Lectures on moduli of principal G-bundles over algebraic curves

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CONTENTS

1. Introduction

These notes are supposed to be an introduction to the moduli of G-bundles on curves. Therefore I will lay stress on ideas in order to make these notes more readable. My presentation of the subject is strongly influenced by the work of several mathematicians as Beauville, Laszlo, Faltings, Beilinson, Drinfeld, Kumar, Narasimhan and others.

In the last years the moduli spaces of G-bundles over algebraic curves have attracted some attention from various subjects like from conformal field theory or Beilinson and Drinfeld's geometric Langlands program [5]. In both subjects it turned out that the "stacky" point of view is more convenient and as the basic motivation of these notes is to introduce to the latter subject our moduli spaces will be moduli stacks (and not coarse moduli spaces). As people may feel uncomfortable with stacks I have included a small introduction to them. Actually there is a forthcoming book of Laumon and Moret-Bailly based on their preprint [15] and my introduction merely does the step -1, *i.e.* explains why we are forced to use them here and recalls the basic results I need later.

So here is the plan of the lectures: after some generalities on G-bundles, I will classify them topologically. Actually the proof is more interesting than the result as it will give a flavor of the basic theorem on G-bundles which describes the moduli stack as a double quotient of loop-groups. This "uniformization theorem", which goes back to A. Weil as a bijection on sets, will be proved in the section following the topological classification.

Then I will introduce two line bundles on the moduli stack: the determinant and the pfaffian bundle. The first one can be used to describe the canonical bundle on the moduli stack and the second to define a square-root of it. Unless G is simply connected the square root depends on the choice of a theta-characteristic. This square root plays an important role in the geometric Langlands program. Actually, in order to get global differential operators on the moduli stack one has to consider twisted differential operators with values in these square-roots.

The rest of the lectures will be dedicated to describe the various objects involved in the uniformization theorem as loop groups or the infinite Grassmannian in some more detail.

2. GENERALITIES ON PRINCIPAL G-BUNDLES

In this section I define principal G-bundles and recall the necessary background I need later. Principal G-bundles were introduced in their generality by Serre in Chevalley's seminar in 1958 [19] based on Weil's "espaces fibrés algébriques" (see remark 2.1.2).

2.1. Basic definitions.
Let Z be a scheme over an algebraically closed field k, G be an affine algebraic group over k.

2.1.1. Definition.
By a G-fibration over Z, we understand a scheme E on which G acts from the right and a G-invariant morphism $\pi : E \to Z$. A morphism between G-fibrations $\pi : E \to Z$ and $\pi' : E' \to Z$ is a morphism of schemes $\varphi : Z \to Z'$ such that $\pi = \pi' \circ \varphi$.

A G-fibration is *trivial* if it is isomorphic to $pr_1 : Z \times G \to Z$, where G acts on $Z \times G$ by $(z, g).\gamma = (z, g\gamma)$.

A *principal G-bundle* in the *Zariski*, resp. *étale*, resp. *fppf*, resp. *fpqc* sense is a G-fibration which is locally trivial in the Zariski, resp. étale, resp. fppf, resp. fpqc topology. This means that for any $z \in Z$ there is a neighborhood U of z such that $E_{|U}$ is trivial, resp. that there is an étale, resp. flat of finite presentation, resp. flat quasi-compact covering $U' \xrightarrow{\varphi} U$ such that $\varphi^*(E_{|U}) = U' \times_U E_{|U}$ is trivial.

2.1.2. *Remark.*
In the above definition, local triviality in the Zariski sense is the strongest whereas in the fpqc sense is the weakest condition. If G is smooth, then a principal bundle in the fpqc sense is even a principal bundle in the étale sense ([9], §6). In the following we will always suppose G to be smooth and we will simply call G-*bundle* a principal G-bundle in the étale sense. If $G = \mathbf{GL}_r$ or if Z is a smooth curve (see Springer's result in [22], 1.9), such a bundle is even locally trivial in the Zariski sense, but it was Serre's observation that in general it is not. He defined those groups for which local triviality in the Zariski sense implies always local triviality in the étale sense to be *special*. Then, for semi-simple G, Grothendieck (same seminar, some exposés later) classified the special groups: these are exactly the direct products of SL_r's and Sp_{2r}'s.

Remark that if the G-bundle E admits a section, then E is trivial. Define the following pointed (by the trivial bundle) set :

$$H^1_{\acute{e}t}(Z, G) = \{\text{G-bundles over } Z\}/\text{isomorphism}.$$

2.2. **Associated bundles.** If F is a quasi-projective k-scheme on which G acts on the left and E is a G-bundle, we can form $E(F) = E \times^G F$ the *associated bundle with fiber F*. It is the quotient of $E \times F$ under the action of G defined by $g.(e, f) = (e.g, g^{-1}f)$. The quasi-projectivity[1] of F is needed in order to assure that this quotient actually exists as a scheme.

There are two important cases of this construction.

2.2.1. *The associated vector bundle.* Let F be a vector space of dimension n. Suppose $G = \mathbf{GL}(F)$. Then G acts on F from the left and we can form for a G-bundle E the associated bundle $V = E(F)$. This is actually a vector bundle of rank n. Conversely, for any vector bundle V of rank n the associated frame bundle E (*i.e.* $\underline{\mathrm{Isom}}_{\mathcal{O}_Z}(\mathcal{O}_Z^n, V)$) is a GL_n-bundle.

2.2.2. *Extension of structure group.* Let $\rho : G \to H$ be a morphism of algebraic groups. Then G acts on H via ρ, we can form the *extension of the structure group* of a G-bundle E, that is the H-bundle $E(H)$. Thus, we have defined a map of pointed sets

$$H^1_{\acute{e}t}(Z, G) \longrightarrow H^1_{\acute{e}t}(Z, H)$$

Conversely, if F is an H-bundle, a *reduction of structure group* is a G-bundle E together with an isomorphism of G-bundles $\tau : E(H) \xrightarrow{\sim} F$.

2.2.3. Lemma. *Suppose $\rho : G \hookrightarrow H$ is a closed immersion. If F is an H-bundle, denote $F(H/G)$ simply by F/G. There is a natural one to one correspondence between sections $\sigma : Z \to F/G$ and reductions of the structure group of F to G.*

Proof. View $F \to F/G$ as a G-bundle and consider for $\sigma : Z \to F/G$ the pullback diagram

$$
\begin{array}{ccc}
\sigma^* F & \longrightarrow & F \\
{\scriptstyle G}\big\downarrow & & \big\downarrow{\scriptstyle G} \\
Z & \xrightarrow{\ \sigma\ } & F/G
\end{array}
$$

which defines the requested reduction of the structure group. $\qquad\square$

---

[1]In fact it is enough to suppose that F satisfies the property that any finite subset of F lies in an affine open subset of F.

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2.3. **G-bundles on a curve.** Let X be a smooth and connected curve. By the above quoted theorem of Springer ([22], 1.9), all G-bundles over X are locally trivial in the Zariski topology, so the reader might ask why I insisted on the étale topology in the above definition. The reason is that in order to study G-bundles on X, we will study families of G-bundles parameterized by some k-scheme S. By definition, these are G-bundles on $X_S = X \times S$, and here is where we need the étale topology.

A warning: it is not a good idea to define families point-wise. Let's look at the example of $\mathbf{O}_r$. Then we may view (considering $\mathbf{O}_r \subset \mathbf{GL}_r$ and using Lemma 2.2.3) an $\mathbf{O}_r$-bundle as a vector bundle E together with an isomorphism $\sigma : E \to E^*$ such that $\sigma = \sigma^*$ (I denote here and later the transposed map of σ by σ^*). The point is as follows. If E is a vector bundle over X_S together with an isomorphism $\sigma : E \to E^*$ such that for all closed point $s \in S$ the induced pair (E_s, σ_s) is an $\mathbf{O}_r$-bundle, this does not imply in general that (E, σ) itself is an $\mathbf{O}_r$-bundle.

3. Algebraic stacks

3.1. **Motivation.** Given a moduli problem such as classifying vector bundles over a curve, there are essentially two approaches to its solution: coarse moduli spaces and algebraic stacks. The former, introduced by Mumford, are schemes and are constructed, after restricting to a certain class of objects such as semi-stable bundles in the above example, as quotients of some parameter scheme by a reductive group using geometric invariant theory. However they do not - in general - carry a universal family and may have artificial singularities coming from the quotient process in their construction. So in order to construct objects on the coarse moduli space, one considers generally first the parameter space (which carries a universal family) and then tries to descend the constructed object to the moduli space which might be tricky or impossible.

In our case of the geometric Langlands program a special line bundle on the moduli space (*i.e.* a certain square root of the dualising sheaf) will play a particular role. However, it can be shown, that even if there is a functorial construction of this line bundle (hence a line bundle on the parameter scheme), it does *not* - for general G - descend to the coarse moduli space of semi-stable G-bundles.

It turns out, for this and other reasons, that in order to study the questions related to the geometric Langlands program, one has to consider the latter, *i.e.* the "stacky" solution to the moduli problem. So in my lectures I will

concentrate on the *moduli stack* of principal G-bundles and as there are not many references for stacks for the moment, I will recall in this section the ideas and properties of stacks I need in order to properly state and prove the basic results for the program.

3.1.1. *The moduli problem.* The basic moduli problem for G-bundles on a projective, connected, and smooth curve X/k is to try to represent the functor which associates to a scheme S/k the set of isomorphism classes of families of G-bundles parameterized by S:

$$\underline{\mathcal{M}_{G,X}} : (\mathbf{Sch}/k)^{op} \longrightarrow \mathbf{Set}$$

$$S \mapsto \left\{ \begin{array}{c} E \\ \downarrow {\scriptstyle G} \\ S \times X \end{array} \right\} / \sim$$

Now, as G-bundles admit in general non trivial automorphisms (the automorphism group of a G-bundle contains the center of G), we can't expect to be able to solve the above problem, *i.e.* find a scheme M that represents the above functor. Loosely speaking, if it would exist we should be able, given any morphism φ from any scheme S to M, to recover uniquely a family E parameterized by S such that the map defined by $s \mapsto [E_s]$ defines the morphism φ. As this should in particular apply to the closed points $\mathrm{Spec}(k) \in M$, the above translates that not only for every G-bundle one is able to choose an element in its isomorphism class with the property that this choice behaves well under families, but also that there is only one such choice with this property. This clearly is an obstruction which makes the existence of M unlikely and can be turned into a rigorous argument.

However, I will not do this here, but rather discuss the first non trivial case of $G = k^*$, *i.e.* the case of rank 1 vector bundles. Then a possible candidate for M is the jacobian $J(X)$. We know that $J(X)$ parameterizes isomorphism classes of line bundles on X and that there is a Poincaré bundle P on $J(X) \times X$. Hence we get, for every $j \in J(X)$, a canonical element in the isomorphism class it represents, namely P_j, and this choice is compatible with families (pullback P to the family). The point in this example is that this choice is not unique as $P \otimes pr_1^*(A)$ is also a Poincaré bundle for $A \in \mathrm{Pic}(J(X))$.

Actually what we can do here is to consider a slightly different functor, by fixing a point $x \in X$ and looking at pairs (L, α) of line bundles together with an isomorphism $\alpha : L_x \xrightarrow{\sim} k$. Such a choice determines uniquely a Poincaré bundle P and $J(X)$ (together with P) actually represents the functor defined

by such pairs. The above process of adding structure to the functor in order
to force the automorphism group of the considered objects to be trivial is
sometimes called to "rigidify" the functor.

Let us return to our original moduli problem. As I explained above, the
main problem is the existence of non trivial automorphisms and there is
nothing much we can do about this, without adding additional structures
which may be complicated in the general case and definitively changes the
moduli problem. Grothendieck's idea to avoid the difficulties posed by the
existence of these non trivial automorphisms is simple : keep them. However,
as we will see, carrying out this idea is technically quite involved.

So how to keep the automorphisms? If we do not want to mod out the
automorphisms what we can do is to replace the set of automorphisms classes
of G-bundles over $S \times X$ by the category which has as objects such bundles
and as morphisms the isomorphisms between them.

By definition, the categories we obtain have the property that all arrows
are invertible; categories with this property are called groupoids. In the
following the category of groupoids will be denoted by **Gpd**. It will be con-
venient to write groupoids and categories in the form {objects} + {arrows}.

Applying the above idea to our moduli problem gives a "functor"

$$\mathcal{M}_{G,X} : \mathbf{Sch}/k^{op} \longrightarrow \mathbf{Gpd}$$
$$S \longmapsto \{G\text{-bundles on } X \times S\} +$$
$$\{\text{isomorphisms of } G\text{-bundles on } X \times S\}$$

Actually this is not really a functor as before, but a broader object, called
a "lax functor": if $f : S' \to S$ is a morphism of k-schemes the pullback
defines a functor $f^* : \mathcal{M}_{G,X}(S) \to \mathcal{M}_{G,X}(S')$. If $g : S'' \to S'$ is another
morphism of k-schemes we get two functors from $\mathcal{M}_{G,X}(S)$ to $\mathcal{M}_{G,X}(S'')$,
namely the composite functor $g^* \circ f^*$ and the functor of the composition
$(f \circ g)^*$. However it does not really make sense to talk about "equality" of
functors here but rather about isomorphisms between them. In our example,
there is a canonical isomorphism

$$\alpha_{g,f} : (f \circ g)^* \xrightarrow{\sim} g^* \circ f^*$$

between them and these satisfy the usual co-cycle condition (see sections
(3.3) and (3.4.1) for a more precise discussion).

What we see in the example is that we get a "2-categorical" gadget: we
have categories (instead of sets), morphisms between categories (instead of

maps between sets), and natural transformations between morphisms (this is new).

3.1.2. *The quotient problem.* Suppose that the linear group H acts on the scheme Z. Suppose moreover that the action is free. Then Z/H exists as a scheme and the quotient morphism $\pi : Z \to Z/H$ is actually an H-bundle. What are the points of Z/H? If $S \xrightarrow{f} Z/H$, we get a cartesian diagram

$$
\begin{array}{ccc}
Z' & \xrightarrow{\ \alpha\ } & Z \\
{\scriptstyle H}\downarrow & & \downarrow{\scriptstyle H} \\
S & \xrightarrow{\ f\ } & Z/H
\end{array}
$$

So f defines an H-bundle $Z' \xrightarrow{H} S$ and an H-equivariant morphism α. If the action of H is not free, the quotient Z/H does not, in general, exist as a scheme, however what we can do is to consider the following lax functor

$$[Z/H] : \mathbf{Sch}/k^{op} \longrightarrow \mathbf{Gpd}$$

$$S \longmapsto \{(Z', \alpha) \ /Z' \xrightarrow{H} S \text{ is a } H\text{-bundle and } \alpha : Z' \xrightarrow{\alpha} Z$$

$$\text{is a } H\text{-equivariant morphism}\} +$$

$$\{\text{isomorphisms of pairs}\}$$

This definition makes sense for any action of H on Z and the "quotient map" $Z \longrightarrow [Z/H]$ (we will see in a moment what this means) behaves like an H-bundle.

3.1.3. The idea then is to *define* a stack exactly as such lax functors, after imposing some natural topological conditions on them. Of course this may seem to be somehow cheating but Grothendieck showed us that one can actually do geometry with a certain class of such stacks which he called algebraic.

After the above motivation, the plan for the rest of the section is:

- Grothendieck Topologies
- k-spaces and k-stacks
- Descent
- Algebraic stacks

3.2. **Grothendieck Topologies.** Sometimes in algebraic geometry we need to use topologies which are finer than the Zariski topology, especially when interested in an analogue of the inverse function theorem. Over $\mathbb{C}$, there is the classical topology, although using it leads to worries about the algebraicity of analytically defined constructions. Otherwise one has to use a Grothendieck topology such as the étale topology.

A Grothendieck topology is a topology on a category. The category might be similar to the category $\mathbf{Zar}(Z)$ of Zariski open sets of a k-scheme Z, or it might be an ambient category like $\mathbf{Sch}/k$ or $\mathbf{Aff}/k$. Grothendieck topologies are most intuitively described using covering families, which describe a basis or a pretopology for the topology.

3.2.1. *Covering families.* In this approach a *Grothendieck topology* (or pretopology) on a category C with fiber products is a function T which assigns to each object U of C a collection $T(U)$ consisting of families $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ of morphisms with target U such that

- if $U' \to U$ is an isomorphism, then $\{U' \to U\}$ is in $T(U)$;
- if $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ is in $T(U)$, and if $U' \to U$ is any morphism, then the family $\{U_i \times_U U' \to U'\}_{i \in I}$ is in $T(U')$;
- if $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ is in $T(U)$, and if for each $i \in I$ one has a family $\{V_{ij} \to U_i\}_{j \in I_i}$ in $T(U_i)$, then $\{V_{ij} \to U_i \to U\}_{i \in I, j \in I_j}$ is in $T(U)$.

The families in $T(U)$ are called *covering families* for U in the T-topology.

A *site* is a category with a Grothendieck topology.

3.2.2. *Small sites.* Let's look at some examples:

(i) If Z is a k-scheme consider the category of Zariski open subsets of Z. A family $\{U_i \subset U\}_{i \in I}$ is a covering family for U if $\bigcup_{i \in I} U_i = U$. The resulting site is the *small Zariski site* or Zariski topology on Z written Z_{Zar}.

(ii) If Z is a k-scheme, let $\mathbf{Et}/Z$ be the category whose objects are étale maps $U \to Z$ and whose morphisms are étale maps $U' \to U$ compatible with the projections to Z. A family $\{U_i \to U\}_{i \in I}$ is a covering family if the union of the images of the U_i is U (such a family is called a surjective family). This is the *small étale site* or étale topology on Z written $Z_{\text{ét}}$.

(iii) Replacing "étale" by "smooth" gives a topology on $\mathbf{Smooth}/Z$ called the smooth topology. The small smooth site on a scheme is Z_{sm}. Using "flat of finite presentation" gives the *fppf topology* and a small

site Z_{fppf}. The letters "fppf" stand for "fidèlement plat de présentation finie." There are also letters "fpqc" standing for "fidèlement plat et quasi-compact." Intuitively, each of these successive topologies is finer than the previous one because there are more open sets.

