

RELATIVE CHARACTER IDENTITIES AND THETA CORRESPONDENCE

WEE TECK GAN AND XIAOLEI WAN

1. Introduction

This paper is inspired by the talk of Yiannis Sakellaridis in the Simons Symposium held at the Schloss Elmau in April 2018. Let us begin by describing the relevant context for his talk.

The study of periods of automorphic forms has been an important theme in the Langlands program, beginning with the early work of Harder-Langlands-Rapoport and Jacquet. In particular, the nonvanishing of certain periods is known to characterize the image of certain Langlands functorial lifting and to be related to the analytic properties of certain automorphic L-functions. An effective approach for proving such results is the technique of relative trace formulae developed by Jacquet. Typically, such an approach involves the comparison of the geometric sides of two relative trace formulae, which results in a global spectral identity and an accompanying family of local relative character identities.

In [35], Sakellaridis and Venkatesh initiated a general framework for treating such period problems in the context of spherical varieties. In particular, to a spherical variety $X = H \backslash G$ over a local field F or a global field k , they associated

- a Langlands dual group X^\vee (at least when G is split), together with a canonical (up to conjugacy) map

$$\iota : X^\vee \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow G^\vee.$$

- a $\frac{1}{2}\mathbb{Z}$ -graded finite-dimensional algebraic representation $V_X = \bigoplus_d V_X^d$ of X^\vee , which gives rise to an L-function

$$L_X(s, \rho) = \prod_d L(s + d, \rho, V_X^d)$$

for each L-parameter ρ valued in X^\vee

They then conjectured, among other things, that representations of G (in the automorphic dual) which have nonzero H -periods are those belonging to A-packets whose associated A-parameters factor through ι . This means roughly that the H -distinguished representations of G are those which are Langlands functorial lift via ι from a (split) group G_X whose dual group G_X^\vee is X^\vee . Experience shows that it is sometimes more pertinent to regard H -distinguished representations of G as lifted from the *Whittaker variety* $(N_X, \psi) \backslash G_X$, as opposed to the group variety G_X itself.

The conjecture of Sakellaridis–Venkatesh can be made on several fronts. We give a brief description of the various incarnations of their conjecture (at least a first approximation), under some simplifying hypotheses and without using the language of A-parameters. Our description is adapted to the needs of this paper. For the conjecture in its most general form (taking into account Vogan L-packets for example), the reader should consult [35].

- (a) In the context of smooth representation theory of $G(F)$ over a local field F , one is interested in determining $\text{Hom}_H(\pi, \mathbb{C})$ for any $\pi \in \text{Irr}(G(F))$. One expects (in the context of this paper) a map

$$\iota_* : \text{Irr}(G_X(F)) \longrightarrow \text{Irr}(G(F)),$$

such that for any $\pi \in \text{Irr}(G(F))$, there is an isomorphism

$$f : \bigoplus_{\sigma: \iota_*(\sigma) = \pi} \text{Hom}_{N_X}(\sigma, \psi) \cong \text{Hom}_H(\pi, \mathbb{C}).$$

In the smooth setting, the Sakellaridis-Venkatesh conjecture thus gives a precise quantitative formulation of the expectation that H -distinguished representations of G are lifted from G_X .

If further ι_* is injective, there is at most one term on the left hand side, and all these Hom spaces are at most one-dimensional (by the uniqueness of Whittaker models). This will be the favourable situation encountered in this paper. In such instances, if $\ell \in \text{Hom}_{N_X}(\sigma, \psi)$, with corresponding $f(\ell) \in \text{Hom}_H(\iota_*(\sigma), \mathbb{C})$, one can define relative characters $\mathcal{B}_{\sigma, \ell}$ and $\mathcal{B}_{\iota_*(\sigma), f(\ell)}$ which are certain equivariant distributions on $(N_X, \psi) \backslash G_X$ and X respectively. In this case, one might expect a relative character identity relating $\mathcal{B}_{\sigma, \ell}$ and $\mathcal{B}_{\iota_*(\sigma), f(\ell)}$.

- (b) In the context of L^2 -representation theory, one is interested in obtaining the spectral decomposition of the unitary representation $L^2(X)$ of G (relative to a fixed G -invariant measure on X). By abstract results of functional analysis, one has a direct integral decomposition

$$L^2(X) \cong \int_{\Omega} \pi_{\omega}^{\oplus m(\omega)} d\mu_X(\omega)$$

where

- $(\Omega, d\mu_X)$ is some measure space;
- $\pi : \omega \mapsto \pi_{\omega}$ is a measurable field of irreducible unitary representations of G defined on Ω , giving rise to a measurable map from Ω to the unitary dual \widehat{G} of G ;
- $m : \Omega \rightarrow \mathbb{N} \cup \{\infty\}$ is a measurable multiplicity function.

There is some fluidity in this direct integral decomposition; for example, given Ω , only the measure class of $d\mu_X$ is well-defined (without explicating the isomorphism).

In this L^2 -setting, the crux of the Sakellaridis-Venkatesh conjecture is to provide a canonical candidate for $(\Omega, d\mu_X, \pi)$. Namely, one expects a map

$$\iota_* : \widehat{G}_X \longrightarrow \widehat{G}$$

associated to ι from the unitary dual of G_X to that of G , so that one has a (unitary) isomorphism

$$L^2(X) \cong \int_{\widehat{G}_X} \iota_*(\sigma)^{\oplus m(\sigma)} d\mu_{G_X}(\sigma),$$

where $d\mu_{G_X}$ denotes the Plancherel measure of G_X and $m(\sigma)$ is a multiplicity space which is typically isomorphic to the dual space of $\text{Hom}_{N_X}(\sigma, \psi)$. In other words, one may take $(\Omega, d\mu_X, \pi)$ to be $(\widehat{G}_X, d\mu_{G_X}, \iota_*)$. One can think of this as saying that the spectral decomposition of $L^2(X)$ is obtained from the Whittaker-Plancherel theorem

$$L^2(N_X, \psi \backslash G_X) \cong \int_{\widehat{G}_X} \sigma^{\oplus m(\sigma)} d\mu_{G_X}(\sigma)$$

by applying ι_* . One consequence of such a spectral decomposition is that it provides a canonical element $\ell_{\iota_*(\sigma)} \in \text{Hom}_H(\iota_*(\sigma), \mathbb{C})$, as we explained in §2, for $m(-) \cdot d\mu_{G_X}$ -almost all σ . Because of the presence of the Plancherel measure $d\mu_{G_X}$ of G_X , only tempered representations σ in \widehat{G}_X need to be considered (though $\iota_*(\sigma)$ may be nontempered).

- (c) Globally, when k is a global field with ring of adeles \mathbb{A} , one considers the global period integral along H :

$$P_H : \mathcal{A}_{cusp}(G) \longrightarrow \mathbb{C}$$

defined by

$$P_H(\phi) = \int_{H(k) \backslash H(\mathbb{A})} \phi(h) dh$$

on the space of cusp forms on G . The restriction of P_H to a cuspidal representation $\Pi = \otimes_v \Pi_v$ of G then defines an element $P_{H,\Pi} \in \text{Hom}_{H(\mathbb{A})}(\Pi, \mathbb{C})$. One is interested in two problems in the global setting:

- (i) characterising those Π for which $P_{H,\Pi}$ is nonzero as functorial lifts from G_X via the map ι ;
- (ii) seeing if $P_{H,\Pi}$ can be decomposed as the tensor product of local functionals.

Such a factorization certainly exists in the instances discussed in this paper since the local Hom spaces $\text{Hom}_{H(F_v)}(\Pi_v, \mathbb{C})$ are at most 1-dimensional for all places v . In (a), we have seen that these Hom spaces are nonzero precisely when $\Pi_v = \iota_*(\sigma_v)$ for some $\sigma_v \in \text{Irr}(G_X(k_v))$. Thus, in the context of the first global problem, one would like to show that, if $P_{H,\Pi} \neq 0$, there exists a cuspidal representation Σ of G_X such that $\Sigma_v \cong \sigma_v$ for all v , so that $\Pi \cong \iota_*(\Sigma)$.

On the other hand, in (b), we have remarked that the spectral decomposition of $L^2(X)$ in the local setting gives rise to a canonical basis element $\ell_{\Pi_v} \in \text{Hom}_{H(F_v)}(\Pi_v, \mathbb{C})$. For the second global problem, it is natural to compare the two elements $P_{H,\Pi}$ and $\prod_v^* \ell_{\Pi_v}$. Here the asterisk in the product indicates that there may be a need to normalize the local functionals ℓ_{Π_v} appropriately to ensure that the Euler product $\prod_v \ell_{\Pi_v}(\phi_v)$ converges. More precisely, to see if the Euler product converges, one would need to evaluate $\ell_{\Pi_v}(\phi_v^0)$ where ϕ_v^0 is a spherical unit vector in Π_v for almost all v . This evaluation has been carried by Sakellaridis in [29, 30] and this is where the L-factor $L_X(s, -)$ associated to the $\frac{1}{2} \cdot \mathbb{Z}$ -graded representation V_X enters the picture. Namely, it turns out that with $\Pi_v \cong \iota_*(\Sigma_v)$ for tempered $\Sigma_v \in \widehat{G}_{X,v}$, one has:

$$|\ell_{\Pi_v}(\phi_v^0)|^2 = L_{X,v}^\#(1/2, \Sigma_v) := \Delta_v(0) \cdot \frac{L_{X,v}(1/2, \Sigma_v)}{L(1, \Sigma_v, \text{Ad})} > 0,$$

where $\Delta_v(s)$ is itself a product of local L-factors which depends only on X and not on the representation Σ_v and $L(s, \Sigma_v, \text{Ad})$ denotes the adjoint L-factor. This necessitates that one defines a normalization of ℓ_{Π_v} by:

$$\ell_{\Pi_v}^\flat = \frac{1}{|L_{X,v}^\#(1/2, \Sigma_v)|^{1/2}} \cdot \ell_{\Pi_v}$$

Then the main issue with the second global problem is to determine the constant $c(\Pi)$ such that

$$|P_{H,\Pi}(\phi)|^2 = c(\Pi) \cdot L_X^\#(1/2, \Pi) \cdot \prod_v |\ell_{\Pi_v}^\flat(\phi_v)|^2 \quad \text{for } \phi = \otimes_v \phi_v \in \otimes_v \Pi_v.$$

Here the global L-function $L_X^\#(s, \Sigma)$ is defined by the Euler product $\prod_v L_{X,v}^\#(s, \Sigma_v)$ for $\text{Re}(s) \gg 0$ and needs to be meromorphically continued so that one can evaluate it at $s = 1/2$.

This concludes our brief and simplified description of the Sakellaridis-Venkatesh conjecture. It is instructive to observe the crucial unifying role played by the typically ignored L^2 -theory, which supplies the canonical basis elements in the relevant local Hom spaces for use in the factorization of the global periods.

We can now describe the content of Sakellaridis' lecture at the Simons Symposium. In a series of recent papers [32, 33, 34], Sakellaridis examined aspects of the above program in the context of rank 1 spherical varieties X . There is a classification of such rank 1 X 's, but a standard example is $X \cong \mathrm{SO}_{n-1} \backslash \mathrm{SO}_n$, i.e. a hyperboloid (or a sphere) in an n -dimensional quadratic space, and a more exotic example is $\mathrm{Spin}_9 \backslash F_4$. In this rank 1 setting, the group G_X is SL_2 or its variants (such as PGL_2 or Mp_2). For example, for $X = \mathrm{SO}_{n-1} \backslash \mathrm{SO}_n$ with n even, $X^\vee \cong \mathrm{PGL}_2(\mathbb{C})$, so that $G_X \cong \mathrm{SL}_2$ and the map ι is given by:

$$\iota : \mathrm{PGL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^2 \times \mathrm{Sym}^{n-4}} \mathrm{SO}_3(\mathbb{C}) \times \mathrm{SO}_{n-3}(\mathbb{C}) \longrightarrow \mathrm{SO}_n(\mathbb{C}).$$

On the other hand, if n is odd, then $X^\vee \cong \mathrm{SL}_2(\mathbb{C})$ and we take $G_X \cong \mathrm{Mp}_2$, with the map ι given by

$$\iota : \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C}) \xrightarrow{\mathrm{Sym}^1 \times \mathrm{Sym}^{n-4}} \mathrm{Sp}_2(\mathbb{C}) \times \mathrm{Sp}_{n-3}(\mathbb{C}) \longrightarrow \mathrm{Sp}_{n-1}(\mathbb{C}).$$

In such rank 1 setting, Sakellaridis developed a theory of transfer of test functions from X to $(N, \psi) \backslash G_X$ as a first step towards establishing local relative character identities and effecting a global comparison of the relative trace formula of X and the Kuznetsov trace formula for G_X . The formula for the transfer map he discovered was motivated by considering an analogous transfer for the boundary degenerations of X and $(N_X, \psi) \backslash G_X$. For the hyperboloid $\mathrm{SO}_{n-1} \backslash \mathrm{SO}_n$, the boundary degeneration is simply the cone of nonzero null vectors in the underlying quadratic space. In any case, the transfer map he wrote down differs from the typical transfer map in the theory of endoscopy in two aspects:

- the spaces of test functions may be larger than the space of compactly supported smooth functions;
- the transfer map in endoscopy is carried out via an orbit-by-orbit comparison, whereas the transfer map in this relative setting is more global in nature, involving an integral kernel transformation reminiscent of the Fourier transform.

An ongoing work [18] of D. Johnstone and R. Krishna establishes the fundamental lemma for the basic functions in the space of test functions; this is necessary for the comparison of relative trace formulae. As an example, in the special case when $n = 4$, one has:

$$X = \mathrm{SO}_3 \backslash \mathrm{SO}_4 \cong \mathrm{PGL}_2 \backslash (\mathrm{SL}_2 \times \mathrm{SL}_2) / \mu_2^\Delta.$$

The relative trace formula for this X is essentially the stable trace formula for SL_2 . Thus, the expected comparison of relative trace formulae is between the stable trace formula for SL_2 and the Kuznetsov trace formula for SL_2 . The local transfer in this case was first investigated in the thesis work of Z. Rudnick. The discussion of these results was the content of Sakellaridis's lecture in the Simons Symposium.

On the other hand, the spectral analysis of $L^2(X)$ when $X = \mathrm{SO}_{n-1} \backslash \mathrm{SO}_n$ or the analysis of the SO_{n-1} -period for representations of SO_n (both locally and globally) is familiar from the theory of theta correspondence. The L^2 -theory was studied in the early work of Strichartz [28] and Howe [17]. In a paper [10] by the first author and R. Gomez, the L^2 -theory was treated using theta correspondence for essentially general rank 1 spherical varieties from the viewpoint of the Sakellaridis-Venkatesh conjecture. For the smooth theory, one can see the recent expository paper [8]. In the case of $X = \mathrm{SO}_{n-1} \backslash \mathrm{SO}_n$, it was known that SO_{n-1} -distinguished representations of SO_n are theta lifts (of ψ -generic representations) from SL_2 or Mp_2 according to whether n is even or odd. Indeed, the theta lifting from SL_2 or Mp_2 to SO_n realises the functorial lifting (at least at the level of unramified representations) predicted by the map $\iota : X^\vee \times \mathrm{SL}_2(\mathbb{C}) \longrightarrow \mathrm{SO}_n(\mathbb{C})$. As such, it is very natural to ask if the results discussed in Sakellaridis' talk can be approached from the viewpoint of the theta correspondence.

This paper is the result of this investigation. In short, its main conclusion is that the theory of transfer developed by Sakellaridis can be very efficiently developed using the theta correspondence. More precisely,

- one can give a conceptual definition of the transfer and the relevant spaces of test functions (Definition 8.1), from which the fundamental lemma (for the basic function and its translate by the spherical Hecke algebra) follows readily (see Lemmas 8.5 and 8.6);
- one can establish the desired relative character identities highlighted in (a) above, without doing a geometric comparison; (see Theorem 9.1)
- one can express this conceptually defined transfer in geometric terms, from which one sees that it agrees with Sakellaridis' formula (see Proposition 11.1);
- one can address the two global problems highlighted in (c) above (see Theorem 12.5).

We leave the precise formulation of the results to the main body of the paper. We would like to remark that, as far as we are aware, the paper [22] of Mao–Rallis is the first instance where one finds a derivation of relative character identities using the theta correspondence; this approach was followed up by the paper [2] of Baruch–Lapid–Mao. The situation treated in this paper is in fact simpler than those in [22] and [2]. In addition, it has been known to practitioners that the theory of theta correspondence is useful for addressing period problems in the smooth local context, the global context, as well as in the local L^2 -context [8, 10], with similar computations and parallel treatment in the various settings. One goal of this paper is to demonstrate how the treatment of the 3 different threads can be synthesised into a rather coherent story.

Here is a short summary of the contents of this paper. In §2, we recall some foundational results of Bernstein [3] on spectral decomposition of $L^2(X)$. These results provide the mechanism for us to navigate between the L^2 -setting and the smooth setting. We illustrate Bernstein's general theory in the setting of the Harish-Chandra-Plancherel formula and the Whittaker-Plancherel formula in §3 and further specialize to the group SL_2 in §4, where we set up some standard conventions and establish some basic results. In §5, we recall the setup of theta correspondence, especially a recent result of Sakellaridis [31] on the spectral decomposition of the Weil representation when restricted to a dual pair. Using the theory of theta correspondence, we address in §6 the local problems (a) and (b), except for the part involving relative character identities. After recalling the notion of relative characters in §7, we come to the heart of the paper (§8-9), where we develop the theory of transfer and establish some of its key properties, culminating in the relative character identity in §9. In §10, we place ourselves in the unramified setting and explicitly determine the local L-factor $L_X(s, -)$ using theta correspondence. We verify that our transfer map is the same as that of Sakellaridis' in §11, where we describe the transfer in geometric terms, as an explicit integral transform. The final §12 discusses and resolves the global problems.

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2. Spectral Decomposition à la Bernstein

Let F be a local field and G a reductive group over F acting transitively on a variety X . We fix a base point $x_0 \in X(F)$, with stabilizer $H \subset G$, so that $g \mapsto g^{-1} \cdot x_0$ gives an identification $H \backslash G \cong X$. For simplicity, we shall write $X = G(F) \cdot x_0 \cong H(F) \backslash G(F)$.

2.1. Direct integral decompositions. Suppose that there is a G -invariant measure dx on X , in which case we may consider the unitary representation $L^2(X, dx)$ of G , with G -invariant inner product

$$\langle \phi_1, \phi_2 \rangle_X = \int_X \phi_1(x) \cdot \overline{\phi_2(x)} \, dx.$$

Such a unitary representation admits a direct integral decomposition

$$(2.1) \quad \iota : L^2(X, dx) \cong \int_{\Omega} \sigma(\omega) \, d\mu(\omega).$$

Here,

- Ω is a measurable space, equipped with a measure $d\mu(\omega)$;
- $\sigma : \omega \mapsto \sigma(\omega)$ is a measurable field of irreducible unitary representations of G over Ω , which we may regard as a measurable map from Ω to the unitary dual \widehat{G} of G (equipped with the Fell topology and the corresponding Borel structure).

In this section, we give an exposition of some results of Bernstein [3] which provide some useful ways of understanding the above direct integral decomposition. This viewpoint of Bernstein underpins the results of this paper.

2.2. Pointwise-defined and fine morphisms. Let $S \subset L^2(X)$ be a subspace which is G -stable. Following Bernstein [3, §1.3], one says that the inclusion $S \hookrightarrow L^2(X)$ is *pointwise-defined* (relative to ι) if there exists a family of G -equivariant (continuous) morphisms $\alpha_{\sigma(\omega)} : S \rightarrow \sigma(\omega)$ for $\omega \in \Omega$ such that for each $\phi \in S$, the element $\iota(\phi)$ in the direct integral decomposition in (2.1) is the measurable section

$$\iota(\phi)(\omega) = \alpha_{\sigma(\omega)}(\phi).$$

In particular, these sections determine the measurable field structure on the right hand side of (2.1). The family $\{\alpha_{\sigma(\omega)} : \omega \in \text{supp}(d\mu)\}$ is essentially unique, in the sense that any two such families differ only on a subset of Ω with measure zero with respect to $d\mu$. Bernstein calls the embedding $S \hookrightarrow L^2(X)$ *fine* if it is pointwise-defined relative to any such isomorphism ι to a direct integral decomposition.

2.3. The maps $\alpha_{\sigma(\omega)}$ and $\beta_{\sigma(\omega)}$. A basic result of Bernstein [3, Prop. 2.3], obtained as an application of the Gelfand-Kostyuchenko method [3, Thm. 1.5], is that the natural inclusion $C_c^\infty(X) \hookrightarrow L^2(X)$ is fine. For any isomorphism ι as in (2.1), we let $\{\alpha_{\sigma(\omega)} : \omega \in \text{supp}(d\mu)\}$ be the associated family of G -equivariant morphisms as above.

The elements in $C_c^\infty(X)$ are G -smooth vectors and so the image of each $\alpha_{\sigma(\omega)}$ is contained in the space $\sigma(\omega)^\infty$ of smooth vectors in $\sigma(\omega)$. As the map $\alpha_{\sigma(\omega)}$ is nonzero for $d\mu$ -almost all ω , its image is dense in $\sigma(\omega)^\infty$, and is in fact equal to $\sigma(\omega)^\infty$ when F is p -adic (where there is no topology considered on $\sigma(\omega)^\infty$). To simplify notation, we shall sometimes write $\sigma(\omega)$ in place of $\sigma(\omega)^\infty$, trusting that the context will make it clear whether one is working with a unitary representation on a Hilbert space or a smooth representation. In particular, $\alpha_{\sigma(\omega)} \in \text{Hom}_G(C_c^\infty(X), \sigma(\omega))$.

If $\alpha_{\sigma(\omega)}$ is nonzero, then by duality, one obtains a G -equivariant embedding

$$\overline{\beta}_{\sigma(\omega)} : \sigma(\omega)^\vee \cong \overline{\sigma(\omega)} \rightarrow C^\infty(X).$$

Here, the isomorphism $\sigma(\omega)^\vee \cong \overline{\sigma(\omega)}$ is induced by the fixed inner product $\langle -, - \rangle_\sigma$ and the duality between $C_c^\infty(X)$ and $C^\infty(X)$ is given by the natural pairing induced by integration with respect to the

G -invariant measure dx . Taking complex conjugate on $C^\infty(X)$, we obtain a G -equivariant linear map

$$\beta_{\sigma(\omega)} : \sigma(\omega)^\infty \longrightarrow C^\infty(X).$$

The maps $\alpha_{\sigma(\omega)}$ and $\beta_{\sigma(\omega)}$ are thus related by the adjunction formula:

$$(2.2) \quad \langle \alpha_{\sigma(\omega)}(\phi), v \rangle_{\sigma(\omega)} = \langle \phi, \beta_{\sigma(\omega)}(v) \rangle_X, \quad \text{for } \phi \in C_c^\infty(X) \text{ and } v \in \sigma(\omega)^\infty.$$

If we compose $\beta_{\sigma(\omega)}$ with the evaluation-at- x_0 map ev_{x_0} , we obtain

$$\ell_{\sigma(\omega)} := ev_{x_0} \circ \beta_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega)^\infty, \mathbb{C}).$$

Thus the direct integral decomposition gives rise to a family of canonical elements $\ell_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega)^\infty, \mathbb{C})$ for $\omega \in \text{supp}(d\mu)$. This family depends on the isomorphism ι in (2.1); changing ι will result in another family which differs from the original one by a measurable function $f : \text{supp}(d\mu) \longrightarrow S^1$. Thus, the family

$$\{\alpha_{\sigma(\omega)} \otimes \overline{\alpha_{\sigma(\omega)}} : \omega \in \text{supp}(d\mu)\}$$

is independent of the choice of the isomorphism ι in (2.1). Likewise, the family

$$\{\beta_{\sigma(\omega)} \otimes \overline{\beta_{\sigma(\omega)}} : \omega \in \text{supp}(d\mu)\}$$

is independent of ι .

2.4. Harish-Chandra–Schwartz space of X . In [3, Pg. 689], Bernstein showed that the space X has a naturally associated Harish-Chandra Schwartz space $\mathcal{C}(X)$ which is G -stable and which contains $C_c^\infty(X)$. Moreover, $\mathcal{C}(X)$ has a natural (complete) topology, such that $C_c^\infty(X)$ is a dense subspace. Indeed, $\mathcal{C}(X)$ is a Frechét space in the archimedean case and is a strict LF space in the non-archimedean case. More importantly, he showed in [3, Thm. 3.2] that the inclusion $\mathcal{C}(X) \hookrightarrow L^2(X)$ is fine. Hence, the maps $\alpha_{\sigma(\omega)} : C_c^\infty(X) \rightarrow \sigma(\omega)$ defined above extend continuously to the larger space $\mathcal{C}(X)$:

$$\alpha_{\sigma(\omega)} : \mathcal{C}(X) \longrightarrow \sigma(\omega)^\infty.$$

The dual map $\beta_{\sigma(\omega)}$ then takes value in the weak Harish-Chandra Schwartz space $\mathcal{C}^w(X) \subset C^\infty(X)$ (see [3] or [5, §2.4] for the group case). The elements $\ell_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega), \mathbb{C})$ are called X -tempered forms and the support of $d\mu$ consists precisely of those representations with nonzero X -tempered forms [3, Pg. 689].

2.5. Inner Product. The direct integral decomposition (2.1) leads to a spectral decomposition of the inner product $\langle -, - \rangle_X$ of X :

$$(2.3) \quad \langle \phi_1, \phi_2 \rangle_X = \int_{\Omega} J_{\sigma(\omega)}(\phi_1, \phi_2) d\mu(\omega),$$

where $J_{\sigma(\omega)}$ is a G -invariant positive-semidefinite Hermitian form on $C_c^\infty(X)$ given by:

$$(2.4) \quad \begin{aligned} J_{\sigma(\omega)}(\phi_1, \phi_2) &= \langle \iota(\phi_1)(\omega), \iota(\phi_2)(\omega) \rangle_{\sigma(\omega)} \\ &= \langle \alpha_{\sigma(\omega)}(\phi_1), \alpha_{\sigma(\omega)}(\phi_2) \rangle_{\sigma(\omega)} \\ &= \langle \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi_1), \phi_2 \rangle_X \end{aligned}$$

for $d\mu$ -almost all ω . In particular, $J_{\sigma(\omega)}$ factors as:

$$J_{\sigma(\omega)} : C_c^\infty(X) \times \overline{C_c^\infty(X)} \xrightarrow{\alpha_{\sigma(\omega)} \otimes \overline{\alpha_{\sigma(\omega)}}} \sigma(\omega) \otimes \overline{\sigma(\omega)} \xrightarrow{\langle -, - \rangle_{\sigma(\omega)}} \mathbb{C}.$$

2.6. Pointwise spectral decomposition. The fact that the morphism $\mathcal{C}(X) \hookrightarrow L^2(X)$ is fine leads to a pointwise spectral decomposition for elements of $\mathcal{C}(X)$. More precisely, for $\phi \in \mathcal{C}(X)$, one has

$$(2.5) \quad \phi(x) = \int_{\Omega} \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi)(x) d\mu(\omega)$$

for any $x \in X$. We give a sketch of the derivation of this when F is non-archimedean. In that case, $\phi \in \mathcal{C}(X)$ is fixed by some open compact subgroup $J \subset G$. The group J also fixes $\beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi)$ for any σ , since $\alpha_{\sigma(\omega)}$ and $\beta_{\sigma(\omega)}$ are G -equivariant. If 1_{xJ} denotes the characteristic function of the open compact subset $xJ \subset X$, then it follows that

$$\langle \phi, 1_{xJ} \rangle_X = \text{Vol}(xJ; dx) \cdot \phi(x) \quad \text{and} \quad \langle \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi), 1_{xJ} \rangle_X = \text{Vol}(xJ; dx) \cdot \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi)(x)$$

where $\text{Vol}(xJ; dx)$ is the volume of $xJ \subset X$ with respect to the measure dx . Now it follows that

$$\begin{aligned} \phi(x) &= \frac{1}{\text{Vol}(xJ; dx)} \cdot \langle \phi, 1_{xJ} \rangle_X \\ &= \frac{1}{\text{Vol}(xJ; dx)} \int_{\Omega} \langle \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi), 1_{xJ} \rangle_X d\mu(\omega) \\ &= \int_{\Omega} \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi)(x) d\mu(\omega), \end{aligned}$$

where the second equality is a consequence of (2.3) and (2.4).

