1. Cover Letter

Dear organizers,

My interests in mathematics revolve around homotopy theory, p-adic geometry, and the geometric Langlands. I am currently working on categorical construction of Whittaker categories in mixed characteristic setting, joint with Ashwin Iyengar (JHU) and Konard Zou (Bonn). As a first step, Ashwin and I have worked out a portion of the geometric Casselman Shalika in mixed characteristic, a long introduction in sec. 2 (feel free to ignore for the experts). Two other ideas I am interested in exploring are the same story but from a deformation and motivic point of view, as I described in Sec. 3.
I strongly believe the workshop will bring new ideas into various aspects of my research. Thank you for considering my application!
Best,
Milton Lin

2. Introduction to mixed characteristic Casselman Shalika

2.1. Classical introduction. Fix a nonarchimedean local field F with ring of integers $\mathcal{O}$, uniformizer ϖ and residue field k of size q. Fix G a split reductive group over F, and choose a Borel subgroup $G = TN$ with T a maximal torus and N the unipotent radical.
Recall that irreducible unramified representations correspond to simple modules for

$$\mathcal{H}_F = \mathcal{C}_c^\infty(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}), \bar{\mathbb{Q}}_\ell),$$

the (commutative) spherical Hecke algebra for G. The Satake isomorphism gives an explicit description of this algebra, via an explicit isomorphism

$$\mathcal{H}_F \xrightarrow{\sim} K_0(\mathrm{Rep}_{\bar{\mathbb{Q}}_\ell}(\widehat{G})) \cong \bar{\mathbb{Q}}_\ell[X_*(T)_+]$$

to the representation ring of G, where $X_*(T)_+$ is the set of dominant coweights.
Write A_λ for the basis vector corresponding to $\lambda \in X_*(T)_+$.
Generic representations embed into the smooth representation $\mathrm{Ind}_{N(F)}^{G(F)}\psi$. We care about unramified representations, so it makes sense to try to understand $(\mathrm{Ind}_{N(F)}^{G(F)})^K$.
In fact, for what we want it turns out we can just look at the compact induction.
It therefore makes sense to try to understand the invariants

$$W = (\mathrm{cInd}_{N(F)}^{G(F)}\psi)^K = \mathcal{C}_c((N(F), \psi)\backslash G(F)/G(\mathcal{O}), \bar{\mathbb{Q}}_\ell)$$

How do you understand this space? There's a decomposition

$$G(F) = \bigsqcup_{\lambda \in X_*(T)} N(F)\lambda(\varpi)G(\mathcal{O}).$$

We can study individual cosets, but one can show that a coset only supports a nonzero function if $\lambda \in X_*(T)_+$ is dominant. Let $\phi_\lambda \in W$ denote the unique nonzero function such that $\phi_0(\lambda(\varpi)) = 1$.

Theorem 2.1 (Shintani, Kato, Casselman–Shalika, Frenkel–Gaitsgory–Kazhdan–Vilonen)**.**

$$\phi_0 \star A_\lambda = q^{(\lambda,\rho)}\phi_\lambda.$$

so W is free of rank 1 over $\mathcal{H}$.

So the action plays very well with respect to the basis. This formula can be used to give an explicit formula for a certain special unramified Whittaker function, namely the one satisfying

(1) $W \cdot A_\lambda = \mathrm{tr}(g(\pi), V_\lambda) \cdot W$.

so that

$$W = \sum_{\lambda \mathrm{dom}} \mathrm{tr}(g(\pi), V_\lambda) \cdot \phi_\lambda$$

2.2. Geometrization. How should we geometrize this? Note the formula amounts to saying that

$$\int_{N(F)} A_\lambda(x^{-1}\nu(\lambda))\psi(x)dx = \begin{cases} 0 & \text{if } \nu \neq \lambda \\ q^{-(\rho,\nu)} & \text{if } \nu = \lambda. \end{cases}$$

