Geometric Satake for GL_2

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This are notes describing, with minimal assumed background, the geometric Satake equivalence for GL_2. The bulk of the material comes from the excellent survey in [5]

1 Preliminaries

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Let K be a nonarchimedian local field with ring of integers $\mathcal{O}$. In fact, we will usually think of K as $k((t))$ and $\mathcal{O}$ as $k[[t]]$ for another field k (at some point I might evener require k finite)

Denote by G the algebraic group GL_2 defined over K and $\hat{G}$ the algebraic group GL_2 defined over $\mathbb{C}$.

This notation makes more sense for classical Satake in more general settings, where H is a reductive algebraic group over K and $\hat{H}$ is the complex dual group, the algebraic group over $\mathbb{C}$ constructed from the dual of the root data for H. Since the root data for GL_2 is self dual we have the above convention.

Fix a maximal torus T of G, contained in a borel B, all defined over $\mathcal{O}$ We likewise define $\hat{T} \subset \hat{B} \subset \hat{G}$.

Further, define the character and cocharacter lattices for T and $\hat{T}$

$$\begin{aligned}
X^\bullet &= X^\bullet(T) = \mathrm{Hom}(T, \mathbb{G}_m) \\
\hat{X}^\bullet &= X^\bullet(\hat{T}) = \mathrm{Hom}(\hat{T}, \mathbb{G}_m) \\
X_\bullet &= X_\bullet(T) = \mathrm{Hom}(\mathbb{G}_m, T) \\
\hat{X}_\bullet &= X_\bullet(\hat{T}) = \mathrm{Hom}(\mathbb{G}_m, \hat{T})
\end{aligned}$$

Note by definition the isomorphism $X^\bullet \simeq \hat{X}_\bullet$

Finally, define $W = N_G(T)/T$. In this case W can be identified with the symmetric group on two letters. I will sometimes use W to denote the Weyl group of $\hat{G}$ instead, but this will hopefully be minimally confusing.

Classical Satake gives aring isomorphism

$$\mathcal{H} \simeq \mathbb{C}[\hat{X}^\bullet]^W$$

Here, $\mathcal{H}$ is the ring of locally constant compactly supported complex functions on $G(K)$ which are invariant by multiplication by $G(\mathcal{O})$ on either side, i.e. $f(x) = f(xk) = f(kx)$ for $k \in G(\mathcal{O})$.

Addition defined in the obvious manner and multiplication defined by a sort of convolution

$$f * g(x) = \int_{G(K)} f(y)g(y^{-1}x)$$

Geometric Satake is a categorification of the above isomoprhism. The main content is the existence of categories Sat_G and $\mathcal{R}$, and an equivalence of categories F between them such that the induced map

$$K(\mathrm{Sat}_G) \otimes \mathbb{C} \simeq K(\mathcal{R}) \otimes \mathbb{C}$$

where $K(-)$ is the grothendieck group of a category, can be identified with the Satake isomorphism listed earlier.

2 $\mathcal{R}$

The category $\mathcal{R}$ is easy to describe. $\mathbb{C}[\hat{X}^\bullet]^W$ is the ring of complex valued "polynomial" functions on $\hat{T}$ that are invariant under the action of W. Such functions can be tautologically extended to conjugation invariant functions on the space of all 2×2 diagonalizable matrices. And since diagonalizable matrices are dense in $GL_2(\mathbb{C})$ there is at most one extension to the set of continuous, conjugation invariant functions on all of $GL_2(\mathbb{C})$, a.k.a the characters of $GL_2(\mathbb{C})$.

This shows that the restriction map $\mathbb{C}[\hat{G}]^{\hat{G}} \to \mathbb{C}[\hat{T}]^W$ is injective. To show surjectivity, note that this isa map of algebras. $\mathbb{C}[\hat{T}]^W$ is generated as a $\mathbb{C}$ algebra by the characters

$$\det : \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mapsto xy$$

$$\mathrm{std} : \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} \mapsto x + y$$

both of which are realized as the restrictions of characters on $\hat{G}$, of the determinant the standard representation.

The ring of characters of $\hat{G}$ can be categorified by realizing it as the grothendieck group of the category of representations of $\hat{G}$. We take this to be the category $\mathcal{R}$.

