AN INTRODUCTION TO GEOMETRIC SATAKE EQUIVALENCE

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ABSTRACT. The geometric Satake equivalence is a categorification of the classical Satake isomorphism, upgrading the ring isomorphism over complex numbers to an equivalence between abelian categories. This equivalence holds even for more general coefficients such as $\mathbb{Z}$ or $\mathbb{F}_p$ as shown in [28]. In this paper, we are trying to sketch the proof of the above theorem.

CONTENTS

1. INTRODUCTION

Let us first look at an imprecise form of the main theorem.

Theorem 1.1 (Geometric Satake Equivalence [28]). *Let G be a complex reductive group, $\Bbbk$ a noetherian ring of finite global dimension*[1]*. Denote $G^\vee_{\mathbf{k}}$ the Langlands dual group of G over $\Bbbk$. Then there exists a complex algebraic "variety" $\mathcal{G}r_G$ with an algebraic group L^+G action on the left, such that we have an equivalence of categories*

$$Rep_{G^\vee_{\Bbbk}} \cong Perv_{L^+G}(\mathcal{G}r_G, \Bbbk).$$

The theorem looks intimidating at first glance, but it is actually very well-motivated. Here are some motivations towards the theorem and constructions.

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[1]This is just a technical assumption to make things reasonable.

First, this theorem can be applied to the representation theory of reductive groups in positive characteristic, which is discussed in [9]. Although semisimplicity vanishes in the modular case, highest weight theory and the linkage principle still work and an analog of the Kazhdan-Lusztig conjecture can be formulated. The earliest proof of the Kazhdan-Lusztig conjecture proceeds by building an equivalence between the category of equivariant $\mathfrak{g}$-modules (with fixed central character) and the category of (possibly twisted) equivariant D-modules and perverse sheaves on the flag manifold. The main theorem does essentially the same[2], giving us powerful geometric tools to analyze representations and multiplicities. Note here $\mathcal{G}r_G$ should not be of finite type, since infinitely many terms are involved the linkage relation in positive characteristic.

Second, if we look at the equivalence in the reverse direction, it gives the only known way to construct the dual group without invoking the complete classification of split reductive groups. In this way, the occurence of Langlands dual group in other contexts should become more natural.

Third, the theorem is a first step towards the geometrization of Langlands program. More precisely, the geometric Satake equivalence is the geometrization of the local unramified Langlands theory over function field. For K a local field and O the ring of integers,the classical Satake isomorphism claims an isomorphism between the spherical Hecke algebra $\mathcal{H} = \mathbb{C}_c[G(O)\backslash G(K)/G(O)]$ and $\mathbb{C}[X_*(T)]^W$. The latter ring is isomorphic to the representation ring of $G^\vee$ by highest weight theory, or to the ring of functions on the quotient $T^\vee/W$. Consider the set of maximal ideals of both sides of the isomorphism, the set of irreducible representations of the spherical Hecke algebra is identified with closed points in $T^\vee/W$ and hence with the set of G-conjugacy orbits of semisimple elements in G. This gives exactly the local unramified Langlands correspondence. The classical Satake isomorphism can be recovered from Theorem 1.1 (or more precisely, its ℓ-adic version) via the sheaf-function correspondence of Grothendieck (for the statement of correspondence, see [21] III.12).

Finally, we mention some recent developments of the geometric Satake equivalence in mixed characteristic. In [40], Xinwen Zhu constructed the Witt vector affine Grassmannian as a moduli space for perfect schemes, and established a geometric Satake equivalence in mixed characteristic. The paper is later applied to Shimura varieties in [37] and can be viewed as a feedback from geometric Langlands to classical Langlands. An integral coefficient improvement of [40] is proved in [38]. In the p-adic geometry setting, Peter Scholze introduces the B_{dR}^+-affine Grassmannian ([33]) and constructs in [12] the Beilinson-Drinfeld Grassmannian over Fargues-Fontaine curve with Fargues. He also created the concepts of diamonds and v-topology, so that the fusion approach to prove geometric Satake equivalence is available[3] in this setting ([12]). The result and language are applied in [36] to solve the $S = T$ conjecture raised in [37]. There is also a comparison result in [2] between the two geometric Satake equivalence functors in [40] and [12].

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[2]The flag manifold has to be upgraded to the affine Grassmannian explained below, whose geometry is more complicated and rich

[3]In comparison, [40] uses the approach in [16] instead of fusion to construct the symmetric convolution product.

In this paper, we show the structure of the proof of Theorem 1.1, focusing more on strategy, geometry and application of sheaf theory. The main references are [28],[31],[39].

This journey is guided by Tannakian reconstruction, which roughly says that some nice abelian category can be realized as a module category and nice monoidal structure turns it into the representation of an affine algebraic group (i.e. Hopf algebra). Keeping the idea of reconstruction in mind, we should imagine $Perv_{L^+G}(\mathcal{G}r_G, \Bbbk)$ to be a representation category and try to (geometrically) construct enough structures and objects in the category to reconstruct group-theoretic data we desire.

More precisely, several important orbits and other geometric constructions are made in Section 2, and they boil down to (compatible) weight spaces, highest weight theory and the monoidal structure on $Perv_{L^+G}(\mathcal{G}r_G, \Bbbk)$ in Section 3. Section 3 also includes the semisimplicity in characteristic 0 which essentially comes from Beilinson-Bernstein-Deligne-Gabber decomposition theorem. In Section 4, we finally realize the Tannakian step and identify the resulting algebraic group with the Langlands dual by reading information of the root systems from highest weight structure. Section 5 contains an application to modular representation theory as mentioned in the motivation.

The reader is supposed to be familiar with basic language of sheaf theory with six functors([14][20][22]), perverse sheaves([3][21]) and reductive groups([18][19][25][26]). However, we do not want to be disturbed too much by sheaf theory details. Therefor, we will work in the complex setting where there exists a sheaf theory with six functors formalism in arbitrary coefficients. A constructible sheaf on some ind-scheme is defined to be one supported on some finite stage. We also assume a theory of equivariant constructible sheaves, which can be found in [5], [23]. The whole story also works for the étale topology over any algebraically closed field instead complex numbers, with coefficient ring over ℓ-adic integers avoiding base characteristic.

We now fix some notations. Let $\Bbbk$ be a commutative noetherian ring of finite global dimension. For any $\Bbbk$-algebra R, denote Mod_R the category of left R-modules that are finitely generated over $\Bbbk$. For a category $\mathscr{C}$ and an object X in $\mathscr{C}$, we write $X \in \mathscr{C}$ instead of $X \in Ob(\mathscr{C})$. When $\mathscr{C}$ is a monoidal category, use $\mathbf{1}$ to denote its unit object. Functors on sheaves are assumed to be derived.

For any ind-scheme X over $\mathbb{C}$, denote $D(X) = D(X, \Bbbk) = D^b_c(X, \Bbbk)$ the bounded derived category of constructible sheaves on the complex analytification of X. Denote $({}^pD^{\leq 0}, {}^pD^{\geq 0})$ the perverse t-structure on $D(X)$ and $Perv(X)$ the category of perverse sheaves. If there is an algebraic group H acting on X, denote $D_H(X)$ the category of H-equivariant derived category and $Perv_H(X)$ the category of H-equivariant perverse sheaves.

Denote $K = \mathbb{C}(\!(t)\!)$, $O = \mathbb{C}[\![t]\!]$. Let G be a reductive group over $\mathbb{C}$ with fixed maximal torus T inside Borel subgroup B. Then we have corresponding weight lattice $X^*(T)$, coweight lattive $X_*(T)$, root system $\Phi = \Phi(G, T)$, positive roots Φ^+, simple roots $\Delta = \Delta(G, B, T)$, dual root system $\Phi^\vee, \Delta^\vee$, partial order given by positive (co)roots, dominant weights and coweights $X^*(T)^+, X_*(T)^+$, root lattice Q and coroot lattice $Q^\vee$, Weyl group W, affine Weyl group $W_{aff} = W \ltimes Q^\vee$, extended affine Weyl group $\widetilde{W}_{aff} = W \ltimes X_*(T)$ and Weyl element $\rho = \frac{1}{2}\sum_{\mu \in \Phi^+} \mu$. The length function l on the affine Weyl group(as a Coxeter group) is extended to a length function on $\widetilde{W}_{aff}$. The longest element in W is denoted by w_0 The Borel B and its unipotent part N have opposite B^- and N^-. For coweight $\lambda \in X_*(T)$, let t^λ denote the image of t of the map $\lambda\colon \mathbb{G}_m(K) \to G(K)$.

2. Geometry of the affine Grassmannian

The construction of the affine Grassmannian breaks the symmetry[4] of the double coset space $G(O)\backslash G(K)/G(O)$ in the classical Satake isomorphism: we consider the left $G(O)$-action on $G(K)/G(O)$. Naively, we can set $\mathcal{G}r_G = G(K)/G(O)$ as a topological quotient. It obtains a scheme structure as shown below.

2.1. **Definition of the affine Grassmannian.**

There are several equivalent descriptions of affine Grassmannian from [39]. We will restrict ourselves to work over the field of complex numbers $\mathbb{C}$ to obtain the analytic topology.

Definition-Theorem 2.1 (the affine Grassmannian)**.** The following moduli problems are represented by the same ind-projective ind-scheme, called the *affine Grassmannian* $\mathcal{G}r_G$ of G.

- (Loop description) Define the *loop space* $LG(R) = G(R(\!(t)\!))$ and the *arc space* $L^+G(R) = G(R[\![t]\!])$. Define $\mathcal{G}r_G$ to be the quotient $[LG/L^+G]$ as fpqc sheaves.
- (Moduli description, local) For any ring R, define *(formal) disk* $D_R = \mathrm{Spec}R[\![t]\!]$ and *punctured (formal) disk* $D_R^* = \mathrm{Spec}R(\!(t)\!)$. Define $\mathcal{G}r_G$ to be the functor sending $\mathbb{C}$-algebra R to the set
$$\{(\mathcal{E}, \beta)|\ \mathcal{E} \text{ is a G-torsor on } D_R \text{ with a trivialization } \beta \text{ on } D_R^*\}.$$
- (Moduli description, global) Let X be a reduced connected curve with smooth closed point x. Denote the complement of x in X by X^*. Define $\mathcal{G}r_G$ to be the functor sending $\mathbb{C}$-algebra R to the set
$$\{(\mathcal{E}, \beta)|\ \mathcal{E} \text{ is a G-torsor on } X_R \text{ with a trivialization } \beta \text{ on } X_R^*\}.$$

Remark 2.2. This definition works for more general reductive groups over $D_{\mathbb{C}}$ or X, which recovers the theory for parahoric groups. For example, if we consider the projection $L^+G \to G$ by $t = 0$, the inverse image of B is the Iwahori subgroup Iw. The quotient $\mathcal{F}l_G = [LG/\mathrm{Iw}]$ is called the *affine flag variety* of G. See [39] for details.

Proof. (Sketch) We first show that three definitions are equivalent to each other. The completion of X at x is identified with $D_{\mathbb{C}}$. Hence any G-torsor on X_R with trivialization outside $\{x\} \times \mathrm{Spec}(R)$ restricts to a G-torsor on the R-disk with trivialization on the punctured disk. Conversely, a G-torsor on the disk is glued with the trivial torsor outside x by their trivializations and Beauville-Laszlo's descent theorem[5]. Thus local and global moduli descriptions are identified, denoted by $\mathcal{G}r_G$. Consider the gluing datum in the formal neighborhood of x, L^+G classifies all trivializations of a torsor on the disk and LG classifies G-torsors with trivializations on the disk and $X - \{x\}$. The projection map $LG \to \mathcal{G}r_G$ is therefore an L^+G-torsor since G-torsors on the disk are étale-locally trivial in R[6]. Hence we get $\mathcal{G}r_G \cong [LG/L^+G]$. All three definitions are identified.

[4]Alternatively, one can work with Hecke stack $[G(O)\backslash G(K)/G(O)]$, but we'd rather remain in the more classical setting.

[5]See Theorem 1.4.3. in [39]

[6]Any G-torsor $P \to X$ is a smooth map as G is smooth, so a section exists étale-locally on the special fiber T=0, then extend the section by formally smoothness

Now we move to representability and properties of $\mathcal{G}r_G$.

We first deal with the case $G = GL_n$. Identifying GL_n-torsors with vector bundles of rank n, the functor $\mathcal{G}r_G$ sends R to all finitely generated projective $R[\![t]\!]$-lattices Λ of $R(\!(t)\!)^n$. Denote the standard lattice $R[\![t]\!]^n$ by Λ_0. Filter $\mathcal{G}r_G$ by closed subschemes $\mathcal{G}r_G^{(N)}(R) = \{\Lambda \text{ as above}, t^N\Lambda_0 \subset \Lambda \subset t^{-N}\Lambda_0\}$. Then $\mathcal{G}r_G = \bigcup \mathcal{G}r_G^{(N)}$,where $\mathcal{G}r_G^{(N)}$ is identified with union of the Grassmannians $\bigsqcup \mathcal{G}r(k, 2Nn)$ modulo $T^N\Lambda_0$. It follows that $\mathcal{G}r_{GL_n}$ is ind-projective.

We move to the general case. There exists a faithful representation $\rho\colon G \to GL_n$ such that the qutient GL_n/G is affine. In this case, the induced morphism $\mathcal{G}r_G \hookrightarrow \mathcal{G}r_{GL_n}$ is a closed embedding, therefore $\mathcal{G}r_G$ is an ind-projective ind-scheme. For more details and references, see [39]. □

Now we are able to state main theorem more precisely.

