Honors Single Variable Calculus 110.113

September 21, 2023

Contents

1 The natural numbers

Lecture 1, Monday, August 28th, Last updated: 01/09/23, dmy.
Reading: [10, Ch.2-3]

We assume the notion of *set*, 2, and take it as a primitive notion to mean a "collection of distinct objects."

Learning Objectives

Next eight lectures:

- To construct the objects:

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}$$

  and define the notion of *sets*, 2.

- To prove properties and reason with these objects. In the process, you will learn various proof techniques. Most importantly, *proof by induction* and *proof by contradiction*.

This lecture:

- how to define the natural numbers, $\mathbb{N}$, and appreciate the role of *definitions*.

- how to apply induction. In particular, we would see that even proving statements as associativity of natural numbers is nontrivial!

Pedagogy

1. $\mathbb{N}$ is presented differently in distinct foundations, such as ZFC or type theory. Our presentation is to be *agnostic* of the foundation. From a working mathematician point of view, it *does not matter*, how the natural numbers are constructed, as long as they obey the properties of the axioms, 1.1.

2. We take the point of view that in mathematics, there are various type of objects. Among all objects studied, some happened to be *sets*. Some presentation of mathematics[a] will regard all objects as sets.

The various types of mathematics are more or less equivalent in our context.

[a]such as ZFC

Why should we delve into the foundations? Two reasons:

1. Foundational language is how many mathematicians do new mathematics. One defines new axioms and explores the possibilities.

2. How can we even discuss mathematics without having a rigorous understanding of our objects?

Discussion

A *natural (counting) number*[a], as we conceived informally is an element of

$$\mathbb{N} := \{0, 1, 2, \ldots\}$$

What is ambiguous about this?

- What does "$\cdots$" mean? How are we sure that the list does not cycle back?

- How does one perform operations?

- What *exactly* is a natural number? What happens if I say

$$\{0, A, AA, AAA, AAAA, \ldots\}$$

  are the numbers?

We will answer these questions over the course.

[a] It does not matter if we regard 0 as a natural number or not. This is a convention.

Forget about the natural numbers we love and know. If one were to define the *numbers*, one might conclude that the numbers are about a concept.

Axioms 1.1. The *Peano Axioms*: [1] Guiseppe Peano, 1858-1932.

1. 0 is a natural number.
$$0 \in \mathbb{N}$$

2. if n is a natural number then we have a natural number, called the *successor of* n, denoted $S(n)$.
$$\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$$

[1] In 1900, Peano met Russell in the mathematical congress. The methods laid the foundation of *Principa Mathematica*

3. 0 is not the successor of any natural number.

$$\forall n \in \mathbb{N}, S(n) \neq 0$$

4. If $S(n) = S(m)$ then $n = m$.

$$\forall n, m \in \mathbb{N}, S(n) = S(m) \Rightarrow n = m$$

5. Principle of induction. Let $P(n)$ be any *property* on the natural number n. Suppose that

   a. $P(0)$ is true.
   b. When ever $P(n)$ is true, so is $P(S(n))$.

   Then $P(n)$ is true for all n natural numbers.

Discussion

What could be meant by a *property?* The principle of induction is in fact an *axiom schema*, consisting of a collection of axioms.

- "n is a prime".
- "$n^2 + 1 = 3$".

We have not yet shown that any collection of object would satisfy the axioms. This will be a topic of later lectures. So we will assume this for know.

Axiom 1.2. There exists a set $\mathbb{N}$, whose elements are the *natural numbers*, for which 1.1 are satisfied.

There can be many such systems, but they are all equivalent for doing mathematics.

Discussion

With only up to axiom 4: This can be *not* so satisfying. What have we done? We said we have a collection of objects that satisfy some concept F="natural numbers". But how do we know, Julius Ceasar does not belong to this concept?

Definition 1.3. We define the following natural numbers:

$$1 := S(0), 2 := S(1) = S(S(0)), 3 := S(2) = S(S(S(0)))$$

$$4 := S(3), 5 := S(4)$$

Intuitively, we want to continue the above process and say that whatever created iteratively by the above process are the *natural numbers*.

Discussion

- Give a set that satisfies axioms 1 and 2 but not 3.
- Give a set that satisfies axioms 1,2 and 3 but not 4.
- Give a set satisfying axioms 1,2,3 and 4, but not 5.

$$\{n/2 : n \in \mathbb{N}\} = \{0, 0.5, 1, 1.5, 2, 2.5, \cdots\}$$

Proposition 1.4. 1 is not 0.