3.2.3. *Big sites.* One can also define a topology on all schemes at once. The category $\mathbf{Sch}/k$ of all k-schemes may be given the Zariski, étale, smooth, fppf, and fpqc topologies. In these topologies the covering families of a scheme U are surjective families $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ of, respectively, inclusions of open subschemes, étale maps, smooth maps, flat maps of finite presentation, and flat quasi-compact maps. Each successive topology has more covering families than the previous one and so is finer.

One can do the same thing to the category $\mathbf{Aff}/k$ of affine k-schemes.

3.2.4. *Sheaves.* A presheaf of sets on a category C with a Grothendieck topology (of covering families) is a functor $F : C^{\mathrm{op}} \to \mathbf{Set}$. A presheaf is separated if for all objects U in C, all $f, g \in F(U)$, and all covering families $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ of U in the topology, the condition $f|_{U_i} = g|_{U_i}$ for all i implies $f = g$. A presheaf is a sheaf if it is separated and in addition, whenever one has a covering family $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ in the topology and a system $\{f_i \in F(U_i)\}_{i \in I}$ such that for all i, j, one has $F(p_{i,j}^1)(f_i) = F(p_{i,j}^2)(f_j)$ in $F(U_i \times_U U_j)$, then there exists an $f \in F(U)$ such that $f|_{U_i} = f_i$ for all i. A compact way to say the above is to say that $F(U)$ is the kernel of the following double arrow

$$\prod_{i \in I} F(U_i) \underset{F(p_{i,j}^2)}{\overset{F(p_{i,j}^1)}{\rightrightarrows}} \prod_{i,j} F(U_i \times_U U_j)$$

3.3. k-spaces and k-stacks. By a k-*space* (resp. k-*group*) we understand a sheaf of sets (rep. groups) over the big site $(\mathbf{Aff}/k)_{fppf}$. A *lax functor* $\mathfrak{X}$ from $\mathbf{Aff}/k^{op}$ to $\mathbf{Gpd}$ associates to any $U \in \mathrm{ob}(\mathbf{Aff}/k)$ a groupoid $\mathfrak{X}(U)$ and to every arrow $f : U' \to U$ in $\mathbf{Aff}/k$ a functor $f^* : \mathfrak{X}(U) \to \mathfrak{X}(U')$ together with isomorphisms of functors $g^* \circ f^* \simeq (f \circ g)^*$ for every arrow $g : U'' \to U'$ in $\mathbf{Aff}/k$. These isomorphisms should satisfy the following compatibility relation: for $h : U''' \to U''$ the following diagram commutes

$$
\begin{array}{ccc}
h^* \circ g^* \circ f^* & \xrightarrow{\ \sim\ } & h^*(f \circ g)^* \\
\wr \downarrow & & \downarrow \wr \\
(g \circ h)^* f^* & \xrightarrow{\ \sim\ } & (f \circ g \circ h)^*
\end{array}
$$

If $x \in \mathrm{ob}(\mathfrak{X}(U))$ and $f : U' \to U$ it is convenient to denote $f^* x \in \mathrm{ob}(\mathfrak{X}(U'))$ by $x_{|U'}$. A lax functor will be called a k-*stack* if it satisfies the following two topological properties:

(i) for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $x, y \in \mathrm{ob}(\mathfrak{X}(U))$ the presheaf

$$\underline{\mathrm{Isom}}(x, y) : \mathbf{Aff}/U \longrightarrow \mathbf{Set}$$
$$(U' \to U) \longmapsto \mathrm{Hom}_{\mathfrak{X}(U')}(x_{|U'}, y_{|U'})$$

is a sheaf (with respect to the *fppf* topology on $\mathbf{Aff}/U$).

(ii) Every descent datum is effective.

Recall that a *descent datum* for $\mathfrak{X}$ for a covering family $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ is a system of the form $(x_i, \phi_{ji})_{i,j \in I}$ with the following properties: each x_i is an object of $\mathfrak{X}(U_i)$, and each $\phi_{ji} : x_{i|U_{ji}} \to x_{j|U_{ji}}$ is an arrow in $\mathfrak{X}(U_{ji})$. Moreover, we have the co-cycle condition

$$\phi_{ki|U_{kji}} = \phi_{kj|U_{kji}} \circ \phi_{ji|U_{kji}}$$

where $U_{ji} = U_j \times_U U_i$ and $U_{kji} = U_k \times_U U_j \times_U U_i$, for all i, j, k.

A descent datum is *effective* if there exists an object $x \in \mathfrak{X}(U)$ and invertible arrows $\phi_i : x|\, U_i \xrightarrow{\sim} x_i$ in $\mathfrak{X}(U_i)$ for each i such that

$$\phi_{j|U_{ji}} = \phi_{ji} \circ \phi_{i|U_{ji}}$$

for all $i, j \in I$.

Any k-space X may be seen as a k-stack, by considering a set as a groupoid (with the identity as the only morphism). Conversely, any k-stack $\mathfrak{X}$ such that $\mathfrak{X}(R)$ is a discrete groupoid (*i.e.* has only the identity as automorphisms) for all affine k-schemes U, is a k-space.

3.3.1. Example. (The quotient stack) Let us consider again the quotient problem of (3.1.2), in the more general setup of a k-group Γ acting on a k-space Z, which we will actually need in the sequel. The quotient stack $[Z/\Gamma]$ is defined as follows. Let $U \in \mathrm{ob}(\mathbf{Aff}/k)$. The objects of $[Z/\Gamma](U)$ are pairs (Z', α) where Z' is a Γ-bundle over U and $\alpha : Z' \to Z$ is Γ-equivariant, the arrows are defined in the obvious way and so are the functors $[Z/\Gamma](U) \to [Z/\Gamma](U')$.

3.4. **Morphisms.** A *1-morphism* $F : \mathfrak{X} \to \mathfrak{Y}$ will associate, for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$, a functor $F(U) : \mathfrak{X}(U) \to \mathfrak{Y}(U)$ and for every arrow $U' \xrightarrow{f} U$ an

isomorphism of functors $\alpha(f) : f_{\mathfrak{X}}^* \circ F(U') \xrightarrow{\sim} F(U) \circ f_{\mathfrak{Y}}^*$

$$
\begin{array}{ccc}
\mathfrak{X}(U) & \xrightarrow{\ F(U)\ } & \mathfrak{Y}(U) \\
{\scriptstyle f_{\mathfrak{X}}^*}\Big\downarrow & {\scriptstyle \alpha(f)} & \Big\downarrow{\scriptstyle f_{\mathfrak{Y}}^*} \\
\mathfrak{X}(U') & \xrightarrow[\ F(U')\ ]{} & \mathfrak{Y}(U')
\end{array}
$$

satisfying the obvious compatibility conditions: (i) if $f = 1_U$ is an identity, then $\alpha(1_U) = 1_{F(U)}$ is an identity and (ii) if f and g are composable, then $F(gf)$ is the composite of the squares $\alpha(f)$ and $\alpha(g)$ further composed with the composition of pullback isomorphisms $g^* \circ f^* \simeq (f \circ g)^*$ for $\mathfrak{X}$ and $\mathfrak{Y}$ (I will not draw the diagram here).

A *2-morphism* between 1-morphisms $\phi : F \to G$ associates for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$, an isomorphism of functors $\phi(U) : F(U) \to G(U)$:

$$
\mathfrak{X}(U) \underset{G(U)}{\overset{F(U)}{\underset{\phi(U)\|}{\rightrightarrows}}} \mathfrak{Y}(U)
$$

There is an obvious compatibility condition which I leave to the reader.

3.4.1. *Remark.* The above definitions of 1- and 2-morphisms make sense for any lax functor. The compatibility conditions, which will be automatically satisfied in our examples, may seem complicated, however can't be avoided with this approach. The point is that typically in nature the pullback objects f^*x for every $x \in \mathrm{ob}(\mathfrak{X})(U)$ and $U' \xrightarrow{f} U$ are well defined up to isomorphism, but that the actual object f^*x is arbitrary in its isomorphism class. Let's have a closer look at our example $\mathcal{M}_{G,X}$. In this case taking the pullback $(f \times \mathrm{id})^* E$ of a G-bundle E on $X \times U$ to $X \times U'$ corresponds to take a tensor product. This is well defined up to canonical isomorphism (it is the solution of a universal problem) and we are so used to choose an element in its isomorphism class that we generally (and safely) forget about this choice. However, when comparing the functors $g^* \circ f^*$ and $(f \circ g)^*$ this choice comes up inherently and we get only something very near to "equality" namely a canonical isomorphism of functors. So once we see that our functors are only lax (as opposed to strict) in general we see that we have to choose these isomorphisms of functors in the definitions and then all sorts of compatibility conditions pop up naturally.

There is another, less intuitive but more intrinsic approach to lax functors using *k-groupoids*. This is an essentially equivalent formalism which avoids the choice of a pullback object, hence reduces the compatibility conditions. As this is the point of view of [15], I will describe briefly the relation between the two (which may also help to facilitate the reading of the first chapter of [15]). I start with a lax functor $\mathfrak{X} : (\mathbf{Aff}/k)^{op} \to \mathbf{Gpd}$ to which I will associate a category $\underline{\mathfrak{X}}$ together with a functor $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ (actually I should denote $\underline{\mathfrak{X}}$ by $\mathfrak{X}$ as well, but here I want to distinguish the two). The objects of $\underline{\mathfrak{X}}$ are

$$\coprod_{U \in \mathrm{ob}(\mathbf{Aff}/k)} \mathrm{ob}\, \mathfrak{X}(U)$$

the morphisms going from $x \in \mathrm{ob}\,\mathfrak{X}(U)$ to $y \in \mathrm{ob}\,\mathfrak{X}(V)$ are pairs (α, f) with $f : U \to V$ an arrow in $(\mathbf{Aff}/k)$ and α an arrow in $\mathfrak{X}(U)$ from x to f^*y. A convenient way to encode these pairs is as follows[2]

$$x \xrightarrow{\quad\alpha\quad} f^*y \dashrightarrow^{f} y$$

With these notations, the composite of two arrows

$$x \xrightarrow{\quad\alpha\quad} f^*y \dashrightarrow^{f} y \xrightarrow{\quad\beta\quad} g^*z \dashrightarrow^{g} z$$

is defined to be

$$x \xrightarrow{\quad\alpha\quad} f^*y \xrightarrow{\quad f^*\beta\quad} f^*g^*z \xrightarrow{\quad\sim\quad} (gf)^*z \dashrightarrow^{gf} z$$

The functor ϕ is defined to send an object of $\mathfrak{X}(U)$ to U and an arrow (α, f) to f. Looking at $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ we see that the categories $\mathfrak{X}(U)$ are the *fiber categories* $\underline{\mathfrak{X}}_U$ with objects the objects x of $\underline{\mathfrak{X}}$ such that $\phi(x) = U$ and arrows the arrows f of $\underline{\mathfrak{X}}$ such that $\phi(f) = 1_{|U}$.

The functor $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ satisfies the following two properties (exercise: prove this)

(i) for every arrow $U' \xrightarrow{f} U$ in $(\mathbf{Aff}/k)$ and every object x in $\underline{\mathfrak{X}}_U$, there is an arrow $y \xrightarrow{u} x$ in $\underline{\mathfrak{X}}$ such that $\phi(u) = f$

(ii) for every diagram $z \xrightarrow{v} x \xleftarrow{u} y$ in $\underline{\mathfrak{X}}$ with image $U'' \xrightarrow{f} U \xleftarrow{g} U'$ in $(\mathbf{Aff}/k)$ there is for every arrow $U'' \xrightarrow{h} U'$ such that $f = gh$ a unique arrow $z \xrightarrow{w} y$ such that $u = vw$ and $\phi(w) = h$.

---

[2]I learned this from Charles Walter's lectures on stacks in Trento some years ago.

A functor $\phi : \underline{\mathfrak{X}} \to (\mathbf{Aff}/k)$ satisfying (i) and (ii) is called a k-groupoid in [15]. So a lax functor defines a k-groupoid. On the other hand, given a k-groupoid we can define a lax functor as follows. To every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ we associate the fiber category $\underline{\mathfrak{X}}_U$. If $U' \xrightarrow{f} U$ is an arrow in $(\mathbf{Aff}/k)$ and $x \in \mathrm{ob}\,\underline{\mathfrak{X}}_U$ then by (i) we know that there is $y \xrightarrow{u} x$ in $\underline{\mathfrak{X}}$. From (ii) it follows that $y \xrightarrow{u} x$ is unique up to isomorphism. Now we *choose* – once and for all – for every f and x such an arrow $y \xrightarrow{u} x$ which we denote by $f^*x \xrightarrow{u} x$. Moreover, for every arrow $x' \xrightarrow{u} x$ in $\underline{\mathfrak{X}}$, we denote by $f^*(u)$ the unique arrow which make the following diagram commutative

$$
\begin{array}{ccc}
f^*x' & \!\!\!\!\!\longrightarrow\!\!\!\!\! & x' \\
{\scriptstyle f^*(u)}\Big\downarrow & & \Big\downarrow{\scriptstyle u} \\
f^*x & \!\!\!\!\!\longrightarrow\!\!\!\!\! & x
\end{array}
$$

We get a functor $f^* : \underline{\mathfrak{X}}_U \to \underline{\mathfrak{X}}_{U'}$, and also for $U'' \xrightarrow{g} U' \xrightarrow{f} U$ an isomorphism of functors $g^* \circ f^* \xrightarrow{\sim} (f \circ g)^*$ satisfying the conditions of a lax functor.

For k-groupoids most of the basic definitions such as 1- and 2-morphisms are more elegant: a 1-morphism is a functor $F : \mathfrak{X} \to \mathfrak{Y}$ strictly compatible with the projection to $(\mathbf{Aff}/k)$; the 2-morphisms are the isomorphisms of 1-morphisms.

3.5. Descent. The word "descent" is just another name for gluing appropriate for situations in which the "open sets" are morphisms (as in the étale topology) rather inclusions of subsets (as in the Zariski topology). The basic descent theorem says that morphisms of schemes can be "glued" together in the flat topology if they agree on the "intersections". The same applies to flat families of quasi-coherent sheaves. Having the notion of a sheaf and a stack to our disposition, faithfully flat descent can be stated as follows:

Theorem. Faithfully flat descent ([SGA 1], VIII 5.1, 1.1 and 1.2):

(i) (Faithfully flat descent for morphisms) For any k-scheme Z the functor of points
$\mathrm{Hom}_{(\mathbf{Aff}/k)}(-, Z) : (\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Set}$ is a k-space.

(ii) (Faithfully flat descent for flat families of quasi-coherent sheaves) For any scheme Z, the lax functor $(\mathbf{Aff}/k)^{\mathrm{op}} \to \mathbf{Gpd}$ defined by

$S \mapsto \{\text{quasi-coherent } \mathcal{O}_{Z \times_k S}\text{-modules flat over} S\} + \{\text{isomorphisms}\}$

is a k-stack.

Other descent results can be derived from these two. For instance, faithfully flat descent for principal G-bundles follows from (ii), *i.e.* the lax functor $\mathcal{M}_{G,X}$ of (3.1.1) is a k-stack.

3.6. Algebraic stacks. I now come to the definition of an algebraic stack, then I will show in the next section that our k-stack $\mathcal{M}_{G,X}$ is actually algebraic.

3.6.1. *The fiber of a morphism of stacks.* Fiber products exist in the category of k-stacks. I will not define them here, but rather explain what is the fiber of a morphisms of stacks, as this is all I need here. Let $F : \mathfrak{X} \to \mathfrak{Y}$ be a morphisms of stacks, let $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and consider a morphism $\eta : U \to \mathfrak{Y}$, that is an object η of $\mathfrak{Y}(U)$. The *fiber* $\mathfrak{X}_\eta$ is the following stack over U:

$$\mathfrak{X}_\eta : \mathbf{Aff}/U \longrightarrow \mathbf{Gpd}$$

$$(U' \to U) \longmapsto \{(\xi, \alpha) \ / \ \xi \in \mathrm{ob}(\mathfrak{X})(U'), \alpha : F(\xi) \xrightarrow{\sim} \eta_{|U'}\} +$$

$$\{(\xi, \alpha) \xrightarrow{f} (\xi', \alpha') \ / \ \xi \xrightarrow{f} \xi' \text{ s.t. } \alpha \circ F(f) = \alpha'\}$$

3.6.2. *Representable morphisms.* The morphism F is *representable* if $\mathfrak{X}_\eta$ is representable as a scheme for all $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $\eta \in \mathrm{ob}\,\mathfrak{Y}(U)$, *i.e.* "the fibers are schemes". All properties P of morphisms of schemes which are stable under base change and of local nature for the *fppf* topology make sense for representable morphisms of stacks. Indeed, one *defines* F to have P if for every $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and every $\eta \in \mathrm{ob}(Y(\eta))$ the canonical morphism of schemes $\mathfrak{X}_\eta \to U$ has P. Examples of such properties are flat, smooth, surjective, étale, etc. ; the reader may find a quite complete list in [15].

3.6.3. *Definition.* A k-stack $\mathfrak{X}$ is *algebraic* if
  (i) the diagonal morphism $\mathfrak{X} \to \mathfrak{X} \times \mathfrak{X}$ is representable, separated and quasi-compact
  (ii) there is a k-scheme P and a smooth, surjective morphism $P \xrightarrow{p} \mathfrak{X}$.

Actually the representability of the diagonal is equivalent to the following: for all $U \in \mathrm{ob}(\mathbf{Aff}/k)$ and all $\eta \in \mathrm{ob}\,\mathfrak{Y}(U)$ the morphism of stacks $U \xrightarrow{\eta} \mathfrak{X}$ is representable. Hence (i) implies that p is representable (and so smoothness and surjectivity of p make sense)

Suppose $F : \mathfrak{X} \to \mathfrak{Y}$ is a *representable* morphism of algebraic k-stacks and that $\mathfrak{Y}$ is algebraic. Then $\mathfrak{X}$ is algebraic also.

