Perverse sheaves and the topology of algebraic varieties
Five lectures at the 2015 PCMI

Mark Andrea A. de Cataldo*

June 5, 2015

Abstract

Contents

*Partially supported by N.S.F. grant DMS-1301761 and by a grant from the Simons Foundation (#296737 to Mark de Cataldo)

Acknowledgments. I would like to thank M. Goresky, L. Migliorini and R. Virk for their very useful comments.

Goal of the lectures. The goal of these lectures is to introduce the novice to the use of perverse sheaves in complex algebraic geometry and to what is perhaps the deepest known fact relating the homological/topological invariants of the source and target of a proper map of complex algebraic varieties, namely the decomposition theorem.

Notation. A variety is a complex algebraic variety, which we do not assume to be irreducible, nor reduced. We work with cohomology with $\mathbb{Q}$-coefficients. $\mathbb{Z}$-coefficients do not fit well in our story. As we rarely focus on a single cohomological degree, for the most part we consider the total, graded cohomology groups, e.g. $H^*(X, \mathbb{Q})$.

Bibliographical references. The main reference is the survey [16] and the extensive bibliography contained in it, most of which is not be reproduced here. This allowed me to try to minimize the continuous distractions related to the peeling apart of the various versions of the results and of the attributions. The reader may also consult the discussions in [17] that did not make it into the very different final version [16].

Style of the lectures and of the lecture notes. I hope to deliver my lectures in a rather informal style. I plan to introduce some main ideas, followed by what I believe to be a striking application, often with an idea of proof. The lecture notes are not intended to replace in any way the existing literature on the subject, they are a mere amplification of what I can possibly touch upon during the five one-hour lectures. As it is usual when meeting a new concept, the theorems and the applications are very important, but I also believe that working with examples, no matter how lowly they may seem, can be truly illuminating and useful in building one own's local and global picture. Because of the time factor, I cannot possibly fit many of these examples in the flow of the lectures. This is why there are plenty of exercises, which are not just about examples, but at time deal head-on with actual important theorems. I could have laid-out several more exercises (you can look at my lecture notes [21], or at my little book [7] for more exercises), but I tried to choose ones that would complement well the lectures; too much of anything is not a good thing anyway. The exercises (temporarily) highlighted in blue are a possible list of the ones to be discussed by Rahbar Virk during his problem sessions.

What is missing from these lectures? A lot! Two related topics come to mind: vanishing/nearby cycles and constructions of perverse sheaves; see the survey above for a quick introduction to both. To compound this infamy, there is no discussion of the equivariant picture [3].

1 Lecture 1: The decomposition theorem

Summary of Lecture 1. Deligne theorem on the degeneration of the Leray spectral sequence for smooth projective maps; this is the 1968 prototype of the 1982 decomposition theorem. Application, via the use of the theory of mixed Hodge structures, to the global invariant cycle theorem, a remarkable topological property enjoyed by families of projective manifolds and compactifications of their total spaces. The main theorem of these lectures, the decomposition theorem, stated in cohomology. Application to a proof of the local invariant cycle theorem, another remarkable topological property concerning the degenerations of families of projective manifolds. Deligne theorem, including semisimplicity of the direct image sheaves, in the derived category. The decomposition theorem: the direct image complex splits in the derived category into a direct sum of shifted and twisted intersection complexes supported on the target of a proper map.

1.1 Deligne theorem in cohomology

Warm-up: the Künneth formula and a question. Let Y, F be varieties. Then

$$H^*(Y \times F, \mathbb{Q}) = \bigoplus_{q \geq 0} H^{*-q}(Y, \mathbb{Q}) \otimes H^q(F, \mathbb{Q}). \tag{1}$$

Note that the restriction map $H^*(Y \times F, \mathbb{Q}) \to H^*(F, \mathbb{Q})$ is surjective.

Question 1.1.1 *Let F be a projective manifold and let $\overline{X}$ be a projective manifold completing $Y \times F$. What can we say about the restriction map $H^*(\overline{X}, \mathbb{Q}) \to H^*(F, \mathbb{Q})$?*

Answer: Theorem 1.2.1 gives an answer in the more general setting of families of projective manifolds and the compactifications of their total spaces.

The decomposition theorem has an important precursor in Deligne theorem, which can be viewed as the decomposition theorem in the absence of singularities of the domain, of the target *and* of the map. We start by stating the cohomological version of his theorem.

Theorem 1.1.2 (**Blanchard-Deligne 1968 theorem in cohomology** [23]) *Let $f : X \to Y$ be a smooth projective map*[1] *of algebraic manifolds. There is an isomorphism*

$$H^*(X, \mathbb{Q}) \cong \bigoplus_{q \geq 0} H^{*-q}(Y, R^q f_* \mathbb{Q}_X). \tag{2}$$

More precisely, the Leray spectral sequence (see §1.7) of the map f is E_2-degenerate.

Proof. Exercise 1.7.3 guides you through Deligne's classical trick (Deligne-Lefschetz criterion) of using the hard Lefschetz theorem on the fibers to force the triviality of the differentials of the spectral sequence. □

[1] smooth: submersion; projective: factors as $X \to Y \times \mathbb{P} \to Y$ (closed embedding, projection)

Compare (1) and (2): both present cohomological shifts; both express the cohomology of the l.h.s. via cohomology groups on Y; in the former case, we have cohomology with constant coefficients; in the latter, and this is crucial, we have cohomology with locally constant coefficients.

Deligne theorem is central in the study of the topology of algebraic varieties. Let us discuss one striking application of this result: the global invariant cycle theorem.

1.2 The global invariant cycle theorem

Let $f : X \to Y$ be a smooth and projective map of algebraic manifolds, let $j : X \to \overline{X}$ be an open immersion into a projective manifold and let $y \in Y$. What are the images of $H^*(X, \mathbb{Q})$ and $H^*(\overline{X}, \mathbb{Q})$ via the restriction maps into $H^*(f^{-1}(y), \mathbb{Q})$? The answer is the global invariant cycle theorem 1.2.1 below.

The direct image sheaf $\mathcal{R}^q := R^q f_* \mathbb{Q}_X$ on Y is the sheaf associated with the pre-sheaf

$$U \mapsto H^q(f^{-1}(U), \mathbb{Q}).$$

In view of Ehresmann lemma, the proper[2]submersion f is a C^∞ fiber bundle. The sheaf $\mathcal{R}^q$ is then locally constant with stalk

$$\mathcal{R}^q_y = H^q(f^{-1}(y), \mathbb{Q}).$$

The fundamental group $\pi_1(Y, y)$ acts via linear transformations on $\mathcal{R}^q_y$: pick a loop $\gamma(t)$ at y and use a trivialization of the bundle along the loop to move vectors in $\mathcal{R}^q_y$ along $\mathcal{R}^q_{\gamma(t)}$, back to $\mathcal{R}^q_y$ (monodromy action for the locally constant sheaf $\mathcal{R}^q$).

The global sections of $\mathcal{R}^q$ identify with the monodromy invariants $(\mathcal{R}^q_y)^{\pi_1} \subseteq \mathcal{R}^q_y$. Note that this subspace is defined topologically. The cohomology group $\mathcal{R}^q_y = H^q(f^{-1}(y), \mathbb{Q})$ has it own Hodge (p, p')-decomposition (pure Hodge structure of weight q), an algebro-geomeric structure.

How is $(\mathcal{R}^q_y)^{\pi_1} \subseteq \mathcal{R}^q_y$ placed with respect to the Hodge structure?

The E_2-degeneration Theorem 1.1.2 yields the following immediate, yet, remarkable, consequence:

$$H^q(X, \mathbb{Q}) \xrightarrow{\text{surj}} (\mathcal{R}^q_y)^{\pi_1} \subseteq \mathcal{R}^q_y, \tag{3}$$

i.e. *the restriction map in cohomology, which automatically factors through the invariants, in fact factors surjectively through them.*

The theory of mixed Hodge structures now tells us that the monodromy invariant subspace $(\mathcal{R}^q_y)^{\pi_1} \subseteq \mathcal{R}^q_y$ (a topological gadget) is in fact a Hodge substructure, i.e. it inherits the Hodge (p, p')-decomposition (the algebro-geometric gadget).

The same mixed theory implies that highly non trivial fact (Exercise 1.7.14) that the images of the restriction maps from $H^*(\overline{X}, \mathbb{Q})$ and $H^*(X, \mathbb{Q})$ (the two may be very different!) into $H^*(f^{-1}(y), \mathbb{Q})$ coincide.

We have reached the following conclusion, proved by Deligne in 1972.

[2]proper := the pre-image of compact is compact; it is the "relative" version of compactness

Theorem 1.2.1 (Global invariant cycle theorem [25]) *Let $f : X \to Y$ be a smooth and projective map of algebraic manifolds, let $j : X \to \overline{X}$ be an open immersion into a projective manifold and let $y \in Y$. Then the images of $H^*(\overline{X}, \mathbb{Q})$ and $H^*(X, \mathbb{Q})$ into $H^*(f^{-1}(y), \mathbb{Q})$ coincide with the subspace of monodromy invariants which is thus a Hodge substructure.*

This theorem provides a far-reaching answer to Question 1.1.1. Note that the Hopf examples in Exercise 1.7.2 show that such a nice answer is not possible outside of the realm of complex algebraic geometry and even there, outside of the realm of proper maps: there are two obstacles: the non E_2-degeneration and then the absence of the global constraints imposed by mixed Hodge structures.

1.3 Cohomological decomposition theorem

The decomposition theorem is a generalization of Deligne theorem 1.1.2 which is valid for any proper map of algebraic varieties: compare (2) and (4). It was first proved by Beilinson-Bernstein-Deligne-Gabber in their monograph [1] (Théorèm 6.2.5) on perverse sheaves.

A possible initial psychological drawback, when compared with Deligne's theorem, is that even if one insists in dealing with maps of projective manifolds, the statement is not about cohomology with locally constant coefficients, but it requires the Goresky-MacPherson intersection cohomology groups with twisted coefficients on various subvarieties of the target of the map. However, this is precisely why this theorem is so striking!

To get to the point, for now we simply say that we have the intersection cohomology groups $IH^*(S, \mathbb{Q}) = H^*(S, \mathcal{IC}_S)$ of an irreducible variety S; they agree with the ordinary cohomology groups when S is nonsingular. The theory is very flexible as it allows for twisted coefficients: given a locally constant sheaf L on a dense open subvariety $S^o \subseteq S_{reg} \subseteq$, we get the intersection cohomology groups $IH^*(S, L)$ of S. We may call such pairs (S, L), enriched varieties (see [39], p.222); this explains the notation $\mathcal{EV}$ below.

Theorem 1.3.1 (Cohomological decomposition theorem) *Let $f : X \to Y$ be a proper map of complex algebraic varieties. For every $q \geq 0$, there is a finite collection $\mathcal{EV}_q$ of pairs (S, L) with $S \subseteq Y$ pairwise distinct closed subvarieties of Y, and an isomorphism*

$$IH^*(X, \mathbb{Q}) \cong \oplus_{q \geq 0, \mathcal{EV}_q} IH^{*-q}(S, L). \tag{4}$$

Note that the same S could appear for distinct q's.

Deligne theorem in cohomology is a special case. In particular, we can deduce an appropriate version of the global invariant cycle theorem [1], 6.2.8. Let us instead focus on its local counterpart.

1.4 The local invariant cycle theorem

The decomposition theorem (4) has a local flavor over the target Y, in both the Zariski and in the classical topology: replace Y by an open set, X by $f^{-1}(U)$ and S by $S \cap U$.

Let us focus on the classical topology. Let X be nonsingular; this is for the sake of our discussion, for then $IH^*(X, \mathbb{Q}) = H^*(X, \mathbb{Q})$.

Let $y \in Y$ be a point and let us pick a small Euclidean "ball" $B_y \subseteq Y$ centered at y, so that (4) reads:

$$H^*(f^{-1}(y), \mathbb{Q}) = H^*(f^{-1}(B_y), \mathbb{Q}) = \oplus_{q, \mathcal{E}\mathcal{V}_q} IH^{*-q}(S \cap B_y, L).$$

Let f be surjective. Let $f^o : X^o \to Y^o$ the restriction of the map f over the open subvariety of Y of regular values for f. Let $y^o \in B_y$ be a regular value for f.

By looking at Deligne theorem for the map f^o it seems reasonable to expect that for every q one of the summands in (4) should be $IH^{*-q}(Y, L_q)$, where L_q is the locally constant $R^q f^o_* \mathbb{Q}$. This is indeed the case.

If follows that for every $q \geq 0$, we have that $IH^0(B_y, L_{q|Y^o \cap B_y})$ is a direct summand of $H^q(f^{-1}(y), \mathbb{Q})$, let us even say that the latter surjects onto the former. Note that we did not assume that $y \in Y^o$.

The intersection cohomology group $IH^0(Y, L_k)$ is the space of monodromy invariants for the representation $\pi_1(Y^o \cap B_y, y^o) \to GL(H^q(f^{-1}(y^o), \mathbb{Q})$. Abbreviate the fundamental group notation to $\pi_{1,loc}$.

We have reached a very important conclusion:

Theorem 1.4.1 (**Local invariant cycle theorem**, [6] and [1], 6.2.9) *Let $f : X \to Y$ be a proper surjective map of algebraic varieties with X nonsingular. Let $y \in Y$ be any point, let B_y be a small Euclidean ball on Y at y, let $y^o \in B_y$ be a regular value of f. Then $H^*(f^{-1}(y), \mathbb{Q}) = H^*(f^{-1}(B_y), \mathbb{Q})$ surjects onto the local monodromy invariants $H^*(f^{-1}(y^o), \mathbb{Q})^{\pi_{1,loc}}$.*

1.5 Deligne theorem

In fact, Deligne proved something stronger than his cohomological theorem, he proved a decomposition theorem for the derived direct image under a smooth proper map.

Pre-warm-up: cohomological shifts. Given a $\mathbb{Z}$-graded object $K = \oplus_{i \in \mathbb{Z}} K^i$, like the total cohomology of a variety, or a complex (of sheaves, for example) on it, or the total cohomology of such a complex, etc., and given an integer $a \in \mathbb{Z}$, we can shift by the amount a and get a new graded object

$$K[a] := \bigoplus_{i \in \mathbb{Z}} K^{i+a}. \tag{5}$$

If $a > 0$, the effect of this operation is to "shift K back by a units." Again, if K has non zero entries contained in an interval $[m, n]$, then $K[a]$ has non zero entries contained in

$[m-a, n-a]$. We have the following basic relation, e.g. for complexes of sheaves

$$\mathcal{H}^i(K[a]) = \mathcal{H}^{i+a}(K).$$

A sheaf F can be viewed as a complex placed in cohomological degree zero; we can then take the $F[a]$'s. We can take a collection of F_q's and form $\oplus_q F_q[-q]$, which is a complex with *trivial differentials.* Then

$$H^*(Y, \oplus_q F_q[-q]) = \oplus_q H^{*-q}(Y, F_q).$$

Warm-up: Künneth for the derived direct image. Let $f : X := Y \times F \to Y$ be the projection. Then there is a canonical isomorphism

$$Rf_*\mathbb{Q}_X = \oplus_{q\geq 0}\underline{H}^q(F)[-q] \qquad (\underline{H}^q(F) \text{ constant sheaf on } Y \text{ with stalk } H^q(F,\mathbb{Q})). \tag{6}$$

The isomorphism takes place in the derived category of the category of sheaves of rational vector spaces on Y; this is where we find the direct image complex $Rf_*\mathbb{Q}_X$, whose cohomology is the cohomology of X: $H^*(Y, Rf_*\mathbb{Q}_X) = H^*(X,\mathbb{Q})$. Exercise 1.7.23 asks you to prove (6).

Now to Deligne's 1968 theorem.

Theorem 1.5.1 (**Deligne 1968 theorem [23]; semisimplicity in 1972 [25], §4.2)** *Let $f : X \to Y$ be a smooth proper map of algebraic varieties. "The derived image complex has trivial differentials", more precisely, there is an isomorphism*

$$Rf_*\mathbb{Q}_X \cong \bigoplus_{q\geq 0} R^q f_*\mathbb{Q}[-q]. \tag{7}$$

Moreover, the locally constant direct image sheaves $R^q f_*\mathbb{Q}_X$ are semisimple.

Theorem 1.5.1(7), which is proved by means of an E_2-degeneration argument[3] along the lines of the one in Exercise 1.7.3, is the "derived" version of (2), which follows by taking cohomology on both sides of (7). In addition to [23], you may want to consult the first two pages of [29]. The semisimplicity result is one of the many amazing applications of the theory of weights (Hodge-theoretic, or Frobenius).

Terminology and facts about semisimple locally constant sheaves. To give a locally constant sheaf on Y is the same as giving a representation of the fundamental group of Y (Exercise 1.7.10). By borrowing from the language of representations, we have the notions of simple (no non trivial locally constant subsheaf; a.k.a. irreducible) and semisimple (direct sum of simples; a.k.a. completely reducible), indecomposable (no non trivial direct sum decomposition) locally constant sheaves.

To give an idea of how powerful is the semisimplicity assertion, let us say this: Deligne gave the first algebro-geometric proof of the hard Lefschetz theorem by making essential use of the semisimplicity of these direct image sheaves in a Lefschetz pencil.

[3] people refer to it as the Deligne-Lefschetz criterion

Once one has semisimplicity, one can decompose further. For a semisimple locally constant sheaf L, we have the canonical isotypical direct sum decomposition

$$L = \oplus_\chi L_\chi, \tag{8}$$

where each summand is the span of all mutually isomorphic simple subobjects, and the direct sum ranges over the set of isomorphism classes of irreducible representations of the fundamental group. In particular, in (7), we have $R^q f_* \mathbb{Q}_X = \mathcal{R}^q = \oplus_\chi \mathcal{R}^q_\chi$.

What is semisimplicity good for? Here is the beginning of an answer: look at Exercise 4.7.4, where it is put to good use to give Deligne's proof of the Hard Lefschetz theorem. In the context of the decomposition theorem, the semisimplicity of the perverse direct images is an essential ingredient in the proof of the relative hard Lefschetz theorem; see [1] and [13] (especially, §5.1 and §6.4).

1.6 The decomposition theorem

As we have seen, Deligne theorem in cohomology has a counterpart in the derived category. The cohomological decomposition Theorem 1.3.1 also has a stronger counterpart in the derived category, i.e Theorem 1.6.1.

In these lectures, we adopt a version of the decomposition theorem that is more general, and simpler to state!, than the one in [1], 6.2.5 (coefficients of geometric origin) and of [47] (coefficients in polarizable variations of pure Hodge structures). The version we adopt is due essentially to T. Mochizuki [42] (with important contributions of C. Sabbah [46]) and it involves semisimple coefficients. [42] works in the context of projective maps of quasi-projective varieties and with $\mathbb{C}$-coefficients; one needs a little bit of tinkering to reach the same conclusions for proper maps of complex varieties with $\mathbb{Q}$-coefficients (to my knowledge, this is not in the literature).

Warning: $\mathcal{IC}$ vs. IC. We are about to meet the main protagonists of our lectures, the intersection complexes $\mathcal{IC}_S(L)$ with twisted coefficients; in fact, the actual protagonists are the shifted (see (5) for the notion of shift):

$$IC_S(L) := \mathcal{IC}_Y(L)[\dim S], \tag{9}$$

which are perverse sheaves on S and on any variety Y for which $S \subseteq Y$ is closed. While $\mathcal{IC}_S(L)$ has non-trivial cohomology sheaves only in the interval $[0, \dim S - 1]$, the analogous interval for $IC_S(L)$ is $[-\dim S, -1]$. Instead of discussing the pro and cons of either notation, let us move on.

Brief on intersection complexes. The intersection cohomology groups of an enriched variety (S, L) are in fact the cohomology groups of S with coefficients in a very special complex of sheaves called the intersection complex of S with coefficients in L and denoted by $\mathcal{IC}_S(L)$: we have $IH^*(S, L) = H^*(S, \mathcal{IC}_S(L))$. If S is nonsingular, and L is constant of rank one, then $\mathcal{IC}_S = \mathcal{IC}_S(\mathbb{Q}) = \mathbb{Q}_S$. The decomposition theorem in cohomology (4) is the shadow in cohomology of a decomposition of the direct image complex $Rf_* \mathcal{IC}_X$ in the derived category of sheaves of rational vector spaces on Y. In fact, the dt holds in the greater generality of semisimple coefficients.

Theorem 1.6.1 (**Decomposition theorem**) *Let $f : X \to Y$ be a proper map of complex algebraic varieties. Let $\mathcal{IC}_X(M)$ be the intersection complex of X with semisimple twisted coefficients M. For every $q \geq 0$, there is a finite collection $\mathcal{EV}_q$ of pairs (S, L) with S pairwise distinct*[4] *and L semisimple, and an isomorphism*

$$Rf_*\mathcal{IC}_X(M) \cong \bigoplus\nolimits_{q\geq 0, \mathcal{EV}_q} \mathcal{IC}_S(L)[-q]. \tag{10}$$

In particular, by taking cohomology:

$$IH^*(X, M) \cong \bigoplus_{q, \mathcal{EV}_q} IH^{*-q}(S, L). \tag{11}$$

We have the isotypical decompositions (8), which can be plugged into what above.

Remark 1.6.2 The fact that there may be summands associated with $S \neq Y$ should not come as a surprise. It is a natural fact due to the singularities (deviation from being smooth) of the map f. One does not need the decomposition theorem to get convinced: the reader can work out the case of the blowing up of the affine plane at the origin; see also Exercise 1.7.20. In general, it is difficult to predict which S will appear in the decomposition theorem; see parts 5 and 7 of Exercise 1.7.21.

[4] the same S could appear for distinct q's

1.7 Exercises for Lecture 1

Exercise 1.7.1 (**Ehresman lemma and local constancy of higher direct images for proper submersions**) Let $f : X \to Y$ be a map of varieties and recall that q-th direct image sheaf $\mathcal{R}^q := R^q f_* \mathbb{Q}_X$ is defined to be the sheafification of the presheaf $Y \supseteq U \mapsto H^q(f^{-1}(U), \mathbb{Q})$. If f admits the structure of a C^∞-fiber bundle, then the sheaves $\mathcal{R}^q$ are locally constant, with stalks the cohomology of the fibers. Give examples of maps where this last statement fails (hint: they cannot be proper). If f is a proper smooth map of complex algebraic varieties, then it admits a structure of C^∞ fiber bundle (Ehresmann lemma). Deduce that nonsingular hypersurfaces of fixed degree in complex projective space are all diffeomorphic to each other. Is the same true in real projective space? Why?

