Clay Mathematics Proceedings
Volume 4, 2005

Harmonic Analysis on Reductive p-adic Groups and Lie Algebras

Robert E. Kottwitz

Contents

Introduction

One of the long term goals in the representation theory of reductive groups over p-adic fields is the local Langlands conjecture, which classifies irreducible representations in terms of Langlands parameters (and auxiliary data on the Langlands dual group). This goal has been achieved for GL_n (see [**Car00**] for a survey of recent work on this problem), but not in general.

One way to approach the classification problem for classical groups is via the twisted Arthur-Selberg trace formula for GL_n, the reason behind this being that all quasi-split classical groups are captured as twisted endoscopic groups for GL_n (strictly speaking, a suitable restriction of scalars of GL_n in the case of unitary groups). The building blocks of representation theory on a p-adic group are the supercuspidal representations, and these show up in the trace formula only through their distribution characters. To use the trace formula successfully it is necessary to know some qualitative facts about these characters.

Here we should pause to recall that for any smooth representation π of our p-adic group G and any $f \in C_c^\infty(G)$ (the space of locally constant and compactly supported functions on G) there is an operator $\pi(f)$ on the underlying vector space V_π of π, defined on $v \in V_\pi$ by

$$\pi(f)(v) := \int_G f(g)\pi(g)(v)\,dg, \tag{0.0.1}$$

dg being some fixed Haar measure on G.

When π is irreducible, it is necessarily admissible (see the survey article [**BZ76**] for this and other basic facts about the representation theory of p-adic groups), which guarantees that $\pi(f)$ has finite rank and hence has a trace. The character Θ_π of π is the distribution on G defined by

$$\Theta_\pi(f) = \operatorname{tr}\pi(f)$$

on each test function $f \in C_c^\infty(G)$.

It is a deep theorem of Harish-Chandra that the distribution Θ_π can be represented by integration against a locally constant function, still denoted Θ_π, on the set G_{rs} of regular semisimple elements in G. Thus, for all $f \in C_c^\infty(G)$ there is an equality

$$\Theta_\pi(f) = \int_G f(g)\Theta_\pi(g)\,dg. \tag{0.0.2}$$

The function Θ_π is independent of the choice of Haar measure, and by comparing equations (0.0.1) and (0.0.2) one sees that formally $\Theta_\pi(g) = \operatorname{tr}\pi(g)$, though of course $\operatorname{tr}\pi(g)$ does not make sense literally when π is infinite dimensional, as is usually the case.

In order to extract information about classical groups from the twisted trace formula for GL_n (see [**Art97**]), one must stabilize the twisted trace formula, which is to say that one must express it as a linear combination of stable trace formulas for the elliptic twisted endoscopic groups H of GL_n. The stabilization process should then yield identities between (suitable linear combinations of) characters on the classical group H and (suitable linear combinations of) twisted characters on GL_n. This has been done in full [**Rog90**] for $GL_3(E)$ and its twisted endoscopic group $U(3)$ (quasi-split unitary group over F coming from a quadratic extension E/F), giving a classification for representations of $U(3)$ in terms of the better understood representations of $GL_3(E)$.

Information about the characters of irreducible representations of p-adic groups is embedded in the spectral side of the trace formula, but in order to carry out the stabilization one must start with the geometric sides of the relevant trace formulas, and so one must study orbital integrals (invariant integrals over conjugacy classes) as well as Arthur's weighted orbital integrals (obtained from certain non-invariant integrals over conjugacy classes, and more generally from certain limits of these).

Thus, in order to use the trace formula, one needs a good understanding of characters, orbital integrals and weighted orbital integrals, and these are precisely the main objects of study in harmonic analysis on G. Many of the most basic (and deepest) results in this area are due to Harish-Chandra, among them his theorem, mentioned above, that the distribution character Θ_π can be represented by a locally integrable function on G, locally constant on G_{rs}.

The main step in Harish-Chandra's proof of this result involves passing to the Lie algebra $\mathfrak{g}$ of G and proving an analogous result there. For this reason, among others, one should study harmonic analysis on $\mathfrak{g}$ along with that on G, and in this article, after introducing some of the key concepts in harmonic analysis on G, we will then concentrate on $\mathfrak{g}$, giving an almost self-contained exposition of Waldspurger's local trace formula on $\mathfrak{g}$ [**Wal95**] as well as many results of Harish-Chandra [**HC78, HC99**]. We should now follow up these rather vague motivational remarks with a more precise discussion of some of the key results in harmonic analysis on G and $\mathfrak{g}$.

Harish-Chandra's first assault [**HC70**] on the problem of representing the distribution character Θ_π by a locally integrable function on G was limited to the case in which π is a supercuspidal representation. Given a vector v in the space of π and a vector $\tilde{v}$ in the space of the contragredient representation $\tilde{\pi}$, we can define a locally constant function $f_{\tilde{v},v}$ on G by

$$f_{\tilde{v},v}(g) := \langle \tilde{v}, \pi(g)v \rangle, \tag{0.0.3}$$

the pairing denoting the value of the linear functional $\tilde{v}$ on the vector $\pi(g)v$. The function $f_{\tilde{v},v}$ is referred to as a matrix coefficient for π. For simplicity let us assume for the remainder of the introduction that the center of G is compact. Our assumption that π is supercuspidal implies [**BZ76**] that all its matrix coefficients are compactly supported functions on G. (Conversely, an irreducible smooth representation whose matrix coefficients are compactly supported is supercuspidal.)

Now choose $v, \tilde{v}$ so that $\langle \tilde{v}, v \rangle$ is the formal degree (see [**HC70**]) of π, and let ϕ denote the matrix coefficient $f_{\tilde{v},v}$. (Thus $\phi(1)$ is the formal degree of π.) Then as an easy first step Harish-Chandra shows that

$$\begin{aligned} \Theta_\pi(f) &= \int_G \Big[\int_G \phi(x) f(g^{-1}xg)\, dx \Big]\, dg \\ &= \int_G \Big[\int_G \phi(gxg^{-1}) f(x)\, dx \Big]\, dg. \end{aligned} \tag{0.0.4}$$

These integrals are convergent only as iterated integrals, and it is not legitimate to interchange the order of integration. However, let's pretend for a moment that we could interchange the order of integration. Then we would conclude that the distribution Θ_π is represented by the function $x \mapsto \int_G \phi(gxg^{-1})\, dg$ on G. This is nonsense since the integral $\int_G \phi(gxg^{-1})\, dg$ diverges unless the centralizer of x is compact. Nevertheless Harish-Chandra shows that the restriction of the distribution Θ_π to G_{rs} is represented by the function

$$x \mapsto \int_G \Big[\int_K \phi(gkxk^{-1}g^{-1})\,dk \Big]\,dg \tag{0.0.5}$$

for any compact open subgroup K of G that we like, and he then goes on to prove the difficult result that the function Θ_π is locally integrable on G and represents the distribution Θ_π on all of G, not just on G_{rs}.

For $x \in G_{\mathrm{rs}}$ the identity component of the centralizer of x is a torus, and when this torus is compact (compact modulo the center, when the center is not assumed to be compact) we say that x is elliptic. For elliptic regular semisimple x Harish-Chandra shows that the order of integration in (0.0.5) can be reversed, so that (for such x) the character value at x is given by the orbital integral

$$\Theta_\pi(x) = \int_G \phi(gxg^{-1})\,dg. \tag{0.0.6}$$

Arthur [**Art87**] has generalized the formula (0.0.6) in a very beautiful way: for any $x \in G_{\mathrm{rs}}$ the character value at x is given by the weighted orbital integral

$$\Theta_\pi(x) = (-1)^{\dim A_M} \int_{A_M\backslash G} \phi(g^{-1}xg)v_M(g)\,d\dot{g} \tag{0.0.7}$$

for a weight function v_M described in 12.6 and a suitably normalized invariant measure $d\dot{g}$ on the homogeneous space $A_M\backslash G$. Here M is a Levi subgroup in which x is elliptic, and A_M is the split component of the center of M.

What happens when our irreducible smooth representation π is not assumed to be supercuspidal? If π is obtained by parabolic induction from a supercuspidal representation σ of a Levi subgroup M of G, then one can easily express the character of π in terms of that of σ, and in this way show that Θ_π is represented by a locally integrable function on G. However, even for GL_2, the Grothendieck group of representations of G having finite length is not spanned by the classes of representations that are parabolically induced from supercuspidal representations of Levi subgroups, so parabolic induction alone does not solve our problem.

To handle the general case Harish-Chandra [**HC78, HC99**] changed his strategy, no longer treating supercuspidal representations separately, and relying even more heavily on passage to the Lie algebra $\mathfrak{g}$. One goal of this article is to work through the main results of [**HC78, HC99**] concerning harmonic analysis on $\mathfrak{g}$. What serve as the Lie algebra analogs of the invariant distributions Θ_π? The answer is quite simple: Fourier transforms of orbital integrals.

Orbital integrals are obtained as follows. For any $X \in \mathfrak{g}$ the adjoint orbit of X can be identified with $G_X\backslash G$ (G_X being the centralizer of X in G) and carries a G-invariant measure $d\bar{g}$; moreover for any $f \in C_c^\infty(\mathfrak{g})$ the integral

$$O_X(f) := \int_{G_X\backslash G} f(g^{-1}Xg)\,d\bar{g}$$

converges, yielding an invariant distribution O_X, called the orbital integral for X.

The Fourier transform $f \mapsto \hat{f}$ for $f \in C_c^\infty(\mathfrak{g})$ is reviewed in 8.2. As usual one extends the notion of Fourier transform to distributions T on $\mathfrak{g}$ by the rule

$$\hat{T}(f) = T(\hat{f}).$$

In particular we can consider the Fourier transform $\hat{O}_X$ of any orbital integral O_X, and, as mentioned above, $\hat{O}_X$ is analogous to the character of an irreducible representation.

For example, if G is split with split maximal torus A and X is a regular element in $\mathrm{Lie}(A)$, then $\hat{O}_X$ is analogous to a principal series character. If $X = 0$, then $\hat{O}_X$ is analogous to the character of the trivial representation. If T is an elliptic maximal torus and X is regular in $\mathrm{Lie}(T)$, then $\hat{O}_X$ is analogous to the character of a supercuspidal representation of G.

Given this analogy, it is perhaps not hard to guess the statement of one of Harish-Chandra's main results in harmonic analysis on $\mathfrak{g}$ (see Theorem 27.8): for every $X \in \mathfrak{g}$ the distribution $\hat{O}_X$ is represented by a locally integrable function on $\mathfrak{g}$, locally constant on $\mathfrak{g}_{\mathrm{rs}}$, the set of regular semisimple elements in $\mathfrak{g}$. In fact Theorem 27.8 says something even more general: for any invariant distribution I on $\mathfrak{g}$ whose support is bounded modulo conjugation (see 15.2) the Fourier transform $\hat{I}$ is represented by a locally integrable function on $\mathfrak{g}$, locally constant on $\mathfrak{g}_{\mathrm{rs}}$.

It is not easy to prove that $\hat{O}_X$ is represented by a function. The essential case is when X lies in $\mathfrak{g}_e$, the open subset of elliptic regular semisimple elements in $\mathfrak{g}$, which Harish-Chandra treats by using Howe's finiteness theorem (see 26.2) to reduce to proving that $\hat{I}_\phi$ is represented by a function for any $\phi \in C_c^\infty(\mathfrak{g}_e)$. Here I_ϕ is the invariant distribution on $\mathfrak{g}$ defined for any cusp form $\phi \in C_c^\infty(\mathfrak{g})$ by the iterated integral

$$I_\phi(f) = \int_G \Big[\int_{\mathfrak{g}} \phi(X) f(g^{-1}Xg)\, dX \Big]\, dg, \tag{0.0.8}$$

the Lie algebra analog of (0.0.4). The integral (0.0.8) is actually convergent as a double integral for the special cusp forms $\phi \in C_c^\infty(\mathfrak{g}_e)$. [Recall that we are assuming that the center of G is compact and hence can be ignored. We should also note that later, when discussing I_ϕ systematically (see 25.2), we will find it convenient to build in a harmless factor $|\mathcal{Z}_G|^{-1}$ (see (25.4.3)).] In particular, we see that ϕ (and more generally any cusp form on $\mathfrak{g}$) behaves analogously to a cusp form on G. (Cusp forms on G turn out to be linear combinations of matrix coefficients of supercuspidal representations of G.)

Now (0.0.8) is something that arises naturally in the context of Waldspurger's local trace formula [**Wal95**] on $\mathfrak{g}$, and following Waldspurger we use (see 25.2) the local trace formula to prove that $\hat{I}_\phi$ is represented by a function, as well as to prove Harish-Chandra's Lie algebra analog of (0.0.6) and Waldspurger's Lie algebra analog of (0.0.7).

Waldspurger uses the exponential map to derive the local trace formula on $\mathfrak{g}$ from Arthur's [**Art91a**] local trace formula on G. A second goal of this article is to write out a direct proof of the local trace formula on $\mathfrak{g}$. For the most part we follow Arthur's treatment of the geometric side of the one on G, the main point being Arthur's key geometric result (Theorem 22.3). However some steps in the proof are handled differently. For example toric varieties are used to pass from weight factors obtained by counting lattice points to weight factors obtained as volumes of convex polytopes; these considerations lead to a variant of the local trace formula taking values in the complexified K-theory of the relevant toric variety.

Shalika germs play an important role in Harish-Chandra's proofs and are used elsewhere in harmonic analysis. The last goal of this article is to give a self-contained treatment of them, including Harish-Chandra's deep linear independence result, which is closely tied to the density of regular semisimple orbital integrals (see Theorem 27.5) in the space of all invariant distributions on $\mathfrak{g}$.

It remains to explain the organization of this article. The first ten sections cover roughly the same material as that presented at the summer school. The first section discusses an abstract form of the local trace formula that one has on any compact group. It provides motivation and a first glimpse of how harmonic analysis works, but is not used again later in the article. The second section discusses the basics of integration on l.c.t.d spaces and proves a rather technical lemma used later in semisimple descent for orbital integrals. The third and fourth sections provide background for the fifth section, which aims to give the reader a feel for orbital integrals on p-adic groups by calculating lots of them for GL_2; a side benefit is that the calculations illustrate the phenomenon of homogeneity that is the subject of DeBacker's article in this volume. The sixth section establishes the existence of the Shalika germ expansion on G. The seventh section proves the Weyl integration formula, a simple but important ingredient in the local trace formula. The eighth section begins our discussion of the local trace formula. The ninth and tenth sections prove the local trace formula on $\mathfrak{g}$ for $G = GL_2$ and derive from it the fact that $\hat{I}_\phi$ is represented by a function for any $\phi \in C_c^\infty(\mathfrak{g}_e)$ (see (25.4.3) in order to understand why the function I_ϕ considered in section 10 is essentially the same as the one defined earlier in this introduction).

The remainder of the article is less elementary, though it is still almost completely self-contained. To keep the structure theory of G as simple as possible, we usually assume that G is split. The eleventh section (on certain convex cones and polytopes in Euclidean space) is quite technical and should be consulted only as needed while reading later sections. The twelfth section proves some basic facts about the weight factors occurring in Arthur's weighted orbital integrals. Once the definitions have been understood, the reader can move on, returning to the lemmas proved in this section when they are referred to later. The next four sections concern descent, both parabolic and semisimple, which is used to perform reduction steps in later proofs. The seventeenth section proves some relatively easy results about Shalika germs on $\mathfrak{g}$: homogeneity (which lets one define Shalika germs as canonical functions on $\mathfrak{g}_{\mathrm{rs}}$, not just germs of functions) and local boundedness (see Theorem 17.9) of normalized Shalika germs. As a consequence we obtain the boundedness (see Theorem 17.10) of the function $X \mapsto I_X(f)$ on $\mathfrak{g}_{\mathrm{rs}}$, where I_X denotes the normalized orbital integral over the orbit of $X \in \mathfrak{g}_{\mathrm{rs}}$.

In the next two sections we study norms on the set $X(F)$ of F-points on a variety X (usually affine) over a field F equipped with a non-trivial absolute value. It is standard practice to use such norms on $\mathfrak{g}$ and G, but it seems useful to study them in greater generality, so that one can also take X to be a G-orbit in $\mathfrak{g}$, for example. Most of this material is very easy, the one result requiring some work being Proposition 18.3. The twentieth section uses this theory of norms to estimate weighted orbital integrals.

The next four sections prove the local trace formula on $\mathfrak{g}$ (for any split group G), including the K-theoretic version (see 24.5) as well as the standard one (Theorem 24.1) involving volumes of convex hulls. The formula simplifies (see (24.10.8) and (24.10.9)) when one of the test functions is a cusp form. In the next section we use the local trace formula to prove (see Theorem 25.1) the facts about $\hat{I}_\phi$ mentioned earlier in this introduction.

The next two sections apply Theorem 25.1, Howe's finiteness theorem (see 26.2) and the elementary part of the theory of Shalika germs in order to prove the rest of the main results in harmonic analysis on $\mathfrak{g}$, namely Theorem 27.5 (linear independence of Shalika germs and density of regular semisimple orbital integrals), Theorem 27.8 ($\hat{I}$ is represented by a function when I is an invariant distribution whose support is bounded modulo conjugation), and Theorem 27.12 (the Lie algebra analog of Harish-Chandra's local character expansion for Θ_π).

The last section is a guide to some of the notation used in this article.

1. Local trace formula for compact groups G

In this section G denotes a compact (Hausdorff) topological group and dg denotes the unique Haar measure on G that gives measure 1 to G.

1.1. Finite dimensional representations of G. We need to spend a moment discussing finite dimensional representations (π, V) of G. In other words we are interested in continuous linear actions $G \times V \to V$, where V is a finite dimensional complex vector space. Continuity means that the map $G \times V \to V$ is continuous; linearity means that for each $g \in G$ the map $v \mapsto gv$ is a linear transformation $\pi(g)$ from V to itself.

We write (π^*, V^*) for the contragredient representation of (π, V). Here V^* is the vector space dual to V, and π^* is the obvious representation of G on V^*, characterized by the equation

$$\langle gv^*, gv\rangle = \langle v^*, v\rangle \tag{1.1.1}$$

for all $v \in V$, $v^* \in V^*$, $g \in G$, the pairing on both sides of this equation being the canonical one given by evaluating linear functionals on vectors.

1.2. Group algebra $C(G)$. For our purposes the best version of the group algebra of G is obtained by taking the vector space $C(G)$ of continuous complex-valued functions on G, viewed as a $\mathbb{C}$-algebra using convolution. Recall that the convolution $f_1 * f_2$ of two functions f_1, $f_2 \in C(G)$ is the function on G whose value at $x \in G$ is given by

$$\int_G f_1(xg^{-1})f_2(g)\,dg. \tag{1.2.1}$$

Let (π, V) be a finite dimensional representation of G. Then the group algebra $C(G)$ acts on V in a natural way. For $f \in C(G)$ we denote by $\pi(f)$ the linear transformation by which f acts on V; it is defined by

$$\pi(f)(v) = \int_G f(g)gv\,dg. \tag{1.2.2}$$

1.3. Characters of finite dimensional representations of G. Let (π, V) be a finite dimensional representation of G. We write Θ_π for the character of π, which is by definition the function on G defined by

$$\Theta_\pi(g) = \operatorname{trace}\pi(g). \tag{1.3.1}$$

Similarly, for $f \in C(G)$ we define a complex number $\Theta_\pi(f)$ by

$$\Theta_\pi(f) = \operatorname{trace}\pi(f). \tag{1.3.2}$$

It is clear from the definitions that

$$\Theta_\pi(f) = \int_G f(g)\Theta_\pi(g)\,dg. \tag{1.3.3}$$

1.4. Function space $L^2(G)$. We use dg to form the space $L^2(G)$ of square-integrable functions on G. The group $G \times G$ acts by unitary transformations on the Hilbert space $L^2(G)$, the action of $(g_1, g_2) \in G \times G$ on $\varphi \in L^2(G)$ being given by the rule

$$\big((g_1, g_2)\varphi\big)(x) = \varphi(g_1^{-1}xg_2). \tag{1.4.1}$$

The $(G \times G)$-module $L^2(G)$ can also be viewed as a $C(G) \otimes_{\mathbb{C}} C(G)$-module, the action of $f_1 \otimes f_2 \in C(G) \otimes_{\mathbb{C}} C(G)$ on $\varphi \in L^2(G)$ being given by the following integrated form of (1.4.1):

$$\big((f_1 \otimes f_2)\varphi\big)(x) = \int_G \int_G f_1(g_1)f_2(g_2)\varphi(g_1^{-1}xg_2)\,dg_1\,dg_2. \tag{1.4.2}$$

In the integral over $g_2 \in G$ we may replace g_2 by $x^{-1}g_1g_2$, obtaining

$$\big((f_1 \otimes f_2)\varphi\big)(x) = \int_G \int_G f_1(g_1)f_2(x^{-1}g_1g_2)\varphi(g_2)\,dg_1\,dg_2, \tag{1.4.3}$$

which shows that $f_1 \otimes f_2$ acts by an integral operator whose kernel function K is given by

$$K(x, y) = \int_G f_1(g)f_2(x^{-1}gy)\,dg. \tag{1.4.4}$$

Clearly this kernel is a continuous (hence square-integrable) function on $G \times G$, so that the action of $f_1 \otimes f_2$ on $L^2(G)$ is given by a Hilbert-Schmidt operator. Similarly (and even more simply) the left-translation (resp., right-translation) action of f_1 (resp., f_2) on $L^2(G)$ is given by a continuous kernel, hence by a Hilbert-Schmidt operator; since the product of the Hilbert-Schmidt operators obtained from f_1 and f_2 separately gives the action of $f_1 \otimes f_2$, we see that $f_1 \otimes f_2$ is a trace class operator whose trace is equal to the integral of the kernel K over the diagonal:

$$\operatorname{trace}(f_1 \otimes f_2; L^2(G)) = \int_G \int_G f_1(g)f_2(x^{-1}gx)\,dg\,dx. \tag{1.4.5}$$

This equation is a preliminary form of the trace formula for the compact group G. We will modify both sides of (1.4.5) in order to get the final form of the trace formula for G. To rewrite the left side we will use the Peter-Weyl theorem.

1.5. Peter-Weyl theorem. The Peter-Weyl theorem (see the book [**Kna86**] by Knapp, for example) tells us that the $(G \times G)$-module $L^2(G)$ is isomorphic to the Hilbert space direct sum

$$\hat{\bigoplus}_{(\pi,V)} V \otimes_{\mathbb{C}} V^*, \tag{1.5.1}$$

where (π, V) runs over a set of representatives for the isomorphism classes of irreducible finite dimensional representations of G. We have already discussed the G-module structure on V^*. We regard $V \otimes_{\mathbb{C}} V^*$ as a $(G \times G)$-module by the following rule:

$$(g_1, g_2)(v \otimes v^*) = g_1v \otimes g_2v^*. \tag{1.5.2}$$

Therefore the left (spectral) side of the trace formula can be rewritten as

$$\sum_{(\pi,V)} \Theta_\pi(f_1)\Theta_{\pi^*}(f_2). \tag{1.5.3}$$

1.6. Final form of the trace formula. Now we manipulate the right (geometric) side of the trace formula. Note that the right side of (1.4.5) can be rewritten as

$$\int_G \int_G f_1(y^{-1}gy) f_2(x^{-1}gx)\, dg\, dx \tag{1.6.1}$$

for any $y \in G$. [Change variables twice: first replace x by $y^{-1}x$, then g by $y^{-1}gy$.] Integrating over y and changing the order of integration, we arrive at the final form of the trace formula:

$$\sum_{(\pi,V)} \Theta_\pi(f_1)\Theta_{\pi^*}(f_2) = \int_G \tilde{f}_1(g)\tilde{f}_2(g)\, dg, \tag{1.6.2}$$

where for any $f \in C(G)$ we define $\tilde{f} \in C(G)$ by

$$\tilde{f}(g) = \int_G f(x^{-1}gx)\, dx. \tag{1.6.3}$$

Thus $\tilde{f}(g)$ is obtained by integrating f over the orbit (or conjugacy class) of g; for this reason $\tilde{f}(g)$ is known as an "orbital integral." Obviously the function $\tilde{f}$ is constant on orbits.

1.7. Algebraic form of the Peter-Weyl theorem. Consider once again the Peter-Weyl theorem isomorphism (of $(G \times G)$-modules)

$$L^2(G) \cong \hat{\bigoplus}_{(\pi,V)} V \otimes_{\mathbb{C}} V^*. \tag{1.7.1}$$

Inside the Hilbert space direct sum on the right side of (1.7.1) we have the algebraic direct sum, which can be characterized as the set of vectors u such that the $(G\times G)$-module generated by u is finite dimensional. Under the Peter-Weyl isomorphism these correspond to functions $\varphi \in L^2(G)$ that are left and right G-finite, in the sense that the $(G \times G)$-submodule of $L^2(G)$ generated by φ is finite dimensional; it turns out that such functions are automatically continuous. Thus we obtain the algebraic form of the Peter-Weyl theorem

$$C(G)_0 \cong \bigoplus_{(\pi,V)} V \otimes_{\mathbb{C}} V^*, \tag{1.7.2}$$

where $C(G)_0$ denotes the space of left and right G-finite continuous functions on G.

We have not yet specified how we are normalizing the Peter-Weyl isomorphism. To do so we note that $V \otimes_{\mathbb{C}} V^*$ is canonically isomorphic to $\mathrm{End}_{\mathbb{C}}(V)$ (even as $(G \times G)$-module). In our normalization of the Peter-Weyl isomorphism a function $f \in C(G)_0$ maps to the family of elements $\pi(f) \in \mathrm{End}_{\mathbb{C}}(V) = V \otimes_{\mathbb{C}} V^*$. In particular for $f \in C(G)_0$ we have $\pi(f) = 0$ (and hence $\Theta_\pi(f) = 0$) for all but finitely many isomorphism classes of irreducible finite dimensional representations (π, V).

1.8. Fourier transforms of orbital integrals. For any irreducible finite dimensional representation (π, V) of G the linear functional $f \mapsto \Theta_\pi(f)$ on $C(G)_0$ is conjugation invariant, and it is clear from the algebraic Peter-Weyl theorem that any conjugation invariant linear functional is an infinite linear combination of these basic ones. In particular this is the case for the orbital integral

$$f \mapsto \int_G f(x^{-1}gx)\,dx. \tag{1.8.1}$$

In fact we have the following formula for any $g \in G$ and $f \in C(G)_0$:

$$\int_G f(x^{-1}gx)\,dx = \sum_{(\pi,V)} \Theta_\pi(g)\Theta_{\pi^*}(f). \tag{1.8.2}$$

How does one prove this formula? Both sides of it are continuous functions of g; to prove that they are equal it is enough to show that they have the same integral against an arbitrary continuous function on G, and this is just a restatement of the preliminary form (1.4.5) of the trace formula.

1.9. Plancherel formula. In the special case $g = 1$ equation (1.8.2) yields the Plancherel formula (valid for any $f \in C(G)_0$)

$$f(1) = \sum_{(\pi,V)} \dim(\pi)\Theta_\pi(f). \tag{1.9.1}$$

(Here we used that π and π^* have the same dimension.)

1.10. Matrix coefficients. Let (π, V) be an irreducible finite dimensional representation of G. For $v \in V$, $v^* \in V^*$ such that $\langle v^*, v\rangle = 1$ we define functions $f_{v^*,v}$ and f_{v,v^*} on G by $f_{v^*,v}(g) = \langle v^*, gv\rangle$ and $f_{v,v^*}(g) = \langle gv^*, v\rangle$. Both functions lie in $C(G)_0$. The function $f_{v^*,v}$ is a *matrix coefficient* for π, while f_{v,v^*} is a matrix coefficient for π^*. The two functions are related by $f_{v,v^*}(g) = f_{v^*,v}(g^{-1})$.

We can use matrix coefficients to give an explicit formula for the inverse β of the isomorphism α appearing in the algebraic Peter-Weyl theorem. Recall that $\alpha(f)$ is the element of

$$\bigoplus_{(\pi,V)} \mathrm{End}_{\mathbb{C}}(V) \tag{1.10.1}$$

whose (π, V)-th component is $\pi(f)$. We will give an explicit formula for β on each summand $V \otimes_{\mathbb{C}} V^* = \mathrm{End}_{\mathbb{C}}(V)$. For $v \otimes v^* \in V \otimes_{\mathbb{C}} V^*$ we claim that $\beta(v \otimes v^*) = \dim \pi \cdot f_{v,v^*}$.

Why is this the right formula for β? The representations $V \otimes_{\mathbb{C}} V^*$ of $G \times G$ are irreducible and pairwise non-isomorphic, and the map $v \otimes v^* \mapsto f_{v,v^*}$ is $(G \times G)$-equivariant and non-zero. Therefore it is clear that there exists a scalar c_π such that $\beta(v \otimes v^*) = c_\pi \cdot f_{v,v^*}$. Taking $f = f_{v,v^*}$ in the Plancherel formula (1.9.1), we see that $c_\pi = \dim \pi$.

1.11. Orbital integrals of matrix coefficients. Let (π, V) be an irreducible finite dimensional representation of G. Let $g \in G$. Then from (1.8.2) it follows easily that

$$\int_G f_{v^*,v}(x^{-1}gx)\,dx = (\dim \pi)^{-1} \cdot \Theta_\pi(g). \tag{1.11.1}$$

Thus the orbital integrals of the matrix coefficient $f_{v^*,v}$ for π give the character values of π (up to the scalar $(\dim \pi)^{-1}$). We have proved this as a consequence of the Peter-Weyl theorem and the trace formula for G, but in fact there is a simple direct proof, as the reader may wish to devise as an exercise. [Hint: Consider the endomorphism $\int_G \pi(x^{-1}gx)\,dx$ of V.]

1.12. Comments. Our goal here has not been to develop harmonic analysis on compact groups in the most efficient way, but rather to concentrate on the trace formula and its relationship to other basic concepts, stressing those, such as orbital integrals, that we will meet again in the non-compact case.

A more standard treatment would emphasize the orthogonality relations (for irreducible characters and for matrix coefficients). The trace formula for G has essentially the same information in it, but packaged in a slightly different way, as we have tried to illustrate.

2. Basics of integration

2.1. l.c.t.d spaces. What kind of spaces will we be integrating over? In this article we are interested in p-adic groups and Lie algebras, so the topological spaces we will encounter will be l.c.t.d spaces (short for locally compact and totally disconnected). Thus an l.c.t.d space is a Hausdorff topological space in which every point has a neighborhood basis of compact open subsets.

On an l.c.t.d space X the most important space of functions is $C_c^\infty(X)$, the space of all locally constant, compactly supported, complex-valued functions on X. Any such function can be written as a linear combination of characteristic functions of compact open subsets of X. This makes integration rather easy, at least in principle: we just need to assign measures to compact open subsets.

Let X be an l.c.t.d space and let Y be a closed subset with complementary open subset U. Then both Y and U are themselves l.c.t.d spaces, and it is an instructive exercise to check that the sequence

$$0 \to C_c^\infty(U) \to C_c^\infty(X) \to C_c^\infty(Y) \to 0 \tag{2.1.1}$$

is exact. (The first map is given by extending by 0, the second by restriction to Y.)

Sometimes it is useful to consider vector-valued functions. For any complex vector space V we write $C_c^\infty(X;V)$ for the space of locally constant, compactly supported functions on X with values in V. It is easy to check that

$$C_c^\infty(X;V) = C_c^\infty(X) \otimes_{\mathbb{C}} V. \tag{2.1.2}$$

LEMMA 2.1. *Let X, Y be l.c.t.d topological spaces. Then the product $X \times Y$ is also a l.c.t.d space, and moreover there are equalities*

$$C_c^\infty(X \times Y) = C_c^\infty(X; C_c^\infty(Y)) = C_c^\infty(X) \otimes_{\mathbb{C}} C_c^\infty(Y).$$

We leave the proof to the reader as another exercise.

2.2. l.c.t.d groups. An l.c.t.d topological group is by definition a locally compact Hausdorff topological group G in which the identity element has a neighborhood basis of compact open subgroups. Clearly G is then a l.c.t.d topological space. For us a typical example is $G(F)$, where F is a p-adic field and G is a linear algebraic group over F. To see that $G(F)$ is a l.c.t.d group we reduce to the case of the general linear group GL_n (by choosing an embedding of G in a general linear group); in $GL_n(F)$ the compact open subgroups

$$K_n = \{g \in GL_n(\mathcal{O}) : g \equiv 1 \mod \pi^n\} \tag{2.2.1}$$

give the desired neighborhood basis at the identity element. Here (and throughout the article) we write $\mathcal{O}$ for the valuation ring in F and π for a generator of the maximal ideal of $\mathcal{O}$.

2.3. Unimodular groups. Any locally compact Hausdorff topological group G admits a left invariant Radon measure dg, known as a (left) Haar measure, and dg is unique up to a positive scalar. Since right translations commute with left translations, a right translate of dg is another Haar measure, hence is a positive multiple of dg; in this way one obtains the *modulus character* δ_G (with values in the multiplicative group of positive real numbers) characterized by the property

$$d(gh^{-1}) = \delta_G(h) \cdot dg \tag{2.3.1}$$

or, equivalently,

$$d(hgh^{-1}) = \delta_G(h) \cdot dg. \tag{2.3.2}$$

When the modulus character is trivial, one says that G is *unimodular*. In this case dg is both left and right invariant and $d(g^{-1}) = dg$.

For a reductive group G over our p-adic field F the group $G(F)$ is always unimodular. This stems from the fact that G acts trivially on the top exterior power of the Lie algebra of G. On the other hand, for any proper parabolic subgroup P of G, the group $P(F)$ is not unimodular.

On a l.c.t.d group G integration is particularly simple. Fix some compact open subgroup K_0. Then there is a unique Haar measure dg giving K_0 measure 1. For any compact open subgroup K of G the measure of K is

$$[K : K \cap K_0] \cdot [K_0 : K \cap K_0]^{-1}. \tag{2.3.3}$$

Moreover for any compact open subset S of G there is a compact open subgroup K small enough that S is a disjoint union of cosets gK, so that the measure of S is the number of such cosets times the measure of K. That's all we need to know about integration on l.c.t.d groups!

2.4. Integration on homogeneous spaces. Let G be a unimodular locally compact Hausdorff topological group and let H be a closed subgroup. Then there exists a right G-invariant Radon measure on $H\backslash G$ if and only if H is unimodular. Assume this is so, and assume also that G (and hence H) is a l.c.t.d group.

Choose Haar measures dg, dh on G, H respectively. Then there is a quotient measure (right G-invariant) dg/dh on $H\backslash G$ characterized by the formula (integration in stages)

$$\int_G f(g)dg = \int_{H\backslash G} \int_H f(hg)\, dh\, dg/dh, \tag{2.4.1}$$

valid for all $f \in C_c^\infty(G)$.

The reason why the integration in stages formula characterizes the invariant integral on $H\backslash G$ is that any function in $C_c^\infty(H\backslash G)$ lies in the image of the linear map

$$C_c^\infty(G) \to C_c^\infty(H\backslash G) \qquad (2.4.2)$$
$$f \mapsto f^\sharp$$

defined by putting

$$f^\sharp(g) = \int_H f(hg)\,dh. \qquad (2.4.3)$$

Again we can see rather concretely how the measure works. Indeed, any compact open subset of our homogeneous space can be written as a disjoint union of ones of the form $H\backslash HgK$ (for some compact open subgroup K of G), and the measure of $H\backslash HgK$ is given by

$$\operatorname{meas}_{dg}(K)/\operatorname{meas}_{dh}(H \cap gKg^{-1}), \qquad (2.4.4)$$

as one sees by applying integration in stages to the characteristic function of gK.

2.5. Integration in stages in reversed order. We continue with G, H as above. Later, when discussing descent for orbital integrals, we will need the integral formula in Lemma 2.3 below. As a warm-up exercise, we first prove a simpler statement, which can be thought of as a version of integration in stages in which the order of integration has in a sense been reversed. This reversed formula involves a compact open subset C of $H\backslash G$. We also need to choose $\alpha \in C_c^\infty(G)$ such that $\alpha^\sharp = 1_C$.

LEMMA 2.2. *For all $f \in C_c^\infty(G)$ such that the image of $\operatorname{Supp}(f)$ under $G \twoheadrightarrow H\backslash G$ is contained in C there is an equality*

$$\int_G f(g)\,dg = \int_H \Big(\int_G f(hg)\alpha(g)\,dg\Big)\,dh. \qquad (2.5.1)$$

PROOF. Change variables in the integral on the right, replacing g by $h^{-1}g$, then reverse the order of integration, then replace h by h^{-1}, noting that $d(h^{-1}) = dh$, a consequence of the unimodularity of H. $\square$

For descent theory we will actually need a variant of (2.5.1), in which we are also given a closed unimodular subgroup I of H and a Haar measure di on I. With C, α as before we then have the following lemma.

LEMMA 2.3. *For any integrable, locally constant function f on $I\backslash G$ such that the image of $\operatorname{Supp}(f)$ in $H\backslash G$ is contained in C there is an equality*

$$\int_{I\backslash G} f(g)\,dg/di = \int_{I\backslash H} \phi(h)\,dh/di \qquad (2.5.2)$$

where ϕ is the integrable, locally constant function on $I\backslash H$ defined by

$$\phi(h) = \int_G f(hg)\alpha(g)\,dg.$$

PROOF. Let $\beta \in C_c^\infty(G)$ and consider $\beta^\sharp \in C_c^\infty(H\backslash G)$ defined as above by

$$\beta^\sharp(g) = \int_H \beta(hg)\,dh = \int_H \beta(h^{-1}g)\,dh.$$

Then, abbreviating dg/di and dh/di to $d\dot{g}$ and $d\dot{h}$ respectively, we have

$$\begin{aligned}\int_{I\backslash G} f(g)\beta^\sharp(g)\,d\dot{g} &= \int_{I\backslash G} f(g)\int_H \beta(h^{-1}g)\,dh\,d\dot{g}\\ &= \int_{I\backslash G} f(g)\int_{I\backslash H}\int_I \beta(h^{-1}i^{-1}g)\,di\,d\dot{h}\,d\dot{g}\\ &= \int_{I\backslash H}\int_{I\backslash G}\int_I f(g)\beta(h^{-1}ig)\,di\,d\dot{g}\,d\dot{h}\\ &= \int_{I\backslash H}\int_G f(g)\beta(h^{-1}g)\,dg\,d\dot{h}\\ &= \int_{I\backslash H}\int_G f(hg)\beta(g)\,dg\,d\dot{h}.\end{aligned}$$

To see that all these integrals are convergent, replace f and β by their absolute values and note that the integral we started with is obviously convergent, since f is integrable and $\beta^\sharp$ is bounded. Fubini's theorem takes care of the rest.

Taking $\beta = \alpha$, we obtain

$$\int_{I\backslash G} f(g)1_C(g)\,d\dot{g} = \int_{I\backslash H}\phi(h)d\dot{h}, \tag{2.5.3}$$

which in view of our assumption on the support of f yields the equality stated in the lemma. We have seen along the way that ϕ is integrable, and it is obviously locally constant. ☐

3. Preliminaries about orbital integrals

3.1. The set-up. We are going to discuss orbital integrals on $G(F)$, where G is a connected reductive group over a p-adic field F. We fix an algebraic closure $\bar{F}$ of F.

3.2. Orbits. Let $\gamma \in G(F)$. We are interested in the orbit $O(\gamma)$ of γ for the conjugation action of G on itself. In other words $O(\gamma)$ is the conjugacy class of γ, a locally closed subset G in the Zariski topology, isomorphic as variety to G/G_γ, where G_γ denotes the centralizer of γ in G (see [**Bor91**, Prop. 6.7]). There is an exact sequence of pointed sets

$$1 \to G_\gamma(F) \to G(F) \to (G/G_\gamma)(F) \to H^1(F, G_\gamma) \to H^1(F, G) \tag{3.2.1}$$

where $H^1(F, G)$ denotes the Galois cohomology set $H^1(\mathrm{Gal}(\bar{F}/F), G(\bar{F}))$, and the boundary map in this sequence induces a bijection from the set of $G(F)$-orbits in $O(\gamma)(F)$ to the set

$$\ker[H^1(F, G_\gamma) \to H^1(F, G)]. \tag{3.2.2}$$

Since $H^1(F, G_\gamma)$ is a finite set (see [**Ser02**, Ch. III,§4]), there are in fact only finitely many such orbits. From the theory of p-adic manifolds (see [**Ser92**]) one knows first of all that each $G(F)$-orbit is open in $O(\gamma)(F)$, hence also closed in $O(\gamma)(F)$, and second of all that the $G(F)$-orbit of γ is isomorphic as p-adic manifold (hence also as topological space) to the homogeneous space $G(F)/G_\gamma(F)$. Since $O(\gamma)$ is locally closed in G, the set $O(\gamma)(F)$ is locally closed in $G(F)$, and it follows that the same is true of each individual $G(F)$-orbit in $O(\gamma)(F)$.

When γ is semisimple, $O(\gamma)$ is closed in G (see [**Bor91**, Thm. 9.2]), hence $O(\gamma)(F)$ and the individual $G(F)$-orbits in it are all closed in $G(F)$.

It is instructive to consider the example of the group GL_2. There is map α of algebraic varieties from GL_2 to the affine plane $\mathbb{A}^2$, defined by $g \mapsto (\mathrm{trace}(g), \det(g))$. On $\mathbb{A}^2$ we have the discriminant function D, defined by $D(b,c) = b^2 - 4c$. All fibers of α are of course closed. The fiber of α over a point (b,c) where D is non-zero consists of a single orbit of regular semisimple elements. (An element in GL_2 is regular semisimple if and only if it has distinct eigenvalues.) The fiber of α over a point (b,c) where D vanishes is the union of two G-orbits, namely those of the matrices

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \text{ and } \begin{bmatrix} a & 1 \\ 0 & a \end{bmatrix},$$

where we have written a for $b/2$. The first matrix is semisimple (but not regular), and its orbit is closed in the fiber. The second matrix is regular (but not semisimple), and its orbit is open in the fiber. A very special feature of the group GL_2 (and, more generally, of GL_n) is that the Galois cohomology set $H^1(F, G_\gamma)$ is always trivial, so that $O(\gamma)(F)$ always consists of a single $G(F)$-orbit.

A map similar to α exists for any G. For GL_n one simply uses all the coefficients of the characteristic polynomial of a matrix to define a map from GL_n to $\mathbb{A}^n$. In general one uses the morphism $G \to G/\mathrm{Int}(G)$, where $G/\mathrm{Int}(G)$ is by definition the affine scheme whose ring of regular functions is the ring of conjugation invariant regular functions on G. Later (see 14.2) we will discuss the analogous construction for the Lie algebra of G in greater detail.

3.3. Definition of orbital integrals. Let $\gamma \in G(F)$. The orbital integral $O_\gamma(f)$ of a function $f \in C_c^\infty(G(F))$ is by definition the integral

$$O_\gamma(f) := \int_{G_\gamma(F)\backslash G(F)} f(g^{-1}\gamma g)\, d\dot{g} \tag{3.3.1}$$

where $d\dot{g}$ is a right $G(F)$-invariant measure on the homogeneous space over which we are integrating. Thus O_γ depends on a choice of measure, but once this choice is made we get a well-defined linear functional on $C_c^\infty(G(F))$. (We are not putting any topology on our function space, so there is no continuity requirement in the definition of linear functional.)

Two comments are needed. First, we need to know that $G_\gamma(F)$ is unimodular in order to ensure that $d\dot{g}$ exists. For semisimple elements γ there is no problem, since then G_γ is reductive. In general, however, G_γ is not reductive, and we need to argue as follows. By the Jordan decomposition (see [**Bor91**]) we can decompose γ uniquely as $\gamma = su = us$ with s semisimple and u unipotent. It follows that $u \in G_s$ and that G_γ coincides with the centralizer of u in the reductive group G_s. This reduces us to the case in which γ is unipotent. Then (since the characteristic of our field is 0) we can write γ as the exponential of a nilpotent element in the Lie algebra over G over F. Using a G-invariant non-degenerate symmetric bilinear form to identify the Lie algebra with its dual, we see that it is enough to prove that the stabilizer (for the coadjoint action) of any element in the dual of the Lie algebra is unimodular. This is equivalent to the statement that every coadjoint orbit carries a $G(F)$-invariant measure, which in turn follows from the fact that every coadjoint orbit admits a G-invariant structure of symplectic manifold and hence admits a G-invariant volume form (which can then be used to construct a

$G(F)$-invariant measure on the F-points of the orbit). See 17.3 for a discussion of the symplectic structure on coadjoint orbits.

Second, we need to know that the integral converges. For semisimple elements there is again no problem, since the orbit is closed, which ensures that the integrand is a compactly supported (and locally constant) function on the homogenous space, which is exactly the sort of function we can integrate. For arbitrary γ the orbit is only locally closed, and while the integrand is still locally constant, it is not necessarily compactly supported. (Of course the integral is still discrete in nature, but it boils down to an infinite series rather than a finite sum, so that convergence is not obvious.) For a proof of convergence see [**Rao72**], and for a slightly different perspective on the geometry involved see [**Pan91**]. The idea is to reduce to the case of nilpotent orbital integrals and then to show that the G-invariant volume form on a nilpotent orbit extends (without singularities) to a suitable desingularization (constructed using the theory of $\mathfrak{sl}(2)$-triples) of the closure of that nilpotent orbit.

3.4. Orbital integrals of characteristic functions of double cosets. We continue with $\gamma \in G(F)$. We are going to lighten notation by writing G and G_γ instead of $G(F)$ and $G_\gamma(F)$. The material in this subsection will be used in section 5, when we calculate orbital integrals of functions in the spherical Hecke algebra of $GL_2(F)$.

Let K be a compact open subgroup of G, and write X for the homogeneous space G/K. Since K is open, X has the discrete topology. We write x_0 for the base point in X (given by the trivial coset of K in G). Given $(x_1, x_2) \in X \times X$ we pick $g_1, g_2 \in G$ such that $x_i = g_i x_0$ for $i = 1, 2$. The double coset $K g_2^{-1} g_1 K$ is well-defined and will be denoted by $\mathrm{inv}(x_1, x_2)$. It follows immediately from these definitions that the map $\mathrm{inv} : X \times X \to K\backslash G/K$ induces a bijection from the set of G-orbits on $X \times X$ to $K\backslash G/K$ (with G acting on $X \times X$ by $g(x_1, x_2) = (gx_1, gx_2)$). Here "inv" is short for "invariant". The reason for this name is that $\mathrm{inv}(x_1, x_2)$ is an invariant measuring the relative position of the two points x_1 and x_2.

For any $a \in G$ we can consider the double coset KaK, a compact open subset of G, as well as its characteristic function 1_{KaK}, which lies in $C_c^\infty(G)$. The orbital integrals of 1_{KaK} can be understood using the action of G on X. Indeed, from (2.4.4) it follows that

$$\int_{G_\gamma\backslash G} 1_{KaK}(g^{-1}\gamma g)\ dg/dg_\gamma = \sum_x \frac{\mathrm{meas}(K)}{\mathrm{meas}(\mathrm{Stab}_{G_\gamma}(x))}, \tag{3.4.1}$$

where the sum on the right side runs over a set of representatives for the G_γ-orbits on the set of elements $x \in X$ such that $\mathrm{inv}(\gamma x, x) = KaK$, and the measures are taken with respect to the Haar measures dg, dg_γ on G, G_γ respectively; $\mathrm{Stab}_{G_\gamma}(x)$ denotes the stabilizer of x in G_γ, a compact open subgroup of G_γ.

We may replace G_γ by any convenient closed subgroup G'_γ such that $G'_\gamma\backslash G_\gamma$ is compact. This multiplies the orbital integral by the factor $\mathrm{meas}(G'_\gamma\backslash G_\gamma)$, but (3.4.1) remains valid, with G'_γ replacing G_γ everywhere. In particular, when G_γ is compact, we may take G'_γ to be the trivial subgroup. Then, if we use the Haar measure dg on G giving K measure 1, we have

$$\int_G 1_{KaK}(g^{-1}\gamma g)\ dg = |\{x \in X : \mathrm{inv}(\gamma x, x) = KaK\}|, \tag{3.4.2}$$

where $|S|$ is being used to denote the cardinality of a finite set S, showing that our orbital integral is the answer to a simple counting problem involving the action of G on X. In section 5 we will solve such counting problems for $G = GL_2(F)$, $K = GL_2(\mathcal{O})$ using the tree for $SL_2(F)$, but first we need to discuss two double coset decompositions.

4. Cartan and Iwasawa decompositions

When calculating orbital integrals for GL_2, we are going to need both the Cartan and Iwasawa decompositions. Later we will need them for all split groups, as this is the context in which we will discuss the Lie algebra version of the local trace formula. In this section we will state both decompositions for all split groups and will sketch proofs for GL_n.

4.1. Notation. Let F be a p-adic field with valuation ring $\mathcal{O}$ and uniformizing element π (so that the valuation of π is 1). For $x \in F^\times$ we denote by $\operatorname{val}(x)$ the valuation of x.

In this section G denotes a split connected reductive group scheme over $\mathcal{O}$. We need to choose various $\mathcal{O}$-subgroup schemes in G: a split maximal torus A, and a Borel subgroup $B = AN$ containing A and having unipotent radical N. We write A_G for the identity component of the center of G. In the case of $G = GL_n$ we make the standard choices: B consists of upper-triangular matrices, A of diagonal matrices, and N of upper-triangular matrices with 1's on the diagonal. We write W for the Weyl group of A.

We write G_{der} for the derived group of the algebraic group G, and we write G_{sc} for its simply connected cover.

We will need the group $X_*(A)$ of cocharacters of A (in other words, the group of homomorphisms from the multiplicative group $\mathbb{G}_m$ to A). The cocharacter group is a free abelian group whose rank is equal to the dimension of A over F. For GL_n we identify $X_*(A)$ with $\mathbb{Z}^n$ as follows: to the n-tuple $(j_1, \dots, j_n)$ corresponds the cocharacter which sends an element z in the multiplicative group to the diagonal matrix whose diagonal entries are $(z^{j_1}, \dots, z^{j_n})$.

To lighten notation we write K for $G(\mathcal{O})$ and then abbreviate $G(F)$, $A(F)$, $B(F)$, $N(F)$ to G, A, B, N respectively.

4.2. The isomorphism $A/A \cap K \simeq X_*(A)$. For GL_n the map which sends a diagonal matrix to the n-tuple of valuations of the diagonal entries induces an isomorphism from $A/A \cap K$ to $\mathbb{Z}^n$. In general we have a canonical isomorphism $A/A \cap K \simeq X_*(A)$, under which a cocharacter μ corresponds to the (class of) the element $\pi^\mu \in A$ obtained by applying the cocharacter to the element π in the multiplicative group.

4.3. Cartan decomposition. The crude version of the Cartan decomposition states simply that $G = KAK$. It is an instructive exercise to prove this for GL_n. [Start with an element in G and modify it by row and column operations coming from K until eventually it is transformed into a diagonal matrix.]

For $a, a' \in A$, when are the double cosets KaK and $Ka'K$ the same? The refined version answers this question. First of all it is evident that $KaK = Ka'K$ if a, a' have the same image in $A/A \cap K$. Second of all, since we can find representatives in K for all elements of the Weyl group W, it is also clear that $KaK = Kw(a)K$ for all $w \in W$ (where $w(a)$ denotes the conjugation action of W on A). In fact these two observations turn out to be the end of the story: $KaK = Ka'K$ if and only if the images of a, a' in $A/A\cap K$ are conjugate under the Weyl group. In view of the discussion in 4.2 we obtain a natural bijection from the set of $K\backslash G/K$ of K-double cosets to the set of orbits of W in $X_*(A)$. The dominant coweights provide a natural set of orbit representatives for this action, so that the set of K-double cosets can also be parametrized by dominant coweights. (Recall that a coweight μ is said to be *dominant* if $\langle\alpha,\mu\rangle \geq 0$ for every positive root α of A.)

For GL_n the dominant coweights are the n-tuples $(j_1,\dots,j_n)\in\mathbb{Z}^n$ such that $j_1 \geq \cdots \geq j_n$, and we conclude that such coweights parametrize the K-double cosets. The following method can be used to prove the refined Cartan decomposition for GL_n. The idea is to construct sufficiently many invariants of K-double cosets. The first idea is to consider the valuation of the determinant; this invariant of a matrix clearly only depends on its K-double coset. A more subtle invariant is to consider the least valuation of all the matrix entries (for this purpose we consider that 0 has valuation $+\infty$). The two procedures can be combined by considering any integer i such that $1 \leq i \leq n$ and considering the least valuation of all the $i\times i$ minors in our matrix. Applying this last invariant to any element in the K-double coset containing the diagonal matrix with diagonal entries $(\pi^{j_1},\dots,\pi^{j_n})$, we obtain the sum of the last i entries in the n-tuple $(j_1,\dots,j_n)$. These sums (for all i) together determine the dominant coweight uniquely. This proves the refined Cartan decomposition for GL_n.

4.4. Iwasawa decomposition. The Iwasawa decomposition states that $G = BK$. This reflects the fact that the flag variety $B\backslash G$ is projective over F, hence satisfies the valuative criterion for properness. For GL_n it is another instructive exercise to prove the Iwasawa decomposition directly. [Start with an element in G and modify it by column operations coming from K until eventually it is transformed into an upper-triangular matrix.]

4.5. Definitions of Λ_G, $H_G : G \to \Lambda_G$, $\mathfrak{a}$, $\mathfrak{a}_G$. We now introduce various objects that will be used throughout this article. For example, there is an obvious surjective homomorphism $GL_n(F)\to\mathbb{Z}$ defined by $g\mapsto \operatorname{val}(\det g)$, which we need to generalize to all split groups.

Let Λ_G denote the quotient of $X_*(A)$ by the coroot lattice for G (by which we mean the subgroup of $X_*(A)$ generated by all the coroots of A). There is a surjective homomorphism $H_G : G\to\Lambda_G$, which by virtue of the Cartan decomposition is characterized by the following two properties. The first property is that the restriction of H_G to the subgroup A is equal to the composition

$$A \twoheadrightarrow A/(A\cap K)\cong X_*(A)\twoheadrightarrow \Lambda_G. \tag{4.5.1}$$

The second property is that the restriction of H_G to K is trivial. Moreover it is true that H_G is also trivial on the image in G of the F-points of G_{sc}. From this it follows that if $g = ank$ with $a\in A$, $n\in N$, $k\in K$ (Iwasawa decomposition), then $H_G(g) = H_G(a)$.

We write $\mathfrak{a}$ for $X_*(A)\otimes_{\mathbb{Z}}\mathbb{R}$ and $\mathfrak{a}_G$ for $X_*(A_G)\otimes_{\mathbb{Z}}\mathbb{R}$. Thus $\mathfrak{a}_G$ can be viewed as a subspace of $\mathfrak{a}$. Moreover the composition $X_*(A_G)\hookrightarrow X_*(A)\twoheadrightarrow\Lambda_G$ induces an isomorphism

$$\mathfrak{a}_G \cong \Lambda_G\otimes_{\mathbb{Z}}\mathbb{R}. \tag{4.5.2}$$

When G_{der} is simply connected as algebraic group, the finitely generated abelian group Λ_G is torsion-free and is therefore a free abelian group. In this case the natural map $\Lambda_G \to \mathfrak{a}_G$ is injective and identifies Λ_G with a lattice in the real vector space $\mathfrak{a}_G$, so that there is no harm in thinking about H_G as being a homomorphism $G \to \mathfrak{a}$ that takes values in the lattice Λ_G in $\mathfrak{a}$. When the derived group is not simply connected, Λ_G has torsion which is lost when one passes to $\Lambda_G \otimes_{\mathbb{Z}} \mathbb{R} = \mathfrak{a}_G$. In order to avoid confusion the reader should be aware that in Arthur's papers H_G denotes the composition of our H_G with the natural map $\Lambda_G \to \mathfrak{a}_G$.

5. Orbital integrals on $GL_2(F)$

5.1. The goal. Our goal in this section is to get a better understanding of orbital integrals by calculating lots of them for the group $GL_2(F)$. As we will see, the phenomenon of homogeneity (covered in DeBacker's course in this summer school), shows up very clearly in these calculations. We follow the exposition in [**Lan80**], using the tree for $SL_2(F)$ as our main computational tool.

As before we consider a p-adic field F with valuation ring $\mathcal{O}$ and uniformizing element π. We write q for the cardinality of the residue field $\mathcal{O}/\pi\mathcal{O}$ of $\mathcal{O}$.

We write G for the group $GL_2(F)$ and write K for its compact open subgroup $GL_2(\mathcal{O})$. We will not consider orbital integrals for arbitrary functions in $C_c^\infty(G)$. We will only consider functions lying in the spherical Hecke algebra $\mathcal{H}$, defined as the subspace of $C_c^\infty(G)$ consisting of functions that are both left and right invariant under K. The multiplication on $\mathcal{H}$ is given by convolution and turns out to be commutative, and there is a simple description of this commutative $\mathbb{C}$-algebra (using the Satake isomorphism). Important as these facts are, they play no role here. Our limited goal is to understand the linear functionals on $\mathcal{H}$ obtained by restriction from the linear functionals O_γ on $C_c^\infty(G)$, and even this will be done only for elements $\gamma \in K$ (which covers orbital integrals for all elements in G whose conjugacy class meets K). So, throughout this section γ will always denote an element of K.

The characteristic functions 1_{KaK} of the double cosets KaK of K in G form a basis for the vector space $\mathcal{H}$. By the Cartan decomposition 4.3 there is bijection from $K\backslash G/K$ to $\{(m,n) \in \mathbb{Z}^2 : m \geq n\}$, which associates to the pair (m,n) the double coset containing the diagonal matrix with diagonal entries (π^m, π^n).

For $(m,n) \in \mathbb{Z}^2$ with $m \geq n$, we write $f_{m,n}$ for the characteristic function of the K-double coset corresponding to (m,n). The functions $f_{m,n}$ form a basis for $\mathcal{H}$, so it is enough to calculate the numbers $O_\gamma(f_{m,n})$. Since $\gamma \in K$, the determinant of γ has valuation 0, which means that $O_\gamma(f_{m,n})$ vanishes unless $m+n=0$. Therefore, it is enough to consider the functions f_m defined by $f_m := f_{m,-m}$ (for $m \geq 0$). We will now compute, case-by-case, the orbital integrals $O_\gamma(f_m)$ for all conjugacy classes meeting K.

To define the orbital integral for γ we need an invariant measure dg/dg'_γ on $G'_\gamma\backslash G$ (see 3.4). We will always use the Haar measure dg on G that gives K measure 1. We will discuss dg'_γ case-by-case below.

5.2. Some useful subgroups of G. Let Z denote the group of non-zero scalar multiples of the identity matrix; in other words, Z is the center of G. Let $B = AN$ be as in 4.1.

