INTEGRAL GROTHENDIECK-RIEMANN-ROCH THEOREM

G. PAPPAS

1. Introduction

Let $f : X \to S$ be a projective morphism between two smooth quasi-projective algebraic varieties defined over the field k. If $\mathcal{F}$ is a vector bundle on X, then the hypercohomology $\mathrm{R}f_*(\mathcal{F})$ of $\mathcal{F}$ is represented by a finite complex $(\mathcal{E}^i)$ of vector bundles on S and we can unambiguously define the class $f_*[\mathcal{F}] := \sum_i (-1)^i[\mathcal{E}^i]$ of $\mathrm{R}f_*(\mathcal{F})$ in the Grothendieck group $\mathrm{K}_0(S)$. The Grothendieck-Riemann-Roch theorem ([BS]) is the identity

$$\mathrm{ch}(f_*[\mathcal{F}]) = f_*(\mathrm{ch}(\mathcal{F})\mathrm{Td}(T_X))\mathrm{Td}(T_S)^{-1} \tag{1.1}$$

in the Chow ring with rational coefficients $\mathrm{CH}^*(S)_{\mathbf{Q}} = \oplus_n \mathrm{CH}^n(S)_{\mathbf{Q}}$. Here ch is the Chern character and $\mathrm{Td}(T_X)$, $\mathrm{Td}(T_S)$ stand for the Todd power series evaluated at the Chern classes of the tangent bundle of X, respectively S. Since both sides of (1.1) take values in $\mathrm{CH}^*(S)_{\mathbf{Q}} := \mathrm{CH}^*(S) \otimes \mathbf{Q}$, only information modulo torsion about the Chern classes of $f_*[\mathcal{F}]$ can be obtained from this identity.

The goal of our paper is to improve on this as follows: Set

$$T_m = \prod_p p^{\left[\frac{m}{p-1}\right]} \tag{1.2}$$

where the bracket denotes the integral part and the product is over prime numbers. The integer T_m is the denominator of the degree m part of the Todd power series; notice that $m!$ divides T_m which divides $T_{m'}$ for $m' \geq m$. Write

$$\mathrm{ch} = \sum_{m\geq 0} \frac{\mathfrak{s}_m}{m!}\ , \qquad \mathrm{Td} = \sum_{m\geq 0} \frac{\mathfrak{Td}_m}{T_m}\ .$$

The numerators $\mathfrak{s}_m$ and $\mathfrak{Td}_m$ of the degree m parts are polynomials with integral coefficients in the Chern classes. We show that, when k has characteristic zero and the relative dimension $d = \dim(X) - \dim(S)$ is non-negative, the Grothendieck-Riemann-Roch (GRR) formula actually applies to calculate

$$\frac{T_{d+n}}{n!} \cdot \mathfrak{s}_n(f_*[\mathcal{F}])$$

in $\mathrm{CH}^n(S)$ (and not just in $\mathrm{CH}^n(S)$ modulo torsion). The point here is that multiplication by T_{d+n} clears all the denominators in the codimension n component of the formula (1.1); we show that the resulting identity is indeed true in $\mathrm{CH}^n(S)$. For example, when $n = 1$,

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$\mathfrak{s}_1 = c_1$ and $\mathfrak{s}_1(f_*[\mathcal{F}])$ is the class of the determinant of cohomology $\det \mathrm{R}f_*(\mathcal{F})$ in $\mathrm{Pic}(S)$. In this case, we obtain

$$(1.3)\qquad T_{d+1}\cdot[\det \mathrm{R}f_*(\mathcal{F})] = \\ = -r(f_*[\mathcal{F}])\cdot\frac{T_{d+1}}{2}\cdot c_1(T_S) + \sum_{m=0}^{d+1}\frac{T_{d+1}}{m!\cdot T_{d+1-m}}\cdot f_*\left[\mathfrak{s}_m(\mathcal{F})\cdot\mathfrak{T}\mathfrak{d}_{d+1-m}(T_X)\right]$$

in $\mathrm{Pic}(S)=\mathrm{CH}^1(S)$. Here $r(f_*[\mathcal{F}]) = \mathfrak{s}_0(f_*[\mathcal{F}]) = \sum_i(-1)^i\mathrm{rank}_{\mathcal{O}_S}(\mathrm{R}^if_*(\mathcal{F}))$ is the (virtual) rank of $\mathrm{R}f_*(\mathcal{F})$ on S. Note that the ratios $T_{d+1}/(m!\cdot T_{d+1-m})$ and $T_{d+1}/2$ are integers (Lemma 2.1).

A few isolated cases of this result were already known: When f is a relative curve, i.e when $d=1$, Mumford has shown, using the moduli space of curves, that the GRR formula applies to calculate $12\cdot[\det \mathrm{R}f_*(\mathcal{F})]$ in $\mathrm{Pic}(S)$. Since $T_2 = 12$, this also follows from (1.3) above, which then generalizes Mumford's result to higher dimensions (but only in characteristic 0). In fact, by applying the integral GRR formula for $d=1$ and $n\geq 2$, we also obtain new integral relations among the pull-backs of tautological classes on S. These relations were known before only up to torsion and refine corresponding (known or conjectural) equations in the integral cohomology of the mapping class group (see §2.d.3). When the morphism f is a finite étale cover, we obtain the Riemann-Roch theorem for covering maps of Fulton-MacPherson. If $f: A\to S$ is an abelian scheme of relative dimension g, then our integral GRR formula implies that the top Chern class of the Hodge bundle over S in $\mathrm{CH}^g(S)$ is annihilated by the integer T_{2g}. This also follows from a (stronger) result of Ekedahl-van der Geer. See §2.d for some more corollaries for families of surfaces.

Our approach was inspired by the classical work of Washnitzer [W] and Fulton [F2] on characterizing the arithmetic genus and by certain constructions in the theory of algebraic cobordism of Levine and Morel [LM]. The crucial ingredients are Hironaka's resolution of singularities and the weak factorization theorem for birational maps of [AKMW] (this is the only ingredient of our proof that has not been available for a long time); the use of these restricts the result to characteristic 0.

Here is an outline of the proof: A refinement of the classical arguments shows that the integral GRR identity holds for projective bundles, for closed immersions and for blow ups along smooth centers. However, contrary to what happens in Grothendieck's approach, the general result does not follow easily from these special cases: Indeed, the use of a projective bundle of dimension higher than that of the variety introduces additional denominators. To show the integral GRR identity in general, we first assume that $\mathcal{F}$ is the structure sheaf $\mathcal{O}_X$: We then prove that if X, X' are smooth linearly equivalent divisors in $W\to S$ with W smooth, then the integral GRR formula holds for X if and only if it holds for X'. In fact, we can extend both sides of the integral GRR formula to general Weil divisors on

W and show that each side respects linear equivalence. We then observe that if the result holds for $X \to S$ then it holds for a projective bundle $\mathbf{P}(\mathcal{E}) \to X \to S$. We also prove, using the factorization theorem and the result for blow ups, that the integral GRR formula holds for $X \to S$ if and only if it holds for any $X' \to S$ which is birationally equivalent to X over S. Now to actually prove the formula for $\mathcal{F} = \mathcal{O}_X$ we argue by double induction, first on n and then on the relative dimension d. The result for $n = 0$ is given by the Hirzebruch-Riemann-Roch theorem since $\mathrm{CH}^0(S) = \mathbf{Z}$ is torsion-free. We then observe that when $f : X \to S$ is not dominant the result follows from the induction hypothesis on n using resolution of singularities, factorization, and integral Riemann-Roch for closed immersions. To show the result for f dominant, we apply induction on d. Since X is birational to the desingularization Y' of a hypersurface $\hat{Y}$ in $\mathbf{P}^{d+1} \times_k S$, by the above, it is enough to deal with such a desingularization. The hypersurface $\hat{Y}$ is linearly equivalent to a sum $\mathrm{pr}_1^{-1}(H) + \mathrm{pr}_2^{-1}(T_1) - \mathrm{pr}_2^{-1}(T_2)$, where H is a smooth hypersurface in $\mathbf{P}_k^{d+1}$ and T_1, T_2 are smooth divisors on S; we eventually reduce to checking the identity in the simple cases that $X = \mathrm{pr}_1^{-1}(H)$, or that $X = \mathrm{pr}_1^{-1}(T_i)$, $i = 1, 2$. The induction hypothesis on d implies that the exceptional locus of the desingularization we employ do not contribute an error to the formula: Indeed, the components of the exceptional locus are birational to projective bundles over varieties of smaller dimension. The argument shows more or less simultaneously that the result is true when $\mathcal{F}$ is a line bundle on X. The case that $\mathcal{F}$ is a general vector bundle follows by using a result of Kleiman which allows us to split $\mathcal{F}$ after a blow-up.

In fact, it turns out that an important part of the proof can also be presented as an application of the "generalized degree formula" in the theory of algebraic cobordism (see Remark 6.2). This gives a somewhat different route toward the main result. We chose the direct and classical argument above to make the paper more self-contained and accessible. However, we feel that this observation establishes an interesting connection which could be important in future developments.

Finally, let us mention that we expect that this modification of Grothendieck's argument can be applied to the proof of other Riemann-Roch type theorems and should also produce versions that capture torsion information. For example, one could attempt to revisit the "functorial" Riemann-Roch of Deligne ([D]) and Franke (unpublished) or Gillet's Riemann-Roch theorem for higher algebraic K-theory ([G]).

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2. Preliminaries

Throughout the paper k is a field of characteristic 0; all algebraic varieties and morphisms are over the field k.

2.a. We start with the following lemma which will be used repeatedly.

Lemma 2.1. *Let m be a positive integer. If $m_1+m_2+\cdots+m_r+m_{r+1}+\cdots+m_{r+s}\le m$ with m_i positive integers, then the product*

$$(m_1+1)!\cdots(m_r+1)!\cdot T_{m_{r+1}}\cdot\cdots\cdot T_{m_{r+s}}$$

divides T_m.

Proof. Recall that if p is a prime number and n an integer with $p^k\le n<p^{k+1}$, then the largest power of p that divides $n!$ is

$$\left[\frac{n}{p}\right]+\left[\frac{n}{p^2}\right]+\cdots+\left[\frac{n}{p^k}\right]\le\left[\frac{n(p^k-1)}{p^k(p-1)}\right]\le\left[\frac{n-1}{p-1}\right].$$

The lemma now follows from this and (1.2). □

2.b. Consider the Todd power series

$$\mathrm{Td}=\prod_{j=1}^{\infty}\frac{x_j}{1-e^{-x_j}}=1+\frac{1}{2}c_1+\frac{1}{12}(c_1^2+c_2)+\frac{1}{24}c_1c_2+\cdots$$

viewed as a formal power series with rational coefficients in the variables c_i (the elementary symmetric functions of x_j) with $\deg(c_i)=i$. For any m, we will consider the degree m part of Td which we will denote by Td_m. (In general, we will denote by P_m the homogeneous degree m part of P.) By [Hi, Lemma 1.7.3], the polynomial

$$\mathfrak{Td}_m=T_m\cdot\mathrm{Td}_m$$

has integral coefficients and is the numerator of the degree m part of Td.

