A. Beilinson - On Langlands Correspondence in the deRham setting I

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$\boxed{\text{Local Picture}}$ today. de Rham version: differs from ℓ-adic setting, use methods unavailable ℓ-adically..

Usual Langlands: relates two seemingly unrelated objects.
G split reductive grp. F = local field $= k((t))$ for us
(usual Langlands: k finite)
Rep theory: $G(F)$ (locally compact grp) & reps of G
Galois theory: $G^\vee$ Langlands dual: dual root data to G
(- consider $/\mathbb{Q}$ or $/\mathbb{C}$ or $/\mathbb{Q}_\ell$.)
Reps $\mathrm{Gal}(F) \longrightarrow G^\vee$ $\Longleftrightarrow$
$G^\vee$ local systems on $\mathrm{Spec}\, F$ (ℓ-adic)

Expect decomposition of reps of $G(F)$ into series labelled by Galois data.
More precise: Bernstein center = End of identity functor of rep category, which is fibered over Spec of the Bernstein center

$\Rightarrow$ expect Spec(Bernstein center) = set of $G^\vee$-local systems

Principal methods are global - no direct purely local relation.

deRham version: k fixed field of char. 0 (eg $\mathbb{C}$)
$G^\vee$-loc systems: now in de Rham sense
= $G^\vee$ bundles with connection on $\mathrm{Spec}\, F$.
- depend on continuous parameters ...
formal differential eqns (no Stokes parameters),
arbitrary irregular singularities allowed!

Rep theory side: $\mathfrak{g}$ f.d reductive Lie algebra
$\mathfrak{g}(F)$ ∞-dim (topological) Lie algebra..
better consider reps of Kac-Moody central extension, at level κ = Ad-invt quadratic form on $\mathfrak{g}$: cocycle
$a, b \longmapsto \mathrm{Res}\, \kappa(b, da)$ giving central extension

Note: everything here will be *purely local* - advantage of de Rham setting
(1 will act as identity for our reps)
Rep theory depends on κ.. should consider special κ: *integral* & *negative* in strong sense (less than critical). & *nondegenerate*
(e.g. $\mathfrak{g}$ torus: nondegen integral scalar product on corresponding lattice)

Format of conjecture: (rough)

$\mathcal{LS}$ = moduli of $G^\vee$-local systems on Spec F - stack over k, *not* algebraic (just know what families mean)

(a) Want to define an associative topological algebra (or $\mathcal{O}_{\mathcal{LS}}$) $\mathcal{A}$ on $\mathcal{LS}$ together with map of Lie algebras
$\mathfrak{g}(F)^\kappa \longrightarrow \mathcal{A}$
Given module over $\mathcal{A}$ $\Longleftrightarrow$ "quasicoherent" sheaf on $\mathcal{LS}$, its global sections carry action of $\mathfrak{g}(F)^\kappa$

$\Rightarrow$ functor $\mathcal{A}$-modules $\xrightarrow{\Gamma}$ $\mathfrak{g}(F)^\kappa$-modules
Want this to be an *equivalence* of *categories*.

... so modules over $\mathfrak{g}(F)^\kappa$ decomposed wrt "spectral parameters" $\mathcal{LS}$.

Very natural construction - (e.g. wrt $\mathfrak{g}(F)$-action)

(b.) Given local system $\in \mathcal{LS}$ can ask which $\mathfrak{g}(F)^\kappa$-modules are supported here? want explicit geometric description - at least for *tame* local systems (regular sings)

Comment a. will come from natural vertex algebra assoc. to G -- e.g. torus $\Rightarrow$ lattice Heisenberg vertex algebra.
- algebra will be equipped with $G^\vee$-action...
so can *twist* vertex algebra by any $G^\vee$-local system, & these are fibers of $\mathcal{A}$!
- can't do on level of plain associative algebras (twisting by $\mathcal{LS}$)

- If moduli of LS happens to be an affine variety this would be same as map $\mathbb{C}[LS] \to$ Center of $\mathfrak{g}(F)^k$ --- but both sides $\mathbb{C}[LS]$, center are trivial! naive version doesn't work.

Def of vertex algebra analogous to getting lattice Heisenberg from plain enveloping algebra - add extra generators.

Center is somehow internal usually - but here construct "external center" .. that with carcass outside ....

