Résolution de Demazure affines et formule de Casselman-Shalika

This is a note on [**14**].

Introduction

Let $G \in \mathrm{AlgGrp}_k^{\mathrm{cn.red.split}}$, $k = \mathbb{F}_q$. For each $\lambda \in X_\bullet(T)_+$, it is possible to construct a projective k-scheme $\bar{\mathrm{Gr}}_\lambda$, whose set of k points is

$$\overline{\mathrm{Gr}^\lambda}(k) := \bigsqcup_{\lambda' \leq \lambda} K\varpi^{\lambda'}K/K$$

of which the group K, viewed as an algebra group over k of infinite dimension, acts through a quotient of finite type. The action induces a stratification of open orbits

$$\overline{\mathrm{Gr}^\lambda} = \bigsqcup_{\lambda' \leq \lambda} \mathrm{Gr}^{\lambda'}$$

The scheme $\overline{\mathrm{Gr}^\lambda}$ is not smooth in general, for a prime $l \neq \mathrm{char}\, k$, it is natural do consider the l-adic IC complex

$$\mathcal{A}_\lambda := \mathrm{IC}(\overline{\mathrm{Gr}^\lambda}, \bar{\mathbb{Q}}_\lambda)$$

which is K-equivariant. The associated function from Frobenius trace

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

defined on the set of k points of $\overline{\mathrm{Gr}^\lambda}$, can be viewed as a function of the unramified Hecke algebra [**7**], of compactly supported functions in $G(F)$ this is biequivariant wrt $G(\mathcal{O})$.
Let $\check{G}$ be the group defined over $\bar{\mathbb{Q}}_l$ whose roots is dual to that of G. In [Sat63], Satake constructed a canonical isomorphic of the hecke algebra $\mathcal{H}$ with the algebra of regular functions on $\check{G}$, which are $\mathrm{Ad}(\check{G})$ equivariant. After Lusztig and Kato, see [**11**], [**??**], the Satake transform of A_λ is equal to, up to a sign, the character of V_λ, irreducible representation of height weight of λ of $\widehat{G}$. More recently, Ginzburg [???], [**12**], has proved a Tannakian equivalence between K equivariant perverse on Gr with the convolution structure, and the algebraic representations of $\check{G}$ with the tensor structure.
The constant terms which are the Fourier coefficients of the functions A_λ are remarkably simple. Let $B := TU$ the a subgroup of Borel of G and ρ the half sum of

roots of T in $\mathrm{Lie}(U)$. After Lustzig and Kato the constant integral term is equal to

$$\int_{U(F)} A_\lambda(x\varpi^\nu)\,dx = (-1)^{2\langle\rho,\nu\rangle} q^{\langle\rho,\nu\rangle} m_\lambda(\nu)$$

where $m_\lambda(\nu)$ is the dimension of the weight space ν in $V(\lambda)$.
Example:
The principle object of this paper is to prove the gometric statement of the above result. For each $\nu \in X_\bullet(T)$ there is a well defined subscheme $S_\nu \subset \mathrm{Gr}$ such that

$$S_\nu(k) := U(F)\varpi^\nu G(\mathcal{O})/G(\mathcal{O})$$

We show that the complex

$$R\Gamma_c(S_\nu \otimes_k \bar{k}, \mathcal{A}_\lambda)$$

is concentrated in degree $2\langle\rho,\nu\rangle$ and that the Frobenius endomoprhism acts on $H^{2\langle\rho,\nu\rangle}$ as multiplication by $q^{2\langle\rho,\nu\rangle}$....
When ν is dominant, we can define a moprhism $h : S_\nu \to \mathbb{G}_a$ such that $\theta(x) = \psi(h(x))$, where $\psi : k \to \bar{\mathbb{Q}}_l^\times$ is a nontrivial additive character on k. We show that the complex

$$R\Gamma_c(S_\nu \otimes_k \bar{k}, \mathcal{A}_\lambda \otimes h^*\mathcal{L}_\psi)$$

Here is the organization of the article. After recalling in 3, known results on affine Grassmanian, we state the principle theorems in 4.1 and 4.2 in **??**. The proof of the theorem occupies the rest of the article. This is based on the study of the geometry of certain resolutions from the simplest $\overline{\mathrm{Gr}^\lambda}$, which corresponds to when λ is minuscule or quasi-minuscule. This strategy is used in [**13**], where the conjecture of [**6**] is proved for GL_n.
In 5 and 6, we prove geometric properties of the intersection $S_\nu \cap \overline{\mathrm{Gr}^\lambda}$, which were probably well known but cannot be found in the literature. 6.2 allows us to show the statements 4.1, 4.2 in the case ν is conjugated by λ by anb element of the Weyl group. We remark on passing, the statement ...
We then study 7, ... , the geometry of $\overline{\mathrm{Gr}^\lambda}$ in the most simple case, that is, when λ is minuscule 7, or when it is quasiminuscule. If λ is minuscule, then $\overline{\mathrm{Gr}^\lambda}$ is equal to Gr^λ and is isomorphic to the scheme G/P of subgroups of G which are conjugate to some parabolic P, further, only the ν which are conjugate to λ are involved, so that 4.1 and 4.2 follows as in the case from 6.2. In section

0.1. Highest weight theory of reductive groups. To motivate: consider G is of *multiplicative type*. This an extension of *Cartier duality*

$$\mathrm{Comm}(\mathrm{FinSch}_k) \xrightarrow{\simeq} \mathrm{Comm}(\mathrm{FinSch}_k)^{\mathrm{op}}$$

This an enlargement of the torus equivalence [**1**, 14.1]

$$\mathrm{Mod}^{\text{fin. gen,op}} \xrightarrow{\simeq} \mathrm{AlgGrp}_k^{\mathrm{diag}}$$

For a triplet (T, B, G),

Theorem 0.1. [**1**, 32.8]

(1) Every irreducible representation has a highest weight, which is dominant.
(2) For all $\lambda \in X_\bullet(T)$, exist as unique $V := V^\lambda \in \mathrm{Rep}_k(G)$ with highest weight λ.

1. Notation

Let k be a finite field of q elements of characteristic p, with algebraic closure $\bar{k}$. Let T be split maximal torus of G and B, B^- be the Borel subgroups such that $B \cap B^- = T$. We denote $\langle -, - \rangle$ the natural paring $X, X^\vee := \mathrm{Hom}(\mathbb{G}_m, T)$. Let $R \hookrightarrow X$ be the system of roots associated to (G, T) and R_+ the roots corresponding to B (resp. B^-) and $\Delta = \{\alpha_1, \ldots, \alpha_r\}$ the set of simple roots. For each $\alpha \in \Phi$, we denote U_α the the root subgroup of G corresponding to α. Let $\Phi^\vee \hookrightarrow X_\bullet$ be the dual roots provided by the bijection

$$\Phi \to \Phi^\vee \quad \alpha \mapsto \alpha^\vee$$

Denote by $\Phi_+^\vee$ the set of positive coroots. Let W be the Weyl group of (G, T). [1] Let

$$\rho := (1/2) \sum_{\alpha \in R_+} \alpha$$

the half sum of positive roots. For each simple root, we have

$$\left\langle \rho, \alpha^\vee \right\rangle = 1$$

We denote $Q^\vee := \mathbb{Z}\Phi^\vee$ (resp. $Q_+^\vee := \mathbb{N}_{\geq 0}\Phi_+^\vee$). We denote by $X_{\bullet,+}$ the cone of dominant cocharacter

$$X_{\bullet,+} := \{\lambda \in X_\bullet \,:\, \langle \alpha, \lambda \rangle \geq 0 \forall \alpha \in \Phi_+\}$$

We consider the partial order on $X_\bullet$ as follows: $\nu \geq \nu'$ if and only if $\nu - \nu' \in Q_+^\vee$. In the case of GL_n, this has a particular simple characterization, see [**13**].

We denote $\check{G}$ the dual group over $\bar{\mathbb{Q}}_l$. It is provided with $\check{T} \hookrightarrow \check{B}$. For each $\lambda \in X_{\bullet,+}$ We denote

$$\Omega(\lambda) := \{\nu \in X_\bullet \,:\, \forall w \in W \quad w\nu \leq \lambda\}$$

[1] The Weyl group is given by $N_G(T)/Z_G(T)$. Typical example to keep in mind is $s := \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$, see [**1**, 26]

This is the set of weight of $\check{T}$ in V_λ, the $\check{G}$-simple $\bar{\mathbb{Q}}_l$ module of highest weight λ. We denote M the set of minimal elements[2] in $X_{\bullet,+} \setminus \{0\}$.

