CHARACTERISTIC CLASSES AND HOMOGENEOUS SPACES, I.*

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Introduction. It is known that the characteristic classes of a real or complex vector bundle may be interpreted as elementary symmetric functions in certain variables, which are 1, 2 or 4 dimensional cohomology classes. If we consider the tangent bundle of the coset space G/U of a compact connected Lie group modulo a closed subgroup, it turns out that these variables may be identified with certain roots of G (or their squares). Our first purpose is to establish this connection between roots and characteristic classes, which is the basis of this paper, and to compute the characteristic classes of certain well-known homogeneous spaces. These results are then applied in particular to G/T (T maximal torus of G), and to other algebraic homogeneous spaces, where they lead to relations between characteristic classes, Betti numbers, the Riemann-Roch theorem and representation theory; they are also used to discuss multiplicative properties of the Todd genus and other genera in fibre bundles with G/U as fibre. As an application, we get a divisibility property of the Chern class of a complex vector bundle over an even dimensional sphere which yields some information about certain homotopy groups of Lie groups.

We now give a summary of the different chapters. For the notions and notations used without further comments, the reader is referred to [2, 19].

Chapter I. The first three Sections give a survey of standard properties of roots and linear representations; § 4 gives two characterizations of systems of positive roots which will be used in Chapter IV. In § 5 we introduce the roots of a Lie group with respect to a commutative subgroup of type $(2, 2, \cdots, 2)$, which will occur in the description of Stiefel-Whitney classes.

Chapter II recalls those concepts of fibre bundle theory which are most often used in this paper, such as restriction and extension of the structural group (with respect to homomorphisms), integration over the fibre, to be denoted by $\natural$, and the bundle of vectors tangent to the fibres of a bundle whose typical fibre admits a differentiable structure invariant under the structural group, to be called hereafter the "bundle along the fibres".

* Received January 29, 1958.

Chapter III starts with a review of the definition by means of symmetric functions of the characteristic classes of a bundle having a classical group as structural group (§ 9). In § 10, we consider the λ-extension η^1 of a principal G-bundle η by means of a unitary representation λ: $G \to U(n)$; the bundle η^1 is a principal $U(n)$-bundle, whose Chern classes are shown to be the elementary symmetric functions in the weights of λ, suitably interpreted as 2-dimensional classes; analogous statements are proved for the real orthogonal representations and Pontrjagin classes (10.3). Now, the real tangent bundle to G/U is the ι-extension of the principal U-bundle $(G, G/U, U)$, with respect to the linear isotropy representation ι of U in the tangent space at a point of G/U fixed under U (Proposition 7.5). Applying 10.3 to this situation yields the relation between roots and characteristic classes mentioned at the beginning of this introduction, which, in fact, holds more generally for the bundle along the fibres of $(E/U, B, G/U)$, where (E, B, G) is a principal G-bundle.

Chapter IV. The homogeneous space G/U admits an invariant almost complex structure J if and only if the isotropy representation ι can be factorized through the standard inclusion of $\boldsymbol{U}(n)$ in $\boldsymbol{SO}(2n)$, $(2n = \dim G/U)$; we obtain in this case a unitary representation ι_c: $U \to \boldsymbol{U}(n)$, whose weights are certain roots of G, to be called the roots of J; they allow us to compute the Chern classes of J using 10.3 and to discuss the integrability of J using the results of § 4; among the applications in § 13 we give new proofs of some results of H. C. Wang.

The invariant complex structures of G/U (where U is the centralizer of a torus in G), can be obtained directly by using the complexification of G; the space G/U is also homogeneous kählerian [5] and even rational algebraic, and there is a close connection between its projective embeddings and the linear representations of G. For later use, we include in § 14 a short discussion of some of these results; moreover, we prove (14.10) that the real cohomology classes of these algebraic homogeneous spaces are all of type (p, p) and for $p = 1$ describe those which are positive in the sense of Kodaira.

Chapter V is devoted to some special cases; in particular, to projective spaces.

Chapter VI.† In § 20 a formula for the homomorphism $\natural$ in the bundle $\xi = (B_T, B_G, G/T, \rho(T, G))$ is established (20.3); it shows, in particular (22.2), that $\natural$, applied to the total Todd class of the bundle along the fibres of ξ endowed with the complex vector bundle structure defined by means of an

† §§ 20 to 30 will be published in a later issue of this Journal.

invariant almost complex structure J on G/T, gives a zero-dimensional term, which is 1 or 0, according to whether J is integrable or not; it follows that the Todd genus $T(G/T)$[1] of G/T with respect to J is 1 or zero respectively and that (22.5) in certain bundles $(E, B, G/T)$, with almost complex E and B the Todd genus, "behaves multiplicatively," i.e., that we have $T(E) = T(B) \cdot T(F)$. These results are generalized to the T_y-genus and to homogeneous almost complex spaces G/U (rank $U = \operatorname{rank} G$).

In § 23 we consider the A-genus of homogeneous spaces G/U with $\operatorname{rank} U = \operatorname{rank} G$ and, in particular, prove it to be equal to 0 when the second Betti number of G/U vanishes; moreover, in a differentiable bundle $(E/U, B, G/U)$ with $A(G/U) = 0$, we also have $A(E) = 0$.

In § 24, G/U ($\operatorname{rank} U = \operatorname{rank} G$) is assumed to be algebraic. The value of the T_y-genus found in § 22 and (14.10) yield a formula for the Betti numbers of G/U in terms of the action of the Weyl groups of G and U on the roots of G. The dimension of the vector space of holomorphic cross-sections of a complex line bundle F on G/U is computed by means of the Riemann-Roch theorem and is shown to be either zero or equal to the degree of the irreducible representation of G having the first Chern class of F as highest weight (24.7); (this fact has led to the results of [7a] and has been further generalized by R. Bott [7b]). The degree (in the sense of algebraic geometry) of the projective embedding of G/U given by this linear representation, or equivalently, by the complete linear system of divisors belonging to F, is also explicitly calculated.

Chapter VII. In § 25, the A-genus is proved to be an integer. More generally, we introduce, for a complex vector bundle η over a differentiable manifold X and for an arbitrary element $d \in H^2(X, \mathbf{Z})$, a rational number $\hat{A}(X, d, \eta)$, in analogy with the Riemann-Roch formula, and prove that it is an integer after multiplication by a suitable power of 2. It follows that the q-th Chern class of a complex vector bundle over the $2q$-dimensional sphere $\mathbf{S}_{2q}$ is divisible by the greatest odd factor of $(q-1)!$; applications of this last fact to the homotopy of Lie groups are given in § 26.

As is well known, the index of a differentiable manifold X equals the L-genus of X, which is a linear combination of Pontrjagin numbers [19]. It was recently proved [12] that the index "behaves multiplicatively" in differentiable bundles. This fact has certain consequences for the Pontrjagin classes of the bundle along the fibres of a differentiable bundle, from which

[1] We allow ourselves to denote by T the Todd genus as well as a maximal torus, since this is unlikely to bring any confusion.

we conclude that the sequence of L-polynomials is essentially uniquely characterized by the property of giving rise to a genus which behaves multiplicatively in differentiable fibre bundles.

In Appendix I, we compare the different definitions of Chern classes known to us, with particular emphasis on the signs.

In Appendix II, it is first proved that the torsion coefficients of $H^*(B_{\boldsymbol{O}(n)}, \boldsymbol{Z})$ and $H^*(B_{\boldsymbol{SO}(n)}, \boldsymbol{Z})$ are all of order 2. This allows us to characterize the universal *integral* Pontrjagin class p_i by its canonical images in $H^{4i}(B_{\boldsymbol{O}(n)}, \boldsymbol{R})$ and $H^{4i}(B_{\boldsymbol{O}(n)}, \boldsymbol{Z}_2)$, the latter being equal to the square of the universal $2i$-th Stiefel-Whitney class. It is also shown that, up to 2-torsion, the integral i-th Pontrjagin class of a principal $\boldsymbol{O}(n)$-bundle ξ can be defined by means of the transgression in a certain bundle η_i associated with ξ; in particular, η_i has the typical fibre $\boldsymbol{O}(n)/\boldsymbol{O}(2i-1)$ when n is odd.

TABLE OF CONTENTS

Chapter I. Compact Lie Groups.

1. Generalities.

1.1. *Coset spaces.* Let G be a Lie group, U a closed subgroup of G, G/U the space of left cosets of $G \bmod U$, π the natural projection of G onto G/U, and $\mathfrak{g}$, $\mathfrak{u}$ the Lie algebras of G, U identified as usual with the tangent spaces at the neutral element. Left translation by $g \in G$ induces a homeomorphism of G/U which will be denoted by the same letter; if $u \in U$, it leaves $o = \pi(e)$ invariant and induces an automorphism $\tilde{u}$ of the tangent space $(G/U)_0$ of G/U at o. The homomorphism $\iota_u : u \to \tilde{u}$ is called the ***isotropy representation*** and its image the *linear isotropy group* $\tilde{U}$. For connected G, its kernel is the subgroup of those elements of G which act trivially on G/U or, also, the largest subgroup of U invariant in G.

$\mathrm{Ad}\, g$ or $\mathrm{Ad}_{\mathfrak{g}}\, g$ will denote the automorphism of $\mathfrak{g}$ induced by the inner automorphism $x \to gxg^{-1}$ of G. If π_e is the differential of π at e, we have clearly

$$\pi_e \circ \mathrm{Ad}_{\mathfrak{g}}\, u = \tilde{u} \circ \pi_e \qquad (u \in U);$$

in particular, since the kernel of π_e is $\mathfrak{u}$, π_e allows us to identify $(G/U)_0$ with any subspace $\mathfrak{m}$ of $\mathfrak{g}$ supplementary to $\mathfrak{u}$, invariant under $\mathrm{Ad}_{\mathfrak{g}}\, u$, in such a way that $\tilde{u}$ is carried over to the restriction of $\mathrm{Ad}_{\mathfrak{g}}\, u$ to $\mathfrak{m}$. If σ is an automorphism of G and σ_e its differential at e, then

$$\mathrm{Ad}\, \sigma(g) = \sigma_e \circ \mathrm{Ad}\, g \circ \sigma_e^{-1}. \tag{1}$$

1.2. From now on, G is a compact Lie group. We recall that its maximal toral subgroups are conjugate to each other by inner automorphisms and are maximal abelian subgroups of G if G is connected; their common dimension is the *rank* of G, to be denoted here by l or $l(G)$. The letter T will be reserved for a maximal torus of G, and S for an arbitrary toral subgroup; V_S will be the universal covering of S and Γ_S the "unit lattice", i.e., the inverse image of the identity element of S. The latter is a free commutative group of rank k ($k = \dim S$), which spans V_S. A real valued linear form on V_S is said to be *integral* if it takes integral values on Γ_S.

1.3. *Roots, diagram.* The representation $s \to \mathrm{Ad}_{\mathfrak{g}}\, s$ of S in $\mathfrak{g}$ is fully reducible and there is a direct sum decomposition

$$\mathfrak{g} = \mathfrak{a}_1 + \cdots + \mathfrak{a}_k + \mathfrak{b} + \mathfrak{s} \qquad (\dim \mathfrak{a}_i = 2) \tag{2}$$

of $\mathfrak{g}$ into subspaces invariant under $\mathrm{Ad}_{\mathfrak{g}}\, S$, where $\mathfrak{b} + \mathfrak{s}$ is the largest subspace on which S operates trivially. We may then write, for $s \in S$,

$$\mathrm{Ad}_{\mathfrak{g}}\, s \mid \mathfrak{a}_i = \begin{pmatrix} \cos 2\pi a_i(s) & -\sin 2\pi a_i(s) \\ \sin 2\pi a_i(s) & \cos 2\pi a_i(s) \end{pmatrix} \tag{3}$$

where $a_i(p(x))$, p the projection of V_S onto S, is a non-zero integral linear form.[2] The linear forms $\pm a_i$ are *the roots of G with respect to S*. We shall be concerned mainly with the case where $S = T$ is a maximal torus. Then $\mathfrak{b} = 0$, and the $2m$ linear forms $\pm a_i$, $(i = 1, \cdots, m;\ \dim. G = l + 2m)$, are simply *the roots* of G.[3] Clearly, if $S \subset T$, the roots relative to S are the restriction to V_S of the roots of G which do not vanish identically on S.

[2] In the sequel, there will be no notational distinction between $a(p(x))$ and $a(t)$.
[3] We call them roots in spite of the facts that the roots in the sense of the

An element $t \in T$ is *singular* if its centralizer has dimension $> l(G)$, *regular* otherwise; in V_T the singular elements are represented by the points of the hyperplanes $a_i \equiv 0 \bmod 1$, $(1 \leqq i \leqq m)$, which form the *diagram of* G.

In case S is a toral subgroup of U, the decomposition (2) may be chosen in such a way that $\mathfrak{u}$ is spanned by a subspace $\mathfrak{b}_1$ of $\mathfrak{b}$, $\mathfrak{s}$ and some of the $\mathfrak{a}_i$, say $\mathfrak{a}_1, \cdots, \mathfrak{a}_q$; the $\pm a_i$ $(q < i \leqq m)$ will be called *the roots* of G *relative* to S *complementary to those of* U, or simply the *complementary roots* if there is no danger of confusion.

1.4. *The Weyl Group.* We choose once and for all a positive definite metric on $\mathfrak{g}$ invariant under $\operatorname{Ad} G$, and consider on V_T the metric which is induced by it in the obvious way; it allows us to define in the standard way a canonical isomorphism between V_T and its dual space $V_T{}^*$ and a metric on $V_T{}^*$; the scalar product on V_T or $V_T{}^*$ will be denoted $(\ ,\ )$. An element $a \in V_T{}^*$ is called *singular* if there exists a root a_i such that $(a, a_i) = 0$; its image in V_T under the canonical isomorphism is then singular in the above sense. Finally, we remark that the symmetry S_a of V_T with respect to the hyperplane $a = 0$ induces a symmetry of $V_T{}^*$, to be denoted also by S_a, defined by

$$S_a(b) = b - 2(a, b)(a, a)^{-1} \cdot a.$$

The Weyl group $W(G)$ of G is the group of automorphisms of T induced by inner automorphisms of G leaving T invariant; it is a finite group and a quotient of N_T/T, where N_T is the normalizer of T in G; it may also be viewed as a group if isometries of V_T leaving Γ_T and the diagram invariant. For connected G, it is isomorphic to N_T/T and is generated by the symmetries to the hyperplanes $a_i = 0$ $(i = 1, \cdots, m)$.

2. Standard properties of roots. G is a compact *connected* Lie group of dimension n and rank l, T a maximal torus, V its universal covering and $\pm a_i$ $(1 \leqq i \leqq m, n = l + 2m)$ are the roots of G. The proofs of the statements in §§ 2, 3 may be found e.g. in [8, 13, 23, 27, 28].

2.1. An element $v \in V$ is in the inverse image of the center of G if and only if $a_i(v) \equiv 0 \bmod 1$ $(i = 1, \cdots, m)$; in particular, G is semi-simple if and only if it has l linearly independent roots.

2.2. a_i and a_j are linearly independent if $i \neq j$.

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infinitesimal theory (i.e., the roots of the Killing equation), are the forms $\pm 2\pi i a_j$ and the zero form; the forms a_j were called "paramètres angulaires" by E. Cartan [8]. Also, unless otherwise specified, the zero form will not be considered as a root.

2.3. The number $2(a,b)\cdot(a,a)^{-1}$ is an integer for any two roots a, b and the linear forms $b - k\cdot a$ are also roots for k integral varying between 0 and $2(a,b)(a,a)^{-1}$.

2.4. *Simple roots.* Let (x^i), $(1 \leqq i \leqq l)$, be a base of V^*. We define a total ordering $\mathcal{S}$ on V^* by saying that $a = a_1x^1 + \cdots + a_lx^l$ is > 0 if its first non vanishing coefficient is > 0 and that $a > b$ if $a - b > 0$. A root is *simple,* relative to $\mathcal{S}$, if it is positive and cannot be represented as the sum of two positive non-zero roots. The simple roots are linearly independent, the scalar product of any two of them is $\leqq 0$, and every root is a linear combination, with integral coefficients of the same sign, of simple roots. It follows, in particular, that, when G is semi-simple, there are l simple roots; also, a simple root is not a linear combination with positive coefficients of other positive roots.

2.5. A set of elements of V or V^* is said to be decomposable if it is the union of two non-empty mutually orthogonal subsets. A semi-simple group G is simple if and only if its root system or the system of its simple roots is indecomposable. Assume G to be simple, let a_i $(1 \leqq i \leqq l)$ be the simple roots and let $b = b_1a_1 + \cdots + b_la_l$ be the highest root with respect to an ordering $\mathcal{S}$. Then we have

$$(b, a_i) \geqq 0, \qquad b_i > 0, \qquad\qquad (i = 1, \cdots, l),$$

and b_i majorizes the coefficient of a_i for all roots.

2.6. *The sign of an element of* $W(G)$. Let a_i $(1 \leqq i \leqq m)$, be the positive roots with respect to $\mathcal{S}$. Since $W(G)$ leaves the diagram invariant, any $w \in W(G)$ induces a permutation of the system $(\pm a_i)$ $(1 \leqq i \leqq m)$, and transforms (a_i) into a system of roots $(\epsilon_i a_i)$ with $\epsilon_i = \pm 1$. We shall denote by $s(w)$ the number of ϵ_i's equal to -1 and by $\operatorname{sgn} w$ the product of the ϵ_i's. We contend that $\operatorname{sgn} w$ is equal to the determinant of w viewed as a linear transformation of V, and, in particular, does not depend on $\mathcal{S}$. In fact, let e_i, e_{-i} be the orthonormal base of $\mathfrak{a}_i$ with respect to which we have (3) of § 1, and let $g \in N_T$ belong to the coset of w. It follows from 2.2 that w permutes the $\mathfrak{a}_i$, and then from (1) § 1 that if $w(a_i) = \epsilon_i a_j$, then $\operatorname{Ad} g(e_i \wedge e_{-i}) = \epsilon_i e_j \wedge e_{-j}$; moreover, the natural isomorphism of $\mathfrak{t}$ on V carries the restriction of $\operatorname{Ad} g$ to $\mathfrak{t}$ over to w; our contention follows readily from this and from the fact that, G being connected, we have $\det \operatorname{Ad} g = 1$.

2.7. *The Weyl chambers.* Let $a_1, \cdots, a_r$ be the simple roots belonging to the order $\mathcal{S}$. A *Weyl chamber* in V or V^* is a connected component of the set of regular elements. In particular, the set of points $v \in V$ (resp. $x \in V^*$), such that $a_i(v) > 0$, (resp. $(x, a_i) > 0$) $(1 \leqq i \leqq r)$, is one and will be called the positive Weyl chamber with respect to $\mathcal{S}$. The Weyl group acts simply transitively on the Weyl chambers, and, in particular, the integer $s(w)$ defined in 2.6 is zero if and only if w is the identity; it is generated by the symmetries to the hyperplanes $a_i = 0$ $(1 \leqq i \leqq r)$.

2.8. *Remark on orderings.* Let $a \in V^*$ be such that $(a, a_i) \neq 0$ for all i $(1 \leqq i \leqq m)$, and let us say that a root is positive if $(a, a_i) > 0$. Then the order relation thus obtained between roots is induced by an ordering on V^* as considered in (2.4) and has, therefore, the properties (2.4), (2.5). To define $\mathcal{S}$, one takes a base (x^i) of V^* dual to a base (e_i) $(1 \leqq i \leqq l)$, of V where e_i $(i \geqq 2)$ is contained in the hyperplane $a = 0$.

2.9. *Classification.* We recall that a compact connected Lie group G has a finite covering $\tilde{G}$ which is the direct product of a torus S by a semi-simple and simply connected group $\tilde{G}'$; the group $\tilde{G}$ is uniquely determined, up to an isomorphism, if we require, moreover, that the kernel of the projection of $\tilde{G}$ onto G intersects S only at the identity. The image of $\tilde{G}'$ in G is the derived group G' of G and is also its largest semi-simple subgroup; the image of S is, of course, the connected identity component of the center of G.

The semi-simple groups are locally isomorphic to products of simple non-commutative groups. For the classification of the Lie algebras of compact simple Lie groups, we refer to [13, 23]. For a list of their roots, see for instance [25a]. For the simple Lie groups and the classical linear groups we follow here the standard notations.

3. Linear representations. a_i $(1 \leqq i \leqq m)$ are the positive roots of the compact connected Lie group G of rank l, with respect to an ordering $\mathcal{S}$, a is the sum of the a_i's and $a_1, \cdots, a_r$ are the simple roots, r being the rank of the semi-simple part G' of G.

3.1. LEMMA. *We have* $(a, a_j) = (a_j, a_j)$, $(1 \leqq j \leqq r)$.

Let S_j be the symmetry with respect to $a_j = 0$. We have

$$S_j a_i = a_i - 2(a_i, a_j) \cdot (a_j, a_j)^{-1} \cdot a_j.$$

Hence $S_j a_i$ and a_i $(j \leqq r)$, when expressed as linear combinations of simple roots, differ at most by the coefficient of a_j; thus, if $a_i \neq a_j$, $S_j a_i$ has at least one positive coefficient, and is a positive root by (2.4); this means that S_j permutes among themselves the positive roots different from a_j. But we have

$$(a_j, S_j a_i + a_i) = 0,$$

whence the lemma.

3.2. To abbreviate, we write $E(b)$, ($b \in V^*$ or $b \in V^* \otimes \boldsymbol{C}$), for

$$\sum_{w \in W(G)} (\operatorname{sgn} w) \exp(2\pi\sqrt{-1} \cdot w(b))$$

and $E(b, x)$ for the value of this function on $x \in V$. If b^* and x_* are the images of b and x under the canonical isomorphism between V and V^* defined by a metric invariant under $W(G)$, we have clearly $E(x_*, b^*) = E(b, x)$.

By a standard result of representation theory, we have

$$E(a/2) = \prod_{i=1}^{i=m} 2\sqrt{-1} \sin \pi a_i. \tag{1}$$

The computation of the m-th orders terms on each side yields the equality

$$m! \prod a_i = \sum_{w \in W(G)} (\operatorname{sgn} w) \cdot w(a/2)^m. \tag{2}$$

Let us denote by $E_x{}^{(m)}(b, 0)$ the value at 0 of the m-th derivative of $E(b, y)$ in the direction x; then we have

$$E_x{}^{(m)}(a/2, 0) = m!(2\pi\sqrt{-1})^m \prod_1^m \langle a_i, x \rangle. \tag{3}$$

It follows directly from the definition that $E(b)$ vanishes identically when b is singular. Conversely, the equality $E(x_*, b^*) = E(b, x)$, recalled above, and (3) imply

$$E_{a^*/2}{}^{(m)}(b, 0) = m!(2\pi\sqrt{-1})^m \prod_1^m \langle a_i, b \rangle = m!(2\pi\sqrt{-1})^m \prod_1^m (a_i, b); \tag{4}$$

hence, if $E(b) = 0$, the element b must be singular.

3.3. *The weights.* An element $b \in V^*$ is called a weight of G if it is integral on the unit lattice of the connected identity component of the center of G and is such that $2(b, a_j) \cdot (a_j, a_j)^{-1}$ is an integer ($j \leqq m$). The weights form a free commutative group of rank l. The previous condition may also be expressed by saying that a weight is a linear form which is integral on the unit lattice Γ_0 corresponding to the covering $\tilde{G}$ of G which has the form $S \times \tilde{G}'$ where S is a torus, $\tilde{G}'$ is semi-simple and simply connected, and such that the kernel of the projection $\tilde{G} \to G$ intersects S only at the identity (2.9).

For $b, c \in V^*$, let us put $q(b, c) = 2 \cdot (b, c)(c, c)^{-1}$. Let S_d be the symmetry to the hyperplane $d=0$. We have $(b, S_d c)=(S_d b, c)$, $(S_d c, S_d c) = (c, c)$ and $S_d b = b - q(b, d)d$, and therefore

$$q(b, S_d c) = q(b, c) - q(b, d) q(d, c). \tag{5}$$

PROPOSITION. *Let b be an element of V^*. Then $q(b, a_j)$ is an integer for $1 \leqq j \leqq m$ if and only if it is so for $1 \leqq j \leqq r$. In particular, $a/2$ is a weight.*

The Weyl group $W(G)$ is generated by the symmetries S_j to the hyperplanes $a_j = 0$ $(1 \leqq j \leqq r)$, and every root is the image of a simple root under some transformation of $W(G)$ (this follows from 2.7). Since $q(d, c)$ is an integer when c and d are roots (2.3), the equality (5) shows that $q(b, S_j a_k)$ is integral if $q(b, a_k)$ and $q(b, a_j)$ are, and our first assertion follows by an obvious induction. The second one is then a consequence of (3.1) and of the fact that a is zero on the identity component of the center of G.

3.4. *Characters.* It follows from the results of H. Weyl and from standard facts about direct products, that the characters of the irreducible representations of the group $\bar{G}$ introduced in 3.3, restricted to the maximal torus $\bar{T}$, are the functions

$$\chi(t) = E(b) \cdot E(a/2)^{-1}, \tag{6}$$

where b runs through the weights contained in the positive Weyl chamber defined by $\mathscr{S}$. In other words, the b's are the weights which verify

$$2(a_j, b) = k_j(a_j, a_j), \qquad (k_j > 0, \text{ integral}, j = 1, \cdots, r).$$

By dividing out in (6), one may write $\chi(t)$ as a finite sum of exponentials $\exp 2\pi\sqrt{-1}\, c_s$, where the c_s's are weights. The highest one is $b - (a/2)$; it has multiplicity one and characterizes the linear representations up to an equivalence. In view of the foregoing, the highest weights are those which satisfy

$$2(a_j, c) = k_j(a_j, a_j), \qquad (k_j \geqq 0, \text{integral}, j = 1, \cdots, r). \tag{7}$$

Assume now G to be semi-simple, hence $r = l = \operatorname{rank} G$. Let ϖ_i be the linear form defined by $q(\varpi_i, a_j) = \delta_{ij}$ $(i = 1, \cdots, l)$. By (3.3), the ϖ_i's are weights, to be called the fundamental weights, and form a basis of the group of weights. By (7), the highest weights are the linear combinations of the ϖ_i's with integral non negative coefficients.

Let again $\bar{G}$ be compact, not necessarily semi-simple. The degree d of the representation with highest weight $b-(a/2)$ is $\chi(0)$. In the right hand side of (6), this appears in the form 0/0, but by taking suitable m-derivatives at the origin, and using (2), (3), (4), one arrives easily at the formulas:

$$(8)\qquad \begin{gathered} d\cdot(m!)\cdot\prod_{1}^{m} a_i=\sum_{w\in W(G)}(\operatorname{sgn} w)\,w(b)^m \\ d=\prod_{1}^{m}(a_i,b)\cdot(a_i,a/2)^{-1}. \end{gathered}$$

Finally, we remark that the representation of $\tilde{G}$ with highest weight $b-(a/2)$ is single-valued on G if and only if $b-(a/2)$ is integral on the unit lattice corresponding to G.

4. Two characterizations of systems of positive roots. We assume here G to be semi-simple, and denote its rank by l, but otherwise follow the notations of § 3. We discuss here two conditions under which given roots are positive relative to a suitable ordering; the first one is the object of (4.3), which will be preceded by two lemmas.

4.1. Lemma. *Let u_j be the integers such that the sum a of the positive roots is equal to $u_1a_1+\cdots+u_la_l$. Then the only solution of the system of inequalities*

$$(1)\qquad (a_j,x)\geqq(a_j,a_j)\,;\ 0\leqq x_j\leqq u_j,\quad (x=x_1a_1+\cdots+x_la_l\,;\,j=1,\cdots,l),$$

is a itself.

That a is a solution follows from (3.1).

Let $y=y_1a_1+\cdots+y_la_l$ be a solution of (1) for which the sum of the y_i is minimum; such a y clearly exists. The y_i's are >0, because if, e.g., $y_k=0$, then we would have $(a_k,y)\leqq 0$ by (2.4).

Put $r_j=(a_j,y)$. Since the a_j's $(1\leqq j\leqq l)$ form a base, it is enough to show that $r_j=(a_j,a_j)$. Suppose otherwise; then there exists a k such that $r_k>(a_k,a_k)$. Consider $z=y-c\cdot a_k$, where c is a small positive constant. Then

$$z=\sum z_ja_j,\quad (z_j=y_j,\ (j\neq k)\,;\,z_k=y_k-c)$$
$$(a_j,z)=(a_j,y)-c(a_j,a_k)=r_j-(a_j,a_k)c.$$

Since $(a_j,a_k)\leqq 0$ for $j\neq k$ by (2.4), z is, for suitably small $c>0$, a solution of (1) for which the sum of the coefficients is strictly smaller than for y, contradicting the latter's definition.

4.2. Lemma. *Let (ϵ_i), $(i=1,\cdots,m)$, be a sequence of integers of* *absolute value* 1, *and let* $a^* = \sum_i \epsilon_i a_i$. *If* $(a^*, a_j) > 0$ *for* $j = 1, \cdots, l$, *then* $a^* = a$ *and* $\epsilon_i = 1$ $(1 \leqq i \leqq m)$.

Let $c = c_1 a_1 + \cdots + c_l a_l$ (resp. $d = d_1 a_1 + \cdots + d_l a_l$), be the sum of the a_i for which $\epsilon_i = 1$ (resp. $\epsilon_i = -1$). We have

$$(2) \quad a^* = c - d;\ a = c + d;\ c_j + d_j = u_j;\ c_j, d_j \geqq 0, \qquad (j = 1, \cdots, l).$$

By assumption,

$$(a_j, c) - (a_j, d) = (a_j, a^*) > 0, \qquad (j = 1, \cdots, l),$$

and by (3.1),

$$(a_j, c) + (a_j, d) = (a_j, a) = (a_j, a_j), \qquad (j = 1, \cdots, l),$$

whence

$$2(a_j, c) > (a_j, a_j), \qquad (j = 1, \cdots, l).$$

But it follows from (2.3) that $2(a_j, c) \cdot (a_j, a_j)^{-1}$ is an integer; therefore the preceding inequality implies that

$$(a_j, c) \geqq (a_j, a_j), \qquad (j = 1, \cdots, l),$$

which, together with (2), shows that c is a solution of (1). By (4.1), this gives $c = a$, $d = 0$, $a = a^*$ and $d = 0$ implies (2.4) that no ϵ_i equals -1.