Proof. Use axiom 3. □

Proposition 1.5. 3 is not equal to 0.

Proof. $3 = S(2)$ by definition, 1.3. If $S(2) = 0$, then we have a contradiction with Axiom 2, 1.1. □

1.1 Addition

Definition 1.6. (Left) Addition. Let $m \in \mathbb{N}$.

$$0 + m := m$$

Suppose, by induction, we have defined $n + m$. Then we define

$$S(n) + m := S(n + m)$$

In the context of 1.13, for each n, our function is $f_n := S : \mathbb{N} \to \mathbb{N}$ is $a_{S(n)} := S(a_n)$ with $a_0 = m$.

Proposition 1.7. For $n \in \mathbb{N}$, $n + 0 = n$.

Proof. Warning: we cannot use the definition 1.6. We will use the principle of induction. What is the *property* here in Axiom 5 of 1.1? The property $P(n)$ is "$0 + n = n$" for each $n \in \mathbb{N}$. We will also have to check the two conditions 5a. and 5b.

a "$P(0)$ *is true.*". People refer to this as the "base case $n = 0$": $0 + 0 = 0$, by 1.6.

b "*If $P(m)$ is true then $P(m+1)$ is true*". The statement "*Suppose $P(m)$ is true*" is often called the "inductive hypothesis". Suppose that $m + 0 = m$. We need to show that $P(S(m))$ *is true*, which is

$$S(m) + 0 = S(m)$$

By def, 1.6,

$$S(m) + 0 = S(m + 0)$$

By hypothesis,

$$S(m + 0) = S(m)$$

By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Such proof format is the typical example for writing inductions, although in practice we will often leave out the italicized part.

Example

Prove by induction

$$\sum_{i=1}^{n} i^2 := 1^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

We observed that we have successfully shown *right* addition with respect to 0 behaves as expected.

Discussion

What should we expect $n + S(m)$ to be?

- Why can't we use 1.6?
- Where would we use 1.7?

Proof is hw.

Proposition 1.8. Prove that for $n, m \in \mathbb{N}$, $n + S(m) = S(n + m)$.

Proof. We induct on n. Base case: $m = 0$.

5b. Suppose $n + S(m) = S(n + m)$. We now prove the statement for

$$S(n) + S(m) = S(S(n) + m)$$

by definition of 1.6,

$$S(n) + S(m) = S(n + S(m))$$

which equals to the right hand side by hypothesis.

□

Proposition 1.9. Addiction is commutative. Prove that for all $n, m \in \mathbb{N}$,

$$n + m = m + n$$

Proof. We prove by induction on n. With m fixed. We leave the base case away. □

Proposition 1.10. Associativity of addition. For all $a, b, c \in \mathbb{N}$, we have

$$(a + b) + c = a + (b + c)$$

Proof. hw. □

Discussion

Can we define "+" on any collection of things? What are examples of operations which are noncommutative and associative? For example, the collection of words?

$$+ : (\text{Seq. English words}) \times (\text{Seq. English words}) \to (\text{Seq. English words})$$

$$\text{"a" , "b"} \mapsto \text{"ab"}$$

This can be a meaningless operation. Let us restrict to the collection of *interpreable* outcomes. In the following examples, there is *structural ambiguity.*

1. (Ice) (cream latte)
2. (British) ((Left) (Waffles on the Falkland Islands))
3. (Local HS Dropouts) (Cut) (in Half)
4. (I ride) (the) (elephant in (my pajamas))
5. (We) ((saw) (the) (Eiffel tower flying to Paris.))

2,3 are actuay news title.

What use is there for addition? We can define the notion of *order* on $\mathbb{N}$. We will see later that this is a *relation* on $\mathbb{N}$.

Definition 1.11. Ordering of $\mathbb{N}$. Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \leq n$ iff there is $a \in \mathbb{N}$, such that $n = m + a$.

1.2 Multiplication

Now that we have addition, we are ready to define multiplication as 1.6.

Definition 1.12.

$$0 \cdot m := 0$$

$$S(n) \cdot m := (n \cdot m) + m$$

1.3 Recursive definition

What does the induction axiom bring us? Please ignore the following theorem on first read.

Theorem 1.13. Recursion theorem. Suppose we have for each $n \in \mathbb{N}$,

$$f_n : \mathbb{N} \to \mathbb{N}$$

Let $c \in \mathbb{N}$. Then we can assign a natural number a_n for each $n \in \mathbb{N}$ such that

$$a_0 = c \quad a_{S(n)} = f_n(a_n) \forall n \in \mathbb{N}$$

Discussion

The theorem seems intuitively clear, but there can be pitfalls.

