ADVANCES IN MATHEMATICS 42, 169–178 (1981)

Green Polynomials and Singularities of Unipotent Classes

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Let X be an irreducible algebraic variety of dimension d over an algebraically closed field.

Deligne [2] has associated to X a complex ${}^{\pi}\mathbb{Q}_l$ of l-adic sheaves (canonical up to quasi-isomorphism) which has constructible cohomology sheaves $\mathscr{H}^i(X)$, which is self-dual in the derived category, which is equivalent to the complex reduced to constant sheaf $\mathbb{Q}_l$ in degree 0 over the smooth part of X, and which has the property: $\mathscr{H}^i(X) = 0$ for $i < 0$, $\mathscr{H}^i(X)$ has support of dimension $\leqslant d - i - 1$ if $i > 0$.

His construction, which is sketched in [8, Sect. 3] is an algebraic analogue of the Goresky and Macpherson middle intersection cohomology theory [3, 4]. We shall call $\mathscr{H}^i(X)$ the DGM sheaves of X.

The purpose of this paper is to describe an application of this theory to the study of irreducible characters of the finite group $GL_n(\mathbb{F}_q)$. Let k be an algebraic closure of $\mathbb{F}_q$. Let $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\ (\geqslant 0))$ be a partition of n: $n = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. We associate to λ the unipotent class $X_\lambda \subset GL_n(k)$ consisting of the unipotent elements which have Jordan blocks of size $\lambda_1, \lambda_2, \dots, \lambda_n$. We also associate to λ the irreducible unipotent representation E_λ of $GL_n(\mathbb{F}_q)$: it is the "biggest" component of the representation induced by the identity representation of the stabilizer of a flag of subspaces of dimensions $\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3, \dots$, in $\mathbb{F}_q^n$. Consider the DGM sheaves $\mathscr{H}^i(\bar{X}_\lambda)$ of the closure of X_λ. In the following theorem the sheaves $\mathscr{H}^i(\bar{X}_\lambda)$ will be regarded as sheaves on the whole variety of unipotent elements in $GL_n(k)$, equal to zero on the complement of $\bar{X}_\lambda$.

THEOREM 1. *Let $u \in GL_n(F_q)$ be a unipotent element. Let $n(\lambda)$ be*

* Supported in part by the National Science Foundation.

0001-8708/81/110169–10$05.00/0

defined by (1.2). *We have*

$$\mathrm{Tr}(u, E_\lambda) = q^{n(\lambda)} \sum_{i \geqslant 0} q^i \dim \mathscr{H}_u^{2i}(\bar{X}_\lambda).$$

Moreover, $\mathscr{H}^i(\bar{X}_\lambda) = 0$ *if* i *is odd.*

The proof will be given in Sections 1 and 2. The idea of the proof is to show that the singularities of the closure of the unipotent class X_λ are a special case of the singularities of a certain generalized Schubert variety (in the sense of [8, Sect. 5]) associated to elements in an affine Weyl group and then to use the results of [8, Sect. 5]. The algebraic formalism we shall use is that of the Hall algebra; an excellent exposition can be found in Macdonald's book [12].

1. The Space of Lattices

Let V be a fixed vector space of dimension n over the field $k((t))$. A $k[[t]]$-submodule of rank n of V is said to be a lattice. Let $\mathscr{L}$ be the set of all lattices in V, and let L_0 be a fixed latice in V. The lattices L such that $L \subset L_0$ form a subset $\mathscr{L}^+$ of $\mathscr{L}$. It can be regarded as a countable disjoint union of (finite dimensional) projective varieties $\mathscr{L}_i^+$ $(i \geqslant 0)$ over k, where $\mathscr{L}_i^+$ is the set of lattices $L \subset L_0$ such that $\dim(L_0/L) = i$. If $L \in \mathscr{L}_i^+$, then $t^{ni}L_0 \subset L$, hence $\mathscr{L}_i^+$ may be identified with the set of all codimension i subspaces of the n^2i-dimensional k-vector space $L_0/t^{ni}L_0$, which are stable under the nilpotent endomorphism t of $L_0/t^{ni}L_0$.