5.3. Tree for $SL_2(F)$. As mentioned before, our main computational tool will be the tree for $SL_2(F)$, which we now need to discuss. A good reference is [**Ser03**]. A tree is the geometric realization of a 1-dimensional simplicial complex that is both connected and simply connected. It can be specified by giving its set V of vertices and saying which pairs of vertices are joined by an edge. The tree of interest here comes equipped with an action of G, and the action is transitive on the set of vertices. In fact the set V of vertices has a base-point v_0 whose stabilizer is KZ, so that V becomes identified with the homogeneous space G/KZ.

Inside the set V we have the orbit of v_0 under A, which can be identified with $A/(A \cap KZ) = A/(A \cap K)Z$. In fact the group $A/(A \cap K)Z$ is isomorphic to $\mathbb{Z}$, via the isomorphism sending a diagonal matrix to the difference of the valuations of its two diagonal entries. Under this isomorphism an integer j then corresponds to the diagonal matrix

$$\begin{bmatrix} \pi^j & 0 \\ 0 & 1 \end{bmatrix} \in A/(A \cap K)Z, \tag{5.3.1}$$

and we write v_j for the corresponding vertex in V. We connect any two successive vertices v_j, v_{j+1} by an edge, obtaining a 1-dimensional simplicial complex whose geometric realization is a real line, with vertices placed at each integer point; this copy of the real line is called the *standard apartment* in our tree.

So far we have just described some of the edges in our tree, namely the ones joining vertices in $A/(A \cap K)Z$. We get all the edges in the tree by using the action of G to move around the edges we have already described (ensuring that the G-action does preserve the set of edges, as desired). It then turns out that the 1-dimensional simplicial complex we have constructed really is a tree, each vertex of which has $q+1$ neighbors (where q is the cardinality of the residue field).

Indeed, because of the G-action it is enough to show that the base-point v_0 has $q+1$ neighbors. It turns out that K (which fixes v_0) acts transitively on the set of neighbors of v_0. One of these neighbors is v_{-1}, and a simple calculation shows that the stabilizer of v_{-1} in K is

$$\begin{bmatrix} \mathcal{O}^\times & \mathcal{O} \\ \pi O & \mathcal{O}^\times \end{bmatrix}. \tag{5.3.2}$$

Thus the K-orbit of v_{-1} is $G(\mathcal{O}/\pi\mathcal{O})/B(\mathcal{O}/\pi\mathcal{O})$, the set of points on the projective line over the residue field, which explains why there are $q+1$ neighbors of v_0.

5.4. Metric on the tree and its relation to $\mathrm{inv}(x_1, x_2)$**.** There is an obvious metric $d(y_1, y_2)$ on the tree. It is G-invariant, and on the standard apartment discussed above it agrees with the usual metric on the real line. For this metric two neighboring vertices have distance 1, and $d(v_m, v_n) = |m - n|$.

As before we write X for G/K and x_0 for the base-point in X. Let $x_1, x_2 \in X$. Since the set V of vertices of the tree is equal to $G/KZ = Z\backslash X$, the images of x_1, x_2 under the canonical surjection $X \to Z\backslash X = V$ are vertices v_1, v_2 in the tree.

As in 3.4 the relative position of x_1, x_2 is measured by $\mathrm{inv}(x_1, x_2) \in K\backslash G/K$. By the refined Cartan decomposition for GL_2 we can view $\mathrm{inv}(x_1, x_2)$ as a pair (m, n) of integers such that $m \geq n$.

It is easy to see that $m + n$ coincides with the valuation of the determinant of any group element g such that $x_1 = gx_2$, and that $m - n$ coincides with the distance between the vertices v_1 and v_2 in the tree.

5.5. Geodesics in the tree, convexity of fixed-point sets. Any two points y_1, y_2 in our tree are joined by a unique shortest path $[y_1, y_2]$, called a geodesic. Inside the standard apartment (identified with $\mathbb{R}$) geodesics are closed intervals. We say that a subset C of the tree is *convex* if for every $c_1, c_2 \in C$ the geodesic $[c_1, c_2]$ is contained in C.

Any element $\gamma \in G$ takes the geodesic joining y_1 and y_2 to the geodesic joining γy_1 to γy_2. Therefore if y_1, y_2 are fixed by γ, so is every point of the geodesic joining them. Thus the fixed point set of γ in the tree is convex.

For elements $\gamma \in K$ we can say more. First of all such an element obviously fixes the base-vertex v_0, so its fixed-point set on the tree is certainly non-empty. Moreover, since the determinant of γ is a unit, the distance $d(\gamma v, v)$ is an even integer for any vertex $v \in V$, and it follows that γ cannot take a vertex to one of its neighbors, and in particular it cannot interchange the two vertices of an edge. Therefore, if γ fixes an interior point of an edge, it actually fixes the entire edge pointwise, and we see that the fixed point set of γ in the tree is simply the union of all the edges both of whose vertices are fixed by γ.

For $\gamma \in K$ these considerations lead to the following simple method for determining $d(\gamma v, v)$ for any vertex v. There is a unique geodesic joining v to a vertex v' fixed by γ and having the additional property that no vertex along this geodesic other than v' is fixed by γ. Equivalently, this geodesic is the shortest possible one joining v to some vertex in the fixed-point set of γ. Applying γ to this geodesic, we see that every point moves but v', so that the geodesic together with its transform form a geodesic from v to γv. This shows that $d(\gamma v, v) = 2d(v, v')$; *in other words, $d(\gamma v, v)$ is twice the distance from v to the fixed point set of γ.* We will use this observation repeatedly below.

5.6. Unipotent orbital integrals restricted to $\mathcal{H}$ (denoted L_0, L_1). In G there are two unipotent orbits, one of them being $\{1\}$, the other being the orbit of the matrix u defined by

$$u = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{5.6.1}$$

In this subsection we compute the restrictions to $\mathcal{H}$ of the orbital integrals O_γ for $\gamma = 1$ and $\gamma = u$.

For $\gamma = 1$ the problem is trivial, since the orbital integral is just evaluation at the identity. Thus the restriction L_0 of O_1 to $\mathcal{H}$ takes the value 1 on f_0 and vanishes on f_m for $m > 0$.

Next we calculate the restriction L_1 of O_u to $\mathcal{H}$. The centralizer G_u is easily seen to be ZN. We identify Z with $F^\times$ in the obvious way, sending a scalar z to the corresponding scalar matrix. We use the Haar measure on Z giving $\mathcal{O}^\times$ measure 1. We identify N with the additive group F in the obvious way, using the upper right matrix entry as our coordinate. We use the Haar measure on F that gives $\mathcal{O}$ measure 1. We use the product measure on ZN, and hence $K \cap NZ$ has measure 1.

From 3.4 we see that

$$L_1(f_m) = \sum_v \operatorname{meas}(\operatorname{Stab}_N(v))^{-1}, \tag{5.6.2}$$

where the sum runs over a set of representatives for the orbits of N on the set of vertices $v \in V$ such that $d(uv, v) = 2m$. Here we used that $NZ\backslash X = N\backslash V$ (clear) and that $\mathrm{Stab}_{NZ}(x) = \mathrm{Stab}_N(v) \cdot (Z \cap K)$ for $x \in X$ mapping to $v \in V$.

Using the Iwasawa decomposition $G = BK = NAK$, we see that the set $N\backslash V$ of orbits of N on V is $A/Z(A \cap K) \simeq \mathbb{Z}$. The vertices v_j ($j \in \mathbb{Z}$) in the standard apartment are a particularly convenient set of orbit representatives. We need to compute $\mathrm{Stab}_N(v)$ for each orbit representative.

This is a simple computation, the result of which is that $\mathrm{Stab}_N(v_j) = \pi^j\mathcal{O}$ (identifying N with F as above), a group whose measure is q^{-j}. Another consequence of this computation is that v_j is fixed by u if and only if $j \leq 0$, and, as we saw in 5.5, this allows us to calculate $d(uv_j, v_j)$ for each $j \in \mathbb{Z}$. Indeed, for $j \leq 0$ the vertex v_j is fixed, so that $d(uv_j, v_j) = 0$. For $j > 0$, the geodesic $[v_j, v_0]$ has v_0 as its unique fixed point, so $d(uv_j, v_j) = 2j$.

Putting all these observations together, we see that

$$L_1(f_0) = 1 + q^{-1} + q^{-2} + \cdots = 1/(1 - q^{-1}) \tag{5.6.3}$$

and that for all $m > 0$

$$L_1(f_m) = q^m. \tag{5.6.4}$$

5.7. O_γ for any γ that is not regular semisimple. For $f \in \mathcal{H}$ it is evident that $O_\gamma(f)$ does not change when γ is multiplied by $z \in Z \cap K$. Any $\gamma \in K$ which is not regular semisimple is conjugate to an element of the form z or zu (for some $z \in Z \cap K$), and therefore O_γ restricted to $\mathcal{H}$ is either L_0 or L_1, as the case may be. It now remains only to consider regular semisimple elements $\gamma \in K$, in other words those whose eigenvalues are distinct.

5.8. Hyperbolic orbital integrals. Next we consider regular semisimple γ whose eigenvalues lie in F. The conjugacy class of such an element meets K if and only if the two eigenvalues are units, and after replacing γ by a conjugate we may assume that

$$\gamma = \begin{bmatrix} a & o \\ 0 & b \end{bmatrix}, \tag{5.8.1}$$

with a, b distinct elements in $\mathcal{O}^\times$. The centralizer G_γ is A, and the most convenient choice for G'_γ is the product of Z and the infinite cyclic subgroup of A generated by

$$\begin{bmatrix} \pi & 0 \\ 0 & 1 \end{bmatrix}. \tag{5.8.2}$$

We use the Haar measure on Z that gives $Z \cap K$ measure 1, and we use the counting measure on the infinite cyclic subgroup $\mathbb{Z}$.

Using 3.4 and 5.5, we see that

$$O_\gamma(f_m) = \sum_v 1, \tag{5.8.3}$$

where the sum runs over a set of representatives for the orbits of $\mathbb{Z}$ on the set of vertices $v \in V$ such that $d(\gamma v, v) = 2m$ (equivalently, such that the distance from v to the fixed point set of γ in the tree is equal to m). Thus $O_\gamma(f_m)$ is the number of orbits of the infinite cyclic subgroup $\mathbb{Z}$ of A on the set of vertices $v \in V$ at distance m from the fixed point set of γ in the tree.

As observed before, since $\gamma \in K$, its fixed point set in the tree is just the union of the edges joining two fixed vertices. Therefore it remains only to understand the set V^γ of vertices fixed by γ. Put $d_\gamma := \mathrm{val}(1 - \frac{a}{b})$, a non-negative integer. We claim that V^γ is the set of vertices $v \in V$ whose distance to the standard apartment is less than or equal to d_γ. By the Iwasawa decomposition we may write $v = anv_0$ with $a \in A$ and $n \in N$. Since the two sets we are trying to prove are equal are both stable under A, it is harmless to suppose that $a = 1$. Thus we need only consider v of the form nv_0.

Let us determine when γ fixes nv_0. Since γ fixes v_0, the condition that γ fix nv_0 is equivalent to the condition that $\gamma n^{-1}\gamma^{-1}n$ fix v_0. But $\gamma n^{-1}\gamma^{-1}n$ lies in N and is easily computed in terms of n and γ. Indeed, identifying N with F as before, so that n becomes an element $y \in F$, we find that $\gamma n^{-1}\gamma^{-1}n$ becomes the element $(1 - \frac{a}{b})y$ of F. Since the stabilizer of v_0 is KZ, it is now clear that γ fixes nv_0 if and only if $y \in \pi^{-d_\gamma}\mathcal{O}$.

To finish proving the claim we now need to compute the distance from nv_0 to the standard apartment in terms of the valuation of y. If $y \in \mathcal{O}$, then nv_0 equals v_0 and hence has distance 0 to the standard apartment. On the other hand, suppose that the valuation of y is negative, say equal to $-r$ for some positive integer r. We saw above that (see 5.6) $\mathrm{Stab}_N(v_j) = \pi^j\mathcal{O}$, from which it follows that the vertex v_j in the standard apartment is fixed by n if and only if $j \leq -r$. Therefore the geodesic $n[v_{-r}v_0]$ meets the standard apartment only at its endpoint v_{-r}, showing that its other endpoint, namely the point nv_0, has distance r to the standard apartment. This completes the proof of the claim.

Having proved the claim, now we can finish the computation of our orbital integral. Working modulo the action of the infinite cyclic subgroup $\mathbb{Z}$, we need to count vertices whose distance to the fixed point set is m. The fixed point set consists of all points whose distance to the standard apartment is less than or equal to d_γ; when $m = 0$ we are simply counting these points. When $m > 0$, a vertex has distance m from the fixed point set if and only if it has distance $m + d_\gamma$ from the standard apartment.

Therefore for any non-negative integer s we need to calculate the number $N(s)$ of orbits of $\mathbb{Z}$ on the set of vertices at distance s from the standard apartment. Clearly $N(s)$ is also equal to the number of vertices v at distance s to the standard apartment and having the additional property that the point in the standard apartment that is closest to v is equal to v_0. Elementary reasoning, using that every vertex has $q + 1$ neighbors, shows that $N(0) = 1$ and $N(s) = q^s - q^{s-1}$ for $s > 0$.

Putting everything together, we now see that

$$O_\gamma(f_0) = 1 + (q-1) + (q^2 - q) + \cdots + (q^{d_\gamma} - q^{d_\gamma - 1}) = q^{d_\gamma} \tag{5.8.4}$$

and that for all $m > 0$

$$O_\gamma(f_m) = q^{m+d_\gamma} - q^{m+d_\gamma-1}, \tag{5.8.5}$$

and comparing this with the computation of L_1 that we made earlier, we obtain

Lemma 5.1. *The restriction to $\mathcal{H}$ of the hyperbolic orbital integral O_γ is equal to $(1 - q^{-1})q^{d_\gamma} \cdot L_1$.*

5.9. Elliptic orbital integrals. The only remaining orbits (among those meeting K) are elliptic. The eigenvalues of an elliptic (regular semisimple) element generate a quadratic extension E of F. How do such elements sit inside G?

Start with a quadratic extension E/F. We can view E as a 1-dimensional E-vector space and as a 2-dimensional F-vector space, and since an E-linear map is necessarily F-linear, we have $GL_E(E) \subset GL_F(E)$. Choosing an F-basis in E, this becomes an embedding of $E^\times$ in G. The image is the set of F-points of a maximal torus T in GL_2.

Using this embedding, we view $\gamma \in E^\times$ as an element of G. Then its two eigenvalues are γ, $\bar{\gamma}$ (using bar to denote the non-trivial element in the Galois group of the quadratic extension), and its determinant is the norm $\gamma\bar{\gamma}$ of γ. If γ is conjugate to an element of K, then its determinant is a unit, and hence γ is a unit in the valuation ring $\mathcal{O}_E$ of E. In order to embed $E^\times$ in G we have to choose an F-basis in E. Let us agree to pick one which is at the same time an $\mathcal{O}$-basis for $\mathcal{O}_E$. Then we will have $E^\times \cap K = \mathcal{O}_E$. In order that γ be regular, we need $\gamma \neq \bar{\gamma}$. Thus the elements of interest are those in $(\mathcal{O}_E)^\times$ but not in $\mathcal{O}^\times$.

For such an element γ the centralizer is $E^\times$, and we are free to take the group G'_γ of 3.4 to be Z. It follows from 3.4 and 5.4 that

$$O_\gamma(f_m) = |\{v \in V : d(\gamma v, v) = 2m\}|. \tag{5.9.1}$$

We are going to calculate these orbital integrals in two steps. First we will compute them in terms of the cardinality of the set V^γ of vertices fixed by γ, then we will calculate $|V^\gamma|$.

Lemma 5.2. *For all elliptic regular semisimple $\gamma \in K$ the restriction of O_γ to $\mathcal{H}$ is equal to*

$$\bigl(2q^{-1} + (1 - q^{-1})|V^\gamma|\bigr) \cdot L_1 - \frac{2}{q-1} \cdot L_0. \tag{5.9.2}$$

Proof. We need to compute $O_\gamma(f_m)$ for all $m \geq 0$. Of course

$$O_\gamma(f_0) = |V^\gamma|. \tag{5.9.3}$$

Now assume that $m > 0$. As we have seen in 5.5, the condition $d(\gamma v, v) = 2m$ is equivalent to the condition that the distance $d(v, V^\gamma)$ from v to the fixed point set V^γ be m. Consider the unique shortest geodesic joining v to the fixed point set, and let w be the unique endpoint of that geodesic lying in V^γ. Then $v \mapsto w$ is a well-defined retraction of V onto V^γ, and thus

$$O_\gamma(f_m) = \sum_{w \in V^\gamma} |\{v \in V : d(v, V^\gamma) = m \text{ and } v \mapsto w\}|. \tag{5.9.4}$$

Given $w \in V^\gamma$, an element $v \in V$ satisfies the two conditions $d(v, V^\gamma) = m$ and $v \mapsto w$ if and only if $d(v, w) = m$ and the unique neighboring vertex of w lying on the geodesic $[w, v]$ is not fixed by γ; the number of such neighbors is $(q + 1) - C_w$, where C_w is the number of neighbors of w fixed by γ. Therefore

$$O_\gamma(f_m) = \sum_{w \in V^\gamma} q^{m-1}\bigl((q + 1) - C_w\bigr). \tag{5.9.5}$$

Summing C_w over all $w \in V^\gamma$, we get $2|E^\gamma|$, where E^γ denotes the set of edges in the tree that are fixed by γ. Now the fixed point set of γ in the tree is convex, hence contractible, and therefore its Euler characteristic $|V^\gamma| - |E^\gamma|$ is 1, which means that the sum over w of C_w is $2(|V^\gamma| - 1)$. Thus we have proved that

$$O_\gamma(f_m) = q^{m-1}\bigl((q - 1)|V^\gamma| + 2\bigr). \tag{5.9.6}$$

The lemma follows from (5.9.3), (5.9.6) and the formulas for L_0 and L_1 that we found before. $\square$

Our next task is to calculate $|V^\gamma|$. The answer depends on whether or not the quadratic extension E/F is ramified.

First we consider the case in which E/F is unramified. In particular the cardinality of the residue field of E is q^2. The tree for $SL_2(F)$ is a subtree of the tree for $SL_2(E)$, and in this bigger tree every vertex has q^2+1 neighbors. Moreover $\mathrm{Gal}(E/F)$ operates on the bigger tree and its set of fixed points is the smaller one.

Inside $GL_2(E)$ our matrix γ is conjugate to the diagonal matrix γ' with diagonal entries $(\gamma, \bar{\gamma})$, so that we are dealing with a hyperbolic element whose fixed point set we already understand. As in the hyperbolic case we define a non-negative integer

$$d_\gamma := \mathrm{val}(1 - \gamma\bar{\gamma}^{-1}).$$

Choose an element of $GL_2(E)$ that conjugates γ' to γ and apply it to the standard apartment, obtaining a non-standard apartment in the bigger tree. From previous work we know that the fixed point set of γ is the set of vertices in the bigger tree whose distance to the non-standard apartment is less than or equal to d_γ. The non-trivial element of $\mathrm{Gal}(E/F)$ preserves this non-standard apartment, flipping it end-to-end, and fixes a unique vertex v' in it. The fixed point set of γ in the smaller tree is precisely the set of vertices $v \in V$ whose distance to v' is less than or equal to d_γ, from which one sees easily that

$$|V^\gamma| = 1 + (q+1)(1 + \cdots + q^{d_\gamma - 1}) = 1 + (q+1)\frac{q^{d_\gamma} - 1}{q-1}. \tag{5.9.7}$$

Combining this with Lemma 5.2, we obtain our final formula

$$(1+q^{-1})q^{d_\gamma} \cdot L_1 - \frac{2}{q-1} \cdot L_0 \tag{5.9.8}$$

for the restriction of O_γ to $\mathcal{H}$ in the unramified case.

Next we consider the case in which E/F is ramified. The tree for $SL_2(F)$ still sits inside the tree for $SL_2(E)$, but in a more complicated way. Since the uniformizing element π for F has valuation 2 in E, the midpoint of each edge in the smaller tree becomes a vertex in the bigger one. Since the residue field does not change, every vertex in the bigger tree still has $q+1$ neighbors. The Galois group of E/F still acts on the bigger tree, and the smaller tree is fixed pointwise by this action, but it does not fill out the whole fixed point set unless E/F is tamely ramified.

Our element $\gamma \in (\mathcal{O}_E)^\times \setminus \mathcal{O}^\times$ still can be diagonalized in $GL_2(E)$, so that we again get a non-standard apartment in the bigger tree. There is a unique edge e in the smaller tree closest to this non-standard apartment, and the shortest path between the edge e and the apartment starts from the midpoint of that edge.

From previous work we know that the fixed point set of γ in the bigger tree consists of all points whose distance d to this non-standard apartment is less than or equal to a certain integer (that depends on γ). For a vertex v in the smaller tree this distance d is twice (because of the subdivision that occurred) the distance from v to the edge e plus a constant (the constant being 1 plus the distance from e to our non-standard apartment). Therefore there exists a non-negative integer d_γ such that the fixed point set of γ in V consists of all vertices whose distance from e

(measured in the smaller tree) is less than or equal to d_γ. In fact one can show that

$$1 + 2d_\gamma = \sup\{\mathrm{val}_E(\gamma - a) : a \in \mathcal{O}^\times\}, \tag{5.9.9}$$

where the valuation val_E being used here assigns 1 to uniformizing elements in E.

It is then easy to see that

$$|V^\gamma| = 2(1 + \cdots + q^{d_\gamma}) = 2\frac{q^{d_\gamma+1} - 1}{q - 1}. \tag{5.9.10}$$

Combining this with Lemma 5.2, we obtain our final formula

$$2q^{d_\gamma} \cdot L_1 - \frac{2}{q-1} \cdot L_0 \tag{5.9.11}$$

for the restriction of O_γ to $\mathcal{H}$ in the ramified case.

5.10. Homogeneity! Our computations have revealed something quite remarkable. The restrictions of the orbital integrals for the unipotent elements 1 and u give us two linear functionals L_0 and L_1 on $\mathcal{H}$. For every other element $\gamma \in K$ the restriction of the orbital integral O_γ to $\mathcal{H}$ is a linear combination of L_0 and L_1. It was by no means obvious *a priori* that this should be the case. In fact this is an example of the deep phenomenon of homogeneity that is the subject of DeBacker's course.

The coefficients of L_0, L_1 in these linear combinations are very interesting functions of γ, called *Shalika germs*, and we will discuss them next.

6. Shalika germs

6.1. The goal. Our goal in this section is to prove the existence of the Shalika germ expansion (see [**Sha72**]). Once the general theory is in place, we will illustrate how it works by re-examining our computations of orbital integrals on GL_2. We now work with any connected reductive group over a p-adic field F, and we once again lighten the notation by writing G for the F-points of our group.

6.2. Notation. Let $\mathcal{U}$ be the set of unipotent elements in G. Then $\mathcal{U}$ is closed, and there are finitely many G-orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ in $\mathcal{U}$. We write $\mu_1, \dots, \mu_r$ for the corresponding unipotent orbital integrals (for some choice of invariant measures on the orbits that we prefer not to encode in the notation). Let T be (the set of F-points of) a maximal torus in G, and let T_{reg} be the subset of regular elements. We are interested in orbital integrals O_γ for variable $\gamma \in T_{\mathrm{reg}}$, so we need a coherent set of choices of invariant measures on the orbits of all such elements γ. This can be done as follows. Once and for all we choose Haar measures dg and dt on G and T respectively. Then for any $\gamma \in T_{\mathrm{reg}}$ we put (for $f \in C_c^\infty(G)$)

$$O_\gamma(f) = \int_{T\backslash G} f(g^{-1}\gamma g)\, dg/dt. \tag{6.2.1}$$

6.3. Distributions. For any l.c.t.d space we have already introduced the space $C_c^\infty(X)$ of locally constant compactly supported functions on X. A *distribution* is by definition any linear functional on $C_c^\infty(X)$ (with no continuity hypothesis since there is no topology on our function space). We write $\mathcal{D}(X)$ for the vector space of all distributions on X.

Let Y be any closed subset of X, and let U be the complementary open subset. Dual to the short exact sequence (2.1.1) is the short exact sequence

$$0 \to \mathcal{D}(Y) \to \mathcal{D}(X) \to \mathcal{D}(U) \to 0. \tag{6.3.1}$$

In other words, given a distribution on X, we can restrict it to U, and among the distributions on X, we have those whose support (see 26.2 for a discussion of the notion of support of a distribution) is contained in Y. Now suppose that some group H acts on X, preserving Y and U. Then, taking invariants under the group action (denoted by a superscript H), we get an exact sequence

$$0 \to \mathcal{D}(Y)^H \to \mathcal{D}(X)^H \to \mathcal{D}(U)^H \tag{6.3.2}$$

but there is no guarantee that the restriction map at the right end is surjective.

From the short exact sequence (2.1.1) we also get an exact sequence

$$C_c^\infty(U)_H \to C_c^\infty(X)_H \to C_c^\infty(Y)_H \to 0, \tag{6.3.3}$$

where the subscript H denotes coinvariants for H. (For an H-module V the space of coinvariants V_H is by definition the biggest quotient of V on which H acts trivially, or, in other words, the quotient of V by the linear span of all vectors of the form $hv - v$ for some $h \in H$, $v \in V$.) The sequence (6.3.2) can also be obtained as the $\mathbb{C}$-dual of the sequence (6.3.3).

6.4. Existence of the Shalika germ expansion. Order the unipotent orbits so that their dimensions increase as i does. Of course there can be several orbits of the same dimension; for these the order is immaterial. By dimension of the orbit we really mean the dimension (as algebraic variety) of the $G(\bar{F})$-orbit containing the given G-orbit. The purpose of this ordering is to guarantee that

$$\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_i \tag{6.4.1}$$

is closed in G for all i.

Inside the space $\mathcal{O}_1 \cup \mathcal{O}_2$ (which is closed in G, hence l.c.t.d) we have the closed subset $\mathcal{O}_1$ and complementary open subset $\mathcal{O}_2$. The group G acts by conjugation on all these spaces. Therefore we get an exact sequence

$$0 \to \mathcal{D}(\mathcal{O}_1)^G \to \mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2)^G \to \mathcal{D}(\mathcal{O}_2)^G. \tag{6.4.2}$$

The spaces $\mathcal{D}(\mathcal{O}_1)^G$ and $\mathcal{D}(\mathcal{O}_2)^G$ are 1-dimensional, spanned by the invariant integrals on the homogeneous spaces $\mathcal{O}_1$ and $\mathcal{O}_2$. Now we need to recall the non-trivial fact that μ_2 is well-defined on $C_c^\infty(G)$ ("convergence of orbital integrals," discussed in 3.3). Therefore μ_2 gives us an element of $\mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2)^G$ that maps to the invariant integral on $\mathcal{O}_2$. We conclude that the map at the right end of the exact sequence above is surjective, and that $\mathcal{D}(\mathcal{O}_1 \cup \mathcal{O}_2)^G$ is 2-dimensional with basis given by (the restrictions of) μ_1 and μ_2. An obvious inductive argument then shows that

$$\mathcal{D}(\mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r)^G$$

is r-dimensional with basis given by (the restrictions of) $\mu_1, \ldots, \mu_r$. Now we are ready for the theorem on germ expansions.

THEOREM 6.1 (Shalika [**Sha72**]). *There exist functions $\Gamma_1, \Gamma_2, \ldots, \Gamma_r$ on T_{reg} having the following property. For every $f \in C_c^\infty(G)$ there exists an open and closed*

G-invariant neighborhood U_f of 1 in G such that

$$O_\gamma(f) = \sum_{i=1}^{r} \mu_i(f) \cdot \Gamma_i(\gamma) \tag{6.4.3}$$

for all $\gamma \in U_f \cap T_{\mathrm{reg}}$. The germs about $1 \in T$ of the functions $\Gamma_1, \dots, \Gamma_r$ are unique. We refer to Γ_i as the Shalika germ for the unipotent orbit $\mathcal{O}_i$ and the torus T.

Proof. Since the unipotent set $\mathcal{U} = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_r$ is closed in G, there is a surjective restriction map $C_c^\infty(G) \to C_c^\infty(\mathcal{U})$, and we may choose functions f_i $(i = 1, \dots, r)$ such that $\mu_i(f_j) = \delta_{ij}$ (Kronecker δ). Inserting the function f_i into (6.4.3), we see that the germ of $\Gamma_i(\gamma)$ must be equal to the germ of $O_\gamma(f_i)$. This already proves the uniqueness assertion in the theorem. It also shows that we may as well take

$$\Gamma_i(\gamma) := O_\gamma(f_i) \tag{6.4.4}$$

as the definition of Γ_i. (There is no need to be troubled by the non-uniqueness of the functions f_i since it is only germs that matter in the theorem.)

However we must still show that (6.4.3) is valid for all functions on G. So let $f \in C_c^\infty(G)$. The function

$$\sum_{i=1}^{r} \mu_i(f) \cdot f_i \tag{6.4.5}$$

obviously has the same unipotent orbital integrals as f does. In other words all unipotent orbital integrals of

$$\phi := f - \sum_{i=1}^{r} \mu_i(f) \cdot f_i \tag{6.4.6}$$

vanish. Choose a neighborhood U of $\mathcal{U}$ as in Lemma 6.2 below. We claim that (6.4.3) holds for the neighborhood $U_f = U$. Indeed, for $\gamma \in U \cap T_{\mathrm{reg}}$ the orbital integral $O_\gamma(\phi)$ vanishes by Lemma 6.2. In view of how ϕ was defined, this establishes (6.4.3). $\square$

Lemma 6.2. *Let $\phi \in C_c^\infty(G)$ and assume that all unipotent orbital integrals of ϕ vanish. Then there is an open and closed conjugation invariant neighborhood U of the unipotent set $\mathcal{U}$ such that $I(\phi) = 0$ for every invariant distribution I supported on U.*

Proof. The dual space to $C_c^\infty(\mathcal{U})_G$ is $\mathcal{D}(\mathcal{U})^G$, which has as basis the unipotent orbital integrals $\mu_1, \dots, \mu_r$, so the vanishing of the unipotent orbital integrals of ϕ is equivalent to the vanishing of the image of ϕ in $C_c^\infty(\mathcal{U})_G$. In order to construct the desired neighborhood U of $\mathcal{U}$, we use that there exists a continuous map α from G to a l.c.t.d space $\mathbb{A}$ such that every fiber of α is a union of conjugacy classes in G and such that there exists $x \in \mathbb{A}$ for which $\alpha^{-1}(x) = \mathcal{U}$. Therefore by the last statement of Lemma 27.1 there exists an open neighborhood ω of x in $\mathbb{A}$ such that the image of ϕ in $C_c^\infty(\alpha^{-1}\omega)_G$ vanishes. Shrinking ω, we may assume that it is compact as well as open, and then $U := \alpha^{-1}\omega$ is the desired neighborhood of $\mathcal{U}$.

It remains to prove the existence of α. In fact the map denoted α in 3.2 does the job. However, we were a bit sketchy about its construction in the general case, and the reader may prefer to use the following cruder version of α, which is however sufficient for our current needs. The cruder version is obtained by choosing an embedding (of algebraic groups) of G into some general linear group, constructing α for the general linear group (using the coefficients of the characteristic polynomial, as in 3.2), and then restricting α to the subgroup G. This works since an element of G is unipotent if and only if the corresponding matrix is unipotent. □

6.5. Back to GL_2. Recall that in $G = GL_2$ there are two unipotent classes and hence two unipotent orbital integrals μ_1, μ_2. Our computations of orbital integrals for GL_2 revealed that for every regular semisimple $\gamma \in K = GL_2(\mathcal{O})$ there are complex numbers $A_1(\gamma)$, $A_2(\gamma)$ such that for every $f \in \mathcal{H}$ the equality

$$O_\gamma(f) = A_1(\gamma)\mu_1(f) + A_2(\gamma)\mu_2(f) \tag{6.5.1}$$

holds. Our computations also showed that the restrictions of μ_1 and μ_2 to $\mathcal{H}$ are linearly independent. Therefore the numbers $A_1(\gamma)$, $A_2(\gamma)$ are uniquely determined, and, moreover, inside the rather small function space $\mathcal{H}$ we can find functions f_1, f_2 satisfying $\mu_i(f_j) = \delta_{ij}$. We have seen that the Shalika germs Γ_1, Γ_2 are obtained by taking the orbital integrals of f_1, f_2, which in view of the equality above means that Γ_i is the germ of A_i $(i = 1, 2)$.

At first glance it might now seem that Shalika germ theory explains (6.5.1) and hence explains why the restrictions to $\mathcal{H}$ of the orbital integrals O_γ for $\gamma \in K$ are all linear combinations of the restrictions of μ_1 and μ_2. This is far from being true, since the Shalika germ expansion is only valid on some (possibly very small) neighborhood of 1 and moreover this neighborhood depends on the function f that we are considering. The amazing thing that has happened here is that there is a *big* neighborhood of 1 (namely K) which works for *all* the functions in $\mathcal{H}$.

It is tempting to refer to A_1, A_2 as "the" Shalika germs for GL_2, since among all possible functions having the correct germs, they are singled out naturally by the property (6.5.1).

7. Weyl integration formula

In this section we work at first with an arbitrary connected reductive group G over our p-adic field F; starting with 7.8 we assume that G is split. We write $\mathfrak{g}$ for the Lie algebra of G. Before we can get down to work on our next topic, the local trace formula, we need to derive the Weyl integration formula for $\mathfrak{g}$, which, roughly speaking, expresses the integral of a function on $\mathfrak{g}$ as an iterated integral, in which one first integrates over the various (adjoint) orbits in $\mathfrak{g}$ and then integrates over the set of orbits.

7.1. Remarks on Weyl groups. When working with maximal tori over non-algebraically closed fields such as F, there are three relevant Weyl groups. In order to explain them clearly we continue to make a notational distinction between the algebraic group G and its group $G(F)$ of F-points.

Let T be a maximal torus in G. The quotient $W := N_G(T)/T$ is a finite algebraic group defined over F. (We are writing $N_G(T)$ for the normalizer in G of T.) We then have inclusions

$$W_T \subset W(F) \subset W(\bar{F}), \tag{7.1.1}$$

where W_T is by definition the quotient $N_G(T)(F)/T(F)$. Of course $W(\bar{F})$ is the absolute Weyl group, and, up to inner automorphisms, is independent of T. The subgroup W_T is the Weyl group needed in the Weyl integration formula.

The main thing we need to know about W_T is that two regular elements $X, X' \in \mathfrak{t}$ are $G(F)$-conjugate if and only if they are conjugate under W_T. Indeed, if $g \in G(F)$ conjugates X to X', then it conjugates the centralizer of X to the centralizer of X'; since the two elements are regular both centralizers are T, and therefore g normalizes T, proving the forward implication. The reverse implication is trivial.

7.2. Calculation of the differential of β. Now we return to our usual practice of abbreviating $G(F)$ to G, $T(F)$ to T, etc. We write $\mathfrak{t}$ for the Lie algebra of T.

Consider the map

$$(T\backslash G) \times \mathfrak{t} \xrightarrow{\beta} \mathfrak{g} \tag{7.2.1}$$

defined by $\beta(g, X) = g^{-1}Xg$. For any $X \in \mathfrak{t}$ the differential $d\beta$ of β at $(1, X) \in (T\backslash G) \times \mathfrak{t}$ is the map $(\mathfrak{g}/\mathfrak{t}) \times \mathfrak{t} \to \mathfrak{g}$ given by $(Y, Z) \mapsto [X, Y] + Z$. The two tangent spaces $(\mathfrak{g}/\mathfrak{t}) \times \mathfrak{t}$, $\mathfrak{g}$ both sit in short exact sequences with $\mathfrak{t}$ as the subspace and $\mathfrak{g}/\mathfrak{t}$ as the quotient space, and $d\beta$ is the identity on $\mathfrak{t}$. Therefore the top exterior powers of the two tangent spaces are canonically isomorphic, and the determinant of $d\beta$ at $(1, X)$ makes sense and is equal to

$$D(X) := \det\bigl(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{t}\bigr). \tag{7.2.2}$$

The map β is G-equivariant (for the translation action on $T\backslash G$, the trivial action on $\mathfrak{t}$, and the adjoint action on $\mathfrak{g}$). Choosing a G-invariant volume form (*i.e.* non-vanishing differential form of top degree) on $T\backslash G$ is the same as choosing a generator of the top exterior power of $\mathfrak{g}/\mathfrak{t}$. Choosing a translation invariant volume form on $\mathfrak{t}$ is the same as choosing a generator of the top exterior power of $\mathfrak{t}$. Make such choices. From them we get a generator of the top exterior power of $\mathfrak{g}$, which we use to get a translation invariant volume form on $\mathfrak{g}$. In this way we get G-invariant volume forms on the source and target of β, and we may use these volume forms to talk about the determinant of $d\beta$, or, in other words, the Jacobian of β. In fact the computation we made above, together with the G-equivariance of β, shows that the Jacobian of β at any point $(g, X) \in (T\backslash G) \times \mathfrak{t}$ is equal to $D(X)$.

7.3. Measures obtained from volume forms. A volume form ω on a p-adic manifold M gives rise to a measure $|\omega|$ on M, just as in the real case. In the end it boils down to assigning a measure $|dx|$ on F to the differential form dx, where x denotes the standard coordinate on F, that is, the identity map on F. In the real case the usual convention is of course that $|dx|$ is Lebesgue measure on $\mathbb{R}$. In the p-adic case one simply agrees $|dx|$ is some Haar measure on F, fixed once and for all. See [**Wei82**] for further details. As in the real case, there is a change of variables formula involving the Jacobian.

The volume forms on $T\backslash G$, $\mathfrak{t}$, $\mathfrak{g}$ chosen above give us a G-invariant measure $d\bar{g}$ on $T\backslash G$ and Haar measures dX, dY on $\mathfrak{t}$, $\mathfrak{g}$ respectively.

7.4. Expression for $D(X)$ in terms of roots. Let R be the set of roots of T in $\mathfrak{g}$. Here we are talking about the absolute root system, a subset of the group $X^*(T)$ of characters on T over $\bar{F}$. The differentials of the roots are linear forms on $\mathfrak{t} \otimes_F \bar{F}$ and hence yield $\bar{F}$-valued functions on $\mathfrak{t}$; these functions on $\mathfrak{t}$ will also be called roots, but no confusion should result from this. It is clear from the definitions that

$$D(X) = \prod_{\alpha \in R} \alpha(X) \tag{7.4.1}$$

for any $X \in \mathfrak{t}$. From this it follows that the differential $d\beta$ is an isomorphism at all points (g, X) such that X is regular. (Recall that a semisimple element in $\mathfrak{g}$ is said to be *regular* if its centralizer in the algebraic group G is a maximal torus, and that $X \in \mathfrak{t}$ is regular in this sense if and only if no root of T vanishes on it.)

7.5. D as polynomial function on $\mathfrak{g}$. Let ℓ denote the absolute rank of G, in other words, the dimension of any maximal torus in G. For any $X \in \mathfrak{g}$ we can consider the characteristic polynomial of the endomorphism $\mathrm{ad}(X)$. Each individual coefficient of this characteristic polynomial is a polynomial function of X, and since generically $\mathrm{ad}(X)$ has the eigenvalue 0 with multiplicity ℓ, we see that the lowest non-vanishing coefficient occurs in front of the ℓ-th power of the variable and is equal to $D(X)$ for $X \in \mathfrak{t}$. In this way we see that the function $D(X)$ defined above for $X \in \mathfrak{t}$ extends to a polynomial function (still denoted by D) on all of $\mathfrak{g}$, which explains why we did not include T in the notation. Note that $D(X) \neq 0$ if and only if X is regular semisimple.

7.6. Decomposition of $\mathfrak{g}_{\mathrm{rs}}$ as a disjoint union of open subsets $\mathfrak{g}_{\mathrm{rs}}^T$. Let $\mathfrak{t}_{\mathrm{reg}}$ be the set of regular elements in $\mathfrak{t}$, let $\mathfrak{g}_{\mathrm{rs}}$ be the set of regular semisimple elements in $\mathfrak{g}$, and let $\mathfrak{g}_{\mathrm{rs}}^T$ be the subset of $\mathfrak{g}_{\mathrm{rs}}$ consisting of all elements that are conjugate under G to some element of $\mathfrak{t}_{\mathrm{reg}}$. Then the map

$$(T \backslash G) \times \mathfrak{t}_{\mathrm{reg}} \to \mathfrak{g} \tag{7.6.1}$$

(obtained from β by restriction) is a local isomorphism of p-adic manifolds and its image, namely $\mathfrak{g}_{\mathrm{rs}}^T$, is open in $\mathfrak{g}$. The fiber of β through $(g, X) \in (T\backslash G) \times \mathfrak{t}_{\mathrm{reg}}$ has $|W_T|$ elements, namely those of the form $(wg, w(X))$ with w ranging through W_T.

To complete our picture of $\mathfrak{g}_{\mathrm{rs}}$, we note that its complement has measure 0, and that

$$\mathfrak{g}_{\mathrm{rs}} = \coprod_T \mathfrak{g}_{\mathrm{rs}}^T, \tag{7.6.2}$$

where the union ranges over a set of representatives T for the set of $G(F)$-conjugacy classes of maximal F-tori in G.

7.7. First form of the Weyl integration formula. These considerations lead to the following formula, known as the Weyl integration formula. Let $f \in C_c^\infty(\mathfrak{g})$. Then

$$\int_{\mathfrak{g}} f(Y)\, dY = \sum_T |W_T|^{-1} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \int_{T\backslash G} f(g^{-1}Xg)\, d\bar{g}\, dX, \tag{7.7.1}$$

where $d\bar{g}$, dY, dX are the measures on $T\backslash G$, $\mathfrak{g}$, $\mathfrak{t}$ respectively that were introduced in 7.3, and where the sum ranges over a set of representatives T for the set of $G(F)$-conjugacy classes of maximal F-tori in G. Since the complement of $\mathfrak{t}_{\mathrm{reg}}$ in $\mathfrak{t}$ has measure 0, we could equally well integrate over $\mathfrak{t}$ instead of $\mathfrak{t}_{\mathrm{reg}}$. Moreover we are not obliged to stick with precisely these measures $d\bar{g}$, dY, dX. Clearly it is only the product $d\bar{g}\, dX$ that matters in the Weyl integration formula, so we are free to multiply $d\bar{g}$ by a constant as long as we divide dX by the same constant, and we are free to multiply dY by a constant as long as we arrange that the product $d\bar{g}\, dX$

is multiplied by the same constant (for all T). For any such choices of measures we say that $d\bar{g}\,dX$ is *compatible* with dY.

Actually there are several useful variants of the Weyl integration formula, one of which is the one we will actually use later. For this we need some further preparation. We again need to maintain a notational distinction between an algebraic group and its group of F-points. We return to assuming that G is split over F.

7.8. Review of Levi subgroups and the definition of $\mathcal{L}$. By a Levi subgroup M of G we mean some Levi component of a parabolic F-subgroup of G. We write A_M for the maximal F-split torus in the center of M. In particular A_G denotes the maximal F-split torus in the center of G. A basic fact about Levi subgroups is that M is the centralizer in G of A_M.

As usual let us fix a split maximal torus A in G. Then A_M is conjugate under $G(F)$ to a subtorus of A. Thus, after replacing M by a conjugate, we may assume that $A_M \subset A$. The condition $A_M \subset A$ is equivalent to the condition $M \supset A$. [Use that M is the centralizer of A_M and that A is its own centralizer.] We write $\mathcal{L} = \mathcal{L}(A)$ for the set of Levi subgroups M of G such that $M \supset A$.

7.9. Definition of $\mathcal{T}_M$. Let T be a maximal F-torus of G, and let A_T denote the maximal F-split subtorus of T. Let M denote the centralizer of A_T in G, a Levi subgroup of G. We claim that $A_M = A_T$. Indeed, it is obvious that A_T is central in M and hence contained in A_M. On the other hand T is contained in M and hence is a maximal torus in M, which implies that T contains the center of M. Therefore A_M is contained in T and hence in A_T.

The reason for introducing M is that T is *elliptic* in M, in the sense that T/A_M is an anisotropic torus over F (which implies that $T(F)/A_M(F)$ is compact). We choose a set $\mathcal{T}_M$ of representatives for the $M(F)$-conjugacy classes of elliptic maximal tori T in M.

7.10. Definition of the positive integer n_T^M. Let M be a Levi subgroup of G and let T be a maximal torus in M. We write $N_{M(F)}(T)$ for the normalizer in $M(F)$ of T. Then $N_{M(F)}(T)/T(F)$ is a finite group, and we write n_T^M for its cardinality.

7.11. Second form of the Weyl integration formula. We return to writing G instead of $G(F)$. Let $f \in C_c^\infty(\mathfrak{g})$. Then

$$\int_{\mathfrak{g}} f(Y)\,dY = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \int_{A_M\backslash G} f(g^{-1}Xg)\,d\dot{g}\,dX, \tag{7.11.1}$$

where W (respectively, W_M) denotes the Weyl group of A in G (respectively, M), and where $d\dot{g}$ is the unique G-invariant measure on $A_M\backslash G$ such that

$$\int_{A_M\backslash G} \varphi(g)\,d\dot{g} = \int_{T\backslash G} \varphi(g)\,d\bar{g} \tag{7.11.2}$$

for every $\varphi \in C_c^\infty(T\backslash G)$. (We used here that $A_M\backslash T$ is compact.) Since we have replaced $d\bar{g}$ by $d\dot{g}$, we need to extend the terminology introduced in 7.7 by now saying that the measure $d\dot{g}\,dX$ is compatible with dY (when $d\dot{g}$ has been obtained from $d\bar{g}$ as above, and $d\bar{g}\,dX$ is compatible with dY).

7.12. Derivation of the second form of the Weyl integration formula from the first. We write $N_{G(F)}(M)$ for the normalizer in $G(F)$ of M. The group $N_{G(F)}(M)/M(F)$ is finite, and we denote by n_M^G its cardinality. We need a couple of lemmas in order to derive the second form of the Weyl integration formula from the first.

LEMMA 7.1. *Let M be a Levi subgroup of G, and let T be an elliptic maximal torus in M. Then the number of $M(F)$-conjugacy classes of maximal tori T' in M such that T' is $G(F)$-conjugate to T is equal to*

$$n_M^G \cdot n_T^M \cdot (n_T^G)^{-1}.$$

PROOF. Let $g \in G(F)$. We claim that $gTg^{-1} \subset M$ if and only if $g \in N_{G(F)}(M)$. Indeed, suppose that $gTg^{-1} \subset M$. Then gTg^{-1} is a maximal torus in M, and therefore its split component gA_Mg^{-1} contains A_M, hence equals A_M (look at dimensions). Thus g normalizes A_M, which implies that it also normalizes the centralizer of A_M, namely M. This proves the forward implication in the claim; the other implication is trivial. A consequence of the claim is that $N_{G(F)}(T)$ normalizes M and hence normalizes $N_{G(F)}(T) \cap M(F) = N_{M(F)}(T)$.

It follows from the claim we just proved that the set of $M(F)$-conjugacy classes of $T' \subset M$ such that T' is $G(F)$-conjugate to T is in natural bijection with the set

$$M(F)\backslash N_{G(F)}(M)/N_{G(F)}(T),$$

and the cardinality of this set is clearly the index of

$$N_{G(F)}(T)/N_{M(F)}(T),$$

a group of order $n_T^G \cdot (n_T^M)^{-1}$, in

$$N_{G(F)}(M)/M(F),$$

a group of order n_M^G. This proves the lemma. □

The next lemma involves the set $\mathcal{L}$ of Levi subgroups of G containing A. For $M, M' \in \mathcal{L}$ we write $M \sim M'$ if M, M' are conjugate under $G(F)$. We write $\mathcal{L}/\sim$ for the set of equivalence classes in $\mathcal{L}$ for the equivalence relation $\sim$. Moreover we fix some Borel subgroup B_0 containing A, and write $\mathcal{P}_0$ for the set of parabolic subgroups of G containing B_0 (called *standard* parabolic subgroups). Also, we write $\mathcal{F}(A)$ for the set of parabolic subgroups of G containing A, and for $P \in \mathcal{F}(A)$ we write M_P for the unique Levi subgroup of P containing A. For a Levi subgroup M we write $\mathcal{P}(M)$ for the set of parabolic subgroups of G having M as Levi component.

LEMMA 7.2. *Let ψ be a function defined on the set $\mathcal{L}$ and assume that $\psi(M) = \psi(M')$ whenever $M \sim M'$. Then*

$$\sum_{M \in \mathcal{L}/\sim} (n_M^G)^{-1}\psi(M) = \sum_{M \in \mathcal{L}} \frac{|W_M|}{|W|}\psi(M) = \sum_{P \in \mathcal{P}_0} |\mathcal{P}(M_P)|^{-1}\psi(M). \tag{7.12.1}$$

PROOF. Let $M \in \mathcal{L}$ and let $g \in G(F)$. We claim that $gMg^{-1} \in \mathcal{L}$ if and only if $g \in N_{G(F)}(A) \cdot M(F)$. Indeed, suppose that $gMg^{-1} \in \mathcal{L}$. Then both $g^{-1}Ag$ and A are split maximal tori in M, from which it follows that they are conjugate under

$M(F)$. Thus there exists $m \in M(F)$ such that mg^{-1} normalizes A. This proves the forward implication in the claim; the reverse implication is clear.

Write $N_{G(F)}(M, A)$ for the intersection of the normalizers in $G(F)$ of M and A, and write $N_W(M)$ for $\{w \in W : wMw^{-1} = M\}$, a subgroup of W that contains W_M as a normal subgroup. As a special case of the claim above we see that

$$N_{G(F)}(M) = N_{G(F)}(M, A) \cdot M(F),$$

and from this it follows that

$$N_{G(F)}(M)/M(F) = N_{G(F)}(M, A)/N_{M(F)}(A) = N_W(M)/W_M. \tag{7.12.2}$$

How many $M' \in \mathcal{L}$ are there such that $M' \sim M$? It follows from the claim proved above that the set of such M' is simply the W-orbit of M in $\mathcal{L}$, and therefore its cardinality is equal to the index $[W : N_W(M)]$, which by (7.12.2) is equal to

$$(n_M^G)^{-1} \frac{|W|}{|W_M|}. \tag{7.12.3}$$

The first equality in the lemma follows from (7.12.3).

Finally, the second sum in the statement of the lemma can obviously be rewritten as

$$\sum_{P \in \mathcal{F}(A)} \frac{|W_{M_P}|}{|W|} |\mathcal{P}(M_P)|^{-1} \psi(M). \tag{7.12.4}$$

Now any $P \in \mathcal{F}(A)$ is conjugate under W to a unique standard parabolic subgroup, and the stabilizer in W of P is W_{M_P}. Therefore (7.12.4) is equal to the third sum in the statement of the lemma. $\square$

Now let's return to the Weyl integration formula. From (7.11.2) and Lemma 7.1 we see that our first form (7.7.1) of that formula can be rewritten as

$$\int_{\mathfrak{g}} f(Y)\, dY = \sum_{M \in \mathcal{L}/\sim} \frac{1}{n_M^G} \sum_{T \in \mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \int_{A_M \backslash G} f(g^{-1}Xg)\, d\dot{g}\, dX, \tag{7.12.5}$$

and then from Lemma 7.2 we see that it can also be rewritten in the form (7.11.1). One could also use Lemma 7.2 to rewrite the Weyl integration formula as a sum over standard parabolic subgroups. Similarly, there are several ways of rewriting the sums in the local trace formula, and the global trace formula as well (see the remarks after Thm. 6.1 in [**Art89a**]).

8. Preliminary discussion of the local trace formula

Now that we have a feel for how orbital integrals work on the group GL_2, it is time to begin a more systematic treatment. Harish-Chandra [**HC78, HC99, HC70**] developed harmonic analysis on the Lie algebra of G and then used the exponential map to climb back to the group itself, following the same path he had taken for real Lie groups. Now in harmonic analysis on the group the two key objects are orbital integrals and irreducible characters. Orbital integrals still make sense on the Lie algebra (integrate over orbits for the adjoint action of G). What about irreducible characters? Have they been irretrievably lost in passing to the Lie algebra? No! Harish-Chandra discovered that the role played by irreducible characters on G is played by Fourier transforms of orbital integrals on the Lie algebra (again he did this first in the real case). Of course Fourier transforms of orbital integrals are in many respects simpler than irreducible characters, and this partly explains why passing to the Lie algebra is so effective.

In any case for most of the rest of this article we are going to work on the Lie algebra rather than the group. We will follow what seems to be the shortest known path through the material, first proving Waldspurger's local trace formula on the Lie algebra [**Wal95**] and then using it as a tool to develop the rest of the theory. This path is not essentially different from the one taken by Harish-Chandra in the papers cited above, and at most key points is exactly the same.

8.1. Local trace formula on the group. We began this article by discussing the trace formula on compact groups. Before passing to the Lie algebra, we should briefly discuss Arthur's local trace formula [**Art76, Art87, Art89b, Art91a, Art91b**] on a p-adic group G. (Actually Arthur also allows real and complex groups.)

Choose a Haar measure dg on G. Just as in the compact case, given $f_1, f_2 \in C_c^\infty(G)$, we get an integral operator on $L^2(G)$ with kernel function

$$K(x,y) = \int_G f_1(g) f_2(x^{-1}gy)\,dg, \tag{8.1.1}$$

a locally constant function on $G \times G$. The restriction of the kernel function to the diagonal will be denoted by $K(x)$ and is given by

$$K(x) = \int_G f_1(g) f_2(x^{-1}gx)\,dg, \tag{8.1.2}$$

Next Arthur uses Harish-Chandra's Plancherel theorem to rewrite $K(x)$ in spectral terms. However, since G is no longer assumed to be compact, the kernel function usually fails to be compactly supported, the integral operator is usually not of trace class, and the integral over G of $K(x)$ is usually divergent. As in the global trace formula, Arthur handles these difficulties by truncating both expressions for $K(x)$ before integrating over G, obtaining in the end a formula with a geometric side involving weighted orbital integrals (which generalize orbital integrals) and a spectral side involving weighted characters (which generalize characters).

8.2. First steps towards the local trace formula on $\mathfrak{g}$. We again write $\mathfrak{g}$ for the Lie algebra of our p-adic group G (the F-points of a connected reductive group over a p-adic field F, just as before). Consider a pair of functions $f_1, f_2 \in C_c^\infty(\mathfrak{g})$ and use them to define a locally constant function $K(x)$ on G by

$$K(x) = \int_{\mathfrak{g}} f_1(Y) f_2(x^{-1}Yx)\,dY, \tag{8.2.1}$$

where dY is a Haar measure on $\mathfrak{g}$. (We are using the expression $x^{-1}Yx$ to denote the adjoint action of x^{-1} on Y.) Clearly this function is the analog for $\mathfrak{g}$ of the function (8.1.2) above that is the starting point for the local trace formula on G. Arthur uses the Plancherel theorem on G to obtain a second expression for (8.1.2); similarly, Fourier theory on the additive group $\mathfrak{g}$ yields an identity

$$\int_{\mathfrak{g}} f_1(Y) f_2(x^{-1}Yx)\,dY = \int_{\mathfrak{g}} \hat{f}_1(Y) \check{f}_2(x^{-1}Yx)\,dY, \tag{8.2.2}$$

where $\hat{f}$ and $\check{f}$ denote the two possible variants of the Fourier transform for $\mathfrak{g}$. In more detail, let us now choose a G-invariant non-degenerate symmetric bilinear form B on $\mathfrak{g}$ and a non-trivial additive character ψ on F. Then for $f \in C_c^\infty(\mathfrak{g})$ we define the first version of the Fourier transform by

$$\hat{f}(Y) = \int_{\mathfrak{g}} f(Z)\psi(B(Y,Z))\,dZ, \tag{8.2.3}$$

where dZ is a self-dual Haar measure on $\mathfrak{g}$. We define $\check{f}$ by the same formula, except that we replace ψ by ψ^{-1}. Thus $f \mapsto \check{f}$ is inverse to $f \mapsto \hat{f}$.

In case G is compact we obtain the local trace formula on $\mathfrak{g}$ simply by regarding both sides of (8.2.2) as functions of x and then integrating over G. In general this integral diverges, and we must truncate before integrating. The truncation needed on the Lie algebra is the same as the one Arthur uses on the group.

8.3. Truncation. In order to keep the structure theory of G as simple as possible (for expository purposes) we assume from now on that G is a split group, and we use the notation $B = AN$ and K of 4.1. We will eventually need to let B vary through the set $\mathcal{B}(A)$ of all Borel subgroups containing A, but it is sometimes convenient to fix one of them, which from now on we will denote by $B_0 = AN_0$. In addition we write A_G for (the F-points of) the identity component of the center of our algebraic group G.

Recall that $\mu \in X_*(A)$ is said to be *dominant* if $\langle \alpha, x\rangle \geq 0$ for every simple root α. There is a standard partial order $\leq$ on $X_*(A)$, defined as follows: $\nu \leq \mu$ means that $\mu - \nu$ is a non-negative integral linear combination of simple coroots. Note that $\nu \leq \mu$ implies that μ and ν have the same image in the quotient Λ_G of $X_*(A)$ introduced in subsection 4.5. Of course all these notions depend on a choice of Borel subgroup, which determines the sets of simple roots and coroots. When we need to stress which Borel subgroup B is being used, we will say B-dominant rather than dominant. However, in the discussion below we will use the fixed Borel subgroup B_0 and say dominant rather than B_0-dominant.

From the Cartan decomposition discussed in 4.3 we have

$$G = \coprod_{\nu} K\pi^{\nu}K, \tag{8.3.1}$$

where ν runs over the set of dominant cocharacters, and where π^ν means (as before) the image of π under the homomorphism $F^\times \to A$ obtained from ν. Each K-double coset is of course a compact subset of G, and therefore the Cartan decomposition gives a very precise way of understanding the non-compactness of G.

We are finally in a position to define the function u^μ that is used to truncate our integral. We need to choose a truncation parameter μ, which is allowed to be any dominant element of $X_*(A)$. Let G^μ denote the subset of G obtained by taking the union of the double cosets $K\pi^\nu K$ for all dominant cocharacters ν such that $\nu \leq \mu$. Since G^μ is a finite union of K-double cosets, it is compact.

We write u^μ for the characteristic function of the subset G^μ of G. For $f_1, f_2 \in C_c^\infty(\mathfrak{g})$ and any truncation parameter μ we put

$$K^\mu(f_1, f_2) := \int_G u^\mu(g) \int_{\mathfrak{g}} f_1(Y) f_2(g^{-1}Yg)\,dY\,dg. \tag{8.3.2}$$

Here we have written dg for the unique Haar measure on G giving K measure 1. It is evident that the integrand of this double integral is compactly supported as well as locally constant, so that the double integral is convergent and can be manipulated in any way we like.

