SCHUBERT CELLS AND COHOMOLOGY OF THE SPACES G/P

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We study the homological properties of the factor space G/P, where G is a complex semisimple Lie group and P a parabolic subgroup of G. To this end we compare two descriptions of the cohomology of such spaces. One of these makes use of the partition of G/P into cells (Schubert cells), while the other consists in identifying the cohomology of G/P with certain polynomials on the Lie algebra of the Cartan subgroup H of G. The results obtained are used to describe the algebraic action of the Weyl group W of G on the cohomology of G/P.

Contents

Introduction

Let G be a linear semisimple algebraic group over the field C of complex numbers and assume that G is connected and simply-connected. Let B be a Borel subgroup of G and $X = G/B$ the fundamental projective space of G.

The study of the topology of X occurs, explicitly or otherwise, in a large number of different situations. Among these are the representation theory of semisimple complex and real groups, integral geometry and a number of problems in algebraic topology and algebraic geometry, in which analogous spaces figure as important and useful examples. The study of the homological properties of G/P can be carried out by two well-known methods. The first of these methods is due to A. Borel [1] and involves the identification of the cohomology ring of X with the quotient ring of the ring of polynomials on the Lie algebra $\mathfrak{h}$ of the Cartan subgroup

$H \subset G$ by the ideal generated by the W-invariant polynomials (where W is the Weyl group of G). An account of the second method, which goes back to the classical work of Schubert, is in Borel's note [2] (see also [3]); it is based on the calculation of the homology with the aid of the partition of X into cells (the so-called Schubert cells). Sometimes one of these approaches turns out to be more convenient and sometimes the other, so naturally we try to establish a connection between them. Namely, we must know how to compute the correspondence between the polynomials figuring in Borel's model of the cohomology and the Schubert cells. Furthermore, it is an interesting problem to find in the quotient ring of the polynomial ring a symmetrical basis dual to the Schubert cells. These problems are solved in this article. The techniques developed for this purpose are applied to two other problems. The first of these is the calculation of the action of the Weyl group on the homology of X in a basis of Schuberts cells, which turns out to be very useful in the study of the representations of the Chevalley groups.

We also study the action of W on X. This action is not algebraic (it depends on the choice of a compact subgroup of G). The corresponding action of W on the homology of X can, however, be specified in algebraic terms. For this purpose we use the trajectories of G in $X \times X$, and we construct explicitly the correspondences on X (that is, cycles in $X \times X$) that specify the action of W on $H_*(X, \mathbf{Z})$. The study of such correspondences forms the basis of many problems in integral geometry.

At the end of the article, we generalize our results to the case when B is replaced by an arbitrary parabolic subgroup $P \subset G$. When $G = GL(n)$ and G/P is the Grassmann variety, analogous results are to be found in [4].

B. Kostant has previously found other formulae for a basis of $H^*(X, \mathbf{Z})$, $X = G/B$, dual to the Schubert cells. We would like to express our deep appreciation to him for drawing our attention to this series of problems and for making his own results known to us.

The main results of this article have already been announced in [13].

We give a brief account of the structure of this article. At the beginning of § 1 we introduce our notation and state the known results on the homology of $X = G/B$ that are used repeatedly in the paper. The rest of § 1 is devoted to a statement of our main results.

In § 2 we introduce an ordering on the Weyl group W of G that arises naturally in connection with the geometry of X, and we investigate its properties.

§ 3 is concerned with the ring R of polynomials on the Lie algebra $\mathfrak{h}$ of the Cartan subgroup $H \subset G$. In this section we introduce the functionals D_w on R and the elements P_w in R and discuss their properties.

In § 4 we prove that the elements D_w introduced in § 3 correspond to the Schubert cells of X.

§ 5 contains generalizations and applications of the results obtained, in particular, to the case of manifolds $X(P) = G/P$, where P is an arbitrary parabolic subgroup of G. We also study in § 5 the correspondences on X and in particular, we describe explicitly those correspondences that specify the action of the Weyl group W on the cohomology of X. Finally, in this section some of our results are put in the form in which they were earlier obtained by B. Kostant, and we also interpret some of them in terms of differential forms on X.

§ 1. Notation, preliminaries, and statement of the main results

We introduce the notation that is used throughout the article.

G is a complex semisimple Lie group, which is assumed to be connected and simply-connected;

B is a fixed Borel subgroup of G;

$X = G/B$ is a fundamental projective space of G;

N is the unipotent radical of B;

H is a fixed maximal torus of G, $H \subset B$;

$\mathfrak{G}$ is the Lie algebra of G; $\mathfrak{h}$ and $\mathfrak{N}$ are the subalgebras of $\mathfrak{G}$ corresponding to H and N;

$\mathfrak{h}^*$ is the space dual to $\mathfrak{h}$;

$\Delta \subset \mathfrak{h}^*$ is the root system of $\mathfrak{h}$ in $\mathfrak{G}$;

Δ_+ is the set of positive roots, that is, the set of roots of $\mathfrak{h}$ in $\mathfrak{N}$, $\Delta_- = -\Delta_+$, $\Sigma \subset \Delta_+$ is the system of simple roots;

W is the Weyl group of G; if $\gamma \in \Delta$, then $\sigma_\gamma : \mathfrak{h}^* \to \mathfrak{h}^*$ is an element of W, a reflection in the hyperplane orthogonal to γ. For each element[1] $w \in W = \mathrm{Norm}(H)/H$, the same letter is used to denote a representative of w in Norm $(H) \subset G$.

$l(w)$ is the length of an element $w \in W$ relative to the set of generators $\{\sigma_\alpha,\ \alpha \in \Sigma\}$ of W, that is, the least number of factors in the decomposition

$$w = \sigma_{\alpha_1}\sigma_{\alpha_2} \ldots \sigma_{\alpha_l}, \quad \alpha_i \in \Sigma. \tag{1}$$

The decomposition (1), with $l = l(w)$, is called reduced; $s \in W$ is the unique element of maximal length, $r = l(s)$;

$N_- = sNs^{-1}$ is the subgroup of G "opposite" to N.

For any $w \in W$ we put $N_w = w\,N_- w^{-1} \cap N$.

HOMOLOGY AND COHOMOLOGY OF THE SPACE X. We give at this point two descriptions of the homological structure of X. The first of these (Proposition 1.2) makes use of the decomposition of X into cells, while the second (Proposition 1.3) involves the realization of two-dimensional cohomology classes as the Chern classes of one-dimensional bundles.

We recall (see [5]) that $N_w = w\,N_- w^{-1} \cap N$ is a unipotent subgroup of

[1] Norm H is the normalizer of H in G.

G of (complex) dimension $l(w)$.

1.1. PROPOSITION (see [5]), *Let $o \in X$ be the image of B in X. The open and closed subvarieties $X_w = Nwo \subset X$, $w \in W$, yield a decomposition of X into N-orbits. The natural mapping $N_w \to X_w$ ($n \mapsto nwo$) is an isomorphism of algebraic varieties.*

Let $\bar{X}_w$ be the closure[1] of X_w in X, $[\bar{X}_w] \in H_{2l(w)}(\bar{X}_w, \mathbf{Z})$ the fundamental cycle of the complex algebraic variety $\bar{X}_w$ and $s_w \in H_{2l(w)}(X, \mathbf{Z})$ the image of $[\bar{X}_w]$ under the mapping induced by the embedding $\bar{X}_w \hookrightarrow X$.

1.2. PROPOSITION (see [2]). *The elements s_w form a free basis of $H_*(X, \mathbf{Z})$.*

We now turn to the other approach to the description of the cohomology of X. For this purpose we introduce in $\mathfrak{h}$ the root system $\{H_\gamma, \gamma \in \Delta\}$ dual to Δ. (This means that $\sigma_\gamma \chi = \chi - \chi(H_\gamma)\gamma$ for all $\chi \in \mathfrak{h}^*$, $\gamma \in \Delta$). We denote by $\mathfrak{h}_{\mathbf{Q}} \subset \mathfrak{h}$ the vector space over $\mathbf{Q}$ spanned by the H_γ. We also set $\mathfrak{h}^*_{\mathbf{Z}} = \{\chi \in \mathfrak{h}^* \mid \chi(H_\gamma) \in \mathbf{Z}$ for all $\gamma \in \Delta\}$ and $\mathfrak{h}^*_{\mathbf{Q}} = \mathfrak{h}^*_{\mathbf{Z}} \otimes_{\mathbf{Z}} \mathbf{Q}$.

Let $R = S(\mathfrak{h}^*_{\mathbf{Q}})$ be the algebra of polynomial functions on $\mathfrak{h}_{\mathbf{Q}}$ with rational coefficients. We extend the natural action of W on $\mathfrak{h}^*$ to R. We denote by I the subring of W-invariant elements in R and set $I_+ = \{f \in I \mid f(0) = 0\}$, $J = I_+ R$.

We construct a homomorphism $\alpha: R \to H^*(X, \mathbf{Q})$ in the following way. First let $\chi \in \mathfrak{h}^*_{\mathbf{Z}}$. Since G is simply-connected, there is a character $\theta \in \mathrm{Mor}(H, \mathbf{C}^*)$ such that $\theta(\exp h) = \exp \chi(h)$, $h \in \mathfrak{h}$. We extend θ to a character of B by setting $\theta(n) = 1$ for $n \in N$. Since $G \to X$ is a principal fibre space with structure group B, this θ defines a one-dimensional vector bundle E_χ on X. We set $\alpha_1(\chi) = c_\chi$, where $c_\chi \in H^2(X, \mathbf{Z})$ is the first Chern class of E_χ. Then α_1 is a homomorphism of $\mathfrak{h}^*_{\mathbf{Z}}$ into $H^2(X, \mathbf{Z})$, which extends naturally to a homomorphism of rings $\alpha: R \to H^*(X, \mathbf{Q})$.

Note that W acts on the homology and cohomology of X. Namely, let $K \subset G$ be a maximal compact subgroup such that $T = K \cap H$ is a maximal torus in K. Then the natural mapping $K/T \to X$ is a homeomorphism (see [1]). Now W acts on the homology and cohomology of X in the same way as on K/T.

1.3. PROPOSITION ([1], [8]). (i) *The homomorphism α commutes with the action of W on R and $H^*(X, \mathbf{Q})$.*

(ii) Ker $\alpha = J$, *and the natural mapping $\bar{\alpha}: R/J \to H^*(X, \mathbf{Q})$ is an isomorphism.*

In the remainder of this section we state the main results of this article.

The integration formula. We have given two methods of describing the

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[1] As X_w is an open and closed variety, its closure in the Zariski topology is the same as in the ordinary topology.

cohomological structure of X. One of the basic aims of this article is to establish a connection between these two approaches. By this we understand the following. Each Schubert cell $s_w \in H_*(X, \mathbf{Z})$ gives rise to a linear functional $\hat{D}_w$ on R according to the formula

$$\hat{D}_w(f) = \langle s_w, \alpha(f)\rangle$$

(where $<\ ,\ >$ is the natural pairing of homology and cohomology). We indicate an explicit form for $\hat{D}_w$.

For each root $\gamma \in \Delta$, we define an operator $A_\gamma : R \to R$ by the formula

$$A_\gamma f = \frac{f - \sigma_\gamma f}{\gamma}$$

(that is, $A_\gamma f(h) = [f(h) - f(\sigma_\gamma h)]/\gamma(h)$ for all $h \in \mathfrak{h}$). Then we have the following proposition.

PROPOSITION. *Let* $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$, $\alpha_i \in \Sigma$. If $l(w) < l$, *then* $A_{\alpha_1} \ldots A_{\alpha_l} = 0$. *If* $l(w) = l$, *then the operator* $A_{\alpha_1} \ldots A_{\alpha_l}$ *depends only on* w *and not on the representation of* w *in the form* $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$; *we put* $A_w = A_{\alpha_1} \ldots A_{\alpha_l}$.