Part (b) (for tame local systems with unipotent monodromy)

$G(F)$ is an ind-scheme from POV of k (inductive limit of affine schemes)

$$\begin{array}{ccccc} G(F) & \supset & G(\mathcal{O}) & \supset & \text{Iwahori} \\ & & {\scriptstyle\text{group scheme}}\downarrow & & \downarrow \\ & & G & \supset & B \end{array}$$

$\phi = G(F)/\text{Iwahori}$: ind-proper ind scheme : Affine flag space

$\Rightarrow$ category $\mathcal{M}(\phi)$ of $\mathcal{D}$-modules on ϕ : [right $\mathcal{D}$-mod] union of f.d varieties wrt closed embeddings, so look at union of $\mathcal{D}$-submodules supported on f.d. piece.
--- right $\mathcal{D}$-mods make sense as sheaves here : these embed into ~~pushforward~~ each other to give unlike left (need to twist).

$$\Gamma : \mathcal{M}(\phi) \longrightarrow \mathfrak{g}(F)\text{-modules}$$

- really should twist by appropriate line bundle!
k defines central extension of $G(F)$ by $\mathbb{G}_m$
$\to G(F)^k$, look at equivariant line bundles for $G(F)^k$ on ϕ : they form a torsor over weight lattice of G (affine condition: 1 acts by 1) - carries action of affine Weyl, fin many orbits.

Pick ample line bundle $\mathcal{L}$ from any W^{aff}-orbit (many ways possible)

$\Rightarrow \quad \Gamma_{\mathcal{L}} : M \longmapsto \Gamma(\phi, M \otimes \mathcal{L})$ $\quad \mathfrak{g}(F)^{\kappa}$-module (exact fully faithful functor)

Wish: $\Gamma_{\mathcal{L}}$ produces equivalence of categories

$$\coprod_{\mathcal{L} \,\in\, W^{aff}\text{-orbits}} \mathcal{M}(\phi) \xrightarrow[\Gamma_{\mathcal{L}}]{\sim} \mathfrak{g}(F)^{\kappa}\text{-modules supported on the nilpotent local systems}$$

– eg all category $\mathcal{O}$, Vermas etc come this way (from tame local systems). Functors $\Gamma_{\mathcal{L}}$ depend on choice of $\mathcal{L}$ but have geometric intertwiners

Case $G = T$ torus

[A = lattice Heisenberg & its twists by local systems
Rep theory side: reps of Lie Heisenberg algebra
– decompose its reps wrt reps of all twisted lattice Heisenberg algebras!]

Very brief introduction to vertex algebras:

Work over a curve X (eventually look a disc)
A = quasicoherent $\mathcal{O}_X$-module

Def: "Factorization structure" on A = a collection of $\mathcal{O}_{X^n}$-modules all n $\{A_{X^n}\}$ with compatibility data

Informally: $A_X = A$, key property: $\forall (x_1, \dots x_n) \in X^n$ consider fiber $A_{(x_1,\dots x_n)}$, demand that it equals $\bigotimes_{x \in \{x_1 \dots x_n\}} A_x$ where we consider $(x_1, \dots x_n)$ as plain subset of X – no multiplicities (one copy for each distinct point).

Precisely on X^2 $\quad A_{X^2} \qquad X \xhookrightarrow{\Delta} X \times X$

$$\begin{array}{c} \uparrow j \\ U = X \times X \setminus \Delta \end{array}$$

demand $\Delta^* A_{X^2} = A$
$\quad j^* A_{X^2} = j^*(A \boxtimes A)$
plus action of switching factors compatible with $A \boxtimes A$

Structure is completely local: gluing of $A\boxtimes A$ off Δ to A on Δ.

Def A chiral algebra structure on A is a factorization structure s.t. 1. all A_{X^n} flat in transversal direction to diagonal 2. A has a unit: global section $1\in\Gamma(X,A)$ s.t. $\forall a\in A$, $a\boxtimes 1$ off Δ extends to diagonal: $a\boxtimes 1\in A_{X^2}\subset j_*j^*A\boxtimes A$ & $\Delta^*(a\boxtimes 1)=a$

Note: such structure yields canonically a $\mathcal{D}$-mod structure on A:

$$\begin{tikzcd} p_1^*A_X \arrow[r] & A_{X^2} & p_2^*A_X \arrow[l] \end{tikzcd}$$

$a\mapsto a\boxtimes 1$ $\qquad$ $1\boxtimes a \leftarrow a$

.. pulled back to diagonal these maps are isomorphisms - so since our objects are flat transversally to diagonal get isomorphism on formal nbhd of diagonal $\Longleftrightarrow$ $\mathcal{D}$ module structure

Operator Product Expansion:

$$\begin{tikzcd}
& A_X\boxtimes A_X \arrow[dl, "{[\,,]}"'] \arrow[dr, "OPE"] \arrow[dr, "a_1\boxtimes a_2\mapsto a_1\circ a_2"'] \arrow[r, hook] & j_*j^*A_X\boxtimes A_X = j_*j^*A_{X^2} \arrow[d] \\
\Delta_*A_X & & A_X^{(n)}((t_1-t_2)) = j_*j^*\hat{A}_{X^2} \arrow[ll]
\end{tikzcd}$$

in local parameter

complete along diagonal & localize wrt equation of Δ

Algebraic part: take only polar part

OPE completely determines A_{X^2} hence everything! just gluing data.