4.3. Theorem. *Let* (ϵ_i), $(i = 1, \cdots, m)$, *be a sequence of integers of absolute value* 1. *Then the set* $(\epsilon_i a_i)$ *is the system of positive roots with respect to some ordering* $\mathcal{S}'$ *if and only if* $a^* = \sum_i \epsilon_i a_i$ *is a regular element.*

Necessity: Suppose that $(\epsilon_i a_i)$ are the positive roots with respect to some ordering; by (2.7), there exists $w \in W(G)$ which sends the $\epsilon_i a_i$ onto the a_i and, therefore, a^* onto a. Since the Weyl group leaves the set of regular elements invariant, it suffices to show that a is regular; but this follows from (3.1).

Sufficiency: Suppose a^* to be regular, and let $\mu_i = \pm 1$ be such that $(a^*, \mu_i a_i) > 0$. Then, as remarked in (2.8), $(\mu_i a_i)$ is the system of positive roots with respect to some ordering $\mathcal{S}_1$, and it will therefore be enough to show that $\mu_i = \epsilon_i$, $(i = 1, \cdots, m)$.

By (2.7), we may find $w \in W(G)$ carrying $(\mu_i a_i)$ onto (a_i) and, therefore, a^* onto $a^{**} = \sum_i \epsilon_i \mu_i a_{\sigma(i)}$, where σ is a permutation of $(1, 2, \cdots, m)$ since $(a^*, \mu_i a_i) > 0$ for all i and since w preserves the scalar product, we have

$(a^{**}, a_j) > 0$, $(j = 1, \cdots, l)$, and (4.2) gives then $a^{**} = a$ and $\mu_i\epsilon_i = 1$, $(i = 1, \cdots, m)$.

4.4. Corollary. *Let J be a non-empty set of integers belonging to the interval $[1, m]$. Suppose we have signs ϵ_j, $(j \in J)$, such that $\sum_{j \in J} \epsilon_j a_j = 0$. Then for any choice of the remaining signs ϵ_i, the form $\sum_1^m \epsilon_i a_i$ is a singular element.*

Suppose otherwise; then by (4.3), the $(\epsilon_i a_i)$ are the positive roots in some suitable ordering, but, obviously, a sum of positive roots cannot vanish.

4.5. Definition. *A set B of roots of G is said to be closed if it contains the sum of any two of its elements whenever this sum is a root of G.*

Our main purpose will be to show that a closed system containing one root from each pair $\pm a_i$ is positive for some ordering.

4.6. Lemma. *Let B be a closed system which for no i $(1 \leqq i \leqq m)$, contains both a_i and $-a_i$. Then if a linear combination b of elements of B with positive integral coefficients is a root, it belongs to B.*

Proof by induction on the sum k of the coefficients of b. For $k = 2$, it is an assumption; assume the lemma to be true for $k - 1$, and let

$$b = c_1 b_1 + \cdots + c_q b_q, \quad (c_i > 0, c_i \text{ integer}, \textstyle\sum c_i = k, b_i \in B).$$

We distinguish two cases. (a) For some $j \leqq q$, $b_1 + b_j$ is a root; it is then in B by definition, and we have

$$b = (b_1 + b_j) + (c_1 - 1) b_1 + c_2 b_2 + \cdots + (c_j - 1) b_j + c_{j+1} b_{j+1} + \cdots + c_q b_q.$$

Therefore, b may also be written as

$$b = c'_1 b'_1 + \cdots + c'_r b'_r, \quad (c'_i > 0, \text{ integral}, \textstyle\sum c'_i = k - 1, b'_i \in B),$$

and is in B by the induction assumption. (b) No element $b_1 + b_j$ is a root; then $(b_1, b_j) \geqq 0$ for $j = 2, \cdots, q$, (see 2.3), whence $(b_1, b) > 0$, and $b - b_1$ is a root. By induction, it is in B, and b then belongs to B by definition.

4.7. Lemma. *We keep the same assumption on B. Then any sum $\sum c_i b_i$ $(b_i \in B, c_i > 0, \text{integral})$ is $\neq 0$.*

Otherwise, we would have

$$-b_1 = (c_1 - 1)b_1 + c_2b_2 + \cdots + c_qb_q,$$

contradicting (4.5) and (4.6).

4.8. A sequence $(b_1, \cdots, b_k)$ of elements of B such that $b_i - b_{i+1} \in B$ for $i = 1, \cdots, k-1$ will be said to be decreasing of length k. The height $h(b)$ of $b \in B$ will be the maximal length of decreasing sequences starting with b. By Lemma 4.7, any two elements in a decreasing sequence are different, and hence $h(b)$ is always finite. Let $k = h(b)$ and $b, b_2, \cdots, b_k$ a decreasing sequence. If we add to b a decreasing sequence for b_2 or for $b - b_2$, we clearly get a decreasing sequence starting with b. Therefore $h(b) = k$ implies that $h(b_2)$, $h(b - b_2)$ are $\leqq k-1$; thus *an element of height $k \geqq 2$ is sum of two elements of heights $\leqq k-1$.*

4.9. THEOREM. *Let B be a closed system of roots which, for each i, $(1 \leqq i \leqq m)$, contains exactly one of the two roots $\pm a_i$. Then B is the set of positive roots for a suitable ordering.*

Let $b_1, \cdots, b_s$ be the elements of height 1 in B. By induction on the height, it follows from the last assertion in (4.8) that every element of B is a linear combination of the b_i's $(i = 1, \cdots, s)$ with positive integral coefficients. Therefore, it suffices to show that the b_i's are linearly independent. Since they span all roots, their rank is l; assume that $b_1, \cdots, b_l$ are independent and that, contrary to our contention, $s \neq l$. We have then a relation

$$(3) \qquad b_{l+1} = c_1b_1 + \cdots + c_lb_l, \quad (c_i \text{ real, not all zero}).$$

The c_i's are also a solution of the linear system

$$(4) \qquad (x_1(b_1, b_j) + \cdots + x_l(b_l, b_j))(b_j, b_j)^{-1} = (b_{l+1}, b_j)(b_j, b_j)^{-1}$$

$(j = 1, \cdots, l)$, whose determinant is, up to a positive factor, the determinant of the products (b_i, b_j) and is therefore $\neq 0$. Thus the c_i's are the unique solution of (4) and are rational numbers since the coefficients are rational by (2.3). The given relation is then equivalent to a relation

$$d_1b_1 + \cdots + d_lb_l + d_{l+1}b_{l+1} = 0 \quad (d_i \text{ integers, } d_{l+1} \neq 0).$$

By (4.7) the coefficients do not all have the same sign, and, after a change in numeration of the b_i $(i \leqq l)$, we arrive at an equality

$$(5) \qquad e_1b_1 + \cdots + e_pb_p = e_{p+1}b_{p+1} + \cdots, + e_{l+1}b_{l+1} = b$$

$(e_i \geqq 0$, integral, $(e_1, \cdots, e_p) \neq (0, \cdots, 0))$. On the other hand, we have $(b_i, b_j) \leqq 0$ for $(1 \leqq i < j \leqq s)$; because otherwise, by (2.3), the elements $\pm (b_i - b_j)$ would be roots, one of them would belong to B and either b_i or b_j would have a height $\geqq 2$. Therefore we have

$$(b, b) = (e_1b_1 + \cdots + e_pb_p, e_{p+1}b_{p+1} + \cdots + e_{l+1}b_{l+1}) \leqq 0$$

and $b = 0$, in contradiction to (4.7), which proves that $l = s$.

4.10. COROLLARY. *Let B be a closed system of roots which for each i, $(1 \leqq i \leqq m)$, contains at least one of the roots $\pm a_i$. Then B contains the set of all positive roots relative to a suitable ordering.*[4]

Let us number the roots in such a way that B consists of $\pm a_1, \cdots, \pm a_q$, $\epsilon_{q+1}a_{q+1}, \cdots, \epsilon_m a_m$ $(\epsilon_i = \pm 1)$. Then the system B' consisting of the a_i $(1 \leqq i \leqq q)$ and the $\epsilon_j a_j$ $(q < j \leqq m)$ is closed; in fact, if $a_s + a_t$ $(s, t \leqq q)$ is a root, it is positive and in B, hence in B'. If $a_s + \epsilon_t a_t = -a_p$ $(s, p \leqq q < t)$, then $a_s + a_p = -\epsilon_t a_t$ and B would not be closed; if $\epsilon_s a_s + \epsilon_t a_t = -a_p$, $(p \leqq q < s, t)$, then $a_p + \epsilon_t a_t = -\epsilon_s a_s$ and B is again not closed. Therefore, by the theorem, B' is the set of positive roots for some ordering.

4.11. *Remark.* Using complex semi-simple Lie algebras, one can also prove more generally than 4.9 that a closed system of roots B which for each i contains at most one of the roots $\pm a_i$ is positive for some ordering. In fact, in the notations of (12.2), it follows readily from (4.7) that the subspace of $\mathfrak{g}^c$ spanned by $\mathfrak{t}^c$ and the $\mathfrak{v}_b$, $(b \in B)$, is a solvable subalgebra. It is then conjugate by an inner automorphism α to a subalgebra of the algebra spanned by $\mathfrak{t}^c$ and the $\mathfrak{v}_{a_i}$ by a result of Morosow (C. R. Acad. Sci. U. R. S. S. (N. S.), 36 (1942), pp. 83-86), also proved in A. Borel, Annals of Math., 64 (1956), pp. 20-80, § 16. Moreover, by the conjugacy of Cartan subalgebras in solvable Lie algebras, we may assume that $\alpha(\mathfrak{t}^c) = \mathfrak{t}^c$ which means that B is transformed onto a subset of the a_i's by an element of the Weyl group.

5. The 2-roots of a compact Lie group.

We define here certain linear forms with values in $\boldsymbol{Z}_2$, analogous to the roots, which are useful in the study of Stiefel-Whitney classes.

5.1. Let G be a Lie group. We denote by Q or Q_s a subgroup of G

[4] Another, completely different, proof of this corollary has been given by Harish-Chandra, *American Journal of Mathematics*, vol. 77 (1955), pp. 743-777, § 2, Lemma 4.

isomorphic to the product of s copies of $\boldsymbol{Z}_2$. The real irreducible linear representations of Q are 1-dimensional, being defined by a character which may be viewed as an element of $\mathrm{Hom}(Q, \boldsymbol{Z}_2)$. Let us now decompose the Lie algebra $\mathfrak{g}$ of G into a direct sum

$$\mathfrak{g} = \mathfrak{b}_1 + \cdots + \mathfrak{b}_n, \qquad (n = \dim \mathfrak{g})$$

of 1-dimensional subspaces invariant under $\mathrm{Ad}_{\mathfrak{g}}\, Q$; the characters $b_1, \cdots, b_n$ corresponding to these subspaces will be called the *2-roots of G with respect to Q*. In case Q is contained in a maximal torus, these are just the restrictions to Q of the roots of G and the zero form with multiplicity $l = \mathrm{rank}\,(G)$, but otherwise, they represent different elements of the group-theoretical structure of G and have a "global" character.

5.2. Let U be a closed subgroup of G containing Q. Then we can choose a decomposition into subspaces $\mathfrak{b}_i$, such that the $n-k$ last ones generate the Lie algebra $\mathfrak{u}$ of U. The 2-roots b_i $(1 \leqq i \leqq k)$ are then the *complementary* 2-roots; or, more explicitly, the 2-roots of G with respect to Q which are complementary to those of U.

Examples.

5.3. $G = \boldsymbol{O}(n)$, $\boldsymbol{SO}(n)$. We consider in $\boldsymbol{O}(n)$ the subgroup $\boldsymbol{Q}$ of diagonal matrices; it is a maximal commutative subgroup of type $(2, 2, \cdots, 2)$, and any subgroup of this type is conjugate to a subgroup of $\boldsymbol{Q}$. We take in $\mathfrak{g}$ the usual basis consisting of the antisymmetric matrices having only two non-vanishing entries, equal to ± 1. Let x_{i1} $(1 \leqq i \leqq n)$ be the diagonal matrix all of whose coefficients are equal to 1, except for the i-th one which is equal to -1, and let (y_i) be the dual basis of $\mathrm{Hom}(\boldsymbol{Q}, \boldsymbol{Z}_2)$. A straightforward computation shows that the basis of $\mathfrak{g}$ mentioned above is invariant under $\mathrm{Ad}_{\mathfrak{g}}\,\boldsymbol{Q}$ and that the 2-roots relative to $\boldsymbol{Q}$ are

$$y_i - y_j, \qquad (1 \leqq i < j \leqq n).$$

In $\boldsymbol{SO}(n)$, the diagonal matrices also form a maximal commutative subgroup $\boldsymbol{Q}'$ of type $(2, 2, \cdots, 2)$, isomorphic to $(\boldsymbol{Z}_2)^{n-1}$. It is convenient to consider it as a subgroup of $\boldsymbol{Q}$, and, therefore, $\mathrm{Hom}(\boldsymbol{Q}', \boldsymbol{Z}_2)$ as a quotient of $\mathrm{Hom}(\boldsymbol{Q}, \boldsymbol{Z}_2)$; it is then generated by n elements y_i subject to the relation $y_1 + \cdots + y_n = 0$, and the 2-roots are again the differences $y_i - y_j$ $(1 \leqq i < j \leqq n)$.

5.4. $G = \boldsymbol{U}(n), \boldsymbol{SU}(n)$. In $\boldsymbol{U}(n)$, all maximal commutative subgroups of type $(2, 2, \cdots, 2)$ are conjugate to the subgroup $\boldsymbol{Q}$ of diagonal matrices with coefficients ± 1, and are therefore contained in maximal tori. The 2-roots are then obtained from the usual roots and, with respect to the standard basis of skew-hermitian matrices, are the zero form with multiplicity n and the differences $y_i - y_j$ $(1 \leqq i < j \leqq n)$, each with multiplicity two.

The 2-roots of $\boldsymbol{SU}(n)$ with respect to the subgroup of the elements of $\boldsymbol{Q}$ having determinant 1 will be the same, the y_i's being subject to the relation $y_1 + \cdots + y_n = 0$, the zero form having multiplicity $n-1$.

5.5. $G = \boldsymbol{Sp}(n)$. Here again, all maximal commutative subgroups of type $(2, 2, \cdots, 2)$ are conjugate to the subgroup $\boldsymbol{Q}$ of diagonal matrices with coefficients ± 1 (G being considered as the group of unitary matrices with quaternionic coefficients) and are isomorphic to $(\boldsymbol{Z}_2)^n$. Since the usual roots are $\pm y_i \pm y_j$ $(1 \leqq i < j \leqq n)$ and $\pm 2y_i$ $(1 \leqq i \leqq n)$, we get as 2-roots: the root zero, with multiplicity $3n$ and $y_i - y_j$ $(1 \leqq i < j \leqq n)$, with multiplicity 4.

In § 17, we shall also discuss the 2-roots of the exceptional group $\boldsymbol{G}_2$ with respect to a subgroup not contained in a maximal torus.

Chapter II. Topological Preliminaries.

6. Fibre bundles.

6.1. *Notations.* p denotes a prime number or zero, K_p a field of characteristic p, $\boldsymbol{Z}_p$ $(p \neq 0)$, $\boldsymbol{Z}_0$, $\boldsymbol{R}$, $\boldsymbol{C}$, the fields of integers mod p, of rational, real, complex numbers respectively.

$H^i(X, A)$ (resp. $H_i(X, A)$) is the i-th singular cohomology (resp. homology) group of the space X with coefficients in the commutative group A, $H^*(X, A)$ (resp. $H_*(X, A)$) the direct sum of the cohomology (resp. homology) groups; for all spaces considered in this paper, $H^i(X, \boldsymbol{Z})$ will be finitely generated and equal to the i-th Alexander-Spanier cohomology group. The map of cohomology (resp. homology) groups induced by a continuous map f is denoted by f^* (resp. f_*).

When dealing with classifying spaces, it will sometimes be convenient to consider formal infinite sums of cohomology elements, and to this effect, we also introduce the *direct product* $H^{**}(X, A)$ of the $H^i(X, A)$; an element $x \in H^{**}(X, A)$ may be identified with a sum $x_0 + \cdots + x_i + \cdots$, with $x_i \in H^i(X, A)$ possibly $\neq 0$ for infinitely many values of i. When A is a ring, $H^{**}(X, A)$ also becomes an associative ring under the cup product. The homomorphism of $H^{**}(Y, A)$ into $H^{**}(X, A)$ induced by $f: X \to Y$ will be denoted by f^{**}.

$A\{X_1, \cdots, X_k\}$ will denote the ring of formal power series in the X_i's, with coefficients in the commutative ring A.

Let U be a closed, connected subgroup of maximal rank of the compact, connected Lie group G, and let T be a maximal torus of U. We have then $H^{**}(B_T, A) = A\{x_1, \cdots, x_l\}$, $(x_i \in H^2(B_T, A),\ 1 \leqq i \leqq l = \operatorname{rank} G)$. The results of [2, §§ 26, 27] imply that $\rho^{**}(T, G)$ maps $H^{**}(B_G, \boldsymbol{Z}_0)$ isomorphically onto the ring of invariants of the Weyl group, and that it is isomorphic to a ring of formal power series in l indeterminates; moreover, $H^{**}(G/U, \boldsymbol{Z}_0)$ is the quotient of $H^{**}(B_U, \boldsymbol{Z}_0)$, regarded as a subring of $H^{**}(B_T, \boldsymbol{Z}_0)$, by the ideal $(I^+_G)^*$ generated in $H^{**}(B_U, \boldsymbol{Z}_0)$ by the (finite or infinite) sums of homogeneous invariants of $W(G)$ with strictly positive degrees. Similar translations in cohomology over $\boldsymbol{R}$, $\boldsymbol{Z}_p$ or $\boldsymbol{Z}$ of the results of [2, § 29] are left to the reader.

6.2. *Fibre bundles.* The fibre bundles occurring in this paper will be locally trivial; we follow the definitions of [19, 26]. We do not require the structural group to act effectively on the fibre [19, § 3.2c)]. A fibre bundle is denoted by (E, B, F, π) or (E, B, F), where E is the total space, B the base space, F the typical fibre, π the propection, or just by one symbol, mostly ξ, η, θ; in the latter case, we often write $E_\xi, B_\xi, F_\xi, \pi_\xi$, G_ξ, τ_ξ for E, B, F, π, the structural group and the transgression in ξ respectively. A bundle with structural group G will also be called a G-bundle.

Let ξ be a principal G-bundle and U a closed subgroup of G. The space of the cosets $x \cdot U$ modulo U $(x \in E_\xi)$, is denoted by E_ξ/U; it is the base space of the principal fibering $(E_\xi, E_\xi/U, U)$ and the total space of the G-bundle $(E_\xi/U, B_\xi, G/U)$. Let F be a space operated upon by G. We denote by $E_\xi \times_G F$ the quotient of $E_\xi \times F$ by the equivalence relation $(x, f) \approx (x \cdot g, g^{-1} \cdot f)$. As is well known, it is the total space of a G-bundle (ξ, F) over B_ξ, with fibre F.

6.3. *Representations of fibre bundles.* Let ξ, η be two fibre bundles. A representation of ξ in η is a continuous map $\phi\colon E_\xi \to E_\eta$ which sends fibres into fibres; it induces then a map $\bar{\phi}\colon B_\xi \to B_\eta$ such that $\bar{\phi} \circ \pi_\xi = \pi_\eta \circ \phi$. We shall use without further comment the fact that ϕ commutes with transgression and, more generally, induces a homomorphism of the spectral sequence of η into that of ξ (see e.g. [2], § 4).

6.4. *Homomorphisms of fibre bundles.* Let G, G' be topological groups, $\lambda\colon G \to G'$ a homomorphism, and F (resp. F'), a space on which G, (resp. G'), operates. A λ-map of F into F' is a continuous map ψ such that

$\psi(g\cdot f)=\lambda(g)\cdot\psi(f)$, (or $\psi(f\cdot g)=\psi(f)\cdot\lambda(g)$ if G, G' operate on the right). Let ξ and η be principal G- and G'-bundles respectively. A homomorphism of ξ into η is a representation induced by a λ-map of E_ξ into E_η; clearly every λ-map defines a homomorphism. A homomorphism of (ξ, F) into (η, F') is a representation defined by two λ-maps of E_ξ and F into E_η and F' respectively.

Let U, U' be closed subgroups of G and G' such that $\lambda(U)\subset U'$. Then we have a commutative diagram

$$\begin{array}{ccccc} E_\xi & \longrightarrow & E_\xi/U & \longrightarrow & B_\xi \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi_2} \\ E_\eta & \longrightarrow & E_\eta/U' & \longrightarrow & B_\eta \end{array}$$

ϕ defines λ-homomorphisms $(E_\xi, E_\xi/U, U)\rightarrow(E_\eta, E_\eta/U', U')$ and $E_\xi, B_\xi, G)\rightarrow(E_\eta, B_\eta, G')$; the map ϕ_1 is a λ-homomorphism of $(E_\xi/U, B_\xi, G/U)$ into $(E_\eta/U', B_\eta, G'/U')$.

6.5. *Restriction and extension of the structural group.* Let ξ and η be two principal bundles over the same base space B, and let λ be a homomorphism of G_ξ into G_η. Assume that there exists a λ-homomorphism of ξ into η which induces the identity of B. Then we say that η is a λ-extension of ξ and that ξ is a λ-restriction of η. We recall that, given ξ and λ, there always exists a λ-extension which is unique up to equivalence, and which will be denoted by $\lambda(\xi)$; whereas given η and λ, a λ-restriction does not always exist and, if it does, is not necessarily unique. The λ-extension η of ξ is defined as follows: $E_\eta=E_\xi\times_G G'$, where G operates on G' by $g\cdot g'=\lambda(g)\cdot g'$; if μ is the projection of $E_\xi\times G'$ onto E_η, then π_η is induced by $\mu(x,g')\rightarrow\pi_\xi(x)$, and the λ-map $\phi: E_\xi\rightarrow E_\eta$ is defined by $\phi(x)=\mu(x,e)$, where e is the neutral element in G'; finally, the principal bundle operations on E_η are introduced by $(x,y)\cdot g'=(x, y\cdot g')$, $(x\in E_\xi; y, g'\in G')$. These notions are defined in the same way for associated bundles; they generalize the standard concepts of extension and restriction of the structural group, to which they reduce when λ is the inclusion map of a closed subgroup. Clearly they can also be formulated for equivalence classes of bundles; if these are identified with the elements of the cohomology sets $H^1(B, G_c)$ and $H^1(B, G'_c)$, in the notations of [19, § 3], then the λ-extension of $\xi\in H^1(B, G_c)$ is its image under the natural coefficient map induced by λ.

6.6. *Characteristic map.* E_G (resp. B_G) is a universal bundle (resp. classifying space) for the compact Lie group G ([26], § 19, [2], § 18; as in

[6] they will usually be taken as universal or classifying for all dimensions). Any principal G-bundle ξ over a base space B belonging to a suitable class of topological spaces is induced from the universal bundle by a map $\sigma: B \to B_G$, defined up to homotopy as the "characteristic map" for ξ [26, § 19].

To a homomorphism λ of G into a compact Lie group G' corresponds a map $\rho(\lambda)$ of B_G into $B_{G'}$, defined up to homotopy, called the characteristic map for the λ-extension of (E_G, B_G, G) (see [6], § 1). It follows immediately from the definitions that a λ-restriction of a principal G'-bundle η exists if and only if the characteristic map σ of η can be written as $\sigma = \rho(\lambda) \circ \sigma'$ where σ' is a map of B into B_G; when λ is an inclusion, $\rho(\lambda)$ reduces to the map $\rho(G, G')$ introduced in [2].

6.7. *Fibre bundle over a fibre bundle.* We discuss here a generalization of the "bundle along the fibres" (see § 7), which allows us to put in its proper setting a useful fact about characteristic maps.

Let ξ, η be fibre bundles, $\bar{\xi}$, $\bar{\eta}$ the corresponding principal bundles. We assume that $F_\xi = B_\eta$ and that G_ξ is also a group of automorphisms of $\bar{\eta}$; the latter condition means that there is a homomorphism $g \to \bar{g}$ of G_ξ in the group of those homeomorphisms of $E_{\bar{\eta}}$ which commute with the operations of G_η, and, of course, such that the induced homeomorphisms of B_η are those which define G_ξ as structural group for ξ. In particular, the homeomorphism $\bar{g} \times \mathrm{Id}$ of $E_{\bar{\eta}} \times F_\eta$ is compatible with the equivalence relation which defines η, hence G_ξ is also a group of homeomorphisms of E_η commuting with π_η. By means of these operations, we define first a bundle $\mu = (E_\mu, B_\xi, E_\eta)$ with structural group G_ξ, and typical fibre E_η, associated to $\bar{\xi}$; its total space is then

$$E_\mu = E_{\bar{\xi}} \times_{G_\xi} E_\eta.$$

Since G_ξ commutes with π_η, this map induces a map λ of E_μ onto $E_{\bar{\xi}} \times_{G_\xi} B_\eta = E_\xi$. Since G_ξ, operating on $E_{\bar{\eta}}$, and G_η commute, the space $E_{\bar{\xi}} \times_{G_\xi} E_{\bar{\eta}}$ can be considered as a principal G_η-bundle over E_ξ, the operations of the group being defined by means of its action on the right factor, and moreover, we have the "associativity law"

$$(E_{\bar{\xi}} \times_{G_\xi} E_{\bar{\eta}}) \times_{G_\eta} F_\eta \approx E_{\bar{\xi}} \times_{G_\xi} (E_{\bar{\eta}} \times_{G_{\bar{\eta}}} F_\eta).$$

From this, it follows immediately that λ is the projection in a fibre bundle $(E_\mu, E_\xi, F_\eta) = \nu$, in which G_η is the structural group, and whose corresponding principle bundle has total space $E_{\bar{\xi}} \times_{G_\xi} E_{\bar{\eta}}$. Therefore, we have obtained a bundle over E_ξ with fibre F_η. It is clear that the inclusion map of a fibre of μ in E_μ may be viewed as a homomorphism of η in ν; it induces a map $i: B_\eta \to E_\xi$ of their base spaces which is the inclusion map of a fibre of ξ.

Therefore, if σ is the characteristic map for ν, then $\sigma \circ i$ is characteristic for η, and the characteristic ring of η is the image of $i^* \circ \sigma^*$. This proves in particular the following:

6.8. PROPOSITION. *Let ξ, η be two bundles with $F_\xi = B_\eta$, satisfying the conditions of 6.7., and let i be the injection of a fibre of ξ. Then the image of $i^*: H^*(E_\xi, A) \to H^*(F_\xi, A)$ contains the characteristic ring of η.*

Let G be a topological group, U be a closed subgroup and θ be a principal G-bundle. Then $\xi = (E_\theta/U, B_\theta, G/U)$ and $\eta = (G, G/U, U)$ satisfy the assumptions of 6.7, η being considered as a principal U-bundle, and $G_\xi = G$ (resp. $G_\eta = U$) acting by means of left (resp. right) translations on G; in this case, μ may be identified with θ and ν with $(E_\theta, E_\theta/U, U)$, and 6.8 reduces to the corollary to Prop. 18.3 of [2]. The application to differentiable bundles will be mentioned in § 7.

7. Vector bundles.

7.1. A real (resp. complex) vector bundle is a fibre bundle with an n-dimensional real (resp. complex) vector space as typical fibre, the structural group operating by means of linear transformations. Most often, we shall identify the typical fibre with $\boldsymbol{R}^n$ or $\boldsymbol{C}^n$, and the structural group with a subgroup of $\boldsymbol{GL}(n, \boldsymbol{R})$ or $\boldsymbol{LG}(n, \boldsymbol{C})$. We refer to [19] for the notions of sub-bundle, quotient bundle of a vector bundle, of Whitney sum $\xi \oplus \eta$ and tensor product $\xi \otimes \eta$ of two vector bundles ξ, η. We recall that a principal bundle with group $\boldsymbol{GL}(n, \boldsymbol{R})$ or $\boldsymbol{GL}(n, \boldsymbol{R})^+$ or $\boldsymbol{GL}(n, \boldsymbol{C})$ has a unique restriction (up to isomorphism) with group $\boldsymbol{O}(n)$ or $\boldsymbol{SO}(n)$ or $\boldsymbol{U}(n)$ [26, § 12].

7.2. *Orientable real vector bundles.* A real vector bundle is orientable if its structural group can be reduced to $\boldsymbol{GL}(n, \boldsymbol{R})^+$ or $\boldsymbol{SO}(n)$. If such a restriction has been made, we then endow each fibre with the orientation which is carried over from a fixed given orientation of the typical fibre V by the allowable homeomorphisms of the bundle structure; the bundle is then said to be oriented; if V has been identified with $\boldsymbol{R}^n$, we always take the natural orientation of $\boldsymbol{R}^n$.

7.3. *Almost complex structures.* A complex vector bundle $(E, B, \boldsymbol{C}^q)$ defines in a natural fashion a real vector bundle $(E', B, \boldsymbol{R}^{2q})$, its λ-extension relative to the standard inclusion λ: $\boldsymbol{GL}(q, \boldsymbol{C}) \to \boldsymbol{GL}(2q, \boldsymbol{R})$; it is oriented. Conversely, if a real vector bundle $(E, B, \boldsymbol{R}^{2q})$ has a λ-restriction, we say that it admits a complex structure and that such a complex restriction is a complex structure of the given bundle. A differentiable manifold admits an almost complex structure (resp. is almost complex) if its tangent bundle admits (resp. has been given) a complex structure.[5] To a complex structure of $\xi = (E, B, \boldsymbol{R}^{2n})$ there is attached a section J in the real vector bundle $\xi^* \otimes \xi = \mathrm{Hom}(\xi, \xi)$, where the value of J_b of J at $b \in B$ is the linear map defined by multiplication by $\sqrt{-1}$; conversely, given a section J of linear maps such that $J_b{}^2 = -\mathrm{Id}$ for all $b \in B$, we introduce on each fibre a complex structure by putting

$$(x + \sqrt{-1}\, y) \cdot v = x \cdot v + y \cdot J_b(v),$$

which gives a complex structure for the given real vector bundle.