The group $GL(L_0)$ of automorphisms of L_0 acts in a natural way on $\mathscr{L}^+$; its orbits are in 1–1 correspondence with the elements of the set

$$\mathscr{P}_n = \{\lambda \in \mathbb{N}^n \mid \lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n\};$$

the orbit $\mathscr{O}_\lambda \subset \mathscr{L}^+$ corresponding to $\lambda \in \mathscr{P}_n$ consists of the lattices $L \subset L_0$ such that the nilpotent transformation t of L_0/L has Jordan blocks of sizes $\lambda_1, \lambda_2, \ldots, \lambda_n$. (In particular, $\mathscr{O}_\lambda \subset \mathscr{L}_{|\lambda|}^+$, where $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$.) The closure of $\mathscr{O}_\lambda$ is the union of the orbits $\mathscr{O}_\mu$ $(\mu \leqslant \lambda)$, where, for $\mu, \lambda \in \mathscr{P}_n$, $\mu \leqslant \lambda$ means that $|\mu| = |\lambda|$ and $\mu_1 \leqslant \lambda_1$, $\mu_1 + \mu_2 \leqslant \lambda_1 + \lambda_2$, $\mu_1 + \mu_2 + \mu_3 \leqslant \lambda_1 + \lambda_2 + \lambda_3$, etc.

Let $\mu \leqslant \lambda$ be two elements of $\mathscr{P}_n$. We wish to compute

$$\Pi_{\mu,\lambda} = \sum_{i \geqslant 0} q^{i/2} \dim \mathscr{H}_x^i(\bar{\mathscr{O}}_\lambda),$$

where $\mathscr{H}_x^i(\bar{\mathscr{O}}_\lambda)$ are the stalks of the DGM sheaves of the variety $\bar{\mathscr{O}}_\lambda$ at a point $x \in \mathscr{O}_\mu$, and $q^{1/2}$ is an indeterminate. Let Λ_n be the algebra of symmetric polynomials in the variables $X=(X_1, X_2,..., X_n)$ with coefficients in $Q(q^{1/2})$. For $\lambda \in \mathscr{P}_n$, we denote by $P_\lambda(X, q) \in \Lambda_n$ the Hall–Littlewood polynomial corresponding to λ [12, III(2.1)]. Then $s_\lambda(X)=P_\lambda(X, 0)$ are the Schur functions [12, I(3.1)]. There are well-defined polynomials (in q) $K_{\lambda,\mu}(q)$ defined for all $\mu \leqslant \lambda$ in $\mathscr{P}_n$ such that

$$s_\lambda(X)=\sum_{\mu \leqslant \lambda} K_{\lambda,\mu}(q)\, P_\mu(X, q) \tag{1.1}$$

for all $\lambda \in \mathscr{P}_n$ [12, III(2.6)]. For each $\lambda \in \mathscr{P}_n$, we set

$$n(\lambda)=\sum_{i \geqslant 1} (i-1)\,\lambda_i; \tag{1.2}$$

cf. [12, I(1.5)].

THEOREM 2. *If $\mu \leqslant \lambda$ are two elements in $\mathscr{P}_n$, we have*

$$\Pi_{\mu,\lambda}=q^{n(\mu)-n(\lambda)} K_{\lambda,\mu}(q^{-1}).$$

Let $\mathscr{B}$ be the set of all sequences $L^0 \supsetneqq L^1 \supsetneqq L^2 \supsetneqq \cdots \supsetneqq L^n=tL^0$, where L^i are lattices in V, and let $L_0 \supsetneqq L_1 \supsetneqq L_2 \supsetneqq \cdots \supsetneqq L_n=tL_0$ be such a fixed sequence (with L_0 as above).

