AN INFORMAL INTRODUCTION TO CATEGORICAL REPRESENTATION THEORY AND THE LOCAL GEOMETRIC LANGLANDS PROGRAM

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ABSTRACT. We provide a motivated introduction to the theory of categorical actions of groups and the local geometric Langlands program. Along the way we emphasize applications, old and new, to the usual representation theory of reductive and affine Lie algebras.

CONTENTS

1. INTRODUCTION

1.1. The theory of categorical actions of groups and the local geometric Langlands correspondence are two important actors in modern representation theory which have undergone rapid development in the past twenty years, with decisive contributions by Beilinson, Drinfeld, Frenkel, and Gaitsgory, and more recently Beraldo, Raskin, and Yang.

Both categorical actions and local geometric Langlands are deeply intertwined with the representation theory of Lie algebras in their history and major applications, and we will emphasize these connections throughout.

On the one hand, the theory of categorical actions provides an efficient framework for treating the representation theory of reductive Lie algebras. This is a convenient entry point into the theory of categorical actions and is how we shall first meet them in the present survey.

On the other hand, and more significantly, the formalism of categorical actions and perspectives from local geometric Langlands have proven essential to the development of the representation theory of affine Lie algebras and related vertex algebras, providing tools to resolve old conjectures and generating new ones. We will reach these developments in the latter half of this survey.

1.2. A wonderful feature of this circle of ideas is that, despite the appearance of fairly sophisticated tools from modern algebra and geometry, the basic results, techniques, and perspectives can be quickly absorbed by a nonspecialist if presented in the right light. It is our hope that the present article can provide such an entry point for the nonexpert reader.

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To this end, our orienting goal has been to survey this area in a manner which is accessible and complements what is already available in the literature. In view of this, we have chosen to emphasize analogies, motivations, and historical precursors over providing a fully rigorous development of the treated material.

1.3. Two immediate problems arise when pursuing such a goal.

The first problem is how much to presume as known by the reader and how much to explain in some detail. We will assume the reader has some familiarity with the basics of the representation theory of simple Lie groups and Lie algebras, algebraic geometry, and homological algebra but not more than what would be covered in an introductory course on these subjects. Certainly these may be further substituted by some mathematical maturity or at least the willingness to take a few assertions on faith. In particular, we will not assume any great familiarity with the tools of geometric representation theory, e.g., D-modules and perverse sheaves, higher algebra, and derived and infinite dimensional algebraic geometry. Instead, we will develop what we need as we go along. In doing so, we have again attempted to provide treatments which convey the basic ideas with minimal fuss, in a way which hopefully provides some orientation to then tackle the relevant literature.

The second problem is that the choice of which ideas, methods, and intuitions to emphasize is necessarily constrained by space as well as the biases and the limitations of the surveyor. Accordingly, we apologize to the many mathematicians and physicists whose contributions were omitted or underemphasized. As a first indication, the extremely rich connections with three- and four-dimensional supersymmetric gauge theory are only touched on briefly towards the end.

1.4. We have endeavored to organize the survey in a such a way that the complexity of the material increases monotonically throughout. In particular, the reader may wish to start at the beginning, skip to where things seem unfamiliar or interesting, and read from there onwards. We hope that the first sections, particularly Section 3, can serve as a light introduction to geometric representation theory for nonspecialists. We also expect that to appreciate every last detail of the final sections, particularly Section 6, would require some degree of prior familiarity.

The precise organization of the article is as follows.

In Section 2 we recall some basic aspects of the representation theory of finite groups to set the stage for their categorical counterparts, which we will encounter later.

In Section 3 we review the localization theorem of Beilinson–Bernstein and see how the resulting correspondence between equivariant D-modules and Harish-Chandra modules for the Lie algebra leads to the idea of a categorical group action.

In Section 4 we then discuss the basics of the theory of categorical group actions for finite dimensional algebraic groups and take some time to explain some of the tools and constructions from modern homological algebra needed to get up and running.

To get to local geometric Langlands, we must pass from the setting of algebraic groups to loop groups. Accordingly, in Section 5, we then set up the basics of the categorical representation theory of loop groups, i.e., review what changes when passing to the infinite dimensional setting.

Finally, in Section 6 we meet the local geometric Langlands correspondence, discuss known and conjectured aspects in some detail, and highlight an important application to localization theory for affine Lie algebras at critical level.

We also provide in Appendix A a concise introduction to D-modules, the Riemann–Hilbert correspondence, and the functions-sheaves correspondence for the benefit of nonspecialists.

1.5. In addition to the references to the literature provided throughout, the reader is encouraged to consult the wonderful earlier survey [Fre08] and book [Fre07a] on local geometric Langlands and the notes [ABC$^+$18]; see also [Fre07b] for an accessible introduction to the global story.

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2. Representations of group algebras and convolution algebras

2.1. Overview.

2.1.1. Given a split reductive group G over the integers,[1] there are instructive analogies between the complex representation theory of its points over finite fields, local fields, and the real numbers. Using these, one can transfer constructions and ideas from one area to the other. Harish-Chandra even dubbed this permeability between the subjects as a Lefschetz principle for representation theory [Lan93].[2]

Similarly, a basic principle in the categorical theory, with its origins in the work of Drinfeld and Laumon on the global geometric Langlands correspondence and Lusztig on character sheaves, is that the categorical representation theory of $G(\mathbb{C})$ should behave similarly to (a part of) the complex representation theory of $G(\mathbb{F}_q)$ and that the categorical representation thory of $G(\mathbb{C}((t)))$ should behave similarly to the complex representation theory of $G(\mathbb{F}_q((t)))$.

2.1.2. With this in mind, before moving to the categorical setting, we first review some salient facts about the usual representations of finite groups and then specialize the discussion to $G(\mathbb{F}_q)$.

2.2. The group algebra via convolution.

2.2.1. Suppose that H is a finite group and k is an algebraically closed field of characteristic zero. Let us write $\operatorname{Rep}(H)^{\heartsuit}$ for the abelian category of representations of H on k vector spaces.

Remark 2.2.2. The superscript on $\operatorname{Rep}(H)^{\heartsuit}$ is to emphasize we are speaking of the abelian category, thought of as the heart of the standard t-structure on the corresponding derived category $\operatorname{Rep}(H)$. Similarly, in what follows, categories of modules and sheaves are derived unless otherwise specified, and the corresponding abelian categories will be denoted with a superscript '$\heartsuit$'.

2.2.3. Recall that to give a representation of H on a vector space V is the data of a map

$$\alpha : H \to \operatorname{End}(V). \tag{2.1}$$

Here $\operatorname{End}(V)$ denotes the k linear endomorphisms of V, and α is unital and intertwines multiplication in H with composition in $\operatorname{End}(V)$. More generally, for any k-algebra A, consider the collection of maps of semigroups

$$H \to A, \tag{2.2}$$

where A is viewed as a semigroup under multiplication. A crucial observation is that, since one is forgetting structure on A, namely addition, this extends to a map from a k-algebra built from H, namely the group algebra. Let us phrase the construction of the group algebra in a way convenient for the categorical variant we shall meet later.

[1]The reader will not lose much by taking G throughout to be the group GL_n of invertible $n \times n$ matrices or even just GL_2.

[2]Recall the usual Lefschetz principle ensures that facts which are true about any complex algebraic variety, e.g., generic smoothness, hold for algebraic varieties over any algebraically closed field of characteristic zero.

2.2.4. For a finite set X, let us write $\mathrm{Fun}(X)$ for the vector space of k-valued functions on X. Let us recall two basic properties of this assignment. First, a factorization of X as a product $X = X_1 \times X_2$ yields a canonical tensor product factorization of its space of functions

$$\mathrm{Fun}(X) \simeq \mathrm{Fun}(X_1) \otimes \mathrm{Fun}(X_2).$$

Second, given a map $f : X \to Y$ of maps of finite sets, one has pushforward and pullback maps

$$f^! : \mathrm{Fun}(Y) \to \mathrm{Fun}(X) \quad \text{and} \quad f_* : \mathrm{Fun}(X) \to \mathrm{Fun}(Y).$$

Explicitly, $f^!$ sends a map $\phi : Y \to k$ to the composition $\phi \circ f$, and f_* integrates a function ψ in $\mathrm{Fun}(X)$ along the fibres of f. I.e., for any element y of Y one has

$$(f_*\psi)(y) := \sum_{x \in f^{-1}(y)} \psi(x).$$

2.2.5. Combining these properties, the group structure on H endows $\mathrm{Fun}(H)$ with the structure of a k-algebra. Namely, if we write $m : H \times H \to H$ for the multiplication map, this yields a convolution map on functions

$$\mathrm{Fun}(H) \otimes \mathrm{Fun}(H) \simeq \mathrm{Fun}(H \times H) \xrightarrow{m_*} \mathrm{Fun}(H). \tag{2.3}$$

Explicitly, if for an element h of H we denote by δ_h the corresponding delta function in $\mathrm{Fun}(H)$, i.e.,

$$\delta_h(h') := \begin{cases} 1 & \text{if } h = h', \\ 0 & \text{otherwise,} \end{cases}$$

then the δ_h, for $h \in H$, form a basis of $\mathrm{Fun}(H)$, and multiply by the rule

$$\delta_{h_1} \star \delta_{h_2} = \delta_{h_1 h_2},$$

where $h_1 h_2$ denotes the product in H. The assignment $h \mapsto \delta_h$ yields a map $H \to \mathrm{Fun}(H)$, with the property that any map of the form (2.2) extends uniquely to a map of k-algebras

$$\mathrm{Fun}(H) \to A.$$

In particular, this yields the familiar equivalence between representations of H and left modules for $\mathrm{Fun}(H)$.

Plainly, one forms $\mathrm{Fun}(H)$ by taking linear combinations of points of H, and it acts in a representation of H on a vector space by taking the corresponding linear combination of endomorphisms.

2.3. **The principal series representation.**

2.3.1. In the remainder of this section, we will recall a basic representation of $\mathbb{F}_q$-points of G, namely the space of functions on its flag manifold. To do so, we first begin with some preliminaries.

2.3.2. As before, H denotes a finite group. If H acts on a finite set X, this induces a natural representation of H on $\mathrm{Fun}(X)$. Namely, given an element h of H and a function ϕ in $\mathrm{Fun}(X)$, the action of h on ϕ yields the element $h \cdot \phi$ given by

$$(h \cdot \phi)(x) := \phi(h^{-1}x), \quad \text{for } x \in X.$$

The corresponding left module structure

$$\mathrm{Fun}(H) \otimes \mathrm{Fun}(X) \to \mathrm{Fun}(X) \tag{2.4}$$

may be described as follows. If we denote the action by $a : H \times X \to X$, then (2.4) is given by the composition

$$\mathrm{Fun}(H) \otimes \mathrm{Fun}(X) \simeq \mathrm{Fun}(H \times X) \xrightarrow{a_*} \mathrm{Fun}(X). \tag{2.5}$$

2.3.3. We recall that $\mathrm{Fun}(X)$ is canonically self dual. Namely, writing $\Delta : X \to X \times X$ for the diagonal and $\pi : X \to \mathrm{pt}$ for the projection, one has the perfect pairing

$$\mathrm{Fun}(X) \otimes \mathrm{Fun}(X) \simeq \mathrm{Fun}(X \times X) \xrightarrow{\Delta^!} \mathrm{Fun}(X) \xrightarrow{\pi_*} \mathrm{Fun}(\mathrm{pt}) \simeq k,$$

Explicitly, this is simply the 'L^2' inner product

$$f \otimes g \mapsto \sum_x f(x) \cdot g(x).$$

2.3.4. In particular, given finite H-sets X and Y, one has a canonical identification

$$\mathrm{Hom}_{\mathrm{Rep}(H)^\heartsuit}(\mathrm{Fun}(X), \mathrm{Fun}(Y)) \simeq (\mathrm{Fun}(X)^* \otimes \mathrm{Fun}(Y))^H \simeq \mathrm{Fun}(X \times Y)^H \simeq \mathrm{Fun}(X \times Y/H),$$

where the superscript 'H' denotes the subspace of H-invariants, and $X \times Y/H$ denotes the orbit set of $X \times Y$ with respect to the diagonal action of H.

Explicitly, given a function $K \in \mathrm{Fun}(X \times Y)$, the corresponding homomorphism $\mathrm{Fun}(X) \to \mathrm{Fun}(Y)$ is given by the integral transform

$$\mathrm{Fun}(X) \xrightarrow{\pi_X^!} \mathrm{Fun}(X \times Y) \xrightarrow{K \cdot -} \mathrm{Fun}(X \times Y) \xrightarrow{\pi_{Y,*}} \mathrm{Fun}(Y),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projection maps, and $K \cdot -$ denotes pointwise multiplication of functions. Moreover, K is H-invariant if and only if the corresponding integral transform is a map of H representations.

2.3.5. Let us specialize the preceding discussion to $G(\mathbb{F}_q)$, where $\mathbb{F}_q$ denotes a finite field with q elements.

Consider the flag manifold Fl_G, which parametrizes the Borel subgroups of G. This is a projective scheme over the integers. The natural action of G on Fl_G by conjugation of Borel subgroups yields on $\mathbb{F}_q$-points an action

$$G(\mathbb{F}_q) \times \mathrm{Fl}_G(\mathbb{F}_q) \to \mathrm{Fl}_G(\mathbb{F}_q).$$

In particular, as in Section 2.3.2 we obtain the unipotent principal series module

$$\mathrm{Fun}(G(\mathbb{F}_q)) \otimes \mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q)) \to \mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q)).$$

Moreover, its algebra of endomorphisms, the *Hecke algebra*, is canonically identified as in Section 2.3.4 with

$$\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q) \times \mathrm{Fl}_G(\mathbb{F}_q)/G(\mathbb{F}_q)).$$

We emphasize that we are not considering the algebras of polynomial functions on the corresponding $\mathbb{F}_q$-varieties, but simply the algebra of k-valued functions on their finite sets of rational points.

2.3.6. Slightly more explicitly, recall that Fl_G parametrizes Borel subgroups of G. These subgroups are all conjugate and self-normalizing in G. So, if one fixes a Borel B, one obtains an identification of the flag manifold with the homogeneous space G/B, and moreover on $\mathbb{F}_q$-points

$$\mathrm{Fl}_G(\mathbb{F}_q) \simeq G(\mathbb{F}_q)/B(\mathbb{F}_q).$$

Similarly, one obtains an identification of $(\mathrm{Fl}_G \times \mathrm{Fl}_G)/G$ with $B\backslash G/B$ and hence an identification

$$\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q) \times \mathrm{Fl}_G(\mathbb{F}_q)/G(\mathbb{F}_q)) \simeq \mathrm{Fun}(B(\mathbb{F}_q)\backslash G(\mathbb{F}_q)/B(\mathbb{F}_q)).$$

This intertwines the algebra structure on the left with the algebra structure on the right given by convolution.

2.3.7. For concreteness, let us discuss the smallest nontrivial example in more detail.

Example 2.3.8. Suppose that $G = GL(V)$ for a free abelian group V of rank two. In this case, Fl_G is the projective line $\mathbb{P}(V)$, which parametrizes rank one quotients of V with its natural action of $GL(V)$. To see this, pick a basis e_1, e_2 of V, and note that the stabilizer of the projection $V \to V/\langle e_2 \rangle$ is the Borel subgroup of upper triangular matrices

$$B = \begin{pmatrix} * & * \\ & * \end{pmatrix},$$

which yields the desired identification with G/B. In particular, $\mathrm{Fl}_G(\mathbb{F}_q)$ has $q+1$ points.

The module $\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q))$ decomposes into two simples as follows. The projection $\pi : \mathrm{Fl}_G \to \mathrm{pt}$ yields a (split) short exact sequence of $G(\mathbb{F}_q)$-modules

$$0 \to \mathrm{St} \to \mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q)) \to \mathrm{Fun}(\mathrm{pt}(\mathbb{F}_q)) \simeq k \to 0,$$

and the kernel St, the Steinberg module, is simple e.g., because the Hecke algebra is two dimensional.

2.3.9. The contents of this section may be summarized as follows. First, we reviewed the construction of the group algebra of a finite group as a convolution algebra. Second, for the finite group $G(\mathbb{F}_q)$, we reviewed its natural action on functions on the flag manifold $\mathrm{Fl}_G(\mathbb{F}_q)$.

3. Beilinson–Bernstein localization and hidden symmetries

3.1. **Overview.**

3.1.1. We considered in the previous section the vector space of functions on the $\mathbb{F}_q$-points of the flag manifold

$$\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q)), \tag{3.1}$$

which naturally carried an action of $G(\mathbb{F}_q)$.

In this section, we will instead be interested in the derived category of D-modules on the complex flag manifold

$$\text{D-mod}(\mathrm{Fl}_G(\mathbb{C})). \tag{3.2}$$

We will begin by recalling the fundamental connection of this category (3.2) to the representation theory of Lie algebras following Beilinson–Bernstein. In reviewing several important properties of this correspondence, we will be led to the analogue of the action of $G(\mathbb{F}_q)$ on $\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q))$. In this way we will meet the idea of categorical actions of groups.

3.2. **The Localization Theorem.**

3.2.1. As the reader may have gleaned from our discussion in Section 3.1, one thinks of the category of D-modules (3.2) as a sort of categorification of (3.1). This analogy is absolutely fundamental in geometric representation theory. For this reason, we have provided a fairly detailed explanation of it, as well as an introduction to D-modules, in Appendix A. The nonspecialist reader may wish to turn there now and return to this section after picking up the basics.

3.2.2. In the remainder of this section, we work over an algebraically closed field k of characteristic zero. For ease of notation, we redefine Fl_G to be the corresponding base change. That is, if we write $\mathrm{Fl}_{G,\mathbb{Z}}$ for the version over Spec $\mathbb{Z}$ discussed in Section 2, we set

$$\mathrm{Fl}_G := \mathrm{Fl}_{G,\mathbb{Z}} \underset{\mathrm{Spec}\,\mathbb{Z}}{\times} \mathrm{Spec}\ k.$$

Of course, the reader may take k to be the complex numbers but should bear in mind that we do not need anything particular about their metric topology.

3.2.3. It turns out that the category of D-modules on the flag variety

$$\text{D-mod}(\mathrm{Fl}_G)$$

has a seemingly different realization, namely as a category of Lie algebra representations. Let us review this remarkable correspondence, which birthed geometric representation theory.

3.2.4. Recall that a D-module $\mathcal{M}$ on a smooth algebraic variety X is a quasicoherent sheaf equipped with a flat connection. Equivalently, $\mathcal{M}$ is equipped with an action of the sheaf of differential operators $\mathcal{D}_X$ extending the action of regular functions $\mathcal{O}_X$. In particular, its global sections will carry an action by global differential operators, i.e., we have a tautological functor

$$\Gamma(X,-) : \text{D-mod}(X) \to \Gamma(X, \mathcal{D}_X)\text{-mod}. \tag{3.3}$$

It is a remarkable theorem of Beilinson–Bernstein that, for $X = \mathrm{Fl}_G$, this functor is an equivalence.

3.2.5. Let us explain why this is surprising.

First, if X is affine, the analogous assertion is true but unsurprising. Indeed, in this case it follows from the fact that the definition of quasicoherent sheaves is rigged so that on an affine variety the functor

$$\Gamma(X,-) : \mathrm{QCoh}(X) \to \Gamma(X, \mathcal{O}_X)\text{-mod} \tag{3.4}$$

is an equivalence.

By constrast, if X is not affine, e.g., projective, the functor (3.4) is hardly ever an equivalence. For example, for Fl_G its only global (derived) functions are scalars, and so (3.4) reduces to the map

$$\Gamma(\mathrm{Fl}_G, -) : \mathrm{QCoh}(\mathrm{Fl}_G) \to \mathrm{Vect},$$

where Vect denotes the derived category of vector spaces. This is far from an equivalence, since Fl_G is not a point! For this reason, it is non-trivial that, upon further equipping quasicoherent sheaves with flat connections, (3.3) is an equivalence for the flag manifold.

We also emphasize that, for a general projective variety, (3.3) is not an equivalence. For example, the reader familiar with algebraic curves may wish to check that the only smooth projective curve for which (3.3) is an equivalence is $\mathbb{P}^1$, i.e., the smallest flag manifold.

3.2.6. Let us turn next to the description of global differential operators

$$\Gamma(\mathrm{Fl}_G, \mathcal{D}_{\mathrm{Fl}_G}).$$

Recall that for any smooth variety X, $\mathcal{D}_X$ is generated as a sheaf of algebras by the functions $\mathcal{O}_X$ and vector fields $\mathcal{T}_X$. Let us see what they contribute to its global sections in the present case of $X = \mathrm{Fl}_G$.

As we mentioned above, the global functions on the flag manifold are simply scalars

$$k \xrightarrow{\sim} \Gamma(\mathrm{Fl}_G, \mathcal{O}_{\mathrm{Fl}_G}).$$

For the vector fields, the natural action of G on Fl_G yields the infinitesimal action of its Lie algebra $\mathfrak{g}$, i.e., a homomorphism of Lie algebras

$$\mathfrak{g} \to \Gamma(\mathrm{Fl}_G, \mathcal{T}_{\mathrm{Fl}_G}). \tag{3.5}$$

To approximate the final answer, we might optimistically hope that this is an isomorphism.

To account for higher order differential operators, we could similarly ask that the resulting map from the enveloping algebra

$$U(\mathfrak{g}) \to \Gamma(\mathrm{Fl}_G, \mathcal{D}_{\mathrm{Fl}_G}) \tag{3.6}$$

be an isomorphism. This would be a strong global analogue of the local generation of sections of $\mathcal{D}_{\mathrm{Fl}_G}$ by functions and vector fields.

3.2.7. As it turns out, both (3.5) and (3.6) are not quite isomorphisms. Instead, they must be corrected by considering the center.

Let us begin with the case of vector fields. Note that the center Z of G acts trivially on Fl_G. So, we need to quotient by the center $\mathfrak{z}$ of $\mathfrak{g}$. After doing so, we do obtain an equivalence

$$\mathfrak{g}/\mathfrak{z} \xrightarrow{\sim} \Gamma(\mathrm{Fl}_G, \mathcal{T}).$$

Similarly, when we pass to all differential operators, we must impose a central quotient. Namely, if we write $Z(\mathfrak{g})$ for the center of $U(\mathfrak{g})$, acting on the trivial representation of $\mathfrak{g}$ yields a character $\chi : Z(\mathfrak{g}) \to k$. Let us denote by $U_0(\mathfrak{g})$ the corresponding central quotient of the entire enveloping algebra

$$U_0(\mathfrak{g}) := U(\mathfrak{g}) \underset{Z(\mathfrak{g})}{\otimes} k_\chi.$$

With this, as in the case of vector fields, (3.6) factors through the resulting quotient, and we obtain an equivalence

$$U_0(\mathfrak{g}) \xrightarrow{\sim} \Gamma(\mathrm{Fl}_G, \mathcal{D}).$$

Example 3.2.8. Let us explicitly describe what $U_0(\mathfrak{g})$ looks like in the case of $\mathfrak{g} = \mathfrak{gl}_2$. We may decompose the Lie algebra as its center and the traceless matrices

$$\mathfrak{gl}_2 = \mathfrak{z} \oplus \mathfrak{sl}_2.$$

This induces decompositions of the enveloping algebra, and hence its center, as

$$U(\mathfrak{gl}_2) \simeq \mathrm{Sym}(\mathfrak{z}) \otimes U(\mathfrak{sl}_2) \quad \text{and} \quad Z(\mathfrak{gl}_2) \simeq \mathrm{Sym}(\mathfrak{z}) \otimes Z(\mathfrak{sl}_2).$$

The center $Z(\mathfrak{sl}_2)$ is a polynomial ring in one variable, generated by the Casimir element Ω of filtered degree two. Explicitly, if we pick a basis element z for $\mathfrak{z}$ and standard generators f, h, e for $\mathfrak{sl}_2$, we have that

$$Z(\mathfrak{gl}_2) \simeq k[z, ef + fe + \frac{1}{2}h^2],$$

where the second generator is Ω. It is straightforward to see both generators act by zero on the trivial representation. In particular, $U_0(\mathfrak{g})$ is the quotient of $U(\mathfrak{gl}_2)$ by the ideal generated by those two elements, i.e.,

$$U_0(\mathfrak{gl}_2) = U(\mathfrak{gl}_2)/(\mathfrak{z}, \Omega) \simeq U(\mathfrak{sl}_2)/(\Omega).$$

3.2.9. Let us denote the derived category of $U_0(\mathfrak{g})$-modules, i.e., representations of $\mathfrak{g}$ with trivial central character, by $\mathfrak{g}\text{-mod}_0$. Summarizing the previous discussion, the theorem of Beilinson–Bernstein reads as follows.

Theorem 3.2.10. (Beilinson–Bernstein [BB81]) The functor of global sections yields a t-exact equivalence

$$\Gamma(\mathrm{Fl}_G, -) : \text{D-mod}(\mathrm{Fl}_G) \to \mathfrak{g}\text{-mod}_0 .$$

It is hard to overstate the significance of this theorem for the development of representation theory. We confine ourselves here to a few remarks.

3.2.11. First, by a version of Schur's lemma due to Quillen, every simple $\mathfrak{g}$-module has a central character, i.e., the center $Z(\mathfrak{g})$ acts by scalars. Therefore, for the study of irreducible modules, working one central character at a time is sufficient.

The above theorem identifies those with the trivial central character and simple D-modules on Fl_G. This allows one to translate problems in the study of such irreducible modules into geometric or topological problems. The latter problems may be approached using the wealth of information known about the topology of algebraic varieties, notably Hodge theory and its singular variants.

The original motivation for the theorem was such an application. Namely, within the category of all $\mathfrak{g}$-modules, to any Borel subgroup B of G one associates the category of $(\mathfrak{g}, B)$-modules, i.e., $\mathfrak{g}$-modules for which the action of the Lie algebra of B is integrable.

A problem which received significant study, beginning with Verma's thesis in 1966 [Ver66], was the determination of the characters of the simple $(\mathfrak{g}, B)$-modules. After their determination in several low rank cases by Jantzen [Jan79] and others, Kazhdan and Lusztig in 1979 formulated a general conjecture for the simple $(\mathfrak{g}, B)$-modules with trivial central character [KL79]. At the time of their conjecture, they knew their formula was intimately related to the Schubert subvarieties of Fl_G, or more precisely with the !-stalks of their intersection cohomology D-modules. However, the connection with $\mathfrak{g}$-modules was not clear.

The work of Beilinson–Bernstein [BB83], and independently Brylinski–Kashiwara [BK81], provided the desired link. Namely, the localization theorem exchanged the simple objects of $(\mathfrak{g}, B)\text{-mod}_0$ with the aforementioned intersection cohomology sheaves, and the character formula followed.

Remark 3.2.12. We have discussed above only the case of the trivial central character. Beilinson–Bernstein in fact proved a similar statement for any regular central character. For variants, including generalized or singular central characters, see [Kas89], [BG99], [BMR06], [BK15], [CD21].

3.2.13. We would like to next describe the behavior of the localization equivalence in three basic examples.

Example 3.2.14. As we described in Section 3.2.7, localization interchanges the algebra of differential operators and the central quotient of the enveloping algebra, i.e.,

$$\Gamma(\mathrm{Fl}_G, \mathcal{D}) \simeq U_0(\mathfrak{g}).$$

Example 3.2.15. As we also discussed, localization interchanges the structure sheaf and the trivial representation, i.e.,

$$\Gamma(\mathrm{Fl}_G, \mathcal{O}) \simeq k.$$

As the reader may be aware, this is the simplest case of the Borel–Weil–Bott theorem, which constructs all the irreducible G-modules as global sections of line bundles on Fl_G. Informally, the work of Beilinson–Bernstein extends this to give a similar construction of all irreducible $\mathfrak{g}$-modules.

Example 3.2.16. Finally, let us consider, for a point x of Fl_G, the image of δ_x. Recall that the stabilizer of x is a Borel subgroup B. With this, writing $\mathfrak{b}$ for the Lie algebra of B, we have that the delta D-module is sent to the corresponding induced module

$$\Gamma(\mathrm{Fl}_G, \delta_x) \simeq U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \det(\mathfrak{g}/\mathfrak{b}). \tag{3.7}$$

Let us sketch a proof of this, which the reader may enjoy thinking through fully. To produce a map

$$U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} k_\chi \to \Gamma(\mathrm{Fl}_G, \delta_x), \tag{3.8}$$

we note that the identification of the generating line of the right hand side with $\det(T_x \mathrm{Fl}_G) = \det(\mathfrak{g}/\mathfrak{b})$ is compatible with the $\mathfrak{b}$-actions, cf. Remark A.2.6. To see the obtained map is an isomorphism, one notes that both sides are freely generated from their determinant lines by the action of any subalgebra $\mathfrak{n}^-$ transverse to $\mathfrak{b}$, i.e., the unipotent radical of an opposite Borel.

3.3. **Equivariance and Harish-Chandra modules.**

3.3.1. Experts immediately understood that the localization theorem carried far more structure than simply an equivalence of categories.

For concrete representation-theoretic applications, the main compatibility was as follows. Fix an algebraic subgroup $H \subset G$, which for simplicity we assume to be connected.

3.3.2. Let us begin with the Lie theoretic side. Associated to H is the full abelian subcategory of Harish-Chandra modules

$$(\mathfrak{g}, H)\text{-mod}^\heartsuit \subset \mathfrak{g}\text{-mod}^\heartsuit.$$

Explicitly, a $\mathfrak{g}$-module is a Harish-Chandra module if the action of the Lie algebra $\mathfrak{h} \subset \mathfrak{g}$ integrates to an action of H. By our assumption that H is connected, this is a property and not a further structure.

Restricting to modules with trivial central character, we obtain a similar full category

$$(\mathfrak{g}, H)\text{-mod}_0^{\heartsuit} \subset \mathfrak{g}\text{-mod}_0^{\heartsuit}.$$

3.3.3. Here are some quick examples.

Example 3.3.4. If $\mathfrak{g}$ is semisimple, the central quotient $U_0(\mathfrak{g})$ is only a Harish-Chandra module for the trivial subgroup .

Example 3.3.5. The trivial module k is a Harish-Chandra module for G itself and in particular for any subgroup.

Example 3.3.6. The Verma module $U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \det(\mathfrak{g}/\mathfrak{b})$ is a $(\mathfrak{g}, B)$-module. More generally, for any representation W of H, the induced module

$$U(\mathfrak{g}) \underset{U(\mathfrak{h})}{\otimes} W$$

is a $(\mathfrak{g}, H)$-module, and any $(\mathfrak{g}, H)$-module is a quotient of such a module.

3.3.7. Let us turn to the geometric side. Associated to H is the full abelian subcategory of H-equivariant D-modules

$$\text{D-mod}(\text{Fl}_G)^{H,\heartsuit} \subset \text{D-mod}(\text{Fl}_G)^{\heartsuit}.$$

Let us review how this is defined in the generality of any algebraic variety X equipped with an action of H.

Remark 3.3.8. For us, varieties are by definition quasicompact, quasiseparated, reduced schemes of finite type over k. Moreover, everything we state in this survey for varieties applies equally well to quasicompact, quasiseparated schemes of finite type over k, i.e., the reducedness plays no role. The reader will lose little by ignoring these technicalities and sticking with e.g. quasi-projective varieties if they are more comfortable doing so.

At first pass, for a D-module M to be H-equivariant roughly means that it is constant along H-orbits within X. To arrive at the actual definition, note that we should therefore have, for any point h of H and x of X, the existence of an isomorphism of stalks

$$i_x^! M \simeq i_{hx}^! M. \tag{3.9}$$

Of course, since our spaces are not discrete, we need to put such statements into a family as we vary h and x, compatibly with parallel transport. To do so, consider the action and projection maps

$$\alpha, \pi : H \times X \to X, \qquad \alpha(h, x) = hx, \quad \pi(h, x) = x. \tag{3.10}$$

With this, an object M of $\text{D-mod}(X)^{\heartsuit}$ is H-equivariant if there exists an isomorphism

$$\alpha^! M \simeq \pi^! M. \tag{3.11}$$

Since we only ask for the existence of an isomorphism, it is clear this is a property of a D-module and not further structure.

It is instructive to compare this to the case of functions. Namely, fix a finite group acting on a finite set, which we suggestively denote by $H(\mathbb{F}_q)$ and $X(\mathbb{F}_q)$, respectively. Note that a function m in $\text{Fun}(X(\mathbb{F}_q))$ is $H(\mathbb{F}_q)$-invariant, i.e., constant along orbits, if and only if it satisfies the equality

$$\alpha^! m = \pi^! m,$$

where $\alpha, \pi : H(\mathbb{F}_q) \times X(\mathbb{F}_q) \to X(\mathbb{F}_q)$ are defined as above, and the !-pullback of functions is as in Section 2.2.4.

3.3.9. Here is a pair of basic but important examples of equivariant D-modules.

Example 3.3.10. Suppose X consists of a single H-orbit, i.e., after choosing a base point, is of the form H/K for some subgroup K of H.

In this case, we claim there is a canonical equivalence between H-equivariant D-modules on X and representations of the component group of K, i.e.,

$$\text{D-mod}(X)^{H,\heartsuit} \simeq \operatorname{Rep}(\pi_0(K))^{\heartsuit}.$$

Indeed, it is straightforward to see that any equivariant D-module on X must be a local system, and the condition (3.11) is equivalent to the pullback along

$$H \to H/K$$

being trivializable. On the other hand, the long exact sequence on homotopy groups associated to the fibration $K \to H \to H/K$ gives in low degrees an exact sequence

$$\pi_1(H) \to \pi_1(H/K) \to \pi_0(K) \to 1,$$

so the claim follows.

Example 3.3.11. Suppose X consists of finitely many H-orbits. In this case, the category of H-equivariant D-modules on X is 'glued' from the individual orbits using recollement.

The details of this are not so important for us. However, let us mention the following basic consequence: the irreducible H-equivariant D-modules on X are in canonical bijection with the irreducible H-equivariant local systems on the individual orbits. Concretely, each irreducible object of

$$\text{D-mod}(X)^{H,\heartsuit}$$

is supported on the closure of a single orbit, and its restriction to the orbit is an irreducible H-equivariant local system. In this way, the classification of simples reduces, by the previous example, to an exercise in finite group theory.

3.3.12. Having introduced the two sides, we may now state the promised compatibility: localization exchanges Harish-Chandra modules and equivariant D-modules.

Theorem 3.3.13 (Beilinson–Bernstein, [BB81]). There is a commutative diagram

$$\begin{tikzcd} \text{D-mod}(\mathrm{Fl}_G)^{H,\heartsuit} \arrow[r, "\sim"] \arrow[d] & (\mathfrak{g}, H)\text{-mod}_0^{\heartsuit} \arrow[d] \\ \text{D-mod}(\mathrm{Fl}_G)^{\heartsuit} \arrow[r, "\sim"] & \mathfrak{g}\text{-mod}_0^{\heartsuit}, \end{tikzcd}$$

where the vertical arrows are the tautological inclusions and the bottom horizontal arrow is localization.

3.3.14. In fact, Theorem 3.3.13 holds, mutatis mutandis, for any H, not necessarily connected and also for the appropriately defined derived categories of equivariant D-modules and Harish-Chandra modules. The latter assertion does not follow from the former, since these typically are not the derived categories of their hearts, but rather something slightly more sophisticated.

Remark 3.3.15. The reader is encouraged to view the aforementioned subtlety about derived categories as a statement of the form 'to get the correct answers, one has to be a bit clever.' However, it does not take much to see why working naively with the derived category of the abelian category of equivariant sheaves can give the wrong answers. The orienting principle here is that H-equivariant D-modules on X should be the same as D-modules on the quotient stack X/H, cf. Remark 3.3.16 below, and should therefore carry the expected relation with its topology. In the simplest case of $X = \mathrm{pt}$, we should in particular want

$$\operatorname{Hom}_{\text{D-mod}(\mathrm{pt})^H}(k,k) \simeq \mathrm{C}^*(\mathrm{pt}/H, k),$$

i.e., the derived endomorphisms of the constant sheaf should be the cohomology of the classifying space. But since on abelian categories

$$\text{D-mod}(\mathrm{pt})^{H,\heartsuit} \simeq \mathrm{Rep}(\pi_0(H)),$$

we simply cannot get *any* higher Exts from the abelian category alone. We will meet the correction, due to Bernstein–Lunts, in Section 4.