Multiplying both sides of (8.2.2) by u^μ and integrating over G, we get a very crude first version of the local trace formula on $\mathfrak{g}$, namely the equality

$$K^\mu(f_1, f_2) = K^\mu(\hat{f}_1, \check{f}_2). \tag{8.3.3}$$

Since both sides of the formula have the same shape (unlike what happens on the group), it is enough to analyze the left side.

8.4. Using the Weyl integration formula to rewrite $K^\mu(f_1, f_2)$. We now use the Weyl integration formula (7.11.1) to rewrite the inner integral in the expression (8.3.2) defining $K^\mu(f_1, f_2)$, obtaining

$$\begin{aligned}\int_{\mathfrak{g}} f_1(Y) f_2(g^{-1}Yg)\, dY = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\ \int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}h^{-1}Xhg)\, d\dot{h}\, dX.\end{aligned} \tag{8.4.1}$$

The notation is the same as in the Weyl integration formula (7.11.1), so that in particular $d\dot{h}\, dX$ is compatible with dY in the sense of 7.11. By adjusting both $d\dot{h}$ and dX in such a way that $d\dot{h}\, dX$ remains unchanged, we now assume that $d\dot{h}$ is the quotient of the Haar measure on $G(F)$ giving K measure 1 by the Haar measure da_M on A_M giving $A_M \cap K = A_M(\mathcal{O})$ measure 1.

Substitute (8.4.1) back into (8.3.2), change the order of integration so that the innermost integral becomes the one taken over G, change variables by replacing g by $h^{-1}g$, and finally do the integration over G in stages, first integrating over A_M and then integrating over $A_M\backslash G$. This yields

$$\begin{aligned}K^\mu(f_1, f_2) = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\ \int_{A_M\backslash G} \int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg) u_M(h, g; \mu)\, d\dot{h}\, d\dot{g}\, dX,\end{aligned} \tag{8.4.2}$$

where

$$u_M(h, g; \mu) := \int_{A_M} u^\mu(h^{-1}a_M g)\, da_M. \tag{8.4.3}$$

To make further progress on the local trace formula we need to analyze the function u_M. In fact, as we will see, the more refined versions of the local trace formula are obtained from our crude one just by replacing u_M by something simpler. We begin by rewriting the definition of u_M in a more convenient form. Recall that we are writing X for the set G/K and x_0 for its base-point. Recall also the function inv from 3.4. It takes values in $K\backslash G/K$, which by the Cartan decomposition we have now identified with the set of dominant coweights in $X_*(A)$. It follows from all our various definitions that $a \mapsto u^\mu(h^{-1}ag)$ is the characteristic function of the set of $a \in A$ such that

$$\mathrm{inv}(h^{-1}agx_0, x_0) \leq \mu. \tag{8.4.4}$$

Putting $x := gx_0$, $y := hx_0$, we conclude that $u_M(h, g; \mu)$ is the measure of the set of $a \in A_M$ such that

$$\mathrm{inv}(ax, y) \leq \mu. \tag{8.4.5}$$

Therefore, in order to understand u_M for any M, we need to understand, for fixed $x, y \in X$, the subset of A consisting of all $a \in A$ satisfying (8.4.5). To do so, it is best to begin with the simplest non-trivial example, that of GL_2. This will be the topic of the next section.

9. Calculation of u_M for $G = GL_2$

In this section G is $GL_2(F)$.

9.1. Variant of truncation for GL_2. In this special case it seems more convenient to do the truncation slightly differently. Recall the function

$$K(x) = \int_{\mathfrak{g}} f_1(Y) f_2(x^{-1}Yx)\, dY$$

on G that we need to truncate. Our method was to multiply this function by the characteristic function of a compact subset of G and then to integrate over G. However, the function $K(x)$ is obviously invariant under translation by A_G, so another perfectly good way to proceed is to multiply $K(x)$ by the characteristic function of a compact subset of G/A_G and then integrate over G/A_G. This is what we will do for $G = GL_2$.

Our truncation parameter will be a non-negative integer D. Given D, we then put

$$G_D := \{g \in G : d(gv_0, v_0) \leq D\}.$$

Here v_0 is the usual base vertex in the tree, and d denotes the usual metric on the tree. Then G_D is the inverse image in G of a compact subset of G/A_G. We write u_D for the characteristic function of the subset G_D of G. Our weight factor will be

$$u_M(h, g; D) = \int_{A_M/A_G} u_D(h^{-1}ag)\, d\dot{a},$$

where $d\dot{a}$ denotes the quotient da_M/da_G of the Haar measures da_M, da_G on A_M, A_G respectively that give measure 1 to their intersections with K.

Putting $v := gv_0$ and $w := hv_0$, we see that $u(h, g; D)$ is the measure of the set of $a \in A_M/A_G$ such that

$$d(av, w) \leq D. \tag{9.1.1}$$

This condition is reminiscent of ones we have seen before and can be understood easily using the geometry of the tree.

9.2. The case $M = A$. We now define $d(v)$ to be the distance from v to the standard apartment (and the same for w). We warn the reader that later on in the article, when we are working with general split groups, we will use the notation $d(\cdot)$ for a different purpose.

Lemma 9.1. *If $D \geq d(v) + d(w)$, then*

$$u_A(h, g; D) = 2\big(D - d(v) - d(w)\big) + 1. \tag{9.2.1}$$

If $D < d(v) + d(w)$, then $u_A(h, g; D)$ is a real number between 0 and 1.

Proof. Consider the shortest path in the tree from the vertex v to the standard apartment, let v' denote the other endpoint of this shortest path (so that v' is some vertex in the standard apartment), and note that $d(v) = d(v, v')$. In the same way, from our other vertex w, we get w' and $d(w) = d(w, w')$.

So long as $v' \neq w'$ it is clear that

$$d(v, w) = d(v) + d(w) + d(v', w'), \tag{9.2.2}$$

and when $v' = w'$ we at least have the inequality

$$d(v, w) \leq d(v) + d(w) \tag{9.2.3}$$

with strict inequality when there is some overlap between the two shortest paths. Thus $d(v)$, $d(w)$ do not quite determine $d(v, w)$. Nevertheless, we can assert that the condition

$$d(v, w) \leq D \tag{9.2.4}$$

is equivalent to the condition

$$d(v) + d(w) + d(v', w') \leq D \tag{9.2.5}$$

so long as $d(v) + d(w) \leq D$.

Of course it is really $d(av, w)$ that we care about. Since the action of A on the tree preserves the standard apartment, it is clear that $d(av) = d(v)$ and $(av)' = av'$. We conclude that, so long as $d(v) + d(w) \leq D$, the condition (9.1.1) is equivalent to the condition

$$d(av', w') \leq D - d(v) - d(w). \tag{9.2.6}$$

The condition (9.2.6) on $a \in A$ depends only upon the image of a under the surjection $A \to \mathbb{Z}$ sending the diagonal matrix with entries (a_1, a_2) to the integer $\mathrm{val}(a_1) - \mathrm{val}(a_2)$, and therefore the measure of the set of $a \in A/A_G$ satisfying (9.2.6) is equal to the number of lattice points u' in the standard apartment whose distance to w' is less than or equal to $D - d(v) - d(w)$, and this number is obviously equal to

$$2\big(D - d(v) - d(w)\big) + 1. \tag{9.2.7}$$

This proves the lemma when $D \geq d(v) + d(w)$.

On the other hand, when $D < d(v) + d(w)$, the condition (9.1.1) implies that $av' = w'$, so that the measure of the set of all such a is a real number between 0 and 1. $\square$

9.3. The case $M = G$. For an elliptic torus T we have $A_T = A_G$, $M = G$, and $u_G(h, g; D) = u_D(h^{-1}g)$, which is equal to 1 if $d(v, w) \leq D$ and is 0 otherwise.

9.4. The functions $\tilde{v}_A$ and $\tilde{v}_G$. The explicit computations done above show that for fixed g, h and for all sufficiently large D, the value of $u_M(h, g; D)$ is given by

$$\begin{aligned} u_A(h, g; D) &= 2\big(D - d(v) - d(w)\big) + 1 \\ u_G(h, g; D) &= 1 \end{aligned} \tag{9.4.1}$$

How large D has to be depends of course on g, h.

We have already mentioned that more refined versions of the local trace formula on $\mathfrak{g}$ will be obtained by replacing u_M by simpler related functions. To keep our notation consistent with that used later in the general case, we will denote the next weight factor to be considered by $\tilde{v}_M(h, g; D)$. In the case of GL_2 we take the right sides of (9.4.1) as our definitions of $\tilde{v}_M$, in other words, we put

$$\begin{aligned} \tilde{v}_A(h, g; D) &= 2\big(D - d(v) - d(w)\big) + 1 \\ \tilde{v}_G(h, g; D) &= 1 \end{aligned} \tag{9.4.2}$$

for all g, h, D.

For later use (in applying Lebesgue's dominated convergence theorem) we note

$$
\begin{aligned}
|u_A(h,g;D) - \tilde{v}_A(h,g;D)| &\le 2\big(d(v) - d(w)\big) \\
|u_G(h,g;D) - \tilde{v}_G(h,g;D)| &\le 1.
\end{aligned} \tag{9.4.3}
$$

There is one final comment to make about $\tilde{v}_M$. Until now D has been the non-negative integer $m-n$. However, as we see from (9.4.2), the definition of $\tilde{v}_M(h,g;D)$ still makes sense for any real number D.

10. The local trace formula for the Lie algebra of $G = GL_2$

In this section G is again $GL_2(F)$.

10.1. Next form of the local trace formula for GL_2. Our preliminary version of the local trace formula for the Lie algebra of GL_2 says that

$$K^D(f_1, f_2) = K^D(\hat{f}_1, \check{f}_2) \tag{10.1.1}$$

with

$$
\begin{aligned}
K^D(f_1,f_2) = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\
\int_{A_M\backslash G}\int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg) u_M(h,g;D)\, d\dot{h}\, d\dot{g}\, dX.
\end{aligned} \tag{10.1.2}
$$

In the case of GL_2 we have also defined functions $\tilde{v}_M$ that are closely related to the functions u_M appearing in (10.1.2). We now define $J^D(f_1,f_2)$ by the formula

$$
\begin{aligned}
J^D(f_1,f_2) = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\
\int_{A_M\backslash G}\int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg) \tilde{v}_M(h,g;D)\, d\dot{h}\, d\dot{g}\, dX.
\end{aligned} \tag{10.1.3}
$$

The only difference between this expression and the previous one is that u_M has been replaced by $\tilde{v}_M$. Recall that $\tilde{v}_M(h,g;D)$ is defined for all $D \in \mathbb{R}$, so the same is true of $J^D(f_1,f_2)$. Looking back at the definition of $\tilde{v}_M$, we see that the convergence of the double integral appearing in (10.1.3) is an immediate consequence of the following two lemmas.

Lemma 10.1. *For any maximal torus T in G the function*

$$X \mapsto |D(X)|^{1/2} \int_{A_T\backslash G} f(g^{-1}Xg)\, d\dot{g} \tag{10.1.4}$$

on $\mathfrak{t}_{\mathrm{reg}}$ is bounded and locally constant on $\mathfrak{t}_{\mathrm{reg}}$. Moreover this function is compactly supported on $\mathfrak{t}$, in the sense that there exists a compact subset C of $\mathfrak{t}$ such that it vanishes off $C \cap \mathfrak{t}_{\mathrm{reg}}$. In particular the integral

$$\int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)|^{1/2} \int_{A_T\backslash G} f(g^{-1}Xg)\, d\dot{g}\, dX \tag{10.1.5}$$

converges.

In the definition of $\tilde{v}_A(h,g;D)$ we were regarding the function $d(v)$ as a function on G. Now we make this more explicit, putting (for $g \in G$) $d(g) := d(gv_0)$, where v_0 is the base vertex in the tree.

LEMMA 10.2. *The function*

$$X \mapsto |D(X)|^{1/2} \int_{A\backslash G} f(g^{-1}Xg)d(g)\,d\dot{g} \tag{10.1.6}$$

on $\mathrm{Lie}(A)_{\mathrm{reg}}$ *is locally constant on* $\mathrm{Lie}(A)_{\mathrm{reg}}$. *Moreover this function is compactly supported on* $\mathrm{Lie}(A)$, *in the sense that there exists a compact subset* C *of* $\mathrm{Lie}(A)$ *such that it vanishes off* $C \cap \mathrm{Lie}(A)_{\mathrm{reg}}$. *Finally, the integral*

$$\int_{\mathrm{Lie}(A)_{\mathrm{reg}}} |D(X)|^{1/2} \int_{A\backslash G} f(g^{-1}Xg)d(g)\,d\dot{g}\,dX \tag{10.1.7}$$

converges.

The first lemma makes sense and is true for any connected reductive G. The same will turn out to be true of the second lemma once we have defined a suitable generalization of the function $d(g)$. We prefer to prove these results in general, and will therefore defer their proofs till later (see Theorems 17.10, 17.11 and 20.6).

Granting the two lemmas, we can now state and prove the second form of the local trace formula for the Lie algebra of GL_2.

THEOREM 10.3. *For all* $f_1, f_2 \in C_c^\infty(\mathfrak{g})$ *and all* $D \in \mathbb{R}$ *there is an equality*

$$J^D(f_1, f_2) = J^D(\hat{f}_1, \check{f}_2). \tag{10.1.8}$$

PROOF. The functions f_1, f_2 will remain fixed throughout the proof. The first thing to note is that the function $D \mapsto J^D(f_1, f_2)$ is affine linear, or, in other words, has the form $D \mapsto aD + b$ for suitable real numbers a, b.

For any non-negative integer D consider the difference $K^D(f_1, f_2) - J^D(f_1, f_2)$, which is given by the expression

$$\sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \int_{A_M\backslash G} \int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg)\cdot \big(u_M(h, g; D) - \tilde{v}_M(h, g; D)\big)\,d\dot{h}\,d\dot{g}\,dX.$$

As $D \to +\infty$ the pointwise limit of the sequence $u_M(h, g; D) - \tilde{v}_M(h, g; D)$ is 0. Moreover the estimate (9.4.3) for $u_M(h, g; D) - \tilde{v}_M(h, g; D)$, in conjunction with the two lemmas above, shows that Lebesgue's dominated convergence theorem can be applied, yielding the conclusion that

$$K^D(f_1, f_2) - J^D(f_1, f_2) \to 0 \text{ as } D \to +\infty. \tag{10.1.9}$$

Now consider the difference $\Delta(D) := J^D(f_1, f_2) - J^D(\hat{f}_1, \check{f}_2)$, which we are trying to prove is zero. On the one hand, we know that $\Delta(D)$ is an affine linear function of $D \in \mathbb{R}$. On the other hand, applying (10.1.9) to both (f_1, f_2) and $(\hat{f}_1, \check{f}_2)$ and using the first form of the local trace formula, we see that the sequence $\Delta(D)$ has limit 0 as $D \to +\infty$. It follows that $\Delta(D)$ is identically zero, as desired. $\square$

10.2. Final form of the local trace formula for GL_2. In the course of proving the theorem above we saw that $J^D(f_1, f_2)$ is an affine linear function of $D \in \mathbb{R}$. Therefore this theorem is actually an equality between two affine linear functions of D, and entails two equalities, one between the linear terms of the two sides, and one between their constant terms. However this is less interesting than it might at first seem, since the equality between linear terms is a consequence of the local trace formula on $\mathrm{Lie}(A)$. (Something similar happens for general G, involving the local trace formulas for the Lie algebras of the various Levi subgroups of G.)

Thus all the real content of the theorem is in the equality of the constant terms of the two sides. These constant terms are obtained by setting D equal to 0. However, we can capture the same information (modulo the local trace formula on $\mathrm{Lie}(A)$) by setting D equal to any constant we like. The simplest result is obtained by taking D to be $-1/2$, as one sees by looking back at how $\tilde{v}_A$ was defined. Doing so (and denoting the new weight factors by $v_M(h,g) := \tilde{v}_M(h,g;-1/2)$) yields the final form for the local trace formula on the Lie algebra of GL_2, namely

THEOREM 10.4. *For all $f_1, f_2 \in C_c^\infty(\mathfrak{g})$ there is an equality*

$$J(f_1,f_2) = J(\hat{f}_1, \check{f}_2), \tag{10.2.1}$$

where $J(f_1,f_2)$ is defined by

$$\begin{aligned} J(f_1,f_2) = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\ \cdot \int_{A_M\backslash G} \int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg) v_M(h,g)\, d\dot{h}\, d\dot{g}\, dX \end{aligned} \tag{10.2.2}$$

with $v_M(h,g)$ given by

$$\begin{aligned} v_A(h,g) &= -2\big(d(g)+d(h)\big) \\ v_G(h,g) &= 1. \end{aligned} \tag{10.2.3}$$

10.3. Invariance versus non-invariance. When $M=G$, in which case v_G is identically 1, the double integral occurring in (10.2.2) is just the product of the orbital integrals (for X) of the functions f_1 and f_2. When $M = A$, the double integral is still taken over the $(G\times G)$-orbit of (X,X) in $\mathfrak{g}\times\mathfrak{g}$, but we are using v_M times the invariant measure on the orbit. Thus we are dealing with what Arthur calls a *weighted* orbital integral on $\mathfrak{g}\times\mathfrak{g}$. We see from the explicit formula for $v_A(h,g)$ that this weighted orbital integral is the sum of two terms, each term being a product of an orbital integral on one of the two factors of $\mathfrak{g}\times\mathfrak{g}$ and a weighted orbital integral on the other factor. This last phenomenon is an especially simple instance of more complicated *splitting formulas* of Arthur (see [**Art81**, Lemma 6.3] for instance) on general groups G .

We can use these remarks to rewrite $J(f_1,f_2)$ in a form more suited to the application we will make in the next section. For any maximal torus T in G and any $X \in \mathfrak{t}_{\mathrm{reg}}$ we use O_X (as usual) to denote the orbital integral

$$O_X(f) = \int_{A_T\backslash G} f(g^{-1}Xg)\, d\dot{g}. \tag{10.3.1}$$

Similarly, for any $X \in \mathrm{Lie}(A)_{\mathrm{reg}}$ we define the *weighted orbital integral* WO_X by

$$WO_X(f) = \int_{A\backslash G} f(g^{-1}Xg) v_A(g)\, d\dot{g}, \tag{10.3.2}$$

where $v_A(g) := 2d(g)$. (Here we should warn the reader that when we define weight factors for general split groups, we will use a different normalization, in which the weight factor v_A for GL_2 turns out to be $d(g)$ rather than $2d(g)$.)

It is then clear from these definitions that

$$
\begin{aligned}
J(f_1, f_2) = &\sum_{T \in \mathcal{T}_G} |W_T^G|^{-1} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| O_X(f_1) O_X(f_2)\, dX \\
&- |W|^{-1} \int_{(\mathrm{Lie}\, A)_{\mathrm{reg}}} |D(X)| O_X(f_1) W O_X(f_2)\, dX \\
&- |W|^{-1} \int_{(\mathrm{Lie}\, A)_{\mathrm{reg}}} |D(X)| W O_X(f_1) O_X(f_2)\, dX.
\end{aligned} \tag{10.3.3}
$$

10.4. Application of the local trace formula for the Lie algebra of GL_2. Suppose that $O_X(f_2) = 0$ for all $X \in (\mathrm{Lie}\, A)_{\mathrm{reg}}$. Then the last term in (10.3.3) vanishes, and the remaining terms can be recombined using the Weyl integration formula. We conclude, for such f_2, that

$$
J(f_1, f_2) = \int_{\mathfrak{g}_{\mathrm{reg}}} f_1(X) F_2(X)\, dX, \tag{10.4.1}
$$

where F_2 is the unique conjugation-invariant function on $\mathfrak{g}_{\mathrm{rs}}$ such that

$$
F_2(X) = \begin{cases} O_X(f_2) & \text{if } X \in \mathfrak{t}_{\mathrm{reg}} \text{ for some } T \in \mathcal{T}_G \\ -W O_X(f_2) & \text{if } X \in \mathrm{Lie}(A)_{\mathrm{reg}}. \end{cases} \tag{10.4.2}
$$

Since F_2 is conjugation-invariant, the distribution $f_1 \mapsto J(f_1, f_2)$ is an invariant distribution on $\mathfrak{g}$. [The group G acts by conjugation on $\mathfrak{g}$, hence on $C_c^\infty(\mathfrak{g})$, hence on $\mathcal{D}(\mathfrak{g})$, and an *invariant* distribution on $\mathfrak{g}$ is one that is fixed by this conjugation action.]

What can we say about the function F_2 (under our assumption on f_2)? It is a consequence of Lemmas 10.1 and 10.2 that F_2 is locally constant on $\mathfrak{g}_{\mathrm{rs}}$. We will often regard F_2 as a function on $\mathfrak{g}$ by extending it by 0 on the complement of $\mathfrak{g}_{\mathrm{rs}}$; this extended function is usually not locally constant on $\mathfrak{g}$. It is obvious that the support of F_2 is *bounded modulo conjugation*, in the sense that there is a compact subset ω in $\mathfrak{g}$ such that $F_2(X) = 0$ unless $X \in \mathfrak{g}$ is G-conjugate to an element in ω (any compact set ω on which f_2 is supported will do). Finally, we claim that F_2 is a locally integrable function on $\mathfrak{g}$.

We pause to recall that a measurable function F on $\mathfrak{g}$ is said to be *locally integrable* if $F(X)f(X)$ is an integrable function on $\mathfrak{g}$ for all $f \in C_c^\infty(\mathfrak{g})$ (equivalently, for all f obtained by taking characteristic functions of compact open subsets of $\mathfrak{g}$). In our case it is the convergence of the integral (10.4.1) for all f_1 which guarantees that F_2 is indeed locally integrable.

What is the most obvious source of functions whose hyperbolic orbital integrals vanish? Recall that $X \in \mathfrak{g}_{\mathrm{rs}}$ is said to be *elliptic* if its centralizer in G is an elliptic maximal torus in G (see 7.9 for the definition of elliptic maximal torus). We denote by $\mathfrak{g}_e$ the set of elliptic elements in $\mathfrak{g}_{\mathrm{rs}}$. It follows from the discussion in 7.6 that $\mathfrak{g}_e$ is open in $\mathfrak{g}$. [In the notation of that subsection the set $\mathfrak{g}_e$ is the union of open sets $\mathfrak{g}_{\mathrm{rs}}^T$ for T ranging through $\mathcal{T}_G$.]

Now suppose that ϕ is a function lying in the subspace $C_c^\infty(\mathfrak{g}_e)$ of $C_c^\infty(\mathfrak{g})$. It is then obvious that the hyperbolic orbital integrals of ϕ vanish. It is less obvious, but true, that the hyperbolic orbital integrals of the Fourier transform $\hat{\phi}$ also vanish. We will prove a suitable generalization of this later (see Lemma 13.5).

As above we now use ϕ to define an invariant distribution I_ϕ on $\mathfrak{g}$ by putting

$$I_\phi(f) := J(f, \phi).$$

We have seen that I_ϕ is represented by the conjugation-invariant function on $\mathfrak{g}_{\mathrm{rs}}$, supported on $\mathfrak{g}_e$, whose value at $X \in \mathfrak{g}_e$ is $O_X(\phi)$. Thus the invariant distribution I_ϕ can be thought of as a "continuous linear combination" of elliptic regular semisimple orbital integrals.

Recall that the Fourier transform $\hat{T}$ of a distribution T on $\mathfrak{g}$ is defined by

$$\hat{T}(f) = T(\hat{f}) \tag{10.4.3}$$

for all $f \in C_c^\infty(\mathfrak{g})$. Let's examine the Fourier transform of I_ϕ. Using the local trace formula for the functions $\hat{f}, \phi$, we see that

$$\hat{I}_\phi(f) = J(\hat{f}, \phi) = J(f, \hat{\phi}),$$

and therefore, since the hyperbolic orbital integrals of $\hat{\phi}$ vanish, we conclude that the distribution $\hat{I}_\phi$ is represented by the locally integrable conjugation-invariant function F on $\mathfrak{g}_{\mathrm{rs}}$ whose values are given by

$$F(X) = \begin{cases} O_X(\phi) & \text{if } X \in \mathfrak{t}_{\mathrm{reg}} \text{ for some } T \in \mathcal{T}_G \\ -WO_X(\phi) & \text{if } X \in \mathrm{Lie}(A)_{\mathrm{reg}}. \end{cases} \tag{10.4.4}$$

This establishes, for the group GL_2, a key special case of a more general result of Harish-Chandra (see Theorem 27.8), which says that for any invariant distribution I on $\mathfrak{g}$ whose support is bounded modulo conjugation, the Fourier transform $\hat{I}$ is represented by a locally integrable function on $\mathfrak{g}$ that is locally constant on $\mathfrak{g}_{\mathrm{rs}}$. The formula for $F(X)$ when X is elliptic is also due to Harish-Chandra [**HC78, HC99**]. The formula for $F(X)$ when X is non-elliptic is due to Waldspurger. Indeed, Waldspurger [**Wal95**] proves a similar result for arbitrary G, and our approach here follows his.

11. Remarks on Euclidean space

Before introducing the weight factors (see section 12) occurring in weighted orbital integrals for general split groups, we need to discuss some elementary (but important) facts about Euclidean space, which will be used again later in the proof of the key geometric result needed for the local trace formula. These results are related to Langlands' combinatorial lemma [**Lan66**], [**Lan76**], [**Art76**, §2], [**Art78**, Lemma 6.3].

11.1. The abstract set-up. Throughout this section V will denote a Euclidean space, in other words a finite dimensional real vector space equipped with a positive definite symmetric bilinear form (v, w). We further suppose that $v_1, \dots, v_n$ is a basis for V such that

$$(v_i, v_j) \le 0 \quad \text{for all } i \ne j, \tag{11.1.1}$$

and we denote by $v_1^*, \dots, v_n^*$ the basis in V dual to $v_1, \dots, v_n$, so that $(v_i, v_j^*) = \delta_{ij}$. (In the example of interest to us $v_1, \dots, v_n$ will be simple roots in a root system.)

We denote by C the cone generated by $v_1^*, \dots, v_n^*$; thus C consists of non-negative linear combinations of the elements $v_1^*, \dots, v_n^*$. We denote by D the cone generated by $v_1, \dots, v_n$. The cones C and D are dual to each other in the sense that $D = \{v \in V : (v, w) \geq 0 \quad \forall\, w \in C\}$ (and the same with the roles of C and D interchanged).

LEMMA 11.1. *Let V and $v_1, \dots, v_n$ be as above. Then*

(1) *For all i, j we have $(v_i^*, v_j^*) \geq 0$. Equivalently, D contains C.*

(2) *For any j the vectors*

$$v_1, \dots, v_j, v_{j+1}^*, \dots, v_n^*$$

form a basis for V; moreover for all k the vector v_k^* is a non-negative linear combination of $v_1, \dots, v_j, v_{j+1}^*, \dots, v_n^*$.

PROOF. We begin by proving the first part of the lemma. Apply the Gram-Schmidt orthonormalization process to $v_1, \dots, v_n$, obtaining an orthonormal basis $e_1, \dots, e_n$. Thus the first basis vector e_1 is the unit vector in the same direction as v_1, the next basis vector e_2 is the unit vector in the same direction as $v_2 - (v_2, e_1)e_1$, and so on.

Claim 1. For all i the vector e_i is a non-negative linear combination of $v_1, \dots, v_i$, and the coefficient of v_i in this combination is strictly positive. [To prove this use induction on i together with (11.1.1).]

Claim 2. For all i the vector v_i^* is a non-negative linear combination of $e_i, \dots, e_n$, and the coefficient of e_i in this combination is strictly positive. [To prove this note that $v_i^* = \sum_j (v_i^*, e_j)e_j$ and then use Claim 1.]

Claim 3. For all i the vector v_i^* is a non-negative linear combination of $v_1, \dots, v_n$. [To prove this combine Claims 1 and 2.]

We are done proving the first statement of the lemma, as it is just a restatement of Claim 3. (For root systems Claim 3 is the familiar fact that the positive Weyl chamber is contained in the cone of elements that are non-negative linear combinations of roots.)

Now we prove the second part of the lemma. The first statement is clear, since $v_{j+1}^*, \dots, v_n^*$ is obviously a basis for the orthogonal complement of the span of $v_1, \dots, v_j$. The second statement is trivial when $k \geq j+1$, so we just need to consider k such that $1 \leq k \leq j$ and show that v_k^* is a non-negative linear combination of $v_1, \dots, v_j, v_{j+1}^*, \dots, v_n^*$.

Let W denote the span of $v_1, \dots, v_j$. We denote by $w_1^*, \dots, w_j^*$ the basis of W dual to $v_1, \dots, v_j$; by the first part of the lemma (in the form of Claim 3) applied to W, we know that w_k^* is a non-negative linear combination of $v_1, \dots, v_j$. Clearly w_k^* is the orthogonal projection of v_k^* on W, and since (as we have already remarked) $v_{j+1}^*, \dots, v_n^*$ is obviously a basis for the orthogonal complement of W, we can write v_k^* as

$$v_k^* = w_k^* + b_{j+1}v_{j+1}^* + \cdots + b_n v_n^* \tag{11.1.2}$$

for real numbers $b_{j+1}, \dots, b_n$. Since we already know that w_k^* is a non-negative linear combination of $v_1, \dots, v_k$, we just need to check that each b_m is non-negative. It follows from (11.1.2) that

$$b_m = (v_k^*, v_m) - (w_k^*, v_m). \tag{11.1.3}$$

The term (v_k^*, v_m) is 0 since $k \neq m$, and $(w_k^*, v_m) \leq 0$ since w_k^* is a non-negative linear combination of $v_1, \dots, v_k$, showing that $b_m \geq 0$, as desired. $\square$

As an immediate consequence of this lemma, we obtain the following corollary about root systems, which will be used later in our proof of the key geometric result needed for the local trace formula. We fix a Borel subgroup B containing A and write Δ for the set of simple roots of A, viewed as linear forms on $\mathfrak{a}$. We also need the fundamental weights $\varpi_\alpha \in (\mathfrak{a}/\mathfrak{a}_G)^* \subset \mathfrak{a}^*$ (indexed by $\alpha \in \Delta$); recall that $\langle \varpi_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ for all $\alpha, \beta \in \Delta$. Finally we consider a parabolic subgroup $P = MU$ containing B (with Levi component M chosen so that $M \supset A$), and write Δ_M for the set of simple roots of A in M and Δ_U for the set of simple roots of A occurring in $\mathrm{Lie}(U)$; thus Δ is the disjoint union of Δ_M and Δ_U.

LEMMA 11.2. *Let $\mu, \nu \in \mathfrak{a}$ and assume that μ, ν have the same image under the canonical surjection $\mathfrak{a} \to \mathfrak{a}_G$. Assume further that $\langle \alpha, \nu \rangle \leq \langle \alpha, \mu \rangle$ for all $\alpha \in \Delta_M$. Then $\nu \underset{B}{\leq} \mu$ if and only if $\langle \varpi_\alpha, \nu \rangle \leq \langle \varpi_\alpha, \mu \rangle$ for all $\alpha \in \Delta_U$.*

PROOF. Since we are given that μ, ν have the same image in $\mathfrak{a}_G$, the condition $\nu \underset{B}{\leq} \mu$ is equivalent to the condition that $\langle \varpi_\alpha, \nu \rangle \leq \langle \varpi_\alpha, \mu \rangle$ for all $\alpha \in \Delta$. Therefore we need only show that the inequalities $\langle \alpha, \nu \rangle \leq \langle \alpha, \mu \rangle$ for all $\alpha \in \Delta_M$ and $\langle \varpi_\alpha, \nu \rangle \leq \langle \varpi_\alpha, \mu \rangle$ for all $\alpha \in \Delta_U$ together imply the inequalities $\langle \varpi_\alpha, \nu \rangle \leq \langle \varpi_\alpha, \mu \rangle$ for all $\alpha \in \Delta$. This follows from the second part of Lemma 11.1. □

11.2. Subspaces V_I of the Euclidean space V. Now let I be a subset of $\{1, \ldots, n\}$ and denote by V_I the linear span of $\{v_i^* : i \in I\}$. The orthogonal complement of V_I has basis $\{v_j : j \notin I\}$. We denote by π_I the orthogonal projection of V onto V_I.

We are going to see that V_I inherits all the structure we have on V. As usual the restriction of the inner product on V makes V_I into a Euclidean space. Obviously $\{\pi_I v_i : i \in I\}$ is the basis in V_I dual to $\{v_i^* : i \in I\}$. We denote by C_I (respectively, D_I) the cone in V_I generated by $\{v_i^* : i \in I\}$ (respectively, $\{\pi_I v_i : i \in I\}$).

LEMMA 11.3. *The following statements hold.*

(1) $C_I = C \cap V_I = \pi_I C$.
(2) $D_I = D \cap V_I = \pi_I D$.
(3) $(\pi_I v_i, \pi_I v_j) \leq 0$ *for all* $i, j \in I$ *with* $i \neq j$.

PROOF. It is clear from the definitions that $C_I = C \cap V_I$ and that $C \cap V_I \subset \pi_I C$. It follows from the second statement of Lemma 11.1 that $C \cap V_I \supset \pi_I C$.

It is clear from the definitions that $D_I = \pi_I D$ and that $D \cap V_I \subset \pi_I D$. It remains only to show that $\pi_I D \subset D$. In other words, we must show that $(\pi_I d, c) \geq 0$ for all $d \in D$ and $c \in C$, and this follows from $(\pi_I d, c) = (d, \pi_I c)$ and $\pi_I C \subset C$.

To prove the last statement of the lemma, we begin by noting that

$$(\pi_I v_i, \pi_I v_j) = (\pi_I v_i, v_j).$$

Next we expand v_i in the basis v_k^*, obtaining

$$v_i = \sum_{k=1}^{n} (v_i, v_k) v_k^*$$

and hence

$$(\pi_I v_i, v_j) = \sum_{k=1}^{n} (v_i, v_k)(\pi_I v_k^*, v_j).$$

In this sum the term indexed by i is zero since $\pi_I v_i^* = v_i^*$ and $i \neq j$. Each remaining term is non-positive; indeed its first factor is non-positive (since $k \neq i$) and its second factor is non-negative (since $\pi_I C \subset C$). Thus $(\pi_I v_i, v_j)$ is non-positive, as desired. $\square$

11.3. The convex polytope $E(x)$. For $x \in C$ we put $E(x) := C \cap (x - D)$. Here $x - D$ has the obvious meaning: it consists of points of the form $x - d$ with $d \in D$ and is a cone with vertex x. Now $E(x)$ is compact (since it is contained in the obviously compact set $D \cap (x - D)$) and is the intersection of finitely many half-spaces; therefore $E(x)$ is a convex polytope. In other words $E(x)$ has finitely many extreme points (also called *vertices*) and is the convex hull of its set of vertices. We are going to determine the set of vertices of $E(x)$, as this is needed in the proof of Lemma 12.2. The reader is encouraged to draw a picture in the 2-dimensional case, where it is evident that $E(x)$ is a quadrilateral.

LEMMA 11.4. *The set of vertices of $E(x)$ is $\{\pi_I x : I \subset \{1, \dots, n\}\}$.*

PROOF. To simplify notation we put $x_I := \pi_I x$, an element in C_I. We begin by noting that $E(x) \cap V_I = C_I \cap (x_I - D_I)$ for any subset $I \subset \{1, \dots, n\}$, or, in other words, $E(x) \cap V_I$ is the analog $E_I(x_I)$ for x_I, V_I of the set $E(x)$ for x, V. Indeed, using that $C \cap V_I = C_I$ and $\pi_I D = D_I$, we reduce to showing that

$$(x - D) \cap V_I = \pi_I(x - D). \tag{11.3.1}$$

It is clear that the left side is contained in the right side, but we must check that $\pi_I(x - D) \subset x - D$. Using again that $\pi_I D = D_I \subset D$, we see that it is enough to show that $x_I \in x - D$, and this follows from the second part of Lemma 11.1.

Now we prove the assertion of the lemma by induction on the dimension of V, the 0-dimensional case being trivial. Let $J \subset \{1, \dots, n\}$ be a subset having $n - 1$ elements. Then $E(x)$ lies on one side of the hyperplane V_J, so that $E_J(x_J) = E(x) \cap V_J$ is a face of the polytope $E(x)$. By our inductive hypothesis the set of vertices of this face is $\{\pi_I x : I \subset J\}$. We have now accounted for all vertices of $E(x)$ lying in one of the codimension 1 faces C_J of the cone C. It remains only to find the vertices of $E(x)$ lying in the interior of the cone C. An interior point of C is clearly extreme in $E(x)$ if and only if it is extreme in $x - D$, and, since $x - D$ is a cone with vertex x, it has a unique extreme point, namely x. Therefore, if x lies in the interior of C, there is exactly one vertex of $E(x)$ in the interior of C, namely x. Otherwise there is no vertex of $E(x)$ lying in the interior of C, but in this case x is equal to x_J for some J as above. In either case we conclude that $\{\pi_I x : I \subset \{1, \dots, n\}\}$ is the set of vertices of $E(x)$. $\square$

12. Weighted orbital integrals in general

We worked out the local trace formula explicitly for GL_2 and found the weight factor $v_A(g) = 2d(g)$ appearing in our weighted orbital integrals. There are similar weight factors v_M in the general case, which we are now going to discuss. Again we prefer to stick to the case of split groups, in order to keep the structure theory of the group as straightforward as possible.

In this section we work with any split connected reductive group G over our p-adic field F. We use the same notation (e.g. A and $K = G(\mathcal{O})$) as in 4.1. In addition, for any Levi subgroup M in G we write $\mathcal{P}(M)$ for the (finite) set of parabolic F-subgroups of G admitting M as Levi component. In the special case $M = A$ we often write $\mathcal{B}(A)$ instead of $\mathcal{P}(A)$; thus $\mathcal{B}(A)$ is the set of Borel subgroups containing A.

We will only be considering Levi subgroups M containing A. In this case A_M is a subgroup of A and the lattice $X_*(A_M)$ is a subgroup of $X_*(A)$. Recall the definitions $\mathfrak{a} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathfrak{a}_M = X_*(A_M) \otimes_{\mathbb{Z}} \mathbb{R}$.

In the rest of this section G will denote the group of F-points of our algebraic group.

12.1. The maps $H_P : G \to \Lambda_M$. Let M be a Levi subgroup of G containing A, and let $P = MU \in \mathcal{P}(M)$, U being the unipotent radical of P. By the Iwasawa decomposition we have $G = MUK$. Therefore we can write any $g \in G$ as $g = muk$ with $m \in M$, $u \in U$, $k \in K$, and the element m we obtain in this way is unique up to right multiplication by an element of $M(\mathcal{O}) = K \cap M$. Recall from 4.5 the homomorphism $H_M : M \to \Lambda_M$, and put $H_P(g) := H_M(m)$. Then H_P is a well-defined map from G to Λ_M. Clearly

$$H_P(mgk) = H_M(m) + H_P(g) \tag{12.1.1}$$

for all $g \in G$, $m \in M$ and $k \in K$.

The most basic case of this construction occurs when $M = A$, in which case the parabolic subgroup in question is a Borel subgroup B containing A, and the map H_B goes from G to $X_*(A)$. It follows from the definitions that there is an important compatibility between the maps H_P and H_B whenever P contains B, namely $H_P(g)$ is the image of $H_B(g)$ under the canonical surjection $X_*(A) \twoheadrightarrow \Lambda_M$.

Of course for a given $P = MU$ there are many Borel subgroups B sandwiched between P and A; via $B \mapsto B \cap M$ these are in one-to-one correspondence with Borel subgroups of M containing A. Moreover, there is another easily verified compatibility, this time between $H_B : G \to X_*(A)$ and $H_{B\cap M} : M \to X_*(A)$, namely for $g = muk$ as above, we have

$$H_B(g) = H_{B\cap M}(m). \tag{12.1.2}$$

This is especially important in the case when M has rank 1 (so that up to isogeny M is the product of SL_2 and a split torus), in which case there are two Borel subgroups B sandwiched between P and A, corresponding to the two Borel subgroups in SL_2 containing the relevant split maximal torus of that group. For this reason, among others, we need to understand H_B for SL_2.

So for a moment we consider the case $G = SL_2$. As usual we take A to be the subgroup of diagonal matrices and $B = AN$ (respectively $\bar{B} = A\bar{N}$) to be the subgroup of upper triangular (respectively, lower triangular) matrices. Let $\alpha^\vee$ be the coroot corresponding to the unique root α occurring in $\mathrm{Lie}(N)$. In order to understand H_B and $H_{\bar{B}}$ for SL_2 it is enough (by (12.1.1) and the Iwasawa decomposition) to compute them on elements $n \in N$. Trivially we have that $H_B(n) = 0$. A simple computation with 2×2 matrices shows that $H_{\bar{B}}(n)$ is $-r\alpha^\vee$, where r is the following non-negative integer. Look at the upper right matrix entry y of n. If $y \in \mathcal{O}$, then $r = 0$. Otherwise, r is the negative of the valuation of y. Identifying $\mathfrak{a}$ with the standard apartment (see 5.3) in the tree, we see that $H_B(n) = v_0$, $H_{\bar{B}}(n) = v_{-2r}$. Looking back at 5.8, we see that v_{-r} is the point in the standard apartment that is closest to nv_0, and that r is the distance from nv_0 to the standard apartment.

From this computation we conclude that for any element $g \in SL_2(F)$

$$H_B(g) - H_{\bar{B}}(g) = r\alpha^\vee \tag{12.1.3}$$

for some non-negative integer r, that r is in fact equal to the distance from gv_0 to the standard apartment, that the point in the apartment closest to gv_0 is the midpoint of the line segment with endpoints at $H_B(g)$ and $H_{\bar{B}}(g)$ (again viewing these as points in the standard apartment), and that the length of this line segment is $2r$. (Since $\langle \alpha, \alpha^\vee \rangle = 2$, translation by $\alpha^\vee$ is translation by 2.) Before going on, we pause to notice that the weight factor $2d(g)$ entering into weighted orbital integrals for GL_2 has now been interpreted in terms of $H_B(g)$ and $H_{\bar{B}}(g)$: it is the length of the line segment joining the two points $H_B(g)$ and $H_{\bar{B}}(g)$ in $\mathfrak{a}$. We will see how this generalizes in a moment.

Now we return to our general split group G. The computation we just made for SL_2, combined with our previous remarks, shows that for any $g \in G$ the family of points $H_B(g)$ indexed by $B \in \mathcal{B}(A)$ is an example of what Arthur calls a positive (G, A)-orthogonal set; we discuss this notion next.

12.2. **(G, A)-orthogonal sets.** A family of points x_B in $X_*(A)$ (respectively, $\mathfrak{a}$), one for each $B \in \mathcal{B}(A)$, is said to be a (G, A)-*orthogonal set* in $X_*(A)$ (respectively, $\mathfrak{a}$) if for every pair B, $B' \in \mathcal{B}(A)$ of adjacent Borel subgroups (meaning that the corresponding Weyl chambers in $\mathfrak{a}$ are adjacent) there exists an integer (respectively, real number) r such that

$$x_B - x_{B'} = r\alpha^\vee, \tag{12.2.1}$$

where $\alpha^\vee$ is the unique coroot for A that is positive for B and negative for B'. (The explanation for the word "orthogonal" is that with respect to a Weyl group invariant inner product on $\mathfrak{a}$, the line segment joining x_B and $x_{B'}$ is orthogonal to the root hyperplane in $\mathfrak{a}$ defined by α.) When all the numbers r are non-negative (resp., non-positive), the (G, A)-orthogonal set is said to be *positive* (resp., *negative*).

Consider once again a parabolic subgroup $P = MU$ with $M \supset A$. For any (G, A)-orthogonal set $(x_B)_{B \in \mathcal{B}(A)}$ it is easy to see that the points

$$(x_B)_{\{B \in \mathcal{B}(A) : B \subset P\}} \tag{12.2.2}$$

form an (M, A)-orthogonal set (identifying $\{B \in \mathcal{B}(A) : B \subset P\}$ with $\mathcal{B}^M(A)$, the analog of $\mathcal{B}(A)$ for the group M); moreover this (M, A)-orthogonal set is positive if the (G, A)-orthogonal set we started with is positive.

We see from (12.1.2) that the set of points $H_B(g)$ ($B \in \mathcal{B}(A)$ with $B \subset P$) is the positive (M, A)-orthogonal set in $X_*(A)$ attached to the element $m \in M$ obtained from the decomposition $g = muk$.

12.3. **Arthur's weight factor in case $M = A$.** Now we define Arthur's weight factor in the case $M = A$. Start with $g \in G$. Obtain from it the positive (G, A)-orthogonal set $H_B(g)$. Take the convex hull in $\mathfrak{a}$ of the points $H_B(g)$ ($B \in \mathcal{B}(A)$). Define the weight factor $v_A(g)$ to be the volume of this convex hull. By volume we mean Lebesgue measure in the real affine space consisting of all points in $\mathfrak{a}$ whose image under $\mathfrak{a} \twoheadrightarrow \mathfrak{a}_G$ is the same as the common image of all the points $H_B(g)$, the Lebesgue measure being normalized so that measure 1 is given to any fundamental domain for the (translation) action of the coroot lattice of G.

In the case of GL_2 the convex hull is the line segment discussed above, and its volume is its length, but measured with respect to the coroot lattice, which has index 2 in $X_*(A)/X_*(A_G)$, so that with our new definition $v_A(g)$ is equal to $d(g)$ instead of $2d(g)$.

Returning to the general case, note that the family $H_B(g)$ depends only on the coset gK, and that if g is multiplied on the left by an element $a \in A$, then the whole family is translated by the vector $H_A(a)$, leaving its volume unchanged. It follows that

$$v_A(agk) = v_A(g) \tag{12.3.1}$$

for all $g \in G$, $a \in A$, and $k \in K$.

The convex hulls of positive (G, A)-orthogonal sets are very beautiful convex polytopes, about which much can be said. In particular, there is an interesting connection with the theory of toric varieties, as we will see in section 23. In a moment we will see how to get a better picture of the shape of these convex polytopes.

12.4. (G, M)**-orthogonal sets.** To define the weight factors in the general case we need to generalize the notion of (G, A)-orthogonal set. Let M be a Levi subgroup containing A. The roots of A in G that are not roots in M have non-zero restrictions to $\mathfrak{a}_M$, hence define hyperplanes (called walls) in $\mathfrak{a}_M$. The connected components of the complement in $\mathfrak{a}_M$ of the union of these hyperplanes are called chambers in $\mathfrak{a}_M$. For $M = A$ these are the usual Weyl chambers. There is a one-to-one correspondence between chambers in $\mathfrak{a}_M$ and the parabolic subgroups $P \in \mathcal{P}(M)$; the chamber corresponding to $P = MU$ is denoted by $\mathfrak{a}_P^+$ and is given by

$$\mathfrak{a}_P^+ := \{x \in \mathfrak{a}_M : \langle \alpha, x \rangle > 0 \quad \forall\, \alpha \in R_U\}, \tag{12.4.1}$$

where R_U denotes the set of roots of A occurring in $\mathfrak{u} := \mathrm{Lie}(U)$.

Consider two adjacent parabolic subgroups $P, P' \in \mathcal{P}(M)$. (By this we mean that the corresponding chambers are adjacent, or, in other words, separated by exactly one wall.) Recall that Λ_M is the quotient of $X_*(A)$ by the coroot lattice for M. Now consider the collection of elements in Λ_M obtained as the images of the coroots $\alpha^\vee$ where α runs through $R_U \cap R_{\bar{U}'}$ (with $\bar{P}' = M\bar{U}'$ denoting the parabolic subgroup in $\mathcal{P}(M)$ opposite P'). We define $\beta_{P,P'}$ to be the unique element in this collection such that all other members of the collection are positive integral multiples of $\beta_{P,P'}$. In case $M = A$, so that P, P' are Borel subgroups, $\beta_{P,P'}$ is the unique coroot of A that is positive for P and negative for P'.

A family of points x_P in Λ_M (respectively, $\mathfrak{a}_M$), one for each $P \in \mathcal{P}(M)$, is said to be a (G, M)*-orthogonal set* in Λ_M (respectively, $\mathfrak{a}_M$) if for every pair $P, P' \in \mathcal{P}(M)$ of adjacent parabolic subgroups there exists an integer (respectively, real number) r (necessarily unique) such that

$$x_P - x_{P'} = r\beta_{P,P'}.$$

When all the numbers r are non-negative (respectively, non-positive), the (G, M)-orthogonal set is said to be *positive* (respectively, *negative*).

It is clear that $\beta_{\bar{P},\bar{P}'} = -\beta_{P,P'}$. Therefore, if $P \mapsto x_P$ is a (G, M)-orthogonal set, then so is $P \mapsto x_{\bar{P}}$, and the one is positive if and only if the other is negative.

12.5. The points x_P **associated to a** (G, A)**-orthogonal set.** Let $P = MU$ be a parabolic subgroup containing A, and let (x_B) be a (G, A)-orthogonal set in $X_*(A)$. We have already observed that the points $(x_B)_{\{B \in \mathcal{B}(A) : B \subset P\}}$ form an (M, A)-orthogonal set in $X_*(A)$. In particular the difference between any two points in this (M, A)-orthogonal set is a sum of coroots for M, which means that they map to the same point in Λ_M. Thus we get a well-defined point $x_P \in \Lambda_M$ as the common image in Λ_M of all the points $\{x_B : B \subset P\}$.

It is easy to see that the points $(x_P)_{P\in\mathcal{P}(M)}$ form a (G, M)-orthogonal set in Λ_M, and that this (G, M)-orthogonal set is positive if (x_B) is positive.

The same things remain true when $X_*(A)$ is replaced by $\mathfrak{a}$ and Λ_M is replaced by $\mathfrak{a}_M$.

12.6. Arthur's weight factor v_M. The weight factor in the general case is defined as follows. Start with $g \in G$. Obtain from it the family of points $H_P(g) \in \Lambda_M$, one for each $P = MU \in \mathcal{P}(M)$. This family $H_P(g)$ $(P \in \mathcal{P}(M))$ is a positive (G, M)-orthogonal set in Λ_M. By definition $v_M(g)$ is the volume of the convex hull of the images in $\mathfrak{a}_M$ of the points $H_P(g)$. (See subsection 24.7 for a precise normalization of the volume.)

The weight factor satisfies

$$v_M(mgk) = v_M(g) \tag{12.6.1}$$

for all $g \in G$, $m \in M$, and $k \in K$.

12.7. Weighted orbital integrals. Now we can define weighted orbital integrals for G. Let T be a maximal torus such that A_T is contained in A. As before, let M be the centralizer of A_T, a Levi subgroup containing A for which $A_M = A_T$. For $X \in \mathfrak{t}_{\mathrm{reg}}$ and $f \in C_c^\infty(G)$ put

$$WO_X(f) := \int_{A_M\backslash G} f(g^{-1}Xg)v_M(g)\,d\dot{g}. \tag{12.7.1}$$

Just as for unweighted orbital integrals, the semisimplicity of X ensures that the integrand is locally constant and compactly supported on $A_M\backslash G$, so that the integral makes sense.

For $G = GL_2$ this agrees with our previous definition, apart from the factor of 2 mentioned in 12.3. When T is elliptic, so that $M = G$, the weight factor is 1 and the weighted orbital integral is actually an orbital integral.

12.8. Weyl group orbits as positive (G, A)-orthogonal sets. As we have seen, each element $g \in G$ gives rise to a positive orthogonal set $H_B(g)$ $(B \in \mathcal{B}(A))$ in $X_*(A)$. However, there is an even simpler way to produce positive orthogonal sets in $X_*(A)$, and this construction is also relevant to the local trace formula.

First recall that an element $\mu \in X_*(A)$ is said to be dominant with respect to a Borel subgroup $B = AN$ if $\langle\alpha, \mu\rangle \geq 0$ for every root of A occurring in Lie(N); since we need to vary B it is best to refer to such an element x as *B-dominant*. The set of B-dominant elements in $X_*(A)$ is the intersection of $X_*(A)$ with the closure of the Weyl chamber corresponding to B (the Weyl chamber itself being defined as the set of $x \in \mathfrak{a}$ for which all the inequalities $\langle\alpha, x\rangle \geq 0$ are strict).

It is a standard fact about root systems that the closure of any Weyl chamber serves as a fundamental domain for the action of the Weyl group W of A on $\mathfrak{a}$. Thus, given $\mu \in X_*(A)$, for any $B \in \mathcal{B}(A)$ there exists a unique $\mu_B \in X_*(A)$ such that μ_B lies in the W-orbit of μ and is B-dominant. Now suppose that $B_1, B_2 \in \mathcal{B}(A)$ are adjacent. Then there is a unique root α which is positive for B_1 and negative for B_2, and the corresponding root hyperplane is the unique one separating the

Weyl chambers corresponding to B_1, B_2. Let $s_\alpha \in W$ be the reflection across this hyperplane. Then $\mu_{B_2} = s_\alpha \mu_{B_1}$, which tells us that

$$\mu_{B_1} - \mu_{B_2} = \langle \alpha, \mu_{B_1} \rangle \alpha^\vee. \tag{12.8.1}$$

Now $\langle \alpha, \mu_{B_1} \rangle \geq 0$ since μ_{B_1} is B_1-dominant and α is positive for B_1, from which we conclude that the family μ_B ($B \in \mathcal{B}(A)$) is a positive (G, A)-orthogonal set.

12.9. Special (G, M)-orthogonal sets. We say that a (G, A)-orthogonal set of points x_B is *special* if x_B is B-dominant for all $B \in \mathcal{B}(A)$. More generally, we say that a (G, M)-orthogonal set of points x_P is *special* if for all $P \in \mathcal{P}(M)$ the image of x_P under $\Lambda_M \to \mathfrak{a}_M$ lies in the closure of the chamber $\mathfrak{a}_P^+$. (This property arises in [**Art91a**] but is not given a name. It seems convenient to have such a name.)

The (G, A)-orthogonal set obtained from the Weyl group orbit of μ as above provides the most obvious example of a special (G, A)-orthogonal set. In this example we saw that the (G, A)-orthogonal set was positive. This is a general phenomenon: any special (G, M)-orthogonal set (x_P) is automatically positive. Indeed, for adjacent P, P' we have

$$x_P - x_{P'} = r\beta_{P,P'}. \tag{12.9.1}$$

Let α be any root in $R_U \cap R_{\bar{U}'}$. Evaluating the root α on both sides of this equation, one gets a non-negative number on the left side, and since $\langle \alpha, \beta_{P,P'} \rangle$ is strictly positive, we conclude that r must be non-negative. (To see painlessly that $\langle \alpha, \beta_{P,P'} \rangle$ is strictly positive, it is convenient to use a Weyl group invariant inner product to identify $\mathfrak{a}$ with its dual, so that $\alpha^\vee$, α become positive multiples of each other.)

Consider a special (G, A)-orthogonal set (x_B) and a Levi subgroup M containing A. Then for any parabolic subgroup $P \in \mathcal{P}(M)$, it is easy to see that the (M, A)-orthogonal set (12.2.2) obtained from (x_B) is special. Moreover the (G, M)-orthogonal set (x_P) in Λ_M obtained from (x_B) (see 12.5) is also special.

12.10. Shape of the convex hull of a positive (G, A)-orthogonal set. Consider a positive (G, A)-orthogonal set of points x_B ($B \in \mathcal{B}(A)$) in $X_*(A)$. Let $B, B' \in \mathcal{B}(A)$. We no longer assume that they are adjacent. However B, B' can be joined by a chain of Borel subgroups (all containing A) such that each consecutive pair in the chain is a pair of adjacent Borel subgroups. Now assuming that the chain is chosen to be as short as possible, the set of root hyperplanes separating the Weyl chambers of B and B' coincides with the set of hyperplanes separating the various consecutive pairs in our chain. In this way we see that $x_B - x_{B'}$ is a non-negative integral linear combination of the coroots $\alpha^\vee$ that are positive for B and negative for B'. In particular we have

$$x_{B'} \underset{B}{\leq} x_B. \tag{12.10.1}$$

What is the meaning of the symbol B below the inequality sign? We have been using the inequality $x \leq y$ to mean that $y - x$ is a non-negative integral linear combination of positive coroots, positive respect to some fixed $B_0 \in \mathcal{B}(A)$. Now we are letting the Borel subgroup vary, and we use the symbol B to indicate which Borel subgroup we are using.

We also need to define $x \underset{B}{\leq} y$ for elements $x, y \in \mathfrak{a}$: say that $x \underset{B}{\leq} y$ if $y - x$ is a non-negative real linear combination of positive coroots.

Since (12.10.1) holds for all B', the convex hull of the points x_B, which we denote by

$$\mathrm{Hull}\{x_B : B \in \mathcal{B}(A)\},$$

is contained in the convex cone

$$C_B^* := \{x \in \mathfrak{a} : x \underset{B}{\leq} x_B\}, \tag{12.10.2}$$

and since this is true for all B, we conclude that the convex hull is contained in the intersection of the cones C_B^*.

LEMMA 12.1. [**Art76**, Lemma 3.2] *Let (x_B) be a positive (G, A)-orthogonal set in $\mathfrak{a}$. Then*

$$\mathrm{Hull}\{x_B : B \in \mathcal{B}(A)\} = \{x \in \mathfrak{a} : x \underset{B}{\leq} x_B \ \forall\, B \in \mathcal{B}(A)\}.$$

PROOF. It remains only to see that the convex hull of the points x_B contains the intersection of the cones C_B^*. From the theory of convex sets we know that the convex hull in question is the intersection of all the halfspaces containing it, so it will suffice to show that any such halfspace contains one of the cones C_B^*. Say the halfspace is given by the set of points $x \in \mathfrak{a}$ such that $\langle \lambda, x\rangle \leq r$ (with $\lambda \in \mathfrak{a}^*$ and $r \in \mathbb{R}$). Choose $B \in \mathcal{B}(A)$ such that λ is dominant for B. The halfspace contains the entire convex hull and thus contains x_B; the dominance of λ then implies that the halfspace contains C_B^*. $\square$

12.11. A property of special (G, A)-orthogonal sets. Let x_B be a special (G, A)-orthogonal set of points in $X_*(A)$. In 12.5 we defined points $x_P \in \Lambda_M$, one for each parabolic subgroup $P = MU$ containing A. Using the canonical map $\Lambda_M \to \mathfrak{a}_M$, we obtain from x_P an element $\bar{x}_P$ of $\mathfrak{a}_M \subset \mathfrak{a}$. We claim that $\bar{x}_P$ lies in the convex hull of the points $\{x_B : B \subset P\}$.

