mathring{\mathrm{Par}}_G$ to Par_G and the calculation in Example 16.3.6 shows that it is an equivalence *if and only if* X is proper.

18.2.3. We summarize these facts by the following chain of functors.

$$\begin{tikzcd}
\mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r, "\Psi_{\mathring{\mathbf{Tw}}}", "\sim"'] & \Theta_G(\Lambda_T; \mathring{\mathbf{Tw}}) \arrow[d, "\otimes\omega_X^{q(\lambda)}", "\cong"'] & & \\
& \Theta_G^+(\Lambda_T; \mathring{\mathbf{Tw}}) \arrow[r, "\sim"] & \mathring{\mathrm{Par}}_G \arrow[r, "(a)"] & \mathrm{Par}_G,
\end{tikzcd}$$

where the last functor (a) is an equivalence if and only if X is proper.

18.2.4. The classification statements for tame gerbes and tame twistings can be summarized by the following commutative cube, where all vertical arrows are equivalence.

$$\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[rr] \arrow[dr] \arrow[dd] & & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[dr] \arrow[dd] & \\
& \mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \arrow[rr] \arrow[dd] & & \mathring{\mathbf{Ge}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[dd] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k \arrow[rr] \arrow[dr] & & \Theta_G(\Lambda_T; \mathring{\mathbf{Tw}}) \arrow[dr] & \\
& \Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k/\mathbb{Z} \arrow[rr] & & \Theta_G(\Lambda_T; \mathring{\mathbf{Ge}})
\end{tikzcd}$$

When $k = \mathbb{C}$, we also have the compatibility between the classifications of factorization tame gerbes and analytic $\mathbb{C}^\times$-gerbes (Chapter 3, §13).

19. Lie-$*$ algebras and factorization twistings

In this section, we return to the study of factorization (usual) twistings. We review central extensions of Lie-$*$ algebras studied by Beilinson–Drinfeld [8] and show that they give rise to factorization twistings on $\mathrm{Gr}_{G,\mathrm{Ran}}$.

The main result is that this "construction functor" $\Xi_{\mathbf{Tw},G}$ is an equivalence for G semisimple, simply connected.

19.1. **Lie-$*$ algebras.**

19.1.1. We fix a connected smooth curve X and consider the *abelian* category $\mathcal{D}_X\text{-Mod}^\heartsuit$ of right $\mathcal{D}$-modules on X. We shall follow the convention of [8] and regard ω_X as an object of $\mathcal{D}_X\text{-Mod}^\heartsuit$, i.e., we absorb the cohomological shift.

19.1.2. *Definition.* A *Lie-$*$ algebra* on X is an object $\mathcal{L} \in \mathcal{D}_X\text{-Mod}^\heartsuit$ equipped with a morphism in $\mathcal{D}_{X^2}\text{-Mod}^\heartsuit$:

$$[-,-] : \mathcal{L} \boxtimes \mathcal{L} \to \Delta_{*,\mathrm{dR}}(\mathcal{L}), \tag{19.1}$$

where the functor $\Delta_{*,\mathrm{dR}}$ is the pushforward of $\mathcal{D}$-modules. The bracket (19.1) is required to satisfy the following properties:

(1) $[a \boxtimes b] = -\sigma_{12}[b \boxtimes a]$;

(2) the following identity holds:

$$[a \boxtimes [b \boxtimes c]] + \sigma_{123}[b \boxtimes [c \boxtimes a]] + \sigma_{123}^2[c \boxtimes [a \boxtimes b]] = 0.$$

Here, σ_{12} denotes the permutation action on $\Delta_{*,\mathrm{dR}}(\mathcal{L})$ and σ_{123} is the permutation $(x,y,z) \rightsquigarrow (y,z,x)$ acting on the pushforward of $\mathcal{L}$ along the main diagonal $X \hookrightarrow X^3$.

19.1.3. Let $\mathfrak{g}$ denote the Lie algebra of G. Then the $\mathcal{D}$-module $\mathfrak{g}_{\mathcal{D}} := \mathfrak{g} \otimes \mathcal{D}_X$ has the structure of a Lie-$*$ algebra. Indeed, the pushforward $\Delta_{*,\mathrm{dR}}\mathcal{D}_X \cong \mathcal{O}_X \underset{\mathcal{O}_X^2}{\otimes} \mathcal{D}_{X^2}$ has a canonical symmetric section, denoted by $\mathbf{1}_{\mathcal{D}}$. We define the map:

$$\mathfrak{g} \boxtimes \mathfrak{g} \to \Delta_{*,\mathrm{dR}}(\mathfrak{g}_{\mathcal{D}}), \quad [\xi \boxtimes \eta] = [\xi, \eta] \otimes \mathbf{1}_{\mathcal{D}},$$

and then extend by $\mathcal{D}_{X^2}$-linearity to obtain a Lie-$*$ bracket on $\mathfrak{g}_{\mathcal{D}}$.

Given any right $\mathcal{D}_X$-module $\mathcal{M}$, there is an associated Zariski sheaf $h(\mathcal{M})$ on X, whose section over U is given by $\mathcal{M} \underset{\mathcal{D}_U}{\otimes} \mathcal{O}_U$ ([8, 2.1.6]). This is the functor of "middle de Rham cohomology." Furthermore, if $\mathcal{L}$ has a Lie-$*$ algebra structure, then $h(\mathcal{L})$ is a Zariski sheaf of Lie algebras. In the example above, $h(\mathfrak{g}_{\mathcal{D}})$ is the Lie algebra $\mathfrak{g} \otimes \mathcal{O}_X$.

19.1.4. *Lie-$*$ modules.* For any Lie-$*$ algebra $\mathcal{L}$, a *Lie-$*$ module* $\mathcal{M}$ over $\mathcal{L}$ is a (right) $\mathcal{D}_X$-module equipped with a map $\mathcal{L} \boxtimes \mathcal{M} \to \Delta_{*,\mathrm{dR}}\mathcal{M}$ satisfying the familiar condition. An important fact about Lie-$*$ action is that it can be characterized purely in terms of the associated Zariski sheaf.

19.1.5. **Lemma.** *Let $\mathcal{L}$ be a Lie-$*$ algebra, and $\mathcal{M}$ a (right) $\mathcal{D}_X$-module. Then the following data are canonically equivalent:*

(1) an $\mathcal{L}$-action on $\mathcal{M}$;

(2) an $h(\mathcal{L})$-action on $\mathcal{M}$ such that for every $m \in \mathcal{M}$, the map:

$$\mathcal{L} \to \mathcal{M}, \quad l \rightsquigarrow \bar{l}m,$$

where $\bar{l}$ denotes the image of l in $h(\mathcal{L})$, is a differential operator.

Proof. This statement is found in [8, 2.5.4] and follows from Lemma 2.2.19 of *loc.cit.*. $\square$

19.2. **Group jet schemes.**

19.2.1. The Lie-$*$ algebra $\mathfrak{g}_{\mathcal{D}}$ arises in a different manner by "differentiating" a certian group $\mathcal{D}$-scheme, i.e., a group prestack which is schematic over X_{dR}. To be more precise, we have a pair of adjunction between schemes equipped with an affine morphism to X, respectively X_{dR}:

$$\mathrm{oblv} : \mathbf{Sch}^{\mathrm{aff}}_{/X_{\mathrm{dR}}} \rightleftarrows \mathbf{Sch}^{\mathrm{aff}}_{/X} : \mathrm{Jet}.$$

Here, oblv is the functor $- \underset{X_{\mathrm{dR}}}{\times} X$, and Jet is the Weil restriction of scalars along $X \to X_{\mathrm{dR}}$. To see that Jet preserves affine morphisms, one only needs to observe that $X \to X_{\mathrm{dR}}$ is surjective on S-points, and the composition $\mathrm{oblv} \circ \mathrm{Jet}$ identifies with the arc scheme construction. On the level of rings, this adjunction is the familiar one between commutative $\mathcal{O}_X$-algebras and commutative $\mathcal{D}_X$-algebras ([8, 2.3]).

19.2.2. The functor Jet is clearly symmetric monoidal. Therefore it sends the group scheme $G \times X \to X$ to a group prestack $\mathrm{Jet}(G \times X)$ affine schematic over X_{dR}. Furthermore, we have a fiber product diagram, where $\mathcal{L}^+_X G$ is the arc group over X:

$$\begin{tikzcd} \mathcal{L}^+_X G \arrow[r] \arrow[d] & \mathrm{Jet}(G \times X) \arrow[d] \\ X \arrow[r] & X_{\mathrm{dR}} \end{tikzcd}$$

19.2.3. To obtain a Lie-$*$ algebra from a smooth group $\mathcal{D}$-scheme $\mathcal{G} \to X_{\mathrm{dR}}$, we first consider the relative differential $\Omega^1_{\mathcal{G}/X_{\mathrm{dR}}}\big|_{X_{\mathrm{dR}}}$, regarded as a quasi-coherent sheaf on X_{dR}, i.e., a left $\mathcal{D}_X$-module. The group structure on $\mathcal{G}$ induces a co-Lie structure on $\Omega^1_{\mathcal{G}/X_{\mathrm{dR}}}\big|_{X_{\mathrm{dR}}}$, so the dualizing procedure of [8, 2.5.7]:

$$\mathcal{N} \rightsquigarrow \mathcal{N}^\vee := \underline{\mathrm{Hom}}_{\mathcal{D}_X}(\mathcal{N}, \mathcal{D}_X)$$

equips $(\Omega^1_{\mathcal{G}/X_{\mathrm{dR}}}\big|_{X_{\mathrm{dR}}})^\vee$ with the structure of a Lie-$*$ algebra. Applying this procedure to the group jet scheme $\mathrm{Jet}(G \times X)$ recovers the co-Lie algebra $\mathfrak{g}^* \otimes \mathcal{D}_X$, hence the Lie-$*$ algebra $\mathfrak{g}_{\mathcal{D}}$.

19.2.4. There is an obvious notion of the action of a group $\mathcal{D}$-scheme $\mathcal{G} \to X_{\mathrm{dR}}$ on a quasi-coherent sheaf $\mathcal{N}$ over X_{dR}. Furthermore, given such an action, we obtain a co-action by the Lie co-algebra:

$$\mathcal{N} \to \mathcal{N} \overset{!}{\otimes} (\Omega^1_{\mathcal{G}/X_{\mathrm{dR}}}\big|_{X_{\mathrm{dR}}}),$$

and dualizing gives a Lie-$*$ module structure on (the right $\mathcal{D}$-module associated to) $\mathcal{N}$. When $\mathcal{G} = \mathrm{Jet}(G \times X)$ for an algebraic group G, let us describe the procedure of differentiating a $\mathcal{G}$-action more explicitly using Lemma 19.1.5. Namely, for any commutative $\mathcal{D}_X$-algebra $\mathcal{A}$, the $\mathcal{G}$-action on $\mathcal{N}$ (seen as a right $\mathcal{D}_X$-module) gives rise to a map:

$$G(\mathcal{O}_X[\varepsilon]/\varepsilon^2) \to \mathrm{End}(\mathcal{N} \underset{\mathcal{D}_X}{\otimes} \mathcal{A}[\varepsilon]/\varepsilon^2).$$

Each element $1 + \varepsilon f$ (with $f \in \mathfrak{g} \otimes \mathcal{O}_X$) induces an endomorphism of $\mathcal{N} \underset{\mathcal{D}_X}{\otimes} \mathcal{A}[\varepsilon]/\varepsilon^2$, which sends $\bar{n} \in \mathcal{N} \underset{\mathcal{D}_X}{\otimes} \mathcal{A}$ to a certain element $\bar{n} + \alpha_f(\bar{n})\varepsilon$. The association $\bar{n} \rightsquigarrow \alpha_f(\bar{n})$ is functorial in the commutative $\mathcal{D}_X$-algebra $\mathcal{A}$, thus defining a $\mathcal{D}_X$-module endomorphism of $\mathcal{N}$. This is precisely the $h(\mathfrak{g}_{\mathcal{D}})$-action on $\mathcal{N}$ describing its Lie-$*$ module structure.

19.2.5. **Example.** We can describe the adjoint action of $\mathrm{Jet}(G \times X)$ on $\mathfrak{g}_{\mathcal{D}}$ as follows. Take a test affine scheme $S \to X_{\mathrm{dR}}$ with $\mathrm{Spec}(\mathcal{A}) = S \underset{X_{\mathrm{dR}}}{\times} X$ (so $\mathcal{A}$ has the structure of a commutative $\mathcal{D}_X$-algebra), we have:

$$\mathrm{Maps}_{/X_{\mathrm{dR}}}(S, \mathrm{Jet}(G \times X)) \cong G(\mathcal{A}), \quad \Gamma(S, \mathfrak{g}_{\mathcal{D}}) \cong \mathfrak{g} \otimes \mathcal{A},$$

and the $G(\mathcal{A})$-action on $\mathfrak{g} \otimes \mathcal{A}$ is the familiar adjoint action.

19.3. **Central extensions of $\mathfrak{g}_{\mathcal{D}}$.**

19.3.1. Given a Lie-$*$ algebra $\mathcal{L}$, a central extension of $\mathcal{L}$ by ω_X is by definition an exact sequence of Lie-$*$ algebras:

$$0 \to \omega_X \to \widehat{\mathcal{L}} \to \mathcal{L} \to 0, \tag{19.2}$$

such that $[\omega_X \boxtimes \widehat{\mathcal{L}}] = 0$. In particular, the Lie-$*$ bracket on $\widehat{\mathcal{L}}$ equips (19.2) with the structure of an exact sequence of $\mathcal{L}$-modules. Let us denote the k-linear groupoid of central extensions by $\mathbf{CExt}(\mathcal{L}, \omega_X)$.

19.3.2. We define the groupoid $\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X)$ to consist of central extensions:

$$0 \to \omega_X \to \widehat{\mathfrak{g}}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{D}} \to 0, \tag{19.3}$$

such that the $\mathfrak{g}_{\mathcal{D}}$-action on (19.3) integrates to a $\mathrm{Jet}(G \times X)$-action.

That the integrability is indeed a condition and not extra data can be justified as follows. Since $\mathrm{Jet}(G \times X)$ is affine over X_{dR}, the datum of its action on any $\mathcal{N} \in \mathrm{QCoh}(X_{\mathrm{dR}})^{\heartsuit}$ is uniquely determined by the pullbacks to X, i.e., by the corresponding $\mathcal{L}^+_X G$-action on $\mathrm{oblv}(\mathcal{N})$. Since $\mathcal{L}^+_X G$ is connected, this is in turn uniquely determined by its differential.

19.4. **Kac–Moody Lie-$*$ algebras.**

19.4.1. Recall the k-linear groupoid Par_G of quantum parameters consisting of pairs (κ, E) (§18.2). We shall construct a functor:

$$\mathrm{Par}_G \to \mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X), \quad (\kappa, E) \rightsquigarrow \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa, E)} \tag{19.4}$$

The Lie-$*$ algebras arising from this construction are called *Kac–Moody* Lie-$*$ algebras.

19.4.2. Let us be given $(\kappa, E) \in \mathrm{Par}_G$. We first define an extension $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ as a $\mathcal{D}$-module. Indeed, decompose $\mathfrak{g}$ into $\mathfrak{z} \oplus \mathfrak{g}_{\mathrm{sc}}$ where $\mathfrak{z}$ is the center and $\mathfrak{g}_{\mathrm{sc}}$ is the semisimple part. We let $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ be the direct sum of the following $\mathcal{D}$-modules:

(1) The push-out E' of the indcued $\mathcal{D}$-module $E_{\mathcal{D}} := E \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X$ along the action map $(\omega_X)_{\mathcal{D}} := \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{D}_X \to \omega_X$;
(2) The induced $\mathcal{D}$-module $(\mathfrak{g}_{\mathrm{sc}})_{\mathcal{D}}$.

Thus we obtain an extension of $\mathcal{D}$-modules:

$$0 \to \omega_X \to \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)} \to \mathfrak{g}_{\mathcal{D}} \to 0 \tag{19.5}$$

which canonically splits over $(\mathfrak{g}_{\mathrm{sc}})_{\mathcal{D}}$. Since the Lie-$*$ bracket of $\mathfrak{g}_{\mathcal{D}}$ takes values in $\Delta_{*,\mathrm{dR}}((\mathfrak{g}_{\mathrm{sc}})_{\mathcal{D}})$, we may define the Lie-$*$ bracket on $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ by adjoining the 2-cocycle:

$$\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\omega_X), \quad \xi \boxtimes \eta \rightsquigarrow \kappa(\xi, \eta)\mathbf{1}'_{\omega},$$

where $\mathbf{1}'_{\omega}$ denotes the canonical anti-symmetric section of $\Delta_{*,\mathrm{dR}}(\omega_X)$—namely, under the equivalence $\Delta_{*,\mathrm{dR}}(\omega_X) \cong \omega_{X^2}(\infty\Delta)/\omega_{X^2}$ given by the Cousin sequence, the section $\mathbf{1}'_{\omega}$ corresponds to $(x-y)^{-2} dx \wedge dy$.

19.4.3. Let us also check the integrability condition. For this, it is best to view (19.5) as an extension of $\mathfrak{g}_{\mathcal{D}}$-modules. It is, in fact, the push-out of an exact sequence of induced $\mathcal{D}_X$-modules:

$$0 \to (\omega_X)_{\mathcal{D}} \to E_{\mathcal{D}} \oplus (\mathfrak{g}_{\mathrm{sc}})_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{D}} \to 0, \tag{19.6}$$

along $(\omega_X)_{\mathcal{D}} \to \omega_X$. The $\mathfrak{g}_{\mathcal{D}}$-action occurs already at the level of (19.6). In terms of Lemma 19.1.5, an element $f \in \mathfrak{g} \otimes \mathcal{O}_X$ centralizes $E_{\mathcal{D}}$, acts as adjoint on $(\mathfrak{g}_{\mathrm{sc}})_{\mathcal{D}}$, and adds the following additional term (after taking the induced $\mathcal{D}$-modules.)

$$E \oplus (\mathfrak{g}_{\mathrm{sc}} \otimes \mathcal{O}_X) \twoheadrightarrow \mathfrak{g} \otimes \mathcal{O}_X \xrightarrow{\kappa(df,-)} \omega_X \hookrightarrow E \oplus (\mathfrak{g}_{\mathrm{sc}} \otimes \mathcal{O}_X).$$

The $\mathrm{Jet}(G \times X)$-action can then be described as follows. For any test commutative $\mathcal{D}_X$-algebra $\mathcal{A}$, a section $g \in G(\mathcal{A})$ acts on the space of horizontal sections $(E \underset{\mathcal{O}_X}{\otimes} \mathcal{A}) \oplus (\mathfrak{g}_{\mathrm{sc}} \otimes \mathcal{A})$: it centralizes $E \underset{\mathcal{O}_X}{\otimes} \mathcal{A}$, acts as adjoint on $\mathfrak{g}_{\mathrm{sc}} \otimes \mathcal{A}$, and adds the following additional term:

$$(E \underset{\mathcal{O}_X}{\otimes} \mathcal{A}) \oplus (\mathfrak{g}_{\mathrm{sc}} \otimes \mathcal{A}) \twoheadrightarrow \mathfrak{g} \otimes \mathcal{A} \xrightarrow{\kappa(g^{-1}dg,-)} \omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{A} \hookrightarrow (E \underset{\mathcal{O}_X}{\otimes} \mathcal{A}) \oplus (\mathfrak{g}_{\mathrm{sc}} \otimes \mathcal{A}).$$

It clearly differentiates to the above $\mathfrak{g}_{\mathcal{D}}$-action.

For semisimple groups, the groupoid $\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X)$ is easy to describe.

19.4.4. **Lemma.** *Suppose G is semisimple. Then* (19.4) *defines an equivalence of k-linear groupoids:*

$$\mathrm{Par}_G \xrightarrow{\sim} \mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X), \quad \kappa \rightsquigarrow \widehat{\mathfrak{g}}^{\kappa}_{\mathcal{D}}. \tag{19.7}$$

Proof. Let $U \subset X$ be an affine open subset. Then any central extension $\widehat{\mathfrak{g}}_{\mathcal{D}}$ over U admits a non-canonical splitting as a $\mathcal{D}$-module. Therefore the groupoid of central extensions of $\mathfrak{g}_{\mathcal{D}}$ by ω_U (without the integrability condition) is equivalent to the truncated Chevalley complex (c.f. [8, §1.4.5]):

$$\mathrm{Hom}(\mathfrak{g}_{\mathcal{D}}, \omega_U) \to \mathrm{Hom}(\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}}, \Delta_{*,\mathrm{dR}}(\omega_U))^{\mathrm{cl}}_{\mathrm{sgn}},$$

where the second abelian group consists of anti-symmetric morphisms $\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\omega_U)$ satisfying the cocycle condition.

We first claim that this groupoid has vanishing π_1. Indeed, given any morphism $\alpha : \mathfrak{g}_{\mathcal{D}} \to \omega_U$ such that the following composition vanishes:

$$\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \xrightarrow{[-,-]} \Delta_{*,\mathrm{dR}}(\mathfrak{g}_{\mathcal{D}}) \xrightarrow{\alpha} \Delta_{*,\mathrm{dR}}(\omega_U),$$

the map α must itself vanish as $[-,-]$ is surjective, by semisimplicity. For π_0, we note that the datum of a map $\mathfrak{g}_{\mathcal{D}} \boxtimes \mathfrak{g}_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\omega_U)$ is equivalent to that of a differential operator:

$$\mathfrak{g} \otimes \mathcal{O}_U \to \mathfrak{g}^* \otimes \omega_U, \tag{19.8}$$

by the procedure of taking de Rham cohomology along the first factor (Lemma [8, 2.2.19]). This morphism is in turn determined by its restriction to the formal punctured disc of any $x \in U$. There, it gives rise to a 2-cocycle of the loop algebra $\mathfrak{g}(\mathbf{F}_x) \otimes \mathfrak{g}(\mathbf{F}_x) \to k$. Under the semisimplicity assumption, such 2-cocycles are classified by W-invariant bilinear forms on $\mathfrak{g}$ ([43, §7]). Thus (19.8) is of the form $\xi \otimes f \rightsquigarrow \kappa(\xi, -) \otimes df$ for some form κ.

Comparing the homotopy groups, we have proved an equivalence over U:

$$\mathrm{Par}_G \xrightarrow{\sim} \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}, \omega_U).$$

The discreteness allows us to globalize this equivalence to X. Finally, since all extensions of the form $\widehat{\mathfrak{g}}^{\kappa}_{\mathcal{D}}$ satisfy the integrability condition, we obtain the desired equivalence (19.7). □

19.5. **Construction of factorization twistings: outline.**

19.5.1. We will now produce a functor:

$$\Xi_{\mathbf{Tw}} : \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}, \omega_X) \to \mathbf{Tw}^{\mathrm{fact}}_{/k}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

By analogy with the functor $\Xi_{\mathbf{Pic}}$ from central extensions of $G \times X$ by $\mathbf{K}_2$, we shall think of $\Xi_{\mathbf{Tw}}$ as the "construction functor" of factorization twistings.

19.5.2. The fucntor $\Xi_{\mathbf{Tw}}$ will be a composition:

$$\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X) \to (\mathrm{A}) \xrightarrow{\sim} (\mathrm{B}) \to \mathbf{Tw}^{\mathrm{fact}}_{/k}(\mathrm{Gr}_{G,\mathrm{Ran}}),$$

where (A) and (B) denote the following k-linear groupoids.

(A) Factorization central extensions of the loop algebra over $\mathrm{Ran}_{\mathrm{dR}}$:

$$0 \to \mathcal{O}_{\mathrm{Ran}_{\mathrm{dR}}} \to \widehat{\mathfrak{g}}_{\mathrm{Ran}_{\mathrm{dR}}} \to \mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g} \to 0, \tag{19.9}$$

equipped with an integration of the $\mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}$-action on (19.9) to $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$. In other words, it is the fiber of the map:

$$\mathbf{CExt}^{\mathrm{fact}}_{\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G}(\mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}, \mathcal{O}_{\mathrm{Ran}_{\mathrm{dR}}}) \to \mathbf{CExt}^{\mathrm{fact}}_{\mathcal{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}G}(\mathrm{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}, \mathcal{O}_{\mathrm{Ran}_{\mathrm{dR}}}).$$

(B) Factorization multiplicative twisting on $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$ equipped with a multiplicative trivialization over $\mathcal{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}G$. In other words, it is the fiber of the map:

$$\mathbf{Tw}^{\times,\mathrm{fact}}_{/\mathrm{Ran}_{\mathrm{dR}}}(\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G) \to \mathbf{Tw}^{\times,\mathrm{fact}}_{/\mathrm{Ran}_{\mathrm{dR}}}(\mathcal{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}G).$$

However, due to the fact that the loop group is of infinite type, some additional care needs to be taken. We use the next section to define the groupoids (A) and (B) in more precise terms. Let us first note the relationship between multiplicative twistings and central extensions of Lie algebras.

19.5.3. **Lemma.** *Let H be a group scheme of finite type over S, and $\mathfrak{h}$ be its Lie algebra. Then the following categories are equivalent:*

(1) a central extension of $\mathcal{O}_S$-linear Lie algebras:

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0,$$

equipped with an integration of the $\mathfrak{h}$-action to H;

(2) a multiplicative twisting on H relative to S.

Proof. The datum of a multiplicative twisting on H is equivalent to a commutative square of *group* prestacks, where we use pt to denote the base scheme S:

$$\begin{tikzcd} H \arrow[r] \arrow[d] & H_{\mathrm{dR}/S} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{B}^2\,\widehat{\mathbb{G}}_m \end{tikzcd}$$

where pt denotes the base scheme S, and $H_{\mathrm{dR}/S}$ denotes the prestack $H_{\mathrm{dR}} \underset{S_{\mathrm{dR}}}{\times} S$. Note that $H_{\mathrm{dR}/S}$ is the quotient of H by the normal subgroup $\exp(\mathfrak{h})$ and $\mathrm{B}^2\,\widehat{\mathbb{G}}_m$ is the quotient of pt by $\mathrm{B}\,\widehat{\mathbb{G}}_m$. (A *normal subgroup* is a map of group prestacks $N \to H$, equipped with an extension of the adjoint H-action to N, whose restriction to N identifies with the tautological action—these data equip the quotient with a group structure.) The above commutative diagram of group prestacks is thus equivalent to a map between the normal subgroups:

$$\exp(\mathfrak{h}) \to \mathrm{B}\,\widehat{\mathbb{G}}_m, \tag{19.10}$$

with the compatibility datum between the H-action on $\exp(\mathfrak{h})$ and the pt-action on $\mathrm{B}\,\widehat{\mathbb{G}}_m$. Furthermore, this datum reduces to the tautological one when we restrict to $\exp(\mathfrak{h})$-action.

The group morphism (19.10) is equivalent to a Lie algebra extension $\widehat{\mathfrak{h}}$ of $\mathfrak{h}$ by $\mathcal{O}_S$. The compatibility datum amounts to an H-action on $\widehat{\mathfrak{h}}$ extending the adjoint action and centralizing $\mathcal{O}_S$. The fact that it reduces to the tautological one on $\exp(\mathfrak{h})$ means that H differentiates to the canonical $\mathfrak{h}$-action on $\widehat{\mathfrak{h}}$. $\square$

19.6. **Remarks on infinite type.**

19.6.1. *Placidity.* Let us recall the notion of *placid ind-schemes* developped in [30]. We work over a base affine scheme S. An ind-scheme Z is *placid* if it admits an ind-pro-presentation:

$$Z \cong \operatorname*{colim}_{\alpha} Z^{\alpha}, \quad Z^{\alpha} \cong \lim_{i} Z_i^{\alpha}, \tag{19.11}$$

where each $Z_i^{\alpha} \twoheadrightarrow Z_j^{\alpha}$ is a smooth surjection of schemes of finite type, and each $Z^{\alpha} \hookrightarrow Z^{\beta}$ is a closed immersion of finite type. In particular, Z^{α} gives a "reasonable" presentation of Z in the sense of [18, §6].

19.6.2. We note that the projection $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G \to \mathrm{Ran}_{\mathrm{dR}}$ is a placid ind-schematic morphism. Indeed, it suffices to check this for the base change $\mathcal{L}_{\mathrm{Ran}}G \to \mathrm{Ran}$, which follows from the fact that $\mathcal{L}^{+}_{\mathrm{Ran}}G \to \mathrm{Ran}$ is placid and schematic, while $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Ran}$ is ind-schematic and of ind-finite type.

19.6.3. *Tate modules.* The theory of (pre-)twistings can be extended formally to placid ind-schemes. Namely, given (19.11), we define:

$$\mathbf{Tw}_{/S}(Z) := \lim_{\alpha} \operatorname*{colim}_{i} \mathbf{Tw}_{/S}(Z_i^{\alpha}),$$

where the transition maps are given by pullbacks of twistings. The proper definition of the k-linear groupoid (B) uses this notion of twisting on the loop and arc groups.

19.6.4. The notion of vector bundles have to be replaced by Tate modules [18, §6]. Indeed, given a ring R, the exact category of Tate R-modules is defined to be the smallest full subcategory of topological R-modules which:

(1) contains projective (discrete) R-modules and their topological duals;
(2) is closed under finite sums and retracts.

Thus every Tate R-module is a retract of some "elementary Tate module" $P \oplus Q^*$ where P, Q are projective R-modules. In particular, the topological dual of a Tate module remains a Tate module.

19.6.5. Given an ind-scheme $Z \to S$ with an R-point $z : \mathrm{Spec}(R) \to Z$, we can define the cotangent space at z as the inverse limit:

$$z^*\Omega_{Z/S} := \lim_{\alpha \gg 0} z^*\Omega_{Z^\alpha/S},$$

where $\alpha \gg 0$ (i.e., sufficiently large with respect to the filtered system) is to guarantee that z factors through $\mathrm{Spec}(R) \to Z^\alpha$. This presentation gives $z^*\Omega_{Z/S}$ the structure of a topologicla R-module. It is clearly compatible with change of (R, z). By [18, Theorem 6.2(iii)], if Z is a formally smooth reasonable ind-scheme, then $z^*\Omega_{Z/S}$ is a Tate R-module. (Taking topological dual, we see that $z^*\mathcal{T}_{Z/S}$ is also a Tate R-module.) This discussion applies, in particular, to formally smooth placid ind-schemes.

19.6.6. We regard $\mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}$ as a Lie algebra of Tate modules over $\mathrm{Ran}_{\mathrm{dR}}$. The precise meaning is as follows. For any test affine scheme $x^I : \mathrm{Spec}(R) \to \mathrm{Ran}_{\mathrm{dR}}$, the space of sections $\mathrm{L}_{x^I}\mathfrak{g}$ is the Tate R-module $\mathfrak{g}\widehat{\otimes}\,\mathrm{H}^0(\mathring{D}_{x^I}, \mathcal{O})$ where the Lie bracket:

$$[-,-] : \mathrm{L}_{x^I}\mathfrak{g} \otimes \mathrm{L}_{x^I}\mathfrak{g} \to \mathrm{L}_{x^I}\mathfrak{g}$$

is continuous in both variables. These data are compatible with respect to change of (R, x^I). The groupoid (A) thus concerns central extensions as Lie algebras of Tate modules over $\mathrm{Ran}_{\mathrm{dR}}$.

Lemma 19.5.3 extends to the infinite type setting.

19.6.7. **Lemma.** *Let H be a group ind-scheme over S which is formally smooth and placid. Let $\mathfrak{h}$ be its Lie algebra of Tate $\mathcal{O}_S$-module. Then the following categories are equivalent:*

(1) a central extension of Lie algebras in Tate $\mathcal{O}_S$-modules:

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{h}} \to \mathfrak{h} \to 0,$$

equipped with an integration of the $\mathfrak{h}$-action to H;

(2) a multiplicative twisting on H relative to S. $\square$

Therefore, the k-linear groupoids (A) and (B) are canonically equivalent.

19.7. **Construction of $\Xi_{\mathbf{Tw}}$.**

19.7.1. Let us now build a functor from $\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X)$ to the groupoid (A). This arises from taking de Rham cohomology on formal punctured discs. More precisely, given a central extension:

$$0 \to \omega_X \to \widehat{\mathfrak{g}}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{D}} \to 0,$$

and an S-point x^I of $\mathrm{Ran}_{\mathrm{dR}}$, consider the punctured formal disc $\mathring{D}_{x^I}$. We obtain an exact sequence on the zeroth de Rham cohomology along the map $\mathring{D}_{x^I} \to S$:

$$0 \to \mathrm{H}^0_{\mathrm{dR}}(\mathring{D}_{x^I}, \omega_X) \to \mathrm{H}^0_{\mathrm{dR}}(\mathring{D}_{x^I}, \widehat{\mathfrak{g}}_{\mathcal{D}}) \to \mathrm{H}^0_{\mathrm{dR}}(\mathring{D}_{x^I}, \mathfrak{g}_{\mathcal{D}}) \to 0. \tag{19.12}$$

Indeed, $\mathrm{H}^{-1}_{\mathrm{dR}}(\mathring{D}_{x^I}, \mathfrak{g}_{\mathcal{D}}) = 0$ because $\mathfrak{g}_{\mathcal{D}}$ is a free $\mathcal{D}_X$-module, and $\mathrm{H}^1_{\mathrm{dR}}(\mathring{D}_{x^I}, \omega_X) = 0$ because $\mathring{D}_{x^I} \to S$ is affine. The push-out of (19.12) along the "sum of residue" map:

$$\mathrm{H}^0_{\mathrm{dR}}(\mathring{D}_{x^I}, \omega_X) \to \mathcal{O}_S \tag{19.13}$$

then produces an exact sequence:

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{g}}_S \to \Gamma(\mathring{D}_{x^I}, \mathfrak{g} \otimes \mathcal{O}_X) \to 0.$$

It has a canonical structure as a central extension of Lie algebras in Tate $\mathcal{O}_S$-modules (see [8, 2.1.13-16]) and the topology on $\Gamma(\mathring{D}_{x^I}, \mathfrak{g} \otimes \mathcal{O}_X)$ identifies with the one coming from the tangent space of the loop group. Furthermore, suppose $\widehat{\mathfrak{g}}_{\mathcal{D}}$ is equipped with a $\mathrm{Jet}(G \times X)$-action. Then the resulting extension $\widehat{\mathfrak{g}}_S$ has an action by $\Gamma(\mathring{D}_{x^I}, G \times X)$, i.e., the S-points of the loop group. The factorization structure is immediate as one varies S.

