Enumeration of Linear Threshold Functions from the Lattice of Hyperplane Intersections

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***Abstract*—We present a method for enumerating linear threshold functions of n-dimensional binary inputs. Our starting point is the geometric lattice $\mathcal{L}_n$ of hyperplane intersections in the dual (weight) space. We show how the hyperoctahedral group O_{n+1}, the symmetry group of the $(n+1)$-dimensional hypercube, can be used to construct a symmetry-adapted poset of hyperplane intersections Λ_n which is much more compact and tractable than $\mathcal{L}_n$. A generalized Zeta function and its inverse, the generalized Möbius function, are defined on Λ_n. Symmetry-adapted posets of hyperplane intersections for three-, four-, and five-dimensional inputs are constructed and the number of linear threshold functions is computed from the generalized Möbius function. Finally, we show how equivalence classes of linear threshold functions are enumerated by unfolding the symmetry-adapted poset of hyperplane intersections into a symmetry-adapted face poset. It is hoped that our construction will lead to ways of placing asymptotic bounds on the number of equivalence classes of linear threshold functions.**

***Index Terms*—Arrangement of hyperplanes, geometric lattice, hyperoctahedral group, linear threshold function, McCulloch-Pitts neuron, Möbius function, symmetry-adapted poset of hyperplane intersections, threshold logic, Zeta function.**

I. Introduction

OUR objective is to enumerate linear threshold functions (LTF's) of n variables, $(x_1, x_2, \cdots x_n)$, defined by

$$\begin{aligned} y &= +1 \quad \text{if } w_0 + \sum_{i=1}^{n} w_i x_i \geq 0 \\ &= -1 \quad \text{otherwise.} \end{aligned} \tag{1}$$

Here w_i (for $i = 1 \cdots n$) is the (real-valued) weight of the variable x_i, and w_0 is the bias. The bias and the weight vector together describe a linear threshold unit (LTU). Like the output y, the inputs x_i are also bipolar, i.e. $x_i = \pm 1, i = 1 \cdots n$. Thus input vectors $\vec{X}^T = (x_1, x_1, \cdots, x_n)$ are disposed at the corners of an n-dimensional hypercube. The output y may be regarded as the class label for the corresponding input $\vec{X}$. This equation also defines the transfer function of McCulloch–Pitts model neurons. The problem is to enumerate distinct LTF's which can be computed by varying the weights and the bias.

This problem was considered independently by several authors in the 1960's. (See Dertouzos [5], Lewis and Coates [7], Muroga *et al.* [9], Muroga [10] and Winder [13], [15].) Our method, however, is based on Zaslavsky's work on the combinatorics of arrangements of hyperplanes which was published in 1975 [16]. The main contribution of our paper is to simplify Zaslavsky's counting theorem for symmetric arrangements and to develop the technique so that it can be used not only for counting linear threshold functions but also listing equivalence classes thereof.[1]

A critical reader may well question the value of yet another method for enumerating linear threshold functions. We would argue that a method which places symmetry at the heart of the enumeration process is likely to offer some advantages over its rivals. First, it may be computationally more efficient than previous methods and may therefore lead to a faster enumeration algorithm which in turn would allow us to extend previous tables of LTF's to more variables.[2] More importantly though, our method could lead to ways of answering certain combinatorial questions about linear threshold functions which have not been addressed before. For example, previous work has derived good asymptotic bounds on the number of linear threshold functions. It seems to us, however, that an asymptotic bound on the number of equivalence classes of linear threshold functions would give a better indication of their proliferation with increasing n. Similarly, the computational power of a linear threshold unit has been conventionally measured as the fraction of Boolean functions which are also LTF's. Once again, we argue that the fraction of equivalence classes of Boolean functions which are also equivalence classes of LTF's is a more meaningful figure of merit.

The first observation regarding the enumeration of LTF's is that it is profitable to analyze the problem geometrically in weight space rather than input space. Each input vector defines a hyperplane in weight space via the equation

$$w_0 + \sum_{i=1}^{n} w_i x_i = 0$$

and bisects it into two half-spaces. The set of input vectors represents an arrangement of hyperplanes in weight space and partitions it into a number of convex regions, each corresponding to a linear threshold function. (All weights in a convex region compute the same LTF.) The problem of enumerating LTF's thus becomes one of enumerating regions into which the weight space is partitioned by a given arrangement of hyperplanes.

At this point, it is useful to embed the n-dimensional input in $(n+1)$-dimensional space by appending the element $x_0 = 1$

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[1]The arrangement of hyperplanes associated with linear threshold functions is highly symmetric. LTF's in an equivalence class are related by a symmetry operation.

[2]Linear threshold functions of upto eight variables have been tabulated in [9].

to the input vector and w_0 to the weight vector.[3] We can then rewrite (1) as

$$y = \operatorname{sign}(\vec{W} \cdot \vec{X}) \tag{2}$$

where $\vec{W}$ and $\vec{X}$ are now $(n+1)$-dimensional vectors and $\operatorname{sign}(0) = 1$. $\vec{X}$ may now be viewed as the direction vector of a hyperplane in weight space passing through its origin. The set of inputs describes an arrangement of hyperplanes in weight space, each passing through the origin. Such an arrangement is called a central arrangement [11].

Zaslavsky [16] has proved that the number of regions in a central arrangement of hyperplanes in Euclidean space may be computed from the Möbius function defined on the geometric lattice of hyperplane intersections. In Section II we summarize Zaslavsky's result and illustrate it with a few examples including a rederivation of the number of linearly separable partitions of p input vectors in a general position in n-dimensional space (Cover's formula [3]).

The second observation regarding the LTF's in (2) is that the distribution of input vectors is highly symmetrical and that LTF's from $\vec{W}$ and $-\vec{W}$ are related by reversal of class labels.[4] Linear threshold functions therefore naturally fall into equivalence classes; LTF's in an equivalence class may be transformed into each other by a symmetry transformation which leaves the input set invariant. The problem of enumerating LTF's thus reduces to that of enumerating symmetry-equivalent classes of LTF's. Once we have a member of a class, other members are generated by symmetry transformations.

The symmetry group of LTF's of n binary variables is O_{n+1}, the hyperoctahedral group in $n+1$ dimensions.[5] To see this, we expand the set of inputs by allowing x_0 to be bipolar (± 1) along with other variables. There are now 2^{n+1} inputs disposed at the corners of an $(n+1)$ dimensional hypercube but the weight space arrangement is still the same because $\pm\vec{X}$ define the same hyperplane through the origin. The O_{n+1} symmetry of the weight-space arrangement is now apparent.

Given an LTF specified by $\vec{W}$, a symmetry transformation $g \in O_{n+1}$ which leaves the set of inputs invariant transforms the above LTF into another one specified by $\vec{W}' = g^{-1}\vec{W}$. Thus the hyperoctahedral group O_{n+1} induces a permutation of linear threshold functions. Our task in this paper is to devise a method for enumerating equivalence classes (orbits) of LTF's of n variables under the action of O_{n+1}.

Neither of the two observations made above is original. The geometric interpretation of linear threshold functions as regions in a partition of the weight space by hyperplanes is very well known and has even been used by Winder [14] to derive some results for the number of LTF's. The credit for realising the O_{n+1} symmetry of n-variable LTF's goes to Chow [2] who exploited this observation to produce compact tables of equivalence clases of linear threshold functions. As we have stated before, the contribution of this paper is to show how Zaslavsky's method is simplified by incorporating O_{n+1} symmetry and then to extend it to list equivalence classes of LTF's.

In Section III, we construct a symmetry-adapted poset of hyperplane intersections (Λ) from a geometric lattice $\mathcal{L}$ which is invariant under the action of a group of permutations of the hyperplanes in the arrangement. A pair of generalized Zeta functions, $\underline{\zeta}(\alpha;\beta)$ and $\bar{\zeta}(\alpha;\beta)$, is defined for $\alpha, \beta \in \Lambda$ and their inverses, generalized Möbius functions $\underline{\mu}(\alpha;\beta)$ and $\bar{\mu}(\alpha;\beta)$, are calculated recursively. We then show that Zaslavsky's formula for counting the number of regions in the partition of weight space can be restated in terms of $\underline{\mu}$. The point in constructing the symmetry-adapted poset of hyperplane intersections is that for highly symmetric arrangements—such as those associated with linear threshold functions—it is much more compact, and therefore much more tractable, than the geometric lattice. The construction of Λ and the computation of the generalized Zeta and Möbius functions is illustrated with a few examples.

