Central Degenerations of Semi-infinite Orbits, MV Cycles and Iwahori Orbits in the Affine Grassmannian of Type A

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Abstract

We study the algebraic geometry and combinatorics of the central degeneration (the degeneration that shows up in Gaitsgory's central sheaves and local models of Shimura varieties) of semi-infinite orbits, MV Cycles, and Iwahori orbits in the affine Grassmannian of type A. We prove that the special fiber limit of a semi-infinite orbit in the affine Grassmannian is a semi-infinite orbit in the affine flag variety. Moreover, we give some bounds for the number of irreducible components in the limits of MV cycles and Iwahori orbits by calculating their moment polytopes. Finally, we relate these special fiber limits to the intersections of Iwahori orbits and semi-infinite orbits in the affine Grassmannian and affine flag variety. These intersections are in turn related to affine Deligne-Lusztig varieties.

Contents

1 Introduction

1.1 Background

Let G be a connected reductive group over a field $k = \mathbb{C}$. Fix a Borel subgroup B and a maximal torus $T \subset B$. W denotes the Weyl group of G. Let $\mathcal{K}$ denote the local field of Laurent series $k((t))$, and $\mathcal{O}$ denote the ring of integers $k[[t]]$. Let $\mathbb{D} = spec(\mathcal{O})$ denote the formal disc, and $\mathbb{D}^* = spec(\mathcal{K})$ denote the punctured formal disc.

Let $G(\mathcal{K})$ be the corresponding group over the field $\mathcal{K}$. It is also called the loop group. Let $G(\mathcal{O})$ denote the corresponding group over $\mathcal{O}$. It is a maximal compact subgroup of $G(\mathcal{K})$, and is called the formal arc group. Let $I \subset G(\mathcal{O})$ denote the Iwahori subgroup.

We can form the two quotients $Gr = G(\mathcal{K})/G(\mathcal{O})$, which is called the affine Grassmannian, and $Fl = G(\mathcal{K})/I$, which is called the affine flag variety. Let Fl_C denote the global affine flag variety over a curve C where the general fibers are isomorphic to $Gr \times G/B$, and the special fiber is isomorphic to Fl.

Note that if G is a complex, connected, reductive group, then its affine Grassmannian is the disjoint union of $\pi_1(G)$ many copies of the affine Grassmannian of the simply-connected semisimple group with the same root system as G.

Let $U = N(\mathcal{K})$, $U^- = N^-(\mathcal{K})$, and $U_w = wUw^{-1}$. The orbits of these groups in the affine Grassmannian or the affine flag variety are called semi-infinite orbits, and are indexed by $T-$fixed points.

$T-$fixed points in the affine Grassmannian are indexed by elements in the coweight lattice, $X_*(T)$, of G. In the affine flag variety, $T-$fixed points are indexed by the affine Weyl group W_{aff}, which is isomorphic to $X_*(T) \rtimes W$.

The central degeneration involves a flat family of ind-schemes in the global affine flag variety. This degeneration is important for the study of local models of Shimura varieties, as well as Gaitsgory's central sheaves in the category of Iwahori-equivariant perverse sheaves on the affine flag variety.

This paper explores this abstract process in a very concrete way. We study the explicit algebraic geometry and combinatorics of this degeneration.

1.2 Main Results

Let S be a subscheme of $Gr \times \{\mathrm{id}\} \subset Gr \times G/B$ that is invariant under $T \subset G$ and $\mathrm{Aut}(\mathbb{D})$, the automorphism group of $\mathbb{D}$. For example, S could be a $G(\mathcal{O})$ orbit, an MV cycle, an Iwahori orbit, a semi-infinite orbit, an orbit of $T(\mathcal{O})$, etc. Then we would like to know the special fiber limit $\tilde{S}$ of S in the affine flag variety. This is an integral family over a curve, and is therefore flat.

When the underlying curve is $\mathbb{A}^1$, $\tilde{S} = \overline{(S \times \{\mathrm{id}\}) \times (\mathbb{A}^1 \backslash \{0\})} \cap Fl$ in $Fl_{\mathbb{A}^1}$. In other words $\tilde{S}$ is the closure in the special fiber of a flat family of schemes isomorphic to S over the $\mathbb{A}^1 \backslash \{0\}$.

We prove that the special fiber limit of the closed orbit of U_w, S_w^{μ}, is the corresponding closed orbit of U_w, $S_w^{(\mu,1)}$ in the affine flag variety. Here μ is a coweight and $(\mu, 1)$ is an element in W_{aff}. Therefore, this central degeneration preserves the semi-infinite/periodic Bruhat order, but not the usual Bruhat order.

Then we study the irreducible components in the special fiber limits of MV cycles and Iwahori orbits, by considering the type A lattice pictures and moment polytopes. We develop the techniques of calculating dimensions through moment polytopes and provide upper bounds and lower bounds for the number of irreducible components of the special fiber limits. We also describe the moment polytopes of some special irreducible components corresponding to the vertices of the moment polytopes.

In addition, we prove that the special fiber limit of an Iwahori orbit is a union of the orbits of different subgroups of the Iwahori group, and is invariant under a small subgroup in the Iwahori group. In particular, it is not invariant under the original Iwahori group.

Given an MV cycle $A = \overline{\cap_{w \in W} S_w^{\mu_w}}$, we conjecture that $\tilde{A}$, the special fiber limit of A, is the intersection of the corresponding closed semi-infinite orbits $\cap_{w \in W} \overline{S_w^{(\mu_w,1)}}$. One possible approach to prove this conjecture is to consider the degenerations of $T-$orbits and $T(\mathcal{O})$ orbits. This approach has been useful in calculating examples in lower dimensions.

For readers who like examples and diagrams, we explicitly describe all the irreducible components of the special fiber limit of an MV cycle and an Iwahori orbit in the $G = SL_3$ case.

Finally, the special fiber limits of MV cycles are contained in the intersections of certain Iwahori orbits and U^-orbits affine flag variety. Such intersections are in turn related to affine Deligne-Lusztig varieties. We prove that the intersections of Iwahori orbits and U^- orbits in the affine Grassmannian are equi-dimensional and give a dimension formula for them. We also discuss some algorithms and dimension bounds for the intersections of Iwahori orbits and U^- orbits in the affine flag variety.

1.3 Relations to Other Work and Future Projects

There are a few different directions that we would like to pursue after this project.

A natural project would be to apply some of the techniques used in this paper to find sharper bounds for the dimensions of affine Deligne-Lusztig varieties.

The beautiful polytopes for the special fiber limits in the affine flag variety inspire further investigation of their properties and connections with tropical geometry.

Gaitsgory constructed some central sheaves on the affine flag variety by considering the nearby cycles of the $G(\mathcal{O})$ equivariant perverse sheaves on the affine Grassmannian. Since this central degeneration behaves well with respect to semi-infinite orbits, we would like to study the U and U^- equivariant sheaves under the nearby cycles functor for the global affine flag variety.

1.4 Organization

The layout of the paper is as follows.

Sections 2-5 consist of introductory materials.

Section 6 discusses the degenerations of one-dimensional subvarieties in the affine Grassmannian, with a special emphasis on the orbits of root subgroups. These orbits are also the one-dimensional orbits of the extended torus.

Section 7 is about the degenerations of a finite product of root subgroups orbits. This is very useful for the subsequent sections.

Section 8 discusses the degenerations of the orbits of $N_w(\mathcal{K})$. It turns out that the central degeneration respects the semi-infinite/periodic Bruhat order.

Section 9 is about the degenerations of MV cycles and Iwahori orbits for $G = SL_2$, a very nice special case.

Section 10 contains a discussion of the special fiber limits of MV cycles, by looking at the moment polytopes of different irreducible components. In particular, we give some bounds on the number of irreducible components of the special fiber limit of a given MV cycle.

Section 11 is about the degenerations of Iwahori orbits. We prove that the special fiber limit of an Iwahori orbit is a union of orbits of different subgroups of the Iwahori group, and is invariant under a small subgroup of the Iwahori group. Moreover, there are similar bounds on the number of irreducible components of the special fiber limits of Iwahori orbits.

Section 12 is a discussion of intersections of Iwahori orbits and U^- orbits in the affine flag variety, which contains the special fibers of MV cycles. This is related to the studies of affine Deligne-Lusztig varieties.

1.5 Acknowledgements

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2 Affine Grassmannian, Affine Flag Variety and Their Global Counterparts

2.1 Loop Groups and Loop Algebra

Let G be a connected, reductive group over the field k, k being $\mathbb{C}$ or $\mathbb{F}_q$. Fix a Borel subgroup B and a maximal torus $T \subset B$. Let $\mathcal{K}$ denote the local field of Laurent series $k((t))$, and $\mathcal{O}$ denote the ring of integers $k[[t]]$.

$G(\mathcal{K})$ is the group scheme that is also called the loop group, as it is the group of analytic maps $\mathbb{C}^* \to G$.

$G(\mathcal{O}) \subset G(\mathcal{K})$ is a maximal compact subgroup of $G(\mathcal{K})$, and is called the formal arc group, as it is the group of analytic maps $\mathbb{C}^* \to G$ that can be extended to $0 \in \mathbb{C}$.

There is a map $ev_0 : G(\mathcal{O}) \to G$ by evaluating at $t = 0$. Let the Iwahori subgroup I denote the pre-image of B under ev_0. Similarly, let I_w denote the pre-image of the conjugate of B wBw^{-1} under ev_0.

Now let $\mathfrak{g}_{\mathbb{C}}$ denote a complexified Lie algebra of G. The complexified Lie algebra of the loop group LG is the loop algebra $L\mathfrak{g}_{\mathbb{C}} = \oplus_{k \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}} \cdot z^k$.

There is a rotation action of $\mathbb{C}^*$ on the loops, and we obtain a semidirect product $\mathbb{C}^* \tilde{\times} LG$.

The complexified Lie algebra of $\mathbb{C}^* \tilde{\times} LG$ decomposes as

$$(\mathbb{C} \oplus \mathfrak{t}_{\mathbb{C}}) \oplus (\oplus_{k \neq 0} t_{\mathbb{C}} z^k) \oplus_{(k,\alpha)} \mathfrak{g}_\alpha z^k$$

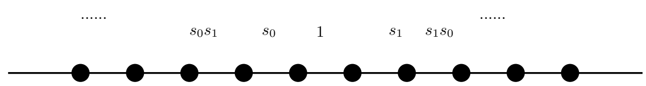
according to the characters of $\mathbb{C}^* \times T$.

The affine Weyl group $W_{aff} = N_G(\tilde{T})/\tilde{T}$ is isomorphic to a semi-direct product of the coweight lattice and the finite Weyl group. It acts transitively on the set of alcoves when G is simply connected, e.g. $G = SL_n(k)$.

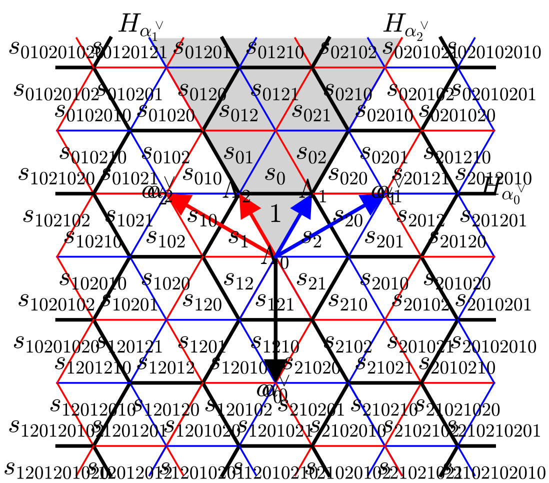
A root subgroup U_α is the one-parameter group $\exp(\eta \cdot e_\alpha)$, where α is an affine root and η is the parameter.

For any w in the finite Weyl group W, the group $U_w = (wNw^{-1})(\mathcal{K})$ is an infinite product of root subgroups in the loop group.

Below is a diagram for the affine A_1 root system with alcoves labelled by elements in the affine Weyl group.



Below is a diagram for the affine A_2 root system with alcoves labelled by elements in the affine Weyl group.



2.2 Affine Grassmannian and Affine Flag Variety

The affine Grassmannian is the ind-scheme $G(\mathcal{K})/G(\mathcal{O})$. There is a sequence of finite type projective schemes Gr^i, $i \in \mathbb{N}$ and closed immersions $Gr^i \hookrightarrow Gr^{i+1}$, $Gr(S) = \lim Hom(S, Gr^i)$.

In type A when $G = GL_n(k)$, the affine Grassmannian $Gr = G(\mathcal{K})/G(\mathcal{O})$ is isomorphic to the moduli space below:

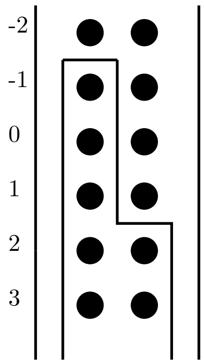
$Gr = \{W \subset \mathcal{K}^n | W \ is \ a \ rank \ n \ \mathcal{O} - module, \ \exists N \gg 0 \ s.t. \ t^N W_0 \subset W \subset t^{-N} W_0\}$, where each W is called a lattice.

When $G = SL_n(k)$, then the affine Grassmannian for G is isomorphic to the moduli spaces of lattices with zero relative dimension. This is related to the fact that matrices in SL_n have determinant one.

Each lattice W could be written as a direct sum $W = \mathcal{O}v_1 \oplus \cdots \mathcal{O}v_n$, where $\{v_1, ..., v_n\}$ is a $\mathcal{K}-$basis of $\mathcal{K}^n$.

By choosing the standard basis $\{e_1, e_2, \cdots, e_n\}$ of $\mathbb{C}^n$, we could represent W in some pictures. Note that $\mathcal{B} = \{t^m \cdot e_i | m \in \mathbb{Z}, i = 1, 2, ..., n\}$ form a $\mathbb{C}-$basis of $\mathbb{K}^n$, in the sense appropriate to a topological vector space with the $t-$adic topology.

For example, when $n = 2$, if $W = \mathcal{O} \cdot t^{-1}e_1 \oplus \mathcal{O} \cdot e_2$, W could be represented as below:



In the picture above, each dot represents an element in $\mathcal{B}$. The straight lines are used to indicate a basis of a the lattice W, which is a rank n $\mathcal{O}-$module. Each basis element is expressed as a linear combination of elements in $\mathcal{B}$.

