Rigid Analytic Geometry

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July 20, 2022

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Introduction and Motivation

This course is an introduction to non-Archimedian geometry. In particular, we will give a thorough introduction to Tate's rigid analytic spaces. These spaces behave quite differently compared to manifolds or schemes over $\mathbf{C}$, so it is essential that one learns the basics well to develop intuition for the subject.

Just for now, though, in the introduction, we'll give a little bit of motivation without going into very much detail. The reader isn't expected to understand everything; this is just an appetizer. Let's get into it!

Let X be a smooth variety over $\mathbf{Q}$ (for example, we could take X to be an elliptic curve). We can consider the extension $\mathbf{C}/\mathbf{Q}$ and the scheme $X_{\mathbf{C}}$. We can associate to $X_{\mathbf{C}}$ a complex manifold, which, by abuse of notation, we refer to as $X(\mathbf{C})$. Now, we can use tools from complex geometry and topology to study X. For example:

1. We get access to topological invariants, like Betti cohomology $H_B^*(X(\mathbf{C}), \mathbf{Q})$.

What algebraic geometers call Betti cohomology is what topologists would call singular cohomology.

2. We get access to Hodge theory: For proper X, we have

$$H_B^n(X(\mathbf{C}), \mathbf{C}) = \bigoplus_{i+j=n} H^i(X, \Omega_X^j).$$

3. If X is an algebraic group, then $X(\mathbf{C})$ is a complex Lie group. This gives us access to the Lie algebra exponential and logarithm.

4. We can use analytic uniformization: If X is an Abelian variety of dimension g, then, for a lattice Λ in $\mathbf{C}^g$, we can write

$$X(\mathbf{C}) = \mathrm{Lie}(X(\mathbf{C}))/H_1(X(\mathbf{C}), \mathbf{Z}) \cong \mathbf{C}^g/\Lambda.$$

   This can be used to describe complex moduli of Abelian varieties. For instance, if X is an elliptic curve, we have for some canonical τ in the complex upper half-plane that X is of the form $\mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$.

5. One can read off the geometric étale fundamental group of X, via

$$\pi_1^{\text{ét}}(X_{\mathbf{C}}) = \pi_1^{\text{top}}(X(\mathbf{C}))^\wedge.$$

Here, the symbol $^\wedge$ denotes profinite completion.

   This lets us deduce that $\pi_1^{\text{ét}}(X)$ is topologically finitely generated.

It would be nice to have a similar "analytic" theory over other so-called "valued fields" (we'll say something more about this shortly) like $\mathbf{Q}_p$ or $\mathbf{C}_p$ or $\mathbf{F}_p((t))$ or $\mathbf{C}((t))$. Not only because it seems intrinsically interesting, but because there are important applications! We will now briefly speak about some of them.

Hasse principle. If we want to study $X(\mathbf{Q})$, the Hasse principle says that it suffices to study $X(\mathbf{R})$ and, for all primes p, the points $X(\mathbf{Q}_p)$. One concrete application of the Hasse principle is the Hasse-Minkowski theorem, which says that a quadratic form over $\mathbf{Q}$ has a nontrivial solution if it has one over $\mathbf{R}$ and, for all primes p, over $\mathbf{Q}_p$.

If you haven't seen quadratic forms before, you can think of things of the form $\sum_{i=1}^n a_i X_i^2$ for $a_i \in \mathbf{Q}^\times$ and indeterminates X_i. The actual definition is slightly different, but every quadratic form can be put into this form.

Langlands program. We want to understand $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. To do this, we naturally try to study its representations. This is difficult, so, for primes p, we look at $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$. From there, we can ask when representations of $\mathrm{Gal}(\overline{\mathbf{Q}_p}/\mathbf{Q}_p)$ "come from geometry", i.e., when they arise in the cohomology of some variety over $\mathbf{Q}_p$.

***p*-adic approximation/lifting.** Given a scheme over $\mathbf{Z}$, it is often easier to deal with points over $X_{\mathbf{F}_p}$ and then try to lift to characteristic 0 via Hensel's lemma. Similarly, sometimes we want to lift a variety over $\mathbf{F}_p$ to characteristic 0.

Function field arithmetic. Start with a variety over $\mathbf{F}_p(t)$. We cannot base change to $\mathbf{C}$, so our only reasonable shot at getting an analytic theory is to look at $\mathbf{F}_p((t))$.

Deformation theory. Moduli spaces are often analytic spaces. In a slightly different direction, if we have a family of varieties over $\mathbf{C}$ parameterized by, say, the affine line, and we have a singular variety at a point in $\mathbf{A}^1_{\mathbf{C}}$, we can remove it by completing and passing to the generic fiber, which leads us to an analytic space over $\mathbf{C}((t))$.

History

Historically, the motivation for rigid analytic spaces was J. Tate's work on elliptic curves. He noticed that if E is an elliptic curve over $\mathbf{C}_p$ satisfying a technical condition, then there is an isomorphism of topological groups

$$E(\mathbf{C}_p) \cong \mathbf{C}_p^\times/q\mathbf{Z}.$$

This looks a lot like the complex case, where we have

$$E(\mathbf{C}) \cong \mathbf{C}/(\mathbf{Z}+\tau\mathbf{Z}) \xrightarrow{\exp(2\pi i\cdot -)} \mathbf{C}^\times/\exp(2\pi i\tau)\mathbf{Z}.$$

But this is *a priori* only an isomorphism of topological groups. It would be much more meaningful to upgrade this to a geometric statement, so that the associated geometric object remembers the elliptic curve in a faithful way. It was this that lead Tate to develop rigid analytic spaces.

Valued Fields

Definition 1. Let K be a field. An **absolute value** on K is a map

$$|\bullet|: K \to \mathbf{R}_{\geq 0}$$

such that for all $a, b \in K$, the following hold:

1. We have $|a| = 0$ if and only if a equals 0.
2. We have $|ab| = |a||b|$.
3. We have $|a + b| \leq |a| + |b|$.

Example 2. The usual absolute value $|\bullet|_\infty$ on $\mathbf{R}$ is an absolute value. Ditto for the usual norm on $\mathbf{C}$.

Example 3. The trivial absolute value sending every element of the field in question to 0 is an absolute value.

Example 4 (p-adic absolute value)**.** The p-adic valuation v_p on $\mathbf{Q}$ is defined as follows: For any $a \in \mathbf{Z} \setminus \{0\}$, if a is equal to $p^n d$ with d coprime to p, we set $v_p(a) = n$. For $a/b \in \mathbf{Q}$, we set $v_p(a/b)$ to be $v_p(a) - v_p(b)$. The p-adic absolute value is $|\bullet|_p : x \mapsto p^{-v_p(x)}$.

Exercise 5. Check that the p-adic absolute value actually is an absolute value.

Definition 6. Two absolute values on $|\bullet|_1$ and $|\bullet|_2$ are called **equivalent** if there exists $c \in \mathbf{R}$ such that we have $|\bullet|_1 = |\bullet|_2^c$.

Theorem 7 (Ostrowski)**.** *Every nontrivial absolute value is equivalent to $|\bullet|_\infty$ or, for some prime p, the p-adic absolute value $|\bullet|_p$.* □

Topology

Lemma 8. *Let K be a field equipped with an absolute value $|\bullet|$. Then, the function*

$$d: K^2 \to \mathbf{R}_{\geq 0} : (x, y) \mapsto |x - y|$$

defines a metric. The induced topology on K makes K into a topological field. □

Exercise 9. Prove lemma 8.

Lemma 10. *Let K be a field. Let $|\bullet|_1$ and $|\bullet|_2$ be two absolute values on K. The following are equivalent:*

1. *The absolute values* $|\bullet|_1$ *and* $|\bullet|_2$ *are equivalent.*

2. *The absolute values* $|\bullet|_1$ *and* $|\bullet|_2$ *define the same topology on* K.

Proof. The proof of (1) $\Rightarrow$ (2) is straightforward.

Let's prove (2) $\Rightarrow$ (1). For $i = 1, 2$ and $x \in K$, we have $|x^n|_i \xrightarrow{n\to\infty} 0$ if and only if we have $|x|_i < 1$. The former statement is equivalent to the statement that the sequence $(x, x^2, x^3, \ldots)$ converges to 0 in K, which is an entirely topological statement. So for any $x \in K$, we have $|x|_1 < 1$ if and only if we also have $|x|_2 < 1$.

Now, for $y, z \in K$ with $|y| \neq 1 \neq |z|$ and $n, m \in \mathbf{Z}$, set $x = y^m z^n$. Applying the conclusion of the previous paragraph to x, we have $m \log |y|_1 + n \log |z|_1 < 0$ if and only if we also have $m \log |y|_2 + n \log |z|_2 < 0$. Rearranging, this shows that we have

$$\frac{\log |y|_1}{\log |z|_1} < \frac{n}{m} \Leftrightarrow \frac{\log |y|_2}{\log |z|_2} < \frac{n}{m}.$$

Since n and m were arbitrary, this shows that we have

$$\frac{\log |y|_1}{\log |z|_1} = \frac{\log |y|_2}{\log |z|_2}.$$

Rearranging yet again, we see that there holds

Scribe's note: I added a few additional details from William Stein's website: https://www.williamstein.org/papers/ant/html/node62.html.

$$\frac{\log |y|_1}{\log |y|_2} = \frac{\log |z|_1}{\log |z|_2}.$$

Since y and z were arbitrary, this shows that there is a constant $c \in \mathbf{R}$ such that for every $w \in K$, we have

$$c = \frac{\log |w|_1}{\log |w|_2}.$$

Rearranging one final time, this shows that for any $w \in K$, we have $|w|_1 = |w|_2^c$. ☐

Completions

Let K be a field and let $|\bullet|$ be an absolute value.

Definition 11. The field K is called **complete** with respect to $|\bullet|$ if every sequence in K that is Cauchy with respect to $|\bullet|$ has a unique limit in K.

Lemma 12. *For any field* K *with absolute value* $|\bullet|$*, there is a field extension* K'/K *together with an absolute value* $|\bullet|'$ *on* K' *extending* $|\bullet|$ *such that* K' *is complete with respect to* $|\bullet|'$ *and* $K \subseteq K'$ *is dense. The field* K' *is initial among continuous morphisms from* K *into complete valued fields.*

This universal property shows that equivalent absolute values have isomorphic completions.

Sketch. Cauchy sequences in K form a K-algebra R. Sequences converging to 0 form a maximal ideal $I \subseteq R$. One can check that

$$|(x_n)_{n\in\mathbf{Z}_{\geq 0}}|' := \lim_{n\to\infty} |x_n|$$

defines an absolute value on R/I. $\square$

Example 13. The field $\mathbf{R}$ is the completion of $\mathbf{Q}$ with respect to the usual absolute value. The field $\mathbf{Q}_p$ is the completion of $\mathbf{Q}$ with respect to the p-adic absolute value $|\bullet|_p$. Like $\mathbf{R}$, the field $\mathbf{Q}_p$ is not algebraically closed. We write $\overline{\mathbf{Q}_p}$ to denote an algebraic closure. In contrast to the real case, the extension $\overline{\mathbf{Q}_p}/\mathbf{Q}$ is infinite.

There are other ways to describe $\mathbf{Q}_p$. For example, we can describe $\mathbf{Q}_p$ as the field of sums of the form $\sum_{n=-\infty}^{\infty} a_n p^n$, where a_n lies in $\{0, 1, \dots, p-1\}$ and vanishes for n sufficiently negative. Yet another way is to first define the p-adic integers $\mathbf{Z}_p$ as $\lim_{n\in\mathbf{Z}_{\geq 0}} \mathbf{Z}/p^n\mathbf{Z}$, and then set $\mathbf{Q}_p$ to be $\mathbf{Z}_p[1/p]$.

Proposition 14. *Let K be a field that is complete with respect to an absolute value $|\bullet|$. Let V be a finite dimensional K-vector space. Then, any two vector space norms on V are equivalent. In fact, if V is isomorphic to K^n, then every norm is equivalent to the norm $(a_1, \dots, a_n) \mapsto \max_i |a_i|$. In particular, every such norm is complete.*

Proposition 15. *Let K be a field that is complete with respect to an absolute value $|\bullet|$. Let L/K be an algebraic extension. There is a unique way to extend $|\bullet|$ to a absolute value $|\bullet|'$ on L. If L/K is finite, then L is complete with respect to $|\bullet|'$ and $|\bullet|'$ admits the following description: for $\alpha \in L$, we have*

$$|\alpha|' = |\mathrm{Nm}_{L/K}(\alpha)|^{1/[L:K]}. \qquad \square$$

Scribe's note: I've altered the statement of Proposition 15 to more or less match the statement given in Brian Conrad's notes: `https://virtualmath1.stanford.edu/~conrad/248APage/handouts/ostrowski.pdf`. The reason is that we need some extension statement for algebraic (but possibly infinite) extensions to extend the p-adic absolute value to an absolute value on $\overline{\mathbf{Q}_p}$.

This implies that we get a unique extension of $|\bullet|_p$ from $\mathbf{Q}_p$ to $\overline{\mathbf{Q}_p}$. Unfortunately, the field $\overline{\mathbf{Q}_p}$ is not complete with respect to $|\bullet|_p$.

Definition 16. Let $\mathbf{C}_p$ denote the completion of $\overline{\mathbf{Q}_p}$ with respect to $|\bullet|_p$.

Proposition 17 (Krasner's Lemma)**.** *The field $\mathbf{C}_p$ is algebraically closed.* $\square$

Strictly speaking, Krasner's Lemma says something more general, but this version suffices for our purposes.

Non-Archimedean Fields

Observe that for $a, b \in \mathbf{Z}$, we have $v_p(a+b) \geq \min(v_p(a), v_p(b))$. This implies that $|a+b|_p$ is no larger than $\max(|a|_p, |b|_p)$. In other words, the p-adic absolute value satisfies a strong version of the triangle inequality called the "ultrametric triangle inequality".

Definition 18. A nontrivial absolute value $|\bullet|$ on a field K is called **non-Archimedean** if it satisfies the **ultrametric triangle inequality**: for all $a, b \in K$, there holds $|a+b| \leq \max(|a|, |b|)$. Otherwise, it is called **Archimedean**.

Theorem 19 (Gelfand-Tornheim-Ostrowski). *If a field K is complete with respect to an Archimedean absolute value, then K is isomorphic to* $\mathbf{R}$ *or* $\mathbf{C}$. □

In contrast, there are many fields that are complete with respect to non-Archimedean absolute values.

Definition 20. A non-Archimedean field is a field K together with an equivalence class of non-Archimedean absolute values with respect to which K is complete.

We take an equivalence class because we want to think of K as a topological field, i.e., we want to emphasize the topology over any specific absolute value inducing it.

Definition 21. Let K be a field. A **valuation** (of rank 1) is a function $v\colon K \to \mathbf{R} \cup \{\infty\}$ such that the following conditions are satisfied for all $a, b \in K$:

1. We have $v(a) = \infty$ if and only if a equals 0.
2. We have $v(ab) = v(a) + v(b)$.
3. We have $v(a+b) \geq \min(v(a), v(b))$.

For any valuation v, we get a non-Archimedean absolute value $x \mapsto \exp(-v(x))$. Conversely, for any non-Archimedean absolute value $|\bullet|$, we get a valuation $v\colon x \mapsto -\log|x|$.

The ultrametric inequality has far-reaching consequences, which we now discuss.

Lemma 22. *Let K be a non-Archimedean field. Let $(a_n)_{n\in\mathbf{Z}_{\geq 0}}$ be a sequence of elements in K. The series $\sum_{n=1}^{\infty} a_n$ converges if and only if the sequence $(a_n)_{n\in\mathbf{Z}_{\geq 0}}$ converges to* 0.

Proof. Let ε be a real number greater than 0. Then, there exists N such that for all $m > N$, we have $|a_m| < \varepsilon$. But then, for any $l \geq m$, we have

$$\left|\sum_{n=m}^{\ell} a_n\right| \leq \max_{n=m}^{\ell} |a_n| < \varepsilon.$$

Thus, the sequence $(\sum_{n=1}^{\ell} a_n)_{l \in \mathbf{Z}_{\geq 0}}$ is Cauchy. □

Definition 23. A valuation ring is an integral domain such that for all nonzero $x \in \operatorname{Frac}(A)$, at least one of x and x^{-1} lies in A.