Proposition 1.1. Let $\mu \in M$. We have the following equivalent:

(1) If $\langle \alpha, \mu \rangle \in \{0, \pm 1\}$ for all $\alpha \in \Phi$, and μ is a minimal element in $X_{\bullet,+}$, then $\Omega(\mu) = W\mu$. In this case, we say that μ is minuscule cocharacter. [3]

(2) Otherwise,[4] there exists a unique root such that $\langle \gamma, \mu \rangle \geq 2$; its a maximal positive root, and we have $\mu = \gamma^\vee$ and $\Omega(\mu) = W\mu \cup \{0\}$. In this case, we say that μ is *quasi-miniscule.*

PROOF. The first [**2**, Chap. VI, Ex. 1.24]. We prove the second. Let $\gamma \in \Phi$ such that $\langle \gamma, \mu \rangle \geq 2$. □

1.0.1. *Quasiminuscule characters in GLn.* Let $G = \mathrm{GL}_n$. Then the set of minimal elements in $X_{\bullet,+} \backslash 0$.

- Characters.

$$(l, \dots, l) \quad l \in \mathbb{Z}$$

- Miniscule + twisted by characters.

$$(l+1, \dots, l+1, l, \dots, l)\, l \in \mathbb{Z}$$

- Quasiminuscle.

$$(1, 0, \dots, 0, -1)$$

$$\begin{tikzcd} \mathrm{GL}_n \arrow[rr] \arrow[dr, "\det"'] & & \mathbb{C}^\times \\ & \mathrm{GL}_n/[GL_n, GL_n] \simeq \mathbb{C}^\times \arrow[ur] & \end{tikzcd}$$

but the det map takes diagonal elements

$$(t_i)_{i=1}^n \mapsto \left(\prod t_i\right) \mapsto \left(\prod t_i\right)^n$$

for $n \in \mathbb{Z}$.

[2] The condition of being minimal: is that there does not exists such that

[3] Take $\mu = (1, 0)$.

[4] In GL_2 there is only *one* positive root. Thus, this criteria simply says that as long as (a, b) satisfies $a \geq b + 2$, then it is not minuscule.

Remarks on the the sets appearing here. If $\lambda \in X_{\bullet,+}$ then

$$W\lambda = \left\{ \mu \in X_\bullet \,:\, L_\mu \in \mathrm{Gr}_G^\lambda \right\}$$

using the Cartan decompostion. Further by the closure relation of the orbits

$$\left\{ \mu \in X_\bullet \,:\, L_\mu \in \overline{\mathrm{Gr}^\lambda} \right\} = \left\{ \mu \in X_\bullet \,:\, \mu^+ \leq \lambda \right\}$$

where μ^+ is unique W-conjugate of μ which is dominant.

2. Remarks on minuscule and quasi-minuscule condition

References: [**15**]. Minuscule representations occur in the study of cohomology of flag varieties [**8**], the classification of Shimura datum, [**4**, 1.2], and representation theory, [**5**].

- lie theoretic point of view: if one considers the induced adjoint representation of $\lambda : \mathbb{G}_m \to T \circlearrowleft \mathfrak{g}$, we have a decomposition
$$\mathfrak{g} \simeq \bigoplus \mathfrak{g}_\lambda(i) \quad \mathfrak{g}_\lambda(i) := \left\{X \in \mathfrak{g} \,:\, \mathrm{Ad}\lambda(a)X = a^i \cdot X\right\}$$
λ is minuscule implies $\mathfrak{g}_\lambda(i) = 0$ for $|i| \geq 2$.
- representation theory: when all weights are conjugate under the Weyl group.[5] being minuscule also implies for the highest weight representation V^λ of $\check{G}$, all weights in V^λ have multiplicity 1.
- context of Shimura varieties
$$\mathrm{Hom}^*(\mathbb{S}, G_\mathbb{R})/G^{\mathrm{ad}}(\mathbb{R}) \simeq \mathrm{Hom}^{*'}(\mathbb{G}_m, G_\mathbb{C})/G(\mathbb{C})$$

A few important points:

- If G is not semisimple the definition of quasiminuscle can be inconsistent among different literature, see. 1.0.1.

2.1. Highest weight á la Bourbaki.

Definition 2.1. A minuscule (resp. quasi-minuscule) repn of a semi-simple Lie algebra is an irreducible representation such that the Weyl group acts transitively on all the weights (resp. nonzero weights).

Let us keep in mind the following example of $\mathfrak{sl}_2(k)$. [**2**, $n°2$, VIII]. It has three distinct elements,
$$X_+, X_-, H$$
Our goal is study $V \in \mathrm{Rep}_e(\mathfrak{sl}_2(k))$, $H \circlearrowleft V$ is diagonalizable. The first representation is the adjoint representation.
$$\mathfrak{sl}_2(k) \circlearrowright \mathfrak{sl}_2(k) \quad g \cdot x := [g, x]$$
In fact, $\mathfrak{h}$, the e-span of H, acts on $\mathfrak{sl}_2(k)$ by commuting operator. This yields the general decomposition
$$\mathfrak{sl}_2(k) \simeq \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi \subset \mathfrak{h}^\vee} \mathfrak{sl}_{2,\alpha}$$

(1) It has an abelian subalgebra of semisimple elements.

This shows the strategy to understand a simple Lie algebra L is

(1) Find an abelian subalgebra H

[5]These are "most" of the small representations. For type A_n: the minuscule representations are the exterior powers. of a group.

For Δ a commutative monoid, let $\mathrm{Fun}(\Delta_{\mathrm{disc}}, \mathrm{Mod}_k)$ of Δ-indexed categories. Then via the composition $\dim : \mathrm{Mod}_k \to \mathbb{Z}$, we obtain

$$\mathrm{ch} : \mathrm{Mod}_k^{\Delta_{\mathrm{disc}}} \to \mathbb{Z}^{\Delta_{\mathrm{disc}}}$$

dim is an additive functor, see [**2**, $n°6$, Exmple, Ch. VIII]. If $\Delta = \mathfrak{h}^\vee$.

3. La Grassmannienne affine

Recall the construction, [**10**]. As *loc. cit.* call a k-space, resp. k-group a sheaf of set, resp. of group over the Alg_k with respect to fppf topology. Consider a the k-group LG and the K-subgroup $L^{\geq 0}G$.
It is clear that $L^{\geq 0}G$. is represented by the projective limit of schemes of finite type

$$R \mapsto G(R[[\varpi]]/\varpi^n)$$

Denote by $L^{(N)}G(R)$ the set of $g \in LG(R)$ such that both the order of the poles of $\rho(g)$ and $\rho(g^{-1})$ does not exceed N. After *loc. cit.* $L^{(N)}(G)$ is representable by a scheme and

$$\mathrm{Gr} \simeq \varinjlim \mathrm{Gr}^{(N)}$$

where $\mathrm{Gr}^{(N)} = L^{(N)}G/L^{\geq 0}G$. Denote $L^{\leq 0}G$ the k group $R \mapsto G(R[\varpi^{-1}])$ [6]and let

$$L^{<0}G := \ker(L^{\leq 0}G \xrightarrow{\varpi^{-1} \mapsto 0} G)$$

Example

$L^{<0}G$ has entries of the form

$$\begin{pmatrix} 1 + \frac{1}{t}p(\frac{1}{t}) & \frac{1}{t}p(\frac{1}{t}) \\ \frac{1}{t}p(1/t) & 1 + \frac{1}{t}p(\frac{1}{t}) \end{pmatrix} \quad p \in k[x]$$

This is a subgroup of LG.

Proposition 3.1. The morphism

$$L^{<0}G \times L^{\geq 0}G \to LG$$

is an open immersion.

We identify $L^{<0}G$ with the open $L^{<0}Ge_0$ where e_0 is a fixed based point of Gr. The Grassmanin Gr is covered by the open tralsates $gL^{<0}Ge_0$. These are easy to study for the local geometry of Gr. For example $L^{<0}G$ is not reduced in general, neither is Gr.
The group $L^{\geq 0}G$ acts naturally on Gr. For all $\lambda \in X_\bullet$ denote e_λ the point $\varpi^\lambda e_0$ of Gr. For $\lambda \in X_{\bullet,+}$ denote Gr^λ the $L^{\geq 0}G$ orbit of e_λ. Denote $\overline{\mathrm{Gr}^\lambda}$ the closure of Gr^λ. Also

$$L^{\geq\lambda}G := \mathrm{ad}\varpi^\lambda L^{\geq 0}G, \quad L^{<\lambda}G := \mathrm{ad}\varpi^\lambda L^{<0}G$$

[6] $L^{\leq 0}G$ is often referred as negative loop group, and is also identified as G^{X-x} where $X = \mathbb{P}^1_k$.