This procedure defines a functor:

$$\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X) \to \mathbf{CExt}^{\mathrm{fact}}_{\mathcal{L}_{\mathrm{Ran}}G}(\mathrm{L}_{\mathrm{Ran}}\mathfrak{g}, \mathcal{O}_{\mathrm{Ran}}). \tag{19.14}$$

19.7.2. The same procedure can be repeated for D_{x^I} instead of $\mathring{D}_{x^I}$. Now, since the map (19.13) vanishes on $\mathrm{H}^0_{\mathrm{dR}}(D_{x^I}, \omega_X)$, we find that (19.14) becomes trivialized when composed with the restriction functor to $\mathbf{CExt}^{\mathrm{fact}}_{\mathcal{L}_{\mathrm{Ran}}G^+}(\mathrm{L}^+_{\mathrm{Ran}}\mathfrak{g}, \mathcal{O}_{\mathrm{Ran}})$. This shows that it canonically lifts to a functor from $\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X)$ to the groupoid (A).

19.7.3. Appealing to the identification between (A) and (B), it remains to construct a functor from (B) to $\mathbf{Tw}^{\mathrm{fact}}_{/k}(\mathrm{Gr}_{G,\mathrm{Ran}})$. We first observe that twistings relative to $\mathrm{Ran}_{\mathrm{dR}}$ (or any de Rham prestack) are equivalent to twistings relative to the base point $\mathrm{Spec}(k)$. Therefore (B) is equivalent to the fiber of the map:

$$\mathbf{Tw}^{\times,\mathrm{fact}}_{/k}(\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G) \to \mathbf{Tw}^{\times,\mathrm{fact}}_{/k}(\mathcal{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}G).$$

There is a canonical functor from this fiber to $\mathbf{Tw}^{\mathrm{fact}}_{/k}(\mathrm{Gr}_{G,\mathrm{Ran}_{\mathrm{dR}}})$, because the multiplicative structure equips a twisting on $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$ with descent data with respect to the $\mathcal{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}G$-action. Finally, we obtain the desired functor by pulling back along:

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}_{\mathrm{dR}}}.$$

19.7.4. **Remark.** By construction, every factorization twisting of the form $\Xi_{\mathbf{Tw}}(\widehat{\mathfrak{g}}_{\mathcal{D}})$ has a connection along Ran as well as a multiplicative structure on its pullback to the loop group.

19.8. Does $\mathbf{Tw}^{\mathrm{fact}}_{/k}(\mathrm{Gr}_{G,\mathrm{Ran}})$ have a combinatorial description?

19.8.1. In this section, we investigate the functor $\Xi_{\mathbf{Tw}}$ in more details. Note that $\mathbf{Tw} := \mathbf{Tw}_{/k}$ fails property (RP1), so it is not possible to define a classification functor using the same approach as for tame twistings. Instead, our analysis relies on the description of twistings with $\mathbb{G}_a$ structure groups in the Zariski topology and an additive variant of the de Rham gerbes.

19.8.2. *Additive de Rham gerbes.* More precisely, we define $\mathbf{Ge}^+_{\mathrm{dR}}$ to be the following presheaf of strict Picard 2-groupoids on $\mathbf{Sch}^{\mathrm{ft}}_{/k}$:

$$\mathbf{Ge}^+_{\mathrm{dR}}(X) := \mathrm{Maps}(X_{\mathrm{dR}}, \mathrm{B}^2_{\mathrm{Zar}}\,\mathbb{G}_a).$$

Therefore, $\mathbf{Ge}^+_{\mathrm{dR}}(X)$ is calculated by the truncated complex $\tau^{\leq 0}\mathrm{R}\Gamma_{\mathrm{Zar}}(X_{\mathrm{dR}}, \mathcal{O}[2])$. The $\mathbf{h}$-descent of perfect complexes (Chapter 1, Lemma 4.2.4) implies that $\mathbf{Ge}^+_{\mathrm{dR}}$ is an h-stack. Indeed, for every h-cover $\widetilde{X} \to X$, the Čech complex of $\widetilde{X}_{\mathrm{dR}} \to X_{\mathrm{dR}}$ is canonically the same whether formed as classical or derived prestacks.

19.8.3. The value group $A(-1)$ canonically identifies with k. The divisor class map:

$$c_1 : \mathbf{Pic} \underset{\mathbb{Z}}{\otimes} k \to \mathbf{Ge}^+_{\mathrm{dR}}, \quad (\mathcal{L}, a) \rightsquigarrow \mathcal{L}^a \tag{19.15}$$

is the usual "first Chern class" construction. Over a smooth scheme X, it is induced from $d\log : \mathcal{O}^\times_X \to \tau^{\leq 2}\Omega^\bullet_X$. For general $X \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$, there is a morphism from $\mathbf{Pic}(X)$ to usual twistings $\mathbf{Tw}_{/k}(X)$ (c.f. §14.2) which has an underlying $\mathbb{G}_a$-gerbe on X_{dR}. One then extends the construction by k-linearity.

19.8.4. **Lemma.** *The sheaf* $\mathbf{Ge}^+_{\mathrm{dR}}$ *and* c_1 (19.15) *define a motivic* h*-theory of gerbes.*

Proof. The properties (RP1), (RP2), and (A) are standard of algebraic de Rham cohomology. To verify (B), we compare $\mathbf{Ge}^+_{\mathrm{dR}}$ with the theory of analytic $\mathbb{C}$-gerbes $\mathbf{Ge}^+_{\mathrm{an}}$ equipped its usual divisor class map. By Grothendieck's comparison theorem, one has an equivalence $\mathbf{Ge}_{\mathrm{dR}}^{+} \cong \mathbf{Ge}_{\mathrm{an}}^{+}$ making the following diagram commute:

$$\begin{tikzcd}
\mathbf{Pic} \underset{\mathbb{Z}}{\otimes} \mathbb{C} \arrow[r] \arrow[d, "\mathrm{id}"] & \mathbf{Ge}_{\mathrm{dR}}^{+} \arrow[d, "\cong"] \\
\mathbf{Pic} \underset{\mathbb{Z}}{\otimes} \mathbb{C} \arrow[r, "2\pi i\cdot"] & \mathbf{Ge}_{\mathrm{an}}^{+}
\end{tikzcd}$$

The theory of gerbes $\mathbf{Ge}_{\mathrm{an}}^{+}$ is motivic, as the same proof of Chapter 3, §13 applies. □

19.8.5. **Corollary.** *There is a canonical equivalence of strict Picard* 2*-groupoids:*

$$\Psi_{\mathrm{dR}}^{+} : \mathbf{Ge}_{\mathrm{dR}}^{+,\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \Theta_G(\Lambda_T; \mathbf{Ge}_{\mathrm{dR}}^{+}),$$

which makes the following diagram commute:

$$\begin{tikzcd}
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}", "\cong"'] & \mathbf{Ge}_{\mathrm{dR}}^{+,\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\Psi_{\mathrm{an}}", "\cong"'] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathbf{Ge}_{\mathrm{dR}}^{+})
\end{tikzcd} \tag{19.16}$$

Proof. This follows from combining Lemma 19.8.4 and Chapter 3, Theorem 10.3.2. □

19.8.6. Now, we note that $\mathbf{Tw}$ admits a canonical map to $\mathbf{Ge}_{\mathrm{dR}}^{+}$, which is compatible with their divisor class maps. In fact, $\mathbf{Tw}$ is precisely the fiber of $\mathbf{Ge}_{\mathrm{dR}}^{+} \to \mathrm{B}_{\mathrm{Zar}}^{2}\,\mathbb{G}_a =: \mathbf{Ge}_{\mathbb{G}_a}$. Therefore, we obtain a functor:

$$\begin{aligned}
\mathbf{Q}_{\mathbf{Tw},G} : \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to & \mathbf{Ge}_{\mathrm{dR}}^{+}(\mathrm{Gr}_{G,\mathrm{Ran}}) \\
& \xrightarrow{\sim} \Theta_G(\Lambda_T; \mathbf{Ge}_{\mathrm{dR}}^{+}) \to \mathcal{Q}(\Lambda_T; k)^W.
\end{aligned} \tag{19.17}$$

Thus, every factorization twisting on $\mathrm{Gr}_{G,\mathrm{Ran}}$ can be given a bilinear form κ. The following Lemma shows that this functor yields the expect answer.

19.8.7. **Lemma.** *Let* $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ *denote the Lie-$*$ central extension attached to the quantum parameter* (κ, E)*. Then the composition:*

$$\mathbf{Q}_{\mathbf{Tw},G} \circ \Xi_{\mathbf{Tw},G}(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) = \kappa.$$

Proof. This is analogous to the proof of Chapter 2, Proposition 7.2.2. It suffices to replace G by its maximal torus T. Then, we consider a point $x \in X$ and the restriction of the central extension of the loop algebra attached to the Lie-$*$ algebra $\widehat{\mathfrak{t}}_{\mathcal{D}}^{(\kappa,E)}$ by the procedure of §19.5, equipped with an $\mathcal{L}_x T$-action:

$$0 \to k \to \widehat{\mathfrak{t}}_{x}^{(\kappa,E)} \to \mathrm{L}_x\mathfrak{t} \to 0.$$

It gives rise to a multiplicative twisting on $\mathcal{L}_x T$ with multiplicative trivialization over $\mathcal{L}_x^+ T$, and thus a $\mathcal{L}_x^+ T$-equivariant twisting on $\mathrm{Gr}_{T,x}$. It is straightforward to verify that the equivariance structure at t^λ, for $\lambda \in \Lambda_T$, is given by the following $\mathbb{G}_a$-torsor on $(\mathcal{L}_x^+ T)_{\mathrm{dR}}$:

$$\kappa(\lambda, -) \in \mathrm{Hom}(\Lambda_T, k) \xrightarrow{\sim} \mathbb{G}_a\text{-}\mathrm{Tors}^\times(T_{\mathrm{dR}}) \xrightarrow{\sim} \mathbb{G}_a\text{-}\mathrm{Tors}^\times((\mathcal{L}_x^+ T)_{\mathrm{dR}}),$$

where $\mathbb{G}_a$-$\mathrm{Tors}^\times$ denotes multiplicative $\mathbb{G}_a$-torsors on a group prestack.

On the other hand, the underlying factorization de Rham $\mathbb{G}_a$-gerbe $\mathcal{G}$ of $\Xi_{\mathbf{Tw},T}(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})$ acquires a strong $\mathcal{L}_{\mathrm{Ran}}^+ T$-equivariance. Thus at $t^\lambda \in \mathrm{Gr}_{T,x}$, this equivariance structure is given by $\kappa'(\lambda, -)$ for κ' the bilinear form attached to $\mathcal{G}$. The identity $\kappa = \kappa'$ thus follows. ∎

The following Theorem shows that for semisimple, simply connected groups, factorization twistings are classified combinatorially.

19.8.8. **Theorem.** *Suppose G is semisimple, simply connected. Then we have equivalences making the following diagram commute:*

$$\begin{tikzcd} \mathbf{CExt}(\mathfrak{g}_{\mathcal{D}}, \omega_X) \arrow[r, "\Xi_{\mathbf{Tw},G}", "\sim"'] \arrow[dr, "\sim", "(a)"'] & \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, "\mathbf{Q}_{\mathbf{Tw},G}", "\cong"'] \\ & \mathcal{Q}(\Lambda_T; k)^W \end{tikzcd}$$

Here, (a) *is the inverse of the equivalence* (19.7)*.*

Proof. The diagram commutes by Lemma 19.8.7. Thus we only need to show that $\mathbf{Q}_{\mathbf{Tw},G}$ is an equivalence for a semisimple, simply connected group G. By Corollary 19.8.5, we have an equivalence:

$$\Psi_{\mathbf{Ge}^+_{\mathrm{dR}},G} : \mathbf{Ge}^{+,\mathrm{fact}}_{\mathrm{dR}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \xrightarrow{\sim} \mathcal{Q}(\Lambda_T; k)^W.$$

Thus it remains to prove that $\mathbf{Ge}^{\mathrm{fact}}_{\mathbb{G}_a}(\mathrm{Gr}_{G,\mathrm{Ran}})$ is contractible.

We consider the projection $\pi : \mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Ran}$, and claim that each $\mathbb{G}_a$-gerbe canonically descends to Ran. Indeed, over $X^I \to \mathrm{Ran}$, we consider the base change of π as a colimit of the Schubert stratification (see Chapter 1) $\pi^{\le\lambda^I} : \mathrm{Gr}^{\le\lambda^I}_{G,X^I} \to X^I$. By the affine Borel–Weil–Bott theorem, we have $\mathrm{H}^i(\mathrm{Gr}^{\le\lambda}_{G,x}, \mathcal{O}) = 0$ for $i \ge 1$ and $\mathrm{H}^0(\mathrm{Gr}^{\le\lambda}_{G,x}, \mathcal{O}) \cong k$ at any k-point $x \in X$ (Chapter 1, Lemma 2.2.10). Thus the same holds for fibers of $\pi^{\le\lambda^I}$ at every k-point. Since $\pi^{\le\lambda^I}$ is proper, flat, and X^I is reduced, the canonical map $\mathcal{O}_{X^I} \to \mathrm{R}\pi^{\le\lambda^I}_* \mathcal{O}_{\mathrm{Gr}}$ is an isomorphism by cohomology and base change. The same argument applies to products of π. Thus pullback defines an equivalence:

$$\mathbf{Ge}^{\mathrm{fact}}_{\mathbb{G}_a}(\mathrm{Ran}) \xrightarrow{\sim} \mathbf{Ge}^{\mathrm{fact}}_{\mathbb{G}_a}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

Finally, we argue that factorization $\mathbb{G}_a$-gerbes on Ran are canonically trivial. By Lemma 19.8.9 below, such a $\mathbb{G}_a$-gerbe $\mathcal{G}$ is pulled back from $\mathcal{G}_1$ along $p : \mathrm{Ran} \to \mathrm{pt}$. We choose distinct k-points $x, y \in X$. The pullbacks $x^*\mathcal{G}$, $y^*\mathcal{G}$, $(x,y)^*\mathcal{G}$ all identify with $\mathcal{G}_1$. However, factorization implies $(x,y)^*\mathcal{G} \xrightarrow{\sim} x^*\mathcal{G} \otimes y^*\mathcal{G}$ so we obtain a trivialization of $\mathcal{G}_1$ which one can see to be canonical. $\square$

We supply the needed calculation of the cohomology of Ran with values in $\mathbb{G}_a$.

19.8.9. **Lemma.** *Pullback along* $\mathrm{Ran} \to \mathrm{pt}$ *induces an isomorphism* $k \xrightarrow{\sim} \mathrm{R}\,\Gamma(\mathrm{Ran}; \mathcal{O})$.

Proof. We note that $\mathrm{R}\Gamma(\mathrm{Ran}; \mathcal{O})$ is by definition $\lim_I \mathrm{R}\Gamma(X^I; \mathcal{O})$. Suppose X is proper. Then each $\mathrm{R}\Gamma(X^I; \mathcal{O})$ is dualizable. Hence we have:

$$\begin{aligned}\mathrm{R}\Gamma(\mathrm{Ran} \times \mathrm{Ran}; \mathcal{O}) &\xrightarrow{\sim} \lim_{I,J} \mathrm{R}\Gamma(X^I \times X^J; \mathcal{O}) \\ &\xrightarrow{\sim} \lim_{I} \mathrm{R}\Gamma(X^I; \mathcal{O}) \otimes \lim_{J} \mathrm{R}\Gamma(X^J; \mathcal{O}) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathrm{Ran}; \mathcal{O}) \otimes \mathrm{R}\Gamma(\mathrm{Ran}; \mathcal{O}).\end{aligned}$$

The argument of [28, §6] thus applies.

Suppose X is affine. Then $\mathrm{R}\Gamma(X^I; \mathcal{O}) \xrightarrow{\sim} \Gamma(X^I; \mathcal{O})$ and the problem reduces to the fact that global functions on Ran are constant ([74, Proposition 4.3.10(1)]). $\square$

19.9. **An "exotic" twisting.**

19.9.1. We will now explain why we do not expect Theorem 19.8.8 to hold for more general groups. Indeed, let us take $X = \mathbb{A}^1 = \mathrm{Spec}(k[x])$, and the group to be $\mathbb{G}_m$. We will first write down a Lie-$*$ central extension:

$$0 \to \omega_X \to \widehat{\mathfrak{t}} \to \mathfrak{t}_{\mathcal{D}} \to 0, \tag{19.18}$$

which is not isomorphic to a central extension of the form $\widehat{\mathfrak{t}}^{(\kappa,E)}$. Indeed, since $\mathfrak{t} \cong k$ and the Lie-$*$ bracket on $\mathfrak{t}_{\mathcal{D}}$ is trivial, the Lie-$*$ 2-cocycles of $\mathfrak{t}_{\mathcal{D}}$ are equivalent to anti-symmetric sections of $\Delta_{*,\mathrm{dR}}(\omega_X)$.

19.9.2. Let us express $\Delta_{*,\mathrm{dR}}(\omega_X)$ more explicity by letting s and t be the coordinates on X^2. Then $\Delta_{*,\mathrm{dR}}(\omega_X)$ identifies with the tensor product $\omega_X \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X^2}/(s-t)\mathcal{D}_{X^2}$, where the $\mathcal{D}_X$-action on $\mathcal{D}_{X^2}/(s-t)\mathcal{D}_{X^2}$ is given by $x \rightsquigarrow s$ (or equivalently t) and $\partial_x \rightsquigarrow \partial_s + \partial_t$. This description translates under the Cousin isomorphism as follows:

$$\omega_{X^2}(\infty\Delta)/\omega_{X^2} \xrightarrow{\sim} \omega_X \underset{\mathcal{D}_X}{\otimes} \mathcal{D}_{X^2}/(s-t)\mathcal{D}_{X^2}, \quad \frac{ds \wedge dt}{s-t} = dx \otimes \mathbf{1}.$$

Recall that the section $\mathbf{1}'_\omega$ is explicity $dx \otimes \partial_t$, so the 2-cocycle defining $\widehat{\mathfrak{t}}_{\mathcal{D}}^{(\kappa,E)}$ is $\kappa \cdot dx \otimes \partial_t$, where κ is viewed as a number.

19.9.3. *2-cocycle.* We will now write down an "exotic" section of $\Delta_{*,\mathrm{dR}}(\omega_X)$, defined by the following equivalent formulas (in Cousin, respectively tensor descriptions):

$$\frac{6t ds \wedge dt}{(s-t)^3} + \frac{6t^2 ds \wedge dt}{(s-t)^4} = dx \otimes (3t\partial_t^2 + t^2\partial_t^3). \tag{19.19}$$

This expression is visibly anti-symmetric, so it defines a Lie-$*$ 2-cocycle $\mathfrak{t}_{\mathcal{D}} \boxtimes \mathfrak{t}_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\omega_X)$, which is *not* cohomologous to the ones defined by κ.

19.9.4. *Integrability.* Let us verify that the Lie-$*$ central extension $\widehat{\mathfrak{t}}$ defined by (19.19) satisfies the integrability condition. As in the Kac–Moody example, we view (19.18) as an extension of $\mathfrak{t}_{\mathcal{D}}$-modules. It is induced from the exact sequence of $\mathfrak{t}_{\mathcal{D}}$-modules along $(\omega_X)_{\mathcal{D}} \to \omega_X$:

$$0 \to (\omega_X)_{\mathcal{D}} \to (\omega_X)_{\mathcal{D}} \oplus \mathfrak{t}_{\mathcal{D}} \to \mathfrak{t}_{\mathcal{D}} \to 0,$$

where the $\mathfrak{t}_{\mathcal{D}}$-action on the middle has the following explicit description using Lemma 19.1.5. For any $f \in \mathfrak{t} \otimes \mathcal{O}_X$, it takes $(\eta, f') \in \omega_X \oplus (\mathfrak{t} \otimes \mathcal{O}_X)$ to:

$$(\eta, f') \rightsquigarrow ((3x\partial_x^2 f + x^2\partial_x^3 f)f' dx, 0).$$

Then one extends by $\mathcal{D}_X$-linearity. To integrate this action to $\mathrm{Jet}(\mathbb{G}_m \times X)$, we consider any test $\mathcal{D}_X$-algebra $\mathcal{A}$. Then we define a $\mathbb{G}_m(\mathcal{A})$-action on $(\omega_X \underset{\mathcal{O}_X}{\otimes} \mathcal{A}) \oplus (\mathfrak{t} \otimes \mathcal{A})$.

Namely, for any $g \in \mathbb{G}_m(\mathcal{A}) \cong \mathcal{A}^\times$, we attach the map:

$$\mathrm{act}_g : (\eta, f') \rightsquigarrow (\eta + (3x\partial_x(g^{-1}\partial_x g) + x^2\partial_x^2(g^{-1}\partial_x g))f'dx, f').$$

Since $\mathrm{act}_{gg'} = \mathrm{act}_g + \mathrm{act}_{g'}$ and $\mathrm{act}_1 = \mathrm{id}$, this formula defines an action of $\mathrm{Jet}(\mathbb{G}_m \times X)$ on $\widehat{\mathfrak{t}}_{\mathcal{D}}$. Hence we obtain a well-defined object $\widehat{\mathfrak{t}}$ of $\mathbf{CExt}_{\mathbb{G}_m}(\mathfrak{t}_{\mathcal{D}}, \omega_X)$.

19.9.5. We claim that the multiplicative factorization twisting on $\mathcal{L}_{\mathrm{Ran}}\mathbb{G}_m$ defined by $\widehat{\mathfrak{t}}_{\mathcal{D}}$ is not equivalent to one defined by some $\widehat{\mathfrak{t}}_{\mathcal{D}}^{(\kappa,E)}$. In fact, this is already the case over a closed point ν on the curve X. The de Rham cohomology of $\widehat{\mathfrak{t}}_{\mathcal{D}}$ over $\mathring{D}_\nu$ yields the central extension of $k(\!(x)\!)$ with Lie algebra 2-cocycle:

$$k(\!(x)\!) \otimes k(\!(x)\!) \to k, \quad f \otimes f' \rightsquigarrow \mathrm{Res}_{x=0}((3x\partial_x^2 f + x^2\partial_x^3 f)f'dx),$$

which is not cohomologous to any Kac–Moody 2-cocycle $f \otimes f' \rightsquigarrow \kappa \cdot \mathrm{Res}_{x=0}((df)f')$.

Of course, (19.19) is only the simplest one in an infinite family of Lie-$*$ cocycles, defined by sections of $\omega_{X^2}(\infty\Delta)$ with poles of order ≥ 3.

19.10. **Summary.**

19.10.1. In the following commutative diagram, we summarize the results about factorization twistings in relation to factorization tame twistings. The dotted arrows denote functors which are not known to exist.

$$\begin{tikzcd}
\mathbf{CExt}(G, \mathbf{K}_2) \underset{\mathbb{Z}}{\otimes} k \arrow[r, dotted] \arrow[d, "\Xi_{\mathbf{Pic}}", "\cong"'] & ? \arrow[r, dotted] \arrow[d, dotted] & \mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X) \arrow[d, "\Xi_{\mathbf{Tw}}"] \\
\mathbf{Pic}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] \arrow[d, "\Psi_{\mathbf{Pic}}", "\cong"'] & \mathring{\mathbf{Tw}}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[r] \arrow[d, "\Psi_{\mathring{\mathbf{Tw}}}", "\cong"'] & \mathbf{Tw}^{\mathrm{fact}}(\mathrm{Gr}_{G,\mathrm{Ran}}) \arrow[d, dotted, "\Psi_{\mathbf{Tw}}"] \\
\Theta_G(\Lambda_T; \mathbf{Pic}) \underset{\mathbb{Z}}{\otimes} k \arrow[r, "c_1"] & \Theta_G(\Lambda_T; \mathring{\mathbf{Tw}}) \arrow[r, "(a)"] & \Theta_G(\Lambda_T; \mathbf{Tw})
\end{tikzcd}$$

We have already seen (§18.2) that the functor (a) is an equivalence *if and only if* X is proper. The results in this section assert that when G is semisimple and simply connected, then $\Psi_{\mathbf{Tw}}$ exists and all solid functors, as well as $\Psi_{\mathbf{Tw}}$, are equivalences. On the other hand, when G is general, we do not expect $\Psi_{\mathbf{Tw}}$ to exist, and we are clueless about the behavior of $\Xi_{\mathbf{Tw}}$.

19.10.2. The diagram also suggests that a functor that goes directly from Brylinski–Deligne data to central extensions of $\mathfrak{g}_{\mathcal{D}}$ might exist. We suspect that it is given by a kind of derivative, but we have not been able to construct it.

Certainly, it would also be nice to have a "construction functor" $\Xi_{\mathring{\mathbf{Tw}}}$ for factorization tame twistings, starting from a 1-full subcategory of $\mathbf{CExt}_G(\mathfrak{g}_{\mathcal{D}}, \omega_X)$ characterized by a bound on the pole order. We hope to address this question in the future.

CHAPTER 5

Compactification

In this final chapter, we address another aspect of metaplectic parameters in the de Rham context. Namely, the k-linear groupoid of quantum parameters Par_G (i.e., enhanced Θ-data for twistings) admits a natural compactification $\overline{\mathrm{Par}}_G$. The resulting space $\overline{\mathrm{Par}}_G$ has the structure of an algebraic stack and is invariant under the change of G by its Langlands dual $\check{G}$.

We will see that an S-point of $\overline{\mathrm{Par}}_G$ defines an S-family of factorization *pre*-twistings on $\mathrm{Gr}_{G,\mathrm{Ran}}$. The DG category of their twisted crystals thus acquires an $\mathcal{O}_S$-linear structure.

The relevance of this compatification for the geometric Langlands program is the following. It realizes all naturally appearing DG categories as a quasi-coherent sheaf of categories over $\overline{\mathrm{Par}}_G$. Furthermore, we will study how these DG categories degenerate at points outside Par_G. This turns out to formalize many expected behaviors of the *quantum* Langlands program as the quantum parameter $c \to \infty$.

This chapter will not use the machinery concerning factorization structures built in the previous chapters. In fact, we will only need the notion of pre-twistings introduced in Chapter 4, so this chapter can be read mostly independently. All the results here have already appeared in [71] and [72]. We continue to fix the ground field $k = \bar{k}$ with $\mathrm{char}(k) = 0$.

20. Compactified quantum parameters

Let us recall the k-linear groupoid of quantum parameters Par_G from Chapter 4, §18.2. It can also be regarded as the k-points of an algebraic stack. In this section, we define the algebraic stack $\overline{\mathrm{Par}}_G$ of compactified quantum parameters. We will define a natural isomorphism $\overline{\mathrm{Par}}_G \xrightarrow{\sim} \overline{\mathrm{Par}}_{\check{G}}$, and explain how $\overline{\mathrm{Par}}_G$ behaves when we change G into the Levi quotient M of a parabolic of G.

20.1. **The space** Par_G.

20.1.1. Let $\mathfrak{g}$ denote the Lie algebra of G. Consider the symplectic form on $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by the pairing:

$$\langle \xi \oplus \varphi, \xi' \oplus \varphi' \rangle := \varphi(\xi') - \varphi'(\xi). \tag{20.1}$$

Let $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ denote the smooth, projective variety parametrizing G-invariant Lagrangian subspaces of $\mathfrak{g} \oplus \mathfrak{g}^*$. We will denote an S-point of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ by $\mathfrak{g}^\kappa$, regarded as a Lagrangian subbundle of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$ stable under the ($\mathcal{O}_S$-linear) G-action.

20.1.2. Clearly, the space $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ of G-invariant symmetric bilinear forms on $\mathfrak{g}$ embeds into $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$, where a form κ, regarded as a linear map $\mathfrak{g} \to \mathfrak{g}^*$, is sent to its graph $\mathfrak{g}^\kappa$. We will use the following notations:

(1) $\mathfrak{g}^\infty$ denotes the k-point $\mathfrak{g}^*$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$;

(2) $\mathfrak{g}^{\mathrm{crit}}$ is the graph of the *critical* form $\mathrm{crit} := -\frac{1}{2}\det$, where det stands for the Killing form of $\mathfrak{g}$.

(3) for every S-point $\mathfrak{g}^\kappa$, the notation $\mathfrak{g}^{\kappa-\mathrm{crit}}$ denotes the Lagrangian subbundle of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$ defined by the property:

$$\xi \oplus \varphi \in \mathfrak{g}^\kappa \iff \xi \oplus (\varphi - \mathrm{crit}(\xi)) \in \mathfrak{g}^{\kappa-\mathrm{crit}}.$$

Note that if $\kappa \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$, then $\mathfrak{g}^{\kappa-\mathrm{crit}}$ is the graph of $\kappa - \mathrm{crit}$, so the above notation is unambiguous. We also have $\mathfrak{g}^{\infty-\mathrm{crit}} = \mathfrak{g}^\infty$. More generally, one may replace $\mathfrak{g}^{\kappa-\mathrm{crit}}$ in the above construction by $\mathfrak{g}^{\kappa+\kappa_0}$ for any $\kappa_0 \in \mathrm{Sym}^2(\mathfrak{g}^*)^G$. This construction defines an action of $\mathrm{Sym}^2(\mathfrak{g}^*)^G$ on $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ that extends addition on $\mathrm{Sym}^2(\mathfrak{g}^*)^G$.

20.1.3. We study the k-points of Par_G a bit more closely. Let $\mathfrak{g} = \mathfrak{z} \oplus \sum_{s\in\mathbf{S}} \mathfrak{g}_s$ be the decomposition of $\mathfrak{g}$ into its center $\mathfrak{z}$ and simple factors $\mathfrak{g}_s$ (for $s \in \mathbf{S}$).

20.1.4. **Lemma.** *Any Lagrangian, G-invariant subspace $L \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ takes the form $L = L_{\mathfrak{z}} \oplus \sum_{s\in\mathbf{S}} L_s$ where:*

(1) $L_{\mathfrak{z}}$ is a Lagrangian subspace of $\mathfrak{z} \oplus \mathfrak{z}^*$;

(2) each L_s is a Lagrangian, G-invariant subspace of $\mathfrak{g}_s \oplus \mathfrak{g}_s^*$.

Proof. The decomposition of $\mathfrak{g}$ induces a decomposition

$$\mathfrak{g} \oplus \mathfrak{g}^* = (\mathfrak{z} \oplus \mathfrak{z}^*) \oplus \sum_{s\in\mathbf{S}} (\mathfrak{g}_s \oplus \mathfrak{g}_s^*),$$

where the summands are mutually orthogonal with respect to the symplectic form (20.1). We may also decompose $L = L_{\mathfrak{z}} \oplus \sum_j L_j$, where $L_{\mathfrak{z}}$ is the G-fixed subspace and each L_j is irreducible. Obviously, the embedding $L \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ sends $L_{\mathfrak{z}}$ into $\mathfrak{z} \oplus \mathfrak{z}^*$ as an isotropic subspace.

We claim that each embedding $L_j \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ factors through $\mathfrak{g}_s \oplus \mathfrak{g}_s^*$ for a unique s. In other words, the composition $L_j \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_s \oplus \mathfrak{g}_s^*$ must vanish for all but one s. Suppose, to the contrary, we have $s \neq s'$ such that both

$$L_j \to \mathfrak{g}_s \oplus \mathfrak{g}_s^*, \quad \text{and} \quad L_j \to \mathfrak{g}_{s'} \oplus \mathfrak{g}_{s'}^*$$

are nonzero. Without loss of generality, we may assume that the projections onto the first factors $L_j \to \mathfrak{g}_s$, $L_j \to \mathfrak{g}_{s'}$ are nonzero. Hence we have

(1) $L_j \cong \mathfrak{g}_s \cong \mathfrak{g}_{s'}$ as G-representations; and

(2) the image of L_j under the projection $\mathfrak{g} \oplus \mathfrak{g}^* \twoheadrightarrow \mathfrak{g}_s \oplus \mathfrak{g}_{s'}$ is a G-invariant subspace with nonzero projection onto both factors.

The second statement implies that this image is the entire space $\mathfrak{g}_s \oplus \mathfrak{g}_{s'}$, contradicting the equality $\dim(L_j) = \dim(\mathfrak{g}_s)$ from the first statement. This prove the claim.