In the next section we construct symmetry-adapted posets of hyperplane intersections (SAPHI) Λ_3, Λ_4 and Λ_5 for linear threshold functions of three, four, and five variables, respectively. The generalized Zeta function, $\underline{\zeta}(\alpha;\beta)$, is tabulated and its inverse is used to count LTF's. Our construction of Λ_n and computation of $\underline{\zeta}(\alpha;\beta)$ is rather *ad hoc* but its correctness is verified by a combinatorial sum rule. We then show how equivalence classes of LTF's are computed from Λ_n. The calculation yields a symmetry-adapted face poset, a set of equivalence classes of regions of dimensions ranging from 0 to $n+1$ and partially ordered by set inclusion. Rank $n+1$ elements of the symmetry-adapted face poset are equivalence classes of linear threshold functions.

In the final section we articulate some questions that have arisen in the course of this research.

II. Partition of $\mathcal{R}^n$ by a Central Arrangement of Hyperplanes

In this section we briefly summarize the exposition of Zaslavsky's original work by Siu *et al.* [12]. Interested readers should consult the latter reference for further details and proofs.

A. Geometric Lattice of Hyperplane Intersections

Let $\mathcal{A}$ be an arrangement of p hyperplanes in n-dimensional Euclidean space of weight vectors $\vec{W}$. It is assumed that all hyperplanes pass through the origin and the direction vectors are $\{\vec{X}_1, \vec{X}_2, \cdots, \vec{X}_p\}$. Such an arrangement is called a central arrangement [11].

Next consider $\mathcal{L}$, the set comprising the entire weight space W and all intersections of the hyperplanes in $\mathcal{A}$. This set is partially ordered with respect to set inclusion, i.e., for $s, t \in \mathcal{L}, s \leq t$ if $s \subseteq t$.

$\mathcal{L}$ is in fact a lattice because every pair of elements $s, t \in \mathcal{L}$ has a unique greatest lower bound and a unique least upper bound, termed, respectively, the meet and the join of the pair. Note that with W excepted, each element $s \in \mathcal{L}$ is the intersection of a subset $\mathcal{A}_s \subseteq \mathcal{A}$ of the hyperplanes in the arrangement. The meet of a pair of such elements, denoted $s \wedge t$, is the intersection of all the hyperplanes in $\mathcal{A}_s \cup \mathcal{A}_t$ whereas the join, denoted

[3] This also removes the artificial distinction between bias and weights.

[4] If $\vec{W} \cdot \vec{X} = 0$ for some inputs, we can always adjust the weights infinitesimally so that $\vec{W} \cdot \vec{X} \neq 0$ for all inputs. We assume without loss of generality that this is the case.

[5] See Coxeter and Moser [4], Humphreys [6, pp. 170–172], and Chen [1] for an introduction to the hyperoctahedral group. It is also briefly discussed in Section IV-A.

$s \vee t$, is the intersection of all the hyperplanes in $\mathcal{A}_s \cap \mathcal{A}_t$. (If $\mathcal{A}_s \cap \mathcal{A}_t$ is empty, $s \vee t = W$.) Also, $s \wedge W = s$ and $s \vee W = W$. More formally, the meet and join of $s, t \in \mathcal{L}$ is defined as follows:

$$s \wedge t = s \cap t;$$
$$s \vee t = \cap\{h \in \mathcal{L} : s \cup t \subseteq h\}.$$

$\mathcal{L}$, being a lattice, has unique minimal and maximal elements (labeled O and W in the rest of this paper). All chains between any two fixed elements have the same length and $\mathcal{L}$ is therefore a geometric lattice. The rank $r(s)$ of $s \in \mathcal{L}$ is defined as the dimension of s; it has the property of being subadditive, i.e.,

$$r(s \wedge t) + r(s \vee t) \leq r(s) + r(t).$$

B. Möbius Function and Zaslavsky's Theorem

The Zeta function for a pair of variables $s, t \in \mathcal{L}$ is defined so that

$$\zeta(s;t) = 1 \quad \text{if } s \leq t$$
$$= 0 \quad \text{otherwise.}$$

We can write the ζ-function as a matrix and arrange its rows and columns so that it has upper triangular form. The Möbius function $\mu(s;t)$ is defined recursively as follows:

$$\mu(t;t) = 1;$$
$$\mu(s;t) = -\sum_{s<u\leq t} \mu(u;t).$$

Notice that the Möbius function is integer-valued and it is defined only for pairs of comparable elements. Its associated matrix also has upper triangular form and is the inverse of the ζ-matrix.

The following theorem is the key combinatorial result for counting the regions into which the weight space is partitioned by $\mathcal{A}$.

Theorem 1: Zaslavsky's Theorem [16]: The number of regions in $\mathcal{R}^n$ partitioned by a set of hyperplanes (all passing through the origin) is given by $\sum_{s\in\mathcal{L}} |\mu(s;W)|$.

C. Examples

In the rest of this section we illustrate Zaslavsky's theorem with some examples.

Example 1: Consider the following direction vectors in three-dimensional weight space:

$$\vec{X}_1^T = (-1, -1, 1)$$
$$\vec{X}_2^T = (-1, 1, 1)$$
$$\vec{X}_3^T = (1, -1, 1)$$
$$\vec{X}_4^T = (1, 1, 1).$$

The geometric lattice and the Möbius function for this arrangement are given in Fig. 1.

Example 2: Add the direction vector $\vec{X}_5^T = (1, 0, 1)$ to the direction vectors of previous example so that $\vec{X}_3, \vec{X}_4$ and $\vec{X}_5$ are on a straight line. Fig. 2 summarizes the associated geometric lattice.

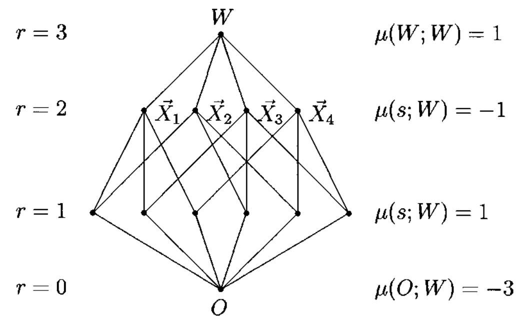


Fig. 1. Geometric lattice for Example 1. The top node represents the entire weight space W; the nodes immediately below the top node represent hyperplanes and the six nodes below that are pairwise intersections of hyperplanes. All hyperplanes pass through the origin O. The nodes at the same "level" have the same rank r which is given on the left. The Möbius function $\mu(s;W)$ depends only on the rank of s and is given on the right hand side of the figure. The number of regions in the weight space is $1 + 4 + 6 + 3 = 14$. Notice that this is a thinly disguised count of linear threshold functions of two variables.

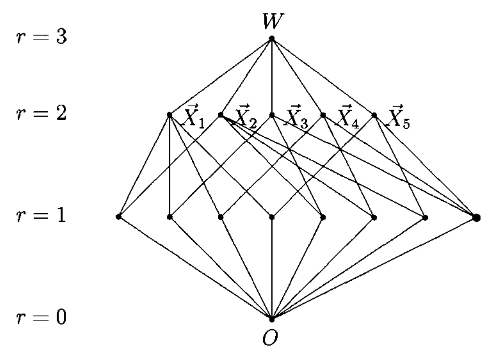


Fig. 2. Geometric lattice for Example 2. Note that $\vec{X}_3, \vec{X}_4$, and $\vec{X}_5$ planes intersect in a line. Apart from $\mu(\vec{X}_3 \cap \vec{X}_4 \cap \vec{X}_5; W) = 2$ and $\mu(O;W) = 5$, $\mu(s;W) = (-1)^{r(W)-r(s)}$ for all other elements. The number of regions in the weight space is 20.