3.6.4. Proposition. *Suppose Z is a k-scheme and H is a linear algebraic group over k acting on Z. Then the quotient k-stack $[Z/H]$ is algebraic:*

Proof. This follows from the definitions : a presentation is given by the morphism $p : Z \to [Z/H]$ defined by the trivial H-bundle on Z. □

3.6.5. Proposition. *The k-stack $\mathcal{M}_{GL_r,X}$ of 3.1.1 is algebraic.*

Proof. ([15],4.14.2.1) □

3.6.6. Corollary. *The k-stack $\mathcal{M}_{G,X}$ of 3.1.1 is algebraic.*

Proof. Choose an embedding $G \subset \mathbf{GL}_r$. Using Lemma 2.2.3 we may (and will) view a G-bundle E over a k-scheme Z as a $\mathbf{GL}_r$-bundle V together with a section $\sigma \in H^0(Z, V/G)$. Consider the morphism of k-stacks

$$\varphi : \mathcal{M}_{G,X} \longrightarrow \mathcal{M}_{GL_r,X}$$

defined by extension of the structure group. The corollary follows from the above proposition and the following remark:

3.6.7. *The above morphism is representable.* Let U be a k-scheme and $\eta : U \to \mathcal{M}_{GL_r,X}$ be a morphism, that is a $\mathbf{GL}_r$-bundle F over $X_U = X \times_k U$. For any arrow $U' \to U$ in $\mathbf{Aff}/k$ the $\mathbf{GL}_r$-bundle F defines a $\mathbf{GL}_r$-bundle over $X_{U'}$ which we denote by $F_{U'}$.

We have to show that the fiber $\mathcal{M}_{G,X}(\eta)$, as defined in (3.6.2), is representable as a scheme over U. As a U-stack, $\mathcal{M}_{G,X}(\eta)$ associates to every arrow $U' \to U$ the groupoid defined on the level of objects by pairs (E, α) where E is a G-bundle over $X_{U'}$ and $\alpha : E(\mathbf{GL}_r) \xrightarrow{\sim} F_{U'}$ is an isomorphism of $\mathbf{GL}_r$-bundles. On the level of morphisms we have the isomorphisms of such pairs, defined as follows: the pair (E_1, α_1) is isomorphic to the pair (E_2, α_2) if there is an isomorphism $\phi : E_1 \to E_2$ such that $\alpha_2 \circ \phi(\mathbf{GL}_r) = \alpha_1$. Such an isomorphism is, if it exists, unique for, since G acts faithfully on $\mathbf{GL}_r$, $\phi(\mathbf{GL}_r) = \alpha_2^{-1} \circ \alpha_1$ uniquely determines ϕ. Therefore, the fiber is a U-space. Moreover, the set of pairs (E, α) is canonically bijective to the set $\mathrm{Hom}_{X_{U'}}(X_{U'}, (F/G)_{U'})$. An easy verification shows that this bijection is functorial, *i.e.* defines an isomorphism between the U-space of the above pairs and the functor which associates to $U' \to U$ the above set of sections. So we are reduced to show that the latter functor is representable. But this follows from Grothendiecks theory of Hilbert schemes ([10], pp. 19–20), once we know that $F/G \to X_U$ is quasi-projective. In order to see this last statement we use Chevalley's theorem on semi-invariants: there is a representation V of $\mathbf{GL}_r$ with a line ℓ such that G is the stabilizer (in $\mathbf{GL}_r$) of ℓ. We get an embedding $\mathbf{GL}_r/G \subset \mathbb{P}(V^*)$, hence the required embedding

$$F/G \subset \mathbb{P}(F(V^*)).$$

(Actually the line bundle on F/G which corresponds to the above embedding is nothing else than the line defined by extension of the structure group of the G-bundle $F \to F/G$ via χ^{-1} where χ is the character defined by the action of G on ℓ.) $\qquad\square$

3.6.8. Proposition. *Suppose G is reductive. The algebraic stack $\mathcal{M}_{G,X}$ is smooth of dimension* $\dim(G)(g-1)$.

This follows from deformation theory. I will be rather sketchy here as rendering precise the arguments below is quite long. Let E be a G-bundle. Consider the action of G on $\mathfrak{g}$ given by the adjoint representation and then the vector bundle $E(\mathfrak{g})$. The obstruction to smoothness of $\mathcal{M}_{G,X}$ lives in $H^2(X, E(\mathfrak{g}))$ which vanishes since X is of dimension 1. The infinitesimal deformations of E are parameterized by $H^1(X, E(\mathfrak{g}))$ with global automorphisms parameterized by $H^0(X, E(\mathfrak{g}))$. Over schemes in order to calculate the dimension we would calculate the rank of its tangent bundle. We can do this here also but on stacks one has to be careful about how one understands the "tangent bundle". We see this readily here: for example for $G = GL_r$ the tangent space $H^1(X, \underline{\mathrm{End}}(E, E))$ is not of constant dimension over the connected components but only over the open substack of simple vector bundles. Of course $\dim H^1(X, \underline{\mathrm{End}}(E, E))$ jumps exactly when $\dim H^0(X, \underline{\mathrm{End}}(E, E))$ jumps, so again one has to take care of global automorphisms. However, we may consider the tangent complex on $\mathcal{M}_{G,X}$. In our case this complex is $Rpr_{1*}(\mathcal{E}(\mathfrak{g}))$ where $\mathcal{E}$ is the universal G-bundle over $\mathcal{M}_{G,X} \times X$, which may be represented by a perfect complex of length one (see section 6.1.1 for this). By definition, the dimension of the stack $\mathcal{M}_{G,X}$ at the point E is the rank of the cotangent complex at E, which is $-\chi(E(\mathfrak{g}))$. If G is reductive there is an isomorphism $\mathfrak{g} \to \mathfrak{g}^*$ of G-modules. Therefore we know that $\deg(E(\mathfrak{g})) = 0$ and then Riemann-Roch gives $\dim \mathcal{M}_{G,X} = \dim(G)(g-1)$. If $g(X) = 0$, then its dimension is $-\dim(G)$, which may be surprising, but which is, in view of the above, the only reasonable result we may ask for (the standard example of a stack with negative dimension is $BG = [\cdot/H]$ which is of dimension $-\dim H$).

4. Topological classification

Here X is a compact connected oriented smooth real surface of genus g and G a connected topological group. A topological G bundle E over X is a topological space E on which G acts from the right together with a G-invariant continuous map $E \xrightarrow{\pi} X$ such that for every $x \in X$ there is an

open neighborhood U of x such that $E_{|U}$ is trivial, *i.e.* isomorphic to $U \times G$ as a G-homogeneous space where G acts on $U \times G$ by right multiplication.

4.1. **Topological loop groups.** Let $x_0 \in X$ and let D be a neighborhood of x homeomorphic to a disc. Define $D^* = D - x_0$ and $X^* = X - x_0$. Associated are the following three groups

$$L^{top} G = \{f : D^* \longrightarrow G / f \text{ is continuous}\}$$
$$L_+^{top} G = \{f : D \longrightarrow G / f \text{ is continuous}\}$$
$$L_X^{top} G = \{f : X^* \longrightarrow G / f \text{ is continuous}\}$$

By definition, we have the following inclusions:

$$L_X^{top} G \subset L^{top} G \supset L_+^{top} G$$

Let $\mathcal{M}_{G,X}^{top}$ be the set of isomorphism classes of topological G-bundles on X.

4.1.1. Proposition. *There is a canonical bijection*

$$L_X^{top} G \backslash L^{top} G / L_+^{top} G \xrightarrow{\sim} \mathcal{M}_{G,X}^{top}$$

Proof. The basic observation is that if E is a topological G-bundle on X then the restrictions of E to D and X^* are trivial. For the restriction to D this is clear, since D is contractible; for the restriction to X^* we view X as a CW-complex of dimension 2 and remark that, since G is connected, there is no obstruction to the existence of a section of a G-bundle on X^*. It follows that if we choose trivialization $\sigma : E_{|D} \xrightarrow{\sim} D \times G$ and $\tau : E_{|X^*} \xrightarrow{\sim} X^* \times G$ then the transition function $\gamma = \tau \circ \sigma_{|D^*}^{-1}$ is an element of $L^{top} G$. On the other hand, we may take trivial bundles on D and X^* and patch them together by γ in order to get a G-bundle E on X. Therefore there is a canonical bijection

$$L^{top} G = \{(E, \sigma, \tau) \ / E \xrightarrow{G} X, \ \sigma : E_{|D} \xrightarrow{\sim} D \times G, \ \tau : E_{|X^*} \xrightarrow{\sim} X^* \times G\}$$

Now, by construction, multiplying $\gamma \in L^{top} G$ from the right by $\alpha \in L_+^{top} G$ corresponds under this bijection to changing the trivialization σ by $\alpha^{\#} \circ \sigma$, where $\alpha^{\#}$ is the map $D \times G \to D \times G$ defined by $(z, g) \mapsto (z, g\alpha(z))$ and analogously multiplying from the left by $\beta^{-1} \in L_X^{top} G$ corresponds to change the trivialization τ. It follows that dividing by $L_+^{top} G$ forgets about the trivialization σ and dividing by $L_X^{top} G$ forgets about the trivialization τ, hence the proposition. $\square$

4.1.2. Corollary. *The set $\mathcal{M}_{G,X}^{top}$ is in bijective correspondence with $\pi_1(G)$.*

Proof. If $\gamma \in L^{top}G$, we denote by $\gamma_* : \pi_1(D^*) \to \pi_1(G)$ the induced map. Let δ be the positive generator of $\pi_1(D^*)$ and consider the map

$$f : L^{top}G \longrightarrow \pi_1(G)$$

$$\gamma \longmapsto \gamma_*(\delta)$$

Now f depends only on the double classes. In order to see this consider for $\alpha \in L_+^{top}G$ and $\beta \in L_X^{top}G$ the element $\beta^{-1}\gamma\alpha$ which we view as an element of $L^{top}G$ as follows: $z \mapsto \beta^{-1}(z)\gamma(z)\alpha(z)$. Then remark that the composite $D^* \to D \xrightarrow{\alpha} G$ is homotopically trivial since it extends to D. For the composite $D^* \to X^* \xrightarrow{\beta} G$ consider the induced map $\pi_1(D^*) \to \pi_1(X^*) \to \pi_1(G)$ and remark (exercise) that the image of $\pi_1(D^*)$ in $\pi_1(X^*)$ has to sit inside the commutator subgroup. It follows that its image in $\pi_1(G)$ is trivial, since $\pi_1(G)$ is abelian. Thus $D^* \to X^* \xrightarrow{\beta} G$ is also homotopically trivial. Therefore $\beta^{-1}\gamma\alpha$ is homotopic to γ, hence f depends only on the double classes. Then it is an easy exercise to see that the induced map on the double quotient is indeed a bijection. $\qquad\square$

5. UNIFORMIZATION

The uniformization theorem is the analogue of proposition 4.1.1 in the algebraic setup. Let k be an algebraically closed field, X be a smooth, connected and complete algebraic curve over k and G be an affine algebraic group over k. We choose a closed point $x_0 \in X$ and consider $X^* = X - \{x_0\}$. Remark that X^* is affine (map X to $\mathbb{P}^1$ using a rational function f with pole of some order at x_0 and regular elsewhere and remark that $f^{-1}(\mathbb{A}^1) = X^*$).

What is the algebraic analogue of the "neighborhood of x_0 homeomorphic to a disc" of section 4? What we can do is to look at the local ring $\mathcal{O}_{X,x_0}$ and then consider its completion $\widehat{\mathcal{O}}_{X,x_0}$. Then $D_{x_0} = \mathrm{Spec}(\widehat{\mathcal{O}}_{X,x_0})$ will be convenient for if we choose a local coordinate z at $x_0 \in X$ then we may identify $\widehat{\mathcal{O}}_{X,x_0}$ with $k[[z]]$, hence D_{x_0} with the "formal disc" $D = \mathrm{Spec}\left(k[[z]]\right)$.

Moreover, $D_{x_0}^* = D - \{x_0\}$ is $\mathrm{Spec}(K_{x_0})$, where K_{x_0} is the field of fractions of $\widehat{\mathcal{O}}_{X,x_0}$. Using our local coordinate z we see that K_{x_0} identifies to $k((z))$, hence $D_{x_0}^*$ to $D^* = \mathrm{Spec}\left(k((z))\right)$.

It will be convenient in the following to introduce the following notations: if $U = \mathrm{Spec}(R)$ then we will denote $D_U^* = \mathrm{Spec}\left(R((z))\right)$, $D_U = \mathrm{Spec}\left(R[[z]]\right)$ and $X_U^* = X^* \times U$.

5.1. **Algebraic loop groups.** The algebraic analogue of the topological loop group $L^{top}G$ is $\mathrm{Hom}_{alg}(D^*, G)$, that is, the points of G with values in

D^*, *i.e.* $G\big(k((z))\big)$. This has to be made functorial so we will consider the functor

$$LG : (\mathbf{Aff}/k) \longrightarrow \mathbf{Grp}$$
$$U = \mathrm{Spec}(R) \longmapsto G\big(R((z))\big)$$

Actually that is a k-group (in the sense of 3.3). We define the k-groups $L_X G$ and $L^+ G$ as well by $U \mapsto G\big(\mathcal{O}(X_U^*)\big)$ and $U \mapsto G\big(R[[z]]\big)$ respectively.

We denote $\mathcal{Q}_G$ the quotient k-space $LG/L^+ G$: this is the sheafification of the presheaf

$$U = \mathrm{Spec}(R) \longmapsto G\big(R((z))\big)/G\big(R[[z]]\big).$$

The k-group $L_X G$ acts on the k-space $\mathcal{Q}_G$; let $[L_X G \backslash \mathcal{Q}_G]$ be the quotient k-stack of 3.3.1.

5.1.1. Theorem. *(Uniformization) Suppose G is semi-simple. Then there is a canonical isomorphism of stacks*

$$\big[{}_{L_X G}\backslash LG/{}_{L^+ G}\big] \xrightarrow{\sim} \mathcal{M}_{G,X}$$

Moreover, the $L_X G$-bundle $\mathcal{Q}_G \xrightarrow{L_X G} \mathcal{M}_{G,X}$ is even locally trivial for the étale topology if the characteristic of k does not divide the order of $\pi_1(G(\mathbb{C}))$.

5.2. **Key inputs.** The theorem has two main inputs in its proof:

- Trivializing G-bundles over X_U^* (for this we need G semi-simple)
- Gluing trivial G-bundles over X_U^* and D_U to a G-bundle over X_U.

Both properties are highly non trivial in our functorial setup where U may be any affine k-scheme, not necessarily noetherian. So I discuss them first.

5.2.1. *Trivializing G-bundles over the open curve.* For general G it is not correct that the restriction of a G-bundle to X^* is trivial. The basic examples are of course line bundles. However, if we consider vector bundles with trivial determinant (*i.e.* $\mathbf{SL}_r$-bundles) then this becomes true. The reason is that a vector bundle E over X^* may be written as the direct sum $\mathcal{O}_{X^*}^{\oplus r} \oplus \det(E_{|X^*})$ (translate to the analogue statement of finite module over a ring and use that $\mathcal{O}(X^*)$ is Dedekind as X^* is a smooth curve). Now if E is a vector bundle with trivial determinant on X_U we may ask whether, locally (for an appropriate topology) on U, the restriction of E to X_U^* is trivial. This is indeed true (for the Zariski topology on U) and the argument proceeds by induction on the rank r of E ([2], 3.5), the rank 1 case being trivial: consider the divisor $d = \{x_0\} \times U$ of X_U and choose an integer n such that $E(nd)$ has

no higher cohomology and is generated by its global sections. Then consider a point $u \in U$ and a nowhere vanishing section s of $E(nd)_{|X \times \{u\}}$ (count dimensions in order to see its existence). Shrinking U, one may suppose that this section is the restriction to $E(nd)$ of a section which does not vanish on X_U. When restricting to X_U^* we get an exact sequence

$$0 \to \mathcal{O}_{X_U^*} \longrightarrow E_{|X_U^*} \longrightarrow F \to 0$$

where F is a vector bundle. But after shrinking U again we may assume that F is trivial by induction and that the sequence splits, hence $E_{|X_U^*}$ is trivial.

The natural guess then is that the above trivialization property is true for semi-simple G at least for the appropriate topology on U. This has been proved by Drinfeld and Simpson.

Theorem (Drinfeld-Simpson). [7] *Suppose G is semi-simple. Let E be a G-bundle over X_U. Then the restriction of E to X_U^* is trivial, locally for the fppf topology over U. If $char(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$, then this is even true locally for the étale topology over U.*

I will not enter into the proof, however I will invite the reader to have a closer look at their note, as it uses some techniques which are quite useful also in other contexts.

5.2.2. *Gluing.* Consider the following cartesian diagram

$$
\begin{array}{ccc}
D_U^* & \!\!\!\!\longrightarrow\!\!\!\! & D_U \\
\big| & & \big| \\
X_U^* & \!\!\!\!\longrightarrow\!\!\!\! & X_U
\end{array}
$$

Given trivial G-bundles on X_U^* and D_U and an element $\gamma \in G\big(R((z))\big)$ we want to glue them to a G-bundle E on X_U. The reader might say that this is easy: just apply what we have learned about descent in section 3. However some care has to be taken here: if U is not noetherian, then the morphism $D_U \to X_U$ is *not* flat! Nevertheless the gluing statement we need is true:

Theorem (Beauville-Laszlo). [3] *Let $\gamma \in G\big(R((z))\big)$. Then there exists a G-bundle E on X_U and trivializations $\sigma : E_{|D_U} \to D_U \times G$, $\tau : E_{|X_U^*} \to X_U^* \times G$. Moreover the triple (E, σ, τ) is uniquely determined up to unique isomorphism.*

Actually the above theorem is proved for vector bundles in [3] but the generalization to G-bundles is immediate. Again, I will not enter into the proof, but invite the reader to have a look at their note.

5.3. **Proof of the uniformization theorem.** Once the above two key inputs are known, the proof of the uniformization theorem is essentially formal.

