Periods and harmonic analysis on spherical varieties.

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ABSTRACT. Given a spherical variety \mathbf{X} for a group \mathbf{G} over a non-archimedean local field k, the Plancherel decomposition for $L^2(X)$ should be related to "distinguished" Arthur parameters into a dual group closely related to that defined by Gaitsgory and Nadler. Motivated by this, we develop, under some assumptions on the spherical variety, a Plancherel formula for $L^2(X)$ up to discrete (modulo center) spectra of its "boundary degenerations", certain \mathbf{G} -varieties with more symmetries which model \mathbf{X} at infinity. Along the way, we discuss the asymptotic theory of subrepresentations of $C^\infty(X)$ and establish conjectures of Ichino–Ikeda and Lapid–Mao. We finally discuss global analogues of our local conjectures, concerning the period integrals of automorphic forms over spherical subgroups.

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1. Introduction

Let $\mathbf{H} \subset \mathbf{G}$ be algebraic groups over a field k. If k is *local*, an important problem of representation theory is to decompose the $\mathbf{G}(k)$ -action on the space of functions on $\mathbf{H} \setminus \mathbf{G}(k)$; if k is *global*, with ring of adeles \mathbb{A}_k , the study of *automorphic period integrals*

$$\varphi \to \int_{\mathbf{H}(k)\backslash \mathbf{H}(\mathbb{A}_k)} \varphi,$$
 (1.1)

where φ is an automorphic form on $\mathbf{G}(k)\backslash\mathbf{G}(\mathbb{A}_k)$, is a central concern of the theory of automorphic forms.

Our goal (continuing the program of [Sak08], [Sak13]) is to relate these questions, and to formulate a unified framework in which they can be studied; the problems discussed here have been previously been studied largely on a case-by-case basis.

We shall set up a general formalism in Part 1 and then, in Parts 2, 3, we give evidence, in the local context, that our formalism is indeed the correct one. In Part 4 we formulate the conjectures and give evidence in the global setting. The resulting circle of ideas could be understood as part of a *relative Langlands program*.

In most cases where (1.1) is related to an L-function, \mathbf{H} acts with an open orbit on the full flag variety of \mathbf{G} ; equivalently, a Borel subgroup of \mathbf{G} acts with an open orbit on $\mathbf{H} \backslash \mathbf{G}$. This leads us to the starting point of the theory, spherical varieties.

1.1. Let G be a reductive group and X a G-variety. In this paper the group will always be split over the base field k, and the base field will be of characteristic zero.

The variety X is called *spherical* if it is normal and a Borel subgroup $B \subset G$ acts with a Zariski dense orbit. It is a remarkable fact that spherical varieties have a uniform structure theory and are classified by combinatorial data. They include all *symmetric* varieties.

We will consider the questions formulated above in the case when $\mathbf{H} \backslash \mathbf{G}$ is a spherical variety under \mathbf{G} . Our goal will be to formulate conjectural answers in terms of the data attached to the spherical variety $\mathbf{H} \backslash \mathbf{G}$.

Throughout we will use the convention of denoting with boldface letters G, X algebraic groups or algebraic varieties, and by G, X, \ldots their points over a local field k.

- **1.2.** As illustrations, we use the following classes of spherical varieties:
 - Symmetric: stabilizers on X are fixed points of an involution on G.
 - Gross-Prasad: $\mathbf{G} = \mathbf{G}_n \times \mathbf{G}_{n+1}$ acting (by right and left multiplication) on $\mathbf{X} = \mathbf{G}_{n+1}$, where $\mathbf{G}_n = \mathbf{SO}_n$ or \mathbf{GL}_n .
 - Whittaker. Here $\mathbf{X} = \mathbf{U} \backslash \mathbf{G}$, where \mathbf{U} is the maximal unipotent subgroup; instead of functions on X we consider sections of a line bundle defined by a nondegenerate additive character of U. (This

does not fall strictly in the framework of spherical varieties, but nonetheless our results and methods apply unchanged to this case - cf. §2.6.)

In the Gross–Prasad and Whittaker cases, it is conjectured that the global automorphic period is related to special values of L-functions; this is also believed in many, but not all, symmetric cases.

1.3. We formulate our main local and global conjectures, and then discuss our results in §1.5. We point the reader to Part 4 for more precise formulations of the conjectures.

Our conjectures are phrased in terms of a dual group \check{G}_X attached to the spherical variety X. This is inspired and motivated by the work of Gaitsgory and Nadler [GN10]. We define the root datum of \check{G}_X in 2.2; the dual group comes equipped with a canonical morphism of the distinguished Cartan subgroup of $\check{G}_X \times \operatorname{SL}_2$ to the distinguished Cartan subgroup of \check{G} . A distinguished morphism is an extension of this to a map

$$\check{G}_X \times \mathrm{SL}_2 \to \check{G};$$
 (1.2)

that satisfies a certain constraint on root spaces formulated in §3.2. We conjecture that such an extension always exists, and prove it (for most spherical varieties, termed "wavefront") assuming that the Gaitsgory-Nadler construction satisfies certain natural axioms e.g. compatibility with boundary degeneration and parabolic induction. We should note here that our definition of the dual group leaves out some varieties – for instance, the \mathbf{GL}_n -variety of non-degenerate quadratic forms in n variables. Our harmonicanalytic results still hold in this case, but formulating a Langlands-type conjecture about the spectrum is a very interesting problem whose answer we do not know.

What is important for applications is that one can rapidly compute (1.2) in any specific case; for example, we give a table of rank one cases in Appendix A.

Now let k be a local field. An Arthur parameter for \mathbf{G} is a homomorphism $\phi: \mathcal{L}_k \times \operatorname{SL}_2 \to \check{G}$, such that the image of the first factor is bounded and the restriction to the second factor is algebraic. Here \mathcal{L}_k is the Weil-Deligne group of k (Weil group in the archimedean case). We say that ϕ is \mathbf{X} -distinguished if it factors through a map $\tilde{\phi}: \mathcal{L}_k \longrightarrow \check{G}_X$, i.e. $\phi(w,g) = \rho(\tilde{\phi}(w),g)$, where ρ is the map (1.2).

1.3.1. Conjecture. The support of the Plancherel measure for $L^2(X)$, as a G-representation, is contained in the union of Arthur packets attached to X-distinguished Arthur parameters.

In fact, we may enunciate a more precise conjecture, predicting a direct integral decomposition:

$$L^{2}(X) = \int_{\phi} \mathcal{H}_{\phi} \mu(\phi), \qquad (1.3)$$

where ψ ranges over \check{G}_X -conjugacy classes of **X**-distinguished Arthur parameters, the Hilbert space \mathcal{H}_{ϕ} is isotypic for a sum of representations belonging to the Arthur packet corresponding to ϕ , and the measure $\mu(\phi)$ is absolutely continuous with respect to the natural "Haar" measure on Arthur parameters. The sharpened conjecture implies that the unitary irreducible G-representations that occur as subrepresentations of $L^2(X)$ – the so-called relative discrete series – are all contained in Arthur packets arising from elliptic parameters $\mathcal{L}_k \longrightarrow \check{G}_X$, i.e. maps that do not factor through a proper Levi subgroup.

1.3.2. EXAMPLE. Let **V** be a 2n-dimensional vector space over k, $\mathbf{G} = \mathbf{GL}(V)$, and **X** the space of alternating forms on **V**. Then **X** is a spherical **G**-variety; the group \check{G}_X is isomorphic to GL_n , and the map

$$\check{G}_X \times \mathrm{SL}_2 = \mathrm{GL}_n \times \mathrm{SL}_2 \longrightarrow \mathrm{GL}_{2n}$$

is the tensor product of the standard representations. The content of the Conjecture is then that the unitary spectrum of $L^2(X)$ are precisely the *Speh representations* $J(2,\sigma)$, where σ is a tempered representation of $\mathrm{GL}(n,k)$; moreover, such a representation embeds into $L^2(X)$ precisely if σ is discrete series. We point to the work of Offen–Sayag [OS07] for work in this direction.

1.3.3. EXAMPLE. For many low-rank spherical varieties, Gan and Gomez have proven this conjecture recently using the theta correspondence, [GG14].

It is desirable to refine Conjecture 1.3.1 to a precise Plancherel formula. We will discuss a more precise version of this conjecture in section 16.

1.4. We now discuss its relationship with a global conjecture about the Euler factorization of periods of automorphic forms (cf. Section 17). Let K be a global field, with ring of adeles \mathbb{A}_K . Let $\mathbf{X} = \mathbf{H} \setminus \mathbf{G}$ be a spherical variety defined over K, and let $\pi = \otimes \pi_v \hookrightarrow C^{\infty}([\mathbf{G}])$ (where $[\mathbf{G}] = \mathbf{G}(K) \setminus \mathbf{G}(\mathbb{A}_K)$) be an irreducible automorphic representation of \mathbf{G} . Under some assumptions on \mathbf{H} (multiplicity-one is clearly sufficient, but not necessary as the work of Jacquet $[\mathbf{Jac01}]$ shows), it is expected that the *period integral* against Tamagawa measure on $[\mathbf{H}]$:

$$I_H: \phi \mapsto \int_{[\mathbf{H}]} \phi(h) dh$$
 (1.4)

(whenever it makes sense) is an *Eulerian* functional on the space of π , i.e. a pure tensor in the restricted tensor product:

$$\operatorname{Hom}_{\mathbf{H}(\mathbb{A}_K)}(\pi,\mathbb{C}) = \otimes'_v \operatorname{Hom}_{\mathbf{H}(K_v)}(\pi_v,\mathbb{C}).$$

We will state a conjecture in the multiplicity-free case, for a treatment of Jacquet's example cf. [FLO12].

The refined Gross-Prasad conjecture by Ichino and Ikeda [II10] gives an explicit Euler factorization of this functional – or rather of the hermitian form $\mathcal{P}^{\text{Aut}} := |I_H|^2$ – in the case of $\mathbf{G} = \mathbf{SO}_n \times \mathbf{SO}_{n+1}$, $\mathbf{H} = \mathbf{SO}_n$, at least

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when π is tempered. They conjecture that (up to explicit, rational global constants) the form \mathcal{P}^{Aut} factorizes as products of the local Hermitian forms

$$\theta_{\pi_v}: (u, \tilde{u}) \in \pi_v \times \overline{\pi_v} \mapsto \int_{\mathbf{H}(K_v)} \langle \pi_v(h)u, \tilde{u} \rangle dh$$

An important motivating observation for us was that these Hermitian forms play an important role in the Plancherel formula for the space $L^2(\mathbf{X}(K_v))$: the Hermitian form θ_{π_v} can be viewed, via Frobenius reciprocity, as a morphism $\pi_v \otimes \overline{\pi_v} \to C^{\infty}(\mathbf{X}(K_v)) \otimes C^{\infty}(\mathbf{X}(K_v))$. Its dual, composed with the unitary pairing $\pi_v \otimes \overline{\pi_v} \to \mathbb{C}$, defines an invariant hermitian form H_{π_v} on $C_c^{\infty}(\mathbf{X}(K_v))$. The L^2 inner product of functions on X is the integral of these Hermitian forms against the standard Plancherel measure on the unitary dual \hat{G} .

Let us note here an important subtlety of the Plancherel formula. In general, the theory of unitary decomposition associates to $L^2(X)$ (here $X = \mathbf{X}(K_v)$ etc.) only a measure class on the unitary dual \widehat{G} . To choose a specific measure μ in this class is essentially equivalent to fixing an embedding $\pi \otimes \overline{\pi} \hookrightarrow C^{\infty}(X \times X)$, for almost all π in the support of that measure. In the group case ($\mathbf{X} = \mathbf{H}^{\text{diag}} \backslash \mathbf{H} \times \mathbf{H}$) there is a canonical normalization of such an embedding, coming from the theory of matrix coefficients.

In general, there is no corresponding normalization; however, our local conjecture gives a natural candidate for μ : If the Plancherel formula can be written in terms of parameters into \check{G}_X , then the measure that one would use for the Plancherel decomposition of $L^2(G_X)$, where G_X is the split group with dual \check{G}_X , seems to be a natural choice. This measure was (conjecturally) described in [HII08] in terms of Langlands parameters, and it only depends on the parameter, up to a rational factor that may show up for ramified representations of exceptional groups. Since our global conjecture is only up to a rational factor, this ambiguity does not concern us here, and we can think of Plancherel measure on G_X as a measure on the set of bounded Langlands parameters into \check{G}_X .

Fixing this measure gives rise to normalized embeddings of $\pi \times \overline{\pi}$ into $C^{\infty}(X \times X)$, for almost every π in the support of Plancherel measure (and, by some continuity property, for all), and by evaluating at the identity we get a H-biinvariant Hermitian form $\mathcal{P}_v^{\text{Planch}}$ on π . We conjecture that this is "the correct normalization for global applications", i.e. whenever \mathcal{P}^{Aut} is Eulerian these local forms are the correct generalization of the forms θ_{π_v} of Ichino and Ikeda:

$$\mathcal{P}^{\text{Aut}} = q \prod_{v}^{\prime} \mathcal{P}_{v}^{\text{Planch}}, \tag{1.5}$$

where q is a nonzero rational factor that we don't specify. The Euler product is typically non-convergent, and the product of all but finitely many factors should be interpreted as a product/quotient of special values of L-functions.

There are many assumptions for this conjecture, besides the local Conjecture 1.3.1, such as a multiplicity-one assumption and the assumption that the space of the automorphic representation in question is the one corresponding to an "X-distinguished (global) Arthur parameter". The existence of these parameters is of course highly conjectural, but in certain cases there are more down-to-earth versions of the conjecture that one can formulate. We point the reader to Section 17.

Clearly, our global conjecture lacks the precision of $[\mathbf{II10}]$, and should be considered as a guiding principle for the time being. In any case, it provides access to the mysterious link between global periods and automorphic L-functions, via a computation of local Plancherel measures that was performed in $[\mathbf{Sak13}]$.

1.5. This paper is divided into four main parts. All four bear on the main conjecture, but the details of individual parts are to a large extent independent and can be read separately.

Some of the main results are Proposition 2.2.2/Theorem 2.2.3 (identification of dual group), Theorem 5.1.1 (asymptotics of representations, implying finite multiplicity), Theorem 6.4.1 (Ichino–Ikeda conjecture), Theorem 9.2.1 (finiteness of discrete series), Theorem 11.1.2 (existence of scattering morphisms), Theorem 14.3.1 (abstract scattering theorem), Theorem 7.3.1 (in many cases, a complete description of scattering) Theorem 15.6.1 (Plancherel decomposition in terms of "normalized Eisenstein integrals") and Theorem 18.4.1 (compatibility of the global conjecture with "unfolding").

Let k be a local non-archimedean field. Practically all of our results are obtained under the assumption that \mathbf{X} is "wavefront" (see §2.1 for the definition). This includes the vast majority of spherical varieties (e.g., in Wasserman's tables [Was96] of rank 2 spherical varieties, only three fail to be wavefront), and in particular covers the Whittaker, Gross–Prasad, and all symmetric cases.

(1) Part 1 (§2 and §3). Dual groups of spherical varieties.

It is primarily concerned with defining the dual group \check{G}_X and establishing – as far as possible – the existence of the morphism $\check{G}_X \times \operatorname{SL}_2 \to \check{G}$. As mentioned, we prove (Theorem 2.2.3) that this morphism exists assuming the compatibility of the Gaitsgory-Nadler construction with certain natural operations, such as boundary degeneration and parabolic induction; a by-product of this proof is an identification of the root system of the Gaitsgory-Nadler dual group.

An important feature of \check{G}_X is its relation to the geometry of \mathbf{X} at ∞ . To each conjugacy class Θ of parabolic subgroups of \check{G}_X , we associate a spherical variety \mathbf{X}_{Θ} (which we call a "boundary degeneration") under \mathbf{G} ; it models the structure of a certain part

of **X** at ∞ . The dual group $\check{G}_{X_{\Theta}}$ to \mathbf{X}_{Θ} is isomorphic to a Levi subgroup of a parabolic subgroup in the class Θ .

The reader more interested in local or global theory could skip most of this section, reading only the parts on the boundary degenerations, and perhaps glancing at the table of examples in the Appendix.

(2) Part 2: Asymptotics and the Ichino-Ikeda conjecture $(\S 5 - 6)$.

We verify (§5) that the multiplicity of any irreducible G-representation in $C^{\infty}(X)$ is finite; we compute (also §5) the asymptotic behavior of "eigenfunctions" (i.e., functions on X whose G-span is of finite length).

The latter result is naturally expressed in terms of an "asymptotics" map

$$e_{\Theta}: C_c^{\infty}(X_{\Theta}) \longrightarrow C_c^{\infty}(X),$$
 (1.6)

see Theorem 5.1.1.

We remark that these results are corollaries to an understanding of the geometry of X at ∞ ; this understanding plays a fundamental role throughout the entire paper.²

By elementary methods, we are able in §6 to completely describe a Plancherel formula (Theorem 6.2.1) for "strongly tempered varieties"; this is a condition that implies $\check{G}_X = \check{G}$ and includes the Gross–Prasad and Whittaker cases, although *not* most symmetric cases. This gives, in particular, a simple derivation of the Whittaker-Plancherel formula for p-adic groups³ (more precisely: a simple reduction to the usual Plancherel formula).

Using these results, we verify conjectures of Ichino–Ikeda (Theorem 6.4.1) and Lapid–Mao (Corollary 6.3.5). We mention only the former: if (H, G) is as in the Gross–Prasad conjecture then for any tempered representation Π of G, the form

$$v \otimes v' \mapsto \int_{h \in H} \langle hv, v' \rangle, \quad v \in \Pi, v' \in \tilde{\Pi}$$

on $\Pi \otimes \tilde{\Pi}$ is nonvanishing if and only if Π is H-distinguished.

(3) Part 3: Scattering theory. (\S 9 – \S 15)

This is the core of the paper; the results of this section are summarized on page 108.

¹Such results on asymptotics, but expressed in the more traditional language of Jacquet modules, were proven for symmetric varieties by Lagier [Lag08], and independently by Kato and Takano [KT08].

²Recently Bezrukavnikov and Kazhdan [BK15] have given a geometric analysis of Bernstein's second adjunction, which is closely related to our analysis in the special case where \mathbf{X} is the group variety $\Delta \mathbf{G} \backslash \mathbf{G} \times \mathbf{G}$. They use, in particular, the structure of the wonderful compactification of \mathbf{G} itself; cf. also §5.5.

³While our paper was being written, a complete description of the Whittaker-Plancherel formula was obtained by Delorme [Del13]. Our proof is rather different.

The set of conjugacy classes of **X**-distinguished Arthur parameters is partitioned into subsets indexed by conjugacy classes of Levi subgroups of \check{G}_X . We give evidence in §11 – §15 that the unitary spectrum of $L^2(X)$ has a corresponding structure; in the most favorable cases (that is to say: satisfying an easy-to-check combinatorial criterion) this amounts to a Plancherel formula modulo the knowledge of discrete (modulo center) series for **X** and all its boundary degenerations \mathbf{X}_{Θ} . (The preceding sections cover preliminary ground: §9 discusses a somewhat subtle issue concerning the fact that one can have continuous families of relative discrete series, and §10 contains some lemmas in linear algebra that are necessary to formulate the scattering arguments).

The main tool is "scattering theory," which relates the spectrum of a space and its boundary. We obtain in Theorem 11.1.2 a canonical G-equivariant map

$$L^2(X_{\Theta}) \xrightarrow{\iota_{\Theta}} L^2(X).$$
 (1.7)

This should be viewed as a unitary analog of the smooth asymptotics map (1.6). We call this map the "Bernstein map", because its existence is essentially equivalent to an unpublished argument of Joseph Bernstein, which proves that the continuous part of the Plancherel formula for X should resemble the Plancherel formula for the boundary degenerations of X as one moves towards infinity. Conjecturally, it corresponds to the evident map on Arthur parameters induced by $\check{G}_{X_{\Theta}} \hookrightarrow \check{G}_{X}$.

parameters induced by $\check{G}_{X_{\Theta}} \hookrightarrow \check{G}_{X}$. Let $L^{2}(X)_{\Theta}$ be the image of $L^{2}(X_{\Theta})_{\text{disc}}$ under ι_{Θ} . We conjecture that $L^{2}(X)_{\Theta} = L^{2}(X)_{\Theta'}$ when Θ, Θ' are associate. In favorable cases we are able to prove this in Theorem 14.3.1, and, in fact, precisely describe the kernel of the morphism $\bigoplus \iota_{\Theta}$.

Finally we discuss in §15 an "explicit description" of ι_{Θ} in terms of Mackey theory. The goal here, which we only partially achieve, is to describe the morphisms ι_{Θ} in terms of explicit intertwining operators, commonly referred to as "Eisenstein integrals". In some combinatorially favorable cases we fully achieve this goal, including many symmetric varieties.

(4) Part 4. Conjectures.

In this part we formulate the local and global conjectures discussed above, and give some evidence for the global ones. The formalism here relies on the local and global Arthur conjectures [Art89]. The local Conjecture 16.2.2 states that the unitary representation $L^2(\mathbf{X}(k))$ (where k is a local field) admits a direct integral decomposition in terms of \mathbf{X} -distinguished Arthur parameters (and

⁴Unfortunately, the term "Bernstein map" is used in [**BK15**] for the smooth asymptotics map (1.6).

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the natural class of measures on them). A finer version (Conjecture 16.5.1) introduces a notion of "pure inner form" for a spherical variety \mathbf{X} , inspired from the relative trace formula and the local Gan–Gross–Prasad conjectures [GGP12].

In Section 17 we formulate the global conjecture on Euler factorization of period integrals (under several assumptions). Finally, in Section 18 we prove the global conjecture in some cases: periods of principal Eisenstein series, Whittaker periods for \mathbf{GL}_n , and all periods that "unfold" to Whittaker periods for \mathbf{GL}_n . Of course, explicit Euler factorizations for these periods have been known in the past; what we do is verify that the local factors are the ones predicted by our conjectures, which are related to local Plancherel formulas. Much of this is known to experts, and our goal is in part to express this computation of local factors in the language of this paper.

- 1.6. Proofs. We outline the ideas behind the results at the heart of this paper, the local Plancherel formula developed in Parts 2 and 3. As we have already mentioned, the basic ingredient in many of the proofs is a good understanding of the geometry of \mathbf{X} at ∞ . We will give briefly some examples of the type of ideas that enter.
- 1.6.1. Geometry at ∞ . A critical fact in the theory of spherical varieties is that there exists a parabolic $\mathbf{P}(\mathbf{X})^-$ (the notation is such because it is in the opposite class of parabolics to one we will denote by $\mathbf{P}(\mathbf{X})$) so that the geometry of \mathbf{X} at ∞ is modelled by a torus bundle \mathbf{Y} over $\mathbf{P}(\mathbf{X})^- \backslash \mathbf{G}$.

For example, the hyperboloid $x^2 - y^2 - z^2 = 1$ is spherical under the group SO_3 ; at ∞ it becomes asymptotic to the cone $x^2 - y^2 - z^2 = 0$, which is a line bundle over the flag variety \mathbb{P}^1 . In terms of the varieties \mathbf{X}_{Θ} previously mentioned, the torus bundle \mathbf{Y} is obtained by taking for Θ the class of Borel subgroups in the dual group \check{G}_X .

In fact, this is an overly simplified view of the geometry of \mathbf{X} at ∞ ; more accurately, the geometry of \mathbf{X} at ∞ is modelled by the so-called wonderful compactification $\overline{\mathbf{X}}$. The \mathbf{G} -orbits on $\overline{\mathbf{X}}$ are canonically in correspondence with conjugacy classes of parabolic subgroups of \check{G}_X ; for each such conjugacy class Θ we call " Θ -infinity" the corresponding orbit at infinity, and define the variety \mathbf{X}_{Θ} as (the open \mathbf{G} -orbit in) the normal bundle to Θ -infinity. In particular, there is – in the sense of algebraic geometry – a degeneration of \mathbf{X} to \mathbf{X}_{Θ} ; in intuitive terms, \mathbf{X}_{Θ} models a part of the geometry of \mathbf{X} at ∞ . The \mathbf{G} -variety \mathbf{X}_{Θ} is "simpler" than \mathbf{X} in a very important way: it carries the additional action of a torus $\mathbf{A}_{X,\Theta}$, generated from the actions of the multiplicative group on the normal bundles to all \mathbf{G} -stable divisors containing Θ -infinity.

1.6.2. Geometry at ∞ over a local field. The discussion above has the following consequence for points over a local field k:

Let J be an open compact subgroup of $G = \mathbf{G}(k)$. We construct a canonical identification of a certain subset of X/J with a certain subset of Y/J, mirroring the fact that X approximates Y at ∞ . This fact remains true (with different subsets) if we replace Y by any X_{Θ} . We call this identification the "exponential map", because it is in fact induced by some kind of exponential map between the (k-points of the) normal bundle and the variety.

In the case of the hyperboloid we may describe this as follows: Write $X = \{(x,y,z) \in k^3 : x^2 - y^2 - z^2 = 1\}$ and $Y = \{(x,y,z) \in k^3 : x^2 - y^2 - z^2 = 0\}$. We declare two J-orbits $xJ \subset X$, $yJ \subset Y$ to be ε -compatible if there exists $x' \in xJ, y' \in yJ$ that are at distance $< \varepsilon$ for the nonarchimedean metric on k^3 . Then, for all sufficiently small ε (this notion depending on J), there exist compact sets $\Omega_X \subset X, \Omega_Y \subset Y$ so that the relation of ε -compatibility gives a bijection between $(X - \Omega_X)/J$ and $(Y - \Omega_Y)/J$.

1.6.3. Asymptotics of eigenfunctions. Call a (smooth) function on X or Y an eigenfunction if its translates under $G = \mathbf{G}(k)$ span a G-representation of finite length. Then the fundamental fact of interest to us is that, for every (J-invariant) eigenfunction f on X, there exists an eigenfunction f_Y on Y so that "f is asymptotic to f_Y ": that is to say, f and f_Y are identified under the "partial bijection" (exponential map) between X/J and Y/J. In fact, this is a fact that does not require admissibility or finite length (although we only need the finite length case for the Plancherel formula): there is a G-morphism: $e_{\Theta}^*: C^{\infty}(X) \to C^{\infty}(X_{\Theta})$ (the dual of (1.6) with the property that functions coincide with their images in neighborhoods of Θ -infinity identified via the exponential map.

1.6.4. The argument of Bernstein. Now we turn our attention to the Plancherel decomposition: its existence and uniqueness is guaranteed by theorems involving C^* -algebras, and by [Ber88] it is known to be supported on "X-tempered" representations. This means that the norm of every Harish-Chandra–Schwartz function f on X admits a decomposition:

$$||f||_{L^2(X)}^2 = \int_{\hat{G}} H_{\pi}(f)\mu(\pi),$$

where the positive semi-definite hermitian forms⁵ H_{π} factor through a quotient G-space of finite length, isomorphic to a number of copies of π . In other words, $H_{\pi}(f)$ can be written as a finite sum of terms of the form $|l_i^*(f)|^2$, where $l_i: \pi \to C^{\infty}(X)$ is an embedding with tempered image.

Given the theory of the exponential map, explained above, and the asymptotic theory for such embeddings, if the support of f is concentrated close to " Θ -infinity" then f can be identified with a function f' on X_{Θ} and the expression $|l_i^*(f)|^2$ can be identified with the same expression for some $l_i': \pi \to C^{\infty}(X_{\Theta})$. In other words, the hermitian form H_{π} give rise to a hermitian form H_{π}' on functions on X_{Θ} ; in precise terms this is simply the pullback of H_{π} under the map $C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X)$ of (1.6).

⁵We feel free to write H(f) := H(f, f) for a hermitian form H.

Could the forms H'_{π} appear in the Plancherel formula for $L^2(X_{\Theta})$? Almost, but not quite. The reason is that the Plancherel formula for X_{Θ} is invariant under the additional torus symmetry group $A_{X,\Theta}$ of X_{Θ} , and this is not, in general, the case with the asymptotic forms H'_{π} . There is, however, an elementary way to extract their " $A_{X,\Theta}$ -invariant part" H^{Θ}_{π} (possibly zero), and then one obtains a Plancherel decomposition for $L^2(X_{\Theta})$:

$$||f'||_{L^2(X_{\Theta})}^2 = \int_{\hat{G}} H_{\pi}^{\Theta}(f')\mu(\pi).$$

We show that this fact is equivalent to the existence of "Bernstein maps":

$$\iota_{\Theta}: L^2(X_{\Theta}) \to L^2(X),$$

which are characterized by the fact that they are "very close" to the smooth asymptotics maps (1.6) for functions supported "close to Θ -infinity".

For readers familiar with the Plancherel decomposition of the space of automorphic functions, we mention that the analog of this map in that case arises as follows: one decomposes a function $f \in L^2(N(\mathbb{A}_K)A(K) \setminus \operatorname{PGL}_2(\mathbb{A}_K))$ (this is the analog of X_{Θ} in that case) as an integral of (unitary) $A(\mathbb{A}_K)$ -eigenfunctions, and then $\iota_{\Theta}f$ will be the corresponding integral of Eisenstein series, i.e., replace each eigenfunction on $N(\mathbb{A}_K)A(K) \setminus \operatorname{PGL}_2(\mathbb{A}_K)$ by the corresponding Eisenstein series on $\operatorname{PGL}_2(K) \setminus \operatorname{PGL}_2(\mathbb{A}_K)$. Notice that this is taken on the tempered line, without any discrete series contributions.

In our case we will define ι_{Θ} in a more abstract way, and only later §15 (and under additional conditions) will we identify it in terms of explicit morphisms ("normalized Eisenstein integrals") analogous to the Eisenstein series. Thus, in the language often used in the literature of harmonic analysis on symmetric spaces, the Bernstein maps can be identified with "normalized wave packets".

1.6.5. *Scattering*. The construction of Bernstein maps also implies that their sum, restricted to discrete spectra:

$$\sum \iota_{\Theta,\mathrm{disc}} : \bigoplus_{\Theta \subset \Delta_X} \iota_{\Theta}(L^2(X_{\Theta})_{\mathrm{disc}}) \to L^2(X)$$

is surjective.

For a full description of $L^2(X)$ in terms of discrete spectra there remains to understand the kernel of this map. Based on the dual group conjecture, we expect the images of $L^2(X_{\Theta})_{\text{disc}}$ and $L^2(X_{\Omega})_{\text{disc}}$ are orthogonal if Θ and Ω are not W_X -conjugate, and coincide otherwise. We prove this under some combinatorial condition ("generic injectivity": §14.2), which is easy to check and is known to hold, at least, for all symmetric varieties.

More precisely, it is easy to show, first, that the images of $L^2(X_{\Theta})_{\text{disc}}$ and $L^2(X_{\Omega})_{\text{disc}}$ are orthogonal unless Θ and Ω are of the same size (i.e. the corresponding orbits of the wonderful compactification are of the same dimension). The combinatorial condition, on characters of the associated

boundary degenerations, is used to rule out the possibility that the images are non-orthogonal when Θ and Ω are non-conjugate. Finally, a delicate analytic argument shows that when they are conjugate the images of $L^2(X_{\Theta})_{\text{disc}}$ and $L^2(X_{\Omega})_{\text{disc}}$ have to coincide. This is encoded by certain maps $L^2(X_{\Theta}) \to L^2(X_{\Omega})$ whenever Θ, Ω are W_X -conjugate, the so-called scattering morphisms. For a more detailed introduction to the results of scattering theory, we point the reader to Section 7.

1.7. Notation and assumptions. We have made an effort to define or re-define most of our notation locally, in order to make the paper more readable. An exception are the notions and symbols introduced in section 2. We note here a few of the conventions and symbols which are used throughout: We fix a locally compact p-adic field k in characteristic zero, with ring of integers \mathfrak{o} ; we denote varieties over k by bold letters and the sets of their k-points by regular font. For example, if \mathbf{Y} is a k-variety, we denote $\mathbf{Y}(k)$ by Y without special remark. On the other hand, for complex varieties (such as dual groups or character groups) we make no notational distinction between the abstract variety and its complex points, and use regular font for both.

We always use the words "morphism" or "homomorphism" in the appropriate category (which should be clear from the context), e.g. for topological groups a "homomorphism" is always continuous, even if not explicitly stated so. We denote by $\mathcal{X}(\mathbf{M}) = \operatorname{Hom}(\mathbf{M}, \mathbf{G}_{\mathrm{m}})$ the character group of any algebraic group \mathbf{M} and for every finitely generated \mathbb{Z} -module R we let $R^* = \operatorname{Hom}(R, \mathbb{Z})$. Normalizers are denoted by \mathcal{N} , or $\mathcal{N}_{\mathbf{G}}$ when we want to emphasize the ambient group; this is not to be confused with the notation $N_{\mathbf{Z}}\mathbf{Y}$, which denotes the normal bundle in a variety \mathbf{Y} of a subvariety \mathbf{Z} .

We denote throughout by \mathbf{G} a connected, reductive, split group over k, and by \mathbf{X} a homogeneous, quasi-affine spherical variety for \mathbf{G} . For most of the paper, we assume this variety to be wavefront (cf. §2.1 – see the list that follows for a full set of assumptions). We fix⁶ a Borel subgroup \mathbf{B} and denote its Zariski open orbit on \mathbf{X} by $\mathring{\mathbf{X}}$. Parabolics containing \mathbf{B} will be called "standard", the unipotent radical of \mathbf{B} will be denoted by \mathbf{U} and the reductive quotient of \mathbf{B} will be denoted by \mathbf{A} , although in some circumstances we identify it with a suitable maximal subtorus of \mathbf{B} . We let Weyl groups act on the left on tori, root systems etc., the action denoted either by an exponent on the left or as w; for example, the action of W=the Weyl group of \mathbf{G} on the character group of \mathbf{A} is given by: ${}^{w}\chi(a) = \chi(w^{-1} \cdot a)$.

As noted above, we feel free to identify the Langlands dual group G of G with its \mathbb{C} -points, and therefore on the dual side we avoid the boldface notation. It comes equipped with a canonical maximal torus A^* .

 $^{^{6}}$ It is actually for convenience of language that we fix a Borel subgroup; one could adopt the language pertinent to "universal Cartan groups" and show that all constructions, such as the quotient torus \mathbf{A}_{X} , are unique up to unique isomorphism because of the fact that Borel subgroups are self-normalizing and conjugate to each other.

To any such variety is attached the following set of data (cf. Section 2):

- $\mathcal{Z}(\mathbf{X}) := \operatorname{Aut}_{\mathbf{G}}(\mathbf{X})^0$, the neutral component of the **G**-automorphism group of **X**;
- a parabolic subgroup $\mathbf{P}(\mathbf{X}) \supset \mathbf{B}$, namely the stabilizer of the open Borel orbit; its reductive quotient (and, sometimes, a suitable Levi subgroup) is denoted by $L(\mathbf{X})$ and the simple roots of \mathbf{A} in $\mathbf{L}(\mathbf{X})$ by $\Delta_{L(X)}$;
- a torus \mathbf{A}_X , which is a quotient of \mathbf{A} ; it is the analog of the universal Cartan for the group.
- a finite group W_X of automorphisms of \mathbf{A}_X , the "little Weyl group;"
- the set $\Delta_X \subset \mathcal{X}(\mathbf{X}) = \operatorname{Hom}(\mathbf{A}_X, \mathbf{G}_{\mathrm{m}})$ of normalized (simple) spherical roots see §2.1 for our normalization of the spherical roots; they are the simple roots of a based root system with Weyl group W_X .
- the valuation cone \mathcal{V} inside $\mathfrak{a}_X := \Lambda_X \otimes \mathbb{Q}$, where $\Lambda_X = \mathcal{X}(\mathbf{X})^* = \operatorname{Hom}(\mathbf{G}_{\mathrm{m}}, \mathbf{A}_X)$. It is the anti-dominant Weyl chamber for the based root system defined by Δ_X , and moreover contains the image of the negative Weyl chamber of the pair (\mathbf{G}, \mathbf{B}) under the projection $\mathfrak{a} \to \mathfrak{a}_X$ (where $\mathfrak{a} = \mathcal{X}(\mathbf{B})^* \otimes \mathbb{Q}$). These anti-dominant chambers will also be denoted by \mathfrak{a}^+ , \mathfrak{a}_X^+ , and the intersection of Λ_X with \mathcal{V} is denoted by Λ_X^+ .
- a submonoid and a subsemigroup $A_X^+, \mathring{A}_X^+ \subset A_X$ defined as:

$$A_X^+ := \{ a \in A_X : |\gamma(a)| \ge 1 \text{ for all } \gamma \in \Delta_X \}$$

and

$$\mathring{A}_X^+ = \{ a \in A_X : |\gamma(a)| > 1 \text{ for all } \gamma \in \Delta_X \}.$$

- The subscript Θ denotes a subset of the set Δ_X of simple spherical roots associated to the spherical variety, and thus corresponds to the face of \mathcal{V} to which it is orthogonal (the word "face" means the intersection of \mathcal{V} with the kernel of a linear functional which is nonnegative on \mathcal{V} hence, it can refer to \mathcal{V} itself). For each such Θ there is a distinguished subtorus $\mathbf{A}_{X,\Theta}$ of \mathbf{A}_X , with cocharacter group the orthogonal complement of Θ in Λ_X (hence, $\mathbf{A}_X = \mathbf{A}_{X,\emptyset}$). We define $A_{X,\Theta}^+$, $\mathring{A}_{X,\Theta}^+$ in a similar way as for A_X (see §2.4.8), using only the elements $\gamma \in \Delta_X \setminus \Theta$. The group $\mathbf{A}_{X,\Theta}$ is canonically isomorphic to the connected component of the \mathbf{G} -automorphism group of the "boundary degeneration" \mathbf{X}_{Θ} of \mathbf{X} (cf. §2.4), and therefore we are invariably using the notations $\mathbf{A}_{X,\Theta}$ and $\mathcal{Z}(\mathbf{X}_{\Theta})$.
- By the phrase "for a sufficiently deep in $A_{X,\Theta}^+$ " we mean, informally, that a is sufficiently far from the "walls" of $A_{X,\Theta}^+$; formally, "there exists T > 1 so that, whenever $|\gamma(a)| \geq T$ for all $\gamma \in \Delta_X \setminus \Theta, \ldots$ "

We use similar phrasing (e.g. "sufficiently large", "sufficiently positive") in many similar contexts.

We follow standard notation for denoting duals of representations, e.g. $\tilde{\pi}$ is the smooth dual of a representation π , \tilde{M} is the adjoint of a morphism M between representations, etc. (In general, we feel free to move between the unitary and smooth categories of representations of G, since it is clear from the context – or unimportant – which of the two we are referring to.)

Induction, the right adjoint functor to restriction, is denoted by Ind, but we usually use the symbol I to denote unitary induction, e.g. $I_P^G(\sigma) = \operatorname{Ind}_P^G(\sigma\delta_P^{\frac{1}{2}})$. Here δ_P is the modular character of the subgroup P, by which we mean the quotient of the right by a left Haar measure. (In the literature, this is sometimes δ_P^{-1} ; e.g. in Bourbaki.) The corresponding algebraic modular character: $\mathbf{P} \to \mathbf{G}_{\mathrm{m}}$, which is equal to the quotient of a right- by a left-invariant volume form, will be denoted by \mathfrak{d}_P .

Similarly, if U is the unipotent radical of P, we use the notation π_U to denote the normalized Jacquet module, that is: as a vector space it consists of the U-coinvariants on G, and we twist the action of P by $\delta_P^{-1/2}$. In particular, there is always a canonical morphism: $(I_P^G(\sigma))_U \to \sigma$.

We will in §2.7 define certain parabolics \mathbf{P}_{Θ} , \mathbf{P}_{Θ}^{-} (where $\Theta \subset \Delta_X$), and we will denote their unipotent radical by \mathbf{U}_{Θ} , \mathbf{U}_{Θ}^{-} and their Levi quotient (or subgroup) by \mathbf{L}_{Θ} . In that case, the corresponding normalized Jacquet module $\pi_{U_{\Theta}}$, $\pi_{U_{\Theta}^{-}}$ of a smooth representation π will also be denoted by π_{Θ} , resp. $\pi_{\Theta^{-}}$. We caution the reader that this notation is only applied when working in the smooth category of representations; thus, the notation $L^{2}(X)_{\Theta}$ is reserved for a different space, a closed subspace of $L^{2}(X)$ defined in Corollary 11.6.2.

When Y is an H-space (where H is a group) endowed with a positive H-eigenmeasure, with eigencharacter η , we define the normalized action of H on functions on Y by:

$$(h \cdot f)(y) = \sqrt{\eta(h)} f(yh).$$

This makes $L^2(Y)$ (with respect to the given measure) a unitary representation, and identifies (in the setting of homogeneous spaces for p-adic groups) the space $C^{\infty}(Y)$ (uniformly locally constant functions on Y, i.e. invariant by an open compact subgroup for G) with the smooth dual of $C_c^{\infty}(Y)$.

For a locally compact group H, we denote by \widehat{H} the unitary dual of H, endowed with the Fell topology. Notice the notational distinction to \check{H} , which is used for the Langlands dual group of H. When H is abelian, \widehat{H} is the Pontryagin dual group of H, and we denote by $\widehat{H}_{\mathbb{C}}$ the group of continuous homomorphisms $\operatorname{Hom}(H, \mathbb{C}^{\times})$, so that $\widehat{H} \subset \widehat{H}_{\mathbb{C}}$.

We slightly abuse the term "wonderful" to apply it to some embeddings of our spherical variety which are smooth but not necessarily wonderful; see $\S 2.3$ for details.

Given a function Q on a subset of $(\mathbb{Z}_+)^r$ we say that Q is "decaying" if it is bounded by a negative exponential: $|Q(x_1,\ldots,x_r)| \leq a^{\sum x_i}$ where a < 1. We apply this term to functions on A_X^+ , A_X^+ etc. by means of the natural

valuation maps, i.e. sending an element in $a \in A_X^+$ to the valuations of all $\gamma(a)$.

We also use the notation $A \ll B$ to mean that there exists a constant c such that $A \leq cB$.

In the final part we use using exponential notation for characters of tori, i.e. for **T** a torus, any character $\chi : \mathbf{T} \to \mathbb{G}_m$ will also be denoted by the symbol e^{χ} when it is more suggestive.

- 1.8. Important assumptions: Later in the paper we introduce further assumptions on the spherical variety X (and its boundary degenerations X_{Θ} , introduced in §2.4), which are used in all theorems without explicit mention:
 - We assume that the action of $\mathcal{Z}(\mathbf{X}) = \operatorname{Aut}_{\mathbf{G}}(\mathbf{X})^0$ is induced by the action of the center of \mathbf{G} (as we may in every case by replacing \mathbf{G} by $\mathcal{Z}(\mathbf{X}) \times \mathbf{G}$).
 - From Section 4 onwards we suppose that X carries a non-zero G-eigenmeasure (which we fix), and endow its "boundary degenerations" X_{Θ} with compatible eigenmeasures (cf. §4.1). As discussed in that section, this is not a significant restriction.
 - From Section 5 onward we assume that X is wavefront, i.e., that it satisfies the condition enunciated in §2.1. This is a genuine restriction, but applies in the vast majority of cases. We also assume from that point on that the connected central torus of G surjects onto $\mathcal{Z}(X)$, which causes no harm in generality.
 - From Section 11 we suppose that the Discrete Series Conjecture 9.4.6 holds for X; this holds in many cases, e.g. X is a symmetric variety (see §9.4.1) and can be checked in many others (possibly in every case) by the methods of §9.5.
 - In Section 15 we assume, in addition, that **X** satisfies the *generic injectivity condition* of §14.2 and is *strongly factorizable*. "Strongly factorizable" is a strong condition that holds, for example, for symmetric spaces but not for most other spherical varieties. The generic injectivity condition holds more generally, again for symmetric varieties (Proposition 14.2.1) and in many other cases (discussion in §14.2).

Notice that the generic injectivity condition is also explicitly assumed in the main scattering theorem 7.3.1, but in its proof (Section 14) it is only used at the very end, so Section 14 is not based on that assumption.

- The main theorems of Section 15, 15.6.1, 15.6.2, are conditional on another combinatorial condition (which, again, is known to hold for symmetric varieties), injectivity of the "small Mackey restriction", but this condition is explicitly stated in the theorems.

1.9. Some open problems. Our work does not resolve the global (respectively local) conjectures about periods (respectively: the support of Plancherel measure for $L^2(X)$) including the need to refine these conjectures. In many cases (whenever the main Scattering Theorem 7.3.1 applies), the local conjecture is essentially reduced, through our work, to discrete spectra. Let us now discuss a few more open questions:

First, although we phrase our results throughout this paper in such a way that they should apply as stated to the non-wavefront cases, most of our proofs break down. There are not many classes of non-wavefront varieties that we know of (the example $\mathbf{GL}_n \backslash \mathbf{SO}_{2n+1}$ is the typical one), but it seems that the non-wavefront case requires, and will lead to, a better understanding of harmonic analysis for p-adic groups and their homogeneous spaces. Notice that the example mentioned is also used by Knop [Kno94b, §10], to illustrate that his powerful theory of invariant differential operators on spherical varieties provides a genuine extension of the action of the center of the universal enveloping algebra.

Secondly, since our Plancherel decomposition is based on the understanding of how discrete series vary with the central character (cf. the Discrete Series Conjecture 9.4.6), it would be desirable to show in general (and not case-by-case, which can be done "by hand") that this conjecture holds, for instance that the unfolding process of §9.5 proves it. This is a problem which pertains to the cases which are not what we call "strongly factorizable"; the explicit Plancherel decomposition of section 15 via "Eisenstein integrals" is also open in those cases.

Third, it would be desirable to settle the combinatorial assumptions of some of the theorems (notably, Theorem 7.3.1) in some generality.

And, finally, a large class of problems has to do with developing Paley-Wiener theorems for spaces of Harish-Chandra–Schwartz or compactly supported functions.

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We are also very grateful to the referees of the manuscript. Their careful reading has greatly improved the paper.

⁷This is not completely the case when there are parameters into \check{G} which admit many lifts to **X**-distinguished parameters; in that case, one needs to know that scattering maps respect whichever parametrization of discrete spectra by **X**-distinguished parameters one has, in order to reduce the conjecture through our work to discrete spectra.

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This article has been written and re-written many times over the past nine years, repeatedly proving our expectations for finishing it unrealistic. Indeed, it has been written so many times that the word "nine" has been repeatedly changed in the prior sentence. Some of the results (those of section 5) were announced as early as in the summer of 2007 at the Hausdorff Center in Bonn; others were based on the assumption of the spectral decomposition of sections 11 and 14, which we only completed in its original form in 2013. The present form of the paper, which includes several corrections and improvements over the previous ones, many suggested by the referees, was completed in 2016. We offer our apologies to anyone who was promised a faster completion of the paper at any stage.

Part 1

The dual group of a spherical variety

2. Review of spherical varieties

The purpose of this section is to collect the necessary facts about spherical varieties. Most of the proofs will be given in the next section.

A spherical variety for a (split, connected) reductive group \mathbf{G} over a field k is a normal variety \mathbf{X} together with a \mathbf{G} -action, such that the Borel subgroup of \mathbf{G} has a dense orbit. We will assume throughout that \mathbf{X} is homogeneous and quasi-affine. The assumption of quasi-affineness is not a serious one, since every homogeneous \mathbf{G} -variety which is not quasi-affine is the quotient of a quasi-affine $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}$ -variety by the action of \mathbf{G}_{m} (and the cover is $\mathbf{G}_{\mathrm{m}} \times \mathbf{G}$ spherical if the original one was \mathbf{G} -spherical). We fix a (complex) dual group \check{G} to \mathbf{G} ; it comes equipped with a canonical maximal torus A^* , and the group \check{G} is canonical up to conjugation by elements of A^* . One of our primary goals is to attach to the spherical variety \mathbf{X} a reductive group \check{G}_X together with a morphism:

$$\check{G}_X \times \mathrm{SL}_2 \to \check{G}.$$
 (2.1)

We shall see that this morphism determines a great deal about the spherical variety, both its geometry and its representation theory.

In §2.1 we introduce basic combinatorial invariants, including the root system associated to a spherical variety by F. Knop. We also give the definition of a wavefront variety; we assume at most points in this text that the varieties under consideration are wavefront.

In §2.2 we modify this root system and discuss the associated reductive group \check{G}_X , which we term the dual group of the spherical variety. This is expected to be related to the group constructed by Gaitsgory and Nadler in [GN10]; in the next section (§3) we shall discuss the morphism (2.1); in particular, we will prove the existence of this morphism if one makes certain natural assumptions regarding the Gaitsgory-Nadler group.

In §2.3 we review the theory of toroidal compactifications, and in §2.4 and §2.5 we study the normal bundles of **G**-orbits in those. This study will be of importance later: we will interpret the asymptotics of special functions on $\mathbf{X}(k)$ using these normal bundles.

In §2.6 we present the modifications needed in order to treat cases such as the Whittaker model.

In §2.7 and 2.8 we introduce some other homogeneous varieties associated to X: Levi varieties and horospherical varieties. The former are closer to the traditional harmonic-analytic approach of studying Levi subgroups of a group, and will not be important for our formulations; they will be useful, however, for some proofs. The latter will be essential in explicating our harmonic-analytic constructions through the language of Eisenstein integrals in Section 15.

Finally, in §2.9 we discuss the example of $\mathbf{X} = \mathbf{PGL}_n$.

2.1. Invariants. We fix throughout a Borel subgroup **B** and denote the open **B**-orbit on **X** by $\mathring{\mathbf{X}}$. (See, however, footnote 6.)

Let **H** be an algebraic group acting on a variety **Y**. The multiplicative group of non-zero rational **H**-eigenfunctions (semiinvariants) will be denoted by $k(\mathbf{Y})^{(\mathbf{H})}$. We will denote the group of **H**-eigencharacters on $k(\mathbf{Y})^{(\mathbf{H})}$ by $\mathcal{X}_H(\mathbf{Y})$, and if **H** is our fixed Borel subgroup **B** then we will denote $\mathcal{X}_B(\mathbf{Y})$ simply by $\mathcal{X}(\mathbf{Y})$. If **Y** has a dense **B**-orbit, then we have a short exact sequence: $1 \to k^{\times} \to k(\mathbf{Y})^{(\mathbf{B})} \to \mathcal{X}(\mathbf{Y}) \to 1$.

As an intrinsic way of understanding the geometry at ∞ of the variety \mathbf{X} , one studies valuations of its function field. A point of basic interest is how the geometry of \mathbf{X} at ∞ interacts with the simple operation of acting on \mathbf{X} by a one-parameter subgroup \mathbf{G}_{m} . The discussion that follows formalizes the study of such matters:

For a finitely generated \mathbb{Z} -module M we denote by M^* the dual module $\operatorname{Hom}_{\mathbb{Z}}(M,\mathbb{Z})$. For our spherical variety \mathbf{X} , we let $\Lambda_X = \mathcal{X}(\mathbf{X})^*$ and $\mathfrak{a}_X = \Lambda_X \otimes_{\mathbb{Z}} \mathbb{Q}$. A **B**-invariant, \mathbb{Q} -valued valuation on $k(\mathbf{X})$ which is trivial on k^* (triviality on k^*) will be implicitly assumed from now on) induces by restriction to $k(\mathbf{X})^{(\mathbf{B})}$ an element of Λ_X . We let $\mathcal{V} \subset \mathfrak{a}_X$ be the cone generated by the images of \mathbf{G} -invariant valuations. (By [Kno91, Corollary 1.8], the map from \mathbf{G} -invariant valuations to \mathfrak{a}_X is injective.) We denote by Λ_X^+ the intersection $\Lambda_X \cap \mathcal{V}$. This is precisely the monoid of \mathbb{Z} -valued valuations. Notice that \mathcal{V} contains the image of the negative Weyl chamber under the natural map $\mathfrak{a} \to \mathfrak{a}_X$, [Kno91, Corollary 5.3]. (To get a sense for some of the geometry here, and in particular why there are "distinguished directions" in A_X at all, the reader may wish to glance at the example in §2.9.1.)

The notation $\mathcal V$ is compatible with the literature on spherical varieties, but in this paper we also denote it, invariably, by $\mathfrak a_X^+$. We say that $\mathbf X$ is a wavefront spherical variety if $\mathcal V$ is precisely equal to the image of the negative Weyl chamber. The terminology is due to the validity of the Wavefront Lemma 5.3.2; this class of varieties was not previously singled out, to our knowledge. Symmetric varieties, in particular, are wavefront [Kno91], but not, for instance, the variety $\mathbf U \setminus \mathbf G$, where $\mathbf U$ is a maximal unipotent subgroup. (However, the latter becomes wavefront if we consider the additional action of a maximal torus "on the left"; i.e., it is wavefront as a homogeneous variety for $\mathbf G \times \mathbf A$. Perhaps the simplest non-wavefront spherical variety which cannot be treated in this way is the variety $\mathbf G \mathbf L_2 \setminus \mathbf S \mathbf O_5$.)

The associated parabolic to \mathbf{X} is the standard parabolic $\mathbf{P}(\mathbf{X}) := \{g \in \mathbf{G} | \mathring{\mathbf{X}} \cdot g = \mathring{\mathbf{X}} \}$. Let us choose a point $x_0 \in \mathring{\mathbf{X}}(k)$ and let \mathbf{H} denote its stabilizer; hence $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$, and $\mathbf{H}\mathbf{B}$ is open in \mathbf{G} . There is the following "good" way of choosing a Levi subgroup $\mathbf{L}(\mathbf{X})$ of $\mathbf{P}(\mathbf{X})$, depending on the choice of x_0 : Pick $f \in k[\mathbf{X}]$, considered by restriction as an element of $k[\mathbf{G}]^{\mathbf{H}}$, such that the set-theoretic zero locus of f is $\mathbf{X} \setminus \mathring{\mathbf{X}}$. Then f is a $\mathbf{P}(\mathbf{X})$ -eigenfunction, but not an eigenfunction for a larger parabolic. Thus, its differential df at $1 \in \mathbf{G}$ defines an element in the coadjoint representation

of **G**, whose centralizer $\mathbf{L}(\mathbf{X})$ is a Levi subgroup of $\mathbf{P}(\mathbf{X})$. The intersection $\mathbf{L}(\mathbf{X}) \cap \mathbf{H}$ is known to contain the derived group of $\mathbf{L}(X)$ [Kno94a, Proposition 2.4].

We fix throughout a maximal torus \mathbf{A} in $\mathbf{B} \cap \mathbf{L}(\mathbf{X})$. We define \mathbf{A}_X to be the torus: $\mathbf{L}(\mathbf{X})/(\mathbf{L}(\mathbf{X}) \cap \mathbf{H}) = \mathbf{A}/(\mathbf{A} \cap \mathbf{H})$; its cocharacter group is the lattice Λ_X defined above. We identify \mathbf{A}_X with a subvariety of $\mathring{\mathbf{X}}$ through the orbit map: $a \mapsto x_0 \cdot a$.

We can also think of \mathbf{A}_X as a canonical, abstract torus associated to \mathbf{X} , in a similar way that the *universal Cartan group* is associated to the group \mathbf{G} . More precisely, if for every Borel subgroup \mathbf{B} we think of \mathbf{A}_X as the quotient by which \mathbf{B} acts on $\mathring{\mathbf{X}}/\mathbf{U}$ (here in the sense of geometric quotient), then for every two choices of Borel subgroups, any element conjugating one to the other induces the same isomorphism between the corresponding \mathbf{A}_X 's.

2.1.1. Remark. A symmetric subgroup \mathbf{H} is usually defined as the fixed point group of an involution θ on \mathbf{G} , and a symmetric variety as the space $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$; therefore it comes with a chosen point $x_0 = \mathbf{H} \cdot 1$. On the other hand, in the treatment of symmetric varieties one usually doesn't choose a Borel subgroup a priori. In that case, for a Borel \mathbf{B} such that $x_0 \cdot \mathbf{B}$ is open, the group $\mathbf{P}(\mathbf{X})$ is what is called a minimal θ -split parabolic. Moreover, the Levi subgroup $\mathbf{L}(\mathbf{X})$ constructed above is the unique θ -stable Levi subgroup of $\mathbf{P}(\mathbf{X})$, $\mathbf{L}(\mathbf{X}) = \mathbf{P}(\mathbf{X}) \cap {}^{\theta}\mathbf{P}(\mathbf{X})$.

The cone $\mathcal{V} = \mathfrak{a}_X^+$ is the fundamental domain for a finite reflection group $W_X \subset \operatorname{End}(\mathfrak{a}_X)$, called the *little Weyl group* of X. If we denote by W, resp. $W_{L(X)}$, the Weyl groups of G and L(X) with respect to A then there is a canonical way to identify W_X with a subgroup of W, normalizing $W_{L(X)}$ (which it intersects trivially) [Kno94b, §6.5]. The set of simple roots of G corresponding to G and the maximal torus G will be denoted by G.

Consider the (strictly convex) cone negative-dual to \mathcal{V} : $\mathcal{V}^{\perp} = \{\chi \in \mathcal{X}(\mathbf{X}) \otimes \mathbb{R} | \langle \chi, v \rangle \leq 0 \text{ for every } v \in \mathcal{V} \}$. The generators of the intersections of its extremal rays with $\mathcal{X}(\mathbf{X})$ are called the *spherical roots* of \mathbf{X} . They are known to form the set of simple roots of a based root system with Weyl group W_X [Kno94a]. This root system will be called the *spherical root system* of \mathbf{X} . We will denote this set of simple roots by Σ_X , following the notation of [Lun01], in order to distinguish it from the set of "(simple) normalized spherical roots", which will be defined later and denoted by Δ_X .

Finally, we discuss the group of **G**-automorphisms of **X**, based on [**Kno96**]: Of course, for any homogeneous variety $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ we have $\mathrm{Aut}_{\mathbf{G}}(\mathbf{X}) = \mathcal{N}(\mathbf{H})/\mathbf{H}$. As it is known [**BP87**, 5.2, Corollaire], the quotient $\mathcal{N}(\mathbf{H})/\mathbf{H}$ is diagonalizable. We will be denoting the torus $\mathrm{Aut}_{\mathbf{G}}(\mathbf{X})^0$ by $\mathcal{Z}(\mathbf{X})$. There is no harm in assuming that the connected center $\mathcal{Z}(\mathbf{G})^0$ of **G** surjects onto

 $^{^{8}}$ This also holds for any quasi-affine spherical variety, if **H** denotes the stabilizer of a point on the open **G**-orbit – cf. the remark after Lemma 6.6 in loc.cit..

⁹ In cases like the Whittaker model, where $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ and we have a morphism $\Lambda : \mathbf{H} \to \mathbf{G}_{\mathbf{a}}$ whose composition with a complex character of k gives rise to the line bundle

 $\mathcal{Z}(\mathbf{X})$ under its natural action on \mathbf{X} (by replacing \mathbf{G} by $\mathcal{Z}(\mathbf{X}) \times \mathbf{G}$, if necessary). This will be a standing assumption from section §5 onward.

Knop defines yet another root system¹⁰ with a distinguished set Σ'_X of simple roots which is proportional to Σ_X (but does not coincide with our Δ_X). Notice that every **G**-automorphism acts by a non-zero constant on each element of $k(\mathbf{X})^{(\mathbf{B})}$, and this defines a map: $\mathrm{Aut}_{\mathbf{G}}(\mathbf{X}) \to \mathrm{Hom}(\mathcal{X}(\mathbf{X}), \mathbf{G}_{\mathrm{m}}) = \mathbf{A}_X$. Knop proves that the corresponding map of character groups belongs to a short exact sequence:

$$0 \to \langle \Sigma_X' \rangle_{\mathbb{Z}} \to \mathcal{X}(\mathbf{X}) = \mathcal{X}(\mathbf{A}_X) \to \mathcal{X}(\operatorname{Aut}_{\mathbf{G}}(\mathbf{X})) \to 0.$$
 (2.2)

In particular, we have:

$$\mathcal{X}(\mathcal{Z}(\mathbf{X})) = \mathcal{X}(\mathbf{X})/(\mathcal{X}(\mathbf{X}) \cap \langle \Sigma_X \rangle_{\mathbb{O}}). \tag{2.3}$$

Finally, we include the following useful lemma of Knop:

2.1.2. LEMMA (Non-degeneracy). For every quasi-affine spherical variety \mathbf{X} , the set of coroots $\check{\alpha}$ of the group such that $\check{\alpha}$ is perpendicular to $\mathcal{X}(\mathbf{X})$ is precisely the set of coroots in the span of $\Delta_{L(X)}$.

PROOF. This is [Kno94a, Lemma 3.1].
$$\Box$$

2.2. The dual group of a spherical variety. Here we formulate, without proofs, our results on the dual group of a spherical variety and the associated "Arthur SL_2 ". All results will be proven in section 3.

Let A^* denote the dual torus of \mathbf{A} . By definition, it is the complex torus whose cocharacter group is $\mathcal{X}(\mathbf{A})$. Similarly, let A_X^* be the dual torus of \mathbf{A}_X . The map $\mathbf{A} \to \mathbf{A}_X$ dualizes to a map $A_X^* \to A^*$ (with finite kernel). Ideally, we would like this to extend to a map of connected reductive groups $\check{G}_X \to \check{G}$, where \check{G}_X has Weyl group W_X . Unfortunately this is not possible in general. For example, in the case of the variety $\mathbf{PO}_2 \setminus \mathbf{PGL}_2$ the kernel of $A_X^* \to A^*$ is of order 2, but the group $\check{G} = \mathrm{SL}_2$ does not have any non-trivial connected covers.

Nonetheless, Gaitsgory and Nadler [GN10] have canonically associated to any spherical affine embedding of \mathbf{X} a reductive subgroup $\check{G}_{X,GN} \subset \check{G}$ whose maximal torus is the image of A_X^* in A^* and whose Weyl group is, conjecturally, equal to W_X . (In the case of $\mathbf{PO}_2 \setminus \mathbf{PGL}_2$, this is just equal to SL_2 .) In our notation we suppress the dependence on the affine embedding: it is expected that $\check{G}_{X,GN}$ is independent of the choice of embedding, and in fact this follows from our arguments below, based on some natural assumptions (GN2)–(GN5) on the Gaitsgory-Nadler dual group.

under consideration, the definition of $\mathcal{Z}(\mathbf{X})$ should be as the connected component of the subgroup of $\mathcal{N}(\mathbf{H})/\mathbf{H}$ which stabilizes Λ , cf. §2.6.

 $^{^{10}}$ We caution the reader against confusing the notation of [Kno96] with ours. We have reserved the letters Σ , Δ to denote sets of simple roots, as is customary in the literature, and more precisely the notation Δ_X for our normalized spherical roots. On the contrary, Knop uses the letters Δ and Δ_X to denote the full sets of roots of the root systems generated by what we denote here by Σ_X , resp. Σ_X' .

In this section, we shall construct – under assumptions on **X** that rule out an example such as $\mathbf{PO}_2 \setminus \mathbf{PGL}_{2^-}$ a slightly different dual group \check{G}_X whose maximal torus is literally A_X^* .

Let γ be a spherical root, i.e. $\gamma \in \Sigma_X$. It is known (either from the work of Akhiezer [Akh83] classifying spherical varieties of rank one, or from the work of Brion [Bri01] for a classification-free argument) that either:

- (1) γ is proportional to a positive root α of \mathbf{G} , or;
- (2) γ is proportional to the sum $\alpha + \beta$ of two positive roots which are orthogonal to each other and part of some system of simple roots (not necessarily the one corresponding to **B**).

Notice that in the second case γ is not proportional to a root of \mathbf{G} , and therefore the two cases are mutually exclusive. In the first case we set $\gamma' = \alpha$, and in the second we set $\gamma' = \alpha + \beta$. Equivalently, γ' is the primitive element in the intersection of the \mathbb{R}_+ -span of γ with the root lattice of \mathbf{G} . The motivation for these choices will be explained in the next section, where we will revisit the work of [**Bri01**]. Notice that in the second case the roots α, β are not unique; however, we will see that there is a canonical choice (whose elements will be called the roots associated to γ). The set $\{\gamma' | \gamma \in \Sigma_X\}$ will be denoted by Δ_X . Let Φ_X denote the set of W_X -translates of the elements in Δ_X . We will see in the next section:

2.2.1. PROPOSITION. The pair (Φ_X, W_X) defines a root system, and Δ_X constitutes a set of simple roots for it.

This root system will henceforth be called the normalized spherical root system of \mathbf{X} , and the elements of Δ_X will be the (simple) normalized spherical roots. When it doesn't matter if we are working with Σ_X or Δ_X (for instance, choosing subsets of either of them), we may be abusing language and talking about "the set of spherical roots" Δ_X . We will use the notation $\check{\Phi}_X$, $\check{\Delta}_X$ – both subsets of $\mathcal{X}(\mathbf{X})^*$ – for the dual root system and the corresponding set of simple coroots.

2.2.2. PROPOSITION. Assume that Σ_X does not contain any elements of the form 2α , where α is a root of \mathbf{G} . Then the set $(\mathcal{X}(\mathbf{X})^*, \check{\Phi}_X, \mathcal{X}(\mathbf{X}), \Phi_X)$ is a root datum.

We refer to the corresponding complex reductive group \check{G}_X with maximal torus A_X^* as the dual group of the spherical variety \mathbf{X} .

The restrictions on Σ_X guarantee, roughly speaking, that the case of $\mathbf{PO}_2 \setminus \mathbf{PGL}_2$ does not "appear" in the "rank-one degenerations" of \mathbf{X} .

Our main result regarding the dual group of the spherical variety relies on the following statements. To formulate them, we must choose an affine embedding \mathbf{X}^a of \mathbf{X} , in order to apply the work of Gaitsgory and Nadler and attach to it a group $\check{G}_{X^a,GN}$; our description (Theorem 2.2.3) of the Gaitsgory-Nadler group based on the following statements will eventually prove that $\check{G}_{X^a,GN}$ is independent of the choice of embedding. The first statement is included in $[\mathbf{GN10}]$, although not explicitly stated. We will

treat the rest – (GN2), (GN3), (GN4) and (GN5) – as axioms. It seems likely that these could be checked, and we discuss them briefly in $\S 3.3$, but, as we are not specialists in the technical details, we prefer to phrase them as hypotheses.

- (GN1) The image of $\check{G}_{X^a,GN}$ commutes with $2\rho_{L(X)}(\mathbb{C}^{\times}) \subset A^*$, where $2 \cdot \rho_{L(X)}$ denotes the sum of positive roots of $\mathbf{L}(\mathbf{X})$, considered as a morphism: $G_{\mathrm{m}} \to A^*$.
- (GN2) The Weyl group of $\check{G}_{X^a,GN}$ equals W_X .
- (GN3) For any $\Theta \subset \Sigma_X$ the dual group of \mathbf{X}_{Θ}^a is canonically a subgroup of $\check{G}_{X^a,GN}$. Here, \mathbf{X}_{Θ}^a is a certain "affine degeneration" of \mathbf{X}^a , to be introduced in §2.5.
- (GN4) If the open **G**-orbit $\mathbf{X} \subset \mathbf{X}^a$ is parabolically induced, $\mathbf{X} = \mathbf{X}_L \times^{\mathbf{P}^-}$ **G**, where \mathbf{X}_L is spherical for the reductive quotient \mathbf{L} of \mathbf{P}^- , then the dual group $\check{G}_{X^a,GN}$ belongs to the standard Levi subgroup \check{L} of \check{G} corresponding to the class of parabolics¹¹ opposite to \mathbf{P}^- . Moreover, if a connected normal subgroup \mathbf{L}_1 of \mathbf{L} acts trivially on \mathbf{X}_L , then $\check{G}_{X^a,GN}$ belongs to the dual group of \mathbf{L}/\mathbf{L}_1 (which is canonically a subgroup of \check{L}).
- (GN5) If \mathbf{X}_1^+ is a spherical homogeneous \mathbf{G} -variety, \mathbf{T} a torus of \mathbf{G} -automorphisms and $\mathbf{X}_2^+ = \mathbf{X}_1^+/\mathbf{T}$, and if $\mathbf{X}_1, \mathbf{X}_2$ are affine embeddings of $\mathbf{X}_1^+, \mathbf{X}_2^+$ with $\mathbf{X}_2 = \operatorname{spec} k[\mathbf{X}_1]^\mathbf{T}$, then there is a canonical inclusion $\check{G}_{X_2,GN} \hookrightarrow \check{G}_{X_1,GN}$ which restricts to the natural inclusion of maximal tori: $A_{X_2,GN}^* \hookrightarrow A_{X_1,GN}^*$ (arising from $\mathcal{X}(\mathbf{X}_2) \hookrightarrow \mathcal{X}(\mathbf{X}_1)$).

The formulation of the result is based on the notion of a "distinguished morphism" from the group \check{G}_X to \check{G} . By a distinguished morphism we mean a morphism which restricts to the canonical map of maximal tori: $A_X^* \to A^*$, and moreover satisfies a condition on the image of simple root spaces, which will be formulated in §3.2. A distinguished morphism from $\check{G}_X \times \mathrm{SL}_2$ to \check{G} is one which restricts to a distinguished morphism on \check{G}_X and, moreover, under the "standard" diagonal embedding of G_{m} to SL_2 it restricts to the map $2\rho_{L(X)}:G_{\mathrm{m}}\to A^*$.

- 2.2.3. THEOREM. Assuming that Σ_X does not contain any elements of the form 2α (where α is a root of \mathbf{G}):
 - (1) There is at most one A^* -conjugacy class of distinguished morphisms

$$\check{G}_X \times \mathrm{SL}_2 \to \check{G}$$
.

(2) Assume Axioms (GN2), (GN3), (GN4), (GN5). Then distinguished morphisms exist.

¹¹We clarify the correspondence between \mathbf{P}^- and \check{L} . Recall that $A^* \subset \check{G}$ is the dual torus of the "universal Cartan" of \mathbf{G} , i.e. the reductive quotient of any Borel subgroup of \mathbf{G} . One chooses a Borel \mathbf{B} opposite to \mathbf{P}^- , and the positive coroots of the Levi \check{L} are the roots in the Lie algebra of $\mathbf{B} \cap \mathbf{P}^-$.

- (3) Assume (GN2), (GN3), (GN4), (GN5). Then the root system of the Gaitsgory-Nadler dual group is given by $(\check{\Phi}_X, W_X)$; in particular, it is independent of the choice of affine embedding of \mathbf{X} .
- 2.2.4. Remark. We stated this theorem for the case that Σ_X does not contain elements of the form 2α . However, in the general case the same statements are true if one replaces \check{G}_X with the abstract group \check{G}_X' defined by the same root system and with maximal torus equal to the image of $A_X^* \to A^*$. In particular, $\check{G}_{X,GN} \simeq \check{G}_X'$. However, this does not seem to be the correct dual group for all purposes of representation theory.

We give a rather clumsy proof of this theorem, which boils down to case-by-case considerations, in the next section.

- 2.2.5. EXAMPLE. For the spherical variety $\mathbf{X} = \mathbf{T} \backslash \mathbf{SL}_2$ (where \mathbf{T} is a non-trivial torus) the dual group is $\check{G}_X = \mathrm{SL}_2$ with its natural isogeny: $\mathrm{SL}_2 \to \check{G} = \mathrm{PGL}_2$. Notice that here the action of \mathbf{SL}_2 on \mathbf{X} factors through the quotient $\mathbf{SL}_2 \to \mathbf{PGL}_2$. This "explains" why the representation theory of \mathbf{X} should be described in terms of SL_2 , which is the dual group of \mathbf{PGL}_2 . However, this argument may not always work when $\check{G}_X \neq \check{G}_{X,GN}$.
- **2.3. Toroidal compactifications.** We will freely use the word "compactification" of X for what should more correctly be termed "spherical embedding" or "partial spherical completion", namely a normal G-variety \bar{X} containing X as a dense G-subvariety. A compactification \bar{X} is said to be *simple* if it contains a unique closed G-orbit, and *toroidal* if no B-stable divisor in X contains a G-orbit of \bar{X} in its closure. To every simple toroidal embedding \bar{X} we associate the cone $C(\bar{X}) \subset \mathcal{V}$ spanned by the valuations defined by all G-stable divisors in \bar{X} . The main theorem in the classification of such embeddings states:
- 2.3.1. Theorem (Luna and Vust, cf. [**Kno91**, Theorem 3.3]). The map $\bar{\mathbf{X}} \mapsto \mathcal{C}(\bar{\mathbf{X}})$ induces a bijection between isomorphism classes of simple toroidal compactifications of \mathbf{X} and strictly convex, finitely generated cones in \mathcal{V} .
- 2.3.2. Remark. Although this theorem, and the other theorems of Brion, Luna, Pauer and Vust which we are going to recall, have been stated for an algebraically closed field in characteristic zero, their proofs carry through over an arbitrary field in characteristic zero, as long as the group ${\bf G}$ is split.

Notice that for every **G**-orbit **Z** in a simple toroidal embedding $\bar{\mathbf{X}}$, the union of all **G**-orbits whose closure contains **Z** is also a simple toroidal embedding. This way, we get a bijection between faces¹² of $\mathcal{C}(\bar{\mathbf{X}})$ and orbits of **G** on **X**

Now observe that when \mathcal{V} itself is strictly convex (equivalently: $\operatorname{Aut}_{\mathbf{G}}(\mathbf{X})$ is finite), this implies the existence of a canonical compactification with

¹²The word "face" is used for the intersection of $\mathcal{C}(\bar{\mathbf{X}})$ with the kernel of a linear functional which is non-negative on $\mathcal{C}(\bar{\mathbf{X}})$; hence $\mathcal{C}(\bar{\mathbf{X}})$ is the face corresponding to the closed orbit, and $\{0\}$ is the face corresponding to \mathbf{X} .

 $\mathcal{C}(\bar{\mathbf{X}}) = \mathcal{V}$. Its existence can also be characterized by the following conditions:

2.3.3. Theorem ([BP87, 5.3, Corollaire]). The following are equivalent:

- (1) There exists a simple complete toroidal compactification of X.
- (2) The **G**-automorphism group $\operatorname{Aut}_{\mathbf{G}}(\mathbf{X}) = \mathcal{N}(\mathbf{H})/\mathbf{H}$ is finite.
- (3) The cone V is strictly convex.

The corresponding complete variety $\bar{\mathbf{X}}$ is sometimes (e.g. in [Kno96]) called the wonderful compactification of \mathbf{X} , though the term "wonderful" ("magnifique") is more often (e.g. in [Lun01]) reserved for the case when $\bar{\mathbf{X}}$ is smooth. To understand the difference between the two, let us recall the Local Structure Theorem of Brion, Luna and Vust: Let $\bar{\mathbf{X}}$ be a simple toroidal embedding of \mathbf{X} . The complement of all \mathbf{B} -stable divisors of $\bar{\mathbf{X}}$ which are not \mathbf{G} -stable is denoted by $\bar{\mathbf{X}}_B$, and it is a \mathbf{B} -stable open affine subvariety. Let \mathbf{Y} be the closure of \mathbf{A}_X in $\bar{\mathbf{X}}_B$; it is the toric compactification of \mathbf{A}_X characterized by the property that for $\lambda \in \Lambda_X = \mathrm{Hom}(\mathbf{G}_{\mathrm{m}}, \mathbf{A}_X)$ we have $\lim_{t\to 0} \lambda(t) \in \mathbf{Y}$ if and only if $\lambda \in \Lambda_X^+$.

2.3.4. Theorem ([BLV86, Théorème 3.5]). The action map

$$\mathbf{Y} \times \mathbf{U}_{P(X)} \to \bar{\mathbf{X}}$$

is an open embedding and its image $\bar{\mathbf{X}}_B$ meets each G-orbit in $\bar{\mathbf{X}}$ along its open B-orbit.

Hence, the variety $\bar{\mathbf{X}}$ will be smooth if and only if the toric variety \mathbf{Y} is. Since \mathcal{V} is a simplicial cone, it follows from this theorem that the compactification of Theorem 2.3.3 has, if any, only finite quotient singularities, and that it is smooth if and only if the monoid Λ_X^+ is generated by its intersections with the extremal rays of \mathcal{V} ; equivalently: if and only if Σ_X generates $\mathcal{X}(\mathbf{X})$. In that case, all \mathbf{G} -orbits are smooth and the complement of the open \mathbf{G} -orbit is a union of \mathbf{G} -stable divisors intersecting transversely.

In this paper we will adopt the following convention: We will, by abuse of language, refer to any smooth, complete, toroidal compactification of X as the "wonderful compactification"; and this term will also be extended to certain non-complete embeddings when we consider Whittaker-type induction in §2.6. We will attach certain "boundary degenerations" to X indexed by subsets of the set of spherical roots, which although by construction seem to depend on the choice of such a compactification, an important result – Proposition 2.5.3 – states that they are actually completely canonical. Our representation-theoretic results, then, will be formulated in terms of these degenerations. Whenever proofs are identical for the wonderful compactification of Theorem 2.3.3 (if it exists and is smooth) and a general smooth, complete, toroidal compactification, for simplicity and clarity we only formulate them for the former; whenever the general case needs extra arguments, we provide them.

Let us now discuss the general case, to which one will necessarily resort when $\operatorname{Aut}_{\mathbf{G}}(\mathbf{X})$ is not finite, or it is finite but the canonical embedding of Theorem 2.3.3 is not smooth.

2.3.5. The non-wonderful case. A toroidal embedding which is not necessarily simple is described by a strictly convex fan, instead of a strictly convex cone, that is: a collection \mathfrak{F} of (distinct) strictly convex subcones of \mathcal{V} as above, such that each face of a cone of \mathfrak{F} is also contained in \mathfrak{F} , and each point of \mathcal{V} belongs to the relative interior of at most one cone in \mathfrak{F} . We point the reader to [Kno91, Theorem 3.3 and discussion after Corollary 5.3] for details.

To get a *complete*, *smooth* toroidal embedding we need to subdivide \mathcal{V} into a finite union of (strictly convex) simplicial cones \mathcal{C}_i , such that $\Lambda_X \cap \mathcal{C}_i$ is a free monoid for every i; then these cones and their faces form the fan of the embedding. Each such embedding is the union of simple (non-complete) smooth toroidal embeddings; hence, the complement of the open **G**-orbit is a divisor with normal crossings.

The Local Structure Theorem 2.3.4, applies equally well to this case with the understanding that the toric variety \mathbf{Y} is that associated to the fan defining the toroidal embedding.

The set of **G**-orbits in such an embedding is in bijection with the set of cones in the fan. The relative interior of the cone corresponding to an orbit **Z** consists of those valuations v whose *center* is the closure of **Z**, that is: $\mathfrak{o}_v \supset \mathfrak{o}_Z$ and $\mathfrak{m}_v \supset \mathfrak{m}_Z$, where by \mathfrak{o} and \mathfrak{m} we denote the subring of $k(\mathbf{X})$ and its ideal, respectively, defined by the valuation v or by the closure of **Z**. This face, in turn, corresponds (non-injectively, in general) to the subset $\Theta \subset \Sigma_X$ of spherical roots to which it is orthogonal. (Since the elements of Σ_X are proportional to the normalized spherical roots, i.e. the elements of Δ_X , we will interchangeably be identifying Θ with a subset of either of the two.) We say that "**Z** corresponds to Θ ".

2.3.6. The notion of Θ -infinity. For each smooth toroidal embedding $\bar{\mathbf{X}}$ and every $\Theta \subset \Delta_X$ (or $\Theta \subset \Sigma_X$) we let ∞_{Θ} , the " Θ -infinity", denote the closure of the union of all \mathbf{G} -orbits which correspond to Θ , i.e. whose corresponding face is orthogonal to the set Θ of spherical roots. We have $\infty_{\Delta_X} = \bar{\mathbf{X}}$. As we will see in a moment, all of those orbits correspond to isomorphic "boundary degenerations".

When working at the level of k-points, we use the notion of "neighborhood of Θ -infinity", which we explicate for clarity: A "neighborhood of Θ -infinity" in X is, by definition, the intersection of X with a neighborhood of ∞_{Θ} in \overline{X} (for the Hausdorff topology induced by that of k). Note that the fact that ∞_{Θ} is the *closure* of the union of all orbits corresponding to Θ affects the meaning of this notion. When the embedding \overline{X} is not explicated, we will mean a "wonderful" embedding (in the above sense). It is clear that this abstract notion of Θ -infinity does not depend on the choice of a wonderful (i.e. complete, toroidal) embedding.

For a given **G**-orbit on $\overline{\mathbf{X}}$, we will say that it "belongs to Θ -infinity" if it is contained in ∞_{Θ} , but not in ∞_{Ω} , for any $\Omega \subseteq \Theta$.

- 2.3.7. Spherical system of a **G**-orbit. For the following proposition we remind that a "color" of a spherical variety is a prime **B**-stable divisor which is not **G**-stable (hence colors are in bijection with **B**-orbits of codimension one in the open **G**-orbit, and by abuse of language we will be calling these **B**-orbits "colors" as well), and that each color **D** defines a "valuation" $\check{v}_D \in \mathcal{X}(\mathbf{X})^*$, defined exactly as in the case of **G**-stable divisors that we discussed above (i.e. by restriction to $k(\mathbf{X})^{(\mathbf{B})}$).
- 2.3.8. Proposition. Let \mathbf{Z} be a \mathbf{G} -orbit in a toroidal compactification, and let Θ be the set of (unnormalized) spherical roots to which the corresponding face \mathcal{F} is orthogonal. Then:
 - (1) $\mathcal{X}(\mathbf{Z}) = \mathcal{F}^{\perp} \subset \mathcal{X}(\mathbf{X});$
 - (2) $\mathbf{P}(\mathbf{Z}) = \mathbf{P}(\mathbf{X});$
 - (3) $\Sigma_{\mathbf{Z}} = \Theta$;
 - (4) for each simple root α of \mathbf{G} , which belongs to Θ , there are precisely two colors $\mathbf{D}_1', \mathbf{D}_2'$ in $\mathring{\mathbf{Z}}\mathbf{P}_{\alpha}$, obtained as the (multiplicity-free) intersection with \mathbf{Z} of the closures of the two colors $\mathbf{D}_1, \mathbf{D}_2$ in $\mathring{\mathbf{X}}\mathbf{P}_{\alpha}$; the valuations $\check{\mathbf{v}}_{D_i'}$ are the images of the valuations $\check{\mathbf{v}}_{D_i}$ under the restriction map: $\mathcal{X}(\mathbf{X})^* \to \mathcal{X}(\mathbf{Z})^*$.

PROOF. The first and second statement follow from the Local Structure Theorem 2.3.4.

The third statement can be proven by induction on the codimension of \mathbf{Z} ; hence, we may assume that \mathcal{F} is a half-line. If this half-line belongs to $\mathcal{V} \cap (-\mathcal{V})$ then the Local Structure Theorem 2.3.4 implies that \mathbf{Z} is isomorphic to the quotient of \mathbf{X} by the subtorus of $\mathcal{Z}(\mathbf{X})$ corresponding to this line. Otherwise, the theory of toroidal embeddings implies that there is a wonderful compactification of $\mathbf{X}/\mathrm{Aut}_{\mathbf{G}}(\mathbf{X})$ which contains a quotient of \mathbf{Z} by its automorphism group, and the statement follows by a characterization of simple spherical roots in terms of the \mathbf{G} -action on the normal bundle to the closed orbit, cf. [Lun01, §1.3].

If α is a simple root of \mathbf{G} which belongs to $\Theta = \Sigma_{\mathbf{Z}}$ then there are two colors in $\mathring{\mathbf{Z}}\mathbf{P}_{\alpha}$, and they are precisely those colors on which the Borel eigenfunction f_{α} of weight α vanishes – the valuation of f_{α} on each of them is one [Lun01, §1.4], [Lun97, Proposition 3.4]. Every nonzero Borel eigenfunction on \mathbf{Z} extends (uniquely) to a Borel eigenfunction on \mathbf{X} [Kno91, Theorem 1.3], and similarly the extension of f_{α} has simple zeroes on the two colors of $\mathring{\mathbf{X}}\mathbf{P}_{\alpha}$. This shows that their closures intersect \mathbf{Z} along the colors of $\mathring{\mathbf{Z}}\mathbf{P}_{\alpha}$ without multiplicity; the fact that all eigenfunctions extend means that the valuations induced by the latter are equal to the valuations induced by the former, restricted to the character group of \mathbf{Z} .

2.4. Normal bundles and boundary degenerations. We are interested in understanding normal bundles of **G**-orbits in wonderful embeddings;

as we will see in Section 5, the asymptotics of eigenfunctions on X can be understood in terms of eigenfunctions on the normal bundles.

Let \mathbf{Z} be a \mathbf{G} -orbit in a smooth toroidal compactification $\bar{\mathbf{X}}$ of \mathbf{X} , corresponding to a subset Θ of the set of spherical roots (§2.3.6; in the propositions that follow, Θ will be considered as a subset of either the unnormalized spherical roots Σ_X or the normalized spherical roots Δ_X , according to what is appropriate in each case). The normal bundle $N_{\mathbf{Z}}\bar{\mathbf{X}}$ of \mathbf{Z} in $\bar{\mathbf{X}}$ carries an action of \mathbf{G} , under which it is spherical; this follows immediately from the Local Structure Theorem 2.3.4. The open \mathbf{G} -orbit on $N_{\mathbf{Z}}\bar{\mathbf{X}}$ will be denoted by \mathbf{X}_{Θ} and called a boundary degeneration of $\bar{\mathbf{X}}$.

We remark that if **X** is wavefront, then so is \mathbf{X}_{Θ} under the action of $\mathcal{Z}(\mathbf{X}_{\Theta}) \times \mathbf{G}$, for every $\Theta \subset \Delta_X$. That will follow from Proposition 2.7.2.

2.4.1. Remark. We notice that in the case of general toroidal compactifications there may be many **G**-orbits corresponding to the same Θ ; as we will see, all varieties \mathbf{X}_{Θ} will be canonically isomorphic to each other, which will allow us to use this notation indistinguishably.

Let **Z** be a **G**-orbit in $\bar{\mathbf{X}}$ corresponding to $\Theta \subset \Sigma_X$. The fact that \mathbf{X}_{Θ} belongs to a normal bundle gives rise to a torus of **G**-equivariant automorphisms of it:

2.4.2. Lemma. Let Γ denote the set of \mathbf{G} -stable divisors in $\bar{\mathbf{X}}$ which contain \mathbf{Z} , then there is a canonical action of the torus \mathbf{G}_{m}^{Γ} on \mathbf{X}_{Θ} by \mathbf{G} -automorphisms, such that the quotient is isomorphic to \mathbf{Z} .

PROOF. Since these **G**-stable divisors intersect transversely, the normal bundle $N_{\mathbf{Z}}\bar{\mathbf{X}}$ splits canonically into a direct sum of line bundles L_{γ} indexed by the spherical roots $\gamma \in \Gamma$. Moreover, \mathbf{X}_{Θ} is isomorphic to the total space of the direct sum $\bigoplus_{\gamma \in \Gamma} L_{\gamma}$, minus the union of the smaller direct sums; this follows, for example, from the Local Structure Theorem 2.3.4. This is a principal bundle over **Z** with structure group the torus \mathbf{G}_{m}^{Γ} .

Proposition 2.5.3 in the next subsection will say that for different orbits \mathbf{Z} in a smooth toroidal compactification, corresponding to the same Θ , the resulting boundary degenerations \mathbf{X}_{Θ} are *canonically* isomorphic. Removing the word "canonically", this could also be inferred from the uniqueness theorem of Losev [Los09, Theorem 1] and the following proposition:

- 2.4.3. PROPOSITION. Let **Z** be a **G**-orbit in a toroidal compactification, and let $\Theta \subset \Sigma_X$ be the set of (unnormalized) spherical roots to which the corresponding face \mathcal{F} is orthogonal, and let \mathbf{X}_{Θ} be defined as above. Then:
 - (1) $\mathcal{X}(\mathbf{X}_{\Theta}) = \mathcal{X}(\mathbf{X})$; more precisely, there is a canonical isomorphism of "universal tori":

$$\mathbf{A}_X \simeq \mathbf{A}_{X_{\Theta}}.\tag{2.4}$$

- (2) $\mathbf{P}(\mathbf{X}_{\Theta}) = \mathbf{P}(\mathbf{X});$
- (3) $\Sigma_{\mathbf{X}_{\Theta}} = \Theta$;

(4) for each simple root α of \mathbf{G} , which belongs to Θ , there are precisely two colors $\mathbf{D}'_1, \mathbf{D}'_2$ in $\mathring{\mathbf{X}}_{\Theta} \mathbf{P}_{\alpha}$, and they induce the same valuations $\check{\mathbf{v}}_{D_i}$ as the two colors in $\mathring{\mathbf{X}} \mathbf{P}_{\alpha}$.

PROOF. Fix a Borel subgroup \mathbf{B} , and consider the local $\mathbf{P}(\mathbf{X})$ -equivariant isomorphism of the Local Structure Theorem 2.3.4. Notice that this isomorphism is not canonical, but its quotient by the action of $\mathbf{U}_{P(X)}$ is. For the smooth toric variety \mathbf{Y} , if $\mathbf{Y}' \subset \mathbf{Y}$ denotes the closure of the orbit corresponding to \mathbf{Z} , there is a unique isomorphism: $\phi: N_{\mathbf{Y}'}\mathbf{Y} \xrightarrow{\sim} \mathbf{Y}$ with the following properties:

- (1) ϕ is $\mathbf{L}(\mathbf{X})$ -equivariant;
- (2) its "partial" differential induces the identity on $N_{\mathbf{Y}'}\mathbf{Y}$.

Restricted to the open $\mathbf{L}(\mathbf{X})$ -orbits, this map gives a canonical isomorphism: $\mathbf{A}_{X_{\Theta}} \xrightarrow{\sim} \mathbf{A}_{X}$. The second statement also follows from Theorem 2.3.4. The third follows from Proposition 2.3.8 and Lemma 2.4.2.

The last assertion is proven as in Proposition 2.3.8, except that now one has to consider an affine degeneration of \mathbf{X} to \mathbf{X}_{Θ} , which will be discussed in §2.5: it is a \mathbf{G} -variety \mathscr{X}^a over a base \mathscr{B} , whose generic fiber is isomorphic to an affine completion of \mathbf{X} and its special fiber is isomorphic to an affine completion of \mathbf{X}_{Θ} . The inclusion of the open Borel orbit on each fiber is a direct product: $\mathring{\mathscr{X}}^a \simeq \mathscr{B} \times \mathring{\mathbf{X}}$, and therefore \mathbf{B} -eigenfunctions extend nontrivially to the special fiber. We omit the details, since this result will not be used in the sequel.

In terms of dual groups, the following is implied by Proposition 2.4.3 under the assumptions of Theorem 2.2.3. (When Σ_X contains elements of the form 2α , the analogous statement applies to $\check{G}_{X,GN}$, cf. Remark 2.2.4.)

- 2.4.4. COROLLARY. The dual group of \mathbf{X}_{Θ} is the Levi of \check{G}_X with simple roots $\check{\Theta}$ (where $\check{\Theta}$ denotes the set of coroots of the elements of Θ now considered as a subset of Δ_X).
- 2.4.5. *Identification of Borel orbits*. Notice that the map (2.4), together with the Local Structure Theorem 2.3.4, gives rise to a **B**-equivariant isomorphism:

$$\mathring{\mathbf{X}}_{\Theta} \xrightarrow{\sim} \mathring{\mathbf{X}} \tag{2.5}$$

inducing the identity on normal bundles. Such isomorphism is not completely canonical, as it depends on the choice involved in the Local Structure Theorem. However, it is canonical up to $\bf B$ -automorphisms of $\dot{\bf X}$ which induce the identity on $\dot{\bf X}/{\bf U}$, which means up to a morphism of the form:

$$(a,u) \mapsto (a,a^{-1}u_1au)$$

(in the setting of Theorem 2.3.4), where $u_1 \in U_{P(X)}$ is fixed by the kernel of the map: $L(X) \to A_X$.

2.4.6. Automorphisms. It follows now from (2.3) that $\mathcal{Z}(\mathbf{X}_{\Theta}) = \operatorname{Aut}_{\mathbf{G}}(\mathbf{X}_{\Theta})^0$ can be canonically identified with the maximal subtorus 13 $\mathbf{A}_{X,\Theta}$ of \mathbf{A}_X whose cocharacter group is perpendicular to Θ . We wish to explicate the embedding $\mathbf{G}_{\mathrm{m}}^{\Delta_X \setminus \Theta} \to \mathbf{A}_{X,\Theta}$ obtained from Lemma 2.4.2 (we restrict ourselves to the wonderful case, where the Γ of Lemma 2.4.2 is equal to $\Delta_X \setminus \Theta$); in fact, it is enough to do so when $\Delta_X \setminus \Theta$ only has one element, since these embeddings are obviously compatible with each other. Choose a Borel subgroup \mathbf{B} and a $\gamma \in \Delta_X$. Let γ' be the corresponding unnormalized spherical root (i.e. $\gamma' \in \Sigma_X$). It follows from the Local Structure Theorem 2.3.4 that the valuation induced by the orbit corresponding to $\Delta_X \setminus \{\gamma\}$ is equal to $-\gamma'^*$, where $-\gamma'^*$ is the element of \mathcal{V} with $\langle \gamma', -\gamma'^* \rangle = -1$ and $\langle \delta, -\gamma'^* \rangle = 0$ for all $\delta \in \Sigma_X \setminus \{\gamma'\}$. (If $\mathcal{Z}(\mathbf{X}) \neq 1$ then we take $-\gamma'^* \in \mathcal{V}'$.) Notice that under our assumption that $\bar{\mathbf{X}}$ is smooth, the elements $-\gamma'^*, \gamma' \in \Sigma_X$ form a basis for the monoid $\Lambda_{\mathbf{X}}^+$. Hence, the action of $m \in \mathbf{G}_{\mathbf{m}}^{\{\gamma\}}$ multiplies a \mathbf{B} -eigenfunction with eigencharacter χ by $\langle \chi, -\gamma'^* \rangle$, and hence we obtain:

2.4.7. LEMMA. The composition of (2.3) with the identification of $\mathcal{Z}(\mathbf{X}_{\Theta})$ with a subtorus of \mathbf{A}_X is given by the maps:

$$-\gamma'^*: \mathbf{G}_{\mathbf{m}}^{\{\gamma\}} \to \mathbf{A}_X. \tag{2.6}$$

2.4.8. Positive elements. For every $\Theta \subset \Delta_X$ we denote:

$$A_{X,\Theta}^{+} := \{ a \in A_{X,\Theta} : |\gamma(a)| \ge 1 \text{ for all } \gamma \in \Delta_X \setminus \Theta \}, \tag{2.7}$$

and:

$$\mathring{A}_{X,\Theta}^{+} = \{ a \in A_{X,\Theta} : |\gamma(a)| > 1 \text{ for all } \gamma \in \Delta_X \setminus \Theta \}.$$
 (2.8)

Let $\Lambda_{X,\Theta}^+$ (resp. $\mathring{\Lambda}_{X,\Theta}^+$) denote the intersection of $\Lambda_X (= \mathcal{X}(\mathbf{X})^*)$ with the face (resp. the relative interior of the face) of the cone \mathcal{V} which is orthogonal to Θ . We can alternatively describe $A_{X,\Theta}^+$ (resp. $\mathring{A}_{X,\Theta}^+$) as the submonoid generated by all m(x), where $m \in \Lambda_{X,\Theta}^+$ is a cocharacter in \mathcal{V} and $x \in k^\times \cap \mathfrak{o}$ (resp. $m \in \mathring{\Lambda}_{X,\Theta}^+$ and $x \in k^\times \cap \mathfrak{p}$). Hence, $A_{X,\Theta}^+$ (resp. $\mathring{A}_{X,\Theta}^+$ is the preimage of $\Lambda_{X,\Theta}^+$ (resp. of $\mathring{\Lambda}_{X,\Theta}^+$) under the "valuation" maps:

$$A_{X,\Theta} \to A_{X,\Theta}/\mathbf{A}_{X,\Theta}(\mathfrak{o}) \simeq \Lambda_{X,\Theta},$$

normalized so that for $\check{\lambda} \in \Lambda_{X,\Theta}$ the valuation of $\check{\lambda}(\varpi)$ is $\check{\lambda}$.

When $\Theta = \emptyset$, $\mathbf{A}_{X,\emptyset} = \mathbf{A}_X$ and the notation is compatible with the notation \mathfrak{a}_X^+ that we have been invariably using for the cone $\mathcal V$ of invariant valuations. In this case we will denote $A_{X,\Theta}^+$, $\mathring{A}_{X,\Theta}^+$ by A_X^+ , \mathring{A}_X^+ (since $A_{X,\emptyset} = A_X$). We remark that under the map: $A \to A_X$, the anti-dominant elements of A are contained in A_X^+ ; indeed, we have already seen that $\mathcal V$ contains the image of the negative Weyl chamber of the group.

The following is an easy corollary of the definitions and of Lemma 2.4.7:

¹³We will be using the notation $\mathcal{Z}(\mathbf{X}_{\Theta})$ and $\mathbf{A}_{X,\Theta}$ interchangeably. On the other hand, $\mathbf{A}_{X,\Theta}$ is not to be confused with $\mathbf{A}_{X_{\Theta}}$; the latter, as we saw in Lemma 2.4.3, is isomorphic to \mathbf{A}_{X} , which allows us never to use the notation $\mathbf{A}_{X_{\Theta}}$ again.

2.4.9. LEMMA. Under the map $\mathbf{G}_{\mathrm{m}}^{\{\gamma\}} \to \mathbf{A}_X$ of Lemma 2.4.7, $A_{X,\Theta}^+$ is precisely the image of $(\mathfrak{o} \setminus \{0\})^{\{\gamma\}}$.

The elements of $\mathring{A}_{X,\Theta}^+$ are precisely those $a \in A_{X,\Theta} = \mathcal{Z}(X_{\Theta})$ with the property that $\lim_{n\to\infty} a^n \cdot x$ is contained in Θ -infinity, for some (hence all) $x \in X_{\Theta}$. The elements of $A_{X,\Theta}^+$ are precisely those $a \in A_{X,\Theta}$ with the property that $\lim_{n\to\infty} a^n \cdot x$ is contained in a G-orbit on \bar{X} whose closure contains Θ -infinity, for some (hence all) $x \in X_{\Theta}$.

2.5. Degeneration to the normal bundle; affine degeneration. It is well-known [Ful84, §2.6] that given a closed embedding $\mathbf{Y} \subset \mathbf{Z}$ of varieties, there is a canonical way to deform \mathbf{Z} over $\mathbf{G}_{\mathbf{a}}$ to the normal cone over \mathbf{Y} . A multi-dimensional version of this, in the case of a simple smooth toroidal embedding $\overline{\mathbf{X}}$ of a spherical variety \mathbf{X} – hence corresponding to a cone $\mathcal{C}(\overline{\mathbf{X}}) \subset \mathcal{V}$ such that $\Lambda_X \cap \mathcal{C}(\overline{\mathbf{X}})$ is a free monoid with basis $(\check{\lambda}_i)_i$ indexed by a set I – is a morphism:

$$\overline{\mathscr{X}}^n \to \mathbf{G}_{\mathbf{a}}^I$$

carrying a $\mathbf{G}_{\mathrm{m}}^{I}$ -action compatible with the action on the basis. Over $\mathbf{G}_{\mathrm{m}}^{I}$ it is canonically $\mathbf{G} \times \mathbf{G}_{\mathrm{m}}^{I}$ -isomorphic to $\overline{\mathbf{X}} \times \mathbf{G}_{\mathrm{m}}^{I}$, and more generally the fiber over a point on the base whose non-zero coordinates are those in the subset $J \subset I$ is canonically isomorphic to the normal bundle of the orbit in $\overline{\mathbf{X}}$ corresponding to J, with the $\mathbf{G}_{\mathrm{m}}^{I \smallsetminus J}$ -factors stabilizing that point acting on the fiber via the inverse of their canonical action on the normal bundle.¹⁴

In fact, we are only interested in the union of open orbits on the fibers. As an analysis starting from the Local Structure Theorem 2.3.4 easily shows, $\overline{\mathcal{X}}^n$ contains an open $\mathbf{P}(\mathbf{X})$ -stable subset which is $\mathbf{P}(\mathbf{X})$ -equivariantly isomorphic to $\mathring{\mathbf{X}} \times \mathbf{G}^I_{\mathbf{a}}$ over the base, which leads to two conclusions:

- 2.5.1. LEMMA. (1) The union of open **G**-orbits in the fibers is an open subset $\mathcal{X}^n \subset \overline{\mathcal{X}}^n$.
- (2) The $\mathbf{G} \times \mathbf{G}_m^I$ -variety \mathscr{X}^n is simple toroidal, with associated cone equal to the diagonal of $\mathcal{C}(\overline{\mathbf{X}})$, where we identify the $\check{\lambda}_i$'s both as elements of Λ_X and as the generators of $\mathbb{N}^I \subset \mathcal{X}(\mathbf{G}_{\mathrm{m}}^I)$.

PROOF. Indeed, the first follows immediately by acting by \mathbf{G} on the open $\mathbf{P}(\mathbf{X})$ -stable subset described before, or just by observing that \mathscr{X}^n is just the union of \mathbf{G} -orbits of maximal dimension on $\overline{\mathscr{X}}^n$, and hence open.

The second follows from an easy inversion of the local structure theorem: there is a map from the toroidal embedding claimed in the lemma to \mathcal{X}^n , and since it is an isomorphism on $\mathbf{P}(\mathbf{X})$ -stable open subsets generating both under the \mathbf{G} -action, it has to be an isomorphism.

¹⁴We clarify the need for "inverse": as we approach a non-open $\mathbf{G}_{\mathrm{m}}^{I}$ -orbit on $\mathbf{G}_{\mathrm{a}}^{I}$, we "stretch" the space \mathbf{X} away from the corresponding \mathbf{G} -orbit closure. It seems actually better to think of the usual $\mathbf{G}_{\mathrm{m}}^{I}$ -action on the normal bundle, and to compactify $\mathbf{G}_{\mathrm{m}}^{I}$ "at ∞ ", instead of at zero; however, this would be cumbersome notationally.

On the other hand, if **X** is spherical and quasi-affine, and **X**^a is any affine embedding of **X**, then $k[\mathbf{X}^a]$ carries a filtration by dominant weights in $\mathcal{X}(\mathbf{X})$, partially ordered with respect to the cone dual to \mathcal{V} [Kno96, §6], more precisely: If we decompose $k[\mathbf{X}^a]$ into a direct sum of highest weight spaces, then the λ -th piece \mathcal{F}_{λ} of the filtration is the sum of spaces with highest weights μ such that $\langle \lambda - \mu, \mathcal{V} \rangle \leq 0$. (Recall that \mathcal{V} contains the image of the negative Weyl chamber.) The important element here is that the maximal possible cone \mathcal{V} with this property is precisely the same cone as that governing embeddings of **X**, namely the cone of invariant valuations.

There is a well-known degeneration of filtered modules to their associated graded, and multidimensional versions of it, which in the literature of spherical varieties are described over various different bases, cf. [Pop86] or [GN10, $\S 5.1$]. We find it preferable to define our preferred degeneration of \mathbf{X}^a as a morphism:

$$\mathscr{X}^a \to \overline{\mathbf{A}_{X,ss}},$$

where $\overline{\mathbf{A}_{X,ss}}$ is the affine embedding of the quotient¹⁵ $\mathbf{A}_{X,ss}$ of \mathbf{A}_X determined by the cone $(-\mathcal{V})$ (and "ss" stands for "semisimple"). In other words, the open orbit in $\overline{\mathbf{A}_{X,ss}}$ is the quotient of \mathbf{A}_X by the subtorus generated by cocharacters in $\mathcal{V} \cap (-\mathcal{V})$; then $(-\mathcal{V})$ maps to a strictly convex cone $(-\overline{\mathcal{V}})$ of cocharacters of this quotient, and $\overline{\mathbf{A}_{X,ss}}$ is the corresponding affine embedding. (It can be non-smooth, but this does not matter for our purposes.)

The variety \mathcal{X}^a is by definition the spectrum of the ring:

$$\bigoplus_{\lambda} t^{\lambda} \mathcal{F}_{\lambda} \subset k[\mathbf{X}^a \times \mathbf{A}_X],$$

where the t^{λ} 's are symbols for the canonical basis of the group ring of $\mathcal{X}(\mathbf{X})$. Notice that the ring contains

$$\bigoplus_{\langle \lambda, \mathcal{V} \rangle \leq 0} t^{\lambda} \mathcal{F}_0,$$

which is the coordinate ring of $\overline{\mathbf{A}_{X,ss}}$. The variety \mathscr{X}^a carries an action of \mathbf{A}_X compatible with its action on the base, and the coordinate ring of the fiber over a point $a \in \overline{\mathbf{A}_{X,ss}}$ fixed under a subtorus $\mathbf{A}_a \subset \mathbf{A}_X$ is graded with respect to the character group of \mathbf{A}_a .

The fiber over $1 \in \mathbf{A}_{X,ss}$ is canonically isomorphic to \mathbf{X}^a , and, more generally, the restriction of the defining map:

$$\mathbf{X}^a \times \mathbf{A}_X \to \mathscr{X}^a \tag{2.9}$$

to $\mathbf{A}_{X,ss}$ is isomorphic to the quotient map $\mathbf{X}^a \times \mathbf{A}_X \to \mathbf{X}^a \times \mathbf{A}_{X,ss}$, although this isomorphism depends on a choice of section $\mathbf{A}_{X,ss} \to \mathbf{A}_X$. On the other hand, completely canonically, we have an isomorphism:

$$\mathscr{X}^a /\!\!/ \mathbf{U} \simeq \mathbf{X}^a /\!\!/ \mathbf{U} \times \overline{\mathbf{A}_{X,ss}},$$
 (2.10)

 $^{^{15}\}mathrm{We}$ thank Jonathan Wang for pointing out a mistake in an earlier version.

simply by embedding the λ -eigenspace of the Borel in $k[\mathbf{X}]$ into the t^{λ} summand of $k[\mathcal{X}^a]$. The map $\mathbf{X} \times \mathbf{A}_X \to \mathcal{X}$ descends to a map

$$\mathbf{X} /\!\!/ \mathbf{U} \times \mathbf{A}_X \to \mathscr{X}^a /\!\!/ \mathbf{U} = \mathbf{X} /\!\!/ \mathbf{U} \times \overline{\mathbf{A}_{X,ss}}$$

which is given by

$$(\bar{x}, a) \mapsto (\bar{x} \cdot a, \bar{a})$$

(where the bar denotes the obvious images in the quotients).

The two degenerations are closely related. More precisely, consider as above a smooth toroidal embedding determined by cocharacters $\check{\lambda}_i$ ($i \in I$). We use the inverses $(-\check{\lambda}_i)$ of these cocharacters to obtain an injective map

$$\mathbf{G}_{\mathrm{m}}^{I}\hookrightarrow\mathbf{A}_{X}.$$

Since the $(-\lambda_i)$'s are in the interior of $(-\mathcal{V})$, the composition with the quotient map $\mathbf{A}_X \to \mathbf{A}_{X,ss}$ extends to a map

$$\mathbf{G}_{\mathrm{a}}^{I} \to \overline{\mathbf{A}_{X,ss}}.$$
 (2.11)

2.5.2. Proposition. The composition

$$\mathbf{X} \times \mathbf{G}_{\mathrm{m}}^{I} \to \mathbf{X} \times \mathbf{A}_{X} \to \mathscr{X}^{a}$$

extends to a morphism:

$$\mathcal{X}^n \to \mathcal{X}^a$$

over (2.11), which identifies:

$$\mathscr{X}^n \simeq \mathscr{X} \times_{\overline{\mathbf{A}_{X,ss}}} \mathbf{G}_{\mathbf{a}}^I,$$

where $\mathscr X$ is the open subset of $\mathscr X^a$ consisting of the union of open $\mathbf G$ -orbits on the fibers.

PROOF. As was the case for \mathscr{X}^n , \mathscr{X}^a also has a $\mathbf{P}(\mathbf{X})$ -stable open subset \mathscr{X} which meets each fiber over $\overline{\mathbf{A}_{X,ss}}$ in precisely the open $\mathbf{P}(\mathbf{X})$ -orbit and is $\mathbf{P}(\mathbf{X})$ -equivariantly isomorphic to $\mathring{\mathbf{X}} \times \overline{\mathbf{A}_{X,ss}}$, cf. [GN10, Proposition 5.2.2]. (The isomorphism depends again on the choice of a section $\mathbf{A}_{X,ss} \to \mathbf{A}_X$.) Thus, the union of its G-translates \mathscr{X} (the open subset consisting of the union of all open G-orbits on the fibers) is a simple toroidal embedding of the open orbit of $\mathbf{G} \times \mathbf{A}_X$ on \mathscr{X}^a . This embedding is described by the cone that describes $\mathring{\mathscr{X}}$ // \mathbf{U} (equivalently: $/\!\!/\mathbf{U}_{P(X)}$) as a toric $(\mathbf{A}_X \times \mathbf{A}_X)/\mathbf{A}_1$ -variety, where \mathbf{A}_1 is the kernel of $\mathbf{A}_X \to \mathbf{A}_{X,ss}$, embedded anti-diagonally. Notice that the restriction of the filtration to $k[\mathbf{X}^a]^{\mathbf{U}}$ is actually a grading. (Thus, if we choose a section $\mathbf{A}_{X,ss} \to \mathbf{A}_X$, we have

$$\mathscr{X}^a /\!\!/ \mathbf{U} \simeq (\mathbf{X}^a /\!\!/ \mathbf{U}) \times \overline{\mathbf{A}_{X,ss}}$$

and

$$\mathring{\mathscr{X}} /\!\!/ \mathbf{U} \simeq (\mathring{\mathbf{X}} /\!\!/ \mathbf{U}) \times \overline{\mathbf{A}_{X,ss}}.)$$

It is easy to see, comparing with (2.9), that the cone of the toric embedding $\mathring{\mathscr{X}} /\!\!/ U$ is equal to the antidiagonal copy of any section of the quotient $\mathcal{V} \to \overline{\mathcal{V}}$ (the latter denoting the image of \mathcal{V} in the cocharacter space of $\mathbf{A}_{X,ss}$); notice

that modulo cocharacters of the diagonal of A_1 , the choice of section does not matter.

It now follows easily from the theory of toroidal embeddings that the map $\mathbf{X} \times \mathbf{G}_{\mathrm{m}}^{I} \to \mathscr{X} \subset \mathscr{X}^{a}$ extends to a morphism $\mathscr{X}^{n} \to \mathscr{X}$; it extends because the cone of \mathscr{X}^{n} (spanned by the antidiagonal of the $\check{\lambda}_{i}$'s) maps to the cone of \mathscr{X} . By the same theory (or, if you prefer, by the description of open $\mathbf{P}(\mathbf{X})$ -orbits), the induced map

$$\mathscr{X}^n \simeq \mathscr{X} \times_{\overline{\mathbf{A}_{X,ss}}} \mathbf{G}^I_{\mathrm{a}}$$

is an isomorphism.

In the previous subsection we associated a boundary degeneration \mathbf{X}_{Θ} to any \mathbf{G} -orbit \mathbf{Z} , belonging to Θ -infinity, of a smooth complete toroidal embedding. The proposition above provides an alternative way to define \mathbf{X}_{Θ} , in terms of the affine degeneration, which allows us to show that the variety \mathbf{X}_{Θ} does not depend on the choice of \mathbf{Z} :

2.5.3. PROPOSITION. For any **G**-orbit **Z** belonging to Θ -infinity, in a smooth toroidal embedding of **X**, the boundary degeneration \mathbf{X}_{Θ} defined in §2.4 is canonically isomorphic to the open **G**-orbit in a fiber of \mathcal{X}^a over a point of $\overline{\mathbf{A}_{X,ss}}$, namely: the point $\lim_{t\to 0} \check{\lambda}(t)$, where $\check{\lambda}$ is any cocharacter into $\mathbf{A}_{X,ss}$ in the interior of the face of $\overline{\mathcal{V}}$ corresponding to Θ .

PROOF. Indeed, if we let $\overline{\mathbf{X}}$ be the simple toroidal subembedding where \mathbf{Z} is the unique closed orbit (i.e. $\overline{\mathbf{X}}$ consists of all orbits in the given embedding that contain \mathbf{Z} in their closure), and apply Proposition 2.5.2 to the corresponding normal bundle degeneration $\overline{\mathcal{Z}}^n$, we obtain an identification of \mathbf{X}_{Θ} with the fiber stated in the proposition. Notice that it does not matter which affine embedding of \mathbf{X} we use to construct the affine degeneration \mathcal{Z}^a , as the statement only uses the open \mathbf{G} -orbit.

Finally, we note that the identification of open Borel orbits (2.5) obtained from the toroidal embedding (canonical modulo U) is compatible with the identification (2.10) obtained from the affine degeneration. We leave the verification to the reader.

2.6. Whittaker-type induction. The harmonic-analytic results of this paper apply equally well to the space of Whittaker functions, and similar spaces induced from complex characters of additive subgroups, although the "dual group" formalism in this case is slightly lacking at the moment. While, for simplicity, we mostly ignore these cases in our notation (for example, we write $C_c^{\infty}(X)$ instead of $C_c^{\infty}(X, \mathcal{L}_{\Psi})$, where \mathcal{L}_{Ψ} could denote the complex line bundle defined by a character of the stabilizers), the arguments carry over verbatim. Therefore, we present here the conventions that need to be used in order to translate our results to that setup.

This is not be the most general setup possible, but we will restrict ourselves to it because we do not know how to describe the "spherical roots" in

all cases. We give ourselves a parabolic subgroup \mathbf{P}^- and a Levi subgroup \mathbf{L} of \mathbf{P}^- . Denote by \mathbf{V} the vector space of homomorphisms:

$$\mathbf{U}_{P^-} o \mathbf{G}_{\mathrm{a}}$$

We give ourselves a *wavefront* homogeneous spherical variety \mathbf{X}^L for \mathbf{L} , together with an equivariant map:

$$\Lambda: \mathbf{X}^L \to \mathbf{V}$$
.

There is a group subscheme $\ker \Lambda$ of $\mathbf{U}_{P^-} \times \mathbf{X}^L$ over \mathbf{X}^L , whose fiber over $x \in \mathbf{X}^L$ is $\ker(\Lambda(x))$.

Finally, we set $\mathbf{X} = \mathbf{X}^L \times^{\mathbf{P}^-} \mathbf{G}$, the spherical variety "parabolically induced" from \mathbf{X}^L to \mathbf{G} . The map Λ defines a principal $\mathbf{G}_{\mathbf{a}}$ -bundle with a compatible \mathbf{G} -action over \mathbf{X} (by induction from the principal $\mathbf{G}_{\mathbf{a}}$ -bundle $(\mathbf{U}_{P^-} \times \mathbf{X}^L)/\ker \Lambda$ over \mathbf{X}^L , which is in fact equipped with an \mathbf{L} -equivariant trivialization). Now we fix a nontrivial additive character $\psi : k \to \mathbb{C}^\times$, which defines a reduction of the $\mathbf{G}_{\mathbf{a}}$ -bundle over \mathbf{X} (or, rather, the associated k-bundle over X) to a \mathbb{C}^\times -bundle, and hence a complex line bundle \mathcal{L}_{Ψ} over X. If $x \in X^L$ and $M \subset L$ denotes its stabilizer, then sections of \mathcal{L}_{Ψ} , restricted to the G-orbit of x, can be identified with functions f on G such that:

$$f(umg) = \psi(\Lambda(x)(u))f(g)$$

for $u \in U_{P^-}$, $m \in M$.

Everything that follows depends on the pair (\mathbf{X}, Λ) , even though there is only \mathbf{X} appearing in the notation; as we will see, the same variety with more degenerate characters will appear as a "boundary degeneration" of the pair (\mathbf{X}, Λ) , so it will be denoted by the letters \mathbf{X}_{Θ} , etc. In other words, the reader should consider \mathbf{X} as a symbol for the pair, not just the variety. In particular, the dual group \check{G}_X that we are about to describe *is not* the dual group of \mathbf{X} viewed as a spherical variety; rather, it is associated to the pair (\mathbf{X}, Λ) .

We let \mathbf{X}_0 denote the *total space* of the \mathbf{G}_{a} -bundle; again, if we fix a point $x \in X^L$ and let \mathbf{M} be its stabilizer in \mathbf{L} and $\mathbf{U}_1 \subset \mathbf{U}_{P_0}$ the kernel of $\Lambda(x)$, then $\mathbf{X}_0 \simeq \mathbf{U}_1 \mathbf{M} \backslash \mathbf{G}$. It is a non-spherical variety. Friedrich Knop has associated in $[\mathbf{Kno94a}]$ a "little Weyl group" to \mathbf{X}_0 , which is a finite crystallographic reflection group of automorphisms of $\mathcal{X}(\mathbf{X})$; we will denote it by W_{X_0} , or by W_X since X really stands for the pair (\mathbf{X}, Λ) .

We will describe a root system associated to this Weyl group. To do that, we recall from [ABS90] a few facts about the L-representation $\mathbf{V} = \mathrm{Hom}(\mathbf{U}_{P^-}, \mathbf{G}_{\mathbf{a}})$. First of all, it is *prehomogeneous*, i.e. L acts with an open orbit. (If the image of the map Λ lies in the open orbit, then Λ is called *generic*.) Secondly, it decomposes as a direct sum of the irreducible modules of L with *lowest* weights $\Delta \setminus \Delta_L$, i.e. the simple roots of G in the unipotent radical of a parabolic opposite to \mathbf{P}^- :

$$\mathbf{V} = \bigoplus_{\alpha \in \Delta \times \Delta_I} V_{-\alpha^{\vee}} \tag{2.12}$$

(where V_{γ} denotes the irreducible **L**-module with highest weight γ , and α^{\vee} is the dual weight to α). Equivalently, the abelianization of $\mathbf{U}_{P^{-}}$ has highest weights $-\Delta \setminus \Delta_{L}$, cf. [ABS90].

2.6.1. Lemma. The following are equivalent for a morphism Λ as above and an $\alpha \in \Delta \setminus \Delta_L$:

- (1) In the decomposition (2.12), the image of Λ has zero component in the α -summand.
- (2) Λ is trivial on the unipotent radical of the parabolic (containing \mathbf{P}^-) whose Levi has simple roots $\Delta \setminus \{\alpha\}$.
- (3) Λ is trivial on the subgroup $\mathbf{U}_{-\alpha}$ of \mathbf{U}_{P^-} on which the center of \mathbf{L} acts by the restriction of the character $-\alpha$.
- (4) Let **B** be a Borel subgroup opposite to \mathbf{P}^- (i.e. \mathbf{BP}^- is open in \mathbf{G}), \mathbf{T} a Cartan subgroup of \mathbf{B} , and let $x \in \mathbf{X}^L$ belong to the open $\mathbf{L} \cap \mathbf{B}$ -orbit. Then the character $\Lambda(x)$ is trivial on $\mathbf{U}_{-\alpha}$, the one-dimensional root subgroup of \mathbf{U}_{P^-} (with respect to the chosen torus) corresponding to the simple root $-\alpha$.

We will say that Λ is α -trivial if the equivalent conditions of this lemma are met, and α -generic if not.

PROOF. The implications $(2) \Rightarrow (3) \Rightarrow (4)$ are immediately clear, since the unipotent subgroup of each statement is contained in the unipotent subgroup of the previous one.

By [ABS90], the subgroups of \mathbf{U}_{P^-} on which the center of \mathbf{L} does not act by the restriction of an element of $-\Delta \smallsetminus \Delta_L$ belong to the derived subgroup of \mathbf{U}_{P^-} . This shows that (3) \Rightarrow (2). Moreover, under the quotient map $\mathbf{U}_{P^-} \to \mathbf{U}_{P^-}^{\mathrm{ab}}$, the subgroup $\widetilde{\mathbf{U}}_{-\alpha}$ maps isomorphically onto the highest weight module $V_{-\alpha}$ (dual to the module $V_{-\alpha^\vee}$ of (2.12)). This shows the equivalence (3) \Leftrightarrow (1).

Finally, $(4) \Rightarrow (3)$ follows from the fact that the stabilizer \mathbf{M} of x in \mathbf{L} stabilizes the additive character $\Lambda(x)$, the Borel $\mathbf{B}_L = \mathbf{B} \cap \mathbf{L}$ normalizes $\mathbf{U}_{-\alpha}$ (indeed, $[U_{-\alpha}, U_{\beta}] = 1$ for all distinct $\alpha, \beta \in \Delta$) and \mathbf{MB}_L is open in \mathbf{L} . Thus, $\Lambda(x)(\ell u \ell^{-1}) = 0$ for all $\ell \in \mathbf{L}$, $u \in \mathbf{U}_{-\alpha}$, and the \mathbf{L} -span of $\mathbf{U}_{-\alpha}$ is $\widetilde{\mathbf{U}}_{-\alpha}$. (Compare with [Sak13, Lemma 6.1.1].) Thus, $\Lambda(x)|_{\widetilde{\mathbf{U}}_{-\alpha}} = 0$, and by homogeneity the same holds for $\Lambda(x')$, for all $x' \in \mathbf{X}^L$.

DEFINITION. Define the set Δ_X of (normalized) spherical roots of **X** (really, of the pair (\mathbf{X}, Λ) or of the non-spherical variety \mathbf{X}_0) as the union of the set of (normalized) spherical roots of \mathbf{X}^L and the set of $\alpha \in \Delta \setminus \Delta_X$ for which Λ is α -generic.

2.6.2. Proposition. The set Δ_X belongs to the character group $\mathcal{X}(\mathbf{X})$, and forms a set of simple positive roots for a root system with little Weyl group W_X , i.e. Knop's Weyl group for the non-spherical variety \mathbf{X}_0 .