Example

$G = \mathrm{GL}_2$, let $\lambda = (a, 0) \in X_{\bullet,+}$ so that $a \in \mathbb{N}_{\geq 0}$. Then

$$\begin{pmatrix} \mathcal{O} & t^a\mathcal{O} \\ \frac{1}{t^a}\mathcal{O} & \mathcal{O} \end{pmatrix}$$

Denote J the prieimage of $U \hookrightarrow B$ under the homomoprhism $L^{\geq 0}G \to G$ deinfed by $\varpi \mapsto 0$. Thus, we have the diagram

$$\begin{tikzcd} J \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^{\geq 0}G \arrow[d] \\ U \arrow[r, hook] & G \end{tikzcd}$$

This is a projective limit of unipotent groups. Denote by

$$J^{\geq\lambda} := J \cap L^{\geq\lambda}G$$

$$J^{\lambda} := J \cap L^{<\lambda}G$$

Example

$G = \mathrm{GL}_2$, then

$$J(k) = \begin{pmatrix} 1 + tk[[t]] & k[[t]] \\ tk[[t]] & 1 + tk[[t]] \end{pmatrix} = \begin{pmatrix} 1 + t\mathcal{O} & \mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix}$$

- $$J^{(1,0)}(k) = k[\frac{1}{t}] \cap k[[t]] = k$$
- Or in general, $\lambda = (a, 0)$. We have
$$L^{<\lambda}(k) = \begin{pmatrix} 1 + \frac{1}{t}p(\frac{1}{t}) & t^a\frac{1}{t}p(\frac{1}{t}) \\ t^{-a}\frac{1}{t}p(\frac{1}{t}) & 1 + \frac{1}{t}p(\frac{1}{t}) \end{pmatrix}$$
$$J^{\lambda}(k) = \mathrm{Span}_k \left\{1, \ldots, t^{a-1}\right\}$$

This is the *finite part* of the decomposition of $L^{<\lambda}G \times L^{\geq\lambda}G \simeq LG$. Don't confuse this with LU!

Let $\alpha \in R$, $i \in \mathbb{Z}$, let $U_{\alpha,i}$ be the image of the homomorphism

$$\mathbb{G}_a \to LG$$

$$x \mapsto U_\alpha(\varpi^i x)$$

The multiplication defines an isomoprhism

$$\prod_{\alpha \in R_+, \langle\alpha,\lambda\rangle > 0} \prod_{i=0}^{\langle\alpha,\lambda\rangle - 1} U_{\alpha,i} \to J^{\lambda}$$

where we made a choice of total order on the set of factors. In particular J^λ is isomorphic to an affine space of dimension $2\langle\rho,\lambda\rangle$.

Example

In the context of GL_n: $\Phi_+ := \{e_i - e_j \,:\, i < j\}$. When $\langle\alpha,\lambda\rangle > 0$, where α is the index of root subgroup. So $\alpha = e_i - e_j$, $\lambda \in X_{\bullet,+}$, the condition means that $\lambda_i > \lambda_j$, i.e. $i > j$.
In the case of $n = 2$, we have $\lambda_1 > \lambda_2$. Thus, this counts the difference between $\lambda_1 - \lambda_2 - 1$. This is the same as that in $L^{<\lambda}(k)$.

Proposition 3.2. The natural morphism

$$J^\lambda \to \mathrm{Gr}^\lambda$$

$$j \mapsto je_\lambda$$

is an open immersion.

Proof. It is clear that multiplication induces an isomorphism

$$J^\lambda \times J^{\geq\lambda} \xrightarrow{\simeq} J$$

It is also clear that the multiplication induces an open immersion

$$J \times B^- \to L^{\geq 0}G$$

Moreover, $J^{\geq\lambda}$ and B^- are subgroups of $L^{\geq\lambda}G$ which fixes e_λ. The lemma follows. □

It follows from 3.2 that Gr^λ is smooth irreducible and of dimension $2\langle\rho,\lambda\rangle$. There exists an embedding $\mathrm{Gr}^\lambda \hookrightarrow \mathrm{Gr}^{(N)}$ for N sufficiently large, hence the closure $\overline{\mathrm{Gr}^\lambda}$ is a porjective scheme, irreducible and stable by the action of $L^{\geq 0}G$. It is well known, see [**11**, 11], that $\overline{\mathrm{Gr}^\lambda}$ is the union of orbits $\mathrm{Gr}^{\lambda'}$ such that $\lambda' \leq \lambda$. In particular, if μ is minuscule [7], then Gr^μ is a smooth projective scheme.
Let [8]

$$L^{>0}G := \ker\left(L^{\geq 0}G \to G\right)$$

Example

$G = \mathrm{GL}_2$, then

$$L^{>0}G = \begin{pmatrix} 1 + t\mathcal{O} & t\mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix}$$

[7]don't we only need being minimal in $X_{\bullet,+}$?
[8]Loops with formal series with no constant terms.

This is a projective limit of unipotent groups. It is clear that for $\lambda \in X_{\bullet,+}$ the moprhism

$$L^{>0}G \cap L^{\geq\lambda}G \times L^{>0}G \cap L^{<\lambda}G \to L^{>0}G$$

is an isomorphism and that [9]

$$L^{>0}G \cap L^{<\lambda}G = \prod_{\alpha\in\Phi_+,\langle\alpha,\lambda\rangle>1} \prod_{i=1}^{\alpha,\lambda-1} U_{\alpha,i}$$

Let P_λ be the parabolic subgroup generated by B^- and by the radical subgroups with $\langle\alpha,\lambda\rangle = 0$, this would be equivalent to the one constructed in 3.1. The Weyl group of W is equal to the stabilizer W_λ of λ. We denote N_λ^+ the opposite unipotent radical of parabolic opposite to P_λ. It is clear that

$$P_\lambda \subset L^{\geq\lambda}G$$

and that

$$J^\lambda = N_\lambda^+ \ltimes L^{>0}G \cap L^{<\lambda}G$$

Example

Proposition 3.3. We have

$$L^+G \cap L^{\geq\lambda}G = P_\lambda \ltimes (L^{>0}G \cap L^{\geq\lambda}G)$$

In particular, the group $L^{\geq 0}G \cap L^{\geq\lambda}G$ is geometrically connected and we have $G \cap L^{\geq\lambda}G = P_\lambda$.

Proof. It sutffices to show that the multiplication morphism

$$\left(L^{>0}G \cap L^{\geq\lambda}G\right) \times P_\lambda \to L^{\geq 0}G \cap L^{\geq\lambda}G$$

□

[9]Taking $\lambda = (1,0)$, whose that the only term that matters is in the top right.

Following Lusztig, Ginzburg, Mkirkovic and Vilonen, we define the convolution product $\mathcal{A}_{\lambda_1} * \mathcal{A}_{\lambda_2}$ for $\lambda_1, \lambda_2 \in X_{\bullet,+}$. Consider the moprhisms

$$\begin{tikzcd} & LG \times \mathrm{Gr} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ \mathrm{Gr} \times \mathrm{Gr} & & \mathrm{Gr} \times \mathrm{Gr} \end{tikzcd}$$

$$\pi_1(g,x) = (ge_0, x) \quad \pi_2(g,x) = (ge_0, gx)$$

The moprhism $\pi - 1$ is the quotient[10] morphism for the action $L^{\geq 0}G$ on $LG \times \mathrm{Gr}$ defined by

$$\alpha_1(h)(g,x) = (gh^{-1}, x)$$

whilst $\pi - 2$ is the quotient morphism of the action of $L^{\geq 0}G$ on $LG \times \mathrm{Gr}$ deinfed by

$$\alpha_2(h)(g,x) = (gh^{-1}, hx)$$

For $\lambda_1, \lambda_2 \in X_{\bullet,+}$ let

$$\overline{\mathrm{Gr}^{\lambda_1}} \bar{\times} \overline{\mathrm{Gr}^{\lambda}_2}$$

be the quotinet of $\pi_1^{-1}(\overline{\mathrm{Gr}^{\lambda_1}} \times \overline{\mathrm{Gr}^{\lambda}_2})$ by $\alpha_2(L^{\geq 0}G)$. The existsence of this question is guaranteed by the local triviality of the moprhism $LG \to \mathrm{Gr}$. More precisely, as the open s of $\overline{\mathrm{Gr}^{\lambda}}$, of the form

$$gL^{<0}Ge_0 \cap \overline{\mathrm{Gr}^{\lambda_1}}$$

the schemes

$$\overline{\mathrm{Gr}^{\lambda_1}} \bar{\times} \overline{\mathrm{Gr}^{\lambda_2}}$$

and

$$\overline{\mathrm{Gr}^{\lambda_1}} \times \overline{\mathrm{Gr}^{\lambda^2}}$$

are isomorphic. Further, these isomoprhisms are clearly compatible with the stratification of $\overline{\mathrm{Gr}^{\lambda}_1} \times \overline{\mathrm{Gr}^{\lambda_2}}$ by the locally closed subsets $\mathrm{Gr}^{\lambda'_1} \times \mathrm{Gr}^{\lambda'_2}$. The projection on second factor defines a moprhism

$$m : \overline{\mathrm{Gr}^{\lambda_1}} \bar{\times} \overline{\mathrm{Gr}^{\lambda_2}} \to \overline{\mathrm{Gr}^{\lambda_1+\lambda_2}}$$

3.0.1. *Some remarks on the twisted products.*

Proposition 3.4. [**18**, 2] $\mathrm{Gr} \tilde{\times} \mathrm{Gr} \cdots \tilde{\times} \mathrm{Gr} \simeq \mathrm{Gr}^n$.

Whenever we have

[10] The terminology is unclear here. Should edit.

3.1. Examples of parabolics. Let $\lambda = (\lambda_1, \lambda_2)$. Generating from roots. For a root α, we can construct

$$\langle B, M_\alpha \rangle$$

where $M_\alpha := Z(T_\alpha), T_\alpha := \ker(T \xrightarrow{\alpha} \mathbb{G}_m)$.