Definition-Theorem 24. *Let K be a field equipped with a non-Archimedean absolute value. Let $\mathcal{O}_K$ denote the ring $\{x \in K : |x| \leq 1\}$. Let $\mathfrak{m}_K$ denote the ideal $\{x \in K : |x| < 1\}$ in $\mathcal{O}_K$.*

Note that there is no completeness assumption.

1. (a) *The ring $\mathcal{O}_K$ is an open subring, called the* ***ring of integers*** *of K. It is a valuation ring.*

   (b) *The maximal ideal of $\mathcal{O}_K$ is $\mathfrak{m}_K$. We sometimes call $\mathfrak{m}_K$ the* ***maximal ideal*** *of K.*

   (c) *We have* $\dim \mathcal{O}_K = 1$.

   Conversely, let R be a 1-dimensional valuation ring. Then, the field $\operatorname{Frac} R$ is a field admitting a non-Archimedean absolute value with R as its ring of integers.

2. *The following are equivalent:*

   (a) *The field K is complete with respect to our choice of non-Archimedean abolute value.*

   (b) *For any $\varpi \in \mathfrak{m}_K$, we have*

$$\mathcal{O}_K = \varprojlim_n \mathcal{O}_K / \varpi^n.$$

Sketch. We do not give a full proof of this theorem. We will, however, say a few words about how to, given a 1-dimensional valuation ring R, construct a non-Archimedean absolute value on $\operatorname{Frac} R$ having R as its ring of integers. We may endow $(\operatorname{Frac} R)^\times / R^\times$ with the structure of a totally ordered set via

$$[x] \leq [y] \Leftrightarrow x^{-1} y \in R.$$

By Proposition 8 in §4.5 of Chapter 6 of Bourbaki's *Commutative Algebra*, there is a map $(\operatorname{Frac} R)^\times / R^\times \to \mathbf{R}$ that, when composed with the natural map $(\operatorname{Frac} R)^\times \to (\operatorname{Frac} R)^\times / R^\times$, yields an absolute value on $\operatorname{Frac} R$ satisfying our requirements. □

Surprising Features of Non-Archimedean Topologies

Let K be a field equipped with a non-Archimedean absolute value.

Lemma 25. *Let $a, b \in K$ be two elements with $|a| \neq |b|$. Then, we have* $|a + b| = \max(|a|, |b|)$.

Geometrically, this means that every triangle in K is isosceles!

Proof. Without loss of generality, we can assume that $|b|$ is less than $|a|$. Then, we have $|a + b| \leq |a|$. But we also have

$$|a| = |a + b - b| \leq \max(|a + b|, |b|) = |a + b|. \quad \square$$

For $r \in \mathbf{R}$ and $a \in K$, let $\mathbf{B}_r(a)$ denote the closed ball $\{x \in K : |x - a| \leq r\}$ of radius r around a. Let $\mathbf{B}_r^-(a)$ denote $\{x \in K : |x - a| < r\}$ and let $\partial\,\mathbf{B}_r(a)$ denote $\mathbf{B}_r(a) - \mathbf{B}_r^-(a)$.

Contrary to popular belief, the ball $\mathbf{B}_r(a)$ is in general not compact.

Lemma 26. *For every $r \in \mathbf{R}$ and every $a \in K$, every point in $\mathbf{B}_r(a)$ is its center. In particular, for $r \in \mathbf{R}$ and $a, b \in K$, the balls $\mathbf{B}_r(a)$ and $\mathbf{B}_r(b)$ are either equal or disjoint.*

Scribe's note: Originally, the first and second sentences in lemma 26 were each their own lemma, but I combined them into one lemma since they're so similar. I also added a few details to the proof.

Proof. Let b be an arbitrary point in $\mathbf{B}_r(a)$. Then, the closed ball $\mathbf{B}_r(b)$ is a subset of $\mathbf{B}_r(a)$; for if c is any point in $\mathbf{B}_r(a)$, we have

$$\begin{aligned} |b - c| &= |b + a - a - c| \\ &= \max(|a - b|, |a - c|) \\ &\leq r. \end{aligned}$$

By symmetry, this shows that $\mathbf{B}_r(a)$ and $\mathbf{B}_r(b)$ are equal. □

Lemma 27. *For any $r \in \mathbf{R}$ and $a \in K$, the ball $\mathbf{B}_r(a)$ is both open and closed. Ditto for $\mathbf{B}_r^-(a)$ and $\partial\,\mathbf{B}_r(a)$.*

Proof. Immediate from lemma 26. □

Theorem 28. *Let K be a equipped with a non-Archimedean absolute value. As a topological space, the field K is totally disconnected.*

To say that a topological space is totally disconnected is to say that any connected subspace consists of at most one element.

Proof. Let A be a subset of K. Suppose that A contains points a, b with $a \neq b$. Let $0 < \partial < d(a, b)$ be a real number. Then, the set $A_1 = \mathbf{B}_\partial(a) \cap A$ is open and closed. Thus, so is $A_2 = A \setminus A_1$, which is nonempty since it contains b. □

Towards Rigid Geometry

The first attempt at analytic geometry over a non-Archimedean field K could be to "copy" the definition of a real or complex manifold.

Definition 29. A **locally K-analytic manifold** is a topological space that is locally isomorphic to $\mathcal{O}_K$.

This is useful in some situations, e.g., for studying K-points of algebraic groups over K. But it isn't well-suited to doing analytic geometry, i.e., studying locally analytic functions on X. To be more precise: lemma 22 suggests a notion of analytic functions on $\mathbf{B}_1(0) \subseteq K$. We could hope to get a well-behaved sheaf of analytic functions on $\mathbf{B}_1(0)$ (or X, for that matter); however, the space $\mathbf{B}_1(0)$ can be written as $\mathbf{B}_1^-(0) \sqcup \partial\,\mathbf{B}_1(0)$, so the space of functions on $\mathbf{B}_1(0)$ should decompose as a product of the spaces of functions on $\mathbf{B}_1^-(0)$ and $\partial\,\mathbf{B}_1(0)$, respectively. This goes against the principle from complex geometry that functions on the closed unit ball should be determined by their behavior in the interior thereof. Even worse: Suppose that K is $\mathbf{C}_p$. This field has $\overline{\mathbf{F}_p}$ as its residue field. For any $a \in \overline{\mathbf{F}_p}$, choose a life $[a] \in \mathbf{C}_p$. Then, we get an *infinite* decomposition

$$\mathbf{B}_1(0) = \mathcal{O}_{\mathbf{C}_p} = \coprod_{a \in \overline{\mathbf{F}_p}} ([a] + \mathfrak{m}_{\mathbf{C}_p}) = \coprod_{a \in \overline{\mathbf{F}_p}} \mathbf{B}_1^-([a]),$$

and the space of functions on $\mathcal{O}_{\mathbf{C}_p}$ decomposes as an infinite product of the spaces of functions on the various $\mathbf{B}_1^-([a])$, for $a \in \overline{\mathbf{F}_p}$.

Rigid analytic geometry solves this by

1. "postulating" that functions admit a global power series expansion, and
2. only allowing certain open covers.

Technically, the definition is closer to the algebraic geometry of varieties than to manifolds.

The Tate Algebra

Let K be a non-Archimedean field with residue field k. Consider the closed unit disc $\mathbf{B}_1(0)$. What should "analytic functions" on $\mathbf{B}_1(0) \subseteq K$ be?

Lemma 30. *Let $(c_n)_{n\in\mathbf{Z}_{\geq 0}}$ be a sequence of elements in K. Then, the function $f = \sum_{n\in\mathbf{Z}_{\geq 0}} c_n X^n \in K[\![X]\!]$ converges on $\mathbf{B}_1(0)$ if and only if the sequence (c_n) converges to 0.*

Proof. Immediate from lemma 22. □

lemma 30 motivates the following definition.

Definition 31. The **Tate algebra** in n variables is the ring

$$T_n := K\langle X_1, \dots, X_n\rangle := \left\{ f = \sum_{i\in\mathbf{Z}^n_{\geq 0}} a_i X^i : a_i \xrightarrow{|i|\to\infty} 0 \right\} \subseteq K[\![X_1, \dots, X_n]\!].$$

Exercise 32. Show that T_n is a K-algebra. If ϖ is a nonzero element of $\mathfrak{m}_K$, show that there is an isomorphism

$$K\langle X_1, \dots, X_n\rangle \cong \mathcal{O}_K[X_1, \dots, X_n]^\wedge_\varpi[1/\varpi].$$

This isomorphism hints at a different approach to rigid analytic geometry, namely that of Raynaud.

It follows from lemma 30 that any $f \in T_n$ defines a "evaluation" morphism $\mathbf{B}^n_1(0) \to K\colon x \mapsto \sum_i a_i x^i$, which, by abuse of notation, we also refer to as f.

Writing

$$\mathcal{O}_K\langle X_1, \dots, X_n\rangle := K\langle X_1, \dots, X_n\rangle \cap \mathcal{O}_K[\![X_1, \dots, X_n]\!],$$

there is a natural reduction map

$$\mathrm{red}\colon \mathcal{O}_K\langle X_1, \dots, X_n\rangle \to k[X_1, \dots, X_n].$$

Exercise 32 suggests that $K\langle X_1, \dots, X_n\rangle$ is "complete", in some way; let's make this more precise.

Definition 33. Let R be a K-algebra. A ***K*-algebra norm** on R is a function $|\bullet|\colon R \to \mathbf{R}_{\geq 0}$ such that for all $c \in K$ and $f, g \in R$, the following conditions hold:

1. There holds $|f| = 0$ if and only if f equals 0.
2. We have $|cf| = |c||f|$.
3. We have $|fg| \leq |f||g|$.
4. We have $|f+g| \leq \max(|f|, |g|)$.

A K-algebra norm is called **multiplicative** if equality holds in (3).

Definition 34. The **Gauss norm** on T_n is the K-algebra norm $\|\bullet\|$ defined by

$$\left\| \sum_{i \in \mathbf{Z}^n_{\geq 0}} c_i X^i \right\| := \sup_{i \in \mathbf{Z}^n_{\geq 0}} \{|c_i|\}.$$

Observe that there holds

$$\{f \in T_n : \|f\| \leq 1\} = \mathcal{O}_K\langle X_1, \dots, X_n\rangle.$$

Lemma 35. *For any nonzero $f \in T_n(K)$, there is $a \in K^\times$ with $\|af\| = 1$.*

Proof. We defined $\|f\|$ to be the supremum of the coefficients in the expansion of f. Since these coefficients converge to 0, the supremum is attained, i.e., there is a coefficient c_i of the expansion of f with $\|f\| = c_i$. Set a to be c_i^{-1}. □

Proposition 36. *The Gauss norm is multiplicative.*

Proof. Let f, g be elements of T_n. By lemma 35, we can assume that there holds $\|f\| = \|g\| = 1$. The kernel $\ker(\text{red})$ of the reduction map is the ideal of elements h with $\|h\| < 1$. Since $k[X_1, \dots, X_n]$ is an integral domain, the ideal $\ker(\text{red})$ is prime. Since neither of f and g lies in $\ker(\text{red})$, the norm $\|fg\|$ of fg must be 1. □

Corollary 37. *An element $f = \sum_{i \in \mathbf{Z}^n_{\geq 0}} c_i X^i$ is a unit if and only if for all i with $|i| > 0$, we have $|c_0| > |c_i|$.*

Proof. First, suppose that f is a unit. By lemma 35, we may assume without loss of generality that there holds $\|f\| = 1$. By Proposition 36, we have $\|f^{-1}\| = 1$. Therefore, the element f^{-1} lies in $\mathcal{O}_K\langle X_1, \dots, X_n\rangle^\times$. This implies that the reduction $\text{red} f$ lies in $k[X_1, \dots, X_n] = k^\times$.

The other direction is left as an exercise. □

Proposition 38 (Maximum Modulus Principle)**.** *For any $f = \sum_{i \in \mathbf{Z}^n_{\geq 0}} c_i X^i \in K\langle X_1, \dots, X_n\rangle$, we have*

$$\|f\| \geq \sup_{x \in \mathbf{B}^n_1(0)} |f(x)|.$$

If K' is an algebraic extension of K with infinite residue field k', then there is a point $x_0 \in (K')^n$ with coordinates lying in the closed unit ball in K' that satisfies $\|f\| = |f(x_0)|$.

Exercise 39. Show that Proposition 38 can fail if k is not assumed to be infinite. Hint: Consider $\mathbf{Q}_p$.

Remember, there is a unique absolute value on K' extending the one on K by Proposition 15. It is with respect to this absolute value that we consider the residue field of K'.

Scribe's note: I've slightly modified the statement and proof of Proposition 38, to strengthen the theorem and avoid mentioning the Tate algebra of $\overline{K}$, which we technically haven't defined (since $\overline{K}$ is generally not complete with respect to the extended absolute value). The statement is now closer to the one given in Bosch's Lectures on Formal and Rigid Geometry.

Proof. For any $x \in \mathbf{B}^n_1(0)$, we have

$$\begin{aligned} |f(x)| &= \left| \sum_{i \in \mathbf{Z}^n_{\geq 0}} c_i x^i \right| \\ &\leq \max_{i \in \mathbf{Z}^n_{\geq 0}} |c_i x^i| \\ &\leq \max_{i \in \mathbf{Z}^n_{\geq 0}} |c_i| \\ &= \|f\|. \end{aligned}$$

We will now show the existence of the point x_0. By lemma 35, we may assume that there holds $\|f\| = 1$. Consider the reduction $\tilde{f} \in k[X_1, \dots, X_n]$. The element $\tilde{f}$ is nonzero, so since k' is infinite, there exists $\tilde{x} \in k'$ with $\tilde{f}(\tilde{x}) \neq 0$. Choose any lift $x_0 \in \mathcal{O}_{K'}$ of $\tilde{x}$. Then, the element $f(x_0)$ has nontrivial image $\tilde{f}(\tilde{x}) \in k'$. Thus, we have $|f(x_0)| = 1$. □

Definition 40. A **Banach K-algebra** is a K-algebra equipped with a K-algebra norm with respect to which it is complete.

Proposition 41. *The algebra T_n is complete with respect to the Gauss norm, i.e., the algebra T_n is a K-Banach algebra.*

Proof. Let $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$ be a Cauchy sequence in T_n. Write

$$f_m = \sum_{i\in\mathbf{Z}^n_{\geq 0}} c_{i,m} X^i.$$

Then, for all $i \in \mathbf{Z}^n_{\geq 0}$ and all $\varepsilon > 0$, there exists N such that for all $m_1, m_2 \geq N$, we have

$$\varepsilon > \|f_{m_1} - f_{m_2}\| = \sup_{i\in\mathbf{Z}^n_{\geq 0}} |c_{i,m_1} - c_{i,m_2}|.$$

The RHS of the preceding expression is no less than $|c_{i,m_1} - c_{i,m_2}|$. This implies that for all i, the sequence $(c_{i,m})_{m\in\mathbf{Z}_{\geq 0}}$ is Cauchy. Since K is complete, it has a unique limit c_i. Set

$$f = \sum_{i\in\mathbf{Z}^n_{\geq 0}} c_i X^i.$$

We need to show that f lies in T_n and that f is the unique limit of the sequence $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$. We first address the former claim. We may assume without loss of generality that for all $m_1 \in \mathbf{Z}_{\geq 0}$ and all $m_2 \geq m_1$, we have $\|f_{m_2} - f_{m_1}\| \leq 1/m_1$. Then, for all $i \in \mathbf{Z}^n_{\geq 0}$, we have $|c_{m_2,i} - c_{m_1,i}| \leq 1/m_1$. Since $|\bullet|$ is continuous, this implies that for all $m \in \mathbf{Z}_{\geq 0}$, we have $|c_i - c_{m,i}| \leq 1/m$. For $m \in \mathbf{Z}_{\geq 0}$, since f_m lies in T_n by assumption, there exists N_m such that for all $i \in \mathbf{Z}^n_{\geq 0}$ with $|i| > N_m$ strict inequality $|c_{m,i}| < 1/m$ holds. By lemma 25, this implies that for all $m \in \mathbf{Z}_{\geq 0}$, there exists $N_m \in \mathbf{Z}_{\geq 0}$ such that for all $i \in \mathbf{Z}^n_{\geq 0}$ with $|i| > N_m$, there holds $|c_i| \leq 1/m$. This verifies that f lies in T_n.