This proposition is proved in § 3 (Theorem 3.4).

The functional $\hat{D}_w$ is easily described in terms of the A_w: we define for each $w \in W$ another functional D_w on R by the formula $D_w f = A_w f(0)$. The following theorem is proved in § 4 (Theorem 4.1).

THEOREM. $D_w = \hat{D}_w$ *for all* $w \in W$.

We can give another more explicit description of D_w (and thus of $\hat{D}_w$). To do this, we write $w_1 \xrightarrow{\gamma} w_2$, $w_1, w_2 \in W$, $\gamma \in \Delta_+$, to express the fact that $w_1 = \sigma_\gamma w_2$ and $l(w_2) = l(w_1) = l + 1$.

THEOREM. *Let* $w \in W$, $l(w) = l$.

(i) *If* $f \in R$ *is a homogeneous polynomial of degree* $k \neq l$, *then* $\hat{D}_w(f) = 0$.

(ii) If $\chi_1, \ldots, \chi_l \in \mathfrak{h}_{\mathbf{Q}}^*$, then $\hat{D}_w(\chi_1 \ldots \chi_l) = \sum \chi_1(H_{\gamma_1}) \ldots \chi_l(H_{\gamma_l})$, where the sum is taken over all chains of the form

$$e = w_0 \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} \ldots \xrightarrow{\gamma_l} w_l = w^{-1}$$

(see Theorem 3.12 (i), (v)).

The next theorem describes the basis of $H^*(X, \mathbf{Q})$ dual to the basis $\{s_w \mid w \in W\}$ of $H^*(X, \mathbf{Z})$. We identify the ring $\bar{R} = R/J$ with $H^*(X, \mathbf{Q})$ by means of the isomorphism $\bar{\alpha}$ of Proposition 1.3. Let $\{P_w \mid w \in W\}$ be the basis of $\bar{R}$ dual to the basis $\{s_w \mid w \in W\}$ of $H_*(X, \mathbf{Z})$. To specify P_w, we note that the operators $A_w : R \to R$ preserve the ideal $J \subset R$ (lemma 3.3 (v)), and so the operators $\bar{A}_w : \bar{R} \to \bar{R}$ are well-defined.

THEOREM. (i) *Let* $s \in W$ *be the element of maximal length*, $r = l(s)$ *Then* $P_s = \rho^r/r!$ (mod J) $= |W|^{-1} \prod_{\gamma \in \Delta_+} \gamma$ (mod J), (*where* $\rho \in \mathfrak{h}_{\mathbf{Q}}^*$ *is half* *the sum of the positive roots and* $|W|$ *is the order of* W)

(ii) *If* $w \in W$, *then* $P_w = \overline{A}_{w^{-1}s}P_s$ (see Theorem 3.15, Corollary 3.16, Theorem 3.14(i)).

Another expression for the P_w has been obtained earlier by B. Kostant (see Theorem 5.9).

The following theorem gives a couple of important properties of the P_w.

THEOREM (i). *Let* $\chi \in \mathfrak{h}^*_{\mathbf{Q}}$, $w \in W$. *Then* $\chi \cdot P_w = \sum\limits_{w \xrightarrow{\gamma} w'} w\chi(H_\gamma)\, P_{w'}$ (see Theorem 3.14 (ii)).

(ii) *Let* $\mathcal{P} : H_*(X, \mathbf{Q}) \to H^*(X, \mathbf{Q})$ *be the Poincaré duality. Then* $\mathcal{P}(s_w) = \overline{\alpha}(P_{ws})$ (see Corollary 3.19).

THE ACTION OF THE WEYL GROUP. The action of W on $H^*(X, \mathbf{Q})$ can easily be described using the isomorphism $\overline{\alpha}$: $R/J \to H^*(X, \mathbf{Q})$, but we are interested in the problem of describing the action of W on the basis $\{s_w\}$ of $H_*(X, \mathbf{Q})$.

THEOREM. *Let* $\alpha \in \Sigma$, $w \in W$. *Then* $\sigma_\alpha s_w = -s_w$ *if* $l(w\sigma_\alpha) = l(w) - 1$ *and* $\sigma_\alpha s_w = -s_w + \sum\limits_{w' \xrightarrow{\gamma} w\sigma_\alpha} w'\alpha(H_\gamma)\, s_{w'}$, *if* $l(w\sigma_\alpha) = l(w) + 1$ (see Theorem 3.12 (iv)).

In § 5 we consider some applications of the results obtained. To avoid overburdening the presentation, we do not make precise statements at this point. We merely mention that Theorem 5.5 appears important to us, in which a number of results is generalized to the case of the varieties $X(P) = G/P$ (P being an arbitrary parabolic subgroup of G), and also Theorem 5.7, in which we investigate certain correspondences on X.

§2. The ordering on the Weyl group and the mutual disposition of the Schubert cells

2.1 DEFINITION (i) *Let* $w_1, w_2 \in W$, $\gamma \in \Delta_+$. *Then* $w_1 \xrightarrow{\gamma} w_2$ *indicates the fact that* $\sigma_\gamma w_1 = w_2$ *and* $l(w_2) = l(w_1) + 1$.

(ii) *We put* $w < w'$ *if there is a chain*

$$w = w_1 \to w_2 \to \dots \to w_k = w'.$$

It is helpful to picture W in the form of a directed graph with edges drawn in accordance with Definition 2.1 (i).

Here are some properties of this ordering.

2.2 LEMMA. *Let* $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_l}$ *be the reduced decomposition of an element* $w \in W$. *We put* $\gamma_i = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}}(\alpha_i)$. *Then the roots* $\gamma_1, \dots, \gamma_l$ *are distinct and the set* $\{\gamma_1, \dots, \gamma_l\}$ *coincides with* $\Delta_+ \cap w\Delta_-$.

This lemma is proved in [6].

2.3 COROLLARY. (i) *Let* $w = \sigma_{\alpha_1} \dots \sigma_{\alpha_l}$ *be the reduced decomposition and let* $\gamma \in \Delta_+$ *be a root such that* $w^{-1}\gamma \in \Delta_-$. *Then for some* i

(2) $$\sigma_\gamma\sigma_{\alpha_1} \dots \sigma_{\alpha_i} = \sigma_{\alpha_1} \dots \sigma_{\alpha_{i-1}}.$$

(ii) *Let* $w \in W$, $\gamma \in \Delta_+$. *Then* $l(w) < l(\sigma_\gamma w)$, *if and only if* $w^{-1}\gamma \in \Delta_+$.

PROOF (i) From Lemma 2.2 we deduce that $\gamma = \sigma_{\alpha_1} \ldots \sigma_{\alpha_{i-1}} (\alpha_i)$ *for some* i, and (2) follows.

(ii) If $w^{-1}\gamma \in \Delta_-$, then by (2) $\sigma_\gamma w = \sigma_{\alpha_1} \ldots a_{\alpha_{i-1}} \sigma_{\alpha_{i+1}} \ldots \sigma_{\alpha_l}$, that is $l(\sigma_\gamma w) < l(w)$. Interchanging w and $\sigma_\gamma w$, we see that if $w^{-1}\gamma \in \Delta_+$, then $l(w) < l(\sigma_\gamma w)$.

2.4 LEMMA. *Let* $w_1, w_2 \in W$, $\alpha \in \Sigma$, $\gamma \in \Delta_+$, *and* $\gamma \neq \alpha$. *Let* $\gamma' = \sigma_\alpha \gamma$. If

(3)
$$\begin{tikzcd} & w_2 \\ \sigma_\alpha w_1 \arrow[ur, "\gamma"] \arrow[dr, "\alpha"'] & \\ & w_1 \end{tikzcd},$$

then

(4)
$$\begin{tikzcd} w_2 \arrow[dr, "\alpha"] & \\ & \sigma_\alpha w_2. \\ w_1 \arrow[ur, "\gamma'"] & \end{tikzcd}$$

Conversely, (3) *follows from* (4).

PROOF. Since $\alpha \in \Sigma$ and $\gamma \neq \alpha$, we have $\gamma' = \sigma_\alpha \gamma \in \Delta_+$. It is therefore sufficient to show that $l(\sigma_\alpha w_2) > l(w_2) = l(w_1)$. This follows from Corollary 2.3, because $\sigma_\alpha w_2 = \sigma_{\gamma'} w_1$ and $(\sigma_\alpha w_2)^{-1}\gamma' = w_2^{-1}\sigma_\alpha\gamma' = w_2^{-1}\gamma \in \Delta_-$ by (3). The second assertion of the lemma is proved similarly.

2.5 LEMMA. *Let* $w, w' \in W$, $\alpha \in \Sigma$ *and assume that* $w < w'$. *Then*

a) *either* $\sigma_\alpha w \leqslant w'$ *or* $\sigma_\alpha w < \sigma_\alpha w'$,

b) *either* $w \leqslant \sigma_\alpha w'$ *or* $\sigma_\alpha w < \sigma_\alpha w'$.

PROOF a) Let

$$w = w_1 \to w_2 \to \ \ldots \ \to w_k = w'.$$

We proceed by induction on k. If $\sigma_\alpha w < w$ or $\sigma_\alpha w = w_2$, the assertion is obvious. Let $w < \sigma_\alpha w$, $\sigma_\alpha w \neq w_2$. Then $\sigma_\alpha w < \sigma_\alpha w_2$ by Lemma 2.4. We obtain a) by applying the inductive hypothesis to the pair (w_2, w').

b) is proved in a similar fashion.

2.6. COROLLARY. *Let* $\alpha \in \Sigma$, $w_1 \xrightarrow{\alpha} w_1'$, $w_2 \xrightarrow{\alpha} w_2'$. *If one of the elements* w_1, w_1' *is smaller (in the sense of the above ordering) than one of* w_2, w_2', *then* $w_1 \leqslant w_2 < w_2'$ *and* $w_1 < w_1' \leqslant w_2'$.

The property in Lemma 2.5 characterizes the ordering $<$. More precisely, we have the following proposition:

2.7 PROPOSITION. *Suppose that we are given a partial ordering* $w \dashv w'$ *on* W *with the following properties:*

a) *If* $\alpha \in \sum$, $w \in W$ *with* $l(\sigma_\alpha w) = l(w) + 1$, *then* $w \dashv \sigma_\alpha w$.

b) *If* $w \dashv w'$, $\alpha \in \Sigma$, *then either* $\sigma_\alpha w \dashv w'$ *or* $\sigma_\alpha w \dashv \sigma_\alpha w'$.

Then $w \dashv w'$ *if and only if* $w \leqslant w'$.

PROOF. Let s be the element of maximal length in W. It follows from a) that $e \dashv w \dashv s$ for all $w \in W$.

I. We prove that $w \leqslant w'$ implies that $w \dashv w'$. We proceed by induction on $l(w')$. If $l(w') = 0$, then $w' = e$, $w = e$ and so $w \dashv w'$. Let $l(w') > 0$ and let $\alpha \in \Sigma$ be a root such that $l(\sigma_\alpha w') = l(w') - 1$. Then by Lemma 2.5 a), either $\alpha_\alpha w \leqslant \sigma_\alpha w'$ or $w \leqslant \sigma_\alpha w'$.

(i) $w \leqslant \sigma_\alpha w' \Rightarrow w \dashv \sigma_\alpha w'$ (by the inductive hypothesis), $\Rightarrow w \dashv w'$. (using a)).

(ii) $\sigma_\alpha w \leqslant \sigma_\alpha w' \Rightarrow \sigma_\alpha w \dashv \sigma_\alpha w'$ (by the inductive hypothesis), $\Rightarrow$ either $w \dashv \sigma_\alpha w'$ or $w \dashv w'$ (applying b) to the pair $(\sigma_\alpha w, \sigma_\alpha w')$), $\Rightarrow w \dashv w'$.

