Local Langlands parameters

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Contents

General set-up and notation: let F be a local field, with ring of integers $\mathcal{O}_F$, uniformizer π_F, and residue field κ_F with $|\kappa_F| = q = p^r$. Let $\mathbb{G}$ be a reductive group over F and $G_F = \mathbb{G}(F)$. Fix a $\mathbb{Z}[1/p]$-algebra Λ (or maybe we should be more restrictive and require it to be a $\mathbb{Z}_p$-algebra); typically one takes $\Lambda = \overline{\mathbb{Q}}_\ell$ where $\ell \neq p$. So far we've been studying $\mathrm{Rep}_\Lambda(G_F)$, the category of *smooth*, or locally constant, representations, where we completely ignore any possible issues of topology on the ring Λ.

Our goal today is to pass to the other ("spectral") side of Langlands duality, and introduce various notions of (local) *Langlands parameters*, roughly some kind of homomorphisms $\phi : \Gamma_F \to \check{G}(\Lambda)$ up to conjugation, where Γ_F is the absolute Galois group (except not really; more on that later). For simplicity, I will require that $\mathbb{G}$ is split over F (so we can work with $\check{G}$ instead of ${}^L G$), and I'll choose a square root of q (to avoid having to think about ${}^C G$); see the references for more general statements. We will consider three kinds.

1. When $\Lambda = \mathbb{Z}_\ell, \overline{\mathbb{Q}}_\ell$, we will *use the topology on* Λ and consider continuous homomorphisms ϕ. These are perhaps most traditionally called Langlands parameters. The notion of continuity doesn't lend well to building moduli spaces (or stacks), but it can be made to work in this case.

2. When Λ is a $\mathbb{Z}[1/p]$-algebra, we can make some choice of generators for the "tame" part and build a moduli stack. In this presentation, the substack of tame parameters will arise naturally.

3. When $\Lambda \supset \mathbb{Q}$, we consider a certain modification of the above called *Weil-Deligne parameters*. In this presentation, the substack of unipotent parameters will arise naturally.

0.1 Acknowledgements

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1 Extensions of local fields

We let Γ_F denote the absolute Galois group of F, equipped with its canonical topology,[1] which is pro-finite. Note: $\overline{F}$ will denote the separable closure, not the algebraic closure. What are some extensions $E \supset F$? Some basic initial facts: every finite extension of E/F is a local field, $\mathcal{O}_E$ is the integral closure of $\mathcal{O}_F$ in E, and we say the *ramification index* of E/F is

$$e_{E/F} := v(\pi_F)/v(\pi_E).$$

1.1 Unramified extensions

An extension E/F is *unramified* if $e_{E/F} = 1$, i.e. if uniformizers π_F of F are also uniformizers of $\mathcal{O}_E$, or equivalently if $[E : F] = [\kappa_E : \kappa_F]$, Morally, these are "extensions coming from the residue field", e.g. $\mathbb{F}_{q^n}((\pi))$ as an extension of $\mathbb{F}_q((\pi))$. Recall that if $\kappa_F = \mathbb{F}_q$, then

$$\Gamma_{\mathbb{F}_q} \simeq \widehat{\mathbb{Z}} \twoheadrightarrow \mathbb{Z}/n\mathbb{Z} \simeq \Gamma_{\mathbb{F}_{q^n}/\mathbb{F}_q}.$$

where $\mathbb{F}_{q^n}/\mathbb{F}_q$ has $[\mathbb{F}_{q^n} : \mathbb{F}_q] = q^{n-1}$ and splitting polynomial $p(x) = x^{p^n} - x$. Note that this factors $x(x^{p-1} - 1)(x^{(n-1)p} + x + \cdots + 1)$, and the factor $x(x^{p-1} - 1)$ is satisfied by every element of $\mathbb{F}_p$, so we can just take the other factor, whose degree implies $|\mathbb{F}_{p^n}| = p^n$.

We have the following. For $\mathbb{F}_q((\pi))$ it is essentially clear what the unramified extensions should be. The pattern turns out to hold in p-adic case as well. We state the following without proof, but it can be found e.g. in [Se79].

Proposition 1.1.1. *The canonical map gives an group isomorphism*

$$\Gamma_F^{ur} \xrightarrow{\simeq} \Gamma_{\kappa_F} \simeq \widehat{\mathbb{Z}}.$$

We can obtain every unramified extension by adjoining roots of 1. Recall the analogous statement in finite fields: $\mathbb{F}_q^\times$ is cyclic,[2] and $\overline{\mathbb{F}}_q$ contains all roots of unity coprime to p,[3] but no extension contains any pth roots of unity other than 1. Thus, $\overline{\mathbb{F}}_p$ is obtained from $\mathbb{F}_p$ by adjoining all roots of unity of order coprime to p.[4] It's hard to say which field extension contains a particular root of unity, since this amounts to factoring $p^n - 1$.

1.2 Tamely ramified extensions

An extension E/F is *tamely ramified* if $p \nmid e_{E/F}$. An extension is *totally ramified* if $\kappa_E = \kappa_F$. Every tamely ramified extension of a local field is obtained by adjoining mth roots for $p \nmid m$.

