DEFORMATION THEORY OF OBJECTS IN HOMOTOPY AND DERIVED CATEGORIES I: GENERAL THEORY

ALEXANDER I. EFIMOV, VALERY A. LUNTS, AND DMITRI O. ORLOV

ABSTRACT. This is the first paper in a series. We develop a general deformation theory of objects in homotopy and derived categories of DG categories. Namely, for a DG module E over a DG category we define four deformation functors $\mathrm{Def}^{\mathrm{h}}(E)$, $\mathrm{coDef}^{\mathrm{h}}(E)$, $\mathrm{Def}(E)$, $\mathrm{coDef}(E)$. The first two functors describe the deformations (and co-deformations) of E in the homotopy category, and the last two - in the derived category. We study their properties and relations. These functors are defined on the category of artinian (not necessarily commutative) DG algebras.

CONTENTS

The first named author was partially supported by grant NSh-1983.2008.1. The second named author was partially supported by the NSA grant H98230-05-1-0050 and CRDF grant RUM1-2661-MO-05. The third named author was partially supported by CRDF grant RUM1-2661-MO-05, grant RFFI 05-01-01034, grant INTAS 05-1000008-8118 and grant NSh-9969.2006.1.

1. Introduction

It is well known (see for example [De1], [De2], [Dr2], [G1], [G2], [H]) that for many mathematical objects X (defined over a field of characteristic zero) the formal deformation theory of X is controlled by a DG Lie algebra $\mathfrak{g}=\mathfrak{g}(X)$ of (derived) infinitesimal automorphisms of X . This is so in case X is an algebra, a compact complex manifold, a principal G -bundle, etc..

Let $\mathcal{M}(X)$ denote the base of the universal deformation of X and $o\in\mathcal{M}(X)$ be the point corresponding to X . Then (under some conditions on $\mathfrak{g}$) the completion of the local ring $\hat{\mathcal{O}}_{\mathcal{M}(X),o}$ is naturally isomorphic to the linear dual of the homology space $H_0(\mathfrak{g})$. The space $H_0(\mathfrak{g})$ is a co-commutative coalgebra, hence its dual is a commutative algebra.

The homology $H_0(\mathfrak{g})$ is the zero cohomology group of $B\mathfrak{g}$ – the bar construction of $\mathfrak{g}$, which is a co-commutative DG coalgebra. It is therefore natural to consider the DG "formal moduli space" $\mathcal{M}^{DG}(X)$, so that the corresponding completion $\hat{\mathcal{O}}_{\mathcal{M}^{DG}(X),o}$ of the "local ring" is the linear dual $(B\mathfrak{g})^*$, which is a commutative DG algebra. The space $\mathcal{M}^{DG}(X)$ is thus the "true" universal deformation space of X ; it coincides with $\mathcal{M}(X)$ in case $H^i(B\mathfrak{g})=0$ for $i\neq 0$. In particular, it appears that the primary object is not the DG algebra $(B\mathfrak{g})^*$, but rather the DG coalgebra $B\mathfrak{g}$ (this is the point of view in [H]). In any case, the corresponding deformation functor is naturally defined on the category of commutative artinian DG algebras (see [H]).

Note that the passage from a DG Lie algebra $\mathfrak{g}$ to the commutative DG algebra $(B\mathfrak{g})^*$ is an example of the Koszul duality for operads [GK]. Indeed, the operad of DG Lie algebras is Koszul dual to that of commutative DG algebras.

Some examples of DG algebraic geometry are discussed in [Ka], [CK1], [CK2].

This paper (and the following papers [ELO2], [ELO3]) is concerned with a general deformation theory in a slightly different context. Namely, we consider deformations of "linear" objects E , such as objects in a homotopy or a derived category. More precisely, E is a right DG module over a DG category $\mathcal{A}$. In this case the deformation theory of E is controlled by $\mathcal{B}=\operatorname{End}(E)$ which is a DG *algebra* (and not a DG Lie algebra). (This works equally well in positive characteristic.) Then the DG formal deformation space of E is the "Spec" of the (noncommutative!) DG algebra $(B\mathcal{B})^*$ – the linear dual of the bar construction $B\mathcal{B}$ which is a DG coalgebra. Again this is in agreement with the Koszul duality for operads, since the operad of DG algebras is self-dual. (All this was already anticipated in [Dr2].)

