# A model for the $\mathbb{E}_3$ fusion-convolution product of constructible sheaves on the affine Grassmannian

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September 29, 2021

#### Abstract

In this paper we provide a detailed construction of an associative and braided convolution product on the category of equivariant constructible sheaves on the affine Grassmannian through derived geometry. This product extends the convolution product on equivariant perverse sheaves and is constructed as an  $\mathbb{E}_3$ -algebra object in  $\infty$ -categories. The main tools amount to a formulation of the convolution and fusion procedures over the Ran space involving the formalism of 2-Segal objects and correspondences from [DK19] and [GR17]; of factorising constructible cosheaves over the Ran space from Lurie, [Lur17, Chapter 5]; and of constructible sheaves via exit paths.

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 $<sup>2020\ \</sup>textit{Mathematics subject classification}.\ \text{Primary 14D24, 57N80; Secondary 18N70, 22E57, 32S60}$ 

Key words and phrases. Affine Grassmannian, Geometric Satake Theorem, constructible sheaves, exit paths, little cubes operads, symmetric monoidal  $\infty$ -categories, stratified spaces, correspondences.

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# 1 Introduction

The content of the present paper is the following: we will provide an extension of the convolution product of equivariant perverse sheaves on the affine Grassmannian, whose definition will be recalled in the first subsection of this Introduction, to the  $\infty$ -category of equivariant constructible sheaves on the affine Grassmannian. We will endow this extension with an  $\mathbb{E}_3$ -algebra structure in  $\infty$ -categories, which is the avatar of Mirkovic and Vilonen's commutativity constraint [MV07, Section 5]. The final result is Theorem 5.9.

**Theorem 1.1** (Theorem 5.9). If G is a reductive complex group and k is a torsion ring, there is an object  $A \in Alg_{\mathbb{E}_3}(\operatorname{Pr}_k^{L,\otimes})$  describing an associative and braided product law on the presentable k-linear  $\infty$ -category  $\operatorname{Cons}_{G_{\mathbb{O}}}(\operatorname{Gr}_G, k)$  of  $G_{\mathbb{O}}$ -equivariant constructible sheaves over the affine Grassmannian (see Section 1.1.3). The restriction of this product law to the abelian category of equivariant perverse sheaves coincides up to shifts with the classical (commutative) convolution product of perverse sheaves [MV07].

#### 1.1 The affine Grassmannian and the Geometric Satake Theorem

#### 1.1.1 Statement of the Geometric Satake Theorem

In order to clarify the aim of the present work, for reasons that will be explained in Section 1.2, we present a short overview of the statement of the Geometric Satake Theorem. The expert reader can skip directly to Section 1.2.

We begin by recalling the statement of the Geometric Satake correspondence in the form proven by I. Mirkovic and K. Vilonen in [MV07]. An excellent survey on the matter is [Zhu16].

**Theorem 1.2** (Geometric Satake Equivalence). Fix a reductive algebraic group G over  $\mathbb{C}$ , and a commutative ring k for which the left-hand-side of the following formula is defined: for example, k could be  $\mathbb{Q}_{\ell}, \mathbb{Z}_{\ell}, \mathbb{Z}/\ell^n \mathbb{Z}, \overline{\mathbb{F}_{\ell}}$ . There is an equivalence of tensor categories

$$\operatorname{Perv}_{G_{\mathfrak{O}}}(\operatorname{Gr}_{G}, k) = \operatorname{Rep}_{\check{G}}^{\operatorname{fin}}(k).$$

Let us explain the meaning of this statement. Here  $\check{G}$  is the Langlands dual of G, obtained by dualising the root datum of the original group G, and the right-hand-side  $\operatorname{Rep}_{\check{G}}^{\operatorname{fin}}(R)$  is the abelian category of finite-dimensional R-representations of  $\check{G}$ , equipped with a tensor (i.e. symmetric monoidal) structure given by the tensor product of representations.

In order to define the left hand side, we need to introduce some further definitions: by  $G_{\mathbb{O}}$  we mean the representable functor  $\mathbb{C}$ -algebras  $\to$  Set,  $R \mapsto \operatorname{Hom}(R[t], G)$  (also denoted by  $G(\mathbb{C}[t])$ ), by  $G_{\mathcal{K}}$  we mean the ind-representable functor  $\mathbb{C}$ -algebras  $\to$  Set,  $R \mapsto \operatorname{Hom}(R((t)), G)$  (also denoted by  $G(\mathbb{C}((t)))$ ), and by  $\operatorname{Gr}_G$  we mean the **affine Grassmannian**, that is the stack quotient  $\operatorname{Gr}_G = G(\mathbb{C}((t)))/G(\mathbb{C}[t])$ . Ind-representability of  $G_{\mathcal{K}}$  (and of Gr by consequence) comes from the fact that there is a natural filtration in finite-dimensional projective schemes  $\operatorname{Gr}_{\leq N}, N \geq 0$ , induced by [Zhu16, Theorem 1.1.3]. We will call this filtration the **lattice filtration**.

**Remark 1.3.** The ind-scheme  $Gr_G$  has a natural stratification in Schubert cells, see [Zhu16, 2.1]. Also, there is an action of  $G_{\mathcal{O}}$  on  $Gr_G$  by left multiplication, and the orbits are precisely the strata of the stratification. The category  $\mathcal{P}erv_{G_{\mathcal{O}}}(Gr_G, k)$  is the abelian category of  $G_{\mathcal{O}}$ -equivariant perverse sheaves on  $Gr_G$  with values in k-modules. This category is defined as

$$\operatorname*{colim}_{N} \operatorname{\mathcal{P}erv}_{G_{\mathfrak{O}}}(\operatorname{Gr}_{\geq N}, k)$$

in the sense of [Zhu16, 5.1 and A.1.4].

The ind-scheme  $\operatorname{Gr}_G$  is related to the theory of curves in the following sense: given a smooth projective complex curve X, the formal neighborhood  $\widehat{X}_x$  at a given closed point x of X is represented by a map  $\phi_x : \operatorname{Spec} \mathbb{C}[\![t]\!] \to X$ . The inclusion  $\mathbb{C}[\![t]\!] \subset \mathbb{C}((t))$  induces a map  $\operatorname{Spec} \mathbb{C}((t)) \to \operatorname{Spec} \mathbb{C}[\![t]\!] \xrightarrow{\phi_x} X$  which is a model for the punctured formal neighbourhood  $\mathring{X}_x$  of x.

We will now recall some basic properties of the affine Grassmannian.

**Definition 1.4.** Let  $\mathbf{Bun}_G$  be the moduli stack of principal G-bundles. If a scheme Z over  $\mathbb{C}$  is given, we define the relative version

$$\mathbf{Bun}_G^Z: \mathbb{C}\mathrm{Alg} \to \mathrm{Grpd}$$

 $R \mapsto \{ \text{principal } G\text{-bundles over } X \times \operatorname{Spec} R, \text{ flat over } \operatorname{Spec} R \}.$ 

In the language of mapping stacks, we can write

$$\mathbf{Bun}_G^Z \simeq \mathbf{Map}_{\mathrm{Stacks}}(Z, \mathbf{Bun}_G).$$

**Proposition 1.5.** For any closed point x of a smooth projective complex curve X, the functor  $Gr_G$  is equivalent to the following:

$$\operatorname{Gr}_{G}^{\operatorname{loc}}: R \to \{ \mathfrak{F} \in \operatorname{Vect}_{n}(\widehat{X}_{x} \times \operatorname{Spec} R), \alpha: \mathfrak{F}|_{\mathring{X}_{x} \times \operatorname{Spec} R} \xrightarrow{\sim} \mathfrak{I}_{R}|_{\mathring{X}_{x} \times \operatorname{Spec} R} \}$$
 (1.1)

where  $\mathfrak{T}_R$  is the trivial G-torsor on  $\widehat{X}_x \times \operatorname{Spec} R$ . In other words,  $\operatorname{Gr}_G$  is equivalent to the fiber at the trivial bundle of the functor  $\operatorname{\mathbf{Bun}}_G^{\widehat{X}_x} \to \operatorname{\mathbf{Bun}}_G^{\widehat{X}_x}$ .

*Proof.* The proof is explained for instance in [Zhu16, Proposition 1.3.6]. 
$$\Box$$

We will need the following important version of the affine Grassmannian as well.

Construction 1.6. Define  $\operatorname{Gr}_G^{\operatorname{glob}}$  as the fiber of the restriction map  $\operatorname{\mathbf{Bun}}_G^X \to \operatorname{\mathbf{Bun}}_G^{X\setminus \{x\}}$ , i.e. as the stack

$$R \to \{\mathfrak{F}, \alpha: \mathfrak{F}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \xrightarrow{\sim} \mathfrak{T}_R|_{(X \setminus \{x\}) \times \operatorname{Spec} R}\}.$$

Indeed, in the diagram

the right-hand square is cartesian by the so-called Formal Gluing Theorem ([HPV16]), extending the theorem of Beauville and Laszlo [BL95]. Since the left-hand square is cartesian by definition, the outer square is cartesian. Therefore,  $\operatorname{Gr}_G^{\operatorname{glob}}$  is isomorphic to the fiber of the restriction map  $\operatorname{\mathbf{Bun}}_{G}^{\widehat{X_x}} \to \operatorname{\mathbf{Bun}}_{G}^{\widehat{X_x}}$ , which is exactly  $\operatorname{Gr}_G^{\operatorname{loc}}$ . For more details, see [Zhu16, Theorem 1.4.2].

## 1.1.2 Convolution product of equivariant perverse sheaves

Now we explain the tensor structure on both sides. The category  $\operatorname{Rep}_{\check{G}}$  is equipped with the standard tensor product of categories; we define now the tensor structure given by **convolution product** on  $\operatorname{Perv}_{G_{\mathcal{O}}}(\operatorname{Gr}_{G})$ . A more detailed account is given in [Zhu16, Section 1, Section 5.1, 5.4]. Consider the diagram

$$G_{\mathcal{K}} \times \operatorname{Gr}_{G} \xrightarrow{q} G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \operatorname{Gr}_{G}$$

$$\operatorname{Gr}_{G} \times \operatorname{Gr}_{G}$$

$$(1.2)$$

where  $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \operatorname{Gr}_{G}$  is the stack quotient of the product  $G_{\mathcal{K}} \times \operatorname{Gr}_{G}$  with respect to the "anti-diagonal" left action of  $G_{\mathcal{O}}$  defined by  $\gamma \cdot (g, [h]) = (g\gamma^{-1}, [\gamma h])$ . The map p is the projection to the quotient on the first factor and the identity on the second one, the map q is the projection to the quotient by the "anti-diagonal" action of  $G_{\mathcal{O}}$ , and the map m is the multiplication map  $(g, [h]) \mapsto [gh]$ . It is important to remark that this construction, like everything else in this section, does not depend on the chosen  $x \in X(\mathbb{C})$ , since the formal neighbourhoods of closed points in a smooth projective complex curve are all (noncanonically) isomorphic.

Note also that the left multiplication action of  $G_{\mathcal{O}}$  on  $G_{\mathcal{K}}$  and on  $\operatorname{Gr}_{G}$  induces a left action of  $G_{\mathcal{O}} \times G_{\mathcal{O}}$  on  $\operatorname{Gr}_{G} \times \operatorname{Gr}_{G}$ . It also induces an action of  $G_{\mathcal{O}}$  on  $G_{\mathcal{K}} \times \operatorname{Gr}_{G}$  given by (left multiplication, id) which canonically projects to an action of  $G_{\mathcal{O}}$  on  $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \operatorname{Gr}_{G}$ . Note that p, q and m are equivariant with respect to these actions.

Now if  $A_1, A_2$  are two  $G_0$ -equivariant perverse sheaves on  $Gr_G$ , one can define a convolution product

$$\mathcal{A}_1 \star \mathcal{A}_2 = m_* \tilde{\mathcal{A}} \tag{1.3}$$

where  $m_*$  is the derived direct image functor, and  $\tilde{\mathcal{A}}$  is any perverse sheaf on  $G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \operatorname{Gr}_{G}$  which is equivariant with respect to the left action of  $G_{\mathcal{O}}$  and such that  $q^*\tilde{\mathcal{A}} = p^*(\mathcal{A}_1 \boxtimes \mathcal{A}_2)$ . (Of course, the tensor product must be understood as a derived tensor product in the derived category.) Note that such an  $\tilde{\mathcal{A}}$  exists because q is the projection to the quotient and  $\mathcal{A}_2$  is  $G_{\mathcal{O}}$ -equivariant.

This is the tensor structure that we are considering on  $\operatorname{Perv}_{G_{\mathcal{O}}}(\operatorname{Gr}_{G})$ .

**Remark 1.7.** Note that  $m_*$  carries perverse sheaves to perverse sheaves: indeed, it can be proven that m is ind-proper, i.e. it can be represented by a filtered colimit of proper maps of schemes compatibly with the lattice filtration. By [KW01, Lemma III.7.5], and the definition of  $\operatorname{Perv}_{G_{\mathfrak{O}}}(\operatorname{Gr}_{G}, k)$  as a direct limit, this ensures that  $m_*$  carries perverse sheaves to perverse sheaves.

Observations similar to Construction 1.6 prove the following:

**Proposition 1.8.** We have the following equivalences of schemes or ind-schemes:

- $G_{\mathcal{O}} \simeq \operatorname{Aut}_{\widehat{X}_x}(\mathfrak{T})$
- $G_{\mathcal{K}}(R) \simeq \{ \mathfrak{F} \in \operatorname{Bun}_G(X \times \operatorname{Spec} R), \alpha : \mathfrak{F}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \simeq \mathfrak{T}|_{(X \setminus \{x\}) \times \operatorname{Spec} R}, \mu : \mathfrak{F}|_{\widehat{X}_x \times \operatorname{Spec} R} \simeq \mathfrak{T}|_{\widehat{X}_x} \times \operatorname{Spec} R \}$

- $(G_{\mathcal{K}} \times \operatorname{Gr}_G)(R) \simeq \{ \mathcal{F}, \alpha : \mathcal{F}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \simeq \mathcal{T}|_{(X \setminus \{x\}) \times \operatorname{Spec} R}, \mu : \mathcal{F}|_{\widehat{X}_x \times \operatorname{Spec} R} \simeq \mathcal{T}|_{\widehat{X}_x \times \operatorname{Spec} R}, \mathcal{G}, \beta : \mathcal{F}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \simeq \mathcal{T}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \}$
- $(G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \operatorname{Gr}_{G})(R) \simeq \{\mathfrak{F}, \alpha : \mathfrak{F}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \simeq \mathfrak{T}|_{(X \setminus \{x\}) \times \operatorname{Spec} R}, \mathfrak{G}, \eta : \mathfrak{F}|_{(X \setminus \{x\}) \times \operatorname{Spec} R} \simeq \mathfrak{G}_{(X \setminus \{x\}) \times \operatorname{Spec} R}\}.$

In this paper, we will fix the reductive group G and denote the affine Grassmannian associated to G simply by Gr.

#### 1.1.3 Equivariant constructible sheaves

Finally, we review the notion of equivariant constructible sheaves on the affine Grassmannian. Recall that the affine Grassmannian admits a stratification in Schubert cells. We can consider the triangulated category of sheaves which are constructible with respect to that stratification, which we denote by  $\operatorname{Cons}(\operatorname{Gr}, \mathscr{S})$ . This category admits a t-structure whose heart is the category of perverse sheaves which are constructible with respect to  $\mathscr{S}$ . We can also consider the category  $\operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr}, \mathscr{S})$  of  $G_{\mathfrak{O}}$ -equivariant constructible sheaves with respect to  $\mathscr{S}$ , defined as

$$\lim \Big( \dots \bigoplus \operatorname{Cons}(G_{\mathfrak{O}} \times \operatorname{Gr}, \mathscr{S}) \bigoplus \operatorname{Cons}(\operatorname{Gr}, \mathscr{S}) \Big), \tag{1.4}$$

where the stratification on  $G_0 \times \cdots \times \ldots G_0 \times Gr$  is trivial on the first factors and  $\mathscr{S}$  on the last one. Note that there exists a notion of category of equivariant constructible sheaves with respect to *some* stratification. In full generality, let us fix a topological group H acting on a topological manifold X, and let us denote by  $\mathscr{S}$  the orbit stratification on X. We have a pullback square of triangulated (or stable  $\infty$ -) categories<sup>2</sup>

$$\operatorname{Cons}_{H}(X, \mathscr{S}) \longrightarrow \operatorname{Cons}(X, \mathscr{S})$$

$$\downarrow \qquad \qquad \downarrow$$

$$\mathfrak{D}_{c,H}(X) \longrightarrow \mathfrak{D}_{c}(X).$$
(1.5)

where

$$\mathcal{D}_{c}(X) = \operatorname*{colim}_{s \text{ stratification of } X} \operatorname{Cons}(X, s)$$

and

$$\mathcal{D}_{c,H}(X) = \lim \Big( \dots \bigoplus \mathcal{D}_{c}(H \times X, \mathscr{S}) \longleftarrow \mathcal{D}_{c}(X, \mathscr{S}) \Big).$$

Now, the horizontal arrows in (1.5) are not equivalences, although they are while restricted to the abelian subcategories of perverse sheaves. Indeed, the forgetful functor

$$\operatorname{\mathcal{P}erv}_{G_{\mathfrak{O}}}(\operatorname{Gr},\mathscr{S}) \to \operatorname{\mathcal{P}erv}(\operatorname{Gr},\mathscr{S})$$

<sup>&</sup>lt;sup>1</sup>This is often denoted by  $Cons_{G_{\mathcal{O}}}(G_{\mathcal{O}})$ , but we will use the terminology Cons and reserve  $\mathcal{D}_{c}(-)$  for the case when the stratification is allowed to vary, see below and Definition A.16.

<sup>&</sup>lt;sup>2</sup>Fully faithfulness of the vertical arrows comes from the fact that the transition maps in the colimits are fully faithful.

is an equivalence (see [BR18, Section 4.4]), but

$$\operatorname{Cons}_{G_{\mathfrak{Q}}}(\operatorname{Gr},\mathscr{S}) \to \operatorname{Cons}(\operatorname{Gr},\mathscr{S})$$

is not: its essential image only generates the target as a triangulated category ([Ric]).

On the contrary, the left vertical arrow of (1.5) is an equivalence: this is shown by reducing to the finite-dimensional case (which can be done by the very definition of category of constructible sheaves on an infinite-dimensional variety), and then by induction on the dimension: indeed, for every G-equivariant complex of sheaves F there is a maximal open subset of X where F is locally constant, and this is G-invariant by maximality, since F is G-equivariant; thus, the complement of this maximal open set is G-invariant as well and one can apply induction.

The aim of the present paper is to extend the convolution product from  $\operatorname{Perv}_{G_{\mathcal{O}}}(\operatorname{Gr},\mathscr{S})$  to  $\operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr},\mathscr{S})$ , together with an additional braided structure. We will now explain briefly why (Section 1.2) and how (Section 1.3).

## 1.2 Motivations and further aims

There are at least two motivations to extend the convolution product from the abelian category of perverse sheaves to the  $\infty$ -category of constructible sheaves. The most immediate one comes from the following observation by D. Arinkin and D. Gaitsgory ([AG15]). As they point out in [AG15, 12.3.2], their functor

$$\operatorname{Sat}^{\operatorname{ren}}: \operatorname{IndCoh}(\operatorname{Hecke}(\check{G})_{\operatorname{spec}}) \simeq \operatorname{Sph}(G, x)^{\operatorname{ren}},$$

which is in some way the derived version of the inverse of the functor appearing in the Geometric Satake Theorem, can be naturally upgraded to an equivalence of factorizable monoidal categories. As they remark, the latter statement is known as "derived Satake theorem"; it was conjectured by V. Drinfeld and proved by J. Lurie and Gaitsgory himself (unpublished, reported in [AG15]) by interpreting  $IndCoh(Hecke(\check{G})_{spec})$  as the  $\mathbb{E}_3$ -center of  $Rep(\check{G})$ , after viewing this latter as an  $\mathbb{E}_2$ -category.

Here

$$\operatorname{Hecke}(\check{G})_{\operatorname{spec}} = \operatorname{pt}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \operatorname{pt}/\check{G}$$

and

$$\operatorname{Sph}(G, x)^{\operatorname{ren}} = \operatorname{Ind}(\operatorname{Sph}(G, x)^{\operatorname{loc.c}}),$$

where  $Sph(G,x)^{loc.c}$  is the category of those  $G_0$ -equivariant D-modules over  $Gr_G$  which become compact objects via the forgetful functor

$$\mathrm{D\text{-}mod}(\mathrm{Gr}_G,x)^{G_{\mathcal{O}}} \to \mathrm{D\text{-}mod}(\mathrm{Gr}_G,x).$$

Now  $\mathrm{Sph}(G,x)$  is related to the category of equivariant constructible sheaves via the Riemann-Hilbert correspondence, which in our case implies the existence a fully faithful embedding  $\mathrm{Cons}_{G_0}(\mathrm{Gr}_G) \to$ 

D-mod( $Gr_G$ ) $^{G_O}$ . We can therefore see  $Sph(G,x)^{ren}$  as a renormalised and enlarged version of  $Cons_O(Gr_G)$ .

The  $\mathbb{E}_3$ -structure on  $\mathrm{Cons}_{G_{\mathfrak{O}}}(\mathrm{Gr})$  that we construct in the present paper is compatible with the one existing on  $\mathrm{Sph}(G,x)^{\mathrm{ren}}$  in that both restrict up to shifts to the convolution of perverse sheaves on the common abelian subcategory  $\mathrm{Perv}_{G_{\mathfrak{O}}}(\mathrm{Gr}_G)$ .

As alternative references for similar phenomena, we recommend to look also at [BZFN10] and [BZNP17].

The second motivation for our construction is related to a possible approach to the Geometric Langlands Program for surfaces. Since the affine Grassmannian is naturally related to the setting of complex smooth projective curves, it is natural to search for versions of the same object in the context of surfaces. For example one could fix a complex smooth projective surface S and a point  $x \in S(\mathbb{C})$ , and define

$$\operatorname{Gr}_G(S,x)(R) = \{ \mathfrak{F} \in \operatorname{Bun}_G(S_R), \alpha : \mathfrak{F}|_{(S \setminus \{x\}) \times \operatorname{Spec} R} \xrightarrow{\sim} \mathfrak{T}_{(S \setminus \{x\}) \times \operatorname{Spec} R} \}.$$

However, one can prove by means of the Hartogs theorem that this functor is "trivial", in the sense that it is equivalent to G (seen as the automorphism group of  $\mathfrak{T}_S$ ).

Alternatively, one could fix an algebraic curve C in S and define

$$\operatorname{Gr}_G(S,C)(R) = \{ \mathfrak{F} \in \operatorname{Bun}_G(S_R), \alpha : \mathfrak{F}|_{(S \setminus C) \times \operatorname{Spec} R} \xrightarrow{\sim} \mathfrak{T}_{(S \setminus C) \times \operatorname{Spec} R} \}.$$

This last definition presents an important difference with respect to the setting of curves when it comes to the convolution product. Indeed, one can figure out suitable versions of the convolution diagram, but the analogous of the map m in (1.2) is not ind-proper ([Kap00, Proposition 2.2.2]), and therefore we are not granted that the pushforward (or the proper pushforward) along that map takes perverse sheaves to perverse sheaves. However,  $m_!$  should preserve constructible sheaves, provided that it carries a sufficiently strong stratified structure.<sup>3</sup> Therefore, although a convolution product of perverse sheaves over  $Gr_G(S,x)$  or  $Gr_G(S,C)$  is probably not well-defined, there is a good chance that it is well-defined at the level of some  $\infty$ -category of (equivariant) constructible sheaves  $\mathcal{C}$ . There is also the chance that techniques analogous to the ones used for the construction of the  $\mathbb{E}_3$ -structure in the present paper can be used to build an  $\mathbb{E}_5$ -structure<sup>4</sup> on  $\mathcal{C}$  or on categories arising from yet other versions of the affine Grassmannian, but this is, at the moment, purely conjectural.

A final comment: some of the techniques used in the present paper are already "folklore" in the mathematical community; for example, application of Lurie's [Lur17, Theorem 5.5.4.10] to the affine

 $<sup>^{3}</sup>$ In the present paper, we do not use properness of m even in the case of curves, see Theorem A.15. We only use that m can be factorised as an open embedding and a proper map.

<sup>&</sup>lt;sup>4</sup>Here  $\mathbb{E}_1$  would stand for an "associativity" coming from some convolution diagram, and  $\mathbb{E}_4$  would come from the real dimension of S, just like  $\mathbb{E}_2$  comes from the real dimension of the curve X in our construction.

Grassmannian appears also in [HY19], though in that paper the Authors are interested in the (filtered) topological structure of the affine Grassmannian and do not take constructible sheaves. Up to our knowledge, the formalism of constructible sheaves via exit paths and exodromy has never been applied to the affine Grassmannian and the Geometric Satake Theorem. Here we use it in order to take into account the homotopy invariance of the constructible sheaves functor, which is strictly necessary for the application of Lurie's [Lur17, Theorem 5.5.4.10].

## 1.3 Overview of the work

## 1.3.1 The Eckmann-Hilton argument

The underlying theoretic argument of this paper is the Eckmann-Hilton Theorem. That theorem states that, given two unitary operations on a set X, such that the two units coincide and the two operations distribute with respect to one another, then the two operations coincide and are commutative.

The strategy is to use a categorical version of this theorem: namely, the set X is replaced by the category of equivariant constructible sheaves on the affine Grassmannian (see (1.4)), with two monoidal structures (essentially, maps  $\operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr}) \otimes \operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr}) \to \operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr})$ , where  $\otimes$  is the Lurie tensor product of  $\infty$ -categories) whose unit is the constant sheaf  $\Lambda$  supported at a point, and such that these structures distribute with respect to one another in some sense.

Such a statement exists, and is formulated in the language of operads. Indeed, the Dunn-Lurie Additivity Theorem [Lur17, 5.1.2.2] is the operadic version of the Eckmann-Hilton theorem, and says that for any symmetric monoidal  $\infty$ -category  $\mathcal{C}$  a functor  $\mathbb{E}_n \times \mathbb{E}_m \to \mathcal{C}^{\otimes}$  which is a map of operads in both variables corresponds to a map of operads  $\mathbb{E}_{n+m} \to \mathcal{C}$ . In other words,

$$\mathrm{Alg}_{\mathbb{E}_n}(\mathrm{Alg}_{\mathbb{E}_m}(\mathfrak{C}^\otimes)) \simeq \mathrm{Alg}_{\mathbb{E}_{n+m}}(\mathfrak{C}^\otimes).$$

In our case, there exists indeed a map of operads

$$\mathbb{E}_1 \times \mathbb{E}_2 \to \Pr_k^{L,\otimes},$$

whose underlying  $\infty$ -category is the  $\infty$ -category of constructible equivariant sheaves on the affine Grassmannian: it thus encodes two different unital monoidal laws on  $\operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr})$  which are functorial with respect to one another. Applying the Dunn-Lurie Theorem one obtains an  $\mathbb{E}_3$ -structure on  $\operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr})$ .

