AS.110.113 HW Week 3

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1) To prove that the relation $\leq$ is transitive for cardinality ($|X| \leq |Y|$, $|Y| \leq |Z|$ implies $|X| \leq |Z|$), we will use set theory principles and cardinality definitions in a more formal proof:

Given:

1. $|X| \leq |Y|$ means there exists an injection $f: X \to Y$.
2. $|Y| \leq |Z|$ means there exists an injection $g: Y \to Z$.

We want to prove:

$|X| \leq |Z|$, which means we need to show that there exists an injection $h: X \to Z$.

Proof:

By definition, $|X| \leq |Y|$ implies there exists an injection $f: X \to Y$.
And $|Y| \leq |Z|$ implies there exists an injection $g: Y \to Z$.

Definition: $f: X \to Y$, is injective if all a,b in X, $f(a)=f(b) \Rightarrow a=b$.

Now, we will define a new function $h: X \to Z$ as follows:
$h(x) = g(f(x))$

We claim that h is an injection from X to Z.
To prove this, we need to show that $h(a) = h(b)$ implies $a = b$ for all a, b in X.

Assume $h(a) = h(b)$, then $g(f(a)) = g(f(b))$.
Since g is an injection, it follows that $f(a) = f(b)$.
And since f is also an injection, it implies $a = b$.

Therefore, we have shown that h is an injection, and thus, $|X| \leq |Z|$.

This completes the proof, demonstrating the transitivity of the relation $\leq$ for cardinality.

3) Let X be any set. Prove that there is no surjection (hence, bijection) between X and {0, 1} ^ X. Deduce that {0, 1} ^ N is uncountable. Argue the first part by contradiction: suppose there exists a surjection

$f : X \to \{0, 1\}^X$

• Consider the set

$A = \{x \in X : x \notin f(x)\}$

• As f is a surjection (write the general definition) there must exists $a \in X$ such that $f(a) = A$. Do case work on whether $a \in A$ or $a \notin A$.

To prove that there is no surjection (and therefore no bijection) between a set X and the set {0, 1} ^ X, we will use a proof by contradiction. We will assume that there exists a surjection from X to {0, 1} ^ X and show that this leads to a contradiction.

Step 1: Assume there exists a surjection $f: X \to \{0, 1\}^X$.

In other words, for each element x in X, there is a function f(x) in {0, 1} ^ X such that $f(x)(x) = 1$ (i.e., the function f(x) maps x to 1).

Step 2: Define the set $A = \{x \in X : x \notin f(x)\}$.

A is the set of elements in X that are not mapped to 1 by their corresponding functions in {0, 1} ^ X.

Step 3: Consider the element $a \in X$ such that $f(a) = A$.

Since f is a surjection, there must exist an element a in X such that $f(a) = A$. In other words, f(a) is the set of all elements in X that are not mapped to 1 by their corresponding functions. R

Now, we have two cases:

Case 1: $a \in A$ ($a \notin f(a)$).

If $a \in A$, by the definition of A, it means that $a \notin f(a)$. However, by the definition of f(a), f(a) should contain all elements in X that are not mapped to 1. This leads to a contradiction because a is in A but not in f(a).

Case 2: $a \notin A$ ($a \in f(a)$).

If $a \notin A$, it means that $a \in f(a)$. This implies that a is not in A because a is in f(a). However, by the definition of A, a should be in A. This again leads to a contradiction because a cannot simultaneously be in A and not in A.

In both cases, we arrive at contradictions. Therefore, our initial assumption that there exists a surjection from X to {0, 1} ^ X is false.

Conclusion:

Since there is no surjection from X to {0, 1} ^ X, it implies that there cannot be a bijection either because a bijection is a surjection and an injection. Therefore, X and {0, 1} ^ X do not have the same cardinality.

Now, let's deduce that {0, 1} ^ N is uncountable. If X were countable, and if there existed a surjection from X to {0, 1} ^ X, then {0, 1} ^ X would also be countable. However, we've just shown that there is no such surjection for X. Therefore, if X = N (the set of natural numbers) were countable, it would imply that {0, 1} ^ N is uncountable.

5) Show that addition, product, and negation are well-defined for rational numbers.

Well-Definedness of Addition:

We want to show that if we have two equivalence classes of rational numbers $[a, b]$ and $[c, d]$ (representing a/b and c/d), then the result of addition is the same regardless of the choice of representatives.