Example

$G = \mathrm{GL}_n$. Let $\lambda = \big(\lambda_1 = \cdots \lambda_{m_1} > \cdots > \lambda_{m_{k-1}+1} = \cdots = \lambda_{m_k}\big)$. The parabolic is of the form:

$$P_\lambda := \begin{pmatrix} \boxed{\mathrm{GL}_{m_1}} & * & * \\ & \ddots & * \\ 0 & & \boxed{\mathrm{GL}_{m_k}} \end{pmatrix}$$

We may consider $\mathrm{ev}_0^{-1}(P_\lambda)$.

Proposition 3.5. [**16**, 2.3.10]

$$\mathrm{ev}_0^{-1}(P_\lambda) \simeq L^{\geq 0}G \cap L^{\geq \lambda}G$$

Proof. Let us consider the $\mathbb{C}$-points. It would be easy to consider the function $\tilde{\lambda}_{(-)} : \{1, \ldots, n\} \to \mathbb{Z}$ as a function given by

$$\tilde{\lambda}_x = \lambda_i \text{ if } 1 \leq x \leq \lambda_{m_i}$$

Then

$$L^{\geq 0}G(\mathbb{C}) \cap L^{\geq \lambda}G(\mathbb{C}) = \left\{ t^{\tilde{\lambda}_i - \tilde{\lambda}_j} a_{ij} \in G(\mathbb{C}[[t]]) \,:\, a_{ij} \in G(\mathbb{C}[t]]) \right\}$$

$\square$

4. Les énconcés principaux

Recall that U denotes the unipotent radical of B associated to R_+. We define LU,

$$L^{\geq 0}U := LU \cap L^{\geq 0}G, \quad L^{\leq 0}U := LU \cap L^{<0}G$$

For each $\nu \in X_\bullet(T)$ we also denote

$$L^{\geq \nu}U := \varpi^{\nu} L^{\geq 0}U\varpi^{-\nu}, L^{<\nu}U := \varpi^{\nu} L^{<0}U\varpi^{-\nu}$$

Example

$G = \mathrm{GL}_2$. $\lambda := (1,0) \in X_{\bullet,+}$. Then

$$L^{\geq\lambda}U = \begin{pmatrix} 1 & tk[[t]] \\ & 1 \end{pmatrix}, L^{<\lambda}U = \begin{pmatrix} 1 & t(1/t)k[1/t] \\ & 1 \end{pmatrix}$$

In general if $\lambda = (a,b)$, then

$$L^{\geq\lambda}U = \begin{pmatrix} 1 & t^{a-b}k[[t]] \\ & 1 \end{pmatrix}, L^{<\lambda}U = \begin{pmatrix} 1 & t^{a-b}(1/t)k[1/t] \\ & 1 \end{pmatrix}$$

For each $\nu \in X_\bullet$, $L^{<\nu}U$ is a closed subgroup of $L^{<\nu}G$ so we can define $L^{<\nu}Ue_\nu$ as a closed subset of the open set $\varpi^\nu L^{<0}Ge_0$. In particular for all $\lambda \in X_{\bullet,+}$ and $\nu \in X_\bullet$, $S_\nu \cap \overline{\mathrm{Gr}}_\lambda$ is a locally closed subscheme, possibly empty, of $\overline{\mathrm{Gr}}_\lambda$. By the Iwaswa decomposition, this yields a stratification of $\overline{\mathrm{Gr}}_\lambda$. We will give a new proof of the following theorem due to Mirkovic and Vilonen in the case $k = \mathbb{C}$, [**12**].

Theorem 4.1. For each $\lambda \in X_{\bullet,+}$, and $\nu \in X_\bullet$ the complex $R\Gamma_c(S_\nu, \mathcal{A}_\lambda)$ is concentrated in degree $2\langle\rho,\nu\rangle$. Further, the endomoprhism Fr_q acts on $H_c^{2\langle\rho,\nu\rangle}(S_\nu, \mathcal{A}_\lambda)$ as $q^{\langle\rho,\nu\rangle}$.

In the previous statement we wrote $R\Gamma_c(S_\nu, \mathcal{A}_\lambda)$ instead of

$$R\Gamma_c((S_\nu \cap \overline{\mathrm{Gr}^\lambda}) \otimes_k \bar{k}, \mathcal{A}_\lambda)$$

for simplicity. We use this notation systematically in the following and does not cause any ambiguity.
For each $\nu \in X_{\bullet,+}, \nu' \in X_\bullet$, choose a total orer of the positive roots and we have an isomoprhism

$$\prod_{\alpha\in R_+} \prod_{\langle\alpha,\nu'\rangle \leq i < \langle\alpha,\nu\rangle} U_{\alpha,i} = L^{<\nu}U \cap L^{\geq\nu'}U$$

For ν fixed ν' more and more anitdominant , this group forms an inductive system for the limit $L^\nu U$.

Example

Use $G = \mathrm{GL}_2$, $\nu_1 = (1, 0)$. Let $\nu'_n := -(n, -n)$, then
$$L^{\geq \nu'} U = \begin{pmatrix} 1 & t^{-2n} k[[t]] \\ & 1 \end{pmatrix}$$

It is then clear that
$$L^{<\nu} = \varinjlim L^{<\nu} U \cap L^{\geq \nu'_n} U$$
For each simple root $\alpha \in \Delta$, denote $u_{\alpha,i}$ the projection over the factor $U_{\alpha,i}$ and
$$h : L^{<\nu} U \cap L^{\geq \nu'} U \to \mathbb{G}_a$$
$$h(x) := \sum_{\alpha \in \Delta} u_{\alpha,-1}(x)$$
Fix a nontrivial additibev character, $\psi : k \to \bar{\mathbb{Q}}_l^{\times}$, and denote $\mathcal{L}_\psi$ the Artin-Schreier sheaf over $\mathbb{G}_a$ associated to ψ. The character $\theta : U(F) \to \bar{\mathbb{Q}}_l$ considered in introduction is the character $x \mapsto \psi(h(x))$. The following statement was a conjecture of [**6**]

Theorem 4.2. For $\nu \neq \lambda$ in $X_{\bullet,+}$ the complex $R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^* \mathcal{L}_\psi)$ is zero. For $\nu = \lambda$ the complex is isomorphic to $\bar{\mathbb{Q}}_l$ provided with the action of Frobenius by $q^{\langle \rho, \lambda \rangle}$, at degree $2 \langle \rho, \lambda \rangle$.

These results imply the statements about constant terms and Fourier coefficients mentioned in the Grothendiecks' function-sheaf dictionary. We will present the proofs of these two theorems in parallel in the rest of the article.

5. L'action du tore T

The torus T normalizes these subgroups $L^{\geq 0}G, L^{<0}G$, $L^{<\nu}G$, ... of LG so that it acts on all the geometric objects we considered. This action provides a valuable tool to study their geometry. Choose once and for all a strictly dominant cocharacter $\phi : \mathbb{G}_m \to T$. The $\mathbb{G}_m$ action we consider follows from the following compositions

$$\mathbb{G}_m \hookrightarrow L^{\geq 0}\mathbb{G}_m \xrightarrow{L^{\geq 0}\phi} L^{\geq 0}G \circlearrowright \mathrm{Gr}$$

Proposition 5.1. For all $\nu \in X_\bullet$ the point e_ν is the fixed point of the action $\mathbb{G}_m \circlearrowleft S_\nu$. Furthermore, it is the attractive fixed point.

Proof. For all $x \in L^{<\nu}U(\bar{k})$ is of the form

$$x = \prod_{\alpha\in\Phi_+} \prod_{i<\langle\alpha,\nu\rangle} U_{\alpha,i}(x_{\alpha,i})$$

where $x_{\alpha,i} \in \bar{k}$ are zero for all but a finite number. Thus, for all $z \in \bar{k}^\times$, we have

$$\phi(z)xe_\nu = \prod_{\alpha\in\Phi_+} \prod_{i<\langle\alpha,\nu\rangle} U_{\alpha,i}(z^{\langle\alpha,i\rangle}x_{\alpha,i})e_\nu$$

$\square$

This lemma shows that e_ν are the only fixed points of the action $\mathbb{G}_m \circlearrowleft \mathrm{Gr}$. Further, it implies following statement

Proposition 5.2. If the intersection $S_\nu \cap \overline{\mathrm{Gr}^\lambda}$ is nonempty, ν belongs $\Omega(\lambda)$.

Proof. $\square$

Proposition 5.3. The Euler-Poincaré characteristic $\chi_c(S_\nu \cap \mathcal{Q}_\lambda)$ is equal to 1 if ν is conjugate to λ by an element of W and 0 otherwise.