II. We now show that $w \dashv w'$ implies that $w \leqslant w'$. We proceed by backward induction on $l(w)$. If $l(w) = r = l(s)$, then $w = s$, $w' = s$, and so $w \leqslant w'$. Let $l(w) < r$ and let α be an element of Σ such that $l(\sigma_2 w) = l(w) + 1$. By b) either $\sigma_\alpha w \dashv w'$ or $\sigma_\alpha w \dashv \sigma_\alpha w'$.

(i) $\sigma_\alpha w \dashv w' \Rightarrow \sigma_\alpha w \leqslant w'$ (by the inductive hypothesis) $\Rightarrow w \leqslant w'$.

(ii) $\sigma_\alpha w \dashv \sigma_\alpha w' \Rightarrow \sigma_\alpha w \leqslant \sigma_\alpha w' \Rightarrow w \leqslant w'$ (by Corollary 2.6). Proposition 2.7 is now proved.

2.8 PROPOSITION. *Let $w \in W$ and let $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$ be the reduced decomposition of w.*

a) *If $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant l$ and*

(5) $$w' = \sigma_{\alpha_{i_1}} \ldots \sigma_{\alpha_{i_k}},$$

then $w' \leqslant w$.

b) *If $w' < w$, then w' can be represented in the form* (5) *for some indexing set $\{i_j\}$.*

c) *If $w' \to w$, then there is a unique index i, $1 \leqslant i \leqslant l$, such that*

(6) $$w' = \sigma_{\alpha_1} \ldots \sigma_{\alpha_{i-1}} \sigma_{\alpha_{i+1}} \ldots \sigma_{\alpha_l}.$$

PROOF. Let us prove c). Let $w' \xrightarrow{\gamma} w$. Then by Lemma 2.2 there is at least one index i for which (6) holds. Now suppose that (6) holds for two indices i, j, $i < j$. Then $\sigma_{\alpha_{i+1}} \ldots \sigma_{\alpha_j} = \sigma_{\alpha_i} \ldots \sigma_{\alpha_{j-1}}$. Thus, $\sigma_{\alpha_i} \ldots \sigma_{\alpha_j} = \sigma_{\alpha_{i+1}} \ldots \sigma_{\alpha_{j-1}}$, which contradicts the assumption that the decomposition $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$ is reduced.

b) follows at once from c) if we take into account the fact that the decomposition (6) is reduced. We now prove a) by induction on l. We treat two cases separately.

(i) $i_1 > 1$. Then by the inductive hypothesis $w' \leqslant \sigma_{\alpha_2} \ldots \sigma_{\alpha_l}$, that is, $w' \leqslant \sigma_{\alpha_1} w < w$.

(ii) $i_1 = 1$. Then, by the inductive hypothesis, $\sigma_{\alpha_1} w' = \sigma_{\alpha_{i_2}} \ldots \sigma_{\alpha_{i_k}} \leqslant \sigma_{\alpha_1} w = \sigma_{\alpha_2} \ldots \sigma_{\alpha_l}$. By Corollary 2.6, $w' \leqslant w$.

Proposition 2.8 yields an alternative definition of the ordering on W (see [7]). The geometrical interpretation of this ordering is very interesting and useful in what follows.

2.9 THEOREM. *Let V be a finite-dimensional representation of a Lie algebra $\mathfrak{G}$ with dominant weight λ. Assume that all the weights $w\lambda$ $w \in W$, are distinct and select for each w a non-zero vector $f_w \in V$ of weight $w\lambda$. Then*

$$w' \leqslant w \Leftrightarrow f_{w'} \in U(\mathfrak{N}) f_w$$

(where $U(\mathfrak{N})$ is the enveloping algebra of the Lie algebra $\mathfrak{N}$).

PROOF. For each root $\gamma \in \Delta$ we fix a root vector $E_\gamma \in \mathfrak{G}$ in such a way that $[E_\gamma, E_{-\gamma}] = H_\gamma$. Denote by $\mathfrak{A}_\gamma$ the subalgebra of $\mathfrak{G}$, generated by E_γ, $E_{-\gamma}$, and H_γ. $\mathfrak{A}_\gamma$ is isomorphic to the Lie algebra $sl_2(\mathbf{C})$. Let $w' \xrightarrow{\gamma} w$ and let $\widetilde{V}$ be the smallest $\mathfrak{A}_\gamma$-invariant subspace of V containing $f_{w'}$.

2.10 LEMMA. *Let $n = w'\lambda(H_\gamma) \in \mathbf{Z}$, $n \geq 0$. The elements $\{E^i_{-\gamma} f_{w'} \mid i = 0, 1, \ldots, n\}$ form a basis of $\widetilde{V}$. Put $\tilde{f} = E^n_{-\gamma} f_{w'}$. Then $E_{-\gamma}\tilde{f} = 0$, $E^n_\gamma \tilde{f} = c' f_{w'}$ $(c' \neq 0)$ and $f_w = c\tilde{f}$ $(c \neq 0)$.*

PROOF. By Lemma 2.2, $w'^{-1}\gamma \in \Delta_+$, hence $E_\gamma f_{w'} = = cE_\gamma w' f_e = cw' E_{w'^{-1}\gamma} f_e = 0$, that is, $f_{w'}$ is a vector of dominant weight relative to $\mathfrak{A}_\gamma$. All the assertions of the lemma, except the last, follow from standard facts about the representations of the algebra $\mathfrak{A}_\gamma \cong sl_2(\mathbf{C})$. Furthermore, $\tilde{f}$ and f_w are two non-zero vectors of weight $w\lambda$ in V, and since the multiplicity of $w\lambda$ in V is equal to 1, these vectors are proportional. The lemma is now proved.

To prove Theorem 2.9 we introduce a partial ordering on W by putting $w' \dashv w$ if $f_{w'} \in U(\mathfrak{N}) f_w$. Since all the weights $w\lambda$ are distinct, the relation $\dashv$ is indeed an ordering; we show that it satisfies conditions a) and b) of Proposition 2.7.

a) Let $\alpha \in \Sigma$ and $l(\sigma_\alpha w) = l(w) + 1$. Then $w \xrightarrow{\alpha} \sigma_\alpha w$, and by Lemma 2.10, $f_w \in U(\mathfrak{N}) f_{\sigma_\alpha w}$, that is, $w \dashv \sigma_\alpha w$.

b) Let $w \dashv w'$. We choose an $\alpha \in \Sigma$ such that $w \xrightarrow{\alpha} \sigma_\alpha w$. Replacing w' by $\sigma_\alpha w'$, if necessary, we may assume that $\sigma_\alpha w' \to w'$. We prove that $\sigma_\alpha w \dashv w'$, that is, $f_{\sigma_\alpha w} \in U(\mathfrak{N}) f_{w'}$. It follows from Lemma 2.10 that $E_{-\alpha} f_{w'} = 0$ and $f_{\sigma_\alpha w} = cE^n_{-\alpha} f_w$. Let $\mathfrak{P}_\alpha$ be the subalgebra of $\mathfrak{G}$ generated by $\mathfrak{N}$, $\mathfrak{h}$ and $\mathfrak{A}_\alpha$. Since $w \dashv w'$, $f_w \in U(\mathfrak{N}) f_{w'}$ and so $f_{\sigma_\alpha w} = cE^n_{-\alpha} f_w = X f_{w'}$, where $X \in U(\mathfrak{P}_\alpha)$. Any element X of $U(\mathfrak{P}_\alpha)$ can be represented in the form $X = \sum_{i=1}^{k} Y_i Y'_i + \widetilde{Y} E_{-\alpha}$, where $Y_i \in U(\mathfrak{N})$, $Y'_i \in U(\mathfrak{h})$, $\widetilde{Y} \in U(\mathfrak{P}_\alpha)$. Therefore, $f_{\sigma_\alpha w} = \sum Y_i Y'_i f_{w'} = \sum c_i Y_i f_{w'} \in U(\mathfrak{N}) f_{w'}$ and Theorem 2.9 is proved.

We use Theorem 2.9 to describe the mutual disposition of the Schubert cells.

2.11. THEOREM (Steinberg [7]). *Let $w \in W$, $X_w \subset X$ a Schubert cell, and $\overline{X}_w$ its closure. Then $X_{w'} \subset \overline{X}_w$ if and only if $w' \leqslant w$.*

To prove this theorem, we give a geometric description of the variety X_w.

Let V be a finite-dimensional representation of G with regular dominant weight λ (that is, all the weights $w\lambda$ distinct). As above, we choose for each $w \in W$ a non-zero vector $f_w \in V$ of weight $w\lambda$. We consider the space $P(V)$ of lines in V; if $f \in V$, $f \neq 0$, then we denote by $[f] \in P(V)$ a line passing through f. Since λ is regular, the stabilizer of the point $[f_e] \in P(V)$ under the natural action of G on $P(V)$ is B. The G-orbit of $[f_e]$ in $P(V)$ is therefore naturally isomorphic to $X = G/B$. In what follows, we regard X as a subvariety of $P(V)$.

For each $w \in W$ we denote by ϕ_w the linear function on V given by $\phi_w(f_w) = 1$, $\phi_w(f) = 0$ if $f \in V$ is a vector of weight distinct from $w\lambda$.

2.12 LEMMA. *Let $f \in V$ and $[f] \in X$. Then*

$$[f] \in X_w \Leftrightarrow f \in U(\mathfrak{N})f_w,\ \varphi(f) \neq 0.$$

PROOF. We may assume that $f = gf_e$ for some $g \in G$.

Let $[f] \in X_w$, that is, $g \in NwB$. Then $f = c_1 \exp(Y)wf_e$ for some $Y \in \mathfrak{N}$, hence $f \in U(\mathfrak{N})f_v$, and $\phi_w(f) \neq 0$.

On the other hand, it is clear that for each $f \in V$ there is at most one $w \in W$ such that $f \in U(\mathfrak{N})f_w$ and $\phi_w(f) \neq 0$. The Lemma now follows from the fact that $X = \bigcup_{w \in W} X_w$.

We now prove Theorem 2.11

a) Let $X_{w'} \subset \overline{X}_w$. Then $[f_{w'}] \in \overline{X}_w$, and by Lemma 2.12, $f_{w'} \in U(\mathfrak{N})f_w$. So $w' \leqslant w$, by Theorem 2.9.

b) To prove the converse it is sufficient to consider the case $w' \xrightarrow{\gamma} w$. Let $n = w\lambda(H_\gamma) \in \mathbf{Z}$. Just as in the proof of Theorem 2.9, a) we can show that $n > 0$, $E_\gamma^n f_w = cf_{w'}$ and $E_\gamma^{n+1} f_w = 0$.

Therefore $\lim_{t \to \infty} t^{-n} \exp(tE_\gamma) f_w = \frac{c}{n!} f_{w'}$, that is, $[f_{w'}] \in X_w$. Hence, $X_{w'} \subset \overline{X}_w$.

§3. Discussion of the ring of polynomials on $\mathfrak{h}$

In this section we study the rings R and $\overline{R}$. For each $w \in W$ we define an element $P_w \in \overline{R}$ and a functional D_w on R and investigate their properties. In the next section we shall show that the D_w correspond to Schubert cells, and that the P_w yield a basis, dual to the Schubert cell basis, for the cohomology of X.

3.1 DEFINITION. (i) *$R = \oplus R_i$ is the graded ring of polynomial functions on $\mathfrak{h}_\mathbf{Q}$ with rational coefficients. W acts on R according to the rule $wf(h) = f(w^{-1}h)$.*

(ii) *I is the subring of W-invariant elements in R,*

$$I_+ = \{f \in I \mid f(0) = 0\}.$$

(iii) *J is the ideal of R generated by I_+.*

(iv) $R = R/J$.