Notice that the character group $\mathcal{X}(\mathbf{X})$ used here is the same as the one we have defined for the spherical variety \mathbf{X} . (The morphism Λ plays no role.)

PROOF. If Λ is α -trivial for some $\alpha \in \Delta \smallsetminus \Delta_L$, the variety \mathbf{X}_0 is parabolically induced from the parabolic containing \mathbf{P}^- whose Levi has simple roots $\Delta \smallsetminus \{\alpha\}$, by property (2) of Lemma 2.6.1. Knop's little Weyl group $W_X = W_{X_0}$ generalizes the little Weyl groups of spherical varieties and, in particular, has the property that if a variety is parabolically induced, it is equal to the little Weyl group of the inducing variety. Thus, we are reduced to the case where Λ is α -generic for all $\alpha \in \Delta \smallsetminus \Delta_L$. In this case, $\Delta_X = \Delta_{X^L} \cup (\Delta \smallsetminus \Delta_L)$.

First we prove that the elements of $\Delta \setminus \Delta_L$ belong to $\mathcal{X}(\mathbf{X})$, i.e. are characters of \mathbf{A}_X . Let \mathbf{B} be a Borel subgroup opposite to \mathbf{P}^- , and $x \in \mathbf{X}^L$ in the open $\mathbf{L} \cap \mathbf{B}$ -orbit. Choose $\mathbf{T} = \mathbf{A}$ a Cartan subgroup chosen as in §2.1, i.e. such that it acts via the quotient \mathbf{A}_X on the orbit of x. By property (4) in Lemma 2.6.1, the character $\Lambda(x)$ is nontrivial on the root subgroup $\mathbf{U}_{-\alpha}$. This character is stabilized by the kernel of $\mathbf{A} \to \mathbf{A}_X$, hence the character α is trivial on this kernel, i.e. $\alpha \in \mathcal{X}(\mathbf{X})$.

To prove that Δ_X forms a set of simple positive roots for a root system with little Weyl group W_X , we first show that the linearly independent set of roots Δ_X determines a Weyl chamber for the action of W_X on $\mathfrak{a}_X = \mathcal{X}(\mathbf{X}) \otimes \mathbb{Q}$.

Friedrich Knop has defined in [Kno95] an action of the full Weyl group on a certain set of **B**-stable subsets of X_0 , which includes X_0 itself. As was remarked in [Sak08, §5.4], in the current setting the simple reflection corresponding to the root α belongs to the stabilizer of X_0 , and more precisely to the little Weyl group W_X . From Knop's construction, it is immediate to see that the same is true for the simple reflections associated to elements of Δ_{X^L} . Thus, a Weyl chamber for W_X is contained in the cone $\mathcal C$ negative-dual to the set of characters Δ_X .

For the converse, since \mathbf{X}^L was assumed to be wavefront, the map of anti-dominant chambers: $\mathfrak{a}_L^+ \to \mathfrak{a}_{X^L}^+$ is surjective; here \mathfrak{a}_L^+ denotes the L-anti-dominant elements of \mathfrak{a} . The subset $\mathfrak{a}^+ \subset \mathfrak{a}_L^+$ of \mathbf{G} -anti-dominant elements is defined by the additional conditions: $\langle \alpha, a \rangle \leq 0$ for all $\alpha \in \Delta \setminus \Delta_L$, and similarly the subcone $\mathcal{C} \subset \mathfrak{a}_{X^L}^+$ is defined by the same additional conditions. Therefore, the map:

$$\mathfrak{a}^+ \to \mathcal{C}$$

is surjective. On the other hand, the image of \mathfrak{a}^+ in \mathfrak{a}_X is contained in a Weyl chamber for the little Weyl group W_X ; therefore, \mathcal{C} coincides with that Weyl chamber.

We have shown that the linearly independent set of roots Δ_X determines a Weyl chamber for the action of W_X on $\mathfrak{a}_X = \mathcal{X}(\mathbf{X}) \otimes \mathbb{Q}$. The fact that the W_X -translates of Δ_X form a root system with this Weyl group follows if we show that whenever an element of W_X carries one of these half-lines

to another, it has to map the corresponding spherical roots to each other. This, in turn, relies on the following fact, which we will use more extensively in §3.1, and therefore we point the reader there for proofs and definitions: each element of Δ_X is either a simple root of \mathbf{G} or the sum of two strongly orthogonal roots. This fact uniquely determines it on the half-line that it spans.

In accordance with this definition of Δ_X we define all the corresponding invariants of \mathbf{X} (that is, of the pair (\mathbf{X}, Λ)) as in §1.7. For example, \mathfrak{a}_X^+ denotes the subset of $\mathcal{X}(\mathbf{X})^* \otimes \mathbb{Q}$ of elements which are anti-dominant with respect to Δ_X ; equivalently, the subset of \mathfrak{a}_{XL}^+ of elements which are ≤ 0 on all $\alpha \in \Delta \setminus \Delta_L$ for which Λ is α -generic. As we saw in the last proof, since $\mathfrak{a}_L^+ \to \mathfrak{a}_{XL}^+$ is assumed to be surjective (wavefront property), the same is true for the map: $\mathfrak{a}^+ \to \mathfrak{a}_X^+$ when Λ is generic (that is, when $\Delta \setminus \Delta_L \subset \Delta_X$), i.e. generic Whittaker-induction in this sense preserves the wavefront property.

Now we discuss "wonderful compactifications". Again, the name will be applied more generally to smooth toroidal embeddings – however, they will not be complete, since, as we will see smooth sections of \mathcal{L}_{Ψ} vanish in certain directions. Hence, in our setting, a "wonderful compactification" of \mathbf{X} (taking into account the character Λ) will be a smooth toroidal embedding $\overline{\mathbf{X}}$ of \mathbf{X} whose fan has support (=the union of its cones) equal to \mathfrak{a}_X^+ . It is easy to see that such an embedding is of the form:

$$\overline{\mathbf{X}} = \overline{\mathbf{X}^L} \times^{\mathbf{P}^-} \mathbf{G},$$

where $\overline{\mathbf{X}^L}$ is a toroidal embedding of \mathbf{X}^L defined by the same fan. The reason for this definition is the following lemma, which will be proven in §5.3.

2.6.3. LEMMA. The support of any element of $C^{\infty}(X, \mathcal{L}_{\Psi})$ has compact closure in \bar{X} , where \bar{X} denotes any "wonderful" embedding as described above.

Hence, for our purposes the space X is as good as a compact space, since the support of smooth sections cannot escape in other directions.

On the other hand, in the other directions the line bundle \mathcal{L}_{Ψ} can be extended to the "wonderful" embedding:

2.6.4. Lemma. The G_a -bundle $X_0 \to X$ extends to a G_a -bundle (with a compatible action of G) over a wonderful embedding \overline{X} .

PROOF. If $\overline{\mathbf{X}^L}$ denotes the closure of \mathbf{X}^L in $\overline{\mathbf{X}}$, then, as mentioned above, $\overline{\mathbf{X}} = \overline{\mathbf{X}^L} \times^{\mathbf{P}^-} \mathbf{G}$. Therefore, it is enough to show that the \mathbf{G}_a -bundle extends to $\overline{\mathbf{X}^L}$ or, equivalently, that the morphism

$$\Lambda: \mathbf{X}^L \to \mathbf{V} = \mathrm{Hom}(\mathbf{U}_{P^-}, \mathbf{G}_{\mathrm{a}})$$

extends.

Let \mathfrak{a}_V be the dual to the vector space spanned by the **B**-eigencharacters on $k(\mathbf{V})$, and let $\mathcal{C} \subset \mathfrak{a}_V$ be the cone dual to the set of eigencharacters of

regular Borel eigenfunctions. This is the cone attached by Luna-Vust theory [**Kno91**] to the spherical embedding **V**, since every **B**-stable divisor on **V** is \mathbf{G}_{m} -stable and hence contains the closed orbit $\{0\}$. By [**Kno91**, Theorem 4.1], to show that the map of spherical varieties $\mathbf{X}^L \to \mathbf{V}$ extends to $\overline{\mathbf{X}^L}$, it is enough to show that the support of the fan of $\overline{\mathbf{X}^L}$, i.e. the cone \mathfrak{a}_X^+ , maps into \mathcal{C} under the natural quotient map: $\mathfrak{a}_X \to \mathfrak{a}_V$.

By the decomposition (2.12), the coordinate ring $k[\mathbf{V}]$ is the symmetric algebra $S^{\bullet}(\bigoplus_{\alpha \in -\Delta \smallsetminus \Delta_L} V_{\alpha})$, where V_{α} is the irreducible **L**-module with highest weight α . Decomposing these symmetric powers in irreducible modules for \mathbf{G} , we get highest weights that belong to the cone spanned by $-\Delta \smallsetminus \Delta_L$ and the relative root cone of \mathbf{L} , i.e. all the highest weights in $k[\mathbf{V}]$ are contained in the negative root cone of \mathbf{G} . Hence, \mathfrak{a}^+ maps into \mathcal{C} , and since \mathfrak{a}^+ surjects onto \mathfrak{a}_X^+ , the same holds for the latter.

Based on this lemma, now, we can define the boundary degenerations \mathbf{X}_{Θ} , which are really symbols for a pair $(\mathbf{X}_{\Theta}, \Lambda_{\Theta})$, where $\mathbf{X}_{\Theta} = \mathbf{X}_{\Theta}^{L} \times^{\mathbf{P}^{-}} \mathbf{G}$ is a homogeneous spherical **G**-variety induced from \mathbf{P}^{-} (possibly: $\mathbf{X}_{\Theta} \simeq \mathbf{X}$ as **G**-varieties), and $\Lambda_{\Theta} : \mathbf{X}_{\Theta}^{L} \to \mathbf{V}$, where **V** is as before.

Namely, as varieties \mathbf{X}_{Θ} and \mathbf{X}_{Θ}^{L} are defined as previously – the open \mathbf{G} orbit, resp. \mathbf{L} -orbit, in the normal bundle to a certain orbit on $\overline{\mathbf{X}}$, resp. $\overline{\mathbf{X}^{L}}$.

If $\mathbf{Z} \subset \overline{\mathbf{X}^{L}}$ is the orbit corresponding to \mathbf{X}_{Θ}^{L} we have a canonical quotient
map (quotient by $\mathbf{A}_{X,\Theta}$): $\mathbf{X}_{\Theta}^{L} \to \mathbf{Z}$, and we let Λ_{Θ} be its composition with
the map $\mathbf{Z} \to \mathbf{V}$ obtained, by the previous lemma, as the extension of Λ .

If $\Omega = \Theta \sqcup \{\alpha\}$, where α is a simple root in the unipotent radical of a parabolic opposite to \mathbf{P}^- , then \mathbf{X}_{Θ}^L and \mathbf{X}_{Ω}^L are isomorphic as varieties, but Λ_{Θ} is trivial on the summand $V_{-\alpha}$ under the decomposition

$$(\mathbf{U}_{P^-})^{\mathrm{ab}} \simeq \oplus_{\beta \in -\Delta \smallsetminus \Delta_L} V_{\beta}$$

dual to (2.12) (equivalently, on the subgroup $\widetilde{\mathbf{U}}_{-\alpha}$ of Lemma 2.6.1.)

2.7. Levi varieties. While the statements of our representation-theoretic results can be formulated without any reference to the internal structure of the boundary degenerations X_{Θ} , the tools that we have at our disposal are unfortunately related to parabolic induction and restriction from Levi subgroups. This is a basic reason why we restrict ourselves to wavefront spherical varieties (s. Proposition 2.7.2 below). We start with a general statement that does not use the wavefront property:

2.7.1. Lemma. Let $\mathbf{L}_{\Theta}, \mathbf{P}_{\Theta}$ denote the standard Levi with simple roots $\tilde{\Theta} := \Delta(\mathbf{X}) \cup \operatorname{supp}(\Theta)$, and the corresponding standard parabolic. Let \mathbf{P}_{Θ}^- denote the parabolic opposite to \mathbf{P}_{Θ} which contains \mathbf{L}_{Θ} . There exists a spherical variety \mathbf{X}_{Θ}^L of \mathbf{L}_{Θ} such that $\mathbf{X}_{\Theta} \simeq \mathbf{X}_{\Theta}^L \times^{\mathbf{P}_{\Theta}^-} \mathbf{G}$.

Such a process of constructing a spherical variety of G from one of a Levi subgroup is called "parabolic induction".

PROOF. By [Lun01, Proposition 3.4], a homogeneous spherical variety \mathbf{Y} for \mathbf{G} is induced from a parabolic \mathbf{P}^- (assumed opposite to a standard parabolic \mathbf{P}) if and only if $\operatorname{supp}(\Delta_Y) \cup \Delta(\mathbf{Y})$ is contained in the set of simple roots of the Levi subgroup of \mathbf{P} .

The variety \mathbf{X}_{Θ}^{L} will be called a *Levi variety* for \mathbf{X} . It is a *canonical* subvariety of \mathbf{X}_{Θ} once the parabolic \mathbf{P}_{Θ}^{-} has been chosen (in particular, once we choose a standard Borel and a Levi $\mathbf{L}(\mathbf{X})$ as explained in 2.1), namely the subvariety of points stabilized by \mathbf{U}_{Θ} . Notice, moreover, that its inclusion in \mathbf{X}_{Θ} , indeed in $\mathring{\mathbf{X}}_{\Theta} \cdot \mathbf{P}_{\Theta}$, allows us to *canonically* identify it with the quotient $\mathring{\mathbf{X}}\mathbf{P}_{\Theta}/\mathbf{U}_{\mathbf{P}_{\Theta}}$. In general, we could have $\mathbf{L}_{\Theta} = \mathbf{G}$; in the wavefront case, however, the boundary degenerations are parabolically induced from sufficiently small Levi subgroups:

2.7.2. PROPOSITION. Assume that X is wavefront and that the map $\mathcal{Z}(G)^0 \to \mathcal{Z}(X)$ is surjective. Then the natural map:

$$\mathcal{Z}(\mathbf{L}_{\Theta})^0 \to \mathbf{A}_{X,\Theta} = \mathrm{Aut}_{\mathbf{G}}(\mathbf{X}_{\Theta})^0$$

("action on the left") is surjective. In particular, for $\Theta \subsetneq \Delta_X$ the Levi \mathbf{L}_{Θ} is proper.

This condition characterizes wavefront spherical varieties: if a spherical variety is not wavefront, then there exists a $\Theta \subset \Delta_X$ such that the above map is not surjective. In particular, if \mathbf{X} is wavefront then \mathbf{X}_{Θ} is also wavefront under the action of $\mathcal{Z}(\mathbf{L}_{\Theta})^0 \times \mathbf{G}$, for every $\Theta \subset \Delta_X$.

PROOF. Since both groups are tori, it suffices to prove surjectivity for the corresponding map of their cocharacter groups, tensored by \mathbb{R} . We will prove somewhat more, namely, the map induces a surjection of positive chambers at this level.

Let \mathfrak{z} (resp. \mathfrak{z}_X) be the kernel of all roots on \mathfrak{a} (resp. the kernel of all spherical roots on \mathfrak{a}_X). The cone \mathfrak{a}^+ (resp. \mathcal{V}) is the preimage in $\mathfrak{a}/\mathfrak{z}$ (resp. $\mathfrak{a}_X/\mathfrak{z}_X$) of the convex hull of the half-lines spanned by the negative dual basis $\{-\alpha^* : \alpha \in \Delta\}$ (resp. $-\gamma^* : \gamma \in \Delta_X$) to Δ (resp. Δ_X).

Since the map $\mathfrak{a}^+ \to \mathfrak{a}_X^+$ is surjective, it follows that for every $\gamma \in \Delta$ there exists $\alpha \in \Delta$ so that $\mathbb{R}\alpha^* \to \mathbb{R}_+\gamma^*$ under the induced $\mathfrak{a}/\mathfrak{z} \to \mathfrak{a}_X/\mathfrak{z}_X$. In fact, this latter condition is *equivalent* to the surjectivity of $\mathfrak{a}^+ \to \mathfrak{a}_X^+$ if we suppose (as we have been doing) that $\mathfrak{z} \to \mathfrak{z}_X$ is surjective; we can rephrase it thus:

For every $\gamma \in \Delta_X$, there exists $\alpha_{\gamma} \in \Delta$ that is contained in the support of γ , but not in the support of any other simple root.

(2.13)

We claim that we may always choose α_{γ} so that it does not belong to $\Delta_{L(X)}$. Suppose to the contrary: Let S be the set of all roots in the support of γ , but not in the support of any other root, and suppose $S \subset \Delta_{L(X)}$. Every $\alpha \in S$ is orthogonal to the support of every spherical $\gamma' \neq \gamma$: Indeed, by non-degeneracy (Lemma 2.1.2), $\langle \alpha^{\vee}, \gamma' \rangle = 0$, but $\langle \alpha^{\vee}, \beta \rangle \leq 0$ for all

 $\beta \in \operatorname{supp}(\gamma')$. Consequently, every $\alpha \in S$ is orthogonal to every element of $\operatorname{supp}(\gamma) - S$. If we write $\gamma = \sum_{\alpha} n_{\alpha} \alpha$, then $\gamma_S := \sum_{\alpha \in S} n_{\alpha} \alpha$ has the property that $\langle \alpha^{\vee}, \gamma_S \rangle = 0$ for all $\alpha \in S$. This contradicts the fact that the matrix $\langle \alpha^{\vee}, \beta \rangle_{\alpha, \beta \in S}$ is nondegenerate.

The claims of the proposition follow immediately. Indeed, the convex hull of $\{\mathbb{R}_+\alpha_{\gamma}^*: \gamma \notin \Theta\}$ is orthogonal to all roots in $\tilde{\Theta}$ and surjects onto $\mathfrak{a}_{X,\Theta}^+$.

On the other hand, if **X** is not wavefront, equivalently: if (2.13) fails, then there is a spherical root γ such that every simple root in the support of γ is also contained in the support of some other spherical root. Hence, $\mathbf{L}_{\Delta_X \setminus \{\gamma\}} = \mathbf{G}$, but $\mathcal{Z}(\mathbf{G})^0$ has image $\mathcal{Z}(\mathbf{X}) \subsetneq \mathcal{Z}(\mathbf{X}_{\Theta})$.

2.8. Horocycle space. Let Φ denote any conjugacy class of parabolic subgroups of \mathbf{G} , whose representatives contain a conjugate of $\mathbf{P}(\mathbf{X})$ as a subgroup. We define the space of Φ -horocycles on \mathbf{X} as the \mathbf{G} -variety \mathbf{X}_{Φ}^{h} classifying pairs:

$$(\mathbf{Q}, \mathfrak{O}),$$

where \mathbf{Q} is a parabolic in the class Φ and \mathfrak{O} is an orbit of \mathbf{U}_Q contained in the open \mathbf{Q} -orbit on \mathbf{X} (it will also be denoted by \mathbf{X}_Q^h). More explicitly, if we choose a Borel and hence a standard representative \mathbf{Q} for the class Φ then, canonically:

$$\mathbf{X}_{Q}^{h} := \mathbf{X}_{\Phi}^{h} \simeq \mathring{\mathbf{X}} \cdot \mathbf{Q} / \mathbf{U}_{Q} \times^{\mathbf{Q}} \mathbf{G},$$

where, as usual, $\mathring{\mathbf{X}}$ denotes the open orbit of the chosen Borel subgroup.

Now let Θ be a subset of spherical roots, take $\mathbf{Q} = \mathbf{P}_{\Theta}$, and denote $\mathbf{X}_{\Theta}^h := \mathbf{X}_Q^h$. The next lemma compares the space of \mathbf{Q} -horocycles with the analogous space for the boundary degeneration \mathbf{X}_{Θ} :

2.8.1. Lemma. If \mathbf{X} is a wavefront spherical variety, then there is a canonical identification:

$$\mathbf{X}_{\Theta}^{h} := \mathbf{X}_{Q}^{h} \stackrel{\sim}{\to} (\mathbf{X}_{\Theta})_{Q}^{h}, \tag{2.14}$$

compatible with the identification of open Borel orbits (2.5).

PROOF. Let us start with $\Theta = \emptyset$, i.e. $\mathbf{Q} = \mathbf{P}_{\Theta} = \mathbf{P}(\mathbf{X})$. Then the Local Structure Theorem 2.3.4 and the canonical identification (2.5) immediately imply that:

$$\mathring{\mathbf{X}}/\mathbf{U}_{\Theta} = \mathring{\mathbf{X}}_{\Theta}/\mathbf{U}_{\Theta}$$

canonically, and hence $\mathbf{X}_Q^h \simeq (\mathbf{X}_{\Theta})_Q^h$.

In the general case, we first would like to exhibit $\mathbf{Y}_{\Theta} := \mathring{\mathbf{X}}_{\Theta} \mathbf{P}_{\Theta} / \mathbf{U}_{\Theta}$ as a boundary degeneration of $\mathbf{Y} := \mathbf{X}_{\Theta}^{L} = \mathring{\mathbf{X}} \mathbf{P}_{\Theta} / \mathbf{U}_{\Theta}$. In a smooth toroidal embedding $\overline{\mathbf{X}}$ of \mathbf{X} , let \mathbf{X}_{1} be the \mathbf{P}_{Θ} -stable subset of those points whose \mathbf{P}_{Θ} -orbit closure contains Θ -infinity. (For simplicity, we assume that Θ -infinity is a unique orbit.) Then \mathbf{U}_{Θ} acts freely on \mathbf{X}_{1} , and $\mathbf{X}_{1} / \mathbf{U}_{\Theta}$ is a simple embedding of \mathbf{Y} such that the open \mathbf{L}_{Θ} -orbit in the normal bundle to the

closed orbit is canonically isomorphic to \mathbf{Y}_{Θ} . In terms of the combinatorial description of spherical embeddings, this is the toroidal embedding of \mathbf{Y} with valuation cone equal to the face of \mathcal{V} that is orthogonal to Θ .

Thus, according to §2.5, \mathbf{Y}_{Θ} is the open \mathbf{L}_{Θ} -orbit in the spectrum of the partial grading of $k[\mathbf{Y}]$ induced by those valuations. But as we saw in Proposition 2.7.2, the grading induced by these valuations coincides (or can be refined by) the grading by characters of $\mathcal{Z}(\mathbf{L}_{\Theta})$. Therefore, $k[\mathbf{Y}]$ is already graded and there is nothing to degenerate: $k[\mathbf{Y}_{\Theta}] \simeq k[\mathbf{Y}]$, and hence $\mathbf{Y}_{\Theta} \simeq \mathbf{Y}$ (canonically). Hence, $\mathbf{X}_{Q}^{h} = (\mathbf{X}_{\Theta})_{Q}^{h}$.

2.9. The example of $\mathbf{PGL}(V)$ as a $\mathbf{PGL}(V) \times \mathbf{PGL}(V)$ variety. We shall describe a basic example and compute much of the foregoing data as an illustration.

Let V be a vector space, and take $\mathbf{X} = \mathbf{PGL}(V)$ considered as $\mathbf{G} = \mathbf{PGL}(V) \times \mathbf{PGL}(V)$ variety via $A \cdot (g,h) = g^{-1}Ah$. Let $x_0 \in \mathbf{X}$ be the homothety class of scalar multiplication. In the discussion that follows, we understand $\mathrm{GL}(V)$ as acting on V on the right.

2.9.1. The case n=2. We use this as an example to explain the behavior of A_X^+ in geometric terms, i.e. in terms of the asymptotic behavior of horocycles.

In this case, the compactification of PGL(2) is simply that induced from its embedding into $\overline{\mathbf{X}} := \mathbb{P}(M_2)$, where M_2 is the algebra of 2×2 matrices.

We take the Borel subgroup $\mathbf{B} \subset \mathbf{G}$ to be $\mathbf{B}^+ \times \mathbf{B}^-$, where + and - denote respectively upper and lower triangular matrices. If we let $x_0 \in \mathbf{X}$ be the identity automorphism, then x_0 is in the open \mathbf{B} -orbit. This orbit consists of the lines of elements $\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in M_2(k)$ with $w \neq 0$. This space is foliated by the U-orbits

$$\mathfrak{h}_c := \left\{ \left(\begin{array}{cc} * & * \\ * & d \end{array} \right) : \frac{d^2}{\det} = c \right\}.$$

In terms of the geometry of $\overline{\mathbf{X}}$, the horocycle \mathfrak{h}_c has quite different behavior as $c \to 0$ and as $c \to \infty$:

- As $c \to \infty$, the entire horocycle \mathfrak{h}_c draws close to the divisor of singular matrices in $\overline{\mathbf{X}}$, which is the closure of the horocycle of (lines of) singular matrices of the form $\begin{pmatrix} * & * \\ * & d \end{pmatrix}$ with $d \neq 0$.
- On the other hand, as $c \to 0$, the horocycle \mathfrak{h}_c converges rather to the **B**-stable divisor (small Bruhat cell) on **X** itself.

In this way, the two "directions" in A_X are distinguished from one another.

More precisely: The action of **B** on these horocycles factors through the quotient $\mathbf{B} \to \mathbf{A} \twoheadrightarrow \mathbf{A}_X$ and defines a faithful \mathbf{A}_X -action; in this case, it is explicitly given by

$$\left(\begin{array}{cc} x & * \\ 0 & 1 \end{array}\right), \left(\begin{array}{cc} y^{-1} & 0 \\ * & 1 \end{array}\right) : \mathfrak{h}_c \mapsto \mathfrak{h}_{c \cdot (xy)}.$$

Thus, the action of a positive one parameter subgroup $\lambda : \mathbf{G}_{\mathrm{m}} \to \mathbf{A}$ (which projects to the *negative* of the valuation cone in $X_*(A_X)$) satisfies $\lambda(t)\mathfrak{h}_c \to \overline{\mathbf{X}} - \mathbf{X}$ as $t \to \infty$, whereas a negative one parameter subgroup (which projects to the valuation cone of $X_*(A_X)$) has the opposite behavior.

2.9.2. The compactification for general n. For n > 2 the wonderful compactification of \mathbf{X} is more subtle. It is classically known as the variety of complete collineations; a modern treatment is given by Thaddeus [**Tha99**] or de Concini and Procesi [**DCP83**].

A set $\Theta \subset \Delta_X$ can be identified with a *flag type* in this case, namely an increasing sequence of dimensions:

$$0 = d_0 < d_1 < \dots < d_k < d_{k+1} = \dim(V). \tag{2.15}$$

Then the "boundary degeneration" \mathbf{X}_{Θ} can be described as classifying triples:

$$(K, I, \phi),$$

where:

- $K: V = K_0 \supset K_1 \supset K_2 \supset \cdots \supset K_k \supset K_{k+1} = V$ is a decreasing flag with $\operatorname{codim} K_i = d_i$ the "kernel flag";
- $I: \{0\} = I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_k \subset I_{k+1} = V$ is an increasing flag with dim $I_i = d_i$ the "image flag";
- ϕ is the homothety class of a graded isomorphism:

$$\phi : \operatorname{gr}^K(V) \to \operatorname{gr}^I(V).$$

Explicitly, ϕ is a collection of linear isomorphisms:

$$\phi_i: K_{i-1}/K_i \to I_i/I_{i-1},$$

determined up to a common scalar multiple. The **G**-automorphism group of \mathbf{X}_{Θ} is generated by scalar multiplications of the individual ϕ_i 's.

On the other hand, the corresponding **G**-orbit \mathbf{Z}_{Θ} of the wonderful compactification can be identified with the variety classifying triples $(K, I, [\phi])$ as before, but with *each* of the morphisms ϕ_i comprising ϕ defined up to homothety (and hence we denote them as $[\phi_i], \phi$).

We describe how these orbits are glued topologically in the wonderful compactification (either in the sense of Zariski topology, or in the sense of the usual topology of points over a p-adic field). It suffices to describe a basis of neighborhoods of a point $z \in \mathbf{Z}_{\Theta}$ inside of $\mathbf{Z}_{\Theta'}$, when \mathbf{Z}_{Θ} is contained in the closure of $\mathbf{Z}_{\Theta'}$ and of codimension one. In other words, if Θ corresponds to a sequence of integers as in (2.15) then Θ' corresponds to the same sequence with a d_i removed. But then, the description of neighborhoods is in complete analogy to the highest-dimensional case, namely when Θ corresponds to a sequence $0 < d < \dim(V)$ and hence $\mathbf{Z}_{\Theta'}$ is the open \mathbf{G} -orbit (i.e. $\mathbf{PGL}(V)$).

In that case, for a point $z = (K, I, [\phi]) \in \mathbf{Z}_{\Theta}$, K is a sequence $V \supset K_1 \supset 0$, and I is a sequence $0 \subset I_1 \subset V$. Moreover the datum $[\phi]$ consists of homothety classes of isomorphisms: $[\phi_1 : V/K_1 \xrightarrow{\sim} I_1]$ and $[\phi_2 : K_1 \xrightarrow{\sim} V/I_i]$.

A neighborhood of z in $\mathbf{PGL}(V)$ then consists of homothety classes [g] of automorphisms $g: V \to V$ with the property that [g] is in a neighborhood, in $\mathbf{P}\operatorname{End}(V)$, of the endomorphism

$$V \to V/K_1 \xrightarrow{\phi_1} I_1 \hookrightarrow V$$
,

(or rather its homothety class), and $[g^{-1}]$ is in a neighborhood, in \mathbf{P} End(V), of the endomorphism:

$$V \to V/I_1 \xrightarrow{\phi_2} K_1 \hookrightarrow V$$

(or rather its homothety class).

To describe the algebraic structure of the whole wonderful variety is a little more complicated, but notice that these codimension-one orbits already show up in the closure of the embedding:

$$\mathbf{PGL}(V) \ni [g] \to ([g], [g^{-1}]) \in \mathbf{P} \operatorname{End}(V) \times \mathbf{P} \operatorname{End}(V).$$

(Notice also that the algebraic structure of the wonderful compactification is easily obtained from a different construction, namely embedding $\mathbf{PGL}(V)$ in $\mathrm{Gr}(\mathfrak{g})$ – the Grassmannian of the Lie algebra of \mathbf{G} – as the \mathbf{G} -orbit of the diagonal: $\mathfrak{pgl}(V) \hookrightarrow \mathfrak{g}$.)

On the other hand, a neighborhood of the above point $z \in \mathbf{Z}_{\Theta}$ in \mathbf{X}_{Θ} can be described as the set of triples (I', K', ϕ') with I', K' in neighborhoods of I and K in the corresponding flag varieties, and $\phi' = (\phi'_1, \phi'_2)$ such that it(s homothety class) is in a neighborhood of (the homothety class of) $(\phi_1, 0)$, and such that (the class of) ${\phi'}^{-1}$ is in a neighborhood of (the class of) $(0, \phi_2^{-1})$. Of course, when varying the point (I', K') we take into account that ϕ' is naturally an element of some bundle over the corresponding product of flag varieties, and hence "in a neighborhood" makes sense for ϕ' .

2.9.3. Identification of orbits. Let k be a p-adic field, and J an open compact subgroup of $\mathbf{G}(k)$. Later in this paper (§4.3) we describe how to identify J-orbits in a neighborhood of Θ -infinity on $\mathbf{X}_{\Theta}(k)$ with J-orbits on a neighborhood of Θ -infinity on $\mathbf{X}(k)$. In preparation for that, let us describe now how to do this explicitly in the example of $\mathbf{X} = \mathbf{PGL}(V)$:

(1) To go from X(k) to $X_{\Theta}(k)$:

Given an element $A \in X(k)$, choose a representative $A \in GL(n,k)$. According to the Cartan decomposition, there exists bases e_1, \ldots, e_n and f_1, \ldots, f_n for \mathfrak{o}^n with the property that $Ae_i = \lambda_i f_i$. We order the λ_i so that

$$|\lambda_1| > |\lambda_2| > \dots |\lambda_n|. \tag{2.16}$$

The bases e_i , f_i are unique up to "corrections" belonging to $GL(n, \mathfrak{o}) \cap D^{\pm 1}$ $GL(n, \mathfrak{o})D^{\mp 1}$, where $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$.

Fix $\Theta \subset \Delta_X$, which we identify as before with a flag type $\{0 = d_0 < d_1 < d_2 < \dots < d_k = n\}$. Say A is Θ -large if $|\lambda_{d_s}/\lambda_{d_s+1}| \geq T$ for each s.

Set

$$K_1 = \langle e_n, \dots, e_{d_1+1} \rangle \supset K_2 = \langle e_n, \dots, e_{d_2} \rangle \supset \dots$$

 $I_1 = \langle f_1, \dots, f_{d_1} \rangle \subset I_2 = \langle f_1, \dots, f_{d_2} \rangle \subset \dots$

and we associate to A the flags K_* , I_* and the natural isomorphism $\operatorname{gr}^K \to \operatorname{gr}^I$ induced by A. Although e_i , f_i are not uniquely defined, the resulting map nonetheless gives a well-defined map

J-orbits on
$$\Theta$$
-large $A \longrightarrow J$ -orbits on X_{Θ} .

so long as T is sufficiently large relative to J; this follows easily from the uniqueness of Cartan decomposition (in the sense noted above).

(2) To go from $X_{\Theta}(k)$ to X(k):

First the notion of " Θ -large" on $X_{\Theta}(k)$: Let $z=(K,I,[\phi]) \in X_{\Theta}(k)$. $[\phi]$ is a class of morphisms up to common homothety; we choose a representative ϕ from this class.

We derive integral structures on K_{i-1}/K_i and I_i/I_{i-1} as the images of $\mathfrak{o}^n \cap K_{i-1}$ and $\mathfrak{o}^n \cap I_i$; now let $\lambda_{\min}(\phi_i)$ and $\lambda_{\max}(\phi_i)$ be the smallest and largest "singular values" of the Cartan decomposition of $\phi_i : K_{i-1}/K_i \to I_i/I_{i-1}$. We then say that z is Θ -large if, for every j,

$$|\lambda_{\min}(\phi_i)| > T \cdot |\lambda_{\max}(\phi_{i+1})|.$$

Choose once and for all a splitting of all flags of type Θ , in such a way that these splittings vary continuously in the (compact) space of such flags under the k-adic topology. The chosen splittings give identifications $\operatorname{gr}^K V \xrightarrow{\sim} V$ and similarly for I. In particular, ϕ induces a linear map $V \to V$, that is to say, an element $\tilde{z} \in X(k)$.

Fix an open compact subgroup J. If we had chosen a different choice of splitting, the resulting elements $\tilde{z}, \tilde{z}' \in X(k)$ nonetheless still lie in the same J orbit so long as T is chosen large enough.

We have therefore obtained a map

$$J$$
-orbits on Θ -large $A \in X_{\Theta}(k) \longrightarrow J$ -orbits on X .

3. Proofs of the results on the dual group

In this section we prove the results of §2.2, including Theorem 2.2.3. We use heavily the work of F. Knop and M. Brion to define \check{G}_X , and the work of Gaitsgory-Nadler to construct a morphism $\check{G}_X \times \mathrm{SL}_2 \to \check{G}$.

3.1. The root datum of a spherical variety. First of all, we recall that F. Knop has defined [Kno95] an action of the Weyl group on the set of Borel orbits on \mathbf{X} ; the stabilizer of the open orbit is equal to the group $W_{(X)} := W_X \ltimes W_{L(X)}$. Here the group W_X , which is originally a subgroup of $\operatorname{End}(\mathcal{X}(\mathbf{X}) \otimes \mathbb{Q})$, is identified with its "canonical lift" to W, which consists of representatives of minimal length modulo $W_{L(X)}$ (the set of those representatives to be denoted by $[W/W_{L(X)}]$). In order to distinguish

between the two, we will denote the subgroup of $\operatorname{End}(\mathcal{X}(\mathbf{X}) \otimes \mathbb{Q})$ by $\overline{W_X}$. Knop's action has the property [**Kno95**, Theorem 4.3] that $\mathcal{X}({}^wY) = w \cdot \mathcal{X}(Y)$ (in characteristic zero).

Recall that two *strongly orthogonal* roots in a root system are two roots whose sum and difference are not roots. We call two roots α and β superstrongly orthogonal if there is a choice of positive roots such that those two roots are simple and orthogonal. This is equivalent to the assertion that the only roots in the linear span of α and β are $\pm \alpha, \pm \beta$. In [Bri01], Brion proves the following:

- 3.1.1. THEOREM (Brion). A set of generators for $W_{(X)}$ consists of elements w which can be written as $w = w_1^{-1} w_2 w_1$ where:
 - $w_1 \mathring{X} =: Y \text{ with } \operatorname{codim}(Y) = l(w_1).$
 - w_2 is either of the following two:
 - (1) equal to the simple reflection w_{α} corresponding to a simple root α such that the \mathbf{PGL}_2 -spherical variety $\mathbf{YP}_{\alpha}/\mathcal{R}(\mathbf{P}_{\alpha})$ is of the form $\mathbf{T} \backslash \mathbf{PGL}_2$ (where \mathbf{T} is a non-trivial torus) or $\mathcal{N}(\mathbf{T}) \backslash \mathbf{PGL}_2$;
 - (2) equal to $w_{\alpha}w_{\beta}$ where α, β are two orthogonal simple roots such that the \mathbf{PGL}_2 -spherical variety $\mathbf{YP}_{\alpha\beta}/\mathcal{R}(\mathbf{P}_{\alpha\beta})$ is of the form $\mathbf{PGL}_2 \setminus \mathbf{PGL}_2 \times \mathbf{PGL}_2$.

When we write, for instance, $\mathbf{Y}\mathbf{P}_{\alpha}/\mathcal{R}(\mathbf{P}_{\alpha})$, we mean simply the homogeneous $\mathbf{P}_{\alpha}/\mathcal{R}(\mathbf{P}_{\alpha})$ -variety where a point stabilizer is given by the projection of a \mathbf{P}_{α} -stabilizer on \mathbf{Y} .

It is explained in [Sak13, §6.2], based on an analysis of low-rank cases, that one can take the set of generators of the above theorem to be the reflections corresponding to (simple) spherical roots. However, we do not need and will not use this result.

- 3.1.2. COROLLARY. Each spherical root $\gamma \in \Sigma_X$ is proportional to:
 - a sum of two superstrongly orthogonal roots of G, or
 - \bullet a root of G.

PROOF. By reduction to the varieties $\mathbf{T} \setminus \mathbf{PGL}_2$, $\mathcal{N}(\mathbf{T}) \setminus \mathbf{PGL}_2$, and $\mathbf{PGL}_2 \setminus \mathbf{PGL}_2 \times \mathbf{PGL}_2$, together with the fact that $\mathcal{X}(^{w_1}\chi) = w_1 \cdot \mathcal{X}(\mathbf{X})$, we deduce that the Weyl group elements described in both cases of the theorem induce hyperplane reflections on $\mathcal{X}(\mathbf{X}) \otimes \mathbb{Q}$, and the vectors $w_1^{-1}\alpha$ (resp. $w_1^{-1}(\alpha + \beta)$) are inverted by those reflections, hence are proportional to the roots of the corresponding root system.

 $^{^{16}}$ Here is how to see the equivalence: If they are simple and orthogonal, the only root system they can generate is $A_1 \times A_1$. Vice versa, if there are no more roots in their linear span, we can find real functionals ℓ_1 and ℓ_2 , such that ℓ_1 is positive on α, β and ℓ_2 is zero on $\pm \alpha, \pm \beta$ and non-zero on all other roots. Then, for $s \gg 0$, the functional $\ell_1 + s\ell_2$ distinguishes a set of positive roots which must have α and β as its simple elements, because it takes larger values on every other positive root. We thank Vladimir Drinfeld for pointing out this equivalence.

Now consider the set \mathcal{T} of all $\overline{W_X}$ -conjugates of reflections thus obtained. The set of roots $\Sigma_{\mathcal{T}}$ associated to elements of \mathcal{T} is a $\overline{W_X}$ -stable subset of Σ_X whose associated reflections generate $\overline{W_X}$. In other words, $\Sigma_{\mathcal{T}}$ is a root subsystem with the same Weyl group. This implies that $\Sigma_{\mathcal{T}} = \Sigma_X$, since root lines are characterized as the -1 eigenspaces of reflections.

The two cases of the corollary are mutually exclusive, since the sum of two superstrongly orthogonal roots of G cannot be proportional to a root. Hence, we can use them to define the type of a root. Notice that, in the first case of the above theorem, if $\mathbf{YP}_{\alpha}/\mathcal{R}(\mathbf{P}_{\alpha})$ is of the form $\mathbf{T} \setminus \mathbf{PGL}_2$ then $w_1^{-1}\alpha \in \mathcal{X}(X)$, while if it is of the form $\mathcal{N}(\mathbf{T}) \setminus \mathbf{PGL}_2$ then $w_1^{-1}\alpha \notin \mathcal{X}(\mathbf{X})$ (while $2w_1^{-1}\alpha \in \mathcal{X}(\mathbf{X})$).

DEFINITION. A spherical root $\gamma \in \Sigma_X$ is said to be:

- of type T if γ is proportional to a root α of G which belongs to $\mathcal{X}(\mathbf{X})$;
- of type N if γ is proportional to a root α of **G** which does not belong to $\mathcal{X}(\mathbf{X})$;
- of type G if γ is proportional to the sum $\alpha + \beta$ of two strongly orthogonal roots of G.

In this notation, the weight α (resp. the weight $\alpha + \beta$ for type G) will be called the *normalized* (simple) spherical root corresponding to γ ; it will sometimes be denoted by γ' . The set of normalized (simple) spherical roots will be denoted by Δ_X .

Notice that there is some issue here with the word "simple": while it should normally be used to distinguish elements of Δ_X from elements of the root system that they generate, it is customary in the theory of spherical varieties to call "spherical roots" only a set of simple roots. Therefore, we adopt this convention and feel free to drop the word "simple" when we talk about the set Δ_X or the set Σ_X .

To say that $\gamma \in \Sigma_X$ is of type N is equivalent to saying that $\gamma = 2\alpha$, where α is a root of G. This follows, for example, from the classification of rank one wonderful varieties [Akh83], but can also be deduced in a classification-free way from Theorem 3.1.1.

In the case of a normalized spherical root of type G, there is a canonical way to choose the roots α and β , which will be a useful fact later.

3.1.3. LEMMA. Consider a generator of $W_{(X)}$ as described in (and with the notation of) Theorem 3.1.1. In cases T and N the root $w_1^{-1}\alpha$ is orthogonal to all the roots of $\mathbf{L}(\mathbf{X})$, while in case G the element $w_1^{-1}w_{\alpha}w_{\beta}w_1$ permutes the positive roots of $\mathbf{L}(\mathbf{X})$. As a corollary, the element $w_1^{-1}w_{\alpha}w_1$, resp. $w_1^{-1}w_{\alpha}w_{\beta}w_1$, belongs to the canonical lift W_X of $\overline{W_X}$.

PROOF. For types T and N, we have $2w_1^{-1}\alpha \in \mathcal{X}(X) \Rightarrow w_1^{-1}\alpha \perp \Delta_{\check{L}(X)}$. For type G, similarly, $w_1^{-1}(\alpha + \beta) \perp \Delta_{L(X)}$ implies that $\langle \alpha, w_1 \check{\delta} \rangle + \langle \beta, w_1 \check{\delta} \rangle = 0$ for every $\check{\delta} \in \Phi_{\check{L}(X)}$. Let $\check{\delta}$ be such, then we claim that $w_1^{-1}w_{\alpha}w_{\beta}w_1\check{\delta} \in \Phi_{\check{L}(X)}$ as well. By non-degeneracy (Lemma 2.1.2), it suffices to show that it is perpendicular to $\mathcal{X}(X)$. Let $\chi \in \mathcal{X}(X)$. Then:

$$\langle \chi, w_1^{-1} w_{\alpha} w_{\beta} w_1 \check{\delta} \rangle = \langle \chi, \check{\delta} \rangle - \langle \alpha, w_1 \check{\delta} \rangle \langle \chi, w_1^{-1} \check{\alpha} \rangle - \langle \beta, w_1 \check{\delta} \rangle \langle \chi, w_1^{-1} \check{\beta} \rangle.$$

From the description of "type G" in Theorem 3.1.1 we see that $\langle \chi, w_1^{-1} \check{\alpha} \rangle = \langle \chi, w_1^{-1} \check{\beta} \rangle$, which together with the above relation imply that the last expression is equal to zero.

Finally, we claim that if $\check{\delta} > 0$ then $w_1^{-1} w_{\alpha} w_{\beta} w_1 \check{\delta} > 0$. Indeed, since $\operatorname{dist}(\mathring{X}, Y) = l(w_1)$ and $W_{P(X)}$ stabilizes \mathring{X} it follows that $w_1 \in [W/W_{P(X)}]$, and hence $w_1 \Phi_{\check{L}(X)}^+ \subset \Phi_{\check{G}}^+ \smallsetminus \{\check{\alpha}, \check{\beta}\}$. Hence, $w_{\alpha} w_1 \Phi_{\check{L}(X)}^+$ and $w_{\beta} w_1 \Phi_{\check{L}(X)}^+$ are contained in $\Phi_{\check{G}}^+$. We have also proved that $w_{\alpha} w_1 \Phi_{\check{L}(X)} = w_{\beta} w_1 \Phi_{\check{L}(X)}^+$, hence $w_{\alpha} w_1 \Phi_{\check{L}(X)}^+ = w_{\beta} w_1 \Phi_{\check{L}(X)}^+$ and therefore $w_1^{-1} w_{\alpha} w_{\beta} w_1 \Phi_{\check{L}(X)}^+ = \Phi_{\check{L}(X)}^+$.

For the final conclusion, we remind that the canonical lift of the coset space $W/W_{L(X)}$ to W consists of those elements which preserve the set of positive roots of $\mathbf{L}(\mathbf{X})$.

3.1.4. COROLLARY. For a spherical root γ of type G, there are precisely two positive roots of G in the (-1)-eigenspace of w_{γ} . They are both orthogonal to the weight $\rho_{L(X)}$.

We will call those the associated roots of γ . For γ of type T or N the associated root will be the unique positive root of \mathbf{G} which is proportional to γ . The second statement of the lemma holds also for the associated root of a spherical root of type T or N for obvious reasons, namely that $\chi(\mathbf{X})$ is orthogonal to all roots of $\mathbf{L}(\mathbf{X})$.

PROOF. The statement does not depend on whether γ is simple or not, so it is enough to show it for generators w_{γ} of the form $w_1^{-1}w_{\alpha}w_{\beta}w_1$ as in the previous lemma. Notice that if α' is an associated root then $w_{\gamma}\alpha' = -\alpha'$, while on the other hand, by the previous lemma, $w_{\gamma}\rho_{L(X)} = \rho_{L(X)}$. Hence,

$$\langle \rho_{L(X)}, \alpha' \rangle = \langle w_{\gamma} \rho_{L(X)}, w_{\gamma} \alpha' \rangle = -\langle \rho_{L(X)}, \alpha' \rangle \Rightarrow \langle \rho_{L(X)}, \alpha' \rangle = 0.$$

Now we return to defining the normalized root system:

3.1.5. PROPOSITION. Under the action of W_X on $\mathcal{X}(\mathbf{X})$, the set Δ_X is the set of simple roots of a root system with Weyl group W_X .

PROOF. Since the same statement is true for Σ_X , it suffices to prove that if $\gamma_1, \gamma_2 \in \Sigma_X$ and $w \in W_X$ are such that $w\gamma_1 = \gamma_2$ then for the corresponding normalized spherical roots γ'_1, γ'_2 we still have: $w\gamma'_1 = \gamma'_2$. But this is obvious from the definitions.

We denote by Φ_X the root system generated by Δ_X .

Now we come to the root data which, conjecturally, correspond to the dual group of Gaitsgory-Nadler $\check{G}_{X,GN} \subset \check{G}$ (to be recalled in the next subsection) and its central isogeny $\check{G}_X \twoheadrightarrow \check{G}_{X,GN}$ (whenever it can be defined). Notice that in the first case, in order for a root datum to define a subgroup of \check{G} , its lattice should be a sublattice of $\mathcal{X}(\mathbf{A})$ without co-torsion.

3.1.6. PROPOSITION. The set $(\mathcal{X}(\mathbf{A}) \cap \mathbb{Q} \cdot \mathcal{X}(\mathbf{X}), \Phi_X, W_X)$ gives rise to to^{17} a root datum. If there are no spherical roots of type N then the set $(\mathcal{X}(\mathbf{X}), \Phi_X, W_X)$ also gives rise to a root datum.

PROOF. By definition, the elements of Φ_X belong to $\mathcal{X}(\mathbf{A}) \cap \mathbb{Q} \cdot \mathcal{X}(\mathbf{X})$, and if there are no spherical roots of type N then they also belong to $\mathcal{X}(\mathbf{X})$ (as we deduce, again, by reduction to the varieties $\mathbf{T} \setminus \mathbf{PGL}_2$ and $\mathbf{PGL}_2 \setminus \mathbf{PGL}_2 \times \mathbf{PGL}_2$ together with the fact that $\mathcal{X}(^{w_1}\chi) = w_1 \cdot \mathcal{X}(\mathbf{X})$). Therefore, it remains to check that the corresponding coroots are integral on the given lattices.

Let $\gamma \in \Phi_X$ correspond to a hyperplane reflection of the form described in the Theorem 3.1.1. The associated coroot equals:

- the image of $w_1^{-1}\check{\alpha}$ in $\mathcal{X}(\mathbf{X})^*$, if $\gamma = w_1^{-1}\alpha$ is of type T or N (in the notation of the theorem);
- the image of $w_1^{-1}\check{\alpha}$ (which coincides with the image of $w_1^{-1}\check{\beta}$), if $\gamma = w_1^{-1}(\alpha + \beta)$ is of type G.

Those are integral on the given lattices, which completes the proof of the proposition. \Box

- **3.2. Distinguished morphisms.** We introduce the following notation:
 - G'_X the abstract complex reductive group defined by the root datum of $(\mathcal{X}(\mathbf{A}) \cap \mathbb{Q} \cdot \mathcal{X}(\mathbf{X}), \Phi_X, W_X)$.
 - If there are no spherical roots of type N, \check{G}_X is the abstract complex reductive group defined by the root datum of $(\mathcal{X}(\mathbf{X}), \Phi_X, W_X)$.

These groups come with preferred maximal tori A_X^* , $A_X'^*$ and are unique up to the inner action of this torus. Moreover, since the root data used to define them are actually based (i.e. have a preferred choice of positive roots), the groups \check{G}_X , \check{G}_X' also have a preferred choice of Borel subgroup containing the canonical maximal tori. The obvious isogeny between their root data gives rise to a canonical central isogeny $\check{G}_X \twoheadrightarrow \check{G}_X'$. We conjecture that the group \check{G}_X' is isomorphic to the one constructed by Gaitsgory and Nadler, which we denote by $\check{G}_{X,GN}$. A priori, the group $\check{G}_{X,GN}$ depends on the choice of an affine embedding of X; conditional on some assumptions on the Gaitsgory-Nadler dual group which will be discussed in §3.3, we will show (Corollary 3.5.2) that $\check{G}_{X,GN}$ is indeed equal to \check{G}_X' and hence independent

¹⁷Usually a root datum is described in terms of a pair L, \check{L} of finite free \mathbb{Z} -modules, together with subsets $\Phi \subset L, \check{\Phi} \subset \check{L}$ of roots and coroots. However, this is determined up to isomorphism by the triple (L, Φ, W_L) , where W_L is the Weyl group.

of the affine embedding. More precisely, we will show that $\check{G}_{X,GN}$ is obtained from a distinguished embedding of \check{G}'_X into \check{G} .

Call a morphism $\check{G}_X \to \check{G}$ distinguished if:

- (1) it extends the canonical map $A_X^* \to A^*$;
- (2) for every simple (normalized) spherical root γ , the corresponding root space of $\check{\mathfrak{g}}_X$ maps into the root spaces of its associated roots.

By Lemma 3.1.4 (and the comment which follows it), the second condition implies that the image of a distinguished morphism commutes with the image of $2\rho_{L(X)}$.

We will call a morphism $\check{G}_X \times \operatorname{SL}_2 \to \check{G}$ distinguished if its restriction to \check{G}_X is distinguished, and its restriction to SL_2 is a *principal* morphism into $\check{L}(X)$:

$$\operatorname{SL}_2 \to \check{L}(X) \subset \check{G}$$

with weight:

$$G_{\mathrm{m}} \stackrel{2\rho_{L(X)}}{\longrightarrow} \check{G},$$

where $G_{\rm m}$ is identified as a subgroup of ${\rm SL}_2$ in the standard way: $a \mapsto \begin{pmatrix} a \\ a^{-1} \end{pmatrix}$.

We apply similar terminology in related situations: for instance, there is a corresponding notion of distinguished map: $\check{G}'_X \to \check{G}$, or a distinguished map when we deal with a standard Levi subgroup of \check{G}_X containing A_X^* .

3.3. The work of Gaitsgory and Nadler. In this subsection we fix an affine embedding X^a of X.

Let us denote by $A_{X,GN}^*$ the image of the dual torus A_X^* inside A^* . Recall as before that we regard the sum $2\rho_{L(X)}$ of positive roots of $\mathbf{L}(\mathbf{X})$ as defining a character $2\rho_{L(X)}: G_{\mathrm{m}} \to A^*$.

3.3.1. Theorem (Gaitsgory and Nadler). To every affine spherical variety \mathbf{X}^a one can associate a connected reductive subgroup $\check{G}_{X^a,GN}$ of \check{G} with maximal torus $A_{X,GN}^*$. The group $\check{G}_{X^a,GN}$ is canonical up to A^* -conjugacy.

This is not very informative as stated, but the assertions (GN1)–(GN5) of §2.2, which we recall here with a few extra comments for the convenience of the reader, give more information about the group $\check{G}_{X,GN}$. To formulate them, let \mathbf{X}_{Θ}^a be, for every $\Theta \subset \Sigma_X$, the affine embedding of \mathbf{X}_{Θ} obtained by partially grading the coordinate ring of \mathbf{X}^a . In terms of the affine degeneration $\mathscr{X}^a \to \overline{\mathbf{A}_{X,ss}}$ discussed in §2.5, it is the fiber over $\lim_{t\to 0} \check{\lambda}(t)$, where $\check{\lambda}$ is any cocharacter in the cone of $\overline{\mathbf{A}_{X,ss}}$ which lies in the interior of the face determined by Θ . Since we are no experts in the technical details of [GN10], we will only prove the first of the following assertions, and treat the remaining as hypotheses:

(GN1) The image of $\check{G}_{X^a,GN}$ commutes with $2\rho_{L(X)}(\mathbb{C}^{\times}) \subset A^*$.

- (GN2) The Weyl group of $\check{G}_{X^a,GN}$ equals W_X . (This is a consequence of [**GN10**, Conjecture 7.3.2], as discussed there.)
- (GN3) For any $\Theta \subset \Sigma_X$ the dual group of \mathbf{X}_{Θ}^a is canonically a subgroup of $\check{G}_{X^a,GN}$. (Our identification of $\check{G}_{X^a,GN}$, based on these axioms, shows that it is the Levi subgroup of $\check{G}_{X^a,GN}$ associated to Θ .)
- (GN4) If the open **G**-orbit $\mathbf{X} \subset \mathbf{X}^a$ is parabolically induced, $\mathbf{X} = \mathbf{X}_L \times^{\mathbf{P}^-}$ **G**, where \mathbf{X}_L is spherical for the reductive quotient \mathbf{L} of \mathbf{P}^- , then the dual group $\check{G}_{X^a,GN}$ belongs to the standard Levi subgroup \check{L} of \check{G} corresponding to the class of parabolic subgroups opposite to \mathbf{P}^- . Moreover, if a connected normal subgroup \mathbf{L}_1 of \mathbf{L} acts trivially on \mathbf{X}_L , then $\check{G}_{X^a,GN}$ belongs to the dual group of \mathbf{L}/\mathbf{L}_1 (which is canonically a subgroup of \check{L}).
- (GN5) If \mathbf{X}_1^+ is a spherical homogeneous \mathbf{G} -variety, \mathbf{T} a torus of \mathbf{G} -automorphisms and $\mathbf{X}_2^+ = \mathbf{X}_1^+/\mathbf{T}$, and if $\mathbf{X}_1, \mathbf{X}_2$ are affine embeddings of $\mathbf{X}_1^+, \mathbf{X}_2^+$ with $\mathbf{X}_2 = \operatorname{spec} k[\mathbf{X}_1]^{\mathbf{T}}$, then there is a canonical inclusion $\check{G}_{X_2,GN} \hookrightarrow \check{G}_{X_1,GN}$ which restricts to the natural inclusion of maximal tori: $A_{X_2,GN}^* \hookrightarrow A_{X_1,GN}^*$ (arising from $\mathcal{X}(\mathbf{X}_2) \hookrightarrow \mathcal{X}(\mathbf{X}_1)$).

We shall give a proof of (GN1) or rather "angle" it out of the articles of Gaitsgory and Nadler. In [GN10] a certain tensor category $\mathbf{Q}(X)$ (denoted $\mathbf{Q}(Z)$ in loc.cit.) is constructed, together with adequate functors:

$$\operatorname{Rep}(\check{G}) \stackrel{\operatorname{Conv}}{\to} \mathbf{Q}(X) \stackrel{\operatorname{fib}}{\to} \operatorname{Vect}.$$
 (3.1)

The first category, as is usual, is constructed as $\mathbf{G}(\mathbb{C}[[[t]]])$ equivariant sheaves on the affine Grassmannian $\mathbf{G}(\mathbb{C}((t)))/\mathbf{G}(\mathbb{C}[[t]])$, and Vect denotes the category of vector spaces. The category $\mathbf{Q}(X)$ is constructed via a certain substitute for $\mathbf{G}(\mathbb{C}[[t]])$ -equivariant sheaves of $X(\mathbb{C}((t)))$.

PROOF OF (GN1). To show that the image of $2\rho_{L(X)}$ commutes with \check{G}_X , it suffices to show that there is a \mathbb{Z} -grading of the tensor category $\mathbf{Q}(X)$ such that under the "convolution" functor $\operatorname{Rep}(\check{G}) \stackrel{\operatorname{Conv}}{\to} \mathbf{Q}(X)$ and the equivalence of $\mathbf{Q}(X)$ with $\text{Rep}(\check{G}_{X,GN})$ the grading corresponds to the decomposition of representations in \mathbb{C}^{\times} -eigenspaces, where \mathbb{C}^{\times} acts via $2\rho_{L(X)}$. This grading is explicit, in the form of a cohomological shift, in [GN09], but implicit in [GN10]. More precisely, it is shown in [GN09][Theorem 1.2.1], where the special case of horospherical varieties is studied, that for a horospherical variety X_0 the irreducible objects of $Q(X_0)$ can be identified with intersection cohomology sheaves of certain strata, shifted in cohomological degree, and the shift is precisely the grading that we want. In [GN10] the authors choose to forget about the cohomological shift, however, this shift has to be compatible with the fiber functor $\mathbf{Q}(X) \to \mathbf{Q}(X_0)$ (where \mathbf{X}_0 is the boundary degeneration that we denoted before by \mathbf{X}_{Θ} for $\Theta = \emptyset$) because the fiber functor is obtained via a nearby cycles functor; hence, the category $\mathbf{Q}(X)$ carries the grading corresponding to $2\rho_{L(X)}$, as well.

3.3.2. Remark. Concerning (GN2) (GN3), (GN4) and (GN5): The statement of (GN2) is easy to deduce from the results of Gaitsgory and Nadler in the most interesting cases. First of all, we claim that $\mathcal{N}_{\check{G}_{X,GN}}(A_X^*)$ is contained in $\mathcal{N}_{\check{G}}(A^*)$. Indeed, $\mathcal{N}_{\check{G}_{X,GN}}(A_X^*)$ centralizes the image of $2\rho_{L(X)}$ (because it belongs to $G_{X,GN}$) and normalizes the centralizer of A_X^* inside of the centralizer of $2\rho_{L(X)}$. By non-degeneracy (Lemma 2.1.2), the common centralizer of A_X^* and $2\rho_{L(X)}$ is A^* , hence $\mathcal{N}_{\check{G}_{X,GN}}(A_X^*) \subset \mathcal{N}_{\check{G}}(A^*)$. Now, the combination of Theorem 4.2.1 and Proposition 5.4.1 of $[\mathbf{GN10}]$ imply that the restriction to A_X^* of any irreducible representation of $\check{G}_{X,GN}$ contains a character in Λ_X^+ . In the cases where W_X coincides with the normalizer of A_X^* in $\mathcal{Z}_{\check{G}}(2\rho_{L(X)})$ (such as for symmetric spaces), it follows immediately that the Weyl group of the dual group of Gaitsgory and Nadler has to be the whole W_X , for otherwise any Weyl chamber of it would be larger than Λ_X^+ . The requirement that $\mathcal{N}_{\mathcal{Z}_{\check{G}}(2\rho_{L(X)})}(A_X^*)/A_X^* = W_X$ can be understood representation theoretically as follows: it was proven in [Sak08] that the multiplicity of a generic unramified representation in the spectrum of X is equal to the product of the "geometric factor" $(N_{\mathcal{Z}_{\check{G}}(2\rho_{L(X)})}(A_X^*)/A_X^*:W_X)$ by the "arithmetic factor" of the number of open $\mathbf{B}(k)$ -orbits on \mathbf{X} . Thus, $\mathcal{N}_{\mathcal{Z}_{\tilde{G}}(2\rho_{L(X)})}(A_X^*)/A_X^* = W_X$ means that the geometric factor of unramified multiplicaties is 1.

(GN3) should follow in the same way as the "fiber functor" construction in [GN10], except that the fiber functor was constructed through a full degeneration of the spherical variety (i.e. a degeneration to X_{\emptyset}), while for (GN3) one would only perform a partial degeneration. (GN4) should also be feasible along the lines of [GN09], by interpreting geometrically the action of the center of L "on the left". Finally, (GN5) should follow from the behavior of intersection cohomology under such quotients by toric actions. However, since we are not specialists in the subject we treat (GN2)–(GN5) as hypotheses.

- **3.4.** Uniqueness of a distinguished morphism. In what follows we denote by $A_{X,GN}^*$ the (canonical) maximal torus of the Gaitsgory-Nadler dual group, i.e. the image of A_X^* in A^* . We fix throughout a standard basis $\{h, e, f\}$ for the Lie algebra \mathfrak{sl}_2 . By the 'weight" of a morphism: $f: \operatorname{SL}_2 \to \check{G}$ we understand either its restriction to G_{m} , or the derivative of this: $\langle h \rangle \simeq \mathfrak{g}_m \to \check{\mathfrak{g}}$. We will repeatedly use the following fact: if f, f' have the same weight, they are conjugate by an element of the centralizer of this weight.
- 3.4.1. Lemma. Let $\gamma \in \Delta_X$, and let \check{G}_{γ} be the corresponding subgroup of \hat{G}'_X , i.e. a connected reductive group with a canonical maximal torus $A^*_{X,GN}$ and unique simple coroot $\gamma : G_m \to A^*_{X,GN}$.
 - (1) A distinguished morphism $\psi : \check{G}_{\gamma} \to \check{G}$, through which the root space of $\check{\gamma}$ maps into the sum of root spaces of the associated coroots, always exists;

- (2) Any two such are conjugate by A^* ;
- (3) If ψ is such, its centralizer in A^* contains the common kernel of the associated roots to γ .

PROOF. Let \mathbf{A}_{γ} be the identity component of the kernel of γ . Then \check{G}_{γ} is the almost-direct product $A_{\gamma}^* \cdot f_{\gamma}(\mathrm{SL}_2)$, where $f_{\gamma} : \mathrm{SL}_2 \to \check{G}_X'$ has weight γ .

Let \check{M} be the centralizer of $A_{\gamma}^* \cdot \operatorname{image}(2\rho_{L(X)})$ inside \check{G} . It is a reductive group.

If ψ is distinguished, the map $\psi \circ f_{\gamma}$ has image in \check{M} and has weight γ ; the association $\psi \mapsto \psi \circ f_{\gamma}$ gives a bijection between distinguished morphisms and the set of:

$$m: SL_2 \to \check{M}, \quad m|G_m = \gamma.$$
 (3.2)

If γ is a root, the existence of m as in (3.2) is clear. Otherwise, it is the sum $\alpha + \beta$ of two superstrongly orthogonal positive roots; choosing a positive system in which α, β are simple, we see there are associated morphisms $f_{\alpha}, f_{\beta} : \operatorname{SL}_2 \to \check{G}$ (corresponding to α, β thought of as coroots of \check{G}). These morphisms have commuting image, since α, β are strongly orthogonal; therefore, we obtain a product morphism

$$f_{\alpha} \times f_{\beta} : \mathrm{SL}_2 \times \mathrm{SL}_2 \longrightarrow \check{G},$$

whose diagonal is a morphism m as in (3.2). This proves the first claim.

To check that any two such morphisms are A^* -conjugate, it suffices to check that any two m as in (3.2) are A^* -conjugate. However, any two such m are conjugate by the centralizer of $m|G_{\rm m}$, i.e., by the centralizer of $\gamma:G_{\rm m}\to \check M$. That is the same as the centralizer of $A^*_{X,GN}\times {\rm image}(2\check\rho_{L(X)})$. Since the spherical variety $\mathbf X$ is quasi-affine and, hence, non-degenerate, this centralizer is equal to A^* .

The final assertion follows from the explicit construction of a distinguished morphism. $\hfill\Box$

3.4.2. Lemma. The associated roots to all $\gamma \in \Delta_X$ are linearly independent.

PROOF (SKETCH). This is a clumsy argument reducing to the low-rank cases: Consider a linear relation between associated roots with non-zero coefficients, let R denote the support of all roots appearing and let $\alpha \in R$ be a simple root which is connected, in the Dynkin diagram, to at most one more element of R, and is not the shorter of the two. (Let's call such a simple root "extreme" for the given collection of associated roots.) Necessarily, the root α has to be contained in the support of at least two associated roots in the linear relation, say γ and δ . These, in turn, should be associated to spherical roots ε and ζ (not necessarily distinct). By inspection of spherical varieties of rank one or two [Was96], we see that an extreme simple root cannot be in the support of two associated roots.

3.4.3. Proposition. Distinguished morphisms $\check{G}'_X \to \check{G}$ and $\check{G}'_X \times \mathrm{SL}_2 \to \check{G}$, if they exist, are unique up to A^* -conjugacy.

PROOF. Let $\psi_1, \psi_2 : \check{G}'_X \to \check{G}$ be distinguished. We have seen that, for every spherical root γ , there exists $a_{\gamma} \in A^*$ so that

$$\operatorname{Ad}(a_{\gamma})\psi_1|_{\check{G}'_{X_{\gamma}}} = \psi_2|_{\check{G}'_{X_{\gamma}}}.$$

The action of A^* on the image of $\check{G}'_{X_{\gamma}}$ factors through the morphism $A^* \to G_{\mathrm{m}}$ or $A^* \to G_{\mathrm{m}}^2$ induced by the associated roots for γ , by part (3) of Lemma 3.4.1. Thus, by Lemma 3.4.2, we may find $a \in A^*$ so that $\mathrm{Ad}(a)$ and $\mathrm{Ad}(a_{\gamma})$ have the same action on $\psi_1|_{\check{G}'_{X_{\gamma}}}$ for all γ . In particular, there exists a so that

$$\operatorname{Ad}(a)\psi_1|_{\check{G}'_{X_{\gamma}}} = \psi_2|_{\check{G}'_{X_{\gamma}}}$$

for all γ . Since the $\check{G}'_{X_{\gamma}}$ generate \check{G}_{X} , it follows that ψ_{1}, ψ_{2} are A^{*} -conjugate, as desired.

This completes the proof of the assertion for \check{G}'_X .

The assertion for $\check{G}'_X \times \operatorname{SL}_2$ follows from the first: Fix a distinguished embedding ψ of \check{G}'_X . Now $2\rho_{L(X)}$ defines a morphism from G_{m} to the connected centralizer of $\psi(\check{G}'_X)$; any two SL_2 -morphisms with this restriction to the diagonal G_{m} must be conjugate under the connected centralizer of $\psi(\check{G}'_X) \times 2\rho_{L(X)}$. The latter is a subgroup of A^* (since X is nondegenerate, Lemma 2.1.2), commuting with $\psi(\check{G}'_X)$.

- **3.5.** The identification of the dual group. In this subsection we will use the axioms (GN) in order to identify the (based) root datum of the Gaitsgory-Nadler dual group with the (based) root datum of the abstract group which we denoted by \check{G}'_X , and in fact to identify $\check{G}_{X,GN}$ as a subgroup of \check{G} uniquely up to A^* -conjugacy. (Notice that in the classical setting, as opposed to the geometric one, the group \check{G} itself is only canonical up to A^* -conjugacy.) Using Proposition 3.4.3, the only thing that we need to prove is that simple roots of $\check{G}_{X,GN}$ and \check{G}'_X have the same length and, in fact, for every simple root γ the embedding of the corresponding standard Levi $\check{G}_{\gamma} \subset \check{G}_{X,GN} \hookrightarrow \check{G}$ is distinguished. The argument will eventually boil down to the classification of rank-one wonderful varieties by Akhiezer [Akh83].
- 3.5.1. PROPOSITION. Assume axioms (GN2)–(GN5). For spherical varieties of rank one there is an isomorphism of Gaitsgory-Nadler dual group with \check{G}'_X , inducing the identity on $A^*_{X,GN}$, and for any such isomorphism the embedding $\check{G}'_X \xrightarrow{\sim} \check{G}_{X,GN} \hookrightarrow \check{G}$ is distinguished.

PROOF. A homogeneous spherical **G**-variety of rank one is of the form:

$$\mathbf{X} = \mathbf{X}_1 \times^{\mathbf{P}} \mathbf{G},\tag{3.3}$$

where:

- (1) \mathbf{X}_1 is a \mathbf{G}_1 -torus bundle ver a variety $\mathbf{H} \backslash \mathbf{G}_1$ from Table 1 of $[\mathbf{Was96}]$ (we denote by \mathbf{G}_1 the group G of loc. cit.);
- (2) **P** is a parabolic subgroup with a homomorphism: $\mathbf{P} \to \operatorname{Aut}(\mathbf{X}_1)$ whose image coincides, up to central subgroups, with the image of \mathbf{G}_1 .

By inspection of this table, and using the axioms (GN1)–(GN5), one can show that the Gaitsgory-Nadler dual group is unambiguously equal to $\check{G}'_{X_{\gamma}}$. The table of Wasserman, together with more details of this argument, are given in Appendix A.

3.5.2. COROLLARY. Assume axioms (GN2)-(GN5). Then there exists a distinguished embedding $\check{G}'_X \hookrightarrow \check{G}$ with image $\check{G}_{X,GN}$. In particular, the group $\check{G}_{X,GN}$ is canonically isomorphic to \check{G}'_X up to $A^*_{X,GN}$ -conjugacy.

PROOF. By (GN2), the coroots of $\check{G}_{X,GN}$ are proportional to elements of the set $W_X \cdot \Delta_X$. (In fact, the lines through coroots are characterized as the -1 eigenspaces of reflections in W_X .) Therefore, there exists a system of simple positive coroots for $\check{G}_{X,GN}$, each of which is proportional to one of the $\gamma \in \Delta_X$.

On the other hand, by (GN3), the group $\check{G}_{X_{\gamma},GN}$ is contained in $\check{G}_{X,GN}$; the former group, as a subgroup of \check{G} , is identified through Lemma 3.4.1 and Proposition 3.5.1. It follows that we may suppose that $\check{G}_{X,GN}$ contains the image of a distinguished homomorphism

$$\check{G}'_{\gamma} \longrightarrow \check{G},$$

as in Lemma 3.4.1. Hence, the coroot of $\check{G}_{X,GN}$ proportional to $\gamma \in \Delta_X$ actually equals γ .

It follows that the coroots of $\check{G}_{X,GN}$ are precisely the elements of the set $W_X \cdot \Delta_X$. Hence, the root data of $\check{G}_{X,GN}$ and \check{G}'_X coincide canonically (recall that from the work of Gaitsgory and Nadler the group $\check{G}_{X,GN}$ is canonical up to A^* -conjugacy) and so the two are canonically isomorphic up to $A^*_{X,GN}$ -conjugacy.

3.6. Commuting SL_2 . In this subsection we will prove that there is a principal SL_2 inside of $\check{L}(X)$ commuting with $\check{G}_{X,GN}$, assuming (GN3) and (GN4). Our proof will be quite clumsy, using combinatorial arguments to reduce the problem to the case of spherical varieties of small rank, where it is checked case-by-case.

The basic result, which will be established case-by-case in Appendix A, is the following:

3.6.1. PROPOSITION. Let X be a spherical variety of rank one and assume (GN4). Then there is a principal map: $\operatorname{SL}_2 \to \check{L}(X)$ which commutes with $\check{G}_{X,GN}$.

¹⁸By G_1 -torus bundle we mean a principal torus bundle with an action of G_1 commuting with the action of the torus.

Using this, we can now show:

3.6.2. Theorem. Assume that there is a distinguished embedding: $\check{G}'_X \to \check{G}$ (as we have proven under the (GN) assumptions in Corollary 3.5.2). Then there is a principal $\mathrm{SL}_2 \to \check{L}(X)$ whose image commutes with \check{G}'_X .

Recall also that, by Proposition 3.4.3, the resulting distinguished morphism: $\check{G}'_X \times \operatorname{SL}_2 \to \check{G}$ is unique up to A^* -conjugacy.

PROOF. Fix a principal SL_2 into $\check{L}(X)$ with weight $2\rho_{L(X)}$, and denote its image by S; all such subgroups are A^* -conjugate. Fix a distinguished embedding of \check{G}'_X into \check{G} . By Proposition 3.6.1, for every $\gamma \in \check{\Delta}_X$ there is an A^* -conjugate of S which commutes with \check{G}_{γ} . Equivalently, there is an A^* -conjugate of \check{G}_{γ} which commutes with S. Arguing as in Proposition 3.4.3, we may find $a \in A^*$ which conjugates all \check{G}_{γ} simultaneously into the centralizer of S.

In the case that there are no spherical roots of type N (equivalently, as we mentioned, no element of Σ_X is of the form 2α , for α some root of \mathbf{G}), composing this with the central isogeny: $\check{G}_X \to \check{G}_X'$ we get the desired distinguished morphism:

$$\check{G}_X \times \mathrm{SL}_2 \to \check{G}.$$
 (3.4)

The proof of Theorem 2.2.3 is now complete.

Part 2

$\begin{array}{c} \textbf{Local theory and the Ichino-Ikeda} \\ \textbf{conjecture} \end{array}$

4. Geometry over a local field

In this section we shall examine certain general features of the geometry of $X = \mathbf{X}(k)$, where k is a p-adic field. In particular, we shall establish the relationship between G-invariant measures (or G-eigenmeasures) on X and X_{Θ} ; this will lead us to fixing compatible measures on X and X_{Θ} for the rest of the paper, as we indicated in §1.8.

More importantly, we shall establish the "exponential map" which relates the structure of X and X_{Θ} near infinity.

4.1. Measures. We may assume, without serious loss of generality, that X carries a positive G-eigenmeasure μ . Indeed, this is the case if the modular character of H (the quotient of its right by its left Haar measures) extends to a character of G. For a given $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$, the algebraic modular character \mathfrak{d}_H of \mathbf{H} is defined over k, and it either has finite image or surjects onto \mathbf{G}_{m} . In the latter case, we may replace \mathbf{H} by the kernel \mathbf{H}_0 of \mathfrak{d}_H ; then there is an $\mathbf{H}/\mathbf{H}_0 \times \mathbf{G}$ -eigen-volume form on $\mathbf{H}_0 \backslash \mathbf{G}$, and its absolute value gives the invariant measure. (Notice also that $\mathbf{H}_0 \backslash \mathbf{G} \to \mathbf{H} \backslash \mathbf{G}$ is surjective on k-points.) In the former case, there is an invariant volume form ω valued in the bundle over $\mathbf{H} \backslash \mathbf{G}$ defined by \mathfrak{d}_H . Since \mathfrak{d}_H has image in the n-th roots of unity μ_n , for some n, the associated complex line bundle is trivial and therefore the absolute value of ω defines an invariant measure on $\mathbf{H} \backslash \mathbf{G}(k)$. (For economy of language, we will never again mention the possibility that our eigenforms are valued in a torsion line bundle, instead of the trivial one.)

We fix from now on such an eigenmeasure μ , i.e. an eigenmeasure which is the absolute value of a volume form. We define $L^2(X) := L^2(X, \mu)$, considered as a *unitary* representation of G by twisting the right regular representation by the square root of the eigencharacter of μ , that is:

$$(g \cdot \Phi)(x) = \sqrt{\eta(g)}\Phi(xg), \tag{4.1}$$

where η denotes the eigencharacter of μ .

- 4.1.1. EXAMPLE. When $\mathbf{X} = \mathbf{U} \backslash \mathbf{G}$, with \mathbf{U} a maximal unipotent subgroup, the action of $A \times G$ on $L^2(X)$ is defined as: $((a,g) \cdot f)(x) = \delta^{-\frac{1}{2}}(a) f(a \cdot x \cdot g)$, where δ is the modular character of the Borel subgroup (the quotient of a right by a left invariant Haar measure).
- **4.2.** The measure on X_{Θ} . Our concern in this section is to relate the measures on X and X_{Θ} . Specifically:
- 4.2.1. PROPOSITION. For every **G**-eigen-volume form ω on **X** there is a canonical $\mathbf{A}_{X,\Theta} \times \mathbf{G}$ -eigen-volume form ω_{Θ} on \mathbf{X}_{Θ} , with the same **G**-eigencharacter, characterized by the property that for every Borel subgroup **B** the restrictions of the forms ω and ω_{Θ} to $\mathring{\mathbf{X}}$, resp. $\mathring{\mathbf{X}}_{\Theta}$ correspond to each other under the isomorphism (2.5).

Again, of course, we will twist the action of $A_{X,\Theta} \times G$ on functions on X_{Θ} as in (4.1).

Indeed, if ω is a differential form of top degree on $\overline{\mathbf{X}}$, then it induces in a natural way a differential form of top degree on the normal bundle to any subvariety. Unfortunately, the **G**-eigen-volume forms are rarely regular at the boundary. Nonetheless (formalizing the intuition that normal bundles model small neighborhoods of submanifolds) we may associate to any rational form on $\overline{\mathbf{X}}$ a rational form on each degeneration \mathbf{X}_{Θ} . We explain how to do this:

4.2.2. Obtaining measures by degeneration/residues. Let $\bar{\mathbf{X}}$ be a smooth variety and let $\mathbf{Z} \subset \bar{\mathbf{X}}$ be a closed subvariety, obtained as the intersection of a fixed set of reduced divisors \mathbf{D}_i : $1 \leq i \leq m$, with simple normal crossings and non-empty intersection. The choice of divisors \mathbf{D}_i makes the normal bundle of \mathbf{Z} into a $\mathbf{T} := \mathbf{G}_{\mathrm{m}}^m$ -space: the associated grading is the decomposition of the normal bundle at a point $z \in \mathbf{Z}$ as the sum of normal bundles at z to each \mathbf{D}_i .

Now suppose ω is a nonzero differential form of top degree on $\mathbf{X} = \overline{\mathbf{X}} \setminus \bigcup_i \mathbf{D}_i$. Let $-n_i - 1$ be the valuation of ω at \mathbf{D}_i ; let $f_i = 0$ be a local equation for \mathbf{D}_i . Then we obtain a rational differential form on the normal bundle via

$$\bar{\omega} := \frac{(\pi^* \operatorname{Res}(\omega \cdot \prod f_i^{n_i})) \wedge df_1' \wedge df_2' \wedge \dots \wedge df_m'}{\prod (f_j')^{n_j + 1}}.$$
(4.2)

where Res denotes the iterated residue, π is the projection from normal bundle to \mathbf{Z} , and f_j' denotes the derivative of f_j , considered as a function on the normal bundle of \mathbf{Z} . It is possible that $\bar{\omega} \equiv 0$: consider the case when $\dim \mathbf{X} = 2$ and $\omega = (f_1^{-1} + f_2^{-1})df_1 \wedge df_2$.

In fact, $\bar{\omega}$ is independent of choices. Indeed, there is a more intrinsic way of understanding (4.2) via degeneration to the normal bundle:

$$\mathscr{X} \to \mathbf{G}_{\mathfrak{d}}^m$$

that we discussed in $\S 2.5$ (where it was denoted by \mathscr{X}^n to distinguish it from the affine degeneration).

Let $\chi_n:(t_1,\ldots,t_m)\mapsto\prod t_i^{n_i}$, and define a differential form $\tilde{\omega}$ on $\mathbf{G}_{\mathrm{m}}^n\times\mathbf{X}\subset\mathcal{X}$ – via

$$\tilde{\omega} := \chi_n(t) \cdot p^* \omega, \tag{4.3}$$

where $p: \mathbf{G}_{\mathrm{m}}^m \times \mathbf{X} \to \mathbf{X}$ is the natural projection. We regard $\tilde{\omega}$ as a rational differential form on \mathscr{X} . Then:

4.2.3. Lemma. The restriction of the form $\tilde{\omega}$ to any fiber of the map: $\mathscr{X} \to \mathbf{G}_{\mathrm{a}}^m$, is well defined as a rational differential form on that fiber. If ω is regular everywhere, then $\tilde{\omega}$ is also regular everywhere. Finally, $\tilde{\omega}$ is an eigenform for the action of $\mathbf{G}_{\mathrm{m}}^m$ on \mathscr{X} , with eigencharacter χ_n^{-1} .

PROOF. This is easy to see for codimension-one orbits of \mathbf{T} in \mathcal{B} , and then by Hartogs' principle it extends to \mathcal{X} . The fact that it is an eigenform for $\mathbf{G}_{\mathrm{m}}^{n}$ is obvious from the definitions.

The restriction to the fiber over zero coincides, up to sign, with the form $\bar{\omega}$ defined previously.

4.2.4. The case of spherical varieties. We now specialize to the case $\mathbf{X} =$ our spherical variety and $\bar{\mathbf{X}} =$ its wonderful compactification. The set Δ parametrizing divisors at infinity can now be identified with Δ_X .

Given a **G**-eigenform ω on **X**, the prior discussion applied to $\overline{\mathbf{X}}, \omega$ and the boundary divisors yields a form $\bar{\omega}$ – henceforth denoted ω_{Θ} on X_{Θ} ; let us verify it has the properties stated in Proposition 4.2.1 and is in particular non-zero. In fact, the only property that needs to be verified is that it "coincides" with ω on open **B**-orbits under an isomorphism as in (2.5); since **B** is arbitrary, this implies that ω_{Θ} is a **G**-eigenform, and the fact that it is an eigenform for $\mathbf{A}_{X,\Theta}$ follows from the fact that $\tilde{\omega}$ is an eigenform for $\mathbf{G}_{\mathrm{m}}^{m}$ and Lemma 2.4.7.

But for the restriction of ω to the open **B**-orbit, it is easy to see that ω_{Θ} coincides with ω under (2.5), using the Local Structure Theorem 2.3.4.

This concludes the proof of Proposition 4.2.1.

4.3. Exponential map. This section plays a critical role in the paper: it shows that the asymptotic geometry of X and X_{Θ} over a local field are "the same" in a suitable sense. Specifically, for any open compact subgroup J of G there is an identification between J-orbits in suitable neighborhoods of Θ -infinity. This is based on the usual idea that a normal bundle of a submanifold is diffeomorphic to a tubular neighborhood of that submanifold.

DEFINITION. Suppose B, C are topological spaces¹⁹.

For any closed subspace $A \subset B$, and an open subset $U_A \subset A$ a germ at U_A of a morphism to C is an equivalence class of pairs (U_B, f) :

$$\{U_B : \text{ neighborhood of } U_A \text{ in } B, f : U_B \to C \text{ morphism}\}$$

under the equivalence relation $(U_B, f) \sim (U_B', f')$ if f and f' agree on a neighborhood of U_A .

The set of such germs, as U_A is varying, forms a sheaf on A, which we will denote by $\underline{\mathrm{Mor}}_A(B,C)$. Its global sections over A will be denoted by $\mathrm{Mor}_A(B,C)$.

In the setting that we are interested in, namely morphisms which are locally p-adic analytic on the p-adic points of smooth varieties, there is actually no difference between global sections of $\underline{\mathrm{Mor}}_A(B,C)$ and germs of morphisms: $U \to C$, where U is a neighborhood of A itself.

Let $\bar{\mathbf{X}}$ be the wonderful compactification of \mathbf{X} or any smooth toroidal embedding (not necessarily complete), and let $\mathbf{Z} \subset \bar{\mathbf{X}}$ be the closure of a Gorbit belonging to Θ -infinity (cf. §2.3.6). In this section, we shall construct a canonical collection of elements

$$\operatorname{\mathfrak{exp}}_{\Theta,J} \in \operatorname{Mor}_{Z/J}(N_Z\bar{X}/J,\bar{X}/J)$$

¹⁹possibly with extra structure, e.g. locally ringed spaces, so that in particular the notion of "morphism" from an open subset of B to C is defined

where "morphisms" means (germs of) measure-preserving, continuous maps and J ranges over all open compact subgroups. In other words, fixing an open compact J, we have a way of transferring J orbits in a neighborhood of Z in $N_Z\bar{X}$, to J-orbits in \bar{X} .

By construction, the restriction of $\exp_{\Theta,J}$ to $X_{\Theta}/J \subset N_Z \bar{X}/J$ will have image in $X/J \subset \bar{X}/J$ (in the sense that any representative of this germ has this property in a neighborhood of Z), and in Proposition 4.3.3 we will see that it is independent from the embedding and choice of orbit closure \mathbf{Z} , in the sense that all these germs of maps, obtained from different embeddings and orbits, glue together to give a well-defined germ of maps from a neighborhood of Θ -infinity (cf. §2.3.6) in X_{Θ}/J to a corresponding neighborhood in X/J. Thus, we have a well-defined element:

$$exp_{\Theta,J} \in Mor_{\infty_{\Theta}}(X_{\Theta}/J, X/J),$$

where now the notation stands for germs of maps defined in a neighborhood of Θ -infinity. This is the sense in which the exponential map will be used in the largest part of the paper, i.e. without reference to a specific embedding, but the geometric picture will be used in the construction and proofs. The collection of the elements $\exp_{\Theta,J}$, as J varies over a set of open-compact neighborhoods of the identity, will be denoted by \exp_{Θ} . We shall informally refer to \exp_{Θ} and the various maps it induces on J-invariants as "the exponential map", because of its construction:

Namely, the germ of $\mathfrak{exp}_{\Theta,J}$ will be induced by any p-adic analytic map $N_Z \bar{X} \to X$ inducing the identity on the normal bundle to Z and respecting G-orbits.

4.3.1. PROPOSITION. Let \mathbf{Z} be the closure of a \mathbf{G} -orbit in $\bar{\mathbf{X}}$, and let \mathbf{X}_1 and \mathbf{X}_2 be either of the varieties $N_{\bar{\mathbf{X}}}(\mathbf{Z})$ or $\bar{\mathbf{X}}$. There are locally p-adic analytic maps²⁰

$$\phi: U_1 \to U_2$$
,

where U_i is a neighborhood of Z in X_i (henceforth called "distinguished"), with the property that ϕ induces the identity between the normal bundles $N_{\mathbf{Z}}\mathbf{X}_1 \simeq N_{\mathbf{Z}}\bar{\mathbf{X}}$ and $N_{\mathbf{Z}}\mathbf{X}_2 \simeq N_{\mathbf{Z}}\bar{\mathbf{X}}$, and that ϕ maps the intersection of every \mathbf{G} -orbit with U_1 to the corresponding \mathbf{G} -orbit on \mathbf{X}_2 .

Any such ϕ has the following property: Given an open compact subgroup $J \subset G$, there are J-invariant neighborhoods $U_1' \subset U_1, U_2' \subset U_2$ of Z such that ϕ descends to a map: $U_1'/J \to U_2'/J$.

Finally, consider the open-compact topology on the space of such maps (with fixed domain). For every compact subset \mathcal{M} of such maps, there are J-invariant neighborhoods U_1'', U_2'' of Z in X_1 , resp. X_2 , such that all $\phi \in \mathcal{M}$ are defined and descend to the same map:

$$U_1''/J \to U_2''/J.$$

 $^{^{20}}$ Recall that this means that, in a neighborhood of every point, the map is given by a convergent power series with respect to systems of local coordinates.

The germ of a map as in the proposition will be denoted by $\mathfrak{exp}_{X_1,X_2,J} \in \operatorname{Mor}_{Z/J}(X_1/J,X_2/J)$. When $\mathbf{X}_1 = N_{\mathbf{\bar{X}}}(\mathbf{Z})$ and $\mathbf{X}_2 = \overline{\mathbf{X}}$, we also denote this by $\mathfrak{exp}_{\Theta,J}$ (where, as before, \mathbf{Z} is the closure of a \mathbf{G} -orbit on $\mathbf{\bar{X}}$ belonging to Θ -infinity).

PROOF. For a fixed neighborhood U_1 of Z in X_1 , consider the set \mathcal{K} of locally p-adic analytic maps $\phi: U_1 \to X_2$ with the properties of the proposition, that is: the differential of ϕ induces the identity on normal bundles, and ϕ maps points of a given \mathbf{G} -orbit on \mathbf{X}_1 to the corresponding orbit on \mathbf{X}_2 . We endow \mathcal{K} with the open compact topology.

4.3.2. Lemma. K is nonempty if U_1 is small enough. Moreover, For every compact open subgroup J of G and any compact subset \mathcal{M} of K, there is a neighborhood $U_1' \subset U_1$ of Z such that the composites: $U_1' \stackrel{\phi}{\rightrightarrows} X_2 \to X_2/J$ coincide, for any $\phi, \phi' \in \mathcal{M}$.

Let us first see why this implies Proposition 4.3.1. Let \mathcal{M} be a compact subset of such maps. The group G acts on such maps by: $g \cdot \phi = g \circ \phi \circ g^{-1}$, and we may assume that \mathcal{M} is J-invariant. But then, for x in a subset U_1' as in the lemma and all $j \in J$, we have: $j \circ \phi \circ j^{-1}(x) \in \phi(x)J = \phi'(x)J$ for all $\phi, \phi' \in \mathcal{M}$, and therefore the restrictions of all elements of \mathcal{M} to U_1' factors through U_1'/J and give rise to the same map: $U_1'/J \to X_2/J$.

We now come to the proof of the lemma: Since we can glue *locally* analytic maps we may replace Z by an arbitrarily small open subset Z' of it. In other words, if we have constructed maps $\phi_i: U_{1,i} \to X_2$ with the desired properties on an open covering of U_1 by open compact sets $U_{1,i}$, then we can refine the $U_{1,i}$ to a partition of U_1 , and glue the restrictions of the ϕ_i to obtain $\phi \in \mathcal{K}$ as desired.

We may assume that there is a Borel subgroup **B** such that Z' is contained in the open **B**-orbit, and use the Local Structure Theorem 2.3.4 to understand neighborhoods of Z'. Finally, we may replace J by the its subgroup $J \cap B$, since it is stronger to prove that the projections to $X_2/J \cap B$ coincide. Let **Y** be the toric variety of the Local Structure Theorem 2.3.4, fix an isomorphism of the distinguished open **B**-subset $\bar{\mathbf{X}}_B$ of $\bar{\mathbf{X}}$ with $\mathbf{Y} \times \mathbf{U}_{P(X)}$, and denote by $\mathbf{Z}' := \bar{\mathbf{X}}_B \cap \mathbf{Z}$ (it is the closure of a **B**-orbit). We are left with proving:

If $\mathbf{X}_1', \mathbf{X}_2'$ are either of the varieties $\mathbf{Y} \times \mathbf{U}_{P(X)}$ or $N_{\mathbf{Y} \times \mathbf{U}_{P(X)}}(\mathbf{Z}')$, then there are locally p-adic analytic maps: $U_1 \to X_2'$, where U_1 is a neighborhood of Z' in X_1' , inducing the identity on normal bundles and preserving the points of corresponding \mathbf{B} -orbits; moreover, for any compact subgroup J_B of B and any compact subset \mathcal{M} of such maps (defined on a fixed U_1) there is a smaller neighborhood $U_1' \subset U_1$ where the compos-

ites:
$$U_1' \stackrel{\phi}{\underset{\phi'}{\Longrightarrow}} X_2' \to X_2'/J_B$$
 coincide for any $\phi, \phi' \in \mathcal{M}$.

The statement is now easily reduced to the analogous statement about smooth toric varieties, indeed eventually to the case where $\mathbf{X}_1'', \mathbf{X}_2''$ are either of the spaces \mathbf{V} or $N_{\mathbf{Z}}(\mathbf{V})$, when \mathbf{V} is an affine space $(\mathbf{A}^1)^n$ and \mathbf{Z} is the intersection of all coordinate hyperplanes, i.e. the origin of \mathbf{V} , and we require the map to preserve all coordinate hyperplanes.

The other assertion – concerning the induced maps $U_1' \to X_2'/J_B$ – reduces similarly to the following assertion: Given a locally analytic morphism $f: k^n \to k^n$ which preserves coordinate axes, so that $f(0,0,\ldots,0) = (0,\ldots,0)$, and so that the derivative of f at zero is the identity map, then f maps each J-orbit near $\underline{0}$ to itself, if $J \subset (k^{\times})^n$ is open compact acting on k^n by coordinate multiplication. Moreover, the notion of "near" can be taken to be uniform if f lies in a compact set of such maps. To see this, one notes that the Taylor expansion $f_j(x_1,\ldots,x_n)=x_j(1+\ldots)$: all higher order terms are divisible by x_j because of preservation of coordinate axes.

(Notice that the requirement that the maps ϕ preserve orbits is made necessary by the fact that open compact subsets of the group do not provide a good uniform structure in the neighborhood of non-open orbits.)

We now come to the properties of $\mathfrak{exp}_{\Theta,J}$, preserving the notation of the previous proposition.

4.3.3. PROPOSITION. Any representative ϕ_J of $\exp_{X_1,X_2,J} \in \operatorname{Mor}_{Z/J}(X_1/J,X_2/J)$ has the following properties:

- (1) it is eventually (that is: after restricting its domain to a smaller neighborhood of Z/J, if necessary) a measure-preserving bijection;
- (2) it is eventually equivariant: for any $h \in \mathcal{H}(G, J)$ (the Hecke algebra of G with respect to J), if $U_1 \subset X_1/J$ denotes the domain of definition of ϕ_J then there is a smaller neighborhood U'_1 of Z/J such that for any J-invariant function f on U'_1 we have:

$$h * \phi_{J*} f = \phi_{J*} (h * f),$$

where $\phi_{J*}f$ denotes the push-forward of f through the bijection ϕ_J . (3) Its restriction to $X_{\Theta} \subset N_Z \bar{X}$ (cf. Proposition 2.5.3) does not depend on the embedding \bar{X} , in the following sense: for any two embeddings \bar{X} , \bar{X}' , orbit closures Z, Z' as before and representatives ϕ_J, ϕ'_J for the corresponding germs $\exp_{\Theta,J}$, there are J-stable neighborhoods N_{Θ}, N'_{Θ} of Z, resp. Z' in X_{Θ} , such that $\phi_J|_{(N_{\Theta} \cap N'_{\Theta})/J} = \phi'_J|_{(N_{\Theta} \cap N'_{\Theta})/J}$. Hence, by working with all orbit closures belonging to Θ -infinity in a wonderful compactification, we get a well-defined germ $\exp_{\Theta,J} \in \operatorname{Mor}_{\infty_{\Theta}}(X_{\Theta}/J, X/J)$ which does not depend on choices.

Notice that, by the third statement, the "eventual equivariance" property of the second statement extends to a neighborhood of Θ -infinity, when applied to smooth functions on X_{Θ} .

PROOF. The preservation of volume follows immediately from the existence of distinguished equivariant morphisms between open **B**-orbits (2.5) and the characterizing property of the measure on X_{Θ} (Proposition 4.2.1).

We come to the proof of eventual equivariance: It is enough to prove it for elements of the Hecke algebra of the form h = JgJ. Write the double coset JgJ as a (finite) disjoint union of right cosets:

$$JgJ = \sqcup_i g_i J.$$

Let $J' = J \cap \bigcap_i g_i J g_i^{-1}$ (an open compact subgroup of G), and choose a distinguished map ϕ giving rise to representatives ϕ_J and $\phi_{J'}$ for $\mathfrak{exp}_{\Theta,J'}$, $\mathfrak{exp}_{\Theta,J'}$.

We obviously have, eventually:

$$\phi_{J*}(h*f) = \phi_{J*}(\sum_{i} g_i \cdot f) = \sum_{i} \phi_{J'*}(g_i \cdot f).$$

Therefore, it is enough to show that

$$\phi_{J'*}q_i \cdot f = q_i \cdot \phi_{J'*}f.$$

But the maps: $\phi^{-1}g_i^{-1}\phi g_i$ are also distinguished, so the result follows from Proposition 4.3.1.

The independence from the orbit closure for a *given* embedding is seen as follows: If two orbit closures \mathbf{Z}, \mathbf{Z}' belonging to Θ -infinity do not intersect, then they have disjoint neighborhoods, so there is nothing to prove. Otherwise, their intersection also belongs to Θ -infinity, and it is therefore enough to prove independence when one (say, \mathbf{Z}) is contained in the other. In that case we have natural identifications of normal bundles:

$$N_{\mathbf{Z}}\overline{\mathbf{X}} = N_{\mathbf{Z}}(N_{\mathbf{Z}'}\mathbf{X}).$$

If we set $\mathbf{Y} = N_{\mathbf{Z}'}\overline{\mathbf{X}}$, there is an "exponential map":

$$\phi: N_{\mathbf{Z}}\mathbf{Y} \to \mathbf{Y},$$

i.e. p-adic analytic map fixing \mathbf{Z} and inducing the identity on its normal bundle, which is the identity on the open \mathbf{G} -orbits, both identified with \mathbf{X}_{Θ} via Proposition 2.5.3; indeed, this can easily be seen by invoking the normal bundle degeneration and the Local Structure Theorem 2.3.4. Composing with an exponential map from \mathbf{Y} to \mathbf{X} we see that in a neighborhood of \mathbf{Z} the two exponential maps coincide.

Thus we have a well-defined germ $\mathfrak{exp}_{\Theta,J}$ of maps in a neighborhood of the Θ -infinity of the given embedding (a priori, depending on the embedding). Given two smooth toroidal embeddings $\overline{\mathbf{X}}, \overline{\mathbf{X}}'$, now, by the Luna-Vust theory we can find a third one $\overline{\mathbf{X}}''$, open \mathbf{G} -invariant subsets $\mathbf{U}, \mathbf{U}' \subset \overline{\mathbf{X}}''$ and proper morphisms: $\mathbf{U} \to \overline{\mathbf{X}}, \mathbf{U}' \to \overline{\mathbf{X}}'$. Indeed, such an embedding can be obtained by constructing a fan, as described in §2.3.5, which contains a partition of \mathcal{C} , for every cone \mathcal{C} in the fan of $\overline{\mathbf{X}}$ or $\overline{\mathbf{X}}'$.

It is easy to see that any representative for $\mathfrak{exp}_{\Theta,J}$ on $\overline{\mathbf{X}}$, resp. $\overline{\mathbf{X}'}$ pulls back to a representative for $\mathfrak{exp}_{\Theta,J}$ on \mathbf{U} , resp. \mathbf{U}' , and their germs glue together to the germ $\mathfrak{exp}_{\Theta,J}$ for $\overline{\mathbf{X}''}$.

4.3.4. Transitivity. Let $\overline{\mathbf{X}}$ be a smooth toroidal embedding of \mathbf{X} , \mathbf{Z} an orbit closure belonging to Θ -infinity, and $\mathbf{Z}' \subset \mathbf{Z}$ an orbit closure belonging to Ω -infinity, for some $\Omega \subset \Theta$. Then $N_{\mathbf{Z}}\overline{\mathbf{X}}$ is a smooth toroidal embedding of \mathbf{X}_{Θ} , and hence it is clear from the identity:

$$N_{\mathbf{Z}'}\left(N_{\mathbf{Z}}\overline{\mathbf{X}}\right) = N_{\mathbf{Z}'}\overline{\mathbf{X}}$$

that \mathbf{X}_{Ω} is canonically identified with the corresponding boundary degeneration of \mathbf{X}_{Θ} . (One can also argue that using the affine degeneration of §2.5.)

If $\phi: N_{Z'}\bar{X} \to X$ is a p-adic analytic map inducing the identity on the normal bundle to Z' and respecting G-orbits, then its partial differential along Z:

$$N_{Z'}\bar{X} \to N_Z\bar{X}$$

also has the same properties. Hence, the exponential maps satisfy the transitivity property:

$$\exp_{\Omega} = \exp_{\Theta} \circ \exp_{\Omega}^{\Theta}, \tag{4.4}$$

where by $\mathfrak{exp}_{\Omega}^{\Theta}$ we denote the corresponding exponential map for X_{Θ} (i.e. a compatible collection, over J, of elements of $\mathrm{Mor}_{Z/J}(X_{\Theta}/J,X/J)$).

5. Asymptotics

From now on we assume that X is *wavefront*, with $\mathcal{Z}(G)^0$ surjecting onto $\mathcal{Z}(X)$.

5.1. The main result. The main goal of this section is to relate the asymptotic behavior of eigenfunctions on X to eigenfunctions on the boundary degenerations. Here we use the word "eigenfunctions" freely, meaning elements of $C^{\infty}(X)$ generating irreducible representations. As we saw in the previous section §4.1, the measure on X canonically induces an $A_{X,\Theta} \times G$ -eigenmeasure, with the same G-eigencharacter, on each boundary degeneration X_{Θ} , which allows us to formulate the main theorem.

Recall that the notion of Θ -infinity has been introduced in §2.3.6.

5.1.1. THEOREM. For every $\Theta \subset \Delta_X$ there exists a unique G-morphism

$$e_{\Theta}: C_c^{\infty}(X_{\Theta}) \longrightarrow C_c^{\infty}(X)$$
 (5.1)

with the property that for every open compact subgroup $J \subset G$ and any representative ϕ_J of $\mathfrak{exp}_{\Theta,J}$ there is a (J-stable) neighborhood N_{Θ} of Θ -infinity such that for all $f \in C_c^{\infty}(N_{\Theta})^J$ we have:

$$e_{\Theta}(f) = \phi_{J*}(f). \tag{5.2}$$

In fact, the morphism is characterized by the validity of (5.2) in a neighborhood of some orbit closure **Z** belonging to Θ -infinity in some smooth toroidal embedding $\overline{\mathbf{X}}$.

The notation e_{Θ} derives from "exponential map", but also from the name "Eisenstein series", because the global analog of e_{Θ} is the construction of pseudo-Eisenstein series. A neighborhood N_{Θ} of Θ -infinity as in the theorem will be called "J-good", and N_{Θ}/J will be identified with a subset of both X/J and X_{Θ}/J via the map ϕ_J as above (which is now a canonical representative of the exponential map, restricted to N_{Θ}/J). We will sometimes, by abuse of language, treat N_{Θ} itself as a subset of both X and X_{Θ} , when the statements that we are making are really about N_{Θ}/J .

We observe that, if N_{Θ} denotes a neighborhood of Θ -infinity for each $\Theta \subset \Delta_X$, then $\bigcup_{\Theta \neq \Delta_X} N_{\Theta}$ necessarily has compact-modulo-center complement inside X, as follows from the compactness of a wonderful embedding \overline{X} . Therefore, the theorem indeed controls the asymptotics in all directions simultaneously.

On the other hand, for given Θ , the last statement of the theorem shows that the map e_{Θ} is characterized by its restriction along a unique direction towards Θ -infinity, in the following sense: Recall from the Luna-Vust theorem 2.3.1 that to any half-line in $\mathcal{V}=$ the cone of invariant valuations for \mathbf{X} we can attach a smooth toroidal embedding (where smoothness follows from the local structure theorem 2.3.4). This embedding has a unique non-open \mathbf{G} -orbit, and by choosing the half-line in the interior of the face corresponding to Θ , this \mathbf{G} -orbit will belong to Θ -infinity.

Dually, we have a morphism:

$$e_{\Theta}^*: C^{\infty}(X) \to C^{\infty}(X_{\Theta}).$$
 (5.3)

Theorem 5.1.1 is equivalent to:

5.1.2. Theorem. There is a unique G-morphism

$$e_{\Theta}^*: C^{\infty}(X) \to C^{\infty}(X_{\Theta})$$

with the property that for every open compact subgroup $J \subset G$ and any representative ϕ_J of $\exp_{\Theta,J}$ there is a (J-stable) neighborhood N_{Θ} of Θ -infinity such that for all $f \in C^{\infty}(X)^J$ we have:

$$e_{\Theta}^{*}(f)|_{N_{\Theta}} = \phi_{J}^{*}(f|_{N_{\Theta}}).$$
 (5.4)

In fact, the morphism is characterized by the validity of (5.4) in a neighborhood of some orbit closure **Z** belonging to Θ -infinity in some smooth toroidal embedding $\overline{\mathbf{X}}$.

For every smooth representation π of G and any G-equivariant map: $M: \pi \to C^{\infty}(X)$, the composition with this morphism gives rise to a G-morphism:

$$\operatorname{Hom}_{G}(\pi, C^{\infty}(X)) \ni M \mapsto M_{\Theta} \in \operatorname{Hom}_{G}(\pi, C^{\infty}(X_{\Theta}))$$
 (5.5)

which will be called the "asymptotics" map. It has the property that for any $v \in \pi^J$ we have:

$$M(v)|_{N_{\Theta}} = M_{\Theta}(v)|_{N_{\Theta}'}. \tag{5.6}$$

Moreover, (5.6) uniquely characterizes M_{Θ} .

We shall give two proofs of this result:

- (1) In §5.2 we formulate a "morally satisfactory" proof based on the "stabilization theorem" of Bernstein and the exp-map;
- (2) In §5.3 we give a "quick and dirty" proof, using the known results about asymptotics of smooth matrix coefficients (of course, this sweeps under the carpet all arguments of the previous approach, which are used to establish the result in the group case). This method requires some additional piece of information, namely a (weak) generalized form of the Cartan decomposition which can be derived from the geometry of the wonderful embedding, in order to show that G_x -invariant functionals (where $x \in X$) can be computed using smooth matrix coefficients.²¹
- 5.1.3. Remark. (i) Recently, Bezrukavnikov and Kazhdan gave a proof of second adjunction [BK15] using the geometry of the wavefront compactification in the group case. In particular, they give in §4 a beautiful abstract approach to essentially the same problem (although phrased in a special case, their method adapts without change to the current situation). In the current context, it gives another proof of asymptotics, without using Bernstein's stabilization theorem. It uses as input certain finite generation statements such as Remark 5.1.7; in our context, we obtain these a posteriori from the asymptotics and the knowledge that X is wavefront. Although the argument can be reordered so that the proof of [BK15] goes through, it does not allow us to bypass the requirement that X be wavefront.
- (ii) Also, we observe that our proofs do not require Bernstein's results if, for instance, one is interested only in the case of π admissible (as is the case with the unitary theory in this paper). In that case, one can easily see that the usual facts about the Jacquet module suffice. Bernstein's results are used to generalize from the case of admissible representations to general smooth representations a generalization that, although conceptually very pleasing, we do not strictly require for the the largest part of the paper (except for section 15).
- 5.1.4. REMARK. If $\Theta \supset \Omega$ then we can apply the theorem to the variety \mathbf{X}_{Ω} in order to get a morphism $e_{\Theta}^{\Omega}: C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X_{\Omega})$. Clearly, we have

 $^{^{21}\}mathrm{This}$ observation has already appeared in work of Lagier [Lag08] and Kato-Takano [KT08].

the transitivity property: $e_{\Omega} \circ e_{\Theta}^{\Omega} = e_{\Theta}$, since the exponential maps have the same transitivity property (4.4).

The theorem implies the following:

5.1.5. THEOREM (Finiteness of multiplicities.). Let π be an irreducible smooth representation of G, and let \mathbf{X} be a wavefront spherical variety with $\mathcal{Z}(\mathbf{G})^0 \to \mathcal{Z}(\mathbf{X})$. Then:

$$\dim \operatorname{Hom}_G(\pi, C^{\infty}(X)) < \infty.$$

PROOF. First, we claim:

For any $\Theta \subset \Delta_X$, we have $\dim \operatorname{Hom}_G(\pi, C^{\infty}(X_{\Theta})) < \infty$ if and only if $\dim \operatorname{Hom}_{A_{X,\Theta} \times G}(\chi \otimes \pi, C^{\infty}(X_{\Theta})) < \infty$ for every character χ of $A_{X,\Theta}$.

Indeed, recall that $C^{\infty}(X_{\Theta})$ is induced from the P_{Θ}^- representation $C^{\infty}(X_{\Theta}^L)$, and by Proposition 2.7.2 the action of $A_{X,\Theta}$ is induced by the center of the corresponding Levi $L_{X,\Theta}$ (up to possibly a finite index due to the fact that the map of k-points: $\mathcal{Z}(L_{\Theta})^0 \to A_{X,\Theta}$ may not be surjective). Since π is irreducible, there is only a finite number of distinct characters χ of $\mathcal{Z}(L_{X,\Theta})^0$ such that π could be embedded in a representation induced via P_{Θ}^- from a representation of L_{Θ} with $\mathcal{Z}(L_{\Theta})^0$ -character χ . Therefore, finiteness for every character χ implies finiteness for π , forgetting the $A_{X,\Theta}$ -action.

Now we may assume, by induction, that the theorem is true for \mathbf{X}_{Θ} under the $\mathbf{A}_{X,\Theta} \times \mathbf{G}$ -action, for every $\Theta \subsetneq \Delta_X$. Recall from Proposition 2.7.2 that \mathbf{X}_{Θ} is also wavefront under this action.

Now, the common kernel of all the asymptotics maps (5.5) (excluding $\Theta = \Delta_X$), consists of morphisms: $\pi \to C^{\infty}(X)$ such that the image of π^J is supported on the complement of all neighborhoods N_{Θ} of Theorem 5.1.1. This complement has a finite number of J-orbits modulo the action of $\mathcal{Z}(X)$, and therefore, since $\mathcal{Z}(G)^0 \twoheadrightarrow \mathcal{Z}(X)$), dim $\operatorname{Hom}_G(\pi, C^{\infty}(X)) < \infty$.

- 5.1.6. REMARK. This proof only gives a bound for the dimension of $\operatorname{Hom}_G(\pi, C^{\infty}(X))$ which depends on the level of π , i.e. on which subgroup J is such that $\pi^J \neq 0$. It is natural to ask whether there exists a bound independent of π . A plausible such upper bound would be the generic multiplicity of unramified principal series, computed in [Sak08, Theorem 5.3.2].
- 5.1.7. Remark. Aizenbud-Avni-Gourevitch [AAG12] have established the following result (using, as input, the finiteness of multiplicities): If J is an open compact subgroup of G, then

 $C_c^\infty(X)^J$ is finitely generated as a module under the J-Hecke algebra $\mathcal{H}(G,J).$

Let us sketch how the theorem may be used to give another proof of this result: one argues just as in the previous result, but using the dual maps $C_c^{\infty}(X) \longrightarrow C_c^{\infty}(X_{\Theta})$, together with the fact (due to Bernstein) [Ber84, 3.11] that if P is a parabolic with associated Levi decomposition P = MN

and J admits the Iwahori decomposition $J = J_-J_MJ_+$, then parabolic induction maps finitely generated $\mathcal{H}(M,J_M)$ -modules to finitely generated $\mathcal{H}(G,J)$ -modules.

Finally, we introduce some language in order to describe another corollary of the asymptotics. This will be only used much later (Proposition 15.3.6) to show meromorphic continuation of certain intertwiners related to "Eisenstein integrals."

A function on an abelian group is said to be *finite* if its translates (under the action of that abelian group on itself by multiplication) span a finite-dimensional vector space. For a normal k-variety \mathbf{V} with a distinguished divisor \mathbf{D} , we will say that a complex-valued function F on V is D-finite if for every $x \in V$ there is a neighborhood of V_x of x, and rational functions f_1, \ldots, f_m , whose zero and polar divisors are contained in \mathbf{D} such that, on $V_x - V_x \cap D$, the function F agrees with the pullback of a finite function on \mathbb{G}_m^m by (f_1, \ldots, f_m) .

The notion of finite function is stable under pullback, i.e. given a morphism $\pi: \mathbf{V}_1 \to \mathbf{V}_2$ of algebraic varieties, a divisor $\mathbf{D}_2 \subset \mathbf{V}_2$, and a function f_2 on V_2 that is D_2 -finite, the pullback $\pi^* f_2$ is D_1 -finite, where $\mathbf{D}_1 = \pi^{-1} \mathbf{D}_2$.

5.1.8. COROLLARY. Let $\bar{\mathbf{X}}$ be a smooth toroidal embedding of \mathbf{X} , and let $\nu: \pi \to C^{\infty}(X)$ be a morphism from an admissible representation. Then the image of ν consists of $(\bar{X} \setminus X)$ -finite functions.

The statement makes sense since, as we have seen from the Local Structure Theorem 2.3.4, the complement of the open **G**-orbit in a smooth toroidal embedding is a union of divisors intersecting transversely.

PROOF. For any point $x \in \bar{X}$, assumed to belong to a **G**-orbit **Z** corresponding to $\Theta \subset \Delta_X$, we can find by (2.5) algebraic identifications of its neighborhoods in X and X_{Θ} which are compatible with the exponential map. Therefore, it is enough to show that the image of $e_{\Theta}^* \circ \nu$ consists of finite functions (with respect to the complement of X_{Θ} in $N_{\bar{Z}}\bar{X}$). But then this notion of finiteness can be seen to be equivalent to finiteness under the action of the torus of Lemma 2.4.2, which is a subgroup of $\mathcal{Z}(X_{\Theta})$, and the image of admissible representations is clearly $\mathcal{Z}(X_{\Theta})$ -finite.

5.2. Proof of asymptotics. In this subsection we prove Theorem 5.1.1, or rather its adjoint Theorem 5.1.2. We fix a $\Theta \subset \Delta_X$, a smooth toroidal embedding $\overline{\mathbf{X}}$ and a \mathbf{G} -orbit closure \mathbf{Z} belonging to Θ -infinity. From now on we will be denoting by \exp_{Θ} a representative for $\exp_{\Theta,J}$, where J is an open compact subgroup under which the functions under consideration are invariant. The dependence on J is suppressed from the notation, since the choice of J does not matter for the statements.

We assume for notational simplicity that G has a unique orbit on X_{Θ} . Fix a parabolic subgroup in the class of \mathbf{P}_{Θ}^- (denoted thus), and recall from §2.7 that this identifies a subvariety of \mathbf{X}_{Θ} as the Levi variety \mathbf{X}_{Θ}^L ; its closure in $\overline{\mathbf{X}}$ intersects \mathbf{Z} along a \mathbf{P}_{Θ}^- -stable set \mathbf{Z}_1 . (Both the \mathbf{X}_{Θ}^L and \mathbf{Z}_1 are the sets of all elements of \mathbf{X}_{Θ} , resp. \mathbf{Z} whose stabilizer \mathbf{G}_z belongs to \mathbf{P}_{Θ}^- , or equivalently: $\mathbf{U}_{\Theta}^- \subset \mathbf{G}_z \subset \mathbf{P}_{\Theta}^-$.) We have a map: $\mathbf{X}_{\Theta}^L \to \mathbf{Z}_1$ arising from the structure of \mathbf{X}_{Θ} as a subset of the normal bundle to \mathbf{Z} , and we denote by fiber(z) the preimage of a given point $z \in Z_1$. Note that fiber(z) is homogeneous under $A_{X,\Theta} = \mathcal{Z}(X_{\Theta})$.

Then Theorem 5.1.2 is equivalent to the following:

5.2.1. Proposition. (Assuming a unique G-orbit on X_{Θ} .)

There is a system $(\mathcal{N}_J)_J$ of $J \cap P_{\Theta}^-$ -stable neighborhoods of Z_1 in 22 X_{Θ}^L , as J varies over a basis of open compact subgroup-neighborhoods of the identity in G, with the following property:

For any z in the open P_{Θ}^- -orbit (equivalently: L_{Θ} -orbit) in Z_1 , there is a G-morphism:

$$C^{\infty}(X) \ni f \mapsto f_{\Theta} \in C^{\infty}(X_{\Theta})$$

(a priori, depending on z) such that, for any J-invariant f,

$$f_{\Theta} = \exp_{\Theta}^* f \text{ on fiber}(z) \cap \mathcal{N}_J.$$
 (5.7)

Moreover, there is a unique G-morphism which has property (5.7) for some z and for some such system of neighborhoods of z in fiber(z).

When there are multiple G-orbits on X_{Θ} or, rather, on the open orbit in Z, one just needs to choose more than one z's representing all orbits.

Of course, the proposition follows from Theorem 5.1.2 (applied to the given embedding $\overline{\mathbf{X}}$ and orbit closure \mathbf{Z}), but we will see that, vice versa, the validity of the proposition for *every* pair $(\overline{\mathbf{X}}, \mathbf{Z})$ implies the full theorem:

PROOF OF THEOREM 5.1.2 ASSUMING PROPOSITION 5.2.1. We first check that the morphism supplied by the proposition is independent of choice of z, so let us temporarily denote the f_{Θ} provided by the proposition by f_{Θ}^z . Notice that P acts transitively on the k-points of the open \mathbf{P} -orbit of \mathbf{Z}_1 , since it is self-normalizing in G (and we are assuming that G acts transitively on X_{Θ} , hence also on the points of the open \mathbf{G} -orbit on \mathbf{Z}). For z':=zg, $g\in P$, and f a $g^{-1}Jg$ -invariant function, the functions $(gf)_{\Theta}^z=gf_{\Theta}^z$ and $\exp_{\Theta}^*(gf)$ coincide on $\mathcal{N}_J \cdot g \cap \mathrm{fiber}(z)$; so f_{Θ}^z and $g^{-1}\exp_{\Theta}^*(gf)$ coincide on $\mathcal{N}_J \cdot g \cap \mathrm{fiber}(z')$. Now, the system $(\mathcal{N}_J \cdot g)_J$ is still a system of neighborhoods of Z_1 , and we can replace it by a system of smaller neighborhoods $(\mathcal{N}_J')_J$ so that $g^{-1}\exp_{\Theta}^*(gf)=f$ there (for any such f). Then f_{Θ}^z and $\exp_{\Theta}^*(f)$ coincide on $\mathcal{N}_J' \cap \mathrm{fiber}(z')$. The uniqueness statement of the Proposition implies that $f_{\Theta}^z=f_{\Theta}^{z'}$.

Now we show that the map $f \mapsto f_{\Theta}$ has the properties of Theorem 5.1.2, first for the given pair $(\overline{\mathbf{X}}, \mathbf{Z})$. First of all, by continuity of the map \exp_{Θ} and the validity of the above proposition in a neighborhood (s. footnote 22) of Z_1 , property (5.7) holds for any $z \in Z_1$, not only in the open orbit. The

²² By this, we more properly mean that \mathcal{N}_J is of the form $\mathcal{N}_J' \cap X_{\Theta}^L$, where \mathcal{N}_J' is a neighborhood of Z in its normal bundle.

space $P_{\Theta}^- \backslash G$ is compact; let $K_1 \subset G$ be a compact preimage of it. Given an open compact subgroup J, let J_0 be the intersection of kJk^{-1} , $k \in K_1$, and denote by $\tilde{\mathcal{N}}_J$ the union of the sets $\mathcal{N}_{J_0}k$, $k \in K_1$; it is a neighborhood of Θ -infinity (the closure of Z) in X_{Θ} . If $f \in C^{\infty}(X)^J$ and $z \in \tilde{\mathcal{N}}_J$, there is a $k \in K_1$ such that $zk^{-1} \in \mathcal{N}_{J_0}$, and the function $k \cdot f$ is kJk^{-1} -invariant, in particular J_0 -invariant. Hence we have:

$$f_{\Theta}(z) = k f_{\Theta}(zk^{-1}) = \exp_{\Theta}^*(kf)(zk^{-1}) = k^{-1} \exp_{\Theta}^*(kf)(z).$$

Again, by the compactness of K_1 we may replace $\tilde{\mathcal{N}}_J$ by a smaller neighborhood of Θ -infinity on which $k^{-1}\exp_{\Theta}^*(kf)=f$ for every $k\in K_1$ and $f\in C^{\infty}(X)^J$. This shows that (5.4) holds in a neighborhood N_{Θ} of the orbit closure \mathbf{Z} in the given embedding \mathbf{X} .

Now, as in the proof of Proposition 4.3.3, for two different pairs $(\overline{\mathbf{X}}, \mathbf{Z})$, $(\overline{\mathbf{X}}', \mathbf{Z}')$ we will work with a third smooth toroidal embedding $\overline{\mathbf{X}}''$ which contains open \mathbf{G} -stable subsets \mathbf{U}, \mathbf{U}' properly dominating $\overline{\mathbf{X}}$, resp. $\overline{\mathbf{X}}'$. If the morphisms $f \mapsto f_{\Theta}$ obtained from different orbit closures belonging to Θ -infinity in $\overline{\mathbf{X}}''$ are equal, then so are the morphisms obtained from the pairs $(\overline{\mathbf{X}}, \mathbf{Z})$, $(\overline{\mathbf{X}}', \mathbf{Z}')$. Thus, the existence statement of the theorem is reduced to the case when $\overline{\mathbf{X}} = \overline{\mathbf{X}}'$ but the orbit closures \mathbf{Z} , \mathbf{Z}' are different. Moreover, again by passing to another embedding, we may assume that Θ -infinity is connected; this amounts to saying that the support of the fan of $\overline{\mathbf{X}}$ with the relative interior of the face of \mathcal{V} (the cone of invariant valuations) corresponding to Θ is connected. This includes, for example, the elementary case $\mathbf{X} = \mathbf{G}_{\mathrm{m}} \subset \mathbf{X} = \mathbf{P}^1$, $\mathbf{Z} = \{0\}$, $\mathbf{Z}' = \{\infty\}$ (where $\Theta = \Delta_X = \emptyset$ and the open orbit \mathbf{G}_m also belongs to Θ -infinity).