7.4. *The bundle along the fibres.* Let ξ be a fibre bundle whose fibre is a differentiable manifold F of dimension n, G_ξ being a group of differentiable homeomorphisms of F; the group G_ξ is then also a group of automorphisms of the tangent bundle $\eta = T(F_\xi)$ to F_ξ and of the bundle of frames $\bar{\eta} = B(F_\xi)$ which have $\boldsymbol{GL}(n, \boldsymbol{R})$ as structural group. We may apply 6.7, and the bundle corresponding to ν of 6.7 will be called *the bundle along the fibres.* It is a real vector bundle over $\boldsymbol{E}_\xi$, whose fibres are the tangent spaces to the fibres of ξ, and will be denoted by $\hat{\xi}$. If F has an almost complex structure which is invariant under G_ξ, in other words, if G_ξ is also a group of automorphisms for a complex structure η' of η, then the construction of 6.7 may also be applied to ξ and η' and yields a complex structure on $\hat{\xi}$ which will then be called a *complex bundle along the fibres of* ξ. Also, if F_ξ carries an orientation invariant under G_ξ, then the structural group of $\hat{\xi}$ may also be reduced to $\boldsymbol{GL}(n, \boldsymbol{R})^+$. Applying 6.8 to the basic elements of the characteristic ring of a $\boldsymbol{O}(n)$, $\boldsymbol{SO}(n)$ or $\boldsymbol{U}(n)$-bundle (see § 9), we obtain the

PROPOSITION. *Let ξ be a fibre bundle whose typical fibre F_ξ has a differentiable structure invariant under G_ξ, and let i be the inclusion map of a fibre in E_ξ. Then the Pontrjagin and Stiefel-Whitney classes of F_ξ, its Euler-Poincaré class with respect to an orientation invariant under G_ξ, and its Chern classes with respect to a G_ξ-invariant almost complex structure are in the image of i^*.*

(A similar remark has already been made in A. Borel, Jour. math. pur. appl. (9) 35, 127-139 (1956), proof of 3.2.)

[5] In this terminology therefore, an *almost complex* structure on a manifold corresponds to a *complex* structure of its tangent bundle.

ξ is said to be differentiable if E_ξ, B_ξ, F_ξ are differentiable manifolds, π_ξ is a differentiable map, and the coordinate functions are differentiable; it follows then that G_ξ is a group of diffeomorphisms of F_ξ. In this case, the fibre of $\hat{\xi}$ over $x \in E_\xi$ may be identified with the subspace of the tangent space of E_ξ at x which is tangent to the fibre of ξ passing through x.

7.5. PROPOSITION. *Let G be a Lie group, U a closed subgroup, $\iota: U \to \boldsymbol{GL}(n, R)$ the isotropy representation* (1.1), *and let ξ be a principal G-bundle. Then the principal bundle η along the fibres of $(E_\xi/U, B_\xi, G/U)$ is the ι-extension of $(E_\xi, E_\xi/U, U)$.*

We have to show the existence of a ι-map: $E_\xi \to E_\eta$ inducing the identity on E_ξ/U.

We recall first that $(E_\xi/U, B_\xi, G/U)$ may be considered as the bundle with typical fibre G/U associated to ξ; more precisely, there is a commutative diagram

$$\begin{array}{ccc} E_\xi & \xrightarrow{\alpha} & E_\xi \times G/U \\ \Big\downarrow{\scriptstyle\gamma} & & \Big\downarrow{\scriptstyle\delta} \\ E_\xi/U & \xrightarrow{\beta} & E_\xi \times_G G/U \end{array}$$

where γ and δ are the natural projections, α is defined by $x \to (x, o)$, the point $o \in G/U$ being the image of U under the projection, β is determined by the other maps and is a homeomorphism. This also allows us to attach to each $x \in E_\xi$ a homeomorphism σ_x of G/U onto the fibre $\gamma(x \cdot G)$ of $\gamma(x)$ in E_ξ/U, defined by

$$\sigma_x(y) = \gamma \cdot \alpha^{-1}(x \cdot g, o), \quad (y \in G/U,\ g \in G \text{ such that } g^{-1}(y) = o).$$

We have

$$\sigma_x(o) = \gamma(x); \quad \sigma_{x \cdot g} = \sigma_x \circ g.$$

All this is well known and easily checked. Let now R_0 be a fixed base of the tangent space to G/U at o. Then $\sigma_x(R_0)$ is a base of the tangent space to the fibre of $(E_\xi/U, B_\xi, G/U)$ at $\gamma(x)$. We define ϕ by $\phi(x) = \sigma_x(R_0)$; from the relation $\sigma_{x \cdot u} = \sigma_x \circ u$, it follows readily that $\phi(x \cdot u) = \phi(x) \cdot \iota(u)$, in other words, that ϕ is a ι-map. Since by construction, ϕ induces the identity on E_ξ/U, our contention is proved.

(7.5) shows in particular that the structural group of the tangent bundle to G/U may be ι-restricted to U. Finally, we mention the following well known elementary fact:

7.6. PROPOSITION. *Let ξ be a differentiable bundle. Then the quotient of the tangent bundle to E_ξ by the bundle along the fibres $\hat{\xi}$ is equivalent to the bundle induced by π_ξ from the tangent bundle to B_ξ.*

In fact, π_ξ induces a bundle map of this quotient onto the tangent bundle to B_ξ, and the proposition follows then from [26, § 10.3].

8. Integration over the fibre.

8.1. Let A be a commutative group, (E, B, F, π) a fibre bundle with connected fibres such that (i) there exists an integer q for which $H^r(F, A) = 0$ for $r > q$ and that (ii) the cohomology groups of the different fibres form a constant sheaf over B.

We want to define, in terms of the spectral sequence of the bundle, a homomorphism

$$\natural\colon H^k(E,A) \rightarrow H^{k-q}(B, H^q(F,A)) \qquad (k = 0, 1, \cdots),$$

the so-called "integration over the fibre". We put, of course, $\natural = 0$ for $k < q$ and assume from now on that $k \geqq q$. By (i), no non-zero element of $E_r^{k-q,q}$, $(r \geqq 2)$, is a coboundary, hence the subgroup of the elements in $E_2^{k-q,q}$ which are cocycles for all differentials is canonically isomorphic to $E_\infty^{k-q,q}$, and we get a natural inclusion map

$$h_1\colon E_\infty^{k-q,q} \rightarrow E_2^{k-q,q} \cong H^{k-q}(B, H^q(F,A)).$$

Let now J^a $(a = 0, 1, \cdots)$, be the decreasing sequence of submodules defining the filtration of $H^*(E, A)$ attached to the fibration, and let us put, as usual, $J^{a,b} = J^a \cap H^{a+b}(E, A)$. Since $E_\infty^{a,b} = 0$ for $b > q$, we have $H^k(E,A) = J^{k-q,q}$, whence

$$E_\infty^{k-q,q} = J^{k-q,q}/J^{k-q+1,q-1} = H^k(E,A)/J^{k-q+1,q-1}$$

and a natural projection

$$h_2\colon H^k(E,A) \rightarrow E_\infty^{k-q,q}.$$

$\natural$ is then defined by $\natural = h_1 \circ h_2$; by linearity it extends to an additive homomorphism of $H^*(E,A)$ into $H^*(B, H^q(F,A))$ and of $H^{**}(E,A)$ into $H^{**}(B, H^q(F,A))$. Whenever $H^q(F,A)$ can be identified with A, for instance, when F is an oriented q-dimensional manifold, we consider it as a map from $H^*(E,A)$ or $H^{**}(E,A)$ to $H^*(B,A)$ or $H^{**}(B,A)$, lowering by q the degree of homogeneous elements.

8.2. Proposition. *Let A be a commutative ring, ξ a bundle satisfying conditions* (i), (ii) *of* (8.1), $\natural$ *the integration over the fibre. Then*

$$(\pi_\xi^*(b)\cdot x)^\natural = b\cdot (x)^\natural, \quad (b\in H^*(B,A), x\in H^*(E,A)).$$

(Here $b\cdot(x)^\natural$ means the product of b and $(x)^\natural$ under the natural pairing of A and $H^q(F_\xi,A)$ to $H^q(F_\xi,A)$.) For the proof, we may assume b and x to be homogeneous of degrees s,t. We identify b with its image in $E_2^{s,0}$ under the canonical isomorphism with $H^s(B_\xi,H^0(F_\xi,A))=H^s(B_\xi,A)$. Then we have

$$h_2(\pi_\xi^*(b)\cdot(x)) = \kappa_\infty^2(b)\cdot h_2(x)$$
$$h_1(\kappa_\infty^2(b)\cdot h_2(x)) = b\cdot(h_1\circ h_2(x))$$

because E_∞ is the graded ring associated to $H^*(E_\xi,A)$ filtered by the J^a, and b is a cocycle for all differentials.

8.3. Proposition. *Let ξ, η be two fibre bundles satisfying the conditions* (i), (ii) *of* (8.1) *and let ϕ be a representation of ξ in η. Let ψ: $H^*(B_\eta,H^q(F_\eta,A))\to H^*(B_\xi,H^q(F_\xi,A))$ be the homomorphism which is induced by the map $\bar\phi\colon B_\xi\to B_\eta$ defined by ϕ, and by the map ν: $H^*(F_\eta,A)\to H^*(F_\xi,A)$ defined by the restriction of ϕ to a fibre.*[6]. *Then the following diagram is commutative*

$$\begin{array}{ccc} H^k(E_\eta,A) & \xrightarrow{\phi^*} & H^k(E_\xi,A) \\ \downarrow{\natural} & & \downarrow{\natural} \\ H^{k-q}(B_\eta,H^q(F_\eta,A)) & \xrightarrow{\psi} & H^{k-q}(B_\xi(H^q(F_\xi,A)). \end{array}$$

This follows from the fact that ϕ induces a homomorphism of the spectral sequence of η into that of ξ, reducing to ψ on the E_2 terms [2, § 4].

Remark. For another discussion of the integration over the fibre, see [11]; it is also proved there, but we shall not need this fact, that in case E, B, F are oriented compact connected manifolds, then $\natural$ is equivalent to the Gysin homomorphism defined by means of π [11, Theorem 3].

8.4. Let ξ be a fibre bundle satisfying (i), (ii) and: (iii) A is a principal ideal ring, $H^*(F_\xi,A)$ is a free A-module of finite rank, $H^q(F_\xi,A)\cong A$, and F_ξ is totally non-homologous to zero in E_ξ.

As is well known, these conditions have the following consequences for the spectral sequence of ξ:

[6] Note that, by assumption (ii), the latter homomorphism has an invariant meaning, independent from the particular fibre to which we restrict ϕ.

$$E_\infty = E_2 = H^*(B_\xi, A) \otimes H^*(F_\xi, A),$$

and π_ξ^* is injective; if $H^*(E_\xi, A)$ is considered as a $H^*(B_\xi, A)$-module by means of the rule $b \cdot x = \pi_\xi^*(b) \cup x$, $(b \in H^*(B_\xi, A)$, $x \in H^*(E_\xi, A))$, and if h_i $(1 \leqq i \leqq m = \text{rank } H^*(F_\xi, A))$ are homogeneous elements of $H^*(E_\xi, A)$ inducing a module basis of $H^*(F_\xi, A)$, then $H^*(E_\xi, A)$ is a free $H^*(B_\xi, A)$-module with base (h_i).

Assume that h_1 induces a generator $\bar{h}_1$ of $H^q(F_\xi, A)$, and use $\bar{h}_1$ to identify $H^q(F_\xi, A)$ with A. Then we have clearly

$$x = \pi_\xi^*(x^\natural) \cdot h_1 + \sum_2^m \pi_\xi^*(b_i) \cdot h_i, \tag{1}$$

and this characterizes $x^\natural$ completely.

Let μ, ν be two fibre bundles with the following properties: $E_\mu = E_\xi$, $B_\mu = E_\nu$, $B_\nu = B_\xi$, $\pi_\xi = \pi_\nu \circ \pi_\mu$, and the restriction of π_μ to a fibre of ξ is the projection map in a fibre bundle $\theta = (F_\xi, F_\nu, F_\mu)$. Assume that ξ, μ, ν, θ satisfy (i), (ii), (iii), (with of course q depending on the fibre bundle); let h_μ, h_ν be homogeneous elements of $H^*(E_\mu, A)$, $H^*(E_\nu, A)$ whose restrictions to a fibre generate the highest non-vanishing cohomology groups. Then $\pi_\mu^*(h_\nu) \cdot h_\mu = h_\xi$ has the same property in ξ. If these elements are used to identify the corresponding cohomology groups of the fibres with A, then it follows immediately from (1) that

$$\natural_\xi = \natural_\nu \circ \natural_\mu. \tag{2}$$

When ξ, μ, ν, θ are fibre bundles satisfying (i), (ii) whose total spaces, fibres and base spaces are compact oriented manifolds, then (2) follows directly from the equivalence with the Gysin homomorphism mentioned in 8.3; it was shown to us to be true in general by Puppe, but since this is not needed here, we shall not reproduce the somewhat longer proof of this fact.

Chapter III. Roots and Characteristic Classes.

9. Characteristic classes. We recall here the definitions of Chern, Stiefel-Whitney and Pontrjagin classes to be used in this paper, i.e., mainly the definitions which use universal bundles and flag manifolds. $S(x_1, \cdots, x_n)$ is the ring of symmetric polynomials in the x_i's, with respect to a ring of coefficients which the context will make precise. $S\{x_1, \cdots, x_n\}$ is the corresponding ring of symmetric formal power series.

9.1. *Chern classes.* Let ξ be a principal $\boldsymbol{U}(n)$-bundle. Its i-th Chern class is denoted by c_i or $c_i(\xi)$, $(c_i \in H^{2i}(B_\xi, \boldsymbol{Z}))$, and c or $c(\xi)$ is the sum of the c_i's. It may be defined as follows: let d_j $(1 \leqq j \leqq n)$ be the complex lines spanned by the canonical basis vectors of $\boldsymbol{C}^n$ and let $\boldsymbol{T}$ be the group of diagonal matrices in $\boldsymbol{U}(n)$; the group $\boldsymbol{T}$ is a maximal torus of $\boldsymbol{U}(n)$, its largest subgroup leaving the d_j's invariant. In the universal covering V of $\boldsymbol{T}$, we introduce coordinates x_j such that $x = (x_1, \cdots, x_n)$ operates on d_j by $z \to z \cdot \exp(2\pi i x_j)$; in other words, x_j is such that, for small positive values of x_j, the product $z \wedge x(z)$ defines the natural orientation of d_j. The restrictions of the x_j to the unit lattice define a basis of $\operatorname{Hom}(H_1(\boldsymbol{T}, \boldsymbol{Z}), \boldsymbol{Z})$ and thus a basis of $H^1(\boldsymbol{T}, \boldsymbol{Z})$, and they are, moreover, permuted by the Weyl group $W(\boldsymbol{U}(n))$ of $\boldsymbol{U}(n)$. Let $y_j = -\tau(x_j)$, where τ is the transgression in $(E_\xi, E_\xi/T, T)$, and let ρ be the projection of $E_\xi/\boldsymbol{T}$ onto B_ξ. Then $y_i \in H^2(E_\xi/T, Z)$ and $c(\xi)$ is defined by

$$\rho^*(c(\xi)) = \prod_1^n (1 + y_i).$$

To legitimize this, we have to know that the right hand side is in the image of ρ^* and that ρ^* is injective. It suffices to prove the first point in the universal bundle, in view of the commutative diagram

$$\begin{tikzcd} E_\xi \arrow[r] \arrow[d] & E_\xi/\boldsymbol{T} \arrow[r, "\rho"] \arrow[d] & B_\xi \arrow[d, "\sigma"] \\ E_{\boldsymbol{U}(n)} \arrow[r] & E_{\boldsymbol{U}(n)}/\boldsymbol{T} \arrow[r, "\rho'"] & B_{\boldsymbol{U}(n)} \end{tikzcd}$$

where σ is a characteristic map, but there it follows from [2, § 29] since ρ is by definition $\rho^*(\boldsymbol{T}, \boldsymbol{U}(n))$. As to the second point, $H^*(\boldsymbol{U}(n)/\boldsymbol{T}, \boldsymbol{Z})$ is equal to its characteristic subalgebra [2, Prop. 29. 2]; hence $\boldsymbol{U}(n)/\boldsymbol{T}$ is totally non-homologous to zero in *any* fibre bundle of the type $(E_\xi/\boldsymbol{T}, B_\xi, \boldsymbol{U}(n)/\boldsymbol{T})$, where ξ is a principal $\boldsymbol{U}(n)$-bundle ([2], Cor. to Prop. 18. 3), and this implies, in particular, that ρ^* is injective.

Let us call here a flag or, more precisely, a complex flag an ordered system of n mutually orthogonal 1-dimensional subspaces of $\boldsymbol{C}^n$. Then $\boldsymbol{U}(n)/\boldsymbol{T}$ is the space of flags and $E_\xi/\boldsymbol{T}$ is the total space of the bundle of flags in the complex vector bundle ξ_1 associated to ξ; the bundle η induced from ξ_1 by ρ, with base space $E_\xi/\boldsymbol{T}$, decomposes into a Whitney sum of n $\boldsymbol{C}^1$-vector-bundles with characteristic classes y_i. Thus the present definition of $c(\xi)$ is quite analogous to that of [19, § 4] and, in fact, will be shown in Appendix I to be equivalent to it.

From the properties of ρ^* quoted above, it follows that $\rho^{**}: H^{**}(B_\xi, \boldsymbol{R}) \to H^{**}(E_\xi/\boldsymbol{T}, \boldsymbol{R})$ is injective and has an image containing the formal power series in the y_i's which are symmetric. Thus we may introduce the *Chern character* $\mathrm{ch}(\xi)$ of ξ as an element of $H^{**}(B_\xi, \boldsymbol{R})$ by

$$\rho^*(ch(\xi)) = \exp y_1 + \cdots + \exp y_n = \sum_{j \geqq 0} (j!)^{-1}(y_1{}^j + \cdots + y_n{}^j).$$

Clearly, $ch(\xi)$ and $c(\xi)$, both regarded as elements of $H^{**}(B_\xi, \boldsymbol{R})$, determine each other; $ch(\xi)$ is denoted by $t(\xi)$ in [19].

9.2. *The Stiefel-Whitney classes mod 2.* Let ξ be a principal $\boldsymbol{O}(n)$-bundle; its i-th Stiefel-Whitney class mod 2 is denoted by w_i or $w_i(\xi)$, $(w_i \in H^i(B_\xi, \boldsymbol{Z}_2))$, and the sum of the w_i by w or $w(\xi)$. By naturality, it is enough to define it in the universal bundle. Let $\boldsymbol{Q}$ be the subgroup of diagonal matrices in $\boldsymbol{O}(n)$; we have $H^*(B_{\boldsymbol{Q}(n)}, \boldsymbol{Z}_2) = \boldsymbol{Z}_2[u_1, \cdots, u_n]$, where the u_i's are 1-dimensional classes, which may be assumed to be permuted among themselves by the normalizer of $\boldsymbol{Q}$ in $\boldsymbol{O}(n)$, and $\rho^*(\boldsymbol{Q}, \boldsymbol{O}(n))$ maps $H^*(B_{\boldsymbol{O}(n)}, \boldsymbol{Z}_2)$ isomorphically onto $S(u_1, \cdots, u_n)$. Then w is defined by

$$\rho^*(\boldsymbol{Q}(n), \boldsymbol{O}(n))(w) = \prod_1^n (1 + u_j)$$

(see [3]). This can also be expressed by means of flags. In fact, $\boldsymbol{O}(n)/\boldsymbol{Q}$ is the space of flags (i.e., of ordered systems of n mutually orthogonal lines) in $\boldsymbol{R}^n$ and $(E_\xi/\boldsymbol{Q}, B_\xi, \boldsymbol{O}(n)/\boldsymbol{Q})$ is the bundle of flags in the vector bundle associated to ξ. Let u'_j be the image of u_j under the characteristic map of $(E_\xi, E_\xi/\boldsymbol{Q}, \boldsymbol{Q})$, and let ρ be the projection of $E_\xi/\boldsymbol{Q}$ on B_ξ. Then

$$\rho^*(w(\xi)) = \prod_1^n (1 + u'_j),$$

and this characterizes $w(\xi)$ since ρ^* is injective by [3], Remark on p. 177, and [2], Cor. to Prop. 18.3.

For an $\boldsymbol{SO}(n)$ bundle, the Stiefel-Whitney classes mod 2 are defined as those of the extension to $\boldsymbol{O}(n)$.

9.3. *The Pontrjagin classes.* Let ξ be a principal $\boldsymbol{O}(n)$- or $\boldsymbol{SO}(n)$-bundle. Its i-th Pontrjagin class p_i or $p_i(\xi)$ is the $2i$-th Chern class of the unitary extension of ξ multiplied by $(-1)^i$, and p or $p(\xi)$ is the sum of the p_i's. It may also be characterized in the following way: for $n = 2m$, $2m + 1$, let d_j be the 2-dimensional subspaces of $\boldsymbol{R}^n$ spanned by the $(2j-1)$-th and $2j$-th canonical basis vectors, and let $\boldsymbol{T}$ be the maximal subgroup of $\boldsymbol{SO}(n)$ leaving the d_j's invariant; it is a maximal torus. We choose coordinates x_j in its universal covering such that $x = (x_1, \cdots, x_m)$ operates on d_j by means of a rotation of angle $2\pi x_j$; for $n = 2m$, we require that for small positive values of the x_j's, the exterior product $v_j \wedge x(v_j)$ $(v_j \in d_j,\ v_j \neq 0,\ j = 1, \cdots, m)$ defines in d_j the same orientation as the $(2j-1)$-th and $2j$-th canonical basis vectors of $\boldsymbol{R}^n$. This determines the x_j's completely. Let us consider the x_j's as a basis of $H^1(\boldsymbol{T}, \boldsymbol{Z})$ and put $y_j = -\tau(x_j)$, where τ is the transgression in the universal bundle. It follows from the definition and from the computations made in [2], proof of Prop. 31.4 (see also 9.4), that

$$\rho_Z{}^*(\boldsymbol{T}, G)(p) = \prod_1^m (1 + y_i{}^2). \tag{1}$$

In § 30, we shall see that

$$\rho_2{}^*(\boldsymbol{T}, G)(p_i) = w_{2i}{}^2, \qquad (G = \boldsymbol{SO}(n), \boldsymbol{O}(n)), \tag{2}$$

and that p is completely characterized by (1) and (2). The Pontrjagin classes mod p $(p \neq 2)$, may also be defined by going over to a bundle of flags. In $\boldsymbol{R}^n$ $(n = 2m, 2m+1)$, we call a 2-flag an ordered system of m mutually orthogonal 2-dimensional oriented subspaces. Then the space of 2-flags in $\boldsymbol{R}^n$ is $\boldsymbol{O}(n)/\boldsymbol{T}$ for n even or $\boldsymbol{O}(n)/\boldsymbol{T}'$ for n odd, where $\boldsymbol{T}'$ is an extension of $\boldsymbol{T}$ by $\boldsymbol{Z}_2$. Let ξ be a principal $\boldsymbol{O}(n)$-bundle and ρ be the projection of $E_\xi/\boldsymbol{T}$ or $E_\xi/\boldsymbol{T}'$ on B_ξ. Then we have

$$\rho^*(p(\xi)) = \prod (1 + \tau(x_i)^2),$$

where τ is the transgression in the canonical principal $\boldsymbol{T}$-bundle over $E_\xi/\boldsymbol{T}$ or $E_\xi/\boldsymbol{T}'$ respectively. This is valid over the integers; however ρ^* is injective in general only for the cohomology mod p $(p \neq 2)$, (again by [2], § 29 and Cor. to Prop. 18.3, since $\boldsymbol{SO}(n)$ and $\boldsymbol{O}(n)$ have no p-torsion for $p \neq 2$).

The $(2i+1)$-th Chern class of the complex extension will be denoted $p_{i+\frac{1}{2}}$; it is an element of order 2, equal to the square of the integral $(2i+1)$-th Stiefel-Whitney class (see Appendix II); $\tilde{p}$ or $\tilde{p}(\xi)$ will be the sum of the p_i and $p_{i+\frac{1}{2}}$. We have $\rho_Z{}^*(\boldsymbol{T}, G)(p) = \rho_Z{}^*(\boldsymbol{T}, G)(\tilde{p})$.

9.4. *Remark on the complex extension.* For computational convenience, we shall take as a complex extension of an $\boldsymbol{O}(n)$-bundle the λ-extension, where $\lambda = \delta \circ \gamma$ is the product of the injection $\gamma : \boldsymbol{O}(n) \to \boldsymbol{U}(n)$ and of the inner automorphism $\delta : x \to gxg^{-1}$, the element g being a direct sum of 2×2 matrices

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}$$

and of (1) for odd n; since it is equivalent to the γ-extension, this does not alter the Chern classes. The maximal torus $\boldsymbol{T}$ of $\boldsymbol{O}(n)$, previously described, is then mapped onto the diagonal matrices with coefficients $\exp(\pm 2\pi i x_j)$, and 1 for odd n. We have

$$\lambda^*(x'_{2j+1}) = -\lambda^*(x'_{2j}) = x_j, \qquad (1 \leqq j \leqq [n/2])$$
$$\lambda^*(x'_n) = 0, \quad (n \text{ odd}),$$

where (x_j), (x'_j) are the bases of the first integral cohomology groups of the standard maximal tori of $\boldsymbol{O}(n)$ and $\boldsymbol{U}(n)$ described before.

9.5. *The Euler-Poincaré class.* Let ξ be an oriented vector bundle with $2m$-dimensional fibre, structural group $\boldsymbol{SO}(2m)$, and let η be the associated bundle of unit spheres. The Euler-Poincaré class $W_{2m}(\xi)$ or W_{2m} of ξ is equal to $-\tau(x)$, where x is the generator of $H^{2m-1}(\boldsymbol{S}_{2m-1}, \boldsymbol{Z})$ defined by the positive orientation, and τ is the transgression in η. In the universal case, it is also characterized by the two properties:

(i) $\qquad W_{2m}$, reduced mod 2, is equal to w_{2m},

(ii) $\qquad \rho_{\boldsymbol{Z}}^*(\boldsymbol{T}, \boldsymbol{SO}(2m))(W_{2m}) = y_1 \cdot \cdots \cdot y_m.$

For the tangent bundle to a differentiable, compact, connected, oriented manifold, W_{2n} is the fundamental class multiplied by the Euler-Poincaré characteristic.

9.6. *Symplectic Pontrjagin classes.* Let ξ be a $\boldsymbol{Sp}(n)$ bundle and η its extension under the standard inclusion of $\boldsymbol{Sp}(n)$ in $\boldsymbol{U}(2n)$. Its i-th symplectic Pontrjagin class $e_i(\xi)$ or e_i is by definition

$$e_i(\xi) = (-1)^i c_{2i}(\xi),$$

and its total symplectic Pontrjagin class $e(\xi)$ or e is the sum of the e_i's. The computations made in the proof of Prop. 31.3 in [2] show then that the universal symplectic Pontrjagin class satisfies

$$\textbf{(3)} \qquad \rho_{\boldsymbol{Z}}^*(T, \boldsymbol{Sp}(n))(e) = \prod_1^n (1 + y_i^2), \qquad (T \text{ a maximal torus of } \boldsymbol{Sp}(n)),$$

where the y_i's form a base of $H^2(B_T, \boldsymbol{Z})$ whose elements are permuted, up to sign, by $W(\boldsymbol{Sp}(n))$. Moreover, it follows from ([2], §9, 29) that $H^*(B_{\boldsymbol{Sp}(n)}, \boldsymbol{Z})$ is the ring of polynomials in the e_i's and that $\rho_{\boldsymbol{Z}}^*(T, \boldsymbol{Sp}(n))$ is injective.

9.7. *The multiplication theorem.* Finally, we recall the Whitney multiplication theorem. Let

$$0 \to \xi' \to \xi \to \xi'' \to 0$$

be an exact sequence of real (resp. complex, resp. quaternionic) vector bundles, with structural group $G=\boldsymbol{O}(n)$ (resp. $\boldsymbol{U}(n)$, resp. $\boldsymbol{Sp}(n)$) for three suitable values of n. Then we have

(4) $$w(\xi)=w(\xi')\cdot w(\xi'') \qquad (G=\boldsymbol{O}(n)),$$

(5) $$\tilde{p}(\xi)=\tilde{p}(\xi')\cdot \tilde{p}(\xi'') \qquad (G=\boldsymbol{O}(n)),$$

(6) $$c(\xi)=c(\xi')\cdot c(\xi'') \qquad (G=\boldsymbol{U}(n)),$$

(7) $$e(\xi)=e(\xi')\cdot e(\xi'') \qquad (G=\boldsymbol{Sp}(n)).$$

(4) and (6) are classical; (5) and (7) follow from (6) and the definitions. We note that, in view of (5), we also have

(8) $$p(\xi)\equiv p(\xi')\cdot p(\xi'') \quad \text{modulo 2-torsion.}$$

These formulae imply, in particular, that w or $\tilde{p}$ (resp. c, resp. e) is invariant under an extension relative to the standard inclusion $\boldsymbol{O}(k)\subset\boldsymbol{O}(m)$ (resp. $\boldsymbol{U}(k)\subset\boldsymbol{U}(m)$, resp. $\boldsymbol{Sp}(k)\subset\boldsymbol{Sp}(m)$), $(m\geqq k)$.

10. Representations and characteristic classes.

10.1. *Integral forms as cohomology classes.* Let T be a torus, V its universal covering, Γ the unit lattice, and $\Gamma^*=\mathrm{Hom}(\Gamma,\boldsymbol{Z})$. Thus $\Gamma^*\cong H^1(T,\boldsymbol{Z})$, and, for any commutative group, $\Gamma^*\otimes A\cong H^1(T,A)$. We shall make this identification and, in particular, identify $H^1(T,\boldsymbol{R})$ with V^* and $H^1(T,\boldsymbol{Z})$ with the integral linear forms on V. Also, *the roots* discussed in Chap. I *will be considered in this way as elements of* $H^1(T,\boldsymbol{Z})$ *or* $H^1(T,A)$.

Let ξ be a principal T-bundle. Then τ_ξ maps all of $H^1(T,A)$ in $H^2(B_\xi,A)$. Unless this may lead to a confusion, *we shall denote by the same symbol* $\omega\in\Gamma^*\otimes A$, *the corresponding element in* $H^1(T,A)$, *and* $-\tau_\xi(\omega)\in H^2(B_\xi,A)$.

