Equivariant generalized cohomology and geometric representation theory

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Abstract. The goal of this series of talks is to describe some relationships between geometric representation theory and generalized equivariant cohomology. The majority of this work is inspired by results of Bezrukavnikov-Finkelberg-Mirkovic [**BFM05**] and Soergel [**Soe90**]. The outline will be:

(a) We begin with a review of the story of equivariant cohomology and its relationship to derived algebraic geometry.
(b) We will then describe some results from geometric representation theory: in particular, the basics of Soergel theory, the statement of geometric Satake, and the Kostant slice.
(c) Finally, we will describe some ideas connecting chromatic homotopy theory to the derived geometric Satake equivalence, and discuss several resulting questions (e.g., a relationship to the Ben-Zvi–Sakellaridis–Venkatesh program). This part is based off the forthcoming work [**Dev23**].

Contents

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1. Equivariant generalized cohomology

Setup 1.1. In this talk, G will be a *compact Lie group*, often assumed to be simply-connected; some parts of the story below have to modified if one wishes to incorporate the case of finite groups.

Definition 1.2. Let G be a compact Lie group. A *finite G-space* X is a space built up from cells of the form $G/H \times D^n$, where $H \subseteq G$ is a closed subgroup (hence compact Lie subgroup). Let $\mathscr{S}(G)$ denote the ∞-category of finite G-spaces and G-equivariant maps between them; this cellular definition can be interpreted as saying that $\mathscr{S}(G)$ is generated under finite colimits by the full subcategory spanned by G-spaces of the form G/H.

If X is a finite G-space, one can construct its homotopy orbits X_{hG}, defined to be the quotient $(EG \times X)/G$, where EG is a contractible space with free G-action. The *Borel equivariant cohomology* $\mathrm{H}^*_G(X; \mathbf{Z})$ of X is defined to be the cohomology of X_{hG}. (There is a more refined notion, known as the *Bredon equivariant cohomology*; we will not need the precise definition here.) If the G-action on X is free, then $X_{hG} \simeq X/G$. In particular:

Example 1.3. There is an isomorphism $\mathrm{H}^*_G(G; \mathbf{Z}) \cong \mathbf{Z}$. Similarly, $\mathrm{H}^*_G(*; \mathbf{Z}) \cong \mathrm{H}^*(BG; \mathbf{Z})$.

The most important example in geometric representation theory is the following (which we will return to in greater detail later):

Example 1.4. Let S^1 act on $S^2 = \mathbf{C}P^1$ by

$$\lambda : [x : y] \mapsto [\lambda x : \lambda^{-1} y].$$

Note that the poles $0 = [0 : 1]$ and $\infty = [1 : 0]$ remain fixed. If we view $S^2 = \mathrm{SU}(2)/S^1$, then the homotopy orbits $S^2_{hS^1}$ can be identified with $\mathrm{BS}^1 \times_{\mathrm{BSU}(2)} \mathrm{BS}^1 = \mathbf{C}P^\infty \times_{\mathbf{H}P^\infty} \mathbf{C}P^\infty$. Since $\mathrm{H}^*(\mathbf{H}P^\infty; \mathbf{Z}) \cong \mathbf{Z}[c^2] \subseteq \mathbf{Z}[c] \cong \mathrm{H}^*(\mathbf{C}P^\infty; \mathbf{Z})$, we find that

$$\mathrm{H}^*_{S^1}(S^2; \mathbf{Z}) \cong \mathbf{Z}[c] \otimes_{\mathbf{Z}[c^2]} \mathbf{Z}[c] \cong \mathbf{Z}[c, d]/(c^2 - d^2),$$

where $|c| = |d| = 2$.

We can also define equivariant versions of other cohomology theories. For instance, if X is a finite G-space, one putative definition for equivariant complex K-theory may be as $\mathrm{KU}^0(X_{hG})$. Although interesting, this turns out to be somewhat lossy. For example, if V is a representation of G, one can view V as defining a map $\rho : BG \to \mathrm{BGL}(V)$, and hence a bundle over BG. This defines a map

$$\{\mathbf{C}\text{-representations of } G\} \to \mathrm{KU}^0(BG) = \mathrm{KU}^0(*_{hG}).$$

If the left-hand side is equipped with the monoid structure given by direct sum of representations, then this map is furthermore a homomorphism. In particular, we obtain a ring map $R(G) \to \mathrm{KU}^0(BG)$, where $R(G)$ is the representation ring of G (defined to be the Grothendieck group of the abelian category of $\mathbf{C}$-representations of G). This map is *not* an equivalence, although it is close.

Example 1.5. If $G = S^1$, then

$$\mathrm{KU}^0(\mathrm{BS}^1) = \mathrm{KU}^0(\mathbf{C}P^\infty) = \mathbf{Z}[\![q - 1]\!],$$

where q denotes the class of the tautological line bundle $\mathscr{O}(1)$ over $\mathbf{C}P^\infty$. On the other hand, $R(S^1) \cong \mathbf{Z}[q^{\pm 1}]$, where q is the canonical weight 1 $\mathbf{C}$-representation of S^1 (given by its action on the complex numbers).

The moral of this example is that the map $R(G) \to \mathrm{KU}^0(BG)$ seems to behave as a *completion*. This is the content of the Atiyah-Segal completion theorem. To state it precisely, we need to recall the definition of equivariant K-theory.

Definition 1.6. Let X be a finite G-space. A *G-equivariant vector bundle* ξ on X is a vector bundle over X equipped with a continuous G-action, such that the map $\xi \to X$ is G-equivariant, and such that $g : \xi_x \to \xi_{g(x)}$ is a linear map of vector spaces. Let $\mathrm{KU}^0_G(X)$ denote the Grothendieck group of the monoid of G-equivariant vector bundles on X (under addition). There is an associated $\mathbf{E}_\infty$-ring KU_G, which we will refer to as *genuine G-equivariant K-theory.*

Example 1.7. Almost by definition, $\mathrm{KU}^0_G(*) = R(G)$. Therefore, if X is any finite G-space, then $\mathrm{KU}^0_G(X)$ admits the structure of a $R(G)$-module. For instance,

$$\mathrm{KU}^0_{\mathrm{SU}(2)}(*) = R(\mathrm{SU}(2)) = R(S^1)^{\mathbf{Z}/2},$$

where $\mathbf{Z}/2$ acts on $R(S^1) = \mathbf{Z}[q^{\pm 1}]$ by $q \mapsto q^{-1}$. The fixed points are therefore

$$\mathrm{KU}^0_{\mathrm{SU}(2)}(*) \cong \mathbf{Z}[q + q^{-1}].$$

Continuing Example 1.4, we have:

Example 1.8. Let S^1 act on S^2 as in Example 1.4. Then there is an isomorphism

$$\mathrm{KU}^0_{S^1}(S^2) \cong \mathbf{Z}[q^{\pm 1}] \otimes_{\mathbf{Z}[q+q^{-1}]} \mathbf{Z}[q^{\pm 1}] \cong \mathbf{Z}[q_1^{\pm 1}, q_2^{\pm 1}]/(q_2 - q_1)(q_1^2 - q_2^2 + 1).$$

Theorem 1.9 (Atiyah-Segal)**.** *Let I denote the augmentation ideal of $R(G)$, defined by the kernel of* $\dim : R(G) \to \mathbf{Z}$. *If X is a finite G-space, there is a natural equivalence* $\mathrm{KU}^0_G(X)^\wedge_I \xrightarrow{\sim} \mathrm{KU}^0(X_{hG})$.

One can reinterpret this story in spectral algebro-geometric terms; this is the subject of Lurie's survey **[Lur09]**. For the purpose of illustration, the reader should pretend that most concepts from classical algebraic geometry can be lifted to the setting of spectral algebraic geometry (e.g., group schemes, formal groups, etc.). In particular, we will use terms such as "$\mathbf{E}_\infty$-ring" below, which one can interpret to mean a multiplicative cohomology theory with additional structure (such as an analogue of Steenrod/Adams operations). Recall Quillen's beautiful picture relating homotopy theory to the theory of formal groups:

Construction 1.10. Let R be a complex-oriented $\mathbf{E}_\infty$-ring, i.e., an $\mathbf{E}_\infty$-ring for which the map $R^2(\mathbf{C}P^\infty) \to R^2(\mathbf{C}P^1) \cong \pi_0(R)$ is surjective. In this case, there is an isomorphism $R^*(\mathbf{C}P^\infty) = \pi_*(R)[\![\hbar]\!]$, where $\hbar \in R^2(\mathbf{C}P^\infty)$ is a lift of $1 \in \pi_0(R)$ along the surjection $R^2(\mathbf{C}P^\infty) \twoheadrightarrow R^2(\mathbf{C}P^1)$. Note that $\hbar$ lives in homological degree -2, and can be viewed as the Chern class $c_1(\mathscr{O}(1))$ of the tautological bundle over $\mathbf{C}P^\infty$. A complex-oriented $\mathbf{E}_\infty$-ring R has Chern classes $c_1^R(\xi)$ for complex vector bundles ξ over compact spaces.

The tensor product of line bundles induces a map

$$\mu^* : R^*(\mathbf{C}P^\infty) = \pi_*(R)[\![\hbar]\!] \to R^*(\mathbf{C}P^\infty \times \mathbf{C}P^\infty) = \pi_*(R)[\![x, y]\!],$$

and the image of $\hbar$ under this map is a 2-variable formal power series $F(x, y)$ with coefficients in $\pi_*(R)$. Explicitly, $F(x, y)$ is the power series such that

$$F(c_1^R(\mathscr{L}), c_1^R(\mathscr{L}')) = c_1^R(\mathscr{L} \otimes \mathscr{L}')$$

for line bundles $\mathscr{L}$ and $\mathscr{L}'$ over compact spaces. This power series is a *formal group law*, i.e., $F(x, y) = x + y +$ higher order terms, and $F(F(x, y), z) = F(x, F(y, z))$.

Suppose that R is *even-periodic*, i.e., that $\pi_*(R) \cong \pi_0(R)[\beta^{\pm 1}]$ for some $\beta \in \pi_2(R)$. Then, R admits a canonical complex orientation (by obstruction theory). In this case, we obtain a formal group $\mathrm{Spf}\, R^0(\mathbf{C}P^\infty) \cong \mathrm{Spf}\, \pi_0(R)[\![\hbar]\!]$, with group structure given by μ^*. This has a lift to a spectral formal group scheme over R, given by

$$\hat{\mathbf{G}}_R := \mathrm{Spf}\, R^{\mathbf{C}P^\infty_+}.$$

Example 1.11. Returning to Example 1.5, recall that $\mathrm{KU}^*(\mathrm{BS}^1) \cong \mathbf{Z}[\![q-1]\!][\beta^{\pm 1}]$, where $|q-1| = 0$ and $|\beta| = 2$. The complex orientation $\hbar$ of KU can be identified with $(q-1)\beta^{-1}$. The associated formal group law is given by

$$F(x,y) = x + y + \beta xy.$$

Indeed, note that

$$\begin{aligned} F(\hbar_1, \hbar_2) &= (q_1 - 1)\beta^{-1} + (q_2 - 1)\beta^{-1} + \beta \cdot (q_1 - 1)(q_2 - 1)\beta^{-2} \\ &= \beta^{-1}(q_1 - 1 + q_2 - 1 + (q_1 - 1)(q_2 - 1)) \\ &= \beta^{-1}(q_1 q_2 - 1). \end{aligned}$$

The associated formal group is

$$\mathrm{Spf}\, \mathrm{KU}^0(\mathrm{BS}^1) = \mathrm{Spf}\, \mathbf{Z}[\![q-1]\!] \cong \hat{\mathbf{G}}_m,$$

i.e., the formal completion $\hat{\mathbf{G}}_m$ of the multiplicative formal group at the identity.

For complex-oriented $\mathbf{E}_\infty$-rings, Borel equivariant cohomology is well-behaved: namely, if X is a finite G-space, then $R^*(X_{hG})$ is often quite understandable. One very natural question to ask is whether there is a genuine G-equivariant analogue of R, which recovers KU_G when $R = \mathrm{KU}$. In this talk, we will only consider the case when G is a torus.

Motivated by the Atiyah-Segal completion theorem and Example 1.11, we are led to view genuine S^1-equivariant K-theory KU_{S^1} as specifying an uncompletion of $\hat{\mathbf{G}}_m$. Therefore, one possible approach to defining a genuine S^1-equivariant analogue of R is as the ring of functions on some uncompletion of the formal group $\hat{\mathbf{G}}_R$ associated to R.

Let $\mathbf{G}$ be a commutative flat 1-dimensional group scheme over R. To be precise, a group scheme in the setting of spectral algebraic geometry simply means a spectral R-scheme whose functor of points is equipped with a lifting

$$\begin{tikzcd} & \mathrm{Mod}_{\mathbf{Z}}^{\geq 0} \simeq s\mathrm{Ab} \arrow[d, "\Omega^\infty\text{, or underlying}"] \\ \{\mathbf{E}_\infty\text{-algebras over } R\} \arrow[r] \arrow[ur, dashed, "\mathbf{G}"] & \mathscr{S} \simeq s\mathrm{Set}. \end{tikzcd}$$

We will let $\mathbf{G}_0$ denote the underlying group scheme of $\mathbf{G}$; this is an ordinary group scheme over $\pi_0(R)$.

Definition 1.12. A *preorientation* of $\mathbf{G}$ is a pointed map $S^2 \to \Omega^\infty \mathbf{G}(A)$, i.e., a class in $\pi_2 \mathbf{G}(A)$. Since $\mathrm{Sym}(S^2) \simeq \mathbf{C}P^\infty$ by Dold-Thom, one can view this as a map $\mathbf{C}P^\infty \to \Omega^\infty \mathbf{G}(A)$ of topological abelian groups.

