Lecture D.4: This is the concluding lecture

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Our Tasks

1. State the *Main Theorem* (14.1 of [G-V])

$$\boxed{\pi_*\mathcal{R}_0 \xrightarrow{\sim} \mathrm{Tor}_*^{S_\infty^\circ}(R_\infty, W).} \quad H_*$$

emphasizing the conjectures it depends on.

2. Mention the *Independence Theorem* (§15 of [G-V])
3. Present a high-level outline of the proof of the Main Theorem.
4. Get into more details relating the derived deformation ring $\mathcal{R}_0$ to the objects of the obstructed Taylor–Wiles method
   - Two technical theorems from §§11-12 of [G-V].

Recollections toward stating the Main Theorem

Base level setting over a number field F:

- A base level K_0 in GL_d/F that is ramified only at $T = S \setminus \{\text{primes over } p\}$
- The "Taylor–Wiles defect" $\ell_0 = r_1(F)\lfloor \frac{d+1}{2} \rfloor + r_2(F)d - 1 \rightsquigarrow$ Lecture D.2.
- $W = W(k)$ with finite residue field k of characteristic p $\mathbb{Z}_p = W(\mathbb{F}_p)$
- The adelic quotient arithmetic manifold $Y_0 = Y(K_0)$ associated to K_0.

  $G = \mathrm{Res}^F_{\mathbb{Q}} \mathrm{GL}_d$ $G(\mathbb{Q}) \backslash G(\mathbb{A}) / A_\infty K_\infty K_0$
- An "anemic" Hecke algebra $\tilde{T}$ acting on $C_*(Y_0, W)$ (in the derived category)

  $\hookrightarrow$ unramified $\mathrm{GL}_d(\mathcal{O}_{F_v}) \subset \mathrm{GL}_d(F_v) \rightsquigarrow$ "$T_{v,i}$".
- A residual Hecke eigensystem $\tilde{T} \twoheadrightarrow k$, with kernel $\mathfrak{m}$
- A Galois representation $\bar{\sigma} : \pi_1 \mathcal{O}_F[1/S] \to \mathrm{GL}_d(k)$ compatible with $\mathfrak{m} \rightsquigarrow$ from Lectures B.2 and D.1 F: totally real, CM. $\to$ abs. irred.
- A *motivically unramified* deformation problem for $\bar{\sigma}$, represented by $\mathcal{R}_0 \in \text{pro-Art}_k \rightsquigarrow$ from Lecture C.4 and D.2
  - Crystalline on $\pi_1 F_v$ $(v|p)$, Hodge-Tate: $\{0, -1, \ldots, -d+1\}$.

    No more ram. than $\bar{\sigma}$ at $v \in T$.

Recollections toward stating the Main Theorem

Taylor–Wiles deformation setting:

- For $n \in \mathbb{Z}_{\geq 1}$, sets of *allowable* Taylor–Wiles primes Q_n, $\#Q_n = q$ constant

  $v \in Q_n$ $\quad N(v) \equiv 1 \ (p^n)$

- Levels $K_n = K_1(Q_n) \subset K_0 \approx K_0(Q_n)$, with $K_0(Q_n)/K_n \twoheadrightarrow \Delta_n = (\mathbb{Z}/p^n\mathbb{Z})^{\oplus s}$,

  difference at Q_n $\quad \Delta_n \subseteq \prod_{v \in Q_n} \mathbb{F}_v^\times \quad s = dq.$

- A Hecke algebra $\tilde{T}_{K_n}$ coming from Hecke actions on $C_*^\Delta(Y(K_n), W)$ in the derived category of $W[\Delta]$-modules
- The residual eigensystem at base level $\tilde{T} \twoheadrightarrow k$ induces a residual eigensystem at level K_n coming from $\tilde{T}_{K_n} \to \tilde{T}$.
- Galois representations $\sigma_n : \pi_1 \mathcal{O}_F[1/SQ_n] \to \mathrm{GL}_d(\tilde{T}_{K_n})$, lifting $\bar{\sigma}$ (Lecture B.2, D.1), that satisfy various local-global compatibility conditions.
- (Classical) Galois deformation rings R_n for Taylor–Wiles level n, with a map from the group algebra of Δ_n corresponding to the torus-valued inertia action at primes in Q_n

  allow ram @ Q_n.

