COMPUTATIONAL AND COGNITIVE NETWORKS THROUGH CATEGORICAL AND GEOMETRIC LENS

MILTON LIN

Personal Experiences

My research, the geometric Langlands program, have been about using algebraic formalism to unviel connections between the discrete (number theory) and continuous (topology). My interests lie in foundational theory: the **extent to which algebraic/categorical models can explain qualitative properties of language models and cognitive networks**, and dually, to what extent **biological realism is reflected in computational processes**.

$$
\begin{tikzcd}
 & \text{Categorical} \arrow[dl, bend right] \arrow[dr, bend left] & \\
\text{Congitive} \arrow[rr, leftrightarrow, bend right] & & \text{Computational}
\end{tikzcd}
$$

We start with associative memory networks (Hopfield Networks) due to their traceability and interpretability. These networks serve as a toy model for exploring how intrinsic properties can explain the emergent behaviors observed in large language models [Kap+20], [SMK23]. The research aims to build connections between cognitive, categorical, and computational frameworks, especially in understanding scaling laws and memory retrieval dynamics. In Section 1, I outline associative memory networks and theoretical and experimental projects. In Section 2, I discuss how I could incorporate mathematics background to the field of study. [1]

1. Associative Memory Networks

Associative Memory Networks, or Hopfield Networks, are well-known models where memories are stored and recalled based on energy functions that govern the network's dynamics [Hop84]. The local minima in the energy landscape correspond to stored memories. Recent models, such as Dense Associative Memories, improve memory capacity exponentially to $2^{d/2}$ by modifying the energy dynamics [KH16; Dem+17]. These have been connected to attention mechanisms in transformers [Ram+21].though their full relations remain largely unexplored.

Date: October 11, 2024.

[1]I am indebted to my discussions with Prof. Daniel Khashabi and Prof. Leyla Isik.

Research Goal: Scaling Properties of Associative Memory and Modern Models. This research aims to explore how associative memory dynamics inform modern deep learning architectures, particularly transformers. The key research areas are:

1. *Polytopal Decomposition of Weight Spaces*: We will extend the polytopal weight space decomposition of memory networks to more complex *simplicial networks* which are memory networks inspired from setwise connection [BF23], building on the work of Curto et al. [CLM20], [CGM23]. Such theory also contributes to a recent field of using spline-theory to understand neural networks, [Bb18]. We will study how the decomposition changes as the *network size increases*, similar studies include, [Bla+22].

2. *Energy Transformer Experiments*: We will evaluate the scaling properties of the *Energy Transformer* [Hoo+23], a model based on dense associative memory principles, across three dimensions: - i) Parameters: we hypothesize increasing parameters will increase the number of attractors (local minima). - ii) Data: we hypothesize scaling data size will refine attractors, resulting in more meaningful minima. - iii) Compute: we hypothesize more compute (depth or iterations) will lead to faster convergence to low-energy retrievals. [2] We will also investigate how *data augmentation* during training affects memory storage and retrieval efficiency, as in prior work on transformers [AL24]. Finally, the end goal is to give both empirical and theoretical comparison with modern networks, as already hinted in [Hoo+24] and various recent works [ND21], [Niu+24], and [CDB24].

2. Categorical Models and Homotopy Theory

Categorical approaches has been surging lately to give a more systematic and structured framework to study computational processes [Gav+24]. This has been particularly successful in the field of geometric deep learning, [Bro+21].

As a first approximation to study networks with algebraic models, we study the *homotopy theory* of networks, this is a finer invariant than *homology*, which is used in topological data analysis (TDA). Both theoretically and empirically, it has been shown that stimulus space can be reconstructed [Man15] up to homotopy. We will thus study Hopfield networks through a recent formalism proposed by Manin et al [MM24], uses summing functors and Gamma spaces to model configuration spaces' resources assignment.

Research Goal: Homotopical Complexity Under Scaling. We will study how **homotopical complexity** (e.g., Betti numbers, simplicial complexes) changes with network size in memory models. Specifically, we will focus on how *memory capacity* correlates with homotopical invariants. Some initial studies has been done in [BF23].

References

[AL24] Allen-Zhu, Zeyuan and Li, Yuanzhi. *Physics of Language Models: Part 3.1, Knowledge Storage and Extraction*. 2024. arXiv: 2309.14316 [cs.CL]. URL: https://arxiv.org/abs/2309.14316 (cit. on p. 2).

[2] We will perform these experiments on standard masked image prediction tasks and assess memory retrieval quality, energy convergence, and the number of distinct attractors.

[Bb18] Balestriero, Randall and baraniuk, richard. "A Spline Theory of Deep Learning". In: *Proceedings of the 35th International Conference on Machine Learning*. Ed. by Jennifer Dy and Andreas Krause. Vol. 80. Proceedings of Machine Learning Research. PMLR, Oct. 2018, pp. 374–383. URL: https://proceedings.mlr.press/v80/balestriero18b.html (cit. on p. 2).