Now, suppose $j \neq j'$ and both embeddings $L_j, L_{j'} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ factor through the same $\mathfrak{g}_s \oplus \mathfrak{g}_s^*$. This is obviously impossible since $L_j \oplus L_{j'} \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ would factor through an isomorphism $L_j \oplus L_{j'} \xrightarrow{\sim} \mathfrak{g}_s \oplus \mathfrak{g}_s^*$, so it is *not* isotropic. We conclude that there is a bijection between the sets $\{L_j\}$ and $\{\mathfrak{g}_s \oplus \mathfrak{g}_s^*\}$ such that each $L_j \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$ factors through the corresponding item $\mathfrak{g}_s \oplus \mathfrak{g}_s^*$.

Finally, since each L_j is an isotropic subspace of $\mathfrak{g}_s \oplus \mathfrak{g}_s^*$, we have:

$$\dim(\mathfrak{g}) = \dim(L_{\mathfrak{z}}) + \sum_j \dim(L_j) \leq \dim(\mathfrak{z}) + \sum_{s \in \mathbf{S}} \dim(\mathfrak{g}_s) = \dim(\mathfrak{g}).$$

Hence the equality is achieved, and each L_j (resp. $L_{\mathfrak{z}}$) is a Lagrangian subspace of $\mathfrak{g}_s \oplus \mathfrak{g}_s^*$ (resp. $\mathfrak{z} \oplus \mathfrak{z}^*$). $\square$

20.1.5. **Corollary.** *Let L be a Lagrangian, G-invariant subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$. Then there is a (non-canonical) isomorphism $L \xrightarrow{\sim} \mathfrak{g}$ of G-representations.* $\square$

Note that we have an obvious morphism:

$$\mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_{s \in \mathbf{S}} \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_s \oplus \mathfrak{g}_s^*) \to \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \tag{20.2}$$

sending a series of vector bundles $\mathfrak{z}^\kappa, \{\mathfrak{g}_s^\kappa\}$ over S to their direct sum $\mathfrak{z}^\kappa \oplus \sum_i \mathfrak{g}_s^\kappa$, which is a subbundle of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$.

20.1.6. **Corollary.** *The morphism* (20.2) *is an isomorphism.*

Proof. Indeed, (20.2) is a proper morphism between smooth schemes. Lemma 20.1.4 shows that it is bijective on k-points, so in particular quasi-finite, and therefore finite (by properness). A finite morphism of degree 1 between smooth schemes is an isomorphism. $\square$

Furthermore, any G-invariant symmetric bilinear form on $\mathfrak{g}_s$ determines an isomorphism $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_s \oplus \mathfrak{g}_s^*) \xrightarrow{\sim} \mathbb{P}^1$. Therefore $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ is *non-canonically* isomorphic to the product of a Lagrangian Grassmannian together with finitely many copies of $\mathbb{P}^1$.

20.1.7. A particular consequence of Corollary 20.1.6 is that we have a morphism given by projection:

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \tag{20.3}$$

Note that $\mathfrak{z}$ identifies with the subspace of G-invariants of $\mathfrak{g}$. Although $\mathfrak{z}^*$ is more naturally the space of G-coinvariants of $\mathfrak{g}^*$, we will identify it with the invariants $(\mathfrak{g}^*)^G$ via the isomorphism $(\mathfrak{g}^*)^G \hookrightarrow \mathfrak{g}^* \twoheadrightarrow \mathfrak{z}^*$.

More intrinsically, the morphism (20.3) is defined on S-points by:

$$\mathfrak{g}^\kappa \rightsquigarrow (\mathfrak{g}^\kappa)^G := \mathfrak{g}^\kappa \cap ((\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S).$$

where $(\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S$ is regarded as a submodule of $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$.

20.1.8. **Remark.** We refer to $(\mathfrak{g}^\kappa)^G$ as the *G-invariants* of $\mathfrak{g}^\kappa$. The same terminology is used in the sequel when we replace G by a different group H and $\mathfrak{g}^\kappa$ by an H-invariant subspace of $V \oplus V^*$, where V is any H-representation for which the composition $(V^*)^H \hookrightarrow V^* \twoheadrightarrow (V^H)^*$ is an isomorphism.

20.1.9. **Remark.** Note that $(\mathfrak{g}^{\kappa-\mathrm{crit}})^G \cong (\mathfrak{g}^\kappa)^G$, since crit vanishes on $\mathfrak{z}$.

Since the embedding $\mathfrak{z} \hookrightarrow \mathfrak{g}$ canonically splits with kernel $\mathfrak{g}_{\mathrm{sc}} := [\mathfrak{g}, \mathfrak{g}]$, there is a surjection $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S \twoheadrightarrow (\mathfrak{z} \oplus \mathfrak{z}^*) \otimes \mathcal{O}_S$. Note that the image of $\mathfrak{g}^\kappa$ identifies with $(\mathfrak{g}^\kappa)^G$, and the composition $(\mathfrak{g}^\kappa)^G \hookrightarrow \mathfrak{g}^\kappa \twoheadrightarrow (\mathfrak{g}^\kappa)^G$ is the identity. In other words,

20.1.10. **Lemma.** *The morphism $(\mathfrak{g}^\kappa)^G \hookrightarrow \mathfrak{g}^\kappa$ canonically splits.* $\square$

We denote the complement of $(\mathfrak{g}^\kappa)^G$ in $\mathfrak{g}^\kappa$ by $\mathfrak{g}^\kappa_{\mathrm{sc}}$; it corresponds to the semisimple part of the Lie algebra $\mathfrak{g}$.

20.1.11. We define the stack $\overline{\mathrm{Par}}_G$ as follows: $\mathrm{Maps}(S, \overline{\mathrm{Par}}_G)$ is the groupoid of pairs $(\mathfrak{g}^\kappa, E)$, where $\mathfrak{g}^\kappa$ is an S-point of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}\oplus\mathfrak{g}^*)$, and E is an extension of $\mathcal{O}_{\mathcal{X}}$-modules:

$$0 \to \omega_{\mathcal{X}/S} \to E \to (\mathfrak{g}^\kappa)^G \boxtimes \mathcal{O}_X \to 0. \tag{20.4}$$

where $\mathcal{X} := S \times X$, and $\omega_{\mathcal{X}/S} \cong \mathcal{O}_S \boxtimes \omega_X$ is the relative dualizing sheaf.

In other words, $\overline{\mathrm{Par}}_G$ is a fiber bundle over $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}\oplus\mathfrak{g}^*)$, whose fiber at a k-point $\mathfrak{g}^\kappa$ is the linear stack $\mathbf{Ext}((\mathfrak{g}^\kappa)^G \boxtimes \mathcal{O}_X, \omega_X)$ of extensions over X. We think of $\mathfrak{g}^\kappa$ as a generalized symmetric bilinear form on $\mathfrak{g}$ and E as an additional quantum parameter.

20.2. Langlands auto-duality of $\overline{\mathrm{Par}}_G$.

20.2.1. We now fix a maximal torus $T \hookrightarrow G$. Let $\check{G}$ be the Langlands dual of G, equipped with the corresponding maximal torus $\check{T}$.

Let $W := N_G(T)/T$ denote the Weyl group of T. It acts on $\mathfrak{t}\oplus\mathfrak{t}^*$ in the standard way. There is a symplectic isomorphism:

$$\mathfrak{t}\oplus\mathfrak{t}^* \xrightarrow{\sim} \check{\mathfrak{t}}\oplus\check{\mathfrak{t}}^*, \quad \xi\oplus\varphi \rightsquigarrow \varphi\oplus(-\xi) \tag{20.5}$$

defined using the canonical identifications $\mathfrak{t}^* \xrightarrow{\sim} \check{\mathfrak{t}}$ and $\mathfrak{t} \xrightarrow{\sim} \check{\mathfrak{t}}^*$. Furthermore, (20.5) intertwines the W and $\check{W}$ actions (again, under the identification $W \xrightarrow{\sim} \check{W}$).

20.2.2. Let $\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}\oplus\mathfrak{t}^*)$ denote the smooth, projective variety parametrizing W-invariant, Lagrangian subspaces of $\mathfrak{t}\oplus\mathfrak{t}^*$. The isomorphism (20.5) induces an isomorphism:

$$\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}\oplus\mathfrak{t}^*) \xrightarrow{\sim} \mathrm{Gr}^{\check{W}}_{\mathrm{Lag}}(\check{\mathfrak{t}}\oplus\check{\mathfrak{t}}^*). \tag{20.6}$$

We denote the image of $\mathfrak{t}^\kappa$ under (20.6) by $\check{\mathfrak{t}}^{\check{\kappa}}$, and view it as the graph associated to the "dual" form.

We define a morphism

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \tag{20.7}$$

by sending an S-point $\mathfrak{g}^\kappa$ to $(\mathfrak{g}^\kappa)^T$, the T-invariants of $\mathfrak{g}^\kappa$. An argument similar to the one before shows that we have a well-defined map $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*)$; it is clear that the image lies in the W-fixed locus.

20.2.3. **Lemma.** *The morphism* (20.7) *is an isomorphism.*

Proof. Indeed, a decomposition of $\mathfrak{g} = \mathfrak{z} \oplus \sum_i \mathfrak{g}_i$ into simple factors induces a decomposition $\mathfrak{t} = \mathfrak{z} \oplus \sum_i \mathfrak{t}_i$, where each $\mathfrak{t}_i$ is the maximal torus of the factor $\mathfrak{g}_i$. Note that $\mathfrak{t}_i$ is irreducible as a W-representation. An analogue of Corollary 20.1.6 asserts an isomorphism $\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}_i \oplus \mathfrak{t}_i^*)$, making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \arrow[r, "(20.7)"] \arrow[d, "\cong"] & \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \arrow[d, "\cong"] \\
\mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_i \oplus \mathfrak{g}_i^*) \arrow[r] & \mathrm{Gr}_{\mathrm{Lag}}(\mathfrak{z} \oplus \mathfrak{z}^*) \times \prod_i \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}_i \oplus \mathfrak{t}_i^*).
\end{tikzcd}$$

Note that the bottom arrow is an isomorphism since the choice of a G-invariant, symmetric bilinear form on $\mathfrak{g}_i$ (hence a W-invariant form on $\mathfrak{t}_i$) identifies both $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g}_i \oplus \mathfrak{g}_i^*)$ and $\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t}_i \oplus \mathfrak{t}_i^*)$ with $\mathbb{P}^1$. $\square$

In particular, we see that the fiber of $\overline{\mathrm{Par}}_G$ over the open locus of bilinear forms in $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ identifies with the k-linear stack associated to Par_G.

20.2.4. **Remark.** Using T, we may also rewrite (20.3) as the two-step procedure of first taking T-invariants and then taking W-invariants:

$$(\mathfrak{g}^\kappa)^G \xrightarrow{\sim} ((\mathfrak{g}^\kappa)^T)^W.$$

This isomorphism again follows from the description of fibers of $\mathfrak{g}^\kappa$ in Lemma 20.1.4.

20.2.5. We will consider a slight variant of the isomorphism (20.7) which takes into account the critical shift:

$$\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \xrightarrow{\sim} \mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*), \quad \mathfrak{g}^\kappa \rightsquigarrow (\mathfrak{g}^{\kappa-\mathrm{crit}})^T. \tag{20.8}$$

There is an isomorphism between $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$ and the corresponding space for $\check{G}$, making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*) \arrow[r, "\sim"] \arrow[d, "(20.8)"] & \mathrm{Gr}^{\check{G}}_{\mathrm{Lag}}(\check{\mathfrak{g}} \oplus \check{\mathfrak{g}}^*) \arrow[d, "(20.8)\text{ for }\check{G}"] \\
\mathrm{Gr}^W_{\mathrm{Lag}}(\mathfrak{t} \oplus \mathfrak{t}^*) \arrow[r, "(20.6)", "\sim"'] & \mathrm{Gr}^{\check{W}}_{\mathrm{Lag}}(\check{\mathfrak{t}} \oplus \check{\mathfrak{t}}^*)
\end{tikzcd}$$

We denote the image of $\mathfrak{g}^\kappa$ in $\mathrm{Gr}^{\check{G}}_{\mathrm{Lag}}(\check{\mathfrak{g}} \oplus \check{\mathfrak{g}}^*)$ by $\check{\mathfrak{g}}^{\check{\kappa}}$.

Since $(\mathfrak{g}^{\kappa-\mathrm{crit}})^G \cong (\mathfrak{g}^\kappa)^G$, there is an isomorphism

$$\mathrm{Par}_G \xrightarrow{\sim} \mathrm{Par}_{\check{G}}, \quad (\mathfrak{g}^\kappa, E) \rightsquigarrow (\mathfrak{g}^{\check{\kappa}}, \check{E}) \tag{20.9}$$

where $\check{E}$ is the extension of $(\check{\mathfrak{g}}^{\check{\kappa}})^{\check{G}} \boxtimes \mathcal{O}_X$ induced from E via the identification of $\mathcal{O}_{S\times X}$-modules:

$$(\mathfrak{g}^\kappa)^G \xrightarrow{\sim} (\mathfrak{g}^{\kappa-\mathrm{crit}})^G \cong (\check{\mathfrak{g}}^{\check{\kappa}-\mathrm{crit}})^{\check{G}} \xleftarrow{\sim} (\check{\mathfrak{g}}^{\check{\kappa}})^{\check{G}}$$

where the middle isomorphism comes from the identification of $(\mathfrak{g}^{\kappa-\mathrm{crit}})^T$ and $(\check{\mathfrak{g}}^{\check{\kappa}-\mathrm{crit}})^{\check{T}}$ under (20.6). We refer to (20.9) as the *Langlands duality* for the parameter space Par_G.

20.2.6. **Example.** Suppose G is simple, and we fix a k-valued parameter $(\mathfrak{g}^\kappa, 0)$ of Par_G corresponding to some bilinear form κ on $\mathfrak{g}$. Then $\kappa = \lambda \cdot \mathrm{Kil}_G$ for some $\lambda \in k$. Write $\lambda = (c - h^\vee)/2h^\vee$ for some $c \in k$, where $h^\vee$ denotes the dual Coxeter number of G.

Assume $c \neq 0$. Then under the isomorphism (20.9):

$$\overline{\mathrm{Par}}_G \xrightarrow{\sim} \overline{\mathrm{Par}}_{\check{G}}, \quad (\mathfrak{g}^\kappa, 0) \rightsquigarrow (\check{\mathfrak{g}}^{\check{\kappa}}, 0),$$

we *claim* that $\check{\mathfrak{g}}^{\check{\kappa}}$ also arises from a bilinear form $\check{\kappa}$, defined by the formulae:

$$\check{\kappa} = \check{\lambda} \cdot \mathrm{Kil}_{\check{G}}, \quad \check{\lambda} = (-\frac{1}{rc} - h)/2h,$$

where $r = 1, 2$ or 3 denotes the maximal multiplicity of arrows in the Dynkin diagram of G. Indeed, one sees this from the fact that $(1/2h^\vee) \cdot \mathrm{Kil}_G$ is the minimal bilinear form and r is the ratio of the square lengths of long and short roots of G.

20.3. **Parabolics and anomaly.**

20.3.1. We now explain how to incorporate, via an additional parameter, the anomaly term that appears in the study of constant term functors (see [20]). This discussion requires further fixing:

1. a Borel subgroup B containing T;
2. a *theta characteristic* on the curve X, i.e., a line bundle θ together with an isomorphism $\theta^{\otimes 2} \xrightarrow{\sim} \omega_X$.

A *standard* parabolic is a parabolic subgroup of G containing B.

20.3.2. Let P be a standard parabolic with Levi quotient M. Then we may regard T as a maximal torus of M via the composition $T \hookrightarrow B \hookrightarrow P \twoheadrightarrow M$. Note that the Weyl group W_M of $T \hookrightarrow M$ is naturally a subgroup of W. The embedding

$$\mathfrak{z} \xrightarrow{\sim} \mathfrak{t}^W \hookrightarrow \mathfrak{t}^{W_M} \xrightarrow{\sim} \mathfrak{z}_M$$

is canonically split; this is because the composition $Z_G^\circ \hookrightarrow G \twoheadrightarrow G/[G,G]$ is an isogeny, giving rise to the projection $\mathfrak{z}_M \to \mathfrak{z}$. It follows that we have a canonical map:

$$\mathfrak{z}_M \oplus \mathfrak{z}_M^* \to \mathfrak{z}_G \oplus \mathfrak{z}_G^* \tag{20.10}$$

from the W_M-invariants of $\mathfrak{t} \oplus \mathfrak{t}^*$ to its W-invariants. In particular, given any Lagrangian, W-invariant subbundle $\mathfrak{t}^\kappa \hookrightarrow (\mathfrak{t} \oplus \mathfrak{t}^*) \otimes \mathcal{O}_S$, we have a morphism

$$(\mathfrak{t}^\kappa)^{W_M} \to (\mathfrak{t}^\kappa)^W \tag{20.11}$$

compatible with (20.10).

20.3.3. There is a *reduction* morphism

$$\mathrm{Gr}_{\mathrm{Lag}}^G(\mathfrak{g} \oplus \mathfrak{g}^*) \to \mathrm{Gr}_{\mathrm{Lag}}^M(\mathfrak{m} \oplus \mathfrak{m}^*) \tag{20.12}$$

given by the composition

$$\mathrm{Gr}_{\mathrm{Lag}}^G(\mathfrak{g} \oplus \mathfrak{g}^*) \xrightarrow{\sim} \mathrm{Gr}_{\mathrm{Lag}}^W(\mathfrak{t} \oplus \mathfrak{t}^*) \hookrightarrow \mathrm{Gr}_{\mathrm{Lag}}^{W_M}(\mathfrak{t} \oplus \mathfrak{t}^*) \xleftarrow{\sim} \mathrm{Gr}_{\mathrm{Lag}}^M(\mathfrak{m} \oplus \mathfrak{m}^*)$$

where the isomorphisms are supplied by (20.8) for G, respectively M. In other words, the image of $\mathfrak{g}^\kappa$ under (20.12) is an S-point $\mathfrak{m}^\kappa$ such that $(\mathfrak{m}^{\kappa-\mathrm{crit}})^T$ and $(\mathfrak{g}^{\kappa-\mathrm{crit}})^T$ are canonically isomorphic as subbundles of $(\mathfrak{t} \oplus \mathfrak{t}^*) \otimes \mathcal{O}_S$.

20.3.4. Let Z_M° denote the neutral component of the center of M. Write $2\check{\rho}_M$ for the character of Z_M° determined by the representation $\det(\mathfrak{n}_P)$, where $\mathfrak{n}_P$ is the Lie algebra of the unipotent part of P.

Let $\check{Z}_M^\circ$ denote the Langlands dual torus of Z_M°. We use $\omega_X^{\check{\rho}_M}$ to denote the $\check{Z}_M^\circ$-bundle on X induced from θ under $2\check{\rho}_M$ (regarded as a cocharacter of $\check{Z}_M^\circ$). Then the Atiyah bundle of $\omega_X^{\check{\rho}_M}$ fits into an exact sequence:

$$0 \to \mathfrak{z}_M^* \otimes \mathcal{O}_X \to \mathrm{At}(\omega_X^{\check{\rho}_M}) \to \mathcal{T}_X \to 0$$

Its monoidal dual gives rise to an extension of $\mathcal{O}_{\mathcal{X}}$-modules for every S (recall the notation $\mathcal{X} := S \times X$):

$$0 \to \omega_{\mathcal{X}/S} \to \mathcal{O}_S \boxtimes \mathrm{At}(\omega_X^{\check{\rho}_M})^* \to (\mathfrak{z}_M \otimes \mathcal{O}_S) \boxtimes \mathcal{O}_X \to 0 \tag{20.13}$$

For each S-point $\mathfrak{m}^\kappa$ of $\mathrm{Gr}^M_{\mathrm{Lag}}(\mathfrak{m} \oplus \mathfrak{m}^*)$, let $E^+_{G\to M}$ denote the extension of $(\mathfrak{m}^\kappa)^M$ induced from (20.13) along the canonical map

$$(\mathfrak{m}^\kappa)^M \hookrightarrow (\mathfrak{z}_M \oplus \mathfrak{z}_M^*) \otimes \mathcal{O}_S \twoheadrightarrow \mathfrak{z}_M \otimes \mathcal{O}_S.$$

The additional parameter $E^+_{G\to M}$ is the *anomaly term* at level $\mathfrak{m}^\kappa$.

20.3.5. The reduction morphism for quantum parameters is defined by

$$\mathrm{Par}_G \to \mathrm{Par}_M, \quad (\mathfrak{g}^\kappa, E) \rightsquigarrow (\mathfrak{m}^\kappa, E_{G\to M}) \tag{20.14}$$

where $\mathfrak{m}^\kappa$ is the image of $\mathfrak{g}^\kappa$ under (20.12), and $E_{G\to M}$ is the Baer sum of the following two extensions of $(\mathfrak{m}^\kappa)^M$:

(1) an extension induced from E (which is an extension of $(\mathfrak{g}^\kappa)^G$) via the map:

$$(\mathfrak{m}^\kappa)^M \xrightarrow{\sim} (\mathfrak{m}^{\kappa-\mathrm{crit}})^M \to (\mathfrak{g}^{\kappa-\mathrm{crit}})^G \xrightarrow{\sim} (\mathfrak{g}^\kappa)^G,$$

where the map in the middle comes from (20.11) for $\mathfrak{t}^\kappa := (\mathfrak{m}^{\kappa-\mathrm{crit}})^T \cong (\mathfrak{g}^{\kappa-\mathrm{crit}})^T$;

(2) the anomaly term $E^+_{G\to M}$ at level $\mathfrak{m}^\kappa$.

20.3.6. **Remark.** The image of $(\mathfrak{g}^\infty, E)$ under (20.14) is simply the unadjusted one $(\mathfrak{m}^\infty, E)$. In particular, we see that (20.14) is *incompatible* with Langlands duality for quantum parameters, i.e., if we let $\check{M}$ be the group dual to M, the following diagram does *not* commute:

$$\begin{tikzcd}
\overline{\mathrm{Par}}_G \arrow[r, "(20.9)"] \arrow[d, "(20.14)"'] & \overline{\mathrm{Par}}_{\check{G}} \arrow[d, "(20.14)"] \\
\overline{\mathrm{Par}}_M \arrow[r, "(20.9)"] & \overline{\mathrm{Par}}_{\check{M}}.
\end{tikzcd}$$

20.3.7. **Remark.** For $P = B$ and $M = T$, the character $2\check{\rho}$ is the sum of positive roots, and splittings of (20.13) form a $\mathfrak{t}^* \otimes \omega_X$-torsor $\mathrm{Conn}(\omega_X^{\check{\rho}})$ known as the *Miura opers.*

20.4. **Structures on $\mathfrak{g}^\kappa$.**

20.4.1. We now note some structures on an S-point $\mathfrak{g}^\kappa$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$, which are functorial in S. First, there is an $\mathcal{O}_S$-bilinear Lie bracket:

$$[-,-] : \mathfrak{g}^\kappa \underset{\mathcal{O}_S}{\otimes} \mathfrak{g}^\kappa \to \mathfrak{g}^\kappa \tag{20.15}$$

defined by the formula (on the ambient bundle $(\mathfrak{g} \oplus \mathfrak{g}^*) \otimes \mathcal{O}_S$):

$$[(\xi \oplus \varphi) \otimes \mathbf{1}, (\xi' \oplus \varphi') \otimes \mathbf{1}] := ([\xi, \xi'] \oplus \mathrm{Coad}_\xi(\varphi')) \otimes \mathbf{1}.$$

One checks immediately that the image lies in $\mathfrak{g}^\kappa$ and the required identities hold. Note that (20.15) factors through the embedding $\mathfrak{g}^\kappa_{\mathrm{sc}} \hookrightarrow \mathfrak{g}^\kappa$. Furthermore, we note that the Lie-bracket factors through an action $[-,-] : \mathfrak{g} \underset{\mathcal{O}_S}{\otimes} \mathfrak{g}^\kappa \to \mathfrak{g}^\kappa$.

20.4.2. Furthermore, there is an $\mathcal{O}_S$-bilinear symmetric pairing:

$$(-,-) : \mathfrak{g}^\kappa \underset{\mathcal{O}_S}{\otimes} \mathfrak{g}^\kappa \to \mathcal{O}_S \tag{20.16}$$

defined by the formula:

$$((\xi \oplus \varphi) \otimes \mathbf{1}, (\xi' \oplus \varphi') \otimes \mathbf{1}) := \varphi'(\xi) \cdot \mathbf{1}.$$

20.4.3. Fixing an S-point $(\mathfrak{g}^\kappa, E)$ of Par_G, there is an extension of $\mathcal{O}_{\mathcal{X}}$-modules:

$$0 \to \omega_{\mathcal{X}/S} \to \widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^\kappa \boxtimes \mathcal{O}_X \to 0. \tag{20.17}$$

induced from (20.4) along $\mathfrak{g}^\kappa \otimes \mathcal{O}_S \to (\mathfrak{g}^\kappa)^G \otimes \mathcal{O}_X$. In other words, $\widehat{\mathfrak{g}}^{(\kappa,E)}$ is the direct sum of E and $\mathfrak{g}^\kappa_{\mathrm{sc}} \boxtimes \mathcal{O}_X$, corresponding to the decomposition $\mathfrak{g}^\kappa \xrightarrow{\sim} \widehat{\mathfrak{g}}^\kappa \oplus \mathfrak{g}^\kappa_{\mathrm{sc}}$.

20.5. Kac–Moody Lie-$*$ algebra.

20.5.1. We now generalize the construction of Chapter 4, §19.4. Suppose $(\mathfrak{g}^\kappa, E)$ is an S-point of $\overline{\mathrm{Par}}_G$. First, we obtain from $\mathfrak{g}^\kappa$ an S-family of Lie-$*$ algebras $\mathfrak{g}^\kappa_{\mathcal{D}} := \mathfrak{g}^\kappa \boxtimes \mathcal{D}_X$ defined by the formula (where the Lie bracket on $\mathfrak{g}^\kappa$ is the one from (20.15)):

$$[-,-] : \mathfrak{g}^\kappa_{\mathcal{D}} \boxtimes \mathfrak{g}^\kappa_{\mathcal{D}} \to \Delta_{*,\mathrm{dR}}(\mathfrak{g}^\kappa_{\mathcal{D}}), \quad [(\xi \oplus \varphi) \boxtimes (\xi' \oplus \varphi')] = [(\xi \oplus \varphi), (\xi' \oplus \varphi')]\mathbf{1}_{\mathcal{D}}.$$

Let $\mathbf{CExt}_G(\mathfrak{g}^\kappa_{\mathcal{D}}, \omega_{\mathcal{X}/S})$ denote the $\mathcal{O}_S$-linear groupoid of central extensions:

$$0 \to \omega_{\mathcal{X}/S} \to \widehat{\mathfrak{g}}_{\mathcal{D}} \to \mathfrak{g}^\kappa_{\mathcal{D}} \to 0 \tag{20.18}$$

of Lie-$*$ algebras relative to S, such that the adjoint $\mathfrak{g}_{\mathcal{D}}$-action on (20.18) integrates to a $\mathrm{Jet}(G \times \mathcal{X})$-action. The groupoid $\mathbf{CExt}_G(\mathfrak{g}^\kappa_{\mathcal{D}}, \omega_{\mathcal{X}/S})$ depends only on $\mathfrak{g}^\kappa$.

20.5.2. We shall construct an object of $\mathbf{CExt}_G(\mathfrak{g}^\kappa_{\mathcal{D}}, \omega_{\mathcal{X}/S})$ making use of the datum E:

$$0 \to \omega_{\mathcal{X}/S} \to \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}} \to \mathfrak{g}^\kappa_{\mathcal{D}} \to 0, \tag{20.19}$$

Namely, $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}$ is the direct sum of $\mathcal{O}_S \boxtimes \mathcal{D}_X$-modules:

(1) The push-out E' of the induced $\mathcal{D}$-module $E \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{D}_{\mathcal{X}/S}$ along the action map $\omega_{\mathcal{X}/S} \underset{\mathcal{O}_{\mathcal{X}}}{\otimes} \mathcal{D}_{\mathcal{X}/S} \to \omega_{\mathcal{X}/S}$;
(2) The induced $\mathcal{D}$-module $(\mathfrak{g}^\kappa_{\mathrm{sc}})_{\mathcal{D}}$.

The 2-cocycle is defined using the pairing (20.16) and the $\mathrm{Jet}_{/S}(G \times \mathcal{X})$-action on (20.19) is defined using the G-action on $\mathfrak{g}^\kappa$ (c.f. Chapter 4, §19.4).

20.5.3. The next step in the construction is taking de Rham cohomology over formal punctured discs. Let us first see how this works for a fixed point $x \in X$. Indeed, x induces a section $\underline{x} : S \to X$. Applying $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, -)$ to the sequence (20.19), we obtain a central extension of Lie algebras in Tate $\mathcal{O}_S$-modules:

$$0 \to \mathcal{O}_S \to \widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^\kappa(\mathbf{F}_x) \to 0, \tag{20.20}$$

which is canonically split over $\mathfrak{g}^\kappa(\widehat{\mathcal{O}}_{X,x})$. The Lie bracket on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ is given by the composition:

$$(\widehat{\mathfrak{g}}^{(\kappa,E)})^{\boxtimes 2} \to (\mathfrak{g}^\kappa(\mathbf{F}_x))^{\boxtimes 2} \to \mathcal{O}_S \oplus \mathfrak{g}^\kappa_{\mathrm{sc}}(\mathbf{F}_x) \to \widehat{\mathfrak{g}}^{(\kappa,E)},$$

where the middle map is defined by

$$(\mu \otimes f) \boxtimes (\mu' \otimes f') \rightsquigarrow (\mu, \mu') \cdot \mathrm{Res}((df)f') + [\mu, \mu'] \otimes ff'.$$

Furthermore, the $\mathfrak{g}(\mathbf{F}_x)$-action on the exact sequence (20.20) integrates to an $\mathcal{L}_x G$-action.

20.5.4. More generally, let us introduce the analogues of the groupoids (A) and (B) of Chapter 4, §19.5. These are $\mathcal{O}_S$-linear groupoids which depend on $\mathfrak{g}^\kappa$.

(A^κ) Factorization central extensions of Lie algebras of Tate modules on $S \times \mathrm{Ran}_{\mathrm{dR}}$:

$$0 \to \mathcal{O}_{S \times \mathrm{Ran}_{\mathrm{dR}}} \to \widehat{\mathfrak{g}} \to \mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}} \mathfrak{g}^\kappa \to 0,$$

equipped with an integration of the $S \times \mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}} \mathfrak{g}$-action to $S \times \mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}} G$, as well as a trivialization of these data over $\mathrm{L}^+_{\mathrm{Ran}_{\mathrm{dR}}} \mathfrak{g}^\kappa$.

We note that $\mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}} \mathfrak{g}^\kappa$ (resp. $\mathrm{L}^+_{\mathrm{Ran}_{\mathrm{dR}}} \mathfrak{g}^\kappa$) defines a multiplicative Lie algebroid on $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}} G$ (resp. $\mathrm{L}^+_{\mathrm{Ran}_{\mathrm{dR}}} G$), and consequently a multiplicative formal moduli problem under it. Let us denote it by $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G^{\kappa}$ (resp. $\mathcal{L}^{+}_{\mathrm{Ran}_{\mathrm{dR}}}G^{\kappa}$). The analogue of groupoid (B) is as follows.

(B^{κ}) Factorization multiplicative pre-twistings on $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$ based at $\mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G^{\kappa}$, equipped with a trivialization of the induced multiplicative pre-twisting on $\mathcal{L}^{+}_{\mathrm{Ran}_{\mathrm{dR}}}G$ based at $\mathcal{L}^{+}_{\mathrm{Ran}_{\mathrm{dR}}}G^{\kappa}$.

20.5.5. The construction of Chapter 4 yields functors:

$$\begin{aligned}\mathbf{CExt}_G(\mathfrak{g}^{\kappa}_{\mathcal{D}}, \omega_{\mathcal{X}/S}) \to &(\mathrm{A}^{\kappa}) \\ &\xrightarrow{\sim} (\mathrm{B}^{\kappa}) \to \mathbf{PTw}^{\mathrm{fact}}_{/k}(\mathrm{Gr}_{G,\mathrm{Ran}}).\end{aligned}$$

We shall denote the image of $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}$ by $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$, in order to distinguish it from a global version which we will explain later. It is clear that $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$ is functor with respect to the S-point (κ, E). Suppose $S = \mathrm{Spec}(k)$ and $\mathfrak{g}^{\kappa}$ belongs to the locus of bilinear forms. Then $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$ identifies with the factorization twisting $\Xi_{\mathbf{Tw}}(\widehat{\mathfrak{g}}^{(\kappa,E)})$ of Chapter 4. This identification is induced from the bijection $\mathfrak{g}^{\kappa} \xrightarrow{\sim} \mathfrak{g}$.

21. Quotients of pre-twistings

This section serves as technical preparation for the next sections, whose purpose is to construct particular examples of DG categories appearing in the quantum Langlands program as quasi-coherent sheaves of categories over $\overline{\mathrm{Par}}_G$.

The problem we are interested in is to explicitly identify these categories at $\kappa = \infty$. It turns out that doing so requires some general results concerning taking quotients of a pre-twisting by a group "inf-scheme." We explain this construction in the present section. The notion of an inf-scheme is due to Gaitsgory–Rozenblyum [36].

