

CYCLES ON SHIMURA VARIETIES VIA GEOMETRIC SATAKE

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ABSTRACT. We construct (cohomological) correspondences between mod p fibers of *different* Shimura varieties and describe the fibers of these correspondences by studying irreducible components of affine Deligne-Lusztig varieties. In particular, in the case of correspondences from a Shimura set to a Shimura variety, we obtain a description of the basic Newton stratum of the latter, and show that the irreducible components of the basic Newton stratum generate all the Tate classes in the middle cohomology of the Shimura variety, under a certain genericity condition. Along the way, we also determine the set of irreducible components of the affine Deligne-Lusztig variety associated to an unramified twisted conjugacy class.

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1. INTRODUCTION

The geometry of Shimura varieties and of moduli spaces of shtukas lies at the heart of the Langlands program. It encodes the deep relation between the world of automorphic representations and the world of Galois representations. In this article, we focus on the relations between the geometry of mod p fibers of *different* Shimura varieties. These relations usually provide a *canonical* Jacquet-Langlands transfer between various Hecke modules in a geometric way, as firstly noticed in the case of modular curves by Ribet and Serre ([Rib89], [Se96]) and then further developed by Ghitza and Helm ([Gh04, Gh06, He10, He12]) in some other cases. Tian and one of the authors (L.X.) took a slightly different viewpoint and constructed characteristic p cycles on the special fiber of some quaternionic and unitary Shimura varieties ([TX16, TX14⁺, HTX14⁺]), parameterized by (the special fiber of) another Shimura variety, as predicted by the Langlands conjecture and the Tate conjecture. In another direction, the recent work of V. Lafforgue ([Laf.V12⁺]) on the Langlands correspondence over global function fields also made use of the relation of cohomology between different moduli of shtukas in an essential way.

In this work, using the geometric Satake for p -adic groups ([Zhu17]) we develop a general strategy that relates the geometry and (ℓ -adic) cohomology of the mod p fibers of different Shimura varieties, which unifies and generalizes some aspects of the above mentioned works. In particular, we give a description of the basic Newton stratum of a Hodge type Shimura variety when its dimension is exactly the half of the dimension of the Shimura variety. Moreover, we prove that the cycle classes of the irreducible components of the basic locus generate Tate classes of the mod p fiber of the Shimura variety, under a certain genericity condition. Note that in this article, we only consider the case when Shimura varieties have good reduction at p and the case when the basic locus is middle dimensional in the ambient Shimura variety, and therefore exclude the modular curve case. We hope to generalize our approach in the future to also take this case into account.

1.1. The main theorem. Let us be more precise about the question we study in this paper. Let (G, X) be a Shimura datum. We fix an open compact subgroup $K \subset G(\mathbb{A}_f)$ sufficiently small, and let

$$\mathbf{Sh}_K(G, X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K$$

denote the corresponding Shimura variety. Recall that $\mathbf{Sh}_K(G, X)$ has a canonical model defined over the reflex field $E = E(G, X) \subset \mathbb{C}$ (e.g. see [Mi05]). Let $d = \dim \mathbf{Sh}_K(G, X)$ denote its

dimension. To motivate our results, we assume that $\mathbf{Sh}_K(G, X)$ is projective in the following discussions.

Let $\mathcal{H}_K := C_c^\infty(K \backslash G(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell)$ denote the $\overline{\mathbb{Q}}_\ell$ -coefficient¹ Hecke algebra of G as usual, and let π_f be an irreducible \mathcal{H}_K -module. It cuts out a “submotive” $\mathbf{Sh}_K(G, X)[\pi_f]$ of $\mathbf{Sh}_K(G, X)$. More precisely, its ℓ -adic realization can be written as

$$W^i(\pi_f, \overline{\mathbb{Q}}_\ell) = \mathrm{Hom}_{\mathcal{H}_K}(\pi_f, H^i(\mathbf{Sh}_K(G, X)_{\overline{\mathbb{Q}}}, \overline{\mathbb{Q}}_\ell)),$$

which is a natural representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/E)$. Here $\overline{\mathbb{Q}}$ denotes the algebraic closure of \mathbb{Q} in \mathbb{C} .

Now let p be an unramified prime for (G, X, K) , namely, $K = K^p K_p \subset G(\mathbb{A}_f^p) \times G(\mathbb{Q}_p)$ with K_p a hyperspecial open compact subgroup of $G(\mathbb{Q}_p)$. In this case, it is expected that $\mathbf{Sh}_K(G, X)$ should have a good reduction $\overline{\mathcal{S}}$ over \mathbb{F}_v , where v is a place of E over p with residue field \mathbb{F}_v . Therefore, the motive $\mathbf{Sh}_K(G, X)[\pi_f]$ should also have good reduction $\overline{\mathcal{S}}[\pi_f]$ at v . In particular, $W^i(\pi_f, \overline{\mathbb{Q}}_\ell)$ is a ϕ_v -module, where ϕ_v denotes the *geometric* Frobenius element of $\mathrm{Gal}(\overline{\mathbb{F}}_v/\mathbb{F}_v)$. If i is even, let

$$T^i(\pi_f, \overline{\mathbb{Q}}_\ell) := \bigcup_{j \geq 1} W^{2i}(\pi_f, \overline{\mathbb{Q}}_\ell)(i)\phi_v^j$$

denote the space of Tate classes for the motive $\overline{\mathcal{S}}[\pi_f]$. According to the Tate conjecture, these classes should come from algebraic cycles on $\overline{\mathcal{S}}$ via the cycle class map. One of the main goals of this work is to verify this for “generic” π_f in many cases when (G, X) is of Hodge type.

To describe our results more precisely, let us first recall the (naive) conjectural description of

$$W(\pi_f, \overline{\mathbb{Q}}_\ell) := \bigoplus_i W^i(\pi_f, \overline{\mathbb{Q}}_\ell)(d/2)$$

as a ϕ_v -module. Here we fix once and for all a square root \sqrt{p} , or equivalently a half Tate twist $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$.

Let $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ be the Langlands dual group of G defined over $\overline{\mathbb{Q}}_\ell$, equipped with a pinning (in particular a Borel subgroup \hat{B} and a maximal torus \hat{T}). Note that our assumption on p implies that the action of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ factors through $\mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ for some finite field \mathbb{F}_{p^m} containing \mathbb{F}_v , and therefore one can define the local Langlands dual group ${}^L G_p := \hat{G} \rtimes \mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$. Let $\mathbb{X}^\bullet(\hat{T}) = \mathrm{Hom}(\hat{T}, \mathbb{G}_m)$ denote the weight lattice of \hat{T} . Let $\{\mu\}$ be the conjugacy class of the one parameter subgroups associated to the Shimura datum, regarded as a dominant weight (with respect to \hat{B}) in $\mathbb{X}^\bullet(\hat{T})$ in the usual way. We write μ^* for $-w_0(\mu)$ where w_0 is the longest element in the Weyl group of \hat{G} , and let V_{μ^*} be the highest weight representation of \hat{G} of highest weight μ^* , which extends canonically to a representation

$$r_{\mu^*} : \hat{G} \rtimes \mathrm{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_v) \rightarrow \mathrm{GL}(V_{\mu^*}).$$

Let $\mathrm{rec}(\pi_{f,p}) : \langle \phi_p \rangle \rightarrow {}^L G_p$ denote the unramified Langlands parameter of the p -component $\pi_{f,p}$ of π_f (which depends on our choice of the half Tate twist). Then ϕ_v acts on V_{μ^*} as $r_{\mu^*}(\mathrm{rec}(\pi_{f,p})(\phi_v))$. This action gives a well-defined isomorphism class in the Grothendieck group of ϕ_v -modules, denoted by $[V_{\mu^*}]$. Then the (naive) Langlands conjecture claims that in the Grothendieck group of ϕ_v -modules,

$$(1.1.1) \quad [W(\pi_f, \overline{\mathbb{Q}}_\ell)] = a_G(\pi_f)[V_{\mu^*}],$$

where $a_G(\pi_f)$ is a certain multiplicity of π_f .

Remark 1.1.1. Here is a more canonical description of $W(\pi_f, \overline{\mathbb{Q}}_\ell)$ as a ϕ_v -module. We consider the conjugation action of \hat{G} on the non-neutral connected component $\hat{G}\phi_p \subset {}^L G_p$. The quotient stack

¹Note that it is possible to work with \mathbb{Q}_ℓ -coefficients instead in many statements below.

$[\hat{G}\phi_p/\hat{G}]$ can be thought as the stack of unramified Langlands parameters, and the \hat{G} -structure on V_{μ^*} defines a vector bundle on $[\hat{G}\phi_p/\hat{G}]$ by descent, denoted by $\widetilde{V_{\mu^*}}$. There is a tautological automorphism γ_{taut} of this vector bundle: it acts on the fiber of $\widetilde{V_{\mu^*}}$ over $\gamma\phi_p$ as $r_{\mu^*}((\gamma\phi_p)^{[v:p]})$. The Langlands parameter of $\pi_{f,p}$ defines a point of this stack. Then $W(\pi_f, \overline{\mathbb{Q}}_\ell)$ equipped with the action of ϕ_v is canonically identified with the fiber of $\widetilde{V_{\mu^*}}$ over this point, equipped with the tautological action.

With the above description of $W(\pi_f, \overline{\mathbb{Q}}_\ell)$ at hand, we can also give a description of $T^{d/2}(\pi_f, \overline{\mathbb{Q}}_\ell)$. For $\lambda \in \mathbb{X}^\bullet(\hat{T})$ and a representation V of \hat{G} , let $V(\lambda)$ denote the λ -weight subspace of V (with respect to \hat{T}). We define the following lattice

$$\Lambda^{\text{Tate}_p} := \left\{ \lambda \in \mathbb{X}^\bullet(\hat{T}) \mid \sum_{i=0}^{m-1} \phi_p^i(\lambda) \in \mathbb{X}_\bullet(Z_G) \right\} \subset \mathbb{X}^\bullet(\hat{T}).$$

Here, Z_G denotes the center of G , and $\mathbb{X}_\bullet(Z_G)$ denotes the coweight lattice of Z_G . If we denote by Z_G° the neutral connected component of Z_G and $\widehat{Z_G^\circ}$ its dual group, then there is a surjective map $\hat{T} \rightarrow \widehat{Z_G^\circ}$ and therefore $\mathbb{X}_\bullet(Z_G) = \mathbb{X}^\bullet(\widehat{Z_G^\circ})$ is a sublattice of $\mathbb{X}^\bullet(\hat{T})$. For a representation V of \hat{G} , we define the following subspace

$$V^{\text{Tate}_p} = \bigoplus_{\lambda \in \Lambda^{\text{Tate}_p}} V(\lambda).$$

This space is usually large. For example, in the case G is an odd unitary group of signature $(i, n-i)$ over a quadratic imaginary field, the dimension of this space at an inert prime p is $\binom{(n-1)/2}{[i/2]}$ (see Remark 4.2.11 for more examples).

We have the following observation (see Lemma 7.4.7 for the proof).

Lemma 1.1.2. *Let $\gamma\phi_p \in \hat{T} \rtimes \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p) \subset \hat{G} \rtimes \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ be an element whose image in $\widehat{Z_G^\circ} \rtimes \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ has finite order. Then*

$$(1.1.2) \quad V_{\mu^*}^{\text{Tate}_p} \subseteq \bigcup_{j \geq 1} (V_{\mu^*})^{r_{\mu^*}((\gamma\phi_p)^{j[v:p]})},$$

and if γ is strongly general with respect to V_{μ^*} in the sense of Definition 1.4.2, the above inclusion is an isomorphism.

Now, up to conjugacy, we may assume that $\text{rec}(\pi_{f,p})(\phi_p) = \gamma_p\phi_p$ for some $\gamma_p \in \hat{T}$ satisfying the condition of the lemma above. Therefore, if (1.1.1) holds, then

$$a_G(\pi_f)V_{\mu^*}^{\text{Tate}_p} \subset T^{d/2}(\pi_f, \overline{\mathbb{Q}}_\ell) \subset W(\pi_f, \overline{\mathbb{Q}}_\ell)$$

and if the local parameter of $\pi_{f,p}$ is general enough, the above inclusion should be an isomorphism.

One of the main goals of our work is to give a uniform construction of those algebraic cycles on $\overline{\mathcal{S}}$ predicted by the Tate conjecture and the Langlands conjecture as above. To state our results, we need the following.

Lemma 1.1.3. *Assume that there is a prime p such that $G_{\mathbb{Q}_p}$ is unramified and $V_{\mu^*}^{\text{Tate}_p} \neq 0$. Then there exists a (unique) inner form G' of G , such that $G'(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$ and $G'_{\mathbb{R}}$ is compact modulo center.*

This is a combination of Corollary 2.1.6 and Proposition 4.2.10.

Now we further assume that (G, X) is of Hodge type and $p > 2$. Let v be a place of E over p . Let $\mathcal{O}_{E,(v)}$ denote the localization of the ring of integers of E at v , \mathbb{F}_v the residue field at v , and $\mathcal{O}_{E,v}$ the completion of $\mathcal{O}_{E,(v)}$. Then by the work of Kisin [Kis10] and Vasiu [Vas07], there is a

canonical smooth integral model of $\mathbf{Sh}_K(G, X)$ over $\mathcal{O}_{E,(v)}$, denoted by $\mathcal{S}_K(G, X)$. It will be more convenient in the sequel to change the notation from $\overline{\mathcal{S}}$ to \mathbf{Sh}_μ to denote (the perfection of) its mod p fiber. There is a natural Newton stratification of \mathbf{Sh}_μ . Let $\mathbf{Sh}_{\mu,b}$ denote the basic Newton stratum, on which \mathcal{H}_K acts by (cohomological) correspondences.²

We need one more notation. For a (not necessarily irreducible) algebraic variety Z of dimension d over an algebraically closed field, let $H_{2d}^{\text{BM}}(Z)$ denote the $(-d)$ -Tate twist of the top degree Borel-Moore homology (with $\overline{\mathbb{Q}}_\ell$ -coefficients), which is the vector space with a basis labeled by the d -dimensional irreducible components of Z . Now for X is a smooth (but not necessarily proper) variety of dimension $d+r$ defined over a finite field \mathbb{F}_q , and a (not necessarily irreducible) projective subvariety $Z \subseteq X_{\overline{\mathbb{F}}_q}$ of dimension d , there is the cycle class map

$$\text{cl} : H_{2d}^{\text{BM}}(Z) \rightarrow T^r(X, \overline{\mathbb{Q}}_\ell) := \bigcup_{j \geq 1} H_c^{2r}(X_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell(r))^{\phi_q^j}.$$

Our main theorem is as follows. (For a more precise form, see Theorem 7.4.6).

Theorem 1.1.4. *Assume that (G, X) is of Hodge type and the center Z_G is connected. Let $K \subset G(\mathbb{A}_f)$ be a (small enough) open compact subgroup. Let $p > 2$ be a prime, such that K_p is hyperspecial and $V_{\mu^*}^{\text{Tate}_p} \neq 0$. Let G' be the inner form of G as in Lemma 1.1.3. Then:*

- (1) $\mathbf{Sh}_{\mu,b}$ is pure of dimension $\frac{d}{2}$. In particular, d is always an even integer. There is an \mathcal{H}_K -equivariant isomorphism

$$H_d^{\text{BM}}(\mathbf{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \cong C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f) / K, \overline{\mathbb{Q}}_\ell) \otimes V_{\mu^*}^{\text{Tate}}.$$

Here (a chosen) isomorphism $G'(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$, we regard K as an open compact subgroup of $G'(\mathbb{A}_f)$, and use $C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f) / K, \overline{\mathbb{Q}}_\ell)$ to denote the space of $\overline{\mathbb{Q}}_\ell$ -valued functions on the finite set $G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f) / K$.

- (2) Let π_f be an irreducible module of \mathcal{H}_K , and let

$$H_d^{\text{BM}}(\mathbf{Sh}_{\mu,b,\overline{\mathbb{F}}_v})[\pi_f] = \text{Hom}_{\mathcal{H}_K}(\pi_f, H_d^{\text{BM}}(\mathbf{Sh}_{\mu,b,\overline{\mathbb{F}}_v})) \otimes \pi_f$$

be the π_f -isotypical component. Then the cycle class map

$$\text{cl} : H_d^{\text{BM}}(\mathbf{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \rightarrow H_c^d(\mathbf{Sh}_{\mu,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d/2))$$

restricted to $H_d^{\text{BM}}(\mathbf{Sh}_{\mu,b,\overline{\mathbb{F}}_v})[\pi_f]$ is injective if the Langlands parameter of $\pi_{f,p}$ (the component of π_f at p) is general with respect to V_{μ^*} .

- (3) Assume that $\mathbf{Sh}_K(G, X)$ is a Kottwitz arithmetic variety (i.e. those considered in [Ko92b]), or assume that G_{der} is simply-connected and anisotropic, and there is a place $p \neq p'$ such that $\pi_{f,p'}$ is an unramified twist of the Steinberg representation. Then the π_f^p -isotypical component of the cycle class map cl is surjective onto

$$\sum_{\pi_p} T^{d/2}(\pi_p \pi_f^p, \overline{\mathbb{Q}}_\ell) \otimes \pi_p \pi_f^p$$

if the Langlands parameters of $\{\pi_p\}$ appearing in the above sum are all strongly general with respect to V_{μ^*} . In particular, the Tate conjecture holds for these π_f^p .

We refer to Definition 1.4.2 and Remark 1.4.3 below for the precise definition and some examples of “general” and “strongly general” with respect to V_{μ^*} .

Remark 1.1.5. (1) We do not need to assume that the Shimura variety $\mathbf{Sh}_K(G, X)$ is compact in Part (1) and (2) of the theorem. Shimura varieties considered in Part (3) are compact.

²The action of the Hecke algebra at p is subtle and can be ignored at the first pass.

- (2) Note that from the dimension formula of the Newton strata of mod p fiber of Shimura varieties of Hodge type ([Ham16⁺, Zha.C15]), it is always the case that $2 \dim \mathrm{Sh}_{\mu,b} \leq \dim \mathrm{Sh}_{\mu}$. It turns out that the equality achieves if and only if $V_{\mu^*}^{\mathrm{Tate}_p} \neq 0$.
- (3) One may replace the sum in Part (3) by a single representation π_f if the following Conjecture 1.3.8 holds. We also conjecture Part (3) of the theorem should be true for general Shimura varieties under the same condition on the Langlands parameter at p .
- (4) Part (2) of Theorem 1.1.4 may be viewed as a geometric realization of Jacquet-Langlands correspondence.

The theorem will be obtained as a combination of another four sets of theorems, which will be discussed below. The first two are of geometric nature, and the last two are of representation theoretical nature. We believe each of them is of independent interests.

1.2. Irreducible components of some affine Deligne-Lusztig varieties. The first ingredient is the study of irreducible components of certain affine Deligne-Lusztig varieties (ADLVs) for unramified elements. We switch to the local setting and will treat the case of equal and mixed characteristics uniformly.³ Let F be a non-archimedean local field with finite residue field $k = \mathbb{F}_q = \mathbb{F}_{p^m}$ and ring of integers \mathcal{O} . We temporarily drop the restriction $p \neq 2$. Let L/F denote the completion of the maximal unramified extension of F , $\mathcal{O}_L \subset L$ its ring of integers, and \bar{k} the residue field. The q -power arithmetic Frobenius in $\mathrm{Gal}(L/F)$ is denoted by σ .

Let G be an unramified reductive group over \mathcal{O} . For simplicity, we assume that its center Z_G is connected in this subsection. Let $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ be the pinned Langlands dual group as before, now equipped with an action of σ . Taking invariants gives a (not necessarily connected) reductive group with a “pinning” $(\hat{G}^{\sigma}, \hat{B}^{\sigma}, \hat{T}^{\sigma}, X)$. As usual, one can define the set of dominant weights $\mathbb{X}^{\bullet}(\hat{T})_{\sigma}^{+} \subset \mathbb{X}^{\bullet}(\hat{T})_{\sigma} = \mathbb{X}^{\bullet}(\hat{T}^{\sigma})$ of \hat{T}^{σ} (with respect to \hat{B}^{σ}), which is canonically bijective to $\mathbb{X}^{\bullet}(\hat{T})_{\sigma}/W_0$, where W_0 denotes the relative Weyl group of G .

Let $B(G)$ denote the set of σ -conjugacy classes of $G(L)$. In this paper, we will mainly consider the subset $B(G)_{\mathrm{unr}}$ of unramified elements of $B(G)$. Recall that an element $b \in B(G)$ is called *unramified* if its twisted centralizer J_b is a Levi subgroup of G , i.e. the F -rank of J_b coincides with the F -rank of G . There is a canonical bijection (see §4.2)

$$B(G)_{\mathrm{unr}} \cong \mathbb{X}^{\bullet}(\hat{T})_{\sigma}^{+}, \quad b \mapsto \lambda_b,$$

which composed with the natural injection $\mathbb{X}^{\bullet}(\hat{T})_{\sigma}^{+} \hookrightarrow \mathbb{X}^{\bullet}(\hat{T})^{\sigma,+} \times \mathbb{X}^{\bullet}(Z(\hat{G})^{\sigma})$ gives the Kottwitz’ invariants of unramified elements. Note that if b is unramified, J_b is an unramified reductive group (but without a preferred choice of \mathcal{O} -structure). Let $\mathbb{H}\mathbb{S}_b$ denote the set of hyperspecial subgroups of $J_b(F)$. Since we have assumed that Z_G is connected, $J_b(F)$ acts transitively on this set by conjugation.

Let $\mathrm{Gr} = LG/L^{+}G$ denote the affine Grassmannian of G , where as usual $L^{+}G$ (resp. LG) denotes the jet group (resp. loop group) of G . For a conjugacy class of cocharacters μ of G and $b \in B(G)$, let

$$X_{\mu}(b) := \{gL^{+}G \in \mathrm{Gr}_G \mid g^{-1}b\sigma(g) \in \overline{L^{+}G\varpi^{\mu}L^{+}G}\}$$

denote the corresponding *affine Deligne-Lusztig variety (ADLV)*, where $\varpi \in \mathcal{O}$ is any choice of uniformizer. This is a closed subvariety of Gr_G , (perfectly) locally of finite type, equipped with an action of $J_b(F)$. The general dimension formula for ADLVs in the affine Grassmannian ([Ham15, Zhu17]) specialized to the case when b is unramified, implies that $X_{\mu}(b)$ is equidimensional of dimension $\langle \rho, \mu - \lambda_b \rangle$. Here, as usual ρ denotes the half sum of positive coroots for $(\hat{G}, \hat{B}, \hat{T})$, which descends to a linear functional on $\mathbb{X}^{\bullet}(\hat{T})_{\sigma}$. For a character $\lambda \in \mathbb{X}^{\bullet}(\hat{T})$, let λ_{σ} denote its image in $\mathbb{X}^{\bullet}(\hat{T})_{\sigma}$.

³This result is new even in the equal characteristic.

The following is the first part of our main results on the geometry of affine Deligne-Lusztig varieties (see Theorem 4.4.14 (3)).

Theorem 1.2.1. *Assume that Z_G is connected. Let μ be a dominant cocharacter of G . Let b be an unramified element. Assume that $k \neq \mathbb{F}_2$ if any of the simple factors of J_τ contains a factor of type B_n, C_n, F_4 , and $k \neq \mathbb{F}_2, \mathbb{F}_3$ if it contains G_2 . Then there is a canonical $J_b(F)$ -equivariant bijection between the set of irreducible components of $X_\mu(b)$ and the set*

$$(1.2.1) \quad \bigsqcup_{\lambda \in \mathbb{X}^\bullet(\hat{T}), \lambda_\sigma = \lambda_b} \text{MV}_\mu(\lambda) \times \mathbb{H}\mathbb{S}_b,$$

where $\text{MV}_\mu(\lambda)$ is the set of so-called Mirković-Vilonen (MV) cycles (which gives a basis of $V_\mu(\lambda)$ via the geometric Satake). In particular, the $J_b(F)$ -orbits of the set of irreducible components of $X_\mu(b)$ give a basis of $V_\mu|_{\hat{G}\sigma}(\lambda_b)$.

Remark 1.2.2. (i) We believe that the restriction on the field k in the theorem is not necessary.

(ii) The above parameterization of the set of irreducible components of $X_\mu(b)$ is independent of the choice of b in the following sense. Let b', b be two representatives of an unramified σ -conjugacy class. Since $\lambda_b = \lambda_{b'}$, the set $\bigsqcup_{\lambda \in \mathbb{X}^\bullet(\hat{T}), \lambda_\sigma = \lambda_b} \text{MV}_\mu(\lambda)$ is independent of the choice of b or b' . Now a choice of g such that $b' = g^{-1}b\sigma(b)$ induces an isomorphism $X_\mu(b) \cong X_\mu(b')$ and $\mathbb{H}\mathbb{S}_b \cong \mathbb{H}\mathbb{S}_{b'}$, and the parameterization in the theorem is compatible with these isomorphisms.

(iii) Note that $\mathbb{H}\mathbb{S}_b$ embeds in the Bruhat-Tits building of J_b as some vertices. This is the point of view adapted in most literatures ([Vo10, VW11, RTW14, HP14, GH15]) to parameterize irreducible components of ADLVs.

Corresponding to each $(\mathbf{a}, x) \in \text{MV}_\mu(\lambda) \times \mathbb{H}\mathbb{S}_b$, there is an irreducible component $X_\mu^{\mathbf{a},x}(b) \subset X_\mu(b)$, under the above parameterization. This is a (perfectly) projective variety with an action of $J_{b,x}(\mathcal{O})$, the hyperspecial subgroup of J_b corresponding to x . The second part of our main results on the geometry of affine Deligne-Lusztig varieties is a construction of (an open subset of) $X_\mu^{\mathbf{a},x}(b)$. Let us briefly describe it and refer to Theorem 4.4.5 for details.

Let $\mathbf{a} \in \text{MV}_\mu(\lambda)$. Under the assumption that Z_G is connected, there is a way to choose a particular pair of dominant weights $(\tau_{\mathbf{a}}, \nu_{\mathbf{a}})$ of \hat{T} (with respect to \hat{B}) so that we can write $\lambda = \tau_{\mathbf{a}} + \sigma(\nu_{\mathbf{a}}) - \nu_{\mathbf{a}}$ (see Lemma 4.4.3(2)). Then up to σ -conjugacy, we can choose $b = \varpi^{\tau_{\mathbf{a}}}$. In this case, $J_b(F) \cap G(\mathcal{O}_L)$ is a hyperspecial subgroup, giving a point $x_0 \in \mathbb{H}\mathbb{S}_b$. This $X_\mu^{\mathbf{a},x_0}(b)$ contains an open subset $\mathring{X}_\mu^{\mathbf{a},x_0}(b)$, which fits into the following Cartesian diagram

$$(1.2.2) \quad \begin{array}{ccc} \mathring{X}_\mu^{\mathbf{a},x_0}(b) & \longrightarrow & \mathring{\text{Gr}}_{(\nu_{\mathbf{a}},\mu)|\tau_{\mathbf{a}}+\sigma(\nu_{\mathbf{a}})}^{0,\mathbf{a}} \\ \downarrow & & \downarrow \\ \mathring{\text{Gr}}_{\nu_{\mathbf{a}}} & \xrightarrow{1 \times b\sigma} & \mathring{\text{Gr}}_{\nu_{\mathbf{a}}} \times \mathring{\text{Gr}}_{\tau_{\mathbf{a}}+\sigma(\nu_{\mathbf{a}})}. \end{array}$$

Here $\mathring{\text{Gr}}_{\nu_{\mathbf{a}}}$ and $\mathring{\text{Gr}}_{\tau_{\mathbf{a}}+\sigma(\nu_{\mathbf{a}})}$ are Schubert cells in Gr , i.e. L^+G -orbits through $\varpi^{\nu_{\mathbf{a}}}$ and $\varpi^{\tau_{\mathbf{a}}+\sigma(\nu_{\mathbf{a}})}$, and $\mathring{\text{Gr}}_{(\nu_{\mathbf{a}},\mu)|\tau_{\mathbf{a}}+\sigma(\nu_{\mathbf{a}})}^{0,\mathbf{a}}$ is a certain L^+G -stable subset in $\mathring{\text{Gr}}_{\nu_{\mathbf{a}}} \times \mathring{\text{Gr}}_{\tau_{\mathbf{a}}+\sigma(\nu_{\mathbf{a}})}$ related to the chosen \mathbf{a} (see (4.4.8)). We refer to §4.4 for more detailed discussions.

Note that the definition of $\mathring{X}_\mu^{\mathbf{a},x_0}(b)$ is similar to the definition of finite Deligne-Lusztig varieties. Indeed, in some cases (e.g. b is basic and $\nu_{\mathbf{a}}$ is minuscule), $\mathring{X}_\mu^{\mathbf{a},x_0}(b)$ is a finite (closed) Deligne-Lusztig variety, on which the action of $J_{b,x_0}(\mathcal{O})$ factors through $J_{b,x_0}(\mathcal{O}/\varpi)$.⁴ However in general for some \mathbf{a} , we cannot choose $\nu_{\mathbf{a}}$ to be minuscule (even if μ is minuscule). Then $X_\mu^{\mathbf{a},x_0}(b)$ belongs

⁴So far most literatures on irreducible components of ADLVs only focus on these cases and we believe our result is the first work that systematically goes beyond these cases.

to a new class of varieties, on which the action of $J_{b,x_0}(\mathcal{O})$ does not factor through $J_{b,x_0}(\mathcal{O}/\varpi)$. For example, if

$$(G, \mu, b) = (U_n, (1^2 0^{n-2}), 1), \quad n > 4,$$

there are many \mathbf{a} 's such that the action of $J_{b,x_0}(\mathcal{O}) \simeq U_n(\mathcal{O})$ on $X_\mu^{\mathbf{a},x_0}(b)$'s factors through $U_n(\mathcal{O}/\varpi^2)$ but not through $U_n(\mathcal{O}/\varpi)$ (in particular they are not finite Deligne-Lusztig varieties and are *different* varieties in mixed and equal characteristics). We refer to [XZ] for a more detailed study of this case. These varieties probably deserve further study.

Remark 1.2.3. Motivated by the above theorem, Miaofen Chen and one of the authors (X.Z.) made a conjecture that gives a similar description of the $J_b(F)$ -orbits of the set of irreducible components of $X_\mu(b)$ for all $b \in B(G)$. Namely, for each σ -conjugacy class $b \in B(G)$, one can attach an element $\lambda_b \in \mathbb{X}^\bullet(\hat{T})_\sigma$, which is the “best integral approximation of” the Newton point ν_b of b .

Conjecture 1.2.4. *The $J_b(F)$ -orbits of the set of irreducible components of $X_\mu(b)$ canonically give a basis of $(V_\mu|_{\hat{G}^\sigma})(\lambda_b)$.*

Besides the cases studied in this paper, this conjecture has also been verified recently by Hamacher and Viehmann [HV17⁺] in some other special cases (which have no overlap with what studied in this paper). Probably combining their techniques and the method of the current paper will give an approach of the conjecture in general and give a method to describe the irreducible components of general $X_\mu(b)$.

Now back to the global situation as in Theorem 1.1.4. We assume $p \neq 2$, and revert μ to μ^* . It turns out under the assumption $V_{\mu^*}^{\text{Tate}_p} \neq 0$, the basic element $b \in B(G_{\mathbb{Q}_p}, \mu^*)$ is unramified and $V_{\mu^*}^{\text{Tate}_p} = V_{\mu^*}|_{\hat{G}^\sigma}(\lambda_b)$ (see §4.2). Then Part (1) of Theorem 1.1.4 follows easily from Theorem 1.2.1 together with the Rapoport–Zink uniformization in this case (see Corollary 7.2.16): there is an isomorphism (which depends on some auxiliary choices)

$$(1.2.3) \quad G'(\mathbb{Q}) \backslash X_{\mu^*}(b) \times G'(\mathbb{A}_f^p)/K^p \simeq \text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}.$$

In addition, the parametrization in (1.2.1) induces a decomposition $X_{\mu^*}(b) = \cup_{\mathbf{a}} X_{\mu^*}^{\mathbf{a}}(b)$ into unions of certain irreducible components of $X_{\mu^*}(b)$, where $\mathbf{a} \in \mathbb{M}\mathbb{V}_{\mu^*}(\tilde{\lambda})$ and $\tilde{\lambda} \in \mathbb{X}^\bullet(\hat{T})$ is a lift of λ_b . This in turn induces a corresponding decomposition $\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v} = \cup_{\mathbf{a}} \text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}^{\mathbf{a}}$ (which is independent of auxiliary choices). Then for the dimension reason, the cycle class map (the Gysin map) induces

$$(1.2.4) \quad \text{Gys}_{\mathbf{a}} : H_d^{\text{BM}}(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}^{\mathbf{a}}) \cong C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell) \rightarrow H_c^d(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d/2)).$$

Dually, there is a restriction map

$$\text{Res}_{\mathbf{b}} : H_c^d(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d/2)) \rightarrow H_c^d(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}^{\mathbf{b}}, \overline{\mathbb{Q}}_\ell(d/2)) \cong C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell).$$

The way we prove Part (2) of Theorem 1.1.4 is to compute the intersection pairing of these cycles, or equivalently the composition of the Gysin and restriction maps $\text{Res}_{\mathbf{b}} \circ \text{Gys}_{\mathbf{a}}$. We will show that the composition

$$\text{Res}_{\mathbf{b}} \circ \text{Gys}_{\mathbf{a}} \in \text{End}(C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell))$$

is induced by an element $f_{\mathbf{a},\mathbf{b}}$ in the local unramified Hecke algebra $C_c^\infty(K_p \backslash G(\mathbb{Q}_p)/K_p)$, i.e.

$$\text{Res}_{\mathbf{b}} \circ \text{Gys}_{\mathbf{a}}(f) = \int_{G(\mathbb{Q}_p)} f(xy^{-1}) f_{\mathbf{a},\mathbf{b}}(y) dy, \quad f \in C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K).$$

It seems in general it is very difficult to compute these elements $\{f_{\mathbf{a},\mathbf{b}}\}$ directly,⁵ especially in the case when some $X_{\mu^*}^{\mathbf{a},x}$ are not finite Deligne-Lusztig varieties. Therefore, we proceed along a

⁵For some explicit computations in some special cases, we refer to [TX16, HTX14⁺] and [XZ].

different way, which makes use of the geometric Satake isomorphism in an essential way. This will be explained in the next subsection.

1.3. Cohomological correspondences between the moduli of local shtukas. Now we explain the key ingredient in the proof of our main theorem. We first work in the local setting and keep the notations from the previous subsection. We introduce some moduli spaces attached to G . For a perfect \mathbb{F}_q -algebra R , let $D_R := \text{Spec} W_{\mathcal{O}}(R)$ denote the disk and $D_R^* := \text{Spec} W_{\mathcal{O}}(R)[1/\varpi]$ the punctured disk parameterized by $\text{Spec} R$. The q -Frobenius of R induces an automorphism of D_R , denoted by σ .

Let $\text{Hk}^{\text{loc}} = [L^+G \backslash \text{Gr}]$ denote the local Hecke stack, whose R -points classify two G -torsors $\mathcal{E}_1, \mathcal{E}_2$ on D_R , and a modification $\beta : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2$ between them (i.e. β is an isomorphism between \mathcal{E}_1 and \mathcal{E}_2 when restricted to D_R^*). Let Sht^{loc} denote the moduli of local G -shtukas “with singularities at the closed point of D ”, i.e. an R -point of Sht^{loc} is an R -point $(\mathcal{E}_1, \mathcal{E}_2, \beta)$ of Hk^{loc} , together with an isomorphism $\mathcal{E}_2 \simeq {}^\sigma \mathcal{E}_1 := \sigma^* \mathcal{E}_1$. Then there is a natural morphism

$$\varphi^{\text{loc}} : \text{Sht}^{\text{loc}} \rightarrow \text{Hk}^{\text{loc}}.$$

Although Hk^{loc} and Sht^{loc} are not algebraic (since they are quotient of ind-schemes by *infinite* dimensional groups), the categories of perverse sheaves $\text{P}(\text{Hk}_{\mathbb{F}_q}^{\text{loc}})$ and $\text{P}(\text{Sht}_{\mathbb{F}_q}^{\text{loc}})$ on these moduli spaces make sense. Roughly speaking, this is because these moduli spaces can be approximated by algebraic stacks so one can define the categories of perverse sheaves on them as certain filtered colimits of categories of perverse sheaves on these approximations. We refer to §5.1.7 and §5.3.10 for the rigorous treatments. In the introduction, it is harmless to pretend that these stacks were finite dimensional quotients so one can naively define the categories.

We also need the moduli space $\text{Hk}(\text{Sht}^{\text{loc}})$, which classify two G -shtukas $(\mathcal{E}_1, \mathcal{E}_2 \simeq {}^\sigma \mathcal{E}_1, \beta)$ and $(\mathcal{E}'_1, \mathcal{E}'_2 \simeq {}^\sigma \mathcal{E}'_1, \beta')$ and a modification $\alpha : \mathcal{E}_1 \dashrightarrow \mathcal{E}'_1$ such that $\beta' \alpha = \sigma^*(\alpha) \beta$. There are two natural projections

$$(1.3.1) \quad \begin{array}{ccc} & \text{Hk}(\text{Sht}^{\text{loc}}) & \\ \swarrow \overleftarrow{h}^{\text{loc}} & & \searrow \overrightarrow{h}^{\text{loc}} \\ \text{Sht}^{\text{loc}} & & \text{Sht}^{\text{loc}}, \end{array}$$

remembering the underlying local G -shtukas. One can regard $\text{Hk}(\text{Sht}^{\text{loc}})$ as the Hecke correspondence of Sht^{loc} . There is a category $\text{P}^{\text{Corr}}(\text{Sht}_{\mathbb{F}_q}^{\text{loc}})$, whose objects are as in $\text{P}(\text{Sht}_{\mathbb{F}_q}^{\text{loc}})$, but whose morphisms roughly speaking are given by cohomological correspondences supported on $\text{Hk}(\text{Sht}^{\text{loc}})$. Again, one needs to approximate $\text{Hk}(\text{Sht}^{\text{loc}})$ by algebraic stacks to make sense of this category. The actual definition of this category is somehow involved, and we refer to §5.3 and §5.4 for a detailed construction of this category. Here we pretend that one can naively define this category and describe (in the naive sense) objects and their endomorphism ring in this category.

Example 1.3.1. For a dominant coweight μ , let $\text{Sht}_{\mu}^{\text{loc}} \subset \text{Sht}^{\text{loc}}$ denote the closed substack classifying those $(\mathcal{E}_1, \mathcal{E}_2 \simeq {}^\sigma \mathcal{E}_1, \beta)$ such that the relative position of the modification β is bounded by μ , and for two coweights μ_1, μ_2 , let

$$\text{Sht}_{\mu_1|\mu_2}^{\text{loc}} = (\overleftarrow{h}^{\text{loc}})^{-1}(\text{Sht}_{\mu_1}^{\text{loc}}) \cap (\overrightarrow{h}^{\text{loc}})^{-1}(\text{Sht}_{\mu_2}^{\text{loc}}) \subset \text{Hk}(\text{Sht}^{\text{loc}}).$$

Note that $\text{Sht}_0^{\text{loc}}$ is isomorphic to the classifying stack $[\text{Spec} k/G(\mathcal{O})]$ of the profinite group $G(\mathcal{O})$ and $\text{Sht}_{0|0}^{\text{loc}} \cong [G(\mathcal{O}) \backslash G(F)/G(\mathcal{O})]$. Therefore, the correspondence

$$[G(\mathcal{O}) \backslash \text{Spec} k] \leftarrow [G(\mathcal{O}) \backslash G(F)/G(\mathcal{O})] \rightarrow [\text{Spec} k/G(\mathcal{O})],$$

is embedded in (1.3.1). Let $\delta_1 \in \mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})$ be the “constant sheaf” on $\mathrm{Sht}_0^{\mathrm{loc}}$. Then by definition elements of $\mathrm{End}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})}(\delta_1)$ should be given by cohomological correspondences supported on the discrete space $\mathrm{Sht}_{0|0}^{\mathrm{loc}}$. Giving such a correspondence should be the same as giving a (compactly supported) function on the double coset space $G(\mathcal{O}) \backslash G(F) / G(\mathcal{O})$. It follows that $\mathrm{End}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})}(\delta_1)$ should be isomorphic to the spherical Hecke algebra $C_c^\infty(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}))$. This is indeed the case, once the precise definitions are given (see Proposition 5.4.4).

On the dual group side, let $\mathrm{Rep}(\hat{G})$ denote the category of finite dimensional representations of \hat{G} (with $\overline{\mathbb{Q}_\ell}$ -coefficients), or equivalently the category of coherent sheaves on the classifying stack $\mathbf{B}\hat{G}$ over $\overline{\mathbb{Q}_\ell}$. Let $[\hat{G}\sigma/\hat{G}]$ denote the stack of unramified Langlands parameters over $\overline{\mathbb{Q}_\ell}$, as introduced in Remark 1.1.1.⁶ There is a natural map $\hat{G}\sigma/\hat{G} \rightarrow \mathbf{B}\hat{G}$ and every representation V of \hat{G} , regarded as a vector bundle on $\mathbf{B}\hat{G}$, pulls back to a vector bundle on $\hat{G}\sigma/\hat{G}$, which is nothing but the previously introduced \tilde{V} . Let $\mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)$ denote the full subcategory of coherent sheaves on the stack $[\hat{G}\sigma/\hat{G}]$ generated by those \tilde{V} . Note that for $V, W \in \mathrm{Rep}(\hat{G})$, the space $\mathrm{Hom}(\tilde{V}, \tilde{W})$ of homomorphisms from \tilde{V} to \tilde{W} as coherent sheaves on $[\hat{G}\sigma/\hat{G}]$ is a module over the ring of regular functions on $[\hat{G}\sigma/\hat{G}]$

$$\mathbf{J} := \Gamma([\hat{G}\sigma/\hat{G}], \mathcal{O}) = \overline{\mathbb{Q}_\ell}[\hat{G}]^{c_\sigma(\hat{G})}.$$

Here $c_\sigma(\hat{G})$ denotes the action of \hat{G} on \hat{G} by $g \bullet h = gh\sigma(g)^{-1}$.

Here is our main local theorem (see Theorem 6.0.1), which is already new in equal characteristic.

Theorem 1.3.2. (1) *There exists a functor $S : \mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma) \rightarrow \mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})$, such that the following diagram is commutative*

$$\begin{array}{ccc} \mathrm{Rep}(\hat{G}) & \xrightarrow{\cong} & \mathrm{P}(\mathrm{Hk}_{\mathbb{F}_q}^{\mathrm{loc}}) \\ \downarrow & & \downarrow \\ \mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma) & \xrightarrow{S} & \mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}_{\mathbb{F}_q}^{\mathrm{loc}}), \end{array}$$

where the top horizontal equivalence is the geometric Satake correspondence, the left vertical functor is $V \mapsto \tilde{V}$ (or pullback of coherent sheaves along $[\hat{G}\sigma/\hat{G}] \rightarrow \mathbf{B}\hat{G}$), and the right vertical functor is the pullback of sheaves along $\varphi_{\mathrm{loc}}^{\mathrm{loc}} : \mathrm{Sht}^{\mathrm{loc}} \rightarrow \mathrm{Hk}^{\mathrm{loc}}$.

(2) *Let $\mathbf{1}$ denote the trivial representation. Then $S(\mathbf{1}) = \delta_1$ and the map*

$$S : \mathbf{J} = \mathrm{End}_{\mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)}(\mathbf{1}) \rightarrow \mathrm{End}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})}(\delta_1) = C_c(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}))$$

coincides with the classical Satake isomorphism.

Remark 1.3.3. The key point of the theorem is a construction of maps between Hom spaces in the two categories. It follows from the theorem that the ring \mathbf{J} acts on every $S(\tilde{V}) \in \mathrm{P}(\mathrm{Sht}^{\mathrm{loc}})$ as cohomological correspondences. These are (local version of) V. Lafforgue’s S -operators, as we shall see below. But the theorem gives more. Namely, the bigger algebra $\mathrm{End}(\tilde{V})$ acts on $S(\tilde{V})$.

Remark 1.3.4. We mention that there is a subtlety when identifying \mathbf{J} with the spherical Hecke algebra at p . This leads the usual switch from μ to μ^* for Galois representations arising from the cohomology of Shimura varieties, and is also related to the canonical isomorphism of stacks $[\hat{G}\sigma/\hat{G}] \cong [\hat{G}\sigma^{-1}/\hat{G}]$. See Remark 3.5.3.

⁶Careful readers may have noticed that we changed the geometric Frobenius ϕ_p to the arithmetic one σ . See Remark 1.3.4 and 3.5.3 for the discussion.

We apply the above theorem in the following way. We come back to the global setting as before. Let (G, X) be a Shimura datum of Hodge type and let p be an unramified prime (i.e. $K_p \subset G(\mathbb{Q}_p)$ is hyperspecial). Let \underline{G} be the reductive \mathbb{Z}_p -model of G determined by K_p , and let Sh_μ denote (the perfection of) the mod p fiber of the corresponding Shimura variety as before. There is a universal local \underline{G} -shtuka on Sh_μ (coming from the crystalline \underline{G} -torsor on Sh_μ), which induces a morphism

$$\mathrm{loc}_p : \mathrm{Sh}_\mu \rightarrow \mathrm{Sht}_\mu^{\mathrm{loc}}.$$

In the Siegel case, loc_p is nothing but the perfection of the morphism sending an abelian variety to its underlying p -divisible group.

Next assume that (G', X') is another Shimura datum such that $G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$. Then the level structure $K \subset G(\mathbb{A}_f)$ can be transported to a level structure of G' . Let $\{\mu'\}$ denote the conjugacy class of the Shimura cocharacters of (G', X') . Let $v \mid p$ be a place of the composite of the reflex field EE' . Let $\mathrm{Sh}_{\mu'}$ denote its mod p fiber over v . We assume that there exists a perfect ind-scheme $\mathrm{Sh}_{\mu'}|_\mu$ that fits into the following diagram with both squares Cartesian

$$(1.3.2) \quad \begin{array}{ccccc} \mathrm{Sh}_{\mu'} & \longleftarrow & \mathrm{Sh}_{\mu'}|_\mu & \longrightarrow & \mathrm{Sh}_\mu \\ \mathrm{loc}_p \downarrow & & \downarrow & & \downarrow \mathrm{loc}_p \\ \mathrm{Sht}_{\mu'}^{\mathrm{loc}} & \longleftarrow & \mathrm{Sht}_{\mu'}^{\mathrm{loc}}|_\mu & \longrightarrow & \mathrm{Sht}_\mu^{\mathrm{loc}}, \end{array}$$

where $\mathrm{Sht}_{\mu'}^{\mathrm{loc}} = (\overleftarrow{h}^{\mathrm{loc}})^{-1}(\mathrm{Sht}_{\mu'}^{\mathrm{loc}}) \cap (\overrightarrow{h}^{\mathrm{loc}})^{-1}(\mathrm{Sht}_\mu^{\mathrm{loc}}) \subset \mathrm{Hk}(\mathrm{Sht}^{\mathrm{loc}})$. We expect such diagram always exists for any such pair (G, X) and (G', X') , at the unramified prime p if

$$\mathrm{Hom}_{\mathrm{Coh}_{\hat{G}_p}(\hat{G}\sigma_p)}(\widetilde{V}_{\mu'}, \widetilde{V}_\mu) \neq 0.$$

Although it seems possible to establish this diagram in greater generality, in this paper, we will only focus on some special cases, which suffices for our applications.

- We show in Proposition 7.3.6 that such diagram exists when $(G, X) = (G', X')$, in which case $\mathrm{Sh}_{\mu'}|_\mu$ is the perfection of the mod p fiber of a natural integral model of the p -power Hecke correspondences of $\mathbf{Sh}_K(G, X)$.
- We establish this diagram using Rapoport-Zink uniformization when $\mathbf{Sh}_K(G', X')$ is a Shimura set (§7.3.4 and Proposition 7.3.5).
- We also prove the existence of this diagram in some PEL case (see §7.3.8 and Proposition 7.3.9).

We leave the general cases to a future work.

Remark 1.3.5. As just mentioned, if $(G, X) = (G', X')$, $\mathrm{Sh}_{\mu'}|_\mu$ is just the mod p fiber of the Hecke correspondence of Sh_μ . In general, $\mathrm{Sh}_{\mu'}|_\mu$ can be regarded as “exotic Hecke correspondences” between mod p fibers of *different* Shimura varieties. These correspondences cannot be lifted to characteristic zero, and give a large class of characteristic p cycles on Shimura varieties.

Now, by pulling back morphisms in $\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})$ along vertical maps in the above diagram gives the following. (See §7.3.11–Proposition 7.3.14 for the proof.)

Theorem 1.3.6. (1) *Let $(G_i, X_i), i = 1, 2, 3$ be a collection of Shimura data, with all $G_i(\mathbb{A}_f)$ isomorphic to each other. We fix a common level structure K , and let p be an unramified prime. In addition, assume that for each pair $(G_i, X_i), (G_j, X_j)$, the Cartesian diagram (1.3.2) exists. Let $\{\mu_i\}$ denote the conjugacy class of Shimura cocharacters of (G_i, X_i) and $d_i = \dim \mathrm{Sh}_K(G_i, X_i)$. Let $V_i = V_{\mu_i}$ be the corresponding highest weight representation of \hat{G} , and \widetilde{V}_i the corresponding vector bundle on $[\hat{G}\sigma_p/\hat{G}]$. Let \mathcal{H}^p denote the prime-to- p Hecke*

algebra. Choose a place $v \mid p$ of the composite of all reflex fields. Then there is a natural action of \mathbf{J} on $H_c^*(\mathrm{Sh}_{\mu_i, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d_i/2))$, and a canonical \mathbf{J} -equivariant map

$$\mathrm{Hom}(\widetilde{V}_i, \widetilde{V}_j) \rightarrow \mathrm{Hom}_{\mathbf{J} \otimes \mathcal{H}^p} (H_c^{*+d_i}(\mathrm{Sh}_{\mu_i, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d_i/2)), H_c^{*+d_j}(\mathrm{Sh}_{\mu_j, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d_j/2))),$$

which is compatible with the natural compositions on both sides. In particular, there is a natural action of the algebra $\mathrm{End}(\widetilde{V}_\mu)$ on $H_c^*(\mathrm{Sh}_{\mu, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell)$.

(2) When $\mathrm{Sh}_{\mu, \overline{\mathbb{F}}_v}$ is a Shimura set, the action of $\mathbf{J} = \mathrm{End}(\widetilde{V}_\mu)$ on

$$H_c^*(\mathrm{Sh}_{\mu, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell) \cong C_c(G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell)$$

coincides with the usual Hecke algebra action.

Remark 1.3.7. (1) The above theorem in particular gives some geometric Jacquet-Langlands transfer, as first studied by Helm [Hel10] in some special cases.

(2) The action of $\mathbf{J} \subset \mathrm{End}(\widetilde{V}_\mu)$ on $H_c^*(\mathrm{Sh}_\mu, \overline{\mathbb{Q}}_\ell)$ in the theorem is the Shimura variety analogue of V. Lafforgue's S -operators. However, the action of the larger algebra $\mathrm{End}(\widetilde{V}_\mu)$ on $H_c^*(\mathrm{Sh}_\mu, \overline{\mathbb{Q}}_\ell)$ is new, even in the function field case (where instead of Shimura varieties one considers the moduli of shtukas). It is easy to deduce the congruence relation conjecture (known as the Blasius-Rogawski conjecture) from the existence of such action and the conjecture below.⁷ We will discuss this in another occasion.

The following conjecture is the analogue of V. Lafforgue's $S = T$ theorem.

Conjecture 1.3.8. *Under the Satake isomorphism, the action of \mathbf{J} in the above theorem coincides with the usual Hecke algebra action on $H_c^*(\mathrm{Sh}_{\mu, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell)$.*

We cannot prove this conjecture in general at the moment except the case of Shimura sets as mentioned above. Fortunately, this suffices for the application to Theorem 1.1.4.

Now, we come back to Theorem 1.1.4. Recall that we assume $V_{\mu^*}^{\mathrm{Tate}_p} \neq 0$, which implies that there exists G' as in Lemma 1.1.3. In particular, we can choose a (weak) Shimura datum (G', X') , such that $\mathrm{Sh}_K(G', X') = G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K$ is a Shimura set. In this set the Shimura cocharacter for (G', X') is central, denoted by $\tau \in \mathbb{X}_\bullet(Z_{G'}) = \mathbb{X}_\bullet(Z_G)$. Then V_{τ^*} is a 1-dimensional representation of \hat{G} .

As we mentioned above, the diagram (1.3.2) exists for the pair (G, X) and (G', X') . Indeed, in this case the diagram $\mathrm{Sh}_\tau \leftarrow \mathrm{Sh}_{\tau|_\mu} \rightarrow \mathrm{Sh}_\mu$ is essentially given by the Rapoport-Zink uniformization (1.2.3). Therefore we obtain a map

$$(1.3.3) \quad C_c(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K) \otimes_{C_c(G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p))} \mathrm{Hom}(\widetilde{V}_\tau, \widetilde{V}_\mu) \rightarrow H_c^d(\mathrm{Sh}_{\mu, \overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d/2)),$$

which turns out to encode all cycle class maps (1.2.4).⁸ To summarize, the intersection matrix $(f_{\mathbf{a}, \mathbf{b}})$, via the Satake isomorphism, now is expressed as the following pairing

$$\mathrm{Hom}(\widetilde{V}_\tau, \widetilde{V}_\mu) \otimes_{\mathbf{J}} \mathrm{Hom}(\widetilde{V}_\mu, \widetilde{V}_\tau) \rightarrow \mathrm{Hom}(\widetilde{V}_\tau, \widetilde{V}_\tau) = \mathbf{J}$$

on the dual group side. So to prove Part (2) of Theorem 1.1.4, it remains to understand this pairing, in particular its determinant, as a divisor over

$$\mathrm{Spec} \mathbf{J} = \hat{G}\sigma // \hat{G},$$

⁷In equal characteristic, V. Lafforgue deduced the congruence relation for moduli of shtukas from the $S = T$ theorem via some ingenious but complicated manipulations in tensor categories. But one can obtain a much simpler proof by using the action of the larger algebra $\mathrm{End}(\widetilde{V}_\mu)$ on the cohomology.

⁸There is a subtlety regarding the choice of τ which we ignore here. See Remark 7.4.4.

the GIT quotient of $\hat{G}\sigma$ by \hat{G} . This turns out to be an interesting question in representation theory, which we discuss in the next subsection.

1.4. Generalized Chevalley restriction map. We now come to another ingredient needed for the main theorem. This will be established in a companion paper [XZ2]. We work on the dual group side. Recall that we assume that Z_G is connected so \hat{G}_{der} is simply-connected.

For a representation V of \hat{G} , we consider the global section

$$\mathbf{J}(V) = \Gamma([\hat{G}\sigma/\hat{G}], \tilde{V}) = (\mathcal{O}_{\hat{G}} \otimes V)^{c_{\sigma}\hat{G}},$$

which can be regarded as the space of \hat{G} -equivariant maps $\hat{G} \rightarrow V$. It follows by definition that

$$\text{Hom}(\tilde{V}, \tilde{W}) \cong \mathbf{J}(V^* \otimes W).$$

Therefore, our goal is to understand the natural pairing

$$(1.4.1) \quad \mathbf{J}(V) \otimes \mathbf{J}(V^*) \rightarrow \mathbf{J}.$$

for a general representation V of \hat{G} .

We have the following theorem. To state it, we need a few notations. Let $S \subset T$ be the split maximal subtorus of T , so its dual group is $\hat{S} = \hat{T}/(\sigma - 1)\hat{T}$, and let Φ_{rel}^{\vee} be the *relative* coroot system of (G, S) . Note that $\Phi_{\text{rel}}^{\vee} \subset \mathbb{X}^{\bullet}(\hat{S}) = \mathbb{X}^{\bullet}(\hat{T})^{\sigma}$. Let W_0 denote Weyl group of Φ_{rel}^{\vee} ; it is the subgroup of the absolute Weyl group W of G fixed by σ . For a character $\lambda \in \mathbb{X}^{\bullet}(\hat{S})$, we write e^{λ} for the function on \hat{S} defined by λ .

Theorem 1.4.1. (1) $\mathbf{J}(V)$ is a finite projective \mathbf{J} -module.

(2) The determinant of the pairing (1.4.1) (which makes sense thanks to (1)) is a divisor on $\hat{G}\sigma//\hat{G} \cong \hat{S}//W_0$ defined by the function

$$\prod_{\alpha' \in \Phi_{\text{rel}}^{\vee}, \frac{\alpha'}{2} \notin \Phi_{\text{rel}}^{\vee}} (e^{\alpha'} - 1)^{\zeta_{\alpha'}} \prod_{\alpha', \frac{\alpha'}{2} \in \Phi_{\text{rel}}^{\vee}} (e^{\alpha'/2} + 1)^{\zeta_{\alpha'}},$$

where

$$\zeta_{\alpha'} = \sum_{n \geq 1} \dim V|_{\hat{G}\sigma}(n\alpha'_{\sigma}),$$

and α'_{σ} denotes a coroot of the absolute system (G, T) such that the sum of its σ -orbits is α' .

Now we can give the promised definition of the “generality” of Satake parameters. Recall that the Frobenius action on \hat{G} factors through the Galois group $\text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$.

Definition 1.4.2. Let V a representation of \hat{G} . An element $\gamma\sigma \in \hat{G}\sigma//\hat{G}$ is called *general with respect to V* (or V -general for short) if it does not belong to the divisor in the above theorem. It is *strongly general with respect to V* if, for every dominant coweight λ of \hat{T}^{σ} appearing in $V|_{\hat{G}\sigma}$, which does not factor through $\widehat{Z}_G^{\circ\sigma}$, we have $\lambda((\gamma\phi_p)^{mn}) \neq 1$ for every $n > 1$.

Remark 1.4.3. Note that if $\gamma\sigma$ is away from the discriminant divisor, i.e. it is regular semisimple, then it is general with respect to any representation V of \hat{G} . However, for some “small” representation V of \hat{G} , the V -general condition might be weaker than regular semisimple condition. For example, let $\hat{G} = \text{GL}_{2n+1}$ on which σ acts via the standard outer automorphism, and let V be the standard representation of GL_n . Let $\{\alpha_1, \dots, \alpha_{2n+1}\}$ be the eigenvalues of γ . Then $\gamma\sigma$ is V -general if and only if $\alpha_i \neq \alpha_{2n+2-i}$ for $i \neq n+1$, while $\gamma\sigma$ is regular semisimple if $\alpha_i \neq \alpha_j$ for all $i \neq j$. Note that also in this case, $\gamma\sigma$ is strongly general with respect to V if any of its power is general with respect to V .

For another example, $\hat{G} = \mathbb{G}_m \times \prod_{i=1}^f \mathrm{GL}_n$ with σ permuting the GL_n -factors. Let $V = \chi \otimes \boxtimes_{i=1}^f \wedge^{a_i} \mathrm{std}$ be the tensor product of exterior powers of the standard representation of GL_n . The condition $V^{\mathrm{Tate}} \neq 0$ is equivalent to $\sum_i a_i \equiv 0 \pmod n$. Then a local Langlands parameter is f diagonal matrices $\{\gamma_i\}$ and a scalar c . We write $\gamma_i = \mathrm{diag}\{\alpha_{i1}, \dots, \alpha_{in}\}$, and put $\beta_j := \prod_i \alpha_{ij}$. Then unless V is one-dimensional, the V -regular condition is equivalent to $\beta_j \neq \beta_{j'}$ whenever $j \neq j'$.

1.5. A Jacquet-Langlands transfer. Finally, we explain the ingredient needed to prove Part (3) of Theorem 1.1.4. First, we restrict ourselves to those (G, X) considered by Kottwitz [Ko92b]. For a Hecke module π_f , and a \mathbb{Q} -rational representation ξ of G , let $m_{G'}(\pi_f \otimes \xi_{\mathbb{C}})$ denote the multiplicity of $\pi_f \otimes \xi_{\mathbb{C}}$ appearing in $C^\infty(G'(\mathbb{Q}) \backslash G'(\mathbb{A})/K)$, and $a_G(\pi, \xi)$ be the integer defined in [Ko92b] (denoted by $a(\pi_f)$, see middle of p.p. 657 of *loc. cit.*). This is the “multiplicity” of π_f appearing in the cohomology of $\mathbf{Sh}_K(G, X)$.

We have the following Jacquet-Langlands type result, proved by a simple comparison of the trace formulas. For a more general statement, see Theorem 2.2.1.

Theorem 1.5.1. *For the π_f and ξ above, we have an equality $a_G(\pi_f, \xi) = m_{G'}(\pi_f \otimes \xi_{\mathbb{C}})$.*

This formula should follow from the forthcoming work of Kaletha-Minguez-Shin on the endoscopic classification of inner forms of unitary group. However, our proof does not relies on any knowledge of Arthur’s multiplicity formula, nor heavy machinery of the endoscopy theory. Indeed, following the idea of Kottwitz [Ko92b], we pseudo-stabilize the trace formulas on G and G' (instead of on the quasi-split inner form), and compare them directly.

Now it is easy to deduce Part (3) from Part (2), together with [Ko92b] and the above theorem. As a corollary, $a_G(\pi, \xi)$ is always a non-negative integer if G' as in Lemma 1.1.3 exists. On the other hand, the computation in Example 2.1.8 (ii) shows that if $\dim \mathbf{Sh}_K(G, X)$ is even, such G' exists. Combining with [Ko92b], we obtain the following result which seems to be new in this generality.

Corollary 1.5.2. *Let $\mathbf{Sh}_K(G, X)$ be a Kottwitz Shimura variety. For an algebraic representation ξ of $G_{\mathbb{C}}$, let \mathcal{L}_ξ denote the local system of \mathbb{C} -vector spaces over $\mathbf{Sh}_K(G, X)$. If $\dim \mathbf{Sh}_K(G, X)$ is even, then the cohomology $H^i(\mathbf{Sh}_K(G, X)_{\mathbb{C}}, \mathcal{L}_\xi) = 0$ for all odd i .*

Remark 1.5.3. For those G arising from a division algebra of dimension 9 over a quadratic imaginary field, the vanishing of H^1 was first observed Rapoport-Zink under an assumption at finite places and proved by Rogawski in general [Ro90, Theorem 15.3.1]. When G is compact at all but one infinite place, this corollary is also a consequence of a result of Clozel (cf. [Cl93, Theorem 3.3, Theorem 3.5]). The above argument is different from the one in [Cl93].

If (G, X) is general, the endoscopy is presented to complicate the picture. But one can still compare the Lefschetz trace formula for $\mathbf{Sh}_K(G, X)$ and the Arthur-Selberg trace formula for G' . As usual, imposing a local condition at $p' \neq p$ will greatly simplify the situation, and one can prove something similar to the above theorem in this case. We refer to §2.3 for precisely statements and details.

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2. PRELIMINARIES ON AUTOMORPHIC REPRESENTATIONS

In this section, we establish a special Jacquet-Langlands type formula, based on results from [Ko86, Ko92b].

In this section, we use \cdot^D to denote the Pontryagin dual of abelian groups. For a torus T over F , let $\mathbb{X}^\bullet(T)$ denote its character group (over \overline{F}) and $\mathbb{X}_\bullet(T)$ its cocharacter group. If G is a reductive

group over a field F , G_{ad} denotes its adjoint form, G_{der} its derived group, and G_{sc} the simply-connected cover of G_{der} . If $T \subset G$ is a maximal torus, its image in G_{ad} is denoted by T_{ad} , and its pre-images in G_{der} and G_{sc} are denoted by T_{der} and T_{sc} respectively. The center of G is denoted by Z_G . Let $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ denote its dual group with a pinning over $\overline{\mathbb{Q}}_\ell$. The center of \hat{G} is denoted by $Z(\hat{G})$. The F -structure on G induces an action $\text{Gal}(\overline{F}/F) \rightarrow \text{Aut}(\hat{G}, \hat{B}, \hat{T}, \hat{X})$.

2.1. A Galois cohomology computation. We recall a result of Kottwitz (cf. [Ko86, §1, §2]).

Theorem 2.1.1 (Kottwitz). *(i) If F is a local field of characteristic zero, and G is a connected reductive group over F , then there is a canonical map*

$$\alpha_G : H^1(F, G) \rightarrow \pi_0(Z(\hat{G})^{\Gamma_F})^D,$$

which is an isomorphism if F is a p -adic field.

(ii) If F is a global field, then there is an exact sequence of pointed set

$$H^1(F, G) \longrightarrow \bigoplus_v H^1(F_v, G) \xrightarrow{\oplus_v \alpha_{G,v}} \pi_0(Z(\hat{G})^{\Gamma_F})^D.$$

We do not need the precise construction of α_G . The following properties of α_G suffice for our purposes.

- If $G = T$ is a torus, then α_T is the given by

$$(2.1.1) \quad H^1(F, T) \cong H^1(F, \mathbb{X}^\bullet(T))^D \cong \pi_0(\hat{T}^{\Gamma_F})^D,$$

where the first isomorphism is the Tate-Nakayama duality, and the second isomorphism is induced by the exponential sequence $0 \rightarrow \mathbb{X}^\bullet(T) \rightarrow \text{Lie } \hat{T} \rightarrow \hat{T} \rightarrow 0$.

- If $T \subset G$ is a maximal torus, then the following diagram is commutative

$$(2.1.2) \quad \begin{array}{ccc} H^1(F, T) & \xrightarrow{\alpha_T} & \pi_0(\hat{T}^{\Gamma_F})^D \\ \downarrow & & \downarrow \\ H^1(F, G) & \xrightarrow{\alpha_G} & \pi_0(Z(\hat{G})^{\Gamma_F})^D, \end{array}$$

where the right vertical map is induced by $Z(\hat{G}) \subset \hat{T}$.

Note that if $G = G_{\text{ad}}$ is an adjoint group, $\pi_0(Z(\hat{G})^{\Gamma_F}) = Z(\hat{G})^{\Gamma_F}$.

We will be interested in using the above result to construct inner forms of a reductive group. For us, an inner form of a connected reductive group G over a field F is a cohomology class of $H^1(F, G_{\text{ad}})$, which as usual can be represented as (G', Ψ) , where G' is an F -form of G , and

$$\Psi : G \rightarrow G'$$

(usually called an inner twist) is a Γ_F -stable $G_{\text{ad}}(\overline{F})$ -orbit of \overline{F} -isomorphisms. Recall that in general, given an F -form G' of G , there may be more than one way to upgrade it to an inner form $\Psi : G \rightarrow G'$ (if such a Ψ exists). I.e., the fibers of $H^1(F, G_{\text{ad}}) \rightarrow H^1(F, \text{Aut}(G))$ may consist of more than one element. However, if G' is quasi-split, or if $F = \mathbb{R}$ and G' is compact modulo center, then the fiber of is a singleton (if non-empty). So we can talk about the quasi-split inner form (or the compact modulo center inner form) of G .

Now let G and G' be two connected reductive group over \mathbb{Q} , and assume that $G \otimes \mathbb{A}_f \simeq G' \otimes \mathbb{A}_f$. Note that they share the same pinned Langlands dual group $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$. In addition, the action of $\text{Gal}(\overline{\mathbb{Q}}_\ell/\mathbb{Q}_\ell)$ on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ induced by G and G' are the same at every finite place ℓ , and therefore the action of $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$ induced by G and G' are the same. It follows that we can choose an inner twist $\Psi : G \rightarrow G'$ so that (G', Ψ) defines an element $H^1(\mathbb{Q}, G_{\text{ad}})$, which is trivial when restricted to all finite places. Then the class $[(G'_{\mathbb{R}}, \Psi_{\mathbb{R}})] \in H^1(\mathbb{R}, G_{\text{ad}})$ maps to zero under

the composition $H^1(\mathbb{R}, G_{\text{ad}} \otimes \mathbb{R}) \rightarrow (Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{R}}})^D \rightarrow (Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}})^D$. Conversely, let G be a connected reductive group over \mathbb{Q} , and let $\Psi_{\mathbb{R}} : G_{\mathbb{R}} \rightarrow G'_{\mathbb{R}}$ be an inner form of $G_{\mathbb{R}}$. Then

Lemma 2.1.2. *If the restriction of the character $\alpha_{G_{\text{ad}}, \infty}[(G'_{\mathbb{R}}, \Psi_{\mathbb{R}})]$ to $Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}} \subset Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{R}}}$ is trivial, then there exists a unique (up to isomorphism) inner form (G', Ψ) of G , which restricts to $[(G'_{\mathbb{R}}, \Psi_{\mathbb{R}})]$ at the infinite place, and which is trivial at all finite places, i.e. Ψ is $G_{\text{ad}}(\bar{\mathbb{A}}_f)$ -conjugate to an isomorphism $G \otimes \mathbb{A}_f \simeq G' \otimes \mathbb{A}_f$.*

Proof. The uniqueness follows from Hasse's principle for adjoint groups (cf. [PR94, Theorem 6.22]). The existence is a direct consequence the above theorem applied to G_{ad} . \square

As we shall see below, there are many examples of $(G, G'_{\mathbb{R}}, \Psi_{\mathbb{R}})$ that satisfy the assumption of the lemma.

2.1.3. We prove a result that makes Lemma 2.1.2 more useful. For the purpose, we first assume that F is real and compute α_G in some cases. Let G be a real reductive group, with \mathfrak{g} its Lie algebra. Let $\mathbb{S} = \text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. Let X be a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h : \mathbb{S} \rightarrow G$ such that for some (and therefore every) $h \in X$,

- (W) under $\text{Ad} \circ h : \mathbb{S} \rightarrow \text{GL}(\mathfrak{g})$, \mathfrak{g} acquires a real Hodge structure of weight zero, i.e. the weight homomorphism $w_h : \mathbb{G}_m \subset \mathbb{S} \xrightarrow{h} G$ factors through the center of G , and
- (P) $\text{adh}(i)$ is a Cartan involution of G_{ad} (note that by (W), $\text{adh}(-1) = 1$ so $\text{adh}(i)$ is an involution).

These two conditions in particular imply that in the inner class of G , there is G_c that is compact modulo center. Indeed,

$$(2.1.3) \quad G_c = \{g \in G(\mathbb{C}) \mid h(i)\iota g = gh(i)\},$$

where ι is the conjugation with respect to the real structure of G . Therefore, G_c determines a class $[G_c] \in H^1(\mathbb{R}, G_{\text{ad}})$.

Let us fix the isomorphism $\mathbb{S}_{\mathbb{C}} = \mathbb{G}_m \times \mathbb{G}_m$ given as follows: For any \mathbb{C} -algebra R ,

$$(2.1.4) \quad (\mathbb{C} \otimes_{\mathbb{R}} R)^{\times} \simeq R^{\times} \times R^{\times}, \quad z \otimes r \mapsto (zr, \bar{z}r).$$

Let $\mathbb{G}_m \rightarrow \mathbb{S}_{\mathbb{C}}$ be the inclusion of the first factor. Then the map $\{h\}$ induces a homomorphism

$$(2.1.5) \quad \mu_h : \mathbb{G}_m \rightarrow \mathbb{S}_{\mathbb{C}} \rightarrow G \otimes_{\mathbb{R}} \mathbb{C}.$$

As h vary in X , μ_h form a conjugacy classes of 1-parameter subgroups of G over \mathbb{C} . In the sequel, μ_h is also denoted by μ for simplicity. Let μ_{ad} denote the composition of μ with the projection $G \rightarrow G_{\text{ad}}$. We regard μ_{ad} as a character of \hat{T}_{sc} , and by restriction to $Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{R}}}$, it defines an element $\bar{\mu}_{\text{ad}} \in (Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{R}}})^D$.

Lemma 2.1.4. *We have $\alpha_{G_{\text{ad}}}[G_c] = \bar{\mu}_{\text{ad}}$.*

Proof. We fix an h and therefore a cocharacter $\mu = \mu_h$ (not just a conjugacy class). Let h_{ad} denote the composition of h with the projection $G \rightarrow G_{\text{ad}}$. We can assume that $h : \mathbb{S} \rightarrow T \subset G$, where T is a fundamental Cartan subgroup of G . Note that $[G_c]$ is represented by the conjugation by $h_{\text{ad}}(i)$, where the element $h_{\text{ad}}(i)$ also represents a cocycle in $H^1(\mathbb{R}, T_{\text{ad}})$, denoted by $[h_{\text{ad}}(i)]$. It is enough to prove that under the isomorphism

$$\alpha_T : H^1(\mathbb{R}, T_{\text{ad}}) \simeq \pi_0(\hat{T}^{\Gamma_{\mathbb{R}}})^D,$$

$$[h_{\text{ad}}(i)] = [\mu_{\text{ad}}].$$

Note that as element in $T_{\text{ad}}(\mathbb{C})$,

$$h_{\text{ad}}(i) = (h_{\text{ad}})_{\mathbb{C}}(i, -i) = h_{\text{ad}}(-1, 1) = \mu_{\text{ad}}(-1),$$

where the first and the third identities are by definition, and the second identity follows from (W).

Note that T_{ad} is a compact torus, and the complex conjugation acts on $\mathbb{X}^\bullet(T_{\text{ad}})$ by -1 . Therefore,

$$\begin{cases} H^1(\mathbb{R}, T_{\text{ad}}) & \simeq (T_{\text{ad}})_2 \simeq \{t \in T_{\text{ad}} \mid t^2 = 1\}, \\ H^1(\mathbb{R}, \mathbb{X}^\bullet(T_{\text{ad}})) & \simeq \mathbb{X}_\bullet(\hat{T}_{\text{sc}})/2, \\ \hat{T}_{\text{sc}}^{\Gamma_{\mathbb{R}}} & = (\hat{T}_{\text{sc}})_2 = \{t \in \hat{T}_{\text{sc}} \mid t^2 = 1\}, \end{cases}$$

The pairing $H^1(\mathbb{R}, T_{\text{ad}}) \times H^1(\mathbb{R}, \mathbb{X}^\bullet(T_{\text{ad}})) \rightarrow \mathbb{Q}/\mathbb{Z}$ is given by

$$(T_{\text{ad}})_2 \times \mathbb{X}^\bullet(T_{\text{ad}})/2 \rightarrow \mathbb{Z}/2, \quad (t, \bar{\lambda}) \mapsto \check{\lambda}(t),$$

where $\check{\lambda} \in \mathbb{X}^\bullet(T_{\text{ad}})$ and $\bar{\lambda}$ denote its image in $\mathbb{X}^\bullet(T_{\text{ad}})/2$. The isomorphism $H^1(\mathbb{R}, \mathbb{X}^\bullet(T_{\text{ad}})) \simeq \pi_0(\hat{T}_{\text{sc}}^{\Gamma_{\mathbb{R}}})$ is given by

$$\mathbb{X}_\bullet(\hat{T}_{\text{sc}})/2 \simeq (\hat{T}_{\text{sc}})_2, \quad \bar{\lambda} \mapsto \check{\lambda}(-1).$$

Putting all above together, we see that for $\bar{\lambda} \in \mathbb{X}_\bullet(\hat{T}_{\text{sc}})/2 \simeq \pi_0(Z(\hat{G}_{\text{sc}}))$,

$$\mu_{\text{ad},1}(\bar{\lambda}) = \mu_{\text{ad}}(\check{\lambda}(-1)),$$

and

$$([h_{\text{ad}}(i)], \bar{\lambda}) = ([\mu_{\text{ad}}(-1)], \bar{\lambda}) = \check{\lambda}(\mu_{\text{ad}}(-1)).$$

Therefore $[h_{\text{ad}}(i)] = \bar{\mu}_{\text{ad}}$ as an element in $\pi_0(\hat{T}_{\text{sc}}^{\Gamma_{\mathbb{R}}})^D$. \square

Alternatively, we may regard G as an inner form of G_c , whose class $[G] \in H^1(\mathbb{R}, G_{c,\text{ad}})$ under the map $\alpha_{G_c} : H^1(\mathbb{R}, G_{c,\text{ad}}) \rightarrow (Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{R}}})^D$ then is given by $-\bar{\mu}_{\text{ad}}$.

Corollary 2.1.5. *Let (G, X) and (G', X') be two pairs, consisting of a real reductive group and a conjugacy class of homomorphisms from \mathbb{S} . Assume that both pairs satisfy (W) and (P) as above, and that the corresponding compact (modulo center) inner forms are isomorphic $G_c \simeq G'_c$. Then there is a canonical inner twist $\Psi : G \rightarrow G'$ such that the image of $[(G', \Psi)] \in H^1(\mathbb{R}, G)$ under α_G is $\bar{\mu}_{\text{ad}} - \bar{\mu}'_{\text{ad}}$, where $\bar{\mu}_{\text{ad}}$ and $\bar{\mu}'_{\text{ad}}$ are the characters of $Z(\hat{G})^{\Gamma_{\mathbb{R}}}$ corresponding to (G, X) and (G', X') as defined above.*

Proof. We choose $h : \mathbb{S} \rightarrow G$, and $h' : \mathbb{S} \rightarrow G'$. Let $T \subset G$ (resp. $T' \subset G'$) be a fundamental Cartan containing $h(\mathbb{C}^\times)$ (resp. $h'(\mathbb{C}^\times)$). Let G_c (resp. G'_c) be the subgroup of $G(\mathbb{C})$ (resp. $G'(\mathbb{C})$) as defined in (2.1.3). By assumption, we can choose an isomorphism $\iota : G_c \simeq G'_c$ sending T to T' . Then we have the inner twist $G_{\mathbb{C}} \cong (G_c)_{\mathbb{C}} \stackrel{\iota}{\simeq} (G'_c)_{\mathbb{C}} \cong G'_{\mathbb{C}}$, which is independent of any choices. Its cohomology class in $H^1(\mathbb{R}, G_{\text{ad}})$ comes from a class in $H^1(\mathbb{R}, T_{\text{ad}})$, which in turn can be represented by $\iota^{-1}(h'_{\text{ad}}(i))h_{\text{ad}}(i) = \iota^{-1}(\mu'_{\text{ad}}(-1))\mu_{\text{ad}}(-1)$. As argued in Lemma 2.1.4, under α_G , this is $\bar{\mu}_{\text{ad}} - \bar{\mu}'_{\text{ad}}$. \square

Combining with Lemma 2.1.2,

Corollary 2.1.6. *Let G be a connected reductive group defined over \mathbb{Q} and assume that there exists a $G(\mathbb{R})$ -conjugacy class X of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying (W) and (P). Let $\mu = \mu_h$ be as in (2.1.5). Let $(G'_{\mathbb{R}}, X')$ be another pair, consisting of a real algebraic group and a conjugacy class of homomorphisms $h' : \mathbb{S} \rightarrow G'_{\mathbb{R}}$, whose compact inner form is isomorphic to the one for $G_{\mathbb{R}}$. Then there is a canonical inner twist $\Psi_{\mathbb{R}} : G_{\mathbb{R}} \rightarrow G'_{\mathbb{R}}$. Let $\mu' = \mu_{h'}$. If $\mu_{\text{ad}}|_{Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}}} = \mu'_{\text{ad}}|_{Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}}}$, then up to isomorphism there exists a unique inner form (G', Ψ) of G that is trivial at all finite places and restricts to $(G'_{\mathbb{R}}, \Psi_{\mathbb{R}})$ at the infinite place.*

In particular, if $\mu_{\text{ad}}|_{Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}}} = 1$, then up to isomorphism there exists a unique inner form (G', Ψ) of G such that $G'_{\mathbb{R}}$ is compact modulo center and Ψ is $G_{\text{ad}}(\overline{\mathbb{A}}_f)$ -conjugate to an isomorphism $G \otimes \mathbb{A}_f \simeq G' \otimes \mathbb{A}_f$.

Remark 2.1.7. In Proposition 4.2.10, we will give a condition at a finite prime p that ensures the last assumption of the above corollary hold.

Example 2.1.8. We give two examples. The first example is well known.

(i) $G = D^\times$, where D is a quaternion algebra over a totally real field F , regarded as a reductive group over \mathbb{Q} . Write $G_{\mathbb{R}} \simeq (\mathrm{GL}_2)^r \times (\mathbb{H})^s$. In this case $Z(\hat{G}_{\mathrm{sc}})^{\Gamma_{\mathbb{R}}} \simeq (\mathbb{Z}/2)^{r+s}$, and $Z(\hat{G}_{\mathrm{sc}})^{\Gamma_{\mathbb{Q}}} \simeq \mathbb{Z}/2$ embeds diagonally into $(\mathbb{Z}/2)^{r+s}$. There is a standard $h : \mathbb{S} \rightarrow (G_{\mathrm{ad}})_{\mathbb{R}}$ given as

$$h(a + bi) = \left(\begin{pmatrix} a & b \\ -b & a \end{pmatrix}^r, 1^s \right).$$

Then it is easy to check $\mu_{\mathrm{ad},1}(a_1, \dots, a_r, a_{r+1}, \dots, a_{r+s}) = a_1 \cdots a_r$. In particular, the value of μ_{ad} on the only non-trivial element $(-1, \dots, -1) \in Z(\hat{G})^{\Gamma_{\mathbb{Q}}}$ is $(-1)^r$. We see that if $r' \equiv r \pmod{2}$, there is a quaternion algebra D' over F , that is isomorphic to D at all the finite places, and split at r' real places. Then the group $G' = D'^{\times}$ is isomorphic to G at all finite places as in Corollary 2.1.6. In particular, if r is even, we have G' that is compact modulo center at the infinity place.

(ii) Let G be a unitary (similitude) group (or its inner form) over a totally real field F , associated to a CM extension E/F . We regard G as a reductive group over \mathbb{Q} . Write $(G_{\mathrm{ad}})_{\mathbb{R}} = \prod_i \mathrm{PU}(p_i, q_i)$. Note that

$$Z(\hat{G}_{\mathrm{sc}})^{\Gamma_{\mathbb{R}}} = \begin{cases} \{1\} & p_i + q_i \text{ is odd,} \\ \prod_i \mathbb{Z}/2 & p_i + q_i \text{ is even.} \end{cases}$$

In the former case, the assumption of Lemma 2.1.2 is empty so there is a unitary group G' , isomorphic to G at all finite places, and $(G'_{\mathrm{ad}})_{\mathbb{R}} = \prod_i \mathrm{PU}(p'_i, q'_i)$ for any signature $\{(p'_i, q'_i)\}$. In particular, there is G' such that $G'_{\mathbb{R}}$ is compact modulo center. In the latter case, $Z(\hat{G}_{\mathrm{sc}})^{\Gamma_{\mathbb{Q}}} \simeq \mathbb{Z}/2$ embeds diagonally into $\prod_i \mathbb{Z}/2$. There is a standard $h : \mathbb{S} \rightarrow (G_{\mathrm{ad}})_{\mathbb{R}}$ given as

$$h(z) = (\mathrm{diag}\{(z/\bar{z})^{p_1}, 1^{q_1}\}, \dots, \mathrm{diag}\{(z/\bar{z})^{p_r}, 1^{q_r}\}),$$

and it is easy to check

$$\mu_{\mathrm{ad}} : Z(\hat{G})^{\Gamma_{\mathbb{R}}} \rightarrow \mathbb{Z}/2, \quad \mu_{\mathrm{ad}}(a_1, \dots, a_r) = \prod a_i^{p_i}.$$

In particular, the value of μ_{ad} on the only non-trivial element $(-1, \dots, -1) \in Z(\hat{G})^{\Gamma_{\mathbb{Q}}}$ is $(-1)^{\sum p_i} = (-1)^{\sum p_i q_i}$. We see that if $\sum p_i q_i = \sum p'_i q'_i$, we have G' as in Corollary 2.1.6 that is isomorphic to G at all finite places, and is of signature $\{(p'_i, q'_i)\}$ at the infinite place. In particular, if $\sum p_i q_i$ is even, there G' , isomorphic to G at all finite places, and is compact modulo center at the infinite places.

2.2. A Jacquet-Langlands transfer in the case without endoscopy. Let G be a connected reductive group over \mathbb{Q} . We fix an inner twist $\Psi : G \rightarrow G'$ over $\overline{\mathbb{Q}}$, and an identification $\theta : G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$ so that

$$(2.2.1) \quad \Psi = \mathrm{Int}(h)\theta$$

for some $h \in G'_{\mathrm{ad}}(\overline{\mathbb{A}}_f)$.

We need to use some trace formulas in a simple way. Here we briefly explain our choice of measures. As usual, we use the Tamagawa measure for connected reductive groups over \mathbb{Q} . We fix compatible Haar measures dx_{∞} on $G(\mathbb{R})$ and on $G'(\mathbb{R})$, and a Haar measure dx^{∞} on $\theta : G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$ so that the Tamagawa measure dx decomposes as $dx^{\infty} dx_{\infty}$. Let $A_G = A_{G'}$ be the maximal split subtorus of $Z_G = Z_{G'}$. We use the canonical Haar measure on $A_G(\mathbb{R})^0$, coming from a basis of the \mathbb{Z} -lattice $\mathbb{X}^{\bullet}(A_G)$.

We fix a few more notations. Let $\mathfrak{a}_G = \mathrm{Hom}(\mathbb{X}^{\bullet}(G)_{\mathbb{Q}}, \mathbb{R})$, where $\mathbb{X}^{\bullet}(G)_{\mathbb{Q}}$ is the group of \mathbb{Q} -rational characters of G , and let $H_G : G(\mathbb{A}) \rightarrow \mathfrak{a}_G$ be the map given by $(H_G(g), \lambda) = \log |\lambda(g)|$. It induces an isomorphism $H_G : A_G(\mathbb{R})^0 \simeq \mathfrak{a}_G$. Let $G(\mathbb{A})^1 = \ker H_G$ and $G(\mathbb{R})^1 = G(\mathbb{R}) \cap G(\mathbb{A})^1$. Then we have the isomorphism $G(\mathbb{A})^1 \times A_G(\mathbb{R})^0 \cong G(\mathbb{A})$ given by multiplication and $G(\mathbb{Q}) \backslash G(\mathbb{A})^1$ has finite volume. For a (quasi)character $\chi : A_G(\mathbb{R})^0 \rightarrow \mathbb{C}^{\times}$, let $L_{\chi}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$ denote the space of

square integrable functions on $G(\mathbb{Q}) \backslash G(\mathbb{A})^1$, regarded as a representation of $G(\mathbb{A})$ on which $G(\mathbb{A})^1$ acts via right translation and $A_G(\mathbb{R})^0$ acts through χ . Similarly, we have $L_\chi^2(G'(\mathbb{Q}) \backslash G'(\mathbb{A}))$. Using $A_G(\mathbb{R})^0 \simeq \mathfrak{a}_G$, we can also regard χ as a character of \mathfrak{a}_G .

Given a smooth function h on $G(\mathbb{A})$ (resp. on $G(\mathbb{R})$), compactly supported modulo $A_G(\mathbb{R})^0$, and transforming by χ^{-1} under $A_G(\mathbb{R})^0$, and a representation π of $G(\mathbb{A})$ (resp. of $G(\mathbb{R})$), on which $A_G(\mathbb{R})^0$ acts through χ , the trace $\text{tr}(h|\pi)$ is understood as

$$\text{tr}(h \cdot (\chi \circ H_G) | \pi \otimes (\chi^{-1} \circ H_G)),$$

where $h \cdot (\chi \circ H_G)$ is regarded as a compactly supported function on $G(\mathbb{A})/A_G(\mathbb{R})^0$ (resp. $G(\mathbb{R})/A_G(\mathbb{R})^0$) that acts on the $G(\mathbb{A})/A_G(\mathbb{R})^0$ (resp. $G(\mathbb{R})/A_G(\mathbb{R})^0$)-representation $\pi \otimes (\chi^{-1} \circ H_G)$. We have similar notations for G' .

We further assume that:

- (1) G and G' are anisotropic modulo center over \mathbb{Q} ;
- (2) G_{der} is simply-connected;
- (3) $(A_G)_{\mathbb{R}}$ is the maximal split subtorus in the center of $G_{\mathbb{R}}$;
- (4) $G_{\mathbb{R}}$ has a compact modulo center inner form;
- (5) The endoscopy of G is trivial in the sense that for every $\gamma \in G(\mathbb{Q})$, the group $\mathfrak{K}(I_\gamma/\mathbb{Q})$ as defined in [Ko86, §4] is trivial.

Let ξ be an irreducible algebraic representation of G (defined over some number field). Let ξ^* denote the dual representation of ξ . Let χ_ξ denote the restriction of $\xi_{\mathbb{C}}^*$ to $A_G(\mathbb{R})^0$. Let Π_ξ be the set of equivalent classes of irreducible representations of $G_{\mathbb{R}}$ with the same central and infinitesimal characters as $\xi_{\mathbb{C}}^*$, and let Π_ξ^0 be the subset of discrete representations. It forms a single L-packet. For $\pi_\xi^0 \in \Pi_\xi^0$, let $f_{\pi_\xi^0}$ denote a pseudo coefficient normalized so that $\text{tr}(f_{\pi_\xi^0} | \pi_\xi^0) = 1$. Let

$$f_\xi = \frac{1}{|\Pi_\xi^0|} \sum_{\pi_\xi^0} f_{\pi_\xi^0}.$$

We similarly have f'_ξ .

Now let π_f be an irreducible $G(\mathbb{A}_f) = G'(\mathbb{A}_f)$ -module. Following Kottwitz ([Ko92b]) let $a_{G,\xi}(\pi_f)$ be the number⁹ defined by

$$a_{G,\xi}(\pi_f) = \sum_{\pi_\infty \in \Pi_\xi} m_G(\pi_f \otimes \pi_\infty) \text{tr}(f_\xi | \pi_\infty),$$

where $m_G(\pi_f \otimes \pi_\infty)$ denotes the multiplicity of $\pi_f \otimes \pi_\infty$ in $L_{\chi_\xi}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))$. Similarly, we have $a_{G',\xi}(\pi_f)$. Note that if $G'_{\mathbb{R}}$ is compact modulo center, $a_{G',\xi}(\pi_f) = m_{G'}(\pi_f \otimes \xi_{\mathbb{C}})$ is the automorphic multiplicity of $\pi_f \otimes \xi_{\mathbb{C}}$ in $L_{\chi_\xi}^2(G'(\mathbb{Q}) \backslash G'(\mathbb{A}))$.

Theorem 2.2.1. *We have $a_{G,\xi}(\pi_f) = a_{G',\xi}(\pi_f)$. In particular, if $G'_{\mathbb{R}}$ is compact modulo center, then $a_{G,\xi}(\pi_f) = m_{G'}(\pi_f \otimes \xi_{\mathbb{C}})$.*

Proof. We use θ to identify $G(\mathbb{A}_f)$ and $G'(\mathbb{A}_f)$. It is enough to prove that the following two distributions on $G(\mathbb{A}_f) \cong G'(\mathbb{A}_f)$ are equal: for $h \in C_c^\infty(G(\mathbb{A}_f))$,

$$(2.2.2) \quad \sum a_{G,\xi}(\pi_f) \text{tr}(h | \pi_f) = \sum a_{G',\xi}(\pi_f) \text{tr}(h | \pi_f).$$

⁹In [Ko92b], this number is simply denoted by $a(\pi_f)$. In addition, Kottwitz define $a(\pi_f)$ using $f_{\pi_\xi^0}$ for a single π_ξ^0 instead of the average f_ξ . But as explained in *loc. cit.*, this does not affect $a(\pi_f)$.

By the trace formula for the compact quotients $G(\mathbb{Q}) \backslash G(\mathbb{A})^1$, the left hand side of (2.2.2) is equal to

$$(2.2.3) \quad \mathrm{tr}(hf_\xi \mid L_{\chi_\xi}^2(G(\mathbb{Q}) \backslash G(\mathbb{A}))) = \sum_{\gamma \in G(\mathbb{Q})/\sim} \tau(I_\gamma) O_\gamma(hf_\xi).$$

Let us explain the notations in the above formula:

- $G(\mathbb{Q})/\sim$ denotes the set of conjugacy classes of $G(\mathbb{Q})$,
- $\tau(I_\gamma) = \mathrm{vol}(I_\gamma(\mathbb{Q}) \backslash I_\gamma(\mathbb{A})/A_G(\mathbb{R})^0)$ is the Tamagawa number for the centralizer I_γ of γ in G ;
- O_γ denotes the usual orbital integral.

As $\mathfrak{K}(I_\gamma/\mathbb{Q})$ is trivial by assumption, by the argument of [Ko86, §9], one can pseudo-stabilize (2.2.3) as

$$\tau(G) \sum_{\gamma \in G(\mathbb{Q})/\mathrm{st}\sim} SO_\gamma(hf_\xi),$$

where $G(\mathbb{Q})/\mathrm{st}\sim$ denote the set of stable conjugacy classes of $G(\mathbb{Q})$. Likewise, the right hand side of (2.2.2) can be written as

$$(2.2.4) \quad \tau(G') \sum_{\gamma \in G'(\mathbb{Q})/\mathrm{st}\sim} SO_\gamma(hf'_\xi).$$

Recall that $\tau(G) = \tau(G')$ by [Ko88]. Also recall the stable orbital integrals of f_ξ , [Ko92b, Lemma 3.1].

$$SO_\gamma(f_\xi) = \begin{cases} \mathrm{tr} \xi_{\mathbb{C}}(\gamma) \cdot \mathrm{vol}(I(\mathbb{R})/A_G(\mathbb{R})^0)^{-1} \cdot e(I) & \gamma \text{ elliptic semisimple} \\ 0 & \gamma \text{ non elliptic,} \end{cases}$$

where I denotes the inner for of the centralizer of γ in G that is anisotropic modulo A_G . In particular, $SO_\gamma(f_\xi) = SO_{\gamma'}(f'_\xi)$ if γ and γ' are associated.

Now the theorem follows from the following lemma. \square

Lemma 2.2.2. *The stable conjugacy classes of $\gamma \in G(\mathbb{Q})$ with γ_∞ elliptic semisimple are bijective to stable conjugacy classes of $\gamma' \in G'(\mathbb{Q})$ with γ'_∞ elliptic semisimple.*

Proof. Using [Ko86, Theorem 6.6], and our assumption $\mathfrak{K}(I_\gamma/\mathbb{Q}) = 0$, it is enough to show that there is a bijection induced by ψ between stable conjugacy classes in $G(\mathbb{A})$ (i.e. elements $\gamma \in G(\mathbb{A})$ up to $G(\mathbb{A})$ -conjugacy) with γ_∞ semisimple elliptic and stable conjugacy classes $\gamma' \in G'(\mathbb{A})$ with γ'_∞ semisimple elliptic. Namely, given such $\gamma \in G(\mathbb{A})$, we claim that there is γ' from $G'(\mathbb{A})$ such that γ' and $\psi(\gamma)$ are conjugate under $G'(\mathbb{A})$ and vice versa. Indeed, for every finite place v , we have $\psi_v = \mathrm{Int}(h_v) \circ \theta_v$ by (2.2.1). Then we take $\gamma'_v = \theta_v(\gamma_v)$. For $v = \infty$, we can assume that $\gamma_\infty \in T(\mathbb{R}) \subset G(\mathbb{R})$, where T is a fundamental Cartan subgroup of $G_{\mathbb{R}}$. Then there exists some $h_\infty \in G'_{\mathrm{ad}}(\mathbb{C})$ such that $\theta_\infty = \mathrm{Int}(h_\infty) \circ \psi_\infty$ induces an embedding $T_{\mathbb{R}} \rightarrow G'_{\mathbb{R}}$. Then we take $\gamma'_\infty = \theta_\infty(\gamma_\infty)$. Clearly, we can reverse the construction to get $\gamma \in G(\mathbb{A})$ from $\gamma' \in G'(\mathbb{A})$ with the condition that γ'_∞ is semisimple elliptic. \square

2.3. A Jacquet-Langlands transfer for Shimura varieties. We retain notations as in the previous subsection, but now only assume (1)-(3). In addition we assume that (G, X) is a Shimura datum of Hodge type and let $\mathbf{Sh}_K(G, X)$ denote the corresponding Shimura variety (with some fixed level K). For an algebraic representation ξ of G , let \mathcal{L}_ξ denote the \mathbb{Q}_ℓ -local system on $\mathbf{Sh}_K(G, X)$. Then it is projective variety. Let p be an unramified prime. Let v be a place of the reflex field over p , and let ϕ_v denote the geometric Frobenius. Assume that the degree of $\mathbb{F}_v/\mathbb{F}_p$ is d . Let

$f^p \in C_c^\infty(G(\mathbb{A}_f^p))$ be a test function away from p . Recall that by the work of Kottwitz ([Ko90]) and Kisin-Shin-Zhu ([KSZ]), the Lefschetz trace formula for the Shimura variety gives

$$(2.3.1) \quad \mathrm{tr}(\phi_v^j \times f^p \mid H^*(\mathbf{Sh}_K(G, X), \mathcal{L}_\xi)) = \sum_{\mathcal{E}} \iota(G, H) ST_e(h), \quad j \gg 0$$

where \mathcal{E} is the set of isomorphism classes of elliptic endoscopic triple (H, s, η) of G , such that H is unramified at p , and $h = h_p h^p h_\infty$ is certain test function on $H(\mathbb{A})$ constructed in [Ko90, §7].

Assume that G has an inner form (G', Ψ) that is trivial at all finite places and $G' \otimes \mathbb{R}$ is compact modulo center. The geometric side of the Arthur-Selberg trace formula for G'

$$(2.3.2) \quad \sum_{\mathcal{E}} \iota(G, H) ST_e(f'^H),$$

where f' is a test function on $G'(\mathbb{A})$, and f'^H is its (usual) transfer to $H(\mathbb{A})$. As before, we identify $G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$ via θ .

Lemma 2.3.1. *If we choose the test function $f' = f'_{p,j} f'^p \Theta_\xi$ such that*

- $f'^p = f^p$,
- $f'_{p,j}$ is the element in the spherical Hecke algebra $H(G(\mathbb{Q}_p), K_p)$ which corresponds $\gamma\phi_p \mapsto \mathrm{tr}((\gamma\phi_p)^{dj}, V_{-\mu})$ under the Satake isomorphism (see §3.5 for a review of the Satake isomorphism),
- and Θ_ξ is the character of $\xi_{\mathbb{C}}$,

then in the two stable trace formula, the parts corresponding to the endoscopic triple $(G^, 1, \mathrm{id})$ coincide.*

Proof. This follows from the explicit construction of h in (2.3.1) as in [Ko90, §7]. □

Now we further assume G_{der} and G'_{der} have no simple factors that are the groups considered in the previous subsection. This in particular implies that for every finite place v of \mathbb{Q} , $G_{\mathrm{der}, \mathbb{Q}_v} \simeq G'_{\mathrm{der}, \mathbb{Q}_v}$ has *no* anisotropic factor.

Now for another place $p' \neq p$, if $f_{p'}$ is the Euler-Poincare function as constructed in [Ko88, Section 2], more precisely, the pullback of the Euler-Poincare function from $\bar{G}_{\mathbb{Q}_{p'}}$, the quotient of $G_{\mathbb{Q}_{p'}}$ by the maximal split torus in the center of $G_{\mathbb{Q}_{p'}}$, then as explained in [KS16⁺, Lemma A.7, Lemma A.12], the trace of f^p on an irreducible representation π^p of $G(\mathbb{A}_f^p)$ appearing in $H^*(\mathbf{Sh}_K(G, X), \mathcal{L}_\xi)$ (or in the spectrum side of the trace formula of G') is not zero only when $\pi_{p'}$ is an unramified character twist of the Steinberg representation. In addition, in both stable trace formulas, only the term corresponding to the endoscopic triple $(G^*, 1, \mathrm{id})$ is non-vanishing.

