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ON CYCLES IN FLAG MANIFOLDS

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0. Introduction.

Let K be a compact connected real Lie group and T a maximal torus in K.

In 1954 R. Bott constructed a Morse function on K/T and showed that K/T was a cell complex with cells in the even dimensions only. That means that the cells considered as cycles give a basis for the homology of K/T. It is easy to calculate these cycles explicitely and in fact they turn out to be the so-called K-cycles of Bott–Samelson [2], which were constructed in 1958 in a more general setting using Morse theory of loop spaces.

The space K/T also appears as G/B, where G is the complexification of K and B is a Borel group in G containing T. In 1954 F. Bruhat discovered that if G was one of the classical Lie groups, G/B had a cell decomposition, each cell being isomorphic as an algebraic variety to $\mathbf{C}^n$. This was soon afterwards proved to be the case for all reductive linear algebraic groups G by Chevalley [3].

The closure of a Bruhat cell can be considered as a cycle (see [4]) and these cycles again generate the homology of G/B.

Now the reductive groups are exactly the complexifications of the compact real groups (see [5]), so we have two decompositions of $K/T = G/B$. We prove that they are identical, and as a consequence of the proof we solve another problem. The closure of a Bruhat cell is in general an algebraic variety with singularities and the construction of the K-cycles can be improved to give a resolution of these singularities.

In section 1 we describe the K-cycles, in section 2 the Bruhat decomposition and in section 3 we show the identity and construct the resolution.

1. The *K*-cycles.

Let K and T be as above. $L(K)$ and $L(T)$ will denote the Lie algebras and $\pm\alpha_i$, $i=1,\ldots,m$, the roots. The hyperplanes through 0 in $L(T)$ given

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by $\alpha_i(x)=0$ are called O_i. The stabilizer of the plane O_i is K_i. The Weyl group $N(T)/T$ is denoted W.

W operates on $L(T)$ by the adjoint action. There is an $r_i \in K_i$, $i=1,2,\ldots,m$, such that $r_i \in W$ and $\mathrm{Ad}(r_i)$ is the reflection in O_i. Corresponding to a choice of fundamental root system we have a fundamental Weyl chamber $\mathscr{F}$ and we keep $w \in W$ fixed. $\mathrm{Ad}(w)$ brings $\mathscr{F}$ to another Weyl chamber $\mathrm{Ad}w(\mathscr{F})$. Let s be a straight line from $\mathscr{F}$ to $\mathrm{Ad}w(\mathscr{F})$ crossing the planes O_i one at a time. We can assume that they are met in the order $O_1,O_2,\ldots,O_k$. It is then clear that $\mathrm{Ad}(r_k \ldots r_2 r_1)$ brings $\mathscr{F}$ to $\mathrm{Ad}w(\mathscr{F})$. Since w operates simply transitively on the Weyl chambers, w must be equal to $r_k \ldots r_2 r_1$.

Now we define

$$\Gamma_w = K_1 \times_T K_2 \times \ldots \times (K_k/T)$$

as the orbit space of the action of $T \times \ldots \times T$ on $K_1 \times \ldots \times K_k$ given by

$$(t_1,t_2,\ldots,t_k)(k_1,\ldots,k_k) = (k_1t_1, t_1^{-1}k_2t_2,\ldots,t_{k-1}^{-1}k_kt_k) .$$

We define $g\colon \Gamma_w \to K/T$ by

$$g[(k_1,k_2,\ldots,k_k)] = k_1k_2\ldots k_kr_k\ldots r_1T .$$

Γ_w is orientable. Let γ_w be a cycle determining the orientation. Then $g_*(\gamma_w)$ is the K-cycle corresponding to w. As shown by Bott–Samelson [2] the set of all $g_*(\gamma_w)$ where $w \in W$, constitute a basis for $H_*(K/T)$.

2. The Bruhat cells.

Following the notation of [1] let G be a reductive complex linear algebraic group with maximal torus $\boldsymbol{T}$ and B a Borel group containing $\boldsymbol{T}$, $B=U\boldsymbol{T}$ where U is the unipotent part of B.

The set of roots is Φ and for each root α the eigenspace $\mathfrak{g}_\alpha$ is the Lie algebra of U_α. $L(\boldsymbol{T})\oplus\mathfrak{g}_\alpha\oplus\mathfrak{g}_{-\alpha}$ is the Lie algebra of the group G_α. The roots fall into two parts, the positive part $\Phi(B)$ and the negative part $-\Phi(B)$, such that the Lie algebra of B is the direct sum of $L(\boldsymbol{T})$ and the eigenspaces corresponding to the positive roots, whereas the sum of $L(\boldsymbol{T})$ and the eigenspaces corresponding to the negative roots is the Lie algebra of $B'=U^-\boldsymbol{T}$, where U^- is the unipotent part of B'.

