ON THE UBIQUITY OF TWISTED SHEAVES

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1. Introduction

In this paper we describe several results, built up over the last decade or so, that use twisted sheaves to study questions in algebra, arithmetic, and cohomology. The paper is for the most part expository and aims to collect in one place what is known about these methods, their applications, and some questions about their future. It also hopes to be a guide to a chunk of the literature, where the reader can find the details of those things only hinted at here.

2. Classical questions

To warm up, here are some questions that one can attack with the methods we will sketch below. Readers unfamiliar with the Brauer group of a field are referred to the masterful exposition in [35].

Question 2.1. Suppose X is a quasi-compact separated scheme. Do the Brauer group and cohomological Brauer group of X coincide?

Question 2.2. Suppose K is a C_d-field. Does every class $\alpha \in \mathrm{Br}(K)$ satisfy

$$\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{d-1}?$$

Question 2.3 (Colliot-Thélène). Suppose X is a geometrically integral geometrically rational variety over a global field k. Is the Brauer-Manin obstruction the only one for 0-cycles on X of degree 1 over k?

Question 2.4. Suppose K is of transcendence degree 1 over a totally imaginary number field. Is the u-invariant of K finite?

Question 2.5. What is the Kodaira dimension of the moduli space of polarized supersingular K3 surfaces?

Question 2.6. Fix a finite field k. Are there only finitely many K3 surfaces defined over k?

This hopefully gives the reader a sense of the broad range of questions and areas that make contact with twisted sheaves. Entirely left out of this exposition is the connection between twisted sheaves and mathematical physics; this is in fact the (modern) birthplace of the theory of twisted sheaves (although traces can be found in France in the 1960s and 1970s [11]). The interested reader is referred to [27, 33, 4, 26, 41, 38, 9, 34, 14, 40, 3] and their references for a start on that story.

3. Twisted sheaves

What are twisted sheaves? They are most profitably approached as sheaves on gerbes (a certain kind stack), but they also have a "softer" description as sheaves with gluing data that aren't quite descent data. We briefly summarize these two descriptions.

Fix an algebraic space X and a G-gerbe $\mathscr{X} \to X$, where $G \hookrightarrow \mathbf{G}_m$ is a closed subgroupscheme (so $G \cong \mathbf{G}_m$ or $G \cong \boldsymbol{\mu}_n$ for some n). (For background on G-gerbes, see [11, 19, 20, 28].) Any sheaf $\mathscr{F}$ on $\mathscr{X}$ gets a natural right G-action $\mathscr{F} \times G \to \mathscr{F}$ in the following way: given an object $f : U \to \mathscr{X}$ and an element $g \in G(U)$, the automorphism g of f induces (by the sheaf property of $\mathscr{F}$) an isomorphism $g^*\mathscr{F}_U \to \mathscr{F}_U$, giving an action on sections $g^* : \mathscr{F}(U) \to \mathscr{F}(U)$.

Definition 3.1. A sheaf of $\mathscr{O}_{\mathscr{X}}$-modules is *$\mathscr{X}$-twisted* if the left action $G \times \mathscr{F} \to \mathscr{F}$ associated to the right action described above is identified with scalar multiplication via the embedding $G \hookrightarrow \mathbf{G}_m$ and the $\mathscr{O}$-module structure on $\mathscr{F}$.

This might seem rather abstract, so let us record the softer definition for the sake of intuition-building. The gerbe $\mathscr{X} \to X$ represents a class in $\mathrm{H}^2(X, G)$. We can also represent such classes by appropriate 2-cocycles for hypercoverings in the fppf topology. Suppose for simplicity that we have an fppf covering $U \to X$ and a 2-cocycle $a \in G(U \times_X U \times_X U)$ (i.e., in the associated Čech hypercoverring) representing the same cohomology class.

Definition 3.2. An *a-twisted sheaf* is a pair $(\mathscr{F}, \varphi)$ where $\mathscr{F}$ is an $\mathscr{O}_{\mathscr{U}}$-module and $\varphi : \mathscr{F} \times_X U \xrightarrow{\sim} U \times_X \mathscr{F}$ is an isomorphism whose coboundary

$$\delta\varphi : \mathscr{F} \times_X U \times_X U \xrightarrow{\sim} \mathscr{F} \times_X U \times_X U$$

is equal to multiplication by a.

For simplicity of notation, we think of everything – U, X, $U \times_X U$, $\mathscr{F}$, etc. – as sheaves on the big fppf site of X. This way $\mathscr{F} \times_X U$ is a sheaf on $U \times_X U$ that is usually written as $(\mathrm{pr}_1)^*\mathscr{F}$, etc. (We will not make much use of this definition or notation in what follows.)

Colloquially, an a-twisted sheaf is a sheaf on U with a gluing datum that is obstructed from being a descent datum by the cocycle a.

Remark 3.3. A warning to the reader who plans to look at the literature on twisted sheaves, especially that arising from the physics side. Early workers on twisted sheaves made an absolutely disastrous decision: when dealing with twisted skyscraper sheaves over a geometric point $x \to X$ (or, more generally, twisted sheaves supported on schemes over which the twisting class is trivial), they used the fact that a is cohomologous to 1 to claim that the "twisted skyscraper sheaf at x" and the (usual) "skyscraper sheaf at

x" are "the same", or that the usual skyscraper sheaf could be "given the structure of twisted sheaf". As is hopefully made clear in Definition 3.1, *these are objects in different categories*. This terminological mistake has had some negative consequences over the years. I beg you to make the distinction if you ever use these things.

Notation 3.4. Write $\mathrm{QCoh}^{\mathrm{tw}}(\mathscr{X})$ for the category of quasi-coherent $\mathscr{X}$-twisted sheaves and $\mathrm{Coh}^{\mathrm{tw}}(\mathscr{X})$ for the subcategory of coherent $\mathscr{X}$-twisted sheaves.

One thing Definition 3.2 makes fairly clear is the following change of structure group statement.

Lemma 3.5. *Let $\mathscr{Y} \to X$ be the $\mathbf{G}_m$-gerbe associated to the G-gerbe $\mathscr{X} \to X$. The natural G-equivariant map $\mathscr{X} \to \mathscr{Y}$ induces by pullback equivalences of abelian categories*

$$\mathrm{QCoh}^{\mathrm{tw}}(\mathscr{Y}) \to \mathrm{QCoh}^{\mathrm{tw}}(\mathscr{X}) \textit{ and } \mathrm{Coh}^{\mathrm{tw}}(\mathscr{Y}) \to \mathrm{Coh}^{\mathrm{tw}}(\mathscr{X}).$$

Thus, from a categorical point of view, $\mathbf{G}_m$-gerbes suffice. We will see in a moment that the proper formulation of standard moduli problems is somewhat sensitive to the structure group.

4. The first appearance of twisted sheaves

Before we dive deeper into the problems of section 2, let us see the most primitive appearance of twisted sheaves: as Picard obstructions.

Example 4.1. Recall that a *Brauer-Severi* variety over a field k is a variety X such that for some $n > 0$

$$X \otimes \overline{k} \cong \mathbf{P}^n_{\overline{k}}.$$

In other words, X is an étale form of projective space. It is a basic results [35, §X.6] that $X \cong \mathbf{P}^n_k$ if and only if X contains a divisor $D \subset X$ that is a hyperplane over $\overline{k}$ if and only if X contains a point defined over k. The canonical example is a plane conic: the conic

$$Z(X^2 + Y^2 + Z^2 = 0) \subset \mathbf{P}^2_{\mathbf{R}}$$

does not contain a rational point, hence is not isomorphic to $\mathbf{P}^1$. More generally, a conic is isomorphic to $\mathbf{P}^1$ if and only if it has a rational point: the isomorphism is given by projection from that point. (In the general case, we can make a rational isomorphism by projecting from the intersection point of several copies of a hyperplane, and we can make a hyerplane by pulling back a hyperplane along such a rational projection, showing that the last two conditions are equivalent. Once we have a hyperplane, the full linear system it defines gives an isomorphism with projective space, showing that the last two conditions are equivalent to the first!)