Under these assumptions, if we consider the equivalence relation between all orbit closures belonging to Θ -infinity generated by: $\mathbf{Z} \sim \mathbf{Z}' \iff \mathbf{Z} \supset \mathbf{Z}'$ or $\mathbf{Z}' \supset \mathbf{Z}$, then all orbits are equivalent, and we are reduced to considering the case of two pairs (\mathbf{X}, \mathbf{Z}) and $(\mathbf{X}, \mathbf{Z}')$ with $\mathbf{Z} \subset \mathbf{Z}'$. We need to prove that the corresponding morphisms: $C^{\infty}(X) \to C^{\infty}(X_{\Theta})$ (temporarily to be denoted by $f \mapsto f_{\Theta}^{Z}$ and $f \mapsto f_{\Theta}^{Z'}$) are identical.

In this case, the validity of the theorem for N_{Θ} = a neighborhood of

In this case, the validity of the theorem for $N_{\Theta} =$ a neighborhood of \mathbb{Z} is evidently weaker than its validity for a neighborhood of \mathbb{Z}' . However, identifying \mathbb{X}_{Θ} with the open \mathbb{G} -orbit in both $N_{\mathbb{Z}}\overline{\mathbb{X}}$ and $N_{\mathbb{Z}'}\overline{\mathbb{X}}$, a neighborhood of Z in X_{Θ} does include a neighborhood of z in fiber(z), for some point z in the open G-orbit on Z'. Thus, the map $f \mapsto f_{\Theta}^{Z}$ satisfies (5.7) in that neighborhood of z in fiber(z), and by the uniqueness statement of Proposition 5.2.1 it has to coincide with the map $f \mapsto f_{\Theta}^{Z'}$. This proves Theorem 5.1.2.

The characterization (uniqueness) statement follows from the uniqueness statement of Proposition 5.2.1.

5.2.2. Setup for the proof of Proposition 5.2.1. Let us fix $z \in Z$. The idea in the proof of the proposition is to replace the action of the monoid

 $A_{X,\Theta}^+$ along the orbit of a point $x \in \text{fiber}(z)$ by the right action of a subtorus of G, or rather a subalgebra of the Hecke algebra of G.

For that reason, choose a Levi subgroup \mathbf{L}_{Θ} of \mathbf{P}_{Θ}^- . We have seen in Proposition 2.7.2 that $\mathcal{Z}(\mathbf{L}_{\Theta})^0$ surjects onto $\mathbf{A}_{X,\Theta}$ and this induces a surjection of positive chambers at the level of Lie algebras (as noted at the end of the proof of Proposition 2.7.2). In what follows we use a,b to denote elements of $\mathcal{Z}(\mathbf{L}_{\Theta})^0$. For $a \in \mathcal{Z}(L_{\Theta})^0$, we write $|a| = \max_{\delta} |\delta(a)|$, the maximum being taken over all negative roots for $\mathcal{Z}(\mathbf{L}_{\Theta})$ with respect to \mathbf{P}_{Θ} , or what is the same, the positive roots with respect to \mathbf{P}_{Θ}^- . We let $\mathcal{Z}(L_{\Theta})^+ \subset \mathcal{Z}(L_{\Theta})^0$ denote the set of elements where $|a| \leq 1$, i.e. $|\chi(a)| \leq 1$ for all negative roots with respect to \mathbf{P}_{Θ} .

Note that $\mathcal{Z}(\mathbf{L}_{\Theta})^0 \twoheadrightarrow \mathbf{A}_{X,\Theta}$ induces $\mathcal{Z}(L_{\Theta})^+ \to A_{X,\Theta}^+$ (this map may not be surjective because of the operation of taking k-points, however).

As elsewhere, we write, for $a \in \mathcal{Z}(L_{\Theta})^+$, that "a is sufficiently deep" in place of "there exists $\varepsilon > 0$ so that, whenever $|a| \leq \varepsilon$, ... " Finally, for $x \in \text{fiber}(z)$ we put

$$x_a := x \cdot a$$
.

5.2.3. Lemma. For every open compact subgroup J with Iwahori factorization with respect to P_{Θ} and P_{Θ}^- , the elements of the Hecke algebra:

$$h_a := 1_{JaJ}, a \in \mathcal{Z}(L_{\Theta}), |a| \leq 1.$$

satisfy: $h_a \star h_b = h_{ab}$ (where b is also an element of $\mathcal{Z}(L_{\Theta})$ satisfying $|b| \leq 1$).

This is straightforward and well-known. As a consequence, the vector space spanned by these elements is a subalgebra \mathcal{H} of the Hecke algebra of G. For now we fix such a subgroup J and the notation of the lemma.

We notice the following fact: for $|a| \leq 1$ and arbitrary b we have $x_b \cdot JaJ = x_{ba}J \subset X_{\Theta}$ because, recalling that x is stabilized by the unipotent radical of \mathbf{P}_{Θ}^- and taking a corresponding Iwahori factorization $J = \underbrace{(J \cap U)}_{J^+} \underbrace{(J \cap P^-)}_{J^-}$, we have $x_b \cdot JaJ = x \cdot bJ^+J^-aJ = x \cdot ba(a^{-1}J^-a)J$, and

we have $a^{-1}J^{-}a \subset J^{-}$ because $|\delta(a^{-1})| \leq 1$ for all negative roots for \mathbf{P}_{Θ}^{-} . Equivalently, if we denote by $h_{a^{-1}}$ the adjoint of h_a (which is equal to the characteristic measure of $J\check{\lambda}(a^{-1})J$) then:

$$h_{a^{-1}} \star 1_{x_b J} = 1_{x_{ab} J}. \tag{5.8}$$

Let us denote by \mathcal{H}' the algebra spanned by the elements $h_{a^{-1}}$. Obviously, the elements $h_{a^{-1}}$ also satisfy the analogous statement of the previous lemma, i.e. $h_{a^{-1}} \star h_{b^{-1}} = h_{a^{-1}b^{-1}}$.

5.2.4. LEMMA. The exponential map is "eventually equivariant" in fiber(Z_1) with respect to the action of the algebra \mathcal{H}' ; that is, there exists a J-stable neighborhood \mathcal{N}_J of Z_1 in fiber(Z_1) $\cdot J$ such that, for any $x \in \mathcal{N}_J$ and any $|b| \leq 1$ we have:

$$\exp_{\Theta}(h_{b^{-1}} \star xJ) = h_{b^{-1}} \star \exp_{\Theta}(xJ).$$

Of course, we have in our notation identified sets with their characteristic functions.

PROOF. This follows from the facts:

- (i) \mathcal{H}' is finitely generated;
- (ii) the eventual equivariance of Proposition 4.3.3;
- (iii) For each $x \in \text{fiber}(Z_1)$ and every $\varepsilon > 0$ the linear span of the characteristic functions of the sets $x_a J$ with $a \le \varepsilon$ is \mathcal{H}' -stable cf. (5.8).

As we have mentioned, the validity of the Proposition is independent of the choice of \mathcal{N}_J , and so we indeed take the neighborhood \mathcal{N}_J so that the prior Lemma is valid.

5.2.5. Inverting elements in the Hecke algebra. Now recall Bernstein's "stabilization theorem" (see [Ber, p 65]): For any smooth representation π and a sufficiently close to zero (how close depends only on J), the action of h_a on π^J is stable, i.e.:

$$\pi^{J} = \ker(\pi(h_a)) \oplus \operatorname{im}(\pi(h_a)). \tag{5.9}$$

This stabilization theorem implies the generalization of "Jacquet's lemma" to the smooth case. Let \mathbf{N}^- denote the unipotent radical of \mathbf{P}_{Θ}^- , and denote by the subscript $_{N^-}$ the Jacquet module (coinvariants) of a representation with respect to N^- . The map

$$\pi^J \to \pi_{N^-}^{J \cap P_\Theta^-}$$

which intertwines the action of h_b with the action²³ of $\pi_{N^-}(b)$ – has for kernel exactly $\ker(\pi(h_a))$, thereby inducing a bijection of $\operatorname{im}(\pi(h_a))$ onto $\pi_{N^-}^{J\cap P_{\Theta}^-}$.

 $\pi_{N^-}^{J\cap P_\Theta^-}$.

The decomposition (5.9) is independent of the choice of a sufficiently small a. It follows that the Hecke elements h_b act invertibly on $\operatorname{im}(\pi(h_a))$ so long as a,b are sufficiently small. We extend this inverse to an operator \tilde{h}_b on π^J by defining \tilde{h}_b to be zero on $\ker(\pi(h_a))$.

Let us denote by \mathfrak{l}_J the inverse of the induced bijection $\operatorname{im}(\pi(h_a)) \to \pi_{N^-}^{J \cap P_{\Theta}^-}$, the "canonical lift" to π^J . In these terms, we have:

$$\tilde{h}_b(f) = \mathfrak{l}_J(\pi_{N^-}(b^{-1})f_{N^-}), f \in \pi^J, \tag{5.10}$$

where f_{N^-} is the image of f in the Jacquet module. Let us note in particular that $\tilde{h}_b(bf)$ is then $\mathfrak{l}_J f_{N^-}$.

²³because we may write $h_b v = (J \cap N^-)b \cdot b^{-1}(J \cap P_{\Theta})b \cdot v$; but $b^{-1}(J \cap P_{\Theta})b$ fixes v for sufficiently small b

5.2.6. The proof of Proposition 5.2.1. We take the subgroups J in our basis to admit Iwahori factorization with respect to P_{Θ} and P_{Θ}^- , and for a given J choose \mathcal{N}_J as in Lemma 5.2.4. Take $x \in \mathcal{N}_J \cap \mathrm{fiber}(z)$; recall that z is stabilized by N^- .

Applying the remarks of §5.2.5 to the representation $\pi = C^{\infty}(X)$, we can define a functional:

$$\Lambda(f) = \lim_{|a| \to 0} (\tilde{h}_a f)(\exp_{\Theta}(x_a J)). \tag{5.11}$$

This functional a priori depends on choices. Our goal is to show that it does not, and that it is in fact G_x -invariant, defining by Frobenius reciprocity the morphism: $e_{\Theta}^*: C^{\infty}(X) \to C^{\infty}(X_{\Theta})$ that we are aiming at. Moreover, the resulting morphism $f \mapsto f_{\Theta}$ has the characterizing property (5.7) if we take $\mathcal{N}_J := a\mathcal{N}_J$ where $a \in \mathcal{Z}(L_{\Theta})$ satisfies $|a| < \varepsilon'$, ε' as in (3) below, and \mathcal{N}_J was the neighborhood from Lemma 5.2.4.

All this follows from the numbered statements following. In what follows, elements a, b are always in $\mathcal{Z}(L_{\Theta})^+$.

(1) The limit (5.11) stabilizes for $|a| < \varepsilon$, where $\varepsilon > 0$ depends only on J.

Indeed, for a small as in Lemma 5.2.4 and any a' with $|a'| \le 1$ we have:

$$h_{aa'}f(\exp_{\Theta}(x_{aa'}J)) =$$

$$\tilde{h}_{a'}\tilde{h}_{a}f(\exp_{\Theta}(h_{a'^{-1}} \star x_{a}J)) =$$

$$h_{a'} \star (\tilde{h}_{a'}\tilde{h}_{a}f)(\exp_{\Theta}(x_{a}J)) =$$

$$\tilde{h}_{a}f(\exp_{\Theta}(x_{a}J)),$$

the first step by (5.8), the second step due to Lemma 5.2.4, the third because $h_{a'}$ and $\tilde{h}_{a'}$ are inverse on the image of h_a .

(2) The limit (5.11) is independent of J, and hence extends to a well-defined functional on $\pi = C^{\infty}(X)$.

Indeed, for $J' \subset J$ we clearly have $\operatorname{im}(\mathfrak{l}_J) \subset \operatorname{im}(\mathfrak{l}_{J'})$, and therefore $\mathfrak{l}_{J'}v = \mathfrak{l}_Jv$ for every $v \in \pi_{N^-}^{J \cap P_{\Theta}^-}$.

(3) For every $f \in \pi^J$ and $|a| < \varepsilon'$ (where $\varepsilon' > 0$ depends only on J) we have:

$$\Lambda(\pi(a)f) = f(\exp_{\Theta}(x_a J)).$$

(This gives (5.7) for a suitable choice of neighborhood, as mentioned above.)

Indeed, by the definition we have (for $|a| < \varepsilon$, where ε is as in (1)):

$$\Lambda(\pi(a)f) = (\operatorname{proj}_{\operatorname{im}(\mathfrak{l}_J)} f)(\exp_{\Theta}(x_a J)),$$

where the projection is with respect to the direct sum (5.9). Hence, if $f \in \text{im}(\mathfrak{l}_J)$ then we are done. For general f, write $\bar{f} := \text{proj}_{\text{im}(\mathfrak{l}_J)} f$

for the projection of f onto the image of the canonical lift; take b small enough so that h_b is stable, and take a with $|a| < \varepsilon$. Then:

$$\Lambda(\pi(ab)f) = \bar{f}(\exp_{\Theta}(x_{ab}J)) = \bar{f}(h_{b^{-1}} \star \exp_{\Theta}(x_{a}J))
= h_{b} \star \bar{f}(\exp_{\Theta}(x_{a}J)) = (\operatorname{proj}_{\operatorname{im}(\mathfrak{l}_{J})}h_{b} \star f)(\exp_{\Theta}(x_{a}J))
= h_{b} \star f(\exp_{\Theta}(x_{a}J)) = f(\exp_{\Theta}(x_{ab}J)),$$

so the statement holds for $\varepsilon' = |b| \cdot \varepsilon$.

(4) Property (3) characterizes Λ among functionals on π which factor through π_{N^-} . (The fact that Λ factors through π_{N^-} follows from (5.10).)

Indeed, let us fix a J in order to show that any Λ' with the same property coincides with Λ on π^J . For any f in the image of π^J and $a \in Z(L_{\Theta})$ with $|a| \leq \varepsilon'$ (where ε' is such that property (3) holds for both Λ and Λ') we have:

$$\Lambda'(\pi(a)f) = f(\exp_{\Theta}(x_a J)) = \Lambda(\pi(a)f).$$

But, by assumption, Λ' factors through π_{N^-} ; since π^J surjects onto $\pi_{N^-}^{P_{\Theta}^- \cap J}$ and $\pi_{N^-}(a)$ acts invertibly on $\pi_{N^-}^{P_{\Theta}^- \cap J}$, it follows that Λ and Λ' coincide there.

(5) The functional Λ is G_x -invariant.

Indeed, for any $g \in G_x$ the functional $f \mapsto \Lambda(\pi(g)f)$ also satisfies (3): We may assume that $g \in L_{\Theta}$ – recall that we are supposing that the stabilizer of x lies between N^- and P_{Θ}^- .

Then:

$$\Lambda(\pi(q)\pi(a)f) = \Lambda(\pi(a)\pi(q)f)$$

(since $a \in \mathcal{Z}(L_{\Theta})$)

$$= (\pi(g)f)(\exp_{\Theta}(x_a(gJg^{-1})))$$
 for small a

(notice that since a belongs to the center of L_{Θ} , $\pi(g)f$ is in the image of the canonical lift for gJg^{-1} if f is in the image of the canonical lift for J, hence the corresponding ε' remains dependent only on the open compact subgroup for such elements)

$$= f(\exp_{\Theta}(x_a g J))$$

(by eventual equivariance; again, the implicit estimate is independent of the particular f)

$$= f(\exp_{\Theta}(x_a J)).$$

This proves Proposition 5.2.1, and hence Theorems 5.1.1, 5.1.2.

We note that the uniqueness statement also follows directly for the following, which is proven like part (4) in the above proof and is more general than the setting of spherical varieties. (It applies if we replace X_{Θ} by $N^-\backslash G$.)

5.2.7. Lemma. If J is an open compact subgroup, and N_{Θ} a J-stable neighborhood of Θ -infinity in X_{Θ} , then the elements of $C_c^{\infty}(X_{\Theta})^J$ which are supported in N_{Θ} generate $C_c^{\infty}(X_{\Theta})^J$ under the Hecke algebra of J-biinvariant functions.

PROOF. Let V denote the quotient of $C_c^{\infty}(X_{\Theta})$ by the G-subspace generated by the elements of $C_c^{\infty}(X_{\Theta})^J$ which are supported on N_{Θ} . If $V^J \neq 0$, then the smooth dual π of V, a subspace of $C^{\infty}(X_{\Theta})$, will also have a nonzero space of J-invariant elements, with the property that, as functions on X_{Θ} , $f(a \cdot x) = 0$ for any x and for $a \in A_{X,\Theta}^+$ "small" enough.

If now x is a point where $f(x) \neq 0$ for some $f \in \pi^J$, N^- denotes the unipotent radical of the parabolic of type P_{Θ}^- contained in the stabilizer of x, and Λ' denotes the functional "evaluation at x", then on one hand $\Lambda'(f) \neq 0$ and on the other $\Lambda'(\pi(a)f') = f'(x_a) = 0$ for every $f' \in \pi^J$ and $a \in \mathcal{Z}(L_{\Theta})$ sufficiently "small" (depending only on x). On the other hand, Λ' factors through the Jacquet module π_{N^-} , the map $\pi^J \to \pi_{N^-}^{J \cap P_{\Theta}^-}$ is surjective, as mentioned above (as a corollary of the stabilization theorem), and a acts invertibly on $\pi_{N^-}^{J \cap P_{\Theta}^-}$, from which we get that $f_{N^-} = \pi_{N^-}(a)f'_{N^-}$ for some $f' \in \pi^J$, and hence

$$\Lambda'(f) = \Lambda'(\pi(a)f') = 0,$$

a contradiction.

5.3. Cartan decomposition and matrix coefficients. The above were just a reformulation of arguments due to Casselman and Bernstein, using the wavefront assumption for the variety X which allowed us to "push" a point on X to infinity using only "anti-dominant" cocharacters (with respect to a parabolic whose open orbit includes this point).

On the other hand, we can also present the above argument in a way where we reduce everything to the (known) case of smooth matrix coefficients on the group. This argument first appeared in [Lag08], [KT08]. The reduction to smooth matrix coefficients is based on the following observation, for which we assume as fixed a Borel subgroup **B** and we consider the "universal Cartan" \mathbf{A}_X of \mathbf{X} as a subvariety of \mathbf{X} as explained in §2.1. We also fix a maximal torus $\mathbf{A} \subset \mathbf{B}$ such that \mathbf{A}_X is an \mathbf{A} -orbit, and denote by A_X^+ the anti-dominant elements of A. We denote by A_X^+ the elements of A_X corresponding to the cone $\mathcal V$ of invariant valuations under the canonical map: $A_X^+ \to \Lambda_X^+$; in the wavefront case, the map: $A_X^+/\mathbf{A}(\mathfrak{o}) \to A_X^+/\mathbf{A}_X(\mathfrak{o})$ is surjective.

5.3.1. Lemma. For any spherical variety X, there is a compact subset $U \subset G$ such that $A_X^+U = X$.

For symmetric spaces see the paper [BO07].

PROOF. For the purpose of this proof, we may replace X by its quotient by $\mathcal{Z}(X)$. After all, if we have found such a set U which works for that

quotient, then any point $x \in X$ differs from an element of A_X^+U by an element of $\mathcal{Z}(X)$, but A_X^+ is $\mathcal{Z}(X)$ -invariant.

Let \bar{X} be a wonderful embedding of X, and let \mathbf{Y} be the toric embedding of \mathbf{A}_X of the Local Structure Theorem 2.3.4. Being a toric variety under the split torus \mathbf{A}_X , it admits a canonical structure over \mathfrak{o} – that is to say, the toric scheme defined by the same combinatorial data – and $\mathbf{Y}(\mathfrak{o}) \cap A_X = A_X^+$.

We may assume that U is open; then, by the Local Structure Theorem again, $\mathbf{Y}(\mathfrak{o})U$ will be open in \bar{X} .

We now prove, for each G-orbit $Z \subset \overline{X}$, that there exists a compact open subset U such that $(\mathbf{Y}(\mathfrak{o}) \cap Z)U = Z$. In particular, taking Z = X, this implies the desired assertion. We proceed by induction on the dimension of Z

The orbit Z of minimal dimension is closed and thus compact. So the assertion is clear in this case.

Take a general orbit Z. By inductive assumption, we may assume that U has been chosen so large that $\mathbf{Y}(\mathfrak{o})U$ contains all orbits Z' of lower dimension. Because $\mathbf{Y}(\mathfrak{o})U$ is open, it contains an open neighborhood of all these Z'. But, because the closure of Z is compact, this means that $Z - \mathbf{Y}(\mathfrak{o})U$ is compact. Enlarging U appropriately, then, we may suppose that $\mathbf{Y}(\mathfrak{o})U$ contains Z, as desired.

In particular, if X is wavefront, there are a finite subset $\{x_1, \ldots, x_n\} \subset \mathring{X}$ and a compact subset $U \subset G$ such that:

$$\cup_i x_i A^+ U = X.$$

Based on this, we can prove:

5.3.2. COROLLARY (The Wavefront Lemma). Let X be a wavefront spherical variety and x_0, x_1, \ldots representatives for the G-orbits on X. Let $o_i : G \to X$ be the corresponding orbit maps. There is a subset $G^+ \subset G$ such that $\cup_i o_i(G^+) = X$ and with the property: For every open $K_1 \subset G$ there is an open $K_2 \subset G$ such that for every i and every $g \in G^+$ we have $o_i(g)K_1 \supset x_iK_2 \cdot g$.

To reformulate, assuming without loss of generality that K_1 and K_2 are subgroups: If we take a double coset K_2gK_1 with $g \in G^+$, its image under o_i consists of a single K_1 -orbit. Informally, the orbit map does not "smear out" a double coset too much. Although in some ways inelegant, this is a very useful tool for reducing questions about X to questions about G.

In particular, o_i defines a map $K_2 \backslash K_2 G^+ K_1 / K_1 \to X / K_1$. In other words, if we enlarge G^+ to be left K_2 -invariant and right K_1 -invariant, the orbit map defines

$$o: K_2 \backslash G^+ / K_1 \longrightarrow X / K_1.$$

Such a result was used by Eskin and McMullen [EM93] – where X is a symmetric variety under a semisimple real group G – in order to establish certain equidistribution and counting results. These arguments partly

inspired an early version of this section and we owe an intellectual debt to these authors.

PROOF OF THE COROLLARY. For notational simplicity, let us assume that there is only one G-orbit represented by $x_0 \in \mathring{X}$, and that the map $A \to A_X$ (hence also the map $A^+ \to A_X^+$, where A^+ denotes anti-dominant elements in A) is surjective. We can take the subset $G^+ = A^+U$ and then it is clearly enough to prove the lemma for $g \in A^+$; indeed, one is reduced to this case by replacing K_1 by the intersection of all of its U-conjugates (which is open since U is compact). By shrinking K_1 further, if necessary, we may assume that it admits a decomposition: $K_1 = N^-M^+$ where N^- belongs to the unipotent radical of the parabolic opposite to B and $M^+ = K_1 \cap B$ is completely decomposable: $M^+ = (M^+ \cap A) \cdot \prod_{\alpha > 0} M_{\alpha}$, with the M_{α} 's belonging to the corresponding root subspaces.

For $g \in A^+$, we have $g^{-1}M^+g \subset M^+$. Therefore, $x_0gK_1 \supset x_0gM^+ \supset x_0M^+g$. Since $x_0\mathbf{B}$ is Zariski open and hence x_0M^+ is open in the Hausdorff topology, we can find a compact open subgroup K_2 of G such that $x_0K_2 \subset x_0M^+$.

We also mention the following strengthening of Lemma 5.3.1, which has been proven under additional assumptions. It generalizes the Cartan and Iwasawa decompositions. In what follows, we assume smooth integral models over \mathfrak{o} for the groups and varieties involved (and reductive, for \mathbf{G} ; in particular, $K = \mathbf{G}(\mathfrak{o})$ is a hyperspecial maximal compact subgroup), and we assume that the point x_0 used to define \mathbf{A}_X as a subset of \mathbf{X} in §2.1 belongs to $\mathring{\mathbf{X}}(\mathfrak{o})$.

5.3.3. Theorem (Under additional assumptions). The set $A_X^+ \subset X$ contains a complete set of representatives for K-orbits on X; elements of A_X^+ which map to distinct elements of Λ_X^+ belong to different K-orbits.

This theorem was proven in [Sak12] using an argument of [GN10], under assumptions that are satisfied at almost every place, if X is defined over a global field. The first part of the theorem was proved in the symmetric case by Delorme and Sécherre [DS11]; see also [BO07] for related results. The implication of Lemma 5.3.1 is obvious, using the fact that wavefront varieties are precisely those for which A_X^+ can be covered by a finite number of orbits of the monoid A^+ .

We are ready to apply the Wavefront Lemma in order to obtain the desired results on asymptotics:

5.3.4. PROPOSITION. Let X be a wavefront spherical variety and x_0, x_1, \ldots representatives for the G-orbits on X. Let $o_i: G \to X$ be the corresponding orbit maps, and assume that K_1, K_2, G^+ are as in Corollary 5.3.2. Let $M: \pi \to C^{\infty}(X)$ be a morphism from a smooth representation of G, and denote by L_i its composition with "evaluation at x_i ", considered as a functional on π . Then for every $v \in \pi^{K_1}$, and for every point $x \in X$ with

 $x = x_i g$, $g \in G^+$ we have $M(v)(x) = \langle \pi(g)v, K_2 * L_i \rangle$, where we denote by $K_2 *$ the operator "convolution by the characteristic measure of K_2 ".

In other words, the morphism M is determined by the smooth functionals K_2*L_i . Using known results about the asymptotics of matrix coefficients, then, we see that we can understand completely the asymptotics of M(x). In principle, this could be used to give a second proof of Theorem 5.1.1; however, as pointed out to us by a referee, this is not a formality. Rather it requires a further analysis of how to choose g when x is near a given "wall" of the compactification. We will not carry this out here, and simply present the Proposition as an alternate way to understand asymptotics, albeit one that is less well adapted to the geometry of the spherical variety.

PROOF. Immediate corollary of the Wavefront Lemma 5.3.2.

Finally, we give the proof of Lemma 2.6.3. We point the reader to §2.6 for the notation.

PROOF OF LEMMA 2.6.3. Let us, for clarity, denote by **Y** the space **X** without the character. Since **Y** is parabolically induced from the variety \mathbf{X}^L , its data $\Lambda_Y^+ \subset \mathfrak{a}_Y^+$, A_Y^+ , its spherical roots etc. are those of the variety \mathbf{X}^L .

We choose a point $x \in \mathbf{X}^L$ with stabilizer $\mathbf{M} \subset \mathbf{L}$ and a torus \mathbf{A} as in §1.7 so that its quotient \mathbf{A}_Y can be identified with the \mathbf{A} -orbit of the point of x. By Lemma 5.3.1, it is enough to show that for every $f \in C^{\infty}(X, \mathcal{L}_{\Psi})$ the support of $f|_{A_{\Psi}^+}$ has compact closure in \overline{X} .

To understand what this support condition means, choose a smooth, complete toroidal embedding $\overline{\mathbf{Y}}$ of \mathbf{Y} which contains $\overline{\mathbf{X}}$, and apply the Local Structure Theorem 2.3.4. The closure of A_Y^+ in \overline{Y} is a compact subset of a smooth toric variety. To describe the subsets which are compact in \overline{X} , we use the "valuation" map: $A_Y^+ \to \Lambda_Y^+$ – notation as in §2.1. The subsets of A_Y^+ which have compact closure in \overline{X} are precisely those whose valuations are contained in a finite number of translates of $\Lambda_X^+ = \Lambda_Y^+ \cap \mathfrak{a}_X^+$.

Recall that the condition defining \mathfrak{a}_X^+ inside of \mathfrak{a}_Y^+ was determined by adding the simple roots of a parabolic opposite to \mathbf{P}^- to the spherical roots of \mathbf{Y} ; thus, a sequence $\check{\lambda}_n$ of elements of Λ_Y^+ which does not belong to any finite number of translates of Λ_X^+ has the property that for some such root α ,

$$\langle \alpha, \check{\lambda}_n \rangle \to \infty.$$
 (5.12)

Fix $f \in C^{\infty}(X, \mathcal{L}_{\Psi})$ which violates the lemma, and let $\check{\lambda}_n$ be such a sequence of elements in the image of the support of f under the valuation map. The **G**-stabilizer of a point on $A_Y \subset Y$ contains the unipotent radical of \mathbf{P}^- . we decompose its Lie algebra into root spaces for the torus **A**. The meaning of (5.12) is that for u in the root space $U_{-\alpha} \simeq k$ in the unipotent radical of P^- :

$$\lim_{n} \check{\lambda}_n(\varpi)^{-1} u \check{\lambda}_n(\varpi) = 0,$$

where ϖ is any uniformizing element. (To be precise, the torus \mathbf{A}_Y does not act on \mathbf{U}_P ; but we saw in the proof of Proposition 2.6.2 that it acts on $\mathbf{U}_{-\alpha}$ through this quotient.)

This implies that f cannot be invariant by a compact open subgroup $U_0 \subset U_{-\alpha}$, because then, for u in that subgroup, denoting for simplicity $\check{\lambda}_n(\varpi)$ by a_n :

$$f(a_n) = f(a_n u) = f((a_n u a_n^{-1}) a_n) = \Psi(a_n u a_n^{-1}) f(a_n),$$

and for large n and some $u \in U_0$, $\Psi(a_n u a_n^{-1}) \neq 1$.

5.4. Mackey theory, the Radon transform and asymptotics. Thus far we have constructed a canonical "asymptotics" map

$$e_{\Theta}: C_c^{\infty}(X_{\Theta}) \longrightarrow C_c^{\infty}(X).$$

Here we will show how, in some instances, this map may be made more explicit.

The basic principle is as follows: In order to write down any "explication", we need, first of all, some common "context" to compare the varieties X and X_{Θ} . Although the varieties X and X_{Θ} look quite different, there is by Lemma 2.8.1 a canonical identification of the space of P_{Θ} -horocycles for them. We shall prove that the adjoint asymptotics map commutes with integration along Θ -horocycles.

The operation of integration along horocycles is a classical concern of integral geometry, at least in certain other contexts, the so-called Radon transform. For example in the case of the quotient of SL_2 by its unipotent subgroup this reduces to the most classical case: integration of a function on an affine plane over lines.

5.4.1. The Radon transform. Let $\Theta \subset \Delta_X$, and recall the horocycle space \mathbf{X}_{Θ}^h defined in §2.8.

Notice that \mathbf{U}_{Θ} acts freely on $\mathbf{X}\mathbf{P}_{\Theta}$ and that, since \mathbf{X} is assumed quasi-affine, the orbits of any unipotent subgroup on \mathbf{X} are all closed. Therefore, we have a well-defined "Radon transform", defined by integration over generic U_{Θ} -orbits:

$$C_c^{\infty}(X) \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^h, \delta_{\Theta}).$$
 (5.13)

Here $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ denotes the space of smooth sections of the complex line bundle δ_{Θ} obtained thus: Let \mathcal{L}_{Θ} be the algebraic line bundle over \mathbf{X}_{Θ}^h whose fiber at a point is the line of invariant volume forms on the corresponding unipotent group (recall there is a defining morphism $\mathbf{X}_{\Theta}^h \to \mathbf{P}_{\Theta} \backslash \mathbf{G}$, which may be thought of as the variety of conjugates of \mathbf{U}_{Θ} ; so to each point of \mathbf{X}_{Θ}^h there is an associated unipotent group.) Now δ_{Θ} is obtained from $\mathcal{L}_{\Theta}^{-1}$ via reduction through $k^{\times} \xrightarrow{|\bullet|} \mathbb{R}_{\perp}^{\times} \hookrightarrow \mathbb{C}^{\times}$.

obtained from $\mathcal{L}_{\Theta}^{-1}$ via reduction through $k^{\times} \xrightarrow{|\bullet|} \mathbb{R}_{+}^{\times} \hookrightarrow \mathbb{C}^{\times}$. Since we twist the action of G on $C_{c}^{\infty}(X)$ by $\sqrt{\eta}$, where η denotes the eigencharacter of the chosen measure on X, we will do the same for $C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})$, in order for R to be equivariant. 5.4.2. Asymptotics via second adjunction. Fix a "standard" parabolic \mathbf{P}_{Θ} and consider the "Levi quotient":

$$\mathbf{X}_{\Theta}^{L} = \mathring{\mathbf{X}} \cdot \mathbf{P}_{\Theta} / \mathbf{U}_{\Theta}. \tag{5.14}$$

Recall that in the wavefront case this is canonically isomorphic to the Levi variety defined in §2.7, which is why we are using the same notation. We will assume that \mathbf{X} is of wavefront type from now on, so that our results on asymptotics hold, and will show how these results translate in terms of Mackey theory and the Radon transform. For simplicity, we denote the Jacquet module of any representation π with respect to U_{Θ} by π_{Θ} .

The Radon transform previously defined gives an identification:

$$C_c^{\infty}(\mathring{X}P_{\Theta})_{\Theta} \stackrel{\sim}{\to} C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta}) \otimes \delta_{\Theta}^{-1}$$
 (5.15)

Here (owing to the normalizing factor in the definition of Jacquet module) we twist the action of L_{Θ} on functions on X_{Θ}^{L} in such a way that $L^{2}(X_{\Theta})$ is unitarily induced from $L^{2}(X_{\Theta}^{L})$, as we are about to explain:

On the space $X_{\Theta}^{L} = \mathring{X}P_{\Theta}/U_{\Theta}$ the measure on X gives rise to an L_{Θ} -eigenmeasure for which the following is true:

$$\int_{\mathring{X}P_{\Theta}} f(x)dx = \int_{X_{\Theta}^{L}} \int_{U_{\Theta}} f(ux)dudx.$$

This depends on the choice of Haar measure on U_{Θ} . The character by which L_{Θ} acts on this measure is $\delta_{\Theta}\eta$ (recall that η is the eigencharacter of the measure on X). Thus, we need to twist the unnormalized action of L_{Θ} on functions by $(\eta\delta_{\Theta})^{\frac{1}{2}}(l)$ in order to obtain a unitary representation.

Another way to describe this twisting is the following: if we identify X_{Θ}^{L} as a subvariety of X_{Θ} as before, and $g \in P_{\Theta}^{-}$ with image $l \in L_{\Theta}^{-}$, then for a function f a function on X_{Θ} we have:

$$l \cdot (f|_{X_{\Theta}^{L}}) := \delta_{\Theta}^{\frac{1}{2}}(l)(g \cdot f)|_{X_{\Theta}^{L}}.$$
 (5.16)

(The twisting by $\sqrt{\eta}$ is already contained in the G-action on X_{Θ} .)

We leave it to the reader to check that there is a choice of invariant measure, valued in the line bundle defined by δ_{Θ} , over $P_{\Theta}^- \backslash G$ such that:

$$L^2(X_{\Theta}) = I_{P_{\Theta}^-}^G \left(L^2(X_{\Theta}^L) \right),$$

where $I_{P_{\Theta}^-}^G$ denotes unitary induction with respect to that measure.

On the other hand, when we consider \mathbf{X}_{Θ}^{L} as a subvariety of \mathbf{X}_{Θ}^{h} and taking into account the twisting of the action we have by restriction of sections a map:

$$C_c^{\infty}(X_{\Theta}, \delta_{\Theta}) \twoheadrightarrow C^{\infty}(X_{\Theta}^L, \delta_{\Theta}) \otimes \delta_{\Theta}^{-1},$$

where $C^{\infty}(X_{\Theta}^{L}, \delta_{\Theta})$ denotes smooth sections of the restriction of the above line bundle to X_{Θ}^{L} . We have a (non-canonical) isomorphism of L_{Θ} -representations:

$$C^{\infty}(X_{\Theta}^{L}, \delta_{\Theta}) \simeq C^{\infty}(X_{\Theta}^{L}) \otimes \delta_{\Theta}.$$

We will denote the representation $C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta}) \otimes \delta_{\Theta}^{-1}$ which appears in (5.15) by $C_c^{\infty}(X_{\Theta}^L)'$; it is non-canonically isomorphic to $C_c^{\infty}(X_{\Theta}^L)$.

Hence we get an embedding:

$$m_{\Theta}: C_c^{\infty}(X_{\Theta}^L)' \hookrightarrow C_c^{\infty}(X)_{\Theta}.$$
 (5.17)

The same considerations for X_{Θ} give an embedding, to be denoted by the same symbol:

$$m_{\Theta}: C_c^{\infty}(X_{\Theta}^L)' \hookrightarrow C_c^{\infty}(X_{\Theta})_{\Theta}.$$
 (5.18)

Recall that the quotients $\mathring{X}P_{\Theta}/U_{\Theta}$ and $\mathring{X}_{\Theta}P_{\Theta}/U_{\Theta}$ are both canonically isomorphic to X_{Θ}^{L} by (2.14).

The analysis of the Jacquet module of $C_c^{\infty}(X)$ in terms of P_{Θ} -orbits is usually called "Mackey theory" or "the geometric lemma", and therefore we will call the embeddings m_{Θ} "Mackey embeddings".

The following result *in principle* identifies the asymptotics (but is rather hard to use in practice):

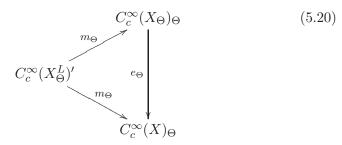
5.4.3. Proposition. (1) Any G-morphism $M: C_c^{\infty}(X_{\Theta}) \to \pi$ is uniquely determined by its "Mackey restriction", i.e. by the induced map:

$$C_c^{\infty}(X_{\Theta}^L)' \to \pi_{\Theta}$$
 (5.19)

obtained as the composition of the Mackey embedding (5.18) with the map of Jacquet modules induced by M. In other words, the Mackey restriction map is injective (in fact, bijective):

$$\operatorname{Hom}_G(C_c^{\infty}(X_{\Theta}), \pi) \to \operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L)', \pi_U).$$

(2) The diagram



commutes, where the slanted maps are as above.

PROOF. Given that $C_c^\infty(X_\Theta) = I_{P_\Theta^-}^G C_c^\infty(X_\Theta^L)$, the first statement is precisely the statement of the second adjunction of Bernstein [Ber, p61]: for any smooth representations σ of L_Θ and π of G, restriction with respect to the Mackey embedding $\sigma \hookrightarrow I_{P_\Theta^-}^G(\sigma)_\Theta$ gives rise to a bijection:

$$\operatorname{Hom}_G(I_{P_{\Theta}^-}^G(\sigma), \pi) \to \operatorname{Hom}_{L_{\Theta}}(\sigma, \pi_{\Theta}).$$

To check commutativity, note that the space $C_c^{\infty}(X_{\Theta}^L)'$ is generated over L_{Θ} by the images of functions $f \in C_c^{\infty}(\mathring{X}P_{\Theta})$ supported "close enough to

 Θ -infinity", i.e. in a "good" neighborhood with respect to their stabilizer. (Indeed, any given compactly supported function on X_{Θ}^L can be translated by the center of L_{Θ} in the desired direction.) It suffices to check commutativity on the images of those elements, which follows from the compatibility (in the sense of Lemma 2.8.1) of the isomorphism: $\mathring{X}P_{\Theta}/U_{\Theta} \xrightarrow{\sim} \mathring{X}P_{\Theta}/U_{\Theta}$ with the exponential map.

5.4.4. Asymptotics and the Radon transform. The dual asymptotics map

$$e_{\Theta}^*: C^{\infty}(X) \to C^{\infty}(X_{\Theta}),$$

when restricted to compactly supported functions, does not, in general, preserve compact support. However, elements in its image are compactly supported along unipotent orbits:

5.4.5. PROPOSITION. There is an affine equivariant embedding $\mathbf{X}_{\Theta} \hookrightarrow \mathbf{Y}$ such that for every $\Phi \in C_c^{\infty}(X)$ the support of $e_{\Theta}^*\Phi$ has compact closure in Y.

This is a result of Bezrukavnikov and Kazhdan in the group case [**BK15**, Prop 7.1], and we prove it for the general case in the next subsection extending their argument. Given that orbits of unipotent groups on affine varieties are closed, it follows that the support of $e_{\Theta}^*\Phi$ intersects each unipotent orbit on a compact set, and hence Radon transform converges absolutely on $e_{\Theta}^*(C_c^{\infty}(X))$. Using this, we will explicate here the dual asymptotics map:

5.4.6. Proposition. The square:

$$C_{c}^{\infty}(X) \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})$$

$$\downarrow e_{\Theta}^{*} \qquad \qquad \downarrow = \qquad (5.21)$$

$$C^{\infty}(X_{\Theta}) \supset e_{\Theta}^{*}(C_{c}^{\infty}(X)) \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})$$

is commutative. Here the right vertical arrow is induced from the canonical identification (2.14) of P_{Θ} -horocycles on X and X_{Θ} .

In principle, one can invert R on the parabolically induced spherical variety X_{Θ} , by using the theory of intertwining operators. We will do this in §15.4.1.

PROOF. To avoid being overwhelmed by Θ -subscripts, we write $\mathbf{Q} = \mathbf{P}_{\Theta}$, with Levi factorization $\mathbf{Q} = \mathbf{L}\mathbf{U}$; let \mathbf{U}^- be opposite to \mathbf{U} . Note that $\mathbf{L} = \mathbf{L}_{\Theta}$ with prior notation.

Fix a compact subgroup J that admits an Iwahori factorization $J = J_{U^-}J_LJ_U$; we have written J_S for $J \cap S$ whenever S is any subgroup of G. We verify the commutativity at the level of J-invariants.

Note that, for any $x \in X$, and any $a \in \mathcal{Z}(L_{\Theta})$ that is sufficiently large (with respect to the roots of \mathbf{P}_{Θ}), we have

$$x(aJ_{U^{-}}a^{-1}) \subset xJ_{Q} \Rightarrow x_{a}J = xaJ_{U^{-}}J_{Q} = xaJ_{U^{-}}a^{-1}aJ_{Q}$$

$$\subset xJ_QaJ_Q\subset x_aa^{-1}J_QaJ_Q\subset x_aJ_Q,$$

where we have written $x_a := xa$ for short. From this we deduce that $\int_U f(x_aku)du$ is independent of $k \in J$, whenever f is itself J-invariant. In particular,

$$\int_{U} f(x_{a}u) = \int_{U} ([JuJ] \star f)(x_{a})du,$$

whenever a is sufficiently large (how large depends on x and J); here [JuJ] is the J-bi-invariant measure of total mass 1 supported on JuJ. The same conclusion holds if $x \in \mathring{X}_{\Theta}$ and f is a J-invariant function on X_{Θ} .

Now let us compare the Radon transforms of $f \in C_c^{\infty}(X)$ and $f_{\Theta} := e_{\Theta}^*(f) \in C_R^{\infty}(X_{\Theta})$. Fix a J-good neighborhood N_{Θ} of Θ -infinity in X, let $x \in N_{\Theta} \cap X$ and let x'J be the corresponding J-orbit on X_{Θ} , i.e. the image under the exp-map. Then, if N_{Θ} is taken sufficiently small the orbits xaJ and x'aJ are also matching under the exp-map for $a \in \mathcal{Z}(L_{\Theta})$ sufficiently large. Indeed, it is enough to show this when J is replaced by any smaller subgroup J' with Iwahori factorization, e.g. a subgroup J' such that $xJ' = xJ'_Q$; then xaJ' = xJ'aJ', and the claim follows from the eventual equivariance (statement (2)) of Proposition 4.3.3 and the fact that the characteristic measures of the sets J'aJ', with a anti-dominant, form a finitely generated subalgebra of the Hecke algebra (so by choosing a finite number of generators, we can find a neighborhood N_{Θ} where the exp-map is equivariant with respect to that subalgebra).

According to what we noted above, for $a \in \mathcal{Z}(L_{\Theta})$ sufficiently large we have:

$$\int_{U} f_{\Theta}(x_a'u) du = \int_{U} ([JuJ] \star f_{\Theta})(x_a') du = \int_{U} ([JuJ] \star f)(x_a) du = \int_{U} f(x_a u) du,$$

the middle step because of the equivariance of the asymptotics map.

Write for $\mathscr{H}(G,J)$ the Hecke algebra of G with respect to J. We have seen that the image of $C_c^{\infty}(X)^J$ under $e_{\Theta}^*R - Re_{\Theta}^*$ is a $\mathscr{H}(G,J)$ -invariant subspace $W \subset C^{\infty}(X_{\Theta}^h, \delta_{\Theta})^J$ with the following property: for any $y \in X_{\Theta,Q}$ there exists a sufficiently positive element z of $A_{X,\Theta}$ (notice that the action of $\mathbf{A}_{X,\Theta} \simeq \mathscr{Z}(\mathbf{X}_{\Theta})$ on \mathbf{X}_{Θ} gives rise to a \mathbf{G} -commuting action on $\mathbf{X}_{\Theta}^h = (\mathbf{X}_{\Theta})_{\mathbf{Q}}^h$, as well) such that $f(z \cdot y) = 0$ for every $f \in W$.

As is the case for X_{Θ} , though, the characteristic functions of the sets $z \cdot yJ$, where y varies over X_{Θ}^h and $z \in A_{X,\Theta}$ is a sufficiently positive element (where the notion of "sufficiently positive" depends on y) generate $C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta})^J$ over $\mathscr{H}(G, J)$. Thus, any $f \in C^{\infty}(X_{\Theta}^h, \delta_{\Theta})^J$ is given by the analogous formula of (5.11): $f(y) = \lim_{z \to 0} (\tilde{h}_z f)(z \cdot y)$ for every $f \in C^{\infty}(X_{\Theta}^h, \delta_{\Theta})^J$.

Therefore,
$$W = 0$$
.

5.5. Support of elements in $e_{\Theta}^*(C_c^{\infty}(X))$. We now prove Proposition 5.4.5.

Let $\mathcal{X}(\mathbf{X})^+$ denote the weights of regular Borel eigenfunctions on \mathbf{X} ; in other words, $\mathcal{X}(\mathbf{X})^+$ is the set of highest weights in the decomposition of k[X] into a multiplicity-free direct sum of highest weight modules. This monoid induces a partial order \succeq on the set $\mathfrak{a}_X = \mathfrak{a}_X = \mathcal{X}(\mathbf{X})^* \otimes \mathbb{Q}$ by: $\mu \succeq \lambda \iff \langle \mu, \chi \rangle \geq \langle \lambda, \chi \rangle$ for all $\chi \in \mathcal{X}(\mathbf{X})^+$. We will extend this order to $\mathfrak{a} = \mathcal{X}(\mathbf{B})^* \otimes \mathbb{Q}$ by pull-back $(\mu \succeq \lambda \iff \bar{\mu} \succeq \bar{\lambda}, \text{ where } \bar{\mu}, \bar{\lambda} \text{ denote the }$ images in \mathfrak{a}_X), and to the tori A, A_X via the natural maps (which we denote by "log"):

$$\log : A \to \mathcal{X}(\mathbf{B})^* \subset \mathfrak{a},$$

$$\log : A_X \to \mathcal{X}(\mathbf{X})^* \subset \mathfrak{a}_X.$$
(5.22)

Since $\mathcal{X}(\mathbf{X})^+$ is contained in the set of dominant weights of $\mathcal{X}(\mathbf{B})$, the set of elements $\succeq 0$ in \mathfrak{a} contains the coroot cone. Finally, we will use, as we have done so far, the notation \mathfrak{a}^+, A^+ (resp. $\mathring{\mathfrak{a}}^+, \mathring{A}^+$ for anti-dominant (resp. strictly anti-dominant) elements, and similarly for \mathfrak{a}_X , $\mathfrak{a}_{X,\Theta}$, etc. (with respect to the root system of X). We are still in the wavefront case, so we have a surjection: $\mathfrak{a}^+ \to \mathfrak{a}_X^+$.

Choose a splitting of the exact sequence of groups:

$$1 \to k^{\times} \to k(\mathbf{X})^{(\mathbf{B})} \to \mathcal{X}(\mathbf{X}) \to 1,$$
 (5.23)

denoted by $\mathcal{X}(\mathbf{X}) \ni \chi \mapsto f_{\chi} \in k(\mathbf{X})^{(\mathbf{B})}$. (Such a splitting exists because $\mathcal{X}(\mathbf{X})$ is free.)

Having fixed a maximal torus $A \subset B$, we choose a special maximal compact subgroup $K \subset G$ stabilizing a special point in the apartment of A in the building of G. Hence, K acts transitively on the points of the flag variety of G and satisfies the Cartan decomposition: $G = K\Lambda^+K$, where Λ^+ , denotes the anti-dominant elements in $\Lambda := \mathcal{X}(\mathbf{B})^*$, considered as elements of G via any choice of uniformizer in F, which leads to a splitting of the log map (5.22).

Define, for every $\chi \in \mathcal{X}(\mathbf{X})^+$, a K-invariant "norm" on X:

$$||x||_{\chi} := \max_{k \in K} |f_{\chi}(xk)|.$$

5.5.1. Lemma. The norms $||x||_{\chi}$ have the following properties:

- (1) $||x||_{\chi_1 + \chi_2} \le ||x||_{\chi_1} \cdot ||x||_{\chi_2};$ (2) For any $a \in A^+$, $||xa||_{\chi} \le |\chi(a)| \cdot ||x||_{\chi}.$

PROOF. The first property is obvious. The second follows by viewing the points of X, via the evaluation maps, as elements in the dual V_{χ}^{*} of the highest weight module V_{χ} spanned by f_{χ} . Then V_{χ}^* can be realized as sections of the line bundle over the flag variety of G induced from the character χ of **B**, that is: regular functions on **G** satisfying $F(bq) = \chi(b)F(q)$ for $b \in \mathbf{B}$, and for such a section F the norm we previously defined is equivalent to the norm: $||F|| = \max_{k \in K} |F(k)|$. From this realization it is easy to see that $||a \cdot F|| \leq |\chi(a)||F||$ for $a \in B$ dominant, where by $a \cdot F$

we denote the right regular action of a on F: This follows from the Bruhat-Tits comparison ([HR10, 10.2.1], cf. [Car79, p. 148]) of the Iwasawa and Cartan decompositions: $Ka_{\lambda}K \subset \cup_{\mu \geq \lambda}Ua_{\mu}K$, where $\mu \geq \lambda$ means that $\lambda - \mu$ belongs to the coroot cone (and recall that every $\chi \in \mathcal{X}(\mathbf{X})^+$ is ≥ 0 on the coroot cone). An equivalent way to state that is that the norms defined extend to K-invariant norms on V_{χ}^* , and the spectral norm of $a \in B$ is $|\chi(a)|$.

For every $\mu \in \mathfrak{a}_X$, let $X_{\succeq \mu}$ denote the set of elements $x \in X$ with $\|x\|_{\chi} \leq q^{-\langle \chi, \mu \rangle}$ for all $\chi \in \mathcal{X}(\mathbf{X})^+$; obviously, by the first statement of the previous lemma, only a finite number of χ 's is needed to define this set. These sets have the following properties:

- (1) $X_{\succeq \mu}$ is a set with compact closure in the points of the affine closure $\overline{\mathbf{X}}^{\mathrm{aff}} := \operatorname{spec} k[\mathbf{X}]$; indeed, the K-translates of a generating set of \mathbf{B} -eigenfunctions generate the coordinate ring $k[\mathbf{X}]$ (since BK = G), and therefore any finite set of generators of $k[\mathbf{X}]$ has bounded evaluations on elements of $X_{\succeq \mu}$.
- (2) The sets $X_{\succeq \mu}$ define a filtration of X by K-invariant sets, decreasing with respect to the \succeq ordering on \mathfrak{a}_X (i.e. $X_{\succeq \mu} \subset X_{\succeq \lambda}$ if $\lambda \preceq \mu$).
- 5.5.2. LEMMA. For any $\lambda \in \Lambda^+$, $\mu \in \mathfrak{a}_X$, we have:

$$X_{\succeq \mu} \cdot K a_{\lambda} K \subset X_{\succeq \mu + \bar{\lambda}},$$

where $\bar{\lambda}$ is the image of λ in \mathfrak{a}_X .

PROOF. In fact, since the sets are K-invariant, this reduces to the statement: $X_{\succeq \mu} \cdot a_{\lambda} \subset X_{\succeq \mu + \bar{\lambda}}$. This follows immediately from the second statement of the previous lemma.

Now we compare these sets for X and X_{Θ} . First of all, we clarify that we will use for the definition of the sets $X_{\Theta,\succeq\mu}$ only the set of weights $\mathcal{X}(\mathbf{X})^+$, despite the fact that X_{Θ} might have more regular functions; this corresponds to the affine embedding of \mathbf{X}_{Θ} obtained from the "affine degeneration" of the affine closure of \mathbf{X} , cf. §2.5. This embedding will play the role of \mathbf{Y} in the proof of Proposition 5.4.5. Secondly, we choose a splitting of the sequence analogous to (5.23) for \mathbf{X}_{Θ} as follows:

Let \mathbf{X}^a denote the affine closure of \mathbf{X} (i.e. $\mathbf{X}^a = \operatorname{spec} k[\mathbf{X}]$), and consider the affine degeneration \mathscr{X}^a of §2.5 over the base $\overline{\mathbf{A}_{X,ss}}$ (same notation as in §2.5). Notice that $\mathscr{X}^a \times_{\overline{\mathbf{A}_{X,ss}}} \mathbf{A}_X \simeq \mathbf{X}^a \times \mathbf{A}_X$ canonically, so we can define the regular function $F_{\chi}: (x,a) \mapsto f_{\chi}(x)\chi(a)$ on it. Choose an affine embedding $\overline{\mathbf{A}_X}$ of \mathbf{A}_X where the kernel \mathbf{A}_1 of $\mathbf{A}_X \to \mathbf{A}_{X,ss}$ acts freely, and such that $\overline{\mathbf{A}_X}/\mathbf{A}_1 = \overline{\mathbf{A}_{X,ss}}$. This corresponds to a lifting of cocharacters from $\mathbf{A}_{X,ss}$ to \mathbf{A}_X , and gives rise to a base change:

$$\widetilde{\mathscr{X}^a} = \mathscr{X}^a \times_{\overline{\mathbf{A}_{X,ss}}} \overline{\mathbf{A}_X}.$$

It follows immediately from the definition of \mathscr{X}^a that the function F_{χ} extends to a regular function on \mathscr{X}^a , and its restriction to \mathbf{X}_{Θ} will be denoted by f_{χ}^{Θ} . By the compatibility between the normal bundle and the affine degenerations (Proposition 2.5.2), it can be seen that the correspondence between f_{χ} and f_{χ}^{Θ} also arises from the almost canonical identification of Borel orbits (2.5).

In the case of X_{Θ} we obviously have:

5.5.3. LEMMA. For every $\mu \in \mathfrak{a}_X$ and every $a \in A_{X,\Theta}$ we have:

$$a \cdot X_{\Theta, \succeq \mu} = X_{\Theta, \succeq \mu + \log a}$$

PROOF. Indeed, for every $\chi \in \mathcal{X}(\mathbf{X})^+$, the function f_{χ}^{Θ} is χ -equivariant with respect to the action of $\mathbf{A}_{X,\Theta}$: $f_{\chi}^{\Theta}(a \cdot x) = \chi(a) f_{\chi}^{\Theta}(x)$.

Now we discuss compatibility of these sets with the exponential map:

5.5.4. LEMMA. For every $\mu \in \mathfrak{a}_X$, and a K-good neighborhood N_{Θ} of Θ -infinity, and for all κ sufficiently deep in $\mathfrak{a}_{X,\Theta}^+$, the sets $X_{\succeq \mu + \kappa} \cap N_{\Theta}$ and $X_{\Theta,\succ \mu + \kappa} \cap N_{\Theta}$ coincide.

Recall that, by abuse of language, we are treating here N_{Θ} as a subset of both X and X_{Θ} , when we really mean that N_{Θ}/K is identified as a subset of both X/K and X_{Θ}/K .

PROOF. Let $\overline{\mathbf{X}}$ be any simple smooth toroidal embedding of \mathbf{X} , and let $\mathbf{Z} \subset \overline{\mathbf{Z}}$ be its closed \mathbf{G} -orbit. The corresponding normal bundle degeneration $\overline{\mathcal{Z}}^n \to \mathbf{G}^I_{\mathrm{m}}$ (discussed in §2.5) has the following property, essentially by construction: For every distinguished (cf. §4.3.1) p-adic analytic map ϕ from the k-points of the normal bundle $N_{\mathbf{Z}}\overline{\mathbf{X}}$ to \overline{X} , every point $x \in N_{\mathbf{Z}}\overline{\mathbf{X}}(k)$, and every "strictly positive" (i.e. positive on every coordinate) cocharacter $\check{\lambda}: \mathbf{G}_{\mathrm{m}} \to \mathbf{G}^I_{\mathrm{m}}$, we have:

$$\lim_{m \to 0} (\phi(\check{\lambda}(m^{-1})x), \check{\lambda}(m)) = x. \tag{5.24}$$

(Recall that the algebrogeometric meaning of $\lim_{m\to 0}$ is that the map extends from \mathbf{G}_{m} to \mathbf{G}_{a} .) While a result of type (5.24) can be proved in the setting of a normal crossing divisor on a general variety, in our current setting it can be proved directly using the Local Structure Theorem 2.3.4 to reduce to the case when $\overline{\mathbf{X}}$ is an affine space and $\overline{\mathbf{Z}}$ is an intersection of coordinate hyperplanes and then computing explicitly.

The property of "distinguished" maps to preserve **G**-orbits is actually irrelevant for this, but if ϕ also preserves **G**-orbits then both sides of the limit will be in the open set denoted by \mathcal{X}^n in §2.5 if the right-hand side is, and by Proposition 2.5.2 they can be regarded as points on the affine degeneration \mathcal{X}^a .

Hence, given such a ϕ , for every $x \in X_{\Theta}$, considered as a subvariety of \mathscr{X}^a , we have:

$$x = \lim_{a \in A_{X,\Theta}^+} (\phi(ax), a^{-1}). \tag{5.25}$$

Now, viewing \mathscr{X}^a as an affine $\mathbf{G} \times \mathbf{A}_X$ -spherical variety, we can define the sets $\mathscr{X}^a_{\succeq \mu}$ as in the case of \mathbf{X} and \mathbf{X}_{Θ} . Here μ can be in the sum $\mathfrak{a}_X \oplus \mathfrak{a}_X$, but we will be interested in the antidiagonal of \mathfrak{a}_X only, so we assume that $\mu \in \mathfrak{a}_X$. The definition is through the functions F_{χ} as above (and their multiples by characters of \mathbf{A}_X extending to the base $\overline{\mathbf{A}_X}$), which specialize to both f_{χ} and f_{χ}^{Θ} .

For any fixed $\mu \in \mathfrak{a}_X$, the set $\mathscr{X}^a_{\succeq \mu}$ is open and compact (as in the discussion prior to Lemma 5.5.2), it intersects X_{Θ} on the set $X_{\Theta,\succeq \mu}$, and it intersects $X \times \{a\}$ on the set $X_{\succeq \mu - \log a} \times \{a\}$. This, together with (5.25), means that for $\log a$ sufficiently deep in $\mathfrak{a}^+_{X,\Theta}$, the point:

$$(\phi(ax), a^{-1})$$

belongs to $X_{\succeq \mu + \log a} \times \{a^{-1}\}$ if and only if $x \in X_{\Theta,\succeq \mu}$.

By Lemma 5.5.3, the point y = ax belongs to $X_{\Theta, \succeq \mu + \log a}$ if and only if $x \in X_{\Theta, \succeq \mu}$. We deduce that:

$$\phi(X_{\Theta,\succeq \mu + \log a}) = X_{\succeq \mu + \log a}$$

as long as log a is sufficiently deep in $\mathfrak{a}_{X,\Theta}^+$. In particular, the intersections of the sets $X_{\Theta,\succeq \mu + \log a}/K$ and $X_{\succeq \mu + \log a}/K$ with N_{Θ}/K (identified as a subset of both X/K and X_{Θ}/K via ϕ) coincide.

We are now ready to prove Proposition 5.4.5 on the support of elements of the form $e_{\Theta}^*\Phi$. As mentioned above, **Y** will be the affine embedding of \mathbf{X}_{Θ} such that the set of highest weights of $k[\mathbf{Y}]$ is $\mathcal{X}(\mathbf{X})^+$. For every $\lambda \in \Lambda^+$, let $\mathcal{H}_{\geq \lambda}$ denote the set of elements of the (full) Hecke algebra of G supported in the union of cosets $Ka_{\mu}K$ of the Cartan decomposition, where $\mu - \lambda$ belongs to the coroot cone. As in [**BK15**, Lemma 8.8], one proves:

5.5.5. LEMMA. Given an open compact subgroup J, there is a finite subset S of $A_{X,\Theta}$ such that for all $f \in C_c^{\infty}(X_{\Theta})^J$ and every $a \in A^+$ with image $\bar{a} \in A_{X,\Theta}^+$ there is an $F \in C_c^{\infty}(X_{\Theta})^J$ whose support lies in $\bar{a}S$ supp f, and a Hecke element $h \in \mathcal{H}_{\geq \log a'}^J$, such that $f = h \star F$. Here $\log a'$ denotes the dual weight of $\log a$ ($= -w \log a$, where w is the longest Weyl group element).

The argument is identical to that of loc.cit, and we omit it. We will now prove a stronger and more precise statement than that of Proposition 5.4.5: Let $\Phi \in C^{\infty}(X)^J$ whose support lies in $X_{\succeq \mu}$ for some μ in \mathfrak{a}_X . If S is as in the previous Lemma and $\lambda \in \mathfrak{a}_X$ is such that $\lambda + \log S \leq \mu$ then we will prove that $e^*_{\Theta}\Phi$ is supported in $X_{\Theta,\succeq \lambda}$.

Indeed, let $f \in C_c^{\infty}(X_{\Theta})^J$ be supported in the complement of $X_{\Theta,\succeq \lambda}$. Choose a J-good neighborhood N_{Θ} of Θ -infinity such that the sets $X_{\succeq \mu + \kappa}$ and $X_{\Theta,\succeq \mu + \kappa}$ coincide for κ deep enough in $\mathfrak{a}_{X,\Theta}^+$, according to Lemma 5.5.4. Choose an element $a \in A^+$ with image $\bar{a} \in \mathring{A}_{X,\Theta}^+$ such that $\bar{a}S \operatorname{supp}(f) \subset N_{\Theta}$, where S is as in Lemma 5.5.5. According to that lemma, $f = h \star F$, where F is supported in $\bar{a}S \operatorname{supp}(f)$ and $h \in \mathcal{H}_{\geq \log a'}^J$. Then the support of F does

not meet $X_{\Theta,\succeq \mu + \log \bar{a}}$, while by Lemma 5.5.2 the support of $h^{\vee} \star \Phi$ (where h^{\vee} denotes the dual Hecke element of h, which belongs to $\mathcal{H}^{J}_{\geq \log a}$) is contained in $X_{\succeq \mu + \log \bar{a}}$. If a has been chosen so that $\log \bar{a}$ is sufficiently deep in $\mathfrak{a}^{+}_{X,\Theta}$ (which we may assume), then $\sup e_{\Theta}F$ does not meet $X_{\succeq \mu + \log \bar{a}}$ by Lemma 5.5.4. Therefore:

$$\langle e_{\Theta}^* \Phi, f \rangle = \langle e_{\Theta}^* \Phi, h \star F \rangle = \langle e_{\Theta}^* h^{\vee} \star \Phi, F \rangle = 0.$$

6. Strongly tempered varieties

It is clear that, if G is a compact group and X a compact homogeneous X-space, a Plancherel formula for $L^2(X)$ is a formal consequence of a Plancherel formula for $L^2(G)$, together with an understanding of which representations are X-distinguished. Indeed, this is so even if we suppose only that point stabilizers are compact.

What is perhaps surprising is that a corresponding phenomenon – the Plancherel measure for X is determined by a Plancherel formula for $L^2(G)$ – persists even when point stabilizers are noncompact, so long as they are "not too big." (As a reference for the Plancherel formula for $L^2(G)$ itself, for G a p-adic group, see Waldspurger's paper [Wal03].)

We term the spherical varieties for which this is so *strongly tempered*, and discuss their general theory. As a consequence of our general discussion, we will prove a conjecture of Ichino and Ikeda, as well as a conjecture of Lapid and Mao. (For the latter, we give a short proof of a Whittaker-Plancherel formula.)

6.1. Abstract Plancherel decomposition. A Plancherel formula for $L^2(X)$ is, by definition, an isomorphism $L^2(X) \cong \int_{\hat{G}} \mathcal{H}_{\pi}\mu(\pi)$ of unitary G-representations; here \hat{G} denotes the unitary dual of G and the Hilbert space \mathcal{H}_{π} is π -isotypic, i.e. isomorphic to a direct sum (in our case, finite) of copies of the unitary representation π . We describe μ as being a Plancherel measure $\mathcal{L}^2(X)$; any other Plancherel measure μ' belongs to the same measure class as μ .

We recall from (cf. [Ber88]) how to describe such a decomposition, and more generally any morphism from $L^2(X)$ to a direct integral of unitary representations. The subspace $C_c^{\infty}(X)$ of smooth, compactly supported, functions is countable-dimensional, so there is a family of morphisms

$$L_{\pi}: C_c^{\infty}(X) \to \mathcal{H}_{\pi}$$

(defined for μ -almost every π) such that $\alpha \mapsto L_{\pi}(\Phi)$ represents Φ for every $\Phi \in C_c^{\infty}(X)$.

 $^{^{24}}$ In some treatments, a Plancherel formula is described by a collection of measures ν_n , for $n \in \{1, 2, ..., \} \cup \{\infty\}$, together with an isomorphism $L^2(X) \cong \sum_n \int \pi^{\oplus n} \nu_n(\pi)$. In this language, μ is in the same measure class as $\sum_n \nu_n$.

By pull-back, we obtain seminorms $\| \bullet \|_{\pi}$ on $C_c^{\infty}(X)$; the spaces \mathcal{H}_{π} can be identified with the completions of $C_c^{\infty}(X)$ with respect to the seminorms $\| \bullet \|_{\pi}$. In particular, the spaces \mathcal{H}_{π} are completions of the spaces of π -coinvariants:

$$C_c^{\infty}(X)_{\pi} := (\operatorname{Hom}_G(C_c^{\infty}(X), \pi))^* \otimes \pi. \tag{6.1}$$

Notice that there is a canonical quotient map: $C_c^{\infty}(X) \to C_c^{\infty}(X)_{\pi}$ (surjectivity follows from the irreducibility of π).

Therefore, by a *Plancherel decomposition* of $L^2(X)$ (or a quotient thereof), we will mean the following set of data: a positive measure²⁵ μ on \hat{G} ; and a measurable set of invariant, non-zero seminorms $\| \bullet \|_{\pi}$ on the spaces $C_c^{\infty}(X)_{\pi}$, for μ -almost every π , so that for every $\Phi \in C_c^{\infty}(X)$:

$$\|\Phi\|^2 = \int_{\hat{G}} \|\Phi\|_{\pi}^2 \mu(\pi). \tag{6.2}$$

(Here we denoted the image of Φ in $C_c^{\infty}(X)_{\pi}$ again by Φ .)

The data μ , $(\| \bullet \|_{\pi})_{\pi}$ are uniquely determined up to the obvious operation of multiplying μ by a non-negative measurable function which is μ -almost everywhere non-zero and dividing $(\| \bullet \|_{\pi})_{\pi}$ by the square root of that function. This is the content of "uniqueness of Plancherel decomposition." In fact, we shall need a slightly stronger uniqueness, even when we allow certain norms to be possibly negative:

6.1.1. PROPOSITION. Suppose given a positive measure μ on \hat{G} as well as a family $\pi \mapsto H_{\pi}$ of Hermitian forms on $C_c^{\infty}(X)_{\pi}$ so that $\Phi \mapsto H_{\pi}(\Phi)$ is μ -measurable, for every Φ in $C_c^{\infty}(X)$, and moreover $\|\Phi\|^2 = \int H_{\pi}(\Phi)\mu(\pi)$ for all Φ . Then H_{π} are positive semidefinite for μ -almost every π , so that (μ, H_{π}) define a Plancherel formula.

PROOF. Let (M, σ) be a Levi subgroup and a supercuspidal representation. Let Y' be the set of all unramified characters of M modulo the finite subgroup of those for which $\sigma \otimes \chi \simeq \sigma$ (it is a complex algebraic variety). Let W_M be the normalizer of M in the Weyl group – it acts on Y', and we set $Y = Y' /\!\!/ W_M = \operatorname{spec} \mathbb{C}[Y']^{W_M}$.

The theory of the Bernstein center has the following consequence: For a "good" basis of compact subgroups J and any $\alpha \in \mathbb{C}[Y]$, there exists an element $f \in \mathcal{H}(G,J)$ (the Hecke algebra of J-biinvariant compactly supported measures on G) so that f acts on the J-invariant space of any representation $i_P^G(\sigma \cdot \chi)$ as scalar multiplication by $\alpha(\chi)$, and f acts on every other Bernstein component as 0.

Let \hat{G}_0 be the subset of \hat{G} consisting of irreducible representations that occur as a subquotient of some $i_P^G(\sigma \otimes \chi)$. Now \hat{G}_0 is closed and open in \hat{G} ; the measure μ induces a measure μ_0 on \hat{G}_0 , and it is sufficient to show that

²⁵More precisely: a positive Borel measure, with respect to the standard Borel structure on \hat{G} ; see [Dix77, Prop. 4.6.1].

 H_{π} is positive semidefinite for μ_0 -almost every π , because the union of sets \hat{G}_0 as we vary M is all of \hat{G} .

If $\pi \in \hat{G}_0$ is a subquotient of $i_P^G(\sigma \otimes \chi)$ then χ is uniquely determined modulo W_M , i.e. the image of χ in Y is uniquely determined. This gives a map $\hat{G}_0 \to Y$. Let Y_0 be the closure (in the usual topology on Y) of the image of \hat{G}_0 . The induced map $\pi : \hat{G}_0 \to Y_0$ is a Borel map.

Let $\bar{\mu}$ be the push-forward of the measure μ_0 to Y. The disintegration of measure implies that we may disintegrate the measure $\mu(\pi)$ as an integral $\int_y \mu_y \cdot d\bar{\mu}(y)$, where $y \mapsto \mu_y$ is a measurable mapping from Y_0 to the space of measures on \hat{G}_0 , and each μ_y is entirely supported on $(\hat{G}_0)_y$, the fiber of the mapping above y. The space $(\hat{G}_0)_y$ is finite, and thus μ_y is nothing more than a function on this finite set.

Fix $\Phi \in C_c^{\infty}$, and let

$$F^{\Phi}(y) = \int H_{\pi}(\Phi) d\mu_y \left(= \sum_{\pi \in (\hat{G}_0)_y} H_{\pi}(\Phi) \mu_y(\{\pi\}) \right).$$

It is measurable on Y_0 . Then for any $z \in \mathbb{C}[Y]$, the theory of the Bernstein center implies

$$\int_{y} |z(y)|^2 F^{\Phi}(y) d\bar{\mu} \ge 0,$$

The Bernstein center induces a *dense* subalgebra of $C(Y_0)$; indeed it separates points on Y and it is closed under complex conjugation. Therefore, $F^{\Phi}(y) \geq 0$ for $\bar{\mu}$ -almost all y. The space C_c^{∞} being of countable dimension, this can be said simultaneously for all Φ . That is to say, away from a set $S \subset Y_0$ with $\mu(S) = 0$, we have:

$$\sum_{\pi \in (\hat{G}_0)_y} H_{\pi}(\Phi) \mu_y(\{\pi\}) \ge 0,$$

for all $\Phi \in C_c^{\infty}(X)$.

The left-hand side is a Hermitian form on the finite length G-representation $\bigoplus_{\pi \in (\hat{G}_0)_y} C_c^{\infty}(X)_{\pi}$. It follows that, whenever $y \notin S$, we have $H_{\pi}(\Phi) \geq 0$ for every π in the fiber above y with $\mu_y(\{\pi\}) > 0$.

Let B be the set of $\pi \in G_0$ for which H_{π} fails to be positive semidefinite. Then B is measurable, since one can test the failure of positive semidefiniteness by a countable number of evaluations. But $\mu(B) = \int_y \mu_y(B) d\bar{\mu}(y)$. According to the discussion above, $\mu_y(B) = 0$ for μ -almost all y, so $\mu(B) = 0$, concluding the proof.

To give invariant norms on $C_c^{\infty}(X)_{\pi}$ is equivalent to giving an equivariant morphism

$$M_{\pi}: C_{\mathfrak{o}}^{\infty}(X \times X) \to \pi \otimes \bar{\pi}$$

(where π is assumed to have a unitary structure $\tilde{\pi} \xrightarrow{\sim} \bar{\pi}$). The hermitian forms associated to the above norms are the so-called *spherical characters*:

$$\theta_{\pi}: C_c^{\infty}(X) \otimes C_c^{\infty}(X) = C_c^{\infty}(X \times X) \to \pi \otimes \bar{\pi} \to \mathbb{C}$$

where the last arrow denotes the unitary pairing. (This map is G-invariant.) Thus, to be explicit, these have the property that:

$$\langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} = \int_{\hat{G}} \theta_{\pi} (\Phi_1 \otimes \Phi_2) \mu(\pi).$$
 (6.3)

Notice that, automatically, for μ -almost every π the spherical characters θ_{π} are positive semi-definite.

6.2. Definition; the canonical hermitian form. In this section we require, for simplicity, that stabilizers of points on X are unimodular; in other words, X admits an invariant measure. The modifications necessary to remove this assumption are straightforward. We say that X is strongly tempered if, for any $x \in X(k)$, the restriction of any (G-)tempered matrix coefficient to the stabilizer H of x in G = G(k) is in $L^1(H)$.

In checking this, the following remark is useful: If π_0 is the normalized induction to G of the trivial representation on a Borel subgroup B, and $v_0 \in \pi_0$ the spherical vector (i.e., K-invariant for K a good maximal compact subgroup of G, satisfying the Iwasawa decomposition G = BK) then every tempered matrix coefficient $\varphi(g)$ is majorized by the spherical one (see [CHH88, Theorem 2])

$$|\varphi(g)| \le c \langle gv_0, v_0 \rangle. \tag{6.4}$$

(The right hand side is positive.) Moreover, if we fix an open compact subgroup U, there exists a constant c = c(U) which works whenever $\varphi(g) = \langle gu_1, u_2 \rangle$ arises from U-invariant u_1, u_2 , with $||u_1|| = ||u_2|| = ||v_0||$.

Let (π, V) be a (G-)tempered representation of G, and assume that X is a strongly tempered variety. We define the morphism: $M_{\pi}: \pi \otimes \tilde{\pi} \to C^{\infty}(X \times X)$ characterized by the property²⁶ that

$$M_{\pi}(v \otimes u)(x,x) = \int_{G_x} \langle \pi(h)v, u \rangle dh.$$

We let

$$\theta_{\pi}: C_c^{\infty}(X \times X) \to \tilde{\pi} \otimes \pi \to \mathbb{C}$$

denote the adjoint composed with the canonical pairing.

Let μ_G denote the canonical Plancherel measure for $L^2(G)$, normalized as usual, i.e. the spherical characters are simply the usual characters. Since the spectrum of G as a $G \times G$ -representation is supported on representations of the form $\pi \otimes \tilde{\pi}$, μ_G will be thought of, as usual, as a measure on \hat{G} .

²⁶This property defines the morphism in the $G \times G$ -orbit of the diagonal $\Delta X \subset X \times X$. We can extend it by zero on the whole space; in fact, the extension plays no role in what follows.

6.2.1. THEOREM. Suppose that (\mathbf{G}, \mathbf{X}) is strongly tempered. Then θ_{π} and μ_{G} define a Plancherel formula for $L^{2}(X)$, in the sense that for $\Phi_{1}, \Phi_{2} \in C_{c}^{\infty}(X)$ we have:

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\hat{G}} \theta_{\pi} (\Phi_1 \otimes \overline{\Phi_2}) \mu_G(\pi).$$
 (6.5)

In particular, $M_{\pi}(v,v)|_{\Delta X} \geq 0$ for every π , and $L^2(X)$ is tempered as a G-representation; its Plancherel measure is absolutely continuous with respect to the group Plancherel measure.

6.2.2. Remark. The positivity assertion states simply that $\int_{h \in G_x} \langle hv, v \rangle \ge 0$ – an assertion that is obvious when G_x is compact.

PROOF. Let π be a unitary representation (endowed with a invariant Hilbert norm, hence with a fixed isomorphism: $\tilde{\pi} \simeq \bar{\pi}$) and let $i_{\pi} : C_c^{\infty}(G) \to \bar{\pi} \otimes \pi$ denote the dual of matrix coefficient m_{π} . (If we identify $\bar{\pi} \otimes \pi$ with a subspace of $\text{End}(\pi)$, the morphism i_{π} simply maps $f \in C_c^{\infty}(G)$ to $\pi(f)$.)

The Plancherel formula on G can be written as:

$$\langle f_1, f_2 \rangle_{L^2(G)} = \int \langle i_\pi(f_1), i_\pi(f_2) \rangle_{HS} \,\mu_G(\pi)$$

where \langle , \rangle_{HS} denotes the Hilbert-Schmidt hermitian form on $\bar{\pi} \otimes \pi \subset \operatorname{End}(\pi)$.

We are going to assume, for simplicity, that G has a single orbit on $X = H \backslash G$, but the general case follows in the identical fashion. Notice that the map:

$$C_c^{\infty}(G) \ni f \mapsto \Phi(x) = \int_H f(hx)dh \in C_c^{\infty}(X)$$

is surjective. Let f and Φ be such, then:

$$\|\Phi\|_{L^{2}(X)}^{2} = \int_{G} \int_{H} f(hg)\bar{f}(g)dhdg = \int_{H} \langle \mathcal{L}_{h^{-1}}(f), f \rangle_{L^{2}(G)} dh$$
 (6.6)

where \mathcal{L}_{\bullet} denotes the left regular representation of G.

We will use the following explication of θ_{π} :

$$\theta_{\pi}(\Phi \otimes \bar{\Phi}) = \int_{H} \langle \pi(h)i_{\pi}(f), i_{\pi}(f) \rangle_{HS}$$
(6.7)

Indeed, suppose that f is J-invariant, and choose dual bases v_1, \ldots, v_n for π^J and v_1^*, \ldots, v_n^* for $(\overline{\pi})^J$. The definition of θ_{π} says that $\theta_{\pi}(\Phi \otimes \overline{\Phi})$ is the integral, over $(g_1, g_2) \in X \times X$, of $\Phi(g_1)\overline{\Phi}(g_2), \int_H \langle \pi(hg_1)v_i, \overline{\pi}(g_2)v_i^* \rangle$. Unfolding, this equals

$$\int_{G\times G} f(g_1)\overline{f(g_2)} \int_{H} \langle \pi(hg_1)v_i, \overline{\pi}(g_2)v_i^* \rangle,$$

which in turn equals $\int_H \langle \pi(h) i_\pi(f) v_i, i_{\overline{\pi}}(f) v_i^* \rangle$ as desired.

Keeping in mind that $i_{\pi}(\mathcal{L}_{h^{-1}f}) = \pi(h^{-1})i_{\pi}(f)$ we get:

$$\|\Phi\|_{L^{2}(X)}^{2} = \int_{H} \int_{\pi} \langle \pi(h)i_{\pi}(f), i_{\pi}(f)\rangle_{HS} \mu(\pi)dh =$$
 (6.8)

$$= \int_{\pi} \int_{H} \langle \pi(h) i_{\pi}(f), i_{\pi}(f) \rangle_{HS} \, \mu(\pi) dh =$$

$$= \int_{\pi} \theta_{\pi} (\Phi \otimes \bar{\Phi}) \mu(\pi).$$

Notice that at all stages these integrals are absolutely convergent, justifying our application of Fubini. Indeed, because of (6.4), the integrand is bounded in absolute value by a constant multiple of $||i_{\pi}(f)||_{HS}^2 \cdot \langle hv_0, v_0 \rangle$. Then $\langle hv_0, v_0 \rangle$ is integrable over H by assumption, and $||i_{\pi}(f)||_{HS}^2$ is μ -integrable by the Plancherel theorem.

In particular, we have established the statement for $\Phi_1, \Phi_2 \in C_c^{\infty}(X)$. That the remaining statements follow is a consequence of Proposition 6.1.1.

6.3. The Whittaker case and the Lapid–Mao conjecture. A case of particular interest which does not literally fall under the strongly tempered is the "Whittaker-Plancherel formula".

We shall now give a short reduction of this formula to the usual Plancherel formula. The proof is largely the same as the previous, but the integrals are only conditionally convergent and we need to interpret them suitably; we therefore treat this case separately.

In fact, the treatment that follows covers many more cases of 'Whittaker-induced" models than the Whittaker model itself. We set up the notation very generally, but the key assumption is given in the paragraph below, and in practice is fulfilled only in "nondegenerate" cases. The general setup is as follows: Let $\mathbf{H} = \mathbf{M} \ltimes \mathbf{U}^-$ be a spherical subgroup, where \mathbf{M} is contained in a Levi subgroup \mathbf{L} of a parabolic \mathbf{P}^- and \mathbf{U}^- is the unipotent radical of this parabolic. Assume that $\Lambda_0: \mathbf{H} \to \mathbf{G}_a$ is a homomorphism and let $\Psi = \Psi_{\Lambda_0}$ be the composition of Λ_0 with a nontrivial unitary character of k; let $\mathbf{H}_0 = \ker \Lambda_0$. Finally, assume that there is a cocharacter $\check{\mu}: \mathbf{G}_m \to \mathcal{Z}(\mathbf{L})$ which is nonpositive under the right adjoint action on \mathfrak{u}^- (that is, its eigencharacters on \mathfrak{u}^- are of the form $a \mapsto a^n$ with $n \leq 0$), normalizes \mathbf{H}_0 , and acts nontrivially on the quotient \mathbf{H}/\mathbf{H}_0 ; identifying this quotient with G_a , the action of $\check{\mu}$ is by a positive character: $x \cdot \check{\mu}(a) = a^{n_0}x$ with $n_0 > 0$. The crucial assumption is:

Let v_s be a holomorphic section in the principal series $I_B^G(\delta_B^s)$ obtained by normalized induction from a power of the modular character of the Borel subgroup. The "Jacquet integral":

$$\int_{H} v_s(h)\Psi^{-1}(h)dh \tag{6.9}$$

is convergent for $\Re s > 0$, and extends continuously to s = 0.

This assumption, and in particular the extension of the integral to s = 0, typically forces some nondegeneracy condition on Λ_0 . One can calculate that the following examples satisfy it:

- (1) **H** is a maximal unipotent subgroup of **G**, $\Psi =$ a generic character.
- (2) $\mathbf{G} = \mathbf{GSp_4}, H = \text{the Bessel/Novodvorsky subgroup } \mathbf{T} \ltimes \mathbf{U}, \text{ where } \mathbf{U} \text{ is the unipotent radical of the Siegel parabolic } (\simeq S^2V, \text{ where } V \text{ is a two-dimensional vector space}), <math>\Lambda : \mathbf{U} \to \mathbf{G_a} \text{ (i.e. } \Lambda \in S^2V^{\vee}) \text{ and } \mathbf{T} \supset \mathbf{SO}(\Lambda) \text{ is the stabilizer of } \Lambda \text{ in a Siegel Levi subgroup.}$
- (3) $\mathbf{G} = \mathbf{SO}_n \times \mathbf{SO}_{n+2m+1}$, $\mathbf{H} = \mathbf{SO}_n \times$ the unipotent radical of the parabolic with Levi $\mathbf{G}_m^m \times \mathbf{SO}_{n+1}$, Λ : a nondegenerate additive character of $\mathbf{H} \cap \mathbf{GL}_m$ (where \mathbf{GL}_m is in the Levi $\mathbf{GL}_m \times \mathbf{SO}_{n+1}$) This case has been considered by Waldspurger [Wall2c, Chapter 5].

For simplicity, we will only present the Whittaker case here, though with notation suggestive of the general case; the steps required for the other cases are completely analogous. Hence, we are in the setting of (1): \mathbf{H} is a maximal unipotent subgroup of \mathbf{G} , $\Psi=$ a generic character, \mathbf{H}_0 is the kernel of the corresponding algebraic morphism into \mathbf{G}_a .

6.3.1. Proposition. The restriction of any tempered matrix coefficient of G to H_0 belongs to $L^1(H_0)$.

PROOF. We fix a Borel subgroup **B** with $\check{\mu}(\mathbf{G}_{\mathrm{m}}) \subset \mathbf{B}$ so that **HB** is open and $\mathbf{H} \cap \mathbf{B}$ is trivial. Let $\pi_s = I_B^G(\delta_B^s)$, the normalized principal series induced from the s-power of the modular character of B. The representation π_s is tempered if $\Re s = 0$. By (6.4), it suffices to prove the proposition for matrix coefficients of π_0 .

We fix the following invariant inner product on π_s ($\Re s = 0$):

$$\langle v_1, v_2 \rangle = \int_H v_1(u) \overline{v_2(u)} du, \tag{6.10}$$

where $v_1, v_2 \in \pi_s$, considered as functions on G. This integral converges; it is the restriction to the open H-orbit of the compact, G-invariant integral of $v_1(u)\overline{v_2(u)}$ over $B \setminus G$.

The underlying vector spaces of all representations π_s can be identified with one another by restriction of functions to K, a maximal compact subgroup satisfying the Iwasawa decomposition G = BK. In particular, K-invariant elements for all representations are identified in this common vector space – let v be a non-zero K-invariant element with $v|_K > 0$, and v_s its "realization" in π_s . (The inner product that we chose on π_s is not compatible with this identification, but of course it varies continuously in s.)

Let f_s be the matrix coefficient $\langle \pi_s(g)v_s, v_s \rangle$, for $\Re s = 0$. To prove convergence of the integral $\int_{H_0} f(n) dn$ we extend f_s to all s with $\Re(s) \geq 0$

using expression (6.10), that is:

$$f_s(g) = \int_H v_s(ug) \overline{v_s(u)} du.$$

(Of course, for $\Re s \neq 0$ this does not represent a matrix coefficient.) It will follow from the argument below that this expression converges for $\Re s \geq 0$, but for the moment we can restrict our attention to positive real s, and treat this as a possibly infinite expression. Notice that for $\Im s = 0$ we have $f_s(g) \geq 0$ for all g; in fact, for such s, we have $v_s(g) > 0$ for all g.

Thus, by expanding the definitions and using Fatou's lemma,

$$\int_{H_0} f_0(n) dn = \int_{H_0} \int_{H} v_0(un) \overline{v_0(u)} du dn \le
\le \lim_{s \to 0^+, s \in \mathbb{R}} \int_{H_0} \int_{H} v_s(un) \overline{v_s(u)} du = \lim_{s \to 0^+, s \in \mathbb{R}} \int_{H_0} f_s(n) dn,$$
(6.11)

be they finite or infinite.

Now, for $\Re s > 0$ the function f_s is absolutely integrable over H (for this statement, the character Λ could be trivial). Indeed, we have $\int_H |f_s(g)| \le \left(\int_H |v_s(u)| du\right)^2$, and the integral $\int_H v_s(u)$ is known to be absolutely convergent from the study of standard intertwining operators. We therefore have

$$\int_{H} f_{s}(g) \Psi_{\Lambda}^{-1}(g) dg = \int_{H} \int_{H} v_{s}(ug) \overline{v_{s}(u)} du \Psi_{\Lambda}^{-1}(g) dg =$$

$$= \left| \int_{H} v_{s}(u) \Psi_{\Lambda}^{-1}(u) du \right|^{2},$$
(6.12)

and

$$W_s(\Lambda, g) = \int_H v_s(ug) \Psi_{\Lambda}^{-1}(u) du$$

is the Jacquet integral which converges absolutely for $\Re s > 0$. (This is one of the assumptions that we made above, and is known to hold in the aforementioned cases, including the Whittaker case.)

We can let Λ vary in the k-points of the one-dimensional vector space $\mathbf{V}^* := \operatorname{Hom}(\mathbf{H}/\mathbf{H}_0, \mathbf{G}_a)$, and then by (6.12) and inverse Fourier transform we get, still for $\Re(s) > 0$,

$$\int_{H_0} f_s(n)dn = \int_{V^*} |W_s(\Lambda, 1)|^2 d\Lambda, \tag{6.13}$$

for a suitable choice of Haar measure $d\Lambda$.

Now – as Λ varies – $W_s(\Lambda, 1)$ may be expressed in terms of the value of W_s for a fixed character at a varying point of the torus; by a routine computation with the known asymptotics of the spherical Whittaker function (s. the remark that follows), the integral (6.13) is uniformly bounded for s in a neighborhood of zero, and the right hand side of (6.11) is finite. (We

discuss this argument at more length below, phrased in a fashion where it generalizes more readily.) \Box

6.3.2. Remark. Let us discuss in more detail how to phrase the final step of the proof – bounding (6.13) – in the language of this paper, so that it may be readily generalized to other settings:

The asymptotics of the "Whittaker function" $W_s(\Lambda, g)$ can be derived from our earlier discussion of asymptotics in section 5, by interpreting the value $W_s(\Lambda, g)$ as an element in a representation induced from a character of a spherical subgroup. Recall the cocharacter $\check{\mu}$ discussed before the statement of the proposition; in the Whittaker case, if **H** is the subgroup corresponding to the *negative* roots for some choice of Borel and Cartan subgroups, then $\check{\mu}$ is (in additive notation) a multiple of $-\check{\rho}$ (minus half the sum of positive coroots). In order to relate $W_s(\Lambda, g)$ to standard Whittaker functions, we only need to notice that changing Λ corresponds to conjugating by an element of $\check{\mu}(k^{\times})$; more precisely:

$$W_s(\Lambda, \check{\mu}(x)) = \delta_B^{s-\frac{1}{2}}(\check{\mu}(x)) \int_H v_s(u) \Psi_{\Lambda}^{-1}(\check{\mu}(x)u\check{\mu}^{-1}(x)) du. \tag{6.14}$$

Indeed (the argument is quite simple, but we formulate it in some language that can be generalized):

- The variety $\mathbf{H}\backslash \mathbf{G}$, equipped with the *trivial* character of \mathbf{H} , is a boundary degeneration of the same variety equipped with the line bundle \mathcal{L}_{Ψ} corresponding to induction from Ψ ; denote that boundary degeneration by \mathbf{X}_{Θ} . (In the Whittaker case, this will be the most degenerate case, so $\Theta = \emptyset$, but not in general.) The left action of the cocharacter $\check{\mu}$ has image in $\mathbf{A}_{X,\Theta}$, with $\check{\mu}(\mathfrak{o})$ mapping to $A_{X,\Theta}^+$.
- We may split the nonzero points of the one-dimensional vector space V^* into a finite number of $G_{\rm m}$ -orbits under the cocharacter $\check{\mu}$; denote them by V_i .
- Let δ_H be the modular character of the k-points of the algebraic group $\mathbf{H}\check{\mu}(\mathbf{G}_{\mathrm{m}})$. We notice that the modular character δ_B on $\check{\mu}(k^{\times})$ is inverse to the modular character δ_H .
- If Λ_i is a representative for V_i^* , then:

which shows (6.14).

$$\begin{split} W_s(\Lambda_i, \check{\mu}(x)) &= \int_H v_s(u\check{\mu}(x)) \Psi_{\Lambda_i}^{-1}(u) du = \\ &= \delta_B^{s+\frac{1}{2}}(\check{\mu}(x)) \int_H v_s(\check{\mu}(x^{-1}) u\check{\mu}(x)) \Psi_{\Lambda_i}^{-1}(u) du = \\ &= \delta_B^{s+\frac{1}{2}}(\check{\mu}(x)) \delta_H(\check{\mu}(x)) \int_H v_s(u) \Psi_{\Lambda_i}^{-1}(\check{\mu}(x) u\check{\mu}^{-1}(x)) du = \\ &\delta_B^{s-\frac{1}{2}}(\check{\mu}(x)) \int_H v_s(u) \Psi_{\Lambda_i}^{-1}(\check{\mu}(x) u\check{\mu}^{-1}(x)) du, \end{split}$$

Therefore for a suitable choice of Haar measures we have:

$$\int_{V_i^*} |W_s(\Lambda, 1)|^2 d\Lambda = \int_{k^*} \delta_B^{1-2s}(\check{\mu}(x)) |W_s(\Lambda_i, \check{\mu}(x))|^2 d(x^{-n_0}) =
= \int_{k^*} \delta_B^{1-2s}(\check{\mu}(x)) |x|^{-n_0} |W_s(\Lambda_i, \check{\mu}(x))|^2 d^*x.$$
(6.15)

Now we notice:

- For $|x| \gg 1$ we have $|W_s(\Lambda_i, \check{\mu}(x))|^2 = 0$.
- Since, by definition, $W_s(\Lambda, \bullet)$ is in the image of an operator: $I_B^G(\delta_B^s) \to C^\infty(H\backslash G, \Psi_\Lambda)$, and $\check{\mu}(\mathfrak{o}) \subset A_{X,\Theta}^+$, by the theory of asymptotics for $|x| \ll 1$ we will have that the function $x \mapsto W_s(\Lambda, \check{\mu}(x))$ is equal to a k^\times -finite function with generalized eigencharacters equal to the (unnormalized) exponents of $I_B^G(\delta_B^s)$. For any given $\varepsilon > 0$ the latter are bounded, for s in a neighborhood of 0, by $\delta_B^{-\frac{1}{2}+\varepsilon}(\check{\mu}(x))$.
- Given this information on generalized eigencharacters to handle |x| small, the vanishing for $|x| \gg 1$ to handle |x| large, and the fact that W_s extends continuously to s=0 (and hence is pointwise bounded for x in a compact set, for s in a neighborhood of zero) to handle the remaining x, it follows that

$$\delta_R^{1-\varepsilon}(\check{\mu}(x))|W_s(\Lambda_i,\check{\mu}(x))|^2$$

is (uniformly for s close to 0) integrable over k^{\times} .

6.3.3. Corollary. For every tempered representation π , there is a canonical normalization of the integral of matrix coefficients:

$$\int_{u}^{*} \left\langle \pi(u)v^{1}, v^{2} \right\rangle \Psi_{\Lambda}(u)^{-1} du \tag{6.16}$$

as the evaluation at Λ of the Fourier transform of the function:

$$u \in H/H_0 \mapsto \int_{H_0} \langle \pi(nu)v^1, v^2 \rangle dn.$$

Indeed, this Fourier transform is a *a priori* a distribution, but it is also invariant under an open compact subgroup of A, so it can be identified with a function in the complement of degenerate characters Λ .

Now, for any non-degenerate Λ the normalized integral above defines a morphism:

$$M_{\pi}^{\Lambda}: \pi \otimes \bar{\pi} \to C^{\infty}(H \backslash G, \Psi_{\Lambda}) \otimes C^{\infty}(H \backslash G, \Psi_{\Lambda}^{-1}), \tag{6.17}$$

characterized by the property that $M_{\pi}^{\Lambda}(v_1 \otimes v_2)(1,1) = \int_H^* \Psi_{\Lambda}^{-1}(u) \langle \pi(u)v_1, v_2 \rangle du$. (This can also be expressed in terms of Jacquet integrals, as we discuss in §6.3.6.) The hermitian dual of this, composed with the unitary pairing between π and $\bar{\pi}$, will be denoted by θ_{π}^{Λ} . Then Theorem 6.2.1 carries over:

6.3.4. THEOREM. The Plancherel formula for $L^2(H\backslash G, \Psi_{\Lambda})$ reads:

$$\langle \Phi_1, \Phi_2 \rangle = \int_{\check{G}} \theta_{\pi}^{\Lambda} (\Phi_1 \otimes \overline{\Phi_2}) \mu_G(\pi).$$

In particular, for every tempered representation π and $v \in \pi$ we have:

$$\int_{H}^{*} \langle \pi(u)v, v \rangle \, \Psi_{\Lambda}(u) du \ge 0,$$

and $L^2(H\backslash G, \Psi_{\Lambda})$ is tempered as a G-representation; its measure is absolutely continuous with respect to the Plancherel measure for G.

PROOF. The proof that we saw in the strongly tempered case carries over almost verbatim, with due care for the regularizations:

Fix $f \in C_c^{\infty}(G)$, and let $\Phi_{\Lambda}(g) = \int_H f(ug)\Psi_{\Lambda}^{-1}(u)du$. Note that Φ_{Λ} is also compactly supported on $H\backslash G$, and so square integrable. Let us consider:

$$\aleph: \Lambda \mapsto \|\Phi_{\Lambda}\|^2 = \int_{H} \Psi_{\Lambda}(u) \int_{\hat{G}} \langle \pi(u) i_{\pi}(f), i_{\pi}(f) \rangle_{HS} \, \mu_{G}(\pi) du. \tag{6.18}$$

The equality here is proved in precisely the same way as (6.6) and (6.8).

(6.18) is indeed a genuine, continuous function of Λ , locally constant on the non-degenerate locus. This statement follows easily from the facts that the support of Φ_{Λ} in $H\backslash G$ is compact, and the integrand involved in the definition of $\Phi_{\Lambda}(g)$ is actually compactly supported (uniformly for g in a compact set). On the other hand, the integral is no longer absolutely convergent in general as a double integral. To push through the previous computations, we shall study \aleph as a distribution in Λ .

Let Q be a smooth function on $V^* = \operatorname{Hom}(\mathbf{H}/\mathbf{H}_0)(k)$, compactly supported away from the degenerate locus, with Fourier transform $u \to \hat{Q}(u)$ (also compactly supported on V and smooth). Then:

$$\int_{V^*} Q(\Lambda) \cdot \aleph(\Lambda) d\Lambda = \int_{H} \hat{Q}(u) \int_{\hat{G}} \langle \pi(u) i_{\pi}(f), i_{\pi}(f) \rangle_{HS} \mu_{G}(\pi) du =$$

$$= \int_{\hat{G}} \int_{H} \hat{Q}(u) \langle \pi(u) i_{\pi}(f), i_{\pi}(f) \rangle_{HS} du \ \mu_{G}(\pi) =$$

$$= \int_{\hat{G}} \int_{V^*} Q(\Lambda) \theta_{\pi}^{\Lambda}(\Phi \otimes \bar{\Phi}) d\Lambda \ \mu_{G}(\pi) = \int_{V^*} Q(\Lambda) \int_{\hat{G}} \theta_{\pi}^{\Lambda}(\Phi \otimes \bar{\Phi}) \mu_{G}(\pi) \ d\Lambda,$$

where $\Phi(g) = \int_H f(ug) \Psi_{\Lambda}^{-1}(u) du$. The first equality on the final line, i.e. the introduction of θ_{π}^{Λ} , is just as in the proof of (6.7).

To deduce the desired result from this, for any given non-degenerate Λ_0 there exists an open neighborhood S of Λ_0 in V^* so that $\aleph(\Lambda)$ and $\theta_\pi^\Lambda(\Phi\otimes\bar{\Phi})^*$ are all constant on S. We then choose Q supported in S to get the desired result.

 $^{^{\}rm 27}{\rm This}$ is easy to deduce directly in the Whittaker case, since U^- is amenable.

We note that a Plancherel formula for the Whittaker model has also been developed by Delorme [Del13], who used different methods. In the archimedean case, the Whittaker-Plancherel formula was developed by Wallach [Wal92].

A corollary to the last sentence of the theorem was conjectured [LM09, Conjecture 3.5] by Lapid and Mao; this also follows from the results of Delorme.

6.3.5. COROLLARY. Suppose that π is a generic irreducible representation of G whose Whittaker functions are square integrable on $U^-\backslash G$; then π is (G-)discrete series. (And similarly for all other cases of Theorem 6.3.4.)

We note that the converse statement is also true: a generic, discrete series representation has Whittaker functions which are square integrable on $U^-\backslash G$. This converse follows from the theory of asymptotics.

6.3.6. Explication. The functionals of the theorem, when π is an induced representation, can be described in terms of Jacquet integrals:

$$\int_{U^{-}} \langle \pi(u)v^{1}, v^{2} \rangle \Psi_{\Lambda}(u) du = W_{0}^{1}(\Lambda) \overline{W_{0}^{2}(\Lambda)}, \tag{6.19}$$

where $W_0^i(\Lambda)$ are constructed from v^1, v^2 by Jacquet integrals in an induced representation. Precisely:

Any tempered representation is a direct summand of a representation of the form $I_P^G(\tau)$, where τ is a discrete series of the Levi quotient of P. We equip it with the unitary structure $\|v\|^2 = \int_{U_M^- \setminus U^-} \|v(u)\|_{\tau}^2 du$ (where $U_M^- = U^- \cap M$, M a Levi subgroup of P).

If π admits a Whittaker functional, then τ does also. In this case, we fix a Whittaker functional (unique up to a scalar of norm one) $\eta_{\Lambda}: \tau \to \mathbb{C}$ so that

$$\begin{split} \eta_{\Lambda}(\tau(u)v) &= \Psi_{\Lambda}(u)\eta_{\Lambda}(v), u \in U_{M}^{-}; \\ \eta_{\Lambda}(v_{1})\overline{\eta_{\Lambda}(v_{2})} &= \int_{U_{M}^{-}}^{*} \left\langle \tau(u)v_{1}, v_{2} \right\rangle \Psi_{\Lambda}(u)du, \quad v_{1}, v_{2} \in \tau. \end{split}$$

(Indeed, that the final integral can be thus factorized follows from multiplicity one for Whittaker functionals, and the non-negativity of Theorem 6.3.4. It can also be verified that the resulting η_{Λ} is nonzero by means of the argument to be presented in the next section §6.4, we briefly sketch the argument – namely, the known theory of asymptotics of Whittaker functions mean that τ occurs inside the L^2 -Whittaker space for M, and then Theorem 6.3.4 implies that the integral defining $\eta_{\Lambda}(v_1)\overline{\eta_{\Lambda}(v_2)}$ must have been nonzero.)

Now define the Jacquet integral on the family of representations $I_P^G(\tau \delta_P^s)$:

$$v \mapsto \int_{U_M^- \setminus U^-} \Psi_{\Lambda}(u) \eta_{\Lambda}(v(u)) du.$$

As before, this defines (after holomorphic continuation) a (U^-, Ψ_{Λ}) -equivariant functional $\Xi_{\Lambda} \in I_P^G(\tau \delta_P^s)^*$ without poles on the tempered axis $\Re s = 0$. Let $v^1, v^2 \in \pi := I_P^G(\tau)$, and let W_0^1, W_0^2 be the corresponding Whittaker functions $W_0^i(g) = \Xi_{\Lambda}(gv^i)$ (we will also write: $W_0^i(\Lambda) := W_0^i(1)$). The product $W_0^1(\Lambda)W_0^2(\Lambda)$ is independent of the choice involved in defining η_{Λ} . Computing formally, the left-hand side of (6.19) equals

$$\int_{(u,u')\in U_M^-\setminus (U^-\times U^-)} \Psi_{\Lambda}(u')\langle v^1(uu'), v_2(u)\rangle \ du \ du'$$

$$= \int_{(u,u')\in U_M^-\setminus U^-\times U_M^-\setminus U^-} \Psi_{\Lambda}(u')\eta_{\Lambda}(v^1(uu'))\overline{\eta_{\Lambda}(v^2(u))} \ du \ du'$$

$$= W_0^1(\Lambda)\overline{W_0^2(\Lambda)}. \quad (6.20)$$

The conversion of this to a formal proof follows along the lines of the previous regularizations carried out in this section, and we omit it.

- **6.4.** The Ichino–Ikeda conjecture. The following establishes a conjecture of Ichino and Ikeda.
- 6.4.1. Theorem. Suppose that X is strongly tempered and wavefront, let H be the stabilizer of a point on X. Then:

$$M_{\pi}: v \otimes \bar{w} \mapsto \int_{H} \langle \pi(h)v, w \rangle dh$$
 (6.21)

defines a nondegenerate Hermitian form on π_H (the H-coinvariants of π), for every G-discrete series representation π .

Moreover, if σ is a discrete series representation of some Levi subgroup and $\pi = I_P^G(\sigma)$ (where P is some corresponding parabolic), then the same expression defines a non-zero, non-negative hermitian form on π_H .

6.4.2. Remark. In the setting of the "Gross-Prasad variety" $SO_n \setminus SO_n \times SO_{n+1}$, it is known by recent work of Waldspurger [Wal12b] that for any tempered L-packet of G there is at most one element π in the L-packet such that $\pi_H \neq 0$. Since any irreducible tempered representation is a subrepresentation of $I_P(\sigma)$ (where σ is a discrete series of the pertinent Levi) and all subrepresentations of that belong to the same L-packet, it follows that in that case (6.21) is non-zero for every irreducible tempered representation π with $\pi_H \neq 0$. This is the conjecture originally formulated by Ichino and Ikeda. In fact, Beuzart-Plessis managed to refine the argument of our theorem [BPar, Proposition 14.2.2] to prove multiplicity one in the induced discrete series, in the setting of the Gross-Prasad conjectures.

The idea of the proof: By our discussion of asymptotics we can show that any embedding $\pi \hookrightarrow C^{\infty}(X/\mathcal{Z}(X), \omega_{\pi})$ (where ω_{π} is the – unitary – central character of π) has image contained in $L^{2+\varepsilon}(X/\mathcal{Z}(X), \omega_{\pi})$. This suggests that any H-distinguished π must contribute to the Plancherel formula. But

we have already computed (Theorem 6.2.1) the Plancherel formula for X and seen that π occurs exactly when M_{π} is nonzero.

PROOF. For G-discrete series π , by our discussion of asymptotics any morphism $\pi \to C^{\infty}(X/\mathcal{Z}(X), \omega_{\pi})$ will have image in $L^{2}(X/\mathcal{Z}(X), \omega_{\pi})$:

Indeed, if $\Theta \subset \Delta_X$ then the $X_{\Theta} = X_{\Theta}^L \times^{P_{\Theta}^-} G$ and the asymptotic morphism $\pi \to C^{\infty}(X_{\Theta}) \to C^{\infty}(X_{\Theta}^L)$ factors through the corresponding Jacquet module π_{Θ^-} . Because π is discrete series, all the characters of this Jacquet module decay on the negative Weyl chamber of the center of L_{Θ} . Since (a finite union of cosets of) this negative Weyl chamber surjects onto $A_{X,\Theta}^+$, by the proof of Proposition 2.7.2, it follows that the image of any $v \in \pi$ "decays in the Θ -direction"; since this is so for all Θ and the exponential map is measure-preserving, this image in fact belongs to L^2 . ²⁸

The claim on nondegeneracy of M_{π} , now, follows from Theorem 6.2.1: indeed, the natural embedding:

$$\pi \otimes \operatorname{Hom}(\pi, C^{\infty}(X)) \hookrightarrow L^{2}(X/\mathcal{Z}(X), \omega_{\pi}),$$

$$v \otimes M \mapsto M(v),$$

endows the space on the left with a non-degenerate hermitian form, of which the Plancherel form θ_{π} of Theorem 6.2.1 is just the hermitian dual (i.e. the dual via the pairing of $\pi \otimes \operatorname{Hom}(\pi, C^{\infty}(X))$ with $C_c^{\infty}(X)_{\tilde{\pi}} = \tilde{\pi} \otimes (\operatorname{Hom}(C_c^{\infty}(X), \tilde{\pi}))^*$). Thus θ_{π} , and hence M_{π} , is non-degenerate.

We now turn to the claim on induced representations, which is more subtle; we need to pass from an "almost-everywhere" statement to a pointwise statement. We do this by establishing a uniform lower bound almost everywhere, and then we can specialize pointwise by a continuity argument.

Let $\pi = I_P^G(\sigma)$ as in the statement with $\pi_H \neq 0$. We establish a series of claims:

• Every H-distinguished direct summand of π belongs to the support of Plancherel measure for $L^2(X)$.

This follows by approximating matrix coefficients: Suppose that π_0 is an H-distinguished direct summand of π , so that we are given a G-morphism $M: \pi_0 \to C^\infty(X)$. The idea is now to approximate the matrix coefficients of π_0 by truncating functions in the image of M. For a fixed open compact subgroup J, partition X into J-fixed subsets:

$$X = \sqcup_{\Theta} N_{\Theta}$$

 $^{^{28}}$ To expand: Recall that, for this argument, the action of $A_{X,\Theta}$ has been twisted already, as per our general notational conventions from §1.7, by the square root of the $A_{X,\Theta}$ -eigencharacter for the measure on X_{Θ} ; thus the normalizations are precisely chosen so that decaying exponents along $A_{X,\Theta}$ force square integrability. We also used the fact that this twist is compatible with the twisting in the definition of the normalized Jacquet module. This follows because of the description of X_{Θ} as a parabolically induced variety, i.e. again from Proposition 2.7.2.

where N_{Θ} belongs to a J-good neighborhood of Θ -infinity, is $A_{X,\Theta}^+$ -stable and has compact image in $X_{\Theta}/A_{X,\Theta}$. We let $\tilde{N}_{\Theta} = \bigcup_{\Omega \subset \Theta} N_{\Omega}$, a J-good, $A_{X,\Theta}^+$ -stable neighborhood of Θ -infinity.

For every $\gamma \in \Delta_X$, set $\hat{\gamma} := \Delta_X \setminus \{\gamma\}$, and choose an element $a_{\gamma} \in \mathring{A}_{X,\hat{\gamma}}^+$. Notice that $\mathring{A}_{X,\hat{\gamma}}/\mathcal{Z}(X)$ is a one-dimensional subtorus of $A_X/\mathcal{Z}(X)$. For $x \in X/\mathcal{Z}(X)$ define a radial function:

$$R(x) := \min\{n \ge 1 | x \notin a_{\gamma}^n \tilde{N}_{\hat{\gamma}} \text{ for any } \gamma \in \Delta_X\}.$$

It is then clear that the sets $X_n := \{x \in X | R(x) \leq n\}$ form an exhaustive, increasing filtration of X by compact-mod- $\mathcal{Z}(X)$ sets. Moreover, they have the following property:

For any compact subset
$$\Omega \subset G$$
, there exists an integer $n \geq 1$ so that $X_k \cdot \Omega \subset X_{k+n}$.

Indeed, since the sets X_n are by definition J-stable, one can replace Ω by a finite subset $\{g_i\}_i$. Then there is an n such that for every γ and i we have: $a_{\gamma}^n \cdot \tilde{N}_{\hat{\gamma}} \cdot g_i^{-1} \subset \tilde{N}_{\hat{\gamma}}$ when $\tilde{N}_{\tilde{\gamma}}$ is considered as a subset of X_{Θ} , and hence also:

$$a_{\gamma}^{n+k} \cdot \tilde{N}_{\hat{\gamma}} \cdot g_i^{-1} \subset a_{\gamma}^k \cdot \tilde{N}_{\hat{\gamma}}$$

for all $k \geq 0$. Hence: $a_{\gamma}^{n+k} \cdot \tilde{N}_{\hat{\gamma}} \cdot \Omega^{-1} \subset a_{\gamma}^{k} \cdot \tilde{N}_{\hat{\gamma}}$, and by the equivariance property of J-good neighborhoods the same is true on X. Therefore, if a point x is in X_k , i.e. does not lie in $a_{\gamma}^k \cdot \tilde{N}_{\hat{\gamma}}$ for any γ , then $x\Omega$ is in X_{k+n} , i.e. does not lie in $a_{\gamma}^{k+n} \cdot \tilde{N}_{\hat{\gamma}}$ for any γ .

We also set $N_{\Theta,k} := N_{\Theta} \cap X_k$ for any k. By the theory of asymptotics, and the fact that π_0 is tempered, the quantity:

$$\langle M(v_1), M(v_2) \rangle_{L^2(N_{\Theta,k}/\mathcal{Z}(X))}$$

either has a limit over k – this cannot happen for all Θ and all v_1, v_2 because π_0 is not discrete – or is "asymptotic to the integral of a generalized $A_{X,\Theta}$ -eigenfunction with trivial generalized eigencharacter". By the latter we mean that the function:

$$A_{X,\Theta}^+ \ni a \mapsto M(v_1)(a \cdot x)\overline{M(v_2)(a \cdot x)} \operatorname{Vol}(a \cdot xJ),$$

for $x \in N_{\Theta}$, is (the restriction to $A_{X,\Theta}^+$ of) a generalized eigenfunction with unitary and subunitary eigencharacters; and hence the integral of $M(v_1)\overline{M(v_2)}$ over $N_{\Theta,k}$, as $k \to \infty$, is dominated by the integral of its summand corresponding to the trivial generalized $A_{X,\Theta}$ -eigencharacter. By an easy calculation over finitely generated abelian groups, this means that there is a nonzero constant $c_{\Theta}(v_1, v_2)$ and a positive integer $r_{\Theta}(v_1, v_2)$ such that:

$$\langle M(v_1), M(v_2) \rangle_{L^2(N_{\Theta,h})} \sim c_{\Theta}(v_1, v_2) k^{r_{\Theta}(v_1, v_2)}$$

Thus, there is a positive integer $r = \max_{\Theta, v_1, v_2} r_{\Theta}(v_1, v_2)$ so that for every triple (Θ, v_1, v_2) the limit:

$$c(v_1, v_2) := \lim_{k} \frac{\langle M(v_1), M(v_2) \rangle_{L^2(X_k)}}{k^r}$$
(6.22)

exists, and moreover it is not zero for all such triples.

Then the limit $c = c(v_1, v_2)$ defines a nonzero Hermitian form on π_0 . We claim that it is G-invariant; indeed, for $g \in \Omega$, we have an inequality

$$\langle M(v_1), M(v_2) \rangle_{L^2(X_{k-n})} \le \langle M(gv_1), M(gv_2) \rangle_{L^2(X_k)} \le \langle M(v_1), M(v_2) \rangle_{L^2(X_{k+n})}$$

from which the invariance follows. Therefore $c(v_1, v_2)$ is a multiple of the inner product.

Now, we can approximate (on compacta) matrix coefficients of π_0 by matrix coefficients of $L^2(X)$: namely, given $v \in \pi_0$ and $g \in \Omega$ we may approximate $\langle gv, v \rangle$, which is a multiple of c(gv, v), by a multiple of $\langle M(gv), M(v) \rangle_{L^2(X_k)}$ for large k. Now this equals

$$\langle gM(v)|_{X_k}, M(v)|_{X_k} \rangle + \langle gM(v)|_{X_k} - M(gv)|_{X_k}, M(v)|_{X_k} \rangle$$

and the second term is bounded by Cauchy-Schwarz by the square root of

$$\langle M(v), M(v) \rangle_{L^2(X_{k+n}-X_{k-n})} \cdot \langle M(v), M(v) \rangle_{L^2(X_k)}$$

which is of lower order than the main term as $k \to \infty$, because of (6.22). Consequently, π_0 belongs to the support of Plancherel measure for $L^2(X)$.

• There is a set of unitary unramified characters of P of positive (Haar) measure such that $M_{\pi_{\chi}} \neq 0$, where $\pi_{\chi} = I_{P}(\sigma \otimes \chi)$.

Indeed, the only tempered representations in a neighborhood of π under the Fell topology are of the form π_{χ} . Since we know (from strong temperedness, i.e. Theorem 6.2.1) that the Plancherel measure for $L^2(X)$ is supported in the set of tempered representations and is absolutely continuous with respect to the natural class of measures on them, it follows that there should be a set of unramified characters χ of positive Haar measure such that the Plancherel morphisms $M_{\pi_{\chi}}$ are non-zero.

As χ varies, we identify the underlying vector spaces of all π_{χ} in the natural way with a fixed vector space V.

• For $v \in V$, the expression $M_{\pi_{\chi}}(v,v)$ is a real analytic function of χ .

Indeed, the integral of the matrix coefficient over H is actually a countable sum of functions polynomial in χ , which converges absolutely and uniformly in χ .

It follows from the last two points that $M_{\pi_{\chi}}$ is non-zero for almost every χ . There remains to show:

• $M_{\pi_{\chi}}$ is non-zero for every χ .

For this, we will treat for simplicity the case of multiplicity one, i.e.

$$M_{\pi_{\chi}}(v \otimes w) = L_{\pi_{\chi}}(v) \otimes \overline{L_{\pi_{\chi}}(v)},$$

where $L_{\pi_{\chi}}: \pi_{\chi} \to \mathbb{C}$, an H-invariant functional. It is easy to see that we may choose $L_{\pi_{\chi}}$ measurable in χ . Moreover, we will assume that G acts transitively on X. The proof is the same in the general case, but its formulation would obscure the argument.

Let K_1 be an open compact subgroup such that $L_{\pi_{\chi}}|_{\pi_{\kappa}^{K_1}} \neq 0$ for a dense set of χ 's. Let $v_{\chi} = \overline{K_1 * L_{\pi_{\chi}}} \in \pi_{\chi}$, i.e. $v_{\chi} \in \pi_{\chi}^{K_1}$ and $\langle u, v_{\chi} \rangle = L_{\pi_{\chi}}(u)$ for $u \in \pi_{\chi}^{K_1}$. In particular, $v_{\chi} \neq 0$ if and only if $M_{\pi_{\chi}}(v, w) \neq 0$ for some $v, w \in \pi_{\chi}^{K_1}$; so, the set of χ for which $||v_{\chi}|| \neq 0$ is of full measure. By Corollary 5.3.4, there exists K_2 such that

$$L_{\pi_{\chi}}(gv_{\chi}) = \langle gv_{\chi}, K_2 * L_{\pi_{\chi}} \rangle \quad (g \in G^+). \tag{6.23}$$

We are going to average both sides of (6.23) over χ , and compute the L^2 norm. The left-hand side can be computed via the Plancherel formula for X; the right hand side, via the Plancherel formula for G. This will lead eventually – to a "almost-everywhere" lower bound on the norm of v_{γ} ; by continuity, we will deduce that $M_{\pi_{\chi}}$ is everywhere nonzero.

Let Z be the set of distinct isomorphism classes of the unitary representations π_{χ} ; it has the canonical structure of an orbifold, and the restriction of the canonical Plancherel measure μ on G is a well defined measure on Z. We will write $\pi_z, M_{\pi_z}, L_{\pi_z}, v_z$ etc.

For any $Z' \subset Z$ we consider the K_1 -invariant L^2 function:

$$\Phi: x \mapsto \int_{Z'} L_{\pi_z}(\pi_z(g)v_z)\mu(z), \quad x = Hg \in H \backslash G$$

on $X = H \backslash G$. For x = Hg, with $g \in G^+$, it coincides with the value of the $K_2 \times K_1$ -invariant function:

$$f: g \mapsto \int_{Z'} \langle \pi_z(g)v_z, K_2 * L_{\pi_z} \rangle \, \mu(z)$$

on G. In what follows, it is harmless to replace G^+ by $K_2 \cdot G^+ \cdot K_1$.

In order to compare norms we need the following fact (see discussion after Corollary 5.3.2):

There is a constant C > 0 such that the (surjective) orbit

$$o: K_2 \backslash G^+/K_1 \to X/K_1$$

satisfies:

$$Vol(o(S)) \le C \cdot Vol(S)$$

for any set S. (Here the volume of a subset $S \subset K_2 \backslash G^+/K_1$ is the volume of the corresponding K_2 , K_1 -invariant subset of G.)

Indeed, writing $x_0 \in X$ for the basepoint that corresponds to the identity coset in our identification $X \simeq H \setminus G$, we have $x_0 g K_1 = x_0 K_2 g K_1$, which is covered by at most $[K_2 \setminus K_2 g K_1]$ translates of $x_0 K_2$, all of which have equal measure. Therefore, the X-volume of the set x_0gK_1 is bounded above by a constant multiple (not depending on $g \in G^+$) of the G-measure of K_2gK_1 .

Consequently we have:

$$\int_{Z'} \|v_z\|^2 \mu(z) \stackrel{(a)}{=} \|\Phi\|_{L^2(X)}^2 = \int_{X/K_1} |\Phi(x)|^2 dx \le$$

$$\stackrel{(b)}{\leq} C \cdot \int_{K_2 \setminus G^+/K_1} |f(g)|^2 dg \le C \cdot \int_G |f(g)|^2 dg =$$

$$\stackrel{(c)}{=} C \cdot \int_{Z'} \|v_z\|^2 \|K_2 * L_{\pi_z}\|^2 \mu(z).$$

Here equality (a) is the Plancherel formula for X; inequality (b) arises from the equality $f(g) = \Phi(Hg)$ for $g \in G^+$ and our previous remark on measures; equality (c) is the Plancherel formula for G.

Because Z' is arbitrary, and the set of z for which $||v_z|| = 0$ has measure 0, it follows that $||K_2 * L_{\pi_z}||^2 \ge C^{-1}$ for almost every z, and because it is continuous ²⁹ in z it follows that the same holds for every z. In particular, $L_{\pi_z} \ne 0$ for every z. This completes the proof.

For the continuity, note that $||K_2*L_{\pi_z}|||^2$ can be expressed as the sum of values of $M_{\pi_z}(w,w)$ over an orthonormal basis w for $\pi_z^{K_2}$. In turn, this is expressed as a finite sum of integrals $\int f_z(h)dh$ over H, where each f_z is a matrix coefficient, varying continuously in z. Using the majorization (6.4) we see that, given $\varepsilon > 0$, we can find a compact subset $\Omega \subset H$ such that $\int_{h\notin\Omega} |f_z(h)|dh < \varepsilon$ for all z. The continuity is now clear.