Construction 1.13. Given a preorientation $S^2 \to \Omega^\infty \mathbf{G}(A)$, we obtain a map $\mathscr{O}_{\mathbf{G}} \to C^*(S^2; R)$ of $\mathbf{E}_\infty$-R-algebras. On π_0, this induces a map $\pi_0 \mathscr{O}_{\mathbf{G}} = \mathscr{O}_{\mathbf{G}_0} \to R^*(S^2)$. However, the target can be identified with the trivial square-zero extension

$\pi_0 A \oplus \pi_{-2} A$, so that the preorientation defines a *derivation* $\mathscr{O}_{\mathbf{G}_0} \to \pi_{-2} A$. By the universal property of Kahler differentials, this defines a map

$$\beta : \omega = \Omega^1_{\mathbf{G}_0/\pi_0 A} \to \pi_{-2} A.$$

Definition 1.14. A preorientation of $\mathbf{G}$ is called an *orientation* if $\mathbf{G}_0$ is smooth of relative dimension 1 over $\pi_0 A$, and the composite

$$\pi_n(A) \otimes_{\pi_0 A} \omega \to \pi_n(A) \otimes_{\pi_0 A} \pi_{-2} A \xrightarrow{\beta} \pi_{n-2} A$$

is an isomorphism for each $n \in \mathbf{Z}$. This forces A to be 2-periodic (but does not force its homotopy to be concentrated in even degrees).

If $\mathbf{G}$ is an oriented group scheme over R, then $\mathscr{O}_{\mathbf{G}}$ is supposed to play the role of the genuine S^1-equivariant cohomology R_{S^1}. Namely, we *define* R_{S^1} via

$$R_{S^1} := \Gamma(\mathbf{G}; \mathscr{O}_{\mathbf{G}}) \in \mathrm{CAlg}_R.$$

Warning 1.15. The group scheme $\mathbf{G}$ need *not* be affine: for example, the motivating example in [**Lur09**] (and [**GM22**]) was the question of defining equivariant elliptic cohomology, in which case $\mathbf{G}$ is taken to be an elliptic curve. In this case, the data of the spectral group scheme $\mathbf{G}$ is more information than R_{S^1}, and is a more convenient/interesting algebro-geometric object to consider.

At this point, it becomes a matter of combinatorics to properly define R_T if T is a compact torus.

Definition 1.16. Let T be a compact torus, and let $\mathbb{X}^*(T) = \mathrm{Hom}(T, S^1)$ be its character lattice. Let $\mathscr{M}_T := \mathrm{Hom}(\mathbb{X}^*(T), \mathbf{G})$ (so that it is noncanonically isomorphic to $\mathbf{G}^{\times \mathrm{rank}(T)}$). Then, define the genuine T-equivariant cohomology R_T via

$$R_T := \Gamma(\mathscr{M}_T; \mathscr{O}_{\mathscr{M}_T}) \in \mathrm{CAlg}_R.$$

Again, $\mathscr{M}_T$ is a more refined object than R_T, at least if $\mathbf{G}$ is not affine.

This defines the T-equivariant cohomology of a *point*; we need to be able to define $R_T(X)$ for finite T-spaces X. If we require that the assignment $X \mapsto R_T(X)$ preserve finite limits, then it suffices to specify $R_T(T/T')$ for subtori $T' \subseteq T$. Such a subtorus defines an injection $\mathbb{X}_*(T') \to \mathbb{X}(T)$, and hence a closed immersion $i : \mathscr{M}_{T'} \hookrightarrow \mathscr{M}_T$. This defines a quasicoherent sheaf $i_* \mathscr{O}_{\mathscr{M}_{T'}} \in \mathrm{QCoh}(\mathscr{M}_T)$, and we define

$$R_T(T/T') := \Gamma(\mathscr{M}_T; i_* \mathscr{O}_{\mathscr{M}_{T'}}) \simeq \Gamma(\mathscr{M}_{T'}; \mathscr{O}_{\mathscr{M}_{T'}}) \in \mathrm{CAlg}_R.$$

Note that this definition is compatible with the idea that $R_T(T/T')$ should be equivalent to $R_{T'}$ (since $T \backslash (T/T') \simeq */T'$).

Remark 1.17. As mentioned above, $\mathscr{M}_T$ is a more refined object than R_T, at least if $\mathbf{G}$ is not affine. Therefore, one might hope that $R_T(X)$ arises as the global sections of some quasicoherent sheaf on $\mathscr{M}_T$. This is indeed true: the assignment $T/T' \mapsto i_* \mathscr{O}_{\mathscr{M}_{T'}}$ extends to a functor

$$\mathscr{S}(T)^{\mathrm{op}} \to \mathrm{QCoh}(\mathscr{M}_T), \ X \mapsto \mathscr{F}_T(X),$$

such that $R_T(X) = \Gamma(\mathscr{M}_T; \mathscr{F}_T(X))$.

Let us illustrate this in the key cases of Example 1.4 and Example 1.8.

Example 1.18. Again, let S^1 act on S^2 in the standard way. Since $T = S^1$, we see that $\mathscr{M}_T = \mathbf{G}$. There is a $\mathbf{Z}/2$-action on $\mathbf{G}$ given by sending x to its inverse under the group structure on $\mathbf{G}$. Let $\mathscr{M}_{\mathrm{SU}(2)} := \mathbf{G}/\!\!/(\mathbf{Z}/2)$ denote the GIT quotient (or the stacky quotient if 2 is a unit in $\pi_0(R)$), so that there is a map $\mathbf{G} \to \mathscr{M}_{\mathrm{SU}(2)}$. This induces a map $f : \mathbf{G} \times_{\mathscr{M}_{\mathrm{SU}(2)}} \mathbf{G} \to \mathbf{G}$, and there is an equivalence

$$\mathscr{F}_{S^1}(S^2) = f_* \mathscr{O}_{\mathbf{G} \times_{\mathscr{M}_{\mathrm{SU}(2)}} \mathbf{G}} = \mathscr{O}_{\mathbf{G}} \otimes_{\mathscr{O}_{\mathscr{M}_{\mathrm{SU}(2)}}} \mathscr{O}_{\mathbf{G}} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathbf{G})).$$

One can check that this recovers the calculations of Example 1.4 and Example 1.8 when $\mathbf{G}$ is $\mathbf{G}_a$ (corresponding to ordinary integral cohomology) and $\mathbf{G}_m$ (corresponding to complex K-theory), respectively.

The Atiyah-Segal completion theorem can be generalized as follows:

Proposition 1.19. *If $\mathscr{I}$ denotes the ideal sheaf of the identity section* $\mathrm{Spec}(R) \hookrightarrow \mathscr{M}_T$ *(induced by the subtorus $\{1\} \subseteq T$), $\hat{\mathscr{M}}_T$ denotes the formal completion of $\mathscr{M}_T$ at the identity section, and X is a finite T-space, there is an equivalence*

$$\Gamma(\hat{\mathscr{M}}_T; \mathscr{F}_T(X)^{\wedge}_{\mathscr{I}}) \simeq C^*(X_{hT}; R).$$

Since it will be useful, let us observe that the above constructions can be generalized even further: instead of just looking at the T-equivariant R-cohomology of X, one can also define the ∞-*category* of T-equivariant local systems of R-modules on X. Namely:

Construction 1.20. Recall that if $T' \subseteq T$ is a subtorus, there is a closed immersion $i : \mathscr{M}_{T'} \hookrightarrow \mathscr{M}_T$. This induces a symmetric monoidal functor $i^* : \mathrm{QCoh}(\mathscr{M}_T) \to \mathrm{QCoh}(\mathscr{M}_{T'})$, which equips $\mathrm{QCoh}(\mathscr{M}_{T'})$ with the structure of a $\mathrm{QCoh}(\mathscr{M}_T)$-module. Roughly, the assignment $T/T' \mapsto \mathrm{QCoh}(\mathscr{M}_{T'})$ extends to a functor from $\mathscr{S}(T)$ to $\mathrm{QCoh}(\mathscr{M}_T)$-modules, which sends X to a symmetric monoidal ∞-category $\mathrm{Loc}_T(X; R)$. There is a symmetric monoidal equivalence $\mathrm{Loc}_T(*; R) \simeq \mathrm{QCoh}(\mathscr{M}_T)$, and pullback along the crushing map $f : X \to *$ induces a symmetric monoidal functor $f^* : \mathrm{QCoh}(\mathscr{M}_T) \to \mathrm{Loc}_T(X; R)$. This functor has a right adjoint f_*, and $\mathscr{F}_T(X) \cong f_* f^* \mathscr{O}_{\mathscr{M}_T}$.

2. Some representation theory

Notation 2.1. Let G be a simply-connected semisimple group over $\mathbf{C}$, and let G_c be a maximal compact subgroup of $G(\mathbf{C})$. Let $B \subseteq G$ be a fixed Borel subgroup, and let $T \subseteq B$ be a fixed maximal torus. Note that there are homotopy equivalences $B(\mathbf{C}) \simeq T(\mathbf{C}) \simeq T_c$. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{t}$ be the Lie algebra of T. The group T admits an action of the Weyl group W of G, which induces an action of W on $\mathfrak{t}$. We will fix an invariant bilinear form on $\mathfrak{g}$, which in particular induces isomorphisms $\mathfrak{g}^* \cong \mathfrak{g}$ and $\mathfrak{t}^* \cong \mathfrak{t}$. We will also write $\mathbf{Z}'$ to denote $\mathbf{Z}[1/|W|]$, where $|W|$ is the order of the Weyl group.

The typical example to keep in mind is $G = \mathrm{SL}_n$, in which case $G_c = \mathrm{SU}(n)$. We can also take B to be the subgroup of upper-triangular matrices in SL_n, and T to be the subgroup of diagonal matrices. In particular, T is abstractly isomorphic to $(\mathbf{C}^\times)^{\times n-1}$. In this case, the Weyl group is the symmetric group Σ_n, and the action of Σ_n on $\mathfrak{t} = \mathbf{C}^{n-1}$ is via the reduced standard representation. For instance, if $n = 2$, then the action of $\Sigma_2 = \mathbf{Z}/2$ on $\mathfrak{t} = \mathbf{C}$ sends $x \mapsto -x$.

Definition 2.2. The *flag variety* $\mathscr{B}$ of G is defined to be the algebraic variety G/B. Its underlying topological space $\mathscr{B}(\mathbf{C})$ is homotopy equivalent to G_c/T_c. For the purpose of this talk, we will abusively write $\mathscr{B} = G_c/T_c$. The flag variety has a G_c-action, given by translation on the left.

Example 2.3. If $G = \mathrm{SL}_n$ and B is the subgroup of upper-triangular matrices, then $\mathscr{B}$ is the space of *complete flags*, i.e., filtrations $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_n$, where V_n is an n-dimensional $\mathbf{C}$-vector space, and V_j is a subspace of dimension j. If $n = 2$, for instance, this is simply the space of lines in a 2-dimensional vector space; in other words, $\mathrm{SL}_2/B \cong \mathbf{C}P^1$. This can be viewed as an algebro-geometric lift of the equivalence $\mathrm{SU}(2)/T_c = \mathrm{SU}(2)/S^1 \simeq S^2$ induced by the Hopf fibration.

To be explicit, if $\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right) \in \mathrm{SL}_2$, and $\left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \in B$, then

$$\begin{pmatrix} x & x' \\ y & y' \end{pmatrix} \cdot \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} = \begin{pmatrix} ax & xb + x'a^{-1} \\ ay & yb + y'a^{-1} \end{pmatrix}.$$

From this, one sees that the orbit of $\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right)$ is determined by the ratio $\frac{x}{y}$, i.e., the map $\mathrm{SL}_2/B \to \mathbf{C}P^1$ sending $\left(\begin{smallmatrix} x & x' \\ y & y' \end{smallmatrix}\right) \mapsto [x : y]$ is an isomorphism. Note that the SL_2-action by left translation is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot [x : y] = [ax + by : cx + dy].$$

In particular, if we view the flag variety $\mathscr{B}$ for SL_2 as $\mathbf{C}P^1 = S^2$, the S^1-action on S^2 from Example 1.4 is simply induced by the left translation action of $S^1 \subseteq \mathrm{SU}(2)$.

A basic principle in geometric representation theory is that "everything is built up from SL_2" (and PGL_2, whose $\mathbf{C}$-points is homotopy equivalent to $\mathrm{SO}(3)$; but we will mainly focus on SL_2). Given the calculation of Example 1.4, one natural question that arises is the following: can one describe the T_c-equivariant cohomology $\mathrm{H}^*_{T_c}(\mathscr{B}; \mathbf{Z})$?

Recollection 2.4. Recall from last time that if A is an even-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}$ is an oriented group scheme over A, then one can define T_c-equivariant cohomology A_T via the global sections of $\mathscr{M}_T := \mathrm{Hom}(\mathbb{X}^*(T_c), \mathbf{G})$. If X is a finite T_c-space, then we defined a quasicoherent sheaf $\mathscr{F}_T(X) \in \mathrm{QCoh}(\mathscr{M}_T)$, and the global sections $\Gamma(\mathscr{M}_T; \mathscr{F}_T(X))$ can be taken to be the definition of the T_c-equivariant cohomology $A_T(X)$. Although we did not define G_c-equivariant cohomology, Lurie **[Lur09]** has sketched a construction (if G_c is connected), which defines a stack $\mathscr{M}_G$ and a quasicoherent sheaf $\mathscr{F}_G(X) \in \mathrm{QCoh}(\mathscr{M}_G)$.

One now can more generally ask if it is possible to describe the T_c-equivariant cohomology $R_{T_c}(\mathscr{B})$, or even $\mathscr{F}_T(\mathscr{B}) \in \mathrm{QCoh}(\mathscr{M}_T)$. As a warmup, observe:

Example 2.5. There is an isomorphism

$$\mathscr{F}_{G_c}(\mathscr{B}) \cong \mathscr{F}_{G_c}(G_c/T_c) \cong \mathscr{F}_{T_c}(*) = \mathscr{O}_{\mathscr{M}_T}.$$

Example 2.6. Let $G = \mathrm{SL}_2$. In Example 1.18, we saw that $\mathscr{F}_{S^1}(\mathscr{B}) \cong \mathscr{O}_{\mathbf{G}} \otimes_{\mathscr{O}_{\mathscr{M}_{\mathrm{SU}(2)}}} \mathscr{O}_{\mathbf{G}}$, so that

$$\mathscr{F}_{S^1}(S^2) = f_* \mathscr{O}_{\mathbf{G} \times_{\mathscr{M}_{\mathrm{SU}(2)}} \mathbf{G}} = \mathscr{O}_{\mathbf{G}} \otimes_{\mathscr{O}_{\mathscr{M}_{\mathrm{SU}(2)}}} \mathscr{O}_{\mathbf{G}} \in \mathrm{CAlg}(\mathrm{QCoh}(\mathbf{G})).$$

This is in fact a special case of the following result, which one can prove by noting that $\mathscr{B}_{hT_c} \simeq BT_c \times_{BG_c} BT_c$ (in orbifolds):

Proposition 2.7. *Let A be an even-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}$ be an oriented group scheme over A. Suppose that $|W|$ is invertible in $\pi_0(A)$. Then taking relative Spec defines an isomorphism*

$$\mathrm{Spec}_{\mathscr{M}_T} \mathscr{F}_{T_c}(\mathscr{B}) \cong \mathscr{M}_T \times_{\mathscr{M}_G} \mathscr{M}_T.$$

Given Proposition 2.7, one is naturally led to wonder if there is a simple description of the ∞-category $\mathrm{Loc}_{T_c}(\mathscr{B}; A)$ of T_c-equivariant local systems of A-modules over $\mathscr{B}$ in terms of $\mathscr{M}_T$ and $\mathscr{M}_G$.