Let us write $H^*(\mathbf{Sh}_K(G, X), \mathcal{L}_\xi) = \sum_{\pi_f} \pi_f^K \otimes W(\pi_f)$ as before. For π_f such that $\pi_{f,p'}$ is a unramified character twist of the Steinberg representation, we choose $f = f_{p'} f'^p$ such that $\mathrm{tr}(f \mid \pi_f) = 1$ and its trace on any other representation appearing in $H^*(\mathbf{Sh}_K(G, X), \mathcal{L}_\xi)$ vanishes. Now, let S_{bad} be the finite set of primes such that if $p \notin S_{\mathrm{bad}}$, then K_p is hyperspecial and f_p is the unite of the unramified Hecke algebra at p . Then by the comparison of the stabilized Lefschetz trace formula at $p \notin S_{\mathrm{bad}}$, and the stable trace formula of G' , we have

$$\mathrm{tr}(f^p \mid \pi_f^p) \mathrm{tr}(\phi_v^j \mid W(\pi_f)) = m_{G'}(\pi_f \otimes \xi_{\mathbb{C}}) \mathrm{tr}(f^p \mid \pi_f) \mathrm{tr}(f'_{p,j} \mid \pi_{f,p})$$

Therefore, we obtain

Corollary 2.3.2. *Let π_f be an irreducible smooth $\theta : G(\mathbb{A}_f) \simeq G'(\mathbb{A}_f)$ representation, which is an unramified character twist of the Steinberg at some place p' . Let $p \notin S_{\mathrm{bad}}$. Then in the Grothendieck group of ϕ_v -modules,*

$$[W(\pi_f)] = m_{G'}(\pi_f \otimes \xi_{\mathbb{C}})[V_{\mu^*}],$$

where $[V_{\mu^*}]$ is the ϕ_v -module defined using the Satake parameter of $\pi_{f,p}$ as in [Ko92b, p.p. 656-657] (see also the introduction).

3. COMPLEMENTS ON THE GEOMETRIC SATAKE

The geometric Satake isomorphism is the main tool for this paper. In this section, we recall its content in a form we need in the sequel. We will also need some results on the crystal structures on Mirković-Vilonen cycles. Our presentation is parallel for both equal and mixed characteristic, following [Zhu17].

Throughout this section, we will use the following notations. Let F be a local field (of either equal or mixed characteristic), with ring of integers \mathcal{O} and residue field $k = \mathbb{F}_q$. Note that in either equal or mixed characteristic case, there is a unique map $W(k) \rightarrow \mathcal{O}$ that induces the identity map of the residue fields.

Let L be the completion of the maximal unramified extension of F , and \mathcal{O}_L its ring of integers with residue field $\bar{k} = \overline{\mathbb{F}_q}$. We fix a uniformizer ϖ of F . Let G be an *unramified* reductive group over \mathcal{O} . We denote by T the abstract Cartan subgroup of G . Recall that it is defined as the quotient of a Borel subgroup $B \subset G$ by its unipotent radical. It turns out that T is independent of the choice of B up to a canonical isomorphism. Let $S \subset T$ denote the maximal split subtorus. When we need to embed T (or S) into G as a (split) maximal torus, we will state it explicitly.

Let $\mathbb{X}^\bullet = \mathbb{X}^\bullet(T)$ denote the weight lattice, i.e. the free abelian group $\text{Hom}(T_{\bar{F}}, \mathbb{G}_m)$, and let $\mathbb{X}_\bullet = \mathbb{X}_\bullet(T)$ denote the coweight lattice, i.e. the dual of $\mathbb{X}^\bullet(T)$. Let $\Phi \subset \mathbb{X}^\bullet$ (resp. $\Phi^\vee \subset \mathbb{X}_\bullet$) denote the set of roots (resp. coroots). The Borel subgroup $B \subset G$ determines the semi-group of dominant coweights $\mathbb{X}_\bullet^+ \subset \mathbb{X}_\bullet$ and the subset of positive roots $\Phi^+ \subset \Phi$, which turn out to be independent of the choice of B . Let $\Delta \subset \Phi^+$ be the set of simple roots and Δ^\vee the corresponding set of simple coroots. The quadruple $(\mathbb{X}^\bullet, \Delta, \mathbb{X}_\bullet, \Delta^\vee)$ is called the *based root datum* of G . The q -power (arithmetic) Frobenius σ acts on $(\mathbb{X}^\bullet, \Delta, \mathbb{X}_\bullet, \Delta^\vee)$, preserving \mathbb{X}_\bullet^+ . Let $\mathbb{X}_{\bullet,\sigma}$ denote the σ -coinvariants of \mathbb{X}_\bullet , and let $\mathbb{X}_{\bullet,\sigma}^+$ denote the image of the map $\mathbb{X}_\bullet^+ \rightarrow \mathbb{X}_{\bullet,\sigma}$. For $\mu \in \mathbb{X}_\bullet$, its image in $\mathbb{X}_{\bullet,\sigma}$ is denoted by μ_σ .

Recall that there is a partial order \preceq on \mathbb{X}_\bullet : $\lambda \preceq \mu$ if $\mu - \lambda$ is a non-negative integral linear combinations of simple coroots; we say $\lambda \prec \mu$ if $\lambda \preceq \mu$ but $\lambda \neq \mu$. The restriction of the partial order to \mathbb{X}_\bullet^+ sometimes is called the *Bruhat order* on \mathbb{X}_\bullet^+ . We will usually denote by $2\rho \in \mathbb{X}^\bullet$ the sum of all positive roots, and put $\rho = \frac{1}{2}(2\rho) \in \mathbb{X}^\bullet(T) \otimes \mathbb{Q}$.

Let W denote the absolute Weyl group, and $W_0 = W^\sigma$ the relative Weyl group of G . There is a canonical bijection $\mathbb{X}_\bullet/W \cong \mathbb{X}_\bullet(T)^+$.

Let $\mathbf{Aff}_k^{\text{pf}}$ denote the category of perfect k -algebras. For a perfect k -algebra R , we define the ring of Witt vectors in R with coefficients in \mathcal{O} as

$$(3.0.1) \quad W_{\mathcal{O}}(R) := W(R) \hat{\otimes}_{W(k)} \mathcal{O} := \varprojlim_n W_{\mathcal{O},n}(R), \quad W_{\mathcal{O},n}(R) := W(R) \otimes_{W(k)} \mathcal{O}/\varpi^n.$$

Consider the usual Technmüller lifting $[-] : R \rightarrow W_{\mathcal{O}}(R)$ sending $r \mapsto [r]$. Note that if $\text{char } F = \text{char } k$, then $W_{\mathcal{O}}(R) \simeq R[[\varpi]]$, and $r \mapsto [r]$ a ring homomorphism. We sometimes write

$$(3.0.2) \quad D_{n,R} = \text{Spec } W_{\mathcal{O},n}(R), \quad D_R = \text{Spec } W_{\mathcal{O}}(R), \quad D_R^* = \text{Spec } W_{\mathcal{O}}(R)[1/\varpi],$$

thought as families of (punctured) discs parameterized by $\text{Spec } R$.

We refer to §A.1 for generalities of perfect algebraic geometry.

3.1. Geometry of Schubert varieties.

3.1.1. *Jet groups and loop groups.* Let H is an affine group scheme of finite type defined over \mathcal{O} , we denote by L^+H (resp. LH) the *jet group* (resp. *loop group*). As a presheaf, we have

$$(3.1.1) \quad L^+H(R) = H(W_{\mathcal{O}}(R)) \quad (\text{resp. } LH(R) = H(W_{\mathcal{O}}(R)[1/\varpi])).$$

It is represented by an affine group scheme (resp. ind-scheme). For $n \geq 0$, let L^nH be the n th jet group, i.e.

$$(3.1.2) \quad L^nH(R) = H(W_{\mathcal{O},n}(R)).$$

Then L^nH is represented by the perfection of an algebraic k -group (the usual Greenberg realization), and $L^+H = \varprojlim L^nH$. Finally, let

$$(3.1.3) \quad L^+H^{(n)} = \ker(L^+H \rightarrow L^nH)$$

be the n th principal congruence subgroup of L^+H .

For $m > n$, let $\pi_{m,n} : L^mH \rightarrow L^nH$ denote the natural projection. The kernel of $\pi_{m,n}$ is denoted by $L^{m-n}H^{(n)}$ so that $L^+H^{(n)}$ can be identified with the inverse limit $\varprojlim_{m \rightarrow \infty} L^{m-n}H^{(n)}$.

3.1.2. *Affine Grassmannians.* We denote by $\text{Gr} := \text{Gr}_G$ the affine Grassmannian of G over k . As a presheaf,

$$\text{Gr}(R) = \left\{ (\mathcal{E}, \beta) \left| \begin{array}{l} \mathcal{E} \text{ is a } G\text{-torsor on } D_R \text{ and} \\ \beta : \mathcal{E}|_{D_R^*} \simeq \mathcal{E}^0|_{D_R^*} \text{ is a trivialization} \end{array} \right. \right\},$$

where \mathcal{E}^0 denotes the trivial G -torsor. It is known that affine Grassmannian Gr is represented as the inductive limit of subfunctors $\text{Gr} = \varinjlim X_i$, with $X_i \rightarrow X_{i+1}$ closed embedding, and X_i being perfections of projective varieties ([BL94, F94, Zhu17, BS15]). In addition, it makes sense to define the category of perverse sheaves (with $\overline{\mathbb{Q}}_\ell$ -coefficients) on Gr as in direct limit

$$\text{P}(\text{Gr}) = \varinjlim \text{P}(X_i),$$

where the connecting functor is given by the pushforward along the closed embedding $X_i \rightarrow X_{i+1}$ (which is independent of the choice of the presentation $\text{Gr} = \varinjlim X_i$ up to a canonical equivalence).

3.1.3. *Bruhat decomposition.* Let K be an algebraically closed field containing \bar{k} . Recall that the Cartan decomposition induces a canonical bijection

$$(3.1.4) \quad G(W_{\mathcal{O}}(K)) \backslash G(W_{\mathcal{O}}(K)[1/\varpi]) / G(W_{\mathcal{O}}(K)) \cong \mathbb{X}_{\bullet}(T)/W \cong \mathbb{X}_{\bullet}(T)^+.$$

The $G(W_{\mathcal{O}}(K))$ -double coset corresponding to $\mu \in \mathbb{X}_{\bullet}(T)^+$ can be concretely realized as follows. We fix embeddings $T \subset B \subset G$. Each coweight $\mu \in \mathbb{X}_{\bullet}(T)$ defines a map $\mu : L^\times \rightarrow T(L) \subset G(L)$, and we denote by $\varpi^\mu = \mu(\varpi)$ the image of the uniformizer ϖ in $G(L) = LG(\bar{k})$. Although ϖ^μ depends on the choice of the embedding $T \subset G$ and the choice of the uniformizer, the double coset $G(W_{\mathcal{O}}(K))\varpi^\mu G(W_{\mathcal{O}}(K))$ does not, and corresponds to μ under the parameterization (3.1.4).

Now let \mathcal{E}_1 and \mathcal{E}_2 be two G -torsors over $D_K = \text{Spec} W_{\mathcal{O}}(K)$, and let $\beta : \mathcal{E}_1|_{D_K^*} \simeq \mathcal{E}_2|_{D_K^*}$ be an isomorphism over D_K^* . Such β is usually called a *modification* from \mathcal{E}_1 to \mathcal{E}_2 , and denoted by

$$\beta : \mathcal{E}_1 \dashrightarrow \mathcal{E}_2.^{10}$$

We attach to β an element

$$\text{Inv}(\beta) \in \mathbb{X}_{\bullet}(T)^+,$$

which we call the *relative position* of β , as follows: by choosing isomorphisms $\phi_1 : \mathcal{E}_1 \simeq \mathcal{E}^0$ and $\phi_2 : \mathcal{E}_2 \simeq \mathcal{E}^0$, one obtains an automorphism of the trivial G -torsor $\phi_2\beta\phi_1^{-1} \in \text{Aut}(\mathcal{E}^0|_{D_K^*})$ and therefore an element in $G(W_{\mathcal{O}}(K)[1/\varpi])$. Different choices of ϕ_1 and ϕ_2 will modify this element by left and right multiplication by elements from $G(W_{\mathcal{O}}(K))$. Therefore, via the bijection (3.1.4),

¹⁰The dotted arrow reminds that there may not be an actual morphism of G -torsors over D_R .

we attach β a dominant coweight $\text{Inv}(\beta) \in \mathbb{X}_\bullet(T)^+$. Note that when $G = \text{GL}_n$, $\text{Inv}(\beta)$ is just the Hodge polygon of the $n \times n$ -matrix determined by β .

Now let \mathcal{E}_1 and \mathcal{E}_2 be two G -torsors over D_R , and $\beta : \mathcal{E}_1|_{D_R^*} \simeq \mathcal{E}_2|_{D_R^*}$ a modification. Then for each geometric point $x \in \text{Spec} R$, by base change we obtain $(\mathcal{E}_1|_{D_x}, \mathcal{E}_2|_{D_x}, \beta_x : \mathcal{E}_1|_{D_x^*} \simeq \mathcal{E}_2|_{D_x^*})$. We write $\text{Inv}(\beta) \preceq \mu$ (resp. $\text{Inv}(\beta) = \mu$) for a dominant coweight μ if $\text{Inv}(\beta_x) \preceq \mu$ (resp. $\text{Inv}(\beta_x) = \mu$) for all geometric points $x \in \text{Spec} R$.

3.1.4. Schubert varieties. We define the (*spherical*) *Schubert variety* Gr_μ as the closed subset

$$\text{Gr}_\mu = \{(\mathcal{E}, \beta) \in \text{Gr} \mid \text{Inv}(\beta) \preceq \mu\},$$

of Gr . It contains the *Schubert cell*

$$\mathring{\text{Gr}}_\mu := \{(\mathcal{E}, \beta) \in \text{Gr} \mid \text{Inv}(\beta) = \mu\} = \text{Gr}_\mu \setminus \bigcup_{\lambda \prec \mu} \text{Gr}_\lambda$$

as an open dense subset¹¹. We will use the same notation to denote the image of $\varpi^\mu \in G(L) = LG(\bar{k}) \rightarrow \text{Gr}(\bar{k})$. Then $\mathring{\text{Gr}}_\mu$ is the L^+G -orbit through ϖ^μ . Note that there is a natural projection

$$(3.1.5) \quad \text{pr}_\mu : \mathring{\text{Gr}}_\mu \rightarrow (\bar{G}/\bar{P}_\mu)^{\text{pf}}, \quad g\varpi^\mu \bmod L^+G \mapsto \bar{g} \bmod \bar{P}_\mu^{\text{pf}},$$

where P_μ is the parabolic subgroup of G generated by root subgroups U_α for those $\langle \alpha, \mu \rangle \leq 0$, and taking bar means reduction modulo ϖ .

The Schubert variety Gr_μ is the perfection of a projective variety defined over the field of definition of the cocharacter μ (which is a finite extension of k).

3.1.5. Convolutional product of affine Grassmannians. For a sequence $\mu_\bullet = (\mu_1, \dots, \mu_t)$ of dominant coweights, we define Gr_{μ_\bullet} as the presheaf which classifies the isomorphism classes of modifications

$$(3.1.6) \quad \mathcal{E}_t \overset{\beta_t}{-} \succ \mathcal{E}_{t-1} \overset{\beta_{t-1}}{-} \succ \dots \overset{\beta_2}{-} \succ \mathcal{E}_1 \overset{\beta_1}{-} \succ \mathcal{E}_0 = \mathcal{E}^0$$

of G -torsors such that each β_i has relative position $\preceq \mu_i$ at all geometric points. For a non-negative integer n (which allows to be ∞), Let $\text{Gr}_{\mu_\bullet}^{(n)}$ denote the natural $L^n G$ -torsor over Gr_{μ_\bullet} classifying the chain of modifications of G -torsors over D_R as above together with an isomorphism $\epsilon_n : \mathcal{E}_t|_{D_{n,R}} \simeq \mathcal{E}^0|_{D_{n,R}}$. When $\mu_\bullet = \mu$, $\text{Gr}_\mu^{(\infty)} \subset LG$ is the preimage of Gr_μ in LG , and

$$(3.1.7) \quad \text{Gr}_\mu^{(n)} = \text{Gr}_\mu^{(\infty)} / L^+G^{(n)}.$$

Note that for every $s \leq t$, there is a projection

$$(3.1.8) \quad \text{pr}_s : \text{Gr}_{\mu_\bullet} \rightarrow \text{Gr}_{\mu_1, \dots, \mu_s}$$

by forgetting $(\mathcal{E}_t \overset{\beta_t}{-} \succ \dots \overset{\beta_{s+1}}{-} \succ \mathcal{E}_{s+1} \overset{\beta_{s+1}}{-} \succ \dots \overset{\beta_t}{-} \succ \mathcal{E}_0 = \mathcal{E}^0)$ in (3.1.6), whose fibers over geometric points (non-canonically) are isomorphic to $\text{Gr}_{\mu_{s+1}, \dots, \mu_t}$. This construction induces an isomorphism

$$(3.1.9) \quad \text{Gr}_{\mu_1, \dots, \mu_t} \cong \text{Gr}_{\mu_1, \dots, \mu_s} \tilde{\times} \text{Gr}_{\mu_{s+1}, \dots, \mu_t} := \text{Gr}_{\mu_1, \dots, \mu_s}^{(\infty)} \times^{L^+G} \text{Gr}_{\mu_{s+1}, \dots, \mu_t},$$

where $\text{Gr}_{\mu_1, \dots, \mu_s}^{(\infty)}$ is the L^+G -torsor over $\text{Gr}_{\mu_1, \dots, \mu_s}$ classifying a point $(\mathcal{E}_s \dashrightarrow \dots \dashrightarrow \mathcal{E}_0 = \mathcal{E}^0)$ of $\text{Gr}_{\mu_1, \dots, \mu_s}$ together with an isomorphism $\mathcal{E}_s \simeq \mathcal{E}^0$. So $\text{Gr}_{\mu_1, \dots, \mu_s} \tilde{\times} \text{Gr}_{\mu_{s+1}, \dots, \mu_t}$ is the twisted product of $\text{Gr}_{\mu_1, \dots, \mu_s}$ and $\text{Gr}_{\mu_{s+1}, \dots, \mu_t}$ (see [Zhu17, §A.1.3] for a more detailed discussion of the twisted product construction). Therefore, $\text{Gr}_{\mu_\bullet} \cong \text{Gr}_{\mu_1} \tilde{\times} \text{Gr}_{\mu_2} \tilde{\times} \dots \tilde{\times} \text{Gr}_{\mu_t}$ is called the *twisted product* of $\text{Gr}_{\mu_1}, \dots, \text{Gr}_{\mu_t}$.

In later sections, we will make use of the following definition.

Definition 3.1.6. For a sequence of dominant coweights μ_\bullet , we say a non-negative integer m is μ_\bullet -large if $m \geq \langle \mu_\bullet, \alpha_h \rangle$, where α_h is the highest root.

¹¹In some literature Schubert varieties are denoted by $\text{Gr}_{\preceq \mu}$ or $\overline{\text{Gr}}_\mu$ while Schubert cells are denoted by Gr_μ . We hope our notation will not cause confusions.

For example, if μ is a dominant minuscule coweight, then 1 is μ -large. Also, if m is μ_\bullet -large, then m is also $\sigma(\mu_\bullet)$ -large and μ_\bullet^* -large. Moreover, if m is μ_\bullet -large and m' is ν_\bullet -large, then $m + m'$ is $(\mu_\bullet, \nu_\bullet)$ -large.

This notion is used the following observation.

Lemma 3.1.7. *If m is μ_\bullet -large, then the left action of $L^+G^{(m+n)}$ on $\mathrm{Gr}_{\mu_\bullet}^{(n)}$ is trivial for every non-negative integer n , or equivalently, the left action of L^+G on $\mathrm{Gr}_{\mu_\bullet}^{(n)}$ factors through the action of $L^{m+n}G$.*

Proof. First we consider the case $\mu_\bullet = \mu$ is single coweight. The lemma follows from

$$\varpi^{-\mu} L^+G^{(m)} \varpi^\mu \subseteq L^+G^{(m-\langle \mu, \alpha_h \rangle)}.$$

To deal with the general case, we can choose a sequence of non-negative integers $m_\bullet = (m_1, \dots, m_t)$ such that $m = \sum m_i$ and m_i is μ_i -large. Since $m' := \sum_{i \geq 2} m_i$ is (μ_2, \dots, μ_t) -large, by induction there is a canonical isomorphism

$$(3.1.10) \quad \mathrm{Gr}_{\mu_1}^{(\infty)} \times^{L^+G^{(m'+n)}} \mathrm{Gr}_{\mu_2, \dots, \mu_t}^{(n)} \cong \mathrm{Gr}_{\mu_1}^{(\infty)} / L^+G^{(m'+n)} \times \mathrm{Gr}_{\mu_2, \dots, \mu_t}^{(n)}.$$

On the one hand, by (3.1.9), $\mathrm{Gr}_{\mu_\bullet}^{(n)}$ is the quotient of by the diagonal action of $L^{m'+n}G$ on the left hand side of (3.1.10). On the other hand, $L^+G^{(m+n)}$ acts trivially on $\mathrm{Gr}_{\mu_1}^{(\infty)} / L^+G^{(m'+n)} = \mathrm{Gr}_{\mu_1}^{(m'+n)}$ by the discussion in the single coweight case. Therefore, $L^+G^{(m+n)}$ acts trivially on $\mathrm{Gr}_{\mu_\bullet}^{(n)}$. \square

3.1.8. *The convolution map.* Let

$$(3.1.11) \quad m_{\mu_\bullet} : \mathrm{Gr}_{\mu_\bullet} \rightarrow \mathrm{Gr}$$

denote the *convolution map*. It sends the sequence of modifications (3.1.6) to the composition $(\mathcal{E}_t, \beta_1 \circ \dots \circ \beta_t) \in \mathrm{Gr}$. Both (3.1.8) and (3.1.11) are perfectly proper morphisms. It is known that m_{μ_\bullet} is a semi-small map (see [MV07, Lemma 4.4], [NP01, Lemma 9.3] and [Zhu17, Proposition 2.3]), i.e.

$$(3.1.12) \quad \dim \mathrm{Gr}_{\mu_\bullet} \times_{\mathrm{Gr}} \mathrm{Gr}_{\mu_\bullet} = \dim \mathrm{Gr}_{\mu_\bullet}.$$

This is equivalent to the fact that $(m_{\mu_\bullet})_* \mathrm{IC}$ is a direct sum of simple perverse sheaves on Gr , each of which is isomorphic to the intersection cohomology sheaf on some Gr_ν .

For λ_\bullet another sequence of dominant coweights, we define

$$\mathrm{Gr}_{\lambda_\bullet | \mu_\bullet}^0 := \mathrm{Gr}_{\lambda_\bullet} \times_{\mathrm{Gr}} \mathrm{Gr}_{\mu_\bullet}$$

where the fiber product is taken with respect to the convolution maps m_{λ_\bullet} and m_{μ_\bullet} . Explicitly, $\mathrm{Gr}_{\lambda_\bullet | \mu_\bullet}^0$ classifies the following commutative diagram of modifications of G -torsors

$$(3.1.13) \quad \begin{array}{ccccccc} \mathcal{E}'_s & \xrightarrow{\beta'_s} & \mathcal{E}'_{s-1} & \xrightarrow{\beta'_{s-1}} & \dots & \xrightarrow{\beta'_2} & \mathcal{E}'_1 & \xrightarrow{\beta'_1} & \mathcal{E}'_0 & \xlongequal{\quad} & \mathcal{E}^0 \\ \parallel & & & & & & & & & & \parallel \\ \mathcal{E}_t & \xrightarrow{\beta_t} & \mathcal{E}_{t-1} & \xrightarrow{\beta_{t-1}} & \dots & \xrightarrow{\beta_2} & \mathcal{E}_1 & \xrightarrow{\beta_1} & \mathcal{E}_0 & \xlongequal{\quad} & \mathcal{E}^0, \end{array}$$

where the top row defines a point of $\mathrm{Gr}_{\lambda_\bullet}$ and the bottom row defines a point of $\mathrm{Gr}_{\mu_\bullet}$.

Let $h_{\lambda_\bullet}^\leftarrow$ and $h_{\mu_\bullet}^\rightarrow$ denote the natural maps from $\mathrm{Gr}_{\lambda_\bullet | \mu_\bullet}^0$ to $\mathrm{Gr}_{\lambda_\bullet}$ and $\mathrm{Gr}_{\mu_\bullet}$ respectively. For the reason that will be clear in the sequel, we call

$$(3.1.14) \quad \mathrm{Gr}_{\lambda_\bullet} \xleftarrow{h_{\lambda_\bullet}^\leftarrow} \mathrm{Gr}_{\lambda_\bullet | \mu_\bullet}^0 \xrightarrow{h_{\mu_\bullet}^\rightarrow} \mathrm{Gr}_{\mu_\bullet}$$

a *Satake correspondence*. Note that $\mathrm{Gr}_{\lambda_\bullet | \mu_\bullet}^0 = \mathrm{Gr}_{\mu_\bullet | \lambda_\bullet}^0$.

Example 3.1.9. If $\lambda_\bullet = \lambda$ is a single dominant coweight, then $\mathrm{Gr}_{\mu_\bullet|\lambda}^0 = m_{\mu_\bullet}^{-1}(\mathrm{Gr}_\lambda)$, where $m_{\mu_\bullet} : \mathrm{Gr}_{\mu_\bullet} \rightarrow \mathrm{Gr}$ is the convolution map.

We need a corollary of (3.1.12). Recall that for a variety X over k , we denote by $H_i^{\mathrm{BM}}(X_{\bar{k}})$ the i th Borel-Moore homology of $X_{\bar{k}}$ (see Notation A.1.16 for our convention).

Proposition 3.1.10. (1) *We have*

$$\dim \mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0 \leq \frac{1}{2} (\dim \mathrm{Gr}_{\lambda_\bullet} + \dim \mathrm{Gr}_{\mu_\bullet}) = \langle \rho, |\lambda_\bullet| + |\mu_\bullet| \rangle.$$

(2) *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{P}(\mathrm{Gr})}((m_{\lambda_\bullet})_* \mathrm{IC}_{\lambda_\bullet}, (m_{\mu_\bullet})_* \mathrm{IC}_{\mu_\bullet}) \cong \mathrm{Corr}_{\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0}((\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC}_{\lambda_\bullet}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC}_{\mu_\bullet})).$$

Under the above isomorphism, the composition

$$(3.1.15) \quad \mathrm{Hom}((m_{\kappa_\bullet})_* \mathrm{IC}_{\kappa_\bullet}, (m_{\lambda_\bullet})_* \mathrm{IC}_{\lambda_\bullet}) \otimes \mathrm{Hom}((m_{\lambda_\bullet})_* \mathrm{IC}_{\lambda_\bullet}, (m_{\mu_\bullet})_* \mathrm{IC}_{\mu_\bullet}) \\ \rightarrow \mathrm{Hom}((m_{\kappa_\bullet})_* \mathrm{IC}_{\kappa_\bullet}, (m_{\mu_\bullet})_* \mathrm{IC}_{\mu_\bullet})$$

corresponds to

$$\mathrm{Corr}_{\mathrm{Gr}_{\kappa_\bullet|\lambda_\bullet}^0}((\mathrm{Gr}_{\kappa_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC})) \otimes \mathrm{Corr}_{\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0}((\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC})) \\ \rightarrow \mathrm{Corr}_{\mathrm{Gr}_{\kappa_\bullet|\mu_\bullet}^0}((\mathrm{Gr}_{\kappa_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC})),$$

obtained by the pushforward of the composition of cohomological correspondences along the perfectly proper morphism

$$\mathrm{Comp} : \mathrm{Gr}_{\kappa_\bullet|\lambda_\bullet}^0 \times_{\mathrm{Gr}_{\lambda_\bullet}} \mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0 \rightarrow \mathrm{Gr}_{\kappa_\bullet|\mu_\bullet}^0.$$

(3) *There is a canonical isomorphism*

$$\mathrm{Hom}_{\mathrm{P}(\mathrm{Gr})}((m_{\lambda_\bullet})_* \mathrm{IC}_{\lambda_\bullet}, (m_{\mu_\bullet})_* \mathrm{IC}_{\mu_\bullet}) \cong H_{\langle 2\rho, |\lambda_\bullet| + |\mu_\bullet| \rangle}^{\mathrm{BM}}(\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0).$$

Proof. These facts are well-known. But due to the importance, we sketch the arguments here.

(1) By the semismallness, the fiber of m_{μ_\bullet} over the point ϖ^ν is $\leq \langle \rho, |\mu_\bullet| - \nu \rangle$. It follows that

$$\dim \mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0 \times_{\mathrm{Gr}} \mathring{\mathrm{Gr}}_\nu \leq \langle \rho, |\lambda_\bullet| - \nu \rangle + \langle \rho, |\mu_\bullet| - \nu \rangle + \dim \mathring{\mathrm{Gr}}_\nu = \langle \rho, |\lambda_\bullet| + |\mu_\bullet| \rangle$$

for any ν . The dimension estimate of $\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0$ then follows.

(2) We write the diagram

$$\begin{array}{ccc} \mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0 & \xrightarrow{h_{\mu_\bullet}^\rightarrow} & \mathrm{Gr}_{\mu_\bullet} \\ h_{\lambda_\bullet}^\leftarrow \downarrow & & \downarrow m_{\mu_\bullet} \\ \mathrm{Gr}_{\lambda_\bullet} & \xrightarrow{m_{\lambda_\bullet}} & \mathrm{Gr}. \end{array}$$

Then by proper base change,

$$\begin{aligned} \mathrm{Hom}_{\mathrm{P}(\mathrm{Gr})}((m_{\lambda_\bullet})_* \mathrm{IC}, (m_{\mu_\bullet})_* \mathrm{IC}) &\cong \mathrm{Hom}_{\mathrm{D}_c^b(\mathrm{Gr}_{\mu_\bullet})}((m_{\mu_\bullet})^*(m_{\lambda_\bullet})_* \mathrm{IC}, \mathrm{IC}) \\ &\cong \mathrm{Hom}_{\mathrm{D}_c^b(\mathrm{Gr}_{\mu_\bullet})}((h_{\mu_\bullet}^\rightarrow)_*(h_{\lambda_\bullet}^\leftarrow)^* \mathrm{IC}, \mathrm{IC}) \\ &\cong \mathrm{Hom}_{\mathrm{D}_c^b(\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0)}((h_{\lambda_\bullet}^\leftarrow)^* \mathrm{IC}, (h_{\mu_\bullet}^\rightarrow)^! \mathrm{IC}) \\ &= \mathrm{Corr}_{\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^0}((\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC})). \end{aligned}$$

The last assertion of (2) follows from the definition of the composition of cohomological correspondences (see Appendix A.2).

(3) Let $\mathring{\mathrm{Gr}}_{\lambda_{\bullet}} \subset \mathrm{Gr}_{\lambda_{\bullet}}$ (resp. $\mathring{\mathrm{Gr}}_{\mu_{\bullet}} \subset \mathrm{Gr}_{\mu_{\bullet}}$) denote the open subset formed by the twisted product of the open Schubert cells, and let $\mathring{\mathrm{Gr}}_{\lambda_{\bullet}|\mu_{\bullet}}^0 = (h_{\lambda_{\bullet}}^{\leftarrow})^{-1}(\mathring{\mathrm{Gr}}_{\lambda_{\bullet}}) \cap (h_{\mu_{\bullet}}^{\rightarrow})^{-1}(\mathring{\mathrm{Gr}}_{\mu_{\bullet}})$. By restriction to the above open subsets, we have

(3.1.16)

$$\mathrm{Corr}_{\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0}((\mathrm{Gr}_{\lambda_{\bullet}}, \mathrm{IC}), (\mathrm{Gr}_{\mu_{\bullet}}, \mathrm{IC})) \rightarrow \mathrm{Corr}_{\mathring{\mathrm{Gr}}_{\lambda_{\bullet}|\mu_{\bullet}}^0}((\mathring{\mathrm{Gr}}_{\lambda_{\bullet}}, \overline{\mathbb{Q}}_{\ell}[\langle 2\rho, |\lambda_{\bullet}| \rangle]), (\mathring{\mathrm{Gr}}_{\mu_{\bullet}}, \overline{\mathbb{Q}}_{\ell}[\langle 2\rho, |\mu_{\bullet}| \rangle])).$$

Note that the latter space is canonically isomorphic to

$$H_{\langle 2\rho, |\lambda_{\bullet}| + |\mu_{\bullet}| \rangle}^{\mathrm{BM}}(\mathring{\mathrm{Gr}}_{\lambda_{\bullet}|\mu_{\bullet}}^0) \cong H_{\langle 2\rho, |\lambda_{\bullet}| + |\mu_{\bullet}| \rangle}^{\mathrm{BM}}(\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0),$$

where the isomorphism follows from Part (1). It remains to show that (3.1.16) is an isomorphism. First, if $\lambda_{\bullet} = \lambda$ is a single coweight, this is standard. For example, see [Zhu16⁺, Corollary 5.1.5] for a proof. The general case can be reduced to the single coweight case as follows. Let $u \in \mathrm{Hom}(\mathrm{IC}_{\lambda}, (m_{\lambda_{\bullet}})_* \mathrm{IC})$, considered as a cohomological correspondence by Part (2). Assume that its restriction to $\mathring{\mathrm{Gr}}_{\lambda_{\bullet}|\mu_{\bullet}}^0$ is zero. Then it is easy to see that for every $v \in \mathrm{Hom}_{\mathrm{P}(\mathrm{Gr})}((m_{\lambda_{\bullet}})_* \mathrm{IC}, (m_{\mu_{\bullet}})_* \mathrm{IC})$, the composition $u \circ v$, when regarded as a cohomological correspondence, becomes zero when restricted to $\mathring{\mathrm{Gr}}_{\lambda_{\bullet}|\mu_{\bullet}}^0$. Therefore $u \circ v$ is zero by the single coweight case. This implies that $u = 0$. Therefore, (3.1.16) is injective. On the other hand,

$$\dim \mathrm{Hom}_{\mathrm{P}(\mathrm{Gr})}((m_{\lambda_{\bullet}})_* \mathrm{IC}, (m_{\mu_{\bullet}})_* \mathrm{IC}) = \sum_{\lambda} \dim \mathrm{Hom}(\mathrm{IC}_{\lambda}, (m_{\lambda_{\bullet}})_* \mathrm{IC}) \dim \mathrm{Hom}(\mathrm{IC}_{\lambda}, (m_{\mu_{\bullet}})_* \mathrm{IC})$$

and

$$\dim H_{\langle 2\rho, |\lambda_{\bullet}| + |\mu_{\bullet}| \rangle}^{\mathrm{BM}}(\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0) = \sum_{\lambda} \dim H_{\langle 2\rho, \lambda + |\lambda_{\bullet}| \rangle}^{\mathrm{BM}}(\mathrm{Gr}_{\lambda|\lambda_{\bullet}}^0) \dim H_{\langle 2\rho, \lambda + |\mu_{\bullet}| \rangle}^{\mathrm{BM}}(\mathrm{Gr}_{\lambda|\mu_{\bullet}}^0).$$

It follows by the single coweight case that the two spaces have the same dimension. Therefore, (3.1.16) is an isomorphism. \square

Definition 3.1.11. An irreducible component of $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0$ of dimension $\langle \rho, |\lambda_{\bullet}| + |\mu_{\bullet}| \rangle$ is called a *Satake cycle*. The set of Satake cycles in $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0$ is denoted by $\mathbb{S}_{\lambda_{\bullet}|\mu_{\bullet}}$. For $\mathbf{a} \in \mathbb{S}_{\lambda_{\bullet}|\mu_{\bullet}}$, the cycle labelled by \mathbf{a} is denoted by $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^{0, \mathbf{a}}$.

Remark 3.1.12. (1) Since $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0 = \mathrm{Gr}_{\mu_{\bullet}|\lambda_{\bullet}}^0$, $\mathbb{S}_{\lambda_{\bullet}|\mu_{\bullet}} = \mathbb{S}_{\mu_{\bullet}|\lambda_{\bullet}}$.

(2) In general, $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0$ is not equi-dimensional (see [Hai06]). So the union of $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^{0, \mathbf{a}}$ for all $\mathbf{a} \in \mathbb{S}_{\lambda_{\bullet}|\mu_{\bullet}}$ need not be the whole $\mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0$.

(3) Later, we will identify the vector space appearing in Part (2) with another vector space, using the geometric Satake isomorphism, which justifies the name.

Now, let ν_{\bullet} and ξ_{\bullet} be two other (not necessarily non-empty) sequences of dominant coweights of length a and b respectively. We define

$$(3.1.17) \quad \mathrm{Gr}_{\nu_{\bullet}; \lambda_{\bullet}|\mu_{\bullet}; \xi_{\bullet}}^0 := \mathrm{Gr}_{\nu_{\bullet}}^0 \tilde{\times} \mathrm{Gr}_{\lambda_{\bullet}|\mu_{\bullet}}^0 \tilde{\times} \mathrm{Gr}_{\xi_{\bullet}}^0.$$

This is a closed subscheme of $\mathrm{Gr}_{(\nu_{\bullet}, \lambda_{\bullet}, \xi_{\bullet})|(\nu_{\bullet}, \mu_{\bullet}, \xi_{\bullet})}^0$, classifying those diagrams (3.1.13) that induce

$$\mathcal{E}_i = \mathcal{E}'_i, \quad 0 \leq i \leq a, \quad \mathcal{E}_{t-j} = \mathcal{E}'_{t-j}, \quad 0 \leq j \leq b,$$

or more explicitly, the diagrams

$$\begin{array}{cccccccccccccccccccc} \mathcal{E}'_{a+b+s} & \rightarrow & \cdots & \rightarrow & \mathcal{E}'_{a+s+1} & \rightarrow & \mathcal{E}'_{a+s} & \rightarrow & \mathcal{E}'_{a+s-1} & \rightarrow & \cdots & \rightarrow & \mathcal{E}'_{a+1} & \rightarrow & \mathcal{E}'_a & \rightarrow & \mathcal{E}'_{a-1} & \rightarrow & \cdots & \rightarrow & \mathcal{E}'_0 = \mathcal{E}^0 \\ \parallel & & \cdots & & \parallel & & \parallel & & & & \parallel & & \parallel & & \parallel & & \parallel & & \cdots & & \parallel & & \parallel \\ \mathcal{E}_{a+b+t} & \rightarrow & \cdots & \rightarrow & \mathcal{E}_{a+t+1} & \rightarrow & \mathcal{E}_{a+t} & \rightarrow & \mathcal{E}_{a+t-1} & \rightarrow & \cdots & \rightarrow & \mathcal{E}_{a+1} & \rightarrow & \mathcal{E}_a & \rightarrow & \mathcal{E}_{a-1} & \rightarrow & \cdots & \rightarrow & \mathcal{E}_0 = \mathcal{E}^0, \end{array}$$

To finish this subsection, we prove the following lemma, which will only be used in §6.3. Recall the notion of cohomologically smooth morphisms in Definition A.1.18.

Lemma 3.1.13. *Assume that n is sufficiently large. Then $gL^+G^{(n)} \mapsto g^{-1}L^+G$ induces a cohomologically smooth morphism $\mathrm{Gr}_\mu^{(n)} \rightarrow \mathrm{Gr}_{\mu^*}$.*

Proof. Note that the fiber over each g is isomorphic to $gL^+Gg^{-1}/L^+G^{(n)} \cong L^+G/g^{-1}L^+G^{(n)}g$, which is a L^+G -homogeneous space and therefore is perfectly smooth. This observation in fact already implies that $\mathrm{rev} : \mathring{\mathrm{Gr}}_\mu^{(n)} \rightarrow \mathring{\mathrm{Gr}}_{\mu^*}$ is perfectly smooth of relative dimension $n \dim G$, where $\mathring{\mathrm{Gr}}_\mu^{(n)}$ is the pre-image of the Schubert cell $\mathring{\mathrm{Gr}}$ in $\mathrm{Gr}_\mu^{(n)}$. So the lemma holds if μ is minuscule.

If $\mu = \mu^*$ is quasi-minuscule, there is a canonical resolution $\widetilde{\mathrm{Gr}}_\mu \rightarrow \mathrm{Gr}_\mu$ as from [Zhu17, Lemma 2.12] (note that Gr_μ was denoted by $\mathrm{Gr}_{\leq \mu}$ there), with geometrically connected fibers.

Now, now let $\mu = \mu_1 + \cdots + \mu_r$, where each μ_i is minuscule or quasi-minuscule. Let $\widetilde{\mathrm{Gr}}_{\mu_i} = \mathrm{Gr}_{\mu_i}$ if μ_i is minuscule. We can form the convolution product $\widetilde{\mathrm{Gr}}_{\mu_\bullet} := \widetilde{\mathrm{Gr}}_{\mu_1} \tilde{\times} \cdots \tilde{\times} \widetilde{\mathrm{Gr}}_{\mu_r}$, which is a resolution of Gr_μ . Write $\widetilde{\mathrm{Gr}}_{\mu_\bullet}^* := \mathrm{Gr}_{\mu_r^*} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{\mu_1^*}$. As in the proof of [Zhu17, Lemma 2.23, Lemma 2.24], the base change of $\mathrm{Gr}_\mu^{(n)} \rightarrow \mathrm{Gr}_{\mu^*}$ along this resolution is $\widetilde{\mathrm{Gr}}_{\mu_\bullet}^{(n)} \rightarrow \widetilde{\mathrm{Gr}}_{\mu_\bullet}^*$, which by Lemma A.1.20 (1) is cohomological smooth. Therefore, by Lemma A.1.20 (2), $\mathrm{Gr}_\mu^{(n)} \rightarrow \mathrm{Gr}_{\mu^*}$ is cohomologically smooth.

Finally, Gr is the union of Schubert varieties Gr_μ with μ a sum of minuscules and quasi-minuscules. Therefore, the lemma holds for all μ . \square

Remark 3.1.14. We expect that the map $\mathrm{Gr}_\mu^{(n)} \rightarrow \mathrm{Gr}_{\mu^*}$ is in fact perfectly smooth. This is the case in equal characteristic, but we are not able to prove it in mixed characteristic.

3.2. Geometry of semi-infinite orbits. We also recall the geometry of semi-infinite orbits in the affine Grassmannians and Mirković-Vilonen cycles. We need to fix embeddings $T \subset B \subset G$.

3.2.1. Mirković-Vilonen cycles. For λ a coweight of G , we use

$$S_\lambda = LU \varpi^\lambda L^+G / L^+G$$

to denote the semi-infinite LU -orbit on the affine Grassmannian.

Note that there is a natural projection

$$(3.2.1) \quad \begin{aligned} \pi_\lambda : LU &\longrightarrow S_\lambda \\ u &\longmapsto \varpi^\lambda u, \end{aligned}$$

realizing LU as an L^+U -torsor over S_λ . In addition, $S_\lambda = \varpi^\lambda S_0 \rightarrow \mathrm{Gr}$ is a locally closed embedding, as explained in [Zhu17, §2.2.1].

Recall the following basic result of Mirković-Vilonen cycles.

Theorem 3.2.2. *For λ and μ two coweights of G with μ dominant, every irreducible component of the intersection $S_\lambda \cap \mathrm{Gr}_\mu$ is of dimension $\langle \rho, \lambda + \mu \rangle$. In addition, the number of its irreducible components equals to the dimension of the λ -weight space $V_\mu(\lambda)$ of the irreducible representation V_μ of \hat{G} of highest weight μ .*

Proof. For the proof, see [GHKR06, Lemma 2.17.4, Proposition 5.4.2] when $\mathrm{char} F > 0$ and [Zhu17, Corollary 2.8] when $\mathrm{char} F = 0$. See also [MV07, Theorem 3.2(a)] for the proof of the dimension formula. \square

Notation 3.2.3. Irreducible components of $S_\lambda \cap \mathrm{Gr}_\mu$ are often referred to as the *Mirković-Vilonen cycles*. We will denote by $\mathbb{M}\mathbb{V}_\mu(\lambda)$ the set of irreducible components of $S_\lambda \cap \mathrm{Gr}_\mu$, and for $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$, write $(S_\lambda \cap \mathrm{Gr}_\mu)^\mathbf{b}$ the irreducible component (a.k.a. MV cycle) labeled by \mathbf{b} .

The following are some additional facts about MV cycles.

- Lemma 3.2.4.** (1) *The point $\varpi^\lambda \in (S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}}$ for every $\mathbf{b} \in \text{MV}_\mu(\lambda)$.*
(2) *The variety $(S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}}$ is stable under the action of L^+B .*
(3) *The set $\text{MV}_\mu(\lambda)$ is independent of the choice of the Borel B up to canonical isomorphisms.*

Proof. Recall that S_λ is the attractor for the \mathbb{G}_m -action on Gr given by $2\rho : \mathbb{G}_m \rightarrow T \rightarrow L^+T$. (1) and (2) follow. For (3), let B_1 and B_2 be two Borel subgroups, and choose $g \in G(\mathcal{O}_L)$ such that $gB_1g^{-1} = B_2$. For a chosen Borel subgroup B , we use S_λ^B and $\text{MV}_\mu^B(\lambda)$ denote the corresponding semi-infinite orbit and set of MV cycles. Then the action of g induces an isomorphism $(S_\lambda^{B_1} \cap \text{Gr}_\mu) \cong (S_\lambda^{B_2} \cap \text{Gr}_\mu)$ and therefore a bijection $\text{MV}_\mu^{B_1}(\lambda) \cong \text{MV}_\mu^{B_2}(\lambda)$. By (2), this bijection is independent of the choice of g . \square

3.2.5. Some explicit description of the MV cycles. Let us recall explicit descriptions of MV cycles in some special cases.

- (1) First, assume that $G = \text{PGL}_2$. Let us identify \mathbb{X}_\bullet with \mathbb{Z} in the usual way (so that $1 \in \mathbb{Z}$ corresponds to the dominant cocharacter $t \mapsto \begin{pmatrix} t & \\ & 1 \end{pmatrix}$). Then $S_\ell \cap \text{Gr}_m$ is non-empty if and only if $|\ell| \leq m$ and $2 \mid m - \ell$. In this case, there is a natural isomorphism

$$(3.2.2) \quad (\mathbb{A}^{\frac{\ell+m}{2}})^{\text{pf}} \simeq S_\ell \cap \text{Gr}_m, \quad (u_{\frac{\ell-m}{2}}, \dots, u_{\ell-1}) \mapsto \begin{pmatrix} 1 & \sum_{\frac{\ell-m}{2} \leq i \leq \ell-1} [u_i] \varpi^i & \\ & 1 & \end{pmatrix} \begin{pmatrix} \varpi^\ell & \\ & 1 \end{pmatrix} \pmod{L^+G}.$$

In particular, $S_\ell \cap \text{Gr}_m$ is irreducible.

- (2) Next, we assume that G is general, but μ is a minuscule or quasi-minuscule cocharacter of G . Recall that a dominant coweight μ of G is called (quasi-)minuscule if all (non-zero) weights of the irreducible representation V_μ of \hat{G} are in a single orbit under the Weyl group. Recall that if G is a simple group not of type A , then the unique quasi-minuscule (but non-minuscule) coweight is the unique *short* dominant coroot. The following statements summarize results of [NP01, §6-§7] and [Zhu17, §2.2.2].

If μ is a minuscule coweight of G , then $\text{Gr}_\mu = \mathring{\text{Gr}}_\mu$ and $S_\lambda \cap \text{Gr}_\mu$ is non-empty if and only if $\lambda = w\mu$ for some $w \in W$. In this case

$$(3.2.3) \quad S_\lambda \cap \text{Gr}_\mu = L^+U\varpi^\lambda L^+G/L^+G \cong L^+U\varpi^\lambda L^+U/L^+U$$

is irreducible, and the projection (3.1.5) maps it isomorphically to $(\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{\text{pf}}$.

Next, we consider the case when μ is quasi-minuscule. Then μ is a dominant coroot. Let Δ_μ denote the set of simple coroots that lie in the W -orbits of μ . Then $\text{Gr}_\mu = \mathring{\text{Gr}}_\mu \sqcup \text{Gr}_0$ and $S_\lambda \cap \text{Gr}_\mu$ is non-empty if and only if $\lambda \in W\mu$ or $\lambda = 0$.

If $\lambda = w\mu$, then

$$(3.2.4) \quad S_\lambda \cap \text{Gr}_\mu = L^+U\varpi^\lambda L^+G/L^+G \cong L^+U\varpi^\lambda L^+U/L^+U$$

is irreducible, and the projection (3.1.5) maps $S_\lambda \cap \text{Gr}_\mu$ onto $(\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{\text{pf}}$. In addition,

- if $\lambda = w\mu \succ 0$ is a positive coroot, then $S_\lambda \cap \text{Gr}_\mu = \text{pr}_\mu^{-1}(\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{\text{pf}}$, and
- if $\lambda = w\mu \prec 0$ is a negative coroot, then $\text{pr}_\mu : S_\lambda \cap \text{Gr}_\mu \rightarrow (\bar{U}w\bar{P}_\mu/\bar{P}_\mu)^{\text{pf}}$ is an isomorphism.

If $\lambda = 0$, then

$$S_0 \cap \text{Gr}_\mu = \bigcup_{\alpha \in \Delta_\mu} (S_0 \cap \text{Gr}_\mu)^\alpha,$$

where $(S_0 \cap \text{Gr}_\mu)^\alpha$ is an irreducible component of $(S_0 \cap \text{Gr}_\mu)$ whose \bar{k} -points can be written as

$$(3.2.5) \quad (S_0 \cap \text{Gr}_\mu)^\alpha = \text{Gr}_0 \bigsqcup \left(\text{pr}_\mu^{-1} \left(\bigsqcup_{w \in W/W_\mu, w\mu \preceq -\alpha} (\bar{U}w\bar{P}_\mu)^{\text{pf}} \right) \setminus \bigsqcup_{w\mu \preceq -\alpha} (S_{w\mu} \cap \text{Gr}_\mu) \right).$$

In this way, we establish a canonical bijection $\mathbb{M}\mathbb{V}_\mu(0) \cong \Delta_\mu$.

3.2.6. Convoluted product of MV cycles. Note that it makes sense to talk about convolutions of semi-infinite orbits. Let $\lambda_\bullet = (\lambda_1, \dots, \lambda_n)$ be a sequence of coweights of G . One can define

$$S_{\lambda_\bullet} := S_{\lambda_1} \tilde{\times} S_{\lambda_2} \tilde{\times} \dots \tilde{\times} S_{\lambda_n} \subset \text{Gr} \tilde{\times} \text{Gr} \tilde{\times} \dots \tilde{\times} \text{Gr}.$$

The formula

$$(3.2.6) \quad (\varpi^{\lambda_1} x_1, \dots, \varpi^{\lambda_n} x_n) \mapsto (\varpi^{\lambda_1} x_1, \varpi^{\lambda_1 + \lambda_2} (\varpi^{-\lambda_2} x_1 \varpi^{\lambda_2}) x_2, \dots)$$

with $x_i \in LU$ defines an isomorphism

$$(3.2.7) \quad \tilde{m} : S_{\lambda_\bullet} \xrightarrow{\cong} S_{\lambda_1} \times S_{\lambda_1 + \lambda_2} \times \dots \times S_{|\lambda_\bullet|},$$

so that the following diagram is commutative

$$\begin{array}{ccc} S_{\lambda_\bullet} & \xrightarrow{\tilde{m}} & S_{\lambda_1} \times S_{\lambda_1 + \lambda_2} \times \dots \times S_{|\lambda_\bullet|} \\ \downarrow & & \downarrow \\ \text{Gr} \tilde{\times} \text{Gr} \tilde{\times} \dots \tilde{\times} \text{Gr} & \xrightarrow{\tilde{m}} & \text{Gr}^n, \end{array}$$

where \tilde{m} on the convoluted affine Grassmannian is the product of the convolution map for the first i factors.

Let $\mu_\bullet = (\mu_1, \dots, \mu_n)$ be a sequence of dominant coweights. Note that each $S_{\lambda_i} \cap \text{Gr}_{\mu_i}$ is L^+U -invariant, so it also makes sense to convolve the L^+U -varieties $S_\lambda \cap \text{Gr}_\mu$, and there is a canonical identification

$$(3.2.8) \quad (S_{\lambda_1} \cap \text{Gr}_{\mu_1}) \tilde{\times} \dots \tilde{\times} (S_{\lambda_n} \cap \text{Gr}_{\mu_n}) \cong S_{\lambda_\bullet} \cap \text{Gr}_{\mu_\bullet}.$$

The set of irreducible components of $(S_{\lambda_1} \cap \text{Gr}_{\mu_1}) \tilde{\times} \dots \tilde{\times} (S_{\lambda_n} \cap \text{Gr}_{\mu_n})$ is given by $\prod_i \mathbb{M}\mathbb{V}_{\mu_i}(\lambda_i)$, and for $\mathbf{b}_\bullet := (\mathbf{b}_1, \dots, \mathbf{b}_n) \in \prod_i \mathbb{M}\mathbb{V}_{\mu_i}(\lambda_i)$, the irreducible component labeled by \mathbf{b}_\bullet is

$$(S_{\lambda_1} \cap \text{Gr}_{\mu_1})^{\mathbf{b}_1} \tilde{\times} \dots \tilde{\times} (S_{\lambda_n} \cap \text{Gr}_{\mu_n})^{\mathbf{b}_n}.$$

Here is the relationship between MV cycles and Satake cycles.

Lemma 3.2.7. *Let $\nu, \mu, \nu + \lambda$ be dominant coweights. There is a unique injective map*

$$i_\nu^{\mathbb{M}\mathbb{V}} : \mathbb{S}_{(\nu, \mu)|\lambda + \nu} \rightarrow \mathbb{M}\mathbb{V}_\mu(\lambda),$$

such that for every $\mathbf{a} \in \mathbb{S}_{(\nu, \mu)|\lambda + \nu}$

$$(3.2.9) \quad \text{Gr}_{(\nu, \mu)|\lambda + \nu}^{0, \mathbf{a}} \cap S_{\nu, \lambda} = (S_\nu \cap \text{Gr}_\nu) \tilde{\times} (S_\lambda \cap \text{Gr}_\mu)^{i_\nu^{\mathbb{M}\mathbb{V}}(\mathbf{a})}$$

as subschemes of $\text{Gr} \tilde{\times} \text{Gr}$.

In addition, this map is compatible with the bijection $\mathbb{M}\mathbb{V}_\mu^{B_1}(\lambda) \cong \mathbb{M}\mathbb{V}_\mu^{B_2}(\lambda)$ in Lemma 3.2.4, for different choices of Borel subgroups.

Proof. To reduce the notation load, we denote $\text{Gr}_{(\nu, \mu)|\lambda + \nu}^{0, \mathbf{a}} \cap S_{\nu, \lambda}$ by Z in the proof.

Let $\text{pr}_1 : \text{Gr} \tilde{\times} \text{Gr} \rightarrow \text{Gr}$ denote the projection to the first factor and $m : \text{Gr} \tilde{\times} \text{Gr} \rightarrow \text{Gr}$ the convolution map. Then we have an isomorphism $(\text{pr}_1, m) : \text{Gr} \tilde{\times} \text{Gr} \simeq \text{Gr} \times \text{Gr}$, given by

$$(3.2.10) \quad (\mathcal{E}_2 \xrightarrow{\beta_1} \mathcal{E}_1 \xrightarrow{\beta_0} \mathcal{E}_0 = \mathcal{E}^0) \mapsto (\mathcal{E}_1 \xrightarrow{\beta_1} \mathcal{E}_0 = \mathcal{E}^0), \quad (\mathcal{E}_2 \xrightarrow{\beta_0 \beta_1} \mathcal{E}_0 = \mathcal{E}^0).$$

Note that by (3.2.7), $S_{\nu,\lambda} \cap (\mathrm{Gr}_\nu \times \mathrm{Gr}_{\lambda+\nu}) = (\mathrm{Gr}_\nu \cap S_\nu) \times (\mathrm{Gr}_{\lambda+\nu} \cap S_{\lambda+\nu})$ may be viewed as an open subspace of $\mathrm{Gr}_\nu \times \mathrm{Gr}_{\lambda+\nu}$. As $\mathrm{Gr}_{(\nu,\mu)|\lambda+\nu}^{0,\mathbf{a}} \subset \mathrm{Gr}_\nu \times \mathrm{Gr}_{\lambda+\nu}$ is closed, the intersection Z is an open subset of $\mathrm{Gr}_{(\nu,\mu)|\lambda+\nu}^{0,\mathbf{a}}$, and therefore is irreducible of dimension $\langle \rho, \lambda + \mu + 2\nu \rangle$ if the intersection is non-empty.

On the other hand by (3.2.8), $Z \subset (\mathrm{Gr}_\nu \cap S_\nu) \tilde{\times} (\mathrm{Gr}_\mu \cap S_\lambda)$, when regarded as subschemes in $\mathrm{Gr} \tilde{\times} \mathrm{Gr}$. It would then follow that if Z is nonempty, it is equal to an irreducible component of $(S_\nu \cap \mathrm{Gr}_\nu) \tilde{\times} (S_\lambda \cap \mathrm{Gr}_\mu)$, which must be of the form $(S_\nu \cap \mathrm{Gr}_\nu) \tilde{\times} (S_\lambda \cap \mathrm{Gr}_\mu)^{i_\nu^{\mathrm{MV}}(\mathbf{a})}$ for some $i_\nu^{\mathrm{MV}}(\mathbf{a}) \in \mathrm{MV}_\mu(\lambda)$. This defines the desired map in the lemma, and it is clear that this map is compatible with the bijection $\mathrm{MV}_\mu^{B_1}(\lambda) \cong \mathrm{MV}_\mu^{B_2}(\lambda)$ in Lemma 3.2.4, for different choices of Borel subgroups.

We shall show that $m_{\nu,\mu}^{-1}(\varpi^{\lambda+\nu}) \cap Z$ is non-empty, where $m_{\nu,\mu} : \mathrm{Gr}_\nu \tilde{\times} \mathrm{Gr}_\mu \rightarrow \mathrm{Gr}$ is the convolution map. First note by the definition of $\mathrm{Gr}_{(\nu,\mu)|\lambda+\nu}^{0,\mathbf{a}}$, $m_{\nu,\mu}^{-1}(\varpi^{\lambda+\nu}) \cap \mathrm{Gr}_{(\nu,\mu)|\lambda+\nu}^{0,\mathbf{a}}$ is a (non-empty irreducible) variety of dimension $\langle \rho, \mu - \lambda \rangle$. Since $\mathrm{Gr}_\nu = \sqcup_{\nu' \preceq \nu} (S_{\nu'} \cap \mathrm{Gr}_\nu)$, it suffices to show that, for $\nu' \prec \nu$, the intersection

$$(3.2.11) \quad (\mathrm{pr}_1)^{-1}(S_{\nu'} \cap \mathrm{Gr}_\nu) \cap m_{\nu,\mu}^{-1}(\varpi^{\lambda+\nu})$$

has dimension $< \langle \rho, \mu - \lambda \rangle$.

Note that $m_{\nu,\mu}^{-1}(\varpi^{\lambda+\nu})$ classifies sequence of modifications (3.2.10) such that $\mathrm{Inv}(\beta_0) \preceq \nu$, $\mathrm{Inv}(\beta_1) \preceq \mu$ and the composition $\beta_1 \circ \beta_0 = \varpi^{\lambda+\nu}$ if \mathcal{E}_2 is properly trivialized. Rearranging the maps gives

$$\mathcal{E}_1 \xleftarrow[\beta_0]{\beta_1^{-1}} \mathcal{E}_0 \xrightarrow[\varpi^{-\lambda-\nu}]{} \mathcal{E}_2.$$

So $m_{\nu,\mu}^{-1}(\varpi^{\lambda+\nu}) \cong \mathrm{Gr}_{\mu^*} \cap \varpi^{-\lambda-\nu} \mathrm{Gr}_\nu$, and under this isomorphism,

$$(3.2.11) \cong \mathrm{Gr}_{\mu^*} \cap \varpi^{-\lambda-\nu} (S_{\nu'} \cap \mathrm{Gr}_\nu) \subseteq \mathrm{Gr}_{\mu^*} \cap S_{-\lambda-\nu+\nu'}$$

is of dimension $< \langle \rho, \mu - \lambda \rangle$. This concludes the proof of the lemma. \square

More generally

Lemma 3.2.8. *Let μ_1, μ_2, μ be dominant. There is a natural injective map*

$$S_{(\mu_1, \mu_2)|\mu} \times \mathrm{MV}_\mu \rightarrow \mathrm{MV}_{\mu_1} \times \mathrm{MV}_{\mu_2}.$$

Proof. Consider $m_{\mu_1, \mu_2} : \mathrm{Gr}_{\mu_1, \mu_2} \rightarrow \mathrm{Gr}$. Let $\mathbf{a} \in \mathrm{MV}_\mu(\lambda)$ and $(S_\lambda \cap \mathrm{Gr}_\mu)^\mathbf{a}$ the corresponding irreducible component. Let $\mathbf{b} \in \mathbb{S}_{(\mu_1, \mu_2)|\mu}$. As $\mathrm{Gr}_{(\mu_1, \mu_2)|\mu}^{0,\mathbf{b}}|_{\mathrm{Gr}_\mu} \rightarrow \mathrm{Gr}_\mu$ is a fibration with (non-canonically) isomorphic fibers of dimension $\langle \rho, \mu_1 + \mu_2 - \mu \rangle$, we see that $C := m_{\mu_1, \mu_2}^{-1}(S_\lambda \cap \mathrm{Gr}_\mu)^\mathbf{a} \cap \mathrm{Gr}_{(\mu_1, \mu_2)|\mu}^{0,\mathbf{b}}|_{\mathrm{Gr}_\mu}$ is irreducible of dimension $\langle \rho, \mu_1 + \mu_2 + \lambda \rangle$. On the other hand, by (3.2.8) and Theorem 3.2.2, every irreducible component of $m_{\mu_1, \mu_2}^{-1}(S_\lambda) \cap \mathrm{Gr}_{\mu_1, \mu_2}$ is of dimension $\langle \rho, \mu_1 + \mu_2 + \lambda \rangle$, and is of the form $(S_{\nu_1} \cap \mathrm{Gr}_{\mu_1})^{\mathbf{c}_1} \tilde{\times} (S_{\nu_2} \cap \mathrm{Gr}_{\mu_2})^{\mathbf{c}_2}$ for unique $\mathbf{c}_i \in \mathrm{MV}_{\mu_i}(\nu_i)$. Therefore, C is an open subset of exactly one of such an irreducible components. We thus construct the desired map given by $(\mathbf{a}, \mathbf{b}) \mapsto (\mathbf{c}_1, \mathbf{c}_2)$. It is clear that this is an injective map \square

Remark 3.2.9. (1) It is easy to see from the proof that under the above map, the image of $S_{(\mu_1, \mu_2)|\mu} \times \mathrm{MV}_\mu(\mu)$ is contained in $\mathrm{MV}_{\mu_1}(\mu_1) \times \mathrm{MV}_{\mu_2}(\mu - \mu_1)$. Therefore, it recovers the map from Lemma 3.2.7.

(2) As we shall see from the proof of Proposition 3.3.17 below, this map is in fact a bijection.

3.2.10. *Relation with Levi subgroups.* Now let $M \subset G$ be a standard Levi subgroup, with the set of simple roots $\Delta_M \subset \Delta$. Let $U^M \subset U$ be the unipotent radical of the parabolic subgroup $P_M = MB$. Then U/U^M is isomorphic to the unipotent radical U_M of the Borel subgroup $B_M = M \cap B$. If $\Delta_M = \{\alpha_i\}$ is a simple root (in which case we call M the *standard Levi corresponding to α_i*), then $U_M \cong U_{\alpha_i}$. Recall that T is a common maximal torus of M and G , so semi-infinite orbits on Gr_M (with respect to U_M) are also parameterized by $\mathbb{X}_\bullet(T)$. Let $S_{M,\lambda}$ be a semi-infinite orbit of Gr_M corresponding to λ . Then there is a natural projection

$$\theta_M : S_\lambda \rightarrow S_{M,\lambda}, \quad \varpi^\lambda LU/L^+U \rightarrow \varpi^\lambda LU_M/L^+U_M.$$

3.3. Crystal structures on the set of MV cycles. In this subsection, we endow a \hat{G} -crystal structure on the set of MV cycles via the Littelmann paths.¹² The main result we need for later application is Proposition 3.3.20.

The \hat{G} -crystal structure on MV_μ was first discovered by Braverman and Gaitsgory [BrGai01], using the geometric Satake equivalence. Our approach is different, and closely follows [NP01]. The connection between Littelmann paths (or some variants) and MV cycles has also been extensively studied in literature (e.g. see [GL05, GL12, BaGau08]). In fact, some of the following results are contained or can be easily deduced from the above mentioned references (we thank J. Kamnitzer to point out this). However, as our proof of Proposition 3.3.20 makes use of the construction of the \hat{G} -crystal structure on MV_μ , we decide to make the exposition self-contained.

We start by recalling the definition of \hat{G} -crystals.

Definition 3.3.1. Recall that Δ (resp. Δ^\vee) denotes the set of simple roots (resp. simple coroots) of G .

- (1) A (normal) \hat{G} -crystal¹³ is a finite set \mathbb{B} , equipped with a map $\text{wt} : \mathbb{B} \rightarrow \mathbb{X}_\bullet(T)$,¹⁴ and operators $e_\alpha, f_\alpha : \mathbb{B} \rightarrow \mathbb{B} \cup \{0\}$ for each $\alpha \in \Delta$, such that
 - (a) for every $\mathbf{b} \in \mathbb{B}$, either $e_\alpha \mathbf{b} = 0$ or $\text{wt}(e_\alpha \mathbf{b}) = \text{wt}(\mathbf{b}) + \alpha^\vee$, and either $f_\alpha \mathbf{b} = 0$ or $\text{wt}(f_\alpha \mathbf{b}) = \text{wt}(\mathbf{b}) - \alpha^\vee$,
 - (b) for all $\mathbf{b}, \mathbf{b}' \in \mathbb{B}$ one has $\mathbf{b}' = e_\alpha \cdot \mathbf{b}$ if and only if $\mathbf{b} = f_\alpha \cdot \mathbf{b}'$, and
 - (c) if $\varepsilon_\alpha, \phi_\alpha : \mathbb{B} \rightarrow \mathbb{Z}$, $\alpha \in \Delta$ are the maps defined by

$$\varepsilon_\alpha(\mathbf{b}) = \max\{n \mid e_\alpha^n \mathbf{b} \neq 0\} \quad \text{and} \quad \phi_\alpha(\mathbf{b}) = \max\{n \mid f_\alpha^n \mathbf{b} \neq 0\},$$

then we require $\phi_\alpha(\mathbf{b}) - \varepsilon_\alpha(\mathbf{b}) = \langle \alpha, \text{wt}(\mathbf{b}) \rangle$.

For $\lambda \in \mathbb{X}_\bullet(T)$, we use $\mathbb{B}(\lambda)$ to denote the set of elements with weight λ for \hat{G} , called the *weight space* with weight λ for \hat{G} .

- (2) Let \mathbb{B}_1 and \mathbb{B}_2 be two \hat{G} -crystals, a morphism $\psi : \mathbb{B}_1 \rightarrow \mathbb{B}_2$ is a map of the underlying sets compatible with wt , e_α , and f_α (whenever it makes sense).
- (3) A \hat{G} -crystal \mathbb{B} is called a *highest weight crystal* of highest weight λ of \hat{G} if there exists $\mathbf{b} \in \mathbb{B}$, satisfying $e_\alpha \mathbf{b} = 0$ for all α , $\text{wt}(\mathbf{b}) = \lambda$, and \mathbb{B} is generated from \mathbf{b} by operators f_α (for all $\alpha \in \Delta$). In this case, such \mathbf{b} is unique and is denoted by \mathbf{b}_λ .
- (4) Let \mathbb{B} be a \hat{G} -crystal. The dual \hat{G} -crystal has underlying set \mathbf{b}^* for each $\mathbf{b} \in \mathbb{B}$ (setting $0^* = 0$), whose maps are given by

$$\text{wt}(\mathbf{b}^*) = -\text{wt}(\mathbf{b}), \quad e_\alpha(\mathbf{b}^*) = (f_\alpha \mathbf{b})^*, \quad \text{and} \quad f_\alpha(\mathbf{b}^*) = (e_\alpha \mathbf{b})^*.$$

¹²Unfortunately, in later section of the paper, the terminology G -crystal will appear with a completely different meaning. Namely, it will be crystals in the sense of Grothendieck. The use of \hat{G} -crystal will be confined in this and the next section. We hope no confusion will arise.

¹³We will not discuss non-normal crystals in this paper.

¹⁴In this section, since we chose to work with \hat{G} -crystals, the target of the weight map is the weight lattice of \hat{G} which is the same as the coweight lattice of G . We shall try to make clear the distinction of weights of G and \hat{G} .

- (5) Let \mathbb{B}_1 and \mathbb{B}_2 be two \hat{G} -crystals. The tensor product $\mathbb{B}_1 \otimes \mathbb{B}_2$ is the \hat{G} -crystal with underlying set $\mathbb{B}_1 \times \mathbb{B}_2$, and $\text{wt}(\mathbf{b}_1 \otimes \mathbf{b}_2) = \text{wt}(\mathbf{b}_1) + \text{wt}(\mathbf{b}_2)$. Following [Ka91] and [Jo95], the operators e_α and f_α are defined by

$$e_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) = \begin{cases} e_\alpha \mathbf{b}_1 \otimes \mathbf{b}_2 & \text{if } \phi_\alpha(\mathbf{b}_1) \geq \varepsilon_\alpha(\mathbf{b}_2) \\ \mathbf{b}_1 \otimes e_\alpha \mathbf{b}_2 & \text{otherwise,} \end{cases}$$

$$f_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) = \begin{cases} f_\alpha \mathbf{b}_1 \otimes \mathbf{b}_2 & \text{if } \phi_\alpha(\mathbf{b}_1) > \varepsilon_\alpha(\mathbf{b}_2) \\ \mathbf{b}_1 \otimes f_\alpha \mathbf{b}_2 & \text{otherwise.} \end{cases}$$

We have

$$(3.3.1) \quad \begin{aligned} \varepsilon_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \max\{\varepsilon_\alpha(\mathbf{b}_1), \varepsilon_\alpha(\mathbf{b}_2) - \langle \alpha, \text{wt}(\mathbf{b}_1) \rangle\} \quad \text{and} \\ \phi_\alpha(\mathbf{b}_1 \otimes \mathbf{b}_2) &= \max\{\phi_\alpha(\mathbf{b}_2), \phi_\alpha(\mathbf{b}_1) + \langle \alpha, \text{wt}(\mathbf{b}_2) \rangle\}. \end{aligned}$$

Taking tensor product of \hat{G} -crystals is associative, making the category of \hat{G} -crystals a monoidal category.

- (6) A family of highest weight \hat{G} -crystals $\{\mathbb{B}_\lambda | \lambda \in \mathbb{X}_\bullet(T)^+\}$ is called *closed* if for every λ, μ , there is a map of \hat{G} -crystals $\mathbb{B}_{\lambda+\mu} \rightarrow \mathbb{B}_\lambda \otimes \mathbb{B}_\mu$ sending $\mathbf{b}_{\lambda+\mu}$ to $\mathbf{b}_\lambda \otimes \mathbf{b}_\mu$. Given such a family, and a sequence of dominant weights μ_\bullet of G , we put $\mathbb{B}_{\mu_\bullet} = \mathbb{B}_{\mu_1} \otimes \cdots \otimes \mathbb{B}_{\mu_n}$.

Recall the following theorem of Joseph ([Jo95]).

Theorem 3.3.2 (Joseph). *Assume that \hat{G} is simply-connected. Then there is a unique (up to unique isomorphism) family of closed highest weight \hat{G} -crystals.*

In general, passing to the adjoint group induces a map $\text{pr} : \mathbb{X}_\bullet(T) \rightarrow \mathbb{X}_\bullet(T_{\text{ad}})$, and \hat{G}_{ad} is simply-connected. Let $\mathbb{B}_\mu = \mathbb{B}_{\text{pr}(\mu)}$. Then $\{\mathbb{B}_\mu\}$ defines a family of closed highest weight \hat{G} -crystals.

3.3.3. Weight subsets as homomorphisms for \hat{G} -crystals. Let $\{\mathbb{B}_\lambda\}$ be the family of a closed highest weight \hat{G} -crystals as constructed above. Let $\lambda \in \mathbb{X}_\bullet(T)$ be a weight of \hat{G} . If $\nu \in \mathbb{X}_\bullet(T)^+$ is a dominant weight of \hat{G} such that $\lambda + \nu$ is also dominant, we have a natural map

$$(3.3.2) \quad i_\nu^\mathbb{B} : \text{Hom}(\mathbb{B}_{\lambda+\nu}, \mathbb{B}_\nu \otimes \mathbb{B}) \rightarrow \mathbb{B}(\lambda),$$

such that $\mathbf{b} = i_\nu^\mathbb{B}(\psi)$ is the unique element in \mathbb{B} satisfying $\psi(\mathbf{b}_{\lambda+\nu}) = \mathbf{b}_\nu \otimes \mathbf{b}$. (Note that $e_\alpha(\mathbf{b}_{\lambda+\nu}) = 0$ for all $\alpha \in \Delta$ forces $\psi(\mathbf{b}_{\lambda+\nu})$ to take such form.) For $\mathbf{b} \in \mathbb{B}(\lambda)$, it is straightforward to check from the definition that

$$(3.3.3) \quad \begin{aligned} \mathbf{b} \in \text{Im } i_\nu^\mathbb{B} &\text{ if and only if } \phi_\alpha(\mathbf{b}_{\nu+\lambda}) = \phi_\alpha(\mathbf{b}_\nu \otimes \mathbf{b}) \text{ for all } \alpha \in \Delta, \\ &\text{ if and only if } \phi_\alpha(\mathbf{b}) \leq \phi_\alpha(\mathbf{b}) - \varepsilon_\alpha(\mathbf{b}) + \phi_\alpha(\mathbf{b}_\nu) \text{ for all } \alpha \in \Delta, \\ &\text{ if and only if } \langle \alpha, \nu \rangle \geq \varepsilon_\alpha(\mathbf{b}) \text{ for all } \alpha \in \Delta. \end{aligned}$$

3.3.4. Littelmann path and \hat{G} -crystals. Given \hat{G} , there are several ways to construct a family of closed highest weight \hat{G} -crystals (although such family is unique up to a unique isomorphism if \hat{G} is simply-connected, by Theorem 3.3.2). Here we recall one construction using Littelmann paths. This construction is easily related to the set of MV cycles. Our presentation follows [NP01]¹⁵. Let

$$\text{Min} \subset \mathbb{X}^\bullet(\hat{T})^+ \setminus \{0\} = \mathbb{X}_\bullet(T)^+ \setminus \{0\}$$

be the set of minimal elements with respect to the Bruhat order. If the group G is simple of type A, Min is the set of minuscule weights of \hat{G} . In general, it is the union of minuscule weights and the quasi-minuscule weights of \hat{G} .

¹⁵Littelmann himself considers more general paths, e.g. what he called LS paths.

Definition 3.3.5. Let $\gamma_i : [0, 1] \rightarrow \mathbb{X}^\bullet(\hat{T}) \otimes \mathbb{R}$, $i = 1, 2$ be two piecewise linear maps. The *concatenation* of γ_1 and γ_2 , denoted by $\gamma_1 * \gamma_2$ is the map given by

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & 0 \leq t \leq 1/2, \\ \gamma_1(1) + \gamma_2(2t - 1) & 1/2 \leq t \leq 1. \end{cases}$$

Two piecewise linear maps $\gamma_1, \gamma_2 : [0, 1] \rightarrow \mathbb{X}^\bullet(\hat{T}) \otimes \mathbb{R}$ are called *reparametrizations* of each other if there exists a strictly increasing piecewise linear function $r : [0, 1] \rightarrow [0, 1]$ such that $r(0) = 0$, $r(1) = 1$, and $\gamma_1(t) = \gamma_2(r(t))$.

Definition 3.3.6. We say a piecewise linear map $\gamma : [0, 1] \rightarrow \mathbb{X}^\bullet(\hat{T}) \otimes \mathbb{R}$ an *elementary Littelmann path* if it is one of the following forms:

- (1) $\gamma(t) = \gamma_\nu(t) := t\nu$, where $\nu \in W\mu$ for some $\mu \in \text{Min}$;
- (2) $\gamma(t) = {}^2\gamma_\alpha(t) := \begin{cases} -t\alpha^\vee & 0 \leq t \leq 1/2, \\ (t-1)\alpha^\vee & 1/2 \leq t \leq 1, \end{cases}$ where α is a simple root such that $\alpha^\vee \in W\mu$ for some $\mu \in \text{Min}$.

A piecewise linear map $\gamma : [0, 1] \rightarrow \mathbb{X}^\bullet(\hat{T}) \otimes \mathbb{R}$ is called a *Littelmann path* if after reparametrization, it is of the form $\gamma = \gamma_1 * \gamma_2 * \cdots * \gamma_m$, where γ_i are elementary Littelmann paths. The set of Littelmann paths up to reparametrization is denoted by Π .

A *dominant Littelmann path* is a Littelmann path γ completely lying in the closure of the dominant Weyl chamber.

Definition 3.3.7. To a Littelmann path γ , we attach a sequence $T(\gamma) = \mu_\bullet = (\mu_1, \dots, \mu_m) \subset \text{Min}$ as follows: If γ is an elementary Littelmann path of type (1), let $T(\gamma) = \mu \in \text{Min}$ such that $\gamma(1) \in W\mu$, if γ is of the type (2), let $T(\gamma) = \mu \in \text{Min}$ such that $-2\gamma(1/2) \in W\mu$. If $\gamma = \gamma_1 * \cdots * \gamma_m$, let $T(\gamma) = (T(\gamma_1), \dots, T(\gamma_m))$. It is easy to see that $T(\gamma)$ is well-defined. We denote m by $\ell(\gamma)$, and call it the length of the Littelmann path γ .

Recall that in [Lit95], Littelmann defined root operators on $\mathbb{Z}\Pi$, the free \mathbb{Z} -module generated by Π .¹⁶ We recall its definition. Let α be a simple root of G , so the corresponding α^\vee is a simple root of \hat{G} . We now define the two root operators $e_\alpha, f_\alpha : \mathbb{Z}\Pi \rightarrow \mathbb{Z}\Pi$ associated to α , starting with e_α .

Let $\gamma = \gamma_1 * \cdots * \gamma_m$, with γ_i elementary. After reparametrization, we can assume that $\gamma_i(t) = \gamma(\frac{i-1+t}{m}) - \gamma(\frac{i-1}{m})$. Let $m_\alpha = \min_{t \in [0, 1]} \{\langle \alpha, \gamma(t) \rangle\}$. If $m_\alpha > -1$, let $e_\alpha \gamma = 0$. If $m_\alpha \leq -1$, let t_1 be minimal among t such that $\langle \alpha, \gamma(t) \rangle = m_\alpha$. Then there are three possibilities.

- (i) If $t_1 = \frac{i}{m}$ for some i , and in addition, $\langle \alpha, \gamma(\frac{i-1}{m}) \rangle = m_\alpha + 1$, we define

$$e_\alpha(\gamma) = \gamma_1 * \gamma_2 * \cdots * \gamma_{i-1} * s_{\alpha^\vee}(\gamma_i) * \gamma_{i+1} * \cdots * \gamma_m,$$

where s_{α^\vee} is the simple reflect on $\mathbb{X}^\bullet(\hat{T})$ given by α^\vee .

- (ii) If $t_1 = \frac{i}{m}$ is some i but $\langle \alpha, \gamma(\frac{i-1}{m}) \rangle = m_\alpha + 2$ (in particular α^\vee is a simple root of \hat{G} that can be conjugated into Min under W , and $\gamma_i(t) = \gamma_{-\alpha}(t)$), we define

$$e_\alpha(\gamma) = \gamma_1 * \gamma_2 * \cdots * \gamma_{i-1} * {}^2\gamma_\alpha(t) * \gamma_{i+1} * \cdots * \gamma_m,$$

- (iii) If $t_1 = \frac{2i-1}{2m}$ for some i (in particular α^\vee is a simple root of \hat{G} that can be conjugated into Min under W , and $\gamma_i(t) = {}^2\gamma_\alpha(t)$), we define

$$e_\alpha(\gamma) = \gamma_1 * \gamma_2 * \cdots * \gamma_{i-1} * \gamma_\alpha(t) * \gamma_{i+1} * \cdots * \gamma_m,$$

The root operator f_α can be defined in a similar way, but we prefer to give another short definition.

Definition 3.3.8. Let $\gamma : [0, 1] \rightarrow \mathbb{X}^\bullet(\hat{T}) \otimes \mathbb{R}$ be a piecewise linear map. We define its dual γ^* as

$$\gamma^*(t) = \gamma(1-t) - \gamma(1).$$

¹⁶In fact, Littelmann defined such operators on the free \mathbb{Z} -modules generated by the set of more general paths.

Then it is clear that the dual of a Littelmann path is a Littelmann path. Indeed, $\gamma_\nu^*(t) = \gamma_{-\nu}(t)$ and ${}^2\gamma_\alpha^*(t) = {}^2\gamma_\alpha(t)$. We define $f_\alpha\gamma = (e_\alpha\gamma^*)^*$.

Here are some simple properties of the root operators ([Lit95, §2]).

Lemma 3.3.9. *Let $\gamma \in \Pi$.*

- (1) *If $e_\alpha(\gamma) \neq 0$, then $e_\alpha(\gamma)(1) = \gamma(1) + \alpha^\vee$ and $T(e_\alpha(\gamma)) = T(\gamma)$. In addition, $f_\alpha e_\alpha(\gamma) = \gamma$. Similar statements hold for f_α .*
- (2) *The path γ is dominant if and only if $e_\alpha(\gamma) = 0$ for all α .*

Now let γ be a dominant Littelmann path, with $\gamma(1) = \mu$ (not necessarily in Min). Let \mathbb{B}_γ denote the set of all Littelmann paths obtained by applying root operators to γ . Recall that in [Lit95], Littelmann proved the following isomorphism theorem.

Theorem 3.3.10 (Littelmann). *Let γ_1 and γ_2 be two dominant Littelmann paths satisfying $\gamma_1(1) = \gamma_2(1)$. Then the assignment $\gamma_1 \rightarrow \gamma_2$ extends to a unique isomorphism $i_{\gamma_1, \gamma_2} : \mathbb{B}_{\gamma_1} \rightarrow \mathbb{B}_{\gamma_2}$ compatible with the action of root operators.*

As a result, for every dominant weight μ of \hat{G} , one can canonically define a set $\mathbb{B}_\mu := \mathbb{B}_\gamma$ for any dominant Littelmann path γ satisfying $\gamma(1) = \mu$. The following theorem can be found in [Jo95].

Theorem 3.3.11. *The collection $\{\mathbb{B}_\mu\}$ defined as above together with the root operators form a closed family of highest weight \hat{G} -crystals.*

Next, we relate the Littelmann paths with the MV cycles.

Proposition 3.3.12. *Given a sequence of dominant weights $\{\mu_\bullet\} \subset \text{Min}$ of \hat{G} and $\lambda \in \mathbb{X}^\bullet(\hat{T})$, there is a canonical bijection between*

$$\{\gamma \in \Pi \mid \gamma(1) = \lambda, T(\gamma) = \mu_\bullet\} \leftrightarrow \{\text{Irreducible components of } m_{\mu_\bullet}^{-1}(S_\lambda) \cap \text{Gr}_{\mu_\bullet}\}.$$

Proof. Recall that

$$m_{\mu_\bullet}^{-1}(S_\lambda) \cap \text{Gr}_{\mu_\bullet} = \bigsqcup_{\nu_\bullet, |\nu_\bullet| = \lambda} S_{\nu_\bullet} \cap \text{Gr}_{\mu_\bullet},$$

$$S_{\nu_\bullet} \cap \text{Gr}_{\mu_\bullet} = (S_{\nu_1} \cap \text{Gr}_{\mu_1}) \tilde{\times} \cdots \tilde{\times} (S_{\nu_n} \cap \text{Gr}_{\mu_n}).$$

By (3.2.3), (3.2.4), and (3.2.5), we have

- if $\nu_i \in W\mu_i$, $S_{\nu_i} \cap \text{Gr}_{\mu_i}$ is irreducible;
- if $\nu_i = 0$, and μ_i is quasi-minuscule, then the irreducible components of $S_0 \cap \text{Gr}_{\mu_i}$ canonically one-to-one correspond to Δ_{μ_i} , given by $(S_0 \cap \text{Gr}_{\mu_i})^\alpha \leftrightarrow \alpha$; and
- if ν_i is not of the above forms, then $S_{\nu_i} \cap \text{Gr}_{\mu_i}$ is empty.

Therefore, given an irreducible component of $m_{\mu_\bullet}^{-1}(S_\lambda) \cap \text{Gr}_{\mu_\bullet}$, we construct a Littelmann path $\gamma = \gamma_1 * \cdots * \gamma_n$, where

- $\gamma_i(t) = \gamma_{\nu_i}(t)$ if $\nu_i \neq 0$;
- $\gamma_i(t) = {}^2\gamma_{\alpha_i}(t)$ if $\nu_i = 0$ and $(S_0 \cap \text{Gr}_{\mu_i})^{\alpha_i}$ with $\alpha_i \in \Delta_{\mu_i}$ appears as a factor in the above factorization of the irreducible component.

It is clear how to construct the inverse map, so we get the bijection as in the proposition. □

For γ a Littelmann path with $\lambda = \gamma(1)$ and $T(\gamma) = \mu_\bullet$, let

$$S_\gamma := (m_{\mu_\bullet}^{-1}(S_\lambda) \cap \text{Gr}_{\mu_\bullet})^\gamma$$

denote the corresponding irreducible component. The following result is proved in [NP01, Lemma 9.5] in the equal characteristic situation. But the proof works for the mixed characteristic situation as well.¹⁷

Proposition 3.3.13. *Assume that λ is dominant and $\gamma(1) = \lambda$. Then $S_\gamma \subset m_{\mu_\bullet}^{-1}(\text{Gr}_\lambda)$ if and only if γ is dominant.*

Here is the relation between root operators and MV cycles.

Lemma 3.3.14. *Let $\gamma \in \Pi$ and assume that $e_\alpha(\gamma) \neq 0$. Then S_γ is contained in the closure of $S_{e_\alpha(\gamma)}$.*

Proof. It is enough to verify this for an elementary Littelmann path. But this is clear by the description of MV cycles in terms of minuscule and quasi-minuscule Schubert varieties given in (3.2.3), (3.2.4), and (3.2.5). \square

Proposition 3.3.15. *Let γ be a dominant Littelmann path and $\gamma(1) = \mu$. Let $\mathbb{B}_\gamma(\lambda) = \{\delta \in \mathbb{B}_\gamma \mid \delta(1) = \lambda\}$. There is a canonical bijection*

$$\phi_\gamma : \mathbb{B}_\gamma(\lambda) \cong \text{MV}_\mu(\lambda),$$

which is compatible with Littelmann's isomorphism theorem (Theorem 3.3.10), i.e., if γ_1 and γ_2 are two dominant Littelmann paths with $\gamma_1(1) = \gamma_2(1) = \mu$, then $\phi_{\gamma_2} i_{\gamma_1, \gamma_2} = \phi_{\gamma_1}$.

Proof. We first construct the bijection as follows. Let $\mu_\bullet = T(\gamma)$. By Proposition 3.3.13, $S_\gamma \subset m_{\mu_\bullet}^{-1}(\text{Gr}_\mu)$. Therefore for any $\delta \in \mathbb{B}_\gamma$, $S_\delta \subset m_{\mu_\bullet}^{-1}(\text{Gr}_\mu)$ by Lemma 3.3.14. In other words,

$$m_{\mu_\bullet}(S_\delta) \subset S_{\delta(1)} \cap \text{Gr}_\mu.$$

We claim that the closure $\overline{m_{\mu_\bullet}(S_\delta)}$ is of dimension $\langle \rho, \delta(1) + \mu \rangle$ and therefore is the closure of an irreducible component of $S_{\delta(1)} \cap \text{Gr}_\mu$. Indeed, if $e_\alpha(\delta) \neq 0$, then $m_{\mu_\bullet}(S_\delta) \subset m_{\mu_\bullet}(\bar{S}_{e_\alpha(\delta)})$ Lemma 3.3.14. But since $m_{\mu_\bullet}(S_{e_\alpha(\delta)}) \subset S_{e_\alpha(\delta)(1)} = S_{\delta(1)+\alpha}$ does not intersect with $m_{\mu_\bullet}(S_\delta) \subset S_{\delta(1)}$, we see that $\dim m_{\mu_\bullet}(\bar{S}_{e_\alpha(\delta)}) \geq \dim m_{\mu_\bullet}(S_\delta) + 1$. But $\dim S_\gamma = \langle 2\rho, \mu \rangle$ and $\dim S_{w_0(\gamma)} = 0$, and for each $\delta \in \mathbb{B}_\gamma$, there is a sequence of root operators in $\{e_\alpha\}$ sending δ to γ and a sequence of root operators in $\{f_\alpha\}$ sending δ to $w_0(\gamma)$. The sum of the lengths of these two sequences is always $\langle 2\rho, \mu \rangle$. This proves the claim.

The claim gives a well-defined map

$$\phi_\gamma : \mathbb{B}_\gamma(\lambda) \rightarrow \text{MV}_\mu(\lambda), \quad \delta \mapsto \overline{m_{\mu_\bullet}(S_\delta)} \cap S_\lambda.$$

The next claim is that this map is injective. Then ϕ_γ is bijective because the two sets have the same cardinality: namely by [Lit95, Theorem 9.1] and Theorem 3.2.2, the cardinality of both sets is $\dim V_\mu(\lambda)$.

Note that the closure of S_γ is exactly an irreducible component of $m_{\mu_\bullet}^{-1}\text{Gr}_\mu$, of dimension $\langle \rho, \mu + |\mu_\bullet| \rangle$. In other words, there is $\mathbf{a} \in \mathbb{S}_{\mu_\bullet|\mu}$ such that $\text{Gr}_{\mu_\bullet|\mu}^{0, \mathbf{a}}$ is the closure of S_γ . We consider

$$m_{\mu_\bullet} : \text{Gr}_{\mu_\bullet|\mu}^{0, \mathbf{a}}|_{\mathring{\text{Gr}}_\mu} \rightarrow \mathring{\text{Gr}}_\mu,$$

which is a fibration, with fibers (non-canonically) isomorphic of each other, of dimension $\langle \rho, |\mu_\bullet| - \mu \rangle$. Now because the generic fibers of $m_{\mu_\bullet} : S_\delta \rightarrow m_{\mu_\bullet}(S_\delta)$ is of dimension $\langle \rho, |\mu_\bullet| - \mu \rangle$, $S_\delta \cap \text{Gr}_{\mu_\bullet|\mu}^{0, \mathbf{a}}|_{\mathring{\text{Gr}}_\mu}$ is dense in $m_{\mu_\bullet}^{-1}(m_{\mu_\bullet}(S_\delta) \cap \mathring{\text{Gr}}_\mu)$, by dimension reasons. This shows that if $\delta \neq \delta'$, $m_{\mu_\bullet}(S_\delta) \cap \mathring{\text{Gr}}_\mu$

¹⁷There is one subtlety that needs one's attention: In the mixed characteristic situation, one cannot use the affine root group such as $U_{\alpha_i, -1}$ in *loc. cit.*. However, in the argument one can replace affine root group such as $U_{\alpha_i, -1}$ by the “compact open” subgroup $U_{\alpha_i, \geq -1} = \prod_{j \geq -1} U_{\alpha_i, j}$, which is defined in the mixed characteristic case.

and $m_{\mu_\bullet}(S_{\delta'}) \cap \text{Gr}_\mu$ can only intersect at a subset of strict smaller dimension. Therefore ϕ_γ is an injection.

Next, we show that the bijection ϕ_γ is compatible with the Littelmann's isomorphism theorem. Let us denote the irreducible component of $(S_\lambda \cap \text{Gr}_\mu)$ corresponding to $\phi_\gamma(\delta) \in \text{MV}_\mu(\lambda)$ by $(S_\lambda \cap \text{Gr}_\mu)^\delta$ instead of $(S_\lambda \cap \text{Gr}_\mu)^{\phi_\gamma(\delta)}$ for simplicity in the sequel.

Given $\mu \in \mathbb{X}_\bullet^+$, let $\ell(\mu)$ denote the minimal length among all dominant Littelmann paths γ with $\gamma(1) = \mu$. We prove the compatibility by induction on $\ell(\mu)$. If $\ell(\mu) = 0$. There is nothing to prove. If $\ell(\mu) = 1$, let γ be an elementary Littelmann path with $\gamma(1) = \mu$, and γ' be another dominant Littelmann path with $\gamma'(1) = \mu$. Let $\delta \in \mathbb{B}^\gamma(\lambda)$ and $\delta' = i_{\gamma, \gamma'}(\delta) \in \mathbb{B}^{\gamma'}(\lambda)$. Let $\mu_\bullet = T(\gamma')$. If μ is minuscule or quasi-minuscule and $\lambda = w\mu$, then as each $S_{w\mu} \cap \text{Gr}_\mu$ is irreducible, the proposition is true in this case. Therefore, we assume that μ is quasi-minuscule, and $\lambda = 0$. By definition, the path ${}^2\gamma_\alpha \in \mathbb{B}_\gamma(0)$ corresponds to the irreducible component $(S_0 \cap \text{Gr}_{\leq \mu})^\alpha$.

Let $\delta_\alpha \in \mathbb{B}_{\gamma'}$ be (the unique) path such that $\delta_\alpha(1) = \alpha$ is a simple coroot (which is conjugate to μ under W), then $m_{\mu_\bullet} : S_{\delta_\alpha} \rightarrow S_\alpha \cap \text{Gr}_\mu$ is dominant. Under Littelmann's isomorphism theorem $f_\alpha(\delta_\alpha)$ maps to the path ${}^2\gamma_\alpha \in \mathbb{B}_\gamma$. Therefore, we need to show that the map $m_{\mu_\bullet} : \text{Gr}_{\mu_\bullet} \rightarrow \text{Gr}_{|\mu_\bullet|}$ restricts to a dominant map

$$m_{\mu_\bullet} : S_{f_\alpha(\delta_\alpha)} \rightarrow (S_0 \cap \text{Gr}_\mu)^\alpha.$$

Note that m_{μ_\bullet} maps $S_{f_\alpha^2(\delta_\alpha)}$ to $(S_{-\alpha} \cap \text{Gr}_\mu)$, which should be in the closure $\overline{m_{\mu_\bullet}(S_{f_\alpha(\delta_\alpha)})}$. But $(S_0 \cap \text{Gr}_\mu)^\alpha$ is the unique irreducible component whose closure contains $(S_{-\alpha} \cap \text{Gr}_\mu)$. Therefore, the compatibility holds in this case.

Now we assume that the case $\ell(\mu) \leq m$ has been proved. Let μ be dominant and $\ell(\mu) = m$. Let $\gamma = \gamma_1 * \dots * \gamma_m$ be a dominant Littelmann path of length m such that $\gamma(1) = \mu$. Write $\mu_\bullet = (\mu_1, \dots, \mu_m) = T(\gamma)$. We can write γ as the concatenation of the path $\gamma_{<m} * \gamma_m$, where $\gamma_{<m} = \gamma_1 * \dots * \gamma_{m-1}$. Let $\delta \in \mathbb{B}_\gamma$ and we also write $\delta = \delta_{<m} * \delta_m$, where δ_m is an elementary Littelmann path. Then it is easy to show that $\delta_{<m-1} \in \mathbb{B}_{\gamma_{<m}}$.

Let γ' be another dominant Littelmann path with $\gamma'(1) = \mu$. We first assume that γ' is of the form

$$\gamma' = \gamma'_{<m} * \gamma_m.$$

Let $\delta' \in \mathbb{B}_{\gamma'}$ be the path corresponding to δ under Littelmann's isomorphism and similarly let $\delta'_{<m} \in \mathbb{B}_{\gamma'_{<m}}$ be the the path corresponding to $\delta_{<m}$. Then it is easy to see that $\delta' = \delta'_{<m} * \delta_m$.

Now, let $\mu_{<m} = \gamma_{<m}(1)$ and $\lambda_{<m} = \delta_{<m}(1)$, and consider

$$\text{Gr}_{\mu_{<m}} \tilde{\times} \text{Gr}_{\mu_m} \rightarrow \text{Gr}_{\mu_{<m} + \mu_m}.$$

By induction, $\overline{m_{\mu_\bullet}(S_{\delta_{<m}})} = \overline{m_{\mu_\bullet}(S_{\delta'_{<m}})} \subset \text{Gr}_{\mu_{<m}} \cap \bar{S}_{\lambda_{<m}}$. Then it is immediate to see that the bijections ϕ_γ and $\phi_{\gamma'}$ are compatible with Littelmann's isomorphism theorem.

Now let γ' be a general dominant Littelmann path with $\gamma'(1) = \mu$. We can consider

$$\gamma'' = \gamma' * \gamma_m^* * \gamma_m.$$

We reduce to show that the bijections $\phi_{\gamma'}$ and $\phi_{\gamma''}$ are compatible with Littelmann's isomorphism theorem. But note that the isomorphism theorem in this case is given by $i_{\gamma', \gamma''}(\delta) = \delta * w_0(\gamma_m^*) * w_0(\gamma_m) \in \mathbb{B}_{\gamma''}$, where w_0 is the longest element in the Weyl group W , which acts on Π via the construction of [Lit95, §8]. It is easy to the compatibility between $\phi_{\gamma'}$ and $\phi_{\gamma''}$ as well. \square

Remark 3.3.16. There is a multi-coweights version of Proposition 3.3.15, which generalizes Proposition 3.3.12. Let μ_\bullet be a sequence of dominant coweights, and γ_\bullet a sequence of dominant Littelmann paths with $\gamma_i(1) = \mu_i$. Let

$$\mathbb{B}_{\gamma_\bullet}(\lambda) = \{\delta = \delta_1 * \dots * \delta_n \mid \delta(1) = \lambda, \delta_i \in \mathbb{B}_{\gamma_i}\},$$

and

$$\mathbb{M}\mathbb{V}_{\mu_\bullet}(\lambda) = \bigsqcup_{\sum \lambda_i = \lambda} \left(\prod_i \mathbb{M}\mathbb{V}_{\mu_i}(\lambda_i) \right).$$

Note that if γ'_\bullet be another set of dominant paths with $\gamma'_i(1) = \mu_i$. Then Littelmann's isomorphism (Theorem 3.3.10) induces $i_{\gamma_\bullet, \gamma'_\bullet} : \mathbb{B}_{\gamma_\bullet}(\lambda) \cong \mathbb{B}_{\gamma'_\bullet}(\lambda)$. Now, there is a canonical bijection

$$\phi_{\gamma_\bullet} : \mathbb{B}_{\gamma_\bullet}(\lambda) \leftrightarrow \mathbb{M}\mathbb{V}_{\mu_\bullet}(\lambda).$$

such that $\phi_{\gamma'_\bullet} i_{\gamma_\bullet, \gamma'_\bullet} = \phi_{\gamma_\bullet}$. Namely, the irreducible component of $m_{\mu_\bullet}^{-1}(S_\lambda)$ corresponding to $\delta_1 * \dots * \delta_n$ is $(S_{\delta_1(1)} \cap \text{Gr}_{\mu_1})^{\delta_1} \tilde{\times} \dots \tilde{\times} (S_{\delta_n(1)} \cap \text{Gr}_{\mu_n})^{\delta_n}$.