Let G be a compact connected Lie group and let T be a maximal torus of G. First assume G to be semi-simple and simply connected. Then the transgression in $(G,G/T,T)$ is an isomorphism of $H^1(T,\boldsymbol{Z})$ onto $H^2(G/T,\boldsymbol{Z})$ since G/T is simply connected. Thus the previous conventions identify $H^2(G/T,\boldsymbol{Z})$ and $H^1(T,\boldsymbol{Z})$ with the weights of G (see 3.3). Let G^* be the quotient of G by a finite invariant subgroup and T^* the image of T under the natural map of G onto G^*. Then G/T is homeomorphic to G^*/T^*, as is well known (see e.g. [2], §26); however the transgression in $(G^*,G^*/T^*,T^*)$ will be an isomorphism of $H^1(T^*,\boldsymbol{Z})$ onto the subgroup of $H^2(G/T,\boldsymbol{Z})$ corresponding to the weights which are integral on the unit lattice of G^*. In the general case, let G_1 be the greatest semi-simple subgroup of G (2.9); since maximal tori are maximal abelian subgroups, $T_1 = G_1 \cap T$ is a maximal torus of G_1, and moreover (2.9), G/T may be identified with G_1/T_1. Since a toral subgroup of a torus is always a direct factor, the map $H^1(T,Z) \to H^1(T_1,Z)$, induced by inclusion, is surjective, and the map: $\mu = (G_1, G_1/T_1, T_1) \to \nu = (G, G/T, T)$, defined by inclusion, shows then that $\tau_\nu(H^1(T,Z)) = \tau_\mu(H^1(T_1,Z))$.

Let now G' be the quotient of G by a closed subgroup of the center, $\pi: G \to G'$ the projection, U a connected subgroup of maximal rank in G, $U_1 = G_1 \cap U$, $U' = \pi(U)$. Then rank $U_1 =$ rank G_1, rank $U' =$ rank G', and it follows from 2.9 that $G/U = G_1/U_1$. Also, the argument of [2, § 26] referred to above shows that U is the full inverse image of U' in G, and, consequently, that $G/U = G'/U'$. Therefore, when we deal with coset spaces G/U (rank $G =$ rank U), there is no loss in generality in assuming that G is semi-simple and simply connected.

10.2. *The weights and the character of a homomorphism.* Let G, G' be two compact Lie groups, $\lambda: G \to G'$ a homomorphism, T and T' toral subgroups of G and G' such that $\lambda(T) \subset T'$, and (x'_i) a base of $H^1(T,\boldsymbol{Z})$. Then λ induces homomorphisms of $H^1(T',Z)$ and V'^* in $H^1(T,\boldsymbol{Z})$ and V^*, both to be denoted by λ^*. The elements $\omega_i = \lambda^*(x'_i)$, viewed either as elements of $H^1(T,\boldsymbol{Z})$ or as integral linear forms, will be called the (T,T')-weights of λ, or simply the weight of λ when T and T' are maximal.[7] The formal power series

$$ch(\lambda) = \sum \exp \omega_i$$

considered as an element of $H^{**}(B_T,\boldsymbol{R})$ or of $H^{**}(B_\xi,\boldsymbol{R})$, where ξ is a principal T-bundle, will be called the *character* of λ.

Assume now T, T' to be maximal and $G' = \boldsymbol{U}(n)$. Then for $t \in T$, the matrix $\lambda(t)$ is diagonal with the coefficients $\exp(2\pi i\omega_j)$; in other words, the ω_j and the sum of the exponentials of the $2\pi i\omega_j$ are, respectively, the weights and the character of the representation λ in the usual sense.

In the case $G' = \boldsymbol{O}(n)$, i.e., of a real linear representation, we have analogously

$$\lambda(x) = \begin{pmatrix} D(2\pi\omega_1) & & & 0 \\ & \cdot & & \\ & & \cdot & \\ 0 & & & D(2\pi\omega_m) \end{pmatrix} \qquad \lambda(x) = \begin{pmatrix} D(2\pi\omega_1) & & & & 0 \\ & \cdot & & & \\ & & \cdot & & \\ 0 & & & D(2\pi\omega_m) & \\ & & & & 1 \end{pmatrix}$$

[7] More precisely, with respect to the basis (x'_i), which is always supposed to be chosen as in § 9 when G' is a classical group.

for $n=2m$ and $n=2m+1$ respectively, where $D(\alpha)$ is a 2-dimensional rotation of angle α; the weights of λ, considered as a representation in $\boldsymbol{U}(n)$, are then the forms $\pm\omega_j$, together with the zero form for odd n.

10.3. THEOREM. *Let G, G' be two compact Lie groups, $\lambda\colon G\to G'$ a homomorphism, T, T' maximal tori of G and G' such that $\lambda(T)\subset T'$, and (ω_j) the weights of λ. Let ξ be a principal G-bundle, η its λ-extension, ρ the projection of E_ξ/T onto B_ξ. Then*

(a) *If $G'=\boldsymbol{U}(m)$, then*

$$\rho^*(c(\eta))=\prod(1+\omega_j)\,;\ \rho^{**}(ch(\eta))=ch\,\lambda.$$

(b) *If $G'=\boldsymbol{SO}(m)$ or $\boldsymbol{O}(m)$, the Pontrjagin class $p(\eta)$ satisfies*

$$\rho^*(p(\eta))=\rho^*(\tilde{p}(\eta))=\prod(1+(\omega_j)^2).$$

(c) *If $G'=\boldsymbol{SO}(2m)$, the Euler-Poincaré class $W_{2m}(\eta)$ satisfies*

$$\rho^*(W_{2m}(\eta))=\prod\omega_j.$$

(a) We have a commutative diagram

$$\begin{tikzcd} E_\xi \arrow[r] \arrow[d, "\phi"'] & E_\xi/T \arrow[r, "\rho"] \arrow[d, "\phi_1"] & B_\xi \arrow[d] \\ E_\eta \arrow[r] & E_\eta/T' \arrow[r, "\rho'"] & B_\xi, \end{tikzcd} \tag{1}$$

where ϕ is a λ-map. By (9.1), putting c' for $c(\eta)$, we have

$$\rho'^*(c')=\prod(1-\tau'(x'_i)),$$

where τ' is the transgression in $(E_\eta, E_\eta/T', T')$; and therefore

$$\rho^*(c')=\phi_1{}^*\cdot\rho'^*(c')=\prod(1-\phi_1{}^*\tau'(x_j')).$$

Since ϕ commutes with transgression, this gives

$$\rho^*(c')=\prod(1-\tau\lambda^*(x'_j))$$

and our assertion follows from the definition of the weights and the notation convention of (10.1). The proofs for (b) and (c) are similar.

10.4. COROLLARY. *Let $G=\boldsymbol{U}(n)$, $G'=\boldsymbol{U}(m)$, $\boldsymbol{T}$ the standard maximal torus of G, $\omega_j=\sum_i a_{ij}x_i$ the weights of λ expressed in terms of the canonical basis of $H^1(\boldsymbol{T},\boldsymbol{Z})$, $(i=1,\cdots,n;j=1,\cdots,m)$. Then*

$$c(\eta)=\prod(1+\sum_i a_{ij}y_i)$$

where the y_i's are formally defined by $c(\xi) = \prod(1+y_i)$. The class $c(\eta)$ is a polynomial with integral coefficients in the classes $c_i(\xi)$.

By (9.1), we have $\rho^*(c(\xi)) = \prod(1-\tau(x_i))$, and our first assertion follows from 10.3 and the fact that ρ^* is injective when G is the unitary group. Moreover, the Weyl group $W(\boldsymbol{U}(n))$ operates in a natural way on the fibration $(E_\xi/\boldsymbol{T}, B_\xi, \boldsymbol{U}(n)/\boldsymbol{T}, \rho)$ and induces the identity on B_ξ (see [2], § 27). Therefore the image of ρ^*, and in particular, $\rho^* c(\eta)$, is made up of invariants of $W(\boldsymbol{U}(n))$; since the latter is the group of permutations of the x_i, or equivalently, of the $\tau(x_i)$, it follows that $c(\eta)$ is a symmetric function in the y_i's, whence our second assertion.

10.5. COROLLARY. *Let $G = \boldsymbol{O}(n)$ or $\boldsymbol{SO}(n)$, G' be $\boldsymbol{O}(m)$ or $\boldsymbol{SO}(m)$. Then $p(\eta)$ reduced* mod p *$(p \neq 2)$, is a polynomial in the $p_i(\xi)$, and if $G = \boldsymbol{SO}(2m)$, in $W_{2m}(\xi)$. If λ can be extended to a homomorphism of $\boldsymbol{U}(n)$ into $\boldsymbol{U}(m)$, then $\tilde{p}(\eta)$ is a polynomial in the classes $p_i(\xi)$, $p_{i+\frac{1}{2}}(\xi)$, and, in particular, $p(\eta)$ reduced mod p $(p \neq 2)$, is a polynomial in the classes $p_i(\xi)$.*

The first assertion is proved in the same way as 10.4, using 9.3, 9.5 and the properties of the invariants of $W(G)$ recalled in 30.2. The second one follows from 10.4 by considering the Pontrjagin classes as the Chern classes of the complex extension.

10.6. *Examples.* (a) Let ξ be a complex vector bundle, ξ^* the dual bundle. Then, if $c(\xi) = \prod(1+x_i)$, we have $c(\xi^*) = \prod(1-x_i)$. In fact, the principal bundle θ associated to ξ^* is the λ-extension of the principal bundle of ξ, where λ is the contragredient representation, whose weights are obviously the forms $-x_i$.

(b) Let $G = \boldsymbol{U}(n)$, j be a positive integer $\leqq n$, and λ the natural representation of $\boldsymbol{U}(n)$ in the j-th exterior power $\bigwedge^j \boldsymbol{C}^n$ of $\boldsymbol{C}^n$. Let (e_i) be the canonical base of $\boldsymbol{C}^n$. Then the products

$$e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \qquad (1 \leqq i_1 < \cdots < i_j \leqq n)$$

form a base of $\bigwedge^j \boldsymbol{C}^n$, and we have

$$\lambda(x)(e_{i_1} \wedge \cdots \wedge e_{i_j}) = \exp[2\pi i(x_{i_1} + \cdots + x_{i_j})](e_{i_1} \wedge \cdots \wedge e_{i_j});$$

i.e., the weights of λ are the sums

$$x_{i_1} + \cdots + x_{i_j}, \qquad (1 \leqq i_1 < \cdots < i_j \leqq n).$$

Here η is the principal bundle associated to the bundle of contravariant p-

vectors in the complex vector bundle associated to ξ. This bundle has, therefore, as Chern class

$$c' = \prod_{1 \leqq i_1 < \cdots < i_j \leqq n} (1 + x_{i_1} + \cdots + x_{i_j}).$$

(c) In the same way, the Chern class of the bundle of contravariant symmetric tensors of degree j will be

$$\prod_{1 \leqq i_1 \leqq \cdots \leqq i_j \leqq n} (1 + x_{i_1} + \cdots + x_{i_j}).$$

(d) Let ξ_i $(i = 1, 2)$, be two complex vector bundles over B and let

$$c_{(i)} = \prod_{j=1}^{n_i} (1 + x_j^{(i)})$$

be formal decompositions of their Chern polynomials. Then

$$c(\xi_1 \otimes \xi_2) = \prod_{i=1}^{n_1} \prod_{j=1}^{n_2} (1 + x_i^{(1)} + x_j^{(2)}).$$

To see this, we take as ξ the principal bundle with group $\boldsymbol{U}(n_1) \times \boldsymbol{U}(n_2)$ associated to the sum $\xi_1 \oplus \xi_2$, whose Chern class is $c_{(1)} \cdot c_{(2)}$ by the multiplication theorem (9.7), and as λ the representation of $\boldsymbol{U}(n_1) \times \boldsymbol{U}(n_2)$ in $\boldsymbol{U}(n_1 \cdot n_2)$ defined by $(g_1, g_2) \rightarrow g_1 \otimes g_2$, considered as an automorphism of $\boldsymbol{C}^{n_1} \otimes \boldsymbol{C}^{n_2}$. The products $e_i \otimes f_j$, where (e_i) and (f_j) are the canonical bases of $\boldsymbol{C}^{n_1}$ and $\boldsymbol{C}^{n_2}$, form a base of $\boldsymbol{C}^{n_1} \otimes \boldsymbol{C}^{n_2}$; hence the weights of λ are the forms $x_i^{(1)} + x_j^{(2)}$, and our contention follows from (10.3) and from the fact that the λ-extension of ξ is the principal bundle of $\xi_1 \otimes \xi_2$.

(e) To compute the Pontrjagin classes of real vector bundles, it is often more convenient to look at the Chern classes of the complex extensions; as an illustration, we take the case where $G = \boldsymbol{SO}(2n)$ and λ is the representation in $\wedge^2 \boldsymbol{R}^{2n}$. Let μ, ν denote the complex extensions of ξ, η (as defined in 9.4), let $\mathbf{T}$, $\mathbf{T}'$ be the standard maximal tori of $\boldsymbol{SO}(2n)$, $\boldsymbol{U}(2n)$, and let (x_i), (x_i') be the canonical bases of $H^1(\boldsymbol{T}, Z)$ and $H^1(\boldsymbol{T}', Z)$. We have a commutative diagram

$$\begin{tikzcd}
E_\xi \arrow[r] \arrow[d, "\phi"'] & E_\xi/\boldsymbol{T} \arrow[r, "\rho"] \arrow[d, "\phi_1"'] & B_\xi \arrow[d, "\mathrm{Id}"] \\
E_\mu \arrow[r] & E_\mu/\boldsymbol{T}' \arrow[r, "\sigma"] & B_\xi,
\end{tikzcd}$$

and it follows from (9.4), (10.3) that

$$x_i = \lambda^*(x'_{2i-1}) = -\lambda^*(x'_{2i}), \qquad (1 \leqq i \leqq n),$$

$$\rho^*(p(\xi)) = \prod_1^n (1 + x_i^2),$$

$$\sigma^*(c(\mu)) = \prod_1^{2n} (1 + x'_j).$$

Now, ν is clearly the extension of μ corresponding to the "complexification of λ," i.e., to the natural representation of $\boldsymbol{U}(2n)$ in $\bigwedge^2 \boldsymbol{C}^{2n}$; therefore, by example (b),

$$\sigma^*(c(\nu)) = \prod_{1 \leqq i < j \leqq 2n} (1 + x'_i + x'_j).$$

This gives

$$\phi^*_1 \cdot \sigma^*(c(\nu)) = \prod_{1 \leqq i < j \leqq n} (1 - (x_i + x_j)^2) \cdot (1 - (x_i - x_j)^2),$$

$$\rho^*(c(\nu)) = \prod_{1 \leqq i < j \leqq n} [(1 - x_i^2 - x_j^2)^2 - 4 \cdot x_i^2 \cdot x_j^2],$$

and finally

$$\rho^*(\tilde{p}(\eta)) = \rho^*(p(\eta)) = \prod_{1 \leqq i < j \leqq n} [(1 + x_i^2 + x_j^2)^2 - 4 \cdot x_i^2 \cdot x_j^2].$$

(f) It may be shown in the same way that if ξ_1, ξ_2 are two real vector bundles over the same base space B with Pontrjagin classes reduced mod p $(p \neq 2)$, equal to

$$p(\xi_1) = \prod_1^m (1 + x_i^2), \qquad p(\xi_2) = \prod_1^n (1 + y_j^2),$$

then

$$p(\xi_1 \otimes \xi_2) = \prod_{i=1}^m \prod_{j=1}^n (1 + (x_i + x_j)^2) \cdot (1 + (x_i - x_j)^2).$$

10.7. THEOREM. *Let G be a compact connected Lie group, U a closed subgroup of G, S a maximal torus of U, and $(\pm b_j)$ $(1 \leqq j \leqq k)$, the roots of G with respect to S which are complementary to those of U. Let ξ be a principal G-bundle, ρ the projection of E_ξ/S onto E_ξ/U, and η the bundle along the fibres (7.4) of $(E_\xi/U, B_\xi, G/U)$. Then $\rho^*(\tilde{p}(\eta)) = \prod (1 + b_j^2)$; if, moreover, U is connected and $\dim G/U = m$ is even, then $\rho^*(W_m(\eta)) = \pm \prod b_j$.*

By (7.5), η is the ι-extension of $(E_\xi, E_\xi/U, U)$, where ι is the isotropy representation; according to the definitions in (1.3) and (10.1), the b_j's are the weights of ι (up to a certain number of zero forms, but this does not alter our formulas), and (10.7) follows then from (10.3). The sign for the Euler-Poincaré class will be determined by the conventions made in 9.5, once the bundle along the fibres has been oriented.

10.8. We keep the previous notations. Let J be an invariant almost complex structure on G/U and η' be the complex vector bundle structure of η constructed by means of J (see 7.4). J is defined by a complex structure on the tangent space $(G/U)_0$, invariant under U; this complex structure gives rise to a linear representation ι_c of U in $\boldsymbol{C}^{m/2}$ ($m = \dim G/U$), which goes over to ι by taking real and imaginary parts. The weights of ι_c are some of the forms $\pm b_j$, and (10.3) also implies:

THEOREM. *We keep the notations of* 10.7; *assume, moreover, that* G/U *has an invariant almost complex structure* J, *and denote by* η' *the complex vector bundle structure of* η *associated to* J. *Then* $\rho^*(c(\eta')) = \prod_{j \in J} (1 + \epsilon_j b_j)$, ($\epsilon_j = \pm 1$), *where* $\epsilon_j b_j$ *runs through the weights of the complex isotropy representation* ι_c *defining* J.

The weights of ι_c will be discussed in detail for the case where $\operatorname{rank} G = \operatorname{rank} U$ in Chapter IV.

10.9. In order to study the tangent bundle to G/U it is usually convenient to consider the bundle $\hat{\theta}$ along the fibres of $(B_U, B_G, G/U, \rho(U, G)) = \theta$ and restrict to a fibre of θ, since this allows one to make use of known results about classifying spaces. We consider here, in particular, the case where $U = T$ is a maximal torus and show the

PROPOSITION. *Let T be a maximal torus of G. Then the total Pontrjagin class* $\tilde{p}(\mu)$ *of the tangent bundle* μ *to* G/T *is* 1.

Let $\theta = (B_T, B_G, G/T)$ and let i be the inclusion map of a fibre. $H^*(G/T, \boldsymbol{Z})$ is torsion free ([5], or R. Bott, *Bull. Soc. Math. France* 84, (1956) 251-281), and therefore the subgroup S of invariants of $W(G)$ in $H^*(G/T, \boldsymbol{Z})$ is a free abelian group; since by a lemma of Leray (see [2], Lemma 27.1), $H^*(G/T, \boldsymbol{R})$ is the space of the regular representation of $W(G)$, it follows that $S = H^0(G/T, \boldsymbol{Z})$ and that the kernel of i^* contains the subgroup I_G^+ of invariants of $W(G)$ in $H^*(B_T, \boldsymbol{Z})$ having strictly positive degrees.

By (10.7) we have

$$\tilde{p}(\hat{\theta}) = \prod (1 + b_j^2),$$

where the $\pm b_j$'s are the roots of G. Since $W(G)$ permutes the b_j^2, it leaves $\tilde{p}(\hat{\theta})$ invariant, whence

$$\tilde{p}(\mu) = i^*(\tilde{p}(\hat{\theta})) = 1.$$

11. Representations and Stiefel-Whitney classes. In this section, Q, Q' denote commutative groups of type $(2,2,\cdots,2)$. The following discussion applies to any arbitrary compact Lie group, but has an interest only for groups in which maximal commutative subgroups of type $(2,\cdots,2)$ are conjugate and play in cohomology mod 2 the role of maximal tori in real cohomology. We therefore assume tacitly that G, G' are products of copies of $\boldsymbol{O}(n)$, $\boldsymbol{SO}(n)$, $\boldsymbol{U}(n)$, $\boldsymbol{SU}(n)$, $\boldsymbol{Sp}(n)$, $\boldsymbol{G}_2$ (see [3]).

11.1. *Characters of Q as cohomology classes.* Q being discrete, $H^*(B_Q,A)$ is the cohomology ring of Q in the sense of Hopf-Eilenberg-MacLane, and, in particular, $H^1(B_Q,A)=\operatorname{Hom}(Q,A)$. Thus $x\in\operatorname{Hom}(Q,A)$ may be considered as a 1-dimensional cohomology element in B_Q or, via the characteristic map, in the base space of any principal bundle (E,B,Q), which will be usually denoted by the same symbol $(x\in H^1(B,A))$ in particular, the 2-roots introduced in § 5 will be considered as elements of $H^1(B,\boldsymbol{Z}_2)$. We note that if $\lambda\colon Q\to Q'$ is a homomorphism, then

$$\rho(\lambda)^*\colon H^1(B_{Q'},A)\to H^1(B_Q,A) \text{ and } \lambda'\colon \operatorname{Hom}(Q',A)\to\operatorname{Hom}(Q,A)$$

are carried into one another by the previous identification.

11.2. *The 2-weights of a homomorphism.* Let $\lambda\colon G\to G'$ be a homomorphism, Q, Q' maximal and such that $\lambda(Q)\subset Q'$, and (x_i), (x'_i) bases of $\operatorname{Hom}(Q,\boldsymbol{Z}_2)$ and $\operatorname{Hom}(Q',\boldsymbol{Z}_2)$, considered as Z_2-modules. Then λ^*: $\operatorname{Hom}(Q',\boldsymbol{Z}_2)\to\operatorname{Hom}(Q,\boldsymbol{Z}_2)$ is characterized by elements $\omega_j=\lambda^*(x'_j)$ $=\sum a_{ij}x_i$, to be called the *2-weights of* λ. Here, also, we assume, in case of an orthogonal group, the basis to be chosen as in (9.2).

11.3. Theorem. *Let G be a compact Lie group, Q a maximal commutative subgroup of type $(2,\cdots,2)$, $\lambda\colon G\to\boldsymbol{O}(n)$ a homomorphism, (ω_j) its 2-weights, ξ a principal bundle, ξ' its λ-extension, and ρ the projection of E/Q onto B. Then $\rho^*(w(\xi'))=\prod(1+\omega_j)$.*

The proof is the same as for (10.3), except that instead of (1) § 10, we use the commutativity of the diagram

$$\begin{tikzcd} E_\xi/Q \arrow[r, "\phi'"] \arrow[d, "\sigma"] & E_\xi/Q' \arrow[d, "\sigma'"] \\ B_Q \arrow[r] & B_{Q'}, \end{tikzcd}$$

where σ and σ' are characteristic maps, and the end remark of (11.1); therefore, we shall not reproduce it here.

Remark. For the groups mentioned at the beginning of § 11, ρ^* is injective [3].

11.4. COROLLARY. *Assume, moreover, that* $G = \boldsymbol{O}(n)$. *Then* $w(\xi') = \prod(1 + \sum a_{ij}x_i)$, *where the* x_i *are the formal roots of* $w(\xi)$. *In particular,* $w(\xi')$ *is a polynomial in the classes* $w_i(\xi)$.

Same proof as for (10.4), except that instead of using the Weyl group, we take the quotient by Q of its normalizer in $\boldsymbol{O}(n)$; its inner automorphisms also induce the group of permutations of the x_i's.

Examples. Computations paralleling those of 10.6, (b), (c), (d) will lead to the same formulas for the Stiefel-Whitney classes of bundles of p-vectors, symmetric tensors, and for tensor products, the x_i and y_j standing now for 1-dimensional classes mod 2. Details are left to the reader.

11.5. THEOREM. *Let* G *be a compact Lie group,* U *a closed subgroup, and* Q *a maximal commutative subgroup of type* $(2, \cdots, 2)$ *of* U. *Let* ξ *be a principal* G*-bundle,* ρ *the projection of* E_ξ/Q *on* E_ξ/U, *and* η' *the bundle along the fibres* G/U. *Then*

$$\rho^*(w(\eta')) = \prod(1 + a_i),$$

where the a_i's *are the 2-roots of* G *with respect to* Q, *complementary to those of* U.

The a_i's are the 2-weights of the isotropy representation; hence (11.5) follows from (7.5) and (11.3).

Applications will be given in Chapter V.

Chapter IV. Roots and Invariant Almost Complex Structures.

In this chapter, G is a compact, connected, semi-simple, Lie group, l its rank, U a proper closed connected subgroup of the same rank, and T a maximal torus of U. If ψ is a set of roots, we put $-\psi = \{-a, a \in \psi\}$.

12. Integrability of invariant almost complex structures. We recall here some known facts in a form convenient for the sequel.

12.1. Let V be a real $2n$-dimensional vector space, endowed with a complex structure defined by a linear transformation J, and let V^c be its complexification. Then

$$V^c = T^+ + T^-, \qquad T^- = \overline{T^+}, \qquad T^+ \cap T^- = (0),$$

where T^+ (resp. T^-) is the eigenspace of the extension J^c of J to V^c corresponding to the eigenvalue $+i$ (resp. $-i$) and where the bar denotes complex conjugation with respect to V. Conversely, given such a decomposition of V^c, we define J^c by $J^c(x) = i \cdot x$ $(x \in T^+)$, $J^c(x) = -ix$, $(x \in T^-)$; then J^c leaves V invariant and induces there a complex structure such that $x \to x + \bar{x}$ is a complex isomorphism $(x \in T^+)$. In particular, given a linear transformation A without real eigenvalues, we define T^+ (resp. T^-) as the sum of the eigenspaces of its semi-simple part corresponding to eigenvalues with positive (resp. negative) imaginary parts, and thus attach to A a complex structure on V.

12.2. The roots of G with respect to T define linear forms on the Lie algebra $\mathfrak{t}$ of T, and it follows from (1.3) and standard facts about the adjoint representation that

$$\operatorname{ad} x \,|_{\mathfrak{a}_i} = \begin{pmatrix} 0 & -2\pi a_i(x) \\ 2\pi a_i(x) & 0 \end{pmatrix}, \quad (x \in \mathfrak{t}),$$

$\operatorname{ad} x$ being defined by $(\operatorname{ad} x)(y) = [x, y]$, $(x, y \in \mathfrak{g})$.

We have then, superscripts denoting complexification, that

$$\mathfrak{g}^c = \mathfrak{t}^c + \mathfrak{a}_1{}^c + \cdots + \mathfrak{a}_m{}^c, \qquad \mathfrak{a}_i{}^c = \mathfrak{v}_{a_i} + \mathfrak{v}_{-a_i},$$

$$[x, e_{\pm a_j}] = \pm 2\pi i a_j(x) e_{\pm a_j}, \qquad (e_{\pm a_j} \in \mathfrak{v}_{\pm a_j});$$

since any two roots are different from each other, the 1-dimensional eigenspaces $\mathfrak{v}_{\pm a_j}$ are well determined by $\mathfrak{t}$. We recall that if α, β are two roots, we have

$$\begin{aligned} &[\mathfrak{v}_\alpha, \mathfrak{v}_\beta] = 0 && \text{if } \alpha + \beta \text{ is not a root and not zero,} \\ (1) \qquad &[\mathfrak{v}_\alpha, \mathfrak{v}_\beta] = \mathfrak{v}_{\alpha+\beta} && \text{if } \alpha + \beta \text{ is a root,} \\ &[\mathfrak{v}_\alpha, \mathfrak{v}_{-\alpha}] \subset \mathfrak{t}^c, \ [\mathfrak{v}_\alpha, \mathfrak{v}_{-\alpha}] \neq 0. && \end{aligned}$$

12.3. Assume now that G/U has been endowed with an invariant almost complex structure and let $\pm b_j$ $(1 \leqq j \leqq k)$ be the complementary roots. The almost complex structure is characterized by a linear transformation J, $(J^2 = -\mathrm{Id})$, of $(G/U)_0$ which commutes with the linear isotropy group (1.1). Since $b_i \neq b_j$ *for* $i \neq j$, J must also leave the subspaces $\mathfrak{b}_i$ invariant and it induces complex structures on them which characterize it completely. Now on each $\mathfrak{b}_j$ there are two complex structures commuting with the isotropy representation of T in $\mathfrak{b}_j$, differing by the orientation they induce; to each b_j we attach a sign ϵ_j, equal to $+1$ (resp. -1), according to whether the ordered pairs $(e, \operatorname{Ad} t(e))$ and $(e, J(e))$ define the same orientation or not

($e \in \mathfrak{b}_j, e \neq 0, t \in T$ such that $0 < b_j(t) < \frac{1}{2}$). The $\epsilon_j b_j$ will be called *the roots of the almost complex structure,* which they describe completely.

We extend J to a linear transformation $\tilde{J}$ of $\mathfrak{g}$ by putting it equal to zero on $\mathfrak{u}$, and to a linear transformation $\tilde{J}^c$ of $\mathfrak{g}^c$; it is readily seen that

$$\tilde{J}^c(e_{\epsilon_j b_j}) = i \cdot e_{\epsilon_j b_j}; \; \tilde{J}^c(e_{-\epsilon_j b_j}) = -i \cdot e_{\epsilon_j b_j} (e_{\pm b_j} \in \mathfrak{v}_{\pm b_j}).$$

The space T^+ of (12.1) may be identified with the space spanned by the $v_{\epsilon_j b_j}$ which, by the foregoing, is invariant under $\mathrm{Ad}_G U$. Since $x \to x + \bar{x}$ is a complex isomorphism of T^+ onto $(G/U)_0$, the previous identification carries the restriction of $\mathrm{Ad}_G U$ onto the complex isotropy representation ι_c defined in § 10, and, therefore, *the $\epsilon_j b_j$ are the weights of ι_c.*

The almost complex structure is integrable, i. e. (since we are in the real analytic case), derives from an automatically invariant complex analytic structure, if and only if

$$\mathfrak{n} = \mathfrak{u}^c + \mathfrak{v}_{\epsilon_1 b_1} + \cdots + \mathfrak{v}_{\epsilon_k b_k}$$

is a Lie algebra [14, § 20]. In view of the properties of the bracket recalled above, this proves the first assertion of:

12. 4. LEMMA. *Let $\mathcal{C}$ be an invariant almost complex structure on G/U, ψ the system of its roots, and Σ the system of roots of U. Then $\mathcal{C}$ is integrable if and only if $\Sigma \cup \psi$ is closed in the sense of § 4. In this case, ψ is closed and contained in a system of positive roots.*

As to the second assertion, we remark that by 4. 10, we have $\Sigma = \theta \cup -\theta$, where $\theta \cup \psi$ is a system of positive roots for some ordering. Since $\Sigma \cup \psi$ and $\theta \cup \psi$ are closed and since $\psi \cap -\psi = \phi$, it follows immediately that ψ is closed.