Part 3

Spectral decomposition and scattering theory

7. Results

This is the core part of the present paper, where we develop the Plancherel decomposition of $L^2(X)$. By this we mean, more precisely, that we reduce the Plancherel decomposition to the understanding of discrete series for X and its boundary degenerations X_{Θ} .

Our approach to this is different to previous works on similar topics – in particular, the series of papers establishing the Plancherel decomposition for real semisimple symmetric spaces [vdBS05a, vdBS05b, Del98]. Our viewpoint is close to that of Lax and Phillips on scattering theory, and further from the viewpoint motivated by global Eisenstein series. The original idea, and a core argument in our approach, is due to Joseph Bernstein.

Our main theorem is Theorem 7.3.1, and our approach to its proof proceeds as follows:

- i. We first derive the existence of morphisms $L^2(X_{\Theta}) \to L^2(X)$, canonically characterized by their asymptotic properties, by using general Hilbert space theory and the existence of asymptotics of eigenfunctions.
- ii. By using a priori information about when a representation Π can occur simultaneously in $L^2(X_{\Theta})$ and $L^2(X_{\Omega})$, where $\Theta \neq \Omega$, we are able to analyze interactions between these maps from (i). This allows us, in particular, to decompose $L^2(X)$ in terms of the discrete parts of $L^2(X_{\Theta})$.
- iii. Finally, we discuss the question of writing an explicit formula for the morphism, using the Radon transform. (This corresponds to the study of "Eisenstein integrals".)

Our results are not complete in general. Step (i) works very generally—it uses only asymptotics of eigenfunctions as an input, and is performed in Sections 10–11. Step (ii), performed in Sections 12–14, requires a few "nonformal" inputs to work. Nonetheless, these non-formal inputs ("generic injectivity", §14.2 and the Discrete Series Conjecture 9.4.6) hold in a wide variety of cases (including symmetric spaces); and we expect the theorem to hold true in general, whether or not these inputs are valid. Step (iii) is performed in Section 15 under much more restrictive conditions, namely that the variety is strongly factorizable; the theory of Eisenstein integrals and their applications is open in the general case.

The reader who wishes to get an idea of our methods without diving into the details may wish to read §8. There we discuss the simple case of linear operators acting on functions on the non-negative integers, and explain how our methods operate in this (very well-known) case.

7.1. Plancherel decomposition and direct integrals of Hilbert spaces. We recall first some generalities about direct integrals and Plancherel decomposition.

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Any (separable) unitary representation $\mathcal H$ of G admits a direct integral decomposition:

$$\int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi) \tag{7.1}$$

where \hat{G} denotes the unitary dual of G, equipped with the Fell topology, and μ is a positive measure on the Borel σ -algebra of \hat{G} . (Note that these are standard Borel spaces, by [Dix77, Proposition 4.6.1], i.e. isomorphic as measurable spaces with the Borel space of a Polish space – which can be taken to be the interval [0,1] – even though the topologies on \hat{G} will be usually non-Hausdorff.)

The Hilbert space \mathcal{H}_{π} is π -isotypic (that is, a finite or countable direct sum of copies of π), and the direct integral makes sense only after we specify a family F of "measurable" sections $\eta_{:}\hat{G} \ni \pi \mapsto \eta_{\pi} \in \mathcal{H}_{\pi}$ satisfying the following axioms:

- (1) A section $\pi \mapsto \xi_{\pi}$ lies in F if and only if for each $\eta \in F$ the function $\pi \mapsto \langle \eta_{\pi}, \xi_{\pi} \rangle$ is measurable.
- (2) There is a countable subfamily $\{\eta_i\}_i \subset F$ such that for all $\pi \in \hat{G}$ the vectors $\{\eta_{i,\pi}\}_i$ span a dense subspace of \mathcal{H}_{π} .

Here and later, we will be using the word "measurable" to mean measurable with respect to the completion of the Borel σ -algebra of \hat{G} with respect to a given measure and a given family of measurable sections into Hilbert spaces, which should be clear from the context; when no measure is present, we mean Borel measurable. We will call this decomposition a *Plancherel* decomposition (and the corresponding measure a Plancherel measure) if μ -almost all spaces \mathcal{H}_{π} are non-zero.

7.2. Discrete spectrum. Before stating our results more precisely, we describe the precise notion of "discrete spectrum" and discuss a difficulty that arises. It should be noted that this difficulty vanishes in the case of symmetric varieties, and thus (to our knowledge) does not arise in prior work.

The space $L^2(X)$ decomposes into a direct sum of a "discrete" and a "continuous" part:

$$L^2(X) = L^2(X)_{\mathrm{disc}} \oplus L^2(X)_{\mathrm{cont}}$$

More precisely, discrete means "discrete modulo center", where "center" is the connected component of $\operatorname{Aut}_G(X)$. More formally, any $f \in L^2(X)$ can be disintegrated as an integral $\int_{\omega} f_{\omega} d\omega$ indexed by characters ω of $\operatorname{Aut}(X)$; here, each $f_{\omega} \in L^2(X; \omega)$.

DEFINITION. $L^2(X)_{\text{disc}}$ consists, by definition, of those f for which almost every f_{ω} belongs to the direct sum of all irreducible subrepresentations of $L^2(X;\omega)$.

It should be noted that it is by no means clear that this defines a closed subspace, although it follows from our later considerations.

Discrete series for a reductive group G come in continuous families, with varying central character, which can be constructed by twisting matrix coefficients by characters of the group. A similar "twisting" is possible for X-discrete series when X is a symmetric variety; however, for a general spherical variety X with infinite automorphism group (hence, in the split case, X-discrete series appearing in continuous families), the variation of X-discrete series with "central character" presents serious challenges, which we analyze in section 9. We expect that in every case the variation of Xdiscrete series can be described in terms of "algebraically twisted" families of representations, and we formulate this expectation as the "Discrete Series Conjecture" 9.4.6. We show how this can proven in individual cases by the method of "unfolding"; however, since we have not proven that this method always applies (although we know of no counterexample), the Discrete Series Conjecture remains a conjecture – but an easy one to check in each individual case. Therefore, formulating theorems which are conditional on this conjecture seems to present no serious harm of generality or applicability.

At the first reading the reader might prefer to skip most of this section ($\S 9$), using it only as a reference for terms and properties encountered in the rest of the paper. In sections 11 and 14 we obtain our main results on the spectral decomposition, which are summarized in the theorem below.

7.3. Main result. For any $\Theta, \Omega \subset \Delta_X$ (possibly the same), set

$$W_X(\Omega, \Theta) = \{ w \in W_X : w\Theta = \Omega \}$$

Let $\mathfrak{a}_{X,\Theta} = \mathcal{X}(\mathbf{A}_{X,\Theta})^* \otimes \mathbb{Q}$, $\mathfrak{a}_{X,\Theta}^*$ its \mathbb{Q} -linear dual. The vector space $\mathfrak{a}_{X,\Theta}$ has a "dominant" chamber $\mathfrak{a}_{X,\Theta}^+$ (namely, its intersection with the dominant chamber of \mathfrak{a}_X , which is a face of the latter), and it is known by properties of root systems that the union of the sets $w^{-1}\mathfrak{a}_{X,\Omega}^+$, $w \in W_X(\Omega,\Theta)$, where Ω varies over all possible subsets of Δ_X with $|\Omega| = |\Theta|$, is a perfect tiling for $\mathfrak{a}_{X,\Theta}$. Here, by "perfect tiling" we mean that these sets cover $\mathfrak{a}_{X,\Theta}$ and their interiors are disjoint, cf. [Art79a, Lemma 1]. ³⁰ Let:

$$c(\Theta)=$$
 the number of those chambers in $\mathfrak{a}_{X,\Theta}=\sum_{\Omega}\#W_X(\Omega,\Theta)$

where the second equality is a consequence of the tiling property.

 $^{^{30}}$ This "perfect tiling" claim is a simple property of root systems; we give a proof. In what follows, we discuss as if the system of spherical roots arises from a genuine Lie algebra; this would be the Lie algebra of the dual group to G_X . Now, any $\lambda \in \mathfrak{a}_{X,\Theta}$ defines a subset of roots, namely those which are non-negative on λ . This corresponds to the set of roots of a parabolic subalgebra \mathfrak{p} , with Levi factor \mathfrak{m} given by the centralizer of λ . Now there is a unique element w of the Weyl group taking \mathfrak{p} to a standard parabolic subalgebra \mathfrak{q} . This element w carries the center \mathfrak{a}_P of \mathfrak{m} to the center \mathfrak{a}_Q of the standard Levi of \mathfrak{q} . Moreover, w is well-defined up to the Weyl group of \mathfrak{q} , which acts trivially on \mathfrak{a}_Q , i.e. the map $\mathfrak{a}_P \to \mathfrak{a}_Q$ is uniquely determined.

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For the statement of the following theorem, recall that while the map: $\mathcal{Z}(\mathbf{L}_{\Theta})^0 \to \mathbf{A}_{X,\Theta}$ is surjective as a map of algebraic tori, it may not be surjective at the level of k-points.

We prove:

7.3.1. THEOREM (Scattering theorem). Suppose that **X** is a wavefront spherical variety and that all degenerations X_{Θ} (including $X_{\Theta} = X$) satisfy the Discrete Series Conjecture 9.4.6. Also assume the condition of "generic injectivity of the map: $\mathfrak{a}_X^*/W_X \to \mathfrak{a}^*/W$ on each face" (cf. §14.2)

Then there exist canonical G-equivariant morphisms ("Bernstein morphisms")

$$\iota_{\Theta}: L^2(X_{\Theta}) \to L^2(X)$$

and $A'_{X,\Theta} \times G$ -equivariant isometries ("scattering morphisms"):

$$S_w: L^2(X_{\Theta}) \longrightarrow L^2(X_{\Omega}), \quad w \in W_X(\Omega, \Theta)$$

where $A_{X,\Theta}$ acts on $L^2(X_{\Omega})$ via the isomorphism: $\mathbf{A}_{X,\Theta} \xrightarrow{\sim} \mathbf{A}_{X,\Omega}$ induced by w and $A'_{X,\Theta}$ denotes the image of $\mathcal{Z}(L_{\Theta})^0$ in $A_{X,\Theta}$, with the following properties:

$$\iota_{\Omega} \circ S_w = \iota_{\Theta}, \tag{7.2}$$

$$S_{w'} \circ S_w = S_{w'w} \text{ for } w \in W_X(\Omega, \Theta), w' \in W_X(Z, \Omega),$$
 (7.3)

$$\iota_{\Omega}^* \circ \iota_{\Theta} = \sum_{w \in W_X(\Omega, \Theta)} S_w. \tag{7.4}$$

Finally, the map:

$$\sum_{\Theta} \frac{\iota_{\Theta, \text{disc}}^*}{\sqrt{c(\Theta)}} : L^2(X) \to \bigoplus_{\Theta \subset \Delta_X} L^2(X_{\Theta})_{\text{disc}}$$
 (7.5)

(where $\iota_{\Theta, disc}^*$ denotes the composition of ι_{Θ}^* with orthogonal projection to the discrete spectrum) is an isometric isomorphism onto the subspace of vectors

$$(f_{\Theta})_{\Theta} \in \bigoplus_{\Theta} L^2(X_{\Theta})_{\text{disc}}$$

satisfying:

$$S_w f_{\Theta} = f_{\Omega}$$
 for every $w \in W_X(\Omega, \Theta)$.

We observe one way in which our approach differs from the usual one: for the proof of this Theorem, we do not write down any explicit formula for ι_{Θ} ; instead, it is entirely characterized in terms of asymptotic properties. The question of writing a formula for ι_{Θ} is the concern of §15; our main results are Theorem 15.6.1 and Theorem 15.6.2. We do not summarize them here because they are more technical. Our results are complete in many cases (including the group case), but not in complete generality.

We do not know if every wavefront spherical variety satisfies either of the two algebraic multiplicity conditions of Theorem 7.3.1, though we know no counterexample. However, they can easily be seen to fail in the case of the non-wavefront variety $\mathbf{GL}_n \backslash \mathbf{SO}_{2n+1}$. As was the case for the theory of asymptotics, we expect our scattering theory to extend to such cases as well:

7.3.2. Conjecture. The conclusions of Theorem 7.3.1 are true for any spherical variety X.

8. Two toy models: the global picture and semi-infinite matrices

We now consider two "toy models" for our reasoning in this paper:

- The first is not really a toy model: it is the global picture for automorphic forms. However, it may be more familiar to readers of this paper than the local setting.

The main point we wish to convey is that the relationship between "smooth" asymptotics $C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X)$ and the L^2 -morphisms $L^2(X_{\Theta}) \to L^2(X)$ corresponds – in this global setting – to the difference to integrating Eisenstein series "far from the unitary axis" and "on the unitary axis." This point of view is not central to us (although we establish, in the case of symmetric varieties and more generally, a corresponding result) but it is closer to other developments and is useful to keep in mind.

- The second is "scattering theory on the non-negative integers": Given a semi-infinite symmetric real matrix A_{ij} with the property that A_{ij} depends only on i-j when i and j are both large, what can we say about its eigenvalues and the corresponding L^2 -spectrum?

Here (unlike the global picture) we sketch proofs, so the reader may get a simple idea of the techniques that we will use in the spherical variety case.

8.1. Global picture. Let \mathbf{G} be a semisimple algebraic group over a number field K; fix a maximal F-split torus with Weyl group W and a minimal K-parabolic containing it. Let $\mathbf{P} = \mathbf{M}\mathbf{N}$ be a standard (i.e., containing the fixed minimal parabolic) parabolic K-subgroup and denote by \mathbf{A}_P the center of \mathbf{M} . We write $w(\mathbf{P})$ for the number of chambers in $\mathrm{Hom}(\mathcal{X}(\mathbf{M}), \mathbb{R})$ (a chamber is a connected component of the space obtained by removing the kernel of roots.) Write $X_{\mathbf{P}} = \mathbf{N}(\mathbb{A}_K)\mathbf{P}(K)\backslash\mathbf{G}(\mathbb{A}_K)$ and set $X = X_{\mathbf{G}}$, so that

$$X = \mathbf{G}(K) \backslash \mathbf{G}(\mathbb{A}_K).$$

By $C^{\infty}(X)$ we shall mean the set of functions that are invariant by an open compact of $\mathbf{G}(\mathbb{A}_{K \text{finite}})$ and are smooth along each translate of $G_{\infty} := \mathbf{G}(K \otimes \mathbb{R})$. By $C^{\text{aut}}(X)$ we mean the space of *automorphic forms* on X (defined in the standard way, see e.g. [**BJ79**]); similarly for $X_{\mathbf{P}}$.

8.1.1. Smooth asymptotics. Recall the constant term:

$$c_{\mathbf{P}}: C^{\infty}(X) \longrightarrow C^{\infty}(X_{\mathbf{P}})$$

obtained by integration along "horocycles": $c_{\mathbf{P}}f(g) = \int_{\mathbf{N}(K)\backslash\mathbf{N}(\mathbb{A}_K)} f(ng) dn$, the integral being taken with respect to the $\mathbf{N}(\mathbb{A}_K)$ -invariant probability measure. This is the analog of our smooth asymptotics map

$$e_{\Theta}^*: C^{\infty}(X) \to C^{\infty}(X_{\Theta});$$

indeed under certain stronger smoothness assumptions on $f \in C^{\infty}(X)$ (for example, L^2 -boundedness of all elements Xf, where X belongs to the universal enveloping algebra of $\mathbf{G}(F \otimes \mathbb{R})$), it is known that f is asymptotically equal to c(f), in a suitable sense and in a particular direction.³¹

In the adjoint direction we have the pseudo-Eisenstein series:

$$e_{\mathbf{P}}: C_c^{\infty}(X_{\mathbf{P}}) \longrightarrow C_c^{\infty}(X)$$

(with rapidly decaying image) defined by the rule $e_{\mathbf{P}}f: g \mapsto \sum_{P(K)\backslash G(K)} f(\gamma g)$. This is the analog of our smooth dual asymptotics map

$$e_{\Theta}: C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X).$$

8.1.2. Eisenstein series. We will be very brief, with the understanding that this section is targeted only at readers who have previous experience with the theory of Eisenstein series.

In what follows, ω will denote a character of $A_P := \mathbf{A}_P(\mathbb{A}_K)/\mathbf{A}_P(K)$, not necessarily unitary; the space $C^{\mathrm{aut}}(X_P)_{\omega}$ denotes the subspace of $C^{\mathrm{aut}}(X_P)$ comprising functions that transform under the character ω under the normalized action of A_P . Also if ω is unitary we may form the space $L^2(X_P)_{\omega}$ of functions f which transform under A_P and so that $|f|^2$ is integrable on X_P/A_P .

We fix a Haar measure on A_P and a dual measure on the Plancherel dual \widehat{A}_P . The set of all, not necessarily unitary, characters of A_P will be denoted by $\widehat{A}_{P\mathbb{C}}$.

For all ω with "sufficiently large real part", and every $f \in C^{\text{aut}}(X)_{\omega}$ the series defining $e_{\mathbf{P}}f$ is absolutely convergent. Moreover, for a Zariski-open set of $\omega \in \widehat{A}_{P\mathbb{C}}$, this series can be regularized by a process of meromorphic continuation. We refer to the result as $e_{\mathbf{P},\omega}$.

The analog of our "Bernstein" L^2 -morphism $\iota_{\Theta}: L^2(X_{\Theta}) \to L^2(X)$ is given by the unitary Eisenstein series:

$$E_{\mathbf{P}}: L^2(X_{\mathbf{P}}) \to L^2(X)$$
, characterized by: (8.1)

$$\int_{\omega \in \widehat{A_P}} f_{\omega} d\omega \quad \mapsto \quad \int_{\omega \in \widehat{A_P}} e_{\mathbf{P},\omega} f_{\omega} d\omega \tag{8.2}$$

³¹This has roughly the same content as the following fact, due to Harish-Chandra and Langlands: If f is an automorphic form on X, then the truncation $\wedge f$ is of rapid decay.

for $f_{\omega} \in C^{\text{aut}}(X_{\mathbf{P}})_{\omega} \cap L^2(X_{\mathbf{P}})_{\omega}$ that varies measurably in ω (suitably interpreted). This map satisfies an asymptotic property analogous to that characterizing $L^2(X_{\Theta}) \to L^2(X)$ (see Theorem 11.1.2).

How does this compare to the "smooth" asymptotics $C_c^{\infty}(X_{\Theta}) \to C^{\infty}(X)$? Given a "sufficiently positive" real character $\omega_0 : A_P \to \mathbb{R}_{>0}$, the morphism $e_{\mathbf{P}}$ is characterized by:

$$e_{\mathbf{P}}: C_c^{\infty}(X_{\Theta}) \rightarrow C^{\infty}(X)$$
 (8.3)

$$\int_{\operatorname{Re}(\omega)=\omega_0} f_{\omega} \quad \mapsto \quad \int_{\operatorname{Re}(\omega)=\omega_0} e_{\mathbf{P},\omega} f_{\omega}. \tag{8.4}$$

whenever the left-hand side belongs to $C_c^{\infty}(X_{\Theta})$. (In fact, a function in $C_c^{\infty}(X_{\Theta})$ may be uniquely expressed as $\int_{\text{Re}(\omega)=\omega_0} f_{\omega}$, so long as the real part of ω_0 is sufficiently positive.)

The difference, then, between smooth and L^2 -asymptotics is (from this point of view) the "line of integration." One can pass from smooth to L^2 by shifting contours: on the other hand, this will introduce extra contributions coming from residues of the map $e_{\mathbf{P},\omega}$.

The maps $E_{\mathbf{P}}$ satisfy an exact analog of Theorem 7.3.1; see, for example, the "main theorem" on page 256 of [Art79b], and for more details [Lan76, MW95].

8.2. Spectra of semi-infinite matrices: scattering theory on \mathbb{N} . Let $C(\mathbb{N})$ and $C_c(\mathbb{N})$ denote the vector spaces of all functions, resp. all compactly supported functions, on the set \mathbb{N} of non-negative integers.

Fix some real numbers c_0, c_1, \ldots, c_K ; we set $c_k = c_{|k|}$ if $|k| \leq K$ and $c_k = 0$ otherwise. We shall consider real self adjoint operators $T : L^2(\mathbb{N}) \to L^2(\mathbb{N})$ that are given by the rule

$$Tf(x) = \sum_{|k| \le K} f(x+k)c_{|k|}$$
 (8.5)

for all large enough x, i.e. there exists M so that this equality holds whenever $x \geq M$. (Such operators are easily seen to exist: for example, (8.5) defines a self-adjoint operator on $L^2(\mathbb{Z})$, and then one composes with the orthogonal projection to $L^2(\mathbb{N})$.)

We denote by T_{∞} the operator on $L^2(\mathbb{Z})$ defined by rule (8.5) for all x; In what follows, let $P \in \mathbb{C}[z, z^{-1}]$ be defined by $\sum_{k=-K}^K c_{|k|} z^k$; we think of this as a meromorphic function of the complex variable z. We denote by $P^{-1}(\lambda)$ the set $\{z \in \mathbb{C} : P(z) = \lambda\}$.

8.2.1. Smooth asymptotics. There is a unique morphism

$$e: C_c(\mathbb{Z}) \longrightarrow C_c(\mathbb{N})$$

that intertwines the T_{∞} and T-action, and carries the characteristic function δ_k of $k \in \mathbb{Z}$ (considered as an element of $C_c(\mathbb{Z})$) to δ_k (now considered as an element of $C_c(\mathbb{N})$), if k is sufficiently large.

Indeed, $v_k := e(\delta_k)$ can be solved for inductively by means of the linear recurrence that they satisfy. Indeed,

$$c_{-K}v_{m-K} + \cdots + c_{K}v_{m+K} = e(T_{\infty}\delta_{m}) = Te(\delta_{m}) = Tv_{m}$$

which is to say that we can determine v_s given knowledge of $v_{s'}$ and $Tv_{s'}$ when s' > s.

The meaning of the phrase "asymptotics" is clearer for the dual morphism $e^*: C(\mathbb{N}) \to C(\mathbb{Z})$. Let $C^{\infty}(\mathbb{N})^{\lambda}$ (resp. $C^{\infty}(\mathbb{Z})^{\lambda}$) be the space of T-eigenfunctions on \mathbb{N} with eigenvalue λ (resp. T_{∞} -eigenfunctions on \mathbb{Z} with eigenvalue λ). Then the dual asymptotics map e^* gives a natural (not always injective) map

$$C^{\infty}(\mathbb{N})^{\lambda} \to C^{\infty}(\mathbb{Z})^{\lambda}$$
 (8.6)

i.e. for any $v \in C^{\infty}(\mathbb{N})^{\lambda}$ there is a unique $w \in C^{\infty}(\mathbb{Z})^{\lambda}$ so that v(n) = w(n) for all large n.

Notice that any eigenfunction of T (and of T_{∞}) necessarily is of the form

$$f(n) = \sum_{i=1}^{k} \alpha_i^n Q_i(n) \quad \text{for } n > M + K,$$
(8.7)

where $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$ and $Q_i(n)$ are nonzero polynomials of total degree $\leq 2K$; also, M is as in (8.5). If there indeed exists such an eigenfunction, the eigenvalue is necessarily given by $P(\alpha_1) = P(\alpha_2) = \cdots = P(\alpha_k)$.

8.2.2. Finiteness of the discrete spectrum. We show now that the eigenfunctions of T in L^2 span a finite-dimensional space. (It is easier to see that the corresponding assertion for $C_c(\mathbb{N})$: indeed $C_c(\mathbb{N})$ is a finitely generated module over $\mathbb{C}[T]$.)

Consider an eigenfunction of T of the form (8.7). If it is to belong to L^2 we must have $|\alpha_i| < 1$ for every i.

Fix for a moment the degrees d_1, \ldots, d_k of the polynomials Q_i . Then the condition that there exists an eigenfunction with the above "asymptotic" is a finite system of linear equations in the coefficients of the Q_i as well as the M+K values $f(0), \ldots, f(M+K)$; the coefficients of this linear system depend algebraically on the α_i . In particular the set of $(\alpha_1, \ldots, \alpha_n)$ for which there exist a solution with $\deg(Q_i) = d_i$ is a constructible subset of \mathbb{C}^n . (As usual, constructible means that it is defined by a finite system of polynomial equalities and inequalities.)

Thus we have a constructible subset $Z \subset \mathbb{C}^k$ whose intersection with $\{\underline{\alpha} : |\alpha_i| < 1\}$ is countable – the corresponding finite-dimensional eigenspaces are orthogonal in the separable Hilbert space $L^2(\mathbb{N})$. (Notice that in the above process we fixed the *degree* of the polynomials, and not just an upper bound; so there are no redundant α_i 's for each eigenfunction.) Therefore, this intersection is in fact finite:

To verify the finiteness, we use the following properties of Zariski-closed subsets $Y \subset \mathbf{C}^k$: There are only finitely many zero-dimensional (Zariski-)irreducible components of Y, and if p does not lie on such a component,

then any neighborhood of p in Y, for the standard topology on \mathbb{C}^k , is uncountable. For the latter assertion we just note that there exists an analytic nonconstant map $f: \mathbb{C} \to Y$ with f(0) = p, which we may verify by slicing by a hyperplane to reduce to the case of $\dim_p(Y) = 1$, i.e. a curve, where we can use the existence of a smooth neighborhood on a desingularization $\tilde{Y} \to Y$. With this in mind, let Z be a constructible set; it is a finite union of sets of the form Y'_i , where each Y'_i is Zariski-open in a Zariski-closed set $Y_i \subset \mathbb{C}^k$. Let B be any open subset of \mathbb{C}^k for the usual topology. If $p \in Z \cap B$ is not one of the finitely many zero-dimensional components of some Y_i , our discussion above has shown that there exists uncountably many points in $Z \cap B$.

8.2.3. L^2 -decomposition. There is a unique bounded map

$$\iota: L^2(\mathbb{Z}) \hookrightarrow L^2(\mathbb{N})$$

(in fact, with image in $L^2(\mathbb{N})_{\mathrm{cts}}$) that intertwines the actions of T and T_{∞} and is "asymptotically the natural identification":

$$\|\iota\delta_k - \delta_k\|_{L^2(\mathbb{N})} \to 0, \ k \to \infty.$$
 (8.8)

We will not give a complete proof of this statement but we will outline the identification between the spectra of T and T_{∞} that is the central ingredient.

According to spectral theory, we may find a measure $\mu(\lambda)$ on the real line together with an isomorphism

$$L^2(\mathbb{N}) \stackrel{\sim}{\to} \int_{\mathbb{R}} \mathcal{H}_{\lambda} \mu(\lambda)$$
 (8.9)

which carries the action of T to multiplication by λ . Here \mathcal{H}_{λ} is a family of Hilbert spaces – finite-dimensional in the present case – over $\lambda \in \mathbb{R}$.

Now let $\nu = \mu - \sum_{\lambda} \mu(\{\lambda\}) \delta_{\lambda}$, the measure obtained from μ by removing all atoms. This yields a corresponding decomposition of $L^2(\mathbb{N})_{\text{cts}}$:

$$L^2(\mathbb{N})_{\mathrm{cts}} \stackrel{\sim}{\to} \int_{\mathbb{R}} \mathcal{H}_{\lambda} \nu(\lambda).$$

In particular, when analyzing $L^2(\mathbb{N})_{\text{cts}}$, we can and do neglect the finite set of λ for which there is a solution to $P(z) = \lambda$ with multiplicity > 1, because that set has ν -measure zero.

Because T acts on \mathcal{H}_{λ} as multiplication by λ , the morphism $C_c(\mathbb{N}) \to \mathcal{H}_{\lambda}$ must necessarily factor through the quotient $C_c(\mathbb{N})_{\lambda}$ of $C_c(\mathbb{N})$ generated by all $Tf - \lambda f$, for $f \in C_c(\mathbb{N})$. In particular, \mathcal{H}_{λ} is finite dimensional, since $C_c(\mathbb{N})$ is a finitely generated $\mathbb{C}[T]$ -module; also, any linear functional on \mathcal{H}_{λ} may be expressed $f \mapsto \int f \phi_{\lambda}$, where $\phi_{\lambda} \in C(\mathbb{N})^{\lambda}$ satisfies $T \phi_{\lambda} = \lambda \phi_{\lambda}$. (Here we write \int for the functional $f \in C_c(\mathbb{N}) \to \sum_{x \in \mathbb{N}} f(x)$).

Because of the asymptotics of eigenfunctions (8.7), the image of f in $C_c(\mathbb{N})_{\lambda}$ depends only on the values $f(0), \ldots, f(M+K)$ together with the values of the "Fourier transform" $\hat{f}(z)$, for $z \in P^{-1}(\lambda)$; here $\hat{f}(z) = \sum_n f(n)z^n$,

the "Fourier transform" of f. It follows that \mathcal{H}_{λ} can be identified with the completion of $C_c(\mathbb{N})$ with respect to a Hermitian form of the type³²

$$H_{\lambda}(f) = \sum_{1 \le n, m \le M + K} b_{n,m}(\lambda) f(n) \overline{f(m)} + \sum_{z,z' \in P^{-1}(\lambda)} a_{z,z'} \hat{f}(z) \overline{\hat{f}(z')}, \quad (8.10)$$

It is not difficult to see that (away from a set of measure zero) $a_{z,z'} = 0$ unless $|z| \leq 1$ and $|z'| \leq 1$. To prove this, one works with a "Schwartz" space slightly larger than $C_c(\mathbb{N})$, allowing functions that have rapid (faster than any polynomial) decay at ∞ ; then H_{λ} must extend continuously to $C_c(\mathbb{N})$, which translates into the desired vanishing. We do not give details here, but the idea is due to Bernstein [Ber88] and the analogous step in our context is in part (2) of Corollary 11.2.2.

The terms $a_{z,z'}$ themselves include the "diagonal" case where $z' = z^{-1} = \bar{z}$; for those, set $a_z := a_{z,z'}$. If we fix a function f and consider an average of the hermitian forms of its (right) translates:

$$\frac{1}{k+1} \sum_{i=l}^{l+k} H_{\lambda}(S^i f),$$

(where S is the translation operator Sf(x) = f(x-1) where we extend f by zero off \mathbb{N} , and l is arbitrary) then in the limit as $k \to \infty$ only the "diagonal" terms survive. The reason for this is all other terms $a_{z,z'}$ occur with a coefficient involving $\frac{1}{k+1} \sum_{i=l}^{l+k} (z\overline{z}')^i$, and this sum goes to zero for large k in the non-diagonal case.

On the other hand:

$$||f||_{L^{2}(\mathbb{N})}^{2} = \frac{1}{k+1} \sum_{i=l}^{l+k} ||S^{i}f||_{L^{2}(\mathbb{N})} = \lim_{k \to \infty} \frac{1}{k+1} \sum_{i=l}^{l+k} ||S^{i}f||_{L^{2}(\mathbb{N})_{cts}} =$$

$$= \lim_{k \to \infty} \frac{1}{k+1} \sum_{i=l}^{l+k} \int_{\mathbb{R}} H_{\lambda}(S^{i}f) \nu(\lambda). \tag{8.11}$$

In the discussion that follows we will justify the fact that one can interchange the limit and the integral³³ in order to arrive at the conclusion:

$$||f||^2 = \int_{\mathbb{R}} \nu(\lambda) \sum_{z \in P^{-1}(\lambda): |z| = 1} |\hat{f}(z)|^2 a_z.$$
 (8.12)

Notice that, if we think of f as a function on $L^2(\mathbb{Z})$, the above expression is invariant under left translation S^{-1} , in particular: it holds for *every* element of $C_c^{\infty}(\mathbb{Z})$. The uniqueness of the spectral decomposition for T_{∞}

³²Warning: Neither $f \mapsto f(n)$, for $n \leq M_K$, nor the functionals $f \mapsto \hat{f}(z)$ $(z \in P^{-1}(\lambda))$ have to descend to the quotient $C_c(\mathbb{N})_{\lambda}$.

³³We thank Joseph Bernstein for pointing out an omission in a previous version.

acting on $L^2(\mathbb{Z})$ implies that it has to coincide with it. More precisely, writing $e(\theta) := e^{2\pi i\theta}$ for $\theta \in \mathbb{R}/\mathbb{Z}$,

$$slope(\theta) := \frac{d}{d\theta} P(e(\theta)),$$

and slope(z) if $z = e(\theta)$, we can write the formula:

$$||f||^2 = \int_{\mathbb{R}/\mathbb{Z}} |\hat{f}(e(\theta))|^2 d\theta,$$

(where the measure on $\theta \in \mathbb{R}/\mathbb{Z}$ is the Haar probability measure) as:

$$||f||^2 = \int_{\mathbb{R}} d\lambda \sum_{e(\theta) \in P^{-1}(\lambda)} |\hat{f}(e(\theta))|^2 |\operatorname{slope}(\theta)|^{-1};$$

this is the spectral decomposition for $L^2(\mathbb{Z})$ under T_{∞} . We conclude that:

$$\int d\lambda \sum_{z \in P^{-1}(\lambda): |z| = 1} |\hat{f}(z)|^2 |\operatorname{slope}(z)|^{-1} = \int \nu(\lambda) \sum_{z \in P^{-1}(\lambda): |z| = 1} |\hat{f}(z)|^2 a_z.$$

and from this we deduce that $\nu(\lambda)$ is absolutely continuous with respect to the Lebesgue measure; without loss of generality, we may take it to equal Lebesgue measure (restricted to the set where the expression on the left is supported), and then

$$a_z = |\operatorname{slope}(z)|^{-1}. (8.13)$$

This analysis shows, in effect, that some part of the Plancherel formula for T acting on $L^2(\mathbb{N})$ is determined by the Plancherel formula for T_{∞} acting on $L^2(\mathbb{Z})$. This is the starting point for the construction of the morphism ι of (8.8), although we will not give details of that construction. We will use this type of idea in §11.

8.2.4. Remark. The Hermitian form H_{λ} in fact satisfies more constraints, which force many values $a_{z,z'}$ to be zero; we don't discuss this here.

We now proceed to justifying the interchange of limit and integral in (8.11). The argument will be more involved than the one used later in Section 11, where we will have a priori knowledge of the fact that the values of z in the above expression for H_{λ} with |z| < 1 are actually uniformly bounded, in absolute value, by a constant c < 1. Here we do not know that, and we will need to separate the set of λ 's in two.

First of all, fix the function f and let us consider the effect of shifting the function f, i.e., replace it by the function $Sf: n \mapsto f(n-1)$; we interpret f(-1), f(-2) etc. as 0.

Associated to f there is a (positive) spectral measure $\mu_f = H_{\lambda}(f)\nu(\lambda)$ on the set of λ 's, where $\nu(\lambda)$ is as in (8.9).

We will need in particular the following key lemma:

Let $L(\delta)$ be the set of eigenvalues λ for which there is $z \in P^{-1}(\lambda)$ such that $1 - \delta < |z| < 1$. Then:

$$\lim_{(\delta,i)\to(0,\infty)} \mu_{S^i f}(L(\delta)) = 0. \tag{8.14}$$

Granted that, we can split the integral in (8.11) into an integral on $\mathbb{R} \setminus L(\delta)$ and an integral on $L(\delta)$, with the contribution of the latter to the right-hand side of the being less than any given $\varepsilon > 0$ as long as δ is small enough and l is large enough. A result of linear algebra (Proposition 10.3.5) allows us to apply the dominated convergence theorem to the former as $k \to \infty$ – cf. also the proof of Theorem 11.3.1. Since ε is arbitrary, we arrive at (8.12).

Finally, we are left with proving (8.14). Since $L(\delta)$ is the intersection with \mathbb{R} of $\{P(z)|\ 1-\delta<|z|<1\}$, it is a semialgebraic set and so a union of intervals (which may or may not contain their endpoints); moreover, the union of the sets $\{\delta\}\times L(\delta)$ is a semialgebraic subset of $(0,1)\times\mathbb{R}$. Also the number of components of $L(\delta)$ is bounded independently of δ , as follows from the following fact, known as Hardt triviality [Har80] (although there is surely an elementary proof):

Given a map $f: X \to Y$ of semialgebraic sets, there exists a finite partition of Y into semi-algebraic subsets so that, on each part Y_i , the map f is (semialgebraically) trivial, i.e., there exists a semialgebraic homeomorphism of $f^{-1}(Y_i) \to Y$ with $F_i \times Y_i \to Y_i$, where F_i is any fiber.

Since $L(\delta)$ is decreasing as $\delta \to 0$ with $\cap L(\delta) = \emptyset$, these intervals must all have length that goes to zero as δ does.

So it is enough to show that, for any $\varepsilon > 0$ there are k and N so that:

(the μ_{S^if} -measure of any interval of the form $(\frac{m}{2^k},\frac{m+1}{2^k})$, with $m\in\mathbb{Z}$) < ε

for all i > N. Notice that the support of all $\mu_{S^i f}$'s lies in a compact set, namely $[-\|T\|, \|T\|]$ where $\|T\|$ is the operator norm of T, so it is enough to consider a finite number of those intervals.

As we will see, the measures μ_{S^if} converge weakly to μ_f^{∞} (the spectral measure of f considered as an element of $L^2(\mathbb{Z})$), which is absolutely continuous with respect to Lebesgue measure. Being, in addition, positive, we can eventually bound the μ_{S^if} -measures of the aforementioned intervals in terms of the μ_f^{∞} -measures of slightly larger intervals, which in turn tend to zero with the length of those.

To show the weak convergence, since these are positive measures whose total mass is uniformly bounded above, it is enough to show it on polynomials. We have:

$$\int \lambda^m \mu_{S^i f} = \langle T^m S^i f, S^i f \rangle$$

which for large i is equal to:

$$\langle T_{\infty}^m S^i f, S^i f \rangle = \langle T_{\infty}^m f, f \rangle = \int \lambda^m \mu_f^{\infty}.$$

This shows that μ_{S^if} converge to μ_f^{∞} , as we just explained above, and concludes our argument for the interchange of limit and integral in (8.11).

9. The discrete spectrum

This section addresses the discrete spectrum of $L^2(X)$, and in particular its variation with central character; see §7.2 for a general discussion of the difficulties involved, and §9.3.1 for more details.

9.1. Decomposition according to the center. We let $\widehat{\mathcal{Z}(X)}_{\mathbb{C}}$ denote the group of complex characters of $\mathcal{Z}(X) := \operatorname{Aut}_G(X)^0$; it is a commutative complex group with infinitely many components (unless $\mathcal{Z}(X)$ is trivial). The identity component is the full torus of unramified characters of $\mathcal{Z}(X)$, and (of course) each connected component is a torsor thereof. The subgroup of unitary unramified characters acts transitively on the unitary characters of each component, inducing a canonical "imaginary" structure on each of them (i.e. the structure of a real algebraic variety which we will call "imaginary"). The subgroup of unitary characters of $\widehat{\mathcal{Z}(X)}_{\mathbb{C}}$ will simply be denoted by $\widehat{\mathcal{Z}(X)}$. We keep assuming, as we have done since Section 5, that $\mathcal{Z}(\mathbf{G})^0$ surjects onto $\mathcal{Z}(\mathbf{X})$. Notice that when we replace \mathbf{X} by a boundary degeneration \mathbf{X}_{Θ} , this means that we also have to "enlarge" the group \mathbf{G} in order to take into account the additional action of $\mathbf{A}_{X,\Theta} = \mathcal{Z}(\mathbf{X}_{\Theta})$.

If we fix a Haar measure on $\mathcal{Z}(X)$, the maps

$$p_{\omega}(\Phi)(x) = \int_{\mathcal{Z}(X)} (z \cdot \Phi)(x) \omega^{-1}(z) dz$$

are surjective maps: $p_{\omega}: C_c^{\infty}(X) \to C_c^{\infty}(X, \omega)$, for every $\omega \in \widehat{\mathcal{Z}(X)}_{\mathbb{C}}$. Here we denote by $C_c^{\infty}(X, \omega)$ the space of smooth functions on X, whose support has compact image under $X \to X/\mathcal{Z}(X)$ and which are eigenfunctions with eigencharacter ω under the (normalized) action of $\mathcal{Z}(X)$. The fixed measures on X and $\mathcal{Z}(X)$ induce a measure on $\mathcal{Z}(X) \setminus X$ and hence L^2 -norms on the spaces: $C_c^{\infty}(X,\omega)$; the corresponding Hilbert space completions will be denoted by $L^2(X,\omega)$.

The Plancherel formula for $L^2(X)$ viewed as a unitary representation of $\mathcal{Z}(X)$ reads:

$$L^{2}(X) = \int_{\widehat{\mathcal{Z}(X)}} L^{2}(X, \omega) d\omega; \qquad (9.1)$$

$$\|\Phi\|_{L^2(X)}^2 = \int_{\widehat{\mathcal{Z}(X)}} \|p_\omega \Phi\|_{L^2(X,\omega)}^2 d\omega, \text{ for all } \Phi \in C_c^\infty(X)$$
 (9.2)

where $d\omega$ is the Haar measure on $\widehat{\mathcal{Z}}(X)$ dual to the chosen measure on $\mathcal{Z}(X)$. From now on we will consider as fixed a measure on $\mathcal{Z}(X)$. A relative discrete series (RDS) representation for X, or an X-discrete series representation, is a pair (π, M) where π is an irreducible smooth representation of G with unitary central character ω and M is a morphism: $\pi \to C^{\infty}(X)$ whose image lies in $L^2(X,\omega)$. The morphism M induces a canonical unitary structure on π , pulled back from $L^2(X,\omega)$. The images of all such M, for given ω , span the discrete spectrum $L^2(X,\omega)_{\text{disc}}$ of $L^2(X,\omega)$.

Assuming that the orthogonal projections: $L^2(X,\omega) \to L^2(X,\omega)_{\text{disc}}$ are measurable in ω (we will prove this in Proposition 9.3.3), they define a direct summand $L^2(X)_{\text{disc}}$, the discrete spectrum of X, which has a Plancherel decomposition according to the center:

$$L^{2}(X)_{\text{disc}} = \int_{\widehat{\mathcal{Z}(X)}} L^{2}(X, \omega)_{\text{disc}} d\omega; \qquad (9.3)$$

It is obvious that X-discrete series belong to $L^2(X)$ in the sense of Fell topology:

9.1.1. LEMMA. If (L, π) is an RDS then π belongs weakly (i.e. under the Fell topology) in $L^2(X)$.

PROOF. Choose a continuous section Y of $X \to X/\mathcal{Z}(X)$, a "radial" function: $s: \mathcal{Z}(X) \to \mathbb{R}_+$ (for example, project $\mathcal{Z}(X)$ to its quotient by its maximal compact, giving a free abelian group Λ , and let s be the restriction of a Euclidean norm on $\Lambda \otimes \mathbb{R}$) and let $\wedge^T: C^\infty(X) \to L^2(X)$ denote "normalized truncation":

$$\wedge^{T}(\Phi)(x) = \begin{cases} \frac{\sqrt{\eta(z)}}{\sqrt{\operatorname{Vol}(s^{-1}[0,T])}} \Phi(x), & \text{if } x = y \cdot z \in Y \cdot s^{-1}[0,T] \\ 0, & \text{otherwise} \end{cases}$$

(Recall that η denotes the eigencharacter of the fixed measure on X.) Then each diagonal matrix coefficient $\langle \pi(g)v,v\rangle$ of π is approximated, uniformly on compacta, by the diagonal matrix coefficients $\langle g\cdot \wedge^T(Lv), \wedge^T(Lv)\rangle$ of $L^2(X)$, as $T\to\infty$.

However, it is not as clear how the relative discrete series vary as one changes the central character ω , and hence how to decompose the space $L^2(X)_{\text{disc}}$. The reader can jump ahead to subsection 9.3.1 for a discussion of the difficulties that one faces. We will first discuss the issue of relative discrete series abstractly, in order to obtain the uniform boundedness of their exponents which we use for the spectral decomposition.

- **9.2.** A finiteness result. The purpose of this section is to show that the problem of constructing relative discrete series for X is, essentially, described by polynomial equations, and to draw certain very general conclusions from this fact, in particular:
- 9.2.1. THEOREM. For a fixed open compact subgroup J of G and a unitary character χ of $\mathcal{Z}(X)$ the space $L^2(X,\chi)^J_{\mathrm{disc}}$ is finite-dimensional.

In the proof we shall use the following simple lemma:

- 9.2.2. LEMMA. Suppose V, W are finite dimensional vector spaces, R a complex algebraic variety (identified with the set of its complex points), and $\{T_i(r)\}_{i\in I} \subset \operatorname{Hom}(V,W) \ (r \in R)$ a (possibly infinite) collection of linear maps which vary polynomially in r (that is, $T_i \in \operatorname{Hom}(V,W) \otimes \mathbb{C}[R]$).
 - (1) For each r, write $W_r := \operatorname{span}\{T_i(r): i \in I\}$. There exists a finite subset $J \subset I$ so that W_r is spanned by $\{T_i(r): j \in J\}$, for all r.
 - (2) Assume that I is finite, and write V_r to be the common kernel of all $T_i(r)$. There exists a constructible partition $R = \bigcup R_{\alpha}$ so that over each R_{α} , V_r is a trivializable sub-bundle of V (i.e. there exists a vector space V_1 and isomorphisms $f_r : V_1 \xrightarrow{\sim} V_r$ varying algebraically with $r \in R_{\alpha}$.).
 - (3) Assume that I is finite, suppose $V \subset W$, and let \mathcal{G} denote the Grassmannian of all subspaces of V. The subset in $R \times \mathcal{G}$ consisting of a point $r \in R$ and a subspace of V stable under each $T_i(r)$ is Zariski-closed.

PROOF. For the first and second assertions: Let r_0 be a point at which rank W_r (resp. rank V_r) is maximized (resp. minimized). It is easy to see that, in the first case, any finite collection of operators whose image spans W_{r_0} at $r = r_0$ also spans W_r for r in a Zariski open neighborhood U of r_0 . In the second case, we can find a set of rational sections of vectors in W which belong to the kernel of all T_i , are regular at r_0 and their specializations form a basis for the common kernel of all $T_i(r_0)$. Then there is a Zariski open neighborhood U of r_0 where all these sections remain regular and linearly independent, and the dimension of the common kernel remains the same. We then replace R by R - U and argue by induction on dimension.

As for the final assertion: the set of r in question is the inverse image in $R \times \mathcal{G}$ of a Zariski-closed subset of $(\operatorname{Hom}(W,V))^J \times \mathcal{G}$, under the map $(r,g) \mapsto (\prod_j T_j(r),g)$.

PROOF OF THE THEOREM. The idea of the proof is to exploit the a priori knowledge of the fact that the set of X-discrete series is countable. On the other hand, we can describe the set of J-fixed vectors in $C^{\infty}(X)$ generating admissible subrepresentations as a constructible set. Constructibility and countability, together, will imply finiteness. See §8.2.2 for this argument in a simpler setting, and in particular see the final paragraph of §8.2.2 for an explanation of why "constructability+countability \Longrightarrow finiteness."

For notational simplicity, assume at first that $\mathcal{Z}(X)$ is trivial. For an assignment

$$R: \Delta_X \supset \Theta \mapsto r_\Theta \in \mathbb{N}$$
,

define the "space of R-exponents":

$$\mathfrak{S}_R = \prod_{\Theta} \widehat{(\mathcal{Z}(X_{\Theta})_{\mathbb{C}}^J)^{r_{\Theta}}}, \tag{9.4}$$

where J denotes "J-fixed."'³⁴ In words: an element of \mathfrak{S}_R is a collection of characters of $\mathcal{Z}(X_{\Theta})$ for all Θ , in fact, r_{Θ} of them for given Θ . We denote an element of \mathfrak{S}_R by $x = (x_j)$, where the subscripts j are of the form (Θ, i) with $i \leq r_{\Theta}$ an integer; we denote by $\chi_{x_{\Theta,1}}, \ldots, \chi_{x_{\Theta,r}}$ for $r \leq r_{\Theta}$ the various characters of $\mathcal{Z}(X_{\Theta})$ indexed by x.

The set \mathfrak{S}_R has the natural structure of a complex variety. Let us fix a J-good neighborhood N'_{Θ} for each Θ , and let $N_{\Theta} = N'_{\Theta} \setminus \cup_{\Omega \subset \Theta} N'_{\Omega}$. This includes $N'_{\Theta} = X$ for $\Theta = \Delta_X$, therefore the sets N_{Θ} cover X. Notice that, if we identify N_{Θ}/J with a subset of X_{Θ}/J via the exponential map, then this subset is compact (finite) modulo the action of $\mathcal{Z}(X_{\Theta})$. For this reason, the set of functions on $\mathcal{Z}(X_{\Theta}) \cdot N_{\Theta}/J$ (as a subset of X_{Θ}/J) annihilated by:

$$\prod_{i=1}^{r_{\Theta}} (z - \chi_{x_{\Theta,i}}(z)), \tag{9.5}$$

for all $z \in \mathcal{Z}(X_{\Theta})$, forms a finite dimensional vector space. (Here we denote, for notational simplicity, simply by z the normalized action of $z \in \mathcal{Z}(X_{\Theta})$ on functions on X_{Θ} , which before was denoted by \mathcal{L}_z .)

Hence, we can form for each $x=(x_j)\in\mathfrak{S}_R$, the finite-dimensional subspace $V_x\subset C(X)^J$ consisting of functions f with the following property: for every Θ , there is a function $f_\Theta\in C(X_\Theta)^J$ that is annihilated, for all $z\in\mathcal{Z}(X_\Theta)$, by (9.5) and such that $f|_{N_\Theta}=f_\Theta|_{N_\Theta}$; here we are using the identification of N_Θ/J with subset of X_Θ/J . We shall say, for simplicity, that elements of V_x are functions "asymptotically annihilated" by the ideal $I_{x,\Theta}\subset\mathbb{C}[\mathcal{Z}(X_\Theta)]$ generated by (9.5).

Notice that for any admissible subrepresentation $\pi \subset C^{\infty}(X)$ there is an R and a $x \in \mathfrak{S}_R$ such that the image of π^J belongs to V_x ; more precisely, the image of π^J is a finite-dimensional subspace of V_x which is stable under the Hecke algebra for $J\backslash G/J$. Vice versa, any $\mathcal{H}(G,J)$ -stable subspace of V_x is a finite module over the Bernstein center and hence generates a finite length (hence admissible) subrepresentation of $C^{\infty}(X)$.

There is a finite subset S of X/J, depending only on R, such that for any given $x \in \mathfrak{S}_R$, each element of V_x is determined by its restriction on S; indeed, the point x determines "characteristic polynomials" for the action of all elements of $\mathcal{Z}(X_{\Theta})$ on the asymptotics of V_x , which amount to recursive relations by which these asymptotics are determined by a finite number of evaluations. In particular, V_x is finite-dimensional; we may indeed consider each V_x as a subspace of \mathbb{C}^S specified by a finite set of linear constraints which vary polynomially in $x \in \mathfrak{S}_R$. An application of Lemma 9.2.2 shows

 $^{^{34}}$ The notion of J-fixed means the group of characters of $\mathcal{Z}(X_{\Theta})$ which appear as eigencharacters on $C^{\infty}(X_{\Theta})^J$. Because our $\mathcal{Z}(X_{\Theta})$ is a quotient of $\mathcal{Z}(L_{\Theta})$, by Proposition 2.7.2, this amounts to the character group of the quotient of $\mathcal{Z}(X_{\Theta})$ by an open compact subgroup; in particular, the set of J-fixed characters is the group of homomorphisms from a finitely generated abelian group to \mathbb{C}^{\times} , and thus has a natural structure of algebraic variety.

that we may partition \mathfrak{S}_R into constructible subsets so that the restriction of $\{V_x\}_x$ to each subset is a trivializable sub-bundle of the trivial bundle $\underline{\mathbb{C}}^S$. Let us denote by \mathcal{G} the Grassmannian of all linear subspaces of \mathbb{C}^S .

Let h_1, \ldots, h_k generate the Hecke algebra for $J \setminus G/J$. Then, by the "eventual equivariance" of Proposition 4.3.3, $h_i V_x \subset \tilde{V}_x$, where \tilde{V}_x is defined as V_x but replacing N_{Θ} by certain smaller neighborhoods \tilde{N}_{Θ} ; in particular, $V_x \subset \tilde{V}_x$, and it still makes sense for a subspace of V_x to be "Hecke stable."

It follows from Lemma 9.2.2 again that the subset $Z_R \subset \mathfrak{S}_R \times \mathcal{G}$ defined as:

$$Z_R = \{(x, M) \in \mathfrak{S}_R \times \mathcal{G} : M \subset V_x, \text{ and } M \text{ is Hecke-stable}\}$$
 (9.6)

is a constructible subset of $\mathfrak{S}_R \times \mathcal{G}$.

By definition, each vector in V_x is "asymptotically annihilated" in the Θ -direction by the ideal $I_{x,\Theta} \subset \mathbb{C}[\mathcal{Z}(X_{\Theta})]$ which is determined by x. We consider the subset $Z_R' \subset Z_R$ consisting of pairs (x,M) where where M is an $\mathcal{H}(G,J)$ -stable subspace of V_x whose annihilator is precisely $I_{x,\Theta}$. This is a constructible set, since it is obtained from Z_R by removing the preimages of the $Z_{R'}$ for all R' < R. (Notice that we can use the same set S for those, so we have a finite number of maps: $\mathfrak{S}_R \times \mathcal{G} \to \mathfrak{S}_{R'} \times \mathcal{G}$ by forgetting certain subsets of the exponents.)

We now introduce the notion of subunitary exponents, which are the exponents that X-discrete series have (see also §10.1.) We say that a (complex) character χ of $A_{X,\Theta}$ is subunitary if $|\chi(a)| < 1$ for all $a \in \mathring{A}_{X,\Theta}^+$. We say that it is strictly subunitary if it is also subunitary when restricted to $A_{X,\Omega}$, for all $\Omega \supset \Theta$ – i.e. when it does not become unitary after restriction to a "wall" of $A_{X,\Theta}^+$.

We say that an element of \mathfrak{S}_R is subunitary if all its components are. We denote by $\mathfrak{S}_R^{\mathrm{su}}$ the semialgebraic, and open in the Hausdorff topology, subset of subunitary exponents.

Let $x \in \mathfrak{S}_R^{\mathrm{su}}$ and let $f \in V_x$ be a function which generates under $\mathcal{H}(G,J)$ a Hecke-stable subspace of V_x . Then we claim that $f \in L^2(X)_{\mathrm{disc}}$. This is a generalization of Casselman's square integrability criterion, [Cas, Theorem 4.4.6].

First of all, we claim that the exponents of f are in fact strictly subunitary, not just unitary, in every direction $\Theta \subsetneq \Delta_X$. Indeed, as mentioned above, f has to generate an admissible subrepresentation of $C^{\infty}(X)$. This implies that for every $\Theta \subset \Omega \subset \Delta_X$, its Ω -exponents contain the restrictions of its Θ -exponents to $A_{X,\Omega}$, and since the Ω -exponents are assumed to be subunitary, for every $\Omega \supset \Theta$, it follows that the Θ -exponents cannot be unitary on any "wall" of $A_{X,\Theta}^+$, i.e. they are strictly subunitary.

This implies that f is in $L^2(X)$: recall from §1.7 that we are using normalized actions to define "exponents", wherein we twist the action of $A_{X,\Theta}$ by the square root of the eigenmeasure, so that the condition of "strictly subunitary" on the $A_{X,\Theta}$ -exponents of every f forces f to be square integrable on N_{Θ} , i.e. the growth of measure on X has been built into the normalization.

Since $f \in L^2(X)$ and generates an admissible subrepresentation of $C^{\infty}(X)$, hence generates a subrepresentation of $L^2(X)$ which belongs to a finite sum of irreducibles, it follows that $f \in L^2(X)_{\text{disc}}$.

Now we claim:

The projection of Z'_R to \mathfrak{S}_R intersects $\mathfrak{S}_R^{\mathrm{su}}$ in a finite set.

Indeed, let $(x, M) \in Z'_R$, with $x \in \mathfrak{S}^{\mathrm{su}}_R$. The space M contains elements of $L^2(X)^J_{\mathrm{disc}}$ which are asymptotically annihilated by the ideal corresponding to x, but not by any larger ideal. Since $L^2(X)_{\mathrm{disc}}$ is a countable direct sum of irreducible subrepresentations, there are only countably many $x \in \mathfrak{S}^{\mathrm{su}}_R$ which admit a subspace M with that property.

On the other hand, the projection of Z'_R to \mathfrak{S}_R is constructible, which implies that its intersection with $\mathfrak{S}_R^{\rm su}$ is either uncountable or finite. So this intersection is finite. Let us denote by $L^2(X)_R^J$ the subspace of $L^2(X)_{\rm disc}^J$ spanned by all such subspaces M.

Finally, observe:

There is a positive integer N such that every $f \in L^2(X)^J_{\text{disc}}$ spanning an irreducible representation is contained in $L^2(X)^J_R$, for some R with |R| < N,

where we write $|R| = \sum_{\Theta} r_{\Theta}$ for short.

Indeed, for an embedding $\pi \to C^{\infty}(X)$ the asymptotics give $\pi \to C^{\infty}(X_{\Theta})$ or, equivalently, an L_{Θ} -equivariant map from the Jacquet module with respect to P_{Θ}^- : $\pi_{\Theta^-} \to C^{\infty}(X_{\Theta}^L)$, where the action of $\mathcal{Z}(X_{\Theta})$ coincides³⁵ with the action of $\mathcal{Z}(L_{\Theta})^0$. The degree of an element of π_{Θ^-} as a finite $\mathcal{Z}(L_{\Theta})^0$ -vector is uniformly bounded as π varies over all irreducible representations, i.e.

For any irreducible representation of G, the length (as M-representation) of the Jacquet functor associated to the parabolic subgroup P = MN is bounded above by the order of the Weyl group of G.

This follows from the exactness of the Jacquet functor, embedding π inside a representation induced from a supercuspidal, and then applying the "geometrical lemma" of Bernstein and Zelevinsky [BZ77, §2].

Therefore, to "detect" exponents of discrete series we only need to work with a finite number of R's. This implies that $L^2(X)^J_{\text{disc}}$ is finite-dimensional.

We have until now discussed only the case where $\mathcal{Z}(X)$ is trivial. In the general case we will denote by \mathfrak{S}_R the tuples of exponents which agree on $\mathcal{Z}(X)$, and repeat the same proof but considering, instead of \mathfrak{S}_R , its fiber $\mathfrak{S}_{R,\chi}$ over a given $\chi \in \widehat{\mathcal{Z}(X)}$.

³⁵Recall that in the wavefront case $\mathcal{Z}(\mathbf{X}_{\Theta})$ is a quotient of $\mathcal{Z}(\mathbf{L}_{\Theta})^0$, Proposition 2.7.2.

9.3. Variation with the central character.

9.3.1. The problem. Now we will discuss the way that relative discrete series vary as their central character varies. Let us first explain through examples the difficulties that one faces:

In the case of a reductive group $(\mathbf{X} = \mathbf{H}, \mathbf{G} = \mathbf{H} \times \mathbf{H})$ we can twist any relative discrete series $\pi \simeq \tau \otimes \tilde{\tau}$ by characters of G of the form $\eta \otimes \eta^{-1}$ (let us say: η is unramified) in order to obtain a "continuous family" of relative discrete series. More precisely, if $M_{\tau}: \tau \otimes \tilde{\tau} \to C^{\infty}(X)$ denotes the "matrix coefficient" map, and we identify the underlying vector spaces of all the representations $\tau \otimes \eta$ (and those of their duals), then M_{τ} lives in the holomorphic (actually, polynomial) family of morphisms $M_{\tau \otimes \eta}$, parametrized by the complex torus D^* of unramified characters of H. Moreover, for η in the real subtorus $D^*_{i\mathbb{R}}$ of unitary characters these morphisms represent relative discrete series for X = H, and notice also that for every value of the parameter they are non-zero. There is a finite-to-one map $D^* \to \widehat{\mathcal{Z}(X)}_{\mathbb{C}}$, and the Plancherel formula for the discrete spectrum of X = H is a sum, over all such orbits $[\tau]$ of discrete series, of terms of the form:

$$\|\Phi\|_{[\tau]}^2 := \int_{D_{i\mathbb{D}}^*} C \circ \tilde{M}_{\tau \otimes \eta}(\Phi \otimes \bar{\Phi}) d\eta,$$

where \tilde{M} denotes the adjoint of matrix coefficients, C denotes the natural contraction map: $(\tau \otimes \eta) \otimes (\tilde{\tau} \otimes \eta^{-1}) \to \mathbb{C}$, and $d\eta$ is a suitable Haar measure on $D_{i\mathbb{R}}^*$. (Notice that it might happen here that a finite subgroup F of $D_{i\mathbb{R}}^*$ stabilizes the isomorphism class of τ ; therefore this integral does not correspond to a direct integral decomposition of the space; for that we would have to write it as an integral over $D_{i\mathbb{R}}^*/F$.)

While in the group case (and more generally, as we shall see, in the case of a symmetric variety) one easily obtains "continuous families" of relative discrete series in this way, let us describe an interesting new issue which arises for general spherical varieties. Consider the action of $\mathbf{G} = \mathbf{G}_{\mathrm{m}} \times \mathbf{PGL}_2$ on $\mathbf{X} = \mathbf{PGL}_2$, where \mathbf{PGL}_2 is acting on itself on the right, whereas $z \in \mathbf{G}_{\mathrm{m}}$ acting via the left action of $\begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$. This is spherical; moreover, $L^2(X)$ decomposes as an integral indexed by unitary characters of $k^{\times} = \mathbf{G}_{\mathrm{m}}(k)$:

$$L^{2}(\mathbf{X}(k)) = \int_{\widehat{k}^{\times}} L_{\omega}^{2} d\omega. \tag{9.7}$$

Here L^2_{ω} is the unitary induction of ω from $\mathbf{G}_{\mathrm{m}}(k)$ to $\mathbf{PGL}_2(k)$. It is known³⁶ that L^2_{ω} possesses discrete series for all ω ; however, a priori, it could vary wildly as ω changes. This leads to a difficulty in analyzing the most discrete part of the spectrum of X.

 $^{^{36}}$ Although this is only relevant as motivation, it follows from Theorem 6.4.1, together with the fact – immediate from the theory of the Kirillov model – that *every* discrete series representation of \mathbf{PGL}_2 is distinguished for (\mathbf{G}_m, χ) .

9.3.2. Algebraicity and measurability. Our goal here is to use similar arguments as in the proof of Theorem 9.2.1 in order to show that $L^2(X,\chi)_{\text{disc}}^J$ varies measurably with $\chi \in \widehat{\mathcal{Z}(X)}$.

Let us specify what this means. The family of spaces $H_{\chi} := L^2(X,\chi)$ form a "measurable family of Hilbert spaces parameterized by χ "; by this we mean that we have a collection of measurable sections $\chi \mapsto f_{\chi} \in H_{\chi}$ satisfying certain natural axioms (cf. [Ber88]). This also gives rise to a notion of measurable sections: $\chi \mapsto T_{\chi} \in \operatorname{End}(H_{\chi})$ as follows: $\chi \mapsto T_{\chi}$ is measurable if and only if for any two measurable sections $\chi \mapsto \eta_{\chi}, \chi \mapsto \eta'_{\chi}$ the inner product $\langle T_{\chi}\eta_{\chi}, \eta'_{\chi} \rangle$ is a measurable function of χ .

9.3.3. Proposition. The projections: $L^2(X,\chi) \to L^2(X,\chi)_{\rm disc}$ are measurable.

PROOF. The idea of the proof is to go over our previous proof of Theorem 9.2.1 and show that (suitably interpreted) the discrete spectrum in fact varies "semi-algebraically" with χ .

Let us make a couple of reductions. First of all:

• It suffices to show that there is a countable number of measurable sections $\chi \mapsto f_i(\chi) \in L^2(X,\chi)_{\text{disc}}$ such that, for almost all χ , $\{f_i(\chi)\}_i$ spans $L^2(X,\chi)_{\text{disc}}$.

This is clear; one may construct the orthogonal projection in a "measurable" fashion from the $f_i(\chi)$, using the Gram-Schmidt process.

Now consider the spaces V_x , $x \in \mathfrak{S}_R$, discussed in the proof of Theorem 9.2.1. Recall (see the last paragraph of the proof of this Theorem) that we denote by \mathfrak{S}_R the subset of $\prod_{\Theta} (\widehat{\mathcal{Z}}(X_{\Theta})_{\mathbb{C}})^{r_{\Theta}}$ such that the restrictions of all factors to $\mathcal{Z}(X)$ coincide; the space V_x then consists of functions on X whose "asymptotics" transform under the component characters of x, and in particular this notion depends on a choice of good neighborhoods of ∞ .

As we remarked, there is a finite subset S of X/J, depending only on R, such that the evaluation maps: $V_x \to \mathbb{C}^S$ are injective, for every x. Hence, we consider the union of the spaces V_x as a subspace of $\mathfrak{S}_R \times \mathbb{C}^S$. It now suffices to show:

• There is a finite number of measurable sections

$$\widehat{\mathcal{Z}(X)} \ni \chi \mapsto f_i(\chi) \in \mathfrak{S}_{R,\chi}^{\mathrm{su}} \times \mathbb{C}^S$$

such that:

- The image lies in the union of the spaces V_x .
- For almost all χ , the $f_i(\chi)$, considered as functions on X, actually lie inside $L^2(X,\chi)^J_{\text{disc}}$ and moreover span this space.

Indeed, if f_i is measurable as a section into $\mathfrak{S}_{R,\chi}^{\mathrm{su}} \times \mathbb{C}^S$ then it is also measurable as a section into $L^2(X,\chi)_{\mathrm{disc}}^J$; this follows by examining the proof ("by linear recurrence") that the map $V_x \to \mathbb{C}^S$ is injective for fixed x. (Also,

in what follows, the fact that the $f_i(\chi)$ actually lie inside $L^2(X,\chi)^J_{\text{disc}}$ will follow as in the arguments on page 124).

Now we use the notation of the proof of Theorem 9.2.1, adjusted to the present setting: Let Z' be the subset of pairs (x, M) such that $x \in \mathfrak{S}_R$ and M an $\mathcal{H}(G, J)$ -stable subspace of V_x , considered as a subspace of \mathbb{C}^S , with the property that M is not asymptotically annihilated by any ideal larger than $I_{x,\Theta}$. ("Hecke stable" means that V_x is Hecke stable when identified as a subspace of $C^{\infty}(X)^J$, cf. discussion before (9.6)). Hence, again, Z' is a constructible subset of $\mathfrak{S}_R \times \mathcal{G}$, where \mathcal{G} denotes the Grassmannian all subspaces of \mathbb{C}^S .

We define another constructible subset $Z'' \subset Z'$ as follows: If $\mathcal{G}_N \subset \mathcal{G}$ denotes the Grassmannian of N-planes, a point $(x, M) \in Z' \cap (\mathfrak{S}_R \times \mathcal{G}_N)$ belongs to Z'' if and only if there is no $(x, M') \in Z' \cap (\mathfrak{S}_R \times \mathcal{G}_{N'})$, with N' > N. In words, Z'' parameterizes pairs (x, M) where M is of maximal dimension among Hecke-stable subspaces of V_x annihilated exactly by $I_{x,\Theta}$.

Since the sum of two Hecke-stable subspaces is also Hecke stable, it is clear that:

The fibers of $Z'' \to \mathfrak{S}_R$ are of size at most one.

Finally, consider Z''^{su} , the intersection of Z'' with $\mathfrak{S}_R^{\text{su}} \times \mathcal{G}$; it is a semi-algebraic set. We have a canonical map: $Z''^{\text{su}} \to \widehat{\mathcal{Z}(X)}$. By what we have already established in §9.2, the fibers of this map are finite.

We now utilize Hardt triviality, recalled in §8.2.3. It provides us with a finite number of semi-algebraic sections

$$Y_i \ni \chi \mapsto g_i(\chi) \in \mathfrak{S}_{R,\chi}^{\mathrm{su}} \times \mathcal{G},$$

where $Y_i \subset \widehat{\mathcal{Z}(X)}$ is semialgebraic. Note that a semialgebraic set is a Polish space. It is known [**Dix81**, Appendix V] that a function $r: P_1 \to P_2$ from a locally compact second-countable space to a Polish space whose graph is a Polish space (more generally, a Souslin set) is Borel measurable.

Choosing, locally on \mathcal{G} , a frame for the corresponding subspace of \mathbb{C}^S , we get a finite number of measurable sections:

$$\widehat{\mathcal{Z}(X)} \ni \chi \mapsto f_i(\chi) \in \mathfrak{S}_{R,\chi}^{\mathrm{su}} \times \mathbb{C}^S$$

(extended by zero away from the sets Y_i) with the property that, as R varies over a finite set, their specializations span $L^2(X,\chi)^J_{\text{disc}}$, for every χ .

9.3.4. COROLLARY. There is a natural measurable structure on the family of Hilbert spaces $\{L^2(X,\chi)_{\mathrm{disc}}\}_{\chi\in\widehat{\mathcal{Z}(X)}}$, which identifies the direct integral:

$$L^{2}(X)_{\text{disc}} := \int_{\widehat{\mathcal{Z}(X)}} L^{2}(X, \chi)_{\text{disc}} d\chi$$
 (9.8)

with a closed subspace of $L^2(X)$.

9.4. Toric families of relative discrete series.

9.4.1. Factorizable spherical varieties. To encode the difference between the examples that we saw in §9.3.1, namely that of the group and of PGL₂ under the $k^{\times} \times \text{PGL}_2$ -action, we call a (homogeneous) spherical variety \mathbf{X} factorizable if the Lie algebra \mathfrak{h} of the stabilizer of a generic point on \mathbf{X} has a direct sum decomposition: $\mathfrak{h} = (\mathfrak{h} \cap \mathcal{Z}(\mathfrak{g})) \oplus (\mathfrak{h} \cap [\mathfrak{g}, \mathfrak{g}])$. We always apply this definition under the assumption that the connected center of the group \mathbf{G} surjects onto $\mathcal{Z}(\mathbf{X})$ – for instance, the spherical variety $\mathbf{SL}_2 \setminus \mathbf{SL}_3$ is not factorizable. We call a spherical variety \mathbf{X} strongly factorizable if for every $\Theta \subset \Delta_X$ the Levi variety \mathbf{X}_{Θ}^L is factorizable. Recall by Proposition 2.7.2 that in the wavefront case the action of $\mathcal{Z}(\mathbf{X}_{\Theta}^L)$ is induced by the connected component of the center of \mathbf{L}_{Θ} – hence "factorizable" in this case means factorizable as an \mathbf{L}_{Θ} -variety. On the other hand, if \mathbf{X} is not wavefront then it cannot be strongly factorizable. Indeed, in that case there is a Levi variety \mathbf{X}_{Θ}^L such that the connected center of \mathbf{L}_{Θ} does not surject onto $\mathcal{Z}(\mathbf{X}_{\Theta}^L)$; but $\mathcal{X}_{\mathbf{L}_{\Theta}}(\mathbf{X}_{\Theta}^L)$ has rank equal at most the rank of the quotient by which $\mathcal{Z}(\mathbf{L}_{\Theta})$ acts on \mathbf{X}_{Θ}^L , which in this case is strictly less than the rank of $\mathcal{Z}(\mathbf{X}_{\Theta}^L)$.

While there are many interesting varieties which are not strongly factorizable, there are also many which are:

9.4.2. Proposition. If X is symmetric, then X is strongly factorizable.

PROOF. First, we notice that the Levi varieties of a symmetric variety are also symmetric; more precisely, given an involution θ on \mathbf{G} and a θ -split parabolic \mathbf{P} (i.e. a parabolic such that \mathbf{P}^{θ} is opposite to \mathbf{P}) the corresponding Levi variety is obtained from the restriction of θ to the Levi $\mathbf{P} \cap \mathbf{P}^{\theta}$. Therefore, it suffices to prove that every symmetric variety is factorizable. But any involution θ preserves the decomposition: $\mathfrak{g} = \mathcal{Z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$, and therefore the fixed subspace of θ on \mathfrak{g} decomposes as the direct sum of $\mathcal{Z}(\mathfrak{g})^{\theta}$ and $[\mathfrak{g}, \mathfrak{g}]^{\theta}$.

A complete classification of strongly factorizable spherical varieties in terms of combinatorial data is given in [DHS].

For factorizable varieties we can describe the discrete spectrum in terms of families of relative discrete series as in the group case. These families have much stronger properties than the measurability of Proposition 9.3.3. We will encode the basic properties of such families in what we will call toric families of relative discrete series, and then try to extend them to the non-factorizable case.

9.4.3. Definition of toric families. In the discussion that follows, we shall often identify complex algebraic tori with their complex points, leaving the algebraic structure implicit. Let us introduce the following convention: By a torus of unramified characters of the k-points of a connected algebraic group \mathbf{O} we will mean the full torus of unramified characters of a torus quotient of \mathbf{O} . More precisely, to any subgroup $\Gamma \subset \mathcal{X}(\mathbf{O})$ is associated a (torus) quotient \mathbf{O}' of \mathbf{O} with coordinate ring $k[\Gamma]$; to this quotient we associate the complex torus D^* of characters of O' that factor through the valuation

map:

$$O' \to \Gamma^* := \operatorname{Hom}(\Gamma, \mathbb{Z}).$$

Hence, the complex points of D^* are equal to $\operatorname{Hom}(\Gamma^*, \mathbb{C}^{\times})$ (here $\Gamma^* = \operatorname{Hom}(\Gamma, \mathbb{Z})$). As noted above, we will be identifying such a torus with its group of \mathbb{C} -points.

A torus of unramified characters for O comes with a \mathbb{Q} -structure, in particular an \mathbb{R} -structure with Lie algebra canonically isomorphic to $\mathrm{Hom}(\Gamma,\mathbb{R})$; explicitly, the \mathbb{R} -points of this structure are generated by characters of O' the form $x\mapsto |\chi(x)|^r$ for $\chi\in\Gamma$ and $r\in\mathbb{R}$. We will be using the notation $D^*_{\mathbb{R}}$ for the group of real points under this structure.

There is a *second* canonical real structure on D^* , whose real points coincide with the maximal compact subgroup $D^*_{i\mathbb{R}}$ of unitary characters, and which we will call the *imaginary* structure on D^* ; explicitly, the real points of this structure are generated by characters of O' of the form $x \mapsto |\chi(x)|^r$ for $\chi \in \Gamma$ and $r \in i\mathbb{R}$.

9.4.4. Remark. By our definition, a torus of unramified characters of **O** is not necessarily a subgroup of the full group of unramified characters of *O*; in general, the map:

$$D^* \to \{\text{unramified characters of } O\}$$

may have finite kernel because the k-points of \mathbf{O} may not surject to the k-points of its torus quotient used to define D^* .

DEFINITION. A toric family of relative discrete series for X consists of the following data: a parabolic subgroup $P \subset G$, an irreducible unitary representation σ of its Levi quotient L, a torus D^* of unramified characters of L and a family of morphisms, defined for almost every $\omega \in D^*_{i\mathbb{R}}$:

$$M_{\omega}: \pi_{\omega}:=I_P^G(\sigma\otimes\omega)\to C^{\infty}(X),$$

where $I_P^G(\bullet) = \operatorname{Ind}_P^G(\delta_P^{\frac{1}{2}} \bullet)$ denotes the normalized induced representation, with the following properties:

- (1) For almost every $\omega \in D^*$ the representation π_{ω} is irreducible.
- (2) The morphism of complex varieties: $D^* \ni \omega \mapsto \chi_\omega \in \widehat{\mathcal{Z}}(X)_{\mathbb{C}}$, obtained by taking the central character³⁷ of π_ω , surjects onto one of the connected components of $\widehat{\mathcal{Z}}(X)_{\mathbb{C}}$.
- of the connected components of $\widehat{\mathcal{Z}(X)}_{\mathbb{C}}$. (3) For $\omega \in D_{i\mathbb{R}}^*$ the image of M_{ω} lies in $L^2(X, \chi_{\omega})$, and varies measurably with ω .

³⁷For notational simplicity, for the rest of this section, we are assuming that the connected center of G surjects onto $\mathcal{Z}(X)$; recall that we are assuming this to be true over the algebraic closure. If this is not true at the level of k-points, then $\mathcal{Z}(X)$ has to be replaced by the image of $\mathcal{Z}(G)^0$. For the purposes of this section, it would also not harm to replace G by its quotient which acts faithfully on X; thus making $\mathcal{Z}(X) = \mathcal{Z}(G)^0$.

9.4.5. Remark. To complete the analogy with the factorizable case, we expect the family of morphisms M_{ω} to extend to a rational family on all of D^* , in the usual sense: if we identify the restrictions of all π_{ω} to K (and hence also the underlying vector spaces) as $V := \operatorname{Ind}_{K \cap P}^K(\delta_P^{\frac{1}{2}\sigma}|_{K \cap P})$ then for all $x \in X$:

$$(v \mapsto M_{\omega}(v)(x)) \in \operatorname{Hom}_{\mathbb{C}}(V, \mathbb{C}(D^*)).$$

However, we will not need the rationality property here (except for a weak version of it which is covered by Theorem 9.4.10 which follows), and therefore we will not impose it or prove it in any case.

We will sometimes condense notation and denote by $(\pi_{\omega}, M_{\omega})_{\omega \in D^*}$ a toric family of relative discrete series as above. The notation conceals the fact that the morphisms are only defined for $\omega \in D^*_{i\mathbb{R}}$ (see, however, the previous remark). The condition "for almost every" means "for almost every with respect to Haar measure on D^* " but also, by means of the theory of reducibility of induced representations, "in the complement of a finite number of divisors".

Our basic goal in the rest of this section will be to prove instances of the following:

9.4.6. DISCRETE SERIES CONJECTURE. Given a spherical variety X, there is a parabolic subgroup $P = L \ltimes U \subset G$, a torus D^* of unramified characters of L, and a countable collection $\{\mathcal{D}_i = (\pi_\omega^i, M_\omega^i)_{\omega \in D^*}\}$ of toric families of relative discrete series representations – all associated to the same L and D^* , and finitely many of them containing non-zero J-fixed vectors, for any fixed open compact subgroup J – such that the norm of $L^2(X)_{\text{disc}}$ admits a decomposition:

$$\|\Phi\|_{\mathrm{disc}}^2 = \sum_i \int_{D_{i\mathbb{R}}^*} \|\tilde{M}_{\omega}^i(\Phi)\|^2 d\omega \tag{9.9}$$

for a suitable Haar measure $d\omega$.

Moreover, notice the canonical maps (with A^* the canonical torus in \check{G})

$$D^* \to (unramified \ characters \ of \ P)^0 \to A^*$$

and

$$\mathcal{Z}(\check{G}_X) \hookrightarrow A_X^* \to A^*,$$

both of which³⁸ have finite kernel. The following is true:

³⁸ The map (unramified characters of P)⁰ $\hookrightarrow A^*$ is injective because, in fact, we can identify the group of unramified characters of P with the center of the dual Levi subgroup $\check{L} \subset \check{G}$ canonically attached to the parabolic P.

There is an isomorphism $D^* \simeq \mathcal{Z}(\check{G}_X)^0$ and an element $w \in \mathcal{N}_W(A_X^*)$ such that the diagram commutes:

$$D^* \longrightarrow A^*$$

$$\downarrow \qquad \qquad \downarrow w$$

$$\mathcal{Z}(\check{G}_X) \longrightarrow A^*$$

$$(9.10)$$

Again, this is not necessarily a direct integral decomposition, as the images of the M_{ω}^{i} 's could be non-orthogonal for different i's and ω 's corresponding to the same central character of G. However, it is easy to see that by an orthogonalization process we can make them orthogonal for different indices i, and as will follow from the proof (in the cases that we establish), the images will be orthogonal for ω 's corresponding to different central characters of L. Of course, if there is multiplicity in the spectrum then this collection of families of RDSs is not unique, though our conjectures for the discrete spectrum in terms of Arthur parameters suggest that there might be a canonical way to pick mutually orthogonal toric families of relative discrete series spanning $L^2(X)_{\text{disc}}$.

9.4.7. Bounds on subunitary exponents. Before we discuss proofs of the Discrete Series Conjecture and related results, let us see a corollary which bounds the possible exponents by which an X_{Ω} -discrete series can embed into X

9.4.8. Proposition (Uniform boundedness of exponents). Let J be a fixed open compact subgroup of G and $\Theta, \Omega \subset \Delta_X$. Assume that X_{Ω} satisfies the Discrete Series Conjecture 9.4.6.

There exists a finite set \mathcal{E} of homomorphisms $A_{X,\Theta} \to \mathbb{R}_+^{\times}$ so that for almost every (with respect to Plancherel measure on X_{Ω}) relative discrete series representation π for X_{Ω} with $\pi^J \neq \{0\}$ and any morphism $M: \pi \to C^{\infty}(X)$ the exponents of $e_{\Theta}^* \circ M$ satisfy:

$$|\chi| \in \mathcal{E}$$
.

In particular, there exists a constant c < 1 so that every exponent with $|\chi| < 1$ on $\mathring{A}^+_{X,\Theta}$ satisfies:

$$|\chi(a)| < c \text{ for } a \in \mathring{A}_{X \Theta}^+.$$

PROOF. Since we are assuming that X_{Ω} satisfies the Discrete Series Conjecture 9.4.6, almost every such π is isomorphic to a subquotient of $I_P^G(\sigma \otimes \omega)$, where $\omega \in D_{i\mathbb{R}}^*$ and $(P, \sigma, D^*, (M_{\omega})_{\omega})$ varies in a *finite* number of toric families of relative discrete series for X_{Ω} .

Now consider the possible exponents of $I_P^G(\sigma \otimes \omega)$ along X_{Θ} . Replacing P by a smaller parabolic, we may assume that σ is supercuspidal with central character χ , and then the exponents of $I_P^G(\sigma \otimes \omega)$ along Θ are contained in the restrictions to $\mathcal{Z}(L_{\Theta})^0$ of the set of characters $\{^w(\chi \cdot \omega)\}$, where w ranges over all elements of the Weyl group which map a Levi subgroup of P into P_{Θ} . The claim follows.

9.4.9. A weaker result on X-discrete series. By using general arguments similar to those in the proof of Theorem 9.2.1 and Proposition 9.3.3, we can easily prove a weaker result than the Discrete Series Conjecture. To formulate it, let us call "algebraic family of representations" any fixed vector space V, with a fixed, admissible action of a maximal compact subgroup K of G, and a family of representations π_{ω} of G on V extending the action of K, such that ω varies on an algebraic variety D^* and the action of $\mathcal{H}(G,J)$ on V^J varies algebraically with ω for every open compact $J \subset K$. (That means, that for every $v \in V^J$ and $h \in \mathcal{H}(G,J)$ we have $\pi_{\omega}(h)v \in V^J \otimes \mathbb{C}[D^*]$.) Such an algebraic family is obtained, for example, by starting from an admissible representation of a Levi subgroup, twisting it by characters of the Levi and parabolically inducing.

Let such an algebraic family $(V, \pi_{\omega})_{\omega \in D^*}$ be given. We assume that all members of the family are irreducible, and ask the question of which members of the family can appear as relative discrete series on X.

9.4.10. THEOREM. Fix an open compact subgroup J. There is a finite set of data (Y_i, M_i) , where Y_i is a semi-algebraic subset of D^* (considered as a real variety by restriction of scalars) and M_i is a semi-algebraic family of relative discrete series $M_{i,\omega}: \pi_\omega \to C^\infty(X)^J$, $\omega \in Y_i$ with the property that for every ω any relative discrete series $\pi_\omega \to C^\infty(X)$ is in the linear span of the $M_{i,\omega}$'s.

We only sketch the main steps of the proof, since the arguments are similar to those encountered in the proof of Theorem 9.2.1:

Sketch of proof. The proof relies on showing that for every open compact $J \subset K$ there is a semi-algebraic subset \mathcal{A} of the product:

$$\operatorname{Hom}(V^J,C(X)^J)\times D^*$$

with the property that $(M, \omega) \in \mathcal{A}$ if and only if M is Hecke equivariant with respect to the π_{ω} -action, and the image is in L^2 -mod-center.

Of course, since the space $\text{Hom}(V^J, C(X)^J)$ is infinite (uncountable) dimensional, we need to fix neighborhoods of infinity and exponents in order to reduce it to finite dimensional spaces, as in the proof of Theorem 9.2.1, in order to talk about a "semialgebraic subset".

Following that, we use again Hardt triviality in order to construct sections from semi-algebraic subsets of D^* to \mathcal{A} . We can construct enough sections so that their specializations at all ω span the space of all relative discrete series from π_{ω} .

9.4.11. Proof of Discrete Series Conjecture 9.4.6 in the factorizable case. Now we focus on the easy case of the Discrete Series Conjecture 9.4.6, and will discuss the general case in the next subsection.

Let L = G, let **O** be the quotient of \mathbf{G}^{ab} by the image of \mathbf{H} (where \mathbf{H} denotes the connected component of the stabilizer of a point $x_0 \in \mathbf{X}$) and let D^* be the torus of unramified characters of O. Hence, $\mathcal{X}(D^*)^* = \mathbf{X}(D^*)$

 $\mathcal{X}_{\mathbf{G}}(\mathbf{X})$. We claim, first of all, that $D^* \simeq \mathcal{Z}(\check{G}_X)^0$ canonically. Indeed, the cocharacter group of $\mathcal{Z}(\check{G}_X)^0$ is the set of all elements in $\mathcal{X}(\mathbf{X})$ which are perpendicular to the spherical coroots $\check{\Delta}_X$. The fact that \mathbf{X} is factorizable implies that $\mathcal{X}_{\mathbf{G}}(\mathbf{X})$ and $\mathcal{X}_{\mathcal{Z}(\mathbf{G})}(\mathbf{X})$ have the same rank – which is the rank of $\mathcal{X}(\mathbf{X}) \cap \check{\Delta}_X^{\perp}$. Since $\mathcal{X}_{\mathbf{G}}(\mathbf{X}) \subset \mathcal{X}(\mathbf{X}) \cap \check{\Delta}_X^{\perp}$, it follows that the two groups are commensurable and that the elements of $\mathcal{X}(\mathbf{X}) \cap \check{\Delta}_X^{\perp}$ are characters of \mathbf{G} . We claim that, in fact, they are trivial on an isotropy group \mathbf{H} or, equivalently, $\mathcal{X}_{\mathbf{G}}(\mathbf{X}) = \mathcal{X}(\mathbf{X}) \cap \check{\Delta}_X^{\perp}$. Indeed, they have to be trivial on the connected component \mathbf{H}^0 of \mathbf{H} since they are commensurable with a group of characters which has this property. Now consider the morphism: $\mathbf{X}^0 := \mathbf{H}^0 \backslash \mathbf{G} \to \mathbf{H} \backslash \mathbf{G} = \mathbf{X}$, an equivariant cover of degree $(\mathbf{H} : \mathbf{H}^0)$. We restrict this cover to the open \mathbf{B} -orbits: $\mathring{\mathbf{X}}^0 \to \mathring{\mathbf{X}}$, and see that \mathbf{H}/\mathbf{H}^0 acts faithfully on the fibers of this map. Therefore, every element of $\mathcal{X}(\mathbf{X})$ which extends to a character of \mathbf{G} has to be trivial on \mathbf{H} . This proves the fact that $\mathcal{X}_{\mathbf{G}}(\mathbf{X}) = \mathcal{X}(\mathbf{X}) \cap \check{\Delta}_X^{\perp}$ and, hence, $D^* = \mathcal{Z}(\check{G}_X)^0$.

The choice of point $x_0 \in X$ defines a map: $X \ni x \mapsto \bar{x} \in O$ (indeed, x_0 defines a morphism of algebraic varieties, and the map $X \to O$ is obtained by taking k-points), and every relative discrete series (π, M) can be twisted by elements of D^* to obtain a toric family of RDS:

$$(M_{\omega})(v)(x) := M(v)\omega(\bar{x}).$$

Moreover the dimension of D^* is equal to the dimension of $\mathcal{Z}(X)$, and therefore D^* surjects onto the identity component of $\widehat{\mathcal{Z}(X)}_{\mathbb{C}}$.

Now choose a unitary central character χ and a direct sum decomposition of $L^2(X,\chi)$ in terms of relative discrete series:

$$L^2(X,\chi) = \bigoplus_i M^i(\pi^i).$$

This gives rise to a direct sum decomposition of $L^2(X, \chi \otimes \chi_{\omega})$, for every $\omega \in D_{i\mathbb{R}}^*$:

$$L^2(X,\chi\otimes\chi_\omega)=\oplus_i(M^i_\omega)(\pi^i\otimes\omega),$$

and therefore from (9.2) we get:

$$\|\Phi\|_{\mathrm{disc}}^2 = \sum_i \int_{D_{i\mathbb{R}}^*} \|\tilde{M}_{\omega}^i(\Phi)\|^2 d\omega$$

where $d\omega$ is a suitable Haar measure.

The next subsection will be devoted to reducing a much more general case to the factorizable one by a method called "unfolding".

9.5. Unfolding. In this section we take the right regular representation of any group on any space of functions to be unnormalized; if we want to consider the normalized action, we will explicitly tensor by a character. As usual, our convention is that the group acts on the right on the given spaces and on the left on function- or measure spaces.

9.5.1. Introduction. Let $\mathbf{G} = \mathbf{PGL}_2$, $\mathbf{T} = \mathbf{a}$ nontrivial split torus in \mathbf{G} . Let χ be a unitary character of T, and consider the representation $L^2(T \setminus G, \chi)$. By imitating a technique which has been used globally, in the theory of period integrals and the Rankin-Selberg method, we will show that there is a unitary, equivariant isomorphism:

$$L^2(T\backslash G,\chi)\simeq L^2(U\backslash G,\psi),$$

where U denotes a nontrivial unipotent subgroup and ψ a nontrivial unitary complex character of it – hence the right-hand-side is a Whittaker model. This is an amazing fact per se: For instance it shows that, as unitary representations, the spaces $L^2(T\backslash G,\chi)$ for all unitary χ are isomorphic! This is what will allow us to pass from a "non-factorizable" to a "factorizable" case: The spaces $L^2(T\backslash G,\chi)$ show up in the spectral decomposition of X=G under the $T\times G$ -action, which is not factorizable. The unfolding technique shows that $L^2(X)\simeq L^2(k^\times\times U\backslash G,\psi)$, which is factorizable. ³⁹ In the general case the passage is not to a factorizable space, but to one which is parabolically induced from a factorizable one.

We give a couple more examples:

9.5.2. Example. For $X = \operatorname{SL}_n \times \operatorname{SL}_n \setminus \operatorname{GL}_{2n}$, as a $k^{\times} \times k^{\times} \times \operatorname{GL}_{2n}$ -space, we have:

$$L^2(X) \simeq L^2(k^{\times}) \otimes L^2(\mathrm{SL}_n^{\mathrm{diag}} \ltimes \mathrm{Mat}_n \setminus \mathrm{GL}_{2n}, \psi),$$

where ψ is a non-degenerate character of Mat_n (the unipotent radical of the $n \times n$ -parabolic) normalized by $\mathrm{GL}_n^{\mathrm{diag}}$. The space on the right is factorizable – it is essentially the "Shalika model".

9.5.3. Example. For $X=\operatorname{SL}_n\backslash\operatorname{Sp}_{2n}$ under the $k^\times\times\operatorname{Sp}_{2n}$ -action, we have:

$$L^2(X) \simeq L^2(k^{\times}) \otimes L^2(O_n \ltimes S^2(k^n) \backslash \operatorname{GL}_{2n}, \psi),$$

where ψ is a non-degenerate character of $S^2(k^n)$ (the unipotent radical of the Siegel parabolic) normalized by O_n . The space on the right is factorizable.

The examples above can be obtained by one step of "unfolding", i.e. one application of inverse Fourier transforms. The following one requires several steps (and we will encounter more of this kind later):

9.5.4. EXAMPLE. For $X = \operatorname{SL}_n \setminus \operatorname{GL}_n \times \operatorname{GL}_{n+1}$ under the $k^{\times} \times \operatorname{GL}_n \times \operatorname{GL}_{n+1}$ action, we have:

$$L^2(X) \simeq L^2(k^{\times}) \otimes L^2(U_n \backslash \operatorname{GL}_n, \psi_n) \otimes L^2(U_{n+1} \backslash \operatorname{GL}_{n+1}, \psi_{n+1}),$$

where the last two factors are Whittaker models for GL_n and GL_{n+1} , respectively.

³⁹The meaning of factorizable here is exactly the same as in the case of the trivial line bundle: the Lie algebra of the stabilizer is a direct sum of its intersections with $[\mathfrak{g},\mathfrak{g}]$ and with the Lie algebra of the center of G; however, notice that the center of X is smaller when we have a non-trivial character, hence, for instance, in the case of the Whittaker model we do not enlarge the group to include the action of $\mathcal{N}(U)/U$ – cf. footnote 9.

9.5.5. Technicalities. To understand some technicalities that arise, replace in the example of $\mathbf{T} \backslash \mathbf{PGL}_2$ the group \mathbf{PGL}_2 by \mathbf{SL}_2 , and consider the space $L^2(T \backslash \mathrm{SL}_2, \chi)$ for some unitary character χ of T. (The reader will notice that the space has not changed, only the group acting has.) Here we have to deal with the fact that the torus acts with multiple open orbits on the character group of a unipotent subgroup that it normalizes. In this case one needs to replace the space $L^2(U \backslash G, \psi)$ by $L^2((\{\pm I\} U \backslash G)(k), \chi|_{\{\pm I\}} \psi)$, which is not precisely the space of an induced representation, because SL_2 does not act with a unique orbit on $(\{\pm I\} \backslash \mathbf{SL}_2)(k)$. (Of course, in this case there is a larger group acting on the space, namely the k-points of $(\mathbf{G}_{\mathrm{m}} \times \mathbf{SL}_2)/\{\pm I\}^{\mathrm{diag}}$, but to deal with the general case we do not want to use this fact.)

Therefore one needs some careful definitions of the spaces that arise in the process of unfolding, not as representations induced from some character but as complex line bundles over the k-points of certain \mathbf{G} -varieties. In the above examples, the space $\mathbf{T}\backslash\mathbf{G}$ will be understood as an affine bundle over $\mathbf{B}\backslash\mathbf{G}$, and the space $\mathbf{U}\backslash\mathbf{G}$ (resp. $\{\pm I\}\mathbf{U}\backslash\mathbf{G}$) as an open subvariety of the total space of a vector bundle over $\mathbf{B}\backslash\mathbf{G}$. The "unfolding" process is a Fourier transform from functions on the former to sections of a certain complex line bundle over the latter – not the trivial line bundle, precisely because $\mathbf{T}\backslash\mathbf{G}$ is an affine bundle and not a vector bundle.

9.5.6. Fourier transform on affine spaces. For a vector space \mathbf{V} (considered as an algebraic variety) over k, Fourier transform is an isomorphism between $L^2(V)$ and $L^2(V^*)$, or $C_c^{\infty}(V)$ and $C_c^{\infty}(V^*)$, depending on a unitary complex character $\psi: k \to \mathbb{C}^{\times}$ and assuming that the spaces are endowed with compatible Haar measures. Under the action of $\mathrm{GL}(V)$ the map is equivariant after we twist by the determinant character:

$$C_c^{\infty}(V) \xrightarrow{\sim} C_c^{\infty}(V^*) \otimes |\det|^{-1}$$

or, if we want to work with unitary representations:

$$L^2(V) \otimes |\det|^{\frac{1}{2}} \xrightarrow{\sim} L^2(V^*) \otimes |\det|^{-\frac{1}{2}}.$$

Here by det we denote the determinant of GL(V) on V, which is inverse to its determinant on V^* .

Now let \mathbf{F} denote an affine space over k, that is, a variety equipped with a equivalence class of isomorphisms with \mathbb{A}^n up to translations and linear automorphisms, for some n. We can still define Fourier transform of functions on F as follows: First of all, let \mathbf{V} denote the unipotent radical of $\mathrm{Aut}(\mathbf{F})$ (the group of affine automorphisms of \mathbf{F}); these are the "translation" automorphisms, i.e. those automorphisms which have no fixed point. The whole affine automorphism group is isomorphic to $\mathrm{GL}(\mathbf{V}) \ltimes \mathbf{V}$, the isomorphism depending on the choice of a point on F.

The dual vector space V^* of V can be thought of as the algebraic variety classifying reductions of F to a G_a -torsor. Therefore, there is a "universal"

principal G_a -bundle over V^* , which we will denote by R. It carries a canonical (right) Aut(F)-action.

Explicitly, we have:

$$\mathbf{R} = (\mathbf{F} \times \mathbf{V}^* \times \mathbf{G}_{\mathbf{a}})/\mathbf{V} \tag{9.11}$$

where V acts on $F \times V^* \times G_a$ as:

$$(f, v^*, x) \cdot u = (f \cdot u, v^*, x + \langle u, v^* \rangle). \tag{9.12}$$

The action of $Aut(\mathbf{F})$ descends to \mathbf{R} from the following action on $\mathbf{F} \times \mathbf{V}^* \times \mathbf{G}_a$:

$$(f, v^*, x) \cdot q = (f \cdot q, v^* \cdot q, x). \tag{9.13}$$

Notice that (9.12) and (9.13) do not coincide on V, i.e. $V \subset \operatorname{Aut}(F)$ acts non-trivially on R.

The choice of a point $f \in F$ defines a trivialization of this bundle: $\mathbf{V}^* \times \mathbf{G}_a \ni (v^*, x) \mapsto [f, v^*, x] \in \mathbf{R}$. We will denote the corresponding trivialized bundle by \mathbf{R}_f . Notice that all these trivializations coincide over $0 \in \mathbf{V}^*$, but not over other points.

Now let $\psi: k \to \mathbb{C}^{\times}$, a unitary character, be given. It defines a reduction of the corresponding k-bundle to a \mathbb{C}^{\times} -bundle over V^* , and hence a complex line bundle over V^* , which we will denote by $R^{\psi} \to V^*$. Again, the choice of a point $f \in F$ defines a trivialization of this bundle, which we will denote by R_f^{ψ} .

Fix throughout dual Haar measures on F and V^* with respect to ψ , where by "Haar" on F we mean a V-invariant measure. Let \mathfrak{d}_V denote the inverse of the character by which $\operatorname{Aut}(\mathbf{V})$ acts on an invariant volume form on \mathbf{V} . Then $L^2(F)\otimes |\mathfrak{d}_V|^{-\frac{1}{2}}$ is a unitary representation of $\operatorname{Aut}(F)$, and notice also that $L^2(V^*,R^\psi)\otimes |\mathfrak{d}_V|^{\frac{1}{2}}$ is also well-defined and unitary, since the character ψ is unitary and hence so are the isomorphisms between the different trivializations: $R_{f_1}^{\psi} \stackrel{Id}{\to} R^{\psi} \stackrel{Id}{\to} R_{f_2}^{\psi}$. (In other words, the line bundle $R^\psi \otimes \overline{R^\psi}$ is canonically trivial.)

Choosing a point $f \in F$ identifies F with V, and hence we can, using the character ψ , define Fourier transform:

$$L^{2}(F) \otimes |\mathfrak{d}_{V}|^{-\frac{1}{2}} \to L^{2}(V^{*}) \otimes |\mathfrak{d}_{V}|^{\frac{1}{2}} = L^{2}(V^{*}, R_{f}^{\psi}) \otimes |\mathfrak{d}_{V}|^{\frac{1}{2}}. \tag{9.14}$$

It is immediate to check that, independently of any choice of point:

9.5.7. Lemma. Fourier transform defines a canonical, Aut(F)-equivariant isomorphism:

$$L^{2}(F) \otimes |\mathfrak{d}_{V}|^{-\frac{1}{2}} \xrightarrow{\sim} L^{2}(V^{*}, R^{\psi}) \otimes |\mathfrak{d}_{V}|^{\frac{1}{2}}. \tag{9.15}$$

Notice also that it preserves smooth, compactly supported sections (here it is preferable to tensor with the modular character only on one side):

$$C_c^{\infty}(F) \stackrel{\sim}{\to} C_c^{\infty}(V^*, R^{\psi}) \otimes |\mathfrak{d}_V|.$$
 (9.16)

9.5.8. Affine bundles and unfolding. We generalize this to affine bundles: Let \mathbf{G} be an algebraic group over k, \mathbf{Y} a homogeneous space for \mathbf{G} (with $\mathbf{Y}(k) \neq \emptyset$) and \mathbf{F} an affine bundle over \mathbf{Y} . Let \mathcal{L} be a complex G-line bundle over \mathbf{Y} (which, at first reading, the reader may take to be trivial). We want to define Fourier transform as an equivariant isomorphism:

$$\mathcal{F}: C_c^{\infty}(F, \mathcal{L}) \otimes \eta^{\frac{1}{2}} \xrightarrow{\sim} C_c^{\infty}(V^*, \mathcal{L} \otimes R^{\psi} \otimes |\mathfrak{d}_V|) \otimes \eta^{\frac{1}{2}}. \tag{9.17}$$

We explain the notation: We denote by V^* the underlying space of the vector bundle over Y whose fiber over $y \in Y$ is the vector spaces of reductions of the corresponding fiber \mathbf{F}_y to a \mathbf{G}_{a} -torsor. The affine bundle \mathbf{F} gives rise to a canonical principal \mathbf{G}_{a} -bundle \mathbf{R} over \mathbf{V}^{*} ; the character ψ defines a reduction of this to a complex line bundle R^{ψ} over V^* . The complex line bundle \mathcal{L} is also considered as a line bundle over F or V^* by pull-back (but denoted by the same letter). We let \mathfrak{d}_V be the inverse of the character by which the stabilizer of a point $y \in \mathbf{Y}$ acts on a translationinvariant volume form on the fiber \mathbf{F}_{y} ; its absolute value defines a complex line bundle over Y (and, by pull-back, over F and V^*), which explains the tensor product with $|\mathfrak{d}_V|$ on the right hand side. Finally, we assume that there is a positive G-eigenmeasure of full support on Y, with eigencharacter η , valued in $(\mathcal{L} \otimes \bar{\mathcal{L}} \otimes |\mathfrak{d}_V|)^{-1}$. This endows $C_c^{\infty}(F, \mathcal{L}) \otimes \eta^{\frac{1}{2}}$ with a G-invariant Hilbert norm; the corresponding Hilbert space completion will be denoted by $L^2(F,\mathcal{L})\otimes \eta^{\frac{1}{2}}$. Recall that in this subsection only we use the unnormalized action of the group on spaces of functions, so the twist by $\eta^{\frac{1}{2}}$ is the one that makes the representation unitary (and is absorbed in the normalized action for the rest of the paper).

We can now apply Fourier transform on compactly supported smooth sections, fiberwise with respect to the maps $\mathbf{F} \to \mathbf{Y}, \mathbf{V}^* \to \mathbf{Y}$. We can make a G-invariant choice of Haar measures⁴⁰ (i.e. translation-invariant measures) on the fibers of $F \to Y$, valued in the line bundle defined by $|\mathfrak{d}_V|$. Hence, if $f \in C_c^{\infty}(F, \mathcal{L})$, then $|f|^2 \in C_c^{\infty}(F, \mathcal{L} \otimes \bar{\mathcal{L}})$ and we can define an "integration against Haar measure on the fiber" morphism:

$$C_c^{\infty}(F, \mathcal{L} \otimes \bar{\mathcal{L}}) \to C_c^{\infty}(Y, \mathcal{L} \otimes \bar{\mathcal{L}} \otimes |\mathfrak{d}_V|)$$
 (9.18)

We claim that $C_c^{\infty}(V^*, \mathcal{L} \otimes R^{\psi} \otimes |\mathfrak{d}_V|) \otimes \eta^{\frac{1}{2}}$ also has a unitary structure. Indeed, for $f \in C_c^{\infty}(V^*, \mathcal{L} \otimes R^{\psi} \otimes |\mathfrak{d}_V|)$ we have $|\underline{f}|^2 \in C_c^{\infty}(V^*, \mathcal{L} \otimes \overline{\mathcal{L}} \otimes |\mathfrak{d}_V|^2)$, where we took into account the fact that $R^{\psi} \otimes \overline{R^{\psi}}$ is the trivial line bundle canonically. Notice also that stabilizers of points act on the Haar measure on fibers of $\mathbf{V}^* \to \mathbf{Y}$ by \mathfrak{d}_V . Therefore, integration along the fibers with respect to Haar measure gives:

$$C_c^{\infty}(V^*, \mathcal{L} \otimes \bar{\mathcal{L}} \otimes |\mathfrak{d}_V|^2) \to C_c^{\infty}(Y, \mathcal{L} \otimes \bar{\mathcal{L}} \otimes |\mathfrak{d}_V|),$$
 (9.19)

 $^{^{40}}$ As usual, we demand that the invariant Haar measures arise from invariant differential forms; thus, the choice of a Haar measure on a fiber fully determines it on all fibers, even if G does not act transitively on Y.

i.e. the same line bundle whose dual we assumed admits a G-eigenmeasure with eigencharacter η .

Having fixed these measures, it is now easy to see that fiberwise Fourier transform indeed represents an equivariant isomorphism as in (9.17). The essence of what we call "the unfolding step" is the following:

9.5.9. Theorem. With suitable choices of invariant Haar measures on the fibers of $F \to Y, V^* \to Y$, Fourier transform (9.17) gives rise to a G-equivariant isometry of unitary representations:

$$\mathcal{F}: L^{2}(F, \mathcal{L}) \otimes \eta^{\frac{1}{2}} \xrightarrow{\sim} L^{2}(V^{*}, \mathcal{L} \otimes R^{\psi} \otimes |\mathfrak{d}_{V}|) \otimes \eta^{\frac{1}{2}}. \tag{9.20}$$

PROOF. Let \mathcal{F}' denote the inverse of \mathcal{F} . Since the spaces of compactly supported smooth sections are dense in the corresponding L^2 -spaces, and are mapped to each other under Fourier transform, the statement is equivalent to the following:

The morphisms $\mathcal{F}, \mathcal{F}'$ are adjoint.

Since the morphisms are defined fiberwise and the L^2 -hermitian forms are computed by "integration on the fibers" followed by (the same) "integration over Y", it suffices to prove this fact fiberwise, where "adjoint" will mean "adjoint with respect to the L^2 (Haar measure)-pairing on the fiber. But this is just Lemma 9.5.7 (ignore tensoring by the characters, since the equivariance of the map has already been established). It is easy to see that there are compatible choices of Haar measures globally.

- 9.5.10. The goal of unfolding. The method of unfolding provides a proof of Conjecture 9.4.6 in all cases that we have examined. However, we have not been able to establish the inductive step abstractly. The idea, in general, is to use this technique in order to prove the following conjecture. Recall that in this subsection we are not normalizing implicitly the right regular representations, so we denote by η the eigencharacter of our measure on X and tensor with its square root to make $L^2(X)$ unitary.
- 9.5.11. Conjecture. Given a homogeneous spherical variety X there exists a Levi subgroup L of a parabolic P of G, a factorizable spherical variety W of L and a complex hermitian line bundle \mathcal{L}_{Ψ} with an L-action over W, together with an isometric isomorphism of unitary representations (depending only on choices of measures on the various spaces):

$$L^2(X) \otimes \eta^{\frac{1}{2}} \xrightarrow{\sim} I_P^G(L^2(W, \mathcal{L}_{\Psi})) \otimes \eta^{\frac{1}{2}}.$$
 (9.21)

A proof of this conjecture proves the first part of the Discrete Series Conjecture 9.4.6. The second part, regarding character groups, can easily be verified in each case, but we have not shown abstractly that the unfolding process establishes it.

9.5.12. The inductive step. The proof of Conjecture 9.5.11 in each particular case goes through a series of "unfolding" steps.

Let a homogeneous spherical variety $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ be given, where we keep assuming that $\mathcal{Z}(\mathbf{G})^0 \twoheadrightarrow \mathcal{Z}(\mathbf{X})$, and let $\mathbf{H} = \mathbf{M} \ltimes \mathbf{N}$ be a Levi decomposition. If \mathbf{X} is not factorizable or parabolically induced from a factorizable spherical variety then there exists⁴¹ a unipotent subgroup \mathbf{U} of \mathbf{G} , normalized by \mathbf{H} and such that $\mathbf{N} \cap \mathbf{U}$ is normal in \mathbf{U} and $\mathbf{U} / \mathbf{N} \cap \mathbf{U}$ is non-trivial and abelian.

Consider the variety $\mathbf{Y} = \mathbf{MNU} \backslash \mathbf{G}$. We will denote by y the point of $\mathbf{MNU} \backslash \mathbf{G}$ represented by "1". Then $\mathbf{X} \to \mathbf{Y}$ has a canonical structure of an affine bundle; indeed, the orbit map for the action of \mathbf{U} on a point over the fiber of \mathbf{MNU} identifies that orbit with the additive group $\mathbf{U}/\mathbf{N} \cap \mathbf{U}$, the choice of point changes this identification by translations and the action of \mathbf{MN} is linear on $\mathbf{U}/\mathbf{N} \cap \mathbf{U}$. Call \mathbf{V}^* the corresponding dual vector bundle, and R^{ψ} the complex line bundle over V^* , as above. Applying Fourier transform, we get according to Theorem 9.5.9 an equivariant isomorphism:

$$L^{2}(X) \otimes \eta^{\frac{1}{2}} \to L^{2}(V^{*}, R^{\psi} \otimes |\mathfrak{d}_{V}|) \otimes \eta^{\frac{1}{2}}. \tag{9.22}$$

This is the first step of the unfolding process. We expect:

9.5.13. Conjecture. V^* is a spherical G-variety.

The L^2 -isomorphism of (9.22), and the implied finiteness of multiplicities, should also imply the spherical property of the above conjecture. Clearly, however, a direct, geometric proof would be more desirable. In any case, since we do not have a complete recipe for proving Conjecture 9.5.11, one needs to check this in any specific case, which is easy to do.

In any case, the fact that \mathbf{X} is spherical implies that \mathbf{M} acts with an open orbit on $\mathbf{X}_y = \mathbf{U}/(\mathbf{U} \cap \mathbf{N})$ (the fiber of \mathbf{X} over the point $y \in Y$).⁴² This implies, in particular, that \mathbf{G} acts with an open orbit $\mathring{\mathbf{V}}^*$ on \mathbf{V}^* ; this open orbit is isomorphic to $(\mathbf{M}\mathbf{N})_{v^*}\mathbf{U}\backslash\mathbf{G}$, where $(\mathbf{H})_{v^*}$ is the stabilizer in \mathbf{H} of a generic element $v \in V_y^*$ (which is the dual of the "translation" automorphism group of X_y).

Recall that R^{ψ} is the complex line bundle over V^* where $H_{v^*}U$ acts on the fiber over $v^* \in V_y^*$ by the character $\Psi := \psi \circ v^*$ (considered as a character of the whole group $H_{v^*}U$), and the same description holds for the other points of V^* . In the case when G acts transitively on \mathring{V}^* we can write:

$$L^{2}(V^{*}, R^{\psi} \otimes |\mathfrak{d}_{V}|) \otimes \eta^{\frac{1}{2}} = L^{2}(H_{v^{*}}U \backslash G, |\mathfrak{d}_{V}|\Psi) \otimes \eta^{\frac{1}{2}}.$$

 $^{^{41}}$ The proof of existence of such a unipotent subgroup departs from the observation that there is a proper parabolic subgroup containing \mathbf{H} ; we omit the details since we do not know how to describe a canonical choice for \mathbf{U} .

⁴² Indeed, if $\widetilde{\mathbf{P}}$ is a parabolic with unipotent radical $\widetilde{\mathbf{U}}$ containing $\mathbf{N}\mathbf{U}$, and with \mathbf{M} in its Levi \mathbf{L} , then since $\mathbf{M}\mathbf{N}$ acts with an open orbit on the flag variety of \mathbf{G} , it has to act with an open orbit on the open $\widetilde{\mathbf{P}}$ -orbit of the flag variety, which is isomorphic to $\mathbf{L}/\mathbf{B}_L \times \widetilde{\mathbf{U}}$. Equivalently, \mathbf{M} acts with an open orbit on $\mathbf{L}/\mathbf{B}_L \times \widetilde{\mathbf{U}}/\mathbf{N}$, and since $\widetilde{\mathbf{U}}/\mathbf{N}$ is fibered over $\widetilde{\mathbf{U}}/\mathbf{U}\mathbf{N}$ with fiber $\mathbf{U}/\mathbf{N} \cap \mathbf{U}$, it follows that \mathbf{M} acts with an open orbit on the latter.

It is easy to see the following:

9.5.14. LEMMA. Fix a point $v^* \in (V^*)_y$ (the fiber of V^* over the point $y \in Y$), and the trivialization of the fiber of R_{ψ} over v^* corresponding to a chosen point $x \in X_y$. (Recall that the choice of a point $x \in X_y$ makes the fiber X_y into a vector space, and hence makes Fourier transform scalar-valued.) Let Ψ denote the character by which the stabilizer $H_{v^*}U$ acts on the fiber of R_{ψ} . Then, for suitable choices of measures, the map:

$$C_c^{\infty}(X) \otimes \eta^{\frac{1}{2}} \to C^{\infty}(V^*, R^{\psi} \otimes |\mathfrak{d}_V|) \otimes \eta^{\frac{1}{2}} \xrightarrow{\operatorname{ev}_{v^*}} \mathbb{C}$$

inducing the above L^2 -isomorphism is given by the integral:

$$\Phi \mapsto \int_{U/U \cap N} \Phi(u) \Psi^{-1}(u) du$$

and its adjoint:

$$C_c^{\infty}(V^*, R^{\psi} \otimes |\mathfrak{d}_V|) \otimes \eta^{\frac{1}{2}} \to C^{\infty}(X) \otimes \eta^{\frac{1}{2}} \xrightarrow{\operatorname{ev}_x} \mathbb{C}$$

is given by the integral:

$$\Phi \mapsto \int_{(\mathbf{H}_{v^*} \setminus \mathbf{H})(k)} \operatorname{ev}_{v^*}(R_h \Phi) dh,$$

where R_h denotes the regular representation.

PROOF. We follow the definitions and constructions that we have presented: The first integral represents, simply, Fourier transform over the fiber $(\simeq (\mathbf{U}/\mathbf{U}\cap\mathbf{N})(k)=U/U\cap N)$ of the affine bundle X(=F) in the notation of Theorem 9.5.9) over y. The fact that the second is its adjoint is completely formal; for simplicity, when G acts with a unique orbit on \mathring{V}^* , and using the isomorphism $(H_{v^*}U\cap H)\backslash H_{v^*}U=U/U\cap N$, when $f\in C_c^\infty(X)$ and $\Phi\in C_c^\infty(V^*,R^\psi\otimes |\mathfrak{d}_V|)=C_c^\infty(H_{v^*}U\backslash G,|\mathfrak{d}_V|\Psi)$:

$$\begin{split} \int_{H_{v^*}U\backslash G} \left(\int_{U/U\cap N} f(ug) \Psi^{-1}(u) du \right) \Phi(g) dg &= \int_{(H_{v^*}U\cap H)\backslash G} f(g) \Phi(g) dg = \\ &= \int_{H\backslash G} \left(\int_{H_{v^*}\backslash H} \Phi(hg) dh \right) f(g) dg. \end{split}$$

If the stabilizer \mathbf{H}_{v^*} of a generic point on the dual of $U/(N \cap U)$, modulo the center of \mathbf{G} , has finite character group, then we are done with the proof of Conjecture 9.5.11 in the given case: the variety to which we have unfolded is factorizable.

If not, it is convenient (for the purpose of this theoretical presentation) to "fold back" in order to eliminate the character ψ , i.e. to do the

following:⁴³ if $(\mathbf{M}\mathbf{N})_{v^*}$ denotes the stabilizer of the line of v^* (necessarily $(\mathbf{M}\mathbf{N})_{v^*}/(\mathbf{M}\mathbf{N})_{v^*} \simeq \mathbf{G}_{\mathrm{m}}$ since \mathbf{G} acts almost transitively on \mathbf{V}^*), and \mathbf{U}_0 denotes the kernel of v^* , let $\mathbf{X}' = (\mathbf{M}\mathbf{N})_{v^*}\mathbf{U}_0\backslash\mathbf{G}$. Now we can "fold back" to \mathbf{X}' , which brings us to the original setup of a spherical variety for \mathbf{G} , without the line bundle defined by an additive character. What we cannot show in general is that the variety \mathbf{X}' is closer to being "factorizable" than the original variety \mathbf{X} ; for instance, that its stabilizers have a larger unipotent part. However, all of the examples that we have examined show that with correct choices this is indeed the case.

9.5.15. EXAMPLE. It is instructive at this point to discuss the unfolding process for the variety $\mathbf{X} = \mathbf{G}_a^2 \backslash \mathbf{SO}_5$ under the $\mathbf{G}_m^2 \times \mathbf{SO}_5$ -action. Here \mathbf{G}_m^2 is a Cartan subgroup and \mathbf{G}_a^2 is the subgroup containing the root subspaces of the two long roots. We want to show:

$$L^2(X) \simeq L^2((k^{\times})^2) \otimes L^2(U \backslash SO_5, \psi),$$

the last factor being a Whittaker model.

One could choose for the first step the group U to be the one corresponding to the sum of the long root spaces and one short root space, that is: the unipotent radical of a parabolic with Levi of type $G_m \times SO_3$. The reader will see that "unfolding" this way will lead to a non-factorizable situation which we cannot unfold further (more precisely, "folding back" as was suggested right above we return to the original space).

On the other hand, one may take for U the unipotent subgroup corresponding to the sum of the short root spaces and one long root space, that is: the unipotent radical of a parabolic with Levi of type GL_2 . The second step now goes through with U= a maximal unipotent subgroup, and leads to the Whittaker model.

We finish by mentioning a few more examples.

9.5.16. EXAMPLE. Generalizing the example of \mathbf{PGL}_2 as a $\mathbf{G}_m \times \mathbf{PGL}_2$ -variety that we mentioned earlier, let $\mathbf{X} = \mathbf{SL}_n \setminus \mathbf{GL}_{n+1}$ as a $\mathbf{G} = \mathbf{G}_m \times \mathbf{GL}_{n+1}$ -space, where $\mathbf{G}_m = \mathbf{GL}_n / \mathbf{SL}_n$, where \mathbf{GL}_n belongs to the mirabolic subgroup of \mathbf{GL}_{n+1} (the stabilizer of a non-zero point under the standard representation). Let \mathbf{N} be the unipotent radical of the mirabolic, and let Ψ be a non-trivial complex character of N. Then we have a G-equivariant isometry (for $n \geq 2$):

$$L^2(\operatorname{SL}_n \setminus \operatorname{GL}_{n+1}) \simeq L^2(P_n \ltimes N \setminus \operatorname{GL}_{n+1}, \Psi)$$

⁴³Without choosing a base point v^* on V^* , the "folding back" process admits the following description: We have obtained sections of a certain complex bundle R^{ψ} over the k-points of a vector bundle \mathbf{V}^* over \mathbf{Y} . Pull back the $\mathbf{G}_{\mathbf{a}}$ -bundle \mathbf{R}^{ψ} (and the corresponding complex vector bundle R^{ψ}) to the blow-up $\mathbb{B}\mathbf{V}^*$ of \mathbf{V}^* along the zero section. Let $\mathbf{Y}' := \mathbb{P}\mathbf{V}^*$ (the projectivization of \mathbf{V}^*); then $\mathbb{B}\mathbf{V}^*$ is a line bundle over $\mathbb{P}\mathbf{V}^*$. There is a $\mathbf{G}_{\mathbf{a}}$ -bundle \mathbf{X}' over \mathbf{Y}' such that the bundle R^{ψ} over $\mathbb{B}\mathbf{V}^*$ is the one obtained by Fourier transforms from the trivial complex bundle over \mathbf{X}' .

where $P_n \subset \operatorname{SL}_n$ denotes the stabilizer of Ψ .

Here (for $n \geq 2$) the group k^{\times} does not act trivially on $L^{2}(P_{n} \ltimes N \setminus \operatorname{GL}_{n+1}, \delta_{N} \Psi)$. However, the variety $P_{n} \ltimes N \setminus \operatorname{PGL}_{n+1}$ is parabolically induced (and the character Ψ is trivial on the unipotent radical of its parabolic), hence:

$$L^{2}(P_{n} \ltimes N \backslash \operatorname{GL}_{n+1}, \Psi) = \operatorname{Ind}_{P}^{G} L^{2}((\operatorname{SL}_{n-1} \times N_{2}) \backslash (\operatorname{GL}_{n-1} \times \operatorname{GL}_{2}), \Psi) =$$

$$= \operatorname{Ind}_{P}^{G} (L^{2}(k^{\times}) \otimes L^{2}(N_{2} \backslash \operatorname{GL}_{2}, \Psi)).$$

Here P denotes the parabolic of type $G_{\rm m} \times \operatorname{GL}_{n-1} \times \operatorname{GL}_2$, N_2 denotes a non-trivial unipotent subgroup of GL_2 , and Ind_P^G denotes unitary induction.

9.5.17. EXAMPLE. Let $G = GL_{2n}$, let Sp_{2n} denote a symplectic subgroup of G and let H be the subgroup of Sp_{2n} stabilizing a point under the standard representation of G. Then by successive applications of "unfolding" one can show that:

$$L^{2}(H\backslash G) = \operatorname{Ind}_{P}^{G} L^{2}(N \times N \backslash \operatorname{GL}_{n} \times GL_{n}, \Psi)$$

where P is the parabolic of type $GL_n \times GL_n$, $N \times N$ is a maximal unipotent subgroup in its Levi and Ψ is a non-degenerate character of this subgroup.

10. Preliminaries to the Bernstein morphisms: "linear algebra"

This section collects some simple results in "linear algebra" (interpreted broadly) which will be used in $\S11$.