Let γ_\bullet and be a sequence dominant Littelmann paths with $\gamma_i(1) = \mu_i$. We denote

$$\mathbb{S}_{\gamma_\bullet|\mu} = \{\delta = \gamma_1 * \delta_2 * \dots * \delta_n \mid \delta(1) = \mu, \delta_i \in \mathbb{B}_{\gamma_i}, \gamma_1 * \delta_2 * \dots * \delta_j \text{ dominant}, j = 2, \dots, n\}$$

Littelmann's decomposition formula says that

$$\mathbb{B}_{\gamma_1, \gamma_2} = \bigsqcup_{\mu} \bigsqcup_{\gamma \in \mathbb{S}_{(\gamma_1, \gamma_2)|\mu}} \mathbb{B}_{\gamma}.$$

Proposition 3.3.17. *There is a canonical bijection*

$$\mathbb{S}_{(\gamma_1, \gamma_2)|\mu} \cong \mathbb{S}_{(\mu_1, \mu_2)|\mu},$$

that induces the following diagram is commutative

$$\begin{array}{ccc} \mathbb{S}_{(\gamma_1, \gamma_2)|\mu} & \longrightarrow & \mathbb{S}_{(\mu_1, \mu_2)|\mu} \\ \downarrow & & \downarrow \\ \mathbb{B}_{\gamma_2}(\mu - \mu_1) & \xrightarrow{\phi_{\gamma_2}} & \mathbb{M}\mathbb{V}_{\mu_2}(\mu - \mu_1) \end{array}$$

where the left vertical arrow is the obvious one, and the right vertical arrow is from Lemma 3.2.7.

Proof. We first construct a more general map $\mathbb{S}_{\gamma_\bullet|\mu} \rightarrow \mathbb{S}_{\mu_\bullet|\mu}$ for a sequence of dominant Littelmann path, where $\mu_i = \gamma_i(1)$. By Proposition 3.3.13, for $\gamma_1 * \delta_2 * \dots * \delta_n \in \mathbb{S}_{\gamma_\bullet|\mu}$,

$$(S_{\mu_1} \cap \text{Gr}_{\mu_1})^{\delta_1} \tilde{\times} (S_{\delta_2(1)} \cap \text{Gr}_{\mu_2})^{\delta_2} \tilde{\times} \dots \tilde{\times} (S_{\delta_n(1)} \cap \text{Gr}_{\mu_n})^{\delta_n} \subset m_{\mu_\bullet}^{-1}(\text{Gr}_\mu)$$

(note that $S_{\mu_1} \cap \text{Gr}_{\mu_1} = (S_{\mu_1} \cap \text{Gr}_{\mu_1})^{\gamma_1}$). As $(S_{\mu_1} \cap \text{Gr}_{\mu_1})^{\delta_1} \tilde{\times} (S_{\delta_2(1)} \cap \text{Gr}_{\mu_2})^{\delta_2} \tilde{\times} \dots \tilde{\times} (S_{\delta_n(1)} \cap \text{Gr}_{\mu_n})^{\delta_n}$ is irreducible of dimension $\langle \rho, |\mu_\bullet| + \mu \rangle$, it is an open subset of one irreducible component of $\text{Gr}_{\mu_\bullet|\mu}^0$. This defines the required map. This map is clearly an injection. In addition, in the case $\gamma_\bullet = (\gamma_1, \gamma_2)$, the diagram is commutative follows directly from the construction here and in Lemma 3.2.7. It remains to show that this map is a bijection.

Note that from the construction, the following diagram is also commutative

$$\begin{array}{ccc} \bigsqcup_{\mu} \bigsqcup_{\gamma \in \mathbb{S}_{(\gamma_1, \gamma_2)|\mu}} \mathbb{B}_{\gamma} & \longrightarrow & \mathbb{B}_{\gamma_1, \gamma_2} \\ \phi_{\gamma} \downarrow & & \downarrow \phi_{(\gamma_1, \gamma_2)} \\ \bigsqcup_{\mu} \mathbb{S}_{(\mu_1, \mu_2)|\mu} \times \mathbb{M}\mathbb{V}_{\mu} & \longrightarrow & \mathbb{M}\mathbb{V}_{\mu_1, \mu_2}, \end{array}$$

where the bottom map is from Lemma 3.2.8, which is injective. The top and the right vertical arrows are bijections. It follows that the left vertical arrow and the bottom arrows are also bijective. It in particular implies that each map $\mathbb{S}_{(\gamma_1, \gamma_2)|\mu} \rightarrow \mathbb{S}_{(\mu_1, \mu_2)|\mu}$ is a bijection for every μ . \square

Putting things together, we have proved the following theorem.

Theorem 3.3.18 (Braverman-Gaitsgory). *The collection $\{\mathbb{M}\mathbb{V}_\mu\}$ for all μ form a closed family of \hat{G} -crystals.¹⁸*

3.3.19. The main result we need in §4.4 is as follows. We recall that $L\mathbb{G}_a$ admits a filtration by closed subgroups $L^+\mathbb{G}_a^{(n)}$, where $L^+\mathbb{G}_a^{(n)}(R) = \varpi^n W_{\mathcal{O}}(R)$. In particular, $L^+\mathbb{G}_a^{(0)} = L^+\mathbb{G}_a$. If U_α is a root subgroup of G , then by choosing $x_\alpha : \mathbb{G}_a \simeq U_\alpha$ over \mathcal{O} and transport of structures, we obtain a filtration $LU_\alpha = \cup_{n \in \mathbb{Z}} L^+U_\alpha^{(n)}$, which is independent of the choice of the isomorphism x_α .

Proposition 3.3.20. *For each simple root α , let M_α denote the standard Levi with roots $\{\pm\alpha\}$. Then the map*

$$\prod \theta_{M_\alpha} : S_\lambda \rightarrow \prod_i S_{M_\alpha, \lambda}$$

defined in §3.2.10 induces a dominant map

$$(S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \rightarrow \prod_{\alpha} S_{M_\alpha, \lambda} \cap \mathrm{Gr}_{M_\alpha, \lambda + \varepsilon_\alpha(\mathbf{b})\alpha^\vee},$$

and its fibers are geometrically connected.

Remark 3.3.21. Note that this proposition refines [BrGai01, Corollary 3.1].

Proof. First, we assume that μ is minuscule. Then $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(w\mu)$ for some $w \in W$. In this case,

$$\varepsilon_\alpha(\mathbf{b}) = \max\{0, -\langle w\mu, \alpha \rangle\}, \quad \phi_\alpha(\mathbf{b}) = \max\{0, \langle w\mu, \alpha \rangle\}.$$

Then the proposition follows from (3.2.3).

Next, we assume that μ is quasi-minuscule. There are two cases.

- If $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(w\mu)$ for some $w \in W$, then still

$$\varepsilon_\alpha(\mathbf{b}) = \max\{0, -\langle w\mu, \alpha \rangle\}, \quad \phi_\alpha(\mathbf{b}) = \max\{0, \langle w\mu, \alpha \rangle\},$$

and the proposition follows from (3.2.4).

- If $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(0)$ corresponding to $\alpha \in \Delta_\mu$ under the canonical bijection $\mathbb{M}\mathbb{V}_\mu(0) \cong \Delta_\mu$, then

$$\varepsilon_\alpha(\mathbf{b}) = \phi_\alpha(\mathbf{b}) = 1, \quad \varepsilon_\beta(\mathbf{b}) = \phi_\beta(\mathbf{b}) = 0, \quad \beta \neq \alpha.$$

But it follows from (3.2.5) that $(S_0 \cap \mathrm{Gr}_\mu)^\alpha$ maps surjectively onto $(S_{M_\alpha, 0} \cap \mathrm{Gr}_{M_\alpha, \alpha^\vee}) \times \prod_{\beta \neq \alpha} (S_{M_\beta, 0} \cap \mathrm{Gr}_{M_\beta, 0})$, with geometrically connected fibers.

In general, $(S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}}$ is dominated by

$$(S_{\lambda_1} \cap \mathrm{Gr}_{\mu_1})^{\mathbf{b}_1} \tilde{\times} (S_{\lambda_2} \cap \mathrm{Gr}_{\mu_2})^{\mathbf{b}_2} \tilde{\times} \cdots \tilde{\times} (S_{\lambda_n} \cap \mathrm{Gr}_{\mu_n})^{\mathbf{b}_n} \rightarrow (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}},$$

for some sequence of minimal elements $(\mu_1, \dots, \mu_n) \subset \mathrm{Min}$, some sequence of weights $\lambda_1, \dots, \lambda_n$, and some elements $\mathbf{b}_j \in \mathbb{M}\mathbb{V}_{\mu_j}(\lambda_j)$ such that $\lambda = \sum \lambda_j$, and $\mathbf{b} = \mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_n$.

Consider the commutative diagram

$$(3.3.4) \quad \begin{array}{ccc} (S_{\lambda_1} \cap \mathrm{Gr}_{\mu_1})^{\mathbf{b}_1} \tilde{\times} (S_{\lambda_2} \cap \mathrm{Gr}_{\mu_2})^{\mathbf{b}_2} \tilde{\times} \cdots \tilde{\times} (S_{\lambda_n} \cap \mathrm{Gr}_{\mu_n})^{\mathbf{b}_n} & \longrightarrow & (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow \\ \prod_{\alpha} (S_{M_\alpha, \lambda_1} \cap \mathrm{Gr}_{M_\alpha, \lambda_1 + \varepsilon_\alpha(\mathbf{b}_1)\alpha^\vee}) \tilde{\times} \cdots \tilde{\times} (S_{M_\alpha, \lambda_n} \cap \mathrm{Gr}_{M_\alpha, \lambda_n + \varepsilon_\alpha(\mathbf{b}_n)\alpha^\vee}) & \twoheadrightarrow & \prod_{\alpha} S_{M_\alpha, \lambda} \cap \mathrm{Gr}_{M_\alpha, \lambda + \varepsilon_\alpha(\mathbf{b})\alpha^\vee} \end{array}$$

¹⁸A priori, the \hat{G} -crystal structures on $\mathbb{M}\mathbb{V}_\mu$ constructed in this way might be different from the one given in [BrGai01]. But it follows from [BaGau08] that they are the same.

Using the minuscule and quasi-minuscule cases discussed above, the left vertical arrow is dominant and has geometric connected fibers. Therefore, it is enough to show that for every $\alpha \in \Delta$, the α -factor of the bottom horizontal map of (3.3.4) is surjective and has geometrically connected fibers. Indeed, the fibers of this map are all affine spaces, so they are geometrically connected.

We now prove the surjectivity. By (3.3.1),

$$\varepsilon_\alpha(\mathbf{b}_1 \otimes \cdots \otimes \mathbf{b}_n) = \max_j \{ \varepsilon_\alpha(\mathbf{b}_j) - \langle \lambda_1 + \cdots + \lambda_{j-1}, \alpha \rangle \}.$$

Therefore, the union of the intervals

$$\bigcup_j [-\varepsilon_\alpha(\mathbf{b}_j) + \langle \lambda_1 + \cdots + \lambda_{j-1}, \alpha \rangle, \langle \lambda_1 + \cdots + \lambda_j, \alpha \rangle - 1]$$

contains $[-\varepsilon_\alpha(\mathbf{b}), \langle \lambda, \alpha \rangle - 1] \cap \mathbb{Z}$. Now we conclude by using the description of $S_\ell \cap \mathrm{Gr}_{\mathrm{PGL}_2, m}$ in (3.2.2). \square

3.4. The geometric Satake.

3.4.1. The geometric Satake. The following theorem of geometric Satake will play the central role in our construction. When $\mathrm{char} F > 0$, this is a result of Lusztig, Drinfeld, Ginzburg, Mirković-Vilonen (see [MV07]) and when $\mathrm{char} F = 0$, this is the main theorem of [Zhu17]. Let $\mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k})$ be the Satake category: the category of $L^+G \otimes \bar{k}$ -equivariant $\overline{\mathbb{Q}}_\ell$ -coefficients perverse sheaves on $\mathrm{Gr} \otimes \bar{k}$. Let us recall the definition of this category in some detail, since similar constructions will repeatedly appear in §5.1.7 and §5.3.10.

We first recall that $\pi_0(\mathrm{Gr} \otimes \bar{k})$ is canonically isomorphic to $\pi_1(G)$. Consider the decomposition $\mathrm{Gr} \otimes \bar{k} = \sqcup_{\zeta \in \pi_1(G)} \mathrm{Gr}_\zeta$ into connected components. For $\nu \in \mathbb{X}_\bullet(T)$, if its image under $\mathbb{X}_\bullet(T) \rightarrow \pi_1(G)$ is ζ , we write $\nu \in \zeta$. Then

$$\mathrm{Gr}_\zeta = \varinjlim_{\mu \in \zeta} \mathrm{Gr}_\mu,$$

where $i_{\mu, \mu'} : \mathrm{Gr}_\mu \rightarrow \mathrm{Gr}_{\mu'}$ is a closed embedding if $\mu \preceq \mu'$. Recall that if m is μ -large, the action of L^+G on Gr_μ factors through the action $L^m G$ (see Lemma 3.1.7). Note that if $m' \geq m \geq \langle \mu, \alpha_h \rangle$, there is a canonical equivalence

$$\mathrm{P}_{L^m G \otimes \bar{k}}(\mathrm{Gr}_\mu) \cong \mathrm{P}_{L^{m'} G \otimes \bar{k}}(\mathrm{Gr}_\mu).$$

Then we define

$$(3.4.1) \quad \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k}) := \bigoplus_{\zeta \in \pi_1(G)} \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr}_\zeta), \quad \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr}_\zeta) = \varinjlim_{(\mu, m)} \mathrm{P}_{L^m G \otimes \bar{k}}(\mathrm{Gr}_\mu),$$

where the limit is taken over $\{(\mu, m) \mid \mu \in \zeta, m \geq \langle \mu, \alpha_h \rangle\}$, with the partial order given by $(\mu, m) \leq (\mu', m')$ if $\mu \preceq \mu'$ and $m \leq m'$, and the connecting functor given the fully faithful embedding

$$\mathrm{P}_{L^m G \otimes \bar{k}}(\mathrm{Gr}_\mu) \cong \mathrm{P}_{L^{m'} G \otimes \bar{k}}(\mathrm{Gr}_\mu) \xrightarrow{i_{\mu, \mu'}, *} \mathrm{P}_{L^{m'} G \otimes \bar{k}}(\mathrm{Gr}_{\mu'}).$$

Note that the limit is filtered (although not directly).

The above defined category has a natural monoidal structure, given by Lusztig's convolution product of sheaves. Let $\mathcal{A}_1, \mathcal{A}_2 \in \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k})$, coming from Gr_{μ_1} and Gr_{μ_2} respectively, we denote by $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ the “external twisted product” of \mathcal{A}_1 and \mathcal{A}_2 on $\mathrm{Gr}_{\mu_1} \tilde{\times} \mathrm{Gr}_{\mu_2}$, whose pullback to $\mathrm{Gr}_{\mu_1}^{(n)} \times \mathrm{Gr}_{\mu_2}^{(n)}$ (for some n that is μ_2 -large) is isomorphic to the external product of the pullback of \mathcal{A}_1 to $\mathrm{Gr}_{\mu_1}^{(n)}$ and \mathcal{A}_2 (this property uniquely characterizes $\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2$ up to a unique isomorphism). For example, if $\mathcal{A}_i = \mathrm{IC}_{\mu_i}$, then $\mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}$ is canonically isomorphic to the intersection cohomology sheaf of $\mathrm{Gr}_{\mu_1} \tilde{\times} \mathrm{Gr}_{\mu_2}$. Similarly, one can define the n -folded “external twisted product” $\mathcal{A}_1 \tilde{\boxtimes} \cdots \tilde{\boxtimes} \mathcal{A}_n$

on $\mathrm{Gr}_{\mu_\bullet}$. If $\mathcal{A}_i = \mathrm{IC}_{\mu_i}$, then this is just the intersection cohomology sheaf $\mathrm{IC}_{\mu_\bullet}$ of $\mathrm{Gr}_{\mu_\bullet}$. Then the convolution product of \mathcal{A}_1 and \mathcal{A}_2 is defined as

$$(3.4.2) \quad \mathcal{A}_1 \star \mathcal{A}_2 := m_!(\mathcal{A}_1 \tilde{\boxtimes} \mathcal{A}_2),$$

where $m : \mathrm{Gr}_{\mu_1} \tilde{\times} \mathrm{Gr}_{\mu_2} \rightarrow \mathrm{Gr}$ is the convolution map (defined by (3.1.11)). A priori, this is an $(L^+G \otimes \bar{k})$ -equivariant ℓ -adic complex on $\mathrm{Gr} \otimes \bar{k}$. But miraculously, it is a perverse sheaf.

This following theorem is usually referred as the geometric Satake isomorphism.

Theorem 3.4.2 (Geometric Satake). *The Satake category is a semisimple tensor category. The hypercohomology functor $H^* : \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k}) \rightarrow \mathrm{Vect}_{\overline{\mathbb{Q}}_\ell}$ lifts to a natural equivalence of tensor categories*

$$(3.4.3) \quad H^* = H^*(\mathrm{Gr}, -) : \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k}) \rightarrow \mathrm{Rep}(\hat{G}_{\overline{\mathbb{Q}}_\ell}).$$

In particular, it sends $\mathrm{IC}_{\mu_1} \star \cdots \star \mathrm{IC}_{\mu_n} = (m_{\mu_\bullet})_* \mathrm{IC}_{\mu_\bullet}$ to the tensor product of highest weight representations V_{μ_\bullet} .

Remark 3.4.3. (1) Indeed, it is more canonical to *define* the dual group \hat{G} of G as the Tannakian group of the Tannakian category $(\mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k}), H^*)$, and *define* V_μ as $H^*(\mathrm{Gr}, \mathrm{IC}_\mu)$, equipped with the tautological action $\hat{G} = \mathrm{Aut}^\otimes(H^*)$. In the rest of the paper, we will take this point of view.

(2) As explained in [Zhu16⁺, §3], the Tannakian group is canonically equipped with a pinning $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$. We briefly recall the construction of (\hat{B}, \hat{T}) and refer to *loc. cit.* for more details. The grading on the cohomology H^* defines a cocharacter $\mathbb{G}_m \rightarrow \hat{G}$, which is regular. Then its centralizer gives a maximal torus \hat{T} , and $\hat{B} \supset \hat{T}$ is the unique Borel in \hat{G} such that this cocharacter is dominant with respect to this Borel.

From now on, we fix an inverse of the equivalence

$$\mathrm{Sat} : \mathrm{Rep}(\hat{G}) \rightarrow \mathrm{P}_{L^+G \otimes \bar{k}}(\mathrm{Gr} \otimes \bar{k}),$$

sending V_μ to IC_μ .

We need the following consequence of the geometric Satake isomorphism.

Corollary 3.4.4. *There is a canonical isomorphism*

$$(3.4.4) \quad \mathrm{Sat} : \mathrm{Hom}_{\hat{G}}(V_{\lambda_\bullet}, V_{\mu_\bullet}) \cong \mathrm{Corr}_{\mathrm{Gr}_{\lambda_\bullet}^0 | \mu_\bullet}((\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC})),$$

such that the composition

$$\mathrm{Hom}_{\hat{G}}(V_{\kappa_\bullet}, V_{\lambda_\bullet}) \otimes \mathrm{Hom}_{\hat{G}}(V_{\lambda_\bullet}, V_{\mu_\bullet}) \rightarrow \mathrm{Hom}_{\hat{G}}(V_{\kappa_\bullet}, V_{\mu_\bullet})$$

corresponding to

$$\begin{aligned} \mathrm{Corr}_{\mathrm{Gr}_{\kappa_\bullet}^0 | \lambda_\bullet}((\mathrm{Gr}_{\kappa_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC})) \otimes \mathrm{Corr}_{\mathrm{Gr}_{\lambda_\bullet}^0 | \mu_\bullet}((\mathrm{Gr}_{\lambda_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC})) \\ \rightarrow \mathrm{Corr}_{\mathrm{Gr}_{\kappa_\bullet}^0 | \mu_\bullet}((\mathrm{Gr}_{\kappa_\bullet}, \mathrm{IC}), (\mathrm{Gr}_{\mu_\bullet}, \mathrm{IC})) \end{aligned}$$

is obtained as the pushforward (in the sense of A.2.6) of the composition of the cohomological correspondences along the perfectly proper morphism

$$\mathrm{Comp} : \mathrm{Gr}_{\kappa_\bullet}^0 | \lambda_\bullet \times_{\mathrm{Gr}_{\lambda_\bullet}^0} \mathrm{Gr}_{\lambda_\bullet}^0 | \mu_\bullet \rightarrow \mathrm{Gr}_{\kappa_\bullet}^0 | \mu_\bullet.$$

3.4.5. Galois-equivariant geometric Satake. As explained in [RZ14] and [Zhu16⁺, §5.5], the Galois group $\text{Gal}(\bar{k}/k)$, acts on $P_{L+G\otimes\bar{k}}(\text{Gr} \otimes k)$ by tensor automorphisms: $\gamma \in \text{Gal}(\bar{k}/k)$ induces $\gamma : \text{Gr} \otimes \bar{k} \cong \text{Gr} \otimes \bar{k}$, and the pullback $\gamma^* : P_{L+G\otimes\bar{k}}(\text{Gr} \otimes \bar{k}) \rightarrow P_{L+G\otimes\bar{k}}(\text{Gr} \otimes \bar{k})$ is a tensor functor. In addition, there is a canonical isomorphism of tensor functors $\alpha_\gamma : H^* \circ \gamma^* \cong H^*$. By general nonsense, this defines an action of $\text{Gal}(\bar{k}/k)$ on \hat{G} : for every $g \in \hat{G}$, regarded as an automorphism of the tensor functor H^* , $\gamma(g)$ is the tensor automorphism of H^* defined as

$$H^* \xrightarrow{\alpha_\gamma} H^* \circ \gamma^* \xrightarrow{g \circ \text{id}} H^* \circ \gamma^* \xrightarrow{\alpha_\gamma^{-1}} H^*.$$

Note that since the Galois action preserves the grading, this action preserves $(\hat{G}, \hat{B}, \hat{T})$.

Remark 3.4.6. However, as explained in *loc. cit.*, this action does not preserve \hat{X} . In our case where k is a finite field, this is a minor issue and can be fixed by choosing a half Tate twist $\overline{\mathbb{Q}}_\ell(1/2)$. We refer *loc. cit.* for detailed discussion.

Notation 3.4.7. For $\gamma \in \text{Gal}(\bar{k}/k)$ and V a representation of \hat{G} , let γV be the representation of \hat{G} obtained as $\hat{G} \xrightarrow{\gamma^{-1}} \hat{G} \rightarrow \text{GL}(V)$, called the γ -twisted of V . In particular, for a dominant coweight μ of G , $\gamma V_\mu \cong V_{\gamma(\mu)}$.

It follows from the above discussion that

$$(3.4.5) \quad \gamma^* \text{Sat}(V) \cong \text{Sat}(\gamma V)$$

for every $V \in \text{Rep}(\hat{G})$.

Let $P_{L+G}^0(\text{Gr}) \subset P_{L+G}(\text{Gr})$ denote the full subcategory of semisimple sheaves of pure of weight zero. For example, by choosing a half Tate twist $\overline{\mathbb{Q}}_\ell(1/2)$, the normalized intersection cohomology sheaf IC_μ^N on Gr_μ is an object in $P_{L+G}^0(\text{Gr})$, where $\text{IC}_\mu^N|_{\text{Gr}_\mu} = \overline{\mathbb{Q}}_\ell[2\langle\rho, \mu\rangle](\langle\rho, \mu\rangle)$. By Corollary 3.1.10(3), and an argument as in [Zhu16⁺, Lemma 5.5.14 (1)], $P_{L+G}^0(\text{Gr})$ is a monoidal subcategory of $P_{L+G}(\text{Gr})$ under the convolution product.

Let ${}^L G = \hat{G} \rtimes \text{Gal}(\bar{k}/k)$ denote the usual (unramified) local Langlands dual group. We may regard ${}^L G$ as a pro-algebraic group and therefore it makes sense to define the category $\text{Rep}({}^L G)$ of algebraic representations of ${}^L G$. In particular, $\text{Rep}(\text{Gal}(\bar{k}/k))$ is the category of representations of finite quotients of $\text{Gal}(\bar{k}/k)$.

Theorem 3.4.8. *The category $P_{L+G}^0(\text{Gr})$ is a tensor subcategory of $P_{L+G}(\text{Gr})$. By choosing a half Tate twist $\overline{\mathbb{Q}}_\ell(1/2)$, the cohomology functor*

$$\bigoplus_i H^i(\text{Gr}, -)(\tfrac{i}{2}) : P_{L+G}^0(\text{Gr}) \rightarrow \text{Rep}(\text{Gal}(\bar{k}/k)),$$

lifts to an equivalence of categories

$$P_{L+G}^0(\text{Gr}) \cong \text{Rep}({}^L G),$$

such that the pullback functor $P_{L+G}^0(\text{Gr}) \rightarrow P_{L+G\otimes\bar{k}}(\text{Gr} \otimes \bar{k})$ corresponds to the restriction functor $\text{Rep}({}^L G) \rightarrow \text{Rep}(\hat{G})$.

Proof. See [RZ14] or [Zhu16⁺, §5.5]. □

3.5. From geometric to classic Satake isomorphism. We need to recall how to deduce the classical Satake isomorphism from the geometric Satake isomorphism via the sheaf-function dictionary. In this subsection, we fix a square root $q^{1/2} \in \overline{\mathbb{Q}}_\ell$ and hence accordingly a half Tate twist $\overline{\mathbb{Q}}_\ell(\frac{1}{2})$ on which the geometric q -Frobenius acts by multiplying with $q^{-1/2}$.

3.5.1. *Review of the classical Satake isomorphism.* There are already some good references for split groups, for example [Gro98]. However, as we need the unramified groups in an essential way, we include a more detailed discussion.

Let us denote by $K = G(\mathcal{O}) \subset G(F)$. We choose an embedding $B \subset G$ but the Satake isomorphism will be independent of the choice. Recall that $T = B/U$ is the abstract Cartan. Let 2ρ be the half sum of positive roots of G with respect to B . Let $S \subset T$ denote the maximal split subtorus. For a coweight $\lambda \in \mathbb{X}_\bullet(T)$, let $d_\lambda = \langle \rho, \lambda \rangle \in \frac{1}{2}\mathbb{Z}$. Recall we denote by σ the q -power Frobenius element, so that $\mathbb{X}_\bullet(S) = \mathbb{X}_\bullet(T)^\sigma$. Let $W_0 = N_G(S)/Z_G(S)$ denote the relative Weyl group of G , which is a subgroup of $W = N_G(T)/T$.

Let $\delta_B : T(F) \rightarrow \mathbb{Z}[q, q^{-1}]$ denote the modular character, i.e. $\delta_B(t) = |\det(\text{Ad}_t|_{\text{Lie}U})|$. Explicitly, $\delta_B|_{T(\mathcal{O})} = 1$ and $\delta_B(\varpi^\lambda) = q^{\langle 2\rho, \lambda \rangle}$ for $\lambda \in \mathbb{X}_\bullet(S)$. With the fixed square root $q^{1/2}$, let $\delta_B^{1/2} : T(F) \rightarrow \mathbb{Z}[q^{1/2}, q^{-1/2}]^\times$ denote the square root of δ_B , i.e. $\delta_B^{1/2}(\varpi^\lambda) = (q^{1/2})^{\langle 2\rho, \lambda \rangle} =: q^{d_\lambda}$.

We normalize the Haar measure on $G(F)$ (and on $U(F)$) so that the volume of K (and $U(\mathcal{O})$) is one. Let H_G denote the spherical Hecke algebra of G with respect to K , i.e. the space of K -bi-invariant, compactly supported, locally constant \mathbb{Z} -valued functions on $G(F)$, with the multiplication given by the convolution

$$(3.5.1) \quad (f * g)(x) = \int_{G(F)} f(xy^{-1})g(y)dy.$$

If $G = T$ is a torus, then H_T is canonically isomorphic to the group algebra $\mathbb{Z}[\mathbb{X}_\bullet(T)^\sigma]$ of the lattice $\mathbb{X}_\bullet(T)^\sigma$, by sending the characteristic function of $\varpi^\lambda T(\mathcal{O})$ to e^λ for $\lambda \in \mathbb{X}_\bullet(T)^\sigma$. Recall the Satake transform

$$\text{CT} : H_G \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \rightarrow H_T \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}], \quad f \mapsto \text{CT}(f),$$

$$\text{CT}(f)(t) := \delta_B^{1/2}(t) \int_{U(F)} f(tu)du.$$

Then the classical Satake isomorphism says that the Satake transform is injective, and induces a canonical isomorphism

$$(3.5.2) \quad \text{CT} : H_G \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \cong H_T^{W_0} \otimes \mathbb{Z}[q^{1/2}, q^{-1/2}] \cong \mathbb{Z}[q^{1/2}, q^{-1/2}][\mathbb{X}_\bullet(T)^\sigma]^{W_0}.$$

See [Ca79, §4.2]. Note that in *loc. cit.*, the Satake isomorphism was stated as an isomorphism of \mathbb{C} -algebras. But a closer look at the proof shows that it is an isomorphism over $\mathbb{Z}[q^{\pm 1/2}]$.

3.5.2. *Reinterpretation in terms of the dual group.* Let us reinterpret the classical Satake isomorphism in terms of the dual group. As we shall see, in fact there are two interpretations.

Recall that $\text{Gal}(\bar{k}/k)$ acts on $(\hat{G}, \hat{B}, \hat{T}, \hat{X})$, and we consider the Langlands dual group ${}^L G := \hat{G} \rtimes \text{Gal}(\bar{k}/k)$ as a pro-algebraic group. Let $\phi = \sigma^{-1}$ denote the geometric q -Frobenius element. Note that the inclusion $S \subset T$ induces $\hat{T} \rightarrow \hat{S}$ identifying \hat{S} as the quotient $\hat{T}/(1-\sigma)\hat{T} = (\hat{T} \times \phi)/\text{Int } \hat{T}$. In addition, $\mathbb{X}_\bullet(T)^\sigma$ the character group of \hat{S} . Recall that the following natural maps are bijective (cf. Gantmacher [Gan39, Theorem 14] and [Bo79, Lemmas 6.4 and 6.5])

$$\hat{S}/W_0 \xleftarrow{\sim} (\hat{T}\phi)/\text{Int } N_0 \xrightarrow{\sim} (\hat{G}\phi)_{ss}/\text{Int } \hat{G}.$$

Therefore, by restricting a function on $\hat{G}\phi$ to $\hat{T}\phi$, there is an isomorphism

$$\Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}) = \mathcal{O}_{\hat{G}}^{c_\phi(\hat{G})} = \mathcal{O}_{\hat{S}}^{W_0} = \overline{\mathbb{Q}}_\ell[\mathbb{X}_\bullet(T)^\sigma]^{W_0}.$$

Combining with (3.5.2), there is a unique isomorphism

$$(3.5.3) \quad \text{Sat}^{cl} := \text{Sat}_G^{cl} : \Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}) \cong H_G \otimes \overline{\mathbb{Q}}_\ell,$$

called the *Satake isomorphism*, making the following diagram commute

$$(3.5.4) \quad \begin{array}{ccc} \Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}) & \xrightarrow[\text{Sat}_G^{cl}]{\cong} & H_G \otimes \overline{\mathbb{Q}}_\ell \\ \text{Res} \downarrow & & \downarrow \text{CT} \\ \Gamma([\hat{T}\phi/\hat{T}], \mathcal{O}) & \xrightarrow[\text{Sat}_T^{cl}]{\cong} & H_T \otimes \overline{\mathbb{Q}}_\ell. \end{array}$$

Let $R({}^L G) = K_0(\text{Rep}({}^L G))$ denote the representation ring of ${}^L G$. If V is an algebraic representation of ${}^L G$, let χ_V denote its character and $[V]$ its class in $R({}^L G)$. Then there is a natural map

$$R({}^L G) \rightarrow \Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}), \quad [V] \mapsto \chi_V|_{\hat{G}\phi}$$

By abuse of notations, we will also use Sat^{cl} to denote the induced composition map

$$\text{Sat}^{cl} : R({}^L G) \rightarrow \Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}) \cong H_G \otimes \overline{\mathbb{Q}}_\ell.$$

Remark 3.5.3. Note that there is another natural map

$$(3.5.5) \quad \text{Sat}^{cl'} : \Gamma([\hat{G}\sigma/\hat{G}], \mathcal{O}) \cong \overline{\mathbb{Q}}_\ell[\mathbb{X}_\bullet(T)^\sigma]^{W_0} \cong H_G \otimes \overline{\mathbb{Q}}_\ell.$$

By restriction of a character χ_V to $\hat{G} \times \sigma \subset {}^L G$, we thus also obtain a map from $R({}^L G)$ to $H_G \otimes \overline{\mathbb{Q}}_\ell$, denoted by the same notation

$$\text{Sat}^{cl'} : R({}^L G) \rightarrow \Gamma([\hat{G}\sigma/\hat{G}], \mathcal{O}) \cong H_G \otimes \overline{\mathbb{Q}}_\ell.$$

However, Sat^{cl} and $\text{Sat}^{cl'}$ are different in general. Indeed, note that the following diagram is commutative

$$\begin{array}{ccccc} & & \Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}) & \xrightarrow[\cong]{\text{Sat}^{cl}} & \overline{\mathbb{Q}}_\ell[\mathbb{X}_\bullet(T)^\sigma]^{W_0} \\ & \nearrow & \downarrow \text{inv} & & \downarrow \lambda \mapsto -\lambda \\ R({}^L G) & & \Gamma([\hat{G}\sigma/\hat{G}], \mathcal{O}) & \xrightarrow[\cong]{\text{Sat}^{cl'}} & \overline{\mathbb{Q}}_\ell[\mathbb{X}_\bullet(T)^\sigma]^{W_0} \end{array}$$

where $\text{inv} : \Gamma([\hat{G}\phi/\hat{G}], \mathcal{O}) \cong \Gamma([\hat{G}\sigma/\hat{G}], \mathcal{O})$ is induced by the map $\hat{G}\phi \mapsto \hat{G}\sigma$, $g\phi \mapsto \sigma(g^{-1})\sigma$. Therefore,

$$\text{Sat}^{cl}([V]) = \text{Sat}^{cl'}([V^*]).$$

It is Sat^{cl} that is directly related to the geometric Satake via Grothendieck's sheaf-function dictionary. However, $\text{Sat}^{cl'}$ will appear in Theorem 6.0.1. This leads to usual switch from V to V^* for Galois representations arising from the cohomology of Shimura varieties.

3.5.4. From geometric to classical Satake isomorphism. Now we review how to deduce the classical Satake isomorphism from the geometric Satake isomorphism (see also [RZ14], [Zhu16⁺, §5.6] where the case $F = k((\varpi))$ was discussed).

For a k -point x of Gr , and a geometric point \bar{x} over it, let ϕ_x denote the geometric Frobenius in $\text{Gal}(k(\bar{x})/k)$. Recall that the Grothendieck fonctions-faisceaux dictionary attaches to every $\mathcal{A} \in \text{P}_{L+G}(\text{Gr})$ a function

$$f_{\mathcal{A}} : \text{Gr}(k) \rightarrow \overline{\mathbb{Q}}_\ell, \quad f_{\mathcal{A}}(x) = \sum_i (-1)^i \text{tr}(\phi_x, H_{\bar{x}}^i(\mathcal{A})),$$

Since \mathcal{A} is L^+G -equivariant, we can regard $f_{\mathcal{A}} \in H_G \otimes_{\mathbb{Z}} \overline{\mathbb{Q}}_\ell$. In addition, it follows from definition that $f_{\mathcal{A}_1 \star \mathcal{A}_2} = f_{\mathcal{A}_1} * f_{\mathcal{A}_2}$ (e.g. see [Zhu16⁺, Lemma 5.6.1]). So $\mathcal{A} \mapsto f_{\mathcal{A}}$ defines a map of algebras

$$\text{Tr} : K_0(\text{P}_{L+G}^0(\text{Gr})) \rightarrow H_G \otimes \overline{\mathbb{Q}}_\ell.$$

On the other hand, taking the K-ring of the equivalence in Theorem 3.4.8 gives a canonical isomorphism

$$K_0(\text{Sat}) : R(^L G) \rightarrow K_0(\mathbf{P}_{L+G}^0(\text{Gr}_G)).$$

Composing these two maps, we thus obtain

$$\text{Sat}^{\text{s-f}} : R(^L G) \rightarrow H_G \otimes \overline{\mathbb{Q}}_\ell.$$

Proposition 3.5.5. $\text{Sat}^{\text{s-f}} = \text{Sat}^{\text{cl}}$.

Proof. We sketch the proof and refer to [RZ14] for more details. First, one shows that the map

$$R(^L G) \rightarrow K_0(\mathbf{P}_{L+G}^0(\text{Gr})) \rightarrow H_G \otimes \overline{\mathbb{Q}}_\ell$$

factors through

$$R(^L G) \rightarrow \Gamma([\hat{G}\phi]/\hat{G}, \mathcal{O}) \rightarrow H_G \otimes \overline{\mathbb{Q}}_\ell.$$

Indeed, the kernel of the map $[V] \mapsto \chi_V|_{\hat{G}\phi}$ is the ideal generated by elements of the form $[V \otimes \psi] - \psi(\phi)[V]$, where $V \in \text{Rep}(^L G)$ and $\psi : \text{Gal}(\bar{k}/k) \rightarrow \overline{\mathbb{Q}}_\ell^\times$ is a character factoring through a finite quotient. On the other hand, the kernel of the map $\text{Tr} : K_0(\mathbf{P}_{L+G}^0(\text{Gr})) \rightarrow H_G \otimes \overline{\mathbb{Q}}_\ell$ is the ideal generated by elements of the form $[\mathcal{A} \otimes \mathcal{L}] - \text{tr}(\phi, \mathcal{L})[\mathcal{A}]$, where $\mathcal{A} \in \mathbf{P}_{L+G}^0(\text{Gr}_G)$, and \mathcal{L} is a rank one local system on Speck , pure of weight zero. But it is clear that these two ideals match under $K_0(\text{Sat})$. The claims follows.

To finish to prove of the proposition, first note that this is clear if $G = T$ is a torus. For general G , one observes that the fiber functor decomposes as a direct sum of weight functors [MV07, §3] and [Zhu17, Corollary 2.10]:

$$\mathcal{CT} : \mathbf{P}_{L+G}^0(\text{Gr}_G) \rightarrow \mathbf{P}_{L+T}^0(\text{Gr}_T), \quad \mathcal{CT}(\mathcal{A}) = \bigoplus_{\lambda} H_c^{d_\lambda}(S_\lambda, \mathcal{A})(d_\lambda).$$

Note that under the sheaf-function dictionary, this corresponds the Satake transform $\text{CT} : H_G \otimes \overline{\mathbb{Q}}_\ell \rightarrow H_T \otimes \overline{\mathbb{Q}}_\ell$. Therefore, both $\text{Sat}^{\text{s-f}}$ and Sat^{cl} are uniquely determined the commutative diagram (3.5.4). The proposition for general G follows. \square

4. IRREDUCIBLE COMPONENTS OF SOME AFFINE DELIGNE-LUSZTIG VARIETIES

This section is devoted to studying the irreducible components of certain affine Deligne-Lusztig varieties in equal and mixed characteristic.

We will continue to use notations from the previous section: let F be a local field (of either equal or mixed characteristic), with ring of integers \mathcal{O} and residue field $k = \mathbb{F}_q$, and let G be an *unramified* reductive group over \mathcal{O} . Let $\mathcal{O}_L = W_{\mathcal{O}}(\bar{k})$ be the completion of the maximal unramified extension of F and $L = \mathcal{O}_L[1/\varpi]$. Let $(\hat{G}, \hat{B}, \hat{T})$ be its based Langlands dual group over $\overline{\mathbb{Q}}_\ell$, constructed from the geometric Satake as in the previous section (see Remark 3.4.3).

For a perfect k -algebra R , and a G -torsor \mathcal{E} over D_R , let ${}^\sigma \mathcal{E}$ denote the G -torsor $(\sigma \otimes \text{id})^* \mathcal{E}$ on $W(R) \otimes_{W(k)} \mathcal{O}$ as the pullback of \mathcal{E} by the Frobenius $\sigma \otimes \text{id}$.

4.1. Definition of ADLVs.

Definition 4.1.1. For $b \in G(L)$ and μ a dominant coweight, we define the (*closed*) *affine Deligne-Lusztig variety* $X_\mu(b)$ as the closed subvariety inside the affine Grassmannian:

$$X_\mu(b) = \{g \in LG/L^+G \mid g^{-1}b\sigma(g) \in \overline{L^+G\varpi^\mu L^+G}\}.$$

In terms of the moduli problem, for every R , $X_\mu(b)(R)$ parametrizes commutative diagrams of modifications of G -torsors on D_R ,

$$(4.1.1) \quad \begin{array}{ccc} \sigma \mathcal{E}_1 & \xrightarrow{\beta_1} & \mathcal{E}_1 \\ \downarrow \sigma(\beta_0) & & \downarrow \beta_0 \\ \sigma \mathcal{E}^0 & \xrightarrow{b \cdot} & \mathcal{E}^0, \end{array}$$

where $b \cdot$ is the modification of the trivial G -torsor $\sigma \mathcal{E}^0 \cong \mathcal{E}^0$ given by multiplication by b , and β_1 has relative position $\leq \mu$.

Note that by the definition, the following diagram is Cartesian

$$(4.1.2) \quad \begin{array}{ccc} X_\mu(b) & \longrightarrow & \mathrm{Gr} \tilde{\times} \mathrm{Gr}_\mu \\ \downarrow & & \downarrow \mathrm{pr} \times m \\ \mathrm{Gr} & \xrightarrow{1 \times b\sigma} & \mathrm{Gr} \times \mathrm{Gr}, \end{array} \quad \begin{array}{ccc} (4.1.1) & \longmapsto & \{\sigma \mathcal{E}_1 \xrightarrow{\beta_1} \mathcal{E}_1 \xrightarrow{\beta_0} \mathcal{E}_0\} \\ \downarrow & & \downarrow \\ \{\mathcal{E}_1 \xrightarrow{\beta_0} \mathcal{E}_0\} & \longmapsto & \{\mathcal{E}_1 \xrightarrow{\beta_0} \mathcal{E}_0, \sigma \mathcal{E}_1 \xrightarrow{b\sigma(\beta_0)} \mathcal{E}_0\}. \end{array}$$

In turn, (4.1.2) can be served as an alternative definition of affine Deligne-Lusztig varieties. Using this point of view, one can replace Gr_μ in (4.1.2) by $\mathrm{Gr}_{\mu_\bullet}$ and therefore obtain an iterated version of affine Deligne-Lusztig varieties.

Let J_b be the twisted centralizer of b , i.e. the algebraic group over F , whose value on an F -algebra R is given by

$$(4.1.3) \quad J_b(R) = \{g \in G(R \otimes_F L) \mid g^{-1}b\sigma(g) = b\}.$$

Then $J_b(F)$ acts on $X_\mu(b)$ naturally by left multiplication.

Finally, recall that the pair $(X_\mu(b), J_b)$ depends only on the σ -conjugacy class of b up to isomorphism.

4.1.2. The set $B(G, \mu)$. Let $B(G)$ denote the set of σ -conjugacy classes of $G(L)$. To study $B(G)$, Kottwitz attached to every element $[b] \in B(G)$ two invariants:¹⁹ the first is the *Newton polygon* of b , which is a σ -invariant conjugacy class of maps

$$\nu_b \in (\mathrm{Hom}(\mathbb{D}, G_L)/G_L)^\sigma \cong \mathbb{X}_\bullet(T)_\mathbb{Q}^{+, \sigma},$$

where \mathbb{D} is the pro-torus with character group \mathbb{Q} ; the second is an element

$$\kappa_G(b) \in \pi_1(G)_\sigma = \mathbb{X}^\bullet(Z(\hat{G})^\sigma).$$

In addition, Kottwitz proved that ν_b and $\kappa_G(b)$ together determine the σ -conjugacy class $[b]$ uniquely. In other words, the map

$$B(G) \rightarrow \mathbb{X}_\bullet(T)_\mathbb{Q}^{+, \sigma} \times \pi_1(G)_\sigma, \quad [b] \mapsto (\nu_b, \kappa_G(b))$$

is injective. The set $B(G)$ naturally forms a poset with $[b] \leq [b']$ if $\kappa_G(b) = \kappa_G(b')$ and $\nu_b \leq \nu_{b'}$. Here, the inequality $\nu_b \leq \nu_{b'}$ means that $\nu_{b'} - \nu_b$ is a non-negative rational linear combination of simple coroots of G . We call an element $[b] \in B(G)$ *basic* if ν_b is central, i.e. it factors as

$$\nu_b : \mathbb{D} \rightarrow Z_G \subseteq G.$$

Or equivalently, $\langle \rho, \nu_b \rangle = 0$. By definition, basic elements are minimal elements in $B(G)$ with respect to the above partial order. It follows from [Ko85, §5] that all minimal elements in $B(G)$ are basic.

¹⁹We remark that Kottwitz' results hold more generally for not necessarily quasi-split groups. But we limit ourselves to only considering unramified groups in the following discussion.

For a dominant coweight μ of G , we define $B(G, \mu) \subset B(G)$ as the subset consisting of those $[b] \in B(G)$ such that

$$(4.1.4) \quad \nu_b \leq \bar{\mu}, \quad \text{and} \quad \kappa_G(b) = [\mu] \text{ in } \pi_1(G)_\sigma,$$

where

$$(4.1.5) \quad \bar{\mu} = \frac{1}{m} \sum_{i=0}^{m-1} \sigma^i(\mu),$$

and m is the degree of a splitting field of G over F . For a coweight μ , $[\mu]$ denotes its image under $\mathbb{X}_\bullet(T) \rightarrow \pi_1(G) \rightarrow \pi_1(G)_\sigma$. By [Ko85, §5], inside $B(G, \mu)$ there is always a unique basic element.

Recall the following result, which is a combination of [RR96, Theorem 4.2] and [Gas10] (see also [Win05] and [GHKR06, Proposition 5.6.1]).

Theorem 4.1.3. *The variety $X_\mu(b)$ is non-empty if and only if $[b] \in B(G, \mu)$.*

4.1.4. *Passing to adjoint groups.* To study ADLV, it is more convenient to assume that G is of adjoint type. We first recall that $\pi_0(LG)$ is canonically isomorphic to $\pi_1(G)$. We denote the map $LG \rightarrow \pi_0(LG) \cong \pi_1(G)$ by w . Note that if $\lambda \in \mathbb{X}_\bullet(T)$, then $w(\varpi^\lambda) = [\lambda]$ is the image of the natural projection $\mathbb{X}_\bullet(T) \rightarrow \pi_1(G)$. Note that if $gL^+G \in X_\mu(b)$, then $w(g)^{-1}w(b)w(\sigma(g)) = w(\varpi^\mu)$. So we have an equality $(\sigma - 1)w(g) = [\mu] - w(b)$ in $\pi_1(G)$. In [CKV15], the authors denote

$$c_{b,\mu}\pi_1(G)^\sigma := \{\gamma \in \pi_1(G) \mid (\sigma - 1)\gamma = [\mu] - w(b)\},$$

which is a $\pi_1(G)^\sigma$ -coset. The above observations imply that there is the following commutative diagram

$$\begin{array}{ccc} X_\mu(b) & \longrightarrow & \text{Gr} \\ \downarrow & & \downarrow \\ c_{b,\mu}\pi_1(G)^\sigma & \longrightarrow & \pi_1(G). \end{array}$$

Now, let $G \rightarrow G'$ be a central isogeny of unramified groups. For a pair (μ, b) , let (μ', b') denote the induced pair for G' . The map $G \rightarrow G'$ induces a natural map $X_\mu(b) \rightarrow X_{\mu'}(b')$. The following lemma was contained in [CKV15, §2.4].

Lemma 4.1.5. *The natural map $X_\mu(b) \rightarrow X_{\mu'}(b')$ induces a Cartesian diagram*

$$\begin{array}{ccc} X_\mu(b) & \longrightarrow & X_{\mu'}(b') \\ \downarrow & & \downarrow \\ c_{b,\mu}\pi_1(G)^\sigma & \longrightarrow & c_{b',\mu'}\pi_1(G')^\sigma. \end{array}$$

4.1.6. *The variety $X_{\mu,\nu}(b)$.* In this paper, we also need to consider the intersection of ADLVs with Schubert varieties in the affine Grassmannian. Let $b \in G(L)$ and μ, ν two dominant coweights, we define

$$(4.1.6) \quad X_{\mu,\nu}(b) = X_\mu(b) \cap \text{Gr}_\nu,$$

i.e. $X_{\mu,\nu}(b)$ classifies those diagrams as in (4.1.1), with the extra condition that $\text{Inv}(\beta_0) \preceq \nu$. Alternatively, we can define it via the Cartesian diagram, similar to (4.1.2)

$$(4.1.7) \quad \begin{array}{ccc} X_{\mu,\nu}(b) & \longrightarrow & \text{Gr}_\nu \tilde{\times} \text{Gr}_\mu \\ \downarrow & & \downarrow \text{pr}_1 \times m_{\nu,\mu} \\ \text{Gr}_\nu & \xrightarrow[47]{1 \times b\sigma} & \text{Gr}_\nu \times \text{Gr}. \end{array}$$

Similarly, for a sequence μ_\bullet of dominant coweights, we can define $X_{\mu_\bullet, \nu}(b)$ by replacing Gr_μ (4.1.7) by Gr_{μ_\bullet} .

Note that $X_{\mu, \nu}(b)$ is equipped with an action of $J_b(F) \cap G(\mathcal{O}_L)$, and the pair $(X_{\mu, \nu}(b), J_b(F) \cap G(\mathcal{O}_L))$ depends on b up to σ -conjugacy by elements of $G(\mathcal{O}_L)$. So it is natural to introduce the set $A(G)$, which is the quotient of $G(L)$ by $G(\mathcal{O}_L)$ by the σ -conjugation action. There is a natural map (the Newton map)

$$(4.1.8) \quad \mathcal{N} : A(G) \rightarrow B(G),$$

sending an element in $G(L)$ up to σ -conjugacy by $G(\mathcal{O}_L)$ to its underlying σ -conjugacy class. In fact $A(G)$ is exactly the set of \bar{k} -points of the moduli of local G -shtukas Sht^{loc} introduced in §5.2. We denote $A(G, \mu)$ to be the preimage of $B(G, \mu)$ under \mathcal{N} . Note that $A(G, \mu)$ can be explicitly presented as the quotient of $\overline{G(\mathcal{O}_L) \varpi^\mu G(\mathcal{O}_L)}$ by σ -conjugation by $G(\mathcal{O}_L)$.

4.2. Unramified elements in $B(G, \mu)$. In this subsection we give a description of the set of unramified elements in $B(G, \mu)$ in terms of the dual group \hat{G} . Although this is just a small subset of $B(G, \mu)$, it already contains many interesting examples (see Remark 4.2.11).

4.2.1. The set of unramified elements. We first give a definition of unramified elements in $B(G)$, different from the one given in the introduction (but the two definitions are the same by Corollary 4.2.12 below). For this purpose, we need to choose a rational Borel $B \subset G$ and fix an embedding of the abstract Cartan $T \subset B$. We call an element $b \in B(G)$ unramified if it is contained in the image of the natural map $B(T) \rightarrow B(G)$, and denote the set of unramified elements in $B(G)$ by $B(G)_{\text{unr}}$. Note that since any two such choices of embeddings are rationally conjugate, this set is independent of the choices.²⁰

Recall that we denote by $\mathbb{X}_\bullet(T)_\sigma$ the σ -coinvariants of $\mathbb{X}_\bullet(T)$. For $\lambda \in \mathbb{X}_\bullet(T)$, let λ_σ denote its projection to $\mathbb{X}_\bullet(T)_\sigma$. The relative Weyl group W_0 of G acts on $\mathbb{X}_\bullet(T)_\sigma$.

Lemma 4.2.2. *The canonical map $\mathbb{X}_\bullet(T) \rightarrow B(G)$, $\lambda \mapsto [\varpi^\lambda]$ induces a bijection*

$$\mathbb{X}_\bullet(T)_\sigma / W_0 \cong B(G)_{\text{unr}}.$$

Proof. Since the map $\mathbb{X}_\bullet(T)_\sigma \rightarrow B(T)$, $\lambda \mapsto [\varpi^\lambda]$ is bijective, the map in the lemma is clearly surjective. Let $\lambda_1, \lambda_2 \in \mathbb{X}_\bullet(T)$, and write $b_i = [\varpi^{\lambda_i}]$. Assume that $b_1 = b_2$. Then since $\kappa_G(b_1) = \kappa_G(b_2)$, $\lambda_1 - \lambda_2 = (\sigma - 1)\nu + \gamma$ for some $\nu \in \mathbb{X}_\bullet(T)$ and $\gamma \in Q^\vee$, where $Q^\vee \subset \mathbb{X}_\bullet(T)$ denotes the coroot lattice of G . Up to σ -conjugation, we may replace λ_1 by $\lambda_1 - (\sigma - 1)\nu$, so assume that $\lambda_1 - \lambda_2 = \gamma$. Since $\nu_{b_1} = \nu_{b_2}$, after conjugation of λ_2 by an element in W_0 we may further assume that $\bar{\lambda}_1 = \bar{\lambda}_2$ (where $\bar{\lambda}_j$ is defined as in (4.1.5)). Therefore, $\sum_{i=0}^{m-1} \sigma^i(\gamma) = 0$. Since σ permutes a basis of the coroot lattice (i.e. the set of the simple coroots), $\gamma = (\sigma - 1)\delta$ for some coroot δ . It follows that the image of λ_1 and λ_2 in $\mathbb{X}_\bullet(T)_\sigma / W_0$ coincide, i.e., the map $\mathbb{X}_\bullet(T)_\sigma / W_0 \rightarrow B(G)_{\text{unr}}$ is also injective. \square

To continue, we recall some combinatorics of $\mathbb{X}_\bullet(T)_\sigma$. Recall that $\Delta \subset \mathbb{X}^\bullet(T)$ (resp. $\Delta^\vee \subset \mathbb{X}_\bullet(T)$) denotes the set of simple roots (resp. coroots) of G . Then we can define the set of dominant elements in $\mathbb{X}_\bullet(T)_\sigma$ as

$$\mathbb{X}_\bullet(T)_\sigma^+ = \left\{ \lambda_\sigma \in \mathbb{X}_\bullet(T)_\sigma \mid \left\langle \lambda, \sum_{i=0}^{m-1} \sigma^i(\alpha) \right\rangle \geq 0 \text{ for every } \alpha \in \Delta \right\}.$$

The natural map $\mathbb{X}_\bullet(T)_\sigma^+ \rightarrow \mathbb{X}_\bullet(T)_\sigma / W_0$ is an isomorphism. In addition, the partial order “ \preceq ” on $\mathbb{X}_\bullet(T)$ descends to a partial order on $\mathbb{X}_\bullet(T)_\sigma$. Explicitly, for $\lambda_\sigma, \mu_\sigma \in \mathbb{X}_\bullet(T)_\sigma$, $\lambda_\sigma \preceq \mu_\sigma$ if $\mu_\sigma - \lambda_\sigma$

²⁰Recall that the set of basic elements in $B(G)$, on the other hand, is the image of $B(T') \rightarrow B(G)$ for some elliptic maximal F -torus of G , see [Ko85, Proposition 5.3].

is a positive *integral* combination of elements in $\Delta^\vee := \text{Im}(\Delta^\vee \rightarrow \mathbb{X}_\bullet(T)_\sigma)$. Note there is a natural partial order preserving injective map

$$\mathbb{X}_\bullet(T)_\sigma^+ \rightarrow \mathbb{X}_\bullet(T)_\mathbb{Q}^{\sigma,+} \times \pi_1(G)_\sigma, \quad \mu_\sigma \mapsto (\bar{\mu}, [\mu]).$$

We can reformulate the above lemma as follows.

Lemma 4.2.3. *The following diagram is commutative, and all maps in the diagram preserve the partial order*

$$\begin{array}{ccc} B(G)_{\text{unr}} & \xrightarrow{\cong} & \mathbb{X}_\bullet(T)_\sigma^+ \\ \downarrow & & \downarrow \\ B(G) & \longrightarrow & \mathbb{X}_\bullet(T)_\mathbb{Q}^{\sigma,+} \times \pi_1(G)_\sigma. \end{array}$$

Corollary 4.2.4. *Under the identifications $\mathbb{X}_\bullet(T)_\sigma^+ \cong \mathbb{X}_\bullet(T)_\sigma/W_0 \cong B(G)_{\text{unr}}$ as above,*

$$B(G)_{\text{unr}} \cap B(G, \mu) = \{\lambda_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+ \mid \lambda_\sigma \preceq \mu_\sigma\}.$$

Remark 4.2.5. Note that the natural map $\mathbb{X}_\bullet(T)^+ \rightarrow \mathbb{X}_\bullet(T)_\sigma^+$ may not be surjective. For example, let $G = (\text{Res}_{E/F} \text{SL}_2)/\mu_2$, where E/F is an unramified quadratic extension and $\mu_2 \subset \text{Res}_{E/F} \mu_2 = Z(\text{Res}_{E/F} \text{SL}_2)$. The coweight lattice of $G_{\text{ad}} = \text{Res}_{E/F} \text{PGL}_2$ is canonically identified with \mathbb{Z}^2 in the usual way such that $\mathbb{Z}_{\geq 0}^2$ is the semigroup of dominant coweights and that the Frobenius σ acts on \mathbb{Z}^2 by permuting the two factors of \mathbb{Z}^2 . Then $\mathbb{X}_\bullet(T)$ is identified with the subgroup of $\mathbb{Z}^2 = \mathbb{X}_\bullet(T_{\text{ad}})$ generated by $(2, 0)$, $(0, 2)$ and $(1, 1)$. Now, let $\tau = (1, -1) \in \mathbb{X}_\bullet(T)$. Then $\tau_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+$ but this element is not in the image of $\mathbb{X}_\bullet(T)^+$.

The failure of the surjectivity of the above map will cause some complications in the sequel. However, if Z_G is connected, there exists fundamental coweights $\omega_i \in \mathbb{X}_\bullet(T)$ such that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ for all $\alpha_j \in \Delta$. Then it follows that the natural map $\mathbb{X}_\bullet(T)^+ \rightarrow \mathbb{X}_\bullet(T)_\sigma^+$ is surjective.

Now let \hat{G}^σ denote the σ -fixed points of \hat{G} . Although this is a possibly non-connected reductive group, its representation theory behaves as if the connected reductive groups: First \hat{G}^σ contains the diagonalizable subgroup \hat{T}^σ , whose character group is $\mathbb{X}_\bullet(T)_\sigma$; Second, the combinatoric structure on $\mathbb{X}_\bullet(T)_\sigma$ as discussed above allows one to talk about the highest weight of a finite dimensional representation of \hat{G}^σ (with respect to \hat{T}^σ). Using the fact that the natural map $\hat{T}^\sigma/\hat{T}^{\sigma,\circ} \rightarrow \hat{G}^\sigma/\hat{G}^{\sigma,\circ}$ is an isomorphism (see [Zhu15, Lemma 4.6]), the usual highest weight theory for representations of a connected reductive group extends to \hat{G}^σ without change (see [Zhu15, Lemma 4.10]). In particular, for every $\mu_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+$, there is a unique irreducible representation of \hat{G}^σ of highest weight μ_σ , and every irreducible representation is of this form. Given $\lambda_\sigma, \mu_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+$, $\lambda_\sigma \preceq \mu_\sigma$ if and only if λ_σ appears as a weight for the representation V_{μ_σ} .

Using the above observations, we can show the following.

Lemma 4.2.6. *The unramified element $[\varpi^\tau]$ belongs to $B(G, \mu)$ if and only if the following subspace of V_μ*

$$(4.2.1) \quad \bigoplus_{\lambda \in \tau + (\sigma - 1)\mathbb{X}_\bullet} V_\mu(\lambda)$$

is nonzero.

Proof. To prove the lemma, we are free to replace τ by $w(\tau)$ for some $w \in W_0$. Therefore, we may assume that $\tau_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+$. The above space (4.2.1) is just the τ_σ -weight space of $V_\mu|_{\hat{G}^\sigma}$, which has μ_σ as its unique highest weight. Therefore, (4.2.1) is non-zero if and only if $\tau_\sigma \preceq \mu_\sigma$, which by Corollary 4.2.4 is equivalent to $[\varpi^\tau] \in B(G, \mu)$. \square

Remark 4.2.7. The above criterion shows that if $[\varpi^\tau] \in B(G, \mu)$, then $[\varpi^{w(\tau)}] \in B(G, \mu)$ for every $w \in W$ (although τ and $w(\tau)$ may not be σ -conjugate if $w \notin W_0$).

4.2.8. *Unramified basic elements.* Next, we discuss when $B(G, \mu) \cap B(G)_{\text{unr}}$ contains the basic element. Recall that we have defined

$$\Lambda = \left\{ \lambda \in \mathbb{X}_\bullet(T) \mid \sum_{i=0}^{m-1} \sigma^i(\lambda) \in \mathbb{X}_\bullet(Z_G) \right\} \quad \text{and} \quad V_\mu^{\text{Tate}} = \bigoplus_{\lambda \in \Lambda} V_\mu(\lambda).$$

Lemma 4.2.9. *There exists some $\tau \in \Lambda$ such that $V_\mu^{\text{Tate}} = \bigoplus_{\lambda \in \tau + (\sigma-1)\mathbb{X}_\bullet} V_\mu(\lambda)$.*

Proof. Let $\lambda_1, \lambda_2 \in \Lambda$ be two weights appearing in V_μ . Then ϖ^{λ_i} represents the basic element in $B(G, \mu)$. Therefore, $\lambda_1 = w(\lambda_2) + (\sigma-1)\nu$ for some $w \in W_0$ and $\nu \in \mathbb{X}_\bullet(T)$ by Lemma 4.2.2. Note that $w(\lambda_2) - \lambda_2$ is an element in the coroot lattice of G and

$$\sum_{i=0}^{m-1} \sigma^i(w(\lambda_2) - \lambda_2) = w\left(\sum_{i=0}^{m-1} \sigma^i(\lambda_2)\right) - \sum_{i=0}^{m-1} \sigma^i(\lambda_2) = 0.$$

Therefore, $w(\lambda_2) - \lambda_2 = (\sigma-1)\nu'$ for some ν' in the coroot lattice. Therefore, $\lambda_1 - \lambda_2 = (\sigma-1)(\nu + \nu')$. The lemma follows. \square

Proposition 4.2.10. *The following are equivalent.*

- (1) *The basic element in $B(G, \mu)$ is unramified;*
- (2) $V_\mu^{\text{Tate}} \neq 0$;
- (3) $\text{def}_G(b) = 0$ *for the basic element $b \in B(G, \mu)$.*
- (3)' $\dim X_\mu(b) = \langle \rho, \mu \rangle$ *for any b representing the basic element of $B(G, \mu)$.*
- (4) μ_{ad} , *regarded as a character of \hat{T}_{sc} , is trivial on $Z(\hat{G}_{\text{sc}})^{\Gamma_F}$.*

If these conditions hold, and in addition Z_G is connected, then the basic element in $B(G, \mu)$ can be represented as ϖ^τ for some $\tau \in \mathbb{X}_\bullet(Z_G)$.

Note that the last statement may fail if Z_G is not connected. See Remark 4.2.5.

Proof. The equivalence of (1) and (2) follows from Lemma 4.2.6 and Lemma 4.2.9. The equivalence of (3) and (3)' follows from the dimension formula of ADLVs in the affine Grassmannian [Ham15, Zhu17] (and the fact the Newton point ν_b of a basic element is central).

Now if $[\varpi^\tau]$ is basic, then $J_{\varpi^\tau} = G$ and therefore, $\text{def}_G(b) = 0$. Conversely, $\text{def}_G(b) = 0$ implies that b_{ad} is σ -conjugate to 1 in $G_{\text{ad}}(L)$. Since $G_{\text{ad}}(L)$ is generated by the image of $G(L) \rightarrow G_{\text{ad}}(L)$ and $T_{\text{ad}}(L)$, b is σ -conjugate to an unramified element. If in addition Z_G is connected, then $G(L) \rightarrow G_{\text{ad}}(L)$ is surjective and therefore b is σ -conjugate to ϖ^τ in $G(L)$ for some $\tau \in \mathbb{X}_\bullet(Z_G)$.

Finally, the identity element represents the basic element in $B(G_{\text{ad}}, \mu_{\text{ad}})$ if and only if $[\mu_{\text{ad}}] = 0$ in $\pi_1(G)_{\Gamma_F} = (Z(\hat{G}_{\text{sc}})^{\Gamma_F})^D$. Therefore, (3) and (4) are equivalent. \square

Remark 4.2.11. Note that if G is a split adjoint group, a dominant coweight satisfying conditions in Proposition 4.2.10 is necessarily in the coroot lattice, and therefore cannot be minuscule. On the other hand, if G is non-split, there are minuscule coweights satisfying these conditions. Here is a list of all such minuscules μ for simple adjoint groups. The group G is necessarily quasi-split but non-split form of type A, D, E₆. In the following two tables, ω_i is the fundamental coweight such that $\langle \omega_i, \alpha_j \rangle = \delta_{ij}$ for simple roots $\{\alpha_j\}$, and our numeration of simple roots follows from Bourbaki [Bou81, pp 250–275].

Type of G	Minuscule coweight μ	$\dim V_\mu$	$\dim V_\mu^{\text{Tate}}$
${}^2\mathbf{A}_n, n > 1 \text{ odd}$	$\omega_i, i = 2, 4, \dots, n-1$	$\binom{n+1}{i}$	$\binom{(n+1)/2}{i/2}$
${}^2\mathbf{A}_n, n \text{ even}$	$\omega_i, i = 1, \dots, n$	$\binom{n+1}{i}$	$\binom{n/2}{\lfloor i/2 \rfloor}$
${}^2\mathbf{D}_n$	ω_1	$2n$	2
${}^3\mathbf{D}_4$	$\omega_1, \omega_3, \omega_4$	8	2
${}^2\mathbf{E}_6$	ω_1, ω_6	27	3

From the table, we see that the dimension of V_μ^{Tate} can be large in the unitary group case, even for minuscule cocharacters. Many more examples that satisfy the equivalent conditions of Proposition 4.2.10 come from the cases when G is a restriction of scalar along an unramified extension. We give some examples.

Let $G = \text{Res}_{F'/F} G'$ where F'/F is a degree d unramified extension and G' an absolute simple adjoint group over F' . We identify

$$\mathbb{X}_\bullet(T) = \prod_{\varphi \in \text{Hom}_F(F', \overline{F})} \mathbb{X}_\bullet(T')$$

in the usual way. Under this identification, we write $\mu \in \mathbb{X}_\bullet(T)$ as $\{\mu_\varphi\} \in \prod_{\varphi: F' \rightarrow \overline{F}} \mathbb{X}_\bullet(T')$. On the dual group side, $\hat{G} = \prod_{\varphi: F' \rightarrow \overline{F}} \hat{G}'$ and the highest weight representation V_μ of \hat{G} is identified with the representation $\boxtimes_\varphi V_{\mu_\varphi}$ of $\prod_\varphi \hat{G}'$. For simplest, we assume that G' is split. Then it is easy to see

$$V_\mu^{\text{Tate}} = \left(\bigotimes_\varphi V_{\mu_\varphi} \right)(0).$$

In particular,

Type of G'	$[F' : F]$	Minuscule coweight $\mu = \{\mu_\varphi\}$	$\dim V_\mu$	$\dim V_\mu^{\text{Tate}}$
A_1	$2g$	$\mu_\varphi = \omega$	2^g	$\binom{2g}{g}$
B_n	2	$\mu_\varphi = \omega_1$	$4n^2$	$2n$
C_n	2	$\mu_\varphi = \omega_n$	2^{2n}	2^n
D_n	2	$\mu_\varphi = \omega_1$	$4n^2$	$2n$

Note that all pairs (G, μ) in the above two tables, except 2E_6 , can appear as the local component of a Shimura datum of abelian type.

We also mention a corollary, which shows the two definitions of the unramified elements, one given in §4.2.1 and one given here, are the same.

Corollary 4.2.12. *The following are equivalent.*

- (1) $b \in B(G)$ can be represented as ϖ^λ for some $\lambda \in \mathbb{X}_\bullet(T)$.
- (2) $\text{def}_G(b) = 0$, i.e. the F -rank of G and J_b coincide.

Proof. Clearly (1) implies (2). Conversely, assume $\text{def}_G(b) = 0$. Since b is σ -conjugate to some basic element in a Levi subgroup of G , we can apply Proposition 4.2.10 to this Levi subgroup to conclude (choose any μ such that $b \in B(G, \mu)$). \square

4.2.13. *Twisted centralizer for unramified elements.* We recall some facts of the twisted centralizer for the unramified elements. We fix \mathcal{O} -embeddings $T \subset B \subset G$, so elements in $B(G)_{\text{unr}}$ can be represented by ϖ^τ for $\tau \in \mathbb{X}_\bullet(T)$. We fix such a representative. For simplicity, we write $J_\tau = J_{\varpi^\tau}$.

It is isomorphic to a Levi subgroup of G , and naturally contains T as a maximal torus. With respect to this torus,

$$(4.2.2) \quad \Delta_\tau = \left\{ \alpha \in \Delta \mid \left\langle \sum_{i=0}^{m-1} \sigma^i(\alpha), \tau \right\rangle = 0 \right\} \subset \Delta$$

form a set of simple roots of J_τ . Let $\Phi_\tau^+ \subset \Phi^+$ denote the corresponding set of positive roots. The relative Weyl group of J_τ (for the torus T) then is

$$(W_\tau)_0 := \{w \in W_0 \mid w(\nu_{\varpi^\tau}) = \nu_{\varpi^\tau}\}.$$

Let B_τ and U_τ denote the corresponding Borel and unipotent radical with respect to this set of simple roots. Then

$$(4.2.3) \quad U_\tau(F) = J_{\varpi^\tau}(F) \cap U(L) = \{u \in U(L) \mid \varpi^\tau \sigma(u) \varpi^{-\tau} = u\}, \quad B_\tau(F) = J_\tau(F) \cap B(L) = T(F)U_\tau(F).$$

We have the Bruhat decomposition

$$(4.2.4) \quad J_\tau(F) = \bigcup_{w \in (W_\tau)_0} B_\tau(F) \dot{w} B_\tau(F),$$

where \dot{w} is a lifting of w to $G(L)$ satisfying $\sigma(\dot{w})\dot{w}^{-1} = \varpi^{w(\tau)-\tau}$.

Now, if we assume further that $\tau \in \mathbb{X}_\bullet(T)^+$,²¹ then (4.2.2) simplifies as

$$\Delta_\tau = \{\alpha \in \Delta \mid \langle \sigma^i \alpha, \tau \rangle = 0 \text{ for all } i\}.$$

In this case, J_τ has a natural hyperspecial integral structure, i.e., the \mathcal{O} -subgroup scheme of G generated by $\{T, U_{\pm\alpha}, \alpha \in \Phi_\tau^+\}$. Then

$$J_\tau(F) = B_\tau(F)J_\tau(\mathcal{O}) = \bigcup_{\chi \in \mathbb{X}_\bullet(T)^\sigma} U_\tau(F) \varpi^\chi J_\tau(\mathcal{O})$$

is an Iwasawa decomposition of J_τ .

4.3. ADLVs via semi-infinite orbits. In this subsection, we fix dominant coweights μ and an unramified element $[\varpi^\tau]$ in $B(G, \mu)$. Then $X_\mu(\varpi^\tau) \neq \emptyset$ by Theorem 4.1.3. By the discussion of the previous subsection, we may assume that $\tau_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+$. We revisit the arguments in [GHKR06, §2]. A more detailed analysis (together with a generalization to unramified groups) will allow us to give a description of the irreducible components of $X_\mu(\varpi^\tau)$ up to the action of $B_\tau(F)$. To simplify the notation, we write $X_\mu(\tau)$ instead of $X_\mu(\varpi^\tau)$.

4.3.1. Let σ denote the q -Frobenius. Let $S_\mu \subset \text{Gr} \otimes k'$ be a semi-infinite orbit, where k' is the field of the definition for μ . Then we have the following diagram

$$(4.3.1) \quad \begin{array}{ccccc} S_\mu & \longrightarrow & S_{\sigma(\mu)} & \longrightarrow & S_\mu \\ \downarrow & & \downarrow & & \downarrow \\ \text{Gr} \otimes k' & \xrightarrow{\sigma \otimes 1} & \text{Gr} \otimes k' & \xrightarrow{1 \otimes \sigma} & \text{Gr} \otimes k'. \end{array}$$

We denote the map $S_\mu \rightarrow S_{\sigma(\mu)}$ by σ for simplicity (since it is induced by the q -Frobenius endomorphism of Gr).

We define

$$Y_\nu := S_\nu \cap X_\mu(\varpi^\tau),$$

²¹As explained in Remark 4.2.5, the map $\mathbb{X}_\bullet(T)^+ \rightarrow \mathbb{X}_\bullet(T)_\sigma^+$ need not be surjective. So this is a genuine assumption.

which is a locally closed sub ind-scheme of $X_\mu(\tau)$. It fits into the following Cartesian diagram

$$(4.3.2) \quad \begin{array}{ccc} Y_\nu & \longrightarrow & (S_\nu \tilde{\times} S_\lambda) \cap (\mathrm{Gr} \tilde{\times} \mathrm{Gr}_\mu) \\ \downarrow & & \downarrow \mathrm{pr}_1 \times m \\ S_\nu & \xrightarrow{1 \times \varpi^\tau \sigma} & S_\nu \times S_{\tau+\sigma(\nu)}, \end{array}$$

where $\lambda = \tau + \sigma(\nu) - \nu$.

Since $(S_\nu \tilde{\times} S_\lambda) \cap (\mathrm{Gr} \tilde{\times} \mathrm{Gr}_\mu) = S_\nu \tilde{\times} (S_\lambda \cap \mathrm{Gr}_\mu)$ by (3.2.8), every $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$ gives a closed subset $Y_\nu^{\mathbf{b}} = Y_\nu^{\mathbf{b}}(\tau)$ of Y_ν that fits into the Cartesian diagram

$$(4.3.3) \quad \begin{array}{ccc} Y_\nu^{\mathbf{b}} = Y_\nu^{\mathbf{b}}(\tau) & \longrightarrow & S_\nu \tilde{\times} (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow \mathrm{pr}_1 \times m \\ S_\nu & \xrightarrow{1 \times \varpi^\tau \sigma} & S_\nu \times S_{\tau+\sigma(\nu)}. \end{array}$$

We quickly mention a couple of simple properties of $Y_\nu^{\mathbf{b}}$.

Lemma 4.3.2. *We fix $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$ with $\lambda = \tau + \sigma(\nu) - \nu$.*

- (1) *Let $\chi \in \mathbb{X}_\bullet(T)^\sigma$ be a σ -invariant coweight of T . Then multiplication (on the left) by $\varpi^\chi \in J_\tau(F)$ induces an isomorphism $Y_\nu^{\mathbf{b}}(\tau) \cong Y_{\nu+\chi}^{\mathbf{b}}(\tau)$.*
- (2) *Let $\tau' = \tau + \delta - \sigma(\delta)$. Then the isomorphism $X_\mu(\tau) \rightarrow X_\mu(\tau')$, $gL^+G \mapsto \varpi^\delta gL^+G$ induces an isomorphism $Y_\nu^{\mathbf{b}}(\tau) \cong Y_{\nu+\delta}^{\mathbf{b}}(\tau')$.*
- (3) *Let $G \rightarrow G'$ be a central isogeny, and (μ', τ', ν') be the image of (μ, τ, ν) under the map $\mathbb{X}_\bullet(T) \rightarrow \mathbb{X}_\bullet(T')$. Then the map $\mathrm{Gr}_G \rightarrow \mathrm{Gr}_{G'}$ induces a natural bijection $\mathbb{M}\mathbb{V}_\mu(\lambda) \cong \mathbb{M}\mathbb{V}_{\mu'}(\lambda')$ and let \mathbf{b}' denote the image of \mathbf{b} . Then the map $\mathrm{Gr}_G \rightarrow \mathrm{Gr}_{G'}$ induces a natural isomorphism $Y_{\nu'}^{\mathbf{b}'}(\tau') \cong Y_\nu^{\mathbf{b}}(\tau)$.*

Proof. Each of the statement is clear from the Cartesian diagram (4.3.3) as the natural morphism induces isomorphisms on all terms in that diagram. \square

Now we simply write $Y_\nu^{\mathbf{b}}$ for $Y_\nu^{\mathbf{b}}(\tau)$. The following proposition gives a key property of $Y_\nu^{\mathbf{b}}$.

Proposition 4.3.3. *We fix $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$.*

- (1) *The subset $Y_\nu^{\mathbf{b}}$ is non-empty if and only if $\lambda = \tau + \sigma(\nu) - \nu$.*
- (2) *The group*

$$U_\tau(F) := J_\tau(F) \cap U(L) = \{u \in U(L) \mid \varpi^\tau \sigma(u) \varpi^{-\tau} = u\}$$

acts transitively on the set of irreducible components of $Y_\nu^{\mathbf{b}}$. Moreover, all irreducible components of $Y_\nu^{\mathbf{b}}$ are of dimension $\langle \rho, \mu - \tau \rangle$.

Proof. The arguments below essentially repeat [GHKR06, §2]. Using Lemma 4.3.2(2)(3), we may pass to the adjoint group of G and therefore (after a σ -conjugation) assume that τ is dominant.

We consider the L^+U -torsors π_λ defined in (3.2.1) and its variants $\nu^\nu L^+U \varpi^{-\nu}$ -torsor π'_ν :²²

$$\begin{array}{ccc} \pi_\lambda : LU & \longrightarrow & S_\lambda \\ u & \longmapsto & \varpi^\lambda u, \end{array} \quad \begin{array}{ccc} \pi'_\nu : LU & \longrightarrow & S_\nu \\ u & \longmapsto & u \varpi^\nu. \end{array}$$

²²One could alternatively present the proof using the L^+U -torsors π_ν and π_λ , but the analogous Lemma 4.3.4 will require some additional effort.

By definition (4.3.3), an element $u \in LU$ belongs to $\tilde{Y}_\nu^{\mathbf{b}} := \pi_\nu'^{-1}(Y_\nu^{\mathbf{b}})$ if and only if there exists $x \in (S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}}$ such that

$$\pi_\nu'(u)x = \varpi^\tau \sigma(\pi_\nu'(u)),$$

or equivalently,

$$x = \pi_\lambda(\varpi^{-\lambda} \varpi^{-\nu} u^{-1} \varpi^\tau \sigma(u) \varpi^{\sigma(\nu)}) = \pi_\lambda(\varpi^{-\tau-\sigma(\nu)} u^{-1} \varpi^\tau \sigma(u) \varpi^{-\tau} \varpi^{\tau+\sigma(\nu)}).$$

In other words, if we define the morphisms

$$(4.3.4) \quad f_\tau : LU \rightarrow LU, \quad u \mapsto u^{-1} \varpi^\tau \sigma(u) \varpi^{-\tau}, \text{ and}$$

$$c_{-\tau-\sigma(\nu)} : LU \rightarrow LU, \quad u \mapsto \varpi^{-\tau-\sigma(\nu)} u \varpi^{\tau+\sigma(\nu)},$$

then

$$(4.3.5) \quad \tilde{Y}_\nu^{\mathbf{b}} := \pi_\nu'^{-1}(Y_\nu^{\mathbf{b}}) = f_\tau^{-1}(c_{-\tau-\sigma(\nu)}^{-1}(\pi_\lambda^{-1}((S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}}))),$$

or equivalently, we have the following Cartesian diagram

$$(4.3.6) \quad \begin{array}{ccccc} Y_\nu^{\mathbf{b}} & \longleftarrow & \tilde{Y}_\nu^{\mathbf{b}} & \longrightarrow & (S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow & & \downarrow \\ S_\nu & \xleftarrow{\pi'_\nu} & LU & \xrightarrow{f_\tau} & LU & \xrightarrow{\pi_\lambda \circ c_{-\tau-\sigma(\nu)}} & S_\lambda. \end{array}$$

The key of the proof lies in the fact that the map f_τ behaves like the Lang map for an algebraic group defined over a finite field. More precisely, we have the following.

Lemma 4.3.4. *Let τ be a dominant coweight.*

- (1) *The morphism f_τ in (4.3.4) restricts to a “pro-étale” surjective morphism $f_\tau : L^+U \rightarrow L^+U$ with Galois group $U_\tau(\mathcal{O})$.*
- (2) *The morphism $f_\tau : LU \rightarrow LU$ is an inductive limit of “pro-étale” surjective maps, and $U_\tau(F)$ acts simply-transitively on the fibers of f_τ by left multiplication.*

Proof. (1) Note that $\varpi^\tau L^+U^{(r)} \varpi^{-\tau} \subseteq L^+U^{(r)}$ for every principle congruence subgroup $L^+U^{(r)}$ of L^+U (see (3.1.3) for the definition). It follows that f_τ induces a morphism $f_\tau : L^+U \rightarrow L^+U$ and a morphism

$$f_{\tau,r} : L^r U \rightarrow L^r U, \quad u \mapsto u^{-1} \varpi^\tau \sigma(u) \varpi^{-\tau}$$

for each r . But $f_{\tau,r}$ is naturally defined before passing to the perfection and its étaleness follows from the fact that the induced tangent map is multiplication by -1 . So $f_{\tau,r}$ is an étale covering with the Galois group $U_\tau(\mathcal{O}/\varpi^r)$. Passing to the inverse limit shows that $f_\tau : L^+U \rightarrow L^+U$ is a “pro-étale” cover with Galois group $U_\tau(\mathcal{O})$.

(2) For a dominant coweight $\chi \in \mathbb{X}_\bullet(T)^{+,\sigma}$, Let $\text{conj}_\chi : LU \rightarrow LU$ be the conjugation map $u \mapsto \varpi^\chi u \varpi^{-\chi}$. It is clear that $f_\tau \circ \text{conj}_\chi = \text{conj}_\chi \circ f_\tau$ forms a Cartesian square. Therefore (1) implies that $f_\tau : \varpi^{-\chi} L^+U \varpi^\chi \rightarrow \varpi^{-\chi} L^+U \varpi^\chi$ is a “pro-étale” map with Galois group $\varpi^{-\chi} U_\tau(\mathcal{O}) \varpi^\chi$. Taking the limit over $\chi \in \mathbb{X}_\bullet(T)^{+,\sigma}$ proves (2). \square

Now we see that $\tilde{Y}_\nu^{\mathbf{b}}$ is the pullback of $(S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}}$ along a sequence of surjective maps. Part (1) of the proposition is clear. In addition, since $(S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}}$ is irreducible, $U_\tau(F)$ acts transitively on the set of irreducible components of $\tilde{Y}_\nu^{\mathbf{b}}$, and hence acts transitively on the set of irreducible components of $Y_\nu^{\mathbf{b}}$.

It remains to show that each irreducible component $Y_\nu^{\mathbf{b}}$ is of dimension $\langle \rho, \mu - \tau \rangle$. For this, we need a “truncated version” of (4.3.6). We pick $r \in \mathbb{Z}$ sufficiently large so that $\varpi^{-\nu} L^+ U^{(r)} \varpi^\nu \subseteq L^+ U$ and $c_{-\tau-\sigma(\nu)}(L^+ U^{(r)}) \subseteq L^+ U$. Then (4.3.6) induces the following Cartesian diagram

$$(4.3.7) \quad \begin{array}{ccccc} Y_\nu^{\mathbf{b}} & \longleftarrow & \tilde{Y}_\nu^{\mathbf{b},(r)} & \longrightarrow & (S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow & & \downarrow \\ S_\nu & \xleftarrow{\pi'_\nu} & LU/L^+ U^{(r)} & \xrightarrow{f_\tau} & LU/L^+ U^{(r)} \xrightarrow{\pi_\lambda \circ c_{-\tau-\sigma(\nu)}} S_\lambda. \end{array}$$

Note that π'_ν is smooth of dimension $r \dim U$, f_τ is étale, and $\pi_\lambda \circ c_{-\tau-\sigma(\nu)}$ is smooth of dimension $\dim L^+ U / c_{-\tau-\sigma(\nu)}(L^+ U^{(r)})$. So we have

$$\begin{aligned} \dim Y_\nu^{\mathbf{b}} &= \dim(S_\lambda \cap \text{Gr}_\mu)^{\mathbf{b}} + \dim \pi_\lambda - \dim \pi'_\nu \\ &= \langle \rho, \mu + \lambda \rangle + (r \dim U - \langle \rho, \tau + \sigma(\nu) \rangle) - r \dim U = \langle \rho, \mu - \tau \rangle. \end{aligned} \quad \square$$

Corollary 4.3.5. *For $\mathbf{b} \in \cup_{\lambda \in \tau + (\sigma-1)\mathbb{X}_\bullet} \mathbb{M}\mathbb{V}_\mu(\lambda)$, let $X_\mu^{\mathbf{b}}(\tau)$ denote the closure of $\cup_{\nu \in \mathbb{X}_\bullet} Y_\nu^{\mathbf{b}}$ in $X_\mu(\tau)$. Then*

- (1) $X_\mu^{\mathbf{b}}(\tau)$ is a union of certain irreducible components of $X_\mu(\tau)$.
- (2) $X_\mu^{\mathbf{b}}(\tau) \neq X_\mu^{\mathbf{b}'}(\tau)$ if $\mathbf{b} \neq \mathbf{b}'$, and $X_\mu(\tau) = \cup_{\mathbf{b}} X_\mu^{\mathbf{b}}(\tau)$.
- (3) The group $B_\tau(F)$ acts transitively on the set of irreducible components of $X_\mu^{\mathbf{b}}(\tau)$.
- (4) $X_\mu^{\mathbf{b}}(\tau)$ is equi-dimensional of dimension $\langle \rho, \mu - \tau \rangle$.

Proof. Since semi-infinite orbits form a partition of the affine Grassmannian into locally closed subsets, $\{Y_\nu^{\mathbf{b}}\}$ form a partition of $X_\mu(\tau)$. Proposition 4.3.3(2) says that every $Y_\nu^{\mathbf{b}}$ is pure of dimension $\langle \rho, \mu - \tau \rangle$; so is $X_\mu^{\mathbf{b}}(\tau)$. It follows that if Z is an irreducible component of $Y_\nu^{\mathbf{b}}$, then its closure \bar{Z} is an irreducible component of $X_\mu(\tau)$. Therefore, $X_\mu^{\mathbf{b}}(\tau)$ is a union of certain irreducible components of $X_\mu(\tau)$, and $X_\mu(\tau) = \cup_{\mathbf{b}} X_\mu^{\mathbf{b}}(\tau)$.

We claim that, if $\mathbf{b} \neq \mathbf{b}'$, then $Y_\nu^{\mathbf{b}} \neq Y_\nu^{\mathbf{b}'}$. Indeed, if $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$ and $\mathbf{b}' \in \mathbb{M}\mathbb{V}_\mu(\lambda')$ for $\lambda \neq \lambda'$, then $Y_\nu^{\mathbf{b}}$ and $Y_\nu^{\mathbf{b}'}$ cannot be both nonempty by Lemma 4.3.2(1). If $\mathbf{b}, \mathbf{b}' \in \mathbb{M}\mathbb{V}_\mu(\lambda)$, then $\pi_\nu'^{-1}(Y_\nu^{\mathbf{b}}) \neq \pi_\nu'^{-1}(Y_\nu^{\mathbf{b}'})$ by (4.3.5). Therefore, $X_\mu^{\mathbf{b}}(\tau) \neq X_\mu^{\mathbf{b}'}(\tau)$ if $\mathbf{b} \neq \mathbf{b}'$.

Finally, by Proposition 4.3.3(2), $B_\tau(F)$ acts transitively on the set of irreducible components of $\cup_\nu Y_\nu^{\mathbf{b}}$, and therefore acts transitively on the set of irreducible components of $X_\mu^{\mathbf{b}}(\tau)$. \square

We end the subsection with the following observations.

Lemma 4.3.6. *We fix $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$ with $\lambda \in \tau + (\sigma-1)\mathbb{X}_\bullet$ and let $X_\mu^{\mathbf{b}}(\tau)$ be defined as above.*

- (1) *Let $G \rightarrow G'$ be a central isogeny, and (μ', τ') be the image of (μ, τ) under the map $\mathbb{X}_\bullet(T) \rightarrow \mathbb{X}_\bullet(T')$. Then one can replace $X_\mu(\mathbf{b})$ and $X_{\mu'}(\mathbf{b}')$ by $X_\mu^{\mathbf{b}}(\tau)$ and $X_{\mu'}^{\mathbf{b}'}(\tau')$ in the diagram in Lemma 4.1.5 and the new diagram is still Cartesian.*
- (2) *Let $\tau' = \tau + \delta - \sigma(\delta)$. Then the isomorphism $X_\mu(\tau) \rightarrow X_\mu(\tau')$, $gL^+G \mapsto \varpi^\delta gL^+G$ induces an isomorphism $X_\mu^{\mathbf{b}}(\tau) \cong X_\mu^{\mathbf{b}}(\tau')$.*

Proof. The map $\text{Gr}_G \rightarrow \text{Gr}_{G'}$ induces a natural bijection $\mathbb{M}\mathbb{V}_\mu(\lambda) = \mathbb{M}\mathbb{V}_{\mu'}(\lambda')$ for any λ , and the isomorphism $Y_\nu^{\mathbf{b}}(\tau) \cong Y_\nu^{\mathbf{b}'}(\tau')$ by Lemma 4.3.2(3). Then (1) follows.

(2) follows from Lemma 4.3.2(2) immediately. \square

4.4. Irreducible components of some ADIVs. In this subsection, we study in more details the irreducible components of $X_\mu(\tau)$. We continue to assume that $\tau_\sigma \in \mathbb{X}_\bullet(T)_\sigma^+$. We fix $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$, where $\lambda \in \tau + (\sigma-1)\mathbb{X}_\bullet$, and let $X_\mu^{\mathbf{b}}(\tau)$ be the union of certain irreducible components of $X_\mu(\tau)$ as constructed in the previous subsection.

Hypothesis 4.4.1. In this section, we assume $k \neq \mathbb{F}_2$ if any of the simple factors of J_τ contains a factor of type B_n, C_n, F_4 , and $k \neq \mathbb{F}_2, \mathbb{F}_3$ if it contains G_2 . Note that this does not depend on the choice of τ in its σ -conjugacy class because all J_τ 's are isomorphic.

Remark 4.4.2. This Hypothesis will only be used to invoke Lemma 4.4.9. On the other hand, we believe that the restriction of the characteristic is not necessary to the validity of Proposition 4.4.7, but currently we are unable to remove this extra assumption.

We need the following lemma. Recall from §3.3 that the set \mathbb{MV}_μ has a \hat{G} -crystal structure and therefore every $\mathbf{b} \in \mathbb{MV}_\mu(\lambda)$ can be attached a collection of non-negative integers $\{\varepsilon_\alpha(\mathbf{b}), \alpha \in \Delta\}$.

Lemma 4.4.3. *Assume that Z_G is connected.*

- (1) *Let $\lambda \in \mathbb{X}_\bullet(T)$ and assume that $\lambda_\sigma = \tau_\sigma$. Then there exists a dominant coweight ν such that $\lambda + \nu - \sigma(\nu)$ is dominant, and that \mathbf{b} lies in the image of the map $i_\nu^{\mathbb{MV}} : \mathbb{S}_{(\nu, \mu)|\lambda + \nu} \rightarrow \mathbb{MV}_\mu(\lambda)$ defined in Lemma 3.2.7.*
- (2) *Among all ν 's satisfying the above property, there is a “minimal” $\nu_{\mathbf{b}}$, unique up to adding by an element in $\mathbb{X}_\bullet(Z_G)$. Here “minimality” means that for any other ν satisfying the above property, $\nu - \nu_{\mathbf{b}}$ is dominant. In addition, for every $\alpha \in \Delta$, at least one of the following inequalities is an equality*

$$\langle \alpha, \nu_{\mathbf{b}} \rangle \geq \varepsilon_\alpha(\mathbf{b}), \quad \langle \alpha_{\sigma(\alpha)}, \nu_{\mathbf{b}} \rangle \geq \varepsilon_{\sigma(\alpha)}(\mathbf{b}), \quad \dots, \quad \langle \alpha_{\sigma^{d-1}(\alpha)}, \nu_{\mathbf{b}} \rangle \geq \varepsilon_{\sigma^{d-1}(\alpha)}(\mathbf{b}),$$

where d is the cardinality of the σ -orbit of α .

Proof. By (3.3.3) and the compatibility Proposition 3.3.17, \mathbf{b} belongs to the image of the map $i_\nu^{\mathbb{MV}}$ if and only if

$$(4.4.1) \quad \langle \nu, \alpha \rangle \geq \varepsilon_\alpha(\mathbf{b}) \quad \text{for all } \alpha \in \Delta.$$

On the other hand, $\lambda + \nu - \sigma(\nu)$ is dominant if and only if, for all simple roots α of G ,

$$\begin{aligned} 0 &\leq \langle \lambda, \alpha \rangle + \langle \nu, \alpha \rangle - \langle \sigma(\nu), \alpha \rangle \\ &= \phi_\alpha(\mathbf{b}) - \varepsilon_\alpha(\mathbf{b}) + \langle \nu, \alpha \rangle - \langle \nu, \sigma^{-1}(\alpha) \rangle. \end{aligned}$$

This is equivalent to, for every σ -orbit $\{\alpha, \sigma(\alpha) \dots, \sigma^d(\alpha) = \alpha\}$ of simple roots of G

$$(4.4.2) \quad \langle \nu, \sigma^j(\alpha) \rangle \geq \langle \nu, \sigma^{j-1}(\alpha) \rangle + \varepsilon_{\sigma^j(\alpha)}(\mathbf{b}) - \phi_{\sigma^j(\alpha)}(\mathbf{b}), \quad j = 1, 2, \dots, d.$$

Since Z_G is connected, there exist coweights $\{\omega_\alpha\}_{\alpha \in \Delta}$ such that $\langle \omega_\alpha, \beta \rangle = \delta_{\alpha\beta}$. Since

$$\sum_j (\varepsilon_{\sigma^j(\alpha)}(\mathbf{b}) - \phi_{\sigma^j(\alpha)}(\mathbf{b})) = \langle -\lambda, \sum_j \alpha_j \rangle = \langle -\tau, \sum_j \alpha_j \rangle \leq 0,$$

one can use $\{\omega_\alpha\}$ to modify ν such that $\{\langle \nu, \sigma^j(\alpha) \rangle\}$ satisfy (4.4.1) and (4.4.2).

Now, let ν_0 be a coweight satisfying (4.4.1) and (4.4.2). Then we can choose $\nu_{\mathbf{b}} \in \nu_0 + \sum_{\alpha \in \Delta} \mathbb{Z}\omega_\alpha$ such that

$$\langle \alpha, \nu_{\mathbf{b}} \rangle = \max_{j=0, \dots, d-1} \{ \varepsilon_{\sigma^j(\alpha)}(\mathbf{b}) + \langle \alpha, \nu_0 \rangle - \langle \alpha_{\sigma^j(\alpha)}, \nu_0 \rangle \}.$$

Such $\nu_{\mathbf{b}}$ is minimal (in the sense of the lemma) among all coweights satisfying (4.4.1) and (4.4.2), and clearly two choices of $\nu_{\mathbf{b}}$ differ by an element in $\mathbb{X}_\bullet(Z_G)$. \square

Remark 4.4.4. (1) Let $\nu = \nu_{\mathbf{b}}$ be a (choice of) minimal coweight as above, (4.4.1) may not be an equality for all $\alpha \in \Delta$.

- (2) For a (choice of) minimal weight $\nu = \nu_{\mathbf{b}}$ above, let $\tau_{\mathbf{b}} = \lambda + \nu_{\mathbf{b}} - \sigma(\nu_{\mathbf{b}})$. Note that $\tau_{\mathbf{b}}$ is well-defined up to adding an element in $(1 - \sigma)\mathbb{X}_\bullet(Z_G)$ (so in particular if G is adjoint, $\tau_{\mathbf{b}}$ is uniquely determined by \mathbf{b}). In any case, it belongs to $\tau + (1 - \sigma)\mathbb{X}_\bullet(T)$, and therefore $\varpi^{\tau_{\mathbf{b}}} \in A(G, \mu)$ is a canonical representative of the unramified element $[b] = [\varpi^\tau] \in B(G, \mu)$

“adapted” to \mathbf{b} . For example, if $[b]$ is basic, then $\tau_{\mathbf{b}}$ is central and $\varpi^{\tau_{\mathbf{b}}}$ usually called superspecial.

- (3) But also note that the class of $\tau_{\mathbf{b}}$ in $\mathbb{X}_{\bullet}(T)/(1-\sigma)\mathbb{X}_{\bullet}(Z_G)$ depends on \mathbf{b} in general. Therefore, if the set $\sqcup_{\lambda_{\sigma}=\tau_{\sigma}} \mathbb{M}\mathbb{V}_{\mu}(\lambda)$ contains more than one element, there may not be a canonical representative of $[b]$. See [XZ] for an example.

Our goal is to study the irreducible components of $X_{\mu}^{\mathbf{b}}(\tau)$. By Lemma 4.3.6(2), $X_{\mu}^{\mathbf{b}}(\tau) \cong X_{\mu}^{\mathbf{b}}(\tau_{\mathbf{b}})$. Therefore, after σ -conjugacy, we may replace τ by $\tau_{\mathbf{b}}$. In particular, τ is assumed to be dominant. Then by §4.2.13, J_{τ} (and U_{τ}) has a natural \mathcal{O} -structure in this case. We denote by $J_{\tau, x_0}(\mathcal{O}) \subset J_{\tau}(F)$ the corresponding hyperspecial subgroup.