Let $[a, b]$ and $[c, d]$ be two equivalence classes of rational numbers, where a, b, c, and d are integers, and $b \neq 0$, $d \neq 0$. These equivalence classes are defined as $[a, b] = \{(x, y) \mid xy = ab, y \neq 0\}$ and $[c, d] = \{(u, v) \mid uv = cd, v \neq 0\}$.

Now, we can perform addition:
$[a, b] + [c, d] = \{(x, y) \mid xy = ab, y \neq 0\} + \{(u, v) \mid uv = cd, v \neq 0\}$

Using the properties of rational numbers, we can find a common denominator and add the fractions:
$[a, b] + [c, d] = \{(ad + bc, bd) \mid y \neq 0, v \neq 0\}$

Now, consider another equivalence class $[c', d']$ such that $c/d = c'/d'$. This means $bc = bc'$, and we can perform addition with this equivalence class:
$[a, b] + [c', d'] = \{(ad + bc', bd) \mid y \neq 0, v \neq 0\}$

Since $c/d = c'/d'$ implies $bc = bc'$, we have $ad + bc = ad + bc'$.
This means $\{(ad + bc, bd) \mid y \neq 0, v \neq 0\} = \{(ad + bc', bd) \mid y \neq 0, v \neq 0\}$.

Therefore, the result of addition is the same regardless of the choice of representatives, which proves the well-definedness of addition for rational numbers.

Well-Definedness of Product:

We want to show that the product of two equivalence classes of rational numbers is well-defined, meaning that it does not depend on the choice of representatives.

Let $[a, b]$ and $[c, d]$ be two equivalence classes of rational numbers, where a, b, c, and d are integers, and $b \neq 0$, $d \neq 0$. These equivalence classes are defined as $[a, b] = \{(x, y) \mid xy = ab, y \neq 0\}$ and $[c, d] = \{(u, v) \mid uv = cd, v \neq 0\}$.

Now, we can perform multiplication:
$[a, b] * [c, d] = \{(x, y) \mid xy = ab, y \neq 0\} * \{(u, v) \mid uv = cd, v \neq 0\}$

Using the properties of rational numbers, we can find a common denominator and multiply the fractions:
$[a, b] * [c, d] = \{(ac, bd) \mid y \neq 0, v \neq 0\}$

Now, consider another equivalence class $[c', d']$ such that $c/d = c'/d'$.
This means $ad = bc'$ (from the definition of equivalence classes).

We can perform multiplication with this equivalence class:
$[a, b] * [c', d'] = \{(ac', bd') \mid y \neq 0, v \neq 0\}$

Since $c/d = c'/d'$ implies $ad = bc'$, we have $ac = ac'$ (from the multiplication).
This means $\{(ac, bd) \mid y \neq 0, v \neq 0\} = \{(ac', bd') \mid y \neq 0, v \neq 0\}$.

Therefore, the result of multiplication is the same regardless of the choice of representatives, which proves the well-definedness of multiplication for rational numbers.

Well-Definedness of Negation:

In order to show that negation is well-defined for rational numbers, we need to demonstrate that it doesn't depend on the choice of representatives within equivalence classes.

Recall that a rational number [a, b] is an equivalence class of ordered pairs (a, b), where a and b are integers and $b \neq 0$, and the equivalence relation is defined as follows: Two ordered pairs (a, b) and (c, d) are equivalent if and only if $ad = bc$ and $d \neq 0$.

Now, let's consider a rational number [a, b], where a and b are integers and $b \neq 0$. This rational number can be represented by multiple ordered pairs, all satisfying the equivalence relation.

For example:
~[a, b] can be represented by (a, b) since $(a)(b) = (a)(b)$, and $b \neq 0$.
~[a, b] can also be represented by (ka, kb) for any nonzero integer k,
since $(ka)(kb) = (k)(a)(kb) = (a)(b)(k) = (a)(b)$, and $kb \neq 0$.

Now, let's consider the negation of [a, b], which we'll denote as -[a, b]. This negation can be represented by multiple ordered pairs as well.