What is the relative Picard scheme $\mathrm{Pic}_{X/k}$? Well,

$$\mathrm{Pic}_{X \otimes \overline{k}/\overline{k}} \cong \mathbf{Z}_{\overline{k}}$$

and the ample generator defines a section that is Galois-invariant. We conclude that

$$\mathrm{Pic}_{X/k} \cong \mathbf{Z}_k,$$

the constant group scheme with value $\mathbf{Z}$.

What is the relative Picard *stack*? Since X is geometrically integral, we know that $\mathscr{P}ic_{X/k}$ is a $\mathbf{G}_m$-gerbe over $\mathrm{Pic}_{X/k}$. In particular, the preimage of the section $1 \in \mathbf{Z}_k$ defines a $\mathbf{G}_m$-gerbe

$$\mathscr{G} \to \operatorname{Spec} k.$$

The restriction of the universal sheaf

$$\mathscr{L} \in \mathscr{P}ic(X \times \mathscr{P}ic_{X/k})$$

yields an invertible sheaf on

$$\mathscr{X} := X \times_{\operatorname{Spec} k} \mathscr{G}.$$

As a first exercise in unwinding definitions, etc., we invite the reader to show that this sheaf $\mathscr{L}$ is an $\mathscr{X}$-twisted sheaf.

Let's consider the gerbe $\mathscr{G}$ itself. Does it admit an invertible twisted sheaf? Suppose it did, say L. Then the sheaf $\mathscr{L} \otimes \mathrm{pr}_2^* L^\vee$ would be an "untwisted" invertible sheaf on X whose image in $\mathrm{Pic}_{X/k}$ is 1. In other words, it would cut out a hyperplane, and X would be trivial (i.e., isomorphic to $\mathbf{P}^n_k$). On the other hand, if X is trivial, then there is a hyperplane, so there is an object of $\mathscr{P}ic_{X/k}$ mapping to 1 and thus a k-map

$$\operatorname{Spec} k \to \mathscr{G}.$$

We conclude that $\mathscr{G}$ is trivial (isomorphic to the classifying stack $\mathrm{B}\mathbf{G}_m$) if and only if X is trivial. Since X has a $k(X)$-point, we see that $\mathscr{G}_{k(X)}$ should be trivial, and we indeed get a trivialization from the universal sheaf $\mathscr{L}$.

This example is less trivial than it may seem. By thinking of Picard obstructions as gerbes with twisted sheaves (geometric objects) instead of cohomology classes, we can study moduli problems associated to these obstructions. These moduli problems play an integral role all but the first question in Section 2, as we will see in a moment.

5. Twisted sheaves and the Brauer group

Let us briefly summarize some aspects of the relation between twisted sheaves and the Brauer group of a scheme. By analogy with the Brauer group of a field, one can define the Brauer group of a scheme as follows [28, Chap. IV], [12]. Fix a scheme X.

Definition 5.1. An *Azumaya algebra of degree n on X* is a sheaf $\mathscr{A}$ of $\mathscr{O}_X$-algebras that is étale-locally isomorphic to $\mathrm{M}_n(\mathscr{O}_X)$. Two Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$ are *Brauer-equivalent* if there are locally free sheaves $\mathscr{V}$ and $\mathscr{W}$ on X with positive rank at every point of X and an isomorphism of $\mathscr{O}_X$-algebras

$$\mathscr{A} \otimes \mathscr{E}nd(\mathscr{V}) \cong \mathscr{B} \otimes \mathscr{E}nd(\mathscr{W}).$$

The *Brauer group* of X is the group whose objects are equivalence classes of Azumaya algebras on X, with group operation induced by tensor product.

An Azumaya algebra of degree n is an étale form of $\mathrm{M}_n(\mathscr{O}_X)$. By the Skolem-Noether theorem and descent theory, there is a corresponding class $[\mathscr{A}] \in \mathrm{H}^1(X, \mathrm{PGL}_n)$. Combining this with the usual exact sequence

$$1 \to \mathbf{G}_m \to \mathrm{GL}_n \to \mathrm{PGL}_n \to 1$$

and Giraud's sequence of non-abelian cohomology yields a class in $\mathrm{H}^2(X, \mathbf{G}_m)$. In fact, this class is n-torsion, as one sees from the analogous sequence

$$1 \to \boldsymbol{\mu}_n \to \mathrm{SL}_n \to \mathrm{PGL}_n \to 1.$$

The basic result is that the map sending $\mathscr{A}$ to $[\mathscr{A}] \in \mathrm{H}^2(X, \mathbf{G}_m)$ gives a group injection

$$\mathrm{Br}(X) \hookrightarrow \mathrm{H}^2(X, \mathbf{G}_m).$$

The following results describe the well-known relationship between twisted sheaves and the Brauer group. The reader is referred to [20] for more details. Nothing here is deep, but the translation into this terminology turns out to have a number of interesting consequences.

- A class $\alpha \in \mathrm{H}^2(X, \mathbf{G}_m)$ lies in the image of $\mathrm{Br}(X)$ if and only if for some (equivalently, any) $\mathbf{G}_m$-gerbe $\mathscr{X} \to X$ representing α there is a locally free $\mathscr{X}$-twisted sheaf of everywhere-positive rank.
- A $\mathbf{G}_m$-gerbe $\mathscr{X} \to X$ represents the trivial Brauer class if and only if there is an invertible $\mathscr{X}$-twisted sheaf.
- Suppose $X = \operatorname{Spec} K$ is a field and $\mathscr{X} \to X$ is a $\mathbf{G}_m$-gerbe representing a Brauer class α. The index of α is the minimal positive rank of an $\mathscr{X}$-twisted sheaf.
- (Generalization) Suppose X is an integral quasi-compact separated scheme with function field K and $\mathscr{X} \to X$ is a $\mathbf{G}_m$-gerbe representing a Brauer class $\alpha \in \mathrm{Br}(X)$. The index of the restriction α_K equals the gcd of the ranks of quasi-coherent $\mathscr{X}$-twisted sheaves of finite rank. Equivalently, the index of α_K equals the minimal positive rank of a perfect complex of $\mathscr{X}$-twisted sheaves.

Recall that the *index* of a Brauer class $\alpha \in \mathrm{Br}(k)$, k a field, is defined to be $\sqrt{\dim_k D}$, with D a finite-dimensional central division algebra over k. Equivalently, if X is a Brauer-Severi variety with class α, the index of α is the minimal positive k-degree of a 0-cycle on X (and this is equal to the gcd of the k-degrees of all 0-cycles on X). The criterion above relates this number to the ranks of twisted sheaves.

This small amount of theory already gives us a way of thinking about Question 2.1. Recall that the *cohomological Brauer group* of X is the torsion subgroup of cohomology

$$\mathrm{Br}'(X) = \mathrm{H}^2(X, \mathbf{G}_m)_{\mathrm{tors}}.$$

We've just seen that there is a natural inclusion

$$(*) : \mathrm{Br}(X) \hookrightarrow \mathrm{Br}'(X).$$

The standard question, first formulated by Grothendieck [13], asks when $(*)$ is an isomorphism. We can rephrase this in twisted sheaves as follows.

Question 5.2. For which schemes X does every $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ (for every n) carry a nowhere vanishing locally free twisted sheaf?