Let $\mathbf{a} \in \mathbb{S}_{(\nu, \mu)|\tau+\sigma(\nu)}$ such that $\mathbf{b} = i_{\nu}^{\mathbb{M}\mathbb{V}}(\mathbf{a})$. By Lemma 3.2.7, such \mathbf{a} is unique and inside $\mathrm{Gr}_{\nu} \times \mathrm{Gr}_{\tau+\sigma(\nu)}$,

$$\mathrm{Gr}_{(\nu, \mu)|\tau+\sigma(\nu)}^{0, \mathbf{a}} \cap ((S_{\nu} \cap \mathrm{Gr}_{\nu}) \times (S_{\nu+\lambda} \cap \mathrm{Gr}_{\tau+\sigma(\nu)})) = (S_{\nu} \cap \mathrm{Gr}_{\nu}) \tilde{\times} (S_{\lambda} \cap \mathrm{Gr}_{\mu})^{\mathbf{b}}.$$

Let us define a closed subset $X_{\mu, \nu}^{\mathbf{a}}(\tau) \subset X_{\mu, \nu}(\tau)$ via the Cartesian square

$$(4.4.3) \quad \begin{array}{ccc} X_{\mu, \nu}^{\mathbf{a}}(\tau) & \longrightarrow & \mathrm{Gr}_{(\nu, \mu)|\tau+\sigma(\nu)}^{0, \mathbf{a}} \\ \downarrow & & \downarrow \\ \mathrm{Gr}_{\nu} & \xrightarrow{1 \times \varpi^{\tau} \sigma} & \mathrm{Gr}_{\nu} \times \mathrm{Gr}_{\tau+\sigma(\nu)}. \end{array}$$

Compare with the definition of $X_{\mu, \nu}(\tau)$ in (4.1.7), we replace $\mathrm{Gr}_{(\nu, \mu)|\tau+\sigma(\nu)}^0$ by one of its irreducible component $\mathrm{Gr}_{(\nu, \mu)|\tau+\sigma(\nu)}^{0, \mathbf{a}}$.

Theorem 4.4.5. *Assume that Z_G is connected. Let $\mathbf{b} \in \mathbb{M}\mathbb{V}_{\mu}(\lambda)$. Let $\nu = \nu_{\mathbf{b}}$ and $\tau = \tau_{\mathbf{b}}$ as in Lemma 4.4.3. Then scheme $X_{\mu, \nu}^{\mathbf{a}}(\tau)$ has a unique irreducible component $X_{\mu}^{\mathbf{b}, x_0}(\tau)$ ²³ of dimension $\langle \rho, \mu - \tau \rangle$ (all other possible irreducible components of would have dimension strictly smaller than $\langle \rho, \mu - \tau \rangle$). In addition, $X_{\mu, \nu}^{\mathbf{a}}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu}$ is irreducible. In particular,*

$$\mathring{X}_{\mu}^{\mathbf{b}, x_0}(\tau) := X_{\mu}^{\mathbf{b}, x_0}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu} = X_{\mu, \nu}^{\mathbf{a}}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu}.$$

All $X_{\mu, \nu}^{\mathbf{a}}(\tau)$, $X_{\mu}^{\mathbf{b}, x_0}(\tau)$, $\mathring{X}_{\mu}^{\mathbf{b}, x_0}(\tau)$ admit an action by $J_{\tau, x_0}(\mathcal{O})$.

Remark 4.4.6. (1) If $\tau \in \mathbb{X}_{\bullet}(Z_G)$ and ν is minuscule, then $\mathrm{Gr}_{\nu} = \mathring{\mathrm{Gr}}_{\nu} = (\bar{G}/\bar{P}_{\mu})^{\mathrm{pf}}$. It follows that in this case

$$X_{\mu, \nu}^{\mathbf{a}}(\tau) = X_{\mu}^{\mathbf{b}, x_0}(\tau)$$

is (the perfection) of a (closed) finite Deligne-Lusztig variety in \bar{G}/\bar{P}_{μ} , and with the action of $J_{\tau, x_0}(\mathcal{O})$ factoring through the usual action of $J_{\tau, x_0}(\mathcal{O}/\varpi)$.

(2) However, if ν is not minuscule, the action $J_{\tau, x_0}(\mathcal{O})$ on $\mathring{X}_{\mu}^{\mathbf{b}, x_0}(\tau)$ does not factor through $J_{\tau, x_0}(\mathcal{O}/\varpi)$. We refer to [XZ] for an example.

(3) We do not know whether $X_{\mu, \nu}^{\mathbf{a}}(\tau)$ is irreducible in general.

The following proposition is the key step towards Theorem 4.4.5.

Proposition 4.4.7. *The scheme $Y_{\nu}^{\mathbf{b}} \cap \mathrm{Gr}_{\nu} = X_{\mu, \nu}^{\mathbf{a}}(\tau) \cap S_{\nu}$ is geometrically irreducible of dimension $\langle \rho, \mu - \tau \rangle$.*

²³Here the superscript x_0 indicates that the stabilizer group of this irreducible component in $X_{\mu}(\tau)$ under the $J_{\tau}(F)$ -action is $J_{\tau, x_0}(\mathcal{O})$.

Proof. The Cartesian diagram (4.3.3) gives

$$\begin{array}{ccc} Y_\nu^{\mathbf{b}} \cap \mathrm{Gr}_\nu & \longrightarrow & (S_\nu \cap \mathrm{Gr}_\nu) \tilde{\times} (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow \mathrm{pr}_1 \times m \\ S_\nu \cap \mathrm{Gr}_\nu & \xrightarrow{1 \times \varpi^\tau \sigma} & (S_\nu \cap \mathrm{Gr}_\nu) \times (S_{\tau+\sigma(\nu)} \cap \mathrm{Gr}_{\tau+\sigma(\nu)}). \end{array}$$

Putting $\tilde{Y}_\nu^{\mathbf{b},x_0} := \pi_\nu'^{-1}(Y_\nu^{\mathbf{b}} \cap \mathrm{Gr}_\nu)$, we claim that it fits into the following Cartesian diagram analogous to (4.3.6)

$$(4.4.4) \quad \begin{array}{ccccc} Y_\nu^{\mathbf{b}} \cap \mathrm{Gr}_\nu & \longleftarrow & \tilde{Y}_\nu^{\mathbf{b},x_0} & \longrightarrow & (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow & & \downarrow \\ S_\nu \cap \mathrm{Gr}_\nu & \xleftarrow{\pi'_\nu} & L^+U & \xrightarrow{f_\tau} & L^+U \xrightarrow{\pi_\lambda \circ c_{-\tau-\sigma(\nu)}} S_\lambda. \end{array}$$

Indeed, we recall that $S_\nu \cap \mathrm{Gr}_\nu = L^+U \varpi^\nu L^+U / L^+U$. From this, we can deduce that

$$(4.4.5) \quad LU \times_{\pi'_{\nu,S_\nu}} (S_\nu \cap \mathrm{Gr}_\nu) = \pi_\nu'^{-1}(S_\nu \cap \mathrm{Gr}_\nu) = L^+U,$$

and the same holds if we replace ν by $\tau+\sigma(\nu)$. It follows immediately that the left square of (4.4.4) is Cartesian. To see the right square of (4.4.4) is Cartesian, it suffices to show that $c_{-\tau-\sigma(\nu)}^{-1} \pi_\lambda^{-1}((S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}})$ is contained in L^+U . Indeed, if an element $u \in LU$ satisfies $\pi_\lambda(c_{-\tau-\sigma(\nu)}(u)) \in (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}}$, or equivalently

$$\varpi^{-\nu} u \varpi^{\tau+\sigma(\nu)} \bmod L^+U \in (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}},$$

then we know that

$$\varpi^\nu \cdot \varpi^{-\nu} u \varpi^{\tau+\sigma(\nu)} \bmod L^+U \in (S_\nu \cap \mathrm{Gr}_\nu) \tilde{\times} (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \xrightarrow{m_{\nu,\mu}} S_{\tau+\sigma(\nu)} \cap \mathrm{Gr}_{\tau+\sigma(\nu)},$$

which forces $u \in L^+U$ by (4.4.5) (with ν replaced by $\tau+\sigma(\nu)$). This concludes the proof of the claim that (4.4.4) is Cartesian.

As in the proof of Proposition 4.3.3, we may choose a congruence subgroup $L^+U^{(r)}$ for some r large enough so that $\varpi^{-\nu} L^+U^{(r)} \varpi^\nu \subseteq L^+U$ and $c_{-\tau-\sigma(\nu)}(L^+U^{(r)}) \subseteq L^+U$. Then passing to the quotient of (4.4.4) by $L^+U^{(r)}$ gives a Cartesian diagram

$$(4.4.6) \quad \begin{array}{ccccc} Y_\nu^{\mathbf{b}} \cap \mathrm{Gr}_\nu & \longleftarrow & \tilde{Y}_\nu^{\mathbf{b},x_0,r} & \longrightarrow & (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}} \\ \downarrow & & \downarrow & & \downarrow \\ S_\nu \cap \mathrm{Gr}_\nu & \xleftarrow{\pi'_\nu} & L^rU & \xrightarrow{f_\tau} & L^rU \xrightarrow{\pi_\lambda \circ c_{-\tau-\sigma(\nu)}} S_\lambda. \end{array}$$

Since this π'_ν is (perfectly) smooth, it is enough to show that $\tilde{Y}_\nu^{\mathbf{b},x_0,r}$ is irreducible. Moreover, since $\pi_\lambda \circ c_{-\tau-\sigma(\nu)}$ is smooth, the fiber product

$$Z_r := L^rU \times_{\pi_\lambda \circ c_{-\tau-\sigma(\nu)}, S_\lambda} (S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}}$$

is irreducible, and $\tilde{Y}_\nu^{\mathbf{b},x_0,r}$ is an étale cover of Z_r with Galois group $U_\tau(\mathcal{O}/\varpi^r)$. Therefore, it is enough to show that the induced representation

$$\rho : \mathrm{Gal}(\bar{\eta}/\eta) \rightarrow U_\tau(\mathcal{O}/\varpi^r)$$

is surjective, where η is the generic point of $Z_r \otimes \bar{k}$.

We will make use of the following easy group theoretical fact.

Lemma 4.4.8. *Let $f : \Gamma_1 \rightarrow \Gamma_2$ be a group homomorphism. Assume that Γ_2 is a nilpotent group. Then f is surjective if and only if the abelianization map $f^{\mathrm{ab}} : \Gamma_1^{\mathrm{ab}} \rightarrow \Gamma_2^{\mathrm{ab}}$ is surjective.*

Proof. The “only if” direction is clear. We prove the other direction. As usual, we denote by $[a, b] = aba^{-1}b^{-1}$ and $a^b = bab^{-1}$.

Let $\gamma_1(\Gamma_i) = \Gamma_i$ and $\gamma_j(\Gamma_i) = [\gamma_{j-1}(\Gamma_i), \Gamma_i]$ be the lower central series. It is enough to show that $\Gamma_1/\gamma_j(\Gamma_1) \rightarrow \Gamma_2/\gamma_j(\Gamma_2)$ is surjective for every j by induction. The case $j = 2$ is our assumption. Suppose that this is true for j . Then it is enough to show that

$$\gamma_j(\Gamma_1)/\gamma_{j+1}(\Gamma_1) \rightarrow \gamma_j(\Gamma_2)/\gamma_{j+1}(\Gamma_2)$$

is surjective. It is enough to show that elements in $\gamma_j(\Gamma_2)$ of the form $[a, b]$ with $a \in \gamma_{j-1}(\Gamma_2)$ and $b \in \Gamma_2$ are contained in the image of the above map.

By inductive hypothesis, there exists $\tilde{a} \in \gamma_{j-1}(\Gamma_1)$ and $\tilde{b} \in \Gamma_1$ such that $f(\tilde{a}) = am, f(\tilde{b}) = bn$, with $m, n \in \gamma_j(\Gamma_2)$. Then

$$f([\tilde{a}, \tilde{b}]) = [am, bn] = [m, b]^a [a, b] [am, n]^b = [a, b] \pmod{\gamma_{j+1}(\Gamma_2)}.$$

The lemma then follows. \square

Therefore, it is enough to show that $\rho : \text{Gal}(\bar{\eta}/\eta) \rightarrow U_\tau(\mathcal{O}/\varpi^r)^{\text{ab}}$ is surjective. The next lemma gives a description of $U_\tau(\mathcal{O}/\varpi^r)^{\text{ab}}$ in most cases.

Lemma 4.4.9. *Let U_H be the unipotent radical of a Borel subgroup B_H of an unramified reductive group H over \mathcal{O} , and for $i \geq 1$, let $\Phi_{H,i}$ denote the set of (absolute) positive roots α of height i , i.e. $\langle \alpha, \rho_H^\vee \rangle = i$, where ρ_H^\vee is the half of the sum of positive coroots. Let $U_{H,i}$ denote the group generated by the coroot groups U_α with $\alpha \in \Phi_{H,i} \cup \Phi_{H,i+1} \cup \dots$. Then we obtain a canonical filtration $U_H = U_{H,1} \supset U_{H,2} \supset \dots$ by normal subgroups.*

Assume that $k \neq \mathbb{F}_2$ if H_{ad} has a factor of type B_n, C_n, F_4 , and $k \neq \mathbb{F}_2, \mathbb{F}_3$ if H_{ad} has a factor of type G_2 , then for every \mathcal{O} -algebra R ,

$$\gamma_2(U_H(R)) = U_{H,2}(R).$$

Proof. The inclusion $\gamma_2(U_H(R)) \subseteq U_{H,2}(R)$ is clear. We prove the other inclusion. We are free to pass to a central isogeny of H , so we may assume that H is adjoint. Then we may assume that H is absolutely simple over \mathcal{O} . For a relative coroot a , let $U_{H,a}$ denote the corresponding root group. Recall that each σ -orbit in $\Phi_H^+ := \cup_{i \geq 0} \Phi_{H,i}$ corresponds to a positive relative root of H . We will say the height of a positive relative root to mean the height of the corresponding absolute roots. Note that every non-simple relative root a of H is either non-reduced (i.e. $a/2$ is also a root) or is contained in a sub root system of rank 2. Then to show $U_{H,a}(R) \subset \gamma_2(U_H(R))$, we may assume H itself is of type BC_1 , or of relative rank two. Explicitly, it is one of the following cases: $PGL_3, PSO_5, G_2, PU_n, n = 3, 4, 5, {}^3D_4$.

We also point out that, to show that $U_{H,2}(R) \subseteq \gamma_2(U_H(R))$, it is enough to show that, for each relative root a of height i , there exist relative roots a_1 and a_2 of heights i_1 and i_2 , such that $a = a_1 + a_2$ (thus $i = i_1 + i_2$) and

(4.4.7)

$$[U_{H,a_1}(R)U_{H,i_1+1}(R)/U_{H,i_1+1}(R), U_{H,a_2}(R)U_{H,i_2+1}(R)/U_{H,i_2+1}(R)] = U_{H,a}(R)U_{H,i+1}(R)/U_{H,i+1}(R).$$

The calculations of the commutator $[x, y]$ for $x \in U_{H,a}(R)$ and $y \in U_{H,b}(R)$ in split case are contained in SGA 3 XXIII, Section 3. Using these formulas and the argument as [Ho74, Lemma 7] give our claim in the split case.

Next we assume that $H = PU_n$ with $n = 3, 4, 5$ for the unramified quadratic extension E/F . Let $\alpha_1, \dots, \alpha_{n-1}$ denote the absolute simple roots, and then each subset of consecutive numbers $I \subseteq \{1, \dots, n-1\}$ corresponds to an absolute positive root. Let $x_{\alpha_i} : \mathbb{G}_a \rightarrow U_{\alpha_i}$ denote the corresponding root morphisms over \mathcal{O}_E , normalized so that $\sigma(x_{\alpha_i}(r)) = x_{\alpha_{n-i}}(\sigma(r))$ for $r \in \mathcal{O}_E$. Then we can normalize other root morphisms so that $[x_{\alpha_i}(r), x_{\alpha_j}(r')] = 0$ unless the concatenation

of I and J (resp. J and I) is exactly another consecutive sequence, in which case $[x_{\alpha_I}(r), x_{\alpha_J}(r')] = x_{\alpha_{I \cup J}}(rr')$ (resp. $[x_{\alpha_I}(r), x_{\alpha_J}(r')] = x_{\alpha_{J \cup I}}(-rr')$).

- (1) When $n = 3$, for the relative root a corresponding to $\alpha_{1,2} = \alpha_1 + \alpha_2$, a straightforward computation shows that²⁴

$$\begin{aligned} & [x_{\alpha_1}(r)x_{\alpha_2}(\sigma(r)), x_{\alpha_1}(r')x_{\alpha_2}(\sigma(r'))] \\ &= x_{\alpha_1}(r)x_{\alpha_2}(\sigma(r))x_{\alpha_1}(r')x_{\alpha_2}(\sigma(r'))x_{\alpha_2}(-\sigma(r'))x_{\alpha_1}(-r')x_{\alpha_2}(-\sigma(r))x_{\alpha_1}(-r) \\ &= x_{\alpha_1}(r)x_{\alpha_1}(r')x_{\alpha_2}(\sigma(r))x_{\alpha_{1,2}}(-\sigma(r)r')x_{\alpha_2}(\sigma(r'))x_{\alpha_2}(-\sigma(r'))x_{\alpha_2}(-\sigma(r))x_{\alpha_1}(-r')x_{\alpha_{1,2}}(r'\sigma(r))x_{\alpha_1}(-r) \\ &= x_{\alpha_{1,2}}(r'\sigma(r) - r\sigma(r')). \end{aligned}$$

It follows that $U_{\alpha_{1,2}}(R) \subseteq \gamma_2(U_H(R))$. (Note that this even holds if $p = 2$.)

- (2) When $n = 4$, the relative positive roots of height ≥ 2 correspond to the σ -orbits $\{\alpha_{1,2}, \alpha_{2,3}\}$ and $\{\alpha_{1,2,3}\}$. We use the following commutator relations (whose computation we omit)²⁵

$$\begin{aligned} & [x_{\alpha_1}(r)x_{\alpha_3}(\sigma(r)), x_{\alpha_2}(r')] = x_{\alpha_{1,2}}(rr')x_{\alpha_{2,3}}(-\sigma(r)r'), \\ & [x_{\alpha_1}(r)x_{\alpha_3}(\sigma(r)), x_{\alpha_{1,2}}(r')x_{\alpha_{2,3}}(-\sigma(r'))] = x_{\alpha_{1,2,3}}(-\sigma(r)r' - r\sigma(r')). \end{aligned}$$

This verifies (4.4.7) for the relative roots $\{\alpha_{1,2}, \alpha_{2,3}\}$ and $\alpha_{1,2,3}$.

- (3) When $n = 5$, the relative positive roots of height ≥ 2 correspond to the σ -orbits $\{\alpha_{1,2}, \alpha_{3,4}\}$, $\{\alpha_{2,3}\}$, $\{\alpha_{1,2,3}, \alpha_{2,3,4}\}$, and $\{\alpha_{1,2,3,4}\}$. The relative root corresponding to $\{\alpha_{2,3}\}$ (resp. $\{\alpha_{1,2,3,4}\}$) is contained in an embedded PU_2 with absolute simple roots α_2 and α_3 (resp. $\alpha_{1,2}$ and $\alpha_{3,4}$), which is covered in (1). The following verifies (4.4.7) for the other two relative positive roots.

$$\begin{aligned} & [x_{\alpha_1}(r)x_{\alpha_4}(\sigma(r)), x_{\alpha_2}(r')x_{\alpha_3}(\sigma(r'))] = x_{\alpha_{1,2}}(rr')x_{\alpha_{3,4}}(-\sigma(r)r'), \\ & [x_{\alpha_1}(r)x_{\alpha_4}(\sigma(r)), x_{\alpha_{2,3}}(r')] = x_{\alpha_{1,2,3}}(rr')x_{\alpha_{2,3,4}}(-\sigma(r)r'). \end{aligned}$$

Finally, we assume that $H = {}^3\text{D}_4$ with respect to the cubic unramified extension E/F . Let $\alpha_1, \alpha_2, \alpha_3$, and α_4 denote the four absolute simple roots; so that σ permutes $\alpha_2, \alpha_3, \alpha_4$. The relative positive roots of height ≥ 2 correspond to the σ -orbits $\{\alpha_{1,2}, \alpha_{1,3}, \alpha_{1,4}\}$, $\{\alpha_{1,2,3}, \alpha_{1,2,4}, \alpha_{1,3,4}\}$, $\{\alpha_{1,2,3,4}\}$, and $\{\alpha_{1,2,3,4}^* = \alpha_{1,2,3,4} + \alpha_1\}$. The relative positive root corresponding to $\{\alpha_{1,2,3,4}^*\}$ (resp. $\{\alpha_{1,2,3}, \alpha_{1,2,4}, \alpha_{1,3,4}\}$) is contained in an embedded PGL_2 (resp. $\text{Res}_{E/F} \text{PGL}_2$) with simple roots α_1 and $\alpha_{1,2,3,4}$ (resp. $\{\alpha_1, \alpha_2, \alpha_3\}$ and $\{\alpha_{1,2}, \alpha_{1,3}, \alpha_{1,4}\}$), which is covered in the split group case. To deal with the other two relative roots, we normalize root morphisms so that $\sigma(x_{\alpha_i}(r)) = x_{\sigma(\alpha_i)}(\sigma(r))$ and $x_{\alpha_{1,i}}(1) = [x_{\alpha_1}(1), x_{\alpha_i}(1)]$ for $i = 2, 3, 4$; $x_{\alpha_{1,i,j}} = [x_{\alpha_{1,i}}(1), x_{\alpha_j}(1)]$ for pairs $(i, j) = (2, 3), (3, 4), (4, 2)$; and $x_{\alpha_{1,2,3,4}}(1) = [x_{\alpha_{1,2,3}}(1), x_{\alpha_4}(1)] = [x_{\alpha_{1,3,4}}(1), x_{\alpha_2}(1)] = [x_{\alpha_{1,4,2}}(1), x_{\alpha_3}(1)]$. The following verifies (4.4.7) for the other relative positive roots.

$$\begin{aligned} & [x_{\alpha_1}(r), x_{\alpha_2}(r')x_{\alpha_3}(\sigma(r'))x_{\alpha_4}(\sigma^2(r'))] = x_{\alpha_{1,2}}(rr')x_{\alpha_{1,3}}(r\sigma(r'))x_{\alpha_{1,4}}(r\sigma^2(r')), \\ & [x_{\alpha_2}(r')x_{\alpha_3}(\sigma(r'))x_{\alpha_4}(\sigma^2(r')), x_{\alpha_{1,2,3}}(r')x_{\alpha_{1,3,4}}(\sigma(r'))x_{\alpha_{1,4,2}}(\sigma^2(r'))] = x_{\alpha_{1,2,3,4}}((1+\sigma+\sigma^2)(-r\sigma(r'))). \end{aligned}$$

This concludes the proof of the lemma. \square

²⁴The relative root group corresponding to $\{\alpha_1, \alpha_2\}$ should be $x_{\alpha_1}(r)x_{\alpha_2}(\sigma(r))x_{\alpha_{1,2}}(-\frac{1}{2}r\sigma(r))$ but its image in $U_{\{\alpha_1, \alpha_2\}}(R)/U_2(R)$ is the same as $x_{\alpha_1}(r)x_{\alpha_2}(\sigma(r))$. Explicitly when $n = 3$, if we write $U_3 = \{A \in \text{GL}_3(\mathcal{O}_E) \mid (\bar{A}^t)^{-1}JA = J\}$ with the anti-diagonal unit matrix J_3 . The commutator subgroup of PU_3 is given by

$$\begin{pmatrix} 1 & r & s \\ & 1 & -\sigma(r) \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & r' & s' \\ & 1 & -\sigma(r') \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -r & \sigma(s) \\ & 1 & \sigma(r) \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & -r' & \sigma(s') \\ & 1 & \sigma(r') \\ & & 1 \end{pmatrix} = \begin{pmatrix} 1 & r'\sigma(r) - r\sigma(r') \\ & 1 & \\ & & 1 \end{pmatrix},$$

where $r\sigma(r) + s + \sigma(s) = 0, r'\sigma(r') + s' + \sigma(s') = 0$.

²⁵Note that the root group for the relative root $\{\alpha_{1,2}, \alpha_{2,3}\}$ is $x_{\alpha_{1,2}}(s)x_{\alpha_{2,3}}(-\sigma(s))$ with the negative sign.

We have a natural projection

$$\mathrm{pr} : L^r U \rightarrow L^r(U/U_2) \rightarrow L^r(U_\tau/U_{\tau,2}).$$

Let

$$\mathbb{L} : L^r(U_\tau/U_{\tau,2}) \rightarrow L^r(U_\tau/U_{\tau,2}), \quad \mathbb{L}(u) = u^{-1}\sigma(u)$$

be (the perfection of) the usual Lang map for the *commutative* algebraic group $L^r(U/U_2)$. Then we have the following commutative, in which the right square is Cartesian

$$\begin{array}{ccccc} L^r U & \longrightarrow & L^r(U/U_2) & \longrightarrow & L^r(U_\tau/U_{\tau,2}) \\ f_\tau \downarrow & & f_{\tau,2} \downarrow & \square & \downarrow \mathbb{L} \\ L^r U & \longrightarrow & L^r(U/U_2) & \longrightarrow & L^r(U_\tau/U_{\tau,2}), \end{array}$$

where $f_{\tau,2} : L^r(U/U_2) \rightarrow L^r(U/U_2)$ is the morphism induced by f_τ .

Let $Z_{r,2}$ be the closure of the image of Z_r under the projection $L^r U \rightarrow L^r(U/U_2)$, whose generic point we denote by η_2 . It follows from this commutative diagram and Lemma 4.4.9 that we have a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(\bar{\eta}/\eta) & \xrightarrow{\rho} & U_\tau(\mathcal{O}/\varpi^r) \\ \downarrow & & \downarrow \\ \mathrm{Gal}(\bar{\eta}_2/\eta_2) & \xrightarrow{\rho_2} & (U_\tau/U_{\tau,2})(\mathcal{O}/\varpi^r) \cong U_\tau(\mathcal{O}/\varpi^r)^{\mathrm{ab}}, \end{array}$$

where the bottom arrow is induced by the Lang map \mathbb{L} . By Lemma 4.4.8, to prove the surjectivity of ρ , it is enough to show the surjectivity of $\mathrm{Gal}(\bar{\eta}/\eta) \rightarrow \mathrm{Gal}(\bar{\eta}_2/\eta_2)$ and the surjectivity of ρ_2 . The former follows from the fact that the map $Z_r \rightarrow Z_{r,2}$ generically has geometrically connected fiber (Proposition 3.3.20). For the latter, we need to show that $Z_{r,2}$ is big enough. We make use of the following lemma.

Lemma 4.4.10. *We have $Z_{r,2} = \coprod L^+U_{\alpha_i}^{(\langle \nu, \alpha_i \rangle - \varepsilon_{\alpha_i}(\mathbf{b}))} / L^+U_{\alpha_i}^{(r)} \subset L^r(U/U_2)$.*

Note that by our assumption of ν and (3.3.3), $\langle \nu, \alpha_i \rangle - \varepsilon_{\alpha_i}(\mathbf{b}) \geq 0$. This is consistent with the fact that $Z_r \subset L^r U$.

Proof. By Proposition 3.3.20, the closure of the projection of $\pi_\lambda^{-1}((S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}}) \subset LU$ to $L(U/U_2)$ is $\prod_i \pi_{M_i, \lambda}^{-1}(S_{M_i, \lambda} \cap \mathrm{Gr}_{M_i, \lambda + \varepsilon_i(\mathbf{b})})$, where $\pi_{M_i, \lambda} : LU_{M_i} = LU_{\alpha_i} \rightarrow S_{M_i, \lambda_i}$ is the projection (3.2.1) for M_i . By (3.2.2) and $\phi_i(\mathbf{b}) - \varepsilon_i(\mathbf{b}) = \langle \lambda, \alpha_i \rangle$, we have

$$\pi_{M_i, \lambda}^{-1}(S_\lambda \cap \mathrm{Gr}_{M_i, \lambda + \varepsilon_i(\mathbf{b})}) = L^+U_{\alpha_i}^{(-\phi_i(\mathbf{b}))}.$$

Now use $Z_r = \varpi^{\sigma(\nu) + \tau} \pi_\lambda^{-1}((S_\lambda \cap \mathrm{Gr}_\mu)^{\mathbf{b}}) \varpi^{-\sigma(\nu) - \tau} / L^+U^{(r)}$ and $\sigma(\nu) + \tau = \nu + \lambda$ to conclude. \square

Note that so far we have not use the fact that $\nu = \nu_{\mathbf{b}}$. Now, assuming this, we see that in each σ -orbit $\{\alpha, \sigma(\alpha), \dots, \sigma^d(\alpha) = \alpha\}$ of Δ , there is some $\sigma^i(\alpha)$ such that $\langle \nu_{\mathbf{b}}, \sigma^i(\alpha) \rangle = \varepsilon_{\sigma^i(\alpha)}(\mathbf{b})$, by Lemma 4.4.3 (2). In particular, $L^r U_{\sigma^i(\alpha)} \subset Z_{r,2}$.

We need a last lemma.

Lemma 4.4.11. *Let H_0 be a connected affine algebraic group over \mathbb{F}_{q^r} and let $H = \mathrm{Res}_{\mathbb{F}_{q^r}/\mathbb{F}_q} H_0$ be its Weil restriction to \mathbb{F}_q . Let $H_0 \rightarrow H \otimes \mathbb{F}_{q^r}$ be the canonical embedding. Let $\mathbb{L}_H : H \rightarrow H$, $h \mapsto h^{-1}\sigma(h)$ denote the Lang map of H , base changed to \mathbb{F}_{q^r} , where σ is the q -Frobenius of H . Then $\mathbb{L}_H : \mathbb{L}_H^{-1}(H_0) \rightarrow H_0$ can be identified with the Lang map for H_0 . In particular, $\mathbb{L}_H^{-1}(H_0)$ is irreducible so the composed map*

$$\pi_1^{\mathrm{et}}(H_0) \rightarrow \pi_1^{\mathrm{et}}(H) \rightarrow H_0(\mathbb{F}_{q^r}).$$

is surjective.

Proof. This follows by definition. \square

Now we finish the proof. Applying this lemma to each σ -orbit $\{\alpha, \sigma(\alpha), \dots, \sigma^d(\alpha) = \alpha\}$ of Δ_τ (using the fact that $Z_{r,2}$ contains $L^r U_{\sigma^i(\alpha)}$ for some i), we see that the image of ρ_2 contains the relative root group corresponding this σ -orbit. Since this is true for all σ -orbit, we conclude that ρ_2 is surjective. This concludes the proof of Proposition 4.4.7. \square

4.4.12. *Proof of Theorem 4.4.5.* Let $\overline{Y_\nu^b \cap \text{Gr}_\nu}$ denote the closure of $Y_\nu^b \cap \text{Gr}_\nu$. By Proposition 4.4.7, it is an irreducible component of $X_\mu^b(\tau)$. Recall that $Y_\nu^b \cap \text{Gr}_\nu = X_{\mu,\nu}^a(\tau) \cap S_\nu$. Therefore, for every $g \in J_\tau(\mathcal{O})$,

$$(Y_\nu^b \cap \text{Gr}_\nu) \cap g(Y_\nu^b \cap \text{Gr}_\nu) = X_{\mu,\nu}^a(\tau) \cap S_\nu \cap gS_\nu.$$

Recall that $S_\nu \cap \text{Gr}_\nu$ is open in Gr_ν . It follows that $(Y_\nu^b \cap \text{Gr}_\nu) \cap g(Y_\nu^b \cap \text{Gr}_\nu)$ is open in $Y_\nu^b \cap \text{Gr}_\nu$. In particular, if this intersection is non-empty, then $gY_\nu^b \cap \text{Gr}_\nu = \overline{Y_\nu^b \cap \text{Gr}_\nu}$ by the irreducibility of $Y_\nu^b \cap \text{Gr}_\nu$ established in Proposition 4.4.7.

Now, by Lemma 3.2.4, $\varpi^\nu \in Y_\nu^b \cap \text{Gr}_\nu$. Let

$$\text{St}_{J_{\tau,x_0}(\mathcal{O})}(\varpi^\nu) = \{g \in J_{\tau,x_0}(\mathcal{O}) \mid g\varpi^\nu = \varpi^\nu \pmod{G(\mathcal{O}_L)}\}.$$

It follows from the above claim that $\overline{Y_\nu^b \cap \text{Gr}_\nu}$ is invariant under the action $\text{St}_{J_{\tau,x_0}(\mathcal{O})}(\varpi^\nu)$. On the other hand, $\overline{Y_\nu^b \cap \text{Gr}_\nu}$ is also invariant under the action of $U_\tau(\mathcal{O})$ (since so is $Y_\nu^b \cap \text{Gr}_\nu$). Since $\text{St}_{J_{\tau,x_0}(\mathcal{O})}(\varpi^\nu)$ and $U_\tau(\mathcal{O})$ generate $J_\tau(\mathcal{O})$ as a group, we see that $\overline{Y_\nu^b \cap \text{Gr}_\nu}$ is invariant under the action by $J_{\tau,x_0}(\mathcal{O})$.

Next, we show that $X_{\mu,\nu}^a(\tau)$ contains a unique irreducible component of dimension $\langle \rho, \mu - \tau \rangle$. Since $\text{Gr}_\nu = \sqcup_{\nu' \prec \nu} (S_{\nu'} \cap \text{Gr}_\nu)$, it is enough to show that $X_{\mu,\nu}^a(\tau) \cap S_{\nu'}$ has dimension strictly smaller than $\langle \mu - \tau \rangle$ if $\nu' \prec \nu$. But by definition, $X_{\mu,\nu}^a(\tau) \cap S_{\nu'}$ fits into the following Cartesian diagram

$$\begin{array}{ccc} X_{\mu,\nu}^a(\tau) \cap S_{\nu'} & \longrightarrow & \text{Gr}_{(\nu,\mu)|\tau+\sigma(\nu)}^{0,a} \cap (S_{\nu'} \cap \text{Gr}_\nu) \times (S_{\tau+\sigma(\nu')} \cap \text{Gr}_{\tau+\sigma(\nu)}) \\ \downarrow & & \downarrow \\ S_{\nu'} \cap \text{Gr}_\nu & \xrightarrow{\text{id} \times \varpi^\tau \sigma} & (S_{\nu'} \cap \text{Gr}_\nu) \times (S_{\tau+\sigma(\nu')} \cap \text{Gr}_{\tau+\sigma(\nu)}). \end{array}$$

We need the following lemma.

Lemma 4.4.13. *Let $\nu' \prec \nu$. Then the intersection $\text{Gr}_{(\nu,\mu)|\tau+\sigma(\nu)}^{0,a} \cap (S_{\nu'} \cap \text{Gr}_\nu) \times (S_{\tau+\sigma(\nu')} \cap \text{Gr}_{\tau+\sigma(\nu)})$ in $\text{Gr}_\nu \times \text{Gr}_{\tau+\sigma(\nu)}$ has dimension $< \langle \rho, \nu + \tau + \sigma(\nu') + \mu \rangle$.*

Proof. Let $(S_{\tau+\sigma(\nu')} \cap \text{Gr}_{\tau+\sigma(\nu)})^c$ be an irreducible component of $S_{\tau+\sigma(\nu')} \cap \text{Gr}_{\tau+\sigma(\nu)}$. Recall from the proof of Lemma 3.2.8 that the preimage of $(S_{\tau+\sigma(\nu')} \cap \text{Gr}_{\tau+\sigma(\nu)})^c \cap \mathring{\text{Gr}}_{\tau+\sigma(\nu)}$ in $\text{Gr}_{(\nu,\mu)|\tau+\sigma(nu)}^{0,a}$ is an open subset $(S_{\eta_1} \times \text{Gr}_\nu)^{c_1} \tilde{\times} (S_{\eta_2} \times \text{Gr}_\lambda)^{c_2}$, where $\eta_1 + \eta_2 = \tau + \sigma(\nu')$, and $(\mathbf{a}, \mathbf{c}) \mapsto (\mathbf{c}_1, \mathbf{c}_2)$ is the map defined in Lemma 3.2.8, which is compatible with Littelmann's decomposition theorem (see Proposition 3.3.17)

We identify the set of MV cycles with the set of \hat{G} -crystals via Proposition 3.3.15. Note that $\mathbf{c} \in \mathbb{M}\mathbb{V}_{\tau+\sigma(\nu)}(\tau + \sigma(\nu'))$ is obtained by the unique element in $\mathbb{M}\mathbb{V}_{\tau+\sigma(\nu)}(\tau + \sigma(\nu))$ by a series of root operators f_α s. Since one of the equalities in Lemma 4.4.3(2) holds, by the definition of the root operators acting on the set of \hat{G} -crystals, we see that

$$\nu' \prec \eta_1, \quad \eta_2 \prec \lambda.$$

This implies that the intersection in the lemma has dimension $< \langle \rho, \nu + \eta_1 + \mu + \eta_2 \rangle = \langle \rho, \nu + \tau + \sigma(\nu') + \mu \rangle$. \square

Now, by the argument similar to Proposition 4.3.3 (in particular, a diagram similar to (4.3.7)), we see that

$$\begin{aligned} \dim X_{\mu,\nu}^{\mathbf{a}}(\tau) \cap S_{\nu'} &= \dim \mathring{\mathrm{Gr}}_{(\nu,\mu)|\tau+\sigma(\nu)}^{0,\mathbf{a}} \cap (S_{\nu'} \cap \mathrm{Gr}_{\nu}) \times (S_{\tau+\sigma(\nu')} \cap \mathrm{Gr}_{\tau+\sigma(\nu)}) - \dim(S_{\tau+\sigma(\nu')} \cap \mathrm{Gr}_{\tau+\sigma(\nu)}) \\ &< \langle \rho, \nu + \tau + \sigma(\nu') + \mu \rangle - \langle \rho, \tau + \sigma(\nu') + \tau + \sigma(\nu) \rangle \\ &= \langle \rho, \mu - \tau \rangle. \end{aligned}$$

We have finished the proof that $X_{\mu}^{\mathbf{b},x_0}(\tau) := \overline{Y_{\nu}^{\mathbf{b}} \cap \mathrm{Gr}_{\nu}}$ is the unique irreducible component of $X_{\mu,\nu}^{\mathbf{a}}(\tau)$ of dimension $\langle \rho, \mu - \tau \rangle$.

Finally, we show that $X_{\mu,\nu}^{\mathbf{a}}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu}$ is irreducible. Let

$$(4.4.8) \quad \mathring{\mathrm{Gr}}_{(\nu,\mu)|\tau+\sigma(\nu)}^{0,\mathbf{a}} := \mathring{\mathrm{Gr}}_{(\nu,\mu)|\tau+\sigma(\nu)}^{0,\mathbf{a}} \cap (\mathring{\mathrm{Gr}}_{\nu} \times \mathring{\mathrm{Gr}}_{\tau+\sigma(\nu)}).$$

Let F be the fiber over ϖ^{ν} of the projection $\mathring{\mathrm{Gr}}_{(\nu,\mu)|\tau+\sigma(\nu)}^{0,\mathbf{a}} \rightarrow \mathring{\mathrm{Gr}}_{\nu}$. This is identified with an irreducible locally closed subset of $\mathrm{Gr}_{\mu} \cap \varpi^{-\nu} \mathring{\mathrm{Gr}}_{\tau+\sigma(\nu)} \subset \mathrm{Gr}$. Let $F^{(\infty)}$ denote its preimage in $LG \rightarrow \mathrm{Gr}$.

Recall that $\mathring{\mathrm{Gr}}_{\nu}$ is a single L^+G -orbit given by $g \mapsto g\varpi^{\nu}$. Then the preimage of $X_{\mu,\nu}^{\mathbf{a}}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu}$ in L^+G is

$$\{g \in L^+G \mid g^{-1}\varpi^{\tau}\sigma(g) \in \varpi^{\nu}F^{(\infty)}\varpi^{-\sigma(\nu)} \subset LG\}.$$

Note that $J_{\tau,x_0}(\mathcal{O})$ acts transitively along the fibers of the map $L^+G \rightarrow LG, g \mapsto g^{-1}\varpi^{\tau}\sigma(g)$. Since $\varpi^{\nu}F^{(\infty)}\varpi^{-\sigma(\nu)}$ is irreducible, it follows that $J_{\tau,x_0}(\mathcal{O})$ acts transitively on the set of irreducible components of $X_{\mu,\nu}^{\mathbf{a}}(\mathcal{O}) \cap \mathring{\mathrm{Gr}}_{\nu}$. Combining with the fact that $X_{\mu}^{\mathbf{b},x_0}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu} \supset Y_{\nu}^{\mathbf{b}} \cap \mathrm{Gr}_{\nu}$ is irreducible in $X_{\mu,\nu}^{\mathbf{a}}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu}$ of dimension $\langle \rho, \mu - \tau \rangle$, we see that

$$X_{\mu}^{\mathbf{b},x_0}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu} = X_{\mu,\nu}^{\mathbf{a}}(\tau) \cap \mathring{\mathrm{Gr}}_{\nu},$$

which is irreducible. The proof of Theorem 4.4.5 now is complete.

Theorem 4.4.14. *Let $b = \varpi^{\tau}$ be an unramified element, and assume that τ_{σ} is dominant. Consider the action of $J_{\tau}(F)$ on the set of irreducible components of $X_{\mu}(\tau)$.*

- (1) *The stabilizer of each irreducible component is a hyperspecial subgroup.*
- (2) *The subset $X_{\mu}^{\mathbf{b}}(\tau) \subset X_{\mu}(\tau)$ is invariant under the action of $J_{\tau}(F)$. The group $J_{\tau}(F)$ acts transitively on the set of irreducible components of $X_{\mu}^{\mathbf{b}}(\tau)$.*
- (3) *Assume that Z_G is connected. There is a canonical $J_{\tau}(F)$ -equivariant bijection between the set of irreducible components of $X_{\mu}^{\mathbf{b}}(\tau)$ and*

$$\sqcup_{\lambda \in \tau + (1-\sigma)\mathbb{X}_{\bullet}} \mathrm{MV}_{\mu}(\lambda) \times \mathrm{HS}_b, \quad (\mathbf{b}, x) \mapsto X_{\mu}^{\mathbf{b},x}(\tau)$$

where HS_b denote the set of hyperspecial subgroups of $J_{\tau}(F)$, and $X_{\mu}^{\mathbf{b},x}(\tau)$ is the irreducible component in $X_{\mu}^{\mathbf{b}}(\tau)$ whose stabilizer is the hyperspecial subgroup of $J_{\tau}(F)$ corresponding to x .

Proof. To proof (1), first assume that Z_G is connected. Then by Proposition 4.4.7, in each $X_{\mu}^{\mathbf{b}}(\tau)$ there is an irreducible component whose stabilizer in $J_{\tau}(F)$ is hyperspecial. Then the statement follows from Corollary 4.3.5(3). In particular, if G is of adjoint type, the statement holds. The general case follows from Lemma 4.1.5.

Recall that $X_{\mu}^{\mathbf{b}}(\tau)$ is $B_{\tau}(F)$ -invariant by Corollary 4.3.5. On the other hand, the stabilizer of each irreducible component is hyperspecial in $J_{\tau}(F)$. Therefore (2) follows from the Iwasawa decomposition if $J_{\tau}(F)$ and Corollary 4.3.5.

If Z_G is connected, then all hyperspecial subgroups of J_τ are conjugate. For every $\mathbf{b} \in \mathbb{M}\mathbb{V}_\mu(\lambda)$, we constructed $X_\mu^{\mathbf{b}, x_0}(\tau)$, and other irreducible components are given by $g \mapsto gX_\mu^{\mathbf{b}, x_0}(\tau)$ for $g \in J_\tau(F)$, with stabilizer group $gJ_{\tau, x_0}(\mathcal{O})g^{-1}$. Part (3) follows. \square

Remark 4.4.15. In general if Z_G is not connected, it seems that J_τ does not have a natural conjugacy class of hyperspecial subgroup. But Theorem 4.4.14 implies that each $\mathbf{b} \in \sqcup_{\lambda \in \tau + (1-\sigma)\mathbb{X}_\bullet} \mathbb{M}\mathbb{V}_\mu(\lambda)$ gives such a conjugacy class, which contains the stabilizers of irreducible components in $X_\mu^{\mathbf{b}}(\tau)$. Note that different \mathbf{b} 's may give different conjugacy classes.

5. THE MODULI OF LOCAL SHTUKAS

The moduli spaces of shtukas were invented by Drinfeld as the function field analogue of the Shimura varieties. However, the moduli space of shtukas can be defined in a much greater generality. In particular, there are notions of moduli spaces of iterated shtukas and partial Frobenius between them, which play a crucial role in realizing the Langlands correspondence over the function fields (see the works of Drinfeld [Dr89], L. Lafforgue [Laf.L02], and more recently V. Lafforgue [Laf.V12⁺]). Unfortunately, so far it is not clear how to construct their Shimura variety counterparts. In this section, we pass to the local situation to give a uniform construction of the moduli spaces of (iterated) local shtukas, for groups over equal or mixed characteristic. In addition, we construct various correspondences between them, including the partial Frobenius operator. Finally, we define certain category $\mathbf{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})$ of perverse sheaves, which will be the main player of the next section.

Let $\mathbf{Aff}_k^{\text{pf}}$ denote the category of perfect rings over k , equipped with the fpqc topology. We continue to use the notations from Section 3, in particular (3.0.1) and (3.0.2). In this section, unless otherwise specified, schemes/algebraic spaces/algebraic stacks mean schemes/algebraic spaces/algebraic stacks in $\mathbf{Aff}_k^{\text{pf}}$. The foundations on perfect algebraic geometry we needed in this paper are recalled in the Appendix A.1, and the results on cohomological correspondences are recalled in A.2.

5.1. The local Hecke stack.

Definition 5.1.1. Let $\mu_\bullet = (\mu_1, \dots, \mu_t)$ be a sequence of dominant coweights (of G). We define the *local Hecke stack* as the prestack

$$\text{Hk}_{\mu_\bullet}^{\text{loc}} = \text{Hk}_{\mu_1, \dots, \mu_t}^{\text{loc}} := [L^+G \backslash \text{Gr}_{\mu_\bullet}]$$

which assigns for every perfect k -algebra R , the groupoid of chains of modifications of G -torsors over $D_R = \text{Spec} W_{\mathcal{O}}(R)$

$$(5.1.1) \quad \mathcal{E}_t \xrightarrow{\beta_t} \mathcal{E}_{t-1} \xrightarrow{\beta_{t-1}} \dots \xrightarrow{\beta_2} \mathcal{E}_1 \xrightarrow{\beta_1} \mathcal{E}_0$$

of relative positions $\preceq \mu_t, \dots$, and $\preceq \mu_1$, respectively. (Comparing to the moduli problem of Gr_{μ_\bullet} in (3.1.6), we drop the trivialization of \mathcal{E}_0 .)

Similarly, we define

$$\text{Hk}_{\lambda_\bullet | \mu_\bullet}^{0, \text{loc}} := [L^+G \backslash \text{Gr}_{\lambda_\bullet | \mu_\bullet}^0], \quad \text{and} \quad \text{Hk}_{\nu_\bullet; \lambda_\bullet | \mu_\bullet; \xi_\bullet}^{0, \text{loc}} := [L^+G \backslash \text{Gr}_{\nu_\bullet; \lambda_\bullet | \mu_\bullet; \xi_\bullet}^0]$$

for sequences of dominant coweights $\lambda_\bullet, \mu_\bullet, \nu_\bullet, \xi_\bullet$. In particular, taking the quotient of (3.1.14) by the left L^+G -action gives the *Satake correspondence* on local Hecke stacks

$$\text{Hk}_{\lambda_\bullet}^{\text{loc}} \xleftarrow{h_{\lambda_\bullet}^{\leftarrow}} \text{Hk}_{\lambda_\bullet | \mu_\bullet}^{0, \text{loc}} \xrightarrow{h_{\mu_\bullet}^{\rightarrow}} \text{Hk}_{\mu_\bullet}^{\text{loc}}.$$

The above definitions are very convenient for the later geometric constructions, but one cannot directly apply the ℓ -adic formalism to it due to the quotient by an infinite group. For this reason, we need a finite dimensional quotient replacement of it.

Definition 5.1.2. Let $\mu_\bullet = (\mu_1, \dots, \mu_t)$ be a sequence of dominant coweights and let m be a μ_\bullet -large integer (see Definition 3.1.6). We define the m -restricted local Hecke stack to be the stack

$$\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)} = \mathrm{Hk}_{\mu_1, \dots, \mu_t}^{\mathrm{loc}(m)} := [L^m G \backslash \mathrm{Gr}_{\mu_\bullet}].$$

We will sometimes write $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(\infty)}$ for $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}}$. Similarly, if m is both $(\nu_\bullet, \lambda_\bullet, \xi_\bullet)$ -large and $(\nu_\bullet, \mu_\bullet, \xi_\bullet)$ -large, we can define

$$\mathrm{Hk}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m)} := [L^m G \backslash \mathrm{Gr}_{\lambda_\bullet | \mu_\bullet}^0], \quad \mathrm{Hk}_{\nu_\bullet; \lambda_\bullet | \mu_\bullet; \xi_\bullet}^{0, \mathrm{loc}(m)} := [L^m G \backslash \mathrm{Gr}_{\nu_\bullet; \lambda_\bullet | \mu_\bullet; \xi_\bullet}^0].$$

In particular, the Satake correspondence (3.1.14) descends to

$$\mathrm{Hk}_{\lambda_\bullet}^{\mathrm{loc}(m)} \xleftarrow{h_{\lambda_\bullet}^\leftarrow} \mathrm{Hk}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m)} \xrightarrow{h_{\mu_\bullet}^\rightarrow} \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)}.$$

5.1.3. *Convolution map.* The convolution map $\mathrm{Gr}_{\mu_\bullet} \rightarrow \mathrm{Gr}_{|\mu_\bullet|}$ induces

$$(5.1.2) \quad m : \mathrm{Hk}_{\nu_\bullet, \mu_\bullet, \xi_\bullet}^{\mathrm{loc}(m)} \rightarrow \mathrm{Hk}_{\nu_\bullet, |\mu_\bullet|, \xi_\bullet}^{\mathrm{loc}(m)},$$

called the convolution map, which is perfectly proper. Here m (which is allowed to be ∞) is $(\nu_\bullet, \mu_\bullet, \xi_\bullet)$ -large.

Notation 5.1.4. We write $\mathbf{BL}^+ G$ (resp. $\mathbf{BL}^m G$ for $m \in \mathbb{Z}_{\geq 0}$) for the moduli stack that classifies for every perfect k -algebra R the groupoid of G -torsors over D_R (resp. $D_{m,R}$). As a stack, it is the same as $[L^+ G \backslash \mathrm{pt}]$ (resp. $[L^m G \backslash \mathrm{pt}]$). For $m \leq m'$, we have quotient maps $L^+ G \rightarrow L^{m'} G \rightarrow L^m G$ which induce

$$\mathbf{BL}^+ G \xrightarrow{\mathrm{res}_{m'} = \mathrm{res}_{m'}^\infty} \mathbf{BL}^{m'} G \xrightarrow{\mathrm{res}_m^{m'}} \mathbf{BL}^m G.$$

We call them the *restriction maps* (restricting the G -torsor from D to $D_{m'}$ and from $D_{m'}$ to D_m). Clearly, for $m \leq m' \leq m''$,

$$(5.1.3) \quad \mathrm{res}_m^{m'} \circ \mathrm{res}_{m'}^{m''} = \mathrm{res}_m^{m''},$$

where we allow m'' to be ∞ . Since $\mathrm{res}_m^{m'}$ is an $L^{m'-m} G^{(m)}$ -gerbe (for $m' < \infty$), it is perfectly smooth.

5.1.5. *Two torsors over restricted local Hecke stacks.* Let $\mu_\bullet = (\mu_1, \dots, \mu_t)$ be a sequence of dominant coweights. There are natural morphisms

$$t_\leftarrow, t_\rightarrow : \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}} \rightarrow \mathbf{BL}^+ G$$

sending (5.1.1) to the *last* torsor \mathcal{E}_t and the *first* torsor \mathcal{E}_0 in the sequence, respectively. In particular, t_\rightarrow comes from taking the quotient of the structure sheaf $\mathrm{Gr}_{\mu_\bullet} \rightarrow \mathrm{pt}$ by the left $L^+ G$ -action.

Similarly, if (m, n) is a pair of non-negative integers and $m - n$ is μ_\bullet -large, by writing

$$[L^m G \backslash \mathrm{Gr}_{\mu_\bullet}] \cong [L^m G \backslash \mathrm{Gr}_{\mu_\bullet}^{(n)} / L^n G] \xrightarrow{t_\leftarrow \times t_\rightarrow} \mathbf{BL}^n G \times \mathbf{BL}^m G,$$

where $\mathrm{Gr}_{\mu_\bullet}^{(n)}$ is the $L^n G$ -torsor over $\mathrm{Gr}_{\mu_\bullet}$ as defined after (3.1.6), we see that over $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)}$, the first map t_\rightarrow defines the $L^m G$ -torsor

$$(5.1.4) \quad \mathrm{Gr}_{\mu_\bullet} \rightarrow [L^m G \backslash \mathrm{Gr}_{\mu_\bullet}],$$

and the second map t_\leftarrow defines the $L^n G$ -torsor

$$(5.1.5) \quad [L^m G \backslash \mathrm{Gr}_{\mu_\bullet}^{(n)}] \rightarrow [L^{m+n} G \backslash \mathrm{Gr}_{\mu_\bullet}].$$

We write $\mathcal{E}_{\rightarrow|D_m}$ and $\mathcal{E}_{\leftarrow|D_n}$ for these two *canonical* torsors. Note that for $m \leq m'$ and $n \leq n'$ ($(m', n') = (\infty, \infty)$ allowed) such that both $m' - n'$ and $m - n$ are μ_\bullet -large, there is a restriction map

$$\mathrm{res}_m^{m'} : \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m')} \rightarrow \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)}$$

such that the following diagram is commutative and the right square is Cartesian.

$$(5.1.6) \quad \begin{array}{ccccc} \mathbf{B}L^{n'}G & \xleftarrow{t_{\leftarrow}} & \mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m')} & \xrightarrow{t_{\rightarrow}} & \mathbf{B}L^{m'}G \\ \downarrow \mathrm{res}_n^{n'} & & \downarrow \mathrm{res}_m^{m'} & & \downarrow \mathrm{res}_m^{m'} \\ \mathbf{B}L^nG & \xleftarrow{t_{\leftarrow}} & \mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m)} & \xrightarrow{t_{\rightarrow}} & \mathbf{B}L^mG. \end{array}$$

In addition, these restrictions maps satisfy the natural compatibility condition (5.1.3).

Remark 5.1.6. On $\mathrm{Hk}_{\lambda_{\bullet}|\mu_{\bullet}}^{0,\mathrm{loc}(m)}$, the pullbacks of the L^mG -torsor $\mathcal{E}_{\rightarrow}|_{D_m}$ and the L^nG -torsor $\mathcal{E}_{\leftarrow}|_{D_n}$ on $\mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m)}$, which we denote by the same notation, are canonically isomorphic to the pullbacks of $\mathcal{E}_{\rightarrow}|_{D_m}$ and $\mathcal{E}_{\leftarrow}|_{D_n}$ from $\mathrm{Hk}_{\lambda_{\bullet}}^{\mathrm{loc}(m)}$, which we denote by $\mathcal{E}'_{\rightarrow}|_{D_{m+n}}$ and $\mathcal{E}'_{\leftarrow}|_{D_n}$, respectively.

5.1.7. *Perverse sheaves on restricted local Hecke stack.* Now we define the category of perverse sheaves on $\mathrm{Hk}^{\mathrm{loc}}$, by essentially repeating the construction of $\mathrm{P}_{L+G\otimes\bar{k}}(\mathrm{Gr}\otimes\bar{k})$. First recall from Notation A.2.1, for a perfectly smooth morphism $f : X \rightarrow Y$ between algebraic stacks of relative dimension d , we write $f^{\star} := f^{\star}[d](\frac{d}{2})$; it takes perverse sheaves to perverse sheaves. Given $m \leq m'$ two μ_{\bullet} -large integers ($m' \neq \infty$), let $\mathrm{Res}_m^{m'} := (\mathrm{res}_m^{m'})^{\star}$. If $m \geq 1$, since $\mathrm{res}_m^{m'}$ is an $L^{m'-m}G^{(m)}$ -gerbe and $L^{m'-m}G^{(m)}$ is the perfection of a unipotent group, the pullback

$$\mathrm{Res}_m^{m'} : \mathrm{P}(\mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m)}) \longrightarrow \mathrm{P}(\mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m')})$$

induces an equivalence of the corresponding categories of perverse sheaves. An inverse equivalence is given by the push-forward $(\mathrm{res}_m^{m'})_!$ up to a shift and a twist. Then similar to (3.4.1), we can define

$$(5.1.7) \quad \mathrm{P}(\mathrm{Hk}_k^{\mathrm{loc}}) := \bigoplus_{\zeta \in \pi_1(G)} \mathrm{P}(\mathrm{Hk}_{\zeta}^{\mathrm{loc}}), \quad \mathrm{P}(\mathrm{Hk}_{\zeta}^{\mathrm{loc}}) = \varinjlim_{(\mu, m) \in \zeta \times \mathbb{Z}_{\geq \langle \mu, \alpha_h \rangle}} \mathrm{P}(\mathrm{Hk}_{\mu}^{\mathrm{loc}(m)})$$

where the connecting functor for the limit above is given the fully faithful embedding

$$\mathrm{P}(\mathrm{Hk}_{\mu}^{\mathrm{loc}(m)}) \xrightarrow[\cong]{\mathrm{Res}_m^{m'}} \mathrm{P}(\mathrm{Hk}_{\mu}^{\mathrm{loc}(m')}) \xrightarrow{i_{\mu, \mu', \star}} \mathrm{P}(\mathrm{Hk}_{\mu'}^{\mathrm{loc}(m')}).$$

Again, note that the limit is filtered. Note that since an $(L^mG \otimes \bar{k})$ -equivariant perverse sheaf on Gr_{μ} descends to a sheaf on $[L^mG \backslash \mathrm{Gr}_{\mu}]$, after shifted by $m \dim G$ and twisted by $\frac{m}{2} \dim G$, there is a natural equivalence of categories

$$(5.1.8) \quad \mathrm{P}_{L^mG}(\mathrm{Gr}_{\mu}) \xrightarrow{\cong} \mathrm{P}(\mathrm{Hk}_{\mu}^{\mathrm{loc}(m)}),$$

which, after passing to the limit and the direct sum over $\pi_1(G)$, gives rise to the natural equivalence of categories.

$$(5.1.9) \quad \mathrm{P}_{L+G\otimes\bar{k}}(\mathrm{Gr}\otimes\bar{k}) \xrightarrow{\cong} \mathrm{P}(\mathrm{Hk}_{\bar{k}}^{\mathrm{loc}}).$$

5.1.8. *Breaking up local Hecke stacks.* Let λ_{\bullet} and μ_{\bullet} be two sequences of dominant coweights. Assume that $m_1 - m_2$ is λ_{\bullet} -large and $m_2 - n$ is μ_{\bullet} -large. By (3.1.10), there is a canonical isomorphism

$$[L^{m_1}G \backslash \mathrm{Gr}_{\lambda_{\bullet}, \mu_{\bullet}}^{(n)}] \cong [L^{m_1}G \backslash \mathrm{Gr}_{\lambda_{\bullet}}^{(m_2)}] \times^{L^{m_2}G} \mathrm{Gr}_{\mu_{\bullet}}^{(n)},$$

which induces an isomorphism

$$(5.1.10) \quad \mathrm{Hk}_{\lambda_{\bullet}, \mu_{\bullet}}^{\mathrm{loc}(m_1)} \cong \mathrm{Hk}_{\mu_{\bullet}}^{\mathrm{loc}(m_2)} \times_{t_{\rightarrow}, \mathbf{B}L^{m_2}G, t_{\leftarrow}} \mathrm{Hk}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1)}.$$

In other words, giving an R -point of $\mathrm{Hk}_{\lambda_\bullet, \mu_\bullet}^{\mathrm{loc}(m_1)}$ is equivalent to giving an R -point x_1 of $\mathrm{Hk}_{\lambda_\bullet}^{\mathrm{loc}(m_1)}$ and an R -point x_2 of $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2)}$, together with an isomorphism of $L^{m_2}G$ -torsors $\mathcal{E}_{x_1, \leftarrow} |_{D_{m_2, R}} \simeq \mathcal{E}_{x_2, \rightarrow} |_{D_{m_2, R}}$.

In a special case, the isomorphism (5.1.10) induces a perfectly smooth morphism

$$(5.1.11) \quad \mathrm{Hk}_{\lambda_\bullet, \mu_\bullet}^{\mathrm{loc}(m_1)} \rightarrow \mathrm{Hk}_{\lambda_\bullet}^{\mathrm{loc}(m_1)} \times \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2)} \rightarrow \mathrm{Hk}_{\lambda_\bullet}^{\mathrm{loc}(m_1-m_2)} \times \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m_2)},$$

where the first map is induced from (5.1.10) (which is perfectly smooth) and the second map is $\mathrm{res}_{m_1-m_2}^{m_1} \times \mathrm{id}$ (which is also perfectly smooth).

5.2. Moduli of local shtukas. In this subsection, we define various moduli of local (iterated) shtukas as prestacks and various correspondences between them.

Definition 5.2.1. Let $\mu_\bullet = (\mu_1, \dots, \mu_t)$ be a sequence of dominant coweights. Let R be a perfect k -algebra. An R -family of *local (iterated) shtukas* of singularities bounded by μ_\bullet consists of

- a point of $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}}$, i.e. a sequence of modifications of G -torsors over D_R of relative positions $(\preceq \mu_t, \preceq \mu_{t-1}, \dots, \preceq \mu_1)$ as in (5.1.1), and
- an isomorphism $\psi : {}^\sigma \mathcal{E}_t \cong \mathcal{E}_0$. (Here the notation ${}^\sigma \mathcal{E}_t$ is defined at the beginning of Section 4.)

The moduli space that assigns every perfect k -algebra R the groupoid of R -families of local Shtukas of singularities bounded by μ_\bullet is denoted by $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}} := \mathrm{Sht}_{\mu_1, \dots, \mu_t}^{\mathrm{loc}}$. In other words, $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}}$ classifies sequences of modifications

$$\mathcal{E}_t \xrightarrow{\beta_t} \mathcal{E}_{t-1} \xrightarrow{\beta_{t-1}} \dots \xrightarrow{\beta_2} \mathcal{E}_1 \xrightarrow{\beta_1} {}^\sigma \mathcal{E}_t.$$

For such an S -point $x \in \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}}(S)$, we write $\mathcal{E}_{x, \leftarrow}$ for \mathcal{E}_t and $\mathcal{E}_{x, \rightarrow}$ for \mathcal{E}_0 .

Note that by definition, there is a forgetful map of prestacks

$$\varphi^{\mathrm{loc}} : \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}} \longrightarrow \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}}.$$

In fact, we can write $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}}$ as a Cartesian product of $\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}}$ with a Frobenius graph

$$(5.2.1) \quad \begin{array}{ccc} \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}} & \xrightarrow{\varphi^{\mathrm{loc}}} & \mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}} \\ \downarrow & & \downarrow t_{\leftarrow} \times t_{\rightarrow} \\ \mathbf{BL}^+G & \xrightarrow{1 \times \sigma} & \mathbf{BL}^+G \times \mathbf{BL}^+G. \end{array}$$

Take two additional (not necessarily non-empty) sequences ν_\bullet, ξ_\bullet of dominant coweights. We similarly define $\mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}}$ and $\mathrm{Sht}_{\nu_\bullet; \lambda_\bullet | \mu_\bullet; \xi_\bullet}^{0, \mathrm{loc}}$ and the *Satake correspondence*

$$\mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}} \xleftarrow{h_{\lambda_\bullet}^{\leftarrow}} \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}} \xrightarrow{h_{\mu_\bullet}^{\rightarrow}} \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}}.$$

For example, $\mathrm{Sht}_{\nu_\bullet; \lambda_\bullet | \mu_\bullet; \xi_\bullet}^{0, \mathrm{loc}}$ classifies the following commutative diagrams of modifications.

$$\begin{array}{ccccccc} & & \mathcal{E}'_{s+a} & \xrightarrow{\beta'_{s+a}} & \dots & \xrightarrow{\beta'_{a+1}} & \mathcal{E}'_a \\ & & \parallel & & & & \parallel \\ \mathcal{E}_{t+a+b} & \xrightarrow{\beta_{t+a+b}} & \dots & \xrightarrow{\beta_{t+a+1}} & \mathcal{E}_{t+a} & \xrightarrow{\beta_{t+a}} & \dots & \xrightarrow{\beta_{a+1}} & \mathcal{E}_a & \xrightarrow{\beta_a} & \dots & \xrightarrow{\beta_1} & \mathcal{E}_0 = {}^\sigma \mathcal{E}_{t+a+b}. \end{array}$$

Remark 5.2.2. (1) In mixed characteristic, a local shtuka $\mathcal{E} \dashrightarrow {}^\sigma \mathcal{E}$ over $\mathrm{Spec} R$ is nothing but an F -crystal with G -structures. In particular, If $G = \mathrm{GL}_n$ and $\mu = \omega_i$ is the i th fundamental coweight of GL_n , by a theorem of Gabber, $\mathrm{Sh}_{\mu}^{\mathrm{loc}}$ can be regarded as the moduli of p -divisible groups of height

n and dimension $n - i$ over k . Therefore, moduli of local shtukas in mixed characteristic can be regarded as vast generalizations of moduli of p -divisible groups in characteristic p .

(2) What we just defined are local shtukas with singularities at the closed point $s \in D$. One can also define local shtukas with singularities at the generic point $\eta \in D$, or even with singularities moving along D . In mixed characteristic, local shtukas with singularities along D are closely related to the Breuil-Kisin modules and their moduli spaces are recently constructed by Scholze ([SW]).

Remark 5.2.3. Our convention to define local iterated shtukas using modifications from \mathcal{E}_t to ${}^\sigma\mathcal{E}_t$ agrees [Va04, Laf.V12⁺], but is incompatible with the definition of affine Deligne-Lusztig varieties in Section 4 (following most literatures on Shimura varieties), where the direction of the modification is from ${}^\sigma\mathcal{E}$ to \mathcal{E} . This will lead a switching from μ to μ^* when relating the two definitions (see (5.2.5) and (5.2.6)). The reason for our choice is limited by the need of considering restricted local shtukas in the next subsection. When comparing that with the de Rham homology of the universal abelian varieties on Shimura varieties, the Grothendieck-Messing theory forces us to use a deeper truncation on ${}^\sigma\mathcal{E}$ than on \mathcal{E} . As we will see in Definition 5.3.1 that the deeper truncation must appear at the target of the modifications, and hence our choice of convention.

Example 5.2.4. If $\mu_\bullet = \mu = 0$, $\text{Sht}_0^{\text{loc}} = \mathbf{BG}(\mathcal{O})$ is the the classifying stack of the profinite group $G(\mathcal{O})$.

Example 5.2.5. Let $\mathcal{O} = W(k)$. Let $G = \text{GL}_n$, and $\mu = \omega_i = 1^i 0^{n-i}$ the i th fundamental coweight. Then by a theorem of Gabber, $\text{Sht}_{\omega_i}^{\text{loc}}$ is isomorphic to the prestack of p -divisible groups of dimension i , and height n over k .

Definition 5.2.6. In addition to the Satake correspondences, there is the following important *partial Frobenius morphism* between the moduli of local iterated shtukas, given by

$$(5.2.2) \quad F_{\mu_\bullet} : \text{Sht}_{\mu_1, \dots, \mu_t}^{\text{loc}} \longrightarrow \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}} \\ (\mathcal{E}_t \xrightarrow{\beta_t} \mathcal{E}_{t-1} \xrightarrow{\beta_{t-1}} \dots \xrightarrow{\beta_2} \mathcal{E}_1 \xrightarrow{\beta_1} {}^\sigma\mathcal{E}_t) \longmapsto (\mathcal{E}_{t-1} \xrightarrow{\beta_{t-1}} \dots \xrightarrow{\beta_2} \mathcal{E}_1 \xrightarrow{\beta_1} {}^\sigma\mathcal{E}_t \xrightarrow{\sigma(\beta_t)} {}^\sigma\mathcal{E}_{t-1}).$$

To motivate later definition of the (inverse) partial Frobenius between moduli of restricted local Shtukas. we explain in steps the inverse $F_{\mu_\bullet}^{-1}$ map on S -points for perfect affine scheme $S = \text{Spec } R$ as follows.

$$\begin{aligned} & \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}(S) \\ & \cong \{ (x, \psi) \mid x \in \text{Hk}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}(S) \text{ and } \psi : {}^\sigma\mathcal{E}_{x, \leftarrow} \simeq \mathcal{E}_{x, \rightarrow} \} & \text{(Definition 5.2.1)} \\ & \cong \left\{ (x_1, x_2, \psi', \psi) \left| \begin{array}{l} x_1 \in \text{Hk}_{\sigma(\mu_t)}(S), \ x_2 \in \text{Hk}_{\mu_1, \dots, \mu_{t-1}}(S), \\ \psi' : \mathcal{E}_{x_1, \leftarrow} \simeq \mathcal{E}_{x_2, \rightarrow}, \text{ and} \\ \psi : {}^\sigma\mathcal{E}_{x_2, \leftarrow} \simeq \mathcal{E}_{x_1, \rightarrow} \end{array} \right. \right\} & \left(\begin{array}{l} \text{Use §5.1.8} \\ \text{to replace } x \\ \text{by } (x_1, x_2, \psi') \end{array} \right) \\ & \cong \left\{ (x'_1, x_2, \psi', \psi'') \left| \begin{array}{l} x'_1 \in \text{Hk}_{\mu_t}(S), \ x_2 \in \text{Hk}_{\mu_1, \dots, \mu_{t-1}}(S), \\ \psi' : {}^\sigma\mathcal{E}_{x'_1, \leftarrow} \simeq \mathcal{E}_{x_2, \rightarrow}, \text{ and} \\ \psi'' : \mathcal{E}_{x_2, \leftarrow} \simeq \mathcal{E}_{x'_1, \rightarrow} \end{array} \right. \right\} & \left(\begin{array}{l} x'_1 = \sigma^{-1}(x_1) \\ \psi'' = \sigma^{-1}(\psi) \end{array} \right) \\ & \cong \left\{ (x', \psi'') \left| \begin{array}{l} x' \in \text{Hk}_{\mu_1, \dots, \mu_t}(S) \text{ and} \\ \psi' : {}^\sigma\mathcal{E}_{x', \leftarrow} \simeq \mathcal{E}_{x', \rightarrow} \end{array} \right. \right\} & \left(\begin{array}{l} \text{Use §5.1.8 to glue} \\ (x_2, x_1, \psi'') \text{ into } x' \end{array} \right) \\ & \cong \text{Sht}_{\mu_1, \dots, \mu_t}(S). & \text{(Definition 5.2.1)} \end{aligned}$$

Similarly, for dominant coweights $\lambda_\bullet, \mu_\bullet, \nu$, there is a natural partial Frobenius morphism

$$F_{\lambda_\bullet | \mu_\bullet; \nu} : \text{Sht}_{\lambda_\bullet | \mu_\bullet; \nu}^{0, \text{loc}} \rightarrow \text{Sht}_{\sigma(\nu); \lambda_\bullet | \mu_\bullet}^{0, \text{loc}}.$$

The morphism F_{μ_\bullet} (and $F_{\lambda_\bullet | \mu_\bullet; \nu}$) is an isomorphism of prestacks (i.e. as functors over *perfect* k -algebras).

Remark 5.2.7. In the equal characteristic case, the partial Frobenius morphism F_{μ_\bullet} can be defined for the imperfect versions of local shtukas, which then is a just universal homeomorphism.

Definition 5.2.8. Let λ_\bullet and μ_\bullet be two sequences of dominant coweights. We are interested in the following correspondence of prestacks

$$\begin{array}{ccc} & \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}} & \\ \swarrow \overleftarrow{h}_{\lambda_\bullet}^{\text{loc}} & & \searrow \overrightarrow{h}_{\mu_\bullet}^{\text{loc}} \\ \text{Sht}_{\lambda_\bullet}^{\text{loc}} & & \text{Sht}_{\mu_\bullet}^{\text{loc}}, \end{array}$$

where $\text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}}$ is the prestack classifying, for each perfect ring R , the following commutative diagram of modifications of G -torsors over D_R :

$$(5.2.3) \quad \begin{array}{ccccccc} \mathcal{E}'_s & - & \xrightarrow{\beta'_s} & \cdots & - & \xrightarrow{\beta'_2} & \cdots & - & \xrightarrow{\beta'_1} & \mathcal{E}'_1 & - & \xrightarrow{\sigma} & \mathcal{E}'_s \\ \downarrow \beta & & & & & & & & & & & \downarrow \sigma(\beta) \\ \mathcal{E}_t & - & \xrightarrow{\beta_t} & \cdots & - & \xrightarrow{\beta_2} & \cdots & - & \xrightarrow{\beta_1} & \mathcal{E}_1 & - & \xrightarrow{\sigma} & \mathcal{E}_t, \end{array}$$

such that the top row (resp. bottom row) defines an R -point of $\text{Sht}_{\lambda_\bullet}^{\text{loc}}$ (resp. $\text{Sht}_{\mu_\bullet}^{\text{loc}}$). We call such a diagram a *Hecke correspondence* from the top row to the bottom row.

If we further require β to have relative position $\preceq \nu$, we get a closed sub-prestack $\text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\nu, \text{loc}}$ of $\text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}}$. Note that if $\nu = 0$, this reduces to the previously defined Satake correspondence so our notations are consistent.

Remark 5.2.9. One should regard $\text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}}$ as the Hecke correspondence between the moduli of local shtukas associated to different (sequences of) coweights. It is non-empty if and only if $B(G, -|\lambda_\bullet|) \cap B(G, -|\mu_\bullet|) \neq \emptyset$.

5.2.10. *Points of Sht^{loc} and the Newton map.* Recall that we introduced the set $A(G)$, which is the quotient of $G(L)$ by σ -conjugation of elements in $G(\mathcal{O}_L)$. We define the bijection

$$(5.2.4) \quad \text{Sht}_\mu^{\text{loc}}(\bar{k}) \cong A(G, \mu)$$

as follows: each \bar{k} -point $x \in \text{Sht}_\mu^{\text{loc}}$, defines a local shtuka

$$x^* \mathcal{E} \dashrightarrow x^*(\sigma \mathcal{E})$$

over $\text{Spec } \bar{k}$: by trivializing $x^* \mathcal{E}$ over \mathcal{O}_L as a G -torsor, the map $x^*(\sigma \mathcal{E}) \dashrightarrow x^* \mathcal{E}^{26}$ is given by an element $b_x \in \overline{G(\mathcal{O}_L)} \varpi^{\mu^*} G(\mathcal{O}_L)$, which changes as $g^{-1} b_x \sigma(g)$ for $g \in G(\mathcal{O}_L)$ when the trivialization of $x^* \mathcal{E}$ changes. Therefore, $[b_x]$ is a well-defined element in $A(G, \mu)$. Conversely, given an element in $A(G, \mu)$, represented by an element in $b \in \overline{G(\mathcal{O}_L)} \varpi^{\mu^*} G(\mathcal{O}_L)$, we can define a local shtuka structure on the trivial G -torsor \mathcal{E}^0 as $b\sigma : \mathcal{E}^0 \dashrightarrow \sigma \mathcal{E}^0$. This gives the bijection.

Then we may reinterpret (4.1.8) as the Newton map

$$(5.2.5) \quad \mathcal{N} : \text{Sht}_\mu^{\text{loc}}(\bar{k}) \rightarrow B(G, \mu^*)$$

According to [RR96], for every $b \in B(G, \mu^*)$, there is a locally closed substack $\text{Sht}_{\mu, b}^{\text{loc}} \subset \text{Sht}_\mu^{\text{loc}}$ such that $\text{Sht}_{\mu, b}^{\text{loc}}(\bar{k}) = \mathcal{N}^{-1}(b)$.

²⁶Note that the arrow is reversed. See also Remark 5.2.3.

5.2.11. *Relation to affine Deligne-Lusztig varieties.* We have the following lemma.

Lemma 5.2.12. *The morphism $\overleftarrow{h}_{\lambda_\bullet}^{\text{loc}} : \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\nu, \text{loc}} \rightarrow \text{Sht}_{\lambda_\bullet}^{\text{loc}}$ is representable by a perfectly proper perfect scheme, and therefore $\overleftarrow{h}_{\lambda_\bullet}^{\text{loc}} : \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}} \rightarrow \text{Sht}_{\lambda_\bullet}^{\text{loc}}$ is ind-representable and ind-proper.*

Proof. For simplicity, we assume that both $\lambda_\bullet = \lambda$ and $\mu_\bullet = \mu$ are single coweight. Let $\text{Spec} R \rightarrow \text{Sht}_\lambda^{\text{loc}}$, giving by a local shtuka $\beta' : \mathcal{E}' \dashrightarrow {}^\sigma \mathcal{E}'$. Then $\text{Spec} R \times_{\text{Sht}_\lambda^{\text{loc}}} \text{Sht}_{\lambda|\mu}^{\text{loc}}$ classifies for every $x : \text{Spec} R' \rightarrow \text{Spec} R$, a modification $\beta : x^* \mathcal{E}' \dashrightarrow \mathcal{E}$ such that $\text{Inv}(\beta) \preceq \nu$ and $\text{Inv}(\sigma(\beta)\beta'\beta^{-1}) \preceq \mu$. All possible modifications β are represented by the relative Schubert variety Gr_ν over $\text{Spec} R$, and the condition $\text{Inv}(\sigma(\beta)\beta'\beta^{-1}) \preceq \mu$ defines a closed subscheme of it. \square

We describe fibers of $\overleftarrow{h}_{\lambda_\bullet}^{\text{loc}}$ over a κ -point x of $\text{Sht}_{\lambda_\bullet}^{\text{loc}}$, where κ is a finite extension (or an algebraic closure) of k . Let $b_x \in A(G, \mu)$ be the element as in (5.2.4). Then

$$(5.2.6) \quad (\overleftarrow{h}_{\lambda_\bullet}^{\text{loc}})^{-1}(x) \simeq X_{\mu^*, \nu^*}(b_x)$$

by sending β to β^{-1} . Similarly, the fiber $(\overleftarrow{h}_{\lambda_\bullet}^{\text{loc}})^{-1}(x) \cap \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\nu, \text{loc}}$ is then identified with $X_{\mu^*, \nu^*}(b_x)$.

In particular, when $\lambda_\bullet = \tau \in \mathbb{X}_\bullet(Z_G)$ is a single central coweight and $\mu_\bullet = \mu$ is a single coweight, $\text{Sht}_\tau^{\text{loc}}$ is identified with $\mathbf{B}J_\tau(\mathcal{O}) = \mathbf{B}G(\mathcal{O})$ as prestacks, and we get the following commutative diagram with the left square Cartesian

$$(5.2.7) \quad \begin{array}{ccccc} \text{pt} & \longleftarrow & X_{\mu^*, \nu^*}(\tau^*) & & \\ \downarrow & & \downarrow & \searrow & \\ \text{Sht}_\tau^{\text{loc}} & \longleftarrow & \text{Sht}_{\tau|\mu}^{\nu, \text{loc}} & \longrightarrow & \text{Sht}_\mu^{\text{loc}} \end{array} .$$

5.2.13. *Composition of isogenies of local shtukas.* There is a natural composition morphism

$$(5.2.8) \quad \text{Comp}^{\text{loc}} : \text{Sht}_{\kappa_\bullet|\lambda_\bullet}^{\text{loc}} \times_{\text{Sht}_{\lambda_\bullet}^{\text{loc}}} \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}} \longrightarrow \text{Sht}_{\kappa_\bullet|\mu_\bullet}^{\text{loc}} .$$

More precisely, a point of $\text{Sht}_{\kappa_\bullet|\lambda_\bullet}^{\text{loc}} \times_{\text{Sht}_{\lambda_\bullet}^{\text{loc}}} \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\text{loc}}$ is represented by the following two commutative squares glued along the middle row, and the map is given by forgetting the middle row but only remembering the composed arrows in left and right columns.

$$(5.2.9) \quad \begin{array}{c} \begin{array}{ccccccc} \mathcal{E}_r'' & \xrightarrow{\beta_r''} & \cdots & \xrightarrow{\beta_2''} & \xrightarrow{\beta_1''} & \xrightarrow{\beta_1''} & \xrightarrow{\sigma} \mathcal{E}_r'' \\ \downarrow \gamma' & & & & & & \downarrow \sigma(\gamma') \\ \mathcal{E}_s' & \xrightarrow{\beta_s'} & \cdots & \xrightarrow{\beta_2'} & \xrightarrow{\beta_1'} & \xrightarrow{\beta_1'} & \xrightarrow{\sigma} \mathcal{E}_s' \\ \downarrow \gamma & & & & & & \downarrow \sigma(\gamma) \\ \mathcal{E}_t & \xrightarrow{\beta_t} & \cdots & \xrightarrow{\beta_2} & \xrightarrow{\beta_1} & \xrightarrow{\beta_1} & \xrightarrow{\sigma} \mathcal{E}_t \end{array} \end{array}$$

Note that Comp^{loc} maps $\text{Sht}_{\kappa_\bullet|\lambda_\bullet}^{\nu', \text{loc}} \times_{\text{Sht}_{\lambda_\bullet}^{\text{loc}}} \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\nu, \text{loc}}$ to $\text{Sht}_{\kappa_\bullet|\mu_\bullet}^{\nu+\nu', \text{loc}}$.

In later sections, we need to replace $\text{Sht}_{\mu_\bullet}^{\text{loc}}$ by its restricted version and to upgrade the previous constructions into cohomological correspondences. For this purpose, we need to discuss other disguises of the Hecke correspondences and their compositions, via Satake correspondences and partial Frobenius morphisms.

The following is a simple yet important observation.

Lemma 5.2.14. *For any dominant coweight η such that $\eta \succeq |\lambda_\bullet| + \sigma(\nu)$ or $\eta \succeq |\mu_\bullet| + \nu$, we have the following commutative diagram of prestacks*

$$(5.2.10) \quad \begin{array}{ccccc} & & \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\nu, \text{loc}} & & \\ & \swarrow & & \searrow & \\ & \text{Sht}_{\lambda_\bullet|(\sigma(\nu^*), \eta)}^{0, \text{loc}} & & \text{Sht}_{(\eta, \nu^*)|\mu_\bullet}^{0, \text{loc}} & \\ & \swarrow \quad \downarrow & & \downarrow \quad \swarrow & \\ \text{Sht}_{\lambda_\bullet}^{\text{loc}} & \text{Sht}_{\sigma(\nu^*), \eta}^{\text{loc}} & \xrightarrow{F_{\eta, \nu^*}^{-1}} & \text{Sht}_{\eta, \nu^*}^{\text{loc}} & \text{Sht}_{\mu_\bullet}^{\text{loc}}, \end{array}$$

where the pentagon is Cartesian when composing F_{η, ν^*}^{-1} with the right vertical arrow.²⁷

In other words, the Hecke correspondence is the composition of a Satake correspondence, the partial Frobenius, and another Satake correspondence.

Proof. Recall that $\text{Sht}_{\lambda_\bullet|\mu_\bullet}^{\nu, \text{loc}}$ classifies the following commutative diagram of modifications of G -torsors

$$\begin{array}{ccccccc} \mathcal{E}'_s & \xrightarrow{\beta'_s} & \cdots & \xrightarrow{\beta'_2} & \mathcal{E}'_1 & \xrightarrow{\beta'_1} & \sigma \mathcal{E}'_s \\ \downarrow \beta & \searrow & & \searrow \alpha & & \searrow & \downarrow \sigma(\beta) \\ \mathcal{E}_t & \xrightarrow{\beta_t} & \cdots & \xrightarrow{\beta_2} & \mathcal{E}_1 & \xrightarrow{\beta_1} & \sigma \mathcal{E}_t, \end{array}$$

where comparing to (5.2.3), we added the diagonal arrow α which is forced to be equal to $\sigma(\beta) \circ \beta'_1 \circ \cdots \circ \beta'_s = \beta_1 \circ \cdots \circ \beta_t \circ \beta$. Since $\eta \succeq |\lambda_\bullet| + \sigma(\nu)$ or $\eta \succeq |\mu_\bullet| + \nu$ by assumption, automatically we have $\text{Inv}(\alpha) \preceq \eta$. This rectangular shape diagram is the same as the two commutative triangles glued along the diagonal map α . We know that

- the upper right triangle represents a point of $\text{Sht}_{\lambda_\bullet|(\sigma(\nu^*), \eta)}^{0, \text{loc}}$, and
- the lower left triangle represents a point of $\text{Sht}_{(\eta, \nu^*)|\mu_\bullet}^{0, \text{loc}}$.

The lemma follows from this. □

The same trick leads the following two lemmas.

Lemma 5.2.15. *Let $\lambda_\bullet, \mu_\bullet, \nu_\bullet, \xi_\bullet, \theta$ be dominant coweights such that $\theta \succeq |\lambda_\bullet| + |\mu_\bullet^*|$ or $\theta \succeq |\xi_\bullet| + |\nu_\bullet^*|$. Then there is a canonical isomorphism*

$$(5.2.11) \quad \text{Sht}_{(\lambda_\bullet, \nu_\bullet)|(\mu_\bullet, \xi_\bullet)}^{0, \text{loc}} \cong \text{Sht}_{\lambda_\bullet|(\mu_\bullet, \theta); \nu_\bullet}^{0, \text{loc}} \times_{\text{Sht}_{\mu_\bullet, \theta, \nu_\bullet}^{\text{loc}}} \text{Sht}_{\mu_\bullet; (\theta, \nu_\bullet)|\xi_\bullet}^{0, \text{loc}}.$$

Proof. The left hand side classifies the following commutative diagram of modifications of G -torsors

$$\begin{array}{ccccccc} \mathcal{E}'_{s+a} & \xrightarrow{\beta'_{s+a}} & \cdots & \xrightarrow{\beta'_{s+1}} & \mathcal{E}'_s & \xrightarrow{\beta'_s} & \cdots & \xrightarrow{\beta'_1} & \mathcal{E}'_0 = \sigma \mathcal{E}'_{s+a} \\ \parallel & & & & & & & & \parallel \\ \mathcal{E}_{t+b} & \xrightarrow{\beta_{t+b}} & \cdots & \xrightarrow{\beta_{t+1}} & \mathcal{E}_t & \xrightarrow{\beta_t} & \cdots & \xrightarrow{\beta_1} & \mathcal{E}_0 = \sigma \mathcal{E}_{t+b}. \end{array}$$

²⁷We could have stated the lemma in terms of the partial Frobenius morphism F_{η, ν^*} (as it is an isomorphism). But we shall later consider a restricted analogue of this lemma (see Definition 5.3.16), where only the analogue of the *inverse* of the partial Frobenius morphism is defined.

The right hand side classifies the following commutative diagram of modifications of G -torsors

$$\begin{array}{ccccccc}
& & & \mathcal{E}'_s & \xrightarrow{\beta'_s} & \cdots & \xrightarrow{\beta'_1} \mathcal{E}'_0 = {}^\sigma \mathcal{E}_{s+a} \\
& & & \parallel & & & \parallel \\
\mathcal{E}'_{s+a} & \xrightarrow{\beta'_{s+a}} \cdots \xrightarrow{\beta'_{s+1}} & \mathcal{E}'_s & \xrightarrow{\gamma} & \mathcal{E}_t & \xrightarrow{\beta_t} \cdots \xrightarrow{\beta_1} & \mathcal{E}_0 = {}^\sigma \mathcal{E}_{t+b} \\
& \parallel & & & \parallel & & \\
\mathcal{E}_{t+b} & \xrightarrow{\beta_{t+b}} \cdots \xrightarrow{\beta_{t+1}} & \mathcal{E}_t & & & &
\end{array}$$

where the top two rows define a point of $\text{Sht}_{\lambda_\bullet | (\mu_\bullet, \theta); \nu_\bullet}^{0, \text{loc}}$, the bottom two rows define a point of $\text{Sht}_{\mu_\bullet | (\theta, \nu_\bullet); \zeta_\bullet}^{0, \text{loc}}$, and they are glued along the middle row which defines a point of $\text{Sht}_{\mu_\bullet, \theta, \nu_\bullet}^{\text{loc}}$.

The two diagrams above differ by the modification $\gamma : \mathcal{E}'_s \dashrightarrow \mathcal{E}_t$ which is required to have relative position $\preceq \theta$ (which is superfluous if $\theta \succeq |\lambda_\bullet| + |\mu_\bullet^*|$ or $\theta \succeq |\zeta_\bullet| + |\nu_\bullet^*|$). Then the isomorphism of the lemma is obtained by adding/removing the modification γ . \square

Lemma 5.2.16. *Let $\lambda_\bullet, \mu_\bullet, \nu_\bullet, \zeta, \zeta_1, \zeta_2$ be dominant coweights such that $\zeta \succeq |\lambda_\bullet| + |\mu_\bullet^*| + |\nu_\bullet^*|$ or $\zeta \succeq \zeta_1 + \zeta_2$. Then there is a canonical isomorphism*

$$(5.2.12) \quad \text{Sht}_{\lambda_\bullet | (\mu_\bullet, \zeta, \nu_\bullet)}^{0, \text{loc}} \times_{\text{Sht}_{\mu_\bullet, \zeta, \nu_\bullet}^{\text{loc}}} \text{Sht}_{\mu_\bullet; \zeta | (\zeta_1, \zeta_2); \nu_\bullet}^{0, \text{loc}} \cong \text{Sht}_{\lambda_\bullet | (\mu_\bullet, \zeta_1, \zeta_2, \nu_\bullet)}^{0, \text{loc}}.$$

Proof. The right hand side classifies the following commutative diagram of modifications of G -torsors

$$\begin{array}{ccccccc}
\mathcal{E}'_s & \xrightarrow{\beta'_s} & \cdots & \cdots & \cdots & \xrightarrow{\beta'_2} & \mathcal{E}'_1 \xrightarrow{\beta'_1} \mathcal{E}'_0 = {}^\sigma \mathcal{E}'_s \\
\parallel & & & & & & \parallel \\
\mathcal{E}_{t+a+2} & \xrightarrow{\beta''_a} & \cdots & \xrightarrow{\beta''_1} & \mathcal{E}_{t+2} & \xrightarrow{\gamma_2} \mathcal{E}_{t+1} & \xrightarrow{\gamma_1} \mathcal{E}_t \xrightarrow{\beta_t} \cdots \xrightarrow{\beta_1} \mathcal{E}_0 = {}^\sigma \mathcal{E}_{t+a+2}.
\end{array}$$

The left hand side classifies the following commutative diagram of modifications of G -torsors

$$\begin{array}{ccccccc}
\mathcal{E}'_s & \xrightarrow{\beta'_s} & \cdots & \cdots & \cdots & \xrightarrow{\beta'_2} & \mathcal{E}'_1 \xrightarrow{\beta'_1} \mathcal{E}'_0 = {}^\sigma \mathcal{E}'_s \\
\parallel & & & & & & \parallel \\
\mathcal{E}_{t+a+2} & \xrightarrow{\beta''_a} & \cdots & \xrightarrow{\beta''_1} & \mathcal{E}_{t+2} & \xrightarrow{\gamma} & \mathcal{E}_t \xrightarrow{\beta_t} \cdots \xrightarrow{\beta_1} \mathcal{E}_0 = {}^\sigma \mathcal{E}_{t+a+2}, \\
& & & & \parallel & & \parallel \\
& & & & \mathcal{E}_{t+2} & \xrightarrow{\gamma_2} & \mathcal{E}_{t+1} \xrightarrow{\gamma_1} \mathcal{E}_t
\end{array}$$

where the top two rows define a point of $\text{Sht}_{\lambda_\bullet | (\mu_\bullet, \zeta, \nu_\bullet)}^{0, \text{loc}}$, bottom two rows define a point of $\text{Sht}_{\mu_\bullet; \zeta | (\zeta_1, \zeta_2); \nu_\bullet}^{0, \text{loc}}$, and they are glued along the middle row which defines a point of $\text{Sht}_{\mu_\bullet, \zeta, \nu_\bullet}^{\text{loc}}$.

The two diagrams above differ by the modification $\gamma : \mathcal{E}_{t+2} \dashrightarrow \mathcal{E}_t$ which is required to have relative position $\preceq \zeta$ (which is superfluous if $\zeta \succeq \zeta_1 + \zeta_2$ or $\zeta \succeq |\lambda_\bullet| + |\mu_\bullet^*| + |\nu_\bullet^*|$). Then the isomorphism of the lemma is obtained by adding/removing the modification γ . \square

Lemma 5.2.17. *Let $\lambda_\bullet, \mu_\bullet, \nu$ be dominant coweights. The following diagram is Cartesian*

$$(5.2.13) \quad \begin{array}{ccc} \mathrm{Sht}_{\sigma(\nu); \lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}} & \xrightarrow{F_{\lambda_\bullet | \mu_\bullet; \nu}^{-1}} & \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet; \nu}^{0, \mathrm{loc}} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{\sigma(\nu), \lambda_\bullet}^{\mathrm{loc}} & \xrightarrow{F_{\lambda_\bullet, \nu}^{-1}} & \mathrm{Sht}_{\lambda_\bullet, \nu}^{\mathrm{loc}} \end{array}$$

Proof. Clear. □

Now we can give another interpretation of $\mathrm{Comp}^{\mathrm{loc}}$.

Lemma 5.2.18. *The composition map $\mathrm{Comp}^{\mathrm{loc}}$ is the composition of the following maps*

$$\begin{aligned} & \mathrm{Sht}_{\kappa_\bullet | \lambda_\bullet}^{\nu', \mathrm{loc}} \times_{\mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}}} \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{\nu, \mathrm{loc}} \\ \cong & \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^*), \eta')}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta', \nu'^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\eta', \nu'^*) | \lambda_\bullet}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}}} \mathrm{Sht}_{\lambda_\bullet | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\eta, \nu^*) | \mu_\bullet}^{0, \mathrm{loc}} \quad (\text{Lemma 5.2.14}) \\ \xrightarrow{(1)} & \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^*), \eta')}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta', \nu'^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\eta', \nu'^*) | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\eta, \nu^*) | \mu_\bullet}^{0, \mathrm{loc}} \\ \cong & \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^*), \eta')}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta', \nu'^*}^{\mathrm{loc}}} \mathrm{Sht}_{\eta' | (\sigma(\nu^*), \theta); \nu'^*}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\sigma(\nu^*), \theta, \nu'^*}^{\mathrm{loc}}} \mathrm{Sht}_{\sigma(\nu^*) | (\theta, \nu'^*) | \eta}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\eta, \nu^*) | \mu_\bullet}^{0, \mathrm{loc}} \quad (\text{Lemma 5.2.15}) \\ \cong & \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^*), \eta')}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\sigma(\nu^*), \eta'}^{\mathrm{loc}}} \mathrm{Sht}_{\sigma(\nu^*) | (\sigma(\nu^*), \theta)}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\theta, \nu'^*, \nu^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\theta, \nu'^*) | \eta; \nu^*}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\eta, \nu^*) | \mu_\bullet}^{0, \mathrm{loc}} \quad (\text{Lemma 5.2.17}) \\ \cong & \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^*), \sigma(\nu^*), \theta)}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\theta, \nu'^*, \nu^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\theta, \nu'^*, \nu^*) | \mu_\bullet}^{0, \mathrm{loc}} \quad (\text{Lemma 5.2.16}) \\ \xrightarrow{(2)} & \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^* + \nu'^*), \theta)}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\theta, \nu^* + \nu'^*}^{\mathrm{loc}}} \mathrm{Sht}_{(\theta, \nu^* + \nu'^*) | \mu_\bullet}^{0, \mathrm{loc}} \\ \cong & \mathrm{Sht}_{\kappa_\bullet | \mu_\bullet}^{\nu + \nu', \mathrm{loc}}. \quad (\text{Lemma 5.2.14}) \end{aligned}$$

Here we choose $\eta' \succeq |\kappa_\bullet| + \sigma(\nu')$, $\eta \succeq |\lambda_\bullet| + \sigma(\nu)$, and $\theta \succeq \eta' + \sigma(\nu)$. The map (1) is induced by the composition of Satake correspondence

$$\mathrm{Sht}_{(\eta', \nu'^*) | \lambda_\bullet}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}}} \mathrm{Sht}_{\lambda_\bullet | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}} \rightarrow \mathrm{Sht}_{(\eta', \nu'^*) | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}},$$

and the map (2) is induced by the convolution of moduli of local shtukas

$$\mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^*), \sigma(\nu^*), \theta)}^{0, \mathrm{loc}} \rightarrow \mathrm{Sht}_{\kappa_\bullet | (\sigma(\nu^* + \nu'^*), \theta)}^{0, \mathrm{loc}}, \quad \mathrm{Sht}_{\theta, \nu'^*, \nu^*}^{\mathrm{loc}} \rightarrow \mathrm{Sht}_{\theta, \nu^* + \nu'^*}^{\mathrm{loc}},$$

$$\text{and } \mathrm{Sht}_{(\theta, \nu'^*, \nu^*) | \mu_\bullet}^{0, \mathrm{loc}} \rightarrow \mathrm{Sht}_{(\theta, \nu^* + \nu'^*) | \mu_\bullet}^{0, \mathrm{loc}}.$$

In particular, $\mathrm{Comp}^{\mathrm{loc}}$ is perfectly proper.