Quick review of the Leray spectral sequence (see Grothendieck's gem "Tohoku"). The Leray spectral sequence for a map $f : X \to Y$ (and for the sheaf $\mathbb{Q}_X$) is a gadget denoted $E_2^{pq} = H^p(Y, R^q f_* \mathbb{Q}_X) \Rightarrow H^{p+q}(X, \mathbb{Q})$. There are the natural differentials $d_r : E_r^{pq} \to E_r^{p+r,q-r+1}$, $d_r^2 = 0$, with $r \geq 2$ and $E_{r+1} = H^*(E_r, d_r)$. E_2-degeneration means that $d_r = 0$ for every $r \geq 2$, so that one has a cohomological decomposition $H^*(X, \mathbb{Q}) \cong \oplus_{q \geq 0} H^{*-q}(Y, R^q f_* \mathbb{Q}_X)$. Note that with $\mathbb{Z}$ coefficients, E_2-degeneration does not imply the existence of an analogous splitting.

Exercise 1.7.2 (**Maps of Hopf-type**) Let $a : \mathbb{C}^2 \setminus o \to \mathbb{P}^1 \cong S^2$ be the usual map $(x, y) \mapsto (x : y)$. It induces two more maps, $b : S^3 \to S^2$ and $c : HS := (\mathbb{C}^2 \setminus o)/\mathbb{Z} \to \mathbb{P}^1$ (where $1 \in \mathbb{Z}$ acts as multiplication by two). These three maps are fiber bundles. Show that there cannot be a cohomological decomposition as in (2). Deduce that their Leray spectral sequence are not E_2-degenerate. Observe that the conclusion of the global invariant cycle theorem concerning the surjectivity onto the monodromy invariants fails in all three cases.

Exercise 1.7.3 (**Proof of the cohomological decomposition (2) via hard Lefschetz**) Let us recall the hard Lefschetz theorem: let X be a projective manifold of dimension d, and let $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of an ample line bundle on X; then for every $q \geq 0$, the iterated cup product maps $\eta^{d-q} : H^q(X, \mathbb{Q}) \to H^{2d-q}(X, \mathbb{Q})$ are isomorphisms. Deduce the primitive Lefschetz decomposition: for every $q \leq d$, set $H^q_{prim} := \mathrm{Ker}\{\eta^{d-q+1} : H^q \to H^{2d-q+2}\}$; then we have, for every $0 \leq q \leq d$ $H^q = \oplus_{j \geq 0} H^{q-2j}_{prim}$, and, for $d \leq q \leq 2d$, we have $H^q = \eta^{q-d} \cup \oplus_{j \geq 0} H^{q-2d-2j}_{prim}$. Let $f : X \to Y$ be as in (2) and let $d := \dim X - \dim Y$. Apply the hard Lefschetz theorem to the fibers of the smooth map f and deduce the analogue of the primitive Lefschetz decomposition for the direct image sheaves $\mathcal{R}^q := R f_* \mathbb{Q}_X$. Argue that in order to deduce (2) it is enough to show the differentials d_r of the spectral sequence vanish on $H^p(Y, \mathcal{R}^q_{prim})$ for every $q \leq d$. Use the following commutative diagram, with some entries left blank on purpose for you to fill-in, to deduce that indeed we have that vanishing:

$$\begin{array}{ccc} H^?(Y, \mathcal{R}^q_{prim}) & \xrightarrow{d_?} & H^?(Y, ?) \\ \downarrow{\scriptstyle \eta^?} & & \downarrow{\scriptstyle \eta^?} \\ H^?(Y, ?) & \xrightarrow{d_?} & H^?(Y, ?). \end{array}$$

(Hint: the right power of η kills a primitive in degree q, but is injective in degree $q-1$.) Remark: the refined decomposition (7) is proved in a similar way by replacing the spectral sequence above with the analogous one for $\mathrm{Hom}(\mathcal{R}^q[-q], Rf_*\mathbb{Q}_X)$: first you prove it is E_2-degenerate; then you lift the identity $\mathcal{R}^q \to \mathcal{R}^q$ to a map in $\mathrm{Hom}(\mathcal{R}^q[-q], Rf_*\mathbb{Q}_X)$ inducing the identity on $\mathcal{R}^q$; see [29].

Heuristics for E_2-degeneration and for semisimplicity of the $R^q f_* \mathbb{Q}_X$ via weights. It seems that Deligne guessed at E_2-degeneration by looking at the same situation over the algebraic closure of a finite field by considerations ("the yoga of weights" [24, 27]) of the size (weight) of the eigenvalues of action of Frobenius on the entries E_r^{pq}: they should have weight something analogous to $\exp(p+q)$, so the Frobenius-compatible differentials must be zero. There is a similar heuristics for the Deligne's theorem to the effect that the $\mathcal{R}^q$ are semisimple: if $0 \to M \to \mathcal{R}^q \to N \to 0$ is a short exact sequence, then Frobenius acts on $Ext^1(N, M)$ with weight $\exp(1)$; take $M \subseteq \mathcal{R}^q$ to be the maximal semisimple subobject; then the corresponding extension is invariant under Frobenius and has weight zero; it follows that the extension splits and the biggest semisimple in $\mathcal{R}^q$ splits off: $\mathcal{R}^q \cong M \oplus N$; if the resulting quotient N were non trivial, then it would contain a non trivial simple that then, by the splitting, would enlarge the biggest semisimple M in $\mathcal{R}^q$; contradiction. This kind of heuristics is now firmly based in deep theorems by Deligne and others [28, 1] for varieties finite fields and their algebraic closure, and by M. Saito [47] in the context of mixed Hodge modules over complex algebraic varieties.

Exercise 1.7.4 (**Rank one locally constant sheaves**) Take $[0,1] \times \mathbb{Q}$ and identify the two ends by multiplication by -1. Interpret this as a rank one locally constant sheaf on S^1 that is not constant. Do the same, but multiply by 2. Do the same, but first replace $\mathbb{Q}$ with $\overline{\mathbb{Q}}$ and multiply by a root of unity. Show that the tensor product operation $(L, M) \to L \otimes M$ induces the structure of an abelian group on the set of isomorphisms classes of rank one locally constant sheaves on a variety Y. Determine the torsion elements of this group when you replace $\mathbb{Q}$ with $\overline{\mathbb{Q}}$. Show that if we replace $\mathbb{Q}$ with $\mathbb{C}$ (this is the character variety for rank one complex representations) we obtain the structure of a complex Lie group.

Exercise 1.7.5 (**Locally constant sheaves and representations of the fundamental group**) A locally constant sheaf (a.k.a. local system) L on Y gives rise to a representation $\rho_L : \pi_1(Y, y) \to GL(L_y)$: pick a loop $\gamma(t)$ at y and use local trivializations of L along the loop to move vectors in L_y along $L_{\gamma(t)}$, back to L_y.

Exercise 1.7.6 (**Representations of the fundamental group and locally constant sheaves**) Given a representation $\rho : \pi_1(Y, y) \to GL(V)$ into a finite dimensional vector space, consider a universal cover $(\widetilde{Y}, \widetilde{y}) \to (Y, y)$, build the quotient space $(V \times \widetilde{Y})/\pi_1(Y, y)$, take the natural map (projection) to Y and take the sheaf of its local sections. Show that this is a locally constant sheaf whose associated representation is ρ.

Exercise 1.7.7 (**Zeroth cohomology of a local system**) Let X be a connected space. Let L be a local system on X, and write for M for the associated $\pi_1(X)$ representation. Show that

$$H^0(X; L) = M^{\pi_1(X)},$$

where the right hand side is the fixed part of M under the $\pi_1(X)$ action.

Exercise 1.7.8 (**Cohomology of local systems on a circle**) Fix an orientation of S^1, i.e., fix a generator $T \in \pi_1(S^1)$. Let L be a local system on S^1 with associated monodromy representation M. Show that

$$H^0(S^1, L) = \ker((T - id)\colon M \to M), \quad H^1(S^1, L) = \operatorname{coker}((T - id)\colon M \to M),$$

and that $H^{>1}(S^1, L) = 0$.
(Hint: one way to proceed is to use Cech cohomology. Alternatively, embed S^1 as the boundary of a disk and use relative cohomology (or dualize and use compactly supported cohomology, where the orientation is easier to get a handle on).

Exercise 1.7.9 (**Fiber bundles over a circle: the Wang sequence**) This is an extension of the previous exercise. Let $f\colon E \to S^1$ be a locally trivial fibration with fibre F and monodromy isomorphism $T\colon F \to F$. Show that the Leray spectral sequence gives rise to a short exact sequence

$$0 \to H^1(S^1, R^{q-1} f_* \mathbb{Q}) \to H^q(E, \mathbb{Q}) \to H^0(S^1, R^q f_* \mathbb{Q}) \to 0.$$

Use the previous exercise to put these together into a long exact sequence

$$0 \longrightarrow H^q(E, \mathbb{Q}) \longrightarrow H^q(F, \mathbb{Q}) \longrightarrow H^q(F, \mathbb{Q}) \to 0$$

where the middle map $H^q(F; \mathbb{Q}) \to H^q(F; \mathbb{Q})$ is given by $T^q - id$. Make a connection with the theory of nearby cycles.

Exercise 1.7.10 (**The abelian category** $Loc(Y)$) Show that the abelian category $Loc\,(Y)$ of locally constant sheaves of finite rank on Y is equivalent to the abelian category of finite dimensional $\pi_1(Y, y)$ representations. Show that both categories are noetherian (acc ok!), artinian (dcc ok!) and have a duality anti-self-equivalence.

Exercise 1.7.11 below is in striking contrast with the category Loc, but also with the one of perverse sheaves, which admits, by its very definition, the anti-self-equivalence given by Verdier duality.

Exercise 1.7.11 (**The abelian category $Sh_c(Y)$ is not artinian.**) Show that in the presence of such a anti-self-equivalence, noetherian is equivalent to artinian. Observe that the category $Sh_c(Y)$ whose objects are the constructible sheaves (i.e. there is a finite partition of $Y = \coprod Y_i$ into locally closed subvarieties to which the sheaf restricts to a locally constant one) is abelian and noetherian, but it is not artinian. Deduce that $Sh_c(Y)$ does not admit an anti-self-equivalence. Give an explicit example of the failure of dcc in $Sh_c(Y)$. Prove that $Sh_c(Y)$ is artinian IFF $\dim Y = 0$.

Exercise 1.7.12 (**Cyclic coverings**) Show that the direct image sheaf sheaf $R^0 f_* \mathbb{Q}$ for the map $S^1 \to S^1$, $t \to t^n$ is a semisimple locally constant sheaf of rank n; find its simple summands (one of them is the constant sheaf $\mathbb{Q}_{S^1}$ and the resulting splitting is given by the trace map). Do the same for locally constant sheaves with $\overline{\mathbb{Q}}$ coefficients.

Exercise 1.7.13 (**Indecomposable non simple**) The rank two locally constant sheaf on S^1 given by the non trivial unipotent 2×2 Jordan block is indecomposable, not simple, not semisimple. Make a connection between this locally constant sheaf and the Picard-Lefschetz formula for the degeneration of a curve of genus one to a nodal curve.

Amusing monodromy dichotomy. There is an important and amusing dichotomy concerning local systems in algebraic geometry (which we state informally): the global local systems arising in complex algebraic are semisimple (i.e. completely reducible; related to Zariski closure the image of the fundamental group in the general linear group being reductive); the restriction of these local systems to small punctured disks with centers at infinity (degenerations), are quasi-unipotent, i.e. unipotent after taking a finite cyclic covering if necessary. This local quasi-unipotency is in some sense the opposite of the global complete reducibility.

Quick review of Deligne's 1972 and 1974 theory of mixed Hodge structures [24, 25, 26]; see also [30]. Deligne discovered the existence of a remarkable structure, a mixed Hodge structure, on the singular cohomology of a complex algebraic variety X: there is an increasing filtration $W_k H^*(X, \mathbb{Q})$ and a decreasing filtration $F^p H^*(X, \mathbb{C})$ (with conjugate filtration denoted by $\overline{F}$) such that the graded quotients $Gr^W_k H^*(X, \mathbb{C}) = \oplus_{p+q=k} H^{pq}_k$, where the splitting is induced by the (conjugate and opposite) filtrations $F, \overline{F}$; i.e. $(W, F, \overline{F})$ induce pure Hodge structure of weight k on Gr^W_k. This structure is canonical and functorial for maps of complex algebraic varieties. Kernels, images, cokernels of pull-back maps in cohomology inherit such a structure. If X is a projective manifold, we get the known Hodge (p,q)-decomposition: $H^i(X, \mathbb{C}) = \oplus_{p+q=i} H^{pq} i(X)$. It is important to take note that for each fixed i we have $H^i(X, \mathbb{C}) = \oplus_k Gr^W_k H^i(X, \mathbb{C}) = \oplus_k \oplus_{p+q=k} H^{pq}_k(X)$ which may admit several non zero k summands for $k \neq i$. In this case, we say that the mixed Hodge structure is mixed. This happens for the projective nodal cubic: $H^1 = H^{0,0}_0$, and for the punctured affine line $H^1 = H^{11}_2$. Here are some "inequalities" for the weight filtration: $Gr^W_k H^d = 0$ for $k \notin [0, 2d]$; if X is complete, then $Gr^W_{k>d} H^d = 0$; if X is nonsingular, then $Gr^W_{k<d} H^d = 0$ and $W_d H^d$ is the image of the restriction map from any nonsingular completion; if $X \to Y$ is surjective and X is complete nonsingular, then the kernel of the pull-back to $H^d(X)$ is $W_{d-1} H^d(Y)$.

Exercise 1.7.14 (**Amazing weights**) Let $Z \to U \to X$ be a closed immersion with Z complete followed by an open dense immersion into a complete nonsingular variety. Using some of the weight inequalities listed above show that the images of $H^*(X, \mathbb{Q})$ and $H^*(U, \mathbb{Q})$ into $H^*(Z, \mathbb{Q})$ coincide. Build a counterexample in complex geometry (Hopf!). Build a counterexample in real algebraic geometry (circle, bi-punctured sphere, sphere).

It is a bit time-consuming to give an explicit example of a projective normal surface having this mixedness property. However, morally speaking, as soon as you leave the world of projective manifolds, "mixedness" is the norm.

It is time-consuming to give an explicit example of a proper map where we fail to have a cohomological decomposition analogous to (2); see Exercise 1.7.22. However, we can produce many by pure-thought using Deligne's theory of mixed Hodge structures. Here is how.

Exercise 1.7.15 (**In general, there is no decomposition** $Rf_*\mathbb{Q}_X \cong \oplus R^q f_* \mathbb{Q}_X[-q]$) Pick a normal projective variety Y whose singular cohomology is a non pure mixed Hodge structure: this is the norm. Resolve the singularities $f : X \to Y$. Use Zariski main theorem to show that $R^0 f_* \mathbb{Q}_X = \mathbb{Q}_Y$. Show that, in view of the the mixed-not-pure assumption, the map of mixed Hodge structures f^* is not injective. Deduce that there cannot be a decomposition $Rf_*\mathbb{Q}_X \cong \oplus R^q f_* \mathbb{Q}_X[-q]$ in this case. (In some sense, the absence of such a decomposition is the norm for proper maps of varieties.)

Exercise 1.7.16 (**The affine cone** Y **over a projective manifold** V) Let $V^d \subseteq \mathbb{P}$ be an embedded projective manifold of dimension d and let $Y^{d+1} \subseteq \mathbb{A}$ be its affine cone with vertex o. Let $j : U := Y \setminus \{o\} \to Y$ be the open embedding. Show that U is the $\mathbb{C}^*$-bundle over V of the dual to the hyperplane line bundle for the given embedding $V \subseteq \mathbb{P}$. Determine $H^*(U, \mathbb{Q})$. Answer: for every for $0 \le q \le d$, $H^q(U) = H^q_{Prim}(V)$ and $H^{1+d+q}(U) = H^{d-q}_{Prim}(V)$. Show that $R^0 j_* \mathbb{Q}_U = \mathbb{Q}_Y$ and that, for $q > 0$, $R^q j_* \mathbb{Q}_U$ is skyscraper at o with stalk $H^q(U, \mathbb{Q})$. Compute $H^q_c(Y, \mathbb{Q})$. Give a necessary and sufficient condition on the cohomology of V that ensures that Y satisfies Poincaré duality $H^q(Y, \mathbb{Q}) \cong H^{2d+2-q}_c(Y, \mathbb{Q})$. Observe that if V is a curve this condition boils down to it having genus zero. Remark: once you know about a bit about Verdier duality, this exercise tells you that the complex $\mathbb{Q}_Y[\dim Y]$ is Verdier self dual iff V meets the condition you have identified above; in particular, it does not if V is a curve of positive genus.

Fact 1.7.17 ($\mathcal{IC}_Y$, Y **a cone over a projective manifold** V) Let things be as in Exercise 1.7.16. By adopting the definition of the intersection complex as an iterated push-forward followed by truncations, as originally given by Goresky-MacPherson, the intersection complex of Y is defined to be $\mathcal{IC}_Y := \tau_{\le d} Rj_* \mathbb{Q}_U$, where we are truncating the complex in the following way: keep the same entries up to degree $d-1$, replace the d-th entry by the kernel of the differential exiting it and setting the remaining entries to be zero; the resulting cohomology sheaves are the same up to degree d included, and zero afterwards. More precisely, the cohomology sheaves of this complex are as follows: $\mathcal{H}^q = 0$ for $q \notin [0,d]$, $\mathcal{H}^0 = \mathbb{Q}_Y$, and for $1 \leq q \leq d$, $\mathcal{H}^i$ is skyscaper at o with stalk $H^q_{Prim}(V,\mathbb{Q})$. Here is a justification for this definition: while $\mathbb{Q}[\dim Y]$ usually fails to be Verdier self-dual, one can verify directly that the intersection complex $IC_Y := \mathcal{IC}_Y[\dim Y]$ is Verdier self-dual. If we were to truncate at any other spot, then we would not get this self-duality behavior (unless we truncate at minus 1 and get zero).

Exercise 1.7.18 (**Example of no decomposition** $Rf_*\mathbb{Q}_X \cong \oplus R^q f_* \mathbb{Q}_X[-q]$) Let things be as in Exercise 1.7.16 and assume that V is a curve of positive genus, so that Y is a surface. Let $f : \widetilde{Y} \to Y$ be the resolution obtained by blowing up the vertex $o \in Y$. Use the failure of the self-duality of $\mathbb{Q}_Y[2]$ to deduce that $\mathbb{Q}_Y$ is not a direct summand of $Rf_*\mathbb{Q}_{\widetilde{Y}}$. Deduce that $Rf_*\mathbb{Q}_X \neq \oplus R^q f_*\mathbb{Q}_X[-q]$. The reader is invited to check out arXiv:math/0504554, §3.1: it is an explicit computation showing that as you try split $\mathbb{Q}_Y$ off $Rf_*\mathbb{Q}_{\widetilde{Y}}$, you meet an obstruction; instead, you end up splitting $\mathcal{IC}_Y$ off $Rf_*\mathbb{Q}_{\widetilde{Y}}$, provided you define it as the correct truncated push-forward as above.

Fact 1.7.19 (**Intersection complexes on curves**) Let Y^o be a nonsingular curve and L be a locally constant sheaf on it. Let $j : Y^o \to Y$ be an open immersion into another curve (e.g. a compactification). Then $IC_Y(L) = j_*L[1]$ (definition of IC via push-forward/truncation). Note that if $y \in Y$ is a nonsingular point, then the stalk $(j_*L)_y$ is given by the local monodromy invariants of L around a small loop about y. The complex $Rj_*L[1]$ may fail to be Verdier self-dual, whereas its truncation $\tau_{\leq -1}Rj_*L[1] = j_*L[1] = IC_Y(L)$ is Verdier self-dual. Note that we have a factorization $Rj_!L[1] \to IC_Y(L) \to Rj_*L[1]$. This is not an "accident": see the end of §2.6.

Exercise 1.7.20 (**Blow-ups**) Compute the direct image sheaves $R^q f_*\mathbb{Q}$ for the blowing-up of $\mathbb{C}^m \subseteq \mathbb{C}^n$(start with $m = 0$; observe that there is a product decomposition of the situation that allows you to reduce to the case $m = 0$). Same question for the composition of the blow up of $\mathbb{C}^1 \subseteq \mathbb{C}^3$, followed by the blowing up of a positive dimensional fiber of the first blow up. Observe that in all cases, one gets an the decomposition $Rf_*\mathbb{Q} \cong \oplus R^q f_*\mathbb{Q}[-q]$. Guess at the shape of the decomposition theorem in both cases.

Exercise 1.7.21 (**Examples of the decomposition theorem**) Guess at the exact form of the cohomological and "derived" decomposition theorem in the following cases: 1) the normalization of a cubic curve with a node and of a cubic curve with a cusp; 2) the blowing up of a smooth subvariety of an algebraic manifold; 3) compositions of various iterations of blowing ups of nonsingular varieties along smooth centers; 4) a projection $F \times Y \to Y$; 5) the blowing up of the vertex of the affine cone over the nonsingular quadric in $\mathbb{P}^3$; 6) same but for the projective cone; 7) blow up the same affine and projective cones but along a plane through the vertex of the cone; 8) The blowing up of the vertex of the affine/projective cone over an embedded projective manifold.

Exercise 1.7.22 (**DT for Lefschetz pencils**) Guess at the shape of the dt for a Lefschetz pencil $f : \widetilde{X} \to \mathbb{P}^1$ on a nonsingular projective surface X. Word out explicitly the invariant cycle theorems in this case. Do the same for a nonsingular projective manifold. Do we get skyscraper contributions?