We could consider the projective schemes Gr^i above as all the lattices W such that $t^iW_0 \subset W \subset t^{-i}W_0$. Then Gr^i is a union of ordinary Grassmannians in $V = (t^{-i}W_0)/(t^iW_0)$ with the extra condition $tL \subset L$ for any subspace L in these ordinary Grassmannians.

The (complete) affine flag variety is the quotient $G(\mathcal{K})/I$. In type A, there is also a lattice picture. It is the space of complete flags of lattices $W. = (W_1 \supset \cdots \supset W_n)$. Each W_i is a lattice such that $W_n \supset tW_1$ and $\dim(W_j/W_{j+1}) = 1$.

There are interpretations of the affine Grassmannian and affine flag variety in terms of principal $G-$bundles, for an algebraic group G of any type, not just type A.

The affine Grassmannian Gr is a functor that associates each scheme S the set of pairs $\mathcal{P}, \phi$, where $\mathcal{P}$ is an $S-$family of $G-$bundles over the formal disc $\mathbb{D}, \phi : \mathcal{P}|_{\mathbb{D}^*} \to \mathcal{P}^0|_{\mathbb{D}^*}$ is a trivialization of $\mathcal{P}$ on $\mathbb{D}^*$.

The (complete) affine flag variety Fl associates to each scheme S all the data above plus a $B-$reduction of the Principal $G-$bundle $\mathcal{P}$ at $\{0\}$.

There is a natural projection map $Fl \to Gr$ with fibers being isomorphic to the flag variety G/B.

2.3 GGMS Strata in the Affine Grassmannian and Affine Flag Variety

The Gelfand-Goresky-Macpherson-Serganova (GGMS) strata on the affine Grassmannian or the affine flag variety are the closures of intersections of some semi-infinite orbits. More precisely, given any collection $\alpha. = (\alpha_w)_{w \in W}$ of elements in the affine Weyl group W_{aff}, we can form the GGMS stratum

$$A(\alpha.) = \overline{\cap_{w \in W} S_w^{\alpha_w}}.$$

In the case of affine Grassmannian, each α_w has to be a coweight.

The moment polytope of the closure of of a nonempty GGMS stratum $A(\alpha.)$ is the convex hull of $\alpha.$. These are called pseudo-Weyl polytopes.

There is a discussion about GGMS strata in the affine Grassmannian in [18] We make a few more comments about GGMS strata in the affine flag variety below.

Lemma 1. *$A(\alpha.)$ is non-empty only if the following two conditions are satisfied:*

(1) $\alpha_v \leq_w \alpha_w$ for all v, w;

(2) If two vertices t^{α_1} and t^{α_2} lie on the same line in the moment polytope of $\overline{A(\alpha.)}$, then there is a one-parameter root subgroup $U_{s\beta}$ such that $\lim_{s\to\infty} U_{s\beta}\cdot\alpha_1 = \alpha_2$.

Moreover, $A(\alpha.)$ is non-empty if and only its projection to every copy of the affine Grassmannian is non-empty.

2.4 MV Cycles and MV Polytopes

Mirkovic-Vilonen cycles (MV cycles) are very interesting projective varieties in the $G(\mathcal{O})$ orbits of the affine Grassmannian. Each MV cycle could be defined as an irreducible component of the closed intersection of a $G(\mathcal{O})$ orbit and a U^- orbit. Alternatively it could be defined as an irreducible component of the closed intersection of a U orbit and a U^- orbit, as well as a GGMS stratum $A(\mu.)$ with some extra conditions on the coweights $\mu.$ involved.

They gave a canonical basis of the highest weight representations of the Langlands dual group.

MV polytopes are the $T-$equivariant moment polytopes of MV cycles, T being the maximal torus in G.

2.5 Iwahori Orbits in the Affine Grassmannian and Affine Flag Variety

Let λ denote a coweight of G and λ_{dom} denote the dominant coweight associated to λ. In the affine Grassmannian, the $G-$orbit of t^λ is the partial flag variety G/P_λ. P_λ denotes the parabolic subgroup of G with a Levi factor associated to the roots α such that $\lambda(\alpha) = 0$.

Let ev_0 denote the map $G(\mathcal{O}) \to G$ by evaluating at $t = 0$. We know that each $G(\mathcal{O})$-orbit Gr^λ in the affine Grassmannian Gr is a vector bundle over a partial flag variety G/P_λ.

The vector bundle projection map is given by $Gr^\lambda \cong G(\mathcal{O})\cdot t^\lambda \to^{ev_0} G/P_{\lambda_{dom}} \cong G\cdot t^\lambda$.

The fibers are isomorphic to $I_1 \cdot t^{\lambda_{dom}}$ as vector spaces. Here I_1 is the subgroup of I that is the pre-image of the identity element under the map ev_0. The dimension of each fiber is $2\cdot ht(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}})$.

Let W denote the Weyl group of G. The partial flag variety G/P_λ has a cell decomposition indexed by elements of the coset $\tilde{W} = W/W_I$, where W_I is the subgroup generated by permutations of the simple roots associated to P_λ. Let X_w and $\overline{X_w}$ denote the open and closed Schubert cell corresponding to $w \in \tilde{W}$. For a unique w in the coset $\tilde{W}$, $\lambda = w\cdot\lambda_{dom}$. Then the Iwahori orbit I^λ in the affine Grassmannian Gr is the pre-image of the open Schubert cell X_w under the map ev_0 above, and is a vector bundle over X_w.

The dimension of the Iwahori orbit in the affine Grassmannian I^λ is $\dim(X_w \subseteq G/P_{\lambda_{dom}}) + 2\,\mathrm{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}})$.

Example 1. *In the case of $G = SL_2(\mathbb{C})$, G/P_λ is $\mathbb{P}^1$ for $\lambda \neq 0$ and is a point for $\lambda = 0$.*

When $\lambda \neq 0$, if λ is dominant, the Iwahori orbit is just a vector bundle over a point with dimension $2\,\mathrm{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}})$; if λ is anti-dominant, the Iwahori orbit is a vector bundle over the dense open cell in $\mathbb{P}^1$, and has dimension $2\,\mathrm{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}}) + 1$.

In the affine flag variety, the Iwahori orbits are affine cells that are indexed by elements in the affine Weyl group W_{aff}. The dimension of each Iwahori orbit is given by the length of the corresponding affine Weyl group element $w \in W_{aff}$.

3 Central Degeneration

3.1 Global Versions of Affine Grassmannian and Affine Flag Variety

Now let X be a smooth curve. We would like to define the global analogs of the affine Grassmannian and the affine flag variety.

The global affine Grassmannian is the functor $Gr_X(S) = \{(y, \mathcal{P}, \phi) | y$ is a point on X, $\mathcal{P}$ is an $S-$family of principal $G-$bundles on X, ϕ is a trivialization of $\mathcal{P}$ on $X \backslash y\}$.

There is an interesting ind-scheme Fl_X over X, constructed as follows: Let x be a distinguished point on X.

$Fl_X(S) = \{(y, \mathcal{P}, \phi, \zeta) | y$ is a point on $X, \mathcal{P}$ is an $S-$family of principal $G-$bundle on X, ϕ is a trivialization of $\mathcal{P}$ on $X \backslash y, \zeta$ is a $B-$reduction of the Principal $G-$bundle at $x \in X\}$.

We have canonical isomorphisms $Fl_{X \backslash \{x\}} \cong Gr_{X \backslash \{x\}} \times G/B$ and $Fl_x \cong Fl$.

Specializing to the curve $\mathbb{A}^1$, we can rewrite the definition as below.

$Fl_{\mathbb{A}^1} = \{(\epsilon \in \mathbb{A}^1, a \in G(k[t, t^{-1}])/I_\epsilon)\}$, where I_ϵ is the pre-image of the Borel subgroup B under the map $G(k[t]) \to G$ by evaluating at $t = \epsilon \in \mathbb{A}^1$.

Topologically I_ϵ are maps $\mathbb{A}^1 \to G$ such that $\{\epsilon\} \hookrightarrow B$.

Each fiber $Fl_{\mathbb{A}^1}|_\epsilon = G(k[t, t^{-1}])/I_\epsilon$ has a map to the affine Grassmannian $Gr = G(k[t, t^{-1}])/G(k[t])$. When $\epsilon \neq 0$, $Fl_{\mathbb{A}^1}|_\epsilon$ has a map to G/B by evaluating at $t = \epsilon$, so it is isomorphic to $Gr \times G/B$. When $\epsilon = 0$, the fiber $G(k[t, t^{-1}])/I_0$ is isomorphic to the affine flag variety.

We have the following commutative diagram:

$$\begin{tikzcd}
Gr \times G/B \arrow[r, hook] \arrow[d] & Gr \times G/B \times (\mathbb{A}^1 \backslash \{0\}) \arrow[r, hook] \arrow[d] & Fl_{\mathbb{A}^1} \arrow[d] & Fl \arrow[l, hook'] \arrow[d] \\
\{\epsilon \neq 0\} \arrow[r, hook] & \mathbb{A}^1 \backslash \{0\} \arrow[r, hook] & \mathbb{A}^1 & \{0\} \arrow[l, hook']
\end{tikzcd}$$

There is also a lattice picture of $Fl_{\mathbb{A}^1}$ of type A. Let $(\epsilon, a) \in Fl_{\mathbb{A}^1}$. $\epsilon \in \mathbb{A}^1$, a defines a lattice L in the affine Grassmannian of type A, plus a flag f in the quotient $L/(t-\epsilon)L$. When $\epsilon = 0$, a defines a point in the affine flag variety of type A.

3.2 Central Degeneration

The Iwahori affine Hecke algebra H_I as a vector space consists of compactly supported bi-I-invariant functions $G(\hat{\mathcal{K}}) \to \overline{\mathbb{Q}}_l$, with the algebra structure given by convolution of functions. The spherical affine Hecke algebra H_{sph} as a vector space consists of compactly supported bi-$G(\hat{\mathcal{O}})$-invariant functions with the algebra structure also given by convolution. The center of the Iwahori Hecke algebra $Z(H_I)$ is isomorphic to H_{sph}.

In [4] Gaitsgory constructed an inverse of π geometrically, $\pi^{-1} : H_{sph} \to Z(H_I)$, through the nearby cycles functor from $P_{G_\mathcal{O}}(Gr)$ to $P_I(Fl)$. We aim to understand this construction more explicitly by examining how some interesting finite-dimensional varieties degenerate.

The central degeneration involves the global affine flag variety constructed above, and over $\mathbb{A}^1$ we would like to take the limit $\epsilon \to 0$ and see how the general fiber transforms to the special fiber.

An interesting basic example was worked out for $G = GL_2(k)$ in Gaitsgory's paper [4].

Consider the closed minuscule $G(\mathcal{O})-$orbit $Y_0 = \{$ lattices L which are contained in the standard lattice $\mathcal{O} \oplus \mathcal{O}$ with $\dim(\mathcal{O} \oplus \mathcal{O} \backslash L) = 1\}$. By construction, Y_0 is isomorphic to $\mathbb{P}_1$.

The special fiber limit of $Y_0 \times \{1\}$ in the affine flag variety Fl is isomorphic to a union of two $\mathbb{P}^1$s intersecting at a point. Locally, this is the family $\{xy = a | a \in k\}$. In later sections we are going to show that all $\mathbb{P}^1$s invariant under the extended torus would degenerate in this way (when the underlying curve is $\mathbb{A}^1$).

3.3 Degenerations of G($\mathcal{O}$)-Orbits

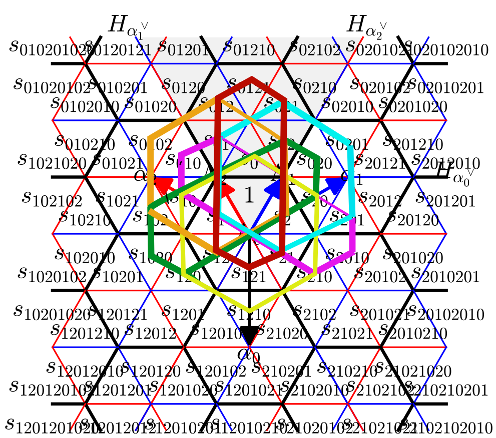
The special fiber limit of a $G(\mathcal{O})$ orbit Gr^λ in the affine Grassmannian is the union of Iwahori orbits I^α, where α is a $\lambda-$admissible element in the affine Weyl group. The term $\lambda-$admissible means that there exists a w in the finite Weyl group such that $\alpha \leq w \cdot \lambda$ in the usual Bruhat order.

The number of irreducible components in the special fiber limit of a $G(\mathcal{O})$ orbit Gr_λ is equal to the size of the quotient of the finite Weyl group $\tilde{W} = W/W_I$ for the (partial) flag variety G/P_λ. When λ is regular, then this is just the size of the Weyl group W.

Pictorially, the number of irreducible components in the special fiber limit is equal to the number of vertices in the $T-$equivariant moment polytope of Gr_λ.

There is a sheaf of groups I_{glob} that maps a scheme S to the group of maps from the localization of $\mathbb{A}^1 \times S$ around $\{\epsilon\} \times S$ to G ($\epsilon \in \mathbb{A}^1$), with the condition that $\{0\} \times S$ maps to B. By taking Taylor expansions at 0, we obtain a map of sheaves $I_{glob} \to \underline{I}$; by taking Taylor expansions at a general point ϵ, we obtain a map of sheaves $I_{glob} \to \underline{G(\mathcal{O})}$(the underline means 'the sheaf represented by'). I_{glob} acts on this global affine flag variety by changing the trivializations of principal $G-$bundles. On the general fiber, its action is compatible with the action of $G(\mathcal{O})$. On the special fiber, its action is compatible with the action of the Iwahori subgroup I.

Example 2. *This illustrates the the $T-$equivariant moment polytopes of the $W-$many irreducible components in the limit of the $G(\mathcal{O})-$orbit $Gr_{\alpha+\beta}$ for $G = SL_3$.*



When $G = GL_n$ and λ is a minuscule coweight, Gr_λ is an ordinary Grassmannian. There are explicit equations of this central degeneration of Gr_λ of type A in [7] .The equations are obtained from studying the following moduli functor.

Consider a functor M_μ such that for any ring R $M_\mu(R)$ is the set of $L. = (L_0, ..., L_{n-1})$ where $L_0, ..., L_{n-1}$ are $R[t]$ submodules of $R[t]^n / t^d \cdot R[t]^n$, d being a positive integer, satisfying the following properties

- as $R-$modules $L_0, ..., L_{n-1}$ are locally direct factors of corank $nd - r$ in $R[t]^n / t^d \cdot R[t]^n$.
- $\gamma(L_i) \subset L_{i+1}(mod\ n)$, where γ is the matrix

$$\begin{bmatrix} 0 & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ t+\epsilon & 0 & \dots & 0 \end{bmatrix}$$

When $d = 1$, we get the case of degenerations of $G(\mathcal{O})$ orbits in the minuscule case, which are exactly ordinary Grassmannians $Gr_r(\mathbb{C}^n)$.