More precisely, let dgart be the category of local artinian (not necessarily commutative) DG algebras and **Gpd** be the 2-category of groupoids. For a right DG module E over a DG category $\mathcal{A}$ we define four pseudo-functors

$$\mathrm{Def}^{\mathrm{h}}(E), \mathrm{coDef}^{\mathrm{h}}(E), \mathrm{Def}(E), \mathrm{coDef}(E) : \mathrm{dgart} \to \mathbf{Gpd}.$$

The first two are the *homotopy* deformation and co-deformation pseudo-functors, i.e. they describe deformations (and co-deformations) of E in the homotopy category of DG $\mathcal{A}^{op}$-modules; and the last two are their *derived* analogues. We prove that the pseudo-functors $\mathrm{Def}^{\mathrm{h}}(E)$, $\mathrm{coDef}^{\mathrm{h}}(E)$ are equivalent and depend only on the quasi-isomorphism class of the DG algebra $\mathrm{End}(E)$. The derived pseudo-functors $\mathrm{Def}(E)$, $\mathrm{coDef}(E)$ need some boundedness conditions to give the "right" answer and in that case they are equivalent to $\mathrm{Def}^{\mathrm{h}}(F)$ and $\mathrm{coDef}^{\mathrm{h}}(F)$ respectively for an appropriately chosen h-projective or h-injective DG module F which is quasi-isomorphic to E (one also needs to restrict the pseudo-functors to the category dgart_- of negative artinian DG algebras).

This first paper is devoted to the study of general properties of the above four pseudo-functors and relations between them. Part 1 of the paper is a rather lengthy review of basics of DG categories and DG modules over them with some minor additions that we did not find in the literature. The reader who is familiar with basic DG categories is suggested to go directly to Part 2, except for looking up the definition of the DG functors i^* and $i^!$.

In the second paper [ELO2] we study the pro-representability of these pseudo-functors. Recall that "classically" one defines representability only for functors with values in the category of sets (since the collection of morphisms between two objects in a category is a set). For example, given a moduli problem in the form of a pseudo-functor with values in the 2-category of groupoids one then composes it with the functor π_0 to get a set valued functor, which one then tries to (pro-) represent. This is certainly a loss of information. But in order to represent the original pseudo-functor one needs the source category to be a bicategory.

It turns out that there is a natural bicategory $2\text{-}\mathrm{adgalg}$ of augmented DG algebras. (Actually we consider two versions of this bicategory, $2\text{-}\mathrm{adgalg}$ and $2'\text{-}\mathrm{adgalg}$, but then show that they are equivalent). We consider its full subcategory $2\text{-}\mathrm{dgart}_-$ whose objects are negative artinian DG algebras, and show that the derived deformation functors can be naturally extended to pseudo-functors

$$\mathrm{coDEF}_-(E) : 2\text{-}\mathrm{dgart}_- \to \mathbf{Gpd}, \quad \mathrm{DEF}_-(E) : 2'\text{-}\mathrm{dgart}_- \to \mathbf{Gpd}.$$

Then (under some finiteness conditions on the cohomology algebra $H(\mathcal{C})$ of the DG algebra $\mathcal{C}=\mathbf{R}\operatorname{Hom}(E,E)$) we prove pro-representability of these pseudo-functors by some local complete DG algebra described by means of A_∞ -structure on $H(\mathcal{C})$.

This pro-representability appears to be more "natural" for the pseudo-functor coDEF_- , because there exists a "universal co-deformation" of the DG $\mathcal{C}^{op}$ -module $\mathcal{C}$. The pro-representability of the pseudo-functor DEF_- may then be formally deduced from that of coDEF_- .

In the third paper [ELO3] we show how to apply our deformation theory of DG modules to deformations of complexes over abelian categories. We also discuss examples from algebraic geometry.

We note that the noncommutative deformations (i.e. over noncommutative artinian rings) of modules were already considered by Laudal in [Lau]. The basic difference between our work and [Lau] (besides the fact that our noncommutative artinian algebras are DG algebras) is that we work in the derived context. That is we only deform the differential in a suitably chosen complex and keep the module structure constant.

It is our pleasure to thank A.Bondal, P.Deligne, M.Mandell, M.Larsen and P.Bressler for useful discussions. We especially appreciate the generous help of B.Keller. We also thank W.Goldman and V.Schechtman for sending us copies of letters [De1] and [Dr2] respectively and W.Lowen for sending us the preprint [Lo]. We also thank J.Stasheff for his useful comments on the first version of this paper.

Part 1. **Preliminaries on DG categories**

2. Artinian DG algebras

We fix a field k . All algebras are assumed to be $\mathbb{Z}$ graded k -algebras with unit and all categories are k -linear. Unless mentioned otherwise $\otimes$ means $\otimes_k$.

For a homogeneous element a we denote its degree by $\bar{a}$.

A *module* always means a (left) graded module.