More precise statements about ψ will be given in 13. 7.

13. Applications.

13. 1. The following known facts will be used in this section. A compact connected Lie group K is semi-simple if and only if $H^1(K, \boldsymbol{R}) = 0$, and then $H^2(K, \boldsymbol{R}) = 0$ (see, e. g., Chevalley-Eilenberg, Trans. Amer. Math. Soc., 63 (1948), 85-124). A simple spectral sequence argument then shows that, if K is compact and semi-simple and L is a closed connected subgroup, the transgression is an isomorphism of $H^1(L, \boldsymbol{R})$ onto $H^2(K/L, \boldsymbol{R})$, and, in particular, that $H^2(K/L, \boldsymbol{R}) = 0$ if and only if L is semi-simple, too.

13. 2. *Coset spaces with second Betti number zero.* As a first application

of §§ 4 and 10, we prove anew a theorem of H. C. Wang [31, Theorem C] to the effect that *a coset space* G/U *with* $\operatorname{rank} G = \operatorname{rank} U$ *and second Betti number zero is not homogeneous complex.*

Assume that G/U has an invariant almost complex structure with roots $(\epsilon_j b_j)$, $(1 \leqq j \leqq k)$; let c_1 be its first Chern class and ρ be the projection of G/T onto G/U. By (10.8) and (12.3),

$$\rho^*(c_1) = -\tau(\epsilon_1 b_1 + \cdots + \epsilon_k b_k)$$

(τ transgression in $(G, G/T, T)$). Since $H^2(G/U, \boldsymbol{R}) = 0$, c_1 must be a fortiori zero as a real cohomology class, and hence, by (13.1), $\sum \epsilon_j b_j = 0$. But then, by § 4, the system $(\epsilon_j b_j)$ does not satisfy the condition of 12.4, and thus the almost complex structure is not integrable.

13.3. *Examples of* (13.2). Now let G be simple and U be a maximal connected subgroup of maximal rank. A complete list of such inclusions is given in [7]; to discuss it, we assume, moreover, the center of G to be reduced to the identity, which is no loss in generality. These inclusions may then be divided into three classes:

(a) U is the connected centralizer of an element of order 2, which generates its center.

(b) U is the centralizer of a one dimensional torus S, and S is the identity component of the center of U.

(c) U is the connected centralizer of an element z of order 3 or 5, which generates its center.

The coset spaces G/U corresponding to the classes (a), (b) are irreducible Riemannian and hermitian symmetric spaces respectively. In the class (c) we find seven spaces, namely $\boldsymbol{G}_2/\boldsymbol{A}_2 = \boldsymbol{S}_6$, $\boldsymbol{F}_4/\boldsymbol{A}_2 \times \boldsymbol{A}_2$, $\boldsymbol{E}_6/\boldsymbol{A}_2 \times \boldsymbol{A}_2 \times \boldsymbol{A}_2$, $\boldsymbol{E}_7/\boldsymbol{A}_2 \times \boldsymbol{A}_5$, $\boldsymbol{E}_8/\boldsymbol{A}_8$, $\boldsymbol{E}_8/\boldsymbol{A}_2 \times \boldsymbol{E}_6$ for z of order 3 and $\boldsymbol{E}_8/\boldsymbol{A}_4 \times \boldsymbol{A}_4$ for z of order 5.

U being the connected centralizer of z, its algebra $\mathfrak{u}$ is the set of fixed points under $\operatorname{Ad} z$; consequently, $\operatorname{Ad} z$ has no real eigenvalues on the complementary subspaces $\mathfrak{b}_j$, and we may attach to it a complex structure on $(G/U)_0$, as recalled in (12.1), which will be invariant under U, since the latter commutes with $\operatorname{Ad} z$, and defines, consequently, an invariant almost complex structure on G/U. Here since U has a discrete center, it is semi-simple, and $H^2(G/U, R) = 0$ (see 13.1). Therefore, by (13.2), we have the

Proposition. *The seven coset spaces of the class* (c) *above are homogeneous almost complex but not homogeneous complex.*

This generalizes a known result for S_6 (Ehresmann Libermann, *C. R. Acad. Sci. Paris* 232 (1951), 1281 [14, § 10]).

13.4. PROPOSITION. *Let G/U be homogeneous almost complex, and let $\iota = \iota_1 + \cdots + \iota_s$ be a decomposition of the isotropy representation into real irreducible representations; then the ι_i are unique, each one has the complex numbers as commuting field, and G/U admits exactly 2^s invariant almost complex structures.*

Let $(G/U)_0 = W_1 + \cdots + W_s$ be a direct sum decomposition of $(G/U)_0$ such that the restriction of ι to W_i is ι_i. Since these subspaces are invariant under T, they are direct sums of subspaces $\mathfrak{b}_j$, and the corresponding roots $\pm b_j$ are the weights of ι_i; since any two roots are different, the complex irreducible components of the ι_i will be pairwise inequivalent, from which follows the uniqueness of the ι_i and of the W_i. Also, a straightforward application of Schur's lemma shows that any linear transformation commuting with ι leaves the W_i's invariant. Since, by assumption, there is at least one transformation without real eigenvalues commuting with ι, we see that the commuting field of ι_i is either the field of complex numbers $\boldsymbol{G}$ or of quaternionic numbers $\boldsymbol{K}$; in any case ι_i is not complex irreducible and its extension to $W_i \otimes \boldsymbol{C}$ decomposes into $\gamma_i + \bar{\gamma}_i$, where γ_i is complex irreducible and $\bar{\gamma}_i$ is the complex conjugate representation of γ_i; the weights of $\bar{\gamma}_i$ are opposite in sign to the weights of γ_i. Since the roots of G are pairwise distinct (§ 2), γ_i is not equivalent to $\bar{\gamma}_i$, and it follows from Schur's lemma again that the commuting field of ι_i is the field of complex numbers. Thus we have on each W_i exactly 2 invariant complex structures, from which our contention follows.

Remark. Let σ be an automorphism of G leaving T and U invariant, $d\sigma$ the induced automorphism of $\mathfrak{g}$; let ψ be the root system of an invariant almost complex structure $\mathcal{C}$ on G/U, and ψ' the transform of ψ under $d\sigma$. From the formula (1) in § 1, it follows readily that the homeomorphism σ' of G/U defined by σ carries $\mathcal{C}$ onto the invariant almost complex structure with roots ψ'. If, in particular, $\sigma(x) = g \times g^{-1}$ with $g \in N_T \cap U$, then σ' reduces to the left translation defined by g and leaves $\mathcal{C}$ invariant; hence the element of $W(U)$ represented by g must leave ψ invariant.

13.5. *Centralizers of tori.* The following proposition is due to H. C. Wang [31]:

PROPOSITION. *G/U (with $\operatorname{rank} U = \operatorname{rank} G$; U connected) is homogeneous complex if and only if U is the centralizer of a torus in G.*

Proof. Let U be the centralizer of a torus S, which we assume, as we may, to be in T, and let $s \in S$ generate an everywhere dense subgroup of S. Then U is the centralizer of s, $\mathfrak{u}$ the space of vectors fixed under $\operatorname{Ad} s$, and we have $b_j(s) \not\equiv 0(1)$ if and only if b_j is complementary. Let $\epsilon_j = \operatorname{sgn}(b_j(s))$ $(1 \leqq j \leqq k)$; since s centralizes U, the $\epsilon_j b_j$ are the roots of an invariant almost complex structure; moreover, being characterized by $\epsilon_j b_j(s) > 0$, these roots satisfy the criterion of 12.4, and the structure is integrable.

Assume now, conversely, that G/U has been endowed with a homogeneous complex structure and let $(\epsilon_j b_j)$ $(1 \leqq j \leqq k)$, be its roots. By (2.9), the group is locally isomorphic to the direct product of its largest semi-simple subgroup U' and of a torus S; moreover, by 13.1 and 13.2, $S \neq \{e\}$. Now let W be the centralizer of S. We have $W = S_1 \cdot W'$, where S_1 is a torus containing S and W' a semi-simple subgroup containing U' and $W' \cap S_1$ is finite. The equalities

$$\operatorname{rank} G = \operatorname{rank} U' + \dim S = \operatorname{rank} W' + \dim S_1$$

show then that $S = S_1$, that $\operatorname{rank} W' = \operatorname{rank} U'$, and that W/U is to be identified with W'/U'; since W' and U' are semi-simple, we have $H^2(W'/U', \boldsymbol{R}) = 0$ and W/U is not homogenous complex (13.1, 13.2).

Let $J \subset [1, k]$ be such that the $\pm b_j$'s, with $j \in J$, are the complementary roots of U in W. The roots $(\epsilon_j b_j)$ $(j \in J)$, define a complex structure on $(W/U)_0$ which is invariant under the linear isotropy representation ι' of U in $(W/U)_0$ since ι' is nothing but the restriction to an invariant subspace of the isotropy representation of U in $(G/U)_0$; moveover, since the system $(\epsilon_j b_j)$ $(1 \leqq j \leqq k)$, satisfies the condition of 12.4, so does $(\epsilon_j b_j)$ $(j \in J)$. Therefore, if $W \neq U$, then we get on W/U an invariant integrable almost complex structure, in contradiction to what has already been proved. Thus $U = W$ and U is the centralizer of the torus S.

13.6. For the sake of completeness, we recall the proof of the following well-known lemma.

LEMMA. *Let U be the centralizer of a torus in G, S the connected center of U and $k = \dim S$. Then, for a suitable ordering $\mathcal{S}$, there are $l - k$ simple roots a_i $(1 \leqq i \leqq l - k)$ vanishing on S and such that the roots of U are exactly the roots of G which are linear combinations of the a_i $(1 \leqq i \leqq l - k)$.*

The roots of U are those of G which vanish on S; since the semi-simple part of U has rank $l - k$, U has $l - k$ independent roots. We consider in the dual space V_T^* of the universal covering V_T of T the lexicographic order which is associated to a base whose first k elements span the covering V_S of S. It is then clear that if a sum of positive linear forms b_i with strictly positive coefficients vanishes on V_S, so does each b_i; the lemma follows readily from this and from the fact that U has $l-k$ linearly independent roots.

Remark. In the ordering $\mathcal{S}$, the complementary roots are linear combinations of the a_j's with at least one of the k last ones appearing with a non-zero coefficient. Therefore, if the sum $a+b$ of a root a of U and of a complementary root b is a root, then it must be a complementary root. Also, the set of positive complementary roots is closed.

13.7. *Number of invariant complex structures.* In this section, we assume G/U to be homogeneous complex; U is then the centralizer of a torus by 13.5, and we keep the notations of 13.6.

PROPOSITION. *Let Θ be a system of positive roots of U. The roots of an invariant complex structure form a closed system Ψ such that $\Theta \cup \Psi$ is a positive system of roots for G. Conversely, a closed set Ψ of complementary roots such that $\Theta \cup \Psi$ is the set of positive roots of G for a suitable ordering is the system of roots of an invariant complex structure of G/U.*

Let Ψ be the root system of an invariant complex structure $\mathcal{C}$. Then (12.4) Ψ is closed and is contained in a system Φ of positive roots of G. Φ is necessarily of the form $\Theta' \cup \Psi$, where Θ' is a system of positive roots for U. There exists, therefore, $w \in W(U)$ which carries Θ' onto Θ; since w leaves Ψ invariant (remark in 13.4), it carries $\Theta' \cup \Psi$ onto $\Theta \cup \Psi$, and the latter is also a positive system.

Let now Ψ be a closed system of complementary roots such that $\Theta \cup \Psi$ is the set of positive roots relative to an ordering $\mathcal{S}'$. The remark in 13.6 and the fact that Ψ is closed show that if $a \in \Theta$ is a sum of two positive roots for $\mathcal{S}'$, then these two roots also belong to Θ; this means that the simple roots of Θ, considered as a positive system for U, are also simple for $\mathcal{S}'$. Let then a_j $(1 \leqq j \leqq l)$ be the simple roots of $\mathcal{S}'$, with $a_j \in \Theta$ for $j \leqq l-k$; the elements of Θ (resp. Ψ) are then linear combinations with positive coefficients of $a_1, \cdots, a_{l-k}$ (resp. $a_1, \cdots, a_l$, where at least one a_j $(j > l-k)$ has a non-vanishing coefficient). This implies first that $\Theta \cup -\Theta \cup \Psi$ is closed and second that there is an $s \in S$ such that $0 < b(s) < \frac{1}{2}$ for all $b \in \Psi$; thus the map of $(G/U)_0 \cong \mathfrak{g}/\mathfrak{u}$ onto itself, defined by $\mathrm{Ad}\, s$, has no real eigenvalues and the complex structure attached to it by the rule of 12.1 has the root system Ψ. Since s commutes with U, this structure is invariant under the isotropy representation, and hence, by (12.4), gives an invariant complex structure on G/U.

13.8. PROPOSITION. *Let G/U be homogeneous complex, let k be the dimension of the center of U, and let l be the rank of G. If $k=1$ (resp. $k=l$, i.e., $U=T$), then the number of invariant complex structures is equal to two (resp. the order of $W(G)$). Given two of them, there is a homeomorphism of G/U induced from an (resp. inner) automorphism of G leaving U invariant and carrying one onto the other.*

Let $k=l$. Then in the notations of 13.7, Θ is empty and the invariant complex structures are in 1-1 correspondence with the different systems of positive roots of G, by 13.7 (or directly by 4.9 and 12.4). Since $W(G)$ operates transitively on the set of systems of positive roots, it is obvious by the remark of 13.4 that the inner automorphisms of G defined by the elements of the normalizer of T induce homeomorphisms of G/T which permute transitively the invariant complex structures.

Now let $k=1$. We first take an ordering $\mathcal{S}$ having the properties mentioned in 13.6. Then the set of positive complementary roots Ψ defines an invariant complex structure by 13.7. Let Ψ' be the root system of another invariant complex structure. As in 13.7, we denote by Θ the set of roots of U which are positive for $\mathcal{S}$ and by a_l the simple root of $\mathcal{S}$ not belonging to Θ; $\Psi' \cup \Theta$ is the set of positive roots for some ordering $\mathcal{S}'$, and the proof of Proposition 13.7 shows that $a_1, \cdots, a_{l-1}$ are also simple for $\mathcal{S}'$. If $a_l \in \Psi'$, then $\Psi' \cup \Theta$ contains all simple roots of $\mathcal{S}'$, and hence $\Psi = \Psi'$. Let now $-a_l \in \Psi'$ and let a'_l be the l-th simple root of $\mathcal{S}'$; $-a_l$ is a linear combination with positive coefficients of $a_1, \cdots, a_{l-1}, a'_l$, and therefore, if we express a'_l as a linear combination of $a_1, \cdots, a_l$, then the root a_l must have coefficient -1. But then the elements of Ψ' which are combinations with positive coefficients of $a_1, \cdots, a_{l-1}, a'_l$, where the last coefficient is $\neq 0$, must also have at least one negative coefficient when expressed as linear combinations of $a_1, \cdots, a_l$; this means that they are negative for $\mathcal{S}$, and therefore that $\Psi' = -\Psi$. Thus we have only two invariant complex structures.

It is known (see [23] or Gantmacher, Rec. Math. Moscou 47 (1939), 101-144) that any automorphism of $\mathfrak{t}$ permuting the roots extends to an automorphism of $\mathfrak{g}$. In particular, there is an automorphism σ carrying each root into its opposite; since we may assume here G to be simply connected (10.1), σ also defines an automorphism of G leaving T invariant; it maps Ψ onto $-\Psi$ and leaves invariant the set of roots of U. Therefore σ leaves U

invariant and defines a homeomorphism of G/U carrying the complex structure with roots Ψ onto the complex structure with roots $-\Psi$.

Remarks. 1) The argument which ends the preceding proof shows more generally the following: let Ψ, Ψ' be the root systems of two invariant complex structures $\mathcal{C}$, $\mathcal{C}'$ on G/U. If there is an automorphism of V_T carrying Ψ onto Ψ' and leaving the root system of U invariant, then $\mathcal{C}$, $\mathcal{C}'$ are equivalent under a differentiable homeomorphism of G/U, which is induced from an automorphism of G leaving U invariant.

2) If Ψ is the set of roots of an invariant complex structure $\mathcal{C}$, then $-\Psi$ is clearly the root system of the "bar structure" or conjugate of Ψ, that is, of the structure in which the vectors of type $(1,0)$ are those of type $(0,1)$ for $\mathcal{C}$. Thus the last part of the above proof shows that on G/U an invariant complex structure and its conjugate are equivalent under an automorphism of G. In the case $\boldsymbol{P}_{n-1}(C) = \mathbf{U}(n)/\boldsymbol{U}(n-1)\times\boldsymbol{U}(1)$, the automorphism may be taken as complex conjugation and therefore has to be an outer automorphism for $n \geqq 3$.

3) The case $k=1$ in 13.8 includes the hermitian symmetric spaces for which our assertion has already been noticed by I. Satake, "A remark on bounded symmetric domains," Sci. Papers Coll. Ed. Gen. Univ. Tokyo 3 (1953), 131-144).

13.9. *Examples of inequivalent structures.* There are cases in which G/U carries at least two invariant complex structures which are not equivalent under a differentiable homeomorphism. For instance, take $G=\boldsymbol{U}(4)$, $U=\boldsymbol{U}(2)\times\boldsymbol{U}(1)\times\boldsymbol{U}(1)$, embedded in the standard fashion. With respect to the standard maximal torus, the roots of $\boldsymbol{U}(4)$ are $\pm(x_i-x_j)$, $(1\leqq i<j \leqq 4)$, and those of U are $\pm(x_1-x_2)$. Let $\mathcal{S}_1$ (resp. $\mathcal{S}_2$) be the ordering defined by $x_4>x_1>x_2>x_3>0$ (resp. $x_1>x_2>x_3>x_4>0$). Then by 13.7, the set Ψ_1 (resp. Ψ_2) formed by x_4-x_1, x_4-x_2, x_4-x_3, x_1-x_3, x_2-x_3 (resp. x_1-x_3, x_1-x_4, x_2-x_3, x_2-x_4, x_3-x_4) is the root system of an invariant complex structure $\mathcal{C}_1$ (resp. $\mathcal{C}_2$). The image in $H^2(\boldsymbol{U}(4)/\boldsymbol{T},\boldsymbol{Z})$ of the first Chern class of $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is, by 10.8, equal to $3(x_4-x_3)$ (resp. $2x_1+2x_2-x_3-3x_4$). Now $\boldsymbol{U}(4)/\boldsymbol{T}=\boldsymbol{SU}(4)/\boldsymbol{T}'$ where $\boldsymbol{T}'\doteq\boldsymbol{T}\cap\boldsymbol{SU}(4)$ is a maximal torus of $\boldsymbol{SU}(4)$. The inclusion map of $\boldsymbol{T}'$ in $\boldsymbol{T}$ identifies $H^1(\boldsymbol{T}',\boldsymbol{Z})$ with the quotient of $H^1(\boldsymbol{T},\boldsymbol{Z})$ by $\boldsymbol{Z}\cdot(x_1+x_2+x_3+x_4)$; since $\boldsymbol{SU}(4)$ is simply connected, the transgression is an isomorphism of $H^1(\boldsymbol{T}',\boldsymbol{Z})$ on $H^2(\boldsymbol{SU}(4)/\boldsymbol{T}',\boldsymbol{Z})$. It follows then that the first Chern class of $\mathcal{C}_1$ (resp. $\mathcal{C}_2$) is divisible (resp. not divisible) by 3. Hence $\mathcal{C}_1$ and $\mathcal{C}_2$ are not equivalent under a differentiable homeomorphism of G/U. Moreover it will be shown in Section 24.14 that the Chern numbers $c_1{}^5$ of $\mathcal{C}_1$ and $\mathcal{C}_2$ are equal to 4860 and 4500 respectively.

The following observation leads to other examples:

PROPOSITION. *Assume that G/U carries two invariant homogeneous structures $\mathcal{C}$, $\mathcal{C}'$ with root systems Ψ, Ψ', and that G^c is the greatest connected group of automorphisms of $\mathcal{C}$ and of $\mathcal{C}'$. Then $\mathcal{C}$ and $\mathcal{C}'$ are equivalent under a differentiable homeomorphism of G/U if and only if there is a linear transformation α of $\mathfrak{t}$ leaving the root system of U invariant and carrying Ψ onto Ψ'.*

The "if" part follows from Remark 1) in 13.8.

Let now β be a differentiable homeomorphism of G/U carrying $\mathcal{C}$ onto $\mathcal{C}'$. By the assumption on G^c, β defines an automorphism of G^c. Using homogeneity and the facts recalled in 14.3, it is then seen that $\mathcal{C}$ and $\mathcal{C}'$ are also equivalent under a homeomorphism γ which is induced from an automorphism of G^c leaving U^c, T^c invariant. Since γ permutes the roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$, and since $\mathfrak{t}$ is characterized as the subset of $\mathfrak{t}^c$ on which the roots are real valued, γ leaves $\mathfrak{t}$ invariant, and its restriction to $\mathfrak{t}$ is the desired α. Q. E. D.

According to Bott (unpublished), G^c satisfies our assumption, for instance, if $G=\boldsymbol{E}_7, \boldsymbol{E}_8$. Moreover, $\boldsymbol{E}_7, \boldsymbol{E}_8$ have no outer automorphisms, hence the automorphisms of $\mathfrak{t}$ keeping the root system of G invariant are just those of the Weyl group. Let now a, b be two different simple roots with respect to an ordering $\mathcal{S}$ and let $w \in W(G)$ be a transformation carrying b onto a. (This exists because the roots of $\boldsymbol{E}_7$ or $\boldsymbol{E}_8$ all have the same length and $W(G)$ is known to be transitive on a set of roots of the same length.)

Let Φ be the set of positive roots for $\mathcal{S}$, and $\Psi=\Phi-a$, $\Psi'=w(\Phi-b)$. The symmetry to $a=0$ carries Φ onto $-a \cup \Psi$ (see the proof of 3.1). Therefore, if there existed an α carrying $\pm a$ onto itself and Ψ onto Ψ', there would also be a $w' \in W(G)$ carrying a onto b and leaving Φ invariant, but this contradicts the fact that $W(G)$ is simply transitive on the Weyl chambers (2.7). Now $\pm a$ is the set of roots of the centralizer U of the singular torus defined by $a=0$. Thus, by our criterion, Ψ and Ψ' are the root systems of two invariant complex structures on G/U which are not equivalent under a differentiable homeomorphism. A similar discussion would also show that the complex structures on $\boldsymbol{U}(4)/\boldsymbol{U}(2)\times \boldsymbol{U}(1)\times \boldsymbol{U}(1)$ discussed above are not equivalent.

14. Complex Lie groups, embeddings, representations, invariant differential forms. We collect here some results to be used in the sequel. For more details about the facts mentioned without proofs in 14. 3, 14. 4 or about related questions, see [5], [7a], Goto "On algebraic homogeneous spaces," *Amer. Jour. Math.* 76 (1954), 811-818, J. Tits, "Sur certaines classes d'espaces homogènes de groupes de Lie," *Mém. Acad. Royale Belgique* 29 (1955), Chapitre III.

14. 1. *Notations.* G is semi-simple and simply connected, $\mathcal{S}$ is an ordering of the roots with respect to T, and $a_1, \cdots, a_l$ are the simple roots for $\mathcal{S}$. If I is a (possibly empty) proper subset of $[1, l]$, we denote by U_I the centralizer of the torus S_I defined by $a_i(t) = 0$ $(i \in I, t \in T)$ and put $M_I = G/U_I$. We consider it to be endowed with the invariant complex structure $\mathcal{C}_I$ defined by the set Ψ_I or Ψ of *positive complementary roots* (whose existence follows from 13. 7). U_I' is the semi-simple part of U_I. Thus we have $U_I = S_I \cdot U_I'$ with $S_I \cap U_I'$ finite.

Remark. Let G be a compact connected Lie group and G_1, G', U, U_1, U' be as in 10. 1. Then $G/U = G'/U' = G_1/U_1$. By a result of Hopf (see e. g. [23, Exp. XXI]), the centralizer of a toral subgroup in a compact connected Lie group is connected; hence, if one of U, U', U_1 is centralizer of a torus, so are the other two. Thus the assumption G semi-simple and simply connected made in § 14, which allows one to avoid some slight irrelevant technical complications, is no real restriction, and the results of this paragraph are valid, with little or no modification, in the general case. In particular, in 14. 4 one has to consider then the representations of the group $\tilde{G}$ mentioned in 2. 9.

14. 2. The natural map $\nu_I \colon G/T \to G/U_I$ is the projection in the fibering $(G/T, G/U_I, U_I/T)$; the spaces $G/T, G/U_I, U_I/T$ have no torsion and have vanishing odd dimensional Betti numbers ([5] or R. Bott, *Bull. Soc. Math. France* 84 (1956), 251-81). Therefore [2, § 4], for any commutative group A of coefficients, the fibre is totally non homologous to zero, $\nu_I{}^*$ is injective, $\nu_I{}^*(H^2(G/U_I, A))$ is the kernel of the map of $H^2(G/T, A)$ into $H^2(U_I/T, A)$ induced by inclusion. It follows that $\nu_I{}^*(H^2(G/U_I, A)$ is a direct summand of $H^2(G/T, A)$; also, since the transgression in $(G, G/U_I, U_I)$ is an isomorphism of $H^1(U_I, \mathbf{Z})$ onto $H^2(G/U_I, \mathbf{Z})$, the former group is free abelian.

LEMMA. *Let A be a principal ideal ring. Then $\nu_I{}^*$ is an isomorphism of $H^2(G/U_I, A)$ onto the submodule of $H^2(G/T, A)$ formed by the elements orthogonal to the a_i's $(i \in I)$, that is, which is spanned by the fundamental weights ϖ_i's $(i \notin I)$.*

By the above and the universal coefficient formula, it is enough to prove the lemma for $A = \boldsymbol{Z}$. The fundamental weights ϖ_i $(1 \leqq i \leqq \operatorname{rank} G)$, form a basis of $H^1(T, \boldsymbol{Z})$, (see 3.4), hence of $H^2(G/T, \boldsymbol{Z})$, and the subgroup B_I spanned by the ϖ_i's $(i \notin I)$ is a direct summand whose rank equals the dimension of S_I.

The group $T' = T \cap U_I'$ is a maximal torus of U_I', whose covering in the universal covering V_T of T is spanned by the contravariant images of the a_i $(i \in I)$. Since U_I' is semi-simple and U_I is locally isomorphic to the product $S_I \times U_I'$, the map $H_1(S_I, \boldsymbol{R}) \to H_1(U_I, \boldsymbol{R})$, (resp. $H_1(T', \boldsymbol{R}) \to H_1(U_I, \boldsymbol{R})$), induced by inclusion, is an isomorphism (resp. has zero image). Since $H^1(U_I, \boldsymbol{Z})$ is free, it follows immediately that $\alpha^* : H^1(U_I, \boldsymbol{Z}) \to H^1(T, \boldsymbol{Z})$ is injective, with its image contained in B_I, and of finite index in B_I, where α is the inclusion of T in U_I.

The projection ν_I defines a representation of the fibering $(G, G/T, T)$ into $(G, G/U_I, U_I)$, whose restriction to a fibre is α. Therefore, using transgression, we see that the image of ν_I^* is a subgroup of finite index of B_I. But we have already shown that it is a direct summand, whence the lemma.

Remark. By transgression, we also see that α^* identifies $H^1(U_I, \boldsymbol{Z})$ with B_I.

14.3. *Complexification.* G^c denotes the complex Lie group containing G, with Lie algebra $\mathfrak{g}^c$, whose existence and uniqueness up to an isomorphism is well known. We use the notation of §§ 1, 12 and, moreover, put

$$\mathfrak{p}_I = \mathfrak{u}_I{}^c + \sum_{-b \in \Psi} \mathfrak{v}_{\mathfrak{b}}.$$

It is a subalgebra which generates a closed, connected, complex analytic subgroup P_I of G^c, equal to its normalizer, such that $P_I \cap G = U_I$; it follows then that G is transitive on G^c/P_I and that there is a natural identification of G/U_I with G^c/P_I which carries $\mathcal{R}_I$ onto the quotient complex structure, as defined in the theory of complex Lie groups (see e.g. [7a]). If, in particular, I is empty, then $\mathfrak{b} = \mathfrak{p}_I$ is solvable and $G^c/B = G/T$, where $B = P_I$.

14.4. *Representations and embeddings.* Let $\mathcal{S}^-$ be the ordering of the roots which is opposite to $\mathcal{S}$, that is, which has the negative roots of $\mathcal{S}$ as positive roots. The highest weights of the irreducible representations of G with respect to $\mathcal{S}^-$ are then the opposite of the highest weights in the order $\mathcal{S}$. If Γ has highest weight ϖ for $\mathcal{S}$ and Γ' highest weight $-\varpi$ for $\mathcal{S}^-$, then Γ' is the contragredient representation to Γ, and its weights are the opposite of those of Γ.

Let Γ be an irreducible representation of degree $q+1$,

$$\varpi = c_1\varpi_1 + \cdots + c_l\varpi_l$$

its highest weight in the ordering $\mathscr{S}$, $\check{\Gamma}$ the contragredient representation, $\check{\Gamma}'$ and Γ' the associated representations by means of projective transformations in $\boldsymbol{P}_q(C)$. Let V be a representation space for $\check{\Gamma}$ and π be the projection of $V - 0$ on $\boldsymbol{P}_q(C)$. There is in V exactly one 1-dimensional subspace W which is invariant under B, and we have

$$\check{\Gamma}(t)(x) = \exp[-2\pi i\varpi(t)]\cdot x, \qquad (t \in T^c, x \in W).$$

$x' = \pi(W-0)$ is then the unique point of $\boldsymbol{P}_q(C)$ fixed under $\check{\Gamma}'(B)$.

