Representation functors

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Some analogies in play

- simplicial commutative ring $\sim$ topological ring
- Simplicial commutative ring : (discrete) commutative ring :: commutative ring : reduced ring.

The classical deformation functor

Fix $\overline{\rho}\colon \Gamma \to G(k)$, and contemplate the functor $\mathrm{F}_{\Gamma}^{\overline{\rho}}$ sending an Artinian algebra A augmented over k to the set of lifts

$$\begin{tikzcd} & G(A)/\sim \arrow[d] \\ \Gamma \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"'] & G(k) \end{tikzcd}$$

Under suitable assumptions, can apply Schlessinger's criterion to pro-represent $\mathrm{F}_{\Gamma}^{\overline{\rho}}$ by the deformation ring $\mathrm{R}_{\Gamma}^{\overline{\rho}}$.

Derived deformation functor

Now, want to define a functor $\mathcal{F}_{\Gamma}^{\overline{\rho}}$, sending an Artinian *simplicial commutative* $A_\bullet$ augmented over k to the *simplicial set* of lifts

$$\begin{tikzcd} & G(A_\bullet)/\sim \arrow[d] \\ \Gamma \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"'] & G(k) \end{tikzcd}$$

Apply derived Schlessinger to get a pro-representing simplicial commutative ring $\mathcal{R}_{\Gamma}^{\overline{\rho}}$.

Warning: Any operations we perform on simplicial objects need to be "derived".

1 $G(A_\bullet)$

2 $\Gamma \longrightarrow G(A_\bullet)$

3 Lift

Desiderata

Homotopy invariance: If

$$A_\bullet \xrightarrow[\sim]{f} A'_\bullet$$

is a weak equivalence respecting augmentations, then

$$\mathcal{F}_\Gamma^{\overline{\rho}}(f)\colon \mathcal{F}_\Gamma^{\overline{\rho}}(A_\bullet) \to \mathcal{F}_\Gamma^{\overline{\rho}}(A'_\bullet)$$

should also be a weak equivalence.

$$G(A_\bullet) = {}^{\Delta}\mathrm{Hom}(\mathcal{O}_G, A_\bullet)$$

$$\mathbb{G}_m(\mathbb{R}) \cong \mathbb{R}^\times \qquad \mathbb{G}_m(\mathbb{C}) \cong \mathbb{C}^\times$$

Compatibility with classical theory:

$$
\begin{tikzcd}
\left\{\begin{array}{c}\text{Artinian local}\\ \text{commutative rings } A\end{array}\right\} \arrow[r, "\mathrm{F}_{\Gamma,\overline{\rho}}"] \arrow[d, hook, "\delta"] & \mathrm{Set} \\
\left\{\begin{array}{c}\text{Artinian local simplicial}\\ \text{commutative rings } A_\bullet\end{array}\right\} \arrow[r, "\mathcal{F}_{\Gamma,\overline{\rho}}"] & \mathrm{sSet} \arrow[u, "\pi_0"']
\end{tikzcd}
$$

$$\Rightarrow \pi_0\left(\mathcal{R}^{\bar{\rho}}_{\Gamma,}\right) = R^{\bar{\rho}}_{\Gamma}$$