- When defining $a_0 = c$, how are we sure this is *not* redefined after some future steps? This is Axiom 3. of 1.1
- When defining $a_{S(n)}$ how are we sure this is not redefined? This uses Axiom 4. of 1.1.
- One rigorous proof is in [4, p48], but requires more set theory.

Proof. The property $P(n)$ of 1.1 is "{ a_n is well-defined}". Start with $a_0 = c$.

- Inductive hypothesis. Suppose we have defined a_n - meaning that there is only one value!
- We can now define $a_{S(n)} := f_n(a_n)$.

□

1.4 References and additional reading

- Nice lecture notes by Robert.
- Russell's book [8, 1,2] for an informal introduction to cardinals.

2 Naïve Set Theory

Week 1, Wednesday, August 30th
As in the construction of $\mathbb{N}$, we will define a *set* via axioms.

Discussion

Why put a foundation of sets?

- This is to make rigorous the notion of a "collection of mathematical objects".
- This has its roots in cardinality. How can you "count" a set without knowing how to define a collection?
- The concept of a set can be used - and is till used in practice - as a practical foundation of mathematics. This forms the basis of *classical mathematics*.

Learning Objectives

In this lecture:

- We discuss *set* in detail. We will need this to construct the integers, $\mathbb{Z}$.
- We illustrate what one *can* and *can not* do with sets.

Pedagogy

Again, we don't say what they *are*. This approach is often taken, such as [4].

Discussion

What object can be called a *set*?

A *set* should be

- determined by a *description of the objects* [a] For example, we can consider

$$E := \text{"The set of all even numbers"}$$

$$P := \text{"The set of all primes"}$$

- If x is an object and A is a set, then we can ask whether $x \in A$ or $x \notin A$. *Belonging* is a primitive concept in sets.

[a] this set consists of all objects satisfying this description and *only those objects*.

In this lecture we will discuss some axioms.

Axiom 2.1. If A is a *set* then A is also a *object*.

Axiom 2.2. Axiom of extension. Two sets A, B are equal if and only if (for all objects x , $(x \in A \Leftrightarrow x \in B)$)

Axiom 2.3. There exist a set $\emptyset$ with no elements. I.e. for any object x, $x \notin \emptyset$.

Proposition 2.4 (Single choice)**.** Let A be nonempty. There exists an object x such that $x \in A$.

Proof. Prove by contradiction. Suppose the statement is false. Then for all objects x, $x \notin A$. By axiom of extension, $A = \emptyset$. $\square$

Discussion

How did we use the axiom of extension? Colloquially, some mathematicians would say "trivially true".

2.1 Subcollections

Definition 2.5. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1, 2\} \subset \{1, 2, 3\}$.

2.2 Comprehension axiom

Definition 2.6. Axiom of Comprehension.

Definition 2.7. *General* comprehension principle. (The paradox leading one). For any property φ, one may form the set of all x with property $P(x)$, we denote this set as

$$\{x \big| P(x)\}$$

Proposition 2.8. Russell, 1901. The general comprehension principle cannot work.

Proof. Let

$$R := \{x : x \text{ is a set and } x \notin x\}$$

This is a set. Then

$$R \in R \Leftrightarrow R \notin R$$

□

Discussion

How is this different from the axiom of specification?

Discussion

How can it even be the case that $x \in x$, for a set? Can this hold for any set x below?

- $\emptyset$
- The set of all primes.
- The set of natural numbers.

The latter two shows that : this set itself is *not even a number*! Indeed, In Zermelo-Frankel set theory foundations it will be proved that $x \notin x$ for all set x. So the set R in 2.8 is the *set of all sets.*

2.3 References

- A nice introduction to set theory is Saltzman's notes [9].
- The relevant section in Tao's notes, [10, 3].
- For the axioms of set theory, an elementary introduction is [4], and also notes by Asaf, [6].

3 Power set construction

Lecture 3: will miss one class due to Labor day.
Reading: [10, Ch.3.1-4], [7, 2].

Learning Objectives

In last lectures, we

- Defined $\mathbb{N}$ axiomatically using the Peano axioms.
- Used induction to prove properties of operations as $+$ and $\times$ on $\mathbb{N}$.