Observation 2.8. If $\mathscr{F} \in \mathrm{Loc}_{G_c}(\mathscr{B}; A)$, then the functor $\mathrm{Loc}_{G_c}(\mathscr{B}; A) \to \mathrm{Mod}_A$ given by pushing forward along the map $\mathscr{B} \to *$ refines to a functor

$$\Gamma : \mathrm{Loc}_{G_c}(\mathscr{B}; A) \to \mathrm{QCoh}(\mathscr{M}_T).$$

This functor is an equivalence, since $\mathrm{Loc}_{G_c}(\mathscr{B}; A) \simeq \mathrm{Loc}_{T_c}(*; A) = \mathrm{QCoh}(\mathscr{M}_T)$ (for the same reason as in Example 2.5). Similarly, if $\mathscr{F} \in \mathrm{Loc}_{T_c}(\mathscr{B}; A)$, then the functor $\Gamma : \mathrm{Loc}_{T_c}(\mathscr{B}; A) \to \mathrm{Mod}_A$ refines to a functor

$$\Gamma : \mathrm{Loc}_{T_c}(\mathscr{B}; A) \to \mathrm{QCoh}(\mathscr{M}_T \times_{\mathscr{M}_G} \mathscr{M}_T).$$

This functor is also an equivalence.

Remark 2.9. The category $\mathrm{Loc}_{T_c}(\mathscr{B}; \mathbf{C})$ therefore turns out to be somewhat easy, but it does not capture most of the interesting geometry of the flag variety. A better object to study is the ∞-category $\mathrm{DMod}_B(\mathscr{B})$ of *equivariant D-modules* on $\mathscr{B}$, which is of great interest in geometric representation theory. One can view this as the ∞-category $\mathrm{Shv}^c_{T_c}(\mathscr{B}; \mathbf{C})$ of T_c-equivariant *constructible sheaves* of $\mathbf{C}$-modules on $\mathscr{B}$; here, the constructibility is taken with respect to the Bruhat stratification of the flag variety. There is still a notion of global sections, which defines a functor

$$\Gamma : \mathrm{Shv}^c_{T_c}(\mathscr{B}; \mathbf{C}) \to \mathrm{QCoh}(\mathfrak{t}^*_{\mathbf{C}} \times_{\mathfrak{t}^*_{\mathbf{C}}/\!\!/W} \mathfrak{t}^*_{\mathbf{C}}).$$

This functor is *not* fully faithful. However, the main result of Soergel theory **[Soe90]** roughly says that a large full subcategory of $\mathrm{DMod}_B(\mathscr{B})$ can be described combinatorially.

Observation 2.8 motivates the geometric Satake equivalence, whose basic goal is to study the story that results when G_c is replaced by the loop group $\mathscr{L}G_c$. There are several motivations for this:

(a) In number theory, one starts with a reductive algebraic group H defined over $\mathbf{Z}_p$, and one is often interested in representations of $H(\mathbf{Q}_p)$. The group $H(\mathbf{Q}_p)$ has a maximal compact subgroup given by $H(\mathbf{Z}_p)$, and the most basic class of $H(\mathbf{Q}_p)$-representations are the "unramified"/"spherical" ones. It turns out that irreducible unramified $H(\mathbf{Q}_p)$-representations correspond to modules over the $\mathbf{C}$-vector space $\mathscr{H} := \mathrm{Fun}(H(\mathbf{Z}_p)\backslash H(\mathbf{Q}_p)/H(\mathbf{Z}_p), \mathbf{C})$ equipped with the algebra structure given by convolution. The algebra $\mathscr{H}$ is called the *Hecke algebra*, and the famous *Satake isomorphism* says that there is an isomorphism of algebras $\mathscr{H} \cong \mathbf{C}[\mathbb{X}_*(T)]^W$.

    The pair $\mathbf{Z}_p \subseteq \mathbf{Q}_p$ behaves similarly to $\mathbf{F}_p[\![t]\!] \subseteq \mathbf{F}_p(\!(t)\!)$ (this is the number field-function field correspondence), which in turn behaves similarly to $\mathbf{C}[\![t]\!] \subseteq \mathbf{C}(\!(t)\!)$. Therefore, if G is a reductive algebraic group over $\mathbf{C}$, one might naturally be interested in $\mathrm{Fun}(G(\mathbf{C}[\![t]\!])\backslash G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!]), \mathbf{C})$. The quotient $G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!]) =: \mathrm{Gr}_G$ is known as the *affine Grassmannian*. It is homotopy equivalent to ΩG_c (by a result of Quillen's), roughly because $\Omega G_c \simeq (\mathscr{L}G_c)/G_c$. In particular, one might view $\mathrm{Fun}(G(\mathbf{C}[\![t]\!])\backslash G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!]), \mathbf{C})$ as $\mathrm{H}^0_{G_c}(\Omega G_c; \mathbf{C})$. This suggests that in this "geometric" setting, there is a natural refinement of the Hecke algebra, given by the entire cohomology ring $\mathrm{H}^*_{G_c}(\Omega G_c; \mathbf{C})$. Even better, there is a *categorification* of this Hecke algebra, given by $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$.

(b) Let C be a smooth projective curve over $\mathbf{C}$. The stack $\mathrm{Bun}_G(C)$ of (algebraic) G-bundles over C is an important object of study in many branches of math. Let us pretend that we are looking at the stack of *topological* G_c-bundles on $C(\mathbf{C})$. Let $x \in C(\mathbf{C})$, and let D be a small disk around x; then, describing G_c-bundles in terms of "transition functions" implies

$$\begin{aligned}\mathrm{Bun}_{G_c}(C(\mathbf{C})) &\cong \mathrm{Map}(C - \{x\}, G_c)\backslash \mathrm{Map}(D - \{x\}, G_c)/\mathrm{Map}(D, G_c) \\ &\cong \mathrm{Map}(C - \{x\}, G_c)\backslash \mathscr{L}G_c/G_c.\end{aligned}$$

A similar result holds in the algebraic setting: for instance, $\mathrm{Bun}_G(\mathbf{P}^1) \cong G(\mathbf{C}[t^{-1}])\backslash G(\mathbf{C}(\!(t)\!))/G(\mathbf{C}[\![t]\!])$. This implies that the first step in understanding $\mathrm{Bun}_G(C)$ is to understand Gr_G.

The discussion in (a) suggests that there should be a version of the Satake equivalence, describing $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$ in simpler terms. To describe this, let us recast the right-hand side $\mathbf{C}[\mathbb{X}_*(T)]^W$ of the Satake isomorphism. Recall that if H is a reductive algebraic group (over $\mathbf{C}$, say) with maximal torus T_H, then representations of H are classified by dominant weights, which means that the representation ring $R(H)$ is isomorphic to $\mathbf{Z}[\mathbb{X}^*(T_H)]^W$. Therefore, if $\mathbb{X}^*(T_H) \cong \mathbb{X}_*(T)$, then $\mathbf{C}[\mathbb{X}_*(T)]^W$ can be interpreted as $R(H) \otimes_{\mathbf{Z}} \mathbf{C}$. The desired H is precisely the *Langlands dual* of G, denoted $\check{G}$: it is an algebraic group whose weight lattice and roots are given by the coweight lattice and coroots of G. For instance, the Langlands dual of $\mathrm{SL}_n = \ker(\det : \mathrm{GL}_n \to \mathbf{G}_m)$ is $\mathrm{PGL}_n = \mathrm{GL}_n/\mathbf{G}_m$.

Remark 2.10. Note that this is a purely combinatorial definition of $\check{G}$, so it is not clear at all how the geometry of G and $\check{G}$ are related. Part of the miracle of

Langlands-esque dualities are that there is in fact a very tight relationship between the two.

In any case, we now see that the Satake isomorphism can be interpreted as an isomorphism

$$\mathscr{H} \cong \mathbf{C}[\mathbb{X}_*(T)]^W \cong K_0(\mathrm{Rep}(\check{G})) \otimes_{\mathbf{Z}} \mathbf{C}, \tag{1}$$

where $K_0(\mathrm{Rep}(\check{G}))$ denotes the Grothendieck group of the category of representations of the Langlands dual group $\check{G}$. Since the left-hand side of (1) is categorified by $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$, and the right-hand side of (1) is categorified by $\mathrm{Rep}(\check{G})$ itself, one can ask whether there is an actual equivalence of categories between $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$ and $\mathrm{Rep}(\check{G})$. This is not quite true: instead, there is a t-structure on $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$, known as the *perverse t-structure*, such that the following holds.

Theorem 2.11 (Mirkovic-Vilonen, [**MV07**])**.** *There is an equivalence of symmetric monoidal abelian categories* $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})^\heartsuit \cong \mathrm{Rep}(\check{G})$.

This is known as the *geometric Satake equivalence*, and plays a central role in geometric Langlands. However, it is somewhat unsatisfactory, since one would naturally want a description of the full ∞-category $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$, and not just its heart. My understanding is that Drinfeld was the first to ask for such a description; it was provided by Bezrukavnikov-Finkelberg in [**BF08**], where the right-hand side now involves derived algebraic geometry.

Theorem 2.12 (Derived geometric Satake, [**BF08**])**.** *There is an $\mathbf{E}_2$-monoidal equivalence* $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C}) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}[2]/\check{G})$ *of $\mathbf{C}$-linear ∞-categories, where* $\check{\mathfrak{g}}[2]$ *is the derived $\mathbf{C}$-scheme* $\mathrm{Spec}\,\mathrm{Sym}_{\mathbf{C}}(\check{\mathfrak{g}}^*[-2])$.

In particular, after 2-periodification, there is an $\mathbf{E}_2$-monoidal equivalence

$$\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}/\check{G}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}].$$

One key ingredient in the proof of this theorem is the calculation of the cohomology $\mathrm{H}^*_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C})$ in terms of the root data of G, which is then reinterpreted in terms of the algebraic geometry of $\check{G}$. In fact, this is a general feature of geometric Langlands-type equivalences: the *algebraic topology* of various geometric objects constructed from G (such as Gr_G) can be reinterpreted in terms of the *algebraic geometry* of $\check{G}$.

This leads to a natural question: if A is an even-periodic $\mathbf{E}_\infty$-ring and $\mathbf{G}$ is an oriented group scheme over A, can one describe $\mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; A)$ in terms of the Langlands dual $\check{G}$? This question turns out to be very interesting, and rather subtle; we will discuss it further in the next talk. For the moment, let us return to the Bezrukavnikov-Finkelberg equivalence, and focus on the simpler subcategory $\mathrm{Loc}_{G_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \subseteq \mathrm{Shv}^c_{G(\mathbf{C}[\![t]\!])}(\mathrm{Gr}_G; \mathbf{C}[\beta^{\pm 1}])$ of locally constant sheaves. This turns out to correspond to a certain *localization* of $\mathrm{QCoh}(\check{\mathfrak{g}}/\check{G})$, which we now describe.

Definition 2.13. An element $x \in \check{\mathfrak{g}}$ is called *regular* if its centralizer $Z_{\check{\mathfrak{g}}}(x)$ has dimension given by the rank of $\check{\mathfrak{g}}$, i.e., its centralizer has the smallest possible dimension. Let $\check{\mathfrak{g}}^{\mathrm{reg}} \hookrightarrow \check{\mathfrak{g}}$ denote the locus of regular elements; this is an open subset with complement of codimension ≥ 2.

Example 2.14. An element $x \in \mathfrak{gl}_n$ is simply an $n \times n$-matrix. It is regular if and only if its characteristic and minimal polynomials agree. If x is diagonalizable, this means that all its eigenvalues are distinct; if x is nilpotent, this means that the Jordan normal form of x has a single Jordan block. For instance, $\left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2$ is regular if $a \neq 0$, and $\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \in \mathfrak{sl}_2$ is regular if $b \neq 0$.

The Bezrukavnikov-Finkelberg equivalence then restricts to an equivalence

$$\mathrm{Loc}_{G_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}/\check{G}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]; \tag{2}$$

in the next talk, we will describe a generalization of this equivalence.