Indeed, we have seen in 12.2 that the points x_B with $B \subset P$ form an (M, A)-orthogonal set. This reduces us to the case in which $P = G$. It is harmless to assume that G is semisimple. Then we must show that the origin lies in the convex hull of the points x_B. This follows from Lemma 12.1 since by hypothesis x_B is dominant for B, whence $x_B \underset{B}{\geq} 0$.

As an easy exercise the reader may wish to verify that a (G, A)-orthogonal set of points x_B is special if and only if it is positive and satisfies the property that for every parabolic subgroup P containing A the point $\bar{x}_P$ lies in the convex hull of the points $\{x_B : B \subset P\}$. (Hint: Use all the parabolic subgroups $P = MU$ for which M has rank 1.)

12.12. Shape of the convex hull of a special (G, A)-orthogonal set. For special (G, A)-orthogonal sets (which are positive, as we have seen), there is an even more useful description of the convex hull, given in terms of the shapes of its intersections with the closures of the various Weyl chambers. We use the same notation C_B^* as above.

LEMMA 12.2. [**Art91a**, Lemma 3.1] *Let H denote the convex hull of a special (G, A)-orthogonal set of points x_B. Let $B \in \mathcal{B}(A)$ and let C_B be the set of B-dominant elements in $\mathfrak{a}$. Then the intersection of H with the cone C_B is equal to the intersection of C_B^* with C_B.*

Proof. It follows from Lemma 12.1 that $H \cap C_B$ is contained in $C_B^* \cap C_B$. It follows from Lemma 11.4 that $C_B^* \cap C_B$ is the convex hull of the points $\{\bar{x}_P : P \supset B\}$; thus, in order to show that $C_B^* \cap C_B$ is contained in $H \cap C_B$, it suffices to show that each $\bar{x}_P$ is in H. This was done in 12.11. $\square$

12.13. Shape of the convex hull of a (G, M)-orthogonal set. Let M be a Levi subgroup of G containing A. For $P = MU \in \mathcal{P}(M)$ we now define a partial order $\underset{P}{\leq}$ on $\mathfrak{a}_M$ by saying that $x \underset{P}{\leq} y$ if $y - x$ is a non-negative linear combination of images under $\mathfrak{a} \twoheadrightarrow \mathfrak{a}_M$ of coroots $\alpha^\vee$ associated to roots α of A in $\mathrm{Lie}(U)$.

We write π_M for the canonical surjection $\mathfrak{a} \twoheadrightarrow \mathfrak{a}_M$. Let $x, y \in \mathfrak{a}$. Clearly, for any $B \in \mathcal{B}(A)$ such that $B \subset P$ we have the implication

$$x \underset{B}{\leq} y \implies \pi_M(x) \underset{P}{\leq} \pi_M(y). \tag{12.13.1}$$

Lemma 12.3. [**Art76**, Lemma 3.2] *Let (x_P) be a positive (G, M)-orthogonal set in $\mathfrak{a}_M$. Then*

$$\mathrm{Hull}\{x_P : P \in \mathcal{P}(M)\} = \{y \in \mathfrak{a}_M : y \underset{P}{\leq} x_P \quad \forall\, P \in \mathcal{P}(M)\}.$$

Proof. This generalizes Lemma 12.1 and can be proved the same way. $\square$

12.14. Another property of special orthogonal sets. We let M and $\pi_M : \mathfrak{a} \twoheadrightarrow \mathfrak{a}_M$ be as in the previous subsection. When $\mathfrak{a}_M$ is identified with a subspace of $\mathfrak{a}$, the map π_M becomes orthogonal projection. Let (x_B) be a positive (G, A)-orthogonal set in $\mathfrak{a}$, and let (x_P) be the positive (G, M)-orthogonal set in $\mathfrak{a}_M$ obtained from (x_B) as in subsection 12.5. Put $\mathrm{Hull} := \mathrm{Hull}\{x_B : B \in \mathcal{B}(A)\}$, and also put $\mathrm{Hull}_M := \mathrm{Hull}\{x_P : P \in \mathcal{P}(M)\}$. The next result is part of Lemmas 3.1 and 3.2 in [**Art81**].

Proposition 12.1 (Arthur). *The image under π_M of* Hull *is* Hull_M. *Moreover, if (x_B) is special, then so is (x_P), and*

$$\mathfrak{a}_M \cap \mathrm{Hull} = \mathrm{Hull}_M\,.$$

Proof. We begin by proving that $\pi_M(\mathrm{Hull}) \subset \mathrm{Hull}_M$. By Lemma 12.3 we must show that $\pi_M(x_B) \underset{P}{\leq} x_P$ for all $B \in \mathcal{B}(A)$ and all $P \in \mathcal{P}(M)$. Choose $B' \in \mathcal{B}(A)$ such that $B' \subset P$. Then $x_B \underset{B'}{\leq} x_{B'}$ (see 12.10), and therefore from (12.13.1) it follows that $\pi_M(x_B) \underset{P}{\leq} \pi_M(x_{B'}) = x_P$.

Next we prove that $\mathrm{Hull}_M \subset \pi_M(\mathrm{Hull})$. For this it is enough to show that $x_P \in \pi_M(\mathrm{Hull})$ for all $P \in \mathcal{P}(M)$. This is clear since $x_P = \pi_M(x_B)$ for any $B \in \mathcal{B}(A)$ such that $B \subset P$. We are now done proving that $\pi_M(\mathrm{Hull}) = \mathrm{Hull}_M$.

Now suppose that (x_B) is special, which means simply that for all $B \in \mathcal{B}(A)$ the point x_B is dominant with respect to B. Let $P \in \mathcal{P}(M)$ and pick $B \in \mathcal{B}(A)$ such that $B \subset P$. By the first part of Lemma 11.3 we see that the point $x_P = \pi_M(x_B)$ lies in the closure of the chamber $\mathfrak{a}_P^+$ in $\mathfrak{a}_M$. (Note that this closure is the intersection of $\mathfrak{a}_M$ with the closed Weyl chamber for B.) Therefore (x_P) is special.

Since we already know that Hull_M is equal to $\pi_M(\mathrm{Hull})$, to prove that $\mathfrak{a}_M \cap \mathrm{Hull}$ coincides with both of them, it is enough to note that

$$\mathfrak{a}_M \cap \mathrm{Hull} \subset \pi_M(\mathrm{Hull})$$

(clear since π_M is the identity on the subspace $\mathfrak{a}_M$) and that

$$\mathrm{Hull}_M \subset \mathfrak{a}_M \cap \mathrm{Hull}$$

(clear since each x_P lies in Hull by the discussion in subsection 12.11). □

13. Parabolic descent and induction

In this section G denotes a connected reductive group over our p-adic field F. Let $P = MU$ be a parabolic subgroup with Levi component M and unipotent radical U. There always exists some compact open subgroup K of G such that $G = PK$; we fix such a subgroup K. We write $\mathfrak{p}$, $\mathfrak{m}$, $\mathfrak{u}$ for the Lie algebras of P, M, U respectively. Thus $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$.

Orbital integrals on $\mathfrak{m}$ can be related to orbital integrals on $\mathfrak{g}$ by Harish-Chandra's dual processes of parabolic descent and parabolic induction, as we will see in this section (which follows [**HC99**]). Often parabolic descent is used to prove statements about general maximal tori T by reducing to the case in which T is elliptic. (Take M to be the centralizer of the split component of T.) We will encounter an application of this kind in 26.1. Parabolic descent will come up again when we are proving the local trace formula.

13.1. Definition of f^P. Given $f \in C_c^\infty(\mathfrak{g})$ we define a function $f^P \in C_c^\infty(\mathfrak{m})$ by

$$f^P(Y) := \int_{\mathfrak{u}} f(Y+Z)\,dZ. \tag{13.1.1}$$

Here $Y \in \mathfrak{m}$ and dZ is Haar measure on $\mathfrak{u}$.

13.2. Definition of cusp forms on $\mathfrak{g}$. A function $f \in C_c^\infty(\mathfrak{g})$ is said to be a *cusp form* if f^P is identically 0 for every parabolic subgroup P of G such that $P \neq G$. Looking ahead to (13.13.2), we see that the Fourier transform of a cusp form is a cusp form.

13.3. Definition of $f^{(P)}$. Given $f \in C_c^\infty(\mathfrak{g})$ we define a function $f^{(P)} \in C_c^\infty(\mathfrak{m})$ by

$$f^{(P)} := \tilde{f}^P, \tag{13.3.1}$$

where $\tilde{f} \in C_c^\infty(\mathfrak{g})$ is defined by

$$\tilde{f}(X) := \int_K f(k^{-1}Xk)\,dk, \tag{13.3.2}$$

dk being the Haar measure on K giving K measure 1. The linear map $f \mapsto f^{(P)}$ depends on the choice of K (and the measure dZ).

13.4. Definition of parabolic induction i_P^G. Let T_M be a distribution on $\mathfrak{m}$. We define a distribution $i_P^G(T_M)$ on $\mathfrak{g}$ as follows: its value on a test function $f \in C_c^\infty(\mathfrak{g})$ is given by

$$i_P^G(T_M)(f) := T_M(f^{(P)}). \tag{13.4.1}$$

LEMMA 13.1. *Suppose that T_M is an invariant distribution on $\mathfrak{m}$. Then $i_P^G(T_M)$ is an invariant distribution on $\mathfrak{g}$ and is independent of the choice of K.*

Proof. Recall from before that we use a subscript G to denote coinvariants under G. Thus the space of invariant distributions on $\mathfrak{g}$ is the $\mathbb{C}$-linear dual of $C_c^\infty(\mathfrak{g})_G$.

Let $f \in C_c^\infty(\mathfrak{g})$. For any $g \in G$ define ${}^g f \in C_c^\infty(\mathfrak{g})$ by ${}^g f(X) = f(g^{-1}Xg)$. Now consider the map $\varphi : G \to C_c^\infty(\mathfrak{m})_M$ defined by

$$\varphi(g) = ({}^g f)^P. \tag{13.4.2}$$

Note that the vector-valued function φ satisfies (for $p \in P$, $g \in G$)

$$\varphi(pg) = \delta_P(p)\varphi(g), \tag{13.4.3}$$

where δ_P is the modulus character on P (see 2.3), given in this case by

$$\delta_P(p) = |\det(\mathrm{Ad}(p); \mathfrak{u})|.$$

Since P is not unimodular, there is no G-invariant measure on the homogeneous space $P\backslash G$. However there is something similar, namely a non-zero G-invariant linear form $\oint_{P\backslash G}$ defined on the space of locally constant $\mathbb{C}$-valued functions ψ on G satisfying

$$\psi(pg) = \delta_P(p)\psi(g). \tag{13.4.4}$$

(The reason this works is that ψ gives a measure on $P\backslash G$, and this measure can then be integrated over $P\backslash G$.) The linear form $\oint_{P\backslash G}$ is unique up to a non-zero scalar. Since $P\backslash G = (P \cap K)\backslash K$ and $\oint_{P\backslash G}$ is G-invariant (hence K-invariant), we see that (for a suitable normalization of $\oint_{P\backslash G}$), we have

$$\oint_{P\backslash G} \psi = \int_K \psi(k)\, dk. \tag{13.4.5}$$

We have the integration-in-stages formula

$$\oint_{P\backslash G} \int_P h(pg)\, dp = \int_G h(g)\, dg \tag{13.4.6}$$

for all $h \in C_c^\infty(G)$, where dp is a left Haar measure on P and dg is a suitable Haar measure on G.

We can apply $\oint_{P\backslash G}$ to our vector-valued function φ, obtaining a well-defined element $\oint_{P\backslash G} \varphi \in C_c^\infty(\mathfrak{m})_M$. Replacing f by a G-conjugate replaces φ by a right translate, hence leaves $\oint_{P\backslash G} \varphi$ unchanged, so that $f \mapsto \oint_{P\backslash G} \varphi$ is a well-defined linear map $C_c^\infty(\mathfrak{g})_G \to C_c^\infty(\mathfrak{m})_M$.

From (13.4.5) we see that $\oint_{P\backslash G} \varphi$ is equal to the image of $f^{(P)}$ under $C_c^\infty(\mathfrak{m}) \to C_c^\infty(\mathfrak{m})_M$. This concludes the proof (and provides a way to define $f^{(P)}$ as an element of $C_c^\infty(\mathfrak{m})_M$ without having to choose K). □

13.5. A variant. This variant will not be used in this article and can be safely skipped. Let ψ be as above. In some contexts it is natural to work with smaller compact open subgroups (for instance an Iwahori subgroup). So, just for this subsection, let K be any compact open subgroup such that ψ is right K-invariant. Then $P\backslash G/K$ is a finite set. For any ψ as above we have (generalizing (13.4.5))

$$\oint_{P\backslash G} \psi = \sum_{x \in P\backslash G/K} \frac{\mathrm{meas}_{dg}(K)}{\mathrm{meas}_{dp}(P \cap xKx^{-1})} \psi(x) \tag{13.5.1}$$

where dp and dg are as in (13.4.6).

Therefore, if $f \in C_c^\infty(\mathfrak{g})$ is $\mathrm{Ad}(K)$-invariant under some compact open subgroup K, then $f^{(P)}$ (viewed in the M-coinvariants) is given by

$$f^{(P)} = \sum_{x \in P\backslash G/K} \frac{\mathrm{meas}_{dg}(K)}{\mathrm{meas}_{dp}(P \cap xKx^{-1})} ({}^x f)^P. \tag{13.5.2}$$

13.6. Dependence on P. We will see later (Corollary 27.7) that for any invariant distribution T_M on $\mathfrak{m}$ the induced distribution $i_P^G(T_M)$ depends only on M, not on the choice of parabolic subgroup having Levi component M (provided that one is careful about the choice of measure dZ).

13.7. Analogous construction on G. We are working on the Lie algebra, but there are analogs of $f^{(P)}$ and i_P^G on the group G. More precisely

$$f^{(P)}(m) := \delta_P^{1/2}(m) \int_U \tilde{f}(mu)\, du \tag{13.7.1}$$

with $\tilde{f}$ again defined by making f conjugation invariant under K.

With a suitable normalization of Haar measures dg, dm, du one has the following basic fact, which explains the significance of parabolic induction of invariant distributions. Let π_M be an irreducible representation of M, and let Θ_M be its distribution character (which depends on dm). Let π be the representation of G obtained from π_M by (unitary) parabolic induction, and let Θ be its distribution character. Then

$$i_P^G(\Theta_M) = \Theta. \tag{13.7.2}$$

13.8. Nice conjugation invariant functions on $\mathfrak{g}$. To state the next result we need the notion of *nice conjugation invariant function* on $\mathfrak{g}$. By this we mean a conjugation invariant function F that is defined and locally constant on $\mathfrak{g}_{\mathrm{rs}}$ and is locally integrable on $\mathfrak{g}$ (after extending it from $\mathfrak{g}_{\mathrm{rs}}$ to $\mathfrak{g}$, say by 0).

The local integrability (a notion reviewed in 10.4) of F guarantees that we get a well-defined distribution

$$f \mapsto \int_{\mathfrak{g}} f(X)F(X)\, dX \tag{13.8.1}$$

on $\mathfrak{g}$, and the conjugation invariance of F implies that this distribution is invariant. We say that F *represents* this distribution. Since nice functions are required to be locally constant on $\mathfrak{g}_{\mathrm{rs}}$, a set whose complement has measure 0, there is at most one nice conjugation invariant function representing a given invariant distribution.

13.9. Parabolic induction of nice invariant distributions. Now we return to the parabolic subalgebra $\mathfrak{p} = \mathfrak{m} \oplus \mathfrak{u}$ of Lie algebra $\mathfrak{g}$. Suppose that F_M is a nice conjugation invariant function on $\mathfrak{m}$, and let T_M be the invariant distribution on $\mathfrak{m}$ that it represents.

LEMMA 13.2. *The parabolically induced distribution $i_P^G(T_M)$ on $\mathfrak{g}$ is represented by the nice function*

$$|D^G(X)|^{-1/2} \sum_Y |D^M(Y)|^{1/2} F_M(Y) \tag{13.9.1}$$

on $\mathfrak{g}_{\mathrm{rs}}$. Here the sum is taken over a set of representatives for the M-conjugacy classes of elements $Y \in \mathfrak{m}$ such that Y is G-conjugate to X. The superscripts on

D are used to distinguish between the functions previously denoted by $D(X)$ on $\mathfrak{g}$ and $\mathfrak{m}$.

PROOF. Use the Weyl integration formula. $\square$

For example suppose that G is a split group with split maximal torus A and Borel subgroup B containing A. Let χ be a quasi-character on A, that is, a continuous homomorphism $A \to \mathbb{C}^\times$. Then we can parabolically induce χ, obtaining a principal series representation (possibly reducible, though often irreducible) of G, whose character is parabolically induced from χ.

What is the analogous situation on the Lie algebra? The analog of χ is a continuous homomorphism $\xi : \mathrm{Lie}(A) \to \mathbb{C}^\times$, and it represents a distribution (still call it ξ) on $\mathrm{Lie}(A)$ that we can parabolically induce to $\mathfrak{g}$, obtaining an invariant distribution $i_B^G(\xi)$ on $\mathfrak{g}$, which by Lemma 13.2 is represented by the nice function on $\mathfrak{g}_{\mathrm{rs}}$ which vanishes on elements not conjugate to something in $\mathrm{Lie}(A)$ and is given by

$$|D^G(X)|^{-1/2} \sum_{w \in W} \xi(w(X)) \tag{13.9.2}$$

for $X \in \mathrm{Lie}(A)_{\mathrm{reg}}$ (with W denoting the Weyl group of A). This formula is completely analogous to the one for the character of a principal series representation, showing that $i_B^G(\xi)$ should be viewed as the Lie algebra analog of the character of a principal series representation.

13.10. The MUK-integration formula. Let $P = MU$ and K be as before (so that $G = PK = MUK$). We have the integration formula (for $f \in C_c^\infty(G)$)

$$\int_G f(g)\,dg = \int_P \int_K f(pk)\,dk\,dp \tag{13.10.1}$$

for suitably normalized Haar measures dg, dp, dk on G, P, K respectively, dp being a left Haar measure.

This formula is not difficult to prove, the main point being that G can be regarded as a homogeneous space for the group $P \times K$ via the action

$$(p,k) \cdot g = pgk^{-1},$$

so that there is a unique (up to a positive constant) measure on G that is left P-invariant and right K-invariant. Both the left and right sides of the equality (13.10.1) provide such measures on G.

Moreover, since P is the semidirect product of M and the normal subgroup U, there is another integration formula (for $f \in C_c^\infty(P)$)

$$\int_P f(p)\,dp = \int_M \int_U f(mu)\,du\,dm \tag{13.10.2}$$

for suitable Haar measures dm, du on the (unimodular) groups M, U respectively. Again one can use that $M \times U$ acts transitively on P via $(m,u) \cdot p = mpu^{-1}$. Note that the order of multiplication matters; if we used $f(um)$ rather than $f(mu)$, we would get a right Haar measure on P. Alternatively, (13.10.2) is an instance of integration in stages, but in a more general case than we considered before, since P is not unimodular.

Combining the two integration formulas, we get the MUK-integration formula (for $f \in C_c^\infty(G)$)

$$\int_G f(g)\,dg = \int_M \int_U \int_K f(muk)\,dk\,du\,dm. \tag{13.10.3}$$

There is also a useful variant involving a unimodular closed subgroup H of M. For any $f' \in C_c^\infty(H\backslash G)$ we then have

$$\int_{H\backslash G} f'(g)\,dg/dh = \int_{H\backslash M} \int_U \int_K f'(muk)\,dk\,du\,dm/dh. \tag{13.10.4}$$

The variant can be derived from the MUK-integration formula, using that any $f' \in C_c^\infty(H\backslash G)$ can be obtained from some $f \in C_c^\infty(G)$ as

$$f'(g) = \int_H f(hg)\,dh. \tag{13.10.5}$$

In other words $f' = f^\sharp$ in the notation of 2.4.

13.11. Parabolic descent for orbital integrals. For $X \in \mathfrak{g}$ we denote by $\mathfrak{g}_X$ the centralizer of X in $\mathfrak{g}$, or, in other words, the kernel of $\mathrm{ad}(X)$. Then (since we are working in characteristic 0) $\mathfrak{g}_X$ is the Lie algebra of the centralizer G_X of X in G.

In this subsection we will need to use the parabolic subgroup $\bar{P} = M\bar{U}$ opposite to $P = MU$ (with Lie algebra $\bar{\mathfrak{p}} = \mathfrak{m} \oplus \bar{\mathfrak{u}}$). We then have the $\mathrm{Ad}(M)$-stable decomposition

$$\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{u} \oplus \bar{\mathfrak{u}}. \tag{13.11.1}$$

Note that as M-module $\bar{\mathfrak{u}}$ is contragredient to $\mathfrak{u}$ (via the Killing form on $\mathfrak{g}$).

Now let $X \in \mathfrak{m}$, and let $X_s \in \mathfrak{m}$ be the semisimple part of (the Jordan decomposition of) X. Put $D_M^G(X) := \det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{m})$. Since $\mathrm{ad}(X)$ preserves the decomposition (13.11.1), we see that $\mathfrak{g}_X \subset \mathfrak{m}$ if and only if $D_M^G(X) \neq 0$. Since $D_M^G(X) = D_M^G(X_s)$, we see also that $\mathfrak{g}_X \subset \mathfrak{m}$ if and only if $\mathfrak{g}_{X_s} \subset \mathfrak{m}$. (To put this in context we should recall that $\mathfrak{g}_X \subset \mathfrak{g}_{X_s}$.)

Now assume that $X \in \mathfrak{m}$ does satisfy the condition $D_M^G(X) \neq 0$. Then X_s satisfies the same condition, so that $\mathfrak{g}_{X_s} \subset \mathfrak{m}$. Since G_{X_s} is connected (see [**Ste75**, Cor. 3.11]), we see that $G_{X_s} \subset M$, and since $G_X \subset G_{X_s}$, we conclude that $G_X = M_X$. Choose a Haar measure dh on $H := G_X$. As Haar measure dZ on $\mathfrak{u}$ (the one we used in 13.1 to define f^P) we now take the one compatible with the Haar measure du used in the MUK-integration formula.

What does compatible mean? For any algebraic group G over our p-adic field there is a notion of compatibility of Haar measures on G and $\mathfrak{g}$. This is because Haar measures can be obtained from invariant volume forms, and we can agree that a (left, say) invariant volume form ω on G is compatible with a translation invariant volume form ω' on $\mathfrak{g}$ if the value of ω at $1 \in G$ is equal to the value of ω' at $0 \in \mathfrak{g}$. (It makes sense to compare the two values since the tangent space in each case is $\mathfrak{g}$.)

LEMMA 13.3. *Assume that $X \in \mathfrak{m}$ satisfies $D_M^G(X) \neq 0$ and put $H := G_X^0$. Let v be any complex-valued function on G that is right invariant under K and left invariant under both U and H. Then for all $f \in C_c^\infty(\mathfrak{g})$ there is an equality*

$$|D_M^G(X)|^{1/2} \int_{H\backslash G} f(g^{-1}Xg)v(g)\,d\dot{g} = \int_{H\backslash M} f^{(P)}(m^{-1}Xm)v(m)\,d\dot{m}, \tag{13.11.2}$$

provided the two integrals converge. Here $d\dot{g} = dg/dh$ and $d\dot{m} = dm/dh$. In case X is regular semisimple in $\mathfrak{g}$, say with centralizer T, the integrals do converge, and we have

$$D_M^G(X) = D^G(X)/D^M(X), \tag{13.11.3}$$

so that the equality above can be rewritten as

$$|D^G(X)|^{1/2} \int_{T\backslash G} f(g^{-1}Xg)v(g)\,d\dot{g} = |D^M(X)|^{1/2} \int_{T\backslash M} f^{(P)}(m^{-1}Xm)v(m)\,d\dot{m}.$$

Proof. Applying the variant MUK-integration formula (13.10.4), we see that we need to compare integrals over the sets $\{u^{-1}Yu : u \in U\}$ and $Y + \mathfrak{u}$, the first using du, the second dZ, where Y is any M-conjugate of X. In fact it is enough to show that $U \to (Y + \mathfrak{u}) \cong \mathfrak{u}$ defined by $u \mapsto u^{-1}Yu$ is an isomorphism of algebraic varieties whose Jacobian (with respect to compatible left-invariant volume forms on U and $\mathfrak{u}$) is the non-zero constant $\det(\mathrm{ad}(X); \mathfrak{u})$. (Here one needs to use that $\bar{\mathfrak{u}}$ is contragredient to $\mathfrak{u}$ as M-module, so that $D_M^G(X) = (-1)^{\dim(U)} \det(\mathrm{ad}(X); \mathfrak{u})^2$.)

By a theorem of Rosenlicht [**Ros61**] the $\mathrm{Ad}(U)$-orbit of Y in $Y + \mathfrak{u}$ is closed. Since the centralizer of Y is contained in M, it intersects U trivially, showing that $u \mapsto u^{-1}Yu$ identifies U with a locally closed subset of $Y + \mathfrak{u}$ having the same dimension as $\mathfrak{u}$ and hence (by closedness) coinciding with $Y + \mathfrak{u}$. Thus our map is a bijective morphism $U \to Y + \mathfrak{u}$, and it remains only to compute its Jacobian.

Identifying all relevant tangent spaces with $\mathfrak{u}$ in the obvious way (using left translations in the case of U), we see that the differential of our morphism at the point $u \in U$ is equal to $\mathrm{ad}(Y^u) : \mathfrak{u} \to \mathfrak{u}$, where $Y^u := u^{-1}Yu$. Since Y^u, X are P-conjugate, we see that the determinant of the differential is $\det(\mathrm{ad}(X); \mathfrak{u})$, as desired.

Since semisimple orbits are closed, it is clear that the integrals converge for semisimple X. When X is not semisimple, convergence will depend on the weight function v. As we have made no assumption on the growth rate of v, we cannot be sure the integrals converge. □

13.12. Parabolic induction for orbital integrals. We continue with the discussion in the last subsection. In particular X still denotes an element of $\mathfrak{m}$ such that $D_M^G(X) \neq 0$. Taking $v = 1$ in Lemma 13.3, we see that

$$i_P^G(O_X^M) = |D_M^G(X)|^{1/2} O_X^G, \tag{13.12.1}$$

where the superscripts on O_X^G and O_X^M are used to distinguish between orbital integrals on G and M.

For regular semisimple $X \in \mathfrak{g}$ it is sometimes more convenient to use the *normalized orbital integral* $I_X = I_X^G$ defined by

$$I_X = |D(X)|^{1/2} O_X. \tag{13.12.2}$$

For $X \in \mathfrak{m}$ such that X is regular semisimple in $\mathfrak{g}$ (and hence in $\mathfrak{m}$ as well), Lemma 13.3 yields the especially simple formula

$$i_P^G(I_X^M) = I_X^G. \tag{13.12.3}$$

13.13. Fourier transform commutes with parabolic induction. Let V be a finite dimensional vector space over a p-adic field, and let V^* be the dual vector space. Then, fixing a non-trivial additive character ψ on F, the Fourier transform $f \mapsto \hat{f}$ from $C_c^\infty(V) \to C_c^\infty(V^*)$ is defined by

$$\hat{f}(v^*) = \int_V f(v)\psi(\langle v^*, v\rangle)\, dv; \tag{13.13.1}$$

it depends on the choice of Haar measure dv on V. For any linear subspace $W \subset V$ we denote by $W^\perp$ the subspace of V^* consisting of all elements $v^* \in V^*$ such that $\langle v^*, w\rangle = 0$ for all $w \in W$. Now suppose that we have two nested subspaces of V, say $V \supset V_1 \supset V_2$. Dually we have nested subspaces $V^* \supset V_2^\perp \supset V_1^\perp$ and a canonical identification $(V_1/V_2)^* \cong V_2^\perp/V_1^\perp$. It is easy to check that the following diagram commutes

$$\begin{array}{ccc} C_c^\infty(V) & \xrightarrow{FT} & C_c^\infty(V^*) \\ \downarrow & & \downarrow \\ C_c^\infty(V_1/V_2) & \xrightarrow{FT} & C_c^\infty(V_2^\perp/V_1^\perp) \end{array}$$

where the horizontal arrows are Fourier transforms, the left vertical arrow is given by restriction to V_1 and integration over the cosets of V_2, and the right vertical arrow is given by restriction to $V_2^\perp$ and integration over the cosets of $V_1^\perp$. Compatible Haar measures are needed: use dual Haar measures on dual vector spaces and build up the Haar measure on V from Haar measures on V/V_1, V_1/V_2, and V_2.

Now return to $\mathfrak{g}$ and consider the nested subspaces $\mathfrak{g} \supset \mathfrak{p} \supset \mathfrak{u}$. We have agreed (in 8.2) to identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ using some fixed G-invariant non-degenerate symmetric bilinear form $B(\cdot,\cdot)$ on $\mathfrak{g}$. With this identification we have $\mathfrak{u}^\perp = \mathfrak{p}$ and $\mathfrak{p}^\perp = \mathfrak{u}$. Therefore the following diagram commutes

$$\begin{array}{ccc} C_c^\infty(\mathfrak{g}) & \xrightarrow{FT} & C_c^\infty(\mathfrak{g}) \\ \downarrow & & \downarrow \\ C_c^\infty(\mathfrak{m}) & \xrightarrow{FT} & C_c^\infty(\mathfrak{m}) \end{array} \tag{13.13.2}$$

where the horizontal maps are again Fourier transforms and the two vertical maps are both equal to the map $f \mapsto f^P$ defined in 13.1.

LEMMA 13.4. *The map $f \mapsto f^{(P)}$ commutes with the Fourier transform, or, in other words, the commutative diagram above continues to commute when the vertical arrows are replaced by the map $f \mapsto f^{(P)}$. Parabolic induction i_P^G of invariant distributions also commutes with the Fourier transform, or, in other words, for any invariant distribution T_M on $\mathfrak{m}$ we have*

$$i_P^G(\hat{T}_M) = \widehat{i_P^G(T_M)}. \tag{13.13.3}$$

PROOF. Recall that $f^{(P)} = (\tilde{f})^P$. We have just shown (see (13.13.2)) that $f \mapsto f^P$ commutes with the Fourier transform. It is clear from the definition that $f \mapsto \tilde{f}$ commutes with the Fourier transform. Therefore $f \mapsto f^{(P)}$ commutes with the Fourier transform. Since i_P^G is dual to $f \mapsto f^{(P)}$, it too commutes with the Fourier transform. $\square$

13.14. Justification of a statement made earlier. Now we are in a position to prove a statement we needed in 10.4. Recall that $\mathfrak{g}_e$ is the (open) subset of elliptic regular semisimple elements in $\mathfrak{g}$.

LEMMA 13.5. *Assume that $P \neq G$ and that $\phi \in C_c^\infty(\mathfrak{g}_e)$. Then $\phi^{(P)}$ and $(\hat{\phi})^{(P)}$ are identically zero. Moreover, for any $X \in \mathfrak{m}$ such that $D_M^G(X) \neq 0$ the integral*

$$\int_{H\backslash G} \hat{\phi}(g^{-1}Xg)\, d\dot{g} \tag{13.14.1}$$

vanishes.

PROOF. Clearly $\tilde{\phi}$ also vanishes off $\mathfrak{g}_e$, and since $\mathfrak{p}$ does not meet $\mathfrak{g}_e$, the function $\phi^{(P)}$ is identically zero. Since $f \mapsto f^{(P)}$ commutes with the Fourier transform, the function $(\hat{\phi})^{(P)}$ is also identically zero. By Lemma 13.3 the integral (13.14.1) vanishes. □

In 10.4 we needed the special case of this lemma in which G is GL_2 and $P = AN$ a Borel subgroup, in order to obtain the vanishing of the hyperbolic orbital integrals of the function $\hat{\phi}$ considered in that subsection.

14. The map $\pi_G : \mathfrak{g} \to \mathbb{A}_G$ and the geometry behind semisimple descent

We have just discussed parabolic descent. When we begin our systematic treatment of Shalika germs in section 17, we will need semisimple descent, to be discussed in section 16. The purpose of this section and the next is to provide the necessary preparation for semisimple descent, beginning with Chevalley's restriction theorem.

We work with a connected reductive group G over an algebraically closed field k of characteristic 0. Let T be a maximal torus in G. We of course write $\mathfrak{g}$ and $\mathfrak{t}$ for the Lie algebras of G and T. We denote by W the Weyl group of T.

14.1. Basic polynomial invariants. The ring of polynomial functions on $\mathfrak{t}$ is the symmetric algebra $S =: S(\mathfrak{t}^*)$. The Weyl group W acts on S, and the ring S^W of invariants is the ring of regular functions on the quotient variety $\mathfrak{t}/W$. Both S and S^W are graded (by polynomial degree). Inside S^W we have the (graded) maximal ideal I comprised of all invariant polynomials with constant term 0. The quotient I/I^2 is a graded vector space. Choose a basis of homogeneous elements in I/I^2 and lift them to homogeneous elements $x_1, \dots, x_n$ in I. Then $n = \dim(\mathfrak{t})$ and S^W is isomorphic to the polynomial ring $k[x_1, \dots, x_n]$ (see [**Bou02**, Ch. 5, §5]). Our choice of $x_1, \dots, x_n$ gives us an isomorphism of varieties from $\mathfrak{t}/W$ to the standard affine n-space $\mathbb{A}^n$. We will sometimes denote $\mathfrak{t}/W$ by $\mathbb{A}_G$. (The notation $\mathbb{A}_G$ is supposed to remind us that we are dealing with an affine space.)

On any affine space $\mathbb{A}^m$ there is an essentially unique volume form (nowhere-vanishing differential form of top degree), unique, that is, up to an element in $k^\times$. Indeed, it is clear that such forms exist, and uniqueness follows from the fact that there are no units in polynomial rings other than constants. Both $\mathfrak{t}$ and $\mathfrak{t}/W$ are affine spaces of dimension $n = \dim(T)$. Pick volume forms on both affine spaces and look at the Jacobian of the canonical surjection $\mathfrak{t} \to \mathfrak{t}/W$. Up to an element of $k^\times$ it is independent of the choice of volume forms. The Jacobian turns out to be

$$\prod_{\alpha \in R_G^+} \alpha \tag{14.1.1}$$

(see [**Bou02**, Ch. 5, §5.5, Prop. 6]), up to an element of $k^\times$. Here $R_G \subset X^*(T)$ is the set of roots of T in $\mathfrak{g}$, and R_G^+ is the subset of positive roots (positive with respect to some Borel subgroup B containing T). As before, the differentials of the roots allow us to view them as linear forms on $\mathfrak{t}$, hence as elements in S.

14.2. Chevalley's restriction theorem. We write $\mathcal{O}_\mathfrak{g}$ for the k-algebra of polynomial functions on $\mathfrak{g}$. Inside $\mathcal{O}_\mathfrak{g}$ we have the subalgebra $(\mathcal{O}_\mathfrak{g})^G$ of conjugation invariant polynomial functions on $\mathfrak{g}$. Chevalley's theorem [**SS70**, 3.17] states that by restricting polynomial functions from $\mathfrak{g}$ to $\mathfrak{t}$ we get an isomorphism from $(\mathcal{O}_\mathfrak{g})^G$ to the algebra S^W of W-invariant polynomial functions on $\mathfrak{t}$. Thus $\operatorname{Spec}(\mathcal{O}_\mathfrak{g})^G \cong \mathfrak{t}/W = \mathbb{A}_G$.

Dual to the inclusion of $(\mathcal{O}_\mathfrak{g})^G$ in $\mathcal{O}_\mathfrak{g}$ is a surjective morphism

$$\pi_G : \mathfrak{g} \to \mathbb{A}_G = \mathfrak{t}/W \tag{14.2.1}$$

which maps $X \in \mathfrak{g}$ to the unique W-orbit in $\mathfrak{t}$ consisting of elements conjugate to the semisimple part of X. Therefore $\pi_G(X) = \pi_G(Y)$ if and only if the semisimple parts of X, Y are G-conjugate. Moreover, the nilpotent cone in $\mathfrak{g}$ equals $\pi_G^{-1}(0)$, where 0 denotes the origin in $\mathbb{A}_G$.

More concretely, we can also view π_G as the map $X \mapsto (P_1(X), \dots, P_n(X))$ from $\mathfrak{g}$ to $\mathbb{A}^n$, where P_i is the homogeneous G-invariant polynomial on $\mathfrak{g}$ corresponding to the element x_i from before. Letting d_i denote the degree of P_i, we define an action of $\mathbb{G}_m$ on $\mathbb{A}^n$ by $\beta \cdot (z_1, \dots, z_n) := (\beta^{d_1} z_1, \beta^{d_2} z_2, \dots, \beta^{d_n} z_n)$ for all $\beta \in \mathbb{G}_m$ and all $(z_1, \dots, z_n) \in \mathbb{A}^n$. For this $\mathbb{G}_m$ action on $\mathbb{A}_G = \mathbb{A}^n$ we have

$$\pi_G(\beta X) = \beta \cdot \pi_G(X). \tag{14.2.2}$$

14.3. Non-algebraically closed fields. When the base field k is not algebraically closed, it is better to define $\mathbb{A}_G$ as $\operatorname{Spec}((\mathcal{O}_\mathfrak{g})^G)$, as this avoids having to choose a maximal k-torus. Note that $(\mathcal{O}_\mathfrak{g})^G$ is still a polynomial ring over k for which we may choose homogeneous generators $P_1, \dots, P_n$. The morphism π_G is defined over k and $\mathbb{G}_m$-equivariant.

14.4. The subgroup H. Now we come to the geometry behind semisimple descent. For this we consider a connected reductive subgroup H of G such that $T \subset H$. For example H might be a Levi subgroup or the centralizer of a semisimple element in $\mathfrak{g}$ (see [**Bor91**, Prop. 13.19] for the fact that such a centralizer is reductive and see [**Ste75**, Cor. 3.11] for the fact that it is connected when the characteristic of the ground field is 0). Taking Lie algebras of the three groups, we have inclusions

$$\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}. \tag{14.4.1}$$

The normalizer of T in H is a subgroup of the normalizer of T in G, so the Weyl group W_H of T in H is a subgroup of the Weyl group W_G of T in G.

14.5. Jacobian of $\rho : \mathbb{A}_H \to \mathbb{A}_G$ and definition of $\mathbb{A}'_H$ and $\mathfrak{h}'$. Using (14.1.1), we see that the Jacobian of the natural finite morphism

$$\rho : \mathfrak{t}/W_H \twoheadrightarrow \mathfrak{t}/W_G$$

is (up to an element in $k^\times$) equal to

$$\prod_{\alpha \in R_G^+ \setminus R_H^+} \alpha. \tag{14.5.1}$$

Let $\mathfrak{t}'$ be the open W_H-stable subvariety of $\mathfrak{t}$ consisting of $X \in \mathfrak{t}$ such that $\alpha(X) \neq 0$ for all $\alpha \in R_G \setminus R_H$, and let $\mathbb{A}'_H$ be the open subvariety of $\mathbb{A}_H$ obtained as the image of $\mathfrak{t}'$ under $\mathfrak{t} \twoheadrightarrow \mathfrak{t}/W_H$. The explicit formula (14.5.1) for the Jacobian of ρ shows that $\mathbb{A}'_H$ is precisely the set of points where the finite morphism $\rho : \mathbb{A}_H \twoheadrightarrow \mathbb{A}_G$ is étale.

Define an open subvariety $\mathfrak{h}'$ in $\mathfrak{h}$ by

$$\begin{aligned} \mathfrak{h}' &:= \{X \in \mathfrak{h} : \det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{h}) \neq 0\} \\ &= \pi_H^{-1}(\mathbb{A}'_H). \end{aligned} \tag{14.5.2}$$

For any $X \in \mathfrak{h}'$ we have $\mathfrak{g}_{X_s} \subset \mathfrak{h}$, so that $G_{X_s} \subset H$ (by the connectedness of G_{X_s}). Since $G_X \subset G_{X_s}$, we also have $G_X \subset H$, so that $H_X = G_X$.

14.6. The morphism β. The morphism $G \times \mathfrak{h}' \to \mathfrak{g}$ defined by $(g, X) \mapsto gXg^{-1}$ is constant on orbits of the right H-action on $G \times \mathfrak{h}'$ given by $(g, X) \cdot h = (gh, h^{-1}Xh)$, and therefore descends to a morphism

$$\beta : G \underset{H}{\times} \mathfrak{h}' \to \mathfrak{g} \tag{14.6.1}$$

from the quotient space for this H-action to $\mathfrak{g}$. Clearly β is G-equivariant for the left translation action of G on the first factor in the source and the adjoint action of G on the target.

We claim that β is étale. Indeed, by G-equivariance it is enough to prove that the differential $d\beta$ is an isomorphism at $(1, X)$ for any $X \in \mathfrak{h}'$. Since $d\beta$ is given by

$$\begin{gathered} \mathfrak{g} \underset{\mathfrak{h}}{\times} \mathfrak{h} \to \mathfrak{g} \\ (\Delta g, \Delta X) \mapsto [\Delta g, X] + \Delta X \end{gathered} \tag{14.6.2}$$

we see that $d\beta$ is surjective (because $\det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{h}) \neq 0$) and hence an isomorphism (look at dimensions).

14.7. Factorization of the morphism β. The morphism β is étale, so general theory says that it can be factorized as an open immersion followed by a finite morphism. In this case there is an obvious way to produce such a factorization. The étale morphism $\mathbb{A}'_H \to \mathbb{A}_G$ factors as

$$\mathbb{A}'_H \xrightarrow{j} \mathbb{A}_H \xrightarrow{\rho} \mathbb{A}_G$$

with j an open immersion and ρ finite. We now perform a base-change, using the morphism $\pi_G : \mathfrak{g} \to \mathbb{A}_G$. We obtain morphisms

$$\mathfrak{g} \times_{\mathbb{A}_G} \mathbb{A}'_H \hookrightarrow \mathfrak{g} \times_{\mathbb{A}_G} \mathbb{A}_H \to \mathfrak{g}$$

with the left arrow an open immersion and the right arrow a finite morphism. We will have the desired factorization of β once we identify $G \underset{H}{\times} \mathfrak{h}'$ with $\mathfrak{g} \times_{\mathbb{A}_G} \mathbb{A}'_H$ over $\mathfrak{g}$.

For this purpose we define a morphism

$$\gamma : G \underset{H}{\times} \mathfrak{h}' \to \mathfrak{g} \times_{\mathbb{A}_G} \mathbb{A}'_H$$

over $\mathfrak{g}$ by putting $\gamma(g, X) := (gXg^{-1}, \pi_H(X))$. It then remains to prove the following lemma.

Lemma 14.1. *The morphism γ is an isomorphism. In particular*

$$(g, X) \mapsto (gXg^{-1}, \pi_H(X))$$

defines a closed immersion

$$G \underset{H}{\times} \mathfrak{h}' \to \mathfrak{g} \times \mathbb{A}'_H. \tag{14.7.1}$$

Proof. We have seen that $G \underset{H}{\times} \mathfrak{h}'$ is étale over $\mathfrak{g}$ (via β). Moreover $\mathfrak{g} \times_{\mathbb{A}_G} \mathbb{A}'_H$ is étale over $\mathfrak{g}$ (since ρ is étale on the open subset $\mathbb{A}'_H$ of $\mathbb{A}_H$). Therefore γ is automatically étale, and it is enough to show that it is bijective on k-points.

First we check surjectivity. Thus we start with a pair $(X, Y) \in \mathfrak{g} \times_{\mathbb{A}_G} \mathbb{A}'_H$, where $Y \in \mathfrak{t}'$ represents an element in $\mathfrak{t}'/W_H = \mathbb{A}'_H$, and we want to show that this pair is in the image of γ. Since X, Y become the same in $\mathbb{A}_G$, we see that the semisimple part X_s of X is G-conjugate to Y, and by the obvious G-equivariance of γ we may assume without loss of generality that $X_s = Y$. Since $Y \in \mathfrak{h}'$, we have $\mathfrak{g}_Y \subset \mathfrak{h}$, and since X commutes with its semisimple part, namely Y, we see that $X \in \mathfrak{h}$. In fact $X \in \mathfrak{h}'$, since an element of $\mathfrak{h}$ lies in $\mathfrak{h}'$ if and only if its semisimple part does. The element $(1, X) \in G \underset{H}{\times} \mathfrak{h}'$ maps to (X, Y) under γ.

Now we check injectivity. Again using the G-equivariance of γ, we see that it is enough to prove that if

$$\gamma(1, X) = \gamma(g, X')$$

then $g \in H$ and $gX'g^{-1} = X$. The second condition is obvious. It follows that $gX'_s g^{-1} = X_s$. Since $X, X' \in \mathfrak{h}'$ become the same in $\mathbb{A}_H$, there exists $h \in H$ such that $hX'_s h^{-1} = X_s$. Therefore $hg^{-1} \in G_{X_s}$. Since $X_s \in \mathfrak{h}'$, we have $G_{X_s} \subset H$, as we noted before. Therefore $hg^{-1} \in H$, proving that $g \in H$, as desired. □

15. Harish-Chandra's compactness lemma; boundedness modulo conjugation

Before moving on to semisimple descent, we discuss some related matters that make use of the map $\pi_G : \mathfrak{g} \to \mathbb{A}_G$ of section 14. In this section F is a local field of characteristic 0.

15.1. Harish-Chandra's compactness lemma. We return to the situation in 14.4, but now we suppose that $H \subset G$ are defined over F. Then we have the following slight generalization of Harish-Chandra's compactness lemma [**HC70**, Lemma 25]. It will be used for semisimple descent.

Lemma 15.1 (Harish-Chandra). *Let $\omega_{\mathfrak{g}}$ be a compact subset of $\mathfrak{g}$ and let ω_H be a compact subset of $\mathbb{A}'_H(F)$. Then*

$$\{g \in G(F)/H(F) : \exists X \in \pi_H^{-1}(\omega_H) \text{ such that } gXg^{-1} \in \omega_{\mathfrak{g}}\} \tag{15.1.1}$$

has compact closure in $G(F)/H(F)$.

Proof. Put $Z := G \underset{H}{\times} \mathfrak{h}'$. On F-points the closed immersion (14.7.1) yields a closed embedding $Z(F) \hookrightarrow \mathfrak{g} \times \mathbb{A}'_H(F)$ that we will denote by i. Projection onto the first factor of $G \times \mathfrak{h}'$ induces a morphism $Z(F) \to (G/H)(F)$ that we will denote by η. Note that $G(F)/H(F)$ sits inside $(G/H)(F)$ as an open and closed subset. The set (15.1.1) is contained in the compact set $\eta i^{-1}(\omega_{\mathfrak{g}} \times \omega_H)$. □

15.2. Boundedness modulo conjugation. Boundedness modulo conjugation came up already in 10.4. We now discuss this notion more systematically. We say that a subset of $\mathbb{A}_G(F)$ is *bounded* if it is contained in some compact subset of $\mathbb{A}_G(F)$. We say that a subset V of $\mathfrak{g}$ is *bounded modulo conjugation* if there exists a compact subset C of $\mathfrak{g}$ such that every element in V is G-conjugate to an element in C (in other words $V \subset \mathrm{Ad}(G)(C)$).

LEMMA 15.2. *Let V be a subset of $\mathfrak{g}$. The following three conditions are equivalent.*

(1) *The set V is bounded modulo conjugation.*
(2) *There exists a compact subset C in $\mathfrak{g}$ such that V is contained in the closure of $\mathrm{Ad}(G)(C)$.*
(3) *The image of V under $\pi_G : \mathfrak{g} \to \mathbb{A}_G(F)$ is bounded in $\mathbb{A}_G(F)$.*

PROOF. The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate. It remains to check that (3) $\Rightarrow$ (1). Let ω be a compact subset of $\mathbb{A}_G(F)$. It is enough to show that $\pi_G^{-1}\omega$ is bounded modulo conjugation.

Let T be a maximal torus in G. We say that a linear subspace $\mathfrak{s}$ of $\mathfrak{t}$ is *special* if it arises as the center of the centralizer of some element in $\mathfrak{t}$. Thus $\mathfrak{s}$ is the intersection of the kernels of some subset of the roots of $\mathfrak{t}$; it follows that $\mathfrak{t}$ has only finitely many special subspaces. For each special subspace $\mathfrak{s}$ let $M_\mathfrak{s}$ denote the centralizer in G of $\mathfrak{s}$ and let $\mathfrak{m}_\mathfrak{s}$ denote the Lie algebra of $M_\mathfrak{s}$. Then $\mathfrak{s}$ coincides with the center of $\mathfrak{m}_\mathfrak{s}$. Let $\mathcal{N}_\mathfrak{s}$ be a set of representatives for the $M_\mathfrak{s}$-conjugacy classes of nilpotent elements in $\mathfrak{m}_\mathfrak{s}$. Now put

$$C := \bigcup_T \bigcup_\mathfrak{s} \bigcup_Y \big((\mathfrak{s} \cap \pi_G^{-1}\omega) + Y\big),$$

where T runs over a set of representatives for the G-conjugacy classes of maximal F-tori in G, $\mathfrak{s}$ runs over the set of special subspaces of $\mathfrak{t} = \mathrm{Lie}(T)$, and Y runs over $\mathcal{N}_\mathfrak{s}$. Each set $(\mathfrak{s} \cap \pi_G^{-1}\omega) + Y$ is compact (since the maps $\mathfrak{s} \hookrightarrow \mathfrak{t} \to \mathbb{A}_G(F)$ are proper) and the union is finite, so the set C is compact. It is clear that $\pi_G^{-1}\omega \subset \mathrm{Ad}(G)(C)$. $\square$

It follows from the lemma that V is bounded modulo conjugation if and only if its closure is.

LEMMA 15.3. *Now we work over a p-adic field F. Let L be a lattice in $\mathfrak{g}$. Then $\mathrm{Ad}(G)(L)$ contains a G-invariant open and closed neighborhood V of the nilpotent cone.*

PROOF. The nilpotent cone equals $\pi_G^{-1}(0)$, where 0 denotes the origin in $\mathbb{A}_G(F)$. Pick any compact open neighborhood ω of 0 in $\mathbb{A}_G(F)$. Then $\pi_G^{-1}(\omega)$ is bounded modulo conjugation, so by the previous lemma there exists an integer m such that $\pi_G^{-1}(\omega) \subset \mathrm{Ad}(G)(\pi^{-m}L)$. Therefore $V := \pi_G^{-1}(\pi^m \cdot \omega)$ does the job. Here we are using the $\mathbb{G}_m$-action on $\mathbb{A}_G$ for which π_G is equivariant (see (14.2.2)). $\square$

16. Semisimple descent

As usual F is a p-adic field. Throughout this section we consider a connected reductive F-group G and a connected reductive F-subgroup H of G that contains some maximal torus of G. Our goal is to compare orbital integrals on G and H. As usual we write G, H for the F-points of these groups. We use notation such as $\mathfrak{h}'$, $\mathbb{A}'_H$, π_H from section 14.

16.1. Associated functions f and ϕ. Let $f \in C_c^\infty(\mathfrak{g})$ and let ω_H be a compact subset of $\mathbb{A}'_H(F)$. By Lemma 15.1 there exists a compact subset C of $H\backslash G$ such that

$$\{g \in G(F)/H(F) : \exists X \in \pi_H^{-1}(\omega_H) \text{ such that } gXg^{-1} \in \operatorname{Supp}(f)\} \tag{16.1.1}$$

is contained in C, and since $H\backslash G$ is an l.c.t.d space, we may even assume that C is open as well as compact. Now choose $\alpha \in C_c^\infty(G)$ such that $\alpha^\sharp = 1_C$ (see 2.4), and define $\phi \in C_c^\infty(\mathfrak{h})$ by

$$\phi(Y) := \int_G f(g^{-1}Yg)\alpha(g)\,dg. \tag{16.1.2}$$

Recall (see 14.5) that for $X \in \mathfrak{h}'$ we have $H_X = G_X$. The next result is contained implicitly in the proof of Lemma 29 in [**HC70**].

LEMMA 16.1. *For any $X \in \pi_H^{-1}(\omega_H)$ there is an equality*

$$\int_{G_X\backslash H} \phi(h^{-1}Xh)\,d\dot{h} = \int_{G_X\backslash G} f(g^{-1}Xg)\,d\dot{g}. \tag{16.1.3}$$

PROOF. Use Lemma 2.3, applied to the function $g \mapsto f(g^{-1}Xg)$ on $G_X\backslash G$. □

16.2. Comparison with parabolic descent. In the special case when H is a Levi subgroup M of G we have already done much better than Lemma 16.1. Indeed, for any $f \in C_c^\infty(\mathfrak{g})$ we produced a function $f^{(P)}$ on $\mathfrak{m}$ having the same orbital integrals as f (up to a Jacobian factor) for all orbits in $\mathfrak{m}'$. The lemma above produces a function ϕ on $\mathfrak{h}$ having the same orbital integrals as f for all orbits in the subset $\pi_H^{-1}(\omega_H)$ of $\mathfrak{h}'$. Of course we are free to take ω_H as large as we like, provided that it stays compact, but if we change ω_H, we will have to change ϕ as well. Nevertheless the lemma above will be enough to prove descent for Shalika germs, as we will see later.

17. Basic results on Shalika germs on $\mathfrak{g}$

In this section F is a p-adic field and G is a connected reductive F-group. As usual we write G for the group of F-points of G.

We defined Shalika germs on G and calculated them for GL_2. Now we begin systematically studying Shalika germs on $\mathfrak{g}$. The discussion will be completed later, in section 27.

17.1. Orbital integrals O_X for regular semisimple X. Let $\mathcal{T}$ be a set of representatives for the G-conjugacy classes of maximal F-tori in G.

Let $T \in \mathcal{T}$. In order to define our orbital integrals for $X \in \mathfrak{t}_{\text{reg}}$ in a coherent way, we fix Haar measures dg, dt on G, T respectively and define (for any $X \in \mathfrak{t}_{\text{reg}}$)

$$O_X(f) := \int_{T\backslash G} f(g^{-1}Xg)\,dg/dt. \tag{17.1.1}$$

The Weyl group W_T (defined in subsection 7.1) acts on $T\backslash G$ by left multiplication, preserving the invariant measure on that homogeneous space, and therefore

$$O_{w(X)}(f) = O_X(f) \tag{17.1.2}$$

for all $f \in C_c^\infty(\mathfrak{g})$, all $X \in \mathfrak{t}_{\text{reg}}$ and all $w \in W_T$.

Now let $Y \in \mathfrak{g}_{\text{rs}}$. There exists unique $T \in \mathcal{T}$ such that Y is G-conjugate to some $X \in \mathfrak{t}$, and moreover X is unique up to the action of W_T. Therefore it is legitimate to put $O_Y := O_X$. Now we have coherent choices for all regular semisimple orbital integrals O_Y.

17.2. Preliminary definition of Shalika germs on $\mathfrak{g}$. There are finitely many nilpotent G-orbits $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_r$ in $\mathfrak{g}$. We write $\mu_1, \dots, \mu_r$ for the corresponding nilpotent orbital integrals. The same reasoning as for unipotent orbital integrals (see 6.4) shows that the distributions $\mu_1, \dots, \mu_r$ are linearly independent.

THEOREM 17.1. *There exist functions $\Gamma_1, \Gamma_2, \dots, \Gamma_r$ on $\mathfrak{g}_{\mathrm{rs}}$ having the following property. For every $f \in C_c^\infty(\mathfrak{g})$ there exists an open neighborhood U_f of 0 in $\mathfrak{g}$ such that*

$$O_X(f) = \sum_{i=1}^{r} \mu_i(f) \cdot \Gamma_i(X) \tag{17.2.1}$$

for all $X \in \mathfrak{g}_{\mathrm{rs}} \cap U_f$. The germs about $0 \in \mathfrak{g}$ of the functions $\Gamma_1, \dots, \Gamma_r$ are unique. We refer to Γ_i as the provisional Shalika germ for the nilpotent orbit $\mathcal{O}_i$.

PROOF. This is proved exactly the same way as the analogous result (Theorem 6.1) on the group. There we worked with one T at a time, but the proof provided a set U_f that works for all T at once. $\square$

A Shalika germ is an equivalence class of functions on $\mathfrak{g}_{\mathrm{rs}}$. As we will see next, the homogeneity of Shalika germs makes it possible to single out one particularly nice function Γ_i within its equivalence class. Once we have done this, Γ_i will from then on denote this function (whose germ about 0 is the old Γ_i). First we need to understand homogeneity of nilpotent orbital integrals themselves.

17.3. Coadjoint orbits as symplectic manifolds. We recall Kirillov's construction of a symplectic structure on coadjoint orbits. We use Ad^* (respectively, ad^*) to denote the coadjoint action of G (respectively, $\mathfrak{g}$) on $\mathfrak{g}^*$.

For $\lambda \in \mathfrak{g}^*$ we let G_λ denote the stabilizer of λ in G (for the coadjoint action); the Lie algebra of G_λ is $\mathfrak{g}_\lambda := \{X \in \mathfrak{g} : \mathrm{ad}^*(X)\lambda = 0\}$. The tangent space at λ to the coadjoint orbit $\mathcal{O}_\lambda$ of λ is $\mathfrak{g}/\mathfrak{g}_\lambda$ (since λ allows us to identify the orbit with G/G_λ).

From λ we get an alternating form ω_λ on $\mathfrak{g}$, defined by

$$\omega_\lambda(X, Y) := \lambda\big([X, Y]\big) = -\langle \mathrm{ad}^*(X)\lambda, Y\rangle. \tag{17.3.1}$$

It is clear from the equality $\omega_\lambda(X, Y) = -\langle \mathrm{ad}^*(X)\lambda, Y\rangle$ that the kernel of the alternating form ω_λ is $\mathfrak{g}_\lambda$; therefore ω_λ can also be viewed as a non-degenerate alternating bilinear form on $\mathfrak{g}/\mathfrak{g}_\lambda$, or, in other words, on the tangent space to $\mathcal{O}_\lambda$ at λ.

Now fix a coadjoint orbit $\mathcal{O}$. Letting λ vary through the orbit $\mathcal{O}$, the construction above yields a G-invariant 2-form ω on $\mathcal{O}$ whose value at λ is ω_λ.

Thus $\mathcal{O}$ is a symplectic G-manifold, and in particular its dimension is even, say $\dim \mathcal{O} = 2d$. The d-fold wedge product $\eta := \omega \wedge \cdots \wedge \omega$ is a G-invariant volume form on $\mathcal{O}$. The associated measure $|\eta|$ is G-invariant, and consequently G_λ is unimodular.

Now suppose that multiplication by $\beta \in F^\times$ preserves the orbit $\mathcal{O}$. (This can happen only if the coadjoint orbit is nilpotent, in the sense that when we identify $\mathfrak{g}^*$ with $\mathfrak{g}$ in the usual way, the corresponding orbit in $\mathfrak{g}$ is nilpotent.) Write $m_\beta : \mathcal{O} \to \mathcal{O}$ for the map $\lambda \mapsto \beta\lambda$.