We can also consider the Chern power series

$$\mathrm{ch}=r+\sum_{j=1}^{\infty}(e^{x'_j}-1)=r+c'_1+\frac{1}{2}(c'^2_1-2c'_2)+\frac{1}{6}(c'^3_1-3c'_1c'_2+3c'_3)+\cdots$$

as a formal power series with rational coefficients in the variables r (rank), c'_i (the elementary symmetric functions of x'_j) with $\deg(r)=0$, $\deg(c'_i)=i$. We set $\mathfrak{s}_m=m!\cdot\mathrm{ch}_m$ for the numerator of the degree m part of ch. Also set

$$\mathfrak{CT}_m=T_m\cdot(\mathrm{ch}\cdot\mathrm{Td})_m=\sum_{j=0}^{m}\frac{T_m}{j!\cdot T_{m-j}}\cdot(\mathfrak{s}_j\cdot\mathfrak{Td}_{m-j})\ .$$

By Lemma 2.1, $\mathfrak{CT}_m$ is a homogeneous polynomial in $\mathbf{Z}[c_1,c_2,\ldots,c_m,r,c'_1,\ldots,c'_m]$.

2.c. Let Y be a variety over k. We will denote by $\mathrm{K}_0(Y)$ the Grothendieck ring of locally free coherent $\mathcal{O}_Y$-sheaves on Y and by $\mathrm{G}_0(Y)$ the Grothendieck group of coherent $\mathcal{O}_Y$-sheaves on Y. Suppose that Y is smooth and quasi-projective. Then the natural map $\mathrm{K}_0(Y) \to \mathrm{G}_0(Y)$ is an isomorphism; we will identify these two groups without further notice. Denote by $\mathrm{CH}^i(Y)$ the Chow group of algebraic cycles of codimension i on Y modulo rational equivalence. There are well-defined intersection pairings $\mathrm{CH}^i(Y) \otimes \mathrm{CH}^j(Y) \to \mathrm{CH}^{i+j}(Y)$ which turn $\mathrm{CH}^*(Y) = \oplus_{i=0}^{\dim(Y)} \mathrm{CH}^i(Y)$ into a graded commutative ring. If $\mathcal{F}$ is a locally free coherent $\mathcal{O}_Y$-sheaf on Y we have the Chern classes $c_i(\mathcal{F}) \in \mathrm{CH}^i(Y)$, $1 \leq i \leq \dim(Y)$. We will denote by $T_Y := (\Omega^1_{Y/k})^\vee$ the tangent sheaf of Y. For $m \geq 0$, we now set

$$\begin{aligned} \mathfrak{CT}_m(\mathcal{F}, Y) &:= \mathfrak{CT}_m(c_1(T_Y), \ldots, c_m(T_Y), r(\mathcal{F}), c_1(\mathcal{F}), \ldots, c_m(\mathcal{F})), \\ \mathfrak{Td}_m(\mathcal{F}) &:= \mathfrak{Td}_m(c_1(\mathcal{F}), \ldots, c_m(\mathcal{F})), \end{aligned}$$

in $\mathrm{CH}^m(Y)$. (We evaluate $\mathfrak{CT}_m$ by setting $c_i = c_i(T_Y)$, $r = \mathrm{rank}(\mathcal{F})$, $c_i' = c_i(\mathcal{F})$, similarly for $\mathfrak{Td}_m(\mathcal{F})$.) Often, we will simply write $\mathfrak{Td}_m(Y)$ instead of $\mathfrak{Td}_m(T_Y)$. It follows from the Whitney sum formula that the functions $\mathfrak{s}_m(-)$, $\mathfrak{Td}_m(-)$ and $\mathfrak{CT}_m(-, Y)$ extend to give well-defined maps $\mathrm{K}_0(Y) \to \mathrm{CH}^m(Y)$. The maps $\mathfrak{s}_m(-)$ and $\mathfrak{CT}_m(-, Y)$ are additive. The multiplicativity of the Chern, resp. Todd, power series implies

$$\mathfrak{s}_m(a \cdot b) = \sum_{i=0}^{m} \frac{m!}{i! \cdot (m-i)!} \cdot \mathfrak{s}_i(a) \cdot \mathfrak{s}_{m-i}(b)\,, \tag{2.1}$$

$$\mathfrak{Td}_m(a + b) = \sum_{i=0}^{m} \frac{T_m}{T_i \cdot T_{m-i}} \cdot \mathfrak{Td}_i(a) \cdot \mathfrak{Td}_{m-i}(b)\,, \tag{2.2}$$

with a, b in the Grothendieck ring $\mathrm{K}_0(Y)$.

Suppose that $f : X \to S$ is a projective morphism between the smooth varieties X and S. Set $d = d_f = \dim(X) - \dim(S)$. There are well-defined push-forward homomorphisms:

$$f_* : \mathrm{CH}^i(X) \to \mathrm{CH}^{i-d}(S), \qquad f_* : \mathrm{K}_0(X) = \mathrm{G}_0(X) \to \mathrm{K}_0(S) = \mathrm{G}_0(S),$$

where for $\mathcal{F}$ a (locally free) coherent $\mathcal{O}_X$-sheaf, we set $f_*[\mathcal{F}] = [\mathrm{R}f_*(\mathcal{F})] = \sum_i (-1)^i [\mathrm{R}^i f_*(\mathcal{F})]$.

Our main result is:

Theorem 2.2. *Suppose k is a field of characteristic 0. Let X and S be smooth quasi-projective varieties over k and let $f : X \to S$ be a projective morphism over k. Set $d = d_f = \dim(X) - \dim(S)$ and suppose that $\mathcal{F}$ is a coherent $\mathcal{O}_X$-sheaf on X.*

a) Suppose $d \geq 0$. Then the identity

$$\frac{T_{d+n}}{T_n} \cdot \mathfrak{CT}_n(f_*[\mathcal{F}], S) = f_*(\mathfrak{CT}_{d+n}(\mathcal{F}, X)) \tag{2.3}$$

holds in $\mathrm{CH}^n(S)$.

b) Suppose $d < 0$. Then the identity

$$\mathfrak{CT}_n(f_*[\mathcal{F}], S) = \frac{T_n}{T_{n+d}} \cdot f_*(\mathfrak{CT}_{n+d}(\mathcal{F}, X)) \tag{2.4}$$

holds in $\mathrm{CH}^n(S)$.

Let $\mathfrak{CT}_m(\mathcal{F}, X/S)$ in $\mathrm{CH}^m(X)$ be the result of evaluating the polynomial $\mathfrak{CT}_m$ by setting $c_i = c_i([T_X] - [f^*T_S])$, $r = \mathrm{rank}(\mathcal{F})$, $c'_i = c_i(\mathcal{F})$. Part (a) implies the following.

Corollary 2.3. *Suppose $d \geq 0$. Then the identity*

$$\frac{T_{d+n}}{n!} \cdot \mathfrak{s}_n(f_*[\mathcal{F}]) = f_*(\mathfrak{CT}_{d+n}(\mathcal{F}, X/S)) \tag{2.5}$$

holds in $\mathrm{CH}^n(S)$.

Proof. Observe that the left hand side of the identity (2.3) can be written

$$\frac{T_{d+n}}{n!} \cdot \mathfrak{s}_n(f_*[\mathcal{F}]) + \sum_{j=1}^{n} \frac{T_{d+n}}{T_{d+n-j} \cdot T_j} \cdot \left\{ \frac{T_{d+n-j}}{(n-j)!} \cdot \mathfrak{s}_{n-j}(f_*[\mathcal{F}]) \right\} \cdot \mathfrak{Td}_j(T_S)\,. \tag{2.6}$$

The statement follows from this observation, the projection formula and (2.3), by induction on n. □

Remark 2.4. a) The image of $f_*(\mathfrak{CT}_{d+n}(\mathcal{F}, X/S))$ in $\mathrm{CH}^n(S) \otimes \mathbf{Q}$ is

$$T_{d+n} \cdot f_*((\mathrm{ch}(\mathcal{F}) \cdot \mathrm{Td}(T_X) \cdot \mathrm{Td}(f^*T_S)^{-1})_{d+n})$$

and so the image of the identity (2.5) in $\mathrm{CH}^n(S) \otimes \mathbf{Q}$ is the identity for $\mathrm{ch}_n(f_*[\mathcal{F}])$ given by the Grothendieck-Riemann-Roch theorem.

b) When the morphism f is a closed immersion of codimension $r = -d$, then (2.4) follows from the "Riemann-Roch without denominators" of Jouanolou [J] (see Theorem 4.3).

2.d. Here we describe some corollaries of this result.

2.d.1. Let $f : X \to Y$ be a finite étale morphism between smooth quasi-projective varieties over k. Then $f^*T_Y \simeq T_X$. Therefore, (2.5) implies

$$\frac{T_n}{n!} \cdot (\mathfrak{s}_n(f_*\mathcal{F}) - f_*(\mathfrak{s}_n(\mathcal{F}))) = 0\,,$$

for any $\mathcal{F}$ on X. As in [FM, Remark 23.8], we see that this immediately implies

$$L_n \cdot (\mathfrak{s}_n(f_*\mathcal{F}) - f_*(\mathfrak{s}_n(\mathcal{F}))) = 0\,, \tag{2.7}$$

where L_n is the product of all primes that divide $T_n/n!$. (The integer $T_n/n!$ is denoted by N_n in loc. cit.) This last identity (2.7) is the integral Riemann-Roch theorem for covering maps of Fulton-MacPherson ([FM, Theorem 23.3]). In the context of group representations and for characteristic classes in integral group cohomology, Evans-Kahn [EK] have shown that, for the (topological) cover given by $BH \to BG$ where H is a subgroup of a finite group G, the integers L_n are the smallest with the property corresponding to (2.7). Using Totaro's construction [T], we can approximate $BH \to BG$ by a finite étale cover of smooth quasi-projective varieties $X \to Y$. Hence, we see that L_n are the smallest integers so that (2.7) holds for all finite étale covers.