This statement can be considered as a geometric interpretation of result of Lusztig, [**11**, 6.1]. Let us use the notation of introduction. Let c_λ be the element of hecke algebra $\mathcal{H}$ defined

$$c_\lambda = (-1)^{2\langle\rho,\lambda\rangle}q^{-\langle\rho,\lambda\rangle}1_\lambda$$

where 1_λ is the characteristic function of $K\varpi^\lambda K$. We know that

$$(c_\lambda) = (K_{\lambda,\mu}(q))^{-1}(A_\lambda)$$

where $K_{\lambda,\mu}(q)$ is the triangular matrices formed the Kazhdan-Lusztig polynomials. The constant terms of the normalizing constants

$$(-1)^{2\langle\rho,\nu\rangle}q^{-\langle\rho,\nu\rangle}\int_{U(F)} c_\lambda(x\varpi^\mu)\,dx$$

6. Les intersections $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$

For all $\lambda \in X_{\bullet,+}$ we considered

$$J^\lambda = \prod_{\alpha \in \Phi_+} \prod_{i=0}^{\langle \alpha, \lambda \rangle - 1} U_{\alpha,i}$$

which is clearly a subgroup of $L^{\geq 0}U$. We also prove that the morphism $J^\lambda \to \overline{\mathrm{Gr}^\lambda}$ is an open immersion. A distinct argument of the content of this section is given in [**3**, 5.2].

Proposition 6.1. Let $\lambda \in X_{\bullet,+}$ induces an isomorphism of J^λ with the open subset $\varpi^\lambda L^{<0}Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$ of $\overline{\mathrm{Gr}^\lambda}$.

PROOF. The image of J^λ is contained in $\varpi^\lambda L^{<0}Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$. By 3.2, it is thus a dense open subset of $\varpi^\lambda L^{<0}Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$. □

Proposition 6.2. Let $\lambda \in X_{\bullet,+}$ for $w \in W$ the moprhism

$$wJ^\lambda w^{-1} \cap LU \to S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$$

defined by

$$j \mapsto je_{w\lambda}$$

is an isomorphism. As a consequence $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is isomorphic to an affine space of dimension $\langle \rho, \lambda + w\lambda \rangle$

PROOF. For $w = 1$, the result follows from the 6.1 due to the following inclusion [11]

$$J^\lambda e_\lambda \subset S^\lambda \cap \overline{\mathrm{Gr}^\lambda} \subset \varpi^\lambda L^{<0}Ge_0 \cap \overline{\mathrm{Gr}^\lambda}$$

□

We can deduce 4.1 in the case $\nu = w\lambda$ and 4.2 in the case $\nu = \lambda$. Indeed the inclusion

$$wJ^\lambda w^{-1} \cap LU \hookrightarrow L^{\geq 0}U \hookrightarrow L^{\geq 0}G$$

implies that $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is contained in the open orbit Gr^λ. Thus the restriction of $\mathcal{A}_\lambda$ to $S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}$ is equal to:

$$\mathcal{A}_\lambda \Big|_{S_{w\lambda} \cap \overline{\mathrm{Gr}^\lambda}} = \bar{\mathbb{Q}}_l[\langle \rho, 2\lambda \rangle](\langle \rho, \lambda \rangle)$$

The statement Thm. 4.1 thus follows. The inclusion $J^\lambda \subset L^{\geq 0}U$ implies that the restriction of h to J^λ is zero. Then 4.2 is true in the case $\nu = \lambda$.

The more general statement below will be needed later. For each $\sigma \in X_{\bullet,+}$ denote

$$h_\sigma : LU \to \mathbb{G}_a \tag{1}$$

the morphism

$$h\mathrm{ad}(\sigma) : x \mapsto h(\varpi^\sigma x \varpi^{-\sigma})$$

---

[11] If $\lambda = (1,0)$, $J^\lambda(k) = \begin{pmatrix} 1 + t\mathcal{O} & \mathcal{O} \\ t\mathcal{O} & 1 + t\mathcal{O} \end{pmatrix} \cap \begin{pmatrix} \frac{1}{t}k[1/t] & t \cdot \frac{1}{t}k[1/t] \\ \frac{1}{t} \cdot \frac{1}{t}k[1/t] & k[1/t] \end{pmatrix}$. For sake

and also the induced homomorphism $h_\sigma : S_\lambda \to \mathbb{G}_a$. Since σ is dominant, the restriction of h_σ to $L^{\geq 0}U$, and a fortiori to J^λ is zero. We thus also have the following

Proposition 6.3. For all $\lambda, \sigma \in X_{\bullet,+}$ we have

$$R\Gamma_c(S_\lambda, \mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_l[-2\langle \rho, \lambda\rangle](-\langle \rho, \lambda\rangle)$$

7. Minuscules

We utilized the notations fixed in 1. Let μ be nonzero minimal [12] element of $X_{\bullet,+}$. By 1.1, we have the following statement

Proposition 7.1. Let μ be minuscule. We have $\Omega(\mu) = W\mu$. For $\alpha \in R$, we have

$$\langle \alpha, \mu\rangle \in \{0, \pm 1\}$$

For example, in the case of GL_n the minuscule ones are precisely those of the form

$$(l+1, l+1, , \ldots, l+1, l, \ldots, l) \quad l \in \mathbb{Z}$$

If μ is minuscule, by minimality, this implies the orbit Gr^μ is closed. Since for all elements ν of $\Omega(\mu)$ is conjugate to μ by anaction of W for 4.1, 4.2 it suffices to verify for the case $\lambda = \mu$ and $\nu \in \Omega(\mu)$.

Proposition 7.2. We have a canonical isomorphism $\mathrm{Gr}^\mu \to G/P$ st.

$$S^{w\mu} \cap \mathrm{Gr}^\mu \simeq UwP/P$$

Proof. Given 3.3 and the two assertions of 7.1, we have that $L^{\geq 0}G \cap L^{\geq \mu}G$ is the inverse image of of P under the homomorphism $\mathrm{ev}_0 : L^{\geq 0}G \to G$. For example, see 3.5.

$$\mathrm{Gr}^\mu = L^{\geq 0}G/(L^{\geq 0}G \cap L^{\geq \mu}G) \simeq G/P_\mu$$

Given, again, 7.1 we knw that $J^\mu = U_\mu^+ = \prod_{\langle \alpha, \mu\rangle = 1} U_\alpha$, which is the unipotent subgroup of the opposite parabolic of P. As a consequence

$$wJ^\mu w^{-1} \cap LU = wU_\mu^+ w^{-1} \cap U$$

The second assersion follow from 6.2. $\square$

[12] why was this necessary again?

8. Quasi-minuscules: étude géométrique

See also exercise of Zhu. Let μ is a quasi-minuscule weight, i.e. a minimal element of $X_{\bullet,+}\setminus\{0\}$, smaller than 0. Recall, that by 1.1 we have

Proposition 8.1. Let μ be quasiminuscule. Then μ is equal to a cocharacter $\gamma^\vee$ associated to a positive maximal root γ.[13] We have $\Omega(\mu) = W\mu \cup \{0\}$. For each root $\alpha \in \Phi\setminus\{\pm\gamma\}$ we have $\langle\alpha,\mu\rangle \in \{0,\pm 1\}$.

Example

Consider the maximal root:

$$e_1 - e_2$$

Then $\langle\mu,\gamma\rangle = 2$, implies that μ is
In fact in general, the maximal root for GL_n us not hard to compute: we can sum up all the positive roots:

$$e_1 - e_n$$

This satisfies that for all other rooots

$$\langle e_1 - e_n, \alpha\rangle \in \{0, \pm 1\}$$

Since 0 is a dominant cocharactere which is smaller smaller than μ, $\overline{\mathrm{Gr}^\mu}$ is the union of Gr^μ and the base point e_0. Denote by P the parabolic subgroup of G generated by T and the subgroup of radical roots U_α such that $\langle\alpha,\gamma^\vee\rangle \leq 0$, see 3.1. Denote

$$V := \mathfrak{h} \oplus \bigoplus_{\alpha\in R\setminus\{\gamma\}} \mathfrak{g}_\alpha$$

where $\mathfrak{h}$ is the Lie algebra of T and where $\mathfrak{g}_\alpha$ are the subspaces of weight α of $\mathfrak{g}$.

Example

For $G = \mathrm{GL}_2$, $\mathfrak{g} = \mathfrak{gl}_2$.

By the preceding lemma V is the sum of weights ν in $\mathfrak{g}$ such that $\langle\gamma,\nu\rangle \leq 1$. It is a result of the definition of P that V is P -stable.
Identify $\mathfrak{g}_\gamma$ with quotient $\mathfrak{g}/V$ with the structure of P-module, we can thus consider the right fibration

$$\begin{array}{c} \mathbb{L}_\gamma \times^P \mathfrak{g}_\gamma \\ \downarrow \\ G/P \end{array}$$

Proposition 8.2. $\mathbb{L}_\gamma \simeq \mathrm{Gr}^\mu$

[13] To have an example, consider the root $(1,-1)$.