Exercise 1.7.23 (**Künneth for the derived image complex**) One needs a little bit of working experience with the derived category to carry out what below. But try anyway. Let $f : X := Y \times F \to Y$. A cohomology class $a_q \in H^q(X, \mathbb{Q})$ is the same thing as a map in the derived category $a_q : \mathbb{Q}_X \to \mathbb{Q}_X[q]$. By pushing forward via Rf_*, by observing that $Rf_* f^* \mathbb{Q}_Y = Rf_* \mathbb{Q}_X$, by pre-composing with the adjunction map $\mathbb{Q}_Y \to Rf_* \mathbb{Q}_X$, we get a map $a_q : \mathbb{Q}_Y \to Rf_* \mathbb{Q}_X[q]$. Take a_q to be of the form $pr_F^* a_q$. Obtain a map $\alpha_q \underline{H}^q(F) \to Rf_* \mathbb{Q}_X[q]$. Shift to get $\alpha_q : \underline{H}^q(F)[-q] \to Rf_* \mathbb{Q}_X$. Show that the map induces the "identity" on the q-th direct image sheaf and zero on the other direct image sheaves. Deduce that $\sum_q \alpha_q : \oplus_q \underline{H}^q(F)[-q] \to Rf_* \mathbb{Q}_X$ is an isomorphism in the derived category inducing the "identity" on the cohomology sheaves. Observe that you did not make any choice in what above.

Exercise 1.7.24 (**Deligne theorem as a special case of the decomposition theorem**) Keeping in mind that if if $S^o = S$, then $\mathcal{IC}_S(L) = L$, recover the Deligne theorem from the dt.

2 Lecture 2: The category of perverse sheaves $P(Y)$

Summary of Lecture 2. The constructible derived category. Definition of perverse sheaves. Artin vanishing and its relation to a proof of the Lefschetz hyperplane theorem for perverse sheaves. The perverse t-structure (really, only the perverse cohomology functors!). Beilinson's and Nori's equivalence theorems. Several equivalent definitions of intersection complexes.

2.1 Three Why? And a brief history of perverse sheaves.

Why intersection cohomology? Let us look at (4) for X and Y nonsingular:

$$H^*(X,\mathbb{Q}) \cong \oplus_{q,\mathcal{EV}_q} IH^{*-q}(S,L),$$

i.e. the l.h.s. is ordinary cohomology, but the r.h.s. is not any kind of ordinary cohomology on Y, we need intersection cohomology to state the decomposition theorem, even when X and Y are nonsingular. The intersection cohomology groups of a projective variety enjoy a battery of wonderful properties (Poincaré-Hodge-Lefschetz package). In some sense, intersection cohomology nicely replaces singular cohomology on singular varieties, but with a funny twist: singular cohomology is functorial, but has no Poincaré duality; intersection cohomology has Poincaré duality, but is not functorial!

Why the constructible derived category? The cohomological Deligne theorem (2) for smooth projective maps is a purely cohomological statement and it can be proved via purely cohomological methods (hard Lefschetz + Leray spectral sequence). The cohomological decomposition theorem (4) is also a cohomological statement. However, there is no known proof of this statement that does not make use of the formalism of the middle perversity t-structure present in the constructible derived category: one proves the derived version (10) and then deduces the cohomological one (4) by taking cohomology. Actually, the definition of perverse sheaves does not make sense if we take the whole derived category, we need to take complexes with cohomology sheaves supported at closed subvarieties (not just classically closed subsets). We thus restrict to an agreeable, yet flexible, class of complexes: the "constructible complexes".

Why perverse sheaves? Intersection complexes, i.e. the objects appearing on both sides of the decomposition theorem (10) are very special perverse sheaves. In fact, in a precise way, they form the building blocks of the category of perverse sheaves: every perverse sheaf is an iterated extension of a collection of intersection complexes. Perverse sheaves satisfy their own set of beautiful properties: Artin vanishing theorem, Lefschetz hyperplane theorem, stability via duality, stability via vanishing and nearby cycle functors. As mentioned above, the known proofs of the decomposition theorem use the machinery of perverse sheaves.

A brief history of perverse sheaves.

Intersection complexes were invented by Goresky-MacPherson as a tool to systematize, strengthen and widen the scope of their own intersection cohomology theory. For example, their original geometric proof of Poincaré duality can be replaced by the self-duality property of the intersection complex.

The conditions leading to the definition of perverse sheaves appeared first in connection with the Riemann-Hilbert correspondence established by Kashiwara and by Mekbouth: their result is an equivalence of categories between the constructible derived category (which we have been procrastinating to define) and the derived category of regular holonomic D-modules (which we shall not define); the standard t-structure, given by the standard truncations met in Exercise 1.7.17, of these two categories do not correspond to each other under the Riemann-Hilbert equivalence; the conditions leading to the "conditions of support" defining of perverse sheaves are the (non trivial) translation in the constructible derived category of the conditions on the D-module side stating that a complex of D-modules has trivial cohomology D-modules in positive degree. It is an un related, yet remarkable and beautiful fact that the conditions of support so-obtained are precisely what makes the Artin vanishing Theorem 2.4.1 work on an affine variety.

As mentioned above, Gelfand-MacPherson conjectured the decomposition theorem for $Rf_*\mathcal{IC}_X$. Meanwhile Deligne had developed a theory of pure complexes for varieties defined over finite fields and established the invariance of purity under push-forward by proper maps. Gabber proved that the intersection complex of a pure local system in that context, is pure. The four authors of [1] introduced and developed systematically the basis for the theory of t-structures, especially with respect to the middle perversity. They then proved that the notions of purity and perverse t-structure are compatible: a pure complex splits over the algebraic closure of the finite field as prescribed by the r.h.s. of (10). The decomposition theorem over the algebraic closure of a finite field follows when considering the purity result for the proper direct image mentioned above. The whole Ch. 6 in [1], aptly named "De $\mathbb{F}$ à $\mathbb{C}$", is devoted to explaining how these kind of results over the algebraic closure of a finite field yield results over the field of complex numbers. This established the original proof of the decomposition theorem over the complex numbers for semisimple complexes of geometric origin (see [1], 6.2.4, 6.2.5), such as $\mathcal{IC}_X$.

M. Saito has developed in [47] the theory of mixed Hodge modules which yields the desired decomposition theorem when M underlies a variation of polarizable pure Hodge structures.

M.A. de Cataldo and L. Migliorini have given a proof based on classical Hodge theory of the decomposition theorem when M is constant [13].

Finally, the decomposition theorem stated in (10) is the most general statement currently available over the complex numbers and is due to work of C. Sabbah [46] and T. Mochizuki [42] (where this is done in the essential case of projective maps of quasi projective manifolds; it is possible to extend it to proper maps of algebraic varieties). The methods (tame harmonic bundles, D-modules) are quite different from the ones discussed in these lectures.

2.2 The constructible derived category $D(Y)$

The decomposition theorem isomorphisms (10) take place in the "constructible derived category" $D(Y)$. It is probably a good time to try and give an idea what this category is.

Constructible sheaf. A sheaf F on Y is *constructible* if there is a finite disjoint union decomposition $Y = \coprod_a S_a$ into locally closed subvarieties such that the $F_{|S_a}$ are locally constant of finite rank. This is a good time to look at Exercise 2.7.1.

Constructible complex. A complex of sheaves C is said to be *constructible* if it is bounded (all but finitely many of its cohomology sheaves are zero) and its cohomology sheaves are constructible sheaves.

Constructible derived category. The definition of $D(Y)$ is kind of a mouthful: *it is the full subcategory of the derived category $D(Sh(Y, \mathbb{Q}))$ of the category of sheaves of rational vector spaces whose objects are the constructible complexes.*

It usually takes time to absorb these notions and to absorb the apparatus it gives rise to. We take a different approach and we try to isolate some of the aspects of the theory that are more relevant to the decomposition theorem. We do not dwell on technical details.

Cohomology. Of course, the first functors to consider are cohomology and cohomology with compact supports $H^i(Y, -), H^i_c(Y, -) : D(Y) \to D(point)$.

Derived direct images $Rf_*, Rf_!$. The derived direct image $Rf_*, Rf_! : D(X) \to D(Y)$, for every map $f : X \to Y$. The first thing to know is that $H^*(X, C) = H^*(Y, Rf_*C)$ and that $H^*_c(X, C) = H^*_c(Y, Rf_*C)$, so that we may view them as generalizing cohomology.

Pull-backs. The pull-back functor f^* is probably the most intuitive one. The extraordinary pull-back functor $f^!$ is tricky and we will not dwell on it. It is the right adjoint to $Rf_!$; for open immersions, $f^! = f^*$; for closed immersions it is the derived versions of the sheaf of sections supported on the closed object; for smooth maps of relative dimension d, $f^! = f^*[2d]$. A down-to-earth reference for $f^!$ and duality I like is [37] (good also, among other things, as an introduction to Borel-Moore homology). I also like [31]. There is also the seemingly inescapable, and nearly encyclopedic [36].

Verdier duality. This is an anti-self equivalence $(-) : D(Y)^{op} \cong D(Y)$. In short, its defining property is the presence of a natural perfect pairing $H^*(Y, C^\vee) \times H^{-*}_c(Y, C) \longrightarrow \mathbb{Q}$, or, equivalently, of a canonical isomorphism

$$H^*(Y, C^\vee) \cong H^{-*}_c(Y, C)^\vee. \tag{12}$$

If Y is nonsingular irreducible, then $\mathbb{Q}^\vee_Y = \mathbb{Q}_Y[2 \dim Y]$, and we get $H^{*+2\dim Y}(Y, \mathbb{Q}) = H^{-*}_c(Y, \mathbb{Q})^\vee$, i.e. Poincaré duality.

Stability of constructibility. It is by no means obvious, nor easy, that if C is constructible, then Rf_*C is constructible. This can be deduced from the Thom isotopy lemmas[5] [34]. This is a manifestation of the important principle that constructibility is

[5]the main two points are: 1) given a map $f : X \to Y$ of complex algebraic varieties, there is a disjoint union decomposition $Y = \coprod_i Y_i$ into locally closed subvarieties such that $X_i := f^{-1}Y_i \to Y_i$ is a topological fiber bundle for the classical topology; 2) algebraic maps can be completed compatibly with the previous assertion (you may want to sit down and come up with a reasonable precise statement yourself)

preserved under all the "usual" operations on the derived category of sheaves on Y (see [4]). The list above is more complete once we include the derived $\mathcal{RH}om$, the tensor product of complexes (it is automatically derived when using $\mathbb{Q}$-coefficients), the nearby and vanishing cycle functors.

Duality exchanges. Verdier duality exchanges Rf_* with $Rf_!$, and f^* with $f^!$, i.e., $Rf_*(C^\vee) = (Rf_! C)^\vee$ and $f^*(K^\vee) = (f^! K)^\vee$. Here is a nice consequence: let f be proper (so that $Rf_* = Rf_!$), then if C is self-dual, then so is $Rf_* C$.

The importance of being proper. Proper maps are important for many reasons: $Rf_! = Rf_*$; the duality exchanges simplify; the proper base change theorem holds, a special case of which tells us that $(R^q f_* \mathbb{Q}_X)_y = H^q(f^{-1}(y), \mathbb{Q})$ (see Exercise 1.7.1); Rf_* preserves pure complexes (Frobenius, mixed Hodge modules); Grothendieck trace formula is about $Rf_!$; the decomposition theorem is about proper maps.

Adjunctions. We have adjoint pairs (f^*, Rf_*) and $(Rf_!, f^!)$, hence natural transformations: $Id \to Rf_* f^*$ $Rf_! f^! \to Id$. By applying cohomology to the first one we get the pull-back map in cohomology, and by applying cohomology with compact supports to the second we obtain the push-forward in cohomology with compact supports. $\mathcal{RH}om$ and $\otimes$ also form an adjoint pair (you should formalize this).

The attaching triangles and the long exact sequences we already know. By combining adjunction maps, we get some familiar situations from algebraic topology. Let $j : U \to Y \leftarrow Z : i$ be a complementary pair of open/closed embeddings. Given $C \in D(Y)$, we have the distinguished triangle $i_* i^! C \to C \to j_* j^* C \to i_! i^! C[1]$ and, by applying sheaf cohomology, we get the long exact sequence of relative cohomology $\ldots H^q(Y, U; C) \to H^q(Y, C) \to H^q(U, C_{|U}) \to H^{q+1}(Y, U) \to \ldots$ If $g : Y \to Z$ is a map, we can push forward the attaching (distinguished) triangle and obtain a distinguished triangle, which will also give rise to hosts of long exact sequences when fed to cohomological functors. By dualizing, we get $j_! j^* C \to C \to i_* i^* C \to j_! j^* C[1]$ and by taking cohomology wit compact supports, we get the long exact sequence of cohomology with compact supports $\ldots H^q_c(U, C) \to H^q_c(Y, C) \to H^q(Z, C_{|Z}) \to H^{q+1}(U, C) \to \ldots$ This is nice, and used very often, because it gives the usual nice relation between the compactly supported Betti numbers of the three varieties (U, Y, Z).

Exercise 2.7.10 asks you to use the first attaching triangle and its push-forward when studying the resolution of singularities of a germ of an isolated surface singularity. This is an important example: it shows how the intersection complex of the singular surface arises; it relates the non degeneracy of the intersection form on the curves contracted by the resolution, to the decomposition theorem for the resolution map. The careful study of this example allows to extract many general ideas an patterns, specifically how to relate the non degeneracy of certain local intersection forms to a proof decomposition theorem.

2.3 Definition of perverse sheaves

Before we define perverse sheaves. The category of perverse sheaves on an algebraic variety is abelian, noetherian, anti-self-equivalent under Verdier duality, artinian. The cohomology groups of perverse sheaves satisfy Poincaré duality, Artin vanishing theorem and the Lefschetz hyperplane theorem. They are stable under the nearby and vanishing cycle functors. The simple perverse sheaves, i.e. the intersection complexes with simple coefficients, satisfy the decomposition theorem and the relative hard Lefschetz theorem, and their cohomology groups satisfy the Hard Lefschetz theorem and, when the coefficients are "Hodge-theoretic", the Hodge-Riemann bilinear relations. Perverse shaves play important roles in the topology of algebraic varieties, arithmetic algebraic geometry, singularity theory, combinatorics, representation theory, geometric Langlands program.

Even though not everyone agrees with this, one may say that perverse sheaves are more natural than constructible sheaves. See Remark 2.4.3. Perverse sheaves on singular varieties are close in spirit to locally constant sheaves on algebraic manifolds.

Perverse sheaves have at least one drawback: they are not sheaves![6]

The conditions of support and of co-support. We say that a constructible complex $C \in D(Y)$ satisfies the conditions of support if $\dim \, supp \, \mathcal{H}^i(C) \leq -i$, for every $i \in \mathbb{Z}$, and that it satisfies the conditions of co-support if its Verdier dual $C^\vee$ satisfies the conditions of support.

Definition 2.3.1 (**Definition of the category $P(Y)$ of perverse sheaves**) We say that $P \in D(Y)$ is a perverse sheaf if P satisfies the conditions of support and of co-support. The category $P(Y)$ of perverse sheaves on Y is the full subcategory of $D(Y)$ with objects the perverse sheaves.

Conditions of (co)support and vectors of dimensions. To fix ideas, let $\dim Y = 4$. The following vector exemplifies the upper bounds for the dimensions of the supports of the cohomology sheaves $\mathcal{H}^i$ for $-4 \leq i \leq 0$ (outside of this interval, the cohomology sheaves of a perverse sheaf can be shown to be zero) $(4, 3, 2, 1, 0)$. For comparison, the analogous vector for an intersection complex of the form $IC_Y(L)$ is $(4, 2, 1, 0, 0)$. For a table giving a good visual for the conditions of support and co-support for perverse sheaves and for intersection complexes, see [16], p.556.

It is important to keep in mind that the support conditions are conditions on the stalks (direct limits of cohomology over neighborhoods) of the cohomology sheaves, whereas the conditions of co-support are conditions on the co-stalk (inverse limits of cohomology with compact supports over neighborhoods) and as such are maybe a bit less intuitive: see Exercise 1.7.16, where this issue is tackled for the constant sheaf on the affine cone over an embedded projective manifold. On the other hand, if for some reason we know that complex $C \in D(Y)$ is Verdier self dual, then it is perverse iff it satisfies the conditions of support. Note that the derived direct image via a proper map of a self-dual complex is self-dual. This simple remark is very helpful in practice.

This is a good time to carry out Exercises 2.7.2 and 2.7.3.

[6] the "sheaves" in perverse sheaves is because the objects and the arrows in $P(Y)$ can be glued from local data, exactly like sheaves; this is false for the derived category and for $D(Y)$!; do you know why? As to the term "perverse", see M. Goresky's post on Math Overflow: What is the etymology of the term perverse sheaf?

2.4 Artin vanishing and Lefschetz hyperplane theorems

Conditions of support and Artin vanishing theorem. The conditions of support seem to have first appeared in the proof of the Artin vanishing theorem for constructible sheaves in SGA 4.3.XIV, Théorème 3.1, p.159: *let Y be affine and $F \in Sh_c(Y)$; then $H^*(Y,F) = 0$ for $* > \dim Y$.* Note that $F[\dim Y]$ satisfies the conditions of support, and that Artin's wonderful proof works for a constructible complex satisfying the conditions of support. Note also that if Y is nonsingular and $F = \mathbb{Q}_Y$, the result is affordable by means of Morse theory[7]!

Since perverse sheaves satisfy the conditions of support and co-support, we see that they automatically satisfy the "improved"[8] version of the Artin vanishing

Theorem 2.4.1 (**Artin vanishing theorem for perverse sheaves**)[9] *Let P be a perverse sheaf on the affine variety Y. Then $H^*(Y,P) = 0$, for $i \neq [-\dim Y, 0]$ and $H^*_c(Y,P) = 0$, for $i \neq [0, \dim Y]$.*

"Proof." See Exercise 2.7.4.

Exactly as in the Morse-theory proof of the weak Lefschetz theorem, we have that the improved Artin vanishing theorem implies

Theorem 2.4.2 (**Lefschetz hyperplane theorem for perverse sheaves**). *Let Y be a quasi projective variety, let $P \in P(Y)$ and let $Y_1 \subseteq Y$ a general hyperplane section. Then $H^*(Y,P) \to H^*(Y_1, P_{|Y_1})$ is an isomorphism for $* \leq -2$, and injective for $* = -1$. There is a similar statement for compactly supported cohomology (guess it!).*

Proof. We give the proof in the case when Y is projective. Let $j : U := Y \setminus Y_1 \to Y \leftarrow Y_1 : i$ be the natural maps. We have the attaching triangle $Rj_!j^*P \to P \to Ri_*i^*P \to$ and the long exact sequence of cohomology (= cohomology with compact supports because Y is projective!)

$$\ldots \to H_c^{-k}(Y, Rj_!j^*P) \to H^{-k}(Y,P) \to H^{-k}(Y_1, P_{|Y_1}) \to H^{-k+1}(Y, Rj_!j^*P) \to \ldots$$

We need to show that $H_c^{-k}(Y, Rj_!j^*P) = H_c^{-k}(U, j^*P) = 0$ for $k < 0$, but this is Artin vanishing for perverse sheaves on the affine U. □

Remark 2.4.3 (**LHT: Perverse sheaves vs. sheaves**) As the proof given above shows, once we assume the projectivity of Y, any hyperplane section Y_1 will do. This is similar to the classical proof of the Lefschetz hyperplane theorem due to Andreotti-Frankel (following a suggestion by Thom) and contained in Milnor's Morse Theory (jewel) book, where if Y

[7] the singular case is one of the reasons for the existence of the book [34]

[8] note that there is also a statement in compactly supported cohomology that does not appear in the statement of the Artin vanishing theorem and it is false in general for sheaves

[9] note that there is no sheaf analogue of Artin vanishing for compactly supported cohomology: the Verdier dual of a constructible sheaf is not a sheaf!

is projective, we only need to pick a hyperplane section that contains all the singularities of Y, so that the desired vanishing stems from Lefschetz duality and from Morse theory. This shows that even the constant sheaf is not well-behaved on singular spaces! If we try and repeat the proof above for the constant sheaf on a singular space, we stumble into the realization that we do not have the necessary Artin vanishing for cohomology with compact supports for the constant sheaf on the possibly singular U. This issue disappears if we use perverse sheaves!

The simple perverse sheaves are the intersection complexes. Since the category $P(Y)$ is artinian, every perverse sheaf $P \in P(Y)$ admits an increasing finite filtration with quotients simple perverse sheaves. The important fact is that the simple perverse sheaves are the intersection complexes $\mathcal{IC}_S(L)$ seen above with L simple! (Exercise 2.7.9). There is a shift involved: $\mathcal{IC}_S(L)$ is not perverse on the nose; we need to shift by $\dim S$, i.e. $IC_S(L) := \mathcal{IC}_S(L)[\dim S]$ and the result is a perverse sheaf on Y. One views these complexes on the closed subvarieties $i : S \to Y$ as complexes on Y supported on S via i_* (we do not do this to simplify the notation).

There is a battery of results in intersection cohomology that generalize to the singular setting the beloved classical results that hold for complex algebraic manifolds. See Exercise 2.7.6.

2.5 The perverse t-structure

The constructible derived category $D(Y)$ comes equipped with the standard t-structure, i.e. the truncation functors are the standard ones, and whose heart is the abelian category $Sh_c(Y) \subseteq D(Y)$ of constructible sheaves. A t-structure on a triangulated category is an abstraction of the notion of standard truncation [1]. A triangulated category may carry several inequivalent t-structures.

The middle perversity t-structure on $D(Y)$. The category of perverse sheaves $P(Y)$ is also the heart of a t-structure on $D(Y)$, the middle-perversity t-structure. Instead of dwelling on the axioms, here is a short discussion.

The perverse sheaf cohomology functors. Every t-structure on a triangulated category comes with its own cohomology functors; the standard one comes with the cohomology sheaves functors. The perverse t-structure then comes with the perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^i : D(Y) \to P(Y)$ which are of course ... cohomological, i.e. turn distinguished triangles into long exact sequences:

$$A \to B \to C \to A[1] \Longrightarrow \ldots \to {}^{\mathfrak{p}}\mathcal{H}^i(A) \to {}^{\mathfrak{p}}\mathcal{H}^i(B) \to {}^{\mathfrak{p}}\mathcal{H}^i(C) \to {}^{\mathfrak{p}}\mathcal{H}^{i+1}(A) \to \ldots, \quad (13)$$

and, moreover, we have:

$${}^{\mathfrak{p}}\mathcal{H}^i(C[j]) = {}^{\mathfrak{p}}\mathcal{H}^{i+j}(C). \quad (14)$$

Let us mention that $C \in D(Y)$ satisfies the conditions of support iff its perverse cohomology sheaves are zero in positive degrees; similarly, for the conditions of co-support (swap positive with negative).