The crux of Bernstein's viewpoint in [3] is that to give the isomorphism ι in the direct integral decomposition (2.1) is equivalent to giving the family $\{\alpha_{\sigma(\omega)} : \omega \in \Omega\}$ (satisfying appropriate properties), together with the measure $d\mu$ on Ω . In the next section, we shall illustrate this in two basic examples.

3. Basic Plancherel Theorems

In this section, we describe two basic Plancherel theorems as an illustration of the abstract theory of Bernstein discussed in the previous section. These are the Harish-Chandra-Plancherel theorem and the Whittaker-Plancherel theorem. The latter will play a crucial role in this paper.

We shall continue to work over a local field F . However, we will implicitly be assuming that F is non-archimedean. In fact, the results of this paper will hold for archimedean local fields as well, but greater care is needed in introducing the various objects (such as various spaces of functions and the topologies they carry) and in formulating the results. Thus, there are analytic and topological considerations that need to be addressed in the archimedean case. We refer the reader to the papers [4, 5] and the thesis of the second author [39] where these issues are dealt with carefully and content ourselves with treating the nonarchimedean case in the interest of efficiency.

3.1. Harish-Chandra-Plancherel Theorem. The most basic example is the regular representation $L^2(G)$ of a semisimple group $G \times G$ (acting by right and left translation):

$$(g_1, g_2)f(g) = f(g_2^{-1} \cdot g \cdot g_1).$$

Here, we have fixed a Haar measure dg on G which defines the inner product on $L^2(G)$.

Now Harish-Chandra's Plancherel theorem [36, 37] asserts that there is an explicitly constructed $G \times G$ -equivariant isomorphism

$$(3.1) \quad L^2(G) \cong \int_{\widehat{G}} \widehat{\sigma} \boxtimes \sigma d\mu_G(\sigma)$$

for a specific measure $d\mu_G$ on \widehat{G} known as the Plancherel measure of G (which depends on the Haar measure dg). The support of this measure is precisely the subset \widehat{G}_{temp} of irreducible tempered representations of G . Thus, in this case, one may take the measurable space Ω to be the unitary dual \widehat{G} and the map $\widehat{G} \rightarrow \widehat{G} \times \widehat{G}$ is given by $\sigma \mapsto \overline{\sigma} \boxtimes \sigma$. Implicit in the theorem is the data of a measurable field of unitary representations over \widehat{G} whose fiber at $\sigma \in \widehat{G}$ is the representation $\overline{\sigma} \boxtimes \sigma$.

One may describe the above direct integral decomposition (including the isomorphism) from Bernstein's viewpoint. The Hilbert space $\overline{\sigma} \boxtimes \sigma$ is naturally identified with the space $\text{End}_{HS}(\sigma)$ of Hilbert-Schmidt operators on σ , equipped with the Hilbert-Schmidt norm, and its space of $G \times G$ -smooth vectors is the space $\overline{\sigma}^\infty \otimes \sigma^\infty = \text{End}_{\text{fin}}(\sigma^\infty)$ of finite rank operators on σ^∞ . To describe the direct integral decomposition, one needs to give the family of maps:

$$\alpha_{\overline{\sigma} \boxtimes \sigma} : C_c^\infty(G) \longrightarrow \text{End}_{\text{fin}}(\sigma^\infty),$$

The map $\alpha_{\overline{\sigma} \boxtimes \sigma}$ is given by

$$\alpha_{\overline{\sigma} \boxtimes \sigma}(f) = \sigma(f) := \int_G f(g) \cdot \sigma(g) dg$$

and the (conjugate) dual map

$$\beta_{\overline{\sigma} \boxtimes \sigma} : \overline{\sigma}^\infty \otimes \sigma^\infty \longrightarrow C^\infty(G)$$

is given by the formation of matrix coefficients. This data characterizes and explicates the measurable field of unitary representations implicit in the Plancherel theorem: the sections $\sigma \mapsto \sigma(f)$ for $f \in C_c^\infty(G)$ generate the family of measurable sections of the Hilbert space bundle $\{\text{End}_{HS}(\sigma) = \overline{\sigma} \boxtimes \sigma\}_{\sigma \in \widehat{G}}$ over \widehat{G} .

The associated inner product $J_{\overline{\sigma} \boxtimes \sigma}$ is given by:

$$J_{\overline{\sigma} \boxtimes \sigma}(f_1, f_2) = \text{Tr}(\sigma(f_1)\sigma(f_2^\vee)),$$

where

$$f_2^\vee(g) = \overline{f_2(g^{-1})}.$$

The $\overline{\sigma} \otimes \sigma$ -component of $f \in C_c^\infty(G)$ in the pointwise spectral decomposition is the function given by

$$(\beta_{\overline{\sigma} \boxtimes \sigma} \circ \alpha_{\overline{\sigma} \boxtimes \sigma})(f)(g) = \text{Tr}(\sigma(g)^{-1} \circ \sigma(f)).$$

In particular,

$$f \mapsto \ell_{\overline{\sigma} \boxtimes \sigma}(\alpha_{\overline{\sigma} \boxtimes \sigma}(f)) = \text{Tr}(\sigma(f))$$

is the Harish-Chandra character distribution of σ . For tempered σ , it extends to the (original) Harish-Chandra Schwartz space $\mathcal{C}(G)$.

It is instructive to take note of how the Plancherel measure $d\mu_G$ depends on the Haar measure dg . If we replace dg by $\lambda \cdot dg$ for some $\lambda \in \mathbb{R}_{>0}^\times$, then we observe that

$$J_{\overline{\sigma} \boxtimes \sigma} \mapsto \lambda^2 \cdot J_{\overline{\sigma} \boxtimes \sigma}, \quad \alpha_{\overline{\sigma} \boxtimes \sigma} \mapsto \lambda \cdot \alpha_{\overline{\sigma} \boxtimes \sigma} \quad \text{and} \quad \beta_{\overline{\sigma} \boxtimes \sigma} \mapsto \beta_{\overline{\sigma} \boxtimes \sigma}.$$

Hence, we have

$$d\mu_G \mapsto \lambda^{-1} \cdot d\mu_G.$$

We have restricted ourselves to semisimple groups in this subsection for simplicity. When G is a reductive algebraic group and $Z \subset G$ the maximal F -split torus in the center of G , then one may fix a unitary character χ of Z and consider the unitary representation $L_\chi^2(G)$ consisting of L^2 -functions f which satisfies $f(zg) = \chi(z) \cdot f(g)$ for $z \in Z$ and $g \in G$ and equipped with the unitary structure determined by a Haar measure dg/dz of G/Z . Moreover, one may also consider nonlinear finite central extensions of $G(F)$ by finite cyclic groups. In all these cases, the Harish-Chandra Plancherel theorem continue to hold.

3.2. Whittaker-Plancherel Theorem. Our second example is the Whittaker-Plancherel theorem (see [5, 7, 35, 36]), which is a variant of the setting discussed above. Let G be a quasi-split semisimple group with N the unipotent radical of a Borel subgroup. Fix a nondegenerate unitary character ψ_N of N . We consider the *Whittaker variety* $(N, \psi_N) \backslash G$ and its associated unitary representation $L^2(N, \psi_N \backslash G)$ (which depends on fixed Haar measures dg on G and dn on N). This extends the setting we discussed above, as one is considering L^2 -sections of a line bundle on the spherical variety $N \backslash G$ instead of L^2 -functions, but it is also covered in [3].

It has been shown (see [7, 35, 38]) that one has a direct integral decomposition

$$(3.2) \quad L^2(N, \psi_N \backslash G) \cong \int_{\widehat{G}} \dim \operatorname{Hom}_N(\sigma, \psi_N) \cdot \sigma \, d\mu_G(\sigma),$$

where we recall that $d\mu_G$ is the Plancherel measure of G (associated to the fixed Haar measure dg). Thus, in this case, we are taking Ω to be \widehat{G} and the map $\Omega \rightarrow \widehat{G}$ is the identity map. The spectral measure $d\mu_{G, \psi_N}$ is equal to $\dim \operatorname{Hom}_N(\sigma, \psi_N) \cdot d\mu_G$, whose support is the subset $\widehat{G}_{temp, \psi_N}$ of ψ_N -generic irreducible tempered representations.

Associated to this direct integral decomposition is the family of morphisms

$$\alpha_\sigma : C_c^\infty(N, \psi_N \backslash G) \longrightarrow \sigma$$

for all $\sigma \in \widehat{G}_{temp, \psi_N}$. Moreover, the map α_σ extends to the Harish-Chandra-Schwarz space $\mathcal{C}(N, \psi_N \backslash G)$. We describe instead the (conjugate) dual map

$$\beta_\sigma \otimes \overline{\beta_\sigma} : \sigma \otimes \overline{\sigma} \longrightarrow C^\infty(N \times N, \psi_N \otimes \overline{\psi_N} \backslash G \times G)$$

as follows. Given $v_1, v_2 \in \sigma$, one has

$$(3.3) \quad \beta_\sigma \otimes \overline{\beta_\sigma}(v_1 \otimes v_2)(g_1, g_2) = \int_N^* \overline{\psi_N(n)} \cdot \langle \sigma(n \cdot g_1)(v_1), \sigma(g_2)(v_2) \rangle_\sigma \, dn$$

where the integral is a regularized one (see [19, Prop. 2.3], [35, §6.3] and [4]). Here, note that $\beta_\sigma \otimes \overline{\beta_\sigma}$ depends on dn , as it should. The composite of this with the evaluation-at-1 map is thus the Whittaker functional

$$(3.4) \quad \ell_\sigma \otimes \overline{\ell_\sigma} : v_1 \otimes v_2 \mapsto \int_N^* \overline{\psi_N(n)} \cdot \langle \sigma(n)(v_1), v_2 \rangle_\sigma \, dn.$$

The associated (positive semidefinite) inner product J_σ on $C_c^\infty(N, \psi_N \backslash G)$ is then given by

$$\begin{aligned} J_\sigma(f_1, f_2) &= \sum_{v \in \operatorname{ONB}(\sigma)} \langle f_1, \beta_\sigma(v) \rangle_{N \backslash G} \cdot \langle \beta_\sigma(v), f_2 \rangle_{N \backslash G} \\ &= \sum_{v \in \operatorname{ONB}(\sigma)} \int_{N \times N \backslash G \times G} f_1(g_1) \cdot \overline{f_2(g_2)} \cdot \ell_\sigma(g_2 v) \cdot \overline{\ell_\sigma(g_1 v)} \frac{dg_1}{dn} \frac{dg_2}{dn} \\ &= \sum_{v \in \operatorname{ONB}(\sigma)} \int_{N \times N \backslash G \times G} f_1(g_1) \cdot \overline{f_2(g_2)} \cdot \left(\int_N^* \overline{\psi_N(n)} \cdot \langle n g_2 v, g_1 v \rangle_\sigma \, dn \right) \frac{dg_1}{dn} \frac{dg_2}{dn}. \end{aligned}$$

The above maps specify on the right hand side of (3.2) the structure of a measurable field of unitary representations on $\widehat{G}_{temp, \psi_N}$ whose fiber at $\sigma \in \widehat{G}_{temp, \psi_N}$ is the representation σ . We can think of this measurable field of unitary representations as a “tautological” or “universal bundle of unitary representations” over the “moduli space” $\widehat{G}_{temp, \psi_N}$ of irreducible ψ_N -generic tempered representations.

It is again useful to take note of how the various quantities change when one replaces the Haar measure dg of G by $\lambda \cdot dg$ for some $\lambda \in \mathbb{R}_{>0}^\times$. In the Whittaker-Plancherel case, one sees from the above formula that β_σ and ℓ_σ are unchanged whereas

$$J_\sigma \mapsto \lambda^2 \cdot J_\sigma \quad \text{and} \quad \alpha_\sigma \mapsto \lambda \cdot \alpha_\sigma,$$

keeping in mind that the Plancherel measure $d\mu_G$ gets replaced by $\lambda^{-1} \cdot d\mu_G$.

As in the case of the Harish-Chandra Plancherel theorem, we could have worked with a reductive algebraic group (in which case we fix a central character χ as before and consider $L_\chi^2(N, \psi_N \backslash G)$) or a nonlinear finite cover thereof. One has the Whittaker-Plancherel theorem in these settings as well, though we take note that uniqueness of Whittaker models fails for nonlinear covering groups in general.

3.3. Continuity properties. We now consider the issue of continuity (in σ) for some of the quantities discussed above. We first need to say a few words about the Fell topology on \widehat{G}_{temp} .

The unitary dual \widehat{G} is typically non-Hausdorff even though it is still a T1 space for the groups considered here. The tempered dual \widehat{G}_{temp} is still not necessarily Hausdorff, but can often be replaced by a substitute which is Hausdorff. Namely, one can work with the space of equivalence classes of induced representations $\tau = \text{Ind}_P^G \pi$ where P is a parabolic subgroup of G and π a discrete series representation of its Levi factor M . This space was variously denoted by \mathcal{T} in [31], $\text{Temp}_{\text{Ind}}(G)$ in [5] and $\mathcal{X}_{temp}(G)$ in [4, 40], so we are spoilt for choices! To add to this galore, we shall denote this space by \widehat{G}_{temp}^{ind} . Then \widehat{G}_{temp}^{ind} has the structure of an orbifold (given by twisting π by unramified unitary characters of M). There is a natural continuous finite-to-one surjective map

$$\widehat{G}_{temp} \longrightarrow \widehat{G}_{temp}^{ind}$$

sending a tempered irreducible representation σ to the unique induced representation $\text{Ind}_P^G \pi$ containing σ . This map is injective outside a subset of \widehat{G}_{temp} which has measure zero with respect to the Plancherel measure $d\mu_G$.

In the setting of the Harish-Chandra-Plancherel theorem of §3.1, one could safely replace the integral over \widehat{G}_{temp} in (3.1) by an integral over \widehat{G}_{temp}^{ind} . Moreover, we have the Hermitian form $J_{\overline{\sigma} \boxtimes \sigma}(\phi_1, \phi_2)$ for $\phi_i \in C_c^\infty(G)$ (or more generally $\mathcal{C}(G)$) and $\sigma \in \widehat{G}_{temp}$. We can similarly define $J_{\overline{\tau} \boxtimes \tau}(\phi_1, \phi_2)$ for $\tau \in \widehat{G}_{temp}^{ind}$. Then we have [5, §2.13]:

Lemma 3.1. *For fixed ϕ_1 and ϕ_2 in $\mathcal{C}(G)$, the maps*

$$\tau \mapsto J_{\overline{\tau} \boxtimes \tau}(\phi_1, \phi_2)$$

is continuous as a \mathbb{C} -valued function on \widehat{G}_{temp}^{ind} . In particular, the map $\sigma \mapsto J_{\overline{\sigma} \boxtimes \sigma}(\phi_1, \phi_2)$ is continuous on the subset of \widehat{G}_{temp} which maps injectively into \widehat{G}_{temp}^{ind} .

In the context of the Whittaker-Plancherel theorem, we are working with the subset $\widehat{G}_{temp, \psi_N}$. When uniqueness of Whittaker models holds (such as for reductive algebraic groups or the metaplectic groups which are two fold covers of symplectic groups), each $\tau = \text{Ind}_P^G \pi$ in \widehat{G}_{temp}^{ind} can have at most one irreducible constituent which is ψ_N -generic. Hence, we see that the composite map

$$\widehat{G}_{temp, \psi_N} \longrightarrow \widehat{G}_{temp} \longrightarrow \widehat{G}_{temp}^{ind}$$

is injective (and continuous). As a consequence of [5, §2.14], we have:

Lemma 3.2. *In the context of the Whittaker-Plancherel theorem, for fixed f_1 and f_2 in $\mathcal{C}(N, \psi_N \backslash G)$, the map*

$$\sigma \mapsto J_\sigma(f_1, f_2)$$

is a continuous \mathbb{C} -valued function on $\widehat{G}_{\text{temp}, \psi_N}$. Likewise, for fixed $f \in \mathcal{C}(N, \psi_N \backslash G)$, the map

$$\sigma \mapsto \beta_\sigma \alpha_\sigma(f)(1)$$

is continuous.

4. GL_2 and SL_2

In this section, we specialize the discussion of the previous section to the case of GL_2 and SL_2 . Since the group SL_2 will feature prominently in the rest of the paper, we also take the opportunity to set up some precise conventions which will be used for the rest of the paper.

4.1. Measures on F and F^\times . Let us first fix a nontrivial additive character

$$\psi : F \longrightarrow S^1.$$

Then ψ determines an additive Haar measure $d_\psi x$ on F , characterized by the requirement that $d_\psi x$ is self-dual with respect to the Fourier transform relative to the pairing $(x, y) \mapsto \psi(xy)$ on F . If F is nonarchimedean with ring of integers \mathcal{O}_F and ψ has conductor \mathcal{O}_F , then the Haar measure $d_\psi x$ gives \mathcal{O}_F volume 1. One also obtains a multiplicative Haar measure on F^\times given by

$$d_\psi^\times x = d_\psi x / |x|.$$

More generally, for any algebraic group G over F , a nonzero element ω of $\wedge^{\text{top}} \text{Lie}(G)^*$ and the additive Haar measure $d_\psi x$ together give rise to a right-invariant (or left-invariant) Haar measure $|\omega|_\psi$. Since ψ is fixed throughout, we will often suppress it from the notation and simply write $|\omega|$.

4.2. The group GL_2 . We now consider the group GL_2 over F . Let \tilde{B} be the upper triangular Borel subgroup with unipotent radical N (the group of upper triangular unipotent matrices) and consider the diagonal maximal torus. We can write the diagonal maximal torus as $Z \cdot S$ where Z is the center of GL_2 (the scalar matrices) and

$$S = \{s(y) = \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} : y \in F^\times\}.$$

By §4.1, we have Haar measures on $N(F) = F$, $Z(F) = F^\times$ and $S(F) = F^\times$ and hence a right-invariant measure on \tilde{B} .

Regard the fixed additive character ψ of F as a character of $N(F)$. For a fixed a unitary character χ of Z , the Whittaker-Plancherel theorem for $L_\chi^2(N, \psi \backslash \text{GL}_2)$ gives a family of GL_2 -equivariant embeddings

$$\tilde{\beta}_{\tilde{\sigma}, \psi} : \tilde{\sigma} \longrightarrow C_\chi^\infty(N, \psi \backslash \text{GL}_2) = \text{Ind}_{Z \cdot N}^{\text{GL}_2} \chi \otimes \psi$$

for irreducible tempered representations $\tilde{\sigma}$ of GL_2 with central character χ . As we have noted, $\tilde{\beta}_{\tilde{\sigma}, \psi}$ only depends on the Haar measure $d_\psi n$ on N (which we have fixed), and does not depend on the choice of the Haar measure on GL_2 which enters into the formulation of the Whittaker-Plancherel theorem.

We may consider the \tilde{B} -equivariant map

$$\text{rest} : C_\chi^\infty(N, \psi \backslash \text{GL}_2) \longrightarrow C_\chi^\infty(N, \psi \backslash \tilde{B}) = \text{Ind}_{Z \cdot N}^{\tilde{B}} \chi \otimes \psi$$

given by the restriction of functions. The Haar measures we have fixed endow the latter space with a unitary structure, whose inner product is given by

$$\langle f_1, f_2 \rangle_{Z \cdot N \backslash \tilde{B}} = \int_{F^\times} f_1 \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix} \cdot \overline{f_2 \begin{pmatrix} y & 0 \\ 0 & 1 \end{pmatrix}} d_\psi^\times y.$$

We now note the following basic result, which is a reformulation of [19, Lemma 4.4] and [5, Prop. 2.14.3]. In these references, this result was shown in the setting of GL_n (with appropriate formulation). In any case, this result is the reason why we consider the case of GL_2 in this section.

Proposition 4.1. *The composite map $\mathrm{rest} \circ \tilde{\beta}_{\tilde{\sigma}, \psi}$ gives an isometric embedding*

$$\mathrm{rest} \circ \tilde{\beta}_{\tilde{\sigma}, \psi} : \tilde{\sigma} \longrightarrow L_\chi^2(N, \psi \backslash \tilde{B}).$$

4.3. The group SL_2 . Now we turn to the group SL_2 . The goal is to deduce from Proposition 4.1 its analog in the setting of SL_2 . We first take the opportunity to introduce some conventions for SL_2 which will be enforced throughout the paper.

We first fix the upper triangular Borel subgroup $B = T \cdot N$, where T is the diagonal torus and N the unipotent radical of B , and a maximal compact subgroup K in good relative position with respect to B . For example, when F is non-archimedean with ring of integers \mathcal{O}_F , we can simply take $K = \mathrm{SL}_2(\mathcal{O}_F)$. We have natural identifications $N(F) \cong F$ and $T(F) \cong F^\times$ such that the modulus character of B is given by $\delta_B(t) = |t|^2$. For $a \in F^\times$ and $b \in N$, we write

$$n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N(F) \quad \text{and} \quad t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T(F).$$

Further, the groups $N(F) = F$ and $T(F) = F^\times$ carry the fixed Haar measure $dn = d_\psi x$ and $dt = d_\psi x/|x|$. We also fix a Haar measure dg on $\mathrm{SL}_2(F)$, which in turn determines a Plancherel measure $d\mu_{\mathrm{SL}_2}$ on the unitary dual $\widehat{\mathrm{SL}_2}$.

Now we come to the analog of Proposition 4.1 in the SL_2 -setting. The main difference between GL_2 and SL_2 is that, whereas there is a unique equivalence class of Whittaker datum in the case of GL_2 , there are $F^\times/F^{\times 2}$ -worth of them in the case of SL_2 . For any $a \in F^\times$, set $\psi_a(x) = \psi(ax)$ so that ψ_a is a nontrivial additive character of F . Then the two Whittaker data (N, ψ_a) and (N, ψ_b) of SL_2 are equivalent if and only if $a/b \in F^{\times 2}$. In formulating an analog of Proposition 4.1 in the SL_2 -setting, it will be necessary to take all the various inequivalent Whittaker data into account.

Henceforth, let us fix a set of representatives $[a]$ for $F^\times/F^{\times 2}$, so that

$$(4.1) \quad F^\times = \bigsqcup_{[a] \in F^\times/F^{\times 2}} aF^{\times 2}.$$

We shall assume $a = 1$ is one of the representatives.

For each $a \in F^\times$, the Whittaker-Plancherel theorem for $L^2(N, \psi_a \backslash \mathrm{SL}_2)$ (relative to the fixed Haar measures dg on SL_2 and $d_\psi x$ on N) then furnishes the maps α_{σ, ψ_a} , β_{σ, ψ_a} and ℓ_{σ, ψ_a} for any irreducible ψ_a -generic tempered representation σ of SL_2 . In particular, if σ is ψ_a -generic, then

$$\beta_{\sigma, \psi_a} : \sigma \longrightarrow C^\infty(N, \psi_a \backslash \mathrm{SL}_2)$$

is an SL_2 -equivariant embedding. As in the GL_2 case, we may consider the restriction of functions

$$C^\infty(N, \psi_a \backslash \mathrm{SL}_2) \longrightarrow C^\infty(N, \psi_a \backslash B).$$

Let us scale this restriction map a little, by setting

$$\text{rest}_a : C^\infty(N, \psi_a \backslash \text{SL}_2) \longrightarrow C^\infty(N, \psi_a \backslash B)$$

to be

$$\text{rest}_a(f) = |a|^{1/2} \cdot f|_B.$$

Combining all these maps together gives us an B -equivariant map

$$(4.2) \quad j_\sigma := \bigoplus_{[a] \in F^\times / F^{\times 2}} \text{rest}_a \circ \beta_{\sigma, \psi_a} : \sigma \longrightarrow \bigoplus_{[a] \in F^\times / F^{\times 2}} C^\infty(N, \psi_a \backslash B)$$

where here, the map β_{σ, ψ_a} is interpreted to be 0 if σ is not ψ_a -generic. Now we have the following analog of Proposition 4.1, which will play a crucial role later on (in the proof of Proposition 6.4)..

Proposition 4.2. *Equip $C^\infty(N, \psi_a \backslash B) \cong C^\infty(T)$ with the unitary structure*

$$\langle f_1, f_2 \rangle_{N \backslash B} = \frac{1}{2} \cdot |2|_F \cdot \int_{F^\times} f_1(t(b)) \cdot \overline{f_2(t(b))} d_\psi^\times b,$$

which is the natural inner product associated to the Haar measures fixed on T scaled by the factor $|2|_F/2$. Then, for any irreducible tempered representation σ of SL_2 , the B -equivariant map defined by (4.2) is an isometry

$$j_\sigma : \sigma \longrightarrow \bigoplus_{[a] \in F^\times / F^{\times 2}} L^2(N, \psi_a \backslash B).$$

Proof. To deduce this proposition from Proposition 4.1, we naturally regard SL_2 as a subgroup of GL_2 . Given an irreducible tempered representation σ of SL_2 , we pick a unitary representation $\tilde{\sigma}$ of GL_2 with unitary central character χ such that $\sigma \subset \tilde{\sigma}$. We may assume that the inner product $\langle -, - \rangle_{\tilde{\sigma}}$ restricts to the inner product $\langle -, - \rangle_\sigma$ on σ .

Consider the ψ -Whittaker functional $\tilde{\ell}_{\tilde{\sigma}, \psi}$ and the associated map $\tilde{\beta}_{\tilde{\sigma}, \psi}$ for $\tilde{\sigma}$. Observe that for any $a \in F^\times$, $\tilde{\ell}_{\tilde{\sigma}, \psi} \circ s(a) \in \text{Hom}_N(\tilde{\sigma}, \psi_a)$ and the following statements are equivalent:

- the restriction of $\tilde{\ell}_{\tilde{\sigma}, \psi} \circ s(a)$ to σ is nonzero;
- σ is ψ_a -generic;
- $s(a)(\sigma) = \sigma$ as a subspace of $\tilde{\sigma}$.

Further, if σ is ψ -generic, then from the formula (3.4), one sees that the restriction of $\tilde{\ell}_{\tilde{\sigma}, \psi}$ to σ is equal to $\ell_{\sigma, \psi}$.

Now take $w_1, w_2 \in \sigma \subset \tilde{\sigma}$. By Proposition 4.1, one has

$$\langle w_1, w_2 \rangle_\sigma = \langle w_1, w_2 \rangle_{\tilde{\sigma}} = \int_{F^\times} \tilde{\beta}_{\tilde{\sigma}}(w_1)(s(x)) \cdot \overline{\tilde{\beta}_{\tilde{\sigma}}(w_2)(s(x))} d_\psi^\times(x),$$

where we have written $\tilde{\beta}_{\tilde{\sigma}}$ in place of $\tilde{\beta}_{\tilde{\sigma}, \psi}$ to simplify notation. To evaluate the latter integral, we decompose the domain of integration into square classes as in (4.1) and uniformize each square class $aF^{\times 2}$ by F^\times , using the map $b \mapsto ab^2$ (which is a 2-to-1 map). Performing the corresponding change of

variables in the integral (i.e. replacing x by ab^2 on $aF^{\times 2}$), we obtain:

$$\begin{aligned} \langle w_1, w_2 \rangle_\sigma &= \sum_{[a] \in F^\times / F^{\times 2}} \frac{1}{2} \cdot |2|_F \cdot \int_{F^\times} \tilde{\beta}_{\tilde{\sigma}}(w_1)(s(ab^2)) \cdot \overline{\tilde{\beta}_{\tilde{\sigma}}(w_2)(s(ab^2))} d_\psi^\times b \\ &= \sum_{[a] \in F^\times / F^{\times 2}} \frac{1}{2} \cdot |2|_F \cdot \int_{F^\times} \tilde{\beta}_{\tilde{\sigma}}(w_1)(s(a) \cdot t(b)) \cdot \overline{\tilde{\beta}_{\tilde{\sigma}}(w_2)(s(a) \cdot t(b))} d_\psi^\times b \end{aligned}$$

This computation is the source of the factor $|2|_F/2$ appearing in the proposition. Let us define a map

$$j_{\sigma,a} : \sigma \mapsto C^\infty(N, \psi_a \setminus B)$$

by

$$j_{\sigma,a}(w)(t(b)) = \tilde{\beta}_{\tilde{\sigma},\psi}(w)(s(a) \cdot t(b)),$$

noting that $j_{\sigma,a}$ is nonzero if and only if σ is ψ_a -generic. Then we have shown that for $w_1, w_2 \in \sigma$,

$$\langle w_1, w_2 \rangle_\sigma = \sum_{[a] \in F^\times / F^{\times 2}} \langle j_{\sigma,a}(w_1), j_{\sigma,a}(w_2) \rangle_{N \setminus B}.$$

In other words, one has an isometry

$$\bigoplus_{[a] \in F^\times / F^{\times 2}} j_{\sigma,a} : \sigma \longrightarrow \bigoplus_{[a] \in F^\times / F^{\times 2}} L^2(N, \psi_a \setminus B)$$

where the unitary structure of the latter spaces are as given in the proposition.