Example 3: Cover's Theorem [3]: Consider an arrangement of p hyperplanes through the origin of n-dimensional weight space. It is assumed that the orientation of the hyperplanes is general, i.e., for $1 \leq d \leq n-1$, any $n-d$ hyperplanes intersect in a d-dimensional linear subspace; the rank of the corresponding element of the geometric lattice is also d. Clearly there are $\binom{p}{n-d}$ elements of rank d. The Möbius function $\mu(s;W)$ depends only on the rank of s, i.e., $\mu(s;W) = f(r(s))$ with $\mu(W;W) = f(n) = 1$. When $1 \leq r(s) < n$, the interval $s < t \leq W$ contains precisely $\binom{n-r(s)}{n-r(t)}$ elements of rank $r(t) > r(s)$. We therefore get the recursion

$$f(r(s)) = -\sum_{r(s)<r(t)\leq n} \binom{n-r(s)}{n-r(t)} f(r(t)) \qquad \text{for } 1 \leq r(s) < n$$

and

$$\mu(O;W) = f(0) = -\sum_{r(t)=1}^{n} \binom{p}{n-r(t)} f(r(t)).$$

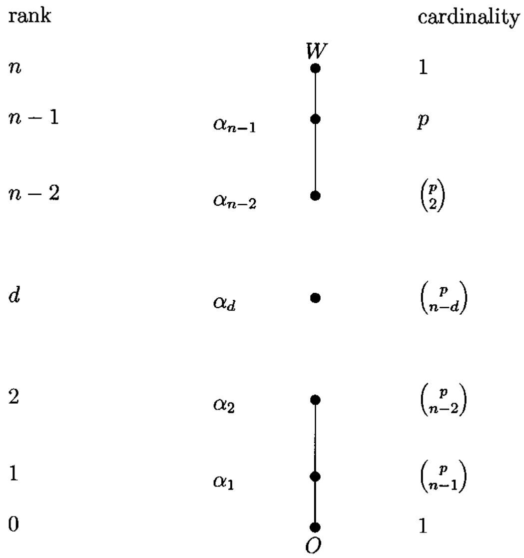


Fig. 3. Symmetry-adapted poset of hyperplane intersections for a central arrangement of p hyperplanes of genereal orientation in n-dimensional space. In this case, the poset is trivially a geometric lattice.

From Zaslavsky's theorem, the number of regions in the partition of the weight space is

$$C(p,n) = f(0) + \sum_{r(t)=1}^{n} \binom{p}{n-r(t)} |f(r(t))|.$$

It is easily verified that

$$f(r(s)) = (-1)^{n-r(s)} \quad \text{for } 1 \leq r(s) \leq n$$

is a solution of the recursion and

$$f(0) = -\sum_{r(t)=1}^{n} \binom{p}{n-r(t)} (-1)^{n-r(t)}.$$

Consequently

$$C(p,n) = \sum_{r(t)=1}^{n} \binom{p}{n-r(t)} \left(1 \pm (-1)^{n-r(t)}\right)$$

where the $+$ sign is to used if $\mu(O;W) < 0$ and the $-$ sign is to be used if $\mu(O;W) > 0$. In either case, one can use standard identities of binomial coefficients to show that

$$C(p,n) = 2\sum_{j=0}^{n-1} \binom{p-1}{j}.$$

This is Cover's celebrated formula for the number of linearly separable partitions of p points in a general position in n-dimensional input space by a hyperplane passing through the origin, or equivalently, the number of regions in a partition of n-dimensional weight space by p hyperplanes of general orientation through the origin [3]. Fig. 3 shows the symmetry-adapted poset of hyperplane intersections for a central arrangement of p hyperplanes of general directions in n-dimensional space.

III. Symmetry Groups and Geometric Lattices

Consider now a permutation σ of the hyperplanes in a central arrangement. The weight space W is, of course, invariant under this permutation. We will restrict ourselves to permutations which 1) induce a permutation of other elements of $\mathcal{L}$ and 2) preserve the order relation, i.e.,

$$s \leq t \Rightarrow \sigma(s) \leq \sigma(t).$$

It follows that these permutations also preserve the rank function and the join and meet of all pairs of elements:

$$\sigma(s) \vee \sigma(t) = \sigma(s \vee t),$$
$$\sigma(s) \wedge \sigma(t) = \sigma(s \wedge t),$$

and

$$r(s) = r(\sigma(s)).$$

Such permutations leave the geometric lattice invariant and their group $\mathcal{G}$ is the symmetry group of $\mathcal{L}$.

A. *Symmetry-Adapted Poset of Hyperplane Intersections (SAPHI)*

Our purpose in considering the symmetry of $\mathcal{L}$ is to simplify its construction and representation. To this end, we construct a symmetry-adapted poset of hyperplane intersections Λ from $\mathcal{L}$ as follows.

1) The action of $\mathcal{G}$ partitions the elements of $\mathcal{L}$ into equivalence classes which we label with Greek letters α, β, etc. The elements of Λ are the equivalence classes of the elements of $\mathcal{L}$.
2) If for $\alpha, \beta \in \Lambda$, there is a corresponding pair of elements $s, t \in \mathcal{L}$ such that $s \in \alpha, t \in \beta$ and $s \leq t$, then $\alpha \leq \beta$; otherwise α and β are incomparable.
3) Λ is a poset but not a lattice because an arbitrary pair $\alpha, \beta \in \Lambda$ does not have a unique greatest lower bound or least upper bound. (For example, in Fig. 6, $\alpha_{1,2}$ and $\alpha_{1,3}$ do not have a unique least upper bound.) It does, however, have unique minimal and maximal elements (O and W, respectively) and all chains between any two fixed elements have the same length. The poset Λ is therefore geometric in character. The rank preserving property of $\mathcal{G}$ ensures that we can define without ambiguity $r(\alpha) = r(s)$ where $s \in \alpha$.

In the next section we will use Λ to compute the Möbius function on $\mathcal{L}$. For this computation we will need

- the cardinality of each equivalence class α;
- for each pair of comparable classes $\alpha \leq \beta$, the number of elements in α with which a typical element $t \in \beta$ can be compared; we will denote this by $\underline{\zeta}(\alpha;\beta)$ and note that $\underline{\zeta}(\alpha;\alpha) = 1$ for all α. Evidently

$$\underline{\zeta}(\alpha;\beta) = \sum_{s\in\alpha} \zeta(s;t)$$

where t is any fixed element in β. Likewise, the number of elements in β with which a typical element $s \in \alpha$ can be compared will be denoted by $\overline{\zeta}(\alpha;\beta)$ with $\overline{\zeta}(\alpha;\alpha)=1$. Once again

$$\overline{\zeta}(\alpha;\beta)=\sum_{t\in\beta}\zeta(s;t)$$

where s is any fixed element in α.

Note that $\underline{\zeta}(\alpha;W)=\overline{\zeta}(O;\alpha)=|\alpha|$ and $\underline{\zeta}(\alpha;\beta)$ and $\overline{\zeta}(\alpha;\beta)$ are related by the simple combinatorial identity

$$|\alpha|\,\overline{\zeta}(\alpha;\beta)=|\beta|\,\underline{\zeta}(\alpha;\beta). \tag{3}$$

The equivalence classes will be arranged so that the matrix $\underline{\zeta}$ is upper triangular. Its inverse, $\underline{\mu}$, which is also upper triangular ($\underline{\mu}(\alpha;\beta)=0$ for $\alpha>\beta$ and $\underline{\mu}(\alpha;\alpha)=1$), will play a key role in the enumeration of the regions of $\mathcal{A}$ via its symmetry-adapted poset of hyperplane intersections. $\underline{\zeta}(\alpha;\beta)$ and $\underline{\mu}(\alpha;\beta)$ are obvious generalizations of the Zeta and Möbius functions of a poset.