2.d.2. Let $f : A \to S$ be an abelian scheme ([CF]) of relative dimension g over the smooth quasi-projective variety S over k. By a result of Grothendieck the morphism f is projective. Using (2.5) we obtain

$$\frac{T_{2g}}{g!} \cdot \mathfrak{s}_g(f_*[\mathcal{O}_A]) = f_*(\mathfrak{CT}_{2g}(\mathcal{O}_A, A/S)) \tag{2.8}$$

in $\mathrm{CH}^g(S)$. The Hodge bundle is the locally free coherent $\mathcal{O}_S$-sheaf $E = s^*(\Omega^1_{A/S})$ where $s : S \to A$ is the zero section; it has rank g and we have $\Omega^1_{A/S} \simeq f^*(E)$. We find

$$f_*(\mathfrak{CT}_{2g}(\mathcal{O}_A, A/S)) = f_*(\mathfrak{Td}_{2g}(f^*(E^\vee))) = f_* f^*(\mathfrak{Td}_{2g}(E^\vee)) = 0$$

in $\mathrm{CH}^g(S)$, while $f_*[\mathcal{O}_A] = \sum_{i=0}^g (-1)^i [\mathrm{R}^i f_*(\mathcal{O}_A)] = \sum_{i=0}^g (-1)^i [\wedge^i(E^\vee)]$. The standard identity [BS, Lemme 18] now gives

$$\mathfrak{s}_g \left(\sum_{i=0}^{g} (-1)^i [\wedge^i(E^\vee)] \right) = g! \cdot c_g(E)\,. \tag{2.9}$$

Therefore, (2.8) implies that $T_{2g} \cdot c_g(E) = 0$ in $\mathrm{CH}^g(S)$. For $g = 1$, we get the classical $12 \cdot c_1(E) = 0$. Ekedahl and van der Geer show that $2(g-1)!\, D_{2g} \cdot c_g(E) = 0$ ([EvdG, Theorem 3.5]) where

$$D_{2g} = \prod_{l \text{ prime},\, l-1 | 2g} l^{1+\mathrm{ord}_l(2g)}\,.$$

By von Staudt's theorem, the number D_{2g} is equal to the denominator of $B_{2g}/2g$ with B_{2g} the Bernoulli number. When $g > 1$, we can see that $2(g-1)!\, D_{2g}$ divides T_{2g} and so this corollary of (2.5) follows from their result.

2.d.3. Assume in addition that $f : X \to S$ is smooth and that the geometric fibers of f are irreducible curves ($d = 1$). Let

$$\kappa_i = f_*(c_1(\Omega^1_{X/S})^{i+1})$$

be Mumford's "tautological" classes in $\mathrm{CH}^i(S)$. Denote by $\omega = \mathrm{R}^0 f_*(\Omega^1_{X/S})$ the Hodge bundle on S. Since $\mathrm{R}^0 f_*(\mathcal{O}_X) \simeq \mathcal{O}_S$ and $\mathrm{R}^1 f_*(\mathcal{O}_X) \simeq \omega^\vee$ (by Serre-Grothendieck duality), we have $\mathfrak{s}_n(f_*[\mathcal{O}_X]) = (-1)^{n-1}\mathfrak{s}_n(\omega)$. Applying (2.5) to $\mathcal{F} = \mathcal{O}_X$ now gives $12 \cdot c_1(\omega) = \kappa_1$ in $\mathrm{Pic}(S)$ (for $n = 1$) and

$$\frac{T_{n+1}}{n!} \cdot \mathfrak{s}_n(\omega) = \begin{cases} 0, & \text{if } n = 2m\,, \\ T_{2m} \dfrac{B_{2m}}{(2m)!} \cdot \kappa_{2m-1}, & \text{if } n = 2m-1\,, \end{cases} \tag{2.10}$$

in $\mathrm{CH}^n(S)$ for $n \geq 2$. The corresponding identity in the integral cohomology of the mapping class group of surfaces is a slightly weakened version of a conjecture of Akita [A]. A corollary of (2.10) is that we have $\kappa_{2m-1} = D_{2m} \cdot \alpha_m + \beta_m$ in $\mathrm{CH}^{2m-1}(S)$, where D_{2m} is the denominator of $B_{2m}/2m$ as above, and β_m is $T_{2m}/(D_{2m} \cdot (2m-1)!)$-torsion. In fact, we conjecture that β_m can be taken to be zero, i.e that κ_{2m-1} is actually D_{2m}-divisible in $\mathrm{CH}^{2m-1}(S)$. (If $k = \mathbf{C}$, see [GMT] for a discussion of the corresponding statements in the integral cohomology $\mathrm{H}^{4m-2}(S(\mathbf{C}), \mathbf{Z})$.)

2.d.4. Let $f : X \to S$ be a relative surface ($d = 2$). Assume that f is smooth and set $\omega_{X/S} = \det(\Omega^1_X) \otimes_{\mathcal{O}_X} \det(f^*\Omega_S)^{-1}$ for the relative dualizing sheaf. For $m \geq 0$, apply (2.5) to $n = 1$ and the sheaves $\mathcal{O}_X$ and $\omega_{X/S}^{\otimes m}$. We deduce the existence of an isomorphism of invertible sheaves on S

$$\Big(\det \mathrm{R}f_*(\omega_{X/S}^{\otimes m}) \otimes_{\mathcal{O}_S} \det \mathrm{R}f_*(\mathcal{O}_X)^{\otimes(2m-1)}\Big)^{\otimes 24} \simeq \langle \omega_{X/S}, \omega_{X/S}, \omega_{X/S}\rangle^{\otimes m(6m-4m^2-2)}$$

where the bracket denotes Deligne's intersection bundle ([D]). (By loc. cit., the class of $\langle \omega_{X/S}, \omega_{X/S}, \omega_{X/S}\rangle$ in $\mathrm{Pic}(S) = \mathrm{CH}^1(S)$ is equal to $f_*(c_1(\omega_{X/S})^3) = -f_*(c_1([T_X] - [f^*T_S])^3)$.) It would be interesting to establish a *canonical* isomorphism as above.

Let us consider an application: Suppose $f : X \to S$ is a family of Enriques surfaces. Then $\omega_{X/S}^{\otimes 2}$ is trivial along the fibers of f. Therefore, $\mathcal{K} := \mathrm{R}^0 f_*(\omega_{X/S}^{\otimes 2})$ is an invertible sheaf on S and we have $\omega_{X/S}^{\otimes 2} \simeq f^*\mathcal{K}$. We also have $\det \mathrm{R}f_*(\mathcal{O}_X) \simeq \mathcal{O}_S$; hence, the projection formula gives $\det \mathrm{R}f_*(f^*\mathcal{K}) \simeq \mathcal{K}$. The above isomorphism for $m = 2$ now gives

$$\mathcal{K}^{\otimes 24} \simeq \langle \omega_{X/S}, \omega_{X/S}, \omega_{X/S}\rangle^{-\otimes 12} \simeq \langle f^*\mathcal{K}, f^*\mathcal{K}, \omega_{X/S}\rangle^{-\otimes 3} \simeq \mathcal{O}_S\,.$$

By [B], such a trivialization of a power of $\mathcal{K}$ can be given explicitly using a Borcherds product on the period domain.

2.e. We will say that integral Riemann-Roch holds for (f, n) when either (2.3) or (2.4) (depending if $d_f \geq 0$ or $d_f < 0$) holds for all $\mathcal{F}$ on X. The following observation will be used repeatedly in our proof of Theorem 2.2.

Proposition 2.5. *Let $f : X \to Y$ and $g : Y \to Z$ be projective morphisms between smooth quasi-projective varieties over k. Suppose that integral Riemann-Roch holds for both $(f, n + d_g)$ and (g, n). Suppose in addition that either $d_f \geq 0$ or $d_g \leq 0$. Then integral Riemann-Roch holds for $(g \cdot f, n)$.*

Proof. This follows easily from the fact that the push-forward homomorphisms (both for Grothendieck groups and Chow groups) satisfy $(g \cdot f)_* = g_* \cdot f_*$. (The assumption on d_f, d_g is needed to guarantee that certain ratios of Todd denominators which are involved in the argument are integers.) □

Remark 2.6. Grothendieck's proof of the Riemann-Roch theorem ([BS]) involves factoring a morphism into a composition of a closed immersion f followed by a projective bundle g. In that case, $d_f < 0$ and $d_g > 0$, and so Proposition 2.5 does not apply.

Proposition 2.7. *Let $f : X \to Y$ be a projective morphism between smooth quasi-projective varieties over k. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-sheaf on X and $\mathcal{G}$ a locally free coherent $\mathcal{O}_Y$-sheaf on Y. Given $n \geq 0$, suppose that the integral Riemann-Roch formula holds for f, $\mathcal{F}$, and all $n' \leq n$. Then it also holds for f, $\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{G}$, and all $n' \leq n$.*

Proof. The proof follows easily from (2.1) and the projection formula. □

3. Divisors

3.a. For $m \geq 1$, let us consider the polynomial

$$Q_m(c_1, \ldots, c_{m-1}, x) = T_{m-1} \cdot ((1 - e^{-x}) \cdot \mathrm{Td})_m$$

in the variables $c_1, \ldots, c_{m-1}$, x, with $\deg(c_i) = i$, $\deg(x) = 1$. By Lemma 2.1, Q_m has integral coefficients.

Suppose that W is a smooth quasi-projective variety of dimension $\delta + 1 \geq 1$ over k. If $[D] \in \mathrm{CH}^1(W)$ is the class of the Weil divisor $D = \sum_i n_i D_i$ of W, we can consider

$$\mathfrak{T}\mathfrak{d}_m(D; W) := Q_m(c_1(T_W), \ldots, c_{m-1}(T_W), [D])$$

in $\mathrm{CH}^m(W)$. Notice that, by its definition, $\mathfrak{T}\mathfrak{d}_m(D; W)$ depends only on the linear equivalence class $[D]$ of D. We also have

$$\frac{T_m}{T_{m-1}} \cdot \mathfrak{T}\mathfrak{d}_m(D; W) = \mathfrak{C}\mathfrak{T}_m([\mathcal{O}_W] - [\mathcal{O}_W(-D)], W)\,, \tag{3.1}$$

where the right hand side is defined in Section 2.

Proposition 3.1. *a) Suppose that D is a smooth divisor and denote by $i : D \hookrightarrow W$ the natural embedding. Then we have*

$$\mathfrak{T}\mathfrak{d}_m(D; W) = i_*(\mathfrak{T}\mathfrak{d}_{m-1}(D)) \tag{3.2}$$

in $\mathrm{CH}^m(W)$.

b) Suppose that $D = D_1 + D_2$ with D_1, D_2 smooth. Suppose also that the scheme theoretic intersection $D_1 \cap D_2$ is smooth and of pure codimension 2. Then

$$\mathfrak{T}\mathfrak{d}_m(D; W) = (i_1)_*\mathfrak{T}\mathfrak{d}_{m-1}(D_1) + (i_2)_*\mathfrak{T}\mathfrak{d}_{m-1}(D_2) - \frac{T_{m-1}}{T_{m-2}} \cdot (i_{12})_*\mathfrak{T}\mathfrak{d}_{m-2}(D_1 \cap D_2), \tag{3.3}$$

where by i_1, i_2, i_{12}, we denote the natural embeddings.

c) Suppose that $D \sim x - y$, with x and y smooth divisors on W. Suppose also that there are smooth divisors y_i, $1 \leq i \leq \delta$, in the same linear equivalence class with y, such that, for each $k = 1, \ldots, \delta$, the scheme theoretic intersections $y_1 \cap \cdots \cap y_k \cap y$, $y_1 \cap \cdots \cap y_k \cap x$ are smooth of pure codimension $k+1$. Denote by $i_x : x \hookrightarrow W$, $i_y : y \hookrightarrow W$,

$i_k : y_1 \cap \cdots \cap y_k \cap x \hookrightarrow W$, $i'_k : y_1 \cap \cdots \cap y_k \cap y \hookrightarrow W$, the natural embeddings. Then, we have

$$
\begin{aligned}
(3.4)\qquad & \mathfrak{T}\mathfrak{d}_m(D;W) = (i_x)_*(\mathfrak{T}\mathfrak{d}_{m-1}(x)) - (i_y)_*(\mathfrak{T}\mathfrak{d}_{m-1}(y)) + \\
& + \sum_{k=1}^{m-1} \frac{T_{m-1}}{T_{m-1-k}} \cdot \left[(i_k)_*(\mathfrak{T}\mathfrak{d}_{m-1-k}(y_1 \cap \cdots \cap y_k \cap x)) - (i'_k)_*(\mathfrak{T}\mathfrak{d}_{m-1-k}(y_1 \cap \cdots \cap y_k \cap y))\right]
\end{aligned}
$$

in $\mathrm{CH}^m(W)$.