It remains then to show that

$$j_{\sigma,a} = \text{rest}_a \circ \beta_{\sigma,\psi_a} \quad \text{for each } a \in F^\times,$$

or equivalently

$$\tilde{\ell}_{\tilde{\sigma}} \circ s(a) = |a|^{1/2} \cdot \ell_{\sigma,\psi_a} \quad \text{on } \sigma \subset \tilde{\sigma}.$$

To see this, for $w_1, w_2 \in \sigma \subset \tilde{\sigma}$, we apply (3.4) to get

$$\begin{aligned} &\tilde{\ell}_{\tilde{\sigma}}(s(a)w_1) \cdot \overline{\tilde{\ell}_{\tilde{\sigma}}(s(a)w_2)} \\ &= \int_F^* \langle n(x) \cdot s(a)w_1, s(a)w_2 \rangle_{\tilde{\sigma}} \cdot \overline{\psi(x)} d_\psi x \\ &= \int_N^* \langle s(a) \cdot n(a^{-1}x)w_1, s(a)w_2 \rangle_{\tilde{\sigma}} \cdot \overline{\psi(x)} d_\psi x \\ &= \int_N^* \langle n(x)w_1, w_2 \rangle_{\tilde{\sigma}} \cdot \overline{\psi(ax)} \cdot |a| d_\psi x \\ &= |a| \cdot \ell_{\sigma,\psi_a}(w_1) \cdot \overline{\ell_{\sigma,\psi_a}(w_2)}. \end{aligned}$$

Here in the penultimate equality, we have applied a change of variables, replacing x by ax and used the unitarity of $\tilde{\sigma}$. We have thus shown that

$$\tilde{\ell}_{\tilde{\sigma}} \circ s(a) = |a|^{1/2} \cdot \ell_{\sigma,\psi_a}$$

at least up to a root of unity (which we may ignore, by absorbing it into $\tilde{\ell}_{\tilde{\sigma}}$). This completes the proof of Proposition 4.2. \square

We can describe the image of the isometry j_σ precisely. Observe that the center $Z_{\mathrm{SL}_2} = \mu_2$ of SL_2 induces a decomposition

$$L^2(N, \psi_a, \backslash B) = L^2(N, \psi_a, \backslash B)^+ \oplus L^2(N, \psi_a, \backslash B)^-$$

into two irreducible B -subrepresentations which are the \pm -eigenspaces of μ_2 . Then one has:

Corollary 4.3. *Let $z_\sigma = \pm$ denote the central character of the irreducible tempered representation σ of SL_2 . The map j_σ in (4.2) defines a B -equivariant isometric isomorphism*

$$j_\sigma : \sigma = \bigoplus_{[a] \in F^\times / F^{\times 2}} \dim \mathrm{Hom}_N(\sigma, \psi_a) \cdot L^2(N, \psi_a \backslash B)^{z_\sigma}$$

where the unitary structure of the right hand side is as defined in Proposition 4.2.

4.4. Harish-Chandra-Schwartz space. Finally, we explicate when a function $f \in C^\infty(N, \psi \backslash \mathrm{SL}_2)$ lies in the Harish-Chandra Schwartz space. The measures dg and dn determine an $\mathrm{SL}_2(F)$ -invariant measure on $N(F) \backslash \mathrm{SL}_2(F)$, which can be described as follows. An element $f \in C^\infty(N, \psi \backslash \mathrm{SL}_2)$ is determined by its restriction to TK , by the Iwasawa decomposition. Then the integral of f with respect to dg/dn is given by

$$f \mapsto \int_T \int_K f(tk) \cdot \delta_B(t)^{-1} dt dk$$

for some Haar measure dk of K .

Given a function $f \in C^\infty(N, \psi \backslash \mathrm{SL}_2)$, the smoothness of f implies that the function $t \rightarrow f(tk)$ on $T \cong F^\times$ is necessarily rapidly decreasing at $|t| \rightarrow \infty$ (indeed, it vanishes on some domain $|t| > C$ in the p -adic case). Thus the analytic properties of f depend on its asymptotics as $|t| \rightarrow 0$. We have the following lemma:

Lemma 4.4. *Let $f \in C^\infty(N, \psi \backslash \mathrm{SL}_2)$ and suppose that there exists $C > 0$ and $d > 0$ such that*

$$\sup_{k \in K} |f(tk)| \leq C \cdot |t|^d \quad \text{as } |t| \rightarrow 0.$$

- (i) *If $d > 1$, then $f \in \mathcal{C}(N, \psi \backslash \mathrm{SL}_2)$.*
- (ii) *If $d > 2$, then $f \in L^1(N, \psi \backslash \mathrm{SL}_2)$.*

5. Theta Correspondence

In this section, we recall the setup of the theta correspondence and recall some results of Sakellaridis [31] on the spectral decomposition of the Weil representation for a dual pair.

5.1. Weil representation. If W is a symplectic vector space and (V, q) a quadratic space over a local field F , then one has a dual reductive pair

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \longrightarrow \mathrm{Sp}(V \otimes W).$$

In this paper, we shall only consider the case where $W = F \cdot e \oplus F \cdot f$ is 2-dimensional with $\langle e, f \rangle_W = 1$. With the Witt basis $\{e, f\}$, we may identify $\mathrm{Sp}(W)$ with $\mathrm{SL}_2(F)$, and we let $B = T \cdot N$ be the Borel subgroup which stabilises the line $F \cdot e$ (so that B is upper triangular). In particular, the conventions we have set up in §4.3 for SL_2 apply to $\mathrm{Sp}(W)$.

Attached to a fixed nontrivial additive character ψ of F and other auxiliary data, this dual pair has a distinguished representation Ω_ψ known as the Weil representation. To be precise, if $\dim V$ is odd, we need to work with the metaplectic double cover $\text{Mp}_2(F)$ of $\text{SL}_2(F)$. To simplify notation, we shall ignore this issue; the reader may assume $\dim V$ is even. We refer the reader to [11, 12] for the metaplectic cases.

To describe the Weil representation Ω_ψ , we first need to endow the vector space V with a Haar measure. Let $\langle -, - \rangle$ be the symmetric bilinear form associated to the quadratic form q on V , so that $\langle v_1, v_2 \rangle = q(v_1 + v_2) - q(v_1) - q(v_2)$. Then one has an S^1 -valued nondegenerate pairing $\psi(\langle -, - \rangle)$ on V . We then equip V with the Haar measure $d_\psi v$ which is self-dual with respect to the Fourier transform defined by this pairing and observe that $d_\psi v$ is $\text{O}(V)$ -invariant. The unitary representation Ω_ψ can be realised on $L^2(f \otimes V) = L^2(V)$, where the inner product is defined using the Haar measure $d_\psi v$. The action of various elements of $\text{SL}_2(F) \times \text{O}(V)$ via Ω_ψ is given as follows:

$$\begin{cases} h \cdot \Phi(v) = \Phi(h^{-1} \cdot v), \text{ for } h \in \text{O}(V); \\ n(b) \cdot \Phi(v) = \psi(b \cdot q(v)) \cdot \Phi(v), \text{ for } n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N; \\ t(a) \cdot \Phi(v) = |a|^{\frac{1}{2} \dim V} \chi_{\text{disc}(V)}(a) \cdot \Phi(a^2 v), \text{ for } t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T. \end{cases}$$

Here $\text{disc}(V) \in F^\times / F^{\times 2}$ is the discriminant of (V, q) and $\chi_{\text{disc}(V)}$ is the associated quadratic character of F^\times . This describes Ω_ψ as a representation of $B \times \text{O}(V)$. To describe the full action of $\text{SL}_2(F)$, one needs to give the action of a nontrivial Weyl group element

$$w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Its action is given by a normalized Fourier transform \mathcal{F} :

$$(5.1) \quad w \cdot \Phi(v) = \mathcal{F}(\Phi)(v) := \gamma_{\psi, q} \cdot \int \Phi(v') \cdot \psi(\langle v, v' \rangle) d_\psi v'$$

where $\gamma_{\psi, q}$ is a root of unity (a Weil index) whose precise value need not concern us here.

One may consider the underlying smooth representation Ω_ψ^∞ which is realized on the subspace $S(V)$ of Schwartz-Bruhat functions on V . Following our convention, we shall use Ω_ψ to denote the Weil representation in both the smooth and L^2 -setting when there is no cause for confusion.

5.2. Smooth Theta correspondence. . The theory of theta correspondence concerns the understanding of the representation Ω_ψ of $\text{SL}_2(F) \times \text{O}(V)$. One can consider this question on the level of smooth representation theory or L^2 -representation theory. In this subsection, we recall the setup of the smooth theory. Henceforth, we shall assume that $\dim V \geq 3$ (and sometimes $\dim V \geq 4$).

For $\sigma \in \text{Irr}(\text{SL}_2)$, the (smooth) big theta lift of σ to $\text{O}(V)$ is:

$$\Theta_\psi(\sigma) := (\Omega_\psi^\infty \otimes \sigma^\vee)_{\text{SL}_2}$$

where we are considering the space of SL_2 -coinvariants. With this definition, we have the natural SL_2 -invariant and $\text{O}(V)$ -equivariant projection map

$$A_\sigma : \Omega_\psi^\infty \otimes \sigma^\vee \longrightarrow \Theta_\psi(\sigma)$$

which gives by duality a canonical $\mathrm{SL}_2 \times \mathrm{O}(V)$ -equivariant map

$$\theta_\sigma : \Omega_\psi^\infty \longrightarrow \sigma \boxtimes \Theta_\psi(\sigma).$$

Likewise, for $\pi \in \mathrm{Irr}(\mathrm{O}(V))$, the (smooth) big theta lift of π to SL_2 is:

$$\Theta_\psi(\pi) := (\Omega_\psi^\infty \otimes \pi^\vee)_{\mathrm{O}(V)}$$

where we are considering the space of $\mathrm{O}(V)$ -coinvariants.

By the Howe duality principle [13], one knows that:

- the representations $\Theta_\psi(\sigma)$ and $\Theta_\psi(\pi)$ are finite length representations which (if nonzero) have unique irreducible quotients $\theta_\psi(\sigma)$ and $\theta_\psi(\pi)$ respectively (known as the small theta lifts);
- for any $\sigma_1, \sigma_2 \in \mathrm{Irr}(\mathrm{SL}_2)$,

$$\theta_\psi(\sigma_1) \cong \theta_\psi(\sigma_2) \neq 0 \implies \sigma_1 \cong \sigma_2.$$

As a consequence, we see that if $\pi := \theta_\psi(\sigma)$, then $\sigma \cong \theta_\psi(\pi)$.

Composing A_σ and θ_σ with the natural projection $\Theta_\psi(\sigma) \twoheadrightarrow \theta_\psi(\sigma)$, we have canonical equivariant maps (still denoted by the same symbols)

$$(5.2) \quad A_\sigma : \Omega \otimes \sigma^\vee \longrightarrow \theta_\psi(\sigma)$$

and

$$(5.3) \quad \theta_\sigma : \Omega_\psi^\infty \longrightarrow \sigma \boxtimes \theta_\psi(\sigma).$$

The theta correspondence for $\mathrm{SL}_2 \times \mathrm{O}(V)$ (when $\dim V$ is even) and $\mathrm{Mp}_2 \times \mathrm{O}(V)$ (when $\dim V$ is odd) was studied in great detail by Rallis [26]. His results were supplemented by later results of J.S. Li [21]. We may summarize their results by:

Proposition 5.1. *(i) Assume that $\dim V \geq 4$ is even and the Witt index $\mathrm{Witt}(V)$ of V is ≥ 2 (so that one is in the stable range). If $\sigma \in \mathrm{Irr}(\mathrm{SL}_2)$ is unitary, then $\theta(\sigma)$ is nonzero unitary, so that one has an injective map*

$$\theta_\psi : \widehat{\mathrm{SL}_2} \longrightarrow \widehat{\mathrm{O}(V)}.$$

In general, the theta correspondence gives a map

$$\theta_\psi : \widehat{\mathrm{SL}_{2\mathrm{temp}}} \longrightarrow \widehat{\mathrm{O}(V)} \cup \{0\}$$

which is injective on that part of the domain outside the preimage of 0.

(ii) Assume that $\dim V \geq 3$ is odd and $\mathrm{Witt}(V)$ is ≥ 2 . If $\sigma \in \mathrm{Irr}(\mathrm{Mp}_2)$ is a unitary genuine representation, then $\theta_\psi(\sigma)$ is nonzero unitary, so that one has an injective map

$$\theta_\psi : \widehat{\mathrm{Mp}_2} \longrightarrow \widehat{\mathrm{O}(V)}$$

where $\widehat{\mathrm{Mp}_2}$ denotes the ψ -generic genuine unitary dual of $\mathrm{Mp}_2(F)$. In general, the theta correspondence gives a map

$$\theta_\psi : \widehat{\mathrm{Mp}_{2\mathrm{temp}}} \longrightarrow \widehat{\mathrm{O}(V)} \cup \{0\}$$

which is injective on that part of the domain outside the preimage of 0.

One can in fact describe the map θ_ψ very explicitly but we will not need this description here.

5.3. Doubling zeta integral. Using the unitary structures of Ω_ψ and σ (and the Haar measure dg on SL_2), the representation $\theta_\psi(\sigma)$ of $\mathrm{O}(V)$ can be given a unitary structure by the local doubling zeta integral. More precisely, for $\Phi_1, \Phi_2 \in S(V)$ and $v_1, v_2 \in \sigma$, the local doubling zeta integral is given by:

$$(5.4) \quad Z_\sigma(\Phi_1, \Phi_2, v_1, v_2) = \int_{\mathrm{SL}_2} \langle g \cdot \Phi_1, \Phi_2 \rangle_\Omega \cdot \overline{\langle \sigma(g) \cdot v_1, v_2 \rangle_\sigma} dg,$$

which converges for tempered σ when $\dim V \geq 3$. It defines a $(\mathrm{SL}_2 \times \mathrm{SL}_2)$ -invariant and $\mathrm{O}(V)^\Delta$ -invariant map

$$Z_\sigma : \Omega_\psi \otimes \overline{\Omega_\psi} \otimes \overline{\sigma} \otimes \sigma \longrightarrow \mathbb{C}.$$

The inner product $\langle -, - \rangle_\sigma$ on σ gives an isomorphism $\overline{\sigma} \cong \sigma^\vee$. Hence, Z_σ factors through the canonical projection map

$$A_\sigma \otimes \overline{A_\sigma} : \Omega_\psi \otimes \overline{\Omega_\psi} \otimes \sigma^\vee \otimes \overline{\sigma^\vee} \longrightarrow \Theta_\psi(\sigma) \otimes \overline{\Theta_\psi(\sigma)}$$

so that

$$(5.5) \quad Z_\sigma(\Phi_1, \Phi_2, v_1, v_2) = \langle A_\sigma(\Phi_1, v_1), A_\sigma(\Phi_2, v_2) \rangle_{\theta(\sigma)}.$$

for some Hermitian form $\langle -, - \rangle_{\theta(\sigma)}$ on $\Theta_\psi(\sigma)$. We have:

Proposition 5.2. *Suppose that σ is an irreducible tempered representation of SL_2 (or Mp_2) such that $\theta_\psi(\sigma) \neq 0$. Then the Hermitian form $\langle -, - \rangle_{\theta(\sigma)}$ on $\Theta_\psi(\sigma)$ descends to a nonzero inner product on $\theta_\psi(\sigma)$.*

Proof. We note:

- If the Witt index of V is ≥ 2 (so that one is in the stable range), this is due to [21].
- In general, it was shown by Rallis [26, Prop. 6.1] that

$$\theta_\psi(\sigma) \neq 0 \iff \langle -, - \rangle_{\theta(\sigma)} \neq 0.$$

- In the archimedean case, it was shown in [16] that $\langle -, - \rangle_{\theta(\sigma)}$ descends to $\theta_\psi(\sigma)$.
- Consider the nonarchimedean case with $\mathrm{Witt}(V) \leq 1$. There are only a few cases of such, all in low rank. In these small number of low rank cases, one can verify that $\Theta_\psi(\sigma)$ is irreducible when σ is tempered.

Taken together, the proposition is proved. \square

Henceforth, we shall equip $\theta_\psi(\sigma)$ with this unitary structure; it depends on dg , $\langle -, - \rangle_\sigma$ and $d_\psi v$. By completion, we may regard $\theta_\psi(\sigma)$ as an irreducible unitary representation of $\mathrm{O}(V)$. Observe that the identity (5.5) may be considered as the local analog of the Rallis inner product formula [11]. A reformulation, using the map θ_σ instead of A_σ is:

$$(5.6) \quad \langle \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\sigma \boxtimes \theta(\sigma)} = \sum_{v \in \mathrm{ONB}(\sigma)} Z_\sigma(\Phi_1, \Phi_2, v, v)$$

where $\mathrm{ONB}(\sigma)$ denotes an orthonormal basis of σ .

5.4. L^2 -theta correspondence. Now we consider the theta correspondence in the L^2 -setting. Though we are not exactly in the setting discussed in §2, Bernstein's theory continues to apply here (see [31]). When $\dim V \geq 3$, it was shown in [10, 31] that one has a direct integral decomposition of $\mathrm{SL}_2(F) \times \mathrm{O}(V)$ -representations:

$$(5.7) \quad \Omega_\psi = L^2(V) \cong \int_{\widehat{\mathrm{SL}_2}} \sigma \boxtimes \theta_\psi(\sigma) d\mu_{\mathrm{SL}_2}(\sigma),$$

where $d\mu_{\mathrm{SL}_2}$ is the Plancherel measure of $\mathrm{SL}_2(F)$ (associated to the fixed Haar measure of SL_2). Hence the spectral measure of Ω_ψ as an SL_2 -module is absolutely continuous with respect to the Plancherel measure. Indeed, by Propositions 5.1 and 5.2, when $\mathrm{Witt}(V) \geq 2$, the support of Ω_ψ as an SL_2 -module is precisely $\widehat{\mathrm{SL}_{2temp}}$.

We need to explicate the spectral decomposition (5.7) here. By the theory of spectral decomposition à la Bernstein, one way to do this is to give a spectral decomposition of the inner product $\langle -, - \rangle_\Omega$. In [31], Sakellaridis showed that for $\Phi_1, \Phi_2 \in S(V)$,

$$\langle \Phi_1, \Phi_2 \rangle_\Omega = \int_{\widehat{\mathrm{SL}_2}} J_\sigma^\theta(\Phi_1, \Phi_2) d\mu_{\mathrm{SL}_2}(\sigma)$$

where

$$(5.8) \quad J_\sigma^\theta(\Phi_1, \Phi_2) = \langle \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\sigma \boxtimes \theta(\sigma)} = \sum_{v \in \mathrm{ONB}(\sigma)} Z_\sigma(\Phi_1, \Phi_2, v, v)$$

where the unitary structure on $\Theta(\sigma)$ is that defined in the previous subsection. What this says is that the family of canonical maps θ_σ defined in (5.3) is precisely the family of maps associated to the direct integral decomposition (5.7) for $\sigma \in \widehat{\mathrm{SL}_{2temp}}$.

Now for $\tau \in \widehat{\mathrm{SL}_{2temp}}^{ind}$, one can define $Z_\tau(\Phi_1, \Phi_2, v_1, v_2)$ by the same formula as in (5.4) and then define J_τ^θ by the formula (5.8). Then it is useful to note [40, Lemma 3.3]:

Lemma 5.3. *For fixed Φ_i the \mathbb{C} -valued function*

$$\tau \mapsto J_\tau^\theta(\Phi_1, \Phi_2)$$

is continuous in $\tau \in \widehat{\mathrm{SL}_{2temp}}^{ind}$.

5.5. The maps A_σ and $B_{\theta(\sigma)}$. We have seen the canonical maps A_σ and θ_σ in (5.2) and (5.3) which intervene in the spectral decomposition (5.7). Identifying σ^\vee with $\bar{\sigma}$ using $\langle -, - \rangle_\sigma$, we may regard A_σ as a map $\Omega_\psi \otimes \bar{\sigma} \rightarrow \theta_\psi(\sigma)$. Then A_σ and θ_σ are related by:

$$A_\sigma(\Phi, v) = \langle \theta_\sigma(\Phi), v \rangle_\sigma.$$

Likewise, we have a $\mathrm{O}(V)$ -invariant and SL_2 -equivariant map

$$B_{\theta(\sigma)} : \Omega_\psi \otimes \overline{\theta_\psi(\sigma)} \rightarrow \sigma$$

characterized by

$$B_{\theta(\sigma)}(\Phi, w) = \langle \theta_\sigma(\Phi), w \rangle_{\theta(\sigma)}.$$

The two maps are related by:

$$(5.9) \quad \langle A_\sigma(\Phi, v), w \rangle_{\theta(\sigma)} = \langle \theta_\sigma(\Phi), v \otimes w \rangle_{\sigma \otimes \theta(\sigma)} = \langle B_{\theta(\sigma)}(\Phi, w), v \rangle_\sigma$$

for $\Phi \in \Omega_\psi$, $v \in \sigma$ and $w \in \theta(\sigma)$. Moreover, the inner product J_σ^θ can be expressed in terms of A_σ and $B_{\theta(\sigma)}$ as follows:

$$J_\sigma^\theta(\Phi_1, \Phi_2) = \langle \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\sigma \otimes \theta(\sigma)} = \sum_{v \in \mathrm{ONB}(\sigma)} \langle A_\sigma(\Phi_1, v), A_\sigma(\Phi_2, v) \rangle_{\theta(\sigma)}.$$

and

$$J_\sigma^\theta(\Phi_1, \Phi_2) = \sum_{w \in \mathrm{ONB}(\theta(\sigma))} \langle B_{\theta(\sigma)}(\Phi_1, w), B_{\theta(\sigma)}(\Phi_2, w) \rangle_\sigma.$$

The maps A_σ and $B_{\theta(\sigma)}$ are local versions of global theta lifting considered in §12.

To summarize, this section discusses the smooth theta correspondence and the L^2 -theta correspondence and the relation between them. In particular, through the theory of the doubling zeta integral, we equip $\theta_\psi(\sigma)$ with a unitary structure so that there is a strong synergy between the smooth theory and the L^2 -theory.

6. Periods

It is a basic principle that theta correspondence frequently allows one to transfer periods on one member of a dual pair to the other member. For an exposition of this in the setting of smooth theta correspondence, the reader can consult [8]. On the other hand, in the setting of L^2 -theta correspondence, this principle has been exploited in [10] to establish low rank cases of the local conjecture of Sakellaridis-Venkatesh on the unitary spectrum of spherical varieties.

In this section, we shall consider the dual pair $\mathrm{SL}_2 \times \mathrm{O}(V)$ and show how the spectral decomposition à la Bernstein allows one to refine the results of [8] and [10].

6.1. Transfer of periods. We first consider periods in smooth representation theory. For $a \in F^\times$, fix a vector $v_a \in V$ with $q(v_a) = a$ (if it exists), so that $V = F \cdot v_a \oplus v_a^\perp$. Set

$$X_a = \{v \in V : q(v) = a\} \subset V,$$

which is a Zariski closed subset of V . By Witt's theorem, $\mathrm{O}(V)$ acts transitively on X_a and the stabilizer of v_a in $\mathrm{O}(V)$ is $\mathrm{O}(v_a^\perp)$. Hence

$$X_a \cong \mathrm{O}(v_a^\perp) \backslash \mathrm{O}(V)$$

via $h \mapsto h^{-1} \cdot v_a$. If v_a does not exist, we understand X_a to be empty (i.e. the algebraic variety has no F -points). To fix ideas, we shall assume that v_1 exists; this is not a serious hypothesis. We also set $\psi_a(x) = \psi(ax)$, so that ψ_a is a nontrivial additive character of F , and write $S(X_a)$ for the space of Schwartz-Bruhat functions on X_a , so that $S(X_a) = C_c^\infty(X_a)$ if F is nonarchimedean.

The following proposition essentially resolves the local problem (a) in the smooth setting for the Sakellaridis-Venkatesh conjecture highlighted in the introduction, except for the part about relative character identities. It is essentially a folklore result and a proof has been written down in [8] in a more general setting. We recount the proof here to explicate the isomorphism f_a in the proposition.

Proposition 6.1. *Let π be an irreducible smooth representation of $\mathrm{O}(V, q)$ and let $\Theta_\psi(\pi) = (\Omega_\psi \otimes \pi^\vee)_{\mathrm{SL}_2}$ be its big theta lift to SL_2 (or Mp_2 if $\dim V$ is odd). For $a \in F^\times$, there is an explicit isomorphism (to be described in the proof)*

$$f_a : \mathrm{Hom}(\Theta_\psi(\pi)_{N, \psi_a}, \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{O}(V)}(S(X_a), \pi) \cong \mathrm{Hom}_{\mathrm{O}(v_a^\perp)}(\pi^\vee, \mathbb{C}),$$

where the second isomorphism is by Frobenius reciprocity. Here, the right hand side is understood to be 0 if X_a is empty. In particular, when π is such that $\sigma := \Theta_\psi(\pi)$ is irreducible, we see that σ is ψ_a -generic if and only if π is $\mathrm{O}(v_a^\perp)$ -distinguished, in which case

$$\dim \mathrm{Hom}_{\mathrm{O}(v_a^\perp)}(\pi^\vee, \mathbb{C}) = 1.$$

Proof. We describe the proof when F is nonarchimedean. The archimedean case is based on the same ideas, and the reader can consult [15, 41] for a careful treatment.

We prove the proposition by computing the space

$$\mathrm{Hom}_{N \times \mathrm{O}(V)}(\Omega_\psi, \psi_a \boxtimes \pi)$$

in two different ways.

On one hand, let us fix any equivariant surjective map

$$\theta : \Omega_\psi \longrightarrow \Theta_\psi(\pi) \boxtimes \pi.$$

Then θ induces an isomorphism

$$\theta^* : \mathrm{Hom}_N(\Theta_\psi(\pi), \psi_a) \cong \mathrm{Hom}_{N \times \mathrm{O}(V)}(\Omega_\psi, \psi_a \otimes \pi).$$

On the other hand, for $a \in F^\times$, consider the surjective restriction map

$$\mathrm{rest} : \Omega_\psi = S(V) \longrightarrow S(X_a).$$

This map induces an equivariant isomorphism

$$\mathrm{rest} : \Omega_{N, \psi_a} \cong S(X_a).$$

Hence, we have an induced isomorphism

$$\mathrm{rest}^* : \mathrm{Hom}_{\mathrm{O}(V)}(C_c^\infty(X_a), \pi) \cong \mathrm{Hom}_{\mathrm{O}(V)}(\Omega_{N, \psi_a}, \pi) \cong \mathrm{Hom}_{N \times \mathrm{O}(V)}(\Omega_\psi, \psi_a \otimes \pi).$$

Since

$$S(X_a) \cong \mathrm{ind}_{\mathrm{O}(v_a^\perp)}^{\mathrm{O}(V)} \mathbb{C},$$

it follows by Frobenius reciprocity that one has the desired isomorphism:

$$f_a^{-1} : \mathrm{Hom}_{\mathrm{O}(v_a^\perp)}(\pi^\vee, \mathbb{C}) \xrightarrow{\mathrm{Frob}} \mathrm{Hom}_{\mathrm{O}(V)}(S(X_a), \pi) \xrightarrow{(\theta^*)^{-1} \circ \mathrm{rest}^*} \mathrm{Hom}_N(\Theta_\psi(\pi), \psi_a).$$

This proves the proposition. \square

The purpose of recounting the proof of the proposition is to bring forth the point that the isomorphism

$$f_a : \mathrm{Hom}(\Theta_\psi(\pi)_{N, \psi_a}, \mathbb{C}) \cong \mathrm{Hom}_{\mathrm{O}(v_a^\perp)}(\pi^\vee, \mathbb{C})$$

essentially depends only on the choice of the equivariant projection map

$$\theta : \Omega_\psi \longrightarrow \Theta_\psi(\pi) \boxtimes \pi.$$

On the other hand, when σ is an irreducible tempered representation of SL_2 with $\theta_\psi(\sigma) \neq 0$, we have seen in (5.7) that there is a canonical map

$$\theta_\sigma : \Omega_\psi \longrightarrow \sigma \boxtimes \theta_\psi(\sigma).$$

Repeating the proof of the proposition using this map θ_σ , we obtain an injective map

$$(6.1) \quad f_a : \mathrm{Hom}_N(\sigma, \psi_a) \longrightarrow \mathrm{Hom}_{\mathrm{O}(V)}(S(X_a), \theta_\psi(\sigma)).$$

It is an isomorphism if $\Theta_\psi(\theta_\psi(\sigma)) \cong \sigma$ (by Proposition 6.1) or if σ is ψ_a -generic (since the target space has dimension at most 1 by [1])

6.2. Unitary structure on $L^2(X_a)$. We may begin our investigation of the local problem (b) in the Sakellaridis-Venkatesh conjecture, i.e. in the L^2 -setting. With $a \in F^\times$, we have seen that

$$X_a = \{v \in V : q(v) = a\} \cong \mathrm{O}(v_a^\perp) \backslash \mathrm{O}(V),$$

under $h^{-1}v_a \longleftrightarrow h$ (assuming $X_a(F)$ has a point v_a). We may equip X_a with an $\mathrm{O}(V)$ -invariant measure and consider the space $L^2(X_a)$; of course the space $L^2(X_a)$ does not depend on the choice of the $\mathrm{O}(V)$ -invariant measure but the unitary structure does. In [10], using the spectral decomposition (5.7) of Ω_ψ , one obtains a spectral decomposition of $L^2(X_a)$. However, we wish to refine the results of [10] by being more precise about the unitary structures and invariant measures used here.