3 Tannakian Formalism

Sat_G has a more complicated definition, and it is what we will spend the rest of the talk attempting to decsribe. I'll start with the full definition

Definition 1. Sat_G *is the category of perverse sheaves on the affine Grassmanian* Gr*, equivariant under the action of* $G(\mathcal{O})$*, with monoidal structure given by convolution product.*

(Some of) these things will be given a better definition, but I'll start with a high level view. Foremostly, what we are doing is constructing a category equivalent to a category of representations of an affine algebraic group. To do this, we use the following theorem. For a full account of the definitions and proof, see [1]

Theorem 1. *Let* $(\mathcal{C}, \otimes)$ *be a rigid abelian tensor category such that* $k \simeq \mathrm{End}(\mathbb{1})$*, and let* $\omega : \mathcal{C} \to \mathrm{Vec}_k$ *be an exact faithful* k*-linear tensor functor. Then,*

1. *The functor* $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ *of* k*-algebras (which sends* R *to the set of natural transformations of the functor* $\omega \otimes R$ *to itself satisfying certain compatibility requirements) is represented by an affine group scheme* G*;*

2. *The functor* $\mathcal{C} \to \mathrm{Rep}_k(G)$ *defined by* ω *is an equivalence of tensor categories*

So Sat_G must be

1. an abelian category with

2. a product structure and

3. a functor to $\mathbb{C}$ vector-spaces

This will immediately let us recognize it as equivalent to the category of representations of some group. It is then more work to show that the group is in fact $\hat{G}$, though that point (among many others!) will be skipped here.

4 The Affine Grassmanian

4.1 As a representation of a functor

Now we begin checking off pieces in the definition. First, we preset two different definitions for Gr

For R a k-algebra, an $R[[t]]$ lattice in $R((t))^2$ is a projective $R[[t]]$ module $\Lambda \subset R((t))^2$ such that $\Lambda \otimes_{R[[t]]} R((t)) = R((t))^2$

The Affine Grassmanian of G, denoted Gr, is the ind-projective k-scheme[1] representing the functor of k-algebras that sends a k-algebra R to the set of $R[[t]]$ lattices of $R((t))^2$.

There is something to prove here, namely that this functor is represented by an ind-scheme. We don't give a full proof, but at least describe the argument on k-points:

[1] not K!

Proof. Define Λ^0 as the lattice $k[[t]]^2 \subset k((t))^2$. For every lattice Λ, there is some positive n such that $t^n\Lambda_0 \subset \Lambda \subset t^{-n}\Lambda_0$.

Now fix n, define $Gr^{(n)} \subset Gr$ as the space of lattices Λ for which the above containment holds. Note $G^{(n)} \subset G^{(n+1)} \subset ....$

$\Lambda/t^n\Lambda_0$ is a subspace of $t^{-n}\Lambda_0/t^n\Lambda_0$, a $4n$ dimensional vector space over k. So there is a map

$$p : Gr^{(n)} \to Gr(t^{-n}\Lambda_0/t^n\Lambda_0) = \sqcup_i Gr(i, t^{-n}\Lambda_0/t^n\Lambda_0)$$

This map is injective, but not surjective. Multiplication by t defines a nilpotent operator on $t^{-n}\Lambda_0/t^n\Lambda^0$, and subspaces in the image of p must be preserved under this operator. This imposes a closed condition on the grassmanian so each $Gr^(n)$ is a projective scheme. It is also clear that $Gr^{(n)} \subset Gr^{(n+1)}$ is a closed embedding. $\square$

We can now directly study the geometry of these $Gr^{(n)}$.

$Gr^{(0)}$ is a point, but $Gr^{(1)}$ is alread interesting. We can take $(1, 0), (0, 1), (t^{-1}, 0), (0, t^{-1}$ as a basis for $t^{-1}\Lambda_0/t\Lambda_0$ so the operator t becomes

$$\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and then study the preserved subspaces by dimension

- There is a single 0 dimensional subspace
- The one dimensional subspaces preserved by t are those in the kernel. The kernel is two dimensional, so we have a $\mathbb{P}^1$ of such spaces
- A two dimensional subspace preserved by t is either $\ker(t)$, or spanned by v and tv for some vector v. $tv \subset \ker(t)$ so the space of choices for tv is $\mathbb{P}^1$. For a given tv there is an $\mathbb{A}^1$ of choices of v generating distinct subspaces. So the scheme in question is a cone over $\mathbb{P}^1$, with the singular point the space corresponding to Λ_0, corresponding to the inclusion $Gr^{(0)} \subset Gr^{(1)}$
- A three dimensional subspace preserved by t must contain $ker(t)$. there are a $\mathbb{P}^1$of such subspaces.
- There is a unique four dimensional subspace preserved by t