Remark 2.15. There is also an analogous description of $\mathrm{Loc}_{T_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \subseteq \mathrm{Shv}_I^c(\mathrm{Gr}_G; \mathbf{C}[\beta^{\pm 1}])$, where $I \subseteq G(\mathbf{C}[\![t]\!])$ is the *Iwahori* subgroup (defined as the preimage of $B(\mathbf{C}) \subseteq G(\mathbf{C})$ under the map $G(\mathbf{C}[\![t]\!]) \to G(\mathbf{C})$). Namely, the Arkhipov-Bezrukavnikov-Ginzburg equivalence **[ABG04]** says that

$$\mathrm{Shv}_I^c(\mathrm{Gr}_G; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{b}}/\check{B}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}],$$

where $\check{B} \subseteq \check{G}$ is the Langlands dual Borel subgroup. This restricts to an equivalence

$$\mathrm{Loc}_{T_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{b}}^{\mathrm{reg}}/\check{B}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]. \tag{3}$$

The equivalences (2) and (3) are closely related to the equivalences $\mathrm{Loc}_{G_c}(*; \mathbf{C}[\beta^{\pm 1}]) \cong \mathrm{QCoh}(\mathfrak{t}^* /\!\!/ W) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$ and $\mathrm{Loc}_{T_c}(*; \mathbf{C}[\beta^{\pm 1}]) \cong \mathrm{QCoh}(\mathfrak{t}^*) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$. For instance, if we are given a T_c-equivariant local system on ΩG_c, we can restrict to the basepoint of ΩG_c to obtain a T_c-equivariant local system on a point. This defines a functor $\mathrm{Loc}_{T_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \to \mathrm{Loc}_{T_c}(*; \mathbf{C}[\beta^{\pm 1}])$. Using (3), we obtain a commutative diagram

$$\begin{tikzcd} \mathrm{Loc}_{T_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \arrow[r, "\mathrm{Res}"] \arrow[d, "\sim"', "(3)"] & \mathrm{Loc}_{T_c}(*; \mathbf{C}[\beta^{\pm 1}]) \arrow[d, "\sim"] \\ \mathrm{QCoh}(\check{\mathfrak{b}}^{\mathrm{reg}}/\check{B}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}] \arrow[r, dashed] & \mathrm{QCoh}(\mathfrak{t}^*) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}], \end{tikzcd}$$

which gives us a dotted functor as indicated. This dotted functor turns out to be given by pullback along an extremely interesting map $\mathfrak{t}^* \cong \check{\mathfrak{t}} \to \check{\mathfrak{b}}^{\mathrm{reg}}/\check{B}$. This map is given by a composite

$$\check{\mathfrak{t}} \xrightarrow{\kappa} \check{\mathfrak{b}}^{\mathrm{reg}} \to \check{\mathfrak{b}}^{\mathrm{reg}}/\check{B}.$$

The map $\kappa : \check{\mathfrak{t}} \to \check{\mathfrak{b}}^{\mathrm{reg}}$ is closely related to the *Kostant section*, which also defines a map

$$\mathfrak{t}^* /\!\!/ W \to \check{\mathfrak{g}}^{\mathrm{reg}} \to \check{\mathfrak{g}}^{\mathrm{reg}}/\check{G}.$$

Let us first describe the case $G = \mathrm{SL}_2$.

Example 2.16. When $G = \mathrm{SL}_2$, we have $\check{G} = \mathrm{PGL}_2$. The Cartan subalgebra $\mathfrak{t}$ is isomorphic to $\mathbf{A}^1$, and $W = \mathbf{Z}/2$ acting on $\mathfrak{t}$ by $x \mapsto -x$; therefore, $\mathfrak{t}^* /\!\!/ W \cong \mathrm{Spec}\, \mathbf{C}[x^2]$. The Kostant section $\mathfrak{t}^* /\!\!/ W \to \mathfrak{pgl}_2^{\mathrm{reg}} \subseteq \mathfrak{pgl}_2$ sends

$$\mathfrak{t}^* /\!\!/ W \ni x^2 \mapsto \kappa(x^2) = \begin{pmatrix} 0 & x^2 \\ 1 & 0 \end{pmatrix} \in \mathfrak{pgl}_2.$$

Note that the minimal and characteristic polynomials of $\kappa(x^2)$ agree; they are given by $t^2 - x^2 = (t + x)(t - x)$. In other words, $\kappa : \mathbf{A}^1 \to \mathfrak{pgl}_2$ lands in the regular locus, and gives a particular section of the map $\chi : \mathfrak{pgl}_2 \to \mathbf{A}^1$ sending a matrix to the constant term of its characteristic polynomial.

The map $\kappa : \check{\mathfrak{t}} \to \check{\mathfrak{b}}^{\mathrm{reg}}$ can be described similarly as follows: it sends

$$\mathfrak{t}^* \ni x \mapsto \kappa(x) = \begin{pmatrix} x & 0 \\ 1 & 0 \end{pmatrix} \in \mathfrak{b}.$$

Note that the characteristic polynomial of $\kappa(x)$ is $t^2 - xt$. For $\check{G} = \mathrm{SL}_2$, the analogue of the map κ instead sends

$$\mathfrak{t}^* \ni x \mapsto \kappa(x) = \begin{pmatrix} x & 0 \\ 1 & -x \end{pmatrix} \in \mathfrak{b}.$$

Definition 2.17. Let $e \in \check{\mathfrak{n}} \subseteq \check{\mathfrak{g}}$ be a principal nilpotent element (e.g., $e = \sum_{\alpha \in \Delta} x_\alpha$, where Δ is the set of simple roots and $x_\alpha \in \check{\mathfrak{g}}_\alpha$ is a nonzero element in the root space), and let $f \in \check{\mathfrak{n}}_- \subseteq \check{\mathfrak{g}}$ be an opposite principal nilpotent. (The element f is determined by e via the Jacobson-Morozov theorem.) The *Kostant slice* $\mathscr{S}$ is defined to be the affine subspace $e + Z_{\check{\mathfrak{g}}}(f) \subseteq \check{\mathfrak{g}}^{\mathrm{reg}} \subseteq \check{\mathfrak{g}}$. Composing the inclusion $\mathscr{S} \subseteq \check{\mathfrak{g}}$ with the characteristic polynomial map $\chi : \check{\mathfrak{g}} \to \mathfrak{t}/\!\!/W$ defines an isomorphism $\mathscr{S} \xrightarrow{\sim} \check{\mathfrak{t}}/\!\!/W$; therefore, the inclusion $\mathscr{S} \subseteq \check{\mathfrak{g}}$ can be regarded as a section $\kappa : \check{\mathfrak{t}}/\!\!/W \to \check{\mathfrak{g}}^{\mathrm{reg}}$ of χ. The map κ is known as the Kostant section.

The other map $\kappa : \check{\mathfrak{t}} \to \check{\mathfrak{b}}^{\mathrm{reg}}$ admits a very similar description: namely, let $\widetilde{\mathscr{S}}$ denote the affine subspace $e + \check{\mathfrak{t}} \subseteq \check{\mathfrak{b}}$. Clearly, the map $x \mapsto e + x$ defines an isomorphism $\check{\mathfrak{t}} \cong \widetilde{\mathscr{S}}$, and the resulting map $\check{\mathfrak{t}} \to \check{\mathfrak{b}}$ is precisely κ. This map, and its "higher chromatic analogue", will play an important role in the next talk.

3. A chromatic analogue

Fix a simply-connected compact Lie group G_c, with associated reductive group G over $\mathbf{C}$; also fix a Borel subgroup $B \subseteq G$. Let $\check{G}$ denote its Langlands dual over $\mathbf{Q}$ (or $\mathbf{C}$). Last time, I stated one consequence of the Bezrukavnikov-Finkelberg/derived geometric Satake equivalence (2): namely, there is an equivalence $\mathrm{Loc}_{G_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{g}}^{\mathrm{reg}}/\check{G}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$. Similarly, one consequence of the Arkhipov-Bezrukavnikov-Ginzburg equivalence **[ABG04]** is the equivalence $\mathrm{Loc}_{T_c}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{QCoh}(\check{\mathfrak{b}}^{\mathrm{reg}}/\check{B}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}]$ of (3). My goal in this talk is to state a generalization of these equivalences to other cohomology theories — the actual *statement* will be easy enough, but I think there are a lot of interesting questions that arise from this work. I have included some ideas for extensions of my work at the end of this section. I'm actively working on some of these ideas, and I would love to talk about them with anyone interested.

In what follows, we will fix an even-periodic $\mathbf{E}_\infty$-ring and an oriented group scheme $\mathbf{G}$ over A which is isomorphic to one of $\mathbf{G}_a$ (really, this is $\hat{\mathbf{G}}_a$ or $\mathbf{G}_a$ with the coordinate in positive degree), $\mathbf{G}_m$, or an elliptic curve. We will write $\mathbf{G}_0$ to mean the underlying group scheme over $\pi_0(A)$. In each of these cases, one can define a "1-shifted Cartier dual" $\mathbf{G}^\vee$ as $\mathrm{Hom}(\mathbf{G}, B\mathbf{G}_m)$; if $\mathbf{G} = \hat{\mathbf{G}}_a$, this is isomorphic to $B\mathbf{G}_a$; if $\mathbf{G} = \mathbf{G}_m$, this is isomorphic to the constant stack $S^1 = B\mathbf{Z}$; and if $\mathbf{G}$ is an elliptic curve E, this is isomorphic to the dual elliptic curve $E^\vee$. Note that $\mathbf{G}_0^\vee$ is a (spectral) stack over $\pi_0 A$, and so upon rationalization defines a stack $(\mathbf{G}_0^\vee)_{\mathbf{Q}}$ over $\pi_0 A_{\mathbf{Q}}$. We will often denote this stack by $\mathbf{G}_{0,\mathbf{Q}}^\vee$.

Remark 3.1. There is a canonical map $\mathbf{G}_0 \times \mathbf{G}_0^\vee \to B\mathbf{G}_m$, which defines a *Poincaré line bundle* $\mathscr{P}$ over $\mathbf{G}_0 \times \mathbf{G}_0^\vee$. In each of the above cases, the Fourier-Mukai functor $\mathrm{QCoh}(\mathbf{G}_0) \to \mathrm{QCoh}(\mathbf{G}_0^\vee)$ sending $\mathscr{F} \mapsto \mathrm{pr}_{2,*}(\mathscr{P} \otimes \mathrm{pr}_1^* \mathscr{F})$ is an equivalence of ∞-categories. More generally, this functor is an equivalence as long as the canonical map $\mathbf{G}_0 \to \mathfrak{D}(\mathbf{G}_0^\vee)$ is an equivalence, and the theorem of the cube[1] holds for $\mathrm{QCoh}(\mathbf{G}_0^\vee)$.

Definition 3.2. Let $\check{B}$ be the Borel subgroup of $\check{G}$ associated to our chosen Borel B. In the above setting, let $\mathrm{Bun}_{\check{B}}^0(\mathbf{G}_{0,\mathbf{Q}}^\vee)$ denote the moduli stack of $\check{B}$-bundles over $\mathbf{G}_{0,\mathbf{Q}}^\vee$ which are of degree zero: this means that the associated $\check{T}$-bundle is a direct sum of line bundles of degree zero on $\mathbf{G}_{0,\mathbf{Q}}^\vee$. Note that the moduli stack of $\check{T}$-bundles of degree 0 along with a trivialization at the origin in $\mathbf{G}_{0,\mathbf{Q}}^\vee$ is isomorphic to $\mathscr{M}_{\check{T}} = \mathrm{Hom}(\mathbb{X}^*(\check{T}), \mathbf{G})$. In particular, there is a canonical map $\mathscr{M}_{\check{T}} \to \mathrm{Bun}_{\check{B}}^0(\mathbf{G}_{0,\mathbf{Q}}^\vee)$.

A $\check{B}$-bundle $\mathscr{P}$ is called *regular* if the dimension of its automorphism group (as a $\check{B}$-bundle) is given by the rank of $\check{G}$, i.e., is as small as possible. Let $\mathrm{Bun}_{\check{B}}^0(\mathbf{G}_{0,\mathbf{Q}}^\vee)^{\mathrm{reg}}$ denote the (open) substack of $\mathrm{Bun}_{\check{B}}^0(\mathbf{G}_{0,\mathbf{Q}}^\vee)$ spanned by the regular $\check{B}$-bundles.

How does this compare to the notion of regularity of a Lie algebra element from Definition 2.13? The key to answering this is the following elementary observation:

Proposition 3.3. *Let X be a scheme over a $\mathbf{Q}$-algebra k. Then $\mathrm{Map}_k(B\mathbf{G}_a, X)$ can be identified with the shifted tangent bundle $T[-1]X = \mathrm{Spec}\,\mathrm{Sym}_{\mathscr{O}_X}(L_X[1])$.*

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[1] In classical algebraic geometry, the theorem of the cube states the following. Let X, Y, and Z be proper schemes over an algebraically closed field with points $x \in X$, $y \in Y$, and $z \in Z$. If $\mathscr{L}$ is a line bundle on $X \times Y \times Z$ such that $\mathscr{L}|_{X \times Y \times \{z\}}$. $\mathscr{L}|_{X \times \{y\} \times Z}$, and $\mathscr{L}|_{\{x\} \times Y \times Z}$ are all trivial, then $\mathscr{L}$ is itself trivial.

Warning 3.4. Proposition 3.3 is false if k is not a $\mathbf{Q}$-algebra; but it continues to hold if $B\mathbf{G}_a$ is replaced by $B\mathbf{G}_a^\sharp$, where $\mathbf{G}_a^\sharp = \operatorname{Spec} k\langle x\rangle$ is the divided power hull of the identity in $\mathbf{G}_a$. Of course, if $\mathbf{Q} \subseteq k$, then x always has divided powers, so $\mathbf{G}_a^\sharp \cong \mathbf{G}_a$. The basic point in the proof of Proposition 3.3 (even away from characteristic zero) is the fact that $B\mathbf{G}_a^\sharp$ is an affine stack, and $\Gamma(B\mathbf{G}_a^\sharp; \mathscr{O}_{B\mathbf{G}_a^\sharp}) \cong k[\epsilon]/\epsilon^2$ with $|\epsilon| = -1$. The result then follows similarly to the observation that the (unshifted) tangent bundle TX can be identified with the mapping stack $\operatorname{Map}_k(\bullet, X)$, where $\bullet = \operatorname{Spec} k[d]/d^2$ is the double point with $|d| = 0$.

Example 3.5. Let $\mathbf{G} = \hat{\mathbf{G}}_a$ (rather, $\mathbf{G}_a$ with coordinate in weight 1), so that $\mathbf{G}_{0,\mathbf{Q}}^\vee \cong B\mathbf{G}_a$. By Proposition 3.3, $\operatorname{Bun}_{\check{B}}(\mathbf{G}_{0,\mathbf{Q}}^\vee) = T[-1](B\check{B})$. But the cotangent complex of $B\check{B}$ is just the Lie algebra $\check{\mathfrak{b}}^*[-1]$ viewed as the coadjoint representation, so that $T[-1](B\check{B}) = \check{\mathfrak{b}}/\check{B}$. It is not difficult to show that every $\check{B}$-bundle is of degree zero, so that $\operatorname{Bun}_{\check{B}}^0(\mathbf{G}_{0,\mathbf{Q}}^\vee) = \check{\mathfrak{b}}/\check{B}$. The condition that an $\check{B}$-bundle is regular is then the same as the regularity condition from Definition 2.13.