In the next two lectures

- Discuss the remaining axioms of set theory. We begin by discussing new notions: *subsets*, 2.1, We end with the construction of the power set.
- Discuss *equivalence relation*, 6, and *ordered pairs*, 6.1. which constructs the integers and the rationals

3.1 Remaining axioms of set theory

Week 2

In this section we continue from previous lecture and discuss remaining axioms from what is known as the *Zermelo-Fraenkel (ZF) axioms of set theory*, due to Ernest Zermelo and Abraham Fraenkel.

Axiom 3.1. Singleton set axiom. If a is an object. There is a set $\{a\}$ consists of just one element.

Axiom 3.2. Axiom of pairwise union. Given any two sets A, B there exists a set $A \cup B$ whose elements which belong to either A or B or both.

Often we would require a stronger version.

Axiom 3.3. Axiom of union. Let A be a set of sets. Then there exists a set

$$\bigcup A$$

whose objects are precisely the elements of the set.

Example

Let

- $A = \{\{1,2\},\{1\}\}$
- $A = \{\{1,2,3\},\{9\}\}$

Discussion

Using the axioms, can we get from $\{1,3,4\}$ to $\{2,4,5\}$?

We will now state the power set axiom for completeness but revisit again.

Axiom 3.4. Axiom of power set. Let X, Y be sets. Then there exists a set Y^X consists of all functions $f : X \to Y$.

We will review definition of function later, 3.11.

3.2 Replacement

If you are an ordinary mathematician, you will probably never use replacement.

Axiom 3.5. Axiom of replacement. For all $x \in A$, and y any object, suppose there is a statement $P(x, y)$ pertaining to x and y. $P(x, y)$ satisfies the property for a given x, there is a *unique* y. There is a set

$$\{y : P(x, y) \text{ is true for some } x \in A\}$$

Discussion

This can intuitively be thought of as the set

$$\{y : y = f(x) \text{ some } x \in A\}$$

That is, *if* we can define a function, then the range of that function is a set. However, $P(x, y)$ described may *not* be a function, see [3, 4.39].

Example

- Assume, we have the set $S := \{-3, -2, -1, 0, 1, 2, 3, \dots\}$, $P(x, y)$ be the property that $y = 2x$. Then we can construct the set

$$S' := \{-6, -4, -2, 0, 2, 4, 6, \dots\}$$

- If x is a set, then so is $\{\{y\} : y \in x\}$. Indeed, we let

$$P(x, y) : \text{"}y = \{x\}\text{"}$$

Again, this is a *schema* as described previously in axiom of comprehension 2.6.

Proposition 3.6. The axiom of comprehension 2.6 follows from axiom of replacement 3.5.

Proof. Let ϕ be a property pertaining to the elements of the set X. We can define the property [2]

$$\psi(x, y) : \begin{cases} y = \{x\} & \text{if } \phi(x) \text{ is true} \\ \emptyset & \text{if } \phi(x) \text{ is false} \end{cases}$$

Let

$$\mathcal{A} := \{y : \exists x, \quad \psi(x, y) \text{ is true}\}$$

be the collection of sets defined by axiom of replacement. Then by union axiom

$$\bigcup \mathcal{A} := \{x \in X : \phi(x) \text{ is true}\}$$

□

3.3 Axiom of regularity (well-founded)

As a first read, you can skip directly and read 3.9. For a set S, and a binary relation, $<$ on S, we can ask if it is *well-founded*. It is well founded when we can do *induction*.

Definition 3.7. A subset A of S is *$<$-inductive* if for all $x \in S$,

$$(\forall t \in S, t < x) \Rightarrow x \in A$$

Definition 3.8. Let X, Y we denote the *intersection of X and Y*[3] as

$$X \cap Y := \{x \in X : x \in X \text{ and } x \in Y\} = \{y \in Y : y \in X \text{ and } y \in Y\}$$

X and Y are *disjoint* if $X \cap Y = \emptyset$.

[2] This can be written in the language of "property" via $(\phi(x) \to y = \{x\}) \wedge (\neg\phi(x) \to y = \emptyset)$

[3] which exists, thanks to axiom of comprehension.

One would ould ask the $\in$ relation on all sets to be inductive. Then what would be required for that $A \notin A$?

Axiom 3.9. Axiom of foundation (regularity) The $\in$ relation is "well-founded". That is for all nonempty sets x, there exists $y \in x$ such that either

- y is not a set.
- or if y is a set, $x \cap y = \emptyset$.

An alternative way to reformulate, is that y is a *minimal element* under $\in$ relation of sets.

Example

- $\{\{1\}, \{1,3\}, \{\{1\}, 2, \{1,3\}\}\}$. What are the $\in$-minimal elements?
- Can I say that there is a "set of all sets"? No, see how.

