Honors Single Variable Calculus 110.113

October 6, 2023

Contents

Just another day of group work

Find yourself ≥ 2 friends. We will review the content of class by working through problems. The following problems is to give you intuition on homework problem 4.

Rules: You may do Q1,4,5 ignoring Q2,3. If I receive no attempts for Q4 points will be deducted.

1. Recall the definition of what it means for a sequence of real numbers to converge. The definition should begin with "There exists some real number $L \in \mathbb{R}$..."

2. Guess the limit of the following sequences. We will be doing these rigorously, but its good to have intuition. Don't spend too long.

   (a) $(\frac{n!}{n^n})_{n=0}^{\infty}$.

   (b) $(\sqrt[n]{n})_{n=1}^{\infty}$.

   (c) $(a^n)_{n=0}^{\infty}$ for $0 < a < 1$.

   (d)
   $$\left(\frac{n}{n+1} - \frac{n+1}{n}\right)_{n=1}^{\infty}$$

   (e)
   $$\left((-1)^n \frac{\sqrt{n}\sin(n^n)}{n+1}\right)_{n=1}^{\infty}$$

3. Which of the following sequences converge to 0

   (a)
   $$\left(\frac{1}{n+1} + \cdots \frac{1}{2n}\right)_{n=1}^{\infty}$$

(b)

$$\left(\frac{1}{n^2} + \cdots \frac{1}{(2n)^2} \right)_{n=1}^{\infty}$$

Bonus: what are the limits?

Of all sequences $\mathrm{Fct}(\mathbb{N}, \mathbb{R}) := \{(a_n)_{n=0}^{\infty} : a_n \in \mathbb{R}\}$ let $\mathrm{Cvg}(\mathbb{N}, \mathbb{R})$ denote the set of converging sequences.

4 Let $L \neq L'$ be distinct real numbers. Such that we cannot have both

$$\lim_{n\to\infty} a_n = L \textit{ and } \quad \lim_{n\to\infty} a_n = L'$$

Let (a_n) be a sequence of real numbers. The notation means that $\lim_n a_n = L$ means "a_n converges to L "

5 The previous problem shows that we can define a function

$$\lim_{n\to\infty} : \mathrm{Cvg}(\mathbb{N}, \mathbb{R}) \to \mathbb{R}$$

The next two problems show that this function is well behaved: Let (a_n) (b_n) be two converging sequences.

(a) Additive.[1] Show that

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$$

(b) Multiplicative. Show that

$$\lim_{n\to\infty} (a_n b_n) = \left(\lim_{n\to\infty} a_n \right) \cdot \left(\lim_{n\to\infty} b_n \right)$$

Hint: check out Tao.

[1] In general, a function $f : X \to \mathbb{R}$ is additive if $f(x+y) = f(x) + f(y)$, when X is a set with a notion of addition $+$.