LECTURE B.4: TAYLOR-WILES METHOD FOR GL_1 IN THE GENERAL CASE $\ell_0 > 0$

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In this talk, we explain the Calegari-Geraghty modifications to the Taylor-Wiles method for the group GL_1. The main references are [C, Part II], [GV, Section 13], and [CG, Section 8].

1. Review of the $\ell_0 = 0$ case

1.1. Set up. We recall the set up from David's talk:

- F is a number field,
- $\mathcal{O}$ is the valuation ring of a finite extension of $\mathbb{Q}_p$, and k its residue field,
- Q is a finite set of primes of F, none of which divide p,
- $\ell_0 = r_1 + r_2 - 1$ is the rank of the unit group of F,
- $\mathrm{RCl}(Q)$ is the ray class group of F with modulus Q,
- Γ_Q is the Galois group of the maximal abelian unramified-outside-Q extension of F,
- $R_Q = \mathcal{O}[\![\Gamma_Q]\!]$ is the unramified-outside-Q deformation ring of the trivial character of G_F,
- $\mathbb{T}_Q$ is the Hecke algebra acting on the space $X_Q = F^\times \backslash \mathbb{A}_F^\times / U_Q$, localized at the appropriate maximal ideal.

We assume that we are given a deformation of the trivial character with values in $\mathbb{T}_Q$ that satisfies a local-global compatibility condition. In particular, we have a homomorphism $R_Q \to \mathbb{T}_Q$.

As in Matt's talk, we can think of $\mathrm{Spec}(R_Q)$ as the space of deformations, and the subset

$$\mathrm{Spec}(\mathbb{T}_Q) \subset \mathrm{Spec}(R_Q)$$

as being the deformations that come from automorphic forms. Our goal is to show that $R_Q \cong \mathbb{T}_Q$, which is to say that all deformations come from automorphic forms.

As David explained, the Hecke algebra $\mathbb{T}_Q$ is the group ring $\mathcal{O}[\mathrm{RCl}(Q)]$ of the ray class group. So the isomorphism $R_Q \cong \mathbb{T}_Q$ amounts to the isomorphism of $\Gamma_Q \cong \mathrm{RCl}(Q)$ of global class field theory. Our proof of this isomorphism is ultimately circular, because we use global duality for Galois cohomology, which requires the full strength of class field theory. The purpose is to illustrate the Calegari-Geraghty method in the simplest case possible.

1.2. The Taylor-Wiles method. We review the Taylor-Wiles method from David's talk, paying special attention to the role of the assumption that ℓ_0 is zero. We specialize to the minimal case and take $R = R_\emptyset$ and $\mathbb{T} = \mathbb{T}_\emptyset$.

One key idea of the Taylor-Wiles method is that we know different kinds of properties about R and about $\mathbb{T}$; if they are to be equal, then both must have both kinds of properties. The properties we know are:

(1): $\mathbb{T}$ is a finite free $\mathcal{O}$-module.

(2): R has as many generators as relations.

Property **(1)** is self-explanatory. To understand property **(2)** we use a variant of the presentation proved in Paulina's talk:

$$R \cong \frac{\mathcal{O}[\![x_1, \dots, x_{h^1_\emptyset}]\!]}{(f_1, \dots, f_{h^2_\emptyset})}. \tag{1.1}$$

Here $h^1_\emptyset$ is the dimension of the Selmer group $H^1_\emptyset(G_F, k)$ from David's talk and $h^2_\emptyset$ is the dimension of a "higher Selmer group" that keeps track of obstructions both to lifting and to making the lift be unramified. Global duality implies that $h^2_\emptyset = \dim_k H^1_{\emptyset^*}(G_F, k(1)) =: h^1_{\emptyset^*}$, and, as in David's talk, the Greenberg-Wiles Euler characteristic formula gives

$$h^1_\emptyset - h^2_\emptyset = -\ell_0.$$

Using the assumption that $\ell_0 = 0$ and (1.1), we see **(2)**.

If $R = \mathbb{T}$, then **(1)** and **(2)** are true, so both R and $\mathbb{T}$ should be LCI rings. It is then natural to try to find a smooth ring in which they are cut out by a regular sequence. The Taylor-Wiles method produces this smooth ring by "patching" together R_Q's for auxiliary sets Q, carefully chosen so that $h^2_Q = 0$, and the regular sequence is provided by the diamond operators for these auxiliary levels. The key point to showing these sequences are regular is:

(1)': $\mathbb{T}_Q$ is a finite free module over the diamond operator algebra $S_Q = \mathcal{O}[\Delta_Q]$.

1.3. **What does wrong when $\ell_0 > 0$?** If $\ell_0 > 0$, the two key properties **(2)** and **(1)'** both fail! We have see why **(2)** fails: $h^1_\emptyset - h^2_\emptyset = -\ell_0$, so R as ℓ_0-many more generators than relations – it has "expected dimension $-\ell_0$".