3.2 DEFINITION. *Let $\gamma \in \Delta$. We specify an operator A_γ on R by the rule*

$$A_\gamma f = \frac{f - \sigma_\gamma f}{\gamma}.$$

$A_\gamma f$ lies in R, since $f - \sigma_\gamma f = 0$ on the hyperplane $\gamma = 0$ in $\mathfrak{h}_{\mathbf{Q}}$.

The simplest properties of the A_γ are described in the following lemma.

3.3 LEMMA.(i) $A_{-\gamma} = -A_\gamma$, $A_\gamma^2 = 0$.

(ii) $wA_\gamma w^{-1} = A_{w\gamma}$.

(iii) $\sigma_\gamma A_\gamma = - A_\gamma \sigma_\gamma = A_\gamma$, $\sigma_\gamma = -\gamma A_\gamma + 1 = A_\gamma \gamma - 1$.

(iv) $A_\gamma f = 0 \Leftrightarrow \sigma_\gamma f = f$.

(v) $A_\gamma J \subset J$.

(vi) *Let $\chi \in \mathfrak{h}_{\mathbf{Q}}^*$. Then the commutator of A_γ with the operator of multiplication by χ has the form $[A_\gamma, \chi] = \chi(H_\gamma)\sigma_\gamma$.*

PROOF. (i) – (iv) are clear. To prove (v), let $f = f_1 f_2$, where $f_1 \in I_+$, $f_2 \in R$. It is then clear that $A_\gamma f = f_1 . A_\gamma f_2 \in J$. As to (vi), since $\sigma_\gamma \chi = \chi - \chi(H_\gamma)\gamma$, we have

$$[A_\gamma, \chi] f = A_\gamma (\chi f) - \chi A_\gamma (f) = \frac{1}{\gamma} (\chi f - \sigma_\gamma \chi \cdot \sigma_\gamma f - \chi f + \chi \sigma_\gamma f) =$$

$$= \frac{\chi - \sigma_\gamma \chi}{\gamma} \cdot \sigma_\gamma f = \chi (H_\gamma) \cdot \sigma_\gamma f.$$

The following property of the A_γ is fundamental in what follows.

3.4 THEOREM. *Let $\alpha_1, \ldots, \alpha_l \in \Sigma$, and put $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$; $A_{(\alpha_1, \ldots, \alpha_l)} = A_{\alpha_1} \ldots A_{\alpha_l}$.*

a) *If $l(w) < l$, then $A_{(\alpha_1, \ldots, \alpha_l)} = 0$.*

b) *If $l(w) = l$, then $A_{(\alpha_1, \ldots, \alpha_l)}$ depends only on w and not on the set $\alpha_1, \ldots, \alpha_l$. In this case we put $A_w = A_{(\alpha_1, \ldots, \alpha_l)}$.*

The proof is by induction on l, the result being obvious when $l = 1$.

For the proof of a), we may assume by the inductive hypothesis that $l(\sigma_{\alpha_1} \ldots \sigma_{\alpha_{l-1}}) = l - 1$, consequently $l(\sigma_{\alpha_1} \ldots \sigma_{\alpha_{l-1}} \sigma_{\alpha_l}) = l - 2$. Then $\sigma_{\alpha_i} \sigma_{\alpha_{i+1}} \ldots \sigma_{\alpha_{l-1}} = \sigma_{\alpha_{i+1}} \ldots \sigma_{\alpha_{l-1}} \sigma_{\alpha_l}$ for some i (we have applied Corollary 2.3 to the case $w = \sigma_{\alpha_{l-1}} \ldots \sigma_{\alpha_1}$, $\gamma = \alpha_l$). We show that $A_{(\alpha_i, \ldots, \alpha_l)} = 0$.

Since $l - i < l$, the inductive hypothesis shows that $A_{\alpha_i} A_{\alpha_{i+1}} \ldots A_{\alpha_{l-1}} = A_{\alpha_{i+1}} \ldots A_{\alpha_{l-1}} A_{\alpha_l}$, and so by lemma 3.3 (i) $A_{\alpha_i} \ldots A_{\alpha_l} = A_{\alpha_{i+1}} \ldots A_{\alpha_l} A_{\alpha_l} = 0$.

To prove b), we introduce auxiliary operators $B_{(\alpha_1, \ldots, \alpha_l)}$, by setting

$$B_{(\alpha_1, \ldots, \alpha_l)} = \sigma_{\alpha_l} \ldots \sigma_{\alpha_1} A_{(\alpha_1, \ldots, \alpha_l)}.$$

We put $w_i = \sigma_{\alpha_l} \ldots \sigma_{\alpha_i}$. Then in view of Lemma 3.3 (ii, iii) we have

$$B_{(\alpha_1, \ldots, \alpha_l)} = A_{\alpha_1}^{w_2} A_{\alpha_2}^{w_3} \ldots A_{\alpha_{l-1}}^{w_l} A_{\alpha_l} \tag{7}$$

(where A_γ^w stands for $wA_\gamma w^{-1}$).

3.5 LEMMA. *Let $\chi \in \mathfrak{h}_{\mathbf{Q}}^*$. The commutator of $B_{(\alpha_1, \ldots, \alpha_l)}$ with the operator of multiplication by χ is given by the following formula:*[1]

$$[B_{(\alpha_1, \ldots, \alpha_l)}, \chi] = \sum_{i=1}^{l} \chi(w_{i+1} H_{\alpha_i}) w_{i+1} w_i^{-1} B_{(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_l)}. \tag{8}$$

PROOF. We have

$$[B_{(\alpha_1, \ldots, \alpha_l)}, \chi] = [A_{\alpha_1}^{w_2} A_{\alpha_2}^{w_3} \ldots A_{\alpha_l}, \chi] =$$
$$= \sum_{i=1}^{l} A_{\alpha_1}^{w_2} A_{\alpha_2}^{w_3} \ldots [A_{\alpha_i}^{w_{i+1}}, \chi] \ldots A_{\alpha_l} = \sum_{i=1}^{l} T_i.$$

By Lemma 3.3 (ii, vi), $[A_{\alpha_i}^{w_{i+1}}, \chi] = \chi(w_{i+1} H_{\alpha_i})\, \sigma_{w_{i+1}\alpha_i}$.

Since $\sigma_{w_{i+1}\alpha_i} = w_{i+1} w_i^{-1}$, we have

$$T_i = \chi(w_{i+1} H_{\alpha_i}) A_{\alpha_1}^{w_2} \ldots A_{\alpha_{i-1}}^{w_i} w_{i+1} w_i^{-1} A_{\alpha_{i+1}}^{w_{i+2}} \ldots A_{\alpha_l}.$$

We want to move the term $w_{i+1} w_i^{-1}$ to the left. To do this we note that for $j < i$

$$A_{\alpha_j}^{w_{j+1}} w_{i+1} w_i^{-1} = w_{i+1} w_i^{-1} (A_{\alpha_j}^{w_{j+1}})^{w_i w_{i+1}^{-1}} = w_{i+1} w_i^{-1} A_{\alpha_j}^{w_i w_{i+1}^{-1} w_{j+1}} =$$
$$= w_{i+1} w_i^{-1} A_{\alpha_j}^{\sigma_{\alpha_l} \ldots \hat{\sigma}_{\alpha_i} \ldots \sigma_{\alpha_{j+1}}}.$$

Therefore,

$$T_i = \chi(w_{i+1} H_{\alpha_i}) w_{i+1} w_i^{-1} A_{\alpha_1}^{\sigma_{\alpha_l} \ldots \hat{\sigma}_{\alpha_i} \ldots \sigma_{\alpha_2}} \ldots A_{\alpha_{i-1}}^{\sigma_{\alpha_l} \ldots \sigma_{\alpha_{i+1}}} A_{\alpha_{i+1}}^{\sigma_{\alpha_l} \ldots \sigma_{\alpha_{i+2}}} \ldots A_{\alpha_l}.$$

By (7), applied to the sequence or roots $(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_l)$, we have

$$T_i = \chi(w_{i+1} H_{\alpha_i}) w_{i+1} w_i^{-1} B_{(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_l)},$$

and Lemma 3.5 is proved.

If $l(\sigma_{\alpha_1} \ldots \hat{\sigma}_{\alpha_i} \ldots \sigma_{\alpha_l}) < l - 1$, then $T_i = 0$ by the inductive hypothesis. If $l(\sigma_{\alpha_1} \ldots \hat{\sigma}_{\alpha_i} \ldots \sigma_{\alpha_l}) = l - 1$, then, putting $w' = \sigma_{\alpha_1} \ldots \hat{\sigma}_{\alpha_i} \ldots \sigma_{\alpha_l}$ and $\gamma = \sigma_{\alpha_1} \ldots \sigma_{\alpha_{i-1}}(\alpha_i)$, we see from Lemma 2.2 that $w' \xrightarrow{\gamma} w$, and also

$$\chi(w_{i+1} H_{\alpha_i}) = w'\chi(w' w_{i+1} H_{\alpha_i}) = w'\chi(\sigma_{\alpha_1} \ldots \sigma_{\alpha_{i-1}} H_{\alpha_i}) = w'\chi(H_\gamma)$$

and

$$w_{i+1} w_i^{-1} B_{(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_l)} = w_{i+1} w_i^{-1} w'^{-1} A_{(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_l)} = w^{-1} A_{(\alpha_1, \ldots, \hat{\alpha}_i, \ldots, \alpha_l)}.$$

[1] $\hat{}$ indicates that the corresponding term must be omitted.

Using Proposition 2.8 c) and the inductive hypothesis, (8) can be rewritten in the following form:

$$[B_{(\alpha_1, \ldots, \alpha_l)}, \chi] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma)\, w^{-1}A_{w'}.$$

The right-hand side of this formula does not depend on the representation of w in the form of a product $\sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$. The proof of theorem 3.4 is thus completed by the following obvious lemma.

3.6. LEMMA. *Let B be an operator in R such that $B(1) = 0$ and $[B, \chi] = 0$ for all $\chi \in \mathfrak{h}_{\mathbf{Q}}^*$. Then $B = 0$.*

3.7. COROLLARY. *The operators A_w satisfy the following commutator relation:*

$$[w^{-1}A_w, \chi] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma)\, w^{-1}A_{w'}.$$

We put $S_i = R_i^*$ (where $R_i \subset R$ is the space of homogeneous polynomials of degree i) and $S = \oplus S_i$. We denote by (,) the natural pairing $S \times R \to \mathbf{Q}$. Then W acts naturally on S.

3.8 DEFINITION. (i) *For any $\chi \in \mathfrak{h}_{\mathbf{Q}}^*$ we let χ^* denote the transformation of S adjoint to the operator of multiplication by χ in R.*

(ii) *We denote by F_γ: $S \to S$ the linear transformation adjoint to A_γ: $R \to R$.*

The next lemma gives an explicit description of the F_γ.

3.9 LEMMA. *Let $\gamma \in \Delta$. For any $D \in S$ there is a $\widetilde{D} \in S$ such that $\gamma^*(\widetilde{D}) = D$. If $\widetilde{D}$ is any such operator, then $\widetilde{D} - \sigma_\gamma \widetilde{D} = F_\gamma(D)$, (in particular, the left-hand side of this equation does not depend on the choice of $\widetilde{D}$).*

PROOF. The existence of $\widetilde{D}$ follows from the fact that multiplication by γ is a monomorphism of R. Furthermore, for any $f \in R$ we have

$$(\widetilde{D} - \sigma_\gamma \widetilde{D}, f) = (\widetilde{D}, f - \sigma_\gamma f) = (\widetilde{D}, A_\gamma f \cdot \gamma) = (\gamma^*(\widetilde{D}), A_\gamma f) = (D, A_\gamma f),$$

hence $\widetilde{D} - \sigma_\gamma \widetilde{D} = F_\gamma$.

