$(\mathfrak{g},K)$-cohomology of tempered reps.

Apr 5, 21

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I. Setup

G connected reductive gp $/\mathbb{R}$. SL_n GL_n $U_{p,q}$

(slightly more general)

$\mathfrak{g} := \mathrm{Lie}\, G$

$K \subset G(\mathbb{R})$ maximal compact. SO_n O_n $U_p \times U_q$

Fix $P_0 \subset G$ minimal parabolic.

A_0 maximal split torus in P_0 $\left(\xleftarrow[\exp]{\simeq} \mathfrak{a}_0 \simeq \mathbb{R}^{\mathrm{rk}\, A} \right)$

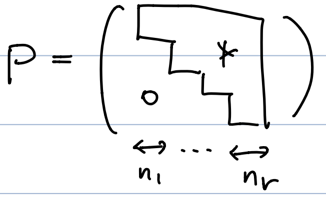
(connected)

(P,A) standard p-pair : $P \supset P_0$, $\boxed{A_0 \supset A}$.

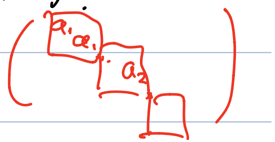
$\rightsquigarrow P = \underset{Z_G(A)}{M} \ltimes N$, $\underline{M = {}^0M \times A}$. $({}^0M \neq M^0)$

Ex $G = GL_n \supset P_0 = \begin{pmatrix} \ddots & * \\ 0 & \ddots \end{pmatrix} \supset A_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}$ in $\mathbb{R}^\times_{>0}$ $\quad a_1 \ldots a_n$

$P =$



$=: P_{n_1,\dots,n_r} \supset A \simeq (\mathbb{R}^\times_{>0})^r$



${}^0M = \begin{pmatrix} \boxed{A_1} & & 0 \\ & \ddots & \\ 0 & & \boxed{A_r} \end{pmatrix}$, $|\det A_i| = 1$. (not an algebraic gp)

Fundamental stuff

(fun.)

Def A Cartan subgp $T \subset G_{/\mathbb{R}}$ is fun.
($\approx$ max. torus)

if $\mathrm{rk}_{\mathbb{R}} T$ is minimal $(= \mathrm{rk}\, G - \mathrm{rk}\, K = \ell_0)$

Fact Such T form a single $G(\mathbb{R})$-conj. class.

Def · P is fun. if minimal among parabolics containing fun. Cartan.

· (P, A) (or P) is cuspidal if $M/A = {}^{\circ}M$ contains cpt Cartan.

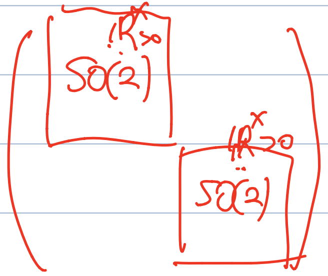
Fact P fun. $\Leftrightarrow$ P cuspidal $+$ ${}^{\circ}M_{ss} \cap G_{reg} \neq \emptyset$.

(P, A) cuspidal $\Leftrightarrow$ ${}^{\circ}M$ has disc. series reps

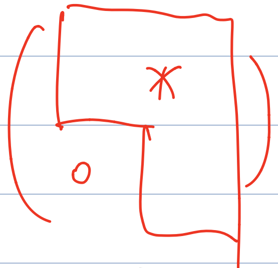
Ex $(G = GL_n)$ P is fun. $\Leftrightarrow$ $\{n_1, \dots, n_r\} = \{2, \dots, 2, 1 \text{ or } 2\}$

cuspidal $\Leftrightarrow$ $\forall i,\ n_i \leq 2$.

fun. Cartan



mod center



= fun. parabolic.

$n_1 = n_2 = 2$

$(G = U_{p,q})$ $P = G$ is fun.

ϟ not all (P, A) are cuspidal.

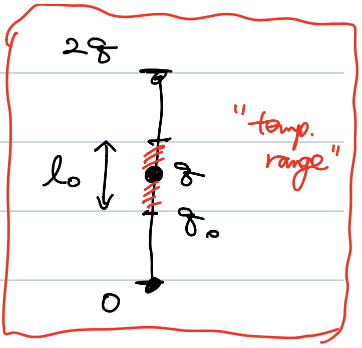
(**Ex** $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} GL_n \subset U_{n,n}$
$\subset P$.

Numerical invariants

$q(G) := \frac{1}{2}(\dim G - \dim K)$

$l_0(G) := \operatorname{rk} G - \operatorname{rk} K \in \mathbb{Z}$

$q_0(G) := q(G) - \frac{1}{2} l_0(G) \in \mathbb{Z}$



Ex	q	l_0	q_0
$G = A$ spl. torus	$\frac{1}{2}\dim A$	$\dim A$	0
SL_n	$\frac{1}{2}\left(\frac{n(n+1)}{2} - 1\right)$	$n - \lfloor \frac{n}{2} \rfloor - 1$	......
GL_n	$\frac{1}{4}n(n+1)$	$n - \lfloor \frac{n}{2} \rfloor$	......
$U_{p,q}$	pq	0	pq

two extremes

II. Statement of Thm

(when $\nexists$ twisting by fin. dim reps)

Main Thm (P, A) cuspidal. $P = M \ltimes N$

σ disc. series rep of ${}^{\circ}M$.

$\nu \in i\mathfrak{a}^*$ $\rightsquigarrow$ $A(\mathbb{R}) \curvearrowright^{\text{unitary}} \mathbb{C}_\nu$

$\boxed{\sigma_\nu} := \sigma \otimes \mathbb{C}_\nu$

$M = {}^{\circ}M \times A$

If $\underline{H^*(\mathfrak{g}, K; I_P(\sigma_\nu)) \neq 0}$.