Let I be the stabilizer of this sequence in $GL(L_0)$. It is known (see, for example [8, Sect. 5]) that $\mathscr{B}$ is in a natural way an infinite dimensional algebraic variety; more precisely it is an increasing union of (finite dimensional) projective varieties. Each orbit of I on $\mathscr{B}$ is isomorphic to an affine space. There are n distinguished orbits $s_1, s_2,..., s_n$ of dimension 1: if $1 \leqslant i \leqslant n-1$, s_i consists of all sequences $L_0 \supsetneqq L_1 \supsetneqq \cdots \supsetneqq L_{i-1} \supsetneqq L^i \supsetneqq L_{i+1} \supset \cdots \supset L_n=tL_0$, $L^i \neq L_i$; if $i=n$, s_n consists of all sequences $L^0 \supsetneqq L_1 \supsetneqq \cdots \supsetneqq L_{n-1} \supsetneqq L^n=tL^0$, $L^0 \neq L_0$. There is also a distinguished orbit τ of dimension 0: it consists of $L_1 \supsetneqq L_2 \supsetneqq \cdots \supsetneqq L_{n-1} \supsetneqq tL_0 \supsetneqq tL_1$. The set $\tilde{W}$ of orbits of I on $\mathscr{B}$ has a natural group structure; it has generators $s_1, s_2,..., s_n, \tau$, where $s_1, s_2,..., s_n$ are the standard generators of an affine Weyl group and τ is an element of infinite order satisfying $\tau s_i=s_{i+1}\tau$ (here i is taken as an integer modulo n). Let l: $\tilde{W} \to \mathbb{N}$ be the length function: $l(w)$ is the dimension of the corresponding orbit in $\mathscr{B}$.

Let H'_n be the Hecke algebra corresponding to $\tilde{W}$: it is an algebra over $Q(q^{1/2})$ with basis T_w ($w \in \tilde{W}$) and multiplication defined by

$$\begin{aligned} T_w T_{w'} &= T_{ww'}, &&\text{if} \quad l(ww')=l(w)+l(w'),\\ (T_{s_i}+1)(T_{s_i}-q) &= 0, &&\text{if} \quad i=1, 2,..., n. \end{aligned}$$

There is a canonical involution of the ring H'_n (see [7, Sect. 1]). It is the unique ring homomorphism $h \to \bar{h}$ of H'_n into itself such that $\overline{q^{1/2}} = q^{-1/2}$ and $\overline{T_w} = T_{w^{-1}}^{-1}$.

If y, w are elements of $\tilde{W}$ such that $y \leqslant w$ (i.e., the orbit defined by y is in the closure of the orbit defined by w) then there is a well-defined polynomial $P_{y,w}$ in q of degree $\leqslant \frac{1}{2}(l(w) - l(y) - 1)$ (if $y < w$) and such that $P_{w,w} = 1$; it is characterized by the identity

$$\overline{q^{-l(w)/2} \sum_{y \leqslant w} P_{y,w} T_y} = q^{-l(w)/2} \sum_{y \leqslant w} P_{y,w} T_y \qquad (\forall\, w \in \tilde{W}); \tag{1.3}$$

see [7, (1.1.6)]. The polynomial $P_{y,w}$ has the interpretation [8, Sect. 5]

$$P_{y,w} = \sum_{i \geqslant 0} q^{i/2} \dim \mathscr{H}^i_\alpha(\bar{w}),$$

where $\mathscr{H}^i_\alpha(w)$ are the stalks of the DGM sheaves of the closure of the orbit defined w at a point α in the orbit defined by y.

Let $\mathscr{B}^+$ be the subset of $\mathscr{B}$ consisting of sequences $L^0 \supsetneqq L^1 \supsetneqq \cdots \supsetneqq L^n = tL^0$ such that $L^0 \subset L_0$. There is a natural map $\mathscr{B}^+ \xrightarrow{\pi} \mathscr{L}^+$ taking $L^0 \supset L^1 \supset \cdots \supset L^n$ to L^0. Its fibres are isomorphic to the flag manifold of an n dimensional vector space over k. The inverse image of $\mathscr{O}_\lambda \subset \mathscr{L}^+$ under this map is a finite union of orbits w of I on $\mathscr{B}^+$; these w form a single double coset with respect to the subgroup $W \subset \tilde{W}$ generated by $s_1, \ldots, s_{n-1}$ ($W \approx \mathfrak{S}_r$). Among these orbits w, there is a unique one of maximal dimension, say, w_λ. If $\mu \leqslant \lambda$, so that $\mathscr{O}_\mu \subset \bar{\mathscr{O}}_\lambda$, then it is clear that $w_\mu \leqslant w_\lambda$ and that the stalk of $\mathscr{H}^i(\bar{\mathscr{O}}_\lambda)$ at a point of $\mathscr{O}_\mu$ is isomorphic to the stalk of $\mathscr{H}^i(\bar{w}_\lambda)$ at a point of w_μ. In particular, we have

$$\Pi_{\mu,\lambda} = P_{w_\mu, w_\lambda}, \tag{1.4}$$

so that $\mathscr{H}^i(\bar{\mathscr{O}}_\lambda) = 0$ for odd i (i.e., $\Pi_{\mu,\lambda}$ is a polynomial in q).