Scribe's note: I added numerous details to the part of the proof of Proposition 41 in which f is shown to lie in T_n and verified to be the unique limit of $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$, following §1.4 in Bosch, Güntzer, and Remmert's Non-Archimedean Analysis.

Finally, we show that f is the unique limit of $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$. For all $m \in \mathbf{Z}_{\geq 0}$, we have $\|f - f_m\| = \sup_{i\in\mathbf{Z}^n_{\geq 0}} |c_i - c_{m,i}| \leq 1/m$. This shows that f is a limit of $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$; and that $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$ must converge coefficient-wise to any limit, so the uniqueness of f follows from the fact that convergent sequences in K have unique limits. □

Exercise 42. Let $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$ be a sequence in T_n. Show that the series $\sum_{m=0}^{\infty} f_m$ converges in T_n if and only if the sequence $(\|f_m\|)_{m\in\mathbf{Z}_{\geq 0}}$ converges to 0.

Weierstrass Preparation and Division

In classical algebraic geometry, varieties are build from maximal spectra of polynomial algebras $k[X_1, \dots, X_n]$ over some field k. The theory relies on the fact that $k[X_1, \dots, X_n]$ has good algebraic properties: it is Noetherian, factorial, Jacobson, and satisfies the Noether normalization and Krull's principal ideal theorems. We would like to have analagous properties for T_n.

Definition 43. Let f be an element of T_n and, for elements $f_j \in T_{n-1}$, write

$$f = \sum_{j=0}^{\infty} f_j X_n^j.$$

The element f is called **distinguished of order *k*** if the following conditions hold:

1. The element f_k is a unit in T_{n-1}.
2. For all $j > k$, we have $\|f_k\| > \|f_j\|$ and $\|f_k\| = \|f\|$.

Example 44. Let $f \in K\langle X\rangle$ be an arbitrary nonzero element. If k is such that there holds $|c_k| = \|f\|$, then f is distinguished of order k.

Theorem 45 (Weierstrass Division)**.** *Let $g \in T_n$ be distinguished of order k. Then, for $f \in T_n$, there is a unique $q \in T_n$ as well as a unique $r \in T_{n-1}[X_n]$ with $\deg r < k$ such that there holds*

$$f = g \cdot q + r.$$

Moreover, we have $\|f\| = \max(\|q\|\|g\|, \|r\|)$.

We will prove Theorem 45 by first proving two intermediate statements.

Scribe's note: I split up the proof of Theorem 45 by pulling out a couple of lemmas. I also added a few details from §1.2 of Bosch's Lectures on Formal and Rigid Geometry.

Lemma 46. *Let $g \in T_n$ be distinguished of order k. Let f be an element of T_n. Suppose that there is $q \in T_n$ as well as $r \in T_{n-1}[X_n]$ with $\deg r < k$ such that there holds*

$$f = g \cdot q + r$$

Then, there must hold $\|f\| = \max(\|q\|\|g\|, \|r\|)$.

Proof. Without loss of generality, we can assume that q and r are both nonzero. By lemma 35, we can assume that we have $\|g\| = 1$ and $\max(\|q\|\|g\|, \|r\|) = 1$. Clearly, we have $\|f\| \leq \max(\|q\|\|g\|, \|r\|) = 1$. Suppose that that inequality is strict. Then, we have

$$\operatorname{red}(qg + r) = \operatorname{red}(f) = 0,$$

and at least one of $\operatorname{red}(q)$ and $\operatorname{red}(r)$ must be nonzero. This contradicts Euclidean division in $k[X_1, \dots, X_n]$. □

Lemma 47. *Let $g \in T_n$ be distinguished of order k with $\|g\| = 1$. Write $g = \sum_{j=0}^{\infty} g_j X_n^j$ for $g_j \in T_{n-1}$. Put $g' = \sum_{j=0}^{k} g_j X_n^j$ and $g'' = \sum_{j=k+1}^{\infty} g_j X_n^j$, so that there holds $g = g' + g''$ and g' is distinguished of order k with $\|g'\| = 1$. Set $\varepsilon = \|g''\| < 1$. Then, for $f \in T_n$, there are elements $q, f_1 \in T_n$ as well as an element $r \in T_{n-1}[X_n]$ with $\deg r < k$ such that the following conditions hold:*

1. *We have $f = qg + r + f_1$.*
2. *We have $\|f_1\| \leq \varepsilon\|f\|$.*

3. *Both* $\|q\|$ *and* $\|r\|$ *are no greater than* $\|f\|$.

Proof. Write $f = \sum_{j=0}^{\infty} f_j X_n^j$ for elements $f_j \in T_{n-1}$. There is $k' \in \mathbf{Z}_{\geq 0}$ such that

$$f'' = \sum_{j=k'+1}^{\infty} f_j X_n^j$$

satisfies $\|f''\| < \varepsilon \|f\|$. Write

$$f' = \sum_{j=0}^{k'} f_j X_n^j.$$

By Euclidean division in $K[X_1, \dots, X_n]$, there exist $q \in T_n$ and $r \in T_{n-1}[X_n]$ with $f' = qg' + r$ and $\deg r < k$. Set $f_1 = -qg'' + f''$, so that there holds

$$f = f' + f'' = qg' + r + f_1 + qg'' = qg + r + f_1,$$

i.e., condition (1) holds. To see that condition (2) holds, note that $\|g''\|$ equals ε, so we have

$$\|f_1\| \leq \max(\|qg''\|, \|f''\|) \leq \varepsilon \|f\|,$$

as desired. The fact that condition (3) holds follows from lemma 46. □

Proof of Theorem 45. lemma 46 shows that if we are given q and r as in the theorem statement, then we have $\|f\| = \max(\|q\|\|g\|, \|r\|)$.

Now, we will show that if there exist elements q and r as in the theorem statement, then q and r are unique. Suppose that q' and r' satisfy $f = gq' + r'$. Then, we have $0 = g(q - q') + (r - r')$. By lemma 46, we have $\max(\|q - q'\|, \|r - r'\|) = 0$ and thus we have $q - q' = r - r' = 0$.

Finally, we show that elements q and r as in the statement of the theorem exist. By lemma 35, we may assume that there holds $\|g\| = 1$. As in the statement of lemma 47, write $g = \sum_{j=0}^{\infty} g_j X_n^j$ for $g_j \in T_{n-1}$ and put $\varepsilon = \|\sum_{j=k+1}^{\infty} g_j X_n^j\|$. Applying lemma 47 inductively, for each $i \in \mathbf{Z}_{\geq 0}$, we find $f_i, q_i \in T_n$ and $r_i \in T_{n-1}[X_n]$ with $f_i = q_i g + r_i + f_{i+1}$ and $|q_i|, |r_i| \leq \varepsilon^i |f|$ and $|f_{i+1}| \leq \varepsilon^{i+1}|f|$. One readily verifies that the elements $q = \sum_{i=0}^{\infty} q_i$ and $r = \sum_{i=0}^{\infty} r_i$ are of the form sought. □

In Example 44, we observed that every nonzero element in T_1 is distinguished of some order. Thus, Theorem 45 yields the following corollary.

Corollary 48. *The algebra* $T_1 = K\langle X \rangle$ *is a Euclidean domain. In particular, it is a PID.*

An analogue of the Weierstrass Preparation Theorem in complex analysis holds for the Tate algebra.

Corollary 49 (Weierstrass Preparation). *Let $g \in T_n$ be distinguished of order k. Then, there is a unique monic polynomial $\omega \in T_{n-1}[X_n]$ of degree k and a unit $e \in T_n$ such that we have $g = e\omega$. The polynomial ω satisfies $\|\omega\| = 1$ and ω is distinguished of order k.*

Analytically, this means that the zero set of g coincides with the zero set of ω. In the case $n = 1$, the element f has only finitely many zeros on $\mathbf{B}_1(0)$. As in complex geometry, this will fail over $B_1^-(0)$.

Proof of Corollary 49. By lemma 35, we can assume that there holds $\|g\| = 1$. Applying Theorem 45 to $f = X_n^k$, we get $q \in T_n$ and $r \in T_{n-1}[X_n]$ with $\deg r < k$ such that there hold $X_n^k = gq + r$ and $\max(\|qg\|, \|r\|) = 1$, which implies in particular that we have $\|q\|, \|r\| \leq 1$. Put $\omega = X_n^k - r$, so we have $qg = \omega$. We have $\|\omega\| = 1$ and ω is distinguished of order k.

Exercise 50. Here, we use Theorem 45 to deduce Corollary 49. Prove that Theorem 45 and Corollary 49 are equivalent by using Corollary 49 to prove Theorem 45.

We claim that q is a unit in T_n. To see this, first note that ω and q and g all lie in $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$. Consider their respective reductions $\tilde{\omega}$ and $\tilde{q}$ and $\tilde{g}$ in $k[X_1, \dots, X_n]$. Since $\tilde{g}$ and $\tilde{\omega}$ are distinguished of order k and have both have Gauss norm 1, they have degree k in X_n. Thus, the element $\tilde{q}$ lies in $k[X_1, \dots, X_n]^\times = k^\times$. This implies that q lies in $T_n^\times$.

Uniqueness is clear from the uniqueness part of Theorem 45. □

Corollary 49 applies only for distinguished elements of T_n. The following lemma shows that this condition is extremely mild in the sense that all elements become distinguished after a suitable change of variables.

Lemma 51. *1. For integers $a_1, \dots, a_{n-1} \in \mathbf{Z}_{\geq 0}$, there is a norm-preserving automorphism σ with*

$$X_i \mapsto \begin{cases} X_i + X_n^{a_i} & i < n \\ X_i & i = n. \end{cases} \tag{$*$}$$

2. Given $f_1, \dots, f_r \in T_n$, there are integers $\alpha_1, \dots, \alpha_{n-1} \in \mathbf{Z}_{\geq 0}$ satisfying the following property: if σ is the automorphism arising from the $\alpha_1, \dots, \alpha_n$ as in (1), then for all j, the element $\sigma(f_j)$ is distinguished of some order in T_n.

Proof. **(1).** There is certainly a map $\sigma_{\text{pre}} \colon K[X_1, \dots, X_n] \to K[X_1, \dots, X_n]$ satisfying the condition ($*$). For all elements $f \in K[X_1, \dots, X_n]$ with $\|f\| = 1$, we have $\|\sigma_{\text{pre}}(f)\| \leq 1$. This shows that σ_{pre} is norm-decreasing, which implies that σ_{pre} is continuous. One readily verifies that σ has an inverse σ_{pre}^{-1} given by

$$X_i \mapsto \begin{cases} X_i - X_n^{a_i} & i < n \\ X_n i = n, \end{cases}$$

which is also norm-decreasing. Thus, the map σ_{pre} is norm-preserving and extends to an automorphism σ of T_n by completeness.

(2). By lemma 35, we may assume that for all j, there holds $\|f_j\| = 1$. For each j, write

$$f_j = \sum_{i \in \mathbf{Z}_{\geq 0}^n} c_{ij} X^i.$$

Scribe's note: I added a few details to the second part of the proof of lemma 51 from §1.2 of Bosch's Lectures on Formal and Rigid Geometry.

Let $N \subseteq \mathbf{Z}_{\geq 0}^n$ denote the finite subset of elements $i \in \mathbf{Z}_{\geq 0}$ for which there exists j such that $\widetilde{c_{ij}}$ does not vanish. Pick t with $t > \max_{i \in N} |i|$, and for $\ell \in \{1, \dots, n-1\}$, put $\alpha_\ell = t^{n-\ell}$. Let σ denote the automorphism arising from $\alpha_1, \dots, \alpha_n$ as in (1) and let $\tilde{\sigma}$ denote the automorphism $k[X_1, \dots, X_n]$ induced by σ. Let $N_j \subseteq N$ denote the subset of elements i for which c_{ij} does not vanish. Since for any $i \in \mathbf{Z}_{\geq 0}$, we have

$$\alpha_1 i_1 + \cdots + \alpha_n i_{n-1} + i_n = t^{n-1} i_1 + \cdots + t i_{n-1} + i_n,$$

the function

$$\varphi_j \colon N_j \to \mathbf{Z}_{\geq 0} \colon i \mapsto \alpha_1 i_1 + \cdots + \alpha_n i_{n-1} + i_n$$

is maximized by a unique element of N_j, namely the multi-index $i_{\max,j}$ in N_j that is largest in the lexicographic order. Thus, for any j, we have

$$\begin{aligned} \tilde{\sigma}(\widetilde{f_j}) &= \sum_{i \in \mathbf{Z}_{\geq 0}^n} \widetilde{c_{ij}} (X_1 + X_n^{\alpha_1})^{i_1} \cdots (X_{n-1} + X_n^{\alpha_1})^{i_{n-1}} X_n^{i_n} \\ &= \sum_{i \in \mathbf{Z}_{\geq 0}} \widetilde{c_{i_{\max,j}j}} X_n^{\max \varphi_j} + O_{X_n}\left(X_n^{\max \varphi_j - 1} \right). \end{aligned}$$

Thus, the element f_j is distinguished of order $\max \varphi_j$. $\square$

Corollary 52. *For every element $g \in T_n$, there is a unit $e \in T_n^\times$ and $\omega \in T_{n-1}[X_n]$ (a polynomial with respect to the variable X_n) such that there holds $g = e\omega$.* $\square$

Applications of Weierstrass Preparation and Division

Proposition 53. *The algebra T_n is Noetherian.*

Proof. We argue by induction. The base case $n = 0$ is trivial. Now, for the inductive step, assume that T_{n-1} is Noetherian. Let $\mathfrak{a} \subseteq T_n$ be any ideal. Choose $g \in \mathfrak{a}$. By lemma 51, we can assume that g is distinguished of order k. By Theorem 45, the quotient $T_n/(g)$ is a finite T_{n-1}-module generated by $1, X_n, \dots, X_n^{k-1}$. In particular, it is Noetherian. Since $\mathfrak{a}$ is an ideal above (g), the ideal $\mathfrak{a}$ is finitely generated. $\square$

Proposition 54. *The algebra T_n is a factorial. In particular, it is normal.*

Proof. We argue by induction. The base case $n = 0$ is trivial. Now, for the inductive step, assume that T_{n-1} is factorial. Then, the algebra $T_1[X]$ is also factorial. Let f be an element of T_n. By lemma 51, we may assume that f is distinguished of order k. Then, by Corollary 49, there is an element $\omega \in T_{n-1}[X]$ such that we have $(f) = (\omega)$. By the inductive hypothesis, it suffices to show that the natural map

$$T_{n-1}[X_n]/(\omega) \to T_n/(\omega)$$

is an isomorphism; to see this, note that both the source and target are generated over T_{n-1} by the elements $1, X_n, \dots, X_n^{k-1}$, then apply Euclidean division on the LHS and Theorem 45 on the RHS. $\square$

Proposition 55 (Noether Normalization). *Let $\mathfrak{a} \subseteq T_n$ be an ideal. Then, there exists $d \in \mathbf{Z}_{\geq 0}$ and a finite injective K-algebra homomorphism $T_d \to T_n/\mathfrak{a}$.*

Scribe's note: I added a few details to the proof of Proposition 55 from §1.2 of Bosch's Lectures on Formal and Rigid Geometry.

Proof. If $\mathfrak{a}$ is (0), this is clear. Otherwise, let g be a nonzero element of $\mathfrak{a}$. By lemma 51, we may assume that g is distinguished. By Theorem 45, the map $T_{n-1} \to T_n \to T_n/(g) \to T_n/\mathfrak{a}$ is finite. Let $\mathfrak{b}$ denote its kernel. If $\mathfrak{b}$ is (0), we are done. If not, we may repeat the above argument with $\mathfrak{b}$ in place of $\mathfrak{a}$ and, continuing inductively in this way, we will find d such that there exists a finite injective map $T_d \to T_n/\mathfrak{a}$ after finitely many steps since T_0 is just the field K. □

Lemma 56. *Let $\mathfrak{m} \subseteq T_n$ be a maximal ideal. Then, the algebra $T_n/\mathfrak{m}$ is a finite field extension of K.*

Proof. By Proposition 55, there is an integer d and a finite injective map $T_d \to T_n/\mathfrak{m}$. Since $T_n/\mathfrak{m}$ is a field, the algebra T_d must be a field, so d is 0 and T_d is K. □

Let $\mathbf{B}^n(\overline{K})$ denote $\{(x_1, \dots, x_n) \in \overline{K} : |x_i| \leq 1\}$. For $x \in \mathbf{B}^n(\overline{K})$, let $\mathfrak{m}_x$ denote the kernel of the natural evaluation map $K\langle X_1, \dots, X_n\rangle \to \overline{K}$.

