DAY I, TALK 3: SINGULAR SUPPORT - I
SINGULAR SUPPORT ON SCHEMES

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Contents

References:

- Detailed treatment in [3], summarized in [4, Sections 2 and 3].
- Less detailed informal introduction: two talks [2].
- Even less detailed version: the talk [1] (unfortunately, the notes are in my handwriting, making them hard to read, but you can watch the video.)

1. Singular support of coherent sheaves via cohomological operations

Let Z be an affine complete intersection over k ($\mathrm{char}(k) = 0$). Thus, $Z = \mathrm{Spec}(A) \subset S = \mathrm{Spec}(R)$, $A = R/(f_1, \ldots, f_n)$, R is a smooth (f.g.) k-algebra. Either need to assume that $(f_1, \ldots, f_n)$ is a regular sequence (i.e., $\mathrm{codim}(Z) = n$), or work in the dg world (where such Z is called 'quasi-smooth').

1.1. **Cohomological operations (Gulliksen, around 1974).** Take $F \in \mathrm{Coh}(Z)$. (For now, F could be a coherent sheaf, but nothing prevents us from looking at an object of the derived category; Coh is what usually is denoted D^b_{coh}.)

Proposition 1. *There are natural operators $\xi_i \in \mathrm{Ext}^2(F, F)$, $i = 1, \ldots, n$; $\xi_i\xi_j = \xi_j\xi_i$.*

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Example. Suppose $n = 1$. Consider $i : Z \hookrightarrow S$. Take Li^*i_*F. From the Koszul resolution, we get an exact triangle

$$F[1] \to Li^*i_*F \to F \to F[2];$$

the corresponding element of $\mathrm{Ext}^2(F,F)$ is ξ. In general, can define each ξ_i in the same way by using the embedding

$$Z \hookrightarrow \operatorname{Spec} R/(f_1,\dots,\hat{f}_i,\dots,f_n)$$

(but in this approach, we need to check that they commute by hand). □

The operators have been used by many people (Eisenbud, especially if n=1; Avramov-Buchweitz, especially if F has finite length).

Consider the graded algebra

$$\mathrm{Ext}^\bullet(F,F) = \bigoplus_i \mathrm{Ext}^i(F,F).$$

It is a (not necessarily commutative) algebra over $A[\xi_1,\dots,\xi_n]$.url in tex

Theorem 2 (Gulliksen)**.** *$\mathrm{Ext}^\bullet(F,F)$ is a f.g. module over $A[\xi_1,\dots,\xi_n]$.*

Proof. Nice exercise in Koszul-type constructions. □

1.2. Singular support.

Definition 3. (used by Avramov-Buchweitz for finite length F, and by Benson-Iyengar-Krause in general)

$$\mathrm{SingSupp}(F) = \mathrm{supp}_{A[\xi_1,\dots,\xi_n]} \mathrm{Ext}^\bullet(F,F) \subset \mathrm{Spec}(A[\xi_1,\dots,\xi_n]) = \mathbb{A}^n \times Z.$$

Some elementary properties of this notion:

- $\mathrm{SingSupp}(F) \subset \mathbb{A}^n \times Z$ is a Zariski-closed conical subset. (Note: we don't try to equip it with a scheme structure.)
- $\mathrm{SingSupp}(F) \cap \{0\} \times Z = pr_2(\mathrm{SingSupp}(F)) = \mathrm{supp}_A \mathrm{Ext}^\bullet(F,F) = \mathrm{supp}(F)$.

Example 4. If F is perfect (= of finite Tor dimension), then $\mathrm{Ext}^k(F,F) = 0$ for $k \gg 0$. The converse is also true (exercise). Since $\mathrm{Ext}^\bullet(F)$ is a finitely generated module, we see that F is perfect $\iff$ $\mathrm{SingSupp}(F) \subset \{0\} \times Z$. In this sense, the singular support is a measure of 'imperfection' of F: its non-zero points appear only over the locus where F is imperfect. However, they also carry some finer information: the 'direction' (or perhaps 'codirection') of imperfection.

1.3. Cohomological operators and Hochschild cohomology.

This section may be skipped[1]. Note that $\xi_i \in \mathrm{Ext}^2(F,F)$ depends naturally on F. In other words, the bifunctor $(F,G) \mapsto \mathrm{Ext}^\bullet(F,G)$ lands in the category of graded $A[\xi_1,\dots,\xi_n]$-modules, and the composition is bilinear. (In particular, $A[\xi_1,\dots,\xi_n]$ maps to the center of $\mathrm{Ext}^\bullet(F,F)$; we already silently used this in the defintion of singular support.)