The hyperboloids X_a are precisely the fibers of the $\mathrm{O}(V)$ -invariant map given by the quadratic form $q : V \longrightarrow F$. This map is submersive at all points of V outside the zero vector. In particular, if we ignore the null cone X_0 and consider the map q over the Zariski open subset $F^\times \subset F$, the Haar measures $d_\psi v$ and $d_\psi x$ we have already fixed for V and F induces an $\mathrm{O}(V)$ -invariant measure $|\omega_a|$ for each fiber X_a (with $a \in F^\times$), characterized by: for any compactly-supported smooth functions f on $V \setminus X_0$,

$$\int_V f(v) d_\psi(v) = \int_{F^\times} \left(\int_{X_a} f \cdot |\omega_a| \right) d_\psi a,$$

where the function of $a \in F^\times$ defined by the inner integral on the right-hand-side is smooth and compactly supported. It is these measures $|\omega_a|$ that we shall use on X_a . Hence, we shall be considering $L^2(X_a, |\omega_a|)$.

The map $q : V \longrightarrow F$ is F^\times -equivariant where $t \in F^\times$ acts by scaling on V and via $x \mapsto t^2 x$ on F . The measure $d_\psi v$ on V is $\mathrm{O}(V)$ -invariant and satisfies: for $b \in F^\times$,

$$d_\psi(bv) = |b|^{\dim V} \cdot d_\psi v.$$

This homogeneity property implies the following property of the family of measures $|\omega_a|$. For $b \in F^\times$, scalar multiplication-by- b gives an isomorphism of varieties $\lambda_b : X_a \longrightarrow X_{ab^2}$ and one may consider the pushforward measure $(\lambda_b)_*(|\omega_a|)$ on X_{ab^2} .

Lemma 6.2. *In the above context, one has:*

$$(\lambda_b)_*(|\omega_a|) = |b|^{2-\dim V} \cdot |\omega_{ab^2}|$$

for any $a, b \in F^\times$.

Proof. For $f \in C_c^\infty(V \setminus X_0)$ and fixed $b \in F^\times$, we have

$$(6.2) \quad \int_V f(bv) dv = |b|^{-\dim V} \cdot \int_V f(v) dv.$$

The left-hand-side of (6.2) is given by

$$\begin{aligned} \int_V f(bv) dv &= \int_{F^\times} \int_{X_a} f(bx) \cdot |\omega_a(x)| d_\psi a \\ &= \int_{a \in F^\times} \int_{X_a} \lambda_b^*(f)(x) \cdot |\omega_a(x)| d_\psi a \\ &= \int_{a \in F^\times} \int_{X_{ab^2}} f(x) \cdot |(\lambda_b)_*(\omega_a)| d_\psi a \\ &= \int_{c \in F^\times} \int_{X_c} f(x) \cdot |(\lambda_b)_*(\omega_{cb^{-2}})| \cdot |b|^{-2} d_\psi c \end{aligned}$$

where in the last equality, we have made a change of variables by setting $c = ab^2$, so that $d_\psi c = |b|^2 \cdot d_\psi a$.

On the other hand, the right hand side of (6.2) is given by

$$|b|^{-\dim V} \cdot \int_V f(v) dv = |b|^{-\dim V} \cdot \int_{c \in F^\times} \int_{X_c} f(x) \cdot |\omega_c(x)| d_\psi c.$$

Comparing the two sides, one obtains:

$$|(\lambda_b)_*(\omega_{cb^{-2}})| = |b|^{2-\dim V} \cdot |\omega_c|$$

which is the desired assertion. \square

Now observe that (on the level of F -valued points),

$$V \setminus X_0 = \bigcup_{[a] \in F^{\times 2} \setminus F^\times} F^\times \cdot X_a \subset V$$

is open dense with complement of measure 0. Hence the measure $d_\psi v$ induces $O(V)$ -invariant measures on each of the open sets $Y_a := F^\times \cdot X_a$ and we have

$$L^2(V) = \bigoplus_{[a] \in F^{\times 2} \setminus F^\times} L^2(Y_a)$$

We would like a more direct description of the unitary structures on the Hilbert spaces $L^2(Y_a)$ in terms of appropriate invariant measures on $F^\times \times X_a$.

Consider the natural surjective map

$$m : F^\times \times X_a \longrightarrow Y_a = F^\times \cdot X_a,$$

defined by $m(t, x) \mapsto t \cdot x$. This map m induces an isomorphism $\mu_2 \backslash (F^\times \times X_a) \cong Y_a$ where $\mu_2 = \{\pm 1\}$ acts diagonally on $F^\times \times X_a$ by scaling on each factor. In terms of the identification $X_A = O(v_a^\perp) \backslash O(V)$, this action of μ_2 on X_a is the left-translation action of $O(v_a) = \mu_2$ (where $O(v_a)$ is the orthogonal group of the 1-dimensional quadratic space $F \cdot v_a$) which commutes with the right-translation action of $O(V)$. In any case, via $f \mapsto m^*(f)$, we may identify functions on Y_a with functions on $F^\times \times X_a$ invariant under the action of μ_2 . Now we note:

Lemma 6.3. *For a smooth compactly supported function f on Y_a , one has:*

$$\int_{Y_a} f(v) d_\psi v = \frac{1}{2} \cdot |2a|_F \cdot \int_{b \in F^\times} \left(\int_{x \in X_a} f(b \cdot x) \cdot |\omega_a(x)| \right) \cdot |b|^{\dim V} \frac{d_\psi b}{|b|}.$$

Hence, if we define

$$\phi_f : F^\times \times X_a \longrightarrow \mathbb{C}$$

by

$$\phi_f(b, x) = \sqrt{|a|} \cdot |b|^{\frac{\dim V}{2}} \cdot f(b \cdot x),$$

then one has

$$\langle f, f \rangle_\Omega = \frac{1}{2} \cdot |2|_F \cdot \int_{F^\times} \int_{X_a} |\phi_f(b, x)|^2 |\omega_a(x)| d_\psi^\times b.$$

In particular, the map $f \mapsto \phi_f$ defines an isometric isomorphism

$$L^2(Y_a, d_\psi v) \longrightarrow L^2(F^\times \times X_a)^{\Delta_{\mu_2}}$$

where the unitary structure on the left is defined by $d_\psi v$ and that on the right is defined by the $O(V)$ -invariant measure $|\omega_a|$ on X_a and the measure $|2|_F/2 \cdot d_\psi^\times t$ of F^\times (defined in §4.3). Further, for $\epsilon = \pm$, if we let $L^2(Y_a)^\epsilon$ and $L^2(F^\times)^\epsilon$ denote the ϵ -eigenspace of the μ_2 -action, then

$$L^2(Y_a) \cong \bigoplus_{\epsilon=\pm} L^2(F^\times)^\epsilon \widehat{\otimes} L^2(X_a)^\epsilon.$$

Proof. We consider $f \in C_c^\infty(Y_a)$. Then

$$\begin{aligned} \int_{Y_a} f(y) dy &= \int_{t \in aF^{\times 2}} \int_{X_t} f(x) \cdot |\omega_t(x)| d_\psi t \\ &= \frac{1}{2} \int_{b \in F^\times} \int_{X_{ab^2}} f(x) \cdot |\omega_{ab^2}(x)| \cdot |2ab|_F d_\psi b \\ &= \frac{1}{2} \cdot |2a|_F \cdot \int_{b \in F^\times} \int_{X_{ab^2}} f(x) \cdot |(\lambda_b)_*(|\omega_a|)| \cdot |b|^{\dim V} \frac{d_\psi b}{|b|} \\ &= \frac{1}{2} \cdot |2a|_F \cdot \int_{b \in F^\times} \int_{X_a} \lambda_b^*(f)(x) \cdot |\omega_a(x)| \cdot |b|^{\dim V} d_\psi^\times b \\ (6.3) \quad &= \frac{1}{2} \cdot |2a|_F \cdot \int_{b \in F^\times} \int_{X_a} f(b \cdot x) \cdot |\omega_a(x)| \cdot |b|^{\dim V} d_\psi^\times b. \end{aligned}$$

Here, in the second equality, we have made a change of variables, replacing t by ab^2 , so that $d_\psi t = |2ab|_F \cdot d_\psi b$, whereas in the third equality, we have applied Lemma 6.2. This establishes the lemma. \square

6.3. Spectral decomposition of $L^2(X_a)$. We are now ready to show the direct integral decomposition of the unitary representation $L^2(X_a, |\omega_a|)$ of $O(V)$, where the unitary structure is determined by the $O(V)$ -invariant measures $|\omega_a|$. Observe that $L^2(X_a, |\omega_a|)$ is in fact a representation of $O(v_a) \times O(V)$, where $O(v_a) \cong \mu_2$ acts by left translation on $O(v_a^\perp) \backslash O(V)$ (this is the action of scaling by -1 on X_a). This gives a decomposition

$$L^2(X_a) = L^2(X_a)^+ \oplus L^2(X_a)^-$$

into $O(V)$ -submodules which are the \pm -eigenspaces of the $O(v_a)$ -action.

As mentioned before, the spectral decomposition of $L^2(X_a)$ as an $O(V)$ -module has been obtained in [10]. The following proposition is a special case of the results in [10]; we recount the proof here to explicate certain isomorphisms used in the course of the proof.

Proposition 6.4. *We have an explicit isomorphism (to be described in the proof):*

$$L^2(X_a, |\omega_a|) \cong \int_{\widehat{\text{SL}}_2} \dim \text{Hom}_N(\sigma, \psi_a) \cdot \theta_\psi(\sigma) d\mu_{\text{SL}_2}(\sigma).$$

Proof. We shall exploit the spectral decomposition of the unitary Weil representation Ω_ψ of $\text{SL}_2(F) \times O(V)$ on $L^2(V)$ given in (5.7). More precisely, we shall consider its restriction to $B \times O(V)$. We have seen that

$$\bigcup_{[a] \in F^{\times 2} \backslash F^\times} Y_a = \bigcup_{[a] \in F^{\times 2} \backslash F^\times} F^\times \cdot X_a \subset V$$

is open dense (with complement of measure 0), so that

$$L^2(V) = \bigoplus_{a \in F^{\times 2} \backslash F^\times} L^2(Y_a)$$

From the formulae for the action of $B \times \mathrm{O}(V)$ on Ω_ψ , one sees that the subspace $L^2(Y_a)$ is stable under the action of $B \times \mathrm{O}(V)$ and thus this is a decomposition of $B \times \mathrm{O}(V)$ -modules. Moreover, with $T \cong F^\times$ acting on V by scaling, we see that $T \times \mathrm{O}(V)$ acts transitively on Y_a and the stabilizer of $v_a \in X_a$ is the subgroup

$$\mu_2^\Delta \times \mathrm{O}(v_a^\perp) \subset Z \times \mathrm{O}(v_a) \times \mathrm{O}(v_a^\perp),$$

where $Z = \mu_2$ is the center of $\mathrm{SL}_2(F)$ and $\mathrm{O}(v_a) = \mu_2$. Thus, the isometry $f \mapsto \phi_f$ described in Lemma 6.3 gives an $B \times \mathrm{O}(V)$ -equivariant isometric map

$$L^2(Y_a, d_\psi v) \cong \bigoplus_{\epsilon=\pm} (\mathrm{ind}_{ZN}^B \epsilon \otimes \psi_a) \widehat{\otimes} L^2(X_a)^\epsilon = \bigoplus_{\epsilon=\pm} L^2(N, \psi_a \setminus B)^\epsilon \widehat{\otimes} L^2(X_a)^\epsilon = L^2(F^\times \times X_a)^{\mu_2},$$

where the unitary structure on $L^2(N, \psi_a \setminus B)^\epsilon$ is given by that defined in Lemma 6.3 or equivalently in Proposition 4.2. Hence, we conclude that

$$(6.4) \quad \Omega_\psi \cong \bigoplus_{a \in F^{\times 2} \setminus F^\times} L^2(F^\times \times X_a)^{\Delta \mu_2} \cong \bigoplus_{a \in F^{\times 2} \setminus F^\times} \bigoplus_{\epsilon=\pm} L^2(N, \psi_a \setminus B)^\epsilon \widehat{\otimes} L^2(X_a)^\epsilon$$

via

$$f \mapsto \left(\phi_f|_{Y_a} \right)_{a \in F^{\times 2} \setminus F^\times}.$$

On the other hand, by (5.7), one has:

$$\iota : L^2(V) \cong \int_{\widehat{\mathrm{SL}_2}} \sigma \otimes \theta_\psi(\sigma) d\mu_{\mathrm{SL}_2}(\sigma)$$

as $\mathrm{SL}_2 \times \mathrm{O}(V)$ -modules. Restricting from SL_2 to B , Corollary 4.3 gives an isometric B -equivariant isomorphism

$$j_\sigma = \bigoplus_{a \in F^\times \setminus F^{\times 2}} j_{\sigma,a} : \sigma|_B \cong \bigoplus_{[a] \in F^{\times 2} \setminus F^\times} \dim \sigma_{N, \psi_a} \cdot L^2(N, \psi_a \setminus B)^{z_\sigma}$$

where $z_\sigma = \pm$ denotes the central character of σ and the unitary structure on the right-hand-side is as in Lemma 6.3. Hence, via $j_\sigma \circ \iota$ for each σ , one has a unitary isomorphism:

$$(6.5) \quad \Omega_\psi \cong \bigoplus_{a \in F^{\times 2} \setminus F^\times} \bigoplus_{\epsilon=\pm} L^2(N, \psi_a \setminus B)^\epsilon \widehat{\otimes} \int_{\widehat{\mathrm{SL}_2}} \dim \sigma_{N, \psi_a} \cdot \theta_\psi(\sigma) \cdot 1(z_\sigma = \epsilon) d\mu_{\mathrm{SL}_2}(\sigma).$$

Comparing the two descriptions of $\Omega = L^2(V)$ as a $B \times \mathrm{O}(V)$ -module given in (6.4) and (6.5), one obtains an isomorphism

$$L^2(X_a)^\epsilon \cong \int_{\widehat{\mathrm{SL}_2}} \dim \sigma_{N, \psi_a} \cdot \theta_\psi(\sigma) \cdot 1(\omega_\sigma = \epsilon) d\mu_{\mathrm{SL}_2}(\sigma).$$

for $\epsilon = \pm$. Summing over ϵ , we obtain the desired isomorphism in the proposition. \square

6.4. A commutative diagram. Examining the proof of Proposition 6.4, the unitary isomorphism there can be explicated as follows. Given $f \in S(X_a)$, we first choose $\Phi \in S(V)$ such that

$$|a|^{1/2} \cdot \mathrm{rest}(\Phi) = f.$$

Then the image of f under the isomorphism of Proposition 6.4 is represented by the measurable section of the direct integral decomposition given by

$$\sigma \mapsto |a|^{1/2} \cdot \ell_{\sigma, \psi_a}(\theta_\sigma(\Phi)),$$

where θ_σ is as given in (5.3) and ℓ_{σ, ψ_a} is the ψ_a -Whittaker functional arising from the Whittaker-Plancherel theorem for $L^2(N, \psi_a \backslash \mathrm{SL}_2)$. In other words, we have:

Proposition 6.5. *For each $\sigma \in \widehat{\mathrm{SL}}_{2\mathrm{temp}, \psi_a}$, there is a commutative diagram:*

$$\begin{array}{ccc}
 & S(V) & \\
 \text{rest} \swarrow & & \searrow \theta_\sigma \\
 S(X_a) & & \sigma \boxtimes \theta(\sigma) \\
 \searrow \alpha_{\theta(\sigma), a} & & \swarrow \ell_{\sigma, \psi_a} \\
 & \mathbb{C}_{\psi_a} \otimes \theta(\sigma) &
 \end{array}$$

where $\alpha_{\theta(\sigma), a}$ is the morphism associated to the direct integral decomposition of Proposition 6.4.

This proposition gives a precise relation between the transfer of periods in the smooth setting and the spectral decomposition of $L^2(N, \psi_a \backslash \mathrm{SL}_2)$, $L^2(X_a, |\omega_a|)$ and Ω_ψ in the L^2 -theory. Indeed, it is fairly clear that one has a commutative diagram as in the proposition up to scalars. The point of the proposition is to explicate the scalar. More precisely, specializing to the case $a = 1$, one has:

Corollary 6.6. *Under the isomorphism*

$$f_\sigma : \mathrm{Hom}_N(\sigma, \psi) \cong \mathrm{Hom}_{\mathrm{O}(V)}(C_c^\infty(X_1), \theta_\psi(\sigma))$$

given in (6.1) for $\sigma \in \widehat{\mathrm{SL}}_{2\mathrm{temp}, \psi}$ (which is induced by the map θ_σ of (5.3)), one has:

$$(6.6) \quad f_\sigma(\ell_\sigma) = \alpha_{\theta(\sigma)}$$

where the Whittaker functional ℓ_σ is the one in (3.4) which intervenes in the Whittaker-Plancherel theorem for $(N, \psi) \backslash \mathrm{SL}_2$ and the morphism $\alpha_{\theta(\sigma)}$ is the one which intervenes in the spectral decomposition of $L^2(X_1, |\omega_1|)$ obtained in Proposition 6.4.

Another way to interpret Corollary 6.6 is that if one defines the elements $\alpha_{\theta(\sigma)}$ by (6.6) or equivalently by requiring that the diagram in Proposition 6.5 be commutative, then the family $\{\alpha_{\theta(\sigma)} : \sigma \in \widehat{\mathrm{SL}}_{2\mathrm{temp}, \psi}\}$ induces the spectral decomposition of $L^2(X_1, |\omega_1|)$ in Proposition 6.4.

We conclude this section with a few formal consequences of Proposition 6.5. The commutative diagram in Proposition 6.5 gives an identity in $\theta(\sigma)$. If we pair both sides of the identity with a vector in $\theta_\psi(\sigma)$, using the inner product on $\theta_\psi(\sigma)$, we obtain:

Corollary 6.7. *For any $\Phi \in S(V) = \Omega_\psi$ and $w \in \theta(\sigma)$, one has*

$$\ell_\sigma(B_{\theta(\sigma)}(\Phi, w)) = \langle \Phi|_X, \beta_{\theta(\sigma)}(w) \rangle_X,$$

where $B_{\theta(\sigma)}$ was defined in §5.5.

Proof. We have

$$\langle \Phi|_X, \beta_{\theta(\sigma)}(w) \rangle_X = \langle \alpha_{\theta(\sigma)}(\Phi|_X), w \rangle_{\theta(\sigma)} = \langle \ell_\sigma(\theta_\sigma(\Phi)), w \rangle_{\theta(\sigma)} = \ell_\sigma(\langle \theta_\sigma(\Phi), w \rangle_{\theta(\sigma)}) = \ell_\sigma(B_{\theta(\sigma)}(\Phi, w)).$$

□

We may also “double-up” the commutative diagram in Proposition 6.5 and contract the resulting doubled identity using the inner product on $\theta_\psi(\sigma)$. This gives:

Corollary 6.8. *For $\Phi_1, \Phi_2 \in S(V) = \Omega_\psi$, one has:*

$$J_{\theta(\sigma)}(\Phi_1|_X, \Phi_2|_X) = \int_N^* \overline{\psi(n)} \cdot J_\sigma^\theta(n \cdot \Phi_1, \Phi_2) \, dn.$$

For fixed $\Phi_1|_X$ and $\Phi_2|_X$ in $S(X)$, the \mathbb{C} -valued function $\sigma \mapsto J_{\theta(\sigma)}(\Phi_1|_X, \Phi_2|_X)$ is continuous in $\sigma \in \widehat{\text{SL}}_{2\text{temp}, \psi}$.

Proof. We have

$$\begin{aligned} & J_{\theta(\sigma)}(\Phi_1|_X, \Phi_2|_X) \\ &= \langle \alpha_{\theta(\sigma)}(\Phi_1|_X), \alpha_{\theta(\sigma)}(\Phi_2|_X) \rangle_{\theta(\sigma)} && \text{(definition of } J_{\theta(\sigma)}\text{)} \\ &= \langle \ell_\sigma(\theta_\sigma(\Phi_1)), \ell_\sigma(\theta_\sigma(\Phi_2)) \rangle_{\theta(\sigma)} && \text{(by Proposition 6.5)} \\ &= \ell_\sigma \otimes \overline{\ell_\sigma} (\langle \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\theta(\sigma)}) && \text{(clear)} \\ &= \int_N^* \overline{\psi(n)} \cdot \langle n \cdot \theta_\sigma(\Phi_1), \theta_\sigma(\Phi_2) \rangle_{\sigma \otimes \theta(\sigma)} \, dn && \text{(formula for } \ell_\sigma \otimes \overline{\ell_\sigma}\text{)} \\ &= \int_N^* \overline{\psi(n)} \cdot J_\sigma^\theta(n \cdot \Phi_1, \Phi_2) \, dn && \text{(definition of } J_\sigma^\theta\text{)} \end{aligned}$$

The continuity of $\sigma \mapsto J_{\theta(\sigma)}(\Phi_1|_X, \Phi_2|_X)$ follows from the above formula, together with Lemma 3.2 and Lemma 5.3. \square

The last two corollaries thus give different variants of the identity in Proposition 6.5.

7. Relative Characters

In this section, we briefly recall the notion of the relative character associated to a period in its various incarnations.

7.1. Relative characters. Suppose that, for $i = 1$ or 2 , $H_i \subset G$ is a subgroup of G and $\chi_i : H_i(F) \rightarrow S^1$ a unitary character of $H_i(F)$. We fix also Haar measures dg on G and dh_i on H_i . For any $\pi \in \widehat{G}$ and $L_i \in \text{Hom}_{H_i}(\pi, \chi_i)$, one can associate a distribution on G as follows. Given $(f_1, f_2) \in C_c^\infty(G) \times C_c^\infty(G)$, one sets:

$$(7.1) \quad \mathcal{B}_{\pi, L_1, L_2}(f_1, f_2) = \sum_{v \in \text{ONB}(\pi)} \overline{L_1(\pi(\overline{f_1})(v))} \cdot L_2(\pi(\overline{f_2})(v))$$

where the sum runs over an orthonormal basis of π . The sum defining $\mathcal{B}_{\pi, L_1, L_2}(f_1, f_2)$ is independent of the choice of the orthonormal basis. It gives a linear map

$$\mathcal{B}_{\pi, L_1, L_2} : C_c^\infty(G) \otimes \overline{C_c^\infty(G)} \longrightarrow \mathbb{C}$$

which is $G(F)^\Delta$ -invariant (and which depends on the Haar measure dg).

The distribution $\mathcal{B}_{\pi, L_1, L_2}$ is called the relative character of π with respect to (L_1, L_2) . Note that, in the literature, it is frequent to find a different convention in the definition of the relative character, using instead the sum

$$\sum_{v \in \text{ONB}(\pi)} L_1(\pi(f_1)(v)) \cdot \overline{L_2(\pi(f_2)(v))}.$$

The difference between the two conventions is merely one of form rather than substance, and it is easy to convert from one convention to the other using complex conjugation. We choose the normalisation given above so as to avoid the appearance of multiple complex conjugations in later formulae.

Now a short computation gives

$$\overline{L_i(\pi(\overline{f_i})v)} = \int_{H_i \backslash G} \overline{L_i(\pi(g_i)(v))} \cdot (f_i)_{H_i, \chi_i}(g) \frac{dg}{dh_i}$$

with

$$(f_i)_{H_i, \chi_i}(g) = \int_{H_i} f(h_i g) \cdot \overline{\chi_i(h_i)} dh_i.$$

Hence one deduces that the linear form $\mathcal{B}_{\pi, L_1, L_2}$ factors as

$$C_c^\infty(G) \otimes \overline{C_c^\infty(G)} \twoheadrightarrow C_c^\infty(H_1, \chi_1 \backslash G) \otimes \overline{C_c^\infty(H_2, \chi_2 \backslash G)} \longrightarrow \mathbb{C}.$$

We may think of $C_c^\infty(H_i, \chi_i \backslash G(F))$ as the space of compactly supported smooth sections of the line bundle on $X_i = H_i \backslash G$ determined by χ_i and denote this space by the alternative notation $C_c^\infty(X_i, \chi_i)$. Then we shall think of $\mathcal{B}_{\pi, L_1, L_2}$ as a G^Δ -invariant linear form on $C_c^\infty(X_1, \chi_1) \otimes \overline{C_c^\infty(X_2, \chi_2)}$; this now depends on the Haar measures dg and dh_i , or rather on the G -invariant measure dg/dh_i on $H_i \backslash G$.

Let us write:

$$f_{L, v}(g) = L(\pi(g)v)$$

for the matrix coefficient associated to $L \in \pi^*$ and $v \in \pi$. Then the distribution $\mathcal{B}_{\pi, L_1, L_2}$ is given by the formula

$$(7.2) \quad \mathcal{B}_{\pi, L_1, L_2}(\phi_1, \phi_2) = \sum_{v \in \text{ONB}(\pi)} \langle \phi_1, f_{L_1, v} \rangle_{X_1} \cdot \langle f_{L_2, v}, \phi_2 \rangle_{X_2},$$

where $\langle -, - \rangle_{X_i}$ is the inner product defined by the measure dg/dh_i .

7.2. Alternative incarnation. We can also give an alternative formulation of the notion of relative characters. Continuing with the context of §7.1, it is not difficult to verify that

$$\mathcal{B}_{\pi, L_1, L_2}(f_1, f_2) = \sum_{v \in \text{ONB}(\pi)} \overline{L_1(\pi(\overline{f_1 * f_2^\vee})(v))} \cdot L_2(v),$$

where

$$f_2^\vee(g) = \overline{f_2(g^{-1})},$$

and

$$(f_1 * f_2)(g) = \int_G f_1(gx^{-1}) \cdot f_2(x) dx$$

is the convolution of f_1 and f_2 .

Thus, we may alternatively define $\mathcal{B}_{\pi, L_1, L_2}$ as a linear form

$$\mathcal{B}_{\pi, L_1, L_2} : C_c^\infty(G) \longrightarrow \mathbb{C}$$

given by the formula

$$\mathcal{B}_{\pi, L_1, L_2}(f) = \sum_{v \in \text{ONB}(\pi)} \overline{L_1(\pi(\overline{f})(v))} \cdot L_2(v).$$

As in §7.1, this linear form factors as:

$$\mathcal{B}_{\pi, L_1, L_2} : C_c^\infty(G) \twoheadrightarrow C_c^\infty(X_1, \chi_1) \longrightarrow \mathbb{C},$$

so that we may regard it as a linear form on $C_c^\infty(X_1, \chi_1)$, given by the formula

$$(7.3) \quad \mathcal{B}_{\pi, L_1, L_2}(\phi) = \sum_{v \in \text{ONB}(\pi)} \langle \phi, f_{L_1, v} \rangle_{X_1} \cdot f_{L_2, v}(1).$$

In fact, it further factors as:

$$\mathcal{B}_{\pi, L_1, L_2} : C_c^\infty(X_1, \chi_1) \twoheadrightarrow C_c^\infty(X_1, \chi_1)_{H_2, \chi_2} \longrightarrow \mathbb{C}.$$

From this alternative description of the relative character, we can recover the previous version discussed in the previous subsection by using the fact that any $f \in C_c^\infty(G)$ can be expressed as a finite linear combination of $f_1 * f_2$. This is clear in the nonarchimedean case and is a result of Dixmier-Malliavin in the archimedean case.

7.3. J_σ as a relative character. We shall now relate the notion of relative character with the theory of direct integral decomposition.