#### 1.3.2 The need for an $\infty$ -categorical setting

The first thing to note is that the construction of the  $\mathbb{E}_2$  "component" of the product requires the use of the  $\infty$ -categorical language. Indeed, it makes use of another theorem by J. Lurie [Lur17, 5.5.4.10], which is one of the prominent theoretical tools used in this paper. That theorem states that we can obtain our desired  $\mathbb{E}_2$ -structure from a very typical property of the affine Grassmannian, namely

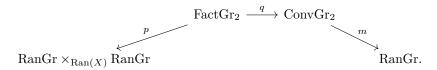
the fact that there is some sort of "cosheaf of categories" over the space  $\operatorname{Ran}(X)$  of finite systems of closed points in the curve X, whose stalk at any singleton  $\{x\} \in \operatorname{Ran}(X)$  is the desired category  $\operatorname{Cons}_{G_0}(\operatorname{Gr})$ , but such that, in addition, some "local constancy" and "factorisation" properties are satisfied.

The "local constancy" property means essentially that the value of the cosheaf of categories on a small open set U of  $\operatorname{Ran}(X)$  is the same as its stalk at any point of U. In our case, this property descends from a purely homotopy-theoretical feature of the affine Grassmannian and of the Ran space, and it requires essentially a treatment from an homotopic point of view: here is where  $\infty$ -categorical tools are necessary to proceed. More precisely, the framework of presentable k-linear  $\infty$ -categories (k being the chosen ring of coefficients for our categories of constructible sheaves), of stratified spaces and exit paths allow to detect homotopy issues and transfer them to constructible sheaves in a functorial way. The correct symmetric monoidal structure in this setting is the so-called Lurie tensor product of prestnable k-linear  $\infty$ -categories. We denote the induced symmetric monoidal  $\infty$ -category of presentable k-linear  $\infty$ -categories is denoted by  $\operatorname{Pr}_k^{\mathrm{L},\otimes}$ .

The second property is better known, as it boils down to the celebrated "factorisation property" of the Beilinson-Drinfeld Grassmannian, see [Zhu16, 3.2].

#### 1.3.3 The convolution product

We sketch now the construction of the two products. The  $\mathbb{E}_1$  "component" is a direct extension of the convolution product of perverse sheaves illustrated in the first section:  $\mathcal{F}\star\mathcal{G}=m_*\mathcal{F}\tilde{\boxtimes}\mathcal{G}$ . The moduli interpretation of the affine Grassmannian allows to formulate a slight variant of the convolution diagram, where Gr is replaced by the moduli space of bundles together with a trivialisation outside a finite system of closed points (of arbitrary finite cardinality). Such systems are, by definition, elements of the Ran space of the curve, and therefore we denote this modification of the affine Grassmannian by RanGr. The modified diagram will have the form:



By remembering only the common "critical locus" of the trivialisations, the vertexes of the modified convolution diagram admit thus a map to the Ran space itself, making the diagram commute. However, if one selects a "singleton" element  $\{x\} \in \text{Ran}(X)$ , corresponding to a closed point in X, and takes the fiber of everything at this point, one recovers the originary convolution diagram. In order to combine this construction with the  $\mathbb{E}_2$  "component" that we are about to describe, it will be convenient to formulate the convolution structure as follows. The convolution diagram is not a correspondence, in that it has four vertexes, and the product is not defined as a "straight" pullback-pushforward procedure. Indeed, as we have seen, it is defined using equivariancy to transfer

sheaves on  $Gr_{Ran} \times Gr_{Ran}$  to sheaves on  $Gr_{Ran} \times Gr_{Ran}$ . We set up a way to use this procedure to recover a straight correspondence (and consequently a straight pullback-pushforward procedure) after having quotiented every vertex of the diagram by a suitable  $G_{\mathcal{O}}$ - or  $G_{\mathcal{O}} \times G_{\mathcal{O}}$ -action: note that taking constructible sheaves on quotients is, by definition, the same as taking equivariant constructible sheaves on the originary spaces, and this is exactly what we need.

The content of the present paragraph is developed in Section 2.

#### 1.3.4 2-Segal objects and correspondences

Therefore, the convolution procedure can be seen as a pullback-pushforward on categories of constructible sheaves on quotients. This admits a functorial formulation with the language of 2-Segal objects as presented and used in [DK19]: the quotiented convolution diagram is the (2,1) level of a 2-Segal semisimplicial object in Sh(StrTSpc), the latter being the category of sheaves with respect to the topology of stratified local homeomorphisms, which is the complex-topological counterpart of the étale topology. This 2-Segal object corresponds to a nonunital  $\mathbb{E}_1$  algebra object in  $\operatorname{Corr}^{\times}(\operatorname{Sh}(\operatorname{StrTSpc}))$  by [DK19, Proposition 8.6.1, Theorem 11.1.6]. Finally, there is a symmetric monoidal functor out of the category of correspondence (with some restrictions on which correspondences are "admissible") to  $\operatorname{Pr}^L{}_k$  which sends a stratified topological space to the  $\infty$ -category of constructible sheaves of k-modules on it and sends an admissible correspondence to the pullback-pushforward functor along that correspondence. The "admissibility" comes from the fact that whereas the pullback of constructible sheaves along stratified maps is always well-defined, we have to ensure that the map along which we want to perform the (derived) pushforward is proper and "cylindrically stratified" (see Definition A.12).

The content of the present paragraph is developed at the beginning of Section 4.

#### 1.3.5 Stratifications

It is crucial to remark that we are interested in the category of (equivariant) constructible sheaves with respect to the stratification in Schubert cells of the affine Grassmannian (see Remark 1.3). We thus need to provide a natural extension of this stratification to the Ran versions of the affine Grassmannian sketched above, and we verify that these are coherent with all the constructions performed. Also, stratified homotopy equivalences between nicely stratified spaces induce equivalences of  $\infty$ -categories between categories of constructible sheaves with respect to those stratifications, and this is not true if the homotopy equivalences are not stratified. We will see in a moment why this behaviour is relevant to us.

The main theoretical tool that we use in order to treat constructible sheaves on stratified spaces is the notion of category of exit paths  $\operatorname{Exit}(X, s)$  associated to a stratified topological space (X, s) ([Lur17, Appendix A], [BGH20]) and the fact that, for a conically stratified space locally of singular shape,

the equivalence of  $\infty$ -categories

$$\operatorname{Fun}(\operatorname{Exit}(X, s), \operatorname{Mod}_{\Lambda}) \simeq \operatorname{Cons}_{\Lambda}(X, s)$$

holds. The content of the present paragraph is developed mostly in Appendix A.

## 1.3.6 The fusion product

The fact that the convolution product can be "extended to" the Ran space is the key for the interaction of the two products. Indeed, the  $\mathbb{E}_2$  "component" of the product will be defined by making significant use of the Ran space (see also [HY19] where the same approach is used in the context of graded ring spectra). One can define a cosheaf of  $\infty$ -categories on the Ran space with an additional operadic structure encoding the fact that the datum of two bundles together with trivialisations outside two disjoint finite systems of points gives rise to a glued bundle together with a trivialisation outside the union of those disjoint systems of points. This construction satisfies the two essential properties mentioned above, relying essentially upon the following facts:

- (Factorisation property) The gluing operation just mentioned is reversible: given a bundle together with a trivialisation outside a union of two disjoints system of points, one can recover two bundles with trivialisation outside the two systems of points respectively, and this construction is inverse to the gluing procedure.
- (Constructibility) The Ran space is locally contractible (it is also contractible, but we do not use this property) and certain "oriented" homotopies inside this space can be lifted to the Ran Grassmannian, thus establishing a homotopy equivalence between the space of bundles trivialised outside a fixed point x (that is, the affine Grassmannian) and the space of bundles trivialised outside a system of points contained in a small open complex disk of X containing x.

These properties allow us to apply Lurie's theorem [Lur17, 5.5.4.10] and recover from this an  $\mathbb{E}_2$ algebra structure for the Ran Grassmannian, which interacts with the  $\mathbb{E}_1$ -convolution structure in
a very clear way (see Remark 3.14). This interaction resides essentially in the fact that the fusion
procedure works for the data encoded in each of the vertexes of the convolution diagram, in the sense
that we can glue data by joining the "critical" sets.

Now we need to transfer these properties to the level of constructible sheaves. We do this by means of the functor Cons, once we have proved that every space under consideration is conically stratified and of singular shape: in this setting, we have the **exodromy equivalence** 

$$Cons(-) \simeq Fun(Exit(-), Mod_{\Lambda})$$

which ensures that both properties are transferred to the level of constructible sheaves. Indeed, symmetric monoidality of the functor  $\operatorname{Fun}(\operatorname{Exit}(-),\operatorname{Mod}_{\Lambda})$  ensures that the operadic structure of our constructions is preserved, and the fact that Cons sends stratified homotopy equivalences to equivalences of  $\infty$ -categories ensures that the constructibility property is transferred as well. The content of the present paragraph is treated in Sections 3 and 4.

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#### 1.3.7 Conclusion

The two procedures result thus combined in an object

$$\mathcal{A}^{\otimes,\mathrm{nu}} \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_2}(\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_1}(\mathrm{Pr}^{\mathrm{L},\otimes}_k)).$$

As we can see, units are missing, but they can easily be added to both structures. We thus finally obtain our object  $\mathcal{A}^{\otimes} \in \mathrm{Alg}_{\mathbb{E}_2}(\mathrm{Alg}_{\mathbb{E}_1}(\mathrm{Pr}_k^{\mathrm{L},\otimes}))$ , and we can apply the Additivity Theorem. This final step is performed at the end of Section 4.

## 1.4 Glossary

We present a list of symbols used throughout the work. Some of them are used in the literature, some others are introduced in the present paper.

- X will always be a complex projective smooth curve, and x a closed point of it.
- G will be a complex reductive group, and  $G_{\mathcal{K}}, G_{\mathcal{O}}$  its loop group functor and arc group functor defined in Section 1.1.1.
- Gr = Gr<sub>G</sub> will be the affine Grassmannian associated to G, namely the étale stack quotient  $G_{\mathcal{K}}/G_{\mathcal{O}}$  (see again Section 1.1.1).
- $\operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr})$  will be the category of equivariant constructible sheaves on the affine Grassmannian defined in Section 1.1.3.
- Ran(X) is the algebraic Ran space associated to the curve X, defined in Definition 2.1.
- RanGr is the Ran Grassmannian associated to X and G, see Definition 2.2.
- If k is a natural number, FactGr<sub>k</sub> is the k-th level of the factorising Grassmannian defined in Definition 2.5. As k varies, these presheaves are organised in a semisimplicial presheaf FactGr<sub>•</sub> described in Construction 2.9.
- FactGr<sub>k,x</sub> is the fiber of FactGr<sub>k</sub> at the point  $\{x\} \in \text{Ran}(X)$ .
- the factorising arc group FactArc• is defined in Definition 2.14.
- FactArc<sub>k,x</sub> is the fiber of FactArc<sub>k</sub> at  $\{x\} \in \text{Ran}(X)$ .
- ConvGr<sub>•</sub> is the Ran version of the equivariant product  $G_{\mathcal{K}} \times^{G_{0}}$  Gr (and of its higher-associativity versions) and is defined in Definition 2.16.
- StrPSh<sub>ℂ</sub> is the category of complex stratified presheaves defined in Construction A.3.
- $\mathbb{X}_{\bullet}(T)$  is the set of coweights of any maximal torus T of G.
- ${\mathscr S}$  is the stratification in Schubert cells of the affine Grassmannian.
- $s_k$  is the stratification of the complex presheaf FactGr<sub>k</sub>, and is defined in Definition 2.27.

- StrTSpc is the category of stratified topological spaces defined in Definition A.1. StrTSpc<sub>csls</sub> is its full subcategory of conically stratified spaces locally of singular shape, see [Lur17, Definitions A.4.15 and A.5.5].
- cnl is the class of maps between stratified topological spaces which are cylindrically stratified, see Definition A.12. comp.cnl is the class of maps between stratified topological spaces that factor uniquely as a stratified open embedding and a proper cylindrically stratified map.
- strtop is the "stratified underlying complex analytic space" functor introduced in Construction A.3.
- For any stratified space locally of singular shape  $(X, s : X \to A)$ , Exit(X, s) is the category of exit paths of X, denoted as  $\text{Sing}^A(X)$  in [Lur17, Definition A.6.2].
- M is the complex analytic manifold of dimension 1 associated to X.
- Ran(M) is the topological Ran space of M, defined in [Lur17, 5.5.1.1]
- $\Re \operatorname{Ran} \operatorname{Gr}_k$ ,  $\operatorname{Fact} \operatorname{Gr}_k$  and  $\operatorname{Fact} \operatorname{Arc}_k$  are the stratified topological spaces associated to  $\operatorname{Ran} \operatorname{Gr}_k$ ,  $\operatorname{Fact} \operatorname{Gr}_k$  and  $\operatorname{Fact} \operatorname{Arc}_k$  respectively.  $\operatorname{Fact} \operatorname{Gr}_{k,x}$  and  $\operatorname{Fact} \operatorname{Arc}_{k,x}$  are those associated to the fibers  $\operatorname{Fact} \operatorname{Gr}_{k,x}$  and  $\operatorname{Fact} \operatorname{Arc}_{k,x}$ , or equivalently, the fibers at  $\{x\} \in M$  of  $\operatorname{Fact} \operatorname{Gr}_k$  and  $\operatorname{Fact} \operatorname{Arc}_k$ .
- she is the class of stratified homotopy equivalences of stratified topological spaces.
- $Sh_{strloc}(StrTSpc_{csls})$  is the  $\infty$ -category of sheaves over  $StrTSpc_{csls}$ , taken with respect to the topology of stratified local homeomorphisms.
- For a topological space X,  $\mathcal{D}_{c}(X)$  is the  $\infty$ -category of constructible sheaves with respect to a non-fixed stratification, see Definition A.16.

## Acknowledgements

I wish to thank my advisors, Mauro Porta (Université de Strasbourg), Gabriele Vezzosi (Università di Firenze) and Angelo Vistoli (Scuola Normale Superiore). I also want to thank Dario Beraldo, Julian Demeio, Ivan Di Liberti, Jeremy Hahn, Andrea Maffei, Valerio Melani, David Nadler, Daniele Palombi, Emanuele Pavia, Michele Pernice, Marco Volpe and Allen Yuan for their suggestions. I specially thank Peter Haine and Morena Porzio for the frequent and fruitful discussions carried out during the work. I also thank the Scuola Normale Superiore and the University of Strasbourg for allowing me to carry on my PhD project in coadvisorship.

# 2 Convolution over the Ran space

The aim of this section is to expand the construction of the Ran Grassmannian defined for instance in [Zhu16, Definition 3.3.2] in a way that allows us to define a convolution product of constructible sheaves in the Ran setting.

## 2.1 The presheaves $FactGr_k$

#### 2.1.1 The Ran Grassmannian

Let us recall the definition of the basic objects that come into play. Let  $\mathbb{C}$ Alg be the category of (discrete) complex algebras and X a smooth projective curve over  $\mathbb{C}$ .

**Definition 2.1.** The algebraic Ran space of X is the presheaf

$$\operatorname{Ran}(X) : \mathbb{C}\operatorname{Alg} \to \operatorname{Set}$$

$$R \mapsto \{\text{finite subsets of } X(R)\}.$$

This is not a sheaf with respect to the étale topology on the category of commutative algebras/affine schemes.

**Definition 2.2.** The Ran Grassmannian of X is the functor RanGr:  $\mathbb{C}Alg \to Grpd$ 

$$R \mapsto \{S \in \operatorname{Ran}(X)(R), \mathfrak{F} \in \operatorname{Bun}_G(X_R), \alpha : \mathfrak{F}|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathfrak{I}|_{X_R \setminus \Gamma_S} \},$$

where  $\mathcal{T}$  is the trivial G-bundle on  $X_R$ ,  $\Gamma_S$  is the union of the graphs of the  $s_i$  inside  $X_R$ ,  $s_i \in S$ , and  $\alpha$  is a trivialisation, i.e. an isomorphism of principal G-bundles with the trivial G-bundle. This admits a natural forgetting map towards  $\operatorname{Ran}(X)$ .

Remark 2.3. Consider the twisted tensor product defined in [Zhu16, (3.1.10)]

$$\operatorname{Gr}_X = \hat{X} \times^{\operatorname{Aut}(\operatorname{Spec} \mathbb{C}[[t]])} \operatorname{Gr}_G,$$

where  $\hat{X}$  is the space of formal parameters defined in *loc. cit.*. This parametrises

$$\operatorname{Gr}_X(R) = \{ x \in X(R), \mathfrak{F} \in \operatorname{Bun}_G(X_R), \alpha : \mathfrak{F}|_{X_R \setminus \Gamma_x} \xrightarrow{\sim} \mathfrak{I}|_{X_R \setminus \Gamma_x} \}.$$

Let  $\mathcal{F}in_{surj}$  be the category of finite sets with surjective maps between them. For each finite set I, there is a multiple version

$$\operatorname{Gr}_{X^I}(R) = \{ (x_1, \dots, x_{|I|}) \in X(R)^I, \mathfrak{F} \in \operatorname{Bun}_G(X_R), \alpha : \mathfrak{F}|_{X_R \setminus \Gamma_{x_1} \cup \dots \cup \Gamma_{x_{|I|}}} \xrightarrow{\sim} \mathfrak{I}|_{X_R \setminus \Gamma_{x_1} \cup \dots \cup \Gamma_{x_{|I|}}} \}.$$

Then we have

$$\operatorname{RanGr} = \operatorname*{colim}_{I \in \mathcal{F} \operatorname{in}_{\operatorname{surj}}} \operatorname{Gr}_{X^I}.$$

**Definition 2.4.** We can define  $G_{\mathcal{K},X^I}$ ,  $G_{\mathcal{O},X^I}$  in the same way. Let also  $\operatorname{Ran}G_{\mathcal{K}}$  be the functor

$$R\mapsto \{S\in \operatorname{Ran}(X)(R), \mathfrak{F}\in \operatorname{Bun}_G(X_R), \alpha: \mathfrak{F}|_{X_R\setminus \Gamma_S}\xrightarrow{\sim} \mathfrak{T}|_{X_R\setminus \Gamma_S},$$

$$\mu: \mathcal{F}|_{\widehat{(X_R)}_{\Gamma_S}} \xrightarrow{\sim} \mathfrak{I}|_{\widehat{(X_R)}_{\Gamma_S}}\} \simeq \mathop{\mathrm{colim}}\nolimits G_{\mathcal{K};X^I}.$$

Let  $RanG_{\mathcal{O}}$  be the functor

$$R\mapsto \{S\in \operatorname{Ran}(X)(R), g\in \operatorname{Aut}_{\widehat{(X_R)_{\Gamma_S}}}(\mathfrak{I})\}\simeq \operatorname{colim}_I G_{\mathfrak{O};X^I}.$$

By means of [Zhu16, Proposition 3.1.9] one has of course that RanGr  $\simeq \text{Ran}G_{\mathcal{K}}/\text{Ran}G_{\mathcal{O}}$  in the sense of a quotient in PSh( $\delta tk_{\mathbb{C}}$ ), i.e.

$$\operatorname{RanGr} \simeq \operatorname{colim}_I G_{\mathcal{K},X^I}/G_{\mathcal{O},X^I}$$

where each term is the étale stack quotient in the category of stacks (recall that  $G_{\mathcal{K}}$  is an ind-scheme and  $G_{\mathcal{O}}$  is a group scheme), and is equivalent to  $\operatorname{Gr}_{X^I}$ .

## 2.1.2 The factorising Grassmannian

**Definition 2.5.** Let the factorising Grassmannian FactGr<sub>k</sub> be the functor  $\mathbb{C}Alg \to Grpd$  defined as

$$\overbrace{\operatorname{Ran}G_{\mathcal{K}}\times_{\operatorname{Ran}(X)}\cdots\times_{\operatorname{Ran}(X)}\operatorname{Ran}G_{\mathcal{K}}}^{k-1\ times}\times_{\operatorname{Ran}(X)}\operatorname{Ran}G_{\mathcal{K}},$$

that is

$$R \mapsto \{S \in \operatorname{Ran}(X)(R), \mathfrak{F}_i \in \operatorname{Bun}_G(X_R),$$

 $\alpha_i$  trivialisation of  $\mathcal{F}_i$  outside  $\Gamma_S$ ,  $i = 1, \ldots, k$ ,

 $\mu_i$  trivialisation of  $\mathcal{F}_i$  on the formal neighborhood of  $\Gamma_S$ ,  $i = 1, \ldots, k-1$ .

We call  $r_k : \operatorname{FactGr}_k \to \operatorname{Ran}(X)$  the natural forgetting map

A priori, RanGr and FactGr<sub>k</sub> are groupoid-valued, because if R is fixed the  $\mathcal{F}_i$ 's may admit nontrivial automorphisms that preserve the datum of the  $\alpha_i$ 's and the  $\mu_j$ 's. Actually, this is not the case, just like for the classical affine Grassmannian which is ind-representable:

**Proposition 2.6.** For any  $k \geq 0$  the functor

$$FactGr_k : \mathbb{C}Alg \to Grpd$$

factorises through the inclusion  $Set \rightarrow Grpd$ .

*Proof.* See Appendix B.1.

It is worthwhile to remark that the map  $\operatorname{FactGr}_k \to \operatorname{Ran}(X)$  is ind-representable, although  $\operatorname{FactGr}_k$  itself is not.

**Definition 2.7.** Let  $x \in X(\mathbb{C})$  be a closed point of X. There is a natural map  $\{x\} \to X \to \operatorname{Ran}(X)$ , represented by the constant functor  $R \mapsto \{x\} \in \operatorname{Set}$ . Let us denote  $\operatorname{FactGr}_{k,x} = \operatorname{FactGr}_k \times_{\operatorname{Ran}(X)} \{x\}$ .

**Proposition 2.8.** Fact $Gr_{k,x}$  is independent from the choice of X and x, and

$$FactGr_{1,x} \simeq Gr_G$$

$$\operatorname{Fact}\operatorname{Gr}_{2,x} \simeq G_{\mathfrak{K}} \times \operatorname{Gr}_{G}.$$

*Proof.* Note first that

$$\operatorname{FactGr}_{k,x}(R) = \left\{ (\mathfrak{F}_i \in \operatorname{Bun}_G(X_R), \alpha_i : \mathfrak{F}_i|_{X_R \setminus (\{x\} \times \operatorname{Spec} R)} \xrightarrow{\sim} \mathfrak{I}|_{X_R \setminus (\{x\} \times \operatorname{Spec} R)}, \right.$$
$$\mu_i : \mathfrak{F}_i|_{\widehat{(X_R)}_{(\{x\} \times \operatorname{Spec} R)}} \xrightarrow{\sim} \mathfrak{I}|_{\widehat{(X_R)}_{(\{x\} \times \operatorname{Spec} R)}})_{i=1,\dots,k} \right\}.$$

By the Formal Gluing Theorem [HPV16] this can be rewritten as

$$\operatorname{FactGr}_{k,x}(R) = \left\{ (\mathfrak{F}_i \in \operatorname{Bun}_G(\widehat{(X_R)}_{\{x\} \times \operatorname{Spec} R}), \right.$$

$$\alpha_i : \mathfrak{F}|_{(\mathring{X_R})_{\{x\} \times \operatorname{Spec} R}} \xrightarrow{\sim} \mathfrak{T}|_{(\mathring{X_R})_{\{x\} \times \operatorname{Spec} R}},$$

$$\mu_i : \mathfrak{F}|_{\widehat{(X_R)}_{(\{x\} \times \operatorname{Spec} R)}} \xrightarrow{\sim} \mathfrak{T}|_{\widehat{(X_R)}_{(\{x\} \times \operatorname{Spec} R)}})_{i=1,\dots,k} \right\} \simeq$$

$$\simeq \left\{ (\mathfrak{F}_i \in \operatorname{Bun}_G(\widehat{X}_{\{x\}} \times \operatorname{Spec} R), \alpha_i : \mathfrak{F}|_{\mathring{X}_{\{x\}} \times \operatorname{Spec} R} \xrightarrow{\sim} \mathfrak{T}|_{\mathring{X}_{\{x\}} \times \operatorname{Spec} R}, \mu_i : \mathfrak{F}|_{\widehat{X}_{\{x\}} \times \operatorname{Spec} R} \xrightarrow{\sim} \mathfrak{T}|_{\widehat{X}_{\{x\}} \times \operatorname{Spec} R})_{i=1,\dots,k} \right\},$$

but  $\widehat{X}_{\{x\}}$  is independent from the choice of the point x, being (noncanonically) isomorphic to Spec  $\mathbb{C}[\![t]\!]$  (and the same for  $\mathring{X}_x$ ). The rest of the statement is clear from the definitions.

## 2.2 The 2-Segal structure

#### 2.2.1 Face maps

We now establish a semisimplicial structure on the collection of the  $FactGr_k$ .

**Construction 2.9.** Let  $\partial_i$  be the face map from [k-1] to [k] omitting i. We define the corresponding map  $\delta_i$ : FactGr<sub>k</sub>  $\to$  FactGr<sub>k-1</sub> as follows. A tuple  $(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_k, \alpha_k)$  is sent to a tuple

$$(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_{i-1}, \alpha_{i-1}, \mu_{i-1},$$
$$\operatorname{Fgl}(\mathcal{F}_i, \mathcal{F}_{i+1}, \alpha_i, \alpha_{i+1}, \mu_i), \mu'_i, \dots, \mathcal{F}_k, \alpha_k),$$

where:

- Fgl( $\mathcal{F}_i, \mathcal{F}_{i+1}, \alpha_i, \alpha_{i+1}, \mu_i$ ) is the pair formed as follows: the Formal Gluing Theorem ([HPV16]) allows us to glue the sheaves  $\mathcal{F}_i|_{X_R \setminus \Gamma_S}$  and  $\mathcal{F}_{i+1}|_{\widehat{(X_R)}_{\Gamma_S}}$  along the isomorphism  $\mu_i^{-1}|_{(\mathring{X_R})_{\Gamma_S}} \circ \alpha_{i+1}|_{(\mathring{X_R})_{\Gamma_S}}$ . This is the first datum of the pair. Also, this inherits a trivialisation over  $(\mathring{X_R})_{\Gamma_S}$  described by  $\alpha_i|_{(\mathring{X_R})_{\Gamma_S}} \mu_i^{-1}|_{(\mathring{X_R})_{\Gamma_S}} \alpha_{i+1}|_{(\mathring{X_R})_{\Gamma_S}}$ , which is the second datum.
- $\mu'_i$  coincides with  $\mu_{i+1}$  via the canonical isomorphism between the glued sheaf and  $\mathcal{F}_{i+1}$  over the formal neighbourhood of  $\Gamma_S$ .