Remark 3.3.16. For the reader unfamiliar with stacks, the following coarse description suffices for our purposes. An Artin stack Y over an algebraically closed field k is, roughly speaking, something like a variety, but with the further property that each k-point y of Y carries a group of 'internal symmetries.'

The most relevant class of Artin stacks for us arise as follows. Given a variety X acted on by an algebraic group H, one can form the quotient stack X/H. The points of X/H are given by H-orbits on X, and for a k-point x of X, the internal symmetry group of the corresponding point

$$H \cdot x \text{ of } X/H$$

is simply the stabilizer H_x of x in H. That is, this k-point of the stack is not the variety pt, i.e., Spec k, but rather its quotient pt/H_x. In general, a k-point y of an Artin stack is the classifying space pt/H_y of an algebraic group H_y, which is the internal symmetry group alluded to above.

As a simple case, consider $X = \mathbb{A}^1$ with the action of $H = \mathbb{Z}/2$ induced by the involution

$$z \mapsto -z.$$

If we pluck out the origin, this restricts to an action of H on $\mathbb{G}_m$. Here, every H orbit is free and, in particular, points of the quotient should have trivial internal symmetries, like an ordinary variety. In fact, the quotient $\mathbb{G}_m/H$ is simply the variety $\mathbb{G}_m$, where the quotient map $\mathbb{G}_m \to \mathbb{G}_m/H$ identifies with the squaring map

$$\mathbb{G}_m \to \mathbb{G}_m, \quad w \mapsto w^2.$$

However, the full quotient $\mathbb{A}^1/H$ is not a variety, as the point corresponding to the origin has been 'orbifolded' into a copy of pt/H. Intuitively, for a general quotient stack, the internal symmetry groups remember the folding that takes place when forming the orbit space X/H from X.

Finally, let us mention that for sheaves on stacks, the basic new feature compared to the case of varieties is that their fibres carry actions by the internal symmetry groups. Namely, for an algebraic group H, we have

$$\mathrm{QCoh}(\mathrm{pt}/H) \simeq \mathrm{Rep}(H) \quad \text{and} \quad \text{D-mod}(\mathrm{pt}/H) \simeq \mathrm{C}_*(H)\text{-mod},$$

where $\mathrm{C}_*(H)$ denotes the dual of the de Rham cohomology of H, and is viewed as an dg-algebra under convolution.

Therefore, for a coherent sheaf on an Artin stack Y, its $*$-fibre at a k-point y carries an action of H_y. Similarly, for a D-module on Y, its !-fibre at y carries an action of $\mathrm{C}_*(H_y)$, which one thinks of as a locally constant action of H_y. In particular, the cohomology groups of the !-fibre carry actions of the component group $\pi_0(H_y)$.

3.3.17. Let us give a basic example of how Theorem 3.3.13 is used.

It follows from 3.3.13 that there exist only finitely many irreducible objects in $(\mathfrak{g}, H)\text{-mod}_0$ if and only if there are finitely many H-orbits on Fl_G. Moreover, in this case, one can explicitly enumerate the simple objects as in Example 3.3.11.

As a concrete example, write $T \subset GL_2$ for a maximal torus, and let us classify the simple objects of

$$(\mathfrak{gl}_2, T)\text{-mod}_0^\heartsuit.$$

To do so, we note that T has three orbits on $\mathbb{P}^1$ – the north pole B_0, the south pole B_∞, and everything else. As T acts with connected stabilizers on all three orbits, it follows that one has three irreducible objects.

Concretely, these are the the simple Verma module of highest weight -2, the simple Verma module of lowest weight 2, and the trivial module, i.e.,

$$U(\mathfrak{gl}_2) \underset{U(\mathfrak{b}_0)}{\otimes} \det(\mathfrak{gl}_2/\mathfrak{b}_0), \quad U(\mathfrak{gl}_2) \underset{U(\mathfrak{b}_\infty)}{\otimes} \det(\mathfrak{gl}_2/\mathfrak{b}_\infty), \quad \text{and} \quad k.$$

Note also that, by similar reasoning, there are infinitely many simple irreducible objects in $(\mathfrak{gl}_n, T)\text{-mod}_0^\heartsuit$ for $n > 2$.

While this discussion was rather soft, we emphasize that subtler questions, such as the determination of simple characters, may be approached geometrically and typically translate to questions regarding the topology of orbit closures.

3.3.18. Finally, let us make a couple orienting remarks about why Theorem 3.3.13 holds. It is clarifying to work in the following generality.

Suppose K is an algebraic group acting on a smooth variety X. As before, if we write $\mathfrak{k}$ for the Lie algebra of K, one has a global sections functor

$$\Gamma(X, -) : \text{D-mod}(X)^\heartsuit \to \mathfrak{k}\text{-mod}^\heartsuit.$$

This admits a left adjoint, given by localization

$$\text{Loc} : \mathfrak{k}\text{-mod}^\heartsuit \to \text{D-mod}(X)^\heartsuit, \qquad M \mapsto \mathcal{D}_X \underset{U(\mathfrak{k})}{\otimes} M.$$

We claim that, for any algebraic subgroup $H \subset K$, this restricts to an adjunction

$$\text{Loc} : (\mathfrak{k}, H)\text{-mod}^\heartsuit \leftrightarrows \text{D-mod}(X)^{H,\heartsuit} : \Gamma(X, -). \tag{3.12}$$

In particular, specializing to $K = G$ and $X = \text{Fl}_G$, this recovers Theorem 3.3.13.

3.3.19. To see why (3.12) holds, we will need to first revisit the definition of an equivariant D-module. Let us follow the notation of Section 3.3.7. For a general group H, not necessarily connected, as part of the data in the definition of an equivariant D-module one asks for a distinguished isomorphism

$$\tau : \alpha^! M \simeq \pi^! M,$$

which is unital and satisfies a cocycle, i.e., associativity, condition on $H \times H \times X$. At the level of stalks, the cocycle condition asks that the two ways to identify

$$i^!_{h_1 h_2 x} M \simeq i^!_x M$$

by applying τ either once to the pair $(h_1 h_2, x)$, or twice to the pair (h_2, x) and then $(h_1, h_2 x)$, coincide. It is a nice fact that, if H is connected, such a datum is unique if it exists, and its existence is equivalent to the a priori weaker statement (3.11).

Remark 3.3.20. For a disconnected group, equivariance truly is a further structure one equips a D-module with. For instance, one can check that if $X = \text{pt}$, then a datum of H-equivariance on an object W of

$$\text{D-mod}(\text{pt})^\heartsuit \simeq \text{Vect}^\heartsuit$$

is the same as a representation of the component group $\pi_0(H)$ on W.

The reader may recognize such representations from Example 3.3.10. This is not a coincidence, and indeed we have

$$\text{D-mod}(H/K)^{H,\heartsuit} \simeq \text{D-mod}(H)^{H \times K,\heartsuit} \simeq \text{D-mod}(\text{pt})^{K,\heartsuit}.$$

In terms of D-modules on stacks, this follows from the identity

$$H \backslash H / K \simeq \text{pt} / K.$$

3.3.21. The rigidified definition of equivariance in Section 3.3.19 may remind the reader of the definition of an algebraic H-module, i.e., an $\mathcal{O}_H$-comodule. This is a vector space W equipped with a coaction map

$$W \to \mathcal{O}_H \otimes W,$$

which satisfies unitality and cocycle conditions.

Remark 3.3.22. In case the reader has not encountered this definition before, in the present setting it reduces to the following, perhaps more familiar description. Namely, W carries an action of the abstract group $H(k)$, is a union of finite dimensional representations M_α, and, with respect to any basis of such a subrepresentation M_α, the matrix coefficients of the action of $H(k)$ are regular functions on H.

Thinking of W as a quasicoherent sheaf on pt, this may be rephrased as the datum of an H-equivariant quasicoherent sheaf on pt, i.e., descent along pt $\to$ pt/H.

With this in mind, here is a reformulation of the data of H-equivariance for a D-module on an H-variety X. First suppose for simplicity that X is affine, and for a D-module M let us denote by the same letter the corresponding D_X-module, i.e., its global sections.

With this, a datum of H-equivariance on M is the same as giving an algebraic action of H on the underlying vector space of M, (i) which is compatible with the action of H on D_X (i.e., one upgrades from a D_X-module to a $H \ltimes D_X$-module), and (ii) such that the induced infinitesimal action of $\mathfrak{h}$ on M agrees with the action by vector fields, i.e., with the composition

$$\mathfrak{h} \to T_X \to D_X. \tag{3.13}$$

I.e., one specifies an integration of the action of $\mathfrak{h}$ from the connection on M to an action of the group H.

For general X, not necessarily affine, one can given a similar definition. Explicitly, equivariance data for a D-module is (i) equivariance data for the underlying quasicoherent sheaf that is compatible with the canonical equivariance data on the quasicoherent sheaf of algebras $\mathcal{D}_X$, which satisfies the analogue of condition (ii) above.

3.3.23. From this formulation, it is clear that global sections restricts to a functor

$$\text{D-mod}(X)^{H,\heartsuit} \to (\mathfrak{k}, H)\text{-mod}^\heartsuit.$$

Let us also sketch why localization restricts to a functor

$$(\mathfrak{k}, H)\text{-mod}^\heartsuit \to \text{D-mod}(X)^{H,\heartsuit}.$$

The starting point here is that $\mathcal{D}_X$ is naturally a H-equivariant quasicoherent sheaf of algebras – we are simply saying functions and vector fields may be moved along the H action. However, $\mathcal{D}_X$ is *not* an H-equivariant D-module. The problem stems from condition (ii), namely the coincidence of the two infinitesimal actions of $\mathfrak{h}$.

To see this, note that the action of $\mathfrak{h}$ on $\mathcal{D}_X$ induced by its left action on itself takes the form $h \otimes P \mapsto h \cdot P$, where h is an element of $\mathfrak{h}$ and P is a local section of $\mathcal{D}_X$. On the other hand, the induced infinitesimal action of $\mathfrak{h}$ on $\mathcal{D}_X$ obtained by H-equivariance must be by derivations of the algebra, so it cannot agree with the left action. Accordingly, it is instead the adjoint action, i.e., takes the form

$$h \otimes P \mapsto h \cdot P - P \cdot h.$$

As the issue lies in the appearance of right multiplication, it should not surprise the reader that tensoring on the right with an H-integrable module should help. Indeed, given a $(\mathfrak{k}, H)$-module M, it is straightforward to see that the underlying quasicoherent sheaf of

$$\text{Loc}(M) = \mathcal{D}_X \underset{U(\mathfrak{k})}{\otimes} M$$

inherits a datum of H-equivariance from the action of H on its tensor factors. For condition (ii), we may note that the derivative of this action takes the form

$$h \otimes (P \otimes m) \mapsto h \cdot P \otimes m - P \cdot h \otimes m + P \otimes h \cdot m,$$

where h and P are as before and m is an element of M. But now the second and third factors cancel, since we tensored over $U(\mathfrak{k})$, and this leaves only the desired first term.

3.4. **Hidden Symmetries.**

3.4.1. In the previous subsection, we saw that localization satisfied an important compatibility – it interchanged equivariant D-modules and Harish-Chandra modules. After some unwinding, the assertion boiled down to a rather concrete and elementary observation about the form of the functors.

However, it is important to recognize that such a statement, especially at the time of its introduction, did not fit easily in a known general formalism in representation theory.

We now would like to explain that this compatibility is in fact a shadow of a richer structure present on both sides of localization.

On the geometric side, the further symmetry will be, if not obvious, not wholly surprising. However, on the representation theoretic side, it is far less obvious. It is reasonable to think of it as a hidden symmetry of the category of Lie algebra representations. As we will explain, however, pieces of it were witnessed and used to great effect on a case by case basis, but the full structure did not emerge for several decades.

3.4.2. Our starting point will be the analogy between

$$\text{D-mod}(\mathrm{Fl}_G) \quad \text{and} \quad \mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q)).$$

Recall the latter carried an action of $G(\mathbb{F}_q)$, which was a linearization of the action of $G(\mathbb{F}_q)$ on $\mathrm{Fl}_G(\mathbb{F}_q)$ itself.

This suggests the following idea – we should view D-mod(Fl_G) as some sort of representation of the group G, again linearizing the action of G on Fl_G. Crucially, the localization equivalence

$$\text{D-mod}(\mathrm{Fl}_G) \simeq \mathfrak{g}\text{-mod}_0$$

should be one of G-modules, and in particular the category of Lie algebra representations should again be a G-module. To make this precise, one needs a good notion of an action of an algebraic group not on a vector space, but on a category.

3.4.3. Let us run a plausibility check on this idea in the somewhat more general setting of K and X as in Section 3.3.18.

Recall the localization and global sections functors

$$\mathrm{Loc} : \mathfrak{k}\text{-mod} \rightleftarrows \text{D-mod}(X) : \Gamma(X, -).$$

Whatever an action of K on a category $\mathcal{C}$ should be, certainly for each element g of K we should have an automorphism of the category

$$g : \mathcal{C} \xrightarrow{\sim} \mathcal{C},$$

and these should be suitably compatible under composition. Let us now find such automorphisms of $\mathfrak{k}$-mod and D-mod(X) in such a way that they are intertwined by localization.

We begin with the geometric side. Since X is a K-variety, acting by g gives an automorphism of the variety itself

$$g : X \xrightarrow{\sim} X$$

and in particular of its category of D-modules

$$g_* : \text{D-mod}(X) \xrightarrow{\sim} \text{D-mod}(X).$$

For example, for any point x of X, this sends δ_x to δ_{gx}. Note that there are, for any g and g' in K, canonical isomorphisms

$$g_* \circ g'_* \simeq (gg')_*,$$

i.e., one has a compatibility between multiplication in the group and composition of endofunctors.

To get a hint of what the corresponding operators on $\mathfrak{k}$-mod should be, let us see what this does to global sections. Note that, for any D-module M on X, the underlying vector spaces of

$$\Gamma(X, M) \quad \text{and} \quad \Gamma(X, g_* M)$$

are canonically identified, but the action of global differential operators is twisted by g.

In view of this, let us denote by

$$\mathrm{Ad}_{g,*} : \mathfrak{k}\text{-mod} \xrightarrow{\sim} \mathfrak{k}\text{-mod}$$

the automorphism which sends a $\mathfrak{k}$-module N with action

$$\mathfrak{k} \otimes N \to N, \quad X \otimes n \mapsto X \cdot n$$

to the $\mathfrak{k}$-module with the same underlying vector space and action $X \otimes n \mapsto \mathrm{Ad}_{g^{-1}}(X) \cdot n$. There again is a compatibility

$$\mathrm{Ad}_{g,*} \circ \mathrm{Ad}_{g',*} \simeq \mathrm{Ad}_{gg',*}\,.$$

3.4.4. By the preceding discussion, we obtain, for every g in K, canonically commuting diagrams

$$\begin{tikzcd} \text{D-mod}(X) \arrow[r, "\Gamma(X,-)"] \arrow[d, "g_*"] & \mathfrak{k}\text{-mod} \arrow[d, "\mathrm{Ad}_{g,*}"] \\ \text{D-mod}(X) \arrow[r, "\Gamma(X,-)"] & \mathfrak{k}\text{-mod}\,. \end{tikzcd}$$

That is, $\Gamma(X, -)$ intertwines the actions of K constructed so far on either category. The reader may wish to check that the same holds for Loc.

3.4.5. Now comes an important point. The amount of structure we have expected so far from a categorical representation of an algebraic group is clearly not enough. For example, we have only used its closed points viewed as an abstract group.

To proceed, we recall that the action of $G(\mathbb{F}_q)$ on $\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q))$ extends to an action of the group algebra $\mathrm{Fun}(G(\mathbb{F}_q))$. Following the functions-sheaves correspondence, we find that the derived category D-mod(K) is monoidal under convolution. Explicitly, given two sheaves $\mathcal{F}$ and $\mathcal{G}$ in D-mod(K), their convolution is given by

$$\mathcal{F} \star \mathcal{G} := m_*\big(p_1^! \mathcal{F} \overset{!}{\otimes} p_2^! \mathcal{G}\big), \tag{3.14}$$

where we denote by the letters $m, p_1, p_2 : K \times K \to K$ the multiplication map and projections onto the first and second factors, respectively, and we use the evident functoriality for morphisms.

We emphasize that formula (3.14) matches the usual multiplication in the group algebra under the functions-sheaves correspondence. Relatedly, the monoidal unit is the delta function δ_e supported at the identity of the group. Somewhat heuristically, one can think of a general D-module as a 'continuous linear combination' of delta D-modules, as in the case of the usual group algebra.

Similarly, the category D-mod(X) is naturally a module category for D-mod(K). Explicitly, given a sheaf $\mathcal{F}$ in D-mod(K) and $\mathcal{M}$ in D-mod(X), their convolution is given by

$$\mathcal{F} \star \mathcal{M} := a_*\big(p_1^! \mathcal{F} \overset{!}{\otimes} p_2^! \mathcal{M}\big), \tag{3.15}$$

where $a, p_1, p_2 : K \times X \to X$ denote the action and projection maps.

By construction, convolution with the delta function supported at a point g of K reproduces the previously discussed symmetry, i.e., one has a canonical isomorphism of functors

$$\delta_g \star (-) \simeq g_* : \text{D-mod}(X) \to \text{D-mod}(X). \tag{3.16}$$

Informally, we have therefore extended the action from individual delta functions to all their linear combinations, i.e., D-mod(K).

Our next task is to find a corresponding action of D-mod(K) on Lie algebra representations. Before doing so, we would like to make some further comments on the D-module side.

Remark 3.4.6. Note that while the discussion of the automorphisms g_* and $\mathrm{Ad}_{g,*}$ worked equally well with abelian or derived categories, for this monoidal structure to behave in the desired way, i.e., to even have a reasonable D-module pushforward, it is essential we work with derived categories. So far, the corresponding triangulated category is enough, but we will be forced to ask for more in the next section.

Note also that to be closed under convolution we immediately must leave the world of finitely generated D-modules. The prototypical example is the convolution of $\mathcal{D}_K$ with itself, namely one can show that

$$\mathcal{D}_K \star \mathcal{D}_K \simeq \mathcal{D}_K \underset{\underline{k}}{\otimes} \mathcal{O}_K.$$

Remark 3.4.7. It is clarifying to rewrite the formulas for convolution and action as

$$\text{D-mod}(K) \otimes \text{D-mod}(K) \simeq \text{D-mod}(K \times K) \xrightarrow{m_*} \text{D-mod}(K) \quad \text{and}$$
$$\text{D-mod}(K) \otimes \text{D-mod}(X) \simeq \text{D-mod}(K \times X) \xrightarrow{a_*} \text{D-mod}(X),$$

exactly as in the function-theoretic case, cf. Equations (2.3) and (2.5). However, to do so we need the tensor product of derived categories, which can only be performed at the level of dg-categories, as discussed in Section 4.

3.4.8. Finally, let us turn to the categorical action of K on $\mathfrak{k}$-mod. To begin, note that the construction of the present action is less immediate than in the case of D-mod(X). Indeed, there we only needed D-modules to act on D-modules, for which we could use geometric functorialities. Here, we need to have D-modules act on Lie algebra representations, which are seemingly more remote from one another.

However, we may proceed by rewriting the category $\mathfrak{k}$-mod as a category of D-modules as follows. Recall that if K acts on a variety X, we saw that the sheaf of differential operators $\mathcal{D}_X$ was not an equivariant D-module. However, it is a prototypical example of a *weakly equivariant* D-modules – the abelian category of such objects, which we denote by

$$\text{D-mod}(X)^{K,w,\heartsuit}$$

is defined exactly as in Section 3.3.21, but without the crucial condition (ii) imposed.

If we take $X = K$, viewed as a K-variety under right multiplication, we claim that one has a canonical equivalence

$$\text{D-mod}(K)^{K,w,\heartsuit} \simeq \mathfrak{k}\text{-mod}^{\heartsuit}. \tag{3.17}$$

Let us sketch a proof for the convenience of the reader. First, one checks that the left hand side has projective generator $\mathcal{D}_K$. This follows from the fact that it corepresents K-invariant global sections, i.e., $\Gamma(K, -)^K$, combined with the tautological isomorphism between K-equivariant quasicoherent sheaves on K and Vect. Second, one concludes by recalling the identification of right invariant differential operators and the enveloping algebra, i.e.,

$$\Gamma(K, \mathcal{D}_K)^K \simeq U(\mathfrak{k}).$$

Summing up, we have found a projective generator with endomorphisms $U(\mathfrak{k})$, so we are done by Morita theory.

The equivalence of Equation (3.17) prolongs to an equivalence of derived categories

$$\text{D-mod}(K)^{K,w} \simeq \mathfrak{k}\text{-mod},$$

and the former carries a D-mod(K) action via left convolution, which completes the construction of the desired action.

3.4.9. With this, the desired upgrade of the localization and global section functors reads as follows.

Theorem 3.4.10 (Beilinson–Drinfeld, [BD]). Let X and K be as in Section 3.3.18. There is a canonical adjunction of D-mod(K) equivariant functors

$$\mathrm{Loc} : \mathfrak{k}\text{-mod} \rightleftarrows \text{D-mod}(X) : \Gamma(X, -).$$

In particular, Beilinson–Bernstein localization, as stated in Theorem 3.2.10, is an equivalence of D-mod(G) representations.[3] We defer an explanation of Theorem 3.4.10, including a discussion of D-mod(K)-equivariance, until Section 4.

3.4.11. Let us indicate the motivating application of this theory. To orient ourselves, consider a subgroup of a finite group, which we suggestively denote by $H(\mathbb{F}_q) \subset K(\mathbb{F}_q)$. Recall that, for any representation π of $K(\mathbb{F}_q)$, its $H(\mathbb{F}_q)$ invariants $\pi^{H(\mathbb{F}_q)}$ carry an action of the convolution algebra

$$\mathrm{Fun}(H(\mathbb{F}_q)\backslash K(\mathbb{F}_q)/H(\mathbb{F}_q)).$$

In a similar way, given an algebraic subgroup H of K, for any categorical representation $\mathcal{C}$ of K one can form its H-invariant objects $\mathcal{C}^H$, and these will carry an action of the convolution category

$$\text{D-mod}(H\backslash K/H).$$

In the current example, we obtain the following upgrade of Theorem 3.3.13 and its derived variants.

Theorem 3.4.12 (Beilinson–Drinfeld [BD]). There is a canonical adjunction of D-mod$(H\backslash K/H)$-equivariant functors

$$\mathrm{Loc} : \mathfrak{k}\text{-mod}^H \leftrightarrows \text{D-mod}(X)^H : \Gamma(X, -).$$

Once the formalism is in place, this is deduced from Theorem 3.4.10 by applying H-invariants. In particular, the general notion of categorical invariants, which we will discuss in Section 4, recovers on the Lie theoretic side the category of Harish-Chandra modules

$$(\mathfrak{k}, H)\text{-mod} \simeq \mathfrak{k}\text{-mod}^H$$

and on the geometric side the category of equivariant D-modules.

3.4.13. Theorem 3.4.12 was proven by Beilinson–Drinfeld in their fundamental work [BD] on quantization of the Hitchin system.[4] In this motivating problem, the compatibility with the action of D-mod$(H\backslash K/H)$ was crucial. Namely, the desired result was the existence of Hecke eigensheaves, certain D-modules with equivariance properties with respect to this action. They were constructed via localization from Lie algebra representations, which were shown directly to have the desired property.

3.4.14. Here is a slightly less sophisticated example. Consider the category of highest weight representations

$$\mathfrak{g}\text{-mod}^B.$$

By the previous formalism, it tautologically carries an action of the Hecke category

$$\text{D-mod}(B\backslash G/B).$$

We stress that, in its historical development, the connection between highest weight modules and D-modules on the flag variety was seen through the localization theorem and, in particular, was highly nonobvious. However, the formalism of categorical actions makes their relation manifest. Moreover, this quickly and conceptually reproduces several important endofunctors of this category.

For example, for each element w of the finite Weyl group, one attaches certain endofunctors

$$j_{w,!} \quad \text{and} \quad j_{w,*} : \mathfrak{g}\text{-mod}^B \to \mathfrak{g}\text{-mod}^B,$$

[3] The careful reader will note that this localization theorem concerned not all of $\mathfrak{g}$-mod but rather $\mathfrak{g}$-mod$_0$. The D-mod(G) action on $\mathfrak{g}$-mod$_0$ formally follows from the appropriate commutation of the $Z(\mathfrak{g})$ and D-mod(G) actions on $\mathfrak{g}$-mod, or can be constructed by hand as in Section 3.4.8.

[4] Strictly speaking, they needed and proved a certain infinite-dimensional analog.

known as the Enright completion and Arkhipov twisting functors, respectively [Enr79], [BB83], [Ark04]. Geometrically, these are given by convolution with the !- and $*$-extensions of the constant D-module on

$$BwB \subset G,$$

and many of their basic properties have simple geometric proofs.

As a related example, modulo nontrivial facts about the intersection cohomology of Schubert varieties, the Kazhdan–Lusztig conjecture follows from the assertion that

$$\mathfrak{g}\text{-mod}_0^B \tag{3.18}$$

is canonically equivalent to $\text{D-mod}(B\backslash G/B)$. This was originally proven by establishing the localization theorem. However, using the action of the Hecke category on (3.18), one can also directly check that convolution with the antidominant Verma module yields an equivalence

$$\text{D-mod}(B\backslash G/B) \simeq \mathfrak{g}\text{-mod}_0^B,$$

from which the fully faithfulness of $\Gamma : \text{D-mod}(\text{Fl}_G) \to \mathfrak{g}\text{-mod}_0$ formally follows, cf. [CD21].

4. Categorical representations of groups

4.1. **Overview.**

4.1.1. In Section 3 we followed the symmetries of Beilinson–Bernstein localization and arrived at the notion of a categorical representation of a group. In this section, we would like to give a more careful introduction to this formalism.

The basic ideas and results here are due to Beilinson–Drinfeld and Frenkel–Gaitsgory. The formalism for discussing the category of all categorical representations and the intertwining functors between them is due largely to Gaitsgory.

This formalism requires some finer points of modern homological algebra to get off the ground, so that is where we begin.

4.2. **From triangulated to dg-categories.**

4.2.1. In Section 3, we treated $\text{D-mod}(K)$ as a monoidal triangulated category, and its modules were triangulated categories. However, to perform many basic manipulations with its modules, the triangulated structure is insufficient.

Here is a basic problem of this form. Given modules $\mathcal{C}_1$ and $\mathcal{C}_2$ for $\text{D-mod}(K)$, one would like to speak about the category of $\text{D-mod}(K)$ equivariant triangulated functors

$$\text{Hom}_{\text{D-mod}(K)\text{-mod}}(\mathcal{C}_1, \mathcal{C}_2).$$

Moreover, we would like these to behave like their function theoretic counterparts. For example, given subgroups $H_1(\mathbb{F}_q)$ and $H_2(\mathbb{F}_q)$ of a finite group $K(\mathbb{F}_q)$, one has a canonical isomorphism

$$\text{Hom}_{K(\mathbb{F}_q)\text{-mod}}\big(\text{Fun}(K(\mathbb{F}_q)/H_1(\mathbb{F}_q)), \text{Fun}(K(\mathbb{F}_q)/H_2(\mathbb{F}_q))\big) \simeq \text{Fun}\big(H_1(\mathbb{F}_q)\backslash K(\mathbb{F}_q)/H_2(\mathbb{F}_q)\big).$$

One would like, for algebraic subgroups H_1 and H_2 of K, a similar isomorphism

$$\text{Hom}_{\text{D-mod}(K)\text{-mod}}\big(\text{D-mod}(K/H_1), \text{D-mod}(K/H_2)\big) \simeq \text{D-mod}(H_1\backslash K/H_2).$$

Unfortunately, the construction of suitable Hom spaces is an issue already if K is the trivial group. In this case we are asking for a reasonable category of triangulated functors from $\mathcal{C}_1$ to $\mathcal{C}_2$. However, this is a well known issue with triangulated categories. For example, given triangulated functors equipped with a morphism $F_1 \to F_2$, it is not clear how to complete it to an exact triangle. Namely, one would like to take cones pointwise, i.e., for each object c of $\mathcal{C}_1$ set

$$\text{Cone}(F_1 \to F_2)(c) \simeq \text{Cone}(F_1(c) \to F_2(c)),$$

but turning this into a functor runs into trouble due to the non-functoriality of cones in $\mathcal{C}_2$.

As a more sophisticated example, one would like to be able to take limits, colimits, and tensor products of D-mod(K) representations, which again are not readily defined when working with triangulated categories.

Happily, all these issues may be addressed by passing from triangulated to dg-categories.

4.2.2. Let us quickly explain the basic idea of dg-categories. Recall that for most triangulated categories appearing in representation theory, such as $\mathfrak{k}$-mod, the homomorphisms between two objects t_1 and t_2 are computed as the zeroth cohomology of a complex of vector spaces. Similarly, the homomorphisms between t_1 and the shift $t_2[i]$, for any integer i, is the i^{th} cohomology of the same complex.

Concretely, for the derived category of an abelian category, one takes either a projective resolution p_1 of t_1 or an injective resolution i_2 of t_2[5] and uses the corresponding complex of homomorphisms

$$\underline{\mathrm{Hom}}(p_1, t_2) \xrightarrow{\sim} \underline{\mathrm{Hom}}(p_1, i_2) \xleftarrow{\sim} \underline{\mathrm{Hom}}(t_1, i_2).$$

We recall that the the above identifications are quasi-isomorphisms, i.e., induce isomorphisms on cohomology, but typically do not identify the entire chain complexes themselves.

The basic idea of dg-categories is to remember not only the cohomology of these complexes but also all the complexes themselves, up to simultaneous quasi-isomorphism. The reader familiar with derived categories should not find this maneuver so surprising. After all, derived categories arise from the usefulness of remembering not only individual derived functors, e.g., Ext^1, but also the complex computing them. Passing from triangulated to dg-categories simply repeats this idea not only for objects, but also for morphisms.

4.2.3. Plainly, a k-linear dg-category $\mathcal{C}$ has a collection of objects. For each pair of objects c_1 and c_2, one has a complex of k vector spaces $\mathrm{Hom}(c_1, c_2)$, and for each triple of objects one has a composition morphism of complexes

$$\mathrm{Hom}(c_1, c_2) \otimes \mathrm{Hom}(c_2, c_3) \to \mathrm{Hom}(c_1, c_3),$$

which is associative in the naive sense.

Here are some basic operations within a dg-category. First, given an object c of $\mathcal{C}$ and an integer i, the shift $c[i]$, if it exists, corepresents the functor

$$\mathrm{Hom}(c[i], -) \simeq \mathrm{Hom}(c, -)[-i].$$

Similarly, given a 0-cycle $f : c_1 \to c_2$, the cone of f, if it exists, corepresents the functor

$$\mathrm{Hom}(\mathrm{Cone}(f), -) \simeq \mathrm{Cone}(\mathrm{Hom}(c_2, -) \to \mathrm{Hom}(c_1, -))[-1],$$

where the cone on the right is the usual one of complexes of vector spaces. Finally, direct sums are defined in the usual way, i.e., when they exist they corepresent the functor

$$\mathrm{Hom}(\bigoplus_i c_i, -) \simeq \prod_i \mathrm{Hom}(c_i, -).$$

A dg-category is said to be cocomplete if it is closed under the above operations, i.e., shifts, cones, and arbitrary direct sums.[6]

4.2.4. Here are two basic examples of cocomplete dg-categories.

Example 4.2.5. The category Vect of k vector spaces is a cocomplete dg-category. Explicitly, objects of Vect are chain complexes of k vector spaces (V_i, d_i), for $i \in \mathbb{Z}$.

Given a pair of complexes (V_i, d_V) and (W_i, d_W), we have

$$\mathrm{Hom}(V_1, V_2) := \underline{\mathrm{Hom}}(V_1, V_2),$$

where $\underline{\mathrm{Hom}}(V_1, V_2)$ is the chain complex with i-chains

$$\underline{\mathrm{Hom}}(V, W)^i := \prod_j \mathrm{Hom}(V_j, W_{i+j}),$$

[5]Of course, strictly speaking, since we are talking about unbounded complexes for an abelian category possibly of infinite cohomological dimension, we mean K-projective and K-injective resolutions in the sense of Bernstein and Spaltenstein [Spa88].

[6]In what follows, we also tacitly assume that any cocomplete dg-category is also presentable. This is a technical assumption about its size satisfied in all relevant examples.

and the usual differential

$$d\phi := d_W \circ \phi - (-1)^i \phi \circ d_V, \quad \text{for } \phi \in \underline{\mathrm{Hom}}(V, W)^i.$$

In particular, i-cycles are simply maps of chain complexes $V \to W[i]$, and i-boundaries are maps which are null-homotopic.

Example 4.2.6. More generally, suppose $\mathcal{A}$ is a k-linear Grothendieck abelian category, such as D-modules on an algebraic variety over k or the left modules for a k-algebra. In this case, the unbounded derived category $\mathrm{D}(\mathcal{A})$ naturally enhances to a dg-category as follows. The objects are given by K-injective complexes of objects of $\mathcal{A}$, and the homomorphisms are defined as in Example 4.2.5.

4.2.7. Given two cocomplete dg-categories $\mathcal{C}_1$ and $\mathcal{C}_2$, one has a cocomplete dg-category

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}_1, \mathcal{C}_2), \tag{4.1}$$

which one thinks of as the functors between them; the subscript '$\mathrm{DGCat}_{\mathrm{cont}}$' will be discussed further in Section 4.2.11 below. The construction of (4.1) is not completely straightforward, as one needs it to be invariant under quasi-equivalences of dg-categories.[7]

However, it may be constructed, as with maps in the derived category, by model category theoretic means as a naive category of dg-functors between suitable replacements of $\mathcal{C}_1$ and $\mathcal{C}_2$; see [Toë11] and references therein.

Let us describe some basic properties of (4.1). First, the set of objects of (4.1) are the quasi-functors in the sense of Keller [Kel94] which are moreover continuous, i.e. commute with arbitrary direct sums. Second, the set of i-cycles in the morphisms between F_1 and F_2 are, up to (co)fibrant replacement, natural transformations $F_1 \to F_2[i]$.

Remark 4.2.8. We should mention that the adjective 'continuous' here refers to commutation with colimits, whereas in other parts of the literature it refers to commutation with limits.

Here are some important examples of such functor categories.

Example 4.2.9. For any cocomplete dg-category $\mathcal{C}$, one has a canonical equivalence

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathrm{Vect}, \mathcal{C}) \simeq \mathcal{C},$$

which exchanges an object c of $\mathcal{C}$ with the dg-functor of tensoring with c, i.e., $V \mapsto V \otimes c$.

Explicitly, this dg-functor assigns a vector space $k^{\oplus I}$, for a set I, the object $c^{\oplus I}$, and given a map $k^{\oplus I} \to k^{\oplus J}$, the resulting map $c^{\oplus I} \to c^{\oplus J}$ is obtained from the map $k \to \mathrm{Hom}_{\mathcal{C}}(c, c)$, i.e. the k-linearity of $\mathcal{C}$. The image of a bounded complex of vector spaces is obtained by taking cones, and for an unbounded complex by taking colimits.

Example 4.2.10. The present example, which is a version of the function theoretic discussion in Section 2.3.4, will be fundamental in what follows and is due to [Toë07] for varieties and extended by [BZFN10] to a broad class of derived stacks.

[7]A naive dg-functor between dg-categories $F : \mathcal{C} \to \mathcal{C}'$ is specified by compatible maps between the sets of objects and chain complexes of morphisms. It is said to be a quasi-equivalence if (i) it induces quasi-isomorphisms on the complexes of homomorphisms and (ii) for any object c' of $\mathcal{C}'$, there exists an object c of $\mathcal{C}$ and a 0-cycle $F(c) \to c'$ inducing quasi-isomorphisms

$$\mathrm{Hom}_{\mathcal{C}'}(-, F(c)) \simeq \mathrm{Hom}_{\mathcal{C}'}(-, c'). \tag{4.2}$$

This may be reformulated as follows. After passing to the zeroth cohomology of all complexes of homomorphisms, a naive dg-functor F induces a functor between the associated 'homotopy' categories

$$\mathrm{Ho}(F) : \mathrm{Ho}(\mathcal{C}) \simeq \mathrm{Ho}(\mathcal{C}').$$

Under the assumption that $\mathcal{C}$ and $\mathcal{C}'$ are closed under shifts, F is a quasi-equivalence if and only if $\mathrm{Ho}(F)$ is an equivalence of ordinary categories.