Next, we note that the elements

$$u_\lambda = \frac{1}{\Phi(q)} \sum_{\substack{w \subset \mathscr{B}^+ \\ \pi(w) \subset \mathscr{O}_\lambda}} T_w \in H'_n \qquad (\lambda \in \mathscr{P}_n), \tag{1.5}$$

where $\Phi(q) = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1})$, span a subspace H_n of H'_n which is closed under multiplication and has identity element u_0. Indeed, H_n could be characterized as the set of elements h in H such that $hu_0 = u_0 h = h$ and such that h is a linear combination of elements T_w ($w \subset \mathscr{B}^+$) (see [1, 2.10]), and hence H_n is an algebra. An argument in [12, V(2.6)] shows that

$$u_\mu \cdot u_\nu = \sum_\lambda g^\lambda_{\mu\nu}(q)\, u_\lambda,$$

where $g^{\lambda}_{\mu\nu}(q)$ are the Hall polynomials (see [12, II, 2]); note that $g^{\lambda}_{\mu\nu}=0$ unless $|\lambda|=|\mu|+|\mu|$.

The involution $h\to\bar{h}$ of H'_n keeps u_0 fixed:

$$\bar{u}_0=\frac{1}{\Phi(q^{-1})}\sum_{w\in W}T^{-1}_{w^{-1}}=\frac{q^{-n(n-1)/2}}{\Phi(q^{-1})}\sum_{w\in W}T_w=\frac{1}{\Phi(q)}\sum_{w\in W}T_w;$$

hence it leaves H_n stable. (From $hu_0=u_0h=h$, it follows that $\bar{h}u_0=u_0\bar{h}=\bar{h}$.)

From (1.3), (1.4), (1.5) and the identity $P_{zw_\mu z',w_\lambda}=P_{w_\mu,w_\lambda}$ ($\forall z, z'\in W$) (see [7, (2.3.g)]), it follows immediately that the identity

$$\overline{q^{-d(\lambda)/2}\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}u_\mu}=q^{-d(\lambda)/2}\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}u_\mu\qquad(\forall\lambda\in\mathscr{P}_n)\tag{1.6}$$

holds in H_n; here $d(\lambda)=l(w_\lambda)-\nu$ is the dimension of $\mathscr{O}_\lambda$.

Next we note that for $1\leqslant r\leqslant n$, the orbit $\mathscr{O}_{(1^r)}$ is just the set of all lattices $L\subset L_0$ of codimension r such that $t=0$ on L_0/L. These are in 1–1 correspondence with the codimension r subspaces of L_0/tL_0, hence form a Grassmanian of dimension $r(n-r)$. In particular, this is a closed orbit. It follows that for $\lambda=(1^r)$, we have $\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}u_\mu=u_\lambda$; hence (1.6) becomes:

$$\overline{q^{-r(n-r)/2}u_{(1^r)}}=q^{-r(n-r)/2}u_{(1^r)}.\tag{1.7}$$

It is known [12, III(3.4)] that there is a unique isomorphism $\Psi: H_n\to\Lambda_n$ of $Q(q^{1/2})$ algebras such that $\Psi(u_\lambda)=q^{-n(\lambda)}P_\lambda(X,q^{-1})$ ($\forall\lambda\in\mathscr{P}_n$).