The reader may wish to refer to the contents only as needed. The main purpose of this section is to separate the "abstract" parts of arguments from the parts that are specific to spherical varieties.

- The first sections ($\S 10.1 \S 10.3$) pertain to the following general question: given a Hermitian form on a vector space V, and a group S acting on V, how can one canonically replace the form by an S-invariant one? Assuming that the forms are S-finite, i.e. generate a finite-dimensional vector space under the action of S, this is possible. These constructions will be used in $\S 11$.
- In section 10.4 we show that, given a family of S-finite linear functionals (or hermitian forms) on V which vary in a measurable way over a parameter space, we may extract their eigenprojections to certain generalized eigencharacters (for instance, unitary ones) and still get a measurable family.
- 10.1. Basic definitions. Suppose S is a finitely generated abelian group together with a finitely generated submonoid $S^+ \subset S$ that generates S.

Thus there is a surjective homomorphism $\mathbb{Z}^k \to S$ so that S^+ is the image of $\mathbb{Z}^k_{>0} := \{\mathbf{x} : x_i \geq 0\}.$

By a locally finite S-vector space V we shall mean a vector space V over $\mathbb C$ equipped with a locally finite action of S (i.e. the S-span of any vector is finite dimensional).

For a vector $v \in V$, an *exponent* of v is any generalized eigencharacter of S on the space of translates; the *degree* of v is the dimension of $\langle Sv \rangle$.

If χ is a character of S, we write $|\chi| \leq c$ (resp. $|\chi| < c$) if $|\chi(s)| \leq c$ (resp. $|\chi(s)| < c$) for all $s \in S^+ \setminus \{0\}$; similarly, we define $|\chi| \geq c$, $|\chi| > c$. Note that if $|\chi| = 1$ (i.e., $|\chi| \leq 1$ and $|\chi| \geq 1$) then χ is unitary. If $|\chi| < 1$ we will say that χ is strictly subunitary. (We will say simply subunitary when $|\chi|$ is < 1 in the interior of S^+ , i.e. its elements which do not lie on the boundary of the cone spanned by S^+ in $S \otimes \mathbb{R}$. In the rank-one case, where this notion will be used in later sections, subunitary and strictly subunitary coincide.)

We warn that $|\chi| < 1$ is a stronger condition than $(|\chi| \le 1 \text{ and } |\chi| \ne 1)$. Indeed the statement $|\chi| < 1$ amounts to asking that χ "decay in all directions", and not merely in *some* directions. In practice, we will always be able to arrange this by shrinking S^+ if necessary.

We often use the following observation: any S^+ -stable subspace of a locally finite S-vector space V is also S-stable. Indeed, the S-span of a vector being finite dimensional implies that the inverse of any invertible operator is a polynomial in the operator.

10.2. Finite and polynomial functions. Now we specialize to the case of functions on S; a function whose S-translates span a finite dimensional vector space will be called a *finite function*. A finite function whose only exponent is the trivial character of S is called a *polynomial*. This coincides with the usual use of "polynomial" when $S = \mathbb{Z}^k$. For any any finite function f, there exists characters χ_i and polynomials P_i so that $f(s) = \sum \chi_i(s) P_i(s)$.

For every finite function f, we refer to the dimension of the space spanned by its S-translates as its degree.

10.2.1. Lemma. A polynomial function that is bounded on S^+ is constant.

PROOF. It is enough to consider the case $S = \mathbb{Z}^k, S^+ = \mathbb{Z}^k_{\geq 0}$. Our assertion reduces to the following: if a polynomial function on \mathbb{R}^k is bounded on $\mathbb{Z}^k_{\geq 0}$, then it is constant.

10.2.2. LEMMA. Let f be any finite function bounded on S^+ . Then there exists a unique S-invariant functional $\langle Sf \rangle \to \mathbb{C}$ that sends the constant function 1 to 1, where $\langle Sf \rangle$ is the span of all translates of f by S.

We refer to this functional as \lim_{S^+} . For instance, if $S = \mathbb{Z}$, $S^+ = \mathbb{Z}_{\geq 0}$, t a nonzero complex number of absolute value ≤ 1 (not equal to 1), and f is the function $n \mapsto 3 + t^n$, then $\lim_{S^+} f = 3$.

PROOF. If f is any finite function, bounded on S^+ , we may write it as a sum of generalized eigenfunctions, $f = \sum f_{\chi}$, where each f_{χ} belongs to the S^+ -span of f and is therefore itself bounded on S^+ . The putative functional must (by invariance) send f_{χ} to zero for $\chi \neq 1$. On the other hand f_1 is

bounded polynomial and thus constant. Therefore, the only possibility for the functional is

$$\sum f_{\chi} \mapsto [f_1],$$

where $[f_1]$ is the constant value of f_1 . It is clear that this functional is S-invariant and has the desired normalization.

We refer to a sequence of positive probability measures ν_i on S^+ , defined for all sufficiently large positive integers i, as an averaging sequence if it is obtained in the following way: Let ℓ_1, ℓ_2 be monotone increasing affine functions $\mathbb{Z} \to \mathbb{Z}$ with $\ell_2 - \ell_1 \to \infty$ as $i \to \infty$; for example, $\ell_1(i) = i, \ell_2(i) = 2i$. Choose a surjection $\mathbb{Z}^k \to S$ mapping $\mathbb{Z}^k_{\geq 0}$ onto S^+ , and let ν_i be the image of the uniform probability measure on $[\ell_1(i), \ell_2(i)]^k$.

In particular, such a sequence has the following properties: For arbitrary $s \in S$, the measure $s\nu_i$ is eventually (i.e. for any large enough i, depending on s) supported on S^+ and the total mass of the difference $|s \cdot \nu_i - \nu_i|(S)$ approaches 0.

- 10.2.3. Lemma. Let f be a finite function and ν_i an averaging sequence.
- (1) If f bounded on S^+ , we have $\int f\nu_i \longrightarrow \lim_{S^+} f$.
- (2) If f is unbounded on S^+ and all exponents χ of f satisfy either $|\chi| = 1$ or $|\chi| < 1$, then $\int |f|^2 \nu_i \longrightarrow \infty$.

The restriction on exponents in the latter part is for simplicity, to avoid situations where f grows in some directions and decays in others. This restriction will be satisfied in our applications.

10.2.4. Remark. The incongruence between the two statements (the first for f and the second for $|f|^2$) will not appear in our applications, as we will apply both to the functions obtained by evaluating S-finite hermitian forms on translates of a vector in an S-vector space; evidently, however, we need a positivity assumption for the second statement to hold.

PROOF. Consider first the case of f bounded. Note that $\int f\nu_i$ are bounded. We may choose a subsequence of the i so that all the integrals $\int g\nu_i$ converge for $g \in \langle Sf \rangle$. Then $g \mapsto \lim_i \int g\nu_i$ defines an S-invariant functional on $\langle Sf \rangle$, which is necessarily \lim_{S^+} . Since the subsequence was arbitrary, the result follows.

Now suppose f is unbounded; write $f = \sum f_{\chi}$, where f_{χ} has generalized character χ . At least one f_{χ} is unbounded. Therefore $|\chi| = 1$, since if $|\chi| < 1$ then certainly f_{χ} must be bounded. Twisting by χ^{-1} , we may suppose that $\chi = 1$, i.e. f_1 is a polynomial.

There exists an element Δ of the group algebra $\mathbb{C}[S]$ so that $\Delta \star f = f_1$. Write $\Delta = \sum a_s s$. Let ν_i' be another averaging sequence, to be chosen momentarily. Let $\nu_i^* = \sum_{a_s \neq 0} s^{-1} \cdot \nu_i'$, a sum of translates of ν_i' . We can and do choose ν_i' in such a way that $\nu_i^* \leq C \cdot \nu_i$ for some positive constant C – choose ν_i' by replacing the linear forms ℓ_1, ℓ_2 used to define ν_i by $\ell_1' = \ell_1 + A, \ell_2' = \ell_2 - A$ for a large enough integer A. The integral $\int |f_1|^2 \nu_i'$ is

a sum of terms of the form $\int (sf)\overline{(s'f)}d\nu'_i$, and by Cauchy-Schwarz we may bound this by $\int |f|^2\nu_i^*$. That is to say,

$$\int |f_1|^2 \nu_i' \le \operatorname{const} \cdot \int |f|^2 \nu_i^*.$$

Visibly, the left-hand side is unbounded (since f_1 is a polynomial). Since $\nu_i^* \leq C\nu_i$ our assertion is proved.

10.2.5. Lemma. Let F(n,c) be the space of all finite functions on S of degree $\leq n$ all of whose exponents satisfy $|\chi| < c$, for some c < 1. Then we can find a finite subset $\Lambda \subset S^+$, depending only on (n,c), so that for all $f \in F(n,c)$:

$$\sup_{k \in S^+} |f(k)| \le \max_{\lambda \in \Lambda} |f(\lambda)|,$$

In fact, there exists a decaying function $Q: S^+ \to \mathbb{R}$ depending only on (n,c) so that, for any $f \in F$:

$$|f(k)| \le Q(k) \max_{\lambda \in \Lambda} |f(\lambda)|. \tag{10.1}$$

10.2.6. Remark. Recall that the notion of "decaying function" was defined in the introduction §1.7. By enlarging Λ we may assume that $|Q| \leq 1$. This way, the first statement becomes a special case of the second.

PROOF. We may again assume that $S = \mathbb{Z}^k, S^+ = \mathbb{Z}^k_{\geq 0}$. Let $s_1 = (1,0,\ldots,0),\ldots,s_k = (0,0,\ldots,1)$ be the standard generators for S^+ . Fix an $f \in F(n,c)$, and let $\mathbf{P} = (P_1,\ldots,P_k)$ be the characteristic polynomials of s_1,\ldots,s_k acting on $\langle Sf \rangle$, each of degree $n_i \leq n$. Note that all the coefficients of all P_i are bounded in terms of (n,c). Let $F(\mathbf{P})$ be the set of functions annihilated by $P_i(s_i)$ for each i.

Put $\Lambda = \prod_{i=1}^k [0, n_i - 1]$. The evaluation map $\mathrm{ev} : F(\mathbf{P}) \to \mathbb{C}^{\Lambda}$ is a linear isomorphism. The action of translation by s_i on $F(\mathbf{P}) \simeq \mathbb{C}^{\Lambda}$ is expressed by a certain endomorphism $A_i \in \mathrm{End}(\mathbb{C}^{\Lambda})$ whose matrix entries are bounded in terms of (n, c). Therefore:

$$f(t_1,\ldots,t_k) = \left(A_1^{t_1}\ldots A_k^{t_k}\operatorname{ev}(f)\right)(\mathbf{0}).$$

The second statement now follows from the following: Suppose that Ω is a compact subset of $\operatorname{End}(\mathbb{C}^{\Lambda})$ so that, for every $A \in \Omega$, all of the eigenvalues of A are $\leq c < 1$; then there exists N so that $\|A^N\| \leq \frac{1}{2}$ for all $A \in \Omega$. Here, $\| \bullet \|$ denotes any norm on $\operatorname{End}(\mathbb{C}^{\Lambda})$. To check that, take 1 > c' > c, use $A^k = \frac{1}{2\pi i} \int_{|z|=c'} \frac{z^k}{z-A} dz$ and the fact that

$$||(z-A)^{-1}||, |z| = c', A \in \Omega,$$

being a continuous function on a compact set, is bounded. As mentioned in the remark, this implies the first assertion, as well (by enlarging Λ).

10.2.7. Lemma. Suppose f is a finite function on S of degree $\leq n$, bounded on S^+ , all of whose exponents are unitary. Then

$$\sup_{k \in S^+} |f|^2 \le n \lim_{S^+} |f|^2.$$

PROOF. The boundedness forces f to be a sum of eigenfunctions (with unitary character, by assumption): if we write $f = \sum_{\chi \in I} f_{\chi}$, then each f_{χ} is bounded. We claim that f_{χ} is proportional to χ . For $\chi = 1$ that is Lemma 10.2.1 and in the general case $\chi = \chi_0$ we apply the same reasoning to $f\chi_0^{-1}$. The result easily follows.

10.3. Hermitian forms. We now turn to the properties of Hermitian forms on S-vector spaces. In what follows, where we speak of "Hermitian forms" we always mean **positive semi-definite** Hermitian forms. If $(u, v) \mapsto H(u, v)$ is a Hermitian form, we use the notation H(v) for H(v, v).

Notice that if H is a hermitian form on a finite-dimensional S-vector space V, then it can be considered as an element of the tensor product representation $V^* \otimes \bar{V}^*$ of S, and hence the form itself is an S-finite vector in an S-vector space; in particular, it makes sense to talk about its exponents. For a hermitian form on a locally finite, possibly infinite-dimensional, S-vector space we call exponents of H the union of its exponents on all S-stable, finite-dimensional subspaces. (The form itself might not be S-finite, in this case.)

Let H be a Hermitian form on a locally finite S-vector space, and put

$$V_f = \{v \in V : H(S^+v) \text{ is bounded.}\}$$

This is an S-invariant subspace of V. That it is a subspace is a consequence of the inequality

$$H(x_1 + \dots + x_m) \le m \sum H(x_i),$$

whereas the S-stability follows from the observation at the end of $\S 10.1$.

The following is an obvious application of Lemma 10.2.2, with positivity following, for instance, from Lemma 10.2.3:

10.3.1. Lemma. Let H be a Hermitian form on a locally finite S-vector space V. Then

$$H^{S}(v) := \lim_{S^{+}} (s \mapsto H(sv)),$$

defines an S-invariant Hermitian form on V_f .

We refer to H^S , extended by ∞ off V_f , as the associated S-invariant form. Then the association $H \mapsto H^S$ is linear, i.e., given Hermitian forms H_1, H_2 and positive scalars a_1, a_2 , we have $(a_1H_1 + a_2H_2)^S = \sum a_iH_i^S$. The following is a corollary to what we have already proved in Lemma 10.2.3, taking into account that we may express H (on any finite dimensional subspace) as a sum of squares of linear forms:

10.3.2. LEMMA. Let H be a Hermitian form on a locally finite S-vector space V. Let ν_i be an averaging sequence; then, for any $v \in V_f$,

$$\int H(av)\nu_i =: \nu_i \star H(v) \longrightarrow H^S(v).$$

The same assertion holds for every $v \in V$ (with possible infinite values on the right hand side) if it is assumed that all exponents of H satisfy $|\chi| = 1$ or $|\chi^{-1}| < 1$ on S^+ .

We shall say that a form H on a locally finite S-vector space is c-good if

Every exponent
$$\chi$$
 of H satisfies $|\chi| = 1$ or $|\chi^{-1}| \le c^2$. (10.2)

The reason for inverting χ is that the exponents of S on linear functionals are inverse to those on the vector space itself; we write c^2 for convenience when comparing to the case of the square of a linear functional.

10.3.3. Remark. Notice that $|\chi^{-1}| \leq c^2$ rules out the possibility of subunitary exponents which are not strictly subunitary. We do that for simplicity, since in later sections we will only use the case where the rank of S is 1, so subunitary and strictly subunitary coincide.

The form H is simply good if it is c-good for some c < 1. In particular, this excludes the possibility of exponents that grow along some "walls" of S^+ but not along other. Similar terminology will be applied to a vector, if it applies to its S-span.

10.3.4. Lemma. Suppose H is c-good. Let R be the sum of all generalized eigenspaces corresponding to characters χ that satisfy neither $|\chi| = 1$ nor $|\chi| \leq c$. Then R lies in the radical of H and H factors through V/R.

In particular one may write (on any finite-dimensional, S-stable subspace): $H = \sum |\ell_i|^2$, where each ℓ_i is linear and each Hermitian form $|\ell_i|^2$ is itself c-good.

PROOF. If $v \in V$ is a eigenvector for S, with eigencharacter χ , then $\langle H^s \text{ considered in } V^* \otimes \overline{V^*}, v \otimes v \text{ considered in } V \otimes \overline{V} \rangle = |\chi(s)|^{-2} H(v).$

That implies that $|\chi|^{-2}$ is an exponent of H if $H(v) \neq 0$. In particular, if χ doesn't satisfy $|\chi| = 1$ or $|\chi| \leq c$, then H(v) = 0, i.e., v lies in the radical of H (because H is semidefinite, $H(v) = 0 \implies H(u+v) = H(u)$ for all u, i.e. v is in the radical).

This conclusion remains valid if v were simply a generalized eigenvector. This proves the first assertion of the lemma.

For the second assertion (concerning linear forms): Choose any expression of H as a sum of squares on the vector space V/R. If ℓ is any linear functional on V/R then all exponents of $\ell \otimes \overline{\ell} \in V \otimes V^*$ are of the form $\chi = (\psi_1 \overline{\psi_2})^{-1}$, where ψ_1, ψ_2 are exponents of S on V/R. In particular, $|\chi^{-1}| = |\psi_1||\psi_2|$ satisfies $|\chi^{-1}| = 1$ or $|\chi^{-1}| \leq c$; that proves the second assertion.

10.3.5. Proposition. Let H be a hermitian form on a locally finite S-vector space V, whose elements have degree bounded by n, and assume that H is c-good.

There exist a finite subset $\Lambda \subset S^+$ and a constant C depending only on (n,c) so that:

$$H^{a} \le C(H^{S} + \sum_{\lambda \in \Lambda} H^{\lambda}), \tag{10.3}$$

for any $a \in S^+$; here $H^a(v) := H(av)$.

PROOF. In fact, we may suppose V to be finite-dimensional and H to be the square of a linear form ℓ , in view of the prior Lemma.

In that case the function $a \mapsto \ell(av)$ is finite for any fixed $v \in V$. Write

$$\ell(av) = f(a) + g(a)$$

where the functions f, g possess only unitary (resp. sub-unitary) exponents, i.e. all exponents of f satisfy $|\chi| = 1$, and all exponents of g satisfy $|\chi| \le c < 1$. The degrees of f, g are both bounded by n.

If f (equivalently f+g) is unbounded on S^+ , then the result is obvious, as the right-hand side of the putative inequality is infinite. We suppose therefore that f is bounded on S^+ . Apply Lemmas 10.2.5 and 10.2.7 (taking Λ as in Lemma 10.2.5):

$$\begin{split} \sup_{S^+} |f+g|^2 &\leq & 2\sup_{S^+} |f|^2 + 2\sup_{S^+} |g|^2 \\ &\leq & 2n\lim_{S^+} |f|^2 + 2\max_{\Lambda} |g|^2 \\ &\leq & 2n\lim_{S^+} |f|^2 + 4\max_{\Lambda} \left(|g+f|^2 + |f|^2\right) \\ &\leq & 6n\lim_{S^+} |f|^2 + 4\max_{\Lambda} |f+g|^2 \\ &= & 6n\lim_{S^+} |f+g|^2 + 4\max_{\Lambda} |f+g|^2, \end{split}$$

the last line since $\lim_{S^+} |f + g|^2 = \lim_{S^+} |f|^2$.

10.3.6. COROLLARY. Notation and assumptions as in the previous lemma, let $\Pi^{<1}$ be the S-equivariant projection of V onto all generalized eigenspaces with subunitary exponent, and $\Pi^{=1}$ the S-equivariant projection onto generalized eigenspaces with unitary exponent. Let $H^{<1} = H \circ \Pi^{<1}$ and $H^{=1} = H \circ \Pi^{=1}$, and denote by n_1 the number of distinct unitary exponents of V.

Then:

$$H^{-1}(v) \leq n_1 H^S(v) \tag{10.4}$$

$$H^{<1}(av) \le Q(a)(H^{S}(v) + \max_{\Lambda} H^{\lambda}(v))$$
 (10.5)

where the decaying function Q depends only on (n, c).

PROOF. Both assertions are linear in H, so we reduce, as in the previous proof, to the case where $H = |\ell|^2$. Note that $\ell = \ell \circ \Pi^{=1} + \ell \circ \Pi^{<1}$; indeed (adopting the notation of Lemma 10.3.4) the functional ℓ may be supposed

to contain R in its kernel, but id $-\Pi^{=1} - \Pi^{<1}$ is a projector to a subspace of R.

Note that the decomposition $\ell = \ell \circ \Pi^{=1} + \ell \circ \Pi^{<1}$ induces the decomposition $\ell(av) = f + g$ of the prior proof, i.e. $H^{=1}(av) = |f|^2$ and $H^{<1}(av) = |g|^2$. For the first bound, now, apply Lemma 10.2.7. (Notice that either the right hand side is infinite, or the eigenprojection of v to the sum of subspaces with unitary exponents has degree bounded by the number of unitary exponents.) For the second, bound $|g|^2$ using (10.1).

10.4. Measurability of eigenprojections. Here we discuss the following issue: Let S^+ be a finitely generated abelian monoid, and let W be an $\mathbb{C}[S^+]$ -module of countable dimension. Let $(L_y)_{y\in Y}$ be a family of linear functionals on W, varying with a parameter y in a measurable space (Y,\mathcal{B}) (i.e. a set equipped with a σ -algebra). Assume that the family L_y is measurable in the sense of evaluations (the function $y \mapsto \langle w, L_y \rangle$ is measurable for every $w \in W$), and that $each\ L_y$ is S^+ -finite, i.e. generates a finite-dimensional $\mathbb{C}[S^+]$ -submodule of W^* .

Hence, each L_y has a decomposition in generalized S^+ -eigenspaces:

$$L_y = \bigoplus_{\chi \in \widehat{S}^+_{\mathbb{C}}} L_y^{\chi}, \tag{10.6}$$

where $\widehat{S^+}_{\mathbb{C}}$ denotes the space of all characters (not necessarily unitary) of S^+ . Let $K \subset \widehat{S^+}_{\mathbb{C}}$ be a measurable subset (with respect to the natural Borel σ -algebra on $\widehat{S^+}_{\mathbb{C}}$), and denote:

$$L_y^K = \bigoplus_{\chi \in K} L_y^{\chi}. \tag{10.7}$$

We want to prove:

10.4.1. Proposition. The functionals L_y^K vary measurably with y.

Again, measurability here is meant with respect to evaluations at every vector of W, like above.

In practice, we will use this proposition to isolate the unitary part of an S^+ -finite functional or a hermitian form.

First of all, we can reduce the proof of the proposition to the case $S^+ = \mathbb{N}$. Indeed, first we replace S^+ by \mathbb{N}^k by taking a surjective map: $\mathbb{N}^k \to S^+$, and then we observe that the Borel σ -algebra of $\widehat{\mathbb{N}}_{\mathbb{C}}$ is the product σ -algebra of the Borel σ -algebras of $\widehat{\mathbb{N}}_{\mathbb{C}}$, and hence we may assume without loss of generality that K is a product subset. (Let us clarify what this means for the functionals L_y^K : writing K as a countable union of subsets K_n corresponds to writing L_y^K as the weak limit of functionals $L_y^{K_n}$, and this limit stabilizes for a given y. Since for the complement K' of K we have $L_y^{K'} = L_y - L_y^K$, the same argument shows that the set of subsets K which satisfy the Proposition is closed under countable unions and intersections, i.e. forms a σ -algebra, and hence it is enough to check for a set of subsets generating the σ -algebra.) Then we can obtain L_y^K in a finite number of steps by restricting

the generalized eigencharacters coordinate-by-coordinate. Hence, from now on we will assume that $S^+ = \mathbb{N}$, and we will denote its generator by x.

The proposition will now be established via the following result: Identify $\mathbb{C}[S^+]$ with the ring $\mathbb{C}[x]$ of polynomials in one variable (where x corresponds to the generator of $S^+ \simeq \mathbb{N}$, and call *minimal polynomial* of L_y the monic generator of its annihilator in $\mathbb{C}[S^+]$. We will denote it by \mathfrak{m}_y .

There is a natural measurable structure on $\mathbb{C}[S^+] \simeq \mathbb{C}[x]$. Namely, a set is measurable if for any d its intersection with polynomials of degree $\leq d$ is Borel-measurable with respect to the standard topological structure on that complex vector space.

10.4.2. Lemma. The minimal polynomials \mathfrak{m}_y vary measurably in $y \in Y$.

Let us see why this implies Proposition 10.4.1.

First of all, by partitioning Y in a countable union we may assume that the degree of \mathfrak{m}_y is constant in Y, say $\mathfrak{m}_y \in \mathbb{C}[S^+]_N$ (degree N). The coefficients of each polynomial are elementary symmetric functions in its roots, and we may pick a measurable section: $\mathbb{C}[S^+]_N \to \mathbb{C}^N$ of the map $\mathbb{C}^N \ni (\alpha_i)_i \mapsto \prod_i (x - \alpha_i) \in \mathbb{C}[S^+]$. (We have continued with the prior notation, so that x is an element of $\mathbb{C}[S^+]$ corresponding to a generator g for S^+). Hence, we may index the roots $(\alpha_{i,y})_i$ of \mathfrak{m}_y in a measurable way. Finally, for given measurable $K \subset \mathbb{C}$, the set $A_y \subset \{1, \ldots, N\}$ of indices such that $\alpha_{i,y} \notin K$ varies measurably with y. Hence, we may write the minimal polynomial \mathfrak{m}_y in a measurable way as a product:

$$\mathfrak{m}_y = \mathfrak{m}_y^1 \mathfrak{m}_y^2,$$

where $\mathfrak{m}_y^1 = \prod_{i \in A_y} (x - a_{i,y})$ and $\mathfrak{m}_y^2 = \prod_{i \notin A_y} (x - a_{i,y})$. The polynomials \mathfrak{m}_y^1 and \mathfrak{m}_y^2 are relatively prime, hence we can find a polynomial \mathfrak{m}_y^3 which is inverse to \mathfrak{m}_y^1 in $\mathbb{C}[x]/\mathfrak{m}_2$. Again, this can be done in a measurable way using the division algorithm. Then:

$$L_y^K = \mathfrak{m}_y^3(x)\mathfrak{m}_y^1(x)L_y. \tag{10.8}$$

Indeed, $\mathfrak{m}_y^1(x)$ annihilates the summands of L_y with exponents outside of K, and since on the rest of the summands $\mathbb{C}[x]$ acts via the quotient $\mathbb{C}[x]/\mathfrak{m}_y^2$, the product $\mathfrak{m}_y^3(x)\mathfrak{m}_y^1(x)$ acts as the identity on them. This shows that L_y^K is measurable, i.e., it concludes the proof that Lemma 10.4.2 implies Proposition 10.4.1.

To prove Lemma 10.4.2, we notice that $\mathfrak{m}_y(x) = x^N + a_{N-1}x^{N-1} + \cdots + a_0$ if and only if:

- (1) for every $w \in W$ we have: $L_y(x^N \cdot w) + a_{N-1}L_y(x^{N-1} \cdot w) + \cdots + a_0L_y(w) = 0$, and
- (2) this is not the case for any polynomial of smaller degree.

We may enumerate a vector space basis $(w_i)_{i\geq 1}$ of W, and for every N, n we consider the following system of linear equations in the unknowns a_0, \ldots, a_{N-1} :

$$\mathbf{S}_{N,n}: \left(L_y(x^N \cdot w_i) + a_{N-1}L_y(x^{N-1} \cdot w_i) + \dots + a_0L_y(w_i) = 0\right)_{i=1}^n. (10.9)$$

The polynomial $x^N + a_{N-1}x^{N-1} + \cdots + a_0$ satisfies the above two conditions (i.e. is the minimal polynomial \mathfrak{m}_y) if its coefficients are the *unique* solution of the system: $\mathbf{S}_{N,\infty} = \bigcup_n \mathbf{S}_{N,n}$.

For given n, N the set of $y \in Y$ such that the system has a solution is measurable; indeed we can attempt to solve the system by row operations, the order of which depends only on whether some coefficients vanish or not (which, of course, depends measurably on $y \in Y$). Among those, uniqueness of the solution is also a measurable condition, for the same reason. Finally, among the latter the unique solution $(a_0, a_1, \ldots, a_{N-1})$ varies measurably in $y \in Y$, again for the same reason.

Hence, for given N the set Y_N of $y \in Y$ such that $\mathbf{S}_{N,\infty}$ has a unique solution is measurable (notice that for given y, if $\mathbf{S}_{N,\infty}$ has a unique solution then so does $\mathbf{S}_{N,n}$ for some n), and the solution varies measurably in $y \in Y$. This proves the lemma.

We will also use this result in the following form:

10.4.3. COROLLARY (Measurability of eigenspaces.). Let W be a finite dimensional vector space and $T(z) \in \operatorname{End}(W)$ a family of matrices varying measurably as z varies in a measurable space Z. Then the T(z)-invariant projection to the generalized 0 eigenspace varies measurably with z.

More generally, suppose that $\alpha_z: \mathbb{Z}^N \to \operatorname{Aut}(W)$ is a measurable family of actions and $\chi: \mathbb{Z}^N \to \mathbb{C}^\times$ a character. Then the canonical $(\alpha_z(\mathbb{Z}^n)$ -invariant) projection of W to the generalized χ -eigenspace for α_z varies measurably with z.

11. The Bernstein morphisms

From now on we assume that the Discrete Series Conjecture 9.4.6 holds for X and all its degenerations X_{Θ} (for example, X is strongly factorizable). Although the structure of discrete series will not be used explicitly in the present section, we will use its corollaries, such as the boundedness of subunitary exponents 9.4.8. Since this will be an ongoing assumption, it will not be explicitly included in the theorems.

- 11.1. In the present section, we construct a canonical morphism $L^2(X_{\Theta}) \to L^2(X)$. We may think of this morphism $L^2(X_{\Theta}) \to L^2(X)$ as
 - the unique morphism asymptotic to the "naive" identification of functions on X_{Θ} and X.

It may be worth beginning our section with the following easy Lemma, which contains the germ of many of the ideas used in this section:

11.1.1. LEMMA. The support of Plancherel measure for $L^2(X_{\Theta})$ is contained in the support of Plancherel measure for $L^2(X)$.

PROOF. It suffices to show that matrix coefficients of the form

$$\langle g \cdot f, f \rangle$$
, (11.1)

where $f \in C_c^{\infty}(X_{\Theta})$ can be approximated, uniformly on compacta, by diagonal matrix coefficients of $L^2(X)$. Assume that f is J-invariant, and given a compact, J-biinvariant subset K of G set $J' = \bigcap_{k \in K} kJk^{-1}$. We may translate f by the action of $\mathcal{Z}(X_{\Theta})$ into a J'-good neighborhood of infinity with the property that the identification with a neighborhood of Θ -infinity on X is equivariant 44 under the action of the elements of $\mathcal{H}(G,J')$ whose support is in K. Then the matrix coefficients (11.1) coincide on K, whether f is considered as a function on X_{Θ} or on X.

We now formulate properties of the morphism more precisely. Recall that

$$e_{\Theta}: C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X)$$
 (11.2)

denotes the equivariant "asymptotics" map which, for J-invariant functions supported in a J-good neighborhood of Θ -infinity, coincides with the identification of J-orbits under the exponential map. The formulations that follow involve, actually, only functions supported close enough to Θ -infinity, and therefore the equivariant extension of this identification is not being used in the statement of the theorem.

We will need to "push" functions towards Θ -infinity; recall that $\mathring{A}_{X,\Theta}^+$ denotes the subset of "strictly anti-dominant" elements of $A_{X,\Theta}$, i.e. those which push points on X_{Θ} towards Θ -infinity.

Denote by \mathcal{L}_a the (normalized) action of $a \in A_{X,\Theta}$ on functions on X_{Θ} , which we understand as $a^{-1} \cdot f$ in the normalization of §4.1; in other words, the \mathcal{L}_a for $a \in \mathring{A}_{X,\Theta}^+$ push functions towards ∞ .

11.1.2. THEOREM. For every $\Theta \subset \Delta_X$ there is a canonical G-equivariant morphism: $\iota_{\Theta} : L^2(X_{\Theta}) \to L^2(X)$, characterized by the property that for any $a \in \mathring{A}_{X,\Theta}^+$ and any $\Psi \in C_c^{\infty}(X_{\Theta})$ we have:

$$\lim_{n \to \infty} (\iota_{\Theta} \mathcal{L}_{a^n} \Psi - e_{\Theta} \mathcal{L}_{a^n} \Psi) = 0.$$
 (11.3)

In words, (11.3) says that ι_{Θ} is approximately equal to the identity furnished by the exponential map on functions supported near Θ -infinity. Although results of this type are present in the scattering theory literature, the idea of the proof that we present here (essentially, the proof of Theorem 11.3.1) is due to Joseph Bernstein and we will consequently refer to ι_{Θ} as the "Bernstein morphism".

11.1.3. REMARK. If we replace e_{Θ} by τ_{Θ} := "truncation in a fixed J-good neighborhood N_{Θ} of Θ -infinity", we can generalize property (11.3) to non-compactly supported smooth L^2 -functions:

$$\lim_{n \to \infty} (\iota_{\Theta} \mathcal{L}_{a^n} \Phi - \tau_{\Theta} \mathcal{L}_{a^n} \Phi) = 0. \tag{11.4}$$

As in the theorem, the limit is taken inside $L^2(X)$, and we identify the function $\tau_{\Theta} \mathcal{L}_{a^n} \Phi$, with a function on X by means of the exponential map.

 $^{^{44}}$ Equivariance implicitly assumes the identification of larger neighborhoods, of course.

The proof is very simple: For given $\varepsilon > 0$ we can find m and a function $\Psi \in C_c^{\infty}(N_{\Theta})^J$ such that $\|\Psi - \mathcal{L}_{a^m}\Phi\| < \varepsilon$ (and hence $\|\mathcal{L}_{a^n}\Psi - \mathcal{L}_{a^{m+n}}\Phi\| < \varepsilon$ for every $n \geq 0$). Then, applying (11.3) to Ψ , we get some n such that:

$$\|\iota_{\Theta}\mathcal{L}_{a^n}\Psi - e_{\Theta}\mathcal{L}_{a^n}\Psi\| < \varepsilon.$$

Therefore, for large enough n:

$$\|\iota_{\Theta}\mathcal{L}_{a^{m+n}}\Phi - \tau_{\Theta}\mathcal{L}_{a^{m+n}}\Phi\| \le \|\iota_{\Theta}\left(\mathcal{L}_{a^{m+n}}\Phi - \mathcal{L}_{a^{n}}\Psi\right)\| + \|\iota_{\Theta}\mathcal{L}_{a^{m+n}}\Phi - \mathcal{L}_{a^{m+n}}\Psi\|$$

$$+\|\iota_{\Theta}\mathcal{L}_{a^{n}}\Psi - e_{\Theta}\mathcal{L}_{a^{n}}\Psi\| + \|e_{\Theta}\mathcal{L}_{a^{n}}\Psi - \tau_{\Theta}\mathcal{L}_{a^{m+n}}\Phi\| < (\|\iota_{\Theta}\| + 2)\varepsilon.$$

For the last inequality, we note that for large enough n that $e_{\Theta}\mathcal{L}_{a^n}\Phi$ is obtained from $\tau_{\Theta}\mathcal{L}_{a^n}\Phi$ via the identification of J-orbits arising from the exponential map; thus the last term has norm bounded by the norm of $\|\tau_{\Theta}\mathcal{L}_{a^n}\Phi - \tau_{\Theta}\mathcal{L}_{a^{m+n}}\Phi\|$, which is at most ε .

However, the proof of Theorem 11.1.2 will be via an estimate:

$$\|\iota_{\Theta} \mathcal{L}_{a^n} \Psi - e_{\Theta} \mathcal{L}_{a^n} \Psi\|^2 \le C_{\Psi} \cdot Q^J(a^n), \tag{11.5}$$

for $\Psi \in C_c^{\infty}(X_{\Theta})^J$, where Q^J is a decaying function on $\mathring{A}_{X,\Theta}^+$ (cf. Lemma 11.5.1) which depends only on the open compact subgroup J, and C_{Ψ} is a constant that depends on Ψ , see Lemma 11.5.1. Such an estimate is not valid for functions which are not compactly supported.

11.2. Harish-Chandra–Schwartz space and temperedness of exponents. In [Ber88] Bernstein explains how to prove that the Plancherel formula for a space of polynomial growth like X is supported on X-tempered representations.

We remind what this means. In order to do this we reprise some of the remarks of §6.1, but replacing the role of $C_c^{\infty}(X)$ by the slightly larger space $\mathscr{C}(X)$ of Harish-Chandra-Schwartz functions on X.

A function $r: X \to \mathbb{R}_+$ is called a radial function if it is positive, locally bounded and proper, i.e. such that the balls $B(a) := \{x \in X | r(x) \le a\}$ are relatively compact, and has the property that for every compact $J \subset G$ there is a constant C > 0 such that $|r(x \cdot g) - r(x)| < C$ for all $x \in X, g \in J$. Two radial functions r and r' are called equivalent if the quotient $\frac{1+r}{1+r'}$ is bounded above and below by absolute positive constants.

A space is called of polynomial growth (for a given radial function) if for some compact $J \subset G$ there is a polynomial $a \mapsto P(a)$ such that for all a > 0 the ball B(a) can be covered by $\leq P(a)$ sets of the form $x \cdot J$. This notion is clearly invariant under equivalence of radial functions.

⁴⁵The reader of [Ber88] will notice that the notion of temperedness obtained there is slightly stronger than temperedness with respect to the Harish-Chandra–Schwartz space; indeed, we can replace that with $L^2(wdx)^{\infty}$ for any summable weight w. However, since summability of weights depends on the rank of the variety, it is more convenient to work with the weaker condition provided by the Harish-Chandra–Schwartz space.

By the generalized Cartan decomposition (see §5.3),⁴⁶ the space X of k-points of our spherical variety possesses a natural equivalence class of radial functions, with respect to which it is of polynomial growth. In fact, if $\mathcal{Z}(X) = 1$, such a radial function R(x) was described in the proof of Theorem 6.4.1. We leave the details of the general case to the reader (the only difference being that one needs to quantify as well the "distance" from all orbits in a smooth toroidal compactification of X, including "orbits belonging to Δ_X -infinity"). We fix such a radial function r.

Then we define the Harish-Chandra-Schwartz space as the Fréchet space:

$$\mathscr{C}(X) = \lim_{\stackrel{\rightarrow}{J}} \bigcap_{d} L^{2}(X, (1+r)^{d} dx)^{J}$$
(11.6)

the limit taken over a basis of neighborhoods of the identity.

Bernstein proves that the embedding $\mathscr{C}(X) \to L^2(X)$ is fine which implies that any Hilbert space morphism to a direct integral of Hilbert spaces:

$$m: L^2(X) \to \int \mathcal{H}_{\alpha} \mu(\alpha)$$
 (11.7)

is pointwise defined on $\mathscr{C}(X)$, i.e. there is a family of morphisms

$$L_{\alpha}: \mathscr{C}(X) \to \mathcal{H}_{\alpha}$$

(defined for μ -almost every α) such that $\alpha \mapsto L_{\alpha}(\Phi)$ represents $m(\Phi)$ for every $\Phi \in \mathscr{C}(X)$.

Notice that such a family of morphisms induces, by pull-back, seminorms $\| \bullet \|_{\alpha}$ on $\mathscr{C}(X)$. If the morphism m is surjective (set-theoretically, hence open) then the spaces \mathcal{H}_{α} can be identified with the completions of $\mathscr{C}(X)$ with respect to the seminorms $\| \bullet \|_{\alpha}$.

A Plancherel decomposition for $L^2(X)$ – or, more generally, for some closed invariant subspace of $L^2(X)$ – can be described by the choice of a measure μ on \hat{G} and a measurably varying family $\| \bullet \|_{\pi}$ of norms on $\mathscr{C}(X)$, with the properties that $\| \bullet \|_{\pi}$ factors through the natural morphism from $\mathscr{C}(X)$ to the space of π -coinvariants⁴⁷

$$\mathscr{C}(X)_{\pi} := (\operatorname{Hom}_{G}(\mathscr{C}(X), \pi))^{*} \otimes \pi. \tag{11.8}$$

and also $\|\Phi\|_{L^2(X)}^2 = \int_{\pi} \|\Phi\|_{\pi}^2 \mu(\pi)$ for every $\Phi \in \mathscr{C}(X)$. (Recall that in the case of wavefront spherical varieties, which we are discussing, the spaces $\operatorname{Hom}_G(\mathscr{C}(X),\pi)$ are finite-dimensional.)

⁴⁶Using the Cartan decomposition is again not necessary: it is enough to know that the union of *J*-good neighborhoods of Θ-infinity, for all $\Theta \subsetneq \Delta_X$, has a compact complement (modulo center).

 $^{^{47}}$ In the case of the Harish-Chandra–Schwartz space, "homomorphism" will always mean "continuous homomorphism". The space of (smooth vectors on) π is endowed with the discrete topology or, what amounts to the same for homomorphisms, the coarsest \mathbb{C} -vector space topology. Similarly, "linear functionals" and "hermitian forms" will always be continuous.

We have a canonical map (recall that the space of π -coinvariants of $C_c^{\infty}(X)$ was defined in a completely analogous way in (6.1)):

$$C_c^{\infty}(X)_{\pi} \twoheadrightarrow \mathscr{C}(X)_{\pi}.$$
 (11.9)

11.2.1. Remark. It is sometimes more convenient to think of the Plancherel formula as giving a "decomposition into eigenfunctions":

The Hermitian norms $\| \bullet \|_{\pi}$ induce $C_c^{\infty}(X)_{\pi} \to \overline{(C_c^{\infty}(X)_{\pi})^{\sim}} = \overline{C^{\infty}(X)^{\pi}}$. For every $f \in C_c^{\infty}(X)$, the conjugate of the image of f under the map⁴⁸

$$C_c^{\infty}(X) \to C_c^{\infty}(X)_{\pi} \to \overline{C^{\infty}(X)^{\pi}}$$

will be denoted by f^{π} .

Then we have a pointwise decomposition:

$$f(x) = \int_{\hat{G}} f^{\pi}(x)\mu(\pi), \tag{11.10}$$

which is another way of writing the Plancherel decomposition for the inner product $\langle f, \operatorname{Vol}(xJ)^{-1} 1_{xJ} \rangle$ for a sufficiently small open compact subgroup J. Note that we also have for $f, g \in C_c^{\infty}(X)$ the equality

$$\langle f, g^{\pi} \rangle_X = \langle f^{\pi}, g \rangle_X \tag{11.11}$$

since both are different ways to express the inner product $H_{\pi}(f,g)$ (where H_{π} is the hermitian form corresponding to $\| \bullet \|_{\pi}$).

We remark, however, that although the image f_{π} of a function f in the space of π -coinvariants is canonically defined, this is not the case for f^{π} , which depends on the choice of Plancherel measure.

By the asymptotics map e_{Θ} (cf. (11.2)) we get a canonical map:

$$C_c^{\infty}(X_{\Theta})_{\pi} \to C_c^{\infty}(X)_{\pi}.$$

The following follows directly from the definitions and the discussion of 10.3. Recall that we always consider the normalized action of $A_{X,\Theta}$ on functions on X_{Θ} , and that the space $C_c^{\infty}(X_{\Theta})_{\pi}$ is $A_{X,\Theta}$ -finite for every irreducible representation π . Hence, we may decompose into sums of generalized eigenspaces:

$$C_c^{\infty}(X_{\Theta})_{\pi} = C_c^{\infty}(X_{\Theta})_{\pi}^{<1} \oplus C_c^{\infty}(X_{\Theta})_{\pi}^{1} \oplus C_c^{\infty}(X_{\Theta})_{\pi}^{\nleq 1}$$

$$(11.12)$$

with exponents, respectively, satisfying⁴⁹ $|\chi^{-1}| < 1$, $|\chi| = 1$ and $|\chi^{-1}| \nleq 1$ on $\mathring{A}_{X,\Theta}^+$.

Here $|\chi^{-1}| \nleq 1$ means that there exists $a \in \mathring{A}_{X,\Theta}^+$ with $|\chi^{-1}(a)| > 1$. The three possibilities are mutually exclusive and one must occur for each

⁴⁸Here $C^{\infty}(X)^{\pi}$ denotes the π -isotypical subspace, which is to say, the image of $\pi \otimes \operatorname{Hom}(\pi, C^{\infty}(X)) \to C^{\infty}(X)$.

⁴⁹Our notation is explained as follows: We denote the sum of eigenspaces of $C_c^{\infty}(X_{\Theta})_{\pi}$ by exponents satisfying $|\chi^{-1}| < 1$ by $C_c^{\infty}(X_{\Theta})_{\pi}^{<1}$ because the hermitian forms will be sub-unitary there. Another way to think of it is the following: if l is a smooth linear functional on $C_c^{\infty}(X_{\Theta})_{\pi}^{<1}$, then via the duality $C_c^{\infty}(X_{\Theta}) \otimes C^{\infty}(X_{\Theta}) \to \mathbb{C}$ it can be considered as an element of $C^{\infty}(X_{\Theta})$. That element will be decaying on $\mathring{A}_{X,\Theta}^+$.

 χ : Recall that $\mathring{A}_{X,\Theta}^+$ denotes the "strict interior" of the cone $A_{X,\Theta}^+$. Now, if the final possibility does not occur, then $|\chi^{-1}| \leq 1$ on $\mathring{A}_{X,\Theta}^+$; then $|\chi^{-1}|$ is bounded above on $A_{X,\Theta}^+$, from which one sees that $|\chi^{-1}| \leq 1$ on $A_{X,\Theta}^+$; but that means that either $|\chi| = 1$ or $|\chi| < 1$ on the "strict interior" $\mathring{A}_{X,\Theta}^+$.

- 11.2.2. LEMMA. Let H_{π} be a G-invariant hermitian form on $C_c^{\infty}(X)_{\pi}$.
- (1) Through the map e_{Θ} it is pulled back to an $A_{X,\Theta}$ -finite, G-invariant Hermitian form $e_{\Theta}^*H_{\pi}$ on $C_c^{\infty}(X_{\Theta})_{\pi}$.
- (2) Suppose the form H_{π} factors through the Harish-Chandra-Schwartz space, i.e. through (11.9). Then, for any Θ , if we decompose as in (11.12), the summand $C_c^{\infty}(X_{\Theta})_{\pi}^{\nleq 1}$ lies inside the radical of $e_{\Theta}^*H_{\pi}$ (in particular, the form vanishes there).

PROOF. For part (1), the only assertion to be proved is the $A_{X,\Theta}$ -finiteness; however, in our present case, the multiplicity of π in $C^{\infty}(X_{\Theta})$ is finite, from which the result follows at once.

For part (2) it is enough to show that $C_c^{\infty}(X_{\Theta})_{\pi}^{\nleq 1}$ lies in the kernel of the composite

$$C_c(X_{\Theta})_{\pi} \longrightarrow C_c^{\infty}(X)_{\pi} \to \mathscr{C}(X)_{\pi}.$$

In other words, given a morphism $\lambda: \mathscr{C}(X) \to \pi$, we need to verify that

$$C_c(X_{\Theta}) \stackrel{e_{\Theta}}{\to} C_c^{\infty}(X) \to \mathscr{C}(X) \stackrel{\lambda}{\to} \pi$$

vanishes on $C_c^{\infty}(X_{\Theta})_{\pi}^{\nleq 1}$.

Suppose that the χ -eigenspace on $C_c^{\infty}(X_{\Theta})_{\pi}$ is nonzero, and that there exists $a \in \mathring{A}_{X,\Theta}^+$ so that $|\chi(a)| > 1$.

Take $\Psi \in C_c^{\infty}(X_{\Theta})^J$. Then (by continuity of λ) the norm of the image of $\mathcal{L}_{a^n}\Psi$ grows at most polynomially in n. But the projection of $\mathcal{L}_{a^n}\Psi$ to this χ - generalized eigenspace – if nonzero – grows as $\chi(a^n)$, at least for a subsequence of n: after replacing this projection by a linear combination of translates by various a^k , we may suppose that it is a nonzero element of the genuine – not just generalized – χ -eigenspace.

- 11.3. Plancherel formula for X_{Θ} from Plancherel formula for X. The theorem below is the heart of Theorem 11.1.2.
- 11.3.1. THEOREM. Consider a Plancherel decomposition for $L^2(X)$ (cf. 6.1):

$$\|\Phi\|^2 = \int_{\hat{G}} H_{\pi}(\Phi)\mu(\pi). \tag{11.13}$$

Fix an open compact subgroup J, and fix a strictly positive cocharacter $s: \mathbb{G}_m \to \mathbf{A}_{X,\Theta}$ (i.e. a cocharacter in the strict interior, in Λ_X^+ , of the face corresponding to Θ), and let $S = s(\varpi^{\mathbb{Z}})$ for ϖ a uniformizer of k.

Consider the pullback $e_{\Theta}^* H_{\pi}$ of H_{π} to $C_c^{\infty}(X_{\Theta})_{\pi}$, and let $(e_{\Theta}^* H_{\pi})^S$ be the associated S-invariant form (by Lemma 10.3.1). Let $\Psi \in C_c^{\infty}(X_{\Theta})^J$; then

$$\|\Psi\|^2 = \int_{\hat{G}} (e_{\Theta}^* H_{\pi})^S (\Psi) \mu(\pi). \tag{11.14}$$

Therefore, the hermitian forms $(e_{\Theta}^* H_{\pi})^S$ define a Plancherel formula for $L^2(X_{\Theta})^J$.

11.3.2. REMARK. In particular, for almost every π the invariant forms $(e_{\Theta}^* H_{\pi})^S$ are $A_{X,\Theta}$ -invariant and finite, and do not depend on the choice of S; we will be denoting them by H_{π}^{Θ} .

Indeed, it is entirely possible that these norms take infinite values for some π ; but this must happen only on a set of measure zero: Clearly, for each individual $\Phi \in C_c(X_{\Theta})$, the set of π for which $e_{\Theta}^*H_{\pi}(\Phi) = \infty$ has measure zero. Since $C_c(X_{\Theta})$ has countable dimension, this implies the stronger statement, because, if a norm takes infinite values, it does so on at least one element of a basis.

11.3.3. Remark. This theorem is roughly the analog of (8.13) from the discussion of the toy model of scattering on \mathbb{N} .

PROOF. First of all, we notice that it suffices to prove the analogous statement to (11.14) for the Hilbert spaces $L^2(X,\chi), L^2(X_{\Theta},\chi)$, for every $\chi \in \widehat{\mathcal{Z}(X)}$, and for a function $\Psi \in C_c^{\infty}(X,\chi)$.

Let S be as in the statement of the theorem, and set $S^+ = S \cap \mathring{A}_{X,\Theta}^+$. In order to simplify notation in what follows, we assume that $\mathcal{Z}(X) = 1$, since the arguments are exactly the same in the general case.

Define for $a \in S^+$ the function $\Phi_a = e_{\Theta} \mathcal{L}_a \Psi \in C_c^{\infty}(X)$. Then, as a approaches infinity inside S^+ , we have $\|\Phi_a\|_{L^2(X)} \to \|\Psi\|_{L^2(X_{\Theta})}$; indeed equality holds for sufficiently large a. Moreover, by definition, $H_{\pi}(\Phi_a) = e_{\Theta}^* H_{\pi}(\mathcal{L}_a \cdot \Psi)$.

The group S acts on $e_{\Theta}^*H_{\pi}$ through a finitely generated quotient, and we may therefore apply the results of §10. Let ν_i be an averaging sequence of probability measures on S^+ (§10.2). Recall each of those is, by construction, of finite support. Then:

$$\lim_{i \to \infty} \int_{\hat{G}} (\nu_i \star e_{\Theta}^* H_{\pi})(\Psi) \mu(\pi) = \lim_{i \to \infty} \int_a \int_{\hat{G}} e_{\Theta}^* H_{\pi}(\mathcal{L}_a \Psi) \mu(\pi) d\nu_i(a) \quad (11.15)$$

$$= \lim_{i \to \infty} \int_a \|\Phi_a\|_{L^2(X)}^2 d\nu_i(a) = \|\Psi\|_{L^2(X_{\Theta})}^2,$$

the last step because $\|\Phi_a\|_{L^2(X)}$ and $\|\Psi\|_{L^2(X_{\Theta})}$ are eventually equal.

Our task is to interchange the limit and the integral on the left hand side of (11.15). By Lemma 10.3.2, $\lim_{i\to\infty}(\nu_i\star e_\Theta^*H_\pi)(\Psi)=(e_\Theta^*H_\pi)^S(\Psi)$.

Applying Fatou's lemma:

$$\int_{\hat{G}} (e_{\Theta}^* H_{\pi})^S (\Psi) \mu(\pi) \le \|\Psi\|^2. \tag{11.16}$$

Before we continue with the proof of Theorem 11.3.1, we draw a corollary from this inequality that will be used in the sequel:

- 11.3.4. COROLLARY. (1) The set of π such that $(e_{\Theta}^* H_{\pi})^S$ takes the value ∞ is of (X-Plancherel = μ) measure zero. Therefore, strengthening Lemma 11.2.2 for the Plancherel forms, for almost all π the restriction of $e_{\Theta}^* H_{\pi}$ to $C_c^{\infty}(X_{\Theta})_{\pi}^1$ (the sum of generalized eigenspaces with unitary exponents) factors through the maximal eigenquotient.
- (2) The restriction of μ to the set of π with $(e_{\Theta}^* H_{\pi})^S \neq 0$ (equivalently: to the set of π for which $e_{\Theta}^* H_{\pi}$ has unitary exponents) is absolutely continuous with respect to Plancherel measure on X_{Θ} .
- (3) The Plancherel measure for X is absolutely continuous with respect to the sum, over all $\Theta \subset \Delta_X$, of G-Plancherel measures for the discrete spectra of X_{Θ} .
- PROOF OF THE COROLLARY. (1) The first statement is clear from (11.16), and the second follows from the fact that S^+ is arbitrary in $\mathring{A}_{X,\Theta}^+$, and that $(e_{\Theta}^*H_{\pi})^S$ takes the value ∞ if its restriction to S-unitary generalized eigenspaces does not factor through the maximal eigenspace quotient.
- (2) Both sides of (11.16) define G-invariant, positive semi-definite hermitian forms on $C_c^{\infty}(X_{\Theta})$, and if \mathcal{H}_l and \mathcal{H}_r (for "left" and "right") denote the corresponding Hilbert spaces then we have a morphism of unitary representations: $\mathcal{H}_r \to \mathcal{H}_l$, necessarily surjective since the image of $C_c^{\infty}(X_{\Theta})$ is dense. By [Dix77, §8], the Plancherel measure for \mathcal{H}_l is absolutely continuous with respect to the Plancherel measure for \mathcal{H}_r .
- (3) If $H_{\pi} \neq 0$ but has only subunitary exponents in all non-trivial directions, then π is an X-discrete series.

Indeed, we show that H_{π} extends continuously to $L^2(X)$. For a fixed function $f \in C_c^{\infty}(X_{\Theta})$ and $a \in A_{X,\Theta}^+$ the quantity $H_{\pi}(e_{\Theta}\mathcal{L}_a f)$ decays rapidly with a, i.e. bounded above by $|\chi^{-1}(a)|$ where $|\chi^{-1}| < 1$ on $\mathring{A}_{X,\Theta}^+$. In fact, more is true, namely $|\chi^{-1}| < 1$ on $A_{X,\Theta}^+$. If not, there is a "wall" of $A_{X,\Theta}$ along which $|\chi| = 1$; this corresponds to an $\Omega \supset \Theta$ for which $e_{\Omega}^* H_{\pi}$ has unitary exponents, contradicting our supposition. Since $|\chi^{-1}| < 1$ on $A_{X,\Theta}^+$, we deduce, by taking f to be the characteristic function of a single J-orbit, that H_{π} is L^2 -bounded on the $A_{X,\Theta}^+$ -span of f; by taking a finite set of such Θ and f, we deduce that H_{π} is bounded on $L^2(X)^J$, which implies that it is also bounded on $L^2(X) - a$ G-invariant Hermitian form on a finite sum of copies of π is uniquely determined by its restriction to J-invariants, for sufficiently small J.

The restriction of Plancherel measure to the set of such π 's is, by definition, the Plancherel measure for $L^2(X)_{\text{disc}}$.

Otherwise, there is a Θ such that π belongs to the set of representations for which $e_{\Theta}^*H_{\pi}$ has unitary exponents. Applying the second statement, for the set of those π the statement is reduced to the analogous statement for the Plancherel measure of X_{Θ} , and the claim follows by induction.

We continue with the proof of Theorem 11.3.1 – we want to upgrade (11.16) to an equality. Let us discuss what might go wrong in order to better understand this. Let us consider an increasing sequence $a_1, a_2, \ldots, a_n, \ldots$ in S that "go to ∞ " inside S^+ ; consider the functions $\Phi_{a_1}, \ldots, \Phi_{a_n}, \cdots \in C_c^\infty(X)$. One could imagine that there existed a sequence of irreducible G-subrepresentations $\pi_1, \pi_2, \cdots \subset L^2(X)$ so that $\Phi_{a_j} \in \pi_j$ (or, more generally, such that a bounded below percentage of the norm of the Φ_{a_j} 's is concentrated on those discrete series). In this case, the left-hand side of (11.16) will be zero (or bounded away from $\|\Psi\|^2$). But we know that this cannot happen precisely because of finiteness of discrete series (Theorem 9.2.1). The input from Section 10 generalizes this result and allows us to show that, in all cases, (11.16) may be replaced by equality:

$$\int_{\hat{G}} (e_{\Theta}^* H_{\pi})^S (\Psi) \mu(\pi) = \|\Psi\|^2.$$
 (11.17)

Indeed, Corollary 11.3.4 and Proposition 9.4.8 imply that there is a uniform bound on the S^+ -subunitary exponents for μ -almost all π with $\pi^J \neq 0$ (for some fixed open compact subgroup J). Moreover, the following easy lemma implies that the degree of elements of $C_c^{\infty}(X_{\Theta})_{\pi}$ as $A_{X,\Theta}$ -finite vectors is also uniformly bounded:

11.3.5. Lemma. There is an integer N (which, in fact, can be taken to be equal to the order of the Weyl group) such that for all irreducible representations π the degree of all elements of $C_c^{\infty}(X_{\Theta})_{\pi}$ as $A_{X,\Theta}$ -finite vectors is $\leq N$.

PROOF OF THE LEMMA. By the definition of $C_c^{\infty}(X_{\Theta})_{\pi}$, this is the same as the degree of elements of $\operatorname{Hom}_G(\tilde{\pi},C^{\infty}(X_{\Theta}))$. Since X_{Θ} is parabolically induced from X_{Θ}^L , this space is isomorphic to $\operatorname{Hom}_{L_{\Theta}}(\tilde{\pi}_{P_{\Theta}^-},C^{\infty}(X_{\Theta}^L))$, and since X is wavefront an L_{Θ} -morphism is also an $A_{X,\Theta}$ -morphism (Proposition 2.7.2). Therefore, the degree is bounded by the $\mathcal{Z}(L_{\Theta})^0$ -degree of elements of the Jacquet module $\tilde{\pi}_{P_{\Theta}^-}$.

But $\tilde{\pi}$ is a subquotient of a parabolically induced irreducible supercuspidal representation; therefore, the degree of any element of any Jacquet module of $\tilde{\pi}$ (with respect to the action of the center of the corresponding Levi) is bounded by the order of the Weyl group.

Therefore, we may apply Proposition 10.3.5 to μ -almost all π with $\pi^J \neq 0$ to deduce:

There are a finite set Λ and a constant C so that, for all indices i:

$$\nu_i \star e_{\Theta}^* H_{\pi}(\Psi) \le C \left((e_{\Theta}^* H_{\pi})^S (\Psi) + \max_{a \in \Lambda} e_{\Theta}^* H_{\pi}(a\Psi) \right).$$

The right-hand side is, by (11.16), integrable. Therefore, we may apply the dominated convergence theorem to (11.15), arriving at the desired conclusion. This finishes the proof of Theorem 11.3.1.

11.4. The Bernstein maps. Equivalence with Theorem 11.1.2. We shall construct the desired maps of Theorem 11.1.2, i.e.

$$\iota_{\Theta}: L^2(X_{\Theta}) \longrightarrow L^2(X)$$

and prove that they have the required properties, using as input Theorem 11.3.1. Note that, although bounded, this map is usually not an isometry; however, see Proposition 11.7.1.

One should like to produce this by completing the asymptotics map

$$e_{\Theta}: C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X)$$

of §5 but it does not, in general, extend continuously to a map $L^2(X_{\Theta}) \to L^2(X)$ and must be modified. Roughly, this modification is to "project out" the part of e_{Θ} which is due to subunitary exponents (such as, but not restricted to, the projection of e_{Θ} to $L^2(X)_{\text{disc}}$).⁵⁰

Fix a Plancherel formula for X as in (11.13). Let \mathcal{H}_{π} denote the completion of $C_c^{\infty}(X)_{\pi}$ under the Plancherel norms. Fix the corresponding Plancherel formula (11.14) for X_{Θ} according to Theorem 11.3.1, and let $\mathcal{H}_{\pi}^{\Theta}$ denote the corresponding completion of $C_c^{\infty}(X_{\Theta})_{\pi}$.

Recall that $C_c^{\infty}(X)_{\pi}$ splits as in (11.12), and obviously (by $A_{X,\Theta}$ -invariance) the norm of $\mathcal{H}_{\pi}^{\Theta}$ factors through projection to $C_c^{\infty}(X_{\Theta})_{\pi}^1$. Consider the map $\iota_{\Theta,\pi}$ obtained as the composition:

$$C_c^{\infty}(X_{\Theta})_{\pi} \xrightarrow{\Pi} C_c^{\infty}(X_{\Theta})_{\pi}^1 \xrightarrow{e_{\Theta,\pi}} C_c^{\infty}(X)_{\pi}$$
 (11.18)

$$\pi \to C^{\infty}(X) \xrightarrow{e_{\Theta}^*} C^{\infty}(X_{\Theta})$$

transforms under $A_{X,\Theta}$ (normalized action) by χ . Suppose, to the contrary, that e_{Θ} were L^2 -bounded, with norm $\|e_{\Theta}\|_{\text{op}}$. Then, for any $g \in C_c^{\infty}(X_{\Theta})$, $f \in \nu(\pi)$, the quantity $\langle f, e_{\Theta}(\mathcal{L}_a g) \rangle$ is bounded independent of $a \in A_{X,\Theta}$; indeed, it is bounded by

$$||e_{\Theta}||_{\text{op}}||f||_{L^{2}(X)}||g||_{L^{2}(X_{\Theta})}.$$

But we may rewrite this expression as $\langle \mathcal{L}_a^{-1} e_{\Theta}^* f, g \rangle$, and this is proportional to $\chi(a^{-1})$, thus unbounded if nonzero. Contradiction. Thus, the failure of the asymptotics map to extend to L^2 is related to the existence of discrete series, and, more generally, representations in $L^2(X)$ with subunitary exponents in the Θ -direction.

⁵⁰Here is some motivation, in a simple situation where there is only one exponent and it occurs with multiplicity onet: Suppose we are given a representation $\pi \stackrel{\nu}{\hookrightarrow} C^{\infty}(X)$ and a non-unitary character χ of $A_{X,\Theta}$ with the property that the embedding

where Π is the $A_{X,\Theta}$ -invariant projection onto $C_c^{\infty}(X_{\Theta})_{\pi}^1$, and we denote by $e_{\Theta,\pi}$ the map induced on π -coinvariants by the asymptotics e_{Θ} .

The square of the norm of $\iota_{\Theta,\pi}$ with respect to $\mathcal{H}_{\pi}^{\Theta}$, \mathcal{H}_{π} is bounded by the number of distinct exponents of $A_{X,\Theta}$ on $C_c^{\infty}(X_{\Theta})_{\pi}^1$. This is a consequence of Corollary 10.3.6, applied to the hermitian forms $H = e_{\Theta}^* H_{\pi}$, $H^S = H_{\pi}^{\Theta}$. In particular, this morphism extends to a bounded map on Hilbert spaces:

$$\iota_{\Theta,\pi}: \mathcal{H}_{\pi}^{\Theta} \to \mathcal{H}_{\pi}.$$
 (11.19)

We now seek to integrate to obtain a map

$$\iota_{\Theta} = \int_{\hat{G}} \iota_{\Theta,\pi}.\tag{11.20}$$

The relevant issues of measurability are handled through a straightforward application of Proposition 10.4.1.

As for norms, recall that the number of generalized $A_{X,\Theta}$ -eigencharacters by which an irreducible representation π can be embedded into $C^{\infty}(X_{\Theta})$ is bounded by the number of generalized eigencharacters in its Jacquet module with respect to a parabolic opposite to P_{Θ} . This is uniformly bounded (see the final quoted result at the end of §9.2 on page 125). Thus the norms of the resulting maps $\mathcal{H}_{\pi}^{\Theta} \to \mathcal{H}_{\pi}$ are uniformly bounded. Therefore, by Corollary 10.3.6, (11.20) gives a G-equivariant bounded map

$$L^2(X_{\Theta}) \to L^2(X)$$
.

11.4.1. REMARK. Let us also discuss the description of the dual Bernstein maps. Take $\Phi \in C_c^{\infty}(X)$.

Recall (Remark 11.2.1) that fixing a Plancherel measure for X induces a pointwise decomposition $\Phi = \int \Phi^{\pi} \mu(\pi)$, where each $\Phi^{\pi} \in C^{\infty}(X)$ is π -isotypical. Similarly we may decompose, using the *same* Plancherel measure:

$$\iota_{\Theta}^* \Phi = \int_{\hat{G}} (\iota_{\Theta}^* \Phi)^{\pi} \mu(\pi).$$

Here we take Φ to be any smooth, L^2 function, and since $\iota_{\Theta}^*\Phi$ may not belong to a "nice" subspace (where the Plancherel decomposition is pointwise defined), for every $x \in X_{\Theta}$ the quantity $(\iota_{\Theta}^*\Phi)^{\pi}(x)$ should be thought of as an element of $L^1(\hat{G}, \mu)$. Then we have:

11.4.2. PROPOSITION. (For μ -almost all π), $(\iota_{\Theta}^*\Phi)^{\pi}$ is image of $e_{\Theta}^*(\Phi^{\pi})$ under the $A_{X,\Theta}$ -invariant projection

$$C^{\infty}(X_{\Theta})^{\pi} \twoheadrightarrow C^{\infty}(X_{\Theta})^{\pi,1}.$$

In words: Take the asymptotics of Φ^{π} , and discard all "decaying" exponents.

PROOF. Indeed, it is clear that ι_{Θ}^* is obtained by integrating the adjoints $\iota_{\Theta,\pi}^*$ of the morphisms $\iota_{\Theta,\pi}$ of (11.19) over \hat{G} .

Now consider our construction of $\iota_{\Theta,\pi}$:

where Π is as before the projection to the some of generalized eigenspaces with unitary exponents. The assertion in question follows by dualizing the entire diagram.

11.5. Property characterizing the Bernstein maps. Let us now verify that ι_{Θ} has the property described in Theorem 11.1.2. More precisely, we will prove:

11.5.1. LEMMA. There is a decaying function Q^J on \mathbb{Z}^+ , depending only on J, such that:

$$\|\iota_{\Theta} \mathcal{L}_{a^n} \Psi - e_{\Theta} \mathcal{L}_{a^n}\| \le C_{\Psi} Q^J(n)$$
(11.21)

for any $\Psi \in C_c^{\infty}(X_{\Theta})^J$ and some constant C_{Ψ} depending on Ψ .

PROOF. Fix $\Psi \in C_c^{\infty}(X_{\Theta})^J$ and for $a \in \mathring{A}_{X,\Theta}^+$, denote $\Psi_a := \mathcal{L}_a \cdot \Psi$. Recall from Lemma 11.2.2 that the pull-back $e_{\Theta}^* H_{\pi}$ factors through the sum: $C_c^{\infty}(X_{\Theta})_{\pi}^{<1} \oplus C_c^{\infty}(X_{\Theta})_{\pi}^1$. Let $H_{\pi}^{<1}$ be the pull-back of H_{π} to $C_c^{\infty}(X_{\Theta})$

$$C_c^{\infty}(X_{\Theta})_{\pi} \twoheadrightarrow C_c^{\infty}(X_{\Theta})_{\pi}^{<1} \hookrightarrow C_c^{\infty}(X_{\Theta})_{\pi} \xrightarrow{e_{\Theta,\pi}} C_c^{\infty}(X)_{\pi}$$
 (11.22)

In other words, if we write Ψ_a^1 for the image of Ψ_a via $C_c^{\infty}(X_{\Theta}) \twoheadrightarrow C_c^{\infty}(X_{\Theta})^1$, we have $H_{\pi}^{<1}(\Psi_a) = e_{\Theta,\pi}^* H_{\pi}(\Psi_a - \Psi_a^1)$. Notice, moreover, that by definition $H_{\pi}(\iota_{\Theta}(\Psi_a)) = e_{\Theta}^* H_{\pi}(\Psi_a^1)$. (But the latter is *not* equal, in general, to $H_{\pi}^{\Theta}(\Psi_a)$ as different unitary eigenspaces of $C_c^{\infty}(X_{\Theta})_{\pi}$ may not be orthogonal under $e_{\Theta,\pi}^* H_{\pi}$.)

Then, by following the definitions:

$$\|\iota_{\Theta}(\Psi_{a}) - e_{\Theta}(\Psi_{a})\|_{L^{2}(X)}^{2} = \int_{\hat{G}} H_{\pi} \left(\iota_{\Theta}(\Psi_{a}) - e_{\Theta}(\Psi_{a})\right) \mu(\pi) =$$
$$\int_{\hat{G}} e_{\Theta}^{*} H_{\pi} \left(\Psi_{a}^{1} - \Psi_{a}\right) \mu(\pi) = \int_{\hat{G}} H_{\pi}^{<1}(\Psi_{a}) \mu(\pi).$$

So, it suffices to check the latter integral is bounded by a multiple of a decaying function Q^{J} . But that follows from (the second inequality of) Corollary 10.3.6. Notice that the corollary applies for the same reasons as in the proof of Theorem 11.3.1, that is because of Proposition 9.4.8 and Lemma 11.3.5. In the present context it asserts that

$$H_{\pi}^{<1}(\Psi_a) \le Q^J(a) \left(H_{\pi}^{\Theta}(\Psi) + \sum_i H_{\pi}(e_{\Theta}\Psi_{a_i}) \right),$$

where the decaying function Q^J depends only on J (since this subgroup determines bounds for the number and growth of subunitary exponents) and for some finite collection $\{a_1, \ldots, a_m\}$. By integrating over π , we get:

$$\|\iota_{\Theta}(\Psi_a) - e_{\Theta}(\Psi_a)\|_{L^2(X)}^2 \le Q^J(a) \cdot \left(\|\Psi\|_{L^2(X_{\Theta})}^2 + \sum_i \|e_{\Theta}\Psi_{a_i}\|_{L^2(X)}^2 \right).$$

To finish the proof of Theorem 11.1.2, there remains to check the *uniqueness* of a map ι_{Θ} with the (weaker than that of the previous lemma) property (11.3).

Suppose ι_{Θ} , ι'_{Θ} were two morphisms with that property. Their difference δ_{Θ} then has the property that:

$$\|\delta_{\Theta}\mathcal{L}_{a^n}\Psi\|_{L^2(X)} \xrightarrow{n} 0,$$

for every $\Psi \in C_c^{\infty}(X_{\Theta})$.

The quantity $\|\delta_{\Theta}\Psi\|_{L^2(X)}$ defines a G-invariant Hilbert seminorm on $L^2(X_{\Theta})$, bounded by $C\|\Psi\|_{L^2(X_{\Theta})}$ for some positive C. We may disintegrate it as $\int_{\pi} N_{\pi}(\Psi)\mu(\pi)$, where N_{π} is a G-invariant square-seminorm on the Hilbert space $\mathcal{H}^{\Theta}_{\pi}$ satisfying $N_{\pi}(\Psi) \leq C \cdot H^{\Theta}_{\pi}(\Psi)$.

Now $\int_{\hat{G}} N_{\pi}(\mathcal{L}_{a^n}\Phi)\mu(\pi) \to 0$. Reasoning as for (11.17), the associated S-invariant norms satisfy $\int_{\hat{G}} N_{\pi}^S(\Psi)\mu(\pi) = 0$, i.e. $N_{\pi}^S = 0$ for almost all π . Since the function $a \mapsto N_{\pi}(\mathcal{L}_a\Psi)$ is bounded (by $C \cdot H_{\pi}^{\Theta}(\mathcal{L}_a\Psi) = C \cdot H_{\pi}^{\Theta}(\Psi)$), we deduce by Lemma 10.2.7 that $N_{\pi}(\Psi) = 0$ for almost all π , as desired.

11.6. Compatibility with composition and inductive structure of $L^2(X)$.

11.6.1. PROPOSITION. For each $\Omega \subset \Theta \subset \Delta_X$, let ι_{Ω}^{Θ} denote the analogous Bernstein morphism: $L^2(X_{\Omega}) \to L^2(X_{\Theta})$. Then:

$$\iota_{\Theta} \circ \iota_{\Omega}^{\Theta} = \iota_{\Omega}. \tag{11.23}$$

PROOF. This follows from the analogous result on the "naive" asymptotics maps:

$$C_c^{\infty}(X_{\Omega}) \xrightarrow{e_{\Omega}^{\Theta}} C_c^{\infty}(X_{\Theta}) \xrightarrow{e_{\Theta}} C_c^{\infty}(X).$$

The composition of these arrows is equal to e_{Ω} , cf. Remark 5.1.4.

Specializing to π -coinvariants, and taking into account that $C_c^{\infty}(X_{\Omega})_{\pi}^1$ maps into $C_c^{\infty}(X_{\Theta})_{\pi}^1$ (the restriction of a unitary character of $A_{X,\Omega}$ to the subtorus $A_{X,\Theta}$ remains unitary), we get the result by the definition (11.18) of $\iota_{\Theta,\pi}$.

11.6.2. COROLLARY. Let $L^2(X)_{\Theta}$ be the image⁵¹ of $L^2(X_{\Theta})_{\text{disc}}$ under ι_{Θ} . Then:

$$\sum_{\Theta \subset \Delta_X} L^2(X)_{\Theta} = L^2(X). \tag{11.24}$$

PROOF. Assume the statement to be true if we replace X by any X_{Θ} , $\Theta \subsetneq \Delta_X$. Then, by Proposition 11.6.1, the orthogonal complement \mathcal{H}' of $\sum_{\Theta \subsetneq \Delta_X} L^2(X)_{\Theta}$ is orthogonal to $\iota_{\Theta} \left(L^2(X_{\Theta}) \right)$ for all $\Theta \subsetneq \Delta_X$. By the definition of ι_{Θ} , this means that the space \mathcal{H}' admits a Plancherel decomposition:

$$\mathcal{H}' = \int_{\hat{G}} \mathcal{H}'_{\pi} \mu(\pi)$$

where the norms for all \mathcal{H}'_{π} are decaying in all directions at infinity (i.e. they have only subunitary, no unitary exponents, for every $\Theta \subsetneq \Delta_X$). But then $\mathcal{H}' \subset L^2(X)_{\text{disc}} = L^2(X)_{\Delta_X}$ by the generalization of Casselman's square integrability criterion, cf. §9.2.

- 11.7. Isometry. As we mentioned, the Bernstein map $i_{\Theta}: L^2(X_{\Theta}) \to L^2(X)$ is not, in general, an isometry; in section 14 we will examine its kernel. However, it is an isometry if we restrict to a small enough subspace of $L^2(X_{\Theta})$:
- 11.7.1. PROPOSITION. Let $\mathcal{H}' \subset L^2(X_{\Theta})$ be an $A_{X,\Theta} \times G$ -stable subspace. Fix a Plancherel measure μ for $L^2(X)$ and corresponding direct integral decompositions for $\mathcal{H} := L^2(X)$ and \mathcal{H}' :

$$\mathcal{H} = \int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi),$$

$$\mathcal{H}' = \int_{\hat{G}} \mathcal{H}'_{\pi} \mu(\pi).$$

Assume that for almost all π the following is true: if $\mathcal{H}'_{\pi} = \bigoplus_{\chi} \mathcal{H}'_{\pi,\chi}$ is a decomposition into $A_{X,\Theta}$ -generalized eigenspaces (necessarily, for almost all π , honest eigenspaces with unitary characters), then for distinct characters χ_i the images of $\mathcal{H}'_{\pi,\chi_i}$ via the Bernstein maps $\iota_{\Theta,\pi}|_{\mathcal{H}'_{\pi}} : \mathcal{H}'_{\pi} \to \mathcal{H}_{\pi}$ are mutually orthogonal.

Then the restriction of the Bernstein map ι_{Θ} to \mathcal{H}' is an isometry onto its image.

11.7.2. REMARK. The proposition applies, in particular, to the case that (almost) every \mathcal{H}'_{π} has a unique exponent. This is the only setting in which we will use it.

 $^{^{51}}$ For L^2 -spaces the index Θ is being used to denote the image of the discrete spectrum of boundary degenerations, while otherwise it is used to denote Jacquet modules. We hope that this will not lead to any confusion, since we do not use Jacquet modules in the category of unitary representations.

PROOF. Having established the existence (and boundedness) of the morphisms ι_{Θ} (and their disintegrations into $\iota_{\Theta,\pi}: \mathcal{H}_{\Theta,\pi} \to \mathcal{H}_{\pi}$, where $\mathcal{H}_{\Theta,\pi}$ is the disintegration of $L^2(X_{\Theta})$ with respect to μ ; it is equipped therefore with a "Plancherel" hermitian norm), we may, a posteriori, rephrase the conclusion of Theorem 11.3.1 in terms of them (here S is as in that theorem):

Let H_{π} denote the Hermitian form on \mathcal{H}_{π} , and $\iota_{\Theta,\pi}^* H_{\pi}$ its pull-back to $\mathcal{H}_{\Theta,\pi}$. The associated S-invariant form $\left(\iota_{\Theta,\pi}^* H_{\pi}\right)^S$ is equal to the "Plancherel" hermitian form on $\mathcal{H}_{\Theta,\pi}$.

Indeed, let us further pull back these norms to $C_c^{\infty}(X_{\Theta})_{\pi}$ via the canonical map: $C_c^{\infty}(X_{\Theta})_{\pi} \to \mathcal{H}_{\Theta,\pi}$. By definition of ι_{Θ} , we have a commutative diagram:

$$\begin{array}{cccc} C_c^{\infty}(X_{\Theta})_{\pi} & \longrightarrow & C_c^{\infty}(X_{\Theta})_{\pi}^{1} & \longrightarrow & \mathcal{H}_{\Theta,\pi} \\ & & & & \downarrow^{\iota_{\Theta,\pi}} & & & \downarrow^{\iota_{\Theta,\pi}} \\ & & & & C_c^{\infty}(X)_{\pi} & \longrightarrow & \mathcal{H}_{\pi} \end{array}$$

Hence, our current pull-backs are obtained from the pull-backs of Theorem 11.3.1 (induced by $e_{\Theta,\pi}: C_c^\infty(X_\Theta)_\pi \to C_c^\infty(X)_\pi$) by composing with the $A_{X,\Theta}$ -equivariant projection to $C_c^\infty(X_\Theta)_\pi^1$. But the process of taking S-invariants also factors through this projection, hence $\left(\iota_{\Theta,\pi}^* H_\pi\right)^S$ coincides with $\left(e_{\Theta,\pi}^* H_\pi\right)^S$ (as a hermitian form on $C_c^\infty(X_\Theta)_\pi$) and hence, by Theorem 11.3.1, with the Plancherel hermitian norm on $\mathcal{H}_{\Theta,\pi}$.

The assumptions on \mathcal{H}' now imply that the restriction of $\iota_{\Theta_{\pi}}^* H_{\pi}$ to \mathcal{H}'_{π} is already $A_{X,\Theta}$ -invariant. Hence, it coincides with the Plancherel hermitian form on \mathcal{H}'_{π} (we implicitly use here that the construction of the "associated invariant norm" from Lemma 10.3.1 is compatible with passage to S-invariant subspaces; this is clear from the definition), and therefore the Bernstein map is an isometry when restricted to \mathcal{H}' .

12. Preliminaries to scattering (I): direct integrals and norms

In this section and the next we gather some useful results, presented in an abstract setting, that will be used in $\S14$.

- §12.1 recalls the general formalism of direct integrals of Hilbert spaces, which is essential for the Plancherel decomposition.
- $\S12.2$ discusses certain norms on direct integrals of Hilbert spaces; these norms will be used extensively in $\S14$, in particular, $\S14.5$.

Roughly speaking, in §14.5, we will have available pointwise bounds on eigenfunctions, and we obtain pointwise bounds on general functions by first decomposing into eigenfunctions and then applying this pointwise bounds; the norms that we discuss are abstractions of this process.

12.1. General properties of the Plancherel decomposition.

12.1.1. Let \mathcal{H} be a unitary representation of G. We discussed in §7.1 the meaning of a Plancherel decomposition:

$$\mathcal{H} = \int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi).$$

By [Dix77][Theorem 8.6.6 and A72], under the assumption that almost all \mathcal{H}_{π} are of finite multiplicity, there is a countable partition of the unitary dual \hat{G} into measurable sets Z_i , and hence a corresponding direct sum decomposition $\mathcal{H} = \bigoplus_i \mathcal{H}_i$, such that each $\mathcal{H}_i \simeq \mathcal{H}'_i \otimes V_i$ as a G-representation, where:

- (1) V_i is a finite dimensional vector space of dimension i, with trivial G-action;
- (2) \mathcal{H}'_i is multiplicity-free, that it it admits a direct integral decomposition: $\mathcal{H}'_i \simeq \int_{Z_i} \mathcal{H}'_{\pi} \mu_i(\pi)$ with \mathcal{H}'_{π} irreducible;
- (3) the measures μ_i are mutually singular;
- (4) the measurable structure is trivializable, ⁵² i.e. there is a Hilbert space H_0 and isomorphisms of Hilbert spaces: $\mathcal{H}'_{\pi} \stackrel{\sim}{\to} H_0$ inducing a bijection between the collection of measurable sections $\pi \mapsto \eta'_{\pi} \in \mathcal{H}'_{\pi}$ and the collection of measurable sections $\pi \mapsto \eta_{\pi} \in H_0$.

The results cited below concerning unitary decomposition may be found in [Dix77], in particular, Theorem 8.5.2 and 8.6.6 (existence and uniqueness of unitary decomposition) and Proposition 8.6.4 (characterization of G-endomorphisms).

12.1.2. Uniqueness of unitary decomposition. Suppose that two unitary representations with Plancherel decompositions: $\int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi)$ and $\int_{\hat{G}} \mathcal{H}'_{\pi} \nu(\pi)$ are isomorphic. Then the measure classes of μ and ν are equal, and moreover, there exists an isometric isomorphism $\mathcal{H}_{\pi} \to \mathcal{H}'_{\pi}$ for μ -almost every (equivalently: ν -almost every) $\pi \in \hat{G}$.

12.1.3. Endomorphisms. Notation as before. A family of (bounded) endomorphisms $\pi \mapsto T_{\pi} : \mathcal{H}_{\pi} \to \mathcal{H}_{\pi}$ is called measurable if for every measurable section $\pi \mapsto \eta_{\pi} \in \mathcal{H}_{\pi}$ the section $\pi \mapsto T_{\pi}\eta_{\pi}$ is measurable (see [Dix77, A78]). Any G-endomorphism f of $\int_{\hat{G}} \mathcal{H}_{\pi}\mu(\pi)$ is "decomposable," that is to say, there is a measurable family of G-endomorphisms f_{π} of \mathcal{H}_{π} such that $f(v) = \int_{\hat{G}} f_{\pi}(v_{\pi})\mu(\pi)$ for $v = \int v_{\pi}\mu(\pi)$. We will symbolically write:

$$f = \int_{\hat{G}} f_{\pi}.$$

 $^{^{52}}$ We are assuming here that all \mathcal{H}'_{π} are infinite-dimensional; in general, the measurable structure is trivializable over the (measurable) subsets where \mathcal{H}'_{π} has fixed dimension.

This assertion follows⁵³ from Proposition 8.6.4 and Theorem 8.6.6 of [Dix77]. We shall refer to this as "disintegration of endomorphisms."

12.1.4. Disintegration of morphisms. Let $\mathcal{J} := \mathcal{H} \oplus \mathcal{H}'$ be the direct sum of two unitary G-representations. Let μ be a Plancherel measure for \mathcal{J} , so we may disintegrate

$$\mathcal{J} = \int_{\hat{G}} \mathcal{J}_{\pi} \mu(\pi).$$

We claim that there are measurable subfields $\mathcal{H}_{\pi}, \mathcal{H}'_{\pi} \subset \mathcal{J}_{\pi}$ (that is, Hilbert subspaces so that the corresponding projections are measurable) so that:

- $\mathcal{J}_{\pi} = \mathcal{H}_{\pi} \oplus \mathcal{H}'_{\pi};$ $\mathcal{H} = \int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi)$ and $\mathcal{H}' = \int_{\hat{G}} \mathcal{H}'_{\pi} \mu(\pi).$

Notice that μ is not necessarily a Plancherel measure for \mathcal{H} or \mathcal{H}' , as the space \mathcal{H}_{π} , \mathcal{H}'_{π} could be zero for π in a non-zero set.

To see that this is true, we disintegrate the projections of \mathcal{J} onto \mathcal{H} and \mathcal{H}' to obtain measurable families of projections of \mathcal{J}_z ; we define \mathcal{H}_z and \mathcal{H}'_z as the images of these projections.

Let us now discuss the analog of §12.1.3 for morphisms with different source and target. Thus let $\mathcal{H}, \mathcal{H}'$ be unitary G-representations and $f: \mathcal{H} \to \mathcal{H}$ \mathcal{H}' a G-morphism. We wish to "disintegrate" f.

This is reduced to the previous setting by replacing f by the endomorphism f+0 of $\mathcal{J}:=\mathcal{H}\oplus\mathcal{H}'$. Then for a decomposition $\mathcal{J}=\int_{\hat{G}}(\mathcal{H}_{\pi}\oplus\mathcal{H}'_{\pi})\mu(\pi)$ as above, we get morphisms $f_{\pi}: \mathcal{J}_{\pi} \to \mathcal{J}_{\pi}$ for almost every π , which disintegrate f + 0. Since, however, \mathcal{H}' is in the kernel of f + 0 and the image is contained in \mathcal{H}' , it follows that for μ -almost every π the morphism f_{π} factors as:

$$f_{\pi}: \mathcal{H}_{\pi} \to \mathcal{H}'_{\pi}.$$
 (12.1)

This is what we will mean by disintegration of a morphism $f: \mathcal{H} \to \mathcal{H}'$; a disintegration with respect to a Plancherel measure for $\mathcal{H} \oplus \mathcal{H}'$.

12.1.5. Remark. It can easily be shown that the class of Plancherel measure of $\mathcal{H} \oplus \mathcal{H}'$ is precisely the class of a sum of Plancherel measures for \mathcal{H} and \mathcal{H}' .

12.2. Norms on direct integrals of Hilbert spaces. ⁵⁴

 $^{^{53}}$ We outline the argument: In the setting of §12.1.1, there are no non-trivial Gmorphisms between the different summands \mathcal{H}_i (loc.cit. Proposition 8.4.7). Now, since V_i is finite-dimensional, we have $\operatorname{End}(\mathcal{H}_i) = \operatorname{End}(\mathcal{H}_i) \otimes \operatorname{End}(V_i)$, and hence $\operatorname{End}(\mathcal{H}_i)^G = \operatorname{End}(\mathcal{H}_i)^G$ $\operatorname{End}(\mathcal{H}'_i)^G \hat{\otimes} \operatorname{End}(V_i)$. It now follows from Proposition 8.6.4 that the first factor consists precisely of the "diagonalizable" endomorphisms, i.e. those which are direct integrals of scalars in the \mathcal{H}'_{π} 's.

⁵⁴We use the word "norm" freely in this section, to include seminorms that are not necessarily bounded – i.e., can be zero or infinite on some nonzero vectors. In other words, a norm on a vector space V is a pair of a subspace $V_f \subset V$ – the "space of vectors of finite norm" – and a seminorm $N: V_f \to \mathbb{R}_{\geq 0}$). By convention, in this setting, we write $||v|| = \infty$ for $v \in V - V_f$.

The reader may wish to postpone this section until reading Lemma 14.5.2, which gives the basic case where the norms discussed here arise.

12.2.1. Basic example. Let (Y, \mathfrak{B}, μ) be a (positive) measure space, denote by F(Y) the space of measurable functions modulo essentially zero functions, and consider the corresponding space $L^2(Y,\mu)$. Every $v \in L^2(Y,\mu)$ corresponds to a signed measure $v\mu$ and a positive measure $|v|\mu$, thus defining an L^1 -seminorm $\|\bullet\|_{L^1(Y,|v|\mu)}$ which is continuous on $L^2(Y,\mu)$. Explicitly, this seminorm is defined by

$$h \in L^2(Y,\mu) \mapsto \int |h| |v| \mu,$$

which is evidently bounded by $||v|| \cdot ||h||$.

We take this observation a step further, and consider instead a direct integral of (non-trivial) Hilbert spaces over a measure space:

$$\mathcal{H} = \int_{Y} \mathcal{H}_{\pi} \mu(\pi).$$

Again, given $v = \int_Y v_\pi \mu(\pi) \in \mathcal{H}$ we can define a corresponding L^1 -norm on \mathcal{H} , namely:

$$\left\| \int_{Y} h_{\pi} \mu(\pi) \right\|_{L_{v}^{1}} := \int_{Y} |\langle h_{\pi}, v_{\pi} \rangle| \mu(\pi).$$
 (12.2)

Again, this is bounded in operator norm by ||v||. On the other hand we have $\langle h, v \rangle \leq ||h||_{L^1}$.

Notice that this norm depends only on $Y, \mathfrak{B}, \mathcal{H}, v$ and not on μ , in the following sense: If we multiply μ by an almost everywhere positive function and divide the hermitian forms on the spaces \mathcal{H}_{π} by the same function, then we have a canonical isomorphism of the new direct integral with \mathcal{H} , and the element corresponding to v defines the same norm on \mathcal{H} .

12.2.2. The case of a G-representation. Suppose \mathcal{H} is an arbitrary unitary G-representation, and (Y, μ) is defined by Plancherel decomposition for \mathcal{H} . In this setting the relative norm can be described thus:

The ring of essentially bounded Borel measurable functions on \hat{G} acts on any unitary G-representation by bounded G-endomorphisms. Then, for $x \in \mathcal{H}$,

$$||x||_{L_v^1} = \sup_E ||\langle x, Ev \rangle||,$$
 (12.3)

where E ranges through Borel measurable functions on \hat{G} satisfying $|E(\pi)| \le 1$ for all $\pi \in \hat{G}$.

This description makes manifest the following: given a bounded G-morphism $f: \mathcal{H}_1 \to \mathcal{H}_2$, then, for any $w \in \mathcal{H}_2$, the pull-back by f of the norm L^1_w is equal to the norm $L^1_{f^*w}$, where $f^*: \mathcal{H}_2 \to \mathcal{H}_1$ denotes the adjoint of f.

12.2.3. Relativization. We also introduce "relative" versions of the above norms. Let p be a morphism of measure spaces:

$$(Y, \mathfrak{B}, \mu) \to (Y', \mathfrak{B}', \nu)$$
 (12.4)

Assume that the direct image of μ under p is absolutely continuous with respect to ν (i.e., μ is zero on inverse images of null sets).

We also need to make certain assumptions on the measures we are dealing with; for the applications that we have in mind, it suffices to assume that μ is compact (see below) and ν is σ -finite.

12.2.4. Remark. (Measure-theoretic details).

Recall from [Fre06, §451] that a compact measure on a given sigmaalgebra is one which is *inner regular* with respect to a *compact class* of subsets. Inner regular means that the measure of a measurable set A is the supremum of the measures of the subsets of A in the given class, and a compact class is a collection K of (measurable) subsets such that $\bigcap_{K \in K'} K \neq$ 0 whenever $K' \subset K$ has the finite intersection property.

In our applications, these conditions will be almost automatic: these measure spaces will arise from a Plancherel decomposition either of a subspace of $L^2(X_{\Theta})$, or of a locally compact abelian group. These are measures on standard Borel spaces (cf. [Dix77, 4.6.1, 7.3.7]). They may be assumed to be finite: by the definition of Dixmier, these measures are σ -finite, and therefore can be replaced by finite measures in the same measure class. Finally, a finite measure on a standard Borel space is automatically inner regular and so compact [Fre06, 434J (g)].

Under the assumptions above, there is a family of measures $\{\mu_{\rho}\}_{{\rho}\in Y'}$ on (Y,\mathfrak{B}) such that $\mu_{\rho}(A)=\mu_{\rho}(A\cap p^{-1}\{\rho\})$ for every $A\in\mathfrak{B}$ and:

$$\int_Y f(\pi)\mu(\pi) = \int_{Y'} \left(\int_Y f(\pi)\mu_\rho(\pi) \right) \nu(\rho).$$

for every measurable function f on Y. This is the disintegration of measures, see, for instance, [Fre06][Theorem 452I].

Let $\mathcal{H} = \int_Y \mathcal{H}_{\pi} \mu(\pi)$ as above, then from Y' we get a coarser decomposition of \mathcal{H} :

$$\mathcal{H} = \int_{Y'} \mathcal{H}_{\rho} \nu(\rho)$$

where $\mathcal{H}_{\rho} = \int_{Y} \mathcal{H}_{\pi} \mu_{\rho}(\pi)$.

Given a vector $v \in \mathcal{H}$ we can now define a norm on \mathcal{H} which is a mixture of the above norms along the fibers of $Y \to Y'$ and the L^2 -norm along Y', more precisely: If $v = \int_{Y'} v_{\rho} \nu(\rho)$ then

$$\left\| \int_{Y'} h_{\rho} \nu(\rho) \right\|_{Y,Y',\nu,v} := \left(\int_{Y'} \|h_{\rho}\|_{L^{1}_{v_{\rho}}}^{2} \nu(\rho) \right)^{\frac{1}{2}}. \tag{12.5}$$

This norm is continuous on \mathcal{H} if $\|v_{\rho}\|_{\mathcal{H}_{\rho}}$ is essentially bounded in ρ – indeed, it is bounded by $\sup_{\rho} \|v_{\rho}\|_{\mathcal{H}_{\rho}}$ times the norm on \mathcal{H} , where \sup

means "essential supremum" – but not in general. Notice also the following: the norms do not depend on v itself, but rather the collection of v_{ρ} up to multiplying each by a scalar of norm one. Finally, they do depend on the choice of ν , albeit not on the choice of μ (in the sense described above, i.e. modifying the measure and the hermitian norms accordingly).

12.2.5. LEMMA. Suppose that each \mathcal{H}_{ρ} , $\rho \in Y'$, is infinite-dimensional; then the norm on \mathcal{H} is not bounded by any finite sum of norms of the form $\| \bullet \|_{Y,Y',\nu,v}$.

PROOF. In fact, given any $v_1, \ldots, v_N \in \mathcal{H}$, we may find a nonzero vector $w \in \mathcal{H}$ with the property that $w_\rho \perp (v_j)_\rho$ for all $\rho \in Y'$. Then $||w||_{Y,Y',\nu,v_j} = 0$ for each j.

12.2.6. The case of $G \times A$ -representations. Our use of relative norms will be in the following situation: G our fixed reductive p-adic group, A a discrete abelian group, and \mathcal{H} a unitary $G \times A$ -representation whose Plancherel measures under $G \times A$ and A satisfy the assumptions for disintegration; we take $Y = \widehat{G \times A}, Y' = \widehat{A}$. For each choice of A-Plancherel measure ν and each $\nu \in \mathcal{H}$, the following is clear:

12.2.7. Lemma. The norm
$$\|\cdot\|_{\widehat{G\times A},\widehat{A},\nu,\nu}$$
 is A-invariant.

We assume that the Plancherel measure for \mathcal{H} as an A-representation is absolutely continuous with respect to Haar measure on \hat{A} , and equip Y' with ν =Haar probability measure.

In this setting, denote the relative norm $\|\cdot\|_{\widehat{G}\times A, \widehat{A}, \nu, v}$ by $\|\cdot\|_{A, v}$ for short. The following Lemma will play a key role in our later proofs.

12.2.8. LEMMA. Given $x_1, \ldots, x_r \in \mathcal{H}$, and any corresponding collection of proper subgroups $T_1, \ldots, T_r \subset A$, the Hilbert norm on \mathcal{H} is not majorized by $\sum \| \bullet \|_{T_j, x_j}$.

PROOF. Assume to the contrary, then by scaling the x_j 's we may assume that $\| \bullet \|_{\mathcal{H}} \leq \sum \| \bullet \|_{T_j,x_j}$. We have already seen that the relative norm $\| \bullet \|_{T_j,x_j}$ is bounded by at most $\sup_{\rho \in \widehat{T}_i} \|x_i\|_{\rho}$ times the norm of \mathcal{H} . Recall that $\int_{\rho \in \widehat{T}_i} \|x_i\|_{\rho}^2 = \|x\|^2$, but this gives no control on the supremum needed to bound the relative norm.

Now let $\mathcal{H}' \subset \mathcal{H}$ be a $G \times A$ -invariant subspace. Then the restriction of $\| \bullet \|_{T_j,x_j}$ to \mathcal{H}' is simply given by $\| \bullet \|_{T_j,\overline{x_j}}$, where $\overline{x_j}$ is the projection of x_j to \mathcal{H} . (This follows from the subsequent Lemma 12.2.9, applied to f the inclusion $\mathcal{H}' \hookrightarrow \mathcal{H}$.)

It follows that it suffices to construct a nonempty $G \times A$ -invariant subspace \mathcal{H}' with the property that

$$\|\overline{x_j}\|_{\rho} < \frac{1}{r}$$
 for all j and all $\rho \in \widehat{T}_j$.

Then we have: $\| \bullet \|_{\mathcal{H}} \leq \sum \| \bullet \|_{T_j,x_j} < r \cdot \frac{1}{r} \| \bullet \|_{\mathcal{H}}$, a contradiction.

The strategy is to take \mathcal{H}' to be the image of the orthogonal projection 1_S induced by a measurable subset $S \subset \widehat{A}$. Then the orthogonal projection $\overline{x_j}$ of x_j to \mathcal{H}' is simply $1_S \cdot x_j$; on the other hand, \mathcal{H}' is nonempty so long as S has positive Haar measure, by virtue of our assumption on the Plancherel measure of \mathcal{H} with respect to A.

It suffices, then, to construct a set S with the property that

$$||1_S x_j||_{\rho}^2 < \frac{1}{r^2} \tag{12.6}$$

for every $1 \leq j \leq r$ and every $\rho \in \widehat{T}_j$.

The function $x_j \mapsto ||x_j||_{\rho}$ is (in the notation of [**Dix77**, Appendix A]) " μ -measurable" where μ is the Haar measure, that is to say, in the complection of the Borel σ -algebra with respect to μ .

Notice that \widehat{A} has the structure of a compact abelian Lie group. Fixing any Riemannian metric on it, we can speak of balls $S(\varepsilon) \subset \widehat{A}$ of radius ε around a point. In what follows, let us fix the Haar measures on \widehat{A} and \widehat{T}_j to be probability measures, and then each fiber of $\widehat{A} \to \widehat{T}_j$ is also endowed with a natural fibral probability measure (indeed, this fiber may be identified with the dual of the discrete group A/T_j in a natural way).

By the Lebesgue differentiation theorem, for Haar-almost every point of \widehat{A} (taken as the center of the balls $S(\varepsilon)$) there is a constant C such that for ε sufficiently small and every j:

$$\int_{S(\varepsilon)} \|x_j\|_{\chi}^2 d\chi \le C \cdot \text{Vol}(S(\varepsilon)). \tag{12.7}$$

The left hand side can be written as:

$$\int_{\widehat{T_j}} \|1_{S(\varepsilon)} x_j\|_{\rho}^2 d\rho = \int_{S(\varepsilon)} \|1_{S(\varepsilon)} x_j\|_{\rho(\chi)}^2 f_j(\varepsilon, \chi)^{-1} d\chi,$$

where $f_j(\varepsilon, \chi)$ is the fibral volume of $S(\varepsilon)$ over $\rho(\chi) \in \widehat{T}_j$. Clearly, $f_j(\varepsilon, \chi) \le C'\varepsilon$ for some constant C', hence:

there is a subset $S \subset S(\varepsilon)$ of positive measure with $\|1_{S(\varepsilon)}x_j\|_{\rho(\chi)} < \frac{1}{r}$ for all $\chi \in S$, $j = 1, \ldots, r$. In particular, $\|1_Sx_j\|_{\rho(\chi)} < \frac{1}{r}$ for all $\chi \in S$, $j = 1, \ldots, r$.

Indeed, the estimates above show that:

$$\sum_{j} \int_{S(\varepsilon)} \|1_{S(\varepsilon)} x_j\|_{\rho(\chi)}^2 d\chi \le rCC' \varepsilon \operatorname{Vol}(S(\varepsilon))$$

and we can choose ε small enough so that $rCC'\varepsilon < \frac{1}{r^2}$. This provides the desired set and proves the lemma.

Now A_1, A_2 be two discrete abelian groups and $T: A_2 \to A_1$ a morphism. Let \mathcal{H}_1 and \mathcal{H}_2 be, respectively, unitary $G \times A_i$ representations with Haar A_i -Plancherel measure, and let $f: \mathcal{H}_1 \to \mathcal{H}_2$ be a morphism which is (G, T, A_2) -equivariant up to a character of A_2 (i.e. $f \circ T(a)$ and $a \circ f$ differ by a – necessarily unitary – character A_2). What can we say about pull-backs of those relative norms? First of all,

12.2.9. LEMMA. Let $w \in \mathcal{H}_2$; then:

$$f^* \| \cdot \|_{A_2, w} = \| \cdot \|_{A_2, f'w}, \tag{12.8}$$

an equality of norms on \mathcal{H}_1 . Here f' denotes the adjoint of f, and in defining the latter norm, we consider \mathcal{H}_1 as an A_2 -representation via T.

PROOF. Decompose $\mathcal{H}_j = \int_{\chi \in \widehat{A}_2} \mathcal{H}_{j,\chi} d\chi$ and $f = \int f_{\chi}$, with $f_{\chi} : \mathcal{H}_{1,\chi} \to \mathcal{H}_{2,\chi}$ (see §12.1.4 for discussion). Then for $v = \int_{\chi} v_{\chi} d\chi \in \mathcal{H}_1$ (and using similar notation for further decompositions of v, w, and f),

$$||f(v)||_{A_{2},w}^{2} = \int_{\chi \in \widehat{A}_{2}} ||f_{\chi}(v_{\chi})||_{L^{1}(w_{\chi})}^{2} d\chi$$

$$= \int_{\chi \in \widehat{A}_{2}} \left| \int_{\hat{G}} |\langle f_{\pi,\chi} v_{\pi,\chi}, w_{\chi,\pi} \rangle | \mu_{\chi}(\pi) \right|^{2} d\chi$$

$$= \int_{\chi \in \widehat{A}_{2}} \left| \int_{\hat{G}} |\langle v_{\chi,\pi}, f'_{\pi,\chi} w_{\pi,\chi} \rangle | \mu_{\chi}(\pi) \right|^{2} d\chi$$

$$= \int_{\chi \in \widehat{A}_{2}} ||v_{\chi}||_{L^{1}(f'w_{\chi})}^{2} d\chi = ||v||_{A_{2},f'w}^{2}.$$
(12.9)

Here μ_{χ} denotes the disintegration of $G \times A_2$ -Plancherel measure on $\mathcal{H}_1 \oplus \mathcal{H}_2$ with respect to the forgetful map $\widehat{G \times A_2} \to \widehat{A_2}$.

The important result will be that if A_1 and A_2 have different rank, the Hilbert space norm on \mathcal{H}_1 cannot be majorized (not even at the level of J-invariants) by any finite sum of pullbacks of mixed norms from \mathcal{H}_2 . We keep assuming, of course, that the A_i -Plancherel measure of \mathcal{H}_i is in the class of Haar measure.

12.2.10. LEMMA. Let notation be as above. Assume that $\dim(A_2 \otimes \mathbb{Q}) < \dim(A_1 \otimes \mathbb{Q})$. Then, for any open compact subgroup $J \subset G$, the Hilbert norm on \mathcal{H}_J^J is not majorized by any finite sum of norms of the form $f^* \| \bullet \|_{A_2,w}$.

More generally, suppose given a finite collection of spaces $\mathcal{H}_2^{(j)}$, for $j=1,2,\ldots,$ together with tori $A_2^{(j)}$ and morphisms $T^{(j)}:A_2^{(j)}\to A_1$. Assume that $\dim(A_2^{(j)}\otimes\mathbb{Q})<\dim(A_1\otimes\mathbb{Q})$ for all j. Let $f_{(j)}:\mathcal{H}_1\to\mathcal{H}_2^{(j)}$ be morphisms as above. Then the Hilbert norm on \mathcal{H}_1^J is not majorized by any finite sum of norms $f_{(j)}^*\|\bullet\|_{A_2^{(j)},w}$.

PROOF. (With a single torus A_2 :) When we decompose \mathcal{H}_1^J over $Y:=\widehat{G\times A_2}$, each fiber $\mathcal{H}_{1,\pi}$ has infinite multiplicity as a G-representation. The conclusion follows from Lemma 12.2.5 and Lemma 12.2.9.

(With multiple tori $A_2^{(j)}$:) We expand on the argument of the previous case. We need to show that the Hilbert norm on \mathcal{H}_1^J is not majorized by a

finite sum of norms of the type $\sum \|\bullet\|_{A_2^{(j)},f'_{(j)}w}$, where we follow the notation of (12.8). Now our claim follows from Lemma 12.2.8.

Finally, a lemma on the A_1 -invariance of the relative norms. Here we use as imput a property of disintegration of a $G \times A_1$ -Plancherel measure with respect to the map $\widehat{G \times A_1} \to \widehat{G \times A_2}$, i.e. as opposed to the previous situation we forget a torus action. The assumption is an injectivity assumption, i.e. that for a given $G \times A_2$ -representation there is a unique $G \times A_1$ -representation appearing (stated measure-theoretically). Notice that in order to be able to disintegrate, we need a σ -finiteness assumption with respect to $G \times A_2$ -Plancherel measure, which in our examples will be provided by Remark 12.2.4.

12.2.11. LEMMA. Assume that for a disintegration $\mu = \int \mu_{\alpha}$ of Plancherel measure on \mathcal{H}_1 with respect to the forgetful map: $\widehat{G} \times \widehat{A}_1 \to \widehat{G} \times \widehat{A}_2$, almost each of the measures μ_{α} is concentrated on one point. Then the pulled-back norms:

$$f^* \| \bullet \|_{A_2, w}$$

are A_1 -invariant.

PROOF. We have seen in (12.9) that

$$f^* \|v\|_{A_2, w}^2 = \int_{\chi \in \widehat{A_2}} \|v_\chi\|_{L^1(f'w_\chi)}^2 d\chi,$$

where the norms $||v_{\chi}||^2_{L^1(f'w_{\chi})}$ are densely defined on the spaces $\mathcal{H}_{1,\chi}$. Clearly, A_2 acts trivially on these norms, so we need to show that they are $\operatorname{coker}(A_2 \to A_1)$ -invariant.

By twisting the space $\mathcal{H}_{1,\chi}$ by a character of A_1 that extends χ , we may suppose that A_2 acts trivially on $\mathcal{H}_{1,\chi}$. In this way we are reduced to the case where A_2 is trivial, and need to prove that under the same assumption for the Plancherel measure of \mathcal{H}_1 with respect to the map:

$$\widehat{G \times A_1} \to \widehat{G},$$

given a vector $w \in \mathcal{H}_1$, the norm:

$$\| \bullet \|_{L^1_w}$$

(defined with respect to a \hat{G} -Plancherel decomposition) is A_1 -invariant. Let $a \in A_1$. By the definition (12.2):

$$||a \cdot v||_{L^1(w)} = \int_{\hat{G}} |\langle a \cdot v_{\pi}, w_{\pi} \rangle| \, \nu(\pi),$$
 (12.10)

where we have used a G-Plancherel decomposition $\mathcal{H}_1 = \int_{\hat{G}} \mathcal{H}_{\pi} \nu(\pi)$.

We may disintegrate the inner product $\langle a \cdot v_{\pi}, w_{\pi} \rangle$ with respect to the A_1 -Plancherel decomposition of \mathcal{H}_{π} in such a way that the corresponding

Plancherel measure $\mu_{\pi}(\chi)$ will be a disintegration of a given Plancherel measure $\mu(\pi,\chi)$ with respect to $\nu(\pi)$:

$$\langle a \cdot v_{\pi}, w_{\pi} \rangle = \int_{\hat{A}_1} \langle a \cdot v_{\pi,\chi}, w_{\pi,\chi} \rangle \, \mu_{\pi}(\chi).$$

But, by assumption, the measures μ_{π} are atomic (for almost all π), i.e. the last integral is equal to a multiple (independent of a) of $\langle a \cdot v_{\pi,\chi}, w_{\pi,\chi} \rangle$ for some χ (depending on π). In particular, $|\langle a \cdot v_{\pi}, w_{\pi} \rangle| = |\chi(a) \langle v_{\pi}, w_{\pi} \rangle| = |\langle v_{\pi}, w_{\pi} \rangle|$, and therefore we get that (12.10) is independent of a.

13. Preliminaries to scattering (II): consequences of the conjecture on discrete series

As the previous section, this section works out certain results needed in $\S14$. The results here all depend on the validity of the Discrete Series Conjecture 9.4.6 for X and its degenerations, as in the statement of Theorem 7.3.1.

Recall that the canonical map:

$$\mathcal{Z}(\mathbf{L}_{\Theta})^0 \to \mathbf{A}_{X,\Theta}$$

is surjective as a morphism of algebraic tori. However, it may not be surjective at the level of k-points; we will thus be denoting by $A'_{X,\Theta}$ the image of:

$$\mathcal{Z}(L_{\Theta})^0 \to A_{X,\Theta}.$$

The space of smooth functions on X_{Θ} varying by a character χ of $A'_{X,\Theta}$ will be denoted by $C^{\infty}(X_{\Theta},\chi)$, and similarly for L^2 -spaces etc. We will analyze when a representation π can occur simultaneously in $C^{\infty}(X_{\Theta},\chi)$ and $C^{\infty}(X_{\Omega},\psi)$. What restrictions does this put on χ and ψ ? In favorable situations they are related by an "affine" map between the character groups of $A'_{X,\Theta}$ and $A'_{X,\Omega}$. In more detail:

- §13.1 discusses the notion of an "affine map" between character groups of k-points of algebraic tori (or finite-index subgroups thereof); this notion will be used in understanding when a representation π can occur simultaneously in $C^{\infty}(X_{\Theta}, \chi)$ and $C^{\infty}(X_{\Omega}, \psi)$.
- §13.2 applies this notion of "affine map" to the problem discussed: If π embeds into $C^{\infty}(X_{\Theta})$ and $C^{\infty}(X_{\Omega})$, what is the relationship between central characters?
- $\S 13.3$ uses the result of $\S 13.2$ to give a canonical decomposition of a morphism

$$L^2(X_{\Theta}) \to L^2(X_{\Omega})$$

into "equivariant" summands (equivariant for suitable actions of $A'_{X,\Theta}$). This is a critical *a priori* input into our analysis of scattering.

- §14.2 shows that the decomposition of §13.3 can be further refined under the assumption that the map $\mathfrak{a}_X^*/W_X \to \mathfrak{a}/W$ is "generically injective."

13.1. Isogenies of tori and affine maps on their character groups. In this section we introduce the notion of an affine map between character groups of tori, which will be used for the canonical decomposition of morphisms in Proposition 13.3.1. We will eventually need to apply this notion to finite-index subgroups of tori, which will not be the points of an algebraic subgroup. In order to not make the notation too heavy, we present the definitions for algebraic tori only; to obtain the general definitions when a torus $\mathbf{A}(k)$ is replaced by a finite index subgroup A', the reader only has to replace:

- any occurrence of $A = \mathbf{A}(k)$ (or its character group) by A' (resp. its character group);
- the maximal compact subgroup $A_0 \subset A$ by the maximal compact subgroup $A' \cap A_0 \subset A'$;
- finally, the notion of "morphisms modulo isogenies" $T: \mathbf{A}_1 \dashrightarrow \mathbf{A}_2$ that we are about to introduce does not change.

Let us remember that the category of algebraic tori is equivalent to the (opposite) category of finitely generated, torsion-free \mathbb{Z} -modules. The functor from the latter to finite dimensional \mathbb{Q} -vector spaces (tensoring by \mathbb{Q} over \mathbb{Z}) corresponds to the semisimple category obtained from tori by inverting isogenies. We will be denoting a morphism in the latter by $T: \mathbf{A}_1 \dashrightarrow \mathbf{A}_2$; explicitly, such a morphism corresponds to an equivalence class of pairs of homomorphisms of tori:

$$(\mathbf{A}_1 \to \mathbf{D}, \mathbf{A}_2 \to \mathbf{D}),$$
 (13.1)

with the second one finite and surjective, where "equivalence" is by passing simultaneously to a further finite quotient of \mathbf{D} .

Each such morphism T defines a canonical subgroup A_1^T of A_1 , as follows: recall that there is a canonical "valuation" map: $A_1 \to \mathcal{X}(\mathbf{A}_1)^*$. The map T induces:

$$\mathcal{X}(\mathbf{A}_1)^* \to \mathcal{X}(\mathbf{A}_2)^* \otimes \mathbb{Q},$$
 (13.2)

and the subgroup A_1^T is defined as the preimage of those elements which map into $\mathcal{X}(\mathbf{A}_2)^*$. In terms of a presentation (13.1), this is equivalent to saying that the elements of A_1^T are those whose images in D have the same "valuation" as elements of A_2 .

Example. If $\mathbf{A}_1 = \mathbf{A}_2 = \mathbb{G}_m$ and T is the isogeny " $x \mapsto x^{2/3}$," described more formally as the diagram $\mathbf{A}_1 \stackrel{2}{\to} \mathbf{D} = \mathbb{G}_m \stackrel{3}{\leftarrow} \mathbb{G}_m$, then

$$A_1^T = \{ \lambda \in A_1 = k^{\times} | \text{ valuation of } \lambda \text{ is divisible by } 3 \}.$$

As one can see from this example, T does not induce a map $A_1^T \to A_2$, but it at least does induces a map:

$$A_1^T/A_{1,0} \to A_2/A_{2,0},$$
 (13.3)

where the index 0 denotes maximal compact subgroup.

In particular, there is a canonical way to pull back any unramified character χ of A_2 to an unramified character $T^*\chi$ of A_1^T ; the fact that χ is unramified will be implicit whenever we write such a pull-back.

A component of A_2 will be a connected component in the natural topology, i.e., the set of all characters with the same restriction to $A_{2,0}$. Every component is a coset for the component of the identity, i.e. the subgroup of unramified characters.

An affine map $\widehat{A}_2 \dashrightarrow \widehat{A}_1$ compatible with the morphism $T: \mathbf{A}_1 \dashrightarrow \mathbf{A}_2$ is a mapping

$$f:$$
 some component of $\widehat{A}_2 \to \widehat{A}_1^T$

which is equivariant with respect to the natural homomorphism of unramified character groups induced by T:

$$\widehat{A}_2^0 \to \widehat{A}_1^T^0$$
.

(In this equation, the superscript 0 denotes connected component of the identity; in this case, it coincides with the group of unramified characters, e.g. \widehat{A}^0 is the dual of A/A_0). The term "affine" is due to the analogy with affine maps between vector spaces (i.e. translates of linear maps).

In other words, for every unramified character χ_2 of A_2 we have:

$$f(\chi_1 \chi_2) = f(\chi_1) T^* \chi_2,$$

when χ_1 is in the component of definition of f.

Explication. For every affine map f compatible with T we may find a character η of A_2 and a character η' of A_1^T with the following property: the domain consists of all characters $\chi \in \widehat{A}_2$ for which $\chi \eta^{-1}$ is unramified, and the map f satisfies

$$f(\chi) = \eta' T^*(\chi \eta^{-1}). \tag{13.4}$$

In the same way, we may define affine maps compatible with T on the space of all (not necessarily unitary) complex characters, without the requirement that they preserve unitarity – they will be denoted as:

$$f:\widehat{A}_{2\mathbb{C}} \dashrightarrow \widehat{A}_{1\mathbb{C}}.$$

As before, we require f to be defined only on one connected component of $\widehat{A}_{2\mathbb{C}}$, have image in $\widehat{A}_{1\mathbb{C}}^{T}$, and be equivariant with respect to the natural homomorphism of the identity components.

We will use this generalization only once.

13.1.1. REMARK. Note the following: If T is defined by the pair of maps $A_1 \to D \leftarrow A_2$ as in (13.1), we have mappings

$$\widehat{A}_2 \leftarrow \widehat{D} \rightarrow \widehat{A}_1 \rightarrow \widehat{A}_1^T$$

this gives us particular a multivalued function $\widehat{A}_2 \longrightarrow \widehat{A}_1^T$, where each "value" is a (possibly empty) finite subset of \widehat{A}_1^T : the image in \widehat{A}_1^T of all preimages in \widehat{D} .

Then this (set-valued) morphism is given, on each component of \widehat{A}_2 , by

$$\chi \in \widehat{A}_2 \mapsto \{f_1(\chi), \dots, f_r(\chi)\}$$

where the f_i are a (possibly empty) collection of affine maps compatible with T.

Indeed, suppose we begin with $\alpha \in \widehat{A}_2$ with nonempty image $\{\psi_1, \dots, \psi_r\} \in \widehat{A}_1^T$. Then it is easy to verify that the image of $\alpha \chi$, for χ an unramified character, is given by

$$\{\psi_1 T^* \chi, \dots, \psi_r T^* \chi\} \in \widehat{A_1^T}.$$

The rule $f_i: \alpha \chi \mapsto \psi_i T^* \chi$ define an affine map from the component of $\widehat{A_2}$ containing α to $\widehat{A_1^T}$, and this collection $\{f_1, \ldots, f_r\}$ has the desired property.

13.1.2. Maps of Hilbert spaces. Notation as above; in particular we have a morphism $T: \mathbf{A}_1 \dashrightarrow \mathbf{A}_2$ in the isogeny category and an affine map $f: \widehat{A}_2 \dashrightarrow \widehat{A}_1$ on character groups that covers T. Choose also η and η' as in 13.4.

Suppose we are given Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ with actions of A_1, A_2 respectively. We say that a mapping

$$S:\mathcal{H}_2\to\mathcal{H}_1$$

is f-equivariant if:

S factors through $(A_{2,0}, \eta)$ -coinvariants⁵⁵ and produces $(A_{1,0}, \eta')$ -invariants (where by the index $_0$ we denote the maximal compact subgroups), and if we define twisted actions of $A_2/A_{2,0}$ and $A_1/A_{1,0}$ on these coinvariant and invariant spaces via the rules:

$$a * v = \eta^{-1}(a)a \cdot v, \ a' * v' = \eta'(a')^{-1}a' \cdot v'$$

then

$$S(T(a') * v) = a' * S(v), \quad a' \in A_1^T / A_{1,0}.$$
(13.5)

Equivalently, S is f-equivariant if we may disintegrate with respect to the A_2 , resp. A_1^T -action:

$$\begin{split} \mathcal{H}_2 &= \int_{\chi} \mathcal{H}_{2,\chi}, \\ \mathcal{H}_1 &= \int_{\chi'} \mathcal{H}_{1,\chi'}, \end{split}$$

and $S = \int_{\chi} S_{\chi}$ with $S_{\chi} = 0$ unless $\chi \in \widehat{A}_2$ belongs to the domain of f, in which case S_{χ} is a morphism: $\mathcal{H}_{2,\chi} \to \mathcal{H}_{1,f(\chi)}$.

 $^{^{55}}$ Because $A_{2,0}$ is compact, with discrete dual, the canonical map from invariants to coinvariants is an isomorphism.

13.2. Relationship between central characters for X_{Θ} and X_{Ω} . Recall that we are assuming the validity of the Discrete Series Conjecture 9.4.6 for all boundary degenerations X_{Θ} . Recall also that the image of the map:

$$\mathcal{Z}(L_{\Theta})^0 \to A_{X,\Theta}$$

is denoted by $A'_{X,\Theta}$.

- 13.2.1. Proposition. Let Θ, Ω be two (possibly the same) subsets of Δ_X .
 - (1) Let J be an open compact subgroup of G. There is a finite collection of morphisms

$$T_i: \mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta},$$
 (13.6)

and 56 affine maps

$$f_i: \widehat{A'_{X,\Theta_{\mathbb{C}}}} \dashrightarrow \widehat{A'_{X,\Omega_{\mathbb{C}}}}$$
 (13.7)

compatible with T_i so that, for almost every $\chi \in \widehat{A'_{X,\Theta}}$ and every representation $\pi \hookrightarrow L^2(X_{\Theta}, \chi)_{\mathrm{disc}}$ with non-zero J-fixed vector the following is true:

If π embeds in $C^{\infty}(X_{\Omega}, \psi)$ then⁵⁷ $\psi = f_i(\chi)$ for some i.

(2) The subcollection of those morphisms (13.6) which are isogenies, and those affine maps (13.7) which preserve unitarity:

$$f_i:\widehat{A'_{X,\Theta}} \dashrightarrow \widehat{A'_{X,\Omega}},$$

is enough in order for the statement to be true for almost all χ and all π (with nonzero J-invariant vectors) which embed both into $L^2(X_{\Theta}, \chi)_{\mathrm{disc}}$ and $L^2(X_{\Omega}, \psi)_{\mathrm{disc}}$.