Kernels, cokernels. Once you have the cohomology functors, you can verify that $P(Y)$ is abelian: take an arrow $a : P \to Q$ in $P(Y)$, form its cone $C \in D(Y)$, and then you need to verify that ${}^{\mathfrak{p}}\mathcal{H}^{-1}(C) \to P$ is the kernel and that $Q \to {}^{\mathfrak{p}}\mathcal{H}^{0}(C)$ is the cokernel. What is the image?

Verdier duality exchange: ($C \cong C^{\vee}$)

$$ {}^{\mathfrak{p}}\mathcal{H}^{i}(C^{\vee}) = {}^{\mathfrak{p}}\mathcal{H}^{-i}(C)^{\vee} \tag{15} $$

If $C \cong C^{\vee}$, then $({}^{\mathfrak{p}}\mathcal{H}^{i}(C))^{\vee} = {}^{\mathfrak{p}}\mathcal{H}^{-i}(C)$ and if, in addition, f is proper, then

$$ (Rf_*C)^{\vee} \cong Rf_*C. \tag{16} $$

The perverse cohomology sheaves of a complex do not determine the complex. However, Exercise 4.7.6 tells us that in the decomposition theorem (10) we may write

$$ Rf_*\mathcal{IC}_X(M) \cong \bigoplus_{c \in \mathbb{Z}} {}^{\mathfrak{p}}\mathcal{H}^{c}(Rf_*\mathcal{IC}_X(M))[-c]. $$

The perverse cohomology sheaf construction is a way to get perverse sheaves out of any complex. So there are plenty of perverse sheaves. In fact, we have the following two rather deep and very (!) surprising facts:

Theorem 2.5.1 (**Derived category in different ways**) *Let Y be a variety.*

1. *$D(Y, \mathbb{Z})$ with its standard t-structure is equivalent to $D^b(Sh_c(Y, \mathbb{Z}))$ with its standard t-structure* (Nori [45][10]).

2. *$D(Y, \mathbb{Q}))$ with its perverse t-structure is equivalent to $D^b(P(Y, \mathbb{Q}))$ with its standard t-structure* ([2])[11]).

Exercise 2.7.8 introduces another construction leading to special perverse sheaves, i.e. the intermediate extension functor $j_{!*}$. This is crucial, in view of the fact that intersection cohomology complex $IC_S(L)$ can be defined as the intermediate extension of $L[\dim S]$ from the open subvariety $S^o \subseteq S_{reg} \subseteq S$ on which L is defined to the whole of S, and thus to any variety that contains S as a closed subvariety.

2.6 Intersection complexes

Recall the conditions of support and co-support for a complex P to be a perverse sheaves: $\dim supp\mathcal{H}^i(C) \leq -i$ and the same for $C^{\vee}$.

The original definition of intersection complex $IC_S(L)$ of an enriched variety (rem: L is a locally constant sheaf on some $S^o \subseteq S_{reg} \subseteq S$) involves repeatedly pushing-forward

[10] this paper contains a lovely proof of the Artin vanishing in characteristic zero

[11] Beilinson also proves his wonderful Lemma 3.3, a strengthening of the Artin vanishing in arbitrary characteristic; Nori calls it the "Basic Lemma" in [45]

and standard-truncating across the strata of a suitable stratification of S, starting from S^o; see [1], Proposition 2.1.11. It is a fact that shrinking S^o does not effect the end result (this is an excellent exercise). The end result can be characterized as follows.

Conditions of (co)support for intersection complexes. The intersection complex $IC_S(L)$ of an enriched variety (S,L) is the complex C, unique up to unique isomorphism subject to the following conditions of support and co-support: $C_{|S^o} = L[\dim S]$ $\dim \mathit{supp}\, \mathcal{H}^i \leq -i-1$ for every $i \neq -\dim S$, and the "same" for $C^\vee$. Recall the two vectors exemplifying the conditions of support in dimension four for perverse sheaves $(4,3,2,1,0)$, and for intersection complexes $(4,2,1,0,0)$.

Another characterization of intersection complexes. The intersection complex $IC_S(L)$ is the unique perverse sheaf extending its own restriction to an open dense subvariety $U \subseteq S$ so that the extension is "minimal" in the following sense: it has no non zero perverse subobject or quotient supported on the boundary $S \setminus U$[12]

Intersection complexes as intermediate extensions. Let $j : U \to S$ be an locally closed embedding. Let $P \in P(U)$. Take the natural map (forget the supports) $Rj_! P \to Rj_*$. Take the map induced at the level of 0-th perverse cohomology sheaves: $a : \mathcal{H}^0(Rj_! P) \to \mathcal{H}^0(Rj_* P)$. Define the intermediate extension of P on U to Y by setting $j_{!*}P := \operatorname{Im} a \in P(S)$. Let (S,L) be an enriched variety and let $j : S^o \to S$; we apply the intermediate extension functor to $P := L[\dim S] \in P(U := S^o)$ and we end up with $IC_S(L)$! The same conclusion holds if we take any U and $P := (IC_S(L))_{|U})$ (the intersection complex is the intermediate extension of its restriction to any dense open subvariety). The intermediate extension functor is not an exact functor (it does not preserve short exact sequences.).

Since $P(Y)$ is Noetherian and closed under Verdier duality, it is artinian, so that the Jordan-Holder theorem holds. Now it is a good time to carry out Exercise 2.7.9, which shows how to produce "explicit" Jordan-Holder decomposition for perverse sheaves. The method also makes it clear that the simple objects in $P(Y)$ are the $IC_S(L)$ with $S \subseteq Y$ a closed subvariety and L simple.

[12] it may have, however, a non zero subquotient supported on the boundary; the formation of this kind of minimal –a.k.a. intermediate extension– is not exact on the relevant abelian categories: it preserves kernels and cokernels, but it does not preserve exact sequences; see [16], p.562.

2.7 Exercises for Lecture 2

Exercise 2.7.1 (**Some very non constructible sheaves**) Use the closed embedding $i : \mathfrak{C} \to \mathbb{A}^1$ of the Cantor set into the complex affine line to show that the direct image sheaf $i_* \mathbb{Q}_{\mathfrak{C}}$ is not constructible. Classify the sheaves of rational vector spaces on $\mathbb{A}^1$ which are both constructible and injective.

Example 2.7.2 (**First (non) examples**) If Y is of pure dimension and $F \in Sh_c(Y)$, then $F[\dim Y]$ satisfies the conditions of support. In general, $F[\dim Y]$ is not perverse as its Verdier dual may fail to satisfy the condition of support. For example, the Verdier dual of $\mathbb{Q}_Y[\dim Y]$ is the shifted dualizing complex $\omega_Y[-\dim Y]$ and the singularities of Y dictate whether or not it satisfies the conditions of support; see [17], §4.3.5-7. If Y is nonsingular of pure dimension, and L is locally constant, then $L[\dim Y]$ is perverse, for its Verdier dual is $L^\vee[\dim Y]$.

Exercise 2.7.3 (**Some perverse sheaves**) The derived direct image of a perverse sheaf via a finite map is perverse. Give examples showing that the derived direct image via a quasi-finite map of a perverse sheaf may fail to be perverse. Let $j : X := \mathbb{C}^* \to \mathbb{C} =: Y$ be the natural open embedding; show that the natural map in $D(Y)$ $Rj_! \mathbb{Q}_X[1] \to Rj_* \mathbb{Q}_X[1]$ is in fact in $P(Y)$; determine kernel, cokernel and image. Show that if we replace $\mathbb{C}^*$ with $\mathbb{C}^n \setminus$, then the map above is not one of perverse sheaves. If instead or removing the origin, we remove a finite configuration of hypersurfaces, then we get a map of perverse sheaves; more generally, an affine open immersion is such that $Rf_!$ and Rf_* preserve perverse sheaves (for these, push-forward and use freely the Stein -instead of affine- version of the Artin vanishing theorem to verify the conditions of support on small ball centered points on the hypersurfaces). The direct image $Rf_* \mathbb{Q}_X[2]$ with $f : X \to Y$ a resolution of singularities of a surface is perverse. Let $f : X^{2d} \to Y^{2d}$ is proper and birational, with X nonsingular, irreducible $y \in Y$ and f is an isomorphism over $Y \setminus y$; give an iff condition that ensures that $Rf_* \mathbb{Q}[\dim X]$ is perverse. Determine the pairs $m \leq n$ such that the blowing up of $f : X \to Y$ of $\mathbb{C}^m \subseteq \mathbb{C}^n =: Y$ is such that $Rf_* \mathbb{Q}_X[n]$ is perverse.

Exercise 2.7.4 (**Artin vanishing: from constructible to perverse sheaves; cohomological dimension**) Assume the Artin vanishing theorem for constructible sheaf and deduce the one for the cohomology of perverse sheaves by use of the Grothendieck spectral sequence $H^p(Y, \mathcal{H}^q(P)) \Longrightarrow H^{p+q}(Y, P)$. (Hint: the supports of the cohomology sheaves are closed <u>affine</u> subvarieties.) Dualize the result to obtain the Artin vanishing theorem for the cohomology with compact support of a perverse sheaf. Use a suitable affine covering of a quasi-projective variety Y to show that the cohomology and cohomology with compact supports of a perverse sheaf live in the interval $[-\dim Y, +\dim Y]$. What about a non quasi projective Y?

Exercise 2.7.5 (**Intersection complex via push-forward and truncation**) The original Goresky-MacPherson's definition of intersection complex involves repeated push-forward and truncation across the strata of a Whitney-stratification. Let us take $j : \mathbb{C}^n \setminus \mathbb{C}^{m=0} =: U \to Y := \mathbb{C}^n$. The formula reads $IC_Y := \tau_{\leq -1} Rj_* \mathbb{Q}_U[n]$. Verify that the result is $\mathbb{Q}_Y[n]$. Do this for other values of m and verify that you get $\mathbb{Q}_Y[n]$, again. Take a complete flag of linear subspaces in $\mathbb{C}^n$, apply the general formula and verify that you get $\mathbb{Q}_Y[n]$. What is your conclusion?

Exercise 2.7.6 (**Hodge-Lefschetz package for intersection cohomology**) Guess the precise statements of the following results concerning intersection cohomology groups: Poincaré duality, Artin vanishing theorem, Lefschetz hyperplane theorem, existence of pure and mixed Hodge structures, hard Lefschetz theorem, primitive Lefschetz decomposition, Hodge-Riemann bilinear relations.

Exercise 2.7.7 (**Injective or surjective?**) Let $j : \mathbb{A}^1 \setminus \{0\} \to \mathbb{A}^1$ be the natural open embedding. Verify that: the natural map $j_! \mathbb{Q} \to j_* \mathbb{Q}$ in $Sh_c(\mathbb{A}^1)$ is injective; the natural map $j_![1]\mathbb{Q} \to j_*\mathbb{Q}[1]$ is ...surjective in $P(\mathbb{A}^1)$.

Exercise 2.7.8 (**Perverse cohomology sheaves and the intermediate extension functor**) Let $j : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n$ be the natural open embedding. Compute ${}^{\mathfrak{p}}\mathcal{H}^i(Rj_!\mathbb{Q})$ and ${}^{\mathfrak{p}}\mathcal{H}^i(Rj_*\mathbb{Q})$. Let $j : U \to X$ be an open embedding and let $P \in P(U)$. Let $a : Rj_!P \to Rj_*P$ be the natural map. Show that the assignment $P \mapsto j_{!*}P := \mathrm{Im}\{{}^{\mathfrak{p}}\mathcal{H}^0(Rj_!P) \to {}^{\mathfrak{p}}\mathcal{H}^0(Rj_*P)\}$ is functorial. This is the intermediate extension functor Is it exact (i.e. does it send short exact sequences in $P(U)$ into ones in $P(X)$? Compute $j_{!*}\mathbb{Q}_U$ when j is the embedding of a Zariski open subset of nonsingular variety. Same for the embedding of affine cones over projective manifolds minus their vertex into the cone. Compute $j_{!*}L[1]$ where $U = \mathbb{C}^*$ and L is a locally constant sheaf on U.

Exercise 2.7.9 (**Jordan-Holder for perverse sheaves**) Do not assume varieties are irreducible, nor pure-dimensional. Let $P \in P(Y)$. Find a non empty open nonsingular irreducible subvariety $j : U \subseteq Y$ such that $Q := j^*P = L[\dim U]$ for a locally constant sheaf on U. Produce a natural commutative diagram with a epimorphic (cokernel is zero)

$$\begin{tikzcd}
 & P \arrow[dr] & \\
{}^{\mathfrak{p}}\mathcal{H}^0(Rj_!Q) \arrow[ur] \arrow[rr] \arrow[dr, "a"'] & & {}^{\mathfrak{p}}\mathcal{H}^0(Rj_*Q) \\
 & IC_{\overline{U}}(L) \arrow[ur] &
\end{tikzcd}$$

Deduce that we have a filtration $\mathrm{Ker} a \subseteq {}^{\mathfrak{p}}\mathcal{H}^0(Rj_!Q) \subseteq P$ with ${}^{\mathfrak{p}}\mathcal{H}^0(Rj_!Q)/\mathrm{Ker} a = IC_{\overline{U}}(L)$. Use noetherian induction to prove that we can refine each side of this two step filtration to a filtration with successive quotients of the form $IC_S(L)$. Each local system admits a finite filtration with simple quotients. Refine further to obtain a finite increasing filtration of P with successive quotients of the form $IC_S(L)$ with L simple. (In the last step you need to know that the intermediate extension functor, while not exact, preserves kernels (and cokernels).)

Exercise 2.7.10 (**Attaching the vertex to a cone**) Use the attaching triangle and resulting long exact sequence of cohomology to study $C = Rf_*\mathbb{Q}_X$, where $f : X \to Y$ is the resolution of the cone (affine and projective) over a nonsingular embedded projective curve obtained by blowing up the vertex.

3 Lecture 3: Semismall maps

Summary of Lecture 3. The material in this section cannot be possibly covered in one lecture! In fact, there are more topics that one could/should add ... I will try to discuss some of the main ideas and leave the rest for a private reading. Definition of semismall map. Hard Lefschetz and Hodge-Riemann bilinear relations for semismall maps. Special form of the decomposition theorem for semismall maps. Hilbert schemes of points on smooth surfaces and the Grojnowski-Nakajima picture. The endomorphism and correspondence algebras are isomorphic and semisimple. Hint of a relation to the Springer picture.

3.1 Semismall maps

Semismall maps are a very special class of maps, e.g. they are necessarily generically finite. On the other hand, the blowing up of point in $\mathbb{C}^3$ is not semismall.Resolutions of singularities are very rarely semismall (except in dimension ≤ 2). It is remarkable that semismall maps appear in important situations, e.g. holomorphic symplectic contractions, quiver varieties, moduli of bundles on surfaces, Springer resolutions, convolution on affine grassmannians, standard resolutions of theta divisors, Hilbert-Chow maps for Hilbert schemes of points on surfaces ... References include [11] and the beautiful book [5].

We now discuss some of their features. To simplify the discussion, we work with proper surjective maps $f : X \to Y$ with X nonsingular and irreducible.

We start with what is likely to be the quickest possible definition of semismall map.

Definition 3.1.1 (**Definition of semismall map**) The map f is said to be semismall if $\dim X \times_Y X = \dim X$.

Quick is good, but not always transparent. The standard definition involves consideration of the dimension of the loci $S_k \subseteq Y$ where the fibers of the map have fixed dimension k: semismallness is the requirement that $\dim S_k + 2k < \dim X$ for every $k > 0$; see Exercise 3.6.1.

Small maps. We say that the map f is small if it is semismall and $X \times_Y X$ has a unique irreducible component of maximal dimension $\dim X$ (which one?). For semismall maps, this is equivalent to having $\dim S_k + 2k < \dim X$ for every $k > 0$.

The blowing ups of $\mathbb{C}^m \subseteq \mathbb{C}^n$, $m \leq n-2$ are semismall iff $m = n-2$. None of these is small. The blowing up of the affine cone over the nonsingular quadric in $\mathbb{P}^3$ along a plane thru the vertex is a small map. The blowing up of the vertex is not.

The Springer resolution of the nilpotent cone in a semisimple Lie algebra is semismall and Grothendieck-Springer simultaneous resolution is small. We shall meet both a bit later and show how they interact beautifully to give us a "decomposition theorem argument" for the presence an action of the Weyl group of the Lie algebra on the cohomology of the fibers of the Springer resolution. The Weyl group does not act on the fibers!

The following beautiful result of D. Kaledin's is a source of a large example of highly non trivial semismall maps.

Theorem 3.1.2 (**Holomorphic symplectic contractions are semismall** [35]) *A projective birational map from a holomorphic symplectic*[13] *nonsingular variety is semismall.*

It is amusing to realize that semismallness and the Hard Lefschetz phenomenon are essentially equivalent. In fact, we have the following

Theorem 3.1.3 (**Hard Lefschetz for semismall maps** [11]) **Hard Lefschetz and semismall maps.** *Let $f : X \to Y$ be a surjective projective map of projective varieties with X nonsingular and let $\eta := f^*L \in H^2(X, \mathbb{Q})$ be the first Chern class of the pull-back of an ample line bundle on Y. pull-back. Then the iterated cup product maps $\eta^r : H^{\dim X - r}(X, \mathbb{Q}) \to H^{\dim X + r}(X, \mathbb{Q})$ are isomorphisms for every $r \geq 0$ iff the map f is semismall. In the semismall case, we have the primitive Lefschetz decomposition and the Hodge-Riemann bilinear relations.*

Hodge-index theorem for semismall maps. There is an important phenomenon concerning projective maps that is worth mentioning, i.e. the signature of certain local intersection forms [13]; for a discussion of these, see [15]. The situation is more transparent in the case of semismall maps, where it is directly related to the Hodge-Riemann bilinear relations associated with of Theorem 3.1.3. To have a clearer picture, let us limit ourselves to state a simple, revealing and important special case. Let $f : X \to Y$ be a surjective semismall projective map with X nonsingular of some even dimension $2d$. Assume that $f^{-1}(y)$ is d-dimensional for some $y \in Y$. By intersecting in X we obtain the refined symmetric intersection pairing $H_{2d}(f^{-1}(y)) \times H_{2d}(f^{-1}(y)) \to \mathbb{Q}$, where we are intersecting the fundamental classes of the irreducible components of this special fiber inside of X. The following is a generalization of a result of Grauert's for $d = 1$.

Theorem 3.1.4 (**Refined intersection forms have a precise sign**) *The refined intersection pairing above is* $(-1)^d > 0$.

By looking carefully at every proper map, similar refined intersection forms appear. One can prove that the DT is (essentially) equivalent to the non-degeneracy of these refined intersection forms together with Deligne's semisimplicity of monodromy theorem; see [13].

Exercise 3.6.2 relates perverse sheaves and semismall maps: if $f : X \to Y$ is semismall, then $Rf_*\mathbb{Q}_X[\dim X]$ is perverse.

The decomposition theorem then tells us that

$$f_*\mathbb{Q}_X[\dim X] = {}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathbb{Q}_X[\dim X]) = \bigoplus_{(S,L)\in EV_0} IC_S(L). \tag{17}$$

Question 3.1.5 What "are" the summands appearing in the decomposition theorem for semismall maps?

[13] i.e. even-dimensional and admitting a closed holomorphic 2-form ω which is non-degenerate, i.e. $\omega^{\frac{\dim X}{2}}$ is nowhere vanishing

3.2 The decomposition theorem for semismall maps

A little bit about stratifications. Even if not logically necessary, it simplifies matters to use the stratification theory of maps to clarify the picture a bit. There is a finite disjoint union decomposition $Y = \coprod_{a \in A} S_a$ into locally closed nonsingular irreducible subvarieties $S_a \subseteq Y$ such that $f^{-1}(S_a) \to S_a$ is locally (for the classical topology) topologically a product over S_a. It is clear that this decomposition refines the one above given by the dimension of the fibers, so that $\dim S_a + 2 \dim f^{-1}(s) \leq \dim X$ for every $s \in S_a$. Since the map is assumed to be proper, it is also clear that all direct image sheaves $\mathcal{R}^q$ restrict to locally constant sheaves on every S_a. We call the S_a the strata (of a stratification of the map f).

Definition 3.2.1 (**Relevant stratum**) We say that S_a is relevant if we have:

$$\dim S_a + 2 \dim f^{-1}(s) = \dim X.$$

We denote by $A_{rel} \subseteq A$ the set of relevant strata.

Exercise 3.6.3 shows that, for each relevant stratum S_a, the direct image sheaf $\mathcal{R}^{\dim X - \dim S_a}$ restricted to S_a is locally constant, semisimple, with finite monodromy. We denote this restriction by L_a.