For any ring R, let $\mathrm{Sp}(R)$ denote its maximal spectrum.

Proposition 57. *There is a bijection*

$$\mathbf{B}^n(\overline{K}) / \operatorname{Gal}(\overline{K}/K) \to \mathrm{Sp}(T_n)$$

Proof. Sending $x \mapsto \mathfrak{m}_x$ defines a map $\phi\colon \mathbf{B}^n(\overline{K}) \to \mathrm{Sp}(T_n)$.

We claim that the map ϕ is surjective. Let $\mathfrak{m} \subseteq T_n$ be a maximal ideal. Then, we can find a K-linear embedding $T_n/\mathfrak{m} \hookrightarrow \overline{K}$. We use this to define a map $\varphi\colon T_n \to T_n/m \to \overline{K}$. We want to see that this is the evaluation map for the point $x = (\varphi(X_1), \dots, \varphi(X_n))$. For this, it suffices to prove that φ is continuous. We will do this by showing that φ is norm-decreasing. We want to show that for any $g \in T_n$, we have $|\varphi(g)| \leq \|g\|$. Suppose not. By lemma 35, we may assume that we have $\|g\| = 1$. Put $a \coloneqq \varphi(g) \in \overline{K}$. Let $f = Y^k + c_1 Y^{k-1} + \dots + c_k$ be its minimal polynomial over K and let L/K denote a splitting field of f in $\overline{K}$. Let $\alpha_1, \dots, \alpha_j$ be the roots of f. By Proposition 15, for every j, there holds

$$|\alpha_j| = |\operatorname{Nm}_{L/K}(\alpha_j)|^{1/[L:K]}.$$

In particular, for all j, we have $|a| = |\alpha_j|$. By the Fundamental Theorem of Symmetric Polynomials, for all i, we have

$$|c_i| \leq |a|^i < |a|^k = |c_k|.$$

This implies that f, viewed as an element of the Tate algebra, is a unit. This implies that $f(a)$ is a unit, a contradiction. This completes the proof that ϕ is surjective.

Now, we claim that ϕ factors through the Galois action. For $\sigma \in \mathrm{Gal}(\overline{K}/K)$ and $x \in \mathbf{B}^n(\overline{K})$, the diagram

$$\begin{tikzcd} \overline{K} \arrow[rr, "\sigma"] & & \overline{K} \\ & T_n, \arrow[ul, "\mathrm{ev}_x"] \arrow[ur, "\mathrm{ev}_{\sigma(x)}"'] & \end{tikzcd}$$

is commutative, and hence the kernels coincide.

Finally, we claim that ϕ is injective. Suppose that $x, y \in \mathbf{B}^n(\overline{K})$ are such that $\mathfrak{m}_x$ and $\mathfrak{m}_y$ coincide. Then, we have $T_n/\mathfrak{m}_x = T_n/\mathfrak{m}_y$. Hence, there exists $\sigma \in \mathrm{Gal}(\overline{K}/K)$ such that the following diagram commutes:

$$\begin{tikzcd} \overline{K} \arrow[rr, "\sigma"] & & \overline{K} \\ T_n/\mathfrak{m}_x \arrow[u, hook] \arrow[rr, bend left, "\sim"] & & T_n/\mathfrak{m}_y \arrow[u, hook] \\ & T_n. \arrow[ul] \arrow[ur] \arrow[uul, bend left, "\mathrm{ev}_x"] \arrow[uur, bend right, "\mathrm{ev}_y"'] & \end{tikzcd}$$

Thus, we have $y = \sigma(x)$. □

Corollary 58. *If K is algebraically closed, then we have $\mathrm{Sp}(T_n) = \mathbf{B}^n(K)$ and every maximal ideal is of the form $(X_1 - x_1, \dots, X_n - x_n)$ for some $x \in \mathbf{B}^n(K)$.* □

Proposition 59. *The algebra T_n is Jacobson.*

Proof. Let $\mathfrak{a}$ be an ideal of T_n.

First, we treat the case $\mathfrak{a} = 0$. Let f be an element of the Jacobson radical of T_n. By Proposition 57, for all $x \in \mathbf{B}^n(\overline{K})$, we have $f(x) = 0$. By applying Proposition 38 to the previous sentence, we see that $\|f\|$ equals 0, which implies that f equals 0.

Now, we treat the case in which $\mathfrak{a}$ is prime, from which the general case follows easily. Our goal is to show that the Jacobson radical $\mathfrak{q}$ of $T_n/\mathfrak{a}$ is (0). By Proposition 55, there exists an integer d and a finite injection $T_d \to T_n/\mathfrak{a}$. The map $\mathrm{Sp}(T_n/\mathfrak{a}) \to \mathrm{Sp}(T_d)$ is surjective, so $\mathfrak{q} \cap T_d$ is contained in the Jacobson radical of T_d, which is (0). Suppose that there is a nonzero element $f \in \mathfrak{q}$. Then, there exist elements $a_1, \dots, a_k \in T_d$ with $a_k \neq 0$ such that there holds $f^k + a_1 f^{k-1} + \dots + a_k = 0$. On the other hand, one may check that we have

$$a_k = -f(f^{k-1} + \dots + c_{k-1}) \in \mathfrak{q} \cap T_d = 0,$$

a contradiction. □

Proposition 60. *Every maximal ideal of T_n has height n and is generated by n elements.* □

Corollary 61. *The ring T_n has Krull dimension n.* □

In conjunction with Proposition 55, Proposition 60 and Corollary 61 yield a good dimension theory for T_n.

Warning 62. In general, there can exist ideals $\mathfrak{a}$ and $\mathfrak{b}$ and injections $T_n/\mathfrak{a} \hookrightarrow T_n/\mathfrak{b}$ that *decrease* dimension.

Functional Analysis on the Tate Algebra

Our goal in this section is to prove the following proposition.

Proposition 63. *Any ideal* $\mathfrak{a} \subseteq T_n$ *is closed.*

There is an elementary proof of Proposition 63 using orthonormal bases. Instead, we will use a very important tool, the Banach Open Mapping Theorem, from functional analysis that we will employ again in the future several times.

As in the previous section, let K be a non-Archimedean field.

Definition 64. A ***K*-Banach space** is a K-vector space V together with a function $\|\bullet\| : V \to \mathbf{R}$ satisfying the following conditions:

1. For all $v \in V$, we have $\|v\| = 0$ if and only if v is 0.
2. For all $c \in K$ and $v \in V$, we have $\|cv\| = |c|\|v\|$.
3. For all $v, w \in V$, we have $\|v + w\| \leq \max(\|v\|, \|w\|)$.
4. The vector space V is complete with respect to $\|\bullet\|$.

Theorem 65 (Banach Open Mapping Theorem)**.** *Any continuous and surjective K-linear homomorphism of K-Banach spaces is open.* ☐

Corollary 66 (Closed Graph Theorem)**.** *A K-linear map $\varphi \colon M \to N$ between K-Banach spaces is continuous if and only if for any sequence $(a_n)_{n \in \mathbf{Z}_{\geq 0}}$ of elements in M, any $a \in M$, and any $b \in N$, if $(a_n)_{n \in \mathbf{Z}_{\geq 0}}$ converges to a and $(f(a_n))_{n \in \mathbf{Z}_{\geq 0}}$ converges to b, then $f(a)$ equals b.*

Proof. First, we prove the reverse direction. Put

$$\Gamma = \{(x, \varphi(x)) : x \in M\} \subseteq M \times N$$

and equip Γ with the subspace topology. The assumption is equivalent to the completeness of Γ; in particular, the space Γ is K-Banach. Consider the natural map $\beta \colon \Gamma \to M \times N \to M$ obtained by composing the inclusion of Γ into $M \times N$ with the projection $M \times N \to M$. The map β is continuous and surjective, so by Theorem 65, it is a homeomorphism. Now observe that φ is the map $M \xrightarrow{\beta^{-1}} \Gamma \to M \times N \to N$.

The forward direction is left as an exercise. ☐

Exercise 67. Prove the forward direction of Corollary 66.

Proof of Proposition 63. Let $\mathfrak{a}' \subseteq T_n$ be the closure of $\mathfrak{a}$. It is easy to check that $\mathfrak{a}'$ is an ideal. Since T_n is Noetherian, the ideal $\mathfrak{a}'$ is finitely generated. Suppose $e_1, \ldots, e_r$ generate $\mathfrak{a}$. Then, the map

$$\varphi \colon T_n^r \to \mathfrak{a}' : (a_i)_{i=1}^r \mapsto \sum_{i=1}^r a_i e_i$$

is a continuous and surjection K-linear homomorphism of K-Banach spaces. By Theorem 65, the map φ is open. In particular, there exists $0 < \varepsilon < 1$ such that there holds

$$\varphi(\mathbf{B}_1(0)) \supset \mathbf{B}_\varepsilon(0) \cap \mathfrak{a}'.$$

Here, the notation $\mathbf{B}_\varepsilon(0)$ refers to the set $\{f \in T_n : \|f\| \leq \varepsilon\}$.

By lemma 35, this means that for all $x \in \mathfrak{a}'$, there exist $a_1, \dots, a_r \in T_n$ such that there hold $\sum_{i=1}^{r} a_i e_i = x$ and $\varepsilon\|a_i\| \leq \|x\|$. Since $\mathfrak{a}$ is dense in $\mathfrak{a}'$, there exist elements $f_1, \dots f_r \in \mathfrak{a}$ with $\|f_i - e_i\| \leq \varepsilon^2$ for all i.

Let x_1 be an element of $\mathfrak{a}'$. Applying the previous paragraph inductively, we obtain elements a_i with $\varepsilon\|a_i\| \leq \|x_i\|$ and

$$x_j = \sum_{i=1}^{r} a_{i,j} e_i = \sum_{i=1}^{r} a_{i,j} f_i + \sum_{i=1}^{r} a_{i,j}(f_i - e_i)$$

and

$$\|x_{j+1}\| \leq \varepsilon^{-1}\|x_j\|\varepsilon^2 = \varepsilon\|x_j\|.$$

For any fixed i, the sequence $(a_{i,j})_{j=1}^{\infty}$ tends to 0. Thus, we have

$$x_1 = \sum_{i=1}^{r} \left(\sum_{j=1}^{\infty} a_{i,j} f_i \right) \in \mathfrak{a}. \qquad \square$$

Affinoid Algebras

Let K be a non-Archimedean field.

Definition 68. A K-algebra A is called **affinoid** if there is a surjective morphism of K-algebras $T_n \twoheadrightarrow A$. A **morphism of affinoid algebras** is just a morphism on the underlying K-algebras. The category of affinoid algebras, which is a full subcategory of the category of K-algebras, is denoted by AffAlg_K.

Equivalently, a K-algebra is called affinoid if it is topologically of finite type over K.

By Propositions 53 and 59, all affinoid algebras are both Noetherian and Jacobson.

Lemma 69. *Let $\varphi\colon B \to A$ be a morphism of affinoid K-algebras. For any $\mathfrak{m} \in \operatorname{Sp} A$, the preimage $\varphi^{-1}(\mathfrak{m})$ is a maximal ideal.*

Proof. There are injections

$$K \hookrightarrow B/\varphi^{-1}(\mathfrak{m}) \hookrightarrow A/\mathfrak{m}.$$

By lemma 56, the field $A/\mathfrak{m}$ is finite over K, and thus $B/\varphi^{-1}(\mathfrak{m})$ is a field. □

Residue Norms and the Supremum Seminorm

Let A be an affinoid K-algebra.

Definition-Theorem 70. *Given any surjection $\alpha\colon T_n \twoheadrightarrow A$, we define the map*

$$|\bullet|_\alpha : A \to \mathbf{R}_{\geq 0} : f \mapsto \inf_{f' \in \alpha^{-1}(f)} \|f'\|.$$

*The map $|\bullet|_\alpha$ is a complete K-algebra norm, making A into a K-Banach algebra. With respect to the topology on A induced by $|\bullet|_\alpha$, the map α is continuous and open. The norm $|\bullet|_\alpha$ is called the **residue norm** of α.*

Proof. We leave the verification that $|\bullet|_\alpha$ is a norm as an exercise.

The map α is easily seen to be norm-decreasing, and thus α is continuous.

Finally, we will show that A is complete with respect to $|\bullet|_\alpha$. Let $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$ be a Cauchy sequence. This is equivalent to the convergence to 0 of the sequence

Exercise 71. Show that for any $\alpha\colon T_n \to A$, the map $|\bullet|_\alpha\colon A \to \mathbf{R}_{\geq 0}$ is a norm. Hint: To show that $f \in A$ is 0 if and only if $|f|_\alpha$ is 0, apply Proposition 63 to $\ker \alpha$.

$(|f_m - f_{m+1}|)_{m\in\mathbf{Z}_{\geq 0}}$. Let $f_0' \in T_n$ be any lift of f_0, and pick lifts $\partial_m \in T_n$ of the various $f_m - f_{m+1}$ such that $(\partial_m)_{m\in\mathbf{Z}_{\geq 0}}$ converges to 0. Put

$$f_m' := f_0' - \sum_{i=0}^{m-1} \partial_m.$$

For every $m \in \mathbf{Z}_{\geq 0}$, the element $f_m' \in T_n$ lifts f_m, and the sequence $(f_m')_{m\in\mathbf{Z}_{\geq 0}}$ converges to some $f' \in T_n$. By the continuity of α, the sequence $(f_m)_{m\in\mathbf{Z}_{\geq 0}}$ converges to $\alpha(f')$.

The openness of α is an consequence of Theorem 65. □

Exercise 72. Show that for any affinoid algebra A and any surjection $\alpha\colon T_n \twoheadrightarrow A$, we have $|1|_\alpha = 1$.

Exercise 73. Consider the surjection $\alpha\colon \mathbf{Q}_p\langle X\rangle \twoheadrightarrow \mathbf{Q}_p\langle X\rangle/(X^2-p)$ of affinoid algebras. Show that $|\sqrt{p}|_\alpha$ equals 1, but $|p|_\alpha$ equals $|p| \neq 1$. This shows that residue norms need not be multiplicative in general.

It is possible to avoid the use of Theorem 65 here. Instead, one can use the following result (that we won't prove):

Proposition 74. *The infimum appearing in the definition of $|\bullet|_\alpha$ is attained. In particular, the image of $|\bullet|_\alpha\colon A \to \mathbf{R}_{\geq 0}$ is $|K|$.* □

Residue norms depend on the surjections defining them; however, this next result will show that the induced topologies don't:

Theorem 75. *Let A and B be two affinoid K-algebras and let $\alpha\colon T_n \twoheadrightarrow A$ and $\beta\colon T_m \twoheadrightarrow B$ be two surjections. With respect to the topologies on A and B induced by the residue norms of α and β, respectively, every homomorphism $\varphi\colon A \to B$ is continuous.*

Corollary 76. *Any two residue norms on A are equivalent.*

Proof. Apply Theorem 75 to the identity on A. □

Proof of Theorem 75. We only treat the case in which B is reduced. Suppose that $(x_k)_{k\in\mathbf{Z}_{\geq 0}}$ is a sequence in A that converges to a point x. Suppose that the sequence $(\varphi(x_k))_{k\in\mathbf{Z}_{\geq 0}}$ converges to some $y \in B$. We will show that y equals x, at which point the claim follows by Corollary 66. Replacing x_k by $x_k - x$ if necessary, we can assume that x is 0, so our goal is to show that y equals 0. For $\mathfrak{m} \in \operatorname{Sp} B$, let $\mathfrak{n}$ denote $\varphi^{-1}(\mathfrak{m})$, which lies in $\operatorname{Sp} A$ by lemma 69. We get an induced map $\overline{\varphi}\colon A/\mathfrak{n} \to B/\mathfrak{m}$ such that the following diagram commutes:

A full proof of Theorem 75 can be found in §1.4 of Bosch's *Lectures on Formal and Rigid Geometry*.

$$\begin{tikzcd}
T_n \arrow[d, two heads] & T_m \arrow[d, two heads] \\
A \arrow[r, "\varphi"] \arrow[d, two heads] & B \arrow[d, two heads] \\
A/\mathfrak{n} \arrow[r, hook, "\overline{\varphi}"] & B/\mathfrak{m}.
\end{tikzcd}$$

The vertical compositions induce residue norms on $A/\mathfrak{n}$ and $B/\mathfrak{m}$; by lemma 56 and Proposition 14, these norms must be respectively equivalent to the unique field norms on $A/\mathfrak{n}$ and $B/\mathfrak{m}$ extending the norm on K, which exist by Proposition 15, and thus $\overline{\varphi}$ is continuous. Thus, for every $\mathfrak{m} \in \operatorname{Sp} A$, the element $(B \to B/\mathfrak{m})(y)$ is 0. Since B is reduced by assumption and Jacobson by Proposition 59, the element y is equal to 0. □

For any $x \in \operatorname{Sp} A$ and any $f \in A$, we write $f(a)$ to denote the element $(A \to A/x)(f) \in A/x$; and, when there is no risk of confusion, we write $|\bullet|$ to denote the unique norm on A/x extending the one on K (the existence of which is implied by Proposition 15).