Proof. The fiber product of the Hecke correspondences in the first row can be decomposed by Lemma 5.2.14, and is the same as the moduli prestack that classifies the following commutative diagram of modifications of G -torsors

$$\begin{array}{ccccccc} \mathcal{E}_r'' & \xleftarrow{\beta_r''} & \cdots & \xleftarrow{\beta_2''} & \mathcal{E}_1'' & \xleftarrow{\beta_1''} & \sigma \mathcal{E}_r'' \\ \downarrow \gamma' & & & \searrow \alpha' & & & \downarrow \sigma(\gamma') \\ \mathcal{E}_s' & \xleftarrow{\beta_s'} & \cdots & \xleftarrow{\beta_2'} & \mathcal{E}_1' & \xleftarrow{\beta_1'} & \sigma \mathcal{E}_s' \\ \downarrow \gamma & & & \searrow \alpha & & & \downarrow \sigma(\gamma) \\ \mathcal{E}_t & \xleftarrow{\beta_t} & \cdots & \xleftarrow{\beta_2} & \mathcal{E}_1 & \xleftarrow{\beta_1} & \sigma \mathcal{E}_t \end{array}$$

The map (1) amounts to forgetting the maps in the middle row $\mathcal{E}'_s \xrightarrow{\beta'_s} \dots \xrightarrow{\beta'_1} \sigma \mathcal{E}'_s$. Next, we use Lemma 5.2.15 to rewrite the moduli problem as the commutative diagram

$$(5.2.14) \quad \begin{array}{ccccc} \mathcal{E}''_r & \xleftarrow{\beta''_r} & \dots & \xleftarrow{\beta''_2} & \mathcal{E}''_1 & \xleftarrow{\beta''_1} & \sigma \mathcal{E}''_r \\ \downarrow \gamma' & \searrow & & \searrow \alpha' & & & \downarrow \sigma(\gamma') \\ & & & \delta & & & \\ \mathcal{E}'_s & \xleftarrow{\gamma} & & \searrow \alpha & & & \sigma \mathcal{E}'_s \\ \downarrow \gamma & & & & & & \downarrow \sigma(\gamma) \\ \mathcal{E}_t & \xleftarrow{\beta_t} & \dots & \xleftarrow{\beta_2} & \mathcal{E}_1 & \xleftarrow{\beta_1} & \sigma \mathcal{E}_t \end{array}$$

by adding the modification $\delta : \mathcal{E}''_r \dashrightarrow \sigma \mathcal{E}_t$. The four triangles (from top to bottom) correspond to the four factors (from left to right) in the product of line 4. After that, we apply Lemma 5.2.17 twice to get

$$\begin{aligned} & - \times_{\mathrm{Sht}^{\mathrm{loc}}_{\eta', \nu'^*}} \mathrm{Sht}^{0, \mathrm{loc}}_{\eta' | (\sigma(\nu^*), \theta); \nu'^*} \times_{\mathrm{Sht}^{\mathrm{loc}}_{\sigma(\nu^*), \theta, \nu'^*}} \mathrm{Sht}^{0, \mathrm{loc}}_{\sigma(\nu^*); (\theta, \nu'^*) | \eta} \\ \cong & - \times_{\mathrm{Sht}^{\mathrm{loc}}_{\sigma(\nu'^*), \eta'}} \mathrm{Sht}^{0, \mathrm{loc}}_{\sigma(\nu'^*); \eta' | (\sigma(\nu^*), \theta)} \times_{\mathrm{Sht}^{\mathrm{loc}}_{\sigma(\nu^*), \theta, \nu'^*}} \mathrm{Sht}^{0, \mathrm{loc}}_{\sigma(\nu^*); (\theta, \nu'^*) | \eta} \quad \text{Lemma 5.2.17 on the first product} \\ \cong & - \times_{\mathrm{Sht}^{\mathrm{loc}}_{\sigma(\nu'^*), \eta'}} \mathrm{Sht}^{0, \mathrm{loc}}_{\sigma(\nu'^*); \eta' | (\sigma(\nu^*), \theta)} \times_{\mathrm{Sht}^{\mathrm{loc}}_{\theta, \nu'^*, \nu^*}} \mathrm{Sht}^{0, \mathrm{loc}}_{(\theta, \nu'^*) | \eta; \nu^*}. \quad \text{Lemma 5.2.17 on the second product} \end{aligned}$$

This does not change the moduli problem above, but only change how the gluing works with the inverse partial Frobenius morphism. Indeed, in line 4, the first and the third fiber product each involves an inverse partial Frobenius; in line 5, only the second product involves the inverse partial Frobenius, applied twice.

Going from line 5 to line 6, we apply Lemma 5.2.16, which amounts to forgetting the modifications α' and α from the diagram (5.2.14). We then arrive at the gluing of two big upper right and big lower left triangles. Finally, we use Satake correspondence pushforward (2) to replace modifications γ and γ' by their composite $\gamma \circ \gamma'$, and hence the modifications $\sigma(\gamma)$ and $\sigma(\gamma')$ by their composite $\sigma(\gamma \circ \gamma')$. We end up with the following diagram of modifications

$$\begin{array}{ccccc} \mathcal{E}''_r & \xleftarrow{\beta''_r} & \dots & \xleftarrow{\beta''_2} & \mathcal{E}''_1 & \xleftarrow{\beta''_1} & \sigma \mathcal{E}''_r \\ \downarrow \gamma\gamma' & \searrow & & \searrow \delta & & & \downarrow \sigma(\gamma\gamma') \\ & & & & & & \\ \mathcal{E}_t & \xleftarrow{\beta_t} & \dots & \xleftarrow{\beta_2} & \mathcal{E}_1 & \xleftarrow{\beta_1} & \sigma \mathcal{E}_t \end{array}$$

which is the same as $\mathrm{Sht}^{\nu + \nu', \mathrm{loc}}_{\kappa_\bullet | \mu_\bullet}$ by Lemma 5.2.14.

We finally remind the reader that the map $\mathrm{Comp}^{\mathrm{loc}}$ is defined, on the level of moduli spaces, to modify the diagram (5.2.9) by removing the middle row and replacing the left and right vertical arrows by their composites. This is exactly what we just explained in the composition of the series of maps of the lemma. The factorization of the map $\mathrm{Comp}^{\mathrm{loc}}$ is clear from this.

The perfect properness of $\mathrm{Comp}^{\mathrm{loc}}$ follows from the perfect properness of the maps (1) and (2), which are in turn consequences of perfect properness of convolution products and Satake correspondences for affine Grassmannians. \square

5.3. Moduli of restricted local shtukas. The definition of the moduli of local (iterated) shtukas in the previous subsection is convenient for various geometric constructions we need. However, as defined in this way, we cannot directly apply the usual ℓ -adic formalism to it. As mentioned in Example 5.2.4, if μ_\bullet is zero or more generally a cocharacter $\tau \in \mathbb{X}_\bullet(Z_G)$, then $\mathrm{Sht}^{\mathrm{loc}}_\tau \cong \mathbf{BG}(\mathcal{O})$ as prestacks, which is not perfectly of finite presentation. So we need an approximation of $\mathrm{Sht}^{\mathrm{loc}}_{\mu_\bullet}$.

Definition 5.3.1. Let μ_\bullet be a sequence of dominant coweights, and let (m, n) be a pair of non-negative integers such that $m - n$ is μ_\bullet -large integer. The moduli stack $\text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)}$ of (m, n) -restricted local iterated shtukas is defined as the stack that classifies for every perfect k -algebra R ,

- an R -point of $\text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$, and
- an isomorphism of $L^n G$ -torsors over $\text{Spec} R$

$$\psi : {}^\sigma(\mathcal{E}_\leftarrow|_{D_n}) \simeq (\mathcal{E}_\rightarrow|_{D_m})|_{D_n},$$

where $\mathcal{E}_\rightarrow|_{D_m}$ and $\mathcal{E}_\leftarrow|_{D_n}$ are the canonical $L^m G$ -torsor and the $L^n G$ -torsor over $\text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$ given by (5.1.4) and (5.1.5), respectively.

Forgetting the isomorphism ψ defines a natural morphism

$$(5.3.1) \quad \varphi^{\text{loc}(m, n)} : \text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)} \rightarrow \text{Hk}_{\mu_\bullet}^{\text{loc}(m)},$$

which exhibits $\text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)}$ as an $\text{Aut}({}^\sigma \mathcal{E}_\leftarrow|_{D_n})$ -torsor over $\text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$. So $\varphi^{\text{loc}(m, n)}$ is a perfectly smooth morphism of relative dimension $n \dim G$. In particular, $\text{Sht}_{\mu_\bullet}^{\text{loc}(m, 0)} = \text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$. Similarly to (5.2.1), we can write $\text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)}$ as a Cartesian product of $\text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$ with a Frobenius graph

$$(5.3.2) \quad \begin{array}{ccc} \text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)} & \xrightarrow{\varphi^{\text{loc}(m, n)}} & \text{Hk}_{\mu_\bullet}^{\text{loc}(m)} \\ \downarrow & & \downarrow t_\leftarrow \times \text{res}_n^m \circ t_\rightarrow \\ \mathbf{B}L^n G & \xrightarrow{1 \times \sigma} & \mathbf{B}L^n G \times \mathbf{B}L^n G. \end{array}$$

Let ν_\bullet, ξ_\bullet be two additional (not necessarily non-empty) sequences of dominant coweights. We similarly define $\text{Sht}_{\lambda_\bullet|\mu_\bullet}^{0, \text{loc}(m, n)}$ (resp. $\text{Sht}_{\nu_\bullet; \lambda_\bullet|\mu_\bullet; \xi_\bullet}^{0, \text{loc}(m, n)}$) to be the stack classifying, for each perfect k -algebra R ,

- an R -point of $\text{Hk}_{\lambda_\bullet|\mu_\bullet}^{0, \text{loc}(m)}$ (resp. $\text{Hk}_{\nu_\bullet; \lambda_\bullet|\mu_\bullet; \xi_\bullet}^{0, \text{loc}(m)}$) ;
- and an isomorphism $\psi : {}^\sigma(\mathcal{E}_\leftarrow|_{D_n}) \simeq (\mathcal{E}_\rightarrow|_{D_m})|_{D_n}$ (or equivalently $\psi : {}^\sigma(\mathcal{E}'_\leftarrow|_{D_n}) \simeq (\mathcal{E}'_\rightarrow|_{D_m})|_{D_n}$) of $L^n G$ -torsors (see Remark 5.1.6).

There is a Satake correspondence for restricted local shtukas that is compatible with the Satake correspondence of restricted local Hecke stacks, as shown by the following Cartesian diagram.

$$(5.3.3) \quad \begin{array}{ccccc} \text{Sht}_{\lambda_\bullet}^{\text{loc}(m, n)} & \xleftarrow{h_{\lambda_\bullet}^\leftarrow} & \text{Sht}_{\lambda_\bullet|\mu_\bullet}^{0, \text{loc}(m, n)} & \xrightarrow{h_{\mu_\bullet}^\rightarrow} & \text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)} \\ \downarrow \varphi^{\text{loc}(m, n)} & & \downarrow \varphi^{\text{loc}(m, n)} & & \downarrow \varphi^{\text{loc}(m, n)} \\ \text{Hk}_{\lambda_\bullet}^{\text{loc}(m)} & \xleftarrow{h_{\lambda_\bullet}^\leftarrow} & \text{Hk}_{\lambda_\bullet|\mu_\bullet}^{0, \text{loc}(m)} & \xrightarrow{h_{\mu_\bullet}^\rightarrow} & \text{Hk}_{\mu_\bullet}^{\text{loc}(m)}. \end{array}$$

In particular, all horizontal maps are perfectly proper, as this is true for the Satake correspondences (3.1.14) between Schubert varieties.

The convolution map of local Hecke stacks induces (5.1.2) induces

$$(5.3.4) \quad m : \text{Sht}_{\nu_\bullet, \mu_\bullet, \lambda_\bullet}^{\text{loc}(m, n)} \rightarrow \text{Sht}_{\nu_\bullet, |\mu_\bullet|, \lambda_\bullet}^{\text{loc}(m, n)},$$

which is perfectly proper

5.3.2. *Group theoretic description of $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)}$.* We give an explicit description of $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)}$ as a quotient stack. Consider the fiber product

$$(5.3.5) \quad \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n),\square} := \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)} \times_{\mathrm{Hk}_{\mu_\bullet}^{\mathrm{loc}(m)}} \mathrm{Gr}_{\mu_\bullet},$$

which is the stack classifying an R -point $x = (\mathcal{E}_t \dashrightarrow \cdots \dashrightarrow \mathcal{E}_0 = \mathcal{E}^0)$ of $\mathrm{Gr}_{\mu_\bullet}$ together with an isomorphism $\psi : \sigma(\mathcal{E}_t|_{D_n}) \simeq \mathcal{E}_0|_{D_n} = \mathcal{E}^0|_{D_n}$ of $L^n G$ -torsors. This is an $L^m G$ -torsor over $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)}$. Sending ψ to $\sigma^{-1}(\psi)$ gives rise to a trivialization of $\mathcal{E}_t|_{D_n}$, and hence a natural isomorphism

$$\begin{aligned} \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n),\square} &\xrightarrow{\cong} \mathrm{Gr}_{\mu_\bullet}^{(n)} \\ (x, \psi) &\longmapsto (x, \sigma^{-1}(\psi) : \mathcal{E}_t|_{D_n} \simeq \sigma^{-1} \mathcal{E}^0|_{D_n} \cong \mathcal{E}^0|_{D_n}). \end{aligned}$$

In addition, the action of $L^m G$ on $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n),\square}$ (to recover $\mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)}$) can be identified with the $L^m G$ -action on $\mathrm{Gr}_{\mu_\bullet}^{(n)}$ given by a twisted conjugation

$$(5.3.6) \quad c_{\sigma^{-1}}(g)(y) = gy\sigma^{-1}(\pi_{m,n}(g^{-1})), \quad \text{for } y \in \mathrm{Gr}_{\mu_\bullet}^{(n)}, \quad g \in L^m G,$$

where $\pi_{m,n} : L^m G \rightarrow L^n G$ is the natural map (see Definition 5.1.1). Taking the quotient with respect to this action, we obtain a canonical isomorphism

$$(5.3.7) \quad \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m,n)} \cong [\mathrm{Gr}_{\mu_\bullet}^{(n)} / c_{\sigma^{-1}} L^m G].$$

By a similar discussion, we can show that

$$\mathrm{Sht}_{\lambda_\bullet|\mu_\bullet}^{0,\mathbf{a},\mathrm{loc}(m,n)} \cong [\mathrm{Gr}_{\lambda_\bullet|\mu_\bullet}^{0,\mathbf{a},(n)} / c_{\sigma^{-1}} L^m G] \quad \text{and} \quad \mathrm{Sht}_{\nu_\bullet;\lambda_\bullet|\mu_\bullet;\xi_\bullet}^{0,\mathrm{loc}(m,n)} \cong [\mathrm{Gr}_{\nu_\bullet} \tilde{\times} \mathrm{Gr}_{\lambda_\bullet|\mu_\bullet} \tilde{\times} \mathrm{Gr}_{\xi_\bullet}^{0,(n)} / c_{\sigma^{-1}} L^m G].$$

Remark 5.3.3. In the equal characteristic case, the affine Grassmannians and the affine Deligne-Lusztig varieties admit canonical deperfections. So we can define an imperfect version of the moduli of restricted local iterated shtukas. For those, a trivialization of \mathcal{E}_\rightarrow for a truncated local shtukas does not give a trivialization $\sigma^{-1}(\psi)$ of $\mathcal{E}_\leftarrow|_{D_n}$ as we cannot use the inverse Frobenius map. We would need to consider the moduli space that classifies trivializations of both $\mathcal{E}_\rightarrow|_{D_m}$ and $\mathcal{E}_\leftarrow|_{D_n}$.

Example 5.3.4. When $\mu = 0$ or more generally $\mu = \tau \in \mathbb{X}_\bullet(Z_G)$, we have $\mathrm{Gr}_\tau^{(n)} \cong L^n G$ and the action (5.3.6) reduces to the σ^{-1} -conjugacy action of $L^m G$ on $L^n G$,

$$(g, h) \mapsto \pi_{m,n}(g)h\sigma^{-1}(\pi_{m,n}(g)^{-1}).$$

By Lang's theorem,

$$\mathrm{Sht}_\tau^{\mathrm{loc}(m,n)} \cong \mathbf{B}L_n^m G,$$

where $L_n^m G$ is the preimage of the discrete group $G(\mathcal{O}/\varpi^n) = L^n G(k) \subset L^n G$ under the natural projection $\pi_{m,n} : L^m G \rightarrow L^n G$.

Remark 5.3.5. The readers can skip this remark, which will only be used in Remark 7.2.5.

As mentioned in Remark 5.2.2, moduli of local shtukas can be regarded as a generalization of the moduli of p -divisible groups with G -structures. It is natural to expect that the moduli of restricted local shtukas generalize moduli of truncated Barsotti-Tate groups. It turns out that they are indeed closely related but the relation is slightly more complicated than the unrestricted case. Here we only explain the relation between $\mathrm{Sht}_\mu^{\mathrm{loc}(2,1)}$ for minuscule μ and the moduli of 1-truncated Barsotti-Tate groups with G -structures, or more precisely, the moduli of G -zips as defined in [PWZ15]. The relation between $\mathrm{Sht}_\mu^{\mathrm{loc}(m,n)}$ with the moduli of n -truncated Barsotti-Tate groups with G -structures is similar.

Let P_μ be the parabolic subgroup of G generated by root subgroups U_α for those $\langle \alpha, \mu \rangle \leq 0$, and $P_{-\mu}$ the opposite parabolic subgroup. Let U_μ and $U_{-\mu}$ be the unipotent radical of P_μ and $P_{-\mu}$

respectively, and let $L_\mu = P_\mu \cap P_{-\mu}$ be the Levi subgroup. Similar to (3.1.5), we have a perfectly smooth $L^+G \times \bar{G}^{\text{pf}}$ -morphism of relative dimension $\dim U_\mu$

$$\text{Gr}_\mu^{(1)} \rightarrow ((\bar{G}/\bar{U}_\mu \times \bar{G}/\bar{U}_{-\mu})/\bar{L}_\mu)^{\text{pf}}, \quad g_1 \varpi^\mu g_2 \mapsto (\bar{g}_1 \bmod \bar{U}_\mu^{\text{pf}}, g_2^{-1} \bmod \bar{U}_{-\mu}^{\text{pf}}) \bmod \bar{L}_\mu^{\text{pf}},$$

where $g_1 \in L^+G$, $g_2 \in \bar{G}^{\text{pf}} = L^1G$, and $\bar{g}_1 = g_1 \bmod L^+G^{(1)}$. Note that this map intertwines the action of L^2G on $\text{Gr}_\mu^{(1)}$ given in (5.3.6), and the left action of \bar{G} on $((\bar{G}/\bar{U}_\mu \times \bar{G}/\bar{U}_{-\mu})/\bar{L}_\mu)^{\text{pf}}$ given by $g \cdot (g_1, g_2) = (gg_1, \sigma^{-1}(g)g_2)$. It follows that there is a perfectly smooth morphism of relative dimension $-\dim P_\mu$

$$\text{Sht}_\mu^{\text{loc}(2,1)} \rightarrow [\bar{G}^{\text{pf}} \backslash (G/\bar{U}_\mu \times \bar{G}/\bar{U}_{-\mu})^{\text{pf}} / \bar{L}_\mu^{\text{pf}}].$$

Note that the codomain of this map is nothing but the perfection of the moduli space G -Zip $_\mu$ of G -zips. We thus obtain the following lemma.

Lemma 5.3.6. *Assume that μ is minuscule. There is a natural perfectly smooth map*

$$\text{Sht}_\mu^{\text{loc}(2,1)} \rightarrow G\text{-Zip}_\mu^{\text{pf}}.$$

of relative dimension $-\dim P_\mu$.

Construction 5.3.7. Let (m', n') and (m, n) be two pairs of non-negative integers such that $m \leq m'$, $n \leq n'$ are both $m' - n'$ and $m - n$ are μ_\bullet -large ($(m', n') = (\infty, \infty)$ allowed). We construct the *restriction morphism*

$$(5.3.8) \quad \text{res}_{m,n}^{m',n'} : \text{Sht}_{\mu_\bullet}^{\text{loc}(m',n')} \longrightarrow \text{Sht}_{\mu_\bullet}^{\text{loc}(m,n)}$$

as follows. We think of (restricted) local shtukas as the fiber product of the local Hecke stack and the Frobenius graph over the classifying space of (restricted) torsors. By (5.1.3) and the right square of (5.1.6), the following diagram is commutative

$$(5.3.9) \quad \begin{array}{ccccc} \text{Hk}_{\mu_\bullet}^{\text{loc}(m')} & \xrightarrow{t_\leftarrow \times t_\rightarrow} & \mathbf{BL}^{n'}G \times \mathbf{BL}^{m'}G & \xrightarrow{\text{id} \times \text{res}_{n'}^{m'}} & \mathbf{BL}^{n'}G \times \mathbf{BL}^{n'}G \\ \text{res}_m^{m'} \downarrow & & \text{res}_n^{n'} \times \text{res}_m^{m'} \downarrow & & \downarrow \text{res}_n^{n'} \times \text{res}_n^{n'} \\ \text{Hk}_{\mu_\bullet}^{\text{loc}(m)} & \xrightarrow{t_\leftarrow \times t_\rightarrow} & \mathbf{BL}^nG \times \mathbf{BL}^mG & \xrightarrow{\text{id} \times \text{res}_n^m} & \mathbf{BL}^nG \times \mathbf{BL}^nG. \end{array}$$

This means that, starting with an R -point x of $\text{Hk}_{\mu_\bullet}^{\text{loc}(m')}$, the canonical $L^{m'}G$ -torsor (5.1.4) and the $L^{n'}G$ -torsor (5.1.5) associated to x restrict to the corresponding torsors associated to the point $\text{res}_m^{m'}(x)$. Thus, an isomorphism $\psi : {}^\sigma(\mathcal{E}_\leftarrow|_{D_{n',R}}) \simeq (\mathcal{E}_\rightarrow|_{D_{m',R}})|_{D_{n',R}}$ (which will lift x to an R -point of $\text{Sht}_{\mu_\bullet}^{\text{loc}(m',n')}$) will naturally give an isomorphism $\psi|_{D_n} : {}^\sigma(\mathcal{E}_\leftarrow|_{D_{n,R}}) \simeq (\mathcal{E}_\rightarrow|_{D_{m,R}})|_{D_{n,R}}$ (which will lift $\text{res}_m^{m'}(x)$ to an R -point of $\text{Sht}_{\mu_\bullet}^{\text{loc}(m,n)}$). This defines the needed map $\text{res}_{m,n}^{m',n'}$. Equivalently, we may define the map as the fiber product of the Cartesian diagram with the Frobenius graph:

$$\begin{aligned} \text{res}_{m,n}^{m',n'} : \text{Sht}_{\mu_\bullet}^{\text{loc}(m',n')} &\stackrel{(5.3.2)}{\cong} \text{Hk}_{\mu_\bullet}^{\text{loc}(m')} \times_{t_\leftarrow \times \text{res}_n^{m'} \circ t_\rightarrow, \mathbf{BL}^{n'}G \times \mathbf{BL}^{n'}G, 1 \times \sigma} \mathbf{BL}^{n'}G \\ &\xrightarrow{\text{res}_m^{m'} \times \text{res}_n^{n'}} \text{Hk}_{\mu_\bullet}^{\text{loc}(m)} \times_{t_\leftarrow \times \text{res}_n^m \circ t_\rightarrow, \mathbf{BL}^nG \times \mathbf{BL}^nG, 1 \times \sigma} \mathbf{BL}^nG \stackrel{(5.3.2)}{\cong} \text{Sht}_{\mu_\bullet}^{\text{loc}(m,n)}. \end{aligned}$$

Note that $\text{res}_{m,0}^{m,n}$ is nothing but the previously introduced map $\varphi^{\text{loc}(m,n)}$ in (5.3.1). In addition, if $(m', n') = (\infty, \infty)$, we denote

$$(5.3.10) \quad \text{res}_{m,n} = \text{res}_{m,n}^{\infty,\infty} : \text{Sht}_{\mu_\bullet}^{\text{loc}} \rightarrow \text{Sht}_{\mu_\bullet}^{\text{loc}(m,n)}.$$

Note that by construction, for $m_1 \leq m_2 \leq m_3$, $n_1 \leq n_2 \leq n_3$ such that $m_i - n_i$ is μ_\bullet -large ($(m_3, n_3) = (\infty, \infty)$ being allowed),

$$(5.3.11) \quad \text{res}_{m_1,n_1}^{m_2,n_2} \circ \text{res}_{m_2,n_2}^{m_3,n_3} = \text{res}_{m_1,n_1}^{m_3,n_3}.$$

Moreover, the restriction morphisms are compatible with the Satake correspondences, i.e., we have the following Cartesian diagram $((m', n') = (\infty, \infty)$ allowed).

$$(5.3.12) \quad \begin{array}{ccccc} \mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}(m', n')} & \longleftarrow & \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m', n')} & \longrightarrow & \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m', n')} \\ \downarrow \mathrm{res}_{m, n}^{m', n'} & & \downarrow \mathrm{res}_{m, n}^{m', n'} & & \downarrow \mathrm{res}_{m, n}^{m', n'} \\ \mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}(m, n)} & \longleftarrow & \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m, n)} & \longrightarrow & \mathrm{Sht}_{\mu_\bullet}^{\mathrm{loc}(m, n)} \end{array}$$

Example 5.3.8. When $\tau \in \mathbb{X}_\bullet(Z_G)$ is a central cocharacter, the restriction map $\mathrm{res}_{m, n} : \mathrm{Sht}_\tau^{\mathrm{loc}} \cong \mathbf{BG}(\mathcal{O}) \rightarrow \mathrm{Sht}_\tau^{\mathrm{loc}(m, n)} \cong \mathbf{BL}_n^m G$ is induced by the natural map $G(\mathcal{O}) \rightarrow L_n^m G$.

Lemma 5.3.9. *There is the following commutative diagram with all squares Cartesian*

$$(5.3.13) \quad \begin{array}{ccccc} \mathrm{Sht}_{\kappa_\bullet | \lambda_\bullet}^{0, \mathrm{loc}} \times_{\mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}}} \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}} & \longrightarrow & \mathrm{Sht}_{\kappa_\bullet | \lambda_\bullet}^{0, \mathrm{loc}(m, n)} \times_{\mathrm{Sht}_{\lambda_\bullet}^{\mathrm{loc}(m, n)}} \mathrm{Sht}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m, n)} & \longrightarrow & \mathrm{Hk}_{\kappa_\bullet | \lambda_\bullet}^{0, \mathrm{loc}(m)} \times_{\mathrm{Hk}_{\lambda_\bullet}^{\mathrm{loc}(m)}} \mathrm{Hk}_{\lambda_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m+n)} \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Sht}_{\kappa_\bullet | \mu_\bullet}^{0, \mathrm{loc}} & \longrightarrow & \mathrm{Sht}_{\kappa_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m, n)} & \longrightarrow & \mathrm{Hk}_{\kappa_\bullet | \mu_\bullet}^{0, \mathrm{loc}(m)} \end{array}$$

Proof. Clear from the definition. \square

5.3.10. *Category of perverse sheaves on local shtukas.* We construct $\mathbf{P}(\mathrm{Sht}_k^{\mathrm{loc}})$ the category of perverse sheaves on the moduli of local shtukas (base changed to \bar{k}), in a way similar to §5.1.7.

Let μ_\bullet is a sequence of dominant coweights. For two pairs (m', n') , (m, n) of non-negative integers, satisfying $m \leq m', n \leq n'$ and both $m' - n'$ and $m - n$ are μ_\bullet -large ($m' \neq \infty$), let $\mathrm{Res}_{m, n}^{m', n'} = (\mathrm{res}_{m, n}^{m', n'})^*$ be the shifted pullback of ℓ -adic sheaves. By (5.3.11), there is a canonical isomorphism of functors

$$\mathrm{Res}_{m, n}^{m', n'} \circ \mathrm{Res}_{m', n'}^{m', n'} \cong \mathrm{Res}_{m, n}^{m', n'}.$$

Same as $\mathrm{Res}_m^{m'}$, the functor $\mathrm{Res}_{m, n}^{m', n'}$ induces an equivalence of categories if $m \geq 1$. On the other hand, $\mathrm{res}_{m', n'}^{m', n'}$ is a torsor under $\mathrm{Ker}(\mathrm{Aut}(\mathcal{E}_{\rightarrow} |_{D_{n'}}) \rightarrow \mathrm{Aut}(\mathcal{E}_{\rightarrow} |_{D_n}))$, which is again the perfection of a unipotent (non-constant) group scheme if $n \geq 1$, therefore, $\mathrm{Res}_{m', n'}^{m', n'}$ is fully faithful.

Then we can define the category of perverse sheaves on $\mathrm{Sht}^{\mathrm{loc}}$ as the direct sum of over the filtered limits

$$(5.3.14) \quad \mathbf{P}(\mathrm{Sht}_k^{\mathrm{loc}}) := \bigoplus_{\zeta \in \pi_1(G)} \mathbf{P}(\mathrm{Sht}_\zeta^{\mathrm{loc}}), \quad \mathbf{P}(\mathrm{Sht}_\zeta^{\mathrm{loc}}) = \varinjlim_{(\mu, m, n)} \mathbf{P}(\mathrm{Sht}_\mu^{\mathrm{loc}(m, n)}),$$

where the limit is taken over the triples $\{(\mu, m, n) \in \zeta \times \mathbb{Z}_{\geq 0}^2 \mid m - n \text{ is } \mu\text{-large}\}$, with the product partial order, and the connecting functor is given by the following composite of fully faithful functors

$$\mathbf{P}(\mathrm{Sht}_\mu^{\mathrm{loc}(m, n)}) \xrightarrow{\mathrm{Res}_{m, n}^{m', n'}} \mathbf{P}(\mathrm{Sht}_\mu^{\mathrm{loc}(m', n')}) \xrightarrow{i_{\mu, \mu', *}} \mathbf{P}(\mathrm{Sht}_{\mu'}^{\mathrm{loc}(m', n')}),$$

and where $i_{\mu, \mu'} : \mathrm{Sht}_\mu^{\mathrm{loc}(m', n')} \rightarrow \mathrm{Sht}_{\mu'}^{\mathrm{loc}(m', n')}$ is the natural closed embedding. Note that compared the situation in §5.1.7, although the first functor is no longer an equivalence, these connecting functors still satisfy natural compatibility conditions given by proper or smooth base change so the limit indeed makes sense.

For each dominant coweight μ and a pair (m, n) such that $m - n$ is μ -large, we have a natural pullback functor

$$(5.3.15) \quad \Phi^{\mathrm{loc}(m, n)} := \mathrm{Res}_{m, 0}^{m, n} : \mathbf{P}(\mathrm{Hk}_\mu^{\mathrm{loc}(m)}) \rightarrow \mathbf{P}(\mathrm{Sht}_\mu^{\mathrm{loc}(m, n)}),$$

which commutes the above connecting morphism by (5.3.11) and the proper smooth base change. Taking the limit and taking the direct sum over $\pi_1(G)$, we obtain a well-defined functor

$$(5.3.16) \quad \Phi^{\text{loc}} : \text{P}(\text{Hk}_k^{\text{loc}}) \rightarrow \text{P}(\text{Sht}_k^{\text{loc}}).$$

Remark 5.3.11. Note that there are many objects in $\text{P}(\text{Sht}_k^{\text{loc}})$ that do not come from $\text{P}(\text{Hk}_k^{\text{loc}})$ under the pullback Φ^{loc} . For example, recall from Example 5.3.4 that $\text{Sht}_0^{\text{loc}(n,n)} = \mathbb{B}G(\mathcal{O}/\varpi^n)$. Therefore, every representation ρ of the finite group $G(\mathcal{O}/\varpi^n)$ defines a local system \mathcal{L}_ρ on $\text{Sht}_0^{\text{loc}(m,n)}$, and therefore an object in $\text{P}(\text{Sht}_0^{\text{loc}(n,n)}) \rightarrow \text{P}(\text{Sht}_k^{\text{loc}})$. This object does not lie in the essential image of Φ^{loc} (as soon as $n > 0$).

Construction 5.3.12. A crucial ingredient we need later is the existence of a “partial Frobenius morphism” between the moduli of *restricted* local iterated shtukas that is compatible with the F_{μ_\bullet} defined in (5.2.2) via the restriction morphism (5.3.8). In fact, for technical reasons (i.e. to apply the formalism of cohomological correspondences), we will instead construct a restricted version of the *inverse* of the partial Frobenius F_{μ_\bullet} .

We fix a sequence of dominant coweights $\mu_\bullet = (\mu_1, \dots, \mu_t)$ as above. A quadruple of non-negative integers (m_1, n_1, m_2, n_2) is said μ_\bullet -*acceptable* if

- (1) $m_1 - m_2 = n_1 - n_2$ are μ_t -large (or equivalently $\sigma(\mu_t)$ -large)²⁸,
- (2) $m_2 - n_1$ is $(\mu_1, \dots, \mu_{t-1})$ -large.

In particular, $m_1 - n_1$ is μ_\bullet -large. We regard $(\infty, \infty, \infty, \infty)$ to be μ_\bullet -acceptable for any μ_\bullet .

Now we fix a μ_\bullet -acceptable (m_1, n_1, m_2, n_2) and construct a natural morphism

$$(5.3.17) \quad F_{\mu_\bullet}^{-1} : \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_1, n_1)} \rightarrow \text{Sht}_{\mu_1, \dots, \mu_t}^{\text{loc}(m_2, n_2)},$$

which resembles Definition 5.2.6. For a perfect affine scheme $S = \text{Spec} R$, the map on the S -points is given as follows.

$$\begin{aligned} & \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_1, n_1)}(S) \\ & \cong \{ (x, \psi) \mid x \in \text{Hk}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_1)}(S) \text{ and } \psi : {}^\sigma \mathcal{E}_{x, \leftarrow} |_{D_{n_1, R}} \simeq \mathcal{E}_{x, \rightarrow} |_{D_{n_1, R}} \} \quad (\text{Definition 5.3.1}) \\ & \cong \left\{ (x_1, x_2, \psi', \psi) \left| \begin{array}{l} x_1 \in \text{Hk}_{\sigma(\mu_t)}^{\text{loc}(m_1)}(S), \ x_2 \in \text{Hk}_{\mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_2)}(S), \\ \psi' : \mathcal{E}_{x_1, \leftarrow} |_{D_{m_2, R}} \simeq \mathcal{E}_{x_2, \rightarrow} |_{D_{m_2, R}}, \text{ and} \\ \psi : {}^\sigma \mathcal{E}_{x_2, \leftarrow} |_{D_{n_1, R}} \simeq \mathcal{E}_{x_1, \rightarrow} |_{D_{n_1, R}} \end{array} \right. \right\} \quad \left(\begin{array}{l} \text{Use §5.1.8} \\ \text{to replace } x \\ \text{by } (x_1, x_2, \psi') \end{array} \right) \\ & \cong \left\{ (x'_1, x_2, \psi', \psi'') \left| \begin{array}{l} x'_1 \in \text{Hk}_{\mu_t}^{\text{loc}(m_1)}(S), \ x_2 \in \text{Hk}_{\mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_2)}(S), \\ \psi' : {}^\sigma \mathcal{E}_{x'_1, \leftarrow} |_{D_{m_2, R}} \simeq \mathcal{E}_{x_2, \rightarrow} |_{D_{m_2, R}}, \text{ and} \\ \psi'' : \mathcal{E}_{x_2, \leftarrow} |_{D_{n_1, R}} \simeq \mathcal{E}_{x'_1, \rightarrow} |_{D_{n_1, R}} \end{array} \right. \right\} \quad \left(\begin{array}{l} x'_1 = \sigma^{-1}(x_1) \\ \psi'' = \sigma^{-1}(\psi) \end{array} \right) \\ & \rightarrow \left\{ (x''_1, x_2, \psi''', \psi'') \left| \begin{array}{l} x''_1 \in \text{Hk}_{\mu_t}^{\text{loc}(n_1)}(S), \ x_2 \in \text{Hk}_{\mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_2)}(S), \\ \psi''' : {}^\sigma \mathcal{E}_{x''_1, \leftarrow} |_{D_{n_2, R}} \simeq \mathcal{E}_{x_2, \rightarrow} |_{D_{n_2, R}}, \text{ and} \\ \psi'' : \mathcal{E}_{x_2, \leftarrow} |_{D_{n_1, R}} \simeq \mathcal{E}_{x''_1, \rightarrow} |_{D_{n_1, R}} \end{array} \right. \right\} \quad \left(\begin{array}{l} x''_1 = \text{res}_{n_1}^{m_1}(x'_1) \\ \psi''' = \psi' |_{D_{n_2, R}} \end{array} \right) \\ & \cong \left\{ (x', \psi'') \left| \begin{array}{l} x' \in \text{Hk}_{\mu_1, \dots, \mu_t}^{\text{loc}(m_2)}(S) \text{ and} \\ \psi'' : {}^\sigma \mathcal{E}_{x', \leftarrow} |_{D_{n_2, R}} \simeq \mathcal{E}_{x', \rightarrow} |_{D_{n_2, R}} \end{array} \right. \right\} \quad \left(\begin{array}{l} \text{Use §5.1.8 to glue} \\ (x_2, x'_1, \psi'') \text{ into } x' \end{array} \right) \\ & \cong \text{Sht}_{\mu_1, \dots, \mu_t}^{\text{loc}(m_2, n_2)}(S). \quad (\text{Definition 5.3.1}) \end{aligned}$$

²⁸In fact it is enough to require both $m_1 - m_2$ and $n_1 - n_2$ are μ_t -large to define $F_{\mu_\bullet}^{-1}$. But this extra generality is not needed in the paper.

Note that the condition that $m_1 - m_2$ are $n_1 - n_2$ are μ_t -large ensured that ψ', ψ''' in the previous line makes sense, namely $\mathcal{E}_{x_1, \leftarrow |D_{m_2, R}}$ and $\mathcal{E}_{x'_1, \leftarrow |D_{n_2, R}}$ is canonically defined; the condition that $m_2 - n_1$ is (μ_1, \dots, μ_t) -large guarantees that the gluing in the last step is valid.

Similar to the construction above, for dominant coweights $\lambda_\bullet, \mu_\bullet, \nu$ and (m_1, n_1, m_2, n_2) which is λ_\bullet -acceptable and μ_\bullet -acceptable, there is a natural inverse partial Frobenius morphism

$$F_{\lambda_\bullet | \mu_\bullet; \nu}^{-1} : \text{Sht}_{\sigma(\nu); \lambda_\bullet | \mu_\bullet}^{0, \text{loc}(m_1, n_1)} \rightarrow \text{Sht}_{\lambda_\bullet | \mu_\bullet; \nu}^{0, \text{loc}(m_2, n_2)}$$

Lemma 5.3.13. *The map $F_{\mu_\bullet}^{-1}$ is equidimensionally perfectly smooth, of relative dimension 0.*²⁹

Proof. The only non-isomorphic arrow is given by a base change of the map $\text{res}_{n_1}^{m_1}$ (which gives a gerbe over a twist of $L^{m_1 - n_1} G^{(n_1)}$) and a restriction of an isomorphism of torsors (which gives a torsor over a twist of $L^{m_2 - n_2} G^{(n_2)}$). \square

Lemma 5.3.14. (1) *The following diagram is commutative (but not a Cartesian)*

$$(5.3.18) \quad \begin{array}{ccccc} \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_1, n_1)} & \xrightarrow{\varphi^{\text{loc}(m_1, n_1)}} & \text{Hk}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_1)} & \xrightarrow{(5.1.11)} & \text{Hk}_{\sigma(\mu_t)}^{\text{loc}(n_1)} \times \text{Hk}_{\mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_2)} \\ F_{\mu_\bullet}^{-1} \downarrow & & & & \uparrow \sigma \times \text{id} \\ \text{Sht}_{\mu_1, \dots, \mu_t}^{\text{loc}(m_2, n_2)} & \xrightarrow{\varphi^{\text{loc}(m_2, n_2)}} & \text{Hk}_{\mu_1, \dots, \mu_t}^{\text{loc}(m_2)} & \xrightarrow{(5.1.11)} & \text{Hk}_{\mu_t}^{\text{loc}(n_1)} \times \text{Hk}_{\mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_2)}. \end{array}$$

(2) *Let $(m'_1, n'_1, m'_2, n'_2) \geq (m_1, n_1, m_2, n_2)$ be two μ_\bullet -acceptable quadruples ($(m'_1, n'_1, m'_2, n'_2) = (\infty, \infty, \infty, \infty)$ being allowed), the following diagram is commutative (but not Cartesian in general)*

$$(5.3.19) \quad \begin{array}{ccc} \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m'_1, n'_1)} & \xrightarrow{F_{\mu_\bullet}^{-1}} & \text{Sht}_{\mu_1, \dots, \mu_t}^{\text{loc}(m'_2, n'_2)} \\ \text{res}_{m_1, n_1}^{m'_1, n'_1} \downarrow & & \downarrow \text{res}_{m_2, n_2}^{m'_2, n'_2} \\ \text{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\text{loc}(m_1, n_1)} & \xrightarrow{F_{\mu_\bullet}^{-1}} & \text{Sht}_{\mu_1, \dots, \mu_t}^{\text{loc}(m_2, n_2)}. \end{array}$$

Proof. Both follow by the construction. \square

Remark 5.3.15. We reinterpret the above construction in terms of group elements. Pick a (non-increasing) sequence of integers $(m_0 = m', m_1 = m, m_2, \dots, m_{t-1}, m_t = n', m_{t+1} = n)$ such that $m_i - m_{i+1}$ is μ_i -large for $i = 1, \dots, t-1$. (This is possible because $m - n'$ is $(\mu_1, \dots, \mu_{t-1})$ -large.)

Consider the action of $\prod_{i=1}^{t+1} L^{m_i} G$ on $\prod_{i=1}^t \text{Gr}_{\mu_i}^{(m_{i+1})}$ given by

$$(5.3.20) \quad (g_1, \dots, g_t, g_{t+1}) \cdot (h_1 L^+ G^{(m_2)}, h_2 L^+ G^{(m_3)}, \dots, h_t L^+ G^{(m_{t+1})}) \\ = (g_1 h_1 g_2^{-1} L^+ G^{(m_2)}, g_2 h_2 g_3^{-1} L^+ G^{(m_3)}, \dots, g_t h_t g_{t+1}^{-1} L^+ G^{(m_{t+1})}).$$

The space $\text{Gr}_{\mu_\bullet}^{(m_{t+1})}$ is the quotient of $\prod_{i=1}^t \text{Gr}_{\mu_i}^{(m_{i+1})}$ by the subgroup $\prod_{i=2}^t L^{m_i} G \subseteq \prod_{i=1}^{t+1} L^{m_i} G$. To get the moduli of truncated local shtukas, we consider an embedding $c_{\sigma^{-1}}^\bullet : \prod_{i=1}^t L^{m_i} G \rightarrow \prod_{i=1}^{t+1} L^{m_i} G$ given by

$$c_{\sigma^{-1}}^\bullet(g_1, \dots, g_t) = (g_1, \dots, g_t, \sigma^{-1}(\pi_{m_1, m_{t+1}}(g_1))).$$

Then we have (compare §5.3.2)

$$\text{Sht}_{\mu_\bullet}^{\text{loc}(m_1, m_{t+1})} \cong \left[\prod_{i=1}^t \text{Gr}_{\mu_i}^{(m_{i+1})} / c_{\sigma^{-1}}^\bullet \left(\prod_{i=1}^t L^{m_i} G \right) \right].$$

²⁹Note that a smooth morphism between stacks of relative dimension 0 need not be étale.

Now, it is straightforward to check that the following pair of morphisms between products of Grassmannians and groups are equivariant for the corresponding actions:

$$\begin{array}{ccc}
(g_0, \dots, g_{t-1}) & \xrightarrow{\quad\quad\quad} & (g_1, \dots, g_{t-1}, \sigma^{-1}(\pi_{m_0, m_t}(g_0))) \\
\prod_{i=0}^{t-1} L^{m_i} G & \xrightarrow{\quad\quad\quad} & \prod_{i=1}^t L^{m_i} G \\
\downarrow \scriptstyle c_{\sigma^{-1}}^{\bullet} & & \downarrow \scriptstyle c_{\sigma^{-1}}^{\bullet} \\
\mathrm{Gr}_{\sigma(\mu_t)}^{(m_1)} \times \prod_{i=1}^{t-1} \mathrm{Gr}_{\mu_i}^{(m_{i+1})} & \xrightarrow{\quad\quad\quad} & \prod_{i=1}^t \mathrm{Gr}_{\mu_i}^{(m_{i+1})} \\
(h_0 L^+ G^{(m_1)}, h_1 L^+ G^{(m_2)}, \dots, h_{t-1} L^+ G^{(m_t)}) & \xrightarrow{\quad\quad\quad} & (h_1 L^+ G^{(m_2)}, \dots, h_{t-1} L^+ G^{(m_t)}, \sigma^{-1}(h_0) L^+ G^{(m_{t+1})}).
\end{array}$$

Taking this quotients by the equivariant action gives the inverse partial Frobenius morphism

$$F_{\mu_{\bullet}}^{-1} : \mathrm{Sht}_{\sigma(\mu_t), \mu_1, \dots, \mu_{t-1}}^{\mathrm{loc}(m_0, m_t)} \rightarrow \mathrm{Sht}_{\mu_1, \dots, \mu_t}^{\mathrm{loc}(m_1, m_{t+1})},$$

which is canonically independent of the choice of numbers m_i .

Lemma 5.2.14 suggests that we can define a restricted version of $\mathrm{Sht}_{\lambda_{\bullet}|\mu_{\bullet}}^{\nu, \mathrm{loc}}$ to be certain fiber product (because a direct definition is unavailable). Here is the precise definition.

Definition 5.3.16. Let $\lambda_{\bullet}, \mu_{\bullet}$ be two sequences of dominant coweights, and ν a dominant coweight. Let (m_1, n_1, m_2, n_2) be a quadruple that is $(|\lambda_{\bullet}| + \nu, \nu)$ -acceptable and $(|\mu_{\bullet}| + \nu, \nu)$ -acceptable. We choose a dominant coweight η such that

- (a) either $\eta \succeq |\lambda_{\bullet}| + \sigma(\nu)$ or $\eta \succeq |\mu_{\bullet}| + \nu$, and
- (b) $m_2 - n_1$ is η -large.

In particular, Condition (b) above and Condition (1) in Construction 5.3.12 imply that $m_1 - n_1 = m_2 - n_2$ is $(\sigma(\nu^*), \eta)$ -large and (η, ν^*) -large. For example, we may take $\eta = |\lambda_{\bullet}| + \sigma(\nu)$ or $|\mu_{\bullet}| + \nu$.

We define the *Hecke correspondence for restricted local shtukas* $\mathrm{Sht}_{\lambda_{\bullet}|\mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)}$ so that the pentagon in the following diagram is Cartesian (when composing the left vertical arrow with F_{η, ν^*}^{-1})

$$\begin{array}{ccccc}
& & \mathrm{Sht}_{\lambda_{\bullet}|\mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)} & & \\
& \swarrow & & \searrow & \\
& \mathrm{Sht}_{\lambda_{\bullet}|\sigma(\nu^*), \eta}^{0, \mathrm{loc}(m_1, n_1)} & & \mathrm{Sht}_{(\eta, \nu^*)|\mu_{\bullet}}^{0, \mathrm{loc}(m_2, n_2)} & \\
& \swarrow \quad \downarrow & & \downarrow \quad \searrow & \\
\mathrm{Sht}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)} & \mathrm{Sht}_{\sigma(\nu^*), \eta}^{\mathrm{loc}(m_1, n_1)} & \xrightarrow{F_{\eta, \nu^*}^{-1}} & \mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}(m_2, n_2)} & \mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_2, n_2)}.
\end{array}$$

The algebraic stack $\mathrm{Sht}_{\lambda_{\bullet}|\mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)}$ is independent of the choice of the coweight η (as long as it satisfies conditions (a) and (b)). In addition, the following Lemma 5.3.17 (see also Lemma 5.3.20) shows that it is also independent of (m, n) (as soon as the conditions (1) and (2) are met), which justifies our notation.

The relation between $\mathrm{Sht}_{\lambda_{\bullet}|\mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)}$ as (m_1, n_1, m_2, n_2) varies is as follows.

Lemma 5.3.17. Assume that $(m'_1, n'_1, m'_2, n'_2) \geq (m_1, n_1, m_2, n_2)$ are two quadruples that are both $(|\lambda_{\bullet}| + \nu, \nu)$ -acceptable and $(|\mu_{\bullet}| + \nu, \nu)$ -acceptable ($(m'_1, n'_1, m'_2, n'_2) = (\infty, \infty, \infty, \infty)$ being allowed).

The following diagram is commutative. Moreover, all squares are Cartesian except the one marked with the letter “X”.

(5.3.22)

$$\begin{array}{ccccccc}
\mathrm{Sht}_{\lambda_{\bullet}}^{\mathrm{loc}(m'_1, n'_1)} & \longleftarrow & \mathrm{Sht}_{\lambda_{\bullet} | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}(m'_1, n'_1)} & \longleftarrow & \mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m'_1, n'_1)} & \longrightarrow & \mathrm{Sht}_{(\eta, \nu^*) | \mu_{\bullet}}^{0, \mathrm{loc}(m'_2, n'_2)} \longrightarrow \mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m'_2, n'_2)} \\
\mathrm{res}_{m'_1, n'_1}^{m'_1, n'_1} \downarrow & & \downarrow & & \mathrm{res}_{m'_1, n'_1}^{m'_1, n'_1} \downarrow & \text{X} & \downarrow \mathrm{res}_{m'_2, n'_2}^{m'_2, n'_2} \\
\mathrm{Sht}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)} & \longleftarrow & \mathrm{Sht}_{\lambda_{\bullet} | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}(m_1, n_1)} & \longleftarrow & \mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)} & \longrightarrow & \mathrm{Sht}_{(\eta, \nu^*) | \mu_{\bullet}}^{0, \mathrm{loc}(m_2, n_2)} \longrightarrow \mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_2, n_2)}.
\end{array}$$

Proof. The commutativity of the diagram is clear. The left square and the right square are Cartesian by (5.3.12). That the second square is also Cartesian follows from a purely formal argument. To reduce the load of notations, we put

$$A = \mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)}, \quad B = \mathrm{Sht}_{\lambda_{\bullet} | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}(m_1, n_1)}, \quad C = \mathrm{Sht}_{(\eta, \nu^*) | \mu_{\bullet}}^{0, \mathrm{loc}(m_2, n_2)}, \quad \text{and} \quad D = \mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}(m_2, n_2)}.$$

Similar, we write A' , B' , C' , and D' when we replace (m_1, n_1, m_2, n_2) by (m'_1, n'_1, m'_2, n'_2) . Then in the following commutative diagram

$$\begin{array}{ccccc}
& & A' & \longrightarrow & B' \\
& \swarrow & \downarrow & & \swarrow \downarrow F^{-1} \\
A & \longrightarrow & B & & \\
\downarrow & & \downarrow & & \downarrow F^{-1} \\
& \swarrow & C' & \longrightarrow & D' \\
\downarrow & & \downarrow & & \downarrow \\
C & \longrightarrow & D & &
\end{array}$$

the front and the back faces are Cartesian by Definition 5.3.16 (or by Proposition 5.2.14 if $(m'_1, n'_1, m'_2, n'_2) = (\infty, \infty, \infty, \infty)$), and the bottom face is Cartesian by (5.3.12). (But note that the right face involving the two partial Frobenius morphisms is not Cartesian.) Now the lemma follows from Lemma A.2.9. \square

5.3.18. *Explicit description of $\mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)}$.* Let us analyze the morphism

$$(5.3.23) \quad \overleftarrow{h}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)} : \mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)},$$

parallel to the discussions in §5.2.11. First, since both morphisms

$$\mathrm{Sht}_{\lambda_{\bullet} | (\sigma(\nu^*), \eta)}^{0, \mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)}, \quad \text{and} \quad \mathrm{Sht}_{(\eta, \nu^*) | \mu_{\bullet}}^{0, \mathrm{loc}(m_2, n_2)} \rightarrow \mathrm{Sht}_{\eta, \nu^*}^{\mathrm{loc}(m_2, n_2)}$$

are perfectly proper (by the comments after (5.3.3)), $\overleftarrow{h}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)}$ is perfectly proper. This is the restricted counterpart of Lemma 5.2.12.

Remark 5.3.19. However, the morphism $\mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_2, n_2)}$ is not perfectly proper (because the restricted inverse partial Frobenius is not perfectly proper). This is the reason that there is an asymmetry in the diagram (5.3.22) above.

Note that since $\mathrm{Sht}_{\lambda_{\bullet} | \mu_{\bullet}}^{\nu, \mathrm{loc}(m_1, n_1)} = \mathrm{Sht}_{\lambda_{\bullet}}^{\mathrm{loc}(m_1, n_1)} \times_{\mathrm{Sht}_{|\lambda_{\bullet}|}}^{\nu, \mathrm{loc}(m_1, n_1)} \mathrm{Sht}_{|\lambda_{\bullet}| | |\mu_{\bullet}|}^{\nu, \mathrm{loc}(m_1, n_1)} \times_{\mathrm{Sht}_{|\mu_{\bullet}|}}^{\mathrm{loc}(m_1, n_1)} \mathrm{Sht}_{\mu_{\bullet}}^{\mathrm{loc}(m_2, n_2)}$, to simplify notations, we assume that $\lambda_{\bullet} = \lambda$ and $\mu_{\bullet} = \mu$ are single coweights in the following lemma. Recall the $L^{m_1}G$ -torsor $\mathrm{Sht}_{\lambda}^{\mathrm{loc}(m_1, n_1), \square} \simeq \mathrm{Gr}_{\lambda}^{(n_1)} \rightarrow \mathrm{Sht}_{\lambda}^{\mathrm{loc}(m_1, n_1)}$ from (5.3.5). Let

$$\mathrm{Sht}_{\lambda | \mu}^{\nu, \mathrm{loc}(m_1, n_1), \square} := \mathrm{Sht}_{\lambda | \mu}^{\nu, \mathrm{loc}(m_1, n_1)} \times_{\mathrm{Sht}_{\lambda}^{\mathrm{loc}(m_1, n_1)}} \mathrm{Sht}_{\lambda}^{\mathrm{loc}(m_1, n_1), \square}$$

be the pullback of the $L^{m_1}G$ -torsor to $\mathrm{Sht}_{\lambda | \mu}^{\nu, \mathrm{loc}(m_1, n_1)}$.

Lemma 5.3.20. *There is a natural $L^{m_1}G$ -equivariant isomorphism*

(5.3.24)

$\mathrm{Sht}_{\lambda|\mu}^{\nu, \mathrm{loc}(m_1, n_1), \square} \cong \{(g_1 L^+ G^{(n_1)}, g_2 L^+ G) \in \mathrm{Gr}_{\lambda}^{(n_1)} \times \mathrm{Gr}_{\sigma(\nu^*)} \mid g_2^{-1} g_1 \sigma^{-1}(g_2) \in [L^+ G \backslash \mathrm{Gr}_{\mu}^{(\infty)} / L^+ G]\}$
such that $\overleftarrow{h}_{\lambda}^{\mathrm{loc}(m_1, n_1), \square} : \mathrm{Sht}_{\lambda|\mu}^{\nu, \mathrm{loc}(m_1, n_1), \square} \rightarrow \mathrm{Sht}_{\lambda}^{\mathrm{loc}(m_1, n_1), \square}$ is identified with the projection to the first factor, where $L^{m_1}G$ acts on $\mathrm{Gr}_{\lambda}^{(n_1)} \times \mathrm{Gr}_{\sigma(\nu^)}$ as follows: it acts on $\mathrm{Gr}_{\lambda}^{(n_1)}$ as in (5.3.6), and on $\mathrm{Gr}_{\sigma(\nu^*)}$ by the usual left multiplication.*

Proof. We may let $\eta = \lambda + \sigma(\nu)$ in the definition of $\mathrm{Sht}_{\lambda|\mu}^{\nu, \mathrm{loc}(m_1, n_1)}$. First note that $\mathrm{Sht}_{\lambda|\mu}^{\nu, \mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{\lambda|(\sigma(\nu^*), \lambda + \sigma(\nu))}^{0, \mathrm{loc}(m_1, n_1)}$ is a closed embedding (since $\mathrm{Sht}_{(\lambda + \sigma(\nu), \nu^*)|\mu}^{0, \mathrm{loc}(m_2, n_2)} \rightarrow \mathrm{Sht}_{\lambda + \sigma(\nu), \nu^*}^{\mathrm{loc}(m_2, n_2)}$ is a closed embedding).

Next, by the discussion from §5.3.2, over $\mathrm{Sht}_{\lambda}^{\mathrm{loc}(m_1, n_1)} \leftarrow \mathrm{Sht}_{\lambda|(\sigma(\nu^*), \lambda + \sigma(\nu))}^{0, \mathrm{loc}(m_1, n_1)} \rightarrow \mathrm{Sht}_{\sigma(\nu^*), \lambda + \sigma(\nu)}^{\mathrm{loc}(m_1, n_1)}$, there are $L^{m_1}G$ -torsors

$$\mathrm{Gr}_{\lambda}^{(n_1)} \leftarrow \mathrm{Gr}_{\lambda|(\sigma(\nu^*), \lambda + \sigma(\nu))}^{0, (n_1)} \rightarrow \mathrm{Gr}_{\sigma(\nu^*), \lambda + \sigma(\nu)}^{(n_1)}.$$

In addition, there is a canonical isomorphism

$$\mathrm{Gr}_{\lambda}^{(n_1)} \times \mathrm{Gr}_{\sigma(\nu^*)} \cong \mathrm{Gr}_{\lambda|(\sigma(\nu^*), \lambda + \sigma(\nu))}^{0, (n_1)},$$

$$(\beta_1 : \mathcal{E}_1 \dashrightarrow \mathcal{E}^0, \epsilon_{n_1} : \mathcal{E}_1|_{D_{n_1}} \simeq \mathcal{E}^0|_{D_{n_1}}, \beta_2 : \mathcal{E}_2 \dashrightarrow \mathcal{E}^0) \rightarrow (\mathcal{E}_1 \xleftarrow[\beta_2^{-1}\beta_1]{\beta_1} \mathcal{E}_2 \xrightarrow[\beta_2]{} \mathcal{E}^0, \epsilon_{n_1}).$$

It is clear that under this isomorphism, the action of $L^{m_1}G$ on the right hand side is identified with the action on the left hand side as described in the lemma. Now the lemma follows from the construction of $F_{\lambda + \sigma(\nu), \nu^*}^{-1}$ (in particular in terms of group elements as in Remark 5.3.15). \square

Corollary 5.3.21. *Assume that $\lambda = \tau \in \mathbb{X}_{\bullet}(Z_G)$ is central. There is a canonical isomorphism*

$$\mathrm{Sht}_{\tau|\mu}^{\nu, \mathrm{loc}(m_1, n_1)} \cong [L_{n_1}^{m_1} G \backslash X_{\mu^*, \nu^*}(\tau^*)],$$

where $L_{n_1}^{m_1}G$ is the group defined in Example 5.3.4. Moreover, if $(m'_1, n'_1, m'_2, n'_2) \geq (m_1, n_1, m_2, n_2)$ is another $(\tau + \nu, \nu)$ -acceptable and $(\mu + \nu, \nu)$ -acceptable quadruple, we have the following natural Cartesian diagram.

$$\begin{array}{ccc} [L_{n'_1}^{m'_1} G \backslash \mathrm{pt}] \cong \mathrm{Sht}_{\tau}^{\mathrm{loc}(m'_1, n'_1)} & \longleftarrow & \mathrm{Sht}_{\tau|\mu}^{\nu, \mathrm{loc}(m'_1, n'_1)} \cong [L_{n'_1}^{m'_1} G \backslash X_{\mu^*, \nu^*}(\tau^*)] \\ \downarrow & & \downarrow \\ [L_{n_1}^{m_1} G \backslash \mathrm{pt}] \cong \mathrm{Sht}_{\tau}^{\mathrm{loc}(m_1, n_1)} & \longleftarrow & \mathrm{Sht}_{\tau|\mu}^{\nu, \mathrm{loc}(m_1, n_1)} \cong [L_{n_1}^{m_1} G \backslash X_{\mu^*, \nu^*}(\tau^*)], \end{array}$$

where the action of $L_{n_1}^{m_1}G$ on $X_{\mu^*, \nu^*}(\tau^*)$ is the σ^{-1} -twist of the usual left multiplication. Here $(m'_1, n'_1, m'_2, n'_2) = (\infty, \infty, \infty, \infty)$ is allowed, in which case $L_{\infty}^{\infty}G$ is interpreted as $G(\mathcal{O})$, and the isomorphism $[G(\mathcal{O}) \backslash \mathrm{pt}] \cong \mathrm{Sht}_{\tau}^{\mathrm{loc}}$ comes from (5.2.7).

Proof. Note that in (5.3.24),

$$\begin{aligned} \{(g_1 L^+ G^{(n_1)}, g_2 L^+ G) \in \mathrm{Gr}_{\lambda}^{(n_1)} \times \mathrm{Gr}_{\sigma(\nu^*)} \mid g_2^{-1} g_1 \sigma^{-1}(g_2) \in [L^+ G \backslash \mathrm{Gr}_{\mu}^{(\infty)} / L^+ G]\} \cong \\ \{(g_1 L^+ G^{(n_1)}, g_2 L^+ G) \in \mathrm{Gr}_{\lambda}^{(n_1)} \times \mathrm{Gr}_{\nu^*} \mid g_2^{-1} g_1^{-1} \sigma(g_2) \in [L^+ G \backslash \mathrm{Gr}_{\mu^*}^{(\infty)} / L^+ G]\}, \end{aligned}$$

where the isomorphism is induced by $\sigma : \mathrm{Gr}_{\nu^*} \simeq \mathrm{Gr}_{\sigma(\nu^*)}$. The corollary follows. \square

The final task of this subsection is to *define* the composition of Hecke correspondences between restricted local shtukas, analogous to Lemma 5.2.18. To slightly reduce the load of notation, we focus on the case that we will need later, namely when κ_\bullet , λ_\bullet and μ_\bullet are single coweights. The general case can be treated in a similar way. We first state the restricted versions of Lemmas 5.2.15, 5.2.16, and 5.2.17.

Lemma 5.3.22. (1) *Let $\lambda, \mu, \nu, \xi, \theta$ be dominant coweights such that $\theta \succeq \lambda + \mu^*$ or $\theta \succeq \xi + \nu^*$, and let (m, n) be a pair of non-negative integers such that $m - n$ is (μ, θ, ν) -large, (λ, ν) -large and (μ, ξ) -large. Then there is a canonical isomorphism*

$$(5.3.25) \quad \text{Sht}_{(\lambda, \nu)|(\mu, \xi)}^{0, \text{loc}(m, n)} \cong \text{Sht}_{\lambda|(\mu, \theta); \nu}^{0, \text{loc}(m, n)} \times_{\text{Sht}_{\mu, \theta, \nu}^{\text{loc}(m, n)}} \text{Sht}_{\mu; (\theta, \nu)|\xi}^{0, \text{loc}(m, n)}.$$

Moreover, the isomorphism obtained by replacing (m, n) by $(m', n') \geq (m, n)$ is the pullback of this isomorphism via the restriction map $\text{res}_{m, n}^{m', n'}$. In particular (5.2.11) is the pull back of (5.3.25) via $\text{res}_{m, n}$.

(2) *Let $\lambda, \mu, \nu, \zeta, \zeta_1, \zeta_2$ be dominant coweights such that $\zeta \succeq \lambda + \mu^* + \nu^*$ or $\zeta \succeq \zeta_1 + \zeta_2$, and let (m, n) be a pair of non-negative integers such that $m - n$ is λ -large, (μ, ζ, ν) -large, and $(\mu, \zeta_1, \zeta_2, \nu)$ -large, then*

$$(5.3.26) \quad \text{Sht}_{\lambda|(\mu, \zeta, \nu)}^{0, \text{loc}(m, n)} \times_{\text{Sht}_{\mu, \zeta, \nu}^{\text{loc}(m, n)}} \text{Sht}_{\mu; \zeta|(\zeta_1, \zeta_2); \nu}^{0, \text{loc}(m, n)} \cong \text{Sht}_{\lambda_\bullet|(\mu, \zeta_1, \zeta_2, \nu)}^{0, \text{loc}(m, n)}.$$

Moreover, the isomorphism obtained by replacing (m, n) by $(m', n') \geq (m, n)$ is the pullback of this isomorphism via the restriction map $\text{res}_{m, n}^{m', n'}$. In particular (5.2.12) is the pull back of (5.3.26) via $\text{res}_{m, n}$.

(3) *Let $\lambda_\bullet, \mu_\bullet, \nu$ be dominant coweights and let (m_1, n_1, m_2, n_2) be a quadruple of nonnegative integers that is $(|\lambda_\bullet| + \nu, \nu)$ -acceptable and $(|\mu_\bullet| + \nu, \nu)$ -acceptable. The following natural diagram is Cartesian*

$$(5.3.27) \quad \begin{array}{ccc} \text{Sht}_{\sigma(\nu); \lambda_\bullet | \mu_\bullet}^{0, \text{loc}(m_1, n_1)} & \xrightarrow{F_{\lambda_\bullet | \mu_\bullet; \nu}^{-1}} & \text{Sht}_{\lambda_\bullet | \mu_\bullet; \nu}^{0, \text{loc}(m_2, n_2)} \\ \downarrow & & \downarrow \\ \text{Sht}_{\sigma(\nu), \lambda_\bullet}^{\text{loc}(m_1, n_1)} & \xrightarrow{F_{\lambda_\bullet, \nu}^{-1}} & \text{Sht}_{\lambda_\bullet, \nu}^{\text{loc}(m_2, n_2)}. \end{array}$$

Moreover, the diagram obtained by replacing (m_1, n_1, m_2, n_2) by (m'_1, n'_1, m'_2, n'_2) is compatible with (5.3.27) via the restriction map $\text{res}_{m_1, n_1}^{m'_1, n'_1}$ and $\text{res}_{m_2, n_2}^{m'_2, n'_2}$. In particular, (5.3.27) is compatible with (5.2.13) via res_{m_1, n_1} and res_{m_2, n_2} .

Proof. (1) and (2) essentially follow from the same proof of Lemmas 5.2.15 and 5.2.16, respectively. More precisely, the proofs work to give corresponding statements when Sht^{loc} is replaced by Gr (with appropriate decorations). Taking the quotient by $L^m G$ under the σ -conjugacy action as in §5.3.2 gives (1) and (2). (3) is clear from the definition. \square

Proposition 5.3.23. *Let $\kappa, \lambda, \mu, \nu, \nu'$ be dominant coweights and let $(m_1, n_1, m_2, n_2, m_3, n_3)$ be a sextuple of non-negative integers satisfying*

- (m_1, n_1, m_2, n_2) is $(\kappa + \nu', \nu')$ -acceptable and $(\lambda + \nu', \nu')$ -acceptable;
- (m_2, n_2, m_3, n_3) is $(\lambda + \nu, \nu)$ -acceptable and $(\mu + \nu, \nu)$ -acceptable.

The following series of maps define a natural perfectly proper morphism

$$(5.3.28) \quad \text{Comp}_{\kappa, \lambda, \mu}^{\text{loc}(m_1, n_1)} : \text{Sht}_{\kappa | \lambda}^{\nu', \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda | \mu}^{\nu, \text{loc}(m_2, n_2)} \longrightarrow \text{Sht}_{\kappa | \mu}^{\nu + \nu', \text{loc}(m_1, n_1)}.$$

$$\begin{aligned}
& \text{Sht}_{\kappa|\lambda}^{\nu', \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_1, n_1)}} \text{Sht}_{\lambda|\mu}^{\nu, \text{loc}(m_2, n_2)} \\
& \cong \text{Sht}_{\kappa|(\sigma(\nu^*), \eta')}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\eta', \nu^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{(\eta', \nu^*)|\lambda}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda|(\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*)|\mu}^{0, \text{loc}(m_3, n_3)} \quad (\text{Defn 5.3.16}) \\
& \xrightarrow{(1)} \text{Sht}_{\kappa|(\sigma(\nu^*), \eta')}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\eta', \nu^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{(\eta', \nu^*)|(\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*)|\mu}^{0, \text{loc}(m_3, n_3)} \\
& \cong \text{Sht}_{\kappa|(\sigma(\nu^*), \eta')}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\eta', \nu^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\eta'|(\sigma(\nu^*), \theta); \nu^*}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\sigma(\nu^*), \theta, \nu^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\sigma(\nu^*); (\theta, \nu^*)|\eta}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*)|\mu}^{0, \text{loc}(m_3, n_3)} \quad (5.3.25) \\
& \cong \text{Sht}_{\kappa|(\sigma(\nu^*), \eta')}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\sigma(\nu^*), \eta'}^{\text{loc}(m_1, n_1)}} \text{Sht}_{\sigma(\nu^*); \eta'|(\sigma(\nu^*), \theta)}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\theta, \nu^*, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\theta, \nu^*)|\eta; \nu^*}^{0, \text{loc}(m_3, n_3)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*)|\mu}^{0, \text{loc}(m_3, n_3)} \quad (5.3.27) \\
& \cong \text{Sht}_{\kappa|(\sigma(\nu^*), \sigma(\nu^*), \theta)}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\theta, \nu^*, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\theta, \nu^*, \nu^*)|\mu}^{0, \text{loc}(m_3, n_3)} \quad (5.3.26) \\
& \xrightarrow{(2)} \text{Sht}_{\kappa|(\sigma(\nu^* + \nu'), \theta)}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\theta, \nu^* + \nu'}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\theta, \nu^* + \nu')|\mu}^{0, \text{loc}(m_3, n_3)} \\
& \cong \text{Sht}_{\kappa|\mu}^{\nu + \nu', \text{loc}(m_1, n_1)} \quad (\text{Defn 5.3.16})
\end{aligned}$$

Here we choose $\eta' \succeq \kappa + \sigma(\nu')$, $\eta \succeq \lambda + \sigma(\nu)$, and $\theta \succeq \eta' + \sigma(\nu)$ such that $m_2 - n_1$ is η' -large, $m_3 - n_2$ is η -large, and $m_3 - n_1$ is θ -large. The map (1) is induced by the composition of Satake correspondence

$$\text{Sht}_{(\eta', \nu^*)|\lambda}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda|(\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)} \rightarrow \text{Sht}_{(\eta', \nu^*)|(\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)},$$

and the map (2) is induced by the convolution map of moduli of local shtukas (5.3.4)

$$\text{Sht}_{\kappa|(\sigma(\nu^*), \sigma(\nu^*), \theta)}^{0, \text{loc}(m_1, n_1)} \rightarrow \text{Sht}_{\kappa|(\sigma(\nu^* + \nu'), \theta)}^{0, \text{loc}(m_1, n_1)}, \quad \text{Sht}_{\theta, \nu^*, \nu^*}^{\text{loc}(m_3, n_3)} \rightarrow \text{Sht}_{\theta, \nu^* + \nu'}^{\text{loc}(m_3, n_3)}, \quad \text{and } \text{Sht}_{(\theta, \nu^*, \nu^*)|\mu}^{0, \text{loc}(m_3, n_3)} \rightarrow \text{Sht}_{(\theta, \nu^* + \nu')|\mu}^{0, \text{loc}(m_3, n_3)}.$$

Moreover, if $(m'_1, n'_1, m'_2, n'_2, m'_3, n'_3) \geq (m_1, n_1, m_2, n_2, m_3, n_3)$ is another sextuple satisfying the same conditions as above ($(m'_1, n'_1, m'_2, n'_2, m'_3, n'_3) = (\infty, \infty, \infty, \infty, \infty, \infty)$ being allowed), then the following diagram is commutative.

$$\begin{array}{ccccc}
& & & \text{Sht}_{\kappa|\mu}^{\nu + \nu', \text{loc}(m'_1, n'_1)} & \\
& & \nearrow \text{Comp}^{\text{loc}(m'_1, n'_1)} & \downarrow & \searrow \\
\text{Sht}_{\kappa}^{\text{loc}(m'_1, n'_1)} & \xleftarrow{\text{Sht}_{\kappa|\lambda}^{\nu', \text{loc}(m'_1, n'_1)} \times \text{Sht}_{\lambda}^{\text{loc}(m'_2, n'_2)}} \text{Sht}_{\lambda|\mu}^{\nu, \text{loc}(m'_2, n'_2)} & \xrightarrow{\text{Sht}_{\lambda}^{\text{loc}(m'_2, n'_2)}} & \text{Sht}_{\mu}^{\text{loc}} & \\
\downarrow \text{res}_{m'_1, n'_1}^{m'_1, n'_1} & \downarrow & & \downarrow \text{res}_{m'_3, n'_3}^{m'_3, n'_3} & \\
\text{Sht}_{\kappa}^{\text{loc}(m_1, n_1)} & \xleftarrow{\text{Sht}_{\kappa|\lambda}^{\nu', \text{loc}(m_1, n_1)} \times \text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda|\mu}^{\nu, \text{loc}(m_2, n_2)} & \xrightarrow{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} & \text{Sht}_{\mu}^{\text{loc}(m_3, n_3)} & \\
& \searrow \text{Comp}^{\text{loc}(m_1, n_1)} & \downarrow & \nearrow & \\
& & \text{Sht}_{\kappa|\mu}^{\nu + \nu', \text{loc}(m_1, n_1)} & &
\end{array}$$

In addition, the middle trapezoid is Cartesian.

Remark 5.3.24. Note that the product $\text{Sht}_{\kappa|\lambda}^{\nu', \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda|\mu}^{\nu, \text{loc}(m_2, n_2)}$ is independent of (m_2, n_2) .

Proof of Proposition 5.3.23. The construction of the map $\text{Comp}^{\text{loc}(m_1, n_1)}$ is clear as explained at each step, and details can be found in the proof of Lemma 5.2.18 (for the unrestricted version).

We will only give a little more detail on the isomorphism between line 4 and line 5 (as it involves changing the depth of the restriction). Indeed, we have natural isomorphisms of functors.

$$\begin{aligned}
& - \times_{\text{Sht}_{\eta', \nu'^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\eta' | (\sigma(\nu^*), \theta); \nu'^*}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\sigma(\nu^*), \theta, \nu'^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\sigma(\nu^*); (\theta, \nu'^*) | \eta}^{0, \text{loc}(m_2, n_2)} \\
& \cong - \times_{\text{Sht}_{\sigma(\nu'^*), \eta'}^{\text{loc}(m_1, n_1)}} \text{Sht}_{\sigma(\nu'^*); \eta' | (\sigma(\nu^*), \theta)}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\sigma(\nu^*), \theta, \nu'^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\sigma(\nu^*); (\theta, \nu'^*) | \eta}^{0, \text{loc}(m_2, n_2)} \quad (5.3.27) \text{ on the first product} \\
& \cong - \times_{\text{Sht}_{\sigma(\nu'^*), \eta'}^{\text{loc}(m_1, n_1)}} \text{Sht}_{\sigma(\nu'^*); \eta' | (\sigma(\nu^*), \theta)}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\theta, \nu'^*, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\theta, \nu'^*) | \eta; \nu^*}^{0, \text{loc}(m_3, n_3)} \quad (5.3.27) \text{ on the second product.}
\end{aligned}$$

The perfect properness of $\text{Comp}^{\text{loc}(m_1, n_1)}$ follows from the perfect properness of the maps (1) and (2), which are in turn consequences of perfect properness of convolution products and Satake correspondences for affine Grassmannians.

The diagram (5.3.29) is clearly commutative by the definition of $\text{Comp}^{\text{loc}(m_1, n_1)}$ (and the description of Comp^{loc} via Lemma 5.2.18 if $(m'_1, n'_1, m'_2, n'_2, m'_3, n'_3) = (\infty, \infty, \infty, \infty, \infty, \infty)$). The trapezoid in (5.3.29) is Cartesian because pulling back along the restriction morphisms $\text{res}_{m'_1, n'_1}^{m'_1, n'_1} : \text{Sht}_{\kappa}^{\text{loc}(m'_1, n'_1)} \rightarrow \text{Sht}_{\kappa}^{\text{loc}(m_1, n_1)}$ of the series of maps for $(m_1, n_1, m_2, n_2, m_3, n_3)$ exactly gives the series of maps for $(m'_1, n'_1, m'_2, n'_2, m'_3, n'_3)$. For example, applying the base change $\text{Sht}_{\kappa}^{\text{loc}(m'_1, n'_1)} \times_{\text{Sht}_{\kappa}^{\text{loc}(m_1, n_1)}}$ to line 2, and repeated applying (5.3.12) and Lemma 5.3.17 to each of the product from the left to right gives

$$\begin{aligned}
& \text{Sht}_{\kappa}^{\text{loc}(m'_1, n'_1)} \times_{\text{Sht}_{\kappa}^{\text{loc}(m_1, n_1)}} \text{Sht}_{\kappa | (\sigma(\nu'^*), \eta')}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\eta', \nu'^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{(\eta', \nu'^*) | \lambda}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda | (\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*) | \mu}^{0, \text{loc}(m_3, n_3)} \\
& \cong \text{Sht}_{\kappa | (\sigma(\nu'^*), \eta')}^{0, \text{loc}(m'_1, n'_1)} \times_{\text{Sht}_{\eta', \nu'^*}^{\text{loc}(m'_1, n'_1)}} \text{Sht}_{(\eta', \nu'^*) | \lambda}^{0, \text{loc}(m'_2, n'_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda | (\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*) | \mu}^{0, \text{loc}(m_3, n_3)} \\
& \cong \text{Sht}_{\kappa | (\sigma(\nu'^*), \eta')}^{0, \text{loc}(m'_1, n'_1)} \times_{\text{Sht}_{\eta', \nu'^*}^{\text{loc}(m'_1, n'_1)}} \text{Sht}_{(\eta', \nu'^*) | \lambda}^{0, \text{loc}(m'_2, n'_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m'_2, n'_2)}} \text{Sht}_{\lambda}^{\text{loc}(m'_2, n'_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\lambda | (\sigma(\nu^*), \eta)}^{0, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m_3, n_3)}} \text{Sht}_{(\eta, \nu^*) | \mu}^{0, \text{loc}(m_3, n_3)} \\
& \cong \text{Sht}_{\kappa | (\sigma(\nu'^*), \eta')}^{0, \text{loc}(m'_1, n'_1)} \times_{\text{Sht}_{\eta', \nu'^*}^{\text{loc}(m'_1, n'_1)}} \text{Sht}_{(\eta', \nu'^*) | \lambda}^{0, \text{loc}(m'_2, n'_2)} \times_{\text{Sht}_{\lambda}^{\text{loc}(m'_2, n'_2)}} \text{Sht}_{\lambda | (\sigma(\nu^*), \eta)}^{0, \text{loc}(m'_2, n'_2)} \times_{\text{Sht}_{\eta, \nu^*}^{\text{loc}(m'_3, n'_3)}} \text{Sht}_{(\eta, \nu^*) | \mu}^{0, \text{loc}(m'_3, n'_3)}.
\end{aligned}$$

The rest can be checked similarly. \square

Lemma 5.3.25. *Let $\mu_1, \mu_2, \mu_3, \mu_4, \nu_1, \nu_2, \nu_3$ be dominant coweights and $(m_1, n_1, m_2, n_2, m_3, n_3, m_4, n_4)$ be an octuple of non-negative integers such that*

- $m_i - m_{i+1} = n_i - n_{i+1}$ is ν_i -large for $i = 1, 2, 3$,
- $m_i - n_{i+1}$ is (μ_i, ν_i) -large and (μ_{i+1}, ν_i) -large.

Then the following diagram commutes

$$\begin{array}{ccc}
\text{Sht}_{\mu_1 | \mu_2}^{\nu_1, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\mu_2 | \mu_3}^{\nu_2, \text{loc}(m_2, n_2)} \times_{\text{Sht}_{\mu_3}^{\text{loc}(m_3, n_3)}} \text{Sht}_{\mu_3 | \mu_4}^{\nu_3, \text{loc}(m_3, n_3)} & \xrightarrow{\text{Comp}_{\mu_1, \mu_2, \mu_3}^{\text{loc}(m_1, n_1)} \times \text{id}} & \text{Sht}_{\mu_1 | \mu_3}^{\nu_1 + \nu_2, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\mu_3}^{\text{loc}(m_3, n_3)}} \text{Sht}_{\mu_3 | \mu_4}^{\nu_3, \text{loc}(m_3, n_3)} \\
\downarrow \text{id} \times \text{Comp}_{\mu_2, \mu_3, \mu_4}^{\text{loc}(m_2, n_2)} & & \downarrow \text{Comp}_{\mu_1, \mu_3, \mu_4}^{\text{loc}(m_1, n_1)} \\
\text{Sht}_{\mu_1 | \mu_2}^{\nu_1, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\mu_2 | \mu_4}^{\nu_2 + \nu_3, \text{loc}(m_2, n_2)} & \xrightarrow{\text{Comp}_{\mu_1, \mu_2, \mu_4}^{\text{loc}(m_1, n_1)}} & \text{Sht}_{\mu_1 | \mu_4}^{\nu_1 + \nu_2 + \nu_3, \text{loc}(m_1, n_1)}.
\end{array}$$

Proof. Using the commutative diagram below (in which the superscripts $\text{loc}(m_i, n_i)$ are omitted), we see that the composition $\text{Comp}_{\mu_1, \mu_2, \mu_4}^{\text{loc}(m_1, n_1)} \circ (\text{id} \times \text{Comp}_{\mu_2, \mu_3, \mu_4}^{\text{loc}(m_2, n_2)})$ is the composition from upper left to lower left by first go to the upper right and then lower right. This is the same as the downwards composition directly from the upper left to lower left. By a symmetric argument, the composition of the downwards left column is the same as the composition $\text{Comp}_{\mu_1, \mu_3, \mu_4}^{\text{loc}(m_1, n_1)} \circ (\text{Comp}_{\mu_1, \mu_2, \mu_3}^{\text{loc}(m_1, n_1)} \times \text{id})$. This then proves the lemma.

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5.4. **The category $P^{\text{Corr}}(\text{Sht}_k^{\text{loc}})$.** We now define the category $P^{\text{Corr}}(\text{Sht}_k^{\text{loc}})$. It has the same objects as $P(\text{Sht}_k^{\text{loc}})$. And we just need to explain the space of morphisms and compositions of them as follows.

5.4.1. *Definition of morphisms.* Recall from §5.3.10 that every object of $P(\text{Sht}_k^{\text{loc}})$ (or equivalently every object of $P^{\text{Corr}}(\text{Sht}_k^{\text{loc}})$) is a direct sum of *connected objects* \mathcal{F} , i.e. objects that appear in the limit of $P(\text{Sht}_\mu^{\text{loc}(m,n)})$ over $(\mu, m, n) \in \zeta \times \mathbb{Z}_{\geq 0}^2$ for some $\zeta \in \pi_1(G)$. Each connected object \mathcal{F} is realized as a perverse sheaf $\mathcal{F}_\mu^{(m,n)} \in P(\text{Sht}_\mu^{\text{loc}(m,n)})$ for some triple (μ, m, n) such that $m - n$ is μ -large. For another triple $(\mu', m', n') \geq (\mu, m, n)$, we identify $\mathcal{F}_\mu^{(m,n)}$ with $\mathcal{F}_{\mu'}^{(m',n')} := i_{\mu,\mu'}^* \text{Res}_{m,n}^{m',n'} \mathcal{F}_\mu^{(m,n)}$.

Recall that for a septuple $(\mu_1, \mu_2, \nu, m_1, n_1, m_2, n_2)$ such that (m_1, n_1, m_2, n_2) is $(\mu_1 + \nu, \nu)$ -acceptable and $(\mu_2 + \nu, \nu)$ -acceptable, there is the correspondence

$$\text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)} \leftarrow \text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m_1, n_1)} \rightarrow \text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}$$

from Definition 5.3.16. If $(\mu'_1, \mu'_2, \nu', m'_1, n'_1, m'_2, n'_2) \geq (\mu_1, \mu_2, \nu, m_1, n_1, m_2, n_2)$ is another septuple satisfying the similar acceptability conditions, there is the following diagram

$$(5.4.1) \quad \begin{array}{ccccc} \text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)} & \xleftarrow{\text{res}_{m_1, n_1}^{m'_1, n'_1}} & \text{Sht}_{\mu_1}^{\text{loc}(m'_1, n'_1)} & \xrightarrow{i_{\mu_1, \mu'_1}} & \text{Sht}_{\mu'_1}^{\text{loc}(m'_1, n'_1)} \\ \uparrow & & \uparrow & & \uparrow \\ \text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m_1, n_1)} & \xleftarrow{\text{res}} & \text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m'_1, n'_1)} & \xrightarrow{i} & \text{Sht}_{\mu'_1 | \mu'_2}^{\nu', \text{loc}(m'_1, n'_1)} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)} & \xleftarrow{\text{res}_{m_2, n_2}^{m'_2, n'_2}} & \text{Sht}_{\mu_2}^{\text{loc}(m'_2, n'_2)} & \xrightarrow{i_{\mu_2, \mu'_2}} & \text{Sht}_{\mu'_2}^{\text{loc}(m'_2, n'_2)} \end{array}$$

where arrows to the left are equidimensional, perfectly smooth, of the same relative dimension, and arrows to the right are closed embeddings. Note that the upper left square is Cartesian by Lemma 5.3.17 and all arrows in the upper right square are perfectly proper.

We define the space $\text{Mor}_{P^{\text{Corr}}(\text{Sht}^{\text{loc}})}(\mathcal{F}_1, \mathcal{F}_2)$ of morphisms between two connected objects $\mathcal{F}_1, \mathcal{F}_2$ ($\mathcal{F}_i \in P(\text{Sht}_{\zeta_i}^{\text{loc}})$) to be

$$(5.4.2) \quad \bigoplus_{\zeta \in \pi_1(G)} \varinjlim_{(\mu_1, \mu_2, \nu, m_1, n_1, m_2, n_2)} \text{Corr}_{\text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m_1, n_1)}}((\text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)}, \mathcal{F}_{1, \mu_1}^{(m_1, n_1)}), (\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}, \mathcal{F}_{2, \mu_2}^{(m_2, n_2)})),$$

where the colimit is taken over all (partially ordered) sextuples $(\mu_1, \mu_2, \nu, m_1, n_1, m_2, n_2)$ satisfying the acceptability conditions as above (so that the space $\text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m_1, n_1)}$ is well-defined) and $\mu_1 \in \zeta_1, \mu_2 \in \zeta_2, \nu \in \zeta$. The connecting map of the filtered colimit is given by $i_! \circ \text{res}^*$, where res^* is the shifted smooth pullback of cohomological correspondences along the map res in (5.4.1) (in the sense of §A.2.11, in particular Notation A.2.12), and $i_!$ is the pushforward of cohomological correspondences along the map i (in the sense of §A.2.6) in (5.4.1). Note that for $(m''_1, n''_1) \geq (m'_1, n'_1)$, the following diagram is Cartesian

$$\begin{array}{ccc} \text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m''_1, n''_1)} & \xrightarrow{i} & \text{Sht}_{\mu'_1 | \mu'_2}^{\nu', \text{loc}(m''_1, n''_1)} \\ \text{res} \downarrow & & \downarrow \text{res} \\ \text{Sht}_{\mu_1 | \mu_2}^{\nu, \text{loc}(m'_1, n'_1)} & \xrightarrow{i} & \text{Sht}_{\mu'_1 | \mu'_2}^{\nu', \text{loc}(m'_1, n'_1)} \end{array}$$

Therefore, by Lemma A.2.17, Lemma A.2.14 and Lemma A.2.8, these connecting maps compose and the filtered colimit is well-defined.

Finally, we define the morphisms between general objects in $\mathbf{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})$ by extending the definition on the connected ones by linearity.

5.4.2. Composition of morphisms. We now define the composition of morphisms. By linearity, it suffices to define this on the connected objects and we may assume that the given morphisms $c_1 : \mathcal{F}_1 \rightarrow \mathcal{F}_2$ and $c_2 : \mathcal{F}_2 \rightarrow \mathcal{F}_3$ are realized as correspondences (as opposed to linear combinations of correspondences)

$$c_1^{(\mu_1, \mu_2, \nu_1, m_1, n_1, m_2, n_2)} \in \text{Corr}_{\text{Sht}_{\mu_1 | \mu_2}^{\nu_1, \text{loc}(m_1, n_1)}}((\text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)}, \mathcal{F}_{1, \mu_1}^{(m_1, n_1)}), (\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}, \mathcal{F}_{2, \mu_2}^{(m_2, n_2)})), \quad \text{and}$$

$$c_2^{(\mu_2, \mu_3, \nu_2, m_2, n_2, m_3, n_3)} \in \text{Corr}_{\text{Sht}_{\mu_2 | \mu_3}^{\nu_2, \text{loc}(m_2, n_2)}}((\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}, \mathcal{F}_{2, \mu_2}^{(m_2, n_2)}), (\text{Sht}_{\mu_3}^{\text{loc}(m_3, n_3)}, \mathcal{F}_{3, \mu_3}^{(m_3, n_3)})),$$

where the septuples satisfy the conditions as above. According to the formalism of the composition of cohomological correspondences (see Definition A.2.2), the composition $c_2^{(\mu_2, \mu_3, \nu_2, m_2, n_2, m_3, n_3)} \circ c_1^{(\mu_1, \mu_2, \nu_1, m_1, n_1, m_2, n_2)}$ is supported on the following correspondence

$$\text{Sht}_{\mu_1}^{\text{loc}(m_1, n_1)} \leftarrow \text{Sht}_{\mu_1 | \mu_2}^{\nu_1, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\mu_2 | \mu_3}^{\nu_2, \text{loc}(m_2, n_2)} \rightarrow \text{Sht}_{\mu_3}^{\text{loc}(m_3, n_3)}.$$

We then define the composition $c_2 \circ c_1$ to be the cohomological correspondence from \mathcal{F}_1 to \mathcal{F}_3 so that $(c_2 \circ c_1)^{(\mu_1, \mu_3, \nu_1 + \nu_2, m_1, n_1, m_3, n_3)}$ is given by the push forward of $c_2^{(\mu_2, \mu_3, \nu_2, m_2, n_2, m_3, n_3)} \circ c_1^{(\mu_1, \mu_2, \nu_1, m_1, n_1, m_2, n_2)}$ along the perfectly proper morphism

$$\text{Sht}_{\mu_1 | \mu_2}^{\nu_1, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{\mu_2}^{\text{loc}(m_2, n_2)}} \text{Sht}_{\mu_2 | \mu_3}^{\nu_2, \text{loc}(m_2, n_2)} \xrightarrow{\text{Comp}^{\text{loc}(m_1, n_1)}} \text{Sht}_{\mu_1 | \mu_3}^{\nu_1 + \nu_2, \text{loc}(m_1, n_1)},$$

in the sense of §A.2.6. To show that $c_2 \circ c_1$ is well-defined, we need to check that if

$$(\mu'_1, \mu'_2, \mu'_3, \nu'_1, \nu'_2, m'_1, n'_1, m'_2, n'_2, m'_3, n'_3) \geq (\mu_1, \mu_2, \mu_3, \nu_1, \nu_2, m_1, n_1, m_2, n_2, m_3, n_3)$$

(in the product partial order), then the connecting map in the colimit (5.4.2),

$$i_!(\text{res}^*((c_2 \circ c_1)^{(\mu_1, \mu_3, \nu_1 + \nu_2, m_1, n_1, m_3, n_3)})) = (c_2 \circ c_1)^{(\mu'_1, \mu'_3, \nu'_1 + \nu'_2, m'_1, n'_1, m'_3, n'_3)}.$$

First, we can apply Lemma A.2.15 and Lemma A.2.10 to see that under the pullback along res and the pushforward along i (in (5.4.1)), $c_2^{(\mu_2, \mu_3, \nu_2, m_2, n_2, m_3, n_3)} \circ c_1^{(\mu_1, \mu_2, \nu_1, m_1, n_1, m_2, n_2)}$ maps to $c_2^{(\mu'_2, \mu'_3, \nu'_2, m'_2, n'_2, m'_3, n'_3)} \circ c_1^{(\mu'_1, \mu'_2, \nu'_1, m'_1, n'_1, m'_2, n'_2)}$. Since the middle trapezoid in (5.3.29) is Cartesian by Proposition 5.3.23, we can apply Lemma A.2.17 to conclude.

Finally, the composition satisfies the natural associativity law $(c_3 \circ c_2) \circ c_1 \cong c_3 \circ (c_2 \circ c_1)$ by Lemma 5.3.25 and Lemma A.2.8. The construction of $\mathbf{P}^{\text{Corr}}(\text{Sht}_k^{\text{loc}})$ now is complete.

5.4.3. Description of morphisms. We describe the endomorphism ring of one object in $\mathbf{P}^{\text{Corr}}(\text{Sht}_k^{\text{loc}})$. Recall by (5.1.9) and (5.3.16), we have a functor

$$\mathbf{P}_{L+G \otimes \bar{k}}(\text{Gr} \otimes \bar{k}) \cong \mathbf{P}(\text{Hk}_k^{\text{loc}}) \xrightarrow{\Phi^{\text{loc}}} \mathbf{P}(\text{Sht}_k^{\text{loc}}).$$

Let $\delta_1 \in \mathbf{P}(\text{Sht}_k^{\text{loc}})$ denote the image of IC_0 . Recall from Example 5.3.4, $\text{Sht}_0^{\text{loc}(m, n)} \cong [L_n^m G \backslash \text{pt}]$ is perfectly smooth. Then δ_1 can be realized as

$$\delta_1^{(m, n)} := \overline{\mathbb{Q}}_\ell[(m - n) \dim G] \left(\frac{m - n}{2} \dim G \right) \in \mathbf{P}(\text{Sht}_{0, \bar{k}}^{\text{loc}(m, n)})$$

for every $m \geq n$.

Proposition 5.4.4. *The endomorphism ring $\text{End}_{\text{P}^{\text{Corr}}(\text{Sht}_k^{\text{loc}})}(\delta_1)$ is isomorphic to the spherical Hecke algebra $H_G \otimes \overline{\mathbb{Q}}_\ell = C_c(G(\mathcal{O}) \backslash G(F) / G(\mathcal{O}))$.*

Proof. We first understand

$$\text{Sht}_0^{\text{loc}(m_1, n_1)} \xleftarrow{\overleftarrow{h}_0^{\text{loc}(m_1, n_1)}} \text{Sht}_{0|0}^{\nu, \text{loc}(m_1, n_1)} \rightarrow \text{Sht}_0^{\text{loc}(m_2, n_2)}$$

for $\nu \in \mathbb{X}_\bullet(T)^{+, \sigma}$.

Note that the affine Deligne-Lusztig variety $X_0(0) \cong \text{Gr}(k)$ is discrete and for ν , $X_{0, \nu^*}(0) \cong \text{Gr}_{\nu^*}(k) = \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O})$. Choose n to be ν -large. By Corollary 5.3.21, the left arrow of the above correspondence is identified with

$$[L_{n_1}^{m_1} G \backslash \text{pt}] \leftarrow [L_{n_1}^{m_1} G \backslash \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O})],$$

with the map is induced by the natural projection $\text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O}) \rightarrow \text{pt}$. It follows from the same considerations as in §A.2.18 (see also Example A.2.3 (5)) that

$$\begin{aligned} & \text{Corr}_{\text{Sht}_{0|0}^{\nu, \text{loc}(m_1, n_1)}}((\text{Sht}_0^{\text{loc}(m_1, n_1)}, \delta_1^{(m_1, n_1)}), (\text{Sht}_0^{\text{loc}(m_2, n_2)}, \delta_1^{(m_2, n_2)})) \\ & \cong H_0^{\text{BM}}([L_{n_1}^{m_1} G \backslash \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O})] \otimes \bar{k}) \cong C(\text{Gr}_{\nu^*}(k))^{G(\mathcal{O} / \varpi^{n_1})} = C(G(\mathcal{O}) \backslash \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O})), \end{aligned}$$

where $C(-)$ denotes the space of $\overline{\mathbb{Q}}_\ell$ -valued functions. More precisely, giving such a function $f : G(F) \rightarrow \overline{\mathbb{Q}}_\ell$, the correspondence $c^{(0, 0, \nu, m_1, n_1, m_2, n_2)}$ at $[g] \in [L_{n_1}^{m_1} G \backslash \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O})]$ is given by multiplying $f(g) : \overline{\mathbb{Q}}_\ell \rightarrow \overline{\mathbb{Q}}_\ell$.

Note that if $(\nu, m_1, n_1, m_2, n_2) \leq (\nu', m'_1, n'_1, m'_2, n'_2)$, the connecting map in (5.4.2) under the above isomorphism is identified with

$$C(G(\mathcal{O}) \backslash \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O})) \rightarrow C(G(\mathcal{O}) \backslash \text{Gr}_{\nu'^*}^{(\infty)}(k) / G(\mathcal{O})),$$

where we regard a function on $\text{Gr}_{\nu^*}^{(\infty)}(k)$ as a function on $\text{Gr}_{\nu'^*}^{(\infty)}(k)$ by extension by zero. Therefore, as a vector space, $\text{End}_{\text{P}^{\text{Corr}}(\text{Sht}_k^{\text{loc}})}(\delta_1)$ is identified with the space of bi- $G(\mathcal{O})$ -invariant, compactly supported functions on $G(F)$. We also need to identify the ring structure.

By the same proof as Lemma 5.3.20 and Corollary 5.3.21, one can identify (5.3.28) ($\kappa = \lambda = \mu = 0$) with

$$[L_{n_1}^{m_1} \backslash (\text{Gr}_{\nu'^*} \tilde{\times} \text{Gr}_{\nu^*})(k)] \rightarrow [L_{n_1}^{m_1} \backslash \text{Gr}_{\nu'^* + \nu^*}(k)],$$

which is induced by the convolution map $\text{Gr}_{\nu'^*} \tilde{\times} \text{Gr}_{\nu^*} \rightarrow \text{Gr}_{\nu'^* + \nu^*}$ (defined in (3.1.11)). Now, let $c_1^{(0, 0, \nu', m_1, n_1, m_2, n_2)}$ (resp. $c_2^{(0, 0, \nu, m_2, n_2, m_3, n_3)}$) be a cohomological correspondence from $\delta_1^{(m_1, n_1)}$ to $\delta_1^{(m_2, n_2)}$ (resp. c from $\delta_1^{(m_2, n_2)}$ to $\delta_1^{(m_3, n_3)}$), corresponding to $f_1 \in C(G(\mathcal{O}) \backslash \text{Gr}_{\nu'^*}^{(\infty)}(k) / G(\mathcal{O}))$ (resp. $f_2 \in C(G(\mathcal{O}) \backslash \text{Gr}_{\nu^*}^{(\infty)}(k) / G(\mathcal{O}))$), under the above identification. The composition $c_2^{(0, 0, \nu, m_2, n_2, m_3, n_3)} \circ c_1^{(0, 0, \nu', m_1, n_1, m_2, n_2)}$ is supported on $\text{Sht}_{0|0}^{\nu', \text{loc}(m_1, n_1)} \times_{\text{Sht}_0^{\text{loc}(m_2, n_2)}} \text{Sht}_{0|0}^{\nu, \text{loc}(m_2, n_2)}$, which again by the same considerations as in §A.2.18 and Example A.2.3 (5), corresponding to a function $G(\mathcal{O})$ -invariant function on $\text{Gr}_{\nu'^*} \tilde{\times} \text{Gr}_{\nu^*}(k)$. It is clear that for $(g_1, g_2) \in \text{Gr}_{\nu'^*}(k) \times \text{Gr}_{\nu^*}(k)$, this function is $f_1(g_1)f_2(g_2)$. Therefore, the function on $\text{Gr}_{\nu'^* + \nu^*}(k)$ corresponding to

$$\text{Comp}_1^{\text{loc}(m_1, n_1)}(c_2^{(0, 0, \nu, m_2, n_2, m_3, n_3)} \circ c_1^{(0, 0, \nu', m_1, n_1, m_2, n_2)})$$

is the convolution $f_1 * f_2$ (defined in (3.5.1)). The proposition follows. \square

Remark 5.4.5. (1) The same statement and argument hold if one replace 0 by a central coweight $\tau \in \mathbb{X}_\bullet(Z_G)$. Namely, $\text{End}_{\text{P}^{\text{Corr}}(\text{Sht}_k^{\text{loc}})}(\Phi^{\text{loc}}(\text{IC}_\tau)) = H_G \otimes \overline{\mathbb{Q}}_\ell$.

(2) Recall that in Remark 5.3.11, for every $n > 0$ we have the object δ_ρ for a representation ρ of $G(\mathcal{O}/\varpi^n)$. If ρ is the regular representation of $G(\mathcal{O}/\varpi^n)$, then by the same argument, one can show that

$$\mathrm{End}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}_k^{\mathrm{loc}})}(\delta_\rho) \cong C_c(K_n \backslash G(F)/K_n),$$

where $K_n = L^+G^{(n)}(k)$ is the n th principal congruence subgroup, and $C_c(K_n \backslash G(F)/K_n)$ denote the corresponding Hecke algebra.

6. COHOMOLOGICAL CORRESPONDENCES BETWEEN MODULI OF LOCAL SHTUKAS

In this section, all (restricted) Hecke stacks and (restricted) moduli of local shtukas are considered over \bar{k} . We explain the proof of Theorem 1.3.2. Let us restate it here.