We say that M_I is *associated* (resp. *strictly associated*) to Γ if $c_i = 0$ for $i \in I$ (resp. and, moreover, if $c_j \neq 0$ for $j \notin I$). If M_I is associated (resp. strictly associated) to Γ, then the map $\phi\colon g \to \check{\Gamma}'(g)\cdot x'$ induces a holomorphic (resp. bijective and bi-holomorphic) map β_I of M_I onto a projective non-singular variety M_Γ. In particular, each irreducible representation yields a holomorphic map β_2 of G/T into some projective space. Since a given M_I is strictly associated to infinitely many representations, it therefore admits projective embeddings.

The cone $\pi^{-1}(M_\Gamma)$ over M_Γ is, in the obvious way, a $\boldsymbol{C}^*$-bundle η over M_Γ; on the other hand, let N be the subgroup of B whose Lie algebra over $\boldsymbol{C}$ is spanned by the $\mathfrak{v}_a$ ($a < 0$ for $\mathscr{S}$). N is the commutator subgroup of B, and $B/N \cong T^c$. The quotient G^c/N can be considered as the total space of a principal T^c-bundle $\xi = (G^c/N, G^c/B, T^c)$. If γ is the representation of T^c with character $\exp(-2\pi i\varpi)$, it is then easily seen that β_2 induces a γ-homomorphism of ξ on η. Therefore, by 10.4, $\varpi = -\beta_2{}^*(c_1(\eta))$. It follows from this and from 14.2 that if M_I is associated to Γ, then $-\varpi$ may be identified with an element of $H^2(M_I, \boldsymbol{Z})$ which is the Chern class of the bundle over M_I induced from η by $\beta_I\colon M_I \to M_\Gamma$. But $c_1(\eta)$ is $-e^*$, where e^* is the dual of the homology class containing a hyperplane section of M_I (see § 29). Thus ϖ is the dual of the homology class of a divisor on M_I, namely, the inverse image of a hyperplane section of M_Γ. It may be shown [7a] that the inverse images of the hyperplane sections of M_Γ form a complete linear system on M_I, and that this system is the only one on which the natural representation of G is Γ'.

14.5. *Positive classes.* Since we want to use some facts about complex Lie algebras, we now identify V_T with its tangent space $\mathfrak{t}$ at e, and assume the invariant metric to be the restriction of $-K$, where K is the Killing form. The roots of $\mathfrak{g}^c$ with respect to $\mathfrak{t}^c$ in the sense of infinitesimal theory are then the forms $2\pi i a$, where a runs through the roots as defined in this paper (or, more precisely, through their extensions to $\mathfrak{t}^c$, since they were originally defined on V_T or $\mathfrak{t}$). If a is a linear form on $\mathfrak{t}^c$, we denote by h_a its contravariant representative with respect to $-K$; if a is real valued on $\mathfrak{t}$, then $h_a \in \mathfrak{t}$. It is known (see, for instance, [23], Exp. 10, 11) that we can find $e_a \in \mathfrak{v}_a$ with the following properties:

$$(1) \qquad [e_a, e_{-a}] = 2\pi i h_a, \qquad K(e_a, e_{-a}) = -1,$$

$\mathfrak{g}$ is spanned over the reals by $\mathfrak{t}$ and the vectors $e_a + e_{-a}$, $i(e_a - e_{-a})$, and the Killing form is the direct sum

$$K = K_1 - \sum_{a>0} x_a x_{-a},$$

where K_1 is the restriction of K to $\mathfrak{t}^c$ and the x_a, x_{-a} form the dual base to (e_a, e_{-a}).

Finally, we recall that if X, Y are left-invariant vector fields and ω is a left-invariant 1-form on a Lie group H, then

$$(2) \qquad d\omega(X,Y) = -\omega([X,Y]) \qquad ([X,Y] = X\cdot Y - Y\cdot X)$$

(see e. g. Chevalley, *Theory of Lie groups I*, Princeton, Chap. V, § 4; it is there stated for real Lie groups, but the proof is also valid in the complex case).

Let us denote by ω_a the left-invariant 1-form on G^c whose restriction to $\mathfrak{g}^c$ is annihilated by $\mathfrak{t}^c$ and which is such that $\omega_a(e_b) = 1$ if $a = b$ and is zero if $a \neq b$. A straightforward computation using (1), (2) and 12.2 yields then the

LEMMA. *Let η_b be the left-invariant 1-form on G^c whose restriction to $\mathfrak{g}^c$ is zero on $\sum \mathfrak{v}_a$ and which induces the linear form b on $\mathfrak{t}^c$. Then*

$$(3) \qquad d\eta_b = -2\pi i \sum_{a>0} (b,a)\,\omega_a \wedge \omega_{-a}.$$

We are interested only in the case where b is real valued on $\mathfrak{t}$; then (b,a) is also real valued. Assume now that b is orthogonal to the roots a_i $(i \in I)$, i. e., that $h_b \in \mathfrak{s}_I$. Then

$$(4) \qquad d\eta_b = -2\pi i \sum_{a \in \Psi_I} (b,a)\,\omega_a \wedge \omega_{-a}.$$

The restriction $\eta_{b|G}$ of η_b to G is left-invariant. By (4), $d\eta_b$ is zero on $\mathfrak{u}$, and hence, by invariance, it vanishes on U; thus $\eta_{b|U}$ and, a fortiori, $\eta_{b|T}$ are closed. Since $\eta_{b|T}$ is clearly in the class $b \in H^1(T,\boldsymbol{R})$, it follows by 14.2

that $\eta_{b|U}$ represents the element of $H^1(U, \boldsymbol{R})$ which we have identified with b. We want to show that $d\eta_{b|G}$ is the inverse image of a 2-form on M_I; for this, it is necessary and sufficient that the 2-form defined by $d\eta_b$ on $\mathfrak{g}$ vanishes whenever one of the arguments is in $\mathfrak{u}_I$ and is invariant under $\mathrm{Ad}_{\mathfrak{g}}\, U_I$ (see e. g. Chevalley-Eilenberg, Trans. A. M. S. 63 (1948), 85-124, Theorem 13. 1). The first property is obvious from (4); as to the second, we may argue as follows: by (4), we have only to show that the restriction of $d\eta_b$ *to* $\sum_{b \in \Psi} \mathfrak{b}$ is invariant under the linear isotropy group $\tilde{U}$. From the properties of the Killing form recalled above, we see that if we take in $\mathfrak{b}$ the real and imaginary parts of x_b as coordinates, then the Killing form is the negative unit form, and therefore, the isotropy group $\tilde{U}$ is orthogonal; this means that $d\eta_b$ is invariant under $\tilde{U}$ if its matrix commutes with $\tilde{U}$. But it follows from (4) and 12. 2 that this matrix is equal to the restriction of the matrix of $\mathrm{ad}_{\mathfrak{g}}\, h_b$. Since $h_b \in \mathfrak{s}_I$, it centralizes $\mathfrak{u}$, and $\mathrm{ad}_{\mathfrak{g}}\, h_b$ does commute with $\mathrm{Ad}_{\mathfrak{g}}\, U$.

By the foregoing and by the definition of transgression, $d\eta_{b|G}$ may be identified with a closed 2-form on G/U_I belonging to the image of $b \in H^1(U, \boldsymbol{R})$ under transgression; in view of the conventions made in 10. 1, this form represents the cohomology class which has been identified with $-b$. Moreover, by the definition of the complex structure $\mathcal{J}_I$ on M_I, the $\mathfrak{v}_a$ $(a \in \Psi_I)$, span the subspace of $M_I \otimes \boldsymbol{C}$ which contains the differentials of local holomorphic functions, and we have $\omega_{-a} = \overline{\omega}_a$ in the standard notations. Thus we have shown the following:

14. 6. PROPOSITION. *We keep the notations of* 14. 1, 14. 2. *Let b be a linear form on V_T orthogonal to the simple roots a_i $(i \in I)$. Then the element of $H^2(G/U_I, \boldsymbol{R})$ identified with b in* 14. 2 *contains the invariant 2-form of type* (1, 1)

$$\omega = 2\pi i \sum_{a \in \Psi_I} (b, a)\, \omega_a \wedge \overline{\omega}_a. \tag{5}$$

ω is the imaginary of the hermitian form

$$4\pi \sum_{a \in \Psi} (b, a)\, \omega_a \cdot \overline{\omega}_a \text{ (symmetric product)}. \tag{6}$$

A 2-dimensional complex cohomology class on a complex manifold M is *positive in the sense of* Kodaira if it contains the imaginary part of a positive non-degenerate hermitian metric, which is then necessarily kählerian. From (5), (6) and the remark in 13. 6, we get:

14. 7. COROLLARY. *b is positive in the sense of Kodaira if $(b, a) > 0$ for $a \in \Psi_I$, that is, if $(b, a_i) > 0$ for $i \notin I$.*

This is in agreement with the fact (14.4) that when b is the highest weight of an irreducible representation Γ to which M_I is strictly associated, then it is dual to the class of a hyperplane section in the projective embedding provided by the representation $\check{\Gamma}$.

14.8. COROLLARY. *The first Chern class c_1 of the tangent bundle to M_I is positive.*

c_1 is the sum of the positive complementary roots (10.7) and, by 13.6, these are the linear combinations of the simple roots in which at least one a_i with $i \notin I$ has a strictly positive coefficient. Thus, if $a \in \Psi_I$, its transform by the symmetry to the plane $a_i = 0$ $(i \in I)$ belongs to Ψ_I. Since $(a_i, S_i a + a) = 0$, it follows that $(a_i, c_1) = 0$. This is also a consequence of 14.2.

In view of (5), (6), we have to show that if $a \in \Psi$, then $(a, c_1) > 0$. Let $b \in \Psi$ and assume that $(a, b) < 0$. Then (§ 2)

$$(7) \qquad b, b + a, b + 2a, \cdots, b + ka \qquad (k = -2(a, b)(a, a)^{-1})$$

are roots of G and, in fact, belong to Ψ, since the latter is a closed system (13.7). We have $(a, b + ka) = (a, a)k/2 > 0$ and

$$(a, b + b + a + \cdots + b + ka) = (k + 1)(a, b) + (a, a)k \cdot (k + 1)/2 = 0.$$

From this we deduce readily that we may represent Ψ as a union of disjoint subsets Ψ_j, where Ψ_j consists either of one root b with $(a, b) \geqq 0$ or of a string of type (7), whose sum is orthogonal to a; in the first category we have the set consisting of a itself, and hence, finally, $(a, c_1) > 0$.

14.9. Using some properties of the constants of structure of $\mathfrak{g}^c$, one can show that 14.6 gives all invariant 2-forms on M_I, as indicated in [5]; this implies that the condition of 14.7 is also necessary for b to be positive as will also follow from § 24.

14.10. We recall that for a kählerian compact manifold M, the d- and the $\bar{\partial}$-cohomology are identical, and that $H^i(M, \mathbf{C})$ is a direct sum of subspaces $H^{p,q}(M)$, $(p + q = i)$, where $H^{p,q}$ is the space of i-dimensional cohomology classes which can be represented by exterior differential forms of type (p, q). This applies, in particular, to the projective variety G/U_I.

PROPOSITION. *In the previous notations, we have $H^{2i+1}(M_I, \mathbf{C}) = 0$ and $H^{2i}(M_I, \mathbf{C}) = H^{i,i}(M_I)$ for all $i \geqq 0$.*

For the first assertion, see [2, Théorème 26.1].

As remarked in 14.2, the projection ν_I of G/T onto G/U_I induces an injective homomorphism of $H^*(G/U_I, \boldsymbol{C})$ in $H^*(G/T, \boldsymbol{C})$. This map is also holomorphic with respect to the complex structures on G/T and G/U_I defined by the positive complementary roots, because, in the notations of 14.3, it can be identified with the projection of G^c/B onto G^c/P_I, both spaces being endowed with the natural quotient complex structures. Thus $\nu_I{}^*$ identifies $H^{p,q}(M_I)$ with a subpace of $H^{p,q}(G/T, \boldsymbol{C})$, and it suffices to prove our contention for G/T. Since $H^*(G/T, \boldsymbol{C})$ is generated by the unit and its 2-dimensional classes [2, § 26], it is enough to show that $H^2(G/T, \boldsymbol{C}) = H^{1,1}(G/T)$, but this follows from 14.6.

Chapter V. Special Cases.

15. Projective spaces.

15.1. *Complex projective spaces.* We wish to apply Theorem 10.8 to the case where $G = \boldsymbol{U}(q)$ and $U = \boldsymbol{U}(1) \times \boldsymbol{U}(q-1)$ and $G/U = \boldsymbol{P}_{q-1}(\boldsymbol{C})$, $(q \geqq 2)$. The imbedding of U in G is the usual one; namely, as follows: $\boldsymbol{U}(q)$ is the group of unitary matrices and $\boldsymbol{U}(1) \times \boldsymbol{U}(q-1)$ is the group of $q \times q$-matrices of the form

$$\begin{pmatrix} a' & 0 \\ 0 & A'' \end{pmatrix},$$

where $a' \in \boldsymbol{U}(1)$ and $A'' \in \boldsymbol{U}(q-1)$, which is a subgroup of maximal rank of G. Let $\boldsymbol{T}$ be the standard maximal torus of diagonal unitary matrices

$$\begin{bmatrix} e^{2\pi i x_1} & & & 0 \\ & \cdot & & \\ & & \cdot & \\ & & & \cdot \\ 0 & & & e^{2\pi i x_q} \end{bmatrix}.$$

$\boldsymbol{T}$ is contained in $\boldsymbol{U}(1) \times \boldsymbol{U}(q-1)$ and plays the role of S in Theorem 10.7. The coordinates $x_1, \cdots, x_q$ are integral linear forms on V_T (see 1.2 and 10.1), and the roots of $\boldsymbol{U}(q)$ with respect to $\boldsymbol{T}$ are $\pm(x_j - x_k)$, where $1 \leqq j < k \leqq q$. The roots of $\boldsymbol{U}(1) \times \boldsymbol{U}(q-1)$ are $\pm(x_j - x_k)$ with $2 \leqq j < k \leqq q$. Hence the roots of $\boldsymbol{U}(q)$ complementary to $\boldsymbol{U}(1) \times \boldsymbol{U}(q-1)$ are $\pm(x_1 - x_j)$ with $2 \leqq j \leqq q$.

The usual invariant complex structure of $\boldsymbol{P}_{q-1}(\boldsymbol{C})$ is given by regarding $\boldsymbol{P}_{q-1}(C)$ as the space of the lines passing through the origin of $\boldsymbol{C}^q$. Let $\boldsymbol{GL}(q, \boldsymbol{C})$ operate in the usual way on $\boldsymbol{C}^q$ and thus on $\boldsymbol{P}_{q-1}(\boldsymbol{C})$. Let $\boldsymbol{GL}(1, q-1; \boldsymbol{C})$ be the subgroup of those elements of $\boldsymbol{GL}(q, C)$ which keep the point $(1, 0, \cdots, 0)$ of $\boldsymbol{P}_{q-1}(\boldsymbol{C})$ fixed. Then

$$\boldsymbol{U}(q)/(\boldsymbol{U}(1)\times\boldsymbol{U}(q-1))=\boldsymbol{GL}(q,C)/\boldsymbol{GL}(1,q-1;C)=\boldsymbol{P}_{q-1}(\boldsymbol{C}).$$

Thus $\boldsymbol{P}_{q-1}(\boldsymbol{C})$ is represented as a quotient of complex Lie groups and is, therefore, endowed with an invariant structure which is just the usual one. The complex isotropy representation ι_c of this complex structure has the weights x_j-x_1 $(j=2,\cdots,q)$, which can be seen as follows: The element $(e^{2\pi i x_1},\cdots,e^{2\pi i x_q})$ of $\boldsymbol{T}$, when operating on the point $(1,z_2,\cdots,z_q)$ of $\boldsymbol{P}_{q-1}(\boldsymbol{C})$, gives the point

$$(1,z_2e^{2\pi i(x_2-x_1)},\cdots,z_qe^{2\pi i(x_q-x_1)})$$

of $\boldsymbol{P}_{q-1}(C)$, which proves the desired result. We may remark here that $\boldsymbol{P}_{q-1}(C)$ admits exactly two invariant complex structures and these are complex conjugate to each other (see 13.8).

Let ξ be a principal $\boldsymbol{U}(q)$-bundle. We consider some associated fibre bundles and their projections according to the following diagram: Let ρ, σ be the natural projections

$$E_\xi/\boldsymbol{T}\xrightarrow{\rho}E_\xi/(\boldsymbol{U}(1)\times\boldsymbol{U}(q-1))\xrightarrow{\sigma}B_\xi \tag{1}$$

and $\pi=\sigma\circ\rho$. Let η be the real vector bundle along the fibres (7.4) of $(E_\xi/(\boldsymbol{U}(1)\times\boldsymbol{U}(q-1)),B_\xi\boldsymbol{P}_{q-1}(\boldsymbol{C}))$ endowed with the complex structure η' coming from the usual invariant complex structure on $\boldsymbol{P}_{q-1}(\boldsymbol{C})$; i.e., η' is defined by the complex isotropy representation ι_c considered above. Then we have for the total Chern class of η'

$$\rho^*c(\eta')=\prod_{j=1}^{q}(1+x_j-x_1).$$

Now let $c_i\in H^{2i}(B_\xi,\boldsymbol{Z})$ be the Chern classes of ξ. Then we have (9.1)

$$\pi^*(1+c_1+c_2+\cdots+c_q)=(1+x_1)(1+x_2)\cdots(1+x_q).$$

Considering z as an indeterminate over $H^*(E_\xi/\boldsymbol{T},\boldsymbol{Z})$, we have the equation

$$z^q+z^{q-1}\pi^*(c_1)+\cdots+\pi^*(c_q)=(z+x_1)(z+x_2)\cdots(z+x_q).$$

Replacing z by $1-x_1$ gives

$$\rho^*c(\eta')=\prod_{j=1}^{q}(1+x_j-x_1)=\sum_{i=0}^{q}(1-x_1)^{q-i}\pi^*(c_i).$$

$x_1,x_2,\cdot\cdot,x_q$ are the first Chern classes of the q principal $\boldsymbol{U}(1)$-bundles $\xi_1,\cdots,\xi_q$, into which the principal bundle $(E_\xi,E_\xi/\boldsymbol{T},\boldsymbol{T})$ splits. The principal bundle $(E_\xi,E_\xi/(\boldsymbol{U}(1)\times\boldsymbol{U}(q-1)),\boldsymbol{U}(1)\times\boldsymbol{U}(q-1))$ splits into a principal $\boldsymbol{U}(1)$-bundle ξ' and a principal $\boldsymbol{U}(q-1)$-bundle ξ''. Obviously,

$\rho^*(\xi')$ is equivalent to ξ_1 and $\rho^*(\gamma_1) = x_1$, where γ_1 denotes the first Chern class of ξ'. Since $\pi^* = \rho^* \circ \sigma^*$ and since ρ^* is injective for integral cohomology, we get

$$c(\eta') = \sum_{i=0}^{q} (1 - \gamma_1)^{q-i} \sigma^*(c_i). \tag{2}$$

Since $c_q(\eta') = 0$, we have

$$\sum_{i=0}^{q} (-\gamma_1)^{q-i} \sigma^*(c_i) = 0. \tag{3}$$

Once γ_1 is defined, the Chern classes of ξ are characterized by (3), a fact due to Hirsch (compare [11]).

To calculate the Chern class of $\boldsymbol{P}_{q-1}(\boldsymbol{C})$, we now specialize to the case where $E_\xi = \boldsymbol{U}(q)$ and where B_ξ is a single point. Then η' is the tangent bundle of $\boldsymbol{P}_{q-1}(\boldsymbol{C})$. Since $c_i = c_i(\xi) = 0$ for $i > 0$, we get

$$c(\eta') = c(\boldsymbol{P}_{q-1}(C)) = (1 - \gamma_1)^q.$$

Now we observe that ξ' corresponds to the Hopf bundle; i.e., $(\boldsymbol{C}^q - \{0\}, \boldsymbol{P}_{q-1}(\boldsymbol{C}), \boldsymbol{C}^*)$ is the extension of ξ' with respect to the natural imbedding of $\boldsymbol{U}(1)$ in $\boldsymbol{C}^*$. Thus $-\gamma_1 = e^*$, where $e^* \in H^2(\boldsymbol{P}_{q-1}(\boldsymbol{C}), \boldsymbol{Z})$ is dual to the hyperplane of $\boldsymbol{P}_{q-1}(\boldsymbol{C})$ (see § 29). Therefore

$$c(\boldsymbol{P}_{q-1}(\boldsymbol{C})) = (1 + e^*)^q.$$

15.2. *Complex projective bundles.* In Sections 15.2 and 15.3, all cohomology groups are taken with real coefficients. The projective unitary group is defined by

$$\boldsymbol{PU}(q) = \boldsymbol{U}(q)/\boldsymbol{D},$$

where $\boldsymbol{D}$ is the 1-dimensional torus of scalar matrices of $\boldsymbol{U}(q)$. Let $\boldsymbol{T}^q$ be the maximal torus of diagonal matrices of $\boldsymbol{U}(q)$. Then $\tilde{\boldsymbol{T}}^{q-1} = \boldsymbol{T}^q/\boldsymbol{D}$ is a maximal torus of $\boldsymbol{PU}(q)$, and we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \boldsymbol{D} & \longrightarrow & \boldsymbol{T}^q & \longrightarrow & \tilde{\boldsymbol{T}}^{q-1} & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \boldsymbol{D} & \longrightarrow & \boldsymbol{U}(q) & \xrightarrow{\alpha} & \boldsymbol{PU}(q) & \longrightarrow & 0 \end{array} \tag{4}$$

which induces a commutative diagram for the classifying spaces (6.6)

$$\begin{array}{ccccc} B_{\boldsymbol{D}} & \longrightarrow & B_{\boldsymbol{T}^q} & \longrightarrow & B_{\tilde{\boldsymbol{T}}^{q-1}} \\ \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle \pi} & & \downarrow \\ B_{\boldsymbol{D}} & \longrightarrow & B_{\boldsymbol{U}(q)} & \xrightarrow{\rho(\alpha)} & B_{\boldsymbol{PU}(q)} \end{array} \tag{5}$$

where the two horizontal lines come from the fibre bundles $(B_{\boldsymbol{T}^q}, B_{\tilde{\boldsymbol{T}}^{q-1}}, B_{\boldsymbol{D}})$ and $(B_{\boldsymbol{U}(q)}, B_{\boldsymbol{PU}(q)}, B_{\boldsymbol{D}})$, see [2]. Then, in the commutative diagram

$$\begin{array}{ccc} H^*(B_{\boldsymbol{T}^q}) & \longleftarrow & H^*(B_{\tilde{\boldsymbol{T}}^{q-1}}) \\ \uparrow \pi^* & & \uparrow \\ H^*(B_{\boldsymbol{U}(q)}) & \xleftarrow{\rho(\alpha)^*} & H^*(B_{\boldsymbol{PU}(q)}) \end{array} \tag{6}$$

which is induced by (5), all arrows indicate injections. If we denote by ξ the universal principal $\boldsymbol{U}(q)$-bundle and by $\tilde{\xi}$ the universal principal $\boldsymbol{PU}(q)$-bundle, then $\rho(\alpha)^*\tilde{\xi}$ is (up to equivalence) the α-extension of ξ. Let $x_1,\cdots,x_q$ be the first Chern classes (with real coefficients) of the q principal $\boldsymbol{U}(1)$-bundles into which $\pi^*\xi$ splits. Then

$$H^*(B_{\boldsymbol{T}^q}) = \boldsymbol{R}[x_1,\cdots,x_q].$$

Using the Weyl group of $\boldsymbol{PU}(q)$ (which is isomorphic to that of $\boldsymbol{U}(q)$), it follows easily from diagram (6) that $\pi^*\rho(\alpha)^*H^*(B_{\boldsymbol{PU}(q)})$ is the subring of those polynomials in $\boldsymbol{R}[x_1,\cdots,x_q]$ which are symmetric in $x_1,\cdots,x_q$ and invariant under the substitution $t\colon x_i \to x_i + b$, where b is an indeterminate. Roughly speaking, for a $\boldsymbol{PU}(q)$-bundle the Chern classes c_j (i.e., the elementary symmetric functions in the x_i) make no sense, but the polynomials in the c_j invariant under t do.

Now let $(L, X, \boldsymbol{P}_{q-1}(\boldsymbol{C}), \sigma)$ be a bundle with $\boldsymbol{PU}(q)$ as a structural group. It is known that σ^* maps $H^*(X)$ isomorphically in $H^*(L)$ (*real* cohomology) and that, for every element $\gamma \in H^2(L)$ whose restriction to the fibre equals the generator e^* of $H^2(\boldsymbol{P}_{q-1}(\boldsymbol{C}))$, there is (8.4) a relation

$$\gamma^q - \sigma^*(d_1)\gamma^{q-1} + \sigma^*(d_2)\gamma^{q-2} - \cdots + (-1)^q\sigma^*(d_q) = 0 \tag{7}$$

with uniquely determined elements $d_i \in H^{2i}(X)$ depending only on γ. Let η' be the complex vector bundle along the fibres of L. We recall that $\boldsymbol{P}_{q-1}(\boldsymbol{C}) = \boldsymbol{PU}(q)/((\boldsymbol{U}(1)\times \boldsymbol{U}(q-1))/\boldsymbol{D})$ and that the complex structure η' comes from the complex isotropy representation ι_c considered in 15.1 (ι_c is trivial on $\boldsymbol{D}$).

15.3. THEOREM. *The Chern class (with real coefficients) of the complex vector bundle η' along the fibres of a fibre bundle $(L, X, \boldsymbol{P}_{q-1}(C), \sigma)$ with $\boldsymbol{PU}(q)$ as structural group is given by the formula*

$$c(\eta') = \sum_{i=0}^{q} (1-\gamma)^{q-i}\sigma^*(d_i), \qquad (d_0 = 1), \tag{8}$$

where $\gamma \in H^2(L)$ is an arbitrary element whose restriction to the fibre gives the generator e^* and where the $d_i \in H^2(X)$ are defined by the relation (7).

For the proof, we introduce an indeterminate z. The polynomial

$$F(z) = \sum_{i=0}^{q} z^{q-i}\sigma^*(d_i), \quad (\text{see } (7)),$$

is then the unique element of $\sigma^*H^*(X)[z]$ which is a polynomial of degree q in z, has the unit 1 as the coefficient of z^q, and for which $F(-\gamma)$ vanishes. If $\tilde{\gamma} = \gamma + \sigma^*(b)$, where $b \in H^2(X)$, then $\tilde{F}(z) = F(z + \sigma^*(b))$ is the analogous unique polynomial with $\tilde{F}(-\tilde{\gamma}) = 0$. Since $F(1-\gamma) = \tilde{F}(1-\tilde{\gamma})$, the right side of (8) is independent of the choice of γ. Taking this into account, (2) and (3) of 15.1 yield our theorem for bundles whose structural groups can be α-reduced to $\boldsymbol{U}(q)$. Furthermore, we see that it is enough to prove the theorem for the universal principal $\boldsymbol{PU}(q)$-bundle $\tilde{\xi}$. Since the theorem is true for $\rho(\alpha)^*\tilde{\xi}$ (see (5)), it follows easily in full generality.

Theorem 15.3 was announced in the first note of [16].

15.4. *Real projective spaces.* In Section 15.4, all cohomology groups are taken with $\boldsymbol{Z}_2$ as coefficients. Let ξ be a principal $\boldsymbol{O}(q)$-bundle and η the vector bundle along the fibres of $(E_\xi/(\boldsymbol{O}(1)\times\boldsymbol{O}(q-1)), B_\xi, \boldsymbol{P}_{q-1}(\boldsymbol{R}))$. Let $\boldsymbol{Q}$ be the group of all diagonal matrices of $\boldsymbol{O}(q)$. Consider the maps

$$E_\xi/\boldsymbol{Q} \xrightarrow{\ \rho\ } E_\xi/(\boldsymbol{O}(1)\times\boldsymbol{O}(q-1)) \xrightarrow{\ \sigma\ } B_\xi,$$

and let w_i $(1 \leqq i \leqq q)$ be the Stiefel-Whitney classes of ξ. Then we have

$$w(\eta) = \sum_{i=0}^{q} (1-\gamma_1)^{q-i}\sigma^*(w_i), \tag{9}$$

where γ_1 is the 1-dimensional characteristic class of the $\boldsymbol{O}(1)$-bundle over $E_\xi/(\mathbf{O}(1)\times\boldsymbol{O}(q-1))$. The class γ_1 induces on each fibre the generator of its cohomology ring. The proof uses 5.3 and 11.5 but is otherwise formally identical to that given in 15.1 (except that the x_i's are now 1-dimensional classes), and is therefore left to the reader. (9) implies the well-known fact that the Stiefel-Whitney class of $\boldsymbol{P}_{q-1}(\boldsymbol{R})$ equals $(1+a)^q$, where a is the generator of $H^*(\boldsymbol{P}_{q-1}(\boldsymbol{R}))$.

15.5. *The Pontrjagin classes of the quaternionic projective spaces.* The treatment of the quaternionic projective spaces

$$\boldsymbol{P}_{q-1}(\boldsymbol{K}) = \boldsymbol{Sp}(q)/(\boldsymbol{Sp}(1)\times\boldsymbol{Sp}(q-1)), \qquad q \geqq 2,$$

is similar to the discussion in 15.1.

$\boldsymbol{Sp}(q)$ is the group of all unitary quaternionic $q\times q$-matrices. $\boldsymbol{Sp}(q)$ contains $\boldsymbol{U}(q)$, and $\boldsymbol{U}(q)$ contains the maximal torus $\boldsymbol{T}$ of 15.1 which is also maximal in $\boldsymbol{Sp}(q)$. When applying Theorem 10.7, we let ξ be the universal principal $\boldsymbol{Sp}(q)$-bundle. Putting

$$G = \boldsymbol{Sp}(q) \quad \text{and} \quad U = \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(q-1),$$

we have the diagram

$$B_{\boldsymbol{T}} = E_\xi / \boldsymbol{T} \xrightarrow{\rho} B_U \xrightarrow{\sigma} B_G, \tag{10}$$

and the integral cohomology ring of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$ has to be identified with $H^{**}(B_U, \boldsymbol{Z})$ (see 6.1) modulo the ideal $(I_G{}^+)^*$. As in 15.1, we have the elements $x_1, \cdots, x_q \in H^1(\boldsymbol{T}, \boldsymbol{Z})$ which, via the negative transgression, are to be regarded as elements of $H^2(B_{\boldsymbol{T}}, \boldsymbol{Z})$.