4 Moment Polytopes

4.1 Torus Action

The maximal torus T in G acts on the entire family $Fl_{\mathbb{A}^1}$. Its action changes the trivializations of principal G bundles away from a point on the curve, as well as the B reductions at 0. This torus action preserves each individual fiber $G[t, t^{-1}]/I_\epsilon$.

Note that the rotation torus $\mathbb{C}^*$ scales the base curve $\mathbb{A}^1$ and moves one fiber to another. Therefore the action of the rotation torus does not preserve individual fibers.

4.2 Moment Polytopes of the Affine Grassmannian and the Affine Flag Variety

Let $\mathcal{L}$ be the determinant bundle on the global affine flag variety and let $\Gamma(Fl_{\mathbb{A}^1}, \mathcal{L}^*)$ be the vector space of global sections.

Then Fl_C embeds in the projective space $\mathbb{P}(V)$, where $V = \Gamma(Fl_C, \mathcal{L})^*$, by mapping $x \in Fl_C$ to the point determined by the line in V dual to the hyperplane $\{s \in \Gamma(Fl_C, \mathcal{L}) | s(x) = 0\}$.

The moment map Φ, for the action of the torus $T \subset G$, is a map from $\mathbb{P}(V)$ to $Lie(T)^*$, which is the same as the lie algebra of the dual torus $Lie(T^\vee)$, which is then isomorphic to $Lie(T^\vee)^*$. We need the data of the weight lattice of $G^\vee$, which is also the data of the coweight lattice of G to connect the algebraic geometry related to G to the representation theory related to $G^\vee$. We can restrict Φ to the global affine flag variety Fl_C since it is $T-$invariant.

The coweight lattice and the affine Weyl group of G embeds in $Lie(T^\vee)^*$. If we choose the properly normalized Fubini-Study form on $\mathbb{P}(V)$, we can arrange the moment map so that when restricted to each fiber Fl_x, Φ maps each fixed point t^α, α being a coweight or an affine Weyl group element, to the image of α in $Lie(T^\vee)^*$. If X is a one-dimensional torus orbit then $\Phi(X)$ is a line segment in a root direction joining the images of the two $T-$fixed end points of X. The moment polytope of a $T-$invariant projective scheme is the convex hull of the images of the $T-$fixed points.

In this paper, we present our moment polytopes in a lattice that is dual to the lattice of alcoves. This lattice embeds in $Lie(T^\vee)^*$. This helps us to see the moment map images of $T-$fixed points more clearly.

Throughout this paper, in a $T-$projective scheme S, we call the $T-$fixed points whose moment map images are the vertices of the moment polytope extremal $T-$fixed points. All the $T-$fixed points in S that are not extremal $T-$fixed points are called internal $T-$fixed points.

5 Flat Families

In this section, we collect a bunch of useful results about flat families of projective schemes.

Lemma 2. *Consider a flat family of projective schemes. Let A and B be two closed subschemes in the general fibers (which are isomorphic). Then the special fiber limit of the intersection of A and B, $A \cap B$, is contained in the intersection of the special fiber limit of A and the special fiber limit of B.*

Lemma 3. *Consider a flat family of projective schemes in which the general fibers are irreducible of dimension d, then the special fiber is equi-dimensional of dimension d.*

A flat family of projective schemes is called $T-$equivariant if a torus T acts on each fiber and its action is preserved by the degeneration.

Lemma 4. *The central degeneration is $T-$equivariant and flat, where T is the maximal torus in G.*

Lemma 5. *Consider a $T-$equivariant flat family of projective schemes, Let A be a $T-$invariant projective scheme A in the general fiber, and let $\tilde{A}$ denote its limit in the special fiber. Then the $T-$equivariant moment polytope of $\tilde{A}$ coincides with that of A.*

Proof. For a $T-$equivariant flat family of projective varieties, the multi-graded Hilbert polynomial is constant. Then the Duistermaat-Heckman measure on $\mathfrak{t}^*$, being the leading order behavior of the multi-graded Hilbert polynomial, also stays constant. The moment polytopes, which is the support of the Duistermaat-Heckman measure on $\mathfrak{t}^*$, is constant as well.

$\square$

Lemma 6. *The central degeneration is $T-$equivariant for the maximal torus $T \subset G$.*

Corollary 1. *The moment polytopes of a flat family of schemes in the global affine flag variety stay the same.*

6 Degenerations of One-Dimensional Subvarieties

We would like to start by considering the degenerations of one-dimensional subvarieties in the affine Grassmannian, with the base curve being $\mathbb{A}^1$.

6.1 Degenerations of Extended-torus Invariant $\mathbb{P}^1$s

In the affine Grassmannian and the affine flag variety, there are discretely many $\mathbb{P}^1$s that are invariant under the action of the extended torus, as expressed in the lemma below.

Lemma 7. *Given an affine root α, we have a one parameter subgroup U_α in the loop group LG generated by the exponential of e_α in the loop algebra $L\mathfrak{g}$.*

In Gr and Fl, every $T \times \mathbb{C}^*-$invariant $\mathbb{P}^1$ is the closure of one orbit of U_α, for some affine root α. In particular, there is a discrete number of $T \times \mathbb{C}^*$ orbits in Gr and Fl.

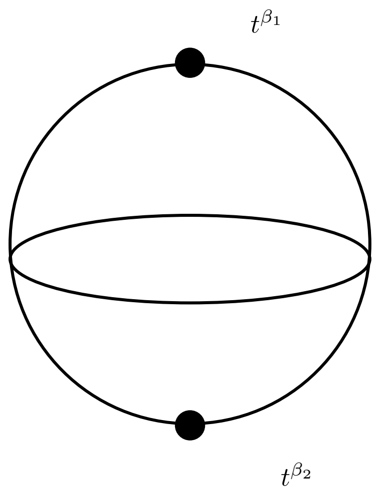
There is an explicit description of the $T-$fixed points in an orbit of a root subgroup U_α.

Lemma 8. *Let $p = t^\gamma$ be a $T-$fixed point in the affine Grassmannian or the affine flag variety, $\alpha = \alpha_0 + k\delta$ be an affine root, where α_0 is a root of G, $k \in \mathbb{Z}$, and δ is the imaginary root.*

Let s_α be the simple reflection in the affine Weyl group that reflects across the hyperplane H_α, where $H_\alpha = \{\beta | < \alpha_0, \beta >= k\}$. $U_\alpha = \exp(\eta \cdot e_\alpha)$ is a one-parameter subgroup, with η being the parameter. Then

$$\lim_{\eta \to \infty} \exp(\eta \cdot e_\alpha) \cdot \gamma = s_\alpha \cdot \gamma.$$

In other words, there is a unique one-dimensional U_α orbit whose closure is a $\mathbb{P}^1$ connecting p and the T-fixed point indexed by $s_\alpha \cdot \gamma$.



We also know how each $T-$fixed point degenerate, as expressed in the lemma below.

Lemma 9. *Under the central degeneration, the image of the $T-$fixed point indexed by β in the affine Grassmannian is the $T-$fixed point indexed by $(\beta, 1)$ in the affine flag variety.*

Now we would like to explain the central degenerations of all the $\mathbb{P}^1$s that are invariant under the extended torus.

Theorem 1. *As $\epsilon \to 0$, the limit of any $T \times \mathbb{C}^*-$invariant $\mathbb{P}^1 \subset Gr \times \{id\}$ in the special fiber is two copies of $\mathbb{P}^1$ intersecting at one $T-$fixed point.*

More explicitly, consider the $T \times \mathbb{C}^*-$invariant $\mathbb{P}^1$ connecting the two $T-$fixed points t^{β_1} and t^{β_2}, where $\beta_1 > \beta_2$ as coweights. This is an orbit of U_α, where $\alpha = \beta_1 - \beta_2 = \alpha_0 + k\delta$ is an affine root.

The special fiber limit of this $\mathbb{P}^1$ are the following two $\mathbb{P}^1$s intersecting at a point:

(1) The $\mathbb{P}^1$ connecting the fixed points $(\beta_1, 1)$ and (β_1, s_{α_0});

(2) The $\mathbb{P}^1$ connecting the fixed points $(\beta_1, s_{\alpha_0}) = s_\alpha \cdot (\beta_2, 1)$ and $(\beta_2, 1)$.

Proof. Consider the $T \times \mathbb{C}^*-$invariant $\mathbb{P}^1$ connecting the two $T-$fixed points t^{β_1} and t^{β_2} in the affine Grassmannian, where $\beta_1 > \beta_2$. This is also the closure of an one-dimensional orbit of the root subgroup U_α^- containing t^{β_2}, where $\alpha = \beta_2 - \beta_1$.

In the moduli interpretation for the central degeneration, when $\epsilon \neq 0$, this a $\mathbb{P}^1$-family of principal $G-$bundles on $\mathbb{A}^1$ together with a trivialization away from $\{\epsilon\}$, times the choice of the standard B-reduction at the point 0. A trivialization of a $G-$bundle away from $\{\epsilon\}$ is the same as a section of the bundle on $\mathbb{A}^1\backslash\epsilon$. We could allow different poles at ϵ.

When $\epsilon = 0$, we still have this $\mathbb{P}^1$ family of $G-$bundles with trivializations away from $\{\epsilon\}$, and we also have additional choices of standard B-reductions at $\epsilon = 0$. The difference is that when ϵ is 0, the B-reduction is no longer applied to the trivial G-bundle.

More explicitly, in the special fiber limit we have the following two $\mathbb{P}^1$s intersecting at a point. Note that we are thinking of $T-$fixed points in the affine flag variety as being indexed by the affine Weyl group, which is a semi-direct product of the coweight lattice and the finite Weyl group.

(1) The $\mathbb{P}^1$ connecting the $T-$fixed points (β_1, s_{α_0}) and $(\beta_2, 1)$; this represents the original family of trivializations away from $\epsilon = 0$. This $\mathbb{P}^1$ in the affine flag variety is the closure of an one-dimensional orbit of the same root subgroup U_α^- mentioned above, where $\alpha = \beta_2 - \beta_1$.

It is natural to also consider the $\mathbb{P}^1$ connecting the $T-$fixed points $(\beta_1, 1)$ and (β_2, s_{α_0}). However, this $\mathbb{P}^1$ is not relevant in this case as the $T-$fixed point (β_2, s_{α_0}) does not lie in the line segment connecting $(\beta_1, 1)$ and $(\beta_2, 1)$ in the $T-$equivariant moment polytope.

(2) The $\mathbb{P}^1$ connecting the fixed points $(\beta_1, 1)$ and (β_1, s_{α_0}). Note that the limit of the $T-$fixed point indexed by β is the $T-$fixed point indexed by $(\beta, 1)$. This extra $\mathbb{P}^1$ represents the extra choices of standard $B-$reductions at 0 for different nontrivial $G-$bundles on the curve.

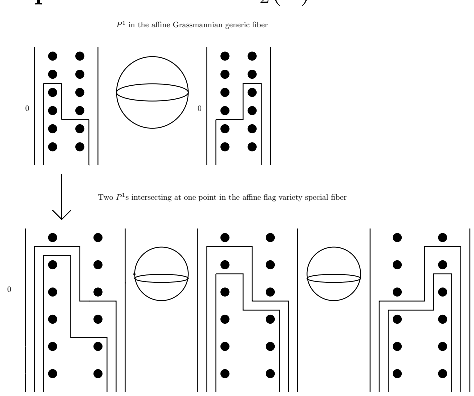
These two different copies of $\mathbb{P}^1$ intersect at the $T-$fixed point (β_1, s_{α_0}).

No other $T-$fixed points in the affine flag variety should be in the special fiber limit. Consider a $T-$fixed point p indexed by $(\gamma, w) \in W_{aff}$. If $\gamma \neq \beta_1$ or β_2, the p will not be in the closure of the limit because those t^γ is not in the closure of the original family to begin with. If $\gamma = \beta_1$ or β_2 but $w \neq 1$ or α_0, then p is not in the special fiber limit. This is because in the moment map image, p does not lie on the same line as the three $T-$fixed points indicated above.

$\square$

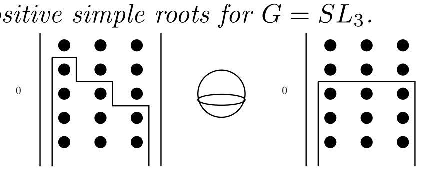
Now let's look at a few examples of the degenerations of $\mathbb{P}^1$s that are invariant under the extended torus.

Example 3. *Let $G = SL_2(\mathbb{C})$. Consider the $\mathbb{P}^1$ connecting the $T-$fixed points t^α and $t^{-\alpha}$.*

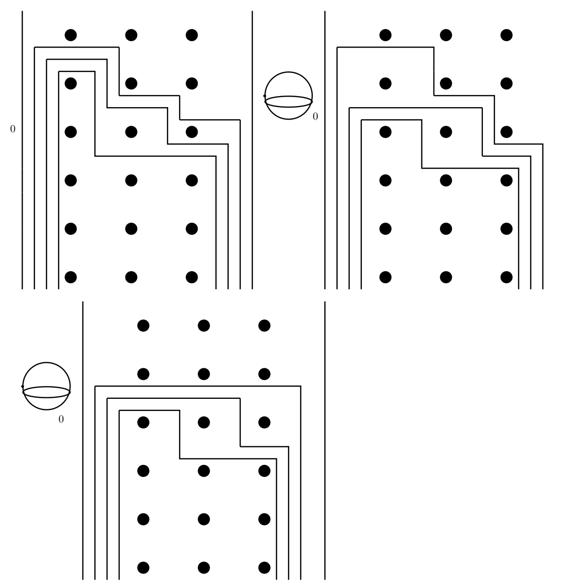


The upper diagram illustrates the $\mathbb{P}^1$ connecting t^α and $t^{-\alpha}$ in the affine Grassmannian. The lower diagram illustrates its limit in the affine flag variety, which consists of three $T-$fixed points *and two $\mathbb{P}^1s$. The three $T-$fixed points are indexed by $(\alpha, 1), (\alpha, \sigma), (-\alpha, 1) \in W_{aff}$, σ being the nontrivial element in the finite Weyl group S_2.*

Example 4. *Let $G = SL_3(\mathbb{C})$. Consider the $\mathbb{P}^1$ connecting $t^{\alpha+\beta}$ to 1. Here α and β are the two positive simple roots for $G = SL_3$.*



A degeneration example for SL_3



The upper diagram illustrates the $\mathbb{P}^1$ in the affine Grassmannian. The two $T-$fixed points are indexed by $\alpha + \beta$ and 1 respectively.