A DG algebra $\mathcal{B}=(\mathcal{B},d_{\mathcal{B}})$ is a (graded) algebra with a map $d=d_{\mathcal{B}}:\mathcal{B}\to\mathcal{B}$ of degree 1 such that $d^2=0$, $d(1)=0$ and

$$d(ab)=d(a)b+(-1)^{\bar{a}}ad(b).$$

Given a DG algebra $\mathcal{B}$ its opposite is the DG algebra $\mathcal{B}^{op}$ which has the same differential as $\mathcal{B}$ and multiplication

$$a\cdot b=(-1)^{\bar{a}\bar{b}}ba,$$

where ba is the product in $\mathcal{B}$. When there is a danger of confusion of the opposite DG algebra $\mathcal{B}^{op}$ with the degree zero part of $\mathcal{B}$ we will add a comment.

We denote by dgalg the category of DG algebras.

A (left) DG module over a DG algebra $\mathcal{B}$ is called a DG $\mathcal{B}$-module or, simply a $\mathcal{B}$-module. A *right* $\mathcal{B}$-module is a DG module over $\mathcal{B}^{op}$.

If $\mathcal{B}$ is a DG algebra and M is a usual (not DG) module over the algebra $\mathcal{B}$, then we say that M^{gr} is a $\mathcal{B}^{\mathrm{gr}}$-module.

An *augmentation* of a DG algebra $\mathcal{B}$ is a (surjective) homomorphism of DG algebras $\mathcal{B} \to k$. Its kernel is a DG ideal (i.e. an ideal closed under the differential) of $\mathcal{B}$. Denote by adgalg the category of augmented DG algebras (morphisms commute with the augmentation).

Definition 2.1. *Let R be an algebra. We call R artinian, if it is finite dimensional and has a (graded) nilpotent two-sided (maximal) ideal $m \subset R$, such that $R/m = k$.*

Definition 2.2. *Let $\mathcal{R}$ be an augmented DG algebra. We call $\mathcal{R}$ artinian if $\mathcal{R}$ is artinian as an algebra and the maximal ideal $m \subset R$ is a DG ideal, i.e. the quotient map $R \to R/m$ is an augmentation of the DG algebra $\mathcal{R}$. Note that a homomorphism of artinian DG algebras automatically commutes with the augmentations. Denote by* dgart *the category of artinian DG algebras.*

Definition 2.3. *An artinian DG algebra $\mathcal{R}$ is called positive (resp. negative) if negative (resp. positive) degree components of $\mathcal{R}$ are zero. Denote by* dgart_+ *and* dgart_- *the corresponding full subcategories of* dgart*. Let* $\mathrm{art} := \mathrm{dgart}_- \cap \mathrm{dgart}_+$ *be the full subcategory of* dgart *consisting of (not necessarily commutative) artinian algebras concentrated in degree zero. Denote by* $\mathrm{cart} \subset \mathrm{art}$ *the full subcategory of commutative artinian algebras.*

Given a DG algebra $\mathcal{B}$ one studies the category $\mathcal{B}$-mod and the corresponding homotopy and derived categories. A homomorphism of DG algebras induces various functors between these categories. We will recall these categories and functors in the more general context of DG categories in the next section.

3. DG categories

In this section we recall some basic facts about DG categories which will be needed in this paper. Our main references here are [BK], [Dr1], [Ke].

A DG category is a k-linear category $\mathcal{A}$ in which the sets $\mathrm{Hom}(A,B)$, $A, B \in Ob\mathcal{A}$, are provided with a structure of a $\mathbb{Z}$-graded k-module and a differential $d : \mathrm{Hom}(A,B) \to \mathrm{Hom}(A,B)$ of degree 1, so that for every $A, B, C \in \mathcal{A}$ the composition $\mathrm{Hom}(A,B) \times \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$ comes from a morphism of complexes $\mathrm{Hom}(A,B) \otimes \mathrm{Hom}(B,C) \to \mathrm{Hom}(A,C)$. The identity morphism $1_A \in \mathrm{Hom}(A,A)$ is closed of degree zero.

The simplest example of a DG category is the category $DG(k)$ of complexes of k-vector spaces, or DG k-modules.

Note also that a DG algebra is simply a DG category with one object.

Using the supercommutativity isomorphism $S \otimes T \simeq T \otimes S$ in the category of DG k-modules one defines for every DG category $\mathcal{A}$ the opposite DG category $\mathcal{A}^{op}$ with $Ob\mathcal{A}^{op} = Ob\mathcal{A}$,

$\mathrm{Hom}_{\mathcal{A}^{op}}(A,B)=\mathrm{Hom}_{\mathcal{A}}(B,A)$. We denote by $\mathcal{A}^{\mathrm{gr}}$ the *graded* category which is obtained from $\mathcal{A}$ by forgetting the differentials on Hom 's.