From the upper triangular form of matrix $\underline{\zeta}$ we immediately obtain the following recursion for the elements of $\underline{\mu}$:

$$\begin{aligned}\underline{\mu}(\alpha;\alpha)&=1,\\ \underline{\mu}(\alpha;\beta)&=-\sum_{\alpha<\gamma\le\beta}\underline{\zeta}(\alpha;\gamma)\underline{\mu}(\gamma;\beta)\end{aligned} \tag{4}$$

whereby all elements of $\underline{\mu}$ are easily computed.

From this and the previous recursion for the Möbius function on $\mathcal{L}(\mathcal{A})$, it is easy to show that

$$\underline{\mu}(\alpha;\beta)=\sum_{s\in\alpha}\mu(s;t)$$

(t is any fixed element of β). Thus the computation of the Möbius function of $\mathcal{L}(\mathcal{A})$ is simplified considerably by computing the closely related generalized Möbius function on the symmetry-adapted poset of hyperplane intersections because the latter has far fewer elements than the former. Zaslavsky's theorem is now restated as follows:

Theorem 2: Zaslavsky's Theorem Restated: The number of regions in $\mathcal{R}^n$ partitioned by a set of hyperplanes (all passing through the origin) is given by $\sum_{\alpha\in\Lambda}|\underline{\mu}(\alpha;W)|$.

In the rest of this section, we construct the SAPHI for a few example cases and compute the combinatorial function $\underline{\mu}(\alpha;W)$.

Example 4: Hyperplanes of General Orientation: It is clear that the geometric lattice of Example 3 is invariant under all permutations of the p hyperplanes; the associated group is the symmetric group S_p. Given two elements s and t of the same rank, there is some permutation σ of the hyperplanes which carries s into t, i.e., $\sigma(s)=t$. Thus all elements of the same rank are in the same equivalence class.

In this case

$$\begin{aligned}\underline{\zeta}(\alpha_i;\alpha_j)&=\binom{p-n+j}{p-n+i},\quad \text{for } 1\le i\le j\le n-1\\ &=0,\quad \text{otherwise}\\ \underline{\zeta}(\alpha_i;W)&=\binom{p}{n-i}\\ \underline{\zeta}(O;\alpha_i)&=1.\end{aligned}$$

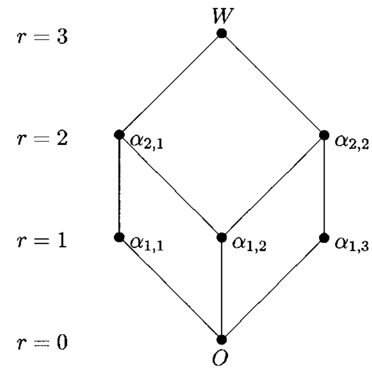


Fig. 4. Symmetry-adapted poset of hyperplane intersections for Example 2. This poset also happens to be a geometric lattice.

Elements of the inverse matrix

$$\underline{\mu}(\alpha_i;W)=(-1)^i\binom{p}{n-i}$$

and Cover's formula follow directly from (4).

Example 5: The geometric lattice of Example 2 is invariant under all permutations of $\{\vec{X}_1,\vec{X}_2\}$ and $\{\vec{X}_3,\vec{X}_4,\vec{X}_5\}$. Its symmetry group is therefore $S_2\times S_3$.

The symmetry adapted poset of hyperplane intersections is drawn in Fig. 4.

The matrix $\underline{\zeta}$ is

$$\underline{\zeta}=\begin{array}{c}\\ O\\ \alpha_{1,1}\\ \alpha_{1,2}\\ \alpha_{1,3}\\ \alpha_{2,1}\\ \alpha_{2,2}\\ W\end{array}\!\!\begin{array}{c}\begin{array}{ccccccc}O & \alpha_{1,1} & \alpha_{1,2} & \alpha_{1,3} & \alpha_{2,1} & \alpha_{2,2} & W\end{array}\\ \begin{pmatrix}1&1&1&1&1&1&1\\0&1&0&0&1&0&1\\0&0&1&0&3&2&6\\0&0&0&1&0&1&1\\0&0&0&0&1&0&2\\0&0&0&0&0&1&3\\0&0&0&0&0&0&1\end{pmatrix}\end{array}.$$

The following matrix elements of $\underline{\mu}$ are easily calculated from recursion 4: $\underline{\mu}(W;W)=1, \underline{\mu}(\alpha_{2,2};W)=-3, \underline{\mu}(\alpha_{2,1};W)=-2, \underline{\mu}(\alpha_{1,3};W)=2, \underline{\mu}(\alpha_{1,2};W)=6, \underline{\mu}(\alpha_{1,1};W)=1$ and the number of regions in the weight space is $\sum_{\alpha\in\Lambda}|\underline{\mu}(\alpha;W)|=20$.

IV. Hyperoctahedral Group and Linear Threshold Functions

In this section we construct symmetry-adapted posets of hyperplane intersections for linear threshold functions of three, four, and five variables. (These are denoted by Λ_3,Λ_4, and Λ_5, respectively; the corresponding geometric lattices are $\mathcal{L}_3,\mathcal{L}_4$, and $\mathcal{L}_5$.) $\mathcal{L}_n$ arises from a central arrangement of 2^n hyperplanes, with direction vectors at the corners of a hypercube, in $n+1$ dimensional weight space. Its symmetry group is O_{n+1}, the $(n+1)$-dimensional hyperoctahedral group. Generalized Zeta and Möbius functions for Λ_3,Λ_4, and Λ_5 are computed and used to count the number of linear threshold functions.

Unlike the illustrative examples of the previous section, symmetry transformations pertinent to linear threshold functions are geometric in the sense that they map the entire weight space onto itself in such a way as to permute the hyperplanes of the arrangement and their intersections. As a result, regions in the partition of weight space are also permuted and fall naturally into equivalence classes. An equivalence class of weight space regions corresponds to an equivalence class of linear threshold functions. At the end of this section, we show how equivalence classes of linear threshold functions are enumerated by unfolding the SAPHI Λ into a symmetry-adapted face poset Π.

A. Hyperoctahedral Group O_n

This is the group of geometric transformations which leave the Cartesian coordinate frame in $\mathcal{R}^n$ invariant [4], [1], and [6, pp. 170–172]. The action of a typical element $g \in O_n$ on a vector $\vec{X}^T = (x_1, x_2, \cdots, x_n)$ is to permute the elements and to change the sign of some of them; elements of O_n have distinct signed-permutation representation. For example

$$g = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 \\ x_2 & -x_3 & x_1 & x_5 & -x_4 \end{pmatrix}$$

is the signed-permutation representation of an element of O_5.

O_n is isomorphic to the wreath product $S_2 \sim S_n$ where S_n is the symmetric group of degree n. The elements of the wreath product are (σ, π) where $\sigma \in S_2^n$ is an n-tuple of the elements of S_2[6] and $\pi \in S_n$ is a permutation of n objects. The rule for composing elements $g_1 = (\sigma_1, \pi_1), g_2 = (\sigma_2, \pi_2), g_1, g_2 \in S_2 \sim S_n$ is

$$g_1 \cdot g_2 = (\sigma_1 \cdot \pi_1\sigma_2, \pi_1 \cdot \pi_2)$$

where $\pi_1\sigma_2$ is the element of S_2^n obtained by permuting the n-tuple of elements of S_2 that is σ_2 according to π_1.

In the signed permutation representation of O_n, there are $n!$ permutations and 2^n possible ways of changing the signs of vector elements. The order of O_n is therefore $2^n n!$.

B. Linear Threshold Functions of Three, Four, and Five Variables

Figs. 5 and 6 show SAPHI Λ_3 and Λ_4 for linear threshold functions of three and four variables, respectively. (Λ_5 is contained implicitly in the genaralized Zeta matrix for five-variable LTF's in the Appendix.) In each case, W is the entire weight space and O is the origin. The generic label for the remaining elements is $\alpha_{r,i}$ where r is the rank and index i distinguishes elements of same rank. Elements of the same rank are placed at the same "level" in the poset and the rank is listed on the left of the figure.