In other words,

$$\int_{N(F)\nu(\varpi)} A_\lambda(x)\psi(x^{-1}t^{-\nu})dx = \begin{cases} 0 & \text{if } \nu \neq \lambda \\ q^{-(\rho,\nu)} & \text{if } \nu = \lambda. \end{cases}$$

In view of the Grothendieck–Lefschetz trace formula, if we can exhibit $N(F)\varpi^\nu$ as the points of some geometric object over $\mathbb{F}_q$, then we can probably geometrize this. So how does this work? Well, one way to geometrize the Hecke algebra is to use the affine Grassmannian. If we now assume that $F = \mathbb{F}_q((\varpi))$, then there exists an ind-scheme Gr_G over $\mathbb{F}_q$, whose $\mathbb{F}_q$-points are exactly $G(F)/G(\mathcal{O})$. Inside of this lives an $\mathbb{F}_q$-points $\nu(\varpi)G(\mathcal{O})$, and the $N(F)$-orbit of this point can also be given an ind-subscheme structure. Moreover, $\mathrm{Gr}_G = \mathrm{colim}\mathrm{Gr}_{\leq\lambda}$ can be written as the colimit over a explicit set of projective varieties over $\mathbb{F}_q$.
The function A_λ is geometrized by a certain $G(\mathcal{O})$-equivariant perverse sheaf $\mathcal{A}_\lambda$ defined over $\mathrm{Gr}_{G,\bar{\mathbb{F}}_q}$ whose Frobenius-traces recover $\mathcal{A}_\lambda$. This sheaf is supported on $\mathrm{Gr}_{\leq\lambda}$.
How do you geometrize ψ?. Well, the $N(F)$-orbit of $\nu(\varpi)$ also admits an ind-scheme structure, and we call this $S_\nu \subset \mathrm{Gr}_G$. Since there's a group homomorphism $\mathbb{F}_q((t)) \to \mathbb{F}_q$ sending $f(x) = \sum c_i x^i \mapsto c_{-1}$, we may as well take ψ to be the map

$$N(F) \xrightarrow{c_{-1}} N(\mathbb{F}_q) \to \mathbb{F}_q \to \bar{\mathbb{Q}}_\ell^\times.$$

This naturally admits a geometrization $h : S_\nu \to \mathbb{A}^1$. But $\mathbb{A}^1$ admits a natural étale $\mathbb{F}_q$-torsor by the Lang isogeny, so one gets an Artin–Schreier sheaf $\mathcal{L}_\psi$ over $\mathbb{A}^1$.
Then:

Theorem 2.2 (Geometric Casselman–Shalika, due to Ngo–Polo, Frenkel–Gaitsgory–Vilonen)**.**

$$R\Gamma_c^{\text{ét}}(S_\nu \cap \mathrm{Gr}_{\leq\lambda}, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi) = \begin{cases} 0 & \text{if } \nu \neq \lambda \\ \bar{\mathbb{Q}}_\ell[2(\rho,\nu)](-(\rho,\nu)) & \text{if } \nu = \lambda. \end{cases}$$

Frenkel–Gaitsgory–Vilonen categorify this picture, by defining a "Whittaker category" and defining an action of the Satake category on the Whittaker category, and studying this action, yielding the theorem as a corollary. Ngo–Polo have a more direct proof.

2.3. Mixed Characteristic. What happens in mixed characteristic?

(1) We can no longer use the usual affine Grassmannian. Instead, we have to use the Witt vector affine Grassmannian, where we will later extend to

B_{dR} -Grassmannian. Before, Gr_G was the sheafification of

$$R \mapsto R((t))/R[[t]].$$

In mixed characteristic, it should be the sheafification of

$$R \mapsto W(R)[1/p]/W(R)$$

except that Witt vectors don't behave well unless R is a perfect $\mathbb{F}_q$-algebra.

Xinwen Zhu fixes this problem by simply restricting to perfect R-algebras, or in other words by taking the perfection. He and Bhatt–Scholze then show that this functor is represented by an ind-(perfection of projective variety).

He defines similar subspaces S_ν and $\mathrm{Gr}_{\leq\lambda}$, and shows that they behave very similarly.