In particular, $\Psi(u_{1^r})=q^{-r(r-1)/2}e_r$, where $e_r\in\Lambda_n$ is the rth elementary symmetric function. Transporting the involution $h\to\bar{h}$ of H_n to Λ_n, via Ψ, we get a ring involution $p\to\bar{p}$ of Λ_n such that $\overline{q^{1/2}}=q^{-1/2}$ and $\overline{q^{-r(r-1)/2}e_r}=q^{-r(n-r)}q^{-r(r-1)/2}e_r$, i.e.,

$$\overline{e_r}=q^{-r(n-1)}e_r\qquad(1\leqslant r\leqslant n).\tag{1.8}$$

Since $P_\lambda(X,q^{-1})$ has total degree of homogeneity $|\lambda|$ in X, it is a linear combination of products $e_{r_1}e_{r_2}\cdots e_{r_s}$, $r_1+\cdots+r_s=|\lambda|$, with coefficients in $\mathbb{Z}[q^{-1}]$, and it follows that

$$\overline{P_\lambda(X,q^{-1})}=q^{-|\lambda|(n-1)}P_\lambda(X,q).\tag{1.9}$$

Thus, applying Ψ to both sides of (1.6) we see that the identity

$$\begin{aligned}&q^{d(\lambda)/2}\sum_{\mu\leqslant\lambda}\bar{\Pi}_{\mu,\lambda}q^{n(\mu)}\cdot q^{-|\mu|(n-1)}P_\mu(X,q)\\&=q^{-d(\lambda)/2}\sum_{\mu\leqslant\lambda}\Pi_{\mu,\lambda}q^{-n(\mu)}P_\mu(X,q^{-1})\qquad(\forall\lambda\in\mathscr{P}_n)\end{aligned}$$

holds in Λ_n. According to [12, V(2.9)] we have

$$n(\lambda) = \frac{|\lambda|\,(n-1)}{2} - \frac{d(\lambda)}{2}$$

hence

$$\sum_{\mu \leqslant \lambda} q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda} P_\mu(X, q)$$
$$= \sum_{\mu \leqslant \lambda} q^{-n(\mu)+n(\lambda)} \Pi_{\mu,\lambda} P_\mu(X, q^{-1}) \qquad (\forall\, \lambda \in \mathscr{P}_n). \tag{1.10}$$

Now, if $\mu < \lambda$, $\Pi_{\mu,\lambda}$ is a polynomial in q of degree $\leqslant \frac{1}{2}(d(\lambda) - d(\mu) - 1) = n(\mu) - n(\lambda) - \frac{1}{2}$; hence $q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda}$ is a polynomial in q without constant term. Thus, the left-hand side of (1.10) is a polynomial in X and in q. This polynomial is invariant under the substitution $q \to q^{-1}$, hence it does not involve q. Hence the left-hand side of (1.10) is equal to its value for $q = 0$. Thus

$$\sum_{\mu \leqslant \lambda} q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda} P_\mu(X, q) = P_\lambda(X, 0) = s_\lambda(X) \qquad (\forall\, \lambda \in \mathscr{P}_n).$$

Comparing with (1.2) it follows that $q^{n(\mu)-n(\lambda)} \bar{\Pi}_{\mu,\lambda} = K_{\lambda,\mu}$ $(\forall\, \mu \leqslant \lambda)$ and Theorem 2 is proved.

COROLLARY 3. *The image of* $c_\lambda = q^{-d(\lambda)/2} \sum_{\mu \leqslant \lambda} \Pi_{\mu,\lambda} u_\mu$ *under the isomorphism*

$$\Psi\colon H_n \overset{\approx}{\to} \Lambda_n$$

is

$$q^{-|\lambda|(n-1)/2} s_\lambda(X).$$

In particular, if $\mu, \nu \in \mathscr{P}_n$, *the product* $c_\mu \cdot c_\nu$ *is a combination with constant coefficients of elements* c_λ. *The multiplication constants are those which give the product of the corresponding Schur functions.*

COROLLARY 4 (see [9; 12, III(6.5)].) *The polynomial* $K_{\lambda,\mu}(q)$ *has* $\geqslant 0$ *coefficients* $(\mu \leqslant \lambda)$.