This section is organized as follows. We first define a "classical action pair" and discuss quotients of Lie algebroids and their central extensions. Then we define a "geometric action pair" and discuss quotients of formal moduli problems and pre-twistings. The main result is that these two procedures agree in overlapping cases. Finally, we discuss "inert pre-twistings" which shall appear at quantum parameter $\kappa = \infty$.

21.1. $(\mathfrak{k}, H)$-Lie algebroids.

21.1.1. Let us start by describing the quotient construction in the classical setting. Throughout, we work over a base classical scheme $S \in \mathbf{Sch}^{\mathrm{ft}}_{/k}$.

A *classical action pair* $(\mathfrak{k}, H)$ consists of an affine group scheme H over S, an $\mathcal{O}_S$-linear Lie algebra $\mathfrak{k}$ acted on by H, as well as a morphism of Lie algebras:

$$\mathfrak{k} \to \mathfrak{h} := \mathrm{Lie}(H) \tag{21.1}$$

with the following properties:

1. (21.1) is H-equivariant, where $\mathfrak{h}$ is equipped with the adjoint H-action;
2. the $\mathfrak{k}$-action on itself induced from (21.1) is the adjoint action.

21.1.2. **Example.** Fix an S-point $\mathfrak{g}^\kappa$ of $\mathrm{Gr}^G_{\mathrm{Lag}}(\mathfrak{g} \oplus \mathfrak{g}^*)$. Then we have a classical action pair $(\mathfrak{g}^\kappa[\![t]\!], S \times G[\![t]\!])$, where the morphism (21.1) is induced from the projection $\mathfrak{g}^\kappa \to \mathfrak{g} \otimes \mathcal{O}_S$. All classical action pairs considered in this paper are variants of $(\mathfrak{g}^\kappa[\![t]\!], S \times G[\![t]\!])$. Note that the group scheme $S \times G[\![t]\!]$ is *not* of finite type.

21.1.3. The notion of a morphism $(\mathfrak{k}^0, H^0) \to (\mathfrak{k}, H)$ of classical action pairs is obvious. We say that $(\mathfrak{k}^0, H^0)$ is a *normal* subpair if $\mathfrak{k}^0 \hookrightarrow \mathfrak{k}$ is an ideal, $H^0 \hookrightarrow H$ is a normal subgroup, the H-action stabilizes $\mathfrak{k}^0$, and H^0 acts trivially on $\mathfrak{k}/\mathfrak{k}^0$. This definition means precisely that a normal subpair fits into an *exact sequence* (in the obvious sense):

$$1 \to (\mathfrak{k}^0, H^0) \to (\mathfrak{k}, H) \to (\mathfrak{k}_0, H_0) \to 1. \tag{21.2}$$

21.1.4. Let $Y \in \mathbf{Sch}_{/S}$ be acted on by H. Recall that every H-equivariant $\mathcal{O}_Y$-module $\mathcal{F}$ admits an $\mathfrak{h}$-action by derivations. Specializing to $\mathcal{O}_Y$ itself, we obtain a canonical map:

$$\mathfrak{h} \otimes \mathcal{O}_Y \to \mathcal{T}_{Y/S}. \tag{21.3}$$

On the other hand, the $\mathcal{O}_Y$-module $\mathcal{T}_{Y/S}$ admits a canonical H-equivariance structure, given by pushforward of tangent vectors.

21.1.5. A $(\mathfrak{k}, H)$-*Lie algebroid* on Y consists of a Lie algebroid $\mathcal{L} \in \mathbf{Lie}_{/S}(Y)$, an H-equivariance structure on the underlying $\mathcal{O}_Y$-module of $\mathcal{L}$, and a morphism $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{L}$ of H-equivariant $\mathcal{O}_Y$-modules, subject to the following conditions:

(1) the H-equivariance structure on $\mathcal{L}$ is compatible with its Lie bracket;

(2) the anchor map σ of $\mathcal{L}$ intertwines the H-equivariance structures on $\mathcal{L}$ and $\mathcal{T}_{Y/S}$;

(3) the following diagram is commutative:

$$\begin{tikzcd} & \mathcal{L} \arrow[dr, "\sigma"] & \\ \mathfrak{k} \otimes \mathcal{O}_Y \arrow[ur, "\eta"] \arrow[dr, "(21.1)"'] & & \mathcal{T}_{Y/S} \\ & \mathfrak{h} \otimes \mathcal{O}_Y \arrow[ur, "(21.3)"'] & \end{tikzcd} \tag{21.4}$$

(4) η is compatible with the Lie bracket on $\mathcal{L}$ in the following sense: given $\xi \in \mathfrak{k} \otimes \mathcal{O}_Y$ and $l \in \mathcal{L}$, there holds:

$$[\eta(\xi), l] = \xi_{\mathfrak{h}} \cdot l \in \mathcal{L} \tag{21.5}$$

where $\xi_{\mathfrak{h}}$ is the image of ξ in $\mathfrak{h} \otimes \mathcal{O}_Y$ along (21.1), and $\xi_{\mathfrak{h}} \cdot l$ denotes the action of $\xi_{\mathfrak{h}}$ on l coming from the equivariance structure.

We will frequently write a $(\mathfrak{k}, H)$-Lie algebroid as $(\mathcal{L}, \eta)$, in order to emphasize the dependence on η. The category of $(\mathfrak{k}, H)$-Lie algebroids on Y is denoted by $\mathbf{Lie}^{(\mathfrak{k},H)}_{/S}(Y)$.

Given another scheme $Y' \in \mathbf{Sch}_{/S}$ acted on by H and an H-equivariant morphism $Y' \to Y$, one can form the pullback of a $(\mathfrak{k}, H)$-Lie algebroid in a way compatible with the forgetful functor to plain Lie algebroids.

Quotient I.

21.1.6. We describe how to form the *quotient* of a $(\mathfrak{k}, H)$-Lie algebroid when the morphism η is *injective*. Denote the category of such $(\mathfrak{k}, H)$-Lie algebroid by $\mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}\,/S}(Y)$.

21.1.7. Suppose $Z \in \mathbf{Sch}_{/S}$ and Y is an H-torsor over Z. Since H is affine, the projection $\pi : Y \to Z$ is an affine, faithfully flat cover (in particular, fpqc). We will define a *quotient* functor:

$$\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}} : \mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}\,/S}(Y) \to \mathbf{Lie}_{/S}(Z) \tag{21.6}$$

on each $(\mathcal{L}, \eta) \in \mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}}(Y/S)$ by the following procedure:

(1) ($\mathcal{O}_Z$*-module and anchor map*) We have a morphism of H-equivariant $\mathcal{O}_Y$-modules:

$$\mathcal{L}/(\mathfrak{k} \otimes \mathcal{O}_Y) \to \mathcal{T}_{Y/S}/(\mathfrak{h} \otimes \mathcal{O}_Y) \xrightarrow{\sim} \pi^*\mathcal{T}_{Z/S}$$

by (21.4). Let $\mathcal{L}_0$ denote the fpqc descent of $\mathcal{L}/(\mathfrak{k} \otimes \mathcal{O}_Y)$ to Z, so we obtain a map of $\mathcal{O}_Z$-modules $\sigma_0 : \mathcal{L}_0 \to \mathcal{T}_{Z/S}$. The image of $(\mathcal{L}, \eta)$ under $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}$ is supposed to have underlying $\mathcal{O}_Z$-module $\mathcal{L}_0$ and anchor map σ_0.

(2) (*Lie bracket*) Since π is affine, it suffices to define an $\mathcal{O}_S$-linear Lie bracket on $\pi^{-1}\mathcal{L}_0$. Consider the embedding:

$$\pi^{-1}\mathcal{L}_0 \hookrightarrow \pi^*\mathcal{L}_0 \xrightarrow{\sim} \mathcal{L}/(\mathfrak{k} \otimes \mathcal{O}_Y).$$

The Lie bracket on $\mathcal{L}$ will induce one on $\pi^{-1}\mathcal{L}_0$ if $[\mathfrak{k} \otimes \mathcal{O}_Y, \pi^{-1}\mathcal{L}_0] = 0$ in $\mathcal{L}$. The latter identity is guaranteed by (21.5).

We omit checking that this procedure gives rise to a well-defined functor $\mathbf{Q}_{\text{inj}}^{(\mathfrak{k},H)}$.

Given a *flat* morphism of schemes $f : Z' \to Z$, we set $Y' := Z' \underset{Z}{\times} Y$ which is an H-torsor over Z'. The map $\tilde{f} : Y' \to Y$ is H-equivariant, and the pullback of $(\mathcal{L}, \eta) \in \mathbf{Lie}_{\text{inj}\,/S}^{(\mathfrak{k},H)}(Y)$ along $\tilde{f}$ lies in $\mathbf{Lie}_{\text{inj}\,/S}^{(\mathfrak{k},H)}(Y')$. Furthermore, $\mathbf{Q}_{\text{inj}}^{(\mathfrak{k},H)}$ is compatible with pullbacks along f and $\tilde{f}$.

21.1.8. **Remark.** Since Lie algebroids are smooth local objects (see [6]) and $\mathbf{Q}_{\text{inj}}^{(\mathfrak{k},H)}$ is compatible with flat pullbacks, we may generalize $\mathbf{Q}_{\text{inj}}^{(\mathfrak{k},H)}$ to the case where $\mathcal{Z} := Y/H$ is representable by an algebraic stack (i.e., smooth locally a scheme).

21.1.9. **Remark.** The special case where the classical action pair is given by $(\mathfrak{h}, H)$ with (21.1) being the identity map, has been studied in [6] under the name *strong quotient*. Note that when H acts freely on Y, the map η is automatically injective.

21.1.10. **Example.** Another instance of the functor (21.6) is the *weak quotient*. This is the case where $\mathfrak{k} = 0$. The only data needed in defining a $(0, H)$-Lie algebroid are a Lie algebroid $\mathcal{L} \in \mathbf{Lie}_{/S}(Y)$, together with an H-equivariance structure on the underlying $\mathcal{O}_Y$-module of $\mathcal{L}$, subject to the first two conditions in the definition of a $(\mathfrak{k}, H)$-Lie algebroid.

Suppose Y/H is representable by an algebraic stack. Then the resulting quotient $\mathbf{Q}_{\text{inj}}^{(0,H)}(\mathcal{L})$ has underlying $\mathcal{O}_{Y/H}$-module the descent of (the $\mathcal{O}_Y$-module) $\mathcal{L}$ along $Y \to Y/H$.

21.1.11. We now characterize the object $\mathbf{Q}_{\text{inj}}^{(\mathfrak{k},H)}(\mathcal{L}) \in \mathbf{Lie}_{/S}(Z)$ by a universal property. Consider an arbitrary Lie algebroid $\mathcal{M} \in \mathbf{Lie}_{/S}(Z)$. We can equip the pullback Lie algebroid $\pi^!_{\mathbf{Lie}}\mathcal{M}$ with the structure of a $(\mathfrak{k}, H)$-Lie algebroid as follows:

(1) regarding $\pi^!_{\mathbf{Lie}}\mathcal{M}$ as the $\mathcal{O}_Y$-module $\pi^*\mathcal{M} \underset{\pi^*\mathcal{T}_{Z/S}}{\times} \mathcal{T}_{Y/S}$, the H-equivariance structure is a combination of the natural H-equivariance structures on $\pi^*\mathcal{M}$ and $\mathcal{T}_{Y/S}$;

(2) the morphism $\eta : \mathfrak{k}\otimes\mathcal{O}_Y \to \pi^!_{\mathbf{Lie}}(\mathcal{M})$ is a combination of the *zero* map $\mathfrak{k}\otimes\mathcal{O}_Y \to \pi^*\mathcal{M}$ and the composition $\mathfrak{k}\otimes\mathcal{O}_Y \to \mathfrak{h}\otimes\mathcal{O}_Y \to \mathcal{T}_{Y/S}$.

Note that $\pi^!_{\mathbf{Lie}}\mathcal{M} \in \mathbf{Lie}^{(\mathfrak{k},H)}_{/S}(Y)$ does *not* belong to $\mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}\,/S}(Y)$ in general.

21.1.12. **Proposition.** *There is a natural bijection:*

$$\mathrm{Maps}_{\mathbf{Lie}_{/S}(Z)}(\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L}),\mathcal{M}) \xrightarrow{\sim} \mathrm{Maps}_{\mathbf{Lie}^{(\mathfrak{k},H)}_{/S}(Y)}(\mathcal{L},\pi^!_{\mathbf{Lie}}\mathcal{M}) \tag{21.7}$$

Proof. A morphism $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L}) \to \mathcal{M}$ is equivalent to an H-equivariant map $\phi : \mathcal{L}/\mathfrak{k}\otimes\mathcal{O}_Y \to \pi^*\mathcal{M}$ preserving the Lie bracket on H-invariant sections. We *claim* that such datum is equivalent to a morphism $\widetilde{\phi} : \mathcal{L} \to \pi^!_{\mathbf{Lie}}\mathcal{M}$ of $(\mathfrak{k},H)$-Lie algebroids.

Indeed, given ϕ, the map $\widetilde{\phi}$ is uniquely determined by the properties that the following diagrams commute:

$$\begin{tikzcd} \mathcal{L} \arrow[r, "\widetilde{\phi}"] \arrow[d] & \pi^!_{\mathbf{Lie}}\mathcal{M} \arrow[d] \\ \mathcal{L}/\mathfrak{k}\otimes\mathcal{O}_Y \arrow[r, "\phi"] & \pi^*\mathcal{M} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L} \arrow[r, "\widetilde{\phi}"] \arrow[dr, "\sigma"'] & \pi^!_{\mathbf{Lie}}\mathcal{M} \arrow[d] \\ & \mathcal{T}_{Y/S}. \end{tikzcd}$$

Furthermore, $\widetilde{\phi}$ preserves the Lie bracket on $\mathcal{L}$, because $\mathcal{L}$ is generated over $\mathcal{O}_Y$ by H-invariant sections and on such sections, the Lie bracket factors through $\mathcal{L}/\mathfrak{k}\otimes\mathcal{O}_Y$ and is preserved by ϕ. Conversely, given $\widetilde{\phi}$, the map ϕ is uniquely determined by the first commutative diagram above. $\square$

21.1.13. Suppose we are given an exact sequence (21.2) of classical action pairs, and an object $(\mathcal{L},\eta) \in \mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}\,/S}(Y)$. Assume also that Y/H is representable by an algebraic stack. Note that:

(1) Y/H^0 admits an H_0-action, realizing it as an H_0-torsor over Y/H (in particular, Y/H^0 is also representable by an algebraic stack);

(2) there is an induced $(\mathfrak{k}_0, H_0)$-Lie algebroid structure on $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^0,H^0)}(\mathcal{L})$, for which the structure map

$$\eta_0 : \mathfrak{k}_0 \otimes \mathcal{O}_{Y/H^0} \to \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^0,H^0)}(\mathcal{L})$$

is again injective, i.e., $(\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^0,H^0)}(\mathcal{L}), \eta_0) \in \mathbf{Lie}_{\mathrm{inj}\,/S}^{(\mathfrak{k}_0,H_0)}(Y/H^0)$.

We have a version of the second isomorphism theorem:

21.1.14. **Proposition.** *There is a natural isomorphism:*

$$\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}_0,H_0)} \circ \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^0,H^0)}(\mathcal{L}) \xrightarrow{\sim} \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L}).$$

Proof. As $\mathcal{O}_{Y/H^0}$-modules, the cokernel of η_0 identifies with the descent of $\mathcal{L}/\mathfrak{k} \otimes \mathcal{O}_Y$ along $Y \to Y/H^0$ since the latter map is faithfully flat. Hence the underlying $\mathcal{O}_{Y/H}$-module of $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}_0,H_0)} \circ \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k}^0,H^0)}(\mathcal{L})$ agrees with that of $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L})$. Identifications of the anchor maps and the Lie brackets are immediate. $\square$

21.1.15. Let us recall the equivalence between classical pre-twistings and central extensions of Lie algebroids by $\mathcal{O}_Y$ (Chapter 4, Proposition 14.3.6).

Suppose we have such an extension, where both Lie algebroids $\widehat{\mathcal{L}}$ and $\mathcal{L}$ have the structure of $(\mathfrak{k}, H)$-algebroids, and $\widehat{\mathcal{L}} \to \mathcal{L}$ is a morphism of such. In particular, the structure map $\widehat{\eta} : \mathfrak{k} \otimes \mathcal{O}_Y \to \widehat{\mathcal{L}}$ is a lift of η. Hence, if $(\mathcal{L}, \eta) \in \mathbf{Lie}_{\mathrm{inj}\,/S}^{(\mathfrak{k},H)}(Y)$, then so does $(\widehat{\mathcal{L}}, \widehat{\eta})$. For fixed $(\mathcal{L}, \eta)$, we denote the category of classical pre-twistings with this additional structure by $\mathbf{PTw}_{/S}^{(\mathfrak{k},H)}(Y/\mathcal{L})$.

Assuming that $\mathcal{Z} := Y/H$ is represented by an algebraic stack. Then the quotient Lie algebroids again form a central extension:

$$0 \to \mathcal{O}_{Y/H} \to \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\widehat{\mathcal{L}}) \to \mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L}) \to 0.$$

Therefore, we may regard $\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}$ as a functor from $\mathbf{PTw}_{/S}^{(\mathfrak{k},H)}(Y/\mathcal{L})$ to $\mathbf{PTw}_{/S}(\mathcal{Z}/\mathbf{Q}_{\mathrm{inj}}^{(\mathfrak{k},H)}(\mathcal{L}))$.

21.1.16. **Remark.** When Y is a placid scheme and $\mathfrak{k}$ is a topological Lie algebra over $\mathcal{O}_S$ (c.f. Chapter 4, §19.6), we can adapt the above definitions to make sense of a Tate $(\mathfrak{k}, H)$-Lie algebroid $\mathcal{L}$ (c.f. §**??**). In particular, η will be a map out of the completed tensor product $\mathfrak{k}\widehat{\otimes}\mathcal{O}_Y \to \mathcal{L}$.

We do not discuss how to keep track of the topology in the (analogously defined) quotient $\mathbf{Q}^{(\mathfrak{k},H)}_{\text{inj}}(\mathcal{L})$, since all quotients considered in this paper have the properties that Y/H is locally of finite type and $\mathbf{Q}^{(\mathfrak{k},H)}_{\text{inj}}(\mathcal{L})$ should be discrete.

21.2. $(H, H^\flat)$-formal moduli problems.

21.2.1. We now study the geometric version of quotient of Lie algebroids. Recall the ∞-category $\mathbf{FMod}_{/S}$ of formal moduli problems in the category of prestacks over S (Chapter 4). We call a group object $(H, H^\flat)$ in $\mathbf{FMod}_{/S}$ a *geometric action pair* if H is a group *scheme* locally of finite type. Explicitly, a geometric action pair consists of a group scheme H, a group prestack $H^\flat \in \mathbf{PStk}^{\text{laft-def}}_{/S}$, and a nil-isomorphism $H \to H^\flat$ that is a group homomorphism.

21.2.2. *Association of action agents.* We will functorially construct a geometric action pair from any classical action pair $(\mathfrak{k}, H)$, where H is locally of finite type:

$$(\mathfrak{k}, H) \rightsquigarrow (H, H^\flat).$$

Indeed, there is a morphism $\exp(\mathfrak{k}) \to H$ coming from the composition $\exp(\mathfrak{k}) \to \exp(\mathfrak{h}) \to H$. Furthermore, the H-action on $\exp(\mathfrak{k})$ equips the prestack quotient $H^\flat := H/\exp(\mathfrak{k})$ with a group structure, such that $H \to H^\flat$ is a group morphism. Note that $H^\flat$ identifies with $\mathrm{B}^f(H \times \exp(\mathfrak{k})^\bullet)$ by formal smoothness of H, where B^f is the quotient functor of Chapter 4, §14.2; in particular, $H^\flat \in \mathbf{PStk}^{\text{laft-def}}_{/S}$, so $(H, H^\flat)$ is a geometric action pair.

21.2.3. **Lemma.** *The category of classical action pairs is the full subcategory of geometric action pairs $(H, H^\flat)$, for which $\mathbb{T}_{H/H^\flat}$ belongs to $\Upsilon_H(\mathrm{QCoh}(H)^\heartsuit)$.*

Proof. We explicitly construct the inverse functor. Given a geometric action pair $(H, H^\flat)$ for which $\mathbb{T}_{H/H^\flat} \in \Upsilon_H(\mathrm{QCoh}(H)^\heartsuit)$, we can functorially associate a classical Lie algebroid $\mathcal{L}$ over H. The following Cartesian diagrams:

$$\begin{tikzcd} H \underset{S}{\times} H \arrow[r] \arrow[d, "m"] & H^\flat \underset{S}{\times} H \arrow[d, "\mathrm{act}"] \\ H \arrow[r] & H^\flat \end{tikzcd} \qquad \begin{tikzcd} H \underset{S}{\times} H \arrow[r] \arrow[d, "m"] & H \underset{S}{\times} H^\flat \arrow[d, "\mathrm{act}"] \\ H \arrow[r] & H^\flat \end{tikzcd}$$

equip the underlying $\mathcal{O}_H$-module of $\mathcal{L}$ with right, respectively left, H-equivariance structures. Hence we may realize $\mathcal{L}$ as $\mathfrak{k} \otimes \mathcal{O}_H$ where $\mathfrak{k}$ is an $\mathcal{O}_S$-module equipped with an H-action. The Lie bracket on $\mathfrak{k}$ comes from the Lie algebroid bracket on $\mathcal{L}$. We omit checking that these data make $(\mathfrak{k}, H)$ into a classical action pair. $\square$

21.2.4. For a geometric action pair $(H, H^\flat)$, we define $\mathbf{FMod}_{/S}^{(H,H^\flat)}$ to be the ∞-category of objects in $\mathbf{FMod}_{/S}$ equipped with an $(H, H^\flat)$-action. Explicitly, an object of $\mathbf{FMod}_{/S}^{(H,H^\flat)}$ consists of the following data:

(1) $\mathcal{Y}, \mathcal{Y}^\flat \in \mathbf{PStk}_{/S}^{\text{laft-def}}$ together with a nil-isomorphism $\mathcal{Y} \to \mathcal{Y}^\flat$;
(2) an H-action on $\mathcal{Y}$, and an $H^\flat$-action on $\mathcal{Y}^\flat$, such that the morphism $\mathcal{Y} \to \mathcal{Y}^\flat$ intertwines them.

Note that there is a functor

$$\mathbf{FMod}_{/S}^{(H,H^\flat)} \to \mathbf{PStk}_{/S}^{\text{laft-def},H}, \quad (\mathcal{Y}, \mathcal{Y}^\flat) \rightsquigarrow \mathcal{Y} \tag{21.8}$$

where $\mathbf{PStk}_{/S}^{\text{laft-def},H}$ denotes the ∞-category of objects in $\mathbf{PStk}_{/S}^{\text{laft-def}}$ equipped with an H-action. The fiber of (21.8) at $\mathcal{Y}$ is denoted by $\mathbf{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y})$. Informally, $\mathbf{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y})$ is the ∞-category of formal moduli problems $\mathcal{Y}^\flat$ equipped with an $H^\flat$-action that extends the H-action on $\mathcal{Y}$.

21.2.5. *Association of action targets.* Suppose $(\mathfrak{k}, H)$ be a classical action pair and $(H, H^\flat)$ be its associated geometric action pair. Let $Y \in \mathbf{Sch}_{/S}^{\text{lft}}$ be acted on by H.

We will now construct a functor:

$$\mathbf{Lie}^{(\mathfrak{k},H)}_{/S}(Y) \to \mathbf{FMod}^{(H,H^\flat)}_{/S}(Y) \tag{21.9}$$

which lifts the association of formal moduli problems to Lie algebroids, in the sense that the following diagram commutes:

$$\begin{tikzcd} \mathbf{Lie}^{(\mathfrak{k},H)}_{/S}(Y) \arrow[r, "(21.9)"] \arrow[d, "\mathrm{oblv}"] & \mathbf{FMod}^{(H,H^\flat)}_{/S}(Y) \arrow[d, "\mathrm{oblv}"] \\ \mathbf{Lie}_{/S}(Y) \arrow[r] & \mathbf{FMod}_{/S}(Y), \end{tikzcd}$$

where the bottom functor is described in Chapter 4, §14.2.

To proceed, let us be given $(\mathcal{L}, \eta) \in \mathbf{Lie}^{(\mathfrak{k},H)}_{/S}(Y)$. We need to construct an $H^\flat$-action on the formal moduli problem $\mathcal{Y}^\flat$ corresponding to $\mathcal{L}$, expressed by some groupoid $\mathcal{Y}^\flat \underset{S}{\times} H^\flat \overset{\mathrm{act}^\flat}{\underset{\mathrm{pr}_1}{\rightrightarrows}} \mathcal{Y}^\flat$, together with a map of simplicial prestacks:

$$\begin{tikzcd} \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & Y \underset{S}{\times} H \underset{S}{\times} H \arrow[r, shift left=2, "\mathrm{act}\times 1"] \arrow[r, "1\times m" description] \arrow[r, shift right=2, "\mathrm{pr}_{12}"'] \arrow[d] & Y \underset{S}{\times} H \arrow[r, shift left, "\mathrm{act}"] \arrow[r, shift right, "\mathrm{pr}_1"'] \arrow[d] & Y \arrow[d] \\ \cdots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & \mathcal{Y}^\flat \underset{S}{\times} H^\flat \underset{S}{\times} H^\flat \arrow[r, shift left=2, "\mathrm{act}^\flat\times 1"] \arrow[r, "1\times m" description] \arrow[r, shift right=2, "\mathrm{pr}_{12}"'] & \mathcal{Y}^\flat \underset{S}{\times} H^\flat \arrow[r, shift left, "\mathrm{act}^\flat"] \arrow[r, shift right, "\mathrm{pr}_1"'] & \mathcal{Y}^\flat. \end{tikzcd} \tag{21.10}$$

Since each formal moduli problem $\mathcal{Y}^\flat \underset{S}{\times} (H^\flat)^\bullet$ arises from the Lie algebroid $\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)^{\oplus \bullet}$ over $Y \underset{S}{\times} H^\bullet$, we only need to

(1) produce a morphism

$$\alpha : \mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H) \to \mathrm{act}^!_{\mathbf{Lie}} \mathcal{L} \tag{21.11}$$

between Lie algebroids over $Y \underset{S}{\times} H$ (which would rise to $\mathrm{act}^\flat$, in a way compatible with the morphism act), and

(2) check that the following diagram:

(21.12)
$$\begin{tikzcd}
\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)^{\oplus 2} \arrow[r, "\mathrm{can}"] \arrow[d, "\mathrm{act}^!_{\mathbf{Lie}}(\alpha)\times 1"] & (1 \times m)^!_{\mathbf{Lie}}(\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)) \arrow[d, "(1\times m)^*_{\mathbf{Lie}}(\alpha)"] \\
\mathrm{act}^!_{\mathbf{Lie}}(\mathcal{L}) \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H) \arrow[d, "\cong"] & (1 \times m)^!_{\mathbf{Lie}} \mathrm{act}^!_{\mathbf{Lie}}(\mathcal{L}) \arrow[d, "\cong"] \\
(\mathrm{act} \times 1)^!_{\mathbf{Lie}}(\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)) \arrow[r, "(\mathrm{act}\times 1)^!_{\mathbf{Lie}}(\alpha)"] & (\mathrm{act} \times 1)^!_{\mathbf{Lie}} \mathrm{act}^!_{\mathbf{Lie}}(\mathcal{L})
\end{tikzcd}$$

of Lie algebroids over $Y \underset{S}{\times} H \underset{S}{\times} H$ is commutative (which would affirm the commutativity of (21.10) up to 2-simplices, but the higher commutativity constraints are satisfied automatically since the corresponding ∞-categories are classical.)

Note that as an $\mathcal{O}_{Y \underset{S}{\times} H}$-module, we have an isomorphism $\mathrm{act}^!_{\mathbf{Lie}}(\mathcal{L}) \xrightarrow{\sim} \mathrm{act}^* \mathcal{L} \underset{\mathrm{act}^* \mathcal{T}_{Y/S}}{\times} \mathcal{T}_{Y \underset{S}{\times} H/S}$ by definition. The required map α is the sum of the following components:

(1) the map $\mathrm{pr}_Y^* \mathcal{L} \to \mathrm{act}^!_{\mathbf{Lie}}(\mathcal{L})$ induced from the H-equivariance structure on $\mathcal{L}$ and the composition

$$\mathrm{pr}_Y^* \mathcal{L} \xrightarrow{\mathrm{pr}_Y^* \sigma} \mathrm{pr}_Y^* \mathcal{T}_{Y/S} \hookrightarrow \mathcal{T}_{Y \underset{S}{\times} H/S},$$

where σ is the anchor map of $\mathcal{L}$;

(2) the map $\mathfrak{k} \otimes \mathcal{O}_H \to \mathrm{act}^!_{\mathbf{Lie}}(\mathcal{L})$ induced from

$$\mathfrak{k} \xrightarrow{\eta} \mathrm{H}^0(Y, \mathcal{L}) \xrightarrow{\mathrm{act}^*} \mathrm{H}^0(Y \underset{S}{\times} H, \mathrm{act}^* \mathcal{L}),$$

and the composition

$$\mathfrak{k} \otimes \mathcal{O}_H \to \mathfrak{h} \otimes \mathcal{O}_H \hookrightarrow \mathcal{T}_{Y \underset{S}{\times} H/S}. \tag{21.13}$$

The following Lemma shows that the functor (21.9) is well-defined.

21.2.6. **Lemma.** *The map α is a morphism of Lie algebroids, and the diagram* (21.12) *commutes.*

Proof. It is obvious that α is compatible with the anchor maps. To show that α preserves the Lie bracket, we check it for sections of $\mathrm{pr}_Y^* \mathcal{L} \oplus \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H)$ of the following types:

(1) $l_1, l_2 \in \mathrm{pr}_Y^{-1} \mathcal{L}$; this follows from the assumptions that the equivariance structure $\theta : \mathrm{pr}_Y^* \mathcal{L} \to \mathrm{act}^* \mathcal{L}$ is compatible with the Lie bracket, and σ is a map of H-equivariant sheaves;

(2) $\xi_1, \xi_2 \in \mathfrak{k}$; this is clear;

(3) $l \in \mathrm{pr}_Y^{-1} \mathcal{L}$ and $\xi \in \mathfrak{k}$; this is a slightly more involved calculation, which we now perform.

Write $\theta(l) = \sum_i f_i \otimes l_i$, where $f_i \in \mathcal{O}_{Y \times H}$ and $l_i \in \mathrm{act}^{-1} \mathcal{L}$. We need to show the vanishing of the following element in $\mathrm{act}^* \mathcal{L} \underset{\mathrm{act}^* \mathcal{T}_{Y/S}}{\times} \mathcal{T}_{Y \underset{S}{\times} H/S}$:

$$[\alpha(l), \alpha(\xi)] = [\sum_i (f_i \otimes l_i) \times \sigma(l), (1 \otimes \eta(\xi)) \times \sigma'(\xi)] \tag{21.14}$$

where σ' denotes the composition (21.13). Note that the $\mathcal{T}_{Y \underset{S}{\times} H/S}$-component of (21.14) vanishes tautologically, so we just need to show the vanishing of its $\mathrm{act}^* \mathcal{L}$-component. The latter is given (using (21.5)) by

$$\sum_i f_i \otimes [l_i, \eta(\xi)] - \sum_i \sigma'(\xi)(f_i) \otimes l_i = -\sum_i (f_i \otimes (\xi_{\mathfrak{h}} \cdot l_i) + (\xi_{\mathfrak{h}} \cdot f_i) \otimes l_i) \tag{21.15}$$

where in the second summand, $\xi_{\mathfrak{h}}$ acts on $f_i \in \mathcal{O}_{Y \underset{S}{\times} H/S}$ by derivation on the $\mathcal{O}_H$-*component.*

Consider the right H-action on $Y \underset{S}{\times} H$, given by $(y, h), h' \rightsquigarrow (y, hh')$; if we equip $\mathrm{act}^* \mathcal{L}$ with the following H-equivariance structure:

$$\mathrm{act}^* \mathcal{L}\big|_{(y,h)} \xrightarrow{\sim} \mathcal{L}\big|_{yh} \xrightarrow{\theta_{(yh,h')}} \mathcal{L}\big|_{yhh'} \xrightarrow{\sim} \mathrm{act}^* \mathcal{L}\big|_{(y,hh')},$$

then (21.15) is the (negative of the) induced action of $\xi_{\mathfrak{h}}$ on the section $\sum_i f_i \otimes l_i = \theta(l)$ in $\mathrm{act}^* \mathcal{L}$. Note that $\mathrm{pr}_Y^* \mathcal{L}$ can also be endowed with an H-equivariance structure:

$$\mathrm{pr}^* \mathcal{L}\big|_{(y,h)} \xrightarrow{\sim} \mathcal{L}\big|_y \xrightarrow{\sim} \mathrm{pr}^* \mathcal{L}\big|_{(y,hh')}$$

such that θ is a map of H-equivariant $\mathcal{O}_{Y \underset{S}{\times} H}$-modules. Hence the element $\xi_{\mathfrak{h}} \cdot \theta(l)$ identifies with $\theta(\xi_{\mathfrak{h}} \cdot l)$. On the other hand, $l \in \mathrm{pr}^{-1} \mathcal{L}$ so $\xi_{\mathfrak{h}} \cdot l = 0$, from which we deduce the required vanishing of (21.15).