$$\mathrm{S}_n^\circ := W[\![\Delta_n]\!] \longrightarrow \mathrm{R}_n \xrightarrow{\rho \nearrow A} \tilde{T}_{K_n}$$

$R_n = \pi R_m.$ $\quad \rho|_{I_v} : I_v \twoheadrightarrow \Delta_n \to T(A) \subset GL_d(A).$ $\quad \underline{R_n = R_{S \cup Q_n}}.$

Recollections toward stating the Main Theorem

Classical patching in the obstructed Taylor–Wiles method (Lecture B.3):

- Power series rings $\mathrm{S}_\infty^\circ = W[\![X_1, \ldots, X_s]\!]$, $\mathrm{R}_\infty = W[\![x_1, \ldots, x_{s-\ell_0}]\!]$
- Ideals $\mathfrak{a}_n := (p^n, (1+X_i)^{p^n} - 1) \subset \mathrm{S}_\infty^\circ \rightsquigarrow$ S_n. R_n. $h^1_{S \cup Q_n}$.
- The patching diagram

$\bar{A}_n := A_n/p^n$.

$$\begin{tikzcd}
S_\infty^\circ \arrow[r, "\sim"] \arrow[d] & R_\infty \arrow[r] \arrow[d] \arrow[l, phantom, "\leftarrow \text{mod } \mathfrak{a}_n \rightarrow" below] & \pi_0 R \arrow[d] \\
S_n^\circ \arrow[r] & R_n \arrow[r, "\otimes_{S_\infty} W"] & \pi_0 R_0/(p^n, m^{k(n)})
\end{tikzcd}$$

kills off ram @ Q_n.

- The patched S_∞°-perfect complex $\mathrm{D}_\infty \rightsquigarrow$ Lecture D.3: Conds [Thm 13.1, G-V].

$H^q(\mathcal{O}_\xi) = H^q(\mathcal{O}_\infty)$: free R_∞-mod. $\qquad D_\infty \otimes_{S_\infty^\circ} W \simeq H_*(Y_0, W)$.

No cong: $H^1_f(\mathrm{Ad}\,\bar{\rho}) = 0$.

$S_\infty^\circ \to R_\infty$ (R_∞ fin gen S_∞°-mod).

$\pi_0 R_0 = W = \mathbb{T}$

Theorem (Calegari–Geraghty)

$H_*(Y_0, W)_\mathfrak{m}$ is a free graded module over $\mathrm{Tor}_*^{\mathrm{S}_\infty^\circ}(\mathrm{R}_\infty, W)$, with generators in degree q. If there are "no congruences," then this graded ring is $\simeq \wedge^*(W^{\oplus \ell_0})$.

§1. Statement of the Main Theorem

Theorem (Galatius–Venaktesh)

There is an isomorphism of graded algebra $\pi_*\mathcal{R}_0 \xrightarrow{\sim} \mathrm{Tor}_*^{S^\circ_\infty}(R_\infty, W)$. *In particular, the* $\mathfrak{m}$*-part of the homology of* Y_0 *is a free graded module over* $\pi_*\mathcal{R}_0$.

Upshot: T–W primes realize classically a derived picture
that's intrinsic to a reln : Gal reps $\leftrightarrow$ H_*(arith groups)

Note: "No congr" not req'd.

We emphasize these questions:

Question 1. What conjectures are we working under?

Question 2. To what extent is the action $\pi_*\mathcal{R}_0 \curvearrowright H_*(Y_0, W)_\mathfrak{m}$ canonical?