In this form, the question is an infantile form the famous question on the resolution property [39]: for which spaces/stacks is every quasi-coherent sheaf of finite presentation a quotient of a locally free sheaf? It is well known that this is always true for schemes admitting an ample family of invertible sheaves. The interesting part of Question 5.2 is that one does not have a "trivial sheaf" to start with, even in the presence of ample sheaves for twisting, trying to produce maps, etc. One is simply trying to prove that a nowhere-zero locally free twisted sheaf even exists at all. What should one expect?

In this thesis [10], Gabber proved that $(*)$ is an isomorphism when X is a separated union of two affine schemes. He subsequently proved the same for any quasi-compact separated scheme admitting an ample invertible sheaf (i.e., an invertible sheaf $\mathscr{L}$ such that the non-vanishing loci of sections of tensor powers of $\mathscr{L}$ generate the Zariski topology of X). De Jong devised a proof of this more general statement [8] that uses twisted sheaves. The idea is this: first one can assume that X is of finite type over $\mathbf{Z}$. Gabber tells us that there are locally free twisted sheaves over every affine, and these can be extended to coherent twisted sheaves. De Jong's argument cleverly leverages these local sheaves to inductively shrink the non-locally free locus (while possibly extending the base ring to a finite flat $\mathbf{Z}$-algebra, something that is immaterial for the result at hand).

The key point? This is "easy" to do locally by taking maps between two well-chosen coherent twisted sheaves. To globalize this, one must produce global maps with good local properties, and this is something one can do by tensoring a global hom-sheaf with a sufficiently ample invertible sheaf and taking sections. As the reader may gather from the verbiage, this is a highly geometric proof of a statement that arises directly from cohomology and K-theory. (Note that it is much easier to show that there is a perfect complex of twisted sheaves of positive rank; the geometry in de Jong's proof might be interpreted as enabling the transition from perfect complexes to locally free sheaves, but this statement is perhaps too fuzzy to be meaningful.)

6. Results on moduli of twisted sheaves

Let us start our brief discussion of the moduli of twisted sheaves by reflecting further on Example 4.1. Let E be a curve of genus 1 over a field K with a rational point $P \in E(K)$, endowed with its canonical structure of elliptic curve (where P is the identity). The Leray spectral sequence for $\mathbf{G}_m$ on the morphism $f : E \to \operatorname{Spec} K$ yields an edge isomorphism

$$\mathrm{H}^1(K, E^\vee) \to \mathrm{H}^1(K, \mathbf{R}^1 f_* \mathbf{G}_m) \to \mathrm{H}^2(E, \mathbf{G}_m)/\operatorname{Br}(K).$$

Geometrically, we can realize this as follows: given a torsor under $E^\vee$, i.e., a form X of $E^\vee$, we get a Brauer class on E by forming the Picard stack $\mathscr{P}ic^1_{X/K} \to \mathrm{Pic}^1_{X/K}$ of degree 1 invertible sheaves (and using the rational point P to canonically identify $\mathrm{Pic}^0_{X/K}$ with $\mathrm{Pic}^1_{X/K}$). If we are a bit careful, we can reverse this construction by making a moduli space of twisted sheaves. Here's how it works.

Fix a $\boldsymbol{\mu}_n$-gerbe $\mathscr{E} \to E$ that has trivial cohomology class in the fiber over P. (One can use $\mathbf{G}_m$-gerbes instead if one is willing to work slightly harder, but for ease of exposition we will not pursue this.) An invertible $\mathscr{E}$-twisted sheaf $\mathscr{L}$ has a degree: the sheaf $\mathscr{L}^{\otimes n}$ is the pullback of a unique invertible sheaf L on E and we can define

$$\deg \mathscr{L} = \frac{1}{n} \deg L.$$

Define a stack $\mathscr{M}_{\mathscr{E}}$ as follows: the objects over a K-scheme T are invertible $\mathscr{E}\times T$-twisted sheaves $\mathfrak{L}$ whose restriction to each geometric fiber $\mathscr{E}_t$ has degree 0.

Lemma 6.1. *The stack $\mathscr{M}_{\mathscr{E}}$ is a trivial $\mathbf{G}_m$-gerbe over an étale form $M_{\mathscr{E}}$ of $E^\vee$. Moreover, the consructions $X \mapsto \mathscr{E}_X$ and $\mathscr{E} \mapsto M_{\mathscr{E}}$ are inverse.*

Sketch of proof. The stack $\mathscr{M}_{\mathscr{E}}$ is algebraic as a consequence of Artin's theorem. We know that $\mathscr{M}_{\mathscr{E}}$ is a $\mathbf{G}_m$-gerbe over an algebraic space $M_{\mathscr{E}}$ because it parametrizes geometrically simple sheaves. Moreover, since $\mathscr{E}$ is trivial over P, restricting a universal sheaf along a map $\operatorname{Spec} K \to \mathscr{E} \otimes \kappa(P)$ trivializes the gerbe $\mathscr{M}_{\mathscr{E}} \to M_{\mathscr{E}}$. The formation of $\mathscr{M}_{\mathscr{E}}$ is also compatible with base change. Tsen's theorem tells us that $\mathrm{Br}(E_{\overline{K}}) = 0$, which allows one to reduce to the case in which $\mathscr{E} \to E$ is trivial. It follows that $M_{\mathscr{E}}$ is a form of $E^\vee$, as claimed. □

Lemma 6.1 has a geometric interpretation: given an elliptic fibration, we can produce new elliptic fibration by taking moduli spaces of twisted sheaves with respect to a suitable Brauer class. This is of course fraught with difficulties (singular fibers, the absence of sections of the fibration, etc., etc.) but it turns out to be surprisingly fruitful, as we will see in Section 9. More generally, we can make forms of the moduli space of sheaves on any curve, something that will prove useful in Section 7.

Proposition 6.2. *Let C/K be a curve over a field with a rational point $P \in C(K)$. Fix a $\boldsymbol{\mu}_n$-gerbe $\mathscr{C} \to C$ (with $n \cdot 1 \in K^\times$) whose restriction to $\kappa(P)$ has trivial cohomology class and such that $[\mathscr{C}] \in \mathrm{H}^2(C_{\overline{K}}, \boldsymbol{\mu}_n) = 0$. Then*

(1) *for any integer d, the stack of geometrically stable $\mathscr{C}$-twisted sheaves of rank n and degree d is a $\mathbf{G}_m$-gerbe over an étale form $M_{\mathscr{C}/K}(n, d)$ of the moduli space $M_{C/K}(n, d)$ of geometrically stable sheaves on C of rank n and degree d;*
(2) *the determinant defines a map $\delta : M_{\mathscr{C}/K}(n, d) \to \mathrm{Pic}^d_{C/K}$ which is an étale form of the classical determinant map $M_{C/K} \to \mathrm{Pic}^d_{C/K}$. In particular, the geometric fibers of δ are unirational.*

(3) *For a given invertible sheaf L on C, write $\mathscr{M}_{C/K}(n, L)$ for the fiber of the determinant map over L (i.e., geometrically stable twisted sheaves with determinants indentified with L), and $M_{C/K}(n, L)$ for its coarse space. The Brauer class*

$$\mathfrak{o} \in \mathrm{Br}(M_{C/K}(n, L))$$

associated to the gerbe

$$\mathscr{M}_{C/K}(n, L) \to M_{C/K}(n, L)$$

generates the Brauer group of $M_{C/K}(n, L)$.

