

CHAPTER 2. POINTS OVER FINITE FIELDS AND THE WEIL CONJECTURES

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In this chapter we will relate the topology of smooth projective varieties over the complex numbers with counting points over finite fields, via the Weil conjectures. If X is a variety defined over a finite field \mathbf{F}_q , one can count its points over the various finite extensions of \mathbf{F}_q ; denote $N_m = |X(\mathbf{F}_q^m)|$ (for instance, if $X \subset \mathbf{A}_{\mathbf{F}_q}^n$ is affine, given by equations f_1, \dots, f_k , then $N_m = |\{x \in \mathbf{F}_q^m \mid f_i(x) = 0, \forall i\}|$). The local Weil zeta function of X ,

$$Z(X; t) := \exp \left(\sum_{m \geq 1} \frac{N_m}{m} \cdot t^m \right) \in \mathbf{Q}[[t]],$$

satisfies a number of fundamental properties, known as the Weil conjectures, which are known to be true mainly by work of Deligne. Some of these are its rationality, a functional equation, and an analogue of the Riemann Hypothesis. Most importantly for this course, for varieties specializing to smooth projective varieties over \mathbf{C} , it is related via its rational representation to the Betti numbers of the latter. My main sources of inspiration for this chapter are [Ha] Appendix C, [Mi], and especially [Mu].

1. VARIETIES OVER FINITE FIELDS

Basics on finite fields. I start by recalling a few facts on finite fields; one standard reference is [La]. Let k be a finite field of characteristic $p > 0$. Then $|k| = p^r$ for some integer $r \geq 1$; for each p and r there exists a unique (up to isomorphism) finite field with this cardinality, which can be described as the splitting field of the polynomial $X^{p^r} - X$ in an algebraic closure $\overline{\mathbf{F}}_p$ (its elements are the roots of this polynomial). This field will be denoted \mathbf{F}_q , with $q = p^r$.

Fix such a $k = \mathbf{F}_q$. If $k \subset K$ is a finite field extension and $[K : k] = m$, then $|K| = q^m$. On the other hand, for any $m \geq 1$, there exists a finite extension $k \subset K$ of degree m . In a fixed algebraic closure there exists a unique such extension, namely \mathbf{F}_{q^m} . If $[K : k] = m$ and $[K' : k] = n$, there exists a morphism of k -algebras $K' \rightarrow K$ if and only if $m|n$.

Fix an algebraic closure $k \subset \bar{k}$. The *Frobenius mapping*

$$\sigma : \mathbf{F}_q \longrightarrow \mathbf{F}_q, \quad x \mapsto x^q$$

can be extended to an element in $\text{Gal}(\bar{k}/k)$, sometimes called the *arithmetic Frobenius* (with the inverse in $\text{Gal}(\bar{k}/k)$ called the *geometric Frobenius*). We can see the unique finite extension of k in \bar{k} of degree m as the field fixed by the m -th power of σ , i.e.

$$\mathbf{F}_{q^m} = \{x \in \bar{k} \mid \sigma^m(x) = x\}.$$

The Galois group $\text{Gal}(\bar{k}/k)$ can be described as follows. First, one can see that the Galois group of a finite extension is cyclic, namely $\text{Gal}(\bar{\mathbf{F}}_{q^m}/\mathbf{F}_q) \simeq \mathbf{Z}/m\mathbf{Z}$. Then one has isomorphisms

$$\text{Gal}(\bar{k}/k) \simeq \varprojlim_m \text{Gal}(\bar{\mathbf{F}}_{q^m}/\mathbf{F}_q) \simeq \varprojlim_m \mathbf{Z}/m\mathbf{Z} = \hat{\mathbf{Z}},$$

the profinite completion of \mathbf{Z} .

Varieties over \mathbf{F}_q . Let X be a reduced scheme of finite type over a field k . We consider and relate various notions of points of X .

Definition 1.1 (Degree of a closed point). Let $x \in X$ be a closed point, with local ring $(\mathcal{O}_{X,x}, \mathfrak{m}_x)$. According to Nullstellensatz, the residue field $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is a finite extension of k . The *degree* of x is

$$\deg(x) := [k(x) : k].$$

Definition 1.2. A K -valued point of X , with $k \subset K$ a field extension, is an element of the set

$$X(K) := \text{Hom}_{\text{Spec } k}(\text{Spec } K, X) = \bigcup_{x \in X} \text{Hom}_k(k(x), K).$$

We relate these notions when $k = \mathbf{F}_q$ is a finite field.

Lemma 1.3. *If X is defined over the finite field $k = \mathbf{F}_q$, and K is an extension of k of degree m , then*

$$|X(K)| = \sum_{d|m} d \cdot |\{x \in X \mid x \text{ closed with } \deg(x) = d\}|.$$

Proof. Let $x \in X$ be the image in X of a K -valued point $\text{Spec } K \rightarrow X$. Then x is a closed point; indeed since the extension $k \subset K$ is algebraic, so is $k \subset k(x)$, hence

$$\dim \overline{\{x\}} = \text{trdeg}_k(k(x)) = 0.$$

Assuming that $[K : k] = m$, we then get

$$X(K) = \bigcup_{\deg(x)|m} \text{Hom}_k(k(x), K).$$

But one can see that if $\deg(x) = d$ with $d|m$, then

$$|\text{Hom}_k(k(x), K)| = d,$$

which finishes the proof. To this end, note that the Galois group $\text{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_q) \simeq \mathbf{Z}/m\mathbf{Z}$ acts transitively on $\text{Hom}_k(k(x), K)$, with the stabilizer of any element isomorphic to $\text{Gal}(\mathbf{F}_{q^m}/\mathbf{F}_{q^d})$, which implies the claim. \square

Note that if $k \subset K$ is a finite extension, there are only finitely many points in $X(K)$. (By taking a finite affine open cover of X , it is enough to see this in the affine case, where things are very explicit: if $X \subset \mathbf{A}_k^n$ is defined by the equations f_1, \dots, f_k , then $X(K)$ is the set of common solutions of these equations in K^n .) By Lemma 1.3, we deduce that for each e there are only finitely many closed points $x \in X$ with $\deg(x) = d$.

A key tool is the interpretation of points over finite extensions of \mathbf{F}_q as being those points fixed by various powers of the Frobenius.

Definition 1.4. The *Frobenius morphism* of X over \mathbf{F}_q is the morphism of ringed spaces

$$\text{Frob}_{X,q} : X \longrightarrow X$$

defined as the identity on the topological space X , and the Frobenius map $a \mapsto a^q$ on the sheaf of rings \mathcal{O}_X . This is a morphism of schemes over \mathbf{F}_q , since $a^q = a$ for any $a \in \mathbf{F}_q$.

Consider now an algebraic closure $\overline{\mathbf{F}}_q$, and let

$$\overline{X} := X \times_{\text{Spec } \mathbf{F}_q} \text{Spec } \overline{\mathbf{F}}_q.$$

Note that \overline{X} is a variety¹ over $\overline{\mathbf{F}}_q$ and $\overline{X}(\overline{\mathbf{F}}_q) = X(\overline{\mathbf{F}}_q)$. There is an induced morphism of schemes over $\overline{\mathbf{F}}_q$:

$$\text{Frob}_{\overline{X},q} = \text{Frob}_{X,q} \times_{\mathbf{F}_q} \text{id}_{\overline{\mathbf{F}}_q} : \overline{X} \longrightarrow \overline{X}.$$

For any $m \geq 1$ this can be composed with itself m times to obtain $\text{Frob}_{\overline{X},q}^m$.

Lemma 1.5. *For any $m \geq 1$, the points in $X(\mathbf{F}_{q^m})$ can be identified with the points of $\overline{X}(\overline{\mathbf{F}}_q)$ fixed by $\text{Frob}_{\overline{X},q}^m$.*

Proof. Since each such point lives in an open set of an affine open cover of X , it is enough to look at the case when $X \subset \mathbf{A}_{\mathbf{F}_q}^n$ is a closed subset. In this case, $\text{Frob}_{\overline{X},q}$ is the restriction of $\text{Frob}_{\mathbf{A}_{\mathbf{F}_q}^n,q}$, which on $\overline{\mathbf{F}}_q$ -points is given by

$$(x_1, \dots, x_n) \mapsto (x_1^q, \dots, x_n^q).$$

According to the description of \mathbf{F}_{q^m} in the previous section, it is clear then that the \mathbf{F}_{q^m} -points are precisely those fixed by the m -th power of this map. \square

¹This is true since \mathbf{F}_q is a perfect field; cf. [Ha] Ch.II, Exercise 3.15. Do this exercise!