The lower diagram illustrates its limit in the affine flag variety, which consists of three $T-$fixed points and the two $T \times \mathbb{C}^*$-invariant $\mathbb{P}^1s$ connecting them. The three $T-$fixed points are indexed by $(\alpha + \beta, 1), (\alpha + \beta, w_0), (1, 1) \in W_{aff}$. w_0 denotes the longest element in the finite Weyl group.

6.2 Degenerations of Other One-dimensional Subschemes

The extended-torus invariant $\mathbb{P}^1$s studied in the previous subsection are very special. In general, the affine Grassmannian is a union of infinitely many $T-$orbits, T being the maximal torus in G.

The degeneration of any one-dimensional subvariety S in a generic $T-$orbit is more complicated. Since this degeneration is $T-$equivariant, the special fiber limit of S would still be a one-dimensional subvariety in a union of $T-$orbits. However, there could be a lot more $T-$fixed points and irreducible components in the limit.

Example 5. $G = SL_2(\mathbb{C})$. *In this case the maximal torus $T \subset G$ is one-dimensional.*

There are infinitely many orbits of the maximal torus $T \subset G$ connecting t^{α} and $t^{-\alpha}$ in the affine Grassmannian, α being a positive coroot. The limit of a generic $T-$orbit has more than two irreducible components as we can choose the coordinate of the original $T-$orbit so that its special fiber limit is arbitrarily close to any $T-$fixed point in between $(\alpha, 1)$ and $(-\alpha, 1)$ in the moment map image.

Therefore, the special fiber limits of some generic $T-$orbits closures have more than three $T-$fixed points and more complicated geometric properties.

7 Degenerations of a Finite Product of Root Subgroups Orbits

After discussing the degenerations of orbits of a single root subgroup, now let's consider things related to finitely many root subgroups.

First we describe the closure of a product of orbits of finitely many root subgroups containing the same $T-$fixed point in the affine Grassmannian and the affine flag variety.

Theorem 2. *Let O_μ be isomorphic to a product of orbits of finitely many root subgroups containing the same $T-$fixed point t^μ, in the affine flag variety (or the affine Grassmannian). Note that the ordering of the product matters. Let$\beta_1, \beta_2, ..., \beta_m$ denote the affine roots involved.*

Then the closure of O_μ, $\overline{O_\mu}$, is an $m-$dimensional projective scheme. It contains all the $T-$fixed points indexed by elements in W_{aff} of the form $s_{\beta_1} \cdots s_{\beta_j} \cdot \mu$, for some $\beta_1, \beta_2, ..., \beta_j \in I$.

$\overline{O_\mu}$ is a union of open orbits of a finite product of root subgroups containing the $T-$fixed points mentioned above.

Each $T-$fixed point of $\overline{O_\mu}$ has an open neighborhood that is isomorphic to a product of root subgroup orbits. If a $T-$fixed point is extremal, i.e. it is one of the vertices of the moment polytope of $\overline{O_\mu}$, then it has an open neighborhood that is isomorphic to a product of root subgroup orbits.

Remark 1. *The geometry of the closure of a product of finitely many root subgroup orbits containing the same $T-$fixed point in the affine Grassmannian or the affine flag variety is completely analogous to a finite-dimensional Schubert variety in an ordinary Grassmannian or flag variety, which is indexed by an element in a quotient of the finite Weyl group.*

Now we would like to discuss the degenerations of the closures of a product of finitely many orbits root subgroups which contain the same $T-$fixed point t^μ.

The theorem below describes an important irreducible component in the special fiber limit. This is the only irreducible component that contains the $T-$fixed point indexed by $(\mu, 1)$, which is the limit of t^μ.

Theorem 3. *Consider in the affine Grassmannian the closure of a finite product of orbits of root subgroups U_α containing the $T-$fixed point t^μ, μ being a coweight. Denote this projective scheme by P. Here $\alpha = \alpha_0 + k\delta$ are in a finite set of affine roots.*

Then $\tilde{P}$, the special fiber limit of P, contains the closure of the product of one-dimensional orbits $U_{\alpha'} \cdot t^{(\mu,1)}$ as one open irreducible component. Here $\alpha' = \alpha$ when α_0 is a positive root, and $\alpha' = \alpha + \delta$ when α_0 is a negative root.

Proof. From the lattice picture of this degeneration, we know that the special fiber limit of a product of m root subgroups orbits containing t^μ would have an $m-$dimensional neighborhood of $t^{(\mu,1)}$ that is invariant under $T \subset G$, and is also a product of root subgroups orbits containing $t^{(\mu,1)}$.

We need to treat the cases of positive α_0 and negative α_0 a bit differently. This is because this process is a flat $T-$equivariant degeneration, so we need to keep the shapes of moment polytopes.

More explicitly, as discussed in the section about one-dimensional orbits of the extended torus, in the affine flag variety, the orbit of the root subgroup $U_\alpha, \alpha = \alpha_0 + k\delta$ connects a translation $T-$fixed point $(\mu', 1)$ to another $T-$fixed point (μ'', s_{α_0}).

When α_0 is positive, $\mu' < \mu''$ and (μ'', s_{α_0}) is contained in the moment polytope line connecting $(\mu, 1)$ and $(\mu'', 1)$.

When α_0 is negative, $\mu' > \mu''$ and (μ'', s_{α_0}) is not contained in the moment polytope connecting $(\mu, 1)$ and $(\mu'', 1)$. On the other hand the orbit of $U_{\alpha+\delta}$ is contained in the limit.

$\square$

Example 6. *Consider the case of the closed orbit of a single root subgroup U_α. This $\mathbb{P}^1$ is a union $U_{-\alpha} \cdot t^{\beta_1} \cup U_\alpha \cdot t^{\beta_2}$.*

The limit contains a union of two $\mathbb{P}^1$s intersecting at a point: $U_{-\alpha+\delta} \cdot t^{(\beta_1,1)} \cup U_\alpha \cdot t^{(\beta_2,1)} \cup point$.

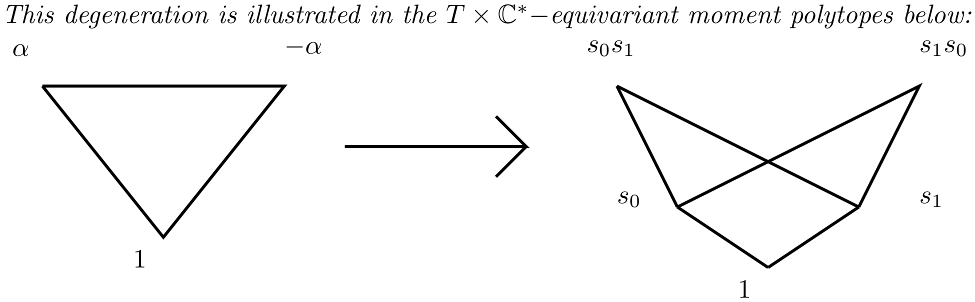
Example 7. *This example is about the limit of the closure of a product of orbits of two root subgroups containing the same $T-$fixed point.*

Consider the $G(\mathcal{O})$ orbit corresponding to α for $G = SL_2$. It is a union of of the products of root subgroup orbits $U_\alpha \cdot t^{-\alpha} \times U_{\alpha+\delta} \cdot t^{-\alpha}$, $U_{-\alpha} \cdot t^{-\alpha} \times U_{-\alpha+\delta} \cdot t^{-\alpha}$, and the point 1 in the affine Grassmannian.

Its special fiber limit consists of two components. The closed Iwahori orbit $s_0 s_1$ is the closure of $U_{\alpha+\delta} \cdot t^{(-\alpha,1)} \times U_{\alpha+2\delta} \cdot t^{(-\alpha,1)}$. The closed Iwahori orbit $s_1 s_0$ is the closure of $U_{-\alpha} \cdot t^{(\alpha,1)} \times U_{-\alpha+\delta} \cdot t^{(\alpha,1)}$.

This agrees with the theorem.

This degeneration is illustrated in the $T \times \mathbb{C}^*-$equivariant moment polytopes below:



Understanding the degeneration of the closure of a finite product of root subgroup orbits helps to understand the central degenerations of many other interesting projective schemes that show up in representation theory and algebraic geometry.

Theorem 4. *Consider a $T-$invariant projective variety A in the affine Grassmannian that satisfies the following condition:*

Each extremal $T-$fixed point $t^{\mu_{ex}}$ has an open neighborhood $O_{\mu_{ex}}$ that is isomorphic to a product of finitely many orbits of distinct root subgroups containing $t^{\mu_{ex}}$.

Examples of such projective varieties in the affine Grassmannian include MV cycles, Iwahori orbits, Borel orbits, etc.

Then for each extremal $T-$fixed point $t^{\mu_{ex}}$, the open set $O_{\mu_{ex}}$ gives rise to a distinct irreducible component in the special fiber limit $\tilde{A}$. Therefore, the number of irreducible components of $\tilde{A}$ is bounded below by the number of vertices of the $T-$equivariant moment polytope of A.

Proof. By the previous theorem, the special fiber limit of the closure of each $O_{\mu_{ex}}$ has a component that is also contained an orbit of a finite product of root subgroups containing the $T-$fixed point $(\mu, 1)$.

For two distinct $O_{\mu_{ex}}$, their limit components described above are distinct cells. The $T-$orbits contained in these cells do not belong to the same component. Therefore these cells corresponding to different extremal $T-$fixed points are distinct generators of the top-dimensional cohomology of $\tilde{A}$. Consequently, the number of irreducible components of $\tilde{A}$ is bounded below by the number of vertices of A.

□

Corollary 2. *The number of irreducible components of the special fiber limits of MV cycles, closed Iwahori orbits, closed Borel orbits are bounded below by the number of vertices in the $T-$equivariant moment polytopes of the original projective varieties in the general fiber.*

Proof. MV cycles, closed Iwahori orbits, closed Borel orbits all satisfy the following property: each extremal $T-$fixed point $t^{\mu_{ex}}$ has an open neighborhood $O_{\mu_{ex}}$ that is isomorphic to a finite product of distinct root subgroups orbits containing $t^{\mu_{ex}}$.

This is because each open Iwahori orbit or Borel orbit is affine, and its closure would satisfy the property above.

An MV cycle is the closure of the intersections of semi-infinite orbits $\overline{\cap_{w \in W} S_w^{\mu_w}}$. The vertices of the corresponding MV polytope are indexed by $\mu_w, w \in W$. Each U_w is an infinite product of root-subgroups. Therefore near each extremal $T-$fixed point, there is a neighborhood that is isomorphic to a finite product of distinct root subgroup orbits containing this $T-$fixed point.

Then by Theorem 4, the number of irreducible components of the special fiber limits of MV cycles, closed Iwahori orbits, closed Borel orbits are bounded below by the number of vertices in their $T-$equivariant moment polytopes.

□

Now let's try to read more geometric information from the moment polytopes of certain $T-$invariant projective schemes in the affine Grassmannian or the affine flag variety. This will be very useful for dimension estimations in subsequent sections.

Theorem 5. *Let A be an irreducible $T-$invariant projective variety in the affine Grassmannian or the affine flag variety such that: (1) each extremal $T-$fixed point in A has an open neighborhood that is a product of root subgroup orbits; (2) A is invariant under* $\mathrm{Aut}(\mathbb{D})$.

Then the dimension of A could be calculated from the moment polytope of A: choose one vertex v. Let S denote the infinite set of all the $T-$fixed points that lie in the closure of one affine root subgroup orbit containing v. The dimension of A is equal to the finite number of elements in S whose moment map images lie in the moment map image of A.

This is independent of the choice of the vertex of the moment polytope.

Proof. Pick any extremal $T-$fixed point p of A and v denote its moment map image, which is a vertex of the moment polytope of A. There is an open neighborhood O_p that is a product of root subgroup orbits $\prod_\gamma U_\gamma \cdot p$.

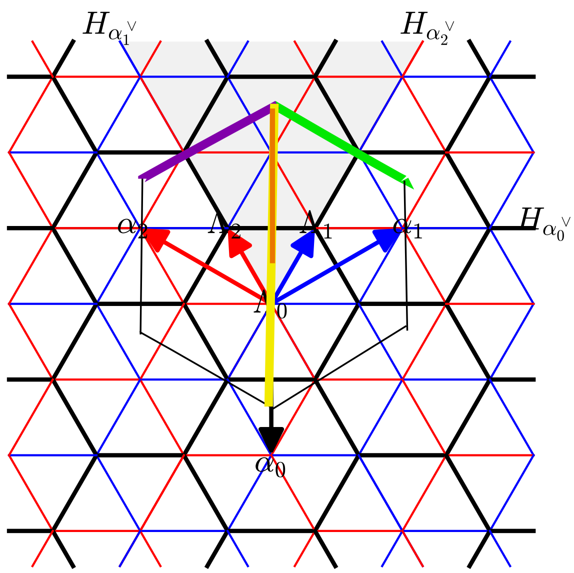
Then for each distinct affine root γ above, the moment map image of the orbit of U_γ containing p is a line segment pointing in the root direction corresponding to γ. It connects v and that of another distinct $T-$fixed point that lies in the closure this orbit.

Since O_p is open in A, the dimension of A is the same as the number of root subgroup orbits in O_p that connects p to another $T-$fixed point in A. This is equal to the number of root directions from v in the moment polytope of A, as A is invariant under $\mathrm{Aut}(\mathbb{D})$.

$\square$

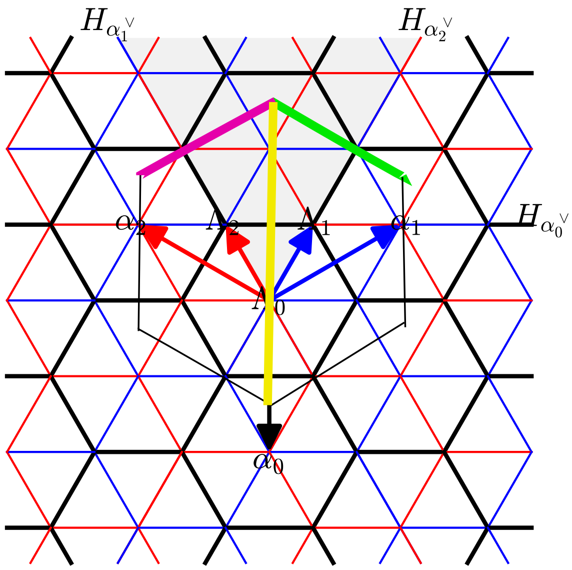
Now let's look at the moment polytopes of a few $T-$invariant projective schemes in the affine Grassmannian for $G = SL_3$. Things work completely analogously for $T-$invariant projective schemes in the affine flag variety.