REMARK. It is often convenient to interpret S as a ring of differential operators on $\mathfrak{h}$ with constant rational coefficients. Then the pairing (,) is given by the formula $(D, f) = (Df)(0)$, $D \in S$, $f \in R$. Also, it is easy to check that $\chi^*(D) = [D, \chi]$, where $\chi \in \mathfrak{h}_{\mathbf{Q}}^*$ and $D \in S$ are regarded as operators on R.

Theorem 3.4 and Corollary 3.7 can be restated in terms of the operators F_γ.

3.10 THEOREM. *Let $\alpha_1, \ldots, \alpha_l \in \Sigma$, $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$.*

(i) *If $l(w) < l$, then $F_{\alpha_l} \ldots F_{\alpha_1} = 0$.*

(ii) *If $l(w) = l$, then $F_{\alpha_l} \ldots F_{\alpha_1}$ depends only on w and not on $\alpha_1, \ldots, \alpha_l$. In this case the transformation $F_{\alpha_l} \ldots F_{\alpha_1}$ is denoted by F_w. (Note that $F_w = A_w^*$).*

(iii) $$[\chi^*, F_w w] = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma) F_{w'} w.$$

3.11. DEFINITION. *We set* $D_w = F_w$ (1).

As we shall show in § 4, the functionals D_w correspond to the Schubert cells in H_* $(X, \mathbf{Q})$ in the sense that $(D_w, f) = \langle s_w, \alpha(f)\rangle$ *for all* $f \in R$.

The properties of the D_w are listed in the following theorem.

3.12. THEOREM. (i) $D_w \in S_{l(w)}$.

(ii) *Let* $w \in W$, $\alpha \in \Sigma$. *Then*

$$F_\alpha D_w = \begin{cases} 0 & \text{if } l(w\sigma_\alpha) = l(w) - 1, \\ D_{w\sigma_\alpha} & \text{if } l(w\sigma_\alpha) = l(w) + 1. \end{cases}$$

(iii) *Let* $\chi \in \mathfrak{h}^*_{\mathbf{Q}}$. *Then*

$$\chi^*(D_w) = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma) D_{w'}.$$

(iv) *Let* $\alpha \in \Sigma$. *Then*

$$\sigma_\alpha D_w = \begin{cases} -D_w, & \text{if } \quad l(w\sigma_\alpha) = l(w) - 1, \\ -D_w + \sum\limits_{w' \xrightarrow{\gamma} w\sigma_\alpha} w'\alpha(H_\gamma) D_{w'} & \text{if } \quad l(w\sigma_\alpha) = l(w) + 1. \end{cases}$$

(v) *Let* $w \in W$, $l(w) = l$, $\chi_1, \ldots, \chi_l \in \mathfrak{h}^*_{\mathbf{Q}}$. *Then* $(D_w, \chi_1, \ldots, \chi_l) = \sum \chi_1(H_{\gamma_1}) \ldots \chi_l(H_l)$, *where the summation extends over all chains*

$$e \xrightarrow{\gamma_1} w_1 \xrightarrow{\gamma_2} w_2 \longrightarrow \ldots \xrightarrow{\gamma_l} w_l = w^{-1}.$$

PROOF. (i) and (ii) follow from the definition of D_w and Theorem 3.10 (i).

(iii) $\chi^*(D_w) = \chi^* F_w w(1) = [\chi^*, F_w w]$ (1) (since $\chi^*(1) = 0$), and (iii) follows from Theorem 3.10 (iii).

It follows from Lemma 3.3 (iii) that $\sigma_\alpha = \alpha^* F_\alpha - 1$. Thus, (iv) follows from (ii) and (iii).

(v) We put $\widetilde{D}_w = D_{w^{-1}}$. Then the $\widetilde{D}_w$ satisfy the relation

(9) $$\chi^*(\widetilde{D}_w) = \sum_{w' \xrightarrow{\gamma} w} \chi(H_\gamma) \widetilde{D}_{w'}.$$

Since $(D, \chi f) = (\chi^*(D), f)$, (v) is a consequence of (9) by induction on l.

Let $\mathscr{H}$ be the subspace of S orthogonal to the ideal $J \subset R$. It follows from Lemma 3.3 (vi) that $\mathscr{H}$ is invariant with respect to all the F_γ. It is also clear that $1 \in \mathscr{H}$. Thus, $D_w \in \mathscr{H}$ for all $w \in H$.

3.13. THEOREM. *The functionals* D_w, $w \in W$, *form a basis for* $\mathscr{H}$.

PROOF. a) We first prove that the D_w are linearly independent. Let $s \in W$ be the element of maximal length and $r = l(s)$. Then, by Theorem 3.12 (v), $D_s(\rho^r) > 0$ and so $D_s \neq 0$. Now let $\sum c_w D_w = 0$ and let $\widetilde{w}$ be one of the elements of maximal length for which $c_w \neq 0$. Put $l = l(\widetilde{w})$.

There is a sequence $\alpha_1, \ldots, \alpha_{r-l}$ for which $\widetilde{w}\sigma_{\alpha_1} \ldots \sigma_{\alpha_{r-l}} = s$. Let $F = F_{\alpha_{r-l}} \ldots F_{\alpha_1}$. It follows from Theorem 3.10 that $FD_{\widetilde{w}} = D_s$ and $FD_w = 0$ if $l(w) \geqslant l$, $w \neq \widetilde{w}$. Therefore $F(\sum c_w D_w) = c_{\widetilde{w}} D_s \neq 0$.

b) We now show that the D_w span $\mathscr{H}$. It is sufficient to prove that if $f \in R$ and $(D_w, f) = 0$ for all $w \in W$, then $f \in J$. We may assume that f is a homogeneous element of degree k. For $k = 0$ the assertion is clear.

Now let $k > 0$ and assume that the result is true for all polynomials f of degree less than k. Then for all $\alpha \in \Sigma$ and $w \in W$, $(D_w, A_\alpha f) = (F_\alpha D_w, f) = 0$, by Theorem 3.10 (i) and (ii). By the inductive hypothesis, $A_\alpha f \in J$, that is, $f - \sigma_\alpha f = \alpha A_\alpha f \in J$. Hence for all $w \in W$, $f \equiv wf \pmod J$. Thus, $|W|^{-1} \sum_{w \in W} wf \equiv f \pmod J$. Since the left-hand side belongs to I_+, we see that $f \in J$. Theorem 3.13 is now proved.

The form (,) gives rise to a non-degenerate pairing between $\overline{R} = R/J$ and $\mathscr{H}$. Let $\{P_w\}$ be the basis of $\overline{R}$ dual to $\{D_w\}$. The following properties of the P_w are immediate consequences of Theorem 3.12.

3.14 THEOREM. (i) *Let* $w \in W$, $\alpha \in \Sigma$. *Then*

$$A_\alpha P_w = \begin{cases} 0 & \text{if } l(w\sigma_\alpha) = l(w) + 1, \\ P_{w\sigma_\alpha} & \text{if } l(w\sigma_\alpha) = l(w) - 1. \end{cases}$$

(ii) $\chi P_w = \sum_{w \xrightarrow{\gamma} w'} w\chi(H_\gamma) P_{w'}$ *for* $\chi \in \mathfrak{h}^*_{\mathbf{Q}}$.

(iii) *Let* $\alpha \in \Sigma$. *Then*

$$\sigma_\alpha P_w = \begin{cases} P_w & \text{if } l(w\sigma_\alpha) = l(w) + 1, \\ P_w - \sum_{w\sigma_\alpha \xrightarrow{\gamma} w'} w\alpha(H_\gamma) P_{w'} & \text{if } l(w\sigma_\alpha) = l(w) - 1. \end{cases}$$

From (i) it is clear that all the P_w can be expressed in terms of the P_s. More precisely, let $w = \sigma_{\alpha_1} \ldots \sigma_{\alpha_l}$, $l(w) = r - l$. Then

$$P_w = A_{\alpha_l} \ldots A_{\alpha_1} P_s.$$

To find an explicit form for the P_w it therefore suffices to compute the $P_s \in \overline{R}$.

3.15 THEOREM. $P_s = |W|^{-1} \prod_{\gamma \in \Delta_+} \gamma \pmod J$.

PROOF. We divide the proof into a number of steps. We fix an element $h \in \mathfrak{h}$ such that all the wh, $w \in W$, are distinct.

1. We first prove that there is a polynomial $Q \in R$ of degree r such that

$$Q(sh) = 1, \quad Q(wh) = 0 \text{ for } w \neq s. \tag{10}$$

For each $w \in W$ we choose in R a homogeneous polynomial $\widetilde{P}_w$ of degree

$l(w)$ whose image in $\overline{R} = R/J$ is P_w. Since $\{P_w\}$ is a basis of R, any polynomial $f \in R$ can be written in the form $f = \sum \widetilde{P}_w f_w$, where $f_w \in I$ (this is easily proved by induction on the degree of f). Now let $Q' \in R$ be an arbitrary polynomial satisfying (10) and let $Q' = \sum \widetilde{P}_w g_w$, $g_w \in I$. It is clear that $Q = \sum_w g_w(h) \widetilde{P}_w$ meets our requirements.

2. Let $\overline{Q}$ be the image of Q in $\overline{R}$, and let $\overline{Q} = \sum c_w P_w$ be the representation of $\overline{Q}$ in terms of the basis $\{P_w\}$ of R. We now prove that $c_s = (-1)^r \prod_{\gamma \in \Delta_+} (\gamma(h))^{-1}$.

To prove this we consider $A_s\overline{Q}$. On the one hand $A_s\overline{Q} = c_s$, by Theorem 3.13 (i); on the other hand, A_sQ is a constant, since Q is a polynomial of degree r. Hence, $A_sQ = c_s$.

We now calculate A_sQ, Let $s = \sigma_{\alpha_1} \dots \sigma_{\alpha_r}$ be the reduced decomposition. We put $w_i = \sigma_{\alpha_i} \dots \sigma_{\alpha_1}$ (in particular, $w_0 = e$), $\gamma_i = w_{i-1}^{-1} \alpha_i$, $Q_i = A_{\alpha_{i+1}} \dots A_{\alpha_r} Q$.

LEMMA. *Q_i is a polynomial of degree i,*

$$Q_i(w_i h) = (-1)^{r-i} \cdot \prod_{r \geqslant j > i} (\gamma_j(h))^{-1}$$

and $Q_i(wh) = 0$ if $w \not\geqslant w_i$.

PROOF. We prove the lemma by backward induction on i. For $i = r$ we have $w_r = s$, $Q_r = Q$, and the assertion of the lemma follows from the definition of Q.

We now assume the lemma proved for Q_i, $i > 0$. In the first place, it is clear that $Q_{i-1} = A_{\alpha_i} Q_i$ is a polynomial of degree $i - 1$.

Furthermore,

$$Q_{i-1}(wh) = A_{\alpha_i} Q_i(wh) = \frac{Q_i(wh) - Q_i(\sigma_{\alpha_i} wh)}{\alpha_i(wh)}.$$

If $w = w_{i-1}$, then $w < w_i$, $\sigma_{\alpha_i} w = w_i$ and $\alpha_i(w_{i-1} h) = (w_{i-1}^{-1} \alpha_i)(h) = -(w_i^{-1} \alpha_i)(h) = -\gamma_i(h)$. Therefore, using the inductive hypothesis, we have

$$Q_{i-1}(w_{i-1}h) = -\frac{Q_i(w_i h)}{\alpha_i(w_{i-1}h)} = (-1)^{r-i+1} \prod_{r \geqslant j > i-1} (\gamma_j(h))^{-1}.$$

But if $w \not\geqslant w_{i-1}$, Corollary 2.6 implies that $w \not\geqslant w_i$ and $\sigma_{\alpha_i} w \not\geqslant w_i$. So $Q_{i-1}(wh) = 0$, and the lemma is proved.