PROOF. The functor $R \mapsto G(R[\varpi]/\varpi^2)$ is TG where

$$\mathrm{TG} \simeq G \ltimes \mathfrak{g}$$

There is a canonical truncation map

$$L^{\geq 0}G \to \mathrm{TG} \simeq G \ltimes \mathfrak{g}$$

By 3.3 and the last statement of 8.1, that we have a fiber

$$\begin{tikzcd} L^{\geq 0}G \cap L^{\geq \mu}G \arrow[r] & L^{\geq 0}G \arrow[d] \\ P \ltimes V \arrow[r] & G \ltimes \mathfrak{g} \end{tikzcd}$$

□

The fiber $\mathbb{L}_\gamma$ compacts in a natural into a straight line fiber of projections. In fact we have

$$\mathbb{L}_\gamma \hookrightarrow \mathrm{Proj}(\mathbb{L}_\gamma \oplus \mathcal{O}_{G/P}) \simeq \mathbb{P}_\gamma$$

we have a natural isomorphism

$$\mathrm{Proj}(\mathbb{L}_\gamma \oplus \mathcal{O}_{G/P}) \simeq \mathrm{Proj}(\mathcal{O}_{G/P} \oplus \mathbb{L}_{-\gamma}) \simeq \mathbb{P}_{-\gamma}$$

we can view $\mathbb{P}\gamma$ as the union of $\mathbb{L}_\gamma$ and $\mathbb{L}_{-\gamma}$. Denote

$$\begin{tikzcd} \mathbb{L}_{\pm\gamma} \arrow[d] \\ G/P \arrow[u, bend left, "\epsilon_{\pm\gamma}"] \end{tikzcd} \tag{2}$$

Proposition 8.3. The isomorphism of Lem. 8.2

$$\begin{tikzcd} \mathbb{L}_\gamma \arrow[r, hook] \arrow[d] & \mathbb{P}_\gamma \arrow[d] \\ \mathrm{Gr}^\mu \arrow[r] & \overline{\mathrm{Gr}^\mu} \end{tikzcd}$$

extends and sends $\epsilon_{-\gamma}(G/P)$ to the point ϵ_0.

Note that

Proposition 8.4. Notation as 2 if $w\gamma \in \Phi_+$ then

$$S_{w\mu} \cap \mathrm{Gr}^{\leq \mu} = \phi_\gamma^{-1}(UwP/P)$$

If $w\gamma \in \Phi^-$ we have

$$S_{w\mu} \cap \overline{\mathrm{Gr}^\mu} = \epsilon_\gamma(UwP/P)$$

Definition 8.5. We denote W_γ the stabilizer of γ in W and Δ_γ the set of simple roots conjugates to γ.

Example

The weyl group of GL_n is S_n. Thus if we consider $W(G,T)$ then this

Proposition 8.6. We have a stratification

$$S_0 \cap \bar{\mathrm{Gr}}_\mu = \{e_0\} \cup \bigcup_{w \in W/W_\gamma} \phi_\gamma^{-1}(UwP/P) \backslash \epsilon_\gamma(UwP/P)$$

9. Quasi-minuscules:étude cohomologique

The notation are as the 8. In particular $\mu = \gamma^\vee$ is quasi-minuscule. The resolution

$$\pi_\gamma : \mathbb{P}_\gamma \to \overline{\mathrm{Gr}^\mu}$$

allows us to compute the local intersection cohomology of A_ν at an isolated singularity e_0. The following statement is due to Kazhdan and Lusztig.
Indeed, in the following situation, the hypothesis is much weaker, and their argument applies. We detail the proof for the convenience of the reader.

Proposition 9.1. Let $d = \langle 2\rho, \mu \rangle$ the dimension $\overline{\mathrm{Gr}^\mu}$. For $i \geq 0$, the group $H^i(\mathcal{A}_\mu)_{e_0}$ is trivial. For $i < 0$, we have the short exact sequence

$$0 \to H^{i+d-2}(G/P)(d/2-1) \to H^{i+d}(G/P)(d/2) \to H^i(\mathcal{A}_\mu)_{e_0} \to 0 \tag{3}$$

Proof. Let $\overline{\mathrm{Gr}_\mu}'$ be th eopen of $\overline{\mathrm{Gr}^\mu}$

$$\overline{\mathrm{Gr}^\mu}' :=$$

we have $\pi_\gamma^{-1}(\overline{\mathrm{Gr}^\mu}') = \mathbb{L}_{-\gamma}$. Denote $\mathcal{A}'_\mu$ the restriction of $\mathcal{A}_\mu$ to this open. Denote the inclusion of the closed point $i : \{e_0\} \to \overline{\mathcal{A}}'_\mu$. The natural morphism

$$\mathcal{A}'_\mu \to i_* i^* \mathcal{A}'_\mu$$

induces a restriction of morphism of cohomology (without support()

$$i^* : R\Gamma(\overline{\mathrm{Gr}^\mu}', \mathcal{A}'_\mu) \to (\mathcal{A}'_\mu)_{e_0}$$

□

Proposition 9.2. Let $\mathcal{C}$ be the factor supported by e_0 in the decomposition

$$R\pi_{\gamma *}\bar{\mathbb{Q}}_l[d](d/2) = \mathcal{A}_\mu \oplus \mathcal{C}$$

For $i < 0$, we have

$$H^i(\mathcal{C}) = H^{i+d-2}(G/P)(d/2-1)$$

For $i \geq 0$ we have

$$H^i(\mathcal{C}) = H^{i+d}(G/P)(d/2)$$

We can now prove statement 4.1 when case λ is a quasiminuscule cocharacter $\mu = \check{\gamma}$. Consider the discussion after 6.2, it reduces to the case $\nu = 0$.

Proposition 9.3. We have isomorphisms

$$R\Gamma_c(S_0, \mathcal{A}_\mu) \simeq \bar{\mathbb{Q}}_l^{|\Delta_\gamma|}$$

where Δ_γ is

Proof. By the theorem for base change of proper morphism, we have

$$R\Gamma_c(\pi_\gamma^{-1}(S_0 \cap \overline{\mathrm{Gr}^\mu}, \bar{\mathbb{Q}}_l)[d](d/2) \simeq R\Gamma_c(S_0, \mathcal{A}_\mu) \oplus \mathcal{C}$$

recall that the stratification obtained in **??**.

$$\pi_\gamma^{-1}(S_0 \cap \bar{\mathrm{Gr}}_\mu) = \bigsqcup_{w \in W/W_\gamma}$$

□

Let us now prove statemet 4.2 in the case $\nu = 0$ and $\lambda = \mu$ quasi-minuscule. We actually prove something more general. Recall that for each $\sigma \in X_\bullet$, we defined a morphism $h_\sigma : S_0 \to \mathbb{G}_a$ see Eq. 1.

Proposition 9.4. For each $\sigma \in X_{\bullet,+}$ we have the isomoprhism

$$R\Gamma_c(S_0, \mathcal{A}_\mu \otimes h_\sigma^* \mathcal{L}_\psi) = \bar{\mathbb{Q}}_l^{|\Delta_\gamma^\sigma|}$$

where Δ_γ^σ is the set of $\alpha \in \Delta_\gamma$ such that $\langle \alpha, \sigma \rangle > 0$.

Example

If $\Delta_\gamma = \Delta$, then $\Delta_\gamma^\sigma = \{\alpha : \langle \alpha, \sigma \rangle > 0\}$ Thus, this counts precisely the number of strictly positive jumps.

The proof of 9.4 is the same as 9.3, which is , a particular case of 9.4. It suffices to prove the following geometric statement.

9.1. Recollection of the work of Kazhdan Lusztig. We refer to [**17**] for a nice introduction. Recall we have the *Bruaht decomposition:*

$$G = \bigsqcup_{W} B\dot{w}B$$

arising fom the action

$$B \times B \circlearrowright G$$

Example: SL_n. Quotients $X = G/B$ are those referred to as *flag varieties.* Again, similar to affine Grassmanian, one has a $T \circlearrowright X$.

- $X_w := \mathrm{im}(B\dot{w}B \to G/B)$. These are the B orbits on X.
- The *Schubert varieties* are $S_w := \overline{X_w} \simeq (X_v)_{v \leq w}$.

Now we can construct another action

$$G \circlearrowright X \times X$$

- The orbits are $\mathcal{O}_w$.

10. Convolution

A better reference is [**19**, 2.1.4].
Let us first recall the construction of twisted product

$$\mathrm{Gr}$$

Recall that M is the minimal cocahracters in $X_{\bullet,+}$. For each $\mu_\bullet = (\mu_1, \dots, \mu_n)$ of elements in M, we can construct the projective subscheme

$$\overline{\mathrm{Gr}^{\mu_\bullet}} = \overline{\mathrm{Gr}^{\mu_1}} \tilde{\times} \cdots \tilde{\times} \overline{\mathrm{Gr}^{\mu_n}} \hookrightarrow_{\mathrm{cl}} \mathrm{Gr}^n$$

The projection of the lass factors of Gr^n defines a proper morphism

$$\overline{\mathrm{Gr}^{\mu_\bullet}} \xrightarrow{m_{\mu_\bullet}} \overline{\mathrm{Gr}^{|\mu_\bullet|}}$$

where $|\mu_\bullet| = \sum_{i=1}^n \mu_i$. Let $\nu_\bullet$ be collection of elements in $X_\bullet$. For $i = 1, \dots, n$, denote $\sigma_i := \nu_1 + \cdots + \nu_i$, we denote

$$S_{\nu_\bullet} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet} := (S_{\sigma_1} \times \cdots \times S_{\sigma_n}) \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet}$$

in Gr^n. It is clear that $S_{\nu_\bullet}$.