Plainly, in practice one does not care about the strict complexes of homomorphisms themselves, but instead the complexes up to quasi-isomorphism. This leads to both the notion of quasi-equivalence and the desire to obtain functor categories invariant under quasi-equivalence.

For algebraic varieties X and Y, one has a canonical equivalence

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \simeq \mathrm{QCoh}(X \times Y),$$

where one associates to sheaf $\mathcal{K}$ on the product the corresponding integral transform, i.e.,

$$\mathrm{QCoh}(X) \xrightarrow{\pi_X^*} \mathrm{QCoh}(X \times Y) \xrightarrow{\mathcal{K} \otimes -} \mathrm{QCoh}(X \times Y) \xrightarrow{\pi_{Y,*}} \mathrm{QCoh}(Y),$$

where $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ denote the projections.

Similarly, one has a canonical equivalence

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\text{D-mod}(X), \text{D-mod}(Y)) \simeq \text{D-mod}(X \times Y),$$

which again sends a D-module $\mathcal{K}$ on the product to the integral transform

$$\text{D-mod}(X) \xrightarrow{\pi_X^!} \text{D-mod}(X \times Y) \xrightarrow{\mathcal{K} \overset{!}{\otimes} -} \text{D-mod}(X \times Y) \xrightarrow{\pi_{Y,*}} \text{D-mod}(Y).$$

4.2.11. To talk about (co)limits of dg-categories, one uses the following structure. There exists an $(\infty, 1)$-category, which we denote by $\mathrm{DGCat}_{\mathrm{cont}}$, whose objects are cocomplete dg-categories and 1-morphisms are continuous quasi-functors; see Chapter 1 of [GR17] and references therein. Here are some of its basic properties.

First, it admits all (co)limits in the ∞-categorical sense. Briefly, an object of a limit of dg-categories $\varprojlim \mathcal{C}_\alpha$ is a homotopy-coherent system of objects in each $\mathcal{C}_\alpha$, and morphisms are the corresponding inverse limit of mapping complexes.[8]

Colimits of diagrams of dg-categories are then characterized by the identity

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\varinjlim \mathcal{C}_\alpha, -) \simeq \varprojlim \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}_\alpha, -)$$

and, relatedly, limits of dg-categories satisfy the identity

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(-, \varprojlim \mathcal{C}_\beta) \simeq \varprojlim \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(-, \mathcal{C}_\beta).$$

It is not easy to describe the objects or morphisms in a general colimit of dg-categories (and indeed, one in general should only expect straightforward descriptions for limits, as they are characterized by a mapping in property). However, to orient the reader, let us describe an important example that admits an explicit description.

Example 4.2.12. Recall that an object c of a dg-category $\mathcal{C}$ is said to be *compact* if

$$\mathrm{Hom}(c, -)$$

commutes with colimits. These should be thought of as small objects, e.g., in Vect an object is compact if and only if it is equivalent to a bounded complex of finite dimensional vector spaces. Relatedly, $\mathcal{C}$ is compactly generated if every object can be written as a colimit of compact objects.

Given a filtered diagram of compactly generated categories $\mathcal{C}_\alpha$, where each 1-morphism $\mathcal{C}_\alpha \to \mathcal{C}_{\alpha'}$ sends compact objects to compact objects, the colimit

$$\varinjlim \mathcal{C}_\alpha$$

admits the following description. It is again compactly generated by insertions of compact objects from each $\mathcal{C}_\alpha$, and homomorphisms between such objects are the filtered colimit of the homomorphisms at each step in the colimit.

[8]We highlight that this is again a basic point where triangulated categories would not suffice. That is, one really needs the complexes, and not simply their cohomology, due to the non-exactness of inverse limits.

4.2.13. A very useful feature of $\mathrm{DGCat}_{\mathrm{cont}}$ is that one can tensor together cocomplete dg-categories

$$\mathcal{C} \text{ and } \mathcal{D} \rightsquigarrow \mathcal{C} \otimes \mathcal{D}.$$

More carefully, $\mathrm{DGCat}_{\mathrm{cont}}$ is a symmetric monoidal ∞-category with respect to the Lurie tensor product.

In many practical examples, the following hands-on description of the tensor product is useful.

Example 4.2.14. Suppose that $\mathcal{C}$ and $\mathcal{D}$ are compactly generated, say by sets of objects c_α and d_β. Then the tensor product is compactly generated by the set of simple tensors $c_\alpha \otimes d_\beta$, with homomorphisms given by the Künneth formula, i.e.,

$$\mathrm{Hom}(c_\alpha \otimes d_\beta, c_{\alpha'} \otimes d_{\beta'}) \simeq \mathrm{Hom}(c_\alpha, c_{\alpha'}) \otimes \mathrm{Hom}(d_\beta, d_{\beta'}).$$

Here are some special cases of the above example, which hopefully convey to the reader the sense in which this tensor product aligns with their intuition from the functions-sheaves correspondence, cf. Section 2.2.4.

Example 4.2.15. Given algebraic varieties X and Y, one has a canonical equivalence of the categories of quasicoherent sheaves

$$\mathrm{QCoh}(X) \otimes \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(X \times Y).$$

Explicitly, given an object $\mathcal{M}$ of $\mathrm{QCoh}(X)$ and $\mathcal{N}$ of $\mathrm{QCoh}(Y)$, the simple tensor $\mathcal{M} \otimes \mathcal{N}$ of the left hand side is sent to the external product object

$$\mathcal{M} \boxtimes \mathcal{N} \simeq \pi_X^* \mathcal{M} \otimes \pi_Y^* \mathcal{N}.$$

Example 4.2.16. Let X and Y be as in the preceding example. One also has a canonical equivalence of the categories of D-modules

$$\text{D-mod}(X) \otimes \text{D-mod}(Y) \simeq \text{D-mod}(X \times Y).$$

Again, given an object $\mathcal{M}$ of D-mod(X) and an object $\mathcal{N}$ of D-mod(Y), the simple tensor $\mathcal{M} \otimes \mathcal{N}$ of the left hand side is sent to the external product object

$$\mathcal{M} \boxtimes \mathcal{N} \simeq \pi_X^! \mathcal{N} \overset{!}{\otimes} \pi_Y^! \mathcal{M}.$$

Example 4.2.17. Given k-algebras A and B, one has a canonical equivalence

$$A\text{-mod} \otimes B\text{-mod} \simeq (A \underset{k}{\otimes} B)\text{-mod}.$$

Note that this generalizes the affine cases of the previous two examples.

Let us mention that the monoidal structure is unital, with unit the category of vector spaces Vect. In particular, for any dg-category $\mathcal{C}$, one has a canonical equivalence

$$\mathrm{Vect} \otimes \mathcal{C} \simeq \mathcal{C}.$$

The reader may wish to check this directly in the case $\mathcal{C}$ is compactly generated.

4.2.18. Given dg-categories $\mathcal{C}$ and $\mathcal{D}$, the category $\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{D})$ of Section 4.2.7 is an internal hom object with respect to the monoidal structure on $\mathrm{DGCat}_{\mathrm{cont}}$.

A technical point is that this is not the same as the category of morphisms between $\mathcal{C}$ and $\mathcal{D}$, viewed as objects of $\mathrm{DGCat}_{\mathrm{cont}}$. Instead, the category of morphisms is the truncation of $\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{D})$ to an ∞-groupoid, i.e., one discards the non-invertible natural transformations. It is expected that one has a canonical enhancement of $\mathrm{DGCat}_{\mathrm{cont}}$ to a symmetric monoidal $(\infty, 2)$-category, where the homomorphisms are indeed $\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(-, -)$, but to our knowledge this is not yet available in the literature.

4.2.19. Using the monoidal structure, one can make sense of algebra objects of $\mathrm{DGCat}_{\mathrm{cont}}$, i.e., monoidal dg-categories. Informally, this consists of a dg-category $\mathcal{A}$ with a multiplication

$$\mathcal{A} \otimes \mathcal{A} \to \mathcal{A},$$

which is associative up to coherent homotopy. We emphasize the coherent homotopy is further structure and is part of the full data of a monoidal dg-category.

Here are some basic examples of monoidal dg-categories.

Example 4.2.20. For an algebraic variety X, $\mathrm{QCoh}(X)$ is a monoidal dg-category under $*$-tensor product, and $\mathrm{D\text{-}mod}(X)$ is a monoidal dg-category under $!$-tensor product.

Given a map $f : X_1 \to X_2$, the corresponding pullback functors

$$f^* : \mathrm{QCoh}(X_2) \to \mathrm{QCoh}(X_1) \quad \text{and} \quad f^! : \mathrm{D\text{-}mod}(X_2) \to \mathrm{D\text{-}mod}(X_1)$$

carry a canonical datum of monoidality.

Example 4.2.21. Suppose a variety Y is an algebra object in the category of varieties. That is, it is equipped with a map

$$\mu : Y \times Y \to Y,$$

which is associative in the evident sense. Then $\mathrm{QCoh}(Y)$ and $\mathrm{D\text{-}mod}(Y)$ both acquire 'convolution' monoidal structures, given by $*$-pushforward along μ.

Given a map $f : Y_1 \to Y_2$ of algebras, the associated pushforward maps

$$f_* : \mathrm{QCoh}(Y_1) \to \mathrm{QCoh}(Y_2) \quad \text{and} \quad f_* : \mathrm{D\text{-}mod}(Y_1) \to \mathrm{D\text{-}mod}(Y_2)$$

carry a canonical datum of monoidality.

As our primary case of interest, we take $Y = K$ to be an algebraic group, with μ its multiplication, and thereby obtain the group algebra $\mathrm{D\text{-}mod}(K)$ we met previously (or rather, its canonical dg-enhancement).

Similarly, given a homomorphism $\phi : K_1 \to K_2$ of groups, one obtains a homomorphism of group algebras

$$\phi_* : \mathrm{D\text{-}mod}(K_1) \to \mathrm{D\text{-}mod}(K_2).$$

Remark 4.2.22. One can replace in the two preceding paragraphs $\mathrm{D\text{-}mod}(K)$ with $\mathrm{QCoh}(K)$ to obtain another monoidal dg-category under convolution. Modules for $\mathrm{D\text{-}mod}(K)$ are sometimes called strong categorical representations and modules for $\mathrm{QCoh}(K)$ weak categorical representations.

It may be clarifying to remark that, given any monoidal functor from varieties to a monoidal ∞-category $(\mathcal{S}, \otimes)$, we obtain a similar convolution algebra linearizing K. In particular, one can speak of further higher categorical versions of $\mathrm{Fun}(K(\mathbb{F}_q))$, e.g., by replacing sheaves with sheaves of categories on K.

4.2.23. Given a monoidal dg-category $\mathcal{A}$, one has associated ∞-categories of its left and right modules, which we denote by

$$\mathcal{A}\text{-mod} \quad \text{and} \quad \text{mod-}\mathcal{A},$$

respectively. Somewhat informally, a left module consists of a cocomplete dg-category $\mathcal{M}$ equipped with an action map

$$\mathcal{A} \otimes \mathcal{M} \to \mathcal{M},$$

which is 'associative' up to coherent homotopy and similarly for a right module. Again, the coherent homotopy is part of the data of specifying a module.

Given two left modules $\mathcal{M}_1$ and $\mathcal{M}_2$, one has the dg-category of $\mathcal{A}$-equivariant functors

$$\mathrm{Hom}_{\mathcal{A}\text{-mod}}(\mathcal{M}_1, \mathcal{M}_2)$$

(and similarly for right modules over $\mathcal{A}$). Explicitly this may be calculated via a bar complex as the limit of the semi-cosimplicial diagram of dg-categories

(4.3)
$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{M}_1, \mathcal{M}_2) \rightrightarrows \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{A} \otimes \mathcal{M}_1, \mathcal{M}_2) \substack{\longrightarrow\\[-1ex] \longrightarrow\\[-1ex] \longrightarrow} \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{A}^{\otimes 2} \otimes \mathcal{M}_1, \mathcal{M}_2) \cdots,$$

where the arrows are induced in the standard pattern of bar resolutions by the action maps of $\mathcal{A}$ on $\mathcal{M}_1$ and $\mathcal{M}_2$ and the multiplication map $\mathcal{A}^{\otimes 2} \to \mathcal{A}$.

In addition, given a right module $\mathcal{M}$ and a left module $\mathcal{N}$, one can form their tensor product

$$\mathcal{M} \underset{\mathcal{A}}{\otimes} \mathcal{N}.$$

As for homomorphisms, this may be computed via a bar resolution as the colimit of the semisimplicial diagram of cocomplete dg-categories

$$\cdots \mathcal{M} \otimes \mathcal{A}^{\otimes 2} \otimes N \rightrightarrows \mathcal{M} \otimes \mathcal{A} \otimes \mathcal{N} \rightrightarrows \mathcal{M} \otimes \mathcal{N} \,. \tag{4.4}$$

Let us now give a few basic examples.

Example 4.2.24. For any left $\mathcal{A}$-module $\mathcal{M}$, there are tautological equivalences

$$\mathrm{Hom}_{\mathcal{A}\text{-mod}}(\mathcal{A}, \mathcal{M}) \simeq \mathcal{M} \simeq \mathcal{A} \underset{\mathcal{A}}{\otimes} \mathcal{M}$$

and similarly for right modules.

Example 4.2.25. Given maps of varieties $X \to Z \leftarrow Y$, as in Example 4.2.20, we obtain canonical actions of $\mathrm{QCoh}(Z)$ on $\mathrm{QCoh}(X)$ and $\mathrm{QCoh}(Y)$. With this, their tensor product is given by

$$\mathrm{QCoh}(X) \underset{\mathrm{QCoh}(Z)}{\otimes} \mathrm{QCoh}(Y) \simeq \mathrm{QCoh}(X \underset{Z}{\times} Y),$$

where the above fibre product is derived, i.e., one obtains a derived scheme.[9] Concretely, the equivalence sends a simple tensor $\mathcal{M} \otimes \mathcal{N}$ to their external product, restricted to $X \underset{Z}{\times} Y$.

Similarly, their homomorphisms are again given by

$$\mathrm{Hom}_{\mathrm{QCoh}(Z)}(\mathrm{QCoh}(X), \mathrm{QCoh}(Y)) \simeq \mathrm{QCoh}(X \underset{Z}{\times} Y).$$

Explicitly, one associates to a quasicoherent sheaf $\mathcal{S}$ on the fibre product the integral transform

$$\mathrm{QCoh}(X) \xrightarrow{\pi_X^*} \mathrm{QCoh}(X \underset{Z}{\times} Y) \xrightarrow{-\overset{*}{\otimes}\mathcal{S}} \mathrm{QCoh}(X \underset{Z}{\times} Y) \xrightarrow{\pi_{Z,*}} \mathrm{QCoh}(Y).$$

The same assertions hold, mutatis mutandis, for D-modules. That is, we have equivalences

$$\text{D-mod}(X) \underset{\text{D-mod}(Z)}{\otimes} \text{D-mod}(Y) \simeq \text{D-mod}(X \overset{\mathrm{cl}}{\underset{Z}{\times}} Y) \text{ and}$$

$$\mathrm{Hom}_{\text{D-mod}(Z)}(\text{D-mod}(X), \text{D-mod}(Y)) \simeq \text{D-mod}(X \overset{\mathrm{cl}}{\underset{Z}{\times}} Y),$$

again given by external tensor product and integral transform. Here the superscript 'cl' denotes the classical, i.e., non-derived, fibre product. Note however that when the derived fibre product has only finitely many nonzero cohomologies in its structure sheaf, e.g., if Z is smooth, due to the invariance of categories of D-modules under nil-thickenings it is unimportant whether one takes the derived or classical fiber product of varieties.

As with other sheaf-theoretic statements we have met, these identities are analogues of the following identity for functions. Given a finite set S, consider $\mathrm{Fun}(S)$ as an algebra under pointwise multiplication. Given finite sets $X \to Z \leftarrow Y$, one then has canonical isomorphisms of vector spaces

$$\mathrm{Fun}(X) \underset{\mathrm{Fun}(Z)}{\otimes} \mathrm{Fun}(Y) \simeq \mathrm{Hom}_{\mathrm{Fun}(Z)}(\mathrm{Fun}(X), \mathrm{Fun}(Y)) \simeq \mathrm{Fun}(X \underset{Z}{\times} Y).$$

4.3. **Basics of categorical representation theory.**

[9]Plainly, this is glued from the derived rings $\mathcal{O}_X \overset{L}{\underset{\mathcal{O}_Z}{\otimes}} \mathcal{O}_Y$ affine by affine, exactly as in the case of the usual fibre product. One thinks of the negative cohomology groups of the structure sheaf as adding 'nilpotent fluff' to the variety, exactly like the perhaps more familiar the non-reducedness of a general scheme. All derived schemes we will meet in this survey arise in this way.

4.3.1. With these generalities in hand, let us describe some of the basic operations in categorical representation theory and thereby recover Theorems 3.4.10 and 3.4.12 encountered in Section 3.

Recall that, for an algebraic group K, we have its associated ∞-category of categorical representations

$$\text{D-mod}(K)\text{-mod}\,.$$

As elsewhere in representation theory, inversion on K induces an isomorphism between D-mod(K) and its reverse monoidal dg-category, and in particular one may freely pass between left and right modules.

4.3.2. A basic family of representations may be constructed geometrically as follows.

Given a variety X with an action of K, one obtains a canonical action of D-mod(K) on D-mod(X). Explicitly, the underlying binary product is given by

$$\text{D-mod}(K) \otimes \text{D-mod}(X) \simeq \text{D-mod}(K \times X) \xrightarrow{a_*} \text{D-mod}(X),$$

exactly as in the function-theoretic case of Section 2.3.2. Given a map $f : X \to X'$ of K-varieties, the functors of $*$-pushforward and !-pullback along f carry canonical data of K-equivariance.

4.3.3. In particular, if we take $X = \text{pt}$, we obtain the trivial representation of K on

$$\text{Vect} \simeq \text{D-mod}(\text{pt}).$$

Concretely, if we write $\pi : K \to \text{pt}$ for the projection, the action map takes the form

$$\text{D-mod}(K) \otimes \text{Vect} \to \text{Vect}, \qquad \mathcal{M} \otimes V \mapsto \pi_*(\mathcal{M}) \underset{k}{\otimes} V.$$

For any representation $\mathcal{C}$ of K, we may accordingly form its invariants and coinvariants

$$\mathcal{C}^K := \text{Hom}_{\text{D-mod}(K)\text{-mod}}(\text{Vect}, \mathcal{C}) \quad \text{and} \quad \mathcal{C}_K := \text{Vect} \underset{\text{D-mod}(K)}{\otimes} \mathcal{C}.$$

These are canonically identified, as in the case of usual representations of finite groups, as follows.

By the smoothness of K, we have a D-mod(K) equivariant adjunction

$$\pi^![-2\dim K] : \text{D-mod}(K) \rightleftarrows \text{Vect} : \pi_*,$$

which upon applying $\text{Hom}_{\text{D-mod}(K)\text{-mod}}(-, \mathcal{C})$ induces an adjunction

$$\text{Oblv} : \mathcal{C}^K \rightleftarrows \mathcal{C} : \text{Av}_*\,.$$

As the notation suggests, one thinks of Oblv as a forgetful functor and Av_* as an averaging functor. For formal reasons, the averaging functor factors through a map out of the coinvariants

$$\text{Av}_* : \mathcal{C}_K \to \mathcal{C}^K.$$

By a very useful theorem of Beraldo and Gaitsgory [Ber17], [Gai20], this is an equivalence.

Let us give some basic examples of (co)invariants.

Example 4.3.4. Given D-mod(K) representations $\mathcal{C}$ and $\mathcal{D}$, one has a canonical equivalence

$$\text{Hom}_{\text{D-mod}(K)\text{-mod}}(\mathcal{C}, \mathcal{D}) \simeq \text{Hom}_{\text{DGCat}_{\text{cont}}}(\mathcal{C}, \mathcal{D})^K,$$

where the invariants are taken with respect to the diagonal action of D-mod(K) on the space of maps.

Example 4.3.5. For a K-variety X, the K-equivariant objects in D-mod(X) canonically identify with the category of D-modules on the quotient stack X/K, i.e.,

$$\text{D-mod}(X)^K \simeq \text{D-mod}(X/K).$$

Plainly, D-modules satisfy smooth descent and the diagram (4.3) simply is the 'Čech complex' for calculating D-modules on X/K via the cover $X \to X/K$.

Example 4.3.6. Under the natural action of D-mod(K) on its category of Lie algebra representations $\mathfrak{k}$-mod, the invariants identify with the algebraic representations of the group itself, i.e.,

$$\mathfrak{k}\text{-mod}^K \simeq K\text{-mod}\,.$$

More generally, given a homomorphism $K \to K'$, the category

$$\mathfrak{k}'\text{-mod}^K$$

canonically identifies with the category of Harish-Chandra modules for the pair $(\mathfrak{k}', K)$.

Example 4.3.7. Here is another class of examples, which in particular gives an alternative perspective on the action on $\mathfrak{k}$-mod. Suppose A is a k-algebra equipped with an action of K by automorphisms. In this case, one obtains an action of QCoh(K) on A-mod.

Upon differentiation at the identity, the action of K on A yields an action α of $\mathfrak{k}$ on A by derivations. Suppose one is further given a trivialization of this action, i.e., a homomorphism of Lie algebras

$$\tau : \mathfrak{k} \to A,$$

such that α agrees with the adjoint action via τ. In this case, one obtains an action of D-mod(K) on A-mod.

Remark 4.3.8. Let us record here a basic point of departure from the case of usual representations. Namely, the operation of taking invariants is not idempotent, i.e., the tautological map

$$\mathrm{Oblv}_{\mathcal{C}^K} : (\mathcal{C}^K)^K \to \mathcal{C}^K$$

is typically not an isomorphism. Instead, one has a canonical equivalence

$$(\mathcal{C}^K)^K \simeq (\mathcal{C}^K) \otimes \text{D-mod}(\mathrm{pt/K}).$$

This happens because, unless K is contractible, the forgetful map $\mathrm{Oblv}_{\mathcal{D}} : \mathcal{D}^K \to \mathcal{D}$ for any (nonzero) representation $\mathcal{D}$ will not be fully faithful, i.e., Exts in $\mathcal{D}^K$ and $\mathcal{D}$ do not agree. Indeed, in the universal case of $\mathcal{D} = \text{D-mod}(K)$, one has

$$\text{D-mod}(K)^K \simeq \text{D-mod}(K/K) \simeq \mathrm{Vect},$$

with generator the constant sheaf. In particular its self Exts in D-mod(K) are $\pi_*(\underline{k})$, i.e., the cohomology of K, while its self Exts in Vect are trivial.

In general, the composition

$$\mathcal{D}^K \xrightarrow{\mathrm{Oblv}_{\mathcal{D}}} \mathcal{D} \xrightarrow{\mathrm{Av}_*} \mathcal{D}^K \xrightarrow{\mathrm{Oblv}_{\mathcal{D}}} \mathcal{D}$$

canonically identifies with $\mathrm{Oblv}_{\mathcal{D}} \otimes \pi_*(\underline{k})$, i.e., tensors the underlying object of a K-equivariant object with the cohomology of K. The reader may wish to compare this, via the functions-sheaves correspondence, to the factor of the cardinality of a finite group that arises in the discussion of averaging for usual representations.

Note, however, that if K is unipotent then Oblv is fully faithful, as K is contractible and hence $\pi_*(\underline{k}) \simeq k$.

4.3.9. Having discussed the functor of invariants, let us indicate how one obtains Theorems 3.4.10 and 3.4.12, beginning with the latter.

Given a map of algebraic groups $H \to K$, one has induction and restriction functors

$$\mathrm{ind}_H^K : \text{D-mod}(H)\text{-mod} \to \text{D-mod}(K)\text{-mod},$$

$$\mathrm{res}_K^H : \text{D-mod}(K)\text{-mod} \to \text{D-mod}(H)\text{-mod}\,.$$

Plainly, res_K^H restricts the action of D-mod(K) on a dg-category $\mathcal{C}$ along the monoidal functor

$$\text{D-mod}(H) \to \text{D-mod}(K),$$

and ind_H^K takes a dg-category $\mathcal{D}$ with an action of D-mod(H) and tensors it up, i.e.,

$$\mathcal{D} \mapsto \text{D-mod}(K) \underset{\text{D-mod}(H)}{\otimes} \mathcal{D}.$$

As with Frobenius reciprocity for finite groups, induction and restriction are canonically both left and right adjoint to one another.

In particular, one obtains a functor of H-invariants on D-mod(K) representations, i.e.,

$$\big(\operatorname{Res}^H_K(-)\big)^H \simeq \operatorname{Hom}_{\text{D-mod}(H)\text{-mod}}(\mathrm{Vect}, \operatorname{res}^H_K -).$$

By adjunction we may rewrite this as

$$\begin{aligned} &\simeq \operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\operatorname{ind}^K_H \mathrm{Vect}, -) \\ &\simeq \operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\text{D-mod}(K) \underset{\text{D-mod}(H)}{\otimes} \mathrm{Vect}, -) \end{aligned}$$

By the identification of H invariants and coinvariants and Example 4.3.5, we may recognize this as

$$\simeq \operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\text{D-mod}(K/H), -).$$

The upshot is that the H-invariants of any D-mod(K) representation carry, by precomposition, a canonical action of

$$\operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\text{D-mod}(K/H), \text{D-mod}(K/H)) \simeq \text{D-mod}(H\backslash K/H),$$

i.e., the Hecke algebra action of Beilinson–Drinfeld we encountered at the end of Section 3.

4.3.10. Let us finally sketch, for a smooth K-variety X, how one obtains the D-mod(K) equivariant adjunction

$$\mathrm{Loc} : \mathfrak{k}\text{-mod} \rightleftarrows \text{D-mod}(X) : \Gamma(X, -).$$

To begin with, induction from $\mathcal{O}_K$-modules to $\mathcal{D}_K$-modules yields a monoidal functor

$$\mathrm{QCoh}(K) \to \text{D-mod}(K).$$

Similarly, one has a canonical action of QCoh(K) on D-mod(X) such that the forgetful functor D-mod$(X) \to$ QCoh(X) carries a canonical datum of QCoh(K)-equivariance. In particular, the functor of global sections, i.e., the composition

$$\text{D-mod}(X) \to \mathrm{QCoh}(X) \xrightarrow{\pi_*} \mathrm{QCoh}(\mathrm{pt}) \simeq \mathrm{Vect},$$

carries a canonical datum of QCoh(K)-equivariance.

However, for any D-mod(K) representation $\mathcal{C}$, one has by adjunction

$$\begin{aligned} \operatorname{Hom}_{\mathrm{QCoh}(K)\text{-mod}}(\operatorname{Res} \mathcal{C}, \mathrm{Vect}) &\simeq \operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\mathcal{C}, \operatorname{Hom}_{\mathrm{QCoh}(K)\text{-mod}}(\text{D-mod}(K), \mathrm{Vect})) \\ &\simeq \operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\mathcal{C}, \mathfrak{k}\text{-mod}), \end{aligned}$$

where for the last equivalence one uses the identification of $\mathfrak{k}$-mod and weakly K-equivariant D-modules on K, cf. Section 3.4.8. This yields the desired D-mod(K) equivariance of the global sections functor. For localization, one can use the nice fact, due to Gaitsgory [Gai20], that any adjoint of a D-mod(K)-mod-equivariant functor inherits a compatible datum of equivariance.

4.4. **Complements I - character sheaves.**

4.4.1. In this section, we began by sketching some preliminary generalities on dg-categories. We then discussed some of the basic definitions and constructions in categorical representation theory, sufficient to recover some results previewed in Section 3.

In the remainder of this section, we would like to survey a few more basic results and constructions. However, the reader may wish to skip directly to Section 5.

4.4.2. Character sheaves are (certain) adjoint equivariant D-modules on groups. Lusztig initiated their study, particularly in the reductive case, in view of their close ties to irreducible characters of finite groups of Lie type [Lus85a, Lus85b, Lus86a, Lus86b]. This is a particularly rich and nontrivial example of the functions-sheaves correspondence.

Of course, the characters of a finite group are not simply adjoint equivariant functions on the group, but rather arise as traces from representations. One can ask whether character sheaves have such an interpretation. Indeed they do, and we turn to this next.

4.4.3. *Duality.* Recall that to define the character of a representation of a finite group, one needs the representation to be finite dimensional so that one may safely take traces. The analogue of being 'small enough' to take traces in the categorical setting is as follows.

Using the monoidal structure on $\mathrm{DGCat}_{\mathrm{cont}}$ one can speak of dualizable dg-categories. Under the analogy between $\mathrm{DGCat}_{\mathrm{cont}}$ and Vect, these correspond to (bounded complexes of) finite dimensional vector spaces, and indeed a dg-category $\mathcal{C}$ is dualizable if and only if the natural map

$$\mathcal{C} \to \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathrm{Vect}), \mathrm{Vect})$$

is an equivalence.

For a dualizable dg-category, one has a canonical equivalence $\mathcal{C}^\vee \simeq \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathrm{Vect})$, i.e., its dual in the monoidal sense is simply its continuous dg-modules. For any $\mathcal{S}$ the natural map

$$\mathcal{S} \otimes \mathcal{C}^\vee \to \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{S})$$

is an equivalence.[10] More generally, there are equivalences

$$\mathrm{Hom}(\mathcal{S} \otimes \mathcal{C}, \mathcal{S}') \simeq \mathrm{Hom}(\mathcal{S}, \mathcal{C}^\vee \otimes \mathcal{S}')$$

for any dg-categories $\mathcal{S}$ and $\mathcal{S}'$.

Let $\mathcal{A}$ be a monoidal dg-category. By functoriality, if $\mathcal{C}$ is a left $\mathcal{A}$-module, then $\mathcal{C}^\vee$ is a right $\mathcal{A}$-module. In particular, for a dualizable D-mod(K) representation $\mathcal{C}$, its dual acquires a canonical action of D-mod(K), i.e., one may form the contragredient representation.

Before we discuss how to define the character sheaf of a dualizable D-mod(K) representation, we would like to discuss some basic examples of dualizable categories and modules. First of all, the following general fact ensures that most dg-categories one encounters in practice are dualizable.

Example 4.4.4. If a category $\mathcal{C}$ is compactly generated, then it is dualizable. Moreover, its dual is compactly generated, and one has a canonical equivalence of non-cocomplete dg-categories between the compact objects of $\mathcal{C}$ and the opposite of the compact objects of $\mathcal{C}^\vee$.

Here are some useful special cases of the previous example.

Example 4.4.5. Given a k-algebra A, its category of left-modules A-mod is dualizable with dual the category of right modules mod-A. Explicitly, the evaluation map

$$\text{mod-}A \otimes A\text{-mod} \to \mathrm{Vect}$$

sends a right module M and left module N to their tensor product $M \underset{A}{\otimes} N$.

Example 4.4.6. Given a variety X, its category of D-modules is made canonically self-dual by the operation of Verdier duality on compact D-modules. Explicitly, the resulting evaluation map is given by the composition

$$\text{D-mod}(X) \otimes \text{D-mod}(X) \simeq \text{D-mod}(X \times X) \xrightarrow{\Delta^!} \text{D-mod}(X) \xrightarrow{\pi_*} \mathrm{Vect},$$

where, exactly as in Section 2.3.3, we denote by $\Delta : X \to X \times X$ the diagonal map and $\pi : X \to \mathrm{pt}$ the projection to the point.

If X is a K-variety, this pairing is canonically D-mod(K) equivariant, i.e., identifies D-mod(X) with its contragredient representation. Similar statements hold, mutatis mutandis, for QCoh(X), viewed as a QCoh(K) representation.

[10] Replacing $\mathcal{C}^\vee$ with $\mathrm{Hom}(\mathcal{C}, \mathrm{Vect})$, this is yet another characterization of dualizable dg-categories.

4.4.7. *Character sheaves.* We are now ready to explain how to take the character sheaf of a representation. An action of D-mod(K) on a category $\mathcal{C}$ is the same as a monoidal functor

$$\text{D-mod}(K) \xrightarrow{\rho} \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{C}).$$

If $\mathcal{C}$ is dualizable, then its endomorphisms $\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{C})$ are again dualizable by the identification $\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{C}) \simeq \mathcal{C}^\vee \otimes \mathcal{C}$. We may therefore dualize the action map to obtain a D-mod$(K \times K)$-equivariant functor of matrix coefficients

$$\mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{C}) \simeq \mathrm{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathcal{C}, \mathcal{C})^\vee \xrightarrow{\rho^\vee} \text{D-mod}(K)^\vee \simeq \text{D-mod}(K).$$

In particular, by passing to diagonal K-invariants, we obtain a functor

$$\mathrm{Hom}_{\text{D-mod}(K)}(\mathcal{C}, \mathcal{C}) \to \text{D-mod}(K/K). \tag{4.5}$$

Applying this to the identity endomorphism $\mathrm{id}_{\mathcal{C}}$, we obtain an adjoint equivariant D-module $\chi_{\mathcal{C}}$ on K, namely the *character sheaf* of $\mathcal{C}$.

4.4.8. Here is a more hands-on description of $\chi_{\mathcal{C}}$. Given a dualizable category $\mathcal{C}$ and an endofunctor $f : \mathcal{C} \to \mathcal{C}$, one can take its trace, which is a complex of vector spaces. Exactly as in linear algebra, this is computed via the composition

$$\mathrm{Vect} \xrightarrow{\mathrm{id}_{\mathcal{C}}} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{f \otimes \mathrm{id}_{\mathcal{C}^\vee}} \mathcal{C} \otimes \mathcal{C}^\vee \xrightarrow{\mathrm{ev}} \mathrm{Vect}\,.$$

That is, the composite endofunctor of Vect is given by tensoring by a vector space, which is by definition $\mathrm{tr}(f, \mathcal{C})$. We remark that, when f is simply the identity endomorphism of $\mathcal{C}$, the resulting complex is known as the Hochschild homology of $\mathcal{C}$.

Having understood how to take traces in the present categorical context, we can now describe the character sheaf fairly explicitly. Namely, its stalks store the traces of the corresponding automorphisms of $\mathcal{C}$, exactly as in usual representation theory. That is, for any closed point $g : \mathrm{pt} \to K$, one has a canonical isomorphism

$$g^!(\chi_{\mathcal{C}}) \simeq \mathrm{tr}(\delta_g \star -, \mathcal{C}).$$

4.4.9. Let us mention a few features and examples.

Example 4.4.10. A D-mod(K) equivariant functor between dualizable modules $\mathcal{C} \to \mathcal{D}$ need not induce a map between their character sheaves. However, if it admits a continuous right adjoint, we do obtain a suitably functorial map $\chi_{\mathcal{C}} \to \chi_{\mathcal{D}}$.

Let us remark that, in practical situations, $\mathcal{C}$ is compactly generated, and the condition of being right adjointable is equivalent to sending compact objects to compact objects, i.e., nothing 'small' becoming 'big'.

Example 4.4.11. The induction and restriction functors on categorical representations preserve the property of dualizability. On characters, they reproduce the usual functors of induction and restriction for character sheaves.

Example 4.4.12. If X is a K-variety, its character sheaf may be described as follows. Consider the closed subscheme Stab of $K \times X$, whose fibre over a point of X is its stabilizer. That is, Stab is the equalizer of the action and projection maps to X. Then under the tautological G-equivariant map

$$\mathrm{Stab} \to G,$$

the pushforward of the dualizing sheaf is $\chi_{\text{D-mod}(X)}$. In particular, by proper base change, the trace of any element of K on D-mod(X) is the Borel–Moore homology of its fixed points. The reader may wish to compare this with the analogous assertion for finite groups and sets, where the trace of an element of the group recovers the number of fixed points.

Example 4.4.13. If $K = G$ is a reductive group, and $\mathcal{C} = \text{D-mod}(\text{Fl}_G)$, then by the preceding example its character sheaf is the Grothendieck–Springer sheaf on G/G. More generally, the functor of matrix coefficients for equivariant endomorphisms, cf. Equation (4.5), takes the form

$$\text{D-mod}(B\backslash G/B) \to \text{D-mod}(G/G).$$

This recovers the Harish-Chandra, or horocycle, transform, i.e., push-pull along the correspondence

$$G/G \leftarrow G/B \to B\backslash G/B,$$

where the left and middle terms are quotients with respect to the adjoint action. The latter was the point of departure for Lusztig's study of character sheaves.

