$\boxed{D3}$ Setup. • F - number field signature (r_1, r_2)

• Symmetric space for $\mathrm{Res}_{F/\mathbb{Q}}(GL_d)$:

$$GL_d(F \otimes_{\mathbb{Q}} \mathbb{R}) / K_\infty^\circ A_\infty^\circ = Y_\infty$$

$$\cong GL_d(\mathbb{R})^{r_1} \times GL_d(\mathbb{C})^{r_2} \Big/ SO(d)^{r_1} \times U(d)^{r_2} \cdot \mathbb{R}_{>0}^{\times}$$

with $\ell_0 = \mathrm{rk}(G_\infty) - \mathrm{rk}(A_\infty K_\infty)$

$$= r_1 \cdot \begin{cases} d/2 & d \text{ even} \\ d+1/2 & d \text{ odd} \end{cases} + r_2 d - 1 .$$

• Locally symmetric spaces: for $K = \prod K_v \subset GL_d(\mathbb{A}_F^\infty)$ compact open,

$$Y(K) = GL_d(F) \backslash (GL_d(\mathbb{A}_F^\infty) \times Y_\infty) / K$$

We'll only consider

$$\begin{pmatrix} \text{pro-}v \\ \text{Iwahori} \end{pmatrix} \subset K_v \subset GL_d(\mathcal{O}_{F_v})$$

$$\| $$

$$\left\{ g \in GL_d(\mathcal{O}_{F_v}) \,\middle|\, g \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod \varpi_v \right\}.$$

- $S \supset \{v \mid p\}$ finite set of primes (and ramif primes)

  Assume $K_v = GL_d(\mathcal{O}_{F_v})$ $\forall v \notin S$ and $\forall v \mid p$.
- $p > d$

Conj from D1 : Consider

$$\mathbb{T}^S(K) \subseteq \mathrm{End}_{D(\mathcal{O})}\big(\underbrace{C_\bullet(Y(K), \mathcal{O})}_{\text{singular chains}}\big)$$

the unramified Hecke algebra.

(Generated by $T_v^{(i)}$, $i = 1, \ldots, d$, $v \notin S$).

Let $\mathfrak{m} \subset \mathbb{T}^S(K)$ be a maximal ideal.

Then $\exists$ $\bar{\rho}_{\mathfrak{m}} : G_{F,S} \longrightarrow GL_d(\underbrace{\mathbb{T}_{\mathfrak{m}}/\mathfrak{m} = k}_{\text{over } \mathbb{F}_p})$

such that $\forall\, v \notin S$

$$\operatorname{char}\left(\bar\rho_{\mathfrak m}(\mathrm{Fr}_v)\right) = \text{Hecke polynomial from D1}$$

arithmetic

$$= X^d - T_v^1 X + \cdots$$

$\cancel{\begin{pmatrix}1&\\&1\end{pmatrix}}$

$\begin{pmatrix}1&&\\&-1&\\&&1\end{pmatrix}$

<u>Moreover:</u>

(∞) $\bar\rho_{\mathfrak m}$ is "as odd as possible":

$\forall$ real $v|\infty$, $\dim H^0(G_{F_v}, \operatorname{ad}(\bar\rho))$ is minimal among all involutions

(p) For $p \gg_d 0$, $\bar\rho_{\mathfrak m}|_{G_{F_v}}$ is Fontaine–Laffaille with "labeled Hodge–Tate weights" $\{0,-1,\ldots,-d+1\}$ $\forall\, v|p$.

<u>And:</u> When $\mathfrak m$ non-Eisenstein,

$$\exists \quad \rho_{\mathfrak m} : G_{F,S} \longrightarrow GL_d(\mathbb{T}_{\mathfrak m})$$

with the analogous local properties

$$\boxed{\bar\rho = \bar\rho_{\mathfrak m}}$$

(absolutely irreducible throughout)

R_S: the universal deformation ring of $\bar{\rho}$ parametrizing (unr. outside S and F-L with the given H-Twts at $v|p$) lifts.

Numerology recap: Greenberg-Wiles then implies

$$h^1_S - h^1_{S^*} = -l_0$$

associated dual Selmer

where

$$h^1_S = \dim\left(\ker\left(H^1(G_{F,S}; \mathrm{ad}\bar{\rho}) \longrightarrow \prod_{v|p} \frac{H^1(G_{F_v}, \mathrm{ad}\bar{\rho})}{\text{F-L tan space}}\right)\right)$$

The task: Study $R_S \longrightarrow \mathbb{T}^S(K)_m$

by studying analogous

$$R_{S\cup Q} \longrightarrow \mathbb{T}_{m_Q}$$

allowing auxiliary ramification at T-W sets Q of primes.