We shall focus on the case when $H_1 = H_2 = H$ and $\chi_1 = \chi_2 = \chi$ and such that $\dim \text{Hom}_H(\pi, \chi) \leq 1$. With $X = H \backslash G$, equipped with a G -invariant measure dx , suppose one has a direct integral decomposition:

$$L^2(X, \chi, dx) := L^2(H, \chi \backslash G) = \int_{\Omega} \sigma(\omega) \, d\mu(\omega)$$

with associated families of maps $\{\alpha_{\sigma(\omega)}\}$ and $\{\beta_{\sigma(\omega)}\}$ (see §2.3) and associated decomposition of inner product as in (2.3):

$$\langle -, - \rangle_X = \int_{\Omega} J_{\sigma(\omega)}(-, -) \, d\mu(\omega).$$

Observe that the positive semidefinite Hermitian form $J_{\sigma(\omega)}$ is a G^Δ -invariant linear form

$$J_{\sigma(\omega)} : C_c^\infty(X, \chi) \otimes \overline{C_c^\infty(X, \chi)} \longrightarrow \mathbb{C}.$$

This suggests that $J_{\sigma(\omega)}$ may be regarded as a relative character according to our definition in §7.1. Indeed, one has:

Lemma 7.1. *One has $J_{\sigma(\omega)} = \mathcal{B}_{\sigma(\omega), \ell_{\sigma(\omega)}, \ell_{\sigma(\omega)}}$, where $\ell_{\sigma(\omega)} = ev_1 \circ \beta_{\sigma(\omega)} \in \text{Hom}_H(\sigma(\omega), \chi)$.*

Proof. Since ω is fixed in the proposition, we shall write $\sigma = \sigma(\omega)$ for simplicity. Now we have:

$$\begin{aligned} J_\sigma(\phi_1, \phi_2) &= \langle \alpha_\sigma(\phi_1), \alpha_\sigma(\phi_2) \rangle_\sigma = \sum_{v \in \text{ONB}(\sigma)} \langle \alpha_\sigma(\phi_1), v \rangle_\sigma \cdot \langle v, \alpha_\sigma(\phi_2) \rangle_\sigma \\ &= \sum_{v \in \text{ONB}(\sigma)} \langle \phi_1, \beta_\sigma(v) \rangle_X \cdot \langle \beta_\sigma(v), \phi_2 \rangle_X. \end{aligned}$$

Noting that

$$\beta_\sigma(v)(g) = ev_1 \circ \beta_\sigma(\sigma(g)(v)) = \ell_\sigma(\sigma(g)(v)) = f_{\ell_\sigma, v}(g),$$

we see that the lemma follows by the equation (7.2). □

We can also work with the alternative context of §7.2. In this incarnation, one has:

Lemma 7.2. *As a linear form on $C_c^\infty(X, \chi)$, one has*

$$\mathcal{B}_{\sigma(\omega), \ell_{\sigma(\omega)}, \ell_{\sigma(\omega)}}(\phi) = \ell_{\sigma(\omega)}(\alpha_{\sigma(\omega)}(\phi)) = \beta_{\sigma(\omega)} \alpha_{\sigma(\omega)}(\phi)(1).$$

Proof. We write $\sigma = \sigma(\omega)$ for simplicity. Then we have

$$\alpha_\sigma(\phi) = \sum_{v \in \text{ONB}(\sigma)} \langle \alpha_\sigma(\phi), v \rangle_\sigma \cdot v = \sum_v \langle \phi, f_{\ell_\sigma, v} \rangle_X \cdot v.$$

Hence

$$\ell_\sigma(\alpha_\sigma(\phi)) = \sum_v \langle \phi, f_{\ell_\sigma, v} \rangle_X \cdot \ell_\sigma(v),$$

so that the lemma follows by equation (7.3). \square

Corollary 7.3. *Let $\mathcal{C}(X, \chi)$ be the Harish-Chandra-Schwarz space of $X = (H, \chi) \backslash G$. Then the relative character $\mathcal{B}_{\sigma, \ell_\sigma, \ell_\sigma}$ extends to $\mathcal{C}(X, \chi)$.*

7.4. Space of orbital integrals. Set

$$\mathcal{I}(X, \chi) := C_c^\infty(X, \chi)_{H, \chi}.$$

We think of this as “the space of orbital integrals” on X . Indeed, given a H -orbit on X , the associated orbital integral factors to $\mathcal{I}(X, \chi)$. As noted above, the relative character $\mathcal{B}_{\sigma, \ell_\sigma, \ell_\sigma}$ factors to give a linear form on $\mathcal{I}(X, \chi)$. Henceforth, we will write $\mathcal{B}_{\sigma, \ell_\sigma}$ in place of $\mathcal{B}_{\sigma, \ell_\sigma, \ell_\sigma}$ to simplify notation.

8. Transfer of Test Functions

If two periods on the two members of a dual pair are related by theta correspondence as in Proposition 6.1, then one might ask if the associated relative characters are related in a precise way. Such a relation is called a relative character identity. To compare the two relative characters in question, which are distributions on different spaces, we first need to define a correspondence of the relevant spaces of test functions.

8.1. A Correspondence of Test Functions. The considerations of the previous sections suggest that one considers the following maps. Set

$$p : S(V) = \Omega^\infty \longrightarrow C^\infty(N, \psi \backslash \text{SL}_2)$$

given by

$$p(\Phi)(g) := (g \cdot \Phi)(v_1).$$

This map is $\text{O}(v_1^\perp)$ -invariant and SL_2 -equivariant. Let us set

$$(8.1) \quad \mathcal{S}(N, \psi \backslash \text{SL}_2) := \text{image of } p,$$

noting that it is a SL_2 -submodule. Likewise, consider the $\text{O}(V) \times (N, \psi)$ -equivariant *surjective* restriction map

$$q = \text{rest} : S(V) \longrightarrow S(X_1)$$

so that

$$q(\Phi)(h) = (h \cdot \Phi)(v_1) = \Phi(h^{-1} \cdot v_1) \quad \text{for } h \in \text{O}(v_1^\perp) \backslash \text{O}(V) \cong X_1.$$

We have already seen and used the map q in the setting of smooth theta correspondence, seeing that it induces an $\text{O}(V)$ -equivariant isomorphism

$$q : S(V)_{N, \psi} \cong S(X_1)$$

Hence we have the diagram:

$$(8.2) \quad \begin{array}{ccc} & S(V) & \\ p \swarrow & & \searrow q \\ \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) & & S(X_1) \end{array}$$

We now make a definition:

Definition 8.1. Say that $f \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$ and $\phi \in S(X_1)$ are in correspondence (or are transfers of each other) if there exists $\Phi \in S(V)$ such that $p(\Phi) = f$ and $q(\Phi) = \phi$.

Our goal in this section is to establish some basic properties of the spaces of test functions and the transfer defined above. We start with the following simple observation.

Proposition 8.2. *Every $f \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$ has a transfer $\phi \in S(X_1)$ and vice versa.*

Proof. This is simply because the maps p and q above are surjective. \square

We also note:

Lemma 8.3. *The space $\mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$ is contained in the Harish-Chandra-Schwarz space of the Whittaker variety $(N, \psi) \backslash \mathrm{SL}_2$. In particular, for any $\sigma \in \widehat{\mathrm{SL}_{2\mathrm{temp}, \psi}}$, the associated relative character $\mathcal{B}_{\sigma, \ell_\sigma}$ extends to a linear form on $\mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$.*

Proof. From the formula defining the Weil representation, we see that for $f = p(\Phi)$,

$$|f(t(a)k)| = |a|^{\dim V/2} \cdot |(k \cdot \Phi)(a \cdot v_1)|.$$

It follows by Lemma 4.4(i) that $f \in \mathcal{C}(N, \psi \backslash \mathrm{SL}_2)$ if $\dim V \geq 3$. \square

8.2. Basic function and fundamental lemmas. We shall now place ourselves in the unramified situation. Namely, let us assume that:

- F is a nonarchimedean local field of residual characteristic different from 2;
- the conductor of the additive character $\psi : F \rightarrow S^1$ is the ring of integers \mathcal{O}_F of F ;
- the quadratic space V contains a self-dual lattice L and $v_1 \in L$.

Under these hypotheses, we have:

- the measure $d_\psi x$ of F is such that the volume of \mathcal{O}_F is 1;
- the measure $d_\psi v$ on V is such that the volume of L is 1;
- the stabilizer $K' = K'_L$ of L in $\mathrm{O}(V)$ is a hyperspecial maximal compact subgroup.

We let $\Phi_0 \in S(V)$ be the characteristic function of L , so that Φ_0 is a unit vector in Ω_ψ . Here is a basic definition:

Definition 8.4. Set

$$f_0 = p(\Phi_0) \quad \text{and} \quad \phi_0 = q(\Phi_0).$$

We call these the basic functions in the relevant space of test functions.

Observe that $\phi_0 = q(\Phi_0)$ is the characteristic function of $X_1(F) \cap L$. On the other hand, f_0 is not compactly supported. Indeed: f_0 is determined by its value on T and we have

$$f_0(t(a)) = (t(a) \cdot \Phi_0)(v_1) = \begin{cases} |a|^{\dim V/2}, & \text{if } |a| \leq 1; \\ 0, & \text{if } |a| > 1. \end{cases}$$

It is also immediate from definition that one has the following “fundamental lemma”:

Lemma 8.5. *The basic functions f_0 and ϕ_0 correspond.*

Let $K = \mathrm{SL}_2(\mathcal{O}_F) \subset \mathrm{SL}_2(F)$ and $K' = K'_L \subset \mathrm{O}(V)$, so that they are hyperspecial maximal compact subgroups which fix the unramified vector Φ_0 . Endow SL_2 and $\mathrm{O}(V)$ with Haar measures such that the volumes of K and K' are 1. Then we have the corresponding spherical Hecke algebras $\mathcal{H}(\mathrm{SL}_2, K)$ and $\mathcal{H}(G, K')$. These are commutative unital algebras whose units are the characteristic functions 1_K and $1_{K'}$ respectively.

It was shown by Howe (see [23, Chap. 5, Thm. I.4, Pg 103] and [23, Pg 107]) that

$$(8.3) \quad \Omega_\psi^K = C_c^\infty(\mathrm{O}(V)) \cdot \Phi_0 \quad \text{and} \quad \Omega_\psi^{K'} = C_c^\infty(\mathrm{SL}_2) \cdot \Phi_0.$$

By applying $1_{K'}$ and 1_K respectively to these two equations, it follows that

$$(\Omega^\infty)^{K \times K'} = \mathcal{H}(\mathrm{SL}_2, K) \cdot \Phi_0 = \mathcal{H}(\mathrm{O}(V), K') \cdot \Phi_0.$$

It also follows from (8.3) that if one has a nonzero equivariant map

$$\Omega^\infty \rightarrow \sigma \otimes \pi \in \mathrm{Irr}(\mathrm{SL}_2 \times \mathrm{O}(V)),$$

then σ is K -unramified if and only if $\pi = \theta_\psi(\sigma)$ is K' -unramified. Indeed, it was shown by Rallis [24, §6] that there is an algebra morphism

$$c : \mathcal{H}(G, K') \longrightarrow \mathcal{H}(\mathrm{SL}_2, K)$$

such that for any $f \in \mathcal{H}(G, K')$, one has

$$f \cdot \Phi_0 = c(f) \cdot \Phi_0.$$

From this, one easily deduces the following “fundamental lemma for spherical Hecke algebras”:

Lemma 8.6. *For any $f \in \mathcal{H}(G, K')$, the element $f \cdot \phi_0 \in C_c^\infty(X_1)$ corresponds to the element $c(f) \cdot f_0 \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$.*

8.3. Relation with Adjoint L-factors. We shall see that the space $\mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$ is intimately related to the standard (degree 3) L-factor of irreducible representations of SL_2 . Let us recall a certain Rankin-Selberg local zeta integral for this particular L-factor, due to Gelbart-Jacquet [14]. It requires the following 3 pieces of data:

- a ψ -generic $\sigma \in \mathrm{Irr}(\mathrm{SL}_2)$ with ψ -Whittaker model \mathcal{W}_σ ,
- the Weil representation ω_ψ of Mp_2 acting on the space $C_c^\infty(F)$ (regarding F as a 1-dimensional quadratic space equipped with the quadratic form $x \mapsto x^2$);
- a principal series representation $I_\psi(\chi, s)$ of Mp_2 , consisting of genuine functions $\phi_s : N \backslash \mathrm{Mp}_2 \rightarrow \mathbb{C}$ such that $\phi_s(t(a)g) = \chi_\psi(a) \cdot \chi(a) \cdot |a|^{1+s} \cdot \phi_s(g)$ (where χ_ψ is a genuine character of the diagonal torus of Mp_2 defined using the Weil index).

Then for $f \in \mathcal{W}_\sigma$, $\varphi \in C_c^\infty(F)$ and a section $\phi_s \in I(s)$, one can consider the local zeta integral

$$Z(f, \varphi, \phi_s) = \int_{N \backslash \mathrm{SL}_2} \overline{f(g)} \cdot (g \cdot \varphi)(1) \cdot \phi_s(g) \frac{dg}{dn}.$$

This converges when $\mathrm{Re}(s) \gg 0$, and when σ is tempered, it converges for $\mathrm{Re}(s) > 0$. Moreover, the GCD of this family of local zeta integrals is used to define the local twisted adjoint L-factor $L(s + \frac{1}{2}, \sigma, \mathrm{Ad} \times \chi) = L(s + \frac{1}{2}, \sigma, \mathrm{std} \times \chi)$.

Hence, the (twisted) adjoint L-value $L(s + \frac{1}{2}, \sigma, \mathrm{Ad} \times \chi)$ is obtained by considering the integrals of $f \in \mathcal{W}_\sigma$ against a space of functions $\mathcal{S}_s(N, \psi \backslash \mathrm{SL}_2)$ of the form

$$g \mapsto \phi_s(g) \cdot (g \cdot \varphi)(1).$$

Moreover, as an SL_2 -module, $\mathcal{S}_s(N, \psi \backslash \mathrm{SL}_2)$ is a quotient of $\omega_\psi \otimes I_\psi(\chi, s)$.

Now let us return to our space of test functions $\mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$. Let us write

$$V = \langle v_1 \rangle \oplus U, \quad \text{with } U = v_1^\perp.$$

Then $C_c^\infty(V) = C_c^\infty(Fv_1) \otimes C_c^\infty(U)$. Here, $C_c^\infty(Fv_1)$ affords the Weil representation ω_ψ of $\mathrm{Mp}_2 \times \mathrm{O}(v_1)$ whereas $C_c^\infty(U)$ afford a Weil representation of $\mathrm{Mp}_2 \times \mathrm{O}(U)$. If $\Phi \in C_c^\infty(V)$ is of the form $\Phi_1 \otimes \Phi'$, then

$$p(\Phi)(g) = (g \cdot \Phi_1)(v_1) \cdot (g \cdot \Phi')(0).$$

The function

$$g \mapsto \phi_{\Phi'}(g) = (g \cdot \Phi')(0)$$

belongs to the principal series $I_\psi(\chi_{\mathrm{disc}(U)}, \frac{1}{2}(\dim V - 3))$. Indeed, by a result of Rallis [25, Thm. II.1.1], the map

$$\Phi' \mapsto \phi_{\Phi'}$$

gives an $\mathrm{O}(U)$ -invariant, Mp_2 -equivariant injective map

$$0 \neq C_c^\infty(U)_{\mathrm{O}(U)} \hookrightarrow I_\psi(\chi_{\mathrm{disc}(U)}, \frac{1}{2}(\dim V - 3)).$$

This result of Rallis underlies the theory of the doubling seesaw and the Siegel-Weil formula. When $\dim V \geq 4$, this injective map is surjective as well. Indeed, when $\dim V > 4$, the relevant principal series $I_\psi(\chi_{\mathrm{disc}(U)}, \frac{1}{2}(\dim V - 3))$ is irreducible. If $\dim V = 4$, the principal series $I_\psi(\chi_{\mathrm{disc}(U)}, 1/2)$ has length 2 with unique irreducible quotient the even Weil representation $\omega_{\psi, \chi_{\mathrm{disc}(U)}}^+$ and unique submodule a (twisted) Steinberg representation. The above map is nonetheless surjective, as the small theta lift of the trivial representation of $\mathrm{O}(U)$ is equal to $\omega_{\psi, \chi_{\mathrm{disc}(U)}}^+$.

To summarise, we have more or less shown:

Proposition 8.7. *When $\dim V \geq 4$, the map p factors as*

$$C_c^\infty(V) \twoheadrightarrow C_c^\infty(V)_{\mathrm{O}(U)} \cong \omega_\psi \otimes I_\psi(\chi_{\mathrm{disc}(U)}, \frac{1}{2}(\dim V - 3)) \twoheadrightarrow \mathcal{S}(N, \psi \backslash \mathrm{SL}_2),$$

so that

$$\mathcal{S}(N, \psi \backslash \mathrm{SL}_2) = \mathcal{S}_{\frac{1}{2}(\dim V - 3)}(N, \psi \backslash \mathrm{SL}_2).$$

Proof. One has an $\mathrm{Mp}_2 \times \mathrm{Mp}_2$ -equivariant isomorphism

$$S(V)_{\mathrm{O}(U)} = S(Fv_1) \hat{\otimes} S(U)_{\mathrm{O}(U)} \cong \omega_\psi \hat{\otimes} I_\psi(\chi_{\mathrm{disc}(U)}, \frac{1}{2}(\dim V - 3)).$$

The rest of the proposition follows from our preceding discussion. \square

Question: Is the surjective map in Proposition 8.7 in fact an isomorphism

$$C_c^\infty(V)_{\mathrm{O}(U)} \cong \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)?$$

8.4. Orbital Integrals. Let us set

$$\mathcal{I}(N, \psi \backslash \mathrm{SL}_2) := \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)_{N, \psi}$$

so that by definition, a linear functional on $\mathcal{I}(N, \psi \backslash \mathrm{SL}_2)$ is a (N, ψ) -equivariant linear form on $\mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$. One may think of $\mathcal{I}(N, \psi \backslash \mathrm{SL}_2)$ as the space of orbital integrals (with respect to the (N, ψ) -period) and write $\mathcal{I}(f)$ for the image of f in $\mathcal{I}(N, \psi \backslash \mathrm{SL}_2)$. Likewise, we set

$$\mathcal{I}(X_1) = \mathcal{S}(X_1)_{\mathrm{O}(U)}.$$

This may be regarded as the space of orbital integrals with respect to the $\mathrm{O}(U)$ -period and we write $\mathcal{I}(\phi)$ for the image of ϕ in $\mathcal{I}(X_1)$.

The following proposition summaries the properties of the transfer of test functions:

Proposition 8.8. *The composite map*

$$S(V) \xrightarrow{p} \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \longrightarrow \mathcal{I}(N, \psi \backslash \mathrm{SL}_2)$$

factors through q , i.e.. it induces a linear map

$$S(X_1) \longrightarrow \mathcal{I}(N, \psi \backslash \mathrm{SL}_2).$$

Hence the transfer correspondence descends to a linear map when one passes to the space of orbital integrals in the target. Indeed, it further descends to give a surjective linear map

$$t_\psi : \mathcal{I}(X_1) \longrightarrow \mathcal{I}(N, \psi \backslash \mathrm{SL}_2).$$

Proof. The composite map in question is (N, ψ) -invariant, and hence factors through $S(V)_{N, \psi} \cong S(X_1)$. But it is also $\mathrm{O}(U)$ -invariant and so further factors through

$$(S(V)_{N, \psi})_{\mathrm{O}(U)} = \mathcal{I}(X_1)$$

as desired. □

Likewise, one may consider the composite

$$S(V) \xrightarrow{q} S(X_1) \longrightarrow \mathcal{I}(X_1)$$

which as above factors through $S(V)_{\mathrm{O}(U)}$. But now we do not know if $S(V)_{\mathrm{O}(U)} \cong \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$; see the Question at the end of the previous subsection. If the answer to that question is Yes, then we will likewise conclude that the above composite map induces a linear map

$$\mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \longrightarrow \mathcal{I}(X_1),$$

which descends further to

$$\mathcal{I}(N, \psi \backslash \mathrm{SL}_2) \longrightarrow \mathcal{I}(X_1),$$

In that case, this linear map will be inverse to the one in the proposition, and hence we will have an isomorphism of vector spaces:

$$t_\psi : \mathcal{I}(N, \psi \backslash \mathrm{SL}_2) \cong \mathcal{I}(X_1).$$

In other words, the transfer correspondence would give an isomorphism of the space of orbital integrals (for the relevant spaces of test functions). As it stands, we only have the surjective transfer map

$$t_\psi : \mathcal{I}(X_1) \twoheadrightarrow \mathcal{I}(N, \psi \backslash \mathrm{SL}_2)$$

given in the above proposition.

9. Relative Character Identities

Finally, we are ready to establish the following relative character identity, which is the main local result of this paper.

Theorem 9.1. *Suppose that*

- $f \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$ and $\phi \in S(X_1)$ are in correspondence;
- $\sigma \in \widehat{\mathrm{SL}_{2\mathrm{temp}, \psi}}$ with (nonzero) theta lift $\theta_\psi(\sigma) \in \widehat{\mathrm{O}(V)}$;
- $\ell_\sigma \in \mathrm{Hom}_N(\sigma, \psi)$ is the canonical element determined in (3.4) by the Whittaker-Plancherel theorem;
- $\ell_{\theta(\sigma)} = f_\sigma(\ell_\sigma) \in \mathrm{Hom}_{\mathrm{O}(v_1^\perp)}(\theta(\sigma), \mathbb{C})$ is the canonical element determined by the spectral decomposition in Proposition 6.4 (which is in turn determined by ℓ_σ and θ_σ).

Then one has the character identity:

$$\mathcal{B}_{\sigma, \ell_\sigma}(f) = \mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}}(\phi).$$

More succinctly, one has the identity

$$\mathcal{B}_{\sigma, \ell_\sigma} \circ t_\psi = \mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}}$$

of linear forms on $\mathcal{I}(X_1)$ or equivalently the identity

$$\mathcal{B}_{\sigma, \ell_\sigma} \circ p = \mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}} \circ q$$

of linear forms on $S(V)$.

See §7, especially §7.4, for the definition of $\mathcal{B}_{\sigma, \ell_\sigma}$ and $\mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}}$.

9.1. Proof of Theorem 9.1. This subsection is devoted to the proof of the theorem. With f and ϕ as given in the theorem, choose $\Phi \in C_c^\infty(V)$ such that $f = p(\Phi)$ and $\phi = q(\Phi)$. We shall now find two different expressions for $\Phi(v_1)$.

On one hand, by the direct integral decomposition given in Proposition 6.4, (2.5) gives

$$\Phi(v_1) = \phi(v_1) = \int_{\widehat{\mathrm{SL}_2}} \beta_{\theta(\sigma)} \alpha_{\theta(\sigma)}(\phi)(v_1) d\mu_{\mathrm{SL}_2, \psi}(\sigma).$$

On the other hand, by the Whittaker-Plancherel theorem for $(N, \psi) \backslash \mathrm{SL}_2$, (2.5) gives

$$\Phi(v_1) = f(1) = \int_{\widehat{\mathrm{SL}_2}} \beta_\sigma \alpha_\sigma(f)(1) d\mu_{\mathrm{SL}_2, \psi}(\sigma).$$

Comparing the two expressions, we deduce that

$$(9.1) \quad \int_{\widehat{\mathrm{SL}_2}} \mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}}(\phi) d\mu_{\mathrm{SL}_2, \psi}(\sigma) = \int_{\widehat{\mathrm{SL}_2}} \mathcal{B}_{\sigma, \ell_\sigma}(f) d\mu_{\mathrm{SL}_2, \psi}(\sigma).$$

We would like to remove the integral sign in the above identity. For this, we will apply a Bernstein center argument.

Given an arbitrary element z in the Bernstein center (or the ring of Arthur multipliers in the archimedean case) of $\mathrm{SL}_2 \times \mathrm{O}(V)$, the element z acts on the irreducible representation $\sigma \boxtimes \theta_\psi(\sigma)$ by a scalar $z(\sigma \boxtimes \theta_\psi(\sigma))$. This implies that one has a commutative diagram

$$(9.2) \quad \begin{array}{ccc} \Omega_\psi & \xrightarrow{z} & \Omega_\psi \\ \theta_\sigma \downarrow & & \downarrow \theta_\sigma \\ \sigma \boxtimes \theta_\psi(\sigma) & \xrightarrow{z(\sigma \boxtimes \theta_\psi(\sigma))} & \sigma \boxtimes \theta_\psi(\sigma) \\ \lambda \downarrow & & \downarrow \lambda \\ \sigma & \xrightarrow{z(\sigma \boxtimes \theta_\psi(\sigma))} & \sigma \end{array}$$

for any linear form λ on $\theta_\psi(\sigma)$. One has an analogous commutative diagram where one takes λ to be any linear form on σ (so the last row of the commutative diagram has $\theta_\psi(\sigma)$ in place of σ).

Now what we would like to show is that there are commutative diagrams

$$(9.3) \quad \begin{array}{ccc} \Omega_\psi & \xrightarrow{z} & \Omega_\psi \\ p \downarrow & & \downarrow p \\ \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) & & \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \\ \alpha_\sigma \downarrow & & \downarrow \alpha_\sigma \\ \sigma & \xrightarrow{z(\sigma \boxtimes \theta_\psi(\sigma))} & \sigma \end{array} \quad \text{and} \quad \begin{array}{ccc} \Omega_\psi & \xrightarrow{z} & \Omega_\psi \\ q \downarrow & & \downarrow q \\ S(X_1) & & S(X_1) \\ \alpha_\sigma \downarrow & & \downarrow \alpha_\sigma \\ \theta_\psi(\sigma) & \xrightarrow{z(\sigma \boxtimes \theta_\psi(\sigma))} & \theta_\psi(\sigma) \end{array}$$

We shall explain how the commutativity of the diagram on the left follows from the commutativity of the diagram in (9.2); a similar argument works for the diagram on the right.

Since the map $\alpha_\sigma \circ p$ is SL_2 -equivariant, it factors through $\theta_\sigma : \Omega_\psi \rightarrow \sigma \boxtimes \theta_\psi(\sigma)$, i.e. there is a $\lambda : \theta_\psi(\sigma) \rightarrow \mathbb{C}$ such that

$$\alpha_\sigma \circ p = \lambda \circ \theta_\sigma.$$

Using this, we see that the desired commutativity of the left diagram in (9.3) is reduced to the commutativity of the diagram in (9.2).

Now we shall apply the identity (9.1) to the pair of test functions arising from $z \cdot \Phi$. Note that

$$\mathcal{B}_{\sigma, \ell_\sigma}(p(z \cdot \Phi)) = \beta_\sigma \alpha_\sigma(p(z \cdot \Phi))(1) = z(\sigma \boxtimes \theta_\psi(\sigma)) \cdot \mathcal{B}_{\sigma, \ell_\sigma}(f)$$

and

$$\mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}} = \beta_{\theta(\sigma)} \alpha_{\theta(\sigma)}(q(z \cdot \Phi))(x_1) = z(\sigma \boxtimes \theta_\psi(\sigma)) \cdot \mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}}(\phi).$$

Hence the identity (9.1), when applied to $z \cdot \Phi$, reads:

$$(9.4) \quad \int_{\widehat{\mathrm{SL}_2}} z(\sigma \boxtimes \theta_\psi(\sigma)) \cdot \left(\mathcal{B}_{\sigma, \ell_\sigma}(f) - \mathcal{B}_{\theta(\sigma), \ell_{\theta(\sigma)}}(\phi) \right) d\mu_{\mathrm{SL}_2, \psi}(\sigma) = 0.$$

Now note that there is a natural homomorphism from the Bernstein center of SL_2 to the Bernstein center for $\mathrm{SL}_2 \times \mathrm{O}(V)$. Hence we may take z to be an element in the (tempered) Bernstein center of SL_2 . Then $z(\sigma \boxtimes \theta_\psi(\sigma)) = z(\sigma)$. When regarded as \mathbb{C} -valued functions on $\widehat{\mathrm{SL}_{2temp, \psi}}$, the elements z of the

(tempered) Bernstein center of SL_2 , are dense in the space of all Schwarz functions on $\widehat{\mathrm{SL}}_{2temp,\psi}$. Hence, (9.4) implies that for $d\mu_{\mathrm{SL}_2,\psi}$ -almost all σ , one has

$$\mathcal{B}_{\sigma,\ell_\sigma}(f) = \mathcal{B}_{\theta(\sigma),\ell_{\theta(\sigma)}}(\phi).$$

To obtain the equality for all $\sigma \in \widehat{\mathrm{SL}}_{2temp,\psi}$, we note that both sides of the identity are continuous as functions of $\sigma \in \widehat{\mathrm{SL}}_{2temp,\psi}$ by Lemma 3.2 and Corollary 6.8. This completes the proof of Theorem 9.1.