A stronger statement can be made: ξ_i are naturally defined as elements in the Hochschild cohomology $\xi_i \in HH^2(Z)$.

[1]or moved to the tutorial

Example 5. Continuation of Example 4. Suppose $\mathrm{SingSupp}(F) \subset \{0\} \times Z$. Then $\xi_i \in \mathrm{Ext}^\bullet(F,F)$ are nilpotent. Therefore, ξ_i also act nilpotently on $\mathrm{Ext}^\bullet(F,G)$ (and on $\mathrm{Ext}^\bullet(G,F)$) for any G: the action agrees with the composition. Since $\mathrm{Ext}^\bullet(F,G)$ is a finitely generated $A[\xi_1,\dots,\xi_n]$-module (follows from 2 applied to $F \oplus G$), we see that $\mathrm{Ext}^k(F,G) = 0$ for $k \gg 0$. This is one way to explain why F must be perfect.

Exercise 1. Show that for any exact triangle

$$\to F_1 \to F_2 \to F_3 \to,$$

$\mathrm{SingSupp}(F_2) \subset \mathrm{SingSupp}(F_1) \cup \mathrm{SingSupp}(F_3)$.

Exercise 2. Show that $\mathrm{SingSupp}(F) - \{0\} \times Z$ depends only on the image of F in the *singularity category* of Z. (D. Orlov proved an equivalent between the singularity category of Z and that of a lci hypersurface; this equivalence can be used to determine $\mathrm{SingSupp}(F)$).

2. Shifted cotangent 'bundle'

Definition 3 has the disadvantage of appearing non-invariant. In fact, it is invariant; let us explain why.

2.1. Definition of shifted cotangent bundle.

Proposition 6. *For any F,*

$$\mathrm{SingSupp}(F) \subset \{(a_1,\dots,a_n,x) : \sum a_i df_i(x) = 0\}.$$

(Conversely, any Zariski-closed conical subset of the R.H.S. may appear as SingSupp *of some coherent sheaf.)*

The R.H.S. of the proposition has the following invariant meaning. Consider T^*Z in the derived sense (the cotangent complex, denoted by $\mathbb{L}Z$). It is represented by the complex

$$\mathcal{O}_Z^n \overset{(df_1,\dots,df_n)}{\to} T^*S|_Z.$$

Then the R.H.S. is the 'kernel' of this map, i.e., it is $H^{-1}T^*Z$. In particular, it depends on Z only, and not on the presentation of Z as a complete intersection. (Rigorously, $H^{-1}T^*Z = \operatorname{Spec}\operatorname{Sym}TZ[1]$.)

We use the notation $\mathrm{Sing}(Z) = H^{-1}T^*Z$, to be consistent with [3, 4]. Note that $\mathrm{Sing}(Z)$ is a kind of 'vector bundle': its fiber over each point of Z is a vector space. However, the dimension of this vector space might vary.

Proposition 7. $\mathrm{SingSupp}(F)$ *is well defined as a subset of* $\mathrm{Sing}(Z)$.

Example 8. Suppose $n = 1$, so Z is a hypersurface. Then the fiber of $\mathrm{Sing}(Z)$ over $x \in Z$ is given by

$$\mathrm{Sing}(Z) = \begin{cases} 0 & \text{if } x \text{ is smooth} \\ \mathbb{A}^1 & \text{if } x \text{ is singular.} \end{cases}$$

By Example 4, we know that the fiber of $\mathrm{SingSupp}(F)$ over x is

$$\mathrm{SingSupp}(F)_x = \begin{cases} \emptyset & \text{if } x \notin \mathrm{supp}(F) \\ 0 & \text{if } F \text{ is perfect at } x \text{ (and non-trivial)} \\ \mathbb{A}^1 & \text{if } F \text{ is not perfect at } x. \end{cases}$$

Remark 9. Note the similarity between the singular support for coherent sheaves and for D-modules: both live in 'cotangent bundles', the only difference is the cohomological shift. (It is possible to develop this analogy further.)