Answer 1. Conjectures about existence of Galois representations and local-global compatibility.

ρ_n val in $\mathbb{T}_{K_n}$.

- $\rho_n|_v : \pi_1 F_v \to GL_d$ crystalline
- unram elsewhere
- complex conj.

p large.

Existence & smoothness of crys def functors.

§2. The *Independence Result*

There is a multitude of choices to set up $\pi_\bullet \mathcal{R}_0 \xrightarrow{\sim} \mathrm{Tor}_\bullet^{\mathrm{S}_\infty^\circ}(\mathrm{R}_\infty, W)$.

Question 2: To what extent is it canonical? → studying $\curvearrowright H_*(Y_0, W)$.

Answer 2, under a "no congruences" condition "$\mathbb{T} = W$": $H^1_f(\mathrm{Ad}\,\bar{\sigma}) = 0$.

$$\text{Let} \quad V := \underset{\text{"}H_f^\bullet\text{"}}{H_f^2(\mathbb{Z}[1/S], \mathrm{Ad}\,\sigma)} \cong H_f^1(\mathbb{Z}[1/S], (\mathrm{Ad}^*\,\sigma)(1))^\vee \quad \text{of } W\text{-rank } \ell_0.$$

W-linear dual.

- Using a derived Hecke algebra, Venkatesh constructed a free action

$$\wedge^\bullet V \curvearrowright H^\bullet(Y_0, W)_{\mathfrak{m}}.$$

V increases deg +1.

- We can also draw a graded isomorphism $\pi_\bullet \mathcal{R}_0 \cong \wedge^\bullet(V^\vee)$, using that

$$\text{no congs.} \Rightarrow \mathrm{t}_{\mathrm{S}_\infty^\circ}/\mathrm{t}_{\mathrm{R}_\infty} \cong V.$$

/W.

Recall, the Main Theorem gave us $\wedge^\bullet(V^\vee) \cong \pi_\bullet \mathcal{R}_0 \curvearrowright H_\bullet(Y_0, W)_{\mathfrak{m}}$.

Theorem (§15 of G-V) → at least w/ free.

These actions are compatible, which implies that one determines the other.

Adjoint-compatibility of exterior actions

Theorem (§15 of G-V)

The actions $\wedge^\bullet V^\vee \curvearrowright H_\bullet(Y_0, W)_{\mathfrak{m}}$ *and* $\wedge^\bullet V \curvearrowright H^\bullet(Y_0, W)_{\mathfrak{m}}$ *are* *compatible.*

What is compatibility?

1. First use Poincaré duality: $\wedge^\bullet V \curvearrowright H^\bullet(Y_0, W)_{\mathfrak{m}} \rightsquigarrow \wedge^\bullet V \curvearrowright H_\bullet(Y_0, W)_{\mathfrak{m}}$
   v. decrease deg −1.
2. We have *compatibility* when the two actions $\wedge^\bullet V, \wedge^\bullet(V^\vee) \curvearrowright H_\bullet$ satisfy

$$v \cdot w \cdot h + w \cdot v \cdot h = \langle v, w \rangle \cdot h$$

for $v \in V$, $w \in V^\vee$, $h \in H_\bullet$.

Example. The natural actions $\wedge^\bullet V, \wedge^\bullet(V^\vee) \curvearrowright \wedge^\bullet(V^\vee)$.

A

Remark: degrees, and quasi-free presentation (over $L = W[1/p]$).

$\sigma : \pi_1 F \to GL_d(F)$ $\qquad$ $R_\sigma \simeq (\widehat{\mathrm{Sym}}(\Sigma_i H^i_f(\mathrm{Ad}\,\sigma)^\vee), d)$

Expect: $H^1_f(\mathrm{Ad}\,\sigma) = 0 \rightsquigarrow \simeq \widehat{\mathrm{Sym}}\, \Sigma_2 H^2_f{}^\vee \simeq \wedge^\bullet (H^2_f)^\vee$ & comm DGA.