The tensor product of DG-categories $\mathcal{A}$ and $\mathcal{B}$ is defined as follows:

(i) $Ob(\mathcal{A}\otimes\mathcal{B}):=Ob\mathcal{A}\times Ob\mathcal{B}$; for $A\in Ob\mathcal{A}$ and $B\in Ob\mathcal{B}$ the corresponding object is denoted by $A\otimes B$;

(ii) $\mathrm{Hom}(A\otimes B,A'\otimes B'):=\mathrm{Hom}(A,A')\otimes\mathrm{Hom}(B,B')$ and the composition map is defined by $(f_1\otimes g_1)(f_2\otimes g_2):=(-1)^{\bar{g}_1\bar{f}_2}f_1f_2\otimes g_1g_2.$

Note that the DG categories $\mathcal{A}\otimes\mathcal{B}$ and $\mathcal{B}\otimes\mathcal{A}$ are canonically isomorphic. In the above notation the isomorphism DG functor ϕ is

$$\phi(A\otimes B)=(B\otimes A),\quad \phi(f\otimes g)=(-1)^{\bar{f}\bar{g}}(g\otimes f).$$

Given a DG category $\mathcal{A}$ one defines the graded category $\mathrm{Ho}^\bullet(\mathcal{A})$ with $Ob\,\mathrm{Ho}^\bullet(\mathcal{A})=Ob\mathcal{A}$ by replacing each Hom complex by the direct sum of its cohomology groups. We call $\mathrm{Ho}^\bullet(\mathcal{A})$ the *graded homotopy category* of $\mathcal{A}$. Restricting ourselves to the 0-th cohomology of the Hom complexes we get the *homotopy category* $\mathrm{Ho}(\mathcal{A})$.

Two objects $A,B\in Ob\mathcal{A}$ are called DG *isomorphic* (or, simply, isomorphic) if there exists an invertible degree zero morphism $f\in\mathrm{Hom}(A,B)$. We say that A,B are *homotopy equivalent* if they are isomorphic in $\mathrm{Ho}(\mathcal{A})$.

A DG-functor between DG-categories $F:\mathcal{A}\to\mathcal{B}$ is said to be a *quasi-equivalence* if $\mathrm{Ho}^\bullet(F):\mathrm{Ho}^\bullet(\mathcal{A})\to\mathrm{Ho}^\bullet(\mathcal{B})$ is an equivalence of graded categories. We say that F is a DG *equivalence* if it is fully faithful and every object of $\mathcal{B}$ is DG isomorphic to an object of $F(\mathcal{A})$. Certainly, a DG equivalence is a quasi-equivalence. DG categories $\mathcal{C}$ and $\mathcal{D}$ are called *quasi-equivalent* if there exist DG categories $\mathcal{A}_1,...,\mathcal{A}_n$ and a chain of quasi-equivalences

$$\mathcal{C}\leftarrow\mathcal{A}_1\to...\leftarrow\mathcal{A}_n\to\mathcal{D}.$$

Given DG categories $\mathcal{A}$ and $\mathcal{B}$ the collection of covariant DG functors $\mathcal{A}\to\mathcal{B}$ is itself the collection of objects of a DG category, which we denote by $\mathrm{Fun}_{\mathrm{DG}}(\mathcal{A},\mathcal{B})$. Namely, let Φ and Ψ be two DG functors. Put $\mathrm{Hom}^k(\Phi,\Psi)$ equal to the set of natural transformations $t:\Phi^{\mathrm{gr}}\to\Psi^{\mathrm{gr}}[k]$ of graded functors from $\mathcal{A}^{\mathrm{gr}}$ to $\mathcal{B}^{\mathrm{gr}}$. This means that for any morphism $f\in\mathrm{Hom}^s_{\mathcal{A}}(A,B)$ one has

$$\Psi(f)\cdot t(A)=(-1)^{ks}t(B)\cdot\Phi(f).$$

On each $A\in\mathcal{A}$ the differential of the transformation t is equal to $d(t(A))$ (one easily checks that this is well defined). Thus, the closed transformations of degree 0 are the DG transformations of DG functors. A similar definition gives us the DG-category consisting of the contravariant DG functors $\mathrm{Fun}_{\mathrm{DG}}(\mathcal{A}^{op},\mathcal{B})=\mathrm{Fun}_{\mathrm{DG}}(\mathcal{A},\mathcal{B}^{op})$ from $\mathcal{A}$ to $\mathcal{B}$.