Also define $U_w^- = U \cap wU^-w^{-1}$ for $w \in W = N(\boldsymbol{T})/\boldsymbol{T}$. With this notation let us recall the Bruhat decomposition theorem [1, p. 347]:

THEOREM 2.1. *G/B is the disjoint union of the U-orbits UwB, $w \in W$. If $w \in W$ the morphism*

$$U_w{}^- \to UwB \quad (u \mapsto uwB)$$

is an isomorphism of varieties.

Moreover $U_w{}^-$ *is the semi-direct product of the* U_α*'s, e.g. it contains the* U_α*'s such that* $\alpha > 0$ *and* $\alpha^w < 0$.

If we consider G as the complexification of K, then K is embedded in G. If T is a maximal torus of K we let B be a Borel group of G containing $\boldsymbol{T}$, the complexification of T.

The Weyl group of G is as in section 1 generated by the r_i, $i=1,\ldots,m$, the action of the Weyl group is the same in the two cases, and the set of roots Φ restricted to T is exactly the roots of K,T.

Let $w \in W$. We saw in section 1 that $w = r_k \ldots r_2 r_1$, where we met $O_1,\ldots,O_k$ successively with a straight line s from $\mathscr{F}$ to $\mathrm{Ad}\,w(\mathscr{F})$. For later use we continue the enumeration of the O_i's beyond $\mathrm{Ad}\,w(\mathscr{F})$ until we meet the opposite Weyl chamber of $\mathscr{F}$.

Lemma 2.2.

$$w(\Phi(B)) = (\Phi(B) \setminus \{\alpha_1,\ldots,\alpha_k\}) \cup \{-\alpha_1,\ldots,-\alpha_k\}\,.$$

Proof. The roots $\Phi(B)$ are the ones taking positive values in $\mathscr{F}$. The roots in $w(\Phi(B))$ are the ones taking positive values in $w(\mathscr{F})$. Now following the line s we first go through O_1 coming to another Weyl chamber. Here all roots in $\Phi(B)$ still take positive value, except α_1, because we passed through the 0-hyperplane O_1 of α_1. But then $-\alpha_1$ takes positive value. An obvious induction now finishes the proof since a root α_i only changes sign along s, when s passes through O_i.

In the following we write U_i for $U\alpha_i$. As a consequence of Lemma 2.2 we get:

Lemma 2.3. *For* $w = r_k \ldots r_2 r_1$ *as above the group* $U_w{}^-$ *equals* $U_k \ldots U_2 U_1$, *the semi-direct product of the* U_i*'s,* $i=1,\ldots,k$.

3. The resolution.

Keeping the notation of section 2 we shall study a typical Bruhat cell of G/B,

$$UwB = U_1 \ldots U_k r_k \ldots r_1 B\,,$$

where $w = r_k \ldots r_2 r_1 \in W$. We want to compare this cell with the set

$$g(\Gamma_w) = K_1 \ldots K_k r_k \ldots r_1 B$$

underlying the K-cycle. In fact the closure of UwB equals $g(\Gamma_w)$. To show this we need a new variety.

Let B_i be the connected subgroup of G with Lie algebra equal to the direct sum of $L(\boldsymbol{T})$ and the eigenspaces of $-\alpha_j$, $j=1,\ldots,i$ and α_j, $j=i+1,\ldots,k$. This is a Lie algebra, since it equals $\mathrm{Ad}\,w(L(B))$. Using this fact for $i=l$ and $i=l-1$ it is easily seen that also the direct sum of $L(\boldsymbol{T})$ and the eigenspaces of $-\alpha_j$, $j=1,\ldots,l$, and α_j, $j=l,\ldots,k$ is a Lie algebra. The corresponding subgroup we denote H_l. In fact $H_i=G_iB_i$.

Definition 3.1. Let

$$M_w = H_1 \times_{B_1} H_2 \times_{B_2} \ldots \times H_k/B_k$$

be the orbit space of the action of $(B_1,\ldots,B_k)$ on $(H_1,\ldots,H_k)$ given by

$$(h_1,\ldots,h_k)(b_1,\ldots,b_k) = (h_1b_1{}^{-1}, b_1h_2b_2{}^{-1},\ldots,b_{k-1}^{-1}h_kb_k)\ .$$

Using [7] it is seen by induction that M_w is a non-singular complex algebraic variety of real dimension $2k$.