Theorem 2.3 (Geometric Satake equivalence [28])**.** *Let G be a complex reductive group, $\Bbbk$ be a noetherian ring of finite global dimension. Denote $G^\vee_{\mathbf{k}}$ the Langlands dual group of G over $\Bbbk$ and define the Satake category $Sat_{G,\Bbbk} := Perv_{L^+G}(\mathcal{G}r_G, \Bbbk)$. Then we have a natural equivalence of symmetric monoidal categories over $\Bbbk$-modules.*

$$\begin{tikzcd} Sat_{G,\Bbbk} \arrow[rr, "\cong"] \arrow[dr, "H^*(-)"'] & & Rep_{G^\vee_\Bbbk} \arrow[dl, "forget"] \\ & Mod_\Bbbk & \end{tikzcd}$$

Here $H^*(-)$ is hypercohomology functor. The Langlands dual group is defined to be the split reductive group with root datum dual to that of G.

2.2. **Geometry of orbits.** Since $\mathcal{G}r_G$ has a natural left LG-action, we are going to consider several subgroups of LG which produce interesting orbits. Conversely, we can have a glimpse of the global geometry of $\mathcal{G}r_G$ by considering it as the union of orbits. With Theorem 2.3 in mind, several structures of the Satake category are reflected in the orbits and corresponding stratification. Very roughly speaking, the proof of Theorem 2.3 is a careful analysis of different orbits, modulo some sheaf techniques and group theory. At the end of this section, we will briefly introduce Bruhat-Tits building to visualize the crazy geometry of affine Grassmannian and show how the building can be used.

To begin with, we recall several standard decompositions.

Theorem 2.4. *We have the following decompositions.*

1. *(Bruhat decomposition)* $G(\mathbb{C}) = \bigsqcup_{w\in W} B(\mathbb{C})wB(\mathbb{C})$
2. *(Bruhat decomposition)* $G(K) = \bigsqcup_{w\in \tilde{W}_{aff}} Iw(\mathbb{C})wIw(\mathbb{C})$
3. *(Cartan decomposition)* $G(K) = \bigsqcup_{\lambda\in X_*(T)^+} G(O)t^\lambda G(O)$
4. *(Birkhoff decomposition)* $G(K) = \bigsqcup_{\lambda\in X_*(T)^+} G(\mathbb{C}[t^{-1}])t^\lambda G(O)$
5. *(Iwasawa decomposition)* $G(K) = B(K)G(O)$

In this theorem, (2)(3)(5) can be shown using the theory of Bruhat-Tits buildings which will be explained later. To prove (4), we identify elements in $G(O)\backslash G(K)/G(O)$ with G-torsors on $\mathbb{P}^1$ and reduce them to T-torsors, see [11], Lemma 4.

For $g \in LG$, denote its image in $\mathcal{G}r_G$ by $[g]$.

2.2.1. *Schubert varieties.* A natural subgroup of LG is L^+G. The corresponding orbits are called Schubert varieties just as for the left B-orbits of the flag manifold G/B. By Theorem 2.4 we can label the orbits more explicitly: for $\lambda \in X_*(T)^+$, the L^+G-orbit containing $[t^\lambda]$ is denoted by $\mathcal{G}r_\lambda$. The following proposition claims these orbits give rise a statification $\mathcal{S}$ of $\mathcal{G}r_G$.

Proposition 2.5.

(1) $\overline{\mathcal{G}r_\lambda} = \mathcal{G}r_{\leq\lambda} := \bigsqcup_{\mu\leq\lambda,\mu\in X_*(T)^+}\mathcal{G}r_\mu$

(2) $\mathcal{G}r_\lambda$ *is an affine bundle over partial flag manifold, and* $\dim \mathcal{G}r_\lambda = \langle 2\rho, \lambda\rangle$.

Proof. (1) First, we show that $\mathcal{G}r_{\leq\lambda}$ is closed. For any dominant weight χ with highest weight representation (ρ_χ, V_χ), viewed as a vector bundle on the disk via G-torsors, consider the locus $\mathcal{G}r_{\leq\lambda,V_\chi}$ where $t^{\langle\chi,\lambda\rangle}\rho_\chi(\beta)$ extends to a morphism of vector bundles on the whole disk. Then $\mathcal{G}r_{\leq\lambda,V_\chi}$ is closed since it is defined by equations, and $\mathcal{G}r_{\leq\lambda} = \bigcap \mathcal{G}r_{\leq\lambda,V_\chi}$ is closed if G^{der} is simply connected. The general case reduces to the special case using z-extension, which claims that we can find a central extension $\widetilde{G}$ of G by a torus such that $\widetilde{G}^{der}$ is simply connected. For more details, see Proposition 3.1. [27].

Now we only need to show that $\mathcal{G}r_\lambda$ is dense in $\mathcal{G}r_{\leq\lambda}$. By induction it suffices to show $t^{\lambda-\alpha} \in \overline{\mathcal{G}r_\lambda}$ for simple coroot $\alpha^\vee$. Denote $i_\alpha\colon SL_2 \to G$ the map corresponding to the line spanned by α, then consider the action of $\left(\begin{smallmatrix} 1 & 0 \\ ct^{\langle\alpha,\lambda\rangle-1} & 1\end{smallmatrix}\right)$, $c \in \mathbb{C}$ on $[t^\lambda] \in \mathcal{G}r_G$, whose limit when $c \to \infty$ gives $\left(\begin{smallmatrix} 0 & -t^{1-\langle\alpha,\lambda\rangle} \\ t^{\langle\alpha,\lambda\rangle-1} & 0\end{smallmatrix}\right)\cdot[t^\lambda] = [t^{\lambda-\alpha^\vee}]$: this can be seen from the unified action of $t^{-\lambda}\left(\begin{smallmatrix} O & tO \\ t^{-1}O & O\end{smallmatrix}\right)t^\lambda$.

(2) We have a well-defined left G-equivariant map

$$\mathcal{G}r_\lambda = L^+G/(L^+G \cap t^\lambda L^+G t^{-\lambda}) \overset{t=0}{\to} G/P_\lambda\,,$$

where P_λ is the parabolic subgroup containing B^- spanned by the negative root perpendicular to λ. The fiber at the trivial coset P_λ is an affine space of dimension $\langle 2\rho, \lambda\rangle - \dim G + \dim P_\lambda$ by computing the matrix coefficient power series. Since G acts transitively on the base, $\mathcal{G}r_\lambda$ is an affine bundle of the desired dimension. □

Corollary 2.6. *The connected components of* $\mathcal{G}r_G$ *are parametrized by* $\pi_1(G) = X_*(T)/Q^\vee$*, while the dimensions of* L^+G*-orbits on the same components have the same parity. Every* L^+G*-orbit is simply connected.*

We now briefly discuss the opposite Schubert varieties and refer to [39] for details.

Definition 2.7. Define $L^-G(R) = G(R[t^{-1}])$, and $\mathcal{G}r^\lambda := L^-G \cdot [t^\lambda], \lambda \in X_*(T)^+$ the opposite Schubert varieties.

The Birkhoff decomposition in Theorem 2.4 again implies we get every L^-G-orbit in this way.

Proposition 2.8.

(1) $\mathcal{G}r^\lambda$ *is a locally closed sub-ind-scheme of* $\mathcal{G}r_G$.

(2) $\overline{\mathcal{G}r^\lambda} = \mathcal{G}r^{\geq\lambda} := \bigsqcup_{\mu\geq\lambda,\mu\in X_*(T)^+}\mathcal{G}r^\mu_G$, codim $\mathcal{G}r^\lambda = \langle 2\rho, \lambda\rangle - \dim G/P_\lambda$.

(3) $\mathcal{G}r^\lambda \cap \mathcal{G}r_\mu \neq$ *if and only if* $\lambda \leq \mu$. *In addition,* $\mathcal{G}r^\lambda \cap \mathcal{G}r_\lambda = G \cdot t^\lambda \cong G/P_\lambda$.

In the proof of Birkhoff decomposition Theorem 2.4, we identify $L^-G(\mathbb{C})\backslash LG(\mathbb{C})/L^+G(\mathbb{C})$ with $Bun_G(\mathbb{P}^1)(\mathbb{C})$. This can be upgraded into an isomorphism of stacks and makes opposite Schubert varieties more interesting.

Theorem 2.9. $[L^-G\backslash LG/L^+G] \cong Bun_G(\mathbb{P}^1)$.

2.2.2. *Semi-infinite orbits.* The occurence of Semi-infinite orbits is a new phenomenon in affine setting. It reflects the weight structure in the representation category of a reductive group.

Definition 2.10. By Iwasawa decomposition, LN acts on the left of $\mathcal{G}r_G$, giving rise to orbit decomposition $\mathcal{G}r_G = \bigsqcup_{\mu \in X_*(T)} LN \cdot [t^\mu]$. Denote $S_\mu = LN \cdot [t^\mu]$. We can also work for its opposite N^-, and define $T_\mu = LN \cdot [t^\mu]$

It might be natural to think weight structure should be related to the T-action on $\mathcal{G}r_G$. Indeed, there is a dynamical description of semi-infinite orbits using T-action as we exaplain below.

Choose any regular dominant coweight η to obtain a $\mathbb{G}_m$-action, then $\mathcal{G}r_G^{\mathbb{G}_m} = \mathcal{G}r_G^T = \{[t^\mu], \mu \in X_*(T)\}$. This can be seen as follows. The $T-$action preserves each L^+G-orbit, so we can look at the G-equivariant map $\pi\colon \mathcal{G}r_\mu \to G/P_\mu$. Note that π maps T-invariant points to T-invariant points, we can show the only T-invariant points of G/P_μ are images of the Weyl group W. Using $N_G(T) \subset G$-equivariance, we reduce to the fiber at trivial coset P_μ. Any $\mathbb{G}_m$-fixed element is represented by a unipotent matrix opposite to P_μ, so the element has to be trivial since $\langle \alpha, \eta \rangle \neq 0, \forall \alpha \in \Phi$.

Proposition 2.11. *S_μ is the attractor of $[t^\mu]$:=$\{x \in \mathcal{G}r_G | \eta(a) \cdot x \to [t^\mu], a \to 0\}$. Dually, $T_\mu = \{x \in \mathcal{G}r_G | \eta(a) \cdot x \to [t^\mu], a \to \infty\}$, the repeller of $[t^\mu]$.*

Proof. It is clear that S_μ is included inside the attractor. The converse inclusion holds since $S_\mu, \mu \in X_*(T)$ form a partition of $\mathcal{G}r_G$. The case of T_μ is similar. □

We now list some properties about S_μ and its interaction with L^+G-orbits. These also hold for T_μ after tiny modifications.

Proposition 2.12.

(1) $\overline{S_\mu} = \bigsqcup_{\nu \leq \mu, \nu \in X_*(T)} S_\nu$.
(2) $\overline{\mathcal{G}r_\lambda} \cap S_\mu \neq \emptyset \iff [t^\mu] \in \overline{\mathcal{G}r_\lambda} \iff \mu \in Conv(W\lambda) \cap (\lambda + Q^\vee)$.
(3) In the case of (2), $\overline{\mathcal{G}r_\lambda} \cap S_\mu$ *is of pure dimension* $\langle \rho, \lambda + \mu \rangle$.
(4) In the case of (2), $\mathcal{G}r_\lambda \cap S_\mu$ *is open dense in* $\overline{\mathcal{G}r_\lambda} \cap S_\mu$.

Proof. This is a sketch of the proof, see [31] for more details.

The proof of (1) requires the following fact: $\mathcal{G}r_G$ has an embedding into some infinite dimensional projective space $\mathbb{P}(V)$ such that $\bigsqcup_{\mu \leq \lambda} S_\mu$ is the inverse image of some linear subspace V_μ and its boundary is cut out by a hyperplane H_μ. The proof of this fact involves constructions related to Kac-Moody groups, so we omit it. Assuming the fact, $\bigsqcup_{\mu \leq \lambda, \mu \in X_*(T)} S_\mu$ is closed and hence $\overline{S_\lambda} \subset \bigsqcup_{\mu \leq \lambda, \mu \in X_*(T)} S_\mu$ holds, while the converse inclusion is an explicit local calculation with SL_2 as what we do in Proposition 2.5(1).

Since $\overline{\mathcal{G}r_\lambda}$ is closed and stable under T-action, (2) follows from Proposition 2.11 by unwinding the definition. For (3), we first show $\dim \overline{\mathcal{G}r_\lambda} \cap \overline{S_\mu} \leq \langle \rho, \lambda + \mu \rangle$ by induction on μ, in a way walking on the lattice $Conv(W\lambda) \cap (\lambda + Q^\vee)$ and adding by one simple coroot at each step. Here by $Conv(W\lambda)$ we mean the convex hull of $W\lambda$. In the case $\mu = w_0\lambda$, $\overline{\mathcal{G}r_\lambda} \cap \overline{S_\mu} = \overline{T_\mu} \cap \overline{S_\mu} = [t^\mu]$ is a point with dimension zero. At each step, $\dim \overline{\mathcal{G}r_\lambda} \cap \overline{S_\mu} \leq \dim \overline{\mathcal{G}r_\lambda} \cap (\overline{S_\mu} \backslash S_\mu) + 1$ by induction hypothesis and using the hyerplane H_μ in (1). Surprisingly, in the maximal case $\mu = \lambda$, the equality holds as we have $\overline{\mathcal{G}r_\lambda} \subset \overline{S_\lambda}$ according to (2) and the dimension calculation in

Proposition 2.5(2). This basically implies all inequalities in the induction steps above should be equality. A more careful analysis will imply all irreducible components of $\overline{\mathcal{G}r_\lambda} \cap S_\mu$ have the same dimension.