We start considering the functor T_G of triples:

$$T_G : (\mathbf{Aff}/k) \longrightarrow \mathbf{Set}$$

$$U \longmapsto \{(E, \tau, \sigma) \ / \ E \xrightarrow{G} X_U \text{ is a } G\text{-bundle with trivializations}$$

$$\sigma : E_{|D_U} \xrightarrow{\sim} D_U \times G, \tau : E_{|X_U^*} \xrightarrow{\sim} X_U^* \times G.\}/ \sim$$

5.3.1. Proposition. *The k-group LG represents the functor T_G.*

Proof. Let (E, τ, σ) be an element of $T_G(U)$. Pulling back the trivializations τ and σ to D_U^* provides two trivializations τ^* and σ^* of the pullback of E over D_U^*: these trivializations differ by an element $\gamma = \tau^{*-1} \circ \sigma^*$ of $G\big(R((z))\big)$ (as usual $U = \mathrm{Spec}(R)$). Conversely, if $\gamma \in G\big(R((z))\big)$, we get an element of $T_G(U)$ by the Beauville-Laszlo theorem. These constructions are inverse to each other by construction. $\qquad\square$

Now consider the functor of pairs P_G:

$$P_G : (\mathbf{Aff}/k) \longrightarrow \mathbf{Set}$$

$$U \longmapsto \{(E, \tau) \ / \ E \xrightarrow{G} X_U \text{ is a } G\text{-bundle with trivialization}$$

$$\tau : E_{|X_U^*} \xrightarrow{\sim} X_U^* \times G.\}/ \sim$$

5.3.2. Proposition. *The k-space $\mathcal{Q}_G$ represents the functor P_G.*

Proof. Let $U = \mathrm{Spec}(R)$ be an affine k-scheme and q be an element of $\mathcal{Q}_G(U)$. By definition of $\mathcal{Q}_G$ as a quotient k-space, there exists a faithfully flat homomorphism $U' \to U$ and an element γ of $G\big(R'((z))\big)$ ($U' = \mathrm{Spec}(R')$) such that the image of q in $\mathcal{Q}_G(U')$ is the class of γ. To γ corresponds by 5.3.1 a triple (E', τ', σ') over $X_{U'}$. Let $U'' = U' \times_U U'$, and let (E_1'', τ_1''), (E_2'', τ_2'') denote the pullbacks of (E', τ') by the two projections of $X_{U''}$ onto $X_{U'}$. Since the two images of γ in $G\big(R''((z))\big)$ differ by an element of $G\big(R''[[z]]\big)$, these pairs are isomorphic. So the isomorphism $\tau_2'' \tau_1''^{-1}$ over $X_{U''}^*$ extends to an isomorphism $u : E_1'' \to E_2''$ over $X_{U''}$, satisfying the usual co-cycle condition (it is enough to check this over X^*, where it is obvious). Therefore

(E', τ') descends to a pair (E, τ) on X_R as in the above statement. Conversely, given a pair (E, τ) as above over X_U, we can find a faithfully flat homomorphism $U' \to U$ and a trivialization σ' of the pullback of E over $D_{U'}$ (after base change, we may assume that the central fiber of the restriction of E to D_U has a section then use smoothness to extend this section to D_U). By 5.3.1 we get an element γ' of $G(R'((z)))$ such that the two images of γ' in $G(R''((z)))$ (with $R'' = R' \otimes_R R'$) differ by an element of $G(R''[[z]])$; this gives an element of $\mathcal{Q}_G(U)$. These constructions are inverse to each other by construction. $\qquad\square$

5.3.3. *End of the proof.* The universal G-bundle over $X \times \mathcal{Q}_G$ (see 5.3.2), gives rise to a map $\pi : \mathcal{Q}_G \to \mathcal{M}_{G,X}$. This map is $L_X G$-invariant, hence induces a morphism of stacks $\overline{\pi} : L_X G \backslash \mathcal{Q}_G \to \mathcal{M}_{G,X}$. On the other hand we can define a map $\mathcal{M}_{G,X} \to L_X G \backslash \mathcal{Q}_G$ as follows. Let U be an affine k-scheme, E a G-bundle over X_U. For any arrow $U' \to U$, let $T(U')$ be the set of trivializations τ of $E_{U'}$ over $X_{U'}^*$. This defines a U-space T on which the group $L_X G$ acts. By Drinfeld-Simpson's theorem, it is an $L_X G$-bundle. To any element of $T(U')$ corresponds a pair $(E_{U'}, \tau)$, hence by 5.3.2 an element of $\mathcal{Q}_G(U')$. In this way we associate functorially to an object E of $\mathcal{M}_{G,X}(U)$ an $L_X G$-equivariant map $\alpha : T \to \mathcal{Q}_G$. This defines a morphism of stacks $\mathcal{M}_{G,X} \to L_X G \backslash \mathcal{Q}_G$ which is the inverse of $\overline{\pi}$. The second assertion means that for any scheme U over k (resp. over k such that $char(k)$ does not divide the order of $\pi_1(G(\mathbb{C}))$) and any morphism $f : U \to \mathcal{M}_{G,X}$, the pullback to U of the fibration π is *fppf* (resp. *étale*) locally trivial, i.e. admits local sections for the *fppf* (resp. *étale*) topology. Now f corresponds to a G-bundle E over X_U. Let $u \in U$. Again by the Drinfeld-Simpson theorem, we can find an *fppf* (resp. *étale*) neighborhood U' of u in U and a trivialization τ of $E_{|X_{U'}^*}$. The pair (E, τ) defines a morphism $g : U' \to \mathcal{Q}_G$ (by 5.3.2) such that $\pi \circ g = f$, that is a section over U' of the pullback of the fibration π.

6. The determinant and the pfaffian line bundles

Let X be a projective curve, smooth and connected over the algebraically closed field k.

6.1. **The determinant bundle.** Let F be a vector bundle over $X_S = X \times_k S$, where S is a locally noetherian k-scheme. As usual we think of F as a family of vector bundles parameterized by S.

6.1.1. *Representatives of the cohomology.* In the following I will call a complex $K^\bullet$ of coherent locally free $\mathcal{O}_S$-modules

$$0 \to K^0 \xrightarrow{\gamma} K^1 \to 0$$

a *representative* of the cohomology of F if for every base change $T \xrightarrow{f} S$

$$
\begin{array}{ccc}
X_T & \xrightarrow{\ g\ } & X_S \\
{\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle p} \\
T & \xrightarrow{\ f\ } & S
\end{array}
$$

we have $H^i(f^*K^\bullet) = R^i u_* g^* F$. In particular, if $s \in S$ is a closed point:

$$H^i(K_s^\bullet) = H^i(X, F_s)$$

Representatives of the cohomology of F are easy to construct in our setup. Indeed, we may choose a resolution

$$0 \to P_1 \longrightarrow P_0 \longrightarrow F \to 0$$

of F by S-flat coherent $\mathcal{O}_{X_S}$-modules such that $p_* P_0 = 0$ (use Serre's theorem A in its relative version to see its existence). Then we have $p_* P_1 = 0$ and, by base change for coherent cohomology, the complex

$$0 \to R^1 p_* P_1 \longrightarrow R^1 p_* P_0 \to 0$$

is convenient. This result is generally quoted as choosing a perfect complex of length one representing $Rp_* F$ in the derived category[3] $\mathcal{D}_c(S)$

6.1.2. *The determinant bundle.* The determinant of a complex $K^\bullet$ of locally free coherent $\mathcal{O}_S$-modules $0 \to K^0 \longrightarrow K^1 \to 0$ if defined by

$$\det(K^\bullet) = \overset{max}{\bigwedge} K^0 \otimes (\overset{max}{\bigwedge} K^1)^{-1}$$

The determinant of our family F of vector bundles parameterized by S is defined by[4]

$$\mathcal{D}_F = \det(Rp_* F)^{-1}$$

---

[3]All the derived category theory I need here and in the proof of 6.2.2 is in ([6],§1). The category of complexes of $\mathcal{O}_S$-modules will be denoted by $\mathcal{C}(S)$; the category with the same objects $\mathcal{C}(S)$ but morphisms homotopy classes of morphisms of $\mathcal{C}(S)$ will be denoted by $\mathcal{K}(S)$. Finally $\mathcal{D}(S)$ is obtained by inverting the quasi-isomorphisms in $\mathcal{K}(S)$. A superscript b (resp. subscript c) means that we consider the full sub-categories of bounded complexes (resp. complexes with coherent cohomology).

[4]The minus sign is chosen in order to get the "positive" determinant bundle.

In general, in order to calculate $\mathcal{D}_F$, we choose a representative $K^\bullet$ of the cohomology of F and then calculate $\det(K^\bullet)^{-1}$. This does not depend, up to canonical isomorphism, on the choice of $K^\bullet$ (and this is the reason why the above definition makes sense) [11].

By construction, the fiber of $\mathcal{D}_F$ at $s \in S$ is given as follows:

$$\mathcal{D}_F(s) = (\bigwedge^{max} H^0(X, F_s))^{-1} \otimes \bigwedge^{max} H^1(X, F_s)$$

We may also twist our family F by bundles coming from X, *i.e.* consider $F \otimes q^*E$ where E is a vector bundle on X. We obtain the line bundle $\mathcal{D}_{F \otimes q^*E}$, and this line bundle actually depends only on the class of E in the Grothendieck group $K(X)$ of X (check this!). It follows that we get a group morphism, Le Potier's determinant morphism [16]

$$\lambda_F : K(X) \longrightarrow \operatorname{Pic}(S)$$

$$u \longmapsto \mathcal{D}_{F \otimes q^*u}$$

If our bundle F comes from a $\mathbf{SL}_r$-bundle, *i.e.* has trivial determinant, twisting F by an element $u \in K(X)$ then taking determinants just means taking the $r(u)$-th power of $\mathcal{D}_F$:

6.1.3. Lemma. *Suppose F is a vector bundle on X_S such that $\bigwedge^{max} F$ is the pullback of some line bundle on X. Then*

$$\mathcal{D}_{F \otimes q^*u} = \mathcal{D}_F^{\otimes r(u)} \ in \ \operatorname{Pic}(S)$$

where $r(u)$ is the rank of u.

Proof. We may suppose that u is represented by a vector bundle L and even – after writing L as an extension – that L is a line bundle. But then it is enough to check it for $L = \mathcal{O}_X(-p)$, for $p \in X$, where it follows, after considering $0 \to \mathcal{O}_X(-p) \to \mathcal{O}_X \to \mathcal{O}_p \to 0$, from the fact $\mathcal{D}_{F \otimes q^*\mathcal{O}_p}$ is trivial under our hypothesis on F. $\qquad\square$

6.1.4. *Theta-functions.* Twisting is particularly useful in order to produce sections of (powers of) the determinant bundle. Suppose S is integral and that F is a vector bundle on X_S with trivial determinant. Choose a vector bundle E such that $F_s \otimes q^*E$ has trivial Euler characteristic for some s. If

$$0 \to K^0 \xrightarrow{\gamma} K^1 \to 0$$

is a representative of the cohomology of $F \otimes q^*E$, then we know that the rank n of K^0 is equal to the rank of K^1, hence γ may be locally represented as a $n \times n$-matrix. We get a section $\theta_E = \det(\gamma)$ of $D_F^{r(E)}$, well defined

up to an invertible function on S: the *theta-function associated to E*. In particular, its divisor Θ_E is well defined with support the points $s \in S$ such that $H^0(F_s \otimes E) \neq 0$.

If we suppose moreover that $F_t \otimes q^*E$ has trivial cohomology for some $t \in S$ then $\Theta_E \neq S$, *i.e.* the section θ_E is non trivial; if there is $t' \in S$ such that $H^0(X, E_{t'} \otimes E) \neq 0$ then $\Theta_E \neq \emptyset$.

6.2. **The pfaffian line bundle.** Suppose $\mathrm{char}(k) \neq 2$ in this subsection. Let F be a vector bundle over $X_S = X \times S$, together with a quadratic non degenerate form σ with values in the canonical bundle ω_X. We will view σ as an isomorphism $F \xrightarrow{\sim} F^\vee$ such that $\sigma = \sigma^\vee$, where $F^\vee = \underline{\mathrm{Hom}}_{\mathcal{O}_{X_S}}(F, q^*\omega_X)$.

6.2.1. Lemma. *If $K^\bullet$ is a representative of the cohomology of F, then $K^{\bullet*}[-1]$ is a representative of the cohomology of $F^\vee$.*

Here[5] $K^{\bullet*}[-1]$ denotes the complex supported in degrees 0 and 1

$$0 \to K^{1*} \xrightarrow{-\gamma^*} K^{0*} \to 0.$$

Proof. In the derived category $\mathcal{D}_c(S)$, we have

$$Rp_*(F^\vee) \xrightarrow{\sim} Rp_*(R\underline{\mathrm{Hom}}_{\mathcal{O}_{X_S}}(F, q^*\omega_X)) \qquad (F \text{ is locally free})$$

$$\xrightarrow{\sim} R\underline{\mathrm{Hom}}(Rp_*(F), \mathcal{O}_S)[-1] \qquad \text{(Grothendieck-Serre duality)}$$

Now if $K^\bullet$ represents the cohomology of F we see that $R\underline{\mathrm{Hom}}(K^\bullet, \mathcal{O}_S)[-1]$ represents the cohomology of $F^\vee$. But this is nothing else than $K^{\bullet*}[-1]$ as the K^i are locally free. $\qquad\square$

6.2.2. Proposition. *There exists, locally for the Zariski topology on S, a representative of the cohomology $K^\bullet$ of F and a symmetric isomorphism:*

$$\tau : K^\bullet \xrightarrow{\sim} K^{\bullet*}[-1]$$

such that τ and σ induce the same map in cohomology.

Proof. Choose a representative $\widetilde{K}^\bullet$ of the cohomology of F and remark that σ induces an isomorphism $\widetilde{\tau}$ in the derived category $\mathcal{D}_c^b(S)$

$$\widetilde{K}^\bullet \xrightarrow{\sim} Rp_*F \xrightarrow{\sigma} Rp_*(F^\vee) \xrightarrow{\sim} \widetilde{K}^{\bullet*}[-1]$$

which is still symmetric (this follows from the symmetry of σ and standard properties of Grothendieck-Serre duality).

---

[5]This is compatible with the usual signs: the dual of $K^\bullet$ is supported in degrees -1 and 0; when translated to the right by 1, the differential acquires a -1 sign.

The problem here is that this isomorphism is only defined in the derived category: the proposition actually claims that we can get a symmetric isomorphism of complexes and this we only get Zariski locally.

First we may suppose that S is affine. Then the category of coherent sheaves on S has enough projectives and as the $\widetilde{K}^i$ are locally free we see that $\widetilde{\tau}$ is an isomorphism in $\mathcal{K}^b_c(S)$. Let φ be a lift of $\widetilde{\tau}$ to $\mathcal{C}^b_c(S)$. We get a morphism of complexes

$$
\begin{array}{ccc}
\widetilde{K}^0 & \xrightarrow{\ \gamma\ } & \widetilde{K}^1 \\
\varphi_0 \downarrow & & \downarrow \varphi_1 \\
\widetilde{K}^{1*} & \xrightarrow{-\gamma^*} & \widetilde{K}^{0*}
\end{array}
$$

which needs neither to be symmetric nor an isomorphism (it is only a quasi-isomorphism). First we symmetrize: $\phi_i = (\varphi_i + \varphi_{1-i}^*)/2$ for $i = 0, 1$. Remark that ϕ is still a quasi-isomorphism, inducing σ in cohomology. Then we fix $s \in S$. A standard argument shows that there is, in a neighborhood of s, another length one complex $K^\bullet$ of free coherent $\mathcal{O}_S$-modules together with a quasi-isomorphism $u : K^\bullet \to \widetilde{K}^\bullet$, such that for the differential d we have $d_{|s} = 0$. Now

$$
\tau = u^*[-1]\phi u : K^\bullet \longrightarrow K^{\bullet *}[-1]
$$

is a symmetric quasi-isomorphism, inducing σ in cohomology, and $\tau_{|s}$ is an isomorphism. Then, in a neighborhood of s, $\tau_{|s}$ will remain an isomorphism which proves the proposition. $\qquad\square$

Let $(K^\bullet, \tau)$ be as in the proposition and consider the following diagram

$$
\begin{array}{ccc}
K^0 & \xrightarrow{\ \gamma\ } & K^1 \\
\tau_0 \downarrow \wr & \searrow^{\alpha} & \wr \downarrow \tau_0^* \\
K^{1*} & \xrightarrow{-\gamma^*} & K^{0*}
\end{array}
$$

It follows that α is *skew-symmetric*. Therefore the cohomology of F may be represented, locally for the Zariski topology on S, by complexes of free coherent $\mathcal{O}_S$-modules

$$
0 \to K \xrightarrow{\alpha} K^* \to 0
$$

with α skew. Such complexes will be called *special* in the following.

An immediate corollary is Riemann's invariance mod 2 theorem:

6.2.3. Corollary. [6] *Let F be a vector bundle on X_S equipped with a non degenerate quadratic form with values in ω_X. Then the function*

$$s \mapsto \dim H^0(X, F_s) \bmod 2$$

is locally constant.

Proof. Locally there is a special representative $K^\bullet$ of the cohomology of F.

$$\dim H^0(X, F_s) = \operatorname{rank} K - \operatorname{rank} \alpha$$

Now use that the rank of α is even as α is skew. □

6.3. The pfaffian bundle. Let F be a vector bundle on X_S equipped with a non degenerate quadratic form with values in ω_X and cover S by Zariski open subsets U_i such that F admits a special representative $K_i^\bullet$ of the cohomology of F on U_i. Over U_i

$$\mathcal{D}_{F|U_i} = \bigwedge^{max} K_i^* \otimes \bigwedge^{max} K_i^*$$

which is a square. It turns out, because the $K^\bullet$ are *special* complexes, that the $\bigwedge^{max} K_i^*$ glue together over S and define a canonical square root of $\mathcal{D}_F$, called the *pfaffian bundle*.

This gluing requires quite some work and is the content of ([14], §7). I will not enter into the proof here: *(loc.cit.)* is self contained.

6.3.1. Theorem. *Let F be a vector bundle over X_S equipped with a non degenerate quadratic form σ with values in ω_X. Then the determinant bundle $\mathcal{D}_F$ admits a canonical square root $\mathcal{P}_{(F,\sigma)}$. Moreover, if $f : S' \to S$ is a morphism of locally noetherian k-schemes then we have $\mathcal{P}_{(f^*F, f^*\sigma)} = f^* \mathcal{P}_{(F,\sigma)}$.*

6.4. The pfaffian bundle on the moduli stack. Let $r \geq 3$ and (F, σ) be the universal $\mathbf{SO}_r$-bundle over $\mathcal{M}_{SO_r, X} \times X$. If we twist by a theta-characteristic κ (*i.e.* a line bundle such that $\kappa \otimes \kappa = \omega_X$), then $F_\kappa = F \otimes q^* \kappa$ comes with a non-degenerate form with values in ω_X. Then we may apply 6.3.1 in order to get the pfaffian bundle $\mathcal{P}_{(F_\kappa, \sigma)}$ which we denote simply by $\mathcal{P}_\kappa$.