(2) A more serious issue is that the additive character

$$\mathbb{F}_q((t)) \to \bar{\mathbb{Q}}_\ell^\times$$

can be chosen to factor through $\mathbb{F}_q$. But if $F = \mathbb{Q}_p$, then any nontrivial $\mathbb{Q}_p \to \bar{\mathbb{Q}}_\ell^\times$ cannot factor through $\mathbb{F}_p$ in any way. In fact, they're all equal to

$$x \mapsto e^{2\pi i a x}.$$

for some $a \in \mathbb{Q}_p$. If we take $a = 1$, then at least it factors through $\mathbb{Q}_p/\mathbb{Z}_p$. But these are exactly the $\mathbb{F}_p$-points of $\mathrm{Gr}_{\mathbb{G}_a}$. So we can still define a morphism

$$h : S_\nu \to \mathrm{Gr}_{\mathbb{G}_a}.$$

But $\mathrm{Gr}_{\mathbb{G}_a}$ is still an infinite-dimensional thing, which makes trying to define some character local system a real headache. But the key observation is that luckily

$$h|_{S_\nu \cap \mathrm{Gr}_{\leq\lambda}}$$

lands in some finite-dimensional sub-(perfect scheme), whose $\mathbb{F}_q$-points are $p^{-r}\mathbb{Z}_p/\mathbb{Z}_p$ (in fact this is the perfection of an affine space).

So

Theorem 2.3 (I.–Lin, in progress)**.** The geometric Casselman–Shalika formula is true in mixed-characteristic.

Our proof essentially blends Ngo–Polo with Zhu's work. The overall strategy is similar, but there are a number of technical differences, due to the fact that h is defined differently, and due to the fact that there is a difference in the way a certain smooth resolution of $\mathrm{Gr}_{\leq\mu}$ is constructed for quasi–minuscule coweights.

2.4. Further questions. We may ask:

(1) To what generalirty can we allow our geometric objects to be? In de Rham sheaf theory: $\mathbb{G}_a$ canont be viewed as a $\mathbb{E}_\infty$-group scheme over $\mathbb{S}$, [10, 1.6.20].

(2) What the appropriate [1] sheaf theory? [2] In the Betti sheaf theory: we need a notion of *constructible sheaves* with *singular support* in Sp. This is unclear but seems more plausible. It may be possible to work out integral case using recent development in *motivic sheaves.*
(3) In defining the Whittaker category when the characteristic is *zero*. One can give a definition of Whittaker category, [6], using the *Kirillov model*. The basic object of study is sheaves on $\mathbb{G}_a \rtimes \mathbb{G}_m$, and the appropriate fourier transform, can we study this in mixed characteristic setting?

[1] A litmus test to a correct sheaf theory, denoted Shv, is the property

$$\mathrm{Whit}(\mathrm{Shv}(G/B;\mathrm{Sp})) \simeq \mathrm{Shv}(T, \mathrm{Sp})$$

for triplet $(B, N, T) = (\text{Borel}, \text{maximal unipotent subgroup}, \text{ maximal torus subgroup})$.

[2] using terminology of [11].

3. Deformations of $\mathrm{Rep}(G_{\mathbb{Z}})$ and integral Whittaker categories

There is an interest in considering *topological twistings*[3] of Langlands, as [7] and [8] in geometric and arithmetic setting respectively. Fundamental to this is an understanding of the moduli of $\mathbb{E}_2$ deformations of the representation category.

$$\mathcal{D}\mathrm{ef}^{(2)}(\mathrm{Rep}_{\mathbb{C}}G) : \mathrm{Art}_{\mathbb{C}}^{(4)} \to \mathcal{S}$$

$$R \mapsto \mathbb{E}_2(\mathrm{LCat}_R) \times_{\mathbb{E}_2(\mathrm{LCat}_{\mathbb{C}})} \{\mathrm{Rep}_{\mathbb{C}}G\}$$

where $\mathrm{Art}_{\mathbb{C}}^{(4)}$ is the category of $\mathbb{E}_4$ Artinian $\mathbb{C}$-algebras. [4] This is classically the moduli problem of *braided tensor deformations*, studied by Yetter and Crane [15], [3]. Its R-points consists of the groupoid of pairs

$$(\mathcal{C}, \alpha) : \mathcal{C} \otimes_R \mathbb{C} \simeq \mathrm{Rep}_{\mathbb{C}}G, \mathcal{C} \in \mathrm{LCat}_R$$