(4) $\mathcal{G}r_\lambda \cap S_\mu \subset \overline{\mathcal{G}r_\lambda} \cap S_\mu$ is open because $\mathcal{G}r_\lambda \subset \overline{\mathcal{G}r_\lambda}$ is by Proposition 2.5(1).Suppose the inclusion doesn't have dense image, then there exists an irreducible component of $\overline{\mathcal{G}r_\lambda} \cap S_\mu$ contained in some $\overline{\mathcal{G}r_\nu} \cap S_\mu, \nu \leq \lambda$. The dimension calculation in (3) gives us a contradiction. □

Corollary 2.13. *Let $X \subset \overline{\mathcal{G}r_\lambda}$ be a closed T-invariant subvariety. Then we have*

$$\dim X \leq \underset{[t^\mu]\in X, \mu \in X_*(T)}{max} \langle \rho, \lambda + \mu \rangle .$$

2.2.3. *Bruhat-Tits building.* We give an informal introduction to Bruhat-Tits building. For more serious surveys, see [34] or [32].

The affine Grassmannian is an analog of the Riemannian symmetric spaces over non-archimedean fields, serving as the quotient of a group by its maximal compact subgroup. Alternatively, it parametrizes all maximal compact subgroups. The main difference between $\mathbb{R}$ and a non-archimedean local field is the topology: the later one admits many small subgroups near the identity. Suppose there is a point moving on the symmetric space. Instead of moving smoothly, it has to jump from here to there since the space is totally disconnected. In order to make the space more connected with nice topology on it, we can add parahoric subgroups into the picture. Then we will get the so-called Bruhat-Tits building.

Let F be a non-archimedean local field with valuation ring O, G be a reductive group over O and denote $V = X_*(T) \otimes_{\mathbb{Z}} \mathbb{R}$. The fastest, but least intuitive way to define the Bruhat-Tits building considers the poset of all parahoric subgroups of $G(K)$ with partial order given by inclusions, and define Bruhat-Tits building as the geometrization of the poset category (considered as a simplicial set). To obtain a more practical definition, we have to introduce valuations into the group theoretic picture. Recall the Weyl group is defined as the quotient $N_G(T)/T$, the extended affine Weyl group can be seen in a similar way: there is a map from K-points of the maximal split torus T to its coweight lattice (viewing as translations on V) which can be extended to an action of N(F) on V Each root corresponds to a embedding $\mathbb{G}_a \hookrightarrow G$ and there is a filtration of $\mathbb{G}_a(K)$ by valuation. There is a parahoric subgroup $\hat{P}_x$ associated to each $x \in V$, generated by the stabilizer of x in N(K) and root datum on different levels of the filtration. The parahoric subgroup only depends on the facet of x under $\widetilde{W}_{aff}$-action and recovers standard parihorics for points in the fundamental alcove in GL_n case. The *Bruhat-Tits building* is defined as $\mathcal{I} = G(K) \times V/\sim$, where

$$(g, x) \sim (h, y) \iff \exists\, n \in N(K), \text{ such that } y = n \cdot x \text{ and } g^{-1}hn \in \hat{P}_x.$$

For $x \in V$, $(g, x) \sim (h, x) \iff g\hat{P}_x = h\hat{P}_x$, since $Stab_{N(K)}(x) \subset \hat{P}_x$ by definition, so the building is indeed a model for all parahorics as in the earlier definition. For fixed g, $(g, x) \sim (g, y)$ implies $x = y$, hence $v \mapsto (g, v)$ is an embedding of V into the building for any $g \in G(K)$. These images are called the *apartments* in the building. The Bruhat-Tits building is called a building in the sense that it satisfies the following axioms:

(I1) For any two facets F, F', there exist an apartment A containing both F, F'.

(I2) For apartments A and $A', A \cap A'$ is a union of facets. For any two facets F, F' in $A \cap A'$, there exists an isomorphism from A to A' fixing (pointwise) F, F', where isomorphism means an isometry that preserves facets.

We can apply these axioms to prove Theorem 2.4. Fix a standard apartment and the fundamental alcove, then the Iwahori Iw is identified with the stabilizer of the fundamental alcove. Any other alcove shares a common apartment with the fundamental one by (I1) and the apartment is transformed to the standard one by an element in Iw by (I2). Hence we have $G(K) = \bigcup_{w \in \widetilde{W}_{aff}} \mathrm{Iw}(\mathbb{C}) w \mathrm{Iw}(\mathbb{C})$. Disjointness can be verified by direct calculation. (3) follows using a similar argument. For (5) we need to compactify the Bruhat-Tits building and realize the strict upper-triangular matrices as the stabilizer of some point on the boundary.

We can visualize the building nicely in SL_2 case. The thick line stands for the standard apartment. In this picture, G(O) is the stabilizer of base point, so G(O)-orbits are circles of different sizes. $N(K)$ (resp.$N^-(K)$) is the stabilizer of the positive (resp. negative) direction (a point at infinity) of standard apartment. So we can consider $N(K)$-orbits as horocycles at the boundary. Geometry happens as if inside the lovable Poincaré hyperbolic disk. Visualization in higher dimension is less-intuitive as shown in the figure.

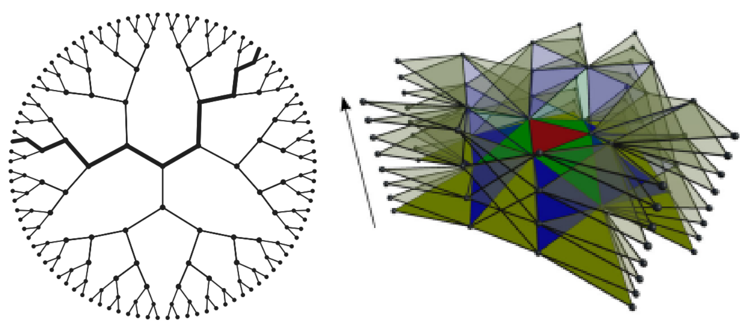

FIGURE 1. Bruhat-Tits building for SL_2[17] and SL_3[4].

2.3. **More about global version of $\mathcal{G}r_G$.** The global moduli description of $\mathcal{G}r_G$ in Definition-Theorem 2.1 can be generalized to Beilinson-Drinfeld Grassmannians. For set I, commutative ring R, smooth curve X, $x = (x_i)_{i \in I} \in X^I(R)$, denote $\Gamma_x = \bigcup_i \Gamma_{x_i}$ where Γ_{x_i} is the graph of $x_i \colon \mathrm{Spec}(R) \to X$.

Definition 2.14 (*Beilinson-Drinfeld Grassmannian*)**.** Define a presheaf

$$\mathcal{G}r_{G,X^I}(R) := \left\{ (x, \mathcal{E}, \beta) \middle| \begin{array}{c} x \in X^I(R),\ \mathcal{E} \text{ is a } G\text{−torsor } on\ X^I_R \\ \text{with trivialization } \beta \text{ on } X^I_R - \Gamma_x \end{array} \right\}$$

with natural projection $f = f^I \colon \mathcal{G}r_{G,X^I} \to X^I$.

This is also an ind-projective ind-scheme over X^I which can be proved similar as before. When $\#I = 1$, the fiber of q^I at any closed point of X is identified with the affine Grassmannian by Definition-Theorem 2.1. In fact, $\mathcal{G}r_{G,X}$ is a product of X and $\mathcal{G}r_G$ twisted by automorphisms of a disk, so if the $Aut(D)$-torsor $\hat{X} := \{(x,\phi)|x \in X$ with identification $\phi\colon \mathrm{Spec}(\mathcal{O}_{X,x})\tilde{\to}D\} \to X$ obtains a section, we can identify $\mathcal{G}r_{G,X}$ with $\mathcal{G}r_G \times X$. This happens when $X = \mathbb{A}^1$ where we can construct the section by hand, which is enough for local arguments.

Things get more interesting when $\#I = 2$. For closed point x of X, the fiber at x is isomorphic to $\mathcal{G}r_G$ if x lies on the diagonal, while it is isomorphic to $\mathcal{G}r_G \times \mathcal{G}r_G$ when x is ouside. Globally, we have

$$\mathcal{G}r_{G,X^2}|_\Delta \cong \mathcal{G}r_{G,X}, \mathcal{G}r_{G,X^2}|_{X^2-\Delta} \cong \mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}|_{X^2-\Delta}$$

This phenomenon continues when I gets larger, and is summarized with name factorization properties.

3. Structures in the Satake category

Recall that we have defined the Satake category $Sat_{G,\Bbbk} := Perv_{L^+G}(\mathcal{G}r_G, \Bbbk)$ in Theorem 2.3. We will omit the $\Bbbk$ in the notation as we are usually fixing such a $\Bbbk$. We state the following theorem to lessen possible concerns about the Satake category. It says once we get a perverse sheaf with respect to the stratification $\mathcal{S}$, we can automatically equip it with a L^+G-action. This will make $Sat_{G,\Bbbk}$ more flexible.

Theorem 3.1. *The forgetful functor induces a natural equivalence*

$$Perv_{L^+G}(\mathcal{G}r_G, \Bbbk)\tilde{\to}Perv_{\mathcal{S}}(\mathcal{G}r_G, \Bbbk).$$

The proof of the above theorem is direct when $\Bbbk$ is a field of characteristic zero because of semisimplicity, which will be discussed in Section 3.3. The general case uses weight spaces introduced in Section 3.1 to induct on the number of orbits involved by taking away a minimal (closed) one each time[7]. To get around semisimplicity, we have to utilize the ramification data to recover global information. For more details, see [31].

3.1. Weight spaces.

In this section, we use the semi-infinite orbits to construct the weight structure in the Satake category. It turns out that taking cohomologies on S_μ is the projection to the weight subspace corresponding to μ, for which we need the following striking result.

Theorem 3.2 (Braden's hyperbolic localization theorem)**.** *There is a canonical isomorphism*

$$H^k(T_\mu, j_\mu^! \mathscr{A}) \cong H_c^k(S_\mu, i_\mu^* \mathscr{A}) \tag{3.3}$$

for any $k \in \mathbb{Z}, \mathscr{A} \in D_{L^+G}(\mathcal{G}r_G), i_\mu\colon S_\mu \hookrightarrow \mathcal{G}r_G, j_\mu\colon T_\mu \hookrightarrow \mathcal{G}r_G$.

Remark 3.4. Braden's theorem actually works for general varieties with a $\mathbb{G}_m$-action, still concerning $\mathbb{G}_m$-fixed points, attractors and repellers. Braden himself dealt with normal varieties in [8] and the theory for general schemes is developed in [10].

Corollary 3.5. *(Purity) If $\mathscr{A}$ is a perverse sheaf, then both sides of (3.3) vanishes unless $k = \langle 2\rho, \mu\rangle$.*

[7]Note this is different from most algebraic geometric proofs where we usually take away a dense open subset each time.

Proof. Suppose $\mathscr{A}$ is a perverse sheaf supported on $\overline{\mathcal{G}r_\lambda}$. Then $\mathscr{A} \in D_c^{\leq -\langle 2\rho,\lambda\rangle}(\mathcal{G}r_G)$ by Proposition 2.5, hence we have $R\Gamma_c(i^*\mathscr{A}) \in D^{\langle 2\rho,\mu\rangle}$ from dimension calculation in Proposition 2.12. This gives the vanishing on one side, while the vanishing on the other side is provided by T_μ by duality. □

Denote $F_\mu(-) = H^{\langle 2\rho,\mu\rangle}(T_\mu, i^!-) = H_c^{\langle 2\rho,\mu\rangle}(S_\mu, i^*-)$, $F(-) = H^*(-)$ for short.

Corollary 3.6 (Weight decomposition)**.** *We have a decomposition*

$$F = \bigoplus F_\mu \colon Sat_G \to Mod_{\Bbbk}.$$

Proof. In fact, we will show $H^k(-) = H_c^k(-) = \bigoplus_{\langle 2\rho,\mu\rangle = k} F_\mu$. We may assume $\mathscr{A} \in Sat_G$ is supported on some $\overline{\mathcal{G}r_\lambda}$, so we have $\mu - \lambda \in Q^\vee$ and $\langle 2\rho, \mu - \lambda\rangle \in 2\mathbb{Z}$ if $\mathscr{A}|_{S_\mu} \neq 0$. Write $S_k = \bigcup_{\langle 2\rho,\mu\rangle = k} S_\mu = \bigsqcup_{\langle 2\rho,\mu\rangle = k} S_\mu$, then we have $\overline{S_k} = \bigsqcup_{l\leq k} S_l$, $\bigoplus_{\langle 2\rho,\mu\rangle = k} F_\mu = H^*(S_k, -)$ and

$$H^l(\overline{S_k}, \mathscr{A}|_{\overline{S_k}}) = \begin{cases} \bigoplus_{\langle 2\rho,\mu\rangle = l} F_\mu(\mathscr{A}), & \text{if } l \leq k \\ 0, & \text{if } l > k \end{cases} \tag{3.7}$$

This is seen by induction on k. For $k \ll 0$, both sides are zero since everything is away from the support of $\mathscr{A}$. If (3.7) holds for k-1, then consider $\overline{S_{k-1}} \overset{i}{\hookrightarrow} \overline{S_k} \overset{j}{\hookleftarrow} S_k$ and apply $R\Gamma_c$ to the extinguished triangle $j_!j^*\mathscr{A}|_{S_k} \to \mathscr{A}|_{\overline{S_k}} \to i_*i^*\mathscr{A}|_{\overline{S_{k-1}}}$. Then (3.7) holds for k by looking at the long exact sequence and using the parity of $\langle 2\rho, \mu\rangle$. Then take k sufficiently large so that $\overline{\mathcal{G}r_\lambda} \subset \overline{S_k}$, so we get the decomposition of cohomology. □

Corollary 3.8 (Forgetful functor)**.** *The functor F is exact and faithful.*

Proof. F is exact since each factor F_μ is. To prove faithfulness, the restriction of some $\mathscr{A}$ to an orbit $\mathcal{G}r_\lambda$ open in support of $\mathscr{A}$ is a local system up to shifts. So the restriction of $\mathscr{A}$ to $\mathcal{G}r_\lambda \cap S_{w_0\cdot\lambda} = \{[t^{w_0\cdot\lambda}]\}$[8] is nonzero, hence has nonzero cohomology. □

Combining Corollary 3.6 and Corollary 3.8, we get the desired weight structure in the Satake category. They correspond to the following operation in a representation category: forgetting the group action, and then splitting into weight spaces.

