RESEARCH STATEMENT

MILTON LIN

Summary of my research

I split my current and future projects into two categories: **core projects**, where I am primarily focused on advancing mixed characteristic and metaplectic aspects of the Langlands program, and additional **ongoing work in related areas**, including categorical deformations, motivic aspects, and stacky approaches.

Core proejcts. In the geometric Langlands program, my graduate work has focused on extensions in the mixed characteristic setting, where joint with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn Univeristy). [ILZ24] we applied the framework of Zhu's perfect geometry [Zhu17] to prove the Casselman-Shalika formula in mixed characteristics [ILZ24]. The Casselman-Shalika formula computes the "fourier coefficients" of automorphic forms and is fundamental to modern works of geometric Langlands, see [FR22]. Moving forward, I will continue this research in two directions:

(1) **Metaplectic aspects of Langlands**, see Section 1 for details, joint with Toan Pham (Johns Hopkins Univeristy) I intend to give a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+24].

(2) **Relative aspects of Langlands**, see Section 2 for details, joint project with Yuta Takaya (University of Tokyo), we aim to explore relative aspects of the Langlands program on the Fargues-Fontaine curve, [FS24], recent conjectures of Ben-Zvi. Sakellaridis, and Venkatesh [BSV], particularly the relationship between period sheaves and L-sheaves as in [FW24].

Related works. In addition to my primary projects in the relative and metaplectic aspects of Langlands, I am equally committed to three other areas of study, each of which contributes to the broader foundation of representation theory. For details of these further works I refer to `https://cwlin4916.github.io/Trees/Application/Postdoc/Research.pdf`.

(1) **Motivic aspects of Langlands** , for details, Building on [RS20], I aim to define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we prove the same statement in [NP01] in the category of mixed Tate motives. The difficulty is that the original argument requires characteristic 0 coefficients, whilst in positive characteristic, there are problems of extensions.

(2) **Stacky Approaches to Periods**, I use recent advancements in formalism of stacks to study periods. I have proven that the unipotent fundamental group associated to a pointed scheme can be recovered via a *stacky approach*, see [GR14], [Toë06], [MR23]: for a given scheme X, there exists a natural stack, $\mathbf{U}(X_{\mathrm{dR}})$ whose fundamental group coincides with the unipotent de Rham fundamental group as studied in [Bro14]. By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham

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fundamental group admits a mixed Hodge structure. [1] We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].

(3) **Categorical deformations of representation category**, this builds upon my current research on the Whittaker category, from the point of view of deformation theory. We will first document a careful proof of Lurie's theorem, [Lur10, Thm 10.10], which describe *formal deformation of categories*, as gerbes see [Lur10, Ch.8-10] for definitions. Then, we will explore deformations of representation of Lusztig's small quantum group, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.

1. Mixed Characteristic Geometry and the Casselman–Shalika formula

Let G be a connected reductive group over a nonarchimedian local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld [AG15], [Gai14]. One fundamental aspect of the program is to understand the *Whittaker Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, as described *op.cit.*, this corresponds to finding the Fourier coefficients of automorphic functions:

Example 1.1. Let $G = \mathrm{PGL}_2$ be the projective linear group over $\mathbb{Q}$. A modular function, f, has an adelic formulation, $\widetilde{f}$ on $G(\mathbb{A}_\mathbb{Q})$. Up to a *normalizing factor*, the Fourier coefficients coincide with the Whittaker coefficients (Jacquet period),

$$a_m \sim \int_{N(\mathbb{Q})\backslash N(\mathbb{A}_\mathbb{Q})} \widetilde{f}(n\alpha_m)\psi(-n)\, dn \quad \text{ for } m \geq 1$$

where $\alpha_m \in T(\mathbb{A}_\mathbb{Q}^{\mathrm{fin}})$ is m considered as a finite idèle and ψ is a standard character on $N(\mathbb{A}_\mathbb{Q}) = \mathbb{A}_\mathbb{Q}$, where N is the radical of standard Borel. For more details, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which joint Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn University) [ILZ24], we replicated a variation: the *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian Gr_G, analogous to the equi-characteristic geometrization carried out in [NP01].

Theorem 1.2. *[ILZ24] The geometric Casselman–Shalika formula holds over the Witt vector Grassmanian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori–Satake equivalence, [Bez+19] without using nearby cycles.

Theorem 1.3 (I.-Lin-Z., in progress)**.** *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [2]

As of writing, D. Hansen, L. Hamman, and L. Mann have announced significant advancements to the global version of the formula. Building on all these progresses, I have developed a strong interest in exploring the geometrization of the Casselman–Shalika formula for covering groups [GGW18].

[1] In Brown's approach, he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

[2] In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

Research Goal A. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula over equal-characteristic local fields. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+20; Bru+24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

Lastly, we would like to highlight that in our recent work [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\leq\mu}$, for minuscule and quasi-minuscule μ. In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal B. Geometry of general perfect Schubert variety *Prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhats-Tits building of G, we have an associated group scheme $\mathcal{G}$ over $\mathcal{O}$.*

Conjecture 1.4. The smooth locus of $\mathrm{Gr}_{\mathcal{G},\leq\mathfrak{s}}$ is $\mathrm{Gr}_{\mathcal{G},\mathfrak{s}}$ in perfect geometry, in the sense of [Zhu17].

2. Relative langlands on the Fargues Fontaine curve

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and L-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with $(\hat{G}, \hat{X})$ of hyperspherical varieties. Let $\Lambda = \bar{\mathbb{Q}}_l$. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_!\Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \hat{\pi}_*\omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. On the automorphic side, Bun_G has a Hardar-Narasimhan straification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G) \simeq \mathbb{Z}$. Interesting phenomena occurs for $n \in \mathbb{Z}_{\geq 0} \subset \mathbb{Z} \simeq B(\mathbb{G}_m)$, and the study of period sheaves reduces to the study of the Abel-Jacobi map previously studied by Fargues [Far20] and Hansen [Han21]. On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles, [FW24, Ch. 6] recently developed by Anschütz and Le Bras [AL21].

Conjecture 2.1. [Lin-T., in progress] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[3] $\mathcal{L}_{\hat{X}}$.

Research Goal C. Relative Langlands on the Fargues Fontaine curve. *Complete Conjecture 2.1 as a first step and then the Hecke case, which classically corresponds to Hecke's integral representation of standard L-function for GL_2. Lastly, one can ask whether on the $\mathcal{B}$-side, the same constructions of [BSV, Ch. 11] works for the p-adic (Emerton-Gee) L-parameter stacks, which potentially give new interpretations to p-adic L-functions.*

[3] One has to take into account shearing, twisting and tensoring. There is also an additional $\mathbb{G}_m := \mathbb{G}_{gr}$ action on X which we do not discuss.

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