9.2. Some consequences. We shall now give some consequences of the relative character identity shown in Theorem 9.1. Let us consider the following diagram:

$$\begin{array}{ccccc}
 & & S(V) & & \\
 & \swarrow q & \downarrow \theta_\sigma & \searrow p & \\
 S(X_1) & & \sigma \boxtimes \theta_\psi(\sigma) & & \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \\
 \downarrow \alpha_{\theta(\sigma)} & \swarrow \ell_\sigma & & \searrow \ell_{\theta(\sigma)} & \downarrow \alpha_\sigma \\
 \theta_\psi(\sigma) & & & & \sigma \\
 & \searrow \ell_{\theta(\sigma)} & & \swarrow \ell_\sigma & \\
 & & \mathbb{C} & &
 \end{array}$$

In this diagram, the rhombus at the bottom is clearly commutative. Now the parallelogram at the upper left side is precisely the commutative diagram in Proposition 6.5. On the other hand, the parallelogram at the upper right side is commutative up to a scalar since

$$\dim \mathrm{Hom}_{\mathrm{O}(v_1^\perp) \times \mathrm{SL}_2}(\Omega_\psi, \mathbb{C} \boxtimes \sigma) = 1 \quad \text{for } \sigma \in \widehat{\mathrm{SL}}_{2temp,\psi}$$

and both $\alpha_\sigma \circ p$ and $\ell_{\theta(\sigma)} \circ \theta_\sigma$ are nonzero elements of this space. We would like to show that it is in fact commutative.

To deduce this, we observe that the composite of the three maps along the left boundary of the hexagon is simply the relative character $\mathcal{B}_{\theta(\sigma)} \circ q$, whereas the composite of the three maps along the right boundary of the hexagon is the relative character $\mathcal{B}_\sigma \circ p$. The relative character identity of Theorem 9.1 says that the boundary of the diagram is commutative! From this, we deduce the following counterpart of Proposition 6.5:

Proposition 9.2. *The following diagram is commutative:*

$$\begin{array}{ccc}
 & S(V) & \\
 \theta_\sigma \swarrow & & \searrow p \\
 \sigma \otimes \theta_\psi(\sigma) & & \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \\
 \ell_{\theta(\sigma)} \searrow & & \swarrow \alpha_\sigma \\
 & \sigma &
 \end{array}$$

Pairing the above identity with an element $v \in \sigma$, we obtain the following counterpart of Corollary 6.7:

Corollary 9.3. *For any $\Phi \in \Omega_\psi$ and $v \in \sigma$, one has:*

$$\ell_{\theta(\sigma)}(A_\sigma(\Phi, v)) = \langle p(\Phi), \beta_\sigma(v) \rangle_{N \backslash \mathrm{SL}_2}$$

where A_σ is the map defined in §5.5.

10. Local L-factor

In this section, we are going to examine the local L-factor $L_X(s, -)$ associated to a spherical variety X . As we explained in the introduction, this local L-factor is associated to a $\frac{1}{2}\mathbb{Z}$ -graded representation $V_X = \oplus_d V_X^d$ of the dual group X^\vee and its value has been computed by Sakellaridis [29, 30] in great generality. However, we shall show that for the particular case treated in this paper, the results developed thus far through theta correspondence can be used to compute $L_X(s, -)$ in terms of the analogous local L-factor for the Whittaker variety $(N, \psi) \backslash \mathrm{SL}_2$.

10.1. Unramified setting. We place ourselves in the unramified setting of §8.2, so that

- F is a nonarchimedean local field of residual characteristic not 2;
- ψ has conductor \mathcal{O}_F , so that the associated measure $d_\psi x$ of F gives \mathcal{O}_F volume 1;
- $L \subset V$ is a self-dual lattice with stabilizer $K' = K'_L$, so that the measure $d_\psi v$ on V gives L volume 1.
- the lattice L endows $X = X_1$ with the structure of a smooth scheme over \mathcal{O}_F ;
- the vector v_1 lies in the lattice L .

Hence, $v_1 \in X(\mathcal{O}_F) = X(F) \cap L$ and we have an orthogonal decomposition

$$L = \mathcal{O}_F v_1 \oplus (v_1^\perp \cap L).$$

The lattices L and $L \cap v_1^\perp$ (which are both self-dual) endow $\mathrm{O}(V)$ and $\mathrm{O}(v_1^\perp)$ with \mathcal{O}_F -structures so that they become smooth group schemes over \mathcal{O}_F ; in particular, $K' = \mathrm{O}(V)(\mathcal{O}_F)$. Then the map $h \mapsto h^{-1} \cdot v_1$ defines an $\mathrm{O}(V)$ -equivariant isomorphism

$$X \longrightarrow \mathrm{O}(v_1^\perp) \backslash \mathrm{O}(V)$$

of smooth schemes over \mathcal{O}_F . Moreover, as a consequence of Hensel's lemma, K' acts transitively on $X(\mathcal{O}_F)$, so that

$$X(\mathcal{O}_F) \cong \mathrm{O}(v_1^\perp)(\mathcal{O}_F) \backslash \mathrm{O}(V)(\mathcal{O}_F).$$

Moreover, the Haar measure $|\omega|$ on X is associated to an $\mathrm{O}(V)$ -invariant differential of top degree which has nonzero reduction on the special fiber. Further, if we equip the smooth group schemes $\mathrm{O}(V)$ and $\mathrm{O}(v_1^\perp)$ over \mathcal{O}_F with invariant differentials $\omega_{\mathrm{O}(V)}$ and $\omega_{\mathrm{O}(v_1^\perp)}$ of top degree with nonzero reduction on the special fibers, then

$$|\omega| = \frac{|\omega_{\mathrm{O}(V)}|}{|\omega_{\mathrm{O}(v_1^\perp)}|}.$$

This means that

$$(10.1) \quad \mathrm{Vol}(X(\mathcal{O}_F); |\omega|) := \int_{X(\mathcal{O})} |\omega| = q^{-\dim X} \cdot \#X(\kappa_F) = q^{-\dim X} \cdot \frac{\#\mathrm{O}(V)(\kappa_F)}{\#\mathrm{O}(v_1^\perp)(\kappa_F)}$$

where κ_F is the residue field of F and $q = \#\kappa_F$.

10.2. The L-factor $L_X^\#$. From the spectral decomposition of $L^2(X, |\omega|)$ obtained in Proposition 6.4, we have a family of $O(v_1^\perp)$ -invariant linear functionals $\ell_{\theta(\sigma)}$ on $\theta_\psi(\sigma)$ for $\sigma \in \widehat{\mathrm{SL}}_{2temp, \psi}$. We remind the reader that even in this unramified setting that we have placed ourselves, the linear functional $\ell_{\theta(\sigma)}$ depends on the Haar measure dg on SL_2 . We have also specified an inner product $\langle -, - \rangle_{\theta(\sigma)}$ in Proposition 5.2 (using the doubling zeta integral) and this depends on the Haar measure dg on SL_2 as well.

Let us assume that $\sigma \in \widehat{\mathrm{SL}}_{2temp, \psi}$ is K -unramified, where $K = \mathrm{SL}_2(\mathcal{O}_F)$. Fix $v_0 \in \sigma$ such that

$$\langle v_0, v_0 \rangle_\sigma = 1 \quad \text{and} \quad K \cdot v_0 = v_0.$$

Then $\theta_\psi(\sigma)$ is K' -unramified and we fix $w_0 \in \theta_\psi(\sigma)$ with

$$\langle w_0, w_0 \rangle_{\theta(\sigma)} = 1 \quad \text{and} \quad K' \cdot w_0 = w_0.$$

We then set

$$L_X^\#(\sigma) := |\ell_{\theta(\sigma)}(w_0)|^2 \in \mathbb{R}_{\geq 0}.$$

Our goal is to determine this non-negative valued function defined on the K -unramified part of $\widehat{\mathrm{SL}}_{2temp, \psi}$, where $K = \mathrm{SL}_2(\mathcal{O}_F)$.

According to the conjecture of Sakellaridis and Venkatesh, one should have

$$L_X^\#(\sigma) = \Delta(0) \cdot \frac{L_X(1/2, \sigma)}{L(1, \sigma, Ad)}$$

where $\Delta(s)$ is a product of local zeta factors (which is independent of the representation σ), $L(s, \sigma, Ad)$ is the adjoint L-factor of σ and

$$L_X(s, \sigma) = \prod_d L(s + d, \sigma, V_X^d)$$

is the L-factor of σ associated to a $1/2 \cdot \mathbb{Z}$ -graded representation $V_X = \bigoplus_d V_X^d$ of X^\vee . This is the essential part of $L_X^\#(\sigma)$. The computation of $L_X(1/2, \sigma)$ is thus equivalent to the precise determination of $L_X^\#(\sigma)$.

10.3. Some constants. We shall determine $L_X^\#(\sigma)$ in terms of the analogous quantity for the Whittaker variety $(N, \psi) \backslash \mathrm{SL}_2$. To this end, let $\Phi_0 = 1_L \in S(V)$ be the characteristic function of L which is a unit vector in Ω_ψ and is $K \times K'$ -invariant. Then under the canonical map

$$\theta_\sigma : \Omega_\psi \longrightarrow \sigma \otimes \theta_\psi(\sigma),$$

we have

$$(10.2) \quad \theta_\sigma(\Phi_0) = c_\sigma \cdot v_0 \otimes w_0 \in \sigma \otimes \theta_\psi(\sigma).$$

for some nonzero constant c_σ .

On the other hand, recall that we have the basic functions

$$p(\Phi_0) = f_0 \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \quad \text{and} \quad q(\Phi_0) = \phi_0 \in S(X).$$

We have observed that

$$\phi_0 = 1_{X(\mathcal{O}_F)}.$$

Now we define a constant $\lambda_{\theta(\sigma)}$ by:

$$(10.3) \quad \alpha_{\theta(\sigma)}(\phi_0) = \lambda_{\theta(\sigma)} \cdot w_0.$$

10.4. Key computations. We now perform the following computations:

(a) Taking inner product of both sides of (10.3) with w_0 gives:

$$\lambda_{\theta(\sigma)} = \langle \alpha_{\theta(\sigma)}(\phi_0), w_0 \rangle_{\theta(\sigma)}.$$

Now the right hand side of this identity is equal to

$$\langle \phi_0, \beta_{\theta(\sigma)}(w_0) \rangle_X = \langle 1_{X(\mathcal{O}_F)}, \beta_{\theta(\sigma)}(w_0) \rangle_X = \beta_{\theta(\sigma)}(w_0)(v_1) \cdot \text{Vol}(X(\mathcal{O}_F), |\omega|).$$

Hence we have

$$\lambda_{\theta(\sigma)} = \ell_{\theta(\sigma)}(w_0) \cdot \text{Vol}(X(\mathcal{O}_F), |\omega|).$$

(b) On the other hand, applying the commutative diagram in Proposition 6.5 to the left hand side of (10.3) and using (10.2) gives:

$$c_\sigma \cdot \ell_\sigma(v_0) \cdot w_0 = \ell_\sigma(\theta_\sigma(\Phi_0)) = \lambda_{\theta(\sigma)} \cdot w_0,$$

so that

$$\lambda_{\theta(\sigma)} = c_\sigma \cdot \ell_\sigma(v_0)$$

(c) Finally, taking inner product of both sides of (10.2) with v_0 gives:

$$A_\sigma(\Phi_0, v_0) = \langle \theta_\sigma(\Phi_0), v_0 \rangle_\sigma = c_\sigma \cdot w_0$$

Computing inner product of both sides gives:

$$|c_\sigma|^2 = \langle A_\sigma(\Phi_0, v_0), A_\sigma(\Phi_0, v_0) \rangle_{\theta(\sigma)} = Z_\sigma(\Phi_0, \Phi_0, v_0, v_0),$$

where the last equality is (5.5) and $Z_\sigma(-)$ is the doubling zeta integral.

Combining the last identities resulting from (a), (b) and (c) above, we obtain:

$$|\ell_{\theta(\sigma)}(w_0)| = |\ell_\sigma(v_0)| \cdot |Z(\Phi_0, \Phi_0, v_0, v_0)|^{1/2} \cdot \text{Vol}(X(\mathcal{O}_F), |\omega|)^{-1}.$$

Hence, it remains to determine the 3 quantities on the right hand side. We have already determined the volume of $X(\mathcal{O}_F)$ in (10.1). On the other hand, we have:

Lemma 10.1. . (i) Suppose $\dim V = 2n \geq 4$ is even. Then

$$|\ell_\sigma(v_0)|^2 = \frac{\zeta_F(2)}{L(1, \sigma, \text{Ad})}$$

and

$$Z(\Phi_0, \Phi_0, v_0, v_0) = \text{Vol}(K; dg) \cdot \frac{L(n-1, \sigma \times \chi_{\text{disc}V}, \text{std})}{\zeta_F(2n-2) \cdot L(n, \chi_{\text{disc}V})}$$

where $\text{Ad} = \text{std}$ is the adjoint = standard L -factor for SL_2 .

(ii) Suppose $\dim V = 2n+1 \geq 3$ is odd, so that one is working with Mp_2 instead of SL_2 . Then

$$|\ell_\sigma(v_0)|^2 = \frac{\zeta_F(2) \cdot L_\psi(1/2, \sigma, \text{std})}{L_\psi(1, \sigma, \text{Ad})}$$

and

$$Z(\Phi_0, \Phi_0, v_0, v_0) = \text{Vol}(K; dg) \cdot \frac{L_\psi(n - \frac{1}{2}, \sigma \times \chi_{\text{disc}V}, \text{std})}{\zeta_F(2n)}$$

Proof. The determination of $|\ell_\sigma(v_0)|$ was carried out in [19, Prop. 2.14 and §2.6] whereas that of the doubling zeta factor can be found in [20, Prop. 3] and [9, Prop. 6.1]. \square

From this lemma, one sees that dependence of $\ell_{\theta(\sigma)}$ on the Haar measure dg . In the unramified setting, it is customary to take the Haar measure dg such that $\text{Vol}(K; dg) = 1$. However, in view of the global applications later on, we prefer to take the Haar measure associated to an invariant differential of top degree on SL_2 over \mathbb{Z} . In that case, we have

$$\text{Vol}(K, dg) = q^{-3} \cdot \#\text{SL}_2(\kappa_F) = \zeta_F(2)^{-1}.$$

Putting everything together, we have shown:

Proposition 10.2. *(i) When $\dim V = 2n$ is even, one has*

$$L_X^\#(\sigma) = |\ell_{\theta(\sigma)}(w_0)|^2 = \frac{L(n-1, \sigma \times \chi_{\text{disc}V}, \text{std})}{L(1, \sigma, \text{Ad})} \cdot \frac{L(n, \chi_{\text{disc}V})}{\zeta_F(2n-2)},$$

taking note that $\text{Ad} = \text{std}$ for SL_2 .

(ii) When $\dim V = 2n+1$ is odd, so that one is working with Mp_2 ,

$$L_X^\#(\sigma) = |\ell_{\theta(\sigma)}(w_0)|^2 = \frac{L_\psi(n - \frac{1}{2}, \sigma \times \chi_{\text{disc}V}, \text{std}) \cdot L_\psi(1/2, \sigma, \text{std})}{L_\psi(1, \sigma, \text{Ad})} \cdot \frac{\zeta_F(2n)}{L(n, \chi_{\text{disc}V})^2}.$$

As an illustration, when $\dim V = 3$ (so $n = 1$) and $\text{disc}(V) = 1$, one gets

$$L_X^\#(\sigma) = \frac{L_\psi(1/2, \sigma, \text{std})^2}{L_\psi(1, \sigma, \text{Ad})} \cdot \frac{\zeta_F(2)}{\zeta_F(1)^2}$$

whereas when $\dim V = 4$ (so $n = 2$) and $\text{disc}(V) = 1$, one has

$$L_X^\#(\sigma) = 1.$$

The reader should compare these values with those given the table (3) in the introduction of [32].

10.5. General case. So far, we have placed ourselves in the unramified setting. We now return to the general setting and define

$$L_X^\# : \widehat{\text{SL}}_2 \longrightarrow \mathbb{C} \quad \text{or} \quad \widehat{\text{Mp}}_2 \longrightarrow \mathbb{C}$$

by using the formulae given in Proposition 10.2, depending on whether $\dim V$ is even or odd.

11. Transfer in Geometric Terms

We have defined the transfer of test functions and established a relative character identity without making any geometric comparison. This is not so surprising, as the theta correspondence is a means of transferring spectral data from one group to another. Nonetheless, one can ask for an explicit formula for the transfer map

$$t_\psi : \mathcal{I}(X_1) \longrightarrow \mathcal{I}(N, \psi \backslash \text{SL}_2).$$

For example, we may wonder if one could describe t_ψ as an integral transform. We shall derive such a formula in this section, assuming that F is nonarchimedean (with ring of integers \mathcal{O}_F and uniformizer ϖ). We also assume for simplicity that the conductor of the additive character ψ is \mathcal{O}_F and the discriminant of V is 1. In particular, the measure $d_\psi x = dx$ on F gives \mathcal{O}_F volume 1.

Recall that we have called the domain and target of t_ψ the spaces of orbital integrals. To describe t_ψ geometrically, we shall appeal to incarnations of these spaces as concrete spaces of functions. Consider

for example the case of $\mathcal{I}(N, \psi \backslash \mathrm{SL}_2) = \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)_{N, \psi}$. Given a function $f \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$, we may consider its literal (N, ψ) -orbital integral:

$$\mathcal{I}(f)(a) = \int_F f(wt(a)n(b)) \cdot \overline{\psi(n)} db.$$

Assuming this converges, it defines a smooth function on the open Bruhat cell NwB which is (N, ψ) -invariant on both sides. Hence it is determined by its value on wT and we may regard it as a function on F^\times . The map \mathcal{I} factors as:

$$\mathcal{S}(N, \psi \backslash \mathrm{SL}_2) \longrightarrow \mathcal{I}(N, \psi \backslash \mathrm{SL}_2) \longrightarrow C^\infty(F^\times),$$

and we view it as giving an incarnation of the elements of $\mathcal{I}(N, \psi \backslash \mathrm{SL}_2)$ as functions on F^\times . Likewise, we shall later see an incarnation of the elements of $\mathcal{I}(X_1)$, as functions on a set of generic $\mathrm{O}(v_1^\perp)$ -orbits on X_1 .

Given $\Phi \in S(V)$, we would thus like to compute the (N, ψ) -orbital integral of $p(\Phi) = f \in \mathcal{S}(N, \psi \backslash \mathrm{SL}_2)$:

$$\mathcal{I}(f)(a) = \int_F f(wt(a)n(b)) \cdot \overline{\psi(n)} db = \int_F (wt(a)n(b) \cdot \Phi)(v_1) \cdot \overline{\psi(b)} db.$$

We should perhaps say a few words about the convergence of this integral. Let us identify $N \backslash \mathrm{SL}_2$ with $W^* = F^2 \setminus O$ (where O is the origin of F^2) via $g \mapsto (0, 1) \cdot g$. Then $|f|$ is a function on W^* which vanishes on a neighbourhood of O . Now the element $Nwt(a)n(b) \in N \backslash \mathrm{SL}_2$ corresponds to the element $(-a, -ab) \in W^*$. For fixed $a \in F^\times$, the function

$$b \mapsto f(wt(a)n(b))$$

is thus not necessarily compactly supported on F . However, if we had assumed that $f \in C_c^\infty(N, \psi \backslash \mathrm{SL}_2)$ (which is a dense subspace of $\mathcal{C}(N, \psi \backslash \mathrm{SL}_2)$), then $|f|$ would in addition vanish outside a compact set of W , so that the above function of b is compactly supported on F and the integral defining $\mathcal{I}(f)(a)$ would have been convergent. This suggests that if we let $U_n = \varpi^{-n} \mathcal{O}_F$ and set

$$\mathcal{I}_n(f)(a) = \int_{U_n} (wt(a)n(b) \cdot \Phi)(v_1) \cdot \overline{\psi(b)} db,$$

then the value $\mathcal{I}_n(f)(a)$ should stabilize for sufficiently large n (and this does happen for $f \in C_c^\infty(N, \psi \backslash \mathrm{SL}_2)$). With this motivation, we shall define

$$\mathcal{I}(f)(a) := \lim_{n \rightarrow \infty} \mathcal{I}_n(f)(a)$$

and shall show below that the right hand side indeed stabilizes.

For this, we will perform an explicit computation:

$$\begin{aligned}
\mathcal{I}_n(f)(a) &= \int_{U_n} (wt(a)n(b) \cdot \Phi)(v_1) \cdot \overline{\psi(b)} db \\
&= \int_{U_n} \mathcal{F}(t(a)n(b) \cdot \Phi)(v_1) \cdot \overline{\psi(b)} db \\
&= \gamma_{\psi,q} \cdot \int_{U_n} \int_V (t(a)n(b) \cdot \Phi)(y) \cdot \psi(\langle v_1, y \rangle) \cdot \overline{\psi(b)} dy db \\
&= \gamma_{\psi,q} \cdot \int_{U_n} \int_V |a|^{\frac{1}{2} \dim(V)} \cdot (n(b) \cdot \Phi)(ay) \cdot \psi(\langle v_1, y \rangle) \cdot \overline{\psi(b)} dy db \\
&= \gamma_{\psi,q} \cdot \int_{U_n} \int_{a^{-1}\omega} |a|^{\frac{1}{2} \dim(V)} \cdot \Phi(ay) \cdot \psi(a^2 bq(y)) \cdot \psi(\langle v_1, y \rangle) \cdot \overline{\psi(b)} dy db \\
&= \gamma_{\psi,q} \cdot \int_{\omega} |a|^{-\frac{1}{2} \dim(V)} \cdot \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) \cdot \left(\int_{U_n} \psi(b(q(x) - 1)) db \right) dx
\end{aligned}$$

where $\omega = \text{supp}(\Phi)$ is compact and we have made the substitution $x = ay$ in the last step. Recall also that \mathcal{F} is a normalized Fourier transform giving the action of the standard Weyl group element w on the Weil representation and $\gamma_{\psi,q}$ is a root of unity (a Weil index).

Now let us consider the inner integral

$$\int_{U_n} \psi(b(q(x) - 1)) db$$

If $q(x) - 1 \notin \varpi^n \mathcal{O}_F$, then the integrand is a nontrivial character of U_n and hence the integral is 0. On the other hand, if $q(x) - 1 \in \varpi^n \mathcal{O}_F$, the integral gives the volume of $U_n = \varpi^{-n} \mathcal{O}_F$. Since ψ is assumed to have conductor \mathcal{O}_F , the volume of U_n with respect to the measure db is q^n (where q is the size of the residue field of F). Hence

$$\int_{U_n} \psi(b(q(x) - 1)) db = q^n \cdot 1(q(x) \in 1 + \varpi^n \mathcal{O}_F)$$

and so

$$\begin{aligned}
\gamma_{\psi,q}^{-1} \cdot \mathcal{I}_n(f)(a) &= q^n \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_V \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) \cdot 1(q(x) - 1 \in \varpi^n \mathcal{O}_F) dx \\
&= q^n \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{q^{-1}(1 + \varpi^n \mathcal{O}_F)} \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) dx.
\end{aligned}$$

Now this last expression is a quantity which appears in the theory of local densities in the theory of quadratic forms over local fields. Indeed, consider the map

$$q : q^{-1}(1 + \varpi^n \mathcal{O}_F) \longrightarrow 1 + \varpi^n \mathcal{O}_F$$

of p -adic manifolds. Since every point in the base is a regular value of the map q , or equivalently q is submersive at every point of the domain, the integral of the compactly supported and locally constant integrand over $q^{-1}(1 + \varpi^n \mathcal{O}_F)$ can be performed by first integrating over the fibers of q followed by integration over the base. Indeed, this was how we had defined the measures $|\omega_a|$ on each fiber X_a (for $a \in F^\times$). In other words for any locally constant compactly supported φ ,

$$\int_{q^{-1}(1 + \varpi^n \mathcal{O}_F)} \varphi(x) dx = \int_{1 + \varpi^n \mathcal{O}_F} q_*(\varphi)(z) dz$$

where

$$q_*(\varphi)(z) = \int_{q^{-1}(z)} \varphi(x) \cdot |\omega_z(x)|$$

But $q_*(\varphi)$ is a locally constant function on the base. Hence for n sufficiently large, the above integral is simply equal to

$$\text{Vol}(\varpi^n \mathcal{O}_F) \cdot q_*(\varphi)(1) = q^{-n} \cdot \int_{X_1} \varphi(x) \cdot |\omega_1(x)|.$$

Applying this to the integral of interest, we thus deduce that the sequence $\mathcal{I}_n(f)(a)$ stabilizes for large n and

$$\mathcal{I}(f)(a) = \gamma_{\psi,q} \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{X_1} \Phi(x) \cdot \psi(\langle v_1, a^{-1}x \rangle) \cdot |\omega_1(x)|$$

Now observe that the map

$$\gamma : X_1 = \text{O}(U) \backslash \text{O}(V) \longrightarrow F$$

given by

$$x = h^{-1}v_1 \mapsto \langle v_1, x \rangle = \langle v_1, h^{-1}v_1 \rangle$$

is $\text{O}(U)$ -invariant (on the right). Moreover, for $\xi \in F$, the preimage of ξ is equal to

$$\gamma^{-1}(\xi) = \{x = \xi \cdot v_1 + v : v \in U \text{ and } q(v) = 1 - \xi^2\}.$$

Outside $\xi^2 = 1$, the map γ is submersive at all points and it follows by Witt's theorem that the fiber $\gamma^{-1}(\xi)$ is a homogeneous space under $\text{O}(U)$. For $x_\xi \in \gamma^{-1}(\xi)$ (with $\xi^2 \neq 1$), its stabilizer in $\text{O}(U)$ is $\text{O}(U \cap x_\xi^\perp)$. Thus, if $F^\heartsuit = F \setminus \{\pm 1\}$ and $X_1^\heartsuit = \gamma^{-1}(F^\heartsuit)$, then

$$X_1^\heartsuit / \text{O}(U) \cong F^\heartsuit.$$

Moreover, the measures $|\omega_1|$ on X_1^\heartsuit and $d\xi$ on F^\heartsuit that we have fixed give rise to measures $|\nu_\xi|$ on the fibers $\gamma^{-1}(\xi)$. The $\text{O}(U)$ -orbital integrals of functions on $S(X_1)$ are thus obtained via integration on the fibres of γ and are smooth functions on F^\heartsuit . These functions give an incarnation of the elements of $\mathcal{I}(X_1) = S(X_1)_{\text{O}(U)}$, so that we have a map

$$\mathcal{I} : S(X_1) \longrightarrow \mathcal{I}(X_1) \longrightarrow C^\infty(F^\heartsuit).$$

Hence, continuing with our computation, we have:

$$\begin{aligned} \mathcal{I}(f)(a) &= \gamma_{\psi,q} \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{X_1^\heartsuit} \Phi(x) \cdot \psi(a^{-1}\gamma(x)) \cdot |\omega_1(x)| \\ &= \gamma_{\psi,q} \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{F^\heartsuit} \int_{\gamma^{-1}(x_\xi)} \Phi(y) \cdot \psi(a^{-1}\xi) \cdot |\nu_\xi(y)| \, d\xi \\ &= \gamma_{\psi,q} \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{F^\heartsuit} \mathcal{I}(\phi)(\xi) \cdot \psi(a^{-1}\xi) \, d\xi \end{aligned}$$

where $\phi = q(\Phi) \in S(X_1)$ (so that ϕ and f are transfers of each other) and $\mathcal{I}(\phi)$ is the orbital integral of ϕ defined by the inner integral over the fibers of γ over F^\heartsuit . We have shown:

Proposition 11.1. *The transfer map $t_\psi : \mathcal{I}(X_1) \longrightarrow \mathcal{I}(N, \psi \backslash \text{SL}_2)$ is given by the integral transform:*

$$t_\psi(\phi)(a) = \gamma_{\psi,q} \cdot |a|^{-\frac{1}{2} \dim(V)} \cdot \int_{F^\heartsuit} \phi(\xi) \cdot \psi(a^{-1}\xi) \, d\xi.$$

where we have regarded $\mathcal{I}(X_1)$ and $\mathcal{I}(N, \psi \backslash \text{SL}_2)$ as spaces of functions on F^\heartsuit and F^\times respectively.