Sketch of proof. This works as above: under the assumption that n is invertible in K, the cohomology class of $\mathscr{C}$ splits over the separable closure of K, which reduces us to the case in which the class of $\mathscr{C} \to C$ in $\mathrm{H}^2(C, \boldsymbol{\mu}_n)$ is trivial. Assuming the class is trivial, there is an isomorphism $\mathscr{C} \cong C \times \mathrm{B}\boldsymbol{\mu}_n$. Letting Λ denote the pullback of the invertible sheaf on $\mathrm{B}\boldsymbol{\mu}_n$ corresponding to the scalar multiplication character $\boldsymbol{\mu}_n \to \mathbf{G}_m$, tensoring by Λ sets up the isomorphism of $M_{C/K} \to M_{\mathscr{C}/K}$. The unirationality was proven by Serre and can be found in [30, Chapter 5]. The last result can be found in [2]. $\square$

So much for curves. What about higher-dimensional base varieties? There are several theorems to describe surfaces and then a large sinkhole opens up. As we will illustrate below, one can in fact use theorems in algebra to prove a meta-theorem: the structure *must* be worse for threefolds.

Proposition 6.3. *Let $\mathscr{X} \to X$ be a $\boldsymbol{\mu}_n$-gerbe on a smooth geometrically connected projective surface over a field K. Fix an invertible sheaf $L \in \mathrm{Pic}(X)$. Given an integer c, let $\mathscr{M}_{\mathscr{X}}(n, L, c)$ be the stack of geometrically stable locally free $\mathscr{X}$-twisted sheaves of rank n with determinant L and second Chern class c. Then*

(1) *The stack $\mathscr{M}_{\mathscr{X}}(L, c)$ is a $\boldsymbol{\mu}_n$-gerbe over an algebraic space $M_{\mathscr{X}}(n, L, c)$ of finite type over K.*
(2) *For any C, there is $c > C$ such that $M_{\mathscr{X}}(n, L, c)$ is non-empty.*
(3) *For all sufficiently large c, the space $M_{\mathscr{X}}(n, L, c)$ is geometrically integral when it is non-empty.*

This is usually described as "asymptotic irreducibility" of the moduli space. The proof, while non-trivial, is very similar in spirit (and often in detail) to that coming from the classical theory of stable vector bundles, and is described in [19].

For more general kinds of surfaces, there is a similar theorem. Let us state it as a vague principle, to be expanded upon below.

Principle 6.4. *Let $\mathscr{X} \to X$ be a smooth proper geometrically connected Deligne-Mumford stack of dimension 2 over a field K that is a $\boldsymbol{\mu}_n$-gerbe over another (therefore smooth proper geom. conn.) Deligne-Mumford stack. The stack $\mathscr{N}$ of torsion free $\mathscr{X}$-twisted sheaves of rank N often contains a geometrically integral locally closed substack.*

The word "often" and the vague integer N make this a Principle rather than a Proposition. The basic structure of an instance of Principle 6.4 is like this. Let us assume that K is perfect for simplicity of exposition (and the typical applications).

(1) Show that the stack $\mathscr{N}$ is non-empty. Note: this is surprisingly hard and relies heavily on the properties of X, N, and $\mathscr{X} \to X$.
(2) Show that the unobstructed locus of $\mathscr{N}$ (i.e., the points $[V]$ whose equideterminantal obstruction space $\mathrm{Ext}^2(V, V)_0$ is 0) is non-empty. This is usually a straightforward deformation theory problem, but *depends crucially on the fact that X is a surface*. We will now replace $\mathscr{N}$ by the unobstructed locus and assume that $\mathscr{N}$ consists of unobstructed sheaves (and is thus smooth).
(3) Now comes a trick originally due to O'Grady that has also appeared in work of de Jong and Starr: $\mathscr{N}$ is naturally organized in a hierarchy via the formation of (partial) reflexive hulls. If a sheaf V is not locally free at a point p, we can locally form $V^{\vee\vee}$ and glue it to V away from p to form a new sheaf (note: this may be obstructed and thus not be in $\mathscr{N}$); conversely, we can start with V and form a subsheaf V' by taking the kernel of any map $V \to \kappa$, where κ is a twisted skyscraper sheaf (now now V' does stay unobstructed). The observation of O'Grady is this: given two connected components N_1 and N_2 of $\mathscr{N}$ and a general point $[V_i]$ of N_i, we can make two cofinite subsheaves $V_1' \subset V_1$ and $V_2' \subset V_2$ that lie in a flat family over a connected base. Moreover, the construction $[V_i] \mapsto [V_i']$ gives a well-defined Galois-equivariant map between connected components!
(4) Now, to produce a geometrically integral locally closed substack of $\mathscr{N}$, we argue as follows: it is enough to find a geometrically connected open substack (as $\mathscr{N}$ is smooth). By bounding (for example) regularity or adding a stability condition, we may replace $\mathscr{N}$ with an open substack of finite type over K, hence posessing only finitely many connected components, say $\xi_1, \ldots, \xi_m$. This give a continuous Galois action on the set $\{\xi_1, \ldots, \xi_m\}$. Applying the trick from the previous step, we can find another Galois set S of connected components of $\mathscr{N}$ and an equivariant map $\{\xi_i\} \to S$ whose image is a singleton. In other words, we have found a Galois fixed point in the set of connected components of $\mathscr{N}$, which gives a geometrially connected open substack, as desired.

What about higher-dimensional ambient spaces? Here the situation is much more mysterious. Things like the Hilbert scheme of finite-length closed subschemes are already a disaster.

Question 6.5. It is impossible for anything like Principle 6.4 to hold for higher-dimensional ambient spaces X in any kind of reasonable generality? Does "any" non-empty moduli space of stable sheaves on a 3-fold contain a geometrically integral locally closed subspace?

While this would not necessarily help us understand things for twisting classes on varieties over algebraically closed fields, it would certainly imply interesting things about unramified classes on varieties over finite fields.

7. Applications to algebra

The modest amount of moduli theory described in Section 6 has a rather outsized payoff in its applications to algebra, in particular to Questions 2.2, 2.3, and 2.4. Let us start with a reminder of a classical problem on algebra known as the *period-index problem*. This is carefully explained in [20] (and goes back to [17, 15] in a slightly different form), so we will be brief.

Fix a Brauer class α over a field K. We can associate to α a finite-dimensional central division algebra D and also a Brauer-Severi variety X. Recall from above that the *index* of α is

$$\operatorname{ind}(\alpha) = \sqrt{\dim_K D} = \min_{[L:K]} X(L) \neq \emptyset.$$

The *period* of α is its order in the Brauer group; in terms of the geometry of X, the period is the smallest positive integer d such that X contains a divisor $D \subset X$ of degree d via any (equivalently: some) identification $X \otimes \overline{K} \xrightarrow{\sim} \mathbf{P}^n_{\overline{K}}$. Colloquially: the index is the smallest degree of a point in X and the period is the smallest degree of an effective divisor in X.

Using basic Galois cohomology, one can show that $\operatorname{per}(\alpha) | \operatorname{ind}(\alpha)$ and that they have the same prime factors, which means that there is some integer ℓ_α such that $\operatorname{ind}(\alpha) | \operatorname{per}(\alpha)^{\ell_\alpha}$. The obvious question, which turns out to be surprisingly deep, is how one can characterize this integer ℓ_α.

- To what extent does ℓ_α depend on α?
- To what extent can ℓ_α be made to depend only on K, or some property of K?

Colliot-Thélène asked if the following is true.

Question 7.1. If K is a C_d-field then must it be true that $\ell_\alpha = d - 1$ always works? In other words, do we always know that $\operatorname{ind}(\alpha) | \operatorname{per}(\alpha)^{d-1}$?

The answer is known to be "yes" in a shockingly small number of cases.