The structure of Λ_3 is further elucidated in Table I. Each element of rank 1 is an equivalence class of lines (through the origin) in weight space. Points on a line are scalar multiples of a weight-vector which can be chosen so that all its elements are integers and their greatest common divisor is 1. The vector $\vec{W}_1$ for an exemplar of its class is given in the first column. Other elements of the class are generated by permuting the elements of $\vec{W}_1$ and changing the sign of a subset of the elements, i.e., by some operation $g \in O_4$. Since $\vec{W}_1$ and $-\vec{W}_1$ define the same element of Λ_3, we avoid double counting by insisting that the first nonzero element of $\vec{W}_1$ is positive.

Elements of higher rank ($r \geq 2$) are equivalence classes of r-dimensional linear subspaces of the weight space. An exemplar of the class is spanned by r basis vectors; we choose the basis set so that each basis vector is an exemplar of a rank 1 element of Λ_3. This choice gives a nonorthogonal basis but facilitates the computation of $\underline{\zeta}$ and $\overline{\zeta}$ matrices.

The direction vectors of the hyperplanes whose intersection gives a member of $\alpha_{r,i}$ are shown diagramatically in the second column of Table I. Once again, of the pair of four-dimensional direction vectors $\pm\vec{X}$ which define the same hyperplane, we show the one for which $x_0 = 1$.

The main body of the table contains elements of the $\underline{\zeta}$ matrix. The last column contains the generalized Möbius function $\underline{\mu}(\alpha; W)$ and the last entry in this column is the sum of its absolute values. This then is the total number of linear threshold functions.

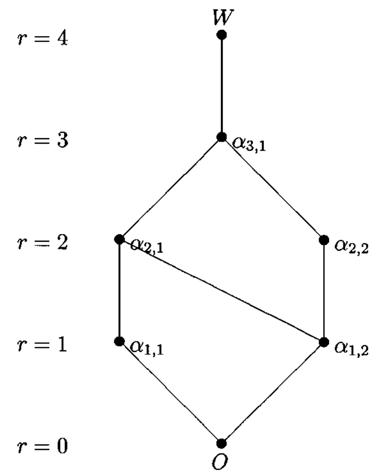


Fig. 5. Symmetry-adapted poset of hyperplane intersections for LTF's of three variables. W is the four-dimensional weight space; each of the other elements is an equivalence class of hyperplane intersections. The generic label for each element is $\alpha_{r,i}$ where r is the rank and i distinguishes elements of same rank. Note that this poset is a geometric lattice as well.

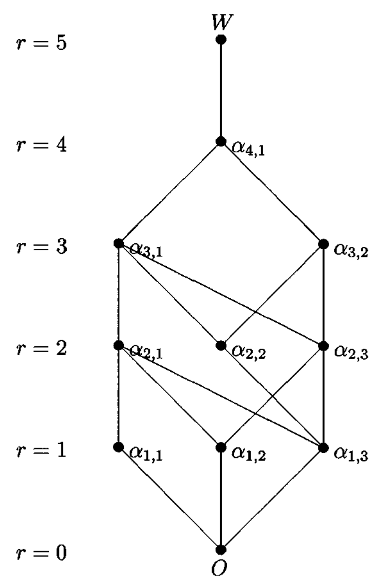
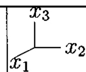
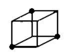
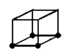
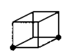
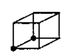
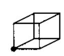


Fig. 6. Symmetry-adapted poset of hyperplane intersections for linear threshold functions of four variables. Its structure is further eleborated in Table II.

[6]S_2^n is the direct product of n copies of S_2.

TABLE I

GENERALIZED ZETA AND MÖBIUS FUNCTIONS FOR THE THE SYMMETRY-ADAPTED POSET OF HYPERPLANE INTERSECTIONS OF Fig. 5. THE DIRECTION VECTORS OF HYPERPLANES WHOSE INTERSECTION GIVES AN EXEMPLAR OF EQUIVALENCE CLASS $\alpha_{r,i}$ ARE SHOWN IN COLUMN 2. (IT IS ASSUMED THAT $x_0 = 1$ IN ALL CASES.) A SET OF r WEIGHT VECTORS WHICH SPAN THE INTERSECTION IS GIVEN IN COLUMN 1. THE MAIN BODY OF THE TABLE CONTAINS THE $\underline{\zeta}$ FUNCTION. THE MÖBIUS FUNCTION $\underline{\mu}(\alpha; W)$ IS LISTED IN THE LAST COLUMN; ITS LAST ENTRY IS $\sum_\alpha |\underline{\mu}(\alpha; W)|$, THE NUMBER OF LINEAR THRESHOLD FUNCTIONS OF THREE VARIABLES

			O	$\alpha_{1,1}$	$\alpha_{1,2}$	$\alpha_{2,1}$	$\alpha_{2,2}$	$\alpha_{3,1}$	W	$\underline{\mu}(\alpha; W)$
		O	1	1	1	1	1	1	1	23
$\vec{W}_1^T = (1111)$		$\alpha_{1,1}$	0	1	0	2	0	3	8	-8
$\vec{W}_1^T = (1001)$		$\alpha_{1,2}$	0	0	1	2	3	6	12	-36
$\vec{W}_1^T = (1001)$ $\vec{W}_2^T = (0110)$		$\alpha_{2,1}$	0	0	0	1	0	3	12	12
$\vec{W}_1^T = (1001)$ $\vec{W}_2^T = (1010)$		$\alpha_{2,2}$	0	0	0	0	1	4	16	16
$\vec{W}_1^T = (1001)$ $\vec{W}_2^T = (1010)$ $\vec{W}_3^T = (1111)$		$\alpha_{3,1}$	0	0	0	0	0	1	8	-8
		W	0	0	0	0	0	0	1	1
										104

The structure of Λ_4 and Λ_5 is similarly elucidated in Tables II and IV.

1) Computing Elements of Λ_n: In this section, we outline our method for computing the elements of Λ_n and the generalized Zeta function. This method is rather *ad hoc* but we are assured of the correctness and completeness of our calculation by a combinatorial sum rule given later. Of course, our results also agree with previous calculations.

In $n+1$-dimensional weight space, a line is at the intersection of n or more hyperplanes. In enumerating rank 1 elements of Λ_n, we therefore look for all possible vectors $\vec{W}_1$ (with interger elements of $gcd = 1$) such that $\vec{W}_1 \cdot \vec{X} = 0$ for n or more direction vectors. The matrix element $\underline{\zeta}(\alpha_{1,i}; W)$, the cardinality of class $\alpha_{1,i}$, is the number of distinct vectors which are generated from $\vec{W}_1$ by the symmetry transformations of O_{n+1}. This is easily calculated.

Rank n elements of $\mathcal{L}_n$ are the hyperplanes and for any given pair (s, t) of these, there is always a symmetry operation $g \in O_{n+1}$ which takes s into t. Therefore there is just one equivalence class of rank n, labeled $\alpha_{n,1}$, and $\underline{\zeta}(\alpha_{n,1}; W) = 2^n$.

It is easy to show that intersections of pairs of hyperplanes are all distinct, i.e., the geometric lattice $\mathcal{L}_n$ has $\binom{2^n}{2}$ elements of rank $n-1$, each identified by a pair of direction vectors. Equivalence classes of rank $n-1$ elements of $\mathcal{L}_n$ are defined by the Hamming distance of the direction vectors: elements $(\vec{X}_1, \vec{X}_2)$ and $(\vec{X}_3, \vec{X}_4)$ are in the same equivalence class if either $H(\vec{X}_1, \vec{X}_2) = H(\vec{X}_3, \vec{X}_4)$ or $H(\vec{X}_1, \vec{X}_2) = (n+1) - H(\vec{X}_3, \vec{X}_4)$, where the second possibility arises because direction vectors $\pm\vec{X}$ refer to the same hyperplane and $H(\vec{X}_3, -\vec{X}_4) = (n+1) - H(\vec{X}_3, \vec{X}_4)$.

Rank 2 elements of Λ_4 and Λ_5 are compiled by considering all possible ways in which basis vectors of rank 1 exemplars may be paired. The computation of $\underline{\zeta}(\alpha_{2,i}; W)$, the cardinality of rank 2 classes, is similar to that of rank 1 classes but requires greater care in avoiding double counting. Rank 3 elements of Λ_5 are similarly computed though the calculation is much more tedious.