By means of (10), the cohomology ring $H^{**}(B_G, \boldsymbol{Z})$ will be considered as a subring of $H^{**}(B_U, \boldsymbol{Z})$, and $H^{**}(B_U, \boldsymbol{Z})$ as a subring of $H^{**}(B_{\boldsymbol{T}}, \boldsymbol{Z})$. The latter ring will be identified with $\boldsymbol{Z}\{x_1, \cdots, x_q\}$. Thus $H^*(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z})$ is the quotient of

$$\boldsymbol{Z}\{x_1{}^2\} \otimes S\{x_2{}^2, \cdots, x_q{}^2\} \tag{11}$$

modulo the ideal $(I_G{}^+)^*$ which is generated by the symmetric power series in $x_1{}^2, \cdots, x_q{}^2$ without constant terms. (We use here essentially, that G and U have no torsion, see [2].)

We restrict ourselves to the calculation of the Pontrjagin classes of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$, i.e., of its tangent bundle. The more general case of the bundle along the fibres is left to the reader.

We do the calculations following a schema which will also be used in other cases.

Roots of $\boldsymbol{Sp}(q)$: $\quad \pm x_i \pm x_j, \pm 2x_k \qquad (1 \leqq i < j \leqq q)$

Roots of $\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(q-1)$: $\quad \pm x_i \pm x_j, \pm 2x_k \qquad (2 \leqq i < j \leqq q)$

Complementary roots:

$$\pm(x_1 - x_2, x_1 - x_3, \cdots, x_1 - x_q; x_1 + x_2, x_1 + x_3, \cdots, x_1 + x_q).$$

We have

$$(1 + x_2{}^2)(1 + x_3{}^2) \cdots (1 + x_q{}^2) = (1 + x_1{}^2)^{-1} \qquad \operatorname{mod} (I_G{}^+)^*.$$

This shows that the r-th elementary symmetric function in the $x_j{}^2$ $(2 \leqq j \leqq q)$ equals $(-1)^r x_1{}^{2r}$ $(\operatorname{mod} (I_G{}^+)^*)$ and that $x_1{}^2$ represents an element $u \in H^4(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z})$ which generates $H^*(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z})$; in particular, u is a generator of the infinite cyclic group $H^4(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z})$.

Introducing an indeterminate z, we have

$$\prod_{j=1}^{q} (z + x_j)(z - x_j) = z^{2q} \qquad \operatorname{mod} (I_G{}^+)^*.$$

Setting $z = 1 + x_1$, this yields

$$(12) \qquad \prod_{j=2}^{q} (1 + x_1 + x_j)(1 + x_1 - x_j) = (1 + x_1)^{2q}(1 + 2x_1)^{-1} \quad \operatorname{mod}(I_G{}^+)^*.$$

Theorem 10.7 shows that the (integral) Pontrjagin class of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$ is represented by the element

$$\prod_{j=2}^{q} (1 + (x_1 + x_j)^2)(1 + (x_1 - x_j)^2).$$

Taking into account that we are dealing with graded rings and that $x_1{}^2$ represents the generator u of $H^4(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z})$, we obtain from (12)

$$(13) \qquad p(\boldsymbol{P}_{q-1}(\boldsymbol{K})) = (1 + u)^{2q}(1 + 4u)^{-1}.$$

15.6. *Application.* Formula (13) implies, in particular, for the first (i.e., four-dimensional) Pontrjagin class of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$ that

$$p_1 = (2q - 4)u.$$

p_1 is different from 0 for $q - 1 > 1$.

Since $\phi^*(p_1) = p_1$ for all diffeomorphisms ϕ of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$ onto itself, we see that, for $q - 1 > 1$, there does not exist a diffeomorphism ϕ with $\phi^*(u) = -u$; i.e., for all ϕ, we have $\phi^*(u) = u$. In particular, all diffeomorphisms preserve orientation ($\phi^*(u^{q-1}) = u^{q-1}$).

This fact on the orientation is obvious for $q - 1 \equiv 0 \pmod 2$ and aribtrary homeomorphisms of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$ onto itself, since then

$$\phi^*(u) = \pm u \text{ implies } \phi^*(u^{q-1}) = u^{q-1}.$$

15.7. *The Stiefel-Whitney class of the quaternionic projective spaces.* We keep essentially the preceding notations and denote by $\boldsymbol{Q}$ the subgroup of elements of order 2 in $\boldsymbol{T}$. For suitable (1-dimensional) generators u_i of $H^*(B_{\boldsymbol{Q}}, \boldsymbol{Z}_2)$, we have [4, § 11]

$$\rho_2{}^*(\boldsymbol{Q}, \boldsymbol{T})(x_i) = u_i{}^2 \qquad (i = 1, \cdots, q),$$

where here the x_i are elements of $H^2(B_{\boldsymbol{T}}, \boldsymbol{Z}_2)$, namely, the reductions mod 2 of the x_i of 15.5. Thus the images of $\rho_2{}^*(\boldsymbol{Q}, \boldsymbol{Sp}(q))$ and $\rho_2{}^*(\boldsymbol{Q}, \boldsymbol{Sp}(1) \times \boldsymbol{Sp}(q-1))$ are, respectively, $S(u_1{}^4, \cdots, u_q{}^4)$ and $\boldsymbol{Z}_2[u_1{}^4] \otimes S(u_2{}^4, \cdots, u_q{}^4)$, and $H^*(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z}_2)$ may be identified with the quotient of the latter ring by the ideal I generated by the symmetric functions in the $u_i{}^4$ of strictly positive degrees; in particular, $u_1{}^4$ represents the generator $\tilde{u}$ of $H^*(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z}_2)$.

By (5.5), the complementary 2-roots are $(u_1 - u_i)$, $(i = 2, \cdots, p)$, each counted with multiplicity 4, and therefore, 11.5 gives for the Stiefel-Whitney class of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$:

$$w = \prod_{i=2}^{q} (1 + u_1 - u_i)^4 = \prod_{i=1}^{q} (1 + u_1{}^4 - u_i{}^4) \bmod I$$

which, modulo I, is equal to $(1 + u_1{}^4)^q$, and hence, finally,

$$w(\boldsymbol{P}_{q-1}(\boldsymbol{K})) = (1 + \tilde{u})^q,$$

where $\tilde{u}$ is the generator of $H^*(\boldsymbol{P}_{q-1}(\boldsymbol{K}), \boldsymbol{Z}_2)$.

The characteristic classes of $\boldsymbol{P}_{q-1}(\boldsymbol{K})$ were calculated in [17] by a different method.

16. Hermitian symmetric spaces.

16.1. We consider here first the homogeneous spaces of the form G/U, where U is the centralizer of a 1-dimensional torus S, has a 1-dimensional center, and G is semi-simple. As was shown in 13.8, such a space admits exactly two invariant complex structures; they are conjugate to each other and equivalent under an automorphism of G.

We may assume that S is defined by $a_1 = \cdots = a_{l-1} = 0$, where $a_1, \cdots, a_l$ are the simple roots with respect to some ordering $\mathcal{S}$. Then (see 13.6, 13.8), the roots of U are the linear combinations of the a_i's with $1 \leqq i \leqq l-1$, and the set Ψ of positive complementary roots is closed and is the root system of one of the two invariant complex structures on G/U, to be denoted by $\mathcal{C}$. Moreover, a root b is in Ψ if and only if, when expressed as a linear combination of the simple roots, the term containing a_l has a strictly positive coefficient. (The space G/U is irreducible hermitian symmetric if and only if G is simple and U is maximal connected. In this case, the coefficients of a_l in the complementary roots are all equal to one and the sum of two elements of Ψ is never a root of G.)

By 14.2, $H^2(G/U, \boldsymbol{Z})$ is infinite cyclic and has a generator g such that $\nu^*(g) = \varpi_l$, where ν is the natural projection of G/T on G/U and ϖ_l is the l-th fundamental weight. If $c_1(G/U)$ denotes the first Chern class of G/U with respect to $\mathcal{C}$, we have then, necessarily, that

$$c_1(G/U) = \lambda(G/U) \cdot g \qquad (\lambda(G/U) \in Z).$$

By 14.7 and 14.8, both g and $c_1(G/U)$ are positive classes, and hence $\lambda(G/U) > 0$. By 10.8, $\nu^*(c_1)$ is equal to the sum of the positive complementary roots, and hence

$$\lambda(G/U) = 2(b, a_l)/(a_l, a_l), \qquad (b = \sum_{a \in \Psi} a).$$

Since the invariant complex structures on G/U are equivalent, $\lambda(G/U)$ does not depend on the choice of the invariant complex structure. Among the spaces considered here are the compact irreducible hermitian symmetric spaces which are divided into six classes, see, e. g., A. Borel, *Bull. Soc. Math. Frances* 80 (1952), 167-182):

I. $\boldsymbol{U}(m+n)/(\boldsymbol{U}(m)\times\boldsymbol{U}(n))$

II. $\boldsymbol{SO}(2n)/\boldsymbol{U}(n)$

III. $\boldsymbol{Sp}(n)/\boldsymbol{U}(n)$

IV. $\boldsymbol{SO}(n+2)/(\boldsymbol{SO}(2)\times\boldsymbol{SO}(n))$, $\qquad (n>2)$

V. $\boldsymbol{E}_6/\boldsymbol{Spin}(10)\times\boldsymbol{T}^1$

VI. $\boldsymbol{E}_7/\boldsymbol{E}_6\times\mathbf{T}^1$.

By the preceding formula, we get for $\lambda(G/U)$ the following values

I. $m+n$, II. $2n-2$, III. $n+1$, IV. n, V. 12, VI. 18.

In the following sections 16.2 to 16.5, we study the non-exceptional types I.-IV. and give formulas for their Chern classes. We also obtain in these cases the values of $\lambda(G/U)$ in a different way. To describe the complex structure on G/U, we choose an ordering having the properties of 13.6. The maximal torus is always chosen in the standard way, i. e., in the cases I, II, III, it is the maximal torus of $\boldsymbol{U}(m+n)$ or $\boldsymbol{U}(n)$ respectively, used in 15.1.

16.2. *The Grassmannian* $\boldsymbol{W}(m,n) = \boldsymbol{U}(m+n)/(\boldsymbol{U}(m)\times\boldsymbol{U}(n))$.

As a system of positive roots of $\boldsymbol{U}(m+n)$, we take

$$\{-x_i + x_j \mid 1 \leqq i < j \leqq m+n\}, \qquad \text{(see 15.1)}.$$

$H^*(\boldsymbol{W}(m,n), \boldsymbol{Z})$ has to be identified with the quotient of

$$(1) \qquad S\{x_1, \cdots, x_m\} \otimes S\{x_{m+1}, \cdots, x_{m+n}\}$$

by the ideal I generated by the symmetric power series in $x_1, \cdots, x_{m+n}$ without constant term. The (total) Chern class of $\boldsymbol{W}(m,n)$ is given by

$$c(\boldsymbol{W}(m,n)) = \prod_{\substack{1\leqq i\leqq m\\ m+1\leqq j\leqq m+n}} (1 - x_i + x_j) \qquad \operatorname{mod} I.$$

In the tensor product (1), we have

$$\prod_{1 \leqq i \leqq m} (1 + x_i)^{-1} = \prod_{m+1 \leqq j \leqq m+n} (1 + x_j) \qquad \bmod I.$$

The r-th elementary symmetric function in the x_i $(1 \leqq i \leqq m)$ represents an element σ_r of $H^*(\boldsymbol{W}(m,n),\boldsymbol{Z})$. The preceding equation shows that the σ_r $(1 \leqq r \leqq m)$ generate $H^*(\boldsymbol{W}(m,n),\boldsymbol{Z})$. (Recall that we are dealing with graded rings.) The element σ_1 generates the infinite cyclic group $H^2(\boldsymbol{W}(m,n),\boldsymbol{Z})$. Using an indeterminate z, we have

$$(2) \qquad \prod_{m+1 \leqq j \leqq m+n} (z + x_j) = z^{m+n} \prod_{1 \leqq i \leqq m} (z + x_i)^{-1}. \qquad \bmod I.$$

By replacing in the preceding formula z by $1 - x_s$ $(1 \leqq s \leqq m)$, respectively, we obtain m equations; multiplying all of these together yields

$$(3) \qquad c(\boldsymbol{W}(m,n)) = \prod_{i=1}^{m} (1 - x_i)^{m+n} \prod_{1 \leqq i \leqq j \leqq m} (1 - (x_i - x_j)^2)^{-1} \qquad \bmod I.$$

We recall that σ_r is the r-th Chern class of the canonical principal $\boldsymbol{U}(m)$-bundle over $\boldsymbol{W}(m,n)$. Formula (3) expresses $c(\boldsymbol{W}(m,n))$ by the σ_r; for example,

$$c_1(\boldsymbol{W}(m,n)) = -(m+n)\sigma_1,$$

$$c_2(\boldsymbol{W}(m,n)) = (C_2^{m+n} + m - 1)\sigma_1^2 + (n-m)\sigma_2.$$

The formula for the first Chern class gives us the value of $\lambda(\boldsymbol{W}(m,n))$ and shows that $-\sigma_1$ is a positive generator of $H^2(\boldsymbol{W}(m,n),\boldsymbol{Z})$ in the sense of 16.1. For $m = 1$, the Grassmannian $\boldsymbol{W}(1,n)$ is the complex projective space $\boldsymbol{P}_n(\boldsymbol{C})$ discussed in 15.1.

16.3. *The space* $\boldsymbol{F}_n = \boldsymbol{SO}(2n)/\boldsymbol{U}(n)$, $(n \geqq 2)$. As a system of positive roots of $\boldsymbol{SO}(2n)$ we take $\{\pm x_i + x_j \mid 1 \leqq i < j \leqq n\}$. We regard the x_i as elements of $H^2(B_{\boldsymbol{T}}, K_p)$. If $p \neq 2$, then $H^*(\boldsymbol{F}_n, K_p)$ may be identified with the quotient of $S\{x_1,\cdots,x_n\}$ by the ideal I generated by the symmetric power series without constant terms in $x_1^2,\cdots,x_n^2$ and by the element $x_1 x_2 \cdots x_n$. All power series under consideration have coefficients in K_p. The total Chern class of $\boldsymbol{F}_n$ is given by

$$(4) \qquad c(\boldsymbol{F}_n) = \prod_{1<j} (1 + x_i + x_j) \qquad \bmod I.$$

The preceding formula expresses the Chern classes of $\boldsymbol{F}_n$ as polynomials in the elementary symmetric functions of the x_i. The coefficients are integral. Let $\sigma_r \in H^{2r}(\boldsymbol{F}_n, \boldsymbol{Z})$ be the r-th Chern class of the canonical $\boldsymbol{U}(n)$-principal bundle over $\boldsymbol{F}_n$. If we reduce σ_r to coefficients K_p $(p \neq 2)$, we get the r-th elementary symmetric function in the x_i $(\bmod I)$. Since $\boldsymbol{F}_n$ has no torsion, (4) gives a formula for integral cohomology if we replace the r-th elementary symmetric function in the x_i by σ_r. For example, we get

$$c_1(\boldsymbol{F}_n) = (n-1)\sigma_1.$$

σ_1 is not a generator of the infinite cyclic group $H^2(\boldsymbol{F}_n, \boldsymbol{Z})$, but σ_1 equals $2\tilde{g}$, where $\tilde{g}$ is a generator. This is true for $n=2$, since $\boldsymbol{F}_2$ is the complex projective line for which $c_1(\boldsymbol{F}_2)$ is twice a generator. It follows then for all n by induction, using the natural imbedding of $\boldsymbol{F}_n$ in $\boldsymbol{F}_{n+1}$ and the fibre bundle $(\boldsymbol{F}_{n+1}, \boldsymbol{S}_{2n}, \boldsymbol{F}_n)$, see [26, § 41.18]. Thus we have

$$c_1(\boldsymbol{F}_n) = (2n-2)\tilde{g};$$

moreover, $\tilde{g}$ is a positive generator of $H^2(\boldsymbol{F}_n, \boldsymbol{Z})$ and $\lambda(\boldsymbol{F}_n) = 2n-2$ (see 16.1).

16.4. *The space* $\boldsymbol{G}_n = \boldsymbol{Sp}(n)/\boldsymbol{U}(n)$. As a system of positive roots of $\boldsymbol{Sp}(n)$ we take

$$\{\pm x_i + x_j (1 \leqq i < j \leqq n), 2x_i\ (1 \leqq i \leqq n)\}.$$

The integral cohomology ring of $\boldsymbol{G}_n$ has to be identified with the quotient of $S\{x_1, \cdots, x_n\}$ by the ideal I generated by the symmetric power series without constant terms in $x_1{}^2, \cdots, x_n{}^2$. The (total) Chern class of $\boldsymbol{G}_n$ is given by

$$c(\boldsymbol{G}_n) = \prod_{1 \leqq i \leqq j \leqq n} (1 + x_i + x_j) \qquad \bmod I.$$

This formula expresses the Chern classes of $\boldsymbol{G}_n$ by the Chern classes σ_r of the canonical $\boldsymbol{U}(n)$-bundle over $\boldsymbol{G}_n$. The element $x_1 + \cdots + x_n$ represents σ_1, and σ_1 is a generator of the infinite cyclic group $H^2(\boldsymbol{G}_n, \boldsymbol{Z})$; we have

$$c_1(\boldsymbol{G}_n) = (n+1)\sigma_1.$$

Thus σ_1 is a positive generator and $\lambda(\boldsymbol{G}_n) = n+1$ (see (16.1)).

16.5. *The complex quadric* $\boldsymbol{Q}_n = \boldsymbol{SO}(n+2)/(\boldsymbol{SO}(2) \times \boldsymbol{SO}(n))$. We distinguish the two cases (a) n is even and (b) n is odd.

(a) $\quad n + 2 = 2k.$

We have the natural imbedding of $\boldsymbol{U}(k)$ in $\boldsymbol{SO}(2k)$ and take for the maximal torus $\boldsymbol{T}$ of $\boldsymbol{SO}(2k)$ the maximal torus of $\boldsymbol{U}(k)$ considered in 15.1. As a system of positive roots of $\boldsymbol{SO}(2k)$, we choose

$$\{x_i \pm x_j \mid 1 \leqq i < j \leqq k\}.$$

The x_i are to be regarded as elements of $H^2(B_T, K_p)$. If $p \neq 2$, then $H^*(\boldsymbol{Q}_n, K_p)$ may be identified with the quotient of the ring V generated by $S\{x_2^2, \cdots, x_k^2\}$ and by the elements $x_2x_3 \cdots x_k$, x_1 in $K_p\{x_1, x_2, \cdots, x_k\}$ by the ideal I generated by the symmetric power series without constant terms in $x_1^2, \cdots, x_k^2$ and by $x_1x_2 \cdots x_k$; i.e.,

$$H^*(\boldsymbol{Q}_n, K_p) = V/I.$$

Using an indeterminate z, we have

$$(5) \qquad \prod_{i=1}^{k} (z - x_i)(z + x_i) = z^{2k} \qquad \operatorname{mod} I.$$

We have also

$$(6) \qquad (1 + x_2^2)(1 + x_3^2) \cdots (1 + x_k^2) = (1 + x_1^2)^{-1} \qquad \operatorname{mod} I.$$

From (6), we see easily that the elements

$$1, x_1, x_1^2, \cdots, x_1^n, \text{ and } x_2x_3 \cdots x_k \in V$$

constitute an additive base of $H^*(\boldsymbol{Q}_n, K_p)$. The Chern class of $\boldsymbol{Q}_n$ is given by

$$c(\boldsymbol{Q}_n) = \prod_{j=2}^{k} (1 + x_1 - x_j)(1 + x_1 + x_j) \qquad \operatorname{mod} I.$$

Replacing in (5) the indeterminate z by $1 + x_1$, yields

$$(7\text{a}) \qquad c(\boldsymbol{Q}_n) = (1 + x_1)^{n+2}(1 + 2x_1)^{-1} \qquad \operatorname{mod} I.$$

$$(\text{b}) \qquad n + 1 = 2k.$$

We have the imbedding of $\boldsymbol{U}(k) = \boldsymbol{U}(k) \times 1$ in $\boldsymbol{SO}(2k+1)$ and take for the maximal torus $\boldsymbol{T}$ of $\boldsymbol{SO}(2k+1)$ the maximal torus of $\boldsymbol{U}(k)$ considered in 15.1. As a system of positive roots of $\boldsymbol{SO}(2k+1)$, we choose

$$\{x_i \pm x_j \ (1 \leqq i < j \leqq k)\,;\, x_i \ (1 \leqq i \leqq k)\}.$$

If $p \neq 2$, then $H^*(\boldsymbol{Q}_n, K_p)$ may be identified with the quotient of

$$K_p\{x_1\} \otimes S\{x_2^2, \cdots, x_k^2\}$$

by the ideal I generated by the symmetric power series without constant terms in $x_1^2, \cdots, x_k^2$. As in the case (a), we see that

$$1, x_1, x_1^2, \cdots, x_1^n$$

constitute an additive base of $H^*(\boldsymbol{Q}_n, K_p)$. The Chern class of $\boldsymbol{Q}_n$ is given by

$$c(\boldsymbol{Q}_n) = (1 + x_1) \prod_{j=2}^{k} (1 + x_1 - x_j)(1 + x_1 + x_j), \qquad \operatorname{mod} I.$$

As in (a), we get

(7b) $$c(\boldsymbol{Q}_n) = (1+x_1)(1+x_1)^{2k}(1+2x_1)^{-1} = (1+x_1)^{n+2}(1+2x_1)^{-1} \mod I.$$

Now we combine again the two cases (a) and (b). Let $\tilde{g}$ be the Euler class of the canonical principal $\boldsymbol{SO}(2)$-bundle over $\boldsymbol{Q}_n = \boldsymbol{SO}(n+2)/(\boldsymbol{SO}(2) \times \boldsymbol{SO}(n))$,

$$\tilde{g} \in H^2(\boldsymbol{Q}_n, \boldsymbol{Z}).$$

If we apply the coefficient homomorphism $\boldsymbol{Z} \to K_p$, the element $\tilde{g}$ goes over into x_1 (mod I). Since $\boldsymbol{Q}_n$ has no torsion, we get from (7a) and (7b)

(8) $$c(\boldsymbol{Q}_n) = (1+\tilde{g})^{n+2}(1+2\tilde{g})^{-1}.$$

For the Pontrjagin class, we obtain

(9) $$p(\boldsymbol{Q}_n) = (1+\tilde{g}^2)^{n+2}(1+4\tilde{g}^2)^{-1}.$$

The Euler number $E(\boldsymbol{Q}_n)$ equals $n+2$ in the case (a), resp. $n+1$ in the case (b). An easy calculation shows that (8) implies

$$2c_n(\boldsymbol{Q}_n) = E(\boldsymbol{Q}_n)\tilde{g}^n.$$

Therefore $\tilde{g}^n$ is twice a generator of $H^n(\boldsymbol{Q}_n, \boldsymbol{Z})$ and it follows that $\tilde{g}$ is a generator of $H^2(\boldsymbol{Q}_n, \boldsymbol{Z})$ for $n > 2$. Formula (8) shows that $\tilde{g}$ is the positive generator g of 16.1 and that, for $n > 2$, $\lambda(\boldsymbol{Q}_n)$ equals n. For $n = 1$ the quadric $\boldsymbol{Q}_n$ is the complex projective line; for $n = 2$ it is reducible, namely the product of two projective lines. For $n \neq 2$, $\boldsymbol{Q}_n$ is irreducible.

(8) can, of course, be derived by other methods (see, e.g., F. Hirzebruch, Proc. Intern. Congress Math. 1954, Vol. III, pp. 457-473).

16.6. The homogeneous space $\boldsymbol{Q}_n = \boldsymbol{SO}(n+2)/(\boldsymbol{SO}(2) \times \boldsymbol{SO}(n))$ can be regarded as the space of oriented planes through the origin of $\boldsymbol{R}^{n+2}$. In this section, we assume $n > 2$. If one attaches to each oriented plane the same plane with the opposite orientation, one gets a one-one real analytic map σ of $\boldsymbol{Q}_n$ onto itself. σ has no fixed points and is involutive ($\sigma\sigma = \mathrm{Id}$). Identifying the points u and $\sigma(u)$ of $\boldsymbol{Q}_n$ gives a manifold

$$\tilde{\boldsymbol{Q}}_n = \boldsymbol{Q}_n/\sigma, \qquad \pi\colon \boldsymbol{Q}_n \to \tilde{\boldsymbol{Q}}_n.$$

Here π denotes the covering map. $\boldsymbol{Q}_n$ is simply connected and is a twofold covering of $\tilde{\boldsymbol{Q}}_n$. The manifold $\tilde{\boldsymbol{Q}}_n$ is the space of all (non-oriented) planes through the origin of $\boldsymbol{R}^{n+2}$, or the space of all projective lines in $\boldsymbol{P}_{n+1}(\boldsymbol{R})$. For $p \neq 2$, it is known that π^* maps $H^*(\tilde{\boldsymbol{Q}}_n, K_p)$ isomorphically onto the subring of those elements of $H^*(\boldsymbol{Q}_n, K_p)$ which are invariant under σ^*. For the Pontrjagin class of $\tilde{\boldsymbol{Q}}_n$, we have

$$\pi^*(p(\tilde{\boldsymbol{Q}}_n)) = p(\boldsymbol{Q}_n) = (1+g^2)^{n+2}(1+4g^2)^{-1}, \tag{10}$$

where g is a generator of $H^2(\boldsymbol{Q}_n, \boldsymbol{Z})$. By (10), the Pontrjagin class of $\tilde{\boldsymbol{Q}}_n$ with coefficients reduced to K_p $(p \neq 2)$ is completely given.

Let a be the following element of $\boldsymbol{SO}(n+2)$: a is a diagonal matrix which has the entry -1 at the first and third places in the diagonal and otherwise $+1$. Then a is in the normalizer of $\boldsymbol{SO}(2) \times \boldsymbol{SO}(n)$. The operation of a by right translation on $\boldsymbol{Q}_n$ is just σ. (We may remark here that σ is a map which carries one of the homogeneous complex structures into the other.) On the other hand, a induces the operation of the Weyl group which maps x_1, x_2 in $-x_1, -x_2$ and keeps the other x_i fixed. Thus we know how σ^* operates on the additive bases of $H^*(\boldsymbol{Q}_n, K_p)$ given in (a), resp. (b). In either case, the ring of invariants of σ^* is generated by $x_1{}^2$. If n is odd, then $\tilde{\boldsymbol{Q}}_n$ is non-orientable. For $n = 2m$, we see that $\tilde{\boldsymbol{Q}}_n$ is orientable and that $H^*(\boldsymbol{P}_m(\boldsymbol{K}), K_p)$ is isomorphic to $H^*(\tilde{\boldsymbol{Q}}_n, K_p)$. The isomorphism

$$\alpha_p \colon H^*(\boldsymbol{P}_m(\boldsymbol{K}), K_p) \to H^*(\tilde{\boldsymbol{Q}}_n, K_p)$$

can be chosen such that

$$(\pi^* \circ \alpha_p)(u_p) = x_1{}^2 (\operatorname{mod} I) = g^2 \ \text{(reduced to } K_p),$$

where u_p is the reduction to coefficients K_p of the generator $u \in H^4(\boldsymbol{P}_m(\boldsymbol{K}), \boldsymbol{Z})$ used in 15.5. Under the isomorphism α_p, the Pontrjagin class (coefficients K_p) of $\boldsymbol{P}_m(\boldsymbol{K})$ is carried over into that of $\tilde{\boldsymbol{Q}}_n$ (see 15.5 and (10)). The value of $\alpha_0(u_0{}^m)$ on the fundamental cycle of $\tilde{\boldsymbol{Q}}_n$ equals ± 1, since g^n takes the value ± 2 on the fundamental cycle of $\boldsymbol{Q}_n$. Therefore, when using proper orientations, the Pontrjagin numbers of $\boldsymbol{P}_m(K)$ equal those of $\tilde{\boldsymbol{Q}}_n$.

The proof in [17] of the fact that $\boldsymbol{P}_m(\boldsymbol{K})$, for $m \neq 2, 3$, does not admit an almost complex structure, works also for $\tilde{\boldsymbol{Q}}_{2m}$ and thus shows that $\tilde{\boldsymbol{Q}}_{2m}$ $(m \neq 2, 3)$ does not admit an almost complex structure (compatible with its usual differentiable structure).

16.7. *Remark.* Let us consider $\boldsymbol{Q}_n$ as imbedded in $\boldsymbol{P}_{n+1}(\boldsymbol{C})$ by the equation

$$z_0{}^2 + z_1{}^2 + \cdots + z_{n+1}{}^2 = 0.$$

The conjugation map

$$z = (z_0, \cdots, z_{n+1}) \to \bar{z} = (z_0, \cdots, z_{n+1})$$

induces a map κ of $\boldsymbol{Q}_n$ onto itself which has no fixed point and which is involutive. If $z \in \boldsymbol{Q}_n$, then the line passing through z, κz is a real line. If we attach to the point z this real line, then we get a homeomorphism between $\boldsymbol{Q}_n/\kappa$ and the space of the projective lines in $\boldsymbol{P}_{n+1}(\boldsymbol{R})$. The map κ corresponds to σ.

17. The Stiefel-Whitney class of $G_2/SO(4)$.

17.1. We recall first some known properties of $\boldsymbol{G}_2$ and of the Cayley numbers. The Cayley-Graves algebra of octonions over the real numbers will be denoted by $\mathfrak{L}$. It is spanned by 1 and seven purely imaginary elements e_i $(i \in \boldsymbol{Z}_7)$ satisfying

$$(1) \qquad e_i \cdot e_{i+1} = e_{i+3}\,;\, e_{i+1} \cdot e_{i+3} = e_i\,;\, e_{i+3} \cdot e_i = e_{i+1} \qquad (i \in \boldsymbol{Z}_7),$$

and, of course, $e_i \cdot e_i = -1$. Thus e_i, e_{i+1}, e_{i+3} generate a subalgebra isomorphic to the field of quaternionic numbers.