§3. High-level proof outline: $\pi_*\mathcal{R}_0 \xrightarrow{\sim} \mathrm{Tor}_*^{\mathrm{S}_\infty^\circ}(\mathrm{R}_\infty, W)$

Notation:

- $\mathcal{R}_n$ / R_n: derived / classical deformation rings of Taylor–Wiles level n
- $\mathcal{S}_n^\circ$ / S_n°: derived / classical def. rings for inertia at Q_n.
- $\overline{\mathrm{R}}_n, \overline{\mathrm{S}}_n^\circ$, W/p^nW: reduction modulo p^n
- All tensor products (of classical rings) are derived

Study the composition:

R_∞

mod $\mathfrak{a}_n$,

$$\mathcal{R}_0 \simeq \mathcal{R}_n \otimes_{\mathcal{S}_n^\circ} W \longrightarrow \mathrm{R}_n \otimes_{\mathrm{S}_n^\circ} W \xrightarrow{\text{mod } p^n} \overline{\mathrm{R}}_n \otimes_{\overline{\mathrm{S}}_n^\circ} W/p^nW$$

limit.

kill ram @ Q_n → levels.

S_∞°

Steps

1. Prove map on t^0 is isom. and t^1 is surjective, for any $n \in \mathbb{Z}_{\geq 1}$
   - Surjectivity: target is "more obstructed" than $\mathcal{R}_0$
2. Compare Euler characteristics after $\boxed{\textit{taking the limit over } n}$
   - There is enough commutation with the limit that the target Euler char. is $\dim \mathrm{t}^0\mathrm{R}_\infty - \dim \mathrm{t}^0\mathrm{S}_\infty^\circ = -\ell_0$.

     → $R_\infty \otimes_{S_\infty^\circ} W$. ← $\mathcal{R}_0$.

   - ... which matches the source Euler characteristic ⇝ Lecture D.2, $\dim H^1_f(\mathbb{Z}[1/S], \mathrm{Ad}\,\rho) - \dim H^2_f(\mathbb{Z}[1/S], \mathrm{Ad}\,\rho) = -\ell_0$.
   - so the resulting isom. on $\mathrm{t}^\bullet$ induces an equivalence of formally cohesive $\mathcal{R}_0 \xrightarrow{\sim} \mathrm{R}_\infty \otimes_{\mathrm{S}_\infty^\circ} W$ ⇝ Lecture C.3.

§4. Two major steps of the proof. Step 1.

$\pi_1 F_v$ $\qquad$ $v \in Q_n.$

Use: Derived refinements $\otimes_{\mathcal{S}_n}\mathcal{S}_n^{\mathrm{ur}}$ of $\otimes_{\mathrm{S}_n}\mathrm{S}_n^{\mathrm{ur}}$, which replaces $\otimes_{\mathrm{S}_n^\circ} W$.

$\pi_1 F_v.$ kill inert $\qquad$ inertia @ Q_n. $\qquad$ kill inertia @ Q_n

Theorem (11.1 of [G-V])

$$\mathcal{R}_0 \; (\simeq \mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}\,) \to \overline{\mathrm{R}}_n \otimes_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}$$

induces an isom. on $\mathfrak{t}^0$ and a surjection on $\mathfrak{t}^1$.