Note that by Lemma 2.2, as i goes from 1 to r, γ_i ranges over all the positive roots exactly once. Therefore

$$c_s = A_s Q = Q_0 = (-1)^r \prod_{\gamma \in \Delta_+} (\gamma(h))^{-1}.$$

3. Consider the polynomial Alt $(Q) = \sum (-1)^{l(w)} wQ$; Alt (Q) is skew-symmetric, that is, σ_α Alt$(Q) = -$Alt(Q) for all $\gamma \in \Delta$. Therefore Alt(Q) is divisible

(in R) by $\prod_{\gamma\in\Delta_+}\gamma$. Since the degrees of $\mathrm{Alt}(Q)$ and $\prod_{\gamma\in\Delta_+}\gamma$ are equal (to r), $\mathrm{Alt}(Q) = \lambda \prod_{\gamma\in\Delta_+}\gamma$. Furthermore, $\mathrm{Alt}(Q)\,(h) = (-1)^r$, so that

(11) $$\mathrm{Alt}\,(Q) = (-1)^r \prod_{\gamma\in\Delta_+} (\gamma(h))^{-1} \prod_{\gamma\in\Delta_+}\gamma.$$

4. We put $\mathrm{Alt}\,(\overline{Q}) = \sum (-1)^{l(w)}\, w\overline{Q}$. By Theorem 3.14 (iii), $\mathrm{Alt}(P_s) = \sum(-1)^{l(w)}\, wP_s = |W|\, P_s$. Therefore $\mathrm{Alt}(\overline{Q}) = c_s\, |W|\, P_s$ + terms of smaller degree. Since $\mathrm{Alt}(Q)$ is a homogeneous polynomial of degree r, we have

(12) $$\mathrm{Alt}\,(\overline{Q}) = c_s\,|W|\,P_s.$$

By comparing (11) and (12) we find that

$$P_s = |W|^{-1} \prod_{\gamma\in\Delta_+}\gamma \pmod J.$$

The theorem is now proved.

3.16 COROLLARY. *Let ρ be half the sum of the positive roots. Then* $P_s = \rho^r/r! \pmod J$.

PROOF. For each $\chi \in \mathfrak{h}^*$ we consider the formal power series $\exp\chi$ on $\mathfrak{h}$ given by

$$\exp\chi = \sum_{n=0}^{\infty} \chi^n/n!.$$

Then we have (see [9])

$$\sum_{w\in W} (-1)^{l(w)} \exp(w\rho) = \prod_{\gamma\in\Delta_+}\left[\exp\frac{\gamma}{2} - \exp\left(-\frac{\gamma}{2}\right)\right].$$

Comparing the terms of degree r we see that

$$\frac{1}{r!}\sum (-1)^{l(w)} (w\rho)^r = \prod_{\gamma\in\Delta_+}\gamma.$$

If $\rho^r \pmod J = \lambda P_s$, $\lambda \in \mathbf{C}$, then $(w\rho^r) \pmod J = \lambda w P_s = \lambda(-1)^{l(w)} P_s$. Thus, $\frac{1}{|W|}\sum (-1)^{l(w)} (w\rho)^r = \lambda P_s \pmod J$. The result now follows from Theorem 3.15.

To conclude this section we prove some results on products of the P_w in $\overline{R}$.

3.17. THEOREM. (i) *Let $\alpha \in \Sigma$, $w \in W$. Then*

$$P_{\sigma_\alpha} P_w = \sum_{w \xrightarrow{\gamma} w'} \chi_\alpha (H_{w^{-1}\gamma})\, P_{w'},$$

where $\chi_\alpha \in \mathfrak{h}_{\mathbf{Z}}^*$ is the fundamental dominant weight corresponding to the root α (that is, $\chi_\alpha\,(H_\beta) = 0$ for $\alpha \neq \beta \in \Sigma$, $\chi_\alpha\,(H_\alpha) = 1$).

(ii) *Let $w_1, w_2 \in W$, $l(w_1) + l(w_2) = r$. Then $P_{w_1} P_{w_2} = 0$ for*

$w_2 \neq w_1 s$, $P_{w_1} P_{w_1 s} = P_s$.

(iii) *Let* $w \in W$, $f \in \overline{R}$. *Then* $f\, P_w = \sum_{w' \geqslant w} c_{w'} P_{w'}$.

(iv) *If* $w_1 \nleqslant w_2 s$, *then* $P_{w_1} P_{w_2} = 0$.

PROOF. (i) By Theorem 3.12 (v), $P_{\sigma_\alpha} = \chi_\alpha \pmod{J}$. Therefore (i) follows from Theorem 3.14 (ii).

(ii) The proof goes by backward induction on $l(w_2)$. If $l(w_2) = r$, then $w_2 = s$, $w_1 = e$ and $P_{w_1} = 1$.

To deal with the general case we find the following simple lemma useful, which is an easy consequence of the definition of the A_γ.

3.18 LEMMA. *Let* $\gamma \in \Delta$, f, $g \in R$. *Then* $A_\gamma(A_\gamma f \cdot g) = A_\gamma f \cdot A_\gamma g$.

Thus, let $w_2 \in W$, $l(w_2) = l < r$, and choose $\alpha \in \Sigma$ so that $w_2 \xrightarrow{\alpha} \sigma_\alpha w_2$. We consider two cases separately.

A) $w_1 \overset{\alpha}{\nrightarrow} \sigma_\alpha w_1$. We observe that the following equation holds for any $w \in W$

$$l(ws) = r - l(w). \tag{13}$$

Since in our case $l(\sigma_\alpha w_2) = l + 1$ and $l(\sigma_\alpha w_1) = r - l + 1$, we see that $\sigma_\alpha w_1 s \neq \sigma_\alpha w_2$, and so $w_1 s \neq w_2$. On the other hand, $P_{w_2} = A_\alpha P_{\sigma_\alpha} w_2$ and $P_{w_1} = A_\alpha P_{\sigma_\alpha} w_1$ by Theorem 3.14 (i). Therefore, an application of Lemma 3.18 shows that

$$P_{w_1} P_{w_2} = A_\alpha P_{\sigma_\alpha w_1} \cdot A_\alpha P_{\sigma_\alpha w_2} = A_\alpha \left(P_{\sigma_\alpha w_1} \cdot A_\alpha P_{\sigma_\alpha w_2}\right) = A_\alpha \left(P_{\sigma_\alpha w_1} \cdot P_{w_2}\right).$$

Since $l(\sigma_\alpha w_1) + l(w_2) = r - l + 1 + l > r$, we have $P_{\sigma_\alpha w_1} P_{w_2} = 0$. Hence $P_{w_1} P_{w_2} = 0$ as well.

B) $\sigma_\alpha w_1 \overset{\alpha}{\nrightarrow} w_1$. In this case, $P_{\sigma_\alpha w_1} = A_\alpha P_{w_1}$ and $P_{w_2} = A_\alpha P_{\sigma_\alpha w_2}$, by Theorem 3.14 (i). Again applying Lemma 3.18, we have

$$A_\alpha \left(P_{w_1} P_{w_2}\right) \doteq A_\alpha \left(P_{w_1} \cdot A_\alpha P_{\sigma_\alpha w_2}\right) = A_\alpha P_{w_1} \cdot A_\alpha P_{\sigma_\alpha w_2} = \tag{14}$$
$$= A_\alpha \left(A_\alpha P_{w_1} \cdot P_{\sigma_\alpha w_2}\right) = A_\alpha \left(P_{\sigma_\alpha w_1} \cdot P_{\sigma_\alpha w_2}\right).$$

Since the P_w form a basis of $\overline{R}$, any element f of degree r in $\overline{R}$ has the form $f = \lambda P_s$, $\lambda \in \mathbf{C}$. Furthermore, $A_\alpha P_s = P_{\sigma_\alpha s} \neq 0$. But $\deg P_{w_1} P_{w_2} = \deg P_{\sigma_\alpha w_1} \cdot P_{\sigma_\alpha w_2} = r$. Therefore (14) is equivalent to

$$P_{w_1} P_{w_2} = P_{\sigma_\alpha w_1} P_{\sigma_\alpha w_2}.$$

Applying the inductive hypothesis to the pair $(\sigma_\alpha w_1, \sigma_\alpha w_2)$, we obtain part (ii) of the theorem.

(iii) is an immediate consequence of Theorem 3.14 (ii).

(iv) follows from (ii) and (iii).

We define the operator $\mathcal{P}: \overline{R} \to \mathcal{H}$ of Poincaré duality by the formula $(\mathcal{P}f)(g) = D_s(fg)$, f, $g \in \overline{R}$, $\mathcal{P}f \in \mathcal{H}$.

3.19. COROLLARY. $\mathcal{P} P_w = D_{ws}$.

§ 4. Schubert cells

We prove in this section that the functionals D_w, $w \in W$ introduced in § 3 correspond to Schubert cells s_w, $w \in W$.

Let $s_w \in H_*(X, \mathbf{Q})$ be a Schubert cell. It gives rise to a linear functional on $H^*(X, \mathbf{Q})$, which, by means of the homomorphism α: $R \to H^*(X, \mathbf{Q})$ (see Theorem 1.3), can be regarded as a linear functional on R. This functional takes the value 0 on all homogeneous components P_k with $k \neq l(w)$, and thus determines an element $\hat{D}_w \in S_{l(w)}$.

4.1. THEOREM. $\hat{D}_w = D_w$ (cf. Definition 3.11).

This theorem is a natural consequence of the next two propositions.

PROPOSITION 1. $\hat{D}_e = 1$, *and for any* $\chi \in \mathfrak{h}^*_{\mathbf{Z}}$

$$\chi^* (\hat{D}_w) = \sum_{w' \xrightarrow{\gamma} w} w'\chi (H_\gamma) \hat{D}_{w'}. \tag{15}$$

PROPOSITION 2. *Suppose that for each* $w \in W$ *we are given an element* $\hat{D}_w \in S_{l(w)}$, *with* $\hat{D}_e = 1$, *for which* (15) *holds for any* $\chi \in \mathfrak{h}^*_{\mathbf{Z}}$. *Then* $\hat{D}_w = D_w$.

Proposition 2 follows at once from Theorem 3.12 (iii) by induction on $l(w)$.

We turn now to the proof of Proposition 1.

We recall (see [10]) that for any topological space Y there is a bilinear mapping

$$H^i (Y, \mathbf{Q}) \times H_j (Y, \mathbf{Q}) \xrightarrow{\cap} H_{j-i} (Y, \mathbf{Q})$$

(the cap-product). It satisfies the condition:

(16) 1. $\langle c \cap y, z \rangle = \langle y, c \cdot z \rangle$

for all $y \in H_j (Y, \mathbf{Q})$, $z \in H^{j-i} (Y, \mathbf{Q})$, $c \in H^i (Y, \mathbf{Q})$.

2. Let f: $Y_1 \to Y_2$ be a continuous mapping. Then

$$f_* (f^*c \cap y) = c \cap f_*y \tag{17}$$

for all $y \in H_j (Y_1, \mathbf{Q})$, $c \in H^i (Y_2, \mathbf{Q})$.

By virtue of (17) we have for any $\chi \in \mathfrak{h}^*_{\mathbf{Z}}$, $f \in R$

$$(\chi^* (\hat{D}_w), f) = (\hat{D}_w, \chi f) = \langle s_w, \alpha_1 (\chi) \alpha (f) \rangle = \langle s_w \cap \alpha_1 (\chi), \alpha (f) \rangle.$$

Therefore (15) is equivalent to the following geometrical fact.