4.2 As a quotient

For a second definition, notice that $GL_2(K)$ acts transitively on the space of lattices and the stabilizer of Λ_0 is $GL_2(\mathcal{O})$, so an alternate definition of Gr could be (brushing past intricacies in the definition of a quotient) $GL_2(K)/GL_2(\mathcal{O})$

We have the decomposition

$$GL_2(K) = \bigsqcup_{a \le b \in \mathbb{Z}} GL_2(\mathcal{O}) \begin{bmatrix} t^a & 0 \\ 0 & t^b \end{bmatrix} GL_2(\mathcal{O})$$

So to understand Gr we can study the quotients

$$GL_2(\mathcal{O}) \begin{bmatrix} t^a & 0 \\ 0 & t^b \end{bmatrix} GL_2(\mathcal{O})/GL_2(\mathcal{O})$$

For any $G \subset H$ and $h \in H$ there is a bijection $GhG/G \simeq G/(G \cap hGh^{-1})$. We realize this by sending the class represented by $g_1 h g_2$ in GhG/G to the class represented by g_1 in $G/(G \cap hGh^{-1})$. Note that

$$g_1 h g_2 = (g_3 h g_4) g$$

is equivalent to

$$g_1 = g_3 (h g_4 g g_2^{-1} h^{-1})$$

So for various integer a and b we need to study schemes of the form

$$S_{a,b} := GL_2(\mathcal{O})/(GL_2(\mathcal{O}) \cap \begin{bmatrix} t^a & 0 \\ 0 & t^b \end{bmatrix} GL_2(\mathcal{O}) \begin{bmatrix} t^{-a} & 0 \\ 0 & t^{-b} \end{bmatrix})$$

Note that $S_{a,b}$ is distinct from, but isomorphic to $S_{a+c,b+c}$ for any constant c, so for the purposes of studying the varieties, we can assume $a = 0$ and $b \ge 0$. Then we have

$$GL_2(\mathcal{O})^{(b)} := GL_2(\mathcal{O}) \cap \begin{bmatrix} t^0 & 0 \\ 0 & t^b \end{bmatrix} GL_2(\mathcal{O}) \begin{bmatrix} t^0 & 0 \\ 0 & t^{-b} \end{bmatrix} = \begin{bmatrix} \mathcal{O} & t^b\mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{bmatrix}$$

So when taking the quotient $GL_2(\mathcal{O})/GL_2(\mathcal{O})^{(b)}$ we can first quotient by powers of t^b and realize it as $GL_2(R_b)/B(R_b)$ where R_b is the ring $k[t]/t^b$ and B is the group of upper triangular matrices.

This quotient is, unsurprisingly, extremely similar to the spaces we have already seen.

An element of $GL_2(R_b)$ can be written as $A + \sum_{i=1}^{b-1} t^i B_i$ for $A \in GL_2(k)$ and $B \in \mathrm{Mat}_{2\times 2}(k)$.

If V is the standard representation of $GL_2(k)$ then we define an action of $GL_2(R_b)$ on $V^{\oplus b}$ where $A \in GL_2(k)$ acts by $A^{\oplus b}$ and t acts as defined earlier.

If W is the subspace of V preserved by $B(k)$, then $B(R_b)$ is the stabilizer of the b dimensional, t-stable space $W^{\oplus b}$ and $G(R_b)$ acts transitively on the space of all b-dimensional t-stable subspaces of $V^{\oplus b}$, so this realizes the quotient as a (possibly no longer closed) subvariety of a grassmanian.

In our earlier language, $S_{a,b}$ is the set of lattices Λ such that $t^a\Lambda_0 \subset \Lambda \subset t^b\Lambda_0$, and the quantity $b - a$ is minimal for Λ.

Once can also show that the closure of a strata $S_{a,b}$ is $\sqcup S_{a+i,b-i}$ for $0 \le i \le \frac{1}{2}|b-a|$

The important quality brought to light by this defintion is that Gr is stratified by varieties corresponding to (positive) cocharacters of the torus $T \subset G$

4.3 Convolution

Finally, we define the "twisted product $Gr\hat{\times}Gr$ as a quotient of $G(K)\times G(K)/G(\mathcal{O})$ where (y,Λ) (z,Λ) if $y\Lambda = z\Lambda$.