**Proposition 2.10.** This construction defines a semisimplicial object  $\operatorname{FactGr}_{\bullet}: \Delta_{\operatorname{inj}}^{\operatorname{op}} \to \operatorname{Fun}(\operatorname{\mathbb{C}Alg},\operatorname{\mathsf{Set}})$  because the given maps satisfy the simplicial identities.

*Proof.* Let k be fixed. We check the face identities  $\delta_i \delta_j = \delta_{j-1} \delta_i$  for i < j.

The essential nontrivial case is when k=3, i=0, j=1 or i=1, j=2 or i=2, j=3. Otherwise the verifications are trivial since, if i < j-1, then the two gluing processes do not interfere with one another. The cases i=0, j=1 and i=2, j=3 are very simple, because there is only one gluing and one forgetting ( $\mathcal{F}_1$  or  $\mathcal{F}_3$  respectively). In the remaining case, we must compare  $\mathcal{F}_{1,23} = \mathrm{Fgl}(\mathcal{F}_1, \mathrm{Fgl}(\mathcal{F}_2, \mathcal{F}_3))$  with  $\mathcal{F}_{12,3} = \mathrm{Fgl}(\mathrm{Fgl}(\mathcal{F}_1, \mathcal{F}_2), \mathcal{F}_3)$ . (we omit the  $S, \alpha_i, \mu_i$  from the notation for short). We have:

- $\mathcal{F}_{1,23}|_{X_R \setminus \Gamma_S} \simeq \mathcal{F}_1|_{X_R \setminus \Gamma_S} \simeq \mathcal{F}_{12}|_{X_R \setminus \Gamma_S} \simeq \mathcal{F}_{12,3}|_{X_R \setminus \Gamma_S}$
- $\bullet \ \ \mathcal{F}_{1,23}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{23}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{2}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{12}|_{\widehat{(X_R)}_{\Gamma_S}} \simeq \mathcal{F}_{12,3}|_{\widehat{(X_R)}_{\Gamma_S}}.$

This tells us that the two sheaves are the same, and from this it is easy to deduce that the same property holds for the trivialisations.  $\Box$ 

We thus have a semisimplicial structure on  $\operatorname{FactGr}_{\bullet}$ , together with maps  $r_k : \operatorname{FactGr}_k \to \operatorname{Ran}(X)$  which commute with the face maps by construction.

## 2.2.2 Verification of the 2-Segal property

The crucial property of the semisimplicial presheaf FactGr<sub>•</sub> is the following:

**Proposition 2.11.** For any  $R \in \mathbb{C}Alg$ , the semisimplicial set  $FactGr_{\bullet}(R)$  enjoys the 2-Segal property, that is the equivalent conditions of [DK19, Proposition 2.3.2].

*Proof. Case* " $0, l \le k$ ". With the natural notations appearing in [DK19, Proposition 2.3.2], there is a map  $\operatorname{FactGr}_{\{0,1,\ldots,l\}} \times_{\operatorname{FactGr}_{\{0,l\}}} \operatorname{FactGr}_{\{0,l,l+1,\ldots,k\}} \to \operatorname{FactGr}_{\{0,\ldots,k\}} = \operatorname{FactGr}_k$  inverse to the natural projection. The map sends

$$(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_l, \alpha_l, \mathcal{F}'_l, \alpha'_l, \mu'_l,$$

$$\xi : (\mathcal{F}'_l, \alpha'_l) \xrightarrow{\sim} \operatorname{Fgl}(\{\mathcal{F}_i, \alpha_i, \mu_j\}_{i=1,\dots,l,j=1,\dots,l-1}), \mathcal{F}'_{l+1}, \alpha'_{l+1}, \mu'_{l+1}, \dots, \mathcal{F}'_k \alpha'_k)$$

$$\mapsto (\mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_l, \alpha_l, \mu''_l, \mathcal{F}'_{l+1}, \alpha'_{l+1}, \mu'_{l+1}, \dots, \mathcal{F}'_k, \alpha'_k)$$

where  $\mu_{l,l+1}''$  is the trivialisation of  $\mathcal{F}_l$  on the formal neighbourhood of x defined as  $\mathcal{F}_l|_{\widehat{(X_R)}_{\Gamma_S}} \xrightarrow{\sim}$ 

$$\operatorname{Fgl}(\{\mathfrak{F}_i,\alpha_i,\mu_j\}_{i=1,\dots,l,j=1,\dots,l-1})|_{\widehat{(X_R)}_{\Gamma_S}}\xrightarrow{\xi^{-1}}\mathfrak{F}'_l|_{\widehat{(X_R)}_{\Gamma_S}}\xrightarrow{\mu'_l}\mathfrak{T}_{\widehat{(X_R)}_{\Gamma_S}}.$$

This map is indeed inverse to the natural map arising from the universal property of the fibered product, thus establishing the 2-Segal property in the case "0, l". The case "l, k" can be tackled in a similar way.

**Notation 2.12.** Given a category or an  $\infty$ -category  $\mathcal{C}$ , we denote by 2-Seg<sup>ss</sup>( $\mathcal{C}$ ) the ( $\infty$ -)category of 2-Segal semisimplicial objects in  $\mathcal{C}$ .

## 2.3 Action of the arc group in the Ran setting

We now introduce analogs of the "arc group"  $G_{\mathcal{O}}$  to our global context, namely group functors over  $\operatorname{Ran}(X)$  denoted by  $\operatorname{FactArc}_k$ , each one acting on  $\operatorname{FactGr}_k$  over  $\operatorname{Ran}(X)$ .

#### 2.3.1 The Ran version of the arc group

Construction 2.13. Consider the functor

$$\operatorname{Ran}G_{\mathcal{O}}: \mathbb{C}\operatorname{Alg} \to \operatorname{Set}$$

defined in the previous subsection. It is immediate to see that this functor takes values in Set just like  $\operatorname{FactGr}_k$ , and admits a map towards  $\operatorname{Ran}(X)$ .

This functor is a group functor over Ran(X) (that is, its fibers have coherent group structures) under the law

$$(S,g) \cdot (S,h) = (S,g \cdot h).$$

There is a semisimplicial group object of  $(PSh_{\mathbb{C}})_{/Ran(X)}$  assigning

$$[k] \mapsto \overbrace{\operatorname{Ran}G_{\mathcal{O}} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{Ran}G_{\mathcal{O}}}^{k \ times}.$$

The face maps are described by

$$\delta_i: (S, g_1, \ldots, g_k) \mapsto (S, g_1, \ldots, \overbrace{g_i g_{i+1}}^i, \ldots, g_k).$$

**Definition 2.14.** We denote this semisimplicial group functor over Ran(X) by  $FactArc_{\bullet}$ .

Note that  $\operatorname{FactArc}_0 \simeq \operatorname{Ran}(X)$ , and  $\operatorname{FactArc}_1 \simeq \operatorname{Ran}G_0$ .

**Remark 2.15.** The functor FactArc<sub>•</sub> enjoys the 2-Segal property. The verification is straightforward thanks to the multiplication structure.

### 2.3.2 The action on FactGr.

The first observation now is that  $\operatorname{FactArc}_{k-1}$  acts on  $\operatorname{FactGr}_k$  on the left over  $\operatorname{Ran}(X)$  in the following way:

$$(S, g_2, \dots, g_k).(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_k, \alpha_k) =$$
  
=  $(S, \mathcal{F}_1, \alpha_1, \mu_1 g_2^{-1}, \mathcal{F}_2, g_2 \alpha_2, \mu_2 g_3^{-1}, \dots, \mathcal{F}_k, g_k \alpha_k)$ 

where  $g_i \alpha_i$  is the modification of  $\alpha_i$  by  $g_i$  on  $(\mathring{X_R})_{\Gamma_S}$  which, by the usual "local-global" reformulation, induces a new trivialisation of  $\mathcal{F}_i$  outside S.

Call  $\Phi_{k-1,k}$  this action of FactArc<sub>k-1</sub> on FactGr<sub>k</sub>.

Call  $\Xi_{1,k}$  the left action of FactArc<sub>1</sub> on FactGr<sub>k</sub> altering the first trivialisation  $\alpha_1$ .

Call  $\Phi_{k,k}$  the left action of FactArc<sub>k</sub> on FactGr<sub>k</sub> obtained as combination of  $\Xi_{1,k}$  and  $\Phi_{k-1,k}$ .

**Definition 2.16.** Define  $\operatorname{ConvGr}_k$ , the **Ran version of the convolution Grassmannian**, as the quotient of  $\operatorname{FactGr}_k$  by the left action  $\Phi_{k-1,k}$  described above. That is,  $\operatorname{ConvGr}_k = \operatorname{colim}_I G_{\mathcal{O},X^I} \setminus (X^I \times_{\operatorname{Ran}(X)} \operatorname{FactGr}_k)$ , where the terms of the colimits are stack quotients with respect to the étale topology and  $G_{\mathcal{O},X^I}$  acts through the pullback of the action  $\Phi_{k-1,k}$ .

**Remark 2.17.** Conv $Gr_k$  is a presheaf which can alternatively be described as follows:

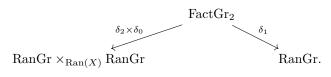
$$R \mapsto \{S \subset X(R), \mathcal{F}_1, \mathcal{G}_2, \dots, \mathcal{G}_k \in \operatorname{Bun}_G(X_R), \alpha_1 : \mathcal{F}_1|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{T}|_{X_R \setminus \Gamma_S},$$
$$\eta_2 : \mathcal{G}_2|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{F}_1|_{X_R \setminus \Gamma_S}, \dots, \eta_k : \mathcal{G}_k|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{G}_{k-1}|_{X_R \setminus \Gamma_S} \}.$$

This is proven in [Rei12, Proposition III.1.10, (1)], because if we take m=k then  $\operatorname{Gr}_p|_{\Delta}$  in loc. cit. is the pullback along  $X^n \to \operatorname{Ran}(X)$  of the functor described in Remark 2.17, and  $\operatorname{Conv}_n^m$  is the pullback along  $X^n \to \operatorname{Ran}(X)$  of our  $\operatorname{ConvGr}_m$ .

## 2.4 The convolution product over Ran(X)

#### 2.4.1 Associative algebra structure on the Hecke stack

**Remark 2.18.** Consider the face maps between  $FactGr_2$  and  $FactGr_1 = RanGr$ . These induce a diagram



This means that the 2-Segal structure induces a nonunital associative algebra structure on RanGr inside the category  $\operatorname{Corr}((\operatorname{PSh}_{\mathbb{C}})_{/\operatorname{Ran}(X)})$  equipped with the Cartesian symmetric monoidal structure. The associativity of the algebra structure should be given by the Segal property: on this matter cfr. [DK19, Chapter 3], and later Chapter 8 and 11 of the same book, where the connection between 2-Segal structures and associative algebras in correspondences is treated from an homotopy-coherent point of view as well.

Therefore, we obtain an object

$$\operatorname{FactGr}^{\times} \in \operatorname{Alg}^{\operatorname{nu}}_{\mathbb{E}_1}(\operatorname{Corr}((\operatorname{PSh}_{\mathbb{C}})_{/\operatorname{Ran}(X)})^{\times}),$$

where  $\mathbb{E}_1^{\otimes} = \mathbf{Assoc}^{\otimes}$  is the associative operad and <sup>nu</sup> stays for "nonunital" (cfr. [Lur17, Section 5.1, Section 5.4.4]).

However, this object has no direct connection with the Mirkovic-Vilonen convolution product. What we need is the following variant.

**Definition 2.19.** Given a category  $\mathcal{C}$  with finite products, we can define a category  $\mathbf{Act}(\mathcal{C})$  having as objects triples  $(C, G, \Phi)$  where  $C \in \mathcal{C}, G \in \mathrm{Grp}(\mathcal{C})$ , and  $\Phi$  is an action of G on C satisfying the usual axioms. We also define the category  $\mathfrak{Act}$  as

$$\dots \Longrightarrow \bullet \Longrightarrow \bullet . \tag{2.1}$$

Note that there is a functor

$$D: \mathbf{Act}(\mathfrak{C}) \to \mathrm{Fun}(\mathfrak{Act}, \mathfrak{C}).$$
 (2.2)

This functor sends a triple  $(C, G, \Phi)$  to the well-known diagram

in  $\mathcal{C}$ . Now if  $\tau$  is a subcanonical Grothendieck topology on the category  $\mathcal{C}$ , then we can consider the embedding  $\mathcal{C} \hookrightarrow \operatorname{Sh}_{\tau}(\mathcal{C})$ . Performing the colimit of the diagram (2.3) in  $\operatorname{Sh}_{\tau}(\mathcal{C})$  is the same as taking the quotient  $C/\Phi$  in the topology  $\tau$ .

Summing up, if we are given a category with finite products  $\mathcal{C}$  and a subcanonical Grothendieck topology  $\tau$ , we have a functor

$$\mathbf{Act}(\mathfrak{C}) \xrightarrow{D} \mathrm{Fun}(\mathfrak{Act}, \mathfrak{C}) \xrightarrow{\mathrm{Yono}} \mathrm{Fun}(\mathfrak{Act}, \mathrm{Sh}_{\tau}\mathfrak{C})) \xrightarrow{\mathrm{colim}} \mathrm{Sh}_{\tau}(\mathfrak{C}).$$

All the functors involved admit a lax monoidal structure with respect to the Cartesian symmetric monoidal structures on all categories. This is indeed a general fact induced by the universal property of the product.

Construction 2.20. Observe now that the same discussion as in Remark 2.18 holds for the 2-Segal object FactArc. This means, in the notations above, that we can assemble the collection of the (FactGr<sub>k</sub>, FactArc<sub>k</sub>,  $\Phi_{k,k}$ ),  $k \in \Delta_{\text{inj}}^{\text{op}}$ , into a nonunital algebra object in

$$\operatorname{Corr}(\operatorname{\mathbf{Act}}((\operatorname{PSh}_{\mathbb{C}})_{/\operatorname{Ran}(X)}))^{\times}.$$

As above, we have the induced diagram

$$\dots \Longrightarrow \operatorname{FactArc}_k \times_{\operatorname{Ran}(X)} \operatorname{FactGr}_k \xrightarrow{\Phi_{k,k}} \operatorname{FactGr}_k .$$
 (2.4)

The standard procedure to compute the quotient  $\Phi_{k,k}\backslash \text{FactGr}_k$  is to compute the colimit of this diagram in the category of étale sheaves, which in the present case is computed as

$$(\operatorname{PSh}(\operatorname{Sh}_{\operatorname{\acute{e}t}}(\operatorname{Sch}_{\mathbb{C}})))_{/\operatorname{Ran}(X)}.$$

However, there is a subtlety. When taking constructible sheaves we need this colimit to be sent to a limit of  $\infty$ -categories (see Section 1.1.3). In order to do so, we need to treat the arrows of the diagram (2.1) with contravariant functoriality (and then rely on the descent properties of the functor Cons). Therefore, we interpret the diagram (2.1) as a functor

$$\mathfrak{Act} \to ((\mathrm{PSh}_{\mathbb{C}})_{/\mathrm{Ran}(X)})^{\mathrm{op}} \hookrightarrow \mathrm{Corr}((\mathrm{PSh}_{\mathbb{C}})_{/\mathrm{Ran}(X)})$$

and then we take the colimit of this diagram in the larger category

$$\operatorname{Corr}((\operatorname{PSh}(\operatorname{Sh}_{\operatorname{\acute{e}t}}(\operatorname{Sch}_{\mathbb{C}})))_{/\operatorname{Ran}(X)}).$$

In the notations of Appendix A.1.3, this provides an algebra object

$$\operatorname{Hck}^{\times} \in \operatorname{Alg}^{\operatorname{nu}}_{\mathbb{R}_1}(\operatorname{Corr}((\operatorname{Sh}_{\operatorname{\acute{e}t}}(\operatorname{PSh}_{\mathbb{C}}))_{/\operatorname{Ran}(X)})^{\times}).$$

We will denote by  $\operatorname{Hck}_k$  the quotient  $\Phi_{k,k}\backslash\operatorname{FactGr}_k$  in the category

$$\mathrm{Sh}_{\mathrm{\acute{e}t}}(\mathrm{PSh}_{\mathbb{C}})_{/\mathrm{Ran}(X)}.$$

This definition is motivated by the fact that, for any  $x \in X(\mathbb{C})$ , the quotient  $\mathfrak{H}\operatorname{ck}_1 \times_{\operatorname{Ran}(X)} \{x\} = G_{\mathbb{O}} \setminus \operatorname{Gr}$  is usually called the **Hecke stack**.

## 2.4.2 Connection with the Mirkovic-Vilonen convolution product

We will now show that the algebra object  $\operatorname{Hck}^{\times}$  describes the Mirkovic-Vilonen convolution diagram. First of all, consider the action of  $\operatorname{FactArc}_k$  on  $\operatorname{ConvGr}_k$  induced by  $\Xi_{1,k}$ , which we still call  $\Xi_{1,k}$  by abuse of notation, and consider also  $\Phi_{1,1}$  as an action of  $\operatorname{Ran}G_{\mathbb{O}}$  on  $\operatorname{Ran}G_{\mathbb{C}}$ . Note that  $\Phi_{1,1}^{\times k}$  is an action of  $\operatorname{Ran}G_{\mathbb{O}}^{\times_{\operatorname{Ran}(X)}k}$  on  $\operatorname{Ran}Gr^{\times_{\operatorname{Ran}(X)}k}$ . The actions  $\Phi_{1,1}^{\times k}$  on  $\operatorname{Ran}Gr^{\times_{\operatorname{Ran}(X)}k}$ ,  $\Phi_{k,k}$  on  $\operatorname{FactGr}_k$ ,  $\Xi_{1,k}$  on  $\operatorname{ConvGr}_k$  and  $\Phi_{1,1}$  on  $\operatorname{Ran}Gr$  are compatible with the k-associative convolution diagram

$$\operatorname{Fact}\operatorname{Gr}_k \xrightarrow{q} \operatorname{Conv}\operatorname{Gr}_k$$

$$\operatorname{Ran}\operatorname{Gr}^{\times_{\operatorname{Ran}(X)}k}$$

$$\operatorname{Ran}\operatorname{Gr}$$

where:

- p is the map that forgets all the trivialisations  $\mu_i$ .
- q is the projection to the quotient with respect to  $\Phi_{k-1,k}$ , alternatively described as follows: we keep  $\mathcal{F}_1$  and  $\alpha_1$  intact, and define  $\mathcal{G}_h$  by induction as the formal gluing of  $\mathcal{G}_{h-1}$  (or  $\mathcal{F}_1$  if h=1) and  $\mathcal{F}_h$  along  $\mu_{h-1}$  and  $\alpha_h$ : indeed,  $\mu_{h-1}$  is a trivialisation of  $\mathcal{G}_{h-1}$  over the formal neighbourhood of  $\Gamma_S$  via the canonical isomorphism between  $\mathcal{G}_{h-1}$  and  $\mathcal{F}_{h-1}$  on that formal neighbourhood. The isomorphism

$$\eta_h: \mathcal{G}_h|_{X_R \setminus \Gamma_S} \xrightarrow{\sim} \mathcal{G}_{h-1}|_{X_R \setminus \Gamma_S}$$

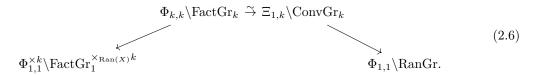
is provided canonically by the formal gluing procedure.

• m is the map sending

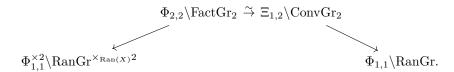
$$(S, \mathcal{F}_1, \alpha_1, \mathcal{G}_2, \eta_2, \dots, \mathcal{G}_k, \eta_k) \mapsto (S, \mathcal{G}_k, \alpha_1 \circ \eta_2 \circ \dots \circ \eta_k).$$

**Remark 2.21.** Consider the special case k = 2. Note first of all that, since this diagram lives over Ran(X), we can take its fiber at  $\{x\} \in Ran(X)$ . Under the identifications of Proposition 1.8, we obtain the diagram (1.2).

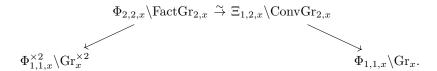
We can consider the diagram



The horizontal map is an equivalence since it exhibits its target as the quotient  $\Xi_{1,k}\Phi_{k-1,k}\backslash \operatorname{FactGr}_k \simeq \Phi_{k,k}\backslash \operatorname{FactGr}_k$ . Therefore, the latter diagram is exactly the one described by  $\operatorname{Hck}^{\times}$  and its face maps. For k=2 one obtains:



**Remark 2.22.** Take again the fiber of this diagram at the point  $\{x\} \in \text{Ran}(X)$ . This results in a diagram of the form



Recall now the identifications of Proposition 1.8.

Here, the action  $\Phi_{1,1,x}$  is the usual left-multiplication action of  $G_{\mathcal{O}}$  over Gr. The action  $\Phi_{2,2,x}$  is the action of  $G_{\mathcal{O}} \times G_{\mathcal{O}}$  on  $G_{\mathcal{K}} \times G_{\mathcal{C}}$  given by  $(g_1, g_2).(h, \gamma) = (g_1hg_2^{-1}, g_2\gamma)$ . Finally, the action  $\Xi_{1,2,x}$  on ConvGr<sub>2,x</sub> is the action of  $G_{\mathcal{O}}$  on  $G_{\mathcal{K}} \times^{G_{\mathcal{O}}}$  Gr given by  $g[h, \gamma] = [gh, \gamma]$ .

Consider now two perverse sheaves  $\mathcal{F}$  and  $\mathcal{G}$  on the quotient  $\Phi_{1,1,x}\backslash \mathrm{Gr}_x$ . This is equivalent to the datum of two  $G_{\mathbb{O}}$ -equivariant perverse sheaves over  $\mathrm{Gr}$ . We can perform the external product  $\mathcal{F}\boxtimes\mathcal{G}$  living over  $\Phi_{1,1,x}^{\times 2}\backslash \mathrm{Gr}^{\times 2}$ , and then pull it back to  $\Phi_{2,2,x}\backslash \mathrm{Fact}\mathrm{Gr}_{2,x}$ . Under the equivalence displayed above, this can be interpreted as a  $\Xi_{1,2}$ -equivariant perverse sheaf over  $\mathrm{Conv}\mathrm{Gr}_2$ . By construction, this is exactly what  $[\mathrm{MV07}]$  call  $\mathcal{F}\boxtimes\mathcal{G}$  (up to shifts). Its pushforward along m is therefore the sheaf  $\mathcal{F}\star\mathcal{G}\in\mathrm{Perv}_{G_{\mathbb{O}}}(\mathrm{Gr}_x)$ .

The same construction can be done with constructible sheaves instead of perverse sheaves. In the next sections, we will describe an algebra structure on the category  $Cons_{G_0}(Gr_G)$  of equivariant constructible sheaves over the affine Grassmannian. This  $\infty$ -category is equivalent to  $Cons(G_0 \backslash Gr_G)$ , and from this point of view the product law is exactly the one described here. Section 1.1.3 tells us that, unlike in the case of perverse sheaves, before considering constructible sheaves one really needs to pay attention to the stratification of the affine Grassmannian and its variants. We do this in the next section.

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#### 2.5 Stratifications

## 2.5.1 Stratification of Gr, RanGr and FactGr<sub>k</sub>

Construction 2.23. Recall from Remark 1.3 that the affine Grassmannian has a stratification in Schubert cells. We have explained in Section 1.1.3 that we are interested in considering constructible sheaves on the affine Grassmannian which are equivariant and constructible with respect to this stratification. We will now extend the stratification to the "global" and "convolution" versions of the affine Grassmannian that we are considering, namely  $FactGr_k$ . We give a definition of stratified schemes and presheaves in Definition A.2 and Construction A.3.

In that formalism, the stratification in Schubert cells can be seen as a continuous map of ind-topological spaces  $\operatorname{zar}(\operatorname{Gr}_G) \to \mathbb{X}_{\bullet}(T)^+$  where  $\operatorname{zar}(\operatorname{Gr}_G)$  is the Zariski ind-topological space associated to  $\operatorname{Gr}_G$ , and  $\mathbb{X}_{\bullet}(T)^+$  is the poset of dominant coweights of any maximal torus  $T \subset G$ . Therefore, the datum  $(\operatorname{Gr}_G, \mathscr{S})$  may be interpreted as an object of  $\operatorname{StrPSh}_{\mathbb{C}}$ . The global version  $\operatorname{Gr}_X$  admits a stratification described in [Zhu16, eq. 3.1.11], which detects the monodromy of the pair (bundle, trivialisation) at the chosen point. By filtering  $\operatorname{Gr}_X$  by the lattice filtration (see discussion after Theorem 1.2) at the chosen point, we can exhibit  $\operatorname{Gr}_X$  as a stratified ind-scheme, or more generally a stratified presheaf, whose indexing poset is again  $\mathbb{X}_{\bullet}(T)^+$ .

**Notation 2.24.** From now on, an arrow of the form  $\mathcal{X} \to P$ , where  $\mathcal{X}$  is a complex presheaf and P is a poset, will denote a geometric morphism  $\operatorname{zar}(\mathcal{X}) \to \operatorname{Sh}(P), \operatorname{zar}(\mathcal{X})$  being  $\operatorname{colim}_{X \to \mathcal{X}} \operatorname{scheme} \operatorname{zar}(X)$ .

Note now that, with the notations introduced in Remark 2.3,  $Gr_{X^I}$  has a stratification by  $(\mathbb{X}_{\bullet}(T)^+)^I$  for any I, where  $(\mathbb{X}_{\bullet}(T)^+)^I$  has the lexicographic order, whose order topology coincides with the product topology. This stratification is defined as follows (see [Zhu16, Proposition 3.1.14]). Recall from [Zhu16] the so-called factorising property of the Beilinson-Drinfeld Grassmannian. For |I| = 2, it is the following:

**Proposition 2.25** ([Zhu16, Proposition 3.1.13]). There are canonical isomorphisms  $\operatorname{Gr}_X \simeq \operatorname{Gr}_{X^2 \times_{X^2,\Delta}} X, c : \operatorname{Gr}_{X^2|_{X^2 \setminus \Delta}} \simeq (\operatorname{Gr}_X \times \operatorname{Gr}_X)|_{X^2 \setminus \Delta}.$ 

For an arbitrary I, the property is stated in [Zhu16, Theorem 3.2.1]. This property allows us to define a stratification on  $Gr_{X^I}$  for an arbitrary I:

**Definition 2.26.** For |I| = 2,  $(\operatorname{Gr}_{X^I})_{(\mu,\lambda)} \subset \operatorname{Gr}_{X^2}$  is defined to be the closure of  $\operatorname{Gr}_X \times \operatorname{Gr}_X \xrightarrow{\sim} (\operatorname{Gr}_{X^2})|_{\Delta}$  inside  $\operatorname{Gr}_{X^2}$ .

For an arbitrary I, the definition uses the small diagonals.