3.1. **DG modules over DG categories.** We denote the DG category $\mathrm{Fun}_{\mathrm{DG}}(\mathcal{A},DG(k))$ by $\mathcal{A}$-mod and call it the category of DG $\mathcal{A}$ -modules. There is a natural covariant DG functor $h:\mathcal{A}\to\mathcal{A}^{op}$-mod (the Yoneda embedding) defined by $h^A(B):=\mathrm{Hom}_{\mathcal{A}}(B,A)$. As in the "classical" case one verifies that the functor h is fully faithful, i.e. there is a natural isomorphism of complexes

$$\mathrm{Hom}_{\mathcal{A}}(A, A') = \mathrm{Hom}_{\mathcal{A}^{op}\text{-mod}}(h^A, h^{A'}).$$

Moreover, for any $M \in \mathcal{A}^{op}$-mod , $A \in \mathcal{A}$

$$\mathrm{Hom}_{\mathcal{A}^{op}\text{-mod}}(h^A, M) = M(A).$$

The DG $\mathcal{A}^{op}$ -modules h^A , $A \in \mathcal{A}$ are called *free.*

For $A \in \mathcal{A}$ one may consider also the covariant DG functor $h_A(B) := \mathrm{Hom}_{\mathcal{A}}(A, B)$ and the contravariant DG functor $h_A^*(B) := \mathrm{Hom}_k(h_A(B), k)$. For any $M \in \mathcal{A}^{op}$-mod we have

$$\mathrm{Hom}_{\mathcal{A}^{op}\text{-mod}}(M, h_A^*) = \mathrm{Hom}_k(M(A), k).$$

A DG $\mathcal{A}^{op}$ -module M is called acyclic, if the complex $M(A)$ is acyclic for all $A \in \mathcal{A}$. Let $D(\mathcal{A}^{op})$ denote the *derived category* of DG $\mathcal{A}^{op}$ -modules, i.e. $D(\mathcal{A}^{op})$ is the Verdier quotient of the homotopy category $\mathrm{Ho}(\mathcal{A}^{op}\text{-mod})$ by the subcategory of acyclic DG-modules. This is a triangulated category.

A DG $\mathcal{A}^{op}$ -module P is called h-*projective* if for any acyclic DG $\mathcal{A}^{op}$ -module N the complex $\mathrm{Hom}(P, N)$ is acyclic. A free DG module is h-projective. Denote by $\mathcal{P}(\mathcal{A}^{op})$ the full DG subcategory of $\mathcal{A}^{op}$-mod consisting of h-projective DG modules.

Similarly, a DG $\mathcal{A}^{op}$ -module I is called h-*injective* if for any acyclic DG $\mathcal{A}^{op}$ -module N the complex $\mathrm{Hom}(N, I)$ is acyclic. For any $A \in \mathcal{A}$ the DG $\mathcal{A}^{op}$ -module h_A^* is h-injective. Denote by $\mathcal{I}(\mathcal{A}^{op})$ the full DG subcategory of $\mathcal{A}^{op}$-mod consisting of h-injective DG modules.

For any DG category $\mathcal{A}$ the DG categories $\mathcal{A}^{op}$-mod , $\mathcal{P}(\mathcal{A}^{op})$, $\mathcal{I}(\mathcal{A}^{op})$ are (strongly) pre-triangulated ([Dr1, BK], also see subsection 3.5 below). Hence the homotopy categories $\mathrm{Ho}(\mathcal{A}^{op}\text{-mod})$, $\mathrm{Ho}(\mathcal{P}(\mathcal{A}^{op}))$, $\mathrm{Ho}(\mathcal{I}(\mathcal{A}^{op}))$ are triangulated.

The following theorem was proved in [Ke].

Theorem 3.1. *The inclusion functors* $\mathcal{P}(\mathcal{A}^{op}) \hookrightarrow \mathcal{A}^{op}$*-mod ,* $\mathcal{I}(\mathcal{A}^{op}) \hookrightarrow \mathcal{A}^{op}$*-mod induce equivalences of triangulated categories* $\mathrm{Ho}(\mathcal{P}(\mathcal{A}^{op})) \simeq D(\mathcal{A}^{op})$ *and* $\mathrm{Ho}(\mathcal{I}(\mathcal{A}^{op})) \simeq D(\mathcal{A}^{op})$ *.*

Actually, it will be convenient for us to use some more precise results from [Ke]. Let us recall the relevant definitions.

Definition 3.2. *A DG* $\mathcal{A}^{op}$ *-module* M *is called relatively projective if* M *is a direct summand of a direct sum of DG* $\mathcal{A}^{op}$ *-modules of the form* $h^A[n]$ *,* $A \in \mathcal{A}$ *,* $n \in \mathbb{Z}$ *. A DG* $\mathcal{A}^{op}$ *-module* P *is said to have property (P) if it admits a filtration*

$$0 = F_{-1} \subset F_0 \subset F_1 \subset ...P$$

such that

(F1) $\cup_i F_i = P$ *;*

(F2) the inclusion $F_i \hookrightarrow F_{i+1}$ *splits as a morphism of graded modules;*