Definition 77. The **supremum seminorm** on A is the seminorm $\|\bullet\|_{\sup}: A \to \mathbf{R}_{\geq 0}$ defined by

$$\|f\|_{\sup} := \sup_{x \in \operatorname{Sp} A} |f(x)|.$$

By Proposition 59, the supremum seminorm on A is a norm if and only if A is reduced.

Warning 78. Some sources refer to the supremum seminorm as the "supremum norm", but, strictly speaking, this is a misnomer.

Exercise 79. Show that the supremum seminorm on T_n coincides with the Gauss norm.

Lemma 82. *For any surjection $\alpha: T_n \twoheadrightarrow A$, we have $\|\bullet\|_{\sup} \leq |f|_\alpha$.*

Proof. Let f' be any lift of f to T_n and let x be any point in $\operatorname{Sp} A$. Then, we have $T_n/\alpha^{-1}(x) = A/x$, which, together with Exercise 79, shows that there holds

$$|f(x)| = |f'(\alpha^{-1}(x))| \leq \|f'\|_{\sup} = \|f'\|.$$

Thus, there holds

$$\|f\|_{\sup} = \sup_{x \in \operatorname{Sp} A} |f(x)| \leq \inf_{f' \in \alpha^{-1}(f)} \|f'\| = |f|_\alpha. \qquad \square$$

Exercise 80. Show that the supremum norm is "power-multiplicative" in the sense that for every $f \in A$, we have

$$\|f^n\|_{\sup} = \|f\|_{\sup}^n.$$

Deduce that an element $f \in A$ is nilpotent if and only if $\|f\|_{\sup}$ equals 0.

Exercise 81. Show that affinoid algebra morphisms decrease the supremum seminorm, i.e., if $\varphi: A \to B$ is a morphism of affinoid algebras and $a \in A$ is any element, then we have

$$\|\varphi(a)\|_{\sup} \leq \|a\|_{\sup}.$$

Recall that, by Proposition 55, we can find a finite injective morphism $T_d \hookrightarrow A$. Our next goal is to relate the supremum seminorm on A to the Gauss norm on T_d.

Proposition 83. *Suppose that A is an integral domain viewed as a finite algebra over T_d via a finite injective morphism $T_d \hookrightarrow A$, the existence of which is guaranteed by Proposition 55. Then, for any $f \in A$, there is a unique monic irreducible polynomial $p(X) = X^n + a_1 X^{n-1} + \cdots + a_n \in T_d[X]$ with $p(f) = 0$. Moreover, we have*

$$\|f\|_{\sup} = \max_i \|a_i\|^{1/i}.$$

In particular, there exists $x \in \operatorname{Sp} A$ with $\|f\|_{\sup} = |f(x)|$.

Lemma 84. *Let $p = X^n + b_1 X^{n-1} + \cdots + b_n \in K[X]$ be a polynomial and let $\beta_1, \ldots, \beta_n \in \overline{K}$ be the roots of p. Then, we have*

$$\max_i |\beta_i| = \max_i |b_i|^{1/i}.$$

Proof. Each coefficient b_j can be expressed as the jth elementary symmetric polynomial in the β_i. Hence, we have $|b_j|^{1/j} \leq \max_i |\beta_i|$.

Suppose that $\max_i |\beta_i|$ is achieved by $\beta_{i_1}, \ldots, \beta_{i_s}$. Then, we have $|b_s| = |\beta_{i_1} \cdots \beta_{i_s}|$. $\square$

Proof of Proposition 83. Consider f as an element of $\operatorname{Frac} A$. Then, the element f is the zero of an irreducible polynomial p over $\operatorname{Frac} T_d$. The coefficients of p are in T_d, as we now explain. Since f is integral over T_d, there exists $h(X) \in T_d[X]$ with $h(f) = 0$. Then, the polynomial p divides h. Since T_d is factorial by

Proposition 54, Gauss's Lemma implies that p lies in $T_d[X]$. The uniqueness of p is clear.

Consider the finite extensions $T_d \to T_d[f] \to A$. The corresponding maps $\operatorname{Sp} A \to \operatorname{Sp} T_d[f] \to \operatorname{Sp} T_d$ are surjective with finite fibers. Then, we have $\|f\|_{\sup,A} = \|f\|_{\sup,T_d[f]}$ so without loss of generality we can assume that $T_d[f]$ is all of A. Let y be a point in $\operatorname{Sp} T_d$ and let $x_1, \dots, x_k \in \operatorname{Sp} A$ be the finitely many points lying over y. Then, letting L denote T_d/y and $\overline{p}$ the reduction of p modulo y, the map $T_d \to A$ reduces to a map $L \to A/y = L[X]/\overline{p}(X)$. Let $\alpha_1, \dots, \alpha_n$ be the roots of $\overline{p}$ in $\overline{K} = \overline{L}$. Then, the x_j are the kernels of the maps $A \to (A/y \to \overline{K} : x \mapsto \alpha_j)$. By lemma 84, we have

$$\max_j |f(x_j)| = \max_j |\alpha_j| = \max_i |a_i(y)|^{1/i}.$$

Thus, we have

$$\begin{aligned} \|f\|_{\sup} &= \sup_{x \in \operatorname{Sp} A} |f(x)| \\ &= \sup_{y \in \operatorname{Sp} T_d} \max_i |a_i(y)|^{1/i} \\ &= \max_i \left(\sup_{y \in \operatorname{Sp} T_d} |a_i(y)| \right)^{1/i} \\ &= \max_i \|a_i\|^{1/i}, \end{aligned}$$

and the supremum is attained by Proposition 38. □

Proposition 85. *Let $A \to B$ be a finite injective morphism of affinoid K-algebras. Then, for every $f \in B$, there exists a polynomial $p(X) = X^n + b_1X^{n-1} + \dots + b_n$ with $p(f) = 0$ and $\|f\|_{\sup}$ equals $\sup_i \|b_i\|_{\sup}^{1/i}$. In particular, there exists $x \in \operatorname{Sp} B$ with $\|f\|_{\sup} = |f(x)|$.* □

Rings of Definition and Power-bounded Elements

As before, we let A be an affinoid K-algebra; and we denote by $\mathcal{O}_K$ the ring of integers of the non-Archimedean field K, by $\mathfrak{m}_K$ the maximal ideal of $\mathcal{O}_K$, and by ϖ a nonzero element of $\mathfrak{m}_K$.

When doing commutative algebra on T_n, we frequently used the integral subring $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$. For general affinoid algebras, we have several ways to get analagous subrings.

Definition 86. A **ring of definition** is an open subring $A_0 \subseteq A$ such that the subspace topology on A_0 coincides with the ϖ-adic topology on A_0.

It makes sense to speak of the ϖ-adic topology on A_0 because A_0, being open, contains some power of ϖ. Note that rings of definition are automatically ϖ-adically complete since open subgroups of topological groups are always also closed.

Example 87. The ring $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$ is a ring of definition for $K\langle X_1, \dots, X_n\rangle$.

We deduce:

Lemma 88. *Let A be an affinoid algebra and let $\alpha : T_n \twoheadrightarrow A$ is a surjective morphism. Then, the ring $A_0 := \{f \in A : |f|_\alpha \leq 1\}$ is a ring of definition for A.*

Of course, the ring A_0 depends on α, and it might therefore be a good idea to include α in the notation; however, it is customary not to do this, and we shall follow this convention.

Proof. By definition, the morphism α restricts to a surjection $\alpha : \mathcal{O}_K\langle X_1, \dots, X_n\rangle \twoheadrightarrow A_0$. We now use that α is a quotient map: this shows that $A_0 \subseteq A$ is open. As the topology on $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$ is the ϖ-adic topology, the same is true for its quotient A_0. ☐

Here is an example of a ring of definition that is not of the form described in lemma 88:

Example 89. The subring $\mathcal{O}_K + \mathfrak{m}\,\mathcal{O}_K\langle X\rangle$ of $\mathcal{O}_K\langle X\rangle$ consisting of those elements $\sum_{n=0}^{\infty} a_n X^n$ such that for all $n \geq 1$, there holds $|a_n| < 1$ is a ring of definition for T_1.

Lemma 90. *Let $A_0 \subseteq A$ be a subring of definition. Then, the ring $A_0[1/\varpi]$ is A.*

Proof. If f is any element of A, then the sequence $(\varpi^n f)_{n\in\mathbf{Z}_{\geq 0}}$ converges to 0. Thus, for n sufficiently large, the open subring A_0 contains $\varpi^n f$. ☐

Definition 91. A subset $S \subseteq A$ is called **bounded** if for any open neighborhood $U \subseteq A$ of 0, there exists $k \in \mathbf{Z}_{\geq 0}$ such that $\varpi^k S$ is contained in U. An element $f \in A$ is called **power-bounded** if the set $\{f^n : n \in \mathbf{Z}_{\geq 0}\}$ is bounded. The subring of power-bounded elements in A is denoted A°.

Exercise 92. Let f be an element of A. Show that the following are equivalent:

1. The element f is power-bounded.
2. There exists a surjective map $\alpha\colon T_n \twoheadrightarrow A$ such that the set $\{|f^n|_\alpha : n \in \mathbf{Z}_{\geq 0}\} \subseteq \mathbf{R}$ is bounded.
3. For all surjective maps $\alpha\colon T_n \twoheadrightarrow A$, the set $\{|f^n|_\alpha : n \in \mathbf{Z}_{\geq 0}\} \subseteq \mathbf{R}$ is bounded.

Proposition 97. *Let $\alpha\colon T_n \twoheadrightarrow A$ be a surjection. Then, the subring $A_0^\alpha := \alpha(\mathcal{O}_K\langle X_1, \dots, X_n\rangle)$ is contained in A°.*

Proof. Let f be an element of A. Then, we have $|f^n|_\alpha \leq |f|_\alpha^n$. Thus, if $|f|_\alpha$ is less than 1, then f lies in A°. ☐

Proposition 98. *Let $\alpha\colon T_n \twoheadrightarrow A$ be a surjection and let f be an element of A. The following are equivalent:*

1. *The element f is power-bounded.*
2. *The element f is integral over A_0.*
3. *There holds $\|f\|_{\sup} \leq 1$.*

Exercise 93. Show that T_n° is $\mathcal{O}_K\langle X_1, \dots, X_n\rangle$.

Exercise 94. Show that A° is an open subring of A and that $A^{\circ}[1/\varpi]$ is A.

Exercise 95. Show that any morphism $A \to B$ of affinoid algebras restricts to a map $A^{\circ} \to B^{\circ}$.

Exercise 96. Show that $(K[X]/X^2)^{\circ}$ is $\mathcal{O}_K + KX$.

Proof. Condition (1) implies condition (3) because, by Exercise 80 and lemma 82, we have

$$\|f\|_{\sup}^n = \|f^n\|_{\sup} \leq |f^n|_\alpha.$$

To see that (3) implies (2), first note that by Proposition 55, we can find an injective and finite map $\varphi\colon T_d \to A$ such that the diagram

$$\begin{tikzcd} & T_n \arrow[dr, two heads, "\alpha"] & \\ T_d \arrow[ur, hook] \arrow[rr, hook, "\varphi"'] & & A \end{tikzcd}$$

commutes. If $\|f\|_{\sup}$ is less than or equal to 1, then by Proposition 85, we can find elements $a_1, \dots, a_n \in T_d$ with

$$f^n + \varphi(a_1) f^{n-1} + \cdots + \varphi(a_n) = 0$$

such that for all i, we have $\|a_i\|_{\sup}^{1/i} \leq 1$. By Exercises 79 and 81, the map $T_d \hookrightarrow T_n$ is decreases Gauss norms. Thus, for all i, we have $|\varphi(a_i)|_\alpha \leq 1$.

Finally, that (3) implies (1) follows from the fact that if $f \in A$ is integral over A_0^α, then $A_0^\alpha[f]$ is a finite A_0^α-module. □

Scribe's note: I added a few details to the second part of the proof of Proposition 98 from §1.4 of Bosch's Lectures on Formal and Rigid Geometry.

Proposition 99. *Let $f_1, \dots, f_n$ be elements of A. The following are equivalent:*

1. *There is a homomorphism $T_n \to A$ sending X_i to f_i.*
2. *For all i, the element f_i is power-bounded.*

In other words, the algebra T_m corepresents the functor $A \mapsto (A^\circ)^m$.

Proof. The fact that (1) implies (2) follows from Exercise 95. Let's show that (2) implies (1). Let

$$F = \sum_{i \in \mathbf{Z}_{\geq 0}^n} a_i X^i$$

be an element of $K\langle X_1, \dots, X_n\rangle$. Then, the sequence $(a_i f^i)_{i \in \mathbf{Z}_{\geq 0}^n}$ converges to 0. Hence, sending F to $F(f_1, \dots, f_m)$ is a well-defined continuous K-algebra homomorphism. □

Corollary 100. *For any $f_1, \dots, f_m \in A^\circ$, the subring $A[f_1, \dots, f_m] \subseteq A$ is a ring of definition associated with the surjection $T_{n+m} \twoheadrightarrow A \colon X_{n+i} \mapsto f_i$ in the sense of lemma 88.* □

Corollary 101. *Let A_0 be the ring of definition arising from a surjection $T_n \twoheadrightarrow A$.*

1. *The ring A° is integral over A_0.*
2. *The ring A° is integrally closed in A.*

In particular, the ring A° is the integral closure of A_0 in A.

Proof. (1) is clear from Proposition 98. For (2), first, let $x \in A$ be such that there exist $a_1, \dots, a_n \in A^\circ$ such that $x^n + a_1 x^{n-1} + \cdots + a_n = 0$. Then, the element x is integral over the ring of definition $A_0[a_1, \dots, a_n]$. Thus, the element f is power-bounded. □

Example 102. For a ring of definition A_0 arising from a surjection $T_n \twoheadrightarrow A$, the ring A° might not be finite over A_0. For example, consider $K = \mathbf{Q}_p(p^{1/p^\infty})^\wedge$. This is an example of a perfectoid field, the definition of which we will not give here. Supposing now that p is not equal to 2, consider the field $L = K(p^{1/2})$. Then, the extension L/K is finite. Letting L_0 denote the ring of definition arising from the surjection $K\langle X\rangle \twoheadrightarrow K\langle X\rangle/(x^2 - p)$, we leave it as an exercise to show that L°/L_0 is not finite.

Exercise 103. In the notation of Example 102, show that L°/L_0 is not finite. Hint: Write $1/(2p^n) = a/2 + b/p^n$ for $a, b \in \mathbf{Z}$.

Theorem 104 (Bosch-Güntzer-Remmert)**.** *Assume that K is a non-Archimedean field that is algebraically closed or discretely valued. Then, for any surjection $T_n \twoheadrightarrow A$, the associated homomorphism $T_n^{\circ} \to A^{\circ}$ is finite.* □

Theorem 105. *Assume that A is reduced.*

1. *All residue norms on A are equivalent to $\|\bullet\|_{\sup}$.*
2. *The ring A° is ϖ-adically complete.* □

Completed Tensor Products

As before, we let ϖ denote a nonzero element of the maximal ideal $\mathfrak{m}_K$.