Corollary 1.6. *Denoting by Δ and Γ_m the diagonal and the graph of $\text{Frob}_{\bar{X},q}^m$ in $\bar{X} \times \bar{X}$, there is a one-to-one correspondence between $X(\mathbf{F}_{q^m})$ and the closed points of $\Delta \cap \Gamma_m$.*

Proposition 1.7. *If X is smooth over \mathbf{F}_q , then the intersection $\Delta \cap \Gamma_m$ is transverse at every point, so that $\Delta \cap \Gamma_m$ consists of a reduced set of points.*

Proof. We first show this when $X = \mathbf{A}_{\mathbf{F}_q}^n$. Write the affine coordinate ring of $\bar{X} \times \bar{X}$ as $\bar{\mathbf{F}}_q[X_1, \dots, X_n, Y_1, \dots, Y_n]$. The diagonal is defined by the ideal $(X_1 - Y_1, \dots, X_n - Y_n)$, while by the discussion above Γ_m is defined by the ideal $(Y_1 - X_1^{q^m}, \dots, Y_n - X_n^{q^m})$. It follows that

$$\Delta \cap \Gamma_m \simeq \prod_{i=1}^n \text{Spec } \mathbf{F}_q[X_i]/(X_i - X_i^{q^m}),$$

which is reduced since the polynomial $X^{q^m} - X$ has no multiple roots.

Consider now the case of an arbitrary smooth X . Let $x \in X$ be a closed point in X corresponding to a point in $X(\mathbf{F}_{q^m})$ as above. Pick a regular system of parameters t_1, \dots, t_n for the regular local ring $\mathcal{O}_{X,x}$, which define an étale map $U \rightarrow \mathbf{A}_{\mathbf{F}_q}^n$ for some Zariski open neighborhood U of X . The restrictions of Δ and Γ_m to $\bar{U} \times \bar{U}$ are precisely the preimages of the analogous sets in $\mathbf{A}_{\bar{\mathbf{F}}_q}^n \times \mathbf{A}_{\bar{\mathbf{F}}_q}^n$ via the induced morphism $\bar{U} \times \bar{U} \rightarrow \mathbf{A}_{\bar{\mathbf{F}}_q}^n \times \mathbf{A}_{\bar{\mathbf{F}}_q}^n$. The statement follows from the case of \mathbf{A}^n , since the preimage of a reduced set via an étale morphism is reduced. \square

2. THE LOCAL WEIL ZETA FUNCTION

Let X be a variety defined over the finite field $k = \mathbf{F}_q$. For every integer $m \geq 1$, we define

$$N_m(= N_m(X)) := |X(\mathbf{F}_{q^m})|.$$

Definition 2.1. The *local Weil zeta function* of X is the formal power series

$$Z(X; t) := \exp \left(\sum_{m \geq 1} \frac{N_m}{m} \cdot t^m \right) \in \mathbf{Q}[[t]].$$

Example 2.2 (Affine space). Let $X = \mathbf{A}_{\mathbf{F}_q}^n$. For each $m \geq 1$, we clearly have $X(\mathbf{F}_{q^m}) = (\mathbf{F}_{q^m})^n$, which is of cardinality q^{nm} . Therefore²

$$Z(\mathbf{A}^n; t) = \exp \left(\sum_{m \geq 1} \frac{q^{nm}}{m} \cdot t^m \right) = \exp(-\log(1 - q^n t)) = \frac{1}{1 - q^n t}.$$

Exercise 2.3. Show that if X is a variety over \mathbf{F}_q , then

$$Z(X \times \mathbf{A}_{\mathbf{F}_q}^n; t) = Z(X; q^n t).$$

²Recall that $\log(1 + t) = \sum_{m \geq 1} \frac{(-1)^{m+1} t^m}{m}$.

Example 2.4 (“Motivic” behavior of the zeta function). This refers to the behavior of the zeta function with respect to the decomposition of varieties into disjoint unions. Consider a variety X over \mathbf{F}_q , with $Y \subset X$ a closed subvariety and $U = X - Y$. Then

$$Z(X; t) = Z(Y; t) \cdot Z(U; t).$$

Indeed, note that by definition $X(\mathbf{F}_{q^m}) = Y(\mathbf{F}_{q^m}) \cup U(\mathbf{F}_{q^m})$, so one can use the multiplicative behavior of the exponential with respect to sums.

Example 2.5 (Projective space). Let $X = \mathbf{P}_{\mathbf{F}_q}^n$. We can write $X = \mathbf{A}_{\mathbf{F}_q}^n \cup \mathbf{P}_{\mathbf{F}_q}^{n-1}$, and continue this decomposition inductively with respect to n . Using Example 2.2 and Example 2.4, we obtain

$$Z(\mathbf{P}^n; t) = \frac{1}{(1-t)(1-qt) \cdots (1-q^nt)}.$$

Equivalently, for all $m \geq 1$,

$$N_m(\mathbf{P}_{\mathbf{F}_q}^n) = 1 + q^m + q^{2m} + \cdots + q^{nm}.$$

Exercise 2.6 (Grassmannians). For any $1 \leq k \leq n-1$, let $G(k, n)$ be the Grassmannian defined over $\text{Spec } \mathbf{Z}$; for each field K , its K -valued points are the k -dimensional linear subspaces in K^n .

(1) Show that $GL_n(\mathbf{F}_q)$ acts transitively on $G(k, n)(\mathbf{F}_q)$, and the stabilizer of each point is isomorphic to $GL_k(\mathbf{F}_q) \times GL_{n-k}(\mathbf{F}_q) \times M_{k, n-k}(\mathbf{F}_q)$.

(2) Show that for each $k \geq 1$ one has

$$|GL_k(\mathbf{F}_q)| = q^{\frac{k(k-1)}{2}} (q^k - 1)(q^{k-1} - 1) \cdots (q - 1).$$

(3) Use the previous parts to show that

$$|G(k, n)(\mathbf{F}_q)| = \frac{(q^n - 1) \cdots (q^{n-k+1} - 1)}{(q^k - 1) \cdots (q - 1)} =: \binom{n}{k}_q,$$

the Gaussian binomial coefficient.

(4) Show that

$$\binom{n}{k}_q = q^k \binom{n-1}{k}_q + \binom{n-1}{k-1}_q$$

and use this to deduce that

$$\binom{n}{k}_q = \sum_{i=0}^{k(n-k)} \lambda_{n,k}(i) q^i,$$

where $\lambda_{n,k}(i)$ can be interpreted as the number of partitions of i into at most $n-k$ parts, each of size at most k .

(5) With the notation in (4), deduce that

$$Z(G(k, n); t) = \prod_{i=0}^{k(n-k)} \frac{1}{(1 - q^i t)^{\lambda_{n,k}(i)}}.$$

Analogy with the Riemann zeta function. This serves to motivate some of the Weil conjectures in the next sections. Recall that, for $s \in \mathbf{C}$, the *Riemann zeta function* is defined as

$$\zeta(s) := \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}.$$

Note that the primes numbers correspond precisely to the closed points of $\text{Spec } \mathbf{Z}$. We can then define analogously, for any scheme X of finite type over \mathbf{Z} , the following zeta function:

$$\zeta_X(s) := \prod \frac{1}{1 - N(x)^{-s}},$$

where the product is taken over all closed points $x \in X$, and $N(x)$ denotes the number of elements of the residue field $k(x)$. Now if $k = \mathbf{F}_q$, using the notation above we can rewrite this as

$$\zeta_X(s) := \prod \frac{1}{1 - (q^{\deg(x)})^{-s}}.$$

On the other hand, we have the following product formula:

Proposition 2.7. *If X is a variety over \mathbf{F}_q , then*

$$Z(X; t) = \prod \frac{1}{1 - t^{\deg(x)}},$$

where the product is taken over all closed points $x \in X$. In particular $Z(X; t) \in 1 + t\mathbf{Z}[[t]]$.

Proof. For $d \geq 1$, write $a_d = |\{x \in X \mid x \text{ closed with } \deg(x) = d\}|$. Recall that Lemma 1.3 says that $N_m = \sum_{d|m} d \cdot a_d$. Note also that the right hand side of the formula in the statement is equal to $\prod_{d \geq 1} (1 - t^d)^{-a_d}$. We then have

$$\begin{aligned} \log Z(X; t) &= \sum_{m \geq 1} \frac{N_m}{m} t^m = \sum_{m \geq 1} \sum_{d|m} \frac{d \cdot a_d}{m} t^m = \sum_{d \geq 1} a_d \cdot \sum_{e \geq 1} \frac{t^{de}}{e} = \\ &= \sum_{d \geq 1} a_d \cdot \log((1 - t^d)^{-1}) = \sum_{d \geq 1} \log((1 - t^d)^{-a_d}) = \log \left(\prod_{d \geq 1} (1 - t^d)^{-a_d} \right). \end{aligned}$$

□

Corollary 2.8. *We have the identification $Z(X; q^{-s}) = \zeta_X(s)$.*

3. STATEMENT OF THE WEIL CONJECTURES

Let X be a smooth projective variety of dimension n over \mathbf{F}_q . The following four theorems are known as the *Weil conjectures*. The first was proved by Dwork [Dwo] (without the assumption that X is smooth and projective) and by Grothendieck [Gro], while the other three were proved by Deligne [De1] (a new proof of the analogue of the Riemann hypothesis was also given by Laumon [Lau]).