Theorem 7.2 (Omnibus period-index theorem)**.** *The answer to Question 7.1 is "yes" in the following cases.*

1. K *is algebraically closed (Gauss)*
2. K *is* C_1 *(Tsen)*
3. K *is of transcendence degree* 1 *over a finite field (Brauer-Hasse-Noether)*
4. K *is of transcendence degree* 2 *over an algebraically closed field (de Jong, de Jong-Starr, Lieblich* [6, 7, 20]*)*
5. K *is of transcendence degree* 2 *over a finite field (*[22]*)*

(6) *K is of transcendence degree at most 1 over a higher local field with residue field of characteristic 0 (*[21]*)*

Sinkhole 7.3. It is remarkable to note that it is not known if one can produce *any* bound at all for the Brauer group of any chosen threefold. For example, we do not know if there is some ℓ (even $2^{10^{200000000000}}$) such that for all $\alpha \in \mathrm{Br}(\mathbf{C}(x, y, z))$ we have $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^{\ell}$. This is deeply disturbing. (However, the result quoted above for surfaces over a finite field does at least give an example of a naturally arising geometric class of C_3-fields for which the answer to Question 7.1 is "yes".) Note that we *do* know that the period and index need not be equal for classes over this field ([5]).

As it turns out, the proof of each statement of Theorem 7.2 can be deduced from the geometry of a suitable moduli space of twisted sheaves. Let us sketch the main ideas, giving references to the literature where appropriate.

Omnibus sketch of proof of Theorem 7.2. The idea is this: given a field K and a class α as in the statement of Theorem 7.2, attach a gerbe $\mathscr{X} \to X$ on some kind of nice algebraic space or stack of dimension at most 2 over a small field $k \subset K$. To bound the index of α by an integer N, one needs to find a coherent $\mathscr{X}$-twisted sheaf of rank N. Stupidly, one is looking for a point on the moduli stack $\mathscr{M}$ of such sheaves.

Strategy 7.4. The fundamental paradigm of algebraic geometry gives us this strategy.

- First study $\mathscr{M} \otimes \overline{k}$. Important: show that it is non-empty! If possible, show that it is integral. (If necessary, restrict to a locally closed substack of $\mathscr{M}$ and start over.)
- Now hope that $\mathscr{M}$ is nice enough and k is small enough that we can establish that $\mathscr{M}$ has a 0-cycle of degree 1. For example, if $k = \mathbf{C}(t)$, it is enough for $\mathscr{M}$ to be unirational. Similarly, if k is finite (or, more generally, QAC), it is enough that $\mathscr{M}$ be geometrically integral.
- Even better: try to package this inductively by fibering X and studying bundles of moduli spaces for lower-dimensional problems.

This strategy works perfectly for the cases enumerated in Theorem 7.2 and then fails miserably (whence the abrupt dropoff in our understanding). Let us illustrate with the first non-trivial case: de Jong's theorem that period and index are equal for Brauer classes over function fields of surfaces (over algebraically closed fields k).

Let us suppose X is a surface and $\alpha \in \mathrm{Br}(X)$ is a Brauer class. (What about clases over $k(X)$ instead of X? This is a very important question, and we will return to it soon.) Choosing a pencil of very ample divisors and blowing up its base locus, we may assume that X admits a generically smooth proper flat morphism $\pi : X \to \mathbf{P}^1$ with at least one section $\sigma : \mathbf{P}^1 \to X$. Suppose $\mathrm{per}(\alpha) = n$. Choose a $\boldsymbol{\mu}_n$-gerbe $\mathscr{X} \to X$ resp-resenting α and let $\mathscr{M} \to \mathbf{P}^1$ be the relative moduli stack parametrizing geometrically stable twisted sheaves of rank n on the fibers of π with determinant identified with

$\mathscr{O}(\sigma(\mathbf{P}^1))$. We know that $\mathscr{M} \to M$ is a $\boldsymbol{\mu}_n$-gerbe over a smooth algebraic space that is proper over the locus $U \subset \mathbf{P}^1$ parametrizing smooth fibers of π.

Claim 7.5. *The fibration $\mathscr{M}_U \to U$ has a section.*

Proof. Proposition 6.2 tells us that M has unirational fibers over U. By the Graber-Harris-Starr Theorem, M has a rational section, i.e., a rational point P over $k(t)$. On the other hand, $\mathscr{M} \to M$ is a $\boldsymbol{\mu}_n$-gerbe, so the obstruction to lifting P to an object of $\mathscr{M}$ (and thus a proof that $\operatorname{ind}(\alpha) = \operatorname{per}(\alpha)$) lies in $\mathrm{H}^2(\operatorname{Spec} k(t), \boldsymbol{\mu}_n) \subset \operatorname{Br}(k(t))$. By Tsen's theorem, the latter group is 0. This establishes the proof. (And – in spite of being the "simplest" case – it already includes all steps of Strategy 7.4: fibering, taking advantage of nice moduli, and using induction on dimensions!) □

What if

$$\alpha \notin \operatorname{Br}(X) \subset \operatorname{Br}(k(X))?$$

There is a special result of working over an algebraically closed field.

Claim 7.6. *Let H be a general sufficiently ample divisor on X such that $D+H$ is ample. Let*

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d] & X \\ \mathbf{P}^1 & \end{tikzcd}$$

be a general pencil of elements of $|D+H|$ with generic fiber

$$C \to \operatorname{Spec} k(t).$$

Then the class α lies in the subgroup

$$\operatorname{Br}(C) \subset \operatorname{Br}(k(X)) = \operatorname{Br}(k(t)(C)).$$

Idea of proof. The proof relies on the theory of ramification of Brauer classes, as described in (for example) [1, §3]. The crux of the matter is this: there is a smallest divisor $D \subset X$ such that α lies in the subgroup

$$\operatorname{Br}(X \setminus D) \subset \operatorname{Br}(k(X)).$$

Moreover, there is a cyclic extension of each generic point of D that measures the failure of α to extend across that divisor, and these cyclic extensions can themselves only ramify over singular points of D. Even better, this whole setup is mildly functorial, in the following sense: given another regular scheme with a morphism

$$f : Z \to X$$

such that $f^{-1}(D)$ is a Cartier divisor, we have that the ramification extensions over $f^{-1}(D)$ are the preimages of the ramification extensions over D.

A general pencil of divisors in $|D+H|$ will have base locus intersecting D in smooth locus D. In particular, the preimage of D in the incidence correspondence

$$\widetilde{X} \to X$$

consists of a union of D and several sections of

$$\widetilde{X} \to \mathbf{P}^1.$$

Moreover, mild functoriality dictates that the ramification extensions on these $\mathbf{P}^1$s are the preimage of the ramification extension of D at the corresponding basepoint, but then they are trivial (as k is algebraically closed). Restricting to C, we see that the ramification is trivial.

We can now use the same argument on the relative moduli space to arrive at a proof that the period and index coincide. □

Now let us follow Strategy 7.4 and consider an *unramified* class $\alpha \in \mathrm{Br}(Z)$, where Z is a smooth projective surface over a *finite* field. Let $\mathscr{Z} \to Z$ be a $\boldsymbol{\mu}_n$-gerbe representing α and let $\mathscr{M} \to M$ be the $\boldsymbol{\mu}_n$-gerbe parametrizing stable locally free $\mathscr{Z}$-twisted sheaves of rank n, determinant $\mathscr{O}$, and sufficiently large second Chern class. By Proposition 6.3, the space M is eventually geometrically integral. But then the Lang-Weil estimates [16] tell us that M has a 0-cycle of degree 1, and Wedderburn's theorem (induction again!) tell us we can lift this 0-cycle to $\mathscr{M}$. Since we really interested in the K-theory of $\mathscr{Z}$, it is enough to find such a 0-cycle to establish that again $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$. (In other words, we can cook up a perfect complex of $\mathscr{Z}$-twisted sheaves that has rank n using the 0-cycle on $\mathscr{M}$. Of course, after the fact one can simply make a twisted sheaf of rank n.)