Here are two related remarks.

Remark 4.4.14. In the present discussion, we treated character sheaves as a direct construction, which takes as input dualizable representations and outputs adjoint equivariant D-modules.

An alternative perspective is to interpret $\text{D-mod}(K/K)$ as the horizontal trace of $\text{D-mod}(K)$. Indeed, for any monoidal dg-category $\mathcal{A}$, the latter is defined as

$$\mathcal{A} \underset{\mathcal{A}\otimes\mathcal{A}^{\text{op}}}{\otimes} \mathcal{A}$$

or, equivalently, as the 2-categorical trace of $\mathcal{A}$-mod, viewed as a dualizable $\text{DGCat}_{\text{cont}}$-linear category. If we were working one categorical level lower, i.e. with an algebra A instead of a monoidal dg-category, the analogue of this discussion is the presentation of the Hochschild homology of A-mod as

$$A \underset{A\otimes A^{\text{op}}}{\otimes} A.$$

In the present case, using the correspondence between categorical coinvariants and D-modules on quotient stacks, we have a tautological identification

$$\text{D-mod}(K) \underset{\text{D-mod}(K\times K)}{\otimes} \text{D-mod}(K) \simeq \text{D-mod}(K \overset{K\times K}{\times} K) \simeq \text{D-mod}(K/K).$$

General formalism assigns to a dualizable $\mathcal{A}$-module an object of its horizontal trace, which recovers the preceding discussion of character sheaves as a special case; see [GKRV21] and [BZN09] for further discussion.

Remark 4.4.15. There are parallel theories to those discussed in this section, where one works with analytic or étale constructible sheaves on K rather than D-modules, cf. Section 6.10 below. When one takes étale sheaves on K defined over a finite field $\mathbb{F}_q$, a version of the functions-sheaves correspondence identifies the category of modules for $K(\mathbb{F}_q)$ with the trace of Frobenius of $\text{Shv}(K)$-mod. This sends categorical representations to usual representations and, compatibly, character sheaves to usual characters; see [Gai16].

4.5. **Complements II: generation.**

4.5.1. Our final topic is several nice theorems about generation. Recall a basic and useful feature of life is Morita theory – for a given ring A, while its category of modules A-mod is compactly generated by A itself, it admits many other compact generators c. For any choice of one, taking its endomorphisms yields an equivalence

$$\text{mod-}\operatorname{End}(c) \simeq A\text{-mod}, \qquad M \mapsto M \underset{\operatorname{End}(c)}{\otimes} c.$$

Said somewhat differently, a given dg-category $\mathcal{C}$ may have many compact generators and correspondingly many presentations as a category of modules. We would like to review here a couple useful statements of this form, which necessarily take place one categorical level higher, for $\text{D-mod}(K)$-mod.

4.5.2. First of all, a nontrivial theorem of Beraldo [Ber17], building on work of Gaitsgory [Gai15], states that D-mod(K) is generated by $\mathfrak{k}$-mod. Note that the endomorphisms of the latter are

$$\begin{aligned}\operatorname{Hom}_{\text{D-mod}(K)\text{-mod}}(\mathfrak{k}\text{-mod}, \mathfrak{k}\text{-mod}) &\simeq \operatorname{Hom}_{\mathrm{DGCat}_{\mathrm{cont}}}(\mathfrak{k}\text{-mod}, \mathfrak{k}\text{-mod})^K \\ &\simeq (\mathfrak{k}\text{-mod}^\vee \otimes \mathfrak{k}\text{-mod})^K \\ &\simeq (\mathfrak{k}\text{-mod} \otimes \mathfrak{k}\text{-mod})^K \\ &\simeq (\mathfrak{k} \oplus \mathfrak{k}\text{-mod})^K.\end{aligned}$$

Stated in words, the equivariant endomorphisms of $\mathfrak{k}$-mod are the category of Harish-Chandra bimodules, i.e., $\mathfrak{k}$-bimodules for which the diagonal action is integrated to the group K.

Remark 4.5.3. We emphasize that this identity is an attractive and useful feature of the formalism. Namely, Harish-Chandra bimodules have been studied and used in representation theory for many decades and have been understood to have many favorable properties; see [BG80] and references therein. The above articulates this in a precise sense – they are exactly the endofunctors of $\mathfrak{k}$-mod which commute with the categorical K-action and vice versa.

The theorem of Beraldo therefore affords an equivalence

$$\text{D-mod}(K)\text{-mod} \simeq (\mathfrak{k} \oplus \mathfrak{k}\text{-mod}^K)\text{-mod}.$$

In particular, for any D-mod(K) representation $\mathcal{C}$ we have a canonical equivalence

$$\mathfrak{k}\text{-mod} \underset{\mathfrak{k}\oplus\mathfrak{k}\text{-mod}^K}{\otimes} \mathcal{C}^{K,w} \simeq \mathcal{C}.$$

That is, recalling the equivalence D-mod$(K)^{K,w} \simeq \mathfrak{k}$-mod, a D-mod$(K)$ representation can be reconstructed from its weak invariants.

4.5.4. We would also like to mention, for a reductive group G, a nice analogue of highest weight theory in the present setting due to Ben-Zvi–Gunningham–Orem [BZGO18]. Namely, they showed that, for any maximal unipotent subgroup N, an arbitrary D-mod(G) representation $\mathcal{C}$ can be reconstructed from its N-invariants, or even the weak Cartan invariants of the latter

$$\mathcal{C}^N \quad \text{and} \quad (\mathcal{C}^N)^{T,w}.$$

Passing to their endomorphisms, which roughly are Hecke categories without prescribed monodromy along T-orbits, we obtain equivalences

$$\text{D-mod}(G)\text{-mod} \simeq \text{D-mod}(N\backslash G/N)\text{-mod} \simeq \text{D-mod}(N\backslash G/N)^{T\times T,w}\text{-mod}$$

and, in particular, identities

$$\text{D-mod}(G/N) \underset{\text{D-mod}(N\backslash G/N)}{\otimes} \mathcal{C}^N \simeq \text{D-mod}(G/N)^{T,w} \underset{\text{D-mod}(N\backslash G/N)^{T\times T,w}}{\otimes} (\mathcal{C}^N)^{T,w} \simeq \mathcal{C}.$$

Remark 4.5.5. The argument of [BZGO18] yields the following useful variation. Given any subgroup H of a group K such that the quotient of K by the normalizer of H is proper, the tautological map

$$\text{D-mod}(K/H) \underset{\text{D-mod}(H\backslash K/H)}{\otimes} \mathcal{C}^K \to \mathcal{C},$$

which need not be essentially surjective, is always a fully faithful embedding. This may be applied to check equivalences of categories on 'highest weight vectors,' which are often much easier to directly analyze, cf. [CD21], [Yan21] for further discussion and examples.

5. Categorical representations of loop groups

5.1. **Overview.**

5.1.1. So far, we have discussed Beilinson–Bernstein localization and categorical representation theory for reductive groups.

In the remainder of this survey, we would like to turn to their affine analogues, i.e., we pass from reductive groups to loop groups. The story here is significantly richer and is the subject of the local geometric Langlands correspondence.

In this section, we will give some introductory comments on loop groups and their categorical representations to prepare for a discussion of the Langlands correspondence in Section 6.

5.2. **Loop groups.**

5.2.1. As before, we continue to denote by G a connected reductive group over an algebraically closed field k of characteristic zero.

The main actor in this section is not G, but rather its loop group LG. At first pass, just as G consists of certain matrices with entries in k, i.e., pick a faithful representation, LG consists of the same form of matrices but with entries in the field of Laurent series $k(\!(t)\!)$. Somewhat more carefully, since we need to make sense of LG as an algebro-geometric object, it is a certain group ind-scheme whose closed points are as we just described.

Example 5.2.2. Here is an explicit construction in the case when G is the (non-reductive) additive group $\mathbb{G}_a$, which contains most of the representative ideas. In this case, we have $\mathbb{G}_a(k) \simeq k$, i.e., it is a one dimensional affine space. As above, we would like $L\mathbb{G}_a(k) \simeq k(\!(t)\!)$. The basic idea is that $k(\!(t)\!)$ is again an infinite dimensional affine space, and, in fact, by bounding the order of the pole of the Laurent series, an ascending union of pro-finite dimensional affine spaces. Explicitly as a vector space

$$k(\!(t)\!) = \varinjlim_n \varprojlim_m t^{-n} \cdot k[[t]]/t^{-n+m} \cdot k[[t]],$$

and we use the same formula to describe its structure as an ind-scheme. That is, fixing n, we obtain a pro-finite dimensional affine space, which explicitly is the spectrum of a polynomial ring in countably many variables, i.e., the functions corresponding to the Taylor coefficients of a series. Allowing n to vary, we obtain an ascending union of infinite dimensional affine spaces along closed embeddings, i.e., an ind-scheme.

Remark 5.2.3. In general, LG admits a succinct description via its functor of points, which is defined as follows. For any variety X, consider the functor LX from affine varieties to sets whose R-points, for any k-algebra R, are

$$LX(R) = X(R(\!(t)\!)).$$

While this is not well-behaved for a general variety, if X is affine, this is representable by an ind-scheme of ind-infinite type. Taking $X = G$ yields the loop group LG.

Having given a formal definition, let us try to give a slightly more hands-on description in the spirit of Example 5.2.2. Recall the analogy between $G(\mathbb{F}_q)$ and $G(k)$ and, more generally, finite $G(\mathbb{F}_q)$-sets and finite dimensional $G(k)$-varieties as discussed in the previous sections. A basic guiding principle for the present situation is an analogy between the p-adic group $G(\mathbb{F}_q(\!(t)\!))$ and the loop group LG.

Remark 5.2.4. While some authors use the term p-adic group to mean the points of G in $\mathbb{Q}_p$ or a finite extension thereof, here by a mild abuse of notation we call the points in any non-Archimedean local field, such as $\mathbb{F}_q(\!(t)\!)$, a p-adic group.

Recall that p-adic groups, with their natural topology, are ind-pro-finite sets. Explicitly, fix any compact open subgroup K, i.e., a subgroup such that $K \cap G(\mathbb{F}_q[[t]])$ is finite index in both K and $G(\mathbb{F}_q[[t]])$. Then K is naturally an inverse limit of finite groups, and the entire group decomposes as a disjoint union of its K cosets. Concretely, for $G(\mathbb{F}_q[[t]])$, the relevant inverse limit comes from the topology on power series, i.e.,

$$G(\mathbb{F}_q[[t]]) = \varprojlim_n G(\mathbb{F}_q[t]/(t^n)). \tag{5.1}$$

In a similar way, one can make sense of compact open subgroups of LG. Namely, one has the arc group L^+G, which is a group scheme of infinite type with k-points $G(k[[t]])$. This is defined as an inverse limit of finite dimensional algebraic groups similarly to (5.1). A sub-group scheme K of LG is said to be compact open if $K \cap L^+G$ is of finite codimension in both K and L^+G. Then K is naturally an inverse limit of algebraic groups, and the coset space

$$LG/K$$

is an ind-scheme of ind-finite type. We emphasize that, under the analogy between finite sets and finite dimensional varieties, this corresponds to being a countable discrete union of cosets in the case of p-adic groups.

5.3. **D-modules on loop groups and categorical actions.**

5.3.1. We would next like to discuss the convolution algebra of D-modules on LG. Since this is an infinite-dimensional object, some care is required.

Its definition may be arrived at as follows. Recall that we thought of modules for D-mod(G) as akin to representations of $G(\mathbb{F}_q)$. In particular, the convolution algebra of D-modules itself was analogous to the group algebra Fun$(G(\mathbb{F}_q))$.

We would similarly like modules for D-mod(LG) to behave like representations of the p-adic group $G(\mathbb{F}_q((t)))$. As the latter is a topological group, one correspondingly works with smooth representations.[11] For this reason, modules for $G(\mathbb{F}_q((t)))$ are not the same as modules for its group algebra, but instead for its Hecke algebra

$$\operatorname{Dist}(G(\mathbb{F}_q((t))))$$

of compactly supported distributions. Concretely, this can be written as the colimit

$$\operatorname{Dist}(G(\mathbb{F}_q((t)))) \simeq \varinjlim_K \operatorname{Dist}(K\backslash G(\mathbb{F}_q((t)))/K), \tag{5.2}$$

where K runs over compact open subgroups and each $\operatorname{Dist}(K\backslash G(\mathbb{F}_q((t)))/K)$ is simply finite linear combinations of delta distributions on double cosets, i.e., distributions that integrate a function over a double coset KgK with respect to some Haar measure.

With this in mind, the category of D-modules on the loop group is defined as

$$\text{D-mod}(LG) \simeq \varinjlim \text{D-mod}(K\backslash LG/K),$$

where K runs over compact open subgroups and the colimit is taken in $\mathrm{DGCat}_{\mathrm{cont}}$ along $*$-pullback functors. It remains then to describe, for a fixed K, the category D-mod$(K\backslash LG/K)$. However, recalling that LG/K is an ind-scheme of ind-finite type, we may moreover write it as an ascending union

$$LG/K = \bigcup_i X_i,$$

where each X_i is a closed subscheme of finite type that is moreover K invariant. As K acts on each X_i via a finite dimensional quotient Q_i with prounipotent kernel, we have

$$\text{D-mod}(X_i)^K \simeq \text{D-mod}(X_i)^{Q_i} \simeq \text{D-mod}(Q_i\backslash X_i),$$

i.e., we are dealing with D-modules on a usual finite dimensional stack. We then set

$$\text{D-mod}(K\backslash LG/K) \simeq \varinjlim \text{D-mod}(Q_i\backslash X_i).$$

The point here is that, roughly speaking, objects in D-mod(LG) resemble compactly supported distributions on $G(\mathbb{F}_q((t)))$. More precisely, by Example 4.2.12, D-mod(LG) is compactly generated, with compact

[11]Explicitly, a representation on a complex vector space V is smooth if the action map $G(\mathbb{F}_q((t))) \times V \to V$ is continuous with respect to the discrete topology on V, i.e., the stabilizer of each vector is open. For the reader unfamiliar with p-adic representation theory, the point here is that this condition mirrors a similar one for Lie group representations, where one asks for the action to be suitably smooth in the sense of manifolds. In the p-adic setting, however, the topologies on the group and on the vector space are, informally speaking, 'coprime.' Practically, smooth representations are what show up in nature when studying automorphic forms.

objects given by $*$-extensions of bounded complexes of coherent D-modules from subvarieties of LG/K, for some K.

While the exact details of this construction are not so important for us, here are a couple orienting remarks.

Remark 5.3.2. First, one can replace all possible K in the above colimit (5.2) with any cofinal subset. A concrete such choice would be the congruence subgroups K_n, for $n \geqslant 1$, of L^+G, which we will simply describe at the level of k-points via the exact sequence

$$1 \to K_n \to L^+G = G(k[[t]]) \to G(k[[t]]/(t^n)) \to 1.$$

Noting that K_1 is the prounipotent radical of L^+G, it follows that any sufficiently small compact open subgroup K is prounipotent. In particular, for two such prounipotent subgroups $K' \subset K$, the pullback

$$\text{D-mod}(K\backslash LG/K) \to \text{D-mod}(K'\backslash LG/K')$$

is a fully faithful embedding, cf. Remark 4.3.8.

Remark 5.3.3. While the above construction may seem somewhat ad hoc, we should mention that it is a special case of a similar definition of D-modules for a larger class of infinite dimensional varieties known as placid ind-schemes [Ras15a].

In fact, a basic subtlety in this infinite dimensional setting is that there are two dual versions of the category of D-modules. One comes with functoriality for $*$-pushforwards and the other with functoriality for !-pullbacks. The presentation above is for the category with $*$-pushforwards. This is the analogue of the distinction between functions and distributions on $G(\mathbb{F}_q((t)))$. Relatedly, the dual categories for LG may be identified after trivializing the dimension torsor, i.e., fixing a compact open subgroup.

5.3.4. As in the finite dimensional case, D-mod(LG) is a monoidal dg-category under convolution. In particular, as before, we can define its category of modules

$$\text{D-mod}(LG)\text{-mod}.$$

Here are some basic examples of representations.

Example 5.3.5. If X is a placid ind-scheme acted on by LG, its category of D-modules D-mod(X) carries a canonical action of D-mod(LG) by convolution.

Example 5.3.6. Given a central extension of the loop group

$$1 \to \mathbb{G}_m \to \widetilde{LG} \to LG \to 1,$$

one obtains upon passing to Lie algebras a central extension of the loop algebra, i.e., an affine Lie algebra

$$0 \to k \cdot 1 \to \widehat{\mathfrak{g}} \to L\mathfrak{g} \to 0.$$

There is a canonical action of D-mod(LG) on the derived category of smooth modules $\widehat{\mathfrak{g}}$-mod, induced by the fact that the adjoint action of $\widetilde{LG}$ on $\widehat{\mathfrak{g}}$ factors through LG. By definition, an object of the corresponding abelian category is a representation of $\widehat{\mathfrak{g}}$ on which 1 acts by the identity and where each vector is annihilated by a compact open subalgebra. The full dg-category, following Frenkel–Gaitsgory [FG09a], is a slight modification of the naive unbounded derived category, such that objects like Verma modules or, more generally, inductions of finite dimensional representations of compact open subalgebras are compact generators.

While the details of this will not be important for us, let us simply remark that the existence and utility of this modification stem from the fact that the abelian category is of infinite cohomological dimension. Relatedly, the renormalized derived category carries a t-structure and canonically identifies on bounded below parts with the usual derived category. That is, the renormalization leads only to differences in cohomological degree $-\infty$. Similarly, the category D-mod(LG) carries a t-structure, which is canonical up to an overall shift,[12] with the property that D-mod(LG) agrees with the naive unbounded derived category of its heart, modulo issues in cohomological degree $-\infty$.

[12]This mirrors the uniqueness of the Haar measure on $G(\mathbb{F}_q((t)))$ up to scaling.

6. Local geometric Langlands and affine Beilinson–Bernstein localization

6.1. **Overview.**

6.1.1. In the previous sections, we discussed localization theory for reductive Lie algebras and explained its relation to the categorical representation theory of reductive groups. In Section 5, we met the loop groups and their categorical representations.

In this final section, we will sketch some of the basic conjectures and results in local geometric Langlands, which may be understood as parametrizing the categorical representations of loop groups. At the end, we will come full circle and meet an analogue of Beilinson–Bernstein localization for loop groups, as developed by Beilinson–Drinfeld, Frenkel–Gaitsgory, and Raskin–Yang.

6.2. **A naive formulation of the correspondence.**

6.2.1. For the p-adic group $G(\mathbb{F}_q(\!(t)\!))$, the local Langlands correspondence (conjecturally) parametrizes its irreducible representations on k vector spaces in terms of Galois theoretic data, to first approximation homomorphisms from the absolute Galois group of $\mathbb{F}_q(\!(t)\!)$ into the Langlands dual $\check{G}(k)$.

In this subsection we would like to review what is expected and what is known about an analogous parametrization of $\text{D-mod}(LG)\text{-mod}$.

6.2.2. Recall that $k(\!(t)\!)$ consists of functions on a formal punctured disk $D^\times$. Following the analogy between Galois representations and local systems, cf. Appendix A, the counterparts to $\check{G}$-valued Galois representations in the present setting are flat $\check{G}$ connections, i.e., $\check{G}$ de Rham local systems on $D^\times$.

Concretely, these are given by connections modulo gauge equivalence, i.e.,

$$\mathrm{LocSys}_{\check{G}} \simeq \{d + \check{\mathfrak{g}}(\!(t)\!)dt\}/\check{G}(k(\!(t)\!)).$$

The same formula defines $\mathrm{LocSys}_{\check{G}}$ further as an object of algebraic geometry, i.e., one takes the quotient prestack of the action of the $L\check{G}$ on the ind-scheme of connection forms $d+\check{\mathfrak{g}}(\!(t)\!)dt$, which is defined similarly to Example 5.2.2; see [Ras15b] for further discussion.

6.2.3. Let us try to arrive at the statement of the local geometric Langlands correspondence in a somewhat heuristic fashion. To first approximation, we can hope that irreducible $\text{D-mod}(LG)$ representations are in bijection with the points of $\mathrm{LocSys}_{\check{G}}$. That is, to each $\check{G}$-connection σ one should be able to canonically attach an irreducible categorical representation $\mathcal{C}_\sigma$ of LG.

Because a general object $\mathcal{C}$ of $\text{D-mod}(LG)\text{-mod}$ should decompose as a direct integral of irreducible representations, it should define a corresponding family of categories over $\mathrm{LocSys}_{\check{G}}$, whose fibres over each point measure the corresponding multiplicity of $\mathcal{C}_\sigma$ in the direct integral.

Since irreducibility is a feature of objects of abelian categories, this discussion is difficult to formulate precisely. However, there is a good notion of a family of categories over a space, namely a quasicoherent sheaf of categories, which we review below.

Modulo corrections, which we will meet duly in Equation (6.19) below, this gives the statement of the local geometric Langlands correspondence. Namely, the conjecture is that there should be an equivalence of $(\infty, 2)$-categories between all categorical representations of the loop group and sheaves of categories on the moduli space of local systems, i.e.,

$$\mathbb{L} : \text{D-mod}(LG)\text{-mod} \simeq \text{2-QCoh}(\mathrm{LocSys}_{\check{G}}). \tag{6.1}$$

Here is a basic orienting remark. The asserted equivalence of 2-categories is not saying that for a given categorical representation $\mathcal{C}$, the underlying dg-categories of $\mathcal{C}$ and $\mathbb{L}(\mathcal{C})$ are equivalent. Rather, it is saying that, for any pair of objects $\mathcal{C}$ and $\mathcal{D}$, there is an equivalence

$$\mathrm{Hom}_{\text{D-mod}(LG)\text{-mod}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}_{\text{2-QCoh}(\mathrm{LocSys}_{\check{G}})}(\mathbb{L}(\mathcal{C}), \mathbb{L}(\mathcal{D})),$$

and these are compatible with composition as one varies $\mathcal{C}$ and $\mathcal{D}$.

Remark 6.2.4. As a toy model, a Morita equivalence for usual algebras

$$A\text{-mod} \simeq B\text{-mod}$$

matches not the underlying vector spaces of modules, but only their homomorphisms. Explicitly, one can consider, for a finite dimensional k vector space V, the Morita equivalence

$$V \otimes - : \mathrm{Vect} \simeq \mathrm{End}(V)\text{-mod}\,.$$

This exchanges k and V, which certainly are not isomorphic, but on endomorphisms we do have

$$\mathrm{Hom}_{\mathrm{Vect}}(k,k) \simeq k \simeq \mathrm{Hom}_{\mathrm{End}(V)}(V,V).$$

Before turning to corrections, we would like to first discuss sheaves of categories and some of the basic compatibilites $\mathbb{L}$ is expected to satisfy.

6.2.5. To discuss sheaves of categories, let us start with the case of an affine variety $X = \mathrm{Spec}\, O_X$. In this case, a quasi-coherent sheaf of categories on X is simply an O_X-linear, as opposed to merely k-linear, dg-category. Since O_X-linearity is the same as being able to tensor by complexes of O_X-modules, we have equivalently that

$$2\text{-QCoh}(X) \simeq \mathrm{QCoh}(X)\text{-mod}\,.$$

In particular 2-QCoh(pt) is simply $\mathrm{DGCat}_{\mathrm{cont}}$, and, for any closed point $\mathrm{pt} \to X$, one has a pullback

$$\mathrm{Vect} \underset{\mathrm{QCoh}(X)}{\otimes} - : 2\text{-QCoh}(X) \to \mathrm{DGCat}_{\mathrm{cont}},$$

i.e., one can speak of the fibres of a sheaf of categories, which are usual dg-categories. In this way, one thinks of a sheaf of categories as a family of categories over X.

Remark 6.2.6. More generally, given any map $f : X \to Y$ pullback defines a monoidal functor $\mathrm{QCoh}(Y) \to \mathrm{QCoh}(X)$, and, in particular, adjoint induction and restriction functors

$$f^* : 2\text{-QCoh}(Y) \leftrightarrows 2\text{-QCoh}(X) : f_*.$$

Here is a basic example of a sheaf of categories.

Example 6.2.7. Given a map $\tilde{X} \to X$ of varieties, $\mathrm{QCoh}(\tilde{X})$ is a $\mathrm{QCoh}(X)$-algebra and, in particular, a sheaf of categories over X. For any point x of X, the fibre of the sheaf of categories is, by Example 4.2.25, simply the quasicoherent sheaves on the actual fibre, i.e.,

$$i_x^* \,\mathrm{QCoh}(\tilde{X}) \simeq \mathrm{QCoh}(\tilde{X} \underset{X}{\times} x).$$

6.2.8. For a general X, not necessarily affine, the definition proceeds by gluing, exactly as in the definition of quasicoherent sheaves. While the details are not so important for us, we should mention the following beautiful phenomenon established by Gaitsgory in [Gai15]. Namely, for general X, one by definition produces sheaves of categories by patching them affine by affine. On the other hand, one also has the global monoidal category of quasicoherent sheaves and hence its category of left modules $\mathrm{QCoh}(X)$-mod. One can therefore ask for the relation between the two. As for usual quasicoherent sheaves, one has a global sections functor

$$\Gamma : 2\text{-QCoh}(X) \to \mathrm{QCoh}(X)\text{-mod}\,. \tag{6.2}$$

For usual sheaves, the analogous map

$$\Gamma : \mathrm{QCoh}(X) \to \Gamma(X, \mathcal{O}_X)\text{-mod}$$

is rarely an equivalence if X is not affine.[13] By contrast, for most finite dimensional X, e.g., any algebraic stack of finite type, the functor (6.2) is an equivalence, a phenomenon called 1-affineness.

A basic conjecture in the subject, due to Gaitsgory, is that $\mathrm{LocSys}_{\check{G}}$ is also 1-affine; see [Ras15b]. This is a subtle assertion, as 1-affineness does not hold in general for ind-schemes or classifying stacks of group

[13]For the reader wondering about the use of 'rarely', the point is that when one considers the full derived algebra of global functions, Γ can still be an isomorphism for non-affine varieties, e.g., for open subvarieties of affine varieties.

Categorical representations of LG	Sheaves of categories on $\mathrm{LocSys}_{\check{G}}$
$\text{D-mod}(\mathrm{Gr}_G)$	$\mathrm{QCoh}(\mathrm{pt}/\check{G})$
$\text{D-mod}(\mathrm{Fl}_G^{\mathrm{aff}})$	$\mathrm{QCoh}(\widetilde{\mathcal{N}}/\check{G})$
$\text{D-mod}(LG/LN, \psi)$	$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$
$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$	$\mathrm{QCoh}(\mathrm{Op}_{\check{G}})$

FIGURE 1. Some objects (conjecturally) exchanged under local geometric Langlands.

schemes of infinite type. So, in some sense, one needs the two types of issues to 'cancel out' for $\mathrm{LocSys}_{\check{G}}$. This was partially addressed in an important work of Raskin [Ras15b], which showed that (6.2) is fully faithful for $X = \mathrm{LocSys}_{\check{G}}$.

6.3. **Properties of the correspondence I.**

6.3.1. Having dispensed with a discussion of definitions, let us get to the heart of the matter and discuss what $\mathbb{L}$ is expected to look like. More precisely, let us list a few basic objects on each side that are expected to be exchanged with one another. Combined with a compatibility with parabolic induction, this is expected to largely pin down the equivalence.

The structure of our discussion will be as follows. We have listed the categories exchanged in Figure 1. For each row in the figure, we will spend a little time describing the objects that appear on either side and motivate why one should expect them to be exchanged. Finally, we will see what matching intertwining operators amounts to in these cases and, in particular, we will meet the promised correction of (6.1).

6.3.2. For the first two examples from Figure 1, it is clarifying to recall some basic features of p-adic representation theory. By smoothness, any representation π of a p-adic group contains nontrivial invariants π^K for all sufficiently small compact open subgroups K. For such a K, if π is irreducible, the fixed vectors π^K will be an irreducible representation of the Hecke algebra $\mathrm{Dist}(K\backslash G(\mathbb{F}_q((t)))/K)$, which naturally acts on it by convolution. In this way, one obtains a bijection between irreducible representations of $G(\mathbb{F}_q((t)))$ containing K-invariant vectors and irreducible representations of this Hecke algebra.

As one shrinks the subgroup K, the corresponding Hecke algebra grows, and this gives a basic measure of the complexity of a representation. Namely, for a given irreducible π, one can ask for a maximal K for which one has nontrivial invariants, and, in some sense, the larger K is, the less complicated π is.

On the other side of local Langlands, the (Weil–Deligne variant of the) absolute Galois group of $\mathbb{F}_q((t))$ carries a canonical filtration by descending normal subgroups called the ramification filtration. For a given $\check{G}(k)$-valued Galois representation, one can measure its complexity by asking through what quotient of this filtration the action factors. A basic and striking organizing principle is that these two measures of complexity are exchanged under local Langlands.[14]

[14]More precisely, the depth of a representation, which is a certain nonnegative rational number defined by Moy–Prasad [MP94], should coincide with the slope of the corresponding Galois representation, as measured by the upper numbering ramification filtration.

6.4. **Properties of the correspondence II: unramified representations.**

6.4.1. From the preceding discussion, it follows that the most easily understood representations of p-adic groups should be those that have fixed vectors with respect to the maximal compact open subgroup $G(\mathbb{F}_q[[t]])$.[15] Similarly, the simplest Galois representations should be those on which the action factors through the final quotient of the ramification filtration, i.e., they extend to $\check{G}$-local systems on the non-punctured disk. Explicitly, these are precisely the representations which factor through the Galois group of the residue field

$$1 \to I \to \mathrm{Gal}(\mathbb{F}_q((t))^s/\mathbb{F}_q((t))) \to \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q) \simeq \widehat{\mathbb{Z}} \to 1, \tag{6.3}$$

where the superscript 's' denotes the separable closure.

Since one actually works with the Weil group, which by definition replaces $\widehat{\mathbb{Z}}$ with $\mathbb{Z}$, unramified Galois representations, up to isomorphism, are specified by elements of $\check{G}$, up to conjugacy. That is, we follow the image of $1 \in \mathbb{Z}$ under a map $\mathbb{Z} \to \check{G}$. In this simplest case, local Langlands reduces to the Satake isomorphism between the spherical Hecke algebra and the character ring of $\check{G}$, i.e.,

$$\mathrm{Dist}(G(\mathbb{F}_q[[t]])\backslash G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]])) \simeq \mathrm{Fun}(\check{G}(k)/\check{G}(k)),$$

where the right hand side denotes the adjoint invariant polynomial functions on $\check{G}$, viewed as a variety over $\mathrm{Spec}\, k$ [Sat63]. By passing to their characters, we obtain that irreducible unramified representations of $G(\mathbb{F}_q((t)))$ are parametrized by (semisimple) conjugacy classes in $\check{G}(k)$, as desired.

Remark 6.4.2. For the reader encountering these ideas for the first time, we emphasize that even the commutativity of the appearing Hecke algebra is not a priori obvious. This is illustrative of how passing to bi-invariant distributions for a large compact subgroup dramatically simplifies life.

6.4.3. To prepare for the geometric version, we should mention that each irreducible unramified representation embeds, uniquely up to scaling, into the universal unramified representation

$$\mathrm{Fun}(G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]])). \tag{6.4}$$

Moreover, to extract each from the universal case, one takes the submodule of (6.4) which transforms by the corresponding character under right convolution by the spherical Hecke algebra.

Finally, let us discuss the analogues of the above in local geometric Langlands. On the automorphic side, we consider the category of D-modules on the affine Grassmannian

$$\text{D-mod}(\mathrm{Gr}_G) = \text{D-mod}(LG/L^+G)$$

as a substitute in a way that is hopefully familiar by now to the above universal unramified representation. On the spectral side, any de Rham local system on the punctured disk that extends to the full disk is trivializable, i.e., we obtain the stacky point

$$\mathrm{pt}/\check{G} \to \mathrm{LocSys}_{\check{G}}$$

and correspondingly view $\mathrm{QCoh}(\mathrm{pt}/\check{G})$ as a sheaf of categories over $\mathrm{LocSys}_{\check{G}}$.

We should also mention that one has analogues of not only the universal case, but also of an individual irreducible unramified representation. We defer a more detailed discussion until Section 6.9.

Remark 6.4.4. We should note that, while the unramified case is the most basic, it is completely fundamental in both arithmetic and geometric Langlands. Namely, in the global Langlands correspondence, which we do not discuss in detail here, any irreducible smooth representation of the adelic group $G(\mathbb{A}_F)$ factorizes as a restricted tensor product of irreducible representations of the p-adic groups $G(F_x)$ at each place, and all but finitely many are unramified. To even make sense of such an infinite tensor product, one makes essential use of the fact that unramified representations have canonical generating lines, namely their spherical invariants. In the global geometric theory, almost all that is known so far is for the analogue of everywhere unramified (or possibly tamely ramified) automorphic representations.

[15] We should note that, for groups other than GL_n, there are other non-conjugate maximal compact subgroups.

6.5. **Properties of the correspondence III: unramified principal series.**

6.5.1. Given the unramified representations of p-adic groups, one can produce further representations as follows. As in other parts of representation theory, one has parabolic induction functors that allow one to build representations of G starting with representations of smaller groups, such as the Cartan T. Feeding unramified representations into this construction yields an important family of modules, the unramified principal series. Their simple subquotients are precisely the simple $G(\mathbb{F}_q(\!(t)\!))$-modules with nontrivial invariants for the compact open Iwahori subgroup I, i.e., the preimage of the Borel of $B(\mathbb{F}_q)$ of $G(\mathbb{F}_q)$ under the projection

$$G(\mathbb{F}_q[[t]]) \to G(\mathbb{F}_q).^{16}$$

That is, applying parabolic induction to unramified representations yields the next simplest family of representations of p-adic groups.

In spectral terms, such irreducible representations of $G(\mathbb{F}_q(\!(t)\!))$ correspond to Galois representations that are tamely ramified with unipotent monodromy. To explain this, recall from the sequence (6.3) the inertia subgroup

$$I \simeq \mathrm{Gal}(\overline{\mathbb{F}}_q(\!(t)\!)^s/\overline{\mathbb{F}}_q(\!(t)\!)),$$

where $\overline{\mathbb{F}}_q$ denotes an algebraic closure of $\mathbb{F}_q$. The next quotient in the ramification filtration splits this as

$$1 \to I_{\mathrm{wild}} \to I \to I_{\mathrm{tame}} \to 1.$$

The basic idea for I_{tame}, the tame inertia, is that it behaves like its characteristic zero counterpart, i.e., covers of a formal punctured disk over k. These covers are classified exactly like the covering spaces of a circle in topology, i.e., one adjoins an n^{th} root of t for a positive integer n. In positive characteristic, these are separable if and only if n is coprime to the characteristic of q, and such covers exhaust all Galois extensions of degree coprime to q. Explicitly, one may choose an isomorphism

$$I_{\mathrm{tame}} \simeq \prod_{\ell} \mathbb{Z}_\ell,$$

where ℓ runs over primes not equal to the characteristic of $\mathbb{F}_q$. A Galois representation is said to be tamely ramified if it factors through the quotient by I_{wild}, and it is further said to be unipotently monodromic if any element of I_{tame}, or equivalently a topological generator, is sent to a unipotent element of $\check{G}$.