Theorem 3.1 (Rationality). *$Z(X; t)$ is a rational function, i.e. $Z(X; t) = \frac{P(t)}{Q(t)}$ with $P(t), Q(t) \in \mathbf{Q}[t]$.*

Theorem 3.2 (Functional equation). Denote by $E = \Delta^2$ the self-intersection number of the diagonal in $X \times X$. Then

$$Z(X; \frac{1}{q^n t}) = \pm q^{nE/2} t^E Z(X; t).$$

Theorem 3.3 (Analogue of the Riemann hypothesis). One can write

$$Z(X; t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}$$

with $P_0(t) = 1 - t$, $P_{2n}(t) = 1 - q^n t$, and for $1 \leq i \leq 2n - 1$, $P_i(t) \in \mathbf{Z}[t]$ with

$$P_i(t) = \prod_j (1 - \alpha_{i,j} t)$$

where $\alpha_{i,j}$ are algebraic integers such that $|\alpha_{i,j}| = q^{i/2}$.

Theorem 3.4 (Betti numbers). Using the notation in Theorem 3.3, define the i -th Betti number of X as $b_i(X) = \deg P_i(t)$. Then

$$E = \sum_{i=0}^{2n} (-1)^i b_i(X).$$

If in addition there exists a finitely generated \mathbf{Z} -algebra R , \mathcal{X} a smooth projective variety over $\text{Spec } R$, and $\mathfrak{p} \subset R$ a maximal ideal such that $R/\mathfrak{p} \simeq \mathbf{F}_q$ and $X = \mathcal{X} \times_{\text{Spec } R} \text{Spec } R/\mathfrak{p}$, then

$$b_i(X) = b_i((\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C})^{\text{an}}).$$

In plain English, this last property says that if X is the reduction mod p of a smooth projective complex variety as in §3, then its Betti numbers coincide with the usual Betti numbers for the singular cohomology of that variety. As formulated here, Theorem 3.4 depends on Theorem 3.3; we will see later that there is a way of formulating it independently as well.

Exercise 3.5. Let X be a smooth projective variety X of dimension n over a field, and let $\Delta \in X \times X$ be the diagonal. Show that

$$\Delta^2 = c_n(T_X) \in A^n(X),$$

i.e. the number E above equals the top Chern class of the tangent bundle of X .

Exercise 3.6. Verify the Weil conjectures for \mathbf{P}^n .

Exercise 3.7. Using Exercise 2.6, verify the Weil conjectures for $G(k, n)$. Use them to deduce that the Betti numbers of the complex Grassmannian are

$$b_{2i+1}(G(k, n)) = 0 \text{ for all } i, \text{ and } b_{2i} = \lambda_{n,k}(i) \text{ for } 1 \leq i \leq k(n - k).$$

Remark 3.8 (Number of points from the rational representation). Given Theorem 3.1, write $Z(X; t) = \frac{P(t)}{Q(t)}$ with $P, Q \in \mathbf{Q}[t]$ normalized (after possibly dividing by powers of t) such that $P(0) = Q(0) = 1$. Write

$$P(t) = \prod_{i=1}^r (1 - \alpha_i t) \text{ and } Q(t) = \prod_{j=1}^s (1 - \beta_j t).$$

Then a simple calculation shows that, for every $m \geq 1$,

$$N_m = \sum_{j=1}^s \beta_j^m - \sum_{i=1}^r \alpha_i^m.$$

Exercise 3.9 (*K3 surfaces*). A smooth projective complex surface S is called a *K3 surface* if $\omega_S \simeq \mathcal{O}_S$ and $H^1(S, \mathcal{O}_S) = 0$. Show the following:

(1) The Hodge diamond of a *K3 surface* is

$$\begin{array}{cccc} & & 1 & \\ & 0 & & 0 \\ 1 & & 20 & \\ & 0 & & 0 \\ & & 1 & \end{array}$$

(2) A smooth quartic hypersurface in \mathbf{P}^3 is a *K3 surface*.

(3) If X is a surface over \mathbf{F}_q which is the reduction mod p of a complex *K3 surface* (as in Theorem 3.4), then

$$||X(\mathbf{F}_q)| - |\mathbf{P}^2(\mathbf{F}_q)|| \leq 23q.$$

[Hint: use (1), the Weil conjectures, and Remark 3.8.] Deduce that every such surface has points over a field with more than 22 elements.

(4) For completeness: there are examples of *K3 surfaces* with no points whatsoever over a particular finite field. For instance, take the Fermat quartic $X_0^4 + X_1^4 + X_2^4 + X_3^4 = 0$. Show that this has no points over \mathbf{F}_5 . Show on the other hand that it has points over every other finite field (use (3) if necessary).

4. SOME PROOFS VIA WEIL COHOMOLOGY THEORIES

Here I will sketch the approach to the proofs of some of the Weil conjectures. Since I will not treat the analogue of the Riemann hypothesis in general, I will start by proving it in the case of curves, where only elementary tools are needed, and then look at the other conjectures in general via the notion of Weil cohomology theory.

The case of curves. Let X be a smooth projective curve over \mathbf{F}_q . Fix an algebraic closure $\overline{\mathbf{F}}_q \subset \overline{\mathbf{F}}_q$. Recall that we denote $\overline{X} = X \times_{\text{Spec } \mathbf{F}_q} \text{Spec } \overline{\mathbf{F}}_q$. We assume that \overline{X} is a smooth projective irreducible curve over $\overline{\mathbf{F}}_q$, of genus

$$g = h^1(X, \mathcal{O}_X) = h^1(\overline{X}, \mathcal{O}_{\overline{X}}).^3$$

Note first that the rationality and the Betti numbers theorems imply that we have

$$Z(X; t) = \frac{P(t)}{(1-t)(1-qt)},$$

³Note that via the natural projection $p: \overline{X} \rightarrow X$, for any quasi-coherent sheaf \mathcal{F} on X we have

$$H^i(X, \mathcal{F}) \otimes_{\mathbf{F}_q} \overline{\mathbf{F}}_q \simeq H^i(\overline{X}, p^* \mathcal{F}).$$

with $P(t) \in \mathbf{Z}[t]$ of degree $2g$. Write $P(t) = \prod_{i=1}^{2g} (1 - \alpha_i t)$, with $\alpha_i \in \mathbf{C}$. The analogue of the Riemann hypothesis amounts to the following

Theorem 4.1. *Using the notation above, the α_i are algebraic integers satisfying $|\alpha_i| = q^{1/2}$ for all i .*

It is clear to begin with that the α_i are algebraic integers: since $P(t)$ has integer coefficients, all symmetric functions $s_j(\alpha_1, \dots, \alpha_{2g})$ are integers, and the α_i are roots of the polynomial $\sum_{j=0}^{2g} (-1)^j s_j(\alpha_1, \dots, \alpha_{2g}) t^{2g-j}$. For the absolute value statement, one can first go through a few simple reduction steps.

First, if $\Delta \subset \overline{X} \times \overline{X}$ is the diagonal, then the self-intersection Δ^2 can be computed using the genus formula for curves on surfaces, i.e.

$$2g - 2 = \Delta^2 + K_{\overline{X} \times \overline{X}} \cdot \Delta.$$

If p_1 and p_2 are the projections to the two factors of $\overline{X} \times \overline{X}$, we have

$K_{\overline{X} \times \overline{X}} = p_1^* K_{\overline{X}} + p_2^* K_{\overline{X}} \equiv (2g - 2) \cdot p_1^* \{\text{pt}\} + (2g - 2) \cdot p_2^* \{\text{pt}\} \equiv (2g - 2)F_1 + (2g - 2)F_2$, where F_1 and F_2 are fibers of the two projections. Since clearly $\Delta \cdot F_1 = \Delta \cdot F_2 = 1$, we get $K_{\overline{X} \times \overline{X}} \cdot \Delta = 4g - 4$, so finally

$$\Delta^2 = 2 - 2g.$$

The functional equation then says in this case

$$Z(X; \frac{1}{qt}) = q^{1-g} t^{2-2g} Z(X; t).$$

A simple calculation shows that this implies

$$\prod_{i=1}^{2g} (1 - \alpha_i t) = q^g \prod_{i=1}^{2g} (t - \frac{\alpha_i}{q}) = \frac{\prod_{i=1}^{2g} \alpha_i}{q^g} \cdot \prod_{i=1}^{2g} (1 - \frac{q}{\alpha_i} t).$$

This has the following consequences:

- $\prod_{i=1}^{2g} \alpha_i = q^g$ (in particular $\alpha_i \neq 0$ for all i).
- the set $\{\alpha_1, \dots, \alpha_{2g}\}$ is invariant under the mapping $\alpha \mapsto q/\alpha$.

The last property shows that it is enough to prove the inequality

$$(1) \quad |\alpha_i| \leq q^{1/2} \quad \text{for all } i.$$

Another standard piece of notation when referring to zeta functions of curves is the following: recalling that $N_m = |X(\mathbf{F}_{q^m})|$, denote

$$a_m = 1 - N_m + q^m.$$

(Note that $|a_m|$ is the difference between the number of points of \mathbf{P}^1 and of X over \mathbf{F}_q .)