Other than constructing the weight structure as expected, faithfulness is very powerful for its own sake. Here are some applications.

First, we can get a criterion for perversity on the affine Grassmannian.

Corollary 3.9. *The converse of Corollary 3.5 holds: purity in cohomology with the desired degree for any μ implies perversity.*

In fact, this follows from the formula $F_\mu({}^p\mathscr{H}^n(\mathscr{A})) = H_c^{\langle 2\rho,\mu\rangle+n}(S_\mu, \mathscr{A})$.

Proof. The formula is proved by induction on cohomological length $:= \#\{n | {}^p\mathscr{H}^n(\mathscr{A}) \neq 0\}$. The length 1 case is exactly Corollary 3.5. If the formula has been proved for complexes with smaller length, apply $R\Gamma_c(S_\mu, -)$ to the distinguished triangle ${}^p\tau^{\leq n}\mathscr{A} \to \mathscr{A} \to {}^p\tau^{>n}\mathscr{A} \to$ after choosing any n making this truncation nontrivial. Take the corresponding long exact sequence and we are done since all nonzero terms are separated. □

[8] The intersection is 0-dimensional by Proposition 2.12 with a T-action. Connectedness of T implies the T-action is trivial, but there is only one T-fiexed point in each semi-infinite orbit

Weight spaces are also useful to analyze standard and costandard objects.

Definition 3.10 (Standard and costandard objects)**.** Denote $j_\lambda\colon \mathcal{G}r_\lambda \to \mathcal{G}r_G$ the embedding. Define

$$\text{standard objects: } \mathcal{J}_!(\lambda,\Bbbk) := {}^p\mathscr{H}^0(j_{\lambda !}\underline{\Bbbk}[\dim \mathcal{G}r_\lambda])$$
$$\text{costandard objects:} \mathcal{J}_*(\lambda,\Bbbk) := {}^p\mathscr{H}^0(j_{\lambda *}\underline{\Bbbk}[\dim \mathcal{G}r_\lambda])$$
$$\text{IC sheaves: } \mathcal{J}_{!*}(\lambda,\Bbbk) := \mathrm{Im}(\mathcal{J}_!(\lambda,\Bbbk) \to \mathcal{J}_*(\lambda,\Bbbk)).$$

The reason why weight spaces are useful for analyzing these objects is because we have the following explicit calculation.

Proposition 3.11. *$F_\mu(\mathcal{J}_!(\lambda,\Bbbk))$(resp. $F_\mu(\mathcal{J}_*(\lambda,\Bbbk))$) is a free $\Bbbk$-module with basis parametrized by connected components*[9] *of $\mathcal{G}r_\lambda \cap S_\mu$(resp. T_μ).*

Proof. By Corollary 3.9, we have

$$\begin{aligned} F_\mu(\mathcal{J}_!(\lambda,\Bbbk)) &= F_\mu({}^p\mathscr{H}^0(j_{\lambda !}\underline{\Bbbk}[\dim \mathcal{G}r_\lambda])) && \text{by Definition 3.10} \\ &= H_c^{\langle 2\rho,\mu\rangle}(S_\mu, j_{\lambda !}\underline{\Bbbk}[\dim \mathcal{G}r_\lambda])) && \text{by Corollary 3.9} \\ &= H_c^{\langle 2\rho,\lambda+\mu\rangle}(\mathcal{G}r_\lambda \cap S_\mu, \underline{\Bbbk}) && \text{by proper base change} \end{aligned}$$

This is a free module with basis parametrized by the connected components of $\mathcal{G}r_\lambda \cap S_\mu$, by the dimension calculation in Proposition 2.12. $F_\mu(\mathcal{J}_*(\lambda,\Bbbk))$ can be dealt with similarly. □

Corollary 3.12.
(1) $\mathcal{J}_!(\lambda,\Bbbk) = \Bbbk \otimes^L_{\mathbb{Z}} \mathcal{J}_!(\lambda,\mathbb{Z})$, $\mathcal{J}_*(\lambda,\mathbb{Z}) = \Bbbk \otimes^L_{\mathbb{Z}} \mathcal{J}_*(\lambda,\Bbbk)$
(2) $\mathcal{J}_!(\lambda,\mathbb{Z}) = \mathcal{J}_{!*}(\lambda,\mathbb{Z})$

Proof. (1) There is a natural map $\mathcal{J}_!(\lambda,\Bbbk) \to \Bbbk \otimes^L_{\mathbb{Z}} \mathcal{J}_!(\lambda,\Bbbk)$ by restricting to $\mathcal{G}r_\lambda$ and adjunction. This is an isomorphism after applying $R\Gamma_c(S_\mu,-)$ for any μ, so $\Bbbk \otimes^L_{\mathbb{Z}} \mathcal{J}_!(\lambda,\Bbbk)$ is a perverse sheaf by Corollary 3.9 and $\mathcal{J}_!(\lambda,\Bbbk) \to \Bbbk \otimes^L_{\mathbb{Z}} \mathcal{J}_!(\lambda,\Bbbk)$ is an isomorphism by Corollary 3.8. Similarly we have $\mathcal{J}_*(\lambda,\mathbb{Z}) = \Bbbk \otimes^L_{\mathbb{Z}} \mathcal{J}_*(\lambda,\Bbbk)$.

(2) By Corollary 3.8, we need to show $F_\mu(\mathcal{J}_!(\lambda,\Bbbk)) \to F_\mu(\mathcal{J}_*(\lambda,\Bbbk))$ is injective for any μ. It suffices to show injectivity after applying $\mathbb{Q} \otimes_{\mathbb{Z}} -$, since $\mathbb{Z}$ is PID. This follows from the fact $\mathcal{J}_!(\lambda,\mathbb{Q}) \cong \mathcal{J}_{!*}(\lambda,\mathbb{Q}) \cong \mathcal{J}_*(\lambda,\mathbb{Q})$, which is the magic of semisimplicity discussed in Section 3.3. □

3.2. **Convolution product.** In this section, we introduce the monoidal structure on Satake category and prove some related properties.

Definition 3.13 (Convolution product)**.** We start from the convolution diagram.

$$\mathcal{G}r_G \times \mathcal{G}r_G \xleftarrow{p} LG \times \mathcal{G}r_G \xrightarrow{q} \mathcal{G}r_G \widetilde{\times} \mathcal{G}r_G := LG \overset{L^+G}{\times} \mathcal{G}r_G \xrightarrow{m} \mathcal{G}r_G$$

where $LG \overset{L^+G}{\times} \mathcal{G}r_G = LG \times \mathcal{G}r_G / L^+G$ is the quotient through the embedding $L^+G \hookrightarrow L^+G^{op} \times L^+G$, which is the inverse map on the first factor and the identity map on the second. This induces the convolution product as the composition:

$$(\mathscr{A}_1,\mathscr{A}_2) \mapsto p^*({}^p\mathscr{H}^0(\mathscr{A}_1 \boxtimes^L_{\Bbbk} \mathscr{A}_2)) \cong q^*(\mathscr{A}_1 \tilde{\boxtimes} \mathscr{A}_2) \leftarrow \mathscr{A}_1 \tilde{\boxtimes} \mathscr{A}_2 \mapsto m_*(\mathscr{A}_1 \tilde{\boxtimes} \mathscr{A}_2) =: \mathscr{A}_1 \star \mathscr{A}_2$$

where we use the equivalence $Perv_{L^+G\times L^+G}(LG \times \mathcal{G}r_G) \cong Perv_{L^+G}(\mathcal{G}r_G \widetilde{\times} \mathcal{G}r_G)$.

[9] These irreducible components are called Mirković–Vilonen cycles.

In fact, convolution exists globally, so we can apply some powerful sheaf machinery to this setting. We first construct the twisted product we need. Define

$$\mathcal{G}r_X \widetilde{\times} \mathcal{G}r_X(R) = \left\{ (x_1, x_2, \mathcal{E}_1, \mathcal{E}, \beta_1, \delta) \middle| \begin{array}{c} x_1, x_2 \in X(R), \mathcal{E}_1, \mathcal{E}\ G-\text{torsors on } X_R \\ \beta_1 \text{ trivialization of } \mathcal{E}_1 \text{ on } X_R - \Gamma_{x_1} \\ \delta \text{ isomorphism } \mathcal{E}_1|_{X_R-\Gamma_{x_2}} \cong \mathcal{E}|_{X_R-\Gamma_{x_2}} \end{array} \right\}$$

$$\widetilde{\mathcal{G}r_x \times \mathcal{G}r_X} = \left\{ (x_1, x_2, \mathcal{E}_1, \mathcal{E}_2, \beta_1, \beta_2, \gamma) \middle| \begin{array}{c} x_1, x_2 \in X(R), \mathcal{E}_1, \mathcal{E}_2\ G-\text{torsors on } X_R \\ \beta_i \text{ trivialization of } \mathcal{E}_i \text{ on } X_R - \Gamma_{x_1} \\ \gamma \text{ isomorphism } \mathcal{E}_1|_{X_R-\Gamma_{x_2}} \cong \mathcal{E}|_{X_R-\Gamma_{x_2}} \end{array} \right\}$$

We are ready to define the convolution diagram.

$$\mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X} \xleftarrow{p} \widetilde{\mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}} \xrightarrow{q} \mathcal{G}r_{G,X} \widetilde{\times} \mathcal{G}r_{G,X} \xrightarrow{m} \mathcal{G}r_{G,X^2}$$

In the diagram, the first arrow p is defined by forgetting γ. The second arrow q gets $\mathcal{E}$ in the target by gluing $\mathcal{E}_1$ and $\mathcal{E}_2$ along the punctured formal disk at x_2. The third arrow forgets $\mathcal{E}_1$ and defines the trivialization as the composition of β_1 and δ. We then obtain the convolution product of sheaves as before [10].

$$(\mathscr{A}_1, \mathscr{A}_2) \mapsto p^*({}^p\mathscr{H}^0(\mathscr{A}_1 \boxtimes^L_{\Bbbk} \mathscr{A}_2)) \cong q^*(\mathscr{A}_1 \widetilde{\boxtimes} \mathscr{A}_2) \leftarrow \mathscr{A}_1 \widetilde{\boxtimes} \mathscr{A}_2 \mapsto m_*(\mathscr{A}_1 \widetilde{\boxtimes} \mathscr{A}_2) =: \mathscr{A}_1 \star_X \mathscr{A}_2$$

Proposition 3.14.

(1) If $\mathscr{A}_1, \mathscr{A}_2 \in Sat_G$, then $\mathscr{A}_1 \star \mathscr{A}_2 \in Sat_G$.
(2) The convolution product satisfies associativity and commutativity constraints. That is, convolution gives rise to a symmetric monoidal structure on Sat_G.
(3) (Compatibility) The forgetful functor $F\colon Sat_G \to X^*(T)$-graded $\Bbbk$-modules is monoidal.

Proof. (1) This is a consequence of the semismallness of m. In order to show m is small, we first construct the stratification $\widetilde{\mathcal{G}r}_{\lambda,\mu} = q(p^{-1}(\mathcal{G}r_\lambda \times \mathcal{G}r_\mu))$ on $\mathcal{G}r_G \widetilde{\times} \mathcal{G}r_G$. That is, $\widetilde{\mathcal{G}r}_{\lambda,\mu}$ is the locus where trivialization β_1 is in position λ and δ is in position μ in the local moduli intepretation[11]. A direct calculation as in Definition-Theorem 2.1 shows $\dim \widetilde{\mathcal{G}r}_{\lambda,\mu} = \langle 2\rho, \lambda + \mu \rangle$. Given $\mathscr{A}_1, \mathscr{A}_2 \in Sat_G$, $\mathscr{A}_1 \widetilde{\boxtimes} \mathscr{A}_2$ is a perverse sheaf with respect to the stratification above.

Now we look at the morphism $m\colon \mathcal{G}r_G \widetilde{\times} \mathcal{G}r_G \to \mathcal{G}r_G$. This is L^+G-equivariant with actions on the left, hence m is a locally trivial fibration on each L^+G-orbit on the target, and $\mathscr{A}_1 \star \mathscr{A}_2$ is constructible with respect to the stratification $\{\mathcal{G}r_\lambda\}$. To verify the semismallness assumption, it suffices to show

$$2\dim(\widetilde{\mathcal{G}r}_{\lambda,\mu} \cap m^{-1}([t^\nu])) \leq \dim \widetilde{\mathcal{G}r}_{\lambda,\mu} - \dim(L^+G \cdot [t^\nu]) = \langle 2\rho, \lambda + \mu + \nu \rangle$$

for any ν negative dominant (i.e. -ν is dominant) by the transitive L^+G-action on the target. Consider the isomorphism $(pr_1, m)\colon \mathcal{G}r_G \widetilde{\times} \mathcal{G}r_G \xrightarrow{\sim} \mathcal{G}r_G \times \mathcal{G}r_G$ to make the situation untwisted. Under this isomorphism, $m^{-1}([t^\nu])$ is identified with $\mathcal{G}r_G \times \{[t^\nu]\}$. We are going to use the dimension estimate Corollary 2.13. Note that the isomorphism (pr_1, m) is T-equivariant and $[t^n u]$ is fixed by T. Carrying

[10]Here $\mathscr{A}_1, \mathscr{A}_2$ are supposed to be perverse "relative to X" so that it behaves well on fibers. This is always satisfied in the case we need and the idea can be made more precise using the notion of universal locally acyclicity.