We would like the category AffAlg_K to have a symmetric monoidal structure, i.e., amalgamated sums.

Proposition 106. *Given a diagram $B \xleftarrow{\varphi_1} A \xrightarrow{\varphi_2} C$ in AffAlg_K, the pushout $B\widehat{\otimes}_A C$ always exists in AffAlg_K.*

Proof. Let $A_0 \subseteq A$ be the ring of definition associated to some surjection $T_n \twoheadrightarrow A$. By Corollaries 100 and 101, we can find rings of definition $B_0 \subseteq B$ and $C_0 \subseteq C$ such that φ_1 and φ_2 restrict to maps $A_0 \to B_0$ and $A_0 \to C_0$, respectively. We now set

$$B_0\widehat{\otimes}_{A_0}C_0 := (B_0 \otimes_{A_0} C_0)^{\wedge}_{(\varpi)}$$

and

$$B\widehat{\otimes}_A C := B_0\widehat{\otimes}_{A_0}C_0\left[\frac{1}{\varpi}\right].$$

We will show that $B\widehat{\otimes}_A C$ satisfies the universal property of pushouts. Let D be an affinoid algebra such that the diagram

$$\begin{tikzcd} A \arrow[r, "\varphi_1"] \arrow[d, "\varphi_2"'] & B \arrow[d] \\ C \arrow[r] & D \end{tikzcd}$$

commutes. By Corollaries 100 and 101, we can find a ring of definition $D_0 \subseteq D$ such that φ_1 and φ_2 restrict to maps $B_0 \to D_0$ and $C_0 \to D_0$, respectively. Thus, we get a map $B_0 \otimes_{A_0} C_0 \to D_0$. Completing and inverting ϖ yields a map $B\widehat{\otimes}_A C \to D$. The uniqueness of this map follows from the density of $B_0 \otimes A_0 \otimes C_0$ in $B_0\widehat{\otimes}_{A_0} \otimes C_0$. It remains to show that $B\widehat{\otimes}_A C$ is affinoid; we leave this as an exercise. □

Exercise 107. In the notation of the proof of Proposition 106, show that $B\widehat{\otimes}_A C$ is affinoid. Hint: First, show that $T_n\widehat{\otimes}_K T_m$ is T_{n+m}. Then, treat the case in which A is K by showing that for ideals $a \subseteq T_n$ and $b \subseteq T_m$, we have $T_n/a\widehat{\otimes}_K T_m/b \cong T_{n+m}/(a,b)$. Finally, treat the general case by showing that there is a surjection $B\widehat{\otimes}_K C \twoheadrightarrow B\widehat{\otimes}_A C$.

Affinoid Spaces

As before, we fix a non-Archimedean field K and an affinoid K-algebra A.

Definition 108. The category of **affinoid spaces**, denoted AffSp_K, is the opposite category of AffAlg_K.

Affinoid Subdomains

Our next goal is to make $\operatorname{Sp} A$ into a ringed space by endowing $\operatorname{Sp} A$ with a topology and a structure sheaf.

There are a few natural candidates for this.

Zariski topology. We could endow $\operatorname{Sp} A$ with the Zariski topology on $\operatorname{Sp} A \subseteq \operatorname{Spec} A$, but this is too coarse for an analytic theory.

Canonical topology. We could endow $\operatorname{Sp} A$ with the topology induced by the non-Archimedean topology on $\operatorname{Sp} A \subseteq \operatorname{Sp} T_n = \mathbf{B}^d(\overline{K}) / \operatorname{Gal}(\overline{K}/K) \subseteq \overline{K}^d$. Concretely, this is generated by the spaces $X(|f| \leq \varepsilon) := \{x \in X : |f(x)| \leq \varepsilon\}$ for $f \in A$ and $\varepsilon \in \mathbf{R}_{>0}$. In fact, it suffices to take $\varepsilon = 1$, since we can restrict attention to those ε equal to $|c|$ for $c \in K^\times$ and then replace f by fc^{-1}. We have seen, though, that the canonical topology is totally discrete, so it is too fine.

We need to find something in between!

Definition 109. A subset $U \subseteq \operatorname{Sp} A$ is called an **affinoid subdomain** if the functor

$$Y \mapsto \{\text{morphisms } Y \to X \text{ that factor (setwise) through } U\}$$

is representable by an affinoid space.

Exercise 110. Suppose that K is algebraically closed and that A is $K\langle T\rangle$. Let U denote the closed ball of radius $|p|$ with origin 0. Show that U is the affinoid subdomain represtented by $B = K\langle T'\rangle$ via $K\langle T\rangle \to K\langle T'\rangle : T \mapsto pT'$. We often denote B by $K\langle T/p\rangle$.

Exercise 111. Adapt Definition 109 to schemes.

Lemma 112. *If $U \subseteq \operatorname{Sp} A$ is an affinoind subdomain with universal morphism $\iota\colon \operatorname{Sp} B \to U$, then ι is a bijection.*

Proof. Let x be a point in U and let $\mathfrak{m} \subseteq A$ be the corresponding maximal ideal. By the universal property, we get a commutative pushout diagram

$$\begin{tikzcd}
B \arrow[r, two heads] \arrow[dr, two heads] & B/\mathfrak{m}B \\
A \arrow[u] \arrow[r, two heads] & A/\mathfrak{m}. \arrow[u, "\sim"']
\end{tikzcd}$$

Thus, we get a fiber diagram

$$\begin{tikzcd} \operatorname{Sp} B \arrow[d] & \operatorname{Sp} B/mB \arrow[l] \arrow[d, "\sim"] \arrow[dl, phantom, "\lrcorner"] \\ \operatorname{Sp} A & \operatorname{Sp} A/\mathfrak{m}, \arrow[l] \arrow[ul] \end{tikzcd}$$

which shows that the fibers of $\operatorname{Sp} B \to \operatorname{Sp} A$ consist of one element. ☐

Lemma 115. *If* $\operatorname{Sp} B \subseteq \operatorname{Sp} A$ *is an affinoid subdomain, then for any* $x \in \operatorname{Sp} B$ *corresponding to the maximal ideal* $m \subseteq A$*, and any* $n \in \mathbf{Z}_{\geq 0}$*, there is an isomorphism* $A/\mathfrak{m}^n \xrightarrow{\sim} B/\mathfrak{m}^n B$*.*

Proof. Essentially the same as that of lemma 112, modulo a slightly more involved diagram chase. ☐

Exercise 113. If U is an affinoid subdomain of $\operatorname{Sp} A$ and V is an affinoid subdomain of U, show that V is an affinoid subdomain of $\operatorname{Sp} A$.

Exercise 114. Show that the pullback of any affinoid subdomain $U \subseteq X$ along a morphism of affinoid spaces is an affinoid subdomain. Conclude that affinoid subdomains are closed under finite intersection.

Proposition 116. *If* $\operatorname{Sp}(B) \subseteq \operatorname{Sp}(A)$ *is an affinoid subdomain, then* $A \to B$ *is flat.*

Proof. It is easy to show that it suffices to prove that for every $\mathfrak{m} \in \operatorname{Sp} B$, the morphism $A \to B_{\mathfrak{m}}$ is flat. For this, we use Bourbaki's local flatness criterion [Bourbaki, Algèbre commutative, Chapitre III, §5, Théorème 1]: since A and B are Noetherian, and for all n the map $A/\mathfrak{m}^n \to B/\mathfrak{m}^n$ is flat, this shows that already $A \to B_{\mathfrak{m}}$ is flat. ☐

Proposition 117. *Every affinoid subdomain* $U \subseteq \operatorname{Sp} A$ *is open in the canonical topology.* ☐

Rational Subdomains

Let X denote $\operatorname{Sp} A$.

The most important kind of affinoid subdomain is:

Definition 118. Let $f_0, f_1, \dots, f_r \in A$ be elements generating the unit ideal of A. We define

$$X\left(\frac{f_1, \dots, f_r}{f_0}\right) := \{x \in X : |f_i(x)| \leq |f_0(x)|, i = 1, \dots, r\}.$$

Any subset of this form is called a **rational subdomain**.

Exercise 119. Show that any rational subdomain is open in the canonical topology.

Exercise 120. Show that the condition that $f_0, f_1, \dots, f_r$ generate the unit ideal implies that there is $\varepsilon > 0$ such that $\varepsilon < \inf_{x \in X} \max_{i=1,\dots,n} |f_i(x)|$. Deduce that if $f_0, \dots, f_r$ generate the unit ideal of A, then for all $x \in X(f_1, \dots, f_r | f_0)$, we have $|f_0(x)| > \varepsilon$, in particular $f_0(x) \neq 0$.

Exercise 121. Show that the intersection of two rational subdomains is a rational subdomain. Hint: If $f_0, \dots, f_r$ and $g_0, \dots, g_s$ both generate the unit ideal, then so does the set of their products $(f_i g_j)_{i=0,\dots,r, j=0,\dots,s}$. Use this to show that $X(\frac{(f_i g_j)_{i,j}}{f_0 g_0})$ is the intersection.

Proposition 122. *Any rational subdomain $U = X(f_1, \dots, f_r | g) \subseteq X$ is an affinoid subdomain. Letting $A\langle T_1, \dots, T_r\rangle$ denote $A \widehat{\otimes}_K T_r$, the algebra representing U is*

$$B := A\left\langle \frac{f_1, \dots, f_r}{g} \right\rangle := A\langle T_1, \dots, T_r\rangle / (gT_1 - f_1, \dots, gT_r - f_r).$$

Proof. Let $\varphi\colon A \to C$ be the homomorphism of affinoid algebras. For each i, let f_i' denote $\varphi(f_i)$, and let g' denote $\varphi(g)$. Then, to say that the map $\operatorname{Sp}\varphi\colon \operatorname{Sp} C \to \operatorname{Sp} A$ factors through U as a set map is to say that for each i and each $x \in \operatorname{Sp} C$, we have $|f_i'(x)| \leq |g'(x)| \neq 0$; the latter condition implies in particular that the element g' is a unit, which implies that for each i, we can consider the fraction f_i'/g' as an element of C.

If $\operatorname{Sp}\varphi$ factors through U, then for all $x \in \operatorname{Sp} C$, we have $|f_i'(x)/g'(x)| \leq 1$, which implies that $\|f_i'/g'\|$ is no greater than 1. By the universal property of Tate algebras, we get a morphism $A\langle T_1, \dots, T_n\rangle \to C\colon T_i \mapsto f_i'/g'$ that sends $gT_i - f_i$ to $g'f_i'/g' - f_i' = 0$ and, as such, factors through $A\langle f_1, \dots, f_r | g\rangle \to C$. This is uniquely determined, as $A[1/g]$ is dense in $A\langle f_1, \dots, f_r | g\rangle$.

On the other hand, suppose that φ factors through a morphism $B \to C$. Let t_i denote the image of T_i in C. By the universal property of the Tate algebra, for all $x \in \operatorname{Sp} C$, we have $|t_i(x)| \leq 1$, which yields

$$|f'(x)| = |g'(x) t_i(x)| \leq |g'(x)| \neq 0.$$

This implies that $\operatorname{Sp}\varphi\colon \operatorname{Sp}(C) \to \operatorname{Sp}(A)$ factors through U. □

Definition 123. For $f \in A$, we write $X(f)$ to denote the rational subdomain $X(f|1)$, and we write $X(f^{-1})$ to denote the rational subdomain $X(1|f)$. Observe that X is covered by $X(f)$ and $X(f^{-1})$; such a covering is called a **simple Laurent covering**. More generally, a **Laurent domain** is an affinoid subdomain of the form $X(f_1) \cap \dots \cap X(f_n) \cap X(g_1^{-1}) \cap \dots \cap X(g_m^{-1})$, for $f_i, g_j \in A$.

Example 124. $X = \operatorname{Sp}(K\langle T\rangle)$.

1. $X(\frac{T}{1})$: Weierstrass domain $X(T)$ represented by $K\langle T, Y\rangle/(Y - T) = K\langle T\rangle$.
2. $X(\frac{T}{p})$: Weierstrass domain $X(p^{-1}T)$ represented by $K\langle T, Y\rangle/(pY - T) = K\langle Y\rangle$
3. $X(\frac{1}{T})$: Laurent domain represented by $K\langle T, Y\rangle/(YT - 1) = K\langle Y^{\pm 1}\rangle$, elements are $\sum_{n\in\mathbf{Z}} a_n Y^n$ with $a_n \to 0, a_{-n} \to 0$ for $n \to \infty$.
4. $X(\frac{p^{-1}}{T})$: represented by $K\langle T, Y\rangle/(YT - p^{-1}) = 0$ because $pYT - 1$ is a unit.
5. $X(\frac{T, T-1}{p})$: represented by

$$K\langle T\rangle\langle\frac{T(T-1)}{p}\rangle = K\langle T, Y_1, Y_2\rangle/(Y_1 p - T, Y_2 p - T - 1) = K\langle Y_1\rangle \times K\langle Y_2\rangle.$$

This is a disjoint unit of two disks $X = X_1 \sqcup X_2$. Note: If we take Spec instead of MaxSpec, then it is not true in that $\operatorname{Spec}(K\langle T, Y\rangle/(YT - p^{-1})) \subseteq \operatorname{Spec}(K\langle T\rangle)$ is injective: The generic points of X_1 and X_2 each go to that of X.

Finally, we mention without proof an important result about affinoid subdomains:

Theorem 125 (Gerritzen–Grauert, 1969). *Every affinoid subdomain of X is a finite union of rational subdomains of X.*

Remark 126. The meaning of this Theorem is the following: It implies that in order to construct and understand the structure sheaf on X, we can reduce to understanding rational subdomains. This is important because for these we have a concrete description of the functions.

Note that in algebraic geometry, we do something similar: We work with standard open subspaces (complements of vanishing sets of a single function) rather than all representable open subspaces. From this perspective, it is actually more natural to develop the whole theory using rational open subdomains as the building blocks of the topology (and in fact, this is the approach taken in adic spaces). The Theorem of Gerritzen Grauert then tells us that this is already the finest topology for which we get representable opens.

Now for historical reasons, and in order to match the definition of the literature, we will still use affinoid subdomains in the following to build rigid spaces. But the upshot of this discussion is that we would get an equivalent definition if we replaced them by rational subdomains everywhere.

Tate's Acyclycity Theorem

Let A be an affinoid K-algebra and let X denote $\operatorname{Sp} A$.

Our goal is to endow X with a structure sheaf. For now, we know what functions should be on rational subdomains. As a first approximation, we define $\mathcal{T}$ to be the category of affinoid subdomains of X with inclusions as morphisms. Then, there is a presheaf $\mathcal{O}_X$ on $\mathcal{T}$ sending a subdomain of X to an algebra representing it.

Definition 127. We say that a set of affinoid subdomains $\mathcal{U} = \{U_i : i \in I\}$ is a **covering** of X if $\bigcup_{i \in I} U$ is all of X. With respect to a fixed ordering of I, we define for a covering $\mathcal{U}$ of X the **Čech complex**

$$\check{\mathcal{C}}(\mathcal{U}, \mathcal{O}_X) := \left[\prod_i \mathcal{O}_X(U_i) \rightrightarrows \mathcal{O}_X(U_i \cap U_j) \substack{\rightarrow\\\rightarrow\\\rightarrow} \prod_{i<j<k} \mathcal{O}_X(U_i \cap U_j \cap U_k) \substack{\rightarrow\\\rightarrow\\\rightarrow\\\rightarrow} \cdots \right].$$

Its cohomology is denoted $\check{H}^\bullet(\mathcal{U}, \mathcal{O}_X)$

Definition 128. A covering $\mathcal{U}$ of X is called **$\mathcal{O}_X$-acyclic** if $\mathcal{O}_X(X) \to \check{\mathcal{C}}(\mathcal{U}, \mathcal{O}_X)$ is a resolution, i.e., if $\check{H}^0(\mathcal{U}, \mathcal{O}_X)$ is A while higher cohomology vanishes. This implies in particular that the sheaf property holds for $\mathcal{U}$.

Theorem 129 (Tate's Acyclicity Theorem). *Any finite covering of X by affinoid subdomains is acyclic.*

Proof. The proof takes an entire lecture.

