Uniformization of Principal Bundles

Minh-Tâm Quang Trinh

Massachusetts Institute of Technology

Plan of talk:

1. Overview
2. Gluing
3. Generic Trivialization
4. Smoothness

Sources:

- Beauville–Laszlo, « Un lemme de descente . . . »
- Beauville–Laszlo, "Conformal Blocks. . . "
- Drinfeld–Simpson, "B-Structures. . . "
- Heinloth, "Uniformization. . . "
- Lurie, Harvard Math 282y S14 lecture notes

§1 Overview

k algebraically closed

G connected reductive algebraic group / k

X smooth projective curve / k

$\mathcal{B}un$ moduli of (fppf) G-torsors over X

Weil observed $\mathcal{B}un(k) \simeq G(F_X)\backslash G(\mathbf{A}_X)/G(\mathbf{O}_X)$.

For $k = \bar{\mathbf{F}}_q$ and G simply-connected semisimple and $v \in X(k)$, strong approximation says

$$G(F_X)G(\hat{F}_v) \text{ is dense in } G(\mathbf{A}_X),$$

which implies the surjectivity of

$$G(\hat{F}_v)/G(\hat{\mathcal{O}}_v) \to \mathcal{B}un(k).$$

$L_v G(R) := G(R \hat{\otimes} \hat{F}_v), \qquad L_v^+ G(R) := G(R \hat{\otimes} \hat{\mathcal{O}}_v)$

Affine Grassmannian: $\mathcal{G}r_v := (L_v G / L_v^+ G)^\sharp$

Thm For simply-connected semisimple G, the map

$$\mathcal{G}r_v \to \mathcal{B}un$$

is étale-surjective. (Also ind-smooth.)

Two steps:

1. Beauville–Laszlo: $\mathcal{G}r_v \simeq \mathcal{G}r_v^{glob}$ as functors.
2. Drinfeld–Simpson: $\mathcal{G}r_v^{glob} \to \mathcal{B}un$ is étale-surjective.

$\Delta_{v,R} := \mathrm{Spec}(R \hat{\otimes} \hat{\mathcal{O}}_v), \qquad \Delta_{v,R}^\times := \mathrm{Spec}(R \hat{\otimes} \hat{F}_v)$

$$\mathcal{G}r_v(R) = \left\{ (E, \alpha) \;\middle|\; \begin{array}{l} E \to \Delta_{v,R} \text{ is a } G\text{-torsor}, \\ \alpha \in \Gamma(\Delta_{v,R}^\times, E) \end{array} \right\}$$

On k-points, $gG(\hat{\mathcal{O}}_v) \rightsquigarrow (E, \alpha) = (G \times \Delta_v, g)$.

$X_R := X \times \mathrm{Spec}(R), \qquad X_R^\times := (X - v) \times \mathrm{Spec}(R)$

$$\mathcal{G}r_v^{glob}(R) := \left\{ (E, \alpha) \;\middle|\; \begin{array}{l} E \to X_R \text{ is a } G\text{-torsor}, \\ \alpha \in \Gamma(X_R^\times, E) \end{array} \right\}$$

$\mathcal{G}r_v^{glob} \to \mathcal{G}r_v$ is restricting to Δ_v.

$\mathcal{G}r_v^{glob} \to \mathcal{B}un$ is forgetting α.

§2 Gluing

Want to show $\mathcal{G}r_v^{glob} \to \mathcal{G}r_v$ is an isomorphism.

Intuitively, if $(E, \alpha) \in \mathcal{G}r_v(R)$, then we use α to glue E to the trivial torsor on $X_R^\times$.

That is, descent along:

$$X_R^\times \sqcup \Delta_{v,R} \to X_R \tag{$\circledast$}$$

Objections:

1. If R is not noetherian, then $\Delta_{v,R} \to X_R$ may not be flat,* so ($\circledast$) may not be fpqc.
2. Not clear that a gluing map over $\Delta_{v,R}^\times$ provides a descent datum for ($\circledast$).

* Stacks Project Tag 0AL8

A 2010 solution by Heinloth:

1. $\mathcal{G}r_v$ and $\mathcal{G}r_v^{glob}$ being of ind-finite type, they are determined by their restrictions to noetherian R.

2. Descent datum for ($\circledast$) is an element of

$$G(X_R^\times) \times G(\Delta_{v,R}^\times) \times G(\Delta_{v,R}^\times) \times G(\Delta_{v,R} \times_{X_R} \Delta_{v,R})$$

satisfying a cocycle condition.