6.5.2. To orient ourselves for the geometric discussion, let us review in slightly more detail how the relevant parabolic induction of p-adic representations goes. Given the Borel B, whose Levi quotient is the abstract Cartan T, one obtains a correspondence on rational points

$$T(\mathbb{F}_q(\!(t)\!)) \leftarrow B(\mathbb{F}_q(\!(t)\!)) \to G(\mathbb{F}_q(\!(t)\!)),$$

where the leftward arrow is a surjection with kernel the rational points of the unipotent radical $N(\mathbb{F}_q(\!(t)\!))$, and the rightward arrow is a closed embedding. This gives rise to a functor of parabolic induction,

$$T(\mathbb{F}_q(\!(t)\!))\text{-mod} \xrightarrow{\mathrm{Res}} B(\mathbb{F}_q(\!(t)\!))\text{-mod} \xrightarrow{\mathrm{Ind}} G(\mathbb{F}_q(\!(t)\!))\text{-mod}. \tag{6.5}$$

Explicitly, one inflates a smooth representation π of $T(\mathbb{F}_q(\!(t)\!))$ to a representation of $B(\mathbb{F}_q(\!(t)\!))$ by having the unipotent radical act trivially, and then one induces up to $G(\mathbb{F}_q(\!(t)\!))$. This induction, as usual, is concretely given by smooth sections of the corresponding vector bundle over the flag manifold, i.e.,

$$G(\mathbb{F}_q(\!(t)\!)) \overset{B(\mathbb{F}_q(\!(t)\!))}{\times} \pi \to G(\mathbb{F}_q(\!(t)\!))/B(\mathbb{F}_q(\!(t)\!)).$$

In particular, it follows that any parabolic induction of an unramified character of T embeds into the parabolic induction of the universal unramified representation, i.e. the universal principal series module

$$\mathrm{Fun}\left(G(\mathbb{F}_q(\!(t)\!))/T(\mathbb{F}_q[[t]]) \cdot N(\mathbb{F}_q(\!(t)\!))\right).$$

[16] This theorem is often attributed to Borel and Matsumoto, and we have also heard it attributed to Bernstein or Casselman; see Proposition 2.4 of [Cas80] and references therein.

Remark 6.5.3. For a reader who has not explicitly encountered parabolic induction before, it may be orienting to realize that the classification of irreducible algebraic representations of reductive groups by their highest weight is an example of parabolic induction.

Indeed, for the algebraic groups G, B, T and N as above, one has adjunctions between their categories of algebraic representations over any field

$$\mathrm{Rep}(G) \underset{\mathrm{Ind}}{\overset{\mathrm{Res}}{\rightleftarrows}} \mathrm{Rep}(B) \underset{\mathrm{Res}}{\overset{\mathrm{Coinv}_N}{\rightleftarrows}} \mathrm{Rep}(T),$$

where Res denotes restriction, Coinv_N denotes the functor of N coinvariants, and Ind is the functor of coinduction. Concatenating these, one obtains the functor of parabolic restriction, which takes a G-module and returns its N-coinvariants, viewed as a T-module, as well as its right adjoint functor of parabolic coinduction, which takes a T-module, extends it to a B-module with a trivial action of N, and coinduces this to G.

With this, the theorem of the highest weight, which can be phrased as each irreducible G-module having a line of N-coinvariants, in particular implies that every irreducible G-module appears as a submodule of a parabolically coinduced module. Moreover, in characteristic zero, by complete reducibility it follows that every irreducible coincides with a parabolically coinduced module. Via the explicit model of the coinduction of a character as sections of a line bundle, this recovers the Borel–Weil theorem about sections of line bundles on G/B.

6.5.4. Let us turn to the geometric case. In this case, the automorphic side will similarly be D-modules on a semi-infinite flag manifold, i.e.,

$$\text{D-mod}(LG/L^+T \cdot LN),$$

which, as before, may be constructed from D-mod(Gr_T) by a suitable functor of parabolic induction similar to (6.5).

Remark 6.5.5. Here is a technical comment regarding this category. It is as yet unknown how to treat the space $LG/L^+T \cdot LN$ as an object of algebraic geometry. However, its category of D-modules may be defined by enforcing smooth descent, i.e., by taking the D-mod($L^+T \cdot LN$) invariants of D-mod(LG). It is known, by an argument due to Raskin [Ras18], that this canonically identifies with the coinvariants and also with the (co)invariants for the Iwahori subgroup, i.e., D-modules on the affine flag variety. This parallels the result of Borel and Matsumoto in the p-adic setting, and in fact is proven similarly.

6.5.6. On the spectral side, one again applies a parabolic induction functor, i.e., pull-push along the correspondence

$$\mathrm{LocSys}_{\check{T}} \leftarrow \mathrm{LocSys}_{\check{B}} \rightarrow \mathrm{LocSys}_{\check{G}}$$

to QCoh(pt $/\check{T}$). Here we mean pullback and pushforward of sheaves of categories, as in Remark 6.2.6.[17]

Let us walk through how to compute this. For the first step, i.e., the pullback, note that we have a Cartesian diagram

$$\begin{tikzcd} \mathrm{LocSys}_{\check{N}} \arrow[r] \arrow[d] & \mathrm{LocSys}_{\check{B}} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{LocSys}_{\check{T}} . \end{tikzcd}$$

This simply says that a $\check{B}$ local system whose associated $\check{T}$ local system is trivialized is the same as an $\check{N}$ local system. Even more colloquially, if the transition matrices have ones along the diagonal, it's not a $\check{B}$-bundle but an $\check{N}$-bundle! By the unipotence of $\check{N}$, the moduli of $\check{N}$-local systems on the punctured disk is simply $\check{\mathfrak{n}}/\check{N}$, via the map

$$\check{\mathfrak{n}}/\check{N} \xrightarrow{\sim} \mathrm{LocSys}_{\check{N}}, \quad X \mapsto d + X \cdot \frac{dt}{t}. \tag{6.6}$$

[17]More carefully, the promised correction of Equation (6.1) will involve passing from quasicoherent sheaves of categories to certain ind-coherent sheaves of categories, but this correction will not affect this calculation.

From this discussion, it follows that the pullback is given by

$$\operatorname{QCoh}(\operatorname{LocSys}_{\check{B}}) \underset{\operatorname{QCoh}(\operatorname{LocSys}_{\check{T}})}{\otimes} \operatorname{QCoh}(\mathrm{pt}/\check{T}) \simeq \operatorname{QCoh}(\operatorname{LocSys}_P \underset{\operatorname{LocSys}_{\check{T}}}{\times} \mathrm{pt}/\check{T}) \simeq \operatorname{QCoh}(\check{\mathfrak{n}}/\check{B}). \tag{6.7}$$

Finally, the pushforward again yields $\operatorname{QCoh}(\check{\mathfrak{n}}/\check{B})$ viewed as a sheaf of categories on $\operatorname{LocSys}_{\check{G}}$ via the projection

$$\check{\mathfrak{n}}/\check{B} \to \operatorname{LocSys}_{\check{B}} \to \operatorname{LocSys}_{\check{G}}. \tag{6.8}$$

It is clarifying to slightly rewrite this purely in terms of $\check{G}$. To do so, we may tautologically rewrite our stack in terms of the cotangent bundle of the flag variety, i.e.,

$$\check{\mathfrak{n}}/\check{B} \simeq (\check{G} \overset{\check{B}}{\times} \check{\mathfrak{n}})/\check{G} =: \widetilde{\mathcal{N}}/\check{G}.$$

Explicitly, $\widetilde{\mathcal{N}}$ is the moduli space of pairs $(\check{B}', X)$, where $\check{B}'$ is a Borel subgroup and X is an element of its unipotent radical. With this, the projection (6.8) may be written via the Springer resolution, i.e., as a composition

$$\widetilde{\mathcal{N}}/\check{G} \to \check{\mathfrak{g}}/\check{G} \to \operatorname{LocSys}_{\check{G}}.$$

Here, the first map sends a pair $(\check{B}', X)$ to X, and the second map is defined similarly to (6.6) (but is very far from an isomorphism).

Remark 6.5.7. As we will review in Section 6.8.25 below, the relationship between principal series representations of G and coherent sheaves on the Springer resolution for $\check{G}$ showed up first in the p-adic theory.

6.5.8. The two preceding examples, namely the first two rows of Figure 1, concern objects of low ramification. These cases of the local geometric Langlands correspondence, while the most basic, are already serious results.

Indeed, on the one hand, as we mentioned before, they are sufficient local inputs for the simplest cases of the (conjectural) global geometric Langlands correspondence, where one studies D-modules on the moduli space of G-bundles on a global curve, possibly with parabolic reductions at finitely many points.

On the other hand, they have important applications within usual representation theory. A notable example is their implications for the representation theory of critical level affine Lie algebras, which we will discuss in more detail in Section 6.9 below. Another key application, after passing to sheaves with modular coefficients,[18] is to the modular representation theory of reductive groups; see [FM99], [MV07], [AR15], [RW18], [BGM+19], [BRR20], [BRar] and references therein, as well as the surveys [Wil17], [CW21].

Given the importance of these developments, it is rather remarkable that they only skim the surface of the full conjecture, i.e., equate only very small subcategories of each side. Indeed, in spectral terms, the previous cases concern the full subcategory of 2-$\operatorname{QCoh}(\operatorname{LocSys}_{\check{G}})$ supported on the formal neighborhood of local systems with regular singularities and unipotent monodromy, which is the smallest in an infinite increasing sequence of loci whose union exhausts $\operatorname{LocSys}_{\check{G}}$.

The next two examples of the correspondence which we discuss, by contrast, concern categorical loop group representations and sheaves of categories on $\operatorname{LocSys}_{\check{G}}$ with arbitrary ramification. As such, these are rather spectacular conjectures, among the deepest currently available in this part of representation theory, and their resolutions likely require new ideas beyond the standard toolkit of geometric representation theory.

6.6. **Properties of the correspondence IV: Whittaker models.**

[18] Strictly speaking, these fit more naturally into the local étale or Betti versions of local geometric Langlands, cf. Section 6.10 below, as they have natural analogues with varying coefficients.

6.6.1. In many situations in representation theory, e.g., algebraic representations of reductive groups or $(\mathfrak{g}, B)$-modules, irreducible representations admit a canonical generating line, namely the highest weight vectors.

By contrast, for the p-adic group $G(\mathbb{F}_q((t)))$, most irreducible modules do not admit a canonical generating line, roughly due to the noncommutativity of $\mathrm{Dist}(K\backslash G/K)$ for all sufficiently small K.

Remark 6.6.2. For unramified representations, there is a canonical generating line, namely the $G(\mathbb{F}_q((t)))$-invariants. This basic miracle, i.e., the commutativity of the spherical Hecke algebra, is crucial in the subject, e.g., for the tensor product factorization of adelic representations and the reduction of the study of unramified automorphic representations to the study of Hecke eigenfunctions and eigensheaves.

However, if one asks instead for a canonical line not inside π, but rather as a quotient of π, the situation is better. Namely, there is a canonical functor from $G(\mathbb{F}_q((t)))$-mod to vector spaces, which sends each irreducible to either a line or zero. This is the functor of Whittaker coinvariants and may be understood as follows.

To approximate the definition, we may first consider a seemingly direct analogue of highest weight theory, namely the Jacquet functor of $N(\mathbb{F}_q((t)))$-coinvariants

$$\mathrm{Jac} : G(\mathbb{F}_q((t)))\text{-mod} \to \mathrm{Vect}, \quad \pi \mapsto \pi_{N(\mathbb{F}_q((t)))}.$$

While this is very useful, it sends many interesting irreducible representations to zero[19] and others to greater than one dimensional vector spaces.

To correct for this, one twists the construction by a generic character of $N(\mathbb{F}_q((t)))$. Explicitly, let us index the simple roots by $i \in I$, and consider the composition

$$\psi : N(\mathbb{F}_q((t))) \to (N/[N,N])(\mathbb{F}_q((t))) \simeq \prod_i \mathbb{F}_q((t)) \xrightarrow{\text{sum}} \mathbb{F}_q((t)) \xrightarrow{\text{res}} \mathbb{F}_q, \tag{6.9}$$

where sum is the addition of Laurent series and res sends a Laurent series to its residue, i.e., the coefficient of t^{-1}. By composing this with a suitably generic character $\mathbb{F}_q \to k^\times$, one obtains a character of $N(\mathbb{F}_q((t)))$, which we again denote by ψ, and the Whittaker coinvariants are given by

$$\mathrm{Whit} : G(\mathbb{F}_q((t)))\text{-mod} \to \mathrm{Vect}, \quad \pi \mapsto \pi_{N(\mathbb{F}_q((t))), \psi}.$$

This sends simple modules to either a line or zero, with the crucial property that many cuspidal representations, e.g., all for $G = GL_n$, are sent to lines.

A basic property here is that Whit admits a right adjoint, given by sending k to the space of Whittaker functions

$$\mathrm{Fun}(G(\mathbb{F}_q((t)))/N(\mathbb{F}_q((t))), \psi),$$

i.e., the coinduction of ψ to $G(\mathbb{F}_q((t)))$.

6.6.3. The above constructions admit the following geometric analog. First of all, the homomorphism of topological groups (6.9) lifts to a map of group ind-schemes

$$\psi : LN \to L(N/[N,N]) \simeq \prod_i L\mathbb{G}_a \to L\mathbb{G}_a \to \mathbb{G}_a.$$

From here, we would like to pull back from $\mathbb{G}_a$ the analogue of a generic character. Recalling the analogy between the trivial representation of $G(\mathbb{F}_q)$ and the tautological categorical action of G on Vect, we therefore need a generic action of $\mathbb{G}_a$ on Vect. An action of an algebraic group H on Vect is the same data as a character sheaf on H, i.e., a D-module χ equipped with a suitably associative isomorphism

$$\mu^! \chi \simeq \chi \boxtimes \chi,$$

where $\mu : H \times H \to H$ denotes the multiplication map. We emphasize that in contrast to the broader usage of character sheaf in Section 4.4, which was the analogue of the character of a finite dimensional representation of

[19]This phenomenon is more or less the existence of cuspidal representations, i.e., of simple $G(\mathbb{F}_q((t)))$-modules that may not be found inside parabolically induced modules. We emphasize that this is a basic fact of life in the p-adic setting and in the simpler setting of finite groups of Lie type, which is absent in the algebraic representation theory of reductive groups.

$G(\mathbb{F}_q)$, the present usage of character sheaf is the analogue of a character, i.e. one dimensional representation, of $G(\mathbb{F}_q)$.

Remark 6.6.4. Slightly more informally, if for any closed point h of H we write χ_h for the corresponding !-stalk of χ, the above is giving isomorphisms of stalks

$$\chi_{h_1 h_2} \simeq \chi_{h_1} \otimes \chi_{h_2}, \quad \text{for } h_1, h_2 \in H.$$

That is, a character sheaf gives a homomorphism into lines, i.e., the invertible elements of $\mathrm{Vect}^{\heartsuit}$, just as a character gives a homomorphism into nonzero scalars, i.e., the invertible elements of k.

With this, the analogue of a generic character of $\mathbb{F}_q$ is the exponential D-module $\exp(z)$ on $\mathbb{G}_a$. Explicitly, if we write z for the standard coordinate on $\mathbb{G}_a \simeq \mathbb{A}^1$, this D-module stores the differential equation satisfied by the exponential function e^z, i.e.,

$$\exp(z) \simeq D_{\mathbb{G}_a}/(D_{\mathbb{G}_a} \cdot (\partial_z - 1)),$$

and its multiplicativity essentially follows from the identity $e^{z_1+z_2} = e^{z_1} \cdot e^{z_2}$. With this, we write ψ for the corresponding character sheaf on LN, i.e., $\psi^! \exp(z)$, and we may again form the functor of Whittaker coinvariants

$$\mathrm{Whit} : \text{D-mod}(LG)\text{-mod} \to \mathrm{Vect}, \quad \mathcal{C} \mapsto \mathcal{C}_{LN,\psi}$$

and the category of Whittaker D-modules

$$\text{D-mod}(LG/LN, \psi). \tag{6.10}$$

Before turning to its spectral counterpart, we would like to make a few orienting comments.

Remark 6.6.5. Let us try to provide three ways of thinking about the analogy between the exponential D-module on $\mathbb{G}_a(k)$ and a generic character of $\mathbb{G}_q(\mathbb{F}_q)$.

The first way is rather down to earth. For simplicity, let us suppose that $\mathbb{F}_q$ coincides with its prime subfield $\mathbb{F}_p$. In this case, if we write $e^{2\pi i/p}$ for a nontrivial pth root of unity in $k^\times$, a concrete choice of generic character is

$$\mathbb{F}_p \to k^\times, \quad a \mapsto e^{a \cdot 2\pi i/p},$$

and any other differs by multiplication by a nonzero element of $\mathbb{F}_p$. That is, a generic character looks like an exponential. The same holds, mutatis mutandis, for a nontrivial extension $\mathbb{F}_q$, where one first uses the trace map $\mathbb{F}_q \to \mathbb{F}_p$.

The second way is also rather practical. Namely, for a k vector space V, its character sheaves are in canonical bijection with closed points of V^*, where to a covector ξ one attaches the character sheaf $\xi^! \exp(z)$. In particular, for $V = \mathbb{G}_a$, there is only one nontrivial choice of character sheaf up to dilation by $\mathbb{G}_m$, just as there is only one nontrivial character for $\mathbb{F}_q$ up to dilation by $\mathbb{F}_q^\times$.

The third way is more general and goes via the functions-sheaves correspondence. To begin, a beautiful feature of positive characteristic is an interesting self-isogeny of any commutative group scheme H over $\mathbb{F}_q$, the Lang isogeny, which takes the form

$$1 \to H(\mathbb{F}_q) \to H \to H \to 1,$$

i.e., it cuts out as its kernel the finite group of rational points. For $H = \mathbb{G}_a$, this is concretely the short exact sequence

$$1 \to \mathbb{G}_a(\mathbb{F}_q) \to \mathbb{G}_a \xrightarrow{x \mapsto x^q - x} \mathbb{G}_a \to 1.$$

Pushing forward the constant sheaf along this isogeny and taking its direct summands yields a family of character sheaves on $\mathbb{G}_a$ indexed by irreducible representations of $\mathbb{G}_a(\mathbb{F}_q)$. Upon taking trace of Frobenius, these character sheaves yield exactly the irreducible characters of $\mathbb{G}_a(\mathbb{F}_q)$. That is, characters of $\mathbb{F}_q$ have canonical lifts to character sheaves on $\mathbb{G}_a$.

For these character sheaves, viewed simply as étale local systems on $\mathbb{A}^1$, one can ask about their behavior at ∞, i.e., after compactification to $\mathbb{P}^1$. One finds that they are wildly ramified at this remaining point. The correspondence with $\exp(z)$ in characteristic zero then fits into a broader analogy between wildly ramified Galois representations and de Rham local systems with irregular singularities.

Remark 6.6.6. Let us comment on the definition of the category of Whittaker D-modules D-mod$(LG/LN, \psi)$. This is defined by taking the (LN, ψ)-coinvariants of D-mod(LG), similarly to the categories of D-modules on semi-infinite flag manifolds. It is also identified with the (LN, ψ)-invariants by an important theorem of Raskin [Ras21]. While an identification of Whittaker invariants and coinvariants does not literally hold in the p-adic setting, it is analogous to the compact approximation of the Whittaker model by Rodier [Rod75].

6.6.7. Let us now describe the expected spectral counterpart to the Whittaker model. This is simply

$$\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}}),$$

i.e., all quasicoherent sheaves on the moduli of local systems.

The basic intuition here again can be taken from the p-adic setting. Namely, each irreducible representation π of $G(\mathbb{F}_q((t)))$ has, at most, a line of Whittaker coinvariants. It will be convenient to work instead with

$$\mathrm{Hom}(\mathrm{ind}_{N(\mathbb{F}_q((t)))}^{G(\mathbb{F}_q((t)))} k_{-\psi}, \pi) =: \ell,$$

where ind is the functor of compactly supported induction. This identifies with the Whittaker coinvariants of the contragredient representation of π and, in particular, is again a line or zero.

On the other hand, under local Langlands one associates to π a Galois representation σ and imagines π as 'sitting over' σ in the space $\mathrm{Gal}_{\check{G}}$ of Galois representations.

In the modern formulations of arithmetic local Langlands as an equivalence of categories, cf. Section 6.10 below, it is expected that any π admitting nonzero Whittaker coinvariants is exchanged with a one dimensional skyscraper sheaf $\mathbb{L}(\pi)$ supported over σ. Moreover, this line canonically associated to π is expected to simply be ℓ. So, combining the identifications

$$\mathrm{Hom}(\mathrm{ind}_{N(\mathbb{F}_q((t)))}^{G(\mathbb{F}_q((t)))} k_{\psi}, \pi) \simeq \ell \simeq \mathrm{Hom}(\mathcal{O}_{\mathrm{Gal}_{\check{G}}}, \mathbb{L}\pi),$$

we see that $\mathbb{L}$ ought to exchange the (compactly) induced Whittaker module and the structure sheaf $\mathcal{O}_{\mathrm{Gal}_{\check{G}}}$.

In the geometric setting, we have already replaced the Whittaker module by the corresponding space of D-modules D-mod$(LG/LN, \psi)$.[20] Similarly, we replace the structure sheaf $\mathcal{O}_{\mathrm{Gal}_{\check{G}}}$ by $\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})$, which again corepresents global sections, but now of sheaves of categories.

As we will shortly discuss when we get to intertwining operators, this correspondence leads to many striking and concrete conjectures in geometry and representation theory. However, for now let us conclude our discussion of this row of Figure 1 with a couple remarks.

Remark 6.6.8. Unlike in the p-adic setting, where not every tempered irreducible representation admits a Whittaker model, the expectation in local geometric Langlands is that every tempered D-mod(LG) representation does.

The basic idea is the following. For a point x of a stack X with isotropy group I_x, the abelian category of quasicoherent sheaves on X scheme theoretically supported at x identifies with $\mathrm{Rep}(I_x)$, and in particular the skyscraper will not generate unless I_x is unipotent. Under local arithmetic Langlands, this matches Shahidi's generic packet conjecture, which states that, within a tempered L-packet,[21] exactly one irreducible representation will admit a Whittaker model [Sha90].

By contrast, the skyscraper module Vect for $\mathrm{QCoh}(X)$ will generate the subcategory of 2-$\mathrm{QCoh}(X)$ supported at x, and in this sense every irreducible tempered module is generic. Despite this, it will not be the case that D-mod$(LG/LN, \psi)$ is a generator for D-mod(LG), for issues having to do with the promised correction to the naive formulation (6.1).

[20] Note that the difference between induction and compact induction disappears, due to Raskin's identification of Whittaker invariants and coinvariants [Ras21].

[21] Beyond the case of GL_n, the map from irreducible representations of $G(\mathbb{F}_q((t)))$ to Galois representations is not a bijection, but instead finite-to-one, and the fibres are called L-packets.

6.7. **Properties of the correspondence V: Kac–Moody representations.**

6.7.1. We have arrived at the last row of Figure 1, and here something rather remarkable occurs. Namely, we will meet a conjectural property of the correspondence, proposed by Frenkel–Gaitsgory [FG06a], which concerns Kac–Moody representations. As such, it has no p-adic counterpart and is an aspect of local Langlands which is special to the geometric context.

Remark 6.7.2. We should mention that the importance of Kac–Moody representations in geometric Langlands first appeared in the global story. Namely, as we alluded to Section 3.4.13, a method of construction for automorphic D-modules, due to Beilinson–Drinfeld, is via localization of Kac–Moody representations. This may be understood as a global analogue of the work of Frenkel–Gaitsgory and was the first instance where the relationship to conformal field theory provided tools unavailable in the arithmetic setting.

6.7.3. We would first like to give the reader a basic feel for Kac–Moody algebras. Their study for a general connected reductive group essentially reduces to the case when $\mathfrak{g}$ is simple. So, to avoid mostly notational distractions, we shall assume that G is a simply connected, almost simple group.

First, we begin with the loop algebra

$$\mathfrak{g}((t)) \simeq \mathfrak{g} \underset{k}{\otimes} k((t)).$$

This is simply the extension of scalars of $\mathfrak{g}$ to the field of Laurent series. Explicitly, the bracket takes the form

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg, \quad X, Y \in \mathfrak{g}, \quad f, g \in k((t)).$$

It is clarifying to think of it as the space of maps from the punctured disk $\mathcal{D}^\times$ to $\mathfrak{g}$, where the Lie bracket is performed pointwise. That is, it is the infinitesimal symmetries of the trivial G-bundle on the punctured disk.

Let us regard the loop algebra as an infinite dimensional Lie algebra over k. A basic fact of life is that many k vector spaces that naively seem like $\mathfrak{g}((t))$-modules in fact carry an action of a central extension, i.e., of an affine Lie algebra

$$0 \to k \cdot \mathbf{1} \to \widehat{\mathfrak{g}} \to \mathfrak{g}((t)) \to 0.$$

Remark 6.7.4. This may be understood as an instance of the general phenomenon that quantization of field theories introduces anomalies for their local symmetries.

Fortunately, all the possible central extensions of the loop algebra admit a simple classification – they are in canonical bijection with the line of invariant bilinear forms on the finite dimensional algebra $\mathfrak{g}$. Explicitly, to such a form $\kappa \in (\mathfrak{g}^* \otimes \mathfrak{g}^*)^G$, one associates the central extension

$$0 \to k \cdot \mathbf{1} \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0,$$

where the bracket of $\mathbf{1}$ with anything is zero, by centrality, and the other brackets are modified by

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg - \kappa(X, Y)\,\mathrm{Res}(f \cdot dg).$$

6.7.5. For a level κ, let us denote the category of smooth $\widehat{\mathfrak{g}}_\kappa$-modules on which the central element $\mathbf{1}$ acts via the identity by $\widehat{\mathfrak{g}}_\kappa$-mod, cf. Example 5.3.6. If κ is integral, meaning the extension of the loop algebra arises from a central extension of LG, then, as discussed in Example 5.3.6, this category carries an action of D-mod(LG), and we therefore shall focus on this case.

Recalling that G was assumed to be simple and simply connected, the integral levels form a lattice

$$\mathbb{Z} \hookrightarrow (\mathfrak{g}^* \otimes \mathfrak{g}^*)^G,$$

generated by the so-called basic form, which explicitly gives the short coroots squared length two.

6.7.6. A basic observation in the study of representations of affine Lie algebras is that as one varies the level κ, the corresponding representation theory has two 'phases.'

Namely, for all sufficiently negative definite κ, the representation theory behaves similarly. For example, all Verma modules have finite length, and the character formulas for their simple quotients across these levels take essentially the same form [KT95].

Likewise, for all positive definite κ, the representation theory behaves similarly as one varies the level. Here, the Verma modules now have infinite length, and the character formulas for their simple quotients do not substantially vary with the level [Kas90].

We may therefore anticipate that there is a special level κ_c, the critical level, where the representation theory undergoes a phase transition. Naively, one might expect this central point to be the zero form. Instead, due to an affine analogue of the ρ-shift in the representation theory of $\mathfrak{g}$, it is minus half of the Killing form, i.e.,

$$\kappa_c = -\frac{1}{2} \cdot \kappa_{\mathrm{Killing}}.$$

6.7.7. As one might expect, the behavior of the representation theory at the phase transition, i.e., at critical level, displays many features not present at other levels.

A fundamental such property was discovered by Feigin and Frenkel [FF91]. Namely, while the center of the enveloping algebra[22] at noncritical levels is trivial, i.e., only scalars, they showed that the center at critical level was nontrivial. Crucially, they further identified it with the algebra of functions on the moduli space $\mathrm{Op}_{\check{G}}$ of $\check{G}$-opers on the punctured disk. While we will momentarily recall its precise definition, at first pass the main point is that representation theory at the critical level has local Langlands duality, particularly the geometry of $\check{G}$-connections, hardwired into it.

6.7.8. To first approximation, a $\check{G}$-oper consists of a $\check{G}$ bundle with a connection and a $\check{B}$ reduction, which are suitably transverse.

Let us spell this out explicitly. Recall that to speak of Langlands dual groups, one fixes pinnings of G and $\check{G}$ and, in particular, Chevalley generators $f_i, \check{\alpha}_i, e_i, i \in I$, of $\check{\mathfrak{g}}$. If we write f for the element of $\check{\mathfrak{g}}$ such that

$$(f, 2\check{\rho}, \sum_i e_i)$$

form an $\mathfrak{sl}_2$ triple, then the space of $\check{G}$-opers is the moduli space of connections

$$\mathrm{Op}_{\check{G}} \simeq \{d + f dt + \check{\mathfrak{b}}(\!(t)\!)dt\}/L\check{N}. \tag{6.11}$$

By thinking of the trivial bundle as equipped with its tautological $\check{B}$-reduction, the term $f dt$ in the above formula implements the aforementioned transversality of the connection and the reduction. For alternative formulations of the definition of an oper, e.g., with the advantage of visibly being independent of the coordinate t, see for example [BD04].

The formula (6.11) may be read as taking a certain affine subspace of all connections $d + \check{\mathfrak{g}}(\!(t)\!)dt$ on the trivial $\check{G}$ bundle and quotienting by the adjoint action of $L\check{N}$. Indeed, (6.11) is the Hamiltonian reduction of the affine space $d + \check{\mathfrak{g}}(\!(t)\!)dt$ with respect to a generic character of $L\check{N}$ and, in particular, carries a canonical Poisson structure. Gelfand–Dikii and Drinfeld–Sokolov [DS84] studied its Poisson geometry in the context of integrable hierarchies prior to its appearance in geometric Langlands; for $\mathfrak{g} = \mathfrak{sl}_2$ this recovers the celebrated KdV hierarchy.

Here are two orienting remarks for a reader who has not encountered this moduli space before.

Remark 6.7.9. Although $\mathrm{Op}_{\check{G}}$ is presented as space of connections modulo the action of a loop group, similarly to $\mathrm{LocSys}_{\check{G}}$, its geometry is astronomically simpler. In fact, it is simply an infinite dimensional affine space, and, in particular, its points do not have automorphisms.

[22] As a technical remark, we mean the center of the appropriate completed enveloping algebra, or equivalently the Bernstein center of the abelian category of smooth $\widehat{\mathfrak{g}}_\kappa$-modules.

Remark 6.7.10. The previous remark is typically proven by showing every $L\check{N}$ orbit is simply transitive, i.e., without nontrivial stabilizers, and exhibiting an explicit slice intersecting each orbit exactly once.

It can be clarifying to note that this whole story, including the above argument, is an affine analogue of a work of Kostant [Kos63]. Namely, he showed that one has a canonical isomorphism

$$\check{\mathfrak{g}}/\!/\check{G} \simeq (f + \check{\mathfrak{b}})/\check{N},$$

i.e., that a similar finite dimensional Hamiltonian reduction of $\check{\mathfrak{g}}$ cuts out a copy of the Poisson center. Moreover, he showed the natural quantizations of both sides are matched, namely that the center $Z(\check{\mathfrak{g}})$ may be cut out from $U(\check{\mathfrak{g}})$ via a quantum Hamiltonian reduction with respect to $\check{N}$.

6.7.11. Via the projection map $\mathrm{Op}_{\check{G}} \to \mathrm{LocSys}_{\check{G}}$, where one passes from an oper to its underlying connection, we may view $\mathrm{QCoh}(\mathrm{Op}_{\check{G}})$ as a sheaf of categories over $\mathrm{LocSys}_{\check{G}}$.

With this, it is expected that $\widehat{\mathfrak{g}}_{\kappa_c}$-mod is exchanged with $\mathrm{QCoh}(\mathrm{Op}_{\check{G}})$ under local geometric Langlands, so that the tautological monoidal functor

$$\mathrm{QCoh}(\mathrm{Op}_{\check{G}}) \to \mathrm{Hom}_{2\text{-}\mathrm{QCoh}(\mathrm{LocSys}_{\check{G}})}(\mathrm{QCoh}(\mathrm{Op}_{\check{G}}), \mathrm{QCoh}(\mathrm{Op}_{\check{G}}))$$

is exchanged, via the Feigin–Frenkel isomorphism $\mathcal{O}_{\mathrm{Op}_{\check{G}}} \simeq Z(\widehat{\mathfrak{g}}_{\kappa_c})$, with the central action

$$Z(\widehat{\mathfrak{g}}_{\kappa})\text{-mod} \to \mathrm{Hom}_{\mathrm{D}\text{-}\mathrm{mod}(LG)\text{-}\mathrm{mod}}(\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}, \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}).$$

In the next subsections, we will discuss several interesting conjectures this correspondence yields about representation theory at critical level.

However, we would like to already mention here one particularly striking consequence. It is known that any $\check{G}$-connection σ admits an oper structure χ, i.e., the projection $\mathrm{Op}_{\check{G}} \to \mathrm{LocSys}_{\check{G}}$ is surjective on k-points [FZ10]. By the above, it formally follows that the 'irreducible' categorical representation $\mathcal{C}_\sigma$ of the loop group, which corresponds to the skyscraper Vect over σ, is given by

$$\widehat{\mathfrak{g}}\text{-mod} \underset{\mathrm{QCoh}(\mathrm{Op}_{\check{G}})}{\otimes} \mathrm{Vect}, \tag{6.12}$$

i.e., the category of Kac–Moody representations with central character χ.

On the one hand, this gives a uniform conjectural construction of all the $\mathcal{C}_\sigma$ via Kac–Moody representations, i.e., the spectral to automorphic direction of the local geometric Langlands correspondence. On the other hand, this predicts that for any two opers χ, χ' lying over σ, the corresponding categories of Kac–Moody modules are equivalent, which is by itself a deep conjecture in representation theory.

Remark 6.7.12. In fact, due to the correction to the naive formulation (6.1) of local geometric Langlands, which we have yet to explain, the category (6.12) is really a full subcategory of $\mathcal{C}_\sigma$. In practice, this means that one obtains $\mathcal{C}_\sigma$ by renormalizing (6.12), cf. Section 6.9 for further discussion.

6.7.13. This completes our introductory discussion of the objects appearing in Figure 1. However, before going further, we should emphasize that there are many more expected compatibilities coming from the arithmetic story and connections with physics, cf. Section 6.10 below.

We cannot help but mention one particularly spectacular example of this. Braverman–Finkelberg have proposed for $G = GL_n$ a candidate for the Langlands dual to the convolution algebra D-mod(LGL_n) itself [BF19]. This is based on physical considerations, ultimately going back to a striking formula for computing S-dual boundary conditions for four dimensional $\mathcal{N} = 4$ Yang–Mills theory discovered by Gaiotto-Witten [GW09]. Explicitly, they conjecture that D-mod(LGL_n) is exchanged under local Langlands duality with coherent sheaves on the moduli space parametrizing degenerate flags of local systems, i.e., local systems $\mathcal{L}_i$ of rank i, $1 \leqslant i \leqslant n$, along with not necessarily injective maps of connections

$$\mathcal{L}_i \to \mathcal{L}_{i+1}, \quad 1 \leqslant i < n.$$

An arithmetic analogue of this conjecture has since been formulated by Zhu [Zhu21].

6.8. **Intertwining operators.** Next, we would like to explain how considering the intertwining operators between the categories appearing in Figure 1 recovers several fundamental theorems and produces many striking conjectures.

At this point, the promised technical correction to the naive formulation (6.1) of the local geometric Langlands correspondence naturally asserts itself in the following manner.

6.8.1. Recall the simplest representations in the p-adic setting are the unramified ones. Pleasantly, we will already meet the necessary correction to the naive formulation (6.1) of the geometric conjecture in the unramified case.

To see this, recall that local Langlands ought to exchange the D-modules on the affine Grassmannian $\text{D-mod}(\mathrm{Gr}_G)$ with quasicoherent sheaves on the trivial bundle $\mathrm{QCoh}(\mathrm{pt}/\check{G})$. In particular, their endomorphisms ought to be canonically identified, so let us see what they are.

On the automorphic side, recalling that Gr_G is the coset space LG/L^+G, as in Section 4 we have

$$\mathrm{Hom}_{\text{D-mod}(LG)\text{-mod}}(\text{D-mod}(\mathrm{Gr}_G), \text{D-mod}(\mathrm{Gr}_G)) \simeq \text{D-mod}(L^+G\backslash LG/L^+G), \tag{6.13}$$

also known as the derived Satake category of bispherical sheaves. As a potential warning for readers encountering this for the first time, this category is not the derived category of the corresponding abelian category, but instead richer.

On the spectral side, using that the map $\check{\mathfrak{g}}/\check{G} \to \mathrm{LocSys}_{\check{G}}$ is formally étale at the trivial bundle, we may compute, as in Example 4.2.25, that

$$\mathrm{Hom}_{2\text{-QCoh}(\mathrm{LocSys}_{\check{G}})}(\mathrm{QCoh}(\mathrm{pt}/\check{G}), \mathrm{QCoh}(\mathrm{pt}/\check{G})) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{G} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{pt}/\check{G}) \simeq \mathrm{QCoh}(\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt}/\check{G}). \tag{6.14}$$

A couple points of explanation are in order. First, for the last equivalence in (6.14), one may use the following general observation. Suppose one has varieties X, Y, and Z acted upon by an algebraic group K. Then, given a K equivariant correspondence

$$X \to Z \leftarrow Y,$$

one has a canonical isomorphism of quotient (derived) stacks

$$X/H \underset{Z/H}{\times} Y/H \simeq (X \underset{Z}{\times} Y)/H.$$

We will use this fact repeatedly going forwards without further comment.