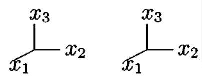
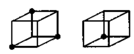
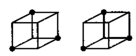
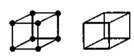
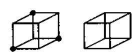
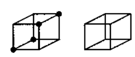
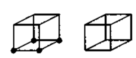
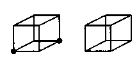
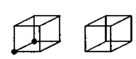
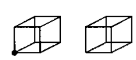
In computing $\underline{\zeta}(\alpha_{r,i}; \alpha_{n,1})$, we note that $\overline{\zeta}(\alpha_{r,i}; \alpha_{n,1})$ (for $r < n$) is just the number of hyperplanes whose intersection gives an exemplar of $\alpha_{r,i}$ and use the combinatorial identity 3. Likewise, $\overline{\zeta}(\alpha_{r,i}; \alpha_{n-1,j})$ is computed by counting the pairs of direction vectors in an exemplar of $\alpha_{r,i}$ whose Hamming distance matches that required for $\alpha_{n-1,j}$ and then using the identity 3. The computation of $\underline{\zeta}(\alpha_{2,i}; \alpha_{3,j})$, elements for Λ_5 was perhaps the most difficult but did not require anything more than very careful counting.

2) A Useful Combinatorial Sum Rule: The following sum rule is invaluable in establishing the completeness of Λ_n and the correctness of the $\underline{\zeta}$ function:

$$\binom{\overline{\zeta}(\alpha; \alpha_{n,1})}{n+1-r} = \sum_\beta \overline{\zeta}(\alpha; \beta) \binom{\overline{\zeta}(\beta; \alpha_{n,1})}{n+1-r}$$

for $\operatorname{rank}(\alpha) < r \le n-1$.

TABLE II
GENERALIZED ZETA MATRIX AND $\underline{\mu}(\alpha; W)$ FOR THE SAPHI FOR FOUR-VARIABLE LTF'S. WEIGHT-SPACE BASIS SETS AND DIRECTION VECTORS OF HYPERPLANES ARE SHOWN IN COLUMNS 1 AND 2, RESPECTIVELY. MATRIX ELEMENTS BETWEEN ELEMENTS OF SAME RANK, $\underline{\zeta}(\alpha_{r,i}; \alpha_{r,j}) = \delta_{i\ j}$, HAVE BEEN SUPPRESSED FOR CLARITY. THE LAST ENTRY IN THE LAST COLUMN IS $\sum_\alpha |\underline{\mu}(\alpha; W)|$

	 $x_4 = -1$ $x_4 = 1$		$\alpha_{2,1}$	$\alpha_{2,2}$	$\alpha_{2,3}$	$\alpha_{3,1}$	$\alpha_{3,2}$	$\alpha_{4,1}$	W	$\underline{\mu}(\alpha; W)$
		O	1	1	1	1	1	1	1	-465
(2 1 1 1 1)		$\alpha_{1,1}$	2	0	0	6	0	20	80	80
(1 1 1 1 0)		$\alpha_{1,2}$	2	2	0	6	3	15	40	280
(1 0 0 0 1)		$\alpha_{1,3}$	1	2	3	4	6	10	20	460
(1 0 0 0 1) (1 1 1 1 0)		$\alpha_{2,1}$				6	0	30	160	-160
(1 0 0 0 1) (1 0 0 1 0)		$\alpha_{2,2}$				3	3	15	60	-180
(1 0 0 0 1) (1 0 1 −1 1)		$\alpha_{2,3}$				1	4	10	40	-120
(1 0 0 0 1) (1 0 0 1 0) (1 1 1 1 0)		$\alpha_{3,1}$						10	80	80
(1 0 0 0 1) (1 0 0 1 0) (1 0 1 −1 1)		$\alpha_{3,2}$						5	40	40
(1 0 0 0 1) (1 0 0 1 0) (1 0 1 0 0) .(1 −1 0 0 0)		$\alpha_{4,1}$							16	-16
		W								1
										1882

The sum on the right–hand side is over all elements $\beta \in \Lambda_n$ which satisfy the following:

$$\text{rank}(\beta) \geq r \tag{5}$$

$$\overline{\zeta}(\beta; \alpha_{n,1}) \geq n + 1 - r \tag{6}$$

$$\beta > \alpha \tag{7}$$

and there is no other $\gamma \in \Lambda_n$ which has the above properties and $\gamma > \beta$. Thus the sum is over the set of maximal elements of all chains whose minimal element is α and other elements have the above properties.

An exemplar of α is the intersection of $\overline{\zeta}(\alpha; \alpha_{n,1})$ hyperplanes and the left–hand side of the above formula gives the number of ways in which a subset of $n + 1 - r$ hyperplanes may be chosen from these. Brief reflection should convince the reader that the right–hand side computes the same number in a different way.

We have found this formula invaluable. For example, $\alpha_{1,1} \in \Lambda_5$ was discovered because the above sum rule had revealed a discrepancy.

3) Equivalence Classes of Linear Threshold Functions: So far we have counted the total number of linear threshold functions of three, four, and five variables. We must, however, go a step further and enumerate equivalence classes of LTF's. We carry out this program by unfolding the SAPHI into a symmetry-adapted face poset. (See Fig. 7 for S-A FP of LTF's of four variables.) The construction is explained below.

In an n-dimensional weight space partitioned by a central arrangement of hyperplanes, each region is an n-dimensional face. An n-dimensional face is bounded by $(n-1)$-dimensional faces which are in turn bounded by $(n-2)$-dimensional faces and so on. (For instance, a three-dimensional region is bounded by planar regions which are bounded by line segments etc.) A partial order of faces is obvious: $f < g$ if face g (or one of its bounding faces) is bounded by f. The topological relationships among the faces of a partition are encapsulated in its face poset (denoted by $\mathcal{F}$ in

TABLE III
WEIGHT-SPACE BASIS SETS FOR EXEMPLAR HYPERPLANE INTERSECTIONS FOR FIVE-VARIABLE LTF'S

α	Basis set	α	Basis set	α	Basis set
$\alpha_{1,1}$	(3 2 2 1 1 1)	$\alpha_{1,2}$	(3 1 1 1 1 1)	$\alpha_{1,3}$	(2 2 1 1 1 1)
$\alpha_{1,2}$	(2 1 1 1 1 0)	$\alpha_{1,5}$	(1 1 1 1 1 1)	$\alpha_{1,6}$	(1 1 1 1 0 0)
$\alpha_{1,7}$	(1 0 0 0 0 1)				
$\alpha_{2,1}$	(1 0 0 0 0 1) (3 1 1 1 2 2)	$\alpha_{2,2}$	(1 1 1 1 0 0) (3 2 2 1 1 1)	$\alpha_{2,3}$	(1 0 0 0 0 1) (2 1 1 1 1 0)
$\alpha_{2,4}$	(1 1 1 1 0 0) (1 1 0 0 1 1)	$\alpha_{2,5}$	(1 0 0 0 0 1) (1 1 1 1 0 0)	$\alpha_{2,6}$	(1 0 0 0 0 1) (0 1 1 1 1 0)
$\alpha_{2,7}$	(1 0 0 0 0 1) (0 0 0 1 1 0)	$\alpha_{2,8}$	(1 0 0 0 0 1) (1 0 0 0 1 0)		
$\alpha_{3,1}$	(1 0 0 0 0 1) (1 1 1 1 0 0) (1 1 0 0 1 1)	$\alpha_{3,2}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (0 1 1 1 1 0)	$\alpha_{3,3}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (0 0 1 1 0 0)
$\alpha_{3,4}$	(1 0 0 0 0 1) (0 1 1 0 0 0) (0 0 0 1 1 0)	$\alpha_{3,5}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (1 0 0 1 0 0)		
$\alpha_{4,1}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (1 0 0 0 1 1) (0 1 1 0 0 0)	$\alpha_{4,2}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (0 1 0 1 0 0) (0 1 1 0 0 0)	$\alpha_{4,3}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (1 0 0 1 0 0) (1 0 1 0 0 0)
$\alpha_{5,1}$	(1 0 0 0 0 1) (1 0 0 0 1 0) (1 0 0 1 0 0) (1 0 1 0 0 0) (1 1 0 0 0 0)				

subsequent discussion). This poset is geometric, i.e., all chains from the minimal element (the origin) to maximal elements (n-dimensional faces) are of the same length and the rank of a face is its dimension. One can think of the face poset as an unfolding of the geometric lattice of hyperplane intersections.