Proposition 1.2.1. *Every tamely ramified extension is of the form $F(a_1^{1/m_1}, \ldots, a_k^{1/m_k})$ for $p \nmid m_i$ and $a_k \in F$.*

Proof. Let E/F be a totally tamely ramified extension, and choose some $\alpha \in E$ and m minimal such that $v_F(\alpha)^m \in \mathbb{Z}$. A priori, we only have $\alpha^m \in F \cdot \mathcal{O}_E$, e.g. if $\alpha = \sqrt{\pi_F} + \pi_F = \sqrt{\pi_F}(1 + \sqrt{\pi_F})$. We want to tweak it so $\alpha^m \in F$. We can write $\alpha^m = u\beta$ where $u \in F^\times$ and $\beta \in \mathcal{O}_E$ where $\beta \equiv \zeta_r \pmod{\mathfrak{m}_E}$ for some rth root of unity where $p \nmid r$ (i.e. because κ_E is obtained from κ_F by adjoining roots of unity as discussed earlier). To get rid of this root of unity, we take instead $\alpha^{mr} = u^r\beta^r$ where now $\beta^r \equiv 1 \pmod{\mathfrak{m}_E}$. Now, by Hensel's lemma[5] applied to the polynomial $p(x) = x^{mr} - \beta^r$ (i.e. take the root 1 of $\overline{p}(x) = x^{mr} - 1 \pmod{\mathfrak{m}_E}$ which is simple since mr is coprime to p), there is $\gamma \equiv 1 \pmod{\mathfrak{m}_E}$ such that $\gamma^{mr} = \beta^r$, and we have $(\alpha/\gamma)^{mr} = u^r \in F^\times$, i.e. $\alpha/\gamma = u^{1/m}$. Now replace F with $F(\alpha/\gamma)$ and continue. $\square$

We note there is no "maximal totally tamely unramified extension." The reason is, for example, suppose $F = \mathbb{F}_q((\pi))$ and that $\epsilon \in \mathbb{F}_q$ has no square root. Then, $F(\sqrt{\epsilon\pi})$ and $F(\sqrt{\pi})$ are totally totally unramified extensions, but $\sqrt{\epsilon} \in F(\sqrt{\epsilon\pi}, \sqrt{\pi})$ comes from an an extension of the residue field.

[1] See Stacks Project.

[2] Because $x^r = 1$ has at most r solutions in any field, so $r \geqslant q - 1$.

[3] The units of $\mathbb{F}_{q^n}^\times$ is a cyclic group of order $q^n - 1$, and for any m coprime to p there is is an n such that $q^n \cong 1 \pmod{m}$.

[4] Formally, this means adding roots of the prime-to-p cyclotomic polynomials, which have integer coefficients.

[5] Let $(R, \mathfrak{m}, v)$ be a $\mathfrak{m}$-adically complete valuation ring, and let $p(x) \in R[x]$ be monic. Let a be a simple root of p modulo $\mathfrak{m}$. This root can be lifted to a unique root in R.

1.3 Wildly ramified extensions

Wildly ramified extensions are those where p divides the ramification index. We won't treat these in detail, since their structure won't be relevant to us; see [La02] for detail. In the tame case, extensions of the equal and mixed characteristic fields have the same structure (in the sense that their tame Galois groups are isomorphic). But in the wild case, the extensions start to look very different.

For one, $\mathbb{Q}_p$ is a perfect field, being characteristic zero, so all extensions are separable. For $F = \mathbb{F}_q((\pi))$ the field extensions $\mathbb{F}_q((\pi^{1/p}))$ are not separable.[6]

Moreover, we have some understanding of the degree p extensions.

Proposition 1.3.1. *The field $F = \mathbb{F}_q((\pi))$ has infinitely many extensions of degree p.*[7]

Proof. Consider the Artin-Schrierer operator

$$\zeta : F \to F, \qquad \zeta(x) = x^q - x$$

which is additive, $\mathbb{F}_q$-linear, with kernel $\ker(\zeta) = \mathbb{F}_q \subset F$. We will consider the splitting polynomials $p_\alpha(x) := \zeta(x) - \alpha$ for $\alpha \in K$. We have:

1. If $\alpha \in \mathrm{im}(\zeta)$, this splits in F (it has a root and its degree is prime).
2. By $\mathbb{F}_q$-linearity, p_α and $p_{u\alpha}$ for $u \in \mathbb{F}_q^\times$ have the same splitting field (i.e. $p_{u_\alpha}(x) = 0$ if and only if $p_\alpha(ux) = 0$).
3. By additivity, p_α and $p_{\alpha+\zeta(\beta)}$ have the same splitting field (i.e. $p_\alpha(x) = 0$ if and only if $p_{\alpha+\zeta(\beta)}(x + \beta) = 0$).

Thus we have an assignment

$$F/(\mathrm{im}(\zeta) \cdot \mathbb{F}_q) - \{0\} \longrightarrow \{\text{degree } p \text{ extensions}\}.$$

The main theorem of Aritin-Schrierer theory says that this map is an isomorphism. ☐

Proposition 1.3.2. *The field $\mathbb{Q}_p$ has finitely many extensions of degree p^n (in particular of any fixed degree).*

Proof. Kummer theory. ☐

1.4 Summary

In summary, for any field extension E/F, we have intermediate field extensions:

$$F \subset E_{ur} \subset E_t \subset E$$

where E_{ur}/F is *unramified*, E/E_{ur} is *totally ramified*, E_t/E_{ur} is *totally tamely ramified*, and E/E_t is *totally wildly ramified*. These can be understood universally in terms of the Galois group, i.e. we have quotients

$$\begin{tikzcd}
\Gamma_F \arrow[r, two heads] \arrow[d, two heads] & \Gamma_F^t \arrow[r, two heads] \arrow[d, two heads] & \Gamma_F^{ur} \arrow[r, two heads] \arrow[d, two heads] & \{1\} \arrow[d, equal] \\
\Gamma_{E/F} \arrow[r, two heads] & \Gamma_{E_t/F} \arrow[r, two heads] & \Gamma_{E_{ur}/F} \arrow[r, two heads] & \Gamma_{F/F}.
\end{tikzcd}$$