Now, by construction,  $\operatorname{FactGr}_k$  is a torsor over  $\operatorname{RanGr} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{RanGr}$ . Each RanGr is the colimit  $\operatorname{colim}_{I \in \mathcal{F}_{\operatorname{in}_{\operatorname{surj}}}} \operatorname{Gr}_{X^I}$ , which by Remark 2.3 inherits from  $\operatorname{Gr}_G^I$  a stratification indexed by the poset  $(\mathbb{X}_{\bullet}(T)^+)^I$ . Therefore RanGr inherits in turn a stratification by the poset  $\operatorname{colim}_{I \in \mathcal{F}_{\operatorname{in}_{\operatorname{surj}}}} (\mathbb{X}_{\bullet}(T)^+)^I = \operatorname{Ran}(\mathbb{X}_{\bullet}(T)^+)$  (whose topology as a poset is the same as the colimit topology). Finally,  $\operatorname{FactGr}_k$  admits a stratification by the poset  $\operatorname{Ran}(\mathbb{X}_{\bullet}(T)^+)^k$ , inherited from the bundle

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map

$$\operatorname{FactGr}_k \to \overbrace{\operatorname{RanGr} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{RanGr}}^k.$$

**Definition 2.27.** We denote the stratification of FactGr<sub>k</sub> over Ran( $(X_{\bullet})^+$ ) by  $s_k$ .

Note that, by construction, the natural map  $r_k : \operatorname{FactGr}_k \to \operatorname{Ran}(X)$  is stratified when we take the cardinality stratification on  $\operatorname{Ran}(X)$ .

## 2.5.2 Interaction with the semisimplicial structure and with the action of FactArc.

Now we want to study the interaction between the stratifications and the semisimplicial structure. In particular, we want to prove that the simplicial maps preserve the stratifications, thus concluding that FactGr<sub>•</sub> upgrades to a semisimplicial object in StrPSh.

**Definition 2.28.** For any I, consider the semisimplicial group

$$\operatorname{Cw}_{ullet}^{(I)}$$

(coweights) defined by

$$\operatorname{Cw}_k^{(I)} = (\mathbb{X}_{\bullet}(T)^+)^{Ik}$$

$$\delta_j: (\mu_1, \dots, \mu_k) \mapsto (\mu_1, \dots, \mu_{j-1}, \mu_j + \mu_{j+1}, \mu_{j+2}, \dots, \mu_k).$$

Thus in general if  $\phi:[h]\to[k],\ i=|I|$ , we have

$$\operatorname{Cw}^{(I)}(\phi) : (\mu_{11}, \dots, \mu_{1i}, \dots, \mu_{k1}, \dots, \mu_{ki}) \mapsto$$

$$(\mu_{11} + \dots + \mu_{\phi(1),1}, \mu_{12} + \dots + \mu_{\phi(1),2}, \dots, \mu_{1i} + \dots + \mu_{\phi(1),i},$$

$$\mu_{\phi(1)+1,1} + \dots + \mu_{\phi(2),1}, \dots, \mu_{\phi(1)+1,i} + \dots + \mu_{\phi(2),i},$$

$$\dots,$$

$$\mu_{\phi(k-1)+1,1} + \dots + \mu_{\phi(k),1}, \dots, \mu_{\phi(k-1)+1,i} + \dots + \mu_{\phi(k),i}$$
.

Define also  $Cw_k = \operatorname{colim}_I Cw_k^{(I)}$ .

By [Zhu16, Proposition 3.1.14], one has that the semisimplicial map  $X^2 \times_{\operatorname{Ran}(X)^2} \operatorname{FactGr}_2 = \operatorname{Gr}_{X^2|\Delta} \to X \times_{\operatorname{Ran}(X)} \operatorname{FactGr}_1 = \operatorname{Gr}_X$  (seen here on the component  $I = \{1\}$  of the colimit) sends the stratum  $(\mu, \lambda)$  to the stratum  $\mu + \lambda$ . For an arbitrary I the argument is the same, and therefore we have that the semisimplicial map  $\operatorname{FactGr}_2 \to \operatorname{FactGr}_1$  sends the stratum  $(\mu_1, \ldots, \mu_i, \lambda_1, \ldots, \lambda_i)$  to the stratum  $(\mu_1 + \lambda_1, \ldots, \mu_i + \lambda_i)$ , where i is the cardinality of the set in  $\operatorname{Ran}(X)$  that we are considering. This means that if  $S: \mathbb{X}_{\bullet}(T)^+ \times \mathbb{X}_{\bullet}(T)^+ \to \mathbb{X}_{\bullet}(T)^+$  is the sum map, the diagram

$$\begin{array}{ccc} \operatorname{FactGr}_2 & \xrightarrow{\delta_1} & \operatorname{FactGr}_1 \\ & & \downarrow_{\tilde{s}_2} & & \downarrow_{\tilde{s}_1} \end{array} \\ \mathbb{X}_{\bullet}(T)^+ \times \mathbb{X}_{\bullet}(T)^+ & \xrightarrow{S} & \mathbb{X}_{\bullet}(T)^+ \end{array}$$

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commutes.

The case of an arbitrary I uses similar arguments. We can therefore say that for any face map  $\phi: [h] \to [k]$  the induced square

$$FactGr_{k} \xrightarrow{FactGr(\phi)} FactGr_{h}$$

$$\downarrow^{\tilde{s}_{k}} \qquad \downarrow^{\tilde{s}_{h}}$$

$$Cw_{k} \xrightarrow{Cw(\phi)} Cw_{h}$$

commutes. Note that the top row is a map of presheaves, so the correct interpretation of this diagram is: the diagrams

$$X^I \times_{\operatorname{Ran}(X)} (\operatorname{FactGr}_k)_{\leq N} \longrightarrow X^I \times_{\operatorname{Ran}(X)} (\operatorname{FactGr}_h)_{\leq N}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow \qquad \qquad ((\mathbb{X}_{\bullet}(T)^+)^I)^h \qquad \longrightarrow ((\mathbb{X}_{\bullet}(T)^+)^I)^h$$

commute for every I and N, where N refers to the lattice filtration.

**Definition 2.29.** Let  $a_k$  be the stratification on FactArc<sub>k</sub> induced by the map to Ran(X) and the cardinality stratification  $\kappa$  of Ran(X).

After endowing  $\operatorname{Ran}(X)$  with the cardinality stratification  $\kappa$  as above, we can consider the category  $(\operatorname{StrPSh}_{\mathbb{C}})_{/\operatorname{Ran}(X)}$ . Note that  $((\operatorname{FactGr}_k, s_k) \to (\operatorname{Ran}(X), \kappa))$  belongs to this category, and  $((\operatorname{FactArc}_k, a_k) \to (\operatorname{Ran}(M), \kappa))$  belongs to  $\operatorname{Grp}((\operatorname{StrPSh}_{\mathbb{C}})_{/\operatorname{Ran}(X)})$ , since it is a group functor *over Ran*.

**Lemma 2.30.** The strata of the stratification  $s_k$  are invariant with respect to the action of FactArc<sub>k</sub> over FactGr<sub>k</sub>.

*Proof.* We know (see for example [Zhu16]) that the orbits of the Schubert stratification of the affine Grassmannian are the orbits of the action of  $G_{\mathbb{O}}$ . The result at the level of FactGr<sub>k</sub> follows straightforwardly from the definitions.

**Remark 2.31.** Note that both (FactGr<sub>•</sub>,  $s_{\bullet}$ ) and (FactArc<sub>•</sub>,  $a_{\bullet}$ ) enjoy the 2-Segal property.

*Proof.* We want to use the unstratified version of the same result, proved in Section 2.2.2. In order to do this, it suffices to prove that the functor  $StrPSh_{\mathbb{C}} \to PSh_{\mathbb{C}}$  preserves and reflects finite limits. We can reduce this statement to the one that  $StrSch_{\mathbb{C}} \to Sch_{\mathbb{C}}$  does. By definition, this follows from Lemma A.4.

We can thus summarise the content of this whole section as follows.

**Theorem 2.32.** There exists a functor

$$\mathbf{Act}\mathrm{FactGr}_{\bullet}: \Delta^{\mathrm{op}}_{\mathrm{inj}} \to \mathbf{Act}((\mathrm{StrPSh}_{\mathbb{C}})_{/(\mathrm{Ran}(X),\kappa)})$$

$$[k] \mapsto (\operatorname{FactGr}_k \to \operatorname{Ran}(X), \operatorname{FactArc}_k \to \operatorname{Ran}(X),$$
  
 $\Phi_{k,k} : \operatorname{FactArc}_k \times_{\operatorname{Ran}(X)} \operatorname{FactGr}_k \to \operatorname{FactGr}_k),$ 

which enjoys the 2-Segal property, and such that:

 $\mathbf{Act}\mathrm{Fact}\mathrm{Gr}_1 = (\mathrm{Ran}\mathrm{Gr} \to \mathrm{Ran}(X), \mathrm{Ran}G_{\mathcal{O}} \to \mathrm{Ran}(X),$  $\Phi_{1,1} : \mathrm{Ran}G_{\mathcal{O}} \times_{\mathrm{Ran}(X)} \mathrm{Ran}\mathrm{Gr} \to \mathrm{Ran}\mathrm{Gr})$ 

• the higher values of the 2-Segal object describe the Mirkovic-Vilonen convolution diagram and its associativity in the sense of Section 2.4.2.

**Definition 2.33.** Recall the definitions in Appendix A.1.3. By abuse of notation, we denote by

$$\operatorname{Hck}^{\times} \in \operatorname{Alg}^{\operatorname{nu}}_{\mathbb{E}_1}(\operatorname{Corr}(\operatorname{Sh}_{\operatorname{str\acute{e}t}}(\operatorname{StrPSh}_{\mathbb{C}})_{/\operatorname{Ran}(M)}))^{\times})$$

the "stratified version" of Construction 2.20.

# 3 Fusion over the Ran space

## 3.1 Analytification

## 3.1.1 Topological versions of $FactGr_k$ and $FactArc_k$

In order to take into account the topological properties of the affine Grassmannian and of its global variants, we will now analytify the construction performed in the previous section. This will allow us to consider the complex topology naturally induced on the analytic analogue of the prestacks  $\operatorname{FactGr}_k$  by the fact that X is a complex curve, as well as a naturally induced stratification on the resulting complex analytic spaces.

In A.1 we describe the stratified analytification functor, which in turn induces a functor between stratified complex presheaves and stratified topological spaces

$$\mathfrak{strtop}: \mathrm{Str}\mathrm{PSh}_{\mathbb{C}} \to \mathrm{Str}\mathrm{TSpc}$$

(see Construction A.3). Recall that this functor preserves finite limits.

Hence, if we precompose  $\mathfrak{strtop}$  with  $\operatorname{FactGr}_{\bullet}:\Delta^{\operatorname{op}}_{\operatorname{inj}}\to\operatorname{StrPSh}_{\mathbb{C}}$  we obtain a 2-Segal semisimplicial object in stratified spaces. Also, since  $\mathfrak{strtop}$  sends stratified étale coverings to stratified coverings in the topology of local homeomorphisms, it extends to a functor between the categories of sheaves.

Notation 3.1. For simplicity, we set  $\Re \operatorname{Ran} \operatorname{Gr} = \operatorname{\mathfrak{strtop}}(\operatorname{Ran} \operatorname{Gr})$ ,  $\operatorname{\mathfrak{F}act} \operatorname{\mathfrak{Gr}}_k = \operatorname{\mathfrak{strtop}}(\operatorname{Fact} \operatorname{Gr}_k)$ , and  $\operatorname{\mathfrak{F}act} \operatorname{\mathfrak{Gr}}_{k,x} = \operatorname{\mathfrak{strtop}}(\operatorname{Fact} \operatorname{Gr}_{k,x}) = \operatorname{\mathfrak{strtop}}(\operatorname{Fact} \operatorname{Gr}_k \times_{\operatorname{Ran}(X)} \{x\})$ .

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This construction admits a relative version over  $\operatorname{Ran}(X)$  which is not exactly the natural one, because of a change of topology on  $\operatorname{top}(\operatorname{Ran}(X))$ . Indeed, the analytification functor induces a well-defined analytification of  $\operatorname{Ran}(X)$ , which has the form  $\operatorname{top}(\operatorname{Ran}(X)) = \operatorname{colim}_I(\operatorname{top}(X))^I$ . In turn this admits a map to another topological space  $\operatorname{Ran}(M)$ :

**Remark 3.2.** Let now  $M = \mathsf{top}(X)$ , which is a real topological manifold of dimension 2. In [Lur17, Definitions 5.5.1.1, 5.5.1.2] J. Lurie defines the Ran space  $\mathrm{Ran}(M)$  of a topological manifold. By definition, there is a map of topological spaces  $\mathsf{top}(\mathrm{Ran}(X)) \to \mathrm{Ran}(M)$ . Indeed,

$$\mathfrak{top}(\operatornamewithlimits{colim}_I X^I) \simeq \operatornamewithlimits{colim}_I \mathfrak{top}(X^I) \simeq \operatornamewithlimits{colim}_I (\mathfrak{top} X)^I = \operatornamewithlimits{colim}_I M^I,$$

because  $\mathfrak{top}$  is a left Kan extension. Now each term of the colimit is the space of I-indexed collections of points in  $X(\mathbb{C})$ , and hence it admits a map of sets towards  $\mathrm{Ran}(M)$ . This is a continuous map: indeed, let  $f: I \to X(\mathbb{C})$  be a function such that  $f(I) \in \mathrm{Ran}(\{U_i\})$  for some disjoint open sets  $U_i$ . Then there is an open set V in  $\mathrm{Map}_{\mathrm{TSpc}}(I, \mathfrak{top}(X))$  containing f and such that  $\forall g \in V, g(I) \in \mathrm{Ran}(\{U_i\})$ : for instance,

$$V = \bigcap_{i} \{g : I \to X \mid g(f^{-1}(f(I) \cap U_i)) \subset U_i\}$$

suffices.

This induces a continuous map from  $\operatorname{colim} M^I$  to  $\operatorname{Ran}(M)$  by the universal property of the colimit topology, and therefore a continuous map  $\operatorname{top}(\operatorname{Ran}(X)^{\operatorname{an}}) \to \operatorname{Ran}(M)$  which is the identity settheoretically.

Construction 3.3. Composition with the map that we have just described yields a functor which we call again

$$\mathfrak{strtop}: (\operatorname{StrPSh}_{\mathbb{C}})_{/\operatorname{Ran}(X)} \to \operatorname{StrTSpc}_{/\operatorname{Ran}(M)}.$$
 (3.1)

In particular, we obtain a map  $\rho_{\bullet}$ :  $\mathfrak{stxtop}(\operatorname{FactGr}_{\bullet}, s_{\bullet}) \to (\operatorname{Ran}(M), \kappa)$ , which we consider as an object of  $2\operatorname{-Seg^{ss}}(\operatorname{StrTSpc}_{/(\operatorname{Ran}(M),\kappa)})$  (we abuse of notation by denoting the cardinality stratification again by  $\kappa$ ).

Analogously, the functor  $\mathbf{Act}$ Fact $\mathrm{Gr}_{\bullet}: \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Act}((\mathrm{StrPSh}_{\mathbb{C}})_{/(\mathrm{Ran}(X),\kappa)})$  induces a functor  $\mathbf{Act}$ Fact $\mathrm{Gr}_{\bullet}: \Delta_{\mathrm{inj}}^{\mathrm{op}} \to \mathbf{Act}(\mathrm{StrTSpc}_{/(\mathrm{Ran}(M),\kappa)})$  and finally a topological analog of the algebra structure on the Hecke stac (see Definition 2.33 and Appendix A.1.3)k:

$$\mathcal{H}ck^{\times} \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_{1}}(\mathrm{Corr}(\mathrm{Sh}_{\mathrm{strloc}}(\mathrm{StrTSpc})_{/(\mathrm{Ran}(M),\kappa)})^{\times}).$$

In the rest of the section we will use the fact that every construction up to now is relative over Ran(M) and transform  $\mathcal{H}ck$  into a double algebra object

$$\mathcal{H}ck^{\times} \in Alg_{Fact(M)\otimes}(Alg_{\mathbb{E}_{1}}^{nu}(Corr(Sh_{strloc}(StrTSpc))^{\times})),$$

where  $\operatorname{Fact}(M)^{\otimes}$  is a suitable operadic structure on the categoy of open sets of  $\operatorname{Ran}(M)$ . However, it will be simpler to perform the needed constructions by manipulating  $\operatorname{Fact}\operatorname{Gr}_{\bullet}$  and  $\operatorname{Fact}\operatorname{Arc}_{\bullet}$  as 2-Segal objects in  $\operatorname{Str}\operatorname{TSpc}$ , and then recover the result for  $\operatorname{Hck}$ .

## 3.1.2 Preimage functors

For every open  $U \subset \text{Ran}(M)$ , there exists a "preimage space"  $(\text{Fact}\mathfrak{Gr}_k)_U \in \text{StrTSpc}$ , whose underlying set can be described by

{tuples in FactGr<sub>k</sub>(
$$\mathbb{C}$$
) such that S lies in U}.

Formally:

**Definition 3.4.** We define functors  $\mathfrak{CGr}_k : \mathfrak{Open}(\mathrm{Ran}(M)) \to \mathrm{StrTSpc}$  as

$$\operatorname{\mathcal{O}pen}(\operatorname{Ran}(M)) \subset \operatorname{TSpc}_{/\operatorname{Ran}(M)} \xrightarrow{\rho_k^{-1}} \operatorname{StrTSpc}_{/\operatorname{Fact}\operatorname{\mathcal{G}r}_k} \to \operatorname{StrTSpc}_{/\operatorname{Ran}(M)}$$

sending U to  $(U, \kappa|_U)$  and finally to  $(\rho_k^{-1}(U) \to \text{Ran}(M), s_k|_{\rho_k^{-1}(U)})$ .

This operation is compatible with the semisimplicial structure, and therefore we obtain a functor:

$$\mathfrak{CGr}_{\bullet}: \mathfrak{O}pen(\mathrm{Ran}(M)) \to 2\text{-Seg}^{\mathrm{ss}}(\mathrm{StrTSpc}_{/\mathrm{Ran}(M)}).$$

We can perform the same restriction construction as above for  $\mathcal{F}act\mathcal{A}rc_k$  and obtain stratified topological groups

$$(\operatorname{\mathcal{F}act} \operatorname{\mathcal{A}rc}_k)_U$$

acting on  $(\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_k)_U \in \operatorname{StrTSpc}$  over  $\operatorname{Ran}(M)$ , functorially in

$$U \in \mathcal{O}pen(Ran(M))$$

and k. We denote the functor  $U \mapsto ((\operatorname{\mathcal{F}act} \operatorname{\mathcal{A}rc}_k)_U \to \operatorname{Ran}(M))$  by

$${\mathcal{CA}\mathrm{rc}}_k: {\mathbb{O}\mathrm{pen}}(\mathrm{Ran}(M)) \to \mathrm{Grp}(\mathrm{Str}\mathrm{TSpc}_{/\mathrm{Ran}(M)}).$$

Again, this is functorial in k.

**Remark 3.5.** The two functors  $\mathfrak{CGr}_k$  and  $\mathfrak{CArc}_k$  are hypercomplete cosheaves with values in  $\mathrm{StrTSpc}_{/(\mathrm{Ran}(M),\kappa)}$ .

## 3.2 Fusion

## Definition 3.6. Let

$$\mathrm{Str}\mathrm{TSpc}^{\odot}_{/(\mathrm{Ran}(M),\kappa)}$$

be the following symmetric monoidal structure on  $\operatorname{StrTSpc}_{/(\operatorname{Ran}(M),\kappa)}$ : if  $\xi: \mathfrak{X} \to \operatorname{Ran}(M), v: \mathfrak{Y} \to \operatorname{Ran}(M)$  are stratified continuous maps, we define  $\xi \odot v$  to be the **disjoint product** 

$$(\mathfrak{X} \times \mathfrak{Y})_{\text{disj}} = \{ x \in \mathfrak{X}, y \in \mathfrak{Y} \mid \xi(x) \cap \upsilon(y) = \varnothing \}$$

together with the map towards Ran(M) induced by the stratified map

$$union: (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}} \to \mathrm{Ran}(M)$$

$$(S,T)\mapsto S\sqcup T.$$

Note that the union map defined on the whole product  $Ran(M) \times Ran(M)$  is not stratified.

Recall the definition of the operad  $\operatorname{Fact}(M)^{\otimes}$  from [Lur17, Definition 5.5.4.9]. The aim of this subsection is to extend the  $\operatorname{\mathcal{CGr}}_k$ 's and the  $\operatorname{\mathcal{CArc}}_k$ 's to maps of operads respectively  $\operatorname{\mathcal{CGr}}_k^{\odot}:\operatorname{Fact}(M)^{\otimes}\to\operatorname{StrTSpc}_{/\operatorname{Ran}(M)}^{\odot}$  and  $\operatorname{\mathcal{CArc}}_k^{\odot}:\operatorname{Fact}(M)^{\otimes}\to\operatorname{Grp}(\operatorname{StrTSpc}_{/\operatorname{Ran}(M)})^{\odot}:$  the idea is that the first one should encode the gluing of sheaves trivialised away from disjoints systems of points, and the second one should behave accordingly.

#### 3.2.1 The gluing map

We turn back for a moment to the algebraic side.

**Definition 3.7.** Let  $(\operatorname{Ran}(X) \times \operatorname{Ran}(X))_{\operatorname{disj}}$  be the subfunctor of  $\operatorname{Ran}(X) \times \operatorname{Ran}(X)$  parametrising those  $S, T \subset X(R)$  for which  $\Gamma_S \cap \Gamma_T = \emptyset$ .

Let also  $(\operatorname{FactGr}_k \times \operatorname{FactGr}_k)_{\operatorname{disj}}$  be the preimage of  $(\operatorname{Ran}(X) \times \operatorname{Ran}(X))_{\operatorname{disj}}$  with respect to the map  $r_k \times r_k : \operatorname{FactGr}_k \times \operatorname{FactGr}_k \to \operatorname{Ran}(X) \times \operatorname{Ran}(X)$ .

**Proposition 3.8.** There is a map of strafied presheaves  $\chi_k$ : (FactGr<sub>k</sub> × FactGr<sub>k</sub>)<sub>disj</sub>  $\rightarrow$  FactGr<sub>k</sub> representing the gluing of sheaves with trivialisations outside disjoint systems of points.

*Proof.* The map  $\chi_k$  is defined as follows: we start with an object

$$(S, \mathcal{F}_1, \alpha_1, \mu_1, \dots, \mathcal{F}_k, \alpha_k), (T, \mathcal{G}_1, \beta_1, \nu_1, \dots, \mathcal{G}_k, \beta_k),$$

where  $S \cap T = \emptyset$ . We want to obtain a sequence  $(P, \mathcal{H}_1, \gamma_1, \zeta_1, \dots, \mathcal{H}_k, \gamma_k)$ . Since the graphs of S and T are disjoint,  $X_R \setminus \Gamma_S$  and  $X_R \setminus \Gamma_T$  form a Zariski open cover of  $X_R$ . Therefore, by the descent property of the stack  $\mathbf{Bun}_G$ , every couple  $\mathcal{F}_i, \mathcal{G}_i$  can be glued by means of  $\alpha_i$  and  $\beta_i$ .

Each of these glued sheaves, that we call  $\mathcal{H}_i$ , inherits a trivialisation  $\gamma_i$  outside  $\Gamma_S \cup \Gamma_T$ , which is well-defined up to isomorphism (it can be seen both as  $\alpha_i|_{X_R\setminus(\Gamma_S\cup\Gamma_T)}$  or as  $\beta_i|_{X_R\setminus(\Gamma_S\cup\Gamma_T)}$ ). Now set  $P=S\cup T$  (in the usual sense of joining the two collections of points).

It remains to define the glued formal trivialisations. However, to define a trivialisation  $\zeta_i$  of  $\mathcal{H}_i$  over the formal neighbourhood of  $\Gamma_P$  amounts to look for a trivialisation of  $\mathcal{H}_i$  on the formal neighbourhood of  $\Gamma_S \sqcup \Gamma_T$ . But the first part of this union is contained in  $X_R \setminus \Gamma_T$ , where  $\mathcal{H}_i$  is canonically isomorphic to  $\mathcal{F}_i$  by construction; likewise, the first part of the union is contained in  $X_R \setminus \Gamma_S$ , where  $\mathcal{H}_i$  is canonically isomorphic to  $\mathcal{G}_i$  by construction. Hence, the originary trivialisations  $\mu_i$  and  $\nu_i$  canonically provide the desired datum  $\zeta_i$ , and the construction of the map is complete.

Moreover, this map is stratified. Indeed, we have the torsor  $\operatorname{FactGr}_k \to \operatorname{RanGr} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{RanGr}$ , and the stratification on  $\operatorname{FactGr}_k$  is the pullback of the one on  $\operatorname{RanGr} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{RanGr}$ . Now for any I, J finite sets, the map  $(\operatorname{Gr}_{X^I} \times \operatorname{Gr}_{X^J})_{\operatorname{disj}} \to \operatorname{Gr}_{X^I \sqcup J}$  is stratified by definition (cfr. Definition 2.26). Therefore,

$$((\overbrace{\operatorname{Gr}_{X^I}\times_{X^I}\cdots\times_{X^I}\operatorname{Gr}_{X^I}}^k)\times(\overbrace{\operatorname{Gr}_{X^J}\times_{X^J}\cdots\times_{X^J}\operatorname{Gr}_{X^J}}^k))_{\operatorname{disj}}\to$$

$$\to \overbrace{\operatorname{Gr}_{X^{I\sqcup J}}\times_{X^{I\sqcup J}}\cdots\times_{X^{I\sqcup J}}\operatorname{Gr}_{X^{I\sqcup J}}}^k$$

is stratified, and taking the colimit for  $I \in \mathcal{F}in_{surj}$ , we obtain that

$$((\operatorname{RanGr}\times_{\operatorname{Ran}(X)}\cdots\times_{\operatorname{Ran}(X)}\operatorname{RanGr})\times(\operatorname{RanGr}\times_{\operatorname{Ran}(X)}\cdots\times_{\operatorname{Ran}(X)}\operatorname{RanGr}))_{\operatorname{disj}}$$

is stratified. Finally, since the stratification on  $\operatorname{FactGr}_k$  is induced by the one on  $\operatorname{RanGr} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{RanGr} \times_{\operatorname{Ran}(X)} \operatorname{RanGr} \times_{\operatorname{Ran}(X)} \operatorname{RanGr}$ , we can conclude.

# **3.2.2** Construction of $\mathfrak{CGr}_k^{\odot}$

**Remark 3.9.** Consider two independent open subsets U and V of Ran(M). We have the following diagram

$$(\mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_{k})_{U} \times (\mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_{k})_{V} \to \mathfrak{strtop}((\mathrm{Fact}\mathrm{Gr}_{k} \times \mathrm{Fact}\mathrm{Gr}_{k})_{\mathrm{disj}}) \to \mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_{k}$$

$$\downarrow^{\pi} \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$U \times V \xrightarrow{\subset} (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}} \xrightarrow{\mathrm{union}} \mathrm{Ran}(M),$$

$$(3.2)$$

where the left hand square is a pullback of topological spaces, and the right top horizontal map is induced by Proposition 3.8 by applying strtop. Here we use that the underlying complex-analytical topological space of  $\operatorname{Ran}(X)$  is - set-theoretically - the space of points of M, and therefore the map  $\operatorname{top}(\operatorname{Ran}(X) \times \operatorname{Ran}(X)) \to \operatorname{Ran}(M) \times \operatorname{Ran}(M)$  restricts to a well-defined map  $\operatorname{top}((\operatorname{Ran}(X) \times \operatorname{Ran}(X))) \to (\operatorname{Ran}(M) \times \operatorname{Ran}(M))_{\operatorname{disi}}$ .