PROOF. a) Since both D and W are smooth, we have $[i^*T_W] = [T_D] + [\mathcal{O}_D(D)]$ in the Grothendieck group $\mathrm{K}_0(D)$. Therefore, by the Whitney sum formula, we obtain

$$
c_j([i^*T_W]) = c_j(T_D) + c_1(\mathcal{O}_D(D)) \cdot c_{j-1}(T_D)\ . \tag{3.5}
$$

For any polynomial P (with integral coefficients) in the Chern classes, a locally free coherent $\mathcal{O}_W$-sheaf $\mathcal{F}$ on W, and $n \geq 1$, we have

$$
[D]^n \cdot P(\mathcal{F}) = i_* \left(c_1(\mathcal{O}_D(D))^{n-1} \cdot P(i^*\mathcal{F})\right). \tag{3.6}
$$

(On the right hand side, the Chern classes and the intersection are in $\mathrm{CH}^*(D)$.) This identity implies

$$
\begin{aligned}
(3.7)\qquad & Q_m(c_1(T_W), \ldots, c_{m-1}(T_W), [D]) = \\
& = i_* \left(\left(T_{m-1} \cdot \left\{ \frac{1 - e^{-x}}{x} \cdot \mathrm{Td} \right\}_{m-1} \right) (c_1(i^*T_W), \ldots, c_{m-1}(i^*T_W), c_1(\mathcal{O}_D(D)) \right),
\end{aligned}
$$

where in the last expression the Chern classes are for bundles on D. The usual expression of the Todd power series in terms of the Chern roots gives that the polynomial

$$
\left\{ \frac{1 - e^{-x}}{x} \cdot \mathrm{Td} \right\}_{m-1} \in \mathbf{Q}[c_1, \ldots, c_{m-1}, x]
$$

is send to $\mathrm{Td}_{m-1}(c_1, \ldots, c_{m-1})$ in $\mathbf{Q}[c_1, \ldots, c_{m-1}]$ under the substitution $c_i \mapsto c_i + x \cdot c_{i-1}$, $x \mapsto x$. This fact, together with (3.5), implies that the expression in (3.7) above equals

$$
i_* \left(\{T_{m-1} \cdot \mathrm{Td}_{m-1}\}(c_1(T_D), \ldots, c_{m-1}(T_D)) \right).
$$

This shows our claim.

b) Consider the identity

$$
1 - e^{-a-b} = (1 - e^{-a}) + (1 - e^{-b}) - (1 - e^{-a})(1 - e^{-b}). \tag{3.8}
$$

Our claim follows from the identity (3.8) and Lemma 2.1 by applying an argument similar to the proof of part (a). (Under our assumptions, $D_1 \cap D_2$ is a smooth divisor in D_2.)

c) Consider the formal identity

$$
1 - e^b = -\sum_{j=1}^{\infty} (1 - e^{-b})^j\ . \tag{3.9}
$$

This together with (3.8) gives

$$1 - e^{-a+b} = (1 - e^{-a}) - (1 - e^{-b}) + \sum_{j=1}^{\infty}((1 - e^{-a})(1 - e^{-b})^j - (1 - e^{-b})^{j+1}). \tag{3.10}$$

Our claim again follows using (3.9) by arguments as in the proofs of (a) and (b) above. □

Proposition 3.2. *a) Under the assumptions of Proposition 3.1 (b), we have*

$$[\mathcal{O}_W] - [\mathcal{O}_W(-D)] = [\mathcal{O}_{D_1}] + [\mathcal{O}_{D_2}] - [\mathcal{O}_{D_1 \cap D_2}].$$

b) Under the assumptions of Proposition 3.1 (c), we have

$$[\mathcal{O}_W] - [\mathcal{O}_W(-D)] = [\mathcal{O}_x] - [\mathcal{O}_y] + \sum_{k=1}^{\delta}([\mathcal{O}_{y_1\cap\cdots\cap y_k\cap x}] - [\mathcal{O}_{y_1\cap\cdots\cap y_k\cap y}])$$

in $\mathrm{K}_0(W)$.

(For simplicity, we omit from the notation the push forward along closed immersions and write for example $\mathcal{O}_{D_1}$, $\mathcal{O}_x$, *instead of* $(i_1)_*(\mathcal{O}_{D_1})$, $(i_x)_*(\mathcal{O}_x)$. *)*

Proof. For simplicity, we write $\mathcal{O} = \mathcal{O}_W$.

a) Under our assumptions, the restriction of $\mathcal{O}(-D_1)$ to D_2 is $\mathcal{O}_{D_2}(-D_1 \cap D_2)$. The result follows using the exact sequences

$$0 \to \mathcal{O}(-D_1 - D_2) \to \mathcal{O}(-D_1) \to \mathcal{O}_{D_2}(-D_1 \cap D_2) \to 0,$$
$$0 \to \mathcal{O}_{D_2}(-D_1 \cap D_2) \to \mathcal{O}_{D_2} \to \mathcal{O}_{D_1 \cap D_2} \to 0.$$

b) The exact sequence

$$0 \to \mathcal{O}(-x + y) \to \mathcal{O}(y) \to \mathcal{O}_x(y) \to 0$$

gives

$$[\mathcal{O}] - [\mathcal{O}(-D)] = [\mathcal{O}_x] + ([\mathcal{O}] - [\mathcal{O}(y)]) - ([\mathcal{O}_x] - [\mathcal{O}_x(y)]). \tag{3.11}$$

The exact sequences

$$0 \to \mathcal{O} \to \mathcal{O}(y) \to \mathcal{O}_y(y) \to 0, \quad 0 \to \mathcal{O}_y \to \mathcal{O}_y(y_1) = \mathcal{O}_y(y) \to \mathcal{O}_{y\cap y_1}(y) \to 0,$$

give

$$[\mathcal{O}] - [\mathcal{O}(y)] = -[\mathcal{O}_y] - [\mathcal{O}_{y_1\cap y}(y)].$$

Inductively, we now obtain

$$[\mathcal{O}] - [\mathcal{O}(y)] = -[\mathcal{O}_y] - \sum_{k=1}^{\delta}[\mathcal{O}_{y_1\cap\cdots\cap y_k\cap y}].$$

(Since $y_1 \cap \cdots \cap y_\delta \cap y$ is 0-dimensional, $[\mathcal{O}_{y_1\cap\cdots\cap y_\delta\cap y}(y)] = [\mathcal{O}_{y_1\cap\cdots\cap y_\delta\cap y}]$.) The same argument also shows

$$[\mathcal{O}_x] - [\mathcal{O}_x(y)] = -\sum_{k=1}^{\delta}[\mathcal{O}_{y_1\cap\cdots\cap y_k\cap x}].$$

The last two equations, combined with (3.11), allow us to conclude the proof. □

Bertini's theorem implies that we can always satisfy the assumptions of Propositions 3.1 (b) and 3.2 (c):

Proposition 3.3. *For any Weil divisor D on W, we can find x, y, very ample smooth divisors on W such that $D \sim x - y$. Given such x and y we can find in addition very ample smooth divisors $y_1, \ldots, y_\delta$ with $y_i \sim y$ which are such that $y_1 \cap \cdots \cap y_k \cap y$, $y_1 \cap \cdots \cap y_k \cap x$, are smooth of (pure) codimension $k+1$, for all $k = 1, \ldots, \delta$.* □

3.b. Now suppose D and D' are two (arbitrary) Weil divisors on W. Apply Proposition 3.3 to D and D'. We can write $D \sim X - Y$, $D' \sim X' - Y'$ with X, Y, X', Y' smooth very ample and find $Y_i \sim Y$, $Y'_i \sim Y'$, $k = 1, \ldots, \delta$, with the properties stated above. In addition, we can arrange so that $X \cap X'$, $Y \cap Y'$ are both smooth and of pure codimension 2. Since $X + X'$ and $Y + Y'$ are also very ample, we can represent them by smooth divisors $U \sim X + X'$, $V \sim Y + Y'$. Proposition 3.1 applied to D, D' and $D + D'$, now gives

$$\begin{aligned}
\mathfrak{Td}_m(D; W) &= (i_X)_*(\mathfrak{Td}_{m-1}(X)) - (i_Y)_*(\mathfrak{Td}_{m-1}(Y)) + A,\\
\mathfrak{Td}_m(D'; W) &= (i_{X'})_*(\mathfrak{Td}_{m-1}(X')) - (i_{Y'})_*(\mathfrak{Td}_{m-1}(Y')) + A',\\
\mathfrak{Td}_m(D + D'; W) &= (i_U)_*(\mathfrak{Td}_{m-1}(U)) - (i_V)_*(\mathfrak{Td}_{m-1}(V)) + A''.
\end{aligned}$$

Similarly, Proposition 3.1 (a) and (b) applied to $U \sim X + X'$, $V \sim Y + Y'$, gives

$$\begin{aligned}
(i_U)_*(\mathfrak{Td}_{m-1}(U)) = \mathfrak{Td}_m(U; W) &= (i_X)_*(\mathfrak{Td}_{m-1}(X)) + (i_{X'})_*(\mathfrak{Td}_{m-1}(X')) + B,\\
(i_Y)_*(\mathfrak{Td}_{m-1}(V)) = \mathfrak{Td}_m(V; W) &= (i_Y)_*(\mathfrak{Td}_{m-1}(Y)) + (i_{Y'})_*(\mathfrak{Td}_{m-1}(Y')) + B'.
\end{aligned}$$

Here A, A', A'' and B, B' are integral linear combinations of classes of the form

$$\frac{T_{m-1}}{T_{m-1-l}} \cdot (i_Z)_*(\mathfrak{Td}_{m-1-l}(Z))$$

with $i_Z : Z \hookrightarrow W$ a smooth subvariety of codimension $l + 1 \geq 2$ in W. By combining the corresponding equations, we obtain

$$\text{(3.12)} \quad \mathfrak{Td}_m(D + D'; W) =$$
$$= \mathfrak{Td}_m(D; W) + \mathfrak{Td}_m(D'; W) + \sum_{l=1}^{m-1} \left[\sum_Z a_Z \frac{T_{m-1}}{T_{m-1-l}} \cdot (i_Z)_* \mathfrak{Td}_{m-l-1}(Z) \right]$$

in $\mathrm{CH}^m(W)$. The sum in the bracket is over the set of smooth subvarieties $i_Z : Z \hookrightarrow W$ of codimension $l + 1$ and a_Z are integers which are almost always 0.