To see why **(1)'** fails, note that, in order to make h^2_Q vanish, one has to have $h^2_\emptyset = h^1_\emptyset + \ell_0$ many primes in the set Q. The patched diamond operator algebra S_∞ has Krull dimension $\#Q + 1 = h^1_\emptyset + \ell_0 + 1$. But the presentation of R implies that the patched Hecke algebra $\mathbb{T}_\infty$ can have Krull dimension at most $h^1_\emptyset + 1$. Hence $\mathbb{T}_\infty$ cannot be finite free over S_∞.

To resolve this issue, we need to find a replacement for **(1)'** that reflects our expectation that R and $\mathbb{T}$ have expected dimension $-\ell_0$. Inspired by derived algebraic geometry, we study this overly intersected situation by replacing modules with complexes.

2. The general case $\ell_0 > 0$

The freeness in **(1)'** comes the definition of $\mathbb{T}_Q$ as an algebra of endomorphisms of a module $M_Q = H^0(X_Q, \mathcal{O})$ and the fact that M_Q is free as a S_Q-module. The idea is to replace M_Q by a complex of free S_Q-modules.

2.1. Commutative algebra.

To know what properties we want from the complex, we need the following lemma from commutative algebra.

Key Lemma. *Let $\delta \geq 0$ and let $A \to B$ be a morphism of regular local rings with $\dim(B) = \dim(A) - \delta$. Let $C^\bullet$ be a complex of finitely generated free A-modules supported in degrees $[0, \delta]$, and suppose that the A-action on $H^*(C^\bullet)$ factors through B. Then $H^*(C^\bullet)$ is concentrated in degree δ and $H^\delta(C^\bullet)$ is a free B-module.*

We apply this lemma to the case:

- $A = S_\infty$, the patched diamond-operator algebra,
- $B = \mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]$, thought of as the numerator in the presentation (1.1) for the patched deformation ring R_∞,
- $\delta = \ell_0$,
- $C^\bullet = M^\bullet_\infty$, the new complex we need to construct.

In addition to $M^\bullet_\infty$ being a complex of finite free S_∞-modules in degrees $[0, \ell_0]$, we want the action of S_∞ on $H^*(M^\bullet_\infty)$ to factor through the maps

$$S_\infty \to \mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!] \twoheadrightarrow R_\infty \twoheadrightarrow \mathbb{T}_\infty. \tag{2.1}$$

The complex $M^\bullet_\infty$ should also have a base change property:

$$M^\bullet_\infty \otimes^{\mathbb{L}}_{S_\infty} \mathcal{O} \simeq M^\bullet_\emptyset. \tag{2.2}$$

for a complex $M^\bullet_\emptyset$ at minimal level (independent of the patching data).

Theorem. *Assume there is a complex $M^\bullet_\infty$ with the above properties. Then the map $R \to \mathbb{T}$ is an isomorphism, and there is an isomorphism*

$$H^*(M^\bullet_\emptyset) \cong \mathrm{Tor}^{S_\infty}_*(M_\infty, \mathcal{O})$$

for a free $\mathbb{T}_\infty$-module M_∞.

Proof. By the Key Lemma, the cohomology of the complex $M^\bullet_\infty$ is concentrated in degree ℓ_0 and $H^{\ell_0}(M^\bullet_\infty)$ is a free $\mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]$-module. Since the $\mathcal{O}[\![x_1, \ldots, x_{h^1_\emptyset}]\!]$-action factors through the surjections in (2.1), we see that $H^{\ell_0}(M^\bullet_\infty)$ is a free $\mathbb{T}_\infty$-module, and that those surjections are isomorphisms. This implies that $R \to \mathbb{T}$ is an isomorphism by base-change.

Letting $M_\infty = H^{\ell_0}(M^\bullet_\infty)$, we see that there is a quasi-isomorphism $M^\bullet_\infty \simeq M_\infty[\ell_0]$. Now applying the base-change isomorphism (2.2) and taking cohomology, we get the desired description of $H^*(M^\bullet_\emptyset)$. □

2.2. **The complex $M^\bullet_\infty$.** The complex $M^\bullet_\infty$ is constructed by patching. For each choice of auxiliary prime set Q, we need

- a complex $M^\bullet_Q$ of finite free S_Q-modules supported in degrees $[0, \ell_0]$,
- a Galois action and Hecke action on $H^*(M^\bullet_Q)$ that are compatible with the given homomorphism $R_Q \to \mathbb{T}_Q$ (in other words, we want the $\mathbb{T}_Q$-valued deformation come from $H^*(M^\bullet_Q)$).

The complex $M^\bullet_Q$ is defined using the singular cohomology complex of the space X_Q. The component group of X_Q is $\mathrm{RCl}(Q)$, and, as explained in Matt's talk, the connect component of the identity is $\mathbb{R} \times (S^1)^{\ell_0}$. The Hecke action comes from the component group. The fact the singular cohomology complex can be represented by a prefect complex of S_Q-modules supported in degrees $[0, \ell_0]$ follows formally from the fact that the cohomological dimension of $(S^1)^{\ell_0}$ is ℓ_0 and that $X_Q \to X_\emptyset$ is a covering space with Galois group Δ_Q.

References

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