Note also that the bottom composition coincides with  $U \times V \to U \star V \hookrightarrow \text{Ran}(M)$ , the first map being the one taking unions of systems of points; hence, by the universal property of the fibered product of topological spaces,  $(\text{Fact}\mathfrak{Gr}_k)_U \times (\text{Fact}\mathfrak{Gr}_k)_V$  admits a map towards  $(\text{Fact}\mathfrak{Gr}_k)_{U\star V} = \text{Fact}\mathfrak{Gr}_k \times_{\text{Ran}(M)} (U \star V)$ , which we call  $p_{U,V,k}$ . Of course the triangle

$$(\operatorname{\mathcal{F}act} \operatorname{\mathcal{A}rc}_k)_U \times (\operatorname{\mathcal{F}act} \operatorname{\mathcal{A}rc}_k)_V \xrightarrow{p_{U,V,k}} (\operatorname{\mathcal{F}act} \operatorname{\mathcal{A}rc}_k)_{U\star V}$$

$$\overset{\operatorname{uniono}\pi}{\longrightarrow} \operatorname{Ran}(M)$$

commutes.

**Proposition 3.10.** Remark 3.9 induces well-defined maps of operads  $CGr_k^{\odot}$ : Fact $(M)^{\otimes} \to StrTSpc_{/Ran(M)}^{\odot}$  encoding the gluing of sheaves trivialised outside disjoint systems of points. That is, we have

•  $\mathfrak{CGr}_k^{\odot}(U)$  is the map  $(\mathfrak{F}actArc_k)_U \to U \hookrightarrow \mathrm{Ran}(M)$  for every U open subset of  $\mathrm{Ran}(M)$ .

• the image of the morphism  $(U, V) \to (U \star V)$  for any independent  $U, V \in \text{Open}(\text{Ran}(M))$  is the commuting triangle

$$(\mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_k)_U\times(\mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_k)_V\xrightarrow{} (\mathfrak{F}\mathrm{act}\mathfrak{G}\mathrm{r}_k)_{U\star V}$$
 
$$\mathrm{Ran}(M)$$

where the top map is the gluing of sheaves trivialised outside disjoints systems of points, and the left map is the map that remembers the two disjoint systems of points and takes their union.

*Proof.* See Appendix B.2. 
$$\Box$$

**Remark 3.11.** The constructions performed in the proof of Proposition 3.10 are compatible with the face maps of  $\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_{\bullet}$ . Indeed, for any two independent open subsets  $U, V \subset \operatorname{Ran}(M)$ , the squares

are commutative because the original diagrams at the algebraic level commute. That is,

$$(\operatorname{FactGr}_k \times \operatorname{FactGr}_k)_{\operatorname{disj}} \longrightarrow \operatorname{FactGr}_k \\ \downarrow \qquad \qquad \downarrow \\ (\operatorname{FactGr}_{k-1} \times \operatorname{FactGr}_{k-1})_{\operatorname{disj}} \longrightarrow \operatorname{FactGr}_{k-1}$$

commutes, since the construction involved in the horizontal maps, as we have seen, does not change the formal trivialisations, and, by the independence hypothesis, the non-formal trivialisations do not change in the punctured formal neighbourhoods involved in the formal gluing procedure.

**Proposition 3.12.** The maps of operads  $\mathfrak{CGr}_k^{\odot} : \operatorname{Fact}(M)^{\otimes} \to \operatorname{StrTSpc}_{/\operatorname{Ran}(M)}^{\odot}$  assemble to a map of operads  $\mathfrak{CGr}_{\bullet}^{\odot} : \operatorname{Fact}(M)^{\otimes} \to (2\operatorname{-Seg}^{\operatorname{ss}}(\operatorname{StrTSpc}))_{/\operatorname{Ran}(M)}^{\odot}$ .

*Proof.* Since we have already noticed that the functor  $\mathfrak{strtop}$  preserve finite limits, the condition that  $\mathfrak{F}act \mathfrak{Gr}_{\bullet}$  is 2-Segal can be recovered from the algebraic setting. Now the map

$$\mathfrak{CGr}_k(U) \to \mathfrak{CGr}_{\{0,\dots,l\}}(U) \times_{\mathfrak{CGr}_{\{0,l\}}(U)} \mathfrak{CGr}_{\{l,\dots,k\}}(U)$$

is the pullback of

$$\mathfrak{F}\mathrm{act} \mathfrak{Gr}_k \to \mathfrak{F}\mathrm{act} \mathfrak{Gr}_{\{0,\dots,l\}} \times_{\mathfrak{F}\mathrm{act} \mathfrak{Gr}_{\{0,l\}}} \mathfrak{F}\mathrm{act} \mathfrak{Gr}_{\{0,l,\dots,k\}}$$

along  $U \to \text{Ran}(M)$ , hence it is a homeomorphism (and the same holds for the  $\{l, k\}$  case).

# 3.2.3 Construction of $CArc_k^{\odot}$

Construction 3.13. We can perform a similar construction for

$$\mathcal{CArc}_k: \mathcal{O}pen(\operatorname{Ran}(M)) \to \operatorname{Grp}(\operatorname{StrTSpc}_{/\operatorname{Ran}(M)})$$

as well. Indeed, we can define

$$(\operatorname{FactArc}_k \times \operatorname{FactArc}_k)_{\operatorname{disj}}(R) = \{ (S, g_1, \dots, g_k) \in \operatorname{FactArc}_k(R),$$
$$(T, h_1, \dots, h_k) \in \operatorname{FactArc}_k(R) \mid \Gamma_S \cap \Gamma_T = \emptyset \}$$

and maps

$$(\operatorname{FactArc}_k \times \operatorname{FactArc}_k)_{\operatorname{disj}}(R) \to \operatorname{FactArc}_k(R)$$
$$((S, g_1, \dots, g_k), (T, h_1, \dots, h_k)) \mapsto (S \cup T, \widetilde{g_1 h_1}, \dots, \widetilde{g_k h_k}),$$

where  $\widetilde{g_ih_i}$  is the automorphism of  $\widehat{\mathfrak{T}_{(X_R)_{\Gamma_S \cup \Gamma_T}}}$  defined separatedly as  $g_i$  and  $h_i$  on the two components, which are disjoint by hypothesis. The rest of the construction is analogous, and provides maps of operads

$$\mathcal{CArc}_k^{\odot}: \mathrm{Fact}(M)^{\otimes} \to \mathrm{Grp}(\mathrm{StrTSpc}_{/\mathrm{Ran}(M)})^{\odot}$$

which are, as usual, natural and 2-Segal in  $k \in \Delta_{\text{ini}}^{\text{op}}$ .

# 3.3 Interaction of convolution and fusion over Ran(M)

**Remark 3.14.** Now we make the two algebra structures interact. Essentially, given the product  $\mathcal{H}ck \times \mathcal{H}ck$ , we have "two ways" of defining an operation:

• restrict to  $\mathcal{H}$ ck  $\times_{\text{Ran}(M)} \mathcal{H}$ ck and consider the correspondence



• restrict to  $(\mathcal{H}\operatorname{ck} \times \mathcal{H}\operatorname{ck})_{\operatorname{disj}}$ , or more precisely to  $\mathcal{H}\operatorname{ck}_U \times \mathcal{H}\operatorname{ck}_V$  for independent open sets  $U, V \subset \operatorname{Ran}(M)$ , and consider the map  $\mathcal{H}\operatorname{ck}_U \times \mathcal{H}\operatorname{ck}_V \to \mathcal{H}\operatorname{ck}U \star V$  induced by Remark 3.9.

Formally, these "restrictions" are obtained by forgetting both structures to StrTSpc. Indeed, the forgetful functor  $\text{StrTSpc}_{/\text{Ran}(M)} \to \text{StrTSpc}$  induces a functor

$$\operatorname{Corr}(\operatorname{Sh}(\operatorname{StrTSpc})_{/\operatorname{Ran}(M)}) \to \operatorname{Corr}(\operatorname{Sh}(\operatorname{StrTSpc}))$$

which is lax monoidal in both variables. Indeed, there are maps

$$\mathcal{H}ck \times_{Ran(M)} \mathcal{H}ck \to \mathcal{H}ck \times \mathcal{H}ck$$

and

$$(\mathcal{H}ck \times \mathcal{H}ck)_{disj} \to \mathcal{H}ck \times \mathcal{H}ck$$

which can be encoded as correspondences from  $\mathcal{H}\text{ck} \times \mathcal{H}\text{ck}$  to  $\mathcal{H}\text{ck} \times_{\text{Ran}(M)} \mathcal{H}\text{ck}$  and  $(\mathcal{H}\text{ck} \times \mathcal{H}\text{ck})_{\text{disj}}$  respectively. Note that the context of correspondences here is very useful to encode this "restriction" procedure.

# 4 Application of Lurie's Theorem

## 4.1 The factorising property

Our aim now is to verify the so-called factorisation property (see [Lur17, Theorem 5.5.4.10]) for the functors

$$\mathfrak{CGr}_k^{\odot}: \mathrm{Fact}(M)^{\otimes} \to \mathrm{StrTSpc}_{/\mathrm{Ran}(M)}^{\odot}$$

and

$$\mathcal{CA}\mathrm{rc}_k^{\odot}: \mathrm{Fact}(M)^{\otimes} \to \mathrm{Grp}(\mathrm{StrTSpc}_{/\mathrm{Ran}(M)})^{\odot}.$$

This will immediatly imply the property also after composing with the forgetful functor as in Remark 3.14, and for  $\mathfrak{H}ck^{\times} : \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}^{\mathrm{nu}}_{1} \to \mathrm{Corr}(\mathrm{Sh}_{\mathrm{strloc}}(\mathrm{StrTSpc}))^{\times}$ .

**Proposition 4.1** (Generalised factorising property). If U, V are independent, then the maps  $\operatorname{CGr}_k(U) \times \operatorname{CGr}_k(V) \to \operatorname{CGr}_k(U \star V)$ , resp.  $\operatorname{CArc}_k(U) \times \operatorname{CArc}_k(V) \to \operatorname{CArc}_k(U \star V)$ , are stratified homeomorphisms over  $\operatorname{Ran}(M)$ , resp. homeomorphisms of topological groups over  $\operatorname{Ran}(M)$ .

Proof. Note that the right-hand square in Diagram (3.2) is Cartesian. Indeed, let us now prove that its algebraic counterpart

$$(\operatorname{Fact} \operatorname{Gr}_k \times \operatorname{Fact} \operatorname{Gr}_k)_{\operatorname{disj}} \longrightarrow \operatorname{Fact} \operatorname{Gr}_k \\ \downarrow \qquad \qquad \downarrow \\ (\operatorname{Ran}(X) \times \operatorname{Ran}(X))_{\operatorname{disj}} \longrightarrow \operatorname{Ran}(X)$$

is cartesian in  $PSh_{\mathbb{C}}$ .

The pullback of the cospan computed in  $\mathrm{PSh}_{\mathbb{C}}$  is, abstractly, the functor parametrising tuples of the form  $(S,T), (P,\mathcal{H}_i,\gamma_i,\zeta_i)$ , where  $(S,T)\in(\mathrm{Ran}(X)\times\mathrm{Ran}(X))_{\mathrm{disj}}(R), P=S\cup T, (P,\mathcal{H}_i,\gamma_i,\zeta_i)\in\mathrm{FactGr}_k(R)$ . From this we can uniquely reconstruct a sequence  $(S,\mathcal{F}_i,\alpha_i,\mu_i,T,\mathcal{G}_i,\beta_i,\nu_i)$  in  $(\mathrm{FactGr}_k\times\mathrm{FactGr}_k)_{\mathrm{disj}}(R)$ . To do so, define  $\mathcal{F}_i\in\mathrm{Bun}_G(X_R)$  as the gluing of  $\mathcal{H}_i$  with the trivial G-bundle around T, which comes with a trivialisation  $\alpha_i:\mathcal{F}_i|_{X_R\setminus\Gamma_S}\stackrel{\sim}{\to} \mathcal{T}|_{X_R\setminus\Gamma_S}$ . We also define  $\mathcal{G}_i$  as the gluing of  $\mathcal{H}_i$  with the trivial G-bundle around S, coming with a trivialisation  $\beta_i$  outside T. As for the formal part of the datum, the  $\zeta_i$ 's automatically restrict to the desired formal neighbourhoods. This construction is inverse to the natural map  $(\mathrm{FactGr}_k\times\mathrm{FactGr}_k)_{\mathrm{disj}}\to\mathrm{FactGr}_k\times_{\mathrm{Ran}(X)}(\mathrm{Ran}(X)\times\mathrm{FactGr}_k)_{\mathrm{disj}}$ 

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 $Ran(X))_{disj}$ . But now, the diagram

$$\begin{array}{ccc} \mathfrak{strtop}(\mathrm{Ran}(X) \times \mathrm{Ran}(X))_{\mathrm{disj}} & \longrightarrow \mathfrak{strtop}(\mathrm{Ran}(X)) \\ & & & \downarrow \\ & & & (\mathrm{Ran}(M) \times \mathrm{Ran}(M))_{\mathrm{disj}} & \longrightarrow & \mathrm{Ran}(M) \end{array}$$

is again Cartesian, since, set-theoretically, the vertical maps are the identity, and we are just performing a change of topology on the bottom map. Hence the right-hand square in Diagram (3.2) is Cartesian, because the functor strtop preserves finite limits.

This concludes the proof since the outer square in (3.2) is Cartesian, and therefore the natural map  $(\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_k)_U \times (\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_k)_V \to (\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_k)_{U\star V}$  is a homeomorphism of topological spaces.

Now we turn to  $\mathcal{CArc}_k$ . It suffices to prove that the square

$$(\operatorname{FactArc}_k \times \operatorname{FactArc}_k)_{\operatorname{disj}} \longrightarrow \operatorname{FactArc}_k$$

$$\downarrow \qquad \qquad \downarrow$$

$$(\operatorname{Ran}(X) \times \operatorname{Ran}(X))_{\operatorname{disj}} \longrightarrow \operatorname{Ran}(X)$$

is Cartesian in  $PSh_{\mathbb{C}}$ . But this is clear once one considers the map

$$(\operatorname{Ran}(X) \times \operatorname{Ran}(X))_{\operatorname{disj}} \times_{\operatorname{Ran}(X)} \operatorname{FactArc}_k \to (\operatorname{FactArc}_k \times \operatorname{FactArc}_k)_{\operatorname{disj}}$$

given by

$$(S,T,\{g_i\in \operatorname{Aut}_{\widehat{(X_R)}_{\Gamma_S\cup\Gamma_T}}(\mathfrak{I})\})\mapsto ((S,\{g_i|_{\widehat{(X_R)}_{\Gamma_S}}\}),(S,\{g_i|_{\widehat{(X_R)}_{\Gamma_T}}\})).$$

Since the graphs are disjoint, this map is an equivalence. This concludes the proof.

## 4.2 Local constancy

The aim of this subsection is to prove that the functors  $\mathfrak{CGr}_k^{\odot}$  and  $\mathfrak{CArc}_k^{\odot}$  satisfy a "local constancy" property in a homotopical sense, which will be used in the following to apply [Lur17, Theorem 5.5.4.10]. First of all, we need a lemma.

**Lemma 4.2.** Let G be a (stratified) group scheme. If  $f: S \to T$  is a morphism of (stratified) schemes which is a (stratified) G-torsor with respect to the étale topology, then  $(\mathfrak{str})\mathsf{top}(f): (\mathfrak{str})\mathsf{top}(S) \to (\mathfrak{str})\mathsf{top}(T)$  is a principal (stratified) G-bundle.

*Proof.* First we perform the proof in the unstratified setting. First of all, to be locally trivial with respect to the étale topology at the algebraic level implies to be locally trivial with respect to the analytic topology at the analytic level (see [Rey71, Section 5] and [Bha, Section 5]). Now the analytic topology on an analytic manifold is the topology whose coverings are jointly surjective families of local homeomorphisms. We want to prove that a trivialising covering for top(f) with respect to

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this topology is the same of a classical trivialising open covering. Any open embedding is a local homeomorphism. Conversely, let us suppose that we have a trivialising local homeomorphism, that is a local homeomorphism of topological spaces  $w:W\to \mathfrak{top}(T)$  such that  $\mathfrak{top}(S)\times_{\mathfrak{top}(T)}W$  is isomorphic to  $\mathfrak{top}(G)\times W$ . Now, by definition of local homeomorphism, for every x in the image of w we have that w restricts to an open embedding on some open set  $U\subset W$  whose image contains x. Moreover,  $U\to W\to \mathfrak{top}(T)$  is trivialising a fortiori, that is  $U\times_{\mathfrak{top}(T)}\mathfrak{top}(S)\simeq U\times \mathfrak{top}(G)$ . In conclusion, if we have a jointly surjective family of trivialising local homeomorphisms, the above procedure yields a covering family of trivialising open sets.

In the stratified setting, the proof is the same, since the relevant maps are étale-stratified on the algebraic side, and therefore become analytic-stratified on the topological side.  $\Box$ 

**Proposition 4.3.** For every nonempty finite collection of disjoint disks  $D_1, \ldots, D_n \subseteq M$  containing open subdisks  $E_1 \subseteq D_1, \ldots, E_n \subseteq D_n$ , the maps

$$\mathfrak{CGr}_k(\operatorname{Ran}(\{E_i\})) \to \mathfrak{CGr}_k(\operatorname{Ran}(\{D_i\}))$$

and

$$\mathcal{CArc}_k(\operatorname{Ran}(\{E_i\})) \to \mathcal{CArc}_k(\operatorname{Ran}(\{D_i\}))$$

are stratified homotopy equivalences.

*Proof.* The factorising property tells us that these maps assume the form  $\prod_{i=1}^{n} \mathcal{CGr}_{k}(\operatorname{Ran}(E_{i})) \to \prod_{i=1}^{n} \mathcal{CGr}_{k}(\operatorname{Ran}(D_{i}))$  and  $\prod_{i=1}^{n} \mathcal{CArc}_{k}(\operatorname{Ran}(E_{i})) \to \prod_{i=1}^{n} \mathcal{CArc}_{k}(\operatorname{Ran}(D_{i}))$  respectively, and hence it suffices to perform the checks term by term, i.e., to assume n=1.

We deal first with the case  $\mathfrak{CGr}_k(\operatorname{Ran}(\{E_i\})) \to \mathfrak{CGr}_k(\operatorname{Ran}(\{D_i\}))$ . Now we observe that we can reduce to the case k=1: indeed,  $\operatorname{FactGr}_k$  is a stratified  $\operatorname{FactArc}_{k-1}$ -torsor over the k-fold product  $\operatorname{RanGr} \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{RanGr}_1 = \operatorname{FactGr}_1 \times_{\operatorname{Ran}(X)} \cdots \times_{\operatorname{Ran}(X)} \operatorname{FactGr}_1$ , and therefore  $\operatorname{FactGr}_k \to \operatorname{FactGr}_1 \times_{\operatorname{Ran}(M)} \cdots \times_{\operatorname{Ran}(M)} \operatorname{FactGr}_1$  is a stratified  $\operatorname{FactArc}_{k-1}$ -principal topological bundle, in particular it is a Serre fibration. Thus, if we prove that  $(\operatorname{FactGr}_1)_{\operatorname{Ran}(E)} \to (\operatorname{FactGr}_1)_{\operatorname{Ran}(D)}$  is a stratified homotopy equivalence, then the same will be true for  $(\operatorname{FactGr}_1)_{\operatorname{Ran}(E)} \times_{\operatorname{Ran}(E)} \cdots \times_{\operatorname{Ran}(E)} (\operatorname{FactGr}_1)_{\operatorname{Ran}(E)} \to (\operatorname{FactGr}_1)_{\operatorname{Ran}(D)} \times_{\operatorname{Ran}(D)} \cdots \times_{\operatorname{Ran}(D)} (\operatorname{FactGr}_1)_{\operatorname{Ran}(D)}$  and therefore at the upper level for  $(\operatorname{FactGr}_k)_{\operatorname{Ran}(E)} \to (\operatorname{FactGr}_k)_{\operatorname{Ran}(D)} \to (\operatorname{FactGr}_k)_{\operatorname{Ran}(D)}$ .

Thus we only need to prove the statement for  $\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_1$ . This is done in [HY19, Proposition 3.17]. More precisely, [HY19, Proposition 3.17] states that

#### (\*) The diagram

is a pullback in StrTSpc and the map  $\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_{1,x} \to (\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_1)_{\operatorname{Ran}(D)}$  is a stratified homotopy equivalence.

where  $\operatorname{FactGr}_{k,x} = \operatorname{FactGr}_k \times_{\operatorname{Ran}(X)} \{x\}$  was defined in Definition 2.7. We thank Jeremy Hahn and Allen Yuan for explaining some details of their proof to us. We report their explanation in Appendix B.4.

An analogous (and simpler) argument works for  $\mathcal{F}actArc_k$ , because

$$\operatorname{Ran}G_{\mathcal{O}}|_{\operatorname{Ran}_{\operatorname{card}_{=k}}(M)} \simeq (X^{\times k})_{\operatorname{disj}} \times G_{\mathcal{O}}^{\times k}$$

(see Appendix B.4).

Note that, by the torsor argument, (\*) implies the following:

Corollary 4.4. If D is a disk around x in M, there is a natural map  $\operatorname{Fact}\operatorname{Gr}_{k,x} \to (\operatorname{Fact}\operatorname{Gr}_k)_{\operatorname{Ran}(D)}$  which are stratified homotopy equivalences.

In a similar way, the natural map  $\operatorname{FactArc}_{k,x} \to (\operatorname{FactArc}_k)_{\operatorname{Ran}(D)}$  is a homotopy equivalence of topological groups.

Notation 4.5. From now on, we will denote the relation "being stratified homotopy equivalent" by  $\overset{\mathfrak{she}}{\sim}$  and "being homotopy equivalent as topological groups" by  $\overset{\mathfrak{ghe}}{\sim}$ .

**Remark 4.6.** In particular,  $\mathfrak{CGr}_{1,x} \stackrel{\mathfrak{she}}{\sim} \mathfrak{F}\mathrm{actGr}_{1,x} = \mathfrak{strtop}(\mathrm{Gr})$  and  $\mathfrak{CArc}_{1,x} \stackrel{\mathfrak{ghe}}{\sim} \mathfrak{F}\mathrm{actArc}_{1,x} = \mathfrak{strtop}(G_{\mathbb{O}})$ .

We can express this property in a more suitable way by considering the  $\infty$ -categorical localisation  $Sh(StrTSpc)[\mathfrak{she}^{-1}]$ , where here  $\mathfrak{she}$  is the smallest saturated class of morphisms of Sh(StrTSpc) containing stratified homotopy equivalences of stratified spaces. Since  $\mathfrak{she}$  is closed under the symmetric monoidal structure of  $Sh(StrTSpc)^{\times}$ , the localisation functor extends to a map of operads

$$\mathrm{Sh}(\mathrm{StrTSpc})^{\times} \to \mathrm{Sh}(\mathrm{StrTSpc})[\mathfrak{she}^{-1}]^{\times}$$

and therefore to

$$\operatorname{Corr}(\operatorname{Sh}(\operatorname{StrTSpc}))^{\times} \to \operatorname{Corr}(\operatorname{Sh}(\operatorname{StrTSpc})[\mathfrak{she}^{-1}])^{\times}.$$

This provides a functor

$$\mathcal{H}\mathrm{ck}^{\times}: \mathrm{Fact}(M)^{\otimes} \times \mathbb{E}^{\mathrm{nu}}_{1} \to \mathrm{Corr}(\mathrm{Sh}(\mathrm{StrTSpc})[\mathfrak{she}^{-1}])^{\times} \tag{4.1}$$

which is a map of operads in both variables, has the factorisation property in the first variable and sends the usual inclusions of Ran spaces of systems of disks to equivalences in the target category.

#### 4.3 The stalk of the factorising cosheaf

We can apply [Lur17, Theorem 5.5.4.10] to the map of operads (4.1), since we have proven in the previous subsections that the hypotheses of the theorem are satisfied. We denote the operads  $\mathbb{E}_n^{\otimes}$ ,  $\mathbb{E}_M^{\otimes}$  by  $\mathbb{E}_n$ ,  $\mathbb{E}_M$ .

Corollary 4.7. The functor  $\mathcal{H}ck^{\times}$  induces a nonunital  $\mathbb{E}_M$ -algebra object

$$\mathcal{H}ck_M^{\times} \in Alg_{\mathbb{E}_M}^{nu}(Alg_{\mathbb{E}_1}^{nu}(Corr(Sh_{strloc}(StrTSpc)[\mathfrak{she}^{-1}])^{\times})).$$

Fix a point  $x \in M$ . The main consequence of the above result is that the stalk of  $\mathcal{H}$ ck at the point  $\{x\} \in \text{Ran}(M)$  inherits an  $\mathbb{E}_2^{\text{nu}}$ -algebra structure in  $\text{Alg}_{\mathbb{E}_1}^{\text{nu}}(\text{Corr}(\text{Sh}(\text{StrTSpc}))^{\times})$ . We will now explain how, running through [Lur17, Chapter 5] again.

Remark 4.8. By [Lur17, Example 5.4.5.3], a nonunital  $\mathbb{E}_M$ -algebra object  $A^{\otimes}$  in a symmetric monoidal  $\infty$ -category  $\mathcal{C}$  induces a nonunital  $\mathbb{E}_n$ -algebra object in  $\mathcal{C}$ , where n is the real dimension of the topological manifold M, by taking the stalk at a point  $x \in M$ . More precisely, there is an object in  $\mathrm{Alg}_{\mathbb{E}_n}(\mathcal{C})$  whose underlying object is  $\lim_{\{x\}\in U\in \mathrm{Open}(\mathrm{Ran}(M))}A(U)$ , which coincides with  $\lim_{x\in D\in \mathrm{Disk}(M)}A(\mathrm{Ran}(D))$ , since the family of Ran spaces of disks around x is final in the family of open neighbourhoods of  $\{x\}$  inside  $\mathrm{Ran}(M)$ . Now each  $A^{\otimes}|_{\mathrm{Fact}(D)^{\otimes}}$  induces a nonunital  $\mathbb{E}_D$ -algebra by Lurie's theorem [Lur17, 5.5.4.10]. But [Lur17, Example 5.4.5.3] tells us that  $\mathbb{E}_D$ -algebras are equivalent to  $\mathbb{E}_n$  algebras. Also, by local constancy (i.e. constructibility), the functor  $D\mapsto A(D)$  is constant over the family  $x\in D\in \mathrm{Disk}(M)$ , and therefore the stalk  $A_x$  coincides with any of those  $\mathbb{E}_n$ -algebras. This also implies that all stalks at points of M are (noncanonically) isomorphic. Also, the content of [Lur17, Subsection 5.5.4] tells us how the  $\mathbb{E}_n$ -multiplication structure works concretely. Choose a disk D containing x. We interpret this as the only object in the  $\langle 1 \rangle$ -fiber of  $\mathbb{E}_D$ . Recall that a morphism in  $\mathbb{E}_D$  lying over the map

$$\langle 2\rangle \to \langle 1\rangle$$

$$1, 2 \mapsto 1$$

is the choice of an embedding  $D \coprod D \hookrightarrow D$ . Call  $n_D$  the unique object lying over  $\langle n \rangle$  in  $\mathbb{E}_D$ .