For example:
~-[a, b] can be represented by (-a, b) since $(-a)(b) = -(a)(b) = -(b)(a) = (b)(-a) = (b)(a) = (a)(b)$ (using properties of integer multiplication), and $b \neq 0$.
~ -[a, b] can also be represented by (a, -b) since $(a)(-b) = -(a)(b) = -(b)(a) = (b)(-a) = (b)(a) = (a)(b)$, and $-b \neq 0$.

The key point here is that regardless of the choice of representatives for [a, b], the negation -[a, b] can be represented consistently with the same equivalence class properties, ensuring that it is well-defined for rational numbers.

In other words, whether you negate (a, b) or (-a, b) or (a, -b) or (-a, -b), you are still representing the same rational number -[a, b], and this representation does not depend on the specific choice of representatives within the equivalence class. Therefore, negation is indeed well-defined for rational numbers.

Tri-weekly diary

Dear Diary,

As I embark on this journey through the labyrinthine realm of mathematics, I find myself oscillating between bewilderment and fascination, grappling with concepts that seem as elusive as they are intriguing. This diary entry serves as a chronicle of my experiences, shedding light on the ebbs and flows of my mathematical odyssey.

Challenging Beginnings

The journey began with a formidable adversary - the axioms of natural numbers, denoted as N. These seemingly innocent integers brought forth a torrent of questions that left me perplexed. The axioms acted as the foundation upon which all mathematical structures are built, and I struggled to grasp their significance. The idea of infinite sets and the Peano axioms, in particular, made my head spin. Why must we define zero and then prove the existence of subsequent numbers? It was a question that danced on the periphery of my understanding, taunting me with its elusiveness.

Curiosity Awakened

However, as I navigated the labyrinth further, a glimmer of curiosity began to illuminate the path. It was during a class on constructing integers from natural numbers that I experienced a mathematical epiphany. This process, a fusion of intuition and logic was akin to building bridges between familiar territories and uncharted lands. I found myself pondering the question: How could something as abstract as mathematics give birth to something as rea as integers? This intersection of theory and practicality ignited a flame of curiosity within me that continues to burn brightly.

Uncharted Territories

Throughout this mathematical journey, there have been topics that I wished were covered in greater detail. One such topic is the enigmatic world of fractals. Their infinite complexity, self-similarity, and paradoxical nature beckon me like an unsolved riddle. I yearn to explore their intricacies, to understand how something so chaotic can exhibit order and beauty. A deeper dive into fractals would surely enrich my understanding of mathematical structures.

The Lure of Boolean Algebra and Probability Theory

Boolean algebra and probability theory have emerged as tantalizing realms, beckoning me with their real-world applications. The concept of manipulating logical expressions using algebraic operations in Boolean algebra has practical implications in computer science and engineering. Likewise, probability theory, with its ability to model uncertainty and randomness, has piqued my interest. I desire a more thorough exploration of these topics, craving a deeper understanding of how they shape our understanding of the world.

People Along the Way

Amidst the equations and theorems, the people I've encountered on this journey have added their unique colors to the canvas of my experience. TA Anna stands out as a guiding light in this labyrinth. Her Friday discussions are a lifeline, where she navigates complex concepts with grace and patience. She has the remarkable ability to transform the most intricate topics into comprehensible fragments, making learning a joyous endeavor. Her sweet disposition is the icing on the mathematical cake.

On the flip side, there's Ethan, who, for reasons unbeknownst to me, manages to strike a chord of unease within. Perhaps it's his cryptic aura or the intimidating depth of his mathematical prowess that leaves me feeling somewhat intimidated. Every interaction with him feels like a challenge, an intellectual duel that I am not sure I am prepared for.

The Journey Continues

In the ever-expanding landscape of mathematics, each day is a new adventure, a chance to unravel the mysteries that lie hidden in equations and proofs. The challenges that once seemed insurmountable are gradually giving way to understanding, and the curiosity that was kindled now burns with an unquenchable fervor.

As I pen down this diary, I am reminded that this mathematical odyssey is not just about numbers and theorems; it's about the pursuit of knowledge, the thrill of discovery, and the camaraderie of fellow explorers. It's a journey that transcends the boundaries of the mind and delves into the heart of human curiosity.

With each page turned and each concept grasped, I move one step closer to the heart of this mathematical labyrinth. It's a journey filled with wonder, challenges, and the promise of boundless knowledge waiting to be unearthed. I can't wait to see what lies around the next corner.

Yours in curiosity,
Haden Gilley