Theorem 6.0.1. (1) *There exists a functor $S : \mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma) \rightarrow \mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}_k^{\mathrm{loc}})$ making the following diagram commutative.*

$$(6.0.1) \quad \begin{array}{ccc} \mathrm{Rep}(\hat{G}) & \xrightarrow{\mathrm{Sat}} & \mathrm{P}(\mathrm{Hk}_k^{\mathrm{loc}}) \\ \downarrow & & \downarrow \Phi^{\mathrm{loc}} \\ \mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma) & \xrightarrow{S} & \mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}_k^{\mathrm{loc}}). \end{array}$$

Here the top arrow is the composition of the usual geometric Satake correspondence and its equivalence from (5.1.9), and the right vertical arrow is the pullback of perverse sheaves as constructed in (5.3.16). $\mathrm{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)$ is defined as the full subcategory of coherent sheaves $[\hat{G}\sigma/\hat{G}]$ spanned by the pullback of coherent sheaves along the natural projection $[\hat{G}\sigma/\hat{G}] \rightarrow \mathbf{B}\hat{G}$ (see Notation 6.1.1 and the left arrow is the pullback functor.

(2) $S(\mathcal{O}_{[\hat{G}\sigma/\hat{G}]}) = \delta_1$ and the map

$$S : \mathbf{J} := \Gamma([G\sigma/\hat{G}], \mathcal{O}) \rightarrow \mathrm{End}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}_k^{\mathrm{loc}})}(\delta_1) \cong H_G \otimes \overline{\mathbb{Q}}_\ell$$

coincides with the Satake isomorphism.

Here the isomorphism $\mathrm{End}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}_k^{\mathrm{loc}})}(\delta_1) \cong H_G \otimes \overline{\mathbb{Q}}_\ell$ in Part (2) is from Proposition 5.4.4, and the Satake isomorphism comes from (3.5.5).

To prove the theorem, we will upgrade the correspondences (5.3.21) between moduli of restricted local shtukas to cohomological correspondences. The essential computation is to verify the compatibility of this functor with respect to the composition law of morphisms. In special cases, we describe these cohomological correspondences more explicitly, by relating them to the cycle class maps or to the Hecke operators. In particular, we give a simpler proof of V. Lafforgue's $S = T$ theorem [Laf.V12⁺]³⁰, which is the content of Part (2) of the theorem.

For the generalities of cohomological correspondences between perfect stacks, we refer to §A.2.

6.1. Preparations. We keep the notation as in the previous section.

Notation 6.1.1. Recall that the Langlands dual group \hat{G} carries an action of (arithmetic) q -Frobenius σ . Consider the conjugation action of \hat{G} on $\hat{G}\sigma \subset {}^L G = \hat{G} \rtimes \langle \sigma \rangle$, or equivalently the twisted conjugation action c_σ of \hat{G} on \hat{G} given by $c_\sigma(g)(h) = gh\sigma(g^{-1})$. We use $\mathrm{Coh}^{\hat{G}}(\hat{G}\sigma)$ to denote the category of coherent sheaves over the stack $[(\hat{G}\sigma)/\hat{G}]$, or equivalently coherent sheaves over $\hat{G}\sigma$ equipped with an equivariant \hat{G} -action.

³⁰In fact, Varshavsky ([Va]) already gave a simpler proof of the $S = T$ theorem, and our proof is a further simplification of Varshavsky's proof.

For each algebraic representation V of \hat{G} , we may associate a vector bundle over $\hat{G}\sigma$ whose global section is $\mathcal{O}_{\hat{G}} \otimes V$ with a \hat{G} -action given by the twisted conjugation c_σ on the first factor and natural action on the latter, that is, for $g \in \hat{G}$ and $f : \hat{G} \rightarrow V$, we have

$$g \bullet f(h) = g(f(g^{-1}h\sigma(g))), \quad h \in \hat{G}.$$

We write $\tilde{V} \in \text{Coh}^{\hat{G}}(\hat{G}\sigma)$ for the resulting equivariant bundle. This defines a natural *faithful* functor $\text{Rep}(\hat{G}) \rightarrow \text{Coh}^{\hat{G}}(\hat{G}\sigma)$, sending V to \tilde{V} . More abstractly, this is the pullback functor for coherent sheaves along the natural projection $[(\hat{G}\sigma)/\hat{G}] \rightarrow \mathbf{B}\hat{G}$.

6.1.2. Hom spaces in $\text{Coh}^{\hat{G}}(\hat{G}\sigma)$. To benefit the readers, we make explicit the above functor on morphisms. For a representation V of G , let $\mathbf{J}(V)$ denote the space of V -valued functions f on \hat{G} such that $f(gh\sigma(g^{-1})) = g(f(h))$ for $g, h \in \hat{G}$, i.e. $\mathbf{J}(V) = (\mathcal{O}_{\hat{G}} \otimes V)^{\hat{G}}$.

For two representations $V_1, V_2 \in \text{Rep}(\hat{G})$, the hom space $\text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\tilde{V}_1, \tilde{V}_2)$ is the subspace of the hom space of coherent sheaves $\tilde{V}_1 \rightarrow \tilde{V}_2$ on $\hat{G}\sigma$, consisting of homomorphisms that respect the \hat{G} -actions. In other words,

$$(6.1.1) \quad \text{Hom}_{\text{Coh}^{\hat{G}}(\hat{G}\sigma)}(\tilde{V}_1, \tilde{V}_2) \cong \text{Hom}_{\mathcal{O}_{\hat{G}\sigma}}(\mathcal{O}_{\hat{G}\sigma} \otimes V_1, \mathcal{O}_{\hat{G}\sigma} \otimes V_2)^{\hat{G}} \cong (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}} \cong \mathbf{J}(V_1^* \otimes V_2).$$

Moreover, for a homomorphism $\mathbf{a} \in \text{Hom}_{\hat{G}}(V_1, V_2)$, its image $\tilde{\mathbf{a}}$ is the constant function with value $\mathbf{a} \in V_1^* \otimes V_2$.

Notation 6.1.3. We write $\text{Coh}_{fr}^{\hat{G}}(\hat{G}\sigma)$ for the full subcategory of $\text{Coh}^{\hat{G}}(\hat{G}\sigma)$ spanned by those \tilde{V} for $V \in \text{Rep}(\hat{G})$.

To define the functor S , we need first make some preparations.

Notation 6.1.4. We fix a half Tate twist $(1/2)$.

Let $\text{Sat} : \text{Rep}(\hat{G}) \rightarrow \text{P}_{L+G \otimes \bar{k}}(\text{Gr} \otimes \bar{k})$ be the geometric Satake equivalence. For $V \in \text{Rep}(\hat{G})$, we write Gr_V for the support of $\text{Sat}(V)$ on Gr . Explicitly, if $\{V_{\nu_i}\}$ are the Jordan-Hölder factors of V , then $\text{Gr}_V = \cup_i \text{Gr}_{V_{\nu_i}}$. Note that Gr_V may not be geometrically connected. An integer m is V -large if it is ν_i -large for all i . For such an integer, we put $\text{Hk}_V^{\text{loc}(m)} = [L^m G \backslash \text{Gr}_V]$ and we write $\text{Sat}(V)^{\text{loc}(m)}$ for the descent of $\text{Sat}(V)$ from Gr_V (via the equivalence (5.1.8)). It represents the object $\text{Sat}(V) \in \text{P}(\text{Hk}^{\text{loc}})$. To simplify notations, sometimes we use $\text{Sat}(V)^{\text{loc}(m)}$ to denote the pair $(\text{Hk}_V^{\text{loc}(m)}, \text{Sat}(V)^{\text{loc}(m)})$, if no confusion is likely to arise.

More generally, for a representation $V_\bullet = V_1 \boxtimes \cdots \boxtimes V_t$ of representations of \hat{G}^t , we define $\text{Gr}_{V_\bullet} = \text{Gr}_{V_1} \tilde{\times} \cdots \tilde{\times} \text{Gr}_{V_t}$ to be the support of the external twisted product

$$\text{Sat}(V_\bullet) := \text{Sat}(V_1) \tilde{\boxtimes} \cdots \tilde{\boxtimes} \text{Sat}(V_t).$$

An integer m is V_\bullet -large if it can be written as a sum $m = m_1 + \cdots + m_t$ such that each m_i is V_i -large. For such an integer, we write $\text{Hk}_{V_\bullet}^{\text{loc}(m)} = [L^m G \backslash \text{Gr}_{V_\bullet}]$ and $\text{Sat}(V_\bullet)^{\text{loc}(m)} \in \text{P}(\text{Hk}_{V_\bullet}^{\text{loc}(m)})$ for the descent of $\text{Sat}(V_\bullet)$ to Gr_{V_\bullet} (after shifted by $-m \dim G$ and twisted by $(-\frac{m}{2} \dim G)$). Moreover, $\text{Sat}(V_\bullet)^{\text{loc}(m)}$ is isomorphic to the \star -pullback (in the sense of Notation A.2.1) of $\boxtimes_i \text{Sat}(V_i)^{\text{loc}(m_i)}$ along the perfectly smooth map $\text{Hk}_{V_\bullet}^{\text{loc}(m)} \rightarrow \prod_i \text{Hk}_{V_i}^{\text{loc}(m_i)}$ from (5.1.11).

We can similarly define $\text{Gr}_{V_\bullet|W_\bullet}^0 = \text{Gr}_{V_\bullet} \times_{\text{Gr}} \text{Gr}_{W_\bullet}$ and $\text{Hk}_{V_\bullet|W_\bullet}^{0, \text{loc}(m)} = [L^m G \backslash \text{Gr}_{V_\bullet|W_\bullet}^0]$. We will occasionally need

$$\text{Hk}_{U_\bullet|V_\bullet|W_\bullet; U'_\bullet}^{\text{loc}(m)} = [L^m G \backslash \text{Gr}_{U_\bullet} \tilde{\times} \text{Gr}_{V_\bullet|W_\bullet} \tilde{\times} \text{Gr}_{U'_\bullet}] \subset \text{Hk}_{U_\bullet \boxtimes V_\bullet \boxtimes U'_\bullet|U_\bullet \boxtimes W_\bullet \boxtimes U'_\bullet}^{\text{loc}(m)}.$$

For a pair of nonnegative integer (m, n) , similarly to Definition 5.3.1, we define the restricted local shtukas $\text{Sht}_{V_\bullet}^{\text{loc}(m, n)}$ and $\text{Sht}_{V_\bullet|W_\bullet}^{0, \text{loc}(m, n)}$ (if $m - n$ is V_\bullet -large and W_\bullet -large), and $\text{Sht}_{U_\bullet; V_\bullet|W_\bullet; U'_\bullet}^{\text{loc}(m, n)} \subset \text{Sht}_{U_\bullet \boxtimes V_\bullet \boxtimes U'_\bullet|U_\bullet \boxtimes W_\bullet \boxtimes U'_\bullet}^{\text{loc}(m, n)}$ if $m - n$ is $U_\bullet \boxtimes V_\bullet \boxtimes U'_\bullet$ -large and $U_\bullet \boxtimes W_\bullet \boxtimes U'_\bullet$ -large. There is the morphism

$$\varphi^{\text{loc}(m, n)} : \text{Sht}_{V_\bullet}^{\text{loc}(m, n)} \rightarrow \text{Hk}_{V_\bullet}^{\text{loc}(m)}.$$

For an algebraic representation $V \in \text{Rep}(\hat{G})$, we define

$$(6.1.2) \quad S(\tilde{V}) := \Phi(\text{Sat}(V)) \in \text{P}(\text{Sht}^{\text{loc}}),$$

where Φ is defined in (5.3.16). More precisely, for any pair (m, n) such that $m - n$ is V -large and $n > 0$, $S(\tilde{V})$ is represented by the perverse sheaf

$$S(\tilde{V})^{\text{loc}(m, n)} := \Phi^{\text{loc}(m, n)}(\text{Sat}(V)^{\text{loc}(m)}) \in \text{P}(\text{Sht}_V^{\text{loc}(m, n)}),$$

where $\Phi^{\text{loc}(m, n)}$ is defined in (5.3.15), namely the \star -pullback (in the sense of Notation A.2.1) of sheaves along $\varphi^{\text{loc}(m, n)}$. Similarly as above, to simplify notations sometimes we use $S(\tilde{V})^{\text{loc}(m, n)}$ to denote the pair $(\text{Sht}_V^{\text{loc}(m, n)}, S(\tilde{V})^{\text{loc}(m, n)})$, if no confusion is likely to arise.

More generally, if $V_\bullet = V_1 \boxtimes \cdots \boxtimes V_t$ is a representation of \hat{G}^t , then for a pair of non-negative integers (m, n) with $n > 0$ and $m - n$ V_\bullet -large, we denote

$$S(\tilde{V}_\bullet)^{\text{loc}(m, n)} := \Phi^{\text{loc}(m, n)}(\text{Sat}(V_\bullet)^{\text{loc}(m)}) \in \text{P}(\text{Sht}_{V_\bullet}^{\text{loc}(m, n)}).$$

Now for a representation $V_\bullet = V_1 \boxtimes \cdots \boxtimes V_t$ of \hat{G}^t and a representation W of \hat{G} . We say a quadruple of nonnegative integers (m_1, n_2, m_1, n_1) is $(V_\bullet \boxtimes W)$ -acceptable if

- $m_1 - m_2 = n_1 - n_2$ is W -large;
- $m_2 - n_1$ is V_\bullet -large.

Then we can adapt Construction 5.3.12 to define an analogous inverse partial Frobenius morphism

$$F_{V_\bullet \boxtimes W}^{-1} : \text{Sht}_{\sigma W \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} \rightarrow \text{Sht}_{V_\bullet \boxtimes W}^{\text{loc}(m_2, n_2)}.$$

Here σW is the σ -twist of W as in Notation 3.4.7.

For representations V_1, V_2 and W of \hat{G} , and nonnegative integers (m_1, m_2, n_1, n_2) that is $((V_1 \otimes W) \boxtimes W^*)$ -acceptable and $((V_2 \otimes W) \boxtimes W^*)$ -acceptable, similarly to Definition 5.3.16, we define

$$\text{Sht}_{V_1|V_2}^{W, \text{loc}(m_1, n_1)} := \text{Sht}_{V_1|\sigma W^* \boxtimes (\sigma W \otimes V_1)}^{0, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{(\sigma W \otimes V_1) \boxtimes W^*}^{\text{loc}(m_2, n_2)}} \text{Sht}_{(\sigma W \otimes V_1) \boxtimes W^*|V_2}^{0, \text{loc}(m_2, n_2)}.$$

Remark 6.1.5. The newly defined spaces $\text{Hk}_{V_\bullet}^{\text{loc}(m)}$ and $\text{Sht}_{V_\bullet}^{\text{loc}(m, n)}$ are simply unions of $\text{Hk}_{\mu_\bullet}^{\text{loc}(m)}$ and $\text{Sht}_{\mu_\bullet}^{\text{loc}(m, n)}$. So we are free to use most of the results from the previous section.

6.1.6. Satake cohomological correspondences. Recall that the geometric Satake (Theorem 3.4.2 and Corollary 3.4.4) in particular gives a natural isomorphism between homomorphisms of \hat{G} -representations and cohomological correspondences of perverse sheaves on Gr supported on the Satake correspondences. This isomorphism (3.4.4) descends to a canonical isomorphism

$$(6.1.3) \quad \mathcal{C}^{\text{loc}(m)} : \text{Hom}_{\hat{G}}(V_\bullet, W_\bullet) \cong \text{Corr}_{\text{Hk}_{V_\bullet|W_\bullet}^{0, \text{loc}(m)}}(\text{Sat}(V_\bullet)^{\text{loc}(m)}, \text{Sat}(W_\bullet)^{\text{loc}(m)}).$$

Here and below, we regard a \hat{G}^t -representation as a \hat{G} -representation by the diagonal restriction along $\hat{G} \subset \hat{G}^t$.

Remark 6.1.7. There are different ways to interpret a tensor product representation of \hat{G} as the diagonal restriction of an exterior tensor product representation of some power of \hat{G} . For example, we may either regard $W_1 \otimes W_2 \otimes W_3$ as the restriction $W_1 \boxtimes W_2 \boxtimes W_3$ as a representation of

\hat{G}^3 , or as the restriction of $(W_1 \otimes W_2) \boxtimes W_3$ as a representation of \hat{G}^2 . Therefore, an element $\mathbf{a} \in \text{Hom}(V, W_1 \otimes W_2 \otimes W_3)$ will induce a correspondence

$$\mathcal{C}^{\text{loc}(m)}(\mathbf{a}) \in \text{Corr}_{\text{Hk}_{V|(W_1 \otimes W_2) \boxtimes W_3}^{\text{loc}(m)}}((\text{Hk}_V^{\text{loc}(m)}, \text{Sat}(V)^{\text{loc}(m)}), (\text{Hk}_{W_1 \boxtimes W_2 \boxtimes W_3}^{\text{loc}(m)}, \text{Sat}(W_1 \boxtimes W_2 \boxtimes W_3)^{\text{loc}(m)}),$$

and a correspondence

$$\mathcal{C}^{\text{loc}(m)}(\mathbf{a}) \in \text{Corr}_{\text{Hk}_{V|(W_1 \otimes W_2) \boxtimes W_3}^{\text{loc}(m)}}((\text{Hk}_V^{\text{loc}(m)}, \text{Sat}(V)^{\text{loc}(m)}), (\text{Hk}_{(W_1 \otimes W_2) \boxtimes W_3}^{\text{loc}(m)}, \text{Sat}((W_1 \otimes W_2) \boxtimes W_3)^{\text{loc}(m)}).$$

The latter is the pushforward of the former along the convolution map

$$\text{Hk}_{W_1 \boxtimes W_2 \boxtimes W_3}^{\text{loc}(m)} \rightarrow \text{Hk}_{(W_1 \otimes W_2) \boxtimes W_3}^{\text{loc}(m)}$$

introduced in (5.1.2). In below, readers should be able to tell from the notations that how we regard a tensor product representation of \hat{G} as the restriction of an exterior tensor product.

Pulling back along the natural restriction morphism $\varphi^{\text{loc}(m,n)}$ (5.3.1), we obtain correspondences on local shtukas.

Lemma 6.1.8. *Let (m, n) be a pair such that $n > 0$ and $m - n$ is V_\bullet -large and W_\bullet -large. There is a natural homomorphism*

$$(6.1.4) \quad \mathcal{C}^{\text{loc}(m,n)} : \text{Hom}_{\hat{G}}(V_\bullet, W_\bullet) \longrightarrow \text{Corr}_{\text{Sht}_{V_\bullet|W_\bullet}^{0,\text{loc}(m,n)}}(S(\widetilde{V}_\bullet)^{\text{loc}(m,n)}, S(\widetilde{W}_\bullet)^{\text{loc}(m,n)}),$$

such that the following diagram is commutative (if $m - n$ is also U_\bullet -large),

$$(6.1.5) \quad \begin{array}{ccc} \text{Hom}_{\hat{G}}(U_\bullet, V_\bullet) \otimes \text{Hom}_{\hat{G}}(V_\bullet, W_\bullet) & \longrightarrow & \text{Corr}_{\text{Sht}_{U_\bullet|V_\bullet}^{0,\text{loc}(m,n)}}(S(\widetilde{U}_\bullet), S(\widetilde{V}_\bullet)) \otimes \text{Corr}_{\text{Sht}_{V_\bullet|W_\bullet}^{0,\text{loc}(m,n)}}(S(\widetilde{V}_\bullet), S(\widetilde{W}_\bullet)) \\ \downarrow & & \downarrow \\ \text{Hom}_{\hat{G}}(U_\bullet, W_\bullet) & \longrightarrow & \text{Corr}_{\text{Sht}_{U_\bullet|W_\bullet}^{0,\text{loc}(m,n)}}(S(\widetilde{U}_\bullet), S(\widetilde{W}_\bullet)), \end{array}$$

where we omit $\text{loc}(m, n)$ in the superscripts, the right vertical map is given by the pushforward of the composition of cohomological correspondences along the perfectly proper morphism

$$\text{Comp}^{\text{loc}(m,n)} : \text{Sht}_{U_\bullet|V_\bullet}^{0,\text{loc}(m,n)} \times_{\text{Sht}_{V_\bullet}^{\text{loc}(m,n)}} \text{Sht}_{V_\bullet|W_\bullet}^{0,\text{loc}(m,n)} \longrightarrow \text{Sht}_{U_\bullet|W_\bullet}^{0,\text{loc}(m,n)}.$$

as in the middle vertical arrow of (5.3.13).

Proof. Recall the commutative diagram from (5.3.3), where the vertical maps between the second and the third arrows are smooth. Then $\mathcal{C}^{\text{loc}(m,n)}$ is the pullback of (6.1.3) along these by the construction in §A.2.11.

To prove the (6.1.5) is commutative, we use the right Cartesian square of (5.3.13) to reduce the question to $\mathcal{C}^{\text{loc}(m)}$ via Lemma A.2.17, which in turn follows from Corollaries 3.1.10 and 3.4.4. \square

Example 6.1.9. For a representation W_\bullet of \hat{G}^t , there is a canonical unit map

$$\delta_{W_\bullet} : \mathbf{1} \rightarrow W_\bullet^* \otimes W_\bullet$$

and a counit map

$$e_{W_\bullet} : W_\bullet \otimes W_\bullet^* \rightarrow \mathbf{1}.$$

The corresponding cohomological correspondence

$$\mathcal{C}^{\text{loc}(m,n)}(\delta_{W_\bullet} \otimes \text{id}_{V_\bullet}) : S(\widetilde{V}_\bullet)^{\text{loc}(m,n)} \rightarrow S(\widetilde{W}_\bullet^* \boxtimes (\widetilde{W}_\bullet \otimes \widetilde{V}_\bullet))^{\text{loc}(m,n)}$$

and

$$\mathcal{C}^{\text{loc}(m,n)}(\text{id}_{V_\bullet} \otimes e_{W_\bullet}) : S((\widetilde{V}_\bullet \otimes \widetilde{W}_\bullet) \boxtimes \widetilde{W}_\bullet^*)^{\text{loc}(m,n)} \rightarrow S(\widetilde{V}_\bullet)^{\text{loc}(m,n)}$$

are the local version of V. Lafforgue's creation and annihilation operators.

The reason we regard $W_\bullet^* \otimes W_\bullet \otimes V_\bullet$ as the diagonal restriction of the \hat{G}^2 -representation $W_\bullet^* \boxtimes (W_\bullet \otimes V_\bullet)$ rather than the diagonal restriction of the \hat{G}^3 -representation $W_\bullet^* \boxtimes W_\bullet \boxtimes V_\bullet$ is to make the support of the cohomological correspondence defined in Construction 6.2.2 below equal to $\text{Sht}_{V_1|V_2}^{W, \text{loc}(m_1, n_1)}$.

Remark 6.1.10. A homomorphism $\mathbf{a} \in \text{Hom}_{\hat{G}}(V_\bullet, W_\bullet)$ induces a homomorphism

$$\text{id}_{U_\bullet} \otimes \mathbf{a} \otimes \text{id}_{U'_\bullet} \in \text{Hom}_{\hat{G}}(U_\bullet \otimes V_\bullet \otimes U'_\bullet, U_\bullet \otimes W_\bullet \otimes U'_\bullet).$$

In this case, the correspondence $\mathcal{C}^{\text{loc}(m, n)}(\text{id}_{U_\bullet} \otimes \mathbf{a} \otimes \text{id}_{U'_\bullet})$ is the pushforward of a natural cohomological correspondence supported on $\text{Sht}_{U_\bullet; V_\bullet|W_\bullet; U'_\bullet}^{0, \text{loc}(m, n)}$. Indeed, it is enough to prove this if \mathbf{a} belongs to the Satake basis. But this follows from the fact that the cohomological correspondence from $(\text{Gr}_{U_\bullet \boxtimes V_\bullet \boxtimes U'_\bullet}, \text{Sat}(U_\bullet \boxtimes V_\bullet \boxtimes U'_\bullet))$ to $(\text{Gr}_{U_\bullet \boxtimes W_\bullet \boxtimes U'_\bullet}, \text{Sat}(U_\bullet \boxtimes W_\bullet \boxtimes U'_\bullet))$ given by $\text{id}_{U_\bullet} \otimes \mathbf{a} \otimes \text{id}_{U'_\bullet}$ (under the isomorphism (3.4.4)) is the pushforward of a natural cohomological correspondence supported on $\text{Gr}_{U_\bullet} \tilde{\times} \text{Gr}_{V_\bullet|W_\bullet}^{0, \mathbf{a}} \tilde{\times} \text{Gr}_{U'_\bullet}$.

In the sequel, when we explicitly write a \hat{G} -homomorphism $U_\bullet \otimes V_\bullet \otimes U'_\bullet \rightarrow U_\bullet \otimes W_\bullet \otimes U'_\bullet$ as $\text{id}_{U_\bullet} \otimes \mathbf{a} \otimes \text{id}_{U'_\bullet}$, we regard $\mathcal{C}^{\text{loc}(m, n)}(\text{id}_{U_\bullet} \otimes \mathbf{a} \otimes \text{id}_{U'_\bullet})$ as a cohomological correspondence supported on $\text{Sht}_{U_\bullet; V_\bullet|W_\bullet; U'_\bullet}^{0, \text{loc}(m, n)}$.

The inverse partial Frobenius morphism $F_{V_\bullet \boxtimes W}^{-1} : \text{Sht}_{\sigma W, V_\bullet}^{\text{loc}(m_1, n_1)} \rightarrow \text{Sht}_{V_\bullet, W}^{\text{loc}(m_2, n_2)}$ between moduli of restricted shtukas can be also upgraded to a cohomological correspondence.

Lemma 6.1.11. *If V_\bullet is a representations of \hat{G}^t and W is a representation of \hat{G} , then there is a natural cohomological correspondence*

$$(6.1.6) \quad \mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W}^{-1}}^* : (\text{Sht}_{\sigma W \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)}, S(\widetilde{\sigma W} \boxtimes \widetilde{V_\bullet})^{\text{loc}(m_1, n_1)}) \longrightarrow (\text{Sht}_{V_\bullet \boxtimes W}^{\text{loc}(m_2, n_2)}, S(\widetilde{V_\bullet} \boxtimes \widetilde{W})^{\text{loc}(m_2, n_2)}).$$

Proof. We construct $\mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W}^{-1}}^*$ as follows. We rewrite commutative diagram (5.3.18) as

$$\begin{array}{ccccc} \text{Sht}_{\sigma W \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} & \xlongequal{\quad} & \text{Sht}_{\sigma W \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} & \xrightarrow{F_{V_\bullet \boxtimes W}^{-1}} & \text{Sht}_{V_\bullet \boxtimes W}^{\text{loc}(m_2, n_2)} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hk}_{\sigma W}^{\text{loc}(n_1)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m_2)} & \xleftarrow{\sigma \times \text{id}} & \text{Hk}_W^{\text{loc}(n_1)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m_2)} & \xlongequal{\quad} & \text{Hk}_{V_\bullet}^{\text{loc}(m_2)} \times \text{Hk}_W^{\text{loc}(n_1)}, \end{array}$$

where vertical maps are perfectly smooth. By (3.4.5), for every representation W of \hat{G} and a W -large integer m , there is a canonical isomorphism

$$\sigma^* \text{Sat}(\sigma W)^{\text{loc}(m)} \cong \text{Sat}(W)^{\text{loc}(m)},$$

as perverse sheaves on $\text{Hk}_W^{\text{loc}(m)}$, which by Example A.2.3 (2) defines a natural correspondence

$$\Gamma_{\sigma \times \text{id}}^* : (\text{Hk}_{\sigma W}^{\text{loc}(m)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}, \text{Sat}(\sigma W)^{\text{loc}(m)} \boxtimes \text{Sat}(V_\bullet)^{\text{loc}(m')}) \rightarrow (\text{Hk}_W^{\text{loc}(m)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}, \text{Sat}(W)^{\text{loc}(m)} \boxtimes \text{Sat}(V_\bullet)^{\text{loc}(m')}),$$

supported on the second row. Then $\mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W}^{-1}}^*$ is the pullback of the correspondence $\Gamma_{\sigma \times \text{id}}^*$ via the construction §A.2.11. \square

Later we will make use of the following two lemmas.

Lemma 6.1.12. *The pushforward of $\mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W_1 \boxtimes W_2}^{-1}} \circ \mathbb{D}\Gamma_{F_{\sigma W_2 \boxtimes V_\bullet \boxtimes W_1}^{-1}} : S(\widetilde{\sigma W_1} \boxtimes \widetilde{\sigma W_2} \boxtimes \widetilde{V_\bullet}) \rightarrow S(\widetilde{V_\bullet} \boxtimes \widetilde{W_1} \boxtimes \widetilde{W_2})$ along the convolution map*

$$\begin{array}{ccccc} \text{Sht}_{\sigma W_1 \boxtimes \sigma W_2 \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} & \xlongequal{\quad} & \text{Sht}_{\sigma W_1 \boxtimes \sigma W_2 \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} & \longrightarrow & \text{Sht}_{V_\bullet \boxtimes W_1 \boxtimes W_2}^{\text{loc}(m_3, n_3)} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Sht}_{\sigma(W_1 \otimes W_2) \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} & \xlongequal{\quad} & \text{Sht}_{\sigma(W_1 \otimes W_2) \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} & \longrightarrow & \text{Sht}_{V_\bullet \boxtimes (W_1 \otimes W_2)}^{\text{loc}(m_3, n_3)} \end{array}$$

is equal to $\mathbb{D}\Gamma_{F_{V_\bullet \boxtimes (W_1 \otimes W_2)}^{-1}}$.

Proof. By definition and Lemma A.2.15, $\mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W_1 \boxtimes W_2}^{-1}} \circ \mathbb{D}\Gamma_{F_{\sigma W_2 \boxtimes V_\bullet \boxtimes W_1}^{-1}}$ is equal to the pullback of the correspondence $\Gamma_{\sigma W_1 \times \sigma W_2 \times \text{id}_{V_\bullet}}^*$, where

$$\sigma W_1 \times \sigma W_2 \times \text{id}_{V_\bullet} : \text{Hk}_{W_1}^{\text{loc}(m_1)} \times \text{Hk}_{W_2}^{\text{loc}(m_2)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')} \rightarrow \text{Hk}_{\sigma W_1}^{\text{loc}(m_1)} \times \text{Hk}_{\sigma W_2}^{\text{loc}(m_2)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}.$$

Note that the map $\text{Sht}_{W_1 \boxtimes W_2 \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} \rightarrow \text{Hk}_{W_1}^{\text{loc}(m_1)} \times \text{Hk}_{W_2}^{\text{loc}(m_2)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}$ factors through $\text{Sht}_{W_1 \boxtimes W_2 \boxtimes V_\bullet}^{\text{loc}(m_1, n_1)} \rightarrow \text{Hk}_{W_1 \boxtimes W_2}^{\text{loc}(m_1)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}$. Then $\mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W_1 \boxtimes W_2}^{-1}} \circ \mathbb{D}\Gamma_{F_{\sigma W_2 \boxtimes V_\bullet \boxtimes W_1}^{-1}}$ is equal to the pullback of the correspondence $\Gamma_{\sigma W_1 \boxtimes W_2 \times \text{id}_{V_\bullet}}^*$, where

$$\sigma W_1 \boxtimes W_2 \times \text{id}_{V_\bullet} : \text{Hk}_{W_1 \boxtimes W_2}^{\text{loc}(m_1)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')} \rightarrow \text{Hk}_{\sigma(W_1 \boxtimes W_2)}^{\text{loc}(m_1)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}.$$

Note that the following diagram is Cartesian

$$\begin{array}{ccc} \text{Sht}_{W_1 \boxtimes W_2 \boxtimes V_\bullet}^{\text{loc}(m, n)} & \longrightarrow & \text{Sht}_{(W_1 \otimes W_2) \boxtimes V_\bullet}^{\text{loc}(m, n)} \\ \downarrow & & \downarrow \\ \text{Hk}_{W_1 \boxtimes W_2}^{\text{loc}(m)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')} & \longrightarrow & \text{Hk}_{W_1 \otimes W_2}^{\text{loc}(m)} \times \text{Hk}_{V_\bullet}^{\text{loc}(m')}. \end{array}$$

Now apply Lemma A.2.17 to conclude. \square

Lemma 6.1.13. (1) Let $\mathbf{a} \in \text{Hom}_{\hat{G}}(U_\bullet, V_\bullet)$, and regard $\mathcal{C}^{\text{loc}(m, n)}(\mathbf{a} \otimes \text{id}_W)$ (resp. $\mathcal{C}^{\text{loc}(m, n)}(\text{id}_{\sigma W} \otimes \mathbf{a})$) as a cohomological correspondence supported on $\text{Sht}_{U_\bullet | V_\bullet; W}^{0, \text{loc}(m, n)}$ (resp. $\text{Sht}_{\sigma W; U_\bullet | V_\bullet}^{0, \text{loc}(m, n)}$) as in Remark 6.1.10. Then

$$\mathcal{C}^{\text{loc}(m, n)}(\mathbf{a} \otimes \text{id}_W) \circ \mathbb{D}\Gamma_{F_{U_\bullet \boxtimes W}^{-1}} = \mathbb{D}\Gamma_{F_{V_\bullet \boxtimes W}^{-1}} \circ \mathcal{C}^{\text{loc}(m, n)}(\text{id}_{\sigma W} \otimes \mathbf{a}).$$

(2) Similarly, let $\mathbf{b} \in \text{Hom}_{\hat{G}}(W, W')$. Then

$$\mathcal{C}^{\text{loc}(m, n)}(\text{id}_{U_\bullet} \otimes \mathbf{b}) \circ \mathbb{D}\Gamma_{F_{U_\bullet \boxtimes W}^{-1}} = \mathbb{D}\Gamma_{F_{U_\bullet \boxtimes W'}^{-1}} \circ \mathcal{C}^{\text{loc}(m, n)}(\text{id}_{U_\bullet} \otimes \sigma \mathbf{b})$$

Proof. We prove (1) and the proof of (2) is similar. By Lemma 5.3.22(3), the supports of this two compositions of cohomological correspondences coincide. Now by definition and by Lemma A.2.15, the statement follows from the evident equality

$$(\mathcal{C}^{\text{loc}(m)}(\mathbf{a}) \times \text{id}_W) \circ \Gamma_{\text{id}_{U_\bullet} \times \sigma}^* = \Gamma_{\text{id}_{V_\bullet} \times \sigma}^* \circ (\mathcal{C}^{\text{loc}(m)}(\mathbf{a}) \times \text{id}_{\sigma W})$$

as cohomological correspondences from $(\text{Hk}_{\sigma W}^{\text{loc}(m')} \times \text{Hk}_{U_\bullet}^{\text{loc}(m)}, \text{Sat}(\sigma W)^{\text{loc}(m')}) \boxtimes \text{Sat}(U_\bullet)^{\text{loc}(m)}$ to $(\text{Hk}_W^{\text{loc}(m')} \times \text{Hk}_{V_\bullet}^{\text{loc}(m)}, \text{Sat}(W)^{\text{loc}(m')}) \boxtimes \text{Sat}(V_\bullet)^{\text{loc}(m)}$. \square

6.2. Cohomological correspondences of perverse sheaves on the moduli of local shtukas.

6.2.1. *Sketch of the construction of S .* We now construct the natural functor

$$S : \text{Coh}_{fr}^{\hat{G}}(\hat{G}\sigma) \rightarrow \text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})$$

so that the diagram (6.0.1) is commutative and hence prove Theorem 1.3.2.

Every object in $\text{Coh}_{fr}^{\hat{G}}(\hat{G}\sigma)$ is of the form \tilde{V} for some $V \in \text{Rep}(\hat{G})$. We define

$$S(\tilde{V}) := \Phi(\text{Sat}(V)) \in \text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}}),$$

which is represented by $\text{Sat}(V)^{\text{loc}(m,n)} \in \text{P}(\text{Sht}_V^{\text{loc}(m,n)})$ for any pair (m, n) of non-negative integers such that m is positive and $m - n$ is V -large. It is clear from the definition that the diagram (6.0.1) of functors is commutative for objects. We are left to define the functor S on the morphisms, that

is, for each pair of representations $V_1, V_2 \in \text{Rep}(\hat{G})$ a natural homomorphism

$$(6.2.1) \quad S : \text{Hom}_{\text{Coh}_{fr}^{\hat{G}}(\hat{G}\sigma)}(\tilde{V}_1, \tilde{V}_2) \stackrel{(6.1.1)}{\cong} \mathbf{J}(V_1^* \otimes V_2) \longrightarrow \text{Mor}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\tilde{V}_1), S(\tilde{V}_2))$$

which is compatible with compositions. Moreover, we need to show the commutativity of functors in (6.0.1) for morphisms. We overview the construction and the proof below, and leave the details to later in this subsection.

For representations $W, V_1, V_2 \in \text{Rep}(\hat{G})$, there is a natural homomorphism

$$\Xi_W : \text{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W^*, V_2) \rightarrow (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}} = \mathbf{J}(V_1^* \otimes V_2) = \text{Hom}_{\text{Coh}_{fr}^{\hat{G}}(\hat{G}\sigma)}(\tilde{V}_1, \tilde{V}_2)$$

sending $\mathbf{a} \in \text{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W^*, V_2)$ to the function

$$(6.2.2) \quad \Xi_W(\mathbf{a}) : g \mapsto \sum_i \mathbf{a}(g \cdot e_i \otimes e_i^*) \in \text{Hom}_{\overline{\mathbb{Q}}_\ell}(V_1, V_2) \cong V_1^* \otimes V_2,$$

where $\{e_i\}_i$ is a basis of $W(= \sigma W)$, and $\{e_i^*\}_i$ its dual basis. Moreover, it follows from the Peter-Weyl theorem that every element of $\mathbf{J}(V_1^* \otimes V_2)$ is in the image of some Ξ_W . So to define S on morphisms, we will first define in Construction 6.2.2 a natural map

$$\mathcal{C}_W : \text{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W^*, V_2) \longrightarrow \text{Mor}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\tilde{V}_1), S(\tilde{V}_2))$$

for each representation W of \hat{G} . Then for every element $\mathbf{c} \in \mathbf{J}(V_1^* \otimes V_2)$, we write it as $\mathbf{c} = \Xi_W(\mathbf{a})$ for some representation W and some element $\mathbf{a} \in \text{Hom}_{\hat{G}}(\sigma W \otimes V_1 \otimes W^*, V_2)$, and define

$$(6.2.3) \quad S(\mathbf{c}) := \mathcal{C}_W(\mathbf{a}) \in \text{Mor}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\tilde{V}_1), S(\tilde{V}_2)).$$

We will show that

- (Lemma 6.2.5) the definition of $S(\mathbf{c})$ is independent of the choice of the representation W and the element \mathbf{a} , and
- (Lemma 6.2.7) this definition of S on morphisms is compatible with compositions of morphisms.

Therefore, we obtained a well-defined functor $S : \text{Coh}_{fr}^{\hat{G}}(\hat{G}\sigma) \rightarrow \text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})$. Finally, we will check in Lemma 6.2.8 that the diagram (6.0.1) of functors is commutative for morphisms.

To sum up, after we construct the map \mathcal{C}_W below, the proof of Theorem 1.3.2 (1) is divided into Lemmas 6.2.5, 6.2.7, and 6.2.8. \square

Construction 6.2.2. We now define the natural map \mathcal{C}_W . For this, we take a quadruple (m_1, n_1, m_2, n_2) that is $(V_1 \otimes W) \boxtimes W^*$ -acceptable and $(V_2 \otimes W) \boxtimes W^*$ -acceptable. Given an element $\mathbf{a} \in \text{Hom}_{\hat{G}}(\sigma W \otimes$

$V_1 \otimes W^*, V_2)$, we define $\mathcal{C}_W(\mathbf{a}) \in \text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2))$ to be the morphism represented by the cohomological correspondence in

$$\text{Corr}_{\text{Sht}_{V_1|V_2}^{W, \text{loc}(m_1, n_1)}}(S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)}, S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)})$$

given by the following composition,

$$\begin{aligned} S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)} &\xrightarrow{\mathcal{C}^{\text{loc}(m_1, n_1)}(\delta_{\sigma W} \otimes \text{id}_{V_1})} S(\widetilde{\sigma W^*} \boxtimes (\widetilde{\sigma W} \otimes \widetilde{V}_1))^{\text{loc}(m_1, n_1)} \\ &\xrightarrow[\mathbb{D}\Gamma_{F(\sigma W \otimes V_1) \boxtimes W^*}^*]{\text{of (6.1.6)}} S((\widetilde{\sigma W} \otimes \widetilde{V}_1) \boxtimes \widetilde{W^*})^{\text{loc}(m_2, n_2)} \xrightarrow{\mathcal{C}^{\text{loc}(m_2, n_2)}(\mathbf{a})} S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}. \end{aligned}$$

The definition of $\mathcal{C}_W(\mathbf{a})$ as a morphism in $\text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2))$ is independent of the choice of (m_1, n_1, m_2, n_2) in a natural way: if $(m'_1, n'_1, m'_2, n'_2) \geq (m_1, n_1, m_2, n_2)$ is another quadruple of nonnegative integers that is $(V_1 \otimes W) \boxtimes W^*$ -acceptable and $(V_2 \otimes W) \boxtimes W^*$ -acceptable, then $\mathcal{C}_W^{\text{loc}(m'_1, n'_1)}(\mathbf{a})$ is the smooth pullback (in the sense of §A.2.11) of $\mathcal{C}_W^{\text{loc}(m_1, n_1)}(\mathbf{a})$ along the following diagram

$$(6.2.4) \quad \begin{array}{ccccc} \text{Sht}_{V_1}^{\text{loc}(m'_1, n'_1)} & \longleftarrow & \text{Sht}_{V_1|V_2}^{W, \text{loc}(m'_1, n'_1)} & \longrightarrow & \text{Sht}_{V_2}^{\text{loc}(m'_2, n'_2)} \\ \downarrow & & \downarrow & & \downarrow \\ \text{Sht}_{V_1}^{\text{loc}(m_1, n_1)} & \longleftarrow & \text{Sht}_{V_1|V_2}^{W, \text{loc}(m_1, n_1)} & \longrightarrow & \text{Sht}_{V_2}^{\text{loc}(m_2, n_2)}, \end{array}$$

where the vertical arrows are perfectly smooth and the left square is Cartesian. This follows from Lemma A.2.15, the Cartesian diagram (5.3.12), and the commutative diagram (5.3.19).

We need to show the definition of S on morphisms using \mathcal{C}_W is independent of choices. First, we have

Lemma 6.2.3. *Let \mathbf{a}' denote the image of \mathbf{a} under the isomorphism $\text{Hom}(\sigma W \otimes V_1 \otimes W^*, V_2) \cong \text{Hom}(V_1, \sigma W^* \otimes V_2 \otimes W)$. Then $\mathcal{C}_W(\mathbf{a})$ can also be computed as*

$$\begin{aligned} S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)} &\xrightarrow{\mathcal{C}^{\text{loc}(m_1, n_1)}(\mathbf{a}')} S(\widetilde{\sigma W^*} \boxtimes (\widetilde{V}_2 \otimes \widetilde{W}))^{\text{loc}(m_1, n_1)} \\ &\xrightarrow[\mathbb{D}\Gamma_{F(V_2 \otimes W) \boxtimes W^*}^*]{\text{of (6.1.6)}} S((\widetilde{V}_2 \otimes \widetilde{W}) \boxtimes \widetilde{W^*})^{\text{loc}(m_2, n_2)} \xrightarrow{\mathcal{C}^{\text{loc}(m_2, n_2)}(\text{id}_{V_2} \otimes e_W)} S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}. \end{aligned}$$

Proof. Indeed, let \mathbf{a}'' be the image of \mathbf{a} under the isomorphism $\text{Hom}(\sigma W \otimes V_1 \otimes W^*, V_2) \cong \text{Hom}(\sigma W \otimes V_1, V_2 \otimes W)$. Then the lemma follows from the following commutative diagram (where we omit subscripts involving $\text{loc}(m_1, n_1)$ and $\text{loc}(m_2, n_2)$).

$$\begin{array}{ccccccc} S(\widetilde{V}_1) & \xrightarrow{\mathcal{C}(\delta_{\sigma W} \otimes \text{id}_{V_1})} & S(\widetilde{\sigma W^*} \boxtimes (\widetilde{\sigma W} \otimes \widetilde{V}_1)) & \xrightarrow[\mathbb{D}\Gamma_{F-1}^*]{} & S((\widetilde{\sigma W} \otimes \widetilde{V}_1) \boxtimes \widetilde{W^*}) & \xrightarrow{\mathcal{C}(\mathbf{a})} & S(\widetilde{V}_2), \\ & \searrow \mathcal{C}(\mathbf{a}') & \downarrow \mathcal{C}(\text{id}_{\sigma W^*} \otimes \mathbf{a}'') & & \downarrow \mathcal{C}(\mathbf{a}'' \otimes \text{id}_W) & \nearrow \mathcal{C}(\text{id}_{V_2} \otimes e_W) & \\ & & S(\widetilde{\sigma W^*} \boxtimes (\widetilde{V}_2 \otimes \widetilde{W})) & \xrightarrow[\mathbb{D}\Gamma_{F-1}^*]{} & S((\widetilde{V}_2 \otimes \widetilde{W}) \boxtimes \widetilde{W^*}) & & \end{array}$$

where the left and the right triangles are commutative due to equalities

$$(\text{id}_{\sigma W^*} \otimes \mathbf{a}'') \circ (\delta_{\sigma W} \otimes \text{id}_{V_1}) = \mathbf{a}' \quad \text{and} \quad (\text{id}_{V_2} \otimes e_W) \circ (\mathbf{a}'' \otimes \text{id}_{W^*}) = \mathbf{a},$$

and the middle square is commutative due to Lemma 6.1.13. \square

Remark 6.2.4. The above proof is similar to an argument in [Laf.V12⁺, Lemma 10.1]. In fact, the proof implies the followings slightly more general statements. Let $f_1 \otimes f_2 : W_1 \otimes W_2 \rightarrow W'_1 \otimes W'_2$ be a homomorphism of $\hat{G} \times \hat{G}$ -modules. Let $\mathbf{b} \in \text{Hom}_{\hat{G}}(V_1, \sigma W_1 \otimes V_2 \otimes W_2)$ and $\mathbf{b}' \in \text{Hom}_{\hat{G}}(V_2 \otimes W'_2 \otimes W'_1, V_2)$. Then

$$(6.2.5) \quad \mathcal{C}(\mathbf{b}' \circ (\text{id}_{V_2} \otimes f_1 \otimes f_2)) \circ \mathbb{D}\Gamma_{F^{-1}}^* \circ \mathcal{C}(\mathbf{b}) = \mathcal{C}(\mathbf{b}') \circ \mathbb{D}\Gamma_{F^{-1}}^* \circ \mathcal{C}((\sigma f_2 \otimes \text{id}_{V_2} \otimes f_1) \circ \mathbf{b})$$

as elements in $\text{Hom}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2))$. Here, as above we omit subscripts involving $\text{loc}(m_1, n_1)$ and $\text{loc}(m_2, n_2)$

Lemma 6.2.5. *The definition of S on morphisms in (6.2.3) is independent of auxiliary choices. More precisely, for $V_1, V_2 \in \text{Rep}(\hat{G})$ and $\mathbf{c} \in J(V_1^* \otimes V_2)$, if we write $\mathbf{c} = \Xi_W(\mathbf{a})$ for some representation W and some element $\mathbf{a} \in \text{Hom}(\sigma W \otimes V_1 \otimes W^*, V_2)$, then the cohomological correspondence $\mathcal{C}_W(\mathbf{a}) \in \text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2))$ defined in Construction 6.2.2 depends only on \mathbf{c} but not on the particular choice of W and \mathbf{a} .*

Proof. We consider $\mathcal{O}_{\hat{G}}$ as an ind-object in $\text{Rep}(\hat{G})$ via the *right translation*. Then for every representation U of \hat{G} , the action map $\hat{G} \times U \rightarrow U$ induces a \hat{G} -equivariant map

$$\text{act}_U : U \rightarrow \mathcal{O}_{\hat{G}} \otimes \underline{U}, \quad u \mapsto \text{act}_U(u)(g) := gu,$$

where \underline{U} means the underlying vector space of U equipped with the *trivial* \hat{G} -action. We have a similar \hat{G} -equivariant map

$$m_U : \underline{U}^* \otimes U \rightarrow \mathcal{O}_{\hat{G}}, \quad (u^*, u) \mapsto m_U(u^*, u)(g) := u^*(gu).$$

Combining the two, we have a natural homomorphism of $\hat{G} \times \hat{G}$ -representations

$$\sigma W^* \otimes V_2 \otimes W \xrightarrow{\text{act}_{\sigma W^*}} \underline{W}^* \otimes \sigma \mathcal{O}_{\hat{G}} \otimes V_2 \otimes W \xrightarrow{m_W} \sigma \mathcal{O}_{\hat{G}} \otimes V_2 \otimes \mathcal{O}_{\hat{G}}.$$

Let \mathbf{a}' denote the image of \mathbf{a} under the isomorphism $\text{Hom}(\sigma W \otimes V_1 \otimes W^*, V_2) \cong \text{Hom}(V_1, \sigma W^* \otimes V_2 \otimes W)$. The map $\hat{G} \times \hat{G} \rightarrow \hat{G}\sigma$ sending $(g, h) \mapsto \sigma(g)^{-1}\sigma(h)\sigma$ induces a natural map $d_\sigma : \mathcal{O}_{\hat{G}\sigma} \rightarrow \sigma \mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}}$, intertwining the (twisted) conjugation action on $\mathcal{O}_{\hat{G}\sigma}$ and the diagonal action on $\sigma \mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}}$. Let $d_\sigma(\mathbf{c}')$ denote the image of \mathbf{c} under the following map

$$J(V_1^* \otimes V_2) \cong (\mathcal{O}_{\hat{G}\sigma} \otimes V_1^* \otimes V_2)^{\hat{G}} \xrightarrow{d_\sigma} (\sigma \mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}} \otimes V_1^* \otimes V_2)^{\hat{G}} \cong \text{Hom}_{\hat{G}}(V_1, \sigma \mathcal{O}_{\hat{G}} \otimes V_2 \otimes \mathcal{O}_{\hat{G}}).$$

It is straightforward to check that the following two diagrams are tautologically commutative.

$$\begin{array}{ccc} V_1 & \xrightarrow{\mathbf{a}'} & \sigma W^* \otimes V_2 \otimes W \\ \parallel & & \downarrow m_W \circ \text{act}_{\sigma W^*} \\ V_1 & \xrightarrow{d_\sigma(\mathbf{c}')} & \sigma \mathcal{O}_{\hat{G}} \otimes V_2 \otimes \mathcal{O}_{\hat{G}} \end{array} \quad \begin{array}{ccc} V_2 \otimes W \otimes W^* & \xrightarrow{\text{id}_{V_2} \otimes e_W} & V_2 \\ \downarrow m_W \circ \text{act}_{W^*} & & \parallel \\ V_2 \otimes \mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}} & \xrightarrow{\text{ev}_{(1,1)}} & V_2, \end{array}$$

where $\text{ev}_{(1,1)}$ is the evaluation map at $(1, 1) \in \hat{G} \times \hat{G}$. Indeed, for $v_1 \in V_1$, its has the same image

$$(g, h) \mapsto \sum_i \langle \mathbf{a}'(v_1), \sigma(g)^{-1}h^{-1} \cdot e_i \otimes e_i^* \rangle$$

under both $d_\sigma(\mathbf{c}')$ and $m_W \circ \text{act}_{\sigma W^*} \circ \mathbf{a}'$, where $\{e_i\}_i$ is a basis of $W(= \sigma W)$ and $\{e_i^*\}_i$ its dual basis.

Now apply Remark 6.2.4 above to the case

$$W_1 \otimes W_2 := W \otimes W^*, \quad W'_1 \otimes W'_2 := \mathcal{O}_{\hat{G}} \otimes \mathcal{O}_{\hat{G}}, \quad f_1 \otimes f_2 := m_W \circ \text{act}_{W^*}, \quad \mathbf{b} := \mathbf{a}', \quad \text{and } \mathbf{b}' := \text{ev}_{(1,1)}.$$

We deduce that

$$\begin{aligned} \mathcal{C}_W(\mathbf{a}) &\stackrel{\text{Lemma 6.2.3}}{=} \mathcal{C}(\text{id}_{V_2} \otimes e_W) \circ \mathbb{D}\Gamma_{F_{(V_2 \otimes W) \boxtimes W^*}}^* \circ \mathcal{C}(\mathbf{a}') \\ &\stackrel{\text{Remark 6.2.4}}{=} \mathcal{C}(\text{ev}_{(1,1)}) \circ \mathbb{D}\Gamma_{F_{(V_2 \otimes \mathcal{O}_{\hat{G}}) \boxtimes \mathcal{O}_{\hat{G}}}}^* \circ \mathcal{C}(d_\sigma(\mathbf{c}')). \end{aligned}$$

But the last expression depends only on \mathbf{c} so the lemma follows. Of course, the map $d_\sigma(\mathbf{c}') : V_1 \rightarrow \sigma\mathcal{O}_{\hat{G}} \otimes V_2 \otimes \mathcal{O}_{\hat{G}}$ only lands in some finite dimensional submodule $\sigma W_2 \otimes V_2 \otimes W_1$ so more precisely we replace $\sigma\mathcal{O}_{\hat{G}} \times \mathcal{O}_{\hat{G}}$ by $\sigma W_2 \otimes W_1$ in the above expression and choose appropriate (m_1, n_1, m_2, n_2) to define the cohomological correspondences. \square

Remark 6.2.6. Of course, one can just define $S(\mathbf{c})$ via the last expression, which then is independent of any choices. But in practice, e.g. in §6.3, we sometimes need to express \mathbf{c} as $\Xi_W(\mathbf{a})$ and realize $S(\mathbf{c})$ as $\mathcal{C}_W(\mathbf{a})$.

Lemma 6.2.7. *For representations V_1, V_2, V_3, W_1, W_2 of \hat{G} , we have the following commutative diagram*

$$(6.2.6) \quad \begin{array}{ccc} \text{Hom}_{\text{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_2) \otimes \text{Hom}_{\text{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_2, \widetilde{V}_3) & \longrightarrow & \text{Hom}_{\text{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)}(\widetilde{V}_1, \widetilde{V}_3) \\ \uparrow \Xi_{W_1} \otimes \Xi_{W_2} & & \uparrow \Xi_{W_2 \otimes W_1} \\ \text{Hom}_{\hat{G}}(\sigma W_1 \otimes V_1 \otimes W_1^*, V_2) \otimes \text{Hom}_{\hat{G}}(\sigma W_2 \otimes V_2 \otimes W_2^*, V_3) & \longrightarrow & \text{Hom}_{\hat{G}}(\sigma W_2 \otimes \sigma W_1 \otimes V_1 \otimes W_1^* \otimes W_2^*, V_3) \\ \downarrow \mathcal{C}_{W_1} \otimes \mathcal{C}_{W_2} & & \downarrow \mathcal{C}_{W_2 \otimes W_1} \\ \text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2)) \otimes \text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_2), S(\widetilde{V}_3)) & \longrightarrow & \text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_3)), \end{array}$$

- where the top horizontal arrow is given by the composition of morphisms in $\text{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma)$ or equivalently natural products of functions on $\hat{G}\sigma$,
- the bottom horizontal arrow is the composition of morphisms in $\text{PCorr}(\text{Sht}^{\text{loc}})$, and
- the middle horizontal arrow is given by sending $\mathbf{a}_1 \otimes \mathbf{a}_2$ to the homomorphism

$$(6.2.7) \quad \sigma W_2 \otimes \sigma W_1 \otimes V_1 \otimes W_1^* \otimes W_2^* \xrightarrow{\text{id}_{\sigma W_2} \otimes \mathbf{a}_1 \otimes \text{id}_{W_2^*}} \sigma W_2 \otimes V_2 \otimes W_2^* \xrightarrow{\mathbf{a}_2} V_3.$$

In particular, the definition of S on morphisms defined in (6.2.3) is compatible with composition of morphisms, i.e. for $\mathbf{a}_1 \in \mathbf{J}(V_1^* \otimes V_2)$ and $\mathbf{a}_2 \in \mathbf{J}(V_2^* \otimes V_3)$, we have

$$(6.2.8) \quad S(\mathbf{a}_2 \circ \mathbf{a}_1) = S(\mathbf{a}_2) \circ S(\mathbf{a}_1) \quad \text{in } \text{Mor}_{\text{PCorr}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_3)).$$

Proof. Once we establish the commutativity of the diagram (6.2.6), the compatibility (6.2.8) of $\widetilde{\text{Sat}}$ with composition of morphisms is clear, as $\mathcal{C}_{W_1} \otimes \mathcal{C}_{W_2}$ (resp. $\mathcal{C}_{W_2 \otimes W_1}$) factors through $\Xi_{W_1} \otimes \Xi_{W_2}$ (resp. $\Xi_{W_2 \otimes W_1}$).

We only establish the commutativity of the bottom square of (6.2.6), as the commutativity of the top square follows from the similar idea but is much easier. We choose $(m_1, n_1, m_2, n_2, m_3, n_3)$ such that

- (m_1, n_1, m_2, n_2) is $(V_1 \otimes W_1) \boxtimes W_1$ -acceptable and $(V_2 \otimes W_1) \boxtimes W_1$ -acceptable;
- (m_2, n_2, m_3, n_3) is $(V_2 \otimes W_2) \boxtimes W_2$ -acceptable and $(V_3 \otimes W_2) \boxtimes W_2$ -acceptable.

For $i = 1, 2$, choose $\mathbf{a}_i \in \text{Hom}_{\hat{G}}(\sigma W_i \otimes V_i \otimes W_i^*, V_{i+1})$, and let

$$\mathbf{a} := \mathbf{a}_2 \circ (\text{id}_{\sigma W_2} \otimes \mathbf{a}_1 \otimes \text{id}_{W_2^*}) \in \text{Hom}_{\hat{G}}(\sigma W_2 \otimes \sigma W_1 \otimes V_1 \otimes W_1^* \otimes W_2^*, V_3)$$

as in (6.2.7). We need to show that

$$(6.2.9) \quad \mathcal{C}_{W_2}^{\text{loc}(m_2, n_2)}(\mathbf{a}_2) \circ \mathcal{C}_{W_1}^{\text{loc}(m_1, n_1)}(\mathbf{a}_1) = \mathcal{C}_{W_2 \otimes W_1}^{\text{loc}(m_1, n_1)}(\mathbf{a})$$

as a cohomological correspondence from $S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)}$ to $S(\widetilde{V}_3)^{\text{loc}(m_3, n_3)}$. By the definition of composition law in the category $\text{P}^{\text{Corr}}(\text{Sht}_k^{\text{loc}})$, the left hand side is given by first forming the composition of cohomological correspondences (Definition A.2.2) $\mathcal{C}_{W_2}^{\text{loc}(m_2, n_2)}(\mathbf{a}_2)$ and $\mathcal{C}_{W_1}^{\text{loc}(m_1, n_1)}(\mathbf{a}_1)$ which is supported on $\text{Sht}_{V_1|V_2}^{W_1, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{V_2}^{\text{loc}(m_2, n_2)}} \text{Sht}_{V_2|V_3}^{W_2, \text{loc}(m_2, n_2)}$, and then pushing it forward (§A.2.6) along the perfectly proper composition map (5.3.28)

$$\text{Comp}^{\text{loc}(m_1, n_1)} : \text{Sht}_{V_1|V_2}^{W_1, \text{loc}(m_1, n_1)} \times_{\text{Sht}_{V_2}^{\text{loc}(m_2, n_2)}} \text{Sht}_{V_2|V_3}^{W_2, \text{loc}(m_2, n_2)} \rightarrow \text{Sht}_{V_1|V_3}^{W_2 \otimes W_1, \text{loc}(m_1, n_1)}.$$

We shall compute this pushforward following the list of isomorphisms and maps given in Proposition 5.3.23.

To simplify the notations, we omit the superscripts $\text{loc}(m_i, n_i)$ everywhere in the following proof; this should not cause any confusion. Our proof relies on the following diagram of cohomological correspondence.

$$\begin{array}{ccccccc}
 S(\widetilde{V}_1) & \xrightarrow{\mathcal{C}(\delta_{\sigma W_1} \otimes \text{id}_{V_1})} & S(\sigma \widetilde{W}_1^* \boxtimes (\sigma \widetilde{W}_1 \otimes \widetilde{V}_1)) & \xrightarrow{\mathbb{D}\Gamma_{F-1}^*} & S((\sigma \widetilde{W}_1 \otimes \widetilde{V}_1) \boxtimes \widetilde{W}_1^*) & \xrightarrow{\mathcal{C}(\mathbf{a}_1)} & S(\widetilde{V}_2) \\
 \parallel & \searrow & \downarrow \mathcal{C}(\text{id}_{\sigma W_1^*} \otimes \delta_{\sigma W_2} \otimes \text{id}_{\sigma W_1^*} \otimes V_1) & \searrow \mathcal{C}(\delta_{W_2} \otimes \text{id}_{\sigma W_1} \otimes V_1 \otimes \text{id}_{W_1^*}) & \downarrow & \searrow \mathcal{C}(\mathbf{a}') & \downarrow \mathcal{C}(\mathbf{a}_2') \\
 & & S(\sigma \widetilde{W}_1^* \boxtimes \sigma \widetilde{W}_2^* \boxtimes (\sigma \widetilde{W}_2 \otimes \sigma \widetilde{W}_1 \otimes \widetilde{V}_1)) & \xrightarrow{\mathbb{D}\Gamma_{F-1}^*} & S(\sigma \widetilde{W}_2^* \boxtimes (\sigma \widetilde{W}_2 \otimes \sigma \widetilde{W}_1 \otimes \widetilde{V}_1) \boxtimes \widetilde{W}_1^*) & \xrightarrow{\mathcal{C}(\mathbf{a}'')} & S(\sigma \widetilde{W}_2^* \boxtimes (\widetilde{V}_3 \otimes \widetilde{W}_2)) \\
 & & \downarrow & \searrow & \downarrow \mathbb{D}\Gamma_{F-1}^* & \searrow \mathcal{C}(\text{id}_{\sigma W_2^*} \otimes \mathbf{a}'') & \downarrow \mathbb{D}\Gamma_{F-1}^* \\
 & & & & S((\sigma \widetilde{W}_2 \otimes \sigma \widetilde{W}_1 \otimes \widetilde{V}_1) \boxtimes \widetilde{W}_1^* \boxtimes \widetilde{W}_2^*) & \xrightarrow{\mathcal{C}(\mathbf{a}'' \otimes \text{id}_{W_2^*})} & S((\widetilde{V}_3 \otimes \widetilde{W}_2) \boxtimes \widetilde{W}_2^*) \\
 & & & & \downarrow & \searrow \mathcal{C}(\text{id}_{V_3} \otimes e_{W_2}) & \downarrow \\
 S(\widetilde{V}_1) & \xrightarrow{\mathcal{C}(\delta_{\sigma W_1} \otimes \sigma W_2 \otimes \text{id}_{V_1})} & S((\sigma \widetilde{W}_1^* \otimes \sigma \widetilde{W}_2^*) \boxtimes (\sigma \widetilde{W}_2 \otimes \sigma \widetilde{W}_1 \otimes \widetilde{V}_1)) & \xrightarrow{\mathbb{D}\Gamma_{F-1}^*} & S((\sigma \widetilde{W}_2 \otimes \sigma \widetilde{W}_1 \otimes \widetilde{V}_1) \boxtimes (\widetilde{W}_1^* \otimes \widetilde{W}_2^*)) & \xrightarrow{\mathcal{C}(\mathbf{a})} & S(\widetilde{V}_3)
 \end{array}$$

Here \mathbf{a}' (resp. \mathbf{a}'') denotes the image of \mathbf{a} under the natural identification of $\text{Hom}(\sigma W_2 \otimes \sigma W_1 \otimes V_1 \otimes W_1^* \otimes W_2^*, V_3)$ with $\text{Hom}(\sigma W_1 \otimes V_1 \otimes W_1^*, \sigma W_2^* \otimes V_3 \otimes W_2)$ (resp. $\text{Hom}(\sigma W_2^* \otimes \sigma W_1 \otimes V_1 \otimes W_1^*, V_3 \otimes W_2)$).

The composition of the first row is $\mathcal{C}_{W_1}(\mathbf{a}_1)$ by Construction 6.2.2 and the composition of the right column is $\mathcal{C}_{W_2}(\mathbf{a}_2)$ by Lemma 6.2.3. The bottom line of the diagram is $\mathcal{C}_{W_2 \otimes W_1}(\mathbf{a})$. We will explain that when moving from the upper right of to lower left of the diagram, one exactly pushes the composition of $\mathcal{C}_{W_1}(\mathbf{a}_1)$ and $\mathcal{C}_{W_2}(\mathbf{a}_2)$ as cohomological correspondences along the morphism defined in Proposition 5.3.23). By definition, this is exactly the composition $\mathcal{C}_{W_2}(\mathbf{a}_2) \circ \mathcal{C}_{W_1}(\mathbf{a}_1)$ in the category $\text{P}^{\text{Corr}}(\text{Sht})$. It then follows that $\mathcal{C}_{W_2}(\mathbf{a}_2) \circ \mathcal{C}_{W_1}(\mathbf{a}_1) = \mathcal{C}_{W_2 \otimes W_1}(\mathbf{a})$ in $\text{P}^{\text{Corr}}(\text{Sht})$.

First, in the triangle in the upper right corner, since $\mathbf{a}' = \mathbf{a}_2' \circ \mathbf{a}_1$, by Lemma 6.1.8, the pushforward of the composition $\mathcal{C}(\mathbf{a}_2')$ and $\mathcal{C}(\mathbf{a}_1)$ as cohomological correspondences along the map (1) in (5.3.28) is $\mathcal{C}(\mathbf{a}')$. Next, the triangle (A) is commutative, by Lemma 5.3.22(1) and Lemma 6.1.8. Here we use the convention as in Remark 6.1.10 so the support of $\mathcal{C}(\delta_{\sigma W_2} \otimes \text{id}_{\sigma W_1} \otimes V_1 \otimes \text{id}_{W_1^*})$ (resp. $\mathcal{C}(\text{id}_{\sigma W_2^*} \otimes \mathbf{a}'')$) is

$$\text{Sht}_{(\sigma W_1 \otimes V_1) | \sigma W_2^* \boxtimes (\sigma W_2 \otimes \sigma W_1 \otimes V_1); W_1^*}^{0, \text{loc}(m_2, n_2)}, \quad (\text{resp. } \text{Sht}_{\sigma W_2^*; (\sigma W_2 \otimes \sigma W_1 \otimes V_1, W_1^*) | (V_3 \otimes W_2)}^{0, \text{loc}(m_2, n_2)}).$$

The two squares involving inverse partial Frobenii are commutative by Lemma 6.1.13.

Now we have the cohomological correspondences along the diagonal of the diagram. We claim its pushforward along the map (2) in Proposition 5.3.23) is exactly the cohomological correspondences along the bottom row. Indeed, the left trapezoid and right triangles are explained by Remark 6.1.7 and the middle pentagon follows from Lemma 6.1.12. This completes the proof of the Lemma. \square

Lemma 6.2.8. *For a morphism $\mathbf{a} \in \text{Hom}_{\hat{G}}(V_1, V_2)$ between two irreducible representations of \hat{G} , we have an equality of morphisms*

$$\Phi^{\text{loc}}(\text{Sat}(\mathbf{a})) = S(\widetilde{\mathbf{a}}).$$

In other words, the diagram (6.0.1) of functors is commutative for morphisms.

Proof. For a morphism $\mathbf{a} \in \text{Hom}_{\hat{G}}(V_1, V_2)$ between two irreducible representations of \hat{G} , $\tilde{\mathbf{a}} \in \text{Hom}_{\text{Coh}_{\hat{G}}(\hat{G}\sigma)}(\widetilde{V_1}, \widetilde{V_2})$ is represented by $\Xi_1(\mathbf{a})$. Since $W = \mathbf{1}$ in this case, we can take $n_1 = n_2$ to be an arbitrary non-negative integer n in Construction 6.2.2. Then $S(\tilde{\mathbf{a}}) = \mathcal{C}_1(\mathbf{a})$ is defined to be the composite

$$\mathcal{C}^{\text{loc}(m,n)}(\mathbf{a}) \circ \mathbb{D}\Gamma_{F_{V_1}^{-1}} \circ \mathcal{C}^{\text{loc}(m,n)}(\delta_1 \otimes \text{id}_{V_1}).$$

But $W = \mathbf{1}$, so the right two maps are trivial and therefore, $S(\tilde{\mathbf{a}}) = \mathcal{C}_1(\mathbf{a})$ is simply equal to the cohomological Satake correspondence $\mathcal{C}^{\text{loc}(m,n)}(\mathbf{a})$. Clearly, if we go right and then down on the diagram (6.0.1), the morphism $\Phi^{\text{loc}}(\text{Sat}(\mathbf{a}))$ will be exactly the one represented by $\mathcal{C}^{\text{loc}(m,n)}(\mathbf{a})$. The lemma is proved. \square

6.3. Cohomology class of the correspondence. In this subsection, we describe the image of S on morphisms explicitly in some special cases, by relating them with cycle class maps or the usual Hecke operators. This will in particular imply Theorem 6.0.1 (2), which essentially is the $S = T$ theorem [Laf.V12⁺, Proposition 0.16] of V. Lafforgue.

Now let $\tau \in \mathbb{X}_{\bullet}(Z_G)$ be central and μ minuscule.³¹ For $\mathbf{a} \in \text{Hom}_{\hat{G}}(\sigma V_{\nu} \otimes V_{\tau} \otimes V_{\nu}^*, V_{\mu})$, recall that we defined $\mathcal{C}_{V_{\nu}}(\mathbf{a})$ in Construction 6.2.2. Concretely, choose (m_1, n_1, m_2, n_2) that is $(\nu + \tau, \nu)$ -acceptable and $(\nu + \mu, \nu)$ -acceptable, then $\mathcal{C}_{V_{\nu}}(\mathbf{a})$ is realized by a cohomological correspondence

$$\mathcal{C}_{V_{\nu}}^{\text{loc}(m_1, n_1)}(\mathbf{a}) \in \text{Corr}_{\text{Sht}_{\tau|\mu}^{\nu, \text{loc}(m_1, n_1)}}(S(\widetilde{V_{\tau}})^{\text{loc}(m_1, n_1)}, S(\widetilde{V_{\mu}})^{\text{loc}(m_2, n_2)}).$$

Recall that from Corollary 5.3.21,

$$\text{Sht}_{\tau|\mu}^{\nu, \text{loc}(m_1, n_1)} \cong [L_{n_1}^{m_1} G \backslash X_{\mu^*, \nu^*}(\tau^*)] =: Y,$$

which is of dimension $\leq \langle \rho, \mu - \tau \rangle = \langle \rho, \mu \rangle$. Thus

$$\text{Corr}_Y(S(\widetilde{V_{\tau}})^{\text{loc}(m_1, n_1)}, S(\widetilde{V_{\mu}})^{\text{loc}(m_2, n_2)}) \cong H^{-\langle 2\rho, \mu \rangle}(Y, \omega_Y(-\langle \rho, \mu \rangle)) \cong H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))^{G(\mathcal{O}/\varpi^{n_1})},$$

where the first isomorphism follows from the same considerations as in §A.2.18. Since $\dim X_{\mu^*, \nu^*}(\tau^*) \leq \langle \rho, \mu \rangle$, the space $H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$ has a basis given by irreducible components of $X_{\mu^*, \nu^*}(\tau^*)$ of dimension $\langle \rho, \mu \rangle$. Therefore, we may regard $\mathcal{C}_{V_{\nu}}^{\text{loc}(m_1, n_1)}(\mathbf{a})$ as an element in $H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$. Note that this element is independent of the choices of (m_1, n_1, m_2, n_2) by (6.2.4), as soon as the choice is $(\nu + \tau, \nu)$ -acceptable and $(\nu + \mu, \nu)$ -acceptable. We thus also write this element as $\mathcal{C}_{V_{\nu}}(\mathbf{a})$. In this subsection, we make this element explicit in two special cases.

Now, we choose n large enough and consider the following commutative diagram of correspondences (6.3.1)

$$\begin{array}{ccccccccc} \text{pt} & \xleftarrow{\text{id} \times \varpi \tau^*} & \text{Gr}_{\sigma(\nu^*)} & \xrightarrow{\text{id} \times \varpi \tau^*} & \text{Gr}_{\sigma(\nu^*)} \times \text{Gr}_{\tau^* + \sigma(\nu^*)} & \xleftarrow{\sigma \times \text{id}} & \text{Gr}_{\nu^*} \times \text{Gr}_{\tau^* + \sigma(\nu^*)} & \xleftarrow{} & \text{Gr}_{(\nu^*, \mu^*)}^0|_{\tau^* + \sigma(\nu^*)} & \xrightarrow{} & \text{pt} \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{Gr}_{\tau}^{(n)} & \xleftarrow{} & (\text{Gr}_{\sigma(\nu^*)}^{(2n)} \times \text{Gr}_{\tau^* + \sigma(\nu)}^{(n)})_{\tau} & \xrightarrow{} & \text{Gr}_{\sigma(\nu^*)}^{(2n)} \times \text{Gr}_{\tau^* + \sigma(\nu)}^{(n)} & \xrightarrow{} & \text{Gr}_{\tau^* + \sigma(\nu)}^{(n)} \times \text{Gr}_{\nu^*} & \xleftarrow{} & (\text{Gr}_{\tau^* + \sigma(\nu)}^{(n)} \times \text{Gr}_{\nu^*})_{\mu} & \xrightarrow{} & \text{Gr}_{\mu} \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \text{Sht}_{\tau}^{\text{loc}(3n, n)} & \xleftarrow{} & \text{Sht}_{\tau|(\sigma(\nu^*), \tau + \sigma(\nu))}^{0, \text{loc}(3n, n)} & \xrightarrow{} & \text{Sht}_{\sigma(\nu^*), \tau + \sigma(\nu)}^{\text{loc}(3n, n)} & \xrightarrow{F^{-1}} & \text{Sht}_{\tau + \sigma(\nu), \nu^*}^{\text{loc}(2n, 0)} & \xleftarrow{} & \text{Sht}_{(\tau + \sigma(\nu), \nu^*)|_{\mu}}^{0, \text{loc}(2n, 0)} & \xrightarrow{} & \text{Sht}_{\mu}^{\text{loc}(2n, 0)}. \end{array}$$

(A) (B) (C) (D) (E)

³¹The assumption on μ is not necessary. But it simplifies the discussion and will be sufficient for our application.

This diagram requires some explanation. The commutative square (D) follows from Remark 5.3.15. We define $(\mathrm{Gr}_{\sigma(\nu^*)}^{(2n)} \times \mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)})_\tau$ and $(\mathrm{Gr}_{\nu^*} \times \mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)})_\mu$ so that the commutative square (C) and (E) are Cartesian. Explicitly,

$$\begin{aligned} (\mathrm{Gr}_{\sigma(\nu^*)}^{(2n)} \times \mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)})_\tau &= \{(g_1 L^+ G^{(2n)}, g_2 L^+ G^{(n)}) \in \mathrm{Gr}_{\sigma(\nu^*)}^{(2n)} \times \mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)} \mid g_1 g_2 \in \varpi^\tau L^+ G^{(n)}\} \cong \mathrm{Gr}_{\sigma(\nu^*)}^{(2n)}, \\ (\mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)} \times \mathrm{Gr}_{\nu^*})_\mu &= \{(g_1 L^+ G^{(n)}, g_2 L^+ G) \in \mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)} \times \mathrm{Gr}_{\nu^*} \mid g_1 g_2 \in L^+ G \varpi^\mu L^+ G\}. \end{aligned}$$

It follows that the commutative square (B) is also Cartesian.

Note that since μ is minuscule, the \star -pullback (in the sense of Notation A.2.1) of IC_{ν^*} along the perfectly smooth morphism $\mathrm{Gr}_{\nu^*} \tilde{\times} \mathrm{Gr}_{\mu^*} \rightarrow \mathrm{Gr}_{\nu^*}$ is $\mathrm{IC}_{\nu^*} \tilde{\boxtimes} \mathrm{IC}_{\mu^*}$ on $\mathrm{Gr}_{\nu^*, \mu^*}$. Then

$$\begin{aligned} & \mathrm{Hom}_{\hat{G}}(V_{\sigma(\nu)} \otimes V_\tau \otimes V_{\nu^*}, V_\mu) \\ & \cong \mathrm{Hom}_{\hat{G}}(V_{\nu^*} \otimes V_{\mu^*}, V_{\tau^*} \otimes V_{\sigma(\nu^*)}) \\ & \stackrel{\mathrm{Sat}}{\cong} \mathrm{Corr}_{\mathrm{Gr}_{(\nu^*, \mu^*)}^0 | \tau^* + \sigma(\nu^*)}((\mathrm{Gr}_{\nu^*, \mu^*}, \mathrm{IC}_{\nu^*} \tilde{\boxtimes} \mathrm{IC}_{\mu^*}), (\mathrm{Gr}_{\tau^* + \sigma(\mu^*)}, \mathrm{IC}_{\tau^* + \sigma(\nu^*)})) \\ & \cong \mathrm{Corr}_{\mathrm{Gr}_{(\nu^*, \mu^*)}^0 | \tau^* + \sigma(\nu^*)}((\mathrm{Gr}_{\nu^*}, \mathrm{IC}_{\nu^*}[-\langle 2\rho, \mu \rangle](-\langle \rho, \mu \rangle)), (\mathrm{Gr}_{\tau^* + \sigma(\mu^*)}, \mathrm{IC}_{\tau^* + \sigma(\nu^*)})) \\ & \cong \mathrm{Corr}_{\mathrm{Gr}_{(\nu^*, \mu^*)}^0 | \tau^* + \sigma(\nu^*)}((\mathrm{Gr}_{\nu^*} \times \mathrm{Gr}_{\tau^* + \sigma(\nu^*)}, \mathrm{IC}_{\nu^*} \boxtimes \mathrm{IC}_{\tau^* + \sigma(\nu^*)}), (\mathrm{pt}, \overline{\mathbb{Q}}_\ell[-\langle 2\rho, \mu \rangle](-\langle \rho, \mu \rangle))), \end{aligned}$$

where the last isomorphism follows from Lemma A.2.4. We denote by $\mathrm{Sat}(\mathbf{a})^\sharp$ the image of $\mathbf{a} \in \mathrm{Hom}_{\hat{G}}(V_{\sigma(\nu)} \otimes V_\tau \otimes V_{\nu^*}, V_\mu)$ under this series of isomorphisms.

Note that all vertical arrows in (6.3.1) are cohomologically smooth by Lemma 3.1.13, and we can apply the formalism of pullback of cohomological correspondences as discussed in §A.2.11.

- The $\langle 3n \dim G \rangle$ -shift of the smooth pullback of $\mathcal{C}^{\mathrm{loc}(3n)}(\mathrm{id}_\tau \otimes \delta_{\sigma(\nu^*)})$ to the middle row is equal to the $\langle n \dim G \rangle$ -shift of the smooth pullback of

$$\mathbb{D}((\Gamma_{\varpi^{\tau^*}}^*)^\sharp) : (\mathrm{pt}, \overline{\mathbb{Q}}_\ell) \rightarrow (\mathrm{Gr}_{\sigma(\nu^*)} \times \mathrm{Gr}_{\tau^* + \sigma(\nu^*)}, \mathrm{IC}_{\sigma(\nu^*)} \boxtimes \mathrm{IC}_{\tau^* + \sigma(\nu^*)}),$$

where $\Gamma_{\varpi^{\tau^*}}^*$ is the cohomological correspondence associated to the canonical isomorphism $(\varpi^{\tau^*})^* \mathrm{IC}_{\tau^* + \sigma(\nu^*)} \cong \mathrm{IC}_{\sigma(\nu^*)}$ as from Example A.2.3 (2), $(\Gamma_{\varpi^{\tau^*}}^*)^\sharp$ is obtained from $\Gamma_{\varpi^{\tau^*}}^*$ by Lemma A.2.4;

- The shifted smooth pullback of $\mathbb{D}F^{-1}$ from the bottom to the middle row is equal to the shifted smooth pullback of $\Gamma_{\sigma \times \mathrm{id}}^*$ from the top to the middle row.
- The shifted smooth pullback of $\mathcal{C}^{\mathrm{loc}(2n)}(\mathbf{a})$ from the bottom to the middle row is equal to the shifted smooth pullback of $\mathrm{Sat}(\mathbf{a})^\sharp$ from the top to the middle row.

It follows from Lemma A.2.15 that the shifted smooth pullback of

$$\mathcal{C}_{V_\nu}^{\mathrm{loc}(3n, n)}(\mathbf{a}) = \mathcal{C}^{\mathrm{loc}(2n, n)}(\mathbf{a}) \circ \mathbb{D}F^{-1} \circ \mathcal{C}^{\mathrm{loc}(3n, n)}(\mathrm{id}_\tau \otimes \delta_{\sigma(\nu^*)})$$

to a cohomological correspondence

$$(6.3.2) \quad (\mathrm{Gr}_\tau^{(n)}, \overline{\mathbb{Q}}_\ell \langle n \dim G \rangle) \rightarrow (\mathrm{Gr}_\mu, \overline{\mathbb{Q}}_\ell\langle 2\rho, \mu \rangle)$$

is equal to the shifted smooth pullback of

$$\mathcal{C}_{V_\nu}^{\mathrm{Gr}}(\mathbf{a}) := \mathrm{Sat}(\mathbf{a})^\sharp \circ \Gamma_{\sigma \times \mathrm{id}}^* \circ \mathbb{D}((\Gamma_{\varpi^{\tau^*}}^*)^\sharp).$$

Let us denote the support of compositions of the correspondences in the upper row (resp. middle row, lower row) of the diagram by C^u (resp. C , C_l). Then

$$C := \{(g_1 L^+ G^{(2n)}, g_2 L^+ G^{(n)}) \in \mathrm{Gr}_{\sigma(\nu^*)}^{(2n)} \times \mathrm{Gr}_{\tau+\sigma(\nu)}^{(n)} \mid g_1 g_2 \in \varpi^\tau L^+ G^{(n)}, g_2 \sigma^{-1}(g_1) \in L^+ G \varpi^\mu L^+ G\}$$

The map $C \rightarrow C^u$ is induced by $g_1 L^+ G^{(2n)} \mapsto \sigma^{-1} g_1 L^+ G$. This realizes C as an $L^{2n}G$ -torsor over $C^u \cong X_{\mu^*, \nu^*}(\tau^*) \subset \mathrm{Gr}_{\nu^*}$.

On the other hand, by the proof of Corollary 5.3.21, further composing this map with

$$X_{\mu^*, \nu^*}(\tau^*) \rightarrow [L_n^{3n} G \backslash X_{\mu^*, \nu^*}(\tau^*)] \cong C_l$$

gives the perfectly smooth morphism $C \rightarrow C_l$.

Now, the cohomological correspondence (6.3.2) is given by a class in $c \in H_{\langle 2\rho, \mu \rangle + 4n \dim G}^{\text{BM}}(C)$, and the cohomological correspondence $\mathcal{C}_{V_\nu}^{\text{Gr}}(\mathbf{a})$ is given by a class in $H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(C^u) = H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$, by the same considerations as in §A.2.18. The above observations imply that under the canonical isomorphisms

$$H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*)) \cong H_{\langle 2\rho, \mu \rangle + 4n \dim G}^{\text{BM}}(C) \cong H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*)),$$

$\mathcal{C}_{V_\nu}(\mathbf{a}) = c = \mathcal{C}_{V_\nu}^{\text{Gr}}(\mathbf{a})$. We thus have proven

Lemma 6.3.1. *The class $\mathcal{C}_{V_\nu}(\mathbf{a}) \in H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$ is equal to the class $\mathcal{C}_{V_\nu}^{\text{Gr}}(\mathbf{a}) \in H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$.*

Assume that

$$\mathbf{a} \in \text{Hom}_{\hat{G}}(\sigma V_\nu \otimes V_\tau \otimes V_{\nu^*}, V_\mu) \cong \text{Hom}(V_{\nu^*} \otimes V_{\mu^*}, V_{\sigma(\nu^*)} \otimes V_{\tau^*})$$

is given by the fundamental class of a Satake cycle $\text{Gr}_{(\nu^*, \mu^*)|\sigma(\nu^*)+\tau^*}^{0, \mathbf{a}}$ via Proposition 3.1.10, and assume that $X_{\mu^*, \nu^*}^{\mathbf{a}}(\tau)$ as defined via (4.4.3). Then it follows from definition that $\mathcal{C}_{V_\nu}(\mathbf{a}) = \mathcal{C}_{V_\nu}^{\text{Gr}}(\mathbf{a})$ in fact lies in the image of $H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}^{\mathbf{a}}(\tau^*)) \rightarrow H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$.

Now we specialize to the cases we need.

Proposition 6.3.2. *Assume that $\mathbf{b} = i^{\mathbb{M}\mathbb{V}}(\mathbf{a}) \in \mathbb{M}\mathbb{V}_\mu(\tau + \sigma(\nu) - \nu)$ satisfies Lemma 4.4.3. Then $\mathcal{C}_{V_\nu}(\mathbf{a}) \in H_{\langle 2\rho, \mu \rangle}^{\text{BM}}(X_{\mu^*, \nu^*}(\tau^*))$ is given by the fundamental class of $X_{\mu^*}^{\mathbf{b}, x_0} \subset X_{\mu^*, \nu^*}(\tau^*)$.*

Proof. By Theorem 4.4.5, $X_{\mu^*, \nu^*}^{\mathbf{a}}(\tau^*)$ has a unique irreducible component of dimension $\langle \rho, \mu \rangle$, namely $X_{\mu^*}^{\mathbf{b}, x_0}(\tau^*)$. So $\mathcal{C}_{V_\nu}(\mathbf{a})$ is some multiple of the fundamental class of $X_{\mu^*}^{\mathbf{b}, x_0}(\tau^*)$. Since the intersection of this component with $\mathring{\text{Gr}}_{\nu^*}$ is non-empty (so open dense in $X_{\mu^*}^{\mathbf{b}, x_0}(\tau^*)$). Then we can restrict every cohomological class to the smooth open locus. The smooth part of the top row of (6.3.1) can be identified with intersection of the diagonal of $\mathring{\text{Gr}}_{\sigma(\nu^*)} \times \mathring{\text{Gr}}_{\sigma(\nu^*)}$ with the following correspondence:

$$C = \mathring{\text{Gr}}_{(\nu^*, \mu^*)|\tau^*+\sigma(\nu^*)}^{0, \mathbf{a}} \xrightarrow{c_1 \times c_2} \mathring{\text{Gr}}_{\sigma(\nu^*)} \times \mathring{\text{Gr}}_{\sigma(\nu^*)},$$

where $\mathring{\text{Gr}}_{(\nu^*, \mu^*)|\tau^*+\sigma(\nu^*)}^{0, \mathbf{a}}$ is as in (4.4.8), and

$$c_1 : \mathring{\text{Gr}}_{(\nu^*, \mu^*)|\tau^*+\sigma(\nu^*)}^{0, \mathbf{a}} \rightarrow \mathring{\text{Gr}}_{\nu^*} \xrightarrow{\sigma} \mathring{\text{Gr}}_{\sigma(\nu^*)}, \quad c_2 : \mathring{\text{Gr}}_{(\nu^*, \mu^*)|\tau^*+\sigma(\nu^*)}^{0, \mathbf{a}} \rightarrow \mathring{\text{Gr}}_{\sigma(\nu^*)+\tau^*} \xrightarrow{\varpi^\tau} \mathring{\text{Gr}}_{\sigma(\nu^*)}.$$

The cohomological correspondence then is identified with $\delta_{\overline{\mathbb{Q}}_\ell} \circ u^\sharp$, where $u : c_1^* \overline{\mathbb{Q}}_\ell[\langle 2\rho, \mu \rangle](\langle \rho, \mu \rangle) \rightarrow c_2^! \overline{\mathbb{Q}}_\ell$ is given by the fundamental class of $\mathring{\text{Gr}}_{(\nu^*, \mu^*)|\tau^*+\sigma(\nu^*)}^{0, \mathbf{a}}$. Choose a deperfection and use the fact that the differential of Frobenius is zero, we see C intersects with the diagonal properly smoothly. The proposition now follows from Lemma A.2.21. \square

In the second case, we assume that $\mu = \tau$, $\sigma(\nu) = \nu$. Let $\mathbf{a} \in \text{Hom}(V_\nu \otimes V_\tau \otimes V_{\nu^*}, V_\tau)$ be the homomorphism induced by $\mathbf{e}_\nu : V_\nu \otimes V_{\nu^*} \rightarrow \mathbf{1}$. Then $\text{Gr}_{(\nu^*, \mu^*)|\sigma(\nu^*)+\tau^*}^{0, \mathbf{a}} = \text{Gr}_{(\nu^*, \mu^*)|\sigma(\nu^*)+\tau^*}^0$ is isomorphic to Gr_{ν^*} , and

$$X_{\mu^*, \nu^*}(\tau^*) = \text{Gr}_{\nu^*}(k)$$

is the set of rational points of Gr_{ν^*} . The top row of (6.3.1) can be identified with

$$\text{pt} \leftarrow \text{Gr}_{\nu^*} \xrightarrow{\Delta} \text{Gr}_{\nu^*} \times \text{Gr}_{\nu^*} \xleftarrow{\sigma \times \text{id}} \text{Gr}_{\nu^*} \times \text{Gr}_{\nu^*} \xleftarrow{\Delta} \text{Gr}_{\nu^*} \rightarrow \text{pt},$$

and the cohomological correspondence can be identified with $\delta_{\mathrm{IC}_{\nu^*}} \circ \Gamma_{\sigma \times \mathrm{id}}^* \circ e_{\mathrm{IC}_{\nu^*}}$, where $e_{\mathrm{IC}_{\nu^*}}$ and $\delta_{\mathrm{IC}_{\nu^*}}$ are as in Example A.2.3(4). Then Lemma A.2.22, together with Proposition 3.5.5, implies the following result, which gives Theorem 6.0.1 (2) (by Remark 3.5.3).

Proposition 6.3.3. *The class $\mathcal{C}_{V_{\nu}}^{\mathrm{loc}(m_1, n_1)}(\mathbf{e}) \in H_0^{\mathrm{BM}}(X_{\mu^*, \nu^*}^{\mathbf{e}}(\tau^*)) = C(\mathrm{Gr}_{\nu^*}(k))$ is the function $\mathrm{Sat}^{\mathrm{cl}}([V_{\nu^*}])$, i.e. its value at $x \in \mathrm{Gr}_{\nu^*}(k)$ is*

$$\mathrm{tr}(\phi_x \mid \mathrm{Sat}(V_{\nu^*})_{\bar{x}}).$$

7. APPLICATION TO SHIMURA VARIETIES

In this section, we prove our main theorem.

Convention 7.0.1. Our convention for Hodge structure, weight maps, and Artin reciprocity map all follow [De79]. But the reciprocity morphism of [De79, §2.2.3] is changed to be the one *without taking the inverse*, as pointed out by [Mi90].

7.1. The canonical integral model of Shimura varieties of Hodge type. In this subsection, we recall the canonical integral models of Shimura varieties of Hodge type and some related constructions, mostly following [Kis10].

7.1.1. The canonical integral model. Let G be a connected reductive group over \mathbb{Q} , and let X be a $G(\mathbb{R})$ -conjugacy class of homomorphisms $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$. Recall that the pair (G, X) is called a *Shimura datum* if for some (and therefore every) $h \in X$,

- (SV1) under $\mathrm{Ad} \circ h : \mathbb{S} \rightarrow \mathrm{GL}(\mathfrak{g})$, \mathfrak{g} acquires a Hodge structure of type $(-1, 1), (0, 0), (1, -1)$,
- (SV2) $\mathrm{adh}(i)$ is a Cartan involution of $G_{\mathbb{R}}^{\mathrm{ad}}$, and
- (SV3) G^{ad} does not contain a \mathbb{Q} -factor whose real points are compact.

In this work, we will also consider groups that do not satisfy (SV3). A reductive group G over \mathbb{Q} together with a set X of a $G(\mathbb{R})$ -conjugacy classes of $h : \mathbb{S} \rightarrow G_{\mathbb{R}}$ satisfying (SV1), (SV2) will be called a *weak Shimura datum*.

The map $w_h : \mathbb{G}_m \subset \mathbb{S} \xrightarrow{h} G_{\mathbb{R}}$ factors through the center of $G_{\mathbb{R}}$, and is therefore independent of the choice of h . We denote it by w and call it the *weight homomorphism*.

For $G = \mathrm{GSp}(V, \psi)$, the similitude group of a symplectic \mathbb{Q} -vector space, we use \mathfrak{H}^{\pm} (instead of X) to denote the Siegel (upper and lower) half spaces of complex structures on V such that $\psi(\cdot, J\cdot)$ is (positive or negative) definite.³² Recall that a (weak) Shimura datum (G, X) is called of *Hodge type* if there is an embedding $\rho : G \rightarrow \mathrm{GSp}(V, \psi)$ that induces an embedding $X \rightarrow \mathfrak{H}^{\pm}$ given by $h \mapsto \rho \circ h$.

Let us fix the isomorphism $\mathbb{S}_{\mathbb{C}} \cong \mathbb{G}_m \times \mathbb{G}_m$ as usual (see (2.1.4)). Let $\mathbb{G}_m \rightarrow \mathbb{S}_{\mathbb{C}}$ denote the inclusion into the first factor. The conjugacy class $\{h\}$ induces a conjugacy class $\mu_h : \mathbb{G}_m \rightarrow \mathbb{S}_{\mathbb{C}} \rightarrow G_{\mathbb{C}}$, which is the same as a conjugacy class of 1-parameter subgroups of G over $\overline{\mathbb{Q}} \subset \mathbb{C}$. This conjugacy class is called the *Shimura cocharacter*, and the field of its definition is called the *reflex field*, denoted by $E = E(G, X)$. In the sequel, we write μ for μ_h to simplify notation.

Let $K \subset G(\mathbb{A}_f)$ be a (sufficiently small) open compact subgroup. Let $\mathbf{Sh}_K(G, X)$ denote the corresponding Shimura variety defined over E . We will fix a prime $p > 2$, such that K_p is a hyperspecial subgroup of $G(\mathbb{Q}_p)$, i.e. $K_p = \underline{G}(\mathbb{Z}_p)$ for some reductive group extension \underline{G} of G to $\mathbb{Z}_{(p)}$. Let v be a place of $E(G, X)$ over p . Let $\mathcal{O}_{E, (v)}$ denote the ring of integers of E localized at v . Recall the following theorem of Kisin [Kis10] and Vasiu [Vas07].

Theorem 7.1.2. *If (G, X) is of Hodge type, there is a smooth integral canonical model $\mathcal{S}_K(G, X)$ of $\mathbf{Sh}_K(G, X)$, defined over $\mathcal{O}_{E, (v)}$.*

³²We used the notation ψ earlier to in the definition of local G -shtukas, but there should be no confusion.

Remark 7.1.3. In fact, the above theorem is more precise. Write $K = K_p K^p$, and consider the pro-scheme $\mathcal{S}_{K_p}(G, X) = \varprojlim_{K^p} \mathcal{S}_{K_p K^p}(G, X)$. It satisfies the extension property: for any regular, formally smooth $\mathcal{O}_{E, (v)}$ -scheme T , a map $S \otimes E \rightarrow \mathcal{S}_{K_p}(G, X)$ extends to a T -point of $\mathcal{S}_{K_p}(G, X)$. This property uniquely characterizes $\mathcal{S}_{K_p}(G, X)$.

The construction of $\mathcal{S}_K(G, X)$ (in the case of Hodge type) is as follows. First, if $(G', X') = (\mathrm{GSp}(V, \psi), \mathfrak{H}^\pm)$ with $\dim V = 2g$, then $\mathbf{Sh}_K(G', X')$ is the moduli space of principally polarized abelian varieties with a K' -level structure. It has a canonical extension to an integral model $\mathcal{A}_{g, K'}$ smooth over $\mathbb{Z}_{(p)}$ as the moduli space of principally polarized abelian schemes, whose definition will be recalled in §7.1.4 below. In general, if (G, X) is of Hodge type, one can choose a faithful representation $i : G \rightarrow \mathrm{GSp}(V, \psi)$ as above such that $i(X) \subset \mathfrak{H}^\pm$, that $\underline{G}(\mathbb{Z}_p) = G(\mathbb{Q}_p) \cap \mathrm{GSp}(V_{\mathbb{Z}_p}, \psi)$ for a self-dual \mathbb{Z}_p -lattice $V_{\mathbb{Z}_p}$ in $V_{\mathbb{Q}_p}$, and that $K^p = G(\mathbb{A}_f^p) \cap K'^p$, where $K'^p \subseteq \mathrm{GSp}(V, \psi)(\mathbb{A}_f^p)$ is a prime-to- p level structure for the Siegel moduli space (cf. [Kis10, Lemma 2.3.1]). Then i induces a closed embedding $i : \mathbf{Sh}_K(G, X) \rightarrow \mathbf{Sh}_{K'}(G', X') \otimes E$ defined over E . We write $\mathcal{S}_K^-(G, X)$ for the closure of $\mathbf{Sh}_K(G, X)$ in $\mathcal{A}_{g, K'}$, and define $\mathcal{S}_K(G, X)$ to be its normalization. By [Kis10, Proposition 2.3.5], the completion at each closed point of $\mathcal{S}_K^-(G, X)$ has *formally smooth* irreducible components over $\mathcal{O}_{E, v}$. So the morphism

$$(7.1.1) \quad i : \mathcal{S}_K(G, X) \rightarrow \mathcal{S}_K^-(G, X) \subseteq \mathcal{A}_{g, K'}$$

induces a surjective map of the completed local rings $\widehat{\mathcal{O}}_{\mathcal{A}_{g, K'}, i(x)} \rightarrow \widehat{\mathcal{O}}_{\mathcal{S}_K(G, X), x}$ at every point $x \in \mathcal{S}_K(G, X)$. In what follows, we will use \mathcal{S}_K to denote the base change of $\mathcal{S}_K(G, X)$ to the completion $\mathcal{O}_{E, v}$.

7.1.4. Abelian schemes up to prime-to- p isogeny. We recall the moduli interpretation of $\mathcal{A}_{g, K'}$. First, recall that for every (quasi-compact) scheme T , there is the category AV^p of abelian schemes on T up to prime-to- p isogenies: objects are abelian schemes over T , and morphisms are the homomorphisms of abelian schemes tensored with $\mathbb{Z}_{(p)}$. Isomorphisms in AV^p are called *prime-to- p quasi-isogenies*. Given A , a *polarization* λ of A is a morphism in $\mathrm{Hom}_{\mathrm{AV}^p}(A, A^\vee)$, such that some of its multiple $n\lambda$ is induced by an ample line bundle. A *principal polarization* is a polarization that is an isomorphism in this category. Then we have the groupoid of principally polarized abelian scheme on T up to prime-to- p isogeny: objects are pairs (A, λ) , consisting of an abelian scheme over T up to prime-to- p isogeny and a principal polarization; morphisms from (A, λ) to (A', λ') are prime-to- p quasi-isogenies $f : A \rightarrow A'$ such that $f^* \lambda' = c\lambda$ for some locally constant function c on T with values in $\mathbb{Z}_{(p)}^\times$.