2. A COMPACTIFICATION OF THE VARIETY OF UNIPOTENT ELEMENTS IN $GL_n(k)$

Let $\bar{V}$ be an n-dimensional k-vector space. Define $E = \bar{V} \oplus \cdots \oplus \bar{V}$ (n copies) and let $t\colon E \to E$ be defined by $t(v_1, \ldots, v_n) = (0, v_1, v_2, \ldots, v_{n-1})$. Let

Y be the variety of all n-dimensional t-stable subspaces of E and let Y_0 be the open subvariety of Y consisting of those subspaces in Y which are transversal to

$$\underbrace{\bar{V}\oplus\cdots\oplus\bar{V}}_{n-1}\oplus 0.$$

To give a subspace $E'\in Y_0$ is the same as to give linear maps $f_1, f_2,..., f_{n-1}: \bar{V}\to\bar{V}$ such that

$$E' = \{(f_{n-1}(v), f_{n-2}(v),..., f_1(v), v) \mid v\in\bar{V}\}.$$

The condition for E' to be t-stable is that $(0, f_{n-1}(v), f_{n-2}(v),..., f_1(v))\in E'$ for all $v\in\bar{V}$, i.e., that $f_2(v)=f_1^2(v)$, $f_3(v)=f_2f_1(v),..., f_{n-1}=f_{n-2}(f_1(v))$, $0=f_{n-1}(f_1(v))$ or equivalently that $f_i=f_1^i$ $(1\leqslant i\leqslant n-1)$ and $f_1^n=0$. Thus, the correspondence

$$f_1\to E' = \{(f_1^{n-1}(v), f_1^{n-2}(v),..., f_1(v), v) \mid v\in V\}$$

gives an isomorphism between the variety X' of nilpotent endomorphisms of $\bar{V}$ and the open subvariety Y_0 of Y.

We shall now identify Y with the variety $\mathscr{L}_n^+$ defined in the previous section. Note that $\mathscr{L}_n^+$ can be identified with the set of t-stable codimension n subspaces of the n^2-dimensional k-vector space L_0/t^nL_0, hence with the set of t-stable n dimensional subspaces of the dual space $(L_0/t^nL_0)^*$. But this last space is isomorphic to $E=\bar{V}\oplus\cdots\oplus\bar{V}$ as a $k[t]$-module, and thus we may identify Y and $\mathscr{L}_n^+$. In this way, the variety X of unipotent elements in $GL(\bar{V})$ (which is canonically isomorphic to X') appears as an open subset of $\mathscr{L}_n^+$. Thus, $\mathscr{L}_n^+$ may be regarded as a compactification of X. The imbedding of X into $\mathscr{L}_n^+$ has the property that the unipotent class $X_\lambda\subset X$ is equal to $\mathscr{O}_\lambda\cap X$ for all $\lambda\in\mathscr{P}_n$, $|\lambda|=n$. In particular for such λ, the DGM sheaf $\mathscr{H}^i(\bar{X}_\lambda)$ of $\bar{X}_\lambda$ (closure in X) is just the restriction of the DGM sheaf $\mathscr{H}^i(\bar{\mathscr{O}}_\lambda)$ of $\bar{\mathscr{O}}_\lambda$ to $\bar{X}_\lambda$. It then follows from Theorem 2 that $\mathscr{H}^i(\bar{X}_\lambda)=0$ for i odd and that

$$\sum_{i\geqslant 0} q^i \dim\mathscr{H}_x^{2i}(\bar{X}_\lambda) = q^{n(\mu)-n(\lambda)}K_{\lambda,\mu}(q^{-1}), \ (x\in X_\mu) \tag{2.1}$$

for all $\mu\leqslant\lambda$ in $\mathscr{P}_n$, $|\lambda|=|\mu|=n$. If we specialize q to be a prime power, and if $u\in GL_n(F_q)$ is a unipotent element with Jordan blocks of sizes $\mu_1\geqslant\mu_2\geqslant\cdots\geqslant\mu_n$, then, by the results of Green [5],

$$q^{n(\mu)}K_{\lambda,\mu}(q^{-1}) = \mathrm{Tr}(u, E_\lambda) \tag{2.2}$$

for all $\mu\leqslant\lambda$ in $\mathscr{P}_n$, $|\lambda|=|\mu|=n$; moreover, $\mathrm{Tr}(u, E_\lambda)=0$ if $\mu\not\leqslant\lambda$.

Now Theorem 1 follows from (2.1) and (2.2).