The elements of the symmetry-adapted face poset (Π) are equivalence classes of faces under the action of the symmetry group of the arrangement. A partial order of elements of Π is defined with reference to the partial order of faces: two equivalence classes of faces are comparable if they contain a pair of comparable faces, i.e., $\alpha \leq \beta, \alpha, \beta \in \Pi$ if there exists a pair of faces $s, t \in \mathcal{F}$ such that $s \in \alpha, t \in \beta$ and $s \leq t$. The symmetry-adapted face poset is also geometric in character. An element of rank r is an equivalence class of regions into which an r dimensional subspace of hyperplane intersections is partitioned by other hyperplanes in the arrangement. The highest rank elements are equivalence classes of linear threshold functions.

Before constructing the symmetry-adapted face poset, we extend Dertouzos's original definition of characteristic vectors of linear threshold functions (highest rank elements of the face poset) to lower rank elements. Characteristic vectors in their canonical form (defined below) are then used to label the elements of Π.

Dertouzos [5] has shown that the vector sum of direction vectors weighted by their class label $(y = \pm 1)$

$$\vec{C} = \sum_i y_i \vec{X}_i$$

is distinct for distinct linear threshold functions and may therefore be used to characterize them. Evidently, if a linear threshold function $f_1(x_0, x_1, \cdots, x_n)$ is transformed into another function $f_2(x_0, x_1, \cdots, x_n)$ by some $g \in O_{n+1}$, i.e., $f_2 = gf_1$, then $\vec{C}_2 = g\vec{C}_1$. Thus equivalence classes of linear threshold functions may be labeled by the characteristic vector of an exemplar. Dertouzos has used the canonical form where all elements of $\vec{C}$ are nonnegative and nonincreasing ($c_j \leq c_i$ if $j > i$) and we will keep to his convension. The number of linear threshold functions in class $\vec{C}$ is the number of distinct vectors that are generated from $\vec{C}$ by permuting its elements and changing the sign of some of them.

Consider $s \in \mathcal{L}$ which is an r dimensional intersection of hyperplanes. It is partitioned into a number of regions by the remaining hyperplanes in the arrangement. Let $\vec{W}$ be some weight vector in one of these regions. Ostensibly, $\vec{W} \cdot \vec{X} = 0$ if $s \subseteq \vec{X}$ and nonzero otherwise. The chracteristic vector for this region of s is defined as

$$\vec{C} = \sum_{\vec{X}_i, \vec{W} \cdot \vec{X}_i \neq 0} \operatorname{sign}(\vec{W} \cdot \vec{X}_i) \vec{X}_i.$$

It is easy to show that this vector is distinct for distinct regions of s and may be used to characterize them. Moreover, if two regions of s are related by a symmetry transformation, then so are their characteristic vectors. Equivalence classes of regions of s, and by extension, equivalence classes of regions of equivalence classes of hyperplane intersections, may therefore be labeled by canonical-form characteristic vectors.

The unfolding of Λ_n into Π_n is carried out step-by-step proceeding from low-rank elements to high-rank elements. Consider $\alpha_{r,i}, \alpha_{r+1,j} \in \Lambda_n$ and $\alpha_{r,i} < \alpha_{r+1,j}$. Let s and t be exemplars of the two classes, i.e. $s, t \in \mathcal{L}_n, s \in \alpha_{r,i}, t \in \alpha_{r+1,j}$ and $s < t$. One can then introduce a basis set $\{\vec{W}_1, \vec{W}_2, \cdots, \vec{W}_r\}$ in s and $\{\vec{W}_1, \vec{W}_2, \cdots, \vec{W}_r, \vec{W}_{r+1}\}$ in t. Let $\vec{C}$ be the characteristic vector of a face in s. Then one can move to two distinct faces in t by moving along $\pm\vec{W}_{r+1}$. The characteristic vectors of these two faces are

$$\vec{C}' = \vec{C} \pm \Delta\vec{C}$$

where

$$\Delta\vec{C} = \sum_{\vec{X}_i,\ s \subseteq \vec{X}_i} \operatorname{sign}(\vec{W}_{r+1} \cdot \vec{X}_i) \vec{X}_i.$$

For instance, let $s \in \alpha_{1,2} \in \Lambda_4$ and $t \in \alpha_{2,1} \in \Lambda_4$ be two hyperplane intersections for LTF's of four variables. A basis set in t comprises

$$\vec{W}_1^T = (1\,1\,1\,1\,0)$$
$$\vec{W}_2^T = (1\,0\,0\,0\,1)$$

whereas $\vec{W}_1$ is a basis in s. In fact, s is just the line of scalar multiples of $\vec{W}_1$. It comprises two faces, positive and negative multiples of $\vec{W}_1$, which are related by inversion. There is only one equivalence class of faces and its characteristic vector is $\vec{C}^T = (6\,6\,6\,6\,0)$. It is easy to compute that

$$\Delta\vec{C}^T = \sum_{\vec{X}_i, \vec{W}_1 \cdot \vec{X}_i = 0, \vec{W}_2 \cdot \vec{X}_i \neq 0} \operatorname{sign}(\vec{W}_1 \cdot \vec{X}_i)\ \vec{X}_i^T$$
$$= (3\,{-1}\,{-1}\,{-1}\,3).$$

TABLE IV

GENERALIZED ZETA MATRIX FOR LTF'S OF FIVE VARIABLES. WE HAVE SUPPRESSED THE ELEMENTS $\zeta(\alpha_{r,i};\alpha_{r,j})=\delta_{ij}$ FOR CLARITY. THE LAST ENTRY IN THE LAST COLUMN IS THE TOTAL NUMBER OF FIVE-VARIABLE LTFs. SAPHI Λ_5 MAY BE CONSTRUCTED FROM THIS TABLE BY NOTING THAT $\alpha_{r,i}\le\alpha_{r',j}$ IF $r\le r'$ AND $\zeta(\alpha_{r,i};\alpha_{r',j})\ne 0$

	$\alpha_{2,1}$	$\alpha_{2,2}$	$\alpha_{2,3}$	$\alpha_{2,4}$	$\alpha_{2,5}$	$\alpha_{2,6}$	$\alpha_{2,7}$	$\alpha_{2,8}$	$\alpha_{3,1}$	$\alpha_{3,2}$	$\alpha_{3,3}$	$\alpha_{3,4}$	$\alpha_{3,5}$	$\alpha_{4,1}$	$\alpha_{4,2}$	$\alpha_{4,3}$	$\alpha_{5,1}$	W	$\underline{\mu}(\alpha;W)$
O	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	27129
$\alpha_{1,1}$	2	4	0	0	0	0	0	0	12	3	0	0	0	32	72	0	300	1920	-1920
$\alpha_{1,2}$	0	0	2	0	0	0	0	0	0	3	0	0	0	8	0	0	30	192	-192
$\alpha_{1,3}$	2	0	0	3	0	2	0	0	5	6	0	12	0	32	15	0	105	480	-3360
$\alpha_{1,4}$	2	2	2	0	2	0	0	0	8	6	6	0	0	24	36	20	120	480	-7200
$\alpha_{1,5}$	0	0	1	0	0	2	0	0	0	3	0	4	0	6	0	0	10	32	-1632
$\alpha_{1,6}$	0	2	0	3	2	1	2	0	6	3	6	6	3	15	18	15	45	120	-14760
$\alpha_{1,7}$	1	0	1	0	1	1	2	3	2	3	4	3	6	7	6	10	15	30	-13950
								$\alpha_{2,1}$	4	3	0	0	0	24	36	0	240	1920	1920
								$\alpha_{2,2}$	4	0	0	0	0	12	36	0	180	1440	1440
								$\alpha_{2,3}$	0	3	0	0	0	12	0	0	60	480	480
								$\alpha_{2,4}$	2	0	0	4	0	12	12	0	75	480	1440
								$\alpha_{2,5}$	4	3	6	0	0	24	36	30	180	960	6720
								$\alpha_{2,6}$	0	3	0	6	0	15	0	0	45	240	1680
								$\alpha_{2,7}$	1	0	3	3	3	9	9	15	45	180	4140
								$\alpha_{2,8}$	0	1	1	0	4	4	2	10	20	80	1840
													$\alpha_{3,1}$	6	16	0	135	1440	-1440
													$\alpha_{3,2}$	8	0	0	60	640	-640
													$\alpha_{3,3}$	4	6	10	60	480	-1440
													$\alpha_{3,4}$	3	0	0	15	120	-360
													$\alpha_{3,5}$	1	0	5	15	120	-360
																$\alpha_{4,1}$	15	240	240
																$\alpha_{4,2}$	10	160	160
																$\alpha_{4,3}$	6	96	96
																	$\alpha_{5,1}$	32	-32
																		W	1
																		$\sum\lvert\underline{\mu}\rvert$	94572