Show that $\Delta_{v,R} \times_{X_R} \Delta_{v,R}$ is the pushout of $\Delta_{v,R}$ along the diagonal $\Delta_{v,R}^\times \to \Delta_{v,R}^\times \times_{X_R} \Delta_{v,R}^\times$.

$$\begin{aligned} g \in G(\Delta_{v,R}^\times) &\rightsquigarrow g^{-1} \boxtimes g \in G(\Delta_{v,R}^\times \times_{X_R} \Delta_{v,R}^\times) \\ &\rightsquigarrow g^{-1} \boxtimes g \in G(\Delta_{v,R} \times_{X_R} \Delta_{v,R}) \end{aligned}$$

The datum is $(1, g, g^{-1}, g^{-1} \boxtimes g)$.

The 1995 solution by Beauville–Laszlo:

Reduce to $G = \mathrm{GL}_n$. Replace $X_R^\times \sqcup \Delta_{v,R} \to X_R$ with

$$\mathrm{Spec}(A[\tfrac{1}{t}]) \sqcup \mathrm{Spec}(\hat{A}) \to \mathrm{Spec}(A)$$

where $t \in A$ is a non-zerodivisor and $\hat{A} = \lim_n A/(t^n)$.

Thm (BL) Fix $M' \in \mathrm{Mod}(A[\frac{1}{t}])$, $M'' \in \mathrm{Mod}(\hat{A})$,

$$\varphi : M' \otimes_A \hat{A} \xrightarrow{\sim} M'' \otimes_A A[\tfrac{1}{t}].$$

If M'' has no t-torsion, then there exist $N \in \mathrm{Mod}(A)$,

$$\psi' : N \otimes_A A[\tfrac{1}{t}] \xrightarrow{\sim} M', \quad \psi'' : N \otimes_A \hat{A} \xrightarrow{\sim} M''$$

all essentially unique, such that φ results from ψ', ψ''.
No noetherian hypotheses.

Proof of existence Let N be the kernel of:

$$(\star) \qquad M' \to M' \otimes \hat{A} \xrightarrow{\varphi} M''[\tfrac{1}{t}] \to (M''[\tfrac{1}{t}])/M''.$$

Tensoring up to $A[\frac{1}{t}] \times \hat{A}$ shows $(\star)$ is surjective, so

$$(\star\star) \qquad 0 \to N \to M' \to (M''[\tfrac{1}{t}])/M'' \to 0$$

is exact.

Tensoring $(\star\star)$ up to $A[\frac{1}{t}]$, *resp.* $\hat{A}$, gives ψ', *resp.* ψ''.

Hard part is ψ'' because $A \to \hat{A}$ may not be flat. But

$$\mathrm{Tor}_1^A(\hat{A}, (M''[\tfrac{1}{t}])/M'') = \varinjlim_n \mathrm{Tor}_1^A(\hat{A}, (\tfrac{1}{t^n} M'')/M'')$$

vanishes, using injectivity of $M'' \xrightarrow{t^n} M''$.

§3 Generic Trivializations

Want to show $\mathcal{G}r_v^{glob} \to \mathcal{B}un$ is étale-surjective.

Fix a Borel $B \subseteq G$. A B-reduction of a G-torsor E is an isomorphism

$$(F \times G)/B \xrightarrow{\sim} E,$$

where F is a B-torsor and $(f, g) \cdot b = (fb, b^{-1}g)$.

Thm (DS) For any G-torsor $E \to X_R$, there is an étale map $R \to R'$ such that $E|_{X_{R'}}$ has a B-reduction.

Thm (DS) Take G simply-connected semisimple.

For any $v \in X(R)$ and G-torsor $E \to X_R$, there is an étale map $R \to R'$ such that $E|_{X_{R'} - v_{R'}}$ trivializes.

Lem A B-reduction of $E \to Y$ is equivalent to a section of the associated bundle

$$E/B := (E \times G/B)/G.$$

Explicitly, if s is such a section, then $F \to Y$ defined by the fiber product

$$\begin{tikzcd} F \arrow[r, "\tilde{s}"] \arrow[d] & E \arrow[d] \\ Y \arrow[r, "s"] & E/B \end{tikzcd}$$

is the B-torsor.

The map $(F \times G)/B \xrightarrow{\sim} E$ sends $[f, g] \mapsto \tilde{s}(f)g$.

Thm A For any G-torsor $E \to X_R$, there is an étale map $R \to R'$ such that $E|_{X_{R'}}$ has a B-reduction.

Proof Fix R and $E \to X_R$. For any R-algebra R', let

$$\mathcal{S}_{R,E}(R') = \Gamma(X_{R'}, (E/B)|_{X_{R'}}).$$

Want to trivialize $p : \mathcal{S}_{R,E} \to \mathrm{Spec}(R)$ étale-locally.