Exercise 4.2. Show that if X is an elliptic curve, then using the notation above

$$Z(X; t) = \frac{1 - a_1 t + q t^2}{(1 - t)(1 - q t)}.$$

Lemma 4.3. *The set of inequalities (1) above is equivalent to the condition*

$$(2) \quad |a_m| \leq 2gq^{m/2} \quad \text{for all } m \geq 1.$$

Proof. The first claim is that for all $m \geq 1$ we have

$$a_m = \sum_{i=1}^{2g} \alpha_i^m.$$

Indeed, note that from the formula for the zeta function we have

$$\begin{aligned} \sum_{m \geq 1} N_m \frac{t^m}{m} &= \sum_{i=1}^{2g} \log(1 - \alpha_i t) - \log(1 - t) - \log(1 - qt) = \\ &= \sum_{m \geq 1} \left(- \sum_{i=1}^{2g} \alpha_i^m + 1 + q^m \right) \frac{t^m}{m}, \end{aligned}$$

which implies what we want. Now if we assume that $|\alpha_i| \leq q^{1/2}$ for all i , then the triangle inequality immediately implies (2).

For the reverse implication, observe that

$$\sum_{m \geq 1} a_m t^m = \sum_{i=1}^{2g} \sum_{m \geq 1} \alpha_i^m t^m = \sum_{i=1}^{2g} \frac{\alpha_i t}{1 - \alpha_i t}.$$

This rational function has a pole at each $t = 1/\alpha_i$. Assuming that we have $|a_m| \leq 2gq^{m/2}$ for all $m \geq 1$, we get

$$\left| \sum_{m \geq 1} a_m t^m \right| \leq 2g \sum_{m \geq 1} (q^{1/2}|t|)^m = \frac{2gq^{1/2}|t|}{1 - q^{1/2}|t|}.$$

This cannot have poles with $|t| < q^{-1/2}$, and therefore we must have $|\alpha_i| \leq q^{1/2}$ as claimed. \square

We finally come to the main content of the proof of Theorem 4.1:⁴ Weil's interpretation of counting rational points by looking at the intersection of the diagonal with the graph of the Frobenius can be combined with the Hodge index theorem for surfaces to give a quick proof of the estimate in (2). Since the same argument works for all \mathbf{F}_{q^m} , it is enough to do this for $m = 1$.

Denote by $\Gamma \subset \overline{X} \times \overline{X}$ the graph of $\text{Frob}_{\overline{X}, q}$. We have seen in Corollary 1.6 and Proposition 1.7 that the intersection $\Delta \cap \Gamma$ is reduced, and that

$$N := |X(\mathbf{F}_q)| = |\Delta \cap \Gamma| = \Delta \cdot \Gamma.$$

We use the following standard application of the Hodge index theorem:

⁴This is essentially [Ha] Ch. V, Exercise 1.10.

Exercise 4.4 ([Ha], Ch. V, Exercise 1.9). Let C_1 and C_2 be two smooth projective curves over $k = \bar{k}$. Denote by F_1 and F_2 any fibers of the projections of $C_1 \times C_2$ onto the respective factors. Then for every divisor D on $C_1 \times C_2$ we have

$$D^2 \leq 2 \cdot (D \cdot F_1) \cdot (D \cdot F_2).$$

We apply this with $C_1 = C_2 = \bar{X}$ and $D = a\Delta + b\Gamma$, for arbitrary $a, b \in \mathbf{Z}$. Recall that $\Delta \cdot F_1 = \Delta \cdot F_2 = 1$ and $\Delta^2 = 2 - 2g$. Note also that clearly $\Gamma \cdot F_1 = 1$, while $\Gamma \cdot F_2 = q$ (the cardinality of the set $\{(x, y) \mid y = x^q\}$). We need to compute Γ^2 as well; for this we again use the genus formula

$$2g - 2 = \Gamma^2 + K_{\bar{X} \times \bar{X}} \cdot \Gamma.$$

We have seen before that $K_{\bar{X} \times \bar{X}} \equiv (2g - 2)(F_1 + F_2)$, and so using the information above we get $\Gamma^2 = q(2 - 2g)$. For $D = a\Delta + b\Gamma$, the inequality in the Exercise can be written (after a small calculation) as

$$ga^2 - (q + 1 - N)ab + gqb^2 \geq 0.$$

Since this holds for all a and b , we must have

$$(q + 1 - N)^2 - 4g^2q \leq 0$$

which after taking square roots gives precisely the inequality we're after.

Weil cohomology theories. Here I am closely following [Mu], which in turn is closely following [deJ] and [Mi]. The main point is the following: Weil realized that the rationality and the functional equation for the zeta function would follow formally from the existence of a cohomology theory in characteristic $p > 0$ with axioms closely resembling those of singular cohomology over \mathbf{C} . More precisely:

Definition 4.5. A *Weil cohomology theory* for varieties over a field k , with coefficients in a field K with $\text{char } K = 0$, is given by the **data**

(D1) A contravariant functor $X \mapsto H^*(X) = \bigoplus_i H^i(X)$, mapping smooth projective varieties over k to graded commutative⁵ K -algebras. We use the cup-product notation $\alpha \cup \beta$ for the product in $H^*(X)$.

(D2) For every such X , a linear trace map $\text{Tr}_X : H^{2\dim X}(X) \rightarrow K$.

(D3) For every such X , and for every closed subvariety $Z \subset X$ of codimension c , a cohomology class $\text{cl}(Z) \in H^{2c}(X)$.

satisfying the following **axioms**

(A1) (Finite dimensionality and vanishing) For every X , all $H^i(X)$ are finite dimensional vector spaces over K , and in addition

$$H^i(X) = 0 \text{ for } i < 0 \text{ and } i > 2\dim X.$$

⁵meaning $ab = (-1)^{\deg(a)\deg(b)}ba$ for every a and b .

(A2) (Künneth formula) For every X and Y , if p_X, p_Y are the projections of $X \times Y$ onto the two factors, then the K -algebra homomorphism

$$H^*(X) \otimes_K H^*(Y) \rightarrow H^*(X \times Y), \alpha \otimes \beta \mapsto p_X^* \alpha \cup p_Y^* \beta$$

is an isomorphism.

(A3) (Poincaré duality) For every X , the trace map Tr_X is an isomorphism, and for every $0 \leq i \leq 2\dim X$, the K -bilinear map

$$H^i(X) \otimes_K H^{2\dim X - i}(X) \rightarrow K, \alpha \otimes \beta \mapsto \text{Tr}_X(\alpha \cup \beta)$$

is a perfect pairing.

(A4) (Trace maps and products) For every X and Y and every $\alpha \in H^{2\dim X}(X)$ and $\beta \in H^{2\dim Y}(Y)$, one has

$$\text{Tr}_{X \times Y}(p_X^* \alpha \cup p_Y^* \beta) = \text{Tr}_X(\alpha) \cdot \text{Tr}_Y(\beta).$$

(A5) (Exterior product of cohomology classes) For every X and Y , and every closed subvarieties $Z \subset X$ and $W \subset Y$, one has

$$\text{cl}(Z \times W) = p_X^* \text{cl}(Z) \cup p_Y^* \text{cl}(W).$$

(A6) (Push-forward of cohomology classes) For every morphism $f : X \rightarrow Y$ and every closed subvariety $Z \subset X$, for every class $\alpha \in H^{2\dim Z}(Y)$ one has

$$\text{Tr}_X(\text{cl}(Z) \cup f^* \alpha) = \deg(Z/f(Z)) \cdot \text{Tr}_Y(\text{cl}(f(Z)) \cup \alpha).$$

(A7) (Pull-back of cohomology classes) For every morphism $f : X \rightarrow Y$ and every closed subvariety $Z \subset Y$ satisfying the conditions

- all irreducible components W_1, \dots, W_r of $f^{-1}(Z)$ have dimension $\dim Z + \dim X - \dim Y$;
- either f is flat in a neighborhood of Z , or Z is generically transverse to f , i.e. $f^{-1}(Z)$ is generically smooth,

assuming that $[f^{-1}(Z)] = \sum_{i=1}^r m_i W_i$ as a cycle ($m_i = 1$ for all i in the generically transverse case), then

$$f^* \text{cl}(Z) = \sum_{i=1}^r m_i \text{cl}(W_i).$$

(A8) (Case of a point) If $x = \text{Spec } k$, then $\text{cl}(x) = 1$ and $\text{Tr}_x(1) = 1$.

As mentioned above, when $k = \mathbf{C}$ and $K = \mathbf{Q}$, singular cohomology provides a Weil cohomology theory. When $\text{char } k = p > 0$, the Weil cohomology theory we will discuss below is ℓ -adic cohomology, with $\ell \neq p$ and $K = \mathbf{Q}_\ell$.

Fix a Weil cohomology theory over the field k . We will need a few extra properties that follow from the axioms.