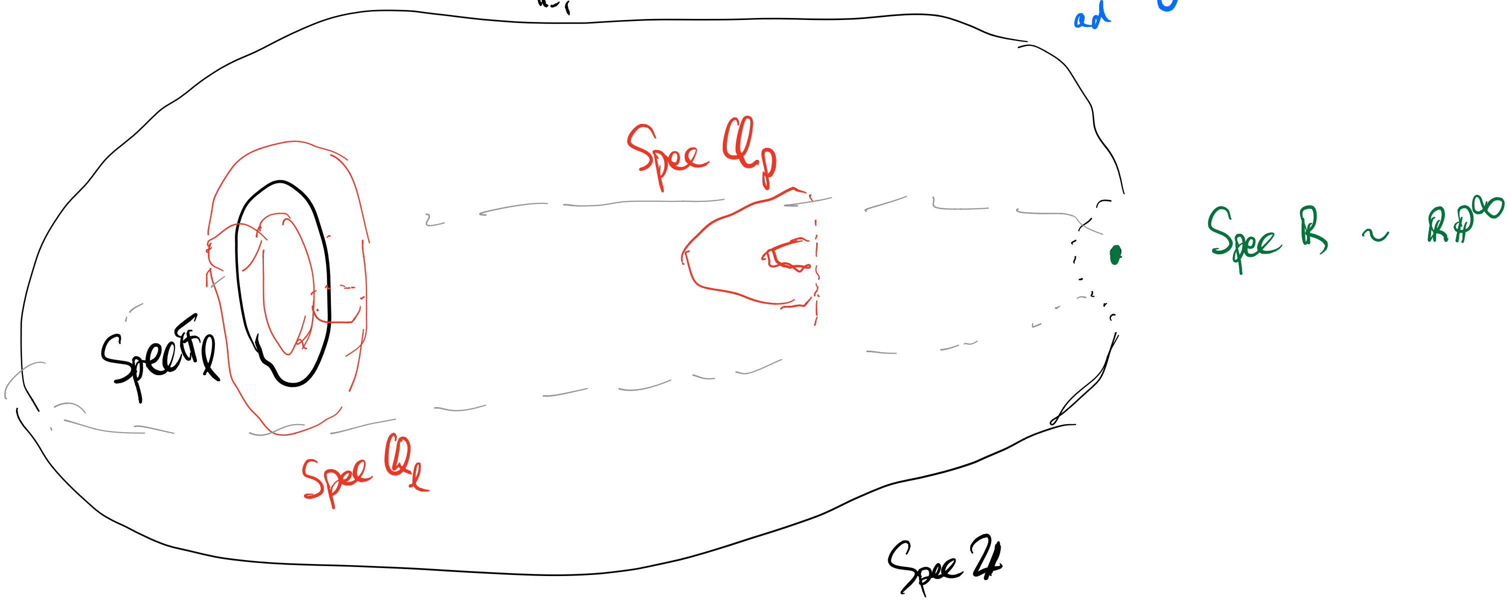
Tangent complex:

The tangent complex of $\mathcal{F}_{\Gamma}^{\overline{\rho}}$ should be $C^{\bullet}(\Gamma, \overline{\rho}^* \mathfrak{g})$; in particular its cohomology groups should be $H^i(\Gamma, \overline{\rho}^* \mathfrak{g})$.

π_{-i}

$+1$

$\Gamma \xrightarrow{\overline{\rho}} \hat{G} \overset{\mathrm{ad}}{\hookrightarrow} \mathfrak{g}$



- This is the source of "quantitative control" on $\mathcal{R}_{\Gamma}^{\overline{\rho}}$ (since "tangent complex detects weak equivalences").

Fact

For any $\mathcal{R} \in \mathrm{Art}_k$, the map $\mathcal{R} \to \pi_0(\mathcal{R})$ induces an isomorphism on $\mathfrak{t}^0$, and an inclusion on $\mathfrak{t}^1$.

Fact

complete local

For any commutative ring R over $\mathbb{Z}_p$, $\dim_k \mathfrak{t}^0(\mathrm{R})$ is the minimal number of generators over $\mathbb{Z}_p$ and $\dim_k \mathfrak{t}^1(\mathrm{R})$ is the minimal number of relations.

$$R = \mathbb{Z}_p[[x_1, \dots, x_n]] / (y_1 \dots y_k)$$

$k \leq n^2$

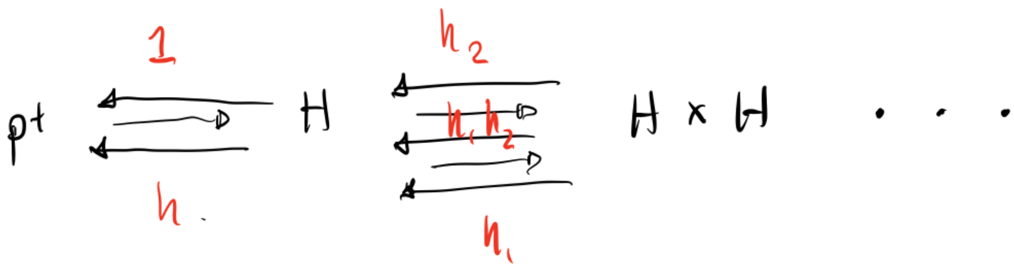
$\Rightarrow k = n^2$

Corollary

$\mathrm{R}_\Gamma^{\overline{\rho}}$ is LCI of dimension $h^1(\Gamma, \overline{\rho}^*\mathfrak{g}) - h^2(\Gamma, \overline{\rho}^*\mathfrak{g})$ over $\mathbb{Z}_p$ if and only if $\mathcal{R}_\Gamma^{\overline{\rho}} \xrightarrow{\sim} \mathrm{R}_\Gamma^{\overline{\rho}}$.