Consider the canonical map  $\mathbb{E}_D \to \mathbb{E}_M$ . If  $A_M$  is the  $\mathbb{E}_M^{\mathrm{nu}}$ -algebra object appearing in the conclusion of Lurie's theorem, call  $A_D$  its restriction to  $\mathbb{E}_D$ . Recall from the proof of Lurie's theorem that  $A_M$  is obtained by operadic left Kan extension of the restriction  $A|_{\mathrm{Disk}(M)}$  along the functor  $\mathrm{Disk}(M)^{\otimes} \to \mathbb{E}_M$ . Then we have that

#### Lemma 4.9.

$$A_D(1_D) = A(\operatorname{Ran}(D)),$$

and

$$A_D(2_D) = \operatorname{colim}\{A(\operatorname{Ran}(E)) \otimes A(\operatorname{Ran}(F)) \mid E, F \in \operatorname{Disk}(M),$$

$$E \cap F = \varnothing, D \coprod D \xrightarrow{\sim} E \coprod F \hookrightarrow D\} \simeq$$

$$\simeq A(\operatorname{Ran}(E_0)) \otimes A(\operatorname{Ran}(F_0))$$

for any choice of an embedding  $D \coprod D \xrightarrow{\sim} E_0 \coprod F_0 \hookrightarrow D$ .

<sup>&</sup>lt;sup>5</sup>Note that this is true only for points of Ran(M) coming from single points of M. If we allow the cardinality of the system of points to vary, stalks may take different values. In fact, the factorisation property tells us that a system of cardinality m will give the m-ary tensor product in  $\mathcal{C}$  of the stalk at the single point.

*Proof.* We need to prove that the colimit degenerates. Take indeed two couples of disks as in the statement, say  $E_1, F_1$  and  $E_2, F_2$ . By local constancy, we can suppose that all four disks are pairwise disjoint. Now we can embed both  $E_1$  and  $E_2$  into some E, and  $F_1, F_2$  into some F, in such a way that  $E \cap F = \emptyset$ . Then we have canonical equivalences

$$A(\operatorname{Ran}(E_1)) \otimes A(\operatorname{Ran}(F_1)) \xrightarrow{\sim} A(\operatorname{Ran}(E)) \otimes A(\operatorname{Ran}(F))$$

and

$$A(\operatorname{Ran}(E_2)) \otimes A(\operatorname{Ran}(F_2)) \xrightarrow{A} (\operatorname{Ran}(E)) \otimes A(\operatorname{Ran}(F)).$$

This discussion implies that the operation  $\mu$  on  $A_x$  encoded by Lurie's theorem has the form

$$A_x \otimes A_x \simeq A(\operatorname{Ran}(D)) \otimes A(\operatorname{Ran}(D)) \xrightarrow{\sim} A(\operatorname{Ran}(E_0)) \otimes A(\operatorname{Ran}(F_0))$$

$$\xrightarrow{\sim} A(\operatorname{Ran}(E_0) \star \operatorname{Ran}(F_0)) \to A(\operatorname{Ran}(D)) \simeq A_x, \tag{4.2}$$

where:

- the first and last equivalences come from local constancy;
- the second equivalence is induced by the chosen embedding  $D \coprod D \xrightarrow{\sim} E_0 \coprod F_0 \hookrightarrow D$ ;
- the third equivalence is the factorisation property.

The discussion about the stalk leads to the main theorem of this section:

**Theorem 4.10.** The stalk at  $x \in M$  of the  $\mathbb{E}_M$ -algebra object  $\mathfrak{H}ck_M$  from Corollary 4.7 can be naturally viewed as an object of

$$\mathcal{H}\mathrm{ck}^\times_x \in \mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_2}(\mathrm{Alg}^{\mathrm{nu}}_{\mathbb{E}_1}(\mathrm{Corr}(\mathrm{Sh}(\mathrm{StrTSpc})[\mathfrak{she}^{-1}])^\times))$$

encoding simultaneously the convolution and fusion procedures on  $\mathcal{H}ck_x = G_{\mathcal{O}} \backslash Gr$ .

# 5 Product of constructible sheaves

#### 5.1 Taking constructible sheaves

We will now check that the hypotheses of Theorem A.25 are satisfied.

**Proposition 5.1.** Every  $\mathfrak{F}act\mathfrak{Gr}_k$  belongs to  $StrTSpc_{csls} \subset StrTSpc$ .

Proof. See Appendix B.3. 
$$\Box$$

As a consequence, the map of operads

$$\mathcal{H}ck_x^{\times}: \mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Corr}(\mathrm{Sh}(\mathrm{StrTSpc})[\mathfrak{she}^{-1}])^{\times}$$

factors through the map induced by  $StrTSpc_{csls} \to StrTSpc$ . This means that we are in the hypotheses of the Exodromy Theorem [Lur17, Theorem A.9.3] and Cons:  $StrTSpc_{csls}^{op} \to Cat_{\infty}$  takes the form  $Fun(Exit(-), Mod_{\Delta})$ .

Recall now the definition of the class of morphisms comp.cnl from Definition A.13 and Appendix A.4.3. We want to prove that  $\mathcal{H}ck_x^{\times}$  takes values in  $corr(Sh(StrTSpc_{csls})_{comp.cnl},\mathfrak{all})$ . For what concerns the variable in  $\mathbb{E}_1^{nu}$ , it suffices to prove that:

**Proposition 5.2.** The map  $\operatorname{Fact} \operatorname{Arc}_{2,x} \backslash \operatorname{Fact} \operatorname{Gr}_{2,x} \to \operatorname{Hck}_x$  belongs to comp.cnl.

Proof. By (2.6), it suffices to prove that the map  $\operatorname{ConvGr}_{2,x} = G_{\mathcal{K}} \times^{G_0} \operatorname{Gr} \to \operatorname{Gr}$  is in comp.cnf after applying the functor strtop. But by [MV07, Proposition 4.4] this map is proper and cylindrically stratified.

Now we need to know that the image of any morphism in  $\mathbb{E}_2^{nu}$  is a ( $\mathfrak{comp.cnl},\mathfrak{all}$ )-correspondence as well. It suffices to prove this for the image of every morphism  $\phi$  in  $\mathbb{E}_2^{nu}$  lying over

$$\alpha:\langle 2\rangle \to \langle 1\rangle$$

$$1, 2 \mapsto 1$$

because on the other hand the maps induced by inclusions of disks are equivalences. Note that by definition, if the entry in  $\mathbb{E}_1^{\text{nu}}$  is fixed, the discussion around Lemma 4.9 tells us that the image of such  $\phi$  is vertical and of the form (4.2), and hence it is a composition of equivalences and stratified open embeddings. Therefore, it belongs to  $\mathsf{comp.cnl}$ .

Therefore:

**Proposition 5.3.** The functor  $\mathcal{H}ck_x^{\times}$  is an  $\mathbb{E}_2^{\text{nu}} \times \mathbb{E}_1^{\text{nu}}$ -double algebra object in

$$\mathrm{Corr}(\mathrm{Sh}(\mathrm{Str}\mathrm{TSpc}_{\mathrm{csls}})[\mathfrak{she}^{-1}])_{\mathfrak{comp.cyl},\mathfrak{all}}^{\times}.$$

Note now that since the functor

$$\mathrm{Cons}_{\mathrm{corr}}^{\otimes}: \mathrm{Corr}(\mathrm{Sh}_{\mathrm{strloc}}(\mathrm{StrTSpc}_{\mathrm{csls}}))_{\mathtt{comp.cuf.aff}}^{\times} \to \mathrm{Pr}_{k}^{\mathrm{L}, \otimes}$$

from Theorem A.25 sends stratified homotopy equivalences to equivalences of  $\infty$ -categories<sup>6</sup>, then it factors through the operadic localisation to  $\mathfrak{she}$  and therefore it can be precomposed with the functor  $\mathfrak{Hck}_x^{\times}$  from Proposition 5.3.

<sup>&</sup>lt;sup>6</sup>This follows immediatly from the use of the Exit Path formalism. For a proof in greter generality, see [Hai20]

**Definition 5.4.** We denote by  $\mathcal{A}^{\otimes,\mathrm{nu}}$  the functor

$$\operatorname{Cons}_{\operatorname{corr}}^{\otimes} \circ \operatorname{\mathcal{H}ck}_{x}^{\times} : \mathbb{E}_{2}^{\operatorname{nu}} \times \mathbb{E}_{1}^{\operatorname{nu}} \to \operatorname{Pr}_{k}^{\operatorname{L}, \otimes}.$$

By construction, thanks to the descent properties of the functor Exit (see [Lur17, Appendix A.3]), the functor Cons takes the space  $\mathcal{H}ck_x = \operatorname{colim}_n(G_0^{\times n} \times \operatorname{Gr})$  to the limit  $\lim_n \operatorname{Cons}(G_0^{\times n} \times \operatorname{Gr}) = \operatorname{Cons}_{G_0}(\operatorname{Gr})$ .

More precisely, the composed functor  $\mathcal{A}^{\otimes,\mathrm{nu}}$  sends

$$(\langle m \rangle, \langle k \rangle) \mapsto \overbrace{\operatorname{Cons}_{G_{\circ}^{\times^{k}}}(G_{\mathcal{K}}^{\times k-1} \times \operatorname{Gr}) \otimes \cdots \otimes \operatorname{Cons}_{G_{\circ}^{\times^{k}}}(G_{\mathcal{K}}^{\times k-1} \times \operatorname{Gr})}^{m}.$$

#### 5.2 Units and the main theorem

**Remark 5.5.** Let us inspect the behaviour of the  $\mathbb{E}_2$  product. We had a map

$$\mu: \mathcal{H}\mathrm{ck}_x \times \mathcal{H}\mathrm{ck}_x \to \mathcal{H}\mathrm{ck}_x$$

from (4.2). By construction, when we apply the functor Cons, this is sent forward with lower shriek functoriality. Therefore, for k = 1, we end up with a map

$$\operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}) \times \operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}) \xrightarrow{\mu_{!}} \operatorname{Cons}_{G_{\mathcal{O}}}.$$

Recovering the original structure of the map  $\mu$ ,  $\mu$ ! decomposes as

$$\operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}) \times \operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr})$$

$$\overset{\sim}{\to} \operatorname{Cons}_{(\operatorname{Ran}G_{\mathcal{O}})_{\operatorname{Ran}(D)}}(\operatorname{RanGr}_{\operatorname{Ran}(D)}) \times \operatorname{Cons}_{(\operatorname{Ran}G_{\mathcal{O}})_{\operatorname{Ran}(D)}}(\operatorname{RanGr}_{\operatorname{Ran}(D)})$$

$$\overset{\sim}{\to} \operatorname{Cons}_{(\operatorname{Ran}G_{\mathcal{O}})_{\operatorname{Ran}(E)}}(\operatorname{RanGr}_{\operatorname{Ran}(E)}) \times \operatorname{Cons}_{(\operatorname{Ran}G_{\mathcal{O}})_{\operatorname{Ran}(F)}}(\operatorname{RanGr}_{\operatorname{Ran}(F)})$$

$$\overset{\sim}{\to} \operatorname{Cons}_{(\operatorname{Ran}G_{\mathcal{O}})_{\operatorname{Ran}(E)} \star \operatorname{Ran}(F)}(\operatorname{RanGr}_{\operatorname{Ran}(E)} \star \operatorname{Ran}(F))$$

$$\to \operatorname{Cons}_{(\operatorname{Ran}G_{\mathcal{O}})_{\operatorname{Ran}(D)}}(\operatorname{RanGr}_{\operatorname{Ran}(D)}) \overset{\sim}{\to} \operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}),$$

where  $j_x : \{x\} \to \text{Ran}(M)$  is the inclusion.

**Proposition 5.6.** Let  $x \in M$  be a point. Consider the functor  $\mathcal{A}_x^{\otimes} : \mathbb{E}_2^{\mathrm{nu}} \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Pr}_k^{\mathrm{L},\otimes}$ . Then this can be upgraded to a map of operads  $\mathbb{E}_2 \times \mathbb{E}_1^{\mathrm{nu}} \to \mathrm{Pr}_k^{\mathrm{L},\otimes}$ .

Proof. We can apply [Lur17, Theorem 5.4.4.5], whose hypothesis is satisfied since for any  $\infty$ -category  $\mathbb C$  the functor  $\mathbb C^{\times} \to \mathcal F$ in<sub>\*</sub> is a coCartesian fibration of  $\infty$ -operads ([Lur17, Proposition 2.4.1.5]): therefore, it suffices to exhibit a quasi-unit for any  $\mathcal A^{\otimes}(-,\langle k\rangle)$ , functorial in  $\langle k\rangle \in \mathbb E_1^{\mathrm{nu}}$ . We can consider the map (natural in k)  $u_k$ : Spec  $\mathbb C \to \mathrm{FactGr}_{k,x}$  represented by the sequence  $(\mathfrak T,\mathrm{id}|_{X\setminus\{x\}},\mathrm{id}|_{\hat X_x},\ldots,\mathfrak T,\mathrm{id}_{X\setminus\{x\}}) \in \mathrm{FactGr}_{k,x}$ . Note now that this induces a map of spaces

$$* \to \mathfrak{F}act \mathfrak{Gr}_{k,x}$$
.

The formal gluing property evidently commutes with this map at the various levels, so this construction is natural in k. As usual, let us denote by  $\mathcal{H}ck_{k,x}$  be the evaluation of  $\mathcal{H}ck_x^{\times}$  at  $\langle 1 \rangle \in \mathbb{E}_2^{nu}, \langle m \rangle \in \mathbb{E}_1^{nu}$ . We have an induced map  $* \to \mathcal{H}ck_{k,x}$  for every k. Now if  $\mu_k : \mathcal{H}ck_{k,x} \times \mathcal{H}ck_{k,x} \to \mathcal{H}ck_{k,x}$  is the multiplication in Sh(StrTSpc), we can consider the composition

$$\operatorname{\mathcal{H}ck}_{k,x} \overset{\mathfrak{she}}{\sim} \operatorname{\mathcal{H}ck}_{k,x} \times * \xrightarrow{\operatorname{id},u_k} \operatorname{\mathcal{H}ck}_{k,x} \times \operatorname{\mathcal{H}ck}_{k,x} \xrightarrow{\mu_k} \operatorname{\mathcal{H}ck}_{k,x}$$

and we find that this composition is the identity. Therefore,  $u_k$  is a right quasi-unit, functorially in k. The condition that it is a left-quasi unit is verified analogously.

**Remark 5.7.** Let us now inspect the behaviour of the  $\mathbb{E}_1$  product. Let us fix the  $\mathbb{E}_2$  entry equal to  $\langle 1 \rangle$  for simpplicity. Then the product law is described by the map

$$\mathcal{A}_{x}(\langle 1 \rangle, \langle 2 \rangle) = \operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}) \otimes \operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}) \xrightarrow{\boxtimes} \operatorname{Cons}_{G_{\mathcal{O}} \times G_{\mathcal{O}}}(\operatorname{Gr} \times \operatorname{Gr})$$

$$\xrightarrow{\mathcal{H}\operatorname{ck}_{x}(\partial_{2} \times \partial_{0})^{*} = p^{*}} \operatorname{Cons}_{G_{\mathcal{O}} \times G_{\mathcal{O}}}(\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_{2,x}) = \operatorname{Cons}(\mathcal{H}\operatorname{ck}_{x}) \cong$$

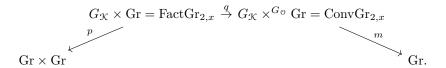
$$\operatorname{Cons}(G_{\mathcal{O}} \setminus (G_{\mathcal{K}} \times^{G_{\mathcal{O}}} \operatorname{Gr})) \xrightarrow{\mathcal{F}\operatorname{act} \operatorname{\mathcal{G}r}(\delta_{1})_{*} = m_{*}} \operatorname{Cons}_{G_{\mathcal{O}}}(\operatorname{Gr}) = \mathcal{A}_{x}(\langle 1 \rangle, \langle 1 \rangle).$$

The "pullback" and "pushforward" steps come from the construction of the functor out of the category of correspondences, which by construction takes a correspondence to the "pullback-pushforward" transform between the categories of constructible sheaves over the bottom vertexes of the correspondence. Note that the most subtle step is the equivalence in the penultimate step. If one is to compute explicitly a product of two constructible sheaves  $F, G \in \mathrm{Cons}_{G_{\mathcal{O}}}(\mathrm{Gr})$ , one must reconstruct the correct equivariant sheaf over  $\mathrm{Conv}\mathrm{Gr}_{2,x}$  whose pullback to  $\mathrm{Fact}\mathrm{Gr}_{2,x}$  is  $p^*(F \boxtimes G)$ , and then push it forward along m (in the derived sense of course). We stress again that this, when restricted to  $\mathrm{Perv}_{G_{\mathcal{O}}}(\mathrm{Gr})$ , is exactly the definition of the convolution product of perverse sheaves from [MV07] (up to shift and t-structure).

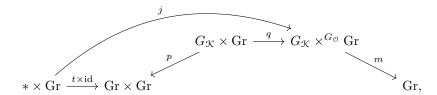
**Proposition 5.8.** The map of operads  $A_x^{\otimes} : \mathbb{E}_2 \times \mathbb{E}_1^{\text{nu}} \to \Pr_k^{L,\otimes}$  can be upgraded to a map of operads  $A_x^{\otimes} : \mathbb{E}_2 \times \mathbb{E}_1 \to \Pr_k^{L,\otimes}$ .

*Proof.* Again, it suffices to exhibit a quasi-unit. In this case, this is represented by the element  $* \xrightarrow{\mathbf{1}} \operatorname{Cons}_{\Lambda,G_{\mathcal{O}}}(\operatorname{Gr})$ . Here **1** is the pushforward along the trivial section  $t: * \to \operatorname{Gr}, t(*) = (\mathfrak{T}, \operatorname{id}|_{X \setminus x})$ , of the constant sheaf with value  $\Lambda$ .

The proof is given in [Rei12, Proposition IV.3.5]. We denote by  $\star$  the  $\mathbb{E}_1$ -product of equivariant constructible sheaves on Gr described by  $\mathcal{A}_x(-)$ . By Remark 5.7 for any  $F \in \mathrm{Cons}_{\Lambda,G_0}(\mathrm{Gr})$  we can compute the product via the convolution diagram



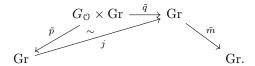
In our specific case, we are given a diagram



where j is the closed embedding  $(\mathcal{F}, \alpha) \mapsto (\mathcal{T}, \mathrm{id}|_{X \setminus x}, \mathcal{F}, \alpha)$  whose image is canonically identified with Gr. Let  $F \in \mathrm{Cons}_{\Lambda, G_{\mathcal{O}}}(\mathrm{Gr})$ . We want to prove that  $\mathbf{1} \widetilde{\boxtimes} F \simeq j_*(\Lambda \boxtimes F)$ , i.e. that

$$q^*j_*(\Lambda \boxtimes F) \simeq p^*(t \times id)_*(\Lambda \boxtimes F).$$

Note that because of the consideration about the image of j the support of both sides lies in  $G_{\mathcal{O}} \times Gr \subset G_{\mathcal{K}} \times Gr$ , and this yields a restricted diagram



This proves the claim. By applying  $m_*$  we obtain

$$1 \star F \simeq m_*(j_*(\Lambda \boxtimes F)) = \Lambda \boxtimes F = F$$

since 
$$mj = id$$
.

Thanks to these results, our functor  $\mathcal{A}_x^{\otimes}$  is promoted to a map of operads  $\mathbb{E}_2 \times \mathbb{E}_1 \to \operatorname{Pr}_k^{L,\otimes}$ . By the Additivity Theorem ([Lur17, Theorem 5.1.2.2]), we obtain an  $\mathbb{E}_3$ -algebra object in  $\operatorname{Pr}_k^{L,\otimes}$ . Summing up:

**Theorem 5.9** ((Main theorem)). There is an object  $\mathcal{A}_x^{\otimes} \in Alg_{\mathbb{E}_3}(\operatorname{Pr}_k^{\mathrm{L},\otimes})$  describing an associative and braided product law on the  $\infty$ -category

$$\operatorname{Cons}_{G_{\mathfrak{O}}}(\operatorname{Gr}_{G},\mathscr{S})$$

of  $G_{\mathbb{O}}$ -equivariant constructible sheaves over the affine Grassmannian. The restriction of this product law to the abelian category of equivariant perverse sheaves coincides, up to shifts, with the classical (commutative) convolution product of perverse sheaves [MV07].

# A Constructible sheaves on stratified spaces: theoretical complements

#### A.1 Stratified schemes and stratified analytic spaces

#### A.1.1 Definitions

The following definitions are particular cases of [BGH20, 8.2.1] and ff. .

**Definition A.1.** Let TSpc be the 1-category of topological spaces. The category of stratified topological spaces is defined as

$$StrTSpc_{\mathbb{C}} = Fun(\Delta^1, TSpc) \times_{TSpc} Poset,$$

where the map  $\operatorname{Fun}(\Delta^1,\operatorname{TSpc})\to\operatorname{TSpc}$  is the evaluation at 1, and Alex: Poset  $\to$  TSpc assigns to each poset P its underlying set with the so-called Alexandrov topology (see [BGH20, Definition 1.1.1]).

**Definition A.2.** Let  $StrSch = Sch \times_{TSpc} StrTSpc$ , where the map  $Sch \to TSpc$  sends a scheme X to its underlying Zariski topological space, and the other map is the evaluation at [0], be the category of stratified schemes, and StrAff its full subcategory of stratified affine schemes.

Analogously, one can define stratified complex schemes  $\operatorname{Str}\operatorname{Sch}_{\mathbb{C}}$  and  $\operatorname{stratified}$  complex affine schemes  $\operatorname{Str}\operatorname{Aff}_{\mathbb{C}}$ . The key point now is that there is an analytification functor an :  $\operatorname{Aff}_{\mathbb{C}} \to \operatorname{Stn}_{\mathbb{C}}$ , the category of Stein analytic spaces. This is defined in [Rey71, Théorème et définition 1.1] (and for earlier notions used there, see also [Gro57, 6]). In this way we obtain a Stein space, which is a particular kind of complex manifold with a sheaf of holomorphic functions. We can forget the sheaf and the complex structure and recover an underlying Hausdorff topological space (which corresponds to the operation denoted by |-| in [Rey71]) thus finally obtaining a functor

$$\mathfrak{top}: \operatorname{Sch}_{\mathbb{C}} \to \operatorname{TSpc}.$$

A reference for a thorough treatment of analytification (also at a derived level) is [HP18].

Let now  $\operatorname{StrStn}_{\mathbb{C}} = \operatorname{Stn}_{\mathbb{C}} \times_{\operatorname{TSpc},\operatorname{ev_0}} \operatorname{StrTSpc}$ . There is a natural stratified version of the functor  $\operatorname{top}$ , namely the one that assigns to a stratified affine complex scheme  $(S,s:\operatorname{zar}(S)\to P)$  the underlying topological space of the associated complex analytic space, with the stratification induced by the map of ringed spaces  $u:S^{\operatorname{an}}\to S$ :

$$\operatorname{StrAff}_{\mathbb{C}} \to \operatorname{StrStn}_{\mathbb{C}} \to \operatorname{StrTSpc}$$
  
 $(S,s) \mapsto (S^{\operatorname{an}}, s \circ u) \mapsto (|S^{\operatorname{an}}|, s \circ u).$ 

Construction A.3. We can define the category  $StrPSh_{\mathbb{C}}$  as  $PSh(StrAff_{\mathbb{C}})$ . Note that StrTSpc is cocomplete, because TSpc,  $Fun(\Delta^1, TSpc)$  and Poset are. By left Kan extension we have a functor

$$\mathfrak{strtop}: \operatorname{StrPSh}_{\mathbb{C}} \to \operatorname{StrTSpc}.$$
 (A.1)

By construction, this functor preserves small colimits and finite limits (since both an :  $Aff_{\mathbb{C}} \to Stn_{\mathbb{C}}$  and  $|-|: Stn_{\mathbb{C}} \to TSpc$  preserve finite limits, see [Rey71]).

#### A.1.2 Pullbacks of stratified spaces

**Lemma A.4.** The forgetful functor  $StrTSpc \rightarrow TSpc$  preserves and reflects finite limits.

*Proof.* Since StrTSpc = Fun( $\Delta^1$ , TSpc)  $\times_{\text{ev}_1,\text{TSpc},\text{Alex}}$  Poset (see Appendix A.1), it suffices to show that:

- the functor  $ev_0 : Fun(\Delta_1, TSpc) \to TSpc$  preserves and reflects pullbacks.
- the functor Alex : Poset  $\rightarrow$  TSpc preserves and reflects pullbacks;

Now, the first point follows from the fact that limits in categories of functors are computed componentwise. The second point can be verified directly, by means of the following facts:

- the functor preserves binary products. Indeed, given two posets P, Q, then the underlying sets of  $P \times Q$  and  $Alex(P) \times Alex(Q)$  coincide. Now, the product topology on  $Alex(P) \times Alex(Q)$  is coarser than the Alexandrov topology  $Alex(P \times Q)$ . Moreover, there is a simple base for the Alexandrov topology of a poset P, namely the one given by "half-lines"  $P_{p_0} = \{p \in P \mid p \geq p_0\}$ . Now, if we choose a point  $(p_0, q_0) \in Alex(P \times Q)$ , the set  $(P \times Q)_{(p_0, q_0)}$  is a base open set for the topology of  $Alex(P \times Q)$ , but it coincides precisely with  $P_{p_0} \times Q_{q_0}$ . Therefore the Alexandrov topology on the product is coarser that the product topology, and we conclude. Note that this latter part would not be true in the case of an infinite product.
- equalizers are preserved by a simple set-theoretic argument. Therefore, we can conclude that finite limits are preserved.
- Alex is a full functor (by direct verification). Since we have proved that it preserves finite limits, then it reflects them as well.