PROPOSITION 3. For all $\chi \in \mathfrak{h}^*_{\mathbf{Z}}$

$$s_w \cap \alpha_1 (\chi) = \sum_{w' \xrightarrow{\gamma} w} w'\chi (H_\gamma) s_{w'}. \tag{18}$$

We restrict the fibering E_χ to $\overline{X}_w \subset X$ and let $c_\chi \in H^2 (\overline{X}_w, \mathbf{Q})$ be the first Chern class of E_χ. By (17) and the definition of the homomorphism α_1: $\mathfrak{h}^*_{\mathbf{Z}} \to H^2 (X, \mathbf{Q})$, it is sufficient to prove that

$$s_w \cap c_\chi = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma)\, s_{w'}. \tag{19}$$

in $H_{2l(w)-2}(\overline{X}_w, \mathbf{Q})$.

To prove (19), we use the following simple lemma, which can be verified by standard arguments involving relative Poincaré duality.

4.2 LEMMA. *Let Y be a compact complex analytic space of dimension n, such that the codimension of the space of singularities of Y is greater than 1. Let E be an analytic linear fibering on Y, and $c \in H^2(Y, \mathbf{Q})$ the first Chern class of E. Let μ be a non-zero analytic section of E and $\sum m_i Y_i$ = div μ the divisor of μ. Then $[Y] \cap c = \sum m_i[Y_i] \in H_{2n-2}(Y, \mathbf{Q})$, where $[Y]$ and $[Y_i]$ are the fundamental classes of Y and Y_i.*

Let $w \in W$, and let $X_w \subset X$ be the corresponding Schubert cell. From Lemma 4.2 and Theorem 2.11 it is clear that to prove Proposition 3 it is sufficient to verify the following facts.

4.3. PROPOSITION. *Let $w' \xrightarrow{\gamma} w$. Then $\overline{X}_w$ is non-singular at points $x \in X_{w'}$.*

4.4. PROPOSITION. *There is a section μ of the fibering E_χ over $\overline{X}_w$ such that*

$$\operatorname{div}\mu = \sum_{w' \xrightarrow{\gamma} w} w'\chi(H_\gamma)\, \overline{X}_{w'}.$$

To verify these facts we use the geometrical description of Schubert cells given in 2.9. We consider a finite-dimensional representation of G on a space V with regular dominant weight λ, and we realize X as a subvariety of $P(V)$. For each $w \in W$ we fix a vector $f_w \in V$ of weight $w\lambda$.

PROOF OF PROPOSITION 4.3. For a root $\gamma \in \Delta_+$ we construct a three-dimensional subalgebra $\mathfrak{A}_\gamma \subset \mathfrak{G}$ (as in the proof of Theorem 2.9). Let i: $SL_2(\mathbf{C}) \to G$ be the homomorphism corresponding to the embedding $\mathfrak{A}_\gamma \to \mathfrak{G}$. Consider in $SL_2(\mathbf{C})$ the subgroups $B' = \left\{\begin{pmatrix} a & b \\ 0 & a^{-1}\end{pmatrix}\right\}$, $H' = \left\{\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right\}$ and $N'_- = \left\{\begin{pmatrix} 1 & 0 \\ x & 1\end{pmatrix}\right\}$ and the element $\sigma = \begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}$. We may assume that $i(H') \subset H$, $i(B') \subset B$.

Let $\widetilde{V}$ be the smallest $\mathfrak{A}_\gamma$-invariant subspace of V containing $f_{w'}$. It is clear that $\widetilde{V}$ is invariant under $i(SL_2(\mathbf{C}))$, and that the stabilizer of the line $[f_{w'}]$ is B'. This determines a mapping δ: $SL_2(\mathbf{C})/B' \to X$. The space $SL_2(\mathbf{C})/B'$ is naturally identified with the projective line $\mathbf{P}^1$. Let $o, \infty \in \mathbf{P}^1$ be the images of $e, \sigma \in SL_2(\mathbf{C})$.

We define a mapping ξ: $N_{w'} \times \mathbf{P}^1 \to X$ by the rule

$$(x, z) \longmapsto x \cdot \delta(z).$$

4.5. LEMMA. *The mapping ξ has the following properties:*

(i) $\xi(N_{w'} \times \{o\}) = X_{w'}$, $\xi(N_{w'} \times (\mathbf{P}^1 \setminus o)) \subset X_w$.

(ii) *The restriction of* ξ *to* $(N_{w'} \times \mathbf{P}^1 \setminus \infty))$ *is an isomorphism onto a certain open subset of* $\overline{X}_w$.

Proposition 4.3 clearly follows from this lemma.

PROOF OF LEMMA 4.5. The first assertion of (i) follows at once from the definition of $X_{w'}$. Since the cell X_w is invariant under N, the proof of the second assertion of (i) is reduced to showing that $\delta(z) \in X_w$ for $z \in \mathbf{P}^1 \setminus o$. Let $h \in SL_2(\mathbf{C})$ be an inverse image of z. Then h can be written in the form $h = b_1 \sigma b_2$, where $b_1, b_2 \in B'$. It is clear that $i(b_2)f_{w'} = c_1 f_{w'}$ and $i(\sigma)f_{w'} = c_2 f_w$, where c_1, c_2 are constants. Therefore $i(h)f_{w'} = c_1 c_2 i(b_1)f_w$, that is, $\delta(z) \in X_w$.

To prove (ii), we consider the mapping

$$w'^{-1} \circ \xi : N_{w'} \times (\mathbf{P}^1 \setminus \infty) \to X.$$

The space $\mathbf{P}^1 \setminus \infty$ is naturally isomorphic to the one-parameter subgroup $N'_- \subset SL_2(\mathbf{C})$.

The mapping $\xi: N_{w'} \times N'_- \to X$ is given by the rule

$$\xi(n, n_1) = ni(n_1)[f_{w'}], \quad n \in N_{w'}, \ n_1 \in N'.$$

Thus,

$$w'^{-1} \circ \xi(n, n_1) = (w'^{-1} n w')(w'^{-1} i(n_1) w')[f_e].$$

We now observe that $w'^{-1} N_{w'} w' \subset N_-$ (by definition of $N_{w'}$), and $w'^{-1} i(N') w' \in N_-$ (since $w'^{-1}\ \gamma \in \Delta_+$). Furthermore, the intersection of the tangent spaces to these subgroups consists only of 0, because $N_{w'} \subset N$, $i(N'_-) \subset N_-$. The mapping $N_- \to X$ ($n \mapsto n[f_e]$) is an isomorphism onto an open subset of X. Therefore (ii) follows from the next simple lemma, which is proved in [5], for example.

4.6. LEMMA. *Let* N_1 *and* N_2 *be two closed algebraic subgroups of a unipotent group* N *whose tangent spaces at the unit element intersect only in* 0. *Then the product mapping* $N_1 \times N_2 \to N$ *gives an isomorphism of* $N_1 \times N_2$ *with a closed subvariety of* N.

This completes the proof of Proposition 4.3.

PROOF OF PROPOSITION 4.4. Any element of $\mathfrak{h}^*_{\mathbf{Z}}$ has the form $\chi = \lambda - \lambda'$, where λ, λ' are regular dominant weights. In this case, $E_\chi = E_\lambda \otimes E_{\lambda'}^{-1}$, and it is therefore sufficient to find a section μ with the required properties in the case $\chi = \lambda$.

We consider the space $P(V)$, where V is a representation of G with dominant weight λ. Let η_V be the linear fibering on $P(V)$ consisting of pairs (P, ϕ), where ϕ is a linear functional on the line $P \subset V$. Then $E_\lambda = i^*(\eta_V)$, where $i: X \to P(V)$ is the embedding described in § 2.

The linear functional ϕ_w on V (see the proof of Theorem 2.11) yields a section of the bundle η. We shall prove that the restriction of μ to this section on $\overline{X}_w$ is a section of the fibering E_λ having the requisite properties.

By Lemma 2.12, $\mu(x) \neq 0$ for all $x \in X_w$. The support of the divisor div μ is therefore contained in $\overline{X}_w \setminus X_w = \bigcup_{w' \xrightarrow{\gamma} w} \overline{X}_{w'}$.

Since $\bar{X}_{w'}$ is an irreducible variety, we see that div $\mu = \sum_{w' \xrightarrow{\gamma} w} a_\gamma \bar{X}_{w'}$, where $a_\gamma \in \mathbf{Z}$, $a_\gamma \geqslant 0$. It remains to show that $a_\gamma = w'\chi(H_\gamma)$.

In view of Lemma 4.5 (i) and (ii), the coefficient a_γ is equal to the multiplicity of zero of the section $\delta^*(\mu)$ of the fibering $\delta^*(E_\lambda)$ on $\mathbf{P}^1$ at the point o, that is, the multiplicity of zero of the function $\psi(t) = \phi_{w'}((\exp tE_{-\gamma})f_{w'})$ for $t = 0$. It follows from Lemma 2.10 that $\psi(t) = ct^n$, hence $a_\gamma = n = w'\chi(H_\gamma)$. This completes the proof of Proposition 4.4 and with it of Theorem 4.1.

§5. Generalizations and supplements

1. Degenerate flag varieties. We extend the results of the previous sections to spaces $X(P) = G/P$, where P is an arbitrary parabolic subgroup of G. For this purpose we recall some facts about the structure of parabolic subgroups $P \subset G$ (see [7]).

Let Θ be some subset of Σ, and Δ_Θ the subset of Δ_+ consisting of linear combinations of elements of Θ. Let G_Θ be the subgroup of G generated by H together with the subgroups $N_\gamma = \{\exp tE_\gamma \mid t \in \mathbf{C}\}$ for $\gamma \in \Delta_\Theta \cup -\Delta_\Theta$, and let N_Θ be the subgroup of N generated by the N_γ for $\gamma \in \Delta_+ \backslash \Delta_\Theta$. Then G_Θ is a reductive group normalizing N_Θ, and $P_\Theta = G_\Theta N_\Theta$ is a parabolic subgroup of G containing B.

It is well known (see [7], for example) that every parabolic subgroup $P \subset G$ is conjugate in G to one of the subgroups P_Θ. We assume in what follows that $P = P_\Theta$, where Θ is a fixed subset of Σ. Let W_Θ be the Weyl group of G_Θ. It is the subgroup of W generated by the reflections σ_α, $\alpha \in \Theta$.

We describe the decomposition of $X(P)$ into orbits under the action of B.

5.1. PROPOSITION. (i) $X(P) = \bigcup_{w \in W} Bwo$, *where $o \in X(P)$ is the image of P in G/P.*

(ii) *The orbits $Bw_1 o$ and $Bw_2 o$ are identical if $w_1 w_2^{-1} \in W_\Theta$ and otherwise are disjoint.*

(iii) *Let W_Θ^1 be the set of $w \in W$ such that $w\Theta \subset \Delta_+$. Then each coset of W/W_Θ contains exactly one element of W_Θ^1. Furthermore, the element $w \in W_\Theta^1$ is characterized by the fact that its length is less than that of any other element in the coset wW_Θ.*

(iv) *If $w \in W_\Theta^1$, then the mapping $N_w \to X(P)$ $(n \to nwo)$ is an isomorphism of N_w with the subvariety $Bwo \subset X(P)$.*

PROOF. (i)–(ii) follow easily from the Bruhat decomposition for G and G_Θ. The proof of (iii) can be found in [7], for example, and (iv) follows at once from (iii) and Proposition 1.1.