Remark 10. (May be skipped) In fact, we have a canonical map $TZ[-1] \to HC^\bullet(Z)$, where $HC^\bullet(Z)$ is the Hochschild complex. (This is part of the HKR theorem.) It induces the map from

$$Sym^k(TZ[1]) \to HC^\bullet(Z)[2k];$$

this is a more canonical way of thinking about the cohomological operations.

3. Ind-coherent sheaves and singular support

The notion of singular support naturally extends to a larger category: the *ind-completion* of $\mathrm{Coh}(Z)$. For now, we allow Z to be any (say, quasicompact separated) (dg) scheme.

3.1. **Ind-coherent sheaves.** So, what is $\mathrm{IndCoh}(Z)$? Informally, $\mathrm{IndCoh}(Z)$ is a version of the quasi-coherent derived category $\mathrm{QCoh}(Z)$ that is compactly generated by $\mathrm{Coh}(Z)$. Recall that $\mathrm{QCoh}(Z)$ is compactly generated by $\mathrm{Perf}(Z) \subset \mathrm{QCoh}(Z)$ (the full subcategory of *perfect* complexes), so $\mathrm{IndCoh}(Z)$ is larger).

There are several equivalent approaches to $\mathrm{IndCoh}(Z)$.

1. (Krause) Homotopy category of unbounded complexes of injectives. (By contrast, $\mathrm{QCoh}(Z)$ is equivalent to the homotopy category of so-called K-injectives, which is smaller.)
2. (Positselski) Co-derived category (modify the definition of derived category by modding out only by some of the acyclic complexes, not all of them).
3. Define it as ind-completion of $\mathrm{Coh}(Z)$.

We prefer the last approach, but it requires working with dg categories (while the other two work fine in the triangulated world). For the rest of this section, all categories will be dg.

Definition 11. $\mathrm{IndCoh}(Z)$ is the derived category of (contravariant) functors $\mathrm{Coh}(Z) \to \mathrm{Vect}$, where Vect is the dg category of (complexes of) vector spaces.

Informally, this means that $F \in \mathrm{IndCoh}(Z)$ is known if we know $\mathrm{Hom}(G, F)$ for all $G \in \mathrm{Coh}(Z)$. We can also describe objects of $\mathrm{IndCoh}(Z)$ as (formal) inductive limits of coherent sheaves.

Remark 12. There are two natural functors relating these categories. On the one hand, the functor $\mathrm{Coh}(Z) \to \mathrm{QCoh}(Z)$ naturally extends to a functor $\Psi : \mathrm{IndCoh}(Z) \to \mathrm{QCoh}(Z)$ (sending 'formal' inductive limit in IndCoh to its value in QCoh). It admits a left adjoint $\Xi : \mathrm{QCoh}(Z) \to \mathrm{IndCoh}(Z)$ that extends the tautological embedding $\mathrm{Perf}(Z) \hookrightarrow \mathrm{Coh}(Z)$. This left adjoint is fully faithful. (One of these functors is easily seen in Positselski's approach, the other in Krause's approach.)

An obvious but important point: the composition $\mathrm{Coh}(Z) \hookrightarrow \mathrm{QCoh}(Z) \hookrightarrow \mathrm{IndCoh}(Z)$ is *not* the same as the tautological embedding $\mathrm{Coh}(Z) \hookrightarrow \mathrm{IndCoh}(Z)$.

Remark 13. Why are Ind-Coherent sheaves important? Some constructions naturally call for coherent sheaves: !-pullback, Serre's duality. (If one tries defining !-pullback on QCoh, it won't be a continuous functor, which we don't allow.) On the other hand, Coh by itself is also not a good category to work in (not cocomplete, !-pullback does not preserve it). IndCoh fixes this. (There are other reasons, e.g., from the world of D-modules.) Some examples to be discussed later.

3.2. **Singular support for Ind-coherent sheaves.** Now return to the case of affine complete intersection Z.

Because an ind-coherent sheaf F is completely determined by $\mathrm{Hom}(G, F)$ for all $G \in \mathrm{Coh}(Z)$, and each G carries the cohomological operations, the definition of the singular support immediately extends to ind-coherent sheaves.

Definition 14. For $F \in \mathrm{IndCoh}(Z)$,

$$\mathrm{SingSupp}(F) = \overline{\left(\bigcup_{G \in \mathrm{Coh}(Z)} \mathrm{supp}_{A[\xi_1,\ldots,\xi_n]} \mathrm{Hom}(G, F) \right)} \subset \mathrm{Sing}(Z).$$

(The closure is there mostly for psychological reasons: we decided that we want the support to be closed.)