[11]We didn't actually define the local version, but we can imitate the construction of global case or simply use the identifications in the global case.

through the isomorphism, the T-fixed points $\overline{\widetilde{\mathcal{G}r}_{\lambda,\mu}}$ are of the form $(t^\alpha, [t^\beta])$ where $W\cdot\alpha \le \lambda, W\cdot\beta \le \mu$. So, by Corollary 2.13, we have

$$\begin{aligned}\dim(\widetilde{\mathcal{G}r}_{\lambda,\mu} \cap m^{-1}([t^\nu])) &\le \underset{\alpha+\beta=\nu}{max} \langle \rho, \lambda+\alpha\rangle \\ &\le \langle \rho, \lambda+\alpha+\beta+\mu\rangle \\ &= \langle \rho, \lambda+\mu+\nu\rangle\end{aligned}$$

as desired.

(2) Associativity can be proved by constructing a three-fold convolution product or immitating the proof of commutativity, so we leave it to the reader. The key to the proof of commutativity is enlarging the formal disk to a global smooth curve, such that sheaf techniques apply.

Let us say $X = \mathbb{A}^1$ to make things simpler, we can identify $\mathcal{G}r_{G,X} \cong \mathcal{G}r_G \times X$. Denote the projection $\mathcal{G}r_{G,X} \to \mathcal{G}r_G$ by τ and $\tau^\circ = \tau^*[1] = \tau^![-1]$. Diagonal embedding $\Delta\colon X \hookrightarrow X^2$ has complement U. Consider the following convolution diagram of Cartesian squares.

$$\begin{tikzcd}
\mathcal{G}r_G \times \mathcal{G}r_G \times X \arrow[d, hook, "i"] & LG \times \mathcal{G}r_G \times X \arrow[l, "p"'] \arrow[r, "q"] \arrow[d, hook, "i"] & \mathcal{G}r_G \widetilde{\times} \mathcal{G}r_G \times X \arrow[r, "m"] \arrow[d, hook, "i"] & \mathcal{G}r_G \times X \arrow[r, "f"] \arrow[d, hook, "i"] & X \arrow[d, hook, "\Delta"] \\
\mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X} & \widetilde{\mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}} \arrow[l, "p"'] \arrow[r, "q"] & \mathcal{G}r_{G,X} \widetilde{\times} \mathcal{G}r_{G,X} \arrow[r, "m"] & \mathcal{G}r_{G,X^2} \arrow[r, "f"] & X^2 \\
\mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}|_U \arrow[u, hook, "j"] & \mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}|_U \times L^+G \arrow[l, "pr"'] \arrow[r, "act"] \arrow[u, hook, "j"] & \mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}|_U \arrow[r, equal] \arrow[u, hook, "j"] & \mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X}|_U \arrow[r, "f"] \arrow[u, hook, "j"] & U \arrow[u, hook, "j"]
\end{tikzcd}$$

Lemma 3.15 (Fusion). *For $\mathscr{A}_1, \mathscr{A}_2 \in Sat_G$,*

(i) $i^*[-1](\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2) \cong \tau^\circ(\mathscr{A}_1 \star \mathscr{A}_2) \cong i^![1](\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2)$

(ii) $\tau^\circ\mathscr{A}_1 \boxtimes \tau^\circ\mathscr{A}_2|_U \cong \tau^\circ\mathscr{A}_1 \widetilde{\boxtimes} \tau^\circ\mathscr{A}_2|_U \cong \tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2|_U$

(iii) $j_{!*}(\tau^\circ\mathscr{A}_1 \boxtimes \tau^\circ\mathscr{A}_2|_U) \cong \tau^\circ(\mathscr{A}_1 \star \mathscr{A}_2)$

Admitting Lemma 3.15, the $\mathbb{Z}/2$-action on $X \times X$ permuting two factors makes $\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2|_U$ an equivariant sheaf on U by permuting two factors. This action extends to the whole $X \times X$ through intermediate extension and restricts to the desired commutativity constraint on the diagonal.

We sketch the proofs of Lemma 3.15[12]. (i) is checked on each block of the diagram above. (ii) follows from the explicit description over the locus U as in the diagram above and the isomorphism is L^+G-equivariance. For (iii), it suffices to check $i^*\tau^\circ(\mathscr{A}_1 \star \mathscr{A}_2) \in {}^pD^{\le -1}$ and $i^!\tau^\circ(\mathscr{A}_1 \star \mathscr{A}_2) \in {}^pD^{\ge 1}$ by (ii). This is verified by the first claim because τ° preserves perversity.

(3) To catch the essence, we assume $\Bbbk$ is a field at the moment. Keeping $X = \mathbb{A}^1$, we consider the global convolution diagram and the direct image $f_*(\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2)$ under the projection $f\colon \mathcal{G}r_{X^2} \to X^2$. Observe that its stalk is isomorphic to $F(\mathscr{A}_1 \star \mathscr{A}_2)$ at point on the diagonal, and to

$$H^*(\mathscr{A}_1 \boxtimes \mathscr{A}_2) = F(\mathscr{A}_1) \otimes F(\mathscr{A}_2) \tag{3.16}$$

at point inside U, up to some shift. So, it suffices to show $f_*(\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2)$ has locally constant cohomology. In fact, we will show that cohomology sheaves of $(f \circ m)_*\mathscr{B}$ are all local systems,for any $\mathscr{B}$ constructible with respect to the

[12]In fact, the first two claims follow from the formalism of universal local acyclicity.

stratification $\{\widetilde{\mathcal{G}r}_{\lambda,\mu}\}$. To see this, we factor $f \circ m$ as the composition

$$\mathcal{G}r_{G,X}\widetilde{\times}\mathcal{G}r_{G,X} \overset{(pr_1,m)}{\longrightarrow} \mathcal{G}r_{G,X} \times \mathcal{G}r_{G,X} \overset{pr_1}{\to} \mathcal{G}r_{G,X} \times X \to X \times X,$$

where the first arrow is isomorphism and the later two are locally trivial fibrations compatible with the stratification. So our result follows[13] from the factorization, dévissage and the fact that local systems is a thick abelian subcategory of ($\Bbbk$-)sheaves.

To add weights into the picture, we should look at the T-action on $\mathcal{G}r_{G,X}, \mathcal{G}r_{G,X^2}$ and $\mathcal{G}r_{G,X}\widetilde{\times}\mathcal{G}r_{G,X}$. The T-action on $\mathcal{G}r_{G,X}$ is horizontal to the base X, so we simply add "$X\times$" to $[t^\mu], S_\mu, T_\mu$. For $\mathcal{G}r_{G,X}\widetilde{\times}\mathcal{G}r_{G,X}$, we use the T-equivariant isomorphism (pr_1, m) again and get T-fixed points $(t^\mu, [t^\nu])$ with attractor $\widetilde{S}_{\mu,\nu}$ and repeller $\widetilde{T}_{\mu,\nu}$. We can see the T-action on $\mathcal{G}r_{G,X^2}$ via $m\colon \mathcal{G}r_{G,X}\widetilde{\times}\mathcal{G}r_{G,X} \to \mathcal{G}r_{G,X^2}$ and the diagram above. This is an isomorphism on U, and maps $(t^\mu, [t^\nu])$ to $[t^{\mu+\nu}]$, hence $\widetilde{S}_{\mu,\nu}$ to $S_{\mu+\nu}$ on the diagonal. So the attractor stratificaiton on $\mathcal{G}r_{G,X^2}$ is given by $S_\mu = \bigcup_{\mu_1+\mu_2=\mu} m(\widetilde{S}_{\mu_1,\mu_2}), \mu \in X_*(T)$. We still have Braden's hyperbolic localization Theorem 3.2 and Corollary 3.6 in this relative setting according to [10]. Denote $s_\mu\colon S_\mu \to \mathcal{G}r_{G,X^2}$ the natural inclusion, then the cohomology sheaves of $(f\circ s_\mu)_! s_\mu^*(\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2)$ are local systems as a direct summand of $f_*(\tau^\circ\mathscr{A}_1 \star_X \tau^\circ\mathscr{A}_2)$, and we get the desired compatibility $F_\mu(\mathscr{A}_1 \star \mathscr{A}_2) = \bigoplus_{\mu_1+\mu_2=\mu} F_{\mu_1}(\mathscr{A}_1) \otimes F_{\mu_2}(\mathscr{A}_2)$ by taking the stalk on and outside the diagonal.

In the general case where $\Bbbk$ is not necessarily a field, the only difference is that $\mathscr{A}_1 \boxtimes \mathscr{A}_2$ need to be replaced by ${}^p\mathscr{H}^0(\mathscr{A}_1 \boxtimes^L_\Bbbk \mathscr{A}_2)$ to work within the category of perverse sheaves, so Künneth formula doesn't directly apply. We need to prove a Künneth formula for ${}^p\mathscr{H}^0(\mathscr{A}_1 \boxtimes^L_\Bbbk \mathscr{A}_2)$. In the case $F(\mathscr{A}_1)$ is flat, there is nothing to worry about by the following lemma.

Lemma 3.17. *If $F(\mathscr{A}_1)$ is flat, then $\mathscr{A}_1 \boxtimes^L_\Bbbk \mathscr{A}_2$ is perverse.*

The lemma directly follows from Corollary 3.9. For general $\mathscr{A}_1$, we need the following fact from Section 4.1: there exists a exact sequence $\mathscr{A}' \to \mathscr{A} \to \mathscr{A}_1 \to 0$ with $F(\mathscr{A}), F(\mathscr{A}_1)$ free. Apply the comparison morphism in Künneth formula to the exact sequence above, then the comparison morphism for $\mathscr{A}_1$ is forced to be an isomorphism because it holds for $\mathscr{A}$ and $\mathscr{A}'$. ☐

3.3. **Semisimplicity in characteristic 0.** In this section, we assume $\Bbbk$ is a field of characteristic 0 so that BBDG decomposition theorem [3] holds.

Theorem 3.18. *The Satake category $Perv_{\mathscr{S}}(\mathcal{G}r_G, \Bbbk)$ is semisimple, i.e. every object is a direct sum of simple objects.*

This follows from the parity of dimension on each connected component and the following claim. Denote IC_λ the intersection cohomology of $\mathcal{G}r_\lambda$ for any $\lambda \in X_*(T)$

Theorem 3.19 (Parity vanishing)**.** $\mathscr{H}^n(IC_\lambda) \neq 0 \Rightarrow n \equiv dim\, \mathcal{G}r_\lambda (mod\ 2)$.

Proof. We first prove Theorem 3.18 with the help of Theorem 3.19.

Since every object is of finite length, we need to show

$$Ext^1(IC_\lambda, IC_\mu) = Hom(IC_\lambda, IC_\mu[1]) = 0, \forall \lambda, \mu \in X_*(T).$$

[13]Reduce to the case where $\mathscr{B}$ is a local system and note that the pushforward of a local system along a smooth projective morphism is still a local system.

We may assume $\overline{\mathcal{G}r_\mu} \nsubseteq \partial\mathcal{G}r_\lambda$ by duality. Consider open immersion $j\colon \mathcal{G}r_\lambda \hookrightarrow \overline{\mathcal{G}r_\lambda}$ and complement $i\colon \overline{\mathcal{G}r_\lambda}\backslash\mathcal{G}r_\lambda \hookrightarrow \overline{\mathcal{G}r_\lambda}$, together with distinguished triangle $j_!j^*IC_\lambda \to IC_\lambda \to i_*i^*IC_\lambda \to$. Applying $Hom(-, IC_\mu[1])$, it suffices to show $Hom(i_*i^*IC_\lambda, IC_\mu[1]) = 0 = Hom(j_!j^*IC_\lambda, IC_\mu[1])$. For the first equality, we notice $i^*IC_\lambda \in {}^pD^{\leq -1}$, so $Hom(i_*i^*IC_\lambda, IC_\mu[1]) = Hom(i_*{}^p\mathscr{H}^{-1}i^*IC_\lambda, IC_\mu) = 0$, since IC_μ has no perverse subsheaf supported on $\overline{\mathcal{G}r_\mu}\backslash\mathcal{G}r_\mu$. For the second one, let $i_\lambda\colon \overline{\mathcal{G}r_\lambda} \hookrightarrow \mathcal{G}r_G$ be the inclusion, we have

$$Hom(j_!j^*IC_\lambda, IC_\mu[1]) = Hom(\Bbbk, j^*i_\lambda^*IC_\mu[1-\dim\mathcal{G}r_\lambda]) = \mathscr{H}^{1-\dim\mathcal{G}r_\lambda}((i_\lambda\circ j)^*IC_\mu)$$

since $(i_\lambda \circ j)^*IC_\mu) \in D^{\leq 1-\dim\mathcal{G}r_\lambda}$ by the definition of perversity. This cohomology vanishes by parity vanishing as λ and μ have to be on the same connected component.