Proposition 10.1. We have a canonical isomorphism

$$S_{\nu_\bullet} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet} \xleftarrow{\simeq} (S_{\nu_1} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_1}) \times \cdots \times \left(S_{\nu_n} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_n}\right)$$

Proof. One can show easily by recurrence that each point

$$(y_1, \dots, y_n) \in S_{\nu_\bullet} \cap \bar{\mathrm{G}}\mathrm{r}^{\mu_\bullet}$$

can be uniquely written as

$$\begin{aligned} y_1 &= x_1 \varpi^{\nu_1} e_0 \\ &\cdots \\ y_n &= x_1 \varpi^{n_1} \cdots x_n \varpi^{\nu_n} e_0 \end{aligned}$$

∎

Example

The decomposition of $y_1, y_2, \dots, y_n$ is an inductive application of the decomposition
$$L^{<\nu_i}N \times L^{\geq \nu_i}N \simeq LN$$
for $i = 1, \dots, n$. In the case of $y_1 \in S_{\nu_1}$, we have
$$\begin{aligned} y_1 &= x\varpi^{\nu_1} \\ &= x_{<\nu_1}\varpi^{\nu_1}x_+ \\ &= x_1\varpi^{\nu_1} \end{aligned}$$
where
$$x = x_{<\nu_1}x_{\geq\nu_1} \in LN, \quad x_{<\nu_1} \in L^{<\nu_1}N, x_{\geq\nu_1} \in L^{\geq\nu_1}N$$
$$x_{\geq\nu_1} = \varpi^{\nu_1}x_+\varpi^{-\nu_1}, x_+ \in L^{\geq 0}N, \quad x_1 := x_{<\nu_1}$$
and equality is taken as coset class.

$$\begin{aligned} y_2 &= x'\varpi^{\sigma_2} \\ &= (x_1\varpi^{\nu_1})(x_1\varpi^{\nu_1})^{-1}x'\varpi^{\nu_1}\varpi^{\nu_2} \\ &= (x_1\varpi^{\nu_1})(\mathrm{ad}((\varpi^{\nu_1})^{-1})(x_1^{-1}x'))\varpi^{\nu_2} \end{aligned}$$
where
$$x' \in LN$$

Proposition 10.2.

Definition 10.3. Let $\mu_\bullet$ denote a sequence of elements in M. Following [**9**], we call a $\mu_\bullet$ -path the following combinatorial data:

- A sequences of vertices in $X_\bullet$ such that for all $i = 1, \dots, n$ we have $\nu_i = \sigma_i - \sigma_{i-1} \in \Omega(\mu_i)$.
- the maps
$$p_i : [0, 1] \to X_\bullet \otimes_{\mathbb{Z}} \mathbb{R}$$
satisfying :
  (1) if σ_{i-1}

By putting the images of p_is at the end points, we get a path in $X_\bullet \otimes_{\mathbb{Z}} \mathbb{R}$ going from 0 to σ_n. The $\mu_\bullet$-path is called *dominant*, if the entire image is contained in the dominant chamber, $(X_\bullet \otimes_{\mathbb{Z}} \mathbb{R})_+$.

Example

Consider 10.1, for all $\nu \in \Omega(|\mu_\bullet|)$ the set of irreducible components of $\pi^{-1}(S_\nu \cap \bar{\mathrm{G}}\mathrm{r}^{|\mu_\bullet|}$ is in canonical bijection with the $\mu_\bullet$ paths χ from 0 to ν.

Proposition 10.4. The convolution product $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$ is a perverse sheaf. It decomposes as a direct sum

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$$

After Lem. 6.2, each $S_{w\mu_i} \cap \bar{\mathrm{G}}\mathrm{r}_{\mu_i}$ is irreducible. Further, by Cor.

Proposition 10.5. For $\nu \in \Omega(|\mu_\bullet|)$ dominant and χ is a $\mu_\bullet$ dominant path starting from 0 to ν, then the component C_χ is contained in $\pi^{-1}(S_\nu \cap \overline{\mathrm{Gr}^\nu})$.

It is not difficult to prove conversely that if the $\mu_\bullet$ path χ is not dominant then $C_\chi \not\subseteq \pi^{-1}(S_\nu \cap \mathrm{Gr}^\mu)$. We leave this to the reader because it is not logically necessary for the rest of the paper. It will only be necessary for us to know that the multiplicity of $\mathcal{A}_\nu$ in $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$, satisfies

$$\dim(V_{|\mu_\bullet|}) \geq |\mu_\bullet\text{-path } \chi \text{starting from 0 to } \nu|$$

Proposition 10.6. For all $\lambda \in\in X_{\bullet,+}$, $\mathcal{A}_\lambda$ is a director factor of a convolution product of the form

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$$

with $\nu_1, \ldots, \mu_n \in M$.

Taken into account 10.4 and 10.5 it suffices to show that there exists a dominant $\mu_\bullet$ path from 0 to ν. We prove this combinatorial statement in 11.

11. Combinatoire

12. Fin des démonstrations

We use the notation of Sec. 10. In particular let $\lambda \in X_{\bullet,+}$ and $\mu_\bullet = (\mu_1, \ldots, \mu_n)$ elements of M such that $\mathcal{A}_\lambda$ is a direct factor of $\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n}$, see 10.6.
Proof: consider the ... it suffices to show that the complex

$$R\Gamma_c(S_\nu, \mathcal{A}_1 * \cdots * \mathcal{A}_{\mu_n}) \simeq R\Gamma_c\left(m_{\mu_\bullet}^{-1}\left(S_\nu \cap \mathrm{Gr}^{\leq \mu_\bullet}\right), \mathrm{IC}(\mathrm{Gr}^{\leq \mu_\bullet})\right)$$

Recall that we have the stratification

$$m_{\mu_\bullet}^{-1}\left(S_\nu \cap \mathrm{Gr}^{\leq \mu_\bullet}\right) = \bigcup_{|\nu_\bullet| = \nu} S_\nu \cap \mathrm{Gr}^{\leq \mu_\bullet}$$

and, after Lemma 10.1, we have an isomorphism

$$S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq \mu_\bullet} \simeq S_{\nu_1} \cap \mathrm{Gr}_{\leq \mu_1} \times \cdots \times (S_{\nu_n} \times \mathrm{Gr}_{\leq \mu_n}) \tag{4}$$

Further this isomorphism induced from the isomorphism of local triviality

$$\varpi^{\mu_1} L^{<0} G e_0 \cap \mathrm{Gr}_{\leq \mu_1}$$

$$R\Gamma_c(S_{\nu_\bullet} \cap \mathrm{Gr}^{\mu_\bullet}, \mathrm{IC}(\mathrm{Gr}^{\leq \mu_\bullet})) \simeq \bigotimes_{i=1}^n R\Gamma_c\left(S_{\nu_i} \cap \mathrm{Gr}^{\leq \mu_i}, \mathcal{A}_{\mu_i}\right)$$

Then result follows from Lem. 6.2 and Lem. 9.4.
Proof of theorem Thm. 4.2 Recall that the easy case when $\nu = \lambda$ was discussed after Lem. 6.2. We now prove the more difficult case $\nu \neq \lambda$.

The sequence $\mu_\bullet$, was chosen so that the multiplicity

$$V^{\lambda}_{\mu\bullet}$$

of $\mathcal{A}_\lambda$ in the decomposition 10.4,

$$\mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \simeq \bigoplus_{\xi \leq |\mu_\bullet|, \xi \in X_{\bullet,+}} \mathcal{A}_\xi \otimes V^{\xi}_{\mu\bullet}$$

We deduce the decomposition equality $V^{\lambda}_{\mu\bullet} \neq 0$ and that $\lambda \neq \nu$ to show that

$$R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^* \mathcal{L}_\psi)$$

it suffices to show that the canonical map

$$R\Gamma_c(S_\nu, \mathcal{A}_\nu \otimes h^* \mathcal{L}_\psi) \otimes V^{\nu}_{\mu_\bullet} \xrightarrow{\simeq} R\Gamma_c(S_\nu, \mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \otimes h^* \mathcal{L}_\psi)$$

which is a quasi isomorphism. Now from the discussion following lemma, 6.2, Combining this with the trivial case we have just proven in Thm 4.1,

$$R\Gamma_c(S_{\nu\bullet} \cap \mathrm{Gr}_{\mu\bullet})$$

Proposition 12.1. If $\sigma \notin X_{\bullet,+}$ we have that

$$R\Gamma_c(S_{\nu'}, \mathcal{A}_{\lambda'} \otimes h^* \mathcal{L}_\psi) = 0$$

Proof. Observe that the $\mathbb{G}_a$ action on S_ν is induced from the constant embedding

$$\mathbb{G}_a \hookrightarrow LN \circlearrowright LN$$

Let $\alpha \in \Phi$ be a simple root such that $\langle \alpha, \sigma \rangle$ is strictly negative. [14] The subgroups

$$\mathbb{G}_a := U_{\alpha, -(\alpha,\sigma)-1}$$

[14] This is the part where we needed σ to be nondominant, this guarantees the embedded copy of $\mathbb{G}_a$ is in the strict upper borel.