Exercise 3. (Essentially did this already) This agrees with the old definition on the image of $\mathrm{Coh}(Z) \hookrightarrow \mathrm{IndCoh}(Z)$.

Exercise 4. The essential image $\Xi(\mathrm{QCoh}(C)) \hookrightarrow \mathrm{IndCoh}(Z)$ is

$$\{F \in \mathrm{IndCoh}(Z) : \mathrm{SingSupp}(F) \subset \text{ zero section}\}.$$

(One inclusion is obvious, the other requires work.)

4. Categories given by singular support

Let us now fix a closed conical set $Y \subset \mathrm{Sing}(Z)$, and put

$$\mathrm{IndCoh}_Y(Z) = \{F \in \mathrm{IndCoh}(Z) : \mathrm{SingSupp}(F) \subset Y\}.$$

This gives us many categories of sheaves to do algebraic geometry in. If Y contains the zero section, $\mathrm{IndCoh}_Y(Z)$ is intermediate between $\mathrm{IndCoh}(Z) = \mathrm{IndCoh}_{\mathrm{Sing}(Z)}(Z)$ and $\mathrm{IndCoh}_0(Z)$, which by Exercise **??** identifies with $\mathrm{QCoh}(Z)$.

Recall that $\mathrm{IndCoh}(Z)$ is compactly generated by $\mathrm{Coh}(Z)$ (by definition). Put

$$\mathrm{Coh}_Y(Z) := \mathrm{Coh}(Z) \cap \mathrm{IndCoh}_Y(Z) = \{F \in \mathrm{Coh}(Z) : \mathrm{SingSupp}(F) \subset Y\}.$$

Clearly, $\mathrm{Coh}_Y(Z)$ is compact in $\mathrm{IndCoh}_Y(Z)$. The following is a special case of a theorem by Benson, Iyengar, and Krause:

Theorem 15. $\mathrm{IndCoh}_Y(Z)$ *is (compactly) generated by* $\mathrm{Coh}_Y(Z)$.

Example 16. $\mathrm{QCoh}(Z) = \mathrm{IndCoh}_0(Z)$ is compactly generated by $\mathrm{Perf}(Z) = \mathrm{Coh}_0(Z)$.

Corollary 17. *The tautological embedding* $\mathrm{IndCoh}_Y(Z) \hookrightarrow \mathrm{IndCoh}(Z)$ *admits a right adjoint* $\mathrm{IndCoh}(Z) \to \mathrm{IndCoh}_Y(Z)$ *('colocalization').*

Remark 18. The following is a version of the results of G. Stevenson (who works with the category of singularities). Let $\mathcal{C} \subset \mathrm{Coh}(Z)$ be any 'reasonable' subcategory: thick full subcategory that is finitely generated. Then $\mathcal{C} = \mathrm{Coh}_Y(Z)$ for unique conical Z. (To accommodate subcategories that are not finitely generated, we'd have to allow Z to be an infinite union of conical closed subsets, i.e., a specialization-closed subset.)

Accordingly, any 'reasonable' (=finitely compactly generated cocomplete) full subcategory of $\mathrm{IndCoh}(Z)$ is of the form $\mathrm{IndCoh}_Y(Z)$ for some Y. In this sense, the singular support is the finest 'reasonable' notion of support for coherent/ind-coherent sheaves.

Topics for tutorial:

Appendix A. The stack of local systems

Appendix B. Geometric Class Field Theory

Appendix C. Koszul transform

References

[1] D. Arinkin. Talk at the Algebraic Geometry in the Northeast (AGNES) Conference, Brown University, 2012. Video and handwritten notes available from http://www.agneshome.org/brown-2012.

[2] D. Arinkin. Two talks at the Spring School on Algebraic Microlocal Analysis, Northwestern University, 2012. Handwritten notes available from http://www.math.northwestern.edu/ tsygan/AMAnotes.html.

[3] D. Arinkin, D. Gaitsgory. Singular support of coherent sheaves, and the geometric Langlands conjecture. arXiv:1201.6343.

[4] D. Gaitsgory. Outline of the proof of the geometric Langlands conjecture for GL(2). arXiv:1302.2506.