We begin the proof of Theorem 3.19. We first prove a similar statement on the affine flag variety in Remark 2.2, which is the affine analog of parity vanishing for the usual flag manifold G/B. The original proof carries with slight modification to fit in the setting. Next, we use the quotient $\mathcal{F}l_G \to \mathcal{G}r_G$ and show why the previous case constrols the situation on $\mathcal{G}r_G$. Recall the orbits of left Iw-action on $\mathcal{F}l_G$ are parametrized by the extended affine Weyl group by Theorem 2.4, say $I_w = \mathrm{Iw}w\mathrm{Iw}$.

Claim. $\mathscr{H}^n(I_w/\mathrm{Iw}, \Bbbk)) \neq 0 \Rightarrow n \equiv l(w)(mod\ 2)$.

Proof of the claim: For $w \in \widetilde{W}_{aff}$, we have a reduced representation $w = s_1s_2...s_kw_0$ where s_i are reflections and w_0 is of length zero. Denote this sequence by s. This gives us the Bott-Samelson (or Demazure) resolution [14]

$$\pi_s : I_{s_1} \times^{\mathrm{Iw}} I_{s_2} \times^{\mathrm{Iw}} ... \times^{\mathrm{Iw}} I_{s_k} \times^{\mathrm{Iw}} I_{w_0}/\mathrm{Iw} \to \overline{I_w/\mathrm{Iw}}\ ,$$

whose fibers are paved by affine spaces[15][15]. As a consequence, the cohomology of fibers concentrate in even dimension and so is $\pi_{s*}\Bbbk$. The claim follows since $IC(I_w/\mathrm{Iw}, \Bbbk)$ is a direct summand of $\pi_{s*}\Bbbk[l(w)]$ by deomposition theorem [3].

Moving back to the affine Grassmannian, consider the quotient morphism $\pi\colon \mathcal{F}l_G \to \mathcal{G}r_G$, which is a locally trivial G/B-bundle and compatible with the stratification on both spaces given by left Iw-action orbits[16]. IC_λ coincides with the intersection cohomology $IC_{w\cdot\lambda}$ of the open dense Iw-orbit $\mathrm{Iw}\cdot[t^{w\cdot\lambda}]$ in $\mathcal{G}r_\lambda$. Similarly, the pullback of $IC_{w\cdot\lambda}$ along π coincides with the intersection cohomology $IC_{\widetilde{w}}$ of an open dense Iw-orbit $\mathrm{Iw}\cdot[\widetilde{w}]$ in $\pi^{-1}(\mathrm{Iw}\cdot[t^{w\cdot\lambda}])$, up to a shift by $\dim G/B{=}l(w_0)$. As a result, parity vanishing is verified since π^* is exact and faithful. $\square$

4. Reconstruction and Identification

4.1. **Reconstruction Step.** We've constructed a monoidal exact faithful functor $F\colon Sat_{G,\Bbbk} \to Mod_\Bbbk$. Now we want to make use these data to realize the Satake category as the category of comodules of a Hopf algebra over $\Bbbk$. Our strategy is to locally recognize $Sat_{G,\Bbbk}$ as a module category first, then take the dual.

We start with two abstract observations.

Theorem 4.1.

(1) *Let R be a $\Bbbk$-algebra.Then R is identified with the opposite of the endomorphism ring of the forgetful functor $Mod_R \to Mod_\Bbbk$.*

[14] It is a resolution in the following sense: the source is smooth , while π_s is proper and birational.

[15] This means the space is disjoint union of locally closed affine spaces.

[16] Notice this stratification on $\mathcal{G}r_G$ is strictly finer than $\mathscr{S}$.

(2) *Let P be an object of abelian category $\mathscr{C}$. Then $Hom_{\mathscr{C}}(P,-)\colon \mathscr{C} \to Mod_{End_{\mathscr{C}}(P)^{op}}$ is an equivalence if and only if P is a compact projective generator.*

A category $\mathscr{C}$ can be equivalent to the module category of many different rings by Morita equivalence, part 1 says that the preferred ring is determined by choice of forgetful functor.

4.1.1. *Tannakian Reconstruction.* We first deal with Tannakian reconstruction for $\Bbbk$ being a field. Let $\mathscr{C}$ be an abelian category with symmetric monoidal structure and $F\colon \mathscr{C} \to Mod_{\Bbbk}$ be an exact faithful monoidal functor $\Bbbk$-modules. Since we are restricted to finite dimensional vector spaces, we cannot expect to find a generator of the whole category. Instead, for any $X \in \mathscr{C}$, define $\langle X\rangle$ to be the full subcategory of $\mathscr{C}$ with objects being subquotients of $X^{\oplus n}$. We will construct $P_X \in \langle X\rangle$ representing F, then $F\colon \langle X\rangle \to Mod_{\Bbbk}$ is identified with the forgetful functor of a module category by Theorem 4.1.

We assume $\mathscr{C} = \langle X\rangle$ in this paragraph. Consider functor category $Hom_{Cat}(\mathscr{C}, Mod_{\Bbbk})$, with typical objects constant functor V for $V \in Mod_{\Bbbk}$ and $Y \in \mathscr{C}$ via the Yoneda lemma. Define $Hom(V,Y)(Z) = Hom_{Mod_{\Bbbk}}(V, Hom_{\mathscr{C}}(Y,Z)), Z \in \mathscr{C}$, which is representable by choosing a basis. Similarly we can define a tensor product: define

$$P_X = \bigcap (Hom(F(X),X) \cap ker(Hom(F(X^{\oplus n}), X^{\oplus n}) \to Hom(F(Y), X^{\oplus n}/Y))),$$

where we intersect over $\{n \geq 0, Y \subset X^{\oplus n}\}$ and Hom(F(X),X) is diagonally embeded in $Hom(F(X^{\oplus n}), X^{\oplus n})$. We apply F and calculate:

$$\begin{aligned} A_X :&= F(P_X) \\ &= \bigcap_{n\geq 0, Y\subset X^{\oplus n}} (End(F(X)) \cap ker(End(F(X^{\oplus n})) \to Hom(F(Y), F(X^{\oplus n})/F(Y)))) \\ &= \{A \in End(F(X)) \mid \forall Y \subset F(X^{\oplus n}), A(F(Y)) \subset F(Y) \text{ as subsets of } F(X^{\oplus n})\} \end{aligned}$$

Observing F automatically factors as $\mathscr{C} \to Mod_{A_X} \to Mod_{\Bbbk}$, we claim the first arrow is an equivalence. Note there is a homomorphism $A_X \to End_{\mathscr{C}}(P_X)^{op}$ by composition so that we can define an inverse by

$$P_X \otimes_{A_X} M := coker(P_X \otimes A_X \otimes M \rightrightarrows P_X \otimes M),$$

where the two arrows are two actions. We can easily check $F(P_X \otimes_{A_X} -) = Id$ and hence F is fully faithful. Lifting the first arrow of a presentation $A_X^m \to A_X^n \to M \to 0$ in Mod_{A_X}, we get F is essentially surjective.

Now we work for general $\mathscr{C}$ again. The second step takes a limit of the local realizations above. However, module category over rings is not compatible with the limiting process we need, because the category of $\varprojlim A_i$-modules is not equivalent to the colimit of category of A_i-modules. To make the limit process correct, we should use comodules instead of modules. So denote $B_X := A_X^\vee$, dual operator induces an equivalence $Comod_{B_X} = Mod_{A_X}$. For $\langle X\rangle \subset \langle X'\rangle$, there is an arrow $A_{X'} \to A_X$ by restriction, so $B_X \to B_{X'}$. We define $B = \varinjlim B_X$ and obtain the equivalence $\mathscr{C} \cong Comod_B$

The third step adds monoidal structure into the picture. Now we've realized $\mathscr{C} \cong Comod_B$, a symmetric monoidal structure $\mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is the same as $Comod_B \times Comod_B \to Comod_B$ and then $B \times B \to B$. This makes B a commutative algebra and bialgebra, or makes Spec$B$ a monoidal scheme in the opposite category.

We want to look for criterions to make Spec$B$ into a group scheme. Recall when we're constructing the dual of a vector space V with group G-action, the inverse operation in G is needed to make G acting on the correct side of $V^\vee$, the dual of V. Actually the converse holds. If "dual" formally exists inside the category (so called "Tannakian category"), we can obtain an antipode map making B a Hopf algebra and $G = \mathrm{Spec}B$ a group scheme. Actually we can make it even simpler: it suffices assume dual exists for "line bundle objects" with respect to F.

Assumption 4.2. For any $X \in \mathscr{C}$ with $\dim_{\Bbbk} F(X) = 1$, there exists $X^\vee \in \mathscr{C}$ and an isomorphism $X \otimes X^\vee \cong \mathbf{1}$. [17]

To show G is a group scheme, we only need to show $G(R)$ is a group for any commutative algebra R over $\Bbbk$. We first show the R-points act by isomorphism on any representation $X \in \mathscr{C}$. If $\dim_{\Bbbk} F(X) = 1$, then R-point simultaneously acts on $X, X^\vee$ and trivially on $\mathbf{1}$. The isomorphism $X \otimes X^\vee \cong \mathbf{1}$ in the assumption the indicates the action on $F(X) \otimes_{\Bbbk} R$ is invertible. For general X, we take the determinant $\bigwedge^{\dim_{\Bbbk} F(X)} X$ to reduce to dimension 1 case. Glueing finite dimensional representations together, $G(R)$ acts on B by isomorphisms, hence acts bijectively on $Hom_{\Bbbk}(B, R) = G(R)$. This implies $G(R)$ is a group since the action coincides with the natural group action.

4.1.2. *Adaption to General Coefficients.* For general $\Bbbk$, the main obstruction happens when we are trying to take the dual of a vector space. To get through this, we have to construct the projective generators P_X above by hand, such that $F(P_X)$ is free. $\langle X \rangle$ is replaced by the subcategory of Sat_G supported on Z, which is a closed union of finitely many L^+G-orbits in $\mathcal{G}r_G$. Use P_Z to denote the projective generator with respect to Z.

We start the construction of P_Z. It is characterized by the object that represents the functor $F = \oplus_\mu F_\mu$, so we construct $P_Z(\mu)$ to represent F_μ and take $P_Z = \oplus_\mu P_Z(\mu)$. Recall F_μ is defined by $H^*(T_\mu, i_\mu^! -)$ for $i_\mu \colon T_\mu \cap Z \hookrightarrow Z$, so it is represented by $i_{\mu !}\Bbbk[-\langle 2\rho, \mu\rangle]$ in $D(Z, \Bbbk)$ by adjunction. We equip it with a L^+G-action by the induction in the following proposition.

Proposition 4.3 (Induction)**.** *If a connected algebraic group H acts on variety X, then the forgetful functor $D_H(X) \to D(X)$ admits a left adjoint Ind_H.*

Proof. Consider the action map $H \times X \to X$ and second projection $H \times X \to X$, then Ind_H is defined as $a_! p^!$. Then

$$\begin{aligned} Hom_{D_H(X)}(Ind_H \mathcal{F}, \mathcal{G}) &= Hom_{D_H(H\times X)}(p^! \mathcal{F}, a^! \mathcal{G}) \\ &= Hom_{D_H(H\times X)}(p^! \mathcal{F}, p^! \mathcal{G}) \\ &= Hom_{D(X)}(\mathcal{F}, \mathcal{G}) \end{aligned}$$

since $p^! \colon D(X) \to D_H(H \times X)$ is an equivalence. □

Finally, we return to perverse sheaves, taking $P_Z(\mu) = {}^p\mathscr{H}^0(Ind_{L^+G} i_{\mu !}\Bbbk[-\langle 2\rho, \mu\rangle])$. Here Ind_{L^+G} makes sense because the L^+G-action on Z factors through a finite dimensional quotient.

[17] Intuitively, the assumptions says if X behaves like a line bundle under F, then it indeed lies in the Picard group of $\mathscr{C}$.

We define $P_Z := \oplus_\mu P_Z(\mu)$, then P_Z represents F and it is automatically a projective generator since F is exact and faithful. Note this is a finite direct sum since only finitely many T_μ intersects with Z.

Compatibility naturally comes from the abstract characterization that it represents F.

Proposition 4.4. *Let Y, Z be two closed unions of L^+G-orbits such that $Y \subset Z$. Then we have $P_Y = {}^p\mathscr{H}^0(i^*P_Z)$ and natural surjection $P_Z \twoheadrightarrow P_Y$.*

Though P_Z is defined abstractly, it is within our reach and behaves nicely, because it is composed of objects that we're familiar with.