There is a "multiplication" map $m : Gr\times Gr \to Gr$ sending $(y,\Lambda)\mapsto y\Lambda$, with the fiber of a lattice Λ isomorphic to $G(K)/\operatorname{Stab}(\Lambda)$, so the fibers are pointwise isomorphic to Gr, but the sturcture isn't actually a product.

5 Intersection Cohomology and Perverse Sheaves

5.1 Definitions

Given a real or complex "manifold" X with singularities that admits a stratification $X - \sqcup X_\alpha$ with each X_α a smooth manifold. Suppose further that X is the closure of a single strata X_0. Intersection homology is a tool for capturing information about X similar to simplicial homology, but more sensitive to the stratification and singularities.

To do this, we choose a *perversity*, a function p from the set of strata of X to the integers. Intersection homology is computed similarly to singular homology, by taking groups of chains and quotienting the cycles by the boundaries, but we require that the chains be closed in X_0, and that for each strata $X_\alpha\subset X\backslash X_0$, the chains of dimension i meet X_α in a space of dimension at most $i-\operatorname{codim}(X_\alpha)+p(X_\alpha)$, with boundary meeting X_α in dimension at most $i-1-\operatorname{codim}(X_\alpha)+p(X_\alpha)$

The groups of i-cycles modulo i-boundaries are denoted ${}^pIC_i(X)$.

Not that if we choose the zero perversity, this is just the requirement that all chains be transverse to the singular strata, and increasing the perversity is a way of allowing more chains and types of intersections. See [2] for examples of interesting computations.

The most interesting perversity seems to be the so called middle perversity, $p(X_\alpha) = \frac{\operatorname{codim}(X_\alpha)-2}{2}$. This cannot be achieved for every stratification, obviously, but will be achievable for us.

Some notes: The above description can be modified just like singular cohomology, to have the chains take values in a local system, rather than constant coefficients. Also, intersection *cohomology* can be defined by having the cochains be functionals on allowable chains.

5.2 IC sheaves

The above definition only makes sense in some topologies, the zariski topology is too coarse to get interesting singular homology groups. But there is an analog with a more algebraic definition.

Given a local system $\mathcal{E}$ on X_0, and a perversity p there is a complex of constructible sheaves on X, unique up to quasi-isomorphism, denoted ${}^p\mathcal{IC}_\bullet(X,\mathcal{E})$ such that

$$\mathbb{H}^i(X,{}^p\mathcal{IC}_\bullet(X,\mathcal{E})) = {}^p H^i(X,\mathcal{E})$$

See [3] for a construction, due to Deligne.

For a quasi-projective scheme X, a perverse sheaf is a complex of constructible sheaves on X satisfying certain properties I won't recount here. This is relevant to the earlier discussion because

1. The category of perverse sheaves on X is a full subcategory of the derived category of constructible sheaves on X, and moreover is abelian.

2. Let $i_*Z \hookrightarrow X$ is a closed embedding and choose a stratfication for Z so that it is the closure of a single strata, and a local system $\mathcal{E}$ on Z_0. Then if p is the middle perversity on Z, $i_*({}^p\mathcal{IC}_\bullet(Z,\mathcal{E})$ is a perverse sheaf on X.

3. The category of perverse sheafs on X is artinian, every element admits a finite length decomposition series into irreducibles. Moreover, the irreducible elements are exactly the perverse sheaves defined in 2

6 Finally, Satake

6.1 Sat_G as tannakian

We can now define Sat_G. I'll repeat the definition

Definition 2. Sat_G *is the category of perverse sheaves on the affine Grassmanian Gr, equivariant under the action of $G(k[[t]])$, with monoidal structure given by convolution product.*

The action of $GL_2(k[[t]])$ on Gr has orbits parametirzed by cocharacters, as we have seen. So requiring equivariance means that the allowable $Z \hookrightarrow X$ are the $\overline{S_{a,b}} \hookrightarrow Gr$.

If μ is a (positive) cocharacter, we can take the constant local system on S_μ, extend it to a perverse sheaf on $\overline{S_\mu}$ and push forward along the inclusion to get a perverse sheaf on Gr, which we will denote IC_μ.

This category has a functor to $\mathbb{C}$-vector spaces by taking a perverse sheaf $\mathcal{F}^\bullet$ to $\mathbb{H}(\mathcal{F}^\bullet) = \oplus_i \mathbb{H}^i(\mathcal{F}^\bullet)$.