Lemma 4.6. *Let X be a smooth projective variety over k , of dimension n . Then:*

- (1) *The morphism $K \rightarrow H^0(X)$ given by the K -algebra structure is an isomorphism.*
- (2) *In $H^0(X)$, one has $\text{cl}(X) = 1$.*
- (3) *If $x \in X$ is a closed point, then $\text{Tr}_X(\text{cl}(x)) = 1$.*
- (4) *Suppose $f : X \rightarrow Y$ is a generically finite surjective morphism of degree d to a smooth projective variety Y . Then*

$$\text{Tr}_X(f^*\alpha) = d \cdot \text{Tr}_Y(\alpha), \quad \forall \alpha \in H^{2n}(Y).$$

Consequently, if $Y = X$, f^ acts as multiplication by d on $H^{2n}(X)$.*

Proof. Using Poincaré duality (A3) with $i = 0$ we obtain that $\dim_K H^0(X) = 1$, which immediately gives (1). Part (2) follows by applying (A7) to the natural morphism $X \rightarrow \text{Spec } k$, combined with (A8). For $x \in X$ a closed point, we can apply (A6) to $X \rightarrow \text{Spec } k$, taking $Z = \{x\}$ and $\alpha = 1$. We get that $\text{Tr}_X(\text{cl}(x)) = \text{Tr}_{\text{Spec } k}(1)$, to which we apply (A8) to get (3).

Consider now $f : X \rightarrow Y$ as in (4), and take a general point $Q \in Y$. We have that as a cycle $[f^{-1}(Q)] = \sum_{i=1}^r m_i P_i$, where P_i are the reduced points of the fiber over Q , and $\sum_{i=1}^r m_i = d$. By generic flatness, since Q is general we have that f is flat over a neighborhood of Q . Therefore by (A7) and (A6) we have

$$\text{Tr}_X(f^*\text{cl}(Q)) = \text{Tr}_X\left(\sum_{i=1}^r m_i \text{cl}(P_i)\right) = d \cdot \text{Tr}_Y(\text{cl}(Q)).$$

This proves (4), since by (3) $\text{cl}(Q)$ generates $H^{2n}(Y)$. □

Definition 4.7 (Push-forward). The data of a contravariant functor guarantees only the existence of a pull-back map in cohomology, given a morphism $f : X \rightarrow Y$. However, the Poincaré duality axiom allows one to define a push-forward in cohomology as well. If $\alpha \in H^i(X)$, then the *push-forward* of α via f is the unique class $f_*\alpha \in H^{2\dim Y - 2\dim X + i}(Y)$ such that

$$\text{Tr}_Y(f_*\alpha \cup \beta) = \text{Tr}_X(\alpha \cup f^*\beta)$$

for every $\beta \in H^{2\dim X - i}(Y)$. This is clearly K -linear, and further properties are collected in the following:

Lemma 4.8. *With the notation above, one has:*

- (1) *(Projection formula) $f_*(\alpha \cup f^*\gamma) = f_*\alpha \cup \gamma$, for any $\alpha \in H^*(X)$ and any $\gamma \in H^*(Y)$.*
- (2) *If $g : Y \rightarrow Z$ is another morphism, then $(g \circ f)_* = g_* \circ f_*$.*
- (3) *If $Z \subset X$ is a closed subvariety, then*

$$f_*\text{cl}(Z) = \deg(Z/f(Z)) \cdot \text{cl}(f(Z)).$$

Proof. Exercise. □

Lemma 4.9. *If X and Y are smooth projective varieties and $\alpha \in H^i(Y)$, then $p_{X*}(p_Y^*\alpha) = \text{Tr}_Y(\alpha)$ if $i = 2\dim Y$, and $p_{X*}(p_Y^*\alpha) = 0$ otherwise.*

Proof. By definition $p_{X*}(p_Y^*\alpha) \in H^{i-2\dim Y}(X)$, so it follows automatically by (A1) that $p_{X*}(p_Y^*\alpha) = 0$ when $i \neq 2\dim Y$. Now if $\alpha \in H^{2\dim Y}(Y)$ and $\beta \in H^{2\dim X}(X)$, then

$$\mathrm{Tr}_X(p_{X*}(p_Y^*\alpha) \cup \beta) = \mathrm{Tr}_{X \times Y}(p_Y^*\alpha \cup p_X^*\beta) = \mathrm{Tr}_Y(\alpha) \cdot \mathrm{Tr}_X(\beta).$$

(The first equality follows by the definition of the trace, and the second by axiom (A4).) This implies that $p_{X*}(p_Y^*\alpha) = \mathrm{Tr}_Y(\alpha)$. \square

Finally, completely analogously to the well-known case of singular cohomology (see [Fu] Ch.19), one can define for each i a *cycle class map*

$$\mathrm{cc} : A^i(X) \longrightarrow H^{2i}(X)$$

from the Chow group of codimension i cycles, such that when putting these together we obtain a ring homomorphism

$$\mathrm{cc} : A^*(X) \longrightarrow H^{2*}(X)$$

compatible with f^* and f_* . This gives in particular the following:

Lemma 4.10. *Let X be a smooth projective variety, and let $\alpha_i \in A^{m_i}(X)$, with $i = 1, \dots, r$, such that $\sum_{i=1}^r m_i = \dim X$. Then*

$$\alpha_1 \cdot \dots \cdot \alpha_r = \mathrm{Tr}_X(\mathrm{cc}(\alpha_1) \cup \dots \cup \mathrm{cc}(\alpha_r)).$$

Proof. Since the cycle class map is a ring homomorphism, it is enough to prove that for $\beta \in Z_0(X)$ one has $\deg(\beta) = \mathrm{Tr}_X(\beta)$. Furthermore, one can assume that β is just a point by additivity, in which case the assertion follows from Lemma 4.6 (3). \square

Trace formula. The key formal result related to Weil cohomology theories that we will use below is the following (for the singular cohomology of complex projective varieties with coefficients in \mathbf{Q} , this is a special case of the famous *Lefschetz fixed point theorem*):

Theorem 4.11 (Trace formula). *Let $\varphi : X \rightarrow X$ be an endomorphism of a smooth projective variety X of dimension n . If $\Delta, \Gamma_\varphi \subset X \times X$ denote the diagonal and the graph of φ respectively, then*

$$\Delta \cdot \Gamma_\varphi = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(\varphi^* | H^i(X)).$$

If Δ and Γ_φ intersect transversely, this number is precisely the cardinality of the fixed point set $\{x \in X \mid \varphi(x) = x\}$.

Applying the Theorem to $\varphi = \mathrm{id}_X$, we obtain a familiar formula:

Corollary 4.12. *If X is a smooth projective variety and $\Delta \subset X \times X$ is the diagonal, then*

$$\Delta^2 = \sum_{i=0}^{2n} (-1)^i h^i(X) =: \chi(X).$$

To prove Theorem 4.11, we need some preparatory lemmas. We use the same notation as in the Theorem, and in addition we denote by p_1, p_2 the projections of $X \times X$ onto the two factors.

Lemma 4.13. *For any $\alpha \in H^*(X)$, we have*

$$p_{1*}(\text{cl}(\Gamma_\varphi) \cup p_2^* \alpha) = \varphi^* \alpha.$$

Proof. Denote by $j : X \hookrightarrow X \times X$ the embedding of the graph of φ . We clearly have $p_1 \circ j = \text{Id}_X$ and $p_2 \circ j = \varphi$. Now $j_* \text{cl}(X) = \text{cl}(\Gamma_\varphi)$ by Lemma 4.8 (3). Therefore

$$p_{1*}(\text{cl}(\Gamma_\varphi) \cup p_2^* \alpha) = p_{1*}(j_* \text{cl}(X) \cup p_2^* \alpha) = p_{1*} j_* (\text{cl}(X) \cup j^* p_2^* \alpha) = p_{1*} j_* (\varphi^* \alpha) = \varphi^* \alpha,$$

where the second identity follows from the projection formula. \square

Lemma 4.14. *Let $\{e_i^r\}_{i=1, \dots, k_r}$ be a basis for $H^r(X)$, for each r . Let $\{f_i^{2n-r}\}_{i=0, \dots, k_r}$ be the dual basis for $H^{2n-r}(X)$ via the Poincaré duality pairing, so that $\text{Tr}_X(e_i^r \cup f_j^{2n-r}) = \delta_{i,j}$. Then*

$$\text{cl}(\Gamma_\varphi) = \sum_{r,i} p_1^* \varphi^* e_i^r \cup p_2^* f_i^{2n-r} \in H^{2n}(X \times X).$$

Proof. The Künneth property (A2) implies that we can write

$$\text{cl}(\Gamma_\varphi) = \sum_{s,j} p_1^* a_j^s \cup p_2^* f_j^{2n-s}$$

for some unique classes $a_j^s \in H^s(X)$ for each s and j . Using Lemma 4.13 and the projection formula we obtain

$$\varphi^* e_i^r = \sum_{s,j} p_{1*} (p_1^* a_j^s \cup p_2^* f_j^{2n-s} \cup p_2^* e_i^r) = \sum_{s,j} a_j^s \cup p_{1*} (p_2^* (f_j^{2n-s} \cup e_i^r)).$$

Now by Lemma 4.9 we have that $p_{1*} (p_2^* (f_j^{2n-s} \cup e_i^r)) = 0$ when $r \neq s$, and $p_{1*} (p_2^* (f_j^{2n-r} \cup e_i^r)) = \text{Tr}_X(f_j^{2n-r} \cup e_i^r)$. But by definition this is zero unless $i = j$, when it is equal to 1. This implies that $a_i^r = \varphi^* e_i^r$. \square

Proof. (of Theorem 4.11) Applying the formula in Lemma 4.14 with $\varphi = \text{Id}$, but with the dual bases $\{f_j^s\}$ and $\{(-1)^s e_j^{2n-s}\}$, we get

$$\text{cl}(\Delta) = \sum_{s,j} (-1)^s p_1^* f_j^s \cup p_2^* e_j^{2n-s}.$$

By Lemma 4.10 we have

$$\begin{aligned} \Delta \cdot \Gamma_\varphi &= \text{Tr}_{X \times X}(\text{cl}(\Delta) \cup \text{cl}(\Gamma_\varphi)) = \text{Tr}_{X \times X} \left(\sum_{r,s,i,j} p_1^* (f_j^s \cup \varphi^* e_i^r) \cup p_2^* (e_j^{2n-s} \cup f_i^{2n-r}) \right) = \\ &= \sum_{r,i} \text{Tr}_X(f_i^{2n-r} \cup \varphi^* e_i^r) \cdot \text{Tr}_X(e_i^r \cup f_i^{2n-r}) = \sum_r (-1)^r \text{Tr}(\varphi^* | H^r(X)). \end{aligned}$$

\square

For later use, let's also note the following formula for the characteristic polynomial of a linear transformation, in terms of traces of iterates.