Let $\mathcal{S}^\circ_{R,E} \subseteq \mathcal{S}_{R,E}$ be the locus where p is smooth.

Suffices to show $p|_{\mathcal{S}^\circ_{R,E}}$ is surjective.

1. Check that p is surjective on k-points.
2. For any $x \in \mathrm{Spec}(R)(k)$, give $y \in p^{-1}(x)$ such that

$$s_y \in \Gamma(x^*(X_R), x^*(E/B)) = \Gamma(X, (x^*E)/B)$$

satisfies $\mathrm{H}^1(X, s_y^* T_{(x^*E)/B \to X}) = 0$.

Claim (1) Equivalent to $R = k$ case of **Thm A**, with $R' = k$ as well.

Take a G-torsor over X. By Steinberg, it trivializes at the generic point η, hence over a dense open $U \subseteq X$.

Pick a section over U. Since G/B is proper, the valuative criterion says it extends to a section over X.

Rem The proof above generalizes to smooth affine algebraic $\mathcal{G} \to X$ with $\mathcal{G}_\eta$ connected reductive. See Lurie's Math 282y S14 notes.

Rem We can prove Steinberg's theorem using the regular centralizer scheme over $\mathfrak{g} /\!\!/ G$. See Gaitsgory's 2009 seminar notes.

Claim (2) Suppose we have

$$\begin{aligned} &E \in \mathcal{B}un(R) && \\ &x \in \operatorname{Spec}(R)(k) && \\ &y \in \mathcal{S}_{R,E}(k) && \text{lifting } x \\ &s_y \in \Gamma(X, (x^*E)/B) && \text{defined by } y \end{aligned}$$

The relative tangent bundle $T_{(x^*E)/B \to X}$ is a vector bundle on $(x^*E)/B$.

$\mathrm{H}^1(X, s_y^* T_{(x^*E)/B\to X})$ controls deformations of s_y:

$$\mathrm{H}^1(X, s_y^* T_{(x^*E)/B\to X}) = 0 \iff y \in \mathcal{S}^\circ_{R,E}(k).$$

For fixed x, must modify y such that LHS holds.

Now we can forget R.

$$E_\circ = X \times G, \qquad E_\bullet = x^*E.$$

Start with any y and set $s = s_y \in \Gamma(X, E_\bullet/B)$.

Since the B-reduction $s^*E_\bullet$ is generically trivial, can find a dense open $U \subseteq X$ and an isomorphism

$$\beta : (E_\circ/B)|_U \xrightarrow{\sim} (E_\bullet/B)|_U$$

such that $\beta \circ \mathbf{1}|_U = s|_U$, where $\mathbf{1}$ is the zero section.

Lem There exist $E_\circ/B \xleftarrow{\phi} M \xrightarrow{\tilde\beta} E_\bullet/B$ and a divisor D supported on $X - U$ such that:

1. ϕ restricts to an isomorphism $(E_\circ/B)|_U \xleftarrow{\sim} M|_U$.
2. If $\sigma \in \Gamma(X, E_\circ/B)$ satisfies $\sigma|_D = \mathbf{1}|_D$, then $\sigma = \phi \circ \tilde\sigma$ for some unique lift $\tilde\sigma \in \Gamma(X, M)$.
3. $T_{M\to X} \simeq \phi^* T_{E_\circ/B\to X}(-D)$.
4. β factors through $\tilde\beta$.

M is the *dilitation* of $E_\circ/B$ along $(\mathbf{1}, D)$.

If $D = \emptyset$, then set $M_\emptyset = E_\circ/B$ and $\mathbf{1}_\emptyset = \mathbf{1}$.

If $D = [p] + D'$, then lift $\mathbf{1}_{D'}$ to $\mathbf{1}_D \in \Gamma(M_{D'})$ and set

$$M_D = \mathrm{Blowup}_{\mathbf{1}_D(p)}(M_{D'}) - \mathrm{Blowup}_{\mathbf{1}_D(p)}(M_{D',p}).$$

The map $M = M_D \to X$ remains smooth.

Pick $\sigma \in \Gamma(X, E_\circ/B)$ as in the lemma.

By (3), $\quad \mathrm{H}^1(\tilde{\sigma}^* T_{M \to X}) \simeq \mathrm{H}^1(\sigma^* T_{E_\circ/B \to X}(-D))$.

By (4), $\quad \mathrm{H}^1(\tilde{\sigma}^* T_{M \to X}) \twoheadrightarrow \mathrm{H}^1(\tilde{\sigma}^* \tilde{\beta}^* T_{E_\bullet/B \to X})$.