We also have a product structure. Given two perverse sheaves$\mathcal{A}$ and $\mathcal{B}$ on Gr, we can take a twisted product to get a perverse sheaf $\mathcal{A}\hat{\boxtimes}\mathcal{B}$on $Gr\hat{\times}Gr$, though the details, and the proof that the construction is perverse, will be omitted. We then define

$$\mathcal{A} * \mathcal{B} = m_!(\mathcal{A}\hat{\boxtimes}\mathcal{B})$$

This is the structure needed to apply the tannakian formalism. A corollary of this should be that when X is a cone over BP^1 stratified at the singular point, and p is the middle perversity (which in this case sends the point to 2) $\oplus_i^p IH^i(X)$ should be an irreducible representation of $GL_2(\mathbb{C})$? I couldn't see this from the computation, or understand the $GL_2(\mathbb{C})$ action.

We will now take on faith that the tannakian formalism yields an isomorphism between Sat_G and $\mathcal{R}$.

6.2 Sat_G sees the Hecke algebra

Given a complex of constructible sheaves $\mathcal{F}^\bullet$ on Gr we get a function $Gr(k) \to \mathbb{C}$ $f_{\mathcal{F}}$

$$f_{\mathcal{F}}(x) = \sum_i (-1)^i \operatorname{Tr}(\sigma_x, H^i_{\overline{x}}(\mathcal{F}^\bullet))$$

Where $\overline{x}$ is a geometric point over x and $\sigma_x \in \mathrm{Gal}(\overline{x}/x)$ is the geometric frobenius.

When The sheaves in Sat_G are left $G(k[[t]])$ invariant, and compactly supported, so when the sheaf-function correspondence is applied they can be thought of as functions in $\mathcal{H}(G) \otimes \mathbb{C}$.

Note that for $x \in Gr(k)$, $m^{-1}(x)$ can be identified with pairs $(y, y^{-1}x)$ for $y \in G(K)/G(\mathcal{O})$

Then

$$f_{\mathcal{A}} * f_{\mathcal{B}}(x) = \int_{G(K)} f_{\mathcal{A}(y)} f_{\mathcal{B}})(y^{-1}x) = \sum_{G(F)/G(\mathcal{O})} f_{\mathcal{A}}(y) f_{\mathcal{B}}(y^{-1}x) = \sum_{z \in m^{-1}(x)(k)} \operatorname{Tr}(\sigma_{\overline{z}}, \mathcal{A} \hat{\boxtimes} \mathcal{B}) = f_{\mathcal{A}*\mathcal{B}}$$

The cheaf function correspondence factors through the grothendieck group, and we in fact have an isomorphism of algebras $K(\mathrm{Sat}_G) \otimes \mathbb{C} \simeq \mathcal{H} \otimes \mathbb{C}$

To actually see what functions the sheaves IC_μ give, we need several results from [4], which I will try to list.

1. $$\dim H^{2i-(2\rho,\mu)}_{t^\lambda} IC_\mu = a_{\mu\lambda,i}$$

   ρ is the sum of all positive roots and $\lambda \leq \mu$, and

   $$a_{\mu\lambda}(x) = \sum_i a_{\mu\lambda,i} x^i$$

   is a Kazhdan Lusztig polynomial.

2. The cohomology sheaves $H^i(IC^\bullet_\mu)$ are zero is i is odd, and when i is even the eigenvalues of frobenius are equal to $q^{i/2}$

which lets us show

$$f_{IC_\mu} = c_\mu + \sum_{\lambda \leq \mu} a_{\mu\lambda}(q) c_\lambda$$

References

[1] P. Deligne and J.S. Milne *Tannakian Categories* https://www.jmilne.org/math/xnotes/tc2018.pdf

[2] Robert Macpherson. *Intersection Homology and Perverse Sheaves* https://www.math.ru.nl/~bmoonen/Seminars/IntersectionHomology.pdf

[3] Bruno Klinger. *Intersection Homology and Perverse Sheaves* https://www2.mathematik.hu-berlin.de/~klingleb/cours/courspervers.pdf

[4] Kazhdan, David and Lusztig, George, *Schubert Varieties and Poincare Duality* https://app.dimensions.ai/details/publication/pub.1089196420

[5] Xinwen Zhu. *An introduction to affine Grassmannians and the geometric Satake equivalence* https://arxiv.org/pdf/1603.05593.pdf