$\boldsymbol{G}_2$ may be defined as the group of automorphism of $\mathfrak{L}$. It is a compact, connected, and simply connected 14-dimensional Lie group of rank 2 and with center reduced to the identity; it leaves invariant the subspace $\mathfrak{M}$ of $\mathfrak{L}$ spanned by the e_i's and thus may be identified with a subgroup of $\boldsymbol{SO}(7)$, whose Lie algebra is the following:

Let G_{ij} $(1 \leqq i, j \leqq 7)$ be the endomorphisms of $\mathfrak{M}$ defined by $G_{ii} = 0$ and

$$G_{ij}(e_j) = e_i\,;\, G_{ij}(e_i) = -e_j\,;\, G_{ij}(e_k) = 0 \qquad (i \neq j\,;\, k \neq i, j).$$

The G_{ij} $(i < j)$ form a basis of the Lie algebra of $\boldsymbol{SO}(7)$. We have

$$(2) \qquad \begin{gathered} G_{ij} + G_{ji} = 0,\, [G_{ij}, G_{jk}] = G_{ik} \qquad (i \neq j\,;\, j \neq k), \\ [G_{ij}, G_{kl}] = 0 \qquad (i, j, k, l \text{ pairwise distinct}). \end{gathered}$$

Using (1) and (2), it is readily seen that the Lie algebra $\mathfrak{g}_2$ of $\boldsymbol{G}_2$, that is, the Lie algebra of derivations of $\mathfrak{L}$, is the direct sum of the seven 2-dimensional commutative subalgebras[8]

$$(3) \qquad \mathfrak{d}_i = \{aG_{i+1,i+3} + bG_{i+2,i+6} + cG_{i+4,i+5}\,;\, a + b + c = 0\}$$

and that we have

$$(4) \qquad [\mathfrak{d}_i, \mathfrak{d}_{i+1}] = \mathfrak{d}_{i+3}\,; \quad [\mathfrak{d}_{i+1}, \mathfrak{d}_{i+3}] = \mathfrak{d}_i\,; \quad [\mathfrak{d}_{i+3}, \mathfrak{d}_i] = \mathfrak{d}_{i+1} \qquad (i \in \boldsymbol{Z}_7);$$

[8] For more details, see H. Freudenthal, "Oktavengeometrie," mimeographed Notes, University of Utrecht. Freudenthal's $e_1, \cdots, e_7$ are replaced here by $e_1, e_4, e_2, e_7, -e_3, e_5, -e_6$ respectively. On page 17, line 14 from the bottom, read G_{76} instead of G_{67}.

The subspace $\mathfrak{u} = \mathfrak{v}_3 + \mathfrak{v}_4 + \mathfrak{v}_6$ is therefore a subalgebra of rank 2 and dimension 6, and hence is isomorphic to the Lie algebra of $\boldsymbol{SO}(4)$, as can also be easily checked directly. Thus the subgroup U of $\boldsymbol{G}_2$ generated by $\mathfrak{u}$ is a compact group locally isomorphic to $\boldsymbol{SO}(4)$, that is, having the product $\boldsymbol{Sp}(1) \times \boldsymbol{Sp}(1)$ as universal covering. It is, in fact, globally isomorphic to $\boldsymbol{SO}(4)$, as can be derived for instance from [7]. In fact, it is proved in [7] that $\boldsymbol{G}_2$ contains exactly one class (relative to inner automorphisms) of subgroups locally isomorphic to $\boldsymbol{SO}(4)$; one of them, say U_1, is the centralizer of a vertex z of order two of a fundamental simplex, and z generates the center of U_1 ([7], Remarque II, p. 220). Looking at the diagram of $\boldsymbol{G}_2$, one sees that the two invariant 3-dimensional subgroups of U_1 are globally isomorphic to $\boldsymbol{Sp}(1)$; since U_1 has a center of order two, it is then isomorphic to $\boldsymbol{SO}(4)$.

17.2. The subgroup $\boldsymbol{Q}$. The relations (1) imply that the linear transformation S_i $(i \in \boldsymbol{Z}_7)$ of $\mathfrak{M}$ which keeps e_{i+1}, e_{i+5}, e_{i+6} fixed and changes the signs of the other e_j's is an automorphism of $\mathfrak{L}$. The seven elements S_i and the identity form a commutative subgroup $\boldsymbol{Q}$ of $\boldsymbol{G}_2$ of type $(2,2,2)$. Moreover, $\boldsymbol{G}_2$ contains no commutative subgroup of type $(2,2,2,2)$ (see A. Borel-J-P. Serre, Comm. Math. Helv. 27 (1953), 128-129 or 17.5).

Proposition. *We keep the previous notations and denote by x_i the element of* $\operatorname{Hom}(\boldsymbol{Q},\boldsymbol{Z}_2)$ *defined by* $x_i(S_j) = \delta_{ij}$ $(1 \leqq i, j \leqq 3)$. *Then $Q \subset U$, and the 2-roots of U (resp. $\boldsymbol{G}_2$) with respect to $\boldsymbol{Q}$ are $x_1 + x_2$, $x_1 + x_3$, $x_2 + x_3$ (resp. together with $x_1, x_2, x_3, x_1 + x_2 + x_3$). Each has multiplicity* 2 *and is the character of $\boldsymbol{Q}$ in one of the $\mathfrak{v}_i$'s.*

It follows from the definition of U that this group leaves invariant the subspaces $\mathfrak{C}$, $\mathfrak{D}$ of $\mathfrak{M}$ spanned, respectively, by e_3, e_4, e_6 and e_1, e_2, e_5, e_7 and that the restriction to $\mathfrak{D}$ of the standard maximal abelian subgroup $\boldsymbol{Q}'$ of type $(2,2,2)$ of U consists of the diagonal matrices of determinant $+1$. On the other hand, it is readily seen that this is also the restriction of $\boldsymbol{Q}$ to $\mathfrak{D}$; since by (1) an automorphism of $\mathfrak{L}$ leaving e_1, e_2, e_5, e_7 fixed must be the identity, we have $\boldsymbol{Q}' = \boldsymbol{Q}$ and $\boldsymbol{Q} \subset U$. The other assertions follow from the fact that the inner automorphism $\operatorname{Ad} S_i$ defined by S_i is the identity on $\mathfrak{v}_{i+1}$, $\mathfrak{v}_{i+5}$, $\mathfrak{v}_{i+6}$ and is $-\mathrm{Id}$ on the other $\mathfrak{v}_j$'s.

17.3. *The cohomology ring* mod 2 *of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$.* The following facts are proved in [3]: $H^*(B_{\boldsymbol{SO}(4)}, \boldsymbol{Z}_2)$ and $H^*(B_{\boldsymbol{G}_2}, \boldsymbol{Z}_2)$ are rings of polynomials in three variables of degrees $2, 3, 4$ and $4, 6, 7$ respectively; the homomorphisms $\rho_2{}^*(\boldsymbol{Q}, \boldsymbol{SO}(4))$ and $\rho_2{}^*(\boldsymbol{Q}, \boldsymbol{G}_2)$ are injective. The ring $H^*(\boldsymbol{G}_2/\boldsymbol{SO}(4), \boldsymbol{Z}_2)$

is the quotient of $H^*(B_{SO(4)},\boldsymbol{Z}_2)$ by the ideal generated by the elements of strictly positive degrees in the image of $\rho_2{}^*(\boldsymbol{SO}(4),\boldsymbol{G}_2)$; the Poincaré polynomial mod 2 of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ is

$$P_2(\boldsymbol{G}_2/\boldsymbol{SO}(4),t) = (1-t^4)(1-t^6)(1-t^7)/(1-t^2)(1-t^3)(1-t^4),$$

and hence

$$P_2(\boldsymbol{G}_2/\boldsymbol{SO}(4),t) = 1+t^2+t^3+t^4+t^5+t^6+t^8.$$

PROPOSITION. *We keep the previous notations and denote by σ_i the i-th elementary symmetric function in the x_i's. Then the image of $\rho_2{}^*(\boldsymbol{Q},U)$ equals $\boldsymbol{Z}_2[u_2,u_3,u_4]$ with $u_2=\sigma_2+\sigma_1{}^2$, $u_3=\sigma_3+\sigma_1\sigma_2$, $u_4=\sigma_1\sigma_3$, and the image of $\rho_2{}^*(\boldsymbol{Q},\boldsymbol{G}_2)$ equals $\boldsymbol{Z}_2[g_4,g_6,g_7]$ with $g_4=u_2{}^2+u_4$, $g_6=u_3{}^2+u_2u_4$ and $g_7=u_4u_3$. Consequently, $H^*(\boldsymbol{G}_2/\boldsymbol{SO}(4),\boldsymbol{Z}_2)$ is generated by two elements u_2,u_3 of degrees 2, 3 with the relations $u_2{}^3=u_3{}^2$ and $u_3u_2{}^2=0$.*

We identify U with $\boldsymbol{SO}(4)$ by means of the representation in $\mathfrak{D}$. Let $\boldsymbol{Q}_1$ be the subgroup of diagonal matrices of $\boldsymbol{O}(4)$ and μ the inclusion of $\boldsymbol{Q}$ in $\boldsymbol{Q}_1$. Then, for an obvious choice of a basis (y_1,y_2,y_3,y_4) of $\mathrm{Hom}(\boldsymbol{Q}_1,\boldsymbol{Z}_2)$, the image of $\rho_2{}^*(\boldsymbol{Q}_1,\boldsymbol{O}(4)$ is the ring of symmetric functions in the y_i (cf. [3]), and the homomorphism μ': $\mathrm{Hom}(\boldsymbol{Q}_1,\boldsymbol{Z}_2)\to\mathrm{Hom}(\boldsymbol{Q},\boldsymbol{Z}_2)$ induced by μ is given by

$$\mu'(y_i)=x_i\ \ (i=1,2,3),\ \ \mu'(y_4)=x_1+x_2+x_3.$$

Therefore, μ' annihilates $y_1+y_2+y_3+y_4$ and maps the i-th elementary symmetric function in the y_j's onto u_i, for $i=2,3,4$. Since $\rho_2{}^*(Q,\boldsymbol{O}(4))$ and $\rho_2{}^*(Q,\boldsymbol{SO}(4))$ have the same image (see [3]), this proves our first assertion.

The image of $\rho_2{}^*(\boldsymbol{Q},\boldsymbol{G}_2)$ is a subring of $\boldsymbol{Z}_2[u_2,u_3,u_4]$ generated by elements of degrees 4, 6, 7 and its elements are invariant under the action of the normalizer of $\boldsymbol{Q}$ in $\boldsymbol{G}_2$, operating in the usual way [3]; therefore, in order to prove the second assertion, it suffices to exhibit an automorphism α of $H^*(B_Q,\boldsymbol{Z}_2)=\boldsymbol{Z}_2[x_1,x_2,x_3]$ induced by an inner automorphism of $\boldsymbol{G}_2$, leaving $\boldsymbol{Q}$ invariant, and for which g_i is the only non-zero invariant of degree i $(i=4,6,7)$.

Let S be the linear transformation of $\mathfrak{M}$ which sends $e_1,\cdots,e_7$ onto $e_5,e_7,-e_3,e_4,e_1,-e_6,e_2$ respectively. It follows from (1) that $S\in\boldsymbol{G}_2$. Moreover, it is seen without difficulty that

$$S\cdot S_1\cdot S=S_1,\ S\cdot S_2\cdot S=S_1\cdot S_3=S_4,\ S\cdot S_3\cdot S=S_1\cdot S_2=S_6$$

19

and, therefore, the automorphism α of $H^*(B_Q, Z_2)$ induced by $\mathrm{Ad}\, S$ satisfies:

$$(5) \qquad \alpha(x_1) = x_1, \quad \alpha(x_2) = x_1 + x_3, \quad \alpha(x_3) = x_1 + x_2,$$

$$(6) \qquad \alpha(\sigma_1) = \sigma_1, \quad \alpha(\sigma_2) = x_1^2 + x_2 x_3, \quad \alpha(\sigma_3) = x_1(x_1^2 + \sigma_2),$$

$$\alpha(u_2) = x_2^2 + x_3^2 + x_2 x_3,$$

$$(7) \qquad \alpha(u_3) = x_2 x_3 (x_2 + x_3),$$

$$\alpha(u_4) = x_1(x_1^2 + \sigma_2)\sigma_1.$$

An element $h \in H^4(B_U, Z_2)$ may be written in the form $h = a \cdot u_4 + b \cdot u_2^2$ $(a, b \in Z_2)$, and hence

$$\alpha(h) = a x_1 (x_1^2 + \sigma_2)\sigma_1 + b x_2^2 x_3^2 + b(x_2^4 + x_3^4).$$

The coefficients of x_1^4 in h and $\alpha(h)$ are b and a respectively; therefore, if $h = \alpha(h)$ with $h \neq 0$ we must have $a = b = 1$ and $h = g_4$. That g_4 is in fact invariant under α can be checked directly, but this is not necessary since we know a priori that $H^4(B_{G_2}, Z_2)$ has dimension one.

The proofs of the invariance of g_6, g_7 under α are quite analogous: an element of degree six may be written

$$h = a u_2^3 + b u_3^2 + c u_2 g_4 \qquad (a, b, c \in Z_2).$$

Using (7), one sees that the coefficients of x_1^6 in h and $\alpha(h)$ are $a + c$ and zero respectively, while those of $x_1^4 \cdot x_2^2$ are $a + b + c$ and c. Thus $\alpha(h) = h$ and $h \neq 0$ imply $a = b = c = 1$ and $h = g_6$. Finally, starting with a general element

$$h = u_3(a u_2^2 + b g_4) \qquad (a, b \in Z_2)$$

of degree seven, we see by looking at the coefficients of $x_1^6 \cdot x_2$ that $h = \alpha(h)$ and $h \neq 0$ imply $a = b = 1$, that is $h = g_7$.

The last assertion follows then from the results recalled at the beginning of 17.3.

17.4. PROPOSITION. *The Stiefel-Whitney classes of* $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ *are non-zero only in the dimensions* 0, 4, 6, 8.

By 11.5 and 17.2, the image under $\rho_2^*(\boldsymbol{Q}, U)$ of the total Stiefel-Whitney class of the bundle along the fibres of $(B_U, B_{G_2}, \boldsymbol{G}/U)$ is

$$w' = (1 + \sigma_1)^2 \cdot \prod_1^3 (1 + x_i)^2;$$

therefore

$$w' = (1+\sigma_1{}^2+\sigma_2{}^2+\sigma_3{}^2)(1+\sigma_1{}^2) = 1+u_2{}^2+u_3{}^2+u_4{}^2,$$

and 17.4 follows now from 17.3.

17.5. *Remarks.* 1) The 2-roots of $\boldsymbol{G}_2$ have been computed with respect to a particular subgroup of type $(2,2,2)$. In fact, these subgroups are conjugate by inner automorphisms, and these 2-roots are therefore invariants of $\boldsymbol{G}_2$. To see this, one uses the fact that given three orthonormal purely imaginary Cayley numbers u, v, w with w also orthogonal to $u\cdot v$, there exists exactly one automorphism of $\mathfrak{K}$ which maps e_1, e_2, e_3 onto u, v, w, respectively (see N. Jacobson, *Duke Math. Jour.* 5 (1939), 776-783). This implies easily that any commutative subgroup of type $(2,2,2)$ of $\boldsymbol{G}_2$ can be put in the diagonal form by means of an automorphism of $\boldsymbol{G}_2$. Moreover, one deduces from (1) that $\boldsymbol{Q}$ contains all diagonal matrices of $\boldsymbol{G}_2$; hence, $\boldsymbol{G}_2$ does not contain commutative subgroups of type $(2,2,2,2)$.

(2) It follows a posteriori from the proof of 17.3 that $\rho_{20}(\boldsymbol{Q},\boldsymbol{G}_2)$ maps $H^*(B_{\boldsymbol{G}_2},\boldsymbol{Z}_2)$ isomorphically onto the ring of invariants of the normalizer of $\boldsymbol{Q}$ in $\boldsymbol{G}_2$. Thus, the analogy between the role of $\boldsymbol{Q}$ in cohomology mod 2 and that of a maximal torus in real cohomology, which is the basis of [3], is also very complete for $\boldsymbol{G}_2$.

18. Some manifolds with Poincaré polynomial $1+t^4+t^8$.

18.1. The quaternionic plane $\boldsymbol{P}_2(\boldsymbol{K})$ is an 8-dimensional manifold with real Poincaré polynomial $1+t^4+t^8$. We know (15.5) that the (integral) Pontrjagin class of $\boldsymbol{P}_2(\boldsymbol{K})$ is given by

$$p = (1+u)^6(1+4u)^{-1}.$$

Thus $p_1 = 2u$ and $p_2 = 7u^2$. The Pontrjagin numbers of $\boldsymbol{P}_2(\boldsymbol{K})$ are

$$p_1{}^2[\boldsymbol{P}_2(\boldsymbol{K})] = 4 \quad \text{and} \quad p_2[\boldsymbol{P}_2(\boldsymbol{K})] = 7.$$

Here we use the orientation defined by $u^2[\boldsymbol{P}_2(\boldsymbol{K})] = 1$.

18.2. The manifolds $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ (see § 17) and $\tilde{\boldsymbol{Q}}_4$ (see 16.6) have the real Poincaré polynomial $1+t^4+t^8$. Their Pontrjagin numbers are (for suitable orientations) the same as those of the quaternionic plane. For $\tilde{\boldsymbol{Q}}_4$, this was proved already in 16.6. In the next section, it will be shown for $\boldsymbol{G}_2/\boldsymbol{SO}(4)$. We do not know whether all differentiable manifolds with real Poincaré polynomial $1 + t^4 + t^8$ have the same Pontrjagin numbers. By the index theorem ([19], p. 85), we know for such a manifold X that (for suitable orientation)

$$(7p_2 - p_1{}^2)[X] = 45, \tag{1}$$

and therefore, it is sufficient to calculate one Pontrjagin number. But, as an example, we shall make the computations without the use of the index theorem in the case of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$.

Remark. Milnor has constructed an 8-dimensional combinatorial manifold with real Poincaré polynomial $1 + t^4 + t^8$ whose Pontrjagin numbers (in the sense of Thom) satisfy (1), but are rational, non-integral numbers. This manifold of Milnor does not admit a differentiable structure compatible with its combinatorial structure.

18.3. By the Hirsch formula, the manifold $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ has the real Poincaré polynomial $(1-t^4)(1-t^{12})(1-t^4)^{-1}(1-t^4)^{-1} = 1 + t^4 + t^8$. We calculate the Pontrjagin class of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ by the schema used in 15.5. All cohomology groups are taken with real coefficients.

Roots of $\boldsymbol{G}_2$:

$$\pm x_1, \pm x_2, \pm(x_1 - x_2), \pm(x_1 - 2x_2), \pm(x_1 - 3x_2), \pm(2x_1 - 3x_2)$$

with respect to a convenient base $x_1, x_2 \in H^2(\boldsymbol{G}_2/T)$ for a maximal torus T of $\boldsymbol{G}_2$.

Following de Siebenthal [25a] we take $\phi_1 = x_2$ and $\phi_2 = x_1 - 3x_2$ as simple roots of $\boldsymbol{G}_2$. The dominant root is then $3\phi_1 + 2\phi_2 = 2x_1 - 3x_2$. By [7, p. 218], we know that there is an imbedding of $\boldsymbol{SO}(4)$ in $\boldsymbol{G}_2$ for which $\pm\phi_1$ and $\pm(3\phi_1 + 2\phi_2)$ are the roots of $\boldsymbol{SO}(4)$.

Complementary roots: $\pm x_1, \pm(x_1 - x_2), \pm(x_1 - 2x_2), \pm(x_1 - 3x_2)$.

We put $y_1 = 2x_1 - 3x_2$ and $y_2 = x_2$.

Invariants of the Weyl group of $\boldsymbol{G}_2$: Since $H^*(B_{\boldsymbol{G}_2})$ is the polynomial ring over $\boldsymbol{R}$ in two indeterminates of degrees 4 and 12, we have only one invariant in dimension 4. Since the dimension of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ equals 8, this is the only invariant we need. An invariant of dimension 4 is always given by the sum of the squares of all roots, which, up to a factor, is, in our case,

$$4(x_1{}^2 - 3x_1x_2 + 3x_2{}^2) = (y_1{}^2 + 3y_2{}^2).$$

It is convenient to express the complementary roots as linear combinations of y_1, y_2. This gives

$$x_1 = \tfrac{1}{2}(y_1 + 3y_2),$$
$$x_1 - x_2 = \tfrac{1}{2}(y_1 + y_2),$$
$$x_1 - 2x_2 = \tfrac{1}{2}(y_1 - y_2),$$
$$x_1 - 3x_2 = \tfrac{1}{2}(y_1 - 3y_2).$$

Euler class W of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$:

$$\pm 16W = (y_1^2 - y_2^2)(y_1^2 - 9y_2^2) = 4y_2^2 \cdot 12y_2^2,$$
$$\pm W = 3y_2^4,$$

the computations being made modulo the invariants. Since the Euler number of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ equals 3, we get

$$y_2^4[\boldsymbol{G}_2/\boldsymbol{SO}(4)] = 1 \tag{2}$$

after choosing the orientation of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ conveniently.

The Pontrjagin class p of $\boldsymbol{G}_2/\boldsymbol{SO}(4)$:

$$\begin{aligned} 1 + 4p_1 + 16p_2 & \\ &= (1 + (y_1 + 3y_2)^2)(1 + (y_1 - 3y_2)^2)(1 + (y_1 - y_2)^2)(1 + (y_1 + y_2)^2) \\ &= (1 + 2(y_1^2 + 9y_2^2) + (y_1^2 - 9y_2^2)^2)(1 + 2(y_1^2 + y_2^2) + (y_1^2 - y_2^2)^2) \\ &= (1 + 12y_2^2 + 144y_2^4)(1 - 4y_2^2 + 16y_2^4). \end{aligned}$$

$$1 + p_1 + p_2 = (1 + 3y_2^2 + 9y_2^4)(1 - y_2^2 + y_2^4),$$
$$p_1 = 2y_2^2, \qquad p_2 = 7y_2^4.$$

(All calculations modulo the invariants.)

By (2), we get for the Pontrjagin numbers (with respect to the orientation defined by (2))

$$p_1^2[\boldsymbol{G}_2/\boldsymbol{SO}(4)] = 4, \qquad p_2[\boldsymbol{G}_2/\boldsymbol{SO}(4)] = 7.$$

19. The Cayley plane.

19.1. The center of the simply connected representative of the local structure $\boldsymbol{F}_4$ consists only of the unit element, as is well known. The structure $\boldsymbol{F}_4$ has, therefore, one and only one representative which we also denote by $\boldsymbol{F}_4$. According to [7], the group $\boldsymbol{F}_4$ contains exactly one class (relative to inner automorphisms) of subgroups with local structure $\boldsymbol{B}_4$. They are, in fact, globally isomorphic to $\boldsymbol{Spin}(9)$. The homogeneous space $\boldsymbol{F}_4/\boldsymbol{Spin}(9)$ has dimension 16 and may be identified with the Cayley plane W, the projective plane over the Cayley-numbers. The Cayley plane $\boldsymbol{W}$, considered as the homogeneous space $\boldsymbol{F}_4/\boldsymbol{Spin}(9)$, was studied, for instance, in [1], see also [2], § 29 and Freudenthal, loc. cit.,[8] § 17. The integral cohomology of $\boldsymbol{W}$ is given by

$$H^0 = H^8 = H^{16} = Z, \qquad H^i = 0 \text{ otherwise.}$$

19.2. Let $\boldsymbol{T}$ be the standard maximal torus of $\boldsymbol{SO}(9)$ with the base $x_1, x_2, x_3, x_4 \in H^1(\boldsymbol{T}, \boldsymbol{Z})$ (see 16.5(b)). Then the roots of $\boldsymbol{SO}(9)$ are

$$(1) \qquad \pm x_i \pm x_j \ (1 \leqq i < j \leqq 4); \ \pm x_1, \pm x_2, \pm x_3, \pm x_4.$$

We have the projection

$$\pi : \boldsymbol{Spin}(9) \to \boldsymbol{SO}(9).$$

$\pi^{-1}(\boldsymbol{T}) = \boldsymbol{T}'$ is a maximal torus of $\boldsymbol{Spin}(9)$. The restriction of π to $\boldsymbol{T}'$ induces an isomorphism of $H^1(\boldsymbol{T}, \boldsymbol{R})$ onto $H^1(\boldsymbol{T}', \boldsymbol{R})$. Thus x_1, x_2, x_3, x_4 may be regarded as elements of $H^1(\boldsymbol{T}', \boldsymbol{R})$. They constitute a base of $H^1(\boldsymbol{T}', \boldsymbol{R})$. By [7, Théorème 4], we can choose an embedding of $\boldsymbol{Spin}(9)$ in $\boldsymbol{F}_4$ such that the roots of $\boldsymbol{F}_4$ with respect to $\boldsymbol{T}'$ (considered as elements of $H^1(\boldsymbol{T}', \boldsymbol{R})$) are those given in (1) together with the following (see, e. g., [25a]):

$$(2) \qquad \tfrac{1}{2}(\pm x_1 \pm x_2 \pm x_3 \pm x_4),$$

which are the roots of $\boldsymbol{F}_4$ complementary to $\boldsymbol{Spin}(9)$. We now regard x_1, x_2, x_3, x_4 as elements of $H^2(B_{\boldsymbol{T}'}, \boldsymbol{R})$. We introduce the elementary symmetric functions $a_1, \cdots, a_4$ in the x_i^2

$$(3) \qquad 1 + a_1 + a_2 + a_3 + a_4 = \prod_{i=1}^{4} (1 + x_i^2).$$

The polynomials

$$(4) \qquad a_1, -6a_3 + a_1 a_2, 12a_4 + a_2^2 - \tfrac{1}{2} a_1^2 a_2$$

are invariants of the Weyl group of $\boldsymbol{F}_4$.

Proof. Since the a_i are invariants of the Weyl group of $\boldsymbol{Spin}(9)$, it suffices to check that the polynomials (4) are invariant under the reflection to the plane

$$x_1 + x_2 + x_3 + x_4 = 0,$$

with respect to the usual Euclidean metric.

19.3. The calculation of the Pontrjagin class of $\boldsymbol{W}$ goes in the same way as for $\boldsymbol{G}_2/\boldsymbol{SO}(4)$ (see § 18). We indicate briefly the various steps of the computation. We put

$$r_i = \tfrac{1}{2}(x_1 \pm x_2 \pm x_3 \pm x_4), \qquad (i = 1, 2, \cdots, 8).$$

For the Pontrjagin class with real coefficients, we get, modulo the invariants (4),

$$p(\boldsymbol{W}) = \prod_{i=1}^{8} (1 + r_i^2) = 1 - a_2 - 13a_4, \text{ i.e.,}$$

$$p_1 = p_3 = 0, \quad p_2 = -a_2, \quad p_4 = -13a_4, \tag{5}$$

$$p_2{}^2 = a_2{}^2 = -12a_4. \tag{6}$$

Let u be a generator of the infinite cyclic group $H^8(\boldsymbol{W}, \boldsymbol{Z})$. Then the Euler class of $\boldsymbol{W}$ equals $\pm 3u^2$. On the other hand, we have, after reducing to real coefficients and modulo the invariants (4),

$$\pm 3u^2 = \prod_{i=1}^{8} r_i = - a_4 = a_2{}^2/12.$$

The preceding equation and (6) yield, since p_2 is a real multiple of u,

$$p_2{}^2 = +36u^2, \qquad p_2 = \pm 6u, \tag{7}$$

$$p_4 = 39u^2. \tag{8}$$

Since $\boldsymbol{W}$ is without torsion, we conclude that (7) and (8) are also true in integral cohomology. We choose the generator u such that $p_2 = 6u$.

19.4. THEOREM. *There exists a generator u of the infinite cyclic group $H^8(\boldsymbol{W}, \boldsymbol{Z})$ such that the integral Pontrjagin classes of W are given by*

$$p_2(\boldsymbol{W}) = 6u, \qquad p_4(\boldsymbol{W}) = 39u^2.$$

Choosing that orientation of $\boldsymbol{W}$ which is defined by u^2, the non-vanishing Pontrjagin numbers of $\boldsymbol{W}$ are

$$p_2{}^2[\boldsymbol{W}] = 36, \qquad p_4[\boldsymbol{W}] = 39. \tag{9}$$

19.5. The manifold $\boldsymbol{W}$, oriented as in 19.4, has the index $\tau(\boldsymbol{W}) = 1$. By the index theorem ([19], Satz 8.2.2), we have

$$(381p_4 - 19p_2{}^2)[\boldsymbol{W}] = 3^4 \cdot 5^2 \cdot 7. \tag{10}$$

We shall see later in this paper by some general arguments that the A-genus ([19], 1.6) of $\boldsymbol{W}$ vanishes. This, together with (10), gives a system of two linear equations for the Pontrjagin numbers from which (9) can also be obtained.

19.6. From Theorem 19.4, we can easily draw the following consequences.

(a) Let $\mathcal{P}_5^1$ be the Steenrod reduced power

$$\mathcal{P}_5^1 \colon H^k(X, \mathbf{Z}_5) \to H^{k+8}(X, \mathbf{Z}_5).$$

For the generator u of 19.4 we have, by [15] (coefficients reduced to $\mathbf{Z}_5$),

$$\mathcal{P}_5^1 u = \tfrac{1}{9}(7p_2 - p_1^2)u = -2p_2 u = -2u^2. \tag{11}$$

(11) implies that, for each homeomorphism ϕ of $\boldsymbol{W}$ onto $\boldsymbol{W}$, we have $\phi^* u = u$.

(b) The manifold $\boldsymbol{W}$ with its usual differentiable structure does not admit an almost complex structure.

Proof.

$$c^2 = (1 + c_4 + c_8)^2 = 1 + 6u + 39u^2$$

would imply

$$c_8 = 15u^2,$$

but, for an almost complex structure, we would have

$$c_8 = \pm 3u^2.$$

(c) The (total) Stiefel-Whitney class w of $\boldsymbol{W}$ is

$$w = 1 + u + u^2 \text{ (coefficients reduced to } \mathbf{Z}_2).$$

Proof. We have (coefficients reduced to $\mathbf{Z}_2$), see 9.2, 9.3 and Appendix II,

$$w_8{}^2 = p_4, \qquad w_{16} = 3u^2 \text{ (Euler class)}.$$

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