Second, we also remind the reader that the appearing fibre products in (6.14) are derived schemes. Concretely, this just means we also remember the Tor, so that the self-intersection

$$\mathrm{pt} \underset{\check{\mathfrak{g}}}{\times} \mathrm{pt}$$

is the spectrum of the exterior algebra $\mathrm{Sym}\, \check{\mathfrak{g}}^*[1]$, by a standard calculation using Koszul resolutions.

We can therefore naively hope for an equivalence of monoidal categories between (6.13) and (6.14). Instead, the derived Satake equivalence, which is an important theorem of Bezrukavnikov–Finkelberg [BF08], identifies (6.13) with an enlargement of (6.14). More precisely, they proved an equivalence

$$\text{D-mod}(L^+G\backslash LG/L^+G) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathrm{pt}/\check{G}), \tag{6.15}$$

where the right hand side is the monoidal category of ind-coherent sheaves with nilpotent singular support.

6.8.2. Let us briefly indicate what is meant by an ind-coherent sheaf, and what it means to impose a singular support condition. These are admittedly somewhat technical corrections to the idea of a quasicoherent sheaf, and may be safely skipped by a reader meeting this circle of ideas for the first time. We begin with ind-coherent sheaves, and then turn to the singular support condition.

In the simplest possible terms, the theory of ind-coherent sheaves stems from the difference between finitely generated modules and finitely generated projective modules, i.e., vector bundles. In the abelian category, say for an affine scheme X, there are certainly fewer vector bundles than finitely generated modules, unless X is a disjoint union of reduced points.

However, in the derived category, the distinction is subtler. Indeed, if X is smooth, then any finitely generated module admits a finite resolution by vector bundles. For this reason, one cannot distinguish between finite complexes of finitely generated modules and finite complexes of vector bundles. If X is not smooth, they again differ in the derived category, as a general finitely generated module now needs an infinite projective resolution. Explicitly, one could take the skyscraper sheaf at any singular point of X. In particular, the distinction between these types of complexes gives a measure of the singularities of X.

6.8.3. Let us treat the preceding paragraphs more carefully, following Krause [Kra05] and Gaitsgory [Gai13]. To begin with, recall that, for an algebraic variety X, its category of quasicoherent sheaves $\mathrm{QCoh}(X)$ is compactly generated by the perfect complexes, i.e., objects locally equivalent to finite complexes of vector bundles. If X is affine, this reduces to the tautology that O_X-mod is generated by the ring of regular functions O_X. For non-affine X, the claimed generation is straightforward for quasi-projective varieties, but is in general a somewhat delicate theorem of Thomason [TT90].

As the global analogue of finite complexes of finitely generated modules, consider the bounded derived category of coherent sheaves

$$\mathrm{Coh}(X)^b \subset \mathrm{QCoh}(X).$$

By definition, $\mathrm{IndCoh}(X)$ is its ind-completion, i.e., $\mathrm{IndCoh}(X)$ is compactly generated by $\mathrm{Coh}(X)^b$. Plainly, objects of $\mathrm{IndCoh}(X)$ are formal colimits of bounded complexes with coherent cohomology, and complexes of homomorphisms are computed as the homotopy limit of complexes

$$\mathrm{Hom}_{\mathrm{IndCoh}(X)}(\varinjlim_i \mathcal{E}_i, \varinjlim_j \mathcal{F}_j) \simeq \varprojlim_i \varinjlim_j \mathrm{Hom}_{\mathrm{QCoh}(X)}(\mathcal{E}_i, \mathcal{F}_j).$$

For any coherent sheaf $\mathcal{E}$, this formula forces $\mathrm{Hom}_{\mathrm{IndCoh}(X)}(\mathcal{E}, -)$ to commute with colimits, which fails in $\mathrm{QCoh}(X)$ unless $\mathcal{E}$ is perfect.

Since perfect complexes are bounded coherent, one obtains upon ind-completing their inclusion functor an adjunction

$$\mathrm{QCoh}(X) \rightleftarrows \mathrm{IndCoh}(X),$$

where the left adjoint is fully faithful and an equivalence if X is smooth. In particular, one thinks of $\mathrm{IndCoh}(X)$ as an enlargement of $\mathrm{QCoh}(X)$, and one has a truncation of any ind-coherent sheaf to a quasi-coherent sheaf. Morever, $\mathrm{IndCoh}(X)$ carries a natural t-structure and agrees with $\mathrm{QCoh}(X)$ on bounded below objects. Informally, one obtains $\mathrm{IndCoh}(X)$ by adjoining objects concentrated in cohomological degree $-\infty$.

Remark 6.8.4. It may be orienting to recall that we already met such renormalizations of derived categories of infinite cohomological dimension when discussing Kac–Moody representations and D-modules on the loop group, cf. Example 5.3.6.

Remark 6.8.5. The idea of systematically distinguishing between perfect and bounded coherent complexes on singular varieties also arose earlier in algebraic geometry, e.g., the quotient $\mathrm{IndCoh}(X)/\mathrm{QCoh}(X)$ is the ind-completion of Orlov's singularity category [Orl04].

6.8.6. Let us next explain the idea of singular support, as developed by Gulliksen [Gul74], Eisenbud [Eis80], Avramov, and Buchweitz [AB00] for complete intersections, and extended by Arinkin–Gaitsgory to derived complete intersections [AG15a]; see also the important [BIK08]. By definition, the difference between ind-coherent and quasicoherent sheaves stems from singularities of X, and, in particular, the quotient

$$\mathrm{IndCoh}(X)/\mathrm{QCoh}(X)$$

will be supported along the singular locus of X. Given a coherent sheaf $\mathcal{F}$, one might like an invariant of it that records, to first approximation, where along the singularities of X it fails to be perfect. Singular support provides a refined answer to this question.

Here is a slightly more precise formulation. Let us suppose X is a quasi-smooth derived scheme, i.e., one whose cotangent complex $T^*(X)$ is supported in cohomological degrees -1 and 0. Then, given an object $\mathcal{F}$ as above, its singular support is a closed conical subset Λ of $H^{-1}T^*(X)$. The singular support measures, affine by affine, the local nilpotence of elements of $\mathcal{O}_X$, recovering the usual support, as well as certain natural degree two self-maps of $\mathcal{F}$. Moreover, given any such subset Λ, one obtains the full subcategory

$$\mathrm{IndCoh}_\Lambda(X) \subset \mathrm{IndCoh}(X)$$

of ind-coherent sheaves with singular support contained in Λ. For example, if Λ is the zero section, this recovers $\mathrm{QCoh}(X)$.

Although we will not provide a general definition of how to compute singular support, we will discuss some special cases momentarily. First, however, we include a few orienting remarks.

Remark 6.8.7. A derived scheme is quasi-smooth if and only if it locally may be written as a fibre product $X \times_Z Y$, where X, Y, and Z are smooth varieties. In this local picture, the cotangent complex takes the following explicit form. Namely, for a point (x, y) of the fibre product with common image z, one has a distinguished triangle of tangent complexes

$$T_{(x,y)}(X \times_Z Y) \to T_x(X) \oplus T_y(Y) \to T_z(Z) \xrightarrow{+1} . \tag{6.16}$$

For the reader unfamiliar with this distinguished triangle, it may be helpful to first understand the case when X and Y map transversally to Z. Here, the fibre product is again smooth, and the above distinguished triangle becomes a hopefully intuitive short exact sequence. For the general case, the point is that the derived structure on the fibre product modifies its tangent complex so that the most naive modification of the transverse situation holds.

Upon dualizing (6.16), one sees that H^{-1} of the cotangent bundle is concretely the codirections $\xi \in T^*_z(Z)$ normal to the images of $T_x(X)$ and $T_y(Y)$ or, even more informally, the remaining directions to walk in Z.

An important special case of the preceding is a fibre product of the form

$$\{0\} \times_{\mathbb{A}^n} Y, \tag{6.17}$$

i.e., the derived vanishing locus of n functions $f_1, \ldots, f_n$ on Y. In this way, quasi-smoothness is an example of the hidden smoothness philosophy in derived algebraic geometry. Namely, while the underlying classical scheme of the fibre product may be arbitrarily singular, it underlies a derived scheme that behaves like a complete intersection, i.e., is only mildly singular.

Let us next explain what singular support amounts to in a few examples.

Example 6.8.8. For a coherent sheaf $\mathcal{F}$, the intersection of its singular support with the zero section is simply the usual support.

Example 6.8.9. Suppose X may be exhibited as a hypersurface in a smooth variety U. In this case,

$$H^{-1}T^*(X)$$

consists of X, thought of as the zero section, along with a line ℓ_x placed at each singular point x of X. Suppose a coherent sheaf $\mathcal{F}$ contains such an x in its usual support. Then its singular support contains ℓ_x if and only if $\mathcal{F}$ is not isomorphic to a perfect complex in any Zariski neighborhood of x.

Example 6.8.10. Suppose X is given, as in the spectral side of derived Satake, as the self-intersection of the origin in a vector space

$$X = 0 \underset{V}{\times} 0.$$

In this case, one has that $T^*_{(0,0)}X \simeq V^*[1]$. Here, singular supports may be understood as usual supports on the other side of Koszul duality. Namely, the skyscraper sheaf k generates $\mathrm{IndCoh}(X)$, which yields upon taking its endomorphisms an equivalence

$$\mathrm{IndCoh}(X) \simeq \mathrm{Sym}\, V[-2]\text{-mod}\,.$$

Via this identity, for any coherent sheaf $\mathcal{F}$, its singular support agrees with the usual support of the corresponding $\operatorname{Sym} V[-2]$-module, ignoring grading shifts. One can replace 0 and V here by any smooth point of any variety.

6.8.11. Having discussed the basics of ind-coherent sheaves and singular support, let us return to the derived Satake equivalence, i.e.,

$$\text{D-mod}(L^+G\backslash LG/L^+G) \simeq \operatorname{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathrm{pt}/\check{G}). \tag{6.18}$$

In this equivalence, the subscript 'nilp' refers to the subcategory of ind-coherent sheaves with singular support lying in the nilpotent cone

$$\mathcal{N} \subset \check{\mathfrak{g}}^* \simeq H^{-1}T^*(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathrm{pt}/\check{G}).$$

6.8.12. As we will see shortly, the pattern encountered in the unramified case, where one has to enlarge the spectral side by allowing ind-coherent sheaves with nilpotent singular support, occurs in several other cases of matching intertwining operators. This necessitates tweaking the spectral side of local geometric Langlands, which is accomplished by passing from quasicoherent sheaves of categories to ind-coherent sheaves of categories with nilpotent singular support

$$\text{2-QCoh}(\mathrm{LocSys}_{\check{G}}) \rightsquigarrow \text{2-IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}_{\check{G}}).$$

We will touch on what this means momentarily, but for now let us say that with this correction, we obtain the current formulation of the local geometric Langlands conjecture, namely an equivalence

$$\text{D-mod}(LG)\text{-mod} \simeq \text{2-IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}_{\check{G}}). \tag{6.19}$$

6.8.13. Let us briefly sketch the theory of ind-coherent sheaves of categories with singular support, which is due to Arinkin [Ari18].

When one increases the categorical level from ind-coherent sheaves to ind-coherent sheaves of categories, to set up a theory of singular support one no longer asks that the base variety X be quasi-smooth, i.e., has its cotangent complex in degrees -1 and 0, but instead that it be smooth, i.e., has its cotangent complex in degree 0. We will see why momentarily.

So, let X be a smooth algebraic variety. Using 1-affineness, one can show that, for any proper map $f : Z \to X$ from a smooth variety, $\operatorname{QCoh}(Z)$ compactly generates the full subcategory of 2-$\operatorname{QCoh}(X)$ consisting of objects supported over $\operatorname{im}(f)$. In particular, this subcategory identifies with modules for the convolution algebra

$$\operatorname{QCoh}(Z \underset{X}{\times} Z).$$

The $(\infty, 2)$-category of ind-coherent sheaves of categories on X has the following basic properties. First, it carries a monoidal action of 2-$\operatorname{QCoh}(X)$, so that one can speak of the supports of objects. Moreover, it is generated by objects of the form $\operatorname{QCoh}(Z)$, for Z as above. However, the endomorphisms of such an object are now increased to

$$\operatorname{IndCoh}(Z \underset{X}{\times} Z),$$

which, roughly speaking, grows larger relative to its quasicoherent analog as $Z \to X$ becomes further from a smooth map. More precisely, as in Remark 6.8.7, the singular support of objects in this convolution category lie along codirections in X normal to the tangent bundle of $Z \times Z$. As a consequence, the less smooth f is over a given point x, the more ind-coherent sheaves of categories supported over x lie in the subcategory of 2-$\operatorname{IndCoh}(X)$ generated by $\operatorname{QCoh}(Z)$.

In particular, $\operatorname{QCoh}(Z)$ will no longer generate a full subcategory of 2-$\operatorname{IndCoh}(X)$ corresponding to objects supported over $\operatorname{im}(f)$, but rather the following microlocal refinement.[23] Since we assumed f is proper, we

[23]Explicitly, to see that a refinement is necessary, note that, in its absence, by taking $Z = X$ we would equate 2-$\operatorname{IndCoh}(X)$ and 2-$\operatorname{QCoh}(X)$.

may form the conormal variety $N^\vee_Z$, i.e., the closed isotropic subvariety of T^*X obtained by projecting the kernel of $f^*T^*X \to T^*Z$ along

$$Z \underset{X}{\times} T^*X \to T^*X.$$

One can associate to any ind-coherent sheaf of categories its support in T^*X, and $\mathrm{QCoh}(Z)$ generates the full subcategory of objects with microsupport lying in $N^\vee_Z$.

More generally, given any closed, conical isotropic subvariety Λ of T^*X, one associates the full subcategory

$$2\text{-}\mathrm{IndCoh}_\Lambda(X) \subset 2\text{-}\mathrm{IndCoh}(X).$$

This is again generated by the (truncations) of $\mathrm{QCoh}(Z)$, with endomorphisms

$$\mathrm{IndCoh}_\Lambda(Z \underset{X}{\times} Z),$$

where now Λ is used to impose a usual singular support restriction on usual ind-coherent sheaves as in Remark 6.8.7. In particular, taking Λ to be the zero section, we recover $2\text{-}\mathrm{QCoh}(X)$.

Remark 6.8.14. An alternative approach to the formalism of ind-coherent sheaves and ind-coherent sheaves of categories, including a theory of singular supports beyond the quasi-smooth and smooth cases, respectively, is under development by di Fiore–Stefanich.

6.8.15. The previous discussion does not literally apply to $\mathrm{LocSys}_{\check{G}}$. This is because, due roughly to its infinite dimensionality, it is not a smooth Artin stack. However, it is expected that it carries a similar theory of ind-coherent sheaves of categories.

Admitting the existence of such a formalism, it only remains from (6.19) to explain what the singular support condition 'nilp' refers to. However, this should not be surprising given what we met in the unramified case. Namely, the cotangent complex to $\mathrm{LocSys}_{\check{G}}$ at a connection $(\check{\mathcal{P}}, \nabla)$ is given by the de Rham complex computing the flat sections on the punctured disk of the coadjoint bundle

$$\check{\mathcal{P}} \overset{\check{G}}{\times} \check{\mathfrak{g}}^*. \tag{6.20}$$

In particular, the cotangent space, i.e., its zeroth cohomology, canonically identifies with the flat sections of (6.20). With this, 'nilp' cuts out within each cotangent space the flat sections which are nilpotent.

6.8.16. Having dealt with the necessary corrections, let us at last have the promised fun and inspect the categories of intertwining operators between the objects in Figure 1. In this way, we will meet many beautiful theorems and conjectures, which will hopefully give the reader more of a hands-on feel for this subject.

On either side, all the appearing categories are self-dual. For the categories of D-modules and quasicoherent sheaves, up to manageable issues of an infinite dimensional nature, this follows from Example 4.4.6. We will discuss the case of Kac–Moody representations in more detail in Section 6.8.35 below. Admitting this self-duality, via the canonical equivalences

$$\mathrm{Hom}_{\mathrm{D\text{-}mod}(LG)\text{-}\mathrm{mod}}(\mathcal{C}, \mathcal{D}) \simeq \mathrm{Hom}_{\mathrm{D\text{-}mod}(LG)\text{-}\mathrm{mod}}(\mathcal{D}^\vee, \mathcal{C}^\vee),$$

and similarly for categories over $\mathrm{LocSys}_{\check{G}}$, we need only write down the categories of intertwining operators 'above the diagonal.'

We record these intertwining operators in Figures 2 and 3. The calculations of these categories on the automorphic and spectral sides essentially follow the pattern of those performed in Section 4, particularly Example 4.2.25 and the discussion of Section 4.3.9. Let us now compare the entries in the two tables and comment on the arising equivalences, both known and unknown.

	D-mod(Gr_G)	D-mod(Fl_G)	D-mod($LG/LN, \psi$)	$\widehat{\mathfrak{g}}_{\kappa_c}$-mod
D-mod(Gr_G)	D-mod($L^+G\backslash LG/L^+G$)	D-mod($L^+G\backslash LG/I$)	D-mod($L^+G\backslash LG/LN, \psi$)	$\widehat{\mathfrak{g}}_{\kappa_c}$-mod$^{L^+G}$
D-mod(Fl_G)		D-mod($I\backslash LG/I$)	D-mod($I\backslash LG/LN, \psi$)	$\widehat{\mathfrak{g}}_{\kappa_c}$-mod^I
D-mod($LG/LN, \psi$)			D-mod($LN, \psi\backslash LG/LN, \psi$)	$\widehat{\mathfrak{g}}_{\kappa_c}$-mod$^{LN,\psi}$
$\widehat{\mathfrak{g}}_{\kappa_c}$-mod				$\widehat{\mathfrak{g}}_{\kappa_c} \oplus \widehat{\mathfrak{g}}_{\kappa_c}$-mod^{LG}

FIGURE 2. Some intertwining operators on the automorphic side of local geometric Langlands.

	QCoh($\mathrm{pt}/\check{G}$)	QCoh($\widetilde{\mathcal{N}}/\check{G}$)	QCoh($\mathrm{LocSys}_{\check{G}}$)	QCoh($\mathrm{Op}_{\check{G}}$)
QCoh($\mathrm{pt}/\check{G}$)	$\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathrm{pt}/\check{G})$	$\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \widetilde{\mathcal{N}}/\check{G})$	QCoh($\mathrm{pt}/\check{G}$)	QCoh($\mathrm{Op}_{\check{G}}^{\mathrm{unramified}}$)
QCoh($\widetilde{\mathcal{N}}/\check{G}$)		$\mathrm{IndCoh}_{\mathrm{nilp}}(\widetilde{\mathcal{N}}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \widetilde{\mathcal{N}}/\check{G})$	QCoh($\widetilde{\mathcal{N}}/\check{G}$)	QCoh($\widetilde{\mathcal{N}}/\check{G} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{Op}_{\check{G}}$)
QCoh($\mathrm{LocSys}_{\check{G}}$)			QCoh($\mathrm{LocSys}_{\check{G}}$)	QCoh($\mathrm{Op}_{\check{G}}$)
QCoh($\mathrm{Op}_{\check{G}}$)				QCoh($\mathrm{Op}_{\check{G}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{Op}_{\check{G}}$)

FIGURE 3. Some intertwining operators on the spectral side of local geometric Langlands.

6.8.17. Let us begin with the unramified case, i.e., the first row of the tables. The first equivalence

$$\text{D-mod}(L^+G\backslash LG/L^+G) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \mathrm{pt}/\check{G}) \tag{6.21}$$

is one we have already met, namely the derived Satake equivalence of Bezrukavnikov–Finkelberg [BF08].

The derived Satake equivalence is t-exact and on abelian categories recovers the geometric Satake equivalence

$$\text{D-mod}(L^+G\backslash LG/L^+G)^{\heartsuit} \simeq \mathrm{QCoh}(\mathrm{pt}/\check{G}) = \mathrm{Rep}(\check{G}),$$

which is an earlier fundamental theorem due to Lusztig [Lus83], Drinfeld, Ginzburg [Gin95], Mirković–Vilonen [MV07]. Both equivalences, but slightly more transparently the abelian one, categorify the analogous assertion in the p-adic setting

$$\mathrm{Dist}(G(\mathbb{F}_q[[t]])\backslash G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]])) \simeq K_0(\mathrm{Rep}(\check{G})) \underset{\mathbb{Z}}{\otimes} k,$$

which, along with its mixed characteristic counterpart, is an isomorphism due to Satake [Sat63].

Let us make two comments about this.

Remark 6.8.18. First, for readers meeting this for the first time, to get some purchase on this equivalence it is instructive to match parameters, i.e., bases for the Grothendieck groups of (almost) compact objects.

The basic point here is that there are only countably many $L^+G \times L^+G$ orbits on LG, and these are naturally indexed by irreducible representations of $\check{G}$. More carefully, any cocharacter

$$\check{\mu} : \mathbb{G}_m \to T$$

yields, upon restriction to the formal disk about the origin, a point of LT, which we denote by $t^{\check{\mu}}$. If we let $\check{\Lambda}^+$ denote the dominant cocharacters, LG is then stratified by the orbits

$$L^+G \cdot t^{\check{\mu}} \cdot L^+G, \quad \text{for } \check{\mu} \in \check{\Lambda}^+.$$

But these are the highest weights of $\check{G}$-modules, and the equivalence exchanges the corresponding intersection cohomology complexes on Gr_G and simple $\check{G}$-modules.

Remark 6.8.19. Second, here is an orienting comment. The underlying space on the automorphic side of derived Satake parametrizes triples $(\mathcal{P}_1, \mathcal{P}_2, \tau)$, where $\mathcal{P}_1$ and $\mathcal{P}_2$ are G-bundles on the disk and τ is an identification of them on the punctured disk.

Similarly, the underlying space on the spectral side parametrizes triples $(\mathcal{E}_1, \mathcal{E}_2, \tau)$, where $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\check{G}$-connections on the formal disk and τ is an isomorphism of their restrictions to the punctured disk. Note that since every $\check{G}$-connection on the formal disk is trivializable, the underlying classical stack is $\mathrm{pt}/\check{G}$. However, this moduli problem is asking us to intersect $\mathrm{pt}/\check{G}$ with itself in $\mathrm{LocSys}_{\check{G}}$, which affords the derived structure crucial to making the equivalence hold.

Thus, somewhat informally speaking, the equivalence exchanges constructible objects on a coherent moduli space for G with coherent objects on an analogous constructible moduli space for $\check{G}$.

6.8.20. The next equivalence of the table, namely

$$\text{D-mod}(L^+G\backslash LG/I) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \widetilde{\mathcal{N}}/\check{G})$$

is also a very nice theorem due to Arkhipov–Bezrukavnikov–Ginzburg [ABG04]. As in Remark 6.8.18, we may count parameters as follows. On the automorphic side, the relevant stratification of LG is now by the strata

$$L^+G \cdot t^{\check{\mu}} \cdot I, \quad \text{for } \check{\mu} \in \check{\Lambda}.$$

That is, to shrink from having L^+G to I on the right, we must grow by a factor of the finite Weyl group, i.e., from dominant cocharacters to all cocharacters.

On the spectral side, recalling that $\widetilde{\mathcal{N}}$ is the cotangent bundle to the (finite) flag variety, we are considering $\check{G}$-equivariant coherent sheaves on a derived thickening of the zero section, i.e., of the flag variety itself. Here, up to ignoring the thickening, the same lattice $\check{\Lambda}$ occurs in

$$\mathrm{QCoh}(\mathrm{Fl}_{\check{G}}/\check{G}) \simeq \mathrm{QCoh}(\check{B}\backslash\check{G}/\check{G}) \simeq \mathrm{QCoh}(\check{B}\backslash \mathrm{pt}) \simeq \mathrm{Rep}(\check{B}),$$

as equivariant line bundles on $\mathrm{Fl}_{\check{G}}$ or, equivalently, the characters of $\mathrm{Rep}(\check{B})$.

6.8.21. We next pair the spherical and Whittaker categories to obtain

$$\text{D-mod}(L^+G\backslash LG/LN, \psi) \simeq \mathrm{QCoh}(\mathrm{pt}/\check{G}).$$

This is also a known and important theorem due to Frenkel–Gaitsgory–Vilonen [FGV01]. It may be understood as a geometric refinement of the Casselman–Shalika formula from p-adic representation theory [CS80], which explicitly determines the Whittaker covector of an unramified representation.

The parameter count here is similar. Namely, LG is stratified by the double cosets

$$L^+G \cdot t^{\check{\mu}} \cdot LN, \quad \text{for } \check{\mu} \in \check{\Lambda},$$

but an orbit supports Whittaker sheaves if and only if the coweight $\check{\mu}$ is antidominant.

Remark 6.8.22. Let us also briefly comment on a basic difference with the bi-spherical situation, i.e., derived Satake. There, the spectral side is a derived enhancement of $\mathrm{Rep}(\check{G})$. Roughly speaking, this occurs since the stalks of the intersection cohomology sheaves on

$$L^+G\backslash LG/L^+G$$

are quite interesting along orbit closures and correspond to q-weight multiplicities for $\check{G}$-representations [Lus83].

In the present case, one has no such derived enhancement. This occurs because the intersection cohomology objects on

$$L^+G\backslash LG/LN, \psi$$

are clean, i.e., have vanishing !-stalks and $*$-stalks along the boundary of the open orbit in their support.

6.8.23. Finally, let us pair spherical vectors and Kac–Moody representations to obtain

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{L^+G} \simeq \mathrm{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{unramified}}),$$

where by definition the latter moduli space parametrizes oper structures on the trivial bundle, i.e.

$$\mathrm{Op}_{\check{G}}^{\mathrm{unramified}} \simeq \mathrm{pt}/\check{G} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{Op}_{\check{G}}.$$

This is also a known theorem, due to Frenkel–Gaitsgory [FG09b], which contains and strengthens many prior results about L^+G-equivariant Kac–Moody representations at critical level due to Feigin–Frenkel, Beilinson–Drinfeld, and others.

In this case, the Grothendieck groups are no longer countable. Instead, one has continuous families of simple modules parametrized by pro-finite dimensional affine spaces. Nonetheless, we can give the following rough count.

On the automorphic side, one has a collection of compact generators, the Weyl modules, which are by definition the parabolic inductions of the irreducible finite dimensional G-modules

$$\mathbb{V}_\lambda := \mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{\kappa_c}}(L_\lambda), \quad \text{for } \lambda \in \Lambda^+.$$

Parabolic induction in the present setting of Lie algebras is given by the correspondence

$$\mathfrak{g} \leftarrow \mathfrak{g}[[t]] \rightarrow \widehat{\mathfrak{g}}_{\kappa_c},$$

i.e. is the composition of restriction and induction functors

$$\mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{\kappa_c}} : \mathfrak{g}\text{-mod} \xrightarrow{\mathrm{Res}} \mathfrak{g}[[t]]\text{-mod} \xrightarrow{\mathrm{ind}} \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}.$$

It is a nontrivial fact that each $\mathbb{V}_\lambda$ has endomorphisms given by a quotient of the center $Z(\widehat{\mathfrak{g}}_{\kappa_c})$, and is moreover projective in $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\lambda^{L^+G}$, where $\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\lambda$ denotes the category of $\widehat{\mathfrak{g}}_{\kappa_c}$-modules on which $Z(\widehat{\mathfrak{g}}_{\kappa_c})$ acts through the same quotient.

Example 6.8.24. If $\lambda = 0$, then we obtain the vacuum module[24] at critical level

$$\mathbb{V}_0 \simeq \mathrm{pind}_{\mathfrak{g}}^{\widehat{\mathfrak{g}}_{\kappa_c}}(k).$$

Its endomorphisms canonically identify, under the Feigin–Frenkel isomorphism, with the algebra of functions on opers on the non-punctured formal disk

$$\mathrm{Op}_{\check{G}}^{\mathrm{reg}} \simeq \{d + f dt + \check{\mathfrak{b}}[[t]]dt\}/\check{N}[[t]].$$

On the spectral side, as we vary λ, these endomorphism algebras correspond to disjoint closed subschemes inside $\mathrm{Op}_{\check{G}}$. The moduli space $\mathrm{Op}_{\check{G}}^{\mathrm{unramified}}$ is a formal thickening of their union, and the theorem further matches these nilpotent normal directions with the self-extensions of Weyl modules.

[24]The terminology 'vacuum' here stems from its origins in conformal field theory. Namely, this is the underlying vector space of a vertex operator algebra, which occurs as local operators in chiral conformal field theories with G-symmetry.

6.8.25. We are ready to move on to the next row of Figures 2 and 3, i.e., the case of tame ramification and unipotent monodromy. We first meet the endomorphisms, i.e., a monoidal equivalence

$$\text{D-mod}(I\backslash LG/I) \simeq \operatorname{IndCoh}_{\mathrm{nilp}}(\widetilde{\mathcal{N}}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \widetilde{\mathcal{N}}/\check{G}). \tag{6.22}$$

This attractive statement is remarkable for many reasons. For one, there is not such a coherent realization of the finite Hecke category, and, in this sense, the equivalence really is something particular to the affine setting and, specifically, its connection with arithmetic.[25] As with other statements of low ramification we have met so far, it is a celebrated theorem due to Bezrukavnikov [Bez16].

As far as we know, this equivalence was first formulated as a conjecture by Ginzburg [CG10]. A major motivation comes from the p-adic setting, where an analogous isomorphism between $\operatorname{Dist}(I\backslash G(\mathbb{F}_q((t)))/I)$, i.e., the affine Hecke algebra, and an appropriate specialization of the equivariant K-theory of the Steinberg variety

$$\widetilde{\mathcal{N}} \underset{\check{\mathfrak{g}}}{\times} \widetilde{\mathcal{N}}$$

is due to Kazhdan–Lusztig [KL87]. They, and independently Ginzburg, used this isomorphism to classify the irreducible representations of affine Hecke algebra and determine their characters. Equivalently, they determined the characters of the simple subquotients of the unramified principal series of $G(\mathbb{F}_q((t)))$.

Let us count parameters. On the automorphic side, the relevant stratification of LG has constituents

$$I \cdot w \cdot t^{\check{\mu}} \cdot I, \quad \text{for } w \in W_f \text{ and } \check{\mu} \in \check{\Lambda}.$$

This is an affine analogue of the Bruhat decomposition of G into the double cosets $B \cdot w \cdot B$, for $w \in W_f$, and indeed the indexing set here is the (extended) affine Weyl group $W_f \ltimes \check{\Lambda}$. Note that, as in Section 6.8.20, we once again shrink from L^+G to I, and therefore our stratification grows by a factor of W_f.

On the spectral side, these parameters arise as follows. To begin, note that the Steinberg variety[26] explicitly parametrizes triples

$$(X, \check{B}', \check{B}''),$$

where X is a nilpotent element of $\check{\mathfrak{g}}$ and $\check{B}'$, $\check{B}''$ are two Borel subgroups containing it. The Steinberg is the union of W_f many locally closed Lagrangians in $\widetilde{\mathcal{N}} \times \widetilde{\mathcal{N}}$, where each Lagrangian is specified by the relative position of $\check{B}_1$ and $\check{B}_2$, i.e., a point of $\check{B}\backslash\check{G}/\check{B}$.

Remark 6.8.26. Recall that we are really interested in the quotient

$$\widetilde{\mathcal{N}}/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \widetilde{\mathcal{N}}/\check{G}.$$

It is clarifying to recognize this as the cotangent (derived stack) of the Hecke stack

$$\check{B}\backslash\check{G}/\check{B} \simeq \operatorname{Fl}_{\check{G}} \times \operatorname{Fl}_{\check{G}}/\check{G} \tag{6.23}$$

in its presentation via Hamiltonian reduction from the right hand side of (6.23). Relatedly, the W_f components in question are simply the conormal bundles to the corresponding Schubert cells.

As in Section 6.8.20, each component admits $\check{\Lambda}$-many equivariant line bundles, given by pullback along either projection to $\widetilde{\mathcal{N}}/\check{G}$,[27] so, summing over components, we get the desired count of $W_f \ltimes \check{\Lambda}$.

[25] Admittedly, the construction via Soergel bimodules can be understood as writing the monodromic Hecke category as the homotopy category of certain coherent sheaves on

$$\check{T} \underset{\check{T}/W_f}{\times} \check{T},$$

cf. [BR18]. However, this identification is somewhat subtler, as it is only fully faithful on tilting objects non-derivedly, i.e., the corresponding Soergel bimodules have nontrivial Exts in the world of bimodules. In any case, another pleasant feature of (6.22) is that the Soergel bimodule presentation of the finite Hecke category is indeed embedded into it.

[26] Although this is the standard terminology, we emphasize that for the monoidal equivalence we really are working instead with the derived enhancement.

[27] The two maps from $\check{\Lambda}$ to its Picard group differ by the action of the corresponding element of w.

6.8.27. Let us now pair the unramified principal series and Whittaker model to obtain an equivalence

$$\text{D-mod}(I\backslash LG/LN, \psi) \simeq \text{QCoh}(\widetilde{\mathcal{N}}/\check{G}). \tag{6.24}$$

This assertion is again known, and is a very nice theorem of Arkhipov–Bezukravnikov [AB09]. Their argument is widely regarded as one of the gems in this subject. The result plays a basic role in Bezrukavnikov's proof of (6.22), in parallel to the role of an analogous identity in K-theory for the identification of the affine Hecke algebra with the equivariant K-theory of Steinberg by Kazhdan–Lusztig.

The parameter count here goes as follows. On the automorphic side, it is slightly more convenient to pass to the semi-infinite variant, i.e.,

$$\text{D-mod}(LN \cdot L^+T\backslash LG/LN, \psi). \tag{6.25}$$

In this presentation, under the stratification of LG by the cosets

$$(LN \cdot L^+T) \cdot w \cdot t^{\check{\mu}} \cdot LN, \quad \text{for } w \in W_f \text{ and } \check{\mu} \in \check{\Lambda},$$

an orbit supports Whittaker sheaves if and only if w equals the longest element $w_\circ$ of W_f. In particular, the set of relevant orbits is canonically parametrized by $\check{\Lambda}$.

On the spectral side, we have

$$\widetilde{\mathcal{N}}/\check{G} = (\check{G} \overset{\check{B}}{\times} \check{\mathfrak{n}})/\check{G} \simeq \check{\mathfrak{n}}/\check{B},$$

and $\check{\Lambda}$ again corresponds to the equivariant line bundles pulled back from $\mathrm{pt}/\check{B}$.

Here are two comments about this equivalence.

Remark 6.8.28. In the p-adic setting, the relevant space of distributions

$$\mathrm{Dist}(I\backslash G(\mathbb{F}_q((t)))/N(\mathbb{F}_q((t))), \psi) \tag{6.26}$$

is an induced representation of the affine Hecke algebra. More precisely, if we write H for the (extended) affine Hecke algebra, H_f for the finite Hecke algebra, and k_{sgn} for its sign representation, then (6.26) is the so-called antispherical representation

$$H \underset{H_f}{\otimes} k_{\mathrm{sgn}}.$$

One has a similar formula in the geometric setting, namely

$$\text{D-mod}(I\backslash LG/LN, \psi) \simeq \text{D-mod}(I\backslash LG/I) \underset{\text{D-mod}(B\backslash G/B)}{\otimes} \text{D-mod}(B\backslash G/N, \psi),$$

cf. [CD21].

Remark 6.8.29. We rewrote the automorphic side of (6.24) in semi-infinite terms in (6.25). This was to minorly simplify the combinatorics of relevant orbits. However, there are less frivolous reasons to bear this alternative expression in mind. Namely, recall that parabolic induction for categorical representations really does produce the semi-infinite version of the affine flag variety, cf. Section 6.5. Relatedly, in the proof of Arkhipov–Bezukravnikov, it is objects of semi-infinite origin, e.g., the Wakimoto sheaves, which play a central role.

6.8.30. By pairing the unramified principal series with Kac–Moody representations, we arrive at an equivalence

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^I \simeq \text{QCoh}(\widetilde{\mathcal{N}}/\check{G} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{Op}_{\check{G}}).$$

This assertion is an extension of the previously discussed results of Frenkel–Gaitsgory in the spherical case, cf. Section 6.8.23. They first conjectured it in [FG06b], and obtained a version of it, after a further specialization of the central character, several years later as a consequence of their work on critical level localization on the affine flag variety [FG09a].[28]

[28] Strictly speaking, in *loc. cit.* one finds an Iwahori monodromic variant.