Proposition 4.5. *Let Z be a closed union of finitely many L^+G-orbits.*

(1) *P_Z admits a filtration with successive subquotients*
$$\{F(J_*(\lambda, \Bbbk))^\vee \otimes J_!(\lambda, \Bbbk), \mathcal{G}r_\lambda \subset Z\}.$$

(2) *$F(P_Z(\mathbb{Z}))$ is a finitely generated free $\mathbb{Z}$-module.*

(3) *$P_Z(\Bbbk)$ is canonically isomorphic to $\Bbbk \otimes^L_{\mathbb{Z}} P_Z(\mathbb{Z})$ and $F(P_Z(\Bbbk)) = \Bbbk \otimes_{\mathbb{Z}} F(P_Z(\mathbb{Z}))$ is free.*

Proof. Induction on the number of L^+G-orbits. Suppose $j\colon \mathcal{G}r_\lambda \hookrightarrow Z$ is an open orbit with complement $i\colon Y \hookrightarrow Z$, then Proposition 4.4 gives us the exact sequence

$$0 \to K(\Bbbk) \to P_Z(\Bbbk) \to P_Y(\Bbbk) \to 0. \tag{4.6}$$

We want to identify $K(\Bbbk)$ with $F(J_*(\lambda, \Bbbk))^\vee \otimes J_!(\lambda, \Bbbk)$, so first we look at how it behaves on $\mathcal{G}r_\lambda$. Note $Mod_\Bbbk$ is identified with equivariant perverse sheaves on $\mathcal{G}r_\lambda$ after shifting by the dimension of $\mathcal{G}r_\lambda$. Pick any $M \in Mod_\Bbbk$ and $\mathcal{M} = M[\langle 2\rho, \lambda\rangle]$ and we have

$$\begin{aligned} Hom(j^*K(\Bbbk), \mathcal{M}) &= Hom(K(\Bbbk), j_*\mathcal{M}) \\ &= Hom(K(\Bbbk), J_*(\lambda, M)) \\ &= Hom(P_Z(\Bbbk), J_*(\lambda, M)) \\ &= F(J_*(\lambda, M)) = F(J_*(\lambda, \Bbbk)) \otimes M, \end{aligned}$$

where the third equality holds because $i^!J_*(\lambda, M) = i^!\,{}^p\tau^{>0}j_*\mathcal{M}[-1] \in {}^pD^{\geq 2}$ and the fifth equality holds by the proof of Proposition 3.11, with $\Bbbk$ replaced by M. As a result, $j^*K(\Bbbk) = F(J_*(\lambda, \Bbbk))^\vee \otimes \Bbbk[\langle 2\rho, \lambda\rangle]$ and we get the desired map

$$\alpha\colon F(J_*(\lambda, \Bbbk))^\vee \otimes J_!(\lambda, \Bbbk) \to K(\Bbbk).$$

Now we want to show α is an isomorphism. It is easier to show the vanishing of the cokernel C of α. Apply $Hom(-, C)$ to (4.6), then $Ext^1(P_Y(\Bbbk), C)$ vanishes beacuse C is supported on Y where P_Y projective. On the other hand, $Hom(P_Y(\Bbbk), C) = Hom(P_Z(\Bbbk), C) = F(C)$ by definition, hence $Hom(K(\Bbbk), C) = 0$, and so $C = 0$.

The vanishing of $ker(\alpha)$ is more involved, where we invoke results from characteristic 0. Let $\Bbbk = \mathbb{Z}$ for now, then Corollary 3.12 (2) implies $J_!(\lambda, \mathbb{Z}) \hookrightarrow J_*(\lambda, \mathbb{Z})$ doesn't have any nontrivial perverse subsheaf supported on Y, so $ker(\alpha) = 0$ in this case. Thus (2) is proved because extension of free $\mathbb{Z}$-modules is free.

Moving back to general $\Bbbk$, $\Bbbk \otimes^L_{\mathbb{Z}} P_Z(\mathbb{Z})$ is perverse beacuse it has a filtration with subquotients $\Bbbk \otimes^L_{\mathbb{Z}} (F(J_*(\lambda, \mathbb{Z}))^\vee \otimes J_!(\lambda, \mathbb{Z})) = F(J_*(\lambda, \Bbbk))^\vee \otimes J_!(\lambda, \Bbbk)$ (by Corollary 3.12) which are all perverse sheaves. It is easy to check $\Bbbk \otimes^L_{\mathbb{Z}} P_Z(\mathbb{Z})$ also represents F, so we get the isomorphism $P_Z(\Bbbk) \cong \Bbbk \otimes^L_{\mathbb{Z}} P_Z(\mathbb{Z})$ in (3) by the Yoneda lemma. As a consequence, $P_Z(\Bbbk)$ inherits the filtration and

$$F(P_Z(\Bbbk)) = \Bbbk \otimes^L_{\mathbb{Z}} F(P_Z(\mathbb{Z})).$$

$\square$

Now we are able to replay the reconstruction steps with our nice objects P_Z. Set $A_Z(\Bbbk) = End(P_Z(\Bbbk))^{op} = F(P_Z(\Bbbk))$ and $B_Z(\Bbbk) = A_Z(\Bbbk)^\vee$. $Y \subset Z$ induces $B_Y(\Bbbk) \to B_Z(\Bbbk)$ and take $B(\Bbbk) = \varinjlim B_Z(\Bbbk)$. The symmetric monoidal structure of $Sat_{G,\Bbbk}$ makes $\mathrm{Spec}B(\Bbbk)$ a monoidal scheme. Proposition 4.5 implies $\widetilde{G}_\Bbbk := \mathrm{Spec}B(\Bbbk) = \mathrm{Spec}\Bbbk \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}B(\mathbb{Z})$ and it becomes a group scheme if Assumption 4.2 is verified (the proof carries because $B(\mathbb{Z})$ is a colimit of free $\mathbb{Z}$-modules).

Lemma 4.7. *If $F(\mathscr{A}) = \mathbb{Z}, \mathscr{A} \in Sat_{G,\mathbb{Z}}$, then there exists $\mathscr{A}^\vee \in Sat_{G,\mathbb{Z}}$ with $\mathscr{A} \star \mathscr{A}^\vee \cong \mathbf{1}$.*

Proof. We first deal with $\mathbb{Q}$-coefficients. Since $F(\mathscr{A} \otimes \mathbb{Q}) = F(\mathscr{A}) \otimes \mathbb{Q} = \mathbb{Q}$, $\mathscr{A} \otimes \mathbb{Q} = J_{!*}(\lambda, \mathbb{Q})$ for $\mathcal{G}r_\lambda = t^\lambda$ a closed point by semisimplicity in Section 3.3 and Proposition 3.11. Now consider the injective map

$$\beta\colon Hom(\mathscr{A}, J_{!*}(\lambda, \mathbb{Z})) \to Hom(F(\mathscr{A}), F(J_{!*}(\lambda, \mathbb{Z}))),$$

whose image is identified with $B(\mathbb{Z})$-module homomorphisms. β becomes an isomorphism after tensoring with $\mathbb{Q}$, so β is nontrivial. The cokernel of β also vanishes because $f\colon F(\mathscr{A}) \to F(J_{!*}(\lambda, \mathbb{Z})))$ is a $B(\mathbb{Z})$-module map if and only if nf is, for n any nonzero integer. Thus β is an isomorphism and hence $\mathscr{A}$ and $J_{!*}(\lambda, \mathbb{Z})$ are isomorphic. We may simply take $\mathscr{A}^\vee = J_{!*}(-\lambda, \mathbb{Z})$. $\square$

Therefore $\widetilde{G}_\Bbbk$ is a group scheme and so is $\widetilde{G}_\Bbbk$.

4.2. **Identification to the dual group.** We've already identify $Sat_{G,\Bbbk}$ with $Rep(\widetilde{G}_\Bbbk)$. So in order to prove the main theorem Theorem 2.3, it remains to identify $\widetilde{G}_\Bbbk$ with the Langlands dual of G with coefficient in $\Bbbk$. More precisely, we need to show the following theorem.

Theorem 4.8. *$\widetilde{G}_\Bbbk$ is a split reductive group with root datum dual to G.*

The idea is exploiting information about the group through categorical properties of the representation category. Since $\widetilde{G}_\Bbbk = \mathrm{Spec}\Bbbk \times_{\mathrm{Spec}\mathbb{Z}} \widetilde{G}_\mathbb{Z}$, we only need to deal with the case $\Bbbk = \mathbb{Z}$. We can further reduce to the case $\Bbbk = \mathbb{Q}$ or $\overline{\mathbb{F}_p}$ with p any prime number and only consider the reduced subscheme by the results of [29]. The proof is divided into steps.

First, we show $\widetilde{G}_\Bbbk$ is a connected group scheme of finite type over $\Bbbk$. This relies on the following result.

Proposition 4.9. *Let H be an affine group scheme over field $\Bbbk$.*

1. *If there exists $X \in Rep(H)$, generating $Rep(H)$ by taking subquotients, direct sums, tensor products and duals, then H is of finite type.*
2. *Assume H is of finite type. If there doesn't exist a nontrivial $X \in Rep(H)$ such that $\langle X \rangle$ is stable under tensor products, then H is connected.*
3. *Assume H is connected of finite type and $char(\Bbbk) = 0$. If $Rep(H_{\overline{\Bbbk}})$ is semisimple, then H is reductive.*

Proof. (1) The representation X is necessarily faithful, hence the map $H \to GL(X)$ is a closed embedding and H is of finite type.

(2) If H is not connected, then the representations which are trivial on the neutral component are stable under tensor products. This category is identified with the representations of $\pi_0(H)$, which is generated by the object $\Bbbk[\pi_0(H)]$

(3) Smoothness is automatic in characteristic 0. By assumption, $H_{\overline{\Bbbk}}$ admits a faithful semisimple representation, hence H is reductive. □

Lemma 4.10. *$\widetilde{G}_{\Bbbk}$ is connected of finite type. It is reductive if $char(\Bbbk) = 0$.*

Proof. Notice that the convolution product corresponds to addition on the coweight lattice. In the case $char(\Bbbk) = 0$, conditions on the representation category as in Proposition 4.9 are translated to the following facts on $X_*(T)$: 1.$X_*(T)$ is finitely generated; 2.nontrivial finite subset of $X_*(T)$ is not closed under addition. They are true simply because $X * (T)$ is a lattice! The positive characteristic case is more involved with the loss of semisimplicity. However, one can still show there is a highest weight theory on $Sat_{G,\Bbbk}$ in the sense of several axioms in [30]. For more details, see [31]. □

Second, the weight structure of F gives a factorization

$$F\colon Rep(\widetilde{(G)}_{\Bbbk}) \cong Sat_{G,\Bbbk} \to Mod_{\Bbbk}[X_*(T)] \cong Rep(T_{\Bbbk}^{\vee}) \to Mod_{\Bbbk}.$$

This gives us an embedding $T_{\Bbbk}^{\vee} \to \widetilde{G}_{\Bbbk}$ by reconstruction. When $\Bbbk = \mathbb{Q}$, this gives us the maximal torus because

$$\dim T_{\Bbbk}^{\vee} = rank_{\mathbb{Z}} X_*(T) = \dim(\mathrm{Spec}\mathbb{C} \otimes K^0(Sat_{G,\Bbbk})) = \dim(\mathrm{Spec}\mathbb{C} \otimes K^0(Rep(\widetilde{G}_{\Bbbk}))) \geq \dim T_{\Bbbk}^{\vee}.$$

For $char(\Bbbk) = p > 0$, we need to consider the embedding

$$T_{\Bbbk}^{\vee} \to (\widetilde{G}_{\Bbbk})_{red} \to R := (\widetilde{G}_{\Bbbk})_{red}/R_u((\widetilde{G}_{\Bbbk})_{red})$$

instead to make use of the theory of reductive groups. Let $Fr\colon \widetilde{G}_{\Bbbk} \to \widetilde{G}_{\Bbbk}^{(1)}$ be the relative Frobenius map, then $Fr^n\colon \widetilde{G}_{\Bbbk} \to \widetilde{G}_{\Bbbk}^{(n)}$ factors through $(\widetilde{G}_{\Bbbk}^{(1)})_{red}$ for n sufficiently large[18]. So we can use the composition $Fr^n\colon \widetilde{G}_{\Bbbk} \to (\widetilde{G}_{\Bbbk})_{red}^{(n)} \to R^{(n)}$ to get[19]

$$\begin{aligned}\dim T_{\Bbbk}^{\vee} &= \dim(\mathrm{Spec}\mathbb{C} \otimes K^0(Rep(\widetilde{G}_{\Bbbk}))) \\ &\geq \dim(\mathrm{Spec}\mathbb{C} \otimes K^0(Rep(R^{(n)}))) \\ &= \dim(\mathrm{Spec}\mathbb{C} \otimes K^0(Rep(R))) \geq \dim T_{\Bbbk}^{\vee}.\end{aligned}$$

That is, $T_{\Bbbk}^{\vee}$ embeds as the maximal torus of R.

Third, we begin to compare the roots. Choose a Borel subgroup $\widetilde{B}$ of R such that $2\rho \in X^*(T) = X_*(T_{\Bbbk}^{\vee})$ is dominant. To unify the notation, denote $R = \widetilde{G}_{\Bbbk}$ when $char(\Bbbk) = 0$. Now we have two root data living on $(X_*(T), X^*(T))$, given by $(R, \widetilde{B}, T_{\Bbbk}^{\vee})$ and (the dual of) (G,B,T). We index them by 1 and 0 respectively.

Lemma 4.11.

(1) $X_*(T)_1^+ = X_*(T)_0^+$. In particular, $\widetilde{B}$ is uniquely defined.
(2) If $char(\Bbbk) = 0$, then $\Delta_0 = \Delta_1$
(3) $Q_1 \subset Q_0$.

[18]See Corollary VI.10.2 of [25].

[19]Here we use some results of representation in characteristic p,see [19].

(4) $W_1 = W_0$ *as subgroups of* $GL(X_*(T))$.
(5) $Q_0^\vee \subset Q_1^\vee$. *If the equality holds, then root data 0 and 1 coincide with each other.*

Proof. (1)We read the information from characters of simple modules. When $char(\Bbbk) = 0$, simple R-modules $L(\mu), \mu \in X_*(T)$ are in bijection with simple perverse sheaves $J_{!*}(\lambda, \Bbbk), \lambda \in X_*(T)$. Here μ (resp.λ) are characterized by the character χ appearing in $L(\mu)$ (resp.$J_{!*}(\lambda, \Bbbk)$), making $\langle 2\rho, \chi\rangle$ maximal. For $char(\Bbbk) = p > 0$, the difficulty occurs in comparing $Rep_{\widetilde{G}_\Bbbk}$ and $Rep(R)$. In one direction, we can restrict $J_{!*}(\lambda, \Bbbk)$ (regarded as representation of $\widetilde{G}_\Bbbk$) on $(\widetilde{G}_\Bbbk)_{red}$ and take a simple factor containing λ. This factor becomes a representation of R and we get $X^*(T)_0^+ \subset X^*(T)_1^+$. In the other direction, for any $\mu \in X^*(T)_1^+$, $L(p^n\mu)$ is a representation of $R^{(n)}$(n as in the second step) and becomes a $\widetilde{G}_\Bbbk$-module. So $p^n\mu \in X^*(T)_0^+$ and then $\mu \in X^*(T)_0^+$.