Example 8. *Consider the $G(\mathcal{O})$ orbit $Gr_{\alpha+\beta}$ in the affine Grassmannian. This is also an MV cycle and an Iwahori orbit. This projective scheme satisfies the hypothesis of the theorem, and we can calculate its dimension by counting the number of root directions in the moment polytope, as shown below:*



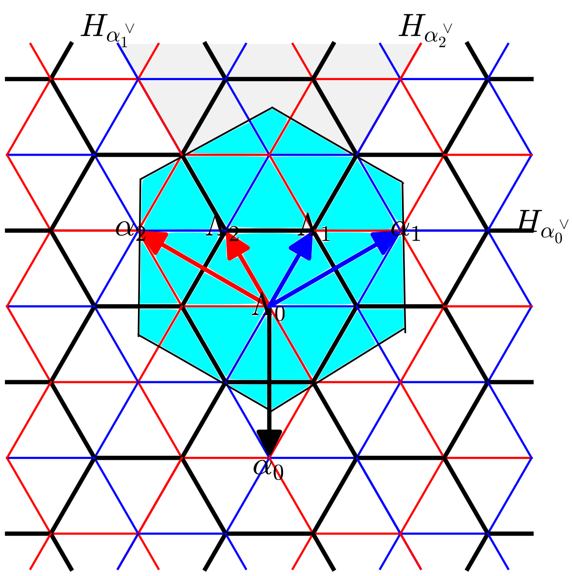
Example 9. *Now consider the G orbit containing the $T-$fixed point $t^{\alpha+\beta}$ in the affine Grassmannian. It is three-dimensional, while the total number of root directions in its moment polytope is four.*

This is because $G-$orbits in the affine Grassmannian or the affine flag variety are not $\mathrm{Aut}(\mathbb{D})$ *invariant.*



Example 10. *Consider the closure of a generic $T-$orbit in the $G(\mathcal{O})$ orbit $Gr_{\alpha+\beta}$ in the affine Grassmannian. They have the same moment polytope, as shown below. However, this $T-$orbit is only two-dimensional.*

This is because the closure of a generic $T-$orbit fails the condition that each extremal $T-$fixed point has a neighborhood being isomorphic to a product of root subgroup orbits. Different root-directions in the same $T-$orbit may not have independent parameters. Another issue is that a $T-$orbit, just like a $G-$orbit, is not $\mathrm{Aut}(\mathbb{D})-$invariant.



8 Degenerations of Semi-infinite Orbits

Let $U = N(\mathcal{K})$, $U_w = wUw^{-1}$ for some $w \in W$, $U^- = w_0 U w_0^{-1}$. Each group U_w is an infinite product of root-subgroups. A semi-infinite orbit S_w^γ is an orbit of U_w for some $w \in W$. Orbits of U_w are indexed by $T-$fixed points.

8.1 Closure Relations of Different Semi-infinite Orbits

We describe the closure relations of semi-infinite orbits in the affine Grassmmaninan and the affine flag variety. The description in the alcove picture works for all types of G, while the description in the lattice picture only works for type A.

We will focus on the orbits of U^-, but the closure relations of orbits of other U_w could be worked out in completely similar ways.

Lemma 10 (Alcove Picture)**.** *The closure relation of U^- orbits (indexed by elements in the affine Weyl group) for SL_2 is given below:*

$$\cdots < s_1 s_0 s_1 < s_1 s_0 < s_1 < 1 < s_0 < s_0 s_1 < s_0 s_1 s_0 < \cdots .$$

For general SL_n, given a positive root γ, $s_{\gamma,1}$ and s_γ generate a copy of the affine symmetric group $\tilde{S}_2$, which is the affine Weyl group for SL_2. Pictorially this generates a line in the diagram of alcoves.

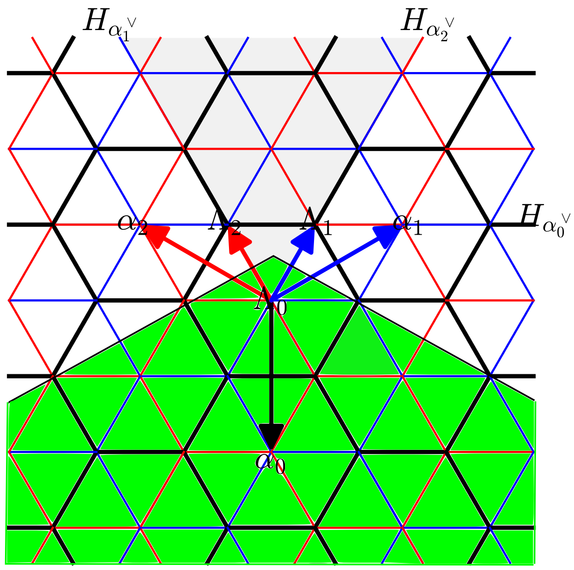
For each $w \in W_{aff}$, $w' \geq w$ iff w' is contained in a closed cone generated by the positive roots, and $w \geq w'$ iff w' is contained in a closed cone generated by the negative roots.

This would also be true for other types of G.

Lemma 11 (Lattice Picture for type A)**.** *Let $G = SL_n(k)$ or $GL_n(k)$. Let $L = (L_0 \supset L_1 \supset \cdots \supset L_{n-1} \supset t \cdot L_0)$ and $L' = (L'_0 \supset L'_1 \supset \cdots \supset L'_{n-1} \supset t \cdot L'_0)$ be two sequences of lattices in the lattice picture for the affine flag variety $Fl = LG/I$. These are two $T-$fixed points in Fl.*

Let U_a^- denote the U^--orbit corresponding to the $T-$fixed point a in the affine flag variety. Then U_L^- is in the closure of $U_{L'}^-$ iff $L_i \leq L'_i$ in the coweight lattice $\forall i = 0, ..., n-1$.

Below is the moment map image of the U^- orbit containing the $T-$fixed point indexed by 1 in the affine flag variety.



8.2 Degenerations

Theorem 6. *Given any closed $U_w = wUw^{-1}$ orbit S_w^μ in the affine Grassmannian, its special fiber limit is the corresponding closed U_w orbit $S_w^{(\mu,1)}$ in the affine flag variety.*

Proof. We would like to show that the limit of the closure of the U^- orbit $\overline{S^\mu}$ in Gr, $\Phi(\overline{S^\mu})$, is the closed U^- orbit $\overline{S^{(\mu,1)}}$ in Fl. Similar techniques should work for the orbits of other U_w.

Note that U^- is an infinite product of root subgroups corresponding to affine roots of the form $\alpha_0 + k\delta$, α_0 being a negative root.

First note that $\Phi(\overline{S^\mu})$ and $\overline{S^{(\mu,1)}}$ contain the same $T-$fixed points and have the same moment polytopes, as this degeneration is flat and T-equivariant.

The closed U^- orbit $\overline{S^\mu}$ is the union of $S^{\mu'}$, where $\mu' \leq \mu$.

By Theorem 4, we know that the limit of the closure of a finite product of root subgroups orbits containing the $T-$fixed point μ has the closure of a finite product of root subgroups orbits containing the $T-$fixed point $(\mu, 1)$ as one irreducible component. Taking an infinite limit of this theorem, we know that $\Phi(\overline{S^\mu})$ contains the closure of $S^{(\mu,1)}$.

Therefore $\overline{S^{(\mu,1)}} \subseteq \Phi(\overline{S^\mu})$.

On the other hand, suppose there is a point $p \in \Phi(\overline{S^\mu})$ such that p is not contained in $\overline{S^{(\mu,1)}}$. Let O_p denote the $T-$orbit containing p. Since $\overline{S^{(\mu,1)}}$ is closed, $\overline{S^{(\mu,1)}} \cap O_p = \emptyset$.

This degeneration is $T-$equivariant, so $\overline{O_p}$ lies in the special fiber limit of the closure of a $T-$orbit O in $\overline{S^\mu} \subset Gr$. There is a U^- orbit S^θ, $\theta \leq \mu \in X_*(T)$, so that $O \subset S^\theta$. The moment polytope of $\overline{O_p}$ is a convex polytope in a regular subdivision of the moment polytope of $\overline{O}$. This implies that there is a $T-$fixed point $q = t^\beta \in \overline{O_p}$ such that $\beta \leq (\theta, 1)$ (according to the semi-infinite Bruhat order), and the moment map image of the O_p is contained in the moment map image of the U^- orbit containing β, S^β.

Since q is contained in $\overline{O_p}$, O_p must be contained in the S^β, which is then contained in $\overline{S^{\mu,1}}$.

We have arrived at a contradiction.

Therefore, $\Phi(\overline{S^\mu}) \subseteq \overline{S^{(\mu,1)}}$.

The argument for the orbits of wUw^{-1} for some other finite Weyl group element w is completely analogous.

□

There is a more structural way to look at the degenerations of orbits of U_w.

Lemma 12. *We could definite a global version of $U_w = N_w(\mathcal{K})$ over a curve C, as follows.*

$$U_{w,glob}(S) = \{p \times S \in C \times S, \text{ the group of maps } f : \hat{p}\backslash\{p\} \times S \to N_w\}.$$

Here S could be any scheme, $\hat{p}\backslash\{p\}$ is isomorphic to the formal punctured disc for each point $p \in C$.

This global group scheme acts on the global affine flag variety $Fl_{\mathbb{A}^1}$. Its action preserves the subfamily in the global affine flag variety whose general fibers are $Gr \times \{id\} \subset Gr \times G/B$, which is what we are interested in.

9 Degenerations of Iwahori Orbits and MV Cycles: SL_2 Case

When $G = SL_2$, MV cycles and Iwahori orbits are a bit simpler compared with the general case.

Each MV cycle is just the closure of $Gr_\lambda \cap S^\mu$ or $S_e^{-\lambda} \cap S^\mu$, where λ is a dominant coweight and μ is a coweight.

There are basically two types of Iwahori orbits I^γ, γ being a coweight. I^γ is the $G(\mathcal{O})$ orbit $Gr_{-\gamma}$ if γ is anti-dominant. It is the closure of a vector space (vector bundle over a point) if γ is dominant. Each Iwahori orbit is also an MV cycle in this case.

Theorem 7. *Let $G = SL_2$ and let A be a $d-$dimensional MV cycle. Then $A = \overline{Gr_\lambda \cap S^\mu}$ or $\overline{S_e^{-\lambda} \cap S^\mu}$, where λ is a dominant coweight and μ is a coweight.*

$\tilde{A}$, the special fiber limit of A, is a union of open intersections of Iwahori orbits I^{w_1} and U^- orbits $S_{w_0}^{w_2} = S^{w_2}$ such that $w_1 \in W_{aff}$ is $\lambda-$admissible, and $w_2 \leq \mu$ in the semi-infinite Bruhat order and is also $\lambda-$admissible. We call each such intersection a stratum.

$\tilde{A}$ consists of $2d+1$ $T-$fixed points, $2(d-k)$ $k-$dimensional strata for $k > 0$. In particular there are two top-dimensional strata whose closures give rise to the two irreducible components.

$\tilde{A}$ is also equal to the intersection of two closed semi-infinite orbits in the affine flag variety for $G = SL_2$, $\overline{S_e^{(-\lambda,1)}} \cap \overline{S^{(\mu,1)}}$. One of its irreducible components contains the $T-$fixed point $(-\alpha, 1)$, and is contained in $S_e^{(-\lambda,1)}$. The other irreducible component contains the $T-$fixed point $(\mu, 1)$, and is contained in $S_{w_0}^{(\mu,1)}$.

Proof. The special fiber limit of A, $\tilde{A}$, is contained in the intersection of the special fiber limits of Gr_λ and S^μ, which we already understand. Therefore, we need to consider the intersections of Iwahori orbits I^{w_1} and U^- orbits S^{w_2} with the extra conditions $w_1 \in W_{aff}$ is $\lambda-$admissible, and $w_2 \leq \mu$ in the semi-infinite Bruhat order and is also $\lambda-$admissible.

Through the technique of alcove walks, we could directly compute the dimensions of all the relevant intersections of Iwahori orbits and U^- orbits and write down explicit dimension formulae for SL_2. There only two relevant intersections of Iwahori orbits with U^- orbits that are of the dimension d. These two top-dimensional intersections must be in the special fiber limit of an MV cycle by Theorem 4. Locally are isomorphic to a product of root subgroup orbits near each of the two extremal $T-$fixed points.

All other such intersections are of smaller dimensions, and are contained in the special fiber limit of smaller MV cycles in the closure of A. Therefore, the special fiber limit of A is equal to the intersection of the intersection of the special fiber limits of Gr_λ and S^μ.

Similarly the special fiber limit of A is contained in the intersection of the special fiber limits of $S_e^{-\alpha}$ and S^μ. Each nonempty intersection of a U orbit in $\overline{S_e^{(-\lambda,1)}}$ and a U^- orbit in $\overline{S^{(\mu,1)}}$ coincide with a relevant intersection of an Iwahori orbit and a U^- orbit discussed in the previous paragraphs. Therefore $\tilde{A} = S_e^{-\alpha} \cap S^\mu$.

□

The above theorem also applies to the degenerations of Iwahori orbits in the affine Grassmannian for $G = SL_2$ as Iwahori orbits are special MV cycles in this case. We can say a bit more about the special fiber limits of Iwahori orbits.

In fact, we can also discuss the special fiber limits of orbits of I_σ, which is the pre-image of the opposite Borel under the map that projects $G(\mathcal{O})$ to G by evaluating at $t = 0$.

Theorem 8. *Let I^γ be a closed Iwahori orbit in the affine Grassmannian. Let I_σ^γ be a closed orbit of I_σ. If I^γ or I_σ^γ is a $G(\mathcal{O})$ orbit, then its special fiber limit is a union of $\gamma-$admissible Iwahori orbits in the affine flag variety.*

If I^γ is not a $G(\mathcal{O})$ orbit, γ is dominant, then its special fiber limit consists of the closure of the orbit of the group generated by $I_2 + (U \cap G(\mathcal{O}))$ containing γ as well as the closure of the Iwahori orbit containing $(s_0s_1)^{\frac{d-1}{2}}s_0$, as the two irreducible components.