One can use axiom of foundation that we cannot have an infinite descending sequence:

Proposition 3.10. There are no infinite descent $\in$-chains. Suppose that (x_n) is a sequence of nonempty sets. Then we cannot have

$$\cdots \in x_{n+1} \in x_n \cdots \in x_1 \in x_0$$

Similarly one can use axiom of replacement for the product, at p32.

3.4 Function

Discussion

How would you intuitively define a function?

Definition 3.11. Let X, Y be two sets. Let

$$P(x, y)$$

be a *property* pertaining to $x \in X$ and $y \in Y$, such that for all $x \in X$, there *exists a unique* $y \in Y$ such that $P(x, y)$ is true. A *function associated to* P is an object

$$f_P : X \to Y$$

such that for each $x \in X$ assigns an output $f_P(x) \in Y$, to be the unique object such that $P(x, f_P(x))$ is true. [4]

[4] We will often omit the subscript of P.

- X is called the *domain*
- Y is called the *codomain*.

Definition 3.12. The *image*...?

Discussion

What kind of properties P does not satisfy the condition of being function?

- "$y^2 = x$".
- "$y = x^2$".

4 The various sizes of infinity

Lecture 4: for competition. We will use our defined notion of, "counting numbers" or "inductive numbers", $\mathbb{N}$ to *count* other sets. This is *cardinality*. In this section, we fix sets X, Y.

Definition 4.1. A function $f : X \to Y$ is

- *injective* if for all $a, b \in X$, $f(a) = f(b)$ implies $a = b$.
- *surjective* if for all $b \in Y$, exists $a \in X$ st. $f(a) = b$.
- *bijective* if f is both injective and bijective.

Example

- the map from $\emptyset \to X$ an injection. The conditions for injectivity vacuously holds.
- $\mathbb{N}$ is in bijection with the set of even numbers,

$$\mathbb{E} := \{n \in \mathbb{N}\,;\, \exists k \in \mathbb{N} \,:\, n = 2k\}$$

- there is no bijection from an empty set to a nonempty set.

Definition 4.2. Two sets X, Y have *equal cardinality* if there is a bijection

$$X \simeq Y$$

- A set is said to have *cardinality* n if

$$\{i \in \mathbb{N} \,:\, 1 \leq i \leq n\} \simeq X$$

  In this case, we say X is *finite*. Otherwise, X is *infinite*.

- A set X is *countably infinite*[5] if it has same cardinality with $\mathbb{N}$.

Definition 4.3. We denote the *cardinality of a set* X by $|X|$. [6]

[5] Or *countable*. Sometimes countable means (finite *and* countably infinite).

[6] This definition does *not make sense yet!*. What if a set has two cardinality? Let us assume this is well-defined first. See question 2.

Discussion

To think about infinity is an interesting problem. Consider Hilbert's Grand Hotel.

- One new guest.
- 1000 guest.
- Hilbert Hotel 2 move over.
- Hilbert chain. Directs customer m in hotel n to position $3^n \times 5^m$. (This shows that $\mathbb{N} \times \mathbb{N} \hookrightarrow \mathbb{N}$.)

Historically, some take *cardinal numbers* as i.e. the equivalence class of bijective sets as the primitive notion.

Definition 4.4. Let X, Y be sets: We denote

- $|X| \leq |Y|$ if there is an injection from X to Y.
- $|X| = |Y|$ if there is a bijection between X and Y.
- $|X| < |Y|$ if $|X| \leq |Y|$ but $|X| \neq |Y|$.

One of the beautiful results in Set theory is the Schroeder Bernstein theorem.

Theorem 4.5. The $\leq$ relation on cardinality, is reflexive: if $|X| \leq |Y|$ and $|Y| \leq |X|$ then $|X| = |Y|$. [7]

Without axiom of choice, one cannot say the following: for all sets X and Y, either $|Y| \leq |X|$ or $|X| \leq |Y|$.

[7] Why is this not obvious? Challenge: google and try to understand the proof.

5 Homework for week 2

Due: Week 3, Friday. All questions in 5.1, Boolean algebra is compulsory. Select 3 other questions to be graded.

Reading: A nice reference in set theory, [2, 4]. We collectively refer to the axioms of set theory we have discussed thus far as the ZF axioms. We did not discuss the axiom of replacement, [10, 3.5] and regularity. This will be left as required reading for certain problems.