Example

Consider the case of GL_n,

- GL_n has root system type A_n. A choice of simple roots is $\{\alpha = e_i - e_{i+1} \,:\, 1 \le i \le n-1\}$. In otherwords
$$U_\alpha \hookrightarrow LN$$
as the offidiagonal entries.
- How does $\mathrm{ad}(t^\sigma)$ act? In the case $n=3$, we can see on the copy of $U_{e_2-e_3}$: for $A \in LN$,
$$ad(t^\sigma)A = \begin{pmatrix} t^{\sigma_1} & & \\ & t^{\sigma_2} & \\ & & t^{\sigma_3} \end{pmatrix} A \begin{pmatrix} t^{-\sigma_1} & & \\ & t^{-\sigma_2} & \\ & & t^{-\sigma_3} \end{pmatrix}$$
Thus on root subgroup $U_{2,3}$, $\mathrm{ad}(t^\sigma)$ scales the corresponding entry in by a power of t given by $\sigma_2 - \sigma_3 = \langle \alpha, \sigma \rangle$.
- Thus,
$$\mathbb{G}_a \simeq U_{\alpha, -\langle \alpha,\sigma\rangle -1} \to LN \xrightarrow{\mathrm{ad}(t^\sigma)} LN$$
embeds a copy of $t^{-1}\mathbb{G}_a$ into LN at position $(i, i+1)$. Under h, the map would be identity.

is contained in $L^{\ge 0}U$ thus act equivaraintly on $(S_\nu, \mathcal{A}_\lambda)$. Thus the restriction of h_σ to the subgroup induces the identity on $\mathbb{G}_a$.
This is equivalent to stating that the existence of commutative diagram.

$$\begin{tikzcd} \mathbb{G}_a \times S_\nu \arrow[r, hook] \arrow[d, "\mathrm{id}\times h_\sigma"] & LU \times S_\nu \arrow[r, "a"] & S_\nu \arrow[d, "h_\sigma"] \\ \mathbb{G}_a \times \mathbb{G}_a \arrow[rr, "a"] & & \mathbb{G}_a \end{tikzcd}$$

Via identifying S_ν as the orbit of $LN \circlearrowright \mathrm{Gr}_G$, this square is equivalent to

$$\begin{tikzcd} \mathbb{G}_a \times LN \arrow[r, hook] \arrow[d, "\mathrm{id}\times h_\sigma"] & LN \times S_\nu \arrow[r, "a"] & LN \arrow[d, "h_\sigma"] \\ \mathbb{G}_a \times \mathbb{G}_a \arrow[rr, "a"] & & \mathbb{G}_a \end{tikzcd}$$

where the bottom map is the additive map, and the upper map is the natural LN action on itself. This diagram implies

$$\mathrm{act}^* h_\sigma^* \mathcal{L}_\psi \simeq h_\sigma^* \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi$$

Thus by monoidality of act^*,

$$\begin{aligned} \mathrm{act}^* \left(\mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi\right) &\simeq \mathrm{act}^* \mathcal{A}_\lambda \otimes \mathrm{act}^* h_\sigma^* \mathcal{L}_\psi \\ &\simeq \mathrm{act}^* \mathcal{A}_\lambda \otimes (\mathrm{id} \times h_\sigma)^* a^* \mathcal{L}_\psi \\ &\simeq \mathrm{act}^* \mathcal{A}_\lambda \otimes (h_\sigma^* \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi) \end{aligned}$$

Now recall that the box tensor product satisfies

$$(A \otimes B) \boxtimes (C \otimes D) \simeq (A \boxtimes C) \otimes (B \boxtimes D)$$

It suffices to apply [**13**, Lemme 3.3]. □

Example

Note that the adjoint action $\mathrm{ad}(t^\sigma)$ on $U_{\alpha,\langle\alpha,\sigma\rangle-1} \hookrightarrow \mathrm{GL}_n$ precisely multiplies $t^{\langle\alpha,\sigma\rangle}$.

We deduce the vanishing

$$R\Gamma_c((S_{\nu_\bullet} \cap \mathrm{Gr}_{\mu\bullet}), \mathrm{IC}_{\mathrm{Gr}_{\leq\mu_\bullet}} \otimes h^*\mathcal{L}_\psi) = 0$$

for the case when $\nu_\bullet$ of which at least one of the partial sums σ_i are non dominant. Let us suppose now $\nu_\bullet$ where each $\nu_i \in \Omega(\mu_i)$ are such that the partial sums are dominant. We say a $\mu_\bullet$ path is of type $\nu_\bullet$ if it has vertices $0, \sigma_1, \ldots, \sigma_n$. Let us observe that the condition $\langle\alpha, \sigma\rangle \geq 1$ in 9.4 is equivalent the Putting together Lem. 6.3 and Lem. 9.4 we arrive the following: for $i \neq \langle 2\rho, \nu\rangle$, we have

$$H^i_c(S_{\nu\bullet} \cap \bar{\mathrm{G}}\mathrm{r}_{\mu_\bullet}, \mathrm{IC}(\bar{\mathrm{G}}\mathrm{r}_{\mu_\bullet}) \otimes h^*\mathcal{L}_\psi) = 0$$

and for $i = 2\langle\rho, \nu\rangle$ we have

$$\dim(V^\nu_{\mu\bullet}) \geq \dim H^i_c(S_\nu, \mathcal{A}_{\mu_1} * \cdots * \mathcal{A}_{\mu_n} \otimes h^*\mathcal{L}_\psi)$$

Recall that in the stratification

$$m_\bullet^{-1} = \textstyle\bigcup_{|\nu_\bullet|} S_{\nu\bullet} \cap \mathrm{Gr}_{\leq\mu_\bullet}$$

each point $(y_1, \ldots, y_n) \in S_{\nu\bullet} \cap \mathrm{Gr}_{\leq\mu_\bullet}$ can be written in the unique form, see 10.1,

$$\begin{aligned} y_1 &= x_1\varpi^{\nu_1}e_0 \\ &\cdots \\ y_n &= x_1\varpi^{\nu_1}\cdots x_n\varpi^{\nu_n}e_0 \end{aligned}$$

For each $\sigma \in X_\bullet$, we denote h_σ as the composition $LU \xrightarrow{\mathrm{ad}(\sigma)} LU \xrightarrow{h} \mathbb{G}_a$, so that $x \mapsto h(\mathrm{ad}(\sigma)x)$. It is clear that

$$h(y_n) = h(x_1) + h_{\sigma_1}(x_2) + \cdots + h_{\sigma_{n-1}}(x_n)$$

which uses the decompostion

$$y_n = x_1\mathrm{ad}(\varpi^{\sigma_1})x_2\cdots\mathrm{ad}(\varpi^{\sigma_{n-1}})x_n\varpi^{\sigma_n}$$

Bibliography

[1] Niven Achenjang, *18.737, Algebraic Groups, notes* (2021).

[2] N Bourbaki, *Groupes et algébres de Lie Chap. IV-VI et VII-VIII* (2007).

[3] Simon Riche, *Geometric Satake* (2018).

[4] Pierre Deligne, *VariÃľtÃľs de Shimura: interprÃľtation modulaire et techniques de construction de modèles canoniques* (1997).

[5] Yunxia Chen, *quasi-minuscule En bundles and level one ...* (2015).

[6] Frenkel, D Gaitsgory, D Kazhdan, and K Vilonen, *Geometric Realization of Whittaker Functions and the Langlands Conjecture* (1998).

[7] B Gross, *On the Satake isomoprhism* (1998), available at `https://people.math.harvard.edu/~gross/preprints/sat.pdf`.

[8] Benedict H Gross, *On minuscule representations and the principal SL2*, Represent. Theory **4** (2000), no. 200, 225–244.

[9] Peter Littleman, *A Littlewood-Richardson rule for symmetrizable Kae-Moody algebras* (1994).

[10] Yves Laszlo and Christopher Sorger, *The line bundles on the moduli of parabolic G-bundles over curves and their sections* (1997).

[11] G Lusztig, *Singularities, character formulas, and a q-analog of weight mul- tiplicities* (1982).

[12] Mirkovic and Vilonen, *Perverse Sheaves on Loop Grassmannians and Langlands Duality* (1997).

[13] Ngô, *Preuve d'une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen* (1998).

[14] Ngô and P. Polo, *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique* (2000).

[15] Masao Oi and Ryotaro Sakamoto and Hiroyoshi Tamori, *Iwahori-Hecke algebra and unramified local L-functions* (2022).

[16] Tim Peerenboom, *The Affine Grassmannian with a View Towards Geometric Satake* (2021).

[17] T.A Springer, *Schubert varieties and generalizations.*

[18] Xinwen Zhu, *An introduction to affine Grassmannians and the geometric Satake equivalence* (2015).

[19] ———, *Affine Grassmannians and the geometric Satake in mixed characteristic* (2014).