Checking the commutativity of (21.12) is not difficult, and we leave it to the reader. □

We now characterize the image of the functor (21.9).

21.2.7. **Proposition.** *The functor* (21.9) *is an equivalence onto the full subcategory:*

$$\mathbf{FMod}_{/S}^{(H,H^\flat)}(Y)^{\mathrm{cl}} \hookrightarrow \mathbf{FMod}_{/S}^{(H,H^\flat)}(Y)$$

that consists of objects $\mathcal{Y}^\flat$ such that $\mathbb{T}_{Y/\mathcal{Y}^\flat}$ lies in $\Upsilon_Y(\mathrm{QCoh}(Y)^\heartsuit)$.

Proof. Indeed, such a formal moduli problems $\mathcal{Y}^\flat$ arises from some Lie algebroid $\mathcal{L}$. Given the additional data of an $(H, H^\flat)$-action, we consider the following commutative diagrams:

$$(21.16)\qquad \begin{tikzcd} Y \underset{S}{\times} H \arrow[rr, "\mathrm{act}"] \arrow[d] & & Y \arrow[d] \\ \mathcal{Y}^\flat \underset{S}{\times} H \arrow[r, "i"] & \mathcal{Y}^\flat \underset{S}{\times} H^\flat \arrow[r, "\mathrm{act}^\flat"] & \mathcal{Y}^\flat \end{tikzcd} \qquad \begin{tikzcd} Y \underset{S}{\times} H \arrow[rr, "\mathrm{act}"] \arrow[d] & & Y \arrow[d] \\ Y \underset{S}{\times} H^\flat \arrow[r, "j"] & \mathcal{Y}^\flat \underset{S}{\times} H^\flat \arrow[r, "\mathrm{act}^\flat"] & \mathcal{Y}^\flat \end{tikzcd}$$

From these diagrams, we obtain two maps between tangent complexes:

$$\mathbb{T}_{Y \underset{S}{\times} H/\mathcal{Y}^\flat \underset{S}{\times} H} \xrightarrow{\mathrm{act}^\flat_* \circ i_*} \mathbb{T}_{Y \underset{S}{\times} H/\mathcal{Y}^\flat} \to \mathbb{T}_{Y/\mathcal{Y}^\flat}\big|_{Y \underset{S}{\times} H},$$

which gives rise to a morphism $\theta : \mathrm{pr}_Y^* \mathcal{L} \to \mathrm{act}^* \mathcal{L}$; and

$$\mathbb{T}_{Y \underset{S}{\times} H / Y \underset{S}{\times} H^\flat} \xrightarrow{\mathrm{act}_*^\flat \circ j_*} \mathbb{T}_{Y \underset{S}{\times} H / \mathcal{Y}^\flat} \to \mathbb{T}_{Y/\mathcal{Y}^\flat}\big|_{Y \underset{S}{\times} H}, \tag{21.17}$$

which gives rise to a map $\widetilde{\eta} : \mathrm{pr}_H^*(\mathfrak{k} \otimes \mathcal{O}_H) \to \mathrm{act}^* \mathcal{L}$; restricting to $Y \underset{S}{\times} \{1\}$, we obtain a map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{L}$.

The functor $\mathbf{FMod}_{/S}^{(H,H^\flat)}(Y)^{\mathrm{cl}} \to \mathbf{Lie}_{/S}^{(\mathfrak{k},H)}(Y)$ inverse to (21.9) is defined by sending $\mathcal{Y}^\flat$ to the Lie algebroid $\mathcal{L}$, equipped with the $(\mathfrak{k}, H)$-structure specified by the above maps θ and η. □

21.2.8. We give an alternative description of the map α that will be convenient later. Consider the commutative diagram:

$$\begin{tikzcd} Y \arrow[rr, "\mathrm{can}"] \arrow[d] & & Y/H \arrow[d] \\ Y \underset{S}{\times} (H^\flat/H) \arrow[r, "\widetilde{j}"] & \mathcal{Y}^\flat \underset{S}{\times} (H^\flat/H) \arrow[r, "\widetilde{\mathrm{act}}^\flat"] & \mathcal{Y}^\flat/H \end{tikzcd} \tag{21.18}$$

which is the "quotient" by H of the right diagram in (21.16). It produces the following map between tangent complexes:

$$\mathbb{T}_{Y/(Y \underset{S}{\times} (H^\flat/H))} \xrightarrow{\widetilde{\mathrm{act}}_*^\flat \circ \widetilde{j}_*} \mathbb{T}_{Y/(\mathcal{Y}^\flat/H)} \to \mathbb{T}_{(Y/H)/(\mathcal{Y}^\flat/H)}\big|_Y \xrightarrow{\sim} \mathbb{T}_{Y/\mathcal{Y}^\flat}. \tag{21.19}$$

We *claim* that (21.19) identifies with the restriction of (21.17) to $Y \underset{S}{\times} \{1\}$. Indeed, this follows from the fact that (21.17) is the pullback of (21.19) along $\mathrm{pr}_Y : Y \underset{S}{\times} H \to Y$, and the composition $Y \underset{S}{\times} \{1\} \hookrightarrow Y \underset{S}{\times} H \xrightarrow{\mathrm{pr}_Y} Y$ is the identity.

21.3. **Quotient II.**

21.3.1. Suppose $(H, H^\flat)$ is a geometric action pair. Let $(\mathcal{Y}, \mathcal{Y}^\flat) \in \mathbf{FMod}_{/S}^{(H,H^\flat)}$. The *quotient* of $(\mathcal{Y}, \mathcal{Y}^\flat)$ by $(H, H^\flat)$ is defined as the quotient in the ∞-category $\mathbf{FMod}_{/S}$.

In other words, it is the geometric realization of the simplicial object $(\mathcal{Y}, \mathcal{Y}^\flat) \times (H, H^\flat)^\bullet$ in $\mathbf{FMod}_{/S}^{(H,H^\flat)}$ characterizing the $(H, H^\flat)$-action on $(\mathcal{Y}, \mathcal{Y}^\flat)$.

21.3.2. **Proposition.** *The quotient of* $(\mathcal{Y}, \mathcal{Y}^\flat)$ *by* $(H, H^\flat)$ *exists.*

Proof. We construct the quotient in the ∞-category $\mathrm{Fun}(\Delta^1, \mathbf{PStk}_{/S}^{\text{laft-def}})$, and then check that the result belongs to the full subcategory $\mathbf{FMod}_{/S}$. Quotient in the above functor category is computed pointwise as follows:

(1) at the vertex [0], we have the prestack quotient $\mathcal{Y}/H$; it is an object of $\mathbf{PStk}_{/S}^{\text{laft-def}}$ because H is a group *scheme* locally of finite type;

(2) at the vertex [1], we assert that the quotient of $\mathcal{Y}^\flat$ by $H^\flat$ exists in $\mathbf{PStk}_{/S}^{\text{laft-def}}$; indeed, it is given by $\mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H)$ where $\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H$ denotes the Hecke groupoid[14] acting on the prestack quotient $\mathcal{Y}^\flat/H$:

$$\cdots \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow}\; \mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat \overset{H}{\underset{S}{\times}} H^\flat/H \;\underset{\mathrm{pr}_{12}}{\overset{\mathrm{act}^\flat\times 1}{\substack{\longrightarrow\\ \xrightarrow{1\times m}\\ \longrightarrow}}}\; \mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H \;\underset{\mathrm{pr}_1}{\overset{\mathrm{act}^\flat}{\rightrightarrows}}\; \mathcal{Y}^\flat/H.$$

Finally, the morphism $\mathcal{Y}/H \to \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H)$ is a nil-isomorphism since it is the composition of nil-isomorphisms $\mathcal{Y}/H \to \mathcal{Y}^\flat/H \to \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H)$. $\square$

Regarding $\mathcal{Y}$ as a fixed prestack acted on by H, we obtain a quotient functor:

$$\mathbf{Q}^{(H,H^\flat)} : \mathbf{FMod}_{/S}^{(H,H^\flat)}(\mathcal{Y}) \to \mathbf{FMod}_{/S}(\mathcal{Y}/H), \quad \mathcal{Y}^\flat \rightsquigarrow \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H). \tag{21.20}$$

21.3.3. Tautologically, the quotient $(\mathcal{Y}/H, \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H))$, equipped with the map from $(\mathcal{Y}, \mathcal{Y}^\flat)$, satisfies the universal property:

$$\mathrm{Maps}_{\mathbf{FMod}_{/S}}((\mathcal{Y}/H, \mathrm{B}_{\mathcal{Y}^\flat/H}(\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H)), (\mathcal{Z}, \mathcal{Z}^\flat)) \xrightarrow{\sim} \mathrm{Maps}_{\mathbf{FMod}_{/S}^{(H,H^\flat)}}((\mathcal{Y}, \mathcal{Y}^\flat), (\mathcal{Z}, \mathcal{Z}^\flat))$$

[14] Suppose $\mathcal{C}$ is an ∞-category with finite products. Let $H \to K$ be a map of group objects in $\mathcal{C}$. Suppose any object in $\mathcal{C}$ with an H-action admits a quotient. Then given an object $Y \in \mathcal{C}$ with a K-action, there exists a *Hecke groupoid* $Y \overset{H}{\times} K/H$ acting on Y/H whose quotient, if exists, agrees with Y/K.

where in the second expression, $(\mathcal{Z}, \mathcal{Z}^\flat)$ is equipped with the trivial $(H, H^\flat)$-action.

Specializing to $\mathcal{Z} = \mathcal{Y}/H$, we see that the object $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat) \in \mathbf{FMod}_{/S}(\mathcal{Y}/H)$ is characterized by the universal property:

$$\operatorname{Maps}_{\mathbf{FMod}_{/S}(\mathcal{Y}/H)}(\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat), \mathcal{Z}^\flat) \xrightarrow{\sim} \operatorname{Maps}_{\mathbf{FMod}^{(H,H^\flat)}_{/S}(\mathcal{Y})}(\mathcal{Y}^\flat, \pi^!_{\mathbf{FMod}}(\mathcal{Z}^\flat)) \tag{21.21}$$

where in the second expression, $\pi^!_{\mathbf{FMod}}\mathcal{Z}^\flat \cong \mathcal{Z}^\flat \underset{(\mathcal{Y}/H)_{\mathrm{dR}}}{\times} \mathcal{Y}_{\mathrm{dR}}$ is acted on by $H^\flat$ through the canonical homomorphism $H^\flat \to H_{\mathrm{dR}}$ on the $\mathcal{Y}_{\mathrm{dR}}$ factor.

21.3.4. **Remark.** Recall the $(\mathfrak{k}, H)$-Lie algebroid structure on $\pi^!_{\mathbf{Lie}}(\mathcal{M})$, where $(\mathfrak{k}, H)$ is any classical action pair and $\mathcal{M}$ is a Lie algebroid on the quotient Y/H. If $H^\flat = H/\exp(\mathfrak{k})$, then the $(H, H^\flat)$-formal moduli problem $\pi^!_{\mathbf{FMod}}(\mathcal{Z}^\flat)$ is precisely the one associated to $\pi^!_{\mathbf{Lie}}(\mathcal{M})$ under the functor (21.9).

21.3.5. Let $(H^0, (H^0)^\flat) \to (H, H^\flat)$ be a morphism of geometric action pairs. We say that $(H^0, (H^0)^\flat)$ is a *normal subpair* of $(H, H^\flat)$ if there is a morphism $(H, H^\flat) \to (H_0, (H_0)^\flat)$ of geometric action pairs whose kernel identifies with $(H^0, (H^0)^\flat)$. In particular, the $(H, H^\flat)$-action on itself extends to $(H^0, (H^0)^\flat)$.

Given a normal subpair $(H^0, (H^0)^\flat)$ of $(H, H^\flat)$, we recover $(H_0, (H_0)^\flat)$ by the isomorphisms:

$$H_0 \xrightarrow{\sim} H/H^0, \quad H_0^\flat \xrightarrow{\sim} \mathbf{Q}^{(H^0,(H^0)^\flat)}(H^\flat).$$

Let $\mathcal{Y}^\flat \in \mathbf{FMod}^{(H,H^\flat)}_{/S}(\mathcal{Y})$. Then the quotient prestack $\mathbf{Q}^{(H^0,(H^0)^\flat)}(\mathcal{Y}^\flat)$ is naturally an object of $\mathbf{FMod}^{(H_0,H_0^\flat)}_{/S}(\mathcal{Y}/H^0)$, and we have a second isomorphism theorem:

21.3.6. **Proposition.** *There is a natural isomorphism:*

$$\mathbf{Q}^{(H_0,H_0^\flat)} \circ \mathbf{Q}^{(H^0,(H^0)^\flat)}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat).$$

Proof. Both sides are the quotient of $(\mathcal{Y}, \mathcal{Y}^\flat)$ by $(H, H^\flat)$ in the ∞-category $\mathbf{FMod}_{/S}$. $\square$

21.3.7. Suppose we have a pre-twisting $\widehat{\mathcal{Y}}^\flat \in \mathbf{PTw}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$, such that $(\mathcal{Y}, \widehat{\mathcal{Y}}^\flat)$ is also an $(H, H^\flat)$-formal moduli problem, and the morphism $\widehat{\mathcal{Y}}^\flat \to \mathcal{Y}^\flat$ preserves this structure. We call pre-twistings with these additional data $(H, H^\flat)$*-pre-twistings* (based at $\mathcal{Y}^\flat$) and denote the category of them by $\mathbf{PTw}^{(H,H^\flat)}_{/S}(\mathcal{Y}/\mathcal{Y}^\flat)$. The quotient $\mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat)$ inherits the structure of a pre-twisting on $\mathcal{Y}/H$ based at $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$. Indeed,

(1) applying $\mathbf{Q}^{(H,H^\flat)}$ to the action groupoid $\widehat{\mathcal{Y}}^\flat \times \mathrm{B}\,\widehat{\mathbb{G}}^\bullet_m$, we obtain a $\mathrm{B}\,\widehat{\mathbb{G}}_m$-action on $\mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat)$, which gives rise to a $\widehat{\mathbb{G}}_m$-gerbe structure;

(2) the section $\mathcal{Y}/H \to \mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat)$ is given by the composition:

$$\mathcal{Y}/H \to \widehat{\mathcal{Y}}^\flat/H \to \mathbf{Q}^{(H,H^\flat)}(\widehat{\mathcal{Y}}^\flat).$$

Therefore, we may view $\mathbf{Q}^{(H,H^\flat)}$ as a functor:

$$\mathbf{PTw}^{(H,H^\flat)}_{/\mathcal{Y}^\flat}(Y/S) \to \mathbf{PTw}_{/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}((Y/H)/S).$$

21.4. Comparison of $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}$ and $\mathbf{Q}^{(H,H^\flat)}$.

21.4.1. Suppose $(\mathfrak{k}, H)$ is a classical action pair and $(H, H^\flat)$ is its associated geometric action pair, and let $Y \in \mathbf{Sch}^{\mathrm{lft}}_{/S}$ be acted on by H. We now show that the two quotient functors constructed above are compatible.

21.4.2. **Proposition.** *The following diagram is commutative:*

$$\begin{tikzcd}
\mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}\,/S}(Y) \arrow[r, hook, "(21.9)"] \arrow[d, "\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}"] & \mathbf{FMod}^{(H,H^\flat)}_{/S}(Y) \arrow[d, "\mathbf{Q}^{(H,H^\flat)}"] \\
\mathbf{Lie}_{/S}(Y/H) \arrow[r, hook] & \mathbf{FMod}_{/S}(Y/H).
\end{tikzcd}$$

Proof. Suppose $(\mathcal{L}, \eta) \in \mathbf{Lie}^{(\mathfrak{k},H)}_{\mathrm{inj}\,/S}(Y)$, i.e., $\mathcal{L}$ is a $(\mathfrak{k}, H)$-Lie algebroid over Y such that the map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{L}$ is injective. Let $\mathcal{Y}^\flat$ be the corresponding formal moduli problem under Y, equipped with the $H^\flat$-action defined by the functor (21.9). Thus $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ satisfies the universal property (21.21) for $\mathcal{Z}^\flat \in \mathbf{FMod}_{/S}(\mathcal{Y}/H)$.

On the other hand, $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{L})$ satisfies the universal property (21.7). Since the essential image of the bottom functor consists of objects $\mathcal{Z}^\flat \in \mathbf{FMod}_{/S}(Y/H)$ such that $\mathbb{T}_{(Y/H)/\mathcal{Z}^\flat} \in \Upsilon_{Y/H}(\mathrm{QCoh}(Y/H)^\heartsuit)$ (Chapter 4, §14.2), it suffices to show that $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ has this property. The result thus follows from the lemma below and the fact that $Y \to Y/H$ is faithfully flat. $\square$

21.4.3. **Lemma.** *Suppose* $(Y, \mathcal{Y}^\flat)$ *is the* $(H, H^\flat)$*-formal moduli problem corresponding to the* $(\mathfrak{k}, H)$*-Lie algebroid* $(\mathcal{L}, \eta)$ *under the functor* (21.9)*. Then there is a canonical isomorphism*

$$\mathcal{T}_{(Y/H)/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}\big|_Y \xrightarrow{\sim} \mathrm{Cofib}(\eta).$$

Proof. We will use the expression of $\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)$ as quotient of the Hecke groupoid $\mathcal{Y}^\flat \overset{H}{\underset{S}{\times}} H^\flat/H$ (see (21.20)). Consider the following commutative diagram extending the diagram (21.18):

$$\begin{tikzcd}
Y \arrow[rr, dotted] \arrow[d, "\mathrm{id}\times\{1\}"'] \arrow[ddr, dotted] & & Y/H \arrow[d, dotted] \\
Y \underset{S}{\times} H^\flat/H \arrow[r, "\widetilde{j}"] \arrow[d, "\mathrm{pr}"'] & \mathcal{Y}^\flat \underset{S}{\times} H^\flat/H \arrow[r, "\widetilde{\mathrm{act}}^\flat"] \arrow[d, "\mathrm{pr}"'] & \mathcal{Y}^\flat/H \arrow[d] \\
Y \arrow[r] & \mathcal{Y}^\flat \arrow[r] \arrow[ur, dotted] & \mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)
\end{tikzcd}$$

where the two lower squares, as well as the dotted quadrilateral, are Cartesian. Thus, we obtain the following commutative diagram of objects in $\mathrm{QCoh}(Y)$, where commutativity of the red (resp. blue) squares is derived from the red (resp. blue) arrows in the above diagram[15]:

$$\begin{tikzcd}
\mathcal{T}_{(Y \underset{S}{\times} H^\flat/H)/Y}\big|_Y[-1] \arrow[r, "\sim"] \arrow[d, "\sim"] & \mathcal{T}_{(\mathcal{Y}^\flat/H)/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}\big|_Y[-1] \arrow[r] \arrow[d] & \mathcal{T}_{(Y/H)/(\mathcal{Y}^\flat/H)}\big|_Y \arrow[r] \arrow[d, "\sim"] & \mathcal{T}_{(Y/H)/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}\big|_Y \arrow[d] \\
\mathcal{T}_{Y/(Y \underset{S}{\times} H^\flat/H)} \arrow[r, "\widetilde{\mathrm{act}}^\flat_* \circ \widetilde{j}_*"] \arrow[drr, bend right=10, "(21.19)"'] & \mathcal{T}_{Y/(\mathcal{Y}^\flat/H)} \arrow[r] & \mathcal{T}_{(Y/H)/(\mathcal{Y}^\flat/H)}\big|_Y \arrow[r] \arrow[d, "\sim"] & \mathcal{T}_{Y/(Y/H)}[1] \\
& & \mathcal{T}_{Y/\mathcal{Y}^\flat} &
\end{tikzcd}$$

Furthermore, the two horizontal red triangles are exact. Note that the composition (21.19) identifies with η, so the upper horizontal triangle identifies $\mathcal{T}_{(Y/H)/\mathbf{Q}^{(H,H^\flat)}(\mathcal{Y}^\flat)}\big|_Y$ with $\mathrm{Cofib}(\eta)$. $\square$

21.5. **Inert pre-twistings.**

21.5.1. We now specialize to Lie algebroids arising from abelian Lie algebras. They give rise to what we call "inert pre-twistings." In the geometric Langlands theory, they arise naturally as degeneration of (non-inert) pre-twistings as the quantum parameter κ tends to ∞.

21.5.2. Recall that over any $\mathcal{Y} \in \mathbf{PStk}^{\text{laft-def}}_{/S}$, there is a functor

$$\mathrm{triv} : \mathrm{IndCoh}(\mathcal{Y}) \to \mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y}))$$

that associates to an ind-coherent sheaf $\mathcal{F}$ the abelian Lie algebra on $\mathcal{F}$, seen as a Lie algebra object in the DG category $\mathrm{IndCoh}(\mathcal{Y})$. More precisely, triv is the right inverse to the forgetful functor; because the latter is conservative and preserves limits, triv also preserves limits.

We also have a pair of adjoint functors:

$$\mathrm{diag}_{\mathcal{Y}} : \mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y})) \rightleftarrows \mathbf{FMod}_{/S}(\mathcal{Y}) : \text{ker-anch}$$

[15]Recall: purple = red + blue.

where $\mathrm{diag}_{\mathcal{Y}}$ preserves fiber products.[16] It follows that the composition $\mathrm{diag}_{\mathcal{Y}} \circ \mathrm{triv}$ preserves fiber products. We call $\mathcal{Y}^\flat := \mathrm{diag}_{\mathcal{Y}} \circ \mathrm{triv}(\mathcal{F})$ the *inert* formal moduli problem on $\mathcal{F}$.

21.5.3. **Remark.** Let Y be a scheme (not necessarily locally of finite type) over S. The classical analogue of the above construction associates to an $\mathcal{O}_Y$-module $\mathcal{F}$ the Lie algebroid on $\mathcal{F}$ with *zero* Lie bracket and anchor map. If $Y \in \mathbf{Sch}^{\mathrm{lft}}_{/S}$, then the formal moduli problem associated to $\mathcal{F}$ agrees with $\mathrm{diag}_{\mathcal{Y}} \circ \mathrm{triv}(\Upsilon_Y(\mathcal{F}))$.

21.5.4. For the remainder of this section, we suppose $Y \in \mathbf{Sch}^{\mathrm{lft}}_{/S}$ is *smooth*. Then the identification $\Upsilon_Y : \mathrm{QCoh}(Y) \xrightarrow{\sim} \mathrm{IndCoh}(Y)$ allows us to view the universal enveloping algebra[17] of an object $\mathcal{Y}^\flat \in \mathbf{FMod}_{/S}(Y)$ as an algebra in $\mathrm{QCoh}(Y)$. If $\mathcal{Y}^\flat = \mathrm{diag}_Y \circ \mathrm{triv}(\Upsilon_Y(\mathcal{F}))$, then it is given by $\mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F})$.

Let $\mathbb{V}(\mathcal{F}) := \underline{\mathrm{Spec}}_Y \, \mathrm{Sym}_{\mathcal{O}_Y}(\mathcal{F})$; it is a stack over Y fibered in linear DG schemes. We have an equivalence of DG categories:

$$\mathrm{IndCoh}(\mathcal{Y}^\flat) \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F})), \tag{21.22}$$

where $\mathbf{oblv} : \mathrm{IndCoh}(\mathcal{Y}^\flat) \to \mathrm{IndCoh}(Y)$ passes to the pushforward functor on QCoh (see [36, VIII.4 §4.1.3, VIII.2 (7.12), and VIII.3 Proposition 5.1.2]).

21.5.5. Suppose, furthermore, that we have a pre-twisting $\widehat{\mathcal{Y}}^\flat \in \mathbf{PTw}_{/S}(Y/\mathcal{Y}^\flat)$ that arises from a triangle $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$ in $\mathrm{QCoh}(Y)$ under the composition of functors $\mathrm{diag}_Y \circ \mathrm{triv} \circ \Upsilon_Y$. We call $\widehat{\mathcal{Y}}^\flat$ the *inert pre-twisting* on the triangle $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$.

The map $\mathcal{O}_Y \to \widehat{\mathcal{F}}$ gives rise to a morphism of DG schemes:

$$\underline{\mathrm{Spec}}_Y \, \mathrm{Sym}_{\mathcal{O}_Y}(\widehat{\mathcal{F}}) \to Y \times \mathbb{A}^1 \tag{21.23}$$

[16] One sees this by identifying $\mathrm{Lie}(\mathrm{IndCoh}(\mathcal{Y}))$ with $\mathbf{FMod}(\mathcal{Y})_{/\mathcal{Y}}$, where $\mathcal{Y}$ is regarded as a formal moduli problem under itself by the identity map. Under this identification, $\mathrm{diag}_{\mathcal{Y}}$ becomes the tautological forgetful functor; see [36, VIII.4].

[17] This is defined as a monad on $\mathrm{IndCoh}(Y)$ in [36, VIII.4.4].

We let $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1}$ be the fiber of (21.23) at $\{1\} \hookrightarrow \mathbb{A}^1$. Note that the analogously defined fiber $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=0}$ identifies with $\mathbb{V}(\mathcal{F})$. There is a canonical equivalence of DG categories:

$$\widehat{\mathcal{Y}}^{\flat}\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1}). \tag{21.24}$$

21.5.6. We now discuss how quotient interacts with inert pre-twistings. Denote by pt the S-scheme S itself. Suppose $(\mathfrak{k}, H)$ is a classical action pair with *zero* map $\mathfrak{k} \to \mathfrak{h}$. Then we have

$$H^{\flat} := H/\exp(\mathfrak{k}) \xrightarrow{\sim} H \ltimes \mathrm{B}\exp(\mathfrak{k}),$$

where the formation of the semidirect product is formed by the H-action on $\mathrm{pt}/\exp(\mathfrak{k})$. Note that the normal subpair $(\mathrm{pt}, \mathrm{pt}/\exp(\mathfrak{k}))$ of $(H, H^{\flat})$ has quotient (H, H), since

$$\mathbf{Q}^{(\mathrm{pt}, \mathrm{B}\exp(\mathfrak{k}))}(H^{\flat}) \xrightarrow{\sim} \mathrm{B}_{H^{\flat}}(H^{\flat} \times (\mathrm{B}\exp(\mathfrak{k}))^{\bullet}) \xrightarrow{\sim} H.$$

21.5.7. We now assume that $\mathfrak{k}$ is also abelian. Suppose the smooth scheme Y admits an H-action, and $\mathcal{Y}^{\flat}$ is the inert formal moduli problem on some H-equivariant sheaf $\mathcal{F} \in \mathrm{QCoh}(Y)^{\heartsuit}$. Suppose we have an H-equivariant map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{F}$, giving rise to an $H^{\flat}$-action on $\mathcal{Y}^{\flat}$. Let $\mathcal{Q} := \mathrm{Cofib}(\eta)$; it is an H-equivariant complex of $\mathcal{O}_Y$-modules, hence descends to an object $\mathcal{Q}^{\mathrm{desc}} \in \mathrm{QCoh}(Y/H)$.

21.5.8. **Proposition.** *The quotient $\mathbf{Q}^{(H,H^{\flat})}(\mathcal{Y}^{\flat})$ identifies with the inert formal moduli problem on $\mathcal{Q}^{\mathrm{desc}} \in \mathrm{QCoh}(Y/H)$.*

Proof. By Proposition 21.3.6, we have

$$\mathbf{Q}^{(H,H^{\flat})}(\mathcal{Y}^{\flat}) \xrightarrow{\sim} \mathbf{Q}^{(H,H)} \circ \mathbf{Q}^{(\mathrm{pt}, \mathrm{B}\exp(\mathfrak{k}))}(\mathcal{Y}^{\flat}) \xrightarrow{\sim} \mathbf{Q}^{(\mathrm{pt}, \mathrm{B}\exp(\mathfrak{k}))}(\mathcal{Y}^{\flat})/H.$$

Note that descent of $\mathcal{O}_Y$-modules corresponds to quotient by H on the inert formal moduli problem. Hence we only need to identify $\mathbf{Q}^{(\mathrm{pt},\mathrm{B}\exp(\mathfrak{k}))}(\mathcal{Y}^\flat)$ as the inert formal moduli problem on $\mathcal{Q}$.

Consider the Čech nerve of $\mathcal{F} \to \mathcal{Q}$ in $\mathrm{QCoh}(Y)$, which identifies with the groupoid $\mathcal{F} \oplus (\mathfrak{k} \otimes \mathcal{O}_Y)^{\oplus \bullet}$. Since the composition $\mathrm{diag}_Y \circ \mathrm{triv}$ preserves fiber products, we see that

$$\mathrm{diag}_Y \circ \mathrm{triv}(\mathcal{F} \oplus (\mathfrak{k} \otimes \mathcal{O}_Y)^{\oplus \bullet}) \xrightarrow{\sim} \mathcal{Y}^\flat \times (\mathrm{B}\exp(\mathfrak{k}))^\bullet$$

identifies with the Čech nerve of the map $\mathcal{Y}^\flat \to \mathrm{diag}_Y \circ \mathrm{triv}(\mathcal{Q})$. The result follows since this is also the Čech nerve of $\mathcal{Y}^\flat \to \mathbf{Q}^{(\mathrm{pt},\mathrm{B}\exp(\mathfrak{k}))}(\mathcal{Y}^\flat)$. $\square$

21.5.9. **Remark.** When Y is any scheme over S (*not* necessarily locally of finite type) but η is injective, we also have an identification of $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{F})$ with the Lie algebroid on $\mathcal{Q}^{\mathrm{desc}}$ with zero Lie bracket and anchor map. This follows immediately from the definition of $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}(\mathcal{F})$.

Geometrically, the datum of η gives rise to a map $\phi : \mathbb{V}(\mathcal{F}) \to Y \underset{S}{\times} \mathfrak{k}^*$, and $\mathbb{V}(\mathcal{Q})$ identifies with its fiber at $\{0\} \hookrightarrow \mathfrak{k}^*$. Hence we have isomorphisms of DG stacks:

$$\mathbb{V}(\mathcal{Q}^{\mathrm{desc}}) \xrightarrow{\sim} \mathbb{V}(\mathcal{Q})/H \xrightarrow{\sim} \phi^{-1}(0)/H. \tag{21.25}$$

21.5.10. Suppose we have an exact sequence of H-equivariant $\mathcal{O}_Y$-modules:

$$0 \to \mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F} \to 0.$$

Let $\widehat{\mathcal{Y}}^\flat \in \mathbf{PTw}_{/\mathcal{Y}^\flat}(Y/S)$ be the corresponding inert pre-twisting. Assume that η lifts to an H-equivariant map $\widehat{\eta} : \mathfrak{k} \otimes \mathcal{O}_Y \to \widehat{\mathcal{F}}$. Then Proposition 21.5.8 shows that the quotient pre-twisting arises from a triangle in $\mathrm{QCoh}(Y/H)$:

$$\mathcal{O}_{Y/H} \to \widehat{\mathcal{Q}}^{\mathrm{desc}} \to \mathcal{Q}^{\mathrm{desc}},$$

where $\widehat{\mathcal{Q}}^{\mathrm{desc}}$ is the descent of $\widehat{\mathcal{Q}} := \mathrm{Cofib}(\widehat{\eta})$ to Y/H. In particular, we have isomorphisms of DG stacks:

$$\mathbb{V}(\widehat{\mathcal{Q}}^{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathbb{V}(\widehat{\mathcal{Q}})_{\lambda=1}/H \xrightarrow{\sim} \widehat{\phi}^{-1}_{\lambda=1}(0)/H \tag{21.26}$$

where $\widehat{\phi}_{\lambda=1}$ is the composition

$$\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1} \hookrightarrow \mathbb{V}(\widehat{\mathcal{F}}) \xrightarrow{\mathbb{V}(\widehat{\eta})} Y \underset{S}{\times} \mathfrak{k}^*.$$

21.5.11. Using (21.24), we obtain the following identifications on module categories:

$$\begin{aligned}\widehat{\mathcal{Y}}^\flat\text{-Mod}^{H^\flat} &\xrightarrow{\sim} (\widehat{\mathcal{Y}}^\flat\text{-Mod}^{\mathrm{B}\exp(\mathfrak{k})})^H \xrightarrow{\sim} (\mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1})^{\mathrm{B}\exp(\mathfrak{k})})^H \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{Q}})_{\lambda=1})^H \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{Q}}^{\mathrm{desc}})_{\lambda=1}),\end{aligned}$$

for $H^\flat := \mathrm{B}\exp(\mathfrak{k}) \rtimes H$. The forgetful functor from $\widehat{\mathcal{Y}}^\flat\text{-Mod}^{H^\flat}$ to $\mathrm{IndCoh}(Y/H)$ passes to the direct image along the projection $\mathbb{V}(\widehat{\mathcal{Q}}^{\mathrm{desc}})_{\lambda=1} \to Y/H$.

21.5.12. **Remark.** In light of (21.25) and (21.26), we would like to think of $\mathbf{Q}^{(H,H^\flat)}$ on inert pre-twistings as an analogue of symplectic reduction where ϕ and $\widehat{\phi}_{\lambda=1}$ play the role of the moment map (but of course, with no symplectic structures involved *a priori*.)

22. Categories of local nature

The goal of this section is to give some examples of categories relevant to the local geometric Langlands program which are naturally defined over $\overline{\mathrm{Par}}_G$. We will make use of the factorization pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$ of §20.5.