Example 3.6. Let $\mathbf{G} = \mathbf{G}_m$, so that $\mathbf{G}_{0,\mathbf{Q}}^\vee \cong S^1$. Then,

$$\operatorname{Bun}_{\check{B}}(\mathbf{G}_{0,\mathbf{Q}}^\vee) = \operatorname{Map}(S^1, B\check{B}) = \check{B}/\check{B},$$

where $\check{B}$ acts on itself by the conjugation action. The condition that an $\check{B}$-bundle is regular is then the same as the condition that the $\check{B}$-conjugacy class contains an element whose centralizer is of minimal dimension.

The main result, then, is the following; the discussion above shows that when $A = \mathbf{Q}[\beta^{\pm 1}]$, we precisely recover the equivalence (3):

Theorem 3.7 (D.)**.** *Under the above assumptions, there is an $\mathbf{E}_2$-monoidal equivalence of $A_\mathbf{Q}$-linear ∞-categories*

$$\operatorname{Loc}_{T_c}(\Omega G_c; A) \otimes \mathbf{Q} \simeq \operatorname{QCoh}(\operatorname{Bun}_{\check{B}}^0(\mathbf{G}_{0,\mathbf{Q}}^\vee)^{\mathrm{reg}}) \otimes \mathbf{Q}[\beta^{\pm 1}].$$

This is technically a slight lie: one has to replace the left-hand side by a categorification of the "associated graded" of the double-speed Postnikov filtration. Since this would take us too deep into technical details, let us pretend that I never mentioned this caveat.

Remark 3.8. Very similar methods can be used to show that Theorem 3.7 also has an analogue for $\operatorname{Loc}_{G_c}(\Omega G_c; A)$ – this requires the setup of *G_c-equivariant elliptic cohomology*, the foundations of which are now described in [**Lur09, Lur18a, Lur18b, Lur19, GM20, GM22**]. The resulting equivalence states that

$$\operatorname{Loc}_{G_c}(\Omega G_c; A) \otimes \mathbf{Q} \simeq \operatorname{QCoh}(\operatorname{Bun}_{\check{G}}^{0,\mathrm{ss}}(\mathbf{G}_{0,\mathbf{Q}}^\vee)^{\mathrm{reg}}) \otimes \mathbf{Q}[\beta^{\pm 1}],$$

where $\operatorname{Bun}_{\check{G}}^{0,\mathrm{ss}}(\mathbf{G}_{0,\mathbf{Q}}^\vee)^{\mathrm{reg}}$ denotes the moduli *stack* of regular semistable $\check{G}$-bundles of degree zero.

Let us sketch the proof of Theorem 3.7. Recall that if X is a T_c-space, $\mathscr{F}_{T_c}(X)$ denotes the $\mathbf{E}_\infty$-algebra in $\operatorname{QCoh}(\mathscr{M}_T)$ which represents the T_c-equivariant A-cochains on X. The ∞-category $\operatorname{Loc}_{T_c}(\Omega G_c; A)$ is equivalent to the ∞-category of $\mathscr{F}_{T_c}(\Omega G_c)^\vee$-comodules in $\operatorname{QCoh}(\mathscr{M}_T)$, so the basic goal is to compute $C_*^{T_c}(\Omega G_c; A)$ as an $\mathbf{E}_2$-algebra in $\mathbf{E}_\infty$-coalgebras. One can do this using an A-analogue of the results of Goresky-Kottwitz-Macpherson. To state the answer, if α is a root, let $\check{T}_\alpha$ denote the kernel of $\alpha : \check{T} \to \mathbf{G}_m$. Let $\mathscr{M}_{T,0}$ denote the scheme underlying $\mathscr{M}_T$, let $\mathfrak{B}$ denote the blowup of $\check{T}_{\pi_0 A} \times \mathscr{M}_{T,0}$ at the union of $\check{T}_\alpha$ and $\mathscr{M}_{T_\alpha,0}$, and let $(T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}$ denote the complement of the proper preimage of $\mathscr{M}_{T_\alpha,0}$ in $\mathfrak{B}$. This is just an affine blowup. Then,

$$\operatorname{Spec}_{\mathscr{M}_{T,0}} \pi_0 \mathscr{F}_{T_c}(\Omega G_c)^\vee \cong (T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}. \tag{4}$$

This is "one half" of the calculation of Theorem 3.7.

Example 3.9. Let us illustrate (4) for $A = \mathbf{Q}[\beta^{\pm 1}], \mathrm{KU}$ and $G = \mathrm{SL}_2, \mathrm{PGL}_2$. In this case, $G_c \cong S^3, \mathrm{SO}(3)$. If $A = \mathbf{Q}[\beta^{\pm 1}]$, we have

$$\pi_0 C^{S^1}_*(\Omega S^3; \mathbf{Q}[\beta^{\pm 1}]) \cong \mathbf{Q}[x, y^{\pm 1}, \tfrac{y-1}{x}],$$
$$\pi_0 C^{S^1}_*(\Omega \mathrm{SO}(3); \mathbf{Q}[\beta^{\pm 1}]) \cong \mathbf{Q}[x, y^{\pm 1}, \tfrac{y^2-1}{2x}].$$

Observe, for instance, that setting $x = 0$ produces nonequivariant cohomology, and in this case we have

$$\operatorname{Spec} \pi_0 C_*(\Omega S^3; \mathbf{Q}[\beta^{\pm 1}]) \cong \mathbf{G}_a \text{ with coordinate } \tfrac{y-1}{x}, \tag{5}$$

$$\operatorname{Spec} \pi_0 C_*(\Omega \mathrm{SO}(3); \mathbf{Q}[\beta^{\pm 1}]) \cong \mathbf{G}_a \times \mu_2 \text{ with coordinates } \tfrac{y^2-1}{2x}, y. \tag{6}$$

This is compatible with the identification $\Omega\mathrm{SO}(3) \simeq \mathbf{Z}/2 \times \Omega S^3$. Similarly, if $A = \mathrm{KU}$, we have

$$\pi_0 C^{S^1}_*(\Omega S^3; \mathrm{KU}) \cong \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \tfrac{y-1}{x-1}],$$
$$\pi_0 C^{S^1}_*(\Omega \mathrm{SO}(3); \mathrm{KU}) \cong \mathbf{Z}[x^{\pm 1}, y^{\pm 1}, \tfrac{y^2-1}{x^2-1}].$$

Observe, for instance, that setting $x = 1$ produces nonequivariant cohomology, and in this case we have

$$\pi_0 C_*(\Omega S^3; \mathrm{KU}) \cong \mathbf{Z}[\tfrac{y-1}{x-1}] \cong \mathscr{O}_{\mathbf{G}_a},$$
$$\pi_0 C_*(\Omega \mathrm{SO}(3); \mathrm{KU}) \cong \mathbf{Z}[y^{\pm 1}, \tfrac{y^2-1}{x^2-1}]/(y^2 - 1) \cong \mathscr{O}_{\mathbf{G}_a \times \mu_2}.$$

The "second half" of Theorem 3.7 is an identification of $(T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}}$ in Langlands dual terms. This relies on an analogue of the Kostant slice. Namely, there is a map $q : \mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}}) \to \mathrm{Bun}^0_{\check{T}}(\mathbf{G}^\vee_{0,\mathbf{Q}})$ given by reducing a $\check{B}$-bundle to a $\check{T}$-bundle, i.e., composition with the map $B\check{B} \to B\check{T}$. When $\mathbf{G} = \hat{\mathbf{G}}_a$, this corresponds to the map $\check{\mathfrak{b}}/\check{B} \to \check{\mathfrak{t}}$. The following result, in a slightly different form, is essentially due to Friedman-Morgan-Witten [**FMW98**]; but as far as I know, the only complete argument seems to be in [**Dav19**].

Proposition 3.10. *For every $\check{T}$-bundle $\mathscr{P}_T$ of degree zero on $\mathbf{G}^\vee_{0,\mathbf{Q}}$, there is a unique* regular *$\check{B}$-bundle $\mathscr{P}_B$ of degree zero on $\mathbf{G}^\vee_{0,\mathbf{Q}}$ such that $\mathscr{P}_B/\check{N} \cong \mathscr{P}_{\check{T}}$. These regular $\check{B}$-bundles fit into a family of $\check{B}$-bundles over $\mathbf{G}^\vee_{0,\mathbf{Q}} \times \mathrm{Bun}^0_{\check{T}}(\mathbf{G}^\vee_{0,\mathbf{Q}})$, and hence define a section $\kappa : \mathrm{Bun}^0_{\check{T}}(\mathbf{G}^\vee_{0,\mathbf{Q}}) \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}})$ of the map $q : \mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}}) \to \mathrm{Bun}^0_{\check{T}}(\mathbf{G}^\vee_{0,\mathbf{Q}})$.*

The map $\kappa : \mathrm{Bun}^0_{\check{T}}(\mathbf{G}^\vee_{0,\mathbf{Q}}) \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}})$ will be called the *Kostant slice*; we will also abusively write κ to denote the composite

$$\mathscr{M}_{\check{T}} \to \mathrm{Bun}^0_{\check{T}}(\mathbf{G}^\vee_{0,\mathbf{Q}}) \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}}).$$

When $\mathbf{G} = \hat{\mathbf{G}}_a$, this is precisely the Kostant slice $\kappa : \check{\mathfrak{t}} \to \check{\mathfrak{b}}/\check{B}$ from the previous talk. When $\mathbf{G}$ is an elliptic curve, the map κ describes some beautiful geometry.

Example 3.11. Let $\mathbf{G} = E$ be an elliptic curve over a commutative ring k, so that $\mathbf{G}_{0,\mathbf{Q}}^\vee$ is the dual elliptic curve. Fixing a point on E allows us to identify $\mathbf{G}_{0,\mathbf{Q}}^\vee \cong E$. Let $\check{G} = \mathrm{SL}_2$, so that a $\check{B}$-bundle is just a rank 2 vector bundle $\mathscr{V}$ such that $\det(\mathscr{V})$ is trivial, along with the data of a line subbundle $\mathscr{L} \subseteq \mathscr{V}$. The pair $(\mathscr{L} \subseteq \mathscr{V})$ can be specified by a class in $\mathrm{Ext}^1(\mathscr{V}/\mathscr{L}, \mathscr{L})$. But since $\det(\mathscr{V}) \cong \mathscr{L} \otimes \mathscr{V}/\mathscr{L}$ is trivial, we can identify $\mathscr{V}/\mathscr{L} \cong \mathscr{L}^{-1}$, which means that $(\mathscr{L} \subseteq \mathscr{V})$ can be specified by a class in $\mathrm{Ext}^1(\mathscr{L}^{-1}, \mathscr{L}) \cong \mathrm{H}^1(E; \mathscr{L}^2)$.

Since E is an elliptic curve, we know that $\mathrm{H}^1(E; \mathscr{L}^2)$ is zero unless $\mathscr{L}^2$ is trivial, in which case it is just isomorphic to k. Therefore, if $\mathscr{L}^2$ is nontrivial, the only $\check{B}$-bundle we can construct sits in a *split* extension

$$0 \to \mathscr{L} \to \mathscr{V} = \mathscr{L} \oplus \mathscr{L}^{-1} \to \mathscr{L}^{-1} \to 0.$$

It is an easy exercise to check that the only automorphisms of $\mathscr{V}$ as a $\check{B}$-bundle (i.e., automorphisms as an SL_2-bundle which preserve the flag $\mathscr{L} \subseteq \mathscr{L} \oplus \mathscr{L}^{-1}$) are given by scaling, and so $\mathrm{Aut}_{\check{B}}(\mathscr{V}) \cong \mathbf{G}_m$. On the other hand, any nontrivial class in $\mathrm{H}^1(E; \mathscr{O}_E) \cong k$ defines a *nonsplit* extension

$$0 \to \mathscr{O}_E \to \mathscr{V} \to \mathscr{O}_E \to 0.$$

If $\mathscr{L}^2$ is trivial, any nontrivial class in $\mathrm{H}^1(E; \mathscr{L}^2) \cong k$ defines the nonsplit extension $\mathscr{V} \otimes \mathscr{L}$ of $\mathscr{L}^{-1}$ by $\mathscr{L}$.

To determine the automorphisms of $\mathscr{V} \otimes \mathscr{L}$ as a $\check{B}$-bundle, note that the associated subgroup of SL_2 will be of the form $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$, where $x \in \mathrm{Hom}(\mathscr{L}, \mathscr{L})$, $y \in \mathrm{Hom}(\mathscr{L}^{-1}, \mathscr{L})$, and $x \in \mathrm{Hom}(\mathscr{L}^{-1}, \mathscr{L}^{-1})$. Not every such matrix defines an automorphism of $\mathscr{V}$; for instance, in order for two maps $x : \mathscr{L} \to \mathscr{L}$ and $z : \mathscr{L}^{-1} \to \mathscr{L}^{-1}$ to define an automorphism of $\mathscr{V} \otimes \mathscr{L}$, we need $x = z \otimes \mathscr{L}^2 = z$. In order for the resulting matrix $\left(\begin{smallmatrix} x & y \\ 0 & z \end{smallmatrix}\right)$ to preserve the trivialization of $\det(\mathscr{V} \otimes \mathscr{L})$, we need $x^2 = 1$; the function y can be arbitrary. This discussion implies that $\mathrm{Aut}_{\check{B}}(\mathscr{V} \otimes \mathscr{L}) \cong \mu_2 \times \mathbf{G}_a$, where the μ_2 encodes x, and $\mathbf{G}_a$ encodes y.