We call

$$I_F = \ker(\Gamma_F \to \Gamma_F^{ur}), \qquad P_F = \ker(\Gamma_F \to \Gamma_F^t)$$

the *inertia subgroup* and *wild inertia subgroup* respectively; they are not open, and do not contain any open subgroup. For a fixed finite extension E/F, we define $I_{E/F}, P_{E/F}$ to be the images in $\Gamma_F \twoheadrightarrow \Gamma_{E/F}$. We summarize the above discussion as follows.

[6] I.e. $p(x) = x^p - \pi = (x - \pi^{1/p})^p$.

[7] Reference: these notes by Brian Conrad.

Proposition 1.4.1. *We have a short exact sequence*

$$1 \longrightarrow I_F/P_F \simeq \prod_{\ell' \neq p} \mathbb{Z}_{\ell'} \longrightarrow \Gamma_F^t \simeq \Gamma_F/P_F \longrightarrow \Gamma_F^{ur} \simeq \Gamma_F/I_F \simeq \widehat{\mathbb{Z}} \longrightarrow 1.$$

Furthermore, P_F is a pro-p group. In particular, I_F is pro-solvable[8] *and P_F is its pro-nilpotent radical.*[9]

2 Langlands parameters

The field $\overline{\mathbb{Q}}_\ell$ is not a local field because its valuation v is not discrete (i.e. we've almost all roots of a uniformizer). It is also not complete. However, it still has a ring of integers $\mathcal{O}_E$ and maximal ideal $\mathfrak{m}_E$ defined

$$\mathcal{O}_E = \{x \in E \mid v(x) \geqslant 0\}, \qquad \mathfrak{m}_E = \{x \in E \mid v(x) > 0\}$$

and its residue field $\kappa_E = \overline{\mathbb{F}}_\ell$.

Recall a Langlands parameter should be something like a continuous map $\Gamma_F \to \check{G}(\overline{\mathbb{Q}}_\ell)$. We start with two basic results; one for passing from $\overline{\mathbb{Q}}_\ell$ to a finite extension $E/\mathbb{Q}_\ell$, and the other establishing that maps from a pro-p group to a pro-ℓ group factor through a finite quotient.

Proposition 2.0.1. *Consider a compact (Hausdorff) subgroup $K \subset GL_n(\overline{\mathbb{Q}}_\ell)$. There is a finite extension E/F such that $K \subset GL_n(E)$.*

Proof. We have

$$\bigcup_{E/F \text{ finite}} K \cap GL_n(E) = K$$

Now, K is a Baire space since it is compact Hausdorff, i.e. every increasing union of subsets with empty interior has empty interior. Since the set of extensions E/F is countable,[10], this means that K has empty interior in K, which cannot be. Thus, one of the $K \cap GL_n(E)$ must be nonempty interior.

Suppose $K \cap GL_n(E)$ has nonempty interior in K, i.e. K contains an open subgroup U; by translating it, we can make it cover K, and by compactness, we can take finitely many translates. That is, $K \cap GL_n(E)$ has finite index in K, so K lives in some finite extension. □

Proposition 2.0.2. *Let Γ be a pro-prime-to-ℓ group. Then, any continuous group homomorphism $\rho : \Gamma \to GL_n(\overline{\mathbb{Q}}_\ell)$ has finite image, i.e. factors through a finite quotient.*

Proof. Since Γ is pro-finite, it is compact. We use Proposition 2.0.1 and replace $\overline{\mathbb{Q}}_\ell$ with a finite extension $E/\mathbb{Q}_\ell$. Then, pick any open pro-ℓ subgroup $K \subset GL_n(E)$, so $\rho^{-1}(K)$ is an open subgroup of Γ. But $\mathring{K}$ is pro-ℓ, and any continuous group homomorphism from a pro-p group to a pro-ℓ-group is trivial. □

So we see the potential for non-finiteness comes from two $\mathbb{Z}_\ell$ in the unramified part $\mathbb{Z}_\ell$ and the tamely ramified part. We will deal with these two separately.

2.1 Galois group to Weil group, and Langlands parameters

Let's start with the unramified part. The fix here is "by hand" but somehow it's the "right" thing to do in Langlands. This story is a bit long to tell, but the rough idea is that various (global and local) Langlands correspondences are required to be compatible with *Hecke operators* under the *Satake isomorphism*:

$$\overline{\mathbb{Q}}_\ell[G(\mathcal{O}_F)\backslash G(F)/G(\mathcal{O}_F)] \simeq \mathcal{O}(\check{G}/\check{G}).$$

[8] I.e. for each quotient, its derived series terminates, or it has a central series with abelian quotients.

[9] It is nilpotent because every pro-p group is pro-nilpotent, i.e. every p-group is nilpotent. To see this, we build the ascending central series $\{1\} \subset Z_1 = Z(G) \subset Z(G/Z_1) \times_{G/Z_1} G \subset \cdots$, which terminates if every p-group has a nontrivial center. If $p \mid |G|$, then write $|G| = |Z(G)| + \sum |C_i|$ where C_i are non-central conjugacy classes to deduce $p \mid |Z(G)|$.