---

[6]In fact the above arguments are valid for any smooth proper morphism $Y \to S$ of relative dimension 1. I only consider the situation of a product $Y = X \times S$ here as this is the one I need in order to define the determinant resp. pfaffian bundles.

6.5. **The square-root of the dualizing sheaf.** Suppose G is semi-simple and consider its action on $\mathfrak{g}$ given by the adjoint representation. It follows from the proof of Proposition 3.6.8, that the dualizing sheaf $\omega_{\mathcal{M}_{G,X}}$ is $\mathcal{D}_{\mathcal{E}(\mathfrak{g})}$ where $\mathcal{E}$ is the universal G-bundle on $\mathcal{M}_{G,X}$. Remark that the bundle $\mathcal{E}(\mathfrak{g})$ comes with a natural quadratic form given by the Cartan-Killing form. Hence the choice of a theta-characteristic κ defines, by the above, a square root $\omega_{\mathcal{M}_{G,X}}^{1/2}(\kappa)$ of $\omega_{\mathcal{M}_{G,X}}$.

6.6. **The pfaffian divisor.** It may seem easier to construct the pfaffian bundle looking at it from a divisorial point of view using smoothness of $\mathcal{M}_{SO_r,X}$. Suppose F is a vector bundle on X_S equipped with a non degenerate quadratic form σ with values in ω_X. Suppose moreover that S is smooth and that there are points $s, t \in S$ such that $H^0(X, F_s) = 0$ and $H^0(X, F_t) \neq 0$. We know from section 6.1.4 that if $K^0 \xrightarrow{\gamma} K^1$ represents the cohomology of F, then $\mathcal{D}_F$ is the line bundle associated to the divisor defined by $\det(\gamma) = 0$. Now, locally γ may be represented by a skew-symmetric matrix α, so we may take its pfaffian. This defines a local equation for a divisor, which will be called the pfaffian divisor, hence, by smoothness of S, our pfaffian line bundle. Of course the preceding sentence has to be made rigorous, but this may seem easier than the (rather formal) considerations of ([14], §7) which lead to Theorem 6.3.1.

However[7], even if the hypothesis of smoothness is satisfied for $\mathcal{M}_{SO_r,X}$, this approach fails to define line bundles $\mathcal{P}_\kappa$ for all theta-characteristics κ.

The point is that the hypotheses of the existence of $s \in S$ such that $H^0(X, F_s) = 0$ is not always satisfied: the equation $\det(\gamma) = 0$ may not define a divisor (but the whole space).

In order to see this, consider the component $\mathcal{M}_{SO_r^0,X}$ of $\mathcal{M}_{SO_r,X}$, containing the trivial $\mathbf{SO}_r$-bundle. Actually we haven't seen yet that the connected components of $\mathcal{M}_{G,X}$ are parameterized by $\pi_1(G)$ (*i.e.* by their topological type), but for the moment let's just use that $\mathcal{M}_{SO_r,X}$ has two components: $\mathcal{M}_{SO_r^0,X}$ and $\mathcal{M}_{SO_r^1,X}$. They are distinguished by the second Stiefel-Whitney class

$$(6.6\,\mathrm{a}) \qquad w_2 : H^1_{\acute{e}t}(X, SO_r) \to H^2_{\acute{e}t}(X, \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z}.$$

Let $r \geq 3$ and (F, σ) be the universal quadratic bundle over $\mathcal{M}_{SO_r^0,X} \times X$. For κ a theta-characteristic, consider the substack Θ_κ of section 6.1.4.

---

[7]There are of course many other reasons to prefer to construct a line bundle directly and not as a line bundle associated to a divisor.

6.6.1. Proposition. *The substack Θ_κ of $\mathcal{M}_{SO_r^0,X}$ is a divisor if and only if r or κ are even.*

Proof. We start with a useful lemma.

6.6.2. Lemma. *Let $A = (E,q)$ be an SO_r-bundle, $r \geq 3$ and κ be a theta-characteristic. Then*

(6.6 b) $$w_2(A) = h^0(E \otimes \kappa) + rh^0(\kappa) \bmod 2.$$

Proof. Indeed, by Riemann's invariance mod 2 theorem, the right-hand side of (6.6 b), denoted $w_2'(A)$ in the following, is constant over the 2 connected components of $\mathcal{M}_{SO_r,X}$. Because (6.6 b) is true at the trivial SO_r-bundle T, it is enough to prove that w_2' is not constant. As $w_2'(T) = 0$, we have to construct an $\mathbf{SO}_r$-bundle A such that $w_2'(A) \neq 0$. In order to do this, let L, M be points of order 2 of the jacobian, such that for the Weil pairing we have $< L, M > = 1$. The choice of a trivialization of their squares defines a non degenerated quadratic form on

$$E = (L \otimes M) \oplus L \oplus M \oplus (r-3)\mathcal{O}_X$$

hence an SO_r-bundle A. By [17], we know that we have $w_2'(A) = < L, M > = 1$ which proves (6.6 b). $\square$

Now choose an ineffective theta-characteristic κ_0 and set $L = \kappa_0 \otimes \kappa^{-1}$. If r is even, there exists a $\mathbf{SO}_r$-bundle $A = (E,q)$ such that $H^0(E \otimes \kappa) = 0$ and $w_2(A) = 0$ (choose $E = rL$ with the obvious quadratic form and use (6.6 b)). If r is odd and κ is even, there exists a $\mathbf{SO}_r$-bundle $A = (E,q)$ such that $H^0(E \otimes \kappa) = 0$ and $w_2(A) = 0$ (by Lemma 1.5 of [1], there is an SL_2-bundle F on X such that $H^0(X, \mathrm{ad}(F) \otimes \kappa) = 0$, then choose $E = \mathrm{ad}(F) \oplus (r-3)L$ with the obvious quadratic form). If r and κ are odd, then $H^0(E \otimes \kappa)$ is odd for all $A \in \mathcal{M}_{SO_r^0,X}$. $\square$

7. Affine Lie algebras and groups

In the following sections I suppose $k = \mathbb{C}$. In order to study the infinite Grassmannian I need some basic material on (affine) Lie algebras which I will recall briefly. I start fixing the notations I will use in the rest of these notes. The reader who is not very familiar with Lie algebras may have a closer look at [8].

7.1. **Basic notations.**

7.1.1. *Lie groups.* From here to the end of these notes, G will be a simple (not necessarily simply connected) complex algebraic group. Let $\widetilde{G} \to G$ be the universal cover of G; its kernel is a subgroup of the center $Z(\widetilde{G}) \subset \widetilde{G}$, canonically isomorphic to $\pi_1(G)$. We will denote the adjoint group $\widetilde{G}/Z(\widetilde{G})$ by $\overline{G}$. We will fix a Cartan subgroup $\overline{H} \subset \overline{G}$ (and denote by H and $\widetilde{H}$ its inverse image in G and $\widetilde{G}$ respectively) as well as a Borel subgroup $\overline{B} \subset \overline{G}$ (and denote by B and $\widetilde{B}$ its inverse image in G and $\widetilde{G}$ respectively).

7.1.2. *Lie algebras.* Let $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{b} = \mathrm{Lie}(B)$ and $\mathfrak{h} = \mathrm{Lie}(H)$. By the roots of $\mathfrak{g}$ we understand the set R of linear forms α on $\mathfrak{h}$ such that $\mathfrak{g}^\alpha = \{X \in \mathfrak{g}/[X, H] = \alpha(H) \; \forall H \in \mathfrak{h}\}$ is non trivial. We have the root decomposition

$$\mathfrak{g} = \mathfrak{h} \oplus \big(\underset{\alpha \in R}{\oplus} \mathfrak{g}^\alpha \big).$$

Let $\Delta = \{\alpha_1, ..., \alpha_r\}$ be the basis of R defined by B and θ be the corresponding highest root; we denote $\alpha_0 = -\theta$. Let $(\, , \,)$ be the Cartan-Killing form, normalized such that $(\theta, \theta) = 2$. Using $(\, , \,)$ we will identify $\mathfrak{h}$ and $\mathfrak{h}^*$ in the sequel. The coroots of $\mathfrak{g}$ are the elements of $\mathfrak{h}$ defined by $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$; they form the dual root system $R^\vee$.

Let $Q(R)$ and $Q(R^\vee)$ be the root and coroot lattices with basis given by $\{\alpha_1, ..., \alpha_r\}$ and $\{\alpha_1^\vee, ..., \alpha_r^\vee\}$ respectively . We denote $P(R)$ and $P(R^\vee)$ the weight and coweight lattices, *i.e.* the lattices dual to $Q(R^\vee)$ and $Q(R)$ respectively. They have basis given by the fundamental weights ϖ_i and coweights $\varpi_i^\vee$ defined by

$$< \varpi_i, \alpha_j^\vee > \, = \, < \varpi_i^\vee, \alpha_j > \, = \delta_{ij}.$$

Note that $Q(R^\vee) \subset Q(R)$ and $P(R^\vee) \subset P(R)$ and that we have equality if all roots are of equal length, *i.e.* if we are in the A-D-E case.

7.1.3. *Representations.* We denote by $P_+ \subset P(R)$ the set of dominant weights and by $\{\varpi_1, \ldots, \varpi_r\}$ the fundamental weights. The set P_+ is in bijection with the set of simple $\mathfrak{g}$-modules; denote by $L(\lambda)$ the $\mathfrak{g}$-module associated to the dominant weight λ.

7.1.4. *The center.* We will identify the quotient $P(R^\vee)/Q(R^\vee)$ with $Z(\widetilde{G})$ through the exponential map; its Pontrjagin dual $\mathrm{Hom}(Z(\widetilde{G}), \mathbf{C}^*)$ identifies to $P(R)/Q(R)$. Recall from ([Bourbaki], VIII, §3, prop. 8) that a system of representatives of $P(R^\vee)/Q(R^\vee)$ is given by the minuscule coweights of

R : these are exactly the fundamental coweights $\varpi_j^\vee$, corresponding to the roots $\alpha_j \in \Delta$ having coefficient 1 when writing

$$\theta = \sum_{\alpha_i \in \Delta} n_i \alpha_i.$$

We will denote $J(\widetilde{G}) = \{i \in \{1,\ldots,r\}/n_i = 1\}$ and $J_0(\widetilde{G}) = J(\widetilde{G}) \cup \{0\}$. Then the set $J_0(\widetilde{G})$ has a natural group structure provided by the group structure on $P(R^\vee)/Q(R^\vee)$ which we will denote additively. Recall, for further reference, that the miniscule coweights are given by

Type of $\mathfrak{g}$	A_r	B_r $(r \geq 2)$	C_r $(r \geq 2)$	D_r $(r \geq 3)$	E_6	E_7	E_8	F_4	G_2
J	$\{1,\ldots,r\}$	$\{1\}$	$\{r\}$	$\{1, r-1, r\}$	$\{1,6\}$	$\{7\}$	$\emptyset$	$\emptyset$	$\emptyset$

For $j \in J_0(\widetilde{G})$ we will denote the corresponding element of $Z(\widetilde{G})$, $\pi_1(\overline{G})$, $P(R)/Q(R)$ or $P(R^\vee)/Q(R^\vee)$ under the above identifications by z_j, τ_j, ϖ_j, $\varpi_j^\vee$ or w_j respectively.

The subgroup of $Z(\widetilde{G})$ corresponding to $\pi_1(G)$ will be denoted by Z, the corresponding subgroup of $J_0(\widetilde{G})$ by J_0 and the lattice generated by $Q(R^\vee)$ and $\varpi_j^\vee$ for $j \in J_0$ by $\Lambda_J(R^\vee)$.

7.1.5. *Dynkin diagrams.* Associated to the Lie algebra $\mathfrak{g}$ is its Cartan matrix A with coefficients $a_{ij} = \langle \alpha_i, \alpha_j^\vee \rangle$, $i,j = 1,\ldots,r$. This matrix is invertible; its determinant is the *connection index* I_c, *i.e.* the index of the root lattice $Q(R)$ in $P(R)$. The associated Dynkin diagram is constructed as follows. The nodes are the simple roots $\alpha_i \in \Pi$; the nodes α_i and α_j are connected by $\max\{|a_{ij}|, |a_{ji}|\}$ lines. Moreover these lines are labeled by ">" if $a_{ij} \neq 0$ and $|a_{ij}| > |a_{ji}|$. These diagrams have various interpretations: the i-th node may be seen representing α_i or ϖ_i or $\alpha_i^\vee$ or been labeled, for example by the dual Coxeter numbers $c_i^\vee$ defined by $\theta^\vee = \sum_{i=1}^r c_i^\vee \alpha_i^\vee$. Note that if ρ is the half sum of the roots $\rho = \sum_{i=1}^r \varpi_i$, then $\langle \rho, \theta^\vee \rangle = \sum_{i=1}^r c_i^\vee$. The number $h^\vee = 1 + \langle \rho, \theta^\vee \rangle$ is called the *dual Coxeter number*. The possible Dynkin diagrams, as well as their connection indexes and dual Coxeter numbers are resumed in table A.

7.2. Affine Lie algebras and groups.

7.2.1. *Loop algebras and central extensions.* Let $L\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}((z))$ be the *loop algebra* of $\mathfrak{g}$. It has a canonical 2-cocycle defined by

$$(7.2\,\mathrm{a}) \qquad \psi_{\mathfrak{g}} : (X \otimes f, Y \otimes g) \mapsto (X, Y) \operatorname*{Res}_{z=0}(g\,df),$$

hence a central extension

$$(7.2\,\mathrm{b}) \qquad 0 \to \mathbb{C} \longrightarrow \widehat{L\mathfrak{g}} \longrightarrow L\mathfrak{g} \to 0$$

In other words, this means that on the level of vector spaces $\widehat{L\mathfrak{g}} = \mathbb{C}c \oplus L\mathfrak{g}$ with Lie bracket given by (c central):

$$(7.2\,\mathrm{c}) \qquad [X \otimes f, Y \otimes g] = [X, Y] \otimes fg + (X, Y) \operatorname*{Res}_{z=0}(g\,df).$$

In the following, we denote $X[f]$ the element $X \otimes f$ of $L\mathfrak{g}$; if $f = z^n$ it is also denoted by $X(n)$. The Lie algebra $L\mathfrak{g}$ has several subalgebras which will be important for us. Define

$$L^+\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} \mathbb{C}[[z]], \ L^{>0}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} z\mathbb{C}[[z]], \ L^{<0}\mathfrak{g} = \mathfrak{g} \otimes_{\mathbb{C}} z^{-1}\mathbb{C}[z^{-1}]$$

These are in fact subalgebras of $\widehat{L\mathfrak{g}}$.

7.2.2. *Irreducible and integrable representations.* In nature, representations of $L\mathfrak{g}$ are projective; this is why we look at (true) representations of $\widehat{L\mathfrak{g}}$. Fix an integer ℓ. Call a representation of $\widehat{L\mathfrak{g}}$ *of level ℓ*, if the center acts by multiplication by ℓ. In order to construct such representations we start with a finite dimensional representation $L(\lambda)$, which we may view as an $L^+\mathfrak{g}$-module by evaluation. As the cocycle (7.2 a) is trivial over $L^+\mathfrak{g}$ the central extension $\widehat{L^+\mathfrak{g}}$ obtained by restriction from (7.2 b) splits. Hence we may consider $L(\lambda)$ as an $\widehat{L^+\mathfrak{g}}$-module of level ℓ by letting the center act by multiplication by ℓ; denote this module $L_\ell(\lambda)$. Now consider the generalized Verma module:

$$M_\ell(\lambda) = \operatorname{Ind}_{\widehat{L^+\mathfrak{g}}}^{\widehat{L\mathfrak{g}}} L_\ell(\lambda) = U(\widehat{L\mathfrak{g}}) \otimes_{U(\widehat{L^+\mathfrak{g}})} L_\ell(\lambda)$$

In the case when ℓ is not the critical level $-h^\vee$ (the dual Coxeter number), $M_\ell(\lambda)$ has a unique irreducible quotient $H_\ell(\lambda)$. Moreover, if $\ell \geq (\lambda, \theta)$ then $H_\ell(\lambda)$ has an important finiteness condition: for all $X \in \mathfrak{g}^\alpha$ and all $f \in \mathbb{C}((z))$ the element $X[f]$ acts locally nilpotent on $H_\ell(\lambda)$, *i.e.* for all $u \in H_\ell(\lambda)$ there is N such that $X[f]^N.u = 0$. Such $\widehat{L\mathfrak{g}}$-modules are called

integrable and it can be shown that all irreducible integrable $\widehat{L\mathfrak{g}}$-modules arise in this way. In view of this it is convenient to define the following set.

$$P_\ell = \{\lambda \in P(R)/(\lambda, \alpha_i) \geq 0 \text{ for } i \in I \text{ and } (\lambda, \theta) \leq \ell\}.$$

In the rest of this subsection we restrict to positive[8] ℓ. Actually if $\lambda \in P_\ell$ then $H_\ell(\lambda)$ is the quotient of $H_\ell(\lambda)$ by the sub-module $Z_\lambda(\ell)$ generated by $X_\theta(-1)^{\ell+1-(\lambda,\theta)} \otimes v_\lambda$, where v_λ is a highest weight vector of $L(\lambda)$. By Poincaré-Birkhoff-Witt, $M_\lambda(\ell) = U(L^{<0}\mathfrak{g}) \otimes_\mathbb{C} L_\lambda$. It follows that we have the exact sequence

(7.2 d) $0 \to Z_\lambda(\ell) \longrightarrow U(L^{<0}\mathfrak{g}) \otimes_\mathbb{C} L_\ell(\lambda) \longrightarrow H_\ell(\lambda) \to 0.$

In other words:

(7.2 e) $[H_\ell(\lambda)]^{L^{>0}\mathfrak{g}} = L_\ell(\lambda) = \{v \in H_\ell(\lambda)/L^{>0}\mathfrak{g}.v = 0\}$

(7.2 f) $H_\ell(\lambda)$ is generated by $L(\lambda)$ over $L^{<0}\mathfrak{g}$ with only one relation:

$$X_\theta(-1)^{\ell+1-(\lambda,\theta)} \otimes v_\lambda = 0.$$

7.3. **Loop groups and central extensions.** We already have defined the loop groups LG and L^+G in 5.1. The Lie algebra of LG is $L\mathfrak{g}$, as the kernel of the homomorphism $LG(R[\varepsilon]) \to LG(R)$ is $L\mathfrak{g}(R) = \mathfrak{g} \otimes_\mathbb{C} R((z))$. For the same reason we have $\text{Lie}(L^+G) = L^+\mathfrak{g}$.