Defn: • A T-W datum is

1) a finite set Q of primes away from S

such that: • $\forall\ v \in Q$, $N(v) \equiv 1 \mod p$

• $\forall\ v \in Q$, $\bar{\rho}(Fr_v)$ has distinct eigenvalues $\in k$

and 2) a choice of ordering $\alpha_{v,1}, \alpha_{v,2}, \ldots, \alpha_{v,d}$ of the e-values of $\bar{\rho}(Fr_v)$ $(\forall\ v \in Q)$.

• An allowable T-W datum is:

1') $\#Q = h^1_{S^*}$ $(\geq l_0)$ ← ind. of Q

1'') $h^1_{(S \cup Q)^*} = (0)$.

here $H^1_{S \cup Q}$ allows any ramification at Q

Numerology: For an allowable T-W datum,

$$h^1_{S \cup Q} - h^1_{(S \cup Q)^*} = -l_0 + \sum_{v \in Q} \left(h^1(G_{F_v}, \mathrm{ad}\,\bar{\rho}) - h^0(G_{F_v}, \mathrm{ad}\,\bar{\rho}) \right)$$

$$= -l_0 + \#Q \cdot \underbrace{d}_{\mathrm{rk}(GL_d)}$$

So $\boxed{h^1_{S \cup Q} = -l_0 + \#Q d}$ ← ind. of Q

<u>Prop</u>: Under appropriate assumptions on $\bar{\rho}(G_{F(\zeta_p)})$, and $\zeta_p \notin F$, for any level n there exist infinitely many allowable Taylor-Wiles data of $\underbrace{\text{level } n}_{N(v) \equiv 1 \bmod p^n}$.

David showed this for GL_1.

To bootstrap to GL_d, think about

$$\begin{tikzcd}[column sep=small]
 & & \text{Fixed field of dual Selmer class} \\
 & F(\mathrm{ad}^0 \bar{\rho}, \zeta_{p^n}) \arrow[ur, dash, "\neq 1"'] & \\
F(\mathrm{ad}^0 \bar{\rho}) \arrow[ur, dash] & & F(\zeta_{p^n}) \arrow[ul, dash] \\
 & F \arrow[ul, dash] \arrow[ur, dash] &
\end{tikzcd}$$

$\leftarrow$ apply Čebotarev here

some Čebotarev condition

split

Local deformation theory at T-W primes

<u>Lemma:</u> Let $v \in Q$. Any lift

$$\begin{tikzcd} & GL_d(A) \arrow[d] \\ G_{F_v} \arrow[ur, "\rho"] \arrow[r, "\bar{\rho}|_{G_{F_v}}"'] & GL_d(k) \end{tikzcd}$$

is $GL_d(A)$-conjugate to

$$\begin{pmatrix} \chi_{v,1} & & & 0 \\ & \chi_{v,2} & & \\ & & \ddots & \\ 0 & & & \chi_{v,d} \end{pmatrix}$$

where $\overline{\chi}_{v,i}(\mathrm{Fr}_v) = \alpha_{v,i}$

<u>The $\chi_{v,i}$</u> thus obtained are unique.

<u>Idea of proof:</u> Inductively use the known structure of $\mathrm{Gal}(F_v^{\mathrm{tame}}/F_v)$ in combination w/ $N(v) \equiv 1 \bmod p$, $\bar{\rho}(\mathrm{Fr}_v)$ regular semisimple.

$\boxed{\text{The } \mathcal{O}\text{-algebra hom } \mathcal{O}[\Delta_Q] \to R_{S \cup Q}}$

Restrict the universal $\rho_{S\cup Q}$ to G_{F_v} for $v \in Q$:

$\rho_{S\cup Q}|_{G_{F_v}}$ is $GL_d(R_{S\cup Q})$-conjugate to

$$\begin{pmatrix} \chi_{v,1} & & \\ & \ddots & \\ & & \chi_{v,d} \end{pmatrix} \quad \text{where } \overline{\chi}_{v,i}(\mathrm{Fr}_v) = \alpha_{i,v}.$$

$$\begin{array}{ccccc} \chi_{v,i} : G_{F_v} & \longrightarrow & & & R_{S\cup Q}^\times \\ \vee & & & & \uparrow \exists \\ I_{F_v} & \xrightarrow[\text{LCFT}]{} & \mathcal{O}_{F_v}^\times & \twoheadrightarrow & k^\times(v)(p) \end{array}$$

max'l p^∞-quotient

$$\rightsquigarrow \prod_{i=1}^{d} \chi_{v,i} : \underbrace{\prod_{i=1}^{d} k^\times(v)(p)}_{\Delta_v} \longrightarrow R_{S\cup Q}^\times$$

$$\rightsquigarrow \quad \Delta_Q = \prod_{v \in Q} \Delta_v \longrightarrow R_{S\cup Q}^\times$$

$$\rightsquigarrow \mathcal{O}[\Delta_Q] \xrightarrow[\mathcal{O}\text{-alg.}]{} R_{S\cup Q}.$$

Getting home:

$\mathfrak{a}_Q$ = augmentation ideal of $\mathcal{O}[\Delta_Q]$

$$\begin{tikzcd}
R_{S\cup Q} \arrow[r] \arrow[d] & R_S \\
R_{S\cup Q}/\mathfrak{a}_Q \arrow[ur, dashed, "\cong"'] &
\end{tikzcd}$$

R_S: quotient of $R_{S\cup Q}$ where $R_{S\cup Q}$ becomes unram at Q

||

$R_{S\cup Q}/\mathfrak{a}_Q$: quotient of $R_{S\cup Q}$ where Δ_Q acts trivially

Level structures at T-W primes

$v \in Q$

pro-v Iwahori

$$\begin{array}{ll}
Y\Big(\prod_{v \notin Q} K_v \times \prod_{v \in Q} Iw_{v,1}\Big) & \\
\downarrow & \\
Y(K_1(Q)) & \\
\downarrow & \\
Y(K_0(Q)) & \\
\downarrow & \\
Y(K) &
\end{array}$$

Galois cover with group $\prod_{v \in Q} T(k(v)) \cong \prod_{v \in Q} (k(v)^\times)^d$

Galois cover with group Δ_Q (p-quotient of $\twoheadleftarrow$)

$$K_0(Q) = \prod_{v \notin Q} K_v \times \underbrace{\prod_{v \in Q} K_0(v)}_{\text{Iwahori (lift Borel mod } v)}$$

Remark: In patching, Carl will consider intermediate covers

$$\begin{array}{l}
Y(K_1(Q)) \\
\downarrow \\
Y(K_n) \\
\downarrow \\
Y(K_0(Q)) \\
\downarrow \\
Y(K)
\end{array}$$

Δ_Q

$\Delta_Q/p^n \cong (\mathbb{Z}/p^n)^{d \cdot \# Q}$

Hecke Algebras

$$
\begin{tikzcd}
C_\bullet(Y(K),\mathcal{O}) & & & C_\bullet(Y(K_0(Q)),\mathcal{O}) \arrow[lll] \arrow[lll, bend right, dashed, "\text{splitting}\ \frac{\text{trace}}{[K:K_0(Q)]}"'] \\
\circlearrowright & & & \circlearrowright \\
\mathbb{T}^S(K) \supseteq \mathbb{T}^{S\cup Q}(K) & & \mathbb{T}^{S\cup Q}(K_0(Q)) \arrow[ll, two heads, "\text{res}"'] & \subset \mathbb{T}^{S\cup Q,\, Q\text{-aug}}(K_0(Q)) \\
& & \mathbb{T}^{S\cup Q}(K_0(Q)/K_1(Q)) \arrow[u, two heads] & \subset \mathbb{T}^{S\cup Q,\, Q\text{-aug}}(K_0(Q)/K_1(Q)) \arrow[u] \\
& & \circlearrowright \text{ in } D(\mathcal{O}[\Delta_Q]) & \\
& & C_\bullet(Y(K_1(Q)),\mathcal{O}) &
\end{tikzcd}
$$

$[K:K_0(Q)] \leftarrow$ prime to p ($N(v)\equiv 1 \bmod p$ and $p>d$)

"Q-aug": include ramified Hecke operators $U^{(i)}_{\varpi_v}$ $i=1,\dots,d$, $v\in Q$

"$K_0(Q)/K_1(Q)$": $\mathcal{O}[\Delta_Q]$-algebra generated by ...