We need to compute the differential of m_β at any point $\lambda \in \mathcal{O}$. This differential is a linear isomorphism from the tangent space $\mathfrak{g}/\mathfrak{g}_\lambda$ at λ to the tangent space $\mathfrak{g}/\mathfrak{g}_{\beta\lambda}$ at $\beta\lambda$. Since m_β is a G-map, we see that its differential $\mathfrak{g}/\mathfrak{g}_\lambda \to \mathfrak{g}/\mathfrak{g}_{\beta\lambda}$ is induced by the identity map on $\mathfrak{g}$ (note that $\mathfrak{g}_{\beta\lambda} = \mathfrak{g}_\lambda$).

Now we can compute the pull-back $m_\beta^*(\omega)$. In fact the computation we just made of the differential of m_β, together with the fact (obvious from the definition of ω) that $\omega_{\beta\lambda} = \beta\omega_\lambda$, shows that $m_\beta^*(\omega) = \beta\omega$. It follows that $m_\beta^*(\eta) = \beta^d\eta$.

Once again identifying $\mathfrak{g}^*$ with $\mathfrak{g}$, we reach the conclusion that for any adjoint orbit $\mathcal{O}$ there exists a G-invariant volume form η on $\mathcal{O}$, and that if multiplication by the scalar β preserves $\mathcal{O}$, yielding a multiplication map $m_\beta : \mathcal{O} \to \mathcal{O}$, then

(17.3.2) $$m_\beta^*(\eta) = \beta^{\dim(\mathcal{O})/2}\eta.$$

17.4. Scaling of functions on $\mathfrak{g}$. For $\beta \in F^\times$ and $f \in C_c^\infty(\mathfrak{g})$ we write f_β for the function on $\mathfrak{g}$ defined by

(17.4.1) $$f_\beta(X) := f(\beta X).$$

17.5. Homogeneity of nilpotent orbital integrals. Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Then for any $\alpha \in F^\times$ multiplication by α^2 preserves $\mathcal{O}$. Indeed, by the Jacobson-Morozov theorem, it is enough to prove this for the group $G = SL_2$, for which the statement is an easy exercise. Let $\mu_\mathcal{O}$ denote the corresponding nilpotent orbital integral, an invariant distribution on $\mathfrak{g}$ whose homogeneity we will now establish.

LEMMA 17.2. *Let $f \in C_c^\infty(\mathfrak{g})$ and let $\alpha \in F^\times$. Then*

(17.5.1) $$\mu_\mathcal{O}(f_{\alpha^2}) = |\alpha|^{-\dim\mathcal{O}}\mu_\mathcal{O}(f).$$

PROOF. This follows from (17.3.2). □

17.6. Behavior of regular semisimple orbital integrals under scaling. It is clear from (17.1.1) that

(17.6.1) $$O_X(f_\beta) = O_{\beta X}(f)$$

for all $X \in \mathfrak{g}_{\mathrm{rs}}$ and all $\beta \in F^\times$.

17.7. Partial homogeneity of our provisional Shalika germs Γ_i. Let $\alpha \in F^\times$. Let $\mathcal{O}_i$ be one of our nilpotent orbits, let μ_i be the corresponding nilpotent orbital integral, and let Γ_i be the corresponding Shalika germ. Put $d_i := \dim\mathcal{O}_i$. We claim that

(17.7.1) $$\Gamma_i(X) = |\alpha|^{d_i}\Gamma_i(\alpha^2 X),$$

where the equality means equality of germs about 0 of functions on $\mathfrak{g}_{\mathrm{rs}}$.

Indeed, as in the proof of existence of Shalika germ expansions, pick a function $f_i \in C_c^\infty(\mathfrak{g})$ such that

(17.7.2) $$\mu_j(f_i) = \delta_{ij}.$$

Then $\Gamma_i(X)$ is the germ about 0 of the function

(17.7.3) $$X \mapsto O_X(f_i)$$

on $\mathfrak{g}_{\mathrm{rs}}$. In fact during the remainder of our discussion of provisional germs, we will use always use (17.7.3) as our choice for a specific function Γ_i having the right germ.

In view of the homogeneity of nilpotent orbital integrals established above, the function $|\alpha|^{d_i} \cdot (f_i)_{\alpha^2}$ also satisfies (17.7.2), so that $\Gamma_i(X)$ is also the germ about 0 of the function

$$X \mapsto O_X(|\alpha|^{d_i} \cdot (f_i)_{\alpha^2}) = |\alpha|^{d_i} \cdot O_{\alpha^2 X}(f_i) \tag{17.7.4}$$

on $\mathfrak{g}_{\mathrm{rs}}$. Comparing (17.7.3), (17.7.4), we see that $\Gamma_i(X)$ and $|\alpha|^{d_i}\Gamma_i(\alpha^2 X)$ have the same germ, as desired.

17.8. Canonical Shalika germs. Let Γ_i be one of our germs. Following Harish-Chandra [**HC78, HC99**], we are going to replace Γ_i by another function Γ_i^{new} on $\mathfrak{g}_{\mathrm{rs}}$ that has the same germ about $0 \in \mathfrak{g}$ and is at the same time homogeneous. Along the way we prove a couple of simple properties of Γ_i^{new}.

LEMMA 17.3. *There is a unique function Γ_i^{new} on $\mathfrak{g}_{\mathrm{rs}}$ which has the same germ about $0 \in \mathfrak{g}$ as Γ_i and which satisfies* (17.7.1) *for all $\alpha \in F^\times$ and all $X \in \mathfrak{g}_{\mathrm{rs}}$. Moreover Γ_i^{new} is real-valued, translation invariant under the center of $\mathfrak{g}$, and invariant by conjugation under G.*

PROOF. Choose a lattice $L \subset \mathfrak{g}$ such that (17.7.1) holds for $\alpha = \pi$ (our chosen uniformizing element in F) and all $X \in L \cap \mathfrak{g}_{\mathrm{rs}}$. Iterating (17.7.1), we see that

$$\Gamma_i(X) = |\pi^k|^{d_i}\Gamma_i(\pi^{2k}X) \tag{17.8.1}$$

for all $k \geq 0$ and all $X \in L \cap \mathfrak{g}_{\mathrm{rs}}$.

For $X \in \mathfrak{g}_{\mathrm{rs}}$ we define $\Gamma_i^{\mathrm{new}}(X)$ by choosing $k \geq 0$ such that $\pi^{2k}X \in L$ and then putting

$$\Gamma_i^{\mathrm{new}}(X) := |\pi^k|^{d_i}\Gamma_i(\pi^{2k}X); \tag{17.8.2}$$

by (17.8.1) Γ_i^{new} is well-defined. This definition is of course forced on us, so Γ_i^{new} is clearly unique.

Next we show that Γ_i^{new} does satisfy (17.7.1). Let $\alpha \in F^\times$. Let L' be a lattice in $\mathfrak{g}$ such that

$$\Gamma_i(X) = |\alpha|^{d_i}\Gamma_i(\alpha^2 X) \tag{17.8.3}$$

for all $X \in L' \cap \mathfrak{g}_{\mathrm{rs}}$. For a given $X \in \mathfrak{g}_{\mathrm{rs}}$ we may pick $k \geq 0$ such that $\pi^{2k}X \in L \cap L'$ and $\pi^{2k}\alpha^2 X \in L$, and we then have

$$\Gamma_i^{\mathrm{new}}(X) = |\pi^k|^{d_i}\Gamma_i(\pi^{2k}X) = |\pi^k|^{d_i}|\alpha|^{d_i}\Gamma_i(\pi^{2k}\alpha^2 X) = |\alpha|^{d_i}\Gamma_i^{\mathrm{new}}(\alpha^2 X), \tag{17.8.4}$$

as desired.

Looking back at how the functions f_i (satisfying $\mu_j(f_i) = \delta_{ij}$) were shown (in 6.4) to exist, we see that they can be chosen to be real-valued functions. Then Γ_i is real-valued and (17.8.2) shows that the same is true of Γ_i^{new}.

The function f_i is translation invariant under some lattice in $\mathfrak{g}$ and hence under some lattice in the center of $\mathfrak{g}$. It follows easily that the provisional germ $\Gamma_i(X) = O_X(f_i)$ is translation invariant under this lattice in the center, and hence (from (17.8.2)) that Γ_i^{new} is translation invariant under the center of $\mathfrak{g}$.

The provisional germ $\Gamma_i(X) = O_X(f_i)$ is clearly invariant under conjugation, from which it follows that the same is true of Γ_i^{new}. $\square$

From now on we replace the germs Γ_i by the functions Γ_i^{new}, but we drop the superscript "new."

We also need a slight strengthening of the fact that Γ_i is translation invariant under the center $\mathfrak{z}$ of $\mathfrak{g}$. Let G' be the derived group of the algebraic group G, and let Z denote the center of G. Then $G(\bar{F}) = G'(\bar{F})Z(\bar{F})$, but for F-points we have only that $G'Z$ is a normal subgroup of finite index in G. We denote by D the finite group $G/G'Z$. Each G-orbit $\mathcal{O}$ in $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{z}$ decomposes as a finite union of G'-orbits $\mathcal{O}'$, permuted transitively by D. We normalize the invariant measures on the orbits in such a way that

$$\int_{\mathcal{O}} = \sum_{x \in D} \int_{x^{-1}\mathcal{O}'x}. \tag{17.8.5}$$

For a nilpotent G-orbit $\mathcal{O}$ (respectively, nilpotent G'-orbit $\mathcal{O}'$) we denote by $\Gamma^G_{\mathcal{O}}$ (respectively, $\Gamma^{G'}_{\mathcal{O}'}$) the corresponding Shalika germ on $\mathfrak{g}_{\mathrm{rs}}$ (respectively, $\mathfrak{g}'_{\mathrm{rs}}$).

LEMMA 17.4. *Let $X \in \mathfrak{g}_{\mathrm{rs}}$ and decompose X as $X' + Z$ with $X' \in \mathfrak{g}'_{\mathrm{rs}}$ and $Z \in \mathfrak{z}$. Then*

$$\Gamma^G_{\mathcal{O}}(X) = \sum_{\mathcal{O}' \subset \mathcal{O}} \Gamma^{G'}_{\mathcal{O}'}(X'). \tag{17.8.6}$$

PROOF. Let $\mathcal{O}'_1, \dots, \mathcal{O}'_s$ be the nilpotent G'-orbits. For each nilpotent G'-orbit $\mathcal{O}'_i$ choose $f'_{\mathcal{O}'_i} \in C^\infty_c(\mathfrak{g}')$ such that

$$\int_{\mathcal{O}'_i} f'_{\mathcal{O}'_j} = \delta_{ij}. \tag{17.8.7}$$

Thus the regular semisimple orbital integrals of $f'_{\mathcal{O}'}$ give the provisional Shalika germ $\Gamma^{G'}_{\mathcal{O}'}$.

For a nilpotent G-orbit $\mathcal{O}$ put

$$f'_{\mathcal{O}} := |D|^{-1} \sum_{\mathcal{O}' \subset \mathcal{O}} f'_{\mathcal{O}'}. \tag{17.8.8}$$

We extend $f'_{\mathcal{O}}$ to a function $f_{\mathcal{O}} \in C^\infty_c(\mathfrak{g})$ by choosing a lattice L in $\mathfrak{z}$ and putting

$$f(X' + Z) = f'(X')1_L(Z) \tag{17.8.9}$$

for any $X' \in \mathfrak{g}'$ and any $Z \in \mathfrak{z}$. Here 1_L denotes the characteristic function of L. It is easy to see that

$$\int_{\mathcal{O}_i} f_{\mathcal{O}_j} = \delta_{ij} \tag{17.8.10}$$

for every pair of nilpotent G-orbits $\mathcal{O}_i$, $\mathcal{O}_j$, so that the regular semisimple orbital integrals of $f_{\mathcal{O}}$ give the provisional Shalika germ $\Gamma^G_{\mathcal{O}}$, and another easy calculation then shows that the provisional Shalika germs for G and G' are related as in the statement of the lemma. By homogeneity the same is true for the Shalika germs themselves. $\square$

17.9. Germ expansions about arbitrary central elements in $\mathfrak{g}$. We have been studying germ expansions about $0 \in \mathfrak{g}$. These involve orbital integrals for the nilpotent orbits $\mathcal{O}_i$. Now we consider germ expansions about an arbitrary element Z in the center of $\mathfrak{g}$. These will involve orbital integrals $\mu_{Z+\mathcal{O}_i}$ for the orbits $Z+\mathcal{O}_i$, but will involve exactly the same germs Γ_i as before.

THEOREM 17.5. *Let Z be an element in the center of $\mathfrak{g}$. For every $f \in C^\infty_c(\mathfrak{g})$ there exists an open neighborhood U_f of Z in $\mathfrak{g}$ such that*

$$O_X(f) = \sum_{i=1}^{r} \mu_{Z+\mathcal{O}_i}(f) \cdot \Gamma_i(X) \tag{17.9.1}$$

for all $X \in U_f \cap \mathfrak{g}_{\mathrm{rs}}$.

Proof. Apply Theorem 17.1 to the translate of f by Z and use that our canonical Shalika germs Γ_i are translation invariant under the center. $\square$

17.10. Germ expansions about arbitrary semisimple elements in $\mathfrak{g}$. We are going to use the descent theory developed in section 16 in order to obtain germ expansions about an arbitrary semisimple element $S \in \mathfrak{g}$. We fix such an element S and let $H := G_S$ denote the centralizer of S, a connected reductive subgroup of G. Then S is contained in the open subset $\mathfrak{h}'$ of $\mathfrak{h}$. We write $\mathfrak{h}'_{\mathrm{rs}}$ for the intersection $\mathfrak{h}' \cap \mathfrak{h}_{\mathrm{rs}}$. Then $\mathfrak{h}'_{\mathrm{rs}} \subset \mathfrak{g}_{\mathrm{rs}}$, since a semisimple element in $\mathfrak{h}'$ is regular in $\mathfrak{h}$ if and only if it is regular in $\mathfrak{g}$.

We will also be concerned with all G-orbits of elements X such that $X_s = S$ (as usual, X_s denotes the semisimple part of X). Such orbits are in one-to-one correspondence with H-orbits of nilpotent elements $Y \in \mathfrak{h}$ (with Y corresponding to $X = S+Y$). Let $Y_1, \dots, Y_s$ be a set of representatives for the nilpotent H-orbits in $\mathfrak{h}$. Let μ_{S+Y_i} denote the orbital integral on $\mathfrak{g}$ obtained by integration over the G-orbit of $S + Y_i$. Let Γ_i^H be the canonical Shalika germ on $\mathfrak{h}_{\mathrm{rs}}$ corresponding to the (nilpotent) H-orbit of Y_i.

Theorem 17.6. *Let S, H be as above. For every $f \in C_c^\infty(\mathfrak{g})$ there exists an open neighborhood U_f of S in $\mathfrak{h}'$ such that*

$$O_X(f) = \sum_{i=1}^{s} \mu_{S+Y_i}(f) \cdot \Gamma_i^H(X) \tag{17.10.1}$$

for all $X \in U_f \cap \mathfrak{g}_{\mathrm{rs}} = U_f \cap \mathfrak{h}_{\mathrm{rs}}$.

Proof. Note that $\pi_H(S)$ lies in $\mathbb{A}'_H(F)$. Let ω_H be a compact open neighborhood of $\pi_H(S)$ in $\mathbb{A}'_H(F)$. By Lemma 16.1 there exists $\phi \in C_c^\infty(\mathfrak{h})$ such that

$$\int_{G_X \backslash H} \phi(h^{-1}Xh)\, d\dot{h} = \int_{G_X \backslash G} f(g^{-1}Xg)\, d\dot{g}. \tag{17.10.2}$$

for any $X \in \pi_H^{-1}(\omega_H)$. Note that $\pi_H^{-1}(\omega_H)$ contains all the elements $S + Y_i$ and is an open neighborhood of S in $\mathfrak{h}'$.

Apply the Shalika germ expansion (Theorem 17.5) to the central element $S \in \mathfrak{h}$ and the function ϕ. Using (17.10.2) to rewrite this expansion in terms of orbital integrals on $\mathfrak{g}$, we obtain the desired result. $\square$

17.11. Normalized orbital integrals and Shalika germs. It is sometimes more convenient (see 13.12, for example) to use the normalized orbital integrals I_X ($X \in \mathfrak{g}_{\mathrm{rs}}$) defined by $I_X = |D(X)|^{1/2} O_X$. When we use I_X instead of O_X, we need to use the normalized Shalika germs

$$\bar{\Gamma}_i(X) := |D(X)|^{1/2}\Gamma_i(X)$$

instead of the usual Shalika germs.

Clearly Theorem 17.1 remains valid when O_X, Γ_i are replaced by I_X, $\bar{\Gamma}_i$ respectively. Now consider the germ expansion about an arbitrary semisimple element $S \in \mathfrak{g}$. As usual put $H := G_S$. The function $X \mapsto \det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{h})$ is non-zero on $\mathfrak{h}'$ and its p-adic absolute value is locally constant on $\mathfrak{h}'$. Moreover

$$D^G(X) = D^H(X) \cdot \det(\mathrm{ad}(X); \mathfrak{g}/\mathfrak{h}).$$

Therefore there is a neighborhood of S in $\mathfrak{h}'$ on which

$$|D^G(X)|^{1/2} = |D^H(X)|^{1/2} \cdot |\det(\mathrm{ad}(S); \mathfrak{g}/\mathfrak{h})|^{1/2}.$$

It then follows from Theorem 17.6 that

$$I_X(f) = |\det(\mathrm{ad}(S); \mathfrak{g}/\mathfrak{h})|^{1/2} \sum_{i=1}^{s} \mu_{S+Y_i}(f) \cdot \bar{\Gamma}_i^H(X) \tag{17.11.1}$$

for all $X \in \mathfrak{g}_{\mathrm{rs}}$ in some sufficiently small neighborhood of S in $\mathfrak{h}'$.

Since $D^G(X)$ is a homogeneous polynomial of degree $\dim(G) - \mathrm{rank}(G)$, where $\mathrm{rank}(G)$ denotes the dimension of any maximal torus in G, we see immediately that the homogeneity property (17.7.1) of the Shalika germs Γ_i implies the following homogeneity property for the normalized Shalika germs $\bar{\Gamma}_i$:

$$\bar{\Gamma}_i(\alpha^2 X) = |\alpha|^{\dim(G_{X_i}) - \mathrm{rank}(G)} \cdot \bar{\Gamma}_i(X) \tag{17.11.2}$$

for all $\alpha \in F^\times$ and all $X \in \mathfrak{g}_{\mathrm{rs}}$. Here we have chosen $X_i \in \mathcal{O}_i$ and introduced its centralizer G_{X_i}. Note that the exponent $\dim(G_{X_i}) - \mathrm{rank}(G)$ appearing in (17.11.2) is always non-negative. This simple observation will play an important role in the proof (to be given shortly) of the boundedness of normalized Shalika germs.

17.12. $\bar{\Gamma}_i$ is a linear combination of functions $\bar{\Gamma}_j^H$ in a neighborhood of S. Again let S be a semisimple element in $\mathfrak{g}$, let H be its centralizer in G, and let T be a maximal torus in H. Consider one of the normalized Shalika germs $\bar{\Gamma}_i$ for G. We are interested in the behavior of $\bar{\Gamma}_i$ on a small neighborhood of S in $\mathfrak{t}$.

Lemma 17.7. *There exists a neighborhood V of S in $\mathfrak{t}$ such that the restriction of $\bar{\Gamma}_i$ to $V \cap \mathfrak{t}_{\mathrm{reg}}$ is a linear combination of restrictions of normalized Shalika germs for H.*

Proof. Pick $f_i \in C_c^\infty(\mathfrak{g})$ such that $\mu_j(f_i) = \delta_{ij}$. By the Shalika germ expansion for f_i there is a lattice L in $\mathfrak{t}$ small enough that

$$\bar{\Gamma}_i(X) = I_X(f_i) \tag{17.12.1}$$

for all regular X in L. Let $\alpha \in F^\times$. By homogeneity of Shalika germs (for both G and H) the lemma holds for S if and only if it holds for αS. Therefore we may assume (by scaling S suitably) that $S \in L$. For some neighborhood V of S in L we also have (by (17.11.1))

$$I_X(f_i) = |\det(\mathrm{ad}(S); \mathfrak{g}/\mathfrak{h})|^{1/2} \sum_{i=1}^{s} \mu_{S+Y_i}(f_i) \cdot \bar{\Gamma}_i^H(X) \tag{17.12.2}$$

for all regular X in V. Combining (17.12.1) and (17.12.2), we get the lemma. □

Corollary 17.8. *Let T be any maximal torus in G. Each normalized Shalika germ $\bar{\Gamma}_i$ is locally constant on $\mathfrak{t}_{\mathrm{reg}}$.*

Proof. Apply the lemma above to any regular element S in $\mathfrak{t}$. Then $H = T$. Furthermore, the only nilpotent element in $\mathfrak{h}$ is 0, and its normalized Shalika germ is constant. Therefore $\bar{\Gamma}_i$ is constant in some sufficiently small neighborhood of S, as was to be shown. □

17.13. Locally bounded functions. We are going to show that the normalized Shalika germs $\bar{\Gamma}_i$ are locally bounded functions on $\mathfrak{t}$. First let's recall what this means. Let f be a complex-valued function on a topological space X. We say that f is *locally bounded* on X if every point $x \in X$ has a neighborhood U_x such that f is bounded on U_x. When X is a locally compact Hausdorff space, f is locally bounded if and only f is bounded on every compact subset of X (easy exercise!).

17.14. Local boundedness of normalized Shalika germs. Let $\bar{\Gamma}_i$ be one of our normalized Shalika germs on $\mathfrak{g}$. Let T be a maximal torus in G. We have just seen that $\bar{\Gamma}_i$ is locally constant on $\mathfrak{t}_{\mathrm{reg}}$; therefore it is locally bounded on $\mathfrak{t}_{\mathrm{reg}}$ for trivial reasons. However we now extend $\bar{\Gamma}_i$ by 0 to a function on $\mathfrak{t}$. We are going to show that $\bar{\Gamma}_i$ is locally bounded as a function on $\mathfrak{t}$ (a result of Harish-Chandra). What this means concretely is that for all $S \in \mathfrak{t}$ there is a neighborhood U of S in $\mathfrak{t}$ such that $\bar{\Gamma}_i$ is bounded on $U \cap \mathfrak{t}_{\mathrm{reg}}$.

THEOREM 17.9. [**HC78, HC99**] *Every normalized Shalika germ $\bar{\Gamma}_i$ is a locally bounded function on $\mathfrak{t}$.*

PROOF. We use induction on the dimension of G, the case $\dim(G) = 0$ being trivial. By Lemma 17.4 we may assume that the center of $\mathfrak{g}$ is 0.

Let S be any non-zero element in $\mathfrak{t}$. Since S is not central, the induction hypothesis applies to the centralizer H of S in G. Now use Lemma 17.7 to conclude that $\bar{\Gamma}_i$ is bounded on some neighborhood of S.

We now know that $\bar{\Gamma}_i$ is locally bounded on $\mathfrak{t} \setminus \{0\}$. Choose any lattice L in $\mathfrak{t}$. Then $L \setminus \pi^2 L$ is a compact subset of $\mathfrak{t} \setminus \{0\}$, and therefore $\bar{\Gamma}_i$ is bounded on $L \setminus \pi^2 L$. By homogeneity (17.11.2) it follows that $\bar{\Gamma}_i$ is bounded on L (by the same bound as on $L \setminus \pi^2 L$).

We have shown that $\bar{\Gamma}_i$ is locally bounded everywhere on $\mathfrak{t}$, and the proof is complete. $\square$

As a consequence of the local boundedness of normalized Shalika germs, we get another result of Harish-Chandra, needed for the local trace formula.

THEOREM 17.10. *Let $f \in C_c^\infty(\mathfrak{g})$ and let T be a maximal torus in G. Then the function $X \mapsto I_X(f)$ on $\mathfrak{t}_{\mathrm{reg}}$ is bounded and locally constant on $\mathfrak{t}_{\mathrm{reg}}$. When extended by 0 to all of $\mathfrak{t}$, the function $X \mapsto I_X(f)$ is compactly supported as well; in other words, there is a compact subset C of $\mathfrak{t}$ such that $I_X(f)$ vanishes for all regular elements of $\mathfrak{t}$ not lying in C. It should be noted that $X \mapsto I_X(f)$ is usually not compactly supported as a function on $\mathfrak{t}_{\mathrm{reg}}$.*

PROOF. Local constancy in a neighborhood of $S \in \mathfrak{t}_{\mathrm{reg}}$ follows from the Shalika germ expansion about S (for which $H = T$, so that there is just one germ, and it is constant).

Boundedness in a sufficiently small neighborhood of any $S \in \mathfrak{t}$ follows from the Shalika germ expansion (17.11.1) about S together with the local boundedness of normalized Shalika germs on $H = G_S$. Thus $X \mapsto I_X(f)$ is locally bounded on $\mathfrak{t}$. Boundedness will then follow once we have proved that $X \mapsto I_X(f)$ is compactly supported on $\mathfrak{t}$.

It now remains only to show that $X \mapsto I_X(f)$ is in fact compactly supported on $\mathfrak{t}$. The support of f is a compact subset of $\mathfrak{g}$, so its image ω under $\pi_G : \mathfrak{g} \to \mathbb{A}_G(F) = (\mathfrak{t}/W)(F)$ is compact. But since $\mathfrak{t} \to \mathfrak{t}/W$ is a proper morphism of algebraic varieties, the map $\mathfrak{t} \to (\mathfrak{t}/W)(F)$ is a proper map between locally compact

Hausdorff spaces. Therefore the inverse image C of ω under $\mathfrak{t} \to (\mathfrak{t}/W)(F)$ is a compact subset of $\mathfrak{t}$. Clearly $I_X(f)$ vanishes off C. $\square$

For weighted orbital integrals we have the following partial generalization of our last theorem.

THEOREM 17.11. *Let $f \in C_c^\infty(\mathfrak{g})$, let T be a maximal torus in G, and let v be a locally constant function on G that is invariant under left translation by T. Then the function*

$$X \mapsto \int_{T\backslash G} f(g^{-1}Xg)v(g)\,d\dot{g}$$

on $\mathfrak{t}_{\mathrm{reg}}$ is locally constant on $\mathfrak{t}_{\mathrm{reg}}$ and, when extended by 0 to $\mathfrak{t}$, is compactly supported on $\mathfrak{t}$.

PROOF. Compact support is established just as in the previous result. Local constancy needs to be proved more directly, as we have not developed a theory of Shalika germs for weighted orbital integrals.

Let $Y \in \mathfrak{t}$. We are going to find a neighborhood U of Y in $\mathfrak{t}_{\mathrm{reg}}$ on which our function is constant. Consider the function ϕ on $\mathfrak{t} \times (T\backslash G)$ defined by $\phi(X, g) := f(g^{-1}Xg)$. Clearly ϕ is locally constant, but it is usually not compactly supported. However, now choosing a compact open neighborhood ω_T of Y in $\mathfrak{t}_{\mathrm{reg}}$, we see from Lemma 15.1 (Harish-Chandra's compactness lemma, applied to $H = T$ and $\omega_{\mathfrak{g}} = \mathrm{Supp}(f)$) that the restriction of ϕ to $\omega_T \times (T\backslash G)$ is compactly supported. By Lemma 2.1 there exists an open neighborhood U of Y in ω_T such that $\phi(X, g) = \phi(Y, g)$ for all $X \in U$, $g \in T\backslash G$. It follows that

$$\int_{T\backslash G} f(g^{-1}Xg)v(g)\,d\dot{g} = \int_{T\backslash G} f(g^{-1}Yg)v(g)\,d\dot{g}$$

for all $X \in U$. $\square$

18. Norms on affine varieties over local fields

The spaces we are working with are usually non-compact, and non-compactly supported functions on them can certainly be unbounded. For various purposes we need a natural way to measure growth rates of such functions. For this we must be able to measure the size of points in the spaces. For instance on the real line one usually uses the absolute value of a real number to measure its size, and one says that a function $f(x)$ on the real line has polynomial growth if there exist $c, R > 0$ such that $|f(x)| \le c|x|^R$ for all $x \in \mathbb{R}$. We want to be able to do something similar on the spaces we are using. For this purpose we now develop a theory of norms on $X(F)$ for any variety (usually affine) over a field F equipped with an absolute value.

Let F be a field equipped with a non-trivial absolute value $|\cdot|$. Thus $|\cdot|$ is a non-negative real-valued function on F such that

(1) $|x| = 0$ if and only if $x = 0$.
(2) $|x + y| \le |x| + |y|$ for all $x, y \in F$.
(3) $|xy| = |x||y|$ for all $x, y \in F$.
(4) There exists $x \in F^\times$ such that $|x| \ne 1$.

As usual $(x, y) \mapsto |x - y|$ defines a metric on F with respect to which F may or may not be complete. Starting with subsection 18.7 we will assume that F is complete.

18.1. Abstract norms. By an *abstract norm* on a set X we mean a real-valued function $\|\cdot\|$ on X such that $\|x\| \geq 1$ for all $x \in X$. Given two abstract norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on X, we say $\|\cdot\|_2$ *dominates* $\|\cdot\|_1$ and write $\|\cdot\|_1 \prec \|\cdot\|_2$ if there exist real numbers $c > 0$ and $R > 0$ such that

$$\|x\|_1 \leq c\|x\|_2^R$$

for all $x \in X$. The relation of dominance is transitive. We say that two norms are *equivalent* if they dominate each other.

For any abstract norm $\|\cdot\|$ we have (by virtue of our requirement that $\|x\| \geq 1$) the inequality

$$c_1\|x\|^{R_1} \leq c_2\|x\|^{R_2}$$

whenever $0 < c_1 \leq c_2$ and $0 < R_1 \leq R_2$. This allows us to increase the constants c, R occurring in the dominance relation whenever it is convenient to do so.

Given two abstract norms $\|x\|_1$ and $\|x\|_2$ on X, the three abstract norms

$$\sup\{\|x\|_1, \|x\|_2\},\ \|x\|_1 + \|x\|_2,\ \|x\|_1 \cdot \|x\|_2$$

on X are equivalent, and their common equivalence class depends only on the equivalence classes of $\|x\|_1$ and $\|x\|_2$.

18.2. Norms on affine varieties over F. Let X be an affine scheme of finite type over F and write $\mathcal{O}_X$ for its ring of regular functions, a finitely generated F-algebra. For any finite set $f_1, \dots, f_m$ of generators for the F-algebra $\mathcal{O}_X$, we define an abstract norm $\|\cdot\|$ on $X(F)$ by

$$\|x\| := \sup\{1, |f_1(x)|, \dots, |f_m(x)|\} \tag{18.2.1}$$

for $x \in X(F)$.

Now let $f \in \mathcal{O}_X$. It is easy to see that there exist $c, R \geq 0$ such that

$$|f(x)| \leq c\|x\|^R \tag{18.2.2}$$

for all $x \in X(F)$. [Indeed, writing f as a polynomial in $f_1, \dots, f_m$, we may take for c the sum of the absolute values of the coefficients in the polynomial, and for R the degree of the polynomial.] Since we are free to increase c, R, we may choose them so that $c, R > 0$ (or even ≥ 1) whenever it is convenient to do so.

Using (18.2.2) for all the members of some other generating set for $\mathcal{O}_X$, we see that the equivalence class of the abstract norm (18.2.1) is independent of the choice of generating set, and by a *norm* on $X(F)$ we mean any abstract norm lying in this equivalence class.

EXAMPLE 18.1. On F^n, the set of F-points of $\mathbb{A}^n$,

$$\|(x_1, \dots, x_n)\| := \sup\{1, |x_1|, \dots, |x_n|\}$$

is a norm. The restriction of this norm to the F-points of any closed subscheme of $\mathbb{A}^n$ is a norm on that set.

EXAMPLE 18.2. On $(F^\times)^n$, the set of F-points of $(\mathbb{G}_m)^n$,

$$\|(x_1, \dots, x_n)\| := \sup\{|x_1|, |x_1|^{-1} \dots, |x_n|, |x_n|^{-1}\}$$

is a norm.

18.3. Bounded subsets. Let X be an affine scheme of finite type over F and let $\|\cdot\|_X$ be a norm on $X(F)$. We say that a subset B of $X(F)$ is *bounded* if the norm function $\|\cdot\|_X$ has an upper bound on B. This notion of boundedness is clearly independent of the choice of norm, so it makes sense to talk about bounded subsets of $X(F)$ without specifying any particular norm.

18.4. Properties of norms. We need to establish various simple results about norms. It is especially important to compare norms on varieties when a morphism between them is given.

PROPOSITION 18.1. *Let X and Y be affine schemes of finite type over F and let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be norms on $X(F)$ and $Y(F)$ respectively.*

(1) *Let $\phi : Y \to X$ be a morphism and denote by $\phi^*\|\cdot\|_X$ the abstract norm on $Y(F)$ obtained by composing $\|\cdot\|_X$ with $\phi : Y(F) \to X(F)$. Then $\|\cdot\|_Y$ dominates $\phi^*\|\cdot\|_X$. If ϕ is finite, then $\|\cdot\|_Y$ is equivalent to $\phi^*\|\cdot\|_X$.*
(2) *Suppose Y is a closed subscheme of X. Then the restriction of $\|\cdot\|_X$ to $Y(F)$ is equivalent to $\|\cdot\|_Y$.*
(3) *If F is locally compact, then a subset of $X(F)$ has compact closure if and only if it is bounded.*
(4) *All three of $\sup\{\|x\|_X, \|y\|_Y\}$, $\|x\|_X + \|y\|_Y$, and $\|x\|_X \cdot \|y\|_Y$ are valid norms on $(X \times Y)(F) = X(F) \times Y(F)$.*
(5) *Let $U := X_f$ denote the principal open subset of X determined by a regular function f on X, so that $U(F) = \{x \in X(F) : f(x) \neq 0\}$. Then $\|u\|_U := \sup\{\|u\|_X, |f(u)|^{-1}\}$ is a norm on $U(F)$.*
(6) *Suppose we are given a finite cover of X by affine open subsets $U_1, \dots, U_r$ as well as a norm $\|\cdot\|_i$ on $U_i(F)$ for each $i = 1, \dots, r$. For $x \in X(F)$ define $\|x\|$ to be the infinum of the numbers $\|x\|_i$, where i ranges over the set of indices for which $x \in U_i(F)$. Then $\|\cdot\|$ is a norm on $X(F)$.*
(7) *Let G be a group scheme of finite type over F, and suppose we are given an action of G on X. Let B be a bounded subset of $G(F)$. Then there exist $c, R > 0$ such that $\|bx\|_X \le c\|x\|_X^R$ for all $b \in B, x \in X(F)$.*

PROOF. We begin by proving the first part of the proposition. Using (18.2.2) for the pull-backs by ϕ of the members of a generating set for the F-algebra $\mathcal{O}_X$, we see that $\|\cdot\|_Y$ dominates $\phi^*\|\cdot\|_X$.

Now suppose that ϕ is finite and let $g \in \mathcal{O}_Y$. Then there exist $n \ge 1$ and $f_1, \dots, f_n \in \mathcal{O}_X$ such that

$$g^n = f_1 g^{n-1} + \cdots + f_{n-1} g + f_n.$$

We claim that for all $y \in Y(F)$ we have the inequality

$$|g(y)| \le \sup\{1, |f_1(\phi(y))| + \cdots + |f_n(\phi(y))|\}. \tag{18.4.1}$$

Indeed, this is trivially true if $|g(y)| \le 1$, and otherwise we have

$$g(y) = f_1(\phi(y)) + f_2(\phi(y))g(y)^{-1} + \cdots + f_n(\phi(y))g(y)^{-(n-1)}$$

and hence

$$|g(y)| \le |f_1(\phi(y))| + |f_2(\phi(y))| + \cdots + |f_n(\phi(y))|.$$

Using (18.2.2) for $f_1, \dots, f_n$, we see from (18.4.1) that there exist $c, R > 0$ such that

$$|g(y)| \le c\|\phi(y)\|_X^R \quad \forall\, y \in Y(F). \tag{18.4.2}$$

Now choose a finite generating set $g_1, \dots, g_m$ for the F-algebra $\mathcal{O}_Y$. As our norm $\|\cdot\|_Y$ we are free to take the one obtained from this generating set (see (18.2.1)). Choosing $c, R \geq 1$ large enough that (18.4.2) holds for all the functions $g_1, \dots, g_m$, we see that $\|\cdot\|_Y$ is dominated by $\phi^*\|\cdot\|_X$. Since we already proved dominance in the other direction, we conclude that $\|\cdot\|_Y$ and $\phi^*\|\cdot\|_X$ are equivalent, as desired.

The second part of the proposition follows from the first, because closed immersions are finite morphisms. Of course a more direct proof can also be given.

As for the third part, we use the second part to reduce to the case of affine space $\mathbb{A}^n$, for which the result is obvious.

For the fourth part we note that it is obvious from the definition of norm that

$$\sup\{\|\cdot\|_X, \|\cdot\|_Y\}$$

is a valid norm on $X(F) \times Y(F)$. It then follows from the discussion at the very end of subsection 18.1 that the other two abstract norms are valid norms as well.

For the fifth part note that the equivalence class of $\|u\|_U = \sup\{\|u\|_X, |f(u)|^{-1}\}$ depends only on that of $\|\cdot\|_X$, so we may suppose that $\|\cdot\|_X$ is the norm (18.2.1) obtained from generators $f_1, \dots, f_m$ of the F-algebra $\mathcal{O}_X$. Then $f^{-1}, f_1, \dots, f_m$ generate $\mathcal{O}_U$, and the norm obtained from this generating set is precisely $\|\cdot\|_U$.

For the sixth part we must show that $\|\cdot\|_X$ is equivalent to $\|\cdot\|$. First we note that $\|\cdot\|_X$ is dominated by $\|\cdot\|$. Indeed, this follows from the first part of this proposition, applied to the morphism

$$\coprod_{i=1}^{r} U_i \to X.$$

It remains to prove that $\|\cdot\|$ is dominated by $\|\cdot\|_X$. Refine the given open cover $U_1, \dots, U_r$ to get an open cover $V_1, \dots, V_s$ by principal open subsets of X (say $V_j = X_{f_j}$ for $f_j \in \mathcal{O}_X$) such that for each index j there exists an index $i(j)$ such that $V_j \subset U_{i(j)}$. By the fifth part of this proposition

$$\|v\|_j := \sup\{\|v\|_X, |f_j(v)|^{-1}\}$$

is a valid norm on $V_j(F)$. By the first part of this proposition (applied to all the inclusions $V_j \hookrightarrow U_{i(j)}$) there exist $d, S \geq 1$ such that for all j

$$\|v\|_{i(j)} \leq d\|v\|_j^S \quad \forall\, v \in V_j(F).$$

Since the principal open subsets V_j cover X, there exist $g_1, \dots, g_s \in \mathcal{O}_X$ such that $\sum_{j=1}^s f_j g_j = 1$. By (18.2.2) there exist $c, R \geq 1$ such that for all j

$$|g_j(x)| \leq c\|x\|_X^R \quad \forall\, x \in X(F).$$

Now let $x \in X(F)$. Then

$$1 = |\sum_{j=1}^{s} f_j(x) g_j(x)| \leq \sum_{j=1}^{s} |f_j(x)| \cdot \big(c\|x\|_X^R\big),$$

and thus there exists j such that $|f_j(x)| \cdot \big(c\|x\|_X^R\big) \geq 1/s$, from which we see that $f_j(x) \neq 0$ (so that $x \in V_j(F) \subset U_{i(j)}(F)$) and that moreover

$$|f_j(x)|^{-1} \leq sc\|x\|_X^R.$$

Thus

$$\|x\| \le \|x\|_{i(j)} \le d\|x\|_j^S = d[\sup\{\|x\|_X, |f_j(x)|^{-1}\}]^S \le d[\sup\{\|x\|_X, sc\|x\|_X^R\}]^S. \tag{18.4.3}$$

But, since $s, c, R \ge 1$, we have $sc\|x\|_X^R \ge \|x\|_X$, so (18.4.3) becomes

$$\|x\| \le d(sc)^S \|x\|_X^{RS},$$

showing that $\|\cdot\|$ is dominated by $\|\cdot\|_X$.

Finally, to prove the seventh part one considers the action morphism $G \times X \to X$ and uses the first and fourth parts of this proposition. □

18.5. Arbitrary schemes of finite type over F. Let X be any scheme of finite type over F. Let $U_1, \dots, U_r$ be any cover of X by affine open subsets. For $i = 1, \dots, r$ let $\|\cdot\|_i$ be any norm on $U_i(F)$. Define an abstract norm $\|\cdot\|$ on $X(F)$ by

$$\|x\| = \inf\{\|x\|_i : i \text{ such that } x \in U_i(F)\}.$$

It is not difficult to show that the equivalence class of $\|\cdot\|$ is independent of all choices, so that we have defined a canonical equivalence class of norms on $X(F)$. When X is affine, we recover our old notion of norm. When X is projective, the constant function 1 is a valid norm on $X(F)$. The reader may enjoy checking these statements as an exercise, but in this article we will only need norms on affine schemes.

18.6. Norm descent property. Let X and Y be affine schemes of finite type over F, let $\|\cdot\|_X$ and $\|\cdot\|_Y$ be norms on $X(F)$ and $Y(F)$ respectively, and let $\phi : Y \to X$ be a morphism. We say that ϕ has the *norm descent property* if the restriction of $\|\cdot\|_X$ to $\operatorname{im}[Y(F) \to X(F)]$ is equivalent to the abstract norm $\phi_*\|\cdot\|_Y$ on $\operatorname{im}[Y(F) \to X(F)]$ whose value at x is equal to the infinum of the values of $\|\cdot\|_Y$ on the fiber of $\phi : Y(F) \to X(F)$ over x. It is easy to see that this condition is independent of the choice of $\|\cdot\|_X$ and $\|\cdot\|_Y$. Moreover, it follows from the first part of Proposition 18.1 that the restriction of $\|\cdot\|_X$ is automatically dominated by $\phi_*\|\cdot\|_Y$; therefore the norm descent property is equivalent to the condition that $\phi_*\|\cdot\|_Y$ be dominated by the restriction of $\|\cdot\|_X$.

LEMMA 18.3. *Let $\phi : Y \to X$ be a morphism of affine schemes over F, and let $\|\cdot\|_Y$ be a norm on $Y(F)$. Then the following two conditions are equivalent.*

(1) *The morphism ϕ satisfies the norm descent property.*
(2) *There exists a norm $\|\cdot\|_X$ on $X(F)$ such that for all $x \in \operatorname{im}[Y(F) \to X(F)]$ there exists $y \in Y(F)$ such that $\phi(y) = x$ and $\|y\|_Y \le \|x\|_X$.*

PROOF. The second condition trivially implies that $\phi_*\|\cdot\|_Y$ is dominated by the restriction of $\|\cdot\|_X$, hence that ϕ has the norm descent property.

Now assume the first condition. Start with any norm $\|\cdot\|_X$ on $X(F)$. Then there exist $c, R \ge 1$ such that $\phi_*\|\cdot\|_Y \le c\|\cdot\|_X^R$ holds on $\operatorname{im}[Y(F) \to X(F)]$. Increasing c, we may improve $\le$ to $<$, and then replacing $\|\cdot\|_X$ by the equivalent norm $c\|\cdot\|_X^R$, we end up with a norm $\|\cdot\|_X$ for which

$$\phi_*\|\cdot\|_Y < \|\cdot\|_Y$$

holds on $\operatorname{im}[Y(F) \to X(F)]$. It is clear that the second condition is satisfied for this choice of $\|\cdot\|_X$. □

LEMMA 18.4. *Consider morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z$ of affine schemes of finite type over F. Put $h = gf : X \to Z$. Assume that the map $f : X(F) \to Y(F)$ is surjective. Then*

(1) *If f, g satisfy the norm descent property, then so does h.*
(2) *If h satisfies the norm descent property, then so does g.*

PROOF. The proof of the first statement uses only Lemma 18.3. The proof of the second statement is similar, but also uses the first part of Proposition 18.1, applied to the morphism f. Details are left to the reader. □

PROPOSITION 18.2. *Let $\phi : Y \to X$ be a morphism of affine schemes of finite type over F. For open U in X write ϕ_U for the morphism $\phi^{-1}U \to U$ obtained by restriction from ϕ.*

(1) *The norm descent property for $\phi : Y \to X$ is local with respect to the Zariski topology on X. In other words, for any cover of X by affine open subsets, the morphism ϕ has the norm descent property if and only if the morphisms ϕ_U have the norm descent property for every member U of the open cover.*
(2) *If the morphism $\phi : Y \to X$ admits a section, then ϕ has the norm descent property. More generally, if $\phi : Y \to X$ admits sections locally in the Zariski topology on X, then ϕ has the norm descent property.*
(3) *Let G be a connected reductive group over F, and let M be a Levi subgroup of G. Then the canonical morphism $G \to G/M$ has the norm descent property.*

PROOF. We begin by proving the first part of the proposition in the special case of principal open subsets. So suppose for the moment that $U = X_f$ is the principal open subset of X defined by $f \in \mathcal{O}_X$. Then $\phi^{-1}X_f = Y_g$, where g is the image of f under the homomorphism $\phi^* : \mathcal{O}_X \to \mathcal{O}_Y$. Assuming the norm descent property for ϕ, we need to prove the norm descent property for $\phi_U : Y_g \to X_f$.

We are free to use any convenient norms on $Y_g(F)$ and $X_f(F)$. Start by picking any norm $\|\cdot\|_Y$ on $Y(F)$. Choose our norm $\|\cdot\|_X$ on $X(F)$ so that the second condition of Lemma 18.3 holds for it. On the principal open subsets Y_g, X_f we use the norms $\|\cdot\|_{Y_g}$, $\|\cdot\|_{X_f}$ obtained from $\|\cdot\|_Y$, $\|\cdot\|_X$ by the construction in the fifth part of Proposition 18.1. With these choices, it is easy to check that $\|\cdot\|_{Y_g}$ and $\|\cdot\|_{X_f}$ satisfy the second condition of Lemma 18.3, proving the norm descent property for ϕ_U.

Next, suppose we have a cover of the affine scheme X by principal affine open subschemes $X_i = X_{f_i}$ $(i = 1, \dots, r)$. Putting $g_i := \phi^*(f_i)$ and $Y_i := Y_{g_i} = \phi^{-1}X_{f_i}$, we get (by restriction from ϕ) morphisms $\phi_i : Y_i \to X_i$. Assuming that each ϕ_i satisfies the norm descent property, we must show that ϕ satisfies the norm descent property.

Again we may use any convenient norms on $X(F)$, $Y(F)$. Choose norms $\|\cdot\|_{Y_i}$ on $Y_i(F)$. Choose norms $\|\cdot\|_{X_i}$ on $X_i(F)$ in such a way that the second condition of Lemma 18.3 holds for $\|\cdot\|_{Y_i}$ and $\|\cdot\|_{X_i}$. Use the construction in the sixth part of Proposition 18.1 to get a norm $\|\cdot\|_Y$ on $Y(F)$ (respectively, $\|\cdot\|_X$ on $X(F)$) from the norms $\|\cdot\|_{Y_i}$ (respectively, $\|\cdot\|_{X_i}$). With these choices it is easy to see that the second condition of Lemma 18.3 holds for $\|\cdot\|_Y$ and $\|\cdot\|_X$, proving the norm descent property for ϕ.

We are now finished with the case of principal affine subsets. Now suppose that we have any cover of X by affine open subsets $U_1, \dots, U_r$. Cover each U_i by principal affine open subsets V_{ij} $(j = 1, \dots, s_i)$ in X. Of course V_{ij} is then also a principal affine open subset of U_i.

By what has already been done we know that the norm descent property for ϕ is equivalent to the norm descent property for all the morphisms $\phi^{-1}V_{ij} \to V_{ij}$ and that this in turn is equivalent to the norm descent property for all the morphisms $\phi^{-1}U_i \to U_i$. This completes the proof of the first part of the proposition.

Now we prove the second part. Assume first that $\phi : Y \to X$ has a section $s : X \to Y$. Note that in this case $Y(F) \to X(F)$ is surjective. To show that ϕ has the norm descent property we must check that $\phi_* \| \cdot \|_Y$ is dominated by $\| \cdot \|_X$. But this follows from the first part of Proposition 18.1, applied to the morphism s. Furthermore, if ϕ admits sections Zariski locally, we see from the first part of Proposition 18.2 that ϕ has the norm descent property.

Now we prove the third part. Choose a parabolic subgroup $P = MU$ with Levi component M and let $\bar{P} = M\bar{U}$ be the opposite parabolic subgroup. From Bruhat theory we know that multiplication induces an open immersion

$$\bar{U} \times U \times M \hookrightarrow G.$$

Thus $G \to G/M$ has a section over the open subset $\bar{U} \times U$ of G/M. Moreover G/M is covered by the $G(F)$-translates of these open sets $\bar{U} \times U$, so we conclude that $G \to G/M$ admits sections locally in the Zariski topology, hence that it has the norm descent property. Note that G/M really is an affine scheme: it can be identified with the G-conjugacy class of any sufficiently general element of the center of M. $\square$

18.7. An additional hypothesis. We now fix an algebraic closure $\bar{F}$ of F. Our given absolute value on F can always be extended to $\bar{F}$, and when F is complete this extension is unique (see [**Lan02**]).

From now on we assume that F is complete, and we continue to denote by $|\cdot|$ the unique extension to $\bar{F}$ of our given absolute value on F. By uniqueness this extension is fixed by any automorphism of $\bar{F}$ over F, and therefore the restriction of our extended absolute value to any finite extension E of F (in $\bar{F}$) of degree n is given by

$$x \mapsto |N_{E/F}(x)|^{1/n}, \qquad (18.7.1)$$

where $N_{E/F}$ denotes the usual norm map of field theory.

18.8. Behavior of norms under algebraic field extensions. Let X be an affine scheme of finite type over F, and let E be a field extension of F. Any finite set of generators for the F-algebra $\mathcal{O}_X$ can also be regarded as a generating set for the E-algebra $E \otimes_F \mathcal{O}_X$ of regular functions on the scheme X_E over E obtained from X by extension of scalars. When E is a subfield of $\bar{F}$ the chosen generating set gives norms on both $X(F)$ and $X(E) = X_E(E)$, and the restriction of the norm on $X(E)$ to the subset $X(F)$ coincides with the norm on $X(F)$, from which it follows that the restriction of any norm on $X(E)$ is a norm on $X(F)$.

For finite separable extensions E/F we use $R_{E/F}$ to denote Weil's restriction of scalars.

LEMMA 18.5. *Let E be a finite separable field extension of F. Then the abstract norm*

$$x \mapsto \sup\{|N_{E/F}(x)|, |N_{E/F}(x)|^{-1}\} \qquad (x \in E^\times) \tag{18.8.1}$$

is a norm on $(R_{E/F}\mathbb{G}_m)(F) = E^\times$.

PROOF. Let I be the set of embeddings of E in $\bar{F}$. Since our torus becomes split over $\bar{F}$, we see from Example 18.2 and the discussion preceding this lemma that

$$x \mapsto \sup\{|\sigma(x)|, |\sigma(x)|^{-1} : \sigma \in I\} \qquad (x \in E^\times) \tag{18.8.2}$$

is a norm on $(R_{E/F}\mathbb{G}_m)(F) = E^\times$. But for any $\sigma \in I$ we have (by (18.7.1))

$$|\sigma(x)| = |N_{E/F}(x)|^{1/[E:F]},$$

showing that the norm (18.8.2) is indeed equivalent to the abstract norm in the statement of the lemma. □

LEMMA 18.6. *Let T be a torus over F, and let S be the biggest split quotient of T, so that $X^*(S)$ is the subgroup of $X^*(T)$ consisting of elements fixed by $\mathrm{Gal}(\bar{F}/F)$, and we have a canonical homomorphism $T \to S$. Then the pullback via $T(F) \to S(F)$ of any norm $\|\cdot\|_S$ on $S(F)$ is a norm on $T(F)$.*

PROOF. One sees easily from Lemma 18.5 that our current lemma is valid for $T = R_{E/F}T_0$ for any finite separable extension E/F and any split torus T_0 over E.

Now consider any torus T and choose a finite Galois extension E/F that splits T. Then T embeds naturally in $R_{E/F}T_E$, a torus for which the lemma is known to be valid, and thus it now suffices to show that if T is a subtorus of a torus T' for which the lemma is known to be valid, then the lemma is valid for T. We of course write S' for the biggest split quotient torus of T'.

Then we have a commutative diagram

$$\begin{array}{ccc} T & \longrightarrow & T' \\ \downarrow & & \downarrow \\ S & \longrightarrow & S'. \end{array}$$

Choose a norm $\|\cdot\|_{S'}$ on $S'(F)$. By our assumption on T' and the second part of Proposition 18.1 (applied to $T \hookrightarrow T'$) we see that the pullback of $\|\cdot\|_{S'}$ to $T(F)$ can serve as our norm on $T(F)$. Going the other way around the commutative square above, we conclude (using the first part of Proposition 18.1 to see that the pullback of $\|\cdot\|_{S'}$ to $S(F)$ is dominated by $\|\cdot\|_S$) that $\|\cdot\|_T$ is dominated by the pullback to $T(F)$ of $\|\cdot\|_S$. But (again by the first part of Proposition 18.1) $\|\cdot\|_T$ also dominates the pullback of $\|\cdot\|_S$, hence is equivalent to it. □

COROLLARY 18.7. *Let T be a torus over F and let n be a positive integer. Then there exists a bounded subset $B \subset T(F)$ such that $T(F) = B \cdot T(F)^n$. Here we are using the superscript n to indicate that we are looking at the subgroup of all n-th powers of elements in $T(F)$.*

PROOF. First we treat the split case. So suppose $T = \mathbb{G}_m^r$ and use the norm $\|\cdot\|_T$ in Example 18.2. Pick $\alpha \in F^\times$ such that $|\alpha| \neq 1$ and put $a := |\alpha|$. The bounded set $\{t \in T(F) : \|t\|_T \leq a^n\}$ does the job.

Now we treat the general case. Let $\phi : T \to S$ be the maximal split quotient of T, and let A_T denote the maximal split subtorus of T. The composition of the inclusion of A_T in T with ϕ yields an isogeny $\psi : A_T \to S$, and therefore there exists a positive integer m such that the m-th power map on S factors through ψ. This guarantees that $\phi(T(F)) \supset S(F)^m$.

By the split case that has already been treated we know that there exists a bounded subset B_0 of $S(F)$ such that $S(F) = B_0 \cdot S(F)^{mn}$. Let B denote the inverse image of B_0 under $\phi : T(F) \to S(F)$; by Lemma 18.6 the set B is bounded in $T(F)$, and it is immediate that $T(F) = B \cdot T(F)^n$. □

18.9. Torsors, quotients and a technical lemma. In this subsection we prove a technical lemma that will be needed in the next subsection. We begin by reviewing some material from SGA 3 on torsors and quotients.

In this subsection absolute values will not appear, and F will denote any field. By a scheme (or morphism) we always mean a scheme (or morphism) over F. Given two schemes X, Y, we denote their product over F simply by $X \times Y$, and we write $X(Y)$ for the set of Y-valued points of X, or, in other words, the set of morphisms from Y to X.

An action of a group scheme G on a scheme X is a morphism $G \times X \to X$ such that for every scheme T the associated map

$$G(T) \times X(T) \to X(T)$$

on T-valued points is an action of the group $G(T)$ on the set $X(T)$.

Now suppose that X is a scheme over another scheme S, so that it comes equipped with a morphism $p : X \to S$. We say that an action of G on X *preserves the fibers of* $p : X \to S$ (or that G acts on X *over* S) if

$$p(gx) = p(x)$$

for all schemes T and all T-valued points $g \in G(T)$, $x \in X(T)$. Given an action of this type there is a canonical morphism

$$G \times X \to X \times_S X \tag{18.9.1}$$

given by $(g, x) \mapsto (gx, x)$ on T-valued points.

By definition a *G-torsor X over S* (for the fpqc topology) is a faithfully flat, quasi-compact morphism $p : X \to S$, together with an action of G on X over S for which (18.9.1) is an isomorphism. The significance of (18.9.1) being an isomorphism is easy to understand: it means that for every scheme T either $X(T)$ is empty or else $G(T)$ acts simply transitively on $X(T)$. Considering the fiber product of p with itself one sees that any property of the morphism $G \to \operatorname{Spec} F$ that is stable under base change and faithfully flat descent will be inherited by the morphism $p : X \to S$; for example, if G is smooth over F, then any G-torsor X over S is smooth over S, and so on.

Let $p : X \to S$ be a G-torsor. By faithfully flat descent for the morphism p, giving a morphism from S to some other scheme S' is the same as giving a morphism $X \to S'$ whose compositions with the two projections $\pi_1, \pi_2 : X \times_S X \to X$ coincide, and since (18.9.1) is an isomorphism, this in turn is the same as giving a morphism $X \to S'$ whose fibers are preserved by the G-action. In other words $p : X \to S$ satisfies the universal property one expects of a quotient of X by G. In particular, given an action of G on X, if there exists a morphism $p : X \to S$ for which X is a

G-torsor over S, then the morphism p is essentially unique, and we will refer to S as the *quotient* of X by G, denoted $G\backslash X$.

An important (and non-trivial) result is that if G is a group scheme of finite type over F, and if H is a closed subgroup scheme of G, then the quotient G/H does exist (see SGA 3, Exp. $\mathrm{VI_A}$) in the sense just described.

Here is a simple example, which we will need later. Let T be a torus over F, and let n be a positive integer. We write $[n] : T \to T$ for the homomorphism given on S-valued points by $t \to t^n$, and we write T_n for the kernel of $[n]$, so that

$$T_n(S) = \{t \in T(S) : t^n = 1\}.$$

When n is not invertible in F, the scheme T_n is not smooth over F. Nevertheless the morphism $[n]$ is faithfully flat, as one can check after extending scalars from F to an algebraic closure $\bar{F}$ of F, so that T splits and one is reduced to the obvious fact that for an indeterminate X the ring $\bar{F}[X, X^{-1}]$ is free of rank n as a module over its subring $\bar{F}[X^n, X^{-n}]$. Therefore the morphism $[n]$ makes T into a T_n-torsor over T and identifies T with the quotient T/T_n, so that the sequence

$$1 \to T_n \to T \xrightarrow{[n]} T \to 1$$

is an exact sequence of sheaves in the fpqc topology. When n is not invertible in F, the sequence above is not exact in the étale topology; indeed, the map $T(F_{\mathrm{sep}}) \to T(F_{\mathrm{sep}})$ is not surjective, F_{sep} being the separable closure of F in $\bar{F}$. This example shows why we are using the fpqc topology.