Classifying spaces

Let H be a discrete group. The classifying space BH is the geometric realization of the simplicial set $N_\bullet(H)$



$N_\bullet(H)$ is the nerve of H viewed as a category (consisting of a single object, with endomorphisms given by H).

Note: If H is not discrete, then we can still form its classifying space H using the same diagram.

If G acts freely on a contractible space EG, then EG/G is a model for the homotopy type of BG.

Analogous to the algebraic geometer's classifying stack $BG^{\mathrm{alg.geom.}} = [\mathrm{pt}\,/\,G]$.

Examples:

$G = \mathbb{Z}$  $\mathbb{R}$ $BG \sim S^1$

$G = \mathbb{C}^\times$ $(\mathbb{C}^\infty) \setminus 0$ $BG \sim \mathbb{C}P^\infty$

$G = \mathbb{Z}/2 = (\mathrm{Gal}(\mathbb{C}/\mathbb{R}))$ G S^∞ $BG \sim \mathbb{R}P^\infty$

G

Let X be a "nice" space (e.g. a simplicial set). Then

$$\begin{aligned} \pi_0 \operatorname{Map}(X, BH) &\leftrightarrow \{\text{“}H\text{-local systems” on } X\} \\ \text{(if } X \text{ connected)} &\leftrightarrow \{\pi_1(X, x) \to H\}/\mathrm{conj}. \end{aligned}$$

"Framed representations", i.e. homomorphisms, correspond to maps of *pointed* spaces: if X connected,

$$\pi_0 \operatorname{Map}_*((X, x), (BH, \mathrm{pt})) \leftrightarrow \{\pi_1(X, x) \to H\}.$$

What is the higher homotopical information in $\mathsf{Map}(X, BH)$? Still assuming H is **discrete**,

- $\pi_1(\mathsf{Map}(X, BH), \rho\colon \pi_1 \to H) \approx Z_G(\mathrm{Im}\ \rho)$
- $\pi_i(\mathsf{Map}(X, BH), \rho) = 0$ for $i \geq 2$.

If X connected,

$$\mathsf{Map}(X, BH) = \coprod_{\rho\colon \pi_1(X,x) \to H/\mathrm{conj}} BZ_G(\rho)$$

If H is not discrete, then $\mathrm{Hom}(\Gamma, H) \to \pi_0 \,\mathrm{Map}_*((B\Gamma, \mathrm{pt}), (BH, \mathrm{pt}))$ is far from a bijection.

Example:

$\Sigma_g =$ surface genus $g \geq 1$

$\Gamma_g =$ fund. group

$B\Gamma_g = \Sigma_g$

$H = \mathbb{C}^\times$

$\pi_0 \,\mathrm{Map}(B\Gamma_g, B\mathbb{C}^\times) =$ "complex line bundles" on Σ_g

$\cong$

$\mathbb{Z}$

$\mathrm{Hom}(\Gamma, \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g}$

Derived classifying spaces

Now let G be an algebraic group over the Witt vectors $W(k)$.

Fact

There exists a functorial cofibrant replacement $c(A_\bullet) \xrightarrow{\sim} A_\bullet$.

Warning 1

We can try to define $G(A_\bullet) = \mathrm{Hom}(c(\mathcal{O}_G^\delta), A_\bullet)$, but this is not a priori a simplicial group, because the functor c is not guaranteed to be monoidal.

More convenient to define $BG(A_\bullet)$ directly, rather than as the classifying space of a simplicial group. $pt \leftleftarrows G \,\substack{\leftarrow\\\leftarrow\\\leftarrow}\, G \times G$

$$c(\mathcal{O}_G) \otimes c(\mathcal{O}_G)$$

$$\text{“}\mathcal{O}_{BG}\text{”} = W(k) \rightrightarrows c(\mathcal{O}_G) \,\substack{\rightarrow\\\rightarrow\\\rightarrow}\, c(\mathcal{O}_G \otimes \mathcal{O}_G) \ldots .$$

$\mathrm{Hom}(\text{“}\mathcal{O}_{BG}\text{”}, A_\bullet)$ is a bisimplicial set $N_\bullet(A_\bullet)$, and we define $BG(A_\bullet)$ to be its geometric realization (i.e. geometric realization of the diagonal simplicial set).

Derived deformations

Let Γ be a discrete group, G an algebraic group.

Let $\overline{\rho} \in \mathrm{Rep}(\Gamma, G(k))$, which defines a 0-simplex of $\mathrm{Map}(B\Gamma, BG(k))$.