Vertical arrows induced by

$$C_\bullet(Y(K_1(Q)),\mathcal{O}) \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \xrightarrow{\sim} C_\bullet(Y(K_0(Q)),\mathcal{O})$$

Maximal Ideals

$$C_\bullet(Y(K),\mathcal{O}) \longleftarrow C_\bullet(Y(K_0(Q)),\mathcal{O})$$

$$\mathbb{T}^S(K) \supseteq \mathbb{T}^{S\cup Q}(K) \xleftarrow{\ \text{res}\ } \mathbb{T}^{S\cup Q}(K_0(Q)) \subset \mathbb{T}^{S\cup Q,\, Q\text{-aug}}(K_0(Q))$$

$$\mathfrak{m} \dashrightarrow \mathfrak{m}^Q \dashrightarrow \mathfrak{m}_0^Q \xrightarrow{\text{aug}} \mathfrak{n}_{0,\alpha}$$

$$\mathfrak{m}_1^Q \qquad \mathbb{T}^{S\cup Q}(K_0(Q)/K_1(Q)) \subset \mathbb{T}^{S\cup Q,\, Q\text{-aug}}(K_0(Q)/K_1(Q))$$

$$\xrightarrow{\text{aug.}} \qquad \mathfrak{n}_{1,\alpha}$$

in $D(\mathcal{O}[\Delta_Q])$

$$C_\bullet(Y(K_1(Q)),\mathcal{O})$$

$\mathfrak{n}_{0,\alpha}$ = ideal generated by $\mathfrak{m}_0^Q$ and $U^i_{\varpi_v} - \alpha_{v,1}\cdot \ldots \cdot \alpha_{v,i}$ $(i=1,\ldots,d,\ v\in Q)$.

$\mathfrak{n}_{1,\alpha}$ – similar construction.

Prop: $\mathfrak{n}_{0,\alpha}$ and $\mathfrak{n}_{1,\alpha}$ are maximal ideals.

Idempotents

$\mathbb{T}^S(K)$ is a finite $\mathcal{O}$-algebra

$$\rightsquigarrow \mathbb{T}^S(K) = \prod_{m'} \mathbb{T}^S(K)_{m'}$$

The corresponding idempotents

$$e_{m'} \in \mathbb{T}^S(K) \subset \mathrm{End}_{D(\mathcal{O})}(C_\bullet(Y(K),\mathcal{O}))$$

induce a direct sum decomposition

$$C_\bullet(Y(K),\mathcal{O}) \cong \bigoplus_{m'} C_\bullet(Y(K),\mathcal{O})_{m'}$$

$$H_*(C_\bullet(Y(K),\mathcal{O})_{m'}) \cong H_*(Y(K),\mathcal{O})_{m'}$$

Likewise obtain:

$$C_\bullet(Y(K_0(Q)),\mathcal{O})_{n_0,\alpha}$$

$$C_\bullet(Y(K_1(Q)),\mathcal{O})_{n_1,\alpha}$$

Level-raising

Analyze the possible congruences to our given Hecke eigenclass at higher level.

$$\begin{tikzcd} C_\bullet(Y(K_1(Q)),\mathcal{O})_{\mathfrak{m}_{1,Q}} \otimes_{\mathcal{O}[\Delta_Q]} \mathcal{O} \arrow[d, "\simeq"', "\text{iso of complexes}"] \\ C_\bullet(Y(K_0(Q)),\mathcal{O})_{\mathfrak{m}_{0,Q}} \arrow[d, "\simeq"', "\text{q-iso}"] \\ C_\bullet(Y(K),\mathcal{O})_{\mathfrak{m}} \end{tikzcd}$$

Equivariant for $T_v^{(i)}$, $v \notin S \cup Q$

Choice of Q $\Longrightarrow$ only tamely ramified principal series level-raising congruences

Conjecture:

$$\exists \quad R_{S\cup Q} \longrightarrow \prod^{S\cup Q,\,Q\text{-aug}} \left(K_0(Q)/K_1(Q) \right)_{\mathfrak{m}_{1,\alpha}}$$

that is moreover a hom. of $\mathcal{O}[\Delta_Q]$-algebras.