3. Some Problems

Let $\mathscr{E}$ be a locally constant l-adic sheaf on an open smooth subset U of an irreducible variety X. Deligne's definition of the complex ${}^{\pi}\mathbb{Q}_l$ is applicable without change to $\mathscr{E}$ instead of the constant sheaf $\mathbb{Q}_l$ and leads to a complex ${}^{\pi}\mathscr{E}$ of l-adic sheaves on X which on U is equivalent to the complex reduced to $\mathscr{E}$ in degree zero. (This construction was used recently by Vogan and by Beilinson and Bernstein in connection with the question of describing the characters of a real semisimple Lie group. This section was influenced by their work.) Let $\mathscr{H}^i({}^{\pi}\mathscr{E})$ denote the cohomology sheaves of ${}^{\pi}\mathscr{E}$. If we are given a finite group W of automorphisms of $\mathscr{E}$ (inducing identity on X) then, by functoriality, W will act on each of the sheaves $\mathscr{H}^i({}^{\pi}\mathscr{E})$, inducing identity on X.

Consider, for example, the case where $X = G$, a reductive connected algebraic group over k. Let U be the open subset of G consisting of all regular semisimple elements in G. There is a canonical principal bundle $\tilde{U} \to^p U$ with group W (where W is the Weyl group of G): $\tilde{U}$ is the set of pairs (s, B), where $s \in U$ and B is a Borel subgroup of G containing s. Let $\mathscr{E} = p_*(\mathbb{Q}_l)$. This is a locally constant sheaf on V, with a natural action of W. (W acts on each stalk by the regular representation.) It follows that W acts naturally on each of the sheaves $\mathscr{H}^i({}^{\pi}\mathscr{E})$. If $g \in G$, the stalk $\mathscr{H}^i_g({}^{\pi}\mathscr{E})$ is naturally isomorphic to $H^i(\mathscr{B}_g, \mathbb{Q}_l)$, where $\mathscr{B}_g$ is the variety of Borel subgroups containing g. Indeed, let $\tilde{G}$ be the set of pairs (g', B), where $g' \in G$ and $B \in \mathscr{B}_g$; and let $p_1 : \tilde{G} \to G$ be the projection $(g', b) \to g'$; then, one can show that ${}^{\pi}\mathscr{E} = (p_1)_* (\mathbb{Q}_l)$. This follows by a general argument of Goresky and Macpherson [4, 4.2] as soon as one checks that there exists a finite partition of G into locally closed, irreducible subsets $G_0, G_1, \dots, G_n$ of G such that $G_0 = U$ and $\dim p_1^{-1}(g) \leqslant \frac{1}{2}$ (codim $G_i - 1$) for all $g \in G_i$ ($i = 1, 2, \dots, n$). The existence of such a partition follows from the finiteness of the number of unipotent classes in a reductive group and from the known inequality $\dim \pi^{-1}(g) \leqslant \frac{1}{2} (\dim Z(g) - \operatorname{rank}(G))$. Thus, we see that there is a natural action of W on $H^i(\mathscr{B}_g, \mathbb{Q}_l)$, for any $g \in G$. (In the case where g is unipotent, and the characteristic of k is not too small this can be identified with Springer's representation [15]; however, our definition seems to be closer to Slodowy's approach [13] to Springer's representation.)

Assume now that G is defined over $\mathbb{F}_q$; let $F: G \to G$ be the corresponding Frobenius map. Then, for any $g \in G$, there is a natural map $F: \mathscr{H}^i_g({}^{\pi}\mathscr{E}) \to \mathscr{H}^i_{F(g)}({}^{\pi}\mathscr{E})$.

Let $w \in W$, and let $\mathscr{B}_{(w)}$ be the variety of Borel subgroup $B \subset G$ such that B, FB are in relative position w; G^F acts on $\mathscr{B}_{(w)}$ by conjugation. In view of [6, 15], it seems natural to state

Conjecture 1.

$$\sum_i (-1)^i \operatorname{Tr}(Fw, \mathscr{H}^i_g({}^{\pi}\mathscr{E})) = \sum_i (-1)^i \operatorname{Tr}(g, H^i(\mathscr{B}_{(w)}, \mathbb{Q}_l)), \textit{for all } g \in G^F.$$

Now let ρ be an irreducible representation of W (over $\mathbb{Q}_l$) and let ρ' be its dual; we define the sheaves $\mathscr{H}^i({}^\pi\mathscr{E})_\rho$ to be $(\mathscr{H}^i({}^\pi\mathscr{E}) \otimes \rho')^W$. We wish to describe the restriction of the sheaf $\mathscr{H}^i({}^\pi\mathscr{E})_\rho$ to the variety $\mathscr{U}$ of unipotent elements in G.