The asssignment

$$\mathscr{L} \mapsto \begin{cases} \mathscr{L} \oplus \mathscr{L}^{-1} & \text{if } \mathscr{L}^2 \text{ is nontrivial,} \\ \mathscr{V} \otimes \mathscr{L} & \text{otherwise} \end{cases}$$

behaves nicely in families; it is precisely the Kostant slice $\kappa : \mathrm{Pic}^0(E) \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}_{0,\mathbf{Q}}^\vee)$. The above discussion lets us calculate the fiber product $X := \mathrm{Pic}^0(E) \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_{0,\mathbf{Q}}^\vee)} \mathrm{Pic}^0(E)$ completely: there is a map $X \to \mathrm{Pic}^0(E)$ whose fibers are a copy of $\check{T} = \mathbf{G}_m$ away from the locus of 2-torsion line bundles; and the fiber over a 2-torsion line bundle is a copy of $\mu_2 \times \mathbf{G}_a$. In fact, X precisely identifies with $(T^*_{\mathbf{G}} \check{T})^{\mathrm{bl}}$. For instance, the above calculation of $\mathrm{Aut}_{\check{B}}(\mathscr{V})$ for $\check{G} = \mathrm{SL}_2$ precisely identifies

$$\mathrm{Spec}\, \pi_0 C_*^{S^1}(\Omega \mathrm{SO}(3); \mathbf{Q}[\beta^{\pm 1}]) \cong \mathrm{Pic}^0(E) \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}_{0,\mathbf{Q}}^\vee)} \mathrm{Pic}^0(E)$$

via (6).

Remark 3.12. The rank 2 vector bundle $\mathscr{V}$ on E is known as the *Atiyah bundle*, and was introduced by Atiyah in the beautiful paper [**Ati57**]. To emphasize how important this construction is, let us just mention a few ways to think about $\mathscr{V}$; these will not be relevant in the discussion below, but may be edifying nonetheless.

- The most classical instantiation of the Atiyah bundle is via the Weierstrass functions. First, note that $\mathrm{H}^1(E; \mathscr{O}_E) = \mathrm{H}^1(E; \mathbf{G}_a)$, so that $\mathscr{V}$ can be viewed as defining a $\mathbf{G}_a$-torsor $\mathscr{A}$ over E; explicitly, this $\mathbf{G}_a$-torsor is the complement of the section at ∞ of the projective line $\mathbf{P}(\mathscr{V})$. If we work complex-analytically, E^{an} can be identified as the quotient $\mathbf{C}/\Lambda$ for some rank 2 lattice $\Lambda \subseteq \mathbf{C}$. Associated to Λ are two Weierstrass functions defined on $\mathbf{C}$:

$$\begin{aligned} \wp(z;\Lambda) &= \frac{1}{z^2} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right), \\ \zeta(z;\Lambda) &= \frac{1}{z} + \sum_{\lambda \in \Lambda - \{0\}} \left(\frac{1}{z-\lambda} + \frac{1}{\lambda} + \frac{z}{\lambda^2} \right). \end{aligned}$$

Note that $\wp(z;\Lambda)$ is doubly-periodic, i.e., $\wp(z+\lambda;\Lambda) = \wp(z;\Lambda)$ for any $\lambda \in \Lambda$. Alternatively, $\wp$ defines a map $\mathbf{C} \to \mathbf{C}$ which factors through a map $\mathbf{C}/\Lambda = E^{\mathrm{an}} \to \mathbf{C}$.

Although $\zeta(z;\Lambda)$ is not doubly-periodic, an easy calculation shows that $\wp(z;\Lambda) = -\partial_z \zeta(z;\Lambda)$; so if $\lambda \in \Lambda$, then $\zeta(z+\lambda;\Lambda) - \zeta(z;\Lambda) = c(\lambda)$ for some constant $c(\lambda)$. The function $\lambda \mapsto c(\lambda)$ is evidently additive, and defines a homomorphism $\Lambda \to \mathbf{C}$, which defines a $\mathbf{C}$-bundle over $E^{\mathrm{an}} = \mathbf{C}/\Lambda$. This $\mathbf{C}$-bundle is precisely the analytification $\mathscr{A}^{\mathrm{an}}$ of the $\mathbf{G}_a$-torsor $\mathscr{A}$. It follows that although ζ is not defined on E^{an}, the torsor $\mathscr{A}^{\mathrm{an}}$ is the universal space over E^{an} on which ζ is defined.

If one prefers, this discussion tells us how to define the total space of the rank 2-bundle $\mathscr{V}^{\mathrm{an}}$ purely analytically. Write $\Lambda = \mathbf{Z} \oplus \tau\mathbf{Z}$. Then $\mathrm{Tot}(\mathscr{V}^{\mathrm{an}})$ is the quotient of $\mathbf{C} \times \mathbf{C}^2$ by the relations $(z,x) \sim (z+1,x)$ and $(z,x) \sim (z+\tau, \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right) x)$. This perspective on $\mathscr{V}^{\mathrm{an}}$ might be more digestible if one is more familiar with the complex-analytic theory of elliptic curves.

- Since the tangent bundle T_E of E is trivial, one can view $\mathscr{V}$ as an extension of T_E by $\mathscr{O}_E$. This gives an identification of the $\mathbf{G}_a$-torsor $\mathscr{A}$ with the moduli space $\mathrm{Loc}_{\mathbf{G}_m}(E)$ of line bundles on E equipped with a flat connection. This $\mathbf{G}_a$-torsor has also been studied in [**Kat77**, Appendix C].
- Let E_{dR} denote the de Rham stack of E, i.e., the stack presented by $(E \times E)^\wedge_\Delta \rightrightarrows E$, where $\Delta \subseteq E \times E$ is the diagonal. Then, there is a canonical quotient map $E \to E_{\mathrm{dR}}$. If $*_{\mathrm{dR}} = * \to E_{\mathrm{dR}}$ is the inclusion of the image of the basepoint of E, the pullback $E \times_{E_{\mathrm{dR}}} *_{\mathrm{dR}}$ is simply the completion $\hat{E}$ of E at the basepoint.

  Let us work in characteristic zero, so that $\hat{E} \cong \hat{\mathbf{G}}_a$. In this case, $\mathfrak{D}(\hat{E}) \cong B\mathbf{G}_a$, and there is a Cartesian square

$$\begin{tikzcd} \mathfrak{D}(E_{\mathrm{dR}}) \arrow[r] \arrow[d] & \mathfrak{D}(E) \cong E^\vee \arrow[d] \\ \mathfrak{D}(*) = * \arrow[r] & \mathfrak{D}(\hat{E}) \cong B\mathbf{G}_a. \end{tikzcd}$$

The right-vertical map $E^\vee \to B\mathbf{G}_a$ is precisely the class in $\mathrm{H}^1(E^\vee; \mathscr{O}_{E^\vee})$ which defines the Atiyah bundle over $E^\vee$. This Cartesian square therefore identifies $\mathfrak{D}(E_{\mathrm{dR}})$ with the $\mathbf{G}_a$-torsor $\mathscr{A}$ over $E^\vee$. Using the discussion in Remark 3.1, one can show that this defines a Fourier-Mukai equivalence

$$\mathrm{DMod}(E) = \mathrm{QCoh}(E_{\mathrm{dR}}) \xrightarrow{\sim} \mathrm{QCoh}(\mathscr{A}) = \mathrm{QCoh}(\mathrm{Loc}_{\mathbf{G}_m}(E^\vee)).$$

The above discussion goes through equally well when E is replaced by an abelian variety, and the resulting Fourier-Mukai equivalence is the *Rothstein-Laumon transform* (see [**Lau96, Rot96**]). This equivalence plays an important role in the geometric Langlands program for a torus (although this appearance of the Atiyah bundle in geometric Langlands is rather disjoint from its appearance in our work).

Remark 3.13. The conclusion of Example 3.11 has an analogue for $\mathbf{G} = \hat{\mathbf{G}}_a$. Namely, recall from Example 2.16 that the map κ sends

$$\mathfrak{t}^* \ni x \mapsto \kappa(x) = \begin{pmatrix} x & 0 \\ 1 & -x \end{pmatrix} \in \mathfrak{b}.$$

We can compute the centralizer of $\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right)$ in $\check{B}$: namely, if $g = \left(\begin{smallmatrix} a & 0 \\ b & a^{-1} \end{smallmatrix}\right)$, then

$$\mathrm{Ad}_g \begin{pmatrix} x & 0 \\ 1 & -x \end{pmatrix} = \begin{pmatrix} x & 0 \\ \frac{2abx+1}{a^2} & -x \end{pmatrix},$$

so g centralizes $\left(\begin{smallmatrix} x & 0 \\ 1 & -x \end{smallmatrix}\right)$ if $\frac{2abx+1}{a^2} = 1$, i.e., $b = \frac{a-a^{-1}}{2x}$. Therefore,

$$\check{\mathfrak{t}} \times_{\check{\mathfrak{b}}/\check{B}} \check{\mathfrak{t}} \cong \mathrm{Spec}\, k[x, a^{\pm 1}, \tfrac{a-a^{-1}}{2x}].$$

Again, the fiber away from the locus $2x = 0$ (analogous to the 2-torsion locus in $\mathrm{Pic}^0(E)$ from Example 3.11) is $\mathbf{G}_m$, while the fiber over the locus $2x = 0$ is $\mathbf{A}^1$.

The conclusion of Example 3.11 turns out to be true in general: namely, the self-intersection of $\kappa : \mathscr{M}_{\check{T}} \to \mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}})^{\mathrm{reg}}$ is isomorphic to $(T^*_{\mathbf{G}}\check{T})^{\mathrm{bl}} \otimes \mathbf{Q}$; the key case is the above calculation for SL_2 and its analogue for PGL_2. If one now chooses an isomorphism $\mathbb{X}^*(T) \cong \mathbf{Z}^{\mathrm{rank}\ G}$, we obtain an isomorphism $\mathscr{M}_{\check{T}} = \mathscr{M}_T$, and hence (4) gives an isomorphism

$$(\mathrm{Spec}_{\mathscr{M}_{T,0}} \pi_0 \mathscr{F}_{T_c}(\Omega G_c)^\vee) \otimes \mathbf{Q} \cong (\mathscr{M}_{T,0})_{\mathbf{Q}} \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}})^{\mathrm{reg}}} (\mathscr{M}_{T,0})_{\mathbf{Q}}.$$

The left-hand side is a group scheme over $(\mathscr{M}_{T,0})_{\mathbf{Q}}$ (where the coalgebra structure on $\mathscr{F}_{T_c}(\Omega G_c)^\vee$ is via the diagonal on ΩG_c), and the right-hand side is also a group scheme over $(\mathscr{M}_{T,0})_{\mathbf{Q}}$ by formal nonsense; the above isomorphism is one of group schemes over $(\mathscr{M}_{T,0})_{\mathbf{Q}}$. Once one has this, Theorem 3.7 is essentially immediate.

Remark 3.14. The fiber product $(\mathscr{M}_{T,0})_{\mathbf{Q}} \times_{\mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}})^{\mathrm{reg}}} (\mathscr{M}_{T,0})_{\mathbf{Q}}$ is a $\mathbf{G}$-analogue of the "regular centralizer". For instance, when $\mathbf{G} = \hat{\mathbf{G}}_a$, this fiber product is $\check{\mathfrak{t}} \times_{\check{\mathfrak{b}}/\check{B}} \check{\mathfrak{t}}$. Let $\widetilde{\check{\mathfrak{g}}}$ denote the Grothendieck-Springer resolution, so that it is the variety of pairs $(x, \check{\mathfrak{b}}')$ where $\check{\mathfrak{b}}'$ is a Borel subalgebra of $\check{\mathfrak{g}}$ and $x \in \check{\mathfrak{b}}'$. Then, there is a canonical map $\nu : \widetilde{\check{\mathfrak{g}}} \to \check{\mathfrak{t}}$ sending $(x, \check{\mathfrak{b}}') \mapsto x \mod [\check{\mathfrak{b}}', \check{\mathfrak{b}}']$. The pullback of $\check{\mathfrak{t}} \times_{\check{\mathfrak{b}}/\check{B}} \check{\mathfrak{t}}$ along ν is closely related to the variety $\mathfrak{Z}$ of triples $(x, \check{\mathfrak{b}}', g)$ where $\check{\mathfrak{b}}'$ is a Borel subalgebra of $\check{\mathfrak{g}}$, $x \in \check{\mathfrak{b}}'$ is a regular element, and $g \in \check{G}$ centralizes x. The variety $\mathfrak{Z}$ is (almost) the group scheme of regular centralizers.

Let us now describe some shortcomings of Theorem 3.7, possible extensions, and relationships to other work. Some of the discussion below will assume more familiarity with geometric representation theory than the preceding lectures might have prepared the reader for; sorry about that!

3.1. Extension 1: loop rotation equivariance. One of the first shortcomings is that the derived geometric Satake and the Arkhipov-Bezrukavnikov-Ginzburg equivalences are concerned with the ∞-category of equivariant *constructible* sheaves on Gr_G. Obviously, it would be very interesting to extend this to other cohomology theories, too. (The reader should keep in mind the caveat mentioned after Theorem 3.7.)

Expectation 3.15. One might expect an $\mathbf{E}_2$-monoidal equivalence of $A_{\mathbf{Q}}$-linear ∞-categories

$$\mathrm{Shv}_I^c(\mathrm{Gr}_G(\mathbf{C}); A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\mathrm{Bun}^0_{\check{B}}(\mathbf{G}^\vee_{0,\mathbf{Q}})) \otimes \mathbf{Q}[\beta^{\pm 1}],$$

and even an $\mathbf{E}_2$-monoidal equivalence of $A_{\mathbf{Q}}$-linear ∞-categories

$$\mathrm{Shv}^c_{G(\mathscr{O})}(\mathrm{Gr}_G(\mathbf{C}); A) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\mathrm{Bun}^{\mathrm{ss}}_{\check{G}}(\mathbf{G}^\vee_{0,\mathbf{Q}})) \otimes \mathbf{Q}[\beta^{\pm 1}].$$

I haven't written out in detail the definition of the ∞-categories $\mathrm{Shv}_I^c(\mathrm{Gr}_G(\mathbf{C}); A)$ and $\mathrm{Shv}^c_{G(\mathscr{O})}(\mathrm{Gr}_G(\mathbf{C}); A)$, but it certainly seems doable. One also expects $\mathrm{Shv}^c_{G(\mathscr{O})}(\mathrm{Gr}_G(\mathbf{C}); A)$ to admit the structure of an $\mathbf{E}_3$-*monoidal* ∞-category such that the equivalence for $\mathrm{Shv}^c_{G(\mathscr{O})}(\mathrm{Gr}_G(\mathbf{C}); A) \otimes \mathbf{Q}$ is $\mathbf{E}_3$-monoidal.