$$\begin{tikzcd}
\mathfrak{t}^0(\overline{\mathrm{R}}_n \otimes_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}) \arrow[r] \arrow[d, "2"] & \mathfrak{t}^0(\overline{\mathrm{R}}_n) \oplus \mathfrak{t}^0(\overline{\mathrm{S}}_n^{\mathrm{ur}}) \arrow[r] \arrow[d, "2"] & \mathfrak{t}^0(\overline{\mathrm{S}}_n) \arrow[r] \arrow[d, "2"] & \mathfrak{t}^1(\overline{\mathrm{R}}_n \otimes_{\overline{\mathrm{S}}_n} \overline{\mathrm{S}}_n^{\mathrm{ur}}) \arrow[r] \arrow[d] & \mathfrak{t}^1(\overline{\mathrm{R}}_n) \oplus \mathfrak{t}^1(\overline{\mathrm{S}}_n^{\mathrm{ur}}) \arrow[d] \\
\mathfrak{t}^0(\mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}) \arrow[r] & \mathfrak{t}^0(\mathcal{R}_n) \oplus \mathfrak{t}^0(\mathcal{S}_n^{\mathrm{ur}}) \arrow[r] & \mathfrak{t}^0(\mathcal{S}_n) \arrow[r] & \mathfrak{t}^1(\mathcal{R}_n \otimes_{\mathcal{S}_n} \mathcal{S}_n^{\mathrm{ur}}) \arrow[r] & \mathfrak{t}^1(\mathcal{R}_n) \oplus \mathfrak{t}^1(\mathcal{S}_n^{\mathrm{ur}}) \arrow[d, "2"] \\
 & & & & \mathfrak{t}^1(\mathcal{S}_n).
\end{tikzcd}$$

"0." $\qquad$ "allowable". $\to$

§4, Step 2: A compactness argument $\rightsquigarrow$ derived patching

Setup. Recall the approximation

$$\mathcal{R}_0 \simeq \mathcal{R}_n \otimes_{\mathcal{S}_n} W \longrightarrow \overline{\mathrm{R}}_n \otimes_{\overline{\mathrm{S}}_n} W/p^n W = (\mathrm{R}_\infty/\mathfrak{a}_n) \otimes_{\mathrm{S}_\infty/\mathfrak{a}_n} W/p^n W =: \mathcal{C}_n.$$

Theorem (12.1 of [G-V])

The pro-objects $\mathcal{R}_0$ and $(n \mapsto \mathcal{C}_n)$ of Art_k represent equivalent functors.

From this we can deduce the main theorem:

$$\pi_\bullet \mathcal{R}_0 \cong \varprojlim \pi_\bullet \mathcal{C}_n = \varprojlim \mathrm{Tor}_\bullet^{\mathrm{S}_\infty^\circ/\mathfrak{a}_n}(\mathrm{R}_\infty/\mathfrak{a}_n, W/p^n) = \mathrm{Tor}_\bullet^{\mathrm{S}_\infty^\circ}(\mathrm{R}_\infty, W).$$

We have a bunch of maps $f_n : \mathcal{R}_0 \longrightarrow \mathcal{C}_n$. Using presentations of $\mathrm{R}_\infty/\mathfrak{a}_n \xrightarrow{\sim} \overline{\mathrm{R}}_n$ and $\overline{\mathrm{S}}_\infty/\mathfrak{a}_n \xrightarrow{\sim} \overline{\mathrm{S}}_n$, we *also* have, for $n > m$,

$$e_{n,m} : \mathcal{C}_n \twoheadrightarrow \mathcal{C}_m,$$

so we have composites

$$f_{n,m} : \mathcal{R}_0 \xrightarrow{f_n} \mathcal{C}_n \xrightarrow{e_{n,m}} \mathcal{C}_m.$$

t^0 isom

t^1 surj.

We want to extract a map $\mathcal{R}_0 \longrightarrow \boxed{\varprojlim} \mathcal{C}_n \simeq \mathrm{R}_\infty \otimes_{\mathrm{S}_\infty^\circ} W$.

$\rightsquigarrow$ Once we have this map, it's an equivalence by the argument (outline) above!