Problems

1. Let A, B, C be sets.

   (a) Prove set inclusion, is reflexive and transitive, i.e.

   $$(A \subseteq B \wedge B \subseteq A) \Rightarrow A = B$$

   $$(A \subseteq B \wedge B \subseteq C) \Rightarrow A \subseteq C$$

   the notation $\wedge$ here reads "and".

   (b) Prove that the union operation $\cup$ on sets 3.2, is associative and commutative:

   $$A \cup (B \cup C) = (A \cup B) \cup C$$
   $$A \cup B = B \cup A$$

2. (**) Let I be a set and that for all $\alpha \in I$, I have a set A_α. [8] Read about the axiom of replacement; see [10, Axiom 3.5] or 3.5.

   (a) Prove that under the ZF axioms, one can form the union of the collection:

   $$\bigcup_{\alpha \in I} A_\alpha := \bigcup \{A_\alpha \, : \, \alpha \in I\}$$

   In particular, explain why the following two objects

   i.

   $$\{A_\alpha \, : \, \alpha \in I\}$$

[8] For example, if $I = \{a, b, c\}$, then I have three sets

$$A_a, A_b, A_c$$

ii.
$$\bigcup \{A_\alpha : \alpha \in I\}$$
are sets.

(b) Give a one line explanation briefly describing why axiom of union 3.3 is insufficient to construct the set $\bigcup_{\alpha \in I} A_\alpha$.

3. The *axiom of regularity* states

Axiom 5.1. [10, 3.9] If A is a nonempty set, then there is at least one element $x \in A$ which is either not a set or, (if it is a set) disjoint from A.

Prove (with singleton set axiom) that for all sets $A, A \notin A$.

4. (***) Let A, B, C, D be sets. This exercise shows that we can actually construct *ordered pairs* using the ZF axioms.[9] Prove

- We can construct the following set [10]
$$\langle A, B\rangle := \{A, \{A, B\}\}$$
from the axioms of set theory.
- $\langle A, B\rangle = \langle C, D\rangle$ if and only if $A = C, B = D$. For this part you will require the *axiom of regularity.* in problem 3. You are free to use the results there.

5. This is a variation of problem 4 [11]. Suppose for two sets A, B we define
$$[A, B] = \{\{A\}, \{A, B\}\}$$
In this case, the problem is a lot easier. Prove $[A, B] = [C, D]$ if and only if $A = C, B = D$.

6. (***) Show that the collection
$$\{Y : Y \text{ is a subset } X\}$$
is a set using the ZF axioms. We denote this as the power set 2^X, where 2 is regarded as the two elements set $\{0, 1\}$. You will need to use the axiom of replacement.
Here are two important remarks on possible false solutions:

[9] Another definition is discussed in or [10, 3.5.1], where they assume this as an axiom.
[10] RIP. So another model of this is $\langle A, B\rangle := \{\{A\}, \{A, B\}\}$
[11] which is what I should have written

(a) (Ryan's) if your property for axiom of replacement $P(x, y) =$ "y is a subset of x" then this is *not correct*. The condition for replacement is that *there is at most one* y, [10, 3.6].

(b) (Kauí's) You cannot use axiom of comprehension, this is similar to Russell's paradox!

As a hint: $\{0,1\}^X$ is a set, by 3.4. For $Y \subseteq X$, $f \in \{0,1\}^X$, let $P(f, Y)$ be the property that

$$Y = f^{-1}(1) := \{x \in X \,:\, f(x) = 1\}$$

5.1 Boolean algebras

This section is compulsory. Boolean algebras form the foundation of probability theory. We will need this later when we get to the projects.

Reading: For some overview of the context, see [1, 1-3], [5, 1], or Tao's Lecture 0 on probability theory.

Definition 5.2. Let Ω be a set. A *Boolean algebra* in Ω is a set $\mathcal{E}$ of subsets of Ω (equivalently, $\mathcal{E} \subseteq 2^{\Omega}$) satisfying

1. $\emptyset \in \mathcal{E}$
2. closed under unions and intersections.

$$E, F \in \mathcal{E} \Rightarrow E \cup F \in \mathcal{E}$$
$$E, F \in \mathcal{E} \Rightarrow E \cap F \in \mathcal{E}$$

3. closed under complements.

A *σ-algebra in* Ω is a Boolean algebra in Ω such that it satisfies

4. Countable[12] closure. If $A_i \in \mathcal{E}$ for $i \in \mathbb{N}$, then $\bigcup A_i \in \mathcal{E}$.

Problems

1. Prove that $\mathcal{E} := \{\emptyset, \Omega\}$ is a σ-algebra.
2. Prove that $2^{\Omega} := \{E \,:\, E \subseteq \Omega\}$ is a σ-algebra.
3. Let $A \subseteq \Omega$, what is the smallest (describe the elements of this σ-algebra) σ-algebra in Ω containing A?