and

$$\begin{aligned}\vec{C}'^T &= \vec{C}^T \pm \Delta\vec{C}^T\\ &= (9\,5\,5\,5\,3)\\ &= (3\,7\,7\,7\,{-3}).\end{aligned}$$

One can work upwards in this manner from O to W and unfold entire Λ_n into Π_n.

V. CONCLUSION

In this paper we have extended Zaslavsky's work [16] on counting the number of regions in a partition of Euclidean space by a central arrangement of hyperplanes by simplifying it for large but symmetric arrangements. We have also shown how the simplified method can be used for enumerating equivalence classes of n-variable linear threshold functions.

Our method is *ad hoc* to the extent that although we are able to check if the complete symmetry-adapted poset of hyperplane intersections has been constructed, we do not yet have a constructive algorithm which guarantees the generation of the complete poset. This is one direction in which the work reported in this paper can be carried forward. It would also be useful to compare the computational cost of our method with that of previous work. All previous algorithms have first generated a larger class of Boolean functions guaranteed to contain all linear threshold functions and then tested them for linear separability. Our method, which generates linear threshold functions directly without considering superfluous Boolean functions, may well turn out to be more efficient.

Second, in threshold logic, there has been considerable interest in placing tight asymptotic lower and upper bounds on $\mathrm{LTF}(n)$, the number of n-variable linear threshold functions. Thus, it has been known for some time that

$$\mathrm{LTF}(n) > 2^{\frac{1}{2}n(n-1)+16}, \quad \text{for } n \ge 8,$$

and

$$\mathrm{LTF}(n) \le 2^{n^2}.$$

(The lower bound was proved in [8] and the upper bound follows from Cover's formula.) More recently, Zuev [17] has improved the lower bound to

$$\log_2 \mathrm{LTF}(n) > n^2(1 - 10/\log n)$$

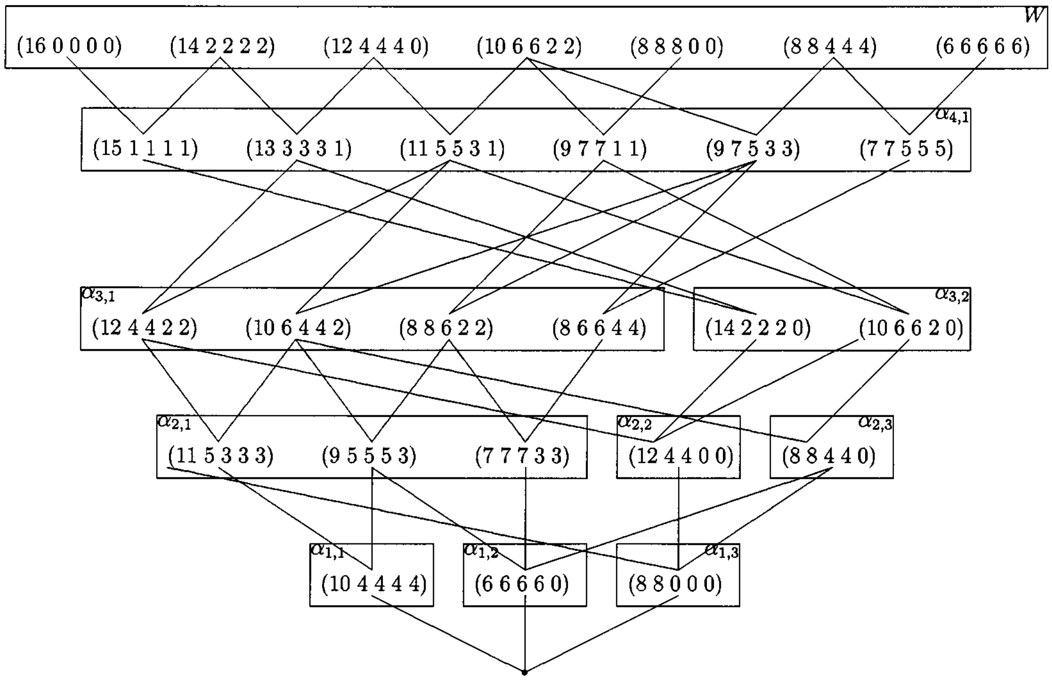


Fig. 7. Symmetry-adapted face poset for four-argument linear threshold functions. Equivalence classes of faces which are obtained by unfolding the same equivalence class of hyperplane intersections of Λ_4 are enclosed in a labeled box.

leading to the conclusion that the previous upperbound was tight, i.e., as $n \to \infty$

$$\frac{1}{n^2}\log_2 \mathrm{LTF}(n) \to 1.$$

We would argue that the asymptotic behavior of $\mathrm{CLTF}(n)$, the number of equivalence classes of linear threshold functions, is a more meaningful measure of how LTF's proliferate with increasing n.[7] It seems to us that the asymptotic behavior of $\mathrm{CLTF}(n)$ was ignored in previous work because previous methods provided no means of analysing it. In contrast, our method suggests how this question may be approached.

Let Ch_n be the number of chains from minimal to maximal elements in the SAPHI Λ_n. In unfolding the SAPHI into a symmetry-adapted face poset, as we step from one element to the next in the chain, the number of characteristic vectors doubles at most. (The characteric vector $\vec{C}$ for an element splits into $\vec{C} \pm \Delta\vec{C}$ for the next element, but the new vectors may be in the same equivalence class. Different chains through the same element may also produce same characteristic vectors.) Thus

$$\mathrm{CLTF}(n) \le 2^n \mathrm{Ch}_n$$

and the problem of placing an upper bound on $\mathrm{CLTF}(n)$ reduces to that of bounding Ch_n.

We next note that the geometric lattice of hyperplane intersections ($\mathcal{L}_n$) associated with LTF's may alternatively be viewed as a sublattice of the lattice of subgroups of O_{n+1}. It follows that the SAPHI Λ_n may be reinterpreted as a subset of a poset whose elements are classes of isomorphic subgroups of O_{n+1} and the number of chains in this larger poset is an upper bound on the number of chains in Λ_n. This observation lifts the problem of bounding $\mathrm{CLTF}(n)$ from its narrow context into the broader realm of group theory. It is our hope that a group theoretical bound on Ch_n, and thereby a bound on $\mathrm{CLTF}(n)$, is realisable.

[7]There is no reason to suppose that the asymptotic behavior of $\mathrm{LTF}(n)$ and $\mathrm{CLTF}(n)$ is the same.

APPENDIX

We present here the results for linear threshold functions of five variables. Exemplar weight-space basis sets for equivalence classes of hyperplane intersections are given in Table III. Direction vectors of associated hyperplanes are easily determined: these are the $\vec{X}s$ for which $\vec{W} \cdot \vec{X} = 0$ for every weight-space basis vector. The generalized Zeta matrix and selected values of the generalized Möbius function are given in Table IV.

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