6.8.31. Let us pass to the third row of Figures 2 and 3 and study intertwiners out of the Whittaker model. We first meet its endomorphisms and an equivalence of monoidal categories

$$\text{D-mod}(LN, \psi \backslash LG/LN, \psi) \simeq \text{QCoh}(\text{LocSys}_{\check{G}}). \tag{6.27}$$

This is a striking and essentially wide open conjecture. Even symmetric monoidality for the left hand side is far from obvious. Morally, it is a manifestation of the uniqueness of Whittaker covectors in the current geometric setting under the usual correspondence between multiplicity freeness and commutative endomorphism algebras.

We emphasize that this equivalence would pin down an enormous swathe of Conjecture (6.19), namely all the sheaves of categories on the spectral side with singular support along the zero section. Such categorical representations are called tempered.

Remark 6.8.32. Although (6.27) is very much open, here is a special case which is known. Namely, by work of Raskin [Ras21], the depth filtration on categorical representations of the loop group gives rise to a filtration on

$$\text{D-mod}(LN, \psi \backslash LG/LN, \psi),$$

with each quotient a monoidal colocalization. The first quotient, corresponding to representations with vectors fixed by the prounipotent radical of an Iwahori subgroup, is the Iwahori–Whittaker category

$$\text{D-mod}(\mathring{I}, \psi \backslash LG/\mathring{I}, \psi).$$

This in turn admits a further quotient, which corresponds to passing from all tamely ramified representations to those with unipotent monodromy, namely the Steinberg–Whittaker category

$$\text{D-mod}(\mathring{I}, \psi, \mathrm{s} \backslash LG/\mathring{I}, \psi, \mathrm{s}).$$

Plainly, this is the full subcategory generated by averages of objects from $\text{D-mod}(I^- \backslash LG/I^-)$, where I^- is an Iwahori subgroup of L^+G defined with respect to a Borel B^- in general position to B, cf. [CD21]. If we write $\mathcal{N}^\wedge$ for the formal completion of $\check{\mathfrak{g}}$ along the nilpotent cone $\mathcal{N}$, we have a monoidal equivalence

$$\text{D-mod}(\mathring{I}, \psi, \mathrm{s} \backslash LG/\mathring{I}, \psi, \mathrm{s}) \simeq \text{QCoh}(\mathcal{N}^\wedge/\check{G}),$$

which is the promised piece of (6.27).

Bezrukavnikov proved the equivalence of affine Hecke categories (6.22) essentially by combining this with the previously mentioned result of Arkhipov–Bezrukavnikov. More precisely, let us write $\widetilde{\mathcal{N}}^\wedge$ for the formal completion of the Grothendieck–Springer variety

$$\widetilde{\check{G}} := \check{G} \overset{\check{B}}{\times} \check{\mathfrak{b}}$$

along the Springer variety. That is, the Grothendieck–Springer variety parametrizes pairs $(X, \check{\mathfrak{b}}')$, where X is an element of $\check{\mathfrak{g}}$ and $\check{\mathfrak{b}}'$ is a Borel subalgebra containing it, and we complete this along the locus where X is nilpotent. Then the desired equivalence (6.22)[29] is obtained via renormalizing the equivalence

$$\begin{aligned}\text{QCoh}(\widetilde{\mathcal{N}}^\wedge/\check{G} \underset{\check{\mathfrak{g}}/\check{G}}{\times} \widetilde{\mathcal{N}}^\wedge/\check{G}) &\simeq \text{QCoh}(\widetilde{\mathcal{N}}^\wedge/\check{G}) \underset{\text{QCoh}(\mathcal{N}^\wedge/\check{G})}{\otimes} \text{QCoh}(\widetilde{\mathcal{N}}^\wedge/\check{G}) \\ &\simeq \text{D-mod}(\mathring{I} \backslash LG/\mathring{I}, \psi, \mathrm{s}) \underset{\text{D-mod}(\mathring{I}, \psi, \mathrm{s} \backslash LG/\mathring{I}, \psi, \mathrm{s})}{\otimes} \text{D-mod}(\mathring{I}, \psi, \mathrm{s} \backslash LG/\mathring{I}).\end{aligned}$$

6.8.33. We next pair the Whittaker model with Kac–Moody representations to obtain an equivalence

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{LN, \psi} \simeq \text{QCoh}(\text{Op}_{\check{G}}). \tag{6.28}$$

This is again a strong statement concerning objects of arbitrary ramification. Unlike other such statements we are encountering, it is in fact a theorem of Raskin [Ras21].

[29] Even more carefully, we mean its monodromic variant, from which the strict statement may be deduced formally.

Namely, the affine Skryabin theorem of Raskin[30] gives, for any level κ, a canonical equivalence

$$\widehat{\mathfrak{g}}_\kappa\text{-mod}^{LN,\psi} \simeq \mathcal{W}_\kappa\text{-mod},$$

where $\mathcal{W}_\kappa$ is the affine W-algebra. On the other hand, at the critical level Feigin–Frenkel had identified $\mathcal{W}_{\kappa_c}$ with the center $Z(\widehat{\mathfrak{g}}_{\kappa_c})$ of the enveloping algebra [FF91], and therefore also with $\mathcal{O}_{\mathrm{Op}_{\check{G}}}$. This in combination with the affine Skyrabin theorem yields (6.28).

Remark 6.8.34. The reader may wish to consult the survey of Arakawa in the present volume for more details about the W-algebra; see also [Ara17], [FBZ04]. Here let us only mention in passing that the affine Skryabin theorem enables the systematic use of techniques from categorical representation theory in the study of representations of W-algebras. As an example, one can see [DR20], which discusses localization theory for highest weight modules for W-algebras.

6.8.35. In the final row, we meet the endomorphisms of Kac–Moody representations. These are identified with affine Harish-Chandra bimodules at critical level as follows.

First, the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod is canonically self-dual as a D-mod(LG) representation and, in particular, as an abstract dg-category.

To see this, it is orienting to begin with a simpler case. For a finite dimensional Lie algebra $\mathfrak{a}$ with a central extension

$$0 \to k \cdot \mathbf{1} \to \widetilde{\mathfrak{a}}_c \to \mathfrak{a} \to 0,$$

consider the category $\widetilde{\mathfrak{a}}_c$-mod of $\widetilde{\mathfrak{a}}_c$-modules on which $\mathbf{1}$ acts via the identity. If we write $\widetilde{\mathfrak{a}}_{-c}$ for the opposite central extension, e.g., given by the additive inverse of any representing 2-cocycle, one has a perfect pairing

$$\widetilde{\mathfrak{a}}_c\text{-mod} \otimes \widetilde{\mathfrak{a}}_{-c}\text{-mod} \to \mathfrak{a}\text{-mod} \to \mathrm{Vect},$$

where the first arrow tensors together representations to cancel the extensions, and the second is Lie algebra homology.

In the affine case, it was shown by Arkhipov–Gaitsgory [AG15b], building on previous work of Frenkel–Gaitsgory [FG06b], that for any κ the composition

$$\widehat{\mathfrak{g}}_\kappa\text{-mod} \otimes \widehat{\mathfrak{g}}_{-\kappa+2\kappa_c}\text{-mod} \to \widehat{\mathfrak{g}}_{2\kappa_c}\text{-mod} \to \mathrm{Vect}$$

is a perfect pairing, where the first arrow tensors representations, and the second arrow is the functor of semi-infinite cohomology. Briefly, this is the natural homology theory for representations of infinite dimensional Lie algebras like $\widehat{\mathfrak{g}}$ (formally, Tate Lie algebras), but it requires a central extension of level $2\kappa_c$. One has such a duality statement for general Tate Lie algebras, cf. [Ras20], [Dhi21].

In particular, one has a self-duality

$$\widehat{\mathfrak{g}}_{\kappa_c}\text{-mod} \otimes \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod} \to \mathrm{Vect},$$

which is another sense in which the critical level is the midpoint of the space of levels. Moreover, as in Section 4.5.2, this identifies the D-mod(LG) equivariant endofunctors of $\widehat{\mathfrak{g}}_{\kappa_c}$-mod with the category of critical level affine Harish-Chandra bimodules

$$\widehat{\mathfrak{g}}_{\kappa_c} \oplus \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{LG}.$$

This is a beautiful, if somewhat elusive, category. For example, all of its objects lie in cohomological degree $-\infty$ with respect to the natural t-structure on $\widehat{\mathfrak{g}}_{\kappa_c} \oplus \widehat{\mathfrak{g}}_{\kappa_c}$-mod, i.e., it consists of homological phantoms introduced through renormalization.

Remark 6.8.36. The affine Harish-Chandra bimodules at other levels are also subtle, with some of the basic works and conjectures due to I. Frenkel–Malikov [FM97] and Gaitsgory [Gai07].

[30]We note in passing that Skryabin is an anagram of 'by Raskin.'

For these bimodules, local geometric Langlands predicts a monoidal equivalence with the spectral convolution algebra

$$\widehat{\mathfrak{g}}_{\kappa_c} \oplus \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{LG} \simeq \mathrm{QCoh}(\mathrm{Op}_{\check{G}} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{Op}_{\check{G}}).$$

This is a striking and essentially wide open conjecture. To our knowledge, even the spherical case and the case of tame ramification with unipotent monodromy are open, though we expect they are within reach using some of the aforementioned work of Frenkel–Gaitsgory, cf. Sections 6.8.23 and 6.8.30, combined with the localization theorem at critical level, which we now turn to.

6.9. **Localization theory.**

6.9.1. We would like to discuss an important application of ideas and techniques from local geometric Langlands to a problem purely within the representation theory of affine Lie algebras. We begin by setting up the context.

6.9.2. Recall that in Section 3 we discussed localization theory for reductive Lie algebras and sketched how it was used to determine the characters of the simple highest weight modules.

For affine Lie algebras, an analogous program was developed at noncritical levels by Kashiwara–Tanisaki in a series of works [Kas90], [KT90], [KT95], [KT96], [KT98], [KT00]. Briefly, at negative level they identified D-modules on the affine flag manifold with certain Kac–Moody representations. At positive level, they did the same, but instead with D-modules on the thick affine flag manifold, a scheme of infinite type parametrizing G-bundles on $\mathbb{P}^1$ with full level structure at zero and a Borel reduction at infinity. As a consequence, they obtained the characters for simple highest weight modules at noncritical levels.

After their work, it was understood that to make inroads on the remaining case, namely the representation theory at critical level, one would need further ideas and tools. We would like to now sketch what localization theory looks like here, following work of Beilinson–Drinfeld, Frenkel–Gaitsgory, and Raskin–Yang.

6.9.3. The basic new complication at critical level is that the geometric side of the localization theorem is not simply D-modules on a space, but a tweak thereof. Pleasantly, one can arrive at the correct statement using only what we have discussed so far about local geometric Langlands. Let us see how.

To begin, recall that for any $\check{G}$ oper χ, with underlying local system σ, local geometric Langlands predicts that the categorical loop group representation $\mathcal{C}_\sigma$ corresponding to $\mathrm{Vect} \simeq \mathrm{QCoh}(\mathrm{pt})$ under the map

$$\sigma : \mathrm{pt} \to \mathrm{LocSys}_{\check{G}}$$

may be obtained by renormalizing

$$\widehat{\mathfrak{g}}_{\kappa}\text{-mod} \underset{\mathrm{QCoh}(\mathrm{Op}_{\check{G}})}{\otimes} \mathrm{Vect},$$

i.e., Kac–Moody representations with central character χ.

Remark 6.9.4. Now that we have discussed ind-coherent sheaves of categories, this renormalization may be understood as a consequence of the formal smoothness of the map

$$\mathrm{Op}_{\check{G}} \to \mathrm{LocSys}_{\check{G}},$$

which, in particular, implies that the sub-2-category generated by $\mathrm{QCoh}(\mathrm{Op}_{\check{G}})$ consists solely of quasi-coherent sheaves of categories. That is, there is no room for nontrivial singular support.

On the Langlands dual side, this predicts that $\widehat{\mathfrak{g}}_{\kappa_c}$-mod is tempered, i.e., that the canonical map

$$\text{D-mod}(LG/LN, \psi) \underset{\text{D-mod}(LN,\psi\backslash LG/LN,\psi)}{\otimes} \mathcal{W}_{\kappa_c}\text{-mod} \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}$$

is an equivalence. This is a remarkable conjecture about Kac–Moody representations at critical level.

6.9.5. Let us specialize the previous discussion, which applies to any local system σ, to the trivial connection. For this, we may tautologically factor σ as

$$\mathrm{pt} \to \mathrm{pt}/\check{G} \to \mathrm{LocSys}_{\check{G}}.$$

We have already discussed in some detail what $\mathrm{QCoh}(\mathrm{pt}/\check{G})$ is meant to correspond to under local geometric Langlands, namely D-modules on the affine Grassmannian. This gives an alternative presentation of $\mathcal{C}_\sigma$, as follows. Since $\mathrm{pt}/\check{G} \to \mathrm{LocSys}_{\check{G}}$ is as far from smooth as possible, we have that Vect lies in the category generated by $\mathrm{QCoh}(\mathrm{pt}/\check{G})$. That is, we find that

$$\begin{aligned}\mathrm{Vect} &\simeq \mathrm{QCoh}(\mathrm{pt}/\check{G}) \underset{\mathrm{Hom}_{2\text{-}\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}_{\check{G}})}(\mathrm{QCoh}(\mathrm{pt}/\check{G}), \mathrm{QCoh}(\mathrm{pt}/\check{G}))}{\otimes} \mathrm{Hom}_{2\text{-}\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}_{\check{G}})}(\mathrm{QCoh}(\mathrm{pt}/\check{G}), \mathrm{QCoh}(\mathrm{pt}))\\ &\simeq \mathrm{QCoh}(\mathrm{pt}/\check{G}) \underset{\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{pt}/\check{G})}{\otimes} \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{pt}).\end{aligned}$$

Therefore, on the automorphic side, we should have that

$$\mathcal{C}_\sigma \simeq \text{D-mod}(\mathrm{Gr}_G) \underset{\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \times_{\mathrm{LocSys}_{\check{G}}} \mathrm{pt}/\check{G})}{\otimes} \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{pt}). \tag{6.29}$$

As tensoring from $\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt}/\check{G} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{pt}/\check{G})$ to $\mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{pt} \underset{\mathrm{LocSys}_{\check{G}}}{\times} \mathrm{pt})$ is simply forgetting equivariance for the action of $\check{G}$, we may rewrite this more plainly as

$$\mathcal{C}_\sigma \simeq \text{D-mod}(\mathrm{Gr}_G) \underset{\mathrm{QCoh}(\mathrm{pt}/\check{G})}{\otimes} \mathrm{QCoh}(\mathrm{pt}). \tag{6.30}$$

Explicitly, this is the category of Hecke eigensheaves on the affine Grassmannian. That is, recall that $\mathrm{QCoh}(\mathrm{pt}/\check{G})$, i.e., $\mathrm{Rep}(\check{G})$, is identified with the abelian Satake category

$$\text{D-mod}(L^+G\backslash LG/L^+G)^{\heartsuit}.$$

This category acts via right convolution on D-modules on the affine Grassmannian, which we denote by

$$- \overset{L^+G}{\star} - : \text{D-mod}(LG/L^+G) \otimes \text{D-mod}(L^+G\backslash LG/L^+G) \to \text{D-mod}(LG/L^+G).$$

Our desired category consists of eigenobjects for this action, i.e., sheaves $\mathcal{F}$ on $\text{D-mod}(\mathrm{Gr}_G)$ equipped with compatible isomorphisms

$$\mathcal{F} \overset{L^+G}{\star} \mathrm{Sat}(V) \simeq \mathcal{F} \otimes \mathrm{Oblv}(V), \quad \text{for } V \in \mathrm{Rep}(\check{G}), \tag{6.31}$$

where $\mathrm{Oblv}(V)$ denotes the underlying vector space of V. A basic observation to make about such an $\mathcal{F}$, for which convolution with any Satake sheaves returns a direct sum of copies of $\mathcal{F}$, is that $\mathcal{F}$ must have infinite dimensional support and, in particular, cannot be a compact object of $\text{D-mod}(\mathrm{Gr}_G)$.

Before comparing the two descriptions of $\mathcal{C}_\sigma$, let us make a couple more orienting comments.

Remark 6.9.6. One arrives at the more down to earth formulation (6.31) of Hecke eigensheaves by dragging the Satake sheaf through the tensor product (6.30), i.e., by considering

$$(\mathcal{F} \overset{L^+G}{\star} V) \otimes k \simeq \mathcal{F} \otimes (V \cdot k) \simeq \mathcal{F} \otimes \mathrm{Oblv}(V),$$

where $\cdot$ denotes the action of $\mathrm{QCoh}(\mathrm{pt}/\check{G})$ on $\mathrm{QCoh}(\mathrm{pt})$.

Remark 6.9.7. The reader meeting these ideas for the first time may find some of the details of the preceding discussion somewhat involved. It is therefore worth emphasizing that one can also predict this by analogy. Namely, in the p-adic case, to extract an irreducible unramified representation π_η of $G(\mathbb{F}_q((t)))$ from

$$\mathrm{Fun}(G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]])),$$

one passes to eigenvectors for the right action of the spherical Hecke algebra with eigenvalue η, i.e., the corresponding homomorphism

$$\eta : \mathrm{Fun}(G(\mathbb{F}_q[[t]])\backslash G(\mathbb{F}_q((t)))/G(\mathbb{F}_q[[t]]))) \to k.$$

In the geometric setting, there is only one possible eigenvalue by Tannaka duality, namely the monoidal functor

$$\mathrm{Oblv} : \mathrm{Rep}(\check{G}) \to \mathrm{Vect},$$

and we are again building the irreducible unramified representation from the universal one by passing to eigenvectors. In particular, the observation we made about the support of Hecke eigensheaves above lifts an analogous assertion about the support of Hecke eigenvectors.

6.9.8. By comparing the two constructions of $\mathcal{C}_\sigma$ in the unramified case, one obtains the following prediction. Namely, for any oper structure χ on the trivial connection, there should be an equivalence

$$\text{D-mod}(\mathrm{Gr}_G) \underset{\mathrm{QCoh}(\mathrm{pt}/\check{G})}{\otimes} \mathrm{QCoh}(\mathrm{pt}) \simeq \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi, \tag{6.32}$$

where the right hand side denotes an appropriate renormalization of the category of Kac–Moody representations with central character χ.

This is in fact an important theorem,[31] whose development involved significant contributions from multiple groups of authors over several decades. A brief history is as follows. In their work on quantization of the Hitchin system [BD], Beilinson–Drinfeld showed that, under the convolution action

$$\text{D-mod}(L^+G\backslash LG/L^+G) \otimes \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{L^+G} \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}^{L^+G},$$

any central quotient of the vacuum module is an eigenobject for the action of the abelian Satake category. This theorem was the major local input into their construction of Hecke eigensheaves in global geometric Langlands corresponding to global opers.

6.9.9. The implications of Beilinson–Drinfeld's work for localization theory at critical level were realized by Frenkel–Gaitsgory, and substantially developed in [FG04], [FG06b], and [FG09c]. To begin, the Hecke eigen-property established by Beilinson–Drinfeld was used by Frenkel–Gaitsgory to construct a functor

$$\text{D-mod}(\mathrm{Gr}_G) \underset{\mathrm{QCoh}(\mathrm{pt}/\check{G})}{\otimes} \mathrm{QCoh}(\mathrm{pt}) \to \widehat{\mathfrak{g}}_{\kappa_c}\text{-mod}_\chi. \tag{6.33}$$

Roughly, this amounted to showing that the global sections of any D-module on Gr_G (twisted by the critical line bundle, so that one obtains a representation at critical level, and with the central quotient corresponding to χ imposed) is naturally a Hecke eigenobject. For delta D-modules, this is exactly the previous theorem of Beilinson–Drinfeld, and the general case may be reduced to this one.

In addition to conjecturing that (6.33) should be a t-exact equivalence, Frenkel–Gaitsgory established many of its fundamental properties. Namely, they proved that it was right t-exact and a fully faithful embedding on bounded below derived categories.[32] They moreover proved that it was an equivalence on I-equivariant objects, i.e., highest weight modules.

Remark 6.9.10. With a little work, one obtains from these results the analog of the Kazhdan–Lusztig conjecture for the characters of simple highest weight modules at critical level. This was conjectured by Feigin–Frenkel, see [AF12], and is proven in [DY].

[31] More carefully, it is a theorem at least under the assumption that χ corresponds to an oper on the formal punctured disk, cf. Example 6.8.24. However, it is expected that the general case may be handled similarly.

[32] The fully faithfulness in general and, in particular, the correct renormalization of the category of Kac–Moody representations, was later obtained by Raskin [Ras22].

6.9.11. After the work of Frenkel–Gaitsgory, the remaining pieces of their conjecture were that (6.33) was t-exact and essentially surjective. The latter is profitably reinterpreted as a generation statement under the categorical action of the loop group as follows. The left hand side of (6.33) is by definition generated by its L^+G-equivariant objects. Therefore, it remained to show the same held for the right hand side.

On the one hand, this is a fairly natural sounding statement. Indeed, the central character imposed is of unramified nature, corresponding to an oper on the formal non-punctured disk, and so it is rather plausible that the category itself should be unramified. However, this proved to be a difficult assertion, in large part due to how little was, and largely still is, explicitly known about Kac–Moody representations beyond tame ramification.

6.9.12. Over a decade later, significant progress was made by Raskin [Ras22], who proved the conjecture in the first nontrivial case of GL_2 and, in fact, for any group of semisimple rank one. To do so, he proved a general assertion about categorical representations of LGL_2, namely their generation by Whittaker and Iwahori invariants, which parallels an earlier and similarly fundamental assertion in the p-adic theory. As (6.33) is an equivalence on such objects, the full equivalence then follows.

The general case of the conjecture was then settled by Raskin–Yang [RY22]. The major new ingredient was the adaptation of Moy–Prasad theory from the p-adic setting by Yang [Yan21]; see also the earlier [CK17]. Crucially, besides the filtration of loop group representations by depth, Moy–Prasad theory moreover provided an explicit list of compact generators for each successive quotient, corresponding to unrefined minimal K-types in the p-adic theory. Their proof then showed by a nontrivial analysis that having a nonzero map from any such generator beyond depth zero is incompatible with an unramified central character, which yielded the desired essential surjectivity.

Remark 6.9.13. In fact, a crucial technical point, which we have swept under the rug in the preceding discussion, is that the category $\widehat{\mathfrak{g}}_{\kappa_c}$-mod$_\chi$ does not carry an a priori categorical action of LG due to its renormalization. This is one of the basic difficulties surmounted in [RY22].

Remark 6.9.14. We would like to make a remark on Moy–Prasad theory in geometric Langlands, which is otherwise independent of our discussion of localization. Namely, an interesting feature of working with $\check{G}$-connections as opposed to Galois representation, due to Bremer–Sage [BS13], [Sag17], [BS18], is that one also has a theory of minimal K-types for $\check{G}$-connections. The interaction with local geometric Langlands and particularly Moy–Prasad theory on the automorphic side is not yet fully understood.

6.9.15. Let us finish our discussion of localization theory at critical level with two comments. First, as the reader may imagine, the above admits an extension from the unramified case to the case of tame ramification, e.g., with unipotent monodromy. This was developed by Frenkel and Gaitsgory in [FG06b] and [FG09a], roughly by passing from the affine Grassmannian to the affine flag variety.

Second, it is rather arresting that, given the remarkable nature of the localization theorem at critical level we have discussed, it is in some sense the tip of the iceberg. That is, it is expected that many of the known constructions of irreducible representations of p-adic groups of higher depth should admit geometric versions, leading to equivalences and, in particular, localization theorems for critical level representations with wildly ramified central characters; see [FG06b] and [KS15] for some discussion and conjectures in this direction.

In particular, the surveyor expects that, for an oper whose underlying local system admits a $\check{B}$-reduction, the corresponding category of Kac–Moody modules localizes as a category of twisted D-modules on the semi-infinite flag manifold. For ramified characters, such a localization is no longer possible also on the affine flag variety as in [FG09a], but instead on the quotient by a smaller compact open subgroup first introduced in the p-adic setting by Roche [Roc98].

6.10. **Siblings.** In this section, we have discussed the local geometric Langlands conjecture, and hopefully the reader has gained a basic feeling for its statement and some of its concrete predictions in representation theory and geometry, both known and conjectural.

Before we finish, we should mention some important variants or, rather, sibling conjectures in nearby areas, as well as some other emerging connections.

6.10.1. *Quasi-split groups.* The first comment we should make is that we have worked throughout this survey with loop groups, i.e., split reductive groups over $k((t))$. However, even for the study of their categorical representations, as in number theory, it is important to also consider the case of twisted loop groups, i.e., the analogues of quasi-split groups in the p-adic setting. Note this allows for a similar picture of the representation theory of twisted affine Lie algebras, and the forms of local geometric Langlands duality discussed below again are expected to extend to the quasi-split case.

6.10.2. *Quantum local geometric Langlands.* Recall that $\widehat{\mathfrak{g}}_\kappa$-mod is a categorical representation of the loop group only for integral levels. For other levels, it instead carries an action of the category of twisted D-modules $\text{D-mod}_\kappa(LG)$. One can ask if there is a form of local Langlands duality for such twisted categorical representations of loop groups. Indeed there is, and this is the subject of quantum geometric Langlands.

To proceed, let us for simplicity assume that $\kappa - \kappa_c$ is a nondegenerate bilinear form. In this case, one may attach a dual form $\check{\kappa}$ for $\check{G}$, as follows. If we write $\kappa_{\check{\mathfrak{g}},c}$ for the critical level for $\check{\mathfrak{g}}$, then $\check{\kappa}$ is defined by the property that

$$\kappa - \kappa_c \text{ and } \check{\kappa} - \kappa_{\check{\mathfrak{g}},c}$$

are dual bilinear forms on $\mathfrak{t}$ and $\check{\mathfrak{t}}$, respectively. That is, up to the critical shift we have repeatedly met throughout, one asks that κ and $\check{\kappa}$ be dual forms.

The local quantum Langlands conjecture, due to Gaitsgory [Gai07], is that one has an equivalence of $(\infty, 2)$-categories

$$\text{D-mod}_\kappa(LG)\text{-mod} \simeq \text{D-mod}_{-\check{\kappa}}(L\check{G})\text{-mod}.$$

This builds on earlier work in the global setting, due to Beilinson–Drinfeld, Feigin–Frenkel, and Stoyanovsky [Sto06].

We confine ourselves here to only a few comments; see however [Gai18b], [ABC+18] for more details.

Remark 6.10.3. A basic thing to notice about the quantum case is that the two sides look more symmetric, as they only concern twisted categorical loop group representations. Moreover, in a precise sense, one recovers the usual statement (6.19) in the limit as $\kappa \to \kappa_c$, and hence $\check{\kappa} \to \infty$, and one recovers its analogue with the roles of G and $\check{G}$ reversed in the limit as $\check{\kappa} \to \kappa_{\check{\mathfrak{g}},c}$, and hence $\kappa \to \infty$.

Remark 6.10.4. The adjective 'quantum' arises as follows. Recall the Feigin–Frenkel isomorphism

$$\mathcal{W}_{\mathfrak{g},\kappa_c} \simeq \mathcal{O}_{\mathrm{Op}_{\check{G}}},$$

which plays a basic role in the story at critical level. In fact, Feigin–Frenkel showed in [FF91] that this is the quasi-classical limit of a family of isomorphisms of affine W-algebras

$$\mathcal{W}_{\mathfrak{g},\kappa} \simeq \mathcal{W}_{\check{\mathfrak{g}},\check{\kappa}}.$$

These isomorphisms, now between non-commutative algebras, deformation quantize the previous isomorphism and play a basic role in the quantum theory.

Relatedly, while at critical level we saw that the representations of the dual group $\mathrm{Rep}(\check{G})$ play a basic role in the unramified cases of geometric Langlands, the analogous role is played in the quantum setting by representations of the quantum group $\mathrm{Rep}_q(\check{G})$.

6.10.5. *Connections with physics.* So far, we have only touched on some of the relations between geometric Langlands and physics and mostly on ties to two-dimensional conformal field theory. However, there is an extremely rich connection with three- and four-dimensional supersymmetric gauge theories, starting with the works [KW07], [Wit08], [GW08], [FW08]. This is the source of many interesting constructions and expected compatibilities in (quantum) geometric Langlands, both local and global. We refer the reader to [Gai18a], [GR19], [BFN18], [Gai19], [CG19], [CG20], [CGL20], [FG20], [BFGT21], [HR21] for a partial indication of recent work, as well as the forthcoming [GY].

Remark 6.10.6. One exciting aspect of some of the above work is its close relation to new developments in the arithmetic Langlands program. Namely, many predictions of Langlands dual categories coming from S-duality of boundary conditions for four-dimensional $\mathcal{N} = 4$ Yang–Mills, as studied by Gaiotto–Witten [GW09], match analogous predictions in arithmetic, notably from the relative Langlands program of Sakellaridis–Venkatesh [SV17]. This connection is developed in ongoing work of Ben-Zvi–Sakellaridis–Venkatesh.

6.10.7. We have arrived at our final point, which we state first somewhat informally. When working with constructible sheaves on algebraic varieties, there are different categories of sheaves one can choose that behave similarly. Plainly, one typically works with either (i) D-modules, for which the corresponding cohomology theory, i.e., global sections of the 'constant sheaf,' is algebraic de Rham cohomology; (ii) ℓ-adic sheaves, for which the corresponding cohomology theory is étale cohomology; and (iii) when working over the complex numbers, constructible sheaves in the analytic topology, for which the corresponding cohomology theory is singular, i.e., Betti cohomology.

In the version of (quantum) geometric Langlands we have discussed, the constructible parts of the correspondence are of de Rham type. That is, on the automorphic side we study D-modules on the moduli of G-bundles, and on the spectral side we study coherent sheaves on the moduli of $\check{G}$-bundles with flat connections, i.e., local systems in the de Rham sense. Relatedly, for a smooth projective curve X over k, the global conjecture, in its modern formulation by Arinkin–Gaitsgory [AG15a], posits an equivalence

$$\text{D-mod}(\mathrm{Bun}_G(X)) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}_{\check{G}}(X)).$$

As we now describe, there are also versions of the theory of étale and Betti flavors.

6.10.8. *Local geometric Langlands with restricted variation.* An important recent development in local arithmetic Langlands has been its formulation in families of representations, i.e. for whole categories and not only irreducible by irreducible [EH14], [Hel21], [Zhu21], [FS21]. That is, its modern formulation is roughly as an equivalence between all representations of $G(\mathbb{F}_q((t)))$ and coherent sheaves on an appropriate moduli space of Galois representations.

Remark 6.10.9. In fact, in the spirit of the work of Zelevinsky, Lusztig, and Vogan [Vog93], the automorphic side is subtler and is instead also glued with the categories of representations of other groups, e.g., certain inner forms of G.

In the series of works [AGK+20b], [AGK+20a], [AGK+21], Arinkin–Gaitsgory–Kazhdan–Raskin–Rozenblyum–Varshavsky have produced a global categorical geometric Langlands conjecture that makes sense both in characteric zero and over function fields. It takes the form

$$\mathrm{Shv}^{\text{étale}}_{\mathrm{nilp}}(\mathrm{Bun}_G(X)) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}(X)). \tag{6.34}$$

On the left hand side, one is considering ind-constructible ℓ-adic sheaves on the moduli of G-bundles on a global curve X with singular support in the global nilpotent cone of $T^*\mathrm{Bun}_G(X)$. On the right hand side, one meets ind-coherent sheaves with nilpotent singular support on an appropriately defined moduli space of ℓ-adic $\check{G}$ local systems on X.

Remark 6.10.10. Very roughly speaking, the basic picture is that, over the complex numbers, the spectral side looks like a 'torn apart' version of the Betti moduli stack of $\check{G}$-local systems, i.e., the moduli space parametrizing representations of the topological fundamental group. That is, one decomposes the Betti stack into many pieces based on the semisimplification of the local system and takes the disjoint union of their formal neighborhoods. For example, the formal neighborhood of an irreducible local system is isolated in $\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}$ and identifies with the corresponding formal completion in the Betti or de Rham moduli spaces. Relatedly, the superscript 'restr' stands for restricted variation.

In the function field setting, upon taking the categorical trace of Frobenius, this was shown to recover (compactly supported) automorphic forms and, in particular, the automorphic-to-Galois direction of unramified global Langlands for cuspidal representations over function fields.

In unpublished work, Gaitsgory has formulated a local version of (6.34). This roughly asserts an equivalence

$$\mathrm{Shv}^{\text{étale}}(G(\overline{\mathbb{F}}_q(\!(t)\!))\text{-mod} \simeq 2\text{-IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}^{\mathrm{restr}}_{\check{G}}),$$

where on the left hand side one really works with a subtler notion of categorical representation to correct the non-essential surjectivity of the fully faithful embedding

$$\mathrm{Shv}^{\text{étale}}(X) \otimes \mathrm{Shv}^{\text{étale}}(Y) \hookrightarrow \mathrm{Shv}^{\text{étale}}(X \times Y)$$

and passes further to a full subcategory of 'spectrally finite' objects.

Via a categorical trace of Frobenius, this is expected in the function field setting to recover a form of the arithmetic local Langlands conjecture in its formulation by Zhu [Zhu21].

6.10.11. *Local Betti geometric Langlands.* Finally, a version of geometric Langlands for Betti sheaves has been formulated by Ben-Zvi–Nadler [BZN18]. Here, the global equivalence, for X a projective algebraic curve over the complex numbers, takes the form

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{nilp}}(\mathrm{Bun}_G(X)) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}(X)).$$

On the left hand side, one is considering Betti sheaves, not necessarily ind-constructible, on the analytification of $\mathrm{Bun}_G(X)$ with singular support in the global nilpotent cone. On the right hand side, one is considering ind-coherent sheaves on the Betti moduli space of local systems, i.e., the natural enhancement of the character variety of the underlying topological space of X to a derived stack.

The local case is not particularly well documented, but let us sketch some basic parts. First, the portion with regular singularities is established by the equivalence

$$\mathrm{Shv}^{\mathrm{Betti}}_{\mathrm{nilp}}(\mathring{I}\backslash LG/\mathring{I}) \simeq \mathrm{IndCoh}_{\mathrm{nilp}}(\widetilde{\check{G}}/\check{G} \underset{\check{G}/\check{G}}{\times} \widetilde{\check{G}}/\check{G}). \tag{6.35}$$

Informally speaking, this is a version of Bezrukavnikov's equivalence (6.22) and its variants for nonunipotent monodromy in families, where the monodromy varies through the adjoint quotient $\check{G}/\check{G}$.[33]

By construction, the modules for the right hand side of (6.35) identify with ind-coherent sheaves of categories with nilpotent singular support on $\mathrm{LocSys}^{\mathrm{Betti}}_{\check{G}}$, i.e., $\check{G}/\check{G}$. The full assertion, i.e., including the counterpart of wild ramification, should relate Betti categorical representations of the loop group, which is to first approximation $\mathrm{Shv}^{\mathrm{Betti}}(LG)$-mod, with ind-coherent sheaves of categories with nilpotent singular support on the moduli of Stokes data.

6.10.12. At this point, we have arrived at the end of the survey. We hope the reader has gained a feel for some of the basic ideas, results, and conjectures in this area, as well as some appreciation for their beauty. The interested reader is encouraged to wade into the literature we have toured, and, as is particularly feasible in an area with so many attractive open problems, further learn by doing.

Appendix A. From functions on $X(\mathbb{F}_q)$ to D-modules on $X(\mathbb{C})$

A.1. **Overview.**

A.1.1. The goal of this appendix, which is a supplement to Section 3, is to explain in what sense passing from functions on the $\mathbb{F}_q$-points of the flag variety to D-modules on the $\mathbb{C}$-points of the flag variety is a natural thing to do.

The short answer is that one thinks of the latter as some sort of categorification of the former. Any explanation of this necessarily involves three topics that play an important role in geometric representation theory – D-modules, the functions-sheaves correspondence, and the Riemann–Hilbert correspondence.