The Bezrukavnikov-Finkelberg equivalence actually admits a refinement, where one considers the ∞-category of $G(\mathscr{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}$-equivariant constructible sheaves on $\mathrm{Gr}_G(\mathbf{C})$ — here, $\mathbf{G}_m^{\mathrm{rot}}$ is the torus acting by loop rotation. Namely, they showed that there is an equivalence

$$\mathrm{Shv}^c_{G(\mathscr{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{C}[\beta^{\pm 1}]) \simeq U_\hbar(\check{\mathfrak{g}})\text{-mod}(\mathrm{Rep}(\check{G})) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}],$$

where $U_\hbar(\check{\mathfrak{g}})$ is the universal enveloping algebra where one sets $xy - yx = \hbar[x, y]$ for $x, y \in \check{\mathfrak{g}}$. There is an analogous description of $\mathrm{Loc}_{G \times S^1_{\mathrm{rot}}}(\Omega G_c; \mathbf{C}[\beta^{\pm 1}])$ as a localization of $U_\hbar(\check{\mathfrak{g}})\text{-mod}(\mathrm{Rep}(\check{G}))$. Note that one can view $U_\hbar(\check{\mathfrak{g}})\text{-mod}(\mathrm{Rep}(\check{G}))$ as $\mathrm{DMod}_\hbar(\check{G})^{\check{G} \times \check{G}}$. Similarly, one has a refinement of the Arkhipov-Bezrukavnikov-Ginzburg equivalence:

Theorem 3.16. *There is an equivalence*

$$\mathrm{Shv}^c_{I \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{DMod}_\hbar(\check{G}/\check{N})^{\check{G} \times \check{T}} \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}].$$

If $\mathrm{Fl}_G = G(\mathscr{K})/I$ *denotes the affine flag variety, there is also an equivalence*

$$\mathrm{Shv}^c_{I \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Fl}_G; \mathbf{C}[\beta^{\pm 1}]) \simeq \mathrm{DMod}_\hbar(\check{N}\backslash\check{G}/\check{N})^{\check{T} \times \check{T}} \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}].$$

Remark 3.17. This result seems to be "known to the experts", in that the essential components — at least for the first equivalence — are in the literature (see [**GR15**] and [**ABG04**]), but haven't been assembled. One can imitate the calculations of [**GR15**] for the affine flag variety, too, but this has not been written down anywhere (as far as I know). I have been discussing these equivalences with Tom Gannon.

Remark 3.18. Beilinson-Bernstein localization implies that $\mathrm{DMod}_\hbar(\check{G}/\check{N})^{\check{G} \times \check{T}}$ is equivalent to the Bernstein-Gelfand-Gelfand (BGG) *category* $\check{\mathscr{O}}$, i.e., the ∞-category of $U_\hbar(\check{\mathfrak{g}})$-modules in $\mathrm{Rep}(\check{T})$ whose $\check{\mathfrak{n}}$-action is locally nilpotent.

Expectation 3.19. I expect there to be an equivalence of $\mathrm{KU}_{\mathbf{Q}} \simeq \mathbf{Q}[\beta^{\pm 1}]$-linear ∞-categories

$$\mathrm{Shv}^c_{G(\mathscr{O}) \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q} \simeq U_q(\check{\mathfrak{g}})\text{-mod}(\mathrm{Rep}_q(\check{G})) \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}],$$

where $\mathrm{Rep}_q(\check{G})$ denotes the ∞-category of representations of the Lusztig quantum group, and $U_q(\check{\mathfrak{g}})$ denotes the quantum enveloping algebra. Similarly, there is a q-analogue $\check{\mathscr{O}}_q$ of category $\check{\mathscr{O}}$, and I expect that

$$\mathrm{Shv}^c_{I \rtimes \mathbf{G}_m^{\mathrm{rot}}}(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q} \simeq \check{\mathscr{O}}_q \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}].$$

Remark 3.20. As a first step towards Expectation 3.19, one can attempt to show that $\mathrm{Loc}_{T \times S^1_{\mathrm{rot}}}(\Omega G_c; \mathrm{KU}) \otimes \mathbf{Q}$ is equivalent to a localization of $\check{\mathscr{O}}_q \otimes_{\mathbf{Q}} \mathbf{Q}[\beta^{\pm 1}]$. Again, the key point is to calculate $C_*^{T \times S^1_{\mathrm{rot}}}(\Omega G_c; \mathrm{KU})$ — this turns out to be doable, and one obtains a "quantization" of $(T^*_{\mathbf{G}} \check{T})^{\mathrm{bl}}$, but I've gotten stuck on interpreting this quantization of $(T^*_{\mathbf{G}} \check{T})^{\mathrm{bl}}$ in Langlands dual terms. This should be approachable, but I'm just not familiar enough with quantum groups yet.

Nevertheless, there *is* a very simple interpretation of the $T \times S^1_{\mathrm{rot}}$-equivariant homology of ΩG_c when $G_c = T_c$; for instance, if $G = T$ is of rank 1, it is not difficult to directly compute that

$$\pi_0 C_*^{T \times S^1_{\mathrm{rot}}}(\Omega T_c; \mathbf{Q}[\beta^{\pm 1}]) \cong \mathbf{Q}[\hbar]\langle x, y^{\pm 1}\rangle / (yx = x(y + \hbar)) \cong \mathscr{D}^{\hbar}_{\check{T}},$$
$$\pi_0 C_*^{T \times S^1_{\mathrm{rot}}}(\Omega T_c; \mathrm{KU}) \cong \mathbf{Z}[q^{\pm 1}]\langle x, y^{\pm 1}\rangle / (yx = qyx) \cong \mathscr{D}^{q}_{\check{T}}.$$

Here, $\mathscr{D}^{\hbar}_{\check{T}}$ and $\mathscr{D}^{q}_{\check{T}}$ are the algebras of ordinary and q-differential operators on the Langlands dual $\check{T}$. More generally, there is a similar description for $\mathscr{F}_{T \times S^1_{\mathrm{rot}}}(\Omega T_c)^\vee$; we refer the reader to [**Dev23, DM23**] for further discussion on these analogues of the Weyl algebra. This calculation affirms Expectation 3.19 in the case when $G = T$, but this obviously sees no interesting representation theory.

3.2. Extension 2: categorification.

Algebraic K-theory is an extremely important, and in some sense, the *universal*, decategorification procedure. In particular, KU can be recovered as the localization of $K(\mathbf{C})$ at the Bott class[2]. This has a natural generalization: namely, if X is a finite space (or anima, whichever term is preferable), then the cochains $C^*(X; \mathrm{KU})$ can be recovered as a localization of the algebraic K-theory of the ∞-category $\mathrm{Loc}(X; \mathrm{Mod}_{\mathbf{C}}) = \mathrm{Fun}(X, \mathrm{Mod}_{\mathbf{C}})$ of vector bundles on X.

The next few sentences are rather imprecise, and I think it would be very interesting to fix that problem. One might hope that the ∞-category $\mathrm{Loc}(X; \mathrm{KU})$ can itself be recovered as the "algebraic K-theory" of some $(\infty, 2)$-category $\mathrm{Loc}^{\mathrm{Cat}}(X; \mathbf{C}) = \mathrm{Fun}(X, \mathrm{LinCat}_{\mathbf{C}})$. Similarly, if G_c is a compact Lie group acting on X, one might hope that the ∞-category $\mathrm{Loc}_{G_c}(X; \mathrm{KU})$ can be recovered as a localization of the "algebraic K-theory" of some $(\infty, 2)$-category $\mathrm{Loc}^{\mathrm{Cat}}_{G_c}(X; \mathbf{C}) = \mathrm{Fun}(X/G_c, \mathrm{LinCat}_{\mathbf{C}})$ of "categorical" vector bundles on the orbifold quotient X/G_c. Even if this is not literally true, one should still view $\mathrm{Loc}_{G_c}(X; \mathrm{KU})$ as some interesting decategorification of $\mathrm{Loc}^{\mathrm{Cat}}_{G_c}(X; \mathbf{C})$. As such, we are led to ask:

[2] This is correct only if $\mathbf{C}$ is viewed as a topological ring. More precisely, the notion of algebraic K-theory can be extended from ordinary commutative rings to the more refined notion of condensed rings (basically, where one keeps track of a topology). Then, the connective complex K-theory spectrum ku can be obtained as the "solidification" of the algebraic K-theory of $\mathbf{C}$, viewed as a condensed ring via its natural topological ring structure; see [**CS20**, Proposition 10.6]. If one instead only considers the algebraic K-theory of $\mathbf{C}$ as a *discrete* ring, then a famous result of Suslin's says that we get ku after profinite completion.

Question 3.21. Is there an interesting description of $\mathrm{Loc}^{\mathrm{Cat}}_{G_c}(\Omega G_c; \mathbf{C})$ or $\mathrm{Loc}^{\mathrm{Cat}}_{T_c}(\Omega G_c; \mathbf{C})$ in Langlands dual terms, whose right-hand side decategorifies (in some appropriate sense) to $\check{G}/\check{G}$ or to $\widetilde{\check{G}}/\check{G}$?

I have almost no ideas about this question (except in the case of a torus). Note that Question 3.21 does not involve any chromatic homotopy theory — as such, it could potentially be addressed just with "classical" representation-theoretic tools.

Example 3.22. Let $G_c = T_c$ be a compact torus. Then, the action of T_c on $\Omega T_c = \mathbb{X}_*(T)$ is trivial, so that we have

$$\mathrm{Loc}^{\mathrm{Cat}}_{T_c}(\Omega T_c; \mathbf{C}) \simeq \bigoplus_{\mathbb{X}_*(T)} \mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}}).$$

Write $\check{T}_{\mathrm{ku}} = \mathrm{Spec}\, \mathrm{ku}[\mathbb{X}^*(\check{T})] = \mathrm{Spec}\, \mathrm{ku}[\mathbb{X}_*(T)]$. Then, for this putative notion of higher algebraic K-theory, we have

$$\begin{aligned} K(\mathrm{Loc}^{\mathrm{Cat}}_{T_c}(\Omega T_c; \mathbf{C})) &\simeq \bigoplus_{\mathbb{X}_*(T)} K(\mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}})) \\ &\simeq \mathrm{QCoh}(B\check{T}_{\mathrm{ku}}) \otimes K(\mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}})). \end{aligned}$$

We therefore need to understand $\mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}})$, i.e., the $(\infty, 2)$-category of $\mathbf{C}$-linear ∞-categories equipped with a T_c-action. There is an equivalence of $(\infty, 2)$-categories

$$\mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}}) \xrightarrow{\sim} \mathrm{Fun}(B\mathbb{X}^*(T), \mathrm{LinCat}_{\mathbf{C}})$$

which sends $\mathscr{C} \mapsto \mathscr{C}^{hT_c}$. Here, the action of $\mathbb{X}^*(T)$ on $\mathscr{C}^{hT_c}$ is defined as follows: an object of $\mathscr{C}^{hT_c}$ is precisely an object $x \in \mathscr{C}$ equipped with coherent equivalences $\alpha_t : x \xrightarrow{\sim} t \cdot x$ for $t \in T_c$. Given a character $\lambda : T_c \to S^1 \subseteq \mathbf{C}^\times$, one can now twist the equivalence α_t by multiplication by $\lambda(t)$; in other words, every $\lambda \in \mathbb{X}^*(T)$ defines a functor $\mathscr{C}^{hT_c} \to \mathscr{C}^{hT_c}$ sending $(x, \alpha_t) \mapsto (x, \lambda(t)\alpha_t)$.

It follows that $K(\mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}})) \simeq K(\mathrm{Fun}(B\mathbb{X}^*(T), \mathrm{LinCat}_{\mathbf{C}}))$. But $B\mathbb{X}^*(T)$ is a *finite* space (namely, the torus $\check{T}_c$), so that the discussion preceding Question 3.21 tells us that this putative notion of "algebraic K-theory" should send

$$\mathrm{Fun}(B\mathbb{X}^*(T), \mathrm{LinCat}_{\mathbf{C}}) \mapsto \mathrm{Loc}(B\mathbb{X}^*(T); \mathrm{ku}) = \mathrm{Loc}(\check{T}_c; \mathrm{ku}).$$

But taking monodromy defines an equivalence $\mathrm{Loc}(\check{T}_c; \mathrm{ku}) \simeq \mathrm{Mod}_{\mathrm{ku}[\mathbb{X}_*(\check{T})]}$, and it follows from the definition of the dual torus over ku that $\mathrm{Loc}(\check{T}_c; \mathrm{ku}) \simeq \mathrm{QCoh}(T_{\mathrm{ku}})$. Therefore:

$$\begin{aligned} K(\mathrm{Loc}^{\mathrm{Cat}}_{T_c}(\Omega T_c; \mathbf{C})) &\simeq \mathrm{QCoh}(B\check{T}_{\mathrm{ku}}) \otimes K(\mathrm{Fun}(BT_c, \mathrm{LinCat}_{\mathbf{C}})) \\ &\simeq \mathrm{QCoh}(B\check{T}_{\mathrm{ku}}) \otimes_{\mathrm{ku}} \mathrm{QCoh}(T_{\mathrm{ku}}). \end{aligned}$$

If we fix an isomorphism $\mathbb{X}_*(\check{T}) \cong \mathbf{Z}^{\mathrm{rank}(\check{T})}$, this can be identified with $\mathrm{QCoh}(\check{T}_{\mathrm{ku}}/\check{T}_{\mathrm{ku}})$.[3] This is the simplest manifestation of Question 3.21, but it does not carry much representation-theoretic meaning. Understanding the analogue of the above discussion when $G = \mathrm{SL}_2$ (for instance) would be very interesting.

[3] Admittedly, picking this trivialization of $\mathbb{X}_*(\check{T})$ is not a very canonical thing to do, and in particular cannot always be done W-equivariantly; but it's the analogue of the identification $\mathfrak{t} \cong \mathfrak{t}^*$ at the level of Lie algebras which is often made in representation theory. Note that $\mathfrak{t}^* = \check{\mathfrak{t}}$, so the identification $\mathfrak{t} \cong \mathfrak{t}^*$ is just an identification $\mathfrak{t} \cong \check{\mathfrak{t}}$.