Define $\mathcal{F}_{\Gamma}^{\overline{\rho}}(A_{\bullet})$ to be the homotopy fiber product

$$\begin{tikzcd}
\mathcal{F}_{\Gamma}^{\overline{\rho}}(A_{\bullet}) \arrow[r] \arrow[d] & \mathrm{Map}(B\Gamma, BG(A_{\bullet})) \arrow[d] \\
\overline{\rho} \arrow[r] & \mathrm{Map}(B\Gamma, BG(k))
\end{tikzcd}$$

Informally, this means lifts

$$\left\{ \begin{tikzcd}
 & BG(A_{\bullet}) \arrow[d, "\epsilon"] \\
B\Gamma \arrow[ur, dashed, "\rho"] \arrow[r, "\overline{\rho}"'] & BG(k)
\end{tikzcd} \right\}.$$

The content of saying "homotopy fiber" instead of "fiber" is intuitively that instead of asking the two maps to agree, we ask for a homotopy between their images.

Example:

$$\begin{tikzcd}
\Omega_{pt} X \arrow[r, hook] \arrow[d] & pt \arrow[d] \\
pt \arrow[r, hook] & X
\end{tikzcd}$$

Onto Galois groups

Previously Γ was discrete. When Γ is profinite, write $\Gamma = \varprojlim \Gamma_\alpha$ where each Γ_α is finite, WLOG assume $\overline{\rho}$ factors over Γ_α.

Define

$$\mathcal{F}_{\Gamma}^{\overline{\rho}} := \varinjlim \mathcal{F}_{\Gamma_\alpha}^{\overline{\rho}}.$$

If $\mathcal{F}_{\Gamma_\alpha}^{\overline{\rho}}$ is pro-represented by $\mathcal{R}_{\Gamma_\alpha}^{\overline{\rho}}$, then $\mathcal{F}_{\Gamma}^{\overline{\rho}}$ will be pro-represented by the pro-system $\mathcal{R}_{\Gamma_\alpha}^{\overline{\rho}}$.

If G is adjoint and $\overline{\rho}$ has trivial centralizer, then derived Schlessinger works out and we get a "derived Galois deformation ring $\mathcal{R}_{\Gamma}^{\overline{\rho}}$" (a pro-system of simplicial commutative rings).

When considering framed deformations, these assumptions are not needed.

Cases of interest are $X = \mathrm{Spec}\ \mathbb{Z}[1/S]$, $X = \mathrm{Spec}\ \mathbb{Q}_{p'}$, $X = \mathrm{Spec}\ \mathbb{Z}_{p'}$, $\Gamma = \pi_1(X, x)$.

framing

- $\mathcal{F}_{\mathbb{Z}[1/S]}^{\overline{\rho}}, \mathcal{F}_{\mathbb{Z}_{p'}}^{\overline{\rho},\square}, \mathcal{F}_{\mathbb{Q}_{p'}}^{\overline{\rho},\square}$.
- $\underline{\mathcal{R}_{\mathbb{Z}[1/S]}^{\overline{\rho}}, \mathcal{R}_{\mathbb{Z}_{p'}}^{\overline{\rho},\square}, \mathcal{R}_{\mathbb{Q}_{p'}}^{\overline{\rho},\square}}$.

Remark 2

The paper of Galatius-Venkatesh avoids taking a (homotopy) limit of this pro-system. However there are reasons one might want to do it, e.g. in order to pass to a characteristic 0 object (over $\mathbb{Q}_p$).

Comparison to classical deformations

Recall that π_0 is left adjoint to the inclusion of commutative rings as simplicial commutative rings. In other words,

$$\mathsf{Hom}_{\mathsf{SCR}}(\mathcal{R}, \mathrm{S}) = \mathsf{Hom}_{\mathsf{CR}}(\pi_0(\mathcal{R}), \mathrm{S}).$$

Hence if S is a classical ring, then $\pi_0(\mathcal{R})$ represents the classical deformation functor.

Instead of $B\Gamma$, [GV] prefer to use the *étale homotopy type* of X (avoids choosing base points).

(Alternatively take the nerve of the groupoid of maximal [insert adjective] extensions.)

Note that the compatibility of the *derived structures* is using that $X = \operatorname{Spec} \mathbb{Q}$, $\operatorname{Spec} \mathbb{Z}[1/S]$, $\operatorname{Spec} \mathbb{Q}_p$, $\operatorname{Spec} \mathbb{Z}_p$, etc. are $K(\pi, 1)$'s.

$X \simeq B\Gamma$

Tangent complexes

We can view the tangent complex of BG as a G-equivariant complex of k-vector spaces.