13.2.2. Proof of Proposition 13.2.1. The Discrete Series Conjecture 9.4.6 applied to the Levi varieties X_{Θ}^L , X_{Ω}^L , together with the finiteness of relative discrete series with J-fixed vectors (Theorem 9.2.1) imply that there is a finite number of triples $(P^-, \sigma, D_{i\mathbb{R}}^*)$ where P^- is a parabolic subgroup of P_{Θ}^- , σ a supercuspidal representation of its Levi quotient L and $D_{i\mathbb{R}}^*$ a torus of unitary unramified characters of P^- (with $D_{i\mathbb{R}}^* \to \widehat{A'_{X,\Theta}}$ finite) such that, for almost every $\chi \in \widehat{A'_{X,\Theta}}$, any representation $\pi \in L^2(X_{\Theta}, \chi)_{\text{disc}}$ which admits a J-invariant vector is a subquotient of $\pi' = I_{P^-}^G(\sigma \otimes \omega)$ for such a triple and some $\omega \in D_{i\mathbb{R}}^*$. For each such triple we have morphisms:

$$\mathcal{Z}(\mathbf{L}_{\Theta})^0 \hookrightarrow \mathcal{Z}(\mathbf{L})^0 \to \mathbf{D}$$

⁵⁶The affine map f_i determines the morphism T_i ; nonetheless, we prefer to keep both in our notation.

⁵⁷(More precisely, given that f_i is defined only on a component of $A_{X,\Theta}$, we should say that (ψ, χ) belongs to the graph of f_i . We shall allow ourself this type of imprecision at several points.)

(here **D** denotes the torus quotient of \mathbf{P}^- defining this toric family of relative discrete series), whose composition is finite and surjective, and such that χ is the twist of η :=the central character of σ (restricted to $\mathcal{Z}(\mathbf{L}_{\Theta})^0$) by the pull-back of an unramified character of \mathbf{D} . Notice that, after possibly replacing \mathbf{D} by a finite quotient, the map $\mathcal{Z}(\mathbf{L}_{\Theta})^0 \to \mathbf{D}$ factors through a map:

$$\mathbf{A}_{X,\Theta} \to \mathbf{D},$$
 (13.8)

since unramified characters of D have to be trivial on the kernel of $\mathcal{Z}(L_{\Theta})^0 \to \mathcal{Z}(X_{\Theta})$. (D is a torus quotient of \mathbf{L} , by definition, and the corresponding representations parabolically induced from \mathbf{P} embed in functions on X_{Θ} ; in particular, their central character must factor through $\mathcal{Z}(L_{\Theta})^0 \to A'_{X,\Theta}$, proving the claim.)

Now, if π embeds in $C^{\infty}(X_{\Omega}, \psi)$ it must be a submodule of a representation parabolically induced from P_{Ω}^- , because of the description of the variety X_{Ω} itself as being parabolically induced (Lemma 2.7.1). In order for $I_{P^-}^G(\sigma \otimes \omega)$ to have a common subquotient with a representation induced from P_{Ω}^- , equivalently with a supercuspidal induced from a parabolic subgroup of P_{Ω}^- , that supercuspidal should be a w-twist of $\sigma \otimes \omega$, for some element $w \in W$, the Weyl group of G. Each $w \in W$ such that $wL \subset L_{\Omega}$ defines a morphism:

$$\mathcal{Z}(\mathbf{L}_{\Omega})^0 \to \mathcal{Z}(\mathbf{L})^0 \to \mathbf{D}.$$
 (13.9)

Let $T: \mathcal{Z}(\mathbf{L}_{\Omega})^0 \dashrightarrow \mathcal{Z}(\mathbf{X}_{\Theta}) = \mathbf{A}_{X,\Theta}$ be the morphism defined by the equivalence class of the pair of maps (13.8),(13.9). The possible $\mathcal{Z}(L_{\Omega})^T$ -characters by which π can be embedded into $C^{\infty}(X_{\Omega})$ are thus "images" of $\chi \in \widehat{A'_{X,\Theta}}$ under the multivalued mapping arising from the diagram:

$$\widehat{A'_{X,\Theta_{\mathbb{C}}}} \leftarrow \widehat{D}_{\mathbb{C}} \rightarrow \widehat{\mathcal{Z}(L_{\Omega})^0}_{\mathbb{C}}$$

which, as we discussed in Remark 13.1.1, can be expressed as an affine mapping on each component of the character groups:

$$\widehat{A'_{X,\Theta_{\mathbb{C}}}} \longrightarrow \widehat{\mathcal{Z}(L_{\Omega})^{0}}_{\mathbb{C}} \tag{13.10}$$

Now we verify that T factors through the quotient: $\mathcal{Z}(\mathbf{L}_{\Omega})^0 \to \mathbf{A}_{X,\Omega}$: If we are given an affine mapping (13.10) compatible with $T: \mathcal{Z}(\mathbf{L}_{\Omega})^0 \longrightarrow \mathbf{A}_{X,\Theta}$, with the property that the image lies within $\widehat{A'_{X,\Omega}}_{\mathbb{C}}$ (considered naturally as a subset of $\widehat{\mathcal{Z}(L_{\Omega})^0}_{\mathbb{C}}$), then in fact T must factor⁵⁸ through

$$\mathbf{K}(k) \cap A_1^T \to A_1^T/A_{1,0} \to X_*(\mathbf{A}_1) \hookrightarrow X_*(\mathbf{A}_2) \otimes \mathbf{Q}$$

⁵⁸Here is the argument in an abstract context: Suppose given an isogeny $T: \mathbf{A}_1 \longrightarrow \mathbf{A}_2$ and a corresponding affine map $f: \widehat{A}_1 \to \widehat{A}_2^T$. (Our affine maps above are with respect to finite-index subgroups, but this doesn't make a difference for the argument.) We suppose that there is a quotient $\mathbf{A}_1 \twoheadrightarrow \mathbf{B}$ such that the image of f is contained in pullbacks of elements of \mathbf{B} . We claim, then, that T factors through \mathbf{B} in the isogeny category. Let \mathbf{K} be the kernel of $\mathbf{A}_1 \to \mathbf{B}$. The assumption forces the pullback of any unramified character of \mathbf{A} to be trivial on \mathbf{K} ; equivalently, the image of $\mathbf{K}(k) \cap A_1^T$ under

the mapping $\mathcal{Z}(L_{\Omega})^0 \to A_{X,\Omega}$, that is to say, T determines a mapping $T: \mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta}$ as desired.

For the second assertion, if π belongs to both $L^2(X_{\Theta},\chi)_{\mathrm{disc}}$ and $L^2(X_{\Omega},\psi)_{\mathrm{disc}}$ then applying the "toric discrete series" assumption to both we get triples $(P^-,\sigma,D^*_{i\mathbb{R}})$ and $(Q^-,\sigma',D'^*_{i\mathbb{R}})$ (with $P^-\subset P^-_{\Theta}$ and $Q^-\subset P^-_{\Omega}$) such that π is a subquotient of the corresponding induced representations. It is known that, should $I_{P^-}(\sigma \cdot \chi)$ have a common subquotient as $I_{Q^-}(\sigma' \cdot \chi')$, this implies that there exists $w \in W$ carrying the Levi subgroup of P^- to the Levi subgroup of Q^- which carries $\sigma \cdot \chi$ to $\sigma' \cdot \chi'$. Our assumption is that such w exists for a set of positive measure; in particular, we may suppose that a particular w works for a set of χ of positive measure. Twisting σ, σ' , we may suppose that $w\sigma = \sigma'$ and that the following is true: For a positive measure set Z of unramified characters $\chi \in D^*_{i\mathbb{R}}$, the character $w\chi$ of wP^- factors through the torus quotient corresponding to $D'^*_{i\mathbb{R}}$.

Now, given a set of positive measure (and thus Zariski-dense) of unramified unitary characters in $D_{i\mathbb{R}}^*$, the intersection of their kernels is necessarily simply the maximal compact subgroup D_0 of the torus quotient $P^- \to D$ corresponding to $D_{i\mathbb{R}}^*$. Thus the intersection of the kernels of all $\chi \in Z$ is simply the preimage of D_0 in P^- . Similarly, the intersection of the kernels of all $w\chi$ ($\chi \in Z$) is the preimage of D_0' in Q^- . So the map w must then carry the preimage of D_0 in P^- into the preimage of D_0' in Q^- . The map w must then carry this into the preimage of D_0' in Q^- . In particular w induces a mapping $D_{i\mathbb{R}}^* \to D_{i\mathbb{R}}'^*$ and this mapping has the property that $\operatorname{ind}(\sigma \cdot \chi)$ and $\operatorname{ind}(\sigma' \cdot w(\chi))$ have a common subconstituent for all χ . Since (as part of the assumption of relative discrete series) we suppose that the maps $D_{i\mathbb{R}}^* \to \widehat{A'_{X,\Theta}}$ and $D_{i\mathbb{R}}'^* \to \widehat{A'_{X,\Omega}}$ are finite and surjective, w induces an isogeny $\mathcal{Z}(X_{\Theta}) \dashrightarrow \mathcal{Z}(X_{\Omega})$. The set of all such isogenies $\mathcal{Z}(X_{\Theta}) \dashrightarrow \mathcal{Z}(X_{\Omega})$ that arise in this fashion from some $w \in W$ then has the property stipulated by the proposition.

13.3. Canonical decomposition of maps $L^2(X_{\Theta}) \to L^2(X_{\Omega})$.

13.3.1. Proposition. (1) Suppose that $S: L^2(X_{\Theta})_{\mathrm{disc}} \to L^2(X_{\Omega})$ is a G-equivariant morphism. Then there exists a unique (up to indexing) decomposition:

$$S = \sum_{i=1}^{\infty} S_i \tag{13.11}$$

is trivial. In other words, the map $\mathcal{X}(A_1)^* \to \mathcal{X}(A_2)^* \otimes \mathbf{Q}$ induced by T is trivial on a finite index subgroup of the subgroup $\mathcal{X}(K)^*$ of $\mathcal{X}(A_1)^*$ corresponding to $\mathbf{K} \subset \mathbf{A}_2$. This means that it is in fact trivial on all of $\mathcal{X}(K)^*$, and so T indeed factors in the isogeny category through the quotient \mathbf{A}_1/\mathbf{K} .

such that each S_i is a nonzero bounded morphism, and equivariant with respect to some (distinct) affine map between central character groups.

In other words, for each i there is a pair (T_i, f_i) (with $f_i \neq f_j$ when $i \neq j$), where

$$T_i: \mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta}$$

is a morphism in the isogeny category of tori and

$$f_i:\widehat{A'_{X,\Theta}} \dashrightarrow \widehat{A'_{X,\Omega}}$$

is an affine map compatible with T_i , such that S_i is f_i -equivariant (see 13.1.2). For each open compact subgroup J only a finite number of summands in (13.11) are non-zero on $L^2(X_{\Theta})^{J}_{\text{disc}}$.

(2) If
$$|\Theta| \neq |\Omega|$$
 then $L^2(X)_{\Theta} \perp L^2(X)_{\Omega}$.

PROOF. Given $S: \mathcal{H}_2 := L^2(X_{\Theta})_{\mathrm{disc}} \to \mathcal{H}_1 := L^2(X_{\Omega})$, let μ be a Plancherel measure for $\mathcal{H}_2 \oplus \mathcal{H}_1$, and let $S = \int S_{\pi}$ be the corresponding decomposition of S, as in §12.1.4, where $\int_{\hat{G}} \mathcal{H}_{2,\pi}\mu(\pi)$ and $\int_{\hat{G}} \mathcal{H}_{1,\pi}\mu(\pi)$ are direct integral decompositions for \mathcal{H}_2 and \mathcal{H}_1 , respectively.

Fix some $T: \mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta}$, and unitary characters η of $A'_{X,\Theta}$ and η' of $A'^T_{X,\Omega}$. Let $\mathcal{H}^{\eta}_2, \mathcal{H}^{\eta'}_1$ be the eigenspaces where the maximal compact subgroups of $A'_{X,\Theta}$ and $A'_{X,\Omega}$ act via the characters η and η' , respectively. The valuation maps give surjections:

$$A_{X,\Omega}^{\prime T} \twoheadrightarrow \Gamma := \mathcal{X}(\mathbf{A}_{X,\Omega})^* \cap T^{-1} \mathcal{X}(\mathbf{A}_{X,\Theta})^*$$
 (13.12)

and:

$$A'_{X,\Theta} \twoheadrightarrow \mathcal{X}(\mathbf{A}_{X,\Theta})^*;$$
 (13.13)

we may let the lattice Γ act on $\mathcal{J}^{\eta,\eta'} := \mathcal{H}_2^{\eta} \oplus \mathcal{H}_1^{\eta'}$ as:

$$\gamma \cdot (h + h') = \eta^{-1}(b)b \cdot h + \eta'^{-1}(a)a \cdot h', \tag{13.14}$$

where a is any lift of γ via (13.12) and b is any lift of $T(\gamma')$ via (13.13). Hence we get a homomorphism:

$$\Gamma \to \operatorname{Aut}_G(\mathcal{J}^{\eta,\eta'}).$$
 (13.15)

Let $\mathcal{J}^{\eta,\eta'} = \int_{\hat{G}} \mathcal{J}_{\pi}^{\eta,\eta'} \mu(\pi)$ be a Plancherel decomposition; we claim that (13.15) decomposes into the direct integral of the analogous maps:

$$\Gamma \to \operatorname{Aut}_G(\mathcal{J}_{\pi}^{\eta,\eta'}),$$
 (13.16)

defined via the same formula (13.14). Indeed, if we decompose (by §12.1.4) the action of $\gamma \in \Gamma$ as an integral of $\gamma_{\pi} : J_{\pi}^{\eta,\eta'} \to J_{\pi}^{\eta,\eta'}$ then it is clear that for

almost⁵⁹ all π the endomorphism γ_{π} of $\mathcal{J}_{\pi}^{\eta,\eta'} = \mathcal{H}_{2,\pi}^{\eta} \oplus \mathcal{H}_{1,\pi}^{\eta'}$ must coincide with the action of the element γ defined as in (13.14).

The inner action of $\operatorname{Aut}_G(\mathcal{J}_{\pi}^{\eta,\eta'})$ on $\operatorname{End}_G(\mathcal{J}_{\pi}^{\eta,\eta'})$ defines by (13.16) an action of Γ on the latter, and it is easy to see by its definition that this action preserves the subspace $\operatorname{Hom}_G(\mathcal{H}_{2,\pi}^{\eta},\mathcal{H}_{1,\pi}^{\eta'})$. In our setting, these spaces are finite-dimensional, and they carry a natural inner product that is preserved by Γ . Let S_{π}^{Γ} be the projection of S_{π} to the eigenspace for the trivial character of Γ . We will show in a moment that the function: $\pi \mapsto \|S_{\pi}^{\Gamma}\|$ is essentially bounded. Assuming that for a moment, by the Proposition 10.4.1 on "measurability of eigenprojections" ⁶⁰ we may integrate the morphisms S_{π}^{Γ} in order to get a morphism:

$$S^{\Gamma} := \int_{\hat{G}} S_{\pi}^{\Gamma} : \mathcal{H}_{2}^{\eta} \dashrightarrow \mathcal{H}_{1}^{\eta'}, \tag{13.17}$$

where the dotted arrow means that it is well-defined on a dense subspace of \mathcal{H}^{η} . By construction, S^{Γ} is equivariant with respect to any affine map: $f_i: \widehat{A'_{X,\Theta_{\mathbb{C}}}} \dashrightarrow \widehat{A'_{X,\Omega_{\mathbb{C}}}}$ which covers T and maps η to η' ; indeed, it is clear by construction that (13.5) is satisfied.

Now, we verify that the function: $\pi \mapsto \|S^{\Gamma}_{\pi}\|$ is essentially bounded. To that end, decompose each $\mathcal{J}^{\eta,\eta'}$ into $A'^T_{X,\Omega} \times A'_{X,\Theta}$ -eigenspaces (these are genuine eigenspaces, rather than generalized ones, because this action preserves the natural inner product). From the Plancherel decomposition of $\mathcal{J}^{\eta,\eta'}$ as an $A'^T_{X,\Omega} \times G$ -representation it follows that the distinct generalized eigenspaces are, for almost all π , honest eigenspaces and orthogonal to each other. Since S^{Γ}_{π} is the sum of some of the operators:

$$\operatorname{pr}_1 \circ S_{\pi} \circ \operatorname{pr}_2$$

where pr_1 and pr_2 vary though all projections to $A_{X,\Omega}^{\prime T} \times A_{X,\Theta}^{\prime}$ - eigenspaces, the norm of S_{π}^{Γ} can be bounded by the norm of S_{π} , multiplied by a number depending only on the number of distinct eigenspaces. Finally, we recall that this number is uniformly bounded by the order of the Weyl group (cf. Lemma 11.3.5).

Now let (T_i, f_i) vary over all those pairs of the first part of Proposition 13.2.1. Notice that they are finitely many if we restrict to representations with non-zero J-invariant vectors, so all together they will be at most countably many. Let Γ_i denote the corresponding finitely generated abelian

 $^{^{59}}$ Recall that every π can only appear with a finite number of $A_{X,\Theta}$ -exponents (cf. the proof of Theorem 5.1.5) in $C^{\infty}(X_{\Theta})$. By the presumed validity of the Discrete Series Conjecture 9.4.6 for X_{Θ} , statements that hold "for almost all π " in the spectrum of $L^2(X_{\Theta})_{\text{disc}}$ also hold "for almost all χ ".

⁶⁰By §12.1.1, by decomposing \hat{G} into a countable union of measurable sets, we may identify the measurable structure of the family of vector spaces $\operatorname{End}_{G}(\mathcal{J}_{\pi}^{\eta,\eta'})$ with that of a trivial family; hence, the Proposition applies.

groups, defined as above. For each i we get an operator $S_i = \int_{\hat{G}} S_{\pi,i}$ as above. We claim:

$$S = \sum_{i} S_i. \tag{13.18}$$

Indeed, for almost all π , by the second part of Proposition 13.2.1 each eigenspace for the $A'_{X,\Theta} \times A'_{X,\Omega}$ -action on $\operatorname{Hom}_G(\mathcal{H}_{2,\pi},\mathcal{H}_{1,\pi})$ is contained in the fixed (eigenvalue=1) subspace of Γ_i , for some i. Moreover, it is clear that for different i's and almost all π (in the setting of Proposition 13.2.1: almost all χ) the fixed subspaces of Γ_i and Γ_j for $i \neq j$ are distinct. If it is not so, there is a positive measure set of $\chi \in \widehat{A'_{X,\Theta}}$ such that $f_i(\chi) = f_j(\chi) \in \widehat{A'_{X,\Omega}}$. This means that f_i, f_j coincide, a contradiction. This implies (13.18).

Regarding uniqueness: For any decomposition $S = \sum_i S_i$, where each S_i is (T_i, f_i) -equivariant, we must have $S_{\pi} = \sum_i S_{\pi,i}$, where $S_{\pi,i}$ is obtained by disintegrating the homomorphism S_i , in the same sense as we have seen above. But then (for almost all χ) $S_{\pi,i}$ is characterized as the Γ_i -fixed part of S (where Γ_i and its action is defined as before), since the measure of the set of χ where $f_i(\chi) = f_i(\chi)$ for any $i \neq j$ is zero.

To prove the second statement, assume that $|\Theta| > |\Omega|$, hence dim $\mathbf{A}_{X,\Theta} < \dim \mathbf{A}_{X,\Omega}$.

From Proposition 13.2.1 it follows that there is a subset Z of $\widehat{A'_{X,\Omega}}$ of measure zero, and a subset Z' of $\widehat{A'_{X,\Theta}}$ of measure zero, such that if $\pi \in L^2(X_\Theta)_{\mathrm{disc}}$ does not have central character in Z' and admits a non-zero morphism into $C^\infty(X_\Omega, \psi)$ for some $\psi \in \widehat{A'_{X,\Omega}}$ then ψ belongs to Z. (After all any morphism: $\mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta}$ has positive-dimensional kernel.) Now, if $L^2(X)_\Omega$ is not orthogonal to $L^2(X)_\Theta$ then we get a non-zero morphism: $\iota^*_\Omega \iota_\Theta : L^2(X_\Theta)_{\mathrm{disc}} \to L^2(X_\Omega)_{\mathrm{disc}}$. But this is impossible: because we are assuming the Discrete Series Conjecture for X_Ω , the $L^2(X_\Omega)_{\mathrm{disc}}$ -Plancherel measure of representations with central character in Z is also zero, and similarly for Θ, Z' . It follows that $L^2(X)_\Omega \perp L^2(X)_\Theta$ if $|\Omega| \neq |\Theta|$.

14. Scattering theory

14.1. In §11 we constructed canonical maps $\iota_{\Theta}: L^2(X_{\Theta}) \to L^2(X)$, and we saw (Corollary 11.6.2) that $\sum_{\Theta} \iota_{\Theta}$ induces a surjection:

$$\bigoplus_{\Theta \subset \Delta_X} L^2(X_\Theta)_{\mathrm{disc}} \twoheadrightarrow L^2(X).$$

We will be denoting the image of $L^2(X_{\Theta})_{\text{disc}}$ by $L^2(X)_{\Theta}$.

The question of scattering is the description of the kernel of this map. Our answer is described in Theorem 7.3.1 (which is conditional on Conjecture 9.4.6 and on generic injectivity, but we expect it to hold in full generality), which implies in particular that $L^2(X)_{\Theta}$ and $L^2(X)_{\Omega}$ coincide

if Θ and Ω are W_X -associates (i.e. there is a $w \in W_X$ such that $w\Theta = \Omega$), and are orthogonal otherwise.

The starting point for the proof of Theorem 7.3.1 is the relatively straightforward statement of Proposition 13.3.1:

Any morphism $L^2(X_{\Theta})_{\mathrm{disc}} \to L^2(X_{\Omega})_{\mathrm{disc}}$ decomposes uniquely as a sum:

$$\sum S_i, \tag{14.1}$$

where the morphism S_i , assumed non-zero, is equivariant with respect to an isogeny $T_i: \mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta}$ and an affine map f_i of character groups compatible with T_i , and the pairs (T_i, f_i) are assumed to be distinct.

Recall that this statement followed, essentially, from the assumption of validity of the Discrete Series Conjecture 9.4.6 and general facts about induced representations.

For $\Theta, \Omega \subset \Delta_X$ and $w \in W_X(\Omega, \Theta)$ (the set of elements of W_X taking Θ to Ω), we say that a pair (T_i, f_i) as above *corresponds* to w if T_i is the isomorphism:

$$\mathbf{A}_{X,\Omega} \to \mathbf{A}_{X,\Theta}$$
 (14.2)

induced by w^{-1} (and hence f_i is the restriction to some component of $\widehat{A'_{X,\Theta}}$ of the map of character groups obtained by T_i). For a morphism $S: L^2(X_{\Theta}) \to L^2(X_{\Omega})$ we will call the w-part of S the sum of those summands in its decomposition (14.1) for which (T_i, f_i) is induced by (14.2). Notice that the w-part of such a morphism is $A'_{X,\Theta}$ -equivariant when $A'_{X,\Theta}$ acts on $L^2(X_{\Omega})$ via $w: A'_{X,\Theta} \to A'_{X,\Omega}$.

Applying this decomposition to the 61 scattering morphisms $i_{\Omega}^*i_{\Theta}$, we additionally need to establish the following:

- the only (T_i, f_i) 's that appear are those corresponding to elements of $W_X(\Omega, \Theta)$;
- for all $w \in W_X(\Omega, \Theta)$ the w-part of $i_{\Omega}^* i_{\Theta}$ is an isometry.

As we will see in $\S14.3$, these two facts imply Theorem 7.3.1.

To establish these two properties we need some algebraic input, encoded the condition of "generic multiplicity one" of Theorem 7.3.1, together with some hard analysis. The analysis leads to Theorem 14.3.1, which should be regarded as the main result of this section; let us first discuss the algebraic condition.

$$\langle \iota_{\Omega}^* \iota_{\Theta} f, \iota_{\Omega'}^{\Omega} f' \rangle = \langle \iota_{\Theta} f, \iota_{\Omega} \iota_{\Omega'}^{\Omega} f' \rangle = \langle \iota_{\Theta} f, \iota_{\Omega'} f' \rangle,$$

and we may now apply the fact that $L^2(X)_{\Theta}$ and $L^2(X)_{\Omega'}$ are perpendicular if $|\Theta| \neq |\Omega'|$, in view of Proposition 13.3.1.

⁶¹A minor remark: The morphism $\iota_{\Omega}^*\iota_{\Theta}$ in fact maps $L^2(X_{\Theta})_{\mathrm{disc}}$ into $L^2(X_{\Omega})_{\mathrm{disc}}$; in particular, $\iota_{\Omega,\mathrm{disc}}^*\iota_{\Theta,\mathrm{disc}}$ and $\iota_{\Omega}^*\iota_{\Theta}$ coincide on $L^2(X_{\Theta})_{\mathrm{disc}}$. Indeed – see the proof of Corollary 11.6.2 – we need only verify that, for $f \in L^2(X_{\Theta})_{\mathrm{disc}}$ that $\iota_{\Omega}^*\iota_{\Theta}f$ is perpendicular to all $\iota_{\Omega'}^{\Omega}f'$, where $f' \in L^2(X_{\Omega'})$, and Ω' contains Ω. But:

14.2. Generic injectivity of the map: $\mathfrak{a}_X^*/W_X \to \mathfrak{a}^*/W$. Denote: $\mathfrak{a}_X^* := \mathcal{X}(\mathbf{X}) \otimes \mathbb{Q} \subset \mathfrak{a}^* := \mathcal{X}(\mathbf{B}) \otimes \mathbb{Q}$. Recall that the subgroup $W_{L(X)}$ of W is the pointwise stabilizer of \mathfrak{a}_X^* , the subgroup W_X normalizes \mathfrak{a}_X^* , and its action on it is generated by simple reflections. Fix a Weyl chamber for W_X on \mathfrak{a}_X^* ; the image of a face \mathcal{F} of that Weyl chamber \mathfrak{a}_X^*/W_X will be called a "face" of \mathfrak{a}_X^*/W_X .

The condition called "generic injectivity of the map: $\mathfrak{a}_X^*/W_X \to \mathfrak{a}^*/W$ on each face" in the statement of Theorem 7.3.1 is the following:

For every integer d, the restriction of the map: $\mathfrak{a}_X^*/W_X \to \mathfrak{a}^*/W$ to the collection of d-dimensional faces of \mathfrak{a}_X^*/W_X is generically injective.

By "generically injective" we mean injective on a subset of full measure, for the natural class of measures, but this is easily seen to be equivalent, in this case, to injectivity outside of the image of a finite number of hyperplanes in \mathfrak{a}_X^* . In other words, outside of a meager set two distinct elements of \mathcal{F}_d (the union of all d-dimensional faces) cannot be W-conjugate.

An equivalent way to formulate this condition, in terms of dual groups, is the following:

For every pair of (standard) Levi subgroups \check{L}_{Θ} , \check{L}_{Ω} of \check{G}_X , and any isomorphism of their centers $\mathcal{Z}(\check{L}_{\Theta}) \to \mathcal{Z}(\check{L}_{\Omega})$ induced by an element of the Weyl group W of \check{G} , there is an element of the little Weyl group W_X which induces the same isomorphism.

It is evident that this condition is very easy to check in each particular case. It is always true in both extreme cases: When the dual group \check{G}_X is isomorphic to SL_2 , and when it is all of \check{G} . Less trivially, Delorme has shown that it holds for all symmetric varieties:

- 14.2.1. Proposition. [Del, Lemma 15] If \mathbf{X} is symmetric, then it satisfies the generic injectivity condition.
- 14.2.2. EXAMPLE. For $\mathbf{X} = \mathbf{Sp}_{2n} \setminus \mathbf{GL}_{2n}$ the dual group is $\check{G}_X = \mathrm{GL}_n \hookrightarrow \mathrm{GL}_{2n} = \check{G}$, with a diagonal element $\mathrm{diag}(\chi_1, \dots, \chi_n)$ embedded as:

$$\operatorname{diag}(\chi_1, \chi_1, \chi_2, \chi_2, \dots, \chi_n, \chi_n).$$

Let Θ, Ω be two subsets of the simple roots of \check{G}_X . Their kernels, are subtori of A_X^* which, when considered as subtori of \check{G} are the connected centers of standard Levi subgroups $\check{L}_{\tilde{\Theta}}, \check{L}_{\tilde{\Omega}}$ corresponding to subsets $\check{\Theta}, \tilde{\Omega}$ of the simple roots of \check{G} . (Explicitly: for the usual numbering $1, \ldots,$ of the roots of GL_n and $\mathrm{GL}_{2n}, \; \check{\Theta}$ is the union of $2 \cdot \Theta$ and all odd roots, and similarly with $\tilde{\Omega}$.) Any isomorphism between $\mathcal{Z}(\check{L}_{\tilde{\Theta}})^0, \mathcal{Z}(\check{L}_{\tilde{\Omega}})^0$ induced by an element of W is actually induced by an element of $W(\tilde{\Omega}, \check{\Theta})$. But $W(\tilde{\Omega}, \check{\Theta}) = W_X(\Omega, \Theta)$, therefore X satisfies the injectivity assumption.

⁶²Recall that a "face" is the intersection of the Weyl chamber with the kernel of a linear functional which is non-negative on it; hence the whole chamber is also a face.

14.2.3. Lemma. Assume that the map:

$$\mathfrak{a}_X^*/W_X \to \mathfrak{a}^*/W$$

is generically injective on each face.

Then – notation as in (14.1) – the only pairs (T_i, f_i) that can appear in the decomposition of a morphism $S: L^2(X_{\Theta})_{\mathrm{disc}} \longrightarrow L^2(X_{\Omega})_{\mathrm{disc}}$ are those corresponding to elements of $W_X(\Omega, \Theta)$.

Hence, any such morphism decomposes as a sum of its w-parts:

$$\sum_{w \in W_X(\Omega,\Theta)} S_w,\tag{14.3}$$

where S_w is the sum of those S_i 's in (14.1) for which the corresonding affine map f_i of character groups is induced (by restriction to a connected component of $\widehat{A'_{X,\Theta}}$) by the element w.

PROOF. Revisiting the proof of the last assertion of Proposition 13.2.1, let us give ourselves two toric families of relative discrete series $(P, \sigma, D_{i\mathbb{R}}^*)$ and $(Q, \sigma', D_{i\mathbb{R}}')$, for X_{Θ} and X_{Ω} respectively, such that $D_{i\mathbb{R}}^*$ is the group of unitary elements in a torus D^* of unramified characters identified with $\mathcal{Z}(\check{L}_{X,\Theta})$ under (9.10) and $D_{i\mathbb{R}}'^*$ is the group of unitary elements of a torus D'^* identified with $\mathcal{Z}(\check{L}_{X,\Omega})$ under (9.10). Recall that the Lie algebras of $\mathcal{Z}(\check{L}_{X,\Theta})$, $\mathcal{Z}(\check{L}_{X,\Omega})$ are the complexifications of the vector spaces $\mathfrak{a}_{X,\Theta}^*$, $\mathfrak{a}_{X,\Omega}^*$, respectively. As we saw in the proof of Proposition 13.2.1, any summand of the scattering map should arise from an element $w \in W$ which takes $\mathfrak{a}_{X,\Theta}^*$ to $\mathfrak{a}_{X,\Omega}^*$.

If the map: $\mathfrak{a}_X^*/W_X \to \mathfrak{a}^*/W$ is generically injective on every face, this means that any element $w \in W$ which carries one family into to the other induces the same map: $\mathfrak{a}_{X,\Theta}^* \to \mathfrak{a}_{X,\Omega}^*$ as an element of $W_X(\Omega,\Theta)$.

This proves the lemma. \Box

14.3. The scattering theorem. Define $L^2(X)_i$ to be the image of

$$\bigoplus_{|\Theta|=i} L^2(X_{\Theta})_{\mathrm{disc}}$$

in $L^2(X)$. Part 2 of Proposition 13.3.1 implies that we have a direct sum decomposition:

$$L^2(X) = \bigoplus_i L^2(X)_i. \tag{14.4}$$

Denote by $\mathfrak{a}_{X,\Theta}$ the space $\mathcal{X}(\mathbf{A}_{X,\Theta})^* \otimes \mathbb{Q} \subset \mathfrak{a}_X$ and by $\mathfrak{a}_{X,\Theta}^+$ its "anti-dominant chamber", i.e. its intersection with the cone \mathcal{V} of invariant valuations. We denote by $\mathring{\mathfrak{a}}_{X,\Theta}^+$ the interior of $\mathfrak{a}_{X,\Theta}^+$. If a morphism $T_i: \mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Theta}$ is an isogeny as in part 2 of Proposition 13.2.1, it induces an isomorphism (again to be denoted by T_i): $\mathfrak{a}_{X,\Omega} \xrightarrow{\sim} \mathfrak{a}_{X,\Theta}$.

Let $\Theta \subset \Delta_X$ and let Ω range over the subsets of Δ_X of the same size as Θ (including Θ). Let \mathcal{H} be any $A_{X,\Theta} \times G$ -invariant closed subspace of

 $L^2(X_{\Theta})_{\text{disc}}$ and consider the "scattering" morphisms $\iota_{\Omega}^*\iota_{\Theta}$ restricted to \mathcal{H} . If in their decomposition (14.1) the summand S_i is non-zero on \mathcal{H} , we will say that the pair (T_i, f_i) (or to be absolutely complete the triple (Ω, T_i, f_i)) appears in the scattering of \mathcal{H} .

The main result of this section is the following:

- 14.3.1. THEOREM (Tiling property of scattering morphisms.). Let $\Theta \subset \Delta_X$ and let \mathcal{H} be a nonzero $A_{X,\Theta} \times G$ -invariant closed subspace of $L^2(X_{\Theta})_{\mathrm{disc}}$.
 - (1) If a triple (Ω, T, f) appears in the scattering of \mathcal{H} and:

$$\mathring{\mathfrak{a}}_{X,\Theta}^{+} \cap T\mathring{\mathfrak{a}}_{X,\Omega}^{+} \neq \emptyset \tag{14.5}$$

then $\Omega = \Theta$, T = Id and f is also the identity (on a connected component of $\widehat{A'_{X \Theta}}$).

(2) If (Ω_i, T_i, f_i) varies in all the triples which appear in the scattering of \mathcal{H} , then:

$$\bigcup_{i} T_{i}(\mathfrak{a}_{X,\Omega_{i}}^{+}) = \mathfrak{a}_{X,\Theta}. \tag{14.6}$$

Indeed, there exists a splitting $\mathcal{H} = \bigoplus \mathcal{H}_{\alpha}$ such that:

- (i) The $A'_{X,\Theta}$ -Plancherel measure for different \mathcal{H}_{α} is mutually singular; in particular, $\operatorname{Hom}_{A_{X,\Theta}}(\mathcal{H}_{\alpha},\mathcal{H}_{\beta})=0$ for $\alpha\neq\beta$;
- (ii) For $a \in \mathfrak{a}_{X,\Theta}$, let J denote the set of indices i such that $a \in T_i(\mathring{\mathfrak{a}}_{X,\Omega}^+)$, and assume that a is generic in the sense that a does not lie on any wall of $T_i\mathfrak{a}_{X,\Omega}^+$. For any α , any $v \in \mathcal{H}_{\alpha}$ and any generic $a \in \mathfrak{a}_{X,\Theta}$ then

$$\sum_{i \in J} \|S_i(v)\|^2 \ge \|v\|^2, \tag{14.7}$$

This will be enough to prove the main Scattering Theorem 7.3.1. Let us first discuss this proof. Actually, we only use the *second* statement of Theorem 14.3.1 in this proof; the first statement won't be used, because it is contained in the "generic injectivity" condition.

PROOF OF THEOREM 7.3.1. The existence and characterization of Bernstein morphisms, together with the fact that the images of their restrictions to discrete spectra span the whole space $L^2(X)$, has already been established in Section 11 (see Corollary 11.6.2). The rest of the statements of the theorem will first be proved by restriction to discrete spectra, i.e. S_w , for $w \in W_X(\Omega, \Theta)$, will first be defined as a morphism with the stated properties from $L^2(X_\Theta)_{\text{disc}}$ to $L^2(X_\Omega)_{\text{disc}}$. Notice that (7.5) involves only discrete spectra. At the end we will extend S_w to the whole $L^2(X_\Theta)$.

Take Ω, Θ with $|\Omega| = |\Theta|$. We write $\iota_{\Theta, \text{disc}}$ for the restriction of ι_{Θ} to $L^2(X_{\Theta})_{\text{disc}}$ and $\iota_{\Theta, \text{disc}}^*$ the adjoint of this restricted map, i.e. it is ι_{Θ}^* followed by the orthogonal projection to discrete spectrum. By Lemma

14.2.3, the morphism $\iota_{\Omega,\mathrm{disc}}^*\iota_{\Theta} = \iota_{\Omega}^*\iota_{\Theta,\mathrm{disc}} = \iota_{\Omega,\mathrm{disc}}^*\iota_{\Theta,\mathrm{disc}}$ (the equalities follow from claim (2) of Proposition 13.3.1⁶³) decomposes as a sum of morphisms $\sum_{w \in W_X(\Omega,\Theta)} S_w$, where each S_w is $A'_{X,\Theta}$ -equivariant, with $A'_{X,\Theta}$ acting on $L^2(X_{\Omega})$ via the isomorphism $A'_{X,\Theta} \stackrel{w}{\to} A'_{X,\Omega}$ induced by w. In particular, $\iota_{\Omega,\mathrm{disc}}^*\iota_{\Theta,\mathrm{disc}}$ is zero unless Θ and Ω are W_X -associate.

Now, root systems have the following tiling property:

The collection of subsets of $\mathfrak{a}_{X,\Theta}$ given by $w^{-1}a_{X,\Omega}^+$, where Ω varies through subsets of Δ_X with $|\Omega| = |\Theta|$, and w ranges through $W_X(\Omega,\Theta)$, form a perfect tiling of $\mathfrak{a}_{X,\Theta}$, i.e. their union is $\mathfrak{a}_{X,\Theta}$ and their interiors are disjoint.

A proof of this property has been indicated in the footnote of §7.3.

In particular, for each Ω and $w \in W_X(\Omega, \Theta)$, if $a \in w^{-1}\mathring{\mathfrak{a}}_{X,\Omega}^+$ then a does not belong to the corresponding set for any other pair $(\Omega', w' \in W_X(\Omega', \Theta))$. So the second statement of Theorem 14.3.1 implies that we may split $L^2(X_{\Theta})_{\text{disc}} = \bigoplus \mathcal{H}_{\alpha}$ in such a way that

$$||S_w v|| \ge ||v|| \tag{14.8}$$

whenever v belongs to any \mathcal{H}_{α} . (Recall that S_w is the sum of S_i 's which are induced from the element w, and those have image on different orthogonal direct summands of $L^2(X_{\Omega})$, corresponding to distinct connected components of the character group $\widehat{A'_{X,\Omega}}$, hence $||S_w v||^2 = \sum_i ||S_i v||^2$, the sum ranging over those i's.)

We can also assume that Proposition 11.7.1 (which can easily be seen to hold when $A_{X,\Theta}$ is replaced by $A'_{X,\Theta}$) applies to any \mathcal{H}_{α} and its image under any S_w . In fact, choose a partition $\widehat{A'_{X,\Theta}} = \coprod U_{\beta}$ up to sets of measure zero, with the property that if if $w \in U_{\beta}$ and $1 \neq w \in W_X(\Theta, \Theta)$ then $w\chi \notin U_{\beta}$. We may assume, without loss of generality, that the decomposition $\mathcal{H} = \bigoplus \mathcal{H}_{\alpha}$ furnished by the Theorem is fine enough that the Plancherel measure for \mathcal{H}_{α} as an $A'_{X,\Theta}$ -representation is wholly supported on some U_{β} . This is enough to ensure that Proposition 11.7.1 applies to \mathcal{H}_{α} and its image under any S_w : our choice and the decomposition (14.1) mean that any G-equivariant map $\mathcal{H}_{\alpha} \to \mathcal{H}_{\alpha}$ is also $A'_{X,\Theta}$ -equivariant, which means that almost all $\pi \in \hat{G}$ for \mathcal{H}_{α} -Plancherel measure have a unique exponent, in the language of Proposition 11.7.1.

Fix $w_0 \in W_X(\Omega, \Theta)$. Let \mathcal{H}'_{α} be the image of \mathcal{H}_{α} under S_{w_0} . Let $p: \mathcal{H}'_{\alpha} \to L^2(X_{\Omega})_{\mathrm{disc}}$ be the natural inclusion. Then $S_{w_0} = p^*S_{w_0}$. We may write

$$(\iota_{\Omega} \circ p)^* \circ \iota_{\Theta} = S_{w_0} + \sum_{w \neq w_0} p^* S_w.$$

⁶³ For example, $\iota_{\Omega,\mathrm{disc}}^*(\iota_{\Theta} - \iota_{\Theta,\mathrm{disc}})$ is zero because $L^2(X)_{\Omega}$ is orthogonal to $\iota_{\Theta}f$ for $f \perp L^2(X_{\Theta})_{\mathrm{disc}}$; and the latter follows because such $\iota_{\Theta}f$ can be expressed as a sum of $\iota_{\Theta'}f'$ for some $|\Theta'| \neq |\Omega|$, by Corollary 11.6.2 and Proposition 11.6.1.

Now $p^*S_w|_{\mathcal{H}_{\alpha}} = 0$ for $w \neq w_0$, as we see by a consideration of $A'_{X,\Omega}$ -character. Indeed choose (as in the discussion after (14.8)) a set $U_{\beta} \subset \widehat{A'_{X,\Theta}}$ containing the $A'_{X,\Theta}$ -support of \mathcal{H}_{α} ; then \mathcal{H}'_{α} is supported on $A'_{X,\Omega}$ -characters in w_0U_{β} , also $S_w\mathcal{H}_{\alpha}$ is supported on $A'_{X,\Omega}$ -characters in wU_{β} , and $w_0U_{\beta} \cap wU_{\beta} = \emptyset$ because of the way the sets U_{β} were chosen (discussion after (14.8)). Therefore:

$$(\iota_{\Omega} \circ p)^* \circ \iota_{\Theta}|_{\mathcal{H}_{\alpha}} = S_{w_0} \tag{14.9}$$

But $\iota_{\Theta}|_{\mathcal{H}_{\alpha}}$ and $\iota_{\Omega} \circ p$ are both isometries according to Proposition 11.7.1. In order for (14.8) to hold, then, the images $\iota_{\Theta}(\mathcal{H}_{\alpha})$ must be contained in $\iota_{\Omega}(\mathcal{H}'_{\alpha})$, and then S_{w_0} is an isometry from \mathcal{H}_{α} to \mathcal{H}'_{α} .

In particular (taking the sum over α) the image $L^2(X)_{\Theta}$ of $L^2(X_{\Theta})_{\text{disc}}$ under ι_{Θ} is contained in $L^2(X)_{\Omega}$; by symmetry, the two coincide:

$$L^2(X)_{\Theta} = L^2(X)_{\Omega}.$$

Also, \mathcal{H}'_{α} and \mathcal{H}'_{β} are mutually orthogonal if $\alpha \neq \beta$: that follows by a consideration of the $A'_{X,\Omega}$ -action, in particular using property 2(i) from Theorem 14.3.1 and the equivariance property of S_{w_0} . Finally, $\sum_{\alpha} \mathcal{H}'_{\alpha} = L^2(X_{\Omega})_{\mathrm{disc}}$: the orthogonal complement $\mathcal{K} \subset L^2(X_{\Omega})_{\mathrm{disc}}$ of all \mathcal{H}'_{α} is an $A'_{X,\Omega} \times G$ -stable space which is exactly the kernel of $S^*_{w_0}$. But it is not hard to see that the adjoint of $S_{w_0}: L^2(X_{\Theta})_{\mathrm{disc}} \to L^2(X_{\Omega})_{\mathrm{disc}}$ is $S_{w_0^{-1}}: L^2(X_{\Omega})_{\mathrm{disc}} \to L^2(X_{\Theta})_{\mathrm{disc}}$; so \mathcal{K} would belong to the kernel of $S_{w_0^{-1}}$; that contradicts the second part of Theorem 14.3.1 applied to $\mathcal{H} = \mathcal{K}$, where we choose a similarly to the discussion before (14.8) to produce a vector v with $||S_{w_0^{-1}}v|| \geq ||v||$.

Therefore, S_{w_0} gives an isometry

$$L^2(X_{\Theta})_{\mathrm{disc}} = \bigoplus \mathcal{H}_{\alpha} \to L^2(X_{\Omega})_{\mathrm{disc}} = \bigoplus \mathcal{H}'_{\alpha}.$$

We note for later use that

The
$$A'_{X,\Omega}$$
-span of $\iota_{\Omega,\mathrm{disc}}^* L^2(X)$ is dense in $L^2(X_{\Omega})_{\mathrm{disc}}$. (14.10)

Indeed the $A'_{X,\Omega}$ -span of $\iota_{\Omega,\mathrm{disc}}^*\iota_{\Theta}\mathcal{H}_{\alpha}$ must contain \mathcal{H}'_{α} .

We also saw above that the images $L^2(X)_{\Theta}$, $L^2(X)_{\Omega}$ are orthogonal if they are not associate, hence we can refine the decomposition (14.4) as:

$$L^{2}(X) = \bigoplus_{\Theta/\sim} L^{2}(X)_{\Theta}, \tag{14.11}$$

where \sim denotes the equivalence relation of being W_X -associate.

Now, with the same notation as what we have just proved:

- (1) $\iota_{\Omega}p^*\iota_{\Omega}^*$ is the identity on $\iota_{\Theta}(\mathcal{H}_{\alpha})$: As we just saw, $\iota_{\Omega} \circ p$ maps \mathcal{H}'_{α} isometrically onto a subspace containing $\iota_{\Theta}(\mathcal{H}_{\alpha})$. Therefore $\iota_{\Omega} \circ p \circ (\iota_{\Omega} \circ p)^*$ is the identity on $\iota_{\Theta}(\mathcal{H}_{\alpha})$, which implies the claim.
- (2) $\iota_{\Omega} \circ S_{w_0} = \iota_{\Theta}$. Indeed, it is enough to check this on each \mathcal{H}_{α} , and there, by (14.9),

$$\iota_{\Omega} \circ S_{w_0}|_{\mathcal{H}_{\alpha}} = \iota_{\Omega} \circ p^* \circ \iota_{\Omega}^* \circ \iota_{\Theta}|_{\mathcal{H}_{\alpha}} = \iota_{\Theta}|_{\mathcal{H}_{\alpha}}$$

by what we just showed.

(3) $S_{w'} \circ S_w = S_{w'w}$ when $w \in W_X(\Omega, \Theta)$ and $w' \in W_X(Z, \Omega)$. Indeed, write w_0 instead of w, for compatibility with our prior notation. Then by what we just showed

$$\iota_Z^* \iota_\Omega S_{w_0} |_{\mathcal{H}_\alpha} = \iota_Z^* \iota_\Theta |_{\mathcal{H}_\alpha},$$

and thus $\iota_Z^* \iota_\Omega S_{w_0} = \iota_Z^* \iota_\Theta$ as morphisms $L^2(X_\Theta)_{\mathrm{disc}} \to L^2(X_Z)_{\mathrm{disc}}$. Hence the w'w-equivariant part of $\iota_Z^* \iota_\Theta$ coincides with the w'-equivariant part of $\iota_Z^* \iota_\Omega$ composed with S_w .

Define the endomorphism of $\sum_{\Omega \sim \Theta} L^2(X_{\Omega})_{\text{disc}}$.

$$S:=\sum_{\Omega_1,\Omega_2\sim\Theta,w\in W_X(\Omega_2,\Omega_1)}S_w.$$

Note that $S^2 = c(\Theta) \cdot S$; here $c(\Theta)$ is as in the statement of Theorem 7.3.1: it is the number of chambers in $\mathfrak{a}_{X,\Theta}$, or, what is the same, the sum $\sum_{\Omega \sim \Theta} \#W_X(\Theta,\Omega)$ (the equality follows from the tiling result, just as in the discussion of §7.3).

In particular,

$$\bar{S} := \frac{S}{c(\Theta)}$$

is a projection. Its image is the " S_w -invariants" (i.e. invariants over all possible S_w between different Ω 's in this associate class).

Set $T = \bigoplus_{\Omega \sim \Theta} \iota_{\Omega, \text{disc}}$ (as an operator from $\sum_{\Omega \sim \Theta} L^2(X_{\Omega})_{\text{disc}}$ to $L^2(X)$), hence $T^* = \bigoplus_{\Omega \sim \Theta} \iota_{\Omega, \text{disc}}^*$. Then $T^*T = S$; it follows then that the image of T^* contains the image of \bar{S} , i.e. the " S_w -invariants." On the other hand, the reverse containment follows from what we have already shown. That is to say, $\bar{S}T^* = T^*$, which follows from (2) above together with the formula $c(\Theta) = \sum_{\Omega \sim \Theta} \#W_X(\Omega, \Theta)$ already noted. So the image of T^* is precisely as stated.

We have proven all statements of the theorem when the S_w 's are defined only on the discrete spectrum. We now turn to "upgrading" them so they apply to the entire spectrum.

We notice the following: S_w can be characterized as the unique $(A'_{X,\Theta}, w)$ -equivariant isometry from $L^2(X_{\Theta})_{\text{disc}}$ to $L^2(X_{\Omega})_{\text{disc}}$ which satisfies: $\iota_{\Theta, \text{disc}} = \iota_{\Omega, \text{disc}} \circ S_w$. Indeed, this condition identifies S_w^* on the image of $\iota_{\Omega, \text{disc}}^*$ and we apply (14.10).

Let now $\Theta \subset \Delta_X$, $Z_1, Z_2 \subset \Theta$ and $w \in W_{X_{\Theta}}(Z_1, Z_2) \subset W_X(Z_2, Z_1)$. In particular, we have scattering morphisms:

$$S_w, S_w^{\Theta}: L^2(X_{Z_1})_{\operatorname{disc}} \to L^2(X_{Z_2})_{\operatorname{disc}},$$

from applying the part of the theorem which is already proven to the varieties X and X_{Θ} , respectively. We claim that they coincide, i.e.

$$S_w = S_w^{\Theta}. (14.12)$$

Indeed, we have $\iota_{Z_i} = \iota_{\Theta} \circ \iota_{Z_i}^{\Theta}$ (Proposition 11.6.1), and the claim follows from the above characterization of S_w :

$$\iota_{Z_1}^{\Theta} = \iota_{Z_2}^{\Theta} \circ S_w^{\Theta} \implies \iota_{Z_1} = \iota_{Z_2} \circ S_w^{\Theta}.$$

Take now $w \in W_X(\Omega, \Theta)$. We may now define S_w in general. We have a decomposition:

$$L^2(X_\Theta) = \bigoplus_{\{Z \mid Z \subset \Theta\}/\sim} L^2(X_\Theta)_Z$$

where $L^2(X_{\Theta})_Z$ is the image of $L^2(X_Z)_{\text{disc}}$ in $L^2(X_{\Theta})$, by the map ι_Z^{Θ} .

We need, then, to define S_w on each space $L^2(X_{\Theta})_Z$. Put $Y = w(Z) \subset \Omega$. We define S_w by requiring the following diagram to commute

$$L^{2}(X_{\Theta})_{Z} \xrightarrow{S_{w}} L^{2}(X_{\Omega})_{Y}$$

$$\iota_{Z}^{\Theta,*} \downarrow \qquad \qquad \iota_{Y}^{\Omega,*} \downarrow \qquad (14.13)$$

$$L^{2}(X_{Z})_{\text{disc}} \xrightarrow{S_{w}} L^{2}(X_{Y})_{\text{disc}}$$

In fact, this diagram can be made to commute: by what we have just proven for discrete spectra (applied to X_{Θ} instead of X), the left-hand vertical arrow identifies $L^2(X_{\Theta})_Z$ with that subspace of $L^2(X_Z)_{\text{disc}}$ that is invariant by all $S_w^{\Theta}, w \in W_{X_{\Theta}}(Z, Z).$

Similarly, the right-hand vertical arrow identifies $L^2(X_{\Omega})_Y$ with that subspace of $L^2(X_Y)_{\text{disc}}$ that is invariant by all S_w^{Ω} , for $w \in W_{X_{\Omega}}(Y,Y)$. Because of the composition property for the maps S_w and (14.12), S_w maps the first space isomorphically to the second.

It remains only to verify that this does not depend on the choice of Zwithin its associate class; this is routine and we omit it.

14.4. Proof of the first part of Theorem 14.3.1. For notational simplicity, in this proof we denote $A'_{X,\Theta}$, $A'_{X,\Omega}$ simply by $A_{X,\Theta}$, $A_{X,\Omega}$.

The basic idea is to consider the inner product $\langle \iota_{\Omega} \Phi, \iota_{\Theta} \Psi \rangle$ for suitable $\Phi \in L^2(X_{\Omega}), \Psi \in L^2(X_{\Theta})$. We then "push" Φ and Ψ towards infinity using suitable elements of $A_{X,\Omega}$ and $A_{X,\Theta}$. The given assumption (after some analysis of scattering) forces this inner product to converge to zero. We then derive a contradiction by comparing with the decomposition of $\iota_{\Omega}^*\iota_{\Theta}$.

Suppose that the first property is false; that is, there exists a triple (Ω, T, f) appearing in the scattering of \mathcal{H} such that: $\mathring{\mathfrak{a}}_{X,\Theta}^+ \cap T\mathring{\mathfrak{a}}_{X,\Omega}^+ \neq \emptyset$.

This means that there is a finitely generated subsemigroup M^+ of $A_{X,\Omega}^T$ with the following properties:

- (1) $M^+ \subset \mathring{A}_{X,\Omega}^+$ and $T(M^+) \subset \mathring{A}_{X,\Theta}^+$; (2) M^+ and $A_{X,\Omega}^0$ generate $A_{X,\Omega}^T$ as a group.

We are slightly abusing notation here, since Ta does not always make sense as an element of $A_{X,\Theta}$ when $a \in A_{X,\Omega}^T$; however, it does make sense

as an element of $A_{X,\Theta}/A_{X_{\Theta}}^0$ via the valuation map and (13.3), therefore the statements above make sense. Let M be the group generated by M^+ . In order to avoid similar clarifications in the rest of the proof, let us choose a homomorphism: $M \to A_{X,\Theta}$ which lifts $T: M \to A_{X,\Theta}/A_{X,\Theta}^0$. By abuse of notation, we will be denoting this homomorphism by T again. If $\Theta = \Omega$, $T = \mathrm{Id}$, we take this homomorphism to be the identity.

Let S_j denote the summand of $\iota_{\Omega}^*\iota_{\Theta}$ corresponding to the quadruple (Ω, T, f) in the decomposition (14.1). If $\Phi \in L^2(X_{\Theta})^J_{\text{disc}}$, then the equivariance property of S_j reads:

$$S_j(T(m) * \Phi) = m * S_j \Phi, \tag{14.14}$$

for $m \in M$, where we have twisted the actions of the tori as in (13.5).

When $m \in M^+$, the fact that $M^+ \subset \mathring{A}_{X,\Omega}^+$ implies, in particular, that the mass of the function $S_j(T(m)^n * \Phi) \in L^2(X_\Omega)_{\mathrm{disc}}^J$ will be "moving towards a J-good neighborhood of infinity in X_Ω as $n \to \infty$." By the quoted phrase, we mean the following: writing $f_j = S_j(T(m)^n * \Phi)$ and N_Ω for the J-good neighbourhood of infinity, the norm $\|f_j\|_{L^2(X-N_\Omega)} \to 0$ as $j \to \infty$. This follows from the fact (in turn from Lemma 2.4.9) that for any compact set $O \subset X_\Omega$ and for $n \gg 1$, we have $m^n O \subset N_\Omega$, and then choosing O so large that the L^2 -norm of f on $X_\Omega - O$ is arbitrarily small.

Now let $\Phi \in L^2(X_{\Theta})_{\text{disc}}^J$, $\Psi \in L^2(X_{\Omega})_{\text{disc}}^J$. Let us choose Φ to be $(A_{X,\Theta}^0, \eta)$ -equivariant and Ψ to be $(A_{X,\Omega}^0, \eta')$ -equivariant, where the restrictions of η and η' to the maximal compact subgroups are those that correspond to the domain and image of f (in particular, $S_j\Phi$ is $(A_{X,\Omega}^0, \eta')$ -equivariant).

Choose an averaging sequence of measures ν_n on M^+ (§10.2) and consider the inner products:

$$P_n := \sum_{M} \nu_n(m) \left\langle \iota_{\Theta}(T(m) * \Phi), \iota_{\Omega}(m * \Psi) \right\rangle. \tag{14.15}$$

where we regard $\nu_n(m) = 0$ off M^+ . The sum is convergent: the inner products that appear are bounded independently of $m \in M^+$, by Cauchy-Schwarz and the boundedness of $\iota_{\Theta}, \iota_{\Omega}$; and $\sum \nu_n(m) = 1$. We now evaluate P_n in two different ways – (i) and (ii) below. We suppose now also that we are *not* in the case $\Omega = \Theta, T = \operatorname{Id}$ and f the identity.

(1). We show that $P_n \to 0$ as $n \to \infty$ (for suitable choice of the groups M, T(M)).

As $n \to \infty$, by the property (11.3) characterizing the Bernstein morphisms we know that for $a \in M^+$ and b = T(a), if Φ was compactly supported then $\|\iota_{\Theta}b^n * \Phi - e_{\Theta}b^n * \Phi\| \to 0$, and similarly for $a^n * \Psi$, where $e_{\Theta} : C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X)$ is the asymptotics map.

We also noted in (11.4) a certain analog of this statement for functions that are not necessarily compactly supported. That implies that, if we fix J-good neighborhoods N_{Θ} , N_{Ω} of Θ - and Ω -infinity, then for any elements in $L^2(X_{\Theta})^J$, resp. $L^2(X_{\Omega})^J$, we get that (14.15) approaches the inner product:

$$P'_{n} := \sum_{M} \nu_{n}(m) \left\langle \tau_{\Theta}(T(m) * \Phi), \tau_{\Omega}(m * \Psi) \right\rangle, \qquad (14.16)$$

Here the meaning of "approaches" is that $\lim_n (P_n - P'_n) = 0$ and τ_{Θ} and τ_{Ω} denote truncation of the given J-invariant functions, even when they are not compactly supported, to J-good neighborhoods of infinity, and identification via the exponential map with functions on X. Indeed, because each ν_n is a probability measure, it is enough to verify that the pointwise difference $\left\langle \tau_{\Theta}(T(m) * \Phi) |_{N_{\Theta}}, \tau_{\Omega}(m * \Psi) |_{N_{\Omega}} \right\rangle - \left\langle \iota_{\Theta}(T(m) * \Phi), \iota_{\Omega}(m * \Psi) \right\rangle$, restricted to the support of ν_n , approaches 0 as $n \to \infty$. In turn, that follows from a slight generalization of (11.4) (from one-parameter groups to several-parameter groups), the fact that the support of ν_n is contained deeper and deeper in the interior of M^+ (more precisely, the first noted property of an averaging sequence, see before Lemma 10.2.3), and the assumed facts that $M^+ \subset \mathring{A}_{X,\Omega}^+$ and $T(M^+) \subset \mathring{A}_{X,\Theta}^+$.

There are now two cases. For what follows, denote by a an element of M^+ , and by b its image under T. If $\Omega = \Theta$ but T is not the identity we can and do choose a so that $b \neq a$.

(i) $\Omega \neq \Theta$, or $\Omega = \Theta$ and $T \neq \mathrm{Id}$, in which case an arbitrarily large percentage of the masses of $\tau_{\Theta}b^n * \Phi|_{N_{\Theta}}$ and $\tau_{\Omega}a^n * \Psi|_{N_{\Omega}}$ is eventually concentrated on disjoint sets. In other words, for every $\varepsilon > 0$, there are subsets $N'_{\Theta} \subset N_{\Theta}$ and $N'_{\Omega} \subset N_{\Omega}$, disjoint when identified with subsets of X, with the property that

$$\|\tau_{\Theta}b^{n} * \Phi\|_{L^{2}(N'_{\Theta})} > (1 - \varepsilon)\|\tau_{\Theta}b^{n} * \Phi\|_{L^{2}(N_{\Theta})}$$
 (14.17)

and similarly for Ω :

$$\|\tau_{\Omega}a^{n} * \Psi\|_{L^{2}(N'_{\Omega})} > (1 - \varepsilon)\|\tau_{\Omega}a^{n} * \Psi\|_{L^{2}(N_{\Omega})}.$$
 (14.18)

This, in turn, comes from the following: Suppose first that $\Theta \neq \Omega$. Take J-stable compact subsets $O \subset X_{\Theta}, O' \subset X_{\Omega}$ which support a sufficiently large percentage of the L^2 -norms of Φ , resp. Ψ . Then a^nO and b^nO' become disjoint for $n \gg 1$ when identified with subsets of X. Indeed, after covering O, O' by finitely many orbits of a compact subgroup of G, it is enough to note that, taking $x \in O$ and $y \in O'$, the limit points of a^nx and b^ny , with this identification, belong to distinct G-orbits on \overline{X} . That follows from Lemma 2.4.9. Next, suppose

that $\Theta = \Omega$ but T is not the identity, so that $b \neq a$. We now use similarly the fact that, for any compact set $O \subset X_{\Theta}$, the sets $a^n O, b^n O$ are eventually disjoint, which is clear, working $A_{X,\Theta}$ -orbit by $A_{X,\Theta}$ -orbit on X_{Θ}/J .

Taking now N'_{Θ} , resp. N'_{Ω} , to be the union of all a^n - (resp. b^n -) translates of O (resp. O', when $\Omega \neq \Theta$) for all large enough n, this shows (14.17), (14.18), where we remind that the point is that N'_{Θ} and N'_{Ω} are disjoint. By the Cauchy-Schwartz inequality, this shows that the inner products averaged in (14.16) become arbitrarily small, for large n.

(ii) $\Omega = \Theta$, T = Id, but $f \neq Id$.

Let us use notation defined before (13.5), in particular the twisted * action that was defined there, which is simply the twist of the usual action by a unitary character. Let us also take a = b. The definition of the * action gives:

$$\langle \tau_{\Theta}(b^n * \Phi), \tau_{\Theta}(a^n * \Psi) \rangle = (\eta' \eta^{-1})(a^n) \cdot \langle \tau_{\Theta} \mathcal{L}_{a^n} \Phi, \tau_{\Theta} \mathcal{L}_{a^n} \Psi \rangle,$$

where η and η' are the (unitary) characters defining the twisted * action of a, as before (13.5). Since $f \neq \text{Id}$, we can choose M so that $(\eta'\eta^{-1})|_M \neq 1$. Then $P'_n \to 0$: the definition of P'_n now takes the shape

$$P'_{n} = \sum_{m \in M} \nu_{n}(m) (\eta' \eta^{-1})(m) \langle \tau_{\Theta} \mathcal{L}_{m} \Phi, \tau_{\Theta} \mathcal{L}_{m} \Psi \rangle$$
 (14.19)

and, for any $\varepsilon > 0$, we may choose n so large so the inner products appearing here are almost constant, i.e. there is a constant A such that $|\langle \tau_{\Theta} \mathcal{L}_m \Phi, \tau_{\Theta} \mathcal{L}_m \Psi \rangle - A| < \varepsilon$ whenever m belongs to the support of ν_n . (Were it not for the truncation operators, we could even take $\varepsilon = 0$, i.e. the inner products would be exactly constant.) Thus, $|P'_n| \leq \varepsilon + \sum_M \nu_n(m)(\eta'\eta^{-1})(m)$. The property of averaging sequences that $S_n := \sum_M \nu_n(m)\chi(m)$ approaches zero as $n \to \infty$ for any nontrivial character χ : to see this, choose $s \in M$ with $\chi(s) \neq 1$ and observe that $(\chi(s)S_n - S_n) = \sum_M (\nu_n(ms^{-1}) - \nu_n(m))\chi(m)$, and the latter sum approaches zero as $n \to \infty$ by the second property of an averaging sequence (see §10.2).

(2) $P_n \to \langle S_j \Phi, \Psi \rangle$.

Each term of the sum (14.15) is equal to:

$$\langle \iota_{\Omega}^* \iota_{\Theta}(T(m) * \Phi), m * \Psi \rangle = \sum_i \langle S_i(T(m) * \Phi), m * \Psi \rangle.$$

We now show that, upon applying the average (14.15) to this expression, only the (T, f)-equivariant summand survives, that is, S_j .

First of all, the $A_{X,\Theta}^0$ and $A_{X,\Omega}^0$ -equivariance properties of Φ and Ψ kill all summands S_i such that the domain or image of f_i is different from that of f.

Next, for those S_i with $T_i \neq T$ one sees by a similar argument to those already given that the inner product $\langle S_i(T(m) * \Phi), m * \Psi \rangle$ eventually approaches zero. Let $U = T_i^{-1}T$, which is defined as a nontrivial isogeny $\mathbf{A}_{X,\Omega} \dashrightarrow \mathbf{A}_{X,\Omega}$. If m lies in the subgroup of M such that U(m) is defined, then $\langle S_i(T(m) * \Phi), m * \Psi \rangle$ has the same absolute value as $\langle \mathcal{L}_{U(m)}S_i\Phi, \mathcal{L}_m\Psi \rangle$, where we no longer twist the actions (twisting only affects the result by a scalar of absolute value 1). Note that $U \neq 1$, and we now reason in exactly the same as at the $\Theta = \Omega$ case after (14.17) on page 194.

Finally, if $T_i = T$, the associated affine map f_i of character groups differs from the affine map f for T simply by multiplication by a character η of $A_{X,\Omega}$, i.e. $f_i(\chi) = f(\chi)\eta$. Then we have, by (13.5),

$$S_i(T(m) * \Phi) = \eta(m)m * S_i(\Phi).$$

By the argument presented after⁶⁴ (14.19) on page 195 the weighted average of terms $\langle S_i T(m) * \Phi, m * \Psi \rangle$ will therefore tend to zero, unless $\eta = 1$ which implies that $(T_i, f_i) = (T, f)$, i.e. S_i is the summand S_i .

But we now see that

$$P_n \longrightarrow \langle S_i \Phi, \Psi \rangle \quad (n \to \infty)$$

because $\langle S_j(T(m) * \Phi), m * \Psi \rangle = \langle S_j \Phi, \Psi \rangle$ by (14.14).

Taking (i) and (ii) together: $\langle S_j \Phi, \Psi \rangle = 0$ for all Φ, Ψ , which is impossible unless $S_j \equiv 0$.

The second part will require a further ingredient: So far, we have not used the fact that the complement of all the neighborhoods of ∞ is actually compact modulo the center. Roughly speaking, we will show that if the second part of the theorem were not the true then we would be able to "push" any function $\Phi \in L^2(X)_{\Theta}$ away from infinity, without changing its L^2 norm and keeping its L^∞ norm under control, which will lead to a contradiction. The argument is delicate, and we complete it in the remaining part of this section.

14.5. Estimates. In this section we fix an open compact subgroup J of G and develop an estimate for the norm on $L^2(X)^J$ in terms of certain norms for the "constant terms" $\iota_{\Omega}^*(\Phi)$ (for all $\Omega \subset \Delta_X$). The only input that we use is the uniform bound of subunitary exponents for $L^2(X)^J$, Proposition 9.4.8, which allows us to apply some results of the "linear algebra" section 10.

We fix, for each $\Omega \subset \Delta_X$, a J-good neighborhood \tilde{N}_{Ω} of Ω -infinity which is stable under $A_{X,\Omega}^+$ (when \tilde{N}_{Ω}/J is identified with a subset of X_{Ω}/J). Let $N_{\Omega} = \tilde{N}_{\Omega} \setminus \bigcup_{\Theta \subseteq \Omega} \tilde{N}_{\Theta}$; then the sets N_{Ω} partition X, and N_{Ω} is represented

 $^{^{64}}$ Our situation is even simpler now, because there are no truncations involved. All that is needed is the final statement of that argument involving S_n .

by a finite number of J-orbits modulo the action of $A_{X,\Omega}^+$. (Indeed, \tilde{N}_{Ω} is the intersection with X of an actual neighborhood \tilde{N}'_{Ω} of ∞_{Ω} in a wonderful compactification \tilde{X} , and if we remove the corresponding neighborhoods for $\Theta \subseteq \Omega$ then the remaining compact subset of \tilde{N}'_{Ω} only intersects ∞_{Ω} along the orbit (or orbits) corresponding to Ω -infinity.)

We denote by τ_{Ω} the operator "truncation to N_{Ω} " (which can be considered as an operator on both $C(X)^J$ and $C(X_{\Omega})^J$). We feel are free to identify functions on N_{Ω}/J as functions on both X/J and X_{Ω}/J , hence expressions of the form $\|\Phi - \tau_{\Omega} \iota_{\Omega}^* \Phi\|_{L^2(X)}$ will make sense for $\Phi \in L^2(X)$.

14.5.1. Bounding the $L^2(X)$ -norm in terms of the asymptotics: the rank one case. Let us first discuss a toy case, namely assume that X is a spherical variety of rank one with $\mathcal{Z}(X)$ trivial. Fix an open compact $J \subset G$ and a J-good neighborhood N_{Θ} of infinity (there is only one nontrivial direction to infinity, which we will denote by Θ), and denote by τ_{Θ} the "truncation" to this neighborhood. Then we claim:

14.5.2. LEMMA. There is a finite set of elements $v_i \in L^2(X)^J$ and a constant C such that, for $\Phi \in L^2(X)^J$ we have:

$$\|\Phi - \tau_{\Theta} \iota_{\Theta}^* \Phi\|_{L^2(X)^J} \le C \sum_i \|\Phi\|_{L^1_{v_i}}.$$
 (14.20)

Recall that the norms that appear in the final term have been defined in $\S12.2.1$.

PROOF. Let Φ_{π} denote the image of $\Phi \in C_c^{\infty}(X)$ in \mathcal{H}_{π} . Fix a Plancherel measure and recall that by $\Phi^{\pi}(x)$ we denote the pairing of Φ_{π} with the characteristic measure of xJ with respect to the corresponding Plancherel form (see Remark 11.2.1).

On N_{Θ} the difference $\Phi - \iota_{\Theta}^* \Phi$ can be expressed pointwise in terms of their spectral decomposition :

$$(\Phi - \iota_{\Theta}^* \Phi)(x) = \int_{\hat{G}} (\Phi^{\pi}(x) - (\iota_{\Theta}^* \Phi)^{\pi}(x)) \mu(\pi),$$

Note that, on the right hand side, we use the identification of J-orbits on X and X_{Θ} to make sense of $\Phi^{\pi}(x)$ and $(\iota_{\Theta}^*\Phi)^{\pi}(x)$ simultaneously. We allow ourselves to abbreviate $(\iota_{\Theta}^*\Phi)^{\pi}$ to $\iota_{\Theta}^*\Phi^{\pi}$ in what follows, to avoid a plethora of bracketing.

Hence:

$$\|\Phi - \iota_{\Theta}^* \Phi\|_{L^2(N_{\Theta})}^2 = \int_{N_{\Theta}} \left| \int_{\hat{G}} (\Phi^{\pi}(x) - \iota_{\Theta}^* \Phi^{\pi}(x)) \mu(\pi) \right|^2 dx.$$

By the asymptotics, $\Phi^{\pi}|_{N_{\Theta}}$ is equal to $e_{\Theta}^{*}\Phi^{\pi}|_{N_{\Theta}}$, and the difference $e_{\Theta}^{*}\Phi_{\pi}(x) - \iota_{\Theta}^{*}\Phi_{\pi}(x)$ can be expressed as a sum of (uniformly, as in Proposition 9.4.8) subunitary exponents – see Remark 11.4.1. Therefore, by Lemma 10.2.5, there is a L^{2} function (see below) Ω on N_{Θ} and a finite set of points

 x_i on N_{Θ} such that for every π, x we have:

$$|\Phi^{\pi}(x) - \iota_{\Theta}^* \Phi^{\pi}(x)| \le \Omega(x) \sum_{i} |\Phi^{\pi}(x_i)|.$$

Note that the important feature of N_{Θ}/J that is used here is that (considered as a subset of X_{Θ}/J): it is $A_{X,\Theta}^+$ -stable and consists of a finite number of $A_{X,\Theta}^+$ -orbits. One applies Lemma 10.2.5 by pulling back to each copy of $A_{X,\Theta}^+$, i.e. apply it to the function $a \cdot e_{\Theta}^* \Phi^{\pi}(x_0)$ for fixed $x_0 \in N_{\Theta}$. Note again that the action of a includes a twist by the square root of the $A_{X,\Theta}$ -eigenmeasure. In particular, the function $\Omega(x)$ actually lies in L^2 , because its exponents in the $A_{X,\Theta}^+$ -direction are subunitary for this twisted action of $A_{X,\Theta}$.

Therefore:

$$\int_{N_{\Theta}} \left| \int_{\hat{G}} (\Phi^{\pi}(x) - \iota_{\Theta}^* \Phi^{\pi}(x)) \mu(\pi) \right|^2 dx \le \|\Omega\|_{L^2(N_{\Theta})}^2 \left(\int_{\hat{G}} \sum_{i} |\Phi^{\pi}(x_i)| \mu(\pi) \right)^2$$

and hence:

$$\|\Phi - \iota_{\Theta}^* \Phi\|_{L^2(N_{\Theta})} \ll \sum_i \int_{\hat{G}} |\Phi^{\pi}(x_i)| \mu(\pi).$$
 (14.21)

If we set v_i =the characteristic function of x_iJ then the integrals appearing on the right hand side are precisely the norms $\|\Phi\|_{L^1_{v_i}}$. We complement this set of v_i 's with the characteristic functions of the J-orbits on the complement of N_{Θ} (there are only finitely many such since we are in the rank one case with $\mathcal{Z}(X)=1$), and then the statement of the lemma is true. \square

Now we allow $\mathcal{Z}(\mathbf{X})$ to be non-trivial, but keeping the rank of X equal to one. We need to modify the statement of the lemma according to the morphism: $\hat{G} \to \widehat{\mathcal{Z}(G)^0}$. To simplify notation, since Plancherel measure is supported in the preimage of $\widehat{\mathcal{Z}(X)} \subset \widehat{\mathcal{Z}(G)^0}$, we feel free to write maps: $\hat{G} \to \widehat{\mathcal{Z}(X)}$, while we should be replacing \hat{G} by the preimage of $\widehat{\mathcal{Z}(X)}$. As we have been doing until now, we fix a Haar measure on $\widehat{\mathcal{Z}(X)}$, which will be used as the Plancherel measure in the definition of the relative norms that appear in the following lemmas and Proposition 14.5.6; this is allowable because clearly the Plancherel measure for X as a $\mathcal{Z}(X)$ -representation lies in the measure class of the Haar measure on $\widehat{\mathcal{Z}(X)}$. (These norms were defined in §12.2.3 and they now depend on the choice of Plancherel measure on $\widehat{\mathcal{Z}(X)}$.)

14.5.3. LEMMA. There is a finite set of elements $v_i \in L^2(X)^J$ such that on $L^2(X)^J$ we have:

$$\|\Phi - \tau_{\Theta} \iota_{\Theta}^* \Phi\|_{L^2(X)^J} \ll \sum_i \|\Phi\|_{\mathcal{Z}(X), v_i}.$$
 (14.22)

PROOF. The proof is like before, except that now we will estimate the norm of $\Phi - \iota_{\Theta}^* \Phi$ on N_{Θ} by first integrating over the action of $\mathcal{Z}(X)$ and then applying the above arguments:

$$\|\Phi - \iota_{\Theta}^* \Phi\|_{L^2(N_{\Theta})}^2 = \int_{\widehat{\mathcal{Z}(X)}} \|\Phi_{\chi} - \iota_{\Theta}^* \Phi\|_{L^2(\mathcal{Z}(X) \setminus N_{\Theta}, \chi)}^2 d\chi.$$

Just as in (14.21), there is a finite set of J-orbits on N_{Θ} (independent of χ) such that $\|\Phi_{\chi} - \iota_{\Theta}^* \Phi\|_{L^2(\mathcal{Z}(X) \setminus N_{\Theta}, \chi)}$ is bounded by a constant times:

$$\sum_{i} \int_{\widehat{G}_{\chi}} |\Phi^{\pi}(x_i)| \mu_{\chi}(\pi),$$

where \widehat{G}_{χ} denotes the fiber of \widehat{G} over χ , and μ_{χ} is the corresponding Plancherel measure on this fiber. Note that Lemma 10.2.5 can be applied uniformly in χ , because it depends only on an upper bound for subunitary exponents (the constant c in that Lemma) and such a bound indeed follows from Proposition 9.4.8.

Therefore:

$$\|\Phi - \iota_{\Theta}^* \Phi\|_{L^2(N_{\Theta})}^2 \ll \int_{\widehat{\mathcal{Z}(X)}} \left(\sum_i \int_{\widehat{G}_{\chi}} |\Phi_{\pi}(x_i)| \mu_{\chi}(\pi) \right)^2 d\chi$$
$$\ll \sum_i \int_{\widehat{\mathcal{Z}(X)}} \left(\int_{\widehat{G}_{\chi}} |\Phi_{\pi}(x_i)| \mu_{\chi}(\pi) \right)^2 d\chi.$$

If we set v_i =the characteristic function of x_iJ then the sum on the right hand side is:

$$\sum_{i} \|\Phi\|_{\widehat{\mathcal{Z}(X)}, v_i}^2.$$

and we have thereby shown:

$$\|\tau_{\Theta}\Phi - \tau_{\Theta}\iota_{\Theta}^*\Phi\|_{L^2(N_{\Theta})} \ll \sum_{i} \|\Phi\|_{\widehat{\mathcal{Z}(X)}, v_i}$$
 (14.23)

Complementing this set of v_i 's by the characteristic functions of a finite set of J-orbits representing all J-orbits in $(X \setminus N_{\Theta})/\mathcal{Z}(X)$, we are done. \square

14.5.4. Bounding the $L^2(X)$ -norm in terms of the asymptotics: the general case. Let **X** be a spherical variety of arbitrary rank now, $J \subset G$ an open compact subgroup.

The analog of (14.23) requires slightly more involved combinatorics because of the possibility of exponents being unitary along a wall.

For (almost all) $\pi \in \hat{G}$ we have a decomposition of the asymptotics $e_{\Theta}^* \Phi^{\pi}$ in the Θ -direction, into (generalized) $A_{X,\Theta}$ -eigencharacters, each of which is either unitary or $A_{X,\Theta}^+$ -subunitary. To any such character χ we attach the subset $\Omega_{\chi} \supset \Theta$ corresponding to the largest "face" of $A_{X,\Theta}^+$ where the restriction of χ is unitary; for example, if χ is unitary then $\Omega_{\chi} = \Theta$, while

if χ is strictly subunitary then $\Omega_{\chi} = \Delta_X$. (Notice that $\chi|_{\mathcal{Z}(X)}$ is necessarily unitary.) Accordingly, we have a decomposition:

$$\Phi^{\pi}|_{N_{\Theta}} = \sum_{\Omega \supset \Theta} \Phi^{\pi,\Omega},$$

where $\Phi^{\pi,\Omega}$ contains the summands with generalized eigencharacter χ such that $\Omega_{\chi} = \Omega$, and integrating over \hat{G} :

$$\Phi|_{N_{\Theta}} = \sum_{\Omega \supset \Theta} \Phi^{\Omega}. \tag{14.24}$$

Moreover, we have

$$(\Phi - \iota_{\Theta}^* \Phi)_{N_{\Theta}} = \sum_{\Omega \supseteq \Theta} \Phi^{\Omega}. \tag{14.25}$$

Now let us extend these notions to all of X at once:

Note (recall the definitions at the start of §14.5) that $\tilde{N_{\Theta}} = \bigcup_{\Omega \subset \Theta} N_{\Omega}$ and the quantity Φ^{Θ} is defined on each N_{Ω} . Thus we may regard Φ^{Θ} as being defined on all of $\tilde{N_{\Theta}}$ and will extend it by zero off $\tilde{N_{\Theta}}$. With this convention, $\Phi = \sum_{\Omega} \Phi^{\Omega}$ on X, and when we restrict to any N_{Θ} only those terms with $\Theta \subset \Omega$ are nonvanishing.

14.5.5. LEMMA. There are a finite number of elements $v_i \in L^2(N_{\Theta})^J$ and a decaying function Q on N_{Θ} such that, for each $\Phi \in C_c^{\infty}(X)^J$, we have

$$|\Phi^{\Delta_X}(x)|^2 \le Q(x) \sum_i \|\Phi\|_{\widehat{\mathcal{Z}(X)}, v_i}^2$$

for $x \in N_{\Theta}$.

In particular, for a suitable constant C

$$\|\Phi^{\Delta_X}\|_{L^2(X)} \le C \sum_i \|\Phi\|_{\widehat{\mathcal{Z}(X)}, v_i}^2$$
 (14.26)

for each such Φ .

Here the meaning of "decaying" for a function Q on N_{Θ} is that the associated function $a \cdot Q(x_0)$ is decaying for $a \in A_{X,\Theta}^+$ and fixed $x_0 \in N_{\Theta}$; note that the action of a is twisted, as always, by the square root of the eigenmeasure.

PROOF. The second estimate (14.26) follows immediately from the first: Compute first the norm on N_{Θ} , and note that a decaying function is square-integrable; then sum over Θ .

The first statement will be proved as in Lemma 14.5.2 and Lemma 14.5.3, but we will replace Lemma 10.2.5 by a slight modification.

Let $J' \subset A_{X,\Theta}$ be an open compact subgroup acting trivially on J-invariant functions on X_{Θ} . Let f be the function on $A_{X,\Theta}^+/J'$ whose value at a is given by evaluating $a \cdot e_{\Theta}^* \Phi^{\pi}$ at any fixed $x_0 \in N_{\Theta}$. Let $S = A_{X,\Theta}/J'$, S^+ the positive cone corresponding to $A_{X,\Theta}^+$. Thus, f is a finite function on S whose degree is absolutely bounded above (see quoted statement from

page 125). Let $f^{<}$ be the projection of f onto all exponents χ for S which are *strictly subunitary* on S^{+} (see §10.1 for definition).

What is needed to adapt the previous statements is precisely:

Each point evaluation of $f^{<}$ is bounded by a finite sum of point evaluations of f, and the constants appearing in this bound can be taken to depend only on the degree n of f as an S-finite function.

However, the projection can be effected by a countour integral, exactly as in in the proof of Lemma 10.2.5. Let us go through that proof with c=1; as in that proof, we obtain a vector space $F(\mathbf{P})$ and an evaluation map $\mathrm{ev}: F(\mathbf{P}) \to \mathbb{C}^{\Lambda}$, together with endomorphisms A_1, \ldots, A_t of $F(\mathbf{P})$, such that

$$f(t_1,\ldots,t_k) = \left(A_1^{t_1}\ldots A_k^{t_k}\operatorname{ev}(f)\right)(\mathbf{0}).$$

Now choose $\delta > 0$ so that any eigenvalue λ_i of any A_i that satisfies $|\lambda_i| < 1$ actually satisfies $|\lambda_i| < 1 - 2\delta$. In our context, such a bound exists because of Proposition 9.4.8, which shows there are only a finite possible number of exponents which can appear for J-invariant functions. Now set $P_i = \int_{|z|=1-\delta} \frac{dz}{z-A_i}$; it gives a projection onto all eigenvalues of A_i which are less than 1 in absolute value. Then $P_1P_2\dots P_t$ furnishes the desired projection to subunitary exponents, and just as in the proof of Lemma 10.2.5 its norm is absolutely bounded in terms of n.

This implies a similar estimate for arbitrary Ω :

$$\|\Phi^{\Omega}\|_{L^{2}(X)} \ll \sum_{i} \|\iota_{\Omega}^{*}\Phi\|_{\widehat{\mathcal{Z}(X_{\Omega})}, v_{i}}.$$
(14.27)

To see this, we just identify \tilde{N}_{Ω} to a subset of X_{Ω} and therefore identify Φ^{Ω} to a function on X_{Ω} . This function is precisely $(\iota_{\Omega}^*\Phi)^{\Omega}$, restricted to \tilde{N}_{Ω} .

Recall our notational convention about relative norms, defined before Lemma 12.2.8. For every $v \in L^2(X_{\Omega})$ we can consider the "relative" norm:

$$\| \bullet \|_{\mathcal{Z}(X_{\Omega}),v}$$

on $L^2(X_{\Omega})$. In what follows, we will use these norms for v=the characteristic function of some J-orbit in $N_{\Omega'}$, $\Omega' \subset \Omega$, hence identified with a function on X_{Ω} .

14.5.6. PROPOSITION. Let $\Phi \in L^2(X)_r^J$ (the image of all $L^2(X_{\Theta})_{\mathrm{disc}}^J$ with $|\Theta| = r$). Then there is a constant C and a finite list of vectors $v_{\Omega,i} \in L^2(X_{\Omega})$ such that

$$\|\Phi\| \le \|\sum_{|\Omega|=r} \tilde{\tau_{\Omega}} \iota_{\Omega}^* \Phi\|_{L^2(\tilde{N}_{\Omega})^J} + C \sum_{|\Omega|>r} \sum_i \|\iota_{\Omega}^* \Phi\|_{\mathcal{Z}(X_{\Omega}), \nu_{\Omega, i}}$$
(14.28)

where the meaning of $\tilde{\tau}_{\Omega}$ is "restriction to \tilde{N}_{Ω} and consider \tilde{N}_{Ω} as a subset of X." (Thus, $\tilde{\tau}_{\Omega}\iota_{\Omega}^*\Phi$ is a function on X, and the Ω -sum is taken on X.)

PROOF. In particular, note that $\Phi^{\Omega} = 0$ if $|\Omega| < r$, because $\iota_{\Omega}^* \Phi = 0$. By the triangle inequality

$$\|\Phi\|_{L^2(X)} \le \|\sum_{\Omega=r} \Phi^{\Omega}\|_{L^2(X)} + \sum_{\Omega>r} \|\Phi^{\Omega}\|_{L^2(X)}.$$

and apply (14.27) to all terms with $|\Omega| > r$. And (tracking through the notation) $\sum_{|\Omega|=r} \Phi^{\Omega}$ is the same as $\sum \tilde{\tau}_{\Omega} \iota_{\Omega}^* \Phi$.

14.6. Proof of the second part of Theorem 14.3.1. Again, for notational simplicity, we denote $A'_{X,\Theta}$, $A'_{X,\Omega}$ simply by $A_{X,\Theta}$, $A_{X,\Omega}$.

Fix a $\Theta \subset \Delta_X$. We may partition $\mathcal{Z}(X_{\Theta})$ into an almost disjoint (i.e. disjoint up to a set of measure zero) union of measurable subsets Y_{β} with the following properties:

- (A) If (T_i, f_i) are the isogenies and affine maps of the decomposition (14.1) for the map $\iota_{\Theta}^* \circ \iota_{\Theta}$, then $Y_{\beta} \cap f_i(Y_{\beta})$ is of measure zero unless T_i is the identity and f_i is also the identity.
- (B) Each Y_{β} is a subset of a single connected component of $\widehat{A_{X,\Theta}}$.

To carry this out, we proceed one connected component at a time. First note that the set of $\chi \in \widehat{A_{X,\Theta}}$ in this component fixed by some f_i is a closed set of measure 0 – it is, in suitable coordinates, a union of translates of sub-tori. Now choose a countable dense set $\{P_i\}$ in the complement of the fixed locus, and for each P_i let B_i be an open ball around P_i that is disjoint from each $f_i(B_i)$. Now take for our partition $B_1, B_2 - B_1, B_3 - (B_1 \cup B_2)$ and so on. In our application, where the isogenies arise from a Weyl group, one can easily in fact give an explicit choice of Y_{β} using positive chambers.

Applying the corresponding idempotents $1_{Y_{\beta}}$ gives rise to a direct sum decomposition of \mathcal{H} into $A_{X,\Theta} \times G$ -stable subspaces for which the assumptions of Proposition 11.7.1 – and, indeed, the stronger assumptions of Remark 11.7.2 – are satisfied. Namely, denote by \mathcal{H}' any one of the resulting $A_{X,\Theta} \times G$ -stable summands of \mathcal{H} , corresponding to the image of $1_{Y_{\beta}}$. Decompose $\mathcal{H}' = \int_{\pi} \mathcal{H}'_{\pi} d\pi$ as G-representation. We need to verify that for almost all π appearing in this decomposition, there is a unique $A_{X,\Theta}$ -exponent on \mathcal{H}'_{π} . After all, choose one such exponent χ , belonging to Y_{β} say; then Proposition 13.2.1 shows that any other exponent must be of the form $f_i(\chi)$ for some (T_i, f_i) , but we know by virtue of assumption (A) above that $f_i(\chi) \notin Y_{\beta}$ unless $f_i(\chi) = \chi$. So χ is actually the unique $A_{X,\Theta}$ -exponent of \mathcal{H}_{π} .

Hence ι_{Θ} is an isometry onto the image when restricted to each of these summands.

Replacing \mathcal{H} by one of these summands, we may suppose that ι_{Θ} is an isometry on our given subspace $\mathcal{H} \subset L^2(X_{\Theta})$. For any $a \in A_{X,\Theta}$, and $v \in L^2(X_{\Theta})^J_{\text{disc}}$ – for some fixed open compact subgroup J – Proposition 14.5.6 applied to the function $\Phi = \iota_{\Theta} \mathcal{L}_{a^n} v$ yields:

$$||v|| = ||\mathcal{L}_{a^n}v|| = ||\iota_{\Theta}\mathcal{L}_{a^n}v||$$

$$\leq \left(\| \sum_{|\Omega|=r} \tilde{\tau}_{\Omega} \iota_{\Omega}^* \iota_{\Theta} \mathcal{L}_{a^n} v \| + C \cdot \sum_{|\Omega|>r} \sum_{j} \| \iota_{\Omega}^* \iota_{\Theta} \mathcal{L}_{a^n} v \|_{\mathcal{Z}(X_{\Omega}), v_{\Omega, j}} \right),$$

where C denotes the implicit constant of Proposition 14.5.6. The relative norms that appear on the right hand side will depend, of course, on J.

Decomposing $\iota_{\Omega}^* \iota_{\Theta}$, for $|\Omega| = r$, according to (14.1), we see that:

$$\lim_{n\to\infty} \|\tilde{\tau}_{\Omega} S_i \mathcal{L}_{a^n} v\| = 0$$

unless $\operatorname{val}(a) \in T_i \mathfrak{a}_{X,\Omega}^+$. Here we denoted by val the natural "valuation" map:

$$A_{X,\Theta} \to \mathcal{X}(\mathbf{A}_{X,\Theta})^* \subset \mathfrak{a}_{X,\Theta}.$$

Hence we get:

$$||v|| \leq \liminf_{n \to \infty} \left(||\sum_{|\Omega| = r, i: T_i \mathfrak{a}_{X,\Omega}^+ \ni \text{val}(a)} \tilde{\tau}_{\Omega} S_i \mathcal{L}_{a^n} v|| + C \cdot \sum_{|\Omega| > r} \sum_{j} ||\iota_{\Omega}^* \iota_{\Theta} \mathcal{L}_{a^n} v||_{\mathcal{Z}(X_{\Omega}), \nu_{\Omega, j}} \right).$$

$$(14.29)$$

We emphasize that the summations over i and j are finite, because the sums in both Proposition 14.5.6 and (14.1) are finite.

Now assume a to be "generic" in the sense that $\operatorname{val}(a) \in T_i \mathfrak{a}_{X,\Omega}^+$ implies $\operatorname{val}(a) \in T_i \mathfrak{a}_{X,\Omega}^+$. As in the proof of the first part of Theorem 14.3.1 (§14.4), it is easy to see that the different summands $\tilde{\tau}_{\Omega} S_i \mathcal{L}_{a^n} v$ become orthogonal in the limit $n \to \infty$, i.e. for any given $\epsilon > 0$ there is an N > 0 such that for all $n \geq N$ and any distinct indices i, j we have:

$$|\langle \tilde{\tau}_{\Omega_i} S_i \mathcal{L}_{a^n} v, \tilde{\tau}_{\Omega_j} S_j \mathcal{L}_{a^n} v \rangle| < \epsilon.$$

Indeed, let (Ω_i, T_i, f_i) (with $|\Omega_i| = r$) be the triples corresponding to the morphisms S_i appearing in the first sum of (14.29). In what follows, we will say that "almost all the mass of a function f is concentrated in a set M" if the square of the L^2 -norm of f restricted to the complement of f is less than a certain multiple of f. (We omit the straightforward task of specifying which multiple is needed.)

For two distinct indices i, j we have the following possibilities:

(1) $\Omega_i \neq \Omega_j$. In this case, let $K_i \subset \infty_{\Omega_i}$, $K_j \subset \infty_{\Omega_j}$ be compact subsets such that almost all the mass of $S_i v$ is concentrated in the preimage of K_i under the quotient map $X_{\Omega_i} \to \infty_{\Omega_i}$ (and similarly

$$\|\tilde{\tau}_{\Omega}^* \mathcal{L}_{b_i^n} S_i f\| \to 0$$

when $b \notin A_{X,\Omega}^+$; in coordinates this amounts to the following: Suppose that $n = n_1 + n_2$. Then, for $g \in L^2(\mathbb{Z}^n)$ and $b = (b_1, \ldots, b_n)$ where some $b_i < 0$ for $1 \le i \le n_1$, we have then

$$\|\text{translate of } g \text{ by } m \cdot b\|_{L^2(\mathbb{N}^{n_1} \times \mathbb{Z}^{n_2})} \stackrel{m \to \infty}{\longrightarrow} 0.$$

⁶⁵Suppose – for simplicity, the modifications in general are not difficult – that the image of a in $A_{X,\Theta}/A_{X,\Theta}^0$ coincides with Tb, where $b \in A_{X,\Omega}^T$ but $b \notin A_{X,\Omega}^+$. By (13.5) it is enough to show that

- for K_j). Recall that ∞_{Ω} denotes " Ω -infinity", i.e. the union of orbits in a toroidal compactification which correspond to Ω . There are neighborhoods of K_i , K_j in X which are disjoint, and for n large enough almost all the mass of $S_i\mathcal{L}_{a^n}v$, $S_j\mathcal{L}_{a^n}v$ is concentrated in the respective neighborhoods.
- (2) $\Omega_i = \Omega_j = \Omega$ but $T_i \neq T_j$. By choosing a section of the quotient map $X_{\Omega} \to \infty_{\Omega}$, we can find a compact subset M of the image of that section and a compact subset N of $A_{X,\Omega}$ such that almost all the mass of $S_i v$ and $S_j v$ is concentrated on $N \cdot M$. Then, for n large enough⁶⁶ the sets $T_i^{-1}a^nN$ and $T_j^{-1}a^nN$ are disjoint; therefore, almost all of the mass of $S_i \mathcal{L}_{a^n} v$, $S_j \mathcal{L}_{a^n} v$ is concentrated on the disjoint sets $T_i^{-1}a^nN \cdot M$, $T_j^{-1}a^nN \cdot M$, respectively.
- (3) $\Omega_i = \Omega_j = \Omega$ and $T_i = T_j$ but $f_i \neq f_j$. Then, by property (A), the $A_{X,\Omega}$ -Plancherel supports of $S_i \mathcal{L}_{a^n} v$ and $S_j \mathcal{L}_{a^n} v$ intersect at a set of Plancherel measure zero, and hence these vectors are orthogonal for every n.

We have thus established the "orthogonality in the limit". Notice also that, in the limit, all the mass of $S_i\mathcal{L}_{a^n}v$ is concentrated in \tilde{N}_{Ω_i} , so we can get rid of the restriction operators $\tilde{\tau}_{\Omega}$. Moreover, using the equivariance property (13.5) (by replacing, if necessary, the element a by a suitable power a^k so that it is of the form $T_i(a')$ in the notation of (13.5), for all i), and the fact that the $L^2(X_{\Omega})$ -norm is $A_{X,\Omega}$ -invariant we can get rid of \mathcal{L}_{a^n} . Therefore, we get:

$$||v|| \leq \left(\sum_{|\Omega|=r,i:\ T_i\mathring{\mathfrak{a}}_{X,\Omega}^+\ni \mathrm{val}(a)} ||S_iv||^2\right)^{\frac{1}{2}} + O\left(\liminf_{n\to\infty} \sum_{|\Omega|>r} \sum_i ||\iota_{\Omega}^*\iota_{\Theta}\mathcal{L}_{a^n}v||_{\mathcal{Z}(X_{\Omega}),v_{\Omega,i}}\right).$$

$$(14.30)$$

To complete the proof, consider the $G \times A_{X,\Theta}$ -invariant Hermitian norm:

$$H(v) := \sum_{|\Omega|=r, i: \ T_i \mathring{\mathfrak{a}}_{X,\Omega}^+ \ni \mathrm{val}(a)} ||S_i v||^2.$$

It is $A_{X,\Theta} \times G$ -invariant, and also absolutely continuous with respect to $\|\cdot\|_{L^2}$, since the S_i are bounded morphisms. By spectral theory, if $H(v) < \|v\|$ for some nonzero v, we may find an $A_{X,\Theta} \times G$ -invariant space $\mathcal{H}' \subset \mathcal{H}$ and $\delta > 0$ with the property that $H(v) < (1 - \delta)\|v\|^2$ on \mathcal{H}' .

⁶⁶Here we are slightly abusing notation and treating the isogenies T_i^{-1} , T_j^{-1} as actual morphisms. In reality, $T_i^{-1}a^n$ is well-defined only for n in a finite-index subgroup of \mathbb{Z} . To be rigorous, enlarge the set N so that $N \cdot M$ also contains most of the mass of $S_i \mathcal{L}_{a^n} v$, $S_j \mathcal{L}_{a^n} v$ for n in a set of representatives of the cosets of this subgroup.

Thus on \mathcal{H}'^J the Hilbert norm $\|v\|$ is bounded by a finite sum of relative norms:

$$||v|| \le C' \liminf_{n \to \infty} \sum_{|\Omega| > r} \sum_{i} ||\iota_{\Omega}^* \iota_{\Theta} \mathcal{L}_{a^n} v||_{\mathcal{Z}(X_{\Omega}), v_{\Omega, i}}, \quad v \in \mathcal{H}'^J.$$

Decomposing $\iota_{\Omega}^*\iota_{\Theta}$ with respect to Proposition 13.3.1, and denoting the summands by $S_{\Omega,j}$, we get:

$$||v|| \le C' \liminf_{n \to \infty} \sum_{|\Omega| > r} \sum_{i,j} ||S_{\Omega,j} \mathcal{L}_{a^n} v||_{\mathcal{Z}(X_{\Omega}), v_{\Omega,i}}, \quad v \in \mathcal{H}'^J.$$

By the equivariance property⁶⁷ of $S_{\Omega,j}$, and Lemma 12.2.11 (which we can apply because, by choice, we are in the "unique exponent" situation of Remark 11.7.2) we get a contradiction to Lemma 12.2.10.

15. Explicit Plancherel formula

In this section we assume that X is strongly factorizable, cf. $\S 9.4.1$, for example: a symmetric variety.

Since the formalism of this section may appear quite involved, we begin by a rough description of its thrust:

We wish to write a formula for the "smooth" and "unitary" asymptotics maps: e_{Θ} and ι_{Θ} . As we have explained in §2.8, the varieties \mathbf{X} and \mathbf{X}_{Θ} look quite different, but there is a canonical identification of their varieties of horocycles \mathbf{X}_{Θ}^h . Our assertion is that the maps e_{Θ} and ι_{Θ} can be obtained by a suitable "interpretation" of the diagram

functions on
$$X \to \text{functions on } X_{\Theta}^h \leftarrow \text{functions on } X_{\Theta}$$
.

where the arrows are obtained by Radon transform. (By "interpretation", we mean, roughly speaking, disintegrating the arrows spectrally and making sense of convergence issues.)

If one carries out the analogs of our constructions in the case where X is a real symmetric space, we arrive at the theory of "Eisenstein integrals" developed by van den Ban, Schlichtkrull, and Delorme [vdBS05a, vdBS05b, Del98]. The relationship between $X, X_{\Theta}, X_{\Theta}^h$ helps to give a geometric interpretation of this theory and, in particular, the correct normalization of Eisenstein integrals.

A notational convention: We have tried (possibly foolishly) to avoid choosing a measure on unipotent radicals in this section. To this end we introduce the following notation: In various contexts we shall denote by V' a space that is non-canonically isomorphic to a space V; but the isomorphism depends on the choice of a measure on a certain unipotent group. To a first

⁶⁷The $S_{\Omega,j}$ are not quite equivariant with respect to a morphism $\mathbf{A}_{X,\Omega} \to \mathbf{A}_{X,\Theta}$; therefore, we are really applying variants of Lemmas 12.2.11 and 12.2.10 which apply to the affine maps of character groups introduced in Section 13; we leave the details to the reader.

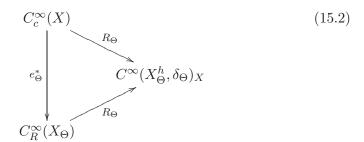
approximation when reading, then, the reader can simply ignore the primes and replace each V' by V.

15.1. Goals. Recall that Radon transform was defined in **5.4.1** as a canonical morphism:

$$C_c^{\infty}(X) \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^h, \delta_{\Theta}),$$
 (15.1)

where by $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ we denote sections of the dual line bundle to the line bundle of Haar measures on the unipotent radical of the parabolic corresponding to each point. Recall that $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ denotes smooth sections of a line bundle over X_{Θ}^h where the stabilizer of each point on X_{Θ}^h (contained in a parabolic of type P_{Θ}) acts on the fiber via the modular character δ_{Θ} , hence the notation. The action of G is twisted by the same character on both $C^{\infty}(X)$ and $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$, in order to make this map equivariant. We will denote by $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$ the Radon transform of the space $C_c^{\infty}(X)$.

Our starting point for the explicit identification of smooth asymptotics is Proposition 5.4.6, which states that the adjoint asymptotics maps e_{Θ}^* commutes with Radon transform, i.e. the following diagram commutes:



where we have used $C_R^{\infty}(X_{\Theta})$ as an *ad hoc* notation for those elements of $C^{\infty}(X_{\Theta})$ for which the transform defining R_{Θ} is absolutely convergent.

This diagram suggests that by "inverting" the lower occurence of Radon transform we can obtain an explicit formula for the smooth asymptotics; we do this in §15.4.1, where we express $e_{\Theta}f$, for every $f \in C_c^{\infty}(X_{\Theta})$, as a "shifted wave packet of normalized Eisenstein integrals".

We then proceed to do the same for unitary asymptotics (the Bernstein maps). In that case, we will "filter out" non-unitary exponents from the inversion of Radon transform, and the expansion of $\iota_{\Theta}f$ will be as a "wave packet" over a set of unitary representations of L_{Θ} . Similar as they may be, the two goals are independent, and the main theorems in the smooth and unitary case do not rely on each other. It should be possible, and would be very interesting, to obtain the expression for ι_{Θ} by shifting the contour in the expression of e_{Θ} , but this is not the direction that we pursue here. For this reason, however, the appearance of Eisenstein integrals in the expression for ι_{Θ} is conditional on some weak "multiplicity one" assumption which we are able to verify in many cases (including symmetric varieties).

Recall from §11.2.1 that, fixing a Plancherel decomposition:

$$L^2(X) = \int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi)$$

we get a pointwise decomposition of (smooth) functions:

$$\Phi(x) = \int_{\hat{G}} \Phi^{\pi}(x) \mu(\pi).$$

Of course, it is more intrinsic to think of $\Phi^{\pi}\mu(\pi)$, which is a measure on \hat{G} valued in functions on X, instead of Φ^{π} , because that doesn't depend on the choice of Plancherel measure.

Our main goal here is to obtain such a decomposition for $\iota_{\Theta} f$ (where $f \in L^2(X_{\Theta})^{\infty}$), but starting from a Plancherel decomposition for $L^2(X_{\Theta})$:

Fix a Plancherel measure ν for $L^2(X_{\Theta}^L)$; as usual, X_{Θ}^L denotes the Levi variety. We will recall the necessary facts about Levi varieties in a moment, including a normalization of their measures such that:

$$L^2(X_{\Theta})_{\text{disc}} = I_{\Theta^-}(L^2(X_{\Theta}^L)_{\text{disc}})$$

(unitary induction). We have a Plancherel decomposition:

$$L^2(X_{\Theta}^L) = \int \mathcal{I}_{\sigma} \nu(\sigma).$$

By induction:

$$L^{2}(X_{\Theta}) = \int_{\widehat{L_{\Theta}}} \mathcal{H}_{\sigma} \nu(\sigma). \tag{15.3}$$

where $\mathcal{H}_{\sigma} = I_{\Theta^{-}} \mathcal{I}_{\sigma}$ is the parabolic induction of \mathcal{I}_{σ} to G.

Our goal is to obtain a formula:

$$\iota_{\Theta}f(x) = \int_{\widehat{I_{\Theta}}} (\iota_{\Theta}^{\sigma}f)(x)\nu(\sigma), \qquad (15.4)$$

for some explicit $\iota_{\Theta}^{\sigma}: \mathcal{H}_{\sigma}^{\infty} \to C^{\infty}(X)$. The abstract existence of such a formula is a tautology: setting $\tilde{1}_{xJ} := \operatorname{Vol}(xJ)^{-1} 1_{xJ}$, one sees formally that $\iota_{\Theta} f(x) = \left\langle f, \iota_{\Theta}^{*} \tilde{1}_{xJ} \right\rangle = \int_{\widehat{L_{\Theta}}} \left\langle f, \iota_{\Theta}^{*} \tilde{1}_{xJ} \right\rangle_{\sigma} \nu(\sigma)$, and hence:

$$\iota_{\Theta}^{\sigma}(f)(x) := \left\langle f, \iota_{\Theta}^{*} \tilde{1}_{xJ} \right\rangle_{\sigma}, \tag{15.5}$$

which clearly factors through \mathcal{H}_{σ} . As the above formula shows, ι_{Θ}^{σ} is adjoint to the map:

$$\iota_{\Theta,\sigma}^*: C_c^{\infty}(X) \to \mathcal{H}_{\sigma}$$
 (15.6)

decomposing ι_{Θ}^* .

We are able to identify some invariant of the morphisms $\iota_{\Theta,\sigma}^*$, the socalled *Mackey restriction* (see Theorem 15.6.3), and under an additional assumption which is known to be true for symmetric varieties through the work of Blanc-Delorme [**BD08**] this is enough to identify them, again, with normalized Eisenstein integrals. Finally, the expression (15.4) can be reinterpreted as a precise and explicit Plancherel formula for $\iota_{\Theta}L^2(X_{\Theta})$, to the extent that the full Scattering Theorem 7.3.1 is known; see Theorem 15.6.2.

- 15.2. Various spaces of coinvariants. In this section we shall, roughly speaking, decompose the Radon transform $C_c^{\infty}(X_{\Theta}) \to C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ over representations of L_{Θ} . More precisely:
- 15.2.1. Goal. Given irreducible unitarizable representation σ of L_{Θ} we shall define two spaces and a morphism between them:

$$RT_{\Theta}: C_c^{\infty}(X_{\Theta})_{\sigma} \longrightarrow C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$$
 (15.7)

The spaces $C_c^{\infty}(X_{\Theta})_{\sigma}$ and $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$ are certain $I_{\Theta^-}(\sigma)$ -isotypical quotients of $C_c^{\infty}(X_{\Theta})$ and $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$. (Recall that $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$ is the image of $C_c^{\infty}(X)$ under Radon transform.)

The definition of these spaces makes sense only for ν -almost every σ . At several points we will abuse notation by omitting the phrase "for ν -almost every σ ."

As for the morphism between these spaces, it is a version of the Radon transform but is also related to the standard intertwining operator:

$$T_{\Theta}: I_{\Theta}(\sigma^{-}) \to \int_{U_{\Theta}} f(u \bullet) du \in I_{\Theta}(\sigma)'.$$
 (15.8)

In fact, there will be a commutative diagram

$$\left(\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma)\right)^{*} \otimes I_{\Theta^{-}}(\sigma) \xrightarrow{\operatorname{id} \otimes T_{\Theta}} \left(\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma)\right)^{*} \otimes I_{\Theta}(\sigma)'$$
(15.9)

The space $I_{\Theta}(\sigma)'$ is isomorphic to $I_{\Theta}(\sigma)$ once one fixes a measure on U_{Θ} , and is defined in 15.2.3. In any case, we will denote the morphism as RT_{Θ} because it can be thought of either as Radon transform or standard intertwining operator.

15.2.2. Parabolics and Levi subgroups. In this section we will not fix parabolics in the classes of \mathbf{P}_{Θ} , \mathbf{P}_{Θ} , except when needed. Hence, we will be thinking of \mathbf{L}_{Θ} as a "universal Levi group of type Θ ", that is, not a subgroup of \mathbf{G} , but rather the reductive quotient of any parabolic in the class of \mathbf{P}_{Θ} , which is a canonical abstract group up to inner automorphisms. We can also define it as the reductive quotient of any parabolic in the class of \mathbf{P}_{Θ}^- , and the two definitions give canonically isomorphic groups, up to inner conjugacy, by identifying the reductive quotients with the intersection $\mathbf{P}_{\Theta} \cap \mathbf{P}_{\Theta}^-$. A "representation of L_{Θ} " will actually be only an isomorphism

class of representations, but when we fix parabolics P_{Θ} and P_{Θ}^- then we will implicitly be fixing (compatible) identifications of their reductive quotients with L_{Θ} , and a realization for the representations of L_{Θ} under consideration.

Similarly, we do not fix a "Levi variety" \mathbf{X}_{Θ}^{L} as a subvariety of \mathbf{X}_{Θ} , except when explicitly saying so. A choice of parabolic in the class of \mathbf{P}_{Θ}^{-} uniquely identifies such a Levi subvariety of \mathbf{X}_{Θ} (consisting of all points whose stabilizers contain the unipotent radical of that parabolic), but in general we are only interested in the isomorphism class of \mathbf{X}_{Θ}^{L} as a homogeneous \mathbf{L}_{Θ} -space.

We let $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ be the torus quotient of \mathbf{L}_{Θ} whose character group consists of all characters of \mathbf{L}_{Θ} which are trivial on the stabilizers of points on \mathbf{X}_{Θ}^{L} . By our assumption that \mathbf{X} is strongly factorizable, the rank of $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ is equal to the rank of $\mathbf{A}_{X,\Theta}$.

15.2.3. Twisting class. The class of representations of L_{Θ} obtained by twisting an irreducible unitary representation σ by all unramified characters of $L_{\Theta,X}^{\rm ab}$ will be called, for short, a twisting class. We will say "for almost every σ " in a twisting class for statements that hold in an Zariski open and dense set of elements of a twisting class; notice the canonical algebraic structure, coming from the torus structure of the set of unramified characters of $L_{\Theta,X}^{\rm ab}$.

15.2.4. Twists and half-twists; normalized and unnormalized induction; intertwiners. We use the symbol $I^{\rm un}$ to denote unnormalized induction. It will be useful to talk about unnormalized induction at first, because some of the algebraic structures are made clearer.

Recall the definition of the space $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ from §5.4.1. First δ_{Θ} denotes the modular character of P_{Θ} , which is inverse to the modular character of P_{Θ}^- ; for example, the usual induction $I_{\Theta}(\sigma)$ is $I_{\Theta}^{\text{un}}(\sigma\delta_{\Theta}^{\frac{1}{2}})$. Secondly, $C^{\infty}(P_{\Theta}\backslash G, \delta_{\Theta})$ denotes smooth sections of the complex line

Secondly, $C^{\infty}(P_{\Theta}\backslash G, \delta_{\Theta})$ denotes smooth sections of the complex line bundle over $P_{\Theta}\backslash G$ whose fiber at a point is dual to the space of Haar measures on the unipotent radical of the corresponding parabolic. The space of sections of this line bundle is non-canonically isomorphic to the representation parabolically induced (unnormalized) from δ_{Θ} . Similarly, $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ denotes sections of the pull-back line bundle on X_{Θ}^h , and $C^{\infty}(X_{\Theta}^L, \delta_{\Theta})$ sections of the restriction of this line bundle to $X_{\Theta}^L \subset X_{\Theta}^h$ (see, however, the discussion below on normalization of the action of L_{Θ}).

With these notations, for instance, the intertwining operator is *canonically*:

$$I_{\Theta^{-}}^{\mathrm{un}}(\sigma) \longrightarrow I_{\Theta}^{\mathrm{un}}(\sigma, \delta_{\Theta})$$

where the right hand side is defined as follows: Tensor the G-linear vector bundle over $P_{\Theta} \backslash G$ corresponding to the induced representation $I_{\Theta}^{\mathrm{un}}(\sigma)$ by the line bundle δ_{Θ} . Smooth sections of this line bundle can be denoted by $I_{\Theta}(\sigma, \delta_{\Theta})$. For the normalized induction the corresponding morphism is:

$$I_{\Theta^{-}}(\sigma) \longrightarrow I_{\Theta}^{\mathrm{un}}(\sigma \delta_{\Theta}^{-\frac{1}{2}}, \delta_{\Theta})$$
 (15.10)

and we abridge the right-hand side to $I_{\Theta}(\sigma)'$; it is isomorphic to $I_{\Theta}(\sigma)$, but it is sometimes clearer not to make this explicit. Similarly the Radon transform is canonically:

$$C_c^{\infty}(X) \longrightarrow C^{\infty}(X_{\Theta}^h, \delta_{\Theta}).$$

We twist the action of L_{Θ} on functions on X_{Θ}^{L} (or sections in $C^{\infty}(X_{\Theta}^{L}, \delta_{\Theta})$) in such a way that $L^{2}(X_{\Theta})$ is the normalized induction of $L^{2}(X_{\Theta}^{L})$ (or, for that matter, $C_{c}^{\infty}(X_{\Theta})$ is the normalized induction of $C_{c}^{\infty}(X_{\Theta}^{L})$). The explicit formula was given in (5.16).

We caution the reader that this may not be the most naturallooking action; for instance, if X has a G-invariant measure and we consider the Levi variety $\mathbf{X}_{\emptyset}^{L} \simeq \mathbf{A}_{X}$, the usual action of A on $C^{\infty}(A_{X})$ is twisted by the square root of the modular character of P(X).

Recall that the *normalized* Jacquet module (which we have been using, by convention, throughout this paper) twists the action of L_{Θ} by $\delta_{\Theta}^{-1/2}$. Therefore, with the definitions above we get a natural inclusion:

$$C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta}) \otimes \delta_{\Theta}^{-1} \hookrightarrow C_c^{\infty}(X)_{\Theta}.$$
 (15.11)

We explicate the map from $C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta})$ to $C_c^{\infty}(X)_{\Theta}$, and then the factor δ_{Θ}^{-1} comes by checking how L_{Θ} acts on both sides: An element $\nu \in C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta})$ assigns to each point of X_{Θ}^L an element ν_x in the dual of the space of Haar measures on U_{Θ} . We send ν to any function in $C_c^{\infty}(\mathring{X}P_{\Theta})$ whose integral over xU_{Θ} against the Haar measure du coincides with $\langle \nu_x, du \rangle$.

We abridge the left-hand side to $C_c^{\infty}(X_{\Theta}^L)'$:

$$C_c^{\infty}(X_{\Theta}^L)' := C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta}) \otimes \delta_{\Theta}^{-1}.$$
(15.12)

A choice of a measure on U_{Θ} identifies $C_c^{\infty}(X_{\Theta}^L)'$ with $C_c^{\infty}(X_{\Theta}^L)$.

15.2.5. The definition of $C_c^{\infty}(X_{\Theta})_{\sigma}$. Let σ be an irreducible representation of L_{Θ} . Any choice of parabolic in the class of P_{Θ}^- , and hence of a Levi subvariety X_{Θ}^L of X_{Θ} , gives rise to a map:

$$\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma) \hookrightarrow \operatorname{Hom}_G(C_c^{\infty}(X_{\Theta}), I_{\Theta^-}(\sigma)),$$

obtained by viewing I_{Θ^-} as a functor, and $C_c^{\infty}(X_{\Theta})$ as $I_{\Theta^-}(C_c^{\infty}(X_{\Theta}^L))$. Note that it is important for the validity of this statement that we twisted the action on $C^{\infty}(X_{\Theta}^L)$.

This embedding gives a corresponding quotient of the $I_{\Theta^-}(\sigma)$ -coinvariants:⁶⁸

$$\begin{array}{lcl} C_c^{\infty}(X_{\Theta})_{I_{\Theta^-}(\sigma)} & = & (\operatorname{Hom}_G(C_c^{\infty}(X_{\Theta}), I_{\Theta^-}(\sigma)))^* \otimes I_{\Theta^-}(\sigma) \\ \\ & \twoheadrightarrow & \left(\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)\right)^* \otimes I_{\Theta^-}(\sigma) \end{array}$$

and the last quotient is what we call the σ -coinvariants $C_c^{\infty}(X_{\Theta})_{\sigma}$.

There should be no confusion with " π "-coinvariants, when π is a representation of G, since the fact that σ is a representation of the Levi suggests

⁶⁸Note that we will prove that the representations $I_{\Theta^-}(\sigma)$ are irreducible for almost every σ in the family, cf Corollary 15.3.5.

that we are using the structure of $C_c^{\infty}(X_{\Theta})$ as an induced representation. As a quotient of $C_c^{\infty}(X_{\Theta})$, the space $C_c^{\infty}(X_{\Theta})_{\sigma}$ does not depend on any of the choices made, though the isomorphism:

$$C_c^{\infty}(X_{\Theta})_{\sigma} \simeq \left(\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)\right)^* \otimes I_{\Theta^-}(\sigma)$$
 (15.13)

does.

We denote the dual of $C_c^{\infty}(X_{\Theta})_{\sigma}$ in $C^{\infty}(X_{\Theta})$ by $C^{\infty}(X_{\Theta})^{\tilde{\sigma}}$, and note that it is isomorphic to (writing $\tau = \tilde{\sigma}$):

$$C^{\infty}(X_{\Theta})^{\tau} \simeq \operatorname{Hom}_{L_{\Theta}}(\tau, C^{\infty}(X_{\Theta}^{L})) \otimes I_{\Theta^{-}}(\tau).$$
 (15.14)

15.2.6. The definition of $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$. Let us recall that $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$ is the image of Radon transform of $C_c^{\infty}(X)$. The space $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$ will be a certain $I_{\Theta}(\sigma)$ -isotypical quotient of that space, defined for almost all σ in each twisting class. As before, the term σ -coinvariants will be used for that quotient.

Choosing a parabolic P_{Θ} gives rise in a similar way as above to a subvariety of X_{Θ}^h which is canonically isomorphic to X_{Θ}^L . Our normalization of the action of L_{Θ} (5.16) implies that the restriction maps give a P_{Θ} -equivariant surjection:

$$C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta}) \to C_c^{\infty}(X_{\Theta}^L)'.$$

(Recall that $C_c^{\infty}(X_{\Theta}^L)'$ was defined in (15.12).) Composing with maps into σ we get a canonical embedding:

$$\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma) \hookrightarrow \operatorname{Hom}_{G}(C_{c}^{\infty}(X_{\Theta}^{h}, \delta_{\Theta}), I_{\Theta}(\sigma)'). \tag{15.15}$$

We leave it to the reader to check the canonicity of this embedding, just recalling here that both $C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta})$ and $I_{\Theta}(\sigma)'$ were defined by pulling back a certain line bundle over $P_{\Theta} \backslash G$.

The morphisms $C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta}) \to I_{\Theta}(\sigma)'$ that arise in the image of (15.15) may not extend to $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$. However, it will follow from Proposition 15.3.6 that they extend for almost every σ ; we state it in a vague form, which will be clarified by Definition 15.3 and Proposition 15.3.6:

15.2.7. PROPOSITION (Proved as Proposition 15.3.6.). The map (15.15) extends naturally, for almost every σ in every twisting class (cf. §15.2.3), to a map:

$$\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma) \hookrightarrow \operatorname{Hom}_G(C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X, I_{\Theta}(\sigma)')$$
 (15.16)

We can finally define the desired quotient $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$ as the image of $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$ under the mapping

$$C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})_{X} \to \left(\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma)\right)^{*} \otimes I_{\Theta}(\sigma)'.$$
 (15.17)

obtained by dualizing (15.16). Note that this quotient is $I_{\Theta}(\sigma)$ -isotypical.

Finally, the combination of (15.13), (15.17) shows that we have a canonical morphism:

$$RT_{\Theta}: C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$$
 (15.18)

induced by the standard intertwining operators (15.8). Indeed, it is immediate to check that this morphism does not depend on the choices of parabolic. These morphisms are defined for almost every σ , and will be seen to be invertible for almost every σ by Corollary 15.3.5.

Thus, by definition, the bottom square of (15.9) commutes. Let us explain also the commutativity of the top square. Because the vertical maps for the bottom square are isomorphisms, it is enough to check that the "big square" commutes, that is to say,

$$C_{c}^{\infty}(X_{\Theta}) \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})_{X}$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow$$

$$\left(\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma)\right)^{*} \otimes I_{\Theta^{-}}(\sigma) \xrightarrow{\operatorname{id} \otimes T_{\Theta}} \left(\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma)\right)^{*} \otimes I_{\Theta}(\sigma)'$$

$$(15.19)$$

Ignoring issues of convergence, directions around this square are given by taking a function on X_{Θ} , integrating along U_{Θ} -orbits on the open orbit, and projecting to σ -coinvariants. In the next section we will see that these integrals are absolutely convergent in a Zariski-open subset of each twisting class (§15.2.3 for definition of a twisting class, Proposition 15.3.6 and Corollary 15.3.5 for the convergence). Hence, in this Zariski open set the diagram commutes, and this extends to all σ for which both composites are defined.