The motivation for Question 3.21 actually comes from mathematical physics (in particular, the work of Kapustin-Witten [**KW07**]), which provides a perspective on the geometric Langlands equivalence as an equivalence of categories of line operators in certain 4-dimensional TQFTs — so I was hoping that the KU-analogue of the geometric Satake equivalence (Theorem 3.7 and Expectation 3.15) is related to a 5-*dimensional* TQFT. Presumably the elliptic cohomology story is related to some sort of "universal" 6-*dimensional* TQFT, but understanding that story seems even further out of reach.

3.3. Extension 3: relationship to Ben-Zvi–Sakellaridis–Venkatesh. Ben-Zvi, Sakellaridis, and Venkatesh have a very exciting program to study a relative version of the Langlands correspondence. Unfortunately, not many details seem to exist yet; but an outline is provided in [**Sak21, BSV21**]. The discussion above suggests a generalization of the Ben-Zvi–Sakellaridis–Venkatesh program. To explain it, let me briefly review my understanding of their story; for simplicity, set $\mathscr{K} = \mathbf{C}(\!(t)\!)$ and $\mathscr{O} = \mathbf{C}[\![t]\!]$. Suppose G is a semisimple algebraic group or a torus over $\mathbf{C}$. For a large class of smooth Hamiltonian G-varieties M of the form T^*X with G acting on X, the program of Ben-Zvi–Sakellaridis–Venkatesh (with previous input by Gaitsgory-Nadler [**GN10**]) conjectures that there is a dual Hamiltonian $\check{G}$-variety $\check{M}$ such that there is an equivalence

$$\mathrm{Shv}(X(\mathscr{K})/G(\mathscr{O}); \mathbf{C}) \simeq \mathrm{QCoh}^{\mathrm{sh}}(\check{M}/\check{G}),$$

where $\mathrm{QCoh}^{\mathrm{sh}}(\check{M}/\check{G})$ denotes the ∞-category of quasicoherent sheaves on a shearing of $\check{M}$. The left-hand side is often known as the "A-side", and the right-hand side is often known as the "B-side". Moreover, this equivalence should satisfy certain compatibility criteria. In many cases, $\check{M}$ is also of the form $T^*\check{X}$ for some *dual* spherical $\check{G}$-variety $\check{X}$. Let us give some examples:

(a) For the pair $(M = T^*G, G \times G)$, the A-side category is

$$\mathrm{Shv}(G(\mathscr{O})\backslash G(\mathscr{K})/G(\mathscr{O}); \mathbf{C}) = \mathrm{Shv}_{G(\mathscr{O})}(\mathrm{Gr}_G).$$

By the derived geometric Satake equivalence, we may identify this with

$$\mathrm{QCoh}(\check{\mathfrak{g}}[2]/\check{G}) \simeq \mathrm{QCoh}(\check{G}\backslash T^*[2](\check{G})/\check{G});$$

the dual pair is therefore $(T^*\check{G}, \check{G} \times \check{G})$.

(b) For the pair $(M = T^*(G/N), G \times T)$, the A-side category is

$$\mathrm{Shv}(G(\mathscr{O})\backslash (G/N)(\mathscr{K})/T(\mathscr{O}); \mathbf{C}).$$

One should interpret this as the ∞-category $\mathrm{Shv}(\mathrm{Fl}_G^{\infty/2}/G(\mathscr{O}); \mathbf{C})$, where $\mathrm{Fl}_G^{\infty/2}$ is the "semi-infinite flag variety" $G(\mathscr{K})/N(\mathscr{K}) \cdot T(\mathscr{O})$. However, in [**Ras14**, Corollary 17.2.3], it was shown that $\mathrm{Shv}(\mathrm{Fl}_G^{\infty/2}/G(\mathscr{O}); \mathbf{C})$ is equivalent to $\mathrm{Shv}(\mathrm{Fl}/G(\mathscr{O}); \mathbf{C}) \simeq \mathrm{Shv}_I(\mathrm{Gr}_G; \mathbf{C})$. The work of Arkhipov-Bezrukavnikov-Ginzburg identifies this with a shearing of $\mathrm{QCoh}(\widetilde{\check{\mathfrak{g}}}/\check{G})$. Now, $\widetilde{\check{\mathfrak{g}}}/\check{G} \cong (\check{\mathfrak{b}}/\check{N})/\check{T}$ is isomorphic to $\check{G}\backslash T^*(\check{G}/\check{N})/\check{T}$, so that the dual pair is $(M = T^*(\check{G}/\check{N}), \check{G} \times \check{T})$.

(c) Let $\psi : N \to \mathbf{G}_a$ be a nondegenerate character (dual to some principal nilpotent $f \in \mathfrak{n}$), such as the composite

$$N \to N/[N,N] \cong \prod_{\alpha \in \Delta} \mathbf{G}_a^{\alpha} \xrightarrow{\sum_\alpha} \mathbf{G}_a.$$

Composition with the map $N(\mathscr{K}) \to N$ given by the residue defines a nondegenerate character of $N(\mathscr{K})$, which we will also denote by ψ. Let $T^*(G/_\psi N)$ denote $(\psi + \mathfrak{b}^*) \times_N G \cong (f + \mathfrak{b}) \times_N G$. For the pair $(M = T^*(G/_\psi N), G)$, the A-side category is $\mathrm{Shv}(G(\mathscr{O})\backslash(G/_\psi N)(\mathscr{K}); \mathbf{C})$, which we may interpret as the Whittaker category $\mathrm{Shv}(\mathrm{Gr}_G; \mathbf{C})^{(N(\mathscr{K}),\psi)}$. By the geometric Casselman-Shalika equivalence of [**FGKV98, FGV01**], we may identify this with $\mathrm{Rep}(\check{G}) \simeq \mathrm{QCoh}(B\check{G})$; the dual pair is therefore $(T^*{*}, \check{G})$.

We summarize these examples in a table, a much-expanded version of which appears at [**Wan22**]:

A-side pair	$(T^*G, G \times G)$	$(T^*(G/N), G \times T)$	$(T^*(G/_\psi N), G)$
B-side pair	$(T^*\check{G}, \check{G} \times \check{G})$	$(T^*(\check{G}/\check{N}), \check{G} \times \check{T})$	$(T^*{*}, \check{G})$

Remark 3.23. One can generalize the above discussion further by adding in loop-rotation equivariance. For instance, for a pair (T^*X, G) with dual pair $(T^*\check{X}, \check{G})$, one predicts an equivalence

$$\mathrm{Shv}_{\mathbf{G}_m^{\mathrm{rot}}}(X(\mathscr{K})/G(\mathscr{O}); \mathbf{C}) \simeq \mathrm{DMod}_\hbar(\check{X})^{\check{G}}.$$

The equivalences of Theorem 3.16 fit into this paradigm.

Given the preceding discussion, one might expect an analogue of the BZSV program to work with $\mathbf{C}$-coefficients replaced by KU-coefficients. For example, when $A = \mathrm{KU}$, Expectation 3.15 suggests that there is an equivalence

$$\mathrm{Shv}^c_{G(\mathscr{O})}(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\check{G}/\check{G}) \otimes \mathbf{Q}[\beta^{\pm 1}] = \mathrm{QCoh}(\check{G}\backslash(\check{G} \times \check{G})/\check{G}) \otimes \mathbf{Q}[\beta^{\pm 1}],$$

and Remark 3.8 says that this equivalence does hold when the left-hand side is replaced by local systems and the right-hand side is localized to the regular locus. Therefore, going from $\mathbf{Q}[\beta^{\pm 1}]$-coefficients to KU-coefficients has the effect of changing $T^*\check{G}$ to $\check{G} \times \check{G}$.

Geometers have already observed that $\check{G} \times \check{G}$ admits a symplectic structure and can be viewed as a *multiplicative* analogue of the cotangent bundle (known as the *Drinfeld double*); see [**AMM98**]. In fact, the paper [**AMM98**] studies a multiplicative analogue of the theory of Hamiltonian $\check{G}$-varieties, which are known as *quasi-Hamiltonian* $\check{G}$-varieties. The definition in [**AMM98**] is somewhat complicated, but Safronov gave a clean algebro-geometric perspective in [**Saf16**]. In what follows, I will use some of the terminology of [**PTVV13**] (in particular, the notion of a shifted symplectic stack and Lagrangian morphisms).

Remark 3.24. One of the main results of [**PTVV13**] establishes that if Y is an "oriented d-stack", then the stack $\mathrm{Bun}_{\check{G}}(Y)$ admits a $(2-d)$-shifted symplectic structure. The stacks $\mathbf{G}_{0,\mathbf{Q}}^\vee = B\mathbf{G}_a$, S^1, and an elliptic curve are prime examples of oriented 1-stacks, so that $\mathrm{Bun}_{\check{G}}(\mathbf{G}_{0,\mathbf{Q}}^\vee)$ admits a 1-shifted symplectic structure. Safronov showed that the data of a Hamiltonian $\check{G}$-variety is equivalent to a Lagrangian morphism to $\mathrm{Bun}_{\check{G}}(B\mathbf{G}_a) = \check{\mathfrak{g}}/\check{G}$: the procedure sends a Hamiltonian $\check{G}$-variety $\check{M}$ to the Lagrangian morphism $\check{M} \to \check{\mathfrak{g}}/\check{G}$ given by taking $\check{G}$-quotients of the moment map $\mu : \check{M} \to \check{\mathfrak{g}}$. Similarly, the data of a quasi-Hamiltonian $\check{G}$-variety is equivalent to a Lagrangian morphism to $\mathrm{Bun}_{\check{G}}(S^1) = \check{G}/\check{G}$. In other words, a Lagrangian morphism $\check{L} \to \check{G}/\check{G}$ defines the quasi-Hamiltonian $\check{G}$-variety

$\check{M} := \check{L} \times_{\check{G}/\check{G}} \check{G}$, and the map $\check{M} \to \check{G}$ is a multiplicative analogue of the moment map.

Based on the preceding discussion, one is led to expect the following generalization of the BZSV program:

Expectation 3.25. For certain spherical varieties X over $\mathbf{C}$, there is a dual quasi-Hamiltonian $\check{G}$-variety $\check{M}_q$ (over $\mathbf{C}$, say) such that there is an equivalence

$$\mathrm{Shv}(X(\mathscr{K})/G(\mathscr{O}); \mathrm{KU}) \otimes \mathbf{C} \simeq \mathrm{QCoh}(\check{M}_q/\check{G}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}];$$

perhaps $\check{M}_q/\check{G}$ even lifts to a KU-stack. This should imply an equivalence of the form

$$\mathrm{Loc}_{G(\mathscr{O})}(X(\mathscr{K}); \mathrm{KU}) \otimes \mathbf{C} \simeq \mathrm{QCoh}(\check{M}_q^{\mathrm{reg}}/\check{G}) \otimes_{\mathbf{C}} \mathbf{C}[\beta^{\pm 1}],$$

where $\check{M}_q^{\mathrm{reg}}$ is the inverse image of $\check{G}^{\mathrm{reg}}$ along the moment map $\check{M}_q \to \check{G}$.

Remark 3.26. For instance, if the usual BZSV dual $\check{M}$ is of the form $T^*\check{X}$ for some spherical $\check{G}$-variety $\check{X}$, one should expect $\check{M}_q$ to be some *multiplicative* analogue of the cotangent bundle of $\check{X}$.

Remark 3.27. There is an obvious variant of Expectation 3.25 where KU is replaced by an even-periodic $\mathbf{E}_\infty$-ring A equipped with an oriented elliptic curve E, where the coherent side is given by quasicoherent sheaves on some stack equipped with a Lagrangian morphism to $\mathrm{Bun}_{\check{G}}(E)$. Such stacks might deserve to be known as "E-Hamiltonian $\check{G}$-varieties". I've only stated Expectation 3.25 for KU because E-Hamiltonian $\check{G}$-varieties do not seem to have been studied much.

Example 3.28. Let $X = G/N$, with $G \times T$ acting on X. Based on the discussion above, one should interpret $\mathrm{Shv}(X(\mathscr{K})/G(\mathscr{O}); \mathrm{KU})$ to mean $\mathrm{Shv}_I(\mathrm{Gr}_G; \mathrm{KU})$. Expectation 3.15 suggests that there is an equivalence

$$\mathrm{Shv}_I^c(\mathrm{Gr}_G(\mathbf{C}); \mathrm{KU}) \otimes \mathbf{Q} \simeq \mathrm{QCoh}(\check{B}/\check{B}) \otimes \mathbf{Q}[\beta^{\pm 1}] = \mathrm{QCoh}(\check{G} \backslash (\check{G} \times^{\check{N}} \check{B})/\check{T}) \otimes \mathbf{Q}[\beta^{\pm 1}],$$

and Theorem 3.7 says that this equivalence does hold when the left-hand side is replaced by local systems and the right-hand side is localized to the regular locus. In particular, the A-side pair $(T^*(G/N), G \times T) = (G \times^N \mathfrak{b}, G \times T)$ admits a "quasi-Hamiltonian/KU-dual" given by $(\check{G} \times^{\check{N}} \check{B}, \check{G} \times \check{T})$. This matches perfectly with the intuition in the literature on quasi-Hamiltonian $\check{G}$-varieties, which suggests viewing $\check{G} \times^{\check{N}} \check{B}$ as a *multiplicative analogue* of the (co[4])tangent bundle of the spherical $\check{G}$-variety $\check{G}/\check{N}$. In other words, Theorem 3.7 is essentially an affirmation of Expectation 3.25 and Remark 3.26 in this case.

Remark 3.29. In order for Expectation 3.25 to even be stated, one needs to answer the following question: what conditions does one need on a spherical $\check{G}$-variety $\check{X}$ in order to define a multiplicative analogue of the cotangent bundle of $\check{X}$ (such that this is a quasi-Hamiltonian $\check{G}$-variety)? For instance, the multiplicative cotangent bundle of $\check{G}$ (resp. $\check{G}/\check{N}$) is $\check{G} \times \check{G}$ (resp. $\check{G} \times^{\check{N}} \check{B}$). This seems to be a rather interesting question, and I'm not sure what generality one should expect an answer in.

[4]These linear duality issues are one of the most pesky/confusing aspects of the quasi-Hamiltonian story. I'm still trying to sort this out for myself.

Expectation 3.25 suggests a lot of Langlands-type equivalences with KU-coefficients, and exploiting concrete consequences of this correspondence seems likely to lead to some rich mathematics.

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