(Use the fact that $(\tilde{\sigma}^* T_{M \to X})|_U \simeq (\tilde{\beta}^* T_{E_\bullet/B \to X})|_U$.)

Remains to pick σ so that $\mathrm{H}^1(\sigma^* T_{E_\circ/B \to X}(-D)) = 0$.

Then $\tilde{s} = \tilde{\beta} \circ \tilde{\sigma} \in \Gamma(X, E_\bullet/B)$ is our modification of s.

The section $\sigma \in \Gamma(X, E_\circ/B)$ is equivalent to a map $g : X \to G/B$.

Lem For any divisor $D \subseteq X$, there is $g : X \to G/B$ such that $g(p) = B$ for all $p \in D$ and

$$\mathrm{H}^1(X, g^* T_{G/B}(-D)) = 0.$$

Proof sketch Let $T \subseteq B$ be a maximal torus.

Let $\deg(g) \in \mathrm{X}_*(T) = \mathrm{Hom}(\mathrm{X}^*(T), \mathbf{Z})$ be the map

$$\mathrm{X}^*(T) = \mathrm{Pic}(pt/B) \xrightarrow{L} \mathrm{Pic}(G/B) \xrightarrow{g^*} \mathrm{Pic}(X) \xrightarrow{\deg} \mathbf{Z}.$$

Via filtering, reduce from $T_{G/B}$ to $L(\lambda)$ with $\lambda \in \Phi_-$.

Reduce to finding g_n such that $\langle \deg(g_n), \lambda \rangle > n$ for all n and $\lambda \in \Phi_-$.

Via a branched cover, reduce to $X = \mathbf{P}^1$ and $n = 0$.

Thm B Take G simply-connected semisimple.

For any $v \in X(R)$ and G-torsor $E \to X_R$, there is an étale map $R \to R'$ such that $E|_{X_{R'} - v_{R'}}$ trivializes.

Etale-locally over $\mathrm{Spec}(R)$, pick a B-reduction F.

Let F' be the extension of F along $B \twoheadrightarrow T \hookrightarrow B$.

Write $B = T \ltimes U$. As $X_R - v$ is affine and U is filtered by copies of $\mathbf{G}_a$, we can show $F'|_{X_R - v} \simeq F|_{X_R - v}$.

So we can assume F has a T-reduction.

Since T is commutative, T-torsors form a group stack.

Suppose that $\check{\lambda} \in \mathrm{X}_*(T)$ and two T-torsors differ by the $\check{\lambda}$-extension of some $\mathbf{G}_m$-torsor.

Suffices to show that the associated G-torsors must be isomorphic étale-locally on $\mathrm{Spec}(R)$.

Since G is simply-connected, it suffices to assume $\lambda^\vee$ is a simple coroot $\check{\alpha}$.

So it suffices to take G generated by T and $r_{\check{\alpha}}(\mathrm{SL}_2)$.

Such a group is the product of SL_2 or GL_2 with some smaller torus.

"In the first case it suffices to show that the restriction [to $X_R - v$] of an SL_2*-bundle on X is trivial locally [over R]. In the second case it is enough to show that that the restriction... of two* GL_2*-bundles on X with the same determinant are isomorphic locally..."*

In fact, Beauville–Laszlo did the SL_n case by induction on n, and the GL_2 case is similar.

Key Idea A high-enough twist at v of the associated vector bundle can be split locally over $\mathrm{Spec}(R)$.

§4 Smoothness

Thm The map $\mathcal{G}r_v^{glob} \to \mathcal{B}un$ is (formally) smooth.

That is: Suppose $R \to R'$ is a square-zero extension of k-algebras and $E \in \mathcal{B}un(R)$ and $E' = E|_{X_{R'}}$. Then

$$\Gamma(X_{v,R}^{\times}, E) \to \Gamma(X_{v,R'}^{\times}, E') \text{ is surjective.}$$

Key idea Below, $X_{v,R'}^{\times} \to X_{v,R}^{\times}$ is square-zero and $E|_{X_{v,R}^{\times}} \to X_{v,R}^{\times}$ is smooth:

$$\begin{tikzcd}
 & E'|_{X^{\times}_{v,R'}} \arrow[d] \\
X^{\times}_{v,R'} \arrow[ur, "\alpha'"] \arrow[d] & E|_{X^{\times}_{v,R}} \arrow[d] \\
X^{\times}_{v,R} \arrow[ur, dashed, "\alpha"] \arrow[r, equal] & X^{\times}_{v,R}
\end{tikzcd}$$

Thank you for listening.