[10] This uses Krasner's lemma and I'll just take it on faith. It also follows from the stronger result in Kummer theory that there are only finitely many extensions of a given degree.

The left evidently has an interpretation as the spherical Hecke algebra for a $G(F)$. On the right, we expect the set of unramified Langlands parameters, i.e. maps $\Gamma_F \to \check{G}(\overline{\mathbb{Q}}_\ell)$ which are trivial on the inertia I_F. However, there is a mismatch: he affine quotient $\check{G}//\check{G}$ parameterizes *semisimple* conjugacy classes of $\check{G}$, while $\mathrm{Hom}_c(\widehat{\mathbb{Z}}, \check{G})/G$ parameterizes torsion conjugacy classes of $\check{G}$. One can imagine fixing this by: (1) replacing $\widehat{\mathbb{Z}}$ with $\mathbb{Z}$ and (2) imposing a semisimplicity condition on the image.

We first deal with the first issue, replacing $\widehat{\mathbb{Z}}$ by $\mathbb{Z}$.

Definition 2.1.1. The *Weil group* W_F of F is

$$W_F := \Gamma_F \times_{\widehat{\mathbb{Z}}} \mathbb{Z}$$

topologized so that the maps in the exact sequence

$$1 \longrightarrow I_F \longrightarrow W_F \longrightarrow \mathbb{Z} \longrightarrow 1$$

are continuous for the discrete topology on $\mathbb{Z}$ and the subspace topology on $I_F \subset \Gamma_F$.[11]

We now define Langlands parameters for a group G split over F.

Definition 2.1.2. Equip Γ_F with its canonical (pro-finite) topology and $\check{G}(\overline{\mathbb{Q}}_\ell)$ with the locally ℓ-adic topology inherted from $\overline{\mathbb{Q}}_\ell$. The set of *Langlands parameters* is

$$L(G) := \mathrm{Hom}_c^{ss}(W_F, \check{G}(\overline{\mathbb{Q}}_\ell))/\check{G}(\overline{\mathbb{Q}}_\ell).$$

i.e. continuous "Frobenius-semisimple" group homomorphisms such that any lift of Frobenius vanishing on maps to a semisimple element.

2.2 Tame monodromy and Weil-Deligne parameters

Some issues with the above definition: (1) we need to define semisimplicity, and (2) the geometry is unclear from this definition. We somewhat address the second problem and clarify some of the geometry by producing a generators-and-relations presentation of the tame part of W_F. We need to introduce a norm on Γ_F coming from local class field theory. There is a *local Artin homomorphism* $\theta : F^\times \to \Gamma_F/[\Gamma_F, \Gamma_F]$ which is an isomorphism after taking pro-finite completion of $F^\times$. The norm is defined to be the image under

$$\Gamma_F \twoheadrightarrow \Gamma_F/[\Gamma_F, \Gamma_F] \xrightarrow{\theta^{-1}} \widehat{F^\times} \xrightarrow{v} \mathbb{Z} \xrightarrow{q^\bullet} \mathbb{Q}.$$

The norm is trivial on I_F, and for any lift of Frobenius σ, we have $||\sigma|| = q$. We also define a norm on W_F in the same way.[12]

We define

$$\begin{tikzcd} I_F \arrow[r, "t"'] \arrow[rr, bend left, "t_\ell"] & I_F/P_F \simeq \prod_{\ell' \neq p} \mathbb{Z}_{\ell'} \arrow[r, two heads] & \mathbb{Z}_\ell \end{tikzcd}$$

i.e. the projection to the tame and pro-ℓ parts, and have the following relationship.

Proposition 2.2.1. *For $w \in \Gamma_F$ and $\tau \in I_F$, we have*

$$t(w\tau w^{-1}) = ||w|| t(\tau) = t(\tau^{||w||}),$$

$$t_\ell(w\tau w^{-1}) = ||w|| t_\ell(\tau) = t_\ell(\tau^{||w||}).$$

[11]Note this is not the subspace topology on $W_F \subset \Gamma_F$; if we took the subspace topology, we would simply be taking a dense subset, which makes no difference when mapping out. In particular, while Γ_F is compact, W_F is only locally compact.

[12]There appears to sometimes be a sign flip when defining the norm here. I'm not sure why, but I think I am accounting for it by an inverse in my definition for the Weil-Deligne group.

Proof. First, P_F is normal, so for $w \in P_F$ we have

$$t(\tau w \tau^{-1}) = t(w) = 1$$

and since $||P_F|| = 1$, this verifies the claim for $w \in P_F$. Likewise, since since the norm is trivial on I_F and I_F/P_F is abelian, we have the same for $w \in I_F$. We only need to check the case where $w = \sigma$ is a lift of Frobenius to Γ_F^t, and $\tau \in I_F/P_F \subset \Gamma_F^t$ of finite order n (automatically, $p \nmid n$), which are dense in I_F/P_F.