Let (A, λ) be a principally polarized abelian scheme of dimension g on T up to prime-to- p isogeny. Then the prime-to- p rational Tate module

$$\mathcal{V}_{\mathrm{et}}(A)^p := \varprojlim_{p \neq n} A[n] \otimes \mathbb{Q}$$

is a \mathbb{A}_f^p -local system on T equipped with an alternating pairing $\mathcal{V}_{\mathrm{et}}(A)^p \otimes \mathcal{V}_{\mathrm{et}}(A)^p \rightarrow \mathbb{A}_f^p(1)$. Let $\mathrm{Isom}(V \otimes \mathbb{A}_f^p, \mathcal{V}_{\mathrm{et}}(A)^p)$ denote the $G'(\mathbb{A}_f^p)$ -torsor of isomorphisms between $V \otimes \mathbb{A}_f^p$ and $\mathcal{V}_{\mathrm{et}}(A)^p$ that respect to the natural pairs up to a $(\mathbb{A}_f^p)^\times$ -scalar³³. Then a K'^p -level structure of (A, λ) is a section η of the étale sheaf $\mathrm{Isom}(V \otimes \mathbb{A}_f^p, \mathcal{V}_{\mathrm{et}}(A)^p)/K'^p$.

Finally, $\mathcal{A}_{g, K'}$ is the moduli functor which assigns every T the groupoid, whose objects are triples (A, λ, η) , where (A, λ) is a principally polarized abelian scheme on T up to prime-to- p isogeny, and η is a K'^p -level structure, and whose morphisms from (A, λ, η) to (A', λ', η') are morphisms $f : (A, \lambda) \rightarrow (A', \lambda')$ as defined above such that $\mathcal{V}_{\mathrm{et}}(f)^p \circ \eta = \eta'$. If K'^p is sufficiently small, this groupoid is discrete (i.e. there is no non-trivial automorphism).

³³See §7.2.2 below for a more canonical formulation.

7.1.5. *Étale, de Rham, and crystalline tensors.* By construction, there is an abelian scheme

$$h : A \rightarrow \mathcal{S}_K(G, X)$$

which is the pullback of the universal one on $\mathcal{A}_{g,K'}$. For our purpose, we need to recall the construction of a collection of tensors in various cohomology of A , following [Kis10].

Notation 7.1.6. Let \mathcal{C} be a rigid tensor additive category equipped with an invertible object \mathbb{T} (in the sequel \mathcal{C} will be categories of “local systems” in various settings, and \mathbb{T} will be the Tate object in the corresponding category). That is, $\mathbb{T} \otimes \mathbb{T}^\vee \rightarrow \mathbf{1}$ is an isomorphism where \mathbb{T}^\vee is the dual of \mathbb{T} and $\mathbf{1}$ is the unit object. We denote by $X(n) = X \otimes \mathbb{T}^{\otimes n}$ if $n \geq 0$ and $X(n) = X \otimes (\mathbb{T}^\vee)^{\otimes n}$ if $n < 0$. If X is an object in \mathcal{C} , equipped with an alternating pairing $\psi : X \otimes X \rightarrow \mathbb{T}$ which induces an isomorphism $X \cong X^\vee(1)$, we denote

$$X^\otimes = \bigoplus_{r \in \mathbb{N}} X^{\otimes 2r}(-r) \cong \bigoplus_{r \in \mathbb{N}} (X^\vee)^{\otimes 2r}(r),$$

as an algebra object in the ind-completion of \mathcal{C} . Note that one can give a natural grading on X^\otimes so that the direct summand $X^{\otimes 2r}(-r) \cong (X^\vee)^{\otimes 2r}(r)$ is of degree r .

Now assume that \mathcal{C} is R -linear, where R is some commutative ring with unit, and let \mathcal{T} be a (not necessarily commutative) graded R -algebra. An object X of \mathcal{C} is said to be *equipped with a \mathcal{T} -structure* if it is given a morphism of graded algebra (in the ind-completion of \mathcal{C})

$$s_X : \mathcal{T} \otimes \mathbf{1} \rightarrow X^\otimes.$$

Given two objects X and Y in \mathcal{C} , both equipped with a \mathcal{T} -structure, we denote the subspace of

$$\mathrm{Hom}((X, s_X), (Y, s_Y)) \subset \mathrm{Hom}_{\mathcal{C}}(X, Y), \quad \text{resp.} \quad \mathrm{Isom}((X, s_X), (Y, s_Y)) \subset \mathrm{Isom}_{\mathcal{C}}(X, Y)$$

consisting of those morphisms (resp. isomorphisms) $f : X \rightarrow Y$ such that $s_Y := f^\otimes \circ s_X$.

Let $V_{\mathbb{Z}(p)}$ be a $\mathbb{Z}(p)$ -lattice of V whose p -adic completion is $V_{\mathbb{Z}_p} \subset V_{\mathbb{Q}_p}$ so ψ induces a perfect pairing of $V_{\mathbb{Z}(p)}$. Let $\mathbb{Z}(p)(1)$ be the rank one $\mathbb{Z}(p)$ -module on which $\mathrm{GSp}(V_{\mathbb{Z}(p)}, \psi)$ acts through the similitudes $\mathrm{GSp}(V_{\mathbb{Z}(p)}, \psi) \rightarrow \mathbb{G}_m$. The symplectic form is regarded as a perfect pairing of $\mathrm{GSp}(V_{\mathbb{Z}(p)}, \psi)$ -representations

$$\psi : V_{\mathbb{Z}(p)} \otimes V_{\mathbb{Z}(p)} \rightarrow \mathbb{Z}(p)(1).$$

We take \mathcal{C} to be the category of representations of $\mathrm{GSp}(V_{\mathbb{Z}(p)}, \psi)$ on finite free $\mathbb{Z}(p)$ -modules and $\mathbb{T} = \mathbb{Z}(p)(1)$. The following lemma slightly refines [Kis10, Proposition 1.3.2].

Lemma 7.1.7. *The group scheme $\underline{G} \subseteq \mathrm{GSp}(V_{\mathbb{Z}(p)}, \psi)$ is the schematic centralizer of a finitely graded $\mathbb{Z}(p)$ -subalgebra $\mathcal{T} \subseteq (V_{\mathbb{Z}(p)}^\otimes)^\underline{G}$.*

Notation 7.1.8. In the sequel, we denote the natural inclusion $\mathcal{T} \subset V_{\mathbb{Z}(p)}^\otimes$ by s . We assume that the element in $V_{\mathbb{Z}(p)}^\vee \otimes V_{\mathbb{Z}(p)}^\vee(1)$ induced by the alternating pairing ψ belongs to \mathcal{T} and is denoted by ψ as well.

If R is a $\mathbb{Z}(p)$ -algebra, we write $V_R := V_{\mathbb{Z}(p)} \otimes_{\mathbb{Z}(p)} R$ and $R(1) = \mathbb{Z}(p)(1) \otimes_{\mathbb{Z}(p)} R$.

In below, let

$$\mathcal{H}_B(A) := R^1 h_{\mathbb{C},*} \mathbb{Q}, \quad \mathcal{H}_{\mathrm{et}}(A) := R^1 h_{\mathrm{et},*} \mathbb{A}_f, \quad \mathcal{H}_{\mathrm{dR}}(A) := R^1 h_{\mathrm{dR},*} \Omega^\bullet$$

denote the first relative Betti, étale, and de Rham cohomology of A (over E). The ℓ -component of $\mathcal{H}_{\mathrm{et}}(A)$ is denoted by $\mathcal{H}_{\mathrm{et}}(A)_\ell$, and the prime-to- p component is denoted by $\mathcal{H}_{\mathrm{et}}(A)^p$. Their duals, the first relative homology, are denoted by $\mathcal{V}_B(A), \mathcal{V}_{\mathrm{et}}(A), \mathcal{V}_{\mathrm{dR}}(A)$. We will apply Notation 7.1.6 in Betti, étale, or de Rham setting. The Tate objects $\mathbb{Q}_B(1), \mathbb{Q}_{\mathrm{et}}(1), \mathbb{Q}_{\mathrm{dR}}(1)$ in each setting are defined

as in [De82, §1]. The ℓ -component and the prime-to- p component of $\mathbb{Q}_{\text{et}}(1)$ are denoted by $\mathbb{Q}_{\text{et}}(1)_{\ell}$ and $\mathbb{Q}_{\text{et}}(1)^p$, respectively.

The representation V of G defines a Betti local system \mathcal{V} on $\text{Sh}_K(G, X)(\mathbb{C})$

$$\mathcal{V} = V \times^{G(\mathbb{Q})} (X \times G(\mathbb{A}_f)/K),$$

which is canonically isomorphic to $\mathcal{V}_B(A)$ of A (because such canonical isomorphism exists over $\mathbf{Sh}_{K'}(\text{GSp}(V), \mathfrak{H}^{\pm})$). Since elements in \mathcal{T} are $G(\mathbb{Q})$ -invariant, they define global sections of \mathcal{V}^{\otimes} , and therefore induce

$$s_B : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathcal{H}_B(A)^{\otimes}.$$

Note that, $s_{B,x}(\mathcal{T})$ is in the subalgebra of Hodge classes of $H^1(A_x, \mathbb{Q})^{\otimes}$, for every $x \in \text{Sh}_K(G, X)(\mathbb{C})$.

By [Kis10, Lemma 2.2.1], there is a map of étale local systems

$$s_{\text{et}} : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathcal{H}_{\text{et}}(A)^{\otimes},$$

which matches s_B under the comparison isomorphism $\mathcal{H}_{\text{et}}(A)_{\ell}|_{\text{Sh}_K(G, X)_{\mathbb{C}}} \cong \mathcal{H}_B(A) \otimes \mathbb{Q}_{\ell}$. In particular, for every extension F/E in \mathbb{C} , and an F -point x of $\text{Sh}_K(G, X)$, the algebra $s_{\text{et},x}(\mathcal{T})$ is in the subalgebra of Tate classes of $H^1_{\text{et}}(A_{x, \bar{F}}, \mathbb{Q}_{\ell})^{\otimes}$.

The relative de Rham cohomology $\mathcal{H}_{\text{dR}}(A)$ is a vector bundle on $\text{Sh}_K(G, X)$ equipped with a flat connection ∇ and a decreasing filtration

$$\mathcal{H}_{\text{dR}}(A) = \text{Fil}^0 \mathcal{H}_{\text{dR}}(A) \supset \text{Fil}^1 \mathcal{H}_{\text{dR}}(A) = (\text{Lie} A / \text{Sh}_K(G, X))^{\vee} \supset \text{Fil}^2 \mathcal{H}_{\text{dR}}(A) = 0.$$

By Deligne's theory of absolute Hodge cycles (cf. [De82] and [Kis10, Corollary 2.2.2]), there is

$$s_{\text{dR}} : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathcal{H}_{\text{dR}}(A)^{\otimes},$$

that match s_B under the comparison isomorphism $\mathcal{H}_{\text{dR}}(A) \otimes \mathbb{C} \cong \mathcal{H}_B(A) \otimes \mathbb{C}$. In addition, $\text{Im}(s_{\text{dR}}) \subset \text{Fil}^0 \mathcal{H}_{\text{dR}}(A)^{\otimes}$.

Next, we move to the integral model. If $\ell \neq p$, $\mathcal{H}_{\text{et}}(A)_{\ell}$ exists as an étale local system on $\mathcal{S}_K(G, X)$, and $s_{\ell} : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathcal{H}_{\text{et}}(A)_{\ell}$ extends to the whole $\mathcal{S}_K(G, X)$, still denoted by s_{ℓ} . The de Rham cohomology $\mathcal{H}_{\text{dR}}(A)$ extends to a vector bundle with a flat connection and a decreasing filtration on $\mathcal{S}_K(G, X)$. It was proved in [Kis10, Corollary 2.3.9] that s_{dR} extends integrally as well, and denoted by the same notation. In addition, integrally $\text{Im}(s_{\text{dR}}) \subset \text{Fil}^0 \mathcal{H}_{\text{dR}}(A)^{\otimes}$.

Let k_v be the residue field of $\mathcal{O}_{E,v}$. We recall the definition of the rigid tensor category $F\text{-Crys}(T)$ of F -crystals on a (quasi-compact) k_v -scheme T . An object is a locally free crystal \mathbb{D} on the big crystalline site $(T/\mathcal{O}_{E,v})_{\text{CRIS}}$, equipped with an quasi-isogeny (the Frobenius map)

$$F : \sigma^* \mathbb{D} \dashrightarrow \mathbb{D},$$

i.e. an element $F \in \text{Hom}_{(T/\mathcal{O}_{E,v})_{\text{CRIS}}}(\sigma^* \mathbb{D}, \mathbb{D}) \otimes \mathbb{Q}$ such that there exists some element $V \in \text{Hom}_{(T/\mathcal{O}_{E,v})_{\text{CRIS}}}(\mathbb{D}, \sigma^* \mathbb{D}) \otimes \mathbb{Q}$ (the Verschiebung) such that $VF = FV = p^{34}$. Morphisms in $F\text{-Crys}(T)$ are morphisms of locally free crystals, compatible with F .

The unit object $\mathbf{1}$ is the structure sheaf of $(T/\mathcal{O}_{E,v})_{\text{CRIS}}$, equipped with F sending 1 to 1. The Tate object $\mathcal{O}_{\text{crys}}(1)$ is the structure sheaf of $(T/\mathcal{O}_{E,v})_{\text{CRIS}}$, equipped with F sending 1 to p .

Now, let A_{k_v} be the mod p fiber of the universal abelian variety, and $\mathbb{D}(A_{k_v})$ its contravariant Dieudonné crystal. This is an F -crystal. In addition, $\mathbb{D}(A_{k_v})(\mathcal{S}_{K,k_v})$ (the value of $\mathbb{D}(A_{k_v})$ at the trivial PD thickening $\mathcal{S}_{K,k_v} \xrightarrow{\text{id}} \mathcal{S}_{K,k_v}$) is canonically isomorphic to $\mathcal{H}_{\text{dR}}(A)|_{\mathcal{S}_{K,k_v}}$ and therefore is equipped with a decreasing filtration. It follows that $\mathbb{D}(A_{k_v})^{\otimes}(\mathcal{S}_{K,k_v})$ is canonically isomorphic to

³⁴Of course, replacing V by $p^{-1}V$, one can require $VF = FV = 1$. We choose this convention so it is compatible with literatures on Dieudonné crystals of p -divisible groups.

$\mathcal{H}_{\text{dR}}(A)^\otimes|_{\mathcal{S}_{K,k_v}}$ and therefore is also equipped with a decreasing filtration. Since \mathcal{S}_K is smooth over $\mathcal{O}_{E,v}$, It follows that the horizontal map $s_{\text{dR}} : \mathcal{T} \rightarrow \mathcal{H}_{\text{dR}}(A)^\nabla$ induces

$$s_0 : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(A)^\otimes,$$

where $\mathbb{D}(A)^\otimes$ is now considered as an object in (the ind-completion) of $F\text{-Isoc}(\mathcal{S}_{K,k_v})$. In addition, $s_0(t)(\mathcal{S}_{K,k_v}) \in \text{Fil}^0(\mathbb{D}(A_{k_v})^\otimes(\mathcal{S}_{K,k_v}))$ for $t \in \mathcal{T}$.

It follows that every characteristic p point x of \mathcal{S}_K , of residue field κ , is canonically equipped with a set of Tate classes $s_{\ell,x}(\mathcal{T}) \subset H^1(A_{x,\bar{\kappa}}, \mathbb{Q}_\ell)^\otimes$, and a set of crystalline classes $s_{0,x}(\mathcal{T}) \subset \mathbb{D}(A_x)^\otimes$.

Remark 7.1.9. In fact, in [Kis10], Kisin first constructed $s_{0,x}$ for every characteristic p point x via the theory of Breuil-Kisin module in order to study the local structure of \mathcal{S}_K , and deduced as a corollary that s_{dR} extends integrally.

7.1.10. The de Rham and crystalline G -torsors. Since $G \otimes \mathbb{Z}_p$ is quasi-split, there is a representative of $\{\mu\}$ defined over E_v , and up to conjugacy, we can assume that it extends to $\mu : \mathbb{G}_m \rightarrow \underline{G} \otimes \mathcal{O}_{E,v}$. Let $\text{Fil}^\bullet(V_{\mathbb{Z}_p}^\vee \otimes \mathcal{O}_{E,v}) = (\text{Fil}^0 \supset \text{Fil}^1 \supset 0)$ be the decreasing filtration on $V_{\mathbb{Z}_p}^\vee \otimes \mathcal{O}_{E,v}$ induced by μ so that the subgroup $P_\mu \subset \text{GL}(V_{\mathbb{Z}_p}^\vee) \times \text{GL}(\mathbb{Z}_p(1))$ that preserves s and the filtration Fil^\bullet , is a parabolic subgroup of $\underline{G} \otimes \mathcal{O}_{E,v}$ determined by μ . By abuse of notation, we still use \underline{G} to denote the base change $\underline{G} \otimes \mathcal{O}_{E,v}$.

Let us also construct the local model diagram for \mathcal{S}_K . Note that local model diagram for Shimura varieties of abelian type with parahoric level structure has been established in [KP15⁺] (under some mild restriction of $G_{\mathbb{Q}_p}$). In the case when K_p is hyperspecial, this is quite easy so we include a proof for completeness. First, we have

Lemma 7.1.11. *The (fppf) sheaf*

$$\mathcal{E}_{\text{dR}} := \text{Isom}(((V_{\mathbb{Z}_p}^\vee \oplus \mathbb{Z}_p(1)) \otimes \mathcal{O}_{\mathcal{S}_K}, s \otimes 1), (\mathcal{H}_{\text{dR}}(A) \oplus \mathbb{Q}_{\text{dR}}(1), s_{\text{dR}})),$$

of isomorphisms of vector bundles $V_{\mathbb{Z}_p}^\vee \otimes \mathcal{O}_{\mathcal{S}_K} \simeq \mathcal{H}_{\text{dR}}(A)$ and $\mathbb{Z}_p(1) \otimes \mathcal{O}_{\mathcal{S}_K} \simeq \mathbb{Q}_{\text{dR}}(1)$ that send $s \otimes 1$ to s_{dR} , is a \underline{G} -torsor on \mathcal{S}_K , and the sheaf

$$\mathcal{P} := \text{Isom}(((V_{\mathbb{Z}_p}^\vee \oplus \mathbb{Z}_p(1)) \otimes \mathcal{O}_{\mathcal{S}_K}, s \otimes 1, \text{Fil}^\bullet), (\mathcal{H}_{\text{dR}}(A) \oplus \mathbb{Q}_{\text{dR}}(1), s_{\text{dR}}, \text{Fil}^\bullet))$$

is a P_μ -torsor on \mathcal{S}_K . There is a canonical isomorphism $\underline{G} \times^{P_\mu} \mathcal{P} = \mathcal{E}_{\text{dR}}$.

Proof. (1) We just consider \mathcal{E}_{dR} , and the statement for \mathcal{P} is similar (by taking account of the filtration).

We regard \mathcal{E}_{dR} as a closed subscheme of the $\text{GL}(V_{\mathbb{Z}_p}^\vee) \times \text{GL}(\mathbb{Z}_p(1))$ -torsor

$$\text{Isom}((V_{\mathbb{Z}_p}^\vee \oplus \mathbb{Z}_p(1)) \otimes \mathcal{O}_{\mathcal{S}_K}, \mathcal{H}_{\text{dR}}(A) \oplus \mathbb{Q}_{\text{dR}}(1)).$$

There is an action of \underline{G} on \mathcal{E}_{dR} and it is clear that the map

$$\underline{G} \times_{\mathcal{S}_K} \mathcal{E}_{\text{dR}} \rightarrow \mathcal{E}_{\text{dR}} \times_{\mathcal{S}_K} \mathcal{E}_{\text{dR}}$$

is an isomorphism. So it remains to show that \mathcal{E}_{dR} is faithfully flat over \mathcal{S}_K . Note that when base change from $\mathcal{O}_{E,v}$ to \mathbb{C} , \mathcal{E}_{dR} is clearly a G -torsor. So we can pass to a formal neighbourhood \hat{U}_x at a closed point x in the special fiber of \mathcal{S}_K . Let κ denote the residue field of x . Let A_x denote the fiber of the above mentioned abelian variety at x .

Attached to x there is a reductive group $G_x \subset \text{GL}(\mathbb{D}(A_x)) \times \text{GL}(\mathcal{O}_{\text{crys}}(1))$ which is the schematically stabilizer of $s_0(\mathcal{T})$. It was shown in [Kis10, §1] that there exists an isomorphism $V_{\mathbb{Z}_p}^\vee \otimes W(\kappa) \simeq \mathbb{D}(A_x)$ intertwining $s : \mathcal{T} \rightarrow L^\otimes$ and $s_0 : \mathcal{T} \rightarrow \mathbb{D}(A_x)^\otimes$. Therefore, G_x is a reductive group isomorphic to $\underline{G} \otimes_{\mathcal{O}_{E,v}} W(\kappa)$.

Let R_{G_x} denote the G_x -adapted deformation ring of $A_x[p^\infty]$ as in ([Kis10, §1.5] and [Kis17, §1.1]). Then by [Kis10, Proposition 2.3.5], there is a natural isomorphism $\hat{U}_x \cong \text{Spf } R_{G_x}$ and

the restriction of $(\mathcal{H}_{\text{dR}}(A), s_{\text{dR}}, \nabla)$ to \hat{U}_x is identified with $(M = \mathbb{D}(A_x) \otimes_{W(\kappa)} R_{G_x}, s_0 \otimes 1, \nabla)$ on $\text{Spf } R_{G_x}$. It follows from the existence of an isomorphism $\mathbb{D}(A_x) \simeq V_{\mathbb{Z}_p}^\vee \otimes W(\kappa)$ as above that the restriction of \mathcal{E}_{dR} to \hat{U}_x is a \underline{G} -torsor, so is faithfully flat over \hat{U}_x . \square

Remark 7.1.12. In fact, \mathcal{E}_{dR} is defined over $\mathcal{S}_K(G, X)$.

Now, let $\widetilde{\mathcal{S}}_K$ denote the scheme over \mathcal{S}_K of the trivialization of the \underline{G} -torsor \mathcal{E}_{dR} .

Corollary 7.1.13. *There is the following local model diagram*

$$\mathcal{S}_K \xleftarrow{\pi} \widetilde{\mathcal{S}}_K \xrightarrow{\tilde{\varphi}} \underline{G}/P_\mu,$$

where π is a \underline{G} -torsor and $\tilde{\varphi}$ is \underline{G} -equivariant smooth of relative dimension $\dim G$. Moreover, if $U \rightarrow \mathcal{S}_K$ is a morphism such that the induced map $U \rightarrow \mathcal{S}_K$ is étale, the induced map $U \rightarrow \underline{G}/P_\mu$ is étale.

Proof. By the previous lemma, the trivial \underline{G} -torsor $\mathcal{E}_{\text{dR}} \times_{\mathcal{S}_K} \widetilde{\mathcal{S}}_K$ is induced by the P_μ -torsor $\mathcal{P} \times_{\mathcal{S}_K} \widetilde{\mathcal{S}}_K$. This defines the map $\tilde{\varphi} : \widetilde{\mathcal{S}}_K \rightarrow \underline{G}/P_\mu$. It follows from the construction that $\tilde{\varphi}$ is \underline{G} -equivariant. It remains to prove the last statement (which implies that $\tilde{\varphi}$ is smooth of relative dimension $\dim G$).

The last statement is clear for the characteristic zero points. Now let y be a characteristic p point of U mapping to $\tilde{x} \in \widetilde{\mathcal{S}}_K$ and $x \in \mathcal{S}_K$. Let \bar{x} denote the image of \tilde{x} in \underline{G}/P_μ . Then the completed local ring $\hat{V}_{\bar{x}}$ at \bar{x} can be identified with the deformation ring R_{G_x} in such a way that the natural filtration on $L^\vee \otimes \hat{V}_{\bar{x}}$ is identified with the Hodge filtration of $\mathbb{D}(A_x) \otimes_{W(\kappa)} R_{G_x}$. Since $\hat{U}_x \simeq \text{Spf } R_{G_x}$, the étaleness of U over $\mathcal{S}_K(G, X)$ at y (or equivalently $\hat{U}_y \simeq \hat{U}_x$) implies that the induced map $U \rightarrow \underline{G}/P_\mu$ takes \hat{U}_y isomorphically to $\text{Spf } R_{G_x} \simeq \text{Spf } \hat{V}_{\bar{x}}$. This verifies the last statement. \square

Another application of Lemma 7.1.11 is as follows.

Corollary 7.1.14. *The sheaf*

$$R \mapsto \text{Isom} \left(((V_R^\vee \oplus R(1)), s), (\mathbb{D}(A)(R) \oplus \mathcal{O}_{\text{crys}}(1)(R), s_0) \right)$$

on $(\mathcal{S}_{K,k_v}/\mathcal{O}_{E,v})_{\text{CRIS}}$, where the isomorphisms are isomorphisms of R -modules $V_R^\vee \simeq \mathbb{D}(A)(R)$ and $R(1) \simeq \mathcal{O}_{\text{crys}}(1)(R)$ sending s to s_0 , defines a \underline{G} -torsor $\mathcal{E}_{\text{cris}}$ on $(\mathcal{S}_{K,k_v}/\mathcal{O}_{E,v})_{\text{CRIS}}$. In addition, there is a map of \underline{G} -torsors $\mathcal{E}_{\text{cris}} \rightarrow {}^\sigma \mathcal{E}_{\text{cris}}$, whose relative position is $\sigma(\mu)$.

Note that by definition there is a canonical isomorphism $\mathcal{E}_{\text{cris}}(\mathcal{S}_K \otimes \mathbb{F}_p) \cong \mathcal{E}_{\text{dR}} \otimes \mathbb{F}_p$ (crystalline-de Rham comparison).

Proof. If x is an R -point of \mathcal{S}_{K,k_v} and $\tilde{R} \rightarrow R$ a PD thickening, we (locally) lift x to $\tilde{x} : \text{Spec } \tilde{R} \rightarrow \mathcal{S}_{K_x}$. Then $\mathcal{E}_{\text{cris}}(\tilde{R}) = \tilde{x}^* \mathcal{E}_{\text{dR}}$, which is independent of the lifting. The F -crystal structure

$$\mathbb{D}(A_x)^\vee \rightarrow \mathbb{D}(A_x^{(p)})^\vee = \sigma^* \mathbb{D}(A_x)^\vee$$

induces a quasi-isogeny $x^* \mathcal{E}_{\text{cris}} \rightarrow x^*({}^\sigma \mathcal{E}_{\text{cris}})$. Note that we use the dual of contravariant Dieudonné modules to define the F -crystal with \underline{G} -structures, as by definition \underline{G} is considered a subgroup of $\text{GSp}(V_{\mathbb{Z}_p}, \psi) \otimes \mathcal{O}_{E,v}$, where V corresponds to homology (as opposed to the cohomology) of the universal abelian variety.

To prove the last statement, we can work pointwise. So assume x is an $\overline{\mathbb{F}}_p$ -point. The relative position of

$$x^*((\sigma^{-1})^* \mathcal{E}_{\text{cris}}) \rightarrow x^* \mathcal{E}_{\text{cris}}$$

is given by μ since the morphism on the associated bundle is the Verschiebung

$$V : (\sigma^{-1})^* \mathbb{D}(A_x)^\vee \rightarrow \mathbb{D}(A_x)^\vee = \mathcal{E}_x \times^{\underline{G}} V_{\mathbb{Z}_p}.$$

\square

Example 7.1.15. We give an example to justify our computation of the relative position: let F be a cubic totally real field and E a CM extension of F . Let p be a prime inert in F and splits as $\mathfrak{p}\bar{\mathfrak{p}}$ in E . Consider the unitary group $G = G(U(1, 2) \times U(0, 3) \times U(0, 3))$. Let τ_1, τ_2, τ_3 be the p -adic/complex embeddings corresponding to \mathfrak{p} . Under the usual identification of $G_{\mathbb{C}}$ with $\mathrm{GL}(3)^{\times 3} \times \mathbb{G}_m$ (where the first three factors use associated complex embeddings τ_1, τ_2, τ_3), $\mu(z)$ is $\mathrm{diag}\{z, 1, 1\}$, id , and id on the first three factors. Then $\dim \mathrm{Lie}(A)_{\tau_i}$ is equal to 1, 0, 0 for $i = 1, 2, 3$, respectively. It follows that the dimension of the cokernel of $V_{\tau_i} : \mathbb{D}(A_x)_{\tau_i}^{\vee} \rightarrow (\sigma^* \mathbb{D}(A_x)_{\tau_i}^{\vee})_{\tau_i}$ has dimension 0, 1, 0, respectively. From this, we see that the associated G -torsor map $\mathcal{E}_{\mathrm{cris}} \rightarrow {}^{\sigma}\mathcal{E}_{\mathrm{cris}}$ has relative position $\sigma(\mu)$.

7.1.16. *Level structure.* Let $x \in \mathcal{S}_K(G, X)(T)$ be a point. Its image on \mathcal{A}_{g, K^p} is represented by a triple (A_x, λ_x, η') , as explained in §7.1.4. In particular,

$$\eta' \in \Gamma(T, \mathrm{Isom}((V \oplus \mathbb{Q}(1)) \otimes \mathbb{A}_f^p, \mathcal{V}_{\mathrm{et}}(A)^p \oplus \mathbb{Q}_{\mathrm{et}}(1)^p) / K'^p).$$

But as explained in [Kis10, §3.4.2], the level structure η' comes from a canonical section

$$\eta_x \in \Gamma(T, \mathrm{Isom}(((V \oplus \mathbb{Q}(1)) \otimes \mathbb{A}_f^p, s \otimes 1), (\mathcal{V}_{\mathrm{et}}(A)^p \oplus \mathbb{Q}_{\mathrm{et}}(1)^p, s_{\mathrm{et}, x})) / K^p).$$

Here $\mathrm{Isom}(((V \oplus \mathbb{Q}(1)) \otimes \mathbb{A}_f^p, s \otimes 1), (\mathcal{V}_{\mathrm{et}}(A)^p \oplus \mathbb{Q}_{\mathrm{et}}(1)^p, s_{\mathrm{et}, x}))$ is the subsheaf of $\mathrm{Isom}((V \oplus \mathbb{Q}(1)) \otimes \mathbb{A}_f^p, \mathcal{V}_{\mathrm{et}}(A)^p \oplus \mathbb{Q}_{\mathrm{et}}(1)^p)$, which intertwines $s : \mathcal{T} \subset V^{\otimes}$ and $s_{\mathrm{et}, x} : \mathcal{T} \otimes \mathbf{1} \rightarrow (\mathcal{H}_{\mathrm{et}}(A)^p)^{\otimes} = (\mathcal{V}_{\mathrm{et}}(A)^p)^{\otimes}$. Note that this is a $G(\mathbb{A}_f^p)$ -torsor so that the quotient by K^p makes sense.

7.2. **The mod p fiber of $\mathcal{S}_K(G, X)$.** We first relate the special fibers of the integral models of Shimura varieties recalled above to the moduli of local shtukas via Proposition 7.2.4. Then we recollect some results about the Rapoport-Zink uniformization of the basic Newton stratum.

7.2.1. *Perfection of the characteristic p fiber of \mathcal{S}_K .* Now we concentrate on mod p fiber. Let k_v denote the residue field of $\mathcal{O}_{E, v}$ and fix \bar{k}_v an algebraic closure of k_v . We denote by

$$\mathrm{Sh}_{\mu, K} := (\mathcal{S}_K \otimes k_v)^{\mathrm{pf}}$$

the perfection of the special fiber of \mathcal{S}_K , or just Sh_{μ} if the level structure K is clear from the context. We also write Sh_{μ, K_p} for $\varprojlim_{K^p} \mathrm{Sh}_{\mu, K_p K^p}$. As suggested by the proof of Proposition 7.1.14, it is convenient to twist the F -crystal $\mathcal{E}_{\mathrm{cris}}$ to consider

$$\mathcal{E} := (\sigma^{-1})^* \mathcal{E}_{\mathrm{cris}}.$$

Then we have the F -crystal of \underline{G} -torsors (i.e. a local \underline{G} -shtuka, see Remark 5.2.2)

$$\beta : \mathcal{E} \dashrightarrow {}^{\sigma}\mathcal{E}$$

over $\mathrm{Sh}_{\mu, K}$, of relative position μ .

Remark 7.2.2. Note that by our definition of \mathcal{E} , it only exists over $\mathrm{Sh}_{\mu, K}$, but not over \mathcal{S}_K . On the other hand, from the theory of Breuil-Kisin module, it is also natural to define \mathcal{E} as $(\sigma^{-1})^* \mathcal{E}_{\mathrm{cris}}$. Namely, Let x be a κ -point of $\mathrm{Sh}_{\mu, K}$ and choose a lifting \tilde{x} of x to a characteristic zero point. Let

$$\mathfrak{M}_{\tilde{x}} = \mathfrak{M}(T_p A_{\tilde{x}}^{\vee}(-1))$$

be the Kisin module associated to the crystalline representation of the (-1) -twist of the p -adic Tate module of the dual abelian variety $A_{\tilde{x}}^{\vee}$ (which is isomorphic to $V_{\mathbb{Z}_p}^{\vee}$). Then $s_{p, \tilde{x}} : \mathcal{T} \rightarrow (T_p A_{\tilde{x}}^{\vee}(-1))^{\otimes}$ induces $s_{\mathrm{BK}} : \mathcal{T} \rightarrow \mathfrak{M}_{\tilde{x}}^{\otimes}$. Recall that $\mathfrak{M}_{\tilde{x}}$ is a free $\mathfrak{S} = W(\kappa)[[u]]$ -module and $\varphi^*(\mathfrak{M}_{\tilde{x}}/u\mathfrak{M}_{\tilde{x}})$ is identified with $\mathbb{D}(A_x)$, where $\varphi : \mathfrak{S} \rightarrow \mathfrak{S}$ is the Frobenius. Therefore, $x^* \mathcal{E}$ is the $\underline{G} \otimes W(\kappa)$ -torsor of isomorphisms $V_{\mathbb{Z}_p}^{\vee} \otimes W(\kappa) \simeq \mathfrak{M}_{\tilde{x}}/u\mathfrak{M}_{\tilde{x}}$ that sends $s(\mathcal{T})$ to the tensors $s_{\mathrm{BK}}(\mathcal{T}) \bmod u$ up to a common scalar.

7.2.3. *The map loc_p .* By only remembering the local \underline{G} -shtuka $\psi : \mathcal{E} \dashrightarrow {}^\sigma \mathcal{E}$, we obtain a morphism of prestacks

$$\text{loc}_p : \text{Sh}_\mu \rightarrow \text{Sht}_\mu^{\text{loc}}.$$

In the Siegel case, loc_p is the perfection of the morphism sending an abelian variety to its underlying p -divisible group. According to Serre-Tate theory, this morphism (before perfection) is formally étale. However formal étaleness is not a good notion for morphisms between perfect schemes, so we will not make use of it. Instead, we will compose it with the morphism from $\text{Sht}_\mu^{\text{loc}}$ to the moduli of restricted local shtukas. Let (m, n) be a pair of non-negative integers such that $m - n$ is μ -large. We set

$$\text{loc}_p(m, n) : \text{Sh}_\mu \xrightarrow{\text{loc}_p} \text{Sht}_\mu^{\text{loc}} \xrightarrow{\text{res}_{m,n}} \text{Sht}_\mu^{\text{loc}(m,n)}.$$

The following result establishes the relation between the special fibers of Shimura varieties and the moduli of local shtukas defined earlier. Recall the definition of perfectly smooth morphisms in Appendix A.1.

Proposition 7.2.4. *The morphism $\text{loc}_p(m, n)$ is perfectly smooth.*

Proof. Let $\text{Sh}_\mu^{(m,n),\square}$ denote the $L^m \underline{G}$ -torsor over Sh_μ that classifies, for each perfect ring R , an R -point x of Sh_μ and a trivialization of the $L^m \underline{G}$ -torsor ${}^\sigma \mathcal{E}_x|_{D_m} \simeq \mathcal{E}^0|_{D_m}$ over $D_{m,R}$. Recall that we have defined its local version, namely the moduli of framed restricted local shtukas $\text{Sht}_\mu^{\text{loc}(m,n),\square}$ in (5.3.5), which is canonically isomorphic to $\text{Gr}_\mu^{(n)}$. We have the following canonical isomorphism

$$(7.2.1) \quad \text{Sh}_\mu^{(m,n),\square} \cong \text{Sh}_{\mu,K} \times_{\text{Sht}_\mu^{\text{loc}(m,n)}} \text{Sht}_\mu^{\text{loc}(m,n),\square}.$$

So to prove that $\text{loc}_p(m, n)$ is perfectly smooth, it suffices to check that the map

$$\text{loc}_p^\square(m, n) : \text{Sh}_\mu^{(m,n),\square} \longrightarrow \text{Sht}_\mu^{\text{loc}(m,n),\square}$$

is perfectly smooth.

Let x^\square be a \bar{k}_v -point of $\text{Sh}_\mu^{(m,n),\square}$, whose image in Sh_μ is denoted by x . By [Zhu17, Corollary A.27], we can choose a small étale neighborhood $a : U = \text{Spec} R \rightarrow \overline{\mathcal{S}}_K$ of x such that both the pullback of the $L^m \underline{G}$ -torsor ${}^\sigma \mathcal{E}|_{D_m}$ to U and the pullback of the $L^n \underline{G}$ -torsor $\mathcal{E}|_{D_n}$ to U are trivial. We fix one such trivialization $\epsilon_m : {}^\sigma \mathcal{E}|_{D_{m,R}} \cong \mathcal{E}^0|_{D_{m,R}}$, which is equivalent to giving a lift $a^{(m)} : U \rightarrow \text{Sh}_{\mu,K}^{(m,n),\square}$ of a . We put

$$a_{\text{loc}}^{(m,n)} := \text{loc}_p^\square(m, n) \circ a^{(m)} : U \rightarrow \text{Sh}_\mu^{(m,n),\square} \rightarrow \text{Sht}_\mu^{\text{loc}(m,n),\square} \cong \text{Gr}_\mu^{(n)}.$$

Then the natural map

$$\begin{aligned} U \times L^m \underline{G} &\longrightarrow \text{Sh}_{\mu,K}^{(m,n),\square} \\ (u, g) &\longmapsto (a^{(m)}(u), \sigma(g)\epsilon_m) \end{aligned}$$

is étale and gives an étale chart near at the point x^\square . So it suffices to show that the composition

$$(7.2.2) \quad U \times L^m \underline{G} \longrightarrow \text{Sh}_\mu^{(m,n),\square} \xrightarrow{\text{loc}_p^\square(m,n)} \text{Sht}_\mu^{\text{loc}(m,n),\square} \cong \text{Gr}_\mu^{(n)}$$

is perfectly smooth. In explicit terms, this map is given by

$$(7.2.3) \quad (u, g) \mapsto \sigma(g) \cdot a_{\text{loc}}^{(m,n)}(u) \cdot \pi_{m,n}(g)^{-1}.$$

See (5.3.6). (Note that g in (5.3.6) is $\sigma(g)$ here.)

We note that the perfection of the local model diagram in Corollary 7.1.13 mod p is exactly

$$\text{Sh}_\mu \leftarrow \text{Sh}_\mu^{(1,0),\square} \rightarrow (\underline{G}/P_\mu \otimes k_v)^{\text{pf}} \cong \text{Gr}_\mu,$$

where the second arrow may be identified with the map $\text{loc}_p^\square(1, 0) : \text{Sh}_\mu^{(1,0),\square} \rightarrow \text{Sht}_\mu^{\text{loc}(1,0),\square} \cong \text{Gr}_\mu$. The lift $a^{(m)} : U \rightarrow \text{Sh}_\mu^{(m,n),\square}$ naturally gives a map $a^{(1)} : U \rightarrow \text{Sh}_\mu^{(1,0),\square}$. The natural compatibility of localization map gives the following commutative diagram

$$\begin{array}{ccccc} U \times L^m \underline{G} & \longrightarrow & \text{Sh}_{\mu,K}^{(m,n),\square} & \xrightarrow{\text{loc}_p^\square(m,n)} & \text{Gr}_\mu^{(n)} \\ \text{proj}_1 \downarrow & & & & \downarrow \\ U & \xrightarrow{a^{(1)}} & \text{Sh}_{\mu,K}^{(1,0),\square} & \xrightarrow{\text{loc}_p^\square(1,0)} & \text{Gr}_\mu. \end{array}$$

The composition of the bottom arrow, namely $a_{\text{loc}}^{(1,0)}$, is étale by Corollary 7.1.13.

This now has become a purely local question: knowing an étale map $a_{\text{loc}}^{(1,0)} : U \rightarrow \text{Gr}_\mu$ lifts to $a_{\text{loc}}^{(m,n)} : U \rightarrow \text{Gr}_\mu^{(n)}$, we shall prove that the associated map $U \times L^m \underline{G} \rightarrow \text{Gr}_\mu^{(n)}$ given by formula (7.2.3) is perfectly smooth. This is equivalent to show that the induced map

$$(7.2.4) \quad U \times L^m \underline{G} \longrightarrow U \times_{\text{Gr}_\mu} \text{Gr}_\mu^{(n)}$$

is perfectly smooth. To prove this, we need to use the imperfect version.

Now, let $L_p^m \underline{G}$ denote the usual Greenberg realization of $\underline{G} \otimes \mathbb{Z}/p^m$ so that $L_p^m \underline{G}$ is a smooth algebraic group over \mathbb{F}_p whose perfection is $L^m \underline{G}$ (cf. [Zhu17, §1]). The étale map $a : U \rightarrow \text{Gr}_\mu$ descends uniquely to an étale morphism $a' : U' \rightarrow \underline{G}/P_\mu \otimes k_v$. If we write $\mathcal{E}_1 \dashrightarrow \mathcal{E}_0 = \mathcal{E}^0$ denote the universal modification over Gr_μ , the $L^n \underline{G}$ -torsor $\mathcal{E}_1|_{D_{n,R}}$ is trivial over U . Thus, if we write $U'^{(n)}$ for the trivial $L_p^n \underline{G}$ -torsor over U' , then

$$(U'^{(n)})^{\text{pf}} \cong U \times_{\text{Gr}_\mu} \text{Gr}_\mu^{(n)}.$$

Now, the map (7.2.4) is the perfection of the map

$$(7.2.5) \quad \begin{aligned} U' \times L_p^m \underline{G} &\longrightarrow U'^{(n)} \\ (u, g) &\longmapsto \sigma(g) a_{\text{loc}}^{(m,n)}(u) \pi'_{m,n}(g)^{-1}, \end{aligned}$$

where $\pi'_{m,n} : L_p^m \underline{G} \rightarrow L_p^n \underline{G}$ is the natural projection. On the level of tangent space, we can ignore the left multiplication by $\sigma(g)$. Hence, the infinitesimal action of $L_p^m \underline{G}$ on $U'^{(n)}$ is entirely along the fiber direction of $U'^{(n)} \rightarrow U'$. So the map (7.2.5) is smooth, and hence (7.2.4) is perfectly smooth. This concludes the proof of the proposition. \square

Remark 7.2.5. The readers can skip this remark. Recall from Lemma 5.3.6 that there is a natural perfectly smooth morphism $\text{Sht}_\mu^{\text{loc}(2,1)} \rightarrow G\text{-Zip}_\mu$. It is easy to see that the composition

$$\text{Sh}_\mu \xrightarrow{\text{loc}_p(2,1)} \text{Sht}_\mu^{\text{loc}(2,1)} \rightarrow G\text{-Zip}_\mu^{\text{pf}}$$

coincides with the perfection of the map defined in [Zha.C13⁺] (which in the case of $G = \text{GSp}(V, \psi)$ is the map sending an abelian variety A to its p -torsion $A[p]$). The following is a corollary of Proposition 7.2.4.

Corollary 7.2.6. *The morphism $\text{Sh}_\mu \rightarrow G\text{-Zip}_\mu^{\text{pf}}$ is perfectly smooth.*

In fact, by an argument similar to the proof of Proposition 7.2.4, one can show that the map $\mathcal{S}_{K,k_v} \rightarrow G\text{-Zip}_\mu$ is already smooth before taking the perfection. See [Zha.C13⁺].

7.2.7. *Newton stratification.* Recall the definition of the Newton map (5.2.5) for the moduli of local shtukas. By composition, one obtains the Newton map for Shimura varieties

$$\mathcal{N} : \mathrm{Sh}_{\mu,K}(\bar{k}_v) \xrightarrow{\mathrm{loc}_R} \mathrm{Sh}_{\mu}^{\mathrm{loc}}(\bar{k}_v) \xrightarrow{\mathcal{N}} B(G_{\mathbb{Q}_p}, \mu^*).$$

For $[b] \in B(G_{\mathbb{Q}_p}, \mu^*)$, let $\mathcal{N}_b = \mathcal{N}_{[b]} := \mathcal{N}^{-1}([b])$ denote the corresponding *Newton stratum*. It is known (e.g. [Wo13⁺, Section 5.2]) that \mathcal{N}_b is a locally closed subset of $\mathrm{Sh}_{\mu,K}$ and its closure $\overline{\mathcal{N}}_b$ is contained in the union $\bigcup_{b' \preceq b} \mathcal{N}_{b'}$. In particular, if $[b] \in B(G_{\mathbb{Q}_p}, \mu^*)$ is basic, \mathcal{N}_b is a closed subset, denoted by $\mathrm{Sh}_{\mu,K,b}$ or just $\mathrm{Sh}_{\mu,b}$ if the level structure K is clear from the context.

Lemma 7.2.8. *The basic Newton stratum $\mathrm{Sh}_{\mu,K,b}$ is non-empty.*

It has been proved by Lee [Le16⁺] that all Newton strata are non-empty (and we learned that Kisin-Madapusi Pera-Shin proved more general statement under weaker assumptions of $G_{\mathbb{Q}_p}$). Since the case of basic Newton stratum is particularly simple, we include an argument here for the basic Newton stratum for readers' convenience.

Proof. We first claim that there is a maximal torus T of G , defined over \mathbb{Q} , such that both $T(\mathbb{R})$ and $T(\mathbb{Q}_p)$ are anisotropic modulo the centers of $G(\mathbb{R})$ and $G(\mathbb{Q}_p)$, respectively. Namely, we apply weak approximation to find a strongly regular semisimple element γ defined over \mathbb{Q} such that γ_{∞} and γ_p are elliptic. Then T is defined as the centralizer of γ .

Next, we claim that there exists $h \in X$ such that $h(\mathbb{C}^{\times}) \subset T(\mathbb{R})$. Indeed, let h' be a point of X . Then $h'(\mathbb{C}^{\times})$ is contained in a maximal torus $T' \subset G_{\mathbb{R}}$, anisotropic modulo center. Therefore, there is some $g \in G(\mathbb{R})$ such that $gT'g^{-1} = T_{\mathbb{R}}$. So $h = gh'g^{-1}$ satisfies the required property. In particular, the pair (T, h) defines a special point of X . By choosing appropriate field of definition and tame level structure, we obtain a point \tilde{x} in $\mathrm{Sh}_K(G, X)$, with a reduction to a point x in Sh_{μ} , such that the Mumford-Tate group of the universal abelian variety $A_{\tilde{x}}$ at \tilde{x} is contained in T . Then $\mathcal{N}(x)$ is contained in the image of $B(T) \rightarrow B(G)$. Since $T_{\mathbb{Q}_p}$ is elliptic, we know that this image must be basic in $B(G)$ (cf. [Ko85, Proposition 5.3]). \square

7.2.9. *The group I_x .* We assume that x is a κ -point of $\mathrm{Sh}_{\mu,K}$, where $\kappa \supset k_v$ is a finite extension, $\sharp \kappa = q = p^r$, and let x be a κ -point of \mathcal{S}_K . We denote $W_x = W(\kappa)$ and $K_x = W_x[1/p]$. Let \bar{x} be the induced $\bar{\kappa}$ -point over x . We similarly have $W_{\bar{x}}$ and $K_{\bar{x}}$. Let π_{A_x} denote the q -Frobenius endomorphism of A_x . Recall that by construction, we have a collection of tensors $s_{\mathrm{et},x} : \mathcal{T} \rightarrow H^1(A_{\bar{x}}, \mathbb{Q}_{\ell})^{\otimes}$ and $s_{0,x} : \mathcal{T} \rightarrow \mathbb{D}(A_x)^{\otimes}$. Recall we assume that \mathcal{T} contains $\psi \in (V_{\mathbb{Z}(p)}^{\vee} \otimes V_{\mathbb{Z}(p)}^{\vee}(1))^{\mathbb{G}}$ corresponds to the alternating pairing. Then $(s_{\mathrm{et},x}(\psi), s_{0,x}(\psi))$ is induced by a principal polarization λ_x of A_x .

We recall the construction of some groups

$$I_{\ell/\kappa}, I_{\ell}, I_{p/\kappa}, I_p, I_{/\kappa}, I,$$

from [Kis17, §2.1.2]. First, let

$$I_{\ell/\kappa} \subset \mathrm{GL}(H^1(A_{\bar{x}}, \mathbb{Q}_{\ell})) \times \mathrm{GL}(\mathbb{Q}_{\mathrm{et}}(1)_{\ell})$$

be the \mathbb{Q}_{ℓ} -subgroup, consisting of elements that fix $s_{\ell,x}$ and commute with the action of π_{A_x} . Replacing κ by a degree n extension κ_n one obtains I_{ℓ/κ_n} . Then sequence $\{I_{\ell/\kappa_n}\}$ is increasing and stabilizes to a \mathbb{Q}_{ℓ} -subgroup $I_{\ell} \subset \mathrm{GL}(H^1(A_{\bar{x}}, \mathbb{Q}_{\ell})) \times \mathrm{GL}(\mathbb{Q}_{\mathrm{et}}(1)_{\ell})$.

Next, $I_{p/\kappa}$ is the \mathbb{Q}_p -group whose R -points consist of $W_x \otimes_{\mathbb{Z}_p} R$ -linear automorphisms in

$$\mathrm{GL}(\mathbb{D}(A_x) \otimes_{\mathbb{Z}_p} R) \times \mathrm{GL}(\mathcal{O}_{\mathrm{crys}}(1)(W_x) \otimes R)$$

that fix $s_{0,x} \otimes 1$ and commute with the Frobenius map $F : \sigma^* \mathbb{D}(A_x) \rightarrow \mathbb{D}(A_x)$. Note that if R is a K_x -algebra, $I_{p/\kappa}(R) \subset \mathrm{GL}(\mathbb{D}(A) \otimes_{W_x} R) \times \mathrm{GL}_1(R)$ is the subgroup of elements that fix $s_0 \otimes 1$ and commute with $\gamma_p = F^r$. By replacing κ by κ_n , one obtains I_{p/κ_n} . Then $I_{p/\kappa_n} \otimes K_{\bar{x}}$ is the centralizer

of γ_p^n in $\mathrm{GL}(\mathbb{D}(A) \otimes_{W_x} K_{\bar{x}})$. Therefore, I_{p/κ_n} stabilizes to a \mathbb{Q}_p -group I_p . Note that if we choose an isomorphism $\mathbb{D}(A_x) \simeq V_{\mathbb{Z}_p}^\vee \otimes W_x$ and $\mathcal{O}_{\mathrm{crys}}(1)(W_x) \simeq \mathbb{Z}_p(1) \otimes W_x$ that sends $s_{0,x}$ to $s \otimes 1$ (i.e. a trivialization of $x^*\mathcal{E}$), then F is represented by $b_x\sigma$ for an element $b_x \in G(W_x)\mu^*(p)G(W_x)$. Then

$$I_{p/\kappa}(R) = \{g \in G(W_x \otimes R) \mid g^{-1}b_x\sigma(g) = b_x\} \subset J_{b_x}(R),$$

where J_{b_x} is the twisted centralizer of b_x as defined in (4.1.3). In particular, $I_p \subset J_{b_x}$.

Finally, we define $I_{/\kappa}$ to be the group whose R -points are elements of $(\mathrm{End}(A_x) \otimes R)^\times \times R^\times$ that preserve the tensors $s_{\ell,x}$ for $\ell \neq p$ and $s_{0,x}$. Define I similarly by replacing $(A_x, (s_{\mathrm{et},x}, s_{0,x}))$ by $(A_{\bar{x}}, (s_{\mathrm{et},\bar{x}}, s_{0,\bar{x}}))$. Note that if we denote by $f \mapsto f^*$ denote the Rosati involution on $\mathrm{End}(A_{\bar{x}}) \otimes \mathbb{Q}$ induced by the polarization on A_x , then $I(R) \subset \{g \in (\mathrm{End}(A_{\bar{x}}) \otimes R)^\times \mid gg^* \in R^\times\}$. In particular, there is a central $\mathbb{G}_m \subset I$ given by the scalar multiplication, and $(I/\mathbb{G}_m)(\mathbb{R})$ is compact. Also note that the Frobenius endomorphism π_{A_x} of A_x is an element in $I_{/\kappa}(\mathbb{Q})$.

Note that I only depends on \bar{x} . Conversely, we can attach to each \bar{k}_v -point of $\mathrm{Sh}_{\mu,K}$ the group I , since every such point is indeed defined over some finite field. To emphasize the dependence of I on the base point \bar{x} , we also denote it by $I_{\bar{x}}$.

Note that by definition, for every ℓ (including $\ell = p$), there are natural inclusions

$$I_{/\kappa} \otimes \mathbb{Q}_\ell \rightarrow I_{\ell/\kappa}, \quad I \otimes \mathbb{Q}_\ell \rightarrow I_\ell,$$

which turn out to be isomorphisms by [Kis17, Corollary 2.3.2]. In addition, it was proved in [Kis17, Corollary 2.3.5] that I is the inner form of a Levi subgroup I_0 of G . More precisely, by choosing an isomorphism $H^1(A_{\bar{x}}, \mathbb{Q}_\ell) \oplus \mathbb{Q}_{\mathrm{et}}(1)_\ell \simeq (V^\vee \oplus \mathbb{Q}(1)) \otimes \mathbb{Q}_\ell$ compatible with $s_{\mathrm{et},x}$ and s , and an isomorphism $\mathbb{D}(A_x) \oplus \mathcal{O}_{\mathrm{crys}}(1)(W_x) \simeq (V_{\mathbb{Z}_p}^\vee \oplus \mathbb{Z}_p(1)) \otimes W_x$ compatible with $s_{0,x}$ and $s \otimes 1$, we may regard π_A as an element $\gamma_\ell \in G(\mathbb{Q}_\ell)$ (including $\ell = p$), whose conjugacy class (twisted conjugacy when $\ell = p$) is independent of the choice. Then by [Kis17, Corollary 2.3.1], there is an element $\gamma_0 \in G(\mathbb{Q})$ which is

- conjugate to γ_ℓ in $G(\mathbb{Q}_\ell)$ for $\ell \neq p$;
- is stably conjugate to γ_p in $G(\overline{\mathbb{Q}_p})$;
- elliptic in $G(\mathbb{R})$.

Let $I_0 \subset G$ be the centralizer of the neutral connected component of the closed subgroup generated by γ_0 . Then it was proved in [Kis17, Corollary 2.3.5] that I is an inner form of I_0 .

7.2.10. Isogeny classes. We say two \bar{k}_v -points x and y of $\mathrm{Sh}_{\mu,K}$ are in the same isogeny class if and only if there is a quasi-isogeny $f : A_x \rightarrow A_y$ that takes $(s_{\mathrm{et},x}, s_{0,x})$ to $(s_{\mathrm{et},y}, s_{0,y})$ up to a common scalar multiple $c \in \mathbb{Q}^\times$. Note that this in particular implies that $f^*\lambda_y = c\lambda_x$. We denote the isogeny class containing x by \mathcal{I}_x . It is clear that such a map $f : A_x \rightarrow A_y$ induces an isomorphism $I_x \simeq I_y$.

Let $x \in \mathrm{Sh}_{\mu,K}$ be a κ -point, and put $W_x = W(\kappa)$. Let $K_x = W_x[1/p]$. If we fix an isomorphism $\mathbb{D}(A_x) \simeq V_{\mathbb{Z}_p}^\vee \otimes W_x$ sending $\{s_{0,x}\}$ to $\{s \otimes 1\}$, or equivalently fix a trivialization of $x^*\mathcal{E}$ (see Remark 7.2.2), we get an honest element $b_x \in [b_x] \cap G(W_x)\mu^*(p)G(W_x) \subset G(K_x)$. Note that b_x is well-defined up to σ -conjugacy by an element in $G(W_x)$. Let $X_{\mu^*}(b_x)$ be the corresponding ADLV. We shall construct a natural morphism

$$X_{\mu^*}(b_x) \rightarrow \mathrm{Sh}_{\mu,K} \otimes \bar{\kappa}$$

whose image are in an isogeny class. The map on the level of closed points was constructed in [Kis17, Proposition 1.4.4]. Recall that the space $X_{\mu^*}(b_x)$ can be viewed as a reduced version of the Rapoport-Zink space. Indeed, it was proved in [Zhu17, §3.2] that $X_{\mu^*}(b_x)$ is the perfection of the Rapoport-Zink space constructed in [Kim15⁺] and [HP17]. Then one can deduce the above desired morphism from [Kim15⁺] and [HP17]. However, we prefer to give a direct construction,

which does not rely on the existence of the Rapoport-Zink space as a formal scheme. It is based on the following lemma, whose proof is inspired by an argument in [HP17].

Lemma 7.2.11. *Let R be a reduced k_v -algebra of finite type, and let $f : \operatorname{Spec} R \rightarrow \mathcal{A}_{g,K'}$ be a morphism given by a principally polarized abelian scheme up to prime-to- p isogeny (B, μ^*) . Let*

$$\iota_{\text{et}} : \mathcal{T} \otimes \mathbf{1} \rightarrow (\mathcal{H}_{\text{et}}(B)^p)^{\otimes}, \quad \iota_0 : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(B)^{\otimes}$$

be two graded algebra homomorphisms. Assume that for every closed point $x \in \operatorname{Spec} R$, there is a point $y \in \mathcal{S}_K$ over $f(x)$ such that $(B_x, \iota_{\text{et},x}, \iota_{0,x})$ is isomorphic to $(A_y, s_{\text{et},y}, s_{0,y})$ in the sense that there is a prime-to- p quasi-isogeny $f : B_x \rightarrow A_y$ sending $(s_{\text{et},y}, s_{0,y})$ to $(\iota_{\text{et},x}, \iota_{0,x})$ (by [Kis17, Corollary 1.3.11], such y if exists is unique). Then there is a unique map $\operatorname{Spec} R \rightarrow \mathcal{S}_K$ such that $(B, \iota_{\text{et}}, \iota_0)$ is the pullback of (A, s_{et}, s_0) on \mathcal{S}_K .

Proof. We define a scheme Y by the following Cartesian diagram

$$\begin{array}{ccc} Y & \xrightarrow{j_Y} & \mathcal{S}_K \\ \text{pr}_Y \downarrow & & \downarrow \\ \operatorname{Spec} R & \longrightarrow & \mathcal{A}_{g,K'} \end{array}$$

Then Y is a scheme finite over $\operatorname{Spec} R$, and therefore is affine of finite type over k_v . By definition, there is a well-defined principally polarized abelian scheme (A_Y, λ_Y) on Y , equipped with two crystalline tensors

$$j_Y^*(s_0) : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(A_Y)^{\otimes}, \quad \text{pr}_Y^*(\iota_0) : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(A_Y)^{\otimes}.$$

Let Y° denote the union of connected components of Y on which there is a \bar{k}_v -point y such that $s_{0,j_Y(y)} = \iota_{0,\text{pr}_Y(y)}$. Then by [MP16, Lemma 5.10],

$$j_Y^*(s_0)|_{Y^\circ} = \text{pr}_Y^*(\iota_0)|_{Y^\circ}.$$

In other words, $Y^\circ(\bar{k}_v)$ is the set of points $y \in Y(\bar{k}_v)$ such that $s_{0,j_Y(y)} = \iota_{0,\text{pr}_Y(y)}$.

Note that by assumption, $Y^\circ \rightarrow \operatorname{Spec} R$ is bijective on \bar{k}_v -points. On the other hand, recall that by definition, the map $\mathcal{S}_K \rightarrow \mathcal{A}_{g,K'}$ is finite, and induces surjective maps between completed local rings at every point. Then $Y^\circ \rightarrow \operatorname{Spec} R$ is a finite morphism of affine schemes of finite type over k_v , and induces bijection on closed points and closed immersion when completed at every points. In addition, R is reduced. Therefore, it is an isomorphism.

Therefore, we obtain the desired map $\operatorname{Spec} R \cong Y^\circ \rightarrow \mathcal{S}_K$. It follows from the construction that ι_0 is the pullback of s_0 . The Chebotarev density implies that the pullback of s_{et} is ι_{et} . \square

Now recall that $X_{\mu^*}(b_x)$ classifies local \underline{G} -Shtukas $\mathcal{E}' \dashrightarrow {}^\sigma \mathcal{E}'$ of relative position $\preceq \mu$ and a modification $\beta : \mathcal{E}' \dashrightarrow x^* \mathcal{E}$ of local \underline{G} -Shtukas. Then

$$\mathbb{D}^\vee := (\mathcal{E}' \times_{\underline{G}} V_{\mathbb{Z}_p}^\vee)^\vee$$

admits a structure of an F -crystal with $p\mathbb{D}^\vee \subset \sigma\mathbb{D}^\vee \subset \mathbb{D}^\vee$. Since the Dieudonné functor is an equivalence of categories over perfect schemes, \mathbb{D}^\vee corresponds to a p -divisible group \mathcal{G} over $X_{\mu^*}(b_x)$ and the modification $\beta : \mathcal{E}' \dashrightarrow x^* \mathcal{E}$ corresponds to a quasi-isogeny

$$(7.2.6) \quad \mathcal{G} \rightarrow A_x[p^\infty] \times_{\kappa} X_{\mu^*}(b_x).$$

From this, we get an abelian variety A' over $X_{\mu^*}(b_x)$ together with a p -power quasi-isogeny

$$A' \rightarrow A_x \times_{\kappa} X_{\mu^*}(b_x),$$

whose induced map on the p -divisible groups is (7.2.6). It follows from the construction that the map

$$\mathcal{T} \otimes \mathbf{1} \xrightarrow{s_{0,x} \otimes \text{id}} (\mathbb{D}(A_x)^\otimes \otimes \mathbf{1})[1/p] \cong \mathbb{D}(A')^\otimes[1/p]$$

factors as $\mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(A')^\otimes$, denoted by s'_0 .

The abelian variety A' comes equipped with a prime-to- p principal polarization as well as a K^p -level structure, inherited from A_x . Using the universal property of Siegel moduli space, we have a morphism

$$X_{\mu^*}(b_x) \longrightarrow \mathcal{A}_{g,K'^p} \otimes \bar{\kappa}.$$

Lemma 7.2.12. *The above map lifts to a unique morphism*

$$\iota_x : X_{\mu^*}(b_x) \longrightarrow \text{Sh}_{\mu,K} \otimes \bar{\kappa},$$

such that for each closed point $y \in X_{\mu^*}(b_x)$, under the identification $\mathbb{D}(A'_y) \cong \mathbb{D}(A_{\iota_x(y)})$, we have

$$s'_{0,y} = s_{0,\iota_x(y)}.$$

Proof. The uniqueness will follow from [Kis17, Corollary 1.3.11], as two maps between perfect schemes coincide if they coincide at the level of points. Therefore, to prove the existence, we can work locally around each point y in $X_{\mu^*}(b_x)$. Let U be an open neighborhood of y , which is assumed to be perfectly of finite presentation, and $U \rightarrow (\mathcal{A}_{g,K'^p} \otimes \bar{\kappa})^{\text{pf}}$ be the morphism. Then we may assume that U is the perfection of a reduced affine scheme U_0 of finite type over κ , and assume that the abelian scheme A' on U with the quasi-isogeny $A' \rightarrow A_x \times U$ is the pullback of a quasi-isogeny of abelian schemes $A_0 \rightarrow A_x \times U_0$ on U_0 . By [Kis17, Proposition 1.4.4], the assumptions of Lemma 7.2.11 are satisfied so we can apply it to conclude. \square

Note that the map from Lemma 7.2.12 is compatible for changing the prime-to- p level and for the tame Hecke algebra action so we can pass to the inverse limit we obtain a $G(\mathbb{A}_f^p)$ -equivariant morphism

$$(7.2.7) \quad \iota_x : X_{\mu^*}(b_x) \times G(\mathbb{A}_f^p) \longrightarrow \text{Sh}_{\mu,K_p} \otimes \bar{\kappa}.$$

Taking quotient by K^p gives

$$(7.2.8) \quad \iota_x : X_{\mu^*}(b_x) \times G(\mathbb{A}_f^p)/K^p \longrightarrow \text{Sh}_{\mu,K} \otimes \bar{\kappa}.$$

It follows from the construction that the image of (7.2.8) is the isogeny class \mathcal{I}_x that x belongs to.

7.2.13. Rapoport-Zink uniformization. Now let x be a \bar{k}_v -point of Sh_{μ,K_p} . Then there is a canonical isomorphism $H^1(A_{\bar{x}}, \mathbb{Q}_\ell) \oplus \mathbb{Q}_{\ell,\text{et}}(1) \simeq (V^\vee \oplus \mathbb{Q}(1)) \otimes \mathbb{Q}_\ell$ for $\ell \neq p$, and therefore there is a canonical injection $I_x(\mathbb{Q}) \subset G(\mathbb{A}_f^p)$. In addition, a choice of the trivialization of $x^*\mathcal{E}$ gives b_x , and an embedding $I_x(\mathbb{Q}) \subset J_{b_x}(\mathbb{Q}_p)$. It is clear that the map (7.2.8) induces

$$(7.2.9) \quad I_x(\mathbb{Q}) \backslash X_{\mu^*}(b_x) \times G(\mathbb{A}_f^p)/K^p \rightarrow \text{Sh}_{\mu,K} \otimes \bar{k}_v.$$

By [Kis17, Proposition 2.1.3], this map is injective at the level of \bar{k}_v -points. Then by an argument as in [RZ96, Chap. 6] (see also [Kim15⁺]), $I_x(\mathbb{Q}) \rightarrow J_{b_x}(\mathbb{Q}_p) \times \prod G(\mathbb{A}_f^p)$ is a discrete subgroup, and the quotient in (7.2.9) on left hand side is representable (if K^p is sufficiently small). In addition, it is independent of the choice of the trivialization of $x^*\mathcal{E}$ up to a canonical isomorphism. We call this map the Rapoport-Zink uniformization of an isogeny class.

Recall that in general $I_x \otimes \mathbb{Q}_p \subset J_{b_x}$, but the inclusion can be strict³⁵. However,

Lemma 7.2.14. *If $[b_x] \in B(G_{\mathbb{Q}_p}, \mu^*)$ is basic, then I_x is an inner form of G . In particular, $I_x \otimes \mathbb{Q}_p \cong J_{b_x}$ is an inner form of $G_{\mathbb{Q}_p}$.*

³⁵We thank Rong Zhou to point this out to us.

If (G, X) is of PEL type, this was contained in [RZ96, Theorem 6.30], which relies on Lemma 6.28 of *loc. cit.*. Our proof, although looks different in disguise, is essentially the same.

Proof. We assume that x is defined over κ and $[\kappa : \mathbb{F}_p] = r$. Recall we write $W_x = W(\kappa)$ and $K_x = W_x \otimes \mathbb{Q}$. Set $\gamma_{x,p} = b_x \sigma_p(b_x) \cdots \sigma_p^{r-1}(b_x) \in G(K_x)$. We need to show that some power of $\gamma_{x,p}$ is in the center of G . It then follows that some power of γ_0 from §7.2.9 is in the center of G . Therefore $I_0 = G$ and I_x is an inner form of G .

First, since I_x is an inner form of a Levi subgroup of G , Z_G canonically embeds into I_x as a subgroup. Let T denote its neutral connected component.

Recall that we have a decomposition $\mathbb{D}(A_x) \otimes K_x = \oplus_i N_i$, where N_i is an isoclinic F -crystal, which is the union of F -stable W_x -lattices M such that $F^{s_i} M = p^{r_i} M$. Replacing s_i by a multiple, we may assume all s_i are equal, denoted by s . The Newton cocharacter $\nu = \nu_{b_x}$ of b_x is defined so that $(s\nu)(p)$ acts on N_i as multiplication by p^{r_i} . Since b_x is basic, $s\nu$ is a cocharacter of T . Then $(s\nu)(p^{-1})$ acts on $\mathbb{D}(A_x)$, commutes with the action of F , and $(s\nu)(p^{-1})F^s$ preserves a lattice in $\mathbb{D}(A_x) \otimes K_x$.

We regard $(s\nu)(p) \in T(\mathbb{A}_f)$ whose p -component is $(s\nu)(p)$ and prime-to- p component is 1. Since $T(\mathbb{Q}) \backslash T(\mathbb{A}_f) / H'$ is always finite for any open compact subgroup H' of $T(\mathbb{A}_f)$, we may replace s by a multiple of it and find some $t \in T(\mathbb{Q})$ such that $t' = (s\nu)(p)t$ is contained in some open compact subgroup of $T(\mathbb{A}_f)$.

Now we consider $g = \pi_{A_x}^s t^r \in I_x(\mathbb{Q})$. Note that g is contained in some open compact subgroup of $H \subset I_x(\mathbb{A}_f)$. Indeed, if $\ell \neq p$, π_{A_x} is contained in some open compact subgroup of $I_x(\mathbb{Q}_\ell)$ (as the Frobenius is an automorphism of the Tate module), therefore g is also contained in some open compact subgroup $H_\ell \subset I_x(\mathbb{Q}_\ell)$. On the other hand, after localized at p , $g = \gamma_{x,p}^s (rs\nu)(p^{-1}) t^r$ that acts on $\mathbb{D}(A_x) \otimes K_x \simeq V_{\mathbb{Z}_p}^\vee \otimes K_x$. As mentioned above, $\gamma_{x,p}^s (rs\nu)(p^{-1})$ fixes a lattice. Therefore g lies in some open compact $H_p \subset I_x(\mathbb{Q}_p)$.

Finally, note that $I_x(\mathbb{Q}) \cap H \subset I_x(\mathbb{A}_f)$ is a finite group since $(I_x / \mathbb{G}_m)(\mathbb{R})$ is compact. It follows that some power g^d is the identity. Therefore, $\gamma_{p,x}^{sd} = t^{-rd}$, which is central. \square

Corollary 7.2.15. *Assume that $[b_x] \in B(G_{\mathbb{Q}_p}, \mu^*)$ is basic. Then there is a unique inner twist $\Psi_x : G \rightarrow I_x$, which is trivial at $\ell \neq p, \infty$, is given by $[(b_x)_{\text{ad}}] \in B(G_{\text{ad}, \mathbb{Q}_p})_{\text{basic}} \cong H^1(\mathbb{Q}_p, G_{\text{ad}})$ at p , and such that $[(I_{\mathbb{R}}, \Psi_{\mathbb{R}})]$ is the unique compact modulo center inner form of G at ∞ .*

Proof. This is the combination of Lemma 7.2.14 and [Kis17, Corollary 2.3.5]. \square

Corollary 7.2.16. *The basic locus $\text{Sh}_{\mu, K_p, b}$ contains a unique isogeny class, and the map (7.2.9) (for a \bar{k}_v -point $x \in \text{Sh}_{\mu, K_p, b}$) factors through the basic locus, and gives the following isomorphism (over \bar{k}_v)*

$$I_x(\mathbb{Q}) \backslash X_{\mu^*}(b_x) \times G(\mathbb{A}_f^p) / K^p \rightarrow \text{Sh}_{\mu, K, b}.$$

As explained in [RZ96] (and [Kim15⁺] in the Hodge type case), the isomorphism is in fact defined over some finite field. We omit the general discussion here. The special case when b is unramified will be essentially treated in Proposition 7.3.5.

Proof. This follows from the previous corollary and [Kis17, Proposition 4.4.13]. Namely, since I_x is an inner form of G , the group defined in [Kis17, (4.4.9)] is trivial. Then this map is bijective at the level of \bar{k}_v -points, and the proof of Lemma 7.2.12 implies that it is an isomorphism. \square

We will be particularly interested in a special case. Recall the notion of unramified elements in $B(G_{\mathbb{Q}_p})$ as discussed in §4.2.

Corollary 7.2.17. *If the basic element of $B(G_{\mathbb{Q}_p}, \mu^*)$ is unramified, then the inner twist $G \rightarrow I_x$ (for a \bar{k}_v -point $x \in \text{Sh}_{\mu, K_p, b}$) is trivial at all finite places, and $I_{x, \mathbb{R}}$ is the unique compact modulo center inner form of $G_{\mathbb{R}}$.*

Remark 7.2.18. Recall that by Corollary 2.1.6 and Proposition 4.2.10, there exists a unique inner form G' of G such that $G'_{\mathbb{R}}$ is compact modulo center and $G'(\mathbb{A}_f) \simeq G(\mathbb{A}_f)$. The above lemma gives another proof of this result in the special case when (G, X) is a Shimura datum of Hodge type.

7.3. Cohomological correspondences between Shimura varieties. In this subsection, we apply Theorem 6.0.1 to construct cohomological correspondences between Shimura varieties. This in particular induces a spectral action on the cohomology of Shimura varieties by the ring of regular functions of the stack of Satake parameters, which is the Shimura variety counterpart of V. Lafforgue's S -operators.

7.3.1. The correspondence. Let (G_1, X_1) and (G_2, X_2) be two Shimura data such that $G_{1, \mathbb{A}_f} \simeq G_{2, \mathbb{A}_f}$. As explained in the paragraph before Lemma 2.1.2, G_1 and G_2 are inner forms. Let $\{\mu_i\}$ denote the conjugacy class of Hodge cocharacters determined by X_i , regarded as a dominant character of \hat{T} . Let $E_i \subset \mathbb{C}$ be the reflex field of (G_i, X_i) .

As explained in Corollary 2.1.5, the Shimura data defines a canonical inner twist $\Psi_{\mathbb{R}} : G_1 \rightarrow G_2$ over \mathbb{C} , whose image under $\alpha_{G, \infty}$ is $\bar{\mu}_{1, \text{ad}} - \bar{\mu}_{2, \text{ad}}$. We assume

$$\mu_{1, \text{ad}}|_{Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}}} = \mu_{2, \text{ad}}|_{Z(\hat{G}_{\text{sc}})^{\Gamma_{\mathbb{Q}}}}.$$

Then by Corollary 2.1.6, this inner twist comes from a global inner twist $\Psi : G_1 \rightarrow G_2$ over $\overline{\mathbb{Q}}$, which is trivial at all finite places, i.e.

$$\Psi = \text{Int}(h) \circ \theta$$

for some $\theta : G_{1, \mathbb{A}_f} \simeq G_{2, \mathbb{A}_f}$ and $h \in G_{2, \text{ad}}(\overline{\mathbb{A}_f})$. Recall that such global inner twist, if exists, is necessarily unique (see Corollary 2.1.6). Examples of such pairs $(G_i, X_i), i = 1, 2$ can be found in Example 2.1.8.

We fix $K_i \subset G_i(\mathbb{A}_f)$ an open compact subgroup which is sufficiently small such that $\theta K_1 = K_2$. Let p be a prime such that $K_{1, p}$ (and therefore $K_{2, p}$) is hyperspecial. Let \underline{G}_i be the integral model of G_{i, \mathbb{Q}_p} over \mathbb{Z}_p determined by $K_{i, p}$. We have the isomorphism $\theta : \underline{G}_1 \simeq \underline{G}_2$.

Let v be a place of $E = E_1 E_2$ lying over p . Let Sh_{μ_i, K_i} denote the mod p fiber of the canonical integral model for $\text{Sh}_{K_i}(G_i, X_i)$, base changed to k_v .

We need a hypothesis, which should always be true and can be proved unconditional in many cases (in particular the cases we are considering). The discussion in §7.3.11 will depend only on this hypothesis.

Hypothesis 7.3.2. (1) The Shimura variety $\text{Sh}_{K_i}(G_i, X_i)$ has a canonical integral model over $\mathcal{O}_{E, (v)}$. Let Sh_{μ_i, K_i} (or Sh_{μ_i} for simplicity if the level structure is clear from the context) denote the perfection of the mod p fiber of the model. Then there is a morphism $\text{loc}_p : \text{Sh}_{\mu_i, K_i} \rightarrow \text{Sht}_{\mu_i}^{\text{loc}}$ such that the composition

$$\text{loc}_p(m, n) := \text{res}_{m, n} \circ \text{loc}_p : \text{Sh}_{\mu_i, K_i} \rightarrow \text{Sht}_{\mu_i}^{\text{loc}(m, n)}$$

is perfectly smooth for any pair (m, n) such that $m - n$ is μ_i -large.

(2) Assume that

$$(7.3.1) \quad \mu_1|_{Z(\hat{G})^{\Gamma_{\mathbb{Q}_p}}} = \mu_2|_{Z(\hat{G})^{\Gamma_{\mathbb{Q}_p}}}.$$

Then there exists a perfect ind-scheme $\text{Sh}_{\mu_1|\mu_2}$ fitting into the following commutative diagram, with both squares Cartesian,

$$(7.3.2) \quad \begin{array}{ccccc} \text{Sh}_{\mu_1, K_1} & \xleftarrow{\overleftarrow{h}_{\mu_1}} & \text{Sh}_{\mu_1|\mu_2} & \xrightarrow{\overrightarrow{h}_{\mu_2}} & \text{Sh}_{\mu_2, K_2} \\ \text{loc}_p \downarrow & & \downarrow & & \downarrow \text{loc}_p \\ \text{Sht}_{\mu_1}^{\text{loc}} & \xleftarrow{\overleftarrow{h}_{\mu_1}^{\text{loc}}} & \text{Sht}_{\mu_1|\mu_2}^{\text{loc}} & \xrightarrow{\overrightarrow{h}_{\mu_2}^{\text{loc}}} & \text{Sht}_{\mu_2}^{\text{loc}} \end{array}.$$

Note that Part (1) of the hypothesis holds for Shimura data of Hodge type, as discussed in the previous subsection. In addition, Part (1) also holds for weak Shimura data (G, X) such that $G_{\mathbb{R}}$ is compact modulo center. Namely, in this case $\text{Sh}_K(G, X) = G(\mathbb{Q}) \backslash G(\mathbb{A}_f) / K$ is a set with an action of $\text{Gal}(\overline{\mathbb{Q}}/E)$ unramified at p , and the Shimura cocharacter $\mu \in \mathbb{X}_{\bullet}(Z_G)$ is central. Then the existence of the canonical integral model and the map loc_p is clear.

We concentrate on Part (2) of the hypothesis. It seems possible to construct the correspondence (7.3.3) in a very general situation. However, to do so it seems best to work in the framework of Shimura varieties of abelian type. To avoid some significant digression, in below we will establish it in some special cases, which suffices for our applications, and leave the general cases in a future work. But let us first make a few remarks on this hypothesis.

Remark 7.3.3. (1) If $(G_1, X_1) = (G_2, X_2) = (G, X)$, $\text{Sh}_{\mu|\mu}$ is the perfection of the mod p fiber of a natural integral model of the Hecke correspondence

$$(7.3.3) \quad \begin{array}{ccc} & G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f^p) / K^p \times G(\mathbb{Q}_p) \times^{K_p} G(\mathbb{Q}_p) / K_p & \\ \swarrow \overleftarrow{h} & & \searrow \overrightarrow{h} \\ G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K & & G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f) / K. \end{array}$$

If $(G_1, X_1) \neq (G_2, X_2)$ but (7.3.1) holds, then $\text{Sh}_{\mu_1|\mu_2}$ can be regarded as “exotic Hecke correspondences” between mod p fibers of *different* Shimura varieties. These correspondences cannot be lifted to characteristic zero, and give a large class of characteristic p cycles on Shimura varieties.

(2) Condition (7.3.1) is equivalent to

$$\text{Hom}_{\text{Coh}_{\hat{G}}(\hat{G}\sigma_p)}(\widetilde{V}_{\mu_1}, \widetilde{V}_{\mu_2}) \neq 0.$$

It is also the necessary and sufficient condition to ensure $\text{Sht}_{\mu_1|\mu_2}^{\text{loc}} \neq \emptyset$.

7.3.4. Examples of exotic Hecke correspondences (7.3.2). We explain that (7.3.2) exists in many cases.

If both $\mathbf{Sh}_{K_1}(G_1, X_1)$ and $\mathbf{Sh}_{K_2}(G_2, X_2)$ are Shimura sets, the top row of (7.3.2) extends naturally to $\mathcal{O}_{E,v}$. For this, we need to check the Shimura’s reciprocity map for weak Shimura data. The geometric Frobenius ϕ_v acts on $\mathbf{Sh}_{K_{i_p}}(G_i, X_i)$ by Hecke translation $p^{\text{Nm}_{E_v/\mathbb{Q}_p}(\mu_i)}$. The condition (7.3.1) ensures that $\text{Nm}_{E_v/\mathbb{Q}_p}(\mu_1 - \mu_2)$ is trivial on $Z(\hat{G})$, which surjects onto the maximal abelian quotient of \hat{G} . This implies that $p^{\text{Nm}_{E_v/\mathbb{Q}_p}(\mu_1)} = p^{\text{Nm}_{E_v/\mathbb{Q}_p}(\mu_2)}$ as an element of $Z_G(\mathbb{Q}_p)$. So (7.3.2) exists in this case.

Next, we assume that $\mathbf{Sh}_{K_1}(G_1, X_1) = G_1(\mathbb{Q}) \backslash G_1(\mathbb{A}_f) / K_1$ is a discrete set and $\mathbf{Sh}_{K_2}(G_2, X_2)$ is of Hodge type. Then G_1 is compact modulo center at the infinite place and μ_1 is central. Then (7.3.1) exactly implies that the basic element in $B(G_{\mathbb{Q}_p}, \mu_2^*)$ can be represented by $p^{\mu_1^*}$. We may choose a \bar{k}_v -point $x \in \text{Sh}_{\mu_2, K}$ and a trivialization of $x^* \mathcal{E}$ such that $b_x = p^{\mu_1^*}$. Recall that in this case the group I_x is equipped with an inner twist $\Psi : G \rightarrow I_x$, which is trivial at all finite places, and is compact modulo center at the infinite place. In particular, there is an isomorphism $G_1 \simeq I_x$,

compatible with the inner twist. We fix such inner twist. We may reinterpret the Rapoport-Zink uniformization as follows.

Proposition 7.3.5. *In the following diagram, both squares are Cartesian.*

(7.3.4)

$$\begin{array}{ccccc}
G_1(\mathbb{Q}) \backslash G_1(\mathbb{A}_f) / K & \xleftarrow{\overleftarrow{h}} & \mathrm{Sh}_{\mu_1|\mu_2} := G_1(\mathbb{Q}) \backslash G(\mathbb{Q}_p) \times^{G(\mathbb{Z}_p)} X_{\mu_2}^*(p^{\mu_1^*}) \times G(\mathbb{A}_f^p) / K^p & \xrightarrow{\overrightarrow{h}} & \mathrm{Sh}_{\mu_2, K} \\
\downarrow & & \downarrow & & \downarrow \mathrm{loc}_p \\
\mathrm{Sht}_{\mu_1}^{\mathrm{loc}} & \xleftarrow{\overleftarrow{h}^{\mathrm{loc}}} & \mathrm{Sht}_{\mu_1|\mu_2}^{\mathrm{loc}} & \xrightarrow{\overrightarrow{h}^{\mathrm{loc}}} & \mathrm{Sht}_{\mu_2}^{\mathrm{loc}}.
\end{array}$$

Moreover, there is a k_v -scheme structure on $G_1(\mathbb{Q}) \backslash G(\mathbb{Q}_p) \times^{G(\mathbb{Z}_p)} X_{\mu_2}^*(p^{\mu_1^*}) \times G(\mathbb{A}_f^p) / K^p$ so that the top row of (7.3.4) is defined over k_v .

Proof. Note that by (5.2.6), $(\overrightarrow{h}^{\mathrm{loc}})^{-1}(\mathrm{loc}_p(x)) = X_{\mu_1^*}(b_x) = X_{\mu_1^*}(p^{\mu_1^*}) = G(\mathbb{Q}_p) / G(\mathbb{Z}_p)$. Then by Corollary 7.2.16,

$$\begin{aligned}
\mathrm{Sht}_{\mu_1|\mu_2}^{\mathrm{loc}} \times_{\mathrm{Sht}_{\mu_2}^{\mathrm{loc}}} \mathrm{Sh}_{\mu_2, K} &= \mathrm{Sht}_{\mu_1|\mu_2}^{\mathrm{loc}} \times_{\mathrm{Sht}_{\mu_2}^{\mathrm{loc}}} \mathrm{Sh}_{\mu_2, K, b} \\
&= \mathrm{Sht}_{\mu_1|\mu_2}^{\mathrm{loc}} \times_{\mathrm{Sht}_{\mu_2}^{\mathrm{loc}}} (I_x(\mathbb{Q}) \backslash X_{\mu_2}^*(b_x) \times G(\mathbb{A}_f^p) / K^p) \\
&\cong G_1(\mathbb{Q}) \backslash G(\mathbb{Q}_p) \times^{G(\mathbb{Z}_p)} X_{\mu_2}^*(p^{\mu_1^*}) \times G(\mathbb{A}_f^p) / K^p = \mathrm{Sh}_{\mu_1|\mu_2}.
\end{aligned}$$

Therefore, we have the right Cartesian square. The left square is Cartesian because the left square of (5.2.7) is Cartesian.

Now we show that the top row of (7.3.4) is defined over k_v in an appropriate sense. We go into the proof of Corollary 7.2.16 when $b_x = p^{\mu_1^*}$. Recall that the point $x \in \mathrm{Sh}_{\mu_2, K_p}$ gives rise to a tuple $(A_x, \eta_x, s_{\mathrm{et}, x}, s_{0, x})$ such that $b_x = p^{\mu_1^*}$ under some trivialization of $x^* \mathcal{E}$. A point $y = (g_p, y_0, g^p)$ in the space $\mathrm{Sh}_{\mu_1|\mu_2}$ gives rise to a modification $\beta_y : \mathcal{E}'_y \xrightarrow{\beta_{y_0}} x^* \mathcal{E} \xrightarrow{g^p} x^* \mathcal{E}$ of local \underline{G} -shtukas, which further gives rise to p -power quasi-isogenies $\beta_y : A'_y \rightarrow A_x$, and a tame level structure η'_y on A'_y given by the image of the section $\eta_x g^p$ under the isomorphism

$$\begin{aligned}
&\mathrm{Isom}((V \otimes \mathbb{A}_f^p \oplus \mathbb{A}_f^p, s \otimes 1), (\mathcal{V}_{\mathrm{et}}(A)^p \oplus \mathbb{A}_f^p(1), s_{\mathrm{et}, x})) / K^p \\
&\xrightarrow{\beta_y^{-1}} \mathrm{Isom}((V \otimes \mathbb{A}_f^p \oplus \mathbb{A}_f^p(1), s \otimes 1), (\mathcal{V}_{\mathrm{et}}(A'_y)^p \oplus \mathbb{A}_f^p, \beta_y^{-1}(s_{\mathrm{et}, x}))) / K^p.
\end{aligned}$$

Then the image of this point y under \overrightarrow{h} in $\mathrm{Sh}_{\mu_2, K}$ is represented by $(A'_y, \eta'_y, \beta_y^{-1}(s_{\mathrm{et}, x}), \beta_y^{-1}(s_{0, x}))$.

We define an action of Frobenius σ_v on $\mathrm{Sh}_{\mu_1|\mu_2}$ by sending such a point (g_p, y_0, g^p) to

$$(p^{\mathbb{N}_v(\mu_1^*)} g_p, \sigma_v(x), g^p), \quad \text{for } \mathbb{N}_v(\mu_1^*) := \sum_{i=1}^{[k_v : \mathbb{F}_p]} \sigma^i(\mu_1^*).$$

This is compatible via \overleftarrow{h} with the Frobenius σ_v action on $\mathrm{Sh}_{\mu_1} = G_1(\mathbb{Q}) \backslash G_1(\mathbb{A}_f) / K$ determined by the Shimura reciprocity law. On the other hand, the image of $z := (p^{\mathbb{N}_v(\mu_1^*)} g_p, \phi_v(x), g^p)$ in $\mathrm{Sh}_{\mu_2, K}$ under \overrightarrow{h} is represented by the abelian variety A'_z , equipped with the p -power isogeny $\beta_z : A'_z \rightarrow A_x$ corresponding to the modification

$$(7.3.5) \quad \beta_z : (1 \times \sigma_v)^*(\mathcal{E}'_y) \xrightarrow{\sigma_v(\beta_y)} (1 \times \sigma_v)^*(x^* \mathcal{E}) \xrightarrow{p^{\mathbb{N}_v(\mu_1^*)}} x^* \mathcal{E}$$

of local \underline{G} -shtukas. In particular, we have a natural isomorphism $A'_z \cong \sigma_v^*(A'_y)$. The last map in (7.3.5) is given by multiplication by $p^{\mathbb{N}_v(\mu_1^*)}$ because the natural map $\sigma \mathcal{E} \rightarrow \mathcal{E}$ is the multiplication by $p^{\mu_1^*}$ (under the chosen trivialization). The tame level structure η'_z on A'_z is given by the image

of $\eta_x g^p$ under the isomorphism induced by β_z^{-1} , which is the same as the image under the following sequence of isomorphisms

$$\begin{array}{ccc}
\eta_x g^p & \in & \text{Isom}((V \otimes \mathbb{A}_f^p \oplus \mathbb{A}_f^p, s \otimes 1), (\mathcal{V}_{\text{et}}(A)^p \oplus \mathbb{A}_f^p, s_{\text{et},x}))/K^p \\
\downarrow \sigma_v & & \downarrow \sigma_v \\
\sigma_v(\eta_x g^p) & \in & \text{Isom}((V \otimes \mathbb{A}_f^p \oplus \mathbb{A}_f^p, s \otimes 1), (\mathcal{V}_{\text{et}}(\sigma_v^*(A_x))^p \oplus \mathbb{A}_f^p, \sigma_v(s_{\text{et},x}))/K^p \\
\downarrow \sigma_v(\beta_y^{-1}) & & \downarrow \sigma_v(\beta_y^{-1}) \\
\sigma_v(\beta_y^{-1})(\sigma_v(\eta_x g^p)) & \in & \text{Isom}((V \otimes \mathbb{A}_f^p \oplus \mathbb{A}_f^p, s \otimes 1), (\mathcal{V}_{\text{et}}(\sigma_v^*(A'_y))^p \oplus \mathbb{A}_f^p, \sigma_v(\beta_y^{-1}(s_{\text{et},x}))/K^p.
\end{array}$$

This exactly says that

$$\vec{h}(p^{\mathbb{N}_v(\mu_1^*)} g_p, \phi_v(x), g^p) = \sigma_v(\vec{h}(g_p, \phi_v(x), g^p))$$

for the natural action σ_v on $\text{Sh}_{\mu_2, K}$. This means that the top row of (7.3.4) can be defined over k_v , using the given Frobenius structure. \square

Finally, we assume that both (G_1, X_1) and (G_2, X_2) are Shimura data of Hodge type. We discuss the case $(G_1, X_1) = (G_2, X_2) = (G, X)$.

Proposition 7.3.6. *There is an ind-scheme $\text{Sh}_{\mu|\mu}$ and morphisms $\overleftarrow{h}, \overrightarrow{h} : \text{Sh}_{\mu|\mu} \rightarrow \text{Sh}_{\mu}$ that fit into the diagram (7.3.2) with both squares Cartesian.*

Proof. Let $\overleftarrow{\text{Sh}}_{\mu|\mu} := \text{Sh}_{\mu} \times_{\text{Sh}_{\mu}^{\text{loc}}, \overleftarrow{h}^{\text{loc}}} \text{Sh}_{\mu}^{\text{loc}}$. The natural projection $\overleftarrow{\text{Sh}}_{\mu|\mu} \rightarrow \text{Sh}_{\mu}$ is denoted by \overleftarrow{h} . Similarly we have $\overrightarrow{h} : \overrightarrow{\text{Sh}}_{\mu|\mu} := \text{Sh}_{\mu}^{\text{loc}} \times_{\overrightarrow{h}^{\text{loc}}, \text{Sh}_{\mu}^{\text{loc}}} \text{Sh}_{\mu} \rightarrow \text{Sh}_{\mu}$. We claim that there is another map

$$\overrightarrow{h} : \overleftarrow{\text{Sh}}_{\mu|\mu} \rightarrow \text{Sh}_{\mu}$$

that induces a map $\overleftarrow{\text{Sh}}_{\mu|\mu} \rightarrow \overrightarrow{\text{Sh}}_{\mu|\mu}$. Indeed, $\overleftarrow{\text{Sh}}_{\mu|\mu}$ classifies a point x of Sh_{μ} and a modification of local \underline{G} -Shtukas $\mathcal{E}' \dashrightarrow x^* \mathcal{E}$. As argued in the paragraphs before Lemma 7.2.12, this provides an abelian scheme A' over $\overleftarrow{\text{Sh}}_{\mu|\mu}$ with a p -power quasi-isogeny $\overleftarrow{h}^* A_x \rightarrow A'$ such that the crystalline tensors $s_{0,x} : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(A_x)^{\otimes}$ induces $s'_0 : \mathcal{T} \otimes \mathbf{1} \rightarrow \mathbb{D}(A')^{\otimes}$. In addition, A' comes equipped with a prime-to- p principal polarization as well as a K^p -level structure, inherited from A_x . Then A' defines a morphism $\overrightarrow{h}' : \overleftarrow{\text{Sh}}_{\mu|\mu} \rightarrow \mathcal{A}_{g, K'^p}$. Note that if y is a closed point Sh_{μ} , the fiber $\overleftarrow{h}^{-1}(y) \cong X_{\mu^*}(b_y)$. Therefore, pointwise, \overrightarrow{h}' lifts to Sh_{μ} matching the s'_0 and s_0 by [Kis17, Proposition 1.4.4]. Now by the same argument as Lemma 7.2.12 (using Lemma 7.2.11), we see \overrightarrow{h}' lifts to $\overrightarrow{h} : \overleftarrow{\text{Sh}}_{\mu|\mu} \rightarrow \text{Sh}_{\mu}$ which induces $\overleftarrow{\text{Sh}}_{\mu|\mu} \rightarrow \overrightarrow{\text{Sh}}_{\mu|\mu}$.

By the same argument, we obtain $\overleftarrow{h} : \overrightarrow{\text{Sh}}_{\mu|\mu} \rightarrow \text{Sh}_{\mu}$ that induces $\overrightarrow{\text{Sh}}_{\mu|\mu} \rightarrow \overleftarrow{\text{Sh}}_{\mu|\mu}$. Clearly, these two maps are inverse to each other. The proposition follows. \square

Remark 7.3.7. For our purpose, the existence of such a self-correspondence between the mod p fiber of the Shimura variety is enough. But it is not difficult to show that $\text{Sh}_{\mu|\mu}$ is the perfection of the mod p fiber of a natural integral model of the p -power Hecke correspondence (7.3.2). This will be discussed in details in another occasion.

7.3.8. Exotic Hecke correspondence for PEL type Shimura varieties. There is another case where (7.3.2) can be constructed relatively easily.

Let $(B, *, \mathcal{O}_B, V_i, (-, -)_i, \Lambda_i)$ (for $i = 1, 2$) be two integral PEL type Shimura data as in [Ko92a] that are unramified at p . Assume that $V_1(\mathbb{A}_f) \simeq V_2(\mathbb{A}_f)$ as skew-Hermitian B -modules. Let G_i denote the group of automorphisms of the skew-Hermitian B -modules V_i (with similitudes), which

is assumed to be connected (in particular the type (D) case is excluded). Assume moreover that the group $\ker^1(\mathbb{Q}, G_i)$ is trivial so that the Shimura variety $\mathcal{S}_K(G_i, X_i)$ is exactly the moduli space of tuples (A, λ, i, η) satisfying the Kottwitz condition as in [Ko92a, §5].

Proposition 7.3.9. *The correspondence (7.3.2) exists between $\mathrm{Sh}_{\mu_1, K}$ and $\mathrm{Sh}_{\mu_2, K}$.*

Proof. The proof is similarly that of Proposition 7.3.6, except using the PEL type moduli interpretation of Shimura varieties $\mathrm{Sh}_{\mu_1, K}$ and $\mathrm{Sh}_{\mu_2, K}$. Namely, a point of $\mathrm{Sh}_{\mu_1, K} \times_{\mathrm{Sht}_{\mu_1}^{\mathrm{loc}}} \mathrm{Sht}_{\mu_1|_{\mu_2}}^{\mathrm{loc}}$ corresponds to (A, λ, i, η) and a modification $\mathcal{E}' \dashrightarrow x^*\mathcal{E}$ of the local Shtuka, where $x^*\mathcal{E}$ arises from $(A[p^\infty], \lambda, i)$ (via the Dieudonné theory). As in Proposition 7.3.6, this modification allows one to produce another abelian scheme A' equipped with a polarization λ' , an endomorphism by \mathcal{O}_B and a level structure η' inherited from A . Since the singularity of \mathcal{E}' is of type μ_2 , the Kottwitz condition on A' is the one imposed for the Shimura variety Sh_{μ_2} . This defines a map $\mathrm{Sh}_{\mu_1, K} \times_{\mathrm{Sht}_{\mu_1}^{\mathrm{loc}}} \mathrm{Sht}_{\mu_1|_{\mu_2}}^{\mathrm{loc}} \rightarrow \mathrm{Sh}_{\mu_2}$. One can define the map in the other direction by the same argument. Then as in the proof of Proposition 7.3.6, we obtain (7.3.2). We leave the details as an exercise for readers. \square

Remark 7.3.10. In the proposition above, it is possible that $\mathrm{Sh}_{\mu_1|_{\mu_2}}$ is empty (when $\mathrm{Sh}_{\mu_1|_{\mu_2}}^{\mathrm{loc}}$ is). For example, if $B = E$ is a CM field with $*$ being the complex conjugation, V_i is a Hermitian space with signature $(p_{i,\tau}, q_{i,\tau})$ at $\tau : F = E^{*=1} \rightarrow \mathbb{R}$ (where $p_{i,\tau} + q_{i,\tau} = n$). Then at an inert prime p of E , $\mathrm{Sh}_{\mu_1|_{\mu_2}}^{\mathrm{loc}}$ is non-empty if and only if $(-1)^{\sum_{\tau} p_{1,\tau} q_{1,\tau}} = (-1)^{\sum_{\tau} p_{2,\tau} q_{2,\tau}}$, which is equivalent to (7.3.1). (Also compare with Example 2.1.8 (ii)).