7.3.1. *The adjoint action.* Let H be an infinite dimensional vector space over $\mathbb{C}$. We define the $\mathbb{C}$-space $\text{End}(H)$ by $R \mapsto \text{End}(H \otimes_\mathbb{C} R)$, the $\mathbb{C}$-group $\mathbf{GL}(H)$ as the group of its units and $\mathbf{PGL}(H)$ by $\mathbf{GL}(H)/G_m$. The $\mathbb{C}$-group $L\widetilde{G}$ acts on $L\mathfrak{g}$ by the adjoint action. We define the adjoint action of $L\widetilde{G}$ on $\widehat{L\mathfrak{g}}$ as follows:

$$\text{Ad}(\gamma).(\alpha', s) = \big(\text{Ad}(\gamma).\alpha', s + \text{Res}_{z=0}(\gamma^{-1}\frac{d}{dz}\gamma, \alpha')\big)$$

where $\gamma \in L\widetilde{G}(R)$, $\alpha = (\alpha', s) \in \widehat{L\mathfrak{g}}(R)$ and $(\,,\,)$ is the $R((z))$-bilinear extension of the Cartan-Killing form. Consider an integral highest weight representation $\bar\pi : \widehat{L\mathfrak{g}} \to \text{End}(H)$. The basic result we will use in the sequel is the following:

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[8]The interesting case for us is actually $\ell = -h^\vee$; I will come back to this later. See also Frenkel's lectures.

7.3.2. Proposition. *(Faltings) Let R be a $\mathbb{C}$-algebra, $\gamma \in L\widetilde{G}(R)$. Locally over $\mathrm{Spec}(R)$, there is an automorphism u_γ of $H_R = H \otimes_{\mathbb{C}} R$, unique up to R^*, such that*

(7.3 a)
$$
\begin{array}{ccc}
H_R & \xrightarrow{\bar{\pi}(\alpha)} & H_R \\
{\scriptstyle u_\gamma}\Big\downarrow & & \Big\downarrow{\scriptstyle u_\gamma} \\
H_R & \xrightarrow[\bar{\pi}(\mathrm{Ad}(\gamma).\alpha)]{} & H_R
\end{array}
$$

for any $\alpha \in \widehat{L\mathfrak{g}}(R)$.

Again, the important fact here is that we work over any $\mathbb{C}$-algebra (and not only over $\mathbb{C}$.) The above proposition is proved in ([2], App. A) in the case SL_r; its generalization to G is straightforward.

7.3.3. *Integration.* An immediate corollary of the above proposition is that the representation $\bar{\pi}$ may be "integrated" to a (unique) *algebraic* projective representation of $L\widetilde{G}$, *i.e.* that there is a morphism of $\mathbb{C}$-groups $\pi : L\widetilde{G} \to \mathbf{PGL}(H)$ whose derivate coincides with $\bar{\pi}$ up to homothety. Indeed, thanks to the unicity property the automorphisms u associated locally to γ glue together to define an element $\pi(\gamma) \in \mathbf{PGL}(H)(R)$ and still because of the unicity property, π defines a morphism of $\mathbb{C}$-groups. The assertion on the derivative is a consequence of (7.3 a).

7.3.4. *Central extensions.* We are now looking for a central extension of $L\widetilde{G}$ such that its derivative is the canonical central extension (7.2 b). In order to do this, we apply the above to the basic representation $H_1(0)$ of $\widehat{L\mathfrak{g}}$. Consider the central extension

(7.3 b) $\qquad 1 \to G_m \longrightarrow \mathbf{GL}(H_1(0)) \longrightarrow \mathbf{PGL}(H_1(0)) \to 1.$

Then the pullback of (7.3 b) to $L\widetilde{G}$ is convenient: it defines a central extension to which we refer to as the *canonical* central extension of $L\widetilde{G}$:

(7.3 c) $\qquad\qquad\qquad 1 \to G_m \longrightarrow \widehat{L\widetilde{G}} \longrightarrow L\widetilde{G} \to 1$

What happens if we restrict to $L^+\widetilde{G}$?

7.3.5. Lemma. *The extension (7.3 c) splits canonically over $L^+\widetilde{G}$.*

Proof. ([14], 4.9) By construction of (7.3 c), it is enough to prove that the representation $\bar{\pi} : L^+\widetilde{G} \to \mathrm{End}(H_1(0))$ integrates to a representation $\pi : L^+\mathfrak{g} \to \mathbf{GL}(H_1(0))$. This will follow from the fact that in the case $\gamma \in$

$L^+\widetilde{G}(R)$ we can *normalize* the automorphism u of Proposition 7.3.2. Indeed, as $L(0) = [H_1(0)]^{L^+\mathfrak{g}}$ by (7.2 e), it follows from (7.3 a) that u maps $L(0)_R$ to $L(0)_R$. Now $L(0)_R$ is a free R-module of rank one, hence we may choose u (in a unique way) such that it induces the identity on $L(0)_R$. $\qquad\square$

8. The infinite Grassmannian

We will now study in some more detail the infinite Grassmannian $\mathcal{Q}_G$ for connected reductive groups over the complex numbers.

8.1. **Ind-schemes.** The category of $\mathbb{C}$-spaces is closed under direct limits. A $\mathbb{C}$-space (resp. $\mathbb{C}$-group) will be called a (strict) *ind-scheme* (resp. *ind-group*) if it is the direct limit of a directed system of quasi-compact $\mathbb{C}$-schemes $(Z_\alpha)_{\alpha\in I}$ such that all the maps $i_{\alpha,\beta} : Z_\alpha \to Z_\beta$ are closed embeddings. Remark that an ind-group is in general not a direct limit of a directed system of *groups*. Any property P of schemes which is stable under passage to closed subschemes make sense for ind-schemes: We say that Z satisfies the ind-property P if each Z_α does. In particular we may define Z to be of ind-finite type or ind-proper.

An ind-scheme is integral (resp. reduced, irreducible) if it is the direct limit of an increasing sequence of integral (resp. reduced, irreducible) $\mathbb{C}$-schemes.

A $\mathbb{C}$-space Z is *formally smooth* if for every $\mathbb{C}$-algebra R and for every nilpotent ideal $I \subset R$ the map $Z(\mathrm{Spec}(R)) \to Z(\mathrm{Spec}(R/I))$ is surjective. If R is an ind-scheme of ind-finite type, then formal smoothness is a local property[9].

8.1.1. **Lemma.** *Let Z be an ind-scheme, direct limit of an increasing sequence of $\mathbb{C}$-schemes. Then the following is true (see [2], 6.3 for a proof).*

(i) *If Z is reduced and is the direct limit of an increasing sequence of $\mathbb{C}$-schemes (Z_n) then $Z = \varinjlim(Z_n)$.*

(ii) *If Z is covered by reduced open sub-ind-schemes, Z is reduced.*

(iii) *The ind-scheme Z is integral if and only if Z is reduced and irreducible.*

(iv) *If U is a $\mathbb{C}$-scheme and $U \times Z$ is integral, Z is integral.*

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[9]Formal smoothness is a weak property in our infinite dimensional setup. For instance, we will see that LGL_r is formally smooth *but not reduced*.

8.2. **The ind-structure of loop groups.** Ind-schemes and ind-groups will be important for us as the $\mathbb{C}$-groups LG, $L_X G$ will be ind-groups, as well as $\mathcal{Q}_G$ which will be an ind-scheme. Actually $L^+ G$ is an infinite product of affine schemes.

8.2.1. Lemma. *The $\mathbb{C}$-group $L^+ GL_r$ is represented by*

$$\mathbf{GL}_r(\mathbb{C}) \times \prod_1^\infty M_r(\mathbb{C})$$

where $M_r(\mathbb{C})$ is the set of $r \times r$-matrices with entries in $\mathbb{C}$.

Proof. This follows from the fact that for any $\mathbb{C}$-algebra R the set $GL_r(R[[z]])$ consists of the matrices of the form $A(z) = \sum_{n=0}^\infty A_n z^n$ with $A_0 \in \mathbf{GL}_r(R)$ and $A_n \in M_r(R)$ for $n \geq 1$. $\square$

Consider more generally the sub-$\mathbb{C}$-space $LGL_r^{(N)}$ of LGL_r defined for any $\mathbb{C}$-algebra R by the set $\mathbf{GL}_r^{(N)}(R)$ of matrices $A(z)$ such that both $A(z)$ and $A(z)^{-1}$ have poles of order $\leq N$. Of course, by definition $LGL_r^{(0)} = L^+ GL_r$.

8.2.2. Lemma. *The $\mathbb{C}$-space $LGL_r^{(N)}$ is representable as an affine scheme.*

Proof. If $M^{(N)}(R)$ is the set of $r \times r$-matrices with coefficients in R, then the corresponding $\mathbb{C}$-space $\mathbf{M}^{(N)}$ is represented by the affine scheme

$$\prod_{n=-N}^\infty M_r(\mathbb{C})$$

Now remark that $LGL_r^{(N)}$ is represented by the closed affine subscheme of $\mathbf{M}^{(N)} \times \mathbf{M}^{(N)}$ of pairs of matrices $(A(z), B(z))$ such that $A(z)B(z) = I$. $\square$

8.2.3. Corollary. *The $\mathbb{C}$-group LGL_r is an ind-group of ind-finite type, direct limit of the increasing sequence of schemes $(LGL_r^{(N)})_{N \geq 0}$*

For a general reductive group choose an embedding $G \subset \mathbf{GL}_r$. Then the ind-structure of LGL_r induces an ind-structure on LG.

8.3. **The ind-structure of the infinite Grassmannian.** The following theorem describes the ind-structure of $\mathcal{Q}_G$

8.3.1. Theorem. *Let G be a connected reductive complex group. Then*
  (i) *The $\mathbb{C}$-space $\mathcal{Q}_G$ is an ind-scheme, ind-proper of ind-finite type.*
  (ii) *The projection $\pi : LG \to \mathcal{Q}_G$ admits, locally for the Zariski topology, a section.*

(iii) *The ind-scheme $\mathcal{Q}_G$ is formally smooth.*

(iv) *The ind-scheme $\mathcal{Q}_G$ is reduced if and only if $\operatorname{Hom}(G, G_m) = 0$.*

Proof. It follows from corollary 8.2.3 that $\mathcal{Q}_{GL_r}$ is an ind-scheme of ind-finite type: if $\mathcal{Q}_{GL_r}^{(N)} = LGL_r^{(N)}/L^+GL_r$,

$$\mathcal{Q}_{GL_r} = \varinjlim \mathcal{Q}_{GL_r}^{(N)}$$

In order to see that $\mathcal{Q}_{GL_r}$ is ind-proper we use the following lattice approach to $\mathcal{Q}_{GL_r}$ (see [2],§2). For any $\mathbb{C}$-algebra R we define a *lattice* in $R((z)))^r$ to be a sub-$R[[z]]$-module W of $R((z)))^r$ such that

$$z^N R[[z]]^r \subset W \subset z^{-N} R[[z]]^r$$

for some integer N and such that $W/z^N R[[z]]^r$ is projective.

8.3.2. Proposition. *The $\mathbb{C}$-space $\mathcal{Q}_{GL_r}$ represents the functor which associates to any affine $\mathbb{C}$-scheme $U = \operatorname{Spec}(R)$ the set of lattices $W \subset R((z)))^r$.*

Proof. This is a consequence of Proposition (5.3.2). Indeed if E is a vector bundle over X_U together with a trivialization $\tau : E_{|X_U^*} \xrightarrow{\sim} U \times \mathbb{C}^r$, we get by restriction an isomorphism $\tau^* : R((z))^r \to H^0(D_U^*, E_{|D_U^*})$. The inverse image W of $H^0(D_U, E_{|D_U})$ is a lattice in $R((z))^r$. On the other hand, given a lattice $W \subset R((z))^r$ we get a vector bundle E_W on X by gluing the trivial bundle over X_U^* with the bundle on D_R associated to the $R[[z]]$-module W; the gluing isomorphism is given by $W \times_{R[[z]]} R((z)) \to R((z))^r$ coming from the inclusion $W \subset R((z))^r$. By definition, E_W comes with a trivialization τ_W and it is easy to see that both constructions are inverse to each other. $\square$

It follows from the above that $\mathcal{Q}_{GL_r}$ is ind-proper as the latter functor is.

Let us look at the special case of $G = \mathbf{SL}_r$. Then we obtain *special* lattices, *i.e.* the lattices such that the projective module $W/z^N R[[z]]^r$ is of rank Nr. Let F_N be the complex vector space $z^{-N}\mathbb{C}[[z]]^r/z^N\mathbb{C}[[z]]^r$. Then $\dim(F_N) = 2Nr$. Multiplication by z induces a (nilpotent) endomorphism ζ_N of F_N. If follows that $1 + \zeta_N$ is an automorphism of F_N, hence we get an automorphism of $\operatorname{Grass}(Nr, 2Nr)$; denote by V_N its fixed points. Then it is easy to see from the above proposition that the $\mathbb{C}$-space $LSL_r^{(N)}$ is isomorphic to the projective variety V_N.

Once we know (i) for $\mathbf{GL}_r$ it follows for a general reductive group after choosing an embedding $G \subset \mathbf{GL}_r$ from the following lemma.

8.3.3. Lemma. *Suppose $G \subset H$ is an inclusion of affine algebraic groups such that H/G is affine and such that $\mathcal{Q}_H$ is an ind-scheme of ind-finite*

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type. Then $\mathcal{Q}_G$ is also an ind-scheme of ind-finite type and the morphism $\mathcal{Q}_G \to \mathcal{Q}_H$ is a closed embedding. In particular, if $\mathcal{Q}_H$ is ind-proper, $\mathcal{Q}_G$ is.

Proof. Left to the reader. □

In order to see (ii), one reduces (use that π is LG-invariant) to show that π admits a section over a Zariski open neighborhood of $[e] \in \mathcal{Q}_G$, which follows from the fact that if $L^{<0}G$ is the $\mathbb{C}$-group defined by $R \mapsto G\big(z^{-1}R[z^{-1}]\big)$, the multiplication map

(8.3 a) $$\mu : L^{<0}G \times L^+G \longrightarrow LG$$

is an open immersion. The last statement is proved in ([2], 1.11) using that $\mathcal{Q}_G$ may be seen as parameterizing G-bundles over $\mathbb{P}^1$ and that if E is a G-bundle over $\mathbb{P}^1_U$ for some U then the set of points $u \in U$ such that E_u is trivial is an open subset of U as $H^1(\mathbb{P}^1, \mathcal{O} \otimes \mathfrak{g}) = 0$.

Formal smoothness of $\mathcal{Q}_G$ follows from formal smoothness of LG and the above, so it remains to prove (iv). Actually to see that $\mathcal{Q}_G$ is reduced whenever $\mathrm{Hom}(G, \mathbb{G}_m) = 0$ is quite delicate. It is proved in ([2],6.4) for SL_r where it is deduced from the corresponding statement for $L_{\mathbb{P}^1}G$ and a direct calculation. For general G it is proved in ([14], 4.6) where it is deduced from a theorem of Šafarevič [24]. I will not enter into the proof here, let me just say why $\mathcal{Q}_{\mathbb{G}_m}$ is *not* reduced. Actually, as $L^+\mathbb{G}_m$ is reduced it is equivalent to show that $L\mathbb{G}_m$ is not reduced. Consider $\mu : \mathbb{G}_m \to \mathbb{G}_m$ defined by $\mu(x) = x^n$ and the induced morphism $L\mathbb{G}_m \to L\mathbb{G}_m$. Then the image is not contained in $(L\mathbb{G}_m)_{red}$, hence $L\mathbb{G}_m$ is not reduced. □

8.4. **The connected components of the infinite Grassmannian.** Suppose in this subsection that G is semi-simple and recall the notations of section 7.1.

8.4.1. **Lemma.** *([4], 1.2)*
  (i) *The group $\pi_0(LG)$ is canonically isomorphic to $\pi_1(G)$.*
  (ii) *The projection $\pi : LG \to \mathcal{Q}_G$ induces a bijection $\pi_0(LG) \to \pi_0(\mathcal{Q}_G)$. Each connected component of $\mathcal{Q}_G$ is isomorphic to $\mathcal{Q}_{\widetilde{G}}$.*

Proof. By [21], there exists a finite family of homomorphisms $x_\alpha : \mathbb{G}_a \to \widetilde{G}$ such that for any extension K of $\mathbb{C}$, the subgroups $x_\alpha(K)$ generate $\widetilde{G}(K)$. Since the ind-group $\mathbb{G}_a(\mathbb{C}((z)))$ is connected, it follows that $L\widetilde{G}$ is connected.

In the general case, consider the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$ as an exact sequence of étale sheaves on $D^* := \mathrm{Spec}\, \mathbb{C}((z))$. Since $H^1(D^*, \widetilde{G})$

is trivial [22], it gives rise to an exact sequence of $\mathbb{C}$-groups

$$(8.4\,\mathrm{a}) \qquad 1 \to L\widetilde{G}/\pi_1(G) \longrightarrow LG \longrightarrow H^1(D^*, \pi_1(G)) \to 1$$

Then (i) follows from the connectedness of $L\widetilde{G}$ and the canonical isomorphism $H^1(D^*, \pi_1(G)) \xrightarrow{\sim} \pi_1(G)$ (Puiseux theorem).