(F3) each quotient F_{i+1}/F_i *is a relatively projective DG* $\mathcal{A}^{op}$ *-module.*

Definition 3.3. *A DG $\mathcal{A}^{op}$ -module M is called relatively injective if M is a direct summand of a direct product of DG $\mathcal{A}^{op}$ -modules of the form $h^*_A[n]$, $A \in \mathcal{A}$, $n \in \mathbb{Z}$. A DG $\mathcal{A}^{op}$ -module I is said to have property (I) if it admits a filtration*

$$I = F_0 \supset F_1 \supset ...$$

such that
(F1') the canonical morphism

$$I \to \lim_{\leftarrow} I/F_i$$

is an isomorphism;
(F2') the inclusion $F_{i+1} \hookrightarrow F_i$ splits as a morphism of graded modules;
(F3') each quotient F_i/F_{i+1} is a relatively injective DG $\mathcal{A}^{op}$ -module.

Theorem 3.4. *(*[Ke]*) a) A DG $\mathcal{A}^{op}$ -module with property (P) is h -projective.*

b) For any $M \in \mathcal{A}^{op}$-mod there exists a quasi-isomorphism $P \to M$, such that the DG $\mathcal{A}^{op}$ -module P has property (P).

c) A DG $\mathcal{A}^{op}$ -module with property (I) is h -injective.

d) For any $M \in \mathcal{A}^{op}$-mod there exists a quasi-isomorphism $M \to I$, such that the DG $\mathcal{A}^{op}$ -module I has property (I).

Remark 3.5. *a) Assume that a DG $\mathcal{A}^{op}$ -module M has an increasing filtration $M_1 \subset M_2 \subset ...$ such that $\cup M_i = M$, each inclusion $M_i \hookrightarrow M_{i+1}$ splits as a morphism of graded modules, and each subquotient M_{i+1}/M_i is h -projective. Then M is h-projective. b) Assume that a DG $\mathcal{A}^{op}$ -module N has a decreasing filtration $N = N_1 \supset N_2 \supset ...$ such that $\cap N_i = 0$, each inclusion $N_{i+1} \hookrightarrow N_i$ splits as a morphism of graded modules, each subquotient N_i/N_{i+1} is h-injective (hence N/N_i is h-injective for each i) and the natural map*

$$N \to \lim_{\leftarrow} N/N_i$$

is an isomorphism. Then N is h-injective.

3.2. **Some DG functors.** Let $\mathcal{B}$ be a small DG category. The complex

$$\mathrm{Alg}_{\mathcal{B}} := \bigoplus_{A,B \in Ob\mathcal{B}} \mathrm{Hom}(A,B)$$

has a natural structure of a DG algebra possibly without a unit. It has the following property: every finite subset of $\mathrm{Alg}_{\mathcal{B}}$ is contained in $e\,\mathrm{Alg}_{\mathcal{B}}\,e$ for some idempotent e such that $de = 0$ and $\bar{e} = 0$. We say that a DG module M over $\mathrm{Alg}_{\mathcal{B}}$ is *quasi-unital* if every element of M belongs to eM for some idempotent $e \in \mathrm{Alg}_{\mathcal{B}}$ (which may be assumed closed of degree 0 without loss of generality). If Φ is a DG $\mathcal{B}$ -module then

$$M_\Phi := \oplus_{A \in Ob\mathcal{B}} \Phi(A)$$

is a quasi-unital DG module over $\mathrm{Alg}_{\mathcal{B}}$. This way we get a DG equivalence between DG category of DG $\mathcal{B}$ -modules and that of quasi-unital DG modules over $\mathrm{Alg}_{\mathcal{B}}$.

Recall that a homomorphism of (unital) DG algebras $\phi : \mathcal{A} \to \mathcal{B}$ induces functors

$$\phi_* : \mathcal{B}^{op}\text{-mod} \to \mathcal{A}^{op}\text{-mod},$$

$$\phi^* : \mathcal{A}^{op}\text{-mod} \to \mathcal{B}^{op}\text{-mod}$$

$$\phi^! : \mathcal{A}^{op}\text{-mod} \to \mathcal{B}^{op}\text{-mod}$$

where ϕ_* is the restriction of scalars, $\phi^*(M) = M \otimes_{\mathcal{A}} \mathcal{B}$ and $\phi^!(M) = \mathrm{Hom}_{\mathcal{A}^{op}}(\mathcal{B}, M)$. The DG functors (ϕ^*, ϕ_*) and $(\phi_*, \phi^!)$ are adjoint: for $M \in \mathcal{A}^{op}\text{-mod}$ and $N \in \mathcal{B}^{op}\text{-mod}$ there exist functorial isomorphisms of complexes

$$\mathrm{Hom}(\phi^* M, N) = \mathrm{Hom}(M, \phi_* N), \quad \mathrm{Hom}(\phi_* N, M) = \mathrm{Hom}(N, \phi^! M).$$