Similarly, the same argument applied to the equations obtained by Proposition 3.2 gives

$$\text{(3.13)} \quad [\mathcal{O}] - [\mathcal{O}(-D - D')] = ([\mathcal{O}] - [\mathcal{O}(-D)]) + ([\mathcal{O}] - [\mathcal{O}(-D')]) + \sum_{l=1}^{\delta} \left(\sum_Z b_Z \cdot [\mathcal{O}_Z] \right)$$

in $\mathrm{G}_0(W) = \mathrm{K}_0(W)$. In fact, the parallel expressions in the statements of Propositions 3.1, 3.2, allow us to observe that if $2 \leq \operatorname{codim}(Z) \leq m$, then $b_Z = a_Z$.

4. Projective bundles, blow-ups and embeddings

In this section, we show that integral Riemann-Roch holds for projective bundles, for closed immersions and for blow ups along smooth centers. The proofs mostly follow the standard arguments of "Riemann-Roch algebra" ([FL]); essentially, we will observe that the integrality of the expressions involved is preserved.

4.a. Suppose that $\mathcal{E}$ is a locally free coherent sheaf of rank $r+1$ over the smooth quasi-projective variety Y and denote by $p : \mathbf{P}(\mathcal{E}) = \mathbf{Proj}(\mathrm{Sym}(\mathcal{E})) \to Y$ the corresponding projective bundle. Denote by $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ the Serre invertible sheaf and set, as usual, $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(a) = \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)^{\otimes a}$. Recall ([FL, V, Theorem 2.3]) that the Grothendieck ring $\mathrm{K}_0(\mathbf{P}(\mathcal{E}))$ is isomorphic to

$$\mathrm{K}_0(Y)[T]/(T^{r+1} - [\mathcal{E}] \cdot T^r + \cdots + (-1)^{r+1}[\wedge^{r+1}(\mathcal{E})]),$$

with $l = [\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)]$ corresponding to the class of the element T in the quotient. Under this isomorphism, the pull-back $p^* : \mathrm{K}_0(Y) \to \mathrm{K}_0(\mathbf{P}(\mathcal{E}))$ is identified with $a \mapsto a \cdot T^0$. Similarly, for the Chow ring we have ([F1, Theorem 3.3]):

$$\mathrm{CH}^*(\mathbf{P}(\mathcal{E})) \simeq \mathrm{CH}^*(Y)[T]/(T^{r+1} - c_1(\mathcal{E})T^r + \cdots + (-1)^{r+1}c_{r+1}(\mathcal{E}))\,. \tag{4.1}$$

Under this isomorphism, the grading of $\mathrm{CH}^*(\mathbf{P}(\mathcal{E}))$ corresponds to the grading given by setting $\deg(a \cdot T^i) = \deg(a) + i$, for $0 \leq i \leq r$. (Here, the class of T corresponds to the first Chern class of $\mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$.) The pull-back $p^* : \mathrm{CH}^*(Y) \to \mathrm{CH}^*(\mathbf{P}(\mathcal{E}))$ is identified with $a \mapsto a \cdot T^0$. By [F1, Prop. 3.1 (a)], the push forward map $p_* : \mathrm{CH}^*(\mathbf{P}(\mathcal{E})) \to \mathrm{CH}^{*-r}(Y)$ is identified, under (4.1), with $p_*(a \cdot T^r) = a$, $p_*(a \cdot T^i) = 0$ if $0 \leq i \leq r-1$.

Proposition 4.1. *We have $p_*[\mathcal{O}_{\mathbf{P}(\mathcal{E})}] = [\mathcal{O}_Y]$, and $p_*[\mathcal{O}_{\mathbf{P}(\mathcal{E})}(a)] = 0$ if $-r \leq a < 0$.*

Proof. This is well-known; see for example [FL, V §2]. □

Theorem 4.2. *Theorem 2.2 is true for the projective bundle $p : \mathbf{P}(\mathcal{E}) \to Y$, i.e*

$$\frac{T_{n+r}}{T_n} \cdot \mathfrak{CT}_n(p_*[\mathcal{F}], Y) = p_*(\mathfrak{CT}_{n+r}(\mathcal{F}, \mathbf{P}(\mathcal{E}))) \tag{4.2}$$

for all $n \geq 0$.

Proof. This closely follows the proof of Riemann-Roch for elementary projections given in [FL, II §2]: The description above implies that $\mathrm{K}_0(\mathbf{P}(\mathcal{E}))$ is generated as a $\mathrm{K}_0(Y)$-module by the classes $l^a = [\mathcal{O}_{\mathbf{P}(\mathcal{E})}(a)]$ for $a = -r, -r+1, \ldots, -1, 0$. Using Proposition

2.7, we see that it is enough to show the identity (4.2) for these classes. By Proposition 4.1, it is enough to show

$$p_*(\mathfrak{CT}_{n+r}(l^a, \mathbf{P}(\mathcal{E}))) = \begin{cases} 0, & \text{if } -r \leq a < 0 \\ \dfrac{T_{n+r}}{T_n} \cdot \mathfrak{Td}_n(T_Y), & \text{if } a = 0. \end{cases} \tag{4.3}$$

There is an exact sequence ([FL, IV Prop. 3.13])

$$0 \to \mathcal{O}_{\mathbf{P}(\mathcal{E})} \to p^*\mathcal{E}^\vee \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1) \to T_{\mathbf{P}(\mathcal{E})} \to p^*T_Y \to 0\ . \tag{4.4}$$

For simplicity, we will denote $p^*\mathcal{E}^\vee \otimes \mathcal{O}_{\mathbf{P}(\mathcal{E})}(1)$ by $\mathcal{E}^\vee(1)$. The exact sequence (4.4) implies that $c_i(T_{\mathbf{P}(\mathcal{E})}) = c_i(\mathcal{E}^\vee(1) + p^*T_Y)$ for all i. This, together with (2.2) and the projection formula, proves that it is enough to show

$$p_*\left[\sum_{j=0}^{n'} \frac{T_{n'}}{j! \cdot T_{n'-j}} \cdot \mathfrak{s}_j(l^a) \cdot \mathfrak{Td}_{n'-j}(\mathcal{E}^\vee(1))\right] = \begin{cases} 0, & \text{if } -r \leq a < 0, \text{ or } n' \neq r \\ T_r \cdot 1, & \text{if } a = 0, \text{ and } n' = r\,. \end{cases} \tag{4.5}$$

The proof can now be completed as in [FL, II §2]: Indeed, the term in the bracket above is the (integral) n'-th degree term

$$T_{n'} \cdot \left\{ e^{aT} \cdot \prod_{i=1}^{r+1} \frac{T - x_i}{1 - e^{-(T - x_i)}} \right\}_{n'}$$

of the (symmetric) formal power series

$$H_a(T, \{x_i\}) = e^{aT} \cdot \prod_{i=1}^{r+1} \frac{T - x_i}{1 - e^{-(T - x_i)}} \in \mathbf{Q}[[x_1, \dots, x_{r+1}, T]]\,,$$

evaluated at $T = c_1(l)$ and $x_i =$ Chern roots of $\mathcal{E}$. By loc. cit. Lemma 2.3 and its proof, the relation $\prod_{i=0}^{r+1}(T - x_i) = 0$ implies that we can write

$$H_a(T, \{x_i\}) = \sum_{j=0}^{r} f_{j,a}(c_1, \dots, c_{r+1}) \cdot T^j,$$

with $f_{j,a} \in \mathbf{Q}[c_1, \dots, c_{r+1}]$ and $f_{r,a} = 1$ if $a = 0$, $f_{r,a} = 0$ if $-r \leq a < 0$. (Here c_i is the i-th elementary symmetric function of $x_1, \dots, x_{r+1}$.) The desired conclusion now follows using the above description of $p_* : \mathrm{CH}^*(\mathbf{P}(\mathcal{E})) \to \mathrm{CH}^{*-r}(Y)$. □

4.b. Let $b : \tilde{X} \to X$ be the blow up of the smooth quasi-projective variety X along the smooth subvariety Z. Denote by $i : Z \hookrightarrow X$ the embedding and by $N = N_{Z|X}$ the normal sheaf of Z in X.

We now recall some aspects of the construction of "deformation to the normal cone" ([F1, 5.1], or [FL, IV §5]). Consider the blow-up of $\pi : W \to X \times \mathbf{P}^1$ along $Z \times \{\infty\}$. Let D be the divisor on W which is the preimage $D = \pi^{-1}(X \times \{0\})$ of $X \times \{0\}$; we can identify D with X. The divisor D is linearly equivalent on W to the divisor D' given by the preimage $\pi^{-1}(X \times \{\infty\})$; D' is the sum $\tilde{X} + \mathbf{P}(N^\vee \oplus \mathcal{O}_Z)$ of two smooth irreducible components: the projective bundle $\mathbf{P}(N^\vee \oplus \mathcal{O}_Z)$ over Z and the blow-up $\tilde{X}$. The scheme theoretic intersection $\tilde{X} \cap \mathbf{P}(N^\vee \oplus \mathcal{O}_Z)$ is the (smooth) projective bundle $\mathbf{P}(N^\vee)$ over Z (this is the exceptional locus of the blow-up $\tilde{X} \to X$).

Using Proposition 3.1 (a) and (b), we obtain

$$\begin{aligned}(4.6)\quad & (i_X)_*(\mathfrak{Td}_m(X)) = \mathfrak{Td}_{m+1}(D;W) = \mathfrak{Td}_{m+1}(D';W) = \\ &= (i_{\tilde{X}})_*\mathfrak{Td}_m(\tilde{X}) + (i_{\mathbf{P}(N^\vee\oplus\mathcal{O}_Z)})_*\mathfrak{Td}_m(\mathbf{P}(N^\vee \oplus \mathcal{O}_Z)) - \frac{T_m}{T_{m-1}} \cdot (i_{\mathbf{P}(N^\vee)})_*\mathfrak{Td}_{m-1}(\mathbf{P}(N^\vee))\end{aligned}$$

in $\mathrm{CH}^{m+1}(W)$. Consider the composition $q := pr_1 \cdot \pi : W \to X \times \mathbf{P}^1 \to X$. Observe that $q \cdot i_X = id_X$, $q \cdot i_{\tilde{X}} = b$. Also $q \cdot i_{\mathbf{P}(N^\vee\oplus\mathcal{O}_Z)}$, $q \cdot i_{\mathbf{P}(N^\vee)}$, are the compositions of the projective bundles $\mathbf{P}(N^\vee \oplus \mathcal{O}_Z) \to Z$, resp. $\mathbf{P}(N^\vee) \to Z$, with $i_Z : Z \hookrightarrow X$. Now apply the push-forward homomorphism q_* to the identity (4.6). Using Proposition 4.1, Theorem 4.2 for $\mathcal{F} =$ the structure sheaf, and the above observations, we obtain

$$\mathfrak{Td}_m(X) = b_*(\mathfrak{Td}_m(\tilde{X})) \tag{4.7}$$

in $\mathrm{CH}^m(X)$, for all $0 \le m \le \dim(X)$.