(*) why this is a form of local-global compatibility at $v \in Q$. Example:

π on $GL_2(\mathbb{A}_Q) \longleftrightarrow \rho_\pi$ with T-W level at $v \in Q$

Check: $\pi_v \cong$ tamely ramified PS $=$ n-$\mathrm{Ind}_B^{GL_2}(\chi_1 \times \chi_2)$

$$\boxed{\begin{array}{l} \mathrm{rec}_v(\pi_v) \longleftrightarrow \rho_\pi|_{G_{F_v}} \\ \quad \shortparallel \\ \chi_1 \oplus \chi_2 \end{array}}$$

$\pi_v^{K_1(v)} = \mathbb{C}\cdot f_1 \oplus \mathbb{C} f_2$

with Hecke eigenvalues

	f_1	f_2
U_{ϖ_v}	$\chi_1(\varpi_v)\cdot(\ )$	$\chi_2(\varpi_v)\cdot(\ )$
$\langle\gamma\rangle = \begin{pmatrix}\gamma & \\ & 1\end{pmatrix}$	$\chi_1(\gamma)$	$\chi_2(\gamma)$

normalized by suitable power of $N(v)$

Localizing at $\mathfrak{m}_{1,\alpha}$ picks out the subspace where U_{ϖ_v} acts by a lift of α_v

Consequence

$$\begin{array}{ccc} R_{S\cup Q} & \longrightarrow & \mathrm{End}_{D(\mathcal{O}[\Delta_Q])}\left(C_\bullet(Y(K_1(Q)),\mathcal{O})_{\mathfrak{m}_{1,\alpha}} \right) \\ \big\downarrow \text{mod } \mathfrak{a}_Q & \circlearrowleft & \big\downarrow \ \otimes^{\mathbb{L}}_{\mathcal{O}[\Delta_Q]} \mathcal{O} \\ R_S & \longrightarrow & \mathrm{End}_{D(\mathcal{O})}\left(C_\bullet(Y(K),\mathcal{O})_{\mathfrak{m}} \right) \end{array}$$

Finally, to patch we need small complexes in place of $C_\bullet(Y(K),\mathcal{O})_{\mathfrak{m}}$

$C_\bullet(Y(K_1(Q)),\mathcal{O})_{\mathfrak{m}_{1,Q}}$

<u>Fact</u>: $C_\bullet(Y(K),\mathcal{O})$, $C_\bullet(Y(K),\mathcal{O})_{\mathfrak{m}}$ are perfect complexes of $\mathcal{O}$-modules.

$C_\bullet(Y(K_1(Q)),\mathcal{O})$, $C_\bullet(Y(K_1(Q)),\mathcal{O})_{\mathfrak{m}_{1,Q}}$ are perfect complexes of $\mathcal{O}[\Delta_Q]$-modules

(use Borel-Serre compactification)

Small replacements

Replace $C_\bullet(Y(K),\Theta)_m$ and

$C_\bullet(Y(K_1(Q)),\Theta)_{n_1,\alpha}$ by

minimal resolutions

$$C_Q \xrightarrow{\text{q-iso}} C_\bullet(Y(K_1(Q)),\Theta)_{n_1,\alpha}$$

$$C_0 \xrightarrow{\text{q-iso}} C_\bullet(Y(K),\Theta)_m$$

Minimal $\Rightarrow$ C_Q and C_0 are supported in the same degrees as

$H_*(Y(K_1(Q)),k)_{n_1,\alpha}$ and

$H_*(Y(K),k)_m$

i.e. conjecturally in degrees $[q_0, q_0+l_0]$.

(Example $l_0 = 0$.)

Patching Preview

$\forall\ n \geq 1$, have allowable T-W data Q_n $(+\ldots)$, $\# Q_n$ independent of Q_n.

$$d \cdot \# Q_n = s \quad \text{" " "}$$

$$\begin{tikzcd}
\mathcal{O}[[X_1, \ldots, X_{s-\ell_0}]] \arrow[r, two heads] & R_{S \cup Q_n} \arrow[r] & \mathrm{End}_{\mathcal{O}[\Delta_{Q_n}]}(C_{Q_n}) \\
 & & \mathcal{O}[\Delta_{Q_n}] \arrow[ul] \arrow[u]
\end{tikzcd}$$

$\Big\downarrow$ "$\lim\limits_{n \to \infty}$"

$$\begin{tikzcd}
\mathcal{O}[[X_1, \ldots, X_{s-\ell_0}]] \arrow[r, two heads] & R_\infty \arrow[r] & \mathrm{End}_{S_\infty}(C_\infty) \\
 & \mathcal{O}[[T_1, \ldots, T_s]] = S_\infty \arrow[u] \arrow[ur] &
\end{tikzcd}$$

- C_∞ is a complex of finite free S_∞-modules in degrees $[q_0, q_0 + \ell_0]$.

- $$\left.\begin{array}{l} R_\infty \otimes_{S_\infty} \mathcal{O} \xrightarrow{\sim} R_S \\[1em] C_\infty \otimes_{S_\infty} \mathcal{O} \xrightarrow{\sim} C_0 \end{array}\right\} \text{compatible}$$