We view τ as an automorphism of the maximal tame extension F^t/F. Choose a uniformizer π of F, choose a fixed nth root $\pi^{1/n}$, and let ζ_n be the nth root of unity such that τ sends $\pi^{1/n} \mapsto \zeta_n \pi^{1/n}$. We know that $\sigma(\pi^{1/n})$ is also an nth root of $\sigma(\pi) = \pi$, so that $\sigma(\pi^{1/n})/\pi^{1/n}$ is an nth root of unity, which is fixed by $\tau \in I_F$. Thus, τ acts on $\sigma(\pi^{1/n})$ by the same ζ_n. Then we have

$$\sigma\tau(\pi^{1/n}) = \sigma(\zeta_n \pi^{1/n}) = \zeta_n^q \sigma(\pi^{1/n}) = \tau^q(\sigma(\pi^{1/n}))$$

as desired. □

Corollary 2.2.2. *There exists an embedding*

$$\iota : W_q := \langle \tau, \sigma \mid \sigma\tau\sigma^{-1} = \tau^q \rangle \hookrightarrow W_F^t$$

determined by a lift of Frobenius σ, and and a topological generator τ of I_F/P_F.

Definition 2.2.3. Fix an embedding ι as in Corollary 2.2.2. We have a short exact sequence

$$1 \longrightarrow P_F \to W_F \longrightarrow W_F^t \longrightarrow 1.$$

We let $W_F^\iota \subset W_F$ denote the preimage of $\iota : W_q \subset W_F^t$, equipped with a topology such the maps in the short exact sequence

$$1 \longrightarrow P_F \longrightarrow W_F^\iota \longrightarrow W_q \longrightarrow 1$$

are continous for the discrete topology on W_q.[13] We define the set of *ι-local Langlands parameters* to be the colimit

$$L^\iota(G) := \mathrm{Hom}_c^{ss}(W_F^\iota, \check{G}(\overline{\mathbb{Q}}_\ell))/\check{G}(\overline{\mathbb{Q}}_\ell) = \operatorname*{colim}_{E/F^t} \mathrm{Hom}^{ss}(W_{E/F}^\iota, \check{G}(\overline{\mathbb{Q}}_\ell))/\check{G}(\overline{\mathbb{Q}}_\ell)$$

of continuous "Frobenius-semisimple" parameters.

2.3 Semisimplicity and Weil-Deligne parameters

We now deal with the issue of Frobenius-semisimplicity using the notion of Weil-Deligne parameters. Naively, we want to require the Frobenius $1 \in \mathbb{Z}$ to act semisimply, but this doesn't make sense for quotients. It's also far too strong to require that lifts act semisimply, since this will be basically everything. Instead, we have some sense that $\mathbb{Z}_\ell$ is contributing the only unipotent part, while everything else acts like a finite group, so we want all lifts that do not involve $\mathbb{Z}_\ell \subset I_F/P_F$ to act semisimply. More precisely, we want to have the ability to "factor out" the nilpotent part, and define Frobenius-semisimplicity by requiring total semisimplicity for the other factor. To do this we need the Grothendieck ℓ-adic monodromy theorem. We define the matrix exponentials and logarithms for $E/\overline{\mathbb{Q}}_\ell$ a finite extension:

$$\exp(x) = \sum_{i=0}^{\infty} \frac{x^i}{i!},$$

$$\log(1+x) = \sum_{i=1}^{\infty} \frac{(-x)^i}{i}.$$

The series $\exp(x)$ converges on matrices with pro-nilpotent elements satisfying $v_E(x) > e/(\ell-1)$, the series $\log(1+x)$ converges on matrices with pro-nilpotent elements satisfying $v_E(x) > 0$, and the two are inverse equivalences where

[13]Following Proposition 2.0.2, this definition is essentially cooked up to make the topology irrelevant.

they are defined.[La94] We also note that exp is well-defined on any (literally, i.e. not pro-)nilpotent element.

Theorem 2.3.1 (Grothendieck ℓ-adic monodromy). *Let $E/\overline{\mathbb{Q}}_\ell$ be a finite extension, and $\rho : W_F \to GL_n(E)$ a continuous homomorphism.*[14] *There is an open*[15] *subgroup $U \subset I_F$ such that*

$$\rho|_U(w) = \exp(t_\ell(w)N)$$

for some uniquely determined nilpotent $N \in \mathfrak{gl}_n(E)$. Furthermore, for $w \in W_F$,

$$\rho(w)N\rho(w)^{-1} = ||w||N.$$

Proof. Since $\ker(t_\ell)$ is pro-p, ρ is trivial on an open subgroup of $\ker(t_\ell)$. Choose an open subgroup $U \subset I_F$ which restricts to it, and consider the factorization of U through

$$U/(U \cap \ker(t_\ell)) \simeq t_\ell(U) \simeq \ell^k\mathbb{Z}_\ell$$

since the compact open subgroups of $\mathbb{Z}_\ell$ are precisely $\ell^k\mathbb{Z}_\ell$. We need to show that $\rho \circ t_\ell$ is given by an exponential. Since I_F is compact, $\rho(I_F)$ is compact, and by a similar argument as in Proposition 2.0.1 $\rho(I_F)$ is contained in a compact open, thus up to conjugation $\mathrm{im}(\rho) \subset GL_n(\mathcal{O}_E)$. Intersect U with a pro-ℓ compact subgroup of $GL_n(\mathcal{O}_E)$ where the logarithm converges, so the map ρ can be written

$$\ell^k\mathbb{Z}_\ell \simeq t_\ell(U) \xrightarrow{\rho} 1 + \ell^j\mathfrak{gl}_n \xrightarrow{\log} \ell^j\mathfrak{gl}_n(\mathcal{O}_E).$$

Every such continuous map is determined by the value of the topological generator $1 \in \mathbb{Z}_\ell$, say $N \in \mathfrak{gl}_n(\mathcal{O}_E)$, and so the map must be $x \mapsto xN$ by uniqueness. For nilpotence, assume the formula; taking w to be a lift of Frobenius, we have the eigenvalues of N are the same as the eigenvalues for qN, i.e. N is nilpotent.