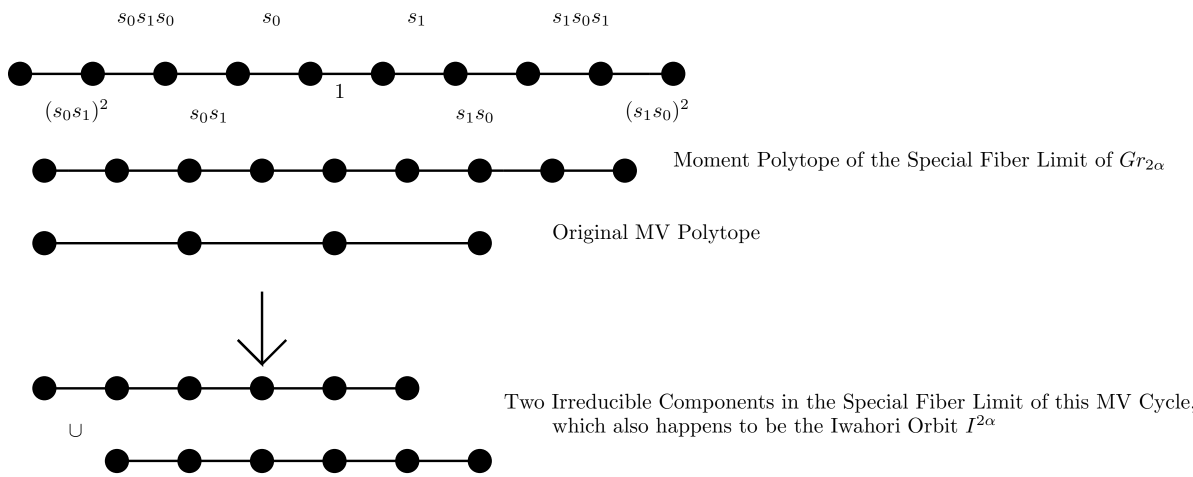
If I_σ^γ is not a $G(\mathcal{O})$ orbit, γ is anti-dominant, then its special fiber limit consists of the closure of the orbit I_1 containing γ and the closure of the Iwahori orbit containing $(s_1s_0)^{\frac{d-1}{2}}s_1$, as the two irreducible components.

Proof. From the discussion about the degenerations of MV cycles above, we can infer that the special fiber limit of an orbit of I or I_σ is the closure of the two irreducible components containing the two extremal $T-$fixed points. Each irreducible component is the closure of the orbit of a subgroup of I or I_σ. The exact descriptions of these components are from the moment polytopes.

□

Example 11. *Consider the MV cycle $\overline{Gr_{2\alpha} \cap S^\alpha}$. This is also the Iwahori orbit $I^{2\alpha}$.*

This degeneration is illustrated in terms of moment polytopes below:



Note that one irreducible component is the orbit of the subgroup H of I. This orbit contains the $T-$fixed point indexed by $s_0s_1s_0s_1$. Each matrix in H has the form: $\begin{bmatrix} 1 & \mathcal{O} \\ t^2 + a_3t^3 + \cdots & 1 \end{bmatrix}$

The other orbit is the Iwahori orbit containing the $T-$fixed point indexed by $s_0s_1s_0$.

10 Degenerations of MV Cycles of Type A

Each MV cycle A could be characterized as one irreducible component of $\overline{Gr^\lambda \cap S^\mu_{w_0}}$, or as one irreducible component of $\overline{S^\lambda_e \cap S^\mu_{w_0}}$, or as a GGMS stratum with some extra conditions.

Then the special fiber limit of an MV cycle is contained in a union of certain intersections of Iwahori orbits and U^- orbits, as well as intersections of closed orbits of different U_w.

Unlike in the SL_2 case, the limit of an MV cycle for general G in type A does not equal the intersection of the limit of a $G(\mathcal{O})$ orbit and the limit of a U^- orbit. One reason is that an MV cycle for general G may only be one of the many irreducible components in the intersection of a $G(\mathcal{O})$ orbit and a U^- orbit in the affine Grassmannian.

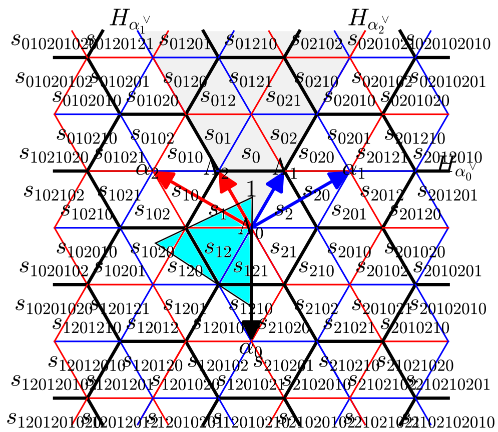
Conjecture 1. *If $Gr_\lambda \cap S^\mu$ has only one irreducible component, the special fiber limit of this MV cycle is the same as the intersection of the special fiber limit of Gr_λ and that of S^μ.*

Example 12. *Consider one of the MV cycles in $Gr_{\alpha+\beta} \cap S^1$ for $G = SL_3$. This MV cycle is two-dimensional.*

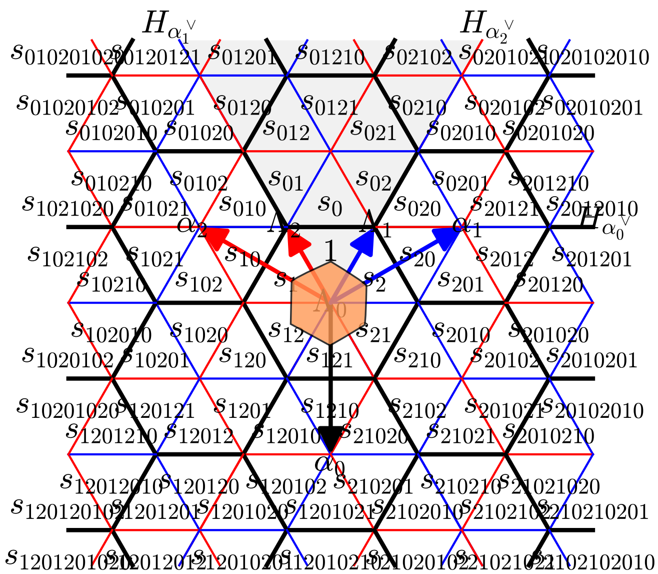
Now consider the intersection of the special fiber limit of $Gr_{\alpha+\beta}$ and the special fiber limit of S^1. Its dimension is strictly bigger than two. In particular, it contains $I^{s_1s_2s_1} \cap S^e_{w_0}$, whose closure is the three-dimensional G/B bundle above t^e in the affine Grassmannian.

Therefore, the special fiber limit of this MV cycle is not the same as the intersection of the special fiber limit of $Gr_{\alpha+\beta}$ and the special fiber limit of S^1. We could also see this from the moment polytopes.

The moment polytope of the special fiber limit of our MV cycle is as below:



The moment polytope of the three-dimensional G/B bundle above 1 *mentioned above is shown below. It does not lie in the moment polytope of the special limit of our MV cycle.*



It is better to relate the special fiber limit of an MV cycle with the intersections of closures of orbits of different U_w. We will adopt this perspective for the rest of this section.

Now let's discuss the number of irreducible components of the limits of MV cycles of type A.

By Corollary 2, the number of irreducible components of the special fiber limit of an MV cycle is bounded below by the number of vertices of the corresponding MV polytope.

After getting a lower bound, let's proceed to find an upper bound for the number of irreducible components in the special fiber limit of an MV cycle $A, \tilde{A}$. Note that by the dimension of each irreducible component in $\tilde{A}$ should be equal to the dimension of A.

General GGMS strata could be very complicated, but the strata that we are interested in are always contained in the intersection of an Iwahori orbit and a U^- orbit in the affine flag variety.

Moreover, they satisfy some nice properties.

Lemma 13. *Consider an irreducible component C in a GGMS stratum in the affine flag variety. When we project C to each copy of affine Grassmannian, its image is an MV cycle.*

Proof. Since the image of an irreducible projective variety is irreducible, the image of C in each copy of the affine Grassmannian is an irreducible component in the intersection $S_e^{\mu_e} \cap S_{w_0}^{\mu_{w_0}}$, which is an MV cycle.

□

As mentioned above, the special fiber limit of an MV cycle $\overline{\cap_{w\in W} S_w^{\mu_w}}$ is contained in the intersections of the limits of the closed semi-infinite orbits $\overline{S_w^{(\mu_w,1)}}$. We would like to look at all the non-empty intersections of W-many open semi-infinite orbits contained in $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$.

Lemma 14. *Consider the moment polytope P of $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$ in the affine flag variety. Each vertex v of P is contained in a unique irreducible component C_v. The number of root directions from v contained in C_v is the maximum number of root directions from v allowed in P. Moreover, this number is equal to the dimension of the corresponding MV cycle $\overline{\cap_{w\in W} S_w^{\mu_w}}$ and C_v is also an irreducible component in the special fiber limit of this MV cycle.*

We call the irreducible components C_v that contains the vertices of P the extremal irreducible components.

Theorem 9. *The dimension of any non-extremal irreducible component in $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$ cannot exceed the dimension of any of the extremal irreducible components.*

Proof. We showed that the image of each irreducible component in $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$ is an MV cycle when projected to each affine Grassmannian. By checking local coordinates, each extremal $T-$fixed point has an open neighborhood that is a product of root subgroups orbits.

Therefore, the dimension of each irreducible component could be calculated by counting the number of root directions from any vertex of the moment polytope of this irreducible component.

Let P' be the moment polytope of a non-extremal irreducible component. Let P again denote the moment polytope of $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$, which is the same as the original MV polytope.

The shape of a convex polytope at a vertex v is determined by the root directions from v.

Now would like to show that any irreducible internal irreducible component in $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$ cannot exceed the dimensions of the extremal irreducible components, which all equal to $\dim(A)$.

This is true for $G = SL_2$. Let's prove this for $G = SL_3$ first.

If the angle of P' at one of its vertices a agrees with the angle of P at one of its vertices v, then we could move P' so that it is contained in the moment polytope of C_v, by translation and rotation according to an element in the Weyl group.

Then it is clear that the number of root directions from a allowed in P' is less than or equal to number of root directions from v allowed in P, which is equal to the dimension of any of the extremal irreducible components.

If the angles of P' at all of its vertices differ from the angles at vertices of P, then the dimension of the MV cycle would be strictly bigger than the number of root directions from any vertex of this polygon. To see this, note that there are only finitely many basic shapes of MV polytopes for SL_3. If this MV polytope P has four sides, then its vertices have all the possible angles allowed and we are back to the previous case. Then the pair (P, P') must be (triangle, hexagon) or (hexagon, triangle). In both cases, P' would be much smaller than P have have strictly fewer number of root directions from any of its vertices.

Now suppose this is not true for SL_n, $n > 3$. This means that there is a non-extremal irreducible component with moment polytope P' such that the number of root directions from any vertex of P' exceeds the dimensions of the extremal irreducible components. Note that the root directions from any vertex is a union of root directions corresponding to different copies of SL_3 in SL_n. Then there is a copy of SL_3 in SL_n such that this still holds for the projections of P' and P to $Lie(T^\vee)^*$ for that copy of SL_3. We have arrived at a contradiction.

□

Now we are ready to give some other bounds for the number of irreducible components in the special fiber limits of an MV cycle.

Theorem 10. *Let A be an MV cycle in the affine Grassmannian such that $A = \overline{\cap_{w \in W} S_w^{\mu_w}}$. Then the special fiber limit of A is contained in the intersection of closed semi-infinite orbits in the affine flag variety $\cap_{w \in W} \overline{S_w^{(\mu_w,1)}}$.*

The number of irreducible components in the special fiber limit of A is bounded above by the total number of irreducible components in non-empty intersections of open semi-infinite orbits $\cap_{w \in W} S_w^{\alpha_w}$ whose dimension is $\dim(A)$.

Another upper bound for the number of irreducible components is given by the number of polytopes in the moment map image contained in the moment polytope of the original MV polytope that satisfy two extra conditions. First, if two vertices of the polytope lie on the same line, then there is an orbit of a root subgroup that connects these two extremal $T-$fixed points. Second, the number of root-directions at each vertex is the same, and is equal to the dimension of A.

This upper bound is greater than or equal to the upper bound given above, but is more visual.

Proof. By the previous theorem, the dimension of any of the irreducible components in all the relevant GGMS strata cannot exceed the dimension of the original MV cycle, and each irreducible component in the relevant GGMS strata contains either one or zero of the irreducible components of the limit of the MV cycle. Therefore we get the first upper bound in the theorem.

For the second upper bound, the number of moment polytopes satisfying the specified conditions is greater than or equal to the total number of irreducible components in $\cap_{w \in W} \overline{S_w^{(\mu_w,1)}}$. The moment polytope of every irreducible component in a relevant GGMS stratum satisfies the conditions specified. However, it is possible that one of such polytopes is not the moment map image of an irreducible component.

□

Now let's illustrate our theorems in some examples.

Example 13. *Let $G = GL_n$. A $G(\mathcal{O})$ orbit Gr_λ, λ being minuscule, is an ordinary Grassmannian. MV cycles in this case are isomorphic to ordinary Schubert varieties.*

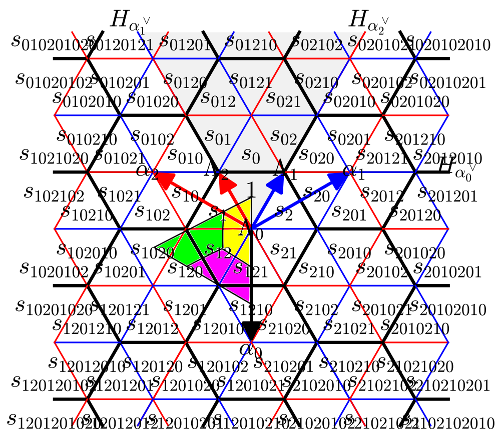
In the special case where $Gr_\lambda \cong \mathbb{P}^n$, all the Schubert varieties are $\mathbb{P}^k, k \leq n$. These are also certain minuscule $G(O)$ orbits for $G = GL_{k+1}$, and we already know how they degenerate. Their special fiber limits have k irreducible components, one for each $T-$fixed point in the original Schubert variety.