Theorem 3.2.2 (**Decomposition theorem for semismall maps**) *Let $f : X \to Y$ be proper surjective semismall with X nonsingular. Then there is a direct sum decomposition*

$$f_* \mathbb{Q}_X[\dim X] = \bigoplus_{a \in A_{rel}} IC_{\overline{S_a}}(L_a).$$

Since the locally constant sheaf L_a is semisimple, it admits the isotypical direct sum decomposition (8), i.e. we have $L_a = \oplus_\chi L_{a,\chi} \otimes M_{a,\chi}$ where χ ranges over a finite set of distinct isomorphism classes of simple locally constant sheaves on S_a and $M_{a,\chi}$ is a vector space of rank the multiplicity $m_{a,\chi}$ of the locally constant sheaf L_χ in L_a. The decomposition theorem then reads

$$f_* \mathbb{Q}_X[\dim X] = \bigoplus_{a,\chi} IC_{\overline{S_a}}(L_{a,\chi} \otimes M_{a,\chi}). \tag{18}$$

3.3 Hilbert schemes of points on surfaces and Heisenberg algebras

An excellent reference is [43]. Let X be a nonsingular complex surface. For every $n \geq 0$ we have the Hilbert scheme $X^{[n]}$ of n points on X. It is irreducible nonsingular of dimension $2n$. There is proper birational surjective map $\pi : X^{[n]} \to X^{(n)}$ onto the n-th symmetric product sending a length n zero dimensional subscheme of X to its support counting multiplicities. There is a natural stratification of the symmetric product of the map: $X^{(n)} = \coprod_{\nu \in P(n)} X^{(n)}_\nu$, where $P(n)$ is the set of partitions $\nu = \{\nu_j\}$ of the integer n (the ν_j are positive integers adding up to n) obtained by taking the locally closed irreducible nonsingular dimension $2l(\nu)$ (l is the length of partition function) sets of points $\sum_i n_i x_i \in X^{(n)}$ with type $\{n_i\}$ given by ν. The remarkable fact is that the fibers $\pi^{-1}(x_\nu)$ of the points $x_\nu \in X_\nu^{(n)}$ are irreducible of dimension $\sum_{\nu_j - 1} = n - l(\nu)$. It follows that the map π is semismall. In fact, $X^{[n]} = \coprod_{P(n)} \pi^{-1}(X_\nu^{(n)}) \to \coprod_{P(n)} X_\nu^{(n)}$ is a stratification of the semismall map π and all the strata are relevant. Since the fibers are all irreducible, the relevant locally constant sheaves are all constant of rank one. In particular, the decomposition theorem for π takes the form: $R\pi_* \mathbb{Q}_{X^{[n]}}[2n] = \oplus_{\nu \in P(n)} IC_{\overline{X_\nu^{(n)}}}$. A second remarkable fact is that the normalization $\overline{X_\nu^{(n)}}$ can be identified with a product of symmetric products $X^{(\nu)} := \prod_{i=1} n X^{(a_i)}$, where a_i is the number of times that i appears in ν. This is a variety obtained by dividing a nonsingular variety by the action of a finite group; in particular, its intersection complex is the constant sheaf. By the IC normalization principle Fact 4.5.3, we see that $IC_{\overline{X_\nu^{(n)}}}$ is the push-forward of the shifted constant sheaf from the normalization. By taking care of shifts, after the dust settles, we obtain the Göttsche formula: (for $n = 0$ take $\mathbb{Q}$ on both sides)

$$\mathbb{H}(X) := \bigoplus_{n \geq 0} H^*(X^{[n]}) = \bigoplus_{n \geq 0} \bigoplus_{\nu \in P(n)} H^{* - cl(\nu)}(X^{(\nu)}, \mathbb{Q}), \tag{19}$$

where the colength $cl(\nu) := n - l(\nu)$.

If we take $X = \mathbb{C}^2$, then something remarkable emerges: look at (20) and (21). The formula above, taken for every $n \geq 0$ gives

$$\sum_{n=0}^{\infty} \dim H^*(\mathbb{C}^{2^{[n]}}, \mathbb{Q}) = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}. \tag{20}$$

Let $R := \mathbb{Q}[x_1, x_2, \ldots]$ be the algebra of polynomials in the infinitely many indeterminates x_i, declared to be of degree i. The infinite dimensional Heisenberg algebra $\mathscr{H}$ is the Lie algebra with underlying rational vector space the one with basis $\{\{d_i\}_{i<0}, c_0, \{m_i\}_{i>0}\}$ and subject to the following relations: c_0 is central, the d_i's commute with each other, the m_i's commute with each other, and $[d_i, m_j] = \delta_{-i,j} c_0$. Then R is an irreducible $\mathscr{H}$-module generated by 1 where d_i acts as formal derivation by x_i and m_j by multiplication by x_j. The dimension a_n of the space of homogeneous polynomials of degree n is given by:

$$\sum_{n=0}^{\infty} a_n q^n = \prod_{j=1}^{\infty} \frac{1}{1 - q^j}. \tag{21}$$

Well, isn't this a coincidence! The operators d_i, m_i change the homogeneous degree of x-monomials by $\pm i$. This, together with the formalism of correspondences in products, suggests that there should be geometrically meaningful cohomology classes in $H^*(\mathbb{C}^{2^{[n]}} \times \mathbb{C}^{2^{[n \pm i]}}, \mathbb{Q})$ that reflect, on the Hilbert scheme side, the Heisenberg algebra action on the polynomial side.

This is indeed the case and it is due to Grojnowski and to Nakajima: they guessed what above, constructed algebraic cycles on the products of Hilbert schemes above that would be good candidates and then verified the Heisenberg Lie algebra relations. In fact, for every nonsingular surface X, there is an associated (Heisenberg-Clifford) "algebra" $\mathscr{H}(X)$ that acts geometrically and irreducibly on $\mathbb{H}(X)$ (19).

3.4 The endomorphism algebra $End(f_*\mathbb{Q}_X)$

An reference here is [12].

Semisimple algebras. A semisimple algebra is an associative artinian (dcc) algebra over a field with trivial Jacobson ideal (the ideal killing all simple left modules). The Artin-Wedderburn theorem classifies the semisimple algebras over a field as the ones which are finite Cartesian products of matrix algebras over finite dimensional division algebras over the field.

Warm-up. Show that $M_{d\times d}(\mathbb{Q})$ is semisimple. Show that the upper triangular matrices do not form a semisimple algebra. Deduce that if $f := pr_2 : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, then $End_{D(\mathbb{P}^1)}(f_*\mathbb{Q})$ is not a semisimple algebra.

Theorem 3.4.1 (**Semismall maps and semisimplicity of the End-algebra**) *Let things be as in Theorem 3.2.2. The endomorphism $\mathbb{Q}$-algebra $End_{D(Y)} f_*\mathbb{Q}_X[\dim X]$ is semisimple.*

Proof. We see more important properties of intersection complexes at play, i.e. the Schur lemma phenomena for simple perverse sheaves.
By simplicity, $Hom(IC_{\overline{S_a}}(L_\chi), IC_{\overline{S_b}}(L_\psi)) = \delta_{\chi,\psi} End(IC_{\overline{S_a}}(L_\chi))$ (no non zero maps if they differ): in fact, look at kernel and cokernel and use simplicity. This leaves us with considering terms of the form $End(IC_{\overline{S_a}}(L_\chi))$ whose elements, for the same reason as above, are either zero, or are isomorphisms. These terms are thus division algebras $D_{a,\chi}$. It follows that $End_{D(Y)} f_*\mathbb{Q}_X[\dim X] = \prod_{a,\chi} M_{d_{a,\chi}\times d_{a,\chi}}(D_{a,\chi})$, which is a semisimple algebra by Artin-Wedderburn. □

The endomorphism algebra as a geometric convolution algebra. An reference here is also [5]. We can realize the algebra $End(f_*\mathbb{Q}_X)$ of endomorphisms in the derived category in geometric terms as the convolution algebra: $H^{BM}_{2\dim X}(X \times_Y X)$, which is thus semisimple. Let us discuss this a bit.

Let X be a nonsingular projective variety. Then, we have an isomorphism of algebras between the first and last term

$$End(H^*(X)) =_1 H^*(X)^\vee \otimes H^*(X) \cong_2 H^*(X) \otimes H^*(X) \cong_3 H^*(X \times X) \cong_4 H_*(X \times X) \tag{22}$$

where: $=_1$: linear algebra; $\cong_2$: Poincaré duality; $\cong_3$: Künneth; $\cong_4$: Poincaré duality; and where: the algebra structure on the last term is the one given by the formalism of composition of correspondences in products (see Exercise 3.6.4).

The classes $\Gamma \in H_\gamma(X \times X)$ appearing in Exercise 3.6.4 are called correspondences. This picture generalizes well, but not trivially, to proper maps $f : X \to Y$ from nonsingular varieties as follows.

Theorem 3.4.2 (**Correspondences and endomaps in the derived category**) *Let $f : X \to Y$ be a proper map from a nonsingular variety. There is a natural isomorphism $End_{D(Y)}(f_*\mathbb{Q}_X) \cong H^{BM}_{2\dim X}(X \times_Y X)$ of $\mathbb{Q}$-algebras.*

Exercise 3.6.7 shows that for our semismall maps the vector space $H^{BM}_{2\dim X}(X \times_Y X)$ has an evident geometric basis.

Since there is a basis of $H^{BM}_{2\dim X}(X \times_Y X) = End_{D(Y)}(Rf_*\mathbb{Q}_X)$ given by algebraic cycles, a formal linear algebra manipulation shows that if X is projective, then decomposition $H^*(X, \mathbb{Q}) = \oplus_{a \in A_a} IH^*(\overline{S_a}, L_a)$ is compatible with the Hodge (p, q)-decomposition, i.e. it is given by pure Hodge substructures; see Exercise 3.6.8. In fact, one even has a canonical decomposition of Chow motives reflecting the decomposition theorem for semismall maps; see [12]. Look at the related Question 5.5.1.

3.5 Geometric realization of the representations of the Weyl group

An excellent reference is [5].

There is a well-developed theory of representations of finite groups G (character theory) into finite dimensional complex vector spaces. In a nearly tautological sense this theory is equivalent to the representation theory of the group algebra $\mathbb{Q}[G]$.

If we take the Weyl group W of any of the usual suspects, e.g. $SL_n(\mathbb{C})$ with Weyl group the symmetric group S_n, then we can ask whether we can realize the irreducible representations of W by using the fact that W is a Weyl group.

Springer realized that this was indeed possible, and in geometric terms! In what follows, we do not reproduce this amazing story, but we limit ourselves to showing how the decomposition theorem allows (there are other ways) to introduce the action of the Weyl group on the cohomology of the Springer fibers. The Weyl group does not act on these fibers!

Take the Lie algebra $sl_n(\mathbb{C})$ of traceless $n \times n$ matrices. Inside of it there is the cone N with vertex the origin given by the nilpotent matrices. Take the flag variety F, i.e. the space of complete flags f in $\mathbb{C}^n$. Set $\widetilde{N} := \{(n, f)|, n \,\text{stabilizes}\, f\} \subseteq N \times F$. Then $\widetilde{N} \to F$ can be shown to the the projection $T^*F \to F$ (and $\widetilde{N}$ is thus a holomorphic symplectic manifold) and the projection $\pi : \widetilde{N} \to N$ is a resolution of the singularities on the nilpotent cone N.

The map π is semismall! We know this, for example, from Kaledin's Theorem 3.1.2. In fact, it was known much earlier, by the work of many. We can partition N according to the Jordan canonical form. This gives rise to a stratification of N and of π. Every stratum is relevant.

It is amusing to realize that the intersection form associated with the deepest stratum (vertex) gives rise to $\pm$ the Euler number of F and that the one associated with the codimension two stratum yields the Cartan matrix for $sl_n(\mathbb{C})$.

The fibers of the map π are called Springer fibers. Springer proved that all the irreducible representations of the Weyl group occur as direct summands of the action of the Weyl group on the homology of the Springer fibers. This beautiful result tells us that indeed one can realize geometrically such representations.

Note that the Weyl group does not act on the Springer fibers. Therefore it is not clear why one should expect (even hope for!)) such a wonderful outcome.

In what follows we aim at explaining how the Weyl group acts on the perverse sheaf $R\pi_*\mathbb{Q}_{\widetilde{N}}$. In turn, by taking stalks, this explains why the Weyl group acts on the homology of the Springer fibers.

Instead of sticking with $\widetilde{N} \to N$, we consider $p : \widetilde{sl_n(\mathbb{C})} \to sl_n(\mathbb{C})$ defined in the same way. The difference is that if we take the Zariski open set U given by diagonalizable matrices with n distinct eigenvalues, then p is a topological Galois cover with group the Weyl group (permutation of eigenvalues). We have a Weyl group action! Unfortunately $N \cap U = \emptyset$!. On the other hand, p is ...small! So $Rp_*\mathbb{Q} = \mathcal{IC}(L)$, where L is the local system on U associated with the Galois cover with group the Weyl group. Then W acts on L. Hence it acts on $IC(L)$ by functoriality of the intermediate extension construction. Since the map p is proper, and it restricts to π over N, we see that the restriction of $Rp_*\mathbb{Q} = IC(L)$ to N is $R\pi_*\mathbb{Q}$ which thus finds itself endowed, almost by the trick of a magician, with the desired W action!

3.6 Exercises for Lecture 3

Exercise 3.6.1 (**Semismallness and fibers**) Show that we always have $\dim X \times_Y X \geq \dim X$. Use Chevalley's result on the upper semicontinuity of the dimensions of the fibers of maps of algebraic varieties to produce a finite disjoint union decomposition $Y = \coprod_{k\geq 0} Y_k$ into locally closed subvarieties with $\dim f^{-1}(y) = k$ for every $y \in Y_k$. Show that f is semismall iff we have $\dim S_k + 2k \leq \dim X$[14] for every $k \geq 0$. Observe that f semismall implies that f is generically finite, i.e. that Y_0 is open and dense. Observe that $f^{-1}(Y_0) \times_{Y_0} f^{-1} Y_0$ has dimension $\dim X$. Give examples of semismall maps where $X \times_Y X$ has at least two irreducible components of dimension $\dim X$.

Exercise 3.6.2 (**Semismal maps and perverse sheaves**) The map f is semismall IFF $f_* \mathbb{Q}_X[\dim X]$ is perverse.

Exercise 3.6.3 (**Relevant locally constant sheaves**) Let $S_a \in A_{rel}$ be relevant. Show that $R^{\dim X - \dim S_a}$ is locally constant with stalks $H^{2\dim f^{-1}(s)}(f^{-1}(s))$. The monodromy of this locally constant sheaf, denoted by L_a, factors through the finite group of symmetries of the set of irreducible components of maximal dimension $\frac{1}{2}(\dim X - \dim S_a)$ of a typical fiber $f^{-1}(s)$. (Note that, a priori, the monodromy could send the fundamental class of such a component to minus itself, thus contradicting the claim just made; that this is not the case follows, for example, from a theorem of Grothendieck's in EGA IV, 15.6.4; see the nice general discussion in B.C. Ngo's paper arxiv0801.0446v3, §7.1.1. In particular, L_a is semisimple. Note that if we switch from $\mathbb{Q}$-coefficients to $\overline{\mathbb{Q}}$-coefficients simple objects may split further. Do they at least stay semisimple?

Exercise 3.6.4 (**Formalism of correspondences in products**) Unwind the isomorphisms (22) to deduce that via such isomorphisms, a class $\Gamma \in H_\gamma(X \times X)$ defines a linear map $\Gamma_* : H^*(X) \to H^{*+\gamma-2\dim X}(X)$ given by $a \mapsto PD(pr_{2*}(pr_1^* a \cap \Gamma))$. Conversely, show that any graded linear map $H^*(X) \to H^{*+\gamma-2\dim X}(X)$ is given by a unique such $\Gamma \in H_\gamma(X \times X)$.

Exercise 3.6.5 (**Sheaf theoretic definition of Borel-Moore homology**) Let X be a variety. Recall (one) definition of Borel-Moore homology: embed X as a closed subvariety of smooth variety Y, then set

$$H_i^{BM} = H^{2\dim Y - i}(Y, Y - X),$$

where the right hand side is relative cohomology. By interpreting relative cohomology sheaf theoretically, give a sheaf theoretic definition of Bore-Moore homology.
(Hint: consider the distinguished triangle $i_* i^! \to \mathrm{id} \to j_* j^* \to$, if you apply this to the constant sheaf and take cohomology (or equivalently push to a point), what long exact sequence do you get?)

[14] think of it as a vary special upper bound on the dimension of the "stratum" where the fibers are k-dimensional

Exercise 3.6.6 (**Bore-Moore homology and Ext/convolution algebras**) Let $\pi\colon X \to Y$ be a proper morphism of varieties, with X smooth. Form a cartesian square

$$\begin{tikzcd} Z \arrow[r, "p_1"] \arrow[d, "p_2"'] & X \arrow[d, "\pi"] \\ X \arrow[r, "\pi"] & Y \end{tikzcd}$$

For sheaves A, B on a variety Z, let $Ext^i(A, B) = Hom_{D^b(Z)}(A, B[i])$. I.e., $Ext^\bullet$ denotes (shifted) Hom in the derived category. Show the following:

1. $Ext^\bullet(R\pi_*\mathbb{Q}, R\pi_*\mathbb{Q}) = Ext^\bullet(\mathbb{Q}, \pi^! R\pi_*\mathbb{Q})$. (Hint: this is matter of remembering some adjointness property).

2. $Ext^\bullet(\mathbb{Q}, \pi^! R\pi_*\mathbb{Q}) = Ext^\bullet(\mathbb{Q}, Rp_{2*}p_1^!\mathbb{Q})$. (Hint: ask your neighbor about proper base change).

3. $Ext^\bullet(\mathbb{Q}, Rp_{2*}p_1^!\mathbb{Q}) = Ext^\bullet(\mathbb{Q}, p_1^!\mathbb{Q})$ (Hint: the right hand side is on the space Z, so something along the lines of (i) must have happened).

4. $Ext^\bullet(\mathbb{Q}, p_1^!\mathbb{Q}) = H^{BM}_{2\dim X - \bullet}(Z)$. (Hint: you need to use the sheaf theoretic definition of Borel-Moore homology. In addition, there is a dimension shift that has happened, this should make you think about duality).

Exercise 3.6.7 (**Geometric basis for $H^{BM}_{2\dim X}(X \times_Y X)$ when f is semismall**) Show that if f is semismall, then the rational vector space $H^{BM}_{2\dim X}(X \times_Y X)$ has a basis formed by the fundamental classes of the irreducible components of $X \times_Y X$ of maximal dimension $\dim X$.

Exercise 3.6.8 (**Hodge-theoretic decomposition theorem for f semismall**) Show that if $f : X \to Y$ is semismall with X projective nonsingular, then the decomposition $H^*(X, \mathbb{Q}) = \oplus_{a \in A_a} IH^*(\overline{S_a}, L_a)$ is one pure Hodge structures (i.e. compatible with the Hodge (p, q)-decomposition of $H^*(X, \mathbb{C})$. (Hint: the projectors onto the direct sums are given by algebraic cycles.)

4 Lecture 4: DT symmetries: VD, RHL. $\mathcal{IC}$ splits off

Summary of Lecture 4. Discussion of the two main symmetries in the decomposition theorem for projective maps: Verdier duality and the relative hard Lefschetz theorem. Hard Lefschetz in intersection cohomology and Stanley's theorem for rational simplicial polytopes. A proof that the intersection complex of the image is always a direct summand. Pure Hodge structures on the intersection cohomology of a projective surface.

Remark 4.0.9 We are going to discuss two symmetries for projective maps: Verdier duality and the relative Hard Lefschetz theorem. Both these statements have to do with the direct image perverse sheaves. In fact, if the target is projective, then we can take the shadow of these two symmetries and notice that there are two additional symmetries: the Verdier duality and Hard Lefschetz theorem on the individual summands $IH*(S, L)$. Exercise 4.7.5 asks you to make an explicit list in a low-dimensional case.

4.1 Verdier duality and the decomposition theorem

Verdier duality and the decomposition theorem. Recall the statement (10) of the decomposition theorem

$$Rf_* \mathcal{IC}_X(M) \cong \bigoplus_{q \geq 0, \mathcal{EV}_q} \mathcal{IC}_S(L)[-q].$$

Switching to the perverse intersection complex. If (S, L) is an enriched variety, then irreducible, then $\mathcal{IC}_S(L)$ is not a perverse sheaf. Recall (9): the perverse object is $IC_S(L) := \mathcal{IC}_S(L)[\dim S]$. In order to emphasize better certain symmetric aspects of the decomposition theorem, we switch to the perverse intersection complex. This entails a a minor headache when re-writing (10), which becomes (verify it as an exercise)

$$Rf_* IC_X(M) \cong \bigoplus_{b \in \mathbb{Z}} \bigoplus_{EV_b} \mathcal{IC}_S(L)[-b], \quad (S, L) \in EV_b \text{ iff } (S, L) \in \mathcal{EV}_{b + \dim X - \dim S}. \tag{23}$$

The perverse cohomology sheaves of the derived direct image. Note that, now, every b-th direct summand above is a perverse sheaf, so that, in view of Exercise 4.7.6, we have that:

$${}^{\mathfrak{p}}\mathcal{H}^b(Rf_* IC_X(M)) = \oplus_{EV_b} IC_S(L). \tag{24}$$

The case when M is self-dual (and semisimple). Let M be self-dual, e.g. a constant sheaf, a polarizable variation of pure Hodge structures, or even the direct sum of any M with its dual; self-dual local systems appear frequently in complex algebraic geometry. Then so is $IC_X(M)$ and, by the duality exchange property for proper maps, so is $Rf_* IC_X(M)$. In view of the duality relation (15) between perverse cohomology sheaves, we see that ${}^{\mathfrak{p}}\mathcal{H}^b(Rf_*IC_X(M))) \cong {}^{\mathfrak{p}}\mathcal{H}^{-b}(Rf_*IC_X(M)))^\vee$. By combining with (24), we get

$$Rf_*IC_X(M) \cong \left(\bigoplus_{b<0\in\mathbb{Z},EV_b} IC_S(L)[-b]\right) \bigoplus \left(\bigoplus_{EV_0} IC_S(L)\right) \bigoplus \left(\bigoplus_{b<0\in\mathbb{Z},EV_b} IC_S(L^\vee)[b]\right). \tag{25}$$

In other words, the direct image is palindromic, i.e. it reads the same, up to shifts and dualities, from right to left and from left to right. Just like the cohomology of a compact oriented manifold.

The defect of semismallness. When trying to determine the precise shape of the decomposition theorem, one important invariant is the minimal interval $[-r,r]$ out of which the perverse cohomology sheaves are zero. In this direction, we have that if M is constant and X is nonsingular, then $r = \dim X \times_Y X - \dim X \geq 0$. This difference is called the defect of semismallness in [13]. In this situation, $r = 0$ iff the map is semismall.

4.2 Verdier duality and the decomposition theorem with large fibers

Here is a nice consequence of Verdier duality, more precisely of (25). It is an observation due to Goresky and MacPherson and it is used by B.C. Ngô in his proof of the support theorem, a key technical and geometric result in his proof of the fundamental lemma in the Langlands' program. See [44], §7.3.