We now prove the formula: choose a nonzero $\tau \in U \cap \mathbb{Z}_\ell$, and apply ρ to the formula in Proposition 2.2.1:

$$\rho(w)\rho(t_\ell(\tau))\rho(w)^{-1} = \rho(t_\ell(\tau))^{||w||}$$

$$\rho(w)\exp(t_\ell(\tau)N)\rho(w)^{-1} = \exp(||w||t_\ell(\tau)N)$$

and apply log (shrinking U if necessary). □

Proposition 2.3.2. *Let ρ be a Langlands parameter. Choose $U \subset I_F$ open and N nilpotent as in Theorem 2.3.1, σ a lift of Frobenius, and define*

$$\rho^{ss}(\sigma^m\tau) = \rho(\sigma^m\tau)\exp(-t_\ell(\tau)N) \qquad \tau \in I_F.$$

Then, $\rho^{ss} : W_F \to \overline{\mathbb{Q}}_\ell$ is a homomorphism, continuous for the discrete topology on $\overline{\mathbb{Q}}_\ell$, and takes values in semisimple elements. Changing the lift of Frobenius gives an conjugate pair (ρ^{ss}, N).

Proof. We first show it is a homomorphism. We write $\sigma^m\tau\sigma^n\tau' = \sigma^{m+n}(\sigma^{-n}\tau\sigma^n)\tau'$, and

$$\begin{aligned}\rho^{ss}(\sigma^m\tau\sigma^n\tau') &= \rho(\sigma^{m+n}\sigma^{-n}\tau\sigma^n)\rho(\tau')\exp(-t_\ell(\sigma^{-n}\tau\sigma^n)N)\exp(-t_\ell(\tau')N)\\ &= \rho(\sigma^m\tau)\rho(\sigma^n\tau')\exp(-q^{-n}t_\ell(\tau)N)\exp(-t_\ell(\tau')N)\\ &= \rho(\sigma^m\tau)\exp(-q^nq^{-n}t_\ell(\tau)N)\rho(\sigma^n\tau')\exp(-t_\ell(\tau')N).\end{aligned}$$

Next, if we change the lift of Frobenius to σ' with $\sigma = \sigma'\tau'$ for $\tau' \in I_F$, then we have

$$(\sigma'\tau')^m = \sigma'\tau'\cdots\sigma'\tau' = \sigma'^m(\sigma^{-m-1}\tau'\sigma^{m+1})\cdots(\sigma^{-1}\tau'\sigma)\tau',$$

[14] This works for Γ_F as well.

[15] I.e. for the subspace topology of I_F, i.e. coming from a finite extension of E.

and that

$$t_\ell((\sigma^{-m-1}\tau'\sigma^{m+1})\cdots(\sigma^{-1}\tau'\sigma)\tau') = (q^{-m-1}+\cdots+q^{-1}+1)t_\ell(\tau') = \frac{q^{-m}-1}{q^{-1}-1}t_\ell(\tau').$$

Then we compute

$$\rho'^{ss}((\sigma'\tau')^m\tau) = \rho(\sigma^m\tau)\exp\left(-\frac{q^{-m}-1}{q^{-1}-1}t_\ell(\tau')N\right)\exp(-t_\ell(\tau)N).$$

We can try conjugating by $\exp(\alpha t_\ell(\tau')N)$:

$$\exp(\alpha t_\ell(\tau')N)\rho(\sigma^m\tau)\exp\left(-\frac{q^{-m}-1}{q^{-1}-1}t_\ell(\tau')N\right)\exp(t_\ell(\tau)N)\exp(-\alpha t_\ell(\tau')N)$$

$$= \rho(\sigma^m\tau)\exp(q^{-m}\alpha t_\ell(\tau')N)\exp\left(-\left(\frac{q^{-m}-1}{q^{-1}-1}+\alpha\right)t_\ell(\tau')N\right)\exp(t_\ell(\tau)N)$$

$$= \rho(\sigma^m\tau)\exp\left(-\left(\frac{q^{-m}-1}{q^{-1}-1}+\alpha(1-q^{-m})\right)t_\ell(\tau')N\right)\exp(t_\ell(\tau)N).$$

Evidently, we can take $\alpha = 1/(q^{-1}-1)$. Next, to see that ρ^{ss} has semisimple values, we note $\rho^{ss}|_{I_F}$ is trivial on U, so it has finite image, thus it has semisimple values in a group of characteristic zero and is continuous in the discrete topology. □

That is we are able to "factor"

$$\rho = \rho^{ss}\rho^u$$

into a "unipotent on I_F" part $\rho^u = \exp(t_\ell(-)N)$ (only defined on I_F) and a "semisimple on I_F" part ρ^{ss} (defined on all of W_F). In particular we can now define Frobenius-semisimplicity by requiring that ρ^{ss} is semisimple on all of W_F, not just I_F. This leads to the following definition.