Now let G be a connected reductive group over F, and let T be a torus in G, in other words, a closed subgroup scheme that is a torus over F. There exists a finite Galois extension F'/F such that T splits over F'; put $n := [F' : F]$ and $T_n := \ker([n] : T \to T)$, as above.

Lemma 18.8. *Assume that F is an infinite field. Then the canonical morphism $f : G/T_n \to G/T$ admits sections locally in the Zariski topology on G/T.*

Proof. We claim that it is enough to show that there exists a non-empty Zariski open subset U in G/T such that $f : G/T_n \to G/T$ has a section over U. Indeed, since f is G-equivariant, it will then have sections over all the open sets gU ($g \in G(F)$), so it is enough to show that $V := \cup_{g\in G(F)} gU$ is equal to G/T. But V is a non-empty $G(F)$-invariant open subset of G/T, so its inverse image V' in G is a non-empty $G(F)$-invariant open subset of G, and since $G(F)$ is Zariski dense in G (since F is infinite, see [**Bor91**, Cor. (18.3)]), it follows that $V' = G$ and $V = G/T$.

Now G/T is connected (since G is connected and $G \to G/T$ is surjective) and smooth (see EGA IV(17.7.7)), hence reduced and irreducible. Let K be the function field of G/T and $\xi : \operatorname{Spec} K \to G/T$ the generic point of G/T. Since $f : G/T_n \to G/T$ is a morphism of finite type, the existence of a section of f over some non-empty open subset of G/T is equivalent to the existence of a section of f over ξ, in other words to the existence of a K-point of G/T_n mapping to ξ under

$$(G/T_n)(K) \to (G/T)(K). \tag{18.9.2}$$

Thus it will suffice to show that the map (18.9.2) is surjective. In fact the map (18.9.2) is surjective for any field extension K of F, as we will now see.

Since F'/F is a Galois extension of degree n, the K-algebra $K \otimes_F F'$ has the form $K' \times \cdots \times K'$ for some finite Galois extension K' of K whose degree divides n. Thus T splits over K' and the Galois cohomology group $H^1(K, T)$ coincides with

$H^1(\mathrm{Gal}(K'/K), T(K'))$, a group killed by the order of the group $\mathrm{Gal}(K'/K)$ and hence killed by n.

When F (and hence K) has characteristic 0, we can then use the exact sequence

$$1 \to T_n(\bar{K}) \to T(\bar{K}) \xrightarrow{n} T(\bar{K}) \to 1 \tag{18.9.3}$$

to see that

$$H^1(K, T_n) \to H^1(K, T)$$

is surjective, from which it follows easily that (18.9.2) is surjective. [Use that the set of $G(K)$-orbits on $(G/T)(K)$ can be identified with the kernel of the map $H^1(K, T) \to H^1(K, G)$ of pointed sets.]

When n is not invertible in K, we need to argue differently, since (18.9.3) is no longer exact when $\bar{K}$ is replaced by the separable closure K_{sep}. To avoid using flat cohomology, we work directly with torsors. The group T acts on the right of G, yielding a T-torsor $f' : G \to G/T$ over G/T. Moreover T/T_n acts on the right of G/T_n, and our morphism $f : G/T_n \to G/T$ is the T/T_n-torsor obtained from f' via the canonical homomorphism $T \twoheadrightarrow T/T_n$.

Using the fpqc exact sequence $1 \to T_n \to T \xrightarrow{n} T \to 1$ to identify T/T_n with T, we see that the $T = T/T_n$-torsor f is obtained from the T-torsor f' via the homomorphism $[n] : T \to T$.

Fortunately, for T-torsors (unlike T_n-torsors) the difference between the fpqc and étale topologies is unimportant: since T is smooth over F, any T-torsor over K is automatically smooth over K, hence has sections étale locally on $\mathrm{Spec}(K)$. Therefore the group of isomorphism classes of T-torsors over K can be identified with the Galois cohomology group $H^1(K, T)$, a group killed by n, as we saw above.

Now we can prove that (18.9.2) is surjective. Consider a K-point of G/T. Pulling back our torsor f to this K-point, we get a $T = (T/T_n)$-torsor over $\mathrm{Spec}\, K$, which we just need to show is trivial (so that it has a section). But, as explained above, the class of this torsor lies in the image of multiplication by n, and since $H^1(K, T)$ is killed by n, every element in the image of multiplication by n is trivial. □

18.10. Another case of the norm descent property. We start with the following lemma which will ensure that the varieties we deal with are affine.

LEMMA 18.9. *Let G be a connected reductive group over a field F and let T be an F-torus in G. Then G/T is affine.*

PROOF. By EGA IV (2.7.1) the property of being an affine morphism is stable under fpqc descent. So we are free to assume that F is algebraically closed. Choose a maximal torus T' of G containing T. Then G/T' can be identified with the G-orbit of any suitably regular element in T', so G/T' is affine. Moreover $G/T \to G/T'$ is a T'/T-torsor and hence is an affine morphism. This proves that G/T is affine. □

Now we again assume that F is equipped with a non-trivial absolute value and that F is complete as a metric space. The next result is related to the corollary on page 112 of [**HC70**] (see also [**Art91a**, Lemma 4.1]).

PROPOSITION 18.3. *Let G be a connected reductive group over F and let T be an F-torus in G. Then $G \to G/T$ has the norm descent property. When T is split, the same is true even if F is not complete.*

Proof. Choose a finite Galois extension F'/F that splits T, put $n := [F' : F]$ and $C := \ker([n] : T \to T)$. Then $G \to G/T$ factorizes as

$$G \to G/C \to G/T.$$

For $g \in G(\bar{F})$ we write $\bar{g}$ (respectively, $\dot{g}$) for the image of g in $(G/C)(\bar{F})$ (respectively, $(G/T)(\bar{F})$).

Since our absolute value is non-trivial, the field F is infinite, and Lemma 18.8 says that $G/C \to G/T$ has sections Zariski locally and hence satisfies the norm descent property. When T is split we may take $F' = F$, so that C is trivial and we are done.

From now on we assume that F is complete. Since $G \to G/C$ is finite, it too satisfies the norm descent property, as is clear from the first part of Proposition 18.1. So $G \to G/T$ is the composition of two morphisms, both of which satisfy the norm descent property. Nevertheless, since $G(F) \to (G/C)(F)$ need not be surjective, we cannot apply Lemma 18.4, and in fact it requires a bit of effort to prove the norm descent property for $G \to G/T$.

In doing so we are free to use any convenient norms on $G(F)$ and $(G/T)(F)$. We begin by picking any norm $\|\cdot\|_{G/C}$ on $(G/C)(F)$. Since $G \to G/C$ is finite, by the first part of Proposition 18.1 the pullback of $\|\cdot\|_{G/C}$ to $G(F)$ can serve as our norm $\|\cdot\|_G$ on $G(F)$. Thus

$$\|g\|_G = \|\bar{g}\|_{G/C} \qquad \forall\, g \in G(F). \tag{18.10.1}$$

Since $G/C \to G/T$ satisfies the norm descent property, we may (by Lemma 18.3) choose our norm $\|\cdot\|_{G/T}$ on $(G/T)(F)$ so that for all y in the image of $(G/C)(F)$ in $(G/T)(F)$ there exists $z \in (G/C)(F)$ such that $z \mapsto y$ and

$$\|z\|_{G/C} \le \|y\|_{G/T}. \tag{18.10.2}$$

When T/C is identified with T via the fpqc exact sequence

$$1 \to C \to T \xrightarrow{[n]} T \to 1,$$

the canonical map $T \to T/C$ becomes the map $[n] : T \to T$. Thus it follows from Corollary 18.7 that there exists a bounded subset $B \subset (T/C)(F)$ such that $(T/C)(F) = B \cdot (T(F)/C(F))$.

From the seventh part of Proposition 18.1 we see that there exist $d, R > 0$ such that

$$\|\bar{g}b^{-1}\|_{G/C} \le d\|\bar{g}\|^R_{G/C} \qquad \forall\, \bar{g} \in (G/C)(F),\, b \in B. \tag{18.10.3}$$

Now we are ready to prove the norm descent property for $G \to G/T$. For this it will suffice to show that for any $g \in G(F)$ there exists $t \in T(F)$ such that

$$\|gt\|_G \le d\|\dot{g}\|^R_{G/T} \tag{18.10.4}$$

(with d, R as chosen above). By (18.10.2) there exists $s \in (T/C)(F)$ such that

$$\|\bar{g}s\|_{G/C} \le \|\dot{g}\|_{G/T}. \tag{18.10.5}$$

Now write

$$s = tb \tag{18.10.6}$$

for some $t \in T(F)/C(F)$ and $b \in B$. Then, using successively (18.10.1), (18.10.6), (18.10.3), (18.10.5), we see that

$$\|gt\|_G = \|\bar{g}t\|_{G/C} = \|\bar{g}sb^{-1}\|_{G/C} \leq d\|\bar{g}s\|_{G/C}^R \leq d\|\dot{g}\|_{G/T}^R$$

as desired. □

COROLLARY 18.10. *Again assume the field F is complete. Let G be a connected reductive group over F, let T be an F-torus in G, and let A_T be the maximal split torus in T. Consider the canonical morphism $\phi : G/A_T \to G/T$. Then for any norm $\|\cdot\|_{G/T}$ on $(G/T)(F)$ the pullback of $\|\cdot\|_{G/T}$ via ϕ is a norm on $(G/A_T)(F)$.*

PROOF. Since $H^1(F, A_T)$ is trivial, the map $G(F) \to (G/A_T)(F)$ is surjective. Moreover, it follows from Proposition 18.3 that $G \to G/T$ satisfies the norm descent property. Therefore, the second part of Lemma 18.4 tells us that $G/A_T \to G/T$ has the norm descent property, which (by Lemma 18.3) means that we can choose our norms $\|\cdot\|_{G/A_T}$ and $\|\cdot\|_{G/T}$ so that for any $y \in (G/T)(F)$ that lies in the image of $(G/A_T)(F)$, there exists $z \in (G/A_T)(F)$ such that $z \mapsto y$ and

$$\|z\|_{G/A_T} \leq \|y\|_{G/T}. \tag{18.10.7}$$

Since the biggest split quotient of T/A_T is trivial, we see from Lemma 18.6 that $(T/A_T)(F)$ is bounded. Therefore the seventh part of Proposition 18.1 tells us that there exist $c, R > 0$ such that

$$\|xu\|_{G/A_T} \leq c\|x\|_{G/A_T}^R \qquad \forall\, x \in (G/A_T)(F),\, u \in (T/A_T)(F). \tag{18.10.8}$$

Since $(T/A_T)(F)$ acts simply transitively on any non-empty fiber of

$$(G/A_T)(F) \to (G/T)(F),$$

we see from (18.10.7) and (18.10.8) that the inequality $\|\cdot\|_{G/A_T} \leq c(\phi^*\|\cdot\|_{G/T})^R$ holds on $(G/A_T)(F)$. Dominance in the other direction follows from the first part of Proposition 18.1. Thus the pullback of $\|\cdot\|_{G/T}$ is equivalent to $\|\cdot\|_{G/A_T}$, as we wished to show. □

18.11. Norms on split p-adic G. We now let G be a split group over a p-adic field F (actually over $\mathcal{O}$). As usual we put $K := G(\mathcal{O})$, fix a split maximal torus A over $\mathcal{O}$, and put $\mathfrak{a} := X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$. We choose a Weyl group invariant inner product on $\mathfrak{a}$, so that $\mathfrak{a}$ becomes a Euclidean space with Euclidean norm $\|\cdot\|_E$. The subscript is supposed to remind us that this is a Euclidean norm, not the sort of norm that we've been discussing in this section.

We define an abstract norm $\|\cdot\|_G$ on $G(F)$ as follows. Let $g \in G(F)$. By the Cartan decomposition there is a unique dominant coweight ν such that $g \in K\pi^\nu K$. Put $\|g\|_G = \exp(\|\nu\|_E)$.

LEMMA 18.11. *The abstract norm $\|\cdot\|_G$ is a valid norm on $G(F)$.*

PROOF. Pick any norm $\|\cdot\|'_G$ on $G(F)$. We must show that $\|\cdot\|_G$ and $\|\cdot\|'_G$ are equivalent. By the seventh part of Proposition 18.1 there exist positive constants c, R such that

$$\|k_1 g k_2\|'_G \leq c(\|g\|'_G)^R \tag{18.11.1}$$

for all $g \in G$, $k_1, k_2 \in K$. Thus for $a \in A(F)$ and $g \in KaK$ there are inequalities

$$\|g\|'_G \leq c(\|a\|'_G)^R \tag{18.11.2}$$

and

$$\|a\|'_G \leq c(\|g\|'_G)^R \tag{18.11.3}$$

In Example 18.2 we wrote down one valid norm on $A(F)$. It is easy to see that this norm is equivalent to the restriction of $\|\cdot\|_G$ to $A(F)$. Therefore (by the second part of Proposition 18.1) the restrictions of $\|\cdot\|'_G$ and $\|\cdot\|_G$ to $A(F)$ are equivalent. This, together with the inequalities (18.11.2) and (18.11.3), shows that $\|\cdot\|_G$ and $\|\cdot\|'_G$ are equivalent. $\square$

19. Another kind of norm on affine varieties over local fields

The norms introduced in section 18 are good for measuring "how big" points are, or, in other words, how close they are to ∞, and can therefore be used to measure growth rates of functions. In this section we discuss another kind of norm, the most important of which (namely $\mathbf{N}_{x_0}$) measures how close a point is to some given point x_0. In order to prove one of the properties of $\mathbf{N}_{x_0}$ it is useful to introduce a more general variant N_Y which measures how close a point is to some given reduced closed subscheme Y.

Again F denotes a field equipped with an absolute value.

19.1. The norm N_Y. Let A be a finitely generated F-algebra and put $X := \mathrm{Spec}(A)$. Let Y be a reduced closed subscheme of X, and let $I = I(Y)$ be the corresponding ideal in A. Thus I is equal to its radical.

Now choose a finite set $f_1, \dots, f_r$ of generators for the ideal I. For $x \in X(F)$ put

$$N_Y(x) := \sup\{|f_1(x)|, \dots, |f_r(x)|\}. \tag{19.1.1}$$

Thus $N_Y(x)$ is a non-negative real-valued function of $x \in X(F)$ which vanishes if and only if $x \in Y(F)$. The size of $N_Y(x)$ measures how far x is from $Y(F)$.

LEMMA 19.1. *Let $f \in I$. There exists a norm $\|\cdot\|_X$ on $X(F)$, of the type considered in the previous section, having the property that*

$$|f(x)| \le \|x\|_X \cdot N_Y(x) \quad \forall\, x \in X(F). \tag{19.1.2}$$

PROOF. Choose elements $g_1, \dots, g_r \in A$ such that $f = g_1 f_1 + \dots + g_r f_r$. Then

$$|f(x)| \le \sum_{i=1}^{r} |g_i(x)||f_i(x)|. \tag{19.1.3}$$

Using (18.2.2) for all the functions g_i, we see that there exists a norm $\|\cdot\|_X$ on $X(F)$ such that

$$|g_i(x)| \le r^{-1}\|x\|_X \quad \forall\, i \in \{1, \dots, r\}. \tag{19.1.4}$$

The inequality (19.1.2) follows directly from the inequalities (19.1.3) and (19.1.4). $\square$

Now suppose that we have two affine schemes X_1, X_2 of finite type over F, as well as two reduced closed subschemes Y_1, Y_2 of X_1, X_2 respectively. Choose finite generating sets for the ideals $I_1 := I(Y_1)$, $I_2 := I(Y_2)$, obtaining in this way $N_1 := N_{Y_1}$, $N_2 := N_{Y_2}$ on $X_1(F)$, $X_2(F)$ respectively. Suppose further that we are given a morphism $\phi : X_1 \to X_2$ of F-schemes such that $\phi(Y_1) \subset Y_2$ (equivalently: $\phi^*(I_2) \subset I_1$).

LEMMA 19.2. *There exists a norm $\|\cdot\|_{X_1}$ on $X_1(F)$, of the type considered in the previous section, having the property that*

$$N_2(\phi(x_1)) \le \|x_1\|_{X_1} \cdot N_1(x_1) \quad \forall\, x_1 \in X_1(F). \tag{19.1.5}$$

PROOF. Say N_1 is obtained from generators $f_1, \dots, f_r$ of I_1 and that N_2 is obtained from generators $g_1, \dots, g_s$ of I_2. Applying Lemma 19.1 to the functions $\phi^*(g_j)$, we see that there exists a norm $\|\cdot\|_{X_1}$ on $X_1(F)$ such that

$$|g_j(\phi(x_1))| \le \|x_1\|_{X_1} \cdot N_1(x_1) \quad \forall\, x_1 \in X_1(F) \tag{19.1.6}$$

for all $j = 1, \dots, s$; since $N_2(\phi(x_1))$ is the maximum of the quantities appearing on the left side of (19.1.6), the lemma is proved. □

19.2. The norm $\mathbf{N}_{x_0}$. We continue with $X = \mathrm{Spec}(A)$ as above. Let $x_0 \in X(F)$ and consider the corresponding reduced closed subscheme $\{x_0\}$ of X, whose ideal is the maximal ideal $\mathfrak{m} := \{f \in A : f(x_0) = 0\}$. We are now interested in generating sets of $\mathfrak{m}$ of the following special type. Consider a finite set $f_1, \dots, f_r$ of generators for the F-algebra A having the property that each f_i lies in the maximal ideal $\mathfrak{m}$. Such generating sets exist, since we can start with an arbitrary generating set and subtract from each generator its value at x_0. It is easy to see that $f_1, \dots, f_r$ necessarily generate the ideal $\mathfrak{m}$. Now define $\mathbf{N}_{x_0}$ to be the function $N_{\{x_0\}}$ obtained from the generating set $f_1, \dots, f_r$. We use the notation $\mathbf{N}_{x_0}$ to keep track of the fact that $f_1, \dots, f_r$ not only generate $\mathfrak{m}$ as ideal, but also A as F-algebra.

Since $f_1, \dots, f_r$ generate A as F-algebra, they define a closed embedding of X into $\mathbb{A}^r$, and from this one sees easily that the sets

$$\{x \in X(F) : \mathbf{N}_{x_0}(x) < \varepsilon\} \tag{19.2.1}$$

for $\varepsilon > 0$ form a neighborhood base at x_0 in $X(F)$.

19.3. An application. Now let G be a reduced affine group scheme of finite type over F, and let H be a closed subgroup scheme of G. We write e_G, e_H for the identity elements of $G(F)$, $H(F)$ respectively. It is evident that $g^{-1}hg$ is close to e_G if h is close to e_H, but the bigger g is, the closer to e_H we must take h to be. In the proof of the key geometric result needed for the local trace formula we are going to need a quantitative version of this qualitative statement, involving the functions $\mathbf{N}_{x_0}$ we have just introduced.

We write $\mathcal{O}_G$, $\mathcal{O}_H$ for the rings of regular functions on G, H respectively. Choose generators $f_1, \dots, f_r$ of the F-algebra $\mathcal{O}_H$ such that $f_i(e_H) = 0$ for all $i = 1, \dots, r$, so that $f_1, \dots, f_r$ also generate the maximal ideal obtained from e_H, and use these generators to get the function $\mathbf{N}_{e_H}$ on $H(F)$. Similarly, choose generators $g_1, \dots, g_s$ of the F-algebra $\mathcal{O}_G$ such that $g_j(e_G) = 0$ for all $j = 1, \dots, s$, and use them to get the function $\mathbf{N}_{e_G}$ on $G(F)$. Finally, let $\|\cdot\|_G$ be any norm on $G(F)$ of the type considered in section 18.

LEMMA 19.3. *Let K be a neighborhood of e_G in $G(F)$. Then there exist positive constants c, R, depending on K, having the following property. For $g \in G(F)$ and $h \in H(F)$ satisfying*

$$\mathbf{N}_{e_H}(h) \le c\|g\|_G^{-R} \tag{19.3.1}$$

the element $g^{-1}hg$ lies in K.

PROOF. Since $f_1, \dots, f_r$ generate the F-algebra $\mathcal{O}_H$, we can also use them to get a norm $\|\cdot\|_H$ on $H(F)$ of the type considered in section 18; comparing the definitions of $\|\cdot\|_H$ and $\mathbf{N}_{e_H}$ one sees immediately that

$$\|h\|_H = \sup\{1, \mathbf{N}_{e_H}(h)\}. \tag{19.3.2}$$

Since G is reduced, the pullbacks of the functions $f_1, \dots, f_r$ to $G \times H$ via the second projection map $(g,h) \mapsto h$ generate the ideal of the closed subset $G \times \{e_H\}$ in $G \times H$, so we can use them to define the function $N_{G\times\{e_H\}}$ on $G(F) \times H(F)$, and it is evident from the definitions that

$$N_{G\times\{e_H\}}(g,h) = \mathbf{N}_{e_H}(h). \tag{19.3.3}$$

Noting that $\|g\|_G\|h\|_H$ is a valid norm on $G(F) \times H(F)$ by Proposition 18.1, and applying Lemma 19.2 to the morphism $G \times H \to G$ defined by $(g,h) \mapsto g^{-1}hg$, we see that there exist $c_1, R > 0$ such that

$$\mathbf{N}_{e_G}(g^{-1}hg) \le c_1(\|g\|_G\|h\|_H)^R \mathbf{N}_{e_H}(h). \tag{19.3.4}$$

Now choose $\varepsilon > 0$ small enough that $\mathbf{N}_{e_G}(g) \le \varepsilon$ implies that $g \in K$. Then

$$c_1(\|g\|_G\|h\|_H)^R \mathbf{N}_{e_H}(h) \le \varepsilon \Longrightarrow g^{-1}hg \in K. \tag{19.3.5}$$

From (19.3.2) we see that $\|h\|_H = 1$ when $\mathbf{N}_{e_H}(h) \le 1$, so that

$$\mathbf{N}_{e_H}(h) \le 1 \text{ and } c_1(\|g\|_G)^R \mathbf{N}_{e_H}(h) \le \varepsilon \Longrightarrow g^{-1}hg \in K, \tag{19.3.6}$$

or, in other words,

$$\mathbf{N}_{e_H}(h) \le \inf\{1, \varepsilon c_1^{-1}\|g\|_G^{-R}\} \Longrightarrow g^{-1}hg \in K. \tag{19.3.7}$$

Letting c be the minimum of 1 and εc_1^{-1} (and remembering that $\|g\|_G \ge 1$), we see that

$$\mathbf{N}_{e_H}(h) \le c\|g\|_G^{-R} \Longrightarrow g^{-1}hg \in K. \tag{19.3.8}$$

□

19.4. Special case of the application above. We will need a more concrete version of the previous lemma that is tailored to the situation we will encounter in proving the local trace formula. We now return to the split reductive group G over the p-adic field F, and we fix a norm $\|\cdot\|_G$ on $G(F)$ as in section 18.

Consider a parabolic subgroup $P = MU$ containing a Borel subgroup $B = AN$, with M containing A. We are going to apply the lemma we just proved to the subgroup U of G. As a variety, U is the product of its root subgroups U_α, where α runs over R_U, the set of roots of A in Lie(U). Fix identifications $U_\alpha \simeq \mathbb{G}_a$, so that we can view an element $u \in U(F)$ as a tuple with components $u_\alpha \in F$, one for each $\alpha \in R_U$. Using the most obvious set of generators, we get $\mathbf{N}_{e_U}$ on $U(F)$, given by

$$\mathbf{N}_{e_U}(u) = \sup\{|u_\alpha| : \alpha \in R_U\}. \tag{19.4.1}$$

Conjugating u by $a \in A(F)$, we get another element aua^{-1} of $U(F)$ whose components are given by $\alpha(a) \cdot u_\alpha$. Therefore

$$\mathbf{N}_{e_U}(aua^{-1}) \le \sup\{|\alpha(a)| : \alpha \in R_U\} \cdot \mathbf{N}_{e_U}(u). \tag{19.4.2}$$

LEMMA 19.4. *Let $K = G(\mathcal{O})$. There exist positive constants D, R, S having the following property. For all $a \in A(F)$, $u \in U(F)$, $g \in G(F)$ satisfying*

$$\inf\{|\alpha(a)|^{-1} : \alpha \in R_U\} \ge D\|u\|_G^R\|g\|_G^S \tag{19.4.3}$$

the element aua^{-1} lies in gKg^{-1}.

PROOF. Using Lemma 19.3 and (19.4.2), we see that there exist positive constants c, S such that for $a \in A(F)$, $u \in U(F)$, $g \in G(F)$

$$\sup\{|\alpha(a)| : \alpha \in R_U\} \cdot \mathbf{N}_{e_U}(u) \leq c\|g\|_G^{-S} \Longrightarrow aua^{-1} \in gKg^{-1}. \tag{19.4.4}$$

As we remarked during the course of the proof of Lemma 19.3, we get a valid norm $\|\cdot\|_U$ on $U(F)$ by putting $\|u\|_U := \sup\{1, \mathbf{N}_{e_U}(u)\}$, and thus (19.4.4) yields

$$\sup\{|\alpha(a)| : \alpha \in R_U\} \leq c\|u\|_U^{-1}\|g\|_G^{-S} \Longrightarrow aua^{-1} \in gKg^{-1}. \tag{19.4.5}$$

Since the restriction of $\|\cdot\|_G$ to $U(F)$ is also a valid norm, and moreover any two norms on $U(F)$ are equivalent, we conclude that there exist positive constants D, R such that

$$\sup\{|\alpha(a)| : \alpha \in R_U\} \leq D^{-1}\|u\|_G^{-R}\|g\|_G^{-S} \Longrightarrow aua^{-1} \in gKg^{-1}, \tag{19.4.6}$$

from which the conclusion of the lemma follows immediately. □

20. Estimates for weighted orbital integrals

In this section we work with a maximal torus T in a connected reductive group G over our p-adic field F. As usual $D(X)$ is the polynomial function on $\mathfrak{g}$ (see 7.5) that turns up in the Weyl integration formula. We are going to prove estimates for weighted orbital integrals. This will use the theory of norms on affine varieties that was developed in section 18.

20.1. Local integrability of various functions on $\mathfrak{t}$. We are interested in the local integrability of various functions on $\mathfrak{t}$ that involve the function $X \mapsto |D(X)|$.

LEMMA 20.1 ([**HC70**, Lemma 44]). *There exists $\epsilon > 0$ such that the function $|D(X)|^{-\epsilon}$ is locally integrable on $\mathfrak{t}$.*

PROOF. The polynomial D is homogeneous of degree d, where $d = d_G$ is the number of roots of T in $\mathfrak{g}$. We claim that $|D(X)|^{-\epsilon}$ is locally integrable on $\mathfrak{t}$ provided that $d\epsilon < 1$. We prove this statement by induction on d, the case $d = 0$ being trivial.

Now we assume that $d > 0$ and that the statement we are trying to prove is true for all smaller d. There is an immediate reduction to the case in which G is semisimple. Let S be any non-zero element in $\mathfrak{t}$, and let H denote its centralizer in G. Since the functions $|D^G|$ and $|D^H|$ are positive multiples of each other in some small neighborhood of S in $\mathfrak{t}$, we conclude by our induction hypothesis (using that $d_H < d_G$ and hence that $d_H\epsilon < 1$) that $|D(X)|^{-\epsilon}$ is locally integrable on this neighborhood of S. Since this is true for all non-zero S, we conclude that $|D(X)|^{-\epsilon}$ is locally integrable on $\mathfrak{t} \setminus \{0\}$.

It remains to show that $|D(X)|^{-\epsilon}$ is integrable on some open neighborhood of 0. For convenience we take this neighborhood to be a lattice L. It is enough to show that

$$\int_L |D(X)|^{-\epsilon}\, dX \tag{20.1.1}$$

is finite. Since D is homogeneous of degree d, this integral is equal to the product of

$$\int_{L\setminus \pi L} |D(X)|^{-\epsilon}\, dX \tag{20.1.2}$$

and the geometric series with ratio $|\pi|^{\dim(\mathfrak{t})-d\epsilon}$. The geometric series is convergent by our assumption that $d\epsilon < 1$, and the integral (20.1.2) is convergent since $|D(X)|^{-\epsilon}$ is locally integrable away from the origin and hence integrable on the complement of πL in L.

It is worth noting that this result of Harish-Chandra can also be derived from rather general results of Igusa [**Igu74, Igu77**] and Denef [**Den84**] on integrals of complex powers of absolute values of p-adic polynomials. $\square$

The next result involves the function

$$X \mapsto \log(\max\{1, |D(X)|^{-1}\})$$

on $\mathfrak{g}_{\mathrm{rs}}$. This function takes non-negative real values and measures how close X is to the singular set $\mathfrak{g} \setminus \mathfrak{g}_{\mathrm{rs}}$: the larger the function value at X, the closer X is to the singular set.

Corollary 20.2. *For every non-negative real number R the function*

$$X \mapsto \big(\log(\max\{1, |D(X)|^{-1}\}\big)^R$$

is locally integrable on $\mathfrak{t}$.

Proof. This follows from the previous result together with the following elementary fact: for every $\epsilon > 0$ and every $R \geq 0$ there exists a positive constant C such that

$$\big(\log(\max\{1, y\})\big)^R \leq C y^\epsilon \tag{20.1.3}$$

for all $y \geq 0$. $\square$

20.2. Estimates for orbital integrals with various weight factors. Now suppose that M is a Levi subgroup of G containing T. Then we have $(M\backslash G)(F) = M(F)\backslash G(F)$ and $(A_M\backslash G)(F) = A_M(F)\backslash G(F)$, so no confusion will result from writing $M\backslash G$ for the set of F-points of the affine algebraic variety $M\backslash G$, and similarly for $A_M\backslash G$. Let $\|\cdot\|_{M\backslash G}$ and $\|\cdot\|_{A_M\backslash G}$ be any norms (as in 18.2) on $M\backslash G$ and $A_M\backslash G$ respectively.

We are also interested in the affine algebraic variety $T\backslash G$, but here we need to be more careful, since $(T\backslash G)(F)$ can be bigger than $T(F)\backslash G(F)$. We let $\|\cdot\|_{T\backslash G}$ be any norm on $(T\backslash G)(F)$. Having warned the reader of the potential for confusion, we nevertheless now write $T\backslash G$ as a convenient abbreviation for $T(F)\backslash G(F)$, an open and closed subset of $(T\backslash G)(F)$. We will only have occasion to use $\|\cdot\|_{T\backslash G}$ on the subset $T\backslash G$.

Before formulating the next results, let's discuss where we're headed. Let $X \in \mathfrak{t}_{\mathrm{reg}}$ and consider the weighted orbital integral

$$\int_{T\backslash G} f(g^{-1}Xg) v_M(g)\, d\bar{g}.$$

Since X is semisimple, its orbit is closed and hence intersects the support of f in a compact subset of the orbit. Thus there is a compact subset C of $T\backslash G$ such that the integrand vanishes unless $g \in C$. The weight factor is left M-invariant, hence left T-invariant, hence remains bounded in absolute value on the compact set C, say by the positive number R. Then the absolute value of the weighted orbital integral is bounded above by R times the orbital integral of $|f|$. Now suppose that we want to estimate the weighted orbital integral for fixed f and variable X. Then we will need to control the size of the compact set C, which of course depends on X. The point is that C grows as X gets closer to the singular set. The theory of norms developed in section 18 makes it easy to get such control, as we will now see.

When we apply the next lemma, the compact set ω will be the support of f, so this is what the reader should have in mind. The lemma is a variant of [**HC70**, Theorem 18] (see also [**Art91a**, Lemma 4.2]).

LEMMA 20.3. *Let ω be a compact subset of $\mathfrak{g}$. Then there exist positive constants c_1, c_2, c_1', c_2' having the following property. For all $X \in \mathfrak{t}_{\mathrm{reg}}$ and all $g \in G$ such that $g^{-1}Xg \in \omega$ there are inequalities*

$$\log \|g\|_{T\backslash G} \le c_1 + c_2 \log \max\{1, |D(X)|^{-1}\}$$

and

$$\log \|g\|_{M\backslash G} \le c_1' + c_2' \log \max\{1, |D(X)|^{-1}\}.$$

PROOF. We again consider the morphism

$$\beta : (T\backslash G) \times \mathfrak{t}_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{rs}}$$

of affine algebraic varieties defined by $\beta(g, X) = g^{-1}Xg$. Choose a norm $\|\cdot\|_{\mathfrak{g}}$ on the affine variety $\mathfrak{g}$. Now $\mathfrak{g}_{\mathrm{rs}}$ is the principal Zariski open subset of $\mathfrak{g}$ defined by the non-vanishing of the polynomial D, so (see Proposition 18.1) as norm $\|\cdot\|_{\mathfrak{g}_{\mathrm{rs}}}$ on $\mathfrak{g}_{\mathrm{rs}}$ we may take

$$\|X\|_{\mathfrak{g}_{\mathrm{rs}}} := \max\{\|X\|_{\mathfrak{g}}, |D(X)|^{-1}\}.$$

Since the morphism β is finite, we may take (see Proposition 18.1) as norm on $(T\backslash G) \times \mathfrak{t}_{\mathrm{reg}}$ the pullback of $\|\cdot\|_{\mathfrak{g}_{\mathrm{rs}}}$ by β. Again by Proposition 18.1 the pullback of the norm $\|\cdot\|_{T\backslash G}$ to $(T\backslash G) \times \mathfrak{t}_{\mathrm{reg}}$ (pull back using the first projection) is dominated by the norm on $(T\backslash G) \times \mathfrak{t}_{\mathrm{reg}}$. Writing out what this means, we see that there are constants $c > 1$ and $R > 0$ such that

$$\|g\|_{T\backslash G} \le c \max\{\|g^{-1}Xg\|_{\mathfrak{g}}, |D(X)|^{-1}\}^R$$

for all $g \in T\backslash G$ and all $X \in \mathfrak{t}_{\mathrm{reg}}$. Now $\|\cdot\|_{\mathfrak{g}}$ remains bounded on the compact set ω; let's choose $d > 1$ that serves as an upper bound. Thus

$$\|g\|_{T\backslash G} \le cd^R \max\{1, |D(X)|^{-1}\}^R \tag{20.2.1}$$

for all $g \in T\backslash G$, $X \in \mathfrak{t}_{\mathrm{reg}}$ such that $g^{-1}Xg \in \omega$. Taking the logarithm of both sides of (20.2.1), we get the first inequality of the lemma. The second inequality can be derived from the first since (again by Proposition 18.1) the pullback of the norm $\|\cdot\|_{M\backslash G}$ to $T\backslash G$ is dominated by $\|\cdot\|_{T\backslash G}$. □

Now we use the lemma to estimate weighted orbital integrals. Actually the proof of the local trace formula involves estimating orbital integrals weighted by various factors other than v_M, but having the same rough growth rate as v_M. Our next result will involve the weight factor $(\log \|\cdot\|_{T\backslash G})^R$, as this will allow us to handle all the weight factors that come up in the proof of the local trace formula.

PROPOSITION 20.1. *Let $f \in C_c^\infty(\mathfrak{g})$ and let R be a non-negative integer. Then the integral*

$$\int_{\mathfrak{t}} |D(X)|^{1/2} \int_{T\backslash G} f(g^{-1}Xg)(\log \|g\|_{T\backslash G})^R \, d\bar{g} \, dX \tag{20.2.2}$$

converges. If T is elliptic in M, then the integral

$$\int_{\mathfrak{t}} |D(X)|^{1/2} \int_{A_M\backslash G} f(g^{-1}Xg)(\log \|g\|_{A_M\backslash G})^R \, d\dot{g}\, dX \tag{20.2.3}$$

converges.

PROOF. The first statement follows from Lemma 20.3 (with $\omega = \mathrm{Supp}(f)$), Corollary 20.2, and the fact that the function $X \mapsto |D(X)|^{1/2} \int_{T\backslash G} |f(g^{-1}Xg)|\, d\bar{g}$ is bounded and compactly supported on $\mathfrak{t}$ (see Theorem 17.10). The second statement follows from the first, together with Corollary 18.10. □

In order to see that the proposition above can be applied to weighted orbital integrals, we need to estimate $v_M(g)$. Before doing so, we discuss the metric on the building of G.

20.3. Metric on X, function $d(x)$ on X, estimate for $\|H_B(g)\|_E$. We now assume that G is split, with split maximal torus A. As in 18.11 we choose a Weyl group invariant Euclidean norm $\|\cdot\|_E$ on $\mathfrak{a}$. From $\|\cdot\|_E$ we get a metric on $\mathfrak{a}$. Viewing $\mathfrak{a}$ as the standard apartment in the building of G, the metric on $\mathfrak{a}$ extends uniquely to a G-invariant metric on the building, denoted by $d(x_1, x_2)$.

As usual we put $X = G/K$ and denote by x_0 the base-point of X. We view X as a subset of the building, so it makes sense to consider the metric $d(x_1, x_2)$ for $x_1, x_2 \in X$. For $x \in X$ we introduce

$$d(x) := d(x, x_0)$$

as a measure of the size of x, and for $g \in G$ we also put

$$d(g) := d(gx_0).$$

The next lemma concerns the maps $H_B : G \to X_*(A) \hookrightarrow \mathfrak{a}$ defined in 12.1.

LEMMA 20.4. *Let $B \in \mathcal{B}(A)$. For all $g \in G$ there is an inequality*

$$\|H_B(g)\|_E \le d(g). \tag{20.3.1}$$

PROOF. It follows from [**BT72**, Section 4.4.4] that if $g \in K\pi^\nu K$ for $\nu \in X_*(A)$, then $H_B(g)$ lies in the convex hull of the Weyl group orbit of ν. Therefore

$$\|H_B(g)\|_E \le \|\nu\|_E, \tag{20.3.2}$$

and it is clear from the definition of the function d that $d(g) = \|\nu\|_E$. □

20.4. Estimate for v_M. Since we have only discussed v_M in the split case, we continue to assume that G is split. Then we have the norm $\|\cdot\|_G$ on $G(F)$ defined in 18.11 using the Euclidean norm $\|\cdot\|_E$ on $\mathfrak{a}$; in terms of the function $d(g)$ introduced above, we have

$$\|g\|_G = \exp(d(g)).$$

By Proposition 18.2 the morphism $G \to M\backslash G$ satisfies the norm descent property. Therefore

$$\|g\|_{M\backslash G} := \inf\{\|mg\|_G : m \in M\} \tag{20.4.1}$$

is a norm on $M\backslash G$. We use this particular norm in the next lemma.

LEMMA 20.5. *Let M be a Levi subgroup of G containing A. Then there exists a positive constant c such that*

$$v_M(g) \leq c(\log \|g\|_{M\backslash G})^{\dim(A_M/A_G)} \tag{20.4.2}$$

for all $g \in G$.

PROOF. Let $P \in \mathcal{P}(M)$. Choose $B \in \mathcal{B}(A)$ such that $B \subset P$. By Lemma 20.4 we have

$$\|H_B(g)\|_E \leq \log \|g\|_G. \tag{20.4.3}$$

As we saw in 12.1, $H_P(g)$ is obtained as the image of $H_B(g)$ under $\mathfrak{a} \twoheadrightarrow \mathfrak{a}_M$. When we view $\mathfrak{a}_M$ as a subspace of $\mathfrak{a}$, the map $\mathfrak{a} \twoheadrightarrow \mathfrak{a}_M$ is given by orthogonal projection. Therefore

$$\|H_P(g)\|_E \leq \log \|g\|_G. \tag{20.4.4}$$

Since $v_M(g)$ is the volume of the convex hull of the points $H_P(g)$ ($P \in \mathcal{P}(M)$), there is a positive constant c such that

$$v_M(g) \leq c(\log \|g\|_G)^{\dim(A_M/A_G)} \tag{20.4.5}$$

for all $g \in G$. But the function $v_M(g)$ is left-invariant under M, so that we may replace $\|g\|_G$ by $\|g\|_{M\backslash G}$ in the last inequality, completing the proof. $\square$

Combining our estimate of v_M with Proposition 20.1, and remembering that the pullback of $\|\cdot\|_{M\backslash G}$ to $T\backslash G$ is dominated by $\|\cdot\|_{T\backslash G}$, we obtain the following result, in which T is any maximal torus in M.

THEOREM 20.6. *Let $f \in C_c^\infty(\mathfrak{g})$. Then the integral*

$$\int_{\mathfrak{t}} |D(X)|^{1/2} \int_{T\backslash G} f(g^{-1}Xg) v_M(g)\, d\bar{g}\, dX \tag{20.4.6}$$

converges.

20.5. Estimate for u_M. We continue to assume that G is split. In the proof of the local trace formula we will also need an estimate for the weight factor u_M appearing in our preliminary form of the local trace formula. Recall that

$$u_M(h, g; \mu) := \int_{A_M} u^\mu(h^{-1}a_M g)\, da_M. \tag{20.5.1}$$

We use the Haar measure da_M on A_M giving $A_M \cap K$ measure 1. Just as in 4.2 we have $A_M/(A_M \cap K) = X_*(A_M)$, and for $a \in A_M$ we denote by ν_a the image of a in $A_M/(A_M \cap K) = X_*(A_M)$. Obviously

$$u_M(h, g; \mu) \leq |\{\nu \in X_*(A_M) : \exists a \in A_M \text{ with } \nu_a = \nu \text{ and } h^{-1}ag \in G^\mu\}|.$$

The previous proof used the left M-invariance of $v_M(g)$. Now the function $(h, g) \mapsto u_M(h, g; \mu)$ is left $(A_M \times A_M)$-invariant, but not $(M \times M)$-invariant, and our first step will be to replace u_M by something larger which is $(M \times M)$-invariant. For this we use the injection $X_*(A_M) \hookrightarrow \Lambda_M$ to see that

$$u_M(h, g; \mu) \leq u'_M(h, g; \mu),$$

with u'_M defined by

$$u'_M(h, g; \mu) := |\{\nu \in \Lambda_M : \exists m \in M \text{ with } H_M(m) = \nu \text{ and } h^{-1}mg \in G^\mu\}|.$$

It is evident that this function of (h, g) is left invariant under $M \times M$.

Put $x = gx_0$ and $y = hx_0$. If $h^{-1}mg \in G^\mu$, then $\mathrm{inv}(mx, y) \le \mu$ and therefore $d(mx, y) \le \|\mu\|_E$, from which it follows (by the triangle inequality) that

$$d(mx_0) \le d(x) + d(y) + \|\mu\|_E, \tag{20.5.2}$$

where d is the function defined in 20.3. Writing ν for $H_M(m)$ and $\bar{\nu}$ for the image of ν under $\Lambda_M \to \mathfrak{a}_M \subset \mathfrak{a}$, one sees easily from the definitions that $\|\bar{\nu}\|_E \le d(mx_0)$. Thus we have shown that

$$u'_M(h, g; \mu) \le |\{\nu \in \Lambda_M : \|\bar{\nu}\|_E \le d(x) + d(y) + \|\mu\|_E\}|, \tag{20.5.3}$$

from which it is clear that there is a positive constant c such that

$$u'_M(h, g; \mu) \le c(1 + d(x) + d(y) + \|\mu\|_E)^{\dim A_M}. \tag{20.5.4}$$

Now recalling that u'_M is invariant under $M \times M$, we see that in the last inequality we may replace $d(x)$ by $\inf\{d(mx) : m \in M\}$, and the same for $d(y)$. Thus we have proved

Lemma 20.7. *Let M be a Levi subgroup of G containing A. Then there exists a positive constant c such that*

$$u_M(h, g; \mu) \le c(1 + \log \|g\|_{M\backslash G} + \log \|h\|_{M\backslash G} + \|\mu\|_E)^{\dim(A_M)} \tag{20.5.5}$$

for all $g, h \in G$.

Of course the exponent $\dim(A_M)$ could easily be improved to $\dim(A_M/A_G)$.

21. Preparation for the key geometric lemma

Now we begin to prepare for the proof of the key geometric result (Theorem 22.3) needed for the local trace formula. Throughout this section we fix a Borel subgroup $B = AN$ containing A, which we use to define positive roots, dominance, and so on. As in 20.3 $\|x\|_E$ is a W-invariant Euclidean norm on $\mathfrak{a}$, which we use to get the metric $d(x, y)$ on the building as well as the function $d(x)$ on X.

21.1. Retractions of the building with respect to an alcove. Given an alcove $\mathbf{a}$ in an apartment in the Bruhat-Tits building of our split group G, there is a retraction $r_{\mathbf{a}}$ of the building into that apartment. As usual, we are mainly interested in the subset $X = G/K$ of the building and the standard apartment (the one coming from A). Inside G/K we have the subset $A/A \cap K$, which we identify with $X_*(A)$ by sending $\mu \in X_*(A)$ to π^μ. Let $\mathbf{a}$ be an alcove in the apartment $\mathfrak{a} = X_*(A) \otimes_{\mathbb{Z}} \mathbb{R}$, and let $I_{\mathbf{a}}$ be the corresponding Iwahori subgroup of G, defined as the pointwise stabilizer of $\mathbf{a}$ in G. From the affine Bruhat decomposition we know that the obvious map $A/A \cap K \to I_{\mathbf{a}}\backslash G/K$ is bijective. We will regard $r_{\mathbf{a}}$ as the retraction of G/K onto $X_*(A)$ defined as follows: given $g \in G/K$ we put $r_{\mathbf{a}}(g) = \mu$ if $g \in I_{\mathbf{a}}\pi^\mu K$. In other words, given a vertex x in the building, $r_{\mathbf{a}}(x)$ is the unique vertex in the standard apartment having the same position relative to $\mathbf{a}$ as x does.

Proposition 21.1. [**BT72**] *The retraction $r_{\mathbf{a}}$ weakly decreases distances. In other words*

$$d(r_{\mathbf{a}}(x_1), r_{\mathbf{a}}(x_2)) \le d(x_1, x_2) \tag{21.1.1}$$

for all $x_1, x_2 \in G/K$.

21.2. An easy fact about root systems. In a moment we will need the following easy result.

LEMMA 21.1. *Let x, y be dominant elements in $\mathfrak{a}$, and let $w \in W$. Then*

$$\|x - y\|_E \leq \|x - wy\|_E. \tag{21.2.1}$$

PROOF. Expanding out the two norms, we find that (21.2.1) is equivalent to the inequality $(x, y - wy) \geq 0$, and this inequality is clear since x is dominant and $y - wy$ is a non-negative linear combination of positive coroots (a standard fact that follows from things we discussed in 12.8 and 12.10). □

21.3. Something like the triangle inequality. Recall that X denotes G/K and x_0 its base-point. Recall also the function inv from 3.4, taking values in $K\backslash G/K$, which by the Cartan decomposition we have identified with the set of dominant coweights in $X_*(A)$. For $x, y \in X$ we can also consider the distance $d(x, y)$, a coarser invariant than $\operatorname{inv}(x, y)$.

LEMMA 21.2. *Let $x, y, x', y' \in X$ and let λ, λ' be the dominant coweights obtained as $\lambda := \operatorname{inv}(x, y)$ and $\lambda' := \operatorname{inv}(x', y')$. Then*

$$\|\lambda - \lambda'\|_E \leq d(x, x') + d(y, y'). \tag{21.3.1}$$

PROOF. The lemma concerns the effect on $\operatorname{inv}(x, y)$ of replacing (x, y) by (x', y'). We can do this replacement in two steps, going from (x, y) to (x, y') to (x', y'). Thus in proving the lemma we may assume that $x = x'$ or $y = y'$, and by symmetry (note that $\operatorname{inv}(y, x) = -\operatorname{inv}(x, y)$) we may as well assume that $y = y'$. We are free to transform all our points by any convenient $g \in G$; doing so, we may assume without loss of generality that x, y both lie in the standard apartment. Inside the standard apartment pick an alcove $\mathbf{a}$ containing y. Then λ, λ' are the unique dominant elements in the Weyl group orbits of $r_{\mathbf{a}}(x)$, $r_{\mathbf{a}}(x')$ respectively, and therefore

$$\|\lambda - \lambda'\|_E \leq d(r_{\mathbf{a}}(x), r_{\mathbf{a}}(x')) \leq d(x, x'), \tag{21.3.2}$$

the first inequality coming from Lemma 21.1, the second from Proposition 21.1. □

The next result is tailor-made for use in the proof of the key geometric result needed for the local trace formula.

COROLLARY 21.3. *Let $x_1, x_2 \in X$. Let ν be a dominant coweight, and let $a \in A$ be an element whose image in $A/A \cap K = X_*(A)$ is ν. Put $\lambda := \operatorname{inv}(ax_2, x_1)$. Then*

$$\|\lambda - \nu\|_E \leq d(x_1) + d(x_2). \tag{21.3.3}$$

PROOF. Since ν is dominant, we have $\operatorname{inv}(ax_0, x_0) = \nu$. Lemma 21.2 then yields

$$\|\lambda - \nu\|_E \leq d(ax_2, ax_0) + d(x_1, x_0) = d(x_2) + d(x_1). \tag{21.3.4}$$

□

22. Key geometric lemma

In this section we are finally going to prove Theorem 22.3, Arthur's key geometric result needed for the local trace formula (see Lemma 4.4 in [**Art91a**]). The reader may wish to skip ahead to the statement of this theorem before trying to digest the lemmas below.

As usual we are working with the split group G over the p-adic field F. Recall that X denotes the set G/K and that x_0 denotes its base-point. As in 20.3, we denote by $d(x,y)$ the metric on the building obtained from some Euclidean norm on $\mathfrak{a}$. For convenience we assume that all roots have length less than or equal to 1 in this Euclidean norm. As in 20.3, for $x \in X$ we put $d(x) := d(x, x_0)$.

22.1. Main steps in the proof. The main steps in the proof of Theorem 22.3 are contained in Lemmas 22.1 and 22.2 below.

Consider a Borel subgroup $B = AN$ and a parabolic subgroup $P = MU$ such that $P \supset B$ and $M \supset A$. Put $B_M := B \cap M$, a Borel subgroup of M. We write Δ for the set of simple roots of A (with respect to B). Then Δ is the disjoint union of Δ_M and Δ_U, where Δ_M is the set of simple roots of A in M, and Δ_U is the set of simple roots of A that occur in $\operatorname{Lie}(U)$. As usual we write R_U for the set of all roots of A that appear in $\operatorname{Lie}(U)$. We write $\bar{P} = M\bar{U}$ for the parabolic subgroup opposite to P.

Recall that we have identified $X_*(A)$ with $A/(A \cap K)$ by sending ν to π^ν. For $a \in A$ we write ν_a for the element of $X_*(A)$ corresponding to the image of a in $A/(A \cap K)$. For $d \geq 0$ we denote by $A(d)$ the set of elements $a \in A$ such that $\langle \alpha, \nu_a \rangle \geq 0$ for all $\alpha \in \Delta_M$ and $\langle \alpha, \nu_a \rangle \geq d$ for all $\alpha \in \Delta_U$. Note that if $a \in A(d)$, then

$$\langle \alpha, \nu_a \rangle \geq d \quad \forall\, \alpha \in R_U, \tag{22.1.1}$$

as follows from the fact that any $\alpha \in R_U$ is a non-negative integral linear combination of roots in Δ with some root in Δ_U occurring with non-zero coefficient.

For $x_1, x_2 \in X$ we have the invariant $\operatorname{inv}(x_1, x_2)$ from 3.4. We now write this invariant as $\operatorname{inv}(x_1, x_2)_B$, to emphasize that within the relevant Weyl group orbit of coweights we are taking the unique one that is dominant with respect to B. We write $\operatorname{inv}(x_1, x_2)_P$ for the image of $\operatorname{inv}(x_1, x_2)$ under the canonical surjection $X_*(A) \to \Lambda_M$ (see 4.5 for a discussion of Λ_M).

Recall also the map $H_P : G \to \Lambda_M$ defined in 12.1. This map descends to a map, also called H_P, from X to Λ_M.

For $x_1, x_2 \in X$ we are going to show that there exists $d \geq 0$ (depending on the points x_1, x_2), such that

$$\operatorname{inv}(ax_2, x_1)_P = \nu_a + H_P(x_2) - H_{\bar{P}}(x_1) \tag{22.1.2}$$

for all $a \in A(d)$. Here we have abused notation slightly by writing ν_a when we really mean its image under the canonical surjection $X_*(A) \to \Lambda_M$. This assertion is the main ingredient in the proof of the key geometric result. However, we need some control on how big d needs to be, and in fact we will show that d grows linearly with $d(x_1)$, $d(x_2)$. More precisely, we have the following lemma.

Lemma 22.1. *There exists $c > 0$ such that for all $x_1, x_2 \in X$*

$$a \in A(d) \Longrightarrow \operatorname{inv}(ax_2, x_1)_P = \nu_a + H_P(x_2) - H_{\bar{P}}(x_1) \tag{22.1.3}$$

so long as $d \geq c(1 + d(x_1) + d(x_2))$.

PROOF. For each $x \in X$ we choose, once and for all, elements $m_x \in M$, $u_x \in U$, $\bar{m}_x \in M$, $\bar{u}_x \in \bar{U}$ such that

$$u_x m_x x_0 = x = \bar{u}_x \bar{m}_x x_0. \tag{22.1.4}$$

The lemma is an easy consequence of the following two statements. These statements involve elements $x_1, x_2 \in X$. To simplify notation we put $\bar{m}_1 := \bar{m}_{x_1}$, $\bar{u}_1 := \bar{u}_{x_1}$, $m_2 := m_{x_2}$, $u_2 := u_{x_2}$, so that

$$x_1 = \bar{u}_1 \bar{m}_1 x_0 \tag{22.1.5}$$

$$x_2 = u_2 m_2 x_0. \tag{22.1.6}$$

Statement 1. There exists $c_1 > 0$ such that for all $x_1, x_2 \in X$

$$a \in A(d) \Longrightarrow \operatorname{inv}(ax_2, x_1)_B = \operatorname{inv}(am_2 x_0, \bar{m}_1 x_0)_B$$

so long as $d \geq c_1(1 + d(x_1) + d(x_2))$.

Statement 2. There exists $c_2 > 0$ such that for all $x_1, x_2 \in X$

$$a \in A(d) \Longrightarrow \operatorname{inv}(am_2 x_0, \bar{m}_1 x_0)_B = \operatorname{inv}^M(am_2 x_0, \bar{m}_1 x_0)_{B_M}$$

so long as $d \geq c_2(1 + d(x_1) + d(x_2))$. Here inv^M is the analog for the group M of inv for G.

First we check that the lemma follows from these two statements. Indeed, take c to be the maximum of c_1 and c_2. Then, so long as $a \in A(d)$ with $d \geq c(1 + d(x_1) + d(x_2))$, we have $\operatorname{inv}(ax_2, x_1)_B = \operatorname{inv}^M(am_2 x_0, \bar{m}_1 x_0)_{B_M}$. Therefore $\operatorname{inv}(ax_2, x_1)_P$ is the image of $\operatorname{inv}^M((am_2 x_0, \bar{m}_1 x_0)_{B_M}$ under $X_*(A) \to \Lambda_M$, namely

$$\begin{aligned} H_M(\bar{m}_1^{-1} a m_2) &= -H_M(\bar{m}_1) + H_M(a) + H_M(m_2) \\ &= -H_{\bar{P}}(x_1) + \nu_a + H_P(x_2). \end{aligned}$$

Next we prove Statement 1. We begin by observing that

$$\operatorname{inv}(ax_2, x_1)_B = \operatorname{inv}(am_2 x_0, \bar{m}_1 x_0)_B \tag{22.1.7}$$

so long as

$$au_2 a^{-1} \text{ fixes } x_1 \tag{22.1.8}$$

and

$$a^{-1} \bar{u}_1 a \text{ fixes } m_2 x_0. \tag{22.1.9}$$

Indeed, the first condition ensures that $\operatorname{inv}(ax_2, x_1)_B = \operatorname{inv}(am_2 x_0, x_1)_B$, while the second ensures that $\operatorname{inv}(am_2 x_0, x_1)_B = \operatorname{inv}(am_2 x_0, \bar{m}_1 x_0)_B$.

If $a \in A(d)$ with $d \gg 0$, then $au_2 a^{-1}$ and $a^{-1}\bar{u}_1 a$ will be very close to the identity element, so (22.1.8) and (22.1.9) will hold. Lemma 19.4 allows us to make this rough statement precise. In that lemma appears a norm $\|\cdot\|_G$ on $G(F)$. It is now convenient to take this norm to be of the special type discussed in 18.11, so that

$$\|g\|_G = \exp d(g) = \exp d(gx_0). \tag{22.1.10}$$

Since P is closed in G, the restriction of $\|\cdot\|_G$ to $P(F)$ is a valid norm on $P(F)$, and the same is true for M and U. Moreover, as a variety P is the product of M and U. Thus both $\|mu\|_G$ and $\sup\{\|m\|_G, \|u\|_G\}$ are valid norms on $P(F)$, and therefore there exist positive constants D_1, R_1 such that

$$\sup\{\|m\|_G, \|u\|_G\} \leq D_1 \|mu\|_G^{R_1} \tag{22.1.11}$$

for all $m \in M$ and $u \in U$.

By Lemma 19.4 there exist positive constants D_2, R_2, S_2 such that (22.1.8) holds so long as

$$\inf\{|\alpha(a)|^{-1} : \alpha \in R_U\} \geq D_2 \|u_2\|_G^{R_2} \|\bar{u}_1 \bar{m}_1\|_G^{S_2}. \tag{22.1.12}$$

Using (22.1.11) and (22.1.12) together, we see that there exist positive constants D, R, S such that (22.1.8) holds so long as

$$\inf\{|\alpha(a)|^{-1} : \alpha \in R_U\} \geq D \|u_2 m_2\|_G^{R} \|\bar{u}_1 \bar{m}_1\|_G^{S}. \tag{22.1.13}$$

The logarithm of the right-hand side of (22.1.13) is

$$\log D + R d(x_2) + S d(x_1).$$

Bearing in mind (22.1.1), we see that there exists $c_3 > 0$ such that (22.1.8) holds for all $a \in A(d)$ so long as

$$d \geq c_3 (1 + d(x_1) + d(x_2)). \tag{22.1.14}$$

A rather similar argument shows that there exists $c_4 > 0$ such that (22.1.9) holds for all $a \in A(d)$ so long as

$$d \geq c_4 (1 + d(x_1) + d(x_2)). \tag{22.1.15}$$

It is now clear that Statement 1 holds for $c_1 = \sup\{c_3, c_4\}$.