$\Rightarrow$
(i) $\nu = 0$ ✓
(ii) P is fun.
(iii) $\chi_\sigma = -s\rho$ for unique $s \in \underline{W^P}$ $(\overset{\text{bij}}{\simeq} \underline{W/W_M})$

and (iv)

$$\dim H^j(\mathfrak{g}, K, I_P(\sigma)) = \binom{l_0}{j - q_0} \quad . \quad j \in \underline{[q_0, q_0 + l_0]}$$

$$= 0 \quad \text{o/w}.$$

Special cases

$\ell_0 = 0$

$\boxed{\text{(DS)}}$ If G is itself fun,

[BW II.5]

$$H^j(\mathfrak{g}, K, \text{temp}) \neq 0$$

$\Leftrightarrow$ temp = disc. series σ as above $(P=G)$,

$j = q = q_0$, $\dim H^q = 1$ (in middle deg)

(SpT) $P = G = A$ split torus. elementary.

$$H^j(\mathfrak{a}, \{1\}, \mathbb{C}) = \wedge^j \mathfrak{a}_{\mathbb{C}}^*$$

The only unitary character with nonzero cohom. is the trivial character.

Proof outline

proved independently $\left\{ \begin{array}{l} \text{(DS)} \\ \text{(SpT)} \end{array} \right. \xRightarrow{\quad}$ Main Thm.

$\uparrow$ standard tools.

(Hochschild-Serre, Shapiro, Künneth, Poincaré duality)

$\boxed{\text{III. Proof}}$

$$H^*(\mathfrak{g}, K; I_P(\sigma_\nu)) \overset{\text{Shapiro}}{=} H^*(\mathfrak{p}, K_P; \sigma_{\rho+\nu})$$

$$I_P(\sigma_\nu) = \text{Ind}_P(\sigma_{\rho+\nu}).$$

means $\rho|_A$.

$*\ \rho|_A = \rho_P.$

$$E_2^{p,q} = H^p(\mathfrak{m}, K_P ; \underbrace{H^q(\mathfrak{n}, \overset{\text{triv. action}}{\sigma_{\rho+\nu}})}_{/\!/\ \text{Künneth}}) \overset{\text{H-S}}{\Rightarrow} H^{p+q}(\mathfrak{p}, K_P, \sigma_{\rho+\nu})$$

$$H^q(\mathfrak{n}, \mathbb{C}) \otimes \sigma_{\rho+\nu}$$

$$\boxed{\underline{\text{Kostant}} \quad H^q(\mathfrak{n}, \mathbb{C}) = \bigoplus_{\substack{s \in W^P \\ \ell(s) = q}} \underset{\uparrow\ \text{highest wt rep.}}{E_{s\rho-\rho}} \quad \text{as } M_{\mathbb{C}}\text{-mod.}}$$

$$M = {}^\circ M \times A \qquad {}^\circ M \qquad A$$

$$H^*(\mathfrak{m}, K_P ; \overset{\curvearrowright}{E_{s\rho-\rho}} \otimes \overset{\curvearrowright}{\sigma} \otimes \overset{\curvearrowright}{\mathbb{C}_{\rho+\nu}})$$

$$\underset{\text{Künneth}}{\simeq} \underbrace{H^*({}^\circ\mathfrak{m}, K_P, \overset{\text{fin. dim.}}{E_{s\rho-\rho}|_{{}^\circ M}} \otimes \overset{\text{DS}}{\sigma})}_{①} \otimes \underbrace{H^*(\mathfrak{a}, \underset{\underset{\mathbb{C}_{s\rho+\nu}.}{\parallel}}{\mathbb{C}_{s\rho-\rho+\rho+\nu}})}_{②}$$

$$① \neq 0 \Rightarrow \overset{\infty\text{'l char.}}{\downarrow}\!\chi_\sigma = -\left((s\rho-\rho)|_{{}^\circ\mathfrak{m}} + \rho_{{}^\circ M}\right) = -\underset{\uparrow\ G\text{-regular.}}{s\rho}|_{{}^\circ M}.$$

$$\Rightarrow \begin{cases} s \in W^P \ (\simeq W/W_M) \text{ is unique.} & \boxed{\text{(iii)}} \\ {}^\circ M_{ss} \cap G_{reg} \neq \phi \Rightarrow P \text{ is fun.} & \boxed{\text{(ii)}} \end{cases}$$

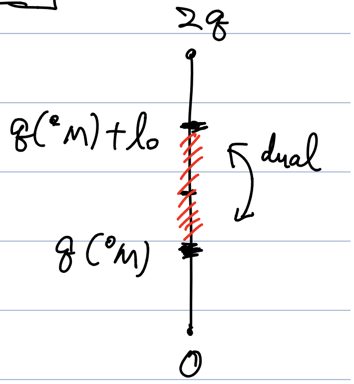
By (DS), $\underline{\underline{① \text{ is concentrated in } H^{q({}^\circ M)},\ \dim = 1.}}$ ✱

$\underline{② \neq 0} \Rightarrow \underset{\substack{\cap \\ i\mathfrak{a}^*}}{\nu} = -\underset{\substack{\cap \\ \mathfrak{a}^*}}{s\rho}|_A. \Rightarrow \nu = -s\rho|_A = 0.$ $\boxed{(i)}$

Now $\underline{② = H^*(\mathfrak{a}, \mathbb{C}) = \Lambda^* \mathfrak{a}_{\mathbb{C}}^* .}$ $\underline{\varepsilon \ \dim = l_0 .}$ ★★

Remains to check : $\boxed{q({}^\circ M) = q_0(G)}$

This follows from ★, ★★
via Poincaré duality.



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