Hints for problems

3. There are 3 cases. What happens $A = \emptyset$ or $A = \Omega$? Now consider the case $A \neq \emptyset$ and $A \neq \Omega$.

[12] A set X is countable if it is in bijection with $\mathbb{N}$. We will explore this word in further detail in the future.

6 Equivalence Relation

Week 3 Reading: [10, Ch.3.5, Ch.4], *On the construction of* $\mathbb{Q}$*, see* [3, 2.4].

Learning Objectives

Last few lectures:

- Defined the natural numbers and sets axiomatically.
- Discussed how *cardinality* came up from "counting" sets.

This and next lecture:

- discuss equivalence relation.
- construct $\mathbb{Z}, \mathbb{Q}$. Extend addition and multiplication in this context.

6.1 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 6.1. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1, 2\} = \{2, 1\}$

In ordered pairs

- $(1, 2) \neq (2, 1)$

Definition 6.2. Let X, Y be two sets. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) : x \in X, y \in Y\}$$

Currently, we can either put the existence of such a set as an axiom, or use the axioms of set theory, this is in hw.

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [4, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 6.3. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$
$$a \sim a$$
- *Transitive*. For all $a, b, c \in A$,
$$a \sim b, b \sim c \Rightarrow a \sim c$$
- *Symmetric*. For all $a, b \in A$,
$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Often times, people do not describe the subset R, but describe it a relation *equivalently* as a rule: saying $a, b \in A$ are related if some property $P(a, b)$ is true. In short hand, one writes

$$a \sim b \text{ iff } \ldots$$

Definition 6.4. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as [13]

$$[x] := \{y \in A \,:\, x \sim y\}$$

An element in such an equivalence is called a *representative* of that class.

Definition 6.5. Quotient set. Given an equivalence relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

Example

Consider $\mathbb{N}$ and the equivalence relation that $a \sim b$ iff a and b have the same parity. [a]

- There are two equivalence classes: the odds and evens.
- For the odd class, a *representative*, or an element in the equivalence class, is 3.

[a] i.e. both or odd or even.

There is a relation between equivalence and partition of sets.

Definition 6.6. A *partition* of a set X is a collection ???

6.2 Integers

What are the integers? It consists of the natural numbers and the negative numbers. What is *subtraction*? We do not know yet. Can we define *negative* numbers without referencing minus sign? Yes, we can. Say

$$-1\text{is "}0-1\text{" is } (0,1)$$

Discussion

Let us say we define the integers as pairs (a, b) where $a, b \in \mathbb{N}$. Would this be our desired

$$\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$$

- How many -1s are there?

But we have a problem, there are multiple ways to express -1. Our system cannot have multiple -1s. What are other ways We can also have $1-2$, or the pair $(1, 2)$.

[13] It does not matter if we write $\{y \in A \,:\, y \sim x\}$ by symmetry condition.

Discussion

Now that we have our $\mathbb{Z}$, how do we define addition? [a]Can we leverage our understanding?

[a]What is addition abstractly? It is an operation $+: X \times X \to X$.

Intuitively, the *integers* is an expression [14] of non-negative integers, (a, b), thought of as $a - b$. Two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 6.7. Let

$$R \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$

consists of all pairs (a, b) and (c, d) such that $a + d = b + c$. Equivalently,

$$R := \{(a, b), (c, d) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \,:\, a + d = b + c\}$$

The *integers* is the set

$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

Definition 6.8. Addition, multiplication. [10, 4.1.2] .

We can now finally define negation.

Definition 6.9. [10, 4.1.4].

Proposition 6.10. Algebraic properties. Let $x, y, z \in \mathbb{Z}$.

- Addition
  - Symmetric $x + y = y + x$.
  - Admits identity element.

6.3 Rational nunbers

Reading: [3, 2.4]
In a similar manner

Definition 6.11. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote a pair (a, b) by a/b.

Again, we need the notion of addition, multiplication, and negation.

[14]Rather than a pair, as an expression has multiple ways of presentation

Definition 6.12. Let $a/b, c/d \in \mathbb{Q}$. Then

1. Addition:
$$a/b + c/d := (ad + bc)/bd$$
2. Multiplication
$$a/b \cdot c/d := (ac)/(bd)$$
3. Negation.
$$-(a/b) := (-a)/b$$

Discussion

Is this definition well defined? What does this mean? This is hw.