CONJECTURE 2. *Given ρ as above, there is a well-defined unipotent class $C \subset G$ and a locally constant, l-adic sheaf $\mathscr{E}$ on C associated to an irreducible representation of the group of components of the centralizer of an element $u \in C$ with the following property: The sheaf $\mathscr{H}^i({}^\pi\mathscr{E})[-b_u]$ on $\bar{C}$, extended by zero on $\mathscr{U} - \bar{C}$ is isomorphic to the restriction of $\mathscr{H}^i({}^\pi\mathscr{E})_\rho$ to $\mathscr{U}$ (where $b_u = \dim \mathscr{B}_u$).*

This conjecture, which would make [15, 6.10] more precise, is supported by Theorem 1.

Remarks. (a) According to an unpublished theorem of Deligne, the variety of all unipotent elements in G is rationally smooth (in the sense of [7, 4.1]), at least in sufficiently large characteristic. It follows that, if $\mathscr{E}$ is the constant sheaf $\mathbb{Q}_l$ on the regular unipotent class, then $\mathscr{H}^i({}^\pi\mathscr{E})$ is the constant sheaf $\mathbb{Q}_l$ for $i = 0$ and it is zero if $i > 0$.

(b) Assume that G is simple and split over $\mathbb{F}_q$. There is a unique unipotent class $C \subset G$ of dimension $2(h-1)$, where h is the Coxeter number. When all root lengths are the same this is the minimal unipotent class not containing the neutral element e; according to an unpublished theorem of Kostant, the number of $\mathbb{F}_q$-rational points of C is given by

$$(q^h - 1)(q^{e_1-1} + q^{e_2-1} + \cdots + q^{e_l-1}), \tag{3.1}$$

where $e_1, \ldots, e_l$ are the exponents of G.

When there are roots of different lengths, C is no longer the minimal unipotent class not containing e. However, one can check, using a case by case analysis, that $\bar{C} - \{e\}$ is rationally smooth (in the sense of [7, A1]) and that the number of $\mathbb{F}_q$-rational points of $\bar{C} - \{e\}$ is again given by (3.1).

Let $\mathscr{E}$ be the constant sheaf $\mathbb{Q}_l$ on C; we consider the corresponding sheaves $\mathscr{H}^i({}^\pi\mathscr{E})$ on $\bar{C}$. Using the method of ([7, Appendix]) it follows that, in general, its stalks at e are described by

$$\sum_{i \geq 0} q^{i/2} \dim \mathscr{H}^i_e({}^\pi\mathscr{E}) = \sum_{i=1}^{l} q^{e_i - 1}. \tag{3.2}$$

(c) Let ρ be a special representation of W (in the sense of [10, 11]), and let C_ρ be the unipotent class in G corresponding to ρ by Conjecture 2. Such a unipotent class is said to be special. This concept was introduced in [10] in a slightly different way, which is however, equivalent to the present definition.

Another definition for special unipotent classes was proposed by Spaltenstein [14]. Following Spaltenstein, we associate to a special unipotent class, C, the subset $\tilde{C} \subset U$ consisting of all elements g in the closure of C which are not in the closure of any special unipotent class $C' \neq C$, $C' \subset \bar{C}$. For example, if C is as in (b), then C is the minimal special unipotent class other than $\{e\}$ and we have $\tilde{C} = \bar{C} - \{e\}$. The results in (b) suggest

CONJECTURE 3. *If C_ρ is any special unipotent class in G, then $\tilde{C}_\rho$ is rationally smooth. If, moreover, G is simple and split over F_q, then the number of F_q-rational points of $\tilde{C}$ is given by a polynomial in q which depends only on ρ (i.e., it is independent of characteristic and is the same for types B_n and C_n).*

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