To prove (ii), we first observe that the group L^+G is connected: for any $\gamma \in L^+G(\mathbb{C})$, the map $F_\gamma : G \times \mathbb{A}^1 \to L^+G$ defined by $F_\gamma(g, t) = g^{-1}\gamma(tz)$ satisfies $F_\gamma(\gamma(0), 0) = 1$ and $F_\gamma(1, 1) = \gamma$, hence connects γ to the origin. Therefore the canonical map $\pi_0(LG) \to \pi_0(\mathcal{Q}_G)$ is bijective. Moreover it follows from (8.4 a) that $(LG)^{\mathrm{o}}$ is isomorphic to $L\widetilde{G}/\pi_1(G)$. $\qquad\square$

9. The ind-group of loops coming from the open curve

Let G be a connected simple complex group, X be a connected smooth projective complex curve. Recall the notations of 7.1.

9.1. The simply connected case.

9.1.1. Proposition. *([14],5.1) The ind-group $L_X\widetilde{G}$ is integral.*

Proof. To see that $L_X\widetilde{G}$ is reduced, consider the morphism $\bar{\pi} : \mathcal{Q}_{\widetilde{G}} \to \mathcal{M}_{\widetilde{G},X}$, which we know to be locally trivial for the étale topology by the uniformization theorem 5.1.1. Hence, locally for the étale topology, $\bar{\pi}$ is $U \times L_X\widetilde{G} \to U$. Now use that $\mathcal{Q}_{\widetilde{G}}$ is reduced (Theorem 8.3.1) and Lemma 8.1.1 (iv).

To prove that $L_X\widetilde{G}$ is irreducible it is enough, as connected ind-groups are irreducible by Proposition 3 of [24], to show that $L_X\widetilde{G}$ is connected. The idea of its proof is due to V. Drinfeld: consider distinct points $p_1, \ldots, p_i$ of X which are all distinct from p. Define $X_i^* = X$ -$\{p, p_1, \ldots, p_i\}$ and, for every affine k-scheme $U = \mathrm{Spec}(R)$, define $X_{i,U}^* = X_i^* \times_k U$. Denote by $A_{X_{i,U}}$ the $\mathbb{C}$-algebra $\Gamma(X_{i,U}^*, \mathcal{O}_{X_{i,U}^*})$ and by $L_X^i\widetilde{G}$ the $\mathbb{C}$-group $R \mapsto \widetilde{G}(A_{X_{i,U}})$. As $L_X\widetilde{G}$, the $\mathbb{C}$-group $L_X^i\widetilde{G}$ is an ind-group. The natural inclusion $A_{X_{i,R}} \subset A_{X_{i+1,R}}$ defines a closed immersion $f : L_X^i\widetilde{G} \to L_X^{i+1}\widetilde{G}$.

9.1.2. Lemma. *([14], 5.3) The map $f : L_X^i\widetilde{G} \to L_X^{i+1}\widetilde{G}$ defines a bijection*

$$(9.1\,\mathrm{a}) \qquad\qquad \pi_0(L_X^i\widetilde{G}) \xrightarrow{\sim} \pi_0(L_X^{i+1}\widetilde{G}).$$

Once we know the lemma, we do the following. Let $g \in L_X\widetilde{G}(\mathbb{C})$ and let K be the field of rational functions on X. Using the fact (*cf.* [21]) that $\widetilde{G}(K)$ is generated by the standard unipotent subgroups $U_\alpha(K)$, $\alpha \in \Delta$, we may suppose that g is of the form $\prod_{j \in J} \exp(f_j n_j)$ where the n_j are nilpotent

elements of $\mathfrak{g}$ and $f_j \in K$. Let $\{p_1, \ldots, p_i\}$ be the poles of the functions $f_j, j \in J$. Then the morphism

$$\mathbb{A}^1 \longrightarrow L_X^i \widetilde{G}$$
$$t \mapsto \prod_{j \in J} \exp(t f_j n_j)$$

is a path from g to 1 in $L_X^i \widetilde{G}$. By the above, the morphism $\pi_0(L_X \widetilde{G}) \to \pi_0(L_X^i \widetilde{G})$ is bijective which shows that g and 1 are in the same connected component of $L_X \widetilde{G}$, hence $L_X \widetilde{G}$ is connected. $\qquad \square$

9.1.3. Corollary. *([14],5.2) Every character $\chi : L_X \widetilde{G} \to G_m$ is trivial.*

Proof. The differential of χ, considered as a function on $L_X \widetilde{G}$, is everywhere vanishing. Indeed, since χ is a group morphism, this means that the deduced Lie algebra morphism $\mathfrak{g} \otimes A_X \to \mathbb{C}$ is zero (with $A_X = \mathcal{O}(X^*)$). The derived algebra $[\mathfrak{g} \otimes A_X, \mathfrak{g} \otimes A_X]$ is $[\mathfrak{g}, \mathfrak{g}] \otimes A_X$ and therefore equal to $\mathfrak{g} \otimes A_X$ (as $\mathfrak{g}$ is simple). Therefore any Lie algebra morphism $\mathfrak{g} \otimes A_X \to k$ is trivial. As $L_X \widetilde{G}$ is integral we can write $L_X \widetilde{G}$ as the direct limit of an increasing sequence of integral varieties V_n. The restriction of χ to V_n has again zero derivative and is therefore constant. For large n, the varieties V_n contain 1. This implies $\chi_{|V_n} = 1$ and we are done. $\qquad \square$

9.2. **The general case.**

9.2.1. Lemma. *([4], 1.2)*

 (i) *The group $\pi_0(L_X G)$ is canonically isomorphic to $H^1(X, \pi_1(G))$.*
 (ii) *The group $L_X G$ is contained in the neutral component $(LG)^o$ of LG.*

Proof. Consider the cohomology exact sequence on X^* associated to the exact sequence $1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$. As $H^1(X^*, \widetilde{G})$ is trivial, we get the following exact sequence of $\mathbb{C}$-groups

$$(9.2\,a) \qquad 1 \to L_X \widetilde{G}/\pi_1(G) \to L_X G \to H^1(X^*, \pi_1(G)) \to 1$$

Now using that the restriction $H^1(X, \pi_1(G)) \to H^1(X^*, \pi_1(G))$ is bijective and that $L_X \widetilde{G}$ is connected by 9.1.1 we get (i).

It follows from (8.4 a) and (9.2 a) that (ii) is equivalent to claim that the restriction map $H^1(X^*, \pi_1(G)) \to H^1(D^*, \pi_1(G))$ is zero. But this is a

consequence of the commutative diagram of restriction maps

$$
\begin{array}{ccc}
H^1(X, \pi_1(G)) & \xrightarrow{\;\sim\;} & H^1(X^*, \pi_1(G)) \\
\Big| & & \Big| \\
H^1(D, \pi_1(G)) & \xrightarrow{\hspace{1.5cm}} & H^1(D^*, \pi_1(G))
\end{array}
$$

and the vanishing of $H^1(D, \pi_1(G))$. $\square$

9.2.2. Corollary. *There is a canonical bijection* $\pi_0(\mathcal{M}_{G,X}) \xrightarrow{\sim} \pi_1(G)$.

Proof. This follows from the uniformization theorem and Lemma 8.4.1, (i), (ii) and Lemma 9.2.1 (iv). $\square$

10. The line bundles on the moduli stack of G-bundles

Let G be a connected simple complex group, X be a connected smooth projective complex curve. Recall the notations of 7.1.

10.1. The line bundles on the infinite Grassmannian.

10.2. A natural line bundle. Consider the canonical central extension (7.3 c) of $L\widetilde{G}$ and its restriction to $L^+\widetilde{G}$. Then we may write

(10.2 a) $$\mathcal{Q}_{\widetilde{G}} = \widehat{L\widetilde{G}} \big/ \widehat{L^+\widetilde{G}}$$

By Lemma 7.3.5 we have a canonical character

(10.2 b) $$\chi : \widehat{L^+\widetilde{G}} \xrightarrow{\sim} \mathbb{G}_m \times L^+\widetilde{G} \xrightarrow{p_1} \mathbb{G}_m,$$

hence a line bundle $\mathcal{L}_{\chi^{-1}}$ on the homogeneous space $\mathcal{Q}_{\widetilde{G}}$.

10.2.1. *A line in the infinite Grassmannian.* Consider the morphism of $\mathbb{C}$-groups $\varphi : SL_2 \to LSL_2$ defined by (for R a $\mathbb{C}$-algebra)

$$\phi : SL_2(R) \longrightarrow SL_2\big(R((z))\big)$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix}$$

and moreover the morphism of $\mathbb{C}$-groups $\psi : LSL_2 \to L\widetilde{G}$ deduced from the map $SL_2 \to \widetilde{G}$ associated to the highest root θ. Let

(10.2 c) $$\varphi = \psi \circ \phi : SL_2 \to L\widetilde{G}.$$

The Borel subgroup $B_2 \subset SL_2$ of upper triangular matrices maps to $L^+\widetilde{G}$ by construction, hence we get a morphism $\overline{\varphi} : \mathbb{P}^1_{\mathbb{C}} \to \mathcal{Q}_{\widetilde{G}}$. An easy calculation

shows that the derivative $\text{Lie}(\varphi)$ maps the standard $\mathfrak{sl}_2$-triplet $\{e,f,h\} = \{X_\theta, X_{-\theta}, H_\theta\}$ to the $\mathfrak{sl}_2$-triplet $\{X_{-\theta}\otimes z, X_\theta \otimes z^{-1}, -H_\theta\}$ of $L\mathfrak{g}$.

10.2.2. Proposition.

(i) *The pullback defines an isomorphism $\overline{\varphi}^* : \text{Pic}(\mathcal{Q}_G) \xrightarrow{\sim} \text{Pic}(\mathbb{P}^1_{\mathbb{C}})$*

(ii) *We have $\overline{\varphi}^*(\mathcal{L}_\chi) = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$, i.e. $\text{Pic}(\mathcal{Q}_G) = \mathbb{Z}\mathcal{L}_\chi$*

Proof. (i) follows from [12]. In order to prove (ii), we use that the restriction of (7.3 c) to SL_2 splits, hence φ lifts to a morphism $\widetilde{\varphi} : SL_2 \to \widehat{L\widetilde{G}}$ and all we have to do is to calculate the character of $B_2 \to \widehat{L^+\widetilde{G}} \xrightarrow{\chi} G_m$. For this it is enough to calculate the character of B_2 on the SL_2-module generated by v_0. By (7.2 f) this is the standard representation, so we are done. $\square$

In the following we denote, in view of the above, $\mathcal{L}_\chi$ by $\mathcal{O}_{\mathcal{Q}_G}(1)$.

10.3. Linearized line bundles on the infinite Grassmannian. Consider the group $\text{Pic}_{L_X G}(\mathcal{Q}_G)$ of $L_X G$-linearized line bundles on $\mathcal{Q}_G$. Recall that a $L_X G$-linearization of $\mathcal{L}$ is an isomorphism $m^*\mathcal{L} \xrightarrow{\sim} pr_2^*\mathcal{L}$, where $m : L_X G \times \mathcal{Q}_G \to \mathcal{Q}_G$ is the action of $L_X G$ on $\mathcal{Q}_G$, satisfying the usual cocycle condition. It follows from the section on stacks that

10.3.1. Proposition. *The map $\pi : \mathcal{Q}_G \to \mathcal{M}_{G,X}$ induces an isomorphism*

$$\pi^* : \text{Pic}(\mathcal{M}_{G,X}) \xrightarrow{\sim} \text{Pic}_{L_X G}(\mathcal{Q}_G).$$

Hence, once we know $\text{Pic}_{L_X G}(\mathcal{Q}_G)$, we know $\text{Pic}(\mathcal{M}_{G,X})$.

10.4. The case of simply connected groups. In order to determine the group $\text{Pic}_{L_X \widetilde{G}}(\mathcal{Q}_{\widetilde{G}})$, consider the forgetful morphism $f : \text{Pic}_{L_X \widetilde{G}}(\mathcal{Q}_{\widetilde{G}}) \to \text{Pic}(\mathcal{Q}_{\widetilde{G}})$.

10.4.1. Proposition. *The map $f : \text{Pic}_{L_X \widetilde{G}}(\mathcal{Q}_{\widetilde{G}}) \to \text{Pic}(\mathcal{Q}_{\widetilde{G}})$ is injective.*

Proof. The kernel of this morphism consists of the $L_X \widetilde{G}$-linearizations of the trivial bundle. Any two such trivializations differ by an automorphism of $pr_2^* \mathcal{O}_{\mathcal{Q}_G}$ that is by an invertible function on $L_X \widetilde{G} \times \mathcal{Q}_{\widetilde{G}}$. Since $\mathcal{Q}_{\widetilde{G}}$ is integral, it is the direct limit of the integral projective varieties and this function is the pullback of an invertible function f on $L_X \widetilde{G}$. The cocycle conditions on the linearizations imply that f is a character, hence $f = 1$ by Corollary 9.1.3. $\square$

Once we know that f is injective, we may ask whether f is surjective, *i.e.* whether $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ admits an $L_X \widetilde{G}$-linearization.

10.4.2. Lemma. *The line bundle $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ admits an $L_X\widetilde{G}$-linearization if and only if the restriction of the central extension* (7.3 c) *to $L_X\widetilde{G}$ splits.*

Proof. Let $\mathrm{Mum}_{L_X\widetilde{G}}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1))$ be the Mumford group of $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ under the action of $L_X\widetilde{G}$ on $\mathcal{Q}_{\widetilde{G}}$. This is the group of pairs (f,g) with $g \in L_X\widetilde{G}$ and $f : g^*\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$. As $\mathcal{Q}_{\widetilde{G}}$ is direct limit of integral projective schemes, we get a central extension

$$(10.4\,\mathrm{a}) \qquad 1 \longrightarrow G_m \longrightarrow \mathrm{Mum}_{L_X\widetilde{G}}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)) \longrightarrow L_X G \longrightarrow 1.$$

In this setup, an $L_X G$-linearization of $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ corresponds to a splitting of (10.4 a). Such a construction works in general[10] and is functorial. Now observe that $\widehat{L\widetilde{G}}$ is $\mathrm{Mum}_{LG}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1))$. It follows that the extension (10.4 a) is the pullback to $L_X\widetilde{G}$ of (7.3 c), which proves the lemma. $\square$

Now our question of the surjectivity of f has a positive answer in view of the following.

10.4.3. Theorem. *The restriction of* (7.3 c) *to $L_X G$ splits.*

Proof. Consider the inclusion $i : L_X\widetilde{G} \hookrightarrow L\widetilde{G}$. The map $\mathrm{Lie}(i) : L_X\mathfrak{g} \hookrightarrow L\mathfrak{g}$ sends $X \otimes f$ to $X \otimes \widehat{f}_{/0}$ where $\widehat{f}_{/0}$ is the Laurent development of f at x_0. By the residue theorem the cocycle (7.2 a) is trivial over $L_X\mathfrak{g}$, hence $L_X\mathfrak{g}$ may be seen as a subalgebra of $\widehat{L\mathfrak{g}}$. Consider the basic highest weight representation $H_1(0)$ of level one of $\widehat{L\mathfrak{g}}$ and take coinvariants:

$$B = [H_1(0)]_{L_X\mathfrak{g}} = {}^{H_1(0)}\big/_{L_X\mathfrak{g}.H_1(0)}.$$

The crucial fact[11] I will use is that $B \neq 0$.

Remark that the commutativity of (7.3 a) implies that for $\gamma \in L_X G(R)$ the associated automorphism u_γ of H maps coinvariants to coinvariants. We get a morphism of $\mathbb{C}$-groups $\pi : L_X G \to \mathbf{PGL}(B)$ hence we may consider the diagram

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & G_m & \longrightarrow & \widehat{L_X\widetilde{G}} & \longrightarrow & L_X\widetilde{G} & \longrightarrow & 1 \\
 & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\
1 & \longrightarrow & G_m & \longrightarrow & \mathbf{GL}(B) & \longrightarrow & \mathbf{PGL}(B) & \longrightarrow & 1
\end{array}
$$

---

[10] The reader may consider to define $\mathbf{GL}_2$ as the Mumford group of $\mathcal{O}_{\mathbb{P}^1}(1)$ under the action of $\mathbf{PGL}_2$ on $\mathbb{P}^1$.

[11] This follows from the decomposition formulas of conformal field theory where B is seen as a space of conformal blocks (see [20]).

By construction, the central extension of $L_X\widetilde{G}$ above coincides with the central extension obtained by restriction of (7.3 c) to $L_X\widetilde{G}$. By definition of B, the derivative of π is trivial. As $L_X\widetilde{G}$ is an *integral* ind-group by proposition 9.1.1 it follows that π has to be the constant map identity. Indeed, write L_XG as the direct limit of integral schemes V_n and remark that π has to be constant on V_n; for large n, as V_n contains 1, this constant is $\pi(1) = 1$. So π being the identity, $\hat{\pi}$ factors through G_m which gives the desired splitting. $\square$

10.4.4. Corollary. *The line bundle $\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(1)$ descends to a line bundle denoted $\mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(1)$ on $\mathcal{M}_{\widetilde{G},X}$. Moreover*

(10.4 b) $$\mathrm{Pic}(\mathcal{M}_{\widetilde{G},X}) = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(1)\mathbb{Z}$$

10.5. **The case of the special linear group.** Now that we know that $\mathrm{Pic}(\mathcal{M}_{\widetilde{G},X}) = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(1)\mathbb{Z}$ we may ask what happens to our determinant bundle $\mathcal{D}$.