[33]This equivalence may not be available in the literature. However, the surveyor has an argument, joint with H. Chen, which hopefully will be recorded soon.

We hope the following presentation of these topics gives enough context and intuition for nonspecialists to appreciate the discussion in the main body of the text. We will defer the suggestion of references to the end of each subsection.

A.2. **Algebraic D-modules.**

A.2.1. In this subsection we would like to discuss the basics of D-modules. We begin with the affine case.

So, suppose X is a smooth and affine algebraic variety over a field k, which for simplicity we take to be of characteristic zero. Informally, a D-module on X is something in which it makes sense to scale by functions on X and to differentiate by vector fields on X.

More formally, let us denote by O_X the algebra of regular functions on X, and by T_X the O_X-module of regular vector fields on X. A D-module M on X is then an O_X-module equipped with a flat connection. That is, M is a vector space equipped with an action of the algebra of functions

$$O_X \otimes M \to M, \quad f \otimes m \mapsto f \cdot m,$$

as well an action by the Lie algebra of vector fields

$$T_X \otimes M \to M, \quad \xi \otimes m \mapsto \xi \cdot m.$$

That the above is a Lie algebra action is exactly the assumption of flatness, i.e., vanishing curvature. These two operations are asked to be compatible with scaling vector fields by functions

$$f \cdot (\xi \cdot m) = (f \cdot \xi) \cdot m$$

and with differentiating functions by vector fields, i.e., to satisfy the 'Leibnitz rule'

$$\xi \cdot (f \cdot m) = (\xi \cdot f) \cdot m + f \cdot (\xi \cdot m),$$

where $\xi \cdot f$ denotes the derivative of f with respect to ξ.

Equivalently, one can form the algebra D_X of polynomial linear differential operators on X, i.e., the subalgebra of all endomorphisms of O_X generated by multiplication by functions and differentiation by vector fields. With this, a D-module is simply a left D_X-module.

Example A.2.2. For the affine space $X = \mathbb{A}^n_k$, the regular functions and vector fields take the form

$$O_X \simeq k[x_1, \ldots, x_n] \quad \text{and} \quad T_X \simeq \bigoplus_i O_X \cdot \partial_i,$$

where the ∂_i, $1 \leqslant i \leqslant n$, denote the standard coordinate vector fields associated to the coordinates x_i. In this case, D_X is generated as a k-algebra by x_i, ∂_j, $1 \leqslant i, j \leqslant n$, subject to the standard commutation relations

$$[x_i, x_j] = [\partial_i, \partial_j] = 0, \qquad [\partial_i, x_j] = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

Explicitly, D_X has as a basis the ordered monomials

$$x_1^{a_1} \cdots x_n^{a_n} \cdot \partial_1^{b_1} \cdots \partial_n^{b_n}, \qquad \text{for } a_i, b_j \in \mathbb{Z}^{\geqslant 0},$$

i.e., consists of linear differential operators with polynomial coefficients.

We write D-mod$(X)^\heartsuit$ for the abelian category of D-modules on X and denote its unbounded derived category by D-mod(X). Let us discuss three basic objects of either category, i.e., examples of D-modules.

Example A.2.3. The algebra of differential operators D_X is naturally a left module over itself, i.e., a D-module.

Next, we turn to a slightly less tautological example of 'global nature.'

Example A.2.4. For any X, the ring of regular functions O_X carries a tautological structure of a D-module. Namely, one multiplies and differentiates functions in the standard way.

This module admits the following presentation via generators and relations. Acting on the constant function 1 yields an isomorphism of D-modules

$$D_X/D_X \cdot T_X \simeq O_X.$$

In particular, giving a map of D-modules $O_X \to M$ is the same as specifying a flat section of M, i.e., an element m annihilated by T_X.

More generally, any vector bundle E equipped with a flat connection ∇ is a D-module, and giving a map of D-modules $E \to M$ is the same as specifying a flat section of the tensor product

$$M \underset{O_X}{\otimes} E^\vee$$

equipped with its natural flat connection.

Finally, we would like to describe a class of examples of 'local nature.'

Example A.2.5. For any closed point x of X, one has an associated delta function D-module. Namely, if we write $\mathfrak{m}_x \subset O_X$ for the maximal ideal of functions vanishing at x, we set

$$\delta'_x := D_X/(D_X \cdot \mathfrak{m}_x).$$

In particular, giving a map of D-modules $\delta'_x \to M$ is the same as specifying a section of M annihilated by $\mathfrak{m}_x$, i.e., scheme-theoretically supported at x.

The D-module δ'_x is an algebro-geometric version of the vector space of distributions supported at x.[34] Relatedly, if $\xi_1, \ldots, \xi_n$ are a collection of vector fields in T_X whose image in the tangent space T_xX at x forms a basis, then the images of the monomials

$$\xi_1^{a_1} \cdots \xi_n^{a_n}, \quad \text{for } a_i \in \mathbb{Z}^{\geqslant 0} \tag{A.1}$$

form a basis for δ'_x. One thinks of the image of 1 from D_X as the usual delta distribution, and the entire module is then spanned by its derivatives.

Here is a somewhat technical point, which the reader may safely ignore at first pass. It turns out to be better to twist the underlying vector space of δ'_x by a line, namely the determinant of the tangent space at x:

$$\delta_x := \delta'_x \underset{k}{\otimes} \det(T_xX).$$

The reason is as follows. Given a map $f : X \to Y$ of smooth affine varieties, one has an associated pushforward of D-modules

$$f_* : \text{D-mod}(X) \to \text{D-mod}(Y),$$

cf. Section A.2.7 below. When we incorporate the twist, we then have canonical isomorphisms

$$f_*(\delta_x) \simeq \delta_{f(x)}, \qquad \text{for } x \in X.$$

That is, delta functions pushforward to delta functions, as one expects from the distributional analogue.

Remark A.2.6. Here is an alternative construction of δ_x, which may help elucidate the appearance of the determinant line.

Consider the Lie algebra of vector fields vanishing at x, i.e., $\mathfrak{m}_x \cdot T_X \subset T_X$. This acts naturally on the quotient $T_X/\mathfrak{m}_x \cdot T_X \simeq T_xX$. Explicitly, the action factors through

$$\mathfrak{m}_x \cdot T_X/\mathfrak{m}_x^2 \cdot T_X \simeq T_x^*X \otimes T_xX \simeq \mathrm{End}(T_x^*X),$$

and the action on T_xX is simply the natural action, for any vector space W, of its Lie algebra of endomorphisms $\mathrm{End}(W)$ on the dual vector space W^*.

[34]This parallel is improved by passing to the corresponding right D-module, but the reader may wish to ignore this point.

Consider the induced action of $\mathfrak{m}_x \cdot T_X$ on the determinant line $\det(T_x X)$. This line also carries a natural action of O_X, and these two actions define an action of the subalgebra $D_{X,x}$ of D_X generated by O_X and $\mathfrak{m}_x \cdot T_X$. A point here is that, before passing to the determinant, the relevant compatibilities would not be satisfied.

Finally, the induced D-module is equipped with a canonical isomorphism

$$D_X \underset{D_{X,x}}{\otimes} \det(T_x X) \simeq \delta_x,$$

which acts as the identity on the generating determinant lines, i.e., we have described the action of $D_{X,x}$ on the span of the delta distribution itself.

As the reader may guess, many D-modules have behavior which interpolates between the last two examples. That is, they resemble an algebraic connection along a subvariety and a delta D-module along its normal directions.

A.2.7. Given a map $X \to Y$ of smooth affine varieties, one has natural pullback and pushforward functors between their categories of D-modules, i.e.,

$$f^! : \text{D-mod}(Y) \to \text{D-mod}(X) \quad \text{and} \qquad f_* : \text{D-mod}(X) \to \text{D-mod}(Y).$$

As for the nature of the functors, for our purposes it suffices to bear the following in mind. On a manifold M, one has the algebra of smooth functions and its dual space of distributions. Given a map $M \to N$, one can pull back functions and dually push forward distributions.

Intuitively, $f^!$ records, for a function on Y satisfying a linear PDE, the linear PDEs satisfied by its pullback. Similarly, f_* records, for a distribution on X satisfying a linear PDE, the linear PDEs satisfied by its pushforward. In particular, $f^!$ and f_* are not always adjoint functors, but rather are dual. This can be made precise using duality of dg-categories as in Section 4.4.3.

We now provide a more detailed discussion of $f^!$ and f_* for the reader's convenience, though they may safely skip this. The pullback $f^!$ is simply given by the quasi-coherent pullback up to a cohomological shift, i.e.,

$$f^!(M) \simeq O_X \underset{O_Y}{\otimes} M[d_X - d_Y],$$

where we denote by d_X and d_Y the dimensions of X and Y, respectively. Here, O_X acts in the tautological way, and T_X acts via the chain rule, i.e., via the map

$$T_X \to O_X \underset{O_Y}{\otimes} T_Y$$

and the natural action of the latter on $f^!(M)$. To describe f_*, note the previous formula for the pullback can be rewritten as tensoring over D_Y with the bimodule

$$O_X \underset{O_Y}{\otimes} D_Y.$$

The same bimodule also defines the pushforward functor f_*. More carefully, it yields a functor between categories of right D-modules

$$f_{\mathrm{r},*} : \text{mod-D}(X) \to \text{mod-D}(Y).$$

The discrepancy is bridged by the fact that left D-modules and right D-modules are canonically identified. Namely, on a smooth affine variety Z, the canonical bundle K_Z carries a natural right action of D_Z. Roughly, one thinks of the canonical bundle as an algebraic substitute for distributions and the action is by integration by parts. Tensoring by it defines the desired equivalence

$$- \underset{O_Z}{\otimes} K_Z : \text{D-mod}(Z) \simeq \text{mod-D}(Z),$$

and applying this on both sides we obtain the pushforward

$$f_* : \text{D-mod}(X) \simeq \text{mod-D}(X) \to \text{mod-D}(Y) \simeq \text{D-mod}(Y).$$

A.2.8. The previous definition of a D-module in the affine case globalizes straightforwardly, as follows. Suppose now X is a smooth algebraic variety over k, not necessarily affine.

A D-module $\mathcal{M}$ on X is a quasi-coherent sheaf $\mathcal{M}$ equipped with a flat connection. That is, if we write $\mathcal{O}_X$ for the sheaf of regular functions, $\mathcal{T}_X$ for the sheaf of regular fields, and $\underline{k}$ for the constant sheaf associated to k, $\mathcal{M}$ is a sheaf of k vector spaces equipped with maps

$$\mathcal{O}_X \underset{\underline{k}}{\otimes} \mathcal{M} \to \mathcal{M} \quad \text{and} \quad \mathcal{T}_X \underset{\underline{k}}{\otimes} \mathcal{M} \to \mathcal{M},$$

satisfying compatibilities as in the previous section. We recall that quasicoherence is the further condition that, for any affine open U and regular function f on U with nonvanishing locus U_f, the natural map

$$\Gamma(U, \mathcal{M}) \underset{O_U}{\otimes} O_{U_f} \xrightarrow{\sim} \Gamma(U_f, \mathcal{M})$$

is an isomorphism. By this assumption, if X is affine, the present sheaf-theoretic category of D-modules is canonically equivalent to the one of the previous subsection, via the functor of taking global sections.

So, somewhat informally, the category of D-modules on a general smooth variety is glued from the previously discussed categories of D-modules on its affine open subvarieties. Similarly, a D-module $\mathcal{M}$ is roughly a sheaf whose local sections one can scale by regular functions and differentiate by regular vector fields.

The examples of the previous subsection carry over to this setting as follows.

Example A.2.9. One has a global sheaf $\mathcal{D}_X$ of differential operators on X, and D-modules are quasicoherent sheaves of $\mathcal{D}_X$-modules.

Example A.2.10. For any X, the sheaf of regular functions $\mathcal{O}_X$ is naturally a D-module, as is the sheaf of sections of any vector bundle with flat connection $(\mathcal{E}, \nabla)$.

Example A.2.11. For an closed point x of X, one again has a delta function D-module supported at x, which we denote by δ_x.

A.2.12. Finally, we note that the D-modules on X, as discussed in Section A.2.8, naturally form an abelian category in such a way that the forgetful functor to sheaves of abelian groups is t-exact. That is, one calculates kernels, cokernels, and direct sums in the usual way for sheaves, and these naturally inherit structures of D-modules and satisfy the correct universal properties.

Definition A.2.13. We write $\text{D-mod}(X)^\heartsuit$ for the abelian category of D-modules on X. We denote its unbounded derived category by $\text{D-mod}(X)$.

Given a map $f : X \to Y$, one again has pushforward and pullback functors

$$f^! : \text{D-mod}(Y) \to \text{D-mod}(X) \quad \text{and} \quad f_* : \text{D-mod}(X) \to \text{D-mod}(Y),$$

defined as in Section A.2.7. We should highlight that in the non-affine case, f_* is the composite of the left derived functor of tensoring with the bimodule

$$\mathcal{O}_X \underset{\mathcal{O}_Y}{\otimes} \mathcal{D}_Y$$

and the right derived functor of pushforward of sheaves along f. As such, it is not the derived functor of a functor between abelian categories of D-modules, and requires working with derived categories from the outset.

A.2.14. For further reading on D-modules, the reader may consult, as some of many possible references, ordered roughly by increasing demands on the reader, [Cou95], [AN12], [Ber82], [Kas03], [Gai05], [HTT08].

A.3. **The functions-sheaves correspondence.**

A.3.1. We have, as promised, made sense of the category of D-modules on the complex flag variety

$$\text{D-mod}(\mathrm{Fl}_G(\mathbb{C})).$$

We would like to next explain the sense in which this is analogous to the vector space of functions

$$\mathrm{Fun}(\mathrm{Fl}_G(\mathbb{F}_q)).$$

A.3.2. For any $\mathbb{F}_q$-variety X and field k of characteristic zero, recall we associated to it

$$\mathrm{Fun}(X(\mathbb{F}_q), k),$$

the vector space of k-valued functions on the finite set of rational points $X(\mathbb{F}_q)$.

A wonderful fact of life is that special functions in $\mathrm{Fun}(X(\mathbb{F}_q), k)$ are often the shadows of special sheaves on X. Let us say this more precisely.

A.3.3. To do so, let us take $k = \overline{\mathbb{Q}}_\ell$, an algebraic closure of the ℓ-adic numbers $\mathbb{Q}_\ell$, where ℓ is any prime number coprime to q.

With this choice of k one has a well-behaved category of étale local systems of k vector spaces on X. This category is equivalent to the category of continuous representations of the étale fundamental group of X on k vector spaces.

Remark A.3.4. Here is an orienting remark for the reader unfamiliar with these notions. Consider a smooth complex algebraic variety X equipped with a local system $\mathcal{L}$, by which we mean a locally constant sheaf of finite dimensional k vector spaces, in the sense of the analytic topology.

The basic observation to make is that one can hardly ever trivialize $\mathcal{L}$ on a Zariski open subset of X. Concretely, the reader may wish to think through the case of $X = \mathbb{G}_m$ and $\mathcal{L}$ a rank one local system whose monodromy around the origin is a root of unity.

However, if the corresponding representation of the fundamental group of X factors through a finite quotient, one can trivialize $\mathcal{L}$ after pulling back to a finite covering space $\tilde{X}$ of X. Crucially, $\tilde{X}$ is again canonically an algebraic variety – we emphasize that this is a nonobvious fact in general. In the previous example, if the monodromy is an n^{th} root of unity, one can take the cover to be

$$\mathbb{G}_m \to \mathbb{G}_m, \qquad z \mapsto z^n.$$

The covering map $\tilde{X} \to X$ is a prototypical example of an étale, i.e., a flat and unramified, map of algebraic varieties. The upshot is that if one allows oneself to check local properties of sheaves on not just Zariski open sets, but also étale covers, one obtains a robust theory of local systems which makes sense for general schemes, not necessarily over the complex numbers.

More generally, if one wants to consider local systems also on locally closed subvarieties of X, these all naturally belong to, and in fact generate, the bounded derived category of constructible étale sheaves $\mathrm{Sh}(X)^b$.

A.3.5. Let us explain what this category looks like in the simplest case of X being a point, i.e., the spectrum of a field κ.

If κ is separably closed, its constructible derived category of sheaves of k vector spaces is simply the bounded derived category of k vector spaces, i.e.

$$\mathrm{Sh}(\mathrm{Spec}\,\kappa)^b \simeq \mathrm{Vect}_k^b\,.$$

Plainly, this is because $\mathrm{Spec}\,\kappa$ does not have interesting étale covers, and this should match the reader's intuition for what happens for usual sheaves on a point in topology or complex geometry. We emphasize that k plays the role of the coefficients for the sheaves, and κ provides the geometric object $\mathrm{Spec}\ \kappa$ on which the sheaves live. In particular, there need be no relation between κ and k.

However, for a general field κ, $\mathrm{Spec}\,\kappa$ does have interesting étale covers. For example, given a separable extension of fields $\kappa \to \kappa'$, the map

$$\mathrm{Spec}\,\kappa' \to \mathrm{Spec}\,\kappa$$

is étale, and any finite étale cover is a disjoint union of such maps. For this reason, writing κ_s for a separable closure of κ and $H := \mathrm{Gal}(\kappa_s/\kappa)$ for its Galois group, one has a canonical equivalence between constructible sheaves on $\operatorname{Spec}\kappa$ and the bounded derived category of finite dimensional continuous representations of H on k vector spaces, i.e.

$$\mathrm{Sh}(\operatorname{Spec}\kappa)^b \simeq \mathrm{Rep}_k(H)^b.$$

In particular, the category of étale sheaves captures arithmetic information about the field κ. An analogue to $\operatorname{Spec}\kappa$ in usual topology would not be a point, but rather the classifying space pt/H of a (profinite) group H. In this analogy, one thinks of the map $\operatorname{Spec}\kappa_s \to \operatorname{Spec}\kappa$ as akin to the map $\mathrm{pt} \to \mathrm{pt}/H$.

A.3.6. Given a map $f : X \to Y$ of schemes, one has pushforward and pullback functors[35]

$$f_* : \mathrm{Sh}(X)^b \to \mathrm{Sh}(Y)^b \quad \text{and} \quad f^! : \mathrm{Sh}(Y)^b \to \mathrm{Sh}(X)^b.$$

Here f_* is the usual pushforward of sheaves, and $f^!$ is the so-called exceptional inverse image.

If f is a proper map (recall this roughly means that the fibres of the map are projective varieties), f_* is left adjoint to $f^!$. In particular, given a closed point x of X, with residue field κ_x, one has an adjunction

$$i_{x,*} : \mathrm{Sh}(\operatorname{Spec}\kappa_x)^b \rightleftarrows \mathrm{Sh}(X)^b : i_x^!.$$

Concretely, $i_{x,*}$ produces skyscraper sheaves supported at x, and its adjoint $i_x^!$ sends a sheaf to its (derived) sections supported at x.

A.3.7. Let X again be a variety over $\mathbb{F}_q$. We are now ready to describe the promised map

$$\mathrm{FF} : \mathrm{Sh}(X)^b \to \mathrm{Fun}(X(\mathbb{F}_q)),$$

which is known as Grothendieck's functions-sheaves correspondence. For each rational point $x : \operatorname{Spec}\mathbb{F}_q \to X$, we therefore need to produce a map

$$\mathrm{Sh}(X)^b \to k,$$

i.e., the values of the associated functions at x. We remind the reader that, as in Section A.3.3, k denotes the algebraic closure $\overline{\mathbb{Q}}_\ell$ of the ℓ-adic numbers. To define the desired evaluation, we first pass to the !-stalk at x, i.e., consider the composition

$$\mathrm{Sh}(X)^b \xrightarrow{i_x^!} \mathrm{Sh}(\operatorname{Spec}\mathbb{F}_q)^b \simeq \mathrm{Rep}_k(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q))^b.$$

From here, we recall that $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ is simply the profinite completion $\widehat{\mathbb{Z}}$ of the integers, with topological generator the Frobenius automorphism Fr.

In particular, by taking the trace of Frobenius, we obtain the desired map

$$\mathrm{Sh}(X)^b \xrightarrow{i_x^!} \mathrm{Rep}_k(\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q))^b \xrightarrow{\mathrm{tr}(\mathrm{Fr},-)} k.$$

Explicitly, this sends a sheaf $\mathcal{S}$ to the number

$$\sum_j (-1)^j \cdot \mathrm{tr}(\mathrm{Fr}, H^j(i_x^! \mathcal{S})).$$

Remark A.3.8. In case this seems remote to the reader, let us pursue the analogy of Section A.3.5. Replacing the completion $\widehat{\mathbb{Z}}$ by the usual integers $\mathbb{Z}$, note that $\mathrm{pt}/\mathbb{Z}$ is simply the circle S^1. That is, one can think of $\operatorname{Spec}\mathbb{F}_q$ as something like a circle.

One can then think of an n-dimensional variety $X \to \mathbb{F}_q$ as akin to a $2n+1$ dimensional real manifold fibred over a circle $\pi : M \to S^1$ and its rational points as a collection of embedded circles in M providing sections of π. The analogue of the trace of Frobenius would send a local system on M to the traces of its monodromies about these circles.

[35] One further has a !-pushforward and a $*$-pullback, but we will not need these.

A.3.9. A basic property of the functions-sheaves correspondence is that it intertwines the $*$-pushforward of functions with the $*$-pushforward of sheaves and similarly for !-pullbacks. For pullbacks, this is by definition, but, for pushforwards, one needs the Lefschetz fixed point formula in étale cohomology.

In addition, since $i^!$ sends distinguished triangles to distinguished triangles,[36] we obtain a map of k vector spaces

$$\mathrm{FF} : K(\mathrm{Sh}(X)^b) \underset{\mathbb{Z}}{\otimes} k \to \mathrm{Fun}(X(\mathbb{F}_q)),$$

where $K(-)$ denotes the Grothendieck group, so that one may reasonably think of $\mathrm{Sh}(X)^b$ as a 'categorification' of the space of functions. This is a useful point of view, although we mildly caution that the above arrow is typically not an isomorphism.

A.3.10. Let us describe two basic examples of the correspondence.

Example A.3.11. Let us denote by $\underline{k}$ the constant sheaf on X. If X is smooth and n-dimensional, then $\mathrm{FF}(\underline{k})$ is the constant function on $X(\mathbb{F}_q)$ with value q^n.

Remark A.3.12. Because we work with !-stalks, the sheaf which 'lifts' the constant function 1 on any variety X is not the constant sheaf but the dualizing sheaf ω_X. This is by definition obtained by pulling back along the projection $\pi : X \to \operatorname{Spec} \mathbb{F}_q$ the trivial representation k of $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, i.e.

$$\omega_X \simeq \pi^!(k).$$

The reader should be aware that many authors work instead with the Verdier dual convention, i.e., with $*$-stalks, when setting up the functions-sheaves correspondence, in which case the constant sheaf, and not the dualizing sheaf, would correspond to the constant function 1.

Example A.3.13. Given a rational point $x : \operatorname{Spec} \mathbb{F}_q \to X$, consider the skyscraper sheaf $i_{x,*}k$, where we view k as a trivial $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$-module, concentrated in cohomological degree zero. Then we have

$$\mathrm{FF}(i_{x,*}k) = \delta_x,$$

i.e., the skyscraper sheaf 'lifts' the indicator function of the point.

Although these simple examples hopefully convey the basic idea to the reader, we should mention that much more interesting numbers and functions, e.g., Gauss sums, Kloosterman sums, and characters of finite groups of Lie type, have sheaf-theoretic counterparts. Another rich example, which gave birth to the geometric Langlands theory, is the lift of unramified Hecke eigenfunctions over function fields to Hecke eigensheaves on moduli spaces of G-bundles on curves over finite fields.

A.3.14. For further details the reader may wish to consult [Mil89], [Mil80], [FK88].

A.4. **The Riemann–Hilbert correspondence.**

A.4.1. Having passed from $\mathrm{Fun}(X(\mathbb{F}_q))$ to $\mathrm{Sh}(X)^b$, we can make sense of the latter for X over any base field and not just $\mathbb{F}_q$.

Over the complex numbers, it will be convenient to work instead with the closely related category of constructible analytic sheaves

$$\mathrm{Sh}(X_{\mathrm{an}})^b.$$

Here X_{an} denotes the analytification of X, i.e., its set of closed points $X(\mathbb{C})$ equipped with their metric topology. Explicitly, $\mathrm{Sh}(X_{\mathrm{an}})^b$ is the full subcategory of the derived category of sheaves of k vector spaces generated by sheaves that are locally constant along a stratification of X into Zariski locally closed subsets.

[36] Recall that this simply means a sequence $A \to B \to C$ of maps in $\mathrm{Sh}(X)^b$, which up to isomorphism comes from a short exact sequence of complexes. More intrinsically, A is the homotopy kernel of $B \to C$, and, equivalently, C is the homotopy cokernel, i.e., cone, of $A \to B$.

Remark A.4.2. The category $\mathrm{Sh}(X_{\mathrm{an}})^b$ is an enlargement of the analogous category of étale sheaves, essentially by the Artin comparison theorem. For example, if $X = \mathbb{G}_m$, a rank one local system in the étale category must have monodromy in the units of the algebraic integers, i.e., $\overline{\mathbb{Z}}_\ell^\times$, whereas in the analytic category the monodromy may be any element of $\overline{\mathbb{Q}}_\ell^\times$.

A.4.3. Finally, let us obtain the desired connection to D-modules. First, fix an isomorphism $\overline{\mathbb{Q}}_\ell \simeq \mathbb{C}$ so that we can identify $\mathrm{Sh}(X_{\mathrm{an}})^b$ with its analogue with coefficients in complex vector spaces.

Recall that $\mathrm{Sh}(X_{\mathrm{an}})^b$ contains objects like local systems supported on subvarieties of X. These may be converted into a full subcategory of D-modules by an elaboration of the operation of passing between systems of differential equations and their local solutions, as follows.

A.4.4. Consider first the subcategory of the constructible derived category consisting of local systems on X. Let us assume that X is smooth. In this case, one has an equivalence of categories

$$\{\text{finite rank local systems on } X_{\mathrm{an}}\} \simeq \{\text{flat holomorphic connections on } X_{\mathrm{an}}\}, \tag{A.2}$$

where the right hand side denotes the category of holomorphic vector bundles $\mathcal{E}$ equipped with a holomorphic flat connection ∇, in the sense of complex geometry.

Let us review how this equivalence works. To go from right to left in (A.2), one sends a connection $(\mathcal{E}, \nabla)$ to its sheaf of flat sections $\mathcal{E}^\nabla$. The point is that for any $x \in X$, any vector in the fibre $\mathcal{E}_x$ extends in a small analytic ball to a unique flat section. Here one uses both that the connection is flat and that one is allowing analytic, and not merely polynomial solutions, to essentially reduce to the case of ordinary differential equations. To go from left to right, one sends a local system $\mathcal{L}$ to the holomorphic bundle

$$\mathcal{L} \underset{\underline{k}}{\otimes} \mathcal{O}_{X_{\mathrm{an}}},$$

with its connection induced by the tautological one on $\mathcal{O}_{X_{\mathrm{an}}}$.

A.4.5. To make contact with algebraic D-modules on X, note that applying $- \underset{\mathcal{O}_X}{\otimes} \mathcal{O}_{X_{\mathrm{an}}}$, i.e., allowing not only polynomial, but also holomorphic sections, yields a natural analytification functor

$$\{\text{flat algebraic connections on } X\} \to \{\text{flat holomorphic connections on } X_{\mathrm{an}}\}. \tag{A.3}$$

We emphasize that, on the left, one is considering objects of algebro-geometric nature and, on the right, objects of a complex analytic nature.

If X is proper then the functor (A.3) is an equivalence. This is a consequence of the so-called GAGA (i.e., géométrie algébrique et géométrie analytique) theorem.

However, if X is non-proper, the functor (A.3) is no longer an equivalence. Namely, it is still essentially surjective, but it typically sends many distinct flat algebraic connections to the same holomorphic connection. The reason is simple and already visible for rank one connections – there are typically many more holomorphic functions than polynomial functions (unlike in the case of X proper), and the differential equations satisfied by them lead to identifications after analytification between distinct algebraic connections.

Example A.4.6. Suppose $X = \mathbb{A}^1$ with coordinate z, and consider the left algebraic D-modules

$$D_X/(D_X \cdot \partial_z) \quad \text{and} \quad D_X/(D_X \cdot (\partial_z - 1)).$$

Explicitly, the underlying O_X-modules both carry canonical identifications with O_X, given by acting on the image of 1 in D_X, and carry the connections

$$\nabla = \partial_z \quad \text{and} \quad \nabla = \partial_z - 1.$$

As algebraic D-modules, they are not isomorphic, but their analytifications are. Indeed, this follows from the fact that e^z is a holomorphic, but not polynomial, function and is, up to scalars, the unique solution to

$$\partial_z u = u,$$

i.e., a flat section of the second connection.

What rectifies the many-to-oneness of (A.3) is the following nontrivial observation. For a given flat holomorphic bundle $\mathcal{E}_{\mathrm{an}}$, among the various algebraic connections $\mathcal{E}$ mapping to it, there will be a unique one whose flat sections do not have 'essential singularities at infinity.' More precisely, an algebraic connection $(\mathcal{E}, \nabla)$ on X is said to be *regular* if, for a smooth compactification

$$j : X \to \overline{X}$$

of X, with boundary $\overline{X} \setminus X$ a simple normal crossings divisor, the connection form $j_*\nabla$ on $j_*\mathcal{E}$ roughly has at most simple poles on the boundary divisor, cf. [HTT08] for a more careful formulation. It turns out this is independent of the choice of $\overline{X}$.

Example A.4.7. In the previous example, we may take the compactification

$$j : \mathbb{A}^1 \to \mathbb{P}^1,$$

i.e., we add the point at infinity. With respect to the coordinate $w = z^{-1}$ near infinity, the two connections take the form

$$\nabla = \partial_w \quad \text{and} \quad \nabla = \partial_w + w^{-2},$$

respectively. In particular, the second connection has a double pole and so is not regular. This is equivalent to the fact that the function e^z has an essential singularity at ∞.

When one restricts to regular connections, the analytification functor

$$\{\text{regular algebraic connections on } X\} \to \{\text{flat holomorphic connections on } X_{\mathrm{an}}\}$$

is an equivalence. The resulting composite equivalence with local systems

$$\nabla : \{\text{regular algebraic connections on } X\} \simeq \{\text{finite rank local systems on } X_{\mathrm{an}}\} \tag{A.4}$$

is explicitly given by taking the flat analytic sections. This is known as the Riemann–Hilbert correspondence for connections.

A.4.8. Finally, let us describe the desired extension of (A.4) to all constructible sheaves on X. Intuitively, we would like to interchange now not only local systems and regular connections, but also those supported on subvarieties.

A first guess would be that one has an equivalence of abelian categories between constructible sheaves and a suitable category of D-modules. After all, the previous correspondence for connections matched the abelian categories. For example, one might expect to exchange the constant sheaf $\mathbb{C}_X$ and the regular functions $\mathcal{O}_X$, and, for any closed point x of X, the rank one skyscraper sheaf $\mathbb{C}_x$ and the delta D-module δ_x. However, this cannot literally hold. Indeed, if X is of dimension $d > 0$, and one looks at the homomorphisms in the abelian category, one has

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\mathrm{an}})^\heartsuit}(\mathbb{C}_X, \mathbb{C}_x) \simeq \mathbb{C} \quad \text{but} \quad \mathrm{Hom}_{\text{D-mod}(X)^\heartsuit}(\mathcal{O}_X, \delta_x) \simeq 0.$$

To correct for this, observe that when passing to derived categories, we have

$$\mathrm{Hom}_{\mathrm{Sh}(X_{\mathrm{an}})}(\mathbb{C}_X, \mathbb{C}_x) \simeq \mathbb{C} \quad \text{and} \quad \mathrm{Hom}_{\text{D-mod}(X)}(\mathcal{O}_X, \delta_x) \simeq \mathbb{C}[-d].$$

The reader is encouraged to check both assertions, e.g., for the affine line $X = \mathbb{A}^1$. In any case, we see the desired extension will need to involve cohomological shifts and, in particular, must be formulated in terms of derived categories.

A.4.9. With this in mind, we are led to the following. The category of regular holonomic D-modules, denoted by $\text{D-mod}^{\mathrm{rh}}(X)$, is the full triangulated subcategory of D-mod(X) generated by $*$-extensions of regular algebraic connections from smooth subvarieties $Z \subset X$.

One has a functor of (derived) flat analytic sections, defined on the category of all D-modules

$$\nabla : \text{D-mod}(X) \to \{\text{sheaves of } \mathbb{C}\text{-vector spaces}\}, \tag{A.5}$$

which we shall describe in more detail momentarily. However, the main result here is that (A.5), when restricted to regular holonomic D-modules, factors through the constructible derived category and yields an equivalence

$$\nabla : \text{D-mod}^{\text{rh}}(X) \simeq \text{Sh}(X_{\text{an}})^b. \tag{A.6}$$

This is known as the Riemann–Hilbert correspondence for D-modules.

A.4.10. Invariantly, (A.5) is defined as follows. The holomorphic canonical bundle $\omega_{X_{\text{an}}}$ is naturally a right D-module, and (A.5) is obtained by tensoring with it:

$$\nabla := \omega_{X_{\text{an}}} \underset{\mathcal{D}_X}{\otimes} -.$$

As with Lie algebra homology, a canonical projective resolution of $\omega_{X_{\text{an}}}$ yields the following presentation. Let us denote by Ω^i_{an}, for $1 \leqslant i \leqslant d$, the sheaves of holomorphic i-forms on X. In particular, note that

$$\Omega^d_{\text{an}} = \omega_{X_{\text{an}}}.$$

With this, (A.5) sends a D-module $\mathcal{M}$, concentrated in cohomological degree zero, to the de Rham complex of sheaves

$$\mathcal{M} \to \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega^1_{\text{an}} \to \cdots \to \mathcal{M} \underset{\mathcal{O}_X}{\otimes} \Omega^d_{\text{an}} \tag{A.7}$$

where $\mathcal{M}$ is placed in cohomological degree $-d$. Explicitly, for a local section m of $\mathcal{M}$, and local coordinates $z_1, \ldots, z_d$ on X, the first differential sends

$$m \mapsto \sum_i (\partial_i \cdot m) \otimes dz_i,$$

and similarly for higher differentials. A complex of D-modules is then sent to the totalization of a corresponding de Rham bicomplex. This construction visibly respects quasi-isomorphisms and yields the desired functor (A.5).

A.4.11. Let us conclude this discussion with some basic examples and properties.

Example A.4.12. The Riemann–Hilbert correspondence sends the structure sheaf to a shift of the constant sheaf, namely

$$\nabla(\mathcal{O}_X) \simeq \mathbb{C}_X[d_X].$$

It sends a delta D-module δ_x to the corresponding skyscraper sheaf, i.e.,

$$\nabla(\delta_x) \simeq \mathbb{C}_x.$$

The reader may wish to check these directly and note in particular their compatibility with the discussion of Section A.4.8.

More generally, for a smooth subvariety $i : Z \to X$ of dimension d_Z and a regular connection $\mathcal{E}$ on Z with flat analytic sections $\mathcal{L}$, we have

$$\nabla(i_* \mathcal{E}) \simeq i_* \mathcal{L}[d_Z].$$

A.4.13. It is also possible to treat D-modules on a singular variety X. For any embedding $X \hookrightarrow S$ into a smooth variety, this category canonically agrees with the full subcategory of D-modules on S set-theoretically supported on X. The Riemann–Hilbert correspondence again holds for singular varieties and may be deduced from the smooth case.

The formulas of Example A.4.12 are then a special case of the following. For any map of algebraic varieties $X \to Y$, the Riemann–Hilbert correspondence exchanges $*$-pushforwards and !-pullbacks of D-modules and constructible sheaves.

A.4.14. Finally, let us discuss abelian categories. The subcategory $\text{D-mod}^{\text{rh}}(X)$ is closed under the standard truncation functors on $\text{D-mod}(X)$ and canonically identifies with the derived category of the corresponding abelian category of regular holonomic D-modules. From Example A.4.12, we see that this abelian category is not exchanged with the standard abelian category of constructible sheaves. Instead, it is identified with the abelian category of perverse sheaves.

A.4.15. For further details, the reader may wish to look at [AN12], [Ber82], [HTT08].

Conflict of interest. There is no conflict of interest.

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