(2) By the representation theory in characteristic 0, the characters involved in a simple module $L(\lambda)$(resp. $J_{!*}(\lambda, \Bbbk)$) are $Conv(W \cdot \lambda) \cap (\lambda + Q_1^\vee)$ $(resp.\ Q_0^\vee)$. Starting from λ, simple roots are precisely the indecomposable elements in the lattice $Conv(W \cdot \lambda) \cap (\lambda + Q^\vee)$, so Δ_0 and Δ_1 are identified.

(3) $Z = \bigcap_{\chi \in Q_0} ker(\chi) \subset T_\Bbbk^\vee$, then $X^*(Z) = X_*(T)/Q_0$ and we have a decomposition of $\widetilde{G}_\Bbbk$-modules $V = \bigoplus_{\chi \in X^*(Z)} V_\chi, \forall V \in Rep(\widetilde{G}_\Bbbk)$, since characters of Z correspond to connected components on $\mathcal{G}r_G$. Therefore Z is central in $\widetilde{G}_\Bbbk$ and hence in R, inducing $X_*(T)/Q_1 \to X_*(T)/Q_0$ and thus $Q_1 \subset Q_0$.

(4) We use the transition between $Rep_{\widetilde{G}_\Bbbk}$ and $Rep(R)$ as in (1). For regular $\lambda \in X_*(T)^+$, we get $\{w \cdot \lambda, w \in W_0\}$ and $\{w \cdot \lambda, w \in W_1\}$ have the same convex hulls. Then we recover $\{w \cdot \lambda, w \in W_0\} = \{w \cdot \lambda, w \in W_1\}$ from the extremal points of the hull and identify simple reflections in W_0 and W_1 by looking at extremal edges of the hull. Then $W_0 = W_1$ since they are generated by simple reflections.

(5) The first claim is the dual of (3) with the help of (4). For $\alpha \in \Delta_0^\vee$, there exists $q \neq 0 \in \mathbb{Q}$ such that $q\alpha \in \Delta_1^\vee$. Because $s_\alpha = s_{q\alpha}$ by (4), $q^{-1}\alpha^\vee = (q\alpha)^\vee \in Q_1 \subset Q_0$, so $q^{-1} \in \mathbb{Z}$. Then we get $\alpha = q^{-1}q\alpha \in Q_1^\vee$. If the equality holds, then q above equals to (1). So $\Delta_0^\vee = \Delta_1^\vee$ and $\Delta_0 = \Delta_1$. Then the root systems match because they are generated by simple roots. □

Finally, we are ready to prove Theorem 4.8.

Proof. For $\Bbbk = \mathbb{Q}$, we use (2) and the proof of (5) of Lemma 4.11 to conclude. For $\Bbbk = \overline{\mathbb{F}}_p$, we have to reduce to the situation where Lemma 4.11 already works.

If G is semisimple of ajoint type, then $X^*(T) = Q_0^\vee \subset Q_0^\vee \subset X^*(T)$. Then $Q_0^\vee = Q_1^\vee$ and we identify the root data from Lemma 4.11(5). Then we have

$$\begin{aligned}
\dim R &= \dim G && \text{(using root datum)}\\
&= \dim \widetilde{G}_{\mathbb{Q}} && \text{(by } \Bbbk = \mathbb{Q} \text{ case)}\\
&\geq \dim \widetilde{G}_\Bbbk && \text{(by lower semicontinuity)}\\
&= \dim (\widetilde{G}_\Bbbk)_{red} \geq \dim R.
\end{aligned}$$

Therefore $(\widetilde{G}_\Bbbk)_{red} = R$ and $(\widetilde{G}_\Bbbk)_{red}$ is reductive.

If G is semisimple, we get its adjoint group G_{ad} by quotienting the center of G. This induces a closed embedding $\mathcal{G}r_G \to \mathcal{G}r_{G_{ad}}$ and then a group homomorphism

$\widetilde{G_{ad\Bbbk}} \to \widetilde{G}_{\Bbbk}$. Denote T_{ad} the image of T in G_{ad}, then G and G_{ad} share the same coroot lattice and we have an exact sequence $0 \to X_*(T) \to X_*(T_{ad}) \to Z \to 0$. $\mathcal{G}r_G$ can be viewed as a union of some connected components in $\mathcal{G}r_{G_{ad}}$ with index parametrized by Z. Hence $\widetilde{G_{ad\Bbbk}} \to \widetilde{G}_{\Bbbk}$ is surjective with finite kernel which is central in $\widetilde{G_{ad\Bbbk}}$. So $\widetilde{G}_{\Bbbk}$ is semisimple because $\widetilde{G_{ad\Bbbk}}$ is and we can identify its root datum with the dual of G from the above picture.

If G is a general reductive group, take $H = Z(G)^\circ$ so that G is the extension of H and G/H, which we both know very well. Functoriality gives $\mathcal{G}r_H \xrightarrow{p} \mathcal{G}r_G \to \mathcal{G}r_{G/H}$ and $\widetilde{G/H}_{\Bbbk} \to \widetilde{G}_{\Bbbk} \to \widetilde{H}_{\Bbbk}$. Recall $\mathcal{G}r_H$ is a union of points parametrized by $X^*(H)$ and p is the quotient map for the free action of $X^*(H)$ on $\mathcal{G}r_G$. From the geometric picture, we see $\widetilde{G/H}_{\Bbbk} \to \widetilde{G}_{\Bbbk}$ is a closed embedding and $\widetilde{G}_{\Bbbk} \to \widetilde{H}_{\Bbbk}$ is surjective. Since $\widetilde{G/H}_{\Bbbk}$ is semisimple and $\widetilde{H}_{\Bbbk}$ is a torus, the unipotent radical lies in the kernel of $\widetilde{G}_{\Bbbk} \to \widetilde{H}_{\Bbbk}$ and its intersection with $\widetilde{G/H}_{\Bbbk}$ is at most a finite group scheme. So $\dim R \geq \dim \widetilde{G/H}_{\Bbbk} + \dim \widetilde{H}_{\Bbbk} = \dim G/H + \dim H = \dim G \geq \dim \widetilde{G}_{\Bbbk} \geq \dim R$. We get $R = (\widetilde{G}_{\Bbbk})_{red}$ is reductive as in the case G is semisimple of adjoint type. This also gives us $|\Phi_1| = |\Phi_0|$ and $\dim R^{der} = |\Phi_1| + |\Delta_1| = |\Phi_0| + |\Delta_0| = \dim \widetilde{G/H}_{\Bbbk}$. Because $\widetilde{G/H}_{\Bbbk}$ is semisimple, it is contained in R^{der} and the equality holds by the previous dimension calculation. Thus we have

$$Q^\vee(R, T^\vee_{\Bbbk}) = Q^\vee(\widetilde{G/H}_{\Bbbk}, (T/H)^\vee_{\Bbbk}) = Q(G/H, T/H) = Q(G, T).$$

We then get $Q_1^\vee = Q_0^\vee$ and conclude by (5) of Lemma 4.10. ☐

As a result, we have finished to proof of Theorem 2.3 and Theorem 1.1.

5. Application to Modular Representation Theory

In the end of this paper, we discuss an application of geometric Satake.

If we consider geometric Satake equivalence as a geometric realization of the representation category of a reductive group, we are then able to apply geometric tools to study representations. This is particularly effective in modular representation theory (i.e. representation in positive characteristic), and the theorem provides a bridge connecting different characteritics. The following example comes from [9].

Let us first introduce some notions in modular representation theory. Let $\Bbbk$ be an algebraically closed field of characteristic $p > 0$, and $H = G^\vee_{\Bbbk}$ a reductive group over $\Bbbk$. For simplicity, assume G is simply connected. Any $\lambda \in X_*(T)^+$ gives rise to a line bundle $O(\lambda)$ on $H/\widetilde{B}_{\Bbbk}$. Denote $\nabla_\lambda := H^0(H/\widetilde{B}_{\Bbbk}, O(\lambda))$. Let $L_\lambda = soc\nabla_\lambda$ be the socale of ∇_λ and $\Delta_\lambda = \nabla^\vee_{-w_0\lambda}$. Then the modules L_λ exactly give us all the simple H-modules. The characters of ∇_λ are computed by a Weyl character formula, and we want to compute the characters of L_λ. To do this, we first decompose $Rep(H)$ into smaller blocks. Note W_{aff} can be seen as the semidirect product of Weyl group and root lattice of H. Note there is a group homomorphism $p\colon W_{aff} \to W_{aff}$, fixing elements in the finite Weyl group W and acting by multiplication by p on the elements in the root lattice. Define the *p-dilated dot action* of W_{aff} by $w \bullet_p \lambda = (pw) \cdot (\lambda + \rho) - \rho$. Then $Rep(H)$ splits into blocks.

Proposition 5.1 (Linkage Principle)**.**

$$Rep(H) = \bigoplus_{\lambda \in X_*(T)/(W_{aff}, \bullet_p)} Rep_\lambda(H).$$

Here $Rep_\lambda(H)$ is the Serre subcategory generated by simple modules

$$\{L_\mu, \mu \in (W_{aff} \bullet_p \lambda) \cap X_*(T)^+\}.$$

Using translation functors between different blocks, we are reduced to analyze the principle block $Rep_0(H)$. Denote $L_x = L_{x\bullet_p 0}$ and $\Delta_x = \Delta x \bullet_p 0$ for $x \in W_{aff}$ such that $x \bullet_p 0 \in X_*(T)^+$.

Conjecture 5.2 (Lusztig[24])**.** *Under certain conditions*[20] *on p and x, we have the following equality in the Grothendieck group of $Rep_0(H)$:*

$$[L_x] = \sum_y (-1)^{l(x)+l(y)} P_{w_0 y, w_0 x}(1)[\Delta_y]$$

To immitate the proof of Kazhdan Lusztig conjecture, we wish that geometric Satake can be upgraded to single out principal block geometrically.

Conjecture 5.3 (Finkelberg-Mirković)**.** *Denote $I_u := ker(Iw \to B \to T)$ the pro-unipotent subgroup of Iwahori. There is an equivalence of abelian categories FM fitting into the following commutative diagram.*

$$\begin{tikzcd}
Rep_0(H) \arrow[r, "\cong", "FM"'] & Perv_{I_u}(\mathcal{G}r_G, \Bbbk) \\
Rep(H) \arrow[u, "Fr"'] \arrow[r, "\cong", "Sat"'] & Perv_{L^+G}(\mathcal{G}r_G, \Bbbk) \arrow[u, "forget"']
\end{tikzcd}$$

Moreover, L_x is mapped to $J_{!*}(x^{-1}, \Bbbk)$ and ∇_x is mapped to $J_*(x^{-1}, \Bbbk)$ by the equivalence FM, where $J_{!*}$ and J_* has similar meaning as in Section 3.1.

Remark 5.4. In the series papers [6][7], Roman Bezrukavnikov, Simon Riche and Laura Rider partially prove the Finkelberg-Mirković conjecture and announce a complete proof in their third paper. More precisely, they've proved the existence of such an equivalence $FM\colon Rep_0(H) \cong Perv_{I_u}(\mathcal{G}r_G, \Bbbk)$ with certain objects having desired image. What they haven't shown is the compatibility with Satake equivalence. In their perspective, this is a result of "modular tamely ramified local geometric Langlands equivalence".

Let us see how to use Finkelberg-Mirković conjecture to deal with Lusztig conjecture. Suppose we have $[L_x] = \sum_y a_{y,x}[\Delta_y]$. By Finkelber-Mirković conjecture, this is equivalent to $[J_{!*}(x^{-1}, \Bbbk)] = \sum_y a_{x,y}[J_*(y^{-1}, \Bbbk)]$. By taking Euler characteristic of costalks at y^{-1}Iw/Iw, we get

$$\begin{aligned}
a_{y,x} &= (-1)^{l(y)}\chi(i^!_{y^{-1}} J_{!*}(x^{-1}, \Bbbk)) \\
&= (-1)^{l(y)}\chi(i^!_{y^{-1}} J_{!*}(x^{-1}, \mathbb{Z}) \otimes^L_{\mathbb{Z}} \Bbbk) \\
&= (-1)^{l(y)}\chi(i^!_{y^{-1}} J_{!*}(x^{-1}, \mathbb{Q})) && \text{If the costalk over } \mathbb{Z} \text{ is p-torsion free.} \\
&= (-1)^{l(x)+l(y)} P_{w_0 y, w_0 x}(1).
\end{aligned}$$

So Lusztig conjecture is converted to a topological problem that whether a costalk contains p-torsion. Intuitively, the torsion should vanished when p is sufficiently large. It's proved in [13] that Lusztig conjecture holds for any p larger than some

[20] We assume $p \geq h$, the Coxeter number of H, $x \bullet_p 0 \in X_*(T)^+$ and x satisfies Jantzen's condition: $\langle x \bullet_p 0 + \rho, \alpha^\vee \rangle \leq p(p - h + 2)$, for any positive root α of H.

$N = N(H)$. On the contrary, there are lots of torsion when p is small. [35] gives such an example, which is also a counterexample for Lusztig conjecture for small p. For more details and history on this topic, see [9].

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