Let $w \in W_\Theta^1$, $X_w(P) = Bwo$, let $\bar{X}_w(P)$ be the closure of $X_w(P)$ and $[\bar{X}_w(P)] \in H_{2l(w)}(\bar{X}_w(P), \mathbf{Z})$ its fundamental class. Let $s_w(P) \in H_{2l(w)}(X(P), \mathbf{Z})$ be the image of $[\bar{X}_w(P)]$ under the mapping induced by the embedding $\bar{X}_w(P) \hookrightarrow X(P)$. The next proposition is an analogue of Proposition 1.2.

5.2. PROPOSITION ([2]). *The elements* $s_w(P)$, $w \in W^1_\Theta$, *form a free basis in* $H_*(X(P), \mathbf{Z})$.

5.3. COROLLARY. *Let* $\alpha_P: X \to X(P)$ *be the natural mapping. Then* $(\alpha_P)_* s_w = 0$ *if* $w \notin W^1_\Theta$, $(\alpha_p)_* s_w = s_w(P)$ *if* $w \in W^1_\Theta$.

5.4. COROLLARY. $(\alpha_P)_*: H_*(X, \mathbf{Z}) \to H_*(X(P), \mathbf{Z})$ *is an epimorphism, and* $(\alpha_P)^*: H^*(X(P), \mathbf{Z}) \to H^*(X, \mathbf{Z})$ *is a monomorphism.*

5.5. THEOREM. (i) $\mathrm{Im}(\alpha_P)^* \subset H^*(X, \mathbf{Z}) = \bar{R}$ *coincides with the set of* W_Θ*-invariant elements of* $\bar{R}$.

(ii) $P_w \in \mathrm{Im}(\alpha_P)^*$ *for* $w \in W^1_\Theta$ *and* $\{(\alpha_P)^{*-1}P_w\}_{w \in W^1_\Theta}$ *is the basis in* $H^*(X(P), \mathbf{Z})$ *dual to the basis* $\{s_w(P)\}_{w \in W^1_\Theta}$ *in* $H_*(X(P), \mathbf{Z})$.

PROOF. Let $w \in W^1_\Theta$. Since $\langle P_w, s_{w_1}\rangle = 0$ for $w_1 \notin W^1_\Theta$, P_w is orthogonal to $\mathrm{Ker}(\alpha_P)_*$, that is, $P_w \in \mathrm{Im}(\alpha_P)^*$. Now (ii) follows from the fact that $\langle(\alpha_P)^* P_w, s_{w'}(P)\rangle = \langle P_w, s_{w'}\rangle$ for $w, w' \in W^1_\Theta$. To prove (i), it is sufficient to verify that the P_w, $w \in W^1_\Theta$, form a basis for the space of W_Θ-invariant elements of $\bar{R}$. We observe that an element $f \in \bar{R}$ is W_Θ-invariant if and only if $A_\alpha f = 0$ for all $\alpha \in \Theta$. Since $w \in W^1_\Theta$ if and only if $l(w\sigma_\alpha) = l(w) + 1$ for all $\alpha \in \Theta$, (i) follows from Theorem 3.14(i).

2. CORRESPONDENCES. Let Y be a non-singular oriented manifold. An arbitrary element $z \in H_*(Y \times Y, \mathbf{Z})$ is called a correspondence on Y. Any such element z gives rise to an operator $z_*: H_*(Y, \mathbf{Z}) \to H_*(Y, \mathbf{Z})$, according to

$$z_*(c) = (\pi_2)_*((\pi_1)^*(\mathcal{D}c) \cap z),\ c \in H_*(Y, \mathbf{Z}),$$

where $\pi_1, \pi_2: Y \times Y \to Y$ are the projections onto the first and second components, and $\mathcal{D}$ is the Poincaré duality operator. We also define an operator $z^*: H^*(Y, \mathbf{Z}) \to H^*(Y, \mathbf{Z})$ by setting $z^*(\xi) = \mathcal{D}[(\pi_1)_*((\pi_2)^*(\xi) \cap z)]$, $\xi \in H^*(Y, \mathbf{Z})$. It is clear that z_* and z^* are adjoint operators.

Let z be assigned to a (possibly singular) submanifold $Z \subset Y \times Y$, in such a way that z is the image of the fundamental cycle $[Z] \in H_*(Z, \mathbf{Z})$ under the mapping induced by the embedding $Z \hookrightarrow Y \times Y$. Then

$$z_*(c) = (\rho_2)_*([Z] \cap (\rho_1)^* \mathcal{D}c),$$

where $\rho_1, \rho_2: Z \to Y$ are the restrictions of π_1, π_2 to Z.

If, in this situation, $\rho_1: Z \to Y$ is a fibering and c is given by a submanifold $C \subset Y$, then the cycle

$$[Z] \cap (\rho_1)^* \mathcal{D}c$$

is given by the submanifold $\rho_1^{-1}(C) \subset Z$.

We want to study correspondences in the case $Y = X = G/B$.

5.6. DEFINITION. *Let* $w \in W$. *We put* $Z_w = \{(gwo, go)\} \subset X \times X$ *and denote by* z_w *the correspondence* $z_w = [\bar{Z}_w] \subset H_*(X \times X, \mathbf{Z})$.

5.7. THEOREM. $(z_w)_* = F_w$.

PROOF. We calculate $(z_w)_*(s_{w'})$.

Since the variety $\overline{Z}_w$ is G-invariant and G acts transitively on X, the mapping $\rho_1 : \overline{Z}_w \to X$ is a fibering. Thus,

$$(z_w)_* (s_{w'}) = (\rho_2)_* [\rho_1^{-1} (\overline{X}_{w'})].$$

It is easily verified that $\rho_1^{-1} (\overline{X}_{w'}) = \overline{\pi_1^{-1} (X_{w'}) \cap Z_w}$. We put $Y = \pi_1^{-1} (X_{w'}) \cap Z_w \subset X \times X$. Then

$$(20) \qquad Y = \{(nw'o,\ nw'bwo) \mid n \in N,\ b \in B\}.$$

Since the dimension of the fibre of $\rho_1 : Z_w \to X$ is equal to $2l(w)$, we see that $\dim Y = 2l(w) + 2l(w')$. It is clear from (20) that

$$\rho_2(Y) = \{nw'bwo \mid n \in N,\ b \in B\} = Bw'Bwo.$$

It is well known (see [6], Ch. IV, § 2.1 Lemma 1) that

$$Bw'Bwo = Bw'wo \cup \Big(\bigcup_{l(w_1) < l(w) + l(w')} Bw_1 o\Big).$$

Thus, two cases can arise.

a) $l(w'w) < l(w') + l(w)$. In this case, $\dim \rho_2 (Y) < 2l(w') + 2l(w)$, and so $(z_w)_*(s_{w'}) = (\rho_2)_*[Y] = 0$.

b) $l(w'w) = l(w') + l(w)$. In this case, $\rho_2 (Y) = X_{w'w} + X'$, where $\dim X' < \dim X_{w'w} = 2l(w') + 2l(w)$. Thus, $(\rho_2)_*[Y] = [\overline{X}_{w'w}]$, that is, $(z_w)_*(s_{w'}) = s_{w'w}$. Comparing the formulae obtained with 3.12 (ii), we see that $(z_w)_* = F_w$.

5.8. COROLLARY. $z_w = \sum s_{w's} \otimes s_{w'w}$, *where the summation extends over those* $w' \in W$ *for which* $l(w'w) = l(w) + l(w')$.

In § 1 we have defined an action of W on $H_*(X, \mathbf{Z})$. This definition depended on the choice of a compact subgroup K. Using Theorem 5.7 we can find explicitly the correspondences giving this action.

In fact, it follows from Lemma 3.3 (iii) that $\sigma_\alpha = \alpha^* F_\alpha - 1$ for any $\alpha \in \Sigma$. The transformation F_α is given by the correspondence Z_{σ_α}. The operator α^* can also be given by a correspondence: if $U_\alpha = \Sigma c_i U_i$ is a divisor in X giving the cycle $\mathscr{P}(\alpha) \in H_{2r-2}(X, \mathbf{Z})$ (for example, $U_\alpha = \sum_{\beta \in \Sigma} \alpha(H_\beta) X_{\sigma_\beta}$), then the cycle $\widetilde{U}_\alpha = \Sigma c_i \widetilde{U}_i$, where $\widetilde{U}_i = \{(x, x) \mid x \in U_i\} \subset X \times X$, determines the correspondence that gives the operator α^*. The operator σ_α in $H_*(X, \mathbf{Z})$ is therefore given by the correspondence $\widetilde{U}_\alpha * Z_{\sigma_\alpha} - 1$ (where $*$ denotes the product of correspondences, as in [11]). Using the geometrical realization of the product of correspondences (see [11]), we can explicitly determine the correspondence S_α that gives the transformation $1 + \sigma_\alpha$ in $H_*(X, \mathbf{Z})$, namely, $S_\alpha = \Sigma c_i \hat{U}_i$ where $\hat{U}_i = \{(x, y) \in X \times X \mid x \in U_i,\ \tilde{x}^{-1}\tilde{y} \in P_{\{\alpha\}}\}$. In this expression, $\tilde{x}, \tilde{y} \in G$ are arbitrary representatives of x, y, and $P_{\{\alpha\}}$ is the parabolic subgroup corresponding to the root α.

3. B. Kostant has described the P_w in another way. We state his result.

Let $h \in \mathfrak{h}_{\mathbf{Q}}^*$ be an element such that $\alpha(h) > 0$ for all $\alpha \in \Sigma$. Let $J_h = \{f \in R \mid f(wh) = 0$ for all $w \in W\}$ be an ideal of R.

5.9. THEOREM. (i) *Let $w \in W$, $l(w) = l$. There is a polynomial $Q_w \in R$ of degree l such that*

(21) $\qquad Q_w(wh) = 1,\ Q_w(w'h) = 0 \quad \text{if} \quad l(w') \leqslant l(w),\ \ w' \neq w.$

The Q_w are uniquely determined by (21) *to within elements of J_h.* (ii) *Let Q_w^0 be the form of highest degree in the polynomial Q_w. The image of Q_w^0 in $\bar{R}$ is equal to* $\prod_{\gamma \in \Delta_- \cap w^{-1}\Delta_+} (\gamma(h))^{-1} \cdot P_w$.

The proof is analogous to that of Theorem 3.15.

4. We choose a maximal compact subgroup $K \subset G$ such that $K \cap B \subset H$ (see § 1). The cohomology of X can be described by means of the K-invariant closed differential forms on X. For let $\chi \in \mathfrak{h}_{\mathbf{Z}}^*$, and let E_χ be the corresponding one-dimensional complex G-fibering on X. Let $\tilde{\omega}_\chi$ be the 2-form on X which is the curvature form of connectedness associated with the K-invariant metric on E_χ (see [12]). Then the class of the form $\omega_\chi \ \frac{1}{2\pi i}\ \bar{\omega}_\chi$ is $c_\chi \in H^2(X, \mathbf{Z})$. The mapping $\chi \to \omega_\chi$ extends to a mapping $\theta\colon R \to \Omega_{ev}^*(X)$, where Ω_{ev}^* is the space of differential forms of even degree on X. One can prove the following theorem, which is a refinement of Proposition 1.3 (ii) and Theorem 3.17.

5.10. THEOREM (i) Ker $\theta = J$, *that is, θ induces a homomorphism of rings $\bar{\theta}\colon \bar{R} \to \Omega_{ev}^*(X)$.* (ii) *Let $w_1, w_2 \in W$, $w_1 \not\leqslant w_2 s$. Then the restriction of the form $\bar{\theta}(P_{w_1})$ to X_{w_2} is equal to* 0. (iii) *Let $w_1, w_2 \in W$, $w_1 \not\leqslant w_2 s$. Then $\bar{\theta}(P_{w_1})\,\bar{\theta}(P_{w_2}) = 0$.*

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