Definition 2.3.3. Consider $\overline{\mathbb{Q}}_\ell$ with the discrete topology. The *Weil-Deligne group* is

$$WD_F := W_F \ltimes \mathbb{G}_a$$

where $w\cdot\lambda = ||w||^{-1}\lambda$, i.e. with group law $(w,\lambda)(w',\lambda') = (ww', ||w||\lambda+\lambda')$, and $||w||$ is obtained via local Artin reciprocity. The set of *Weil-Deligne parameters* is

$$WD(G) := \mathrm{Hom}_c^{ss}(WD_F, \check{G}(\overline{\mathbb{Q}}_\ell))/\check{G}(\overline{\mathbb{Q}}_\ell)$$

$$= \{(\rho^{ss}, N) \in \mathrm{Hom}_c^{ss}(W_F, \check{G}(\overline{\mathbb{Q}}_\ell)) \times \mathcal{N}_{\check{G}}(\overline{\mathbb{Q}}_\ell) \mid \psi(w)N\psi(w)^{-1} = ||w||N\}/\check{G}(\overline{\mathbb{Q}}_\ell)$$

where ρ^{ss} is *semisimple* in the sense that its image consists entirely of semisimple elements.

2.4 Moduli stack of Langlands parameters

Let Γ be a discrete group and G an affine algebraic group. The moduli stacks will make use of *representation schemes* $\mathrm{Hom}(\Gamma, G)$. These schemes have natural stacky enhancements via the adjoint G-action, and a re-interpretation in terms of moduli stacks of Betti local systems

$$\mathrm{Loc}_G(X) = \mathrm{Map}_{\mathbf{St}}(X, BG) = \mathrm{Hom}(\Gamma, G)/G$$

where X is any topological space (i.e. homotopy type, or anima) with $\pi_1(X) \simeq \Gamma$. These in turn have natural derived enhancements by taking derived mapping stacks of the Eilenberg-Maclane space

$$\mathrm{Loc}_G(K(\Gamma,1)) = \mathrm{Loc}_G(B\Gamma) = \mathrm{Map}_{\mathbf{DSt}}(B\Gamma, BG).$$

Alternatively, there are algebraic constructions due to Yuri Berest, Giovanni Felder, and Ajay Ramadoss under the monikers representation homology, derived representation schemes, et cetera.

Given a cell presentation of X, i.e. via pushouts diagrams involving n-spheres and disks, one can inductively compute $\mathrm{Loc}_G(X)$ by the formula

$$\mathrm{Loc}_G(\operatorname{colim} X_\alpha) = \mathrm{Map}(\operatorname{colim} X_\alpha, BG) = \lim \mathrm{Map}(X_\alpha, BG).$$

Let's do some examples.

1. The scheme $\mathrm{Hom}(\mathbb{Z}, G) = G$. There is no derived structure. The stacky version is $\mathrm{Loc}_G(S^1) = G/G$.

2. More generally, $\mathrm{Hom}(F_k, G) = G^n$, where F_k is the free group on k generators. The stacky version is $\mathrm{Loc}_G(\bigvee_k S^1) \simeq G^n/G$.

3. One can compute that $\mathrm{Loc}_G(S^n) = \Omega^{n-1}G$, i.e the based loop space at the identity, inductively using the presentation of S^n as S^{n-1} with two n-cells attached. For example, for S^2, we have

$$\begin{array}{ccc} \mathrm{Loc}_G(S^2) & \longrightarrow & \mathrm{Loc}_G(*) = \{e\}/G \\ \downarrow & & \downarrow \\ \mathrm{Loc}_G(*) = \{e\}/G & \longrightarrow & \mathrm{Loc}_G(S^1) = G/G. \end{array}$$

   The underlying scheme of $\Omega^{n-1}G$ is just a point (the identity), so this is all about derived structure.

4. The scheme $\mathrm{Hom}(\mathbb{Z}/n\mathbb{Z}, G)$ is the subscheme of n-torsion points of G. The underlying reduced scheme is discrete. Taking $n = 2$, the derived stacky version is $\mathrm{Loc}_G(\mathbf{RP}^\infty)$. Note that this is different from $\mathrm{Loc}_G(\mathbf{RP}^2)$, which is computed via its presentation using a 0-cell, a 1-cell, and a 2-cell:

$$\begin{array}{ccc} \mathrm{Loc}_G(\mathbf{RP}^2) & \longrightarrow & G/G \\ \downarrow & & \downarrow{\scriptstyle (-)^2} \\ \{e\}/G & \longrightarrow & G/G \end{array}$$

   Very often, $\mathrm{Loc}_G(\mathbf{RP}^2)$ is non-derived. However, to go from $\mathbf{RP}^2$ to $\mathbf{RP}^\infty$ requires attaching 1 cells in every higher dimension, which potentially introduces derived structure.

These representation schemes (or moduli stacks of local systems) can be defined for any pro-discrete group by taking colimits. We can also take Γ to be any algebraic group.

Definition 2.4.1. We define three versions of the moduli stack of Langlands parameters over various coefficient fields Λ. They are essentially the same as the three notions of Langlands parameters we have already defined, except that they fit into moduli stacks, and the Frobenius-semisimplicity condition is removed. We note that the Langlands dual group $\check{G}$ can be defined over $\mathbb{Z}$, thus any coefficient ring Λ.

1. $\mathrm{Loc}_{\check{G}}(F)$ is a stack over $\operatorname{Spec} \mathbb{Z}_\ell$, thus can be defined for any $\mathbb{Z}_\ell$-algebra (including $\overline{\mathbb{Q}}_\ell$). The general challenge to defining this due to continuity issues, and is handled in [Zh21] and [FS21].