Comparing with the formula for the transfer defined in [32], we see that our transfer map t_ψ essentially agrees with that of [32]. In particular, our approach gives an alternative proof of the transfer theorem of [32] in the setting of hyperboloids.

We close this section with another remark. As mentioned in the introduction, the transfer map in [32] was first defined and studied on the level of the boundary degenerations of the rank 1 spherical varieties and then one uses essentially the same formula in the setting of the spherical varieties themselves. For the case treated in this paper, the boundary degeneration of X_1 is simply the nullcone (with vertex removed)

$$X_0 = \{0 \neq x \in V : q(x) = 0\}$$

of nonzero isotropic vectors. This is a homogeneous $O(V)$ -variety and one can carry out essentially all the analysis of the earlier sections with X_0 in place of X_a with $a \neq 0$. One would then be describing the spectrum of X_0 in terms of the spectrum of the basic affine space $N \backslash \mathrm{SL}_2$. Indeed, since the map $q : V \setminus \{0\} \rightarrow F$ is submersive at all points, the derivation of the formula for the transfer map given in this section can be carried out essentially uniformly for X_a with any $a \in F$. In other words, the Weil representation allows one to construct a coherent family of transfer maps relating $\mathcal{S}(N, \psi_a \backslash \mathrm{SL}_2)$ and $S(X_a)$ varying smoothly with $a \in F$ (though $S(X_0) \neq C_c^\infty(X_0)$ in the nonarchimedean case), which explains in some sense why “the same formula works”. We leave the analysis of the transfer map for the boundary degeneration X_0 as an exercise for the interested reader.

12. Factorization of Global Periods

In this final section, we turn to the global setting, where we examine the question of factorisation of global period integrals, in the context of the periods considered in the earlier sections. We first need to introduce the global analogs of various constructions encountered in the local setting.

12.1. Tamagawa measures. Let k be a number field with ring of adèles \mathbb{A} . We fix a nontrivial unitary character

$$\psi : k \backslash \mathbb{A} \rightarrow S^1.$$

This has a factorization $\psi = \prod_v \psi_v$ where ψ_v is a nontrivial character of the local field k_v for each place v of k . Then ψ_v determines a self-dual Haar measure $d_{\psi_v}x$ on k_v such that for almost all v , the volume of the ring of integers \mathcal{O}_{k_v} relative to $d_{\psi_v}x$ is 1. The product measure $dx := \prod_v d_{\psi_v}x$ then gives a measure on \mathbb{A} . This is the Tamagawa measure of \mathbb{A} : it is independent of ψ (by the Artin-Whaples product formula) and satisfies

$$\int_{F \backslash \mathbb{A}} dx = 1.$$

If G is a (smooth) algebraic group over k , we may consider the adelic group $G(\mathbb{A})$. It is a restricted direct product $\prod'_v G(k_v)$, taken with respect to a sequence $\{K_v = G(\mathcal{O}_{k_v})\}$ of open compact subgroups determined by any \mathcal{O}_k -structure on G . For almost all v , K_v is a hyperspecial maximal compact subgroup of $G(k_v)$.

Now suppose ω_G is a nonzero invariant differential of top degree on G over k . Then for each place v of k , the pair (ω_G, ψ_v) determines a Haar measure $|\omega_G, \psi_v|_v$ of $G(k_v)$. We would like to consider the product measure $\prod_v |\omega_G, \psi_v|_v$ on $G(\mathbb{A})$. For this, one needs to assume that $\prod_v \mathrm{Vol}(K_v; |\omega_G, \psi_v|_v)$ is finite. This is the case for unipotent groups or semisimple groups. If the infinite product is not convergent (e.g. if $G = \mathbb{G}_m$), one can still deal with this by introducing “normalization factors”; we will not go into this well-documented story here. In any case, this product measure on $G(\mathbb{A})$ is independent of ψ and ω_G . It is the so-called Tamagawa measure of G . When $G = \mathbb{G}_a$ is the additive group, the Tamagawa measure

on $G(\mathbb{A}) = \mathbb{A}$ is precisely the measure $dx = \prod_v d\psi_v x$ defined above (so that the terminology is used consistently).

More generally, let $X = H \backslash G$ be a homogeneous G -variety over k (with G acting on the right). Assume for simplicity that $X(k_v) = H(k_v) \backslash G(k_v)$ for each place v of k and $X(\mathbb{A}) = H(\mathbb{A}) \backslash G(\mathbb{A})$. Suppose further that ω_X is a nonzero G -invariant differential form of top degree on X over k . Then for each v , one has a $G(k_v)$ -invariant measure $|\omega_{X,\psi_v}|_v$ on $X(k_v)$. We shall call the product measure $\prod_v |\omega_{X,\psi_v}|_v$ (when it makes sense) the Tamagawa measure of $X(\mathbb{A})$. It is independent of ψ and ω_X . Moreover, it is simply the quotient of the Tamagawa measures of $G(\mathbb{A})$ and $H(\mathbb{A})$. Indeed, one can construct an invariant differential ω_X of top degree as a quotient of (right-)invariant differentials ω_G and ω_H of top degree on G and H .

In short, when working with adelic groups or the adelic points of homogeneous G -varieties, we shall always use such Tamagawa measures.

12.2. Automorphic Forms. For a reductive group G defined over k , we shall write $[G]$ for the quotient $G(k) \backslash G(\mathbb{A})$ and equip it with its Tamagawa measure dg (divided by the counting measure on the discrete subgroup $G(k)$).

Let $C_{mod}^\infty([G])$ denote the space of smooth functions on $[G]$ which are of (uniform) moderate growth. It is a representation of $G(\mathbb{A})$ containing the $G(\mathbb{A})$ -submodule $\mathcal{A}(G)$ of (smooth) automorphic forms on G , which in turn contains the submodule $\mathcal{A}_{cusp}(G)$ of cusp forms:

$$\mathcal{A}_{cusp}(G) \subset \mathcal{A}(G) \subset C_{mod}^\infty([G]).$$

When the group G has a nontrivial split torus in its center, we shall fix a unitary automorphic central character χ and consider the space $\mathcal{A}_\chi(G)$ of automorphic forms with central character χ ; we shall suppress this technical issue in the following discussion.

On $\mathcal{A}_{cusp}(G)$, we have the Petersson inner product $\langle -, - \rangle_G$ (defined using the Tamagawa measure dg). Indeed, the Petersson inner product defines a pairing between $\mathcal{A}_{cusp}(G)$ and the larger space $C_{mod}^\infty([G])$. Hence, we have a canonical projection map

$$C_{mod}^\infty([G]) \longrightarrow \mathcal{A}_{cusp}(G).$$

In particular, for an irreducible cuspidal representation $\Sigma \subset \mathcal{A}_{cusp}(G)$, we have a projection

$$\text{pr}_\Sigma : C_{mod}^\infty([G]) \longrightarrow \Sigma.$$

We denote the restriction of the Petersson inner product on Σ by $\langle -, - \rangle_\Sigma$.

12.3. Global periods. Let $H \subset G$ be a subgroup so that $X = H \backslash G$ is quasi-affine. Fix a unitary Hecke character χ of H . Then we may consider the global (H, χ) -period:

$$P_{H,\chi} : \mathcal{A}_{cusp}(G) \longrightarrow \mathbb{C}$$

defined by

$$P_{H,\chi}(\phi) = \int_{[H]} \overline{\chi(h)} \cdot \phi(h) dh$$

where dh is the Tamagawa measure of $H(\mathbb{A})$. For a cuspidal representation $\Sigma \subset \mathcal{A}_{cusp}(G)$, we may thus consider the restriction of $P_{H,\chi}$ to Σ , denoting it by $P_{H,\chi,\Sigma}$.

12.4. The maps α_{Aut} and β_{Aut} . We shall now introduce the global analog of the maps α_σ and β_σ introduced in §2.3 in the local setting. Set $X_{\mathbb{A}} = H(\mathbb{A}) \backslash G(\mathbb{A})$, equipped with its Tamagawa measure (which is the quotient of the Tamagawa measures of $G(\mathbb{A})$ and $H(\mathbb{A})$). We have a $G(\mathbb{A})$ -equivariant map

$$\theta : C_c^\infty(X_{\mathbb{A}}, \chi) \longrightarrow C_{\text{mod}}^\infty([G])$$

defined by

$$\theta(f)(g) = \sum_{x \in X_k} f(x \cdot g).$$

The map θ is called the formation of theta series. Hence, we may define a composite map

$$\alpha_\Sigma^{\text{Aut}} : C_c^\infty(X_{\mathbb{A}}, \chi) \xrightarrow{\theta} C_{\text{mod}}^\infty([G]) \xrightarrow{\text{pr}_\Sigma} \Sigma.$$

Concretely, we have:

$$\alpha_\Sigma^{\text{Aut}}(f) = \sum_{\phi \in \text{ONB}(\Sigma)} \langle \theta(f), \phi \rangle_G \cdot \phi.$$

On the other hand, we have the $G(\mathbb{A})$ -equivariant map

$$\beta_\Sigma^{\text{Aut}} : \Sigma \longrightarrow C^\infty(X_{\mathbb{A}})$$

defined by

$$\beta_\Sigma^{\text{Aut}}(\phi)(g) = P_{H, \chi}(g \cdot \phi).$$

One has the following adjunction formula, which is the global analog of (2.2):

Lemma 12.1. *For $f \in C_c^\infty(X_{\mathbb{A}}, \chi)$ and $\phi \in \Sigma$, one has*

$$\langle \alpha_\Sigma^{\text{Aut}}(f), \phi \rangle_G = \langle f, \beta_\Sigma^{\text{Aut}}(\phi) \rangle_X$$

Proof. We have:

$$\begin{aligned} \langle \alpha_\Sigma^{\text{Aut}}(f), \phi \rangle_G &= \int_{[G]} \theta(f)(g) \cdot \overline{\phi(g)} dg = \int_{[G]} \sum_{\gamma \in H(k) \backslash G(k)} f(\gamma g) \cdot \overline{\phi(g)} dg \\ &= \int_{H(k) \backslash G(\mathbb{A})} f(g) \cdot \overline{\phi(g)} dg = \int_{X_{\mathbb{A}}} \int_{[H]} f(hg) \cdot \overline{\phi(hg)} dh \frac{dg}{dh} \\ &= \int_{X_{\mathbb{A}}} f(x) \cdot \overline{P_{H, \chi}(\phi)(x)} dx = \langle f, \beta_\Sigma^{\text{Aut}}(\phi) \rangle_X, \end{aligned}$$

as desired. □

12.5. Global Relative Characters. We may also introduce the global analog of the inner product J_σ :

$$J_\Sigma^{\text{Aut}}(\phi_1, \phi_2) := \langle \alpha_\Sigma^{\text{Aut}}(\phi_1), \alpha_\Sigma^{\text{Aut}}(\phi_2) \rangle_\Sigma = \langle \beta_\Sigma^{\text{Aut}} \alpha_\Sigma^{\text{Aut}}(\phi_1), \phi_2 \rangle_X.$$

Then

$$\begin{aligned} J_\Sigma^{\text{Aut}}(\phi_1, \phi_2) &= \sum_{f \in \text{ONB}(\Sigma)} \langle \alpha_\Sigma^{\text{Aut}}(\phi_1), f \rangle_\Sigma \cdot \langle f, \alpha_\Sigma^{\text{Aut}}(\phi_2) \rangle_\Sigma \\ &= \sum_{f \in \text{ONB}(\Sigma)} \langle \phi_1, \beta_\Sigma^{\text{Aut}}(f) \rangle_X \cdot \langle \beta_\Sigma^{\text{Aut}}(f), \phi_2 \rangle_X. \end{aligned}$$

By analog with the local case, we may introduce the global relative character $\mathcal{B}_\Sigma^{\text{Aut}}$ as an equivariant distribution on $C_c^\infty(X_\mathbb{A}, \chi)$, defined by

$$\mathcal{B}_\Sigma^{\text{Aut}}(f) = \beta_\Sigma^{\text{Aut}}(\alpha_\Sigma^{\text{Aut}}(f))(1) = \sum_{\phi \in \text{ONB}(\Sigma)} \langle f, \beta_\Sigma^{\text{Aut}}(\phi) \rangle_X \cdot P_{H, \chi}(\phi)$$

for $f \in C_c^\infty(X_\mathbb{A}, \chi)$. When pulled back to give a distribution on $G(\mathbb{A})$, one has

$$\mathcal{B}_\Sigma^{\text{Aut}}(f) = \sum_{\phi \in \text{ONB}(\Sigma)} \overline{P_{H, \chi}(\Sigma(\bar{f})\phi)} \cdot P_{H, \chi}(\phi),$$

for $f \in C_c^\infty(G(\mathbb{A}))$.

12.6. Quadratic spaces and hyperboloids. Suppose now that (V, q) is a quadratic space over k . Then as an additive group scheme over k , $V \cong \mathbb{G}_a^k$ and so $V(\mathbb{A})$ has its canonical Tamagawa measure. We would like to compare this Tamagawa measure with the measures we considered in the local setting.

If $\langle -, - \rangle$ is the symmetric bilinear form associated to q and $\psi = \prod_v \psi_v$ is our fixed additive character of $F \backslash \mathbb{A}$, then the pair $(\langle -, - \rangle, \psi_v)$ determines a Haar measure $d_{\psi_v} v$ on $V_v = V \otimes_k k_v$ (the self-dual measure with respect to the pairing $\psi(\langle -, - \rangle)$). This is the measure on V_v that we have been using in the local setting. If $L \subset V$ is any \mathcal{O}_k -lattice, which endows V with an \mathcal{O}_k -structure, then for almost all places v , the volume of $L_v = L \otimes_{\mathcal{O}_k} \mathcal{O}_{k_v}$ with respect to $d_{\psi_v} v$ is 1. We may thus consider the product measure

$$dv_\mathbb{A} = \prod_v d_{\psi_v} v \quad \text{on } V_\mathbb{A}.$$

As the notation suggests, it is independent of the choice of ψ . Moreover, $dv_\mathbb{A}$ is equal to the Tamagawa measure on $V_\mathbb{A}$.

Suppose that the quadratic space (V, q) contains a vector $v_1 \in V_k$ with $q(v_1) = 1$. By changing L if necessary, we may assume that v_1 lies in the lattice L . Let

$$X_1 = \{x \in V : q(x) = 1\}$$

be a hyperboloid. Then the map $h \mapsto h^{-1} \cdot v_1$ gives an isomorphism

$$\text{O}(v_1^\perp) \backslash \text{O}(V) \cong X_1$$

of $\text{O}(V)$ -varieties over k . Moreover, in this case, one has (by Witt's theorem):

$$\text{O}(v_1^\perp)(\mathbb{A}) \backslash \text{O}(V)(\mathbb{A}) \cong X_1(\mathbb{A}).$$

Recall that we have equipped both sides with their Tamagawa measures which are respected by this isomorphism. Now we would like to relate the Tamagawa measure on $X_1(\mathbb{A})$ with the measures we have been using in the local case.

As noted, the additive character $\psi = \prod_v \psi_v$ gives us decompositions of Tamagawa measures

$$dv_\mathbb{A} = \prod_v d_{\psi_v} v \quad \text{and} \quad dx_\mathbb{A} = \prod_v d_{\psi_v} x$$

on $V_\mathbb{A}$ and \mathbb{A} respectively. Using the submersive map

$$q : V \setminus X_0 \longrightarrow F \setminus \{0\},$$

the local measures $d_{\psi_v}v$ and $d_{\psi_v}x$ determine an $O(V_v)$ -invariant measure $|\omega_{1,v}|$ on $X_1(k_v)$: this is the measure on $X_1(k_v)$ that we have been using in the local setting. We observe that the product measure

$$|\omega_{1,\mathbb{A}}| := \prod_v |\omega_{1,v}|$$

is equal to the Tamagawa measure of $X_1(\mathbb{A}) = \prod'_v X_1(k_v)$ where the restricted direct product is taken with respect to the family $\{X_1(\mathcal{O}_{k_v}) = X_1(k_v) \cap L_v\}$ for almost all v .

12.7. Global Weil representation. We now consider the dual pair $SL_2 \times O(V)$ and recall its global Weil representation. We have fixed an \mathcal{O}_k -lattice $L \subset V$. For almost all v , $L_v = L \otimes_{\mathcal{O}_k} \mathcal{O}_{k_v}$ is a self dual lattice of volume 1 with respect to $d_{\psi_v}v$. Let

$$K'_v = \text{stabilizer of } L \text{ in } O(V_v) \quad \text{and} \quad \Phi_{0,v} = 1_{L_v} \in S(V_v).$$

For each v , we have the (smooth) Weil representation Ω_{ψ_v} of $SL_2(k_v) \times O(V_v)$ realized on the space $S(V_v)$ of Schwarz-Bruhat functions on $V_v = V \otimes_k k_v$. For almost all v , $\Phi_{0,v}$ is a unit vector which is fixed by $K_v \times K'_v = SL_2(\mathcal{O}_{k_v}) \times K'_v$. The restricted tensor product

$$\Omega_{\psi} = \otimes'_v \Omega_{\psi_v}$$

with respect to the family of vectors $\Phi_{0,v}$ is the Weil representation of the adelic dual pair $SL_2(\mathbb{A}) \times O(V)(\mathbb{A})$. It is realised on the space

$$S(V_{\mathbb{A}}) = S(V_{\infty}) \otimes (\otimes'_{v<\infty} S(V_v))$$

of Schwarz-Bruhat functions on $V_{\mathbb{A}}$ (where $V_{\infty} = V \otimes_{\mathbb{Q}} \mathbb{R}$).

The Weil representation Ω_{ψ} has a canonical automorphic realization

$$\theta : \Omega_{\psi} = S(V_{\mathbb{A}}) \longrightarrow C_{mod}^{\infty}([SL_2 \times O(V)])$$

defined by

$$\theta(\Phi)(g, h) = \sum_{v \in V_k} (\Omega_{\psi}(g, h)\Phi)(v).$$

12.8. Global theta lifting. For an irreducible cuspidal representation $\Sigma \subset \mathcal{A}_{cusp}(SL_2)$ of SL_2 , we may consider its global theta lift to $O(V)$. More precisely, given $\Phi \in \Omega_{\psi}$ and $f \in \Sigma$, one defines the $SL_2(\mathbb{A})$ -invariant and $O(V)(\mathbb{A})$ -equivariant map

$$A_{\Sigma}^{\text{Aut}} : \Omega_{\psi} \otimes \overline{\Sigma} \longrightarrow \mathcal{A}(O(V))$$

by

$$A_{\Sigma}^{\text{Aut}}(\Phi, f)(h) = \int_{[SL_2]} \theta(\Phi)(gh) \cdot \overline{f(g)} dg.$$

The image of A_{Σ}^{Aut} is the global theta lift of Σ , which we denote by

$$\Pi = \Theta^{\text{Aut}}(\Sigma) \subset \mathcal{A}(O(V)).$$

If Π is cuspidal and nonzero, then it follows by the Howe duality conjecture that Π is an irreducible cuspidal representation.

Conversely, assume that $\Pi \subset \mathcal{A}_{cusp}(O(V))$ is irreducible cuspidal. Then we may consider the global theta lift of Π to SL_2 . More precisely, given $\Phi \in \Omega_{\psi}$ and $\phi \in \Pi$, one defines the $O(V)(\mathbb{A})$ -invariant and $SL_2(\mathbb{A})$ -equivariant map

$$B_{\Pi}^{\text{Aut}} : \Omega_{\psi} \otimes \overline{\Pi} \longrightarrow \Sigma \subset \mathcal{A}(SL_2)$$

by

$$B_{\Pi}^{\text{Aut}}(\Phi, \phi)(g) = \int_{[\mathcal{O}(V)]} \theta(\Phi)(gh) \cdot \overline{\phi(h)} dh.$$

By computing constant term, one can show that the image of B_{Π}^{Aut} is necessarily cuspidal.

12.9. The maps A_{Σ}^{Aut} and $B_{\Theta(\Sigma)}^{\text{Aut}}$. We continue with the setting of the previous subsection. If $\Pi = \Theta^{\text{Aut}}(\Sigma)$ is nonzero cuspidal, then the image of $B_{\Theta(\Sigma)}^{\text{Aut}}$ is Σ (because the cuspidal spectrum of SL_2 is multiplicity-free [27]). In this case, the maps A_{Σ}^{Aut} and $B_{\Theta(\Sigma)}^{\text{Aut}}$ are global analogs of the maps A_{σ} and $B_{\Theta(\sigma)}$ introduced in §5.5. By an exchange of the order of integration, we have the following global analog of (5.9):

$$(12.1) \quad \langle A_{\Sigma}^{\text{Aut}}(\Phi, f), \phi \rangle_{\Theta(\Sigma)} = \langle B_{\Theta(\Sigma)}^{\text{Aut}}(\Phi, \phi), f \rangle_{\Sigma}$$

for $\Phi \in \Omega_{\psi}$, $f \in \Sigma$ and $\phi \in \Theta^{\text{Aut}}(\Sigma)$.

12.10. Global transfer of periods. For $\Phi \in \Omega_{\psi}$ and $\phi \in \Pi$, we may compute the ψ -Whittaker coefficient of $B_{\Pi}^{\text{Aut}}(\Phi, \phi)$. One has the following global analog of Corollary 6.7:

Proposition 12.2. *For $\Phi \in \Omega_{\psi}$ and $\phi \in \Pi$,*

$$P_{N, \psi}(B_{\Pi}^{\text{Aut}}(\Phi, \phi)) = \int_{H(\mathbb{A}) \backslash G(\mathbb{A})} \Phi(g^{-1}v_1) \cdot P_{\mathcal{O}(v_1^{\perp})}(\phi)(g) \frac{dg}{dh} = \langle \Phi|_X, \beta_{\Pi}^{\text{Aut}}(\phi) \rangle_{X_{\mathbb{A}}}.$$

In particular, a cuspidal representation Π of $\mathcal{O}(V)$ has nonzero $\mathcal{O}(v_1^{\perp})$ -period if and only if its global theta lift to SL_2 is globally ψ -generic.

We omit the proof as it is based on a standard computation.

12.11. Decompositions of unitary representations. Suppose now that $\Sigma \subset \mathcal{A}_{\text{cusp}}(\text{SL}_2)$ is an irreducible *tempered* ψ -generic cuspidal representation of SL_2 and $\Theta^{\text{Aut}}(\Sigma) \subset \mathcal{A}_{\text{cusp}}(\mathcal{O}(V))$ is a nonzero (irreducible) cuspidal representation of $\mathcal{O}(V)$. In this case, $\Theta^{\text{Aut}}(\Sigma)$ is globally $\mathcal{O}(v_1^{\perp})$ -distinguished. We would like to factor the global $\mathcal{O}(v_1^{\perp})$ -period of $\Theta^{\text{Aut}}(\Sigma)$ as a product of the local functionals constructed in the earlier part of the paper. To carry this out, we need to set things up precisely and systematically.

On the side of SL_2 , we begin by fixing a decomposition of the Tamagawa measures

$$dg = \prod_v dg_v$$

To be concrete, we take an invariant differential ω_{SL_2} of top degree on SL_2 over \mathbb{Z} (which is well-defined up to ± 1), which together with the $d_{\psi_v}x$ on k_v gives a measure

$$dg_v := |\omega_{\text{SL}_2, \psi_v}|_v \quad \text{on } \text{SL}_2(k_v),$$

for which

$$\int_{\text{SL}_2(\mathcal{O}_{k_v})} dg_v = \zeta_{k_v}(2)^{-1}.$$

This is the measure we used on $\text{SL}_2(k_v)$ in §10 when we computed the local L-factor $L_X^{\#}$. The product of these measures is then equal to the Tamagawa measure dg .

We also fix an isomorphism

$$(12.2) \quad \Sigma \cong \otimes'_v \Sigma_v$$

and a decomposition of the Petersson inner product

$$(12.3) \quad \langle -, - \rangle_\Sigma = \prod_v \langle -, - \rangle_{\Sigma_v}.$$

This equips Σ_v with a unitary structure $\langle -, - \rangle_{\Sigma_v}$. Alternatively, we could work with the tautological measurable field of unitary representations provided by the local Whittaker-Plancherel theorem (so that each Σ_v comes equipped with a unitary structure already), in which case one would require the isomorphism in (12.2) to be an isometry, so that one has (12.3) as a consequence. These are two viewpoints with no difference in content. The restricted tensor product in (12.2) is with respect to a family of unit vectors $v_{0,v}$ which is fixed by $K_v = \mathrm{SL}_2(\mathcal{O}_{k_v})$.

For the Weil representation, we have already seen in §12.7 the decompositions:

$$(12.4) \quad S(V_{\mathbb{A}}) = S(V_\infty) \otimes (\otimes'_{v<\infty} S(V_v))$$

and

$$dv_{\mathbb{A}} = \prod_v d\psi_v v \quad \text{of Haar measures on } V_{\mathbb{A}},$$

inducing compatible unitary structures on the global and local Weil representations. In particular, the restricted direct product in (12.4) is with respect to the family of unit vectors $\Phi_{0,v} = 1_{L_v}$ for almost all v and

$$\langle -, - \rangle_{\Omega_\psi} = \prod_v \langle -, - \rangle_{\Omega_{\psi_v}}.$$

Once these decompositions are fixed as above, we see that for each place v , we have the local big theta lift

$$\Theta_{\psi_v}(\Sigma_v) := (\Omega_{\psi_v} \otimes \Sigma_v^\vee)_{\mathrm{SL}_2(k_v)}$$

and its unique irreducible quotient

$$\theta_{\psi_v}(\Sigma_v) \in \mathrm{Irr}(\mathrm{O}(V_v))$$

Moreover, $\theta_{\psi_v}(\Sigma_v)$ inherits an inner product $\langle -, - \rangle_{\theta(\Sigma_v)}$ defined via the local doubling zeta integral. We let $w_{0,v} \in \theta_{\psi_v}(\Sigma_v)$ be a unit vector which is fixed by K'_v for almost all v . Then we may form the abstract global theta lift

$$\Theta^{\mathbb{A}}(\Sigma) = \otimes'_v \theta_{\psi_v}(\Sigma_v)$$

where the restricted tensor product is with respect to the family of unit vectors $w_{0,v}$. The abstract global theta lift $\Theta^{\mathbb{A}}(\Sigma)$ inherits a unitary structure

$$\langle -, - \rangle_{\Theta^{\mathbb{A}}(\Sigma)} = \prod_v \langle -, - \rangle_{\theta(\Sigma_v)}$$

from that of its local components. Hence we may fix an isometric isomorphism

$$\Theta^{\mathrm{Aut}}(\Sigma) \cong \Theta^{\mathbb{A}}(\Sigma)$$

so that

$$\langle -, - \rangle_{\Theta^{\mathrm{Aut}}(\Sigma)} = \prod_v \langle -, - \rangle_{\theta(\Sigma_v)}$$

where the inner product on the left is that defined by the Petersson inner product.

12.12. Adelic periods. Having fixed the various decompositions in the previous subsection, we can now introduce the adelic versions of various period maps. We have introduced various global or automorphic quantities associated to Σ and $\Theta^{\text{Aut}}(\Sigma)$, namely

- the maps $\alpha_{\Sigma}^{\text{Aut}}, \beta_{\Sigma}^{\text{Aut}}, P_{N,\psi,\Sigma}$ and J_{Σ}^{Aut} related to the global Whittaker period with respect to (N, ψ) ;
- the maps $\alpha_{\Theta(\Sigma)}^{\text{Aut}}, \beta_{\Theta(\Sigma)}^{\text{Aut}}, P_{O(v_1^\perp), \Theta(\Sigma)}$ and $J_{\Theta(\Sigma)}^{\text{Aut}}$ related to the $O(v_1^\perp)$ -period;
- the maps $A_{\Sigma}^{\text{Aut}}, B_{\Theta(\Sigma)}^{\text{Aut}}$ related to global theta lifting.

All the above global objects have local counterparts, relative to the decompositions fixed in the previous subsection. Namely, for each place v of k , we have:

- the maps $\alpha_{\Sigma_v}, \beta_{\Sigma_v}, \ell_{\Sigma_v}$ and J_{Σ_v} given by the Whittaker-Plancherel theorem;
- the maps $\alpha_{\Theta(\Sigma_v)}, \beta_{\Theta(\Sigma_v)}, \ell_{\Theta(\Sigma_v)}$ and $J_{\Theta(\Sigma_v)}$ given by the spectral decomposition of $L^2(X_v, |\omega_v|)$;
- the maps A_{Σ_v} and $B_{\Theta(\Sigma_v)}$ given by the spectral decomposition of the Weil representation Ω_{ψ_v} .