Lemma 4.15. *Let V be a vector space over the field K , and $\varphi : V \rightarrow V$ a K -linear transformation. Then*

$$\det(\text{Id} - t\varphi) = \exp \left(- \sum_{m \geq 1} \text{Tr}(\varphi^m) \cdot \frac{t^m}{m} \right).$$

Proof. By extending everything to \overline{K} , we can assume that K is algebraically closed. Then there exists a basis for V in which the matrix of φ is upper triangular. If the entries on the diagonal are a_1, \dots, a_n , then

$$\det(\text{Id} - t\varphi) = \prod_{i=1}^n (1 - a_i t).$$

On the other hand

$$\begin{aligned} \exp \left(- \sum_{m \geq 1} \text{Tr}(\varphi^m) \cdot \frac{t^m}{m} \right) &= \exp \left(- \sum_{m \geq 1} \sum_{i=1}^n \frac{a_i^m t^m}{m} \right) = \\ &= \exp \left(\sum_{i=1}^n \log(1 - a_i t) \right) = \prod_{i=1}^n (1 - a_i t). \end{aligned}$$

□

Rationality. Assume that there exists a Weil cohomology theory for varieties over $\overline{\mathbf{F}}_p$, with p a prime. We will use this to deduce the rationality Theorem 3.1 for varieties over \mathbf{F}_q , with $q = p^r$. Denote $F := \text{Frob}_{\overline{X}, q} : \overline{X} \rightarrow \overline{X}$. More precisely, we have

Theorem 4.16. *Let X be a smooth projective geometrically connected variety over \mathbf{F}_q , of dimension n . Then*

$$Z(X; t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}$$

where $P_i(t) = \det(\text{Id} - tF^*|H^i(\overline{X}))$ for all $0 \leq i \leq 2n$. In particular, $Z(X; t) \in \mathbf{Q}(t)$.

Proof. We have seen in Proposition 1.7 that Δ and Γ_m are transverse in $\overline{X} \times \overline{X}$, and $N_m = \Delta \cdot \Gamma_m$. Theorem 4.11 implies then

$$N_m = \sum_{i=0}^{2n} (-1)^i \text{Tr}((F^m)^*|H^i(\overline{X})).$$

Using Lemma 4.15, we get

$$Z(X; t) = \exp \left(\sum_{m \geq 1} \sum_{i=0}^{2n} (-1)^i \text{Tr}((F^m)^*|H^i(\overline{X})) \frac{t^m}{m} \right) = \prod_{i=0}^{2n} \det(\text{Id} - tF^*|H^i(\overline{X}))^{(-1)^{i+1}},$$

which gives the first part. Since the cohomology theory has coefficients in K , this gives $Z(X; t) \in K(t) \cap \mathbf{Q}[[t]]$, which in turn implies $Z(X; t) \in \mathbf{Q}(t)$ by the Exercise below. □

Exercise 4.17. Let L be a field, and $f = \sum_{m \geq 1} a_m t^m \in L[[t]]$. Then $f \in L(t)$ if and only if there exist natural numbers m, n such that the linear span of the vectors

$$\{(a_i, a_{i+1}, \dots, a_{i+n}) \in L^{n+1} \mid i \geq m\}$$

is a proper subspace of L^{n+1} . In particular, for any field extension $L \subset L'$, $f \in L(t)$ if and only if $f \in L'(t)$.

Functional equation. We will need another linear algebra lemma (cf. [Ha] Appendix C, Lemma 4.3).

Lemma 4.18. Let $\varphi : V \times W \rightarrow K$ be a perfect pairing of vector spaces of dimension r over a field K . Let $f \in \text{End}_K(V)$, $g \in \text{End}_K(W)$ and $\lambda \in K^*$ be such that $\varphi(f(v), g(w)) = \lambda \varphi(v, w)$ for all $v \in V$ and $w \in W$. Then

$$\det(\text{Id} - tg) = \frac{(-1)^r \lambda^r t^r}{\det(f)} \cdot \det(\text{Id} - \lambda^{-1} t^{-1} f)$$

and

$$\det(g) = \frac{\lambda^r}{\det(f)}.$$

Proof. Again we may assume, by extending scalars, that K is algebraically closed. We can then put the matrix of f in upper triangular form; in other words, there exists a basis e_1, \dots, e_r of V such that $f(e_i) = \sum_{j=1}^r a_{i,j} e_j$, and $a_{i,j} = 0$ for $i > j$. By the perfect pairing property, there exists a basis e'_1, \dots, e'_r of W such that $\varphi(e_i, e'_j) = \delta_{i,j}$.

The hypothesis implies that f and g are invertible. Let's check this for g : if $g(w) = 0$, then $\varphi(f(v), g(w)) = \lambda \varphi(v, w) = 0$ for all $v \in V$, so $w = 0$. Write now $g^{-1}(e'_j) = \sum_{l=1}^r b_{j,l} e'_l$. Then $b_{j,i} = 0$ for $i > j$; indeed, note that since $\varphi(f(e_i), e'_j) = 0$ for $j < i$, we also have $\varphi(e_i, g^{-1}(e'_j)) = 0$. We can relate the diagonal entries as well:

$$a_{i,i} = \varphi(f(e_i), e'_i) = \lambda \varphi(e_i, g^{-1}(e'_i)) = \lambda b_{i,i}.$$

The second identity follows by noting that $\det(f) = \prod_{i=1}^r a_{i,i}$ and $\det(g) = \prod_{i=1}^r b_{i,i}^{-1} = \lambda^r / \prod_{i=1}^r a_{i,i}$. For the first, note that

$$\begin{aligned} \det(\text{Id} - tg) &= \det(g) \cdot \det(g^{-1} - t\text{Id}) = \frac{\lambda^r}{\det(f)} \cdot \prod_{i=1}^r (a_{i,i} \lambda^{-1} - t) = \\ &= \frac{(-1)^r \lambda^r t^r}{\det(f)} \cdot \prod_{i=1}^r (1 - a_{i,i} \lambda^{-1} t^{-1}) = \frac{(-1)^r \lambda^r t^r}{\det(f)} \cdot \det(\text{Id} - \lambda^{-1} t^{-1} f). \end{aligned}$$

□

We can now deduce the functional equation for the zeta function.

Proof. (of Theorem 3.2.) We will apply Lemma 4.18 to the perfect pairing given by Poincaré duality (axiom (A3)):

$$\varphi_i : H^i(\overline{X}) \otimes H^{2n-i}(\overline{X}) \rightarrow K, \quad \varphi_i(\alpha \otimes \beta) = \text{Tr}_{\overline{X}}(\alpha \cup \beta),$$

with f, g taken to be F^* acting on the respective cohomology groups. Note that by Lemma 4.6, F^* acts on $H^{2n}(\bar{X})$ as multiplication by q^n , since $F : \bar{X} \rightarrow \bar{X}$ is a finite morphism of degree q^n (check!). For every $\alpha \in H^i(\bar{X})$ and $\beta \in H^{2n-i}(\bar{X})$ we have

$$\varphi_i(F^*\alpha, F^*\beta) = \text{Tr}_{\bar{X}}(F^*(\alpha \cup \beta)) = \text{Tr}_{\bar{X}}(q^n(\alpha \cup \beta)) = q^n \varphi_i(\alpha, \beta).$$

Denote $b_i = \dim_K H^i(\bar{X})$ and $P_i(t) = \det(\text{Id} - tF^*|H^i(\bar{X}))$. Lemma 4.18 implies then that

$$\det(F^*|H^{2n-i}(\bar{X})) = \frac{q^{nb_i}}{\det(F^*|H^i(\bar{X}))}$$

and

$$P_{2n-i}(t) = \frac{(-1)^{b_i} q^{nb_i} t^{b_i}}{\det(F^*|H^i(\bar{X}))} \cdot P_i(1/q^n t).$$

Finally, recall that by Corollary 4.12 that $E = \sum_{i=0}^{2n} (-1)^i b_i$. Using the two identities above and Theorem 4.16, we obtain

$$\begin{aligned} Z(X; \frac{1}{q^n t}) &= \prod_{i=0}^{2n} P_i(1/q^n t)^{(-1)^{i+1}} = \prod_{i=0}^{2n} P_{2n-i}(t)^{(-1)^{i+1}} \cdot \frac{(-1)^E q^{nE} t^E}{\prod_{i=0}^{2n} \det(F^*|H^i(\bar{X}))^{(-1)^i}} = \\ &= \pm Z(X; t) \cdot \frac{q^{nE} t^E}{q^{nE/2}} = \pm q^{nE/2} t^E Z(X; t). \end{aligned}$$

□

It follows from the proof that one can make the sign in the formula more precise: it is $(-1)^E$ if $\det(F^*|H^n(\bar{X})) = q^{nb_n/2}$ and $(-1)^{E+1}$ if $\det(F^*|H^n(\bar{X})) = -q^{nb_n/2}$. Note also that our discussion for P_1 in the case of curves can be generalized: if $P_n(t) = \prod_{i=1}^{b_n} (1 - \alpha_i t)$, then the second identity above (for $i = n$) implies that the set $\{\alpha_1, \dots, \alpha_{b_n}\}$ is invariant under the operation $\alpha \mapsto q^n/\alpha$, and $\prod_{i=1}^{b_n} \alpha_i = \det(F^*|H^n(\bar{X}))$, computed as above.