Comparing the two situations shows us that there is a basic interplay at work:

Interplay 7.7. In the modular interpretation of the period-index problem, we can arrive at affirmative answers by *simple geometry over larger fields* or *hard geometry over smaller fields*.

Given the nature of the proof and Principle 6.4, one might think that $\mathrm{per}(\alpha) = \mathrm{ind}(\alpha)$ for any class α in the Brauer group of a field of transcendence degree 2 over a finite field! Unfortunately, this is not true: as proven in [5], for any odd prime power q and any element

$$a \in \mathbf{F}_q^\times \setminus (\mathbf{F}_q^\times)^2,$$

the class $(x,y)\otimes(a,1-x)$ (tensor product of quaternion algebras over $\mathbf{F}_q(x,y)$) has period 2 (it's a sum of classes of period 2!) and index 4 (it is itself a division algebra.). But this class is *ramified*: it does not lie in $\mathrm{Br}(\mathbf{P}^2_{\mathbf{F}_q}) = 0$.

Let's think more about this. Fix a class α in the Brauer group of the function field of a surface X over a finite field. Interpreting the theory of ramification a certain way tells us that while the class α might not live in $\mathrm{Br}(X)$, there is a *stacky* surface $\mathcal{X}$ with the same function field and such that $\alpha \in \mathrm{Br}(\mathcal{X})$. (The Brauer group of a stack is defined using Azumaya algebras in precisely the same way as for schemes.) More precisely, let us suppose that there is an snc divisor

$$D = D_1 + \cdots + D_m \subset X$$

such that α is ramified precisely along D. Let us suppose in addition that $n = \ell$ is a prime invertible on X. Then we have the following: there is a stack $\mathcal{X} \to X$ of "ℓth roots of each D_i". Its étale-local structure around a point $x \in X$ is the same as

(1) $\operatorname{Spec} k[s,t]$ if $x \in X \setminus D$;
(2) $[\operatorname{Spec} k[s,t]/\boldsymbol{\mu}_\ell]$ with $\zeta \cdot (s,t) = (\zeta s, t)$ if x lies in precisely one component of D;
(3) $[\operatorname{Spec} k[s,t]/\boldsymbol{\mu}_\ell \times \boldsymbol{\mu}_\ell]$ with $(\zeta, \eta) \cdot (s,t) = (\zeta s, \eta t)$ if x is a singular point of D.

A sheaf on a stack has representations of stabilizer groups as fibers (rather than simply vector spaces). Roughly speaking, we can think about twisted sheaves the same way. In particular, we must specify the representations that occur as part any moduli problem of twisted sheaves. As it turns out, the only consistent way to do this that works for all Brauer classes requires a multiple of the "regular representation" at points of type (2) above, and for certain points of type (3) above, this is only possible along both branches if the sheaves have rank ℓ^2 (essentially, the representations are given by tensoring the regular representation of each factor group). This is a stacky interpretation of Saltman's well-known meteorology of ramification points [32] and is explained in [21, 22].

In other words, there is a *local* obstruction to the equality of period and index for ramified classes on surfaces over finite fields. Once the local obstruction has been erased by a suitable definition of the moduli problem, we can indeed follow Principle 6.4 (and the usual proofs work, for the most part). □

8. Applications to arithmetic

Applications to algebra are amusing, but the usefulness of twisted sheaves does not stop there. There is an interesting interaction between the moduli theory of twisted sheaves, Colliot-Thélène's conjectures on 0-cycles and Brauer-Manin obstruction, and various properties of finitely generated fields. There is also an interplay between the derived category of twisted sheaves on a K3 surface, Fourier-Mukai equivalences, and the Tate conjecture for K3 surfaces over finite fields.

Let us start by recalling the statement of the (mildest form of) Colliot-Thélène conjecture.

Conjecture 8.1 (Colliot-Thélène)**.** *If X is a smooth projective geometrically rational variety over a global field k then the Brauer-Manin obstruction is the only obstruction to the existence of 0-cycles on X of degree 1 over k.*

The Brauer-Manin obstruction is described in [36].

In [18] (part of which improves results of [21]), the reader will find proofs of the following results.

Proposition 8.2. *Let k be a function field with scheme of integers V and K/k be a finitely generated field extension of transcendence degree 1. Suppose $X \to V$ is regular proper model of K over V. If Colliot-Thélène's conjecture holds over extensions of k then*

1. *every Brauer class $\alpha \in \mathrm{Br}(X)$ satisfies $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^2$;*
2. *every class $\alpha \in \mathrm{Br}(K)$ satisfies $\mathrm{ind}(\alpha) | \mathrm{per}(\alpha)^5$.*

While we already know (more than) this when k has positive characteristic (as explained in the previous section), this is totally foreign over number fields. The powerful techniques of the last section – compactifying, transforming a curve over a function field into a surface, etc. – are unavailable over a number field. The crux of the proof is an analysis of ramification of p-torsion classes at places lying over p. This turns out to be surprisingly subtle, reliant on the work of Bloch and Kato, and special to the case of surfaces. Low dimensionality again rears its ugly (beguiling?) head.

Idea of proof. We proceed as in Section 7: let C/k be the proper smooth curve with function field K. After passing to the algebraic closure of k in K, we may assume that C is geometrically connected. (Note: this is the only place where we need to make a non-trivial extension of k.) Write $\mathscr{M}$ (resp. M) for the stack (resp. coarse space) of geometrically stable twisted sheaves on C of rank ℓ and trivial determinant.

Assume α is unramified on all of X. Given a place v of k with complete local ring $\mathscr{O}_v$, we know that $\mathrm{Br}(X_{\mathscr{O}_v}) = 0$; it follows from the deformation theory of twisted sheaves that there is a stable locally free twisted sheaf with trivial determinant and rank ℓ. In other words,

$$\mathscr{M}(\mathbf{A}_k) \neq \emptyset.$$

On the other hand, $\mathscr{M} \to M$ generates $\mathrm{Br}(M)$, so the resulting elements of $M(\mathbf{A}_k)$ lie in $M(\mathbf{A}_k)^{\mathrm{Br}}$. Choosing a smooth compactification of M, we find that the same holds. Applying Colliot-Thélène's conjecture and a simple moving argument, we see that there is a 0-cycle of degree 1 in M. Since the obstruction to lifting this to a 0-cycle in $\mathscr{M}$ is in the Brauer group of k (and its finite extensions), we see that this introduces at most one more factor of ℓ into the estimate: after making an extension of the 0-cycle of degree at most ℓ we can lift to the stack. This shows the desired statement.

To get to the ramified case one can split the ramification by making a suitable field extension. The methods involved are beyond the scope of this survey; the reader is referred to [18] for more details. □

Building on this result, one can also prove the following. Recall that the u-invariant of a field K is the largest dimension of an anisotropic quadratic form over K. In general, questions about the u-invariant turn out to be quite subtle. For example, the u-invariant can take any power of 2 as a value, but not 3, 5, or 7. Famous results of Merkurjev (resp. Izhboldin) gives a field with u-invariant 6 (resp. 9). It is not clear how the u-invariant should behave in field extensions (unlike, say the C_d-property).

Theorem 8.3. *Suppose K/k is a field extension of transcendence degree 1 over a totally imaginary number field. If Conjecture 8.1 holds then the u-invariant of K is finite.*

The proof of Theorem 8.3 would take us rather far afield. Suffice it to say that the key is to study representations of elements of $\mathrm{H}^3(K, \boldsymbol{\mu}_2)$ by sums of symbols, and in so doing to reduce to questions about H^2, where one can apply period-index results.