- 15.3. Convergence issues and affine embeddings. Fix a $\Theta \subset \Delta_X$. We consider a G-stable class \mathcal{F} of smooth functions f on X or X_{Θ} with the following properties (stated here with respect to \mathbf{X}):
 - (1) There is an affine embedding $\overline{\mathbf{X}}$ of \mathbf{X} such that the support of all $f \in \mathcal{F}$ has compact closure in \overline{X} . In a slight variation of the language of $[\mathbf{BK15}]$, having implicitly fixed the affine embedding $\overline{\mathbf{X}}$, we will say that such functions have bounded support.
 - (2) For a given compact open $J \subset G$, the elements of \mathcal{F}^J are uniformly of moderate growth; i.e.: There is a completion $\widetilde{\mathbf{X}}$ of \mathbf{X} , a finite open cover (in the Hausdorff topology) $\widetilde{X} = \bigcup_i U_i$, and, for each i, a rational function F_i which is regular on $U_i \cap X$, such that each $f \in \mathcal{F}^J$ satisfies:

$$|f| \le C_f |F_i|$$

on U_i , where C_f is a constant that depends on f.

Note, in particular, that condition (1) guarantees that the Radon transform Rf is defined for $f \in \mathcal{F}$, since the orbits of a unipotent group on an affine variety are Zariski closed.

For a complex character ω of the k-points of an algebraic group \mathbf{M} we can write its absolute value in terms of absolute values of algebraic characters of \mathbf{M} :

$$|\omega| = \prod_{i} |\chi_i|^{s_i}$$

where each $\chi_i: \mathbf{M} \to \mathbb{G}_m$ is algebraic; then we define the real part of ω :

$$\Re\omega := \sum_{i} \Re s_i \chi_i \in \mathfrak{m}^* := \mathcal{X}(\mathbf{M}) \otimes \mathbb{R}$$
 (15.20)

which is independent of choices.

Recall that $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ be the torus quotient of \mathbf{L}_{Θ} whose character group consists of all characters of L_{Θ} which are trivial on the stabilizers of points on X_{Θ}^{L} . Let us choose a parabolic P_{Θ} , giving rise to the quotient map:

$$\mathring{\mathbf{X}}\mathbf{P}_{\Theta}/\mathbf{U}_{\Theta} \simeq \mathbf{X}_{\Theta}^{L}.\tag{15.21}$$

Also, choose a base point on X_{Θ}^L in order to identify characters of $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ with functions on \mathbf{X}_{Θ}^{L} ; our statements will not depend on any of these choices.

15.3.1. Lemma. Consider an algebraic character $\chi \in \mathcal{X}(\mathbf{L}_{\Theta,X}^{\mathrm{ab}})$ as a function on $\mathring{\mathbf{X}} \cdot \mathbf{P}_{\Theta}$ via the quotient map (15.21). Let $\overline{\mathbf{X}}$ be an affine embedding of \mathbf{X} . Then for χ in an open subcone⁶⁹ of $\mathcal{X}(\mathbf{L}^{ab}_{\Theta,X})$ this function extends to a regular function on $\overline{\mathbf{X}}$ which vanishes on $\overline{\mathbf{X}} \setminus \mathring{\mathbf{X}} \cdot \mathbf{P}_{\Theta}$.

Proof. Consider the quotient map of L_{Θ} -spaces:

$$\overline{\mathbf{X}} \to \overline{\mathbf{X}} /\!\!/ \mathbf{U}_{\Theta} = \operatorname{spec} k[\overline{\mathbf{X}}]^{\mathbf{U}_{\Theta}}.$$

We claim that \mathbf{X}_{Θ}^{L} embeds as the open \mathbf{L}_{Θ} -orbit in $\overline{\mathbf{X}} /\!\!/ \mathbf{U}_{\Theta}$, and its preimage is precisely $\mathbf{X} \cdot \mathbf{P}_{\Theta}$.

If $k[\overline{\mathbf{X}}] = \bigoplus_{\lambda \in \mathcal{X}(\mathbf{X})^+} V_{\lambda}$ denotes the decomposition of $k[\overline{\mathbf{X}}]$ into a (multiplicityfree) sum of irreducible subrepresentations, where $\mathcal{X}(\mathbf{X})^+$ is a saturated (by normality), generating (by quasi-affineness) submonoid of $\mathcal{X}(\mathbf{X})$, depending on $\overline{\mathbf{X}}$, then highest weight theory implies that $k[\overline{\mathbf{X}}]^{\mathbf{U}_{\Theta}}$ has the following multiplicity-free decomposition into irreducible \mathbf{L}_{Θ} -representations:

$$k[\overline{\mathbf{X}}]^{\mathbf{U}_{\Theta}} = \bigoplus_{\lambda \in \mathcal{X}(\mathbf{X})^{+}} V_{\lambda}^{\mathbf{U}_{\Theta}}.$$
 (15.22)

In particular, it is finitely generated: indeed, it is generated by the sum of $V_{\lambda}^{\mathbf{U}_{\Theta}}$'s for a set of λ 's generating $\mathcal{X}(\mathbf{X})^{+}$.70

On the other hand, we have a decomposition:

$$k[\mathbf{X}_{\Theta}^L] = \bigoplus_{\lambda \in \mathcal{X}(\mathbf{X}_{\Theta}^L)^+} V_{\lambda}',$$

where V'_{λ} now denotes the highest weight module of weight λ for \mathbf{L}_{Θ} , and $\mathcal{X}(\mathbf{X})^+ \subset \mathcal{X}(\mathbf{X}_{\Theta}^L)^+ \subset \mathcal{X}(\mathbf{X})$. By choosing a finite set of generators of $\mathcal{X}(\mathbf{X}_{\Theta}^{L})^{+}$ and a suitable – finite – set of G-translates of the corresponding highest weight vectors, we obtain a finite set of elements in the fraction field of $k[\overline{\mathbf{X}}]^{\mathbf{U}_{\Theta}}$ generating $k[\mathbf{X}_{\Theta}^{L}]$; hence, the morphism: $\mathbf{X}_{\Theta}^{L} \to \overline{\mathbf{X}}/\mathbf{U}_{\Theta}$

⁶⁹i.e., in a generating, saturated submonoid of $\mathcal{X}(\mathbf{L}_{\Theta,X}^{\mathrm{ab}})$ – the intersection of $\mathcal{X}(\mathbf{L}_{\Theta,X}^{\mathrm{ab}})$

with an open subcone of $\mathcal{X}(\mathbf{L}_{\Theta,X}^{\mathrm{ab}}) \otimes \mathbb{R}$ To see that, note that if Λ is such a set of λ s, and $\chi = \sum_{\lambda \in \Lambda} n_{\lambda} \lambda$, then, for v_{λ} as highest weight vector in V_{λ} , the product $\prod_{\lambda} v_{\lambda}^{n_{\lambda}}$ is a **B**-invariant vector of weight χ ; that shows that $V_{\chi}^{\mathbf{U}_{\Theta}}$ is contained is contained in the image of a suitable tensor product of $V_{\lambda}^{\mathbf{U}_{\Theta}}\mathbf{s}$ by a multiplication map.

is birational and dominant, and since \mathbf{X}_{Θ}^{L} is homogeneous it is an open embedding.

On the other hand, let us verify that the preimage of the open L_{Θ} -orbit (call it O) in $\overline{X} /\!\!/ U_{\Theta}$ is precisely $\mathring{X} \cdot P_{\Theta}$. Were this not so, there is another P_{Θ} -orbit on \overline{X} whose image is equal to O; in particular, there is some B-orbit $Z \subset \overline{X}$, disjoint from \mathring{X} , whose image contains the open $B \cap L_{\Theta}$ orbit on $\overline{X} /\!\!/ U_{\Theta}$. Then the map $Z \to \overline{X} /\!\!/ U_{\Theta}$ is dominant; that would imply that no non-zero element of $k[\overline{X} /\!\!/ U_{\Theta}]^{(B)} = k[\overline{X}]^{(B)}$ vanishes on Z. This cannot be the case: the complement of the open B-orbit in \overline{X} is a closed, B-stable subvariety; consider the B-stable ideal of regular functions vanishing on it – it must contain non-trivial B-semiinvariants.

Now, it suffices to prove that for any affine embedding \mathbf{Y} of a factorizable spherical \mathbf{L}_{Θ} -variety \mathbf{X}_{Θ}^L the cone of characters of $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ which vanish in the complement of the open orbit is non-trivial and, actually, of full rank. This is the case, of course, for affine toric varieties, and we will reduce to this case using the quotient map:

$$\mathbf{Y} \to \mathbf{Y} \ /\!\!/ \ [\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}].$$

All we need to prove is that the preimage of the open \mathbf{L}_{Θ} -orbit on $\mathbf{Y} /\!\!/ [\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]$ is not larger than \mathbf{X}_{Θ}^{L} . Recall that, when a reductive group acts on an affine variety, any two closed sets are separated by an invariant function (see $[\mathbf{MFK94}]$). It suffices, then, to show that all $[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]$ -orbits on \mathbf{X}_{Θ}^{L} are closed in \mathbf{Y} . We claim that these are spherical $[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]$ -varieties with a finite number of automorphisms – then by $[\mathbf{Kno94a}, \mathbf{Corollary} 7.9]$ they have no non-trivial affine embeddings, hence have to be closed in \mathbf{Y} . Finally, to show that the $[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]$ -orbits on \mathbf{X}_{Θ}^{L} are spherical without continuous group of automorphisms, we use the hypothesis that \mathbf{X} is strongly factorizable – hence \mathbf{X}_{Θ}^{L} is factorizable under the \mathbf{L}_{Θ} -action. Recall that the connected \mathbf{L}_{Θ} -automorphism group of \mathbf{X}_{Θ}^{L} is induced by the action of $\mathcal{Z}(\mathbf{L}_{\Theta})$; hence, factorizability means that $\mathbf{H} \cap [\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]$ is spherical in $[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]$ and coincides with the connected component of its normalizer there.

15.3.2. Remark. A knowledge of the combinatorial data describing $\overline{\mathbf{X}}$ allows to read off the precise cone of characters which vanish on the complement of $\mathring{\mathbf{X}}\mathbf{P}_{\Theta}$. Indeed, the above proof shows that these are precisely the characters which vanish in the complement of the open orbit of the toric $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ -variety:

$$\overline{\mathbf{X}} /\!\!/ [\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}] \mathbf{U}_{\Theta} = \operatorname{spec} k[\overline{\mathbf{X}}]^{[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}] \mathbf{U}_{\Theta}} = \operatorname{spec} k[\mathcal{X}(\mathbf{X})^{+} \cap \mathcal{X}(\mathbf{L}_{\Theta})].$$

Hence, the monoid of characters which extend to the complement is the set of those elements of $\mathcal{X}(\mathbf{X})^+$ which are characters of \mathbf{L}_{Θ} , and the characters that vanish on the complement are those in the "interior" of the monoid.

Let us see what this lemma implies. Again, if we fix opposite parabolics $\mathbf{P}_{\Theta}, \mathbf{P}_{\Theta}^{-}$, we can regard \mathbf{X}_{Θ}^{L} as a subvariety of both \mathbf{X}_{Θ} and \mathbf{X}_{Θ}^{h} . Fix a point

 $x_0 \in X_{\Theta}^L$ in order to define a quotient map:

$$\mathbf{X}_{\Theta}^{L} \to \mathbf{L}_{\Theta,X}^{\mathrm{ab}},$$
 (15.23)

and use it to consider characters of $L_{\Theta,X}^{\mathrm{ab}}$ as functions on X_{Θ}^{L} . If $M: C_{c}^{\infty}(X_{\Theta}^{L}) \to \sigma$ is a morphism of L_{Θ} -representations and ω is a character of $L_{\Theta,X}^{\mathrm{ab}}$, we get a morphism:

$$\omega^{-1}M:\omega^{-1}\otimes C_c^{\infty}(X_{\Theta}^L)\to\omega^{-1}\sigma$$

simply by twisting by ω . Explicitly, if $f \in \omega^{-1} \otimes C_c^{\infty}(X_{\Theta}^L)$ we have:

$$\omega^{-1}M(f) = M(\omega f) \in \omega^{-1}\sigma,$$

where the underlying vector spaces of $C_c^{\infty}(X_{\Theta}^L)$ and $\omega^{-1} \otimes C_c^{\infty}(X_{\Theta}^L)$, as well as those of σ and $\omega^{-1}\sigma := \omega^{-1} \otimes \sigma$ have been identified.

The statement of the following corollary will make use of the concept of "extension of a morphism by a convergent series", by which we mean the following: Let $\mathbf{Y} \subset \overline{\mathbf{Y}}$ be smooth varieties with an action of a group \mathbf{L} , with \mathbf{Y} open, and let $M: C_c^\infty(Y) \to \sigma$ be a morphism to a smooth representation σ . We say that it "extends by a convergent series" to $C_c^\infty(\overline{Y})$ if for every $\Phi \in C_c^\infty(\overline{Y})$ and $v \in \tilde{\sigma}$ (the smooth dual of σ) the inner product $\left\langle \Phi, \tilde{M}(v) \right\rangle$ (where \tilde{M} is the adjoint of M) converges absolutely, defining a morphism: $C_c^\infty(\overline{Y}) \to \tilde{\sigma}$. (In the corollary, σ is admissible so $\tilde{\sigma} = \sigma$.) Here we think of $\tilde{M}(v)$ as a smooth measure on Y, hence one could equivalently write Φ as a convergent sum $\sum_i \Phi_i$ with $\Phi_i \in C_c^\infty(Y)$ and require that $\sum_i \left\langle \Phi_i, \tilde{M}(v) \right\rangle$ converges, hence the language.

15.3.3. COROLLARY. Let \mathcal{F} be a class of functions as on p. 212. Let $M: C_c^{\infty}(X_{\Theta}^L) \to \sigma$ be a morphism to an admissible L_{Θ} -representation. Then:

- (1) If \mathcal{F} consists of functions on X_{Θ} , for $\Re(\omega)$ in a translate of an open cone as in the previous lemma, the morphism $\omega^{-1}M$ can be extended by a convergent series to functions of the form: $f|_{X_{\Theta}^L}$, $f \in \mathcal{F}$.
- (2) If \mathcal{F} consists of either functions on X or on X_{Θ} , the analogous statement holds for the Radon transform R_{Θ} of \mathcal{F} : the morphism $\omega^{-1}M \otimes \delta_{\Theta} : \omega^{-1} \otimes C_c^{\infty}(X_{\Theta}^L, \delta_{\Theta}) \to \sigma'$ can be extended by a convergent series to functions of the form: $R_{\Theta}f|_{X_{\Theta}^L}$.
- (3) In the case of X_{Θ} , if $M: C_c^{\infty}(X_{\Theta}^L) \to \sigma$ is a morphism to an irreducible unitary representation of L_{Θ} and $T_{\Theta}: I_{\Theta^{-}}(\omega^{-1}\sigma) \to I_{\Theta}(\omega^{-1}\sigma)'$ is the standard intertwining operator, then T_{Θ} is defined by a convergent integral and the following diagram commutes:

$$\mathcal{F} \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})$$

$$I_{\Theta^{-}}(\omega^{-1}M) \downarrow \qquad \qquad \downarrow I_{\Theta}(\omega^{-1}M) \qquad (15.24)$$

$$I_{\Theta^{-}}(\omega^{-1}\sigma) \xrightarrow{T_{\Theta}} I_{\Theta}(\omega^{-1}\sigma)'$$

Here $\omega^{-1}M$ denotes the twist of M defined above, and $I_{\Theta}(\omega^{-1}M)$, $I_{\Theta^{-}}(\omega^{-1}M)$ are the maps obtained by functoriality of induction. (The right vertical arrow is defined on the image of \mathcal{F} by the previous statement, not on the whole space $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$.)

In the arguments that follow, we will implicitly use the following easy fact: If \mathbf{Y} is a homogeneous variety for a group \mathbf{H} , and the set Y of its k-points is equipped with an H-eigenmeasure dy, if $\overline{\mathbf{Y}}$ is an embedding of \mathbf{Y} and if P is a regular function on $\overline{\mathbf{Y}}$ which vanishes on $\overline{\mathbf{Y}} \setminus \mathbf{Y}$, then for any moderate-growth function f on Y whose support has compact closure in \overline{Y} , the function $|P^n|f$ is in $L^1(Y,dy)$ for sufficiently large n. The reason is that the eigenmeasure itself is of moderate growth, i.e. in a neighborhood of a point of \overline{Y} , choosing local coordinates for the etale topology, the measure can be written as h(x)dx, where dx is the usual Lebesgue measure in these coordinates and h is a function of moderate growth. Replacing $\overline{\mathbf{Y}}$ by a blowup $\widetilde{\mathbf{Y}}$, so that f and h are bounded, locally in a (compact, without loss of generality) neighborhood U_i around any point $y \in \widetilde{Y} \setminus Y$ by a rational function F_i defined on $U_i \cap Y$ as in the definition of "moderate growth" (§15.3), we get that the integral of $|P^n|f$ on U_i is bounded by:

$$\int |P^n F_i|(x) dx, \tag{15.25}$$

and for large enough n the function P^nF_i has no poles (and hence is bounded) on U_i .

PROOF OF THE COROLLARY. The first two statements follow from the fact that the support of the functions $f \in \mathcal{F}$ is contained in a compact subset of an affine embedding of \mathbf{X} or \mathbf{X}_{Θ} , and that they are uniformly of moderate growth. (Indeed, their asymptotics in every direction are governed by exponents, by the theory of asymptotics that we have developed and the fact that the Jacquet functor preserves admissibility.)

More precisely, let χ be an algebraic character of $\mathbf{L}_{\Theta,X}^{\mathrm{ab}}$ as in Lemma 15.3.1, considered as a function on $\mathring{\mathbf{X}}\mathbf{P}_{\Theta}$ or $\mathring{\mathbf{X}}_{\Theta}\mathbf{P}_{\Theta}$ by fixing a base point and extending by zero to $\overline{\mathbf{X}}$. The given morphism $M: C_c^{\infty}(X_{\Theta}^L) \to \sigma$ has an adjoint $\tilde{M}: \tilde{\sigma} \to C^{\infty}(X_{\Theta}^L)$ and, because σ is admissible, the functions in its image are also of uniformly moderate growth on X_{Θ}^L . Notice that X_{Θ}^L is closed in $\mathring{X} \cdot P_{\Theta}$. Let f_1 be in the image of \tilde{M} , and let $f_2 \in \mathcal{F}$.

For the first claim, applying the remark before the proof to $\mathbf{Y} = \mathbf{X}_{\Theta}^{L}$ and $\overline{\mathbf{Y}}$ =its closure in the given (from the properties of \mathcal{F}) affine embedding $\overline{\mathbf{X}}_{\Theta}$ of \mathbf{X}_{Θ} , we get that the product:

$$|\chi^n|f_1f_2$$

is an integrable function on X_{Θ}^{L} for $n \gg 0$. This proves the first claim.

Similarly, for the second claim, if $f_2 \in \mathcal{F}$, $v_1 \in \tilde{\sigma}$ and $\omega = |\chi|^n$, the pairing:

$$\langle \omega^{-1} M \otimes \delta_{\Theta} \circ R_{\Theta}(f_2), v_1 \rangle$$
 (15.26)

can be written as the integral over $X_{\Theta}^L \subset X_{\Theta}^h$ of $R_{\Theta}|\chi|^n f_1$ against an element $f_1 \in C^{\infty}(X_{\Theta}^L, \delta_{\Theta}^{-1})$ of moderate growth or, equivalently (using the definition of Radon transform), as the integral of f_2 over $\mathring{X}P_{\Theta}$ (or $\mathring{X}_{\Theta}P_{\Theta}$) of f_2 against a function of moderate growth on $\mathring{X}P_{\Theta}$ (or $\mathring{X}_{\Theta}P_{\Theta}$). The same argument now shows that this is convergent for $n \gg 0$.

For the third statement, let us first verify that T_{Θ} is absolutely convergent. This is the case for $\Re(\omega)$ in a translate of the dominant cone inside of $\mathcal{X}(\mathbf{L}_{\Theta}) \otimes \mathbb{R}$. We need to show that $\mathcal{X}(\mathbf{L}_{\Theta,X}^{\mathrm{ab}})$ intersects the interior of that cone nontrivially.

The interior of the dominant cone of $\chi \in \mathcal{X}(\mathbf{L}_{\Theta})$ consists of \mathbb{R}_+ -multiples of precisely those algebraic characters χ for which the function

$$f_{\chi}: u^{-}lu \in \mathbf{U}_{\Theta}^{-} \times \mathbf{L}_{\Theta} \times \mathbf{U}_{\Theta} \mapsto \chi(l)$$

extends to a regular function on G that vanishes on the complement of the "open cell" $U_{\Theta}^{-}L_{\Theta}U_{\Theta}$. (Without the vanishing condition, we do not get strictly dominant.)

But elements of $\mathcal{X}(\mathbf{L}_{\Theta,X})$ which belong to \mathcal{T} have that property: We have seen that (considering ω can be regarded as a function on the open \mathbf{P}_{Θ} orbit on \mathbf{X}_{Θ}) that ω extends to a regular function on \mathbf{X}_{Θ} . Now if we consider the orbit map:

$$\mathbf{G} \ni g \mapsto x \cdot g \in \mathbf{X}_{\Theta} \to \simeq \mathbf{X}_{\Theta}^{L} \times_{\mathbf{P}_{\Theta}^{-}} \mathbf{G},$$

where $x \in X_{\Theta}$ is a point mapping to the chosen point x_0 of X_{Θ}^L , the function ω on \mathbf{X}_{Θ} pulls back to the function f_{ω} on \mathbf{G} . So f_{ω} extends to a function on \mathbf{G} , and so ω is dominant. But even better: because the preimage of the open orbit in \mathbf{X}_{Θ} is the open cell $\mathbf{U}_{\Theta}^{-}\mathbf{L}_{\Theta}\mathbf{U}_{\Theta}$, and so f_{ω} vanishes on the complement of that open cell, and so ω is strictly dominant. This concludes the proof that T_{Θ} is absolutely convergent.

The rest of the third statement will be formal after we unwind its meaning; it is simply the fact that the standard intertwining operator is given (in the appropriate context) by a Radon transform.

Indeed, we have:

$$I_{\Theta}(\omega^{-1}M) \circ R_{\Theta}(f)(1) = \omega^{-1}M \left(R_{\Theta}f|_{X_{\Theta}^{L}} \right) = M \left(\omega R_{\Theta}f|_{X_{\Theta}^{L}} \right) =$$

$$= M \left(\omega \lim_{n} R_{U_{n}}f|_{X_{\Theta}^{L}} \right).$$

Here U_n denotes a sequence of compact subgroups exhausting U_{Θ} , and R_{U_n} denotes the partial Radon transform: $R_{U_n}f(x) = \int_{U_n} f(x \cdot u) du$. For x in any compact subset of X_{Θ}^L , this limit eventually stabilizes because unipotent orbits in affine varieties are closed, and f has compact support in some affine embedding.

Next, we can interchange M and the limit here. To check that, we must give a bound on $M(\omega(R_{U_n} - R_{U_\infty}f))$ in absolute value that goes to zero with n. But this follows by examining the reasoning by which we verified

the convergence in the first place: As in (15.26) we must bound the integral of $(R_{U_n} - R_{U_\infty} f)|_{X_\Theta^L} \cdot \omega$ against a function of moderate growth on X_Θ^L , and this bound should go to zero with n. As we noted above, the quantity $(R_{U_n} - R_{U_\infty} f)|_{X_\Theta^L} \cdot \omega$ vanishes on any fixed compact subset of X_Θ^L for large enough n. Now, $R_{U_n} f$ is bounded in absolute value by $R_{U_\infty} |f|$; and the rest follows just as in (15.26), using again the discussion around (15.25).

Thus, interchanging M and the limit, we get

$$I_{\Theta}(\omega^{-1}M) \circ R_{\Theta}(f)(1) = \lim_{n} M\left(\omega R_{U_{n}} f|_{X_{\Theta}^{L}}\right) =$$

$$= \lim_{n} \int_{U_{n}} M\left(\omega \cdot (u \cdot f)|_{X_{\Theta}^{L}}\right) du = \lim_{n} T_{U_{n}} \circ I_{\Theta^{-}}(\omega^{-1}M)(f) (1),$$

where T_{U_n} is the analogous partial version of T_{Θ} . Since we verified that T_{Θ} is absolutely convergent, the last integral equals

$$T_{\Theta} \circ I_{\Theta^{-}}(\omega^{-1}M)(f)(1).$$

15.3.4. REMARK. The reader will notice from the proofs of Lemma 15.3.1 and Corollary 15.3.3 that the subcone of $\mathcal{X}(\mathbf{L}_{\Theta,X}^{ab}) \otimes \mathbb{R}$ of the corollary depends only on the toric variety $\mathbf{Y} /\!\!/ [\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}] \mathbf{U}_{\Theta}$, where \mathbf{Y} is the affine \mathbf{G} -variety containing the support of elements of the class \mathcal{F} .

In particular, for the cases $\mathcal{F} = C_c^{\infty}(X)$ and $\mathcal{F} = C_c^{\infty}(X_{\Theta})$, where **Y** can be taken to be the affine closure of **X**, resp. \mathbf{X}_{Θ} , the same cone \mathcal{T} will work. Indeed, there is an affine embedding of \mathbf{X}_{Θ} whose coordinate ring is, as a **G**-module, isomorphic to $k[\mathbf{X}]$ (§2.5), and then the corresponding categorical quotients by $[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}]\mathbf{U}_{\Theta}$ will coincide.

We will write $\omega \gg 0$ for a character ω as in the Corollary, when $\mathcal{F} = C_c^{\infty}(X)$ or $C_c^{\infty}(X_{\Theta})$.

DEFINITION. For $\omega \gg 0$, we define the extension of (15.15) to a map:

$$\operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \omega^{-1}\sigma) \hookrightarrow \operatorname{Hom}_{G}(C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})_{X}, I_{\Theta}(\omega^{-1}\sigma)'), \quad (15.27)$$
 by inducing from the extension of a morphism $\omega^{-1}M$ to $R_{\Theta}f|_{X_{\Theta}^{L}}$ as guaranteed by Corollary 15.3.3.

It is this extension which enabled us to define the coinvariant space $C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\omega^{-1}\sigma}$ for $\omega \gg 0$. The extension to almost every ω will follow from Proposition 15.3.6.

15.3.5. COROLLARY. For σ in a dense, Zariski open subset of a twisting class:

- i. the induced representation $I_{\Theta^-}(\sigma)$ is irreducible;
- ii. T_{Θ} is an isomorphism;

PROOF. It is enough to show that conditions (i) and (ii) are verified at a single point, because the twisting class is an irreducible variety and

if conditions (i) or (ii) hold at a point of such a variety, they hold at a Zariski-open set.⁷¹

We have already seen that the natural map

$$\mathcal{X}(\mathbf{L}_{\Theta,X}) \otimes \mathbb{R} \to \mathcal{X}(\mathbf{L}_{\Theta}) \otimes \mathbb{R}$$

carries the cone \mathcal{T} of Corollary 15.3.3 into the strongly dominant cone (see p. 217). But it is well-known [Cas, Theorem 6.6.1] that, if one twists σ by a sufficiently dominant character, the induced representation is irreducible and the intertwining operator is an isomorphism.

15.3.6. Proposition. Let $\mathcal{F} = C_c^{\infty}(X)$ or $C_c^{\infty}(X_{\Theta})$, and let M be an element of $\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)$. The composition of:

$$\mathcal{F} \xrightarrow{R_{\Theta}} C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta}) \xrightarrow{I_{\Theta}(\omega^{-1}M)} I_{\Theta}(\omega^{-1}\sigma)', \tag{15.28}$$

(the second arrow being defined only on the image of the first), which converges for $\omega \gg 0$ by Corollary 15.3.3, extends to a rational family of morphisms for all ω .

For $\mathcal{F} = C_c^{\infty}(X)$ this composition is just the unnormalized Eisenstein integrals of the literature (or rather, their adjoints). We also note that, in the case of X symmetric, this Proposition proven by Blanc and Delorme [BD08] (see also discussion of their paper in §15.5.5).

PROOF. We will refer to X in our notation, but the argument for X_{Θ} is verbatim the same.

Let us choose a measure on U_{Θ} in order to identify $I_{\Theta}(\omega^{-1}\sigma)'$ with $I_{\Theta}(\omega^{-1}\sigma)$. For any $v^* \in \tilde{\sigma}$ the composition of these arrows with "restriction to the coset $P_{\Theta}1$ " and "pairing with v^* ":⁷²

$$\mathcal{F} \to I_{\Theta}(\omega^{-1}\sigma) \to \omega^{-1}\sigma \xrightarrow{\langle \bullet, v^* \rangle} \mathbb{C}$$

is given formally by an integral

$$f \mapsto \int_X f \cdot F_\omega dx$$

(resp. such an integral on X_{Θ}), and that the function F_{ω} is the product of a fixed, locally constant, U_{Θ} -invariant function F on $\mathring{X}P_{\Theta}$ (namely, M^*v^* , via the map $\mathring{X}P_{\Theta} \to X_{\Theta}^L$), and the character ω .

Consider the quotient: $\mathbf{X} \to \mathbf{X} /\!\!/ \mathbf{U}_{\Theta}$, which induces an affine embedding of \mathbf{X}_{Θ}^L . Let $\mathbf{Y} \to \mathbf{X} /\!\!/ \mathbf{U}_{\Theta}$ be a proper surjective morphism, where \mathbf{Y}

 $^{^{71}}$ Indeed, irreducibility amounts to asking that certain elements of a Hecke algebra, with respect to an open compact J, generate the endomorphisms of J-fixed vectors; and if an algebraic family of matrices has full rank at a point, it has full rank at a Zariski-open set. Similarly for (ii): once the representation is irreducible, the map T_{Θ} is an isomorphism if and only if it is nonzero, which can be checked on a single J-fixed vector.

⁷²Again, we are implicitly identifying the underlying vector spaces of all representations in the family $\omega^{-1}\sigma$ (as ω varies), and hence v^* lives in the dual of all of them.

is a smooth toroidal embedding of \mathbf{X}_{Θ}^{L} . Such an embedding and morphism always exist; indeed, if $\mathcal{C}(\mathbf{X}/\!\!/\mathbf{U}_{\Theta})$ is the cone of the affine embedding $\mathbf{X}/\!\!/\mathbf{U}_{\Theta}$ of \mathbf{X}_{Θ}^{L} (we point to [Kno91, page 8] for the definition of this cone), then \mathbf{Y} will be described by a fan whose support is $\mathcal{C}(\mathbf{X}/\!\!/\mathbf{U}_{\Theta}) \cap \mathcal{V}$ (where \mathcal{V} denotes the cone of invariant valuations of \mathbf{X}_{Θ}^{L} , cf. Section 2). By [Kno91] we have a morphism $\mathbf{Y} \to \mathbf{X}/\mathbf{U}_{\Theta}$, which is proper by loc.cit. Theorem 4.2 and surjective by loc.cit. Lemma 3.2.

In particular, we know from the Local Structure Theorem 2.3.4 that the complement of \mathbf{X}_{Θ}^{L} in \mathbf{Y} is a union of divisors intersecting transversely. Let $\overline{\mathbf{Z}}$ be the closure of the image of the map:

$$\mathring{\mathbf{X}}\mathbf{P}_{\Theta} o \mathbf{X} imes \mathbf{Y}$$

(natural inclusion times projection to \mathbf{X}_{Θ}^{L}). It comes with morphisms: $\overline{\mathbf{Z}} \to \mathbf{X}$ and $\overline{\mathbf{Z}} \to \mathbf{Y}$, the former surjective and proper. (In fact, $\overline{\mathbf{Z}}$ is contained in $\mathbf{X} \times_{\mathbf{X}/\mathbf{U}_{\mathbf{P}_{\Theta}}} \mathbf{Y}$, and the latter has a proper map to \mathbf{X}). Applying resolution of singularities, we may replace $\overline{\mathbf{Z}}$ by a nonsingular variety \mathbf{Z} equipped with a proper birational morphism $\mathbf{Z} \to \overline{\mathbf{Z}}$. In particular the induced map $\mathbf{Z} \to \mathbf{X}$ is proper also.

We notice that a proper morphism of algebraic varieties over k induces a proper map of the corresponding topological spaces of k-points, see [Con12, Proposition 4.4]. That means that the pull-back of f is a locally constant, compactly supported function on Z.

Hence it is enough to show that the integral of the pull-back of $F_{\omega}dx$ over a compact open neighborhood in Z is rational in ω . Here, we make sense of the "pullback of dx" because dx is the measure obtained as the absolute value of a volume form, which can be pulled back to \mathbf{Z} . As for F_{ω} , we extend it first of all by zero off $\mathring{\mathbf{X}}\mathbf{P}_{\Theta}$ and then pull back. Note that, fixing ω_0 , we have, by definition,

$$F_{\omega} = F_{\omega_0} \cdot \prod |f_i|^{s_i(\omega)},$$

where the f_i are rational functions on **Z** and the exponents $s_i(\omega)$ vary linearly with ω . This is a matter of the definitions: the f_i are obtained by pulling back the coordinate functions on \mathbb{G}_m^t under the maps

$$\mathring{\mathbf{X}}\mathbf{P}_{\Theta}/\mathbf{U}_{\Theta} \to \mathbf{X}_{\Theta}^{L} \to \mathbf{X}_{\Theta}^{L}/[\mathbf{L}_{\Theta}, \mathbf{L}_{\Theta}] = \mathbf{L}_{\Theta, \mathbf{X}}^{\mathrm{ab}} \simeq \mathbb{G}_{m}^{t}.$$

For a normal k-variety \mathbf{V} with a distinguished divisor \mathbf{D} , we have defined before Corollary 5.1.8 the notion of a function on V being "D-finite." By that Corollary 5.1.8, F is a D-finite function on Y, where \mathbf{D} is the complement of \mathbf{X}_{Θ}^{L} , and therefore (see discussion prior to quoted Corollary) its pull-back to \mathbf{Z} is also so (with respect to the complement of $\mathring{X}P_{\Theta}$).

We may now apply the following consequence of Igusa theory ([**Igu00**], see in particular Theorem 8.2.1 and the proof that follows; that reference deals with a special case of a single f_i , but for a discussion of the modification necessary for many f_i one can proceed in the fashion of [**Den84**, p. 5])

Consider the integral

$$I(\omega) := \int_{V_0} F \cdot |\Omega| \cdot \prod_i |f_i|^{s_i(\omega)},$$

where:

- V_0 is an open compact subset of V:
- F is a D-finite function;
- Ω an algebraic volume form, with polar divisor in **D**;
- f_i rational functions with polar divisor in **D**;
- The exponents $s_i(\omega)$ vary linearly in ω .

If $I(\omega)$ converges for a open set of the parameters in the Hausdorff topology, then it is rational in ω .

This establishes the proposition.

15.4. Normalized Eisenstein integrals and smooth asymptotics.

15.4.1. Definition of Eisenstein integrals. We now define our normalized version of Eisenstein integrals. Recall that for σ an irreducible unitarizable representation of L_{Θ} , we have a diagram

$$C_{c}^{\infty}(X_{\Theta}) \qquad C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})_{X} \xleftarrow{R} C_{c}^{\infty}(X)$$

$$\downarrow \qquad \qquad \downarrow \qquad \qquad (15.29)$$

$$C_{c}^{\infty}(X_{\Theta})_{\sigma} \xleftarrow{RT_{\Theta}^{-1}} C^{\infty}(X_{\Theta}^{h}, \delta_{\Theta})_{X,\sigma}$$

Since RT_{Θ}^{-1} and $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_{X,\sigma}$ make sense only for generic σ in a twisting class, we implicitly assume that we are referring to such σ .

Note that the Radon transform maps $C_c^{\infty}(X_{\Theta})$ into $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})$, but we do not know, a priori, that the image is contained in $C^{\infty}(X_{\Theta}^h, \delta_{\Theta})_X$. This is why we omit an arrow at the upper left. (It can be shown that the image is indeed in this space.)

We denote by

$$R_{\Theta,\sigma}: C_c^\infty(X) \to C^\infty(X_\Theta^h, \delta_\Theta)_{X,\sigma}, \quad E_{\Theta,\sigma}^*: C_c^\infty(X) \to C_c^\infty(X_\Theta)_\sigma,$$

the morphisms obtained by following the arrows in the above diagram. We refer to $E_{\Theta,\sigma}^*$ as the adjoint normalized Eisenstein integral. (The adjoints of unnormalized Eisenstein integrals, already encountered in (15.28), are essentially the operators above without the last arrow representing RT_{Θ}^{-1} .) The established language in the harmonic analysis of real symmetric spaces would suggest calling it (normalized) Fourier transform, but we prefer to reserve the term "Fourier transform" for additive groups. Now $E_{\Theta,\sigma}^*$ is defined a priori in a Zariski-dense subset of each twisting class; let us note that this automatically means that it is defined in a full measure subset of the unitary axis of the twisting class.

The normalized Eisenstein integral is the adjoint map:

$$E_{\Theta,\sigma}: C^{\infty}(X_{\Theta})^{\tilde{\sigma}} \to C^{\infty}(X),$$
 (15.30)

where $C^{\infty}(X_{\Theta})^{\tilde{\sigma}}$ denotes the dual of $C_c^{\infty}(X_{\Theta})_{\sigma}$, considered as a subspace of $C^{\infty}(X_{\Theta})$.

Note that our notion of Eisenstein integral involves in a crucial way the "intermediary" of X_{Θ}^h between X and X_{Θ} . From our point of view, this is closely related to the appearance of factors related to the action of intertwining operators in the Plancherel formula.

The main theorem of this subsection is the following: The Plancherel formula for X_{Θ} gives rise, for every $f \in L^2(X_{\Theta})^{\infty}$, to a $C^{\infty}(X_{\Theta})$ -valued measure $f^{\sigma}\nu(\sigma)$ on \widehat{L}_{Θ} , by the Plancherel formula:

$$\langle f, \Phi \rangle_{L^2(X_{\Theta})} = \int_{\widehat{L_{\Theta}}} \int_{X_{\Theta}} f^{\sigma}(x) \overline{\Phi(x)} dx \nu(\sigma)$$
 (15.31)

(for all $\Phi \in C_c^{\infty}(X_{\Theta})$).

For almost every σ , the function f^{σ} belongs to $C^{\infty}(X_{\Theta})^{\sigma}$. If $f \in C_c^{\infty}(X_{\Theta})$, this measure has a natural "translation" to any translate of $\widehat{L_{\Theta}}$ (see the discussion following Lemma 15.4.5), and we have:

$$\langle f, \Phi \rangle_{L^{2}(X_{\Theta})} = \int_{\omega^{-1}\widehat{L_{\Theta}}} \int_{X_{\Theta}} f^{\sigma}(x) \overline{\Phi(x)} dx \nu(\omega \sigma)$$
 (15.32)

for every character ω of $L_{\Theta,X}^{\mathrm{ab}}$.

15.4.2. THEOREM. For any $\omega \gg 0$, if $f \in C_c^{\infty}(X_{\Theta})$ admits the decomposition (15.32) then:

$$e_{\Theta}f(x) = \int_{\omega^{-1}\widehat{L_{\Theta}}} E_{\Theta,\sigma}f^{\tilde{\sigma}}(x)\nu(\omega\sigma). \tag{15.33}$$

Recall that in our notation $E_{\Theta,\sigma}: C^{\infty}(X_{\Theta})^{\tilde{\sigma}} \to C^{\infty}(X)$; if $\sigma \in \omega^{-1}\widehat{L_{\Theta}}$ then $\tilde{\sigma} \in \omega \widehat{L_{\Theta}}$. We proceed with the proof of the theorem in several steps, including the explanation of (15.32).

15.4.3. Moderate growth.

PROPOSITION. The image of $e_{\Theta}^*: C_c^{\infty}(X) \to C^{\infty}(X_{\Theta})$ is a space \mathcal{F} of functions satisfying the assumptions of §15.3; namely, for any open compact subgroup J the J-invariants are of uniformly moderate growth, and their support has compact closure in an affine embedding of X_{Θ} .

PROOF. The statement on the support is Proposition 5.4.5. For the moderate growth, we may partition X_{Θ} into the union of subsets N_{Ω} belonging to J-good neighborhoods of Ω -infinity, for all $\Omega \subset \Theta$, so that N_{Ω} is compact modulo $A_{X,\Omega}$. It is then the case that the functions $e_{\Theta}^*\Phi$, $\Phi \in C_c^{\infty}(X)^J$, are of uniformly moderate growth if and only if the same is true for the functions $e_{\Omega}^*\Phi|_{N_{\Omega}}$, $\Omega \subset \Theta$. Indeed, it is easy to see that "moderate growth" is compatible with our exponential map, i.e. the exponential map of §4.3 between neighborhoods of Ω -infinity of X_{Θ}/J and X_{Ω}/J carries functions of uniformly moderate growth in a neighborhood of a point of Ω -infinity to functions of uniformly moderate growth.

Thus, assuming that uniformly moderate growth has been proven for all $\Omega \subseteq \Theta$, it now suffices to prove that the restriction of $e_{\Omega}^*\Phi$, $\Phi \in C_c^{\infty}(X)^J$ on $A_{X,\Theta}$ -orbits is of moderate growth.

Thus, it suffices to show the following: given $x \in X_{\Theta}$ and an open compact $J \subset G$ there is a finite number of regular functions ω_i of $\mathbf{A}_{X,\Theta}$ such that for all $\Phi \in C_c^{\infty}(X)^J$ we have:

$$|e_{\Theta}^*\Phi(a\cdot x)| \ll \sum_i |\omega_i(a)|.$$

Indeed, the pairs $(U_i, F_i = \omega_i)$, where U_i are the open-closed subsets $U_i = \{a \in A_{X,\Theta} | |\omega_i(a)| \geq |\omega_j(a)| \text{ for all } j\}$ provide a cover of $A_{X,\Theta}$ as in the definition of "moderate growth", showing that the pull-back of $e_{\Theta}^* \Phi$ to $A_{X,\Theta}$ under the action map is of moderate growth.

Using a Plancherel decomposition:

$$\|\bullet\|^2 = \int_{\hat{G}} H_{\pi}\mu(\pi)$$

for $L^2(X)$, we get:

$$e_{\Theta}^*\Phi(a\cdot x) = \operatorname{Vol}(axJ)^{-1} \langle \Phi, e_{\Theta} 1_{a\cdot xJ} \rangle_{L^2(X)} = \operatorname{Vol}(axJ)^{-1} \int_{\hat{G}} H_{\pi}(\Phi, e_{\Theta} 1_{a\cdot xJ}) \mu(\pi).$$

The sesquilinear forms:

$$C_c^{\infty}(X) \otimes \overline{C_c^{\infty}(X_{\Theta})} \ni \Phi \otimes \overline{f} \mapsto H_{\pi}(\Phi, e_{\Theta}f) \in \mathbb{C}$$
 (15.34)

are $A_{X,\Theta}$ -finite with respect to the action of $A_{X,\Theta}$ on the second variable; this is because they factor through the π -coinvariants, which are of finite length (by finiteness of multiplicity, i.e. Theorem 5.1.5).

Now recall (Corollary 11.6.2) that for almost every π there exists an $\Omega \subset \Delta_X$ such that π is a relative discrete series for X_{Ω} , and the conclusion of Proposition 9.4.8 is satisfied: the absolute value of the exponents of any $\pi \to C^{\infty}(X)$ belong to a finite set of homomorphisms: $A_{X,\Theta} \to \mathbb{R}_{+}^{\times}$. By an analog of Lemma 10.2.5, this implies that there is a finite subset $\Lambda \subset A_{X,\Theta}$ and a finite set of characters ω_i of $\mathbf{A}_{X,\Theta}$ such that:

$$|H_{\pi}(\Phi, e_{\Theta} 1_{a \cdot xJ})| \le \left(\sum_{\lambda \in \Lambda} |H_{\pi}(\Phi, e_{\Theta} 1_{\lambda \cdot xJ})| \right) \cdot \sum_{i} |\omega_{i}(a)|$$

for all $a \in A_{X,\Theta}$; thus:

$$|e_{\Theta}^*\Phi(a\cdot x)| \le \left(\sum_{\lambda \in \Lambda} |e_{\Theta}^*\Phi(\lambda \cdot x)|\right) \cdot \sum_i |\omega_i(a)|.$$

15.4.4. Plancherel decomposition of moderate growth functions with bounded support. Fix a parabolic in the class P_{Θ}^- , hence a subspace X_{Θ}^L of X_{Θ} , and compatible measures so that:

$$\int_{X_{\Theta}} f(x)dx = \int_{P_{\Theta}^{-} \backslash G} \int_{X_{\Theta}^{L}} (g \cdot f)|_{X_{\Theta}^{L}} f(x)dxdg.$$

Also fix a point on X_{Θ}^{L} in order to consider characters of $L_{\Theta,X}^{ab}$ as functions on X_{Θ}^{L} , as before.

Let us fix a Plancherel decomposition for X_{Θ}^{L} :

$$\int_{X_{\Theta}^{L}} f_1(x) f_2(x) dx = \int_{\widehat{L_{\Theta}}} H'_{\sigma}(f_1, f_2) \nu(\sigma)$$

for X_{Θ}^L ; we have written the inner product as a bilinear pairing, so the forms H_{σ} are bilinear pairings of the spaces of coinvariants:

$$H'_{\sigma}: C_c^{\infty}(X_{\Theta}^L)_{\sigma} \otimes C_c^{\infty}(X_{\Theta}^L)_{\tilde{\sigma}} \to \mathbb{C}.$$

We can "twist" the forms H_{σ} to forms:

$$H'_{\omega\sigma}: C_c^{\infty}(X_{\Theta}^L)_{\omega\sigma} \otimes C_c^{\infty}(X_{\Theta}^L)_{\omega^{-1}\tilde{\sigma}} \to \mathbb{C},$$

for characters ω of $L_{\Theta,X}^{ab}$ which are not necessarily unitary, simply by setting:

$$H'_{\omega\sigma}(f_1, f_2) = H_{\sigma}(\omega^{-1}f_1, \omega f_2).$$

This definition is consistent (for unitary ω) if and only if the Plancherel measure ν chosen is $\widehat{L_{\Theta,X}^{ab}}$ -invariant (which we can assume).

The Plancherel formula for $L^2(X_{\Theta})$ will involve the forms "induced" from the H'_{σ} :

$$\int_{X_{\Theta}} f_1(x) f_2(x) dx = \int_{\widehat{L_{\Theta}}} H_{\sigma}(f_1, f_2) \nu(\sigma),$$

$$H_{\sigma}(f_1, f_2) = \int_{P_{\Theta}^- \setminus G} H_{\sigma}'(g \cdot f_1|_{X_{\Theta}^L}, g \cdot f_2|_{X_{\Theta}^L}) dg.$$

15.4.5. LEMMA. Let \mathcal{F} be a class of functions on X_{Θ} as before, then for $\omega \gg 0$, $f_1 \in \mathcal{F}$ and $f_2 \in C_c^{\infty}(X_{\Theta})$ we have a "Plancherel" decomposition of the inner product:

$$\int_{X_{\Theta}} f_1 \cdot f_2 = \int_{\omega^{-1} \widehat{L_{\Theta}}} H_{\sigma}(f_1, f_2) \nu(\omega \sigma), \tag{15.35}$$

for $\omega \gg 0$. Notice that the image of f_1 in $C_c^{\infty}(X_{\Theta})_{\sigma}$ makes sense for $\omega \gg 0$ by Corollary 15.3.3, thus the right hand side is well-defined.

PROOF. We compute:

$$\int_{X_{\Theta}} f_1 \cdot f_2 = \int_{P_{\Theta}^- \setminus G} \int_{X_{\Theta}^L} (g \cdot f_1)(x)(g \cdot f_2)(x) dx dg =$$

$$= \int_{P_{\Theta}^- \setminus G} \int_{X_{\Theta}^L} \omega(x)(g \cdot f_1)(x)\omega^{-1}(x)(g \cdot f_2)(x) dx dg.$$

For $\omega \gg 0$ both factors of the integrand are in $L^2(X_{\Theta}^L)$, so applying the Plancherel decomposition we get:

$$\int_{P_{\Theta}^{-}\backslash G} \int_{\widehat{L_{\Theta}}} H'_{\sigma}(\omega \cdot (g \cdot f_{1}), \omega^{-1} \cdot (g \cdot f_{2}))\nu(\sigma)dg =$$

$$= \int_{\widehat{L_{\Theta}}} \int_{P_{\Theta}^{-}\backslash G} H'_{\sigma}(\omega \cdot (g \cdot f_{1}), \omega^{-1} \cdot (g \cdot f_{2}))dg\nu(\sigma) =$$

$$= \int_{\omega^{-1}\widehat{L_{\Theta}}} H_{\sigma}(f_{1}, f_{2})\nu(\omega\sigma).$$

An alternative way to state that is: Recall that $C^{\infty}(X_{\Theta})^{\sigma}$ is the dual of $C_c^{\infty}(X_{\Theta})_{\tilde{\sigma}}$, considered as a subspace of $C^{\infty}(X_{\Theta})$. The Plancherel formula for X_{Θ} gives rise, for every $f \in C_c^{\infty}(X_{\Theta})$, to a $C^{\infty}(X_{\Theta})$ -valued measure $f^{\sigma}\nu(\sigma)$ on $\widehat{L_{\Theta}}$, defined by (15.31). For almost every σ , the function f^{σ} belongs to $C^{\infty}(X_{\Theta})^{\sigma}$. The above discussion shows that the definition of this measure can be extended to translates of $\widehat{L_{\Theta}}$ by characters of $L_{\Theta,X}^{ab}$: simply replace (15.31) by the analogous expression coming from (15.35) (valid for any ω , if we take f_1 and f_2 to be compactly supported).

Then the above result amounts to saying that the expression (15.32) is valid for $f \in \mathcal{F}$, as long as $\omega \gg 0$. For such ω , and $\sigma \in \omega^{-1}\widehat{L}_{\Theta}$, the map: $\mathcal{F} \to C_c^{\infty}(X_{\Theta})_{\sigma}$ (or, equivalently, to $C^{\infty}(X_{\Theta})^{\sigma}$) is defined by the "convergent series" extension of morphisms of Corollary 15.3.3.

15.4.6. Proof of Theorem 15.4.2. Let $\Phi \in C_c^{\infty}(X)$. By Propositions 5.4.5 and 15.4.3 we may apply Lemma 15.4.5 to the function $e_{\Theta}^*\Phi$; if $f \in C_c^{\infty}(X_{\Theta})$, we then get:

$$\left\langle \Phi, \overline{e_{\Theta}f} \right\rangle_{L^{2}(X)} = \left\langle e_{\Theta}^{*}\Phi, \overline{f} \right\rangle_{L^{2}(X_{\Theta})} = \int_{\omega^{-1}\widehat{L_{\Theta}}} H_{\sigma}(e_{\Theta}^{*}\Phi, f) \nu(\omega\sigma).$$

By the third statement of Corollary 15.3.3 and the definition of the normalized Eisenstein integrals, the image of $e_{\Theta}^*\Phi$ in $C_c^{\infty}(X_{\Theta})_{\sigma}$ is equal to the adjoint normalized Eisenstein integral $E_{\Theta,\sigma}^*\Phi$. Therefore:

$$\langle \Phi, \overline{e_{\Theta}f} \rangle_{L^{2}(X)} = \int_{\omega^{-1}\widehat{L_{\Theta}}} H_{\sigma}(E_{\Theta,\sigma}^{*}\Phi, f) \nu(\omega\sigma) =$$

$$= \int_{\omega^{-1}\widehat{L_{\Theta}}} \int_{X_{\Theta}} (E_{\Theta,\sigma}^{*}\Phi)(x) f^{\tilde{\sigma}}(x) dx \nu(\omega\sigma)$$

(by the definition of $f^{\tilde{\sigma}}$ in (15.31))

$$= \int_{\omega^{-1}\widehat{L_{\Theta}}} \int_{X} \Phi(x) (E_{\Theta,\sigma} f^{\tilde{\sigma}})(x) dx \nu(\omega \sigma) =$$

$$= \int_{X} \Phi(x) \int_{\omega^{-1}\widehat{L_{\Theta}}} (E_{\Theta,\sigma} f^{\tilde{\sigma}})(x) \nu(\omega \sigma) dx.$$

This proves Theorem 15.4.2.

15.5. The canonical quotient and the small Mackey restriction.

We will now discuss certain technical prerequisites for the explicit decomposition of unitary asymptotics (i.e. the Bernstein maps). The basic issue here is the absence of a diagram analogous to (15.2); for this reason, Eisenstein integrals do not appear explicitly a priori, and their relevance has to be established via their properties – more precisely, their asymptotics, and the notion of "small Mackey restriction" that we are about to define.

15.5.1. The canonical quotient of an induced representation. If τ is a smooth admissible representation of L_{Θ} , and the intertwining operator T_{Θ} : $I_{\Theta}^G - \tau \to I_{\Theta}^G (\tau)'$ is regular at τ , we can obtain a certain canonical quotient of the (normalized) Jacquet module of the (normalized) induced representation $I_{\Theta}^G - \tau$ as the composition:

$$I_{\Theta^{-}}^{G}(\tau)_{\Theta} \xrightarrow{T_{\Theta}} I_{\Theta}^{G}(\tau)_{\Theta}' \to \tau'$$

where τ' denotes a representation that is isomorphic to τ once a measure on U_{Θ} is fixed. We call this "the canonical quotient," even though it is defined only when the intertwining operator is regular.

Note that there is a canonical inclusion $\tau' \hookrightarrow I_{\Theta^-}^G(\tau)_{\Theta}$ (by considering those elements of $I_{\Theta^-}^G(\tau)$ which are supported on the open P_{Θ^-} -orbit), and when composed with the canonical quotient this gives the identity, i.e. the composite

$$\tau' \hookrightarrow I_{\Theta^-}^G(\tau)_{\Theta} \to \tau'$$
 (15.36)

is the identity.

In the case of $C_c^{\infty}(X_{\Theta})_{\sigma}$ (abstractly isomorphic to an induced admissible representation; see (15.13)) we denote the corresponding quotient by $C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma]$:

$$(C_c^{\infty}(X_{\Theta})_{\sigma})_{\Theta} \to C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma]. \tag{15.37}$$

Again, it is defined only for the set of σ for which the intertwining operator $I_{\Theta^-}(\sigma) \to I_{\Theta}(\sigma)'$ is an isomorphism; but we have seen in Corollary 15.3.5 that this includes ν_{disc} -almost every σ . We have a canonical isomorphism:

$$C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma] = \left(\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)\right)^* \otimes \sigma'. \tag{15.38}$$

Later we shall use the following property of the canonical quotient. To state it, note that the action of $\mathcal{Z}(L_{\Theta})$ on the flag variety $P_{\Theta}^- \backslash G$ induces an action of $\mathcal{Z}(L_{\Theta})$ by equivalences on the functor I_{Θ^-} . In this way, we obtain an action of $\mathcal{Z}(L_{\Theta})$ on $I_{\Theta^-}(\tau)$. Therefore, the Jacquet module $I_{\Theta^-}(\tau)_{\Theta}$ has a $\mathcal{Z}(L_{\Theta}) \times L_{\Theta}$ -action.

Consider the morphism

$$I_{\Theta^-}(\tau)_{\Theta} \to \tau'.$$
 (15.39)

15.5.2. LEMMA. For generic τ , the antidiagonal copy $\mathcal{Z}(L_{\Theta}) \hookrightarrow \mathcal{Z}(L_{\Theta}) \times L_{\Theta}$ acts trivially on the quotient τ' of (15.39) in other words: the $\mathcal{Z}(L_{\Theta})$ -action on $I_{\Theta^-}(\tau)_{\Theta}$ commutes with the $\mathcal{Z}(L_{\Theta}) \hookrightarrow L_{\Theta}$ -action on τ' .

PROOF. This is just a consequence of the fact that (15.36) is the identity, with its first arrow being $\mathcal{Z}(L_{\Theta})$ -equivariant and the second being L_{Θ} -equivariant.

15.5.3. The small Mackey restriction. Fix a parabolic in the class P_{Θ} , and recall (5.19) that for a representation π of G, "Mackey restriction" is the natural morphism:

$$\operatorname{Hom}_G(C_c^{\infty}(X), \pi) \to \operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L)', \pi_{\Theta})$$

obtained by applying the Jacquet functor and identifying $C_c^{\infty}(X_{\Theta}^L)'$ with a subspace of the Jacquet module of $C_c^{\infty}(X)$ as in (15.12).

We will use the term "small Mackey restriction" in the following situation: Suppose that π is endowed with an isomorphism to an induced representation $I_{\Theta^-}(\tau)$ and the intertwining operator $T_{\Theta}:I_{\Theta^-}(\tau)\to I_{\Theta}(\tau)'$ is an isomorphism. In that case, composition with the canonical quotient gives:

$$Mac: \operatorname{Hom}_G(C_c^{\infty}(X), \pi) \to \operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L)', \tau') = \operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \tau).$$
(15.40)

and this morphism will be, by definition, the "small Mackey restriction."

We have proved that for almost all τ in a given twisting class (§15.2.3) the intertwining map $I_{\Theta^-}(\tau) \to I_{\Theta}(\tau)'$ is an isomorphism (see Corollary 15.3.5) and, therefore, the notion of "small Mackey restriction" makes sense for $\pi = I_{\Theta^-}(\tau)$ at least for τ generic in a twisting class.

In particular, given an irreducible representation σ of L_{Θ} , and any morphism (not necessarily the canonical one):

$$M: C_c^{\infty}(X_{\Theta}) \to \pi := C_c^{\infty}(X_{\Theta})_{\sigma},$$

and taking into account that the right-hand side has the structure of an induced representation with canonical quotient $\tau' = C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma]$ (see (15.37)) the small Mackey restriction of M is a morphism:

$$Mac(M): C_c^{\infty}(X_{\Theta}^L)' \to C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma].$$
 (15.41)

This enjoys the following property:

15.5.4. Lemma. For any G-morphism:

$$M: C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X_{\Theta})_{\sigma},$$

the small Mackey restriction (15.41) is $A_{X,\Theta}$ -invariant.

PROOF. In fact, we claim that the $A_{X,\Theta}$ -action on source and target of (15.41) coincides with the restriction of the L_{Θ} -action to the center $\mathcal{Z}(L_{\Theta})$, by means of the surjection $\mathcal{Z}(L_{\Theta}) \twoheadrightarrow A_{X,\Theta}$. That will prove the lemma, because (15.41) is certainly L_{Θ} -equivariant.

For the source, this is easy to see from the definitions.

For the target, this is Lemma 15.5.2.

15.5.5. The work of Blanc and Delorme. We now translate a very useful result of Blanc and Delorme [BD08] in the symmetric case, into our language.

They prove that, if \mathbf{X} is a *symmetric* variety, the small Mackey restriction:

$$\operatorname{Hom}_{G}(C_{c}^{\infty}(X), I_{\Theta^{-}}(\sigma)) \to \operatorname{Hom}_{L_{\Theta}}(C_{c}^{\infty}(X_{\Theta}^{L}), \sigma)$$
 (15.42)

is *injective*, generically for σ within a twisting class.

The main theorem of Delorme and Blanc actually concerns the composite

$$\operatorname{Hom}(C_c^{\infty}(X), I_{\Theta}(\sigma)) \to \operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X)_{\Theta}, I_{\Theta}(\sigma)_{\Theta}) \to \operatorname{Hom}(C_c^{\infty}(X_{\Theta}^L)', \sigma).$$

It asserts that the composite map is an isomorphism. In fact, their paper analyzes this in terms of distributions on the flag variety invariant by a point stabilizer on X, but this is easily translated to the stated claim using the isomorphism: $\operatorname{Hom}_G(C_c^{\infty}(X), \pi) \simeq \operatorname{Hom}_G(\tilde{\pi}, C^{\infty}(X))$ for admissible representations π . One passes to (15.42) by applying the intertwining operator.

- 15.6. Unitary asymptotics (Bernstein maps). Our main theorem for unitary asymptotics is the following; its formulation uses the $C^{\infty}(X_{\Theta})$ -valued measure $f^{\sigma}\nu(\sigma)$ on $\widehat{L_{\Theta}}$, attached to $f \in L^{2}(X_{\Theta})^{\infty}$ as in (15.31). This was also used in the explicit description of smooth asymptotics, Theorem 15.4.2, except that there we were restricting ourselves to $f \in C_{c}^{\infty}(X_{\Theta})$, and here we will not need to translate off the unitary spectrum. We confine ourselves to describing the restriction of ι_{Θ} to $L^{2}(X_{\Theta})_{\text{disc}}$, since the space $L^{2}(X)$ is built out of the images of those spaces; so, ν_{disc} will denote the "discrete" part of a Plancherel measure for $L^{2}(X_{\Theta}^{L})$. Restricting to discrete spectra will simplify somehow the proof of Theorem 15.6.3 in §15.6.4.
- 15.6.1. Theorem. Assume that for $\nu_{\rm disc}$ -almost all σ the small Mackey restriction map (15.40):

$$\operatorname{Hom}_G(C_c^{\infty}(X), I_{\Theta^-}(\sigma)) \to \operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)$$

is injective (cf. §15.5.5 for its validity for symmetric varieties).

Then for a function $f \in L^2(X_{\Theta})^{\infty}_{\mathrm{disc}}$ with pointwise Plancherel decomposition:

$$f(x) = \int_{\widehat{L_{\Theta}}} f^{\sigma}(x) \nu_{\text{disc}}(\sigma)$$
 (15.43)

(where $f^{\sigma} \in C^{\infty}(X_{\Theta})^{\sigma}$), its image under the Bernstein morphism is given by:

$$\iota_{\Theta}f(x) = \int_{\widehat{L_{\Theta}}} E_{\Theta,\sigma} f^{\sigma}(x) \nu(\sigma). \tag{15.44}$$

The decomposition (15.43) is analogous to (11.10), where the space $C^{\infty}(X_{\Theta})^{\sigma}$ now denotes the dual of the space of σ -coinvariants $C_c^{\infty}(X_{\Theta})_{\sigma}$. Thus, the normalized Eisenstein integral $E_{\Theta,\sigma}$ (the dual of $E_{\Theta,\sigma}^*$) maps $C^{\infty}(X_{\Theta})^{\sigma}$ into $C^{\infty}(X)$.

In combination with the scattering theorem 7.3.1, this implies the following explicit Plancherel decomposition; recall that under Theorem 7.3.1,

 $L^2(X)$ is a direct sum of the spaces $L^2(X)_{\Theta}$, where Θ ranges over associate classes of subsets of Δ_X (that is, conjugacy classes of Levi subgroups of \check{G}_X).

15.6.2. THEOREM. Under the assumptions of Theorem 15.6.1, the norm on $L^2(X)_{\Theta}$ admits a decomposition:

$$\|\Phi\|_{\Theta}^2 = \frac{1}{|W_X(\Theta, \Theta)|} \int_{\widehat{L_{\Theta}}} \|E_{\Theta, \sigma}^* \Phi\|_{\sigma}^2 \nu_{\operatorname{disc}}(\sigma). \tag{15.45}$$

The measure and norms here are the discrete part of the Plancherel decomposition (15.3) for $L^2(X_{\Theta})$.

The real content of Theorem 15.6.1 is an unconditional statement about the "Mackey restrictions" of the morphisms that decompose the Bernstein map ι_{Θ} (the morphisms $\iota_{\Theta,\sigma}^*: C_c^{\infty}(X) \to \mathcal{H}_{\sigma}$ from (15.6)). Notice that \mathcal{H}_{σ} is canonically an induced representation from a unitary representation of L_{Θ} (since $L^2(X_{\Theta}) = I_{\Theta^-}L^2(X_{\Theta}^L)$). Therefore, its "canonical quotient":

$$\mathcal{H}_{\sigma}^{\infty} \to H_{\sigma}^{\infty}[\sigma]$$

is well-defined, cf. $\S15.5.1$.

Moreover, since \mathcal{H}_{σ} is a completion of the σ -coinvariants $C_c^{\infty}(X_{\Theta})_{\sigma}$, we have a commuting diagram of canonical quotients:

$$C_c^{\infty}(X_{\Theta})_{\sigma} \longrightarrow C_c^{\infty}(X_{\Theta})[\sigma]$$

$$\downarrow \qquad \qquad \downarrow^{\text{compl}}$$

$$\mathcal{H}_{\sigma}^{\infty} \longrightarrow \mathcal{H}_{\sigma}^{\infty}[\sigma].$$

$$(15.46)$$

The label "compl" denotes "completion".

For the maps $\iota_{\Theta,\sigma}^*$ and $E_{\Theta,\sigma}^*$ we have the notion of "small Mackey restriction", with images, respectively, in $C_c^{\infty}(X_{\Theta})[\sigma]$ and $\mathcal{H}_{\sigma}^{\infty}[\sigma]$. Our (unconditional) assertion is that these two maps coincide after completion:

15.6.3. THEOREM. The small Mackey restrictions of both $\iota_{\Theta,\sigma}^*$ and $E_{\Theta,\sigma}^*$ coincide (for ν_{disc} -almost every σ) after composing the latter with the map "compl" of (15.46).

As a prelude to the proof, let us actually compute this small Mackey restriction for $E_{\Theta,\sigma}^*$: Recalling that the canonical quotient of $C_c^{\infty}(X_{\Theta})_{\sigma}$ is identified with:

$$\left(\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)\right)^* \otimes \sigma'$$

(cf. (15.38)), the small Mackey restriction of $E_{\Theta,\sigma}^*$ is the natural projection:

$$C_c^{\infty}(X_{\Theta}^L) \to \left(\operatorname{Hom}_{L_{\Theta}}(C_c^{\infty}(X_{\Theta}^L), \sigma)\right)^* \otimes \sigma.$$
 (15.47)

This is just obtained by tracing the definitions. An equivalent formulation is the following: The composite of the maps defining the small Mackey restriction of $E_{\Theta,\sigma}^*$:

$$C_c^{\infty}(X_{\Theta}^L)' \longrightarrow C_c^{\infty}(X_{\Theta})_{\Theta} \longrightarrow (C_c^{\infty}(X_{\Theta})_{\sigma})_{\Theta} \longrightarrow C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma],$$

$$(15.48)$$

(we recall that the first arrow is the canonical embedding defining Mackey restriction, the second is induced by $E_{\Theta,\sigma}^*$, and the third is the canonical quotient) coincides with the natural projection; this comes down to (15.36).

15.6.4. *Proofs.* Our goal here is to prove Theorem 15.6.3, from which Theorem 15.6.1 follows by a formal argument (which we omit).

Thus, the statement of Theorem 15.6.3 is that the compositions of the following arrows coincide:

$$C_c^{\infty}(X_{\Theta}^L)' \xrightarrow{\sim} C_c^{\infty}(\mathring{X}P_{\Theta})_{\Theta} \hookrightarrow C_c^{\infty}(X)_{\Theta} \xrightarrow[\text{complo}E_{\Theta,\sigma}^*]{\iota_{\Theta,\sigma}^*} (\mathcal{H}_{\sigma}^{\infty})_{\Theta} \to \mathcal{H}_{\sigma}^{\infty}[\sigma].$$
(15.4)

Recall that the index " Θ " denotes Jacquet module with respect to P_{Θ} . Here, the maps we have denoted $\iota_{\Theta,\sigma}^*$ and $E_{\Theta,\sigma}^*$ are more precisely the Jacquet functors applied to these maps.

These morphisms are, a priori, just L_{Θ} -equivariant. However, Lemma 15.5.4 implies that their composition is also $A_{X,\Theta}$ -equivariant. In what follows, we denote by $\langle \ , \ \rangle_{\sigma}$ the hermitian inner product on \mathcal{H}_{σ} .

15.6.5. Lemma. Consider the bilinear form on $C_c^{\infty}(X_{\Theta})$ given by the formula:

$$(f_1, \overline{f_2}) \mapsto \langle f_1, \iota_{\Theta}^* e_{\Theta} f_2 \rangle_{\sigma}.$$
 (15.50)

It carries a finite⁷³ diagonal action of $A_{X,\Theta}$; the $A_{X,\Theta}$ -invariant part⁷⁴ of this pairing is equal (for ν_{disc} -almost all σ) to the pairing:

$$(f_1, f_2) \mapsto \langle f_1, f_2 \rangle_{\sigma}. \tag{15.51}$$

Here is an equivalent phrasing: the quotient map:

$$C_c^{\infty}(X_{\Theta}) \to C_c^{\infty}(X_{\Theta})_{\sigma} \to \mathcal{H}_{\sigma}^{\infty}$$

coincides with the $A_{X,\Theta}$ -invariant part of

$$C_c^{\infty}(X_{\Theta}) \ni \Phi \mapsto \text{ the image of } \iota_{\Theta}^* e_{\Theta} \Phi \text{ in } \mathcal{H}_{\sigma}.$$

PROOF. Write $\tau = I_{\Theta^-}(\sigma)$. Recall that τ is irreducible for almost every σ . Let Π be the natural projection $C_c^{\infty}(X_{\Theta})_{\tau} \to C_c^{\infty}(X_{\Theta})_{\sigma}$ – indeed the latter space is τ -isotypical by construction, and therefore a quotient of the former.

Let $f_1, f_2 \in C_c^{\infty}(X_{\Theta})$.

$$\langle f_1, \iota_{\Theta}^* e_{\Theta} f_2 \rangle_{\sigma} = \langle f_1, \iota_{\Theta}^* \iota_{\Theta} f_2 \rangle_{\sigma} + \langle f_1, \iota_{\Theta}^* (e_{\Theta} - \iota_{\Theta}) f_2 \rangle_{\sigma} = \langle f_1, (\iota_{\Theta}^* \iota_{\Theta} f_2)_{\text{disc}} \rangle_{\sigma} + \langle f_1, \iota_{\Theta}^* (e_{\Theta} - \iota_{\Theta}) f_2 \rangle_{\sigma}.$$
 (15.52)

⁷³Indeed, in *both* arguments, it factors through the quotient $C_c^{\infty}(X_{\Theta})_{I_{\Theta}^{-\sigma}}$.

⁷⁴i.e., the $A_{X,\Theta}$ -equivariant projection to the generalized eigenspace with eigenvalue 1 which, a posteriori, is $A_{X,\Theta}$ -invariant for almost all σ

since $\langle \ , \ \rangle_{\sigma}$ is a Plancherel Hermitian form for the discrete spectrum of X_{Θ} . Let us examine the second term. Fix a Plancherel formula for $L^2(X)$:

$$L^2(X) = \int_{\hat{G}} \mathcal{H}_{\pi} \mu(\pi)$$

and the corresponding Plancherel formula for X_{Θ} according to Theorem 11.3.1:

$$L^2(X_{\Theta}) = \int_{\hat{G}} \mathcal{H}_{\pi}^{\Theta} \mu(\pi).$$

Then, by the uniqueness of Plancherel decomposition, $\langle \ , \ \rangle_{\sigma}$ factors (for $\nu_{\rm disc}$ -almost every σ) through a norm on $\mathcal{H}_{\tau}^{\Theta}$. Consequently, we may reexpress this second term as a linear function of

(image of
$$f_1$$
 in $\mathcal{H}_{\tau}^{\Theta}$) \otimes (image of $\iota_{\Theta}^*(e_{\Theta} - \iota_{\Theta})f_2$ in $\mathcal{H}_{\tau}^{\Theta}$). (15.53)

Now, in (15.53), the action of $A_{X,\Theta}$ on the first argument here is through unitary exponents (being a unitary action on a Hilbert space).

On the other hand, the map

$$f_2 \mapsto \text{ image of } \iota_{\Theta}^*(e_{\Theta} - \iota_{\Theta}) f_2 \in \mathcal{H}_{\tau}^{\Theta}$$

is given (for almost all τ) by the composite $\iota_{\Theta,\tau}^* \circ (e_{\Theta,\tau} - \iota_{\Theta,\tau})$ where we regard $e_{\Theta,\tau}$ and $\iota_{\Theta,\tau}$ as mappings $C_c^{\infty}(X_{\Theta})_{\tau} \to \mathcal{H}_{\tau}$ and we regard $\iota_{\Theta,\tau}^*$ as a mapping $\mathcal{H}_{\tau} \to \mathcal{H}_{\tau}^{\Theta}$. By (11.22), for every $a \in A_{X,\Theta}^+$ we have

$$\|(e_{\Theta,\tau} - \iota_{\Theta,\tau})(a^n \cdot f_2)\|_{\mathcal{H}_{\tau}} \to 0,$$

for $f_2 \in C_c^{\infty}(X_{\Theta})^J$. Since $\iota_{\Theta,\tau}^*$ is bounded, we have a similar property for $\iota_{\Theta,\tau}^*(e_{\Theta,\tau}-\iota_{\Theta,\tau})(a^n\cdot f_2)$: it converges to zero inside $\mathcal{H}_{\tau}^{\Theta}$. That shows the action of $A_{X,\Theta}$ on the second argument must be through non-unitary exponents.

We have now established that the $A_{X,\Theta}$ -invariant part of the second term of (15.52) is zero.

Consider the first term of (15.52). We may decompose according to Proposition 13.3.1

$$(\iota_{\Theta}^* \iota_{\Theta} f_2)_{\text{disc}} = \sum_i S_i(f_2)_{\text{disc}},$$

where the morphisms S_i are equivariant with respect to isogenies $T_i: \mathbf{A}_{X,\Theta} \to \mathbf{A}_{X,\Theta}$, cf. Proposition 13.3.1. Note that, by the footnote on page 185, one could replace $(\iota_{\Theta}^* \iota_{\Theta} f_2)_{\text{disc}}$ by $\iota_{\Theta, \text{disc}}^* \iota_{\Theta, \text{disc}} f_{2, \text{disc}}$ and therefore we only need the statement for the discrete spectrum.

Therefore:

$$\langle f_1, \iota_{\Theta}^* \iota_{\Theta} f_2 \rangle_{\sigma} = \sum_i \langle f_1, S_i(f_2)_{\text{disc}} \rangle_{\sigma}.$$

Now $S_{\rm id}$ is equal to the identity. Although that certainly follows from Theorem 7.3.1 when generic injectivity is known, this does not require generic injectivity; it is a direct consequence of Proposition 11.7.1, using a decomposition as in the start of §14.6.

Therefore, the $A_{X,\Theta}$ -invariant summand of this pairing is equal to:

$$\langle f_1, (f_2)_{\text{disc}} \rangle_{\sigma} = \langle f_1, f_2 \rangle_{\sigma}$$
.

15.6.6. COROLLARY. Given a choice of pararabolic in the class P_{Θ} the two P_{Θ} -equivariant maps: $C_c^{\infty}(X_{\Theta}^L)' \to C_c^{\infty}(X_{\Theta})_{\sigma}[\sigma]$ obtained from following diagram are identical, for almost all irreducible representations σ of L_{Θ} within a fixed twisting class:

$$C_c^{\infty}(X)_{\Theta}$$

$$\uparrow^{e_{\Theta}} \qquad \downarrow^{\iota_{\Theta,\sigma}^*}$$

$$C_c^{\infty}(X_{\Theta}^L)' \longrightarrow C_c^{\infty}(X_{\Theta})_{\Theta} \longrightarrow (\mathcal{H}_{\sigma}^{\infty})_{\Theta} \longrightarrow \mathcal{H}_{\sigma}^{\infty}[\sigma].$$

$$(15.54)$$

Here the arrows on the horizontal row are as follows: the first arrow is the canonical embedding defining Mackey restriction, the second arrow is induced by passing to σ -coinvariants and completing, and the third is the canonical quotient.

Again, in the above diagram, the arrows denoted e_{Θ} and $\iota_{\Theta,\sigma}^*$ are more precisely the Jacquet functors applied to those morphisms; we do not denote this explicitly for typographical reasons.

PROOF. We will use Lemma 15.5.4.

It follows from Lemma 15.6.5 that the quotient map: $C_c^{\infty}(X_{\Theta}) \to \mathcal{H}_{\sigma}^{\infty}$ is simply the $A_{X,\Theta}$ -equivariant part of the map $\iota_{\Theta,\sigma}^* \circ e_{\Theta}$. In other words, the "upper" and "lower" composite

$$C_c^{\infty}(X_{\Theta})_{\Theta} \to (\mathcal{H}_{\sigma}^{\infty})_{\Theta}$$

both have the same $A_{X,\Theta}$ -invariant part.

But Lemma 15.5.4 implies that the composite map from $C_c^{\infty}(X_{\Theta}^L)'$ (either "lower" or "upper") to $\mathcal{H}_{\sigma}^{\infty}[\sigma]$ is $A_{X,\Theta}$ -invariant.

Therefore the difference of the two maps in (15.54) is, on the one hand, $A_{X,\Theta}$ -invariant; on the other hand, its $A_{X,\Theta}$ -invariant part is zero. Therefore this difference is zero, i.e., the two maps of (15.54) coincide.

Finally, we recall by Proposition 5.4.3 that the Mackey embeddings: $C_c^{\infty}(X_{\Theta}^L)' \hookrightarrow C_c^{\infty}(X_{\Theta})_{\Theta}$, $C_c^{\infty}(X_{\Theta}^L)' \hookrightarrow C_c^{\infty}(X)_{\Theta}$ commute with the map e_{Θ} . Hence, altogether we get a diagram of Jacquet modules:

$$C_{c}^{\infty}(X_{\Theta}^{L})' \longrightarrow C_{c}^{\infty}(X)_{\Theta}$$

$$\downarrow^{e_{\Theta}} \qquad \downarrow^{\iota_{\Theta,\sigma}^{*}}$$

$$C_{c}^{\infty}(X_{\Theta})_{\Theta} \longrightarrow (\mathcal{H}_{\sigma}^{\infty})_{\Theta} \longrightarrow \mathcal{H}_{\sigma}^{\infty}[\sigma],$$

$$(15.55)$$

where the composed morphisms: $C_c^{\infty}(X_{\Theta}^L)' \to \mathcal{H}_{\sigma}^{\infty}[\sigma]$ agree. The "lower" map is the Mackey restriction of E_{Θ}^* composed with completion, by the

discussion of (15.48); the upper map is the Mackey restriction of $\iota_{\Theta,\sigma}^*$; their agreement is the assertion of Theorem 15.6.3.

15.7. The group case. The case when X = H, a (split) connected reductive group under the action of $G = H \times H$, is a multiplicity-free (and symmetric) example, therefore Theorem 15.6.1 holds. The more familiar form of the Plancherel formula in this case is obtained by relating normalized Eisenstein integrals to the duals of matrix coefficients:

Let L be a Levi subgroup of H; the corresponding boundary degeneration X_{Θ} is isomorphic to $L\setminus (U\setminus H\times U^-\setminus L)$, where P=LU and $P^-=LU^-$ are two opposite parabolics with Levi L.

Let τ be an irreducible representation of L, and $\sigma := \tau \otimes \tilde{\tau}$, a representation of $L_{\Theta} = L \times L$. The matrix coefficient map is $M_{\tau} : \sigma \otimes \delta_{P_{\Theta}}^{-\frac{1}{2}} \to C^{\infty}(L)$, (where $P_{\Theta} = P^{-} \times P$), and by applying the induction functor we get a G-morphism:

$$I_{P \times P^{-}}(\sigma) \xrightarrow{I_{\Theta^{-}} M_{\tau}} C^{\infty}(X_{\Theta}).$$
 (15.56)

The image lies in what was previously denoted by $C^{\infty}(X_{\Theta})^{\sigma}$. Finally, we may apply the normalized Eisenstein integral $E_{\Theta,\tilde{\sigma}}$ to this to get a map into $C^{\infty}(X)$:

$$I_{P \times P^{-}}(\sigma) \xrightarrow{E_{\Theta, \tilde{\sigma}} \circ M_{\tau}} C^{\infty}(H).$$
 (15.57)

On the other hand, we may choose an invariant measure on the suitable line bundle over $P \setminus G$ in order to identify the representation $\rho = I_P(\tilde{\tau})$ with the dual of $I_P(\tau)$, and then we have a matrix coefficient map

$$M_{\rho}: I_{P}(\tau) \otimes I_{P}(\tilde{\tau}) \to C^{\infty}(H).$$
 (15.58)

The question is what is the relationship between (15.57) and (15.58). In order to formulate the answer, let T_2 denote the standard intertwining operator in the second factor (similarly, T_1 will denote the corresponding operator in the first factor):

$$I_{P\times P}(\sigma)\to I_{P\times P^-}(\sigma).$$

To define it, we use the measure on U^- which corresponds to the chosen measure on $P \setminus G$, i.e. such that if $f \in \operatorname{Ind}_P^G(\delta_P)$ then:

$$\int_{P\backslash G}f(g)dg=\int_{U^{-}}f(u)du.$$

15.7.1. Lemma. The map (15.57) is the composition of (15.58) with T_2^{-1} .

PROOF. We add to the picture a third map,

$$I_{P^- \times P}(\sigma) \xrightarrow{I_{\Theta} M_{\tau}} C^{\infty}(X_{\Theta}^h),$$
 (15.59)

arising as well from induction of matrix coefficients. Note that the space $C^{\infty}(X_{\Theta}^h)$ is dual to $C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta})$, non-canonically. Everything is very explicit here, and isomorphisms can be chosen compatibly, so we will not worry about

distinguishing $I_{\Theta}(\sigma)$ from $I_{\Theta}(\sigma)'$, $C^{\infty}(X_{\Theta}^h)$ from the dual of $C_c^{\infty}(X_{\Theta}^h, \delta_{\Theta})$, etc, leaving the details to the reader.

Following the definitions, and using the fact that the adjoint of a standard intertwining operator $T: I_P(\tau) \to I_{P^-}(\tau)'$ is again T, this time as a map $I_{P^-}(\tilde{\tau}) \to I_P(\tilde{\tau})'$, one can see that the following diagram commutes:

$$C^{\infty}(H) \stackrel{M_{\rho}}{\longleftarrow} I_{P}(\tau) \otimes I_{P}(\tilde{\tau})$$

$$E_{\Theta,\tilde{\sigma}} \qquad \qquad \uparrow_{T_{1}}$$

$$C^{\infty}(X_{\Theta}^{h})^{\sigma} \stackrel{I_{\Theta}M_{\tau}}{\longleftarrow} I_{P^{-}\times P}(\tau \otimes \tilde{\tau})$$

$$T_{\Theta}^{-1} \qquad \qquad \uparrow_{T_{\Theta}^{-1}}$$

$$C^{\infty}(X_{\Theta})^{\sigma} \stackrel{I_{\Theta}-M_{\tau}}{\longleftarrow} I_{P\times P^{-}}(\tau \otimes \tilde{\tau}).$$

$$(15.60)$$

Composing T_1 with T_{Θ}^{-1} we get T_2^{-1} , which implies the claim of the lemma.

The compatibility of measures on X = H and X_{Θ} can be expressed as follows: one chooses measures on U^-, L and U such that $d(u^-)dldu$, as a measure on the open Bruhat cell U^-LU , is equal to the measure on H. Then one defines a measure on $X_{\Theta} = L \times {}^{P \times P^-}(H \times H)$ by considering the measures on U^-, U as measures on $P \setminus G, P^- \setminus G$, respectively, and integrating the measure on L over $P \setminus G \times P^- \setminus G$. Given the Plancherel formula for $L^2(L)$ (with respect to this measure):

$$||f||^2 = \int_{\hat{I}} ||M_{\tau}^* f||_{\tau}^2 \nu(\tau), \tag{15.61}$$

we get a Plancherel formula for X_{Θ} :

$$||f||^2 = \int_{\hat{L}} ||f||_{\tau}^{2} \nu(\tau), \tag{15.62}$$

where the norms $||f||'^2$ are obtained from the dual of the induced matrix coefficient $I_{\Theta^-}M_{\tau}$ (the last horizontal arrow of diagram (15.60)) and the unitary structure on $I_{P\times P^-}(\tilde{\tau}\otimes\tau)$ obtained from the unitary structure on τ and the given measures on U^-, U .

For those fixed measures, let $c(\tau)$ denote the constant which makes the following diagram commute:

$$I_{P^{-}}^{H}(\tau) \otimes I_{P^{-}}^{H}(\tilde{\tau}) \xrightarrow{T \otimes T} I_{P}^{H}(\tau) \otimes I_{P}^{H}(\tilde{\tau})$$

$$\downarrow C \qquad \qquad \downarrow C \qquad (15.63)$$

$$\mathbb{C} \qquad \xrightarrow{\cdot c(\tau)} \qquad \mathbb{C}.$$

Then from Theorem 15.6.2 we deduce the Plancherel formula for the group up to discrete Plancherel measures:

15.7.2. THEOREM. There is a direct sum decomposition: $L^2(H) \simeq \bigoplus_{L/\sim} L^2(H)_L$, where the sum is taken over conjugacy classes of Levi subgroups, and a Plancherel decomposition for $L^2(H)_L$:

$$\|\Phi\|^2 = \int_{\hat{L}_{\mathrm{disc}}/W(L,L)} \|\Phi\|_{I_P(\tau)}^2 c(\tau)^{-1} \nu_{\mathrm{disc}}(\tau).$$

Here the measure $\nu_{\rm disc}$ is the above Plancherel measure for $L^2(L)_{\rm disc}$, and the norm $\|\Phi\|_{I_P(\tau)}$ is the Hilbert-Schmidt norm of Φdg acting by convolution on $I_P(\tau)$, for any parabolic P with Levi subgroup L.

Notice that this Hilbert-Schmidt norm is precisely the norm obtained by the adjoint of M_{ρ} (first horizontal arrow of the diagram (15.60) and the unitary structure for $I_{P}(\tilde{\tau}) \otimes I_{P}(\tau)$.

Part 4 Conjectures

16. The local X-distinguished spectrum

Let K be a number field. In principle, the discussion of this section should be valid for global function fields, but since we will appeal to results of the rest of this paper, which used theorems on the structure of spherical varieties that have been proven only in characteristic zero, we restrict ourselves to number fields. We adopt the following notation: for an algebraic group G over K, we denote by [G] the adelic quotient $G(K)\backslash G(\mathbb{A}_K)$. We keep assuming that G is split over the global or the local field, unless otherwise stated.

16.1. Recollection of the Arthur conjectures [Art89]. To each local field k (resp. global field K) one associates a locally compact group \mathcal{L}_k (resp. \mathcal{L}_K) (the "Langlands group"), together with morphisms that fit into the following diagram:

$$\mathcal{L}_{K_v} \longrightarrow \mathcal{W}_{K_v} \longrightarrow \mathbb{R}_{>0}
\downarrow \qquad \qquad \downarrow \qquad \qquad \downarrow
\mathcal{L}_K \longrightarrow \mathcal{W}_K \longrightarrow \mathbb{R}_{>0}$$
(16.1)

where W denotes the Weil group. If k is nonarchimedean, the image of the map $\mathcal{L}_k \to \mathbb{R}_{>0}$ takes values in $q_k^{\mathbb{Z}}$, where q_k is the cardinality of the residue field of k. There is as yet no fully satisfactory definition of \mathcal{L}_K in the case of a number field; we use it primarily for motivational purposes.

For a summary of these these conjectural L-groups we refer to [Art02]. In particular, in the case of a local field, \mathcal{L}_k can be taken to be the Weil group of k in the archimedean case, and its product with SU_2 in the nonarchimedean case.

Functoriality implies the following property of \mathcal{L}_K :

By this we mean the following: Given any morphism $\varphi : \mathcal{L}_K \to \mathrm{GL}_m(\mathbb{C})$, the associated morphism $\mathcal{L}_{K_v} \to \mathcal{L}_K \to \mathrm{GL}_m(\mathbb{C})$ factors through $q_v^{\mathbb{Z}}$ for almost all places v; the image of a generator is a *Frobenius at* v, and these are necessarily Zariski dense in image(φ).

(1) A local, resp. global Arthur parameter is a homomorphism $\psi: \mathcal{L}_k \times \operatorname{SL}_2(\mathbb{C}) \to \check{G}$, resp. $\psi: \mathcal{L}_K \times \operatorname{SL}_2(\mathbb{C}) \to \check{G}$, so that the restriction to \mathcal{L}_k (\mathcal{L}_K) has bounded image and the restriction to SL_2 is algebraic. Let us call the restriction of the Arthur parameter to $\operatorname{SL}_2(\mathbb{C})$ the type or SL_2 -type of the Arthur parameter.

Given an Arthur parameter, its composition with the morphism

$$\mathcal{L}_k \to \mathcal{L}_k \times \mathrm{SL}_2(\mathbb{C}): \ w \mapsto \left(w \times \left(\begin{array}{cc} |w|^{\frac{1}{2}} & 0\\ 0 & |w|^{-\frac{1}{2}} \end{array}\right)\right) \text{ defines a Lang-}$$

lands parameter, which we shall refer to as the *associated* Langlands parameter.

- (2) Conjecturally, to each \check{G} -conjugacy class of local Arthur parameters $[\psi]$ we may "naturally" associate a finite set of unitary representations of G(k), the Arthur packet of ψ . These representations should all have the same *infinitesimal character* (using the notation of [Vog93a, §8]: in the nonarchimedean case two representations π, π' are said to have the same infinitesimal character if and only if the restrictions of their Langlands parameters to the Weil group coincide) and contain the L-packet of the associated Langlands parameter. They behave as expected with respect to parabolic induction: if the parameter ψ has image in a Levi M of the dual group, the associated packet consists of the irreducible summands of (unitarily) parabolically induced representations from the corresponding packet of M (where M is a Levi subgroup of G whose conjugacy class corresponds to the conjugacy class of M) [Art89, p.44. Notice that local Arthur packets are not, in general, mutually disjoint.
- (3) Over a local non-archimedean field k fix a hyperspecial maximal compact subgroup $G_0 \subset G$, if such exists. By the theory of principal series and the Satake transform, different isomorphism classes of G_0 -unramified (for short: "unramified") representations have different infinitesimal characters. Therefore, every Arthur packet contains at most one unramified representation.

In the reverse direction, suppose that ψ_1, ψ_2 are Arthur parameters whose associated packets contain the *same* unramified representation. Then $\psi_1 | \operatorname{SL}_2$ and $\psi_2 | \operatorname{SL}_2$ are conjugate:

In fact, set α_i to be the restriction of ψ_i to $G_m \subset SL_2$. We may suppose that $\alpha_i(G_m) \subset A^*$. It suffices to check that the derivative $d\alpha_1$ is conjugate to $d\alpha_2$. The assertion about "infinitesimal character" in A-packets shows that there exists bounded elements $g_i \in \check{G}$ (i.e. elements spanning relatively compact subgroups) so that $g_1\alpha_1(q^{1/2})$ is conjugate to $g_2\alpha_2(q^{1/2})$.

Let W be the Weyl group of $A^* \subset \check{G}$, and $\mathfrak{a}^* := \mathcal{X}(A^*)^* \otimes \mathbb{R}$. There is a natural projection eig: $\check{G} \to \mathfrak{a}^*/W$: it is the unique conjugacy-invariant continuous assignment that coincides with the natural projection $A^* \stackrel{H}{\to} \mathfrak{a}^* \to \mathfrak{a}^*/W$, where H is the "logarithm map" characterized by

$$|\alpha(t)| = e^{\langle \alpha, H(t) \rangle}, \quad \alpha \in \mathcal{X}(A^*), \quad t \in A^*.$$

Note also that if g_1, s are semisimple commuting elements and $g_1^{\mathbb{Z}}$ is relatively compact, then $\operatorname{eig}(g_1s) = \operatorname{eig}(s)$; this follows from the equality $H(g) = \frac{H(g^n)}{n}$ for elements of A^* . We conclude that $\operatorname{eig}(\alpha_1(q^{1/2})) = \operatorname{eig}(\alpha_2(q^{1/2}))$, whence the conclusion.

(4) To every \check{G} -conjugacy class of global Arthur parameters $[\psi]$ one should be able to associate a subspace $\mathcal{A}_{[\psi]}$ of the space of automorphic forms, such that

$$L^{2}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}_{K})) = \int \mathcal{A}_{[\psi]}\mu(\psi),$$

where the measure class of the above direct integral is the natural measure class⁷⁵ on the set of conjugacy classes of Arthur parameters.

We note that Arthur did not phrase the conjectures in terms of the subspaces $A_{[\psi]}$; however, other formulations e.g. [**GG05**] phrase the global conjecture in such a fashion, and the work of V. Lafforgue (see [**Lafb**, §2.2], which summarizes results proved in [**Lafa**]) gives further evidence for it over a global function field.

By means of Langlands' work on the spectral decomposition, this conjecture is equivalent to a description of the discrete spectrum (modulo center): Fix a unitary central idele class character Ω and consider the set of conjugacy classes of Arthur parameters which correspond to this idele class character and, moreover, their image does not lie in any proper Levi subgroup of \check{G} . Then we should have:

$$L^{2}(\mathbf{G}(K)\mathbf{Z}(\mathbb{A}_{K})\backslash\mathbf{G}(\mathbb{A}_{K}),\Omega)_{\mathrm{disc}} = \oplus L^{2}_{[\psi]}$$
(16.3)

where $L^2_{[\psi]} = \mathcal{A}_{[\psi]}$, with $[\psi]$ ranging over these classes. (For simplicity we will drop the brackets from $[\psi]$ from now on.)

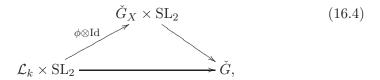
The spaces $\mathcal{A}_{[\psi]}$ have the following properties: each irreducible subquotient $\pi \subset \mathcal{A}_{[\psi]}$ factors as a restricted tensor product $\otimes_v' \pi_v$, and π_v belongs to the local Arthur packet associated to the pullback ψ_v of ψ to \mathcal{L}_{K_v} . For almost all v, π_v is the unramified representation corresponding to the associated Langlands parameter of ψ_v .

Of course, the above is by no means a description of all the properties that the Arthur packets are expected to have; just those which we will use in this paper.

16.2. The conjecture on the local spectrum (weak form). Let us recall that, to every quasi-affine spherical variety without roots of type N, we have associated a distinguished class of morphisms $f: \check{G}_X \times \mathrm{SL}(2) \to \check{G}$; the restriction $f|\mathrm{SL}(2)$ is a principal Levi for $\check{L}(X)$, and $f(\check{G}_X)$ is in the conjugacy class of the Gaitsgory-Nadler dual group $\check{G}_{X,GN}$.

 $^{^{75}\}mathrm{An}$ Arthur parameter ψ can be twisted by parameters into the centralizer of its image. Those form a locally compact abelian group, and the natural measure class on the orbit of ψ is the class of Haar measure.

Definition. An X-distinguished Arthur parameter is a commutative diagram of the form:



where ϕ is a tempered (i.e. bounded on \mathcal{L}_k) Langlands parameter into \check{G}_X and the right slanted arrow is the "canonical" one.

16.2.1. REMARK. Notice that an "**X**-distinguished Arthur parameter" is not really an Arthur parameter into \check{G} , but rather such a parameter together with a lift of it to $\check{G}_X \times \mathrm{SL}_2$. However, given an Arthur parameter into \check{G} we will say that it is **X**-distinguished if it admits such a lift, and we will also call its Arthur packet **X**-distinguished.

If X has roots of type N, we may, for the purposes of the weak conjecture, replace \check{G}_X in the above definition by the Gaitsgory-Nadler group $\check{G}_{X,GN}$; however, this is not appropriate for the more refined Conjecture 16.5.1.

The group \check{G}_X acts by conjugacy on the set of **X**-distinguished Arthur parameters, and, as happens with Langlands parameters, there is a natural class of measures on the set of \check{G}_X -conjugacy classes of **X**-distinguished Arthur parameters. Thus, we may for instance talk about "**X**-discrete parameters", meaning those diagrams (16.4) for which they image of ϕ does not lie in a proper Levi subgroup of \check{G}_X .

16.2.2. Conjecture (Local Conjecture – weak form). Let k be a local field. There is a direct integral decomposition:

$$L^{2}(\mathbf{X}(k)) = \int_{[\psi]} \mathcal{H}_{\psi}\mu(\psi), \qquad (16.5)$$

where:

- $[\psi]$ varies over \check{G}_X -conjugacy classes of **X**-distinguished Arthur parameters;
- μ is in the natural class of measures for X-distinguished Arthur parameters modulo conjugacy;
- H_ψ is isomorphic to a (possibly empty) direct sum of irreducible representations belonging to the Arthur packet associated to the image of ψ in Ğ.

This conjecture states only *necessary* conditions for a representation to belong weakly to $L^2(\mathbf{X}(k))$ (namely, it has to belong to the Fell closure of **X**-distinguished Arthur packets), and it also postulates that the X-discrete spectrum belongs to Arthur packets with **X**-discrete parameter. It may be,

 $^{^{76}}$ Recall from Theorem 2.2.3 that it is canonical up to \check{A}^* -conjugacy.

though, that the Hilbert space corresponding to such a parameter is zero; the refined Conjecture 16.5.1 will address that issue.

As far as we know, this conjecture was not anticipated elsewhere, e.g. in the (fairly extensive) study of the spectrum of symmetric varieties. The conjecture above gives (another) sense in which the Arthur packets are natural in representation theory.

REMARKS. (1) A corollary of the conjecture is this: the support of the discrete spectrum $L^2_{disc}(\mathbf{X}(k))$ is contained in Arthur packets associated to discrete series parameters $\mathcal{L}_k \to \check{G}_X$. More colloquially, if \mathbf{G}_X is a split k-group with dual group \check{G}_X , there should be a "lifting" from the discrete spectrum $L^2_{disc}(\mathbf{X}(k))$ to the discrete series of $\mathbf{G}_X(k)$.

We have, however, no understanding of which portion of the relatively discrete spectrum is in fact relatively supercuspidal, i.e. has compactly supported image in $C^{\infty}(X)$.

- (2) Another subtle issue is which elements of the Arthur packet for ψ actually show up in \mathcal{H}_{ψ} above; the conjecture as written gives no information on this. For a start of a discussion of this point, see §16.5.
- (3) It is essential that the conjecture was formulated with Arthur packets, rather than the associated L-packets. It is possible, for instance, for the Arthur type to be nontrivial but yet $L^2(\mathbf{X}(k))$ contains weakly a tempered representation; an example is given by \mathbf{G} of type G_2 acting on the level set of an invariant quadratic form in its standard (7-dimensional) representation.
- (4) The conjecture predicts that $L^2(X)$ is tempered when $\check{G}_X = \check{G}$. Although we do not have a general proof, Theorem 6.2.1 proves this in many cases.
- (5) Suppose that (G, X) is symmetric over $k = \mathbb{R}$. Let \mathfrak{a} be a "maximally σ -split torus" and let \mathfrak{l} be its centralizer, the Lie algebra of the Levi subgroup associated to X. It is known by the work of Flested-Jensen and Matsuki, Oshima that any discrete series for $L^2(X)$ is the cohomological induction from \mathfrak{l} of a one-dimensional representation (see [Vog88] for a summary of these results.)

The results of Adams and Johnson [AJ87] likely could be used to give evidence that these indeed belong to Arthur packets as predicted by the conjecture, but we have not verified the details.

- (6) Gan and Gomez have shown that the Howe duality correspondence can be used to exhibit establish the conjecture for certain pairs (G, X) ([GG14]).
- (7) Conjecture 16.2.2 addresses the *unitary* spectrum; what of other **X**-distinguished representations? In other words, what if we want to decompose $C^{\infty}(X)$ rather than $L^{2}(X)$? This is a more subtle issue; the trivial representation is always **X**-distinguished locally

and globally, but does not belong to the Arthur parameter for an X-distinguished Arthur parameter (except, of course, when X = a point).

16.3. A global to local argument. In this section, we admit the Arthur conjectures as formulated in $\S16.1$, and establish some evidence – Theorem 16.3.1 below – for our Conjecture 16.2.2, using a local-global argument.

We continue to denote by K a global field, with ring of adeles \mathbb{A}_K . We will only discuss here the case where \check{G}_X is a subgroup of \check{G} , i.e. $\check{G}_X = \check{G}_{X,GN}$, so an **X**-distinguished Arthur parameter is indeed an Arthur parameter into \check{G} :

$$\psi: \mathcal{L}_{(k \text{ or } K)} \times \operatorname{SL}_2 \to \check{G}$$

factoring through $\check{G}_X \times \mathrm{SL}_2$ in the specified way.

Call such a local Arthur parameter ψ weakly X-distinguished if:

- (1) $\psi | SL_2$ is conjugate to the SL_2 -type of **X**;
- (2) The semisimple part of every $\psi(g)$, $g \in \mathcal{L}_k \times \mathrm{SL}_2$, is conjugate to an element in the image of $\check{G}_X \times \mathrm{SL}_2$.

In general this is strictly weaker than \mathbf{X} -distinguished, but in some instances is equivalent to it. A typical example is: $\mathbf{G} = \mathbf{GL}_{2n}$ acting on $\mathbf{X} = \{\text{alternating 2-forms.}\}$ In this case, $\check{G}_X = \check{G}_{X,GN}$ is the centralizer of the SL_2 -type of \mathbf{X} , and so requirement (1) above already guarantees that any weakly distinguished parameter is \mathbf{X} -distinguished.

Let $\widehat{\mathbf{G}(K_v)}_{X-wkdist}$ be the set of unitary representations of $\mathbf{G}(K_v)$ that belong to a weakly **X**-distinguished Arthur packet.

The following theorem will require some local input from [Sak13]: It was proven in [Sak13, Theorem 9.0.1] that, under certain combinatorial assumptions on the spherical variety \mathbf{X} , its unramified L^2 -spectrum is supported on, what turns out to be, the set of \mathbf{X} -distinguished unramified Arthur parameters. This is a result of explicit computation, and we do not know a conceptual proof or reasoning for it that does not invoke the Langlands dual. Similar results have been obtained in other (including non-split) cases by Hironaka, Offen and others, e.g. [Hir99, Off04].

If we invoke Langlands duality, though, it is easy to see on the L-group side why this is so. The case $\check{G}_X \subset \check{G}$ that we are considering is particularly straightforward, because for the image of an X-distinguished Arthur parameter into \check{G} to be unramified, the parameter itself had to be unramified in the first place. But an unramified parameter $\mathcal{L}_k \to \check{G}_X$ has image in a Cartan subgroup, hence unramified representations can only appear in the most continuous part of the spectrum.

Theorem 9.0.1 in *loc.cit*. shows that, under assumptions on **X**, the Plancherel measure for $L^2(\mathbf{X}(k))^{K_G}$ where k is a non-archimedean place and K_G is a hyperspecial maximal compact subgroup of $\mathbf{G}(k)$, is supported on the set of unramified representations which are subquotients of $I_{P(X)}(\chi)$,

the normalized-induced representation obtained from a (unitary) character $\chi \in \check{A}_X$ of P(X). The conditions include that \mathbf{X} is homogeneous affine (that is, $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$ with \mathbf{H} reductive) or Whittaker-induced from a homogeneous affine variety (in the sense of our §18.3). There are also certain combinatorial conditions to be checked, which are probably satisfied automatically under the previous conditions; a table of some varieties satisfying those appears at the end of *loc. cit.*

In terms of Arthur parameters, this theorem states that there is a Plancherel decomposition the unramified spectrum of $\mathbf{X}(k)$ in terms of representations with \mathbf{X} -distinguished unramified Arthur parameter. Notice that by property (3), §16.1 of Arthur packets, this is the *only* Arthur parameter that these representations admit.

16.3.1. Theorem. Assume the Arthur conjectures (as formulated in $\S 16.1$). Suppose that:

- (i) the set of points in $\mathbf{X}(K)$ with anisotropic stabilizers is dense in $\mathbf{X}(K_w)$;
- (ii) \mathbf{X}_{K_v} satisfies the conditions of [Sak13, Theorem 9.0.1] (see discussion before Theorem) for almost all places v.

Then the support of $L^2(\mathbf{X}(K_w))$ is contained in the closure of $\widehat{\mathbf{G}}(K_w)_{X-wkdist}$ in the Fell topology.

Note that assumption (i) is trivially satisfied if \mathbf{G} is anisotropic (for this theorem we don't need to assume that \mathbf{G} itself is split); as we will see in the course of proof, it could also be replaced by either of the following two assumptions:

- there exists a place w for which there is a **X**-distinguished super-cuspidal representation;
- the convolution of the invariant measure on $[\mathbf{G}_{x_0}] \subset [\mathbf{G}]$ (where $x_0 \in \mathbf{X}(K)$) with a compactly supported measure on $\mathbf{G}(\mathbb{A}_K)$ is given by an L^2 -density on $[\mathbf{G}]$.

The proof is inspired by the Burger-Sarnak principle as well as work of Clozel. It is based on a globalization result that is perhaps of independent interest (although it is very closely related to other results, it has a slightly different range of applicability, since it is based on the use of the Fell topology).

Let us denote by **Z** the center of **G**. An automorphic discrete series is an irreducible unitary representation π of $\mathbf{G}(\mathbb{A}_K)$ together with a non-zero morphism $\nu : \pi \to L^2(\mathbf{G}(K)\mathbf{Z}(\mathbb{A}_K)\backslash \mathbf{G}(\mathbb{A}_K),\Omega)$ (where Ω is the central character of π , an idele class character). Let $x_0 \in \mathbf{X}(K)$. An automorphic discrete series (π, ν) is (\mathbf{X}, x_0) -distinguished if the functional

$$f \mapsto \int_{[\mathbf{G}_{x_0}]} \nu(f)$$

is non-zero on the subspace of smooth vectors of π .⁷⁷ The restriction to smooth vectors is an important technical point: The elements of π are L^2 -functions on the automorphic space; but smooth vectors are given by genuine functions, and are defined on measure zero subsets such as $[\mathbf{G}_{\pi_0}]$.

If S is any finite set of places of K, we write K_S for $\prod_{v \in S} K_v$ and K^S for the restricted direct product (with respect to integers) $\prod_{v \notin S}' K_v$.

16.3.2. THEOREM. Suppose that, for some $x_0 \in \mathbf{X}(K)$, \mathbf{G}_{x_0} is anisotropic; write $X_S^{(0)}$ for $x_0 \cdot \mathbf{G}(K_S) \subset \mathbf{X}(K_S)$. Suppose that σ is an irreducible representation of $\mathbf{G}(K_S)$ which belongs to the support of Plancherel measure for $L^2(X_S^{(0)})$. Then there exist a sequence of (\mathbf{X}, x_0) -distinguished automorphic representations π_i whose restrictions $\pi_{i,S}$ to $\mathbf{G}(K_S)$ converge, in the Fell topology, to σ .

We now prove that Theorem $16.3.2 \implies$ Theorem 16.3.1, and return to the proof of Theorem 16.3.2 in §16.4.

The key point of the argument is due to L. Clozel ([Clo04]) and lies in the beautiful idea of considering two places simultaneously. Indeed, let $S = \{v, w\}$ comprise two places, so that v is a "good place" for (\mathbf{G}, \mathbf{X}) , i.e. a place where [Sak13, Theorem 9.0.1] holds. That theorem computes precisely the decomposition of the unramified part of $L^2(\mathbf{X}(K_v))$.

Now let σ_w be an arbitrary unitary representation that occurs weakly in $L^2(\mathbf{X}(K_w))$. Let σ_v be an unramified representation occuring weakly in $L^2(\mathbf{X}(K_v))$. Then $\sigma_S := \sigma_v \otimes \sigma_w$ occurs weakly in $L^2(\mathbf{X}(K_S))$. By Theorem 16.3.2, there exists a sequence of \mathbf{X} -distinguished automorphic forms π_j whose local constituent at v, w approach σ_v, σ_w respectively, in the Fell topology. Choose an Arthur parameter ψ_j with the property that the global packet \mathcal{A}_{ψ_j} contains a representation isomorphic to π_j .

Fix now $\psi = \psi_j$. Let Q be the Zariski-closure of the image of $\psi | \mathcal{L}_K$. The map

$$\psi: \mathcal{L}_K \to \mathcal{L}_K \times \mathrm{SL}_2 \longrightarrow \check{G}$$
 (16.6)

visibly factors through

$$\left(\psi, \left(\begin{array}{cc} |\cdot|^{1/2} & 0\\ 0 & |\cdot|^{-1/2} \end{array}\right)\right) : \mathcal{L}_K \longrightarrow \check{G}$$
 (16.7)

where we regard the target $G_{\rm m}$ as the torus in SL_2 .

By property (3), §16.1 of Arthur packets, together with the theorem of [Sak13] that computes the unramified Plancherel measure, the SL_2 -type of ψ_j (for large enough j) is the same as the SL_2 -type $\iota: SL_2 \to \check{G}$ associated to the spherical variety (viz. the principal SL_2 for $\check{\mathbf{L}}_{\mathbf{X}}$).

⁷⁷For this definition we need, of course, the integrals to converge. Ignoring analytic difficulties, we could also describe the **X**-distinguished spectrum as follows: There is a canonical intertwiner: $C_c^{\infty}(\mathbf{X}(\mathbb{A}_K)) \to C^{\infty}(\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}_K))$ (summation over $\mathbf{X}(K)$). It seems natural to define *globally distinguished* as "the image of the adjoint morphism." However, this leads to difficulties when not all point stabilizers on **X** are anisotropic.

Here, we have used a strengthening of (3) on page 239, given a sequence of unramified representations σ_j of G belonging to Arthur packets with a given SL_2 -type ι_1 , and another unramified representation σ_{∞} which is a limit point for the Fell topology of the σ_j , then any Arthur packet containing σ_{∞} also has SL_2 -type ι . Indeed, it follows (see e.g. [Tad88, Theorem 2.2]) the Satake parameters of σ_j approach that of σ_{∞} , and then we argue as on page 239, where now equality is replaced by "almost equality," which is enough for our purposes because the set of possibilities for "eig $(\alpha_i(q^{1/2}))$ " (as in the last line of the argument on page 239 is discrete.

Let Fr_s denote the image, under (16.7), of a Frobenius element at the unramified place s. The Fr_s , when we vary the place s, are Zariski-dense in $Q \times G_{\mathrm{m}}$. Indeed, by assumption (16.2), their Zariski closure gives a subgroup $Q' \subset Q \times G_{\mathrm{m}}$ projecting onto Q; if $Q' \neq Q \times G_{\mathrm{m}}$, then there is an integer $m \geq 1$ and a character $\chi: Q \to G_{\mathrm{m}}$ so that $Q' = \{(q, x): x^m = \chi(q)\}$. But this cannot be: if we write (q_s, x_s) for the image of Fr_s in $Q(\mathbb{C}) \times G_{\mathrm{m}}(\mathbb{C})$, then $q_s^{\mathbb{Z}}$ is precompact whereas $x_s^{\mathbb{Z}}$ is not.

On the other hand, the aforementioned theorem of [Sak13] shows more precisely that the image of every Fr_s inside \check{G} , under (16.6), is conjugate to an element of $f(\check{G}_X \times \operatorname{SL}_2)$. (Here, $f: \check{G}_X \times \operatorname{SL}_2 \to \check{G}$ is the distinguished morphism associated to X.)

Therefore, a Zariski-dense set of elements in the image Q^* of $Q \times G_{\mathrm{m}}$ inside \check{G} are conjugate to elements of $f(\check{G}_X \times \mathrm{SL}_2)$.

We have just seen that any conjugacy-invariant function on \check{G} , zero on $f(\check{G}_X \times \operatorname{SL}_2)$, must be identically zero on Q^* . Since conjugacy-invariant algebraic functions separate semisimple conjugacy classes, we conclude that every semisimple element in Q^* is conjugate to $f(\check{G}_X \times \operatorname{SL}_2)$. The claimed assertion follows.

16.4. Proof of Theorem 16.3.2. The theorem follows immediately from:

Let $\Omega \subset G_S = \mathbf{G}(K_S)$ be compact and $f \in C_c(X_S^{(0)})$. Then $y \mapsto \langle yf, f \rangle$, for $y \in \Omega$, is a convex combination of diagonal matrix coefficients of (\mathbf{X}, x_0) -distinguished automorphic representations.

Indeed, the quoted statement means that $L^2(X_S^{(0)})$, considered as a G_S -representation, is "weakly contained" in a direct sum $\bigoplus \pi_i$ of the various (\mathbf{X}, x_0) -distinguished automorphic representations. According to $[\mathbf{Dix77}, \mathbf{Proposition} \ 8.6.8]$ – see also Theorem 3.4.4, *loc. cit.* for the definition of weak containment – this means that the support of Plancherel measure for $L^2(X_S^{(0)})$ is contained in the support of Plancherel measure for $\bigoplus \pi_i$, in particular, in the closure of the set of restrictions $\pi_i|G_S$.

PROOF. We write the proof in the S-arithmetic language, rather than adelically. Fix a function $f \in C_c(X_S^{(0)})$.

Let H_S be the point stabilizer of $x_0 \in X_S^{(0)}$ in G_S , so that $H_S = \mathbf{H}(K_S)$. Choose a congruence subgroup Γ of $\mathbf{G}(K_S) = G_S$ and U a compact subset of G_S so that, with $\Gamma_H := \mathbf{H}(K) \cap \Gamma$, we have:

- (i) $U = U^{-1}$ and $\Omega \subset U$;
- (ii) x_0U contains the support of gf for any $g \in \Omega$;
- (iii) $\Gamma_H \cdot U \supset H_S$
- (iv) $U_4 \cdot H_S \cap \Gamma = \Gamma_H$, where we write $U_2 = U \cdot U, U_3 = U \cdot U \cdot U$ and so on.

This can be done: First of all choose Γ and choose U satisfying (i) – (iii). Now we may shrink Γ , leaving Γ_H unchanged, so that (iv) is satisfied, by passing from Γ to a subgroup of the form $\pi_N^{-1}\pi_N\Gamma_H$, where π_N is the "reduction modulo N" map, for a suitable large ideal N.

Now fix $q \in \Omega$ and set

$$F(g) := \sum_{\gamma \in \Gamma_H \setminus \Gamma} f(x_0 \gamma g),$$

a compactly supported function on $\Gamma \backslash G_S$. (Indeed, its support is contained in $\Gamma \backslash \Gamma U_2$).

We are going to show that

$$\langle yf, f \rangle_{X_0^{(S)}} = c \langle yF, F \rangle_{\Gamma \backslash G_S}, \quad (y \in \Omega).$$
 (16.8)

where the positive constant c depends only on normalization of measure. This will conclude the proof:

If π is an irreducible G_S -subrepresentation of functions on $\Gamma \backslash G_S$ (we do not require square integrability!) and there exists $v \in \pi$ such that

$$\langle F, v \rangle_{\Gamma \backslash G_S} \neq 0,$$

then π is distinguished, because

$$\int_{\Gamma \backslash G_S} F(g) \overline{v(g)} dg = \int_{\Gamma_H \backslash G_S} f(x_0 g) \overline{v(g)} dg = \int_{h \in \Gamma_H \backslash H_S} \int_{g \in H_S \backslash G_S} f(x_0 g) \overline{v(hg)} dg,$$

so some translate v^g has nonvanishing period over $\int_{\Gamma_H \backslash H_S}$. In particular, the expression $\langle gF, F \rangle$ is in fact – after spectrally expanding F – a convex combination of diagonal matrix coefficients of (\mathbf{X}, x_0) -distinguished representations, as required.

Thus indeed (16.8) will conclude the proof.

We compute, by unfolding, that

$$\langle yF, F \rangle_{L^{2}(\Gamma \backslash G_{S})} = \int_{g \in \Gamma \backslash G_{S}} yF(g) \overline{\sum_{\gamma \in \Gamma_{H} \backslash \Gamma} f(x_{0}\gamma g)} dg$$

$$= \int_{g \in \Gamma_{H} \backslash G_{S}} yF(g) \overline{f(x_{0}g)} dg$$

$$= \int_{\Gamma_{H} \backslash G_{S}} dg \ f(x_{0}gy) \overline{f(x_{0}g)} + \int_{\Gamma_{H} \backslash G_{S}} dg \ \sum_{\gamma \in \Gamma_{H} \backslash \Gamma - \{1\}} f(x_{0}\gamma gy) \overline{f(x_{0}g)}$$

$$(16.9)$$

We claim that the final term is zero: If not, there exists $\gamma \in \Gamma - \Gamma_H$ and $g \in G_S$ such that $f(x_0 \gamma gy)$ and $f_2(x_0 g)$ are both nonzero. In particular, by (ii)

$$\gamma gy \in H_SU$$
 and $g \in H_SU$.

Adjusting γ on the left by Γ_H and using (iii), we may suppose that $\gamma gy \in U_2$. Therefore,

$$\gamma = (\gamma g y) \cdot y^{-1} \cdot g^{-1} \in U_4 H_S,$$

a contradiction to (iv).

16.4.1. Remark. This has the following corollary:

If σ is automorphically isolated as well as in the support of $L^2(X_S^{(0)})$, it is the local constituent of an (\mathbf{X}, x_0) -distinguished global representation.

Here, we say that a unitary irreducible representation σ of $\mathbf{G}(K_S)$ is automorphically isolated if there do not exist a sequence σ_i of unitary $\mathbf{G}(K_S)$ -representations, each of which occurs as the local constitutent of an automorphic representation, which converge to σ in the Fell topology.

Results of a similar nature are well-known; see e.g. [PSP08] when the σ are supercuspidal. The condition noted above (automorphically isolated) is very slightly weaker. For instance, every discrete series for GL_n is automorphically isolated, and it seems likely that discrete series representations are *always* automorphically isolated (although we do not know how to prove it – it would be interesting to verify that this is a consequence of Arthur's conjectures, or to verify it for the other classical groups).

16.5. Pure inner forms. We now formulate a refined version of the prior Local Conjecture 16.2.2.

For the purposes of the present subsection we assume:

- (1) The spherical variety \mathbf{X} has no roots of type N; in particular, its dual group \check{G}_X is defined.
- (2) The center of G acts faithfully on X. (If this is not the case, one should replace G by its quotient by the kernel of the action of $\mathcal{Z}(G)$.)

By a "pure inner form" of **G** we understand an isomorphism class α of left **G**-torsors. We call an isomorphism class α of left **G**-torsors, with the property that $\mathbf{X} \times^{\mathbf{G}} \mathbf{T}(k) \neq \emptyset$ for **T** in the class α , a "pure inner form" of **X**. We denote by \mathbf{G}^{α} the automorphism group of a torsor in the class α , and we denote by G^{α} its k points. We give further discussion and examples (see Examples 16.5.5–16.5.7) after we formulate the conjecture.

Let us recall that Vogan [Vog93b] has proposed a version of Arthur's conjectures whereby the Arthur packet should be considerd, in fact, as a collection of representations of varying pure inner forms of \mathbf{G} . In particular each element of the Arthur packet defines a representation of the group $\prod_{\beta} G^{\beta}$, where β ranges over pure inner forms of G, and the representation is understood to be nontrivial on only one direct factor.

16.5.1. Conjecture – strong form). There is a direct integral decomposition:

$$\bigoplus_{\alpha} L^{2}(X^{\alpha}) = \int_{[\psi]} \mathcal{H}_{\psi} \mu(\psi), \qquad (16.10)$$

where

- α parametrizes pure inner forms of X;
- we regard both sides as representations of the product:

$$\prod_{\beta} G^{\beta} \tag{16.11}$$

of all pure inner forms of G – the right-hand side as discussed above, and the left-hand side by means of the evident map from pure inner forms of X to pure inner forms of G;

- $[\psi]$ varies over G_X -conjugacy classes of **X**-distinguished Arthur parameters;
- μ is in the natural class of measures for X-distinguished Arthur parameters modulo conjugacy;
- H_ψ is isomorphic to a multiplicity-free direct sum of irreducible representations belonging to the (Vogan) Arthur packet associated to the class [ψ];
- for μ -almost all ψ , the spaces \mathcal{H}_{ψ} are non-zero.

Note that, when multiple inner forms of X correspond to the same inner form of G, the *same* irreducible representations of this inner form may appear multiple times on both sides of (16.10); this doesn't contradict the multiplicity-freeness requirement, since we consider elements of the Vogan-Arthur packet as representations of (16.11) (where the isomorphic inner forms can appear as distinct factors).

In comparison to the weak version 16.2.2, the present form states that the condition on Arthur parameters to be X-distinguished is also *sufficient* for the A-packet to be distinguished, as long as we take pure inner forms into consideration.

We have also postulated that the spaces \mathcal{H}_{ψ} should be multiplicity-free. In the case of GL_n , where A-packets are singletons, this means that the multiplicity of a given representation on $\bigoplus_{\alpha} L^2(X^{\alpha})$, at least generically in the sense of Plancherel measure, is given by the number of lifts of its Arthur parameter to an **X**-distinguished Arthur parameter. For example, the multiplicity statement is true for the most continuous spectrum of X under the assumptions of the Scattering Theorem 7.3.1: indeed, the spectrum of the most degenerate boundary degeneration X_{\emptyset} is a multiplicity-free direct integral over Arthur parameters with "Langlands part" into the maximal torus A_X^* of \check{G}_X , and the corresponding "most continuous spectrum" $L^2(X)_{\emptyset}$ is a multiplicity-free direct integral over W_X -conjugacy classes of such parameters. However, we should point out that there is not enough evidence about whether the multiplicity statement is correctly formulated for ramified representations in the case of nontrivial Arthur-SL₂.

In the remainder of this section, we examine more carefully the notion of pure inner form of X:

Consider the quotient stack: $[\mathbf{X} \times \mathbf{X}/\mathbf{G}]$ (we understand the diagonal action of \mathbf{G} without putting brackets). We denote by $[\mathbf{X} \times \mathbf{X}/\mathbf{G}](k)$ the set of isomorphism classes of k-objects of the stack. By abuse of language, we will be calling them "k-points". They consist of isomorphism classes of diagrams:

$$T \rightarrow X \times X$$
,

where **T** is a (right) **G**-torsor and the map is **G**-equivariant. Two such diagrams are isomorphic if there is an isomorphism of torsors which makes the composite commute.