Theorem 4.2.1 *Let $f : X \to Y$ be proper with X nonsingular and equidimensional fibers of dimension d. Assume a subvariety S appears in the decomposition theorem (10) for $Rf_*\mathbb{Q}_X$. Then codim$(Z) \leq d$.*

Proof. There is a maximum index $b_S^+ \in \mathbb{Z}$ such that a term of the form $IC_S(L)[-b_S^+]$ appears. By the palindromicity (25), we may assume that $b_S^+ \geq 0$. Recall that L is defined on some open dense $S^o \subseteq S$. Let $U \subseteq Y$ be open such that its trace on S is S^o. Replace Y with U. Denote by $i : S^o \to Y$ the closed embedding. Then $Rf_*\mathbb{Q}[\dim X]$ admits $i_*L[\dim S][-b_S^+]$ as a direct summand. Then i_*L is a non trivial direct summand of $R^{\dim X - \dim S + b_S^+} f_*\mathbb{Q}$. Since the fibers have dimension d and $b_S^+ \geq 0$, we have that $\dim X - \dim S \leq \dim X - \dim S + b_S^+ \leq 2d$. Since $\dim X = \dim Y + d$, the conclusion follows. □

4.3 The relative hard Lefschetz theorem

Poincaré duality vs. hard Lefschetz. Let X be a projective manifold and let $\eta \in H^2(X,\mathbb{Q})$ be the class of a hyperplane section. Then we have two separate phenomena:

$$H^{\dim X - r}(X,\mathbb{Q}) = H^{\dim X + r}(X,\mathbb{Q})^\vee \quad \text{(Poincaré duality)},$$

$$\eta^r : H^{\dim X - r}(X,\mathbb{Q}) \cong H^{\dim X + r}(X,\mathbb{Q})^\vee \text{(hard Lefschetz)}.$$

The first statement is that the pairing $\int_X - \wedge -$ between cohomology in complementary degrees is non-degenerate. The second one is that the pairing $\int_X \eta^r - \wedge -$ on cohomology group $H^{\dim X - r}$ is non-degenerate. They both imply the usual symmetry of Betti numbers. The latter implies also their unimodality, i.e. $b_d \geq b_{d-2} \geq \ldots$, where $d = \dim X, \dim X - 1$ (compare with the Hopf surface, which has the former, but not the latter).

Exercise 4.7.4 discusses two proofs of the hard Lefschetz theorem. Both proofs generalize and, with some work, afford proofs of the relative hard Lefschetz theorem. In particular, they yield proofs of the hard Lefschetz theorem in intersection cohomology.

We have mentioned how, in the context of singular varieties, Poincaré duality in cohomology is lost but found again in intersection cohomology. The same is true for the hard Lefschetz theorem for the intersection cohomology of projective varieties! This brings us back to the theme harping the importance of the derived category and of perverse sheaves: the statement of the hard Lefschetz for intersection cohomology is cohomological, but there is no known proof that avoids perverse sheaves.

Let $f : X \to Y$ be a map of varieties and let $\eta \in H^2(X, \mathbb{Q})$ be a cohomology class. It is a general fact that η induces, $\eta : C \to C[2]$, which induces $\eta : Rf_* C \to Rf_* C[2]$, which, by taking perverse cohomology sheaves, induces $\eta : {}^{\mathfrak{p}}\mathcal{H}^b(C) \to {}^{\mathfrak{p}}\mathcal{H}^C(b+2)$. It follows that, for every $b \geq 0$, we obtain maps $\eta^b : {}^{\mathfrak{p}}\mathcal{H}^{-b}(C) \to {}^{\mathfrak{p}}\mathcal{H}^b(C)$ in $P(Y)$.

Let $f : X \to Y$ be a projective map with and let $IC_X(M) \in P(X)$. Let $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of a line bundle on X which is ample on every fiber of f. It is a general fact that $\eta \in H^2(X, \mathbb{Q})$ induces, $\eta : IC_X(M) \to IC_X(M)[2]$, which induces $\eta : Rf_* IC_X(M) \to Rf_* IC_X(M)[2]$, which, by taking perverse cohomology sheaves, induces $\eta : {}^{\mathfrak{p}}\mathcal{H}^b \to {}^{\mathfrak{p}}\mathcal{H}^{b+2}$. It follows that, for every $b \geq 0$, we obtain maps $\eta^b : {}^{\mathfrak{p}}\mathcal{H}^{-b} \to {}^{\mathfrak{p}}\mathcal{H}^b$ in $P(Y)$.

Theorem 4.3.1 (**Relative hard Lefschetz** [1, 47, 13, 46, 42]) *Let $f : X \to Y$ be a projective map and $\eta \in H^2(X, \mathbb{Q})$ be the first Chern class of an f-ample line bundle on X*[15]*. Let $IC_X(M)$ be semisimple, i.e. X irreducible and M semisimple. For every $b \geq 0$ the iterated cup product map $\eta^b : {}^{\mathfrak{p}}\mathcal{H}^{-b}(Rf_* IC_X(M)) \to {}^{\mathfrak{p}}\mathcal{H}^b(Rf_* IC_X(M))$ is an isomorphism.*

Hard Lefschetz for the intersection cohomology of projective varieties. The special case when Y is a point and M is constant yields the hard Lefschetz theorem for the intersection cohomology groups of a projective variety.

The same Deligne-Lefschetz criterion employed in the proof of the derived Deligne theorem, allows to deduce formally a first approximation to the decomposition theorem (this argument does not afford the semisimplicity part of the decomposition theorem)

$$Rf_* IC_X(M) \cong \bigoplus_b {}^{\mathfrak{p}}\mathcal{H}^b(Rf_* IC_X(M))[-b]. \tag{26}$$

Exercise 4.7.3 gets you a bit more acquainted with the primitive Lefschetz decompositions. Exercise 4.7.4 draws a parallel between the classical inductive approach to the Hard

[15] here is one such line bundle: since f is projective, there is a factorization of f as follows $X \to Y \times \mathbb{P} \to Y$ (closed embedding, followed by the projection); pull-back the hyperplane bundle from $\mathbb{P}$ to $Y \times \mathbb{P}$ and restrict to X

Lefschetz theorem via the Lefschetz hyperplane section theorem and the semisimplicity of monodromy for the family of hyperplane sections; see Deligne's second paper on the Weil Conjectures [28], §4.1.

4.4 Application of RHL: Stanley's theorem

An excellent reference is [48]. For more details, see [16].

A convex polytope is the convex hull of a finite set in real Euclidean space. It is said to be simplicial if all its faces are simplices. Example: a triangle. Non-example: a square. Example: to square-based pyramids joined at the bases. Let P be a d-dimensional simplicial convex polytope with f_i i-dimensional faces, $0 \leq i \leq d-1$. The f-vector (f for faces) of P is the vector $f(P) = (f_0, \ldots, f_{d-1})$. The h-vector of P is defined by setting $h(P) = (h_0, \ldots, h_d)$ with

$$h_i = \sum_{j=0}^{i} \binom{d-j}{d-i} (-1)^{i-j} f_{j-1} \quad (f_{-1} := 1)$$

The f and h-vectors determine each other.

Question 4.4.1 *When is a vector $(f_0, \ldots, f_{d-1})$ an $f(P)$-vector for some P?*

(A reformulation of) P. McMullen's 1971 conjecture. (Of course, to give f is the same as giving h.) A vector f is an $f(P)$-vector for some P iff 1) $h_i = h_{d-i}$ and 2) there is a graded commutative $\mathbb{Q}$-algebra $R = \oplus_{i \geq 0} R_i$, with $R_0 = \mathbb{Q}$, generated by R_1, and with $\dim R_i = h_i - h_{i-1}$, $1 \leq i \leq \lfloor d/2 \rfloor$. In particular, $h_0 \leq h_1 \leq \ldots h_d$.

Associated simplicial toric variety. Stanley himself writes: "we are led to suspect the existence of a smooth d-dimensional projective variety $X(P)$ for which (the Betti numbers) $b_{2i} = h_i$", for which $H^{even}(X, \mathbb{Q})$ is generated by $H^2(X, \mathbb{Q})$ and we can take for $R := H^{even}(X, \mathbb{Q})/(\eta)$ (quotient by ideal generated by hyperplane class).

Then 1) above would be Poincaré duality and 2) would be a direct consequence of hard Lefschetz on the smooth projective X.

This is what happens.

The combinatorial data of the simplicial P gives rise to a simplicial toric variety $X(P)$.

P simplicial means that $X(P)$, while possibly singular, has singularities of the type "vector space modulo a finite group".

Necessity of the conditions. It is a fact that $H^*(X(P), \mathbb{Q}) = H^{even}(X(P), \mathbb{Q})$ and that $b_{2i} = h_i$. The basic idea is that: faces give rise to torus orbits; torus orbits assemble into cells with the shape of affine spaces modulo finite groups; then $X(P)$ is a disjoint union of such cells; since the cells automatically of even real dimension, the cohomology has graded bases these cells; the only issue is to count these cells properly; this is indeed the explanation of the relation $f(P) \leftrightarrow h(P)$ (in the simplicial case).

Exercise 4.7.9 tells us that $\mathcal{IC}_{X(P)} = \mathbb{Q}_{X(P)}$. It follows that $H^*(X(P), \mathbb{Q})$ satisfies Poincaré duality. We thus get the necessity of 1) in McMullen's conjecture.

The necessity of 2) would follow if we knew the hard Lefschetz theorem for $H^*(X(P), \mathbb{Q})$. But we do, because we know it for the intersection cohomology groups, which, by $\mathcal{IC} = \mathbb{Q}$, are the cohomology groups.

We thus have.

Theorem 4.4.2 (**Simplicial polytopes: iff for f being an $f(P)$ vector**) *The McMullen conditions are necessary (Stanley: discussion above) and sufficient (Billera and Lee: construction).*

4.5 Intersection cohomology of the target as direct summand

A funny situation. Singular cohomology is functorial, but, in general, f^* is not injective, not even if f is proper[16]. On the other hand, intersection cohomology is not functorial, but for proper surjective maps, the DT exhibits the intersection cohomology of the target as a direct summand of the intersection cohomology of the source.

The following theorem is one of the most striking and useful applications of the decomposition theorem. It is usually used, stated and proved in the context of proper birational maps. The proof in the presence of generic large fibers is not more difficult. In fact, we give a proof as it is also a chance to meet and use some very useful general principles of the theory we have been talking about in these lectures.

Theorem 4.5.1 (**Intersection complex as a direct summand**) *Let $f : X \to Y$ be a proper map of irreducible varieties with image Y'. Then $IH^*(Y')$ is a direct summand of $IH^*(X)$. More precisely, $\mathcal{IC}_{Y'}$ is a direct summand of $f_*\mathcal{IC}_X$.*

Let us state three general and useful principles. Recall that for a given (S, L), the locally constant sheaf L is only defined on a suitable S^o open dense subvariety of S_{reg} and that one can shrink S^o.

Fact 4.5.2 (*IC* **Localization Principle**) *Let $IC_S(L) \in P(Y)$, so that $S \subseteq Y$ is closed, and let $U \subseteq Y$ be open. Then $\mathcal{IC}_S(L)_{|U} = \mathcal{IC}_{S \cap U}(L_{|S^o \cap U})$.*

Fact 4.5.3 (*IC* **Normalization Principle**) *Let $\nu : \hat{Y} \to Y$ be the normalization of a variety. Then $\nu_*\mathcal{IC}_{\hat{Y}}(L) = \mathcal{IC}_Y(L)$ (here, ν is finite, so that $R\nu_* = \nu_*$, derived=underived).*

These first two principles hold because intersection complexes with coefficients are characterized by the strengthened conditions of support and by restricting to the locally constant sheaf on some Zariski dense open subset of the regular part, and both conditions are preserved under restriction to any open set and under a finite birational map. See also Exercise 4.7.10.

Fact 4.5.4 (**DT Localization Principle**) *A summand $\mathcal{IC}_S(L)$ appears in the decomposition theorem on Y iff there is an open $U \subseteq Y$ meeting S such that the restriction $\mathcal{IC}_S(L)_{|U}$ appears in the DT on U.*

[16] if is so for f proper of algebraic manifolds (trace map)

This last principle is very important. It fails, for example, for the map from the Hopf surface to $\mathbb{P}^1$ in the following sense: there is no decomposition theorem over $\mathbb{P}^1$ (else we would have E_2-degeneration of the LSS), but the Hopf map is locally trivial over any open proper subset $U \subseteq \mathbb{P}^1$, so that the decomposition theorem holds there (Künneth). It is important because when looking for summands in the decomposition theorem, it may be easier to detect them over some Zariski open subset. For example, if X is nonsingular, given $f : X \to Y$, there is the open subset $Y_{reg(f)} \subseteq Y$ of regular values of f. Let $\mathcal{R}^q$ be the locally constant sheaves given by the cohomology of the fibers of f over $Y_{reg(f)}$. Deligne's theorem applies to the map over $Y_{reg(f)}$. The reader can now observe that the DT localization principle allows us to deduce that all the $\mathcal{IC}_Y(R^q)[-q]$ are direct summands of $f_*\mathbb{Q}_X$.

The principle follows from the validity of the decomposition theorem on Y and on every U and from the fact that the summands of the decomposition theorem over U are uniquely determined (this is left as Exercise 4.7.7).

In the proof of Theorem 4.5.1 we shall also make use of a simple fact concerning topological coverings that we leave as Exercise 4.7.11

Proof of the Theorem 4.5.1

- WLOG, we may assume that $Y' := f(X) = Y$, i.e. that f is surjective.
- We have $f_*\mathcal{IC}_X \cong \oplus_{q\geq 0} \oplus_{(S,L)\in EV_q} \mathcal{IC}_S(L)[-q]$.
- By the two localization principles above, we can replace Y with any of its Zariski-dense open subsets.
- We may thus assume that there are no enriched proper subvarieties in the DT:
$$f_*\mathcal{IC}_X \cong \oplus_{q\geq 0}\mathcal{IC}_Y(L_q)[-q].$$
- By constructibility, and by further shrinking if necessary, may also assume that $\mathcal{IC}_Y(L_q) = L_q$ is locally constant and we get
$$Rf_*\mathcal{IC}_X \cong \bigoplus_{q\geq 0} R^q f_*\mathcal{IC}^q[-q] = \bigoplus_{q\geq 0} L_q[-q].$$
- We may also assume that $\mathcal{IC}_Y = \mathbb{Q}_Y$.
- WLOG, we may assume that X is normal. In fact, take the normalization $\nu : X' \to X$; by the IC Normalization Principle, we have $R\nu_*\mathcal{IC}_{X'} = \mathcal{IC}_X$; on the other hand, we have $R(f\circ\nu)_* = Rf_* \circ R\nu_*$ (ν is finite, so $R\nu_* = R^0\nu_*$; but we do not need this here).
- FACT: regardless of normality, there is always a natural map $\mathbb{Q}_X \to \mathcal{IC}_X$ in place. Since X is normal, this map induces $\mathbb{Q}_X \cong \mathcal{H}^0(\mathcal{IC}_X)$, i.e. we have a distinguished triangle $\mathbb{Q}_X \to \mathcal{IC}_X \to \tau_{\geq 1}\mathcal{IC}_X \xrightarrow{[1]}$. We push it forward and deduce
$$R^0 f_*\mathbb{Q}_X = R^0 f_*\mathcal{IC}_X.$$

- We are thus reduced to showing that, up to further-shrinkage, $\mathbb{Q}_Y$ is a direct summand of $L_0 = R^0 f_* \mathcal{IC}_X = R^0 f_* \mathbb{Q}_X$.

- Stein factorize $f := h_{finite} \circ g_{connected\, fibers} =: X \overset{g\, conn. fib.}{\longrightarrow} Y' \overset{h\, fin.}{\longrightarrow} Y$.

- Because of connected fibers, we have $R^0 g_* \mathbb{Q}_X = \mathbb{Q}_Y$. We are thus reduced to showing that, after shrinking, $\mathbb{Q}_Y$ is a direct summand of $R^0 h_* \mathbb{Q}_{Y'}$.

- Since h is finite, by shrinking the target if necessary, h becomes a covering map.

- The desired assertion is now a standard trace argument (Exercise 4.7.11). □

4.6 Pure Hodge structure in intersection cohomology

M. Saito's theory of mixed Hodge modules [47] endows the intersection cohomology groups of complex varieties with a mixed Hodge structure.

Let us use the intersection complex as a direct summand theorem 4.5.1 to endow the intersection cohomology of a complete surface with a pure Hodge structure. This is merely to illustrate the method, which works for any algebraic variety [18, 9, 10]. This special case is simple because the resolution is semismall, yet illuminating because its simplified set-up allows to focus on the main ideas without distractions.

Let Y be a complete surface. We are interested in $IH^*(Y, \mathbb{Q})$, so that we may assume that Y is normal, for the intersection cohomology groups do not change under normalization. In particular, Y has isolated singularities. Let S be the finite set of singular points. Pick a resolution of the singularities $f : X \to Y$ that leave Y_{reg} untouched. The decomposition theorem has the form $Rf_* \mathbb{Q}_X[2] = IC_Y \oplus \oplus_b V_S^b[-b]$, where V_S^b is a skyscraper sheaf at S. Since all fibers have dimension ≤ 1, we have $R^{q \geq 3} f_* \mathbb{Q}_X = 0$. It follows that $V_S^{b \geq 1} = 0$. By the symmetries of Verdier duality, we have that $V_S^{b \leq -1} = 0$ and we have $Rf_* \mathbb{Q}_X[2] \cong IC_Y \oplus V_\Sigma^0$. This is perverse and self-dual. The associated pairing yields the intersection pairing on $H^*(X, \mathbb{Q}) = IH^*(Y, \mathbb{Q}) \oplus V_S$. The two summands are orthogonal for this pairing (there are no maps $IC_Y \to V_S^0$!). The l.h.s. is a pure Hodge structure. The pairing is a map of pure Hodge structures. In order to conclude that $IH^*(Y, \mathbb{Q})$ is a pure Hodge substructure (our goal!), we need to show that $V_S^0 \subseteq H^2(X, \mathbb{Q})$ is a pure Hodge substructure. By the support conditions for IC, $\mathcal{H}^0(IC_Y) = 0$. It follows that $V_S^0 = R^2 f_* \mathbb{Q}_X = H^2(f^{-1}(S), \mathbb{Q})$, which is generated by the fundamental classes of the curve fibers, which are of (p, q)-type $(1, 1)$, i.e. they form a pure Hodge structure of weight two.

4.7 Exercises for Lecture 4

Exercise 4.7.1 (**Failure of local Poincaré duality; $\mathbb{Q}_Y$ not a direct summand**) Let Y be the affine cone over a nonsingular embedded projective curve of genus $g \geq 1$. Use the defining property ω_Y to show that $\omega_Y \neq \mathbb{Q}_Y[4]$ so that the usual local Poincaré duality fails. Take of of the usual resolutions $f: X \to Y$. Use the fundamental relation $(Rf_*C)^\vee = (Rf_*C)^\vee$ and deduce, by using the failure of Poincaré duality in neighborhoods of the vertex, that $\mathbb{Q}_Y$ is not a direct summand of $f_*\mathbb{Q}_X$.

Exercise 4.7.2 (**Goresky-MacPherson's estimate**) Let $f: X \to Y$ be proper with X nonsingular (or at least with $\mathcal{IC}_X = \mathbb{Q}_X$. Assume that $S \subseteq Y$ appears in the decomposition theorem for $Rf_*\mathbb{Q}_X[\dim X]$. Show that

$$\dim X - \dim S \leq 2 \dim f^{-1}(s), \quad \forall s \in S$$

Observe that equality implies that S appears only in perversity zero (i.e. with shift $b = 0$ only). Deduce from this that, for example, in the decomposition theorem for the mall resolution of the affine cone over $\mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$, the vertex does not contribute direct summands. What happens if the map f is of pure relative dimension 1?(Relate the answer to the number of irreducible components in the fibers and consider what kind of very special property the intersection complexes appearing in the decomposition theorem should enjoy). What happens if, in addition to having f of relative dimension one, Y is also nonsingular, or at least has $\mathcal{IC}_Y = \mathbb{Q}_Y$?

Exercise 4.7.3 (**Primitive Lefschetz decomposition**) Familiarize yourself with the primitive Lefschetz decomposition (PLD) associated with the hard Lefschetz theorem for compact Kähler manifolds. Observe that the same proof of PLD holds if you start with a graded object H^* in an abelian category with $H^b = 0$ for $|b| \gg 0$ and endowed with a degree two operator η satisfying $\eta^b : H^{-b} \cong H^b$ for every $b \geq 0$. Deduce the appropriate PLD for the graded object ${}^{\mathfrak{p}}\mathcal{H}^*$ arising from the relative hard Lefschetz theorem 4.3.1. State an appropriate version of the unimodality of the Betti numbers of Kähler manifolds in the context of the graded object ${}^{\mathfrak{p}}\mathcal{H}^* \in P(Y)$.