8.4. There is another amusing application of the theory of twisted sheaves to a certain kind of arithmetic question: *given a finite field k, are there only finitely many K3 surfaces over k?* This is a simple question in a long line of similar questions (are there finitely curves of genus g? finitely many abelian varieties of dimension g? etc.) It turns out that this question is intimately related to the Tate conjecture.

Theorem 8.5 ([25])**.** *Fix a finite field k. The Tate conjecture holds for all K3 surfaces over finite extensions of k if and only if there are only finitely many isomorphism classes of K3 surfaces over each finite extension of k.*

Idea of proof. The relevant direction of this theorem is how one might deduce that the Tate conjecture holds for K3 surfaces if there are only finitely many K3 surfaces over each finite field. (The other direction is more believable: the Tate conjecture hands control over the Picard group to the Galois action on the étale and crystalline cohomology. Using semisimplicity of Frobenius and a bit of elbow grease, one can thus control the degrees of divisors, bounding the degree of an ample divisor. This embeds all of the K3s over k into a fixed projective space, so there can be only finitely many.)

If one starts with a K3 surface X over k with infinite Brauer group $\mathrm{Br}(X)$, one would like to show that there are infinitely many K3 surfaces over k. If $\mathrm{Br}(X)$ is infinite, then in fact there is a prime ℓ and a sequence of classes α_n such that $\ell\alpha_n = \alpha_{n-1}$ and α_n has order exactly ℓ^n. From this sequence we can construct infinitely many K3 surfaces as moduli spaces of stable twisted sheaves! This is familiar from the classical theory of Mukai [29]: given a K3 surface, one can make many new K3 surfaces M as moduli spaces of sheaves on X. Moreover, there are equivalences of derived categories of coherent sheaves $\mathrm{D}(X) \cong \mathrm{D}(M)$.

The idea is to apply precisely the same "numerical technology" to show that many moduli spaces of twisted sheaves are themselves also K3 surfaces. So, associated to α_n we get M_n and equivalences of derived categories

$$D(M_n) \cong D(X, \alpha_n),$$

the latter being used (abusively) to denote the derived category of twisted sheaves on a gerbe representing α_n. If there are finitely many K3s over k, then infinitely many of the M_n coincide. Flipping this around, there is one K3 surface with infinitely many "twisted partners". By carefully lifting to characteristic 0 and using a twisted variant of the classical Torelli theorem, one can show that this is impossible. The reader is referred to [25] for details. □

Without going into details, the proof of Theorem 8.5

9. Applications to geometry

Before concluding, let us briefly mention a recent application of the ideas found in Section 6 and Section 8 to the geometry of the moduli space of supersingular K3 surfaces in characteristic p. These are K3 surfaces with Picard number 22. They also have a remarkable property: some very large, positive-dimensional, cohomology with finite coefficients. This is a theorem first proven by Artin, but the proof seems to have escaped publication. A proof will appear in [23], along with the details of everything else in this section.

Proposition 9.1 (Artin). *Let $f : X \to \operatorname{Spec} k$ be a supersingular K3 surface. The fppf sheaf $\mathrm{R}^2 f_* \boldsymbol{\mu}_p$ is representable by a smooth group scheme locally of finite type over k whose connected component containing 0 is isomorphic to $\mathbf{G}_a$.*

Suppose k is algebraically closed. Playing with the Leray spectral sequence, one finds that there is a universal class

$$\alpha \in \mathrm{H}^2(X \times \mathbf{A}^1, \boldsymbol{\mu}_p)$$

giving rise to the immersion

$$\mathbf{G}_a \hookrightarrow \mathbf{R}^2 f_* \boldsymbol{\mu}_p.$$

We can choose a $\boldsymbol{\mu}_p$-gerbe $\mathscr{X} \to X \times \mathbf{A}^1$ representing α. This gives a continuous p-torsion deformation of the trivial Brauer class over $\mathbf{A}^1$. (In fact, the existence of such a deformation characterizes supersingular K3 surfaces.) Using moduli spaces of twisted sheaves and equivalences between their derived categories, we can transform this moving Brauer class into a moving surface. While there are many ways to do this, there is one in particular that links this back up nicely to Example 6.1. Suppose given an elliptic pencil $X \to \mathbf{P}^1$ (supersingular K3s always have infinitely many of these, although they form finitely many orbits under the automorphism group [24]).

Definition 9.2. Define $M \to \mathbf{P}^1 \times \mathbf{A}^1$ to be the moduli space of stable sheaves of rank 1 and degree 1 on the fibers of

$$X \times \mathbf{A}^1 \to \mathbf{P}^1 \times \mathbf{A}^1$$

(with respect to a fixed sufficiently general polarization).

This construction is very natural and geometric. Even better, it is precisely a geometrization of the Artin-Tate isomorphism [37] between $\mathrm{Br}(X)$ and $\text{Ш}(\mathrm{Jac}(X_\eta))$, where X_η is the generic fiber of the elliptic pencil:

Proposition 9.3. *Given a field K and a point $\operatorname{Spec} K \to \mathbf{A}^1$, let $\beta \in \mathrm{Br}(X_K)$ denote the pullback of α. Then the fibration*

$$M_K \to \mathbf{P}^1_K$$

is an étale form of $X_K \to \mathbf{P}^1_K$ and there is an isomorphism of $\mathbf{Q}$-lattices

$$H(X) \otimes \mathbf{Q} \xrightarrow{\sim} H(M_K) \otimes \mathbf{Q}.$$

Moreover, it is precisely the translate of the constant form by the element of $Ш(X_\eta)$ *associated to* β*. In particular, if* β *is not defined over* k*, neither is* M_K*.*

Here, the lattice $H(X)$ is formed by taking the Chow theory

$$\mathrm{CH}(X) = \mathrm{CH}^0(X) \oplus \mathrm{CH}^1(X) \oplus \mathrm{CH}^2(X)$$

and imposing a different intersection form, namely that

$$(a, b, c) \cdot (a', b', c') = bb' - ac' - a'c \in \mathrm{CH}^2(X) = \mathbf{Z}.$$

One can in fact think of this lattice isomorphism as a reflection of something motivic, and one can see it in the derived category in the form of an equivalence between two suitable derived categories of twisted sheaves.

The upshot of Proposition 9.3 is that the formation of the space of stable twisted sheaves transforms the curve $\mathbf{G}_a$ in cohomology into a family of K3 surfaces over $\mathbf{A}^1$ whose fiber at 0 is X and whose geometric generic fiber cannot be defined over k. In other words, it is a truly moving family of K3 surfaces. We will now see how to make this a curve in a moduli space.

In his deep study of the crystalline cohomology of supersingular K3 surfaces [31], Ogus produced a period domain, analogous to the classical period domain coming Hodge theory for complex K3 surfaces. This space parametrizes pairs $(X, \iota : N \to \mathrm{Pic}(X))$, where X is a supersingular K3 surface, N is one of 10 lattices, and ι is a lattice embedding into the Picard group. Of particular interest is the generic case (so-called "Artin invariant 10"). Here, N has discriminant $-p^{20}$ and ι is an isomorphism. Ogus showed that this defines a dense open subspace of the period space. Let us call this subspace U. The theorem we have proven is following.

Theorem 9.4. *For any point* (X, ν) *of* U*, the construction of Proposition 9.3 yields a non-constant morphism*

$$\mathbf{A}^1 \to U$$

sending 0 *to* (X, ν)*.*

All that remains to explain is why the marking should deform along with the surface. This comes from the isomorphism between lattices: since $H(X)$ is invariant under field extension, there can be no monodromy action on $H(M) \otimes \mathbf{Q}$ (i.e., no non-trivial action of the Galois group of $k((t))$ on $\mathrm{Pic}(M_{k((t))}) \otimes \mathbf{Q}$), and thus none on $H(M)$.