10.5.1. Lemma. *Let $\mathcal{D}$ be the determinant line bundle on $\mathcal{M}_{SL_r,X}$. Then*

$$\mathcal{D} = \mathcal{O}_{\mathcal{M}_{SL_r,X}}(1)$$

Proof. Consider the morphism $\overline{\varphi}$ of 10.2.1:

$$
\begin{array}{ccc}
\mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\ \overline{\varphi}\ } & \mathcal{Q}_{SL_r} \\
& {\scriptstyle \gamma}\searrow & \Big\downarrow{\scriptstyle \pi} \\
& & \mathcal{M}_{SL_r,X}
\end{array}
$$

Using γ, we get a family E of SL_r-bundles parameterized by $\mathbb{P}^1_{\mathbb{C}}$ and, by the above, we have to show that the determinant line bundle of this family is $\mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$. By definition of φ it is enough to treat the rank 2 case in which this family is easily identified: if we think of $\mathcal{Q}_{SL_2}$ as parameterizing special lattices as in Proposition 8.3.2 and the remarks following it. Then $E_{[a:c]}$ is defined by the inclusion

$$W = \begin{pmatrix} d & cz^{-1} \\ bz & a \end{pmatrix} (\mathbb{C}[[z]] \oplus \mathbb{C}[[z]]) \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z)).$$

As the lattice

$$V = z^{-1}\mathbb{C}[[z]] \oplus \mathbb{C}[[z]] \hookrightarrow \mathbb{C}((z)) \oplus \mathbb{C}((z))$$

defines the rank 2-bundle $F = \mathcal{O}_X(p) \oplus \mathcal{O}_X$, we may view, via the inclusion $W \subset V$, the family $E_{[a:c]}$ as the kernel of the morphism $F \to \mathbb{C}_p$ which maps

the local sections $(z^{-1}f, g)$ to $af(p) - cg(p)$. But then it is easy to see that $\mathcal{D}_E = \mathcal{O}_{\mathbb{P}^1_{\mathbb{C}}}(1)$ ([1], 3.4). $\qquad\square$

10.6. The Dynkin index. Let $\rho : \widetilde{G} \to \mathbf{SL}_r$ be a representation. By extension of structure group we get a morphism of stacks $f_\rho : \mathcal{M}_{\widetilde{G}, X} \to \mathcal{M}_{SL_r, X}$, hence by pullback

$$f_\rho^* : \mathrm{Pic}(\mathcal{M}_{SL_r, X}) \longrightarrow \mathrm{Pic}(\mathcal{M}_{\widetilde{G}, X}).$$

As we have seen, both groups are canonically isomorphic to $\mathbb{Z}$, so f_ρ^* is an injection. The index d_ρ of f_ρ^* is called the *Dynkin index* of ρ. It has been introduced to the theory of G-bundles over curves by Kumar, Narasimhan and Ramanathan [12].

This index may be calculated as follows. Looking at the commutative diagram

$$
\begin{array}{ccc}
\mathcal{Q}_{\widetilde{G}} & \xrightarrow{\widetilde{f}_\rho} & \mathcal{Q}_{SL_r} \\
\big| & & \big| \\
\mathcal{M}_{\widetilde{G}, X} & \xrightarrow{f_\rho} & \mathcal{M}_{SL_r, X}
\end{array}
$$

we see that $\widetilde{f}_\rho^*(\mathcal{O}_{\mathcal{Q}_{SL_r}}(1)) = \mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(d_\rho)$. As the canonical central extension (7.3 c) is $\mathrm{Mum}_{LSL_r}(\mathcal{O}_{\mathcal{Q}_{SL_r}}(1))$, by functoriality of the Mumford group the restriction of (7.3 c) to $L\widetilde{G}$ under $L\rho : L\widetilde{G} \to LSL_r$ defines the Mumford group $\widetilde{L\widetilde{G}} = \mathrm{Mum}_{L\widetilde{G}}(\mathcal{O}_{\mathcal{Q}_{\widetilde{G}}}(d_\rho))$. Looking at the differentials we see that if we restrict the canonical central extension (7.2 b) to $L\mathfrak{g}$

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widetilde{L\mathfrak{g}} & \longrightarrow & L\mathfrak{g} & \longrightarrow & 0 \\
& & \| & & \big| & & \big| L\rho & & \\
0 & \longrightarrow & \mathbb{C} & \longrightarrow & \widehat{L\mathfrak{sl}_r} & \longrightarrow & L\mathfrak{sl}_r & \longrightarrow & 0
\end{array}
$$

all we have to do is to determine the extension $\widetilde{L\mathfrak{g}}$, *i.e.* calculate its cocycle.

10.6.1. Lemma. *Let $\rho : \mathfrak{g} \to \mathfrak{sl}(V)$ be a representation of $\mathfrak{g}$ and consider the central extension obtained by restriction of (7.2 b) to $L\mathfrak{g}$. Then, if $V = \sum_\lambda n_\lambda e^\lambda$ is the formal character of V, its cocycle is given by*

$$(10.6\ a) \qquad\qquad \Big(\frac{1}{2} \sum_\lambda n_\lambda \lambda(H_\theta)^2\Big) \psi_{\mathfrak{g}}$$

where $\psi_{\mathfrak{g}}$ is the cocycle of (7.2 a).

50

Proof. By definition the cocycle is given by $\text{Tr}(\rho(X_\theta)\rho(X_{-\theta}))\psi_{\mathfrak{g}}$, so all we have to do is to calculate this number. For this, decompose the $\mathfrak{sl}_2(\theta)$-module V as $\oplus_i V^{(d_i)}$, where $V^{(d_i)}$ is the standard irreducible $\mathfrak{sl}_2$-module with highest weight d_i. As usual, we may realize $V^{(d_i)}$ as the vector space of homogeneous polynomials in 2 variables x and y of degree d_i. Then X_θ acts as $x\partial/\partial y$, and $X_{-\theta}$ as $y\partial/\partial x$. Using the basis $x^l y^{d_i-l}, l = 0,\ldots,d_i$ of $V(d_i)$, we see

$$\text{Tr}(\rho(X_\theta)\rho(X_{-\theta})) = \sum_i \sum_{k=0}^{d_i} k(d_i + 1 - k).$$

The formal character of the $\mathfrak{sl}_2(\theta)$-module $V^{(d)}$ is $\sum_{k=0}^d e^{d\rho_\theta - k\alpha_\theta}$ where α_θ is the positive root of $\mathfrak{sl}_2(\theta)$ and $\rho_\theta = \frac{1}{2}\alpha_\theta$. Therefore we are reduced to prove the equality

$$\sum_{k=0}^d k(d+1-k) = \frac{1}{2}\sum_{k=0}^d \big((d\rho_\theta - k\alpha_\theta)(H_\theta)\big)^2 = \frac{1}{2}\sum_{k=0}^d (d - 2k)^2$$

which is easy. $\qquad\square$

Define the Dynkin index $d_{\mathfrak{g}}$ of $\mathfrak{g}$ itself by $\gcd(d_\rho)$ where ρ runs over all representations of $\mathfrak{g}$. The Dynkin indices of the fundamental and the adjoint representations, as well as of $\mathfrak{g}$ itself are listed in Table B. If ρ is a representation of $\mathfrak{g}$, we denote by $\mathcal{D}_\rho$ the pullback of the determinant bundle under the morphism $\mathcal{M}_{\widetilde{G},X} \to \mathcal{M}_{SL_r,X}$. Let $\text{Pic}_{\det}(\mathcal{M}_{\widetilde{G},X})$ be the subgroup generated by the $\mathcal{D}_\rho$, where ρ runs over all representations of $\mathfrak{g}$.

10.6.2. Corollary. *The index of* $\text{Pic}_{\det}(\mathcal{M}_{\widetilde{G},X})$ *in* $\text{Pic}(\mathcal{M}_{\widetilde{G},X})$ *is* $d_{\mathfrak{g}}$.

If $\widetilde{G}$ is of type B,D or G_2, choosing a theta-characteristic κ defines a square-root $\mathcal{P}_\kappa$ of the determinant bundle $\mathcal{D} = \mathcal{D}_{\varpi_1}$ (see section 6). As the Picard group is $\mathbb{Z}$ for simply connected groups we see that $\mathcal{P}_\kappa$ does not depend on κ in this case, hence we may denote it simply by $\mathcal{P}$. Looking at Table B, we see that $d_{\mathfrak{g}}$ is 2 in the B,D or G_2, hence

10.6.3. Corollary. *Suppose* $\widetilde{G}$ *is of type* B,D *or* G_2. *Then*

$$\text{Pic}(\mathcal{M}_{\widetilde{G},X}) = \mathbb{Z}\mathcal{P}$$

In particular, in the B,D or G_2 case there are no other line bundles than (powers of) the determinant and the pfaffian line bundles.

We saw in 6.5 that $\omega_{\mathcal{M}_{\widetilde{G},X}} = \mathcal{D}_{\text{Ad}}^{-1}$ admits a square-root $\omega^{\frac{1}{2}}_{\mathcal{M}_{\widetilde{G},X}}(\kappa)$. Again, in the simply connected case, this square root does not depend on κ. Looking at the Dynkin index of the adjoint representation in Table B, we see

10.6.4. Corollary. *Let $\omega_{\mathcal{M}_{\widetilde{G},X}}$ be the dualizing sheaf, $\omega_{\mathcal{M}_{\widetilde{G},X}}^{\frac{1}{2}}$ its canonical (G is simply connected) square root. Then*

$$(10.6\ \mathrm{b}) \qquad\qquad \omega_{\mathcal{M}_{\widetilde{G},X}}^{\frac{1}{2}} = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(-h^{\vee})$$

where $h^{\vee}$ is the dual Coxeter number of $\mathfrak{g}$.

10.7. The non simply connected case. In the non simply connected case, $\mathcal{M}_{G,X}$ acquires $\pi_1(G)$ connected components. I will restrict[12] myself here, for simplicity of the notations, to the component containing the trivial bundle $\mathcal{M}_{G,X}^0$.

10.7.1. *The basic index.* We start by defining a number which will be useful in the sequel. Define the *basic index* $\ell_b(G)$ of G to be the smallest positive integer such that $\ell_b(\varpi_j^{\vee}, \varpi_{j'}^{\vee})$ is an integer for all $j, j' \in J_0$ (recall the notations of 7.1). An easy calculation (see [23], Proposition 2.6.3), shows that this number is given by Table C. In order to state the next theorem correctly, I have to modify one of these numbers[13]: define $\ell_b(\mathbf{SO}_{4m}^{\pm}) = 2$ if m is even.

If A is a finite abelian group, denote $A^{\wedge} = \mathrm{Hom}(A, \mathbb{G}_m)$ its Pontrjagin dual.

10.7.2. Theorem. *([4]) Suppose $g(X) \geq 1$.*

(i) *Let $\mathrm{Pic}_t(\mathcal{M}_{G,X}^0)$ be the torsion subgroup of $\mathrm{Pic}(\mathcal{M}_{G,X}^0)$. Then we have the canonical isomorphism*

$$\mathrm{Pic}_t(\mathcal{M}_{G,X}^0) \xrightarrow{\sim} H^1(X, \pi_1(G))^{\wedge}$$

(ii) *The quotient $\mathrm{Pic}(\mathcal{M}_{G,X}^0)/\mathrm{Pic}_t(\mathcal{M}_{G,X}^0)$ is infinite cyclic. For its positive generator $\mathcal{L}$ we have*

$$f_{\pi}^* \mathcal{L} = \mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(\ell_b)$$

where $\pi : \widetilde{G} \to G$ and $f_{\pi} : \mathcal{M}_{\widetilde{G},X} \to \mathcal{M}_{G,X}^0$ is the morphism defined by extension of the structure group.

Proof. It follows from the proof of Proposition 10.4.1 that the kernel of the forgetful map $f : \mathrm{Pic}_{L_X\widetilde{G}}(\mathcal{Q}_{\widetilde{G}}) \to \mathrm{Pic}(\mathcal{Q}_{\widetilde{G}})$ identifies to the character group

---

[12]This is not really a restriction: actually the result is the same for the other components.

[13]This is related to the fact that the center of $\mathbf{Spin}_{4m}$ is $\mathbb{Z}_2 \times \mathbb{Z}_2$ which has a non trivial central extension.

$\mathcal{X}(L_X G)$ of $L_X G$. Then (i) follows from (9.2 a) and the fact that $L_X \widetilde{G}$ has no non trivial characters by Corollary 9.1.3.

I will not prove (ii) here. Actually one shows, using central extensions of LG, that an obstruction to the existence of $\mathcal{L}$ is that the following pairing (recall the notations of section 7.1.4)

$$c : Z \times Z \longrightarrow \mathbb{G}_m$$
$$(z_j, z'_j) \longmapsto e^{2\pi i (\varpi_j^\vee, \varpi_{j'}^\vee)}$$

is trivial (see also [18], 4.6.3).

Once we know that we can't do better than ℓ_b, we have to show that $\mathcal{O}_{\mathcal{M}_{\widetilde{G},X}}(\ell_b)$ actually descends. This may be easy, as for $G = \mathbf{PGL}_r$, where a pfaffian of $\mathcal{D}_{Ad}$ is convenient (just look at the numbers of Tables B and C) or more complicated, as for SL_r/μ_s with $s \mid r$ (see [13]). $\square$

10.8. **The case of the special orthogonal group.** We close the section by looking in more detail at $G = \mathbf{SO}_r$. According to Theorem 10.7.2, there is a canonical exact sequence

(10.8 a) $$0 \to J_2 \xrightarrow{\lambda} \mathrm{Pic}(\mathcal{M}_{SO_r^0, X}) \longrightarrow \mathbb{Z} \to 0 \ ,$$

where the torsion free quotient is generated by any of the $\mathcal{P}_\kappa$'s.

Denote by $\theta(X) \subset \mathrm{Pic}(X)$ the subgroup of $\mathrm{Pic}(X)$ generated by the theta-characteristics; it is an extension of $\mathbb{Z}$ by J_2.

10.8.1. Proposition. *The map* $\kappa \mapsto \mathcal{P}_\kappa$ *defines an isomorphism*

(10.8 b) $$\mathcal{P} : \theta(X) \xrightarrow{\sim} \mathrm{Pic}(\mathcal{M}_{SO_r^0, X}) \ ,$$

which coincides with λ *on* J_2.

This means that we have a canonical isomorphism of extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & J_2 & \longrightarrow & \theta(X) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \| & & \downarrow{\scriptstyle \mathcal{P}} & & \| & & \\
0 & \longrightarrow & J_2 & \xrightarrow{\lambda} & \mathrm{Pic}(\mathcal{M}_{SO_r^0, X}) & \longrightarrow & \mathbb{Z} & \longrightarrow & 0
\end{array}
$$

Proof. ([4], 5.2) $\square$

TABLE A

Type	Dual Coxeter (*Coxeter*) numbers	I_c	$h^\vee$
A_r	─── ─── ─── ········ ─── ─── 1 1 1 1 1	$r+1$	$r+1$
B_r	─── ─── ─── ········ ───═⟩═ 1 2 2 2 1(2)	2	$2r-1$
C_r	─── ─── ─── ········ ───═⟨═ 1(2) 1(2) 1(2) 1(2) 1	2	$r+1$
D_r	1 ─── ─── ─── ·········· ─── 1 2 2 2 1	4	$2r-2$
E_6	2 │ ─── ─── ─── ─── 1 2 3 2 1	3	12
E_7	2 │ ─── ─── ─── ─── ─── 2 3 4 3 2 1	2	18
E_8	3 │ ─── ─── ─── ─── ─── ─── 2 4 6 5 4 3 2	1	30
F_4	───═⟩═ ─── 2 3 2(4) 1(2)	1	4
G_2	≡⟨≡ 1(3) 2	1	9

TABLE B

Type	Dynkin index	d_{Ad}	$d_{\mathfrak{g}}$
A_r	$\binom{r-1}{0}\quad\binom{r-1}{1}\quad\binom{r-1}{2}\quad\cdots\cdots\quad\binom{r-1}{r-2}\quad\binom{r-1}{r-1}$	$2r+2$	1
B_r	$2\binom{2r-1}{0}\ 2\binom{2r-1}{1}\ 2\binom{2r-1}{2}\quad\cdots\cdots\quad 2\binom{2r-1}{r-2}\Rightarrow 2^{(r-2)}$	$4r-2$	2
C_r	$\binom{2r-2}{0}\quad\binom{2r-2}{1}\quad\binom{2r-2}{2}-\binom{2r-2}{0}\quad\cdots\cdots\quad \binom{2r-2}{r-2}-\binom{2r-2}{r-4}\ \Leftarrow\ \binom{2r-2}{r-1}-\binom{2r-2}{r-3}$	$2r+2$	1
D_r	$2\binom{2r-2}{0}\quad 2\binom{2r-2}{1}\quad 2\binom{2r-2}{2}\quad\cdots\cdots\quad 2\binom{2r-2}{r-3}\ <\ {2^{(r-3)} \atop 2^{(r-3)}}$	$4r-4$	2
E_6	$\begin{array}{c} 24 \\ 6\quad 150\quad 1800\quad 150\quad 6 \end{array}$	24	6
E_7	$\begin{array}{c} 360 \\ 36\quad 4680\quad 297000\quad 17160\quad 648\quad 12 \end{array}$	36	12
E_8	$\begin{array}{c} 85500 \\ 5292000\quad 141605100 \\ 1500\quad 8345660400\quad 1778400\quad 14700\quad 60 \end{array}$	60	60
F_4	$18\quad 882\Rightarrow 126\quad 6$	8	6
G_2	$2\ \Lleftarrow\ 8$	18	2

TABLE C

Type	$Z(\widetilde{G})$	Z	ℓ_b	$G = \widetilde{G}/Z$
A_r	$\mathbf{Z}_{r+1}$	$\mathbf{Z}_{r+1}$	$r+1$	$\mathbf{PGL}_{r+1}$
		$\mathbf{Z}_s,\ s|(r+1)$	smallest k s.t. $\frac{k(r+1)r}{s^2} \in \mathbf{Z}$	
B_r	$\mathbf{Z}_2$	$\mathbf{Z}_2$	1	$\mathbf{SO}_{2r+1}$
C_r	$\mathbf{Z}_2$	$\mathbf{Z}_2$	1 for r even, 2 for r odd	$\mathbf{PSp}_{2r}$
D_{2m}	$\mathbf{Z}_2 \times \mathbf{Z}_2$	$\mathbf{Z}_2^0$	1	$\mathbf{SO}_{4m}$
		$\mathbf{Z}_2^+$	1 for m even, 2 for m odd	$\mathbf{SO}_{4m}^+$
		$\mathbf{Z}_2^-$	1 for m even, 2 for m odd	$\mathbf{SO}_{4m}^-$
		$\mathbf{Z}_2 \times \mathbf{Z}_2$	2	$\mathbf{PSO}_{4m}$
D_{2m+1}	$\mathbf{Z}_4$	$\mathbf{Z}_2$	1	$\mathbf{SO}_{4m+2}$
		$\mathbf{Z}_4$	4	$\mathbf{PSO}_{4m+2}$
E_6	$\mathbf{Z}_3$	$\mathbf{Z}_3$	3	$\mathbf{PE}_6$
E_7	$\mathbf{Z}_2$	$\mathbf{Z}_2$	2	$\mathbf{PE}_7$

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