Finally, we prove Statement 2. Put

$$\lambda := \operatorname{inv}(a m_2 x_0, \bar{m}_1 x_0)_B \in X_*(A) \tag{22.1.16}$$

$$\lambda_M := \operatorname{inv}^M(a m_2 x_0, \bar{m}_1 x_0)_{B_M} \in X_*(A). \tag{22.1.17}$$

We need to prove that $\lambda = \lambda_M$ when $a \in A(d)$ with d sufficiently large. Note that λ, λ_M lie in the same orbit of the Weyl group W, and that λ is dominant for B. Thus, in order to ensure that $\lambda_M = \lambda$, it is enough to ensure that λ_M is dominant for B. It is automatic that λ_M is dominant for B_M, so we just need to ensure that

$$\langle \alpha, \lambda_M \rangle \geq 0 \quad \forall\, \alpha \in \Delta_U. \tag{22.1.18}$$

Recall that we have used our chosen Euclidean norm on $\mathfrak{a}$ to get a metric $d(x, y)$ on X. Of course this can be done for M as well as G, so that we also get a metric $d_M(x_M, y_M)$ on $X_M := M/M \cap K$. The set X_M can be identified with a subset of X, and the metric on X extends the one on X_M.

Using Lemma 21.2 for the group M, we see that λ_M lies in the closed ball of radius $d(\bar{m}_1 x_0, x_0)$ about $\operatorname{inv}^M(a m_2 x_0, x_0) = \operatorname{inv}^M(m_2 x_0, a^{-1} x_0)$, which in turn lies in the closed ball of radius $d(m_2 x_0, x_0)$ about $\operatorname{inv}^M(x_0, a^{-1} x_0) = \nu_a$. Recall that we are assuming that all roots have norm less than or equal to 1 in the Euclidean norm on $\mathfrak{a}$. Therefore for any $\alpha \in \Delta_U$ we have

$$|\langle \alpha, \lambda_M \rangle - \langle \alpha, \nu_a \rangle| \leq d(\bar{m}_1 x_0, x_0) + d(m_2 x_0, x_0). \tag{22.1.19}$$

Using (22.1.11) (and its analog for $\bar{P}$), we see that there exists $c_2 > 0$ such that

$$|\langle \alpha, \lambda_M \rangle - \langle \alpha, \nu_a \rangle| \leq c_2 \bigl(1 + d(x_1) + d(x_2)\bigr). \tag{22.1.20}$$

Thus (22.1.18) will hold so long as

$$\langle \alpha, \nu_a \rangle \geq c_2 \bigl(1 + d(x_1) + d(x_2)\bigr) \tag{22.1.21}$$

for all $\alpha \in \Delta_U$, proving that Statement 2 holds for the constant c_2 that we have constructed. $\square$

In the next lemma we use the usual partial order $\leq$ on $X_*(A)$ determined by our choice of Borel subgroup B. Thus $x \leq y$ if and only if $y - x$ is a non-negative integral linear combination of simple coroots.

LEMMA 22.2. *There exists $c_B > 0$ having the following property. For any $x_1, x_2 \in X$ and any $\mu \in X_*(A)$ satisfying*

$$\langle \alpha, \mu \rangle \geq c_B\big(1 + d(x_1) + d(x_2)\big) \quad \forall \alpha \in \Delta \tag{22.1.22}$$

the following two statements hold.

(1) *The coweight $\mu - H_B(x_2) + H_{\bar{B}}(x_1)$ is dominant for B.*
(2) *For $a \in A$ such that ν_a is dominant for B the condition* $\mathrm{inv}(ax_2, x_1)_B \leq \mu$ *is equivalent to the condition $\nu_a \leq \mu - H_B(x_2) + H_{\bar{B}}(x_1)$.*

PROOF. We are going to show that we can take c_B to be $1 + c$, where c is the positive constant appearing in the statement of Lemma 22.1, but chosen large enough to work for all parabolic subgroups P containing B. Consider x_1, x_2, μ satisfying the hypothesis (22.1.22).

It follows from easily from Lemma 20.4 that the first conclusion of the lemma holds. It remains to verify the second conclusion, so now consider an element $a \in A$ such that ν_a is dominant for B. To simplify notation we put $\lambda := \mathrm{inv}(ax_2, x_1)_B$ and we abbreviate $d(x_1)$, $d(x_2)$ to d_1, d_2 respectively.

The parabolic subgroups P containing B are in one-to-one correspondence with subsets of Δ (by making $P = MU$ correspond to the subset Δ_M). Now take $P = MU$ to be the unique parabolic subgroup containing B for which

$$\Delta_M = \{\alpha \in \Delta : \langle \alpha, \nu_a \rangle \leq c(1 + d_1 + d_2)\}, \tag{22.1.23}$$

with c as chosen above. By Lemma 22.1 we then have the equality

$$\lambda = \nu_a + H_P(x_2) - H_{\bar{P}}(x_1) \tag{22.1.24}$$

in Λ_M (with λ and ν_a being regarded as elements in Λ_M via the canonical surjection $X_*(A) \to \Lambda_M$), or, equivalently,

$$\lambda \text{ and } \nu_a + H_B(x_2) - H_{\bar{B}} \text{ have the same image in } \Lambda_G \tag{22.1.25}$$

and

$$\langle \varpi_\alpha, \lambda \rangle = \langle \varpi_\alpha, \nu_a + H_B(x_2) - H_{\bar{B}}(x_1) \rangle \quad \forall\, \alpha \in \Delta_U. \tag{22.1.26}$$

Here ϖ_α is the fundamental weight corresponding to α (see the discussion preceding Lemma 11.2).

We are going to apply Lemma 11.2, and in order to do so we first need to verify two inequalities:

$$\langle \alpha, \lambda \rangle \leq \langle \alpha, \mu \rangle \quad \forall\, \alpha \in \Delta_M \tag{22.1.27}$$

$$\langle \alpha, \nu_a + H_B(x_2) - H_{\bar{B}}(x_1) \rangle \leq \langle \alpha, \mu \rangle \quad \forall\, \alpha \in \Delta_M. \tag{22.1.28}$$

In view of our hypothesis on μ, it is enough to check that the left sides of both these inequalities are less than or equal to $c_B(1 + d_1 + d_2)$. For the first inequality this follows from Corollary 21.3 and the definition of Δ_M, and for the second inequality it follows from Lemma 20.4 and the definition of Δ_M.

Now using Lemma 11.2 together with (22.1.25), (22.1.26), (22.1.27), (22.1.28), we see that $\lambda \leq \mu$ if and only if $\nu_a + H_B(x_2) - H_{\bar{B}}(x_1) \leq \mu$, which finishes the proof of the lemma. $\square$

22.2. Key geometric result. In this section we no longer fix the Borel subgroup B, as we need to consider all $B \in \mathcal{B}(A)$ at once. However, it will be convenient to pick one such Borel subgroup and call it B_0. We use B_0 to define dominance and the partial order $\leq$ on coweights.

In Lemma 22.2 there appears a positive constant c_B. We now put $c := \sup\{c_B : B \in \mathcal{B}(A)\}$ in order to get a constant that works for all B at once. This is the positive constant c appearing in our next result, the key geometric result needed for the local trace formula, which, in view of its importance, we give the status of a theorem.

THEOREM 22.3 (Arthur [**Art91a**]). *Let $x_1, x_2 \in X$ and let μ be a dominant coweight satisfying the inequality*

$$\langle \alpha, \mu \rangle \geq c\big(1 + d(x_1) + d(x_2)\big) \tag{22.2.1}$$

for every root α of A that is simple for B_0. Then $\mu_B - H_B(x_2) + H_{\bar{B}}(x_1)$ is a special (G, A)-orthogonal set. Here μ_B denotes the unique element in the Weyl group orbit of μ that is dominant with respect to B.

Moreover for any $a \in A$ the inequality $\operatorname{inv}(ax_2, x_1) \leq \mu$ *is satisfied if and only if the following two conditions hold:*

(1) *ν_a lies in the convex hull H of the set $\{\mu_B - H_B(x_2) + H_{\bar{B}}(x_1) : B \in \mathcal{B}(A)\}$.*
(2) *In Λ_G there is an equality $\nu_a = \mu - H_G(x_2) + H_G(x_1)$.*

In the second condition we have written simply ν_a and μ when we really mean their images under the canonical surjection $X_*(A) \twoheadrightarrow \Lambda_G$.

PROOF. We begin by noting that μ_B is a positive orthogonal set (see 12.8). Moreover $H_B(x_2)$ is a positive orthogonal set (see 12.1), and $H_{\bar{B}}(x_1)$ is a negative orthogonal set (see the end of 12.4). So $\mu_B - H_B(x_2) + H_{\bar{B}}(x_1)$ is the difference of the positive orthogonal set μ_B and the negative orthogonal set $H_B(x_2) - H_{\bar{B}}(x_1)$ and in general is neither positive nor negative. However it will be special (hence positive) when μ is big enough. In fact our assumption on μ does guarantee that μ is big enough, since from the first part of Lemma 22.2, we see that for each $B \in \mathcal{B}(A)$ the coweight $\mu_B - H_B(x_2) + H_{\bar{B}}(x_1)$ is dominant for B.

Choose $B \in \mathcal{B}(A)$ such that ν_a is dominant for B. From Lemma 12.2 it follows that ν_a satisfies conditions (1) and (2) in the theorem if and only if

$$\nu_a \underset{B}{\leq} \mu_B - H_B(x_2) + H_{\bar{B}}(x_1), \tag{22.2.2}$$

and by the second part of Lemma 22.2 this happens if and only if

$$\operatorname{inv}(ax_2, x_1)_B \underset{B}{\leq} \mu_B,$$

or, equivalently, if and only if

$$\operatorname{inv}(ax_2, x_1)_{B_0} \underset{B_0}{\leq} \mu_{B_0}.$$

□

23. The weight factors $\tilde{u}_M$ and $\tilde{v}_M$

We are almost ready to prove the local trace formula. Before we can do so we need to introduce some more weight factors and relate them to toric varieties.

23.1. Weight factor $\tilde{u}_A$. Recall the definition of the weight factor u_A occurring in our preliminary form of the local trace formula: $u_A(g_1, g_2; \mu)$ is the measure of the set of $a \in A$ such that

$$\mathrm{inv}(ax_2, x_1) \le \mu, \tag{23.1.1}$$

where $x_1 = g_1 x_0$, $x_2 = g_2 x_0$. We use the Haar measure on A giving measure 1 to $A \cap K$. From the key geometric result (Theorem 22.3) we see that when the dominant coweight μ is big enough relative to g_1, g_2, the (G, A)-orthogonal set

$$B \mapsto \mu_B - H_B(g_2) + H_{\bar{B}}(g_1)$$

is positive and the weight factor $u_A(g_1, g_2; \mu)$ is equal to the number of coweights $\nu \in X_*(A)$ satisfying the following two conditions:

(1) ν lies in the convex hull of the points $\mu_B - H_B(g_2) + H_{\bar{B}}(g_1)$,
(2) in Λ_G the elements ν and $\mu - H_G(g_2) + H_G(g_1)$ are equal.

It is well-known that such counting problems for lattice points in convex polyhedra arise naturally in the theory of toric varieties. Fulton's book [**Ful93**] is an excellent reference for all that we will need about toric varieties.

What torus do we need? We write $\hat{A}$ for the Langlands dual torus of A; it is a complex torus characterized by the property that $X^*(\hat{A}) = X_*(A)$ (and hence $X_*(\hat{A}) = X^*(A)$). We let $\hat{G} \supset \hat{A}$ be a Langlands dual group for G: the roots (respectively, coroots) of $\hat{A}$ in $\hat{G}$ are the coroots (respectively, roots) of A in G. We write $Z(\hat{G})$ for the center of $\hat{G}$; thus the adjoint group of $\hat{G}$ is $\hat{G}/Z(\hat{G})$ with maximal torus $\hat{A}/Z(\hat{G})$.

The toric variety $V = V^G$ we need is a toric variety for the torus $\hat{A}/Z(\hat{G})$. To specify V we must say which fan we are using. We take the Weyl fan in $X_*(\hat{A}/Z(\hat{G})) \otimes_{\mathbb{Z}} \mathbb{R}$. This is the fan determined by the root hyperplanes in this vector space. Thus the cones of maximal dimension in our fan are the closures of the Weyl chambers in $X_*(\hat{A}/Z(\hat{G})) \otimes_{\mathbb{Z}} \mathbb{R}$, and there is one cone in the fan for each $P \in \mathcal{F}(A)$, the set of parabolic subgroups P of G such that $P \supset A$. The toric variety V is projective, and since $\hat{G}/Z(\hat{G})$ is adjoint, it is also non-singular.

The torus $\hat{A}/Z(\hat{G})$ acts on V and hence $\hat{A}$ also acts on V (through $\hat{A} \twoheadrightarrow \hat{A}/Z(\hat{G})$). The $\hat{A}$-orbits in V are in one-to-one correspondence with cones in the Weyl fan, that is, with parabolic subgroups $P \in \mathcal{F}(A)$; we write V_P for the orbit of $\hat{A}$ indexed by P. Each orbit has a natural base point, and in fact

$$V_P = \hat{A}/Z(\hat{M})$$

where M is *the* Levi component of P (that is, the unique Levi component of P that contains A), and $\hat{M}$ is the corresponding Levi subgroup of $\hat{G}$ containing $\hat{A}$ (the one whose roots are the coroots of M). The closure $\overline{V}_P$ of V_P is

$$\bigcup_{Q : Q \subset P} V_Q$$

and is the toric variety V^M associated to the Weyl fan for $(\hat{M}/Z(\hat{M}), \hat{A}/Z(\hat{M}))$.

Let $\mathcal{L}$ be an $\hat{A}$-equivariant line bundle on V. At each $\hat{A}$-fixed point in V the torus $\hat{A}$ acts by a character on the line (in our line bundle) at that fixed point. There is one fixed point for each $B \in \mathcal{B}(A)$ (namely the single point in the orbit $V_B = \hat{A}/\hat{A}$), so for each $B \in \mathcal{B}(A)$ we get a character $x_B \in X^*(\hat{A})$, or, in other words, a cocharacter $x_B \in X_*(A)$.

Since $Z(\hat{G})$ acts trivially on V, there is a single character of $Z(\hat{G})$ by which $Z(\hat{G})$ acts on every line in our line bundle; therefore all the elements $x_B \in X_*(A)$ have the same image in the quotient Λ_G of $X_*(A)$. (Note that Λ_G can be identified with $X^*(Z(\hat{G}))$. But much more is true. For any $P \in \mathcal{F}(A)$ the restriction of $\mathcal{L}$ to $\overline{V}_P = V^M$ is an equivariant line bundle on the toric variety V^M for M; therefore, applying what has already been said to M rather than G, we see that the points x_B for all $B \in \mathcal{B}(A)$ such that $B \subset P$ have the same image in Λ_M; thus (x_B) is a (G, A)-orthogonal set in $X_*(A)$. In fact $\mathcal{L} \mapsto (x_B)$ is an isomorphism from the group of isomorphism classes of $\hat{A}$-equivariant line bundles on V to the group of (G, A)-orthogonal sets in $X_*(A)$. Restriction of equivariant line bundles from V to $\overline{V}_P = V^M$ corresponds to sending the orthogonal set $(x_B)_{B\in\mathcal{B}(A)}$ to the (M, A)-orthogonal set $(x_B)_{B\in\mathcal{B}(A):B\subset P}$, the operation on orthogonal sets discussed in 12.2.

If the orthogonal set is positive, all the higher cohomology groups $H^i(V, \mathcal{L})$ $(i > 0)$ vanish, and as an $\hat{A}$-module $H^0(V, \mathcal{L})$ is multiplicity free and contains the character $x \in X^*(\hat{A}) = X_*(A)$ if and only if the following two conditions hold:

(1) x lies in the convex hull of $\{x_B : B \in \mathcal{B}(A)\}$,

(2) the image of x in Λ_G coincides with the common image of the points x_B.

For any line bundle $\mathcal{L}$ on V we put

$$EP(\mathcal{L}) := \sum_i (-1)^i \dim H^i(V, \mathcal{L}).$$

For any $g_1, g_2 \in G$ and any dominant coweight μ we let $\mathcal{L}_{(g_1,g_2;\mu)}$ be an equivariant line bundle on V such that the associated (G, A)-orthogonal set in $X_*(A)$ is

$$B \mapsto \mu_B - H_B(g_2) + H_{\bar{B}}(g_1),$$

and we put

$$\tilde{u}_A(g_1, g_2; \mu) := EP(\mathcal{L}_{(g_1,g_2;\mu)}). \tag{23.1.2}$$

It then follows from Theorem 22.3 that when μ is big compared to g_1, g_2, the weight factors $\tilde{u}_A(g_1, g_2; \mu)$ and $u_A(g_1, g_2; \mu)$ coincide. In our next version of the local trace formula (see 24.3), the weight factor u_A will be replaced by the more pleasant weight factor $\tilde{u}_A$. Before we can carry this out, we need to introduce modified weight factors $\tilde{u}_M$ for all Levi subgroups M containing A.

23.2. Weight factor $\tilde{u}_M$. We have treated the case $M = A$. The general case is similar, though slightly more complicated, as we will now see. The toric variety $Y_M = Y_M^G$ we need is a non-singular projective toric variety for the torus $Z(\hat{M})/Z(\hat{G})$. Note that the quotient $Z(\hat{M})/Z(\hat{G})$ really is a torus, although in general $Z(\hat{G})$ and $Z(\hat{M})$ are only diagonalizable groups. Since $Z(\hat{M})/Z(\hat{G})$ is the center of the Levi subgroup $\hat{M}/Z(\hat{G})$ in the adjoint group $\hat{G}/Z(\hat{G})$, we may as well temporarily simplify notation by assuming that $\hat{G}$ is adjoint (or, equivalently, that G is semisimple and simply connected).

Since we are now assuming that $\hat{G}$ is adjoint, the group $Z(\hat{M})$ is a torus, and in fact is a subtorus of $\hat{A}$, so that $X_*(Z(\hat{M}))$ is a subgroup of $X_*(\hat{A})$. Inside the real vector space $X_*(\hat{A})_{\mathbb{R}}$ obtained from $X_*(\hat{A})$ by tensoring over $\mathbb{Z}$ with $\mathbb{R}$ we have the Weyl fan, and the subspace $X_*(Z(\hat{M}))_{\mathbb{R}}$ of $X_*(\hat{A})_{\mathbb{R}}$ is a union of cones in the Weyl fan; thus, the collection of cones in the Weyl fan that happen to lie inside the subspace $X_*(Z(\hat{M}))_{\mathbb{R}}$ gives us a fan in $X_*(Z(\hat{M}))_{\mathbb{R}}$, hence a toric variety $Y_M = Y_M^G$ for $Z(\hat{M})$, which is obviously complete and easily seen to be non-singular (again because $\hat{G}$ is adjoint) and projective.

The index set for the cones in this fan in $X_*(Z(\hat{M}))_{\mathbb{R}}$ is $\mathcal{F}(M)$, the set of parabolic subgroups Q of G such that $Q \supset M$. Thus the decomposition of Y_M as a union of $Z(\hat{M})$-orbits is given by

$$Y_M = \bigcup_{Q \in \mathcal{F}(M)} Z(\hat{M})/Z(\hat{L}_Q),$$

where L_Q denotes the unique Levi component of Q that contains M.

We need to understand how Y_M is related to V. Since the fan used to produce Y_M can also be viewed as a fan in $X_*(\hat{A})_{\mathbb{R}}$ whose support is the subspace $X_*(Z(\hat{M}))_{\mathbb{R}}$, it also produces a toric variety U_M for $\hat{A}$, sitting inside V as an $\hat{A}$-stable open subvariety. The decomposition of U_M as a union of $\hat{A}$-orbits is

$$U_M = \bigcup_{Q \in \mathcal{F}(M)} V_Q = \bigcup_{Q \in \mathcal{F}(M)} \hat{A}/Z(\hat{L}_Q).$$

Moreover Y_M sits inside V as a closed $Z(\hat{M})$-stable subspace, and the multiplication map $\hat{A} \times Y_M \to V$ has image U_M and induces an isomorphism

$$\hat{A} \underset{Z(\hat{M})}{\times} Y_M \simeq U_M. \tag{23.2.1}$$

(The space on the left side of the identification is the quotient of $\hat{A} \times Y_M$ by the equivalence relation $(az, y) \sim (a, zy)$.)

Consider any $Q = LU \in \mathcal{F}(A)$ (with L chosen so that $L \supset A$, as usual). If $Q \notin \mathcal{F}(M)$, then $\overline{V}_Q$ does not meet Y_M. On the other hand, if $Q \in \mathcal{F}(M)$, so that $L \supset M$, then

$$U_M \cap \overline{V}_Q = \bigcup_{\{Q' \in \mathcal{F}(M) : Q' \subset Q\}} V_{Q'},$$

and since $\{Q' \in \mathcal{F}(M) : Q' \subset Q\}$ can be identified with $\mathcal{F}^L(M)$, the set of parabolic subgroups of L containing M, we see that

$$U_M \cap \overline{V}_Q = \hat{A} \underset{Z(\hat{M})}{\times} Y_M^L.$$

From these considerations we obtain the following result.

LEMMA 23.1. *Let M be a Levi subgroup of G containing A and let $Q = LU$ be a parabolic subgroup of G whose Levi component L contains A. Recall that $\overline{V}_Q$ can be identified with the toric variety V^L. If $Q \notin \mathcal{F}(M)$, then $\overline{V}_Q$ does not meet Y_M. Otherwise L contains M, and the non-singular closed subvarieties $\overline{V}_Q = V^L$ and Y_M of V intersect transversely, their intersection being the non-singular closed subvariety Y_M^L in V^L.*

We have now completed our discussion of Y_M in the case $\hat{G}$ is adjoint, so we return to the case of a general split group G and Levi subgroup $M \in \mathcal{L}(A)$. We write G_0 for the simply connected cover of the derived group of G, and M_0 for the Levi subgroup in G_0 obtained as the inverse image of M under $G_0 \to G$; thus $\hat{G}_0 = \hat{G}/Z(\hat{G})$, $\hat{M}_0 = \hat{M}/Z(\hat{G})$, and $Z(\hat{M}_0) = Z(\hat{M})/Z(\hat{G})$.

We have already defined the toric variety $Y_{M_0}^{G_0}$ for $Z(\hat{M}_0) = Z(\hat{M})/Z(\hat{G})$. Using the canonical surjection $Z(\hat{M}) \twoheadrightarrow Z(\hat{M}_0)$, we now view $Y_{M_0}^{G_0}$ as a space on which $Z(\hat{M})$ acts, and rename it Y_M^G. Thus, as a space, Y_M^G depends only on G_0, but $Z(\hat{M})$ and its action on Y_M^G reflect G and M.

23.3. Equivariant line bundles on Y_M. The decomposition of Y_M into $Z(\hat{M})$-orbits

$$Y_M = \bigcup_{Q \in \mathcal{F}(M)} Z(\hat{M})/Z(\hat{L}_Q),$$

proved in the case $\hat{G}$ is adjoint, obviously remains valid in the general case, since

$$Z(\hat{M})/Z(\hat{L}_Q) = (Z(\hat{M})/Z(\hat{G}))/(Z(\hat{L}_Q)/Z(\hat{G})).$$

Thus the fixed points of $Z(\hat{M})$ in Y_M are indexed by $\mathcal{P}(M)$. Given $P = MU \in \mathcal{P}(M)$, the fixed point in Y_M indexed by P is the unique point in the 1-element set $\overline{V}_P \cap Y_M$. The character group of the diagonalizable group $Z(\hat{M})$ is Λ_M, as we noted before. A $Z(\hat{M})$-equivariant line bundle $\mathcal{M}$ on Y_M gives us a (G, M)-orthogonal set of points $y_P \in \Lambda_M$, with y_P defined as the character through which $Z(\hat{M})$ acts on the line (in $\mathcal{M}$) at the fixed point indexed by P, and in this way we get an isomorphism from the group of isomorphism classes of $Z(\hat{M})$-equivariant line bundles on Y_M to the group of (G, M)-orthogonal sets in Λ_M. Note that all the points y_P (with P ranging through $\mathcal{P}(M)$) have the same image in Λ_G.

23.4. Restriction of equivariant line bundles from V to Y_M. Now suppose that $\mathcal{L}$ is an $\hat{A}$-equivariant line bundle on V. From $\mathcal{L}$ we obtain a (G, A)-orthogonal set of points $x_B \in X_*(A)$. The restriction $\mathcal{L}|_{Y_M}$ of $\mathcal{L}$ to the subspace Y_M is a $Z(\hat{M})$-equivariant line bundle on Y_M, hence yields a (G, M)-orthogonal set of points y_P in Λ_M. Now y_P is the character on which $Z(\hat{M})$ acts on the line in $\mathcal{L}$ at the unique point in $\overline{V}_P \cap Y_M$, but since $Z(\hat{M})$ acts trivially on $\overline{V}_P = V^M$, it acts by a single character on the lines in $\mathcal{L}$ at all points in $\overline{V}_P$ (as we have already discussed in 23.1), showing that y_P is the common image x_P of the points x_B ($B \in \mathcal{B}(A)$ such that $B \subset P$) under $X_*(A) \twoheadrightarrow \Lambda_M$. In other words $(y_P) = (x_P)$, where (x_P) is the (G, M)-orthogonal set in Λ_M obtained from the (G, A)-orthogonal set (x_B) by the procedure in 12.5.

23.5. Euler characteristics of $Z(\hat{M})$-equivariant line bundles on Y_M. Let $\mathcal{M}$ be a $Z(\hat{M})$-equivariant line bundle on Y_M with associated (G, M)-orthogonal set $(y_P)_{P \in \mathcal{P}(M)}$ in Λ_M. We put

$$EP(\mathcal{M}) := \sum_i (-1)^i \dim H^i(Y_M, \mathcal{M}). \tag{23.5.1}$$

More generally, for $s \in Z(\hat{M})$ we put

$$EP(s, \mathcal{M}) := \sum_i (-1)^i \operatorname{tr}\bigl(s; H^i(Y_M, \mathcal{M})\bigr), \tag{23.5.2}$$

so that we recover $EP(\mathcal{M})$ when $s = 1$.

If the (G, M)-orthogonal set (y_P) associated to $\mathcal{M}$ is positive, the higher cohomology groups $H^i(Y_M, \mathcal{M})$ $(i > 0)$ vanish, and the representation of $Z(\hat{M})$ on $H^0(Y_M, \mathcal{M})$ is multiplicity free, with the character $y \in X^*(Z(\hat{M})) = \Lambda_M$ appearing in $H^0(Y_M, \mathcal{M})$ if and only if

(1) The image $\bar{y}$ of y under $\Lambda_M \to \mathfrak{a}_M$ lies in the convex hull of the points $\bar{y}_P$ obtained as images under $\Lambda_M \to \mathfrak{a}_M$ of the points y_P in our orthogonal set.
(2) The image of y in Λ_G coincides with the common image in Λ_G of the points y_P.

Thus, when (y_P) is positive, the number of points $y \in \Lambda_M$ satisfying the two conditions above is equal to $EP(\mathcal{M})$.

However the weight factor $u_M(g_1, g_2; \mu)$ (for large μ) involves counting points in $X_*(A_M)$ rather than Λ_M, a circumstance which must be be taken into account. Recall the canonical injective homomorphism $X_*(A_M) \hookrightarrow \Lambda_M$, which we use to identify $X_*(A_M)$ with a subgroup of finite index in Λ_M. We write $\mathcal{Z}_M$ for the Pontryagin dual group

$$\mathcal{Z}_M := \mathrm{Hom}(\Lambda_M/X_*(A_M), \mathbb{C}^\times) \tag{23.5.3}$$

of $\Lambda_M/X_*(A_M)$. It is easy to see that $\mathcal{Z}_M$ can be identified with the subgroup of $Z(\hat{M}) = \mathrm{Hom}(\Lambda_M, \mathbb{C}^\times)$ obtained as the center of the derived group of $\hat{M}$.

When (y_P) is positive, Fourier analysis on the finite abelian group $\mathcal{Z}_M$ shows that the number of points $y \in X_*(A_M)$ satisfying conditions (1) and (2) above is equal to

$$|\mathcal{Z}_M|^{-1} \sum_{s \in \mathcal{Z}_M} EP(s, \mathcal{M}). \tag{23.5.4}$$

23.6. Definition of the weight factors $\tilde{u}_M$ and $\tilde{v}_M$. Let $g_1, g_2 \in G$ and let μ be a dominant coweight. As in 23.1 we get an $\hat{A}$-equivariant line bundle $\mathcal{L}_{(g_1,g_2;\mu)}$ on V, which we restrict to the subspace Y_M, obtaining $\mathcal{L}_{(g_1,g_2;\mu)}|_{Y_M}$.

Define the weight factors $\tilde{u}_M$ and $\tilde{v}_M$ by

$$\tilde{u}_M(g_1, g_2; \mu) := |\mathcal{Z}_M|^{-1} \sum_{s \in \mathcal{Z}_M} EP(s, \mathcal{L}_{(g_1,g_2;\mu)}|_{Y_M}), \tag{23.6.1}$$

$$\tilde{v}_M(g_1, g_2; \mu) := |\mathcal{Z}_M|^{-1} EP(\mathcal{L}_{(g_1,g_2;\mu)}|_{Y_M}). \tag{23.6.2}$$

23.7. Agreement of u_M and $\tilde{u}_M$ when μ is big. In this subsection we will check that

$$u_M(g_1, g_2; \mu) = \tilde{u}_M(g_1, g_2; \mu)$$

when μ is big enough relative to g_1, g_2.

Indeed, let $g_1, g_2 \in G$, let $x_1, x_2 \in X$ be the transforms of the basepoint $x_0 \in X = G/K$ under g_1, g_2 respectively, and let μ be a dominant coweight big enough that the conclusions of Theorem 22.3 hold for x_1, x_2, μ. Put

$$x_B := \mu_B - H_B(x_2) + H_{\bar{B}}(x_1),$$

so that (x_B) is a special (G, A)-orthogonal set in $X_*(A)$, and let $(x_P)_{P \in \mathcal{P}(M)}$ be the (G, M)-orthogonal set in Λ_M obtained from (x_B) as in 23.4 (and 12.5). From (8.4.5) and Theorem 22.3, we see that $u_M(g_1, g_2; \mu)$ is the number of points $x \in X_*(A_M)$ such that

(1) The point x lies in the convex hull of the points $\{x_B : B \in \mathcal{B}(A)\}$.
(2) The image of x in Λ_G coincides with the common image in Λ_G of the points x_B.

On the other hand, we have designed our definitions so that $\tilde{u}_M(g_1, g_2; \mu)$ is equal to the number of points $x \in X_*(A_M)$ such that

(1′) The image $\bar{x}$ of x under $X_*(A_M) \hookrightarrow \Lambda_M \to \mathfrak{a}_M$ lies in the convex hull of the points $\bar{x}_P$ obtained as images under $\Lambda_M \to \mathfrak{a}_M$ of the points x_P.

(2′) The image of x in Λ_G coincides with the common image in Λ_G of the points x_P.

Clearly conditions (2) and (2′) are equivalent. Moreover (1) and (1′) are equivalent by Proposition 12.1. Therefore

$$u_M(g_1, g_2; \mu) = \tilde{u}_M(g_1, g_2; \mu),$$

as desired.

23.8. Qualitative behavior of $EP(s, \mathcal{M})$. The group $E := \mathrm{Pic}_{Z(\hat{M})}(Y_M)$ of isomorphism classes of $Z(\hat{M})$-equivariant line bundles $\mathcal{M}$ on Y_M is a finitely generated abelian group, isomorphic to the group of (G, M)-orthogonal sets $(y_P)_{P\in\mathcal{P}(M)}$ in Λ_M. There is an obvious embedding $\Lambda_M \hookrightarrow E$, obtained by using $y \in \Lambda_M = X^*(Z(\hat{M}))$ to define a $Z(\hat{M})$-equivariant line bundle on a point, and then pulling this back to Y_M; the corresponding orthogonal set is the one for which $y_P = y$ for all $P \in \mathcal{P}(M)$.

The quotient E/Λ_M is isomorphic to $\mathrm{Pic}(Y_M) \simeq H^2(Y_M, \mathbb{Z})$, known to be a free abelian group (whose rank is easy to compute [**Ful93**]). It is also known that there exists (an obviously unique) polynomial F of degree $\dim Y_M = \dim(A_M/A_G)$ on the $\mathbb{Q}$-vector space $E/\Lambda_M \otimes_{\mathbb{Z}} \mathbb{Q} = H^2(Y_M, \mathbb{Q})$ such that

$$EP(\mathcal{M}) = F(\mathcal{M}). \tag{23.8.1}$$

(In the right side of this equality we are using $E \twoheadrightarrow E/\Lambda_M$ to view F as a function on E.)

Slightly more generally, now consider $s \in Z(\hat{G})$. Since $Z(\hat{G})$ acts trivially on Y_M, it acts on all lines in $\mathcal{M}$ by the same character, namely the character $z \in \Lambda_G = X^*(Z(\hat{G}))$ obtained as the common image in Λ_G of the points y_P. Therefore

$$EP(s, \mathcal{M}) = \langle s, z\rangle F(\mathcal{M}) \tag{23.8.2}$$

for the same polynomial F as before.

However, we need to understand the qualitative nature of the function $\mathcal{M} \mapsto EP(s, \mathcal{M})$ for any $s \in Z(\hat{M})$. For this it is convenient to use the localization theorem for equivariant K-theory [**Nie74, BFQ79**] (see also [**Bri88**]), which expresses $EP(s, \mathcal{M})$ in terms of contributions from the various connected components of the fixed point set Y_M^s of s on Y_M. Each of these connected components is a nonsingular projective toric variety for some quotient of $Z(\hat{M})$, and this leads to the conclusion that $EP(s, \mathcal{M})$ can be expressed as a finite sum of the form

$$\sum_{P\in\mathcal{P}(M)} \langle s, y_P\rangle F_{s,P}(\mathcal{M}) \tag{23.8.3}$$

where $F_{s,P}$ is some polynomial function on $(E/\Lambda_M) \otimes_{\mathbb{Z}} \mathbb{Q}$ of degree no bigger than $\dim(A_M/A_G)$. Note (although it causes no trouble) that the polynomials $F_{s,P}(\mathcal{M})$ are not unique (unless s is generic in $Z(\hat{M})$), since the characters $(y_P) \mapsto \langle s, y_P\rangle$ on E (one for each $P \in \mathcal{P}(M)$) need not be distinct.

Consequently (bearing in mind Lemma 20.4), we see that the weight factors $\tilde{u}_M$, $\tilde{v}_M$ both satisfy the same kind of estimate (see 20.5) as u_M, namely, there exists a positive constant c such that

$$\tilde{u}_M(g_1, g_2; \mu) \le c(1 + \log \|g_1\|_{M\backslash G} + \log \|g_2\|_{M\backslash G} + \|\mu\|_E)^{\dim(A_M)} \tag{23.8.4}$$

(and the same for $\tilde{v}_M$).

24. Local trace formula

24.1. The goal. In this section we will prove various versions of the local trace formula for our split group G. All versions will have the following shape. Recall that $\mathcal{L} = \mathcal{L}(A)$ denotes the set of Levi subgroups M of G such that $M \supset A$. For each $M \in \mathcal{L}$ we will have a weight factor $w_M(g_1, g_2)$ on $G \times G$, left invariant under $A_M \times A_M$ (and, in some cases, even under $M \times M$) and right invariant under $K \times K$. Given such a family $w = (w_M)_{M\in\mathcal{L}}$, we can define a distribution T_w on $\mathfrak{g} \times \mathfrak{g}$ by

$$\begin{aligned} T_w(f_1, f_2) = \sum_{M\in\mathcal{L}} \frac{|W_M|}{|W|} \sum_{T\in\mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\ \int_{A_M\backslash G} \int_{A_M\backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg) w_M(h, g)\, d\dot{h}\, d\dot{g}\, dX, \end{aligned} \tag{24.1.1}$$

f_1, f_2 being two functions in $C_c^\infty(\mathfrak{g})$, so long as all these triple integrals make sense. Here the notation is the same as in our second form of the Weyl integration formula (see 7.11).

For each of the weight factors we will consider, the integrals will make sense, and we will have a version of the local trace formula, namely the equality

$$T_w(f_1, f_2) = T_w(\hat{f}_1, \check{f}_2). \tag{24.1.2}$$

24.2. Weight factors u_M. We already know (see 8.4) that for any dominant coweight μ the local trace formula holds for the weight factors $u_M(g_1, g_2, \mu)$. However, it seems that these weight factors are too complicated to be of much use.

24.3. Weight factors $\tilde{u}_M$. Again let μ be a dominant coweight. Let us now check that the local trace formula holds for the weight factors $\tilde{u}_M(g_1, g_2; \mu)$. To this end we need to choose an auxiliary dominant regular coweight μ_1 (so that $\langle\alpha, \mu_1\rangle > 0$ for every simple root α). Replacing μ_1 by $N\mu_1$ for some positive integer N, we may assume that

$$\langle s, \mu_1\rangle = 1 \tag{24.3.1}$$

for all $M \in \mathcal{L}$ and all $s \in \mathcal{Z}_M$.

Fix $f_1, f_2 \in C_c^\infty(\mathfrak{g})$ and define complex valued functions $\varphi(d)$, $\tilde{\varphi}(d)$ on the set of non-negative integers d by the following rules:

$$\varphi(d) := T_w(f_1, f_2) \text{ with } w_M(g, h) = u_M(g, h; \mu + d\mu_1), \tag{24.3.2}$$

$$\tilde{\varphi}(d) := T_w(f_1, f_2) \text{ with } w_M(g, h) = \tilde{u}_M(g, h; \mu + d\mu_1). \tag{24.3.3}$$

In view of the discussion in 23.8 our assumption (24.3.1) on μ_1 guarantees that $\tilde{\varphi}$ is a polynomial function of d.

We claim that $\tilde{\varphi}(d)-\varphi(d) \to 0$ as $d \to +\infty$. Obviously $\tilde{\varphi}(d)-\varphi(d) = T_{w_d}(f_1, f_2)$ for the weight factors w_d defined by

$$(w_d)_M(g, h) := \tilde{u}_M(g, h; \mu + d\mu_1) - u_M(g, h; \mu + d\mu_1),$$

and for fixed g, h we know that this difference is 0 once d is sufficiently big. Therefore the integrands in the integrals defining T_{w_d} approach 0 pointwise, and to conclude that $\tilde{\varphi}(d)-\varphi(d) \to 0$, it is enough to justify the application of Lebesgue's dominated convergence theorem.

For this we need an estimate for $(w_d)_M$ that is independent of d, unlike the estimates we already have for the two terms we took the difference of to get w_d, which of course do depend on d.

However, our first step is to use the estimates we already have (see Lemma 20.7 and the inequality (23.8.4)) for u_M and $\tilde{u}_M$ to conclude that there exists a positive constant c such that

$$|(w_d)_M(g,h)| \leq c(1+\log\|g\|_{M\backslash G}+\log\|h\|_{M\backslash G}+d)^{\dim(A_M)}. \tag{24.3.4}$$

Noting that for any simple root α there is an inequality $\langle\alpha, \mu+d\mu_1\rangle \geq d$, we see from Theorem 22.3 and the discussion in 23.7 that there exists a positive constant c' such that $(w_d)_M(g,h)=0$ unless

$$d \leq c'(1+d(g)+d(h))$$

(with $d(g) := d(gx_0, x_0) = \log\|g\|_G$). Since $(w_d)_M(g,h)$ is left $(A_M \times A_M)$-invariant, it follows that $(w_d)(g,h)=0$ unless

$$d \leq c'(1+d_{A_M\backslash G}(g)+d_{A_M\backslash G}(h)),$$

where

$$d_{A_M\backslash G}(g) := \inf\{d(a_M g) : a_M \in A_M\},$$

Also, since $\log\|g\|_{M\backslash G} = \inf\{d(mg) : m \in M\}$, we trivially have

$$\log\|g\|_{M\backslash G} \leq d_{A_M\backslash G}(g).$$

We conclude that for all $d \geq 0$ there is an inequality

$$|(w_d)_M(g,h)| \leq c\big(1+d_{A_M\backslash G}(g)+d_{A_M\backslash G}(h)+c'(1+d_{A_M\backslash G}(g)+d_{A_M\backslash G}(h))\big)^{\dim A_M},$$

which can be simplified to an inequality of the form

$$|(w_d)_M(g,h)| \leq c''\big(1+d_{A_M\backslash G}(g)+d_{A_M\backslash G}(h)\big)^{\dim A_M}.$$

Consider the right side of this estimate as a family of weight factors w_{est}. The integrals appearing in $T_{w_{est}}(f_1, f_2)$ are all convergent by Proposition 20.1. Therefore we have justified the application of Lebesgue's dominated convergence theorem.

We can summarize what we have done so far as follows: $\tilde{\varphi}(d)$ is a polynomial function of d such that $\tilde{\varphi}(d)-\varphi(d) \to 0$ as $d \to +\infty$. We used f_1, f_2 to define φ, $\tilde{\varphi}$; we now indicate this dependence by writing φ_{f_1,f_2} and $\tilde{\varphi}_{f_1,f_2}$. The local trace formula for the weight factors u_M tells us that $\varphi_{f_1,f_2} = \varphi_{\hat{f}_1,\check{f}_2}$. Therefore $\tilde{\varphi}_{f_1,f_2}(d) - \tilde{\varphi}_{\hat{f}_1,\check{f}_2}(d)$ is a polynomial function of d that approaches 0 as $d \to +\infty$, which obviously implies that it is identically 0, or, in other words, that

$$\tilde{\varphi}_{f_1,f_2}(d) = \tilde{\varphi}_{\hat{f}_1,\check{f}_2}(d)$$

for all $d \geq 0$; taking $d=0$ we conclude that the local trace formula holds for the weight factors $\tilde{u}_M(g_1, g_2; \mu)$, as desired.

24.4. Weight factors $\tilde{v}_M$. Recall that $\tilde{u}_M$ was defined by the equality

$$\tilde{u}_M(g_1, g_2; \mu) := |\mathcal{Z}_M|^{-1} \sum_{s \in \mathcal{Z}_M} EP(s, \mathcal{L}_{(g_1,g_2;\mu)}|_{Y_M}). \tag{24.4.1}$$

It follows from (23.8.3) that the function

$$\mu \mapsto EP(s, \mathcal{L}_{(g_1,g_2;\mu)}|_{Y_M}) \tag{24.4.2}$$

on the monoid of dominant coweights μ has the form

$$\sum_{w \in W} \langle w(s), \mu \rangle F_w \tag{24.4.3}$$

for some collection of polynomial functions F_w of μ. (As usual W denotes the Weyl group of A.) Applying linear independence of characters on the monoid of dominant coweights, we conclude that the local trace formula holds for the weight factors $\tilde{v}_M(g_1, g_2; \mu)$ (since these were obtained from the weight factors $\tilde{u}_M$ by extracting the contribution of the trivial character on the monoid of dominant coweights).

24.5. Weight factors $\tilde{\mathbf{v}}_M$. So far all of our weight factors have been numbers. We now consider weight factors $\tilde{\mathbf{v}}_M$ (closely related to $\tilde{v}_M$) taking values in $K(V)_{\mathbb{C}}$, the complexification of the Grothendieck group $K(V)$ of vector bundles (in the sense of algebraic geometry) on our toric variety $V = V^G$. The Grothendieck groups $K(Y_M)$ will also play an auxiliary role. Since V, Y_M are non-singular projective varieties, we may also view these K-groups as Grothendieck groups of coherent sheaves.

Consider the closed immersion $i_M : Y_M \hookrightarrow V$. Thinking of our K-groups in terms of vector bundles, we have a restriction (pull-back) map

$$i_M^* : K(V) \to K(Y_M), \tag{24.5.1}$$

and thinking of our K-groups in terms of coherent sheaves, we have a push-forward map

$$(i_M)_* : K(Y_M) \to K(V). \tag{24.5.2}$$

We now define our K-theoretic weight factor $\tilde{\mathbf{v}}_M$ as follows:

$$\tilde{\mathbf{v}}_M(g_1, g_2) := |\mathcal{Z}_M|^{-1} (i_M)_* i_M^* [\mathcal{L}_{(g_1,g_2)}] \in K(V)_{\mathbb{C}}, \tag{24.5.3}$$

where $[\mathcal{L}_{(g_1,g_2)}]$ denotes the class in K-theory of the line bundle $\mathcal{L}_{(g_1,g_2)}$ on V obtained by taking $\mathcal{L}_{(g_1,g_2;\mu)}$ for $\mu = 0$.

Pushing forward from V to a point, we get a homomorphism

$$EP : K(V) \to \mathbb{Z}, \tag{24.5.4}$$

whose value on the class of a coherent sheaf $\mathcal{F}$ is

$$\sum_i (-1)^i \dim H^i(V, \mathcal{F}).$$

Our definition of $\tilde{\mathbf{v}}_M$ was designed so that

$$EP(\tilde{\mathbf{v}}_M(g_1, g_2)) = \tilde{v}_M(g_1, g_2; 0). \tag{24.5.5}$$

Since the local trace formula holds for the weight factors $\tilde{v}_M(g_1, g_2; 0)$, it is reasonable to hope that it will also hold for the weight factors $\tilde{\mathbf{v}}_M(g_1, g_2)$ (as an equality between two elements in $K(V)_{\mathbb{C}}$), and we will now check that this really is the case.

For this we need to check that for all linear functionals $\lambda : K(V)_{\mathbb{C}} \to \mathbb{C}$, the local trace formula holds for the weight factors v_M^λ defined by

$$v_M^\lambda(g_1, g_2) := \langle \lambda, \tilde{\mathbf{v}}_M(g_1, g_2)\rangle.$$

So far we know only that this is true for $\lambda = EP$.

Of course we only need to consider a collection of linear functionals λ that spans the vector space dual to $K(V)_{\mathbb{C}}$. We now define such a collection of linear functionals λ_P, one for each parabolic subgroup $P = LU \in \mathcal{F}(A)$ (with L being the unique Levi component of P that contains A). Recall that inside V we have the non-singular closed subvariety $\overline{V}_P = V^L$, the toric variety associated to L. We define λ_P (as the complex linear extension of)

$$K(V) \to K(\overline{V}_P) \to \mathbb{Z},$$

where the first map is pull-back (restriction) from V to $\overline{V}_P$ and the second is push-forward from $\overline{V}_P = V^L$ to a point.

It follows from Lemma 23.1 that

$$v_M^{\lambda_P}(g_1, g_2) = \begin{cases} \tilde{v}_M^L(l_1, l_2; 0) & \text{if } M \subset L \\ 0 & \text{otherwise} \end{cases} \tag{24.5.6}$$

where $\tilde{v}_M^L$ is the weight factor for the Levi subgroup M of L, and where we have used the Iwasawa decomposition to decompose g_i $(i = 1, 2)$ as $g_i = l_i u_i k_i$ for $l_i \in L$, $u_i \in U$, $k_i \in K$.

Using Lemma 13.3 and applying the local trace formula for L (with weight factors $\tilde{v}_M^L(g_1, g_2; 0)$) to the functions $f_1^{(P)}$ and $f_2^{(P)}$ on $\mathrm{Lie}(L)$, we see that the local trace formula holds for the weight factors $v_M^{\lambda_P}$ and hence for the K-theoretic weight factors $\tilde{\mathbf{v}}_M$. Actually, for this we should check that the various measures we are using in our integrals on L (and in the definition of $f \mapsto f^{(P)}$) are compatible with the ones we are using in our integrals on G, but we are going to omit this point.

24.6. Weight factors $\mathbf{v}_M$ and v_M. Finally we come to the weight factors v_M we really want, those defined using volumes of convex hulls. These are related to our K-theoretic weight factors in the following way. The Chern character ch induces an isomorphism (of $\mathbb{C}$-algebras)

$$ch : K(V)_{\mathbb{C}} \simeq H^\bullet(V, \mathbb{C}).$$

We write $\mathbf{v}_M(g_1, g_2) \in H^\bullet(V, \mathbb{C})$ for the image of the weight factor $\tilde{\mathbf{v}}_M(g_1, g_2) \in K(V)_{\mathbb{C}}$ under the Chern character isomorphism ch. Since the local trace formula holds for the weight factors $\tilde{\mathbf{v}}_M$, it also holds for the weight factors $\mathbf{v}_M$.

Consider the linear functional λ on $H^\bullet(V, \mathbb{C})$ projecting $H^\bullet(V, \mathbb{C})$ onto its summand $H^{2r}(V, \mathbb{C}) = \mathbb{C}$ of top degree (with $r = \dim V = \dim(A/A_G)$). Define yet another weight factor by

$$v_M(g_1, g_2) := \langle \lambda, \mathbf{v}_M(g_1, g_2)\rangle. \tag{24.6.1}$$

Obviously the local trace formula holds for the weight factors v_M. Our next goal is to express v_M in terms of volumes of convex hulls. We cannot do this without a better understanding of (24.6.1) when $M \neq A$, so we will rewrite (24.6.1) in terms of the cohomology ring $H^\bullet(Y_M,\mathbb{C})$ of Y_M. Consider the diagram

$$\begin{array}{ccc} K(Y_M)_{\mathbb{C}} & \xrightarrow{ch_M} & H^\bullet(Y_M,\mathbb{C}) \\ {\scriptstyle (i_M)_*}\big\downarrow & & {\scriptstyle (i_M)_*}\big\downarrow \\ K(V)_{\mathbb{C}} & \xrightarrow{ch} & H^\bullet(V,\mathbb{C}) \end{array} \tag{24.6.2}$$

in which the horizontal arrows are Chern character isomorphisms and the right vertical map is the usual Gysin push-forward map on cohomology, coming from the natural map $(i_M)_* : H_\bullet(Y_M,\mathbb{C}) \to H_\bullet(V,\mathbb{C})$ on homology after we use Poincaré duality to identify the cohomology of V with its homology, and the same for Y_M. We claim that the diagram (24.6.2) is commutative. For this we need to consider the Riemann-Roch theorem for the morphism $i_M : Y_M \hookrightarrow V$. Let N denote the normal bundle to Y_M in V. To show that (24.6.2) commutes it is enough to show that the Todd class $td(N)$ is 1 (see the proof of [**Ful98**, Theorem 15.2]). In fact more is true: the normal bundle itself is trivial, as one sees from (23.2.1), which shows that the open neighborhood U_M of Y_M in V is isomorphic to the product $S \times Y_M$ for any subtorus S of $\hat{A}/Z(\hat{G})$ complementary to the subtorus $Z(\hat{M})/Z(\hat{G})$.

Consider the linear functional λ_M on $H^\bullet(Y_M,\mathbb{C})$ projecting $H^\bullet(Y_M,\mathbb{C})$ onto its summand $H^{2s}(Y_M,\mathbb{C}) = \mathbb{C}$ of top degree (with $s = \dim Y_M = \dim(A_M/A_G)$). Then, as a consequence of the commutativity of the diagram (24.6.2), we have the equality

$$v_M(g_1,g_2) = |\mathcal{Z}_M|^{-1}\langle \lambda_M, ch_M[\mathcal{M}_{(g_1,g_2)}]\rangle, \tag{24.6.3}$$

where $\mathcal{M}_{(g_1,g_2)}$ is the restriction of $\mathcal{L}_{(g_1,g_2)}$ to Y_M, so that the corresponding (G,M)-orthogonal set in Λ_M is $H_{\bar{P}}(g_1) - H_P(g_2)$.

24.7. Volumes of positive orthogonal sets. We need to specify the measures with respect to which we take our volumes. Consider a positive (G,M)-orthogonal set of points $(y_P)_{P\in\mathcal{P}(M)}$ in Λ_M. The points y_P all have the same image in Λ_G. Pick $y \in \Lambda_M$ having the same image in Λ_G as all the points y_P. Then the translated points $y_P - y$ all lie in the subgroup $\Lambda_M^0 := X^*(Z(\hat{M})/Z(\hat{G}))$ of $X^*(Z(\hat{M})) = \Lambda_M$. Since Λ_M^0 is a free abelian group, there is a canonical translation invariant measure on $\Lambda_M^0 \otimes_{\mathbb{Z}} \mathbb{R}$, namely the one that gives measure 1 to any fundamental domain for Λ_M^0. By definition, we take the volume of the convex hull of the points y_P to be the volume in $\Lambda_M^0 \otimes_{\mathbb{Z}} \mathbb{R}$ (for the measure we just defined) of the translated points $y_P - y$. (Clearly this is independent of the choice of y.)

24.8. Computation of $\langle\lambda_M, ch_M[\mathcal{M}]\rangle$ for certain line bundles $\mathcal{M}$. Let $\mathcal{M}$ be a $Z(\hat{M})$-equivariant line bundle on Y_M with associated (G,M)-orthogonal set $(y_P)_{P\in\mathcal{P}(M)}$ in Λ_M. It is known (see [**Ful93**]) that when the orthogonal set y_P is positive, $\langle\lambda_M, ch_M[\mathcal{M}]\rangle$ is equal to the volume of the convex hull of the points y_P.

The map $\mathrm{Pic}(Y_M) \to K(Y_M)_{\mathbb{C}} \xrightarrow{ch_M} H^\bullet(Y_M,\mathbb{C}) \xrightarrow{\lambda_M} \mathbb{C}$ is a homogeneous polynomial function of degree $\dim Y_M = \dim(A_M/A_G)$. Therefore, if y_P is a *negative* orthogonal set (in the sense that $-y_P$ is a positive orthogonal set), $\langle\lambda_M, ch_M[\mathcal{M}]\rangle$ is equal to $(-1)^{\dim(A_M/A_G)}$ times the volume of the convex hull of the points y_P. (There is no need to replace the points by their negatives since this does not affect the volume.)

24.9. Computation of v_M in terms of volumes of convex hulls. Since $H_{\bar{P}}(g_1) - H_P(g_2)$ is a negative orthogonal set, we conclude that

$$v_M(g_1, g_2) = (-1)^{\dim(A_M/A_G)} |\mathcal{Z}_M|^{-1} \operatorname{vol}\bigl(\operatorname{Hull}\{H_{\bar{P}}(g_1) - H_P(g_2) : P \in \mathcal{P}(M)\}\bigr).$$

Thus, for these weight factors v_M we have the final version of the local trace formula:

THEOREM 24.1 (Waldspurger [**Wal95**]). *Let $f_1, f_2 \in C_c^\infty(\mathfrak{g})$. Then $T(f_1, f_2) = T(\hat{f}_1, \check{f}_2)$, where*

$$\begin{aligned} T(f_1, f_2) = \sum_{M \in \mathcal{L}} \frac{|W_M|}{|W|} \sum_{T \in \mathcal{T}_M} \frac{1}{n_T^M} \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)| \cdot \\ \int_{A_M \backslash G} \int_{A_M \backslash G} f_1(h^{-1}Xh) f_2(g^{-1}Xg) v_M(h, g) \, d\dot{h} \, d\dot{g} \, dX. \end{aligned} \tag{24.9.1}$$

24.10. Splitting. Recall that $\mathcal{M}_{(g_1,g_2)}$ is the $Z(\hat{M})$-equivariant line bundle on Y_M associated to the negative (G, M)-orthogonal set $P \mapsto H_{\bar{P}}(g_1) - H_P(g_2)$. Thus it is natural to introduce (for $g \in G$) the $Z(\hat{M})$-equivariant line bundles $\mathcal{M}'_g$ and $\mathcal{M}_g$ associated to the negative (G, M)-orthogonal sets $B \mapsto H_{\bar{P}}(g)$ and $P \mapsto -H_P(g)$ respectively, as well as the elements

$$\mathbf{v}_M(g) := ch_M[\mathcal{M}_g] \in H^\bullet(Y_M, \mathbb{C}), \tag{24.10.1}$$

$$\mathbf{v}'_M(g) := ch_M[\mathcal{M}'_g] \in H^\bullet(Y_M, \mathbb{C}). \tag{24.10.2}$$

These definitions are set up so that

$$\mathbf{v}_M(g_1, g_2) = |\mathcal{Z}_M|^{-1} (i_M)_* \bigl(\mathbf{v}'_M(g_1) \cdot \mathbf{v}_M(g_2)\bigr), \tag{24.10.3}$$

the product on the right being taken in the cohomology ring $H^\bullet(Y_M, \mathbb{C})$.

Now let $T \in \mathcal{T}_M$ and let $X \in \mathfrak{t}_{\mathrm{reg}}$. Then for $f \in C_c^\infty(\mathfrak{g})$ we can define normalized weighted orbital integrals $\mathcal{J}_X(f) = \mathcal{J}_X^G(f)$ and $\mathcal{J}'_X(f) = (\mathcal{J}')_X^G(f)$ taking values in $H^\bullet(Y_M, \mathbb{C})$ by putting

$$\mathcal{J}_X(f) := |D(X)|^{1/2} \int_{A_M \backslash G} f(g^{-1}Xg) \mathbf{v}_M(g) \, d\dot{g}, \tag{24.10.4}$$

$$\mathcal{J}'_X(f) := |D(X)|^{1/2} \int_{A_M \backslash G} f(g^{-1}Xg) \mathbf{v}'_M(g) \, d\dot{g}. \tag{24.10.5}$$

These definitions are set up so that the expression

$$\mathcal{J}_X(f_1, f_2) := |D(X)| \int_{A_M \backslash G} \int_{A_M \backslash G} f_1(g_1^{-1}Xg_1) f_2(g_2^{-1}Xg_2) v_M(g_1, g_2) \, d\dot{g}_1 \, d\dot{g}_2$$

occurring in Theorem 24.1 is given by

$$\mathcal{J}_X(f_1, f_2) = |\mathcal{Z}_M|^{-1} \langle \lambda_M, \mathcal{J}'_X(f_1) \cdot \mathcal{J}_X(f_2) \rangle, \tag{24.10.6}$$

the product $\mathcal{J}'_X(f_1) \cdot \mathcal{J}_X(f_2)$ being taken in $H^\bullet(Y_M, \mathbb{C})$.

By parabolic descent (see Lemma 13.3) for any parabolic subgroup $P = LU \in \mathcal{F}(M)$ (with $L \supset M$), the image of $\mathcal{J}_X(f) \in H^\bullet(Y_M, \mathbb{C})$ under the map

$$H^\bullet(Y_M, \mathbb{C}) \to H^\bullet(Y_M^L, \mathbb{C})$$

induced by $Y_M^L = \overline{V}_P \cap Y_M \hookrightarrow Y_M$ is equal to $\mathcal{J}_X^L(f^{(P)}) \in H^\bullet(Y_M^L, \mathbb{C})$, and the analogous statement holds for $\mathcal{J}'_X$.