Topological setup for derived patching

1. Enrich pro-Art_k over sSets:

$$\text{pro-Art}_k(A, B) = \lim_i \operatorname{colim}_j \mathrm{Art}_k(A_j, B_i)$$

2. In nice cases, we can understand pro-$\mathrm{Art}_k(A, B)$ well.

def

$$\operatorname{holim}_i \operatorname{colim}_j \mathrm{Art}_k(A_j, B_i) \xrightarrow{\sim} \text{pro-Art}_k(A, B).$$

$$\text{assuming} \begin{cases} \textbf{Cofibrant} & A_i \text{ and } B_i, \\ \textbf{Fibrations} & B_j \to \varprojlim_{i<j} B_i \end{cases}$$

3. Assuming $\mathfrak{t}^\bullet A$ is finite-dimensional, pro-$\mathrm{Art}_k(A, B_i)$ has finite π_i.

4. Write $[A, B'] := \pi_0(\text{pro-Art}_k(A, B'))$. Apply to $B' = B$, $B' = B_i$ and get

$$[A, B] = \lim_i \underbrace{[A, B_i]}_{\text{finite}}.$$

profinite.

Apply it! $\boxed{[A, B] = \lim_i [A, B_i]}$

Application: $\underline{A = \mathcal{R}_0}$, $B = \mathcal{C}_n := (\mathrm{R}_\infty/\mathfrak{a}_n) \otimes_{\mathrm{S}^\circ_\infty/\mathfrak{a}_n} W/p^n W$.

Let $B_i \simeq \tau_{\leq i} \left((\mathrm{R}_\infty/\mathfrak{a}_n) \otimes_{(\mathrm{S}^\circ_\infty/\mathfrak{a}_n)} c(W/p^n W) \right)$

Recall from Lecture C.2: $B_{i+1} = B_i \underbrace{\times^h_{k \oplus k[i+2]} k}_{\text{finitely many times}}$

So because $\mathfrak{t}^\bullet \mathcal{R}_0 \neq 0 \implies \bullet \in \{0, 1\}$, we get

$$[\mathcal{R}_0, B_i] = \underline{[\mathcal{R}_0, B_{i+1}]} \qquad \text{for } i \geq 1.$$

Upshot: $[\mathcal{R}_0, \mathcal{C}_n]$ is finite !

$= [\mathcal{R}_0, B_1]$.

The subsets of the finite sets $[\mathcal{R}_0, \mathcal{C}_n]$ that we care about

Let $X_n := \left\{ [f] \in [\mathcal{R}_0, \mathcal{C}_n] : \mathrm{t}^\bullet f \text{ is } \begin{cases} \textit{isom.} & \text{for } \bullet = 0 \\ \textit{surj.} & \text{for } \bullet = 1 \end{cases} \right\}$

We can make $(X_n)_{n \geq 1}$ an inverse subsystem of $[\mathcal{R}_0, \mathcal{C}_n]_{n \geq 1}$ using $e_{n,n-1} : \mathcal{C}_n \twoheadrightarrow \mathcal{C}_{n-1}$. preserves: $X_n \to X_{n-1}$.

$R_\infty \to R_n$
$S_\infty^\circ \to S_n^\circ$.

Topology (compactness): Because the $[\mathcal{R}_0, \mathcal{C}_n]$ are finite and $[f_n] \in X_n$, the limit

$$\lim_n X_n$$

non-empty!

is **non-empty!** This implies:

- there are $g_n : \mathcal{R}_0 \to \mathcal{C}_n$ and paths (1-simplices) connecting

$$e_{n+1,n} \circ g_{n+1} \rightsquigarrow g_n.$$

- We have a map of deformation functors

$$\operatorname{hocolim}_n \operatorname{Hom}(\mathcal{C}_n, -) \to \operatorname{Hom}(\mathcal{R}_0, -).$$

$R_\infty \otimes_{S_\infty} W$.
$\uparrow 2$
$\mathcal{R}_0 \Rightarrow \lim \mathcal{C}_n$.
$\mathcal{C}_n = \bar{R}_n \otimes_{\bar{S}_n^\circ} W/p^n$.

- Map in hand, the summary proof above establishes weak equivalence. Done!