[BF23] Burns, Thomas F and Fukai, Tomoki. "Simplicial Hopfield networks". In: *The Eleventh International Conference on Learning Representations*. 2023. URL: https://openreview.net/forum?id=_QLsH8gatwx (cit. on p. 2).

[Bla+22] Black, Sid et al. *Interpreting Neural Networks through the Polytope Lens*. 2022. arXiv: 2211.12312 [cs.LG]. URL: https://arxiv.org/abs/2211.12312 (cit. on p. 2).

[Bro+21] Bronstein, Michael M. et al. *Geometric Deep Learning: Grids, Groups, Graphs, Geodesics, and Gauges*. 2021. arXiv: 2104.13478 [cs.LG]. URL: https://arxiv.org/abs/2104.13478 (cit. on p. 2).

[CDB24] Cabannes, Vivien, Dohmatob, Elvis, and Bietti, Alberto. *Scaling Laws for Associative Memories*. 2024. arXiv: 2310.02984 [stat.ML]. URL: https://arxiv.org/abs/2310.02984 (cit. on p. 2).

[CGM23] Curto, Carina, Geneson, Jesse, and Morrison, Katherine. *Stable fixed points of combinatorial threshold-linear networks*. 2023. arXiv: 1909.02947 [q-bio.NC]. URL: https://arxiv.org/abs/1909.02947 (cit. on p. 2).

[CLM20] Curto, Carina, Langdon, Christopher, and Morrison, Katherine. *Combinatorial Geometry of Threshold-Linear Networks*. 2020. arXiv: 2008.01032 [math.CO]. URL: https://arxiv.org/abs/2008.01032 (cit. on p. 2).

[Dem+17] Demircigil, Mete et al. "On a Model of Associative Memory with Huge Storage Capacity". In: *Journal of Statistical Physics* 168.2 (May 2017), pp. 288–299. ISSN: 1572-9613. URL: http://dx.doi.org/10.1007/s10955-017-1806-y (cit. on p. 1).

[Gav+24] Gavranović, Bruno et al. *Position: Categorical Deep Learning is an Algebraic Theory of All Architectures*. 2024. arXiv: 2402.15332 [cs.LG]. URL: https://arxiv.org/abs/2402.15332 (cit. on p. 2).

[Hoo+23] Hoover, Benjamin et al. *Energy Transformer*. 2023. arXiv: 2302.07253 [cs.LG]. URL: https://arxiv.org/abs/2302.07253 (cit. on p. 2).

[Hoo+24] Hoover, Benjamin et al. *Memory in Plain Sight: Surveying the Uncanny Resemblances of Associative Memories and Diffusion Models*. 2024. arXiv: 2309.16750 [cs.LG]. URL: https://arxiv.org/abs/2309.16750 (cit. on p. 2).

[Hop84] Hopfield, John J. "Neurons with graded response have collective computational properties like those of two-state neurons." In: *Proceedings of the national academy of sciences* 81.10 (1984), pp. 3088–3092 (cit. on p. 1).

[Kap+20] Kaplan, Jared et al. *Scaling Laws for Neural Language Models*. 2020. arXiv: 2001.08361 [cs.LG]. URL: https://arxiv.org/abs/2001.08361 (cit. on p. 1).

[KH16] Krotov, Dmitry and Hopfield, John J. *Dense Associative Memory for Pattern Recognition*. 2016. arXiv: 1606.01164 [cs.NE]. URL: https://arxiv.org/abs/1606.01164 (cit. on p. 1).

[Man15] Manin, Yuri I. *Neural codes and homotopy types: mathematical models of place field recognition*. 2015. arXiv: 1501.00897 [math.HO]. URL: https://arxiv.org/abs/1501.00897 (cit. on p. 2).

[MM24] Manin, Yuri and Marcolli, Matilde. "Homotopy Theoretic and Categorical Models of Neural Information Networks". In: *Compositionality* Volume 6 (2024) (Sept. 2024). ISSN: 2631-4444. URL: http://dx.doi.org/10.46298/compositionality-6-4 (cit. on p. 2).

[ND21] Nichol, Alex and Dhariwal, Prafulla. *Improved Denoising Diffusion Probabilistic Models*. 2021. arXiv: 2102.09672 [cs.LG]. URL: https://arxiv.org/abs/2102.09672 (cit. on p. 2).

[Niu+24] Niu, Xueyan et al. *Beyond Scaling Laws: Understanding Transformer Performance with Associative Memory*. 2024. arXiv: 2405.08707 [cs.LG]. URL: https://arxiv.org/abs/2405.08707 (cit. on p. 2).

[Ram+21] Ramsauer, Hubert et al. "Hopfield Networks is All You Need". In: *International Conference on Learning Representations*. 2021. URL: https://openreview.net/forum?id=tL89RnzIiCd (cit. on p. 1).

[SMK23] Schaeffer, Rylan, Miranda, Brando, and Koyejo, Sanmi. *Are Emergent Abilities of Large Language Models a Mirage?* 2023. arXiv: 2304.15004 [cs.AI]. URL: https://arxiv.org/abs/2304.15004 (cit. on p. 1).