In what follows we will denote isomorphism classes of \mathbf{G} -torsors by small Greek exponents (they correspond bijectively to elements of $H^1(k, \mathbf{G})$), and the exponent will appear on the opposite side from which \mathbf{G} acts. For instance, ${}^{\alpha}\mathbf{T}$ denotes a right \mathbf{G} -torsor "in the class α "; by composing with the inverse map we get a left \mathbf{G} -torsor which will is denoted \mathbf{T}^{α} . The \mathbf{G} -automorphism group of a torsor is an inner form \mathbf{G} ; for the torsors ${}^{\alpha}\mathbf{T}$ and \mathbf{T}^{α} , this form will be denoted by \mathbf{G}^{α} and will act on the left, resp. on the right. Notice the canonical $\mathbf{G}^{\alpha} \times \mathbf{G}^{\alpha}$ -equivariant isomorphism: ${}^{\alpha}\mathbf{T} \times {}^{\mathbf{G}}\mathbf{T}^{\alpha} \simeq \mathbf{G}^{\alpha}$ (here, unlike the rest of the paper, the left multiplication of \mathbf{G}^{α} on itself is defined as a left action).

Given a **G**-variety **V** and a left **G**-torsor \mathbf{T}^{α} , we denote by \mathbf{V}^{α} the \mathbf{G}^{α} -variety: $\mathbf{V} \times^{\mathbf{G}} \mathbf{T}^{\alpha}$. The following is easy to see by applying the $\times^{\mathbf{G}} \mathbf{T}^{\alpha}$ operation:

16.5.2. LEMMA. The set of isomorphism classes of **G**-morphisms: ${}^{\alpha}\mathbf{T} \to \mathbf{V}$ is in natural bijection with the set of $\mathbf{G}^{\alpha}(k)$ -orbits on $\mathbf{V}^{\alpha}(k)$. In particular, the existence of such a morphism is equivalent to the statement: $\mathbf{V}^{\alpha}(k) \neq \emptyset$.

To apply this to $\mathbf{V} = \mathbf{X} \times \mathbf{X}$, where \mathbf{X} is our spherical \mathbf{G} -variety and \mathbf{G} acts diagonally on \mathbf{V} , we notice that $\mathbf{V}^{\alpha} = \mathbf{X}^{\alpha} \times \mathbf{X}^{\alpha}$. Indeed, if \mathbf{V} carries

an action of a larger group $\tilde{\mathbf{G}} \supset \mathbf{G}$ and $\mathbf{F}^{\alpha} := \tilde{\mathbf{G}} \times^{\mathbf{G}} \mathbf{T}^{\alpha}$, a $\tilde{\mathbf{G}}$ -torsor, then obviously $\mathbf{V} \times^{\mathbf{G}} \mathbf{T}^{\alpha} = \mathbf{V} \times^{\tilde{\mathbf{G}}} \mathbf{F}^{\alpha}$; in our case, $\tilde{\mathbf{G}} := \mathbf{G} \times \mathbf{G}$ and $\mathbf{F}^{\alpha} = \mathbf{T}^{\alpha} \times \mathbf{T}^{\alpha}$, so we get:

16.5.3. LEMMA. The set of isomorphism classes of \mathbf{G} -morphisms: ${}^{\alpha}\mathbf{T} \to \mathbf{X} \times \mathbf{X}$ is in natural bijection with the set of $\mathbf{G}^{\alpha}(k)$ -orbits on $\mathbf{X}^{\alpha}(k) \times \mathbf{X}^{\alpha}(k)$. In particular, the existence of such a map is equivalent to the statement: $\mathbf{X}^{\alpha}(k) \neq \emptyset$.

16.5.4. COROLLARY. We have:

$$[\mathbf{X} \times \mathbf{X}/\mathbf{G}](k) = \sqcup_{\alpha} \mathbf{X}^{\alpha}(k) \times \mathbf{X}^{\alpha}(k)/\mathbf{G}^{\alpha}(k), \tag{16.12}$$

where α runs over all pure inner forms of X.

16.5.5. EXAMPLE. If $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$, the pure inner forms of \mathbf{X} correspond to \mathbf{G} -torsors obtained by reduction of \mathbf{H} -torsors, i.e. to the image of the map: $H^1(k, \mathbf{H}) \to H^1(k, \mathbf{G})$.

In fact, we have an isomorphism of stacks: $\mathbf{X} \times \mathbf{X}/\mathbf{G} \simeq \mathbf{H}\backslash\mathbf{G}/\mathbf{H}$ and \mathbf{H} has a fixed point on $\mathbf{H}\backslash\mathbf{G}$, one gets from Lemma 16.5.2:

$$[\mathbf{X} \times \mathbf{X}/\mathbf{G}](k) = \sqcup_{\beta \in H^1(k,\mathbf{H})} (\mathbf{H}^{\beta} \backslash \mathbf{G}^{\beta})(k) / \mathbf{H}^{\beta}(k). \tag{16.13}$$

Here, similarly, we denote by \mathbf{H}^{β} the automorphism group of an \mathbf{H} -torsor ${}^{\beta}\mathbf{S}$ in the class of β and by \mathbf{G}^{β} the isomorphism class of its reduction to a \mathbf{G} -torsor ${}^{\beta}\mathbf{T}$ (i.e. ${}^{\beta}\mathbf{T} = {}^{\beta}\mathbf{S} \times^{\mathbf{H}} \mathbf{G}$); the action of \mathbf{H}^{β} on ${}^{\beta}\mathbf{S}$ gives rise to a natural injection: $\mathbf{H}^{\beta} \hookrightarrow \mathbf{G}^{\beta}$.

- 16.5.6. EXAMPLE. The pure inner forms of $\mathbf{X} = \mathbf{a}$ point coincide with the isomorphism classes of \mathbf{G} -torsors, despite the fact that all varieties \mathbf{X}^{α} are isomorphic.
- 16.5.7. EXAMPLE. Let $V \subset W$ be two non-degenerate quadratic spaces of codimension one in each other and let $\mathbf{X} = \mathbf{H} \backslash \mathbf{G} = \mathbf{SO}(V) \backslash (\mathbf{SO}(V) \times \mathbf{SO}(W))$.

Isomorphism classes of $\mathbf{SO}(V)$ -torsors correspond canonically to isomorphism classes of quadratic spaces of the same dimension and discriminant as V, and similarly for $\mathbf{SO}(W)$ -torsors. The reduction of a $\mathbf{SO}(V)$ -torsor to an $\mathbf{SO}(W)$ -torsor corresponds to the operation $V^{\alpha} \mapsto V^{\alpha} \oplus k$ (orthogonal direct sum), where V^{α} is a quadratic space corresponding to the given torsor. Pure inner forms of \mathbf{X} correspond to isomorphism classes of quadratic spaces $V^{\alpha} \subset W^{\alpha}$, where V^{α} has the same dimension and discriminant as V and $W^{\alpha} \simeq V^{\alpha} \oplus k$. This is the setting of the Gross-Prasad conjectures [GP92]; these conjectures have been now largely established through the work of Waldspurger and (for the unitary analogue) Beuzart-Plessis ([Wal12b, Wal12c, Wal12a, BPar, BP]).

16.5.8. Remark. Suppose that $\mathbf{X} = \mathbf{H} \backslash \mathbf{G}$, and that $\mathbf{H} = \mathbf{M} \ltimes \mathbf{U}$ is the Levi decomposition of \mathbf{H} . Let $\Lambda : \mathbf{U} \to \mathbf{G}_a$ be a homomorphism which is fixed by \mathbf{M} . Extend it to a homomorphism $\Lambda : \mathbf{H} \to \mathbf{G}_a$ by making it

trivial on **M**. We are interested in defining "pure inner forms" for the space $L^2(X, \mathcal{L}_{\psi})$, where $\psi : k \to \mathbb{C}^{\times}$ is a character and \mathcal{L}_{ψ} is the complex line bundle defined by $\psi \circ \Lambda$.

Since unipotent groups in characteristic zero have trivial Galois cohomology, all pure inner forms of \mathbf{X} correspond to \mathbf{M} -torsors. Let $c: \operatorname{Gal}(\bar{k}/k) \to \mathbf{M}(\bar{k})$ be a cocycle; by inner automorphisms, it defines an inner twist \mathbf{H}^{α} of \mathbf{H} . Since the action of \mathbf{M} preserves the morphism $\Lambda: \mathbf{U} \to \mathbf{G}_{\mathbf{a}}$ (where \mathbf{M} acts trivially on $\mathbf{G}_{\mathbf{a}}$), the "inner twist" of Λ by c is defined over k, that is, we have a morphism $\Lambda^{\alpha}: \mathbf{H}^{\alpha} \to \mathbf{G}_{\mathbf{a}}$.

With this convention, then, the Conjecture also applies to "Whittaker-type" induction.

17. Speculation on a global period formula

Throughout this section we adopt the following notation: For K a global field – which we understand as fixed – and \mathbf{G} any algebraic group over K, we denote by $[\mathbf{G}]$ the adelic quotient $\mathbf{G}(K)\backslash\mathbf{G}(\mathbb{A}_K)$.

In this section we wish to discuss a potential generalization of the work of Ichino–Ikeda [II10] to all spherical varieties. Our central conjecture, speaking somewhat imprecisely, gives a link between the *local Plancherel formula* and *global periods*.

We assume that **X** is *homogeneous affine*, i.e. the stabilizers of points on **X** are reductive; the discussion is also valid for "parabolically-induced" or "Whittaker-induced" varieties of this form. For a discussion of the general non-affine case, cf. [Sak12]; our conjectures naturally extend to this more general case, but because of the speculative nature of this case we will ignore it in our formulations and only appeal to a special case in Theorem 18.4.1.

We will consider automorphic representations that embed weakly in $L^2([\mathbf{G}])$, hence: abstract irreducible unitary representations $\pi \simeq \otimes'_v \pi_v$ of $\mathbf{G}(\mathbb{A}_K)$ which admit a tempered embedding:

smooth subspace of
$$\pi \hookrightarrow C^{\infty}([\mathbf{G}])$$
,

i.e. the image is in $L^{2+\varepsilon}$ for every $\varepsilon > 0$. The embedding will not be part of the data, but it will be assumed to be unitary whenever the image is discrete modulo center. The normalization of embeddings corresponding to the continuous spectrum will be discussed in §17.5.

For the automorphic representations π that we will encounter, we make the following multiplicity-one assumption:

For all places v of K, we have dim $\operatorname{Hom}_{\mathbf{G}(K_v)}(\pi_v, C^{\infty}(\mathbf{X}(K_v))) \leq 1$.

Strictly we should write π_v^{∞} above, instead of π_v – which is by definition unitary. We will allow ourselves this imprecision at some points below.

As the work of Jacquet [Jac01] shows (see also [FLO12]), the multiplicity-one assumption is too restrictive – one could have Euler products even without it; however, we will contend ourselves to provide some conjectures in this more restrictive setting.

17.1. Tamagawa measure. We use throughout Tamagawa measures for [G], $G(\mathbb{A}_K)$ and, more generally, the adelic and local points of smooth homogeneous varieties.

To define Tamagawa measure we proceed as follows: Let μ be the measure on \mathbb{A}_K that assigns mass 1 to the quotient \mathbb{A}_K/K , and fix a factorization $\mu = \mu_v$, where μ_v is a measure on K_v . Fix a K-rational top differential form ω . Then (using the choice of μ_v) we obtain a volume form $|\omega|_v$ on $\mathbf{G}(K_v)$. For v a finite place, set c_v to be the $|\omega_v|$ -mass of $\mathbf{G}(\mathfrak{o}_v)$ and take $c_v = 1$ otherwise. For all but finitely many places, c_v is the value of a certain local L-factor, and we can interpret the (non-convergent, in general) Euler product $C = \prod_v c_v$ accordingly. If \mathbf{G} has trivial k-character group, then $C \neq 0$ and we define the Tamagawa measure:

$$C^{-1} \cdot \left(\prod_{v} c_v |\omega_v| \right).$$

(If G has a nontrivial character group, one usually regularizes the situation by multiplying or dividing by the appropriate power of the (correspondingly partial) Dedekind zeta function. We will discuss in $\S17.5$ how our conjecture is independent of such a choice.)

We fix factorizations of the Tamagawa measures, e.g. if dx denotes the invariant Tamagawa measure on $\mathbf{X}(\mathbb{A}_K)$, we fix an Euler product: $dx = \prod_v dx_v$, where dx_v is a measure on $\mathbf{X}(K_v)$ and $dx_v(\mathbf{X}(\mathfrak{o}_v)) = 1$ for almost all v.

17.2. Factorization and the Ichino–Ikeda conjecture. Pick $x_0 \in \mathbf{X}(K)$; let **H** be the stabilizer of x_0 . We assume throughout that the connected component of the center of **G** acts faithfully on **X**.

Let $\nu : \pi = \otimes \pi_v \hookrightarrow C^{\infty}([\mathbf{G}])$ be an automorphic representation, together with an embedding. By multiplicity one, the global "period" Hermitian form

$$\mathcal{P}^{\mathrm{Aut}}: \varphi \longrightarrow \left| \int_{[\mathbf{H}]} \nu(\varphi) \right|^2$$
 (17.1)

factorizes as a product of local $\mathbf{H}(K_v)$ -biinvariant Hermitian forms \mathcal{P}_v on the representations π_v . Of course, this period integral is not always convergent, and has to be suitably regularized. In the discussion that follows we assume such a regularization.

Basic question. Is it possible to give a purely local expression for \mathcal{P}_v ?

Let us make this more precise by formulating the answer given by Ichino–Ikeda [II10], based on the results of Waldspurger [Wal85] and others. First, assume that \mathbf{X} is "strongly tempered", and let $\mathcal{P}_v^{\text{Planch}}$ be the "canonical hermitian form" discussed in §6.2:

$$\mathcal{P}_{v}^{\text{Planch}}(u_1, u_2) = \int_{\mathbf{H}(K_v)} \langle \pi_v(h) u_1, u_2 \rangle dh. \tag{17.2}$$

The case under consideration in [II10] is $\mathbf{G} = \mathbf{SO}_V \times \mathbf{SO}_{V \oplus G_a}$, where V is a nondegenerate quadratic space, and \mathbf{H} =the diagonal copy of \mathbf{SO}_V in \mathbf{G} . If π is a tempered and cuspidal automorphic representation of \mathbf{G} (in this case there is a unique, up to scalar, embedding of π in $C^{\infty}([\mathbf{G}])$), and we fix a unitary such embedding, then by the Arthur conjectures it should be attached to a "global Arthur parameter" ϕ (cf. §16.1) whose centralizer in the connected dual group \check{G} is a finite 2-group S_{ϕ} , and which is trivial on the "Arthur SL_2 " – i.e., it is a global Langlands parameter. In that case, Ichino and Ikeda conjecture:

$$\mathcal{P}^{\text{Aut}} = \frac{1}{|S_{\phi}|} \prod_{v}' \mathcal{P}_{v}^{\text{Planch}}, \tag{17.3}$$

This Euler product is not absolutely convergent, and the symbol \prod' denotes that it should be understood "in the sense of L-functions": more precisely, it is computed in [II10] that for local unramified data the local factors are equal to a certain quotient of special values of L-functions, and the meaning of \prod' is that one should replace almost all Euler factors by the corresponding quotient of special values of the (analytically continued) partial L-functions.

The conjecture has been verified for $n \leq 3$, and special cases in higher rank. There is also evidence that exactly the same conjecture applies to the Whittaker period, with the regularized Plancherel hermitian forms that we defined in §6.3; in the case of $G = \operatorname{GL}_n$ the validity of the conjecture for the Whittaker case is known to experts, and we will recall the argument in Section 18; Lapid and Mao have recently proven it for automorphic representations of the double metaplectic cover of $\operatorname{Sp}_{2n}[\operatorname{LM}]$. As we will see, by our interpretation of "unfolding" (§9.5) the conjecture also holds whenever we can "unfold" the period integral to a known case, such as in the case of the Rankin-Selberg integral on $\operatorname{GL}_n \times \operatorname{GL}_{n+1}$ (i.e. the space $\operatorname{GL}_n \setminus \operatorname{GL}_n \times \operatorname{GL}_{n+1}$).

In the general case, the form (17.2) does not converge, nor can it be regularized by the methods of §6.3. A typical example where it diverges is the Sp_{2n} -period inside GL_{2n} , which has been studied by Jacquet–Rallis [JR92] and Offen [Off06]. However, by Proposition 6.2.1, the Hermitian form $\mathcal{P}_v^{\operatorname{Planch}}$ is intrinsically characterized – at least off a set of Plancherel measure zero – by its role in a Plancherel formula for $L^2(\mathbf{X}(K_v))$. This suggests a reformulation of the definition of $\mathcal{P}_v^{\operatorname{Planch}}$ which has the possibility of working even when (17.2) is divergent.

17.3. Local prerequisites for the conjecture. We keep assuming multiplicity one at all places, a corollary of which is that $\check{G}_X \subset \check{G}$. We feel free to assume the validity of all conjectures in this paper for X, in particular

Conjecture 16.2.2 on the local L^2 -spectrum of X. Hence,

$$L^{2}(\mathbf{X}(K_{v})) = \int_{\{\phi\}/\sim} \mathcal{H}_{\phi}\mu(\phi)$$
 (17.4)

is a decomposition of $L^2(\mathbf{X}(K_v))$ in terms of \check{G}_X -conjugacy classes of X-distinguished Arthur parameters, where the measure μ belongs to the natural class of measures on Arthur parameters. The spaces \mathcal{H}_{ϕ} here may be zero.

We would like to fix a Plancherel measure in this class, in order to fix the associated norms on \mathcal{H}_{ϕ} . The idea is to "fix the natural Plancherel measure for $\mathbf{G}_X(K_v)$ ", where \mathbf{G}_X is the split group with dual group $\check{\mathbf{G}}_X$. This is slightly problematic, in the sense that the Plancherel measure for $\mathbf{G}_X(K_v)$ is, according to the conjectures of Hiraga–Ichino–Ikeda [HII08], not quite a measure on the set of (tempered) Arthur parameters into $\check{\mathbf{G}}_X$, but also depends on the representation in the corresponding packet. However, for the purposes of the present discussion, where we formulate our conjecture up to \mathbb{Q}^{\times} , this will not matter.

More precisely, it is expected that there is a measure μ_v on the set of local, tempered Arthur parameters (i.e. bounded Langlands parameters) into \check{G}_X modulo conjugacy, such that $L^2(\mathbf{G}_X(K_v))$ admits a direct integral decomposition analogous to (17.4), where for every unitary, tempered representation τ the Plancherel norm on the space spanned by its matrix coefficient is a multiple of the canonical (Hilbert-Schmidt) norm by an integer (which can be bounded independently of the representation). In fact, there is a minimal such choice in the \mathbb{Q}^{\times} class of μ_v , in the sense that with that choice for some representations in the packet (those, conjecturally, corresponding to characters of the component group of the centralizer of the parameter) we will not need to multiply by an integer. Of course, this measure depends on the choice of a measure on $G_X(K_v)$, but again we may choose Tamagawa measures globally to eliminate the dependence on local choices in the conjecture that follows. A good choice of local Tamagawa measures is described in [Gro97], and for this choice there is a very precise conjecture on Plancherel measures in [HII08]. For discrete series:

$$\mu_v^{\text{Planch}}(\tau) = \frac{\langle 1, \pi \rangle}{|S_{\phi}^{\natural}|} \cdot |\gamma(0, \tau, \text{Ad}, \psi)|$$
 (17.5)

hence in that case we would take the measure for the corresponding Langlands parameter ϕ to be:

$$\mu_v(\phi) = \frac{1}{|S_{\phi}^{\natural}|} \cdot |\gamma(0, \tau, \mathrm{Ad}, \psi)|.$$

We refer the reader to [HII08, p. 287] for the notation and normalization.

By the obvious bijection between (local) X-distinguished Arthur parameters into \check{G} and tempered Langlands parameters into \check{G}_X we can consider this measure as a measure on the set of \check{G}_X -conjugacy classes of X-distinguished Arthur parameters. To this choice of measure corresponds, for almost every $\tau \in L^2(\mathbf{X}(K_v))$, a generalized character:

$$\theta_v^{\text{Planch}}: C_c^{\infty}(\mathbf{X}(K_v) \times \mathbf{X}(K_v)) \to \tau \otimes \bar{\tau} \to \mathbb{C}$$
 (17.6)

or dually (and composing with evaluation at the chosen point x_0) a hermitian form:

$$\mathcal{P}_v^{\text{Planch}} : \tau \otimes \bar{\tau} \to C^{\infty}(\mathbf{X}(K_v) \times \mathbf{X}(K_v)) \xrightarrow{\text{ev}_{(x_0, x_0)}} \mathbb{C}.$$
 (17.7)

Although this is, a priori, not well defined at any specific τ which is not in the discrete spectrum of $\mathbf{X}(K_v)$, it is expected to be rational in τ ; this follows from Theorem 15.6.1 under the assumptions of that theorem. Therefore, $\mathcal{P}_v^{\text{Planch}}$ is uniquely defined wherever it is regular.

 $\mathcal{P}_{v}^{\mathrm{Planch}}$ is uniquely defined wherever it is regular.

Question: Is it true that $\mathcal{P}_{v}^{\mathrm{Planch}}$ is regular on the set of X-tempered representations?

From now on we will assume this to be so, or we will assume the local components of our global representations to be on the regular set. The last local piece of input that we need to discuss is the value of $\mathcal{P}_v^{\text{Planch}}$ on normalized unramified data.

Let v be a non-archimedean place of K, unramified over \mathbb{Q} . Assume that G and X carry an integral model at v with $x_0 \in \mathbf{X}(\mathfrak{o}_v)$ and such that $\mathbf{X}(K_v)$ satisfies the "generalized Cartan decomposition" [Sak13, Axiom [2.4.1] with respect to the hyperspecial maximal compact subgroup $\mathbf{G}(\mathfrak{o}_v)$ – this is the case at almost every place under the multiplicity-one assumption. Let π be an irreducible, unramified (with respect to $\mathbf{G}(\mathfrak{o}_v)$) representation in $L^2(\mathbf{X}(K_v))$, which is isomorphic to the unramified subquotient of the representation $I_{P(X)}^G(\chi)$. Then, up to a combinatorial condition which is easy to check and which is expected to hold for all affine homogeneous spherical varieties (s. the statement of [Sak13, Theorem 7.2.1] – from now on we assume this condition to hold for our spherical variety), the value of $\mathcal{P}_{v}^{\text{Planch}}(u \otimes \bar{u})$ where $u \in \pi^{\mathbf{G}(\mathfrak{o}_{v})}$ with ||u|| = 1 follows from the Plancherel formula of [Sak13, Theorem 9.0.1]. More precisely, it is the quotient L_X of L-values attached to the spherical variety X in loc.cit., divided by the "Plancherel measure for G_X ". Hence, ⁷⁸ in the notation of [Sak13, Definition 7.2.3] (in particular: using exponential notation $e^{\check{\gamma}}$ instead of $\check{\gamma}$ for characters of tori), but adding the index v:

$$L_{X,v}^{\sharp}(\pi) := \mathcal{P}_{v}^{\text{Planch}}(u \otimes \bar{u}) =$$

$$= \frac{c_{v}^{2}}{Q_{v}^{P(X)}} \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}_{X}} (1 - q_{v}^{-1} e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta} (1 - \sigma_{\check{\theta}} q_{v}^{-r_{\check{\theta}}} e^{\check{\theta}})} (\chi).$$

$$(17.8)$$

⁷⁸See the table at the end of [Sak13] for some examples.

We recall that $Q_v^{P(X)} = [\mathbf{G}(\mathfrak{o}_v) : \mathbf{P}(\mathbf{X})^-(\mathfrak{o}_v)\mathbf{P}(\mathbf{X})(\mathfrak{o}_v)]$, where $\mathbf{P}(\mathbf{X})^-$ is a parabolic opposite to $\mathbf{P}(\mathbf{X})$. It is also equal to:

$$Q_v^{P(X)} = \prod_{\check{\alpha} \in \mathfrak{u}_{P(X)}} \frac{1 - q_v^{-1} e^{\check{\alpha}}}{1 - e^{\check{\alpha}}} (\delta^{\frac{1}{2}})$$
 (17.9)

(the product over all roots in the unipotent radical of the parabolic dual to P(X)). For affine varieties the constant c_v is:

$$\frac{\prod_{\check{\theta}>0} (1 - \sigma_{\check{\theta}} q_v^{-r_{\check{\theta}}} e^{\check{\theta}})}{\prod_{\check{\gamma}>0} (1 - e^{\check{\gamma}})} (\delta_{P(X)}^{\frac{1}{2}})$$
 (17.10)

but for a variety which is "Whittaker-induced" from a homogeneous affine spherical variety \mathbf{X}' of a Levi subgroup, the constant c_v is the same as for \mathbf{X}' . For example, for the Whittaker model itself we have $c_v = 1$.

We define $Q_v^{P(X)}(s)$ by replacing $\delta^{\frac{1}{2}}$ by $\delta^{\frac{1}{2}+s}$ in (17.9), $c_v(s)$ by replacing $\delta^{\frac{1}{2}}_{P(X)}$ by $\delta^{\frac{1}{2}+s}_{P(X)}$ in (17.10) and:

$$L_{X,v}^{\sharp}(\pi,s) := \frac{c_v(s)^2}{Q_v^{P(X)}(s)} \cdot \frac{\prod_{\check{\gamma} \in \check{\Phi}_X} (1 - q_v^{-1-s} e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta} (1 - \sigma_{\check{\theta}} q_v^{-r_{\check{\theta}} - s} e^{\check{\theta}})} (\chi). \tag{17.11}$$

17.3.1. Remark. The purpose of introducing the parameter s is to make sense of the Euler product of the $L_{X,v}^{\sharp}$'s as the analytic continuation of a quotient of L-functions. (We understand the existence of such an analytic continuation as part of the conjecture.) If the pertinent (global) L-functions turn out to have zeroes or poles when s=0 the way we have chosen the s-parameter plays an important role; for instance, changing some occurences of s by 2s could introduce a power of 2 as an extra factor. (Of course, this wouldn't matter at present, since we are only formulating conjectures up to \mathbb{Q}^{\times} .) It appears by [II10] that the definitions we have given here are the correct ones.

We explain how to deduce (17.8) from [Sak13]: By definition, $\mathcal{P}_v^{\text{Planch}}(u \otimes \bar{u})$ is the quotient of "Plancherel measure for $\mathbf{X}(K_v)$ " by "Plancherel measure for $\mathbf{G}_X(K_v)$ ", where by "Plancherel measure" we mean, as in *loc.cit*. the Plancherel measure corresponding to Hecke eigenfunctions normalized to have value 1 at x_0 . Notice that since we are using Tamagawa measures, the formulas that follow will differ from those of *loc.cit*. by a factor of $(1-q^{-1})^{-\operatorname{rk} X}$, though this factor actually doesn't play a role since it is cancelled upon division. By Theorem 9.0.1 of *loc.cit*. the unramified Plancherel measure for $\mathbf{X}(K_v)$, considered as a measure on A_X^*/W_X , is:

$$\frac{1}{Q_{v,G}^{P(X)}(1-q^{-1})^{\operatorname{rk}X}} L_{X,v}(\chi) d\chi$$

where we write $Q_{v,G}^{P(X)}$ to emphasize that this factor is defined with the group \mathbf{G} in mind. On the contrary, for $\mathbf{G}_X(K_v)$ the corresponding factor will be

defined with respect to the group $\mathbf{G}_X \times \mathbf{G}_X$, and hence we will denote it by $\left(Q_{v,G_X}^{P(X)}\right)^2$. Hence, the unramified Plancherel measure for $\mathbf{G}_X(K_v)$ is:

$$\frac{1}{\left(Q_{v,G_X}^{P(X)}\right)^2 (1 - q^{-1})^{\operatorname{rk} X}} L_{G_X,v}(\chi) d\chi$$

which is equal to:

$$\prod_{\check{\gamma}\in\check{\Phi}_X}\frac{1-e^{\check{\gamma}}}{1-q_v^{-1}e^{\check{\gamma}}}(\chi).$$

By the definition of $L_{X,v}$ in 7.2.3 of *loc.cit.*, the claim of (17.8) follows.

17.4. Global conjecture. Let $\mathcal{A}_{[\psi]}$ be the subspace of the space of automorphic forms corresponding to an X-distinguished Arthur parameter. The notion of X-distinguished, here, is the same as locally: the restriction of the parameter to the hypothetical Langlands group is bounded and factors through \check{G}_X , while the SL₂-type is the X-distinguished SL₂-type. Hence, this is the analog of the "tempered" hypothesis in the Ichino–Ikeda conjecture (although it does not imply that the automorphic representations are tempered; rather they are "tempered relative to X.")

One can speculate about extending all that follows to the general case along the lines of [II10], but we have no reason to get into that here. The space $\mathcal{A}_{[\psi]}$ is a unitary space; for the discrete-modulo-center space we fix norms by integrating over $[\mathbf{G}/\mathbf{Z}]$ (where \mathbf{Z} denotes the center) against Tamagawa measure, and we will explain how to fix norms on the continuous spectrum in §17.5, after we give a rough statement of the conjecture.

The following should be viewed more as a working hypothesis, rather than a solid conjecture. We feel more confident about it in the case where there is at most one X-distinguished representation in each local A-packet, which should be the case if and only if the relative trace formula for $H\backslash G/H$ is stable. A good conjecture, including an understanding of the unspecified rational constants, should be the result of a theory of endoscopy for the relative trace formula.

To formulate it, let $\mathcal{A}'_{[\psi]}$ denote a subspace of $\mathcal{A}_{[\psi]}$ with the properties:

- $\mathcal{A}'_{[\psi]}$ contains with multiplicity one all irreducible automorphic representations which occur in $\mathcal{A}_{[\psi]}$;
- the restriction of the hermitian form \mathcal{P}^{Aut} to the orthogonal complement of $\mathcal{A}'_{[\psi]}$ is zero.

Such a subspace exists by the assumption that X is multiplicity-free. It is not unique, as we may arbitrarily choose its component inside isotypic subspaces of $\mathcal{A}_{[\psi]}$ where $\mathcal{P}^{\mathrm{Aut}}$ is identically zero. However, such a choice allows us to formulate the conjecture uniformly.

17.4.1. Conjecture (Period conjecture). Let ψ denote an X-distinguished global Arthur parameter. For each irreducible $\nu: \pi = \otimes_v \pi_v \hookrightarrow \mathcal{A}'_{[\psi]}$ there is

a rational number q such that:

$$\mathcal{P}^{\text{Aut}}\big|_{\nu(\pi)} = q \cdot \prod_{v}' \mathcal{P}_{v}^{\text{Planch}}.$$
 (17.12)

Here $\mathcal{P}_v^{\text{Planch}}$ are the $\mathbf{H}(K_v)$ -biinvariant forms on π_v "normalized according to $\mathbf{G}_X(K_v)$ -Plancherel measure", as explained in §17.3, and conventions for the interpretation of the Euler product and Tamagawa measures will be explained in the next subsection.

17.5. How to understand the Euler product. Again, the Euler product (17.12) should be understood in the sense of L-functions; namely, for every $u = \otimes u_v \in \pi = \otimes' \pi_v$ there will be a large enough set T of places (including the archimedean ones and the places of ramification of K over \mathbb{Q}) such that:

- there is (and we fix) a smooth integral model for **G** and **X** outside of T, with $x_0 \in \mathbf{X}(\mathfrak{o}_T)$ (where \mathfrak{o}_T denotes the ring of T-integers);
- the formula of [Sak13, Theorem 7.2.1] for eigenvectors of the spherical Hecke algebra $\mathcal{H}(\mathbf{G}(K_v), \mathbf{G}(\mathfrak{o}_v))$ on $\mathbf{X}(K_v)$ holds for $v \notin T$.
- $u_v \in \pi_v^{\mathbf{G}(\mathfrak{o}_v)}$ for $v \notin T$.

Then for $v \notin T$ we have:

$$P_v^{\text{Planch}}(u_v) = L_{X,v}^{\sharp}(\pi).$$

The equality (17.12) should be thought of as a *formal equality*, whose real meaning is:

$$\left| \int_{[\mathbf{H}]} \nu(u)(h) |\omega|(h) \right|^2 = q L_X^{\sharp (T)}(\pi) \cdot \prod_{v \in T} \mathcal{P}_v^{\text{Planch}}(u_v)$$
 (17.13)

where $L_X^{(T)}(\pi)$ is the value at s=0 of the analytic continuation of the quotient of partial L-functions (outside of T) whose Euler factors are (17.11). Again, we emphasize that the existence of such an analytic continuation should be considered as part of the conjecture.

Here on the left hand side we have explicitly integrated against a rational differential form in order to make the point that one may have to divide the whole expression by some zeta factors to make the two sides finite. Tamagawa measures are, by definition, defined by taking the absolute value of invariant, rational, volume forms, times local "convergence factors" which are cancelled, globally, by multiplying by a special value of a partial L-function. However, if a group **H** has nontrivial k-character group, then the corresponding partial L-function has a pole at the desired point of evaluation. This is usually resolved by multiplying, instead, by the leading coefficient of its Laurent expansion, which leads to a non-canonical but quite standard choice. In that case, we expect that $L_X^{\sharp}(T)(\pi)$ will also have a pole of at most the same order, and should be replaced by its leading term. Equivalently, one should treat both sides of the above equation as formal

Euler products and "cancel" the same power of the (partial) Dedekind zeta function of k from both.

17.5.1. EXAMPLE. Let $\mathbf{G} = \mathbf{PGL}_2$, $\mathbf{H} = \mathbf{A} \subset \mathbf{G}$ a split torus. This example is the original "Hecke integral," which was reinterpreted adelically in the work of $[\mathbf{JL70}]$ of Jacquet and Langlands.

Then $\mathbf{A}(\mathfrak{o}_v) = 1 - q_v^{-1}$ at almost every place, and therefore the "formal" measure of a set $S = \prod_v S_v$ with $S_v = \mathbf{A}(\mathfrak{o}_v)$ outside of a finite set of places T is:

$$|\omega|(S) = \frac{1}{\zeta_K^{(T)}(1)} \prod_{v \in T} |\omega|_v(S_v),$$

which is zero. Therefore, the period integral should be computed with respect to the measure:

$$\zeta_K^{(T)}(1)|\omega|(S):=\prod_{v\in T}|\omega|_v(S_v).$$

(Notice that it is not standard to multiply by a partial ζ -function, but in fact the conjecture is independent of how exactly one chooses to normalize the Tamagawa measure!)

On the right hand side, correspondingly, we have, outside of a finite set of places:

$$L_{X,v}^{\sharp}(\pi) = \frac{(1 - q_v^{-1})^2}{1 - q_v^{-2}} \prod_{\check{\alpha} \in \check{\Phi}_G} \frac{1 - q_v^{-1} e^{\check{\alpha}}}{(1 - q_v^{-\frac{1}{2}} e^{\frac{\check{\alpha}}{2}})^2} (\chi_v) = \frac{1 - q_v^{-1}}{1 - q_v^{-2}} \cdot \frac{(L_v(\pi_v, \frac{1}{2}))^2}{L_v(\pi_v, \mathrm{Ad}, 1)}.$$

Notice that the *same* globally problematic factor of $\frac{1}{\zeta_{K,v}(1)}$ appears on the right hand side, as well!

Therefore, for a cuspidal representation $\pi = \otimes' \pi_v$ the conjecture says that the period integral with respect to the measure $\zeta_K^{(T)}(1)|\omega|(S)$ is equal, up to a rational factor, to:

$$\zeta_K^{(T)}(2) \frac{(L^{(T)}(\pi, \frac{1}{2}))^2}{L^{(T)}(\pi, \operatorname{Ad}, 1)} \cdot \prod_{v \in T} \mathcal{P}_v^{\operatorname{Planch}}.$$

Of course, this is known to hold, with the implicity rational factor equal to 1. We will explain the meaning of the period integral on other parts of the spectrum below.

17.5.2. Remark. It is not necessary that there exists a meaningful, finite regularization of the period integral for *every* representation. For example, in the case of $\mathbf{A} \subset \mathbf{PGL}_2$ and $\pi = 1$, the trivial representation, it is reasonable to think of the "correct" value of the period integral (with respect to a non-zero finite measure, such as $\zeta_K^{(T)}(1)|\omega|(S)$) as being "infinity". This is reflected on the right hand side, as well: indeed, for the trivial representation

the partial L-factors on the right hand side are:

$$\frac{(\zeta_K^{(T)}(1))^2}{\zeta_K^{(T)}(2)},$$

which is infinite.

Finally, we explain how to make sense of the conjecture for the continuous spectrum. Recall that we are assuming an appropriate regularization of period integrals, so we will only explain how to choose a norm on the spaces of unitary Eisenstein series. The idea here is that the Eisenstein series morphism:

$$\mathcal{E}_P: \operatorname{Ind}_{\mathbf{P}(\mathbb{A}_K)}^{\mathbf{G}(\mathbb{A}_K)}(\delta_P^{\frac{1}{2}}\sigma) \to C^{\infty}([\mathbf{G}]),$$

where σ is a discrete automorphic representation for the pertinent Levi subgroup L, should be an isometry. However, on $\mathbf{P}\backslash\mathbf{G}$ we have again the issue of making sense, globally, of Tamagawa measures. More precisely, let ω be a K-rational invariant volume form on $\mathbf{P}\backslash\mathbf{G}$ valued in the line bundle defined by \mathfrak{d}_P^{-1} . Then, locally (having fixed good integral models outside of a finite set of places T, and taking $v \notin T$ such that σ_v is unramified), we consider the induced square-norm:

$$\int_{\mathbf{P}\backslash\mathbf{G}(K_v)} \|\phi_v^0(g)\|^2 |\omega|_v(g) \tag{17.14}$$

where the unramified vector ϕ_v^0 is defined by choosing $u \in \sigma_v^{\mathbf{L}(\mathfrak{o}_v)}$ with ||u|| = 1 and setting $\phi_v^0(pk) = \delta_P^{\frac{1}{2}}\sigma(p)u$ $(p \in \mathbf{P}(K_v), k \in \mathbf{G}(\mathfrak{o}_v))$. Then one computes that this integral, using measures coming from integral, residually non-vanishing volume forms, is equal to:

$$Q_v^P = [\mathbf{G}(\mathfrak{o}_v): \mathbf{P}^-(\mathfrak{o}_v)\mathbf{P}(\mathfrak{o}_v)] = \prod_{\check{\alpha} \in \mathfrak{u}_P} \frac{1 - q_v^{-1} e^{\check{\alpha}}}{1 - e^{\check{\alpha}}} (\delta^{\frac{1}{2}}).$$

The Euler product of the Q_v^P 's, understood as the quotient of special values of zeta functions, is "infinite". Therefore, for the conjecture to make sense we need to redefine the norm (17.14) of the "standard vector" ϕ_v^0 to be equal to 1 outside of a finite set T of places and, correspondingly, divide the Euler factors on the right hand side, for $v \notin T$, by Q_v^P ; that is:

$$\mathcal{P}^{\mathrm{Aut}}(\mathcal{E}_{P}(\prod_{v \notin T} \phi_{v}^{0} \cdot \prod_{v \in T} \phi_{v})) = \prod_{v \notin T}' \frac{L_{X,v}^{\sharp}(\pi_{v})}{Q_{v}^{P}} \cdot \prod_{v \in T} \mathcal{P}_{v}^{\mathrm{Planch}}(u_{v}),$$

where the partial Euler product on the right is now expected to make sense as a quotient of L-values.

17.5.3. REMARK. If **P** is not a self-associate parabolic, then the variety $\mathbf{Y} = \mathbf{U}_P \backslash \mathbf{G}$ is (spherical and) multiplicity-free for the group $\mathbf{L} \times \mathbf{G}$, and the requirement that the Eisenstein morphism \mathcal{E}_P be an isometry is equivalent to the validity of our conjecture for the variety **Y** (i.e. for the constant term

of the Eisenstein series. If \mathbf{P} is self-associate then \mathbf{Y} is not multiplicity-free, and our conjecture holds, tautologically, for the "first summand" of the constant term of the Eisenstein series.

- 17.6. Everywhere discrete or unramified. Because of the meager state of knowledge about the Arthur conjectures in general, it is useful to discuss a specific case which can be formulated without reference to them. For any representation of $\mathbf{G}(\mathbb{A}_K)$ and a large enough collection of places T, write π_T for the vectors that are unramified outside T. Here by "large enough" we mean, as before:
 - T includes the archimedean places;
 - K is unramified over \mathbb{Q} outside of T;
 - there is (and we fix) a smooth integral model for **G** and **X** outside of T, with $x_0 \in \mathbf{X}(\mathfrak{o}_T)$;
 - the formula of [Sak13, Theorem 7.2.1] for eigenvectors of the spherical Hecke algebra $\mathcal{H}(\mathbf{G}(K_v), \mathbf{G}(\mathfrak{o}_v))$ on $\mathbf{X}(K_v)$ holds for $v \notin T$.

"Unramified", of course, means "fixed by $\mathbf{G}(\mathfrak{o}_v)$ ". Following standard notation, we denote $K_T = \prod_{v \in T} K_v$, and $L^{(T)}$ a partial L-function outside of T

The conjecture that follows is stated with the help of a modification L_X^{\flat} of L_X^{\sharp} , which will be defined afterwards:

17.6.1. Conjecture (X-variant). Endow $\mathbf{X}(K_T)$ with the invariant measure μ_T such that $\mu^T \cdot \mu_T = Tamagawa$ measure, where μ^T is the invariant measure on $\prod_{v \notin T} \mathbf{X}(K_v)$ such that $\mu^T \left(\prod_{v \notin T} \mathbf{X}(\mathfrak{o}_v)\right) = 1$.

If $\pi \in L^2([\mathbf{G}])$ is irreducible and $l_T : \pi_T \hookrightarrow L^2(\mathbf{X}(K_T))$ is an isometric embedding (i.e. π_T is an X-discrete series), then for $\phi \in \pi_T$:

$$\left| \int_{[\mathbf{H}]} \phi \right|^2 \in \mathbb{Q}^{\times} \cdot L_X^{\flat(T)}(\pi) \cdot |l_T(\phi)(x_0)|^2.$$

This does not follow in an entirely routine way from Conjecture 17.4.1, because $|\operatorname{ev}_{x_0} \circ l_T|^2$ differs from $\prod_{v \in T} \mathcal{P}_v^{\operatorname{Planch}}$ by a factor which takes into account the normalization of Plancherel measures. According to the conjectural formula (17.5), the Plancherel measure used to define $\mathcal{P}_v^{\operatorname{Planch}}$ for $v \in T$ is, up to a rational number, equal to an adjoint γ -factor for the group G_X . Since we do not want to use any functoriality assumptions in order to formulate the conjecture at this point, we will substitute the product of these γ -factors at places $v \in T$ by the inverse of the corresponding partial gamma factor away from T, since we expect the product of these gamma factors over all places to be equal to 1. We also have to take into account that the measure for $v \notin T$ is normalized, here, to give mass one to $\mathbf{X}(\mathfrak{o}_v)$, while "local Tamagawa measure" gives mass:

$$\frac{Q_v^{P(X)} \cdot (1 - q^{-1})^{\text{rk}X}}{c_v} \tag{17.15}$$

to $\mathbf{X}(\mathfrak{o}_v)$ [Sak13, Theorem 9.0.3].

Therefore, formally the local unramified factor for $L_{X,v}^{\flat}$ is:

$$L_{X,v}^{\flat}(\pi) := \frac{Q_v^{P(X)} \cdot (1 - q^{-1})^{\text{rk}X}}{c_v} \cdot L_{X,v}^{\sharp}(\pi) \cdot \gamma_{\check{G}_X}(\pi, \text{Ad}, 0), \tag{17.16}$$

where $\gamma_{\check{G}_X}$ denotes the adjoint gamma factor for the unramified representation π regarded as a semisimple conjugacy class in \check{G}_X . However, the latter is zero (since it has a numerator has the factor $(1-q^s)^{\operatorname{rk} X}$ evaluated at zero). But again, we are only interested in making sense of $L_{X,v}^{\flat}$ globally (i.e. making sense of the partial quotient of L-functions $L_X^{\flat}(T)$), and the corresponding partial γ -factor $\gamma_{\check{G}_X}^{(T)}(\pi,\operatorname{Ad},0)$ should be finite and nonzero, since the factors for $v\in T$ are all assumed to correspond to discrete parameters. Hence, we define:

$$L_{X,v}^{\flat}(\pi,s) := c_v(s) \cdot \frac{(1 - q_v^{-s})^{\text{rk}A_X} \prod_{\check{\gamma} \in \check{\Phi}_X} (1 - q_v^{-s} e^{\check{\gamma}})}{\prod_{\check{\theta} \in \Theta} (1 - \sigma_{\check{\theta}} q_v^{-r_{\check{\theta}} - s} e^{\check{\theta}})} (\chi), \qquad (17.17)$$

and let $L_X^{\flat(T)}(\pi)$ denote the value, at s=0, of the (conjectural) meromorphic continuation of:

$$L_X^{\flat,(T)}(\pi,s) = \prod_{v \notin T} L_{X,v}^{\flat}(\pi_v,s)$$
 (17.18)

where, again, if we have to modify the left-hand-side of the conjecture to make sense of the global Tamagawa measure, then we also have to modify $L_X^{\flat\,(T)}$ by the appropriate factors.

18. Examples

We finally outline some examples where the period conjectures of the prior section can be verified. We do not make any claim to originality: many of the results are known to experts. The material of §18.1 is related to regularization of Eisenstein periods, a topic which has been developed in the works [JLR99, LR01, LR03]. The results about the Whittaker period are established already in the paper [LM15] of Lapid and Mao, and the results of Theorem 18.4.1 concern periods whose Euler factorization is already known. Thus, our main concern has been to show that the local factors are equal to the "Plancherel" factors predicted by the Period Conjecture 17.4.1, thus illustrating the compatibility of known methods with the framework of this paper. In particular – see §18.4 – the formulation of "unfolding" as an isometry between local L^2 -spaces arises naturally in the evaluation of the global period.

We will use \int^* to denote a regularized integral. This notation will often be omitted for integral expressions which depend meromorphically on a parameter in some region of convergence and are meromorphically continued to other values of the parameter; those will generally be denoted by \int .

An expression of the form $\lim_{s\to 0} I(s)$, where I(s) is an expression which literally makes sense only for $\Re s \gg 0$, means the value at s=0 of the meromorphic continuation of I(s), if it exists.

For normalizations of Tamagawa measure, etc. we refer to §17.1.

18.1. Principal Eisenstein periods. Let $X = H \setminus G$ be a multiplicity-free spherical variety, and let us assume, for simplicity, that P(X) = B. Assume that B and H have been chosen so that BH is open in G; the multiplicity-free assumption, together with the assumption that P(X) = B, implies that $H \cap B$ is a torus and that there is a unique open $B(k_v)$ -orbit on $X(k_v)$, for every completion v; moreover, that $X(k_v)$ is a unique $G(k_v)$ -orbit, cf. [Sak08]. We will assume, as we have done throughout, that the connected component of the center of G acts faithfully on X.

Let $\pi = I(\chi) = I_{\mathbf{B}(\mathbb{A}_K)}^{\mathbf{G}(\mathbb{A}_K)}(\chi)$ be a unitary principal series representation with **X**-distinguished parameter, i.e. the idele class character χ corresponds by class field theory to a homomorphism with bounded image: $\mathcal{W}_K \to A_X^*$ (where \mathcal{W}_K denotes the Weil group of K). For the purpose of regularization, however, we should at first drop the requirement of "bounded image", i.e. the assumption that χ is unitary, and consider all idele class characters χ of **B** which are trivial on $(\mathbf{B} \cap \mathbf{H})(\mathbb{A}_K)$. We would like to compute the (regularized) period integral of $\mathcal{E}_B(u)$, for every $u \in \pi$, where \mathcal{E}_B denotes, as before, the Eisenstein series morphism.

Consider the operator:

$$\Delta_{\chi}: I(\chi) \ni u \mapsto \Phi(\mathbf{H}g) = \int_{(\mathbf{H} \cap \mathbf{B}) \backslash \mathbf{H}(\mathbb{A}_K)} u(hg) dh \in C^{\infty}(\mathbf{X}(\mathbb{A}_K)) \quad (18.1)$$

which was called "unnormalized Eisenstein integral" in section 15; we take our measures to be given by volume forms with the understanding, as was explained in §17.5, that if they have to be modified by convergence factors to make sense of them, then the same modification will be applied to the results. The operator Δ_{χ} converges absolutely when $\chi^{-1} \gg 0$ in the notation introduced after Corollary 15.3.3.

It is reasonable to postulate that for almost all unitary χ the correct normalization of the integral:

$$\int_{[\mathbf{H}]} \mathcal{E}_B(h) dh$$

is obtained as the analytic continuation (assuming it exists) of the evaluation at $\mathbf{H} \cdot 1$ of the operator Δ_{χ} . This is easier to justify in the case that $\mathbf{H} \cap \mathbf{B}$ is trivial, where the period integral of a pseudo-Eisenstein series $\Phi = \int \mathcal{E}_B(u_\chi) d\chi$ (where $u_\chi \in I(\chi)$ and the integral is taken over a suitable translate of the set of unitary idele class characters) over $[\mathbf{H}]$ has an expression whose main term is:

$$\int \Delta_{\chi}(u_{\chi})(1)d\chi,$$

integrated over the same set, cf. [Sak13, Section 10]. In the general case the analogous expression for $\int_{[\mathbf{H}]} \Phi$ is over a smaller set of characters (corresponding to X-distinguished Arthur parameters), so the period integral should not give a function on the space of Eisenstein series, but a distribution (or, rather, a generalized function). Thus, the value of the period in this case should be thought to be "infinity"; this is indeed the case with Δ_X , if it is globally defined by invariant volume forms (or volume forms modified by the appropriate local factors for the measure on $[\mathbf{H}]$ to make sense, as mentioned before): the volume of $(\mathbf{H} \cap \mathbf{B}) \setminus \mathbf{H}(\mathbb{A}_K)$ is infinite. Thus, in this case our calculations should be seen as formal manipulations – both the period and the result will be "infinite" but "with the same order of $\zeta(1)$ appearing". One could dwell on the issue of how to make a rigorous statement out of this (how to describe the period as a generalized function on the space of χ 's, for example), but we will not get into that now (again, cf. the literature on regularized Eisenstein periods, in particular $[\mathbf{JLR99}, \mathbf{LR01}, \mathbf{LR03}]$).

Let us therefore explain how this matches the Period Conjecture 17.4.1. We fix a completion k_v , and start denoting by regular font the points of various varieties over k_v . Instead of the local factor Δ_{χ_v} of the operator Δ_{χ} , we might consider the adjoint:

$$C_c^{\infty}(X) \ni \Phi_v \mapsto \Delta_{\chi_v}^*(\Phi)(g) = \int_{B \cap H \setminus B} \Phi_v(Hbg) \chi_v \delta^{-\frac{1}{2}}(b) db \in I_B^G(\chi_v^{-1})$$

in order to show that the corresponding Hermitian form:

$$\|\Phi_v\|_{\chi_v}^2 := \|\Delta_{\chi_v}^*(\Phi_v)\|^2$$

where the norm on the right hand side is that on $I_{B^-}^G(\chi_v^{-1})$, is the form $\mathcal{P}_v^{\text{Planch}}$ predicted by the Period Conjecture 17.4.1; we will recall what this means. In fact, this will not quite be the case: what we will show is that there are local factors γ_v (depending on χ_v) with the properties:

- (1) $\gamma_v \|\Delta_{\chi_v}^*(\Phi_v)\|^2 = \mathcal{P}_v^{\text{Planch}}(\Phi_v);$
- (2) for almost all v, γ_v can be identified with a quotient of local L-values, and:

$$\prod_{v}' \gamma_v = 1.$$

The product here is taken over all places, and understood as in $\S17.5$, i.e. as a partial L-value times a finite number of factors.

We drop the index v from χ and Φ from now on.

Recall that both the norm on $I_{B^-}^G(\chi^{-1})$ and the form $\mathcal{P}_v^{\text{Planch}}$ depend on k-rational volume forms used to define measures on $X, B \setminus G$ and G_X (the split group with dual \check{G}_X); we will see that these volume forms can be chosen compatibly.

First of all, fix a k-rational \mathbf{G} -eigen-volume form ω on \mathbf{X} that will be used to define measures on the points $\mathbf{X}(k_v)$ over each completion. For simplicity, let us actually assume that the form is \mathbf{G} -invariant. We recall

from Proposition 4.2.1 that this induces an invariant volume form on each boundary degeneration \mathbf{X}_{Θ} , and the latter was used to fix a measure on the points $\mathbf{X}_{\Theta}(k_v)$. Clearly, the volume form on \mathbf{X}_{Θ} provided by Proposition 4.2.1 is k-rational if ω is so.

Recall that \mathbf{X}_{\emptyset} is the "most degenerate" boundary degeneration of \mathbf{X} . We will recall the explicit Plancherel Theorem 15.6.2 for the most continuous part $L^2(X)_{\emptyset}$ of the spectrum (where $X = \mathbf{X}(k_v)$ for some completion), in a formulation that is suitable for our present purposes. The variety \mathbf{X}_{\emptyset} here is isomorphic to: $\mathbf{T}\mathbf{U}^-\backslash\mathbf{G}$, where \mathbf{U}^- is the maximal unipotent subgroup of \mathbf{G} (taken opposite to the chosen Borel \mathbf{B}) and \mathbf{T} is the subtorus of \mathbf{A} such that $\mathbf{A}/\mathbf{T} = \mathbf{A}_X$. We fix such an isomorphism over k. We have a Plancherel decomposition for $L^2(X_{\emptyset})$:

$$L^{2}(X_{\emptyset}) = \int_{\widehat{A_{X}}} \mathcal{H}_{\chi} \nu(\chi),$$

where $\nu(\chi)$ is in the class of Haar measure. The precise measure ν and the square of the norm on \mathcal{H}_{χ} are not canonical, of course, but their product is.

There is an action of the little Weyl group W_X on the unitary dual A_X , and in this (multiplicity-free) case it identifies almost every point of the (settheoretic) quotient \widehat{A}_X/W_X with a subset of the unitary dual \widehat{G} of G. At the same time, it is identified with a subset of the unitary dual of G_X , the (k_v -points of the) split group with dual \check{G}_X . Hence we have maps, defined off a set of measure zero:

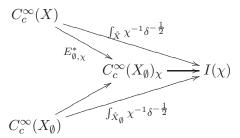
$$\widehat{A_X}/W_X \dashrightarrow \widehat{G},$$

$$\widehat{A_X}/W_X \dashrightarrow \widehat{G_X}.$$

Let us fix a measurable subset S of $\widehat{A_X}$ where these maps are injective; hence, we may identify S as a subset of \widehat{G} . Theorem 15.6.2 states that the corresponding part of the Plancherel formula for $L^2(X)$ is given by:

- normalized adjoint Eisenstein integrals: $E_{\emptyset,\chi}^*: C_c^{\infty}(X) \to \mathcal{H}_{\chi};$
- the restriction of the measure $\nu(\chi)$ to S.

It is particularly easy in this case to describe the normalized adjoint Eisenstein integrals $E_{\emptyset,\chi}^*$: by identification of $\mathring{\mathbf{X}}/\mathbf{U}$ with $\mathring{\mathbf{X}}_{\emptyset}/\mathbf{U}$ over k (2.5), choosing corresponding k-points on each of them we may identify a character of A_X as a function on X/U or X_{\emptyset}/U . The following diagram, then, where the integrals are obtained by analytic continuation and by restriction of the fixed measures on X and X_{\emptyset} obtained from the aforementioned volume forms, should commute, cf. (15.2):



Recall that $C_c^{\infty}(X_{\emptyset})_{\chi}$ denotes simply the quotient through which the lower arrow factors, which in this (multiplicity-free) case coincides with the space of smooth vectors of \mathcal{H}_{χ} , for almost all χ . Notice that the top arrow is the morphism $\Delta_{\chi^{-1}}^*$ that we encountered above.

Finally, we may fix k-rational identifications: $\mathbf{X}_{\emptyset} = \mathbf{T}\mathbf{U}^{-}\backslash\mathbf{G}$, $\mathring{\mathbf{X}}_{\emptyset} = \mathbf{A}_{X} \times \mathbf{U}$, hence $\mathbf{A}_{X}\backslash\mathbf{X}_{\emptyset} = \mathbf{B}^{-}\backslash\mathbf{G}$. We fix a corresponding factorization of the volume form on $\mathring{\mathbf{X}}_{\emptyset}$ into a product of invariant volume forms on $\mathbf{A}_{X}, \mathbf{U}$, inducing Haar measures da, du on the k_{v} -points of these spaces, as well as a δ^{-1} -valued measure on the quotient $A_{X}\backslash X$, and let $d\chi$ denote the corresponding dual measure on \widehat{A}_{X} . Then we can also identify:

$$\mathcal{H}_{\chi}^{\infty} \simeq I_{B^{-}}^{G}(\chi)$$

where the quotient $C_c^{\infty}(X) \to I_{B^-}^G(\chi)$ is given by the integral:

$$\Phi \mapsto \int_{A_X} \Phi(a \bullet) \chi^{-1} \delta^{-\frac{1}{2}} da$$

with norm on $I_{B^-}^G(\chi)$ obtained from the aforementioned measure on $A_X \backslash X = B^- \backslash G$:

$$||u||^2 = \int_{B^- \setminus G} |u^2(g)| dg.$$

The Plancherel measure corresponding to this norm is the Haar measure $d\chi$ dual to da. This is *not* yet the Plancherel measure that we need to use by the Period Conjecture 17.4.1, but now the adjoint normalized Eisenstein morphism can be identified with the composition of the maps:

$$E_{\emptyset,\chi}^*:C_c^\infty(X) \xrightarrow{\Delta_\chi^*} I_B^G(\chi) \xrightarrow{T_\emptyset} I_{B^-}^G(\chi) \ ,$$

where both arrows (the second represents the "standard" intertwining operator) are defined using global volume forms, and so is the norm on $I_{B^-}^G(\chi)$. Thus, we have shown that for the application of the explicit Plancherel theorem 15.6.2 we can use fixed volume forms defined over k.

If the hermitian forms $\Phi \mapsto \|E_{\emptyset,\chi}^*(\Phi)\|_{I_{B^-}^G(\chi)}^2$ correspond to Haar Plancherel measure $d\chi$ on the set S, then the forms $\Phi \mapsto \|\Delta_\chi^*(\Phi)\|_{I_B^G(\chi)}^2$ correspond to Plancherel measure:

$$c(\chi)^{-1}d\chi$$

where $c(\chi)$ was defined in (15.63). We emphasize once more that everthing here is defined by measures obtained from globally defined, k-rational volume forms.

We will now compare this measure with the Plancherel measure that corresponds to the Plancherel forms $\mathcal{P}_{v}^{\text{Planch}}$ of Conjecture 17.4.1. We will see that the quotient of the two is given by scalars γ_v with the aforementioned properties. Already, we notice that the scalars $c(\chi)$ (let us write $c_v(\chi_v)$ now to distinguish global from local) have the properties stated for the factors γ_v : their "global Euler product" is trivial for unitary idele class characters χ . We put quotation marks here because there is no convergent Euler product, not even in a certain region for χ ; instead, all but finitely many factors can be interpreted as quotients of (abelian, here) L-factors, and we replace the infinite product by the corresponding values of L-functions. The statement of triviality of the global $c(\chi)$ is the statement that for unitary idele class characters, the intertwining operator T_{\emptyset} is an isometry, as long as global volume forms are used to define them and the norms on principal series. Thus, for the purpose of factorizing the global H-period on principal Eisenstein series, there is no difference whether we use the normalized or the unnormalized Eisenstein integrals $E_{\emptyset,\chi}$ resp. Δ_{χ} - or whether we use local Plancherel measures $d\chi_v$ or $c_v(\chi_v)d\chi_v$.

The Plancherel measure that corresponds to the Plancherel forms $\mathcal{P}_v^{\text{Planch}}$ of Conjecture 17.4.1 is the restriction to $S \subset \widehat{G_X}$ of standard Plancherel measure for G_X . This standard Plancherel measure is the one corresponding to a Haar measure obtained by a global invariant volume form on G_X . Let us see how this Plancherel measure compares to the measure $d\chi$ that we discussed before; again, we fix a completion k_v and drop the index v when not necessary. Also, recall that the definition of $d\chi$ arises from a fixed invariant volume form on A_X .

The volume form on \mathbf{G}_X induces, again, an invariant volume form on $\mathbf{G}_{X,\emptyset}$ (the most degenerate boundary degeneration of \mathbf{G}_X). To describe the most continuous part of the Plancherel formula of G_X , one could use again normalized Eisenstein integrals and the measure $d\chi$ on $\widehat{A_X}$ (everything with volume forms defined over k, just by replacing \mathbf{X} in the above discussion by \mathbf{G}_X), or matrix coefficients and the measure $c_{G_X}(\chi)^{-1}d\chi$, cf. Theorem 15.7.2. (We introduced the index G_X here, in order to distinguish from the factor $c(\chi)$ above: while the $c(\chi)$ are defined by the diagram (15.63) for the group G, $c_{G_X}(\chi)$ is defined by the same diagram for the group G_X .)

The important point here is the observation made on p. 233, that in order to define the intertwining operator used to define $c_{G_X}(\chi)$ one needs to use the measure on N^- (there: U^-) which corresponds to the chosen measure on $P\backslash G$. In particular, if we use a global, k-rational volume form to define local norms on the principal series, the measure on N^- is also defined by a global, k-rational volume form. Hence, again, the constants $c_{G_X}(\chi)$ are globally trivial.

To summarize:

- the hermitian forms $\Phi_v \mapsto \|\Delta_{\chi_v}^*(\Phi_v)\|_{I_B^G(\chi_v)}^2$ correspond to Plancherel measure $c(\chi_v)^{-1}d\chi_v$;
- measure $c(\chi_v)^{-1}d\chi_v$; • the hermitian forms $\mathcal{P}_v^{\text{Planch}}$ correspond to Plancherel measure $c_{G_X}(\chi_v)^{-1}d\chi_v$.

Thus:

$$\gamma_v \|\Delta_{\chi_v}^*(\Phi_v)\|_{L^2_B(\chi_v)}^2 = \mathcal{P}_v^{\mathrm{Planch}}(\Phi_v)$$

with:

$$\gamma_v = \frac{c_{G_X}(\chi_v)}{c(\chi_v)},$$

which are globally trivial, in the above sense.

This shows that the analytic continuation of the integral (18.1) (evaluated at q = 1) satisfies the Period Conjecture 17.4.1.

18.2. Parabolic periods. The computation of period integrals in the examples that follow is based on the "trick" of representing the constant function on [H] as the residue of an Eisenstein series on [H], thus effectively replacing the H-period integral by a period integral over a parabolic subgroup of H. Therefore, we develop here the basic result that we will use. (We do not actually prove any instances of the conjecture in the current subsection; but this basic result will be applied in §18.3 and §18.4 to prove instances of it).

Let **H** be a semisimple group and **P** a parabolic subgroup of **H**; let Δ_P be the set of simple roots of **H** belonging to the unipotent radical of **P**. For any automorphic function ϕ or rapid decay⁷⁹ on [**H**], we have:

$$\int_{[\mathbf{H}]} \phi(h)dh = \prod_{\alpha \in \Delta_P} \langle \delta_P, \alpha^{\vee} \rangle \lim_{s \to 0} \frac{\int_{[\mathbf{P}]} \phi(p) \delta_P^s(p) \prod_v \zeta_v(1)^{\# \Delta_P} d_v p}{\zeta(1+s)^{\# \Delta_P}}.$$
 (18.2)

where the local measures $d_v p$ are the measures defined by a right-invariant differential form on \mathbf{P} . Note that the abelianization of \mathbf{P} has rank equal to $\#\Delta_P$, and hence the Euler product of measures $\prod \zeta_v(1)^{\#\Delta_P} d_v p$ converges to a nonzero right-invariant measure on $\mathbf{P}(\mathbb{A}_K)$. As usual, δ_P denotes the modular character of P. The pairing \langle , \rangle is the canonical linear pairing between the vector spaces spanned by roots and coroots; for example, $\langle \delta_P, \check{\alpha} \rangle = n$ when \mathbf{P} is the parabolic of type $\mathbf{GL}_{n-1} \times \mathbf{G}_{\mathbf{m}}$ in \mathbf{GL}_n and α is the unique simple root in its unipotent radical. Note

⁷⁹This can be relaxed, of course

that the integral $\int_{[\mathbf{P}]} \phi(p) \delta^s(p)$ is absolutely convergent for s > 0 if ϕ is of rapid decay⁸⁰.

We will denote the right-hand side of (18.2) by $\int_{[\mathbf{P}]}^*$, the "regularized integral over \mathbf{P} ," so that the formula asserts simply that

$$\int_{[\mathbf{H}]} \phi(h)dh = \int_{[\mathbf{P}]}^* \phi(p). \tag{18.3}$$

We will actually prove the statement in a somewhat more intrinsic formulation; in particular, in place of $\zeta(1+s)$, we will use a general meromorphic function of the same general type to regularize.

Let c(s) be any meromorphic function of s which has a pole at s=0 of order $\#\Delta_P$, and for $\Re s\gg 0$ admits an Euler product decomposition: $\prod_v c_v(s)$, with $c_v(0)\in\mathbb{C}^\times$ and the property that the Euler product of measures:

$$d'p := \prod_{v} c_v(0)dp_v \tag{18.4}$$

(where dp_v is the measure obtained from a K-rational right invariant volume form on \mathbf{P}) is convergent.

Then the measure:

$$d'g := \prod_{v} c_v(0)^{-1} dg_v \tag{18.5}$$

on $\mathbf{P}\backslash\mathbf{H}(\mathbb{A}_K)$, valued in the line bundle defined by δ_P^{-1} (so that the composition of the two is Tamagawa measure on $\mathbf{H}(\mathbb{A}_K)$), will also converge.

There is another invariant integral we can define on sections of this line bundle, i.e. on the (unnormalized) induced space $\operatorname{Ind}_{\mathbf{P}(\mathbb{A}_K)}^{\mathbf{H}(\mathbb{A}_K)}(\delta_P^1)$: Given a section f_0 , we extend it to a continuous section $f_s \in \operatorname{Ind}_{\mathbf{P}(\mathbb{A}_K)}^{\mathbf{H}(\mathbb{A}_K)}(\delta_P^{1+s})$ and form $\lim_{s\to 0} c(s)^{-1} \int_{\mathbf{U}^-(\mathbb{A}_K)} f_s(u) du$ where du is Tamagawa measure on $\mathbf{U}^-(\mathbb{A}_K)$, and \mathbf{U}^- is the unipotent radical of a K-rational parabolic opposite to \mathbf{P} .

Let C be the scalar quotient of the functionals that we just defined:

$$C = \frac{f_0 \mapsto \int_{\mathbf{P}\backslash\mathbf{G}(\mathbb{A}_K)} f_0(g) \prod_v c_v(0)^{-1} dg_v}{f_0 \mapsto \lim_{s \to 0} c(s)^{-1} \int_{\mathbf{U}^-(\mathbb{A}_K)} f_s(u) du},$$

where $f_s \in \operatorname{Ind}_{\mathbf{P}(\mathbb{A}_K)}^{\mathbf{H}(\mathbb{A}_K)}(\delta_P^{1+s})$ (unnormalized induction) is any continuous section that specializes to f_0 , \mathbf{U}^- is the unipotent radical of a K-rational parabolic opposite to \mathbf{P} and du is Tamagawa measure on $\mathbf{U}^-(\mathbb{A}_K)$.

⁸⁰ Indeed, choose any linear algebraic representation of **H** on a K-vector space V, and let $v_0 \in V$; then the definition of rapid decays shows that $|\varphi(h)| \leq \|hv_0\|_{\mathbf{A}}^{-N}$. for any "adelic norm" on V. But it is possible to such a vector v_0 that is a **P**-eigencharacter, and moreover the eigencharacter may be any dominant character of **P**; that is to say, $|\varphi(p)| \ll_N \|\chi(p)\|_{\mathbf{A}}^{-N}$ for any dominant character χ . That is to say, φ decays in all directions when $|\chi| > 1$ for some dominant character χ ; and if $|\chi| \leq 1$ for all dominant characters, then in particular $|\delta_P| \leq 1$.

18.2.1. Proposition.

$$\int_{[\mathbf{H}]} \phi(h)dh = \lim_{s \to 0} C \cdot c(s)^{-1} \cdot \int_{[\mathbf{P}]} \phi(p)\delta^{s}(p) \prod_{v} c_{v}(0)dp_{v},$$

where the integrals are with respect to Tamagawa measure, and the right-hand integral is regularized as in the prior section.

If we take, for example, $c_v(s) = \zeta_v(1+s)^{\#\Delta_P}$, then one may evaluate $C = \prod_{\alpha \in \Phi} \langle \delta_P, \alpha^\vee \rangle$, thus obtaining (18.2).

PROOF. Let $f_s \in \operatorname{Ind}_{\mathbf{P}(\mathbb{A}_K)}^{\mathbf{H}(\mathbb{A}_K)}(\delta_P^{1+s})$ be any continuous section; then $\lim_{s\to 0} c(s)^{-1}\mathcal{E}_P(f_s)$ is a constant function on $[\mathbf{H}]$, where \mathcal{E}_P denotes the Eisenstein series intertwiner whose value at the identity is given by $\sum_{\mathbf{P}(K)\backslash\mathbf{G}(K)}$. Moreover, taking constant term along \mathbf{U}^- and taking residue as $s\to 0$, we deduce

$$\lim_{s \to 0} \frac{\mathcal{E}_P(f_s)(g)}{\int_{\mathbf{U}^-(\mathbb{A}_K)} f_s(ug) du} = 1$$

for any g, and therefore:

$$\int_{[\mathbf{H}]} \phi(h)dh = \frac{\lim_{s \to 0} c(s)^{-1} \int_{[\mathbf{H}]} \phi \mathcal{E}_P(f_s)}{\lim_{s \to 0} c(s)^{-1} \int_{\mathbf{U}^-(\mathbb{A}_K)} f_s(ug) du}.$$
 (18.6)

The numerator is equal to:

$$\lim_{s\to 0} c(s)^{-1} \int_{\mathbf{P}(K)\backslash \mathbf{H}(\mathbb{A}_K)} \phi f_s = \int_{\mathbf{P}\backslash \mathbf{H}(\mathbb{A}_K)} f_0(g) \left(\lim_{s\to 0} c(s)^{-1} \int_{[\mathbf{P}]} \phi(pg) \delta_P^s(p) d'p \right) d'g.$$

Here we have denoted by d'p, d'g the modified measures defined as in the statement of the proposition. Now, as we just saw, this defines a functional in f_s which is in fact invariant under $\mathbf{H}(\mathbb{A}_K)$ -translation; that shows that the inner expression $\lim_{s\to 0} c(s)^{-1} \int_{[\mathbf{P}]} \phi(pg) \delta_P^s(p)$ is in fact constant as a function of $g \in \mathbf{H}(\mathbb{A}_K)$.

Therefore the numerator of (18.6) equals

$$\int_{\mathbf{P}\backslash\mathbf{H}(\mathbb{A}_K)} f_0(g) d'g \cdot \lim_{s\to 0} c(s)^{-1} \int_{[\mathbf{P}]} \phi(p) \delta_P^s(p) d'p$$

and the claim follows.

- 18.3. The Whittaker case for GL_n . We denote by \mathbf{P}_n the mirabolic subgroup of $\mathbf{G} = \mathbf{GL}_n$, i.e. the stabilizer of a vector under the standard representation. We now repeat the argument of Jacquet [Jac01] to precisely compute the absolute value of the Whittaker period; this result has appeared already in [LM15], although the regularization of local periods has a slightly different definition there. Notice that the constant q is nontrivial here, as opposed to [LM15], because we are computing the norms of cusp forms as integrals over [PGL_n], as opposed to $\mathbf{GL}_n(\mathbb{A}_K)^1$.
- 18.3.1. THEOREM. Conjecture 17.4.1 is true for the Whittaker period of cuspidal representations of GL_n , with the constant q equal to n^{-1} .

Notice that the local Plancherel formula for the Whittaker model was established in Theorem 6.3.4, thus the forms $P_v^{\rm Planch}$ of Conjecture 17.4.1 are given by the regularized integrals of matrix coefficients of Corollary 6.3.3. This, of course, assumes that the local components of the cuspidal representation are tempered, as we have assumed the Arthur conjectures for the formulation of the Period Conjecture, but even without assuming this the following proof can give some Euler factorization by "meromorphically continuing" these local factors to the nontempered spectrum.

PROOF. For simplicity, we choose a factorization of Tamagawa measure for \mathbb{A}_K (which, we recall, is define so that the measure of \mathbb{A}_K/K is 1) into self-dual measures on K_v with respect to the characters ψ_v .

By Proposition 18.2.1, we have:

$$\int_{[\mathbf{PGL}_n]} |\phi(g)|^2 dg = \int_{[\mathbf{P}_n]}^* |\phi(p)|^2 dp,$$

where by a slight abuse of notation, we will also denote by \mathbf{P}_n the image of the mirabolic inside \mathbf{PGL}_n ; this image is a parabolic subgroup.

For any nonzero invariant measure d'p on $[\mathbf{P}_n]$ and any s with $\Re s \gg 0$ the integral:

$$\int_{[\mathbf{P}_n]} |\phi(p)|^2 \delta_{P_n}^s(p) d' p$$

"unfolds" to the Whittaker model, i.e. is equal to:

$$\int_{\mathbf{U}\backslash\mathbf{P}_n(\mathbb{A}_K)} |W_{\phi}(p)|^2 \delta_{P_n}^s(p) d' p,$$

where $W_{\phi}(g) = \int_{[\mathbf{U}]} \phi(u) \psi^{-1}(u) du$. This unfolding process is a sequence of inverse Fourier transforms, and is compatible with the Tamagawa (hence self-dual with respect to the given characters) measures that we are using on the adelic points of additive groups.

Hence:

$$\int_{[\mathbf{PGL}_n]} |\phi|^2 = \int_{\mathbf{U}\backslash \mathbf{P}_n(\mathbb{A}_K)}^* |W_{\phi}(p)|^2 dp, \tag{18.7}$$

where the regularization should be understood exactly as in (18.3), but with integrals over $[\mathbf{P}_n]$ replaced by integrals over $\mathbf{U}\backslash\mathbf{P}_n(\mathbb{A}_K)$.

Now we will write the right-hand side as an Euler product. Fix local measures $dp'_v = \zeta_v(1)dp_v$, so that their Euler product is convergent, and factorize $W_\phi = \prod_v W_{\phi_v}$. The local factors:

$$\int_{U \setminus P_n(K_v)} |W_{\phi_v}(p_v)|^2 \, \delta_P^s(p_v) \zeta_v(1) dp_v$$

are, by Rankin-Selberg theory, almost everywhere equal to $\zeta_v(1+s)$ times a factor whose Euler product is analytic at s=0.

Therefore, by (18.2):

$$\int_{\mathbf{U}\backslash\mathbf{P}_n(\mathbb{A}_K)}^* |W_{\phi}(p)|^2 dp = n \cdot \lim_{s \to 0} \prod_v \int_{U\backslash P_n(K_v)} |W_{\phi_v}|^2(p_v) \delta_P^s(p_v) dp_v.$$

Notice that there is no factor $\zeta_v(0)$ in front of the measure on the right hand side.

We will now see that the local Euler factor:

$$\int_{U \setminus P_n(K_v)} |W_{\phi_v}(p_v)|^2 dp_v, \qquad (18.8)$$

is as predicted by Conjecture 17.4.1, that is: the "adjoint" of the regularized form:

$$\int_{\mathbf{U}(K_v)}^* \langle \pi_v(u)\phi_v, \phi_v \rangle \, \psi^{-1}(u) du \tag{18.9}$$

that we constructed in §6.3.⁸¹ More precisely, we would like to show that if $\phi_v \in \pi_v$ and $\phi_v \mapsto W_{\phi_v}(1)$ is a Whittaker functional with the property that $\|\phi_v\|^2$ is given by (18.8), then $|W_{\phi_v}(1)|^2$ is given by (18.9).

We show this first for the case of GL_2 , which is simpler. By the definition of \int^* as a Fourier transform, we have for every vector $\phi_v \in \pi_v$:

$$\langle \phi_v, \phi_v \rangle = \int_{\mathbf{U} \backslash \mathbf{P}_n(K_v)} \int_{\mathbf{U}(K_v)}^* \langle \pi(u) \phi_v, \phi_v \rangle \, \psi(pup^{-1}) du \delta_{P_n}(p) dp.$$

Here the inner integral is regularized, and the outer integral is in fact absolutely convergent.

To see this, recall that our measures are always supposed to be given by invariant differential forms, defined globally; notice that in the case of GL_2 we have $\mathbf{U}\backslash \mathbf{P}_n(K_v)\simeq K_v^\times$, and the measure $\delta_{P_n}(p)dp$ can be thought of as additive measure on K_v , restricted to K_v^\times . Moreover, it is easy to see that the restriction of the matrix coefficient to $\mathbf{U}(K_v)$ is L^2 – in particular, its Fourier transform is a function and does not include any distribution supported at $0\in K_v$. Hence, the above equation is just duality for the Fourier transform, using the self-duality of the chosen factorization of measures.

This can be re-written as:

$$\int_{\mathbf{U}\backslash\mathbf{P}_n(K_v)} \int_{\mathbf{U}(K_v)}^* \langle \pi(u)\pi(p)\phi_v, \pi(p)\phi_v \rangle \psi(u) du dp.$$

If the image of $\phi_v \otimes \overline{\phi}_v$ under the morphism: $\pi_v \otimes \overline{\pi_v} \to C^{\infty}(\mathbf{U} \backslash \mathbf{G}(K_v), \psi) \otimes C^{\infty}(\mathbf{U} \backslash \mathbf{G}(K_v), \psi^{-1})$ defined by the regularized integral of the matrix coefficient is denoted by $W(g) \otimes \overline{W(g)}$ then the last integral can be written:

$$\int_{\mathbf{U}\backslash\mathbf{P}_n(K_v)} |W(p)|^2 dp,$$

and we have finished the proof for $G = GL_2$.

⁸¹Literally speaking, the regularization constructed there made use of the fact that we were working over a nonarchimedean field – see after Corollary 6.3.3; but it is simple to extend it to the archimedean case.

The general case will be proven by an inductive application of the above argument. To state it, let \mathbf{U}_i denote the unipotent radical of the parabolic corresponding to the first i nodes of the Dynkin diagram (hence, with Levi $\mathbf{GL}_i \times \mathbf{G}_{\mathrm{m}}^{n-i}$), and let $\tilde{\mathbf{P}}_i$ be the preimage, in that parabolic, of the mirabolic subgroup of \mathbf{GL}_i under the natural quotient map. In particular, $\mathbf{U}_1 = \mathbf{U} = \tilde{\mathbf{P}}_1$ and $\tilde{\mathbf{P}}_n = \mathbf{P}_n$. Denote by \mathbf{N} the commutator subgroup of \mathbf{U} , and by \mathbf{N}_i its intersection with \mathbf{U}_i ; in particular, $\mathbf{N}_1 = \mathbf{N}_2 = \mathbf{N}$.

Denote by f(g) the tempered matrix coefficient $\langle \pi(g)\phi_v(g), \phi_v(g)\rangle$. We start denoting K_v -points of algebraic groups by regular font, as we have done in previous sections. As in Proposition 6.3.1, one can show that f is integrable over N; indeed, that Proposition established integrability over the kernel H_0 of a generic character, but one has:

$$\int_{H_0} f(h)dh = \int_{H_0/N} \int_N f(hn)dndh,$$

which means that the inner integral is finite for almost all h; but it is also locally constant in h, which shows that it is finite for every h. The same argument shows that f is integrable over N_i for every i; the function $f(g) = \int_{N_i} f(n_i g) dn_i$ will be denoted by f_{N_i} (typically considered as a function on $g \in U_i$ or U_{i-1}).

Similarly, consider the integral of $|W|^2$ over $U \setminus P_n$, where W is a Whittaker function for a tempered representation. The asymptotics of Whittaker functions make it easy to see that the integral is absolutely convergent. But this integral can be written as a consecutive application of integrals:

$$\int_{U \setminus P_n} |W|^2 = \int_{\tilde{P}_{n-1} \setminus P_n} \int_{\tilde{P}_{n-2} \setminus \tilde{P}_{n-1}} \cdots \int_{U \setminus \tilde{P}_2} |W|^2,$$

and by the same argument all of the integrals are absolutely convergent.

18.3.2. LEMMA. The restriction of f to U_i is in $L^{1+\varepsilon}$, for every i and every $\varepsilon > 0$.

PROOF. We will use the Cartan and Iwasawa decompositions, with $K = \mathbf{GL}_n(\mathfrak{o}_v)$.

Recall that $f(k_1ak_2) \ll \delta^{-\frac{1}{2}}(a)$ when $k_1, k_2 \in K$ and $a \in A^+ \subset A$, i.e. is *B-anti-dominant*.

For an element $g \in G$ write it in terms of the Cartan and Iwasawa decompositions with respect to the *opposite* Borel $B^- = AU^-$:

$$q = k_1 a_c(q) k_2, \quad q = u^- a_i(q) k,$$

with $a_c(g) \in A^+/A_0$, $u^- \in U^-$, $a_i(g) \in A/A_0$, where A_0 denotes the maximal compact subgroup of A.

It is known that $\log a_i(g) - \log a_c(g)$ (the same log maps as in (5.22)) is in the cone spanned by positive coroots. In particular, $\delta(a_i)(g) \leq \delta(a_c)(g)$. Hence:

$$\int_{U_i} |f(u)|^{1+\varepsilon} du \ll \int_{U_i} \delta^{-\frac{1+\varepsilon}{2}} (a_c(u)) du \le$$

$$\le \int_{U_i} \delta^{-\frac{1+\varepsilon}{2}} (a_i(u)) du.$$

The last integral represents the value on the spherical vector for the standard intertwining operator:

$$I_{B^-}(\delta^{-\frac{\varepsilon}{2}}) \to I_{B'}(\delta^{-\frac{\varepsilon}{2}}),$$

where B' is the Borel subgroup obtained from B^- by inverting the opposites of the roots in the Lie algebra \mathfrak{u}_i . This intertwining operator is known to converge absolutely for $\varepsilon > 0$.

Recall that:

$$|W(1)|^2 = \int_U^* \langle \pi_v(u)\phi_v, \phi_v \rangle \psi^{-1}(u)du$$

was defined as the value at ψ of Fourier transform of f_N , the latter considered as a function on U/N:

$$|W(1)|^2 = \widehat{f_N}(\psi).$$

To be precise, it was defined in terms of Fourier transform on U/H_0 (where H_0 denotes the kernel of ψ), but since the Fourier transform of f_N is locally constant on the nondegenerate locus, its value on ψ coincides with the one previously defined. More generally, we will make use of the following:

18.3.3. Lemma. Let $V \subset W$ be two vector spaces and f a function on W which is integrable over preimages of compact subsets of W/V, and such that its product with Lebesgue measure is a tempered distribution (in the archimedean case). Then:

$$\widehat{f}|_{V^{\perp}} = \widehat{f_V},\tag{18.10}$$

where:

- $f_V(w) = \int_V f(w+v)dv$, considered as a function on W/V;
- measures have been chosen compatibly on V, W and W/V for defining Fourier transform between (tempered generalized) functions and for the definition of f_V;
- the meaning of "restriction" of Fourier transform to $V^{\perp} \subset W^*$ is the following: Let $W = V \oplus V'$ be a decomposition, and consider an approximation of the delta measure at the identity on $(V')^{\perp}$ by Schwartz measures μ_n on $(V')^{\perp}$. Then, by definition:

$$\hat{f}|_{V^{\perp}} = \lim_{n} \left(\mu_n \star \hat{f}|_{V^{\perp}} \right) \Big|_{V^{\perp}}$$

as tempered generalized functions, provided that this limit is independent of choices (i.e. the independence is part of the above assertion). The proof of this fact is easy and left to the reader.

Going back to the notation that we introduced above, denote by V_i the quotient of U_i by U_{i+1} . As in the case of \mathbf{GL}_2 , the integral of $|W|^2$ over $U \setminus P_2$ can be identified with the integral of \widehat{f}_N over the subset $\psi + V_1^* \subset \widehat{(U/N)}$ – indeed, by Lemma 18.3.2, f_N is a locally integrable function, so again the integral over an open dense subset of $\psi + V_1^*$, which is represented by the $U \setminus P_2$ integration, is the same as the integral over the whole set.

Applying Lemma 18.3.3 to the function $\widehat{f_N}$ with V replaced by V_1^* , $(\widehat{f_N})|_{V_*^*}$, which is locally constant around ψ , is the Fourier the function transform of the restriction of f_N to U_2/N . (Notice that this restriction is smooth under translation by elements of U, which proves that f_N is indeed integrable over preimages of compact subsets of $(U_2/N)^*$, as required by the Lemma.)

Now we repeat the same step, this time over the vector space U_2/N_3 on which the group P_3 (and, in fact, its normalizer – a parabolic subgroup) acts. The orbit under this parabolic of the restriction of a nondegenerate character of U is open in (U_2/N_3) ; this shows that the Fourier transform of f_{N_3} is locally constant around nondegenerate elements of the subspace $(U_2/N)^*$, and by applying Lemma 18.3.3 we see:

$$\widehat{f_{N_3}}(\psi) = \widehat{f_N}(\psi)$$

for such characters ψ . Now, the integral of $\widehat{f}_{N_3}(\psi)$ over the action of $\widetilde{P}_2 \setminus \widetilde{P}_3$ can be identified with its integral over ψ +characters of U_2/U_3 so we get, as before, the Fourier transform of the restriction of f_{N_3} to U_3 .

Thus, inductively, in the end we see that $\int_{U\setminus P_n} |W(p)|^2 dp = f(1) =$ $\langle \phi_v, \phi_v \rangle$. This completes the proof of the theorem.

18.4. Compatibility of the conjecture with unfolding. Our local interpretation of the "unfolding" process as an isometry between different L^2 -spaces (Theorem 9.5.9) allows us to prove Conjecture 17.4.1 when one period integral "unfolds" to another, e.g.:

18.4.1. Theorem. Conjecture 17.4.1 is true for cuspidal representations, with the given value of q, for the following spherical varieties:

- SL^{diag}_n\GL_n × GL_{n+1} under the action of G_m × GL_n × GL_{n+1} (q⁻¹ = n · (n + 1));
 P_n^{diag}_n\GL_n × GL_n (the classical Rankin-Selberg integral, q⁻¹ =
- \bullet $\mathbf{SL}_n \times \mathbf{P}_n \setminus \mathbf{GL}_{2n}$ under the action of $\mathbf{G}_m \times \mathbf{GL}_{2n}$ where $\mathbf{G}_m =$ $\mathbf{GL}_n^{\mathrm{ab}}$ (the Rankin-Selberg integral of Bump-Friedberg [**BF90**], $q^{-1} =$ 2n).

As before, \mathbf{P}_n denotes the mirabolic subgroup. We should clarify the meaning of the conjecture when the stabilizer \mathbf{H} of a point on \mathbf{X} is not reductive, at least in the cases above. Notice that the above examples actually correspond to periods against *characters* of spherical subgroups, but with the character expressed as a character of the group; for instance, in the first case the automorphic representation is of the form $\pi = \chi \otimes \pi_1 \otimes \pi_2$, where χ denotes an idele class character of $\mathbf{G}_{\mathrm{m}} = \mathbf{GL}_n^{\mathrm{ab}}$, and the period of an element of π over $[\mathbf{H}]$ is the same as the $[\mathbf{GL}_n]^{\mathrm{diag}}$ -period of an element of $\pi_1 \otimes \pi_2$ against the character χ .

Since **X** is always quasi-affine, we will say that a character χ of $\mathbf{G}(\mathbb{A}_K)$ which is trivial on $\mathbf{H}(\mathbb{A}_K)$ is "sufficiently X-positive" if it is of the form:

$$\omega = \chi \cdot |\mathfrak{c}|^s,$$

where χ is unitary, \mathfrak{c} is an algebraic character vanishing on $\overline{\mathbf{X}}^{\mathrm{aff}} \setminus \mathbf{X}$ (where $\overline{\mathbf{X}}^{\mathrm{aff}}$ denotes the affine closure) and $\Re(s)$ is sufficiently large. The same notion will be applied to characters of $G(K_v)$, of course.

We will also apply this notion to *central* characters of $\mathbf{G}(\mathbb{A}_K)$, provided they are of the form (a unitary character) \times (a sufficiently X-positive character of $\mathbf{G}(\mathbb{A}_K)$).

Now, for a unitary cuspidal representation π of \mathbf{G} the integral over $[\mathbf{H}]$ is not in general convergent; the reason is, of course, that its elements are not rapidly decaying, but only rapidly decaying modulo the center, and $[\mathbf{H}]$ does not have finite volume. By the way, if \mathbf{H} is not reductive then $[\mathbf{H}]$ is not necessarily closed in $[\mathbf{G}]$, which is another way to see the lack of convergence. However, it is easy to show that the $[\mathbf{H}]$ -period is convergent on elements of $\pi \otimes \omega$, where ω is any sufficiently X-positive idele class character of \mathbf{G} . Thus, given a cusp form ϕ with unitary central character we can interpret:

$$\int_{[\mathbf{H}]} \phi(h)dh = \lim_{s \to 0} \int_{[\mathbf{H}]} \phi \cdot \omega^{s}(h)dh,$$

where ω is a sufficiently X-positive idele class character and the limit denotes, as before, the value at s=0 of the meromorphic continuation of the given expression. The ability to continue meromorphically, of course, comes in question, but in the cases above it is not an issue as the above period integrals can be interpreted as inner products against Eisenstein series; we leave the details to the reader.

Finally, we should remind that since the invariant measure on $[\mathbf{H}]$ defined by a right-invariant volume form dh is not well-defined (does not correspond to a convergent Euler product) when \mathbf{H}^{ab} has nonzero split rank, one should heed the conventions of §17.5 in order to make sense of the conjecture: both the local measures and the local Euler factors on the right hand side of (17.12) should be multiplied by the same local factors so that the Euler products become convergent.

PROOF. Fix an invariant volume form dh on \mathbf{H} and let $d'h_v = \zeta_v(1)^{\mathrm{rk}\mathbf{H}^{\mathrm{ab}}}dh_v$ so that the Euler product of measures converges. In the first two examples above $\mathrm{rk}\mathbf{H}^{\mathrm{ab}} = 1$, while in the third $\mathrm{rk}\mathbf{H}^{\mathrm{ab}} = 2$. We let $d'h = \prod_v d'h_v$, a measure on $\mathbf{H}(\mathbb{A}_K)$ and $[\mathbf{H}]$.

Fix a unitary cuspidal representation π of $\mathbf{G}(\mathbb{A}_K)$, identified with a space of functions on $[\mathbf{G}]$, and let $\phi \in \pi$. In all of these cases the period integral of cusp forms "unfolds" to the Whittaker model, i.e.:

$$\int_{[\mathbf{H}]} \phi \cdot \omega^{s}(h) d'h = \int_{\mathbf{U} \cap \mathbf{H} \setminus \mathbf{H}(\mathbb{A}_{K})} \int_{[\mathbf{U}]} \phi \cdot \omega^{s}(nh) \psi^{-1}(n) du d'h$$

for sufficiently X-positive idele class characters ω^s of G.

As in the previous proof, choose a morphism $\phi_v \mapsto W_v$ into the Whittaker space of π_v in such a way that:

$$|W_v(1)|^2 = \int_{\mathbf{U}(K_v)}^* \langle \pi_v(u)\phi_v, \phi_v \rangle \, \psi^{-1}(u) du.$$
 (18.11)

Since the conjecture holds for the Whittaker model, Theorem 18.3.1, we can write:

$$\left| \int_{[\mathbf{H}]} \phi \cdot \omega^s(h) d'h \right|^2 = q \left| \int_{\mathbf{U} \cap \mathbf{H} \setminus \mathbf{H}(\mathbb{A}_K)} \prod_v W_v \cdot \omega_v^s(h) d'h \right|^2,$$

where q is as stated in the theorem.

Now we can take analytic continuation of both sides to s=0; this is compatible with the Period Conjecture, where Euler products where interpreted by means of analytic continuation.

Thus, it remains to verify that the local factors:

$$\phi_v \mapsto \lim_{s \to 0} \left| \int_{U \cap H \setminus H} W_v \cdot \omega^s(h) d' h_v \right|^2$$
 (18.12)

(where we started again denoting by U, H etc. the K_v -points of the corresponding groups) are equal to $\zeta_v(1)^{2\mathrm{rk}\mathbf{H}^{\mathrm{ab}}}$ (because we modified the measures) times the "Plancherel" hermitian forms $\mathcal{P}_v^{\mathrm{Planch}}$ of (17.12). Equivalently, that the same local factors with measures dh_v are equal to $\mathcal{P}_v^{\mathrm{Planch}}$.

In other words, we need to verify that if $\Phi \in C_c^{\infty}(X)$ and we consider the adjoints of the maps (18.12) as morphisms:

$$I_{\pi_v}: C_c^{\infty}(X) \to \pi_v$$

then we have a Plancherel formula:

$$\|\Phi\|_{L^2(X)}^2 = \int_{\widehat{\mathbf{G}(K_v)}} \|I_{\pi_v}(\Phi)\|^2 \mu_{G_v}(\pi_v), \tag{18.13}$$

where μ_{G_v} is Plancherel measure on $\widehat{\mathbf{G}}(K_v)$. Here we remind that all measures (including the measure on $\widehat{\mathbf{G}}(K_v)$) and hence the Plancherel measure on $\widehat{\mathbf{G}}(K_v)$) are chosen by global volume forms, which we factorize at will. For any choice of local measures on G and G (and, compatibly, on G),

the Plancherel formula of Theorem 6.3.4 holds for the Whittaker model – i.e. the analog of (18.13) when $L^2(X)$ is replaced by L^2 of the Whittaker model and $||I_{\pi_v}||^2$ is replaced by the adjoint of the map $\phi_v \mapsto W_v$.

Now recall our local interpretation of "unfolding" in §9.5 as an isometry:

Unf:
$$L^2(X) \to L^2(U \backslash G, \psi)$$
. (18.14)

This isometry, restricted to $C_c^{\infty}(X)$, was given by a series of Fourier transforms over subgroups of U. We claim that we can factorize global Tamagawa measures to obtain local measures on G, U, H etc. (and compatibly on the corresponding homogeneous spaces) so that this series of Fourier transforms does indeed give rise to an isometry (18.14) when these measures are used.

Let us use the notation preceding Theorem 9.5.9, according to which a step of the unfolding process consists in applying Fourier transform between sections of suitable complex line bundles along the fibers of the maps: $F \to Y$ and $V^* \to Y$. We remind that \mathbf{F} is the total space of an affine bundle over a variety \mathbf{Y} and \mathbf{V}^* is the total space of its dual vector bundle; all of \mathbf{F} , \mathbf{V}^* and \mathbf{Y} carry compatible, homogeneous (or almost homogeneous, for \mathbf{V}^*) actions of our group \mathbf{G} . In our example at the beginning of the unfolding process we have $\mathbf{F} \simeq \mathbf{X}$, while at the end of the unfolding process we have $\mathbf{V}^* \simeq$ a partial compactification of $\mathbf{U} \setminus \mathbf{G}$. (And, more generally, by "folding back" after each step we may assume that at every step we have a homogeneous space \mathbf{F} for \mathbf{G} and a "Whittaker-type" space \mathbf{V}^* where \mathbf{G} acts with an open orbit \mathbf{V}^{*+} .)

Theorem 9.5.9 holds, locally, for measures on F, V^* which can be written as the composition of dual Haar measures on the fibers of $F \to Y$ and $V^* \to Y$ with a G-eigenmeasure on Y valued in a suitable line bundle. If we take Haar measures coming from invariant differential forms on the fibers, over a fixed k-point, of the map: $\mathbf{F}(\mathbb{A}_K) \to \mathbf{Y}(\mathbb{A}_K)$ and $\mathbf{V}^*(\mathbb{A}_K) \to \mathbf{Y}(\mathbb{A}_K)$ then these measures are dual to each other by global additive duality. Notice that the measure on \mathbf{Y} obtained from suitable eigenforms over K is infinite for the adelic points of \mathbf{Y} ; as a result, the resulting measure on $\mathbf{F}(\mathbb{A}_K)$ does not make sense, although the measure on $\mathbf{V}^{*+}(\mathbb{A}_K)$ does.

We return to the proof of (18.13). Since the unfolding map (18.14) is an isometry, we have $\|\Phi\|_{L^2(X)}^2 = \|\Phi\|_{L^2(U\setminus G,\psi)}$. It is easy to see that the inverse of the unfolding map preserves compact support:

$$\operatorname{Unf}^{-1}: C_c^{\infty}(U \backslash G, \psi) \to C_c^{\infty}(X)$$

(in the nonarchimedean case; in the arhimedean the image will lie in the Schwartz space). Let \mathcal{U} denote the image; for Φ lying in the image the decomposition (18.13) holds.

To show the validity for all $\Phi \in C_c^{\infty}(X)$ there are several ways to proceed, none of which is very pleasant: One needs to describe the correct morphisms: $C_c^{\infty}(X) \to \pi_v$ which restrict to the morphisms I_{π_v} on \mathcal{U} and appear in the Plancherel formula. First, one can appeal to a multiplicity-one statement for μ_{G_v} -almost all representations, whenever it is available, or try

to prove it by analyzing the precise image of $C_c^{\infty}(X)$ under the unfolding map. Secondly, one can try to show that the image of $C_c^{\infty}(X)$ under Unf is in the Harish-Chandra Schwartz space $\mathscr{C}(U\backslash G,\psi)$ of the Whittaker model, and hence the morphisms I_{π_v} are the correct ones as the continuous extension of the Plancherel norms from $C_c^{\infty}(U\backslash G,\psi)$ to $\mathscr{C}(U\backslash G,\psi)$ (with respect to the topology of the latter; in particular: the integral (18.12) is convergent for s=0). We will appeal to a more direct but less informative argument, proving directly:

18.4.2. Lemma. The integral (18.12) is convergent for s = 0, uniformly in terms of the asymptotics of the Whittaker function.

In particular, we may approximate an element of $C_c^{\infty}(X)$ in the L^2 -norm by elements of \mathcal{U} , and then the integrands on the right hand side of (18.13) converge uniformly, proving the validity of the formula.

The lemma itself is an easy application of the Iwasawa decomposition, the asymptotics of Whittaker functions, and straightforward volume computations, and is left to the reader. This completes the proof of the theorem.

Appendix A. Prime rank one spherical varieties

A.1. Goals. In this appendix, we compute the dual group and the commuting (Arthur) SL_2 for every affine spherical variety of rank one, proving Propositions 3.5.1 and 3.6.1.

As in the text, we use the following terminology: for a morphism $f: \mathrm{SL}_2 \to \check{G}$, we shall call the restriction $f|_{G_{\mathrm{m}}}$ (or to the Lie algebra \mathfrak{g}_m) the weight of f.

Let us recall that the notion of a normalized spherical root is defined in 3.1: It is a character of \mathbf{A}_X that is either a root or a sum of two superstrongly orthogonal roots; it is denoted below by γ . In particular, for each spherical variety of rank one we have well-defined maps:

$$G_{\mathrm{m}} \xrightarrow{\gamma} A_{X}^{*} \to A^{*}.$$

Recall also that $A_{X,GN}^*$ is, by definition, the image of A_X^* in A^* . Moreover, the sum $2\rho_{L(X)}$ of positive roots of the Levi $\mathbf{L}(X)$ defines another map $2\rho_{L(X)}: G_{\mathrm{m}} \to A^*$.

We shall check two assertions for rank one spherical varieties, which we term "existence" and "uniqueness."

Existence: For each spherical variety X of rank one and normalized spherical root γ , there exists a morphism:

$$f_X \times f_A : \operatorname{SL}_2 \times \operatorname{SL}_2 \longrightarrow \check{G}$$
 (A.1)

such that the weight of f_X is γ , and f_A is principal into $\check{L}(X)$ with weight equal to $2\rho_{L(X)}$.

To describe the uniqueness assertion, recall that Gaitsgory and Nadler have associated to every affine spherical variety a group $\check{G}_{X,GN}$, which we suppose to satisfy axioms (GN1) – (GN5) from §3.3. In the case of rank one it is necessarily the image of a morphism: $f_{GN}: \left(A_{X,GN}^*\right)^{W_X} \times \operatorname{SL}_2 \to \check{G}$ which is the identity on $\left(A_{X,GN}^*\right)^{W_X}$ and has weight positively proportional to γ (by (GN2)). We will show:

Uniqueness: Assuming (GN1) – (GN5), the restriction of f_{GN} to SL_2 has weight γ .

A.2. Lie algebra versions. Note that both the "Existence" and "Uniqueness" assertions can be checked at the level of Lie algebras, namely:

Existence – **Lie:** For each spherical variety **X** of rank one and normalized spherical root γ , there exists a morphism:

$$f_X \times f_A : \mathfrak{sl}_2 \times \mathfrak{sl}_2 \longrightarrow \check{\mathfrak{g}}$$
 (A.2)

such that the weight of f_X is γ , and f_A is principal into $\check{\mathfrak{l}}(X)$ (with weight equal to $2\rho_{L(X)}$).

Uniqueness – Lie: Let \mathbf{X} be an affine spherical variety of rank one, and let $f_{GN}: \left(\mathfrak{a}_{X,GN}^*\right)^{w\gamma} \times \mathfrak{sl}_2 \to \check{\mathfrak{g}}$ be the Gaitsgory-Nadler morphism having image equal to $\check{g}_{X,GN}$ and weight positively proportional to γ (which is possible by (GN2)). Assuming (GN1)–(GN5), it actually has weight equal to γ .

- **A.3.** Reductions for the existence statement. We first make several reductions:
- A.3.1. **X** is homogeneous. The existence statement depends only on the data γ , $\check{L}(X)$, and hence only on the open **G**-orbit (which may not be affine).
- A.3.2. Parabolic induction. Suppose that \mathbf{P} is a parabolic subgroup of \mathbf{G} , with Levi quotient \mathbf{L} , and \mathbf{X}_1 is a spherical variety for \mathbf{L} . Suppose moreover that \mathbf{X} is isomorphic to $\mathbf{X}_1 \times^{\mathbf{P}^-} \mathbf{G}$.

The Levi $\check{L}(X)$, as well as the normalized spherical root, coincide for X and X_1 (more precisely: they are related by means of the canonical inclusion $\check{L} \to \check{G}$) therefore the existence statement is reduced to the case where X is not parabolically induced.

- A.3.3. Group surjection. Let \mathfrak{g}_1 be a (Lie algebra) direct summand of \mathfrak{g} such that the corresponding normal, connected subgroup \mathbf{G}_1 acts trivially on \mathbf{X} . The normalized spherical root is independent of whether we consider \mathbf{X} as a \mathbf{G} -variety or as a \mathbf{G}/\mathbf{G}_1 -variety, and $\check{l}(X) = \check{l}'(X) \oplus \check{g}_1$, where $\check{l}'(X)$ is the analog of $\check{l}(X)$ when \mathbf{X} is considered as a \mathbf{G}/\mathbf{G}_1 -variety. (Here \check{g}_1 does not quite make sense as a subalgebra of \check{g} , but its sum with the center does.) It is clear that the existence statement is equivalent whether we are talking about $\check{l}(X)$ or $\check{l}'(X)$, which reduces the problem to the case where the action is infinitesimally faithful.
- A.3.4. Quotient by the connected component of the center. Let $\overline{\mathbf{X}}$ be the quotient of \mathbf{X} by $\mathcal{Z}(\mathbf{X})$. The spherical roots of \mathbf{X} and $\overline{\mathbf{X}}$ coincide, as do the associated Levi subgroups. Therefore, the existence statement is reduced to the case that $\mathcal{Z}(\mathbf{X})$ is trivial. In this case, \mathbf{X} admits a wonderful compactification.
- A.3.5. Passage to a simply connected cover. By the previous two reductions, \mathbf{G} is semisimple. Notice that the existence statement for Lie algebras does not depend on whether we consider \mathbf{X} as a spherical variety for \mathbf{G} or \mathbf{G}^{sc} , the simply connected cover of \mathbf{G} . Therefore, altogether, we are reduced to the case of a pair (\mathbf{G}, \mathbf{X}) such that:
 - **G** is semisimple simply connected;
 - X is homogeneous, not parabolically induced (in any nontrivial way) and with $\mathcal{Z}(X)$;
 - there is no direct factor of **G** acting trivially.

In particular, since \mathbf{X} has rank one, its isotropy groups are *prime* in the sense of [Was96, Definition 2.3], and its wonderful embedding is included in [Was96, Table 1].

A.3.6. Quotient by a finite automorphism group. The normalized spherical root and the associated Levi subgroup do not change if we divide **X** by a finite group of **G**-automorphisms. Therefore, it is enough to prove the existence statement for a class of representatives for the varieties of Table 1 of [**Was96**] modulo the operation of taking quotients by finite subgroups of the automorphism group. We present here a table of such representatives:

| | $X = H \backslash G$ | $\mathbf{L}(\mathbf{X})$ | γ | Root type. |
|------|--|--------------------------|---|------------|
| 1. | $\operatorname{GL}_n \setminus \operatorname{SL}_{n+1}$ | [2, n-1] | $\sum_{1}^{n} \alpha_{i}$ | Τ |
| 2. | $\operatorname{SL}_2 \setminus \operatorname{SL}_2^2$ | Ø | $\alpha_1 + \alpha_1'$ | G |
| 3. | $\operatorname{Sp}_4 \setminus \operatorname{SL}_4$ | $\{1,3\}$ | $\alpha_1 + 2\alpha_2 + \alpha_3$. | G |
| 4. | $\operatorname{Spin}_{2n} \backslash \operatorname{Spin}_{2n+1}$ | [2, n] | $\sum_{i=1}^{n} \alpha_i$ | Τ |
| 5. | $\operatorname{Spin}_{2n-1} \backslash \operatorname{Spin}_{2n} \ (n \ge 3)$ | [2, n] | $2\sum_{1}^{n-2}\alpha_{i} + \alpha_{n-1} + \alpha_{n}$ | G |
| 6. | $\mathrm{Spin}_7 \backslash \mathrm{Spin}_8$ | $\{1, 3\}$ | $\alpha_1 + 2\alpha_2 + \alpha_3$ | G |
| 7. | $\operatorname{SL}_n \times G_{\mathrm{m}} \ltimes \wedge^2 G_{\mathrm{a}}^n \setminus$ | [2, n] | $\sum_{i=1}^{n} \alpha_i$ | Τ |
| | $\operatorname{Spin}_{2n+1}$ | | | |
| 8. | $G_2 \backslash \mathrm{Spin}_7$ | [1, 2] | $\alpha_1 + 2\alpha_2 + 3\alpha_3$ | G |
| 9. | $\operatorname{SL}_2 \times \operatorname{Sp}_{2n-2} \setminus$ | $\{1\} \cup [3,n]$ | $\alpha_1 + \alpha_n +$ | Τ |
| | $\operatorname{Sp}_{2n} (n \ge 2)$ | | $2\sum_{i=1}^{n-1}\alpha_i$ | |
| 10. | $\operatorname{Spin}_9 \backslash F_4$ | [1, 3] | $\alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4$ | Т |
| 11. | $\operatorname{SL}_3 \setminus G_2$ | {2} | $2\alpha_1 + \alpha_2$ | T |
| 12 . | $G_{\mathrm{m}} \times \mathrm{SL}_2 \ltimes (G_{\mathrm{a}} \oplus G_{\mathrm{a}}^2) \setminus$ | Ø | $\alpha_1 + \alpha_2$ | Τ |
| | G_2 | | | |

Notational conventions: Simple roots are labelled according to Bourbaki. We parameterize $\mathbf{L}(\mathbf{X})$ by using the standard numbering of simple roots and giving the set of simple roots of \check{G} (= simple cooots of \mathbf{G}) contained in it: the notation [a,b] is taken to mean the set of all integers i with $a \leq i \leq b$. (Recall that, in the quasi-affine case, these are exactly the same as those simple coroots which are orthogonal to γ).

A routine verification shows that the (Existence) assertion holds for all the varieties in the above list. We refer to $\S A.5$ for further discussion of this and the following result:

A.3.7. LEMMA. In cases (2) – (11), as well as case (1) with n odd, there is a unique morphism $SL_2 \to \check{G}$ which commutes with $2\rho_{L(X)}$ and has weight proportional to γ .

A.4. Reductions for the uniqueness statement. In the case of the uniqueness statement, we should like to repeat the same reductions; but we face the problem that, a priori, the Gaitsgory-Nadler dual group depends on the choice of affine embedding.

Thus, for example, in the case when \mathbf{X}^+ (the open \mathbf{G} -orbit) is parabolically induced, we may not immediately replace \mathbf{X} by a spherical variety for the Levi – (GN4) only gives us some information about the image of the dual group (namely, that it lies in the corresponding Levi subgroup).

We know, by the prior discussion, that to any rank one variety \mathbf{X} we may find:

- a parabolic subgroup **P**⁻, with Levi quotient **L**;
- a spherical L-variety X_1 , such that the action of L on X factors through some quotient L' whose derived group is simple (since this is, by inspection, the case of all wonderful varieties of rank one);
- write: $\mathbf{L}'' = \text{the simply connected cover of the adjoint group of } \mathbf{L}',$ then the variety $\mathbf{X}_1/\mathcal{Z}(\mathbf{X}_1)$, as a variety under \mathbf{L}'' , appears in the prior table (up to the operation of taking a finite quotient, cf. §A.3.6.);

in such a way that such that the open orbit on X is isomorphic to $X_1 \times^{\mathbf{P}^-} \mathbf{G}$. In this way, we regard X as belonging to one of twelve "types" indexed by the table above.

Using notation as in §A.3.2, by (GN4) the dual group of **X** is contained in \check{L} , the dual Levi corresponding to the class opposite to \mathbf{P}^- . In fact, again by (GN4), $\check{g}_{X,GN}$ lives in the dual Lie algebra of \mathfrak{l}' (which is canonically a subalgebra of $\check{\mathfrak{l}}$).

We call a spherical variety (\mathbf{G}, \mathbf{X}) of rank one *good* if there exists data as above and a *unique* morphism $f: \mathfrak{sl}_2 \to \check{\mathfrak{l}}'$ with weight proportional to γ that commutes both with the image of $2\rho_{L(X)}$ and $(\mathfrak{a}_X^*)^{W_X}$.

The uniqueness statement is evidently valid for any good (G, X) (because of the existence statement, which is already proven in the prior section.) That the uniqueness statement holds for any X of rank one follows from the two Lemmas that follow:

A.4.1. Lemma. Any affine spherical variety of rank one, except possibly type (1) for n even are good.

PROOF. Note the following sufficient criterion for (\mathbf{G}, \mathbf{X}) to be good: for $(\mathbf{G}', \mathbf{X}')$ the corresponding entry in the table of types: there exists a unique morphism $\mathfrak{sl}_2 \to \check{G}'$ commuting with $2\rho_{L(X')}$. In particular, in all cases but (12), the assertion follows from Lemma A.3.7.

For case (12): Since **X** is affine, its homogeneous part $\mathbf{X}_1 \times^{\mathbf{P}^-} \mathbf{G}$ must be quasi-affine. This forces \mathbf{X}_1 to be quasi-affine, also. 82

Now we claim that \mathfrak{a}_X^* , considered inside \mathfrak{l} , actually projects to a full Cartan subalgebra of \mathfrak{l}'' ; the result follows easily from there. This is equivalent to saying that if $\mathfrak{a}_1 \subset \mathfrak{a}_{L'}$ is the Lie algebra of the stabilizer of a generic point of \mathbf{X}_1 in the Borel of \mathbf{L}' modulo its unipotent radical (and $\mathfrak{a}_{L'}$ denotes the universal Cartan algebra of \mathbf{L}'), then $a_1 \cap [\mathfrak{l}', \mathfrak{l}']$ is trivial.

Notice that in this case \mathbf{X}_1 is, up to finite quotient, the quotient of \mathbf{L}' (whose derived group is \mathbf{G}_2) by $\mathbf{T} \cdot \mathbf{SL}_2 \cdot (\mathbf{G}_a \oplus \mathbf{G}_a^2)$, where \mathbf{T} is a torus in \mathbf{L}' commuting with $\mathbf{SL}_2 \cdot (\mathbf{G}_a \oplus \mathbf{G}_a^2)$. From this it is easy to see that $a_1 \cap [\mathfrak{l}', \mathfrak{l}']$ is the Lie algebra of $\mathbf{T} \cap [\mathbf{L}', \mathbf{L}']$, which is at most equal to \mathbf{G}_m . But it is

⁸²This follows from the fact that an orbit of a linear algebraic group on an affine variety is quasi-affine.

easy to see that $\mathbf{G}_{\mathrm{m}} \cdot \mathbf{SL}_{2} \cdot (\mathbf{G}_{a} \oplus \mathbf{G}_{a}^{2}) \backslash \mathbf{G}_{2}$ is not quasiaffine, which implies what we want.

Finally, if **X** is of type (1) we show in $\S A.6$:

A.4.2. LEMMA. Suppose **X** of type (1). Then, with the above notation, there is a good **L**-spherical variety \mathbf{Y}_1 , in fact a torus bundle over \mathbf{X}_1 , and an affine spherical embedding **Y** of $\mathbf{Y}^+ := \mathbf{Y}_1 \times^{\mathbf{P}^-} \mathbf{G}$ such that $k[\mathbf{X}] = k[\mathbf{Y}]^{\mathbf{T}}$ (where **T** is the torus of automorphisms of that bundle).

Then the uniqueness statement follows, in this final case, from (GN5).

A.5. Further discussion of the existence result. We now give a few details, or at least a table of some useful data, related to Lemma A.3.7 and the prior existence assertion.

We describe, in the case where G is a classical group, the representation of $\mathfrak{sl}_2 \times \mathfrak{sl}_2$ arising as the composite of $f_X \times f_A$ with the classical representation of \mathfrak{g} . We denote the n-dimensional irreducible representation of SL_2 by ρ_n .

- 1. $\rho_2 \otimes \rho_1 \oplus \rho_1 \otimes \rho_{n-1}$.
- 2. ρ_A is trivial and ρ_{GN} is the diagonal morphism $\mathfrak{sl}_2 \to \mathfrak{sl}_2^2$.
- 3. $\rho_2 \otimes \rho_2$.
- 4. $\rho_1 \otimes \rho_{2n-2} \oplus \rho_2 \otimes \rho_1$.
- 5. $\rho_1 \otimes \rho_{2n-3} \oplus \rho_3 \otimes \rho_1$.
- 6. Related to case 5 via triality of spin₈.
- 7. Identical to (4).
- 8. $\rho_2 \otimes \rho_3$.
- 9. $\rho_1 \otimes \rho_{2n-3} \oplus \rho_2 \otimes \rho_2$.
- 10. In a suitable system of coordinates, we have $\alpha_1 = e_1 e_2$, $\alpha_2 = e_2 e_3$, $\alpha_3 = e_3$, $\alpha_4 = \frac{-e_1 e_2 e_3 e_4}{2}$. (The roots consist of all vectors of norm 1 or 2 in $\mathbb{Z}^4 \cup (\mathbb{Z}^4 + \frac{1}{2}(1,1,1,1))$.)

The normalized spherical root is $\gamma = -e_4$, and $\rho_{L(X)} = e_3 + 3e_2 + 5e_1$.

Now $\check{\mathfrak{l}}(X)$ is the Levi subalgebra of $\check{\mathfrak{f}}_4=\mathfrak{f}_4$ obtained by deleting the left-most vertex of the Dynkin diagram. The centralizer \mathfrak{s} of $2\rho_{L(X)}$, considered as a cocharacter into $\check{\mathfrak{l}}(X)$, has semisimple rank 1. This shows that $(F_4, \mathrm{Spin}_9 \backslash F_4)$ is good in the sense previously discussed. But also $[\mathfrak{s}, \mathfrak{s}]$ commutes with $\check{\mathfrak{l}}(X)$.

Taking the principal \mathfrak{sl}_2 inside $\mathfrak{l}(X)$ gives a morphism

$$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \to [\mathfrak{s},\mathfrak{s}] \times \check{\mathfrak{l}}(X) \to \mathfrak{f}_4,$$

which verifies uniqueness.

- 11. γ corresponds to a *short coroot* of $\check{\mathfrak{g}}$, and the $\mathrm{SL}_2 \times \mathrm{SL}_2$ is that associated to the orthogonal pair (long root, short root).
- 12. Again γ is a short coroot.

A.6. Proof of Lemma **A.4.2.** We recall the situation:

 \mathbf{X} is an affine spherical \mathbf{G} -variety whose open orbit is isomorphic to $\mathbf{X}^+ \simeq \mathbf{X}_1 \times^{\mathbf{P}^-} \mathbf{G}$, where \mathbf{X}_1 is a torus bundle over $\mathbf{GL}_n \setminus \mathbf{SL}_{n+1}$ (or a finite quotient thereof). By (GN4), the image of \mathfrak{sl}_2 under the map $(\mathfrak{a}_X^*)^{W_X} \times \mathfrak{sl}_2 \to \mathfrak{g}$ is contained in a well-defined direct summand of a Levi subalgebra \mathfrak{l} of \mathfrak{g} which is isomorphic to \mathfrak{pgl}_{n+1} . The weight of such a morphism satisfying axioms (GN1) and (GN2) is not uniquely defined by the requirement that it commutes with $(\mathfrak{a}_X^*)^{W_X}$ if and only if n is even and one of the following equivalent conditions hold:

- $(\mathfrak{a}_X^*)^{W_X}$ has trivial image under the projection to the summand: $\check{\mathfrak{l}} \to \mathfrak{pgl}_n$:
- the stabilizer of a point in X_1 under the action of SL_{n+1} contains GL_n ;
- the valuations induced (on $k(\mathbf{X})^{(\mathbf{B})}$) by the two colors (=**B**-stable divisors) contained in $\mathbf{X}_1 \cdot \mathbf{B}$ are equal.

LEMMA. In the above setting, there is a torus bundle $\mathbf{Y}_1 \to \mathbf{X}_1$ which does not satisfy the equivalent conditions above, and an affine spherical embedding \mathbf{Y} of $\mathbf{Y}^+ := \mathbf{Y}_1 \times^{\mathbf{P}^-} \mathbf{G}$ such that $k[\mathbf{X}] = k[\mathbf{Y}]^{\mathbf{T}}$ (where \mathbf{T} is the torus of automorphisms of that bundle).

PROOF. There are clearly many possible choices for \mathbf{Y}_1 . Choosing any of them, we can describe the isomorphism class of a simple spherical embedding \mathbf{Y} of $\mathbf{Y}^+ := \mathbf{Y}_1 \times^{\mathbf{P}^-} \mathbf{G}$ by a pair $(\mathcal{C}(\mathbf{Y}), \mathcal{F}(\mathbf{Y}))$, where $\mathcal{F}(\mathbf{Y})$ is the set of colors of \mathbf{Y}^+ which contain the closed \mathbf{G} -orbit in their closure and $\mathcal{C}(\mathbf{Y})$ is the cone in $\mathrm{Hom}(\mathcal{X}(\mathbf{Y}), \mathbb{Q})$ generated by the valuations induced by all \mathbf{B} -invariant (including the \mathbf{G} -invariant) divisors in \mathbf{Y} containing the closed \mathbf{G} -orbit [$\mathbf{Kno91}$, Theorem 3.1]. The embedding is affine if there is a hyperplane containing $\mathcal{C}(\mathbf{X})$ and strictly separating $\mathcal{V}_{\mathbf{X}} \cup \mathcal{C}(\mathbf{X})$ from the set of valuations induced by colors not in $\mathcal{F}(\mathbf{X})$ [$\mathbf{Kno91}$, Theorem 6.7].

Since there is a bijection between colors of \mathbf{X} and colors of \mathbf{Y} , we may choose for $\mathcal{F}(\mathbf{Y})$ the preimage of $\mathcal{F}(\mathbf{X})$. Moreover, for any extremal ray of $\mathcal{C}(\mathbf{X})$ which does not contain the image of a color (and hence is generated by an element of \mathcal{V}_X), we can choose a non-zero element of \mathcal{V}_Y in its preimage, and hence obtain a cone $\mathcal{C}(\mathbf{Y})$ generated by those and the images of elements in $\mathcal{F}(\mathbf{Y})$. The cone is strictly convex, since $\mathcal{C}(\mathbf{Y})$ was, and the pair $(\mathcal{C}(\mathbf{Y}), \mathcal{F}(\mathbf{Y}))$ satisfies the criterion for affinity, since the corresponding pair for \mathbf{X} does. Thus, we have an affine spherical variety \mathbf{Y} , and the map $\mathbf{Y}^+ \to \mathbf{X}^+$ extends to: $\mathbf{Y} \to \mathbf{X}$ [Kno91, Theorem 4.1].

If we decompose $k[\mathbf{Y}]$ into highest weight spaces, the weights that will appear are precisely the elements of $\mathcal{X}(\mathbf{Y})$ which are ≥ 0 on $\mathcal{C}(\mathbf{Y})$ and the valuations induced by colors. Those which restrict to the trivial character for \mathbf{T} are precisely the elements of $\mathcal{X}(\mathbf{X})$ which are ≥ 0 on $\mathcal{C}(\mathbf{X})$ and the valuations induced by colors, hence: $k[\mathbf{X}] = k[\mathbf{Y}]^{\mathbf{T}}$. This proves the claim.

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