Similarly, we can define also define order relation.

Definition 6.13. Let $x \in \mathbb{Q}$,

- x is *positive* iff $x = a/b$ where a, b are positive integers, we often denote positive integers as $\mathbb{Z}_{>0}$.
- x is *negative* iff $x = -y$ where y is some positive rational.

With the notion of positive rationals[15] from def. 6.13, we can define order relation $<, \leq$ on $\mathbb{Q}$.

Definition 6.14. Let $x, y \in \mathbb{Q}$, then we say

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is zero or positive.

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 6.15. $\sqrt{2}$ is not rational.

Proof. ??? □

[15] The same trick is used to define order in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

7 Homework for week 3

Due: Week 4, Saturday. You will select 3 problems to be graded.
Problems 1-3 are on cardinality. Problem 4 is on a general construction of equivalence relations. Problems 5-7 is about addition, multiplication, and division on $\mathbb{Z}$ and $\mathbb{Q}$.

1. Show that the relation $\leq$ is transitive, i.e. $|X| \leq |Y|, |Y| \leq |Z|$ then $|X| \leq |Z|$.

2. (**) Prove that $\mathbb{N} \times \mathbb{N}$ is countably infinite. [16] Prove that $\mathbb{Q}$ is countably infinite. *You are free to use results from previous problems and theorems stated in lectures.*

3. (**) Let X be any set. Prove that there is no surjection (hence, bijection) between X and $\{0,1\}^X$. Deduce that $\{0,1\}^{\mathbb{N}}$ is uncountable. Argue the first part by contradiction:

   - Consider the set
   $$A = \{x \in X \,:\, x \notin f(x)\}$$
   - As f is a surjection (write the general definition) there must exists $a \in X$ such that $f(a) = A$. Do case work on whether $a \in A$ or $a \notin A$. [17]

4. (**) Let X be any set. Recall that a binary relation on X, is any subset $R \subseteq X \times X$. We define $R^{(n)}$ as follows

   - For $n = 0$,
   $$R^{(0)} = \{(x,x) \,:\, x \in X\}$$
   - Suppose $R^{(n)}$ has been defined.
   $$R^{(n+1)} := \left\{(x,y) \in X \times X \,:\, \exists z \in X, (x,z) \in R^{(n)}, (z,y) \in R\right\}$$

   (a) Show that
   $$R^t := \bigcup_{n \geq 1} R^{(n)} = R^{(1)} \cup R^{(2)} \cup \cdots$$
   defines a *smallest* transitive relation on X containing R. i.e. if Y is any other transitive relation on X containing R, then $R^t \subseteq Y$.

[16] Knowing the Cartesian product is required for this problem, skip 5. and 6. if unfamiliar.
[17] The argument is similar to that of Russell's argument.

(b) Show that

$$R^{tr} := \bigcup_{n\geq 0} R^{(n)} = R^{(0)} \cup R^{(1)} \cdots$$

is the *smallest* reflexive and transitive relation on X. i.e. if Y is any other transitive and reflexive relation on X containing R, then $R^{st} \subseteq Y$.

5. (***) Show that addition, product, and negation are well-defined for rational numbers; see def. 6.11 or [10, 4.2]. You are free to use any facts and properties you know about $\mathbb{Z}$.

6. (*) Let $x, y, z \in \mathbb{Z}$. Use the definition of addition and multiplication from 6.8, or [10, 4.1], show :

   (a) $x(y + z) = xy + xz$.

   (b) $x(yz) = (xy)z$.

   You are free to use any facts and properties you know about $\mathbb{N}$.

7. Let $x, y \in \mathbb{Z}$. You are free to use any facts you know about $\mathbb{N}$, in particular, it would be helpful to use the following the result: [10, 2.3.3]: *Let* $n, m \in \mathbb{N}$. *Then* $n \times m = 0$ *if and only if at least one of* n, m *is equal to zero.* Show that if $xy = 0$ then $x = 0$ or $y = 0$.

7.1 Tri-weekly diary

8. (**) Write a 800-1000 words diary or story. Pen down a diary on your experiences with the course topics and experiences so far, focusing particularly on:

   - Concepts or ideas that you initially found challenging or confusing. For example, the axioms of natural numbers $\mathbb{N}$, set theory, etc.
   - Topics that have piqued (if any, XD) your curiosity.
   - Topics that you wanted to be covered, and why.
   - Topics that you would like further elaboration.
   - People you find fun to be with (or scared of)!

   + (*) points for the best diary.

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