Step 1: the first sheaf axiom for $\mathfrak{U}$ (local vanishing implies vanishing)

Lemma 130. *The map $A \to \prod_i \mathcal{O}_X(U_i)$ is injective.*

Proof. Let f be in the kernel. Let $x \in \operatorname{Sp}(A)$, $\mathfrak{m}$ the corresponding ideal, then $x \in U_i$ for some i. Let $B = \mathcal{O}_X(U_i)$ and consider the diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & A_{\mathfrak{m}} \arrow[r, hook] \arrow[d] & A_m^\wedge \arrow[d] \\
B \arrow[r] & B_{\mathfrak{m}} \arrow[r] & B_{\mathfrak{m}^\wedge}
\end{tikzcd}$$

The map on the right is isomorphism (last time). The top right morphism is injective by Krull's intersection Theorem. Thus x goes to 0 under each $A \to A_{\mathfrak{m}}$, this implies $x = 0$. $\square$

Step 2: Simple Laurent coverings:

Proposition 131. *For any $f \in A$, the following sequence of A-modules is split exact:*

$$0 \to A \to A\langle f\rangle \times A\langle f^{-1}\rangle \to A\langle f, f^{-1}\rangle \to 0$$

In other words, any simple Laurent cover $\mathfrak{U} = \{X(f), X(f^{-1})\}$ is acyclic.

Proof. By definition of these respective algebras, we have a commutative diagram

$$\begin{tikzcd}
 & & (T-f)A\langle T\rangle \times (1-T^{-1}f)A\langle T^{-1}\rangle \arrow[r, "\beta"] \arrow[d] & (T-f)A\langle T^{\pm 1}\rangle \arrow[d] & \\
0 \arrow[r] & A \arrow[r] \arrow[d] & A\langle T\rangle \times A\langle T^{-1}\rangle \arrow[r, "\alpha"] \arrow[d] & A\langle T^{\pm 1}\rangle \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & A \arrow[r] & A\langle f\rangle \times A\langle f^{-1}\rangle \arrow[r] & A\langle f, f^{-1}\rangle \arrow[r] & 0.
\end{tikzcd}$$

The columns are short exact: Here in the top right, we use that $1 - T^{-1}f = T^{-1}(T-f)$.

The middle row is split exact: Left-exactness is clear, and a splitting of α is given by sending $g = \sum_{n\in\mathcal{N}} a_n T^n$ to $(s_1(g), s_2(g))$ where

$$s_1(g) := \sum_{n\geq 0} a_n T^n, \quad s_2(g) := \sum_{n<0} a_n T^n.$$

By Step 1, the bottom left map is injective. It thus suffices to prove that β is surjective. For this let $g = (T-f)h$ where $h = \sum_{n\in\mathbf{Z}} a_n T^n$ and write this as $g = g_1 + g_2$ where

$$g_1 := (T-f)s_1(h), \quad g_2 := (1-T^{-1}f)\cdot T \cdot s_2(h).$$

Then $(g_1, g_2) \mapsto g$. $\square$

Step 3: Any Laurent covering

Corollary 132. *Let $f_1, \dots, f_n$ in A and consider the Laurent covering given by the set $\mathfrak{U}$ of all Laurent domains $X(f_1^{e_i}, \dots, f_n^{e_i})$ for any $e_1, \dots, e_n \in \{\pm 1\}$. Then $\mathfrak{U}$ is $\mathcal{O}_X$-acyclic.*

Such a $\mathfrak{U}$ is called a Laurent covering.

Proof. The Čech complex for $\mathfrak{U}$ is the completed tensor product of complexes

$$\widehat{\otimes}_A [A\langle f_i \rangle \times A\langle f_i^{-1} \rangle \to A\langle f_i, f_i^{-1} \rangle]$$

Since split exactness is preserved by the additive functor ${}_A$, this is still acyclic. □

Corollary 133. *Let $\mathfrak{U} = \{U_1, \dots, U_n\}$ be a cover of X and let $f \in A$ be such that the induced covers $\mathfrak{U} \cap Y$ on $Y = X(f), X(f^{-1}), X(f, f^{-1})$ are $\mathcal{O}_Y$-acyclic. Then $\mathfrak{U}$ is $\mathcal{O}_X$-acyclic.*

Proof. By the Proposition, the sequence of Čech complexes for the sheaf $\mathcal{O}_X$

$$0 \to \check{C}^*(\mathfrak{U}) \to \check{C}^*(\mathfrak{U} \cap X(f)) \times \check{C}^*(\mathfrak{U} \cap X(f^{-1})) \to \check{C}^*(\mathfrak{U} \cap X(f, f^{-1})) \to 0$$

is a short exact sequence of complexes. The statement follows from the long exact sequence of cohomology for a sequence of complexes. □

Step 4: Refinements: Let $\mathfrak{V} = \{V_j, j \in J\}$ and $\mathfrak{U} = \{U_i.i \in I\}$ be coverings of X. Then $\mathfrak{V}$ is called a refinement of $\mathfrak{U}$ if each V_j is contained in some U_i.

Lemma 134. *Let $\mathfrak{U}$ be a covering of X. Let $\mathfrak{V}$ be a covering refining $\mathfrak{U}$, and such that for each $U_{i_1}, \dots, U_{i_r}$ with $U_{i_1,\dots,i_r} := U_{i_1} \cap \dots \cap U_{i_r}$, the covering $\mathfrak{V} \cap U_{i_1,\dots,i_r}$ is $\mathcal{O}_{U_{i_1,\dots,i_r}}$-acyclic. Then $\mathfrak{U}$ is $\mathcal{O}_X$-acyclic.*

Proof. This is formal and has nothing to do with rigid geometry, see [BGR, §8.1.4 Corollary3]. □

Note: If $\mathfrak{U}$ is a Laurent covering of X, and $V \subseteq X$ is affinoid open, then $V \cap \mathfrak{U}$ is a Laurent covering of V.

Step 5: Rational coverings:

Definition 135. Let $f_0, \dots, f_n \in A$ generate the unit ideal. Then the sets

$$U_i := \{x \in X \mid |f_j(x)| \le |f_i(x)| \text{ for } j = 1, \dots, n\}$$

clearly form a covering of X. We call this a "rational covering".

Lemma 136. *Rational coverings are $\mathcal{O}_X$-acyclic.*

Proof. We argue by induction on n, and for each n by induction on the number of f_i that are non-units:

If all f_i are units, then $\mathfrak{U}$ is refined by the Laurent covering defined by the $g_{ij} := f_i / f_j$. In this case, steps 3 and 4 give the result.

Otherwise, suppose that f_n is a non-unit. By the exercise, we can find $a \in K^\times$ such that

$$|a| < \inf_{x \in X} \max\{|f_i(x)|\}.$$

Let $f := f_n/a$ and consider the Laurent covering $X = X(f) \cup X(f^{-1})$.

By Step 3, it suffices to prove that for $Y = X(f), X(f^{-1})$ or $X(f, f^{-1})$, the cover $\mathfrak{U} \cap Y$ is $\mathcal{O}_Y$-acyclic.

On $Y = X(f^{-1})$ or $X(f, f^{-1})$, the function f^{-1} becomes a unit, so this holds by induction hypothesis.

On $Y = X(f)$, we have

$$|f_n(x)| \leq |a| < \max |f_i(x)|,$$

hence the rational covering $\mathfrak{U}_n$ given by the $f_1, \dots, f_{n-1}$ is a refinement of that for $f_1, \dots, f_n$ (because the condition added by f_n is automatic). By induction hypothesis and refinement Lemma, it follows that $\mathfrak{U} \cap Y$ is $\mathcal{O}_Y$-acyclic. □

This finishes the proof of Tate's Acyclicity Theorem. □

Rigid spaces

Affinoid rigid spaces

Starting from Tate's Acyclicity Theorem, the idea is now that the presheaf $\mathcal{O}_X$ is a "sheaf with respect to finite covers". In order to make this precise, we need to use a notion that is more general than topological spaces and allows for a more restrictive notion of covers:

Grothendieck topologies

The most powerful framework to deal with "things more general than topological spaces" are sites. For our purposes, a slightly weaker notion is more convenient:

Definition 137. A "Grothendieck topological space", for short "G-topological space", is a triple $(X, \mathcal{C}, \mathrm{Cov}(X))$ where X is a set, $\mathcal{C}$ is a full subcategory of the subsets of X with inclusions as morphisms and $\mathrm{Cov}(X)$ is a family of coverings $\mathfrak{U} = \{U_i, i \in I\}$ of subsets $U = \cup U_i$ in $\mathcal{C}$ by $U_i \in \mathcal{C}$ satisfies the following axioms:

1. For any $U \in \mathcal{C}$, the trivial covering $\mathfrak{U} = \{U\}$ is in $\mathrm{Cov}(X)$.
2. If $\{U_i | i \in I\} \in \mathrm{Cov}(X)$, and for each $i \in I$ we have a covering $\{V_{ij} | i \in J_i\}$ of U_i in $\mathrm{Cov}(X)$, then also $\{V_{ij} | i, j\} \in \mathrm{Cov}(X)$.
3. If $\{U_i\} \in \mathcal{C}$ and $V \in \mathcal{C}$, then $\{U_i \cap V\}$ is a cover of V in $\mathrm{Cov}(X)$.

We call $U \in \mathcal{C}$ the "admissible opens" and the covers in $\mathrm{Cov}(X)$ the "admissible covers".

Example 138. Let X be a topological space. This induces a G-topological space where $\mathcal{C}$ is the category of open subsets of X, and $\mathrm{Cov}(X)$:= the set of coverings.

As usual, we often drop $\mathcal{C}, \mathrm{Cov}(X)$ from notation and just refer to X as a G-top space.

(Note: The definition of a sheaf on a topological space only uses the axioms of a G-topological space:)

Definition 139. A sheaf $\mathcal{F}$ on a G-topological space $(X, \mathcal{C}, \mathrm{Cov}(X))$ is a contravar functor

$$\mathcal{F} : \mathcal{C} \to \mathrm{Ab}/\mathrm{Rings}/\ldots$$

such that for any covering $\{U_i\} \in \mathrm{Cov}(X)$ of $U \in \mathscr{C}$, the following sequence is exact

$$0 \to \mathscr{F}(U) \to \prod \mathscr{F}(U_i) \to \prod_{i,j} \mathscr{F}(U_i \cap U_j)$$

Definition 140. Let X be an affinoid space. Let $\mathscr{T}$ be the category of affinoid subdomains of X, with morphisms the open inclusions. Let $\mathrm{Cov}(X)$ be the collection of finite coverings $\{U_1, \dots, U_n\}$ of affinoid subspaces by affinoid subspaces. This is a G-topological space. We call this the "weak Grothendieck topology" on X.

Exercise 141. Check that this satisfies the axioms.

We can now rephrase Tate acyclicity in the following way: $\mathscr{O}_X$ is a sheaf on $\mathscr{T}$.

Next goal (like in algebraic geometry): extend $\mathscr{O}_X$ to functions on "reasonable" open subsets of X that are not affinoid, for example the open unit disc inside $\mathbf{B}^1$.

Idea: test by affinoid subdomains.

First observe: The refinement lemma shows that we can more generally allow also infinite coverings that admit a *refinement* by a finite cover by affinoid subdomains! Let us call such a cover "admissible".

Definition 142. Let X be an affinoid space.

1. Any subset $U \subseteq X$ is called "admissible open" if there is a (not nec finite) covering $U = \cup_{i \in I} U_i$ by affinoid subdomains such that for every morphism $f : Z \to X$ of affinoid spaces such that $f(Z) \subseteq U$, the induced cover $Z = \cup_{i \in I} f^{-1}(U_i)$ is admissible, i.e. admits a refinement by a finite cover by affinoids.

2. A covering $U = \cup_{i \in I} U_i$ of an admissible open subset by admissible open subsets of X is called admissible if for every morphism $f : Z \to X$ of affinoid spaces, the covering $Z = \cup_{i \in I} f^{-1}(U_i)$ is admissible, i.e. admits a refinement by a finite cover by affinoids.

Upshot: "open/cover is admissible if it looks admissible to affinoid spaces"

Example 143. Let $U \subseteq \mathbf{B}^1 = \mathrm{Sp}(K\langle T\rangle)$ be the open unit disc, consisting of $x \in \mathbf{B}^1$ such that $|T| < 1$. Let $\varpi \in K^\times$, $\varpi < 1$, then we have an infinite cover by affinoid subdomains

$$\mathbf{B}^1 = \bigcup_{n \in \mathcal{N}} \mathbf{B}^1(|T| \leq |\varpi^{1/n}|).$$

Let $\varphi : Z \to \mathbf{B}^1$ be any morphism of affinoid spaces that factors through U. Let $t \in \mathscr{O}(Z)$ be the image of T, then $|t| < 1$. But then $a := \|t\|_{\sup} < 1$ by the Maximum Modulus Principle, hence Z is covered by the one set $\varphi^{-1}(\mathbf{B}^1(|T| \leq |\varpi^{1/n}|))$ for any n such that $a \leq |\varpi^{1/n}|$.

Hence the cover is admissible

Example 144. Let $X = \mathbf{B}^1$ and consider the cover $X = X(|T| < 1) \sqcup X(|T| = 1)$. This is a finite cover, but it is not admissible: Testing by the identity $Z = X \to X$, we would have to give a finite refinement by affinoids. But on any finite cover of $X(|T| < 1)$ by affinoids, T would attain a maximum.

Example 145. In contrast, $X = X(|T| < 1) \sqcup X(|T| \geq |p|)$ is admissible.

Definition 146. Let $\mathfrak{T}$ be the category of admissible opens in X and let $\mathrm{Cov}(X)$ be the admissible covers. Then $(X, \mathfrak{T}, \mathrm{Cov}(X))$ is called the "strong Grothendieck topology on X".

Proposition 147. $(X, \mathfrak{T}, \mathrm{Cov}(\mathfrak{T}))$ *is a G-top space satisfying the following additional "completeness axioms":*

(G_0) *$\emptyset$ and X are admissible open.*

(G_1) *If $(U_i)_{i\in I}$ is an admissible cover of U and $V \subseteq U$ is any subset such that each $V \cap U_i$ is admissible, then V is admissible open (i.e. admissibility can be checked locally)*

(G_2) *A set-theoretic covering $(U_i)_{i\in I}$ of an admissible open U by admissible opens that can be refined by an admissible covering is admissible.*

Exactly as for schemes, we can now extend the structure sheaf: Let $U \subseteq X$ be any admissible open. Let $U = \cup U_i$ be an admissible cover by affinoids. Then we set

$$\mathcal{O}_X(U) := \ker\Big(\prod_i \mathcal{O}_X(U_i) \to \prod_{i,j} \mathcal{O}_X(U_i \cap U_j)\Big).$$

Proposition 148. *$\mathcal{O}_X$ is a sheaf with respect to the strong Grothendieck topology.*

Proof. Only need to check that two admissible covers give the same result. For this use:

Exercise 149. If $\mathfrak{U}$ and $\mathfrak{V}$ are admissible, then $\mathfrak{U} \cap \mathfrak{V} = \{U \cap V, U \in \mathfrak{U}, V \in \mathfrak{V}\}$ is admissible.

For refinements, the statement follows as before. □

Remark 150. In fact, this is really just a formality about sites: What we have really done is consider the morphism of sites $i : \mathfrak{T} \to \mathcal{T}$ and defined $\mathcal{O}_X := i^* \mathcal{O}_X$.

An advantage of G-ringed spaces over sites is that we have a straightforward notion of stalks:

Definition 151. Let X be a G-ringed space. Then for any $x \in X$, the stalk at x is

$$\mathcal{O}_{X,x} := \varinjlim_{x \in U \subseteq X} \mathcal{O}_X(U)$$

where U ranges through the admissible opens.