A similar argument, using Proposition 3.2 (a) and Proposition 4.1, gives

$$[\mathcal{O}_X] = b_*[\mathcal{O}_{\tilde{X}}]\,. \tag{4.8}$$

(This also follows from [SGA6, VII, Prop. 3.6].)

4.c. Let $i : Z \hookrightarrow X$ be as above. Set $r = -d_i = \dim(X) - \dim(Z)$.

Theorem 4.3. *Given $n \ge 0$, and $\mathcal{F}$ a locally free coherent $\mathcal{O}_Z$-sheaf on Z, we have*

$$\frac{T_n}{T_{n-r}} \cdot i_*(\mathfrak{CT}_{n-r}(\mathcal{F}, Z)) = \mathfrak{CT}_n(i_*[\mathcal{F}], X)\,. \tag{4.9}$$

Proof. This can be deduced from the Riemann-Roch theorem "without denominators" for regular immersions ([J]). Here we give a direct argument using the technique of deformation to the normal cone. We have the formal identity

$$\sum_{j=0}^{r}(-1)^j \sum_{i_1<\cdots<i_j} e^{-x_{i_1}-\cdots-x_{i_j}} = \prod_{i=1}^{r}(1-e^{-x_i}) = x_1\cdots x_r \cdot \prod_{i=1}^{r}\frac{1-e^{-x_i}}{x_i}\,. \tag{4.10}$$

The degree m part of the left hand side is zero when $m < r$ and equal to

$$x_1\cdots x_r \cdot \left\{\prod_{i=1}^{r}\frac{1-e^{-x_i}}{x_i}\right\}_{m-r}$$

if $m \ge r$. The denominator of the degree m part of the left hand side of (4.10) divides $m!$ and so

$$\mathfrak{Td}^{inv}_{m-r} := m! \cdot \left\{\prod_{i=1}^{r}\frac{1-e^{-x_i}}{x_i}\right\}_{m-r}$$

is a symmetric homogeneous polynomial with integral coefficients. As such, it can be expressed as an integral polynomial in the elementary symmetric functions $c_1, \ldots, c_r$ of the variables $x_1, \ldots, x_r$. Denote by $\mathfrak{Td}^{inv}_{m-r}(\mathcal{G})$ the result of evaluating $\mathfrak{Td}^{inv}_{m-r}$ at the Chern classes $c_i = c_i(\mathcal{G})$.

Proposition 4.4. *Recall that N is the normal sheaf of Z in X. We have*

$$\mathfrak{s}_m(i_*[\mathcal{F}]) = \sum_{l=r}^{m} \frac{m!}{(m-l)! \cdot l!} \cdot i_*(\mathfrak{s}_{m-l}(\mathcal{F}) \cdot \mathfrak{Td}^{inv}_{l-r}(N)) \tag{4.11}$$

in $\mathrm{CH}^m(X)$, *for all* $m \geq r$, *while* $\mathfrak{s}_m(i_*[\mathcal{F}]) = 0$ *for* $m < r$.

Proof. This follows the proof of Riemann-Roch theorem for regular immersions in [F1, §15.2] mutatis-mutandis. As in loc. cit. p. 287-288, we see that it is enough to prove the statement in the model situation in which $X = \mathbf{P}(N^\vee \oplus \mathcal{O}_Z)$ and $i : Z \hookrightarrow \mathbf{V}(N) = \mathbf{Spec}(\mathrm{Sym}(N^\vee)) \subset \mathbf{P}(N^\vee \oplus \mathcal{O}_Z) = X$, where $Z \hookrightarrow \mathbf{V}(N)$ is the zero section of the bundle $\mathbf{V}(N) \to Z$. The argument for showing (4.11) in this case, is similar to the argument in loc. cit., p. 282-283: Let $p : X = \mathbf{P}(N^\vee \oplus \mathcal{O}_Z) \to Z$ be the projection, let Q be the universal kernel sheaf on $\mathbf{P}(N^\vee \oplus \mathcal{O}_Z)$ and let s be the section of $\mathbf{V}(Q^\vee)$ determined by the projection of Q to the trivial factor of $p^*(N^\vee \oplus \mathcal{O}_Z)$. The (scheme) theoretic zero locus of s is Z. Since we have $\mathcal{F} \simeq i^*(p^*\mathcal{F})$, by using the projection formula and (2.1), we see that it is enough to show the claim for $\mathcal{F} = \mathcal{O}_Z$. The coherent sheaf $i_*\mathcal{O}_Z$ on X has a Koszul resolution

$$0 \to \wedge^r Q \to \cdots \to Q \xrightarrow{s^*} \mathcal{O}_X \to i_*\mathcal{O}_Z \to 0\,. \tag{4.12}$$

This, together with the identity (4.10), now gives

$$\mathfrak{s}_m(i_*[\mathcal{O}_Z]) = \sum_{j=0}^{r} (-1)^j \mathfrak{s}_m(\wedge^j Q) = \begin{cases} c_r(Q) \cdot \mathfrak{Td}^{inv}_{m-r}(Q), & \text{if } m \geq r, \\ 0, & \text{if } m < r. \end{cases} \tag{4.13}$$

Observe that $c_r(Q)$ is represented by the zero locus Z of the section $s : X \to \mathbf{V}(Q^\vee)$. Therefore, we have

$$c_r(Q) \cdot \mathfrak{Td}^{inv}_{m-r}(Q) = i_*(\mathfrak{Td}^{inv}_{m-r}(i^*Q)) \tag{4.14}$$

in $\mathrm{CH}^m(X)$. By our construction, i^*Q is isomorphic to N and our claim for $\mathcal{F} = \mathcal{O}_Z$ and $i : Z \to X = \mathbf{P}(N^\vee \oplus \mathcal{O}_Z)$ follows from (4.13) and (4.14). By the above discussion, the proof of the proposition also follows. □

Let us now complete the proof of Theorem 4.3. Observe that $[N] + [T_Z] = [i^*T_X]$ in $\mathrm{K}_0(Z)$. This, together with the definition of $\mathfrak{Td}^{inv}_l(N)$ via the inverse of the Todd power series, implies

$$\frac{T_m}{T_{m-r}} \cdot \mathfrak{Td}_{m-r}(T_Z) = \sum_{j=0}^{m-r} \frac{T_m}{(j+r)! \cdot T_{m-r-j}} \cdot \mathfrak{Td}^{inv}_j(N) \cdot \mathfrak{Td}_{m-r-j}(i^*T_X)\,. \tag{4.15}$$

The identity (4.9) of Theorem 4.3 now follows from Proposition 4.4 by using (4.15) and the projection formula. □

4.d. We continue with the assumptions and notations of the previous paragraphs. In particular, $b : \tilde{X} \to X$ is the blow-up of X along Z.

Theorem 4.5. *Let $\mathcal{F}$ be a locally free coherent $\mathcal{O}_{\tilde{X}}$-sheaf on $\tilde{X}$. Then for $n \geq 0$ we have*

$$b_*(\mathfrak{CT}_n(\mathcal{F}, \tilde{X})) = \mathfrak{CT}_n(b_*[\mathcal{F}], X) \tag{4.16}$$

in $\mathrm{CH}^n(X)$.

Proof. We have the (commutative) blow-up diagram

$$\begin{array}{ccc} \tilde{Z} = \mathbf{P}(N^\vee) & \xrightarrow{j} & \tilde{X} \\ q \downarrow & & \downarrow b \\ Z & \xrightarrow{i} & X\,. \end{array} \tag{4.17}$$

It follows from (4.7), (4.8) and Proposition 2.7 that (4.16) is true for $\mathcal{F}$ of the form $b^*\mathcal{G}$, with $\mathcal{G}$ a locally free coherent $\mathcal{O}_X$-sheaf on X. By [SGA6, VII, Th. 3.7], each element of $\mathrm{K}_0(\tilde{X}) = \mathrm{G}_0(\tilde{X})$ can be written in the form $a = b^*(a') + j_*(z)$ with $a' \in \mathrm{K}_0(X)$, $z \in \mathrm{K}_0(\tilde{Z})$. Since both sides of (4.16) are additive, it remains to prove the equality (4.16) for $\mathcal{F} = j_*\mathcal{H}$, where $\mathcal{H}$ is a locally free coherent $\mathcal{O}_{\tilde{Z}}$-sheaf on $\tilde{Z}$. Using $b_* \cdot j_* = (b \cdot j)_* = (i \cdot q)_* = i_* \cdot q_*$, Theorem 4.3 for the embeddings j and i, and Theorem 4.2 for $q : \tilde{Z} = \mathbf{P}(N^\vee) \to Z$, we obtain

$$\begin{aligned} b_*(\mathfrak{CT}_n(j_*\mathcal{H}, \tilde{X})) = b_*\left(\frac{T_n}{T_{n-1}} \cdot j_*(\mathfrak{CT}_{n-1}(\mathcal{H}, \tilde{Z}))\right) &= \\ = \frac{T_n}{T_{n-1}} \cdot i_*\left(q_*(\mathfrak{CT}_{n-1}(\mathcal{H}, \tilde{Z}))\right) &= \\ = \frac{T_n}{T_{n-1}} \cdot i_*\left(\frac{T_{n-1}}{T_{n-1-(r-1)}} \cdot \mathfrak{CT}_{n-1-(r-1)}(q_*[\mathcal{H}], Z)\right) &= \\ = \frac{T_n}{T_{n-r}} \cdot i_*\left(\mathfrak{CT}_{n-r}(q_*[\mathcal{H}], Z)\right) &= \\ = \mathfrak{CT}_n(i_*q_*[\mathcal{H}], X) = \mathfrak{CT}_n(b_*[j_*\mathcal{H}], X). & \end{aligned}$$

□

5. Factorization

Let $f : Y \to S$ and $f' : Y' \to S$ be projective morphisms between smooth quasi-projective varieties. We will say that f, f' *are birationally isomorphic over* S if the following is true: We have $f(Y) = f'(Y')$ and there is an isomorphism of the function fields $a : k(Y') \xrightarrow{\sim} k(Y)$ which commutes with the $k(f(Y)) = k(f'(Y'))$-algebra structures given by f', f.

Now assume that f, f' are birationally isomorphic over S and let us write $\phi : Y - \to Y'$ for the corresponding birational map. Let $U \subset Y$ be the largest open subscheme of Y such that $\phi_{|U} : U \to \phi(U)$ is an isomorphism; in what follows, we will implicitly identify U and $\phi(U)$.