7.3.11. *The cohomological correspondence.* Let $(G_i, X_i), i = 1, 2, 3$ be three Shimura data, equipped with isomorphisms $G_{1,\mathbb{A}_f} \simeq G_{2,\mathbb{A}_f} \simeq G_{3,\mathbb{A}_f}$, and a level structure K . Let p be an unramified prime, such that (7.3.1) holds for each pair of them. We assume Hypothesis 7.3.2 holds for each pair of them. Let \mathcal{H}^p be the prime-to- p Hecke algebra. We also fix a square root \sqrt{p} in \mathbb{Q}_ℓ , i.e. a half Tate twist $\overline{\mathbb{Q}}_\ell(1/2)$. To simplify notations, for a smooth variety X of pure dimension d over \bar{k}_v , we write $\langle d \rangle = [d](d/2)$ as usual, and

$$H_c(X) = H_c^*(X, \overline{\mathbb{Q}}_\ell \langle d \rangle).$$

Recall that for every representation V of \hat{G} , there is a vector bundle \tilde{V} on the stack $\hat{G}\sigma/\hat{G}$ of unramified Langlands parameters. Let $d_i = \langle 2\rho, \mu_i \rangle = \dim \mathrm{Sh}_K(G_i, X_i)$. Write $V_i = V_{\mu_i}$. Let $S(\tilde{V}_i) \in \mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})$, which can be represented as a perverse sheaf $S(\tilde{V}_i)^{\mathrm{loc}(m,n)}$ on $\mathrm{Sht}_{\mu_i}^{\mathrm{loc}(m,n)}$ for some pair non-negative integers (m, n) with m being positive and $m - n$ being μ_i -large. Note that since μ_i is minuscule, the pullback $\mathrm{loc}_p(m, n)^* S(\tilde{V}_i)$ to Sh_{μ_i} is canonically isomorphic to $\overline{\mathbb{Q}}_\ell \langle d_i \rangle$.

We first construct a map

$$(7.3.6) \quad \mathrm{Hom}_{\mathrm{P}^{\mathrm{Corr}}(\mathrm{Sht}^{\mathrm{loc}})}(S(\tilde{V}_1), S(\tilde{V}_2)) \rightarrow \mathrm{Hom}_{\mathcal{H}^p}(H_c(\mathrm{Sh}_{\mu_1, \bar{k}_v}), H_c(\mathrm{Sh}_{\mu_2, \bar{k}_v}))$$

as follows. Choose a dominant coweight ν and a quadruple (m_1, n_1, m_2, n_2) that is $(\mu_1 + \nu, \nu)$ -acceptable and $(\mu_2 + \nu, \nu)$ -acceptable. Then we have

$$(7.3.7) \quad \begin{array}{ccccc} \mathrm{Sh}_{\mu_1} & \longleftarrow & \mathrm{Sh}_{\mu_1|_{\mu_2}}^\nu & \longrightarrow & \mathrm{Sh}_{\mu_2} \\ \downarrow & & \downarrow \mathrm{loc}_p^\nu & & \downarrow \\ \mathrm{Sht}_{\mu_1}^{\mathrm{loc}} & \longleftarrow & \mathrm{Sht}_{\mu_1|_{\mu_2}}^{\nu, \mathrm{loc}} & \longrightarrow & \mathrm{Sht}_{\mu_2}^{\mathrm{loc}} \\ \downarrow & & \downarrow & \text{X} & \downarrow \\ \mathrm{Sht}_{\mu_1}^{\mathrm{loc}(m_1, n_1)} & \longleftarrow & \mathrm{Sht}_{\mu_1|_{\mu_2}}^{\nu, \mathrm{loc}(m_1, n_1)} & \longrightarrow & \mathrm{Sht}_{\mu_2}^{\mathrm{loc}(m_2, n_2)} \end{array}$$

Note that all squares except the one marked with “X” are Cartesian: that the upper two squares being Cartesian follows from Hypothesis 7.3.2(2) and that the lower left square is Cartesian follows from Lemma 5.3.17. The map $\text{loc}_p(m_i, n_i) : \text{Sh}_{\mu_i} \rightarrow \text{Sht}_{\mu_i}^{\text{loc}(m_i, n_i)}$ are perfectly smooth by Proposition 7.2.4. Therefore, the composition of the vertical maps in the middle column, denoted by $\text{loc}_p^\nu(m_1, n_1)$, is also perfectly smooth. Also recall that the map $\text{Sh}_{\mu_1|\mu_2}^\nu \rightarrow \text{Sh}_{\mu_1}$ is perfectly proper.

By the formalism of pullback cohomological correspondences along perfectly smooth morphisms (§A.2.11), and proper pushforward of cohomological correspondences (§A.2.6), we thus obtain a map

$$(7.3.8) \quad \begin{aligned} & \text{Corr}_{\text{Sht}_{\mu_1|\mu_2}^{\nu, \text{loc}(m_1, n_1)}} (S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)}, S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}) \\ & \xrightarrow{\text{loc}_p^\nu(m_1, n_1)^*} \text{Corr}_{\text{Sh}_{\mu_1|\mu_2}^\nu} ((\text{Sh}_{\mu_1}, \overline{\mathbb{Q}}_\ell \langle d_1 \rangle), (\text{Sh}_{\mu_2}, \overline{\mathbb{Q}}_\ell \langle d_2 \rangle)) \\ & \xrightarrow{H_c} \text{Hom}_{\mathcal{H}^p} (H_c(\text{Sh}_{\mu_1, \bar{k}_v}), H_c(\text{Sh}_{\mu_2, \bar{k}_v})). \end{aligned}$$

Note that it follows from the diagram (6.2.4) that $\text{loc}_p^\nu(m_1, n_1)^*$ is compatible with changing (m_1, n_1, m_2, n_2) in a natural way. If $\nu \preceq \nu'$ and if (m_1, n_1, m_2, n_2) is $(\mu_1 + \nu', \nu')$ -acceptable and is $(\mu_2 + \nu', \nu')$ -acceptable, then proper smooth base change implies the commutativity of the following diagram

$$\begin{array}{ccc} \text{Corr}_{\text{Sht}_{\mu_1|\mu_2}^{\nu, \text{loc}(m_1, n_1)}} (S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)}, S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}) & \longrightarrow & \text{Corr}_{\text{Sht}_{\mu_1|\mu_2}^{\nu', \text{loc}(m_1, n_1)}} (S(\widetilde{V}_1)^{\text{loc}(m_1, n_1)}, S(\widetilde{V}_2)^{\text{loc}(m_2, n_2)}) \\ \downarrow & & \downarrow \\ \text{Corr}_{\text{Sh}_{\mu_1|\mu_2}^\nu} ((\text{Sh}_{\mu_1}, \overline{\mathbb{Q}}_\ell \langle d_1 \rangle), (\text{Sh}_{\mu_2}, \overline{\mathbb{Q}}_\ell \langle d_2 \rangle)) & \longrightarrow & \text{Corr}_{\text{Sh}_{\mu_1|\mu_2}^{\nu'}} ((\text{Sh}_{\mu_1}, \overline{\mathbb{Q}}_\ell \langle d_1 \rangle), (\text{Sh}_{\mu_2}, \overline{\mathbb{Q}}_\ell \langle d_2 \rangle)). \end{array}$$

Therefore, when composed with H_c , by Lemma A.2.8, the composition map (7.3.8) factors through $\text{Hom}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2))$ and gives the desired (7.3.6).

Now assume we have three Shimura data of Hodge type (G_i, X_i) , and we assume that Hypothesis 7.3.2 holds for each pair out of three.

Lemma 7.3.12. *The following diagram is commutative.*

$$\begin{array}{ccc} \text{Hom}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_2)) \otimes \text{Hom}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_2), S(\widetilde{V}_3)) & \longrightarrow & \text{Hom}_{\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})}(S(\widetilde{V}_1), S(\widetilde{V}_3)) \\ \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{H}^p} (H_c^*(\text{Sh}_{\mu_1, \bar{k}_v}), H_c^*(\text{Sh}_{\mu_2, \bar{k}_v})) \otimes \text{Hom}_{\mathcal{H}^p} (H_c^*(\text{Sh}_{\mu_2, \bar{k}_v}), H_c^*(\text{Sh}_{\mu_3, \bar{k}_v})) & \longrightarrow & \text{Hom}_{\mathcal{H}^p} (H_c^*(\text{Sh}_{\mu_1, \bar{k}_v}), H_c^*(\text{Sh}_{\mu_3, \bar{k}_v})). \end{array}$$

Proof. First, it is clear that from the hypothesis, there exists $\text{Comp} : \text{Sh}_{\mu_1|\mu_2}^\nu \times_{\text{Sh}_{\mu_2}} \text{Sh}_{\mu_2|\mu_3}^{\nu'} \rightarrow \text{Sh}_{\mu_1|\mu_3}^{\nu+\nu'}$ making the following diagram commutative.

$$(7.3.9) \quad \begin{array}{ccccc} & & & \text{Sh}_{\mu_1|\mu_3}^{\nu+\nu'} & \\ & \nearrow & & \downarrow & \searrow \\ \text{Sh}_{\mu_1} & \xleftarrow{\quad} & \text{Sh}_{\mu_1|\mu_2}^\nu \times_{\text{Sh}_{\mu_2}} \text{Sh}_{\mu_2|\mu_3}^{\nu'} & \xrightarrow{\quad} & \text{Sh}_{\mu_3} \\ & \downarrow \text{loc}_p & \downarrow & \downarrow & \downarrow \text{loc}_p \\ \text{Sht}_{\mu_1}^{\text{loc}} & \xleftarrow{\quad} & \text{Sht}_{\mu_1|\mu_2}^{\nu,\text{loc}} \times_{\text{Sht}_{\mu_2}^{\text{loc}}} \text{Sht}_{\mu_2|\mu_3}^{\nu',\text{loc}} & \xrightarrow{\quad} & \text{Sht}_{\mu_3}^{\text{loc}} \\ & \searrow & \downarrow \text{Comp}^{\text{loc}} & \downarrow & \nearrow \\ & & \text{Sht}_{\mu_1|\mu_3}^{\nu+\nu',\text{loc}} & & \end{array}$$

In addition, the middle trapezoid is a Cartesian.

Combining the above diagram with the diagram in Proposition 5.3.23, we obtain a diagram similar to (7.3.9), with the bottom two rows replaced by moduli of restricted shtukas. Then lemma follows from the compatibility of pullback and pushforward of cohomological correspondence as in Lemma A.2.17, and the definition of compositions of morphisms in $\text{P}^{\text{Corr}}(\text{Sht}^{\text{loc}})$ and Lemma A.2.10. \square

Now combining the above considerations and Theorem 6.0.1, we see that there is a canonical map

$$(7.3.10) \quad \text{Spc} : \text{Hom}_{\text{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma_p)}(\tilde{V}_i, \tilde{V}_j) \rightarrow \text{Hom}_{\mathcal{H}^p}(\text{H}_c(\text{Sh}_{\mu_i, \bar{k}_v}), \text{H}_c(\text{Sh}_{\mu_j, \bar{k}_v}))$$

compatible with compositions. In particular, the ring

$$\mathbf{J} := \Gamma([\hat{G}\sigma_p/\hat{G}], \mathcal{O}) = \overline{\mathbb{Q}}_\ell[\hat{G}]^{c_\sigma(\hat{G})}$$

of regular functions on the stack acts on every $\text{H}_c(\text{Sh}_{\mu_i, \bar{k}_v})$, and the above map is upgraded to a \mathbf{J} -module homomorphism

$$\text{Spc} : \text{Hom}_{\text{Coh}_{f_r}^{\hat{G}}(\hat{G}\sigma_p)}(\tilde{V}_i, \tilde{V}_j) \rightarrow \text{Hom}_{\mathcal{H}^p \otimes \mathbf{J}}(\text{H}_c(\text{Sh}_{\mu_i, \bar{k}_v}), \text{H}_c(\text{Sh}_{\mu_j, \bar{k}_v})),$$

compatible with compositions.

This action is the Shimura variety counterpart of V. Lafforgue's S -operator. Recall that the classical Satake isomorphism is a canonical isomorphism

$$\mathbf{J} \cong \mathbf{J}_{\hat{T}} \cong \overline{\mathbb{Q}}_\ell[\mathbb{X}_\bullet(T)^\sigma]^{W_0} \cong C_c(G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p)).$$

On the other hand, the Hecke algebra $C_c(G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p))$ naturally acts on $\text{H}_c(\text{Sh}_{\mu, \bar{k}_v})$: the existence of smooth toroidal compactification ([Lan13] [MP15]) implies that after choosing a specialization map $\bar{k}_v \rightarrow \bar{E}_v$, there is an isomorphism

$$\text{H}_c(\text{Sh}_{\mu, \bar{k}_v}) \simeq \text{H}_c(\text{Sh}_K(G, X)_{\bar{E}_v}),$$

and since the action of the Hecke algebra $C_c(G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p) / G(\mathbb{Z}_p))$ on the right hand side commutes with the action of the Galois group, such action translates to an action of the left hand side.

Conjecture 7.3.13. *The action of \mathbf{J} on $\text{H}_c(\text{Sh}_{\mu, \bar{k}_v})$ constructed above coincides with the usual Hecke algebra action, via the Satake isomorphism.*

This conjecture would be the analogue of V. Lafforgue's $S = T$ theorem in the Shimura variety setting. In this direction, we have

Proposition 7.3.14. *If $\mathrm{Sh}_K(G, X)$ is a Shimura set, then Conjecture 7.3.13 holds.*

Proof. Let $f \in \mathbf{J}$, given by the restriction of a character χ_V of a representation of ${}^L G$ to $\hat{G}\sigma$. As discussed in Example A.2.3(5), in this case the cohomological correspondence $\mathrm{Spc}(f)$ is given by a function on

$$\mathrm{Sh}_{\mu|\mu} = G(\mathbb{Q}) \backslash (G(\mathbb{A}_f^p)/K^p \times G(\mathbb{Q}_p) \times {}^{G(\mathbb{Z}_p)} G(\mathbb{Q}_p)/G(\mathbb{Z}_p)),$$

which in turn is the pullback of a function on $\mathrm{Sh}_{\mu|\mu}^{\mathrm{loc}} = G(\mathbb{Z}_p) \backslash G(\mathbb{Q}_p)/G(\mathbb{Z}_p)$ by construction. But from Proposition 6.3.3, this function is

$$\mathrm{Sat}^{cl'}([V]) = \mathrm{Sat}^{cl}([V^*]).$$

The proposition follows. \square

7.4. Cycle classes of the basic Newton stratum. In this subsection, we combine all the previous results to compute the determinant of the intersection matrix of cycles coming from the irreducible components of the basic locus of the special fiber of the Shimura variety, and prove the main Theorem 1.1.4.

We keep the notation as in the previous sections. In addition, we assume that $V_{\mu^*}^{\mathrm{Tate}_p} \neq 0$. Recall from Proposition 4.2.10, the basic element $b \in B(G_{\mathbb{Q}_p}, \mu^*)$ is unramified. Let $\lambda_b \in \mathbb{X}_{\bullet}(T)_{\sigma_p}$ be the element associated to b via Lemma 4.2.3. Let

$$\mathbb{M}\mathbb{V}_{\mu^*}^{\mathrm{Tate}_p} := \bigsqcup_{\lambda, \lambda_{\sigma_p} = \lambda_b} \mathbb{M}\mathbb{V}_{\mu^*}(\lambda).$$

By Corollary 2.1.6 and Proposition 4.2.10, there is the inner form $G \rightarrow G'$, which is trivial at all finite places and such that $G'_{\mathbb{R}}$ is compact modulo center.

7.4.1. Irreducible components of basic locus. Let $\mathrm{Sh}_{\mu,b} \subset \mathrm{Sh}_{\mu}$ denote the basic Newton stratum.

Let x be a \bar{k}_v -point in the basic locus of Sh_{μ, K_p} so we obtain $\mathrm{loc}_p(x) \in \mathrm{Sh}_{\mu}^{\mathrm{loc}}(\bar{k}_v) = A(G_{\mathbb{Q}_p}, \mu^*)$ as from §7.2.7. Let $b_x \in G(L)$ be a representative of $\mathrm{loc}_p(x)$. Recall that we have a canonical decomposition

$$X_{\mu^*}(b_x) = \bigcup_{\mathbf{b} \in \mathbb{M}\mathbb{V}_{\mu^*}^{\mathrm{Tate}}} X_{\mu^*}^{\mathbf{b}}(b_x),$$

and $J_{b_x}(\mathbb{Q}_p) \cong G(\mathbb{Q}_p)$ acts on each $X_{\mu^*}^{\mathbf{b}}(b_x)$ (Theorem 4.4.14). Therefore, by the Rapoport-Zink uniformization (Corollary 7.2.16), we can decompose the basic Newton stratum $\mathrm{Sh}_{\mu,b}$ as the union of

$$\mathrm{Sh}_{\mu,b}^{\mathbf{b}} := I_x(\mathbb{Q}) \backslash X_{\mu^*}^{\mathbf{b}}(b_x) \times G(\mathbb{A}_f^p)/K^p,$$

for $\mathbf{b} \in \mathbb{M}\mathbb{V}_{\mu^*}^{\mathrm{Tate}}$. Note that this decomposition is independent of the choice of x and the representative b_x .

Applying Theorem 4.4.14, we obtain

Proposition 7.4.2. *Every irreducible component of $\mathrm{Sh}_{\mu, K, b} \otimes \bar{k}_v$ is of dimension $d/2$. There is a bijection between the set of irreducible components of $\mathrm{Sh}_{\mu, K, b} \otimes \bar{k}_v$ and $G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K \times \mathbb{M}\mathbb{V}_{\mu^*}^{\mathrm{Tate}_p}$.*

7.4.3. *Cycle classes coming from irreducible components of basic locus.* The cycle class map induces (see §A.2.18)

$$(7.4.1) \quad \text{cl}(\mathbf{b}) := \text{cl}(\text{Sh}_{\mu, \bar{k}_v}^{\mathbf{b}}) : H_d^{\text{BM}}(\text{Sh}_{\mu, \bar{k}_v}^{\mathbf{b}}) \cong C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K) \rightarrow H_c^d(\text{Sh}_{\mu, \bar{k}_v}, \overline{\mathbb{Q}}_\ell(d/2)).$$

Putting them together, we have

$$(7.4.2) \quad \text{cl}(\text{Sh}_{\mu, \bar{k}_v}) : \bigoplus_{\mathbf{b}} H_d^{\text{BM}}(\text{Sh}_{\mu, \bar{k}_v}^{\mathbf{b}}, \mathbb{Q}_\ell) \rightarrow H_c^d(\text{Sh}_{\mu, \bar{k}_v}, \overline{\mathbb{Q}}_\ell(d/2))$$

On the other hand, we may represent the basic element of $B(G_{\mathbb{Q}_p}, \mu^*)$ as p^{τ^*} for some $\tau^* \in \mathbb{X}_\bullet(Z_G)$ (under our assumption that Z_G is connected). Note that then (7.3.1) holds, and we have

$$(7.4.3) \quad \text{JL}_{\tau, \mu} : H_c(\text{Sh}_{\tau, \bar{k}_v}) \otimes_{\mathbf{J}} \text{Hom}(\widetilde{V}_\tau, \widetilde{V}_\mu) \rightarrow H_c^d(\text{Sh}_{\mu, \bar{k}_v}, \overline{\mathbb{Q}}_\ell(d/2))$$

from (7.3.10). We relate these two maps.

For each $\mathbf{b} \in \text{MV}_{\mu^*}^{\text{Tate}}$, recall that by Lemma 4.4.3, we can find $\tau_{\mathbf{b}}^* \in \mathbb{X}_\bullet(Z_G)$ and a dominant cocharacter $\nu_{\mathbf{b}}^*$ such that

- \mathbf{b} belongs to the image of $i_{\nu_{\mathbf{b}}^*}^{\text{MV}} : \mathbb{S}_{(\nu_{\mathbf{b}}^*, \mu^*)|_{\tau_{\mathbf{b}}^* + \sigma(\nu_{\mathbf{b}}^*)}} \rightarrow \text{MV}_{\mu^*}(\tau_{\mathbf{b}}^* + \sigma(\nu_{\mathbf{b}}^*) - \nu_{\mathbf{b}}^*)$, and
- $\nu_{\mathbf{b}}^*$ in minimal among all possible ν^* 's such that the above condition holds.

Then $\mathbf{a} := (i_{\nu_{\mathbf{a}}^*}^{\text{MV}})^{-1}(\mathbf{b})$ determines a Satake cycle $\text{Gr}_{(\nu_{\mathbf{b}}^*, \mu^*)|_{\tau_{\mathbf{b}}^* + \sigma(\nu_{\mathbf{b}}^*)}}^{0, \mathbf{a}}$, which via Proposition 3.1.10 and the geometric Satake, determines

$$\mathbf{a}_{\text{in}} \in \text{Hom}(V_{\nu_{\mathbf{b}}^*} \otimes V_{\mu^*}, V_{\tau_{\mathbf{b}}^*} \otimes V_{\sigma(\nu_{\mathbf{b}}^*)}) \cong \text{Hom}(V_{\sigma(\nu_{\mathbf{b}}^*)} \otimes V_{\tau_{\mathbf{b}}^*} \otimes V_{\nu_{\mathbf{b}}^*}, V_{\mu}),$$

and therefore an element $\Xi_{V_{\nu_{\mathbf{b}}^*}}(\mathbf{a}_{\text{in}}) \in \text{Hom}(\widetilde{V}_{\tau_{\mathbf{b}}^*}, \widetilde{V}_\mu)$ (see (6.2.2)). But by Proposition 6.3.2, we have the following commutative diagram

$$\begin{array}{ccc} H_c(\text{Sh}_{\tau_{\mathbf{b}}^*, \bar{k}_v}) & \xrightarrow{\cong} & H_d^{\text{BM}}(\text{Sh}_{\mu, \bar{k}_v}^{\mathbf{b}}) \\ & \searrow \text{JL}_{\tau_{\mathbf{b}}^*, \mu}(\mathbf{a}_{\text{in}}) & \swarrow \text{cl}(\mathbf{b}) \\ & H_c^d(\text{Sh}_{\mu, \bar{k}_v}, \overline{\mathbb{Q}}_\ell(d/2)) & \end{array}.$$

In particular

$$\text{Im cl}(\mathbf{b}) = \text{Im JL}_{\tau_{\mathbf{b}}^*, \mu}(\mathbf{a}_{\text{in}}).$$

Remark 7.4.4. Again we remark that the above chosen $\tau_{\mathbf{b}} \in \mathbb{X}_\bullet(Z_G)$ depends on \mathbf{b} .

Dually, there is the map

$$(7.4.4) \quad H_c^d(\text{Sh}_{\mu, \bar{k}_v}, \overline{\mathbb{Q}}_\ell(d/2)) \rightarrow H^0(\text{Sh}_{\tau_{\mathbf{b}}^*, \bar{k}_v}),$$

dual to the cycle class map, see §A.2.18. In addition, the Satake cycle $\text{Gr}_{(\nu_{\mathbf{b}}^*, \mu^*)|_{\tau_{\mathbf{b}}^* + \sigma(\nu_{\mathbf{b}}^*)}}^{0, \mathbf{a}}$ also induces

$$\mathbf{a}_{\text{out}} \in \text{Hom}(V_{\tau_{\mathbf{b}}^*} \otimes V_{\sigma(\nu_{\mathbf{b}}^*)}, V_{\nu_{\mathbf{b}}^*} \otimes V_{\mu^*}) \cong \text{Hom}(V_{\sigma(\nu_{\mathbf{b}}^*)} \otimes V_{\mu} \otimes V_{\nu_{\mathbf{b}}^*}, V_{\tau_{\mathbf{b}}^*})$$

Then (7.4.4) coincides with $\text{JL}_{\mu, \tau_{\mathbf{b}}^*}(\mathbf{a}_{\text{out}})$. We have proven the following lemma.

Lemma 7.4.5. *Let $\{\mathbf{b}_1, \dots, \mathbf{b}_r\}$ be the elements in $\text{MV}_{\mu^*}^{\text{Tate}_p}$. The intersection matrix of the cycle classes from (7.4.2) is given by a matrix with (i, j) -entry*

$$(7.4.5) \quad \text{JL}_{\mu, \tau_{\mathbf{b}_j}}(\mathbf{a}_{j, \text{out}}) \circ \text{JL}_{\tau_{\mathbf{b}_i}, \mu}(\mathbf{a}_{i, \text{in}})$$

As the matrix of an endomorphism of the vector bundle $\oplus_i \widetilde{V_{\tau_{\mathbf{b}_i}}}$ on $[\hat{G}\sigma_p/\hat{G}]$, its determinant \mathcal{D} is a regular function on the stack $[\hat{G}\sigma_p/\hat{G}]$. Let $\hat{G}' = \hat{G}_{\text{der}} \times Z(\hat{G})^\circ$, and $\Delta = \hat{G}_{\text{der}} \cap Z(\hat{G})^\circ$. Note that σ_p acts on \hat{G}' and Δ . The exact sequence $1 \rightarrow \Delta \rightarrow \hat{G}' \rightarrow \hat{G} \rightarrow 1$ induces an isomorphism

$$\Gamma([\hat{G}\sigma_p/\hat{G}], \mathcal{O}) \rightarrow \Gamma([\hat{G}'\sigma_p/\hat{G}'], \mathcal{O})^{\Delta_{\sigma_p}},$$

where $\Delta_{\sigma_p} = \Delta/(1 - \sigma_p)\Delta$ is the group of coinvariants. It is enough to calculate \mathcal{D} as a function on $[\hat{G}'\sigma_p/\hat{G}']$. Let $\tau \in Z(\hat{G})^\circ$ be the central character of V_μ . Restriction of $\lambda = \tau_{\mathbf{b}_i} + (\sigma - 1)\nu_{\mathbf{b}_i}$ to $Z(\hat{G})^\circ$ gives $\tau = \tau_{\mathbf{b}_i} + \sigma(\eta_i) - \eta_i$, where $\eta_i = \nu_{\mathbf{b}_i}|_{Z(\hat{G})^\circ}$, which can be regarded as a character of \hat{G}' . It follows that for G' , we can use a single τ for all $\mathbf{b} \in \mathbb{M}_{\mu^*}^{\text{Tate}}$ in Lemma 4.4.3. In addition, over $\hat{G}'\sigma_p/\hat{G}'$, we have the isomorphism

$$\widetilde{V_\tau} \simeq \widetilde{V_{\tau_{\mathbf{b}_i}}}, \quad f \mapsto fe^{\eta_i}.$$

Denote this map by \mathbf{c}_i . The collection $\{\mathbf{a}_{i,\text{in}} \circ \mathbf{c}_i : \widetilde{V_\tau} \rightarrow \widetilde{V_\mu}\}$ form a basis of $\text{Hom}_{[\hat{G}'\sigma_p/\hat{G}']}(\widetilde{V_\tau}, \widetilde{V_\mu})$, and a similarly dual statement holds. Therefore, the determinant of the intersection matrix (7.4.5) is identified with the determinant of the map

$$\text{Hom}_{[\hat{G}'\sigma_p/\hat{G}']}(\widetilde{V_\tau}, \widetilde{V_\mu}) \otimes_{\mathbf{J}_{\hat{G}'}} \text{Hom}_{[\hat{G}'\sigma_p/\hat{G}']}(\widetilde{V_\mu}, \widetilde{V_\tau}) \rightarrow \mathbf{J}_{\hat{G}'},$$

which can be rewritten as the pairing of $\mathbf{J}_{\hat{G}'}$ -modules

$$(7.4.6) \quad \mathbf{J}_{\hat{G}'}(V_\mu \otimes V_\tau^*) \otimes_{\mathbf{J}_{\hat{G}'}} \mathbf{J}_{\hat{G}'}(V_\tau \otimes V_\mu^*) \rightarrow \mathbf{J}_{\hat{G}'}.$$

Finally, we (re)state and prove the main theorem of the paper.

Theorem 7.4.6. *Assume that (G, X) is of Hodge type and the center Z_G is connected. Let $K \subset G(\mathbb{A}_f)$ be a (small enough) open compact subgroup. Let $p > 2$ be a prime, such that K_p is hyperspecial and $V_{\mu^*}^{\text{Tate}_p} \neq 0$. Then:*

- (1) *There is a unique inner form G' of G , that is trivial at all finite places, and is compact modulo center at the infinite place.*
- (2) *$\text{Sh}_{\mu,b}$ is pure of dimension $\frac{d}{2}$. In particular, d is always an even integer. There is an \mathcal{H}_K -equivariant isomorphism*

$$\text{H}_d^{\text{BM}}(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \cong C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell) \otimes_{\mathbf{J}} V_{\mu^*}^{\text{Tate}_p}.$$

Here via the isomorphism $G'(\mathbb{A}) \simeq G(\mathbb{A})$, we regard K as an open compact subgroup of $G'(\mathbb{A}_f)$, and we use $C(G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K, \overline{\mathbb{Q}}_\ell)$ to denote the space of $\overline{\mathbb{Q}}_\ell$ -valued functions on the finite set $G'(\mathbb{Q}) \backslash G'(\mathbb{A}_f)/K$.

- (3) *Let π_f be an irreducible module of \mathcal{H}_K , and let*

$$\text{H}_d^{\text{BM}}(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v})[\pi_f] = \text{Hom}_{\mathcal{H}_K}(\pi_f, \text{H}_d^{\text{BM}}(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \otimes \overline{\mathbb{Q}}_\ell) \otimes \pi_f$$

be the π_f -isotypical component. Then the cycle class map

$$\text{cl} : \text{H}_d^{\text{BM}}(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v}) \otimes \overline{\mathbb{Q}}_\ell \rightarrow \text{H}_{\text{et},c}^d(\text{Sh}_{\mu,\overline{\mathbb{F}}_v}, \overline{\mathbb{Q}}_\ell(d/2))$$

restricted to $\text{H}_d^{\text{BM}}(\text{Sh}_{\mu,b,\overline{\mathbb{F}}_v})[\pi_f]$ is injective if the Satake parameter of $\pi_{f,p}$ (the component of π_f at p) is general with respect to V_μ .

- (4) *Assume that $\mathbf{Sh}_K(G, X)$ is a Kottwitz arithmetic variety (i.e. those considered in [Ko92b]), or assume that G_{der} is simply-connected and anisotropic, and there is a place $p \neq p'$ such that $\pi_{f,p'}$ is an unramified twist of Steinberg representation. Then the π_f^p -isotypical component of the cycle class map cl is surjective to*

$$\sum_{\pi_p} T^d(\pi_p \pi_f^p, \overline{\mathbb{Q}}_\ell) \otimes \pi_p \pi_f^p$$

if the Satake parameters of $\{\pi_p\}$ are all strongly general with respect to V_μ . In particular, the Tate conjecture holds for these π_f^p .

Proof. Part (1) follows from the combination of Corollary 2.1.6 and Proposition 4.2.10. Part (2) follows from Theorem 7.2.16 and Theorem 4.4.14. Since we can write the intersection matrix of the cycle classes as (7.4.6), Part (3) follows Theorem 1.4.1. Finally, Part (4) follows from Part (3), Theorem 2.2.1 in the first situation and Corollary 2.3.2 in the second situation, and the following lemma.

Lemma 7.4.7. *Let $\gamma\phi_p \in \hat{T} \rtimes \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p) \subset \hat{G} \rtimes \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ be an element whose image in $\widehat{Z}_G^\circ \rtimes \text{Gal}(\mathbb{F}_{p^m}/\mathbb{F}_p)$ has finite image. Then*

$$V_{\mu^*}^{\text{Tate}_p} \subseteq \bigcup_{j \geq 1} (V_{\mu^*})^{r_{\mu^*}((\gamma\phi_p)^{j[v:p]})},$$

and if γ is strongly general in the sense of Definition 1.4.2, the above inclusion is an isomorphism.

Proof. We can rewrite V^{Tate_p} alternatively as

$$V^{\text{Tate}_p} = \bigoplus_{\lambda \in \mathbb{X}^\bullet(\widehat{Z}_G^\circ)^\sigma} V|_{\widehat{G}^\sigma}(\lambda).$$

For $\lambda \in \mathbb{X}^\bullet(\widehat{Z}_G^\circ)^\sigma$, $r_{\mu^*}((\gamma\phi_p)^{jm})$ acts trivially on $V|_{\widehat{G}^\sigma}(\lambda)$ because $\gamma\phi_p$ has finite image in \widehat{Z}_G° . This implies the inclusion (1.1.2). On the other hand, if $\lambda \in \mathbb{X}^\bullet(\widehat{T}^\sigma)$ and it does not factor through \widehat{Z}_G° , $V|_{\widehat{G}^\sigma}(\lambda)$ has no invariants under $r_{\mu^*}((\gamma\phi_p)^{jm})$ for any $j > 1$ when $\gamma\sigma$ is strongly general with respect to V . In this case, the inclusion (1.1.2) is an equality. \square

\square

APPENDIX A. PRELIMINARIES FROM ALGEBRAIC GEOMETRY.

A.1. (Pre)stacks in \mathbf{Aff}^{pf} . Let k be a field. Recall that in the usual formalism of algebraic geometry, one considers the category of k -algebras \mathbf{Aff}_k as the test objects. The category of \mathbf{Aff}_k can be endowed with various topology (Zariski, étale, fpqc, etc.). Then one can define the category of schemes (resp. algebraic spaces, resp. algebraic stacks) as certain full subcategory of the category of sheaves (resp. stacks) on \mathbf{Aff}_k . We can mimic this approach to do algebraic geometry over perfect rings.

Let k be a perfect field of characteristic $p > 0$. We denote by $\mathbf{Aff}_k^{\text{pf}}$ the category of perfect k -algebras.

Definition A.1.1. A *presheaf* is a covariant functor from $\mathbf{Aff}_k^{\text{pf}}$ to the category of sets. A *prestack* is a covariant 2-functor from $\mathbf{Aff}_k^{\text{pf}}$ to the 2-category of groupoids. By regarding a set as a discrete groupoid, we regard a presheaf as a prestack.

We remark that we considered (pre)stacks in the paper, like $\text{Hk}_\mu^{\text{loc}}$ and $\text{Sht}_\mu^{\text{loc}}$. But we only used ℓ -adic formalism for algebraic stacks, which we define in the sequel.

We can equip $\mathbf{Aff}_k^{\text{pf}}$ with Zariski, étale or fpqc topology.³⁶ Then we have the notion of sheaves (resp. stacks) in $\mathbf{Aff}_k^{\text{pf}}$. A sheaf in fpqc topology is sometimes also called a space. Note that all the mentioned topology are subcanonical.

In [BS15], Bhatt-Scholze also defined the v -topology on $\mathbf{Aff}_k^{\text{pf}}$, which is finer than any of the above mentioned topology but is still subcanonical. We will not make use of this topology in the paper.

³⁶Note that however we avoid fppf topology since perfect rings over k are almost never of finite type.

Definition A.1.2. An affine scheme $\mathrm{Spec} R$ in $\mathbf{Aff}_k^{\mathrm{pf}}$ is a presheaf on $\mathbf{Aff}_k^{\mathrm{pf}}$ of the form $\mathrm{Hom}_{\mathbf{Aff}_k^{\mathrm{pf}}}(R, -)$. A scheme in $\mathbf{Aff}_k^{\mathrm{pf}}$ is Zariski sheaf that admits a Zariski cover by affine schemes.

As mentioned above, affine schemes are Zariski sheaves and therefore are schemes. The category of schemes in $\mathbf{Aff}_k^{\mathrm{pf}}$ admits finite products. One can define the usual Zariski, étale, fpqc topology on a scheme. In addition, it makes sense to define schematic morphisms of presheaves in $\mathbf{Aff}_k^{\mathrm{pf}}$.

Definition A.1.3. An *algebraic space* X in $\mathbf{Aff}_k^{\mathrm{pf}}$ is an étale sheaf such that the diagonal is schematic, and there is an étale surjection from a scheme $U \rightarrow X$.

The product of two algebraic spaces in $\mathbf{Aff}_k^{\mathrm{pf}}$ exists as an algebraic space in $\mathbf{Aff}_k^{\mathrm{pf}}$. As usual, it makes sense to define representable morphisms (by algebraic spaces) of presheaves in $\mathbf{Aff}_k^{\mathrm{pf}}$. For a representable morphism of presheaves, the usual topological notions for algebraic spaces make sense, such as quasi-compact, quasi-separated, separated, connected, irreducible, etc.

Remark A.1.4. The category of schemes in $\mathbf{Aff}_k^{\mathrm{pf}}$ is a full subcategory of the category of algebraic spaces in $\mathbf{Aff}_k^{\mathrm{pf}}$. The latter is a full subcategory of the category of fpqc sheaves on $\mathbf{Aff}_k^{\mathrm{pf}}$ ([Stack, Tag03W8]). In fact, algebraic spaces in $\mathbf{Aff}_k^{\mathrm{pf}}$ are even v -sheaves in the sense of [BS15].

Definition A.1.5. Let \mathcal{F} be a presheaf on \mathbf{Aff}_k . Its *perfection*, denoted by $\mathcal{F}^{\mathrm{pf}}$, is its restriction to $\mathbf{Aff}_k^{\mathrm{pf}} \subset \mathbf{Aff}_k$.

It is easy to see that the perfection of a scheme (resp. algebraic space) in \mathbf{Aff}_k is represented by a scheme (resp. algebraic space) in $\mathbf{Aff}_k^{\mathrm{pf}}$.

On the other hand, recall in usual algebraic geometry, there is another definition of perfection of a scheme (resp. algebraic space) in \mathbf{Aff}_k : a scheme X (resp. an algebraic space) in \mathbf{Aff}_k is called perfect if its Frobenius endomorphism σ_X is an isomorphism. Given a scheme (resp. algebraic space) X , the perfection of X is defined as

$$X^{\mathrm{pf}} := \varprojlim_{\sigma_X} X.$$

The functor $X \mapsto X^{\mathrm{pf}}$ is right adjoint to the forgetful functor from the category of perfect schemes (resp. algebraic spaces) to the category of all schemes (resp. algebraic spaces). The following lemma is essentially [Zhu17, Lemma A.10], which explains that the two notions of perfections are essentially equivalent.

Lemma A.1.6. *The functor from the category of perfect schemes (resp. perfect algebraic spaces) in \mathbf{Aff}_k to the category of schemes (resp. algebraic spaces) in $\mathbf{Aff}_k^{\mathrm{pf}}$ given by restriction is an equivalence.*

From now on, by abuse of language, we also call a scheme (resp. algebraic spaces) in $\mathbf{Aff}_k^{\mathrm{pf}}$ a perfect scheme (resp. perfect algebraic space).

Definition A.1.7. A perfect algebraic space X is *perfectly locally of finite type* if there exists an étale affine cover $\{U_i\}$ of X such that each U_i is the perfection of an affine scheme of finite type over k . It is called *perfectly of finite type* if it is perfectly locally of finite type and quasi-compact. It is called *perfectly of finite presentation* (pfp for short) if it is perfectly of finite type and quasi-separated (i.e. diagonal is quasi-compact).

A morphism $f : X \rightarrow Y$ between pfp perfect algebraic spaces is called *perfectly proper* if it is separated and universally closed.

Recall the following results ([Zhu17, §A]).

- Proposition A.1.8.** (1) *A perfect algebraic space is pfp if and only if it is the perfection of an algebraic space of finite presentation over k .*
- (2) *Any morphism $f : X \rightarrow Y$ between two pfp perfect algebraic spaces is the perfection of a morphism $f' : X' \rightarrow Y'$ between two algebraic spaces (in \mathbf{Aff}_k) of finite presentation.*
- (3) *A perfectly proper morphism between pfp perfect algebraic spaces is the perfection of a proper morphism between algebraic spaces of finite presentation.*

For a pfp algebraic space X , an algebraic space X' of finite presentation such that $X = X'^{\text{pf}}$ is called a *deperfection* of X .

Definition A.1.9. Let $f : X \rightarrow Y$ be a map between two algebraic spaces. We say that f is *perfectly smooth* at $x \in X$ if there exists an étale atlas $U \rightarrow X$ at x and an étale atlas $V \rightarrow Y$ at $f(x)$, such that the map $U \rightarrow Y$ factors as $U \xrightarrow{h} V \rightarrow Y$ and h factors as $h = \text{pr} \circ h'$, where $h' : U \rightarrow V \times (\mathbb{A}^n)^{\text{pf}}$ is étale and $\text{pr} : V \times (\mathbb{A}^n)^{\text{pf}} \rightarrow V$ is the projection. We say that f is *perfectly smooth* if it is perfectly smooth at every point of X . We say that X is perfectly smooth (at x) if $X \rightarrow \text{Spec } k$ is perfectly smooth (at x).

Now we define algebraic stacks. First, it makes sense to define the notion of prestacks in $\mathbf{Aff}_k^{\text{pf}}$, and stacks with respect to étale or fpqc topology of $\mathbf{Aff}_k^{\text{pf}}$.

Definition A.1.10. An *algebraic stack* X in $\mathbf{Aff}_k^{\text{pf}}$ is a stack with respect to the fpqc topology, such that (i) the diagonal is represented by an algebraic space in $\mathbf{Aff}_k^{\text{pf}}$; (ii) there exists a perfectly smooth surjective map $U \rightarrow X$ from an algebraic space. Such a U is called a *smooth atlas*.

Definition A.1.11. An algebraic stack X in $\mathbf{Aff}_k^{\text{pf}}$ is called *quasi-compact* if there exists a smooth atlas $U \rightarrow X$ with U quasi-compact.

An algebraic stack X in $\mathbf{Aff}_k^{\text{pf}}$ is called *locally of finite type* (resp. *perfectly of finite type*, resp. *pfp*) if there exists one smooth atlas $U \rightarrow X$, where U is perfectly locally of finite type (resp. perfectly of finite type, resp. pfp).

A morphism of algebraic stack $f : X \rightarrow Y$ in $\mathbf{Aff}_k^{\text{pf}}$ is called

- *separated* if the diagonal map $X \rightarrow X \times_Y X$ (which is always representable by an algebraic space) is perfectly proper;
- *quasi-separated* if the diagonal is quasi-compact and quasi-separated;
- *DM* if the diagonal is étale.

Definition A.1.12. A morphism $f : X \rightarrow Y$ of pfp algebraic stacks in $\mathbf{Aff}_k^{\text{pf}}$ is called *perfectly proper* if f is separated, perfectly of finite type, and universally closed.

Definition A.1.13. A morphism $f : X \rightarrow Y$ of pfp algebraic stacks in $\mathbf{Aff}_k^{\text{pf}}$ is called *perfectly smooth* if there is a perfectly smooth surjective morphism $U \rightarrow X$ from an algebraic space U such that the composition $U \rightarrow X \rightarrow Y$ is perfectly smooth.

Example A.1.14. Let G be a perfect algebraic group (i.e. the perfection of an algebraic group) over k . Then $\mathbf{BG} \rightarrow \text{Spec } k$ is perfectly smooth.

As before, for a prestack \mathcal{F} in \mathbf{Aff}_k , its restriction to $\mathbf{Aff}_k^{\text{pf}} \subset \mathbf{Aff}_k$ is called the *perfection* of \mathcal{F} . It is clear that the perfection of algebraic stacks in \mathbf{Aff}_k are algebraic stacks in $\mathbf{Aff}_k^{\text{pf}}$. Note however, for a usual algebraic stack X in \mathbf{Aff}_k , the Frobenius endomorphism σ_X may not be representable and therefore the inverse limit $\varprojlim_{\sigma_X} X$ may not exist as an algebraic stack in \mathbf{Aff}_k .

A.1.15. *The ℓ -adic formalism.* Let $\ell \neq p$. As explained in [Zhu17, §A.3], for a pfp algebraic space X in $\mathbf{Aff}_k^{\text{pf}}$, the ℓ -adic derived category $D(X, \overline{\mathbb{Q}}_\ell)$ is well-defined, and there exists the six operation

formalism. Thanks to Proposition A.1.8, no additional effort is needed to construct this theory. In addition, the proper base change isomorphism (for perfectly proper morphisms) and the smooth base change isomorphism (for perfectly smooth morphisms) hold. In particular, the category of perverse sheaves $P(X)$ is well-defined, and is the heart of a t-structure of $D(X, \overline{\mathbb{Q}}_\ell)$.

We can endow the category of pfp algebraic spaces with smooth topology (the covering maps being perfectly smooth surjective maps). Then as usual, perverse sheaves form a stack of abelian categories with respect to this topology. Therefore, given a pfp algebraic stack X in $\mathbf{Aff}_k^{\text{pf}}$, the category of perverse sheaves $P(X)$ on X is well-defined.

Note that as explained in [LZ12⁺, LZ14⁺], the six operation formalism and the base change isomorphism (and their enhancements to the ∞ -categorical level) can be expressed as certain functor from a simplicial set (built out of separated perfectly of finite type schemes over k) to the category of presentable ∞ -categories sending X to $D(X, \overline{\mathbb{Q}}_\ell)$.

Now the same techniques as developed in [LO08a, LO08b, LZ12⁺, LZ14⁺] allows one to define the ℓ -adic derived category $D(X, \overline{\mathbb{Q}}_\ell)$ for pfp algebraic stacks in $\mathbf{Aff}_k^{\text{pf}}$. For a smooth atlas $U \rightarrow X$ with U a pfp algebraic space, let $U^{\bullet/X} \rightarrow X$ denote its Čech nerve. Then $D(X, \overline{\mathbb{Q}}_\ell)$ is the limit (as ∞ -categories) of $D(U^{\bullet/X}, \overline{\mathbb{Q}}_\ell)$. As usual, $P(X)$ forms the heart of a t-structure of $D(X, \overline{\mathbb{Q}}_\ell)$. In fact, we just need quotient stack in the paper. If $X = [Y/G]$, then we can define $D(X, \overline{\mathbb{Q}}_\ell)$ as the equivariant derived category $D_G(Y, \overline{\mathbb{Q}}_\ell)$ and $P(X)$ as $P_G(Y)$ (up to a shift).

Notation A.1.16. For a perfect pfp algebraic space X over k , we denote by

$$H_i^{\text{BM}}(X_{\bar{k}}) := H^{-i}(X_{\bar{k}}, \omega_X(-i/2))$$

the i th *Borel-Moore homology* of $X_{\bar{k}}$. Note that our definition is different from the standard one by a Tate twist. The reason is with this normalization, if $\dim X = d$, then $H_{2d}^{\text{BM}}(X_{\bar{k}})$ is the $\overline{\mathbb{Q}}_\ell$ -vector space with a basis given by its irreducible components of dimension d .

A.1.17. *The trace map.* Recall that as explained in [Zhu17, §A.3], if X is a pfp algebraic space of dimension d , then the fundamental class

$$[X] \in H_{2d}^{\text{BM}}(X_{\bar{k}}) \cong H_c^{2d}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d))^*$$

of X (which is the sum of the fundamental classes of the irreducible components of $X_{\bar{k}}$ of dimension d) defines the trace map $\text{Tr} : H_c^{2d}(X_{\bar{k}}, \overline{\mathbb{Q}}_\ell(d)) \rightarrow \overline{\mathbb{Q}}_\ell$. As in [LZ12⁺, LZ14⁺], this trace map extends to pfp algebraic stacks. In addition, if $f : X \rightarrow Y$ a equidimensional perfectly smooth morphism of pfp algebraic stacks, of relative dimension d , then the fiberwise trace maps can be organized into a family, i.e. there is the trace map

$$\text{Tr}_f : R^{2d}f_!\overline{\mathbb{Q}}_\ell(d) \rightarrow \overline{\mathbb{Q}}_\ell,$$

which induces the fiberwise trace map by restriction. By the projection formula, it induces a natural map $f^*[2d](d) \rightarrow f^!$ which is an isomorphism. In practice, it is convenient to introduce the following condition on morphisms, which is weaker than perfectly smooth but share the same cohomological properties.

Definition A.1.18. An equidimensional morphism $f : X \rightarrow Y$ of pfp algebraic stacks in $\mathbf{Aff}_k^{\text{pf}}$, of relative dimension d , is called *cohomologically smooth* if there is a trace map

$$\text{Tr}_f : R^{2d}f_!\overline{\mathbb{Q}}_\ell(d) \rightarrow \overline{\mathbb{Q}}_\ell,$$

which restricts to the usual trace map along each geometric fiber $f_y : X_{\bar{y}} \rightarrow \bar{y}$ for $y \in Y$, and which induces an isomorphism $f_y^*[2d](d) \cong f_y^!$.

Note that trace map, if exists, is unique. In addition, cohomologically smooth morphisms are stable under base change. Part (1) of the following statement justifies the terminology.

Proposition A.1.19. (1) Let $f : X \rightarrow Y$ be cohomologically smooth. Then the trace map induces $f^*[2d](d) \rightarrow f^!$, which is an isomorphism. (2) The composition of cohomologically smooth maps are cohomologically smooth.

Proof. Part (1) is clear and Part (2) follows from Part (1) directly. \square

Here is an easy but useful criterion for a morphism to be cohomologically smooth.

Lemma A.1.20. (1) Let $f : X \rightarrow Y$ be a morphism of perfectly smooth pfp schemes, and assume that each fiber of f is perfectly smooth, equidimensional of dimension d . Then f is cohomologically smooth.

(2) Let $f : X \rightarrow Y$ be a morphism of pfp schemes. Assume that there is a perfectly proper morphism $g : \tilde{Y} \rightarrow Y$ with connected geometric fibers such that the base change $\tilde{X} := X \times_Y \tilde{Y} \rightarrow \tilde{Y}$ is cohomologically smooth, then f is cohomologically smooth.

Note that Part (2) of the lemma essentially says that cohomological smoothness can be checked locally in the v -topology introduced in [BS15].

Proof. (1) Since X and Y are perfectly smooth,

$$f^! \overline{\mathbb{Q}}_\ell[2d](d) \cong f^! \omega_Y[2d - 2d_Y](d - d_Y) \cong \omega_X[-2d_X](-d_X) \cong \overline{\mathbb{Q}}_\ell,$$

where $d_X = \dim X$, $d_Y = \dim Y$, and ω_X and ω_Y are the dualizing sheaves. This isomorphism induces $f_! \overline{\mathbb{Q}}_\ell[2d](d) \rightarrow \overline{\mathbb{Q}}_\ell$ which in turn induces $\mathrm{Tr}_f : R^{2d} f_! \overline{\mathbb{Q}}_\ell(d) \rightarrow \overline{\mathbb{Q}}_\ell$ since $R^i f_! \overline{\mathbb{Q}}_\ell(d) = 0$ for $i > 2d$. It follows from the construction that fiberwise, Tr_f restricts to the usual trace map. Since fibers are perfectly smooth, $f_y^*[2d](d) \cong f_y^!$.

(2) Let us denote $\tilde{X} \rightarrow X$ by g_X and $\tilde{Y} \rightarrow Y$ by g_Y . Let $\tilde{f} : \tilde{X} \rightarrow \tilde{Y}$ be the base change of f . Then we have

$$g_Y^* f_! \overline{\mathbb{Q}}_\ell[2d](d) \cong \tilde{f}_! \overline{\mathbb{Q}}_\ell[2d](d) \xrightarrow{\mathrm{Tr}_{\tilde{f}}} \overline{\mathbb{Q}}_\ell,$$

which induces $f_! \overline{\mathbb{Q}}_\ell[2d](d) \rightarrow (g_Y)_* \overline{\mathbb{Q}}_\ell$. Since $f_! \overline{\mathbb{Q}}_\ell[2d](d)$ lives in cohomological degree ≤ 0 and $(g_Y)_* \overline{\mathbb{Q}}_\ell$ in cohomological degree ≥ 0 . This gives

$$\mathrm{Tr}_f : R^{2d} f_! \overline{\mathbb{Q}}_\ell(d) \rightarrow R^0 (g_Y)_* \overline{\mathbb{Q}}_\ell = \overline{\mathbb{Q}}_\ell$$

since the fibers of g_Y are geometrically connected. It is clear that Tr_f restricts to the fiberwise trace map. The isomorphism $f_y^*[2d](d) \cong f_y^!$ is clear. \square

We would like to have a base change homomorphism (but not necessarily an isomorphism) in non-Cartesian case.

Definition A.1.21. A commutative square of perfect pfp algebraic stacks

$$(A.1.1) \quad \begin{array}{ccc} D & \xrightarrow{p} & C \\ b \downarrow & & \downarrow a \\ Y & \xrightarrow{f} & X \end{array}$$

is called *base changeable* if the induced morphism $h : D \rightarrow C \times_X Y$ is representable by perfectly proper algebraic spaces.

The following lemma is straightforward.

Lemma A.1.22. (1) A commutative diagram as (A.1.1) is base changeable if

- either p is representable by perfectly proper algebraic spaces and f is separated;
- or b is representable by perfectly proper algebraic spaces and a is separated.

(2) In the commutative diagram

$$(A.1.2) \quad \begin{array}{ccccc} E & \xrightarrow{q} & D & \xrightarrow{p} & C \\ c \downarrow & & b \downarrow & & \downarrow a \\ Z & \xrightarrow{g} & Y & \xrightarrow{f} & X \end{array}$$

if both inner squares are base changeable, so is the outer rectangle.

This name is justified by the following fact.

Lemma A.1.23. Assume we have a base changeable commutative diagram of perfect pfp algebraic stacks as (A.1.1). Then there is a natural base change homomorphism

$$(A.1.3) \quad \mathrm{BC}_!^* : a^* f_! \rightarrow p_! b^*,$$

which is an isomorphism if (A.1.1) is Cartesian. If p and f are separated, it fits into the commutative diagram

$$\begin{array}{ccc} a^* f_! & \longrightarrow & p_! b^* \\ \downarrow & & \downarrow \\ a^* f_* & \longrightarrow & p_* b^*, \end{array}$$

where the bottom arrow is the natural adjunction.

In addition, given (A.1.2) with both squares base changeable, then the base change homomorphism $a^*(fg)_! \rightarrow (pq)_! c^*$ is equal to the composition

$$a^*(fg)_! = a^* f_! g_! \xrightarrow{\mathrm{BC}_!^*} f_! b^* g_! \xrightarrow{\mathrm{BC}_!^*} f_! g_! c^* = (fg)_! c^*.$$

Similarly, $(fg)^* a_! \rightarrow c_!(pq)^*$ is equal to the composition

$$(fg)^* a_! = g^* f^* a_! \xrightarrow{\mathrm{BC}_!^*} g^* b_! f^* \xrightarrow{\mathrm{BC}_!^*} c_! g^* f^* = c_!(fg)^*.$$

Proof. This follows from the previously mentioned properties of the base change isomorphism, together with the natural adjunction $\mathrm{id} \rightarrow h_* h^* \cong h_! h^*$ where $h : D \rightarrow C \times_X Y$. \square

Remark A.1.24. Given a diagram (A.1.1), base changeable or not, there is always a homomorphism

$$(A.1.4) \quad p_! b^! \rightarrow a^! f_!$$

by adjunction. In addition, given (A.1.2), the map $(pq)_! c^! \rightarrow a^!(fg)_!$ is equal to

$$(pq)_! c^! = p_! q_! c^! \rightarrow p_! b^! g_! \rightarrow a^! f_! g_! = a^!(fg)_!,$$

and the map $c_!(pq)^! \rightarrow (fg)^! a_!$ is equal to

$$c_!(pq)^! = c_! q^! p^! \rightarrow g^! b_! p^! \rightarrow g^! f^! a_! = (fg)^! a_!.$$

Now assume that (A.1.1) is Cartesian. Then the base change isomorphism $\mathrm{BC}_!^*$ also induces (by adjunction)

$$(A.1.5) \quad \mathrm{BC}^*! : b^* f^! \rightarrow p^! a^*.$$

If f is equidimensionally cohomologically smooth of relative dimension d , since the trace map is compatible with the base change, (A.1.5) coincides with

$$(A.1.6) \quad b^* f^! \cong b^* f^*[2d](d) = p^* a^*[2d](d) \cong p^! a^*$$

A.2. Review of cohomological correspondence. We recall the formalism of cohomological correspondences, partially following [Va07]. We will assume that k is a perfect field of characteristic $p > 0$. Below, stacks are algebraic stacks, perfectly of finite presentation in $\mathbf{Aff}_k^{\text{pf}}$.³⁷ We write pt for $\text{Spec } k$. We fix $\ell \neq p$, and fix a half Tate twist $\overline{\mathbb{Q}}_\ell(1/2)$. As usual, we write $\langle d \rangle$ for $[d](d/2)$.

Notation A.2.1. For an equidimensional cohomologically smooth morphism $f : X \rightarrow Y$ between algebraic stacks of relative dimension d , we write $f^* := f^*\langle d \rangle$; it takes perverse sheaves to perverse sheaves.

If X is a stack, let $D_c^*(X) \subset D(X, \overline{\mathbb{Q}}_\ell)$ ($*$ = +, −, b , \emptyset) denote its constructible derived categories of ℓ -adic sheaves. Let $\omega_X \in D_c^b(X)$ denote the dualizing sheaf on X . For $\mathcal{F} \in D_c^b(X)$, let $\mathbb{D}\mathcal{F} = R\text{Hom}(\mathcal{F}, \omega_X)$ denote the Verdier dual. By definition, there are canonical (unit and counit) maps

$$(A.2.1) \quad e : \mathcal{F} \otimes \mathbb{D}\mathcal{F} \rightarrow \omega_X, \quad \delta : \overline{\mathbb{Q}}_\ell \rightarrow \mathcal{F} \otimes^! \mathbb{D}\mathcal{F} := \Delta^!(\mathcal{F} \boxtimes \mathbb{D}\mathcal{F})$$

Definition A.2.2. Let (X_i, \mathcal{F}_i) for $i = 1, 2$ be two pairs, where X_i are stacks, and $\mathcal{F}_i \in D(X_i, \overline{\mathbb{Q}}_\ell)$. A *cohomological correspondence* $(C, u) : (X_1, \mathcal{F}_1) \rightarrow (X_2, \mathcal{F}_2)$ is a stack $C \xrightarrow{c_1 \times c_2} X_1 \times X_2$, together with a morphism $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$. We call C the *support* of (C, u) . For simplicity, (C, u) is sometimes denoted by C or by u . There is an obvious notion of (iso)morphisms between two cohomological correspondences (C, u) and (C', u') from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) . The set of isomorphism classes of cohomological correspondences from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) supported on C is denoted by $\text{Corr}_C((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2))$ and by definition

$$\text{Corr}_C((X_1, \mathcal{F}_1), (X_2, \mathcal{F}_2)) \cong \text{Hom}_C(c_1^* \mathcal{F}_1, c_2^! \mathcal{F}_2).$$

When c_1 is proper, a cohomological correspondence u will naturally induce a homomorphism $H_c(u) : H_c^*(X_1 \otimes \bar{k}, \mathcal{F}_1) \rightarrow H_c^*(X_2 \otimes \bar{k}, \mathcal{F}_2)$, given as the composition of the following maps

$$H_c^*(X_1 \otimes \bar{k}, \mathcal{F}_1) \rightarrow H_c^*(C \otimes \bar{k}, c_1^* \mathcal{F}_1) \xrightarrow{H_c^*(u)} H_c^*(C \otimes \bar{k}, c_2^! \mathcal{F}_2) \rightarrow H_c^*(X_2 \otimes \bar{k}, \mathcal{F}_2).$$

In fact, this is a special case of the push-forward cohomological correspondences to be reviewed below.

If $(C, u) : (X_1, \mathcal{F}_1) \rightarrow (X_2, \mathcal{F}_2)$ and $(D, v) : (X_2, \mathcal{F}_2) \rightarrow (X_3, \mathcal{F}_3)$ are two cohomological correspondences, we define their composition to be $(C \times_{X_2} D, v \circ u) : (X_1, \mathcal{F}_1) \rightarrow (X_3, \mathcal{F}_3)$, where $v \circ u$ is the following composition

$$p^* c_1^* \mathcal{F}_1 \xrightarrow{p^* u} p^* c_2^! \mathcal{F}_2 \xrightarrow{\text{BC}^{*!}} q^! d_1^* \mathcal{F}_2 \xrightarrow{q^! v} q^! d_2^! \mathcal{F}_3,$$

$\text{BC}^{*!}$ is the base change isomorphism as in (A.1.5), and p, q are the projections from $C \times_{X_2} D$ to C and to D respectively. For the induced homomorphism on the cohomology groups, we have $H_c(v \circ u) = H_c(v) \circ H_c(u)$, as maps $H_c^*(X_1 \otimes \bar{k}, \mathcal{F}_1) \rightarrow H_c^*(X_3 \otimes \bar{k}, \mathcal{F}_3)$, see Lemma A.2.10 below.

Here are examples we use in the paper.

Example A.2.3. (1) For a morphism $f : X \rightarrow Y$ and $\mathcal{F} \in D(X, \overline{\mathbb{Q}}_\ell)$, there is a cohomological correspondence $(X \xleftarrow{\text{id}} X \xrightarrow{f} Y, u : \mathcal{F} \rightarrow f^! f_! \mathcal{F})$ from (X, \mathcal{F}) to $(Y, f_! \mathcal{F})$, called the *pushforward correspondence*, and denoted by $(\Gamma_f)!$ for simplicity.
(2) For a morphism $f : X \rightarrow Y$ and $\mathcal{F} \in D(Y, \overline{\mathbb{Q}}_\ell)$, there is a natural cohomological correspondence $(Y \xleftarrow{f} X \xrightarrow{\text{id}} X, u := \text{id} : f^* \mathcal{F} \rightarrow f^* \mathcal{F})$ from (Y, \mathcal{F}) to $(X, f^* \mathcal{F})$, called the *pullback correspondence*, and denoted by Γ_f^* for simplicity.
(3) If $(C, u) : (X_1, \mathcal{F}_1) \rightarrow (X_2, \mathcal{F}_2)$ is a cohomological correspondence, with $\mathcal{F}_i \in D_c^b(X_i)$, then $(C, \mathbb{D}u) : (X_2, \mathbb{D}\mathcal{F}_2) \rightarrow (X_1, \mathbb{D}\mathcal{F}_1)$ is a cohomological correspondence, called the *dual correspondence*.

³⁷Of course, the same discussions will carry through when we work with algebraic stacks in \mathbf{Aff}_k .

- (4) Let (X, \mathcal{F}) be a pair with $\mathcal{F} \in D_c^b(X)$. Then (A.2.1) gives a cohomological correspondence and its dual

$$e_{\mathcal{F}} \in \text{Corr}_X((X \times X, \mathcal{F} \boxtimes \mathbb{D}\mathcal{F}), (\text{pt}, \overline{\mathbb{Q}}_\ell)), \quad \delta_{\mathcal{F}} \in \text{Corr}_X((\text{pt}, \overline{\mathbb{Q}}_\ell), (X \times X, \mathcal{F} \boxtimes \mathbb{D}\mathcal{F})).$$

- (5) Assume that $k = \bar{k}$. Let X_1, X_2 be perfectly smooth algebraic spaces, of pure dimension, and let $X_1 \leftarrow C \rightarrow X_2$ be a correspondence. Then

$$(A.2.2) \quad \begin{aligned} \text{Corr}_C((X_1, \overline{\mathbb{Q}}_\ell\langle d_1 \rangle), (X_2, \overline{\mathbb{Q}}_\ell\langle d_2 \rangle)) &= \text{Hom}_{D_c^b(C)}(\overline{\mathbb{Q}}_\ell\langle d_1 \rangle, \omega\langle d_2 - 2 \dim X_2 \rangle) \\ &= H_{2 \dim X_2 + d_1 - d_2}^{\text{BM}}(C). \end{aligned}$$

So if $2 \dim C = 2 \dim X_2 + d_1 - d_2$, $\text{Corr}_C((X_1, \overline{\mathbb{Q}}_\ell\langle d_1 \rangle), (X_2, \overline{\mathbb{Q}}_\ell\langle d_2 \rangle))$ can be regarded as space of functions on the set of irreducible components of C of maximal dimension. In particular, If X_1, X_2, C are finite sets, regarded as varieties over k , $\text{Corr}_C((X_1, \overline{\mathbb{Q}}_\ell), (X_2, \overline{\mathbb{Q}}_\ell))$ can be identified with the space of $\overline{\mathbb{Q}}_\ell$ -valued functions on C .

We will also make use of the following construction.

Lemma A.2.4. *Assume that $\mathcal{F}_i \in D_c^b(X_i)$. The set of cohomological correspondences from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) is canonically bijective to the set of cohomological correspondences from $(X_1 \times X_2, \mathcal{F}_1 \boxtimes \mathbb{D}\mathcal{F}_2)$ to $(\text{pt}, \overline{\mathbb{Q}}_\ell)$. We denote this bijection as $(C, u) \leftrightarrow (C, u)^\sharp$ or simply $u \leftrightarrow u^\sharp$, called the sharp correspondence. Moreover, $(\mathbb{D}u)^\sharp$ is the same as u^\sharp if we naturally identify $X_1 \times X_2$ with $X_2 \times X_1$.*

Proof. Explicitly, given (C, u) , u^\sharp can be constructed as

$$(A.2.3) \quad (c_1 \times c_2)^* \mathcal{F}_1 \boxtimes \mathbb{D}\mathcal{F}_2 \cong c_1^* \mathcal{F}_1 \otimes c_2^* \mathbb{D}\mathcal{F}_2 \xrightarrow{u \otimes \text{id}} c_2^! \mathcal{F}_2 \otimes \mathbb{D}c_2^! \mathcal{F}_2 \xrightarrow{(A.2.1)} \omega_C.$$

It is clear that taking the dual on u does not change the sharp correspondence (after swapping the two factors). \square

Example A.2.5. (1) Under the sharp correspondence, the identity cohomological correspondence id becomes $\text{id}^\sharp = e_{\mathcal{F}} \in \text{Corr}_X((X \times X, \mathcal{F} \boxtimes \mathbb{D}\mathcal{F}), (\text{pt}, \overline{\mathbb{Q}}_\ell))$ from Example A.2.3(4).

(2) In case when $X_1 = X_2 = \text{pt}$ and $\mathcal{F}_1 = \mathcal{F}_2 = \overline{\mathbb{Q}}_l$, u and u^\sharp both viewed as homomorphisms $\overline{\mathbb{Q}}_{l,C} \rightarrow \omega_C$ are the same.

A.2.6. Pushforward of cohomological correspondences. Now assume that we have the following commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 \\ f_1 \downarrow & & \downarrow f & & \downarrow f_2 \\ Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 \end{array}$$

and let $(C, u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2)$ be a cohomological correspondence from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) . Assume that the left commutative square is base changeable (Definition A.1.21), then we have

$$d_1^*(f_1)_! \mathcal{F}_1 \xrightarrow{\text{BC}_1^*} f_! c_1^* \mathcal{F}_1 \xrightarrow{f_! u} f_! c_2^! \mathcal{F}_2 \xrightarrow{(A.1.4)} d_2^!(f_2)_! \mathcal{F}_2.$$

By abuse of notation, we denote the above map still by $f_!(u)$. Then $(D, f_!(u))$ is a cohomological correspondence from $(Y_1, (f_1)_! \mathcal{F}_1)$ to $(Y_2, (f_2)_! \mathcal{F}_2)$. We call this the *pushforward of the cohomological correspondence*.

Remark A.2.7. If f_1 is separated and f is representable by perfectly proper algebraic spaces (so the left square is base changeable), then $f_!(u)$ factors as the composition of the natural map $(f_1)_! \mathcal{F}_1 \rightarrow (f_1)_* \mathcal{F}_1$ and

$$d_1^*(f_1)_* \mathcal{F}_1 \rightarrow f_* c_1^* \mathcal{F}_1 \rightarrow f_! c_2^! \mathcal{F}_2 \rightarrow d_2^!(f_2)_! \mathcal{F}_2.$$

The latter map is the pushforward of cohomological correspondences considered in [Gr77]. However, in most places of this paper, we are in the situation where d_1 is separated and c_1 is representable by perfectly proper algebraic spaces.

In particular, if $(Y_1 \xleftarrow{d_1} D \xrightarrow{d_2} Y_2)$ is $(S = S = S)$ for some base scheme S , and c_1 is proper (so the left square is base changeable), we get a homomorphism of ℓ -adic sheaves

$$f_!(u) : (f_1)_! \mathcal{F}_1 \rightarrow (f_2)_! \mathcal{F}_2.$$

More specifically, if $S = \text{Spec } k$ is algebraically closed, and c_1 is proper, we obtain the previously mentioned $H_c(u) : H_c^*(X_1, \mathcal{F}_1) \rightarrow H_c^*(X_2, \mathcal{F}_2)$.

The pushforward of cohomological correspondences behaves well under compositions.

Lemma A.2.8. *Let*

$$\begin{array}{ccccc} X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 \\ f_1 \downarrow & & \downarrow f & & \downarrow f_2 \\ Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 \\ g_1 \downarrow & & \downarrow g & & \downarrow g_2 \\ Z_1 & \xleftarrow{e_1} & E & \xrightarrow{e_2} & Z_2 \end{array}$$

be a commutative diagram, such that the left two squares are base changeable. Let $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$ be a cohomological correspondence from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) . Then the left outer rectangle is base changeable and $(E, (gf)_!(u))$ is equal to $(E, g_!(f_!(u)))$ as cohomological correspondences from $(Z_1, (g_1 f_1)_! \mathcal{F}_1 \cong (g_1)_!(f_1)_! \mathcal{F}_1)$ to $(Z_2, (g_2 f_2)_! \mathcal{F}_2 \cong (g_2)_!(f_2)_! \mathcal{F}_2)$.

Proof. Using Lemma A.1.23 and Remark A.1.24, this is straightforward. \square

Before proceeding, let us state is usual lemma, which will be used several times below.

Lemma A.2.9. *In a cubic commutative diagram as below,*

$$\begin{array}{ccccc} & & A' & \longrightarrow & B' \\ & \swarrow & \downarrow & \swarrow & \downarrow \\ A & \longrightarrow & B & & \\ \downarrow & & \downarrow & & \downarrow \\ & & C' & \longrightarrow & D' \\ \downarrow & \swarrow & \downarrow & \swarrow & \downarrow \\ C & \longrightarrow & D & & \end{array}$$

if the front, the back, and the bottom faces are Cartesian, so is the top face.

Proof. Go through the definitions. \square

The pushforward cohomological correspondence is compatible with composition of cohomological correspondences.

Lemma A.2.10. *Suppose we have a commutative diagram*

$$\begin{array}{ccccccc} X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 & \xleftarrow{c'_1} & C' & \xrightarrow{c'_2} & X_3 \\ f_1 \downarrow & & \downarrow f & & \downarrow f_2 & & \downarrow f' & & \downarrow f_3 \\ Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 & \xleftarrow{d'_1} & D' & \xrightarrow{d'_2} & Y_3 \end{array}$$

with the first and the third square base changeable, c_1, c'_1 proper and d_1, d'_1 separated, and let $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$ and $v : c'_1^* \mathcal{F}_2 \rightarrow c'_2^! \mathcal{F}_3$ be cohomological correspondences from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) and

from (X_2, \mathcal{F}_2) to (X_3, \mathcal{F}_3) , respectively. Set $\tilde{C} := C \times_{X_2} C'$ and $\tilde{D} := D \times_{Y_2} D'$ and let $\tilde{f} : \tilde{C} \rightarrow \tilde{D}$ denote the naturally induced morphism. Then the following statements hold.

(1) The right square in the diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{c_1} & C & \xleftarrow{p_C} & \tilde{C} \\ f_1 \downarrow & & \downarrow f & & \downarrow \tilde{f} \\ Y_1 & \xleftarrow{d_1} & D & \xleftarrow{p_D} & \tilde{D} \end{array}$$

is base changeable. Note that this ensures that $\tilde{f}_!(v \circ u)$ is well defined by Lemma A.1.22 (1).

(2) We have an equality of cohomological correspondences from $(Y_1, (f_1)_! \mathcal{F}_1)$ to $(Y_3, (f_3)_! \mathcal{F}_3)$ supported on \tilde{D} :

$$\tilde{f}_!(v \circ u) = f'_!(v) \circ f_!(u).$$

Proof. (1) Using Lemma A.2.9, one sees that $(C \times_D \tilde{D}) \simeq C \times_{X_2} (X_2 \times_{Y_2} D')$. Since $C' \rightarrow X_2 \times_{Y_2} D'$ is perfectly proper by definition, $\tilde{C} \rightarrow C \times_D \tilde{D}$ is also perfectly proper.

(2) Let $\tilde{c}_1 = c_1 p_C$ and $\tilde{d}_1 = d_1 p_D$. We write $q_C : \tilde{C} \rightarrow C'$, $\tilde{c}_2 = c_2 q_C$, $q_D : \tilde{D} \rightarrow D'$, and $\tilde{d}_2 = d_2 q_D$. Proving the lemma amounts to proving the commutativity of the following diagram.

$$\begin{array}{ccccccccccc} \tilde{d}_1^* f_{1,!} \mathcal{F}_1 & \xrightarrow{\text{BC}_!^*} & \tilde{f}_! \tilde{c}_1^* \mathcal{F}_1 & \xrightarrow{=} & \tilde{f}_! p_C^* c_1^* \mathcal{F}_1 & \xrightarrow{u} & \tilde{f}_! p_C^* c_2^* \mathcal{F}_2 & \longrightarrow & \tilde{f}_! q_C^! c_1'^* \mathcal{F}_2 & \xrightarrow{v} & \tilde{f}_! q_C^! c_2'^* \mathcal{F}_3 & \xrightarrow{=} & \tilde{f}_! \tilde{c}_2^! \mathcal{F}_3 & \longrightarrow & \tilde{d}_2^! f_{3,!} \mathcal{F}_3 \\ \parallel & & \nearrow \text{BC}_!^* & & \nearrow & & \nearrow & & \searrow & & \searrow & & \parallel \\ p_D^* d_1^* f_{1,!} \mathcal{F}_1 & \xrightarrow{\text{BC}_!^*} & p_D^* f_! c_1^* \mathcal{F}_1 & \xrightarrow{u} & p_D^* f_! c_2^! \mathcal{F}_2 & \longrightarrow & p_D^* d_2^! f_{2,!} \mathcal{F}_2 & \longrightarrow & q_D^! d_1'^* f_{2,!} \mathcal{F}_2 & \longrightarrow & q_D^! f_! c_1'^* \mathcal{F}_2 & \xrightarrow{v} & q_D^! f_! c_2'^! \mathcal{F}_3 & \longrightarrow & q_D^! d_2'^! f_{3,!} \mathcal{F}_3 \end{array}$$

The commutativity of the left pentagon follows from Lemma A.1.23, and the commutativity of the right pentagon follows from Remark A.1.24. We remain to explain the commutativity of the middle hexagon. First notice that in the following cubic commutative diagram, the top and the bottom faces are Cartesian, and the left and the right faces are base changeable.

$$\begin{array}{ccccc} & & \tilde{C} & \xrightarrow{q_C} & C' \\ & \swarrow p_C & \downarrow & \swarrow & \downarrow f' \\ C & \xrightarrow{\quad} & X_2 & & D' \\ \downarrow f & & \downarrow \tilde{D} & \xrightarrow{\quad} & \downarrow d_1' \\ D & \xrightarrow{d_2} & Y_2 & & \end{array}$$

Applying Lemma A.1.23 to the composition $\tilde{C} \rightarrow D'$ and $C \rightarrow Y_2$, we obtain

$$\begin{array}{ccccc} & & q_{D,!} \tilde{f}_! p_C^* & \xrightarrow{=} & f_! q_C^! p_C^* \\ & \nearrow \text{BC}_!^* & & & \nearrow \text{BC}_!^* \\ q_{D,!} p_D^* f_! & \xrightarrow{\quad} & & & f_! c_1'^* c_{2,!} \\ & \searrow \text{BC}_!^* & & & \searrow \text{BC}_!^* \\ & & d_1'^* d_{2,!} f_! & \xrightarrow{=} & d_1'^* f_{2,!} c_{2,!} \end{array}$$

By adjunction of pairs $(q_D, !, q_D^!)$ and $(c_2, !, c_2^!)$, we obtain

$$\begin{array}{ccccc}
 & & \tilde{f}_! p_C^* c_2^! & \longrightarrow & q_D^! f'_! q_C, ! p_C^* c_2^! \\
 & \nearrow & & & \searrow \\
 p_D^* f_! c_2^! & & & & q_D^! f'_! c_1^* \\
 & \searrow & & & \nearrow \\
 & & q_D^! d_1^* d_2, ! f_! c_2^! & \longrightarrow & q_D^! d_1^* f_2, !
 \end{array}$$

It remains to notice that both diagrams

$$\begin{array}{ccc}
 \tilde{f}_! p_C^* c_2^! & \longrightarrow & q_D^! f'_! q_C, ! p_C^* c_2^! \\
 \text{BC}^{*!} \downarrow & & \downarrow \\
 \tilde{f}_! q_C^! c_1^* & \xrightarrow{(A.1.4)} & q_D^! f'_! c_1^*
 \end{array}
 \quad
 \begin{array}{ccc}
 p_D^* f_! c_2^! & \longrightarrow & q_D^! d_1^* d_2, ! f_! c_2^! \\
 (A.1.4) \downarrow & & \downarrow \\
 p_D^* d_2^! f_2, ! & \xrightarrow{\text{BC}^{*!}} & q_D^! d_1^* f_2, !
 \end{array}$$

are commutative. \square

A.2.11. Pullback of cohomological correspondences. Assume that we have the following commutative diagram

$$\begin{array}{ccccc}
 Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 \\
 f_1 \downarrow & & \downarrow f & & \downarrow f_2 \\
 X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2
 \end{array}$$

and assume that f is cohomologically smooth, equidimensional, of relative dimension d_f . Given $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$ so that (C, u) is a cohomological correspondence from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) , then we have a cohomological correspondence $(D, f^*(u))$ from $(Y_1, f_1^* \mathcal{F}_1)$ to $(Y_2, f_2^! \mathcal{F}_2 \langle -2d_f \rangle)$ defined as follows:

$$f^*(u) : d_1^* f_1^* \mathcal{F}_1 \cong f^* c_1^* \mathcal{F}_1 \xrightarrow{f^* u} f^* c_2^! \mathcal{F}_2 \cong f^! c_2^! \mathcal{F}_2 \langle -2d_f \rangle \cong d_2^! f_2^! \mathcal{F}_2 \langle -2d_f \rangle.$$

We call this the *smooth pullback of the cohomological correspondence*.

Notation A.2.12. If f , f_1 , and f_2 are all smooth and equidimensional of dimension $d = d_f = d_{f_1} = d_{f_2}$, then we shift $f^* \langle d \rangle$ to get the cohomological correspondence from $f_1^* \mathcal{F}_1$ to $f_2^* \mathcal{F}_2$, denoted by $f^*(u)$, and called the *shifted smooth pullback*.

Remark A.2.13. There are another two situations one can pullback of cohomological correspondences. Namely, if the right inner square of the above diagram is Cartesian, then for a (C, u) is a cohomological correspondence from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) , we can define

$$f^*(u) : d_1^* f_1^* \mathcal{F}_1 = f^* c_1^* \mathcal{F}_1 \xrightarrow{f^* u} f^* c_2^! \mathcal{F}_2 \xrightarrow{\text{BC}^{*!}} d_2^! f_2^* \mathcal{F}_2.$$

which, by (A.1.6), coincides with the above defined smooth pullback of the cohomological correspondence if in addition f is cohomologically smooth. Similarly, if the left inner square of the above diagram is Cartesian, then we have

$$f^!(u) : d_1^* f_1^! \mathcal{F}_1 \xrightarrow{\text{BC}^{*!}} f^! c_1^* \mathcal{F}_1 \xrightarrow{f^! u} f^! c_2^! \mathcal{F}_2 = d_2^! f_2^! \mathcal{F}_2,$$

which coincides with the above defined smooth pullback of the cohomological correspondence if in addition f is cohomologically smooth. The following Lemma A.2.14 and Lemma A.2.15 have similar counterparts in these situations. But we will not need them in this paper.

Lemma A.2.14. *Let*

$$\begin{array}{ccccc}
Z_1 & \xleftarrow{e_1} & E & \xrightarrow{e_2} & Z_2 \\
f_1 \downarrow & & \downarrow f & & \downarrow f_2 \\
Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 \\
g_1 \downarrow & & \downarrow g & & \downarrow g_2 \\
X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2
\end{array}$$

be a commutative diagram, such that f (resp. g) are cohomologically smooth, equidimensional of dimension d_f (resp. d_g), and let $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$ be a cohomological correspondence from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) . Then

$$(gf)^*(u) = f^*(g^*(u))$$

as cohomological correspondences from $(g_1 f_1)^* \mathcal{F}_1 \cong f_1^*(g_1^* \mathcal{F}_1)$ to $(g_2 f_2)^! \mathcal{F}_2 \langle -2d_{gf} \rangle \cong f_2^!(g_2^! \mathcal{F}_2 \langle -2d_g \rangle) \langle -2d_f \rangle$.

Proof. Straightforward. \square

Similar to pushforward cohomological correspondences, smooth pullback of cohomological correspondence is also compatible with composition of cohomological correspondences in certain situations.

Lemma A.2.15. *Suppose we have a commutative diagram*

$$\begin{array}{ccccccccc}
Y_1 & \xleftarrow{d_1} & D & \xrightarrow{d_2} & Y_2 & \xleftarrow{d'_1} & D' & \xrightarrow{d'_2} & Y_3 \\
f_1 \downarrow & & \downarrow f & & \downarrow f_2 & & \downarrow f' & & \downarrow f_3 \\
X_1 & \xleftarrow{c_1} & C & \xrightarrow{c_2} & X_2 & \xleftarrow{c'_1} & C' & \xrightarrow{c'_2} & X_3
\end{array}$$

with f, f_2 cohomologically smooth equidimensional, of relative dimension d_f, d_{f_2} . Assume that the third inner square is Cartesian. Set $\tilde{C} := C \times_{X_2} C'$ and $\tilde{D} := D \times_{Y_2} D'$, and let $\tilde{f} : \tilde{D} \rightarrow \tilde{C}$ be the natural map. Let $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$ and $v : c'_1{}^* \mathcal{F}_2 \rightarrow c'_2{}^! \mathcal{F}_3$ be cohomological correspondences from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) and from (X_2, \mathcal{F}_2) to (X_3, \mathcal{F}_3) , respectively. Then cohomological correspondence $\tilde{f}^*(v \circ u) : (Y_1, f_1^* \mathcal{F}_1) \rightarrow (Y_3, f_3^! \mathcal{F}_3 \langle 2d_f \rangle)$ is equal to

$$(Y_1, f_1^* \mathcal{F}_1) \xrightarrow{f^*(u)} (Y_2, f_2^! \mathcal{F}_2 \langle 2d_f \rangle \cong f_2^* \mathcal{F}_2 \langle 2d_f - 2d_{f_2} \rangle) \xrightarrow{f'^*(v)} (Y_3, f_3^! \mathcal{F}_3 \langle 2d_f \rangle).$$

Remark A.2.16. Varshavsky ([Va]) proved a stronger statement, where instead of requiring that the third square is Cartesian, it imposes the assumption that both f' and \tilde{f} are equidimensional smooth of relative dimension $d_{f'}$ and $d_{\tilde{f}}$, and that $d_{\tilde{f}} + d_{f_2} = d_f + d_{f'}$. For the applications in this paper, the weaker statement is enough.

Proof. Since the third square is Cartesian, $\tilde{D} = D \times_{Y_2} D' \cong D \times_{X_2} C' = D \times_C C \times_{X_2} C' = D \times_C \tilde{C}$. So $\tilde{f} : \tilde{D} \rightarrow \tilde{C}$ is smooth of relative dimension d_{f_2} . We write $p_C : \tilde{C} \rightarrow C, q_C : \tilde{C} \rightarrow C', \tilde{c}_1 : \tilde{C} \rightarrow X_1, \tilde{c}_2 : \tilde{C} \rightarrow X_3, p_D : \tilde{D} \rightarrow D, q_D : \tilde{D} \rightarrow D', \tilde{d}_1 : \tilde{D} \rightarrow Y_1$, and $\tilde{d}_2 : \tilde{D} \rightarrow Y_3$ for the natural maps. The decomposition of the cohomological correspondences is equivalently to the commutativity of the following diagram

$$\begin{array}{ccccccccccccccc}
p_D^* d_1^* f_1^* \mathcal{F}_1 & = & \tilde{f}^* p_C^* c_1^* \mathcal{F}_1 & \xrightarrow{u} & \tilde{f}^* p_C^* c_2^! \mathcal{F}_2 & \xrightarrow{\cong} & \tilde{f}^! p_C^* c_2^! \mathcal{F}_2 \langle -2d_f \rangle & \longrightarrow & \tilde{f}^! q_C^* c_1^* \mathcal{F}_2 \langle -2d_f \rangle & \xrightarrow{v} & \tilde{f}^! q_C^* c'_2{}^! \mathcal{F}_3 \langle -2d_f \rangle & = & \tilde{d}_2^! f_3^! \mathcal{F}_3 \langle -2d_f \rangle \\
\parallel & & & & \nearrow & \nearrow & \nearrow & & \nearrow & & \nearrow & & \parallel \\
p_D^* f^* c_1^* \mathcal{F}_1 & \xrightarrow{u} & p_D^* f^* c_2^! \mathcal{F}_2 & \xrightarrow{\cong} & p_D^* f^! c_2^! \mathcal{F}_2 \langle -2d_f \rangle & = & p_D^* d_2^! f_2^! \mathcal{F}_2 \langle -2d_f \rangle & \longrightarrow & q_D^! d_1^* f_2^! \mathcal{F}_2 \langle -2d_f \rangle & \xrightarrow{\cong} & q_D^! f'^! c_1^* \mathcal{F}_2 \langle -2d_f \rangle & \xrightarrow{v} & q_D^! f'^! c'_2{}^! \mathcal{F}_3 \langle -2d_f \rangle
\end{array}$$

Only the commutativity of the middle trapezoid needs explanation. Note that in the following cubic commutative diagram, the top and the bottom faces are Cartesian, and all vertical maps are perfectly smooth

$$\begin{array}{ccccc}
 & \tilde{D} & \xrightarrow{q_D} & D' & \\
 p_D \swarrow & \downarrow & & \swarrow & \\
 D & \xrightarrow{\quad} & Y_2 & & \\
 \downarrow f & & \downarrow & & \downarrow f' \\
 & \tilde{C} & \xrightarrow{\quad} & C' & \\
 & \swarrow & & \swarrow & \\
 C & \xrightarrow{c_2} & X_2 & &
 \end{array}$$

Let $\phi = c_2 \circ f = f_2 \circ d_2 : D \rightarrow X_2$ and $\psi = f' \circ q_D = q_C \circ \tilde{f}$. Then we need to show that the two maps

$$p_D^* \phi^! \xrightarrow{\cong} \tilde{f}^! p_C^* c_2^! \xrightarrow{\text{BC}^*!} \psi^! c_1^*, \quad p_D^* \phi^! \xrightarrow{\text{BC}^*!} q_D^! d_1^* f_2^! \xrightarrow{\cong} \psi^! c_1^*$$

are the same. Since the right face of the cube (i.e. the third inner square of the diagram in the lemma) is Cartesian, the isomorphisms $p_D^* \phi^! \cong \tilde{f}^! p_C^* c_2^!$ and $q_D^! d_1^* f_2^! \cong \psi^! c_1^*$ coincide with the base change map $\text{BC}^*!$ (see (A.1.6)). Then the claims follows from the fact that the composition of the base change maps is the base change map. \square

Finally, the smooth pullback of cohomological correspondence is compatible with pushforward correspondence in the Cartesian case (the base change for cohomological correspondences).

Lemma A.2.17. *Assume that we have the following commutative diagram*

$$\begin{array}{ccccccc}
 & & X'_1 & \xleftarrow{c'_1} & C' & \xrightarrow{c'_2} & X'_2 \\
 & \swarrow \pi_{X_1} & \downarrow f'_1 & & \swarrow \pi_C & \downarrow f' & \swarrow \pi_{X_2} \\
 X_1 & \xleftarrow{\quad} & C & \xrightarrow{\quad} & X_2 & & \\
 \downarrow f_1 & & \downarrow c_1 & & \downarrow c_2 & & \downarrow f_2 \\
 & & Y'_1 & \xleftarrow{d'_1} & D' & \xrightarrow{d'_2} & Y'_2 \\
 & \swarrow \pi_{Y_1} & \downarrow d_1 & & \swarrow \pi_D & \downarrow d_2 & \swarrow \pi_{Y_2} \\
 Y_1 & \xleftarrow{\quad} & D & \xrightarrow{\quad} & Y_2 & &
 \end{array}$$

in which f_1, f'_1, f, f' are representable by perfectly proper algebraic spaces, the middle sectional square with vertices C, D, C', D' is Cartesian, and π_C and π_D are smooth of dimension d . Let $u : c_1^* \mathcal{F}_1 \rightarrow c_2^! \mathcal{F}_2$ be a cohomological correspondence from (X_1, \mathcal{F}_1) to (X_2, \mathcal{F}_2) . Then we have the following commutative diagram of cohomological correspondences

$$\begin{array}{ccc}
 (Y'_1, \pi_{Y'_1}^* f_{1,!} \mathcal{F}_1) & \xrightarrow{\pi_D^* f_1(u)} & (Y'_2, \pi_{Y'_2}^! f_{2,!} \mathcal{F}_2(-2d)) \\
 \downarrow & & \uparrow \\
 (Y'_1, f'_{1,!} \pi_{X_1}^* \mathcal{F}_1) & \xrightarrow{f'_1 \pi_C^*(u)} & (Y'_2, f'_{2,!} \pi_{X_2}^! \mathcal{F}_2(-2d)).
 \end{array}$$

Proof. For simplicity, we write $\tilde{\mathcal{F}}_2$ for $\mathcal{F}_2\langle -2d \rangle$. The condition on properness implies that both f and f' are proper. The lemma amounts to the commutativity of the following diagram

$$\begin{array}{ccccccccccc}
d_1'^* \pi_{Y_1}^* f_{1,!} \mathcal{F}_1 & = & \pi_D^* d_1^* f_{1,!} \mathcal{F}_1 & \rightarrow & \pi_D^* f_{1!} c_1^* \mathcal{F}_1 & \xrightarrow{u} & \pi_D^* f_{1!} c_2^! \mathcal{F}_2 & \xrightarrow{\cong} & \pi_D^! f_{1!} c_2^! \tilde{\mathcal{F}}_2 & \rightarrow & \pi_D^! d_2^! f_{2,!} \tilde{\mathcal{F}}_2 & = & d_2'^! \pi_{Y_2}^! f_{2,!} \tilde{\mathcal{F}}_2 \\
\downarrow & & \downarrow & & \downarrow \cong & & \downarrow \cong & & \uparrow & & \uparrow & & \uparrow \\
d_1'^* f_{1,!} \pi_{X_1}^* \mathcal{F}_1 & \rightarrow & f_{1!}' c_1'^* \pi_{X_1}^* \mathcal{F}_1 & = & f_{1!}' \pi_C^* c_1^* \mathcal{F}_1 & \xrightarrow{u} & f_{1!}' \pi_C^* c_2^! \mathcal{F}_2 & \xrightarrow{\cong} & f_{1!}' \pi_C^! c_2^! \tilde{\mathcal{F}}_2 & = & f_{1!}' c_2'^! \pi_{X_2}^! \tilde{\mathcal{F}}_2 & \rightarrow & d_2'^! f_{2,!}' \pi_{X_2}^! \tilde{\mathcal{F}}_2.
\end{array}$$

The left two squares are commutative by Lemma A.1.23 and the right two squares commutative by Remark A.1.24. The third square is clearly commutative, and the commutativity of the fourth square follows from the smooth proper base change. \square

A.2.18. *Cycle class maps and cohomological correspondences.* Assume $k = \bar{k}$. Let $f : X \rightarrow Y$ be a morphism of pfp algebraic spaces with Y smooth. Set $d_X = \dim X$, $d_Y = \dim Y$, and $d = \dim Y - \dim X$. If X is proper over k , we have the usual cycle class map as

$$\text{cl}_X : H_{2d_X}^{\text{BM}}(X) \rightarrow H_c^{2d}(Y, \mathbb{Q}_\ell(d)).$$

On the other hand, let S be the set of irreducible components of X of dimension d_X , and let \tilde{X} denote the disjoint union of irreducible components of X of dimension d_X . Then there is a correspondence $S \leftarrow \tilde{X} \rightarrow Y$. By (A.2.2)

$$\text{Corr}_{\tilde{X}}((S, \mathbb{Q}_\ell), (Y, \mathbb{Q}_\ell\langle 2d \rangle)) = H_{2d_X}^{\text{BM}}(\tilde{X}) \cong H_{2d_X}^{\text{BM}}(X).$$

In particular, the fundamental class $[\tilde{X}] \in H_{2d_X}^{\text{BM}}(\tilde{X})$ (which is the sum of the fundamental classes of each irreducible component of \tilde{X}) defines a cohomological correspondence from $(S, \mathbb{Q}_\ell) \rightarrow (Y, \mathbb{Q}_\ell\langle 2d \rangle\langle d \rangle)$, denoted by $c_{\tilde{X}}$. If X is proper, it induces

$$H(c_{\tilde{X}}) : H^0(S, \mathbb{Q}_\ell) \rightarrow H_c^{2d}(Y, \mathbb{Q}_\ell(d)).$$

Note the $H^0(S, \mathbb{Q}_\ell)$ is just the space of \mathbb{Q}_ℓ -valued functions on the set S , and we have the natural identification $H^0(S, \mathbb{Q}_\ell) = H_{2d_X}^{\text{BM}}(X)$, $f \mapsto \sum_{s \in S} f(s)[X_s]$, where X_s is the irreducible component labelled by s . It follows from definition that under this identification,

$$(A.2.4) \quad \text{cl}_X = H(c_{\tilde{X}}).$$

We also have the dual correspondence

$$\mathbb{D}c_{\tilde{X}} : (Y, \mathbb{Q}_\ell\langle 2d_X \rangle) \rightarrow (S, \mathbb{Q}_\ell).$$

If the map $X \rightarrow Y$ is proper, it induces

$$H(\mathbb{D}c_{\tilde{X}}) : H_c^{2d_X}(Y, \mathbb{Q}_\ell(d_X)) \rightarrow H^0(S, \mathbb{Q}_\ell).$$

Now if X and X' are perfectly proper algebraic spaces and Y is a separated perfectly smooth algebraic space such that $\dim X + \dim X' = \dim Y$, and if we have morphisms $X \rightarrow Y$ and $X' \rightarrow Y$, then the composition gives a map

$$H(\mathbb{D}c_{\tilde{X}'}) \circ H(c_{\tilde{X}}) : H^0(S) \rightarrow H^0(S').$$

which can be represented by a kernel function $I : S \times S' \rightarrow \overline{\mathbb{Q}}_\ell$ such that

$$H(\mathbb{D}c_{\tilde{X}'}) \circ H(c_{\tilde{X}})(f)(s') = \sum_{s \in S} I(s, s') f(s').$$

we will call I the *intersection number* of X and X' for the following reasons. If Y is a perfectly proper scheme, and X_s and $X'_{s'}$ are closed subschemes of Y , then $H(\mathbb{D}c_{\tilde{X}'})$ is dual to $H(c_{\tilde{X}'})$ under the Poincaré duality $H_c^{2d_{X'}}(Y, \mathbb{Q}_\ell(d_{X'})) = H^{2d_{X'}}(Y, \mathbb{Q}_\ell(d_{X'})) \cong H^{2d_X}(Y, \mathbb{Q}_\ell(d_X))$. It follows that $I(s, s')$ is the intersection number of X_s and $X'_{s'}$ (as algebraic cycles on Y), and therefore

is an integer. If Y is not necessarily proper, but admits a perfectly an open embedding into a perfectly proper and smooth scheme \bar{Y} , and X, X' are proper over k . Then the following diagram is commutative

$$\begin{array}{ccccc} H^0(S, \mathbb{Q}_\ell) & \xrightarrow{H(c_{\bar{X}})} & H_c^{2d}(Y, \mathbb{Q}_\ell(d)) & & \\ & \searrow & \downarrow & \searrow H(\mathbb{D}c_{\bar{X}'}) & \\ & & H^{2d}(\bar{Y}, \mathbb{Q}_\ell(d)) & \longrightarrow & H^0(S', \mathbb{Q}_\ell) \end{array}$$

It follows that $I(s, s')$ is the intersection number of X_s and $X'_{s'}$ in \bar{Y} . (Note however that this number is independent of the choice of the compactification \bar{Y} .)

A.2.19. Trace formula. We will consider the following situation. Assume $k = \bar{k}$. Let $c := (c_1, c_2) : C \rightarrow X \times X$ be a pfp self-correspondence of the pfp algebraic space X , and let $\text{Fix}(c) := X \times_{\Delta, X \times X} C$ be the fixed point of the correspondence. Let $\mathcal{F} \in D_c^b(X)$. We construct a map

$$(A.2.5) \quad \text{Corr}_C((X, \mathcal{F}\langle d \rangle), (X, \mathcal{F})) \rightarrow H_d^{\text{BM}}(\text{Fix}(c)).$$

as follows. Let $u : c_1^* \mathcal{F}\langle d \rangle \rightarrow c_2^! \mathcal{F}$ be a cohomological correspondence, which by Lemma A.2.4 can be regarded as $u^\sharp : (X \times X, \mathcal{F} \boxtimes \mathbb{D}\mathcal{F}) \rightarrow (\text{pt}, \overline{\mathbb{Q}}_\ell\langle -d \rangle)$. Then the support of the correspondence

$$(\text{pt}, \overline{\mathbb{Q}}_\ell) \xrightarrow{\delta_{\mathcal{F}}} (X \times X, \mathcal{F} \boxtimes \mathbb{D}\mathcal{F}) \xrightarrow{u^\sharp} (\text{pt}, \overline{\mathbb{Q}}_\ell\langle -d \rangle)$$

is $\text{Fix}(c)$ and by (A.2.2) is given by a class of $H_d^{\text{BM}}(\text{Fix}(c))$. Note that if $d = 0$, (A.2.5) recovers the trace map defined by Varshavsky [Va07, §1.2.2].

Remark A.2.20. If $f : X \rightarrow X'$ is representable by perfectly proper algebraic spaces, then the sharp correspondence $u \leftrightarrow u^\sharp$ is compatible with the pushforward of cohomological correspondence. Then it follows from Lemma A.2.10 that (A.2.5) is compatible with the pushforward along the proper morphism, giving the Lefschetz trace formula for correspondences.

The following lemma follows directly from the purity.

Lemma A.2.21. *Assume that X, C are irreducible perfectly smooth, and $d = \dim C - \dim X$. Assume that $\text{Fix}(c) \subset C$ is perfectly smooth of codimension d , i.e. C intersects with the diagonal properly smoothly. Assume that $u : c_1^* \overline{\mathbb{Q}}_\ell\langle 2d \rangle \rightarrow c_2^! \overline{\mathbb{Q}}_\ell$ is given by the fundamental class $[C]$. Then the image of u under (A.2.5) is the fundamental class $[\text{Fix}(c)]$.*

Another case we need is as follows, which is due to Braverman-Varshavsky [BV06].

Lemma A.2.22. *Let (X, \mathcal{F}) be a pair over $k = \mathbb{F}_q$, with $\mathcal{F} \in D_c^b(X)$. Let $u = \Gamma_{\sigma_q}^*$ be the cohomological correspondence. Then $\delta_{\mathcal{F}} \circ u^\sharp$ is supported on $X(\mathbb{F}_q)$, given by the function on $X(\mathbb{F}_q)$ (as in Example A.2.3 (5) or (A.2.5)) defined as*

$$f_{\mathcal{F}}(x) = \text{tr}(\phi_x, \mathcal{F}_{\bar{x}}).$$

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