Lemma 152. *Let X be an affinoid space with the strong G-topology and $x \in X$. Then $\mathcal{O}_{X,x}$ is a local ring.*

Proof. The point x defines a maximal ideal $\mathfrak{m}_{U,x} \subseteq \mathcal{O}_X(U)$ for each U such that

$$\mathcal{O}_X(U)/\mathfrak{m}_{U,x} = \mathcal{O}_X(X)/\mathfrak{m}_{X,x} =: K'$$

is a field. We thus have a quotient map $\mathcal{O}_{X,x} \to K'$. Let $\mathfrak{m}_x$ be its maximal ideal. If $f \in \mathcal{O}_{X,x} \setminus \mathfrak{m}_x$, then there is $x \in U \subseteq X$ such that $f \in \mathcal{O}_X(U)$ and $|f(x)| \neq 0$. Hence $|f(x)| > \varepsilon$ for some $\varepsilon > 0$. Then $U(|f| \geq \varepsilon)$ is also an affinoid neighbourhood of x, and f is invertible, hence is invertible in $\mathcal{O}_{X,x}$. □

Definition 153. A locally G-ringed K-space is a pair $(X, \mathcal{O}_X)$ where X is a G-topological space and $\mathcal{O}_X$ is a sheaf of K-algebras on X such that all stalks $\mathcal{O}_{X,x}$ are local rings.

A morphism of locally G-ringed K-spaces $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a pair (φ, φ^*) where $\varphi : X \to Y$ is a morphism of G-topological spaces and $\varphi^* : \varphi_* \mathcal{O}_Y \to \mathcal{O}_X$ is a morphism of sheaves of K-algebras such that for each $x \in X$, the induced morphism

$$\mathcal{O}_{Y,\varphi(x)} \to \mathcal{O}_{X,x}$$

is local, i.e. sends the maximal ideal into the maximal ideal.

Proposition 154. *Any affinoid space X equipped with the strong Grothendieck topology defines a locally G-ringed K-space $(X, \mathcal{O}_X)$.*

More precisely, sending A to $(X = \mathrm{Sp}(A), \mathcal{O}_X)$ defines a fully faithful contravariant functor

$$\{\textit{affinoid } K\textit{-algebras}\} \to \{\textit{locally } G\textit{-ringed } K\textit{-spaces}\}.$$

We call a locally ringed G-space an affinoid rigid space if it is in the essential image.

Proof. Exercise to fill in the details. As usual, the "locally ringed" is required to make this fully faithful, i.e. to ensure that $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ uniquely determines $X \to Y$. □

Definition 155. A rigid analytic space is a locally G-ringed K-space $(X, \mathcal{O}_X)$ satisfying completeness axioms (G_1), (G_2) such that X admits an admissible cover $X = \cup_{i \in I} U_i$ for which each $(U_i, \mathcal{O}_{X|U_i})$ is an affinoid rigid space. We recall the axioms:

(G_1) If $(U_i)_{i \in I}$ is an admissible cover of $U \subseteq X$ and $V \subseteq X$ is such that each $V \cap U_i$ is admissible, then V is admissible open.

(G_2) A set-theoretic covering $(U_i)_{i \in I}$ of an admissible open U by admissible opens that can be refined by an admissible covering is admissible.

Exercise 156. Show that a rigid space automatically satisfies (G_0).

Exercise 157. Any affinoid space X is quasi-compact wrt the strong Grothendieck topology, in the following sense: If $X = \cup_{i \in I} U_i$ is an admissible cover of X, then there is a finite $J \subseteq I$ such that $X = \cup_{j \in J} U_j$ is an admissible cover.

Remark 158. There are various other names in the literature, but they all refer to the same definition: "Rigid spaces" = "Rigid K-spaces" = "Rigid analytic spaces", and even just "analytic spaces" in some of the very early articles on the subject. If people say "rigid analytic varieties", they sometimes make an additional assumption that X is connected and one has a cover by affinoid spaces whose affinoid algebras are integral domains. We now discuss some examples of rigid spaces. The first is admissible open subspaces of affinoid spaces:

Lemma 159. *Let $(X, \mathcal{O}_X)$ be a rigid space. Then any admissible open subspace $U \subseteq X$ inherits the structure of a rigid space $(U, \mathcal{O}_{X|U})$. We call this a rigid open subspace.*

Proof. Define admissible open in U to be admissible open in X. □

Example 160. 1. The open disc $U \subseteq \mathbf{B}^1 = \mathrm{Sp}(K\langle T\rangle)$ defined by $|T| < 1$ is a non-affinoid rigid space. Explicitly, it admits an admissible cover by affinoid rigid spaces

$$U = \cup_{n\in\mathcal{N}} \mathbf{B}^1(|T| < |\varpi^{1/n}|).$$

2. The Zariski-open complement of a function $f \in \mathbf{B}^n = \mathrm{Sp}(K\langle T_1, \dots, T_n\rangle)$:

$$\mathbf{B}^n(f \neq 0) = \cup_{n\in\mathcal{N}} \mathbf{B}^n(|f| > |\varpi^{1/n}|)$$

is a rigid open subspace. This is usually not affinoid.

Second, we can glue rigid spaces:

Lemma 161. *Let $(X_i)_{i\in I}$ be rigid spaces. Assume we are given for each $i, j \in I$ an admissible open $V_{i,j} \subseteq X_i$ and isomorphisms $\varphi_{ij} : V_{i,j} \to V_{j,i}$ of rigid spaces such that the cocycle condition holds. Then there is a unique (up to isomorphism) rigid space X such that the X_i form an admissible cover of X with intersections $X_i \cap X_i = V_{ij}$.*

Proof. We define a subspace of X to be admissible if its intersection with every X_i is admissible, and similarly for covers. Exercise: well-defined and unique. NOTE: We crucially need to use the axioms G1,G2! □

Analytification

Proposition 162. *Let X be any scheme of locally finite type over K. Then there is a rigid space X^{an} and a morphism of locally G-ringed K-spaces*

$$X^{\mathrm{an}} \to X$$

such that for any rigid space Y and any morphism of locally G-ringed K-spaces $Y \to X$, there is a unique morphism $Y \to X^{\mathrm{an}}$ making the obvious diagram commute.

Moreover, $X^{\mathrm{an}} \to X$ induces a bijection between X^{an} and the closed points of X.

Definition 163. We call X^{an} the rigid analytification of X.

Proof. It suffices to consider $X = \operatorname{Spec}(A)$ where $A = K[T_1, \dots, T_n]/I$ and $Y = \operatorname{Sp}(B)$, the universal property will do the rest using the gluing lemma. First try:

$$X_0^{\mathrm{an}} := \operatorname{Sp}(K\langle T_1, \dots, T_n\rangle/I),$$

but by the universal property of the Tate algebra this only works if the images $t_i \in B$ of T_i satisfy $\|t_i\| \leq 1$ on Y. The trick is therefore to consider for any $m \in \mathcal{N}$ the space

$$X_m^{\mathrm{an}} := \operatorname{Sp}(K\langle \varpi^m T_1, \dots, \varpi^m T_n\rangle/I),$$

Then $Y \to X$ factors uniquely through X_m^{an} for any m such that $\|t_i\| \leq |\varpi|^{-m}$. Moreover, for any $k \in \mathcal{N}$,

$$X_m^{\mathrm{an}} \subseteq X_{m+k}^{\mathrm{an}}$$

is the affinoid subdomain defined by $|T_i| \leq |\varpi|^{-m}$. It follows that we can glue the X_m^{an} to a rigid space

$$X^{\mathrm{an}} = \cup_{m \in \mathcal{N}} X_m^{\mathrm{an}}.$$

This now has the desired universal property. The description of the points follows from testing by $Y = \operatorname{Sp}(L)$ for finite field extensions $L|K$. $\square$

Example 164. Let $X = \mathbf{A}^n$. Then $X^{\mathrm{an}} = \cup_{m \in \mathcal{N}} \mathbf{B}_m^n$ is the rigid affine space: A union of closed discs of increasing radius. Note: $|T|$ is not bounded on X, so X cannot be affinoid!

Let's calculate the global sections: We have

$$\mathcal{O}(\mathbf{A}^{1,\mathrm{an}}) = \cap_m \mathcal{O}(X_m) = \varprojlim K\langle \varpi^m T\rangle = \left\{ \sum a_n T | a_n x^n \to 0 \text{ for all } x \in \mathbf{R}_{\geq 0} \right\}.$$

An example for an element that is not a polynomial is $\sum \varpi^{n^2} X^n$.

Recall that $X = \mathbf{A}^1$ represents the functor $(Y, \mathcal{O}_Y) \to \mathcal{O}_Y(Y)$ on locally ringed spaces. It follows that for any rigid space Y,

$$\operatorname{Mor}(Y, \mathbf{A}^1) = \mathcal{O}(Y)$$

Exercise 165. Show this directly from the explicit description of $\mathbf{A}^1$.

Example 166. For example, over $K = \mathbf{Q}_p$, we can make sense of the p-adic logarithm

$$\log(1+x) = -\sum_{n=1}^{\infty} \frac{(-x)^n}{n} : U \to \mathbb{A}^1$$

from the open unit disc around 1 as a morphism of rigid spaces.

Remark 167. WARNING: Note in particular that the morphism

$$\operatorname{Mor}(\mathbb{A}^1, \mathbb{A}^1) \to \operatorname{Mor}(\mathbb{A}^{1,\mathrm{an}}, \mathbb{A}^{1,\mathrm{an}})$$

is not surjective! SO analytification is not fully faithful.

This in itself is not necessarily a problem, but what would be really awkward is if this made separated things non-quasi-separated! As it turns out, this is not a problem for separated spaces, whose analytification is again separated. But the analytification of a quasi-separated space can be non-quasi-separated.

Example 168. For $X = \mathbf{P}^n$, we get the rigid projective space $\mathbf{P}^{n,\mathrm{an}}$. By construction of $-^{\mathrm{an}}$, this is glued from $n+1$ copies of $\mathbb{A}^n$. But one can also give a *finite* admissible affinoid cover: Let $x_0, \dots, x_n$ be homogeneous coordinates on $\mathbf{P}^n$, then set

$$V_i := \mathbf{P}^{n,\mathrm{an}}(|x_j| \leq |x_i| \text{ for all } j) = \mathrm{Sp}(K\langle X_1, \dots, X_n\rangle)$$

where we think of $X_i = \frac{x_i}{x_j}$. Then $\mathbf{P}^n = V_0 \cup \dots \cup V_n$.

For example, for $n = 1$, this means that we can either construct $\mathbf{P}^1$ by glueing two copies of $\mathbb{A}^{1,\mathrm{an}}$ along $\mathbb{A}^{1,\mathrm{an}}(X \neq 0)$ via $X \mapsto X^{-1}$. Or we can take two affinoid closed discs $\mathbf{B}^1$ and glue them along the boundary $\mathbf{B}^1(|X| = 1)$. In particular, $\mathbf{P}^n$ is quasi-compact:

Definition 169. A rigid space is called quasi-compact if it admits an admissible cover by finitely many affinoid spaces.

rigid GAGA

Theorem 170. *Let X, Y be proper K-schemes. Then* $\mathrm{Mor}(X, Y) \to \mathrm{Mor}(X^{\mathrm{an}}, Y^{\mathrm{an}})$ *is bijective. (Moreover, analytication identifies categories of coherent modules on X and X^{an}).*

Definition 171. A Zariski-closed subspace $Z \subseteq \mathbf{P}^{n,\mathrm{an}}$ is called a projective rigid space. This means that for each i, the map $Z \cap V_i \to V_i$ is defined by a surjection $\mathcal{O}(V_i) \to \mathcal{O}(V_i \cap Z)$.

Corollary 172 (Chow's Theorem)**.** *Any projective rigid space is the analytification of a projective K-scheme.*

WARNING: This fails for proper K-schemes!

Say I care a lot about projective varieties. Does that mean that rigid geometry doesn't give us anything new? Quite the opposite! We can use rigid geometry to gain new insights using analytic methods, and then translate them back to schemes. Historically, the most important example is the following, which was literally what rigid spaces were invented for:

The Tate curve

Let E be an elliptic curve over $\mathbf{C}$. Then we have a complex uniformisation

$$E^{\mathrm{an}} = E(\mathbf{C}) = \mathbf{C}/\Lambda \text{ where } \Lambda = \mathbf{Z} \oplus \tau\mathbf{Z}.$$

Recall: over alg closed field, an elliptic curve is uniquely determined by its j-invariant $j(E)$. One can recover the algebraic curve E from τ in terms of the j-invariant $j(\tau)$, a holomorphic function $j : \mathbb{H} \to \mathbf{C}$ that is $\mathrm{SL}_2(\mathbf{Z})$-invariant. Alternatively, we can rewrite

$$\mathbf{C}/\Lambda \xrightarrow{x \mapsto \exp(2\pi i x)} \mathbf{C}^\times/q^{\mathbf{Z}} \text{ where } q =: \exp(2\pi i\tau).$$

Note: j has q-expansion $j = 1/q + 744 + 196884q + \cdots \in \mathbf{Z}((q))$. Conversely, we can recover $j(E)$ from q by this formula, $q = 1/j + 744\frac{1}{j^2} \ldots$.

We can now have a (partial) analogue in p-adic geometry!

Theorem 173. *Assume that K is algebraically closed, e.g. $K = \mathbf{C}_p$. Let E be an elliptic curve over K such that $j(E) \in K$ satisfies $|j(E)| > 1$. Then there is a canonical isomorphism*

$$E^{\mathrm{an}} \cong \mathbb{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$$

where $q = 1/j + 744\frac{1}{j^2} \ldots$.

This is super cool! It's literally the reason why Tate invented rigid spaces. Namely, Tate saw that the q-periodic meromorphic functions on $\mathbb{G}_m^{\mathrm{an}}$ form an elliptic function field. Example application:

Corollary 174. *There is a short exact sequence*

$$0 \to \mu_N \to E[N] \to \mathbf{Z}/N\mathbf{Z} \to 0.$$

The quotient $\mathbb{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ is something we cannot take in schemes, but we can in rigid space:

Construction: First observe that $0 < |q| < 1$, so $q^{\mathbf{Z}} \subseteq \boldsymbol{G}_m$ is discrete. Let $\boldsymbol{G}_m = K[X^{\pm}]$. For any $n \in \mathbf{Z}$, let $U_n := \boldsymbol{G}_m^{\mathrm{an}}(|q|^{\frac{n+1}{2}} \leq |X| \leq |q^{\frac{n}{2}}|)$. This is an affinoid subspace

$$U_n = \mathrm{Sp}(K\langle \frac{X}{q^{\frac{n+1}{2}}}, \frac{q^{\frac{n}{2}}}{X}\rangle),$$

and $\boldsymbol{G}_m = \cup_{n\in\mathcal{N}} U_n$ is an admissible cover (exercise). Let

$$U_n^- := U_n(|q^{\frac{n+1}{2}}| = |X|), \quad U_n^+ := U_n(|q^{\frac{n}{2}}| = |X|).$$

Then we can reconstruct $\boldsymbol{G}_m^{\mathrm{an}}$ from the U_n by glueing U_n^- to U_{n+1}^+ via the map $X \mapsto X$.

Now let's furthermore glue U_n to U_{n+2} via the map

$$U_n \xrightarrow{\sim} U_{n+2}, \quad qX \leftarrow\!\shortmid X$$

and call the result U. Then the natural maps $U_n \subseteq U$ glue to a map

$$q : \boldsymbol{G}_m^{\mathrm{an}} \to U$$

and for any $x \in \boldsymbol{G}_m^{\mathrm{an}}$, we have $q^{-1}(q(x)) = q^{\mathbf{Z}}x$. So this deserves to be called

$$U = \boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}.$$

Observe: Already U_0 and U_1 cover U, namely we can construct U by glueing

$$U_1^+ \xrightarrow{\sim} U_0^-, \quad X \leftarrow\!\shortmid X$$

$$U_1^- \xrightarrow{\sim} U_0^+, \quad qX \leftarrow\!\shortmid X$$

Hence $\boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ is quasi-compact! The group structure on $\boldsymbol{G}_m^{\mathrm{an}}$ induces on $\boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ the structure of a "rigid group variety" (i.e. a group object in rigid spaces).

Sketch of proof of Theorem: Can cook up a line bundle L on $\boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ by descent from $\boldsymbol{G}_m^{\mathrm{an}}$. Use a version of Riemann–Roch to show that $\dim H^0(\boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}, L) = 3$ and defines an embedding $\boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}} \subseteq \mathbf{P}^2$. Then $\boldsymbol{G}_m^{\mathrm{an}}/q^{\mathbf{Z}}$ is projective $\Longrightarrow$ is analytification of projective curve C. By GAGA, get a group structure on $C \Longrightarrow$ is an elliptic curve E. Show that $j(E) = j(q)$.