This generalizes to a DG functor $F : \mathcal{A} \to \mathcal{B}$ between DG categories. We obtain DG functors

$$F_* : \mathcal{B}^{op}\text{-mod} \to \mathcal{A}^{op}\text{-mod},$$

$$F^* : \mathcal{A}^{op}\text{-mod} \to \mathcal{B}^{op}\text{-mod}.$$

$$F^! : \mathcal{A}^{op}\text{-mod} \to \mathcal{B}^{op}\text{-mod}.$$

Namely, the DG functor F induces a homomorphism of DG algebras $F : \mathrm{Alg}_{\mathcal{A}} \to \mathrm{Alg}_{\mathcal{B}}$ and hence defines functors F_* , F^* between quasi-unital DG modules as above. (These functors F_* and F^* are denoted in [Dr1] by Res_F and Ind_F respectively.) The functor $F^!$ is defined as follows: for a quasi-unital $\mathrm{Alg}_{\mathcal{A}}^{op}$ -module M put

$$F^!(M) = \mathrm{Hom}_{\mathrm{Alg}_{\mathcal{A}}^{op}}(\mathrm{Alg}_{\mathcal{B}}, M)^{\mathrm{qu}},$$

where $N^{\mathrm{qu}} \subset N$ is the *quasi-unital* part of a $\mathrm{Alg}_{\mathcal{B}}^{op}$ -module N defined by

$$N^{\mathrm{qu}} := \mathrm{Im}(N \otimes_k \mathrm{Alg}_{\mathcal{B}} \to N).$$

The DG functors (F^*, F_*) and $(F_*, F^!)$ are adjoint.

Lemma 3.6. *Let $F : \mathcal{A} \to \mathcal{B}$ be a DG functor. Then*

a) F_* preserves acyclic DG modules;

b) F^* preserves h-projective DG modules;

c) $F^!$ preserves h-injective DG modules.

Proof. The first assertion is obvious and the other two follow by adjunction. □

By Theorem 3.1 above the DG subcategories $\mathcal{P}(\mathcal{A}^{op})$ and $\mathcal{I}(\mathcal{A}^{op})$ of $\mathcal{A}^{op}\text{-mod}$ allow us to define (left and right) derived functors of DG functors $G : \mathcal{A}^{op}\text{-mod} \to \mathcal{B}^{op}\text{-mod}$ in the usual way. Namely for a DG $\mathcal{A}^{op}$ -module M choose quasi-isomorphisms $P \to M$ and $M \to I$ with $P \in \mathcal{P}(\mathcal{A}^{op})$ and $I \in \mathcal{I}(\mathcal{A}^{op})$. Put

$$\mathbf{L}G(M) := G(P), \qquad \mathbf{R}G(M) := G(I).$$

In particular for a DG functor $F:\mathcal{A}\to\mathcal{B}$ we will consider derived functors $\mathbf{L}F^*: D(\mathcal{A}^{op})\to D(\mathcal{B}^{op})$, $\mathbf{R}F^!: D(\mathcal{A}^{op})\to D(\mathcal{B}^{op})$. We also have the obvious functor $F_*: D(\mathcal{B}^{op})\to D(\mathcal{A}^{op})$. The functors $(\mathbf{L}F^*, F_*)$ and $(F_*, \mathbf{R}F^!)$ are adjoint.

Proposition 3.7. *Assume that the DG functor $F:\mathcal{A}\to\mathcal{B}$ is a quasi-equivalence. Then*

a) $F^*:\mathcal{P}(\mathcal{A}^{op})\to\mathcal{P}(\mathcal{B}^{op})$ is a quasi-equivalence;
b) $\mathbf{L}F^*: D(\mathcal{A}^{op})\to D(\mathcal{B}^{op})$ is an equivalence;
c) $F_*: D(\mathcal{B}^{op})\to D(\mathcal{A}^{op})$ is an equivalence.
d) $\mathbf{R}F^!: D(\mathcal{A}^{op})\to D(\mathcal{B}^{op})$ is an equivalence.
e) $F^!:\mathcal{I}(\mathcal{A}^{op})\to\mathcal{I}(\mathcal{B}^{op})$ is a quasi-equivalence.

Proof. a) is proved in [Ke] and it implies b) by Theorem 3.1. c) (resp. d)) follows from b) (resp. c) by adjunction. Finally, e) follows from d) by Theorem 3.1. □

Given DG $\mathcal{A}^{op}$-modules M, N we denote by $\mathrm{Ext}^n(M,N)$ the group of morphisms $\mathrm{Hom}^n_{D(\mathcal{A})}(M,N)$.