Exercise 4.7.4 (**Proofs of hard (vache (!), in French) Lefschetz**) Let $i: Y \to Y$ be a nonsingular hyperplane section. Use Poincaré duality and the slogan "cup product in cohomology = transverse intersection in homology, to show that we have commutative diagrams for every $r \geq 1$ (for $r = 1$ we get a triangle!)

$$\begin{array}{ccc} H^{\dim X - r}(X, \mathbb{Q}) & \xrightarrow{\eta^r} & H^{\dim X - r}(X, \mathbb{Q}) \\ \text{restriction} \Big\downarrow i^* & & i_! \Big\uparrow \text{Gysin} \\ H^{\dim Y - r + 1}(Y, \mathbb{Q}) & \xrightarrow{\eta_{|Y}^{r-1}} & H^{\dim Y + r - 1}(X, \mathbb{Q}) \end{array} \tag{27}$$

Assume the Hard Lefschetz for Y (induction). Use the Lefschetz hyperplane theorem and deduce the hard Lefschetz for X, but only for $r \geq 2$ (for $r = 0$ it is trivial). We have the commutative triangle with i^* injective and $i_!$ surjective

$$\begin{tikzcd} H^{\dim X - 1}(X, \mathbb{Q}) \arrow[rr, "\eta"] \arrow[dr, "i^*"', "\text{restriction}"] & & H^{\dim X + 1}(X, \mathbb{Q}) \\ & H^{\dim Y}(Y, \mathbb{Q}). \arrow[ur, "i_!"', "\text{Gysin}"] & \end{tikzcd} \tag{28}$$

Hard Lefschetz boils down to the statement that $\operatorname{Im} i^* \cap \operatorname{Ker} i_! = \{0\}$. Show that hard Lefschetz is equivalent to the statement: $(*)$ the non-degenerate intersection form on $H^{\dim Y}(Y, \mathbb{Q})$ stays non-degenerate when restricted to $i^* H^{\dim X - 1}(X, \mathbb{Q})$. At this point, we have two options. Option 1: use the Hodge-Riemann bilinear relations for Y: a class $i^* a \neq 0$ in the intersection would be primitive and the same would be true for its (p, q)-components; argue that we may assume a to be of type (p, q); the Hodge-Riemann relations would then imply $0 \neq \int_Y (i^* a)^2 = \int_X \eta \wedge a^2 = 0$, a contradiction. Option 2: put Y in a pencil $\widetilde{X} \to \mathbb{P}^1$ with smooth total space (blow up X); let Σ be the set of critical values of f; use the global invariant cycle theorem and the Deligne semisimplicity to show that the irreducible $\pi_1(\mathbb{P}^1 \setminus \Sigma)$-module $H^{\dim Y}(Y, \mathbb{Q})$ has $i^* H^{\dim Y}(X, \mathbb{Q})$ as its module of invariants. Conclude by first proving and then by using the following lemma (Deligne's Weil II, p.218): let V a completely reducible linear representation of a group π, endowed with a π-invariant and non-degenerate bilinear form Φ; then the restriction of Φ to the invariants V^π is non degenerate.

Exercise 4.7.5 **(Four symmetries)** Let $f : X \to Y$ be a projective map of projective varieties of with X nonsingular and Y of dimension 4. Assume that the direct image perverse cohomology sheaves ${}^{\mathfrak{p}}\mathcal{H}^b$ of $Rf_* \mathbb{Q}_X[\dim X]]$ live in the interval $[-2, 2]$ and that the enriched varieties appearing in the decomposition theorem are supported at set of finite points S^0, curves S^1 and surfaces S^2 on Y. Denote the b-th graded spaces of the perverse filtration on $H^*(X, \mathbb{Q})$ as $H^*_b(X, \mathbb{Q})$. Write out the decompositions $H^*_b(X, \mathbb{Q}) = \oplus_k H^*_{b, S^k}(X, \mathbb{Q})$. List the four kind of symmetries among the various $H^*_{b, S^k}(X, \mathbb{Q})$: Verdier Duality and Relative Hard Lefschetz for b and $-b$; Verdier duality and Hard Lefschetz in intersection cohomology for the for the same b. (See [13], §2.4.)

Exercise 4.7.6 **(Perverse cohomology sheaves and the decomposition theorem)** Show that if $P \in P(Y)$, then $P = {}^{\mathfrak{p}}\mathcal{H}^0(P)$. Show that if $C = \bigoplus P_b[-b]$ with $P_b \in P(Y)$, then ${}^{\mathfrak{p}}\mathcal{H}^b(C) = P_b$. Let $j : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}^n$ be the natural open embedding. Compute ${}^{\mathfrak{p}}\mathcal{H}^i(Rj_! \mathbb{Q})$ and ${}^{\mathfrak{p}}\mathcal{H}^i(Rj_* \mathbb{Q})$. Let $j : U \to X$ be an open embedding and let $P \in P(U)$. Deduce that in the decomposition theorem for $C := Rf_* IC_S(L)$, we have that ${}^{\mathfrak{p}}\mathcal{H}^b(C) = \oplus_{EV_b} IC_S(L)$, whereas in the one for $K := Rf_* \mathcal{IC}_S(L)$ we have a less simple expression (involving the dimensions of the varieties S).

Exercise 4.7.7 (**The summands in the DT are uniquely determined**) Prove that the direct summands in the DT are uniquely determined. How non-unique is the isomorphism in the statement of the decomposition theorem?

Exercise 4.7.8 (**No extra supports**) Let $f : X \to Y$ be a proper map with X a nonsingular surface and Y a curve. Use the proof of Theorem 4.2.1 to show that if all fibers are irreducible, then the enriched varieties appearing in the decomposition theorem for $Rf_*\mathbb{Q}_X$ are supported on Y.

Exercise 4.7.9 ($\mathcal{IC} = \mathbb{Q}$ **for** $\mathbb{C}^n/G$) Let G be a finite group acting on a complex vector space V and let $f : X := V \to Y := V/G$ be the resulting finite quotient map. Show that the natural map $\mathbb{Q}_Y \to R^0 f_*\mathbb{Q}_X$ splits. Deduce that $\mathbb{Q}_Y[\dim Y]$ is Verdier self-dual. Deduce that $\mathbb{Q}_Y[\dim Y]$ is perverse. Show that moreover, it satisfies the conditions of support that characterize the intersection complex $\mathcal{IC}_Y$.

Exercise 4.7.10 (IC **stable under finite maps**) Prove that the direct image of an intersection complex under a finite map is an intersection complex.

Exercise 4.7.11 (**Coverings traces**) Let $f : X \to Y$ be a proper submersion of fiber dimension zero (finite topological covering). Prove (without using the fancy semisimplicity results seen above, that $L := f_*\mathbb{Q}_X$ is a semisimple locally constant sheaf admitting $\mathbb{Q}_Y$ as a direct summand (trace map). If the covering is normal (a.k.a. Galois), use the language of representations of the fundamental group to reach the same conclusion.

5 The perverse filtration

Summary of Lecture 5. The classical "topologists" Leray spectral sequence for fiber bundles. Grothendieck Spectral sequence, Leray as a special case. Verdier's spectral objects. Geometric description of the perverse Leray filtration. Relation to the topologists' point of view. Hodge-theoretic applications. The P=W theorem and conjecture in non abelian Hodge theory. A sample perversity calculation. A motivic question on the projectors that can be associated with the decomposition theorem.

5.1 The perverse spectral sequence and the perverse filtration

It is important to keep in mind that given $f : X \to Y$, the Leray filtration L on $H^*(X, \mathbb{Q})$ is defined a priori, independently of the Leray spectral sequence. The latter is machinery that tells you something about the graded pieces $Gr^L_i H^d(X, \mathbb{Q})$ of the Leray filtration, i.e. $Gr^L_i H^d(X, \mathbb{Q}) = E^{d-i,i}_\infty$.

The Leray spectral sequence for a fiber bundle. You can consult Spanier's or Hatcher's algebraic topology textbooks Let $p : E \to B$ be a topological fiber bundle with fiber F. Assume you are given a cell complex structure $B_\bullet$ on B: in short, we have the p-th skeleta B_p, $B_p \setminus B_{p-1}$ is a disjoint union of p-cells, $H^r(B_p, B_{p-1}) = \widetilde{H}^r(\text{Bouquet of } p \text{ spheres}) \cong \delta_{rp}\mathbb{Q}^\#$ (we call this the cellularity condition), etc. The cohomology of the complex $H^p(B_p, B_{p-1})$ with differential given by consideration of the coboundary operators in the long exact sequence of the triples (B_p, B_{p-1}, B_{p-2}), computes $H^*(B, \mathbb{Q})$. In fact, there is a spectral sequence $E^{s,t}_1 = H^{s+t}(B_s, B_{s-1})$, but the cellularity condition reduces the spectral sequence to a complex. The same kind of spectral sequence for the pre-images $E_\bullet$ of $B_\bullet$ reads $E^{st}_1 = H^{s+t}(E_s, E_{s-1})$ and it does not reduce to a complex. The bundle structure and the cellularity condition tell us that $E^{pq}_1 = H^p(B_p, B_{p-1}) \otimes H^q(F)$. One then argues that $E^{pq}_2 = H^p(B, \mathcal{R}^q)$. This is probably close in spirit to the original way of viewing the Leray spectral sequence for a fiber bundle. The increasing Leray filtration on the cohomology of the total space is given by the kernel of the restriction maps to pre images of the skeleta $\mathrm{Ker} H^?(E, \mathbb{Q}) \to H^?(E_?, \mathbb{Q})$. Let us not worry about indexing schemes.

Grothendieck's Leray spectral sequence. You can consult Grothendieck "Tohoku" paper. Grothendieck gave a sheaf-theoretic approach to this picture: start with a complex of sheaves C on Y; take a Cartan-Eilenberg resolution for C, i.e. an injective resolution $C \to I$ that "is" also an injective resolution for the truncated complexes $\tau_{\leq i}C$ and for the cohomology sheaves $\mathcal{H}^i(C)$; the complex of global sections $\Gamma(Y, I)$ if filtered; the Grothendieck spectral sequence is the spectral sequence for this filtered complex, and it abuts to the standard (Grothendieck) filtration given by $\mathrm{Im}\, H^*(Y, \tau_{\leq i}C) \subseteq H^*(Y, C)$. Given $f : X \to Y$ and $C \in D(X)$, the (Grothendieck-)Leray spectral sequence is the Grothendieck spectral sequence for Rf_*C, and the (Grothendieck)-Leray filtration is the standard filtration for Rf_*C.

Perverse and perverse Leray spectral sequences and filtrations. The main input leading to the machinery above, is the use of injective resolutions together with the system of standard truncation maps. If we replace the standard truncation with the perverse truncation maps, we obtain the perverse and perverse Leray spectral sequences and the perverse and perverse Leray filtrations. At the end, given $C \in D(Y)$, the perverse filtration is given by setting $\mathscr{P}_b H^*(Y,C) := \operatorname{Im} H^*(Y, {}^{\mathfrak{p}}\tau_{\leq b} C) \subseteq H^*(Y,C)$. Similarly for the perverse Leray filtration $\mathscr{P}_b H^*(X, Rf_*K) := \operatorname{Im} H^*(Y, {}^{\mathfrak{p}}\tau_{\leq b} Rf_*K) \subseteq H^*(Y, Rf_*K) = H^*(X,K)$. In fact, every t-structure on $D(Y)$, and there are many different ones!, gives rise to the same kind of picture outlined above. The Grothendieck and Leray filtrations correspond to the standard t-structure on $D(Y)$.

Verdier's spectral objects. There is at least another convenient way to view these mechanisms, one that, once you are given a cohomological functor, avoids injective resolutions, namely Verdier's spectral objects [29]: the input is a cohomological functor on a t-category; the output is a spectral sequence abutting to the filtration defined in cohomology by the t-truncations. Even if not logically necessary, this is a useful tool when there are no injectives (e.g. the category of perverse sheaves does not have enough injectives), and it is a , yet another!, cool way to look at spectral sequences.

Why bother with the perverse Leray filtration? Because in the context of the decomposition theorem $C := Rf_* IC_X(M) \cong \oplus_b \oplus_{EV_b} IC_S(L)[-b] = \oplus_b {}^{\mathfrak{p}}\mathcal{H}^b(C)[-b]$, the perverse Leray spectral sequence is E_2-degenerate, the graded pieces of the perverse Leray filtration are the cohomology groups $H^*(Y, {}^{\mathfrak{p}}\mathcal{H}^b(C))$ the cohomological decomposition theorem gives splittings of the perverse Leray filtration. In particular, every cohomology class in $IH^*(X,M)$ has b-components, which split further into EV_b-components. In particular, if you know that the perverse filtration has some property, e.g. it carries a Hodge structure, then the graded b-pieces inherit such a structure as well, and, maybe, so will the individual EV_b-pieces. See Corollary 5.3.1 and Remark 5.3.2

Scheleta in algebraic geometry? Let us go back to Leray for fiber bundles. In that topological context, it is natural to work with a cell complex structure. We can do that with smooth projective maps $f: X \to Y$ in complex algebraic geometry (varieties can be triangulated), but the skeleta will not be algebraic subvarieties. It is hard to predict the properties of the Leray filtration if it is described as a kernel of a restriction map to some closed subspace that is not a subvariety. We may say that in the context of topological fiber bundles, the Leray filtration is described geometrically (using the geometry at hand) via the kernels of the pull-back maps to pre-images of skeleta. In the context of maps of complex algebraic varieties, we are interested in a geometric description of the perverse and perverse Leray filtrations, but the skeleta of a cell-decomposition do not seem to be immediately helpful.

5.2 Geometric description of the perverse filtration

Question 5.2.1 *Can we describe the perverse filtration $\mathscr{P}_b H^*(Y,C)$ geometrically?*

Let us approach this problem in the special case that is reminiscent to the decomposition theorem, i.e. let us assume that $C = \oplus_b P_b[-b] \in D(Y)$ with $P_b \in P(Y)$. In this case, we have that the perverse filtration is comes to us already canonically split:

$$\mathscr{P}_b H^*(Y,C) = \mathrm{Im}\left(\oplus_{b' \leq b} H^{*-b}(Y,P_b)\right) \subseteq H^*(Y,C). \tag{29}$$

It is a good place to remark that the decomposition theorem asserts the existence of a direct sum decomposition, not that one can find a natural one. In general, there is no such thing. A relatively ample line bundle provides you with the possibility of choosing some distinguished splittings; see [29] and [10].

Exercise 5.6.1 proves half of the following fact: let P be a perverse sheaf on a quasi projective variety Y and let $Y_\bullet$ be a general flag of linear sections of Y for some embedding in projective space; here $Y_k \subseteq Y$ has codimension k in Y; then $P_{|Y_k}[-k] \in P(Y_k)$.

Exercise 5.6.2 first asks you to apply repeatedly the Lefschetz hyperplane theorem for perverse sheaves to the elements of the flag $Y_\bullet$ to show that the restriction maps $H^*(Y,P) \to H^*(Y_k, P_{|Y_k})$ are injective for every $* \leq -k$. Next, it asks you to specialize the situation to the case when Y is affine, to use Artin vanishing theorem and deduce that, for Y affine, the restriction maps $H^*(Y,P) \to H^*(Y_k, P_{|Y_k})$ maps are zero for $* > -k$.

We conclude that, when $C = \oplus_b P_b[-b] \in D(Y)$, with $P_b \in P(Y)$ and Y is affine, we have a geometric description of the perverse filtration

$$\mathrm{Ker}(H^*(Y,C) \to H^*(Y_k, C_{|Y_k})) = \mathscr{P}_{*+k-1} H^*(Y,C). \tag{30}$$

In other words, we for cohomological purposes, we may consider the $Y_\bullet$ as the skeleta of a "cell" decomposition; the term cells now refer to the fact that the relative cohomology groups $H^*(Y_k, Y_{k+1}, P)$ are non zero in at most one cohomological degree, which is reminiscent of the analogous fact for cell complexes (vanishing for bouquet of spheres).

By renumbering 30, we get

$$\mathscr{P}_b H^*(Y,C) = \mathrm{Ker}(H^*(Y,C) \to H^*(Y_{b-*+1}, C_{|Y_{b-*+1}})). \tag{31}$$

What if $C \in D(Y)$ is not split and Y is not affine?

If Y is affine, then exact same description, but with a different proof, remains valid for every $C \in D(Y)$.

If Y is quasi projective, then we can use the Jouanolou trick (Exercise 5.6.3), to reduce to the case to the affine situation; see [9]. The use of this trick is not necessary and one can work directly on Y, but has to use a general pair of flags coming from a suitable embedding in projective space.

Let us state the end result for the perverse Leray filtration in the special, but key case of a map to an affine variety. Note that the map needs not to be proper and that it applies to every complex, not just one whose direct image splits as above.

Theorem 5.2.2 (**Geometric description of the perverse Leray filtration** [19]) *Let $f: X \to Y$ be a map of varieties with Y affine and let $K \in D(X)$. Then there is a flag $Y_\bullet \subseteq Y$, with pre-image flag $X_\bullet \subseteq X$, such that*

$$\mathscr{P}_b H^*(X,K) = \mathrm{Ker}\left\{(H^*(X,K) \to H^*(X_{b-*+1}, K_{|X_{b-*+1}})\right\}.$$

5.3 Hodge-theoretic consequences

Here is a corollary that exemplifies the utility of having a geometric description of the perverse filtration as the kernels of restriction maps.

Corollary 5.3.1 (**Perverse Leray and MHS**) *Let $f : X \to Y$ be a map of varieties. Then the subspaces $\mathscr{P}_b \subseteq H^*(X, \mathbb{Q})$ of the perverse Leray filtration on the cohomology of the domain are mixed Hodge substructures. In particular, the graded pieces carry a natural mixed Hodge structure.*

Remark 5.3.2 (**Streamlined Hodge-theoretic proof of the decomposition theorem**) Corollary 5.3.1 can be used as the basis for a streamlined Hodge-theoretic proof of the decomposition and allied results for the push-forward of the intersection complex of a variety. This would shorten considerably the proofs in [13].

5.4 Character variety and Higgs moduli: $P = W$

There is a version of the story that follows for any complex reductive group; one can even consider higher dimensional projective manifolds instead of just curves. However, we stick with $GL/SL/PGL(2, \mathbb{C})$ and curves.

Fix a smooth projective curve X of genus $g \geq 2$ a point $x \in X$.

Character variety M_B. Let M_B be the moduli space of irreducible representations of $\pi_1(X \setminus x) \to GL(2, \mathbb{C})$ subject to the condition that a small loop circuiting x maps to $-Id$. This is a nonsingular affine variety of dimension $2a := 8g - 6$.

Higgs moduli space and Hitchin map. Let M_D be the moduli space of stable rank two and degree one Higgs bundles (E, ϕ) on X, where ϕ is a one form with coefficients in $End(E)$. This is a nonsingular quasi projective variety of the same dimension $2a = 8g - 6$. It is quasi projective, neither affine, nor quasi projective: it carries the projective Hitchin map $h : M_D \to A \cong \mathbb{C}^{a=4g-3}$, $(E, \phi) \mapsto (\mathrm{trace}(\phi), \det(\phi))$ (sections of T^*X and of its tensor square), and the general fiber is an abelian variety of dimension $4g - 3$.

Non abelian Hodge theorem. Part of the non abelian Hodge theorem states that these two two varieties are diffeomorphic. They are not biholomorphic. There is also a third moduli space in the picture, related to flat connections; but we stick to our limited set-up above.

Mixed Hodge structures on $H^*(M_D)$ and $H^*(M_B)$. The mixed Hodge theory of both sides is relatively well-understood: the one for M_D is pure (Exercise 5.6.5), so that the weight filtration W_D on $H^*(M_D, \mathbb{Q})$ is the filtration by cohomological degree: $0 = W_{D,d-1}H^d \subsetneq W_{D,d}H^d = H^d$, or $Gr^{W_D}_d H^d = H^d$. The mixed Hodge structure on $H^*(M_B)$ is way more interesting: the odd graded pieces $Gr^{W_B}_{odd} H^d = 0$, while the even ones $Gr^{W_B}_{2b} H^d$ are of pure type (b, b) (Hodge-Tate) and the non-purity, here, refers to the fact that we do have degrees d for which $H^d = \oplus_b Gr^{W_B}_{2b} H^d$ with more than one non trivial summand; the non trivial graded pieces live in the interval $[0, 4a]$.

W_B and W_D do not match. By what above, it is clear that the weight filtrations W_B (mixedness) and W_D (purity) do not correspond under the diffeomorphism $M_B \cong M_D$.

The curious hard Lefschetz phenomenon on $H^*(M_B, \mathbb{Q})$). There is a distinguished cohomology class $\alpha \in H^2(M, \mathbb{Q})$ which is linked to a curious phenomenon concerning the M_B-side, i.e. there is a sort of hard Lefschetz statement of the form:

$$\alpha^k \cup - : Gr^{W_B}_{2a-2b} H^*(M_B) \xrightarrow{\cong} Gr^{W_B}_{2a+2b} H^{*+2b}(M_B). \tag{32}$$

It is called the curious hard Lefschetz because it looks like a hard Lefschetz-kind of statement, but remember that M_B is affine and that α_B is not even a $(1,1)$ class!

Question 5.4.1 *Via the non abelian Hodge theorem isomorphism $H^*(M_B, \mathbb{Q}) \cong H^*(M_D, \mathbb{Q})$, what corresponds to W_B together with its curious hard Lefschetz, on the $H^*(M_D, \mathbb{Q})$-side?*

To answer this question, we first normalize the perverse Leray filtration $\mathscr{P}$ on $H^*(M_D, \mathbb{Q})$ for the Hitchin map and so that its graded pieces are trivial outside the interval $[0, 2a]$. We denote the result by $\mathscr{P}_D$. This means that instead of working with $Rf_*\mathbb{Q}_{M_D}$, we work with $Rf_*\mathbb{Q}_{M_D}[a]$ (Exercise 5.6.6).

Recall that the analogous interval for the weight filtration W_B is $[0, 4a]$ and that the odd graded pieces are trivial. This makes the following answer to Question 5.4.1 "numerically" plausible: intervals match by halving, in view of $[0, 2a]$ and $[0, 4a]$ and $Gr^{W_B}_{odd} = 0$. On the CHL/RHL-side, the answer is also made plausible by the fact that the class α is ample on the fibers of the Hitchin map, so that indeed it gives rise to a RHL.

Theorem 5.4.2 (**P=W** [20]) *For $G = GL/SL/PGL(2, \mathbb{C})$, via the non abelian Hodge theorem $M_B \cong M_D$, we have:*

$$W_{B,2b} \longleftrightarrow \mathscr{P}_{D,b} \quad \forall b, \qquad CHL \longleftrightarrow RHL.$$

Of course, even if numerically plausible, the fact that the subspaces of the filtrations match and that CHL turns into RHL seems striking to some of us.

The proof of Theorem 5.4.2 makes an essential use of the geometric description of the perverse filtration for the Hitchin map based on Theorem 5.2.2: generators and relations for the cohomology ring $H^*(M_B, \mathbb{Q})$ are known (Hausel-Thaddeus, Hausel-Rodriguez Villegas); the generators are of pure type (p, p), hence live in $W_{B,2p}$; every cohomology class is a sum of monomials in these generators; such monomials have type which is the sum of the types of the factors and their level in W_B is the sum of the levels of the factors; the proof then hinges on the verification that all monomials of level $2b$ in W_B live in $\mathscr{P}_{D,b}$; in turn this follows by verifying that the generators have this property and, critically, that $\mathscr{P}_D$ is multiplicative.