Theorem 5.1. *([AKMW]) There is a finite sequence of birational maps between smooth quasi-projective varieties*

$$Y = Y_0 - \xrightarrow{\phi_1} Y_1 - \xrightarrow{\phi_2} \cdots - \xrightarrow{\phi_{n-1}} Y_{n-1} - \xrightarrow{\phi_n} Y_n = Y'$$

over k such that:

(a) $\phi = \phi_n \cdot \cdots \cdot \phi_2 \cdot \phi_1$,
(b) *the ϕ_i's are isomorphisms on U,*
(c) *for each i, either $\phi_i : Y_{i-1} - \to Y_i$ or $\phi_i^{-1} : Y_i - \to Y_{i-1}$ is obtained by blowing up a nonsingular subscheme disjoint from U,*
(d) *there is an index i_0 such that for all $i \leq i_0$, resp. $i \geq i_0$, the birational map $\phi(i) := \phi_1^{-1} \cdot \cdots \cdot \phi_2^{-1} \cdot \phi_i^{-1} : Y_i - \to Y_0 = Y$, resp. $\phi(i) := \phi_i \cdot \phi_{i+1} \cdot \cdots \cdot \phi_n : Y_i - \to Y_n = Y'$, is a projective morphism,*
(e) *the varieties Y_i support projective morphisms $f_i : Y_i \to S$ such that, for each i, the blow-up morphism $\phi_i : Y_{i-1} - \to Y_i$ or $\phi_i^{-1} : Y_i - \to Y_{i-1}$ given by item (c) is a morphism over S (i.e commutes with f_i and f_{i-1}).*

Proof. This follows from the "functorial weak factorization theorem" (Theorem 0.3.1, cf. Remark (1)) of [AKMW]. The result in loc. cit. gives varieties Y_i and birational maps ϕ_i with the properties (a)-(d). It remains to observe that item (e) follows from (a)-(d). Indeed, we can define an S-structure on Y_i as follows, using (d): If $i \leq i_0$, then we set $f_i = f \cdot \phi(i)$. If $i \geq i_0$, then we set $f_i = f' \cdot \phi(i)$. We can see that these f_i satisfy the requirements of item (e). □

Corollary 5.2. *If f, f' are birationally isomorphic over S, then integral Riemann-Roch holds for (f, n) if and only if it holds for (f', n).*

Proof. Using Theorem 5.1, we see that it is enough to show the statement under the additional assumption that the birational map $\phi : Y' \to Y$ is obtained by blowing-up Y along a smooth center: By Theorem 4.5, integral Riemann-Roch holds for ϕ. If integral Riemann-Roch holds for (f, n) then, by Proposition 2.5, integral Riemann-Roch holds for $(f \cdot \phi, n) = (f', n)$. To show the converse, let $\mathcal{F}$ be a locally free coherent $\mathcal{O}_Y$-sheaf on Y. We will show that the integral Riemann-Roch identity for $\phi^*\mathcal{F}$ and (f', n) implies the integral Riemann-Roch identity for $\mathcal{F}$ and (f, n). Suppose first $d = d_f = d_{f'} \geq 0$. Since $f'_* = f_* \cdot \phi_*$, we obtain:

$$\frac{T_{n+d}}{T_n} \cdot \mathfrak{CT}_n(f_*\phi_*[\phi^*\mathcal{F}], S) = f_*\phi_*(\mathfrak{CT}_{n+d}(\phi^*\mathcal{F}, Y')).$$

However, we have $\phi_*[\phi^*\mathcal{F}] = [\mathcal{F}]$ by [SGA6, VII Prop. 3.6.], while $\phi_*(\mathfrak{CT}_{n+d}(\phi^*\mathcal{F}, Y')) = \mathfrak{CT}_{n+d}(\phi_*[\phi^*\mathcal{F}], Y)$ by Theorem 4.5. Therefore, we obtain that integral Riemann-Roch holds for $\mathcal{F}$ and (f, n). The argument for $d < 0$ is similar. □

6. Completion of the proof

We will now complete the proof of Theorem 2.2 by induction on the degree n.

When $n = 0$, since $\mathrm{CH}^0(S) = \mathbf{Z}$ is torsion-free, Theorem 2.2 follows from the standard Grothendieck-Riemann-Roch theorem. (More directly, we can deduce this case from Theorems 4.2 and 4.3 as in Grothendieck's proof.) We suppose that Theorem 2.2 is known for all f, and for all degrees $n' < n$.

6.a. Let $f : X \to S$ be a *non-dominant* projective morphism between the smooth quasi-projective varieties X and S. We will show that integral Riemann-Roch for (f, n) follows from our inductive hypothesis.

Denote by T the image $f(X)$; this is a subvariety of S of codimension $r > 0$. By embedded resolution of singularities ([H]), we can find a projective birational morphism $b : \tilde{S} \to S$, which is obtained as a succession of blow-ups along smooth centers, so that the strict transform $\tilde{T} \subset \tilde{S}$ of T is smooth. Consider now the base change $X \times_T \tilde{T}$ of $X \to T$ along $b : \tilde{T} \to T$. By applying resolution of singularities to $X \times_T \tilde{T}$, we can find a projective birational morphism $\phi : \tilde{X} \to X \times_T \tilde{T} \to X$ such that $\tilde{X}$ is smooth. Thus, we obtain a commutative diagram

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\tilde{g}} & \tilde{T} & \xrightarrow{\tilde{i}} & \tilde{S} \\ \phi\downarrow & & \downarrow b & & \downarrow b \\ X & \xrightarrow{g} & T & \xrightarrow{i} & S \end{array}$$

with $i \cdot g = f$. Set $\tilde{f} = \tilde{i} \cdot \tilde{g}$. By Theorem 4.3, integral Riemann-Roch holds for $(\tilde{i}, n)$. By our induction hypothesis, it also holds for $(\tilde{g}, n - r)$. Proposition 2.5 applied to the composition $\tilde{f} = \tilde{i} \cdot \tilde{g}$ allows us to conclude that integral Riemann-Roch holds for $(\tilde{f}, n)$. Theorem 4.5 and Proposition 2.5 applied to the composition $b \cdot \tilde{f}$ now imply that integral Riemann-Roch also holds for $(b \cdot \tilde{f}, n)$. Since $f \cdot \phi = b \cdot \tilde{f}$, the morphisms $b \cdot \tilde{f} : \tilde{X} \to S$ and $f : X \to S$ are birationally isomorphic over S via ϕ. Corollary 5.2 now shows that integral Riemann-Roch for (f, n) follows.

6.b. We will now deduce integral Riemann-Roch for (f, n) from our inductive hypothesis on n, by using induction on the relative dimension $d = d_f$. If $d < 0$, then f is not dominant and the result follows by the previous paragraph. Hence, we can assume that $d \geq 0$. We now suppose that integral Riemann-Roch for (g, n) holds for all g of relative dimension $d_g < d$; recall that we also assume that integral Riemann-Roch for n' holds for all morphisms, provided that $n' < n$.

For a fixed smooth quasi-projective variety S over k, let $\mathcal{S}_d$ be the set of S-isomorphism classes $[f]$ of projective k-morphisms $f : X \to S$ of relative dimension $d \geq 0$ with X smooth over k. We will first show that integral Riemann-Roch holds for (f, n) and the structure sheaf $\mathcal{F} = \mathcal{O}_X$, for all such f. In other words, we will show that the "error" function $E : \mathcal{S}_d \to \mathrm{CH}^n(S)$ given by

$$E([f : X \to S]) = \frac{T_{d+n}}{T_n} \cdot \mathfrak{C}\mathfrak{T}_n(f_*[\mathcal{O}_X], S) - f_*(\mathfrak{T}\mathfrak{d}_{d+n}(X)) \tag{6.1}$$

vanishes. The induction hypothesis and our results in the previous sections give:

(i) If $f : X \to S$ is S-isomorphic to a composition of the form $X = \mathbf{P}(\mathcal{E}) \to Y \to S$, with $\mathbf{P}(\mathcal{E}) \to Y$ a projective bundle and $\dim(Y) < \dim(X)$, then $E([f : X \to S]) = 0$. (This follows from the induction hypothesis, Proposition 4.1 and Theorem 4.2.)

(ii) If Z is a smooth subvariety of X,

$$E([f : X \to S]) = E([\tilde{f} : \tilde{X} \to S]),$$

with $\tilde{X}$ the blow-up of X along Z. (This follows from (4.7) and (4.8).)

In fact, Theorem 5.1 allows us to strengthen (ii) (cf. proof of Corollary 5.2):

(ii)$'$ If $f : X \to S$ and $f' : X' \to S$ are birationally isomorphic over S then

$$E([f : X \to S]) = E([f' : X' \to S]).$$

(iii) Suppose that $g : W \to S$ is in $\mathcal{S}_{d+1}$ and restrict E to the smooth codimension 1 closed subschemes of W, i.e to $f = g \cdot i$ with $i : X \hookrightarrow W$ of codimension 1. Proposition 3.1 (a) implies that we can extend E to Weil divisors by defining

$$E^g(D) = \frac{T_{d+n}}{T_n} \cdot \mathfrak{C}\mathfrak{T}_n(g_*([\mathcal{O}_W] - [\mathcal{O}_W(-D)]), S) - g_*(\mathfrak{T}\mathfrak{d}_{d+1+n}(D; W)). \tag{6.2}$$

We can see that E^g respects linear equivalence. Using (3.12) and (3.13) for $m = d+n+1$ (including the fact that $b_Z = a_Z$) and our induction hypothesis, we see that

$$E^g : \mathrm{Pic}(W) \to \mathrm{CH}^n(S)$$

is a group homomorphism.

(iv) If f is *not* dominant then $E([f : X \to S]) = 0$.