Example 14. *Let's consider all the MV cycles in $Gr_{\alpha+\beta} \cap S^{\mu}$ for some relevant coweight μ for $G = SL_3$.*

When $\mu = -(\alpha+\beta)$, we see one $T-$fixed point μ degenerates to the $T-$fixed point $(\mu, 1)$ in the affine flag variety.

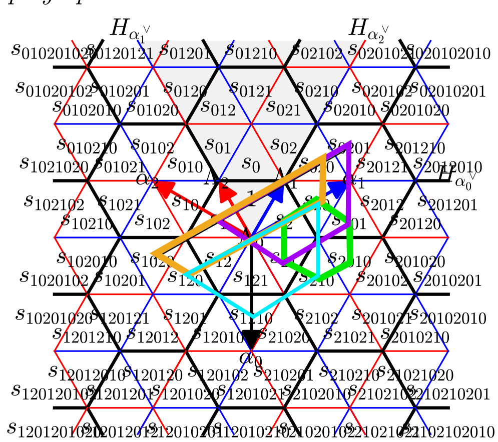
When $\mu = -\alpha$ or $-\beta$, we have one $\mathbb{P}^1$ degenerates to two $\mathbb{P}^1$s.

When $\mu = 1$, each of the two two-dimensional MV cycles degenerates like $\mathbb{P}^2$ locally:

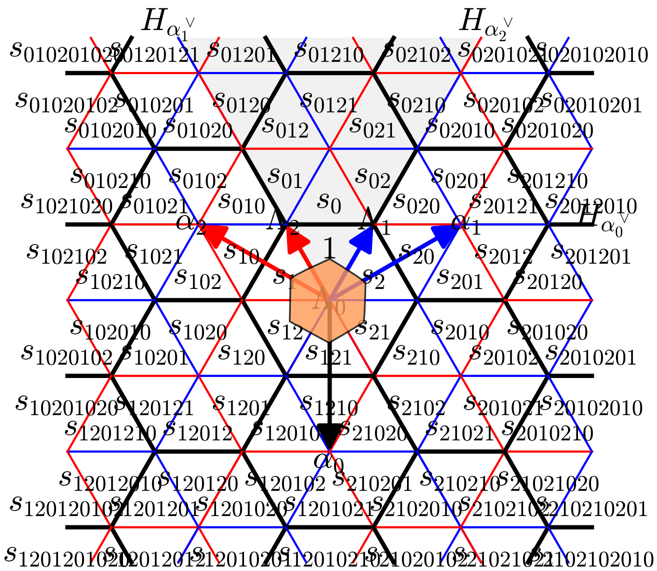


When $\mu = \alpha$ (or β), the MV cycle is three-dimensional.

The irreducible components in the special limit corresponding to the four vertices of the original MV polytope are shown below:



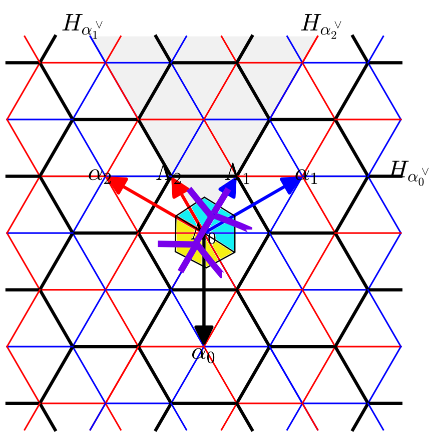
There is one additional irreducible component in the intersection of the corresponding closed semi-infinite orbits in the affine flag variety. This is the three-dimensional G/B bundle above 1 if we think of the affine flag variety as a G/B bundle over the affine Grassmannian.

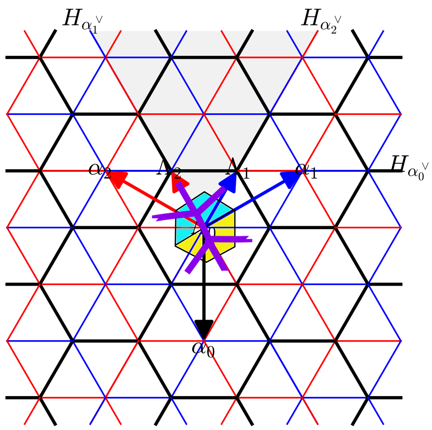


To see that this internal irreducible component with hexagonal moment polytope must be present, we consider the degenerations of the closures of all the generic $T-$orbits in the MV cycle. This forms a continuous $\mathbb{A}^1$ family. The data of a toric equivariant degeneration of a toric variety is given by a regular subdivision of the moment polytope, as well as the polyhedral complex dual to this regular subdivision.

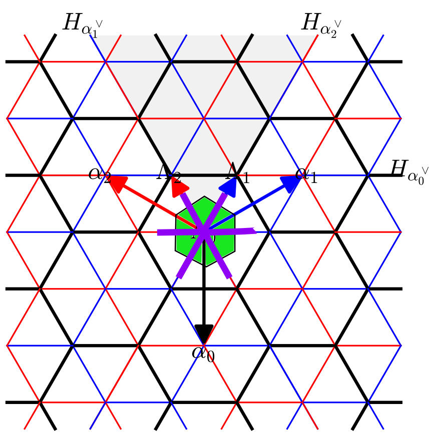
We restrict our attention to the possible regular subdivisions of this hexagon under this degeneration of our continuous family of generic toric varieties.

Due to the shapes of the moment polytopes of the extremal irreducible components, there exist generic toric varieties in the original MV cycle whose special fiber limits divide this hexagon in the following two ways. We show the regular subdivision of this hexagon as well as the associated polyhedral complexes.



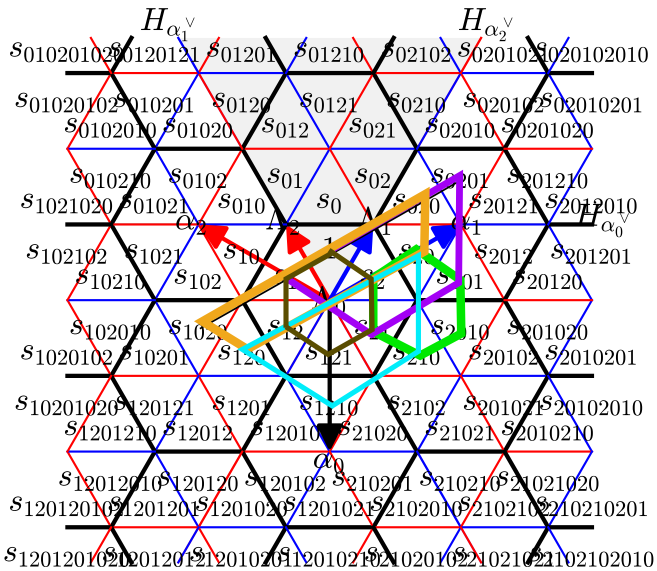


Since these generic toric varieties in the original MV cycle form a continuous family, there must be some generic toric varieties whose special fiber limits do not subdivide this hexagon. We show this moment polytope with the associated polyhedral compolex below:

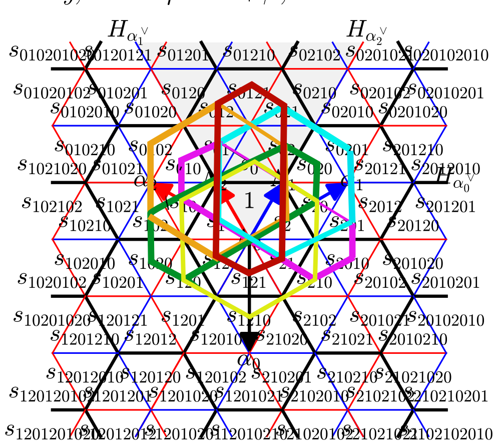


This implies that some generic $T-$orbits in this G/B bundle above 1 must be in the special fiber limit of some generic toric varities in the original MV cycle. Therefer, the G/B bundle above 1, which is also a GGMS stratum in the affine flag variety and the closed Iwahori orbit $I^{s_1s_2s_1}$, must be an irreducible component of the special fiber limit of this MV cycle.

So now all the five irreducible components in the special fiber limit of an MV cycle in this case is illustrated below:



Finally, When $\mu = \alpha + \beta$, we are back in the picture of the degeneration of the entire $G(O)$ orbit.



We have a conjecture which gives an explicit description of the special limit of an MV cycle.

Conjecture 2. *Let A be an MV cycle in the affine Grassmannian, and A could be represented as a GGMS stratum, $\overline{\cap_{w\in W} S_w^{\mu_w}}$. Then the special fiber limit of A, $\tilde{A}$, is the intersection of closed semi-infinite orbits in the affine flag variety $\cap_{w\in W}\overline{S_w^{(\mu_w,1)}}$.*

We already have some upper bounds and lower bounds on the number of irreducible components of the special fiber limit of an MV cycle. One possible way to prove this conjecture by considering the degenerations of a continuous family of some smaller sub-schemes in the MV cycle, like a continuous family of the orbits of T or $T(\mathcal{O})$. We used the degenerations of a one-parameter family of toric varieties to fully work out the example above.

11 Degenerations of Iwahori Orbits in the Affine Grassmannian of Type A

Consider generalized Iwahori subgroups of $G(\mathcal{O})$, $I_w = ev_0^{-1}(wBw^{-1}), w \in W$. In this section we are going to discuss the central degeneration of the orbits of $I_w, w \in W$.

Theorem 11. *Let I_w^λ be an orbit of I_w, where λ is a coweight.*

The special fiber limit of the Iwahori orbit I_w^λ is the union $\cup_{x \in J} A_x \cdot x$. J is the set of $T-$fixed points that lie in the limit of the original moment polytope of I_w^λ. A_x is a particular subgroup of I_w that contains $I_{ht(\lambda_{dom})}$ as a proper subgroup.

Moreover, the special fiber limit of any orbit of I_w is invariant under $I_{\text{height}(\lambda_{dom})}$.

Proof. $I-$orbits are the same as the orbits of $ev_0^{-1}(N)$, which is a subgroup of $U \times U^-$. From our knowledge of the degenerations of the orbits of U and U^-, we know that the special fiber limit of an Iwahori orbit is a finite union of orbits of subgroups of the Iwahori group. More exact combinatorial information about the subgroups involved are obtained from the moment polytopes.

The arguments for the degenerations of the orbits of other $I_w, w \in W$ are completely analogous.

□

Remark 2. *The degenerations of Iwahori orbits do not have a nice structural description because the Iwahori group is not a factorization scheme. We cannot define a global group scheme which acts on our family with the Iwahori group being the general fibers. On the other hand, we can definite such global groups schemes for $G(\mathcal{O})$ and U_w.*

Theorem 12. *The number of irreducible components of the special fiber limit of the orbit of I_w, I_w^λ, is bounded below by the number of vertices of the moment polytope of I_w^λ.*

Moreover, it is bounded above by the number of certain convex polytopes in $Lie(T^\vee)^*$ that satisfy a few extra conditions.

Firstly, each polytope is contained in the limit of the moment polytope of the closure of I_w^λ. Secondly, if two vertices of the polytope lie on the same line, then there is an orbit of a root subgroup that connects these two extremal $T-$fixed points. Lastly, the number of root-directions at each vertex is the same, and is equal to the dimension of A.

Proof. The lower bound follows from Corollary 2.

The special fiber limit of I_w^λ is a union of orbits of subgroups of I_w contained in certain $\lambda-$admissible $I-$orbit. Each such an orbit is an affine scheme. Its closure satisfies the property that each extremal $T-$fixed point a has a neighborhood isomorphic to a product of root-subgroup orbit containing a.

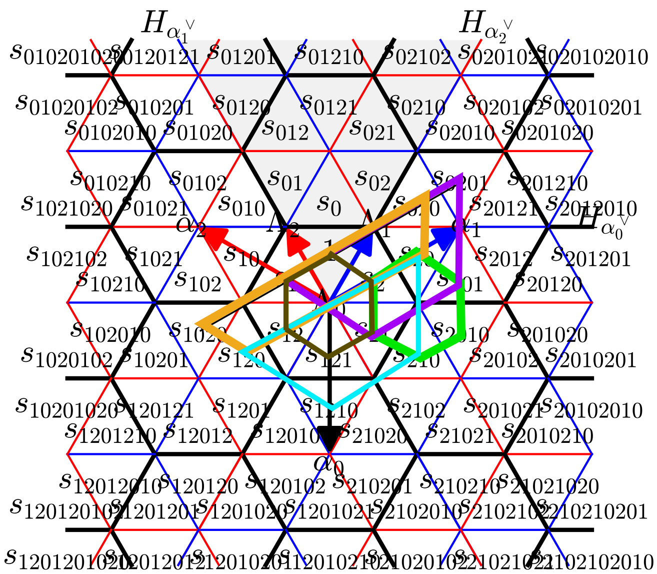
Therefore, the dimension of each irreducible component in the special fiber limit is equal to the number of root directions from any vertex of its moment polytope.

Any irreducible component is the closure of an orbit of a subgroup of I_w. It is therefore necessary that its moment polytope satisfies the three conditions: Firstly, each polytope is contained in the limit of the moment polytope of the closure of I_w^λ. Secondly, if two vertices of the polytope lie on the same line, then there is an orbit of a root subgroup that connects these two extremal $T-$fixed points. Lastly, the number of root-directions at each vertex is the same, and is equal to the dimension of A.

As a result, an upper bound for the number of irreducible components of the special fiber limit of I_w^λ is the number of convex polytopes that satisfy the conditions above.

□

Example 15. *Consider the Iwahori orbit I^α in the affine Grassmannian for $G = SL_3$. This Iwahori orbit happens to be the MV cycle $\overline{Gr_{\alpha+\beta} \cap S^\alpha}$. So its special fiber limit has five irreducible components, as illustrated below:*



One of these five irreducible components is the closure of the Iwahori orbit $I^{s_1 s_2 s_1}$. The other four are closures of orbits of distinct subgroups of the Iwahori group. $I_{\text{height}(\alpha+\beta)} = I_2$ acts on the special fiber limit of this Iwahori orbit.

12 Relations with Affine Deligne-Lusztig Varieties

As discussed in the previous sections, the special fiber limits of MV cycles are contained in a union of intersections of Iwahori orbits with U^-orbits in the affine flag variety. On the other hand, intersections of Iwahori orbits and U^- orbits in the affine Grassmannian and affine flag variety are closely related to affine Deligne-Lusztig varieties, as explained in [10] and [11].

In this section, we would first prove that the intersections of Iwahori orbits and U^- orbits in the affine Grassamnnian are equi-dimensional, and give an explicit dimension formula. Then we would proceed to discuss some algorithms and dimension bounds for the intersections of Iwahori orbits and U^- orbits in the affine flag variety.