### A.1.3 Topologies

In the analytic setting, there are two specially relevant Grothendieck topologies to consider: the étale topology on the algebraic side and the topology of local homeomorphisms on the topological side (which has however the same sheaves as the topology of opens embeddings). We have thus sites  $Aff_{\mathbb{C}}$ , ét and TSpc, loc. We can therefore consider the following topoi:

- $Sh_{\acute{e}t}(Sch_{\mathbb{C}});$
- $PSh(Sh_{\acute{e}t}(\mathcal{S}ch_{\mathbb{C}}))$ , which we interpret as "étale sheaves over the category of complex presheaves";
- Sh<sub>loc</sub>(TSpc). This last topos is indeed equivalent to the usual topos of sheaves over the category
  of topological spaces and open covers.

Now, there are analogs of both topologies in the stratified setting. Namely, we can define strét as the topology whose coverings are stratification-preserving étale coverings, and strloc as the topology whose coverings are jointly surjective families of stratification-preserving local homemorphisms. Therefore, we have well-defined stratified analogs:

- Sh<sub>strét</sub>(StrSch<sub>ℂ</sub>)
- $Sh_{str\acute{e}t}(StrPSh_{\mathbb{C}}) := PSh(Sh_{str\acute{e}t}(StrSch_{\mathbb{C}}))$
- Sh<sub>strloc</sub>(StrTSpc).

The stratification functor  $\mathfrak{strtop}$ :  $\operatorname{StrSch}_{\mathbb{C}} \to \operatorname{StrTSpc}$  sends stratified étale coverings to stratified coverings in the topology of local homeomorphism, and thus induces a functor

$$\operatorname{Sh}_{\operatorname{str\acute{e}t}}(\operatorname{StrPSh}_{\mathbb{C}}) \to \operatorname{Sh}_{\operatorname{strloc}}(\operatorname{StrTSpc}).$$

#### A.2 Symmetric monoidal structures on the constructible sheaves functor

# A.2.1 Constructible sheaves on conically stratified spaces locally of singular shape

Fix a torsion ring  $\Lambda$ . In order to treat p-adic fields and other coefficient fields, it would be more convenient to talk about diagrams of sheaves of rings, which is done in [LZ17]. We treat the torsion case for simplicity, referring to the more elaborated approach of [LZ17] for the general case. Alternatively, our construction can be carried out with l-adic constructible sheaves in place of constructible sheaves, starting from [GL18, Definition 2.3.2.1] instead of [GL18, Definition 2.2.6.3].

Remark A.5. Let (X, s) be a stratified topological space, and  $\Lambda$  be a torsion ring. Suppose that (X, s) is conically stratified and X is locally of singular shape (see [Lur17, Definition A.5.5 and A.4.15 resp.]). By [Lur17, Theorem A.9.3] (recently extended to the case of arbitrary posets in [Lej21])the  $\infty$ -category of constructible sheaves on X with respect to s with coefficients in  $\Lambda$ , denoted by  $\operatorname{Cons}_{\Lambda}(X, s)$  is equivalent to the  $\infty$ -category

$$\operatorname{Fun}(\operatorname{Exit}(X,s),\operatorname{Mod}_{\Lambda}).$$

Here  $\operatorname{Exit}(X,s)$  is the  $\infty$ -category of exit paths on (X,s) (see [Lur17, Definition A.6.2], where it is denoted by  $\operatorname{Sing}^A(X)$ , A being the poset associated to the stratification). We will often write Cons instead of  $\operatorname{Cons}_{\Lambda}$ .

Corollary A.6. Let (X, s) be a conically stratified space, and  $\Lambda$  be a torsion ring. Then  $Cons_{\Lambda}(X, s)$  is a presentable stable  $\Lambda$ -linear category.

Therefore, the  $\infty$ -category  $\operatorname{Pr}^L_{\Lambda}$  of presentable stable  $\Lambda$ -linear  $\infty$ -categories will be our usual environment from now on.

Note that the category  $StrTSpc_{csls}$  admits finite products because the product of two cones is the cone of the join space. Therefore, there is a well-defined symmetric monoidal Cartesian structure  $StrTSpc_{csls}^{\times}$ .

Consider the restricted functor Cons :  $\operatorname{StrTSpc}_{\operatorname{csls}} \to \operatorname{Cat}_{\infty}$ . We will often use the fact that the equivalence  $\operatorname{Cons} = \operatorname{Fun}(\operatorname{Exit}(-), \operatorname{Mod}_{\Lambda})$  holds.

#### A.2.2 Symmetric monoidal structure

Lemma A.7. The functor

$$\operatorname{Exit}: \operatorname{StrTSpc}_{\operatorname{csls}} \to \operatorname{Cat}_{\infty}$$
$$(X, s: X \to P) \mapsto \operatorname{Exit}(X, s) = \operatorname{Sing}^P(X)$$

carries a symmetric monoidal structure when we endow both source and target with the Cartesian symmetric monoidal structure. That is, it extends to a symmetric monoidal functor

$$\operatorname{StrTSpc}_{\operatorname{csls}}^{\times} \to \operatorname{Cat}_{\infty}^{\times}.$$

*Proof.* Given two stratified topological spaces  $X, s: X \to P, Y, t: Y \to Q$ , in the notations of [Lur17, A.6], consider the commutative diagram of simplicial sets

$$\operatorname{Sing}^{P \times Q}(X \times Y) \longrightarrow \operatorname{Sing}(X \times Y) \xrightarrow{\sim} \operatorname{Sing}(X) \times \operatorname{Sing}(Y)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$N(P \times Q) \longrightarrow \operatorname{Sing}(P \times Q)$$

$$\sim \downarrow \qquad \qquad \qquad \qquad \downarrow$$

$$N(P) \times N(Q) \longrightarrow \operatorname{Sing}(P) \times \operatorname{Sing}(Q)$$

The inner diagram is Cartesian by definition. Therefore the outer diagram is Cartesian, and we conclude that  $\operatorname{Sing}^{P \times Q}(X \times Y)$  is canonically equivalent to  $\operatorname{Sing}^{P}(X) \times \operatorname{Sing}^{Q}(Y)$ . Since  $\operatorname{Sing}^{P}(X)$  models the  $\infty$ -category of exit paths of X with respect to s, and similarly for the other spaces, we conclude.

**Remark A.8.** There exists a functor  $\mathcal{P}^{(*)}: \operatorname{Cat}_{\infty}^{\times} \to \operatorname{Pr}^{L^{\otimes}}$  sending an  $\infty$ -category  $\mathcal{P}(\mathcal{C})$ , a functor  $F: \mathcal{C} \to \mathcal{D}$  to the functor  $F^*: \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C})$  given by given by restriction under F.

We would like to set a symmetric monoidal structure on this functor. However, this is slightly complicated. Up to our knowledge, the symmetric monoidal structure is well-studied on its "covariant version", in the following sense.

**Lemma A.9** ([Lur17, Remark 4.8.1.8 and Proposition 4.8.1.15]). There exists a symmetric monoidal functor  $\mathcal{P}_{(!)}: \operatorname{Cat}_{\infty}^{\times} \to \operatorname{Pr}^{L^{\otimes}}$  sending an  $\infty$ -category  $\mathfrak{C}$  to the  $\infty$ -category of  $\mathfrak{S}$ -valued presheaves  $\mathcal{P}(\mathfrak{C})$ , a functor  $F: \mathfrak{C} \to \mathfrak{D}$  to the functor  $\mathcal{P}(\mathfrak{D}) \to \mathcal{P}(\mathfrak{C})$  given by  $F_! = \operatorname{Lan}_F(-)$ .

*Proof.* The existence of an oplax-monoidal structure follows from [Lur17, 4.8.1]. As for symmetric monoidality, apparently, a detail in the proof of [Lur17, Proposition 4.8.1.15] needs to be fixed: for any pair of  $\infty$ -categories  $\mathcal{C}, \mathcal{D}$ , the equivalence  $\mathcal{P}(\mathcal{C}) \times \mathcal{P}(\mathcal{D})$  follows from the universal property of the tensor product of presentable categories, and not from [Lur17, Corollary 4.8.1.12]. Indeed, for any cocomplete  $\infty$ -category  $\mathcal{E}$  one has:

$$\begin{split} \operatorname{Cocont}(\operatorname{\mathcal{P}}(\operatorname{\mathcal{C}}\times\operatorname{\mathcal{D}}),\operatorname{\mathcal{E}})&\simeq\operatorname{Fun}(\operatorname{\mathcal{C}}\times\operatorname{\mathcal{D}},\operatorname{\mathcal{E}})\simeq\operatorname{Fun}(\operatorname{\mathcal{C}},\operatorname{Fun}(\operatorname{\mathcal{D}},\operatorname{\mathcal{E}}))\simeq\\ &\simeq\operatorname{Fun}(\operatorname{\mathcal{C}}_0,\operatorname{Cocont}(\operatorname{\mathcal{P}}(\operatorname{\mathcal{D}}),\operatorname{\mathcal{E}}))\simeq\operatorname{Cocont}(\operatorname{\mathcal{P}}(\operatorname{\mathcal{C}}),\operatorname{Cocont}(\operatorname{\mathcal{P}}(\operatorname{\mathcal{D}}),\operatorname{\mathcal{E}}))\simeq\\ &\simeq\operatorname{Bicocont}(\operatorname{\mathcal{P}}(\operatorname{\mathcal{C}})\times\operatorname{\mathcal{P}}(\operatorname{\mathcal{D}}),\operatorname{\mathcal{E}}). \end{split}$$

Corollary A.10. There is a well-defined symmetric monoidal functor

$$\operatorname{Cons}_{(!)}^{\otimes} : \operatorname{StrTSpc}_{\operatorname{csls}}^{\times} \to \operatorname{Pr}_{\Lambda}^{\operatorname{L}, \otimes}$$

$$(X, s) \mapsto \operatorname{Cons}_{\Lambda}(X, s)$$

$$f \mapsto f_{!}^{\operatorname{formal}} = \operatorname{Lan}_{\operatorname{Exit}(f)}.$$

*Proof.* The previous constructions provide us with a symmetric monoidal functor

$$StrTSpc^{\times} \xrightarrow{Exit(-)} \mathfrak{C}at_{\infty}^{\times} \xrightarrow{op} \mathfrak{C}at_{\infty}^{\times} \xrightarrow{\mathcal{P}(-)} Pr^{L, \otimes}$$

sending

$$(X, s) \mapsto \operatorname{Fun}(\operatorname{Exit}(X, s), \mathbb{S}),$$
  
$$f \mapsto \operatorname{Lan}_{\operatorname{Exit}(f)}.$$

But now, with the notations of [Lur17, Subsection 1.4.2], for any ∞-category C we have

$$\operatorname{Fun}(\mathfrak{C},\operatorname{Sp})=\operatorname{Sp}(\operatorname{Fun}(\mathfrak{C},\mathfrak{S})).$$

Then we can apply [Rob14, Remark 4.2.16] and finally [Rob14, Theorem 4.2.5], which establish a symmetric monoidal structure for the functor  $\mathrm{Sp}(-): \mathrm{Pr}^{\mathrm{L}, \otimes} \to \mathrm{Pr}^{\mathrm{L}, \otimes}_{\mathrm{stable}}$ . The upgrade from  $\mathrm{Sp}$  to  $\mathrm{Mod}_{\Lambda}$  is straightforward and produces a last functor  $\mathrm{Pr}^{\mathrm{L}, \otimes}_{\mathrm{stable}} \to \mathrm{Pr}^{\mathrm{L}, \otimes}_{\Lambda}$  (the  $\infty$ -category of presentable stable  $\Lambda$ -linear categories).

From now on, we will often omit the "linear" part of the matter and prove statements about the functor Cons:  $StrTSpc_{csls}^{\times} \to Pr^{L,\otimes}$  and its variations, because the passage to the stable  $\Lambda$ -linear setting is symmetric monoidal.

**Remark A.11.** We denote the functor  $\operatorname{Lan}_{\operatorname{Exit}(f)}$  by  $f_!^{\operatorname{formal}}$  because, in general, it does not coincide with the proper pushforward of sheaves. We will see in the next subsection that it does under some hypothesis on f.

We will also see that, as a corollary of Corollary A.10, there exists a symmetric monoidal structure on the usual contravariant version

$$\operatorname{Cons}^{(*)} : \operatorname{StrTSpc}^{\operatorname{op}}_{\operatorname{csls}} \to \operatorname{Pr}^{R, \otimes}$$

$$(X, s) \mapsto \operatorname{Cons}(X, s)$$

$$f \mapsto f^* = - \circ \operatorname{Exit}(f)$$

as well.

#### A.3 The Beck-Chevalley property for constructible sheaves

**Definition A.12.** A cylindrically stratified map is a stratified map f between stratified topological spaces  $(X, s) \to (Y, t)$  such that Condition (ii) in  $[B^+84, 10.12]$  holds:

• Let  $y \in Y$ , and let S be its stratum. Then there exists a neighbourhood U of y in S, a stratified space F and a stratification preserving homeomorphism  $F \times U \to f^{-1}(U)$  which transforms the projection to U in f.

**Definition A.13.** Let comp.cnl be the class of compactifiably cylindrically stratified morphisms, that is of cylindrically stratified morphisms that factor as a stratified open embedding (which is automatically cylindrical) and a proper cylindrically stratified map.

Conjecture A.14. Every cylindrically stratified morphism of stratified topological spaces is compactifiable in the above sense.

We believe that this conjecture is reasonable, but we will not need it to hold true. Our aim is to prove the following theorem (we will indeed prove a more powerful version, see Theorem A.25).

**Theorem A.15.** There is an  $\infty$ -functor

$$Cons_{corr} : Corr(StrTSpc_{csls})_{comp.cnl,all} \rightarrow Cat_{\infty}$$

that coincides with Cons when restricted to

$$\operatorname{StrTSpc}^{\operatorname{op}} \hookrightarrow \operatorname{Corr}(\operatorname{StrTSpc}_{\operatorname{csls}})_{\operatorname{\mathfrak{comp.cnf}},\mathfrak{all}}.$$

It sends morphisms in horiz to pullback functors along those morphisms, and morphisms in comp.chl to proper pushforward functors along those morphisms.

#### A.3.1 Unstratified Beck-Chevalley condition

First of all, we need to recall some properties of the category of constructible sheaves with respect to an unspecified stratification.

**Definition A.16.** Let X be a topological space. Then there is a well-defined  $\infty$ -category of constructible sheaves with respect to a non-fixed stratification

$$\mathcal{D}_c(X) = \operatorname*{colim}_{s:X \to P} \operatorname*{colim}_{s: tatification} \operatorname{Cons}(X, s).$$

where the colimit is taken over the category  $\operatorname{StrTSpc} \times_{\operatorname{TSpc}} \{X\}$  of stratifications of X and refinements between them.

Unlike in the stratified case (i.e. the case when the stratification is fixed at the beginning, treated in the previous subsections), there is a well-defined six functor formalism. In particular

**Lemma A.17.** For any continuous map  $f: X \to Y$ , there are well defined functors  $f^*, f^!: \mathcal{D}_{\mathbf{c}}(Y) \to \mathcal{D}_{\mathbf{c}}(X)$ . Moreover,  $f^*$  has a right adjoint  $Rf_*$  and  $f^!$  has a left adjoint  $Rf_!$ .

From now on, we will write  $f_*$  for  $Rf_*$  and  $f_!$  for  $Rf_!$ . With this notations, the Proper Base Change Theorem (stated e.g. in [Kim15, Theorem 6]) holds:

**Theorem A.18** (Proper Base Change theorem). For any Cartesian diagram of unstratified topological spaces

$$X' \xrightarrow{g'} X$$

$$\downarrow^{f'} \qquad \downarrow^{f}$$

$$Y' \xrightarrow{g} Y$$
(A.2)

there is a canonical transformation of functors  $\mathcal{D}_c(X) \to \mathcal{D}_c(Y')$ 

$$f_1'g'^* \rightarrow g^*f_1$$

which is an equivalence of functors.

#### A.3.2 Stratified Beck-Chevalley property

We would like to prove an analogous property in the stratified case, that is for the functor Cons. First of all, the functor Cons does not have a well-behaved  $-_*$  or  $-_!$  functoriality.<sup>7</sup> In general, pushforward or proper pushforward along stratified maps do not send constructible sheaves with respect to the given stratification on the source to constructible sheaves with respect to the given stratification on the target. This is instead true if one allows refinements on the target, which is the reason why the

<sup>&</sup>lt;sup>7</sup>We should point out that the abstract procedure of left Kan extension is indeed available, as explained in Appendix A.2.2, but it does not coincide with the actual proper pushforward of sheaves in general. It does in the case of cylindrically stratified maps, as we are about to see.

situation is much better behaved when the stratification is allowed to vary, that is while dealing with  $\mathcal{D}_{c}$ . However, in our situation this problem does not exist.

**Proposition A.19** ([B+84, Theorem 10.17]). If  $f:(X,s) \to (Y,t)$  is a cylindrically stratificated map, then there are well-defined functors<sup>8</sup>

$$f_*, f_! : \operatorname{Cons}(X, s) \to \operatorname{Cons}(Y, t).$$

We want to transfer the results obtained for  $\mathcal{D}_c$  to the level of Cons. Given a stratified space (X, s), there is a natural functor  $\operatorname{Cons}(X, s) \to \mathcal{D}_c(X)$ .

**Lemma A.20.** The functor  $Cons(X, s) \to \mathcal{D}_c(X)$  is fully faithful.

*Proof.* It suffices to prove that the transition maps in the diagram defining  $\mathcal{D}_{c}(X)$  are fully faithful. This is true because for any refinement  $(X,t) \to (X,s)$  the map  $\operatorname{Fun}(\operatorname{Exit}(X,t),\mathbb{S}) \to \operatorname{Fun}(\operatorname{Exit}(X,s),\mathbb{S})$  is fully faithful by the proof of [AFR19, Theorem 3.3.12].

Suppose now that we have a Cartesian diagram in StrTSpc

$$(X',s') \xrightarrow{g'} (X,s)$$

$$\downarrow_{f'} \qquad \qquad \downarrow_{f}$$

$$(Y',s') \xrightarrow{g} (Y,s)$$
(A.3)

where the map f is cylindrically stratified. A simple verification shows that f' is also cylindrically stratified by base change. Now observe that

**Lemma A.21.** Let  $F: \mathcal{C} \to \mathcal{D}$  be a functor between  $\infty$ -categories, which restricts to a functor  $F_0$  between full subcategories  $\mathcal{C}_0 \subset \mathcal{C}$  and  $\mathcal{D}_0 \subset \mathcal{D}$ . Suppose that F has a right adjoint G sending  $\mathcal{D}_0$  to  $\mathcal{C}_0$ . Then  $G_0 = G|_{\mathcal{D}_0}$  is right ajoint to  $F_0$ .

Remark A.22. If f is a cylindrically stratified map between (X, s) and (Y, t), thanks to the previous discussion we can apply this lemma (functorially) to  $\mathcal{C} = \mathcal{D}_{c}(Y)$ ,  $\mathcal{D} = \mathcal{D}_{c}(X)$ ,  $F = f^{*}$ ,  $\mathcal{C}_{0} = \operatorname{Cons}(Y, t)$ ,  $\mathcal{D}_{0} = \operatorname{Cons}(X, s)$ ,  $G = f_{*}$ , and obtain that  $f_{*}|_{\operatorname{Cons}(X, s)}$  is right adjoint to  $f^{*}|_{\operatorname{Cons}(Y, t)}$ . Also, from the unstratified case we know that the natural transformation  $f'_{!}g'^{*} \to g^{*}f_{!}$  is an equivalence, which is therefore still true after restricting to sheaves constructible with respect to the given stratifications.

In particular, if f in Diagram (A.3) is both proper and cylindrically stratified, then we have the adjunctions  $f^* \dashv f_* = f_!, f'^* \dashv f_* = f_!$  and the equivalence  $f'_*g'^* \xrightarrow{\sim} g^*f_*$ .

<sup>&</sup>lt;sup>8</sup>As usual, we denote by  $f_*$  and  $f_!$  what Borel calls  $Rf_*$  and  $Rf_!$ .

<sup>&</sup>lt;sup>9</sup>Note that, for our purposes, the hypothesis of "conically smoothness" appearing in the statement of that theorem can be dropped.

#### A.4 Proof of Theorem A.15

We are now ready to prove Theorem A.15. We follow the approach used by D. Gaitsgory and N. Rozenblyum in [GR17, Chapter 5] to construct IndCoh as a functor out of the category of correspondences. As in these works, we need a theory of  $(\infty, 2)$ -categories. In particular, we need to extend  $\operatorname{Cat}_{\infty}$  to an  $(\infty, 2)$ -category, which we shall denote by  $\operatorname{Cat}_{\infty}^{2-\operatorname{cat}}$ : this is done in [GR17], informally by allowing natural transformations of functors which are not natural equivalences. We proceed by steps.

#### A.4.1 Step 1: $(f^*, f_*)$ adjunction and proper base change.

The previous subsection tells us that  $\operatorname{Cons}^{\operatorname{op}}:\operatorname{StrTSpc}\to\operatorname{Cat}_{\infty}^{2-\operatorname{cat},\operatorname{op}}$  satisfies the *left Beck-Chevalley condition* [GR17, Chapter 7, 3.1.5] with respect to  $\operatorname{\mathfrak{adm}}=\operatorname{\mathfrak{vert}}=\operatorname{\mathfrak{all}},\operatorname{\mathfrak{horij}}=\operatorname{\mathfrak{proper}}\cap\operatorname{\mathfrak{cyl}}$ , taking  $\Phi=\operatorname{Cons}:\operatorname{StrTSpc}^{\operatorname{op}}\to\operatorname{Cat}_{\infty}$ . Indeed, for any  $f\in\operatorname{\mathfrak{horij}}=\operatorname{\mathfrak{proper}}\cap\operatorname{\mathfrak{cyl}}$  we set  $\Phi^!(f)=f_*=f_!$ , which is right adjoint to  $f^*=\Phi(f)$ . Finally, the base change property for the diagram (A.3) completes the proof of the left Beck-Chevalley property.

Therefore by [GR17, Chapter 7, Theorem 3.2.2.(a)] the functor  $Cons^{op}: StrTSpc \to Cat_{\infty}^{2-cat,op}$  extends to a functor

$$(\operatorname{Cons}^{\operatorname{op}})_{\mathfrak{all};\mathfrak{proper}\cap\mathfrak{cnl}}:\operatorname{Corr}(\operatorname{StrTSpc})_{\mathfrak{all};\mathfrak{proper}\cap\mathfrak{cnl}}^{\mathfrak{proper}\cap\mathfrak{cnl}}\to\operatorname{Cat}_{\infty}^{2-\operatorname{cat},\operatorname{op}},$$

or equivalently a functor

$$\mathrm{Cons}_{\mathfrak{all};\mathfrak{proper}\cap\mathfrak{cpl}}:(\mathrm{Corr}(\mathrm{StrTSpc})^{\mathfrak{proper}\cap\mathfrak{cpl}}_{\mathfrak{all};\mathfrak{proper}\cap\mathfrak{cpl}})^{\mathrm{op}}\to\mathfrak{C}\mathrm{at}_{\infty}^{2-\mathrm{cat}}.$$

Now we restrict this functor to the  $(\infty, 1)$ -category

$$(\operatorname{Corr}(\operatorname{StrTSpc}^{\operatorname{op}})^{\mathfrak{isom}}_{\mathfrak{all};\mathfrak{proper}\cap\mathfrak{enl}})^{\operatorname{op}},$$

which is equivalent to the more familiar  $\operatorname{Corr}(\operatorname{StrTSpc})^{\mathfrak{isom}}_{\mathfrak{proper}\cap\mathfrak{cyl};\mathfrak{all}}$  (horizontal and vertical maps are interchanged while considering the opposite of the correspondence category). We have an  $(\infty, 1)$ -functor

$$Cons_{\mathfrak{proper}\cap\mathfrak{cpl},\mathfrak{all}}: Corr(StrTSpc)_{\mathfrak{proper}\cap\mathfrak{cpl},\mathfrak{all}}^{\mathfrak{isom}} \to \mathfrak{C}at_{\infty}. \tag{A.4}$$

#### A.4.2 Step 2: Nagata compactification and proper pushforward

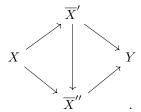
Consider the classes of morphisms in StrTSpc given by  $\mathfrak{hori3} = \mathfrak{all}$ ,  $\mathfrak{vert} = \mathfrak{comp.cyl}$ ,  $\mathfrak{adm} = \mathfrak{open} \subset \mathfrak{hori3}$ ,  $\mathfrak{co} - \mathfrak{adm} = \mathfrak{proper} \cap \mathfrak{cyl} \subset \mathfrak{vert}$  (note that we need  $\mathfrak{adm} \subset \mathfrak{hori3}$  and  $\mathfrak{co} - \mathfrak{adm} \subset \mathfrak{vert}$  instead of the converse, because we are performing a kind of construction dual to that used for IndCoh in [GR17]). We want to apply [GR17, Chapter 7, Theorem 5.2.4] and extend our functor to  $\operatorname{Corr}(\operatorname{StrTSpc})^{\mathfrak{open}}_{\mathfrak{comp.cyl,all}} \supset \operatorname{Corr}(\operatorname{StrTSpc})^{\mathfrak{isom}}_{\mathfrak{proper} \cap \mathfrak{cyl,all}}$ . We perform the verifications necessary to the application of that theorem:

• The pullback and 2-out-of-3 properties for  $\mathfrak{horij} = \mathfrak{all}, \mathfrak{co} - \mathfrak{adm} = \mathfrak{proper} \cap \mathfrak{cyl}, \mathfrak{isom}$ , which are immediatly verified.

- Every map in  $\mathfrak{adm} \cap \mathfrak{co} \mathfrak{adm} = \mathfrak{open} \cap \mathfrak{proper} \cap \mathfrak{cyl}$  should be a monomorphism. But the class  $\mathfrak{open} \cap \mathfrak{proper} \cap \mathfrak{cyl}$  consists of embeddings of unions of connected components.
- For any  $\alpha: X \to Y$  in  $\mathfrak{vert} = \mathfrak{comp.cyl}$ , consider the ordinary category  $Factor(\alpha)$ , whose objects are

$$X \xrightarrow{\varepsilon} \overline{X} \xrightarrow{\gamma} Y$$
,

where  $\varepsilon \in \mathfrak{adm} = \mathfrak{open}$  and  $\gamma \in \mathfrak{co} - \mathfrak{adm} = \mathfrak{proper} \cap \mathfrak{cyl}$ , and whose morphisms are commutative diagrams



Consider the  $(\infty, 1)$ -category N(Factor $(\alpha)$ ). We require N(Factor $(\alpha)$ ) to be contractible for any  $\alpha \in \mathfrak{vert}$ . But this is exactly the compactifiability hypothesis, i.e. the existence of the factorisation (see also [GR17, Chapter 5, Proposition 2.1.6]).