A brief introduction to ℓ -adic cohomology. This is just a very quick review, necessary to at least define and mention some properties of ℓ -adic cohomology; details can be found for instance in Milne's book [Mi]. Let X be a Noetherian scheme. We consider the category $\acute{\text{E}}\text{t}(X)$ of all étale morphisms $f : Y \rightarrow X$ from a scheme to X . We think of this as the analogue of the category of open subsets of a topological space: an object can be thought of as an *étale open subset* of X .⁶ For instance, inclusions of open subsets correspond here to morphisms in $\acute{\text{E}}\text{t}(X)$: for any étale schemes Y and Z over X , any morphism $Y \rightarrow Z$ over X is étale. Intersections of open sets correspond to fiber products, which exist in $\acute{\text{E}}\text{t}(X)$. An *étale open cover* in $\acute{\text{E}}\text{t}(X)$ is a family of étale morphisms $f_i : U_i \rightarrow U$ in $\acute{\text{E}}\text{t}(X)$ such that $U = \cup_i f_i(U_i)$. The set of all such étale covers of U is denoted $\text{Cov}(U)$.

This data defines the *étale topology* on X . It is an example of a *Grothendieck topology*, in the sense that it satisfies the following properties:

(0) (Fiber products) Fiber products exist in $\acute{\text{E}}\text{t}(X)$.

⁶Recall for instance that an étale morphism of complex algebraic varieties is the same thing as a local analytic isomorphism in the classical topology.

- (1) (Isomorphisms) If $f : U \rightarrow V$ is an isomorphism in $\acute{\text{E}}\text{t}(X)$, then $(f) \in \text{Cov}(V)$.
- (2) (Local character) If $(U_i \rightarrow U) \in \text{Cov}(U)$ and for every i we have $(U_{i,j} \rightarrow U_i) \in \text{Cov}(U_i)$, then $(U_{i,j} \rightarrow U)_{i,j} \in \text{Cov}(U)$.
- (3) (Stability under base change) If $(U_i \rightarrow U) \in \text{Cov}(U)$ and $V \rightarrow U$ is an morphism in $\acute{\text{E}}\text{t}(X)$, then we have $(U_i \times_U V \rightarrow V) \in \text{Cov}(V)$.

Sometimes $\acute{\text{E}}\text{t}(X)$ together with the above Grothendieck topology is called the *étale site* of X . The main point is that in order to develop a sheaf theory and a cohomology theory for sheaves, using for instance injective resolutions according to Grothendieck's recipe, one need only have families of coverings for the objects of a category, satisfying the properties above, i.e. a Grothendieck topology.

We consider the category of sheaves on the étale site, or sheaves in the étale topology, denoted $\text{Sh}(X_{\acute{\text{E}}\text{t}})$. Concretely, an étale presheaf, say of abelian groups, on X is a contravariant functor $\acute{\text{E}}\text{t}(X) \rightarrow \text{AbGps}$. It is a sheaf if in addition it satisfies the gluing property: for every $U \in \acute{\text{E}}\text{t}(X)$ and every $(U_i \rightarrow U) \in \text{Cov}(U)$, the complex of abelian groups

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \longrightarrow \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

is exact. Note that every such \mathcal{F} defines a sheaf \mathcal{F}_U in the usual sense on the domain U of each object in $\acute{\text{E}}\text{t}(X)$, but there is more information contained in the definition of an étale sheaf. It can be shown that $\text{Sh}(X_{\acute{\text{E}}\text{t}})$ is an abelian category with enough injectives, and therefore we can consider the right derived functors of the left exact functor $\mathcal{F} \mapsto \mathcal{F}(X)$. These are called the *étale cohomology groups* of \mathcal{F} on X , denoted by $H_{\acute{\text{E}}\text{t}}^i(X, \mathcal{F})$ for $i \geq 0$.

Constant sheaves. Let G be any abelian group. The étale constant sheaf on X , denoted by G as well, is the functor

$$\acute{\text{E}}\text{t}(X) \rightarrow \text{AbGps}, (U \rightarrow X) \mapsto G^{\pi_0(U)},$$

where $\pi_0(X)$ is the number of connected components of U . We will be especially interested in the cohomology groups $H_{\acute{\text{E}}\text{t}}^i(X, \mathbf{Z}/n\mathbf{Z})$, for $n \geq 1$. For instance, when $n = 1$ we will see a rather elementary interpretation of these groups below.

\mathcal{O}_X -modules. Let \mathcal{F} be a quasi-coherent sheaf of \mathcal{O}_X -modules. One defines the associated $W(\mathcal{F}) \in \text{Sh}(X_{\acute{\text{E}}\text{t}})$ as follows: for every $U \rightarrow X$ in $\acute{\text{E}}\text{t}(X)$, $W(\mathcal{F})(U) := f^*\mathcal{F}(U)$. The fact that this is an étale sheaf follows from the theory of faithfully flat descent.⁷ One can show that for all i there are canonical isomorphisms

$$H_{\acute{\text{E}}\text{t}}^i(X, W(\mathcal{F})) \simeq H^i(X, \mathcal{F}).$$

⁷First, one easily reduces to checking the sheaf condition only for Zariski open covers, and for étale covers consisting of a single map $V \rightarrow U$ with V and U both affine. This then becomes equivalent to checking the exactness of the natural sequence

$$M \rightarrow B \otimes_A M \rightarrow B \otimes_A B \otimes_A M$$

(with $B \rightarrow B \otimes_A B$ given by $b \mapsto 1 \otimes b - b \otimes 1$) for every ring homomorphism $A \rightarrow B$ corresponding to an étale surjective morphism $\text{Spec}(A) \rightarrow \text{Spec}(B)$, and every A -module M . For this it is in fact enough to assume that $A \rightarrow B$ is *faithfully flat*, a weaker condition. Cf. [Mi] p.51.

Group schemes. Let \mathbf{G} be an abelian group scheme over X . We can consider \mathbf{G} as an étale sheaf on X , via the functor

$$(U \rightarrow X) \mapsto \mathbf{G}(U) := \mathrm{Hom}_X(U, \mathbf{G}).$$

Once more, this presheaf is a sheaf due to faithfully flat descent. As an important example, consider the multiplicative group scheme over X ,

$$\mathbf{G}_m := X \times_{\mathrm{Spec} \mathbf{Z}} \mathrm{Spec} \mathbf{Z}[t, t^{-1}].$$

For every $U \rightarrow X$ in $\acute{\mathrm{E}}\mathrm{t}(X)$, we have that $\mathbf{G}_m(U) = \mathcal{O}_U(U)^*$. We can also consider the closed sub-group scheme corresponding to n -th roots of unity

$$\mu_n := X \times_{\mathrm{Spec} \mathbf{Z}} \mathrm{Spec} \mathbf{Z}[t]/(t^n - 1) \subset \mathbf{G}_m,$$

where $\mu_n(U) = \{u \in \mathcal{O}_U(U) \mid u^n = 1\}$. If X is defined over a field k which is separably closed, then for every integral k -algebra A one has $\{u \in A \mid u^n = 1\} \subset k$. Hence if X is integral, any choice of an n -th root of unity in k determines an isomorphism of étale sheaves $\mu_n \simeq \mathbf{Z}/n\mathbf{Z}$.