3.3. **DG category $\mathcal{A}_{\mathcal{R}}$.** Let $\mathcal{R}$ be a DG algebra. We may and will consider $\mathcal{R}$ as a DG category with one object whose endomorphism DG algebra is $\mathcal{R}$. We denote this DG category again by $\mathcal{R}$. Note that the DG category $\mathcal{R}^{op}$-mod is just the category of right DG modules over the DG algebra $\mathcal{R}$.

For a DG category $\mathcal{A}$ we denote the DG category $\mathcal{A}\otimes\mathcal{R}$ by $\mathcal{A}_{\mathcal{R}}$. Note that the collections of objects of $\mathcal{A}$ and $\mathcal{A}_{\mathcal{R}}$ are naturally identified. A homomorphism of DG algebras $\phi:\mathcal{R}\to\mathcal{Q}$ induces the obvious DG functor $\phi=\mathrm{id}\otimes\phi:\mathcal{A}_{\mathcal{R}}\to\mathcal{A}_{\mathcal{Q}}$ (which is the identity on objects), whence the DG functors ϕ_*, ϕ^*, $\phi^!$ between the DG categories $\mathcal{A}^{op}_{\mathcal{R}}$-mod and $\mathcal{A}^{op}_{\mathcal{Q}}$-mod. For $M\in\mathcal{A}^{op}_{\mathcal{R}}$-mod we have

$$\phi^*(M)=M\otimes_{\mathcal{R}}\mathcal{Q}.$$

In case $\mathcal{Q}^{\mathrm{gr}}$ is a finitely generated $\mathcal{R}^{\mathrm{gr}}$-module we have

$$\phi^!(M)=\mathrm{Hom}_{\mathcal{R}^{op}}(\mathcal{Q},M).$$

In particular, if $\mathcal{R}$ is augmented then the canonical homomorphisms of DG algebras $p:k\to\mathcal{R}$ and $i:\mathcal{R}\to k$ induce functors

$$p:\mathcal{A}\to\mathcal{A}_{\mathcal{R}},\quad i:\mathcal{A}_{\mathcal{R}}\to\mathcal{A},$$

such that $i\cdot p=\mathrm{Id}_{\mathcal{A}}$. So for $S\in\mathcal{A}^{op}$-mod and $T\in\mathcal{A}^{op}_{\mathcal{R}}$-mod we have

$$p^*(S)=S\otimes_k\mathcal{R},\quad i^*(T)=T\otimes_{\mathcal{R}}k,\quad i^!(T)=\mathrm{Hom}_{\mathcal{R}^{op}}(k,T).$$

For an artinian DG algebra $\mathcal{R}$ we denote by $\mathcal{R}^*$ the DG $\mathcal{R}^{op}$-module $\mathrm{Hom}_k(\mathcal{R},k)$. This is a left $\mathcal{R}$-module by the formula

$$rf(q):=(-1)^{(\bar f+\bar q)\bar r}f(qr)$$

and a right $\mathcal{R}$ -module by the formula

$$fr(p) := f(rp)$$

for $r, p \in \mathcal{R}$ and $f \in \mathcal{R}^*$. The augmentation map $\mathcal{R} \to k$ defines the canonical (left and right) $\mathcal{R}$ -submodule $k \subset \mathcal{R}^*$. Moreover, the embedding $k \hookrightarrow \mathcal{R}^*$ induces an isomorphism $k \to \operatorname{Hom}_{\mathcal{R}}(k, \mathcal{R}^*)$.

Definition 3.8. *Let $\mathcal{R}$ be an artinian DG algebra. A DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module M is called graded $\mathcal{R}$ -free (resp. graded $\mathcal{R}$ -cofree) if there exists a DG $\mathcal{A}^{op}$ -module K such that $M^{\mathrm{gr}} \simeq (K \otimes \mathcal{R})^{\mathrm{gr}}$ (resp. $M^{\mathrm{gr}} \simeq (K \otimes \mathcal{R}^*)^{\mathrm{gr}}$). Note that for such M one may take $K = i^*M$ (resp. $K = i^!M$).*

Lemma 3.9. *Let $\mathcal{R}$ be an artinian DG algebra.*

a) The full DG subcategories of DG $\mathcal{A}^{op}_{\mathcal{R}}$ -modules consisting of graded $\mathcal{R}$ -free (resp. graded $\mathcal{R}$ -cofree) modules are DG isomorphic. Namely, if $M \in \mathcal{A}^{op}_{\mathcal{R}}$-mod is graded $\mathcal{R}$ -free (resp. graded $\mathcal{R}$ -cofree) then $M \otimes_{\mathcal{R}} \mathcal{R}^*$ (resp. $\operatorname{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, M)$) is graded $\mathcal