- Cons $|_{\text{StrTSpc}}$  should satisfy the *right* Beck-Chevalley condition with respect to  $\mathfrak{adm} = \mathfrak{open} \subset \mathfrak{horij} = \mathfrak{all}$ . This is true, because for every  $f \in \mathfrak{open}$  we have that  $f^* = f^!$ : now this admits a left adjoint  $f_!$ , and the Base Change Theorem holds.
- Given a Cartesian diagram

$$\begin{array}{ccc} X & \stackrel{\epsilon_0}{\longrightarrow} Y \\ \downarrow^{\gamma_1} & & \downarrow^{\gamma_0} \\ Z & \stackrel{\epsilon_1}{\longrightarrow} W \end{array}$$

with  $\epsilon_i \in \mathfrak{open}$  and  $\gamma_i \in \mathfrak{proper} \cap \mathfrak{cpl}$ , the Beck-Chevalley condition is satisfied (every functor from correspondences must satisfy it; this is the "easy" part of the extension theorems). Hence there is an equivalence

$$(\gamma_1)_* \epsilon_0^* \simeq \epsilon_1^* (\gamma_0)_*$$
.

Now we use the *other* Beck-Chevalley condition, the one introduced and checked in the previous point, using the fact that  $\epsilon_1 \in \mathfrak{open}$ . This new set of adjunctions gives us a morphism

$$(\epsilon_1)_!(\gamma_1)_* \to (\gamma_0)_*(\epsilon_0)_!.$$

We require that this morphism is an equivalence. But since  $\gamma_i \in \mathfrak{proper} \cap \mathfrak{cnl}$ , we have that  $(\gamma_i)_* \simeq (\gamma_i)_!$  and we conclude by commutativity of the diagram and functoriality of the proper pushforward.

Note that we have used exactly once that, respectively, for an open embedding  $f^* \simeq f^!$  and for a proper morphism  $f_* = f_!$ .

This completes the proof of Theorem A.15: the application of [GR17, Chapter 7, Theorem 5.2.4] provides us with an  $(\infty, 2)$ -functor from

$$\operatorname{Corr}(\operatorname{StrTSpc})^{\mathfrak{open}}_{\mathfrak{comp.cnl,all}}$$

to  $\operatorname{\mathtt{Cat}}^{2-\operatorname{\mathtt{cat}}}_{\infty}$  which we restrict to an  $(\infty,1)\text{-functor from}$ 

$$\operatorname{Corr}(\operatorname{StrTSpc})^{\mathfrak{isom}}_{\mathfrak{comp.cyl,all}}$$

to  $Cat_{\infty}$ .

**Remark A.23.** Note also that the adjunction properties proven in these pages tell us that in the cylindrically stratified case the functors  $f_!$  and  $Cons_{(!)}(f) = f_!^{formal}$  from Corollary A.10 coincide.

#### A.4.3 Symmetric monoidality and extension to sheaves

The functor Cons<sub>corr</sub> from Theorem A.15 carries a symmetric monoidal structure. That is:

**Proposition A.24.** The functor  $Cons_{corr}: Corr(StrTSpc_{csls})_{comp.cnf.all} \to Cat_{\infty}$  carries a symmetric monoidal structure with respect to the Cartesian structure on correspondences and the Lurie tensor product on  $Cat_{\infty}$ .

Proof. Let  $(StrTSpc_{csls})_{comp.cnf}$  be the category obtaining by restricting the class of morphisms in  $StrTSpc_{csls}$  to comp.cnf. Consider the embedding  $(StrTSpc_{csls})_{comp.cnf} \rightarrow Corr(StrTSpc)_{comp.cnf,aff}$ . Now we have a functor  $Cons_{(!)}^{\otimes} : StrTSpc_{csls}^{\times} \rightarrow Pr_{\Lambda}^{L,\otimes}$  from Corollary A.10, whose restriction to  $(StrTSpc_{csls})_{comp.cnf}$  coincides with  $Cons_{corr}|_{(StrTSpc_{csls})_{comp.cnf}}$  by Remark A.23. We can thus apply [GR17, Chapter 9, Proposition 3.1.9] with

$$vert = comp.cnl, co - adm = isom,$$

and obtain the desired symmetric monoidal functor

$$\mathrm{Cons}_{\mathrm{corr}}^{\otimes}: \mathrm{Corr}(\mathrm{StrTSpc}_{\mathrm{csls}})_{\mathfrak{comp.cyl,all}} \to \mathrm{Pr}_{\Lambda}^{\mathrm{L}, \otimes}.$$

In particular, the functor

Cons: 
$$\operatorname{StrTSpc}_{\operatorname{csls}}^{\operatorname{op}} \to \operatorname{Cat}_{\infty}$$
  
 $(X,s) \mapsto \operatorname{Cons}(X,s)$   
 $f \mapsto f^* = - \circ \operatorname{Exit}(f)$ 

is symmetric monoidal, as we had announced in Remark A.11.

Let now  $\mathrm{Sh_{strloc}}(\mathrm{StrTSpc_{csls}})$  be the  $\infty$ -category of sheaves on  $\mathrm{StrTSpc_{csls}}$  with the topology of local homeomorphisms (or open embeddings: the two sites are Morita-equivalent). We can define the class

$$comp.cnl \subset Mor(Sh(StrTSpc_{csls}))$$

to be the smallest saturated class containing the originary

$$comp.cnl \subset Mor(StrTSpc_{csls}).$$

**Theorem A.25.** In the above situation, there is a well-defined functor

$$\mathrm{Cons}_{\mathrm{corr}}^{\otimes}: \mathrm{Corr}(\mathrm{Sh}_{\mathrm{strloc}}(\mathrm{StrTSpc}_{\mathrm{csls}}))_{\mathtt{comp.cnf.alf}}^{\times} o \mathrm{Pr}_{\Lambda}^{\mathrm{L},\otimes}$$

which extends the functor

Cons: 
$$\operatorname{StrTSpc}_{\operatorname{csls}}^{\operatorname{op}} \to \operatorname{Cat}_{\infty}$$
  
 $(X, s) \mapsto \operatorname{Fun}(\operatorname{Exit}(X, s), \operatorname{Mod}_{\Lambda})$ 

 $\textit{along the embedding} \; \mathrm{StrTSpc}^{\mathrm{op}}_{\mathrm{csls}} \to \mathrm{Corr}(\mathrm{StrTSpc}_{\mathrm{csls}})_{\mathfrak{comp.cyl},\mathfrak{all}}.$ 

*Proof.* We argue as in [GR17, Chapter 5, 3.4]. By right Kan extension of Cons along the Yoneda embedding we obtain a functor  $\mathcal{P}(\text{StrTSpc}_{\text{csls}})^{\text{op}} \to \text{Cat}_{\infty}$ . By using the same arguments as in the proof of [GR17, Chapter 5, Theorem 3.4.3], Theorem A.15 provides us with an extension to

To replace  $\mathcal{P}(\mathrm{StrTSpc_{csls}})$  with the category of sheaves we use the descent properties of the functor Exit (see [Lur17, Appendix A.3]) and the fact that the functor  $\mathcal{P}: \mathrm{Cat}_{\infty}^{\mathrm{op}} \to \mathcal{P}r^{\mathrm{L}}$  sends colimits to limits.

Finally, the symmetric monoidal structure for the extended functor comes from arguments similar to the ones used in [GR17, Chapter 5, 4.1.5], that is essentially [GR17, Chapter 9, Proposition 3.2.4], whose hypotheses are trivially verified since  $\mathfrak{hori3} = \mathfrak{all}$ .

# B Omitted proofs and details

# B.1 Proof of Proposition 2.6

*Proof.* Fix a discrete complex algebra R, and let  $\xi = (S, \mathcal{F}_i, \alpha_i, \mu_i)_i$  be a vertex of the groupoid  $\operatorname{FactGr}_k(R)$ . We must prove that  $\pi_1(\operatorname{FactGr}_k(R), \xi) = 0$ . We know that  $\operatorname{Ran}(X)$  is a presheaf of sets over complex algebras. Therefore it suffices to prove that for every  $S \in \operatorname{Ran}(X)(R)$ , the fiber of  $\operatorname{FactGr}_k \to \operatorname{Ran}(X)$  at S is discrete.

Consider then an automorphism of a point  $(S, \mathcal{F}_i, \alpha_i, \mu_i)$ : this is a sequence of automorphisms  $\phi_i$  for each bundle  $\mathcal{F}_i$ , such that the diagrams

$$\begin{array}{ccc} \mathcal{F}_i|_{X_R \backslash \Gamma_S} & \xrightarrow{\alpha_i} & \mathcal{T}|_{X_R \backslash \Gamma_S} \\ \phi_i|_{X_R \backslash \Gamma_S} & & & & \\ & \mathcal{F}_i|_{X_R \backslash \Gamma_S} & & & & \end{array}$$

(for  $i = 1, \ldots, k$ ) and

$$\begin{array}{ccc} \mathcal{F}_i|_{\widehat{(X_R)}_{\Gamma_S}} & \xrightarrow{\mu_i} \mathcal{I}|_{\widehat{(X_R)}_{\Gamma_S}} \\ & \phi_i|_{\widehat{(X_R)}_{\Gamma_S}} & & & \\ & \mathcal{F}_i|_{\widehat{(X_R)}_{\Gamma_S}} & & & \end{array}$$

(for i = 1, ..., k - 1) commute. (Actually, only the commutation of the former set of diagrams is relevant to the proof.)

The first diagram implies that  $\phi_i$  is the identity over  $X_R \setminus \Gamma_S$ . We want to show that  $\phi_i$  is the identity. In order to show this, we consider the relative spectrum  $\underline{\operatorname{Spec}}_{X_R}(\operatorname{Sym}(\mathcal{F}_i))$  of  $\mathcal{F}_i$ , which comes with a map  $\pi: Y = \underline{\operatorname{Spec}}_{X_R}(\operatorname{Sym}(\mathcal{F}_i)) \to X_R$ . An automorphism of  $\mathcal{F}_i$  corresponds to an automorphism  $f_i$  of Y over  $X_R$ , which in our case is the identity over the preimage of  $X_R \setminus \Gamma_S$  inside Y. Let U be the locus  $\{f_i = \operatorname{id}\}$ . This set is topologically dense, because it contains the preimage of the dense open set  $X_R \setminus \Gamma_S$ . We must see that it is schematically dense, that is the restriction map  $\mathcal{O}_Y(Y) \to \mathcal{O}_Y(U)$  is injective. If we do so, then  $\phi_i = \operatorname{id}$  globally.

The remaining part of the proof was suggested to us by Angelo Vistoli.

We may suppose that R (and therefore Y and  $X_R$ ) are Noetherian. Indeed, we can reduce to the affine case and suppose  $X_R = \operatorname{Spec} P$  and Y of the form  $\operatorname{Spec} P[t_1, \ldots, t_n]$  from the Noether Lemma (observe that Y is finitely presented over  $X_R$ ). Any global section f of  $\operatorname{Spec} P[t_1, \ldots, t_n]$  lives in a smaller Noetherian subalgebra  $P'[t_1, \ldots, t_n]$ , because it has a finite number of coefficients in P. Analogously, we can suppose U to be a principal open set of  $\operatorname{Spec} P[t_1, \ldots, t_n]$  and thus  $f|_U$  can be seen as a section in some noetherian subalgebra of  $P[t_1, \ldots, t_n]_g$ , with g a polynomial in  $P[t_1, \ldots, t_n]$ . Therefore we conclude that the proof that f is zero can be carried out over a Noetherian scheme.

Let us recall the following facts.

• ([Mat89, page 181]) If  $A \to B$  is a flat local homomorphism of local noetherian rings, then

$$\operatorname{depth} B = \operatorname{depth} A + \operatorname{depth} B/\mathfrak{m}B,$$

where  $\mathfrak{m}$  is the maximal ideal of A.

• If  $f: S \to T$  is a flat morphism of noetherian schemes,  $p \in S$ , then p is associate in S if and only if p is associate in the fiber of f(p) and f(p) is associate in T.

Let now S = Y and  $T = X_R$ . First of all, if we consider the composition  $Y^{\text{red}} \to X_R$  we have that  $U^{\text{red}} = Y^{\text{red}}$ , because two morphisms between separated and reduced schemes coinciding on an open dense set coincide everywhere.

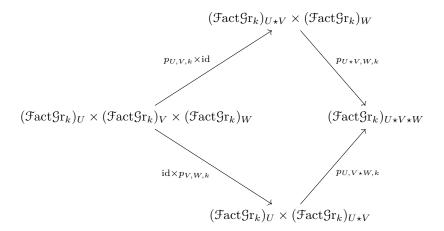
Now we note that U contains the generic points of every fiber. Indeed, every  $f^1(x) \subset Y$  factors through  $Y^{\text{red}} \to Y$  because the fibers are integral, and hence through  $U^{\text{red}} = Y^{\text{red}} \to Y$ .

Now if y is an associate point in Y then it is associate in  $f^{-1}(f(y))$ . Therefore it is a generic point of  $f^{-1}(f(y))$ , because every fiber of a principal G-bundle is isomorphic to G, which is integral. But U contains all generic points of the fibers, which are their associated points because the fibers are integral.

This implies that U is schematically dense.

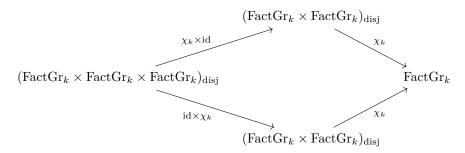
# B.2 Proof of Proposition 3.10

*Proof.* Since the inclusions  $U \hookrightarrow V$  of open sets in  $\operatorname{Ran}(X)$  induce inclusions  $(\operatorname{\mathcal{F}act}\operatorname{\mathcal{G}r}_k)_U \hookrightarrow (\operatorname{\mathcal{F}act}\operatorname{\mathcal{G}r}_k)_V$  and do not alter the datum of  $(\operatorname{\mathcal{F}act}\operatorname{\mathcal{G}r}_k)_U$ , it suffices to prove that the maps  $p_{U,V,k}$  make the diagram



commute in StrTSpc. Now this is true because of the following. Define  $(\text{Ran}(X) \times \text{Ran}(X) \times \text{Ran}(X))$  as the subfunctor of  $\text{Ran}(X) \times \text{Ran}(X) \times \text{Ran}(X)$  parametrising those  $S, T, P \subset X(R)$  whose graphs are pairwise disjoint in  $X_R$ . Let  $(\text{FactGr}_k \times \text{FactGr}_k)$  be its preimage

under  $r_k \times r_k \times r_k$ : FactGr<sub>k</sub> × FactGr<sub>k</sub> × FactGr<sub>k</sub> → Ran(X) × Ran(X) × Ran(X). Then the diagram



commutes because the operation of gluing is associative, as it is easily checked by means of the defining property of the gluing of sheaves.

Note also that everything commutes over Ran(M).

Finally, to prove that the functor defined in Remark 3.9 is a map of operads, we use the characterisation of inert morphisms in a Cartesian structure provided by [Lur17, Proposition 2.4.1.5]. Note that:

- An inert morphism in  $\operatorname{Fact}(M)^{\otimes}$  is a morphism of the form  $(U_1, \ldots, U_m) \to (U_{\phi^{-1}(1)}, \ldots U_{\phi^{-1}(n)})$  covering some inert arrow  $\phi : \langle m \rangle \to \langle n \rangle$  where every  $i \in \langle n \rangle^{\circ}$  has exactly one preimage  $\phi^{-1}(i)$ .
- An inert morphism in StrTSpc<sup>×</sup> is a morphism of functors  $\bar{\alpha}$  between  $f: \mathscr{P}(\langle m \rangle^{\circ})^{\mathrm{op}} \to \mathrm{TSpc}$  and  $g: \mathscr{P}(\langle n \rangle^{\circ})^{\mathrm{op}} \to \mathrm{TSpc}$ , covering some  $\alpha: \langle m \rangle \to \langle n \rangle$ , and such that, for any  $S \subset \langle n \rangle$ , the map induced by  $\bar{\alpha}$  from  $f(\alpha^{-1}S) \to g(S)$  is an equivalence in StrTSpc.

By definition,  $\mathfrak{CGr}_k^{\times}((U_1,\ldots,U_m))$  is the functor f assigning  $T\subset \langle m\rangle^{\circ}\mapsto \prod_{j\in T}(\mathfrak{F}\mathrm{act}\mathfrak{Gr}_k)_{U_j}$ , and analogously  $\mathfrak{CGr}_k^{\times}((U_{\phi^{-1}(1)},\ldots,U_{\phi^{-1}(m)}))$  is the functor g assigning  $S\subset \langle n\rangle^{\circ}\mapsto \prod_{i\in S}(\mathfrak{F}\mathrm{act}\mathfrak{Gr}_k)_{U_{\phi^{-1}(i)}}$ . But now, if  $\alpha=\phi$  and  $T=\phi^{-1}(S)$ , we have the desired equivalence.

#### B.3 Proof of Proposition 5.1

*Proof.* It is sufficient to prove that each  $\mathcal{F}$ act $\mathcal{A}$ rc $_k$  is conically stratified and locally of singular shape. Indeed, each  $(\mathcal{F}$ act $\mathcal{A}$ rc $_k)_U$ , being an open set of  $\mathcal{F}$ act $\mathcal{A}$ rc $_k$  with the induced stratification, will be conically stratified and locally of singular shape as well.

Moreover, it suffices to show that  $\mathfrak{strtop}(RanGr)$  is conically stratified and locally of singular shape. Indeed, this will imply the same property for the k-fold-product of copies of  $\mathfrak{strtop}(RanGr)$  over Ran(M), and  $\mathfrak{F}act\mathcal{A}rc_k$  is a principal bundle over this space, with unstratified fiber. This consideration implies the property for  $\mathfrak{F}act\mathcal{A}rc_k$ .

Let us then prove the property for strtop(RanGr). First of all, the Ran Grassmannian is locally of singular shape because of the following argument.

**Proposition B.1.** Let  $\mathcal{U}: \Delta^{\mathrm{op}} \to \mathrm{Open}(X)$  be a hypercovering. Then  $\mathrm{Sing}(X) \simeq \mathrm{colim} \, \mathrm{Sing}_{n \in \Delta^{\mathrm{op}}}(\mathcal{U}_n)$ .

*Proof.* We use [Lur17, Theorem A.3.1]. Condition (\*) in *loc.cit*. is satisfied for the following modification of  $\mathcal{U}$ . Since  $\mathcal{U}$  is a hypercovering, one can choose for any [n] a covering  $(U_n^i)_i$  of  $\mathcal{U}_n$ ,

functorially in n. We can thus define a category  $\mathcal{C} \to \Delta^{\mathrm{op}}$  as the unstraightening of the functor

$$\mathbf{\Delta}^{\mathrm{op}} 
ightarrow \mathtt{Set}$$

$$[n] \mapsto \{(U_n^i)_i\}.$$

Then there is a functor  $\tilde{\mathcal{U}}: \mathcal{C} \to \mathcal{O}\mathrm{pen}(X)$ ,  $([n], U_n^i) \mapsto U$ . This functor satisfies (\*) in [Lur17, Theorem A.3.1], and therefore  $\mathrm{colim}_{([n],U)\in\mathcal{C}}\mathrm{Sing}(U)\simeq\mathrm{Sing}(X)$ . Now note that  $\mathcal{U}$  is the left Kan extension of  $\tilde{\mathcal{U}}$  along  $\mathcal{C} \to \Delta^{\mathrm{op}}$ . Therefore,

$$\operatorname{Sing}(X) \simeq \operatorname{colim}_{\mathfrak{C}} \operatorname{Sing}(U) \simeq \operatorname{colim}_{\Delta^{\operatorname{op}}} \operatorname{Sing}(\mathfrak{U}_n).$$

This allows us to apply the proof of [Lur17, Theorem A.4.14] to any hypercovering of Ran(M). Now, the space Ran(M) is of singular shape because is contractible and homotopy equivalences are shape equivalences. Therefore, so are elements of the usual prebase of its topology, namely open sets of the form  $\prod_i \text{Ran}(D_i)$ . One can construct a hypercovering of Ran(M) by means of such open subsets, and therefore we can conclude by applying the modified version of [Lur17, Theorem A.4.14] that we have just proved.

It remains to prove that the Ran Grassmannian is conically stratified. Indeed, it is Whitney stratified. The proof of such property has been suggested to us by David Nadler [Nad], and uses very essential properties of the Ran Grassmannian. Consider two strata X and Y of RanGr. We want to prove that they satisfy Whitney's conditions A and B; that is:

- for any sequence  $(x_i) \subset X$  converging to y, such that  $T_{x_i}X$  tends in the Grassmanian bundle to a subspace  $\tau_y$  of  $\mathbb{R}^m$ ,  $T_yY \subset \tau_y$  (Whitney's Condition A for  $X, Y, (x_i), y$ );<sup>10</sup>
- when sequences  $(x_i) \subset X$  and  $(y_i) \subset Y$  tend to y, the secant lines  $x_i y_i$  tend to a line v, and  $T_{x_i} X$  tends to some  $\tau_y$  as above, then  $v \in \tau_y$  (Whitney's Condition B for  $X, Y, (x_i), (y_i), y$ ).<sup>11</sup>

The only case of interest is when  $\overline{X} \cap Y$  is nonempty. Observe that, when the limit point  $y \in Y$  appearing in Whitney's conditions is fixed, conditions A and B are local in Y, i.e. we can restrict our stratum Y to an (étale) neighbourdhood U of the projection of y in Ran(M). Also, both Y and the  $y_i$  in Condition B live over some common stratum  $Ran_n(M)$  of Ran(M). Using the factorisation property, which splits components and tangent spaces, we can suppose that n = 1. Therefore, X projects onto the "cardinality 1" component of Ran(M), that is M itself. By the locality of conditions A and B explained before, we can suppose that y and the  $(y_i)$  involved in Whitney's conditions live over  $\mathbf{A}^1$ . There, the total space is  $Gr_{\mathbb{A}^1}$  and this is simply the product  $Gr \times \mathbb{A}^1$  because on the affine line the identification is canonical. From this translational invariance it follows that we can suppose

 $^{11}$ Idem.

 $<sup>^{10}</sup>X$  and Y are said to satisfy Whitney's condition A if this is satisfied for any  $(x_i) \subset X$  tending to  $y \in Y$ . The space is said to satisfy Whitney's condition A if every pair of strata satisfies it.

our stratum Y concentrated over a fixed point  $0 \in \mathbb{A}^1$ , that is: both y and the  $y_i$  can be canonically (thus simultaneously) seen inside  $\operatorname{Gr}_0 \subset \operatorname{Gr}_{\mathbb{A}^1}$ .

Now, by [Kal05, Theorem 2] we know that there exists at least a point  $y \in \overline{X} \cap Y$  such that, for any  $(x_i) \to y$  as in the hypothesis of Whitney's Condition A, Whitney's Condition A is satisfied, and the same for Whitney's condition B. In other words, the space

$$\operatorname{Sing}(X,Y) = \{ y \in \overline{X} \cap Y \mid y \text{ does not satisfy either Whitney's } \}$$

Condition A or B for some choice of 
$$(x_i), (y_i)$$

does not coincide with the whole  $\overline{X} \cap Y$ . Let  $\pi : \Re \operatorname{AnGr} \to \operatorname{Ran}(M)$  be the natural map. Note that X and Y are acted upon by  $\Re \operatorname{AnArc} \times_{\operatorname{Ran}(M)} \pi(X)$  and  $\Re \operatorname{AnArc} \times_{\operatorname{Ran}(M)} \pi(Y) \simeq G_{\mathcal{O}}$  respectively (recall that Y is a subset of  $\operatorname{Gr} = \operatorname{Gr}_0 \subset \operatorname{Gr}_{\mathbb{A}^1}$ ), and this action is transitive on the fibers over any point of  $\operatorname{Ran}(M)$ . Now, the actions of  $\operatorname{RanArc}$  and  $G_{\mathcal{O}}$  take Whitney-regular points with respect to X to Whitney-regular points with respect to X, since it preserves all strata. Therefore, if Y is a "regular" point as above, the whole Y is made of regular points, and we conclude.

# B.4 Details from the proof of [HY19, Proposition 3.17]

The following proof has been communicated to us by Jeremy Hahn and Allen Yuan, expanding the one contained in [HY19]. We report the details, as communicated by them, because we need them for our purposes.

• We adapt to the notations of loc. cit., so that

$$\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_{1,x} = \operatorname{Gr}(\{x\}), (\operatorname{\mathcal{F}act} \operatorname{\mathcal{G}r}_1)_{\operatorname{Ran}^{\leq n}(D)} = \operatorname{Gr}(D),$$
  
$$p: \operatorname{Gr}(D) \to \operatorname{Ran}^{\leq n}(D), i: \operatorname{Gr}(\{x\}) \to \operatorname{Gr}(D).$$

- First of all, note that every connected component of Gr({x}) is simply connected. Indeed, there are at least two ways to see this. One is to note that there is an explicit Iwahori cell decomposition, by even cells. So as a CW complex Gr({x}) is built only out of 0-cells, 2-cells, 4-cells, etc. and therefore has trivial fundamental group at any arbitrary basepoint. A second way is to note that the affine Grassmannian is the based loop space of a Lie group G<sup>an</sup>, and therefore its fundamental group at any basepoint corresponds to the second homotopy group of the Lie group at a corresponding basepoint (well-defined up to homotopy). However, π<sub>2</sub> of any Lie group is trivial. The Iwahori cell decomposition is a refinement of the filtration we consider, so Gr({x}) is just an explicit skeleton which again has only even cells.
- Let  $\operatorname{Ran}^n(D)$  be the open subset of  $\operatorname{Ran}^{\leq n}(D)$  consisting of those subsets od D having exactly n elements, and let  $\operatorname{Gr}^n(D)$  be its preimage along p. Now suppose n=1. Then  $\operatorname{Gr}^1(D)$  is (the underlying complex topological space of) the so-called Beilinson-Drinfeld Grassmannian, restricted from M to D. Call  $\operatorname{Gr}^1(M)$  the Beilinson-Drinfeld Grassmannian over the whole M.

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Then  $\operatorname{Gr}^1(M) \to \operatorname{Ran}^1(M) = M$  is a fiber bundle, because following [Zhu16, Equation 3.1.10]  $\operatorname{Gr}^1(M)$  can be rewritten as the underlying complex topological space of  $\hat{X} \times^{G_{\mathcal{O}}} \operatorname{Gr}(\{x\})$ . Now  $\hat{X}$  is a  $G_{\mathcal{O}}$ -torsor over X, hence the corresponding map of topological spaces  $\hat{M} \to M$  is a locally trivial fibration (a fiber bundle). Therefore by [AP12, Proposition 2.6.4]  $\operatorname{Gr}(M) \to M$  is a locally trivial fibration as well, that is a fiber bundle with fiber  $\operatorname{Gr}(\{x\})$ .

• The map  $Gr(Ran(M)) \to Ran(M)$  is a stratified fibration with respect to the stratification  $Ran^{\leq n}(M)$ . This comes from the previous step and from the fact that homotopies of paths can be lifted along that map.

As for the map  $CArc_k(Ran(\{E_i\})) \to CArc_k(Ran(\{D_i\}))$ , by similar arguments we can reduce ourselves to the case k = 1. But since the datum of  $G_{\mathcal{O}}$  is completely local, the verification boils down to the fact that the map  $Ran(\{E_i\}) \to Ran(\{D_i\})$  is a homotopy equivalence.

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