The materials in this section lack precision with regard to infinite type issues. Namely, we need the theory built in §21 to extend to schemes of infinite type (replacing "smooth" by "formally smooth and placid"), where all Lie algebroids carry a topology. We have not worked out the details, but hope to address them systematically in the future.

22.1. **Kac–Moody representations.**

22.1.1. Let us fix an S-point $(\mathfrak{g}^\kappa, E)$ of $\overline{\mathrm{Par}}_G$. The construction of §20.5 yields a factorization central extension of Lie algebras of Tate modules on $S \times \mathrm{Ran}_{\mathrm{dR}}$:

$$0 \to \mathcal{O}_{S\times\mathrm{Ran}_{\mathrm{dR}}} \to \widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}^\kappa \to 0, \tag{22.1}$$

equipped with an integration of the $S \times \mathrm{L}_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}$-action to $S \times \mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$ and a trivialization over $\mathrm{L}^+_{\mathrm{Ran}_{\mathrm{dR}}}\mathfrak{g}^\kappa$. The $\mathcal{O}_{S\times\mathrm{Ran}_{\mathrm{dR}}}$-linear category $\widehat{\mathfrak{g}}^{(\kappa,E)}$-Mod consists of those $\widehat{\mathfrak{g}}^{(\kappa,E)}$-modules on which $\mathbf{1} \in \mathcal{O}_{S\times\mathrm{Ran}_{\mathrm{dR}}}$ acts identically. Namely,

$$\widehat{\mathfrak{g}}^{(\kappa,E)}\text{-Mod} := U(\widehat{\mathfrak{g}}^{(\kappa,E)})/(1-\mathbf{1})\text{-Mod}.$$

Note that the $S \times \mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$-action on $\widehat{\mathfrak{g}}^{(\kappa,E)}$ induces an $S \times \mathcal{L}_{\mathrm{Ran}_{\mathrm{dR}}}G$-action on the DG category $\widehat{\mathfrak{g}}^{(\kappa,E)}$-Mod.

22.1.2. Let us specialize the above construction to the k-point $(\mathfrak{g}^\infty, 0)$ of $\overline{\mathrm{Par}}_G$. For simplicity, we shall also pick a k-point $x \in X$ instead of working with the prestack $\mathrm{Ran}_{\mathrm{dR}}$. Thus (22.1) becomes a central extension of topological Lie algebras:

$$0 \to k \to \widehat{\mathfrak{g}}^{(\infty,0)}_x \to \mathfrak{g}^\infty(\mathbf{F}_x) \to 0.$$

Let $\mathrm{Conn}(\mathring{D}_x)$ denote the space of connections on $\mathring{D}_x$, regarded as ind-affine space (of infinite type). It is equipped with an $\mathcal{L}_xG$-action by gauge transformations.

22.1.3. **Lemma.** *There is a canonical isomorphism of topological associative algebras acted on by $\mathcal{L}_xG$.*

$$U(\widehat{\mathfrak{g}}^{(\infty,0)})/(1-\mathbf{1}) \xrightarrow{\sim} \mathrm{H}^0(\mathrm{Conn}(\mathring{D}_x), \mathcal{O}).$$

Proof. This follows from a direct calculation. ☐

It follows from Lemma 22.1.3 that we have an $\mathcal{L}_x G$-equivariant equivalence:

$$\widehat{\mathfrak{g}}_x^{(\infty,0)}\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x)). \tag{22.2}$$

This is the first example of a family of DG categories over $\overline{\mathrm{Par}}_G$ with an explicit degeneration behavior "at ∞."

22.2. **Kazhdan–Lusztig categories.**

22.2.1. Let us analyze further the $S \times \mathcal{L}_{\mathrm{Ran_{dR}}} G$-action on (22.1). Since the sequence splits over $\mathrm{L}^+_{\mathrm{Ran_{dR}}} \mathfrak{g}^\kappa$, the $S \times \mathcal{L}^+_{\mathrm{Ran_{dR}}} G$-action on the corresponding pre-twisting factors through

$$(S \times \mathcal{L}^+_{\mathrm{Ran_{dR}}} G)^\kappa := S \times \mathcal{L}^+_{\mathrm{Ran_{dR}}} G / \exp(\mathrm{L}^+_{\mathrm{Ran_{dR}}} \mathfrak{g}^\kappa).$$

This is (the pro-finite generalization) of the paradigm in §21.2, where we regard $(\mathrm{L}^+_{\mathrm{Ran_{dR}}} \mathfrak{g}^\kappa, S \times \mathcal{L}^+_{\mathrm{Ran_{dR}}} G)$ as a classical action pair. In particular, it makes sense to take the quotient of the module category:

$$\mathrm{KL}_G^{(\kappa,E)} := \widehat{\mathfrak{g}}^{(\kappa,E)}\text{-Mod}^{(S \times \mathcal{L}^+_{\mathrm{Ran_{dR}}} G)^\kappa}.$$

This is the factorization Kazhdan–Lusztig category at level $(\mathfrak{g}^\kappa, E)$.

22.2.2. Let us again specialize to the k-point $(\mathfrak{g}^\infty, 0)$ and $x \in X$, and see what the resulting DG category $\mathrm{KL}_{G,x}^{(\infty,0)}$ looks like. We first note the isomorphism:

$$(\mathcal{L}^+_x G)^\infty \xrightarrow{\sim} \mathrm{B}\exp(\mathfrak{g}^*(\mathbf{O}_x)) \rtimes \mathcal{L}^+_x G,$$

where the semi-direct product is formed using the co-adjoint action. Applying results of §21.5 and (22.2), we obtain an identification:

$$\begin{aligned}\mathrm{KL}_{G,x}^{(\infty,0)} := \widehat{\mathfrak{g}}^{(\infty,0)}\text{-Mod}^{(\mathcal{L}^+_x G)^\infty} &\xrightarrow{\sim} (\mathrm{QCoh}(\mathrm{Conn}(\overset{\circ}{D}_x))^{\mathrm{B}\exp(\mathfrak{g}^*(\mathbf{O}_x))})^{\mathcal{L}^+_x G} \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Conn}(D_x))^{\mathcal{L}^+_x G} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{pt}/G),\end{aligned}$$

where the last isomorphism is due to the identification of $\mathrm{Conn}(D_x)/\mathcal{L}_x^+ G$ with pt/G. In other words, the Kazhdan–Lusztig category degenerates to G-representations "at ∞."

22.2.3. *Variant.* Let us also note the degeneration behavior of twisted crystals on $\mathrm{Gr}_{G,\mathrm{Ran}}$. Indeed, recall the factorization pre-twisting $\mathcal{T}_{\mathrm{loc}}^{(\kappa,E)}$ over $\mathrm{Gr}_{G,\mathrm{Ran}}$ (§20.5). We may use it to define:

$$\mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,\mathrm{Ran}}) := \mathcal{T}_{\mathrm{loc}}^{(\kappa,E)}\text{-Mod}.$$

Then for a point $x \in X$, the DG category $\mathcal{D}\text{-Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x})$ identifies with $\mathrm{QCoh}(\mathrm{Gr}_{G,x,\mathrm{conn}})$, where $\mathrm{Gr}_{G,x,\mathrm{conn}}$ classifies a point $(\mathcal{P}_G, \alpha)$ of $\mathrm{Gr}_{G,x}$ together with a connection on $\mathcal{P}_G$.

22.3. **Whittaker category.**

22.3.1. In this subsection, we study the degeneration behavior of Whittaker sheaves on the affine Grassmannian. Since the notations involved are already heavy, we will work over a point $x \in X$ throughout the discussion. The result arrives in Lemma 22.3.18, where we identify the fiber of the Whittaker category "at ∞" as quasi-coherent sheaves on the space of unramified opers.

22.3.2. *Twist by $\mathring{\omega}_x^{1/2}$.* In order to be completely canonical in defining the Whittaker category, we need to introduce a twist by the theta characteristic. From now on, we fix a square root of ω_x and call it $\omega_x^{1/2}$. We let ω_x^{ρ} denote the T-bundle induced from $\omega_x^{1/2}$ along $2\rho \in \Lambda_T$. As usual, its sections over the formal punctured disc will be denoted by $\mathring{\omega}_x^{\rho}$.

We let $\mathcal{L}_x N_\omega$ denote the group scheme over $\mathring{D}_x$ which classifies automorphisms of the induced B-bundle $(\mathring{\omega}_x^{\rho})_B$, which preserve the further induced T-bundle $((\mathring{\omega}_x^{\rho})_B)_T \xrightarrow{\sim} \mathring{\omega}_x^{\rho}$. Here are some variants of the geometric objects considered above:

(1) $\mathcal{L}_x G_\omega$ (respectively $\mathcal{L}_x^+ G_\omega$) denotes sections of $(\mathring{\omega}_x^{\rho})_G$ (respectively $(\omega_x^{\rho})_G$);

(2) $\mathrm{Gr}_{G,x,\omega}$ classifies a G-bundle over D_x, together with an isomorphism $\mathcal{P}_G\big|_{\mathring{D}_x} \xrightarrow{\sim} (\mathring{\omega}_x^\rho)_G$.

22.3.3. We can still realize $\mathrm{Gr}_{G,x,\omega}$ as the quotient $\mathcal{L}_x G_\omega / \mathcal{L}_x^+ G_\omega$. There is an analogue of the central extension (22.1) (at the point $x \in X$), denoted by:

$$0 \to k \to \widehat{\mathfrak{g}}_\omega^{(\kappa,E)} \to \mathrm{L}_x \mathfrak{g}_\omega^\kappa \to 0.$$

It is formed by taking the $(\omega_X^\rho)_G$-twist of the Lie-$*$ algebra extension $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}$ and then taking de Rham cohomology over $\mathring{D}_x$.

In particular, $\mathrm{L}_x \mathfrak{g}_\omega^\kappa$ can be realized as sections of the twisted bundle $(\mathfrak{g}^\kappa)_{\mathring{\omega}_x^\rho}$, where we regard $\mathfrak{g}^\kappa$ as a T-representation.

Similar notations $\mathrm{L}_x(\cdot)_\omega$ and $\mathrm{L}_x^+(\cdot)_\omega$ will be employed for any T-representation. As a particular example, we have the twisted loop algebra $\mathrm{L}_x\mathfrak{g}_\omega$, which identifies with the Lie algebra of the group scheme $\mathcal{L}_x G_\omega$.

22.3.4. We have $\mathring{\omega}_x^{1/2}$-twisted analogues of the above categories:

(1) $\mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathcal{L}_x G_\omega) := U(\widehat{\mathfrak{g}}_\omega^{(\kappa,E)} \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G})/(1-\mathbf{1})\text{-Mod}$;

(2) $\mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega}) := \mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathcal{L}_x G_\omega)^{(\mathcal{L}_x^+ G_\omega)^\kappa}$.

The analogues of their degeneration behavior continue to hold. More precisely, we have:

$$\mathcal{D}\text{-Mod}^{(\infty,0)}(\mathcal{L}_x G_\omega) \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(\mathring{D}_x)),$$

where $\mathrm{Conn}_\omega(\mathring{D}_x)$ denotes the space of connections on the G-bundle $(\mathring{\omega}_x^\rho)_G$. We use the notation $\mathrm{Conn}_\omega(D_x)$ in a similar way, and there holds:

$$\begin{aligned}\mathcal{D}\text{-Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x,\omega}) &\xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x))^{\mathcal{L}_x^+ G_\omega} \\ &\xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,x,\omega,\mathrm{conn}}),\end{aligned}$$

where $\mathrm{Gr}_{G,x,\omega,\mathrm{conn}}$ classifies the data of $\mathrm{Gr}_{G,x,\omega}$ together with a connection on $\mathcal{P}_G$.

22.3.5. We now analyze the Whittaker condition. Suppose $\mathfrak{g}^\kappa$ is a Lagrangian, G-invariant subspace of $\mathfrak{g} \oplus \mathfrak{g}^*$. Associated to $\mathfrak{g}^\kappa$ is a subspace:

$$\mathfrak{n}^\kappa := \mathfrak{g}^\kappa \cap (\mathfrak{n} \oplus \mathfrak{b}^\perp) \hookrightarrow \mathfrak{n} \oplus \mathfrak{b}^\perp$$

where $\mathfrak{b}^\perp := (\mathfrak{g}/\mathfrak{b})^* \subset \mathfrak{g}^*$ consists of linear functionals vanishing on $\mathfrak{b}$. We write $\mathfrak{n}^\kappa_{(1)}$ for the subspace $[\mathfrak{n}, \mathfrak{n}^\kappa] \hookrightarrow \mathfrak{n}^\kappa$. Note that $\mathfrak{n}^\kappa_{(1)}$ is also the intersection of $\mathfrak{n}^\kappa$ with $\mathfrak{n}_{(1)} \oplus (\mathfrak{b}_{(-1)})^\perp$, where $\mathfrak{b}_{(-1)}$ is the sum of $\mathfrak{b}$ with the negative simple root spaces.

22.3.6. The weights of the $\mathfrak{t}$-action on $\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)}$ identify with the simple roots $\{\check{\alpha}_i\}_{i\in\Delta}$.[18] Thus we may form the "canonical" character:

$$\chi : \mathcal{L}_x(\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)})_\omega \xrightarrow{\sim} \bigoplus_{i\in\Delta} (\mathring{\omega}_x^{1/2})^{\langle 2\rho, \check{\alpha}_i\rangle} \xrightarrow{\sim} \bigoplus_{i\in\Delta} \mathring{\omega}_x \xrightarrow{\sum \mathrm{Res}} k. \tag{22.3}$$

where $\sum \mathrm{Res}$ denotes the "sum of residue" map. The precomposition of (22.3) with the projection map $\mathcal{L}_x(\mathfrak{n}^\kappa)_\omega \to \mathcal{L}_x(\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)})_\omega$ will again be denoted by χ (as no confusion should arise!)

22.3.7. **Example.** At the fully degenerate point $\mathfrak{g}^\infty$, we have:

$$\mathfrak{n}^\infty/\mathfrak{n}^\infty_{(1)} \xrightarrow{\sim} \mathfrak{b}^\perp/(\mathfrak{b}^\perp_{(-1)}) \xrightarrow{\sim} (\mathfrak{b}_{(-1)}/\mathfrak{b})^*$$

so χ defines an element in $\mathrm{Hom}_c(\mathcal{L}_x(\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega, k)$ that we may call the "canonical" element.

22.3.8. Define a group prestack $(\mathcal{L}_x N_\omega)^\kappa$ by the quotient:

$$(\mathcal{L}_x N_\omega)^\kappa := \mathcal{L}_x N_\omega / \exp(\mathcal{L}_x \mathfrak{n}^\kappa_\omega),$$

where we use the tautological action of $\mathcal{L}_x N_\omega$ on $\mathcal{L}_x \mathfrak{n}^\kappa_\omega$.

[18]One may be tempted to fix "Chevalley generators" $\{e_i\}_{i\in\Delta}$ as a $\mathfrak{t}$-eigenbasis of $\mathfrak{n}^\kappa/\mathfrak{n}^\kappa_{(1)}$. However, this *cannot* be done compatibly over the entire space $\overline{\mathrm{Par}}_G$. For example, when $G = \mathrm{SL}_2$, such a choice amounts to a nonvanishing global section of $\mathcal{O}_{\mathbb{P}^1}(-1)$.

22.3.9. Let us note the "1-categorical level down" version of Chapter 4, Lemma 19.5.3. Suppose H is a group object in $\mathbf{PStk}_{/k}^{\text{laft-def}}$ and $\mathfrak{k}$ is a Lie algebra together with a morphism $\mathfrak{k} \to \mathfrak{h}$. Suppose the H-action on $\mathfrak{h}$ extends to $\mathfrak{k}$ and furthermore the $\mathfrak{k}$-action induced from $\mathfrak{k} \to \mathfrak{h}$ is the adjoint action. Then $H/\exp(\mathfrak{k})$ acquires a group structure such that $H \to H/\exp(\mathfrak{k})$ is a group homomorphism.

In this setting, the following data are equivalent:

(1) H-equivariant Lie algebra characters of $\mathfrak{k}$;

(2) multiplicative line bundles on $H/\exp(\mathfrak{k})$ equipped with a trivialization over H.

22.3.10. By the analogous fact in infinite type situation, the character χ (22.3) determines a multiplicative line bundle $(\mathcal{L}_x N_\omega)^\kappa$ together with a trivialization over $\mathcal{L}_x N_\omega$. Hence, if we have a map of prestacks $\mathcal{Y} \to \mathcal{Y}^\flat$ acted on compatibly by the group schemes $\mathcal{L}_x N_\omega \to (\mathcal{L}_x N_\omega)^\kappa$, we may form the category of $(\mathcal{L}_x N_\omega)^\kappa$-equivariant sheaves $\mathrm{IndCoh}(\mathcal{Y}^\flat)^{(\mathcal{L}_x N_\omega)^\kappa,\chi}$ against the character χ; it is equipped with a forgetful functor:

$$\mathrm{oblv} : \mathrm{IndCoh}(\mathcal{Y}^\flat)^{(\mathcal{L}_x N_\omega)^\kappa,\chi} \to \mathrm{IndCoh}(\mathcal{Y})^{\mathcal{L}_x N_\omega}.$$

22.3.11. **Example.** Suppose $\mathfrak{g}^\kappa$ is the graph of a bilinear form. Then we have an isomorphism $(\mathcal{L}_x N_\omega)^\kappa \xrightarrow{\sim} (\mathcal{L}_x N_\omega)_{\mathrm{dR}}$; thus the datum on the right is precisely a multiplicative local system on $\mathcal{L}_x N_\omega$ whose underlying line bundle is trivialized. The local system determined by (22.3) identifies with the pullback of exp under:

$$(\mathcal{L}_x N)_\omega \to (\mathcal{L}_x N)_\omega/[(\mathcal{L}_x N)_\omega, (\mathcal{L}_x N)_\omega] \xrightarrow{\sim} \bigoplus_{i\in\Delta} \mathring{\omega}_x \xrightarrow{\sum \mathrm{Res}} \mathbb{G}_a.$$

[19] Indeed, this follows from the fact that $\mathrm{id} : \mathrm{Lie}(\mathbb{G}_a) \to k$ determines the exponential local system on $\mathbb{G}_a$, and the equivalence of §22.3.9 is functorial.

[19]The isomorphism in the middle is constructed as follows. Consider the exact sequence of B-representations (where $\mathfrak{n}_{\check{\alpha}_i}$ is the simple root space corresponding to $\check{\alpha}_i$, regarded as a *quotient* of $\mathfrak{n}/\mathfrak{n}^{(1)}$):

$$0 \to \mathfrak{n}_{\check{\alpha}_i} \to (k\mathbf{1} \oplus \mathfrak{n}_{\check{\alpha}_i}) \to k\mathbf{1} \to 0.$$

22.3.12. Let $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G$ be a (say k-valued, for simplicity) quantum parameter. Recall that the category $\mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})$ from §22.3.4. It is equipped with a $(\mathcal{L}_x G_\omega)^\kappa$-action. We define

$$\mathrm{Whit}_{G,x}^{(\kappa,E)} := \mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})^{(\mathcal{L}_x N_\omega)^\kappa,\chi}$$

i.e., the category of objects in $\mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})$ that are $(\mathcal{L}_x N_\omega)^\kappa$-equivariant against χ. From Example 22.3.11, we have:

22.3.13. **Lemma.** *Suppose* $(\mathfrak{g}^\kappa, E) \in \mathrm{Par}_G^\circ$. *Then* $\mathrm{Whit}_{G,x}^{(\kappa,E)}$ *identifies with the usual Whittaker category* $\mathcal{D}\text{-Mod}^{(\kappa,E)}(\mathrm{Gr}_{G,x,\omega})^{\mathcal{L}_x N_\omega,\chi}$. □

22.3.14. *Unramified opers.* We recall the definition of the placid ind-scheme $\mathrm{Op}_{G,x}^{\mathrm{unr}}$. It classifies quadruples $(\mathcal{P}_G, \nabla, \mathcal{P}_B, \alpha)$ where:

(1) $\mathcal{P}_G$ is a G-bundle over D_x, and ∇ is a connection on it;
(2) $\mathcal{P}_B$ is a reduction of $\mathcal{P}_G$ to B *over* $\mathring{D}_x$, and α is an isomorphism of its induced T-bundle $(\mathcal{P}_B)_T \xrightarrow{\sim} \mathring{\omega}_x^\rho$.

These data are suppose to satisfy the following *oper* condition. To state it, we note first that α gives rise to an isomorphism for each simple root $\check{\alpha}_i$:

$$\mathcal{P}_B^{\check{\alpha}_i} \xrightarrow{\sim} (\mathcal{P}_B)_T^{\check{\alpha}_i} \xrightarrow{\sim} \mathring{\omega}_x^{\langle\rho,\check{\alpha}_i\rangle} \xrightarrow{\sim} \mathring{\omega}_x. \tag{22.4}$$

On the other hand, we may consider the composition:

$$\mathcal{T}_{\mathring{D}_x} \xrightarrow{\nabla} \mathrm{At}(\mathcal{P}_G) \to \mathrm{At}(\mathcal{P}_G)/\,\mathrm{At}(\mathcal{P}_B) \xrightarrow{\sim} (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}. \tag{22.5}$$

We require that

(1) the image lands in $(\mathfrak{b}_{(-1)}/\mathfrak{b})_{\mathcal{P}_B}$, and

---

After we twist it by the B-bundle $(\mathring{\omega}_x^\rho)_B$, the first term becomes $\mathring{\omega}_x$ and the last term becomes $\mathcal{K}_x$. An element of $(\mathcal{L}_x N)_\omega$ thus determines a "shearing" map $\mathcal{K}_x \to \mathring{\omega}_x$, i.e., a section of $\mathring{\omega}_x$.

(2) the projection to each negative simple root space

$$\mathcal{T}_{\overset{\circ}{D}_x} \to (\mathfrak{b}_{-\check{\alpha}_i}/\mathfrak{b})_{\mathcal{P}_B} \xrightarrow{\sim} \mathcal{P}_B^{-\check{\alpha}_i} \tag{22.6}$$

is the monoidal dual of (22.4).

22.3.15. **Remark.** If G is of adjoint type, then we may drop α from the definition, and simply require the maps (22.6) to be isomorphisms. Indeed, we may recover α as follows: the isomorphisms (22.6) tell us what $(\mathcal{P}_B)_T^{\check{\alpha}_i}$ is for each simple root, and the adjoint type hypothesis says that the simple roots span $\check{\Lambda}_T$.

22.3.16. We introduce a piece of notation. Given the data $(\mathcal{P}_G, \nabla, \mathcal{P}_B)$, we may form the composition (22.5). It is $\mathbf{F}_x$-linear, so may be regarded as an object $\nabla_{/\mathcal{P}_B}$ in any of the following vector spaces:

$$\begin{aligned}\nabla_{/\mathcal{P}_B} \in \operatorname{Hom}_{\mathcal{K}_x}(\mathcal{T}_{\overset{\circ}{D}_x}, (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{K}_x}((\mathfrak{b}^{\perp})_{\mathcal{P}_B}, \overset{\circ}{\omega}_x) \\ &\xrightarrow{\sim} \operatorname{Hom}_c((\mathfrak{b}^{\perp})_{\mathcal{P}_B}, k).\end{aligned}$$

Given the additional datum α, the above requirements can be rephrased as:

(1) $\nabla_{/\mathcal{P}_B}$ belongs to the subspace $\operatorname{Hom}_c((\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\mathcal{P}_B}, k)$;

(2) since the B action on $(\mathfrak{b}_{(-1)}/\mathfrak{b})^*$ factors through T, we have

$$(\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\mathcal{P}_B} \xrightarrow{\sim} (\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\omega}$$

so we require $\nabla_{/\mathcal{P}_B}$ to identify with the "canonical" element in $\operatorname{Hom}_c((\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\omega}, k)$ (see Example 22.3.7).

22.3.17. **Remark.** Of course, we can combine the two requirements into saying that $\nabla_{/\mathcal{P}_B}$ identifies with the "canonical" element in $\operatorname{Hom}_c((\mathfrak{b}_{(-1)}/\mathfrak{b})^*_{\omega}, k) \hookrightarrow \operatorname{Hom}_c((\mathfrak{g}/\mathfrak{b})^*_{\mathcal{P}_B}, k)$.

We can now state the degeneration result:

22.3.18. **Lemma.** *There is a canonical equivalence of DG categories:*

$$\mathrm{Whit}^{(\infty,0)}_{G,x} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Op}^{\mathrm{unr}}_{G,x}). \tag{22.7}$$

We first note from §22.3.4 the isomorphisms:

$$\mathcal{D}\text{-Mod}^{(\infty,0)}(\mathrm{Gr}_{G,x,\omega}) \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x))^{\mathcal{L}^+_x G} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,x,\omega,\mathrm{conn}})$$

so we tautologically have:

$$\mathrm{Whit}^{(\infty,0)}_{G,x} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}_{G,\nabla,\omega})^{\mathcal{L}_x N^\infty_\omega,\chi} \xrightarrow{\sim} (\mathrm{QCoh}(\mathrm{Gr}_{G,x,\omega,\mathrm{conn}})^{\mathrm{B}\exp(\mathcal{L}_x \mathfrak{n}^\infty_\omega),\chi})^{\mathcal{L}_x N_\omega}.$$

22.3.19. We define the following auxiliary objects:

(1) let $\mathrm{Conn}^{\mathrm{Op}}_\omega(D_x)$ be the closed subscheme of $\mathrm{Conn}_\omega(D_x)$ consisting of connections ∇ on $(\omega^\rho_x)_G$ whose restriction to $\overset{\circ}{D}_x$ satisfies the oper condition.
(2) let $\mathrm{Gr}^{\mathrm{Op}}_{G,x,\omega,\mathrm{conn}}$ be the closed subscheme of $\mathrm{Gr}_{G,x,\omega,\mathrm{conn}}$, where the connection ∇ on $\mathcal{P}_G$ restricts to one on $\mathcal{P}_G\big|_{\overset{\circ}{D}_x} \xrightarrow{\sim} (\omega^\rho_x)_G$ that satisfies the oper condition (as above).

Clearly, we have a Cartesian square:

$$\begin{tikzcd}
\mathcal{L}_x G_\omega \times \mathrm{Conn}^{\mathrm{Op}}_\omega(D_x) \arrow[r, hook] \arrow[d, "\mathcal{L}^+_x G_\omega"] & \mathcal{L}_x G \times \mathrm{Conn}_\omega(D_x) \arrow[d, "\mathcal{L}^+_x G_\omega"] \\
\mathrm{Gr}^{\mathrm{Op}}_{G,x,\omega,\mathrm{conn}} \arrow[r, hook] & \mathrm{Gr}_{G,x,\omega,\mathrm{conn}}
\end{tikzcd}$$

where the vertical maps are $\mathcal{L}^+_x G_\omega$-torsors.

22.3.20. On the other hand, $\mathcal{L}_x N_\omega$ acts on $\mathrm{Gr}^{\mathrm{Op}}_{G,x,\omega,\mathrm{conn}}$, and there is a canonical isomorphism:

$$\mathcal{L}_x N_\omega \backslash \mathrm{Gr}^{\mathrm{Op}}_{G,x,\omega,\mathrm{conn}} \xrightarrow{\sim} \mathrm{Op}^{\mathrm{unr}}_{G,x}.$$

Thus we have reduced the statement of Lemma 22.3.18 to an $\mathcal{L}_x N_\omega$-equivariant equivalence:

$$\text{(22.8)} \qquad \mathrm{QCoh}(\mathrm{Gr}_{G,x,\omega,\mathrm{conn}})^{\mathrm{B}\exp(\mathcal{L}_x\mathfrak{n}_\omega^\infty),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{Gr}^{\mathrm{Op}}_{G,x,\omega,\mathrm{conn}}).$$

The equivalence (22.8) will in turn follow from an $(\mathcal{L}_x N_\omega, \mathcal{L}_x^+ G_\omega)$-bi-equivariant equivalence:

$$\text{(22.9)} \qquad \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x))^{\mathrm{B}\exp(\mathfrak{n}_\omega^\infty),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega^{\mathrm{Op}}(D_x)).$$

22.3.21. *Module categories against a character.* To prove (22.9), we need a generalization of the equivalences of module categories in §21.5.11. Let us be given a smooth scheme Y, a quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(Y)$, and a vector space $\mathfrak{k}$ equipped with a map $\eta : \mathfrak{k} \otimes \mathcal{O}_Y \to \mathcal{F}$. These data determine an action of $\mathrm{B}\exp(\mathfrak{k})$ on the inert formal moduli problem $Y^\flat$ associated to $\mathcal{F}$.

Let us now include the additional datum of a character (of abelian Lie algebras) $\chi : \mathfrak{k} \to k$. Note that the map η gives rise to a morphism:

$$\mathrm{char} : \mathbb{V}(\mathcal{F}) \to \mathfrak{k}^* \times Y \xrightarrow{\mathrm{pr}} \mathfrak{k}^*,$$

We let $\mathbb{V}(\mathcal{F})_{\mathrm{char}=\chi}$ denote its fiber at $\{\chi\}$.

Then there is a canonical equivalence of DG categories:

$$\text{(22.10)} \qquad \mathrm{IndCoh}(Y^\flat)^{\mathrm{B}\exp(\mathfrak{k}),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F})_{\mathrm{char}=\chi})$$

(Recall that $\mathrm{IndCoh}(Y^\flat) \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\mathcal{F}))$, so we have an easy way to calculate its $\mathrm{B}\exp(\mathfrak{k})$-invariants against a character.) There is also a twisted version of (22.10) which asserts an equivalence of DG categories:

$$\text{(22.11)} \qquad \widehat{Y}^\flat\text{-}\mathrm{Mod}^{\mathrm{B}\exp(\mathfrak{k}),\chi} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1,\mathrm{char}=\chi}),$$

where we recall $\widehat{Y}^\flat\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1})$.

22.3.22. We now apply (22.11) to the following situation (assuming that a generalization to the infinite type situation exists):

(1) Y is the loop group $\mathcal{L}_x G_\omega$;
(2) the central extension of inert Lie algebroids $\mathcal{O}_Y \to \widehat{\mathcal{F}} \to \mathcal{F}$ is given by:

$$\mathcal{O}_{\mathcal{L}_x G} \to \widehat{\mathfrak{g}}^\infty_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G_\omega} \to \mathfrak{g}^\infty_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G_\omega}$$

(3) $\mathfrak{k} = \mathfrak{n}^\infty_\omega \xrightarrow{\sim} \mathfrak{b}^\perp_\omega$;
(4) χ is the "canonical" element of $\mathrm{Hom}_c(\mathcal{L}_x(\mathfrak{b}_{(-1)}/\mathfrak{b})^*_\omega, k)$ (see Example 22.3.7), embedded in $\mathrm{Hom}_c(\mathcal{L}_x \mathfrak{b}^\perp_\omega, k)$.
(5) the $\mathrm{B}\exp(\mathfrak{k})$-action is supplied by the inclusion $\eta : \mathfrak{b}^\perp_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G} \to \widehat{\mathfrak{g}}^\infty_\omega \widehat{\otimes} \mathcal{O}_{\mathcal{L}_x G}$.

In particular, the morphism $\mathrm{char} : \mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1} \to \mathfrak{k}^*$ is given by:

$$\mathcal{L}_x G_\omega \times \mathrm{Conn}_\omega(D_x) \to \mathrm{Hom}_c(\mathfrak{b}^\perp_\omega, k), \quad (g, \nabla) \rightsquigarrow \nabla_{/\mathcal{P}_B}.$$

Hence the object $\mathbb{V}(\widehat{\mathcal{F}})_{\lambda=1,\mathrm{char}=\chi}$ identifies with $\mathcal{L}_x G_\omega \times \mathrm{Conn}^{\mathrm{Op}}_\omega(D_x)$. The equivalence (22.10) then gives produces (22.9). We omit checking that it is equivariant with respect to both $\mathcal{L}_x N_\omega$ and $\mathcal{L}^+_x G_\omega$-actions. □(Lemma 22.3.18)

23. The global Kac–Moody pre-twisting

In Section 20.5, we have produced a factorization pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$ over $\mathrm{Gr}_{G,\mathrm{Ran}}$ for every S-point $(\mathfrak{g}^\kappa, E)$ of the compactified space of quantum parameters $\overline{\mathrm{Par}}_G$. In this section, we will descend $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$ to the moduli stack of G-bundles Bun_G. In other words, we set X to be proper, and our goal is to produce a pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ on Bun_G whose pullback to $\mathrm{Gr}_{G,\mathrm{Ran}}$ identifies with $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$.

The pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ will be responsible for the degeneration of $\mathcal{D}$-modules on Bun_G into quasi-coherent sheaves on LocSys_G.

23.1. **The classical pre-twisting $\tilde{\mathcal{T}}^{(\kappa,E)}_{\mathrm{glob}}$ over** $\mathrm{Bun}_{G,\infty x}$.

23.1.1. Let $\mathrm{Bun}_{G,\infty x}$ denote the stack classifying pairs $(\mathcal{P}_G, \alpha)$ where $\mathcal{P}_G$ is a G-bundle on X and $\alpha : \mathcal{P}_G\big|_{D_x} \xrightarrow{\sim} \mathcal{P}^0_G$ is a trivialization over D_x. The (right) $\mathcal{L}^+_x G$-action on $\mathrm{Bun}_{G,\infty x}$ by changing α realizes $\mathrm{Bun}_{G,\infty x}$ as a $\mathcal{L}^+_x G$-bundle over Bun_G, locally trivial in the étale topology. In particular, $\mathrm{Bun}_{G,\infty x}$ is placid.