In the $GL_2/SL_2/PGL_2$-case, it is not hard to verify that the generators have the required property. The heart of the proof of the P=W Theorem 5.4.2 in [20] consists of showing that the perverse Leray filtration for the Hitchin map is multiplicative with respect to the cup product. This is automatic for the Leray filtration of any map, but fails in general for the perverse Leray filtration (Exercise 5.6.7).

Let us illustrate the use of Theorem 5.2.2 with a calculation whose result tells us that a certain generator, let us call it $\beta \in H^4(M, \mathbb{Q})$, of type $(2,2)$ in $W_{B,4}H^4(M_B, \mathbb{Q})$, in fact lies in $\mathscr{P}_{D,2}H^4(M_D, \mathbb{Q})$. See [20], §3.1.

By keeping in mind the normalization above of the perverse filtration, the geometric description of the perverse filtration Theorem 5.2.2 requires us to verify that the class β vanishes over the pre-image of a generic affine line in $A \cong \mathbb{C}^a$ (end of Exercise 5.6.6).

The class $\beta \in H^4(M_D, \mathbb{Q})$ is known to be a multiple of the second Chern class of the tangent bundle of M_D. Since the generic fiber is an abelian variety, it is clear that β vanishes over the pre-image of a generic point. More is true: every linear function on A gives rise to a Hamiltonian vector field tangent to the fibers of the Hitchin map; since the tangent bundle of $M_{D,reg}$ (pre-image of regular values A_{reg} of h) is an extension of the pull-back of the (trivial) tangent bundle of A_{reg} by the relative tangent bundle (also trivialized by the Hamiltonian vector fields above), we have that in fact β is trivial on $M_{D,reg}$.

Ngô's striking support theorem [44] tells us that there is a Zariski dense open set $A^{ell} \subseteq A$ with closed complement of codimension $> g-2$ such that the decomposition theorem over A^{ell} is of the form $Rh^{ell}_* \mathbb{Q} \cong \oplus_{q \geq 0} \mathcal{IC}_{A^{ell}}(R^q)[-q]$. A generic line will avoid the small closed complement (at least if $g \geq 3$; $g = 2$ can be dealt with separately), where R^q are the locally constant direct image sheaves over the regular part.

Pick a generic line Λ and observe that the decomposition theorem for the Hitchin map restricted over the line reads:

$$Rh^{\Lambda}_* \mathbb{Q} \cong \bigoplus_{q \geq 0} \mathcal{IC}_{\Lambda}(R^q)[-q];$$

this is because the restriction of an (un-shifted) intersection complex $\mathcal{IC}$ to a general linear section is an (un-shifted) intersection complex.

Let $j : \Lambda_{reg} \to \Lambda$ be the open immersion of the set of regular values of h^{Λ}. Since Λ is a nonsingular curve, we know that

$$Rh^{\Lambda}_* \mathbb{Q} \cong \bigoplus_{q \geq 0} j_* R^q[-q];$$

this is because intersection complexes on nonsingular curves are obtained via the ordinary sheaf-theoretic push-forward (Fact 1.7.19).

Note that our perverse sheaves are now just sheaves (up to shift). It follows that the perverse spectral sequence for h^{Λ} is just the ordinary Leray spectral sequence and the same holds for map $h^{\Lambda_{reg}}$ over the set of regular values of h^{Λ}.

By the functoriality of the Leray spectral sequence and by Artin vanishing on the affine curves Λ and Λ_{reg} ($H^{>1} = 0!$), we have a commutative diagram of short exact sequences

(the edge sequences for the Leray spectral sequences for the maps h)

$$\begin{tikzcd} 0 \arrow[r] & H^1(\Lambda, j_* R^3) \arrow[r] \arrow[d] & H^4(M_\Lambda) \arrow[r] \arrow[d] & H^0(A, j_* R^4) \arrow[r] \arrow[d, "="] & 0 \\ 0 \arrow[r] & H^1(\Lambda_{reg}, R^3) \arrow[r] & H^4(M_{\Lambda_{reg}}) \arrow[r] & H^0(R^4) \arrow[r] & 0, \end{tikzcd} \tag{33}$$

where the first vertical map is injective (edge sequence for the Leray spectral sequence for the map j), and the third is an isomorphism (definition of direct image sheaf).

A simple diagram chase, tells us that the vertical restriction map in the middle of (33) is also injective.

On the other hand, the class $\beta_{|M_\Lambda} \mapsto \beta_{|M_{\Lambda_{reg}}} = 0$ by what seen earlier (β restricts to zero over A_{reg}, hence over Λ_{reg}).

By the injectivity statement above, we see that β vanishes over the generic line and we deduce that $\beta \in \mathscr{P}_{D,2} H^4(M_D, \mathbb{Q})$, as predicted by $P = W$.

Question 5.4.3 *We can formulate $P = W$ for every complex reductive group. Does it hold, at least for GL_n?*

There are indications that this should be ok for GL_n, n small.

5.5 Let us conclude with a motivic question

Let $f : X \to Y$ be a projective map of projective varieties with X nonsingular. By the decomposition decomposition theorem, there is an isomorphism

$$\phi : H^*(X, \mathbb{Q}) \cong \oplus_{q, \mathcal{EV}_q} IH^{*-q}(S, L). \tag{34}$$

This implies that for each (S, L) in $\mathcal{EV}_q$ we obtain a projector (map that squares to itself) on $H^*(X, \mathbb{Q})$ with image $\phi(IH^{*-q}(S, L))$. We view this projector as a cohomology class $\pi_\phi := H^{2\dim X}(X \times X, \mathbb{Q})$.

It is possible to endow each term on the r.h.s. of (34) with a natural pure Hodge structure and then to choose an isomorphism ϕ (34) that is an isomorphism of pure Hodge structures. This implies that π_ϕ is rational and of (p, q)-type $(\dim X, \dim X)$, i.e. it is a Hodge class.

According to the Hodge conjecture, π_ϕ should be algebraic (cohomology class of an algebraic cycle in $X \times X$).

Question 5.5.1 (**Motivic decomposition theorem**) *Can we chose ϕ so that the resulting projectors π_ϕ are given by algebraic cycles?*

The answer is positive for semismall maps (Exercise 3.6.8). We have no idea if/why this should be true. We can prove something much weaker: the projectors are absolute Hodge (in the sense of Deligne), even motivated (in the sense of André); see [22].

If it were true, then, by applying this to the blowing up of the projective cone over an embedded projective manifold, it would imply the Grothendieck standard conjecture of Lefschetz type (the inverse to the Hard Lefschetz isomorphisms are induced by algebraic cycles in the product).

5.6 Exercises for Lecture 5

Exercise 5.6.1 (**Restricting perverse sheaves to general linear sections**) Let Y be quasi projective and $P \in P(Y)$. Show that if $i : Y_k \to Y$ is a codimension k general linear section of Y relative to any fixed embedding in $Y \to \mathbb{P}^N$, then $P_{|Y_k}[-k] \in P(Y_k)$. To do so first verify directly that $P_{|Y_k}[-k]$ satisfies the conditions of support (use the Bertini theorem to cut down the supports of cohomology sheaves). It not so trivial to verify the conditions of co-support, i.e. the conditions of support for the dual of $P_{|Y_k}[-k]$. Here, we simply say that we can choose Y_k general, depending on P, so that $i^! = i^*[-2k]$ and that the desired conditions follow formally from this and from the duality exchange property: $(i^*(P[-k]))^\vee = i^! P^\vee[k] = i^* P^\vee[-k]$.

Exercise 5.6.2 (**Restriction to general linear sections maps in cohomology**) Let Y be quasi projective , let $P \in D(Y)$ and let Y_k be the complete intersection of k general linear sections of Y relative to any embedding of Y in some projective space. Use the Lefschetz hyperplane theorem to show that the restriction maps $H^*(Y, P) \to H^*(Y_k, P_{|Y_k})$ are injective for every $* \leq -k$. Assume in addition that Y is affine. Use the Artin vanishing theorem for perverse sheaves and Exercise 5.6.1 to show that the same restriction maps are zero for $* > -k$.

Exercise 5.6.3 (**Jouanolou trick**) Let Y be quasi projective. There is a map $p : \mathcal{Y} \to Y$ with $\mathcal{Y}$ affine and that is a Zariski locally trivial bundle with fiber affine spaces $\mathbb{A}^m$. There are various different ways to do this; following is a series of hits to achieve the goal. Choose a projective completion $Y \subseteq \overline{Y}$ so that the embedding is affine (e.g. with boundary a divisor). Argue that we may assume that Y is projective. Pick a suitably positive rank $\dim Y + 1$ vector bundle E on Y, where suitably positive $:= \mathcal{O}_{\mathbb{P}(E)}(1)$ is very ample and has a section s that is not identically zero on any projective fiber. Show that if you take $\mathbb{P}^1 \times \mathbb{P}^1 \setminus \Delta \to \mathbb{P}^1$ you land in a special case of the construction above. Show that $\mathcal{Y} := \mathbb{P}(E) \setminus (s = 0)$ does the job. The usefulness of this trick in our situation is that if we start with $f : X \to Y$, Y quasi projective, we can base change to $g : \mathcal{X} \to \mathcal{Y}$ so that now the target is affine and the properties of p (smooth map with "contractible fibers) allow us to prove the assertions on f by first proving them for g and then "descending" them to f.

Exercise 5.6.4 (**Leray and perverse Leray**) In this exercise use the following: if $j : S^o \to S$ is an open embedding of nonsingular curves and L is alocally constant sheaf on S^o, then $\mathcal{IC}_S(L) = R^0 j_* L$. Let $p : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ be a projection and let $b : X \to \mathbb{P}^1 \times \mathbb{P}^1$ be the blowing up at a point. Let $f := p \circ b$. Determine and compare the Perverse Leray and the Leray filtrations for f on $H^*(X, \mathbb{Q})$. (Renumber the perverse Leray one so that $1 \in \mathscr{P}_0 \setminus \mathscr{P}_{-1}$; this way they both "start" at the same "time".) Do the same thing, but for a Lefschetz pencil of plane curves. Note how the graded spaces for the two filtrations differ in the "middle".

Exercise 5.6.5 (**$H^*(M_D)$ is pure**). There is a natural $\mathbb{C}^*$-action on M_D obtained by multiplying the Higgs field ϕ by a scalar. Show that the Hitchin map is $\mathbb{C}^*$-equivariant. Use the $\mathbb{C}^*$-action and the properness of the Hitchin map to show that the closed embedding of the fiber $h^{-1}(0) \to M_D$ induces an isomorphism in cohomology. Use the weight inequalities listed at the end of our quick review of mixed Hodge theory in §1.7, to deduce the $H^j(M_D)$ is pure of weight j for every j.

Exercise 5.6.6 (**Normalizing the perverse filtration**) Assume that $C = \oplus_b P_b[-b] \in D(Y)$ with $P_b \in P(Y)$. Show that $\mathscr{P}_b H^*(Y, C[m])) = \mathscr{P}_{b+m} H^{*+m}(Y, C)$ (in fact, this is true in general, i.e. without assuming that C splits). Deduce that if $H^*(Y, C) \neq 0$, then $\exists!\, m \in \mathbb{Z}$ such that $Gr^{\mathscr{P}}_{b<0} H^*(Y, C[m]) = 0$ and $Gr^{\mathscr{P}}_0 H^*(Y, C[m]) \neq 0$. The Hitchin map is of pure relative dimension a with a nonsingular domain; use these two facts, and the decomposition theorem, to deduce that if $C = Rf_* \mathbb{Q}_{M_D}$, then the $m = a$, i.e. the perverse Leray filtration for $H^*(A, Rf_*\mathbb{Q}_{M_D}[a])$ "starts" at level zero, and that it "ends" at level $2a$ (trivial graded pieces after that). Reality check: verify that a class lives in $\mathscr{P}_{D,2} H^4(M_D, \mathbb{Q})$ iff it restricts to zero over a general line $\Lambda^1 \subseteq A^a$.

Exercise 5.6.7 (**The perverse filtration is not multiplicative in general**) Let $X' = S \times C$ (surface times curve, both projective and nonsingular), let X be the blowing up of a point in X' and let $f : X \to C$ be the natural map (blow-down followed by projection. This is a flat map of relative dimension 2. Let $\mathscr{P}$ be the perverse Leray filtration on $H^*(X, \mathbb{Q}) = H^{*-2}(C, Rf_*\mathbb{Q}_X[2])$. Verify that it lives in the interval $[0, 4]$. Verify that the class e of the exceptional divisor lies in $\mathscr{P}_1 \setminus \mathscr{P}_0$ and that e^2 lives in $\mathscr{P}_3 \setminus \mathscr{P}_2$. Deduce that the perverse filtration is not multiplicative in general (multiplicative:= $\mathscr{P}_i \cup \mathscr{P}_j \to \mathscr{P}_{i+j}$).

Exercise 5.6.8 (**When Question 5.5.1 has an easy answer**) List some classes of proper maps $f : X \to Y$ such that Question 5.5.1 has an affirmative answer.

References

[1] A.A. Beilinson, J.N. Bernstein, P. Deligne, *Faisceaux pervers*, Astérisque **100**, Paris, Soc. Math. Fr. 1982. 6, 7, 9, 12, 19, 24, 26, 41

[2] A.A. Beilinson, "On the derived category of perverse sheaves," *K-theory, arithmetic and geometry (Moscow, 1984–1986)*, pp.27–41, Lecture Notes in Math., 1289, Springer 1987. 25

[3] J. Bernstein, V. Lunts, *Equivariant sheaves and functors,* Lecture Notes in Mathematics, **1578**. Springer-Verlag, Berlin, 1994. 3

[4] A. Borel et al., *Intersection Cohomology*, Progress in Mathematics Vol. **50**, Birkhäuser, Boston Basel Stuttgart 1984. 21

[5] N. Chriss, V. Ginzburg, *Representations theory and Complex Geometry,* Birkhäuser, Boston, 1997. 30, 34, 35

[6] C.H. Clemens, "Degeneration of Kähler manifolds," Duke Math.J. **44** (1977), 215-290. 7

[7] M.A. de Cataldo, *The Hodge Theory of projective manifolds,* Imperial College Press, London, 2007. 3

[8] M.A. de Cataldo, "The standard filtration on cohomology with compact supports with an appendix on the base change map and the Lefschetz hyperplane theorem," Contemporary Mathematics, 496 (2009), Interactions of Classical and Numerical Algebraic Geometry, Bates et al. Eds., 199-220.

[9] M.A. de Cataldo, "The perverse filtration and the Lefschetz Hyperplane Theorem, II," J. Algebraic Geometry 21 (2012) 305-345. 45, 51

[10] M.A. de Cataldo, "Hodge-theoretic splitting mechanisms for projective maps (in appendix, a letter from P. Deligne)," J. Singul. 7 (2013), 134-156. 45, 51

[11] M.A. de Cataldo, L. Migliorini, "The Hard Lefschetz Theorem and the topology of semismall maps," Ann.Scient.Ec.Norm.Su., 4^e série, t.35, 2002, 759-772. 30, 31

[12] M. de Cataldo, L. Migliorini, "The Chow motive of semismall resolutions," Math.Res.Lett. **11** (2004), 151-170. 34, 35

[13] M.A. de Cataldo, L. Migliorini, "The Hodge Theory of Algebraic maps," Ann. Scient. Éc. Norm. Sup., 4^e série, t. **38**, (2005), 693-750. 9, 19, 31, 40, 41, 47, 52

[14] M. de Cataldo, L. Migliorini, "The Hodge Theory of Algebraic maps," arXiv:math.AG/0306030v1 (2003).

[15] M. de Cataldo, L. Migliorini, "Intersection forms, algebraic maps and motivic decomposition for resolution of threefolds," in *Algebraic Cycles and Motives,* London Math.Soc. Lecture Note Series, n.343, vol.1, pp.102-137, Cambridge University Press, Cambridge, UK, 2007. 31

[16] M. de Cataldo, L. Migliorini, "The Decomposition Theorem and the topology of algebraic maps," Bulletin of the A.M.S., Vol. 46, n.4, (2009), 535-633. 3, 22, 26, 42

[17] M. de Cataldo, L. Migliorini, "The Decomposition Theorem and the topology of algebraic maps," "version one", arXiv:0712.0349v1. 3, 27

[18] M. de Cataldo, L. Migliorini, "Hodge-theoretic aspects of the Decomposition Theorem ," Algebraic Geometry, Seattle 2005, Proceedings of Symposia in Pure Mathematics, Vol. 80.2, 2009. 45

[19] M. de Cataldo, L. Migliorini, "The perverse filtration and the Lefschetz Hyperplane Section Theorem," Annals of Mathematics, Vol. 171, No. 3, 2010, 2089-2113. 51

[20] M. de Cataldo, T. Hausel, L. Migliorini, "Topology of Hitchin systems and Hodge theory of character varieties: the case A1," Annals of Mathematics 175 (2012), 1329-1407. 53, 54

[21] M.A. de Cataldo, L. Migliorini, "The Hodge theory of maps. Lectures 4-5 by de Cataldo." Math. Notes, 49, Hodge theory, 273?296, Princeton Univ. Press, Princeton, NJ, 2014. 3

[22] M.A. de Cataldo, L. Migliorini, "The projectors of the decomposition theorem are motivated," to appear in M.R.L. 55

[23] P. Deligne, "Théorème de Lefschetz et critères de dégénérescence de suites spectrales," Publ.Math. IHES **35** (1969), 107-126. 4, 8

[24] P.Deligne, "Théorie de Hodge. I," Actes du Congrès International des Mathématiciens (Nice, 1970), Tome 1, pp. 425?430. Gauthier-Villars, Paris, 1971. 12, 14

[25] P. Deligne, "Théorie de Hodge, II," Publ.Math. IHES **40** (1971), 5-57. 6, 8, 14

[26] P. Deligne, "Théorie de Hodge, III," Publ.Math. IHES **44** (1974), 5-78. 14

[27] P. Deligne, "Poids dans la cohomologie des variétés alg?briques," Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), Vol. 1, pp. 79?85. Canad. Math. Congress, Montreal, Que., 1975. 12

[28] P. Deligne, "La conjecture de Weil, II," Publ.Math. IHES **52** (1980), 138-252. 12, 42

[29] P. Deligne, "Décompositions dans la catégorie Dérivée", Motives (Seattle, WA, 1991), 115–128, Proc. Sympos. Pure Math., **55**, Part 1, Amer. Math. Soc., Providence, RI, 1994. 8, 12, 50, 51

[30] A.H. Durfee, "A naive guide to mixed Hodge theory," *Singularities,* Part 1 (Arcata, Calif., 1981), 313–320, Proc. Sympos. Pure Math., 40, Amer. Math. Soc., Providence, RI, 1983. 14

[31] S. Gelfand, Y.I. Manin, *Methods of homological algebra,* Second edition. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2003. 20

[32] R. Godement, *Topologie algébrique et théorie des faisceaux,* Publications de l'Institut de Mathmatique de l'Universit de Strasbourg, XIII. Actualits Scientifiques et Industrielles, No. 1252. Hermann, Paris, 1973.

[33] M. Goresky, R. MacPherson, "Intersection homology. II," Invent. Math. **72** (1983), no. 1, 77–129.

[34] M. Goresky, R. MacPherson, *Stratified Morse Theory*, Ergebnisse der Mathematik, und ihrer Grenzgebiete 3.folge. Band 2, Springer-Verlag, Berlin Heidelberg 1988. 20, 23

[35] D. Kaledin, "Symplectic singularities from the Poisson point of view", J. Reine Angew. Math. 600 (2006), 135?156. 31

[36] M. Kashiwara, P. Schapira *Sheaves on manifolds*, Grundlehren der mathematischen Wissenschaften. Vol. 292, Springer-Verlag, Berlin Heidelberg 1990. 20

[37] B. Iversen, *Cohomology of Sheaves*, Universitext, Springer-Verlag, Berlin Heidelberg 1986. 20

[38] S.L. Kleiman "The development of Intersection Homology Theory," Pure and Appl. Math. Quart. **3** no. 1 (2007) Special issue in honor of Robert MacPherson, 225-282.

[39] R. MacPherson, "Global questions in the topology of singular spaces," Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 213-235, PWN, Warsaw, 1984. 6

[40] R. MacPherson, *Intersection homology and perverse sheaves,* Hermann Weyl Lectures, manuscript, 1990, http://faculty.tcu.edu/gfriedman/notes/ih.pdf

[41] J. Milnor, *Morse Theory,* Annals of Mathematics Studies, **51,** Princeton University Press, Princeton, N.J. 1963.

[42] T. Mochizuki, "Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules, I, II, Mem. Amer. Math. Soc. **185** (2007), no. 869, 870. 9, 19, 41

[43] H. Nakajima, *Lectures on Hilbert schemes of points on surfaces*, University Lecture Series, 18. American Mathematical Society, Providence, RI, 1999. xii+132 pp. 32

[44] B.C. Ngô, "Le lemme fondamental pour les alg?bres de Lie," Publ. Math. Inst. Hautes ?tudes Sci. No. 111 (2010), 1?169. 40, 54

[45] M. Nori, "Constructible sheaves," Algebra, arithmetic and geometry, Part I, II (Mumbai, 2000), 471?491, Tata Inst. Fund. Res. Stud. Math., 16, Tata Inst. Fund. Res., Bombay, 2002. 25

[46] C. Sabbah, "Polarizable twistor D-modules, Astérisque No. **300** (2005). 9, 19, 41

[47] M. Saito, "Mixed Hodge modules," Publ. Res. Inst. Math. Sci. **26** (1990), no. 2, 221–333. 9, 12, 19, 41, 45

[48] R. Stanley, "Combinatorial applications of the hard Lefschetz theorem," Proceedings of the International Congress of Mathematicians, Vol. 1, 2 (Warsaw, 1983), 447?453, PWN, Warsaw, 1984. 42

[49] J.L. Verdier, *Des catégories dérivées des catégories abéliennes,* Astérisque, 239, Soc. Math. France, Paris, 1983.

Authors' addresses:

Mark Andrea A. de Cataldo, Department of Mathematics, SUNY at Stony Brook, Stony Brook, NY 11794, USA. e-mail: *mark.decataldo@stonybrook.edu*