Now assume that f is a cusp form (see 13.2), so that $f^P = 0$ (and hence $f^{(P)} = 0$) for all $P \neq G$. Then, since the fundamental classes of $\overline{V}_P \cap Y_M$ ($P \in \mathcal{F}(M)$, $P \neq G$) in the homology groups $H_\bullet(Y_M, \mathbb{C})$ span (see [**Ful93**]) the subspace

$$\sum_{i=0}^{2\dim Y_M - 1} H_i(Y_M, \mathbb{C})$$

of $H_\bullet(Y_M, \mathbb{C})$, we see that for such f the weighted orbital integrals $\mathcal{J}_X(f)$, $\mathcal{J}'_X(f)$ lie in the top degree subspace $H^{2\dim(Y_M)}(Y_M, \mathbb{C}) = \mathbb{C}$ of $H^\bullet(Y_M, \mathbb{C})$.

Therefore for any $f_1 \in C_c^\infty(\mathfrak{g})$ and any cusp form $f_2 \in C_c^\infty(\mathfrak{g})$ the product $\mathcal{J}'_X(f_1)\mathcal{J}_X(f_2)$ is equal to $\mathcal{J}_X(f_2)$ times the projection of $\mathcal{J}'_X(f_1)$ on $H^0(Y_M, \mathbb{C}) = \mathbb{C}$. Since the projection of $\mathbf{v}'_M(g_1)$ on $H^0(Y_M, \mathbb{C})$ is obviously $1 \in \mathbb{C} = H^0(Y_M, \mathbb{C})$, we conclude that

$$\mathcal{J}'_X(f_1)\mathcal{J}_X(f_2) = I_X(f_1)\mathcal{J}_X(f_2), \tag{24.10.7}$$

where $I_X(f)$ is the normalized orbital integral

$$I_X(f) := |D(X)|^{1/2} \int_{A_M \backslash G} f(g^{-1}Xg)\, d\dot{g}.$$

From (24.10.6) and (24.10.7) we conclude that when f_2 (and hence $\check{f}_2$) is a cusp form, the local trace formula (see Theorem 24.1) reduces to the statement that

$$T_{\text{cusp}}(f_1, f_2) = T_{\text{cusp}}(\hat{f}_1, \check{f}_2), \tag{24.10.8}$$

with T_{cusp} defined by

$$\begin{aligned} T_{\text{cusp}}(f_1, f_2) := \sum_{M \in \mathcal{L}} (-1)^{\dim(A_M/A_G)} |\mathcal{Z}_M|^{-1} \frac{|W_M|}{|W|} \sum_{T \in \mathcal{T}_M} \frac{1}{n_T^M} \cdot \\ \int_{\mathfrak{t}_{\text{reg}}} |D(X)| O_X(f_1) WO_X(f_2)\, dX, \end{aligned} \tag{24.10.9}$$

where

$$O_X(f) = \int_{A_M \backslash G} f(g^{-1}Xg)\, d\dot{g}, \tag{24.10.10}$$

$$WO_X(f) = \int_{A_M \backslash G} f(g^{-1}Xg) v_M(g)\, d\dot{g}, \tag{24.10.11}$$

$v_M(g)$ being (as in earlier sections of this article) the volume of the convex hull of $\{H_P(g) : P \in \mathcal{P}(M)\}$.

25. An important application of the local trace formula

Following Waldspurger [**Wal95**], we are going to use the local trace formula on the Lie algebra to strengthen a result of Harish-Chandra [**HC78**] that is the first key step in his proof that the distribution characters of irreducible representations of G are represented by locally integrable functions.

25.1. Definition of support of a distribution. Let's recall the notion of support of a distribution on an l.c.t.d space X. For this we need to remember that for open subsets $U \subset V$ of X there is a restriction map

$$\mathcal{D}(V) \to \mathcal{D}(U)$$

on distributions that is dual to the inclusion

$$C_c^\infty(U) \to C_c^\infty(V).$$

With these restriction maps $U \mapsto \mathcal{D}(U)$ is a sheaf of vector spaces on X. In particular, given a distribution D on X, there is a biggest open subset U of X such that the restriction of D to U is zero. The complement Y of U is called the *support* of D. Thus Y is the smallest closed subset of X for which D is in the image of the embedding $\mathcal{D}(Y) \hookrightarrow \mathcal{D}(X)$.

25.2. Definition of the invariant distribution I_ϕ on $\mathfrak{g}$. Let $\phi \in C_c^\infty(\mathfrak{g})$ be a cusp form. For any $f \in C_c^\infty(\mathfrak{g})$ put

$$I_\phi(f) := T_{\text{cusp}}(f, \phi), \tag{25.2.1}$$

with T_{cusp} as in (24.10.9) above. Thus I_ϕ is a well-defined distribution on $\mathfrak{g}$. It is clear from the definition that I_ϕ is invariant and supported on the closure of the set of G-conjugates of elements in the compact set $\operatorname{Supp}(\phi)$. In particular the support of I_ϕ is bounded modulo conjugation (see 15.2 for this notion).

Recall that the Fourier transform $\hat{T}$ of a distribution T on $\mathfrak{g}$ is defined so that $\hat{T}(f) = T(\hat{f})$ for all test functions $f \in C_c^\infty(\mathfrak{g})$. Since the Fourier transform commutes with adjoint G-action, it takes invariant distributions to invariant distributions.

The next result makes use of the notion of nice conjugation invariant function on $\mathfrak{g}$ (discussed in 13.8).

THEOREM 25.1. *Let ϕ be a cusp form on $\mathfrak{g}$. Then $\hat{\phi}$ is also a cusp form, and there is an equality*

$$\hat{I}_\phi = I_{\hat{\phi}}. \tag{25.2.2}$$

Moreover the distribution I_ϕ is represented by the nice conjugation invariant function F_ϕ on $\mathfrak{g}$ which is 0 off $\mathfrak{g}_{\text{rs}}$ and whose value at any $X \in \mathfrak{t}_{\text{reg}}$ for any $M \in \mathcal{L}$ and $T \in \mathcal{T}_M$ is given by

$$F_\phi(X) = (-1)^{\dim(A_M/A_G)} |\mathcal{Z}_M|^{-1} WO_X(\phi). \tag{25.2.3}$$

PROOF. We observed long ago that the Fourier transform of a cusp form is a cusp form (see 13.2). The equality (25.2.2) follows from (24.10.8). The second statement of the theorem follows from (24.10.9), the Weyl integration formula, and the local constancy of $WO_X(\phi)$ as a function of $X \in \mathfrak{t}_{\text{reg}}$ (see Theorem 17.11). □

25.3. Remarks. This theorem can be regarded as a Lie algebra analog of a result of Arthur [**Art87**] which says that (up to a sign) the weighted orbital integrals of a matrix coefficient of a supercuspidal representation give the character values of that representation. In particular a cusp form ϕ on $\mathfrak{g}$ should be regarded as being analogous to a matrix coefficient for a supercuspidal representation of G.

25.4. The special case in which $\phi \in C_c^\infty(\mathfrak{g}_e)$. As before (see 10.4), we write $\mathfrak{g}_e$ for the open subset of $\mathfrak{g}$ consisting of elements whose centralizers are elliptic maximal tori in G. Let $\phi \in C_c^\infty(\mathfrak{g}_e)$. Clearly ϕ is a cusp form on $\mathfrak{g}$, so we can consider the invariant distribution I_ϕ of 25.2. As an immediate consequence of Theorem 25.1 we obtain the following result of Harish-Chandra and Waldspurger.

THEOREM 25.2 ([**HC78, Wal95**]). *The invariant distribution $\hat{I}_\phi$ is represented by a nice conjugation invariant function whose value at any $X \in \mathfrak{t}$ for any $M \in \mathcal{L}$, $T \in \mathcal{T}_M$ is given by*

$$(-1)^{\dim(A_M/A_G)}|\mathcal{Z}_M|^{-1}WO_X(\hat{\phi}). \tag{25.4.1}$$

Harish-Chandra proved that $\hat{I}_\phi$ is represented by a nice conjugation invariant function and proved the formula above for its value at elliptic elements $X \in \mathfrak{g}$, for which the weighted orbital integral reduces to an ordinary orbital integral. The formula (25.4.1) in the case of non-elliptic elements is due to Waldspurger.

We should note that because ϕ is supported on the elliptic set $\mathfrak{g}_e$, we get from (24.10.9) the following simple formula

$$I_\phi(f) = |\mathcal{Z}_G|^{-1} \sum_{T \in \mathcal{T}_G} \frac{1}{n_T^G} \cdot \int_{\mathfrak{t}_{\mathrm{reg}}} |D(X)|O_X(\phi)O_X(f)\, dX, \tag{25.4.2}$$

which can also be rewritten (using the Weyl integration formula) as

$$I_\phi(f) = |\mathcal{Z}_G|^{-1} \int_{A_G\backslash G} \int_{\mathfrak{g}} \phi(X) f(g^{-1}Xg)\, dX\, d\dot{g}. \tag{25.4.3}$$

26. Niceness of $\hat{O}_X$ for $X \in \mathfrak{g}_{\mathrm{rs}}$.

26.1. Goal. We see from (25.4.2) that the distribution I_ϕ is an integral of distributions of the form O_X for $X \in \mathfrak{g}_e$. By varying ϕ we get many such integrals, and for each one we know that its Fourier transform $\hat{I}_\phi$ is represented by a nice conjugation invariant function. This suggests that for any $X \in \mathfrak{g}_e$ the Fourier transform $\hat{O}_X$ of O_X is represented by a nice conjugation invariant function on $\mathfrak{g}$. In fact this is true, and is the main step in the proof of the following more general result of Harish-Chandra (which in turn is a special case of the yet more general result Theorem 27.8, again due to Harish-Chandra):

THEOREM 26.1 ([**HC78, HC99**]). *For every $X \in \mathfrak{g}_{\mathrm{rs}}$ the Fourier transform $\hat{O}_X$ of the orbital integral O_X is represented by a nice conjugation invariant function on $\mathfrak{g}$.*

For the time being we remark only that it suffices to prove the theorem in case X is elliptic. The general case will then follow from Lemma 13.2, Lemma 13.4, and equation (13.12.1). In 26.5 we will use Theorem 25.2 to treat the elliptic case. This will require Howe's finiteness theorem, to be discussed next.

26.2. Howe's finiteness theorem. Before stating Howe's finiteness theorem for $\mathfrak{g}$, we need a few preliminary remarks.

Let V be a subset of $\mathfrak{g}$ that is conjugation invariant and bounded modulo conjugation (see 15.2 for this notion). We denote by $J(V)$ the space of invariant distributions on $\mathfrak{g}$ whose support is contained in V.

Now let L be a lattice in $\mathfrak{g}$. Inside $C_c^\infty(\mathfrak{g})$ we have the subspace $C_c(\mathfrak{g}/L)$ consisting of functions that are compactly supported and translation invariant under L. There is of course a restriction map

$$\mathcal{D}(\mathfrak{g}) \to C_c(\mathfrak{g}/L)^* \tag{26.2.1}$$

where $C_c(\mathfrak{g}/L)^*$ denotes the vector space dual to $C_c(\mathfrak{g}/L)$.

Now we can state Howe's finiteness theorem, proved by Howe [**How73**] for GL_n and by Harish-Chandra in the general case [**HC78**]. (There is an analogous result for G, known as Howe's conjecture, which was proved by Clozel [**Clo89**].)

THEOREM 26.2. *For any lattice L in $\mathfrak{g}$ and any subset V of $\mathfrak{g}$ that is conjugation invariant and bounded modulo conjugation, the image of $J(V)$ under the restriction map* (26.2.1) *is finite dimensional.*

We will use this theorem without proving it. For additional insight into why Howe's finiteness theorem is useful, see DeBacker's article in this volume.

We also need to understand what the theorem says in the Fourier transformed picture. The Fourier transform gives an isomorphism

$$C_c^\infty(\mathfrak{g}) \xrightarrow{FT} C_c^\infty(\mathfrak{g}),$$

and this isomorphism restricts to an isomorphism

$$C_c(\mathfrak{g}/L) \xrightarrow{FT} C_c^\infty(L^\perp),$$

where $L^\perp$ is the lattice in $\mathfrak{g}$ that is Pontryagin dual to L. (When we view elements of $\mathfrak{g}$ as characters on $\mathfrak{g}$, the lattice $L^\perp$ consists of all those characters that are trivial on L.) Since any lattice arises as the Pontryagin dual of some other lattice, Howe's theorem can be reformulated as follows.

THEOREM 26.3. *For any lattice L in $\mathfrak{g}$ and any subset V of $\mathfrak{g}$ that is conjugation invariant and bounded modulo conjugation, the image of $J(V)$ under the composed map*

$$\mathcal{D}(\mathfrak{g}) \xrightarrow{FT} \mathcal{D}(\mathfrak{g}) \xrightarrow{\text{res}} \mathcal{D}(L) \tag{26.2.2}$$

is finite dimensional. The first map is the Fourier transform on distributions, and the second map is given by restriction of distributions from $\mathfrak{g}$ to its open subset L.

26.3. Topology on V^*. When using Howe's finiteness theorem, it is useful to topologize $\mathcal{D}(\mathfrak{g})$. The topology we use is of the following type.

Let V be any complex vector space (not assumed to be finite dimensional as the example we have in mind is $C_c^\infty(\mathfrak{g})$). For any subspace U of V we let $U^\perp$ denote the subspace of V^* consisting of all linear forms that vanish on U. Similarly, for any subspace W of V^* we let $W^\perp$ denote the subspace of V consisting of all vectors v that vanish on W.

We write V as the direct limit of its finite dimensional subspaces U. The dual space V^* is then the projective limit of the dual spaces $U^* = V^*/U^\perp$. We give each dual space U^* the discrete topology and then use the projective limit topology on V^*. In the topological vector space V^* the subgroups $U^\perp$ are open and form a neighborhood base at 0. Thus two linear forms are close to each other if they agree on a large finite dimensional subspace of V. This topology is obviously Hausdorff. When V is finite dimensional, V^* has the discrete topology.

Taking $V = C_c^\infty(\mathfrak{g})$, we get the desired topology on $\mathcal{D}(\mathfrak{g}) = V^*$.

LEMMA 26.4. *The topology above has the following properties.*

(1) *For any linear map $f : V_1 \to V_2$ the dual map $f^* : V_2^* \to V_1^*$ is continuous, and its kernel and image are closed. If f is surjective, then f^* is a homeomorphism of V_2^* onto a closed subspace of V_1^*.*
(2) *Let W be any subspace of V^*. Then the closure of W is $W^{\perp\perp}$.*
(3) *Any finite dimensional subspace W of V^* is closed, and the topology it inherits from V^* is discrete.*

PROOF. We leave the first two items as exercises for the reader. To prove the last item, first note that the natural map $V \to W^*$ is surjective, then apply the second statement of the first item to this surjection. □

Combining the last statement of the lemma above with the Fourier transformed version of Howe's theorem (Theorem 26.3), we get the following useful result.

PROPOSITION 26.1. *Let V be a conjugation invariant subset of $\mathfrak{g}$ that is bounded modulo conjugation, and let $U_1 \subset U_2$ be subspaces of $J(V)$ such that U_2 is contained in the closure of U_1. Let L be any lattice in $\mathfrak{g}$. Then U_1 and U_2 have the same image under the composed map*

$$\mathcal{D}(\mathfrak{g}) \xrightarrow{FT} \mathcal{D}(\mathfrak{g}) \xrightarrow{\text{res}} \mathcal{D}(L). \tag{26.3.1}$$

26.4. Elliptic regular orbital integrals as limits of distributions I_ϕ. The next result illustrates how the topology on $\mathcal{D}(\mathfrak{g})$ works. It will be needed to complete the proof of Theorem 26.1.

LEMMA 26.5. *Let T be an elliptic maximal torus in G, let $X \in \mathfrak{t}_{\mathrm{reg}}$, and let ω_T be a compact open neighborhood of X in $\mathfrak{t}_{\mathrm{reg}}$. Then O_X lies in the closure of the linear subspace*

$$\{I_\phi : \phi \in C_c^\infty(\mathrm{Ad}(G)(\omega_T))\}$$

of $\mathcal{D}(\mathfrak{g})$.

PROOF. We may shrink ω_T as needed. Recall the map

$$(T\backslash G) \times \mathfrak{t}_{\mathrm{reg}} \to \mathfrak{g}_{\mathrm{rs}} \tag{26.4.1}$$

(sending (g, Y) to $g^{-1}Yg$) that we used when proving the Weyl integration formula. Its differential is an isomorphism at all points. Shrinking ω_T, we can find a compact open neighborhood $\omega_{T\backslash G}$ of $1 \in T\backslash G$ and a compact open neighborhood ω_G of X in $\mathfrak{g}_{\mathrm{rs}}$ such that the map (26.4.1) restricts to an isomorphism (of p-adic manifolds)

$$\omega_{T\backslash G} \times \omega_T \to \omega_G. \tag{26.4.2}$$

For $\phi \in C_c^\infty\big(\mathrm{Ad}(G)(\omega_T)\big)$ and any $f \in C_c^\infty(\mathfrak{g})$, we see from (25.4.2) that

$$I_\phi(f) = |\mathcal{Z}_G|^{-1} \int_{\omega_T} |D(Y)| O_Y(f) O_Y(\phi)\, dY. \tag{26.4.3}$$

Suppose that $f \in C_c^\infty(\mathfrak{g})$ is such that $I_\phi(f) = 0$ for all $\phi \in C_c^\infty\big(\mathrm{Ad}(G)(\omega_T)\big)$. In order to prove the lemma, we need to check that $O_X(f) = 0$. In fact we will check that $O_Y(f) = 0$ for all $Y \in \omega_T$. Indeed, since the integral in (26.4.3) vanishes for all $\phi \in C_c^\infty\big(\mathrm{Ad}(G)(\omega_T)\big)$, it is enough to convince ourselves that every locally constant function φ on ω_T arises as $Y \mapsto O_Y(\phi)$ for some $\phi \in C_c^\infty\big(\mathrm{Ad}(G)(\omega_T)\big)$. But this is clear: pull φ back to $\omega_{T\backslash G} \times \omega_T$ using the second projection, view this pullback as a function on ω_G using the isomorphism (26.4.2), and then divide by

$\operatorname{meas}(\omega_{T\backslash G})$ to get a function $\phi \in C_c^\infty(\omega_G)$ that does the job. (Note that ω_G is an open subset of $\operatorname{Ad}(G)(\omega_T)$.) $\square$

26.5. Proof of Theorem 26.1. As we already remarked, it suffices to prove that $\hat{O}_X$ is represented by a nice conjugation invariant function in the case when X lies in $\mathfrak{t}_{\mathrm{reg}}$ for some elliptic maximal torus T. Let ω_T be a compact open neighborhood of X in $\mathfrak{t}_{\mathrm{reg}}$. Put $\omega := \operatorname{Ad}(G)(\omega_T)$, a subset which is clearly bounded modulo conjugation.

By Lemma 26.5 O_X lies in the closure of the space of distributions I_ϕ with $\phi \in C_c^\infty(\omega)$. Moreover the distributions O_X and I_ϕ all lie in $J(\omega)$. Applying Proposition 26.1 to $J(\omega)$, we see that there exists $\phi \in C_c^\infty(\omega)$ such that $\hat{O}_X$ and $\hat{I}_\phi$ have the same restriction to L. From Theorem 25.2 we know that the restriction of $\hat{I}_\phi$ to L is represented by an integrable function that is locally constant on $L \cap \mathfrak{g}_{\mathrm{rs}}$. Therefore the same is true of $\hat{O}_X$. This proves the theorem since the collection of all lattices covers $\mathfrak{g}$.

27. Deeper results on Shalika germs; Lie algebra analog of the local character expansion

Harish-Chandra's Theorem 26.1, together with Howe's finiteness theorem, will allow us to prove quite a number of deep results in harmonic analysis on $\mathfrak{g}$.

27.1. Density of orbital integrals. Our next main goal is to prove the linear independence of Shalika germs. This is closely related, as we will see, to the density of regular semisimple orbital integrals. The first step is the density of all orbital integrals. What do we mean by this? Recall that we have topologized the space of distributions on $\mathfrak{g}$. We consider the subspace $\mathcal{D}(\mathfrak{g})^G$ of invariant distributions with its inherited topology. Inside of $\mathcal{D}(\mathfrak{g})^G$ we have the linear subspace $\mathcal{D}(\mathfrak{g})_{\mathrm{orb}}$ spanned by all orbital integrals O_X ($X \in \mathfrak{g}$). Now we can state the density result, but one should bear in mind that it is only of temporary interest, since we will soon prove a stronger (and more difficult) statement.

PROPOSITION 27.1 ([**HC78**]). *The subspace $\mathcal{D}(\mathfrak{g})_{\mathrm{orb}}$ is dense in $\mathcal{D}(\mathfrak{g})^G$.*

PROOF. As we saw when we discussed (in 26.3) the topology on duals of vector spaces, the statement we need to prove can be reformulated as follows. Let $f \in C_c^\infty(\mathfrak{g})$. If $O_X(f) = 0$ for all $X \in \mathfrak{g}$, then $I(f) = 0$ for every invariant distribution I on $\mathfrak{g}$. In terms of coinvariants $C_c^\infty(\mathfrak{g})_G$, this can in turn be reformulated as the statement that if $O_X(f) = 0$ for all $X \in \mathfrak{g}$, then the image of f in $C_c^\infty(\mathfrak{g})_G$ is 0.

So we need a better understanding of $C_c^\infty(\mathfrak{g})_G$. For this we will again use the map $\pi_G : \mathfrak{g} \to \mathbb{A}_G(F)$. For $x \in \mathbb{A}_G(F)$ we denote by $\mathfrak{g}_x$ the fiber $\pi_G^{-1}(x)$ over x. The conjugation action of G preserves $\mathfrak{g}_x$, so we can also consider the coinvariants $C_c^\infty(\mathfrak{g}_x)_G$. Restriction of functions to the fiber induces a surjective map

$$C_c^\infty(\mathfrak{g})_G \to C_c^\infty(\mathfrak{g}_x)_G, \tag{27.1.1}$$

and these can be assembled to give a map

$$C_c^\infty(\mathfrak{g})_G \to \prod_x C_c^\infty(\mathfrak{g}_x)_G, \tag{27.1.2}$$

where x runs over all points $x \in \mathbb{A}_G(F)$. It follows from Lemma 27.1 below that the map (27.1.2) is injective.

When x is the origin in the affine space, the fiber $\mathfrak{g}_x$ is the nilpotent cone in $\mathfrak{g}$, and we already have a good understanding of $C_c^\infty(\mathfrak{g}_x)_G$: its dimension is the number of nilpotent orbits, and the integrals over the nilpotent orbits provide a basis for the dual of $C_c^\infty(\mathfrak{g}_x)_G$

The situation for arbitrary x is quite similar. The fiber $\mathfrak{g}_x$ is a finite union of G-orbits, and the integrals over these orbits provide a basis for the dual of $C_c^\infty(\mathfrak{g}_x)_G$. This is proved the same way as for the nilpotent cone, so we will not discuss it any further.

Now return to our function f. Since all orbital integrals of f vanish by hypothesis, the image of f under (27.1.2) is 0. Since (27.1.2) is injective, f is 0 in $C_c^\infty(\mathfrak{g})_G$, and we are done. □

The next lemma is similar to the material in section 2.36 of [**BZ76**].

LEMMA 27.1. *Let X and Y be l.c.t.d spaces, and let $f : X \to Y$ be a continuous map. For $y \in Y$ we denote by X_y the fiber $f^{-1}(y)$. Suppose that an abstract group G acts on X, preserving the fibers of f. Restriction of functions from X to X_y induces a map*

$$C_c^\infty(X)_G \to C_c^\infty(X_y)_G, \tag{27.1.3}$$

and these can be assembled to give a map

$$C_c^\infty(X)_G \to \prod_{y\in Y} C_c^\infty(X_y)_G. \tag{27.1.4}$$

The map (27.1.4) *is injective.*

Moreover, for any open neighborhood U of $y \in Y$ there is a surjective restriction map

$$C_c^\infty(f^{-1}U)_G \to C_c^\infty(X_y)_G,$$

and these fit together to give an isomorphism

$$\varinjlim_U C_c^\infty(f^{-1}U)_G \cong C_c^\infty(X_y)_G, \tag{27.1.5}$$

where the colimit is taken over the set of open neighborhoods U of y.

PROOF. Replacing Y by its 1-point compactification (which is again a l.c.t.d space), we may assume without loss of generality that Y is compact.

Suppose that we have a decomposition of Y as a disjoint union of open (hence closed) subsets Y_i $(i \in I)$. Then

$$C_c^\infty(X) = \bigoplus_{i\in I} C_c^\infty(Y_i)$$

and therefore

$$C_c^\infty(X)_G = \bigoplus_{i\in I} C_c^\infty(Y_i)_G. \tag{27.1.6}$$

For any open neighborhood U of $y \in Y$ there is a surjective restriction map

$$C_c^\infty(f^{-1}U) \to C_c^\infty(X_y),$$

and these fit together to give an isomorphism

$$\varinjlim_U C_c^\infty(f^{-1}U) \cong C_c^\infty(X_y), \tag{27.1.7}$$

where the colimit is taken over the set of open neighborhoods U of y. Surjectivity is clear, but let's check injectivity. So suppose that we have a function $\phi \in C_c^\infty(f^{-1}U)$ whose restriction to X_y is 0. Then the support S of ϕ is a compact set disjoint from X_y, so its image $f(S)$ does not contain y. Let V be the open subset of U obtained by removing all points in the compact set $f(S)$. Then ϕ becomes 0 in $C_c^\infty(f^{-1}V)$, hence in the colimit.

Taking coinvariants in (27.1.7), we get the isomorphism (27.1.5) mentioned in the last statement of the lemma. (Coinvariants commute with arbitrary colimits.)

Now we finish the proof. Let $\phi \in C_c^\infty(X)$ and suppose that the image of ϕ under (27.1.4) is 0. By (27.1.5) for every $y \in Y$ there exists a compact open neighborhood U_y of y such that ϕ is 0 in $C_c^\infty(f^{-1}U_y)_G$. Since Y is compact, it can be covered by finitely many compact open subsets $U_1, \dots, U_n$ such that ϕ is 0 in $C_c^\infty(f^{-1}U_i)_G$ for all i. Now put

$$Y_1 = U_1,\, Y_2 = U_2 \setminus U_1, \dots, Y_n = U_n \setminus (U_1 \cup \dots \cup U_{n-1}).$$

Thus we have written Y as a disjoint union of open subsets Y_i such that ϕ is 0 in $C_c^\infty(f^{-1}Y_i)_G$ for all i. It follows from (27.1.6) that ϕ is 0 in $C_c^\infty(X)_G$, as desired. □

27.2. Preliminary remarks regarding linear independence of Shalika germs. We will soon be proving that the Shalika germs $\Gamma_1, \dots, \Gamma_r$ (attached to the nilpotent orbits $\mathcal{O}_i$) are linearly independent functions on $\mathfrak{g}_{\mathrm{rs}}$.

LEMMA 27.2. *Assume that the Shalika germs $\Gamma_1, \dots, \Gamma_r$ are linearly independent functions on $\mathfrak{g}_{\mathrm{rs}}$. Then for any open neighborhood U of 0 in $\mathfrak{g}$, the restrictions of $\Gamma_1, \dots, \Gamma_r$ to $U \cap \mathfrak{g}_{\mathrm{rs}}$ remain linearly independent.*

PROOF. Without loss of generality we may assume that U is a lattice in $\mathfrak{g}$. Now we use homogeneity of Shalika germs. The additive semigroup of non-negative integers acts on $U \cap \mathfrak{g}_{\mathrm{rs}}$, with j acting by multiplication by the scalar $\pi^{2j} \in F^\times$, and therefore acts on the space of functions on $U \cap \mathfrak{g}_{\mathrm{rs}}$ (the action of j transforming a function $F(X)$ into $F(\pi^{2j}X)$).

By homogeneity of Shalika germs (see (17.7.1)) the restriction of Γ_i to $U \cap \mathfrak{g}_{\mathrm{rs}}$ transforms under the character

$$j \mapsto q^{j \dim \mathcal{O}_i}$$

on our semigroup. But in any representation of our semigroup, vectors transforming under distinct characters are linearly independent. Thus, in order to prove linear independence of the restrictions of Shalika germs to $U \cap \mathfrak{g}_{\mathrm{rs}}$, it is enough to fix a non-negative integer d and prove linear independence of the restrictions of the Shalika germs for all nilpotent orbits of dimension d. But all these germs are homogeneous of the same degree, namely d, so it is clear that any dependence relation that holds on the subset $U \cap \mathfrak{g}_{\mathrm{rs}}$ will also hold on the whole set $\mathfrak{g}_{\mathrm{rs}}$. □

Next we relate linear independence of Shalika germs to the problem of writing nilpotent orbital integrals as limits (for our usual topology on $\mathcal{D}(\mathfrak{g})$, see 26.3) of linear combinations of regular semisimple orbital integrals.

LEMMA 27.3. *The functions $\Gamma_1, \dots, \Gamma_r$ on the set $\mathfrak{g}_{\mathrm{rs}}$ are linearly independent if and only if all nilpotent orbital integrals μ_i lie in the closure of the linear span of the subset*

$$\{O_X : X \in \mathfrak{g}_{\mathrm{rs}}\}$$

of $\mathcal{D}(\mathfrak{g})$.

Proof. First we need to recall that the closure occurring in the statement of the lemma is equal to the set of all distributions I such that $I(f) = 0$ for all $f \in C_c^\infty(\mathfrak{g})$ such that $O_X(f) = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}}$.

($\Rightarrow$) Suppose that the functions Γ_i are linearly independent. Given $f \in C_c^\infty(\mathfrak{g})$ such that $O_X(f) = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}}$, we must show that $\mu_i(f) = 0$ for all i. By Shalika germ theory there exists an open neighborhood U of 0 in $\mathfrak{g}$ such that

$$O_X(f) = \sum_{i=1}^{r} \mu_i(f)\Gamma_i(X) \quad \forall\ X \in U \cap \mathfrak{g}_{\mathrm{rs}}. \tag{27.2.1}$$

Since the function $X \mapsto O_X(f)$ on $\mathfrak{g}_{\mathrm{rs}}$ is identically zero, and since the restrictions of the functions Γ_i to the subset $U \cap \mathfrak{g}_{\mathrm{rs}}$ remain linearly independent, we see that $\mu_i(f) = 0$ for all i, as desired.

($\Leftarrow$) Consider a dependence relation $a_1\Gamma_1 + \cdots + a_r\Gamma_r = 0$. By linear independence of the distributions μ_i there exists $f \in C_c^\infty(\mathfrak{g})$ such that $\mu_i(f) = a_i$ for all i. By Shalika germ theory there exists an open neighborhood U of 0 in $\mathfrak{g}$ such that

$$O_X(f) = 0 \quad \forall\ X \in U \cap \mathfrak{g}_{\mathrm{rs}}. \tag{27.2.2}$$

It follows from Lemma 15.3 that $\mathrm{Ad}(G)(U)$ contains a G-invariant open and closed neighborhood V of the nilpotent cone. Multiplying f by the characteristic function of V, we obtain a function $f' \in C_c^\infty(\mathfrak{g})$ such that

$$O_X(f') = \begin{cases} O_X(f) & \text{if } X \in V \\ 0 & \text{if } X \notin V. \end{cases} \tag{27.2.3}$$

Combining (27.2.2) with (27.2.3), we see that $O_X(f') = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}}$. Therefore, since we are assuming that nilpotent orbital integrals are in the closure of the span of the regular semisimple orbital integrals, we conclude that $\mu_i(f') = 0$ for all i. But, again using (27.2.3), we find that $a_i = \mu_i(f) = \mu_i(f')$, and we are done. $\square$

Now let S be a semisimple element of $\mathfrak{g}$, and let $H = G_S$, $\mathfrak{h}$, $\mathfrak{h}'$ be as in 17.10. For nilpotent $Y \in \mathfrak{h}$ we write μ_{S+Y} for integration over the G-orbit of $S+Y$ in $\mathfrak{g}$. We write $Y_1, \dots, Y_s$ for representatives of the H-orbits of nilpotent elements in $\mathfrak{h}$.

Lemma 27.4. *Assume that the Shalika germs $\Gamma_1^H, \dots, \Gamma_s^H$ for H are linearly independent functions on $\mathfrak{h}_{\mathrm{rs}}$. Then for every $X \in \mathfrak{g}$ whose semisimple part is S, the distribution O_X lies in the closure of the linear span of the subset*

$$\{O_{X'} : X' \in \mathfrak{g}_{\mathrm{rs}}\}$$

of $\mathcal{D}(\mathfrak{g})$.

Proof. This proof is almost the same as that of half of the previous lemma. Given $f \in C_c^\infty(\mathfrak{g})$ such that $O_X(f) = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}}$, we must show that $\mu_{S+Y_i}(f) = 0$ for all i. By Theorem 17.6 there exists an open neighborhood U of S in $\mathfrak{h}'$ such that

$$O_{X'}(f) = \sum_{i=1}^{s} \mu_{S+Y_i}(f) \cdot \Gamma_i^H(X') \tag{27.2.4}$$

for all $X' \in U \cap \mathfrak{h}_{\mathrm{rs}} = U \cap \mathfrak{g}_{\mathrm{rs}}$. Since the function $X' \mapsto O_{X'}(f)$ on $\mathfrak{g}_{\mathrm{rs}}$ is identically zero, and since the restrictions of the functions Γ_i^H to the subset $U \cap \mathfrak{h}_{\mathrm{rs}}$ remain linearly independent, we see that $\mu_{S+Y_i}(f) = 0$ for all i, as desired. $\square$

27.3. Linear independence of Shalika germs and density of linear combinations of regular semisimple orbital integrals. Now we are ready to prove Harish-Chandra's theorem stating that Shalika germs are in fact linearly independent.

THEOREM 27.5 ([**HC78, HC99**]). *The Shalika germs $\Gamma_1, \dots, \Gamma_r$ are linearly independent functions on $\mathfrak{g}_{\mathrm{rs}}$. Indeed they remain linearly independent when restricted to $U \cap \mathfrak{g}_{\mathrm{rs}}$ for any open neighborhood U of 0 in $\mathfrak{g}$. Moreover every invariant distribution on $\mathfrak{g}$ lies in the closure of the linear span of the subset*

$$\{O_X : X \in \mathfrak{g}_{\mathrm{rs}}\}$$

of $\mathcal{D}(\mathfrak{g})$.

PROOF. We reproduce Harish-Chandra's beautiful proof, which uses just about everything we have done. By Lemma 27.2 the second statement of the theorem follows from the first. We prove the first and last statements of the theorem by induction on the dimension of $\mathfrak{g}$, the case when $\dim(G) = 0$ being trivial. Assuming the theorem is true for all connected reductive H with $\dim(H) < \dim(G)$, we must show that it is true for G.

We claim that the first statement of the lemma holds for G if and only if the last statement of the theorem holds for G. Indeed, if the last statement is true, then the first statement is true by Lemma 27.3. Now assume that the first statement is true. Then for every semisimple element S in $\mathfrak{g}$, the Shalika germs for G_S are linearly independent. By Lemma 27.4, for every $X \in \mathfrak{g}$ the distribution O_X is in the closure of the linear span of the subset

$$\{O_{X'} : X' \in \mathfrak{g}_{\mathrm{rs}}\}$$

of $\mathcal{D}(\mathfrak{g})$. This, together with Proposition 27.1, shows that the last statement of the theorem is true.

Using Lemma 17.4, we reduce to the case in which the center of $\mathfrak{g}$ is trivial. It remains to verify the last statement of the theorem. For this we need to consider the two subspaces $C_2 \subset C_1$ of $C_c^\infty(\mathfrak{g})$ defined by

$$C_1 := \{f \in C_c^\infty(\mathfrak{g}) : O_X(f) = 0 \quad \forall X \in \mathfrak{g}_{\mathrm{rs}}\}$$
$$C_2 := \{f \in C_c^\infty(\mathfrak{g}) : O_X(f) = 0 \quad \forall X \in \mathfrak{g}\}.$$

Using our induction hypothesis and Lemma 27.4, we see that

$$C_1 = \{f \in C_c^\infty(\mathfrak{g}) : O_X(f) = 0 \quad \forall X \in \mathfrak{g} \text{ such that } X \text{ is not nilpotent}\},$$

from which we see that C_1/C_2 is finite dimensional and that the dual space $(C_1/C_2)^*$ is spanned by the images of the nilpotent orbital integrals $\mu_1, \dots, \mu_r$.

It follows from Proposition 27.1 that

$$C_2 = \{f \in C_c^\infty(\mathfrak{g}) : I(f) = 0 \text{ for every invariant distribution } I\},$$

from which it is clear that the Fourier transform takes C_2 isomorphically onto itself. We also see that in order to prove that the last statement of the theorem is true, we must show that $C_1/C_2 = 0$.

We claim that the Fourier transform also takes C_1 isomorphically onto itself. It is enough prove that the Fourier transform $f \mapsto \hat{f}$ carries C_1 into itself, since then the same will be true of the inverse Fourier transform $f \mapsto \check{f}$. So let $f \in C_1$. We must show that $O_X(\hat{f}) = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}}$. But $O_X(\hat{f}) = \hat{O}_X(f)$, and we know from Theorem 26.1 that $\hat{O}_X$ is represented by a nice conjugation invariant function on $\mathfrak{g}$. Since all regular semisimple orbital integrals of f vanish, it is then clear that $\hat{O}_X(f)$ vanishes.

Thus the Fourier transform on functions induces an isomorphism

$$C_1/C_2 \xrightarrow{FT} C_1/C_2, \tag{27.3.1}$$

and the Fourier transform on distributions induces an isomorphism

$$(C_1/C_2)^* \xrightarrow{FT} (C_1/C_2)^*. \tag{27.3.2}$$

Thus the Fourier transforms $\hat{\mu}_1, \dots, \hat{\mu}_r$ of the nilpotent orbital integrals also span $(C_1/C_2)^*$.

Now we use homogeneity of nilpotent orbital integrals (see Lemma 17.2):

$$\mu_{\mathcal{O}}(f_{\alpha^2}) = |\alpha|^{-\dim \mathcal{O}} \mu_{\mathcal{O}}(f). \tag{27.3.3}$$

An easy calculation shows that $(f_\beta)\hat{} = |\beta|^{-\dim(G)}(\hat{f})_{\beta^{-1}}$; therefore the Fourier transform $\hat{\mu}_{\mathcal{O}}$ is also homogeneous:

$$\hat{\mu}_{\mathcal{O}}(f_{\alpha^2}) = |\alpha|^{\dim \mathcal{O} - 2\dim(G)}\, \hat{\mu}_{\mathcal{O}}(f). \tag{27.3.4}$$

Let D be the set of integers that arise as the dimension of some nilpotent orbit for G. Assuming that $\dim(G) \neq 0$, as we may, then

$$d < \dim G \tag{27.3.5}$$

for all $d \in D$.

Therefore, there is one basis for $(C_1/C_2)^*$ in which each basis element scales by the factor $|\alpha|^{-d}$ for some $d \in D$, and another in which each scales by the factor $|\alpha|^{d-2\dim(G)}$ for some $d \in D$. By (27.3.5) $-d \neq d' - 2\dim G$ for all $d, d' \in D$. Therefore, by linear independence of characters, we have $C_1/C_2 = 0$, and this establishes that the last statement of the theorem holds for G. □

Corollary 27.6. *Let C be an open and closed G-invariant subset of $\mathfrak{g}$. Then every invariant distribution on $\mathfrak{g}$ supported in C lies in the closure of the linear span of the subset*

$$\{O_X : X \in \mathfrak{g}_{\mathrm{rs}} \cap C\}$$

of $\mathcal{D}(\mathfrak{g})$.

Proof. Let $f \in C_c^\infty(\mathfrak{g})$ and suppose that $O_X(f) = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}} \cap C$. We must show that $I(f) = 0$ for every invariant distribution supported on C. Put $f_0 := f 1_C$, where 1_C denotes the characteristic function of C. Clearly $I(f) = I(f_0)$. Moreover it is clear that $O_X(f_0) = 0$ for all $X \in \mathfrak{g}_{\mathrm{rs}}$. By the theorem above $I(f_0) = 0$. □

Now return to the situation in 13.6.

Corollary 27.7. *Parabolic induction i_P^G is independent of the choice of parabolic subgroup P having Levi component M.*

Proof. Let P, P' be two parabolic subgroups with Levi component M. We want to show that

$$I(f^{(P)}) = I(f^{(P')}) \tag{27.3.6}$$

for every invariant distribution I on $\mathfrak{m}$. By the theorem above (applied to M), it is enough to show that

$$O_X^M(f^{(P)}) = O_X^M(f^{(P')}) \tag{27.3.7}$$

for all $X \in \mathfrak{m}_{\mathrm{rs}}$. For $X \in \mathfrak{m} \cap \mathfrak{g}_{\mathrm{rs}}$ this follows from Lemma 13.3, and by continuity (use the local constancy statement in Theorem 17.10) it then follows for all elements $X \in \mathfrak{m}_{\mathrm{rs}}$. $\square$

27.4. Niceness of Fourier transforms of invariant distributions whose support is bounded modulo conjugation. So far we know (see Theorem 26.1) that $\hat{O}_X$ is represented by a nice conjugation invariant function on $\mathfrak{g}$ for any $X \in \mathfrak{g}_{\mathrm{rs}}$. In fact the same is true for any $X \in \mathfrak{g}$ (the case of nilpotent X being especially interesting). Our next theorem, also due to Harish-Chandra, says that an even stronger and more general result is true.

Let ω be a compact open subset of $\mathbb{A}_G(F)$ and put $C := \pi_G^{-1}(\omega)$. Let $J(C)$ be the space of invariant distributions supported in C. By Corollary 27.6 the linear span of the set

$$\{O_X : X \in \mathfrak{g}_{\mathrm{rs}} \cap C\} \tag{27.4.1}$$

is dense in $J(C)$.

THEOREM 27.8 ([**HC78**]). *Let $I \in J(C)$, and let L be any lattice in $\mathfrak{g}$. Then there exists a linear combination I' of elements in the set* (27.4.1) *such that the distributions $\hat{I}$ and $\hat{I}'$ have the same restriction to L. Consequently, for any invariant distribution I on G whose support is bounded modulo conjugation, the Fourier transform $\hat{I}$ is represented by a nice conjugation invariant function on $\mathfrak{g}$.*

PROOF. The first statement follows from Proposition 26.1. Now we derive the second statement from the first. By Lemma 15.2 I is contained in $J(C)$ for suitably big ω. Since the distributions $\hat{I}'$ appearing in the first statement of the theorem are nice by Theorem 26.1, we see that the restriction of $\hat{I}$ to any lattice L is nice. This finishes the proof, since the collection of all lattices is an open cover of $\mathfrak{g}$. $\square$

It follows from Theorem 27.8 that the Fourier transform $\hat{O}_X$ of any orbital integral O_X is represented by a nice conjugation invariant function, which we will also denote by $\hat{O}_X$. Context will determine whether we are thinking about $\hat{O}_X$ as a distribution or as a nice function on $\mathfrak{g}$. The same goes for $\hat{\mu}_i$.

27.5. Uniformity of Shalika germ expansions. By the Shalika germ expansion, for any $f \in C_c^\infty(\mathfrak{g})$ there is a lattice L' in $\mathfrak{g}$ such that

$$O_X(f) = \sum_{i=1}^{r} \mu_i(f) \cdot \Gamma_i(X) \tag{27.5.1}$$

for all $\in \mathfrak{g}_{\mathrm{rs}} \cap L'$. The lattice L' depends on f. Of course, given finitely many functions f, we can find a single lattice that works for all of them at once, but there is no guarantee that we can do so for an infinite collection of functions. Nevertheless, we will now see that Howe's finiteness theorem implies that for any lattice L in $\mathfrak{g}$, we can find a lattice L' that works for all the functions in $C_c(\mathfrak{g}/L)$.

PROPOSITION 27.2. *Let L be a lattice in $\mathfrak{g}$. Then there exists a lattice L' in $\mathfrak{g}$ such that*

$$O_X(f) = \sum_{i=1}^{r} \mu_i(f) \cdot \Gamma_i(X) \tag{27.5.2}$$

for all $f \in C_c(\mathfrak{g}/L)$ and all $X \in \mathfrak{g}_{\mathrm{rs}} \cap L'$.

PROOF. Pick a compact open neighborhood ω of 0 in $\mathbb{A}_G(F)$ and put $C := \pi_G^{-1}(\omega)$, an open neighborhood of the nilpotent cone. Note that $O_X \in J(C)$ for all $X \in C$. By Howe's theorem the image of $J(C)$ in $C_c(\mathfrak{g}/L)^*$ is finite dimensional. The subspace W in $C_c(\mathfrak{g}/L)$ consisting of all functions annihilated by all distributions in $J(C)$ therefore has finite codimension, so that we can choose finitely many functions $f_1, \dots, f_m \in C_c(\mathfrak{g}/L)$ that together with W span $C_c(\mathfrak{g}/L)$. For each f_j there is a neighborhood U_j of 0 in $\mathfrak{g}$ such that the Shalika germ expansion for f_j works on U_j. The Shalika germ expansion for each $f \in W$ works on the open neighborhood C of 0, since both sides of (27.5.2) vanish for such f. Therefore for any lattice L' contained in $C \cap U_1 \cap \cdots \cap U_m$ the Shalika germ expansion works on L' for all $f \in C_c(\mathfrak{g}/L)$. □

PROPOSITION 27.3. *Let L be a lattice in $\mathfrak{g}$. Then there exists a lattice L' in $\mathfrak{g}$ such that for all $X \in \mathfrak{g}_{\mathrm{rs}} \cap L'$ and all $Y \in \mathfrak{g}_{\mathrm{rs}} \cap L$ there is an equality*

$$\hat{O}_X(Y) = \sum_{i=1}^{r} \Gamma_i(X) \cdot \hat{\mu}_i(Y). \tag{27.5.3}$$

PROOF. This proposition is the Fourier transform of the previous one. □

COROLLARY 27.9. *Let L' be a lattice in $\mathfrak{g}$. Then there exists a lattice L in $\mathfrak{g}$ such that for all $X \in \mathfrak{g}_{\mathrm{rs}} \cap L'$ and all $Y \in \mathfrak{g}_{\mathrm{rs}} \cap L$ there is an equality*

$$\hat{O}_X(Y) = \sum_{i=1}^{r} \Gamma_i(X) \cdot \hat{\mu}_i(Y). \tag{27.5.4}$$

PROOF. An easy calculation shows that $\hat{O}_X(Y) = \hat{O}_{\beta X}(\beta^{-1}Y)$ for all $\beta \in F^\times$ and all $X, Y \in \mathfrak{g}_{\mathrm{rs}}$. Moreover the right side of (27.5.4) does not change when (X, Y) is replaced by $(\alpha^2 X, \alpha^{-2} Y)$ (for $\alpha \in F^\times$), because of the homogeneity properties of Shalika germs (17.7.1) and Fourier transforms of nilpotent orbital integrals (27.3.4). Therefore the equality (27.5.4) holds for all $X \in \mathfrak{g}_{\mathrm{rs}} \cap L'$ and all $Y \in \mathfrak{g}_{\mathrm{rs}} \cap L$ if and only if it holds for all $X \in \mathfrak{g}_{\mathrm{rs}} \cap \alpha^2 L'$ and all $Y \in \mathfrak{g}_{\mathrm{rs}} \cap \alpha^{-2} L$. From the previous proposition there exists some pair of lattices L_0, L_0' on which the equality (27.5.4) holds. Pick α such that $L' \subset \alpha^2 L_0'$. Then the statement of the corollary holds for $L := \alpha^{-2} L_0$. □

27.6. Linear independence of the restrictions of nilpotent orbital integrals to $C_c(\mathfrak{g}/L)$. We have already observed that the nilpotent orbital integrals $\mu_1, \dots, \mu_r$ are linearly independent distributions. Now let L be any lattice in $\mathfrak{g}$.

LEMMA 27.10. *The restrictions of $\mu_1, \dots, \mu_r$ to $C_c(\mathfrak{g}/L)$ are linearly independent.*

PROOF. Since $\mu_1, \dots, \mu_r$ are linearly independent, there exists some lattice L' for which the lemma is true. There exists $\alpha \in F^\times$ such that $L \subset \alpha^2 L'$. The distributions $\mu_1', \dots, \mu_r'$ obtained from $\mu_1, \dots, \mu_r$ by scaling by α^2 remain linearly independent on $C_c(\mathfrak{g}/\alpha^2 L')$ and hence on the bigger space $C_c(\mathfrak{g}/L)$ as well. But by homogeneity of nilpotent orbital integrals, μ_i' is a positive multiple of μ_i. This proves the lemma. □

COROLLARY 27.11. *For any lattice L in $\mathfrak{g}$ the restrictions to L of the nice functions $\hat{\mu}_1, \dots, \hat{\mu}_r$ are linearly independent.*

PROOF. This statement is the Fourier transform of the statement in the lemma. □

27.7. Lie algebra analog of the local character expansion. First we explain the statement of Harish-Chandra's local character expansion. Harish-Chandra proved [**HC78**] that the distribution character of any irreducible admissible representation of G is represented by a locally constant function Θ on G_{rs} that is locally integrable on G. Use the exponential function to identify a suitable open neighborhood of 0 in $\mathfrak{g}$ with an open neighborhood of 1 in G. Then use the exponential function to transport the nice functions $\hat{\mu}_i$ to this neighborhood of 1. Harish-Chandra then proved that there are unique constants c_i such that

$$\Theta(g) = \sum_{i=1}^{r} c_i \hat{\mu}_i(g) \tag{27.7.1}$$

for all regular semisimple g in some suitably small neighborhood of 1 in G. How small the neighborhood has to be depends on the representation of G that we started with.

His proof uses the Lie algebra analog of this statement. We have already made the point that Fourier transforms of orbital integrals are the Lie algebra analogs of irreducible characters on G. Therefore we would expect Fourier transforms of orbital integrals to appear on the left side of the Lie algebra analog of (27.7.1). Actually a more general statement is true: the Fourier transform of any invariant distribution whose support is bounded modulo conjugation has a local character expansion. In the case of Fourier transforms of regular semisimple orbital integrals, one even knows what the constants c_i are: they are Shalika germs. Here is the precise statement of Harish-Chandra's Lie algebra analog of the local character expansion.

THEOREM 27.12 ([**HC78**]). *Let ω be any compact open subset of $\mathbb{A}_G(F)$ and let $C := \pi_G^{-1}(\omega)$, a closed and open G-invariant subset of $\mathfrak{g}$ that is bounded modulo conjugation. There exists a lattice L in $\mathfrak{g}$ such that the following two statements hold.*

(1) *For all $X \in C \cap \mathfrak{g}_{\mathrm{rs}}$ there is an equality*

$$\hat{O}_X = \sum_{i=1}^{r} \Gamma_i(X)\hat{\mu}_i$$

of functions on $L \cap \mathfrak{g}_{\mathrm{rs}}$.

(2) *For all $I \in J(C)$ there exist unique complex numbers $c_1, \dots, c_r$ such that*

$$\hat{I} = \sum_{i=1}^{r} c_i \hat{\mu}_i$$

on $L \cap \mathfrak{g}_{\mathrm{rs}}$.

PROOF. By Lemma 15.2 there exists a lattice L' such that $C \subset \mathrm{Ad}(G)(L')$. Thus the first statement follows from Corollary 27.9. In view of Theorem 27.8 the second statement follows from the first. □

28. Guide to notation

- See 4.1 for F, $\mathcal{O}$, π, $\mathrm{val}(x)$, G, $B = AN$, W, K, G_{der}, G_{sc}, $X_*(A)$.
- See 4.5 for Λ_G, H_G, $\mathfrak{a}$, $\mathfrak{a}_G$.
- See 7.8 for $\mathcal{L} = \mathcal{L}(A)$.
- See 7.12 for $\mathcal{F}(A)$, $\mathcal{P}(M)$.
- See 8.3 for $\mathcal{B}(A)$, $B_0 = AN_0$. See 14.2 for $\pi_G : \mathfrak{g} \to \mathbb{A}_G$.
- See 23.2 for $\mathcal{F}(M)$.

References

[Art76] J. Arthur, *The characters of discrete series as orbital integrals*, Invent. Math. **32** (1976), 205–261.

[Art78] J. Arthur, *A trace formula for reductive groups. I. Terms associated to classes in $G(\mathbb{Q})$*, Duke Math. J. **45** (1978), 911–952.

[Art81] J. Arthur, *The trace formula in invariant form*, Ann. of Math. (2) **114** (1981), 1–74.

[Art87] J. Arthur, *The characters of supercuspidal representations as weighted orbital integrals*, Proc. Indian Acad. Sci. Math. Sci. **97** (1987), 3–19.

[Art89a] J. Arthur, *The L^2-Lefschetz numbers of Hecke operators*, Invent. Math. **97** (1989), 257–290.

[Art89b] J. Arthur, *Towards a local trace formula*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 1–23.

[Art91a] J. Arthur, *A local trace formula*, Inst. Hautes Études Sci. Publ. Math. **73** (1991), 5–96.

[Art91b] J. Arthur, *Some problems in local harmonic analysis*, Harmonic analysis on reductive groups (Brunswick, ME, 1989), Progr. Math., vol. 101, Birkhäuser Boston, Boston, MA, 1991, pp. 57–78.

[Art97] J. Arthur, *The problem of classifying automorphic representations of classical groups*, Advances in mathematical sciences: CRM's 25 years (Montreal, PQ, 1994), CRM Proc. Lecture Notes, vol. 11, Amer. Math. Soc., Providence, RI, 1997, pp. 1–12.

[BFQ79] P. Baum, W. Fulton, and G. Quart, *Lefschetz-Riemann-Roch for singular varieties*, Acta Math. **143** (1979), no. 3-4, 193–211.

[Bor91] A. Borel, *Linear algebraic groups*, Graduate Texts in Mathematics, vol. 126, Springer-Verlag, New York, 1991.

[Bou02] N. Bourbaki, *Lie groups and Lie algebras. Chapters 4–6*, Elements of Mathematics (Berlin), Springer-Verlag, Berlin, 2002.

[Bri88] M. Brion, *Points entiers dans les polyèdres convexes*, Ann. Sci. École Norm. Sup. (4) **21** (1988), no. 4, 653–663.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local. I*, Inst. Hautes Études Sci. Publ. Math. **41** (1972), 5–251.

[BZ76] I. N. Bernšteĭn and A. V. Zelevinskiĭ, *Representations of the group $GL(n, F)$, where F is a local non-Archimedean field*, Uspehi Mat. Nauk **31** (1976), no. 3(189), 5–70.

[Car00] H. Carayol, *Preuve de la conjecture de Langlands locale pour* GL_n*: travaux de Harris-Taylor et Henniart*, Astérisque (2000), no. 266, Exp. No. 857, 4, 191–243.

[Clo89] L. Clozel, *Orbital integrals on p-adic groups: a proof of the Howe conjecture*, Ann. of Math. (2) **129** (1989), no. 2, 237–251.

[Den84] J. Denef, *The rationality of the Poincaré series associated to the p-adic points on a variety*, Invent. Math. **77** (1984), no. 1, 1–23.

[Ful93] W. Fulton, *Introduction to toric varieties*, Ann. of Math. Studies, No. 131, Princeton University Press, 1993.

[Ful98] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], vol. 2, Springer-Verlag, Berlin, 1998.

[HC70] Harish-Chandra, *Harmonic analysis on reductive p-adic groups. Notes by G. van Dijk*, Lecture Notes in Mathematics, No. 162, Springer-Verlag, 1970.

[HC78] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups*, Proceedings of the 1977 annual seminar of the Canadian Mathematical Congress (W. Rossmann, ed.), Queen's Papers in Pure and Applied Mathematics, no. 48, 1978, pp. 281–347.

[HC99] Harish-Chandra, *Admissible invariant distributions on reductive p-adic groups. Notes by S. DeBacker and P.J. Sally, Jr.*, University Lecture Series, vol. 16, AMS, 1999.

[How73] R. Howe, *Two conjectures about reductive p-adic groups*, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), Amer. Math. Soc., Providence, R.I., 1973, pp. 377–380.

[Igu74] J.-I. Igusa, *Complex powers and asymptotic expansions. I. Functions of certain types*, J. Reine Angew. Math. **268/269** (1974), 110–130.

[Igu77] J.-I. Igusa, *Some observations on higher degree characters*, Amer. J. Math. **99** (1977), no. 2, 393–417.

[Kna86] A. Knapp, *Representation theory of semisimple groups*, Princeton University Press, Princeton, N.J., 1986.

[Lan66] R. P. Langlands, *Eisenstein series*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 235–252.

[Lan76] R. P. Langlands, *On the functional equations satisfied by Eisenstein series*, Lecture Notes in Mathematics, No. 544, Springer-Verlag, 1976.

[Lan80] R. P. Langlands, *Base change for $GL(2)$*, Ann. of Math. Studies 96, Princeton University Press, 1980.

[Lan02] S. Lang, *Algebra*, Graduate Texts in Mathematics, vol. 211, Springer-Verlag, New York, 2002.

[Nie74] H. Nielsen, *Diagonalizably linearized coherent sheaves*, Bull. Soc. Math. France **102** (1974), 85–97.

[Pan91] D. Panyushev, *Rationality of singularities and the Gorenstein property of nilpotent orbits*, Funct. Anal. Appl. **25** (1991), 76–78.

[Rao72] R. Rao, *Orbital integrals in reductive groups*, Ann. of Math. (2) **96** (1972), 505–510.

[Rog90] J. Rogawski, *Automorphic representations of unitary groups in three variables*, Annals of Mathematics Studies, vol. 123, Princeton University Press, Princeton, NJ, 1990.

[Ros61] M. Rosenlicht, *On quotient varieties and the affine embedding of certain homogeneous spaces*, Trans. Amer. Math. Soc. **101** (1961), 211–223.

[Ser92] J-P. Serre, *Lie algebras and Lie groups*, Lecture Notes in Mathematics, vol. 1500, Springer-Verlag, Berlin, 1992.

[Ser02] J-P. Serre, *Galois cohomology*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2002.

[Ser03] J-P. Serre, *Trees*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[Sha72] J. Shalika, *A theorem on semi-simple $\mathcal{P}$-adic groups*, Ann. of Math. (2) **95** (1972), 226–242.

[SS70] T. Springer and R. Steinberg, *Conjugacy classes*, Seminar in Algebraic Groups and Related Finite Groups, Lecture Notes in Math., 131, Springer Verlag, 1970, pp. 167–266.

[Ste75] R. Steinberg, *Torsion in reductive groups*, Advances in Math. **15** (1975), 63–92.

[Wal95] J.-L. Waldspurger, *Une formule des traces locale pour les algèbres de Lie p-adiques*, J. Reine Angew. Math. **465** (1995), 41–99.

[Wei82] A. Weil, *Adeles and algebraic groups*, Progress in Mathematics, vol. 23, Birkhäuser Boston, Mass., 1982.

Department of Mathematics, University of Chicago, 5734 University Ave., Chicago, IL 60637

E-mail address: `kottwitz@math.uchicago.edu`