6.c. We continue with our inductive proof. The argument here was inspired by the calculation of the cobordism ring of a point by Levine and Morel [LM, Theorem 4.3.7]. Let $f : X \to S$ be a projective morphism as above which is dominant. Then the generic fiber X_K of f is a smooth projective variety over $K = k(S)$ of dimension d. Using the primitive element theorem (recall that $\mathrm{char}(k) = 0$), we see that X_K is birationally isomorphic to a closed irreducible hypersurface $Y \subset \mathbf{P}^{d+1}_K$. Denote by $\hat{Y}$ the Zariski closure of Y in $\mathbf{P}^{d+1}_S = \mathbf{P}^{d+1}_k \times_k S$; this is a divisor in $\mathbf{P}^{d+1}_S$ and affords a dominant projective morphism $\hat{Y} \to S$. For simplicity of notation, we set $P = \mathbf{P}^{d+1}_k \times_k S$. There is an integer $m > 0$ and a line bundle $\mathcal{L}$ on S such that

$$\mathcal{O}_P(\hat{Y}) \simeq \mathrm{pr}_1^*(\mathcal{O}_{\mathbf{P}^{d+1}_k}(m)) \otimes_{\mathcal{O}_P} \mathrm{pr}_2^*(\mathcal{L}).$$

If $d = 0$, take $H \subset \mathbf{P}^1_k$ to be the union of m distinct points. If $d > 0$, take $H \subset \mathbf{P}^{d+1}_k$ to be a smooth irreducible hypersurface of degree m. Set $D = H \times_k S \subset P$ for the corresponding smooth "horizontal" divisor in P. We can assume that D intersects $\hat{Y}$ properly. Using Bertini's theorem, we can write $\mathcal{L} \simeq \mathcal{O}_S(T_1 - T_2)$, where T_1 and T_2 are both smooth very ample divisors on S. Set $F_1 = \mathrm{pr}_2^{-1}(T_1)$, $F_2 = \mathrm{pr}_2^{-1}(T_2)$. Then we have

$$\hat{Y} \sim D + F_1 - F_2 \tag{6.3}$$

on P. Let U be an open subset of P which contains the generic points of D, F_1, F_2, $\hat{Y}$, and is such that the divisor $U \cap (D + F_1 + F_2 + \hat{Y})$ has strict normal crossings on U. The work of Hironaka on resolution of singularities now implies:

Theorem 6.1. *There is birational morphism $\beta : \tilde{P} \to P = \mathbf{P}^{d+1} \times_k S$ over S, which is obtained as a succession of blow-ups along smooth centers lying over $P - U$, such that the following is true: Let T denote one of the divisors D, $\hat{Y}$, F_1, F_2. Then for each such choice of T:*

a) the strict transform T' of T in $\tilde{P}$ is smooth,

b) the total transform $\beta^*(T)$ is a divisor with strict normal crossings and we can write

$$\beta^*(T) = T' + \sum_i n_i \cdot Z_i \ , \tag{6.4}$$

where $n_i \in \mathbf{Z}$ and the exceptional divisors Z_i are birationally isomorphic over S to projective bundles over smooth quasi-projective varieties of dimension $< \dim(X)$.

Proof. This can be deduced from [H, Theorem $\mathrm{I}_2^{N,n}$, pg. 170] by taking $N = \dim(P)$, $n = N - 1$, and $(\mathfrak{R}_I^{N,N-1}, U)$ the resolution datum $((|D + F_1 + F_2|; P; \hat{Y}), U)$. (The last statement about the divisors Z_i follows from the fact that, in this desingularization procedure, the blow-up centers are always transverse to the exceptional locus.) □

By construction, $\tilde{P}$ is a smooth variety that supports a projective morphism $g : \tilde{P} \to S$. By (6.3) and Theorem 6.1, we now obtain

$$Y' \sim D' + F_1' - F_2' + \sum_j m_j Z_j \ , \tag{6.5}$$

where the smooth divisors Z_j are birationally isomorphic over S to projective bundles over smooth quasi-projective varieties of dimension $< \dim(X)$.

Now let us use (i), (ii)$'$, (iii), (iv) of §6.b (these hold under our induction hypothesis): Apply the homomorphism $E^g : \mathrm{Pic}(\tilde{P}) \to \mathrm{CH}^n(S)$ to the identity (6.5). Since the Z_j's are smooth and are birationally isomorphic over S to projective bundles over smooth varieties of smaller dimension, (iii), (i), and (ii)$'$, give $E^g(Z_j) = E(Z_j \to S) = 0$. Using (iii) and (iv), we obtain $E^g(F_i') = Err(F_i' \to S) = 0$, for $i = 1, 2$. On the other hand, (ii)$'$ gives $E^g(D') = E(D' \to S) = E(D \to S)$. Thus, the identity (6.5) implies

$$E^g(Y') = E(D \to S)\,. \tag{6.6}$$

Now by its construction, Y' is smooth and birationally isomorphic to X over S. Hence, (iii) and (ii)$'$ imply $E^g(Y') = E(Y' \to S) = E(X \to S)$. Therefore, by (6.6), to show $E(X \to S) = 0$, it is enough to show that $E(D \to S) = 0$. Recall that $D = H \times_k S$. We have $(pr_2)_*[\mathcal{O}_{H\times_k S}] = \chi(H, \mathcal{O}_H) \cdot [\mathcal{O}_S]$ in $\mathrm{K}_0(S)$. On the other hand, we find

$$(pr_2)_*(\mathfrak{T}\mathfrak{d}_{n+d}(H \times_k S)) = \frac{T_{d+n}}{T_d \cdot T_n} \cdot \deg(\mathfrak{T}\mathfrak{d}_d(H)) \cdot \mathfrak{T}\mathfrak{d}_n(S)$$

in $\mathrm{CH}^n(S)$. Therefore, since the Hirzebruch-Riemann-Roch for H gives $\chi(H, \mathcal{O}_H) = \deg(\mathfrak{T}\mathfrak{d}_d(H))/T_d$, we obtain $E(D \to S) = 0$. We conclude that integral Riemann-Roch theorem for $\mathcal{F} = \mathcal{O}_X$ holds for all (f, n) with $d_f = d$.

Now suppose that $\mathcal{F} = \mathcal{L}$ is an invertible sheaf on X. We claim that

$$\frac{T_{d+n}}{T_n} \cdot \mathfrak{C}\mathfrak{T}_n(f_*[\mathcal{L}], S) = f_*(\mathfrak{C}\mathfrak{T}_{d+n}(\mathcal{L}, X))\,. \tag{6.7}$$

Set $\mathcal{L} = \mathcal{O}_X(-F)$, with F a Weil divisor on X. Let us write

$$\begin{aligned} \mathfrak{C}\mathfrak{T}_{n+d}(\mathcal{L}, X) &= \mathfrak{C}\mathfrak{T}_{n+d}(\mathcal{O}_X, X) - \mathfrak{C}\mathfrak{T}_{n+d}([\mathcal{O}_X] - [\mathcal{O}_X(-F)], X) = \\ &= \mathfrak{T}\mathfrak{d}_{n+d}(X) - \frac{T_{n+d}}{T_{n+d-1}} \cdot \mathfrak{T}\mathfrak{d}_{n+d}(F; X) \end{aligned} \tag{6.8}$$

(using (3.1).) Apply Bertini's theorem (Proposition 3.3) to the divisor F on X. Using Proposition 3.1 (c) for $m = n+d$ and (6.8), we can express $\mathfrak{C}\mathfrak{T}_{n+d}(\mathcal{L}, X)$ in terms of integral Todd classes of smooth varieties of dimension $\leq \dim(X)$. Correspondingly, Proposition 3.2 allows us to obtain a similar expression for the class $[\mathcal{L}] = [\mathcal{O}_X(-F)]$. Since, by the above, integral Riemann-Roch holds for the structure sheaf when the relative dimension is $\leq d$, our claim for $\mathcal{F} = \mathcal{L}$ follows by comparing these two expressions.

Finally, suppose that $\mathcal{F}$ is an arbitrary locally free coherent $\mathcal{O}_X$-sheaf on X. By a result of Kleiman [K, Theorem 4.7 (b)], there is a birational morphism $b : \tilde{X} \to X$ which is obtained by successive blow-ups along smooth centers, such that $b^*\mathcal{F}$ has a filtration whose graded pieces are line bundles $\mathcal{L}_i$ on $\tilde{X}$. In particular

$$[b^*\mathcal{F}] = [\mathcal{L}_1] + \cdots + [\mathcal{L}_r]$$

in the Grothendieck group $\mathrm{K}_0(\tilde{X})$. Since both sides of the Riemann-Roch equation are additive, our previous arguments give that the integral Riemann-Roch identity holds for $b^*\mathcal{F}$ and $(f \cdot b, n)$. The argument in the proof of Corollary 5.2 now gives the integral Riemann-Roch identity (2.3) for $\mathcal{F}$ and (f, n). This and our induction, allows us to conclude the proof of Theorem 2.2. $\square$

Remark 6.2. Here we briefly sketch how a part of the argument can be recast using certain constructions in the theory of algebraic cobordism [LM] and in particular the "generalized degree formula". (We follow the notations of §6.b and loc. cit.)

As a first step, one shows that the function $E : \mathcal{S}_d \to \mathrm{CH}^n(S)$ given as in (6.1) gives a group homomorphism $E : \Omega^{-d}(S) \to \mathrm{CH}^n(S)$. In fact, both terms of the difference (6.1) that defines the error function E extend to group homomorphisms from the cobordism group $\Omega^{-d}(S)$: For the first term, this follows using the transformation between cobordism and K-theory ([LM, 4.2]). For the second, this can be seen using the transformation between cobordism and a suitable twisted Chow cohomology theory (see the argument in the construction of the Conner-Floyd Chern classes [LM, 7.3.3], also [LM, 4.4.19]).

Then, by the generalized degree formula [LM, Theorem 1.2.14], we can find for $i = 0, \dots, r$, classes $a_i \in \Omega^{-d_i}(k)$, and for $i = 1, \dots, r$, irreducible subvarieties Z_i of S together with morphisms $\tilde{Z}_i \to S$ from smooth varieties $\tilde{Z}_i$ with image Z_i and $\tilde{Z}_i \to Z_i$ birational, such that

$$[f : X \to S] = a_0 \cdot [\mathrm{id} : S \to S] + \sum_{i=1}^{r} a_i \cdot [\tilde{Z}_i \to S] \tag{6.9}$$

in $\Omega^{-d}(S)$. Here $d_0 = d$ and a_0 is the degree of $[f]$ in the terminology of loc. cit. In addition, by using embedded resolution for $Z_i \subset S$, the argument in the proof of [LM, Theorem 1.2.14] allows us to choose $\tilde{Z}_i \to S$ that factor as a composition $\tilde{Z}_i \hookrightarrow \tilde{S}_i \to S$ where $\tilde{S}_i$ is obtained by a successive blow-up of S along smooth centers. We can also assume that for each i, the class a_i is an integral linear combination of the classes of smooth projective varieties $Y_{ij} \to \mathrm{Spec}\,(k)$. Hence, (6.9) implies that integral Riemann-Roch for f and the structure sheaf will follow from integral Riemann-Roch (for the structure sheaf) for $Y_{0j} \times S \to S$ and for the compositions

$$Y_{ij} \times \tilde{Z}_i \xrightarrow{\mathrm{pr}} \tilde{Z}_i \hookrightarrow \tilde{S}_i \to S$$

with $i = 1, \dots, r$. Integral Riemann-Roch for a projection $Y \times Z \xrightarrow{\mathrm{pr}} Z$ now follows (as before) from Hirzebruch-Riemann-Roch for $Y \to \mathrm{Spec}\,(k)$ and the projection formula. On the other hand, integral Riemann-Roch for $\tilde{Z}_i \hookrightarrow \tilde{S}_i$ and $\tilde{S}_i \to S$ can be shown using Theorem 4.3, and Theorem 4.5 and Proposition 2.5 respectively. Integral Riemann-Roch for (f, n) and the structure sheaf now follows from the above and Proposition 2.5.

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Dept. of Mathematics, Michigan State University, E. Lansing, MI 48824-1027, USA,
email: pappas@math.msu.edu