2. $\mathrm{Loc}^\iota_{\check{G}}(F)$ is a stack over $\mathbb{Z}[1/p]$, but requires a choice of ι. It is:

$$\mathrm{Loc}^\iota_{\check{G}}(F) := \operatorname*{colim}_{E \supset F^t} \mathrm{Hom}(W^\iota_{E/F}, \check{G})/\check{G}.$$

   Note that $\mathrm{Hom}(\Gamma, G)$ for Γ a discrete group is always representable by a finite-type stack. The (open and closed, non-connected) substack of *tame Langlands parameters* is defined by taking $E = F^t$, i.e. requiring factorization through $W^\iota_{F^t/F} \simeq W^\iota_F/P_F \simeq W_q$:

$$\mathrm{Hom}(W_q, \check{G})/\check{G} = \{(\sigma, \tau) \in \check{G} \times \check{G} \mid \sigma\tau = \tau^q\sigma\} = \mathrm{Loc}_{\check{G}}(T_q)$$

   where T_q is the "q-twisted torus" obtained by gluing the ends of a cylinder by a degree q map. As long as $q \neq \pm 1$, one can show this stack has no derived structure.

3. $\mathrm{Loc}^{WD}_{\check{G}}(F)$ is a stack over $\mathbb{Q}$. For $E \supset F^{ur}$ we let $WD_{E/F} = W_{E/F} \ltimes \mathbb{G}_a$ (i.e. the norm vanishes on $\ker(W_F \to W_{E/F})$), and define

$$\mathrm{Loc}^{WD}_{\check{G}}(F) := \operatorname*{colim}_{E \supset F^{ur}} \mathrm{Hom}(WD_{E/F}, \check{G})/\check{G}.$$

One can define the (open and closed, connected) substack of *unipotent Langlands parameters* by taking $E = F^{ur}$, i.e. requiring factorization $W^{WD}_{E^{ur}/E} \simeq W^{WD}_F/(I_F \ltimes \mathbb{G}_a) \simeq \mathbb{Z} \ltimes \mathbb{G}_a$:

$$\mathrm{Hom}(\mathbb{Z} \ltimes \mathbb{G}_a, \check{G})/\check{G} = \mathrm{Loc}_{\check{G}}(S^1 \ltimes B\mathbb{G}_a).$$

Note that since Γ involves a $\mathbb{G}_a$, a priori this stack can have infinitesimal and derived structure. One can show that, essentially thanks to the twist by q (i.e. the semidirect product), that it does not.

We now state a few results from [Zh21] that establishes the compatibility between the above notions of moduli stacks.

Proposition 2.4.2. *The maps $\iota : W^{\iota}_F \to W_F$ induces natural isomorphisms*

$$\mathrm{Loc}_{\check{G}}(F) \xrightarrow{\simeq} \mathrm{Loc}^{\iota}_{\check{G}}(F) \otimes_{\mathbb{Z}[1/p]} \mathbb{Z}_\ell \qquad \mathrm{Loc}^{\iota}_{\check{G}}(F) \otimes_{\mathbb{Z}[1/p]} \mathbb{Q}_\ell \xleftarrow{\simeq} \mathrm{Loc}^{WD}_{\check{G}}(F) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

inducing isomorphisms

$$\mathrm{Loc}_{\check{G}}(F) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell \xrightarrow{\simeq} \mathrm{Loc}^{\iota}_{\check{G}}(F) \otimes_{\mathbb{Z}[1/p]} \mathbb{Q}_\ell \xleftarrow{\simeq} \mathrm{Loc}^{WD}_{\check{G}}(F) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell.$$

Furthermore, the ind-scheme $\mathrm{Loc}^{\iota}_{\check{G}}(F)$ is a disjoint union of classical reduced finite-type affine schemes, flat over $\mathbb{Z}[1/p]$, equidimensional of dimension $\dim(\check{G})$, a local complete intersection, and Calabi-Yau. Thus, the same is true of $\mathrm{Loc}_{\check{G}}(F)$ and $\mathrm{Loc}^{WD}_{\check{G}}(F)$ (over their respective coefficients).

Remark 2.4.3. The stack $\mathrm{Loc}^{\iota}_{\check{G}}(F)$ could have been defined over $\mathbb{Z}$. However, it does not give the "correct" answer when $\ell = p$ (in that the above proposition is compatible with the "right" definition of $\mathrm{Loc}_{\check{G}}(F)$ over $\mathbb{Z}_p$), so we define it over $\mathbb{Z}[1/p]$. Likewise, the stack $\mathrm{Loc}^{WD}_{\check{G}}(F)$ can be defined over $\mathbb{Z}$, but in order to relate it to any of the others we need the exponential map, so we define it over $\mathbb{Q}$.

References

[DKHM20] Jean-François Dat, David Helm, Robert Kurinczuk and Gilbert Moss, Moduli of Langlands Parameters. Preprint: https://arxiv.org/abs/2009.06708 (2020).

[FS21] Laurent Fargues and Peter Scholze, Geometrization of the local Langlands correspondence. Preprint: https://arxiv.org/abs/2102.13459 (2021).

[La94] Serge Lang, *Algebraic Number Theory.* 2nd edition, Graduate Texts in Mathematics Vol. 110, Springer (1994).

[La02] Serge Lang, *Algebra.* 3rd edition, Graduate Texts in Mathematics Vol. 211, Springer (2002).

[Se79] Jean-Pierre Serre, *Local Fields.* Graduate Texts in Mathematics Vol. 67, Springer (1979).

[Zh21] Xinwen Zhu. Coherent sheaves on the stack of Langlands parameters. Preprint: https://arxiv.org/abs/2008.02998 (2021).