For future projects, we would like to employ some of the moment polytopes techniques used in the previous sections to give sharper bounds on the dimensions of affine Deligne-Lusztig varieties.

12.1 Intersections of the Iwahori and U^- Orbits in the Affine Grassmannian

Let's first consider the intersection of the I-orbits with the U^--orbits in the affine Grassmannian. This is the same as the intersections of the Iwahori orbits with the open MV cycles in the $G(\mathcal{O})$ orbits.

Theorem 13. *Let G be a connected reductive algebraic group. Let λ and μ be coweights of G, and let λ_{dom} be the dominant coweight associated to λ.*

Let $\tilde{W} = W/W_I$ denote the quotient of the finite Weyl group associated to the partial flag variety $G/P_{\lambda_{dom}}$.

Let $\lambda = w \cdot \lambda_{dom}$ for a unique $w \in \tilde{W}$.

The intersection of the U^- orbit S^μ with the Iwahori orbit I^λ is equidimensional and of dimension

$$\text{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$$

when $\lambda \leq \mu \leq \lambda_{dom}$, and is $\emptyset$ otherwise. (X_w is the Schubert variety for $w \in W_I$.)

Proof. $G(\mathcal{O})$ acts transitively on Gr^λ. Consider the two subgroups I and $U^-_{\mathcal{O}} = U^- \cap G(\mathcal{O})$. Given a $T-$fixed point t^γ, there is an inclusion $U^-_{\mathcal{O}} \cdot t^\gamma \hookrightarrow S^\gamma \cap Gr^\gamma$.

First consider the case when $\mu = \lambda_{dom}$.

Claim 1: $S^{\lambda_{dom}} \cap Gr^\lambda = U^-_{\mathcal{O}} \cdot t^{\lambda_{dom}} \cap Gr^\lambda = J^- \cdot t^{\lambda_{dom}}$. Here $J^- = ev_0^{-1}(N^-)$, where N^- is the unipotent radical of the opposite Borel B^- in G. This equality holds only at a dominant coweight. This is the pre-image of the open Schubert cell X_1^- under the map ev_0, and is a vector bundle over the opposite open Schubert cell X_1^-.

Claim 2: $S^{\lambda_{dom}} \cap I^\lambda$ equals $J^- \cdot t^{\lambda_{dom}} \cap I^\lambda$. It is the pre-image of the open Richardson variety $X_1^- \cap X_w$ under the map ev_0. The open Richardson variety $X_1^- \cap X_w$ has dimension $l(w), w \in \tilde{W}$ and is dense in X_w. Therefore, $S^{\lambda_dom} \cap I^\lambda$ is dense in the Iwahori orbit I^λ and only has one component.

Then it follows that $S^\mu \cap I^\lambda$ is dense in I^λ with only one irreducible component. The dimension of the intersection is the same as the dimension of the Iwahori orbit I^λ itself, namely $2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$.

On the other hand, let's consider the case $\mu = \lambda$.

Claim 3: $S^\lambda \cap Gr^\lambda = U^-_{\mathcal{O}} \cdot t^\lambda \cap Gr^\lambda$.

Proof of Claim 3:

U^- could be written as a product of some root subgroups U_a^-, a being an affine root of the form $-\alpha + j\delta$, for some $\alpha \in R^+_{Re}$.

If $j \geq 0$, U_a^- is a subgroup of $U^-_{\mathcal{O}}$;

If $j < 0$, then $U_a^- \cdot t^\lambda \cap Gr^\lambda = t^\lambda$. This could also be seen from the lattice picture.

Therefore $S^\lambda \cap I^\lambda = U^-_{\mathcal{O}} \cdot t^\lambda \cap I^\lambda$.

$G(\mathcal{O})$ acts on Gr^λ transitively. The tangent space at t^λ in Gr^λ is isomorphic to the following quotient of the loop subalgebra $L^+\mathfrak{g} = \oplus_{k\geq 0}\mathfrak{g}t^k$,

$L^+\mathfrak{g}/Z$, where $Z = \{a \in L^+_{\mathfrak{g}} | a \cdot t^\lambda = t^\lambda \cdot h \ for\ some\ h \in L^+_{\mathfrak{g}}\}$.

There is a decomposition of the Lie algebra $\mathfrak{g} = \mathfrak{t} \oplus (\oplus_\alpha(\mathfrak{e}_\alpha \oplus \mathfrak{e}_{-\alpha}))$, where α ranges over all the positive roots.

The tangent space at t^λ in Gr^λ generated by the orbits of the Iwahori subgroup I is $\mathfrak{t} \oplus (\oplus_\alpha(\mathfrak{e}_\alpha)) \oplus_{k>0} \mathfrak{g}t^k/Z$. The tangent space at t^λ generated by the orbits of the subgroup $U^-_{\mathcal{O}}$ is $\oplus_{\alpha,k\geq 0}\mathfrak{e}_{-\alpha} \cdot t^k/Z$. As a result, the Iwahori and U^- orbits generate the tangent space at t^λ in the $G(\mathcal{O})$ orbit Gr^λ.

Therefore the intersection $I^\lambda \cap S^\lambda$ is transverse. By a transversality theorem of Kleiman, it is equi-dimensional and each component has dimension $(\text{height}(\lambda_{dom} + \lambda)) + (2 \cdot \text{height}(\lambda_{dom}) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)) - 2 \cdot \text{height}(\lambda_{dom}) = \text{height}(\lambda_{dom} + \lambda) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$.

Now we know the theorem holds for the extreme cases $\mu = \lambda_{dom}$ and $\mu = \lambda$. Let's consider coweights μ such that $\lambda < \mu < \lambda_{dom}$.

In the affine Grassmannian Gr, $\overline{S^{\gamma}} = \cup_{\eta \leq \gamma} S^{\eta}$. Given a projective embedding of Gr, for each semi-infinite orbit S^{μ}, its boundary is given by a hyperplane section H_{μ}.

This means that given two coweights μ_1 and μ_2 such that height$(\mu_2) =$ height$(\mu_1) - 1$, and given any irreducible component C_2 of $I^{\lambda} \cap S^{\mu_2}$, there is an irreducible component C_1 of $I^{\lambda} \cap S^{\mu_1}$ such that $C_2 \cap H_{\mu}$ is dense in C_1. Dimension of C_1 is bigger than or equal to $\dim(C_2) - 1$, as C_1 is cut out by a hyperplane.

The difference of $\dim(S^{\lambda_{dom}} \cap I^{\lambda})$ and $\dim(S^{\lambda} \cap I^{\lambda})$ is exactly height$(\lambda_{dom} - \lambda)$. For $\lambda < \mu \leq \lambda_{dom}$, whenever the height of μ decreases by 1, the dimension of the intersection $S^{\mu} \cap I^{\lambda}$ has to also decrease by 1.

As a result, we know that for $\lambda \leq \mu \leq \lambda_{dom}$, the intersections $S^{\mu} \cap I^{\lambda}$ are equidimensional and we have the dimension formula as stated in the theorem.

□

Note that in [26], it was shown the number of top-dimensional irreducible components in the intersections of Iwahori orbits and U^- orbits in the affine Grassmannian is equal to the dimensions of different weight spaces in some Demzure modules for the Langlands dual group. The main original contribution here is the proof for equi-dimensionality, and the methods used to derive a dimension formula is different.

12.2 Intersections of Iwahori Orbits with U^- Orbits in the Affine Flag Variety

In this section, we are going to discuss the intersections of Iwahori orbits and U^- orbits in the affine flag variety of type A. The arguments here could be generalized to algebraic groups of other types.

Locally the affine flag variety is a vector bundle, and also an iterated sequence of line bundles, over the affine Grassmannian. We could understand more about the intersections of Iwahori orbits and U^- orbits in the affine flag variety using our results in the affine Grassmannian.

Theorem 14. *Let $w_1 = (L_0, L_1, ..., L_{n-1})$ and $w_2 = (L'_0, L'_1, ..., L'_{n-1})$ be two elements in the affine Weyl group of type A. Each L_i and L'_i is a coweight for GL_n.*

Then the intersection $I^{w_1} \cap S^{w_2} \neq \emptyset$ if and only if $I^{L_i} \cap S^{L'_i} \neq \emptyset$ in the affine Grassmannian for all $i = 0, ..., n-1$.

If the intersection is indeed nonempty, then it is contained in the G/B-bundle above the intersection $I^{L_0} \cap S^{L'_0}$ in the affine Grassmannian.

Proof. First, the affine flag variety has natural projection maps $p_i, i \in \{0, 1, \cdots, n-1\}$ to the n copies of affine Grassmannians $Gr_i, i \in \{0, 1, \cdots, n-1\}$, with different relative positions. If $I^{w_1} \cap S^{w_2} \neq \emptyset$, then its image under the projection map p_i should be nonempty too.

Now suppose $I^{L_i} \cap S^{L'_i} \neq \emptyset$ in the affine Grassmannians Gr_i for all $i = 0, ..., n-1$, then these intersections would build up the intersection $I^{w_1} \cap S^{w_2}$. Let A_j denote the restriction of $I^{w_1} \cap S^{w_2}$ in the affine flag variety to the first j affine Grassmannians. Then locally A_{j+1} is a line bundle over A_j.

For the last statement, both I^{w_1} and S^{w_2} lie in the G/B bundle above I^{L_0} and $S^{L'_0}$, therefore the intersection $I^{w_1} \cap S^{w_2}$ lies in the G/B bundle above $I^{L_0} \cap S^{L'_0}$.

□

The intersection may have multiple irreducible components. In the theorem we give a basis of an open coordinate chart for each component. This gives a way to understand each irreducible component explicitly.

Theorem 15. *Let $w_1 = (L_0, L_1, ..., L_{n-1})$ and $w_2 = (L'_0, L'_1, ..., L'_{n-1})$ be two elements in the affine Weyl group of type A. Each L_i and L'_i is a coweight for GL_n.*

Let C be an irreducible component in the intersection $I^{w_1} \cap S^{w_2}$.

Then C has an open affine chart that is generated by the following basis $\beta = \beta_0 \cup \cdots \cup \beta_n$.

Here β_0 is a basis that generates one open irreducible component in $I^{L_0} \cap S^{L'_0}$;

β_1 is a basis that is obtained from a basis for the open chart of one irreducible component in $I^{L_1} \cap S^{L'_1}$ minus any basis elements that belong to the 0th affine Grassmannian Gr_0.

Similary β_i is obtained from a basis that generates one component in $I^{L_i} \cap S^{L'_i}$ excluding any elements that belong to the $Gr_0 \cup \cdots \cup Gr_{i-1}$. Discard β_i if the intersection $I^{L_i} \cap S^{L'_i}$ is zero-dimensional.

Proof. The affine flag variety has n projections to n different affine Grassmannians. The algorithm above basically calculates the basis elements of the intersection on the projected image of each of these affine Grassmannians, adding them up but also taking into account of over-counting. □

Corollary 3. *Let $w_1 = (w \cdot \lambda_{dom}, w')$, $w_2 = (\mu, w'')$ be two elements in the affine Weyl group for G. Let $\tilde{W}$ denote the Weyl group associated with the partial flag variety $G/P_{\lambda_{dom}}$. w', w'' are in the finite Weyl group W, $w \in \tilde{W}$, λ_{dom} is a dominant coweight and μ is a coweight.*

The intersection $I^{w_1} \cap S^{w_2}$ consists of the intersection $I^{L_0} \cap S^{L'_0}$ in the affine Grassmannian as well as some intersections in the G/B fiber.

The dimension of the intersection of the U^- orbit S^{w_2} with the Iwahori orbit I^{w_1} is bounded above by

$$\mathrm{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w) + \dim(G/B)$$

.

Proof. The intersection of an Iwahori orbit and a U^- orbit consists of intersections on the affine Grassmannian and intersections on the G/B fiber above the affine Grassmannian. □

Below is a sharper bound on the dimensions of the intersection of an Iwahori orbit and a U^- orbit in the affine flag variety.

Theorem 16. *Let $w_1 = (w \cdot \lambda_{dom}, w')$, $w_2 = (\mu, w'')$ be two elements in the affine Weyl group for G. Let $\tilde{W}$ denote the Weyl group associated with the partial flag variety $G/P_{\lambda_{dom}}$. w', w'' are in the finite Weyl group W, $w \in \tilde{W}$, λ_{dom} is a dominant coweight and μ is a coweight.*

The dimension of $I^{\alpha_1} \cap S^{\alpha_2}$ in the affine flag variety for G is less than or equal to

$$\mathrm{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w) + \begin{cases} \dim(G/B) - l(w') & \text{for} \quad w' = w'' \in W \\ l(w') - l(w'') & \text{for} \quad w' \geq w'' \in W \end{cases}$$

Proof. The intersection $I^{w_1} \cap S^{w_2}$ is contained in the G/B-bundle above the intersection $I^{L_0} \cap S^{L'_0}$ in the affine Grassmannian. The dimension of $I^{L_0} \cap S^{L'_0}$ is $\mathrm{height}(\lambda_{dom} + \mu) - \dim(G/P_{\lambda_{dom}}) + \dim(X_w)$.

In the G/B bundle above $I^{L_0} \cap S^{L'_0}$, a U^- orbit is the same as the B^- orbit containing w_2, and an I orbit is a subset of the product of the B^- orbit and Borbit containing w_1.

When $w_1 = w_2$, the intersection in the G/B fiber is a subset of the B^- orbit containing the T-fixed point indexed by $w_1 = w_2$, whose dimension is given by $\dim(G/B) - l(w_1)$. When $w_1 > w_2$, the intersection in the G/B fiber is a subset of the Richardson variety $X_{w_1} \cap X^-_{w_2}$, whose dimension is $l(w_1) - l(w_2)$. When $w_1 < w_2$, the intersection is empty.

□

So far we study the intersections of the Iwahori orbits and U^- orbits in the affine flag variety by focusing on the fact that the affine flag variety is a G/B bundle over the affine Grassmannian.

On the other hand, we could also think more about the picture of alcoves. In previous sections, we developed some techniques of calculating/estimating the dimensions of certain projective varieties in the affine Grassmannian or the affine flag variety by looking at $T-$equivariant moment polytopes.

In some low-dimensional examples, such techniques are useful for estimating and comparing the dimensions of the intersections of Iwahori orbits with U^- orbits in the affine flag variety.

As a next step, we hope to employ more moment polytope techniques to study affine Deligne-Lusztig varieties.

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