We will now describe this moduli problem from the perspective of geometric representation theory. The *Whittaker construction/category* yields an equivalence of *stable (triangulated) categories*, [4],

$$\mathrm{Rep}_{\mathbb{C}}(G) \simeq \mathrm{Whit}(\mathrm{Dmod}_{\mathbb{C}}(\mathrm{Gr}_{\check{G}})) \tag{1}$$

The Grothendieck group of the right hand side is the module of Whittaker functions, in the sense of B. Casselman and J. Shalika, [2].

Consider the topological parameter space of *multiplicative* $\mathbb{G}_m$-gerbes:

$$\mathrm{Ge}_{\mathbb{G}_m}(\mathrm{Gr}_{\check{G}}) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(\mathrm{Gr}_{\check{G}}, B^2\mathbb{G}_m(R))$$

where $\mathbb{G}_m(R) := (\Omega^\infty R)^\times$, the groupoid of invertible elements.

Theorem 3.1. [9, 10] The map on R-points given by *twisting* [7, 2]

$$\eta \mapsto \mathrm{Whit}^\eta(\mathrm{Dmod}_{\mathbb{C}}(\mathrm{Gr}_{\check{G}}))$$

yields an equivalence

$$\widehat{\mathrm{Ge}_{\mathbb{G}_m}(\mathrm{Gr}}_{\check{G}}) \simeq \mathcal{D}\mathrm{ef}^{(2)}\mathrm{Rep}_{\mathbb{C}}G$$

where the left hand side is the formal completion at the trivial gerbe.

[3] there are *twists* of different flavours:

	algebro-geometric	differential geometric	Topological
Parameter Type	K-theoretic, [14]	Quantum, [5]	Metaplectic, [7].

The metaplectic version gives rise to *gerbes*, which exists in most sheaf theoretic context, see [16].

[4] The $\mathbb{E}_4$ condition is technical and it can be ignored: it is so that LCat_R is an $\mathbb{E}_2$ monoidal category.

3.1. Question: is there an integral version of whittaker category? Can one give

- an *integral* Whittaker category/ integral version of (1)?
- a topological/geometric description of the $\mathbb{E}_2$ deformations of $\mathrm{Rep}_{\mathbb{Z}} G_{\mathbb{Z}}$?

Hints of this *rationally* are in the work of T. Richarz and J. Scholbach [12]. To a prestack X, one constructs presentable stable categories $\mathrm{DM}(X, \mathbb{Q})$ of motives, extending the theory of Ayoub and Cisinki-Déglise.

When $X = \mathrm{LG}^+_{\mathbb{Z}} \backslash LG_{\mathbb{Z}} / \mathrm{LG}^+_{\mathbb{Z}}$ is the automoprhic Hecke stack, X is a stratified space. One has a full subcategory of *stratified Tate motives* $\mathrm{DTM}(X; \mathbb{Q})$. This has a t structure whose heart is the *mixed (stratified) Tate motives*:

Theorem 3.2. [13, Thm. C] For each finite field $\mathbb{F}_q$, there is a symmetric monoidal equivalence,

$$\mathrm{MTM}(\mathrm{LG}^+_{\mathbb{F}_q} \backslash \mathrm{LG}_{\mathbb{F}_q} / \mathrm{LG}^+_{\mathbb{F}_q}) \simeq \mathrm{Rep}_{\mathbb{Q}}(\widehat{G}_1)^{\heartsuit}$$

where $\widehat{G}_1$ is the *modified Langlands dual*, [1, 5].

Our goal is to give a Whittaker construction/category in this context, obtaining a similar equivalence as (1) at the level of *stable (triangulated) categories*.

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