23.1.2. The Beauville-Laszlo theorem shows that $\mathrm{Bun}_{G,\infty x}$ also classifies pairs $(\mathcal{P}_{G,\Sigma}, \alpha)$, where $\mathcal{P}_{G,\Sigma}$ is a G-bundle on $\Sigma := X - \{x\}$ and $\alpha : \mathcal{P}_{G,\Sigma}\big|_{\overset{\circ}{D}_x} \xrightarrow{\sim} \mathcal{P}^0_G$ is a trivialization over $\overset{\circ}{D}_x$. This alternative description shows that the $\mathcal{L}^+_x G$-action on $\mathrm{Bun}_{G,\infty x}$ extends to an $\mathcal{L}_x G$-action.

Fix an S-point $(\mathfrak{g}^\kappa, E)$ of Par_G. We apply the construction of §20.5 to the relative curve

$$\tilde{\mathcal{X}} := S \times \mathrm{Bun}_{G,\infty x} \times X \quad \text{over} \quad \tilde{\mathcal{S}} := S \times \mathrm{Bun}_{G,\infty x},$$

and obtain a central extension in $\mathbf{Lie}^*(\tilde{\mathcal{X}}/\tilde{\mathcal{S}})$:

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}} \to \mathfrak{g}^\kappa_{\mathcal{D}} \to 0. \tag{23.1}$$

In other words, (23.1) is the image of Kac-Moody extension (20.19) under the base change functor $- \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} : \mathbf{Lie}^*(\mathcal{X}/S) \to \mathbf{Lie}^*(\tilde{\mathcal{X}}/\tilde{\mathcal{S}})$.

Let $\underline{\tilde{x}} : \tilde{\mathcal{S}} \hookrightarrow \tilde{\mathcal{X}}$ (resp. $\underline{x} : S \hookrightarrow \mathcal{X}$) denote the section given by $x \in X$. Let $\tilde{\mathcal{P}}_G$ be the tautological G-bundle over $\tilde{\mathcal{X}}$ equipped with the trivialization α over $D_{\underline{\tilde{x}}}$. Since $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}$ and $\mathfrak{g}^\kappa_{\mathcal{D}}$ are equipped with $\mathcal{G}$-actions, we can form the $\tilde{\mathcal{P}}_G$-twist of (23.1):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G} \to (\mathfrak{g}^\kappa_{\mathcal{D}})_{\tilde{\mathcal{P}}_G} \to 0. \tag{23.2}$$

23.1.3. **Remark.** (1) Since $\mathfrak{g}^\kappa_{\mathcal{D}}$ is the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module induced from $\mathfrak{g}^\kappa \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x} \times X}$ and the $\mathcal{G}$-action comes from one on $\mathfrak{g}^\kappa \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x} \times X}$, we see that $(\mathfrak{g}^\kappa_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}$ is the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module induced from $\mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}$.

(2) the datum of α gives an isomorphism between (23.1) and (23.2) when restricted to $D_{\tilde{x}}$.

We apply the functors $\Gamma_{\mathrm{dR}}(\Sigma, -)$ and $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, -)$ to (23.2). Using the two observations above, we obtain a morphism between two triangles in $\mathrm{QCoh}^{\mathrm{Tate}}(\tilde{\mathcal{S}})$:

$$
\begin{tikzcd}
\Gamma_{\mathrm{dR}}(\Sigma, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) \arrow[r] \arrow[d] & \Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}) \arrow[r] \arrow[d] & \Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G}) \arrow[d, "\gamma"] \arrow[dl, dotted, "\widehat{\gamma}"'] \\
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) \arrow[r] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \arrow[r] & \mathfrak{g}^{\kappa}(\mathbf{F}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}
\end{tikzcd}
\tag{23.3}
$$

where $\mathfrak{g}^{\kappa}(\mathbf{F}_x)$ is (as before) an object of $\mathrm{QCoh}^{\mathrm{Tate}}(S)$.

By residue theory, the first vertical map in (23.3) vanishes. Hence we obtain a splitting $\widehat{\gamma}$ as depicted. Note that γ (hence $\widehat{\gamma}$) is injective, so we may define two Tate $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules by cokernels without running into DG issues:

$$\widehat{\mathcal{L}}^{(\kappa,E)} := \mathrm{Coker}(\widehat{\gamma}), \quad \mathcal{L}^{\kappa} := \mathrm{Coker}(\gamma).$$

Since $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})$ is canonically isomorphic to $\mathcal{O}_{\tilde{\mathcal{S}}}$, we arrive at an exact sequence of Tate $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules:

$$0 \to \mathcal{O}_{\tilde{\mathcal{S}}} \to \widehat{\mathcal{L}}^{(\kappa,E)} \to \mathcal{L}^{\kappa} \to 0 \tag{23.4}$$

23.1.4. **Notation.** In what follows, we will show that (23.4) has the structure of a classical pre-twisting on Tate modules over $\tilde{\mathcal{S}}$ (relative to S), denoted by $\tilde{\mathcal{T}}^{(\kappa,E)}_{\mathrm{glob}}$.

23.1.5. We (temporarily) use the notation $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D},\mathcal{X}}$ to denote the Kac-Moody Lie-$*$ algebra over $\mathcal{X}$, constructed using the recipe in §20.5 for the relative curve $\mathcal{X} \to S$. The isomorphism $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}} \xrightarrow{\sim} \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D},\mathcal{X}} \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ gives rise to an isomorphism in $\mathrm{QCoh}^{\mathrm{Tate}}(\tilde{\mathcal{S}})$:

$$\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \xrightarrow{\sim} \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D},\mathcal{X}})\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \cong \widehat{\mathfrak{g}}^{(\kappa,E)}\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \tag{23.5}$$

Observe that the $\mathcal{L}_x G$-action on $\mathrm{Bun}_{G,\infty x}$ gives rise to a $\mathfrak{g}(\mathbf{F}_x)$-action[20] on $\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ by derivations. Hence, the Lie algebroid bracket on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ can be defined using the $\mathcal{O}_S$-linear Lie bracket on $\widehat{\mathfrak{g}}^{(\kappa,E)}$:

$$[\mu \boxtimes f, \mu' \boxtimes f'] := [\mu, \mu'] + \overline{\mu}(f') \cdot \mu' - \overline{\mu}'(f) \cdot \mu.$$

where $\overline{\mu}$ denotes the image of $\mu \in \widehat{\mathfrak{g}}^{(\kappa,E)}$ along $\widehat{\mathfrak{g}}^{(\kappa,E)} \to \mathfrak{g}^{\kappa}(\mathbf{F}_x) \to \mathfrak{g}(\mathbf{F}_x) \widehat{\boxtimes} \mathcal{O}_S$, which acts on $\mathcal{O}_{\tilde{\mathcal{S}}}$ by $\mathcal{O}_S$-linear derivations. The anchor map $\widehat{\sigma}$ of $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ is defined by the composition:

$$\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \xrightarrow{(23.5)} \widehat{\mathfrak{g}}^{(\kappa,E)} \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \mathfrak{g}(\mathcal{K}_x) \widehat{\boxtimes} \mathcal{O}_{\tilde{\mathcal{S}}} \to \mathcal{T}_{\tilde{\mathcal{S}}/S}. \tag{23.6}$$

We have thus equipped $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ with the structure of a Lie algebroid. The following lemma, whose proof is deferred to §23.1.6, extends this Lie algebroid structure to its quotient $\widehat{\mathcal{L}}^{(\kappa,E)}$:

23.1.6. **Lemma.** *The morphism $\widehat{\gamma}$ realizes $\Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G})$ as an ideal of $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$.*

In an analogous way, we turn $\mathfrak{g}^{\kappa}(\mathbf{F}_x) \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ into an object of $\mathbf{Lie}_{/S}(\tilde{\mathcal{S}})$, and the map $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \to \mathfrak{g}^{\kappa}(\mathbf{F}_x) \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$ in (23.3) is a morphism of such. Lemma 23.1.6 shows that γ also realizes $\Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G})$ as an ideal of $\mathfrak{g}^{\kappa}(\mathbf{F}_x) \widehat{\boxtimes} \mathcal{O}_{\mathrm{Bun}_{G,\infty x}}$. Hence the cokernels (23.4) is a central extension of Lie algebroids.

Proof of Lemma 23.1.6. We first give an alternative description of the Lie bracket on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$. Indeed, from the identification in (23.5) and the $\mathfrak{g}(\mathbf{F}_x)$-action on $\widehat{\mathfrak{g}}^{(\kappa,E)}$, we obtain an action of $\mathfrak{g}(\mathbf{F}_x) \widehat{\boxtimes} \mathcal{O}_{\tilde{\mathcal{S}}}$ on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ by $\mathcal{O}_S$-linear derivations. It follows that the Lie bracket on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})$ agrees with the composition:

$$\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})^{\boxtimes 2} \xrightarrow{(\mathrm{pr},\mathrm{id})} (\mathfrak{g}(\mathcal{K}_x) \widehat{\boxtimes} \mathcal{O}_{\tilde{\mathcal{S}}}) \boxtimes \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \xrightarrow{\mathrm{act}} \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}), \tag{23.7}$$

[20] Unlike the Tate $\mathcal{O}_S$-module $\mathfrak{g}^{\kappa}(\mathbf{F}_x)$, the notation $\mathfrak{g}(\mathbf{F}_x)$ is reserved for the Tate vector space $\mathfrak{g} \otimes \mathbf{F}_x$ (similarly for the notation $\mathfrak{g}(\mathbf{O}_x)$.)

where pr denotes the composition of the first two maps in (23.6).

Therefore, it suffices to show that the Tate $\mathcal{O}_{\tilde{S}}$-submodule:

$$\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\tilde{\mathcal{P}}_G}) \hookrightarrow \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) \tag{23.8}$$

is invariant under the aforementioned $\mathfrak{g}(\mathbf{F}_x)\widehat{\boxtimes}\mathcal{O}_{\tilde{S}}$-action. Note that by construction, this action arises from the $S \times \mathcal{L}_x G$-equivariance structure on $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})$. The following claim is immediate:

23.1.7. **Claim.** There is also an $S\times\mathcal{L}_xG$-equivariance structure on $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\tilde{\mathcal{P}}_G})$, defined at every T-point $(s, \mathcal{P}_{G,\Sigma}, \alpha, g)$ of $S \times \mathrm{Bun}_{G,\infty x} \times \mathcal{L}_x G$ (for $T \in \mathbf{Sch}^{\mathrm{aff}}_{/k}$) by:

(1) first identifying the fiber of $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\tilde{\mathcal{P}}_G})$ at both of the T-points

$$(s, \mathcal{P}_{G,\Sigma}, \alpha), \text{ and } (s, \mathcal{P}_{G,\Sigma}, g\cdot\alpha), \quad g \in \mathrm{Maps}(T, \mathcal{L}_x G),$$

with $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}})$;[21] and then

(2) relating the above two fibers via the identity map on $\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}})$. $\square$

So we have reduced the problem to showing that (23.8) preserves the $S \times \mathcal{L}_x G$-equivariance structure. In other words, the following diagram in $\mathrm{QCoh}^{\mathrm{Tate}}(T)$ needs to commute:

(23.9)

$$\begin{tikzcd}
\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}) \arrow[r, "\sim"] \arrow[d, "\mathrm{id}"] & \Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_G})\big|_{(s,\mathcal{P}_{G,\Sigma},\alpha)} \arrow[r, "(23.8)"] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}) \arrow[d, "g\cdot"] \\
\Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}) \arrow[r, "\sim"] & \Gamma_{\mathrm{dR}}(\Sigma, (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_G})\big|_{(s,\mathcal{P}_{G,\Sigma},g\cdot\alpha)} \arrow[r, "(23.8)"] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)}).
\end{tikzcd}$$

Here, the two horizontal compositions express the procedure of

(1) first restricting a flat section of $(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}$ to $\overset{\circ}{D}_{\underline{\tilde{x}}} \hookrightarrow T \times \Sigma$;

[21] We are slightly abusing the notation $(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,E)})_{\mathcal{P}_{G,\Sigma}}$, since this is now the Kac-Moody extension associated to the parameter $T \xrightarrow{s} S \xrightarrow{(\mathfrak{g}^\kappa,E)} \mathrm{Par}_G$, twisted by $\mathcal{P}_{G,\Sigma}$ on the open curve $T \times \Sigma$.

(2) then using the trivialization α (respectively, $g\cdot\alpha$) to identify it with a section of $\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}$.

However, the following diagram is tautologically commutative:

$$\begin{tikzcd}
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\mathcal{P}_{G,\Sigma}}) \arrow[r, "\alpha_*"] \arrow[d, "\mathrm{id}"] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \arrow[d, "g\cdot"] \\
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, (\widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}})_{\mathcal{P}_{G,\Sigma}}) \arrow[r, "(g\cdot\alpha)_*"] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}),
\end{tikzcd}$$

so we obtain the commutativity of (23.9). □

23.2. **Descent to** Bun_G.

23.2.1. We continue to fix the S-point $(\mathfrak{g}^\kappa, E)$ of $\overline{\mathrm{Par}}_G$. The goal of this section is to "descend" the classical pre-twisting $\tilde{\mathcal{T}}^{(\kappa,E)}_{\mathrm{glob}}$ to Bun_G. Recall the action of $H := S\times\mathcal{L}^+_xG$ on $\tilde{\mathcal{S}} = S\times\mathrm{Bun}_{G,\infty x}$, whose quotient is given by $\tilde{\mathcal{S}}/H \xrightarrow{\sim} S\times\mathrm{Bun}_G$. Let $\mathfrak{k} := \mathfrak{g}^\kappa(\mathbf{O}_x)$. Then $(\mathfrak{k}, H)$ forms a classical action pair.

23.2.2. We now equip (23.4) with the structure of a $(\mathfrak{k}, H)$-action. Indeed, applying the functor $\Gamma(D_{\underline{\tilde{x}}}, -)$ to (23.2) and using $\Gamma_{\mathrm{dR}}(D_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) = 0$, we obtain a commutative diagram:

(23.10)
$$\begin{tikzcd}
 & \Gamma_{\mathrm{dR}}(D_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \arrow[r, "\sim"] \arrow[d] & \Gamma(D_{\underline{\tilde{x}}}, \mathfrak{g}^\kappa\boxtimes\mathcal{O}_{\mathrm{Bun}_{G,\infty x}\times X}) \arrow[d, "\eta"] \arrow[dl, dotted, "\widehat{\eta}"'] \\
\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}) \arrow[r] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \arrow[r] & \mathfrak{g}^\kappa(\mathbf{F}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}}
\end{tikzcd}$$

where the splitting $\widehat{\eta}$ exists for obvious reasons. Since $\Gamma(D_{\underline{\tilde{x}}}, \mathfrak{g}^\kappa\boxtimes\mathcal{O}_{\mathrm{Bun}_{G,\infty x}\times X})$ is canonically isomorphic to $\mathfrak{k}\widehat{\otimes}\mathcal{O}_{\tilde{\mathcal{S}}}$, we obtain the $(\mathfrak{k}, H)$-action datum on $\widehat{\mathcal{L}}^{(\kappa,E)}$ via the composition:

$$\mathfrak{k}\widehat{\otimes}\mathcal{O}_{\tilde{\mathcal{S}}} \xrightarrow{\widehat{\eta}} \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{\tilde{x}}}, \widehat{\mathfrak{g}}^{(\kappa,E)}_{\mathcal{D}}) \to \widehat{\mathcal{L}}^{(\kappa,E)},$$

which we again denote by $\widehat{\eta}$.

23.2.3. **Remark.** Ideally, we would like to define $\mathcal{T}^{(\kappa,E)}$ as the quotient $\mathbf{Q}^{(\mathfrak{k},H)}(\tilde{\mathcal{T}}^{(\kappa,E)})$. However, we run into problems because $\tilde{\mathcal{S}}$ is not locally of finite type (so we cannot use $\mathbf{Q}^{(H,H^\flat)}$ (21.20)), and $\widehat{\eta}$ is not injective (so we cannot use $\mathbf{Q}^{(\mathfrak{k},H)}_{\mathrm{inj}}$ (21.6)). In what follows, we circumvent this technical problem using a combination of the two functors.

23.2.4. For each integer $n \geq 0$, let $\mathrm{Bun}_{G,nx}$ denote the stack classifying pairs $(\mathcal{P}_G, \alpha_n)$ where $\mathcal{P}_G$ is a G-bundle on X and $\alpha_n : \mathcal{P}_G\big|_{\mathrm{Spec}(\mathcal{O}_x^{(n)})} \xrightarrow{\sim} \mathcal{P}_G^0$ is a trivialization over the nth infinitesimal neighborhood $\mathrm{Spec}(\mathcal{O}_x^{(n)})$ of x. Then $\mathrm{Bun}_{G,nx}$ is an $\mathcal{L}_{nx}G$-torsor over Bun_G, where $\mathcal{L}_{nx}G$ classifies maps from $\mathrm{Spec}(\mathcal{O}_x^{(n)})$ to G. In particular, $\mathcal{L}_{nx}G$ is a group scheme of finite type.

23.2.5. Set $H_n := S \times \mathcal{L}_{nx}G$, and we have an exact sequence of group schemes over S:

$$1 \to H^n \to H \to H_n \to 1.$$

Define $\mathfrak{k}^n := \mathfrak{k} \otimes \mathfrak{m}_x^n$, and $\mathfrak{k}_n := \mathfrak{k}/\mathfrak{k}^n \cong \mathfrak{k} \otimes \mathcal{O}_x^{(n)}$. Then the above sequence extends to an exact sequence of classical action pairs:

$$1 \to (\mathfrak{k}^n, H^n) \to (H, \mathfrak{k}) \to (H_n, \mathfrak{k}_n) \to 1. \tag{23.11}$$

23.2.6. We briefly review the Harder-Narasimhan truncation of Bun_G. For this, we need to fix a Borel $B \hookrightarrow G$, whose quotient torus is denoted by T. There are canonical maps

$$\begin{tikzcd} & \mathrm{Bun}_B \arrow[dl, "\mathfrak{p}"'] \arrow[dr, "\mathfrak{q}"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_T . \end{tikzcd}$$

Let Λ_G denote the coweight lattice of G, and $\Lambda_G^+, \Lambda_G^{\mathrm{pos}} \subset \Lambda_G$ denote the submonoid of dominant coweights, respectively the submonoid generated by positive simple coroots. Denote by $\Lambda_G^{+,\mathbb{Q}}$ and $\Lambda_G^{\mathrm{pos},\mathbb{Q}}$ the corresponding rational cones. There is a partial ordering on $\Lambda_G^{\mathbb{Q}}$, given by:

$$\lambda_1 \underset{G}{\leq} \lambda_2 \iff \lambda_2 - \lambda_1 \in \Lambda_G^{\mathrm{pos},\mathbb{Q}}.$$

Given $\lambda \in \Lambda_G^{\mathbb{Q}}$, define Bun_B^{λ} as the pre-image of λ under the composition:

$$\mathrm{Bun}_B \xrightarrow{\mathfrak{q}} \mathrm{Bun}_T \xrightarrow{\deg} \Lambda_T^{\mathbb{Q}} \cong \Lambda_G^{\mathbb{Q}}.$$

For each $\theta \in \Lambda_G^{+,\mathbb{Q}}$, define $\mathrm{Bun}_G^{(\leq\theta)}$ as the substack of Bun_G classifying G-bundles $\mathcal{P}_G$ with the following property:

– for each B-bundle $\mathcal{P}_B \in \mathrm{Bun}_B^{\lambda}$ with $\mathfrak{p}(\mathcal{P}_B) \cong \mathcal{P}_G$, we have $\lambda \underset{G}{\leq} \theta$.

23.2.7. **Lemma.** $\mathrm{Bun}_G^{(\leq\theta)}$ *is an open, quasi-compact substack of* Bun_G*.*

Proof. This is in [19]. □

23.2.8. **Remark.** The definition of $\mathrm{Bun}_G^{(\leq\theta)}$ in [19] refers to all standard parabolics P of G, rather than just the Borel. However, the two definitions are equivalent; see the discussion in §7.3.3 in *loc.cit.*

23.2.9. For each integer $n \geq 0$ (as well as $n = \infty$), we let $\mathrm{Bun}_{G,nx}^{(\leq\theta)}$ denote the preimage of $\mathrm{Bun}_G^{(\leq\theta)}$ under the canonical map $\mathrm{Bun}_{G,nx} \to \mathrm{Bun}_G$. We denote the universal G-bundle over $\mathrm{Bun}_G^{(\leq\theta)} \times X$ by $\mathcal{P}_G$, and that over $\mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \times X$ by $\tilde{\mathcal{P}}_G$; their pullbacks to $S \times \mathrm{Bun}_G^{(\leq\theta)} \times X$ and $S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \times X$ are denoted by the same characters.

The key technical assertion we need is:

23.2.10. **Proposition.** *For each $\theta \in \Lambda_G^{+,\mathbb{Q}}$, there exists an integer $N(\theta)$ such that whenever $n \geq N(\theta)$, we have*

$$(\mathfrak{g}^{\kappa}(\mathfrak{m}_x^n)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}^{(\leq\theta)}}) \cap \Gamma(\Sigma, \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G}) = 0$$

as submodules of $\mathfrak{g}^{\kappa}(\mathbf{F}_x)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}^{(\leq\theta)}}$ (via η and γ).

Proof. Fix $\theta \in \Lambda_G^{+,\mathbb{Q}}$. For each integer $n \geq 0$, we have an isomorphism:

$$(\mathfrak{g}^\kappa(\mathfrak{m}_x^n)\widehat{\boxtimes}\mathcal{O}_{\mathrm{Bun}_{G,\infty x}^{(\leq\theta)}}) \cap \Gamma(\Sigma, \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}) \xrightarrow{\sim} \mathrm{R}^0(\mathrm{pr}_{\infty x})_* \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}(-nx),$$

where $\mathrm{pr}_{\infty x}$ is the projection map in the following Cartesian diagram:

$$\begin{tikzcd} S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \times X \arrow[r] \arrow[d, "\mathrm{pr}_{\infty x}"] & S \times \mathrm{Bun}_G^{(\leq\theta)} \times X \arrow[d, "\mathrm{pr}"] \\ S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \arrow[r] & S \times \mathrm{Bun}_G^{(\leq\theta)}. \end{tikzcd}$$

Since $\tilde{\mathcal{P}}_G$ is the pullback of the universal G-bundle $\mathcal{P}_G$ over $S \times \mathrm{Bun}_G^{(\leq\theta)} \times X$, it suffices to show that $\mathrm{R}^0(\mathrm{pr})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ vanishes for sufficiently large n (relative to θ).[22] We shall choose n such that $\mathrm{H}^0(X, \mathfrak{g}_{\underline{\mathcal{P}}_G}(-nx))$ vanishes for all $\underline{\mathcal{P}}_G \in \mathrm{Bun}_G^{(\leq\theta)}$.

23.2.11. **Claim.** For such n, $\mathrm{R}^0(\mathrm{pr})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ is a vector bundle.

Indeed, representing $\mathrm{pr}_*\, \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ by a two-term complex of vector bundles, it suffices to show that $\mathrm{R}^1(\mathrm{pr})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ is flat. However, its fiber at a k-point $(\kappa, \underline{\mathcal{P}}_G)$ is given by:

$$\mathrm{H}^1(\mathrm{L}\,\iota^*_{(\kappa,\underline{\mathcal{P}}_G)} \circ \mathrm{R}\,\mathrm{pr}_*\, \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)) \cong \mathrm{H}^1(X, \mathfrak{g}_{\underline{\mathcal{P}}_G}(-nx)),$$

since $\kappa \cong \mathfrak{g}$ as a G-representation (Corollary 20.1.5). The Riemann-Roch theorem shows:

$$\dim \mathrm{H}^1(X, \mathfrak{g}_{\underline{\mathcal{P}}_G}(-nx)) = -\deg(\mathfrak{g}_{\underline{\mathcal{P}}_G}(-nx)) - \dim(\mathfrak{g}) \cdot (1-g) = \dim(\mathfrak{g})(n+g-1),$$

which is constant as the k-point $(\kappa, \underline{\mathcal{P}}_G)$ varies.

[22]Identification of $\mathrm{R}^0(\mathrm{pr}_{\infty x})_* \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}(-nx)$ with the pullback of $\mathrm{R}^0(\mathrm{pr})_* \mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ follows from flatness of the projection $S \times \mathrm{Bun}_{G,\infty x}^{(\leq\theta)} \to S \times \mathrm{Bun}_G^{(\leq\theta)}$.

Now that $\mathrm{R}^0(\mathrm{pr})_*\mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)$ is a vector bundle, its fiber at any k-point $(\kappa, \underline{\mathcal{P}}_G)$ can be computed as follows:

$$\mathrm{R}^0(\mathrm{pr})_*\mathfrak{g}^\kappa_{\mathcal{P}_G}(-nx)\big|_{(\kappa,\underline{\mathcal{P}}_G)} \xrightarrow{\sim} \mathrm{H}^0(X, \mathfrak{g}_{\underline{\mathcal{P}}_G}(-nx)) \cong 0.$$

This establishes the required vanishing. $\square$

It follows from Proposition 23.2.10 that the $(\mathfrak{k}, H)$-algebroid $\mathcal{L}^\kappa$ (hence also $\mathcal{L}^{(\kappa,E)}$) is an object of $\mathbf{Lie}^{(\mathfrak{k}^n,H^n)}_{\mathrm{inj}}(S \times \mathrm{Bun}^{(\leq\theta)}_G/S)$ whenever $n \geq N(\theta)$.

23.2.12. For each $\theta \in \Lambda^{+,\mathbb{Q}}_G$, denote by $\tilde{\mathcal{T}}^{(\leq\theta)}_{\mathrm{glob}}$ the restriction of the classical pre-twisting $\tilde{\mathcal{T}}^{(\kappa,E)}_{\mathrm{glob}}$ to $S \times \mathrm{Bun}^{(\leq\theta)}_{G,\infty x}$.[23] Given $n \geq N(\theta)$, we can define a pre-twisting over $S \times \mathrm{Bun}^{(\leq\theta)}_G$ by the formula:

$$\mathcal{T}^{(\leq\theta)}_{\mathrm{glob},n} := \mathbf{Q}^{(H_n,H^\flat_n)} \circ \mathbf{Q}^{(\mathfrak{k}^n,H^n)}_{\mathrm{inj}}(\tilde{\mathcal{T}}^{(\leq\theta)}_{\mathrm{glob}}), \tag{23.12}$$

where $H^\flat_n$ denotes the quotient $H_n/\exp(\mathfrak{k}_n)$.

23.2.13. **Remark.** Note that $\mathbf{Q}^{(\mathfrak{k}^n,H^n)}_{\mathrm{inj}}(\mathcal{T}^{(\leq\theta)}_{\mathrm{glob}})$ is well-defined as a classical pre-twisting over $S \times \mathrm{Bun}^{(\leq\theta)}_{G,nx}$, equipped with a $(\mathfrak{k}_n, H_n)$-action. Since the stack $S \times \mathrm{Bun}^{(\leq\theta)}_{G,nx}$ is locally of finite type, any classical pre-twisting gives rise to a pre-twisting, and the $(\mathfrak{k}_n, H_n)$-action induces an $(H_n, H^\flat_n)$-action. Hence the formula (23.12) makes sense.

23.2.14. Suppose $n_1 \geq n_2 \geq N(\theta)$. We would like to construct a canonical isomorphism of pre-twistings

$$\mathcal{T}^{(\leq\theta)}_{\mathrm{glob},n_1} \xrightarrow{\sim} \mathcal{T}^{(\leq\theta)}_{\mathrm{glob},n_2}. \tag{23.13}$$

Indeed, let $(\mathfrak{k}', H')$ be the kernel of the map $(\mathfrak{k}_{n_1}, H_{n_1}) \to (\mathfrak{k}_{n_2}, H_{n_2})$. In particular, H' is of finite type. Furthermore, we have an exact sequence of classical action pairs:

$$1 \to (\mathfrak{k}^{n_1}, H^{n_1}) \to (\mathfrak{k}^{n_2}, H^{n_2}) \to (\mathfrak{k}', H') \to 1.$$

[23]We temporarily suppress the notational dependence on the parameter $(\mathfrak{g}^\kappa, E)$.

Hence, there are isomorphisms:

$$\mathcal{T}^{(\leq\theta)}_{\mathrm{glob},n_1} \xrightarrow{\sim} \mathbf{Q}^{(H_{n_2},H^\flat_{n_2})} \circ \mathbf{Q}^{(H',(H')^\flat)} \circ \mathbf{Q}^{(\mathfrak{k}^{n_1},H^{n_1})}_{\mathrm{inj}}(\tilde{\mathcal{T}}^{(\leq\theta)}_G)$$
$$\xrightarrow{\sim} \mathbf{Q}^{(H_{n_2},H^\flat_{n_2})} \circ \mathbf{Q}^{(\mathfrak{k}',H')}_{\mathrm{inj}} \circ \mathbf{Q}^{(\mathfrak{k}^{n_1},H^{n_1})}_{\mathrm{inj}}(\tilde{\mathcal{T}}^{(\leq\theta)}_G) \xrightarrow{\sim} \mathcal{T}^{(\leq\theta)}_{\mathrm{glob},n_2},$$

using Propositions 21.3.6, 21.4.2, and 21.1.14. In light of the isomorphism (23.13), we may let $\mathcal{T}^{(\leq\theta)}_{\mathrm{glob}}$ denote the pre-twisting $\mathcal{T}^{(\leq\theta)}_{\mathrm{glob},n}$ over $S \times \mathrm{Bun}^{(\leq\theta)}_G$ for any $n \geq N(\theta)$.

23.2.15. Finally, we check that the pre-twistings $\mathcal{T}^{(\leq\theta)}_{\mathrm{glob}}$ glue along various Harder-Narasimhan truncations. Indeed, suppose $\theta_1, \theta_2 \in \Lambda^{+,\mathbb{Q}}_G$. Then we have isomorphisms:

$$\mathcal{T}^{(\leq\theta_1)}_{\mathrm{glob},n}\big|_{S\times(\mathrm{Bun}^{(\leq\theta_1)}_G \cap \mathrm{Bun}^{(\leq\theta_2)}_G)} \xrightarrow{\sim} \mathbf{Q}^{(H_n,(H_n)^\flat)} \circ \mathbf{Q}^{(\mathfrak{k}^n,H^n)}_{\mathrm{inj}}(\mathcal{T}_{\infty x}\big|_{S\times(\mathrm{Bun}^{(\leq\theta_1)}_{G,\infty x} \cap \mathrm{Bun}^{(\leq\theta_2)}_{G,\infty x})})$$
$$\xrightarrow{\sim} \mathcal{T}^{(\leq\theta_2)}_{\mathrm{glob},n}\big|_{S\times(\mathrm{Bun}^{(\leq\theta_1)}_G \cap \mathrm{Bun}^{(\leq\theta_2)}_G)},$$

whenever $n \geq N(\theta_1), N(\theta_2)$. Therefore we obtain a pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ on $S \times \mathrm{Bun}_G$ (relative to S) whose restriction to each $S \times \mathrm{Bun}^{(\leq\theta)}_G$ agrees with $\mathcal{T}^{(\leq\theta)}_{\mathrm{glob}}$.

23.2.16. **Notation.** We write $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}} = \mathbf{Q}^{(\mathfrak{g}^\kappa(\mathbf{O}_x),\mathcal{L}^+_x G)}(\tilde{\mathcal{T}}^{(\kappa,E)}_{\mathrm{glob}})$, although it is tacitly understood that the construction of $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ requires two quotient steps and gluing. In a similar way, we write:

$$\mathcal{T}^{(\kappa,E)}_{\mathrm{glob},n} := \mathbf{Q}^{(\mathfrak{g}^\kappa(\mathfrak{m}^{(n)}_x),H^n)}(\tilde{\mathcal{T}}^{(\kappa,E)}_{\mathrm{glob}}), \tag{23.14}$$

for the corresponding pre-twisting on $S \times \mathrm{Bun}_{G,nx}$.

Since the construction of $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ (resp. $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob},n}$) is functorial in S, we obtain a *universal* pre-twisting $\mathcal{T}^{\mathrm{univ}}_{\mathrm{glob}}$ over $\mathrm{Par}_G \times \mathrm{Bun}_G$ (resp. $\mathcal{T}^{\mathrm{univ}}_{\mathrm{glob},n}$ over $\mathrm{Par}_G \times \mathrm{Bun}_{G,nx}$.)

23.2.17. **Remark.** Note that the DG category $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$-Mod is naturally a QCoh(S)-module. Again from the functoriality in maps $(\mathfrak{g}^\kappa, E) : S \to \mathrm{Par}_G$, we obtain a sheaf of DG categories over Par_G, denoted by $\mathcal{T}^{\mathrm{univ}}_{\mathrm{glob}}$-Mod.

23.2.18. **Lemma.** *The pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ satisfies:*

(1) *It is well-defined independently of the chosen point* $x \in X$*;*

(2) *Its pullback along the map* $\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Bun}_G$ *identifies with the pre-twisting* $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$.