Assume now X is defined over a field k such that $\mathrm{char}(k)$ does not divide n . Then there is an exact sequence of étale sheaves called the *Kummer sequence*:

$$0 \longrightarrow \mu_n \longrightarrow \mathbf{G}_m \longrightarrow \mathbf{G}_m \longrightarrow 0,$$

where the morphism $\mathbf{G}_m \rightarrow \mathbf{G}_m$ is given by $u \mapsto u^n$.⁸ Note that there is an isomorphism similar to the isomorphism $H^1(X, \mathcal{O}_X^*) \simeq \mathrm{Pic}(X)$ in the Zariski topology, namely

$$H_{\acute{\mathrm{e}}\mathrm{t}}^1(X, \mathbf{G}_m) \simeq \mathrm{Pic}(X).$$

(This is sometimes called *Hilbert's Theorem 90*; see [Mi] Proposition 4.9.) Making all the assumptions above, namely that X is an integral scheme over a separably closed field k with $\mathrm{char}(k)$ not dividing n , by passing to cohomology in the Kummer sequence we obtain an exact sequence

$$(3) \quad \Gamma(X, \mathcal{O}_X)^* \xrightarrow{(\cdot)^n} \Gamma(X, \mathcal{O}_X)^* \rightarrow H_{\acute{\mathrm{e}}\mathrm{t}}^1(X, \mathbf{Z}/n\mathbf{Z}) \rightarrow \mathrm{Pic}(X) \xrightarrow{(\cdot)^n} \mathrm{Pic}(X) \rightarrow H_{\acute{\mathrm{e}}\mathrm{t}}^2(X, \mathbf{Z}/n\mathbf{Z}).$$

The case of curves. The étale cohomology of constant sheaves associated with finite abelian groups can often be described by reducing to the case of curves. In that case, one has the following general result:

Theorem 4.19. *Let X be a smooth projective curve of genus g over an algebraically closed field k . Let n be a natural number not divisible by $\mathrm{char}(k)$. Then one has:*

- $H_{\acute{\mathrm{e}}\mathrm{t}}^0(X, \mathbf{Z}/n\mathbf{Z}) \simeq \mathbf{Z}/n\mathbf{Z}$.
- $H_{\acute{\mathrm{e}}\mathrm{t}}^1(X, \mathbf{Z}/n\mathbf{Z}) \simeq \{L \in \mathrm{Pic}^0(X) \mid L^n \simeq \mathcal{O}_X\} \simeq (\mathbf{Z}/n\mathbf{Z})^{2g}$.
- $H_{\acute{\mathrm{e}}\mathrm{t}}^2(X, \mathbf{Z}/n\mathbf{Z}) \simeq \mathbf{Z}/n\mathbf{Z}$.
- $H_{\acute{\mathrm{e}}\mathrm{t}}^i(X, \mathbf{Z}/n\mathbf{Z}) = 0$ for $i > 2$.

⁸This is surjective due to the fact that for any k -algebra A and every $a \in A$, the natural morphism $A \rightarrow A[t]/(t^n - a)$ is étale and surjective, and the image of a via this morphism is an n -th power.

The main technical point for proving the result above is to show that

$$H_{\text{ét}}^i(X, \mathbf{G}_m) = 0 \text{ for all } i \geq 2.$$

(This is a consequence of Tsen's theorem saying that a non-constant homogeneous polynomial of degree $< n$ in $k(X)[X_1, \dots, X_n]$ must have a non-trivial zero; cf. [Mi] III.2.22(d).) Using the long exact sequence in cohomology associated to the Kummer sequence, one immediately obtains the first and last statement, together with the fact that the sequence (3) is exact on the right. Note now that multiplication by n is surjective on $\text{Pic}^0(X)$, which means that $H_{\text{ét}}^2(X, \mathbf{Z}/n\mathbf{Z})$ is the same as the cokernel of multiplication by n on the Neron-Severi group of X , hence isomorphic to $\mathbf{Z}/n\mathbf{Z}$. Note also that if $L \in \text{Pic}^0(X)$ satisfies $L^n \simeq \mathcal{O}_X$, then we must have $L \in \text{Pic}^0(X)$. Therefore we are looking at the subgroup of n -torsion points of the abelian variety $\text{Pic}^0(X)$ defined over k , and given the assumption on the characteristic it is well-known that this is isomorphic to $(\mathbf{Z}/n\mathbf{Z})^{2g}$.

Exercise 4.20. Revisit the Weil conjectures for curves in view of Theorem 4.19 and the section below (cf. also [Mi] V.2).

ℓ -adic cohomology. Let now k be an algebraically closed field. If $\text{char}(k) = p > 0$, consider a prime $\ell \neq p$. As above, given any $m \geq 1$, the étale cohomology $H_{\text{ét}}^i(X, \mathbf{Z}/\ell^m\mathbf{Z})$ is a $\mathbf{Z}/\ell^m\mathbf{Z}$ -module, and there are natural maps

$$H_{\text{ét}}^i(X, \mathbf{Z}/\ell^{m+1}\mathbf{Z}) \longrightarrow H_{\text{ét}}^i(X, \mathbf{Z}/\ell^m\mathbf{Z})$$

forming an inductive system.

Definition 4.21. The i -th ℓ -adic cohomology of X is

$$H_{\text{ét}}^i(X, \mathbf{Z}_\ell) := \varprojlim_m H_{\text{ét}}^i(X, \mathbf{Z}/\ell^m\mathbf{Z}).$$

This has a natural structure of \mathbf{Z}_ℓ -module, where \mathbf{Z}_ℓ is the ring of ℓ -adic integers. We also consider

$$H_{\text{ét}}^i(X, \mathbf{Q}_\ell) := H_{\text{ét}}^i(X, \mathbf{Z}_\ell) \otimes_{\mathbf{Z}_\ell} \mathbf{Q}_\ell.$$

One of the main facts on ℓ -adic cohomology is that, when restricted to smooth projective varieties over k , it forms a Weil cohomology theory with coefficients in \mathbf{Q}_ℓ (see [Mi] Ch. VI). Theorem 4.16 applied in this setting implies that if X is smooth, projective, geometrically connected over \mathbf{F}_q , $q = p^r$, and $\ell \neq p$, then

$$Z(X; t) = \frac{P_1(t) \cdot P_3(t) \cdots P_{2n-1}(t)}{P_0(t) \cdot P_2(t) \cdots P_{2n}(t)}$$

where $P_i(t) = \det(\text{Id} - tF^*|H_{\text{ét}}^i(\overline{X}, \mathbf{Q}_\ell))$ for all $0 \leq i \leq 2n$.

Along the same lines, one can give a proof of rationality for arbitrary varieties (i.e. not necessarily smooth or projective) over \mathbf{F}_q . The only difference is that one needs to consider instead ℓ -adic cohomology with compact supports, denoted $H_c^i(\overline{X}, \mathbf{Q}_\ell)$. With this modification, the exact same formula as above holds.

Going back to the case of smooth projective varieties, as we saw above the functional equation for the zeta function follows from Poincaré duality for ℓ -adic cohomology. Finally,

the hardest conjecture, the analogue of the Riemann Hypothesis, is verified in [De1], where it is shown that

$$P_i(t) = \det(\text{Id} - tF^*|H_{\text{ét}}^i(\overline{X}, \mathbf{Q}_\ell)) = \prod_i (1 - \alpha_i t) \in \mathbf{Z}[t],$$

and $|\alpha_i| = q^{i/2}$ for all i , for any choice of isomorphism $\mathbf{Q}_\ell \simeq \mathbf{C}$. This is substantially more complicated than the proofs of the other conjectures explained above.

Betti numbers. The proof of Theorem 3.4 now follows from a general theorem comparing étale and singular cohomology in characteristic 0: if X is a smooth complex variety, then the étale and singular cohomology of X with coefficients in a finite abelian group⁹ are isomorphic (see e.g [Mi] Theorem III.3.12).

Assume now that there exists a finitely generated \mathbf{Z} -algebra R , \mathcal{X} a scheme which is smooth and projective over $\text{Spec } R$, and $\mathfrak{p} \subset R$ a maximal ideal such that $R/\mathfrak{p} \simeq \mathbf{F}_q$ and

$$X = \mathcal{X} \times_{\text{Spec } R} \text{Spec } R/\mathfrak{p}.$$

By the comparison theorem mentioned above, one gets for every m :

$$H_{\text{ét}}^i(\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C}, \mathbf{Z}/\ell^m \mathbf{Z}) \simeq H^i((\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C})^{\text{an}}, \mathbf{Z}/\ell^m \mathbf{Z}).$$

Passing to the limit and tensoring with \mathbf{Q}_ℓ , we obtain

$$H_{\text{ét}}^i(\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C}, \mathbf{Q}_\ell) \simeq H^i((\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C})^{\text{an}}, \mathbf{Q}_\ell).$$

On the other hand, the smooth base change theorem for étale cohomology and the proper smooth base change theorem for locally constant sheaves (see [Mi] VI. §4) imply that

$$H_{\text{ét}}^i(\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C}, \mathbf{Q}_\ell) \simeq H_{\text{ét}}^i(X, \mathbf{Q}_\ell).$$

Putting these two facts together, we obtain a comparison result between the ℓ -adic cohomology of X and the singular cohomology of the complex points of its lifting,

$$H_{\text{ét}}^i(X, \mathbf{Q}_\ell) \simeq H^i((\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C})^{\text{an}}, \mathbf{Q}_\ell),$$

and consequently

$$b_i(X) = \deg P_i(t) = b_i((\mathcal{X} \times_{\text{Spec } R} \text{Spec } \mathbf{C})^{\text{an}}).$$

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⁹The result is not necessarily true otherwise, for instance for \mathbf{Z} -coefficients.

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