We may take the Euler product of the above local quantities. As an example, we set:

$$\beta_{\Sigma}^{\mathbb{A}} := \prod_v^* \beta_{\Sigma_v} \quad \text{and} \quad \ell_{\Sigma}^{\mathbb{A}} := \prod_v^* \ell_{\Sigma_v}.$$

Here, the Euler product has to be understood as a regularized product, via meromorphic continuation if necessary, as discussed in the introduction. Let us illustrate this in three instances, assuming that $\dim(V) = 2n$ and $\text{disc}(V) = 1 \in k^\times/k^{\times 2}$ (for simplicity):

- Suppose we want to define $\ell_{\Sigma}^{\mathbb{A}}$. Given a decomposable vector $u = \otimes_v u_v \in \Sigma$, with $u_v = v_{0,v}$ for almost all v , we need to examine the convergence of the product

$$\prod_v \ell_{\Sigma_v}(u_v).$$

This is determined by the value $\ell_{\Sigma_v}(v_{0,v})$ for almost all v . In Lemma 10.1(i), we have noted that

$$|\ell_{\Sigma_v}(v_{0,v})|^2 = \frac{\zeta_{k_v}(2)}{L(1, \Sigma_v, Ad)}.$$

Thus the Euler product may not converge because of the denominator. However, we may normalize ℓ_{Σ_v} by setting

$$\ell_{\Sigma_v}^b = \lambda_v^{-1} \cdot \ell_{\Sigma_v} \quad \text{for each place } v$$

with

$$|\lambda_v|^2 = \frac{\zeta_{k_v}(2)}{L(1, \Sigma_v, Ad)}$$

and

$$\ell_{\Sigma_v}^b(v_{0,v}) = 1 \quad \text{for almost all } v.$$

Then the product $\prod_v \ell_{\Sigma_v}^b(u_v)$ certainly converges and we set

$$\ell_{\Sigma}^{\mathbb{A}} := \prod_v^* \ell_{\Sigma_v} := \left(\frac{|\zeta_k(2)|}{|L(1, \Sigma, Ad)|} \right)^{1/2} \cdot \prod_v \ell_{\Sigma_v}^b$$

where the global L-value is defined by meromorphic continuation of the global L-function.

- Suppose we want to define

$$A_{\Sigma}^{\mathbb{A}} : \Omega_{\psi} \otimes \overline{\Sigma} \longrightarrow \Theta^{\mathbb{A}}(\Sigma).$$

Hence we need to consider the infinite product

$$\prod_{v \notin S} A_{\Sigma_v}(\Phi_{0,v}, v_{0,v})$$

for some finite set S of places of k . By Lemma 10.1(i), we see that for almost all v ,

$$A_{\Sigma_v}(\Phi_{0,v}, v_{0,v}) = \frac{L(n-1, \Sigma_v, std)}{\zeta_{k_v}(2) \cdot \zeta_{k_v}(n) \cdot \zeta_{k_v}(2n-2)} \cdot w_{0,v} \in \theta_{\psi_v}(\Sigma_v)$$

where $\dim V = 2n$ and $w_{0,v}$ is a unit vector. Since Σ is tempered, the relevant Euler product actually converges absolutely when $n > 2$, in which case we can simply define

$$A_{\Sigma}^{\mathbb{A}} = \prod_v A_{\Sigma_v}.$$

In general, we would set

$$A_{\Sigma_v}^b = \lambda_v^{-1} \cdot A_{\Sigma_v}$$

with

$$|\lambda_v|^2 = \frac{|L(n-1, \Sigma_v, std)|}{|\zeta_{k_v}(2)| \cdot |\zeta_{k_v}(n)| \cdot |\zeta_{k_v}(2n-2)|} \quad \text{for all } v,$$

and

$$A_{\Sigma_v}^b(\Phi_{0,v}, v_{0,v}) = w_{0,v} \quad \text{for almost all } v.$$

Then we set

$$A_{\Sigma}^{\mathbb{A}} = \left(\frac{|L(n-1, \Sigma, std)|}{|\zeta_k(2)| \cdot |\zeta_k(n)| \cdot |\zeta_k(2n-2)|} \right)^{1/2} \cdot \prod_v A_{\Sigma_v}^b.$$

- Suppose we want to define $\ell_{\Theta(\Sigma)}^{\mathbb{A}} : \Theta^{\mathbb{A}}(\Sigma) \longrightarrow \mathbb{C}$. Then we need to consider the Euler product

$$\prod_{v \notin S} \ell_{\theta(\Sigma_v)}(w_{0,v}) \quad \text{for some finite set } S.$$

In Proposition 10.2, we have seen that

$$|\ell_{\theta(\Sigma_v)}(w_{0,v})|^2 = \frac{L(n-1, \Sigma_v, std)}{L(1, \sigma, Ad)} \cdot \frac{\zeta_{k_v}(n)}{\zeta_{k_v}(2n-2)}$$

for almost all v (with $\dim V = 2n$). If $n = 2$, the right hand side is equal to 1, so the relevant Euler product converges. For $n > 2$, we set

$$\ell_{\theta(\Sigma_v)}^b = \lambda_v^{-1} \cdot \ell_{\theta(\Sigma_v)}$$

with

$$|\lambda_v|^2 = \frac{L(n-1, \Sigma_v, std)}{L(1, \Sigma_v, Ad)} \cdot \frac{\zeta_{k_v}(n)}{\zeta_{k_v}(2n-2)} \quad \text{for all } v$$

and

$$\ell_{\theta(\Sigma_v)}^b(w_{0,v}) = 1 \quad \text{for almost all } v.$$

Then we set

$$\ell_{\Theta^{\mathbb{A}}(\Sigma)}^{\mathbb{A}} = \left(\frac{|L(n-1, \Sigma, std)|}{|L(1, \Sigma, Ad)|} \cdot \frac{|\zeta_k(n)|}{|\zeta_k(2n-2)|} \right)^{1/2} \cdot \prod_v \ell_{\theta(\Sigma_v)}^b.$$

After these three illustrative examples, we leave the precise definition of other adelic period maps to the reader.

12.13. Comparison of automorphic and adelic periods. We can now compare the various adelic period maps with their automorphic counterparts, using the decompositions

$$\Omega_\psi \cong \otimes'_v \Omega_\psi, \quad \Sigma \cong \otimes'_v \Sigma_v \quad \text{and} \quad \Theta^{\text{Aut}}(\Sigma) \cong \Theta^{\mathbb{A}}(\Sigma) := \otimes'_v \theta_{\psi_v}(\Sigma_v)$$

fixed in §12.11.

Since both $\ell_\Sigma^{\mathbb{A}}$ and $P_{\Sigma, N, \psi}$ are nonzero elements of the 1-dimensional space $\text{Hom}_{N(\mathbb{A})}(\Sigma, \psi)$, there is a constant $c(\Sigma) \in \mathbb{C}^\times$ such that

$$P_{\Sigma, N, \psi} = c(\Sigma) \cdot \ell_\Sigma^{\mathbb{A}},$$

so that

$$\beta_\Sigma^{\text{Aut}} = c(\Sigma) \cdot \beta_\Sigma^{\mathbb{A}}$$

Likewise, we have $c(\Theta(\Sigma)) \in \mathbb{C}^\times$ such that

$$P_{\Theta(\Sigma), \mathcal{O}(v_1^\perp)} = c(\Theta(\Sigma)) \cdot \ell_{\Theta(\Sigma)}^{\mathbb{A}}$$

so that

$$\beta_{\Theta(\Sigma)}^{\text{Aut}} = c(\Theta(\Sigma)) \cdot \beta_{\Theta(\Sigma)}^{\mathbb{A}},$$

Similarly, we have $a(\Sigma)$ and $b(\Sigma) \in \mathbb{C}^\times$ such that

$$A_\Sigma^{\text{Aut}} = a(\Sigma) \cdot A_\Sigma^{\mathbb{A}} \quad \text{and} \quad B_{\Theta(\Sigma)}^{\text{Aut}} = b(\Sigma) \cdot B_{\Theta(\Sigma)}^{\mathbb{A}}$$

12.14. Global result. The main global problem is to determine the constant $|c(\Theta(\Sigma))|^2$. We shall resolve this by relating $c(\Theta(\Sigma))$ to the other constants $c(\Sigma)$, $a(\Sigma)$, and $b(\Sigma)$.

Proposition 12.3. *We have:*

$$\overline{c(\Theta(\Sigma))} = c(\Sigma) \cdot b(\Sigma).$$

Proof. This follows by combining the global Proposition 12.2 and the local Corollary 6.7. □

It remains then to compute $c(\Sigma)$ and $b(\Sigma)$.

Proposition 12.4. *We have:*

$$a(\Sigma) = b(\Sigma) \quad \text{and} \quad |a(\Sigma)| = 1.$$

Moreover,

$$c(\Sigma) = \begin{cases} 1, & \text{if } \Sigma \text{ is non-endoscopic;} \\ 1/2, & \text{if } \Sigma \text{ is an endoscopic lift from } \mathcal{O}_2 \times \mathcal{O}_1; \\ 1/4, & \text{if } \Sigma \text{ is an endoscopic lift from } \mathcal{O}_1 \times \mathcal{O}_1 \times \mathcal{O}_1. \end{cases}$$

Proof. The equality of $a(\Sigma)$ and $b(\Sigma)$ follows by the global equation (12.1) and the local equation (5.9). On the other hand, the Rallis inner product formula [11] gives:

$$\langle A_\Sigma^{\text{Aut}}(\Phi, f), A_\Sigma^{\text{Aut}}(\Phi, f) \rangle_{\Theta(\Sigma)} = Z_\Sigma(\Phi, \Phi, f, f) = \langle A_\Sigma^{\mathbb{A}}(\Phi, f), A_\Sigma^{\mathbb{A}}(\Phi, f) \rangle$$

where $Z_\Sigma(-)$ is the global doubling zeta integral (evaluated at the point $s = (\dim V - 3)/2$, where it is holomorphic). Combining this with the local equation (5.5), we deduce that $|a(\Sigma)| = 1$.

Finally, the value of $c(\Sigma)$ (for the group SL_2) was determined in [19, Cor. 4.3 and §6.1]. □

As a consequence, we have:

Theorem 12.5. *Let Σ be a globally ψ -generic cuspidal representation of SL_2 such that $\Pi = \Theta(\Sigma) \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{O}(V))$. Then $|c(\Theta(\Sigma))| = |c(\Sigma)|$, so that*

$$|P_{\Pi, \mathrm{O}(v_1^\perp)}(\phi)|^2 = |c(\Sigma)|^2 \cdot |\ell_\Pi^\mathbb{A}(\phi)|^2$$

for all $\phi \in \Pi$.

12.15. Avoiding Rallis inner product. In proving Theorem 12.5, we have pulled the Rallis inner product formula out of the hat to deduce that $|a(\Sigma)| = 1$ in Proposition 12.4. In fact, it is possible to avoid the Rallis inner product formula, as we briefly sketch in this subsection.

Just as Proposition 12.2 is the global analog of the local Corollary 6.7, one can establish a global analog of Corollary 9.3, namely:

Proposition 12.6. *For $\Phi \in \Omega_\psi$ and $f \in \Sigma$, one has*

$$P_{\mathrm{O}(v_1^\perp)}(A_\Sigma^{\mathrm{Aut}}(\Phi, f)) = \langle p(\Phi), \beta_\Sigma^{\mathrm{Aut}}(f) \rangle_{N \backslash \mathrm{SL}_2}.$$

Proof. The proof relies on the see-saw diagram

$$\begin{array}{ccc} \mathrm{Mp}_2 \times \mathrm{Mp}_2 & & \mathrm{O}(V) \\ | & \searrow & | \\ \mathrm{SL}_2 & & \mathrm{O}(v_1) \times \mathrm{O}(v_1^\perp) \end{array}$$

which gives rise to a global see-saw identity. More precisely, if we take

$$\Phi = \Phi_1 \otimes \Phi' \in \mathcal{S}(\mathbb{A}v_1) \otimes \mathcal{S}((v_1^\perp)_\mathbb{A}),$$

then the see-saw identity reads:

$$P_{\Pi, \mathrm{O}(v_1^\perp)}(A_\Sigma^{\mathrm{Aut}}(\Phi, f)) = \int_{[\mathrm{SL}_2]} \overline{f(g)} \cdot \theta(\Phi_1)(g) \cdot I(\Phi')(g) dg$$

where

- $\theta(\Phi_1)$ is the theta function associated to $\Phi_1 \in \mathcal{S}(\mathbb{A}v_1)$ which affords the Weil representation (associated to ψ) for $\mathrm{Mp}_2 \times \mathrm{O}(v_1)$;
- $I(\Phi')$ is the theta integral

$$I(\Phi')(g) := \int_{[\mathrm{O}(v_1^\perp)]} \theta(\Phi')(gh) dh$$

which belongs to the global theta lift of the trivial representation of $\mathrm{O}(v_1^\perp)$ to Mp_2 .

The theta integral converges absolutely when $\dim V > 4$ (it is in the so-called Weil's convergence range) or when $\mathrm{O}(v_1^\perp)$ is anisotropic. When $\dim V = 4$ and $\mathrm{O}(v_1^\perp)$ is split, one needs to regularise the theta integral following Kudla-Rallis (see [11, §3]). Since our intention here is to indicate an alternative approach to a result which we have shown, we will ignore this analytic complication in the following exposition.

In §8.3, we have seen that the map $\phi' \mapsto \phi_{\Phi'}$, where

$$\phi_{\Phi'}(g) = (g \cdot \Phi')(0),$$

gives an isomorphism

$$\mathcal{S}((v_1^\perp)_\mathbb{A})_{\mathrm{O}(v_1^\perp)} \longrightarrow I_\psi(\chi_{\mathrm{disc}(v_1^\perp)}, (\dim V - 3)/2)$$

of the $O(v_1^\perp)$ -coinvariant of the Weil representation of $O(v_1^\perp) \times \mathrm{Mp}_2$ to a principal series representation of Mp_2 . Now the Siegel-Weil formula shows that

$$I(\Phi') = E(\phi_{\Phi'})$$

where $E(\phi_{\Phi'})$ is the Eisenstein series associated to $\phi_{\Phi'}$. Again, when $\dim V > 4$, the sum defining the Eisenstein series is convergent, but when $\dim V = 4$, it is defined by meromorphic continuation. Further, if $O(v_1^\perp)$ is also split, then the Eisenstein series does have a pole at the point of interest, and we need to invoke the second term identity of the Siegel-Weil formula [11]. As mentioned before, we omit these extra (though interesting) details in this proof.

Hence, we have the following identity:

$$P_{\Pi, O(v_1^\perp)}(A_\Sigma^{\mathrm{Aut}}(\Phi, f)) = \int_{[\mathrm{SL}_2]} \overline{f(g)} \cdot \theta(\Phi_1)(g) \cdot E(\phi_{\Phi'}) dg.$$

Now the right hand side is the value at $s = (\dim V - 3)/2$ of the global zeta integral

$$Z(f, \Phi_1, \phi_s) = \int_{[\mathrm{SL}_2]} \overline{f(g)} \cdot \theta(\Phi_1)(g) \cdot E(\phi_s) dg$$

for $\phi_s \in I_\psi(\chi_{\mathrm{disc}(v_1^\perp)}, s)$. This is the global analog of the local zeta integrals we discussed in §8.3 and represents the (twisted) adjoint L-function of Σ . The unfolding of this global zeta integral, for $\mathrm{Re}(s)$ sufficiently large, gives:

$$Z(s, f, \Phi_1, \phi) = \int_{N(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A})} \overline{P_{N, \psi}(g \cdot f)} \cdot (g \cdot \Phi_1)(v_1) \cdot \phi_s(g) dg.$$

Specializing to $s = (\dim V - 3)/2$ gives the Proposition. \square

By combining Proposition 12.6 and Corollary 9.3, we deduce:

Corollary 12.7. *One has:*

$$\overline{c(\Sigma)} = c(\Theta(\Sigma)) \cdot a(\Sigma).$$

Combining Corollary 12.7 with Propositions 12.3 and 12.4, we see that

$$\overline{c(\Sigma)} = c(\Theta(\Sigma)) \cdot a(\Sigma) = \overline{c(\Sigma)} \cdot \overline{b(\Sigma)} \cdot a(\Sigma) = \overline{c(\Sigma)} \cdot |a(\Sigma)|^2$$

from which we deduce that

$$|a(\Sigma)| = 1.$$

There is a good reason for avoiding the use of the Rallis inner product formula in the treatment of the global problem. Indeed, the viewpoint and techniques developed in this paper should carry over to essentially all the low rank spherical varieties treated in [10]. Many of these (such as $\mathrm{Spin}_9 \backslash F_4$, $G_2 \backslash \mathrm{Spin}_8$ or $F_4 \backslash E_6$ to name a few) would involve the exceptional theta correspondence. Unfortunately, in the setting of the exceptional theta correspondence, an analog of the Rallis inner product formula is not known. The argument in this subsection, however, shows that this lack need not be an obstruction in the exceptional setting.

12.16. Global Relative Character Identity. We can also establish the global analog of the relative character identity. One has the diagram

$$\begin{array}{ccc} & S(V_{\mathbb{A}}) & \\ p \swarrow & & \searrow q \\ \mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A})) & & S(X_{\mathbb{A}}) \end{array}$$

which is the adelic analog of the diagram (8.2). The space $\mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A}))$ (which is the image of p) is the restricted tensor product of the local spaces $\mathcal{S}(N(k_v), \psi_v \backslash \mathrm{SL}_2(k_v))$ of test functions defined in (8.1), where the restricted tensor product is with respect to the family of basic functions $\{f_{0,v}\}$ given in Definition 8.4. Likewise, the space $S(X_{\mathbb{A}})$ is the restricted tensor product of $S(X_{k_v})$ (which is $C_c^\infty(X_{k_v})$ at finite places) with respect to the family of basic functions $\{\phi_{0,v}\}$ in Definition 8.4. As in the local case, one says that $f \in \mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A}))$ and $\phi \in S(X_{\mathbb{A}})$ are in correspondence or are transfers of each other if there exists $\Phi \in S(V_{\mathbb{A}})$ such that $f = p(\Phi)$ and $\phi = q(\Phi)$. The fundamental Lemma 8.5 ensures that every f has a transfer ϕ and vice versa.

Before formulating the global relative character identity, we need to address an additional subtle point here. In our general discussion in §12.4, we have considered the maps

$$\alpha_{\Sigma}^{\mathrm{Aut}} : C_c^\infty(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A})) \longrightarrow \Sigma \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{SL}_2)$$

and

$$\alpha_{\Theta^{\mathrm{Aut}}(\Sigma)}^{\mathrm{Aut}} : C_c^\infty(X(\mathbb{A})) \longrightarrow \Theta^{\mathrm{Aut}}(\Sigma) \subset \mathcal{A}_{\mathrm{cusp}}(\mathrm{O}(V)).$$

The point here is that their domains consist of smooth compactly supported functions on the adelic points of the relevant spherical varieties. Likewise, in §12.5, the global relative character is given as a distribution on the space of smooth compactly supported functions. Now in the case of the hyperboloid X , this is fine since the basic function $\phi_{0,v}$ belongs to $C_c^\infty(X(k_v))$. However, this is not sufficient for the case of the Whittaker variety $(N, \psi) \backslash \mathrm{SL}_2$ since the basic function $f_{0,v}$ is not compactly supported. In particular, while it is true that

$$C_c^\infty(N(k_v), \psi_v \backslash \mathrm{SL}_2(k_v)) \subset \mathcal{S}(N(k_v), \psi_v \backslash \mathrm{SL}_2(k_v)) \quad \text{for each place } v,$$

one has in the adelic setting:

$$C_c^\infty(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A})) \not\subset \mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A})).$$

Indeed, these adelic spaces have nothing much to do with each other.

To define global relative characters for the Whittaker variety, we thus need to ensure that the map $\alpha_{\Sigma}^{\mathrm{Aut}} = \mathrm{pr}_{\Sigma} \circ \theta$ can be defined on $\mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A}))$.

The main issue is to ensure that the formation of theta series θ can be applied to $f = \otimes_v f_v \in \mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2)$. Recall that

$$(12.5) \quad \theta(f)(g) = \sum_{\gamma \in N(k) \backslash \mathrm{SL}_2(k)} f(\gamma g) \quad \text{for } g \in \mathrm{SL}_2(\mathbb{A}).$$

Hence, if $f = p(\Phi)$ with $\Phi \in S(V_{\mathbb{A}})$, we would like to show the convergence of the sum

$$\sum_{\gamma \in B(k) \backslash \mathrm{SL}_2(k)} \sum_{\lambda \in k^\times} |f(t(\lambda)\gamma g)| = \sum_{\gamma \in B(k) \backslash \mathrm{SL}_2(k)} \sum_{\lambda \in k^\times} |(\gamma g \cdot \Phi)(\lambda \cdot v_1)|.$$

Let us denote the inner sum by

$$F_\Phi(g) := \sum_{\lambda \in k^\times} |(g \cdot \Phi)(\lambda \cdot v_1)|.$$

which is certainly convergent since Φ is a Schwartz function on $V_\mathbb{A}$. Then we need to show the convergence of

$$(12.6) \quad \sum_{\gamma \in B(k) \backslash \mathrm{SL}_2(k)} F_\Phi(\gamma g).$$

This looks very much like the sum defining an Eisenstein series on SL_2 . Indeed, observe that F_Φ is a function on $T(k) \cdot N(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A})$ and for $a \in \mathbb{A}^\times$ and $k \in K = \prod_v K_v$, we have:

$$F_\Phi(t(a)k) = |a|_{\mathbb{A}}^{\frac{\dim V}{2}} \cdot \sum_{\lambda \in k^\times} |(k \cdot \Phi)(a\lambda \cdot v_1)|.$$

To understand the asymptotic of F_Φ as $|a|_{\mathbb{A}}$ tends to 0 or ∞ , we note:

Lemma 12.8. *Let $\phi \in S(\mathbb{A})$ and define a function Ψ on \mathbb{A}^\times by*

$$\Psi(a) = \sum_{\lambda \in k^\times} \phi(a\lambda).$$

Then Ψ is rapidly decreasing as $|a|_{\mathbb{A}} \rightarrow \infty$. Moreover, as $|a|_{\mathbb{A}} \rightarrow 0$,

$$|\Psi(a)| \leq C \cdot |a|_{\mathbb{A}}^{-1} \quad \text{for some constant } C.$$

Proof. The rapid decrease of $\Psi(a)$ as $|a|_{\mathbb{A}} \rightarrow \infty$ follows from the fact that $\phi \in S(V_\mathbb{A})$. For the asymptotic as $|a|_{\mathbb{A}} \rightarrow 0$, we need to apply the Poisson summation formula to the sum defining Ψ . Writing $\widehat{\phi}$ for the Fourier transform of ϕ (with respect to $d_\psi(x)$), we have:

$$\begin{aligned} \Psi(a) &= -\phi(0) + \sum_{\lambda \in k} \phi(a\lambda) \\ &= -\phi(0) + |a|_{\mathbb{A}}^{-1} \cdot \sum_{\lambda \in k} \widehat{\phi}(a^{-1} \cdot \lambda) \\ &= -\phi(0) + |a|_{\mathbb{A}}^{-1} \cdot \widehat{\phi}(0) + |a|_{\mathbb{A}}^{-1} \cdot \sum_{\lambda \in k^\times} \widehat{\phi}(a^{-1} \cdot \lambda) \end{aligned}$$

As $|a|_{\mathbb{A}}^{-1} \rightarrow \infty$, the last sum tends to 0 rapidly, so the third term is bounded. Hence we see that the asymptotic of $\Psi(a)$ as $|a|_{\mathbb{A}} \rightarrow 0$ is governed by the second term in the last expression. \square

Applying the lemma to $\phi(a) = (k \cdot \Phi)(a \cdot v_1)$, we deduce that $F_\Phi(t(a)k)$ is rapidly decreasing as $|a|_{\mathbb{A}} \rightarrow \infty$ and

$$|F_\Phi(t(a)k)| \leq C \cdot |a|_{\mathbb{A}}^{\frac{\dim V}{2}-1} \quad \text{as } |a|_{\mathbb{A}} \rightarrow 0$$

for some C which can be taken to be independent of $k \in K$. In other words, the sum in (12.6) is dominated by the sum defining a spherical Eisenstein series associated to the principal series representation $I(\frac{\dim V}{2} - 2)$ of SL_2 . Hence, when $\dim V > 6$, the above sum does converge to give a smooth function on $[\mathrm{SL}_2]$ of moderate growth (c.f. [6, Thm. 11.2]), so that (12.5) defines a $\mathrm{SL}_2(\mathbb{A})$ -equivariant map

$$\theta : \mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2) \longrightarrow C_{mod}^\infty([\mathrm{SL}_2]).$$

One then has the map $\alpha_\Sigma^{\mathrm{Aut}} = \mathrm{pr}_\Sigma \circ \theta$ such that one still has the adjunction formula

$$\langle \alpha_\Sigma^{\mathrm{Aut}}(f), \phi \rangle_\Sigma = \langle f, \beta_\Sigma^{\mathrm{Aut}}(\phi) \rangle_{N(\mathbb{A}) \backslash \mathrm{SL}_2(\mathbb{A})},$$

for $f \in \mathcal{S}(N(\mathbb{A}), \psi \backslash \mathrm{SL}_2(\mathbb{A}))$ and $\phi \in \Sigma$.

Presumably, one can show that θ and $\alpha_{\Sigma}^{\text{Aut}}$ are also defined when $3 \leq \dim V \leq 6$ by a more careful analysis, involving the meromorphic continuation of pseudo-Eisenstein series, but we have not pursued this further. One now has the following global relative character identity.

Theorem 12.9. *Assume that $\dim V > 6$. If $f \in \mathcal{S}(N(\mathbb{A}), \psi \backslash \text{SL}_2(\mathbb{A}))$ and $\phi \in S(X_{\mathbb{A}})$ are transfer of each other, then for a cuspidal representation Σ of SL_2 with cuspidal theta lift $\Theta^{\text{Aut}}(\Sigma)$ on $\text{O}(V)$, one has:*

$$\mathcal{B}_{\Sigma}^{\text{Aut}}(f) = \mathcal{B}_{\Theta(\Sigma)}^{\text{Aut}}(\phi).$$

Proof. We have defined $c(\Sigma)$ by

$$\beta_{\Sigma}^{\text{Aut}} = c(\Sigma) \cdot \beta_{\Sigma}^{\mathbb{A}}.$$

By the adjunction formulae for the pairs $(\alpha_{\Sigma}^{\text{Aut}}, \beta_{\Sigma}^{\text{Aut}})$ and $(\alpha_{\Sigma}^{\mathbb{A}}, \beta_{\Sigma}^{\mathbb{A}})$, we deduce that

$$\alpha_{\Sigma}^{\text{Aut}} = \overline{c(\Sigma)} \cdot \alpha_{\Sigma}^{\mathbb{A}}.$$

Hence, one has

$$\mathcal{B}_{\Sigma}^{\text{Aut}}(f) = (\beta_{\Sigma}^{\text{Aut}} \alpha_{\Sigma}^{\text{Aut}}(f))(1) = |c(\Sigma)|^2 \cdot (\beta_{\Sigma}^{\mathbb{A}} \alpha_{\Sigma}^{\mathbb{A}}(f))(1) = |c(\Sigma)|^2 \cdot \mathcal{B}_{\Sigma}^{\mathbb{A}}(f).$$

Likewise, we have

$$\mathcal{B}_{\Theta(\Sigma)}^{\text{Aut}}(\phi) = |c(\Theta(\Sigma))|^2 \cdot \mathcal{B}_{\Theta(\Sigma)}^{\mathbb{A}}(\phi).$$

Since $|c(\Sigma)| = |c(\Theta(\Sigma))|$, the desired result follows from the local relative character identity of Theorem 9.1. \square

12.17. End remarks. We end this paper with some comparisons with the relative trace formula approach. The spectral side of a relative trace formula is essentially a sum of the relevant global relative characters over all cuspidal representations. One then hopes to separate the different spectral contributions by using the action of the spherical Hecke algebra at almost all places. The main global output of a comparison of (the geometric side of) two such relative trace formulae is typically a global relative character identity as in Theorem 12.9, as a consequence of which one deduces Proposition 12.2 and the local relative character identities in Theorem 9.1, which in turn implies Proposition 6.1. It is interesting to compare this with the approach via theta correspondence which we have pursued in this paper.

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DEPARTMENT OF MATHEMATICS, NATIONAL UNIVERSITY OF SINGAPORE, 10 LOWER KENT RIDGE ROAD, SINGAPORE 119076

E-mail address: `matgwt@nus.edu.sg`

E-mail address: `xiaoleiwan@u.nus.edu`