Proof. Part (1) follows from repeating the construction for multiple points. Part (2) is immediate from comparing the constructions of $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ and $\mathcal{T}^{(\kappa,E)}_{\mathrm{loc}}$. $\square$

24. $\lim_{c\to\infty} \mathcal{D}\text{-Mod}^c(\mathrm{Bun}_G) \cong \mathrm{QCoh}(\mathrm{LocSys}_G)$

In this section, we show that at level $\mathfrak{g}^\kappa = \mathfrak{g}^\infty$, the pre-twisting $\mathcal{T}^{(\kappa,E)}_{\mathrm{glob}}$ constructed in the previous section recovers the DG algebraic stack LocSys_G in the following sense: $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}$ is the *inert* pre-twisting on some triangle

$$\mathcal{O}_{\mathrm{Bun}_G} \to \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}} \to \mathcal{Q}^{(\infty,0)}_{\mathrm{desc}}$$

in $\mathrm{QCoh}(\mathrm{Bun}_G)$. Furthermore, the corresponding stack $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1}$ over Bun_G identifies with LocSys_G, so we obtain an equivalence of DG categories $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_G)$. Then we comment on the role of certain additional parameters E when $\mathfrak{g}^\kappa = \mathfrak{g}^\infty$.

24.1. **The underlying $\mathcal{O}_{S\times\mathrm{Bun}_G}$-modules of $\mathcal{T}^{(\kappa,0)}_{\mathrm{glob},n}$.**

24.1.1. We adopt the following notations from the previous section: let $\mathcal{S}_n := S \times \mathrm{Bun}_{G,nx}$, and $\mathcal{X}_n := S \times \mathrm{Bun}_{G,nx} \times X$ which is a curve over $\mathcal{S}_n$. The tautological G-bundle over $\mathcal{X}_n$ is denoted by $\mathcal{P}^{(n)}_G$. Write $\widetilde{S} := S \times \mathrm{Bun}_{G,\infty x}$ and similarly for $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{P}}_G$.

24.1.2. Recall the pre-twisting $\mathcal{T}^{(\kappa,0)}_{\mathrm{glob},n}$ and $\mathcal{T}^{(\kappa,0)}_{\mathrm{glob}} = \mathcal{T}^{(\kappa,0)}_{\mathrm{glob},0}$ which are special cases of (23.14) for the S-valued parameter $(\mathfrak{g}^\kappa, 0)$. Suppose $\mathcal{T}^{(\kappa,0)}_{\mathrm{glob},n}$ is expressed as a map of some formal moduli problems $\widehat{\mathcal{S}}^\flat_n \to \mathcal{S}^\flat_n$ under $\mathcal{S}_n$.

24.1.3. Since $\mathcal{T}^{(\kappa,0)}_{\mathrm{glob},n}$ is the quotient of $\tilde{\mathcal{T}}^{(\kappa,0)}_{\mathrm{glob}}$ by the pair $(\mathfrak{g}^\kappa(\mathfrak{m}_x^n), H^n)$, the underlying ind-coherent sheaves of $\widehat{\mathcal{S}}^\flat_n$ and $\mathcal{S}^\flat_n$ arise from a triangle in $\mathrm{QCoh}(\mathcal{S}_n)$:

$$\mathcal{O}_{\mathcal{S}_n} \to \widehat{\mathcal{Q}}^{(\kappa,0)}_{n,\mathrm{desc}} \to \mathcal{Q}^\kappa_{n,\mathrm{desc}}, \tag{24.1}$$

where $\widehat{\mathcal{Q}}^{(\kappa,0)}_{n,\mathrm{desc}}$ is the descent of the H^n-equivariant complex of $\mathcal{O}_{\tilde{\mathcal{S}}}$-modules

$$\widehat{\mathcal{Q}}^{(\kappa,0)}_n := \mathrm{Cofib}(\mathfrak{g}^\kappa(\mathfrak{m}_x^n) \boxtimes \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathcal{L}}^{(\kappa,0)}),$$

and similarly for $\mathcal{Q}^\kappa_{n,\mathrm{desc}}$.

24.1.4. The Atiyah bundle construction gives rise to a triangle $\omega_{\mathcal{X}_n/\mathcal{S}_n} \to \mathrm{At}(\mathcal{P}^{(n)}_G)^* \to \mathfrak{g}^*_{\mathcal{P}^{(n)}_G}$ over $\mathcal{X}_n$. Its pullback along the projection $\mathfrak{g}^\kappa_{\mathcal{P}^{(n)}_G} \to \mathfrak{g}^*_{\mathcal{P}^{(n)}_G}$ is denoted by:

$$\omega_{\mathcal{X}_n/\mathcal{S}_n} \to \mathcal{E}^\kappa(\mathcal{P}^{(n)}_G) \to \mathfrak{g}^\kappa_{\mathcal{P}^{(n)}_G}. \tag{24.2}$$

Note that there is a canonical isomorphism $\mathcal{Q}^\kappa_{n,\mathrm{desc}} \xrightarrow{\sim} \mathrm{R}\Gamma(X, \mathfrak{g}^\kappa_{\mathcal{P}^{(n)}_G}(-nx))[1]$.

24.1.5. **Proposition.** *The triangle* (24.1) *identifies with the push-out of*

$$\mathrm{R}\Gamma(X, \omega_{\mathcal{X}_n/\mathcal{S}_n}(-nx))[1] \to \mathrm{R}\Gamma(X, \mathcal{E}^\kappa(\mathcal{P}^{(n)}_G)(-nx))[1] \to \mathrm{R}\Gamma(X, \mathfrak{g}^\kappa_{\mathcal{P}^{(n)}_G}(-nx))[1] \tag{24.3}$$

along the trace map $\mathrm{R}\Gamma(X, \omega_{\mathcal{X}_n/\mathcal{S}_n}(-nx))[1] \to \mathcal{O}_{\mathcal{S}_n}$.

24.1.6. The remainder of this subsection is devoted to the proof of Proposition 24.1.5. Since both triangles in question are descent of triangles over $\tilde{\mathcal{S}}$, we ought to establish an H^n-equivariant isomorphism between the triangle:

$$\mathcal{O}_{\tilde{\mathcal{S}}} \to \widehat{\mathcal{Q}}^{(\kappa,0)}_n \to \mathcal{Q}^\kappa_n \tag{24.4}$$

and the push-out of the analogous triangle:

$$\mathrm{R}\Gamma(X, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}(-nx))[1] \to \mathrm{R}\Gamma(X, \mathcal{E}^\kappa(\tilde{\mathcal{P}}_G)(-nx))[1] \to \mathrm{R}\Gamma(X, \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G}(-nx))[1] \tag{24.5}$$

under the trace map $\mathrm{R}\Gamma(X, \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}(-nx))[1] \to \mathcal{O}_{\tilde{\mathcal{S}}}$.

24.1.7. We describe more explicitly the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-modules underlying the extension sequence of Lie-$*$ algebras (23.2):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to (\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,0)})_{\tilde{\mathcal{P}}_G} \to (\mathfrak{g}_{\mathcal{D}}^{\kappa})_{\tilde{\mathcal{P}}_G} \to 0,$$

in the case where the $E = 0$. Namely, consider the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-modules induced from the sequence (24.2) (where we use $\tilde{\mathcal{X}}$ instead of $\mathcal{X}^{(n)}$ in the Atiyah bundle construction):

$$0 \to (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} \to \mathcal{E}^{\kappa}(\widetilde{\mathcal{P}}_G)_{\mathcal{D}} \to (\mathfrak{g}_{\mathcal{D}}^{\kappa})_{\tilde{\mathcal{P}}_G} \to 0$$

Let $\mathcal{E}^{\kappa}(\widetilde{\mathcal{P}}_G)_{\mathcal{D}}^{\mathrm{push}}$ be the push-out along $\mathrm{act} : (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$ of the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module $\mathcal{E}^{\kappa}(\tilde{\mathcal{P}}_G)_{\mathcal{D}}$.

24.1.8. **Lemma.** *The $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-module underlying the extension $(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,0)})_{\tilde{\mathcal{P}}_G}$ identifies with $\mathcal{E}^{\kappa}(\widetilde{\mathcal{P}}_G)_{\mathcal{D}}^{\mathrm{push}}$.*

Proof. Recall that $(\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,0)})_{\tilde{\mathcal{P}}_G}$ is the $\tilde{\mathcal{P}}_G$-twist of the trivial extension $\widehat{\mathfrak{g}}_{\mathcal{D}}^{(\kappa,0)} \xrightarrow{\sim} \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus \mathfrak{g}_{\mathcal{D}}^{\kappa}$. Consider the push-out diagram:

$$\begin{tikzcd}
(\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} \arrow[r] \arrow[d, "\mathrm{act}"] & (\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^{\kappa} \otimes \mathcal{O}_{\tilde{\mathcal{X}}}))_{\mathcal{D}} \arrow[d] \\
\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \arrow[r] & \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus \mathfrak{g}_{\mathcal{D}}^{\kappa}.
\end{tikzcd} \tag{24.6}$$

Note that the entire diagram is acted on by the sheaf of groups $\mathcal{G}$, as described below:

(1) the $\mathcal{G}$-actions on $(\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}}$ and $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$ are trivial, and the action on $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus \mathfrak{g}_{\mathcal{D}}^{\kappa}$ is the one given in §20.5;

(2) the $\mathcal{G}$-action on $(\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^{\kappa} \otimes \mathcal{O}_{\tilde{\mathcal{X}}}))_{\mathcal{D}}$ is the $\mathcal{D}_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$-linear extension of the following $\mathcal{G}$-action on $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^{\kappa} \otimes \mathcal{O}_{\tilde{\mathcal{X}}})$ centralizing $\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}}$:

$$g_{\mathcal{U}} \cdot (\xi \oplus \varphi) = \varphi(g_{\mathcal{U}}^{-1} d g_{\mathcal{U}}) + (\mathrm{Ad}_{g_{\mathcal{U}}}(\xi) \oplus \mathrm{Coad}_{g_{\mathcal{U}}}(\varphi)) \tag{24.7}$$

where $g_{\mathcal{U}} \in \mathcal{G}(\mathcal{U})$ and $\xi \oplus \varphi \in \mathfrak{g}^\kappa \otimes \mathcal{O}_{\mathcal{U}}$.

If we twist the trivial $\mathcal{O}_{\tilde{\mathcal{X}}}$-module extension equipped with the $\mathcal{G}$-action (24.7):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \oplus (\mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}}) \to \mathfrak{g}^\kappa \otimes \mathcal{O}_{\tilde{\mathcal{X}}} \to 0$$

by the G-bundle $\tilde{\mathcal{P}}_G$, we obtain precisely the Atiyah sequence (pulled back along $\mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G} \to \mathfrak{g}^*_{\tilde{\mathcal{P}}_G}$):

$$0 \to \omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \to \mathcal{E}^\kappa(\tilde{\mathcal{P}}_G) \to \mathfrak{g}^\kappa_{\tilde{\mathcal{P}}_G} \to 0.$$

Therefore, twisting the diagram (24.6) by $\tilde{\mathcal{P}}_G$, we obtain a push-out diagram:

$$\begin{tikzcd}
(\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}})_{\mathcal{D}} \arrow[r] \arrow[d, "\mathrm{act}"] & \mathcal{E}^\kappa(\tilde{\mathcal{P}}_G)_{\mathcal{D}} \arrow[d] \\
\omega_{\tilde{\mathcal{X}}/\tilde{\mathcal{S}}} \arrow[r] & (\widehat{\mathfrak{g}}^{(\kappa,0)}_{\mathcal{D}})_{\tilde{\mathcal{P}}_G}.
\end{tikzcd}$$

This proves the Lemma. $\square$

By construction of $\widehat{\mathcal{Q}}^{(\kappa,0)}_n$ and $\mathcal{Q}^\kappa_n$, the required isomorphism shall follow from a general claim. We first explain the set-up (which is quite involved): let $\mathcal{S}$ be a scheme, and $\mathcal{X} := X \times \mathcal{S}$ with section $\underline{x}$ given by the closed point $x \in X$. Suppose we have an exact sequence of $\mathcal{O}_{\mathcal{X}}$-modules:

$$0 \to \omega_{\mathcal{X}/\mathcal{S}} \to \mathcal{E} \to \mathcal{F} \to 0.$$

Let $\mathcal{E}_{\mathcal{D}}$ denote the induced $\mathcal{D}$-module of $\mathcal{E}$ and $\mathcal{E}^{\mathrm{push}}_{\mathcal{D}}$ its push-out along $\mathrm{act} : (\omega_{\mathcal{X}/\mathcal{S}})_{\mathcal{D}} \to \omega_{\mathcal{X}/\mathcal{S}}$.

Then we may form a map between exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & \Gamma_{\mathrm{dR}}(\Sigma, \omega_{\mathcal{X}/\mathcal{S}}) \arrow[r] \arrow[d, "0"] & \Gamma_{\mathrm{dR}}(\Sigma, \mathcal{E}^{\mathrm{push}}_{\mathcal{D}}) \arrow[r] \arrow[d] & \Gamma(\Sigma, \mathcal{F}) \arrow[r] \arrow[d, "\gamma"] \arrow[dl, dotted, "\widehat{\gamma}"'] & 0 \\
0 \arrow[r] & \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \omega_{\mathcal{X}/\mathcal{S}}) \arrow[r] & \Gamma_{\mathrm{dR}}(\mathring{D}_{\underline{x}}, \mathcal{E}^{\mathrm{push}}_{\mathcal{D}}) \arrow[r] & \Gamma(\mathring{D}_{\underline{x}}, \mathcal{F}) \arrow[r] & 0,
\end{tikzcd}$$

as well as a section $\widehat{\gamma}$ from the residue theorem. On the other hand, let $\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}(\mathfrak{m}^{(n)})$ denote the $\mathcal{O}_{\mathcal{S}}$-submodule of $\Gamma_{\mathrm{dR}}(D_{\underline{x}}, \mathcal{E}_{\mathcal{D}}^{\mathrm{push}})$ annihilated by the restriction to $D_{\underline{x}}^{(n)}$; we use the notation $\mathcal{F}(\mathfrak{m}^{(n)})$ for a similar meaning. We have a triangle:

$$\mathcal{O}_{\mathcal{S}} \to \widehat{\mathcal{Q}} \to \mathcal{Q} \tag{24.8}$$

where:

(1) $\widehat{\mathcal{Q}} := \mathrm{Cofib}(\Gamma(\Sigma, \mathcal{F}) \to \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \mathcal{E}_{\mathcal{D}}^{\mathrm{push}})/\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}(\mathfrak{m}^{(n)}))$;
(2) $\mathcal{Q} := \mathrm{Cofib}(\Gamma(\Sigma, \mathcal{F}) \to \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)}))$.

24.1.9. **Remark.** For $\mathcal{S} := \tilde{\mathcal{S}}$, $\mathcal{E} := \mathcal{E}^{\kappa}(\tilde{\mathcal{P}}_G)$, and $\mathcal{F} := \mathfrak{g}^{\kappa}_{\tilde{\mathcal{P}}_G}$, we see from the construction of (24.4) that it identifies with the triangle (24.8).

24.1.10. **Claim.** The triangle (24.8) identifies with the push-out of the canonical triangle:

$$\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathcal{E}(-nx))[1] \to \mathrm{R}\,\Gamma(X, \mathcal{F}(-nx))[1] \tag{24.9}$$

along the trace map $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1] \to \mathcal{O}_{\mathcal{S}}$.

Proof. Recall the identification:

$$\mathcal{Q} = \mathrm{Cofib}(\Gamma(\Sigma, \mathcal{F}) \to \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)})) \xrightarrow{\sim} \mathrm{R}\,\Gamma(X, \mathcal{F}(-nx))[1],$$

which is also valid when $\mathcal{F}$ is replaced by any $\mathcal{O}_{\mathcal{X}}$-module. It suffices to produce a morphism of triangles from (24.9) to (24.8), whose first and third terms are the trace map, respectively the above isomorphism.

Consider the diagram defining $\mathcal{E}_{\mathcal{D}}^{\mathrm{push}}$:

$$\begin{tikzcd}
0 \arrow[r] & (\omega_{\mathcal{X}/\mathcal{S}})_{\mathcal{D}} \arrow[r] \arrow[d] & \mathcal{E}_{\mathcal{D}} \arrow[r] \arrow[d] & \mathcal{F}_{\mathcal{D}} \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \omega_{\mathcal{X}/\mathcal{S}} \arrow[r] & \mathcal{E}_{\mathcal{D}}^{\mathrm{push}} \arrow[r] & \mathcal{F}_{\mathcal{D}} \arrow[r] & 0.
\end{tikzcd}$$

Using the functors $\Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, -)$ and $\mathcal{M} \rightsquigarrow \mathcal{M}(\mathfrak{m}^{(n)})$, we obtain a diagram:

$$\begin{tikzcd}
0 \arrow[r] & \overset{\circ}{\omega}/\omega(\mathfrak{m}^{(n)}) \arrow[r] \arrow[d, "\mathrm{res}"] & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{E})/\mathcal{E}(\mathfrak{m}^{(n)}) \arrow[r] \arrow[d] & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)}) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathcal{O}_{\tilde{\mathcal{S}}} \arrow[r] & \Gamma_{\mathrm{dR}}(\overset{\circ}{D}_{\underline{x}}, \mathcal{E}^{\mathrm{push}}_{\mathcal{D}})/\mathcal{E}^{\mathrm{push}}_{\mathcal{D}}(\mathfrak{m}^{(n)}) \arrow[r] & \Gamma(\overset{\circ}{D}_{\underline{x}}, \mathcal{F})/\mathcal{F}(\mathfrak{m}^{(n)}) \arrow[r] & 0
\end{tikzcd}$$

where the rows are still exact sequences by the Snake lemma. We now take cofibers of the map from the triangle $\Gamma(\Sigma, \omega) \to \Gamma(\Sigma, \mathcal{E}) \to \Gamma(\Sigma, \mathcal{F})$ to the top row, and the cofibers of the map from $0 \to \Gamma(\Sigma, \mathcal{F}) \to \Gamma(\Sigma, \mathcal{F})$ to the bottom row:

$$\begin{tikzcd}
\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1] \arrow[r] \arrow[d] & \mathrm{R}\,\Gamma(X, \mathcal{E}(-nx))[1] \arrow[r] \arrow[d] & \mathrm{R}\,\Gamma(X, \mathcal{F}(-nx))[1] \arrow[d, "\cong"] \\
\mathcal{O}_{\mathcal{S}} \arrow[r] & \widehat{\mathcal{Q}} \arrow[r] & \mathcal{Q}
\end{tikzcd}$$

This is a morphism between triangles. Finally, we make the observation that the residue morphism from $\overset{\circ}{\omega}/\omega(\mathfrak{m}^{(n)})$ passes to the trace map from $\mathrm{R}\,\Gamma(X, \omega_{\mathcal{X}/\mathcal{S}}(-nx))[1]$.

□

We have now constructed an isomorphism from (24.4) to the push-out of (24.5) along the trace map $\mathrm{R}\,\Gamma(X, \omega_{\widetilde{\mathcal{X}}/\widetilde{\mathcal{S}}}(-nx))[1] \to \mathcal{O}_{\widetilde{\mathcal{S}}}$. We omit checking that this map is compatible with the H^n-equivariance structure. □(Proposition 24.1.5)

24.2. **An alternative description of** LocSys_G.

24.2.1. Recall that LocSys_G is defined as the mapping stack $\underline{\mathrm{Maps}}(X_{\mathrm{dR}}, \mathrm{B}\,G)$; it is represented by a DG algebraic stack ([1, §10]). We give an alternative description of LocSys_G in terms of "G-bundles with connections." This latter description is the form in which LocSys_G will appear at level ∞.

24.2.2. Let LocSys'_G denote the prestack over Bun_G such that for every affine DG scheme S, the groupoid $\mathrm{Maps}(S, \mathrm{LocSys}'_G)$ classifies:

(1) a G-bundle $\mathcal{P}_G$ over $S \times X$;

(2) a splitting of the canonical triangle:

$$\mathfrak{g}_{\mathcal{P}_G} \to \operatorname{At}(\mathcal{P}_G) \to \mathcal{T}_{S\times X/S}. \tag{24.10}$$

It is not hard to see that LocSys'_G is represented by a DG algebraic stack.

24.2.3. Note that a lift of $\mathcal{P}_G$ to an S-point of LocSys_G supplies the dotted arrow in the following commutative diagram:

$$\begin{tikzcd} S\times X \arrow[r, "\mathcal{P}_G"] \arrow[d] & S\times \mathrm{B}G \arrow[d] \\ S\times X_{\mathrm{dR}} \arrow[r] \arrow[ur, dotted] & S \end{tikzcd}$$

This arrow gives rise to a splitting of (24.10) because $\mathcal{T}_{S\times X/S\times X_{\mathrm{dR}}}$ is isomorphic to $\mathcal{T}_{S\times X/S}$. In other words, we have a morphism of stacks over Bun_G:

$$\operatorname{LocSys}_G \to \operatorname{LocSys}'_G\,. \tag{24.11}$$

24.2.4. **Proposition.** *The morphism* (24.11) *is an isomorphism.*

Proof. Recall that any eventually co-connective affine DG scheme S fits into a sequence:

$$S_0 \to S_1 \to \cdots \to S_n = S$$

where S_0 is a classical affine DG scheme, and each morphism $S_i \to S_{i+1}$ is a square-zero extension ([36, Chapter 1, Proposition 5.5.3]). Furthermore, for any prestack $\mathcal{Y}$ admitting deformation theory and a point $y : S \to \mathcal{Y}$, a lift of y along a square-zero extension $S \to S'$ is governed by maps out of the cotangent complex $\mathcal{T}^*_{\mathcal{Y}}\big|_y$ (see [36, Chapter 1, 0.1.5]). Hence Proposition 24.2.4 reduces to the following two steps.

Step 1: The morphism (24.11) is an isomorphism when restricted to classical test (affine) schemes. Indeed, this follows from the classical statement that any connection over a curve is automatically flat.

Step 2: The morphism (24.11) identifies the relative cotangent complexes $\mathcal{T}^*_{\mathrm{LocSys}_G\,/\,\mathrm{Bun}_G}$ and $\mathcal{T}^*_{\mathrm{LocSys}'_G\,/\,\mathrm{Bun}_G}$ at any S-point. Indeed, given an S-point of LocSys_G represented by the G-bundle $\mathcal{P}_G$ over $S \times X_{\mathrm{dR}}$, the cotangent complex $\mathcal{T}^*_{\mathrm{LocSys}_G\,/\,\mathrm{Bun}_G}\big|_S$, regarded as a functor $\mathrm{QCoh}(S)^{\leq 0} \to \infty\text{-}\mathbf{Gpd}$, is given by:[24]

$$\begin{aligned}\mathcal{M} \rightsquigarrow \tau^{\leq 0}\,\mathrm{Fib}(\mathrm{R}\,\Gamma(S \times X_{\mathrm{dR}}, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}_G}[1]) \to \mathrm{R}\,\Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}_G}[1])) \\ \xrightarrow{\sim} \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X).\end{aligned}$$

On the other hand, the cotangent complex $\mathcal{T}^*_{\mathrm{LocSys}'_G\,/\,\mathrm{Bun}_G}\big|_S$ sends $\mathcal{M} \in \mathrm{QCoh}(S)^{\leq 0}$ to the ∞-groupoid $\mathrm{Maps}_{S//\,\mathrm{Bun}_G}(\widetilde{S}, \mathrm{LocSys}'_G)$, where $\widetilde{S}$ is the split square-zero extension of S by $\mathcal{M}$, equipped with the canonical map $\widetilde{\mathcal{P}}_G : \widetilde{S} \to S \to \mathrm{Bun}_G$. The ∞-groupoid $\mathrm{Maps}_{S//\,\mathrm{Bun}_G}(\widetilde{S}, \mathrm{LocSys}'_G)$ classifies null-homotopies of the Atiyah sequence of $\widetilde{\mathcal{P}}_G$, regarded as an element in

$$\begin{aligned}&\mathrm{Fib}(\tau^{\leq 0}\,\mathrm{R}\,\Gamma(\widetilde{S} \times X, \mathfrak{g}_{\widetilde{\mathcal{P}}_G} \otimes \omega_X[1]) \to \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X[1])) \\ &\quad \xrightarrow{\sim} \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X[1]),\end{aligned}$$

where the isomorphism uses the facts that $\tau^{\leq 0}$ commutes with limits, and that the ideal sheaf of the embedding $\iota : S \hookrightarrow \widetilde{S}$ agrees with $\iota_*\mathcal{M}$. However, the Atiyah sequence of $\widetilde{\mathcal{P}}_G$ already admits a splitting by the map $\widetilde{S} \to S \to \mathrm{LocSys}'_G$. Hence $\mathrm{Maps}_{S//\,\mathrm{Bun}_G}(\widetilde{S}, \mathrm{LocSys}'_G)$ classifies null-homotopies of the zero element in $\tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X[1])$, i.e., elements of $\tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{M} \otimes \mathfrak{g}_{\mathcal{P}_G} \otimes \omega_X)$.

We conclude the proof of Proposition 24.2.4. □

[24] We implicitly use the Dold-Kan equivalence between connective complexes of abelian groups and group objects in $\infty\text{-}\mathbf{Gpd}$.

24.2.5. **Remark.** If one appeals to [36, Chapter 1, Proposition 8.3.2], it would suffice to check Step 2 only for classical affine schemes S.

24.3. Identification of the fiber at ∞.

24.3.1. We now specialize to the parameter $(\mathfrak{g}^\infty, 0) : \mathrm{pt} \to \mathrm{Par}_G$, where $\mathfrak{g}^\infty$ identifies with the subspace $\mathfrak{g}^* \hookrightarrow \mathfrak{g} \oplus \mathfrak{g}^*$. The pre-twisting $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}$ over Bun_G is obtained as the quotient of $\widetilde{\mathcal{T}}^{(\infty,0)}_{\mathrm{glob}}$ (i.e., (23.4) at parameter $(\mathfrak{g}^\infty, 0)$) by the pair $(\mathfrak{g}^\infty(\mathbf{O}_x), \mathcal{L}^+_x G)$ along the $\mathcal{L}^+_x G$-torsor $\mathrm{Bun}_{G,\infty x} \to \mathrm{Bun}_G$.

24.3.2. **Proposition.** *The following statements hold for* $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}$.

(1) $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}$ *is the inert pre-twisting associated to the triangle* (24.1) *(for* $n = 0$*):*

$$\mathcal{O}_{\mathrm{Bun}_G} \to \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}} \to \mathcal{Q}^{\infty}_{\mathrm{desc}} \tag{24.12}$$

(2) there is a canonical isomorphism of DG stacks:

$$\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}_G\,.$$

Combined with (21.24), we obtain an equivalence of DG categories:

$$\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}\text{-Mod} \xrightarrow{\sim} \mathrm{QCoh}(\mathrm{LocSys}_G).$$

Proof of Proposition 24.3.2. It is clear from the construction that the classical pre-twisting $\widetilde{\mathcal{T}}^{(\infty,0)}_{\mathrm{glob}}$ is given by the central extension of Lie algebroids (with zero Lie bracket and anchor map)

$$0 \to \mathcal{O}_{\mathrm{Bun}_{G,\infty x}} \to \widehat{\mathcal{L}}^{(\infty,0)} \to \mathcal{L}^\infty \to 0.$$

Since $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}$ arises from the quotient of $\widetilde{\mathcal{T}}^{(\infty,0)}_{\mathrm{glob}}$ by $(\mathfrak{g}^\infty(\mathbf{O}_x), \mathcal{L}^+_x G)$, the pre-twisting $\mathcal{T}^{(\infty,0)}_{\mathrm{glob}}$ is the inert pre-twisting on the triangle (24.12).

For the second statement, note that we have a push-out diagram in $\mathrm{QCoh}(\mathrm{Bun}_G)$:

$$\begin{tikzcd} \mathrm{R}\,\Gamma(X, \mathcal{O}_{\mathrm{Bun}_G \times X})^* \arrow[r] \arrow[d] & \mathrm{R}\,\Gamma(X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)^* \arrow[d] \\ \mathcal{O}_{\mathrm{Bun}_G} \arrow[r] & \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}}, \end{tikzcd}$$

by Proposition 24.1.5 and Serre duality. Hence $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1}$ fits into the commutative diagram:

$$\begin{tikzcd} \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathcal{O}_{\mathrm{Bun}_G \times X})^*) & \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)^*) \arrow[l] \\ \mathrm{Bun}_G \arrow[u, "\{1\}"] & \mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1}. \arrow[l] \arrow[u] \end{tikzcd}$$

For any derived scheme S mapping to Bun_G (represented by the G-bundle $\mathcal{P}_G$ over $S \times X$), a computation using the projection formula shows:

(1) $\mathrm{Maps}_{\mathrm{Bun}_G}(S, \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)^*)) \xrightarrow{\sim} \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X)$, and

(2) $\mathrm{Maps}_{\mathrm{Bun}_G}(S, \mathbb{V}(\mathrm{R}\,\Gamma(X, \mathcal{O}_{\mathrm{Bun}_G \times X})^*)) \xrightarrow{\sim} \tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{O}_{S \times X})$.

Hence $\mathrm{Maps}_{\mathrm{Bun}_G}(S, \mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1})$ identifies with the ∞-groupoid

$$\tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathrm{At}(\mathcal{P}_G) \otimes \omega_X) \underset{\tau^{\leq 0}\,\mathrm{R}\,\Gamma(S \times X, \mathcal{O}_{S \times X})}{\times} \{1\}$$

i.e., the ∞-groupoid of splittings of the Atiyah sequence $\mathfrak{g}_{\mathcal{P}_G} \to \mathrm{At}(\mathcal{P}_G) \to \mathcal{T}_{S \times X/S}$. We obtain an isomorphism $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}'_G$ so the result follows from Proposition 24.2.4. $\square$

24.3.3. **Remark.** An alternative argument (one that avoids using the results of §24.1) runs as follows: by a local computation, one identifies the universal envelope of the classical pre-twisting (23.4) with the (topological) ring of functions over $\mathrm{LocSys}_{G,\infty x}(\Sigma)$, the stack classifying $(\mathcal{P}_G, \alpha) \in \mathrm{Bun}_{G,\infty x}$ together with a connection over $\mathcal{P}_G\big|_{\Sigma}$. One then shows that the closed subscheme $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)})_{\lambda=1}$ identifies with

$\mathrm{LocSys}_{G,\infty x}$, and (21.26) gives rise to isomorphisms:

$$\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}_{G,\infty x}/\mathcal{L}^+_x G \xrightarrow{\sim} \mathrm{LocSys}_G\,.$$

24.3.4. Finally, we comment on the role of *integral* additional parameters at ∞, i.e., the ones arising from Z_G-bundles. More precisely, let $E := \mathrm{At}(\mathcal{P}_{Z_G})^*$ for some Z_G-bundle $\mathcal{P}_{Z_G}$. Then E is an extension of $\mathfrak{z}^*_G \otimes \mathcal{O}_X$ by ω_X, so $(\mathfrak{g}^\infty, E)$ is a well defined k-point of $\overline{\mathrm{Par}}_G$.

24.3.5. **Proposition.** *Let* $E = \mathrm{At}(\mathcal{P}_{Z_G})^*$ *for a* Z_G*-bundle* $\mathcal{P}_{Z_G}$*. Then there is a canonical isomorphism of derived stacks:*

$$\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathrm{LocSys}_G \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_G, \tag{24.13}$$

where the second map $\mathrm{Bun}_G \to \mathrm{Bun}_G$ *is the central shift* $-\otimes \mathcal{P}_{Z_G}$*.*

Proof. Note that the $\mathcal{D}_{\mathrm{Bun}_{G,\infty x}\times X/\mathrm{Bun}_{G,\infty x}}$-module (23.2) at parameter $(\mathfrak{g}^\infty, E)$ is induced from the following sequence:

$$0 \to \omega_{\mathrm{Bun}_{G,\infty x}\times X/\mathrm{Bun}_{G,\infty x}} \to \mathrm{At}(\mathcal{P}_{Z_G}\otimes \mathcal{P}_G)^* \to \mathfrak{g}^*_{\mathcal{P}_G} \to 0$$

via the functor $(-)_{\mathcal{D}}$ and pushing out (see §24.1). An argument similar to above shows that $\mathcal{T}^{(\infty,E)}_G$ is the inert pre-twisting associated to the triangle in $\mathrm{QCoh}(\mathrm{Bun}_G)$:

$$\mathcal{O}_{\mathrm{Bun}_G} \to \widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}} \to \mathcal{Q}^{\infty}_{\mathrm{desc}},$$

where we have a canonical isomorphism $\widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}}\big|_{\mathcal{P}_G} \xrightarrow{\sim} \widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}}\big|_{\mathcal{P}_{Z_G}\otimes\mathcal{P}_G}$. Hence the result follows from Proposition 24.3.2. $\square$

24.3.6. **Remark.** A connection on $\mathcal{P}_{Z_G}$ gives rise to a splitting of E, hence an isomorphism $\mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,E)}_{\mathrm{desc}})_{\lambda=1} \xrightarrow{\sim} \mathbb{V}(\widehat{\mathcal{Q}}^{(\infty,0)}_{\mathrm{desc}})$. Geometrically, this corresponds to a lift of the isomorphism $-\otimes\mathcal{P}_{Z_G} : \mathrm{Bun}_G \xrightarrow{\sim} \mathrm{Bun}_G$ to LocSys_G.

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