Lemma 3.2. *The map induced by inclusion*

$$i\colon K_1 \times_T K_2 \times \ldots \times K_k/T \to H_1 \times_{B_1} H_2 \times \ldots \times H_k/B_k$$

is a homeomorphism.

Proof. i is one–one, since $K_i \cap B_i = T$. But Γ_w and M_w are manifolds of the same dimension, hence the conclusion.

Now consider the commutative diagram

$$\begin{array}{ccc} M_w = H_1 \times_{B_1} H_2 \times \ldots \times H_k/B_k & \xrightarrow{\varphi} & G/B \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ \Gamma_w = K_1 \times_T K_2 \times \ldots \times K_k/T & \xrightarrow{g} & K/T \end{array}$$

where g was defined in section 1 and φ is defined similarly by

$$\varphi[(h_1,\ldots,h_k)] = h_1h_2\ldots h_kr_k\ldots r_1B\ .$$

The K-cycle $g(\Gamma_w)$ is now seen to be the same as $\varphi(M_w)$, but

$$\varphi(M_w) = H_1\ldots H_kr_k\ldots r_1B$$

obviously contains $U_1\ldots U_kr_k\ldots r_1B$. Moreover, according to Theorem 2.1 the dimension of UwB is $2k$ and the dimension of $\varphi(M_w)$ is not greater. Now Γ_w is compact, which ensures us that $\varphi(M_w)$ is compact and thus closed. More precisely, $\varphi(M_w)$ contains the closure of UwB in the strong topology and therefore also in the Zariski topology, because UwB is constructible (cf. [6]).

Since closed subvarieties of an algebraic variety always have strictly smaller dimension we can conclude that $\varphi(M_w)$ equals the closure of UwB. We have thus proved:

THEOREM 3.3. *The sets underlying the K-cycles of Bott–Samelson are the closures of the Bruhat cells.*

We have seen that it suffices to take representatives for elements in M_w from $K_1 \times K_2 \ldots \times K_k$. More illuminating is the following:

LEMMA 3.4. *Elements in M_w can be represented by elements of the form $(v_1,\ldots,v_k)$, where $v_i \in U_i \cup \{r_i\}$.*

PROOF. Let $[(h_1,h_2,\ldots,h_k)]$ be an arbitrary element in M_w. We shall find $(b_1,\ldots,b_k) \in (B_1,\ldots,B_k)$ such that

$$(h_1 b_1, b_1^{-1} h_2 b_2, \ldots, b_{k-1}^{-1} h_k b_k) = (v_1,\ldots,v_k)$$

where $v_i \in U_i \cup \{r_i\}$. Assume inductively that we found $(b_1,\ldots,b_j)$ such that

$$(h_1 b_1, \ldots, b_{j-1}^{-1} h_j b_j) = (v_1,\ldots,v_j), \quad v_i \in U_i \cup \{r_i\} .$$

Now using Theorem 2.1 on G_i we obtain

$$G_{\alpha_i} = U_{-\alpha_i} U_{-\alpha_i} \boldsymbol{T} \cup U_{-\alpha_i} r_i U_{-\alpha_i} \boldsymbol{T}$$

and therefore

$$G_{\alpha_i} = r_i U_{-\alpha_i} \boldsymbol{T} \cup U_{\alpha_i} U_{-\alpha_i} \boldsymbol{T} .$$

Hence

$$H_i = G_{\alpha_i} B_i = r_i B_i \cup U_i B_i \quad \text{for } i=1,\ldots,k .$$

Since $b_j^{-1} h_{j+1} \in H_{j+1}$, we can thus find $b_{j+1} \in B_{j+1}$ such that

$$b_j^{-1} h_{j+1} b_{j+1} \in U_{j+1} \cup \{r_j\} ,$$

and the induction step is concluded.

THEOREM 3.5. *$\varphi\colon M_w \to G/B$ is a resolution of the closure of UwB.*

PROOF. We have only left to show that φ is one–one when restricted to $\varphi^{-1}(UwB)$. By 2.1 we know that

$$i\colon\ U_1 \times \ldots \times U_k \to U_1 \ldots U_k r_k \ldots r_i B$$

is a homeomorphism. So according to Lemma 3.4 we only have to show that elements outside of $[U_1 \times \ldots \times U_k]$ of the form $[(v_1, \ldots, v_k)]$, where $v_i = r_i$ for at least one i, map outside of UwB by φ. But such elements are in the boundary of $[U_1 \times \ldots \times U_k]$ in M_w, and therefore the images are in the boundary of UwB, which is disjoint from UwB since it consists of other Bruhat cells.

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