Of course, the space U is highly non-separated, and generally pathological. If instead one looks at the supersingular locus in any moduli space of polarized K3 surfaces, one will find precisely the same phenomenon – a covering by rational curves – since one can dominate these loci by U in all cases.

References

[1] M. Artin and D. Mumford. Some elementary examples of unirational varieties which are not rational. *Proc. London Math. Soc. (3)*, 25:75–95, 1972.

[2] Vikraman Balaji, Indranil Biswas, Ofer Gabber, and Donihakkalu S. Nagaraj. Brauer obstruction for a universal vector bundle. *C. R. Math. Acad. Sci. Paris*, 345(5):265–268, 2007.

[3] Vasile Brînzănescu and Ruxandra Moraru. Twisted Fourier-Mukai transforms and bundles on non-Kähler elliptic surfaces. *Math. Res. Lett.*, 13(4):501–514, 2006.

[4] Andrei Căldăraru. Derived categories of twisted sheaves on elliptic threefolds. *J. Reine Angew. Math.*, 544:161–179, 2002.

[5] Jean-Louis Colliot-Thélène. Exposant et indice d'algèbres simples centrales non ramifiées. *Enseign. Math. (2)*, 48(1-2):127–146, 2002. With an appendix by Ofer Gabber.

[6] A. J. de Jong. The period-index problem for the Brauer group of an algebraic surface. *Duke Math. J.*, 123(1):71–94, 2004.

[7] A. J. de Jong, Xuhua He, and Jason Michael Starr. Families of rationally simply connected varieties over surfaces and torsors for semisimple groups. *Publ. Math. Inst. Hautes Études Sci.*, (114):1–85, 2011.

[8] Aise Johan de Jong. A result of Gabber. Preprint.

[9] Ron Donagi and Tony Pantev. Torus fibrations, gerbes, and duality. *Mem. Amer. Math. Soc.*, 193(901):vi+90, 2008. With an appendix by Dmitry Arinkin.

[10] Ofer Gabber. Some theorems on Azumaya algebras. In *The Brauer group (Sem., Les Plans-sur-Bex, 1980)*, volume 844 of *Lecture Notes in Math.*, pages 129–209. Springer, Berlin, 1981.

[11] Jean Giraud. *Cohomologie non abélienne*. Springer-Verlag, Berlin, 1971. Die Grundlehren der mathematischen Wissenschaften, Band 179.

[12] Alexander Grothendieck. Le groupe de Brauer. I. Algèbres d'Azumaya et interprétations diverses. In *Dix Exposés sur la Cohomologie des Schémas*, pages 46–66. North-Holland, Amsterdam, 1968.

[13] Alexander Grothendieck. Le groupe de Brauer. II. Théorie cohomologique. In *Dix Exposés sur la Cohomologie des Schémas*, pages 67–87. North-Holland, Amsterdam, 1968.

[14] Daniel Huybrechts and Paolo Stellari. Proof of Căldăraru's conjecture. Appendix: "Moduli spaces of twisted sheaves on a projective variety" [in *moduli spaces and arithmetic geometry*, 1–30, Math. Soc. Japan, Tokyo, 2006] by K. Yoshioka. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 31–42. Math. Soc. Japan, Tokyo, 2006.

[15] Serge Lang and John Tate. Principal homogeneous spaces over abelian varieties. *Amer. J. Math.*, 80:659–684, 1958.

[16] Serge Lang and André Weil. Number of points of varieties in finite fields. *Amer. J. Math.*, 76:819–827, 1954.

[17] Stephen Lichtenbaum. The period-index problem for elliptic curves. *Amer. J. Math.*, 90:1209–1223, 1968.

[18] Lieblich, Parimala, and Suresh. Colliot-Thélène's conjecture and finiteness of u-invariants, year = 2012, note = Preprint in preparation.

[19] Max Lieblich. Moduli of twisted sheaves. *Duke Math. J.*, 138(1):23–118, 2007.

[20] Max Lieblich. Twisted sheaves and the period-index problem. *Compos. Math.*, 144(1):1–31, 2008.

[21] Max Lieblich. Period and index in the Brauer group of an arithmetic surface. *J. Reine Angew. Math.*, 659:1–41, 2011. With an appendix by Daniel Krashen.

[22] Max Lieblich. The period-index problem for fields of transcendence degree 2. 2011.

[23] Max Lieblich. The moduli space of supersingular K3 surfaces is uniruled. 2012. Preprint in preparation.

[24] Max Lieblich and Davesh Maulik. A note on the cone conjecture for k3 surfaces in positive characteristic. 2011.

[25] Max Lieblich, Davesh Maulik, and Andrew Snowden. Finiteness of k3 surfaces and the tate conjecture. 2012.

[26] Shouhei Ma. On the 0-dimensional cusps of the Kähler moduli of a $K3$ surface. *Math. Ann.*, 348(1):57–80, 2010.

[27] Dimitri Markushevich. Rational Lagrangian fibrations on punctual Hilbert schemes of $K3$ surfaces. *Manuscripta Math.*, 120(2):131–150, 2006.

[28] James S. Milne. *Étale cohomology*, volume 33 of *Princeton Mathematical Series*. Princeton University Press, Princeton, N.J., 1980.

[29] S. Mukai. On the moduli space of bundles on $K3$ surfaces. I. In *Vector bundles on algebraic varieties (Bombay, 1984)*, volume 11 of *Tata Inst. Fund. Res. Stud. Math.*, pages 341–413. Tata Inst. Fund. Res., Bombay, 1987.

[30] P. E. Newstead. *Introduction to moduli problems and orbit spaces*, volume 51 of *Tata Institute of Fundamental Research Lectures on Mathematics and Physics*. Tata Institute of Fundamental Research, Bombay, 1978.

[31] Arthur Ogus. A crystalline Torelli theorem for supersingular $K3$ surfaces. In *Arithmetic and geometry, Vol. II*, volume 36 of *Progr. Math.*, pages 361–394. Birkhäuser Boston, Boston, MA, 1983.

[32] David J. Saltman. Division algebras over surfaces. *J. Algebra*, 320(4):1543–1585, 2008.

[33] Justin Sawon. Derived equivalence of holomorphic symplectic manifolds. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 193–211. Amer. Math. Soc., Providence, RI, 2004.

[34] Justin Sawon. Lagrangian fibrations on Hilbert schemes of points on $K3$ surfaces. *J. Algebraic Geom.*, 16(3):477–497, 2007.

[35] Jean-Pierre Serre. *Local fields*, volume 67 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1979. Translated from the French by Marvin Jay Greenberg.

[36] Alexei Skorobogatov. *Torsors and rational points*, volume 144 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2001.

[37] John Tate. On the conjectures of Birch and Swinnerton-Dyer and a geometric analog. In *Séminaire Bourbaki, Vol. 9*, pages Exp. No. 306, 415–440. Soc. Math. France, Paris, 1995.

[38] Yukinobu Toda. Deformations and Fourier-Mukai transforms. *J. Differential Geom.*, 81(1):197–224, 2009.

[39] Burt Totaro. The resolution property for schemes and stacks. *J. Reine Angew. Math.*, 577:1–22, 2004.

[40] Kōta Yoshioka. Moduli spaces of twisted sheaves on a projective variety. In *Moduli spaces and arithmetic geometry*, volume 45 of *Adv. Stud. Pure Math.*, pages 1–30. Math. Soc. Japan, Tokyo, 2006.

[41] Kōta Yoshioka. Fourier-Mukai transform on abelian surfaces. *Math. Ann.*, 345(3):493–524, 2009.