$pt \xrightarrow{G} BG$

Theorem

The tangent complex of BG is the adjoint representation $\mathfrak{g}$ concentrated in cohomological degree -1 (homotopy degree 1).

$$\begin{tikzcd} \Omega BG \simeq G \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & BG \end{tikzcd} \qquad \begin{tikzcd} T_G \arrow[r] \arrow[d] & 0 \arrow[d] \\ 0 \arrow[r] & T_{BG} \end{tikzcd}$$

Exercise X, Y stacks $f \in {}^{\mathbb{L}}\mathrm{Map}(X, Y)$

$T_f\, {}^{\mathbb{L}}\mathrm{Map}(X, Y) = C^\bullet(X, f^* T_Y)$

Theorem

The tangent complex of $\mathrm{Map}^{\overline{\rho}}(X, BG)$ *is* $C^\bullet(X, \overline{\rho}^* TBG) \approx C^{\bullet+1}(X, \overline{\rho}^*\mathfrak{g})$.

We can compute $\mathfrak{t}^n\mathcal{F}$ by evaluating $\mathcal{F}(k \oplus k[n])$. $K(\mathfrak{g}, n+1)$

$$\begin{tikzcd} & BG(k \oplus k[n]) \arrow[d] \\ X \arrow[ur, dashed] \arrow[r, "\overline{\rho}"] & BG(k) \end{tikzcd} \quad \sim K(G(k), 1)$$

$$\begin{tikzcd} H \arrow[r] \arrow[d] & pt \arrow[d] \\ pt \arrow[r] & BH \end{tikzcd}$$

Using the homotopy fiber sequence $\underbrace{H \to \mathrm{pt} \to BH}$, we see that

- $BG(k)$ is a $K(\pi, 1)$,

- $BG(k \oplus k[n])$ has $\pi_1 = G(k)$ and $\pi_{n+1} = \mathfrak{g}$, $\pi_i = 0$ otherwise, i.e. "homotopy fiber of $BG(k \oplus k[n]) \to BG(k)$ is a $K(\mathfrak{g}, n+1)$".

Other approaches to the derived deformation functor

$$\left(\text{Classical rings}\right)^{op} \longrightarrow \begin{array}{c}\infty\text{-cat of}\\ \text{C-rings}\end{array}$$

[Toën, Zhu] First define the framed functor $\mathcal{F}_\Gamma^{\overline{\rho},\square}$ by making sense of the expression

$$\text{“}\mathrm{Hom}(\Gamma, \mathrm{Hom}(\mathcal{O}_G, A))\text{”} \quad / G$$

by taking all maps in the appropriate ∞-categories. (Then define the unframed version by quotienting.)

Comparison with our definition comes from

$$\mathrm{Hom}(\Gamma, G(A)) = \mathrm{Hom}_*((B\Gamma, \mathrm{pt}), (BG(A), e))$$

(“full faithfulness of the bar construction on grouplike monoids”).

Warning: this notation hides that "$\mathrm{Hom}(\Gamma, H)$" needs to be "derived", and even $\pi_0 \mathrm{Hom}(\Gamma, H)$ looks nothing like the set of homomorphisms from Γ to H.

$$\Sigma_g \qquad \Gamma_g = \left\langle \begin{matrix} a_1, \dots, a_g \\ b_1, \dots, b_g \end{matrix} \;\middle|\; \cancel{[a_1, b_1][a_2, b_2] = 1} \right\rangle$$

$$[a_1, b_1][a_2, b_2] = 1 \rangle$$

$$\left(F \rightrightarrows F^{2g}\right) \to \Gamma_g$$

$$\begin{array}{ccc} {}^{\text{"}}\mathrm{Hom}(\Gamma, \mathbb{C}^\times) & \longrightarrow & {}^{\text{"}}\mathrm{Hom}(F^{2g}, \mathbb{C}^\times) = (\mathbb{C}^\times)^{2g} \\ \downarrow & & \downarrow \text{constant to } e \\ e & \longrightarrow & {}^{\text{"}}\mathrm{Hom}(F, \mathbb{C}^\times) = \mathbb{C}^\times \end{array}$$

$$(\mathbb{C}^\times)^{2g} \times \underbrace{\Omega \mathbb{C}^\times}_{\mathbb{Z}} \;\wr\!\wr$$

Thanks for listening!

$R^{\hat{\rho}, \mathrm{crys}}$ $\ell_0 > 0$ never LCI.

$R_{\bar{\rho}}^{\bar{\rho}}$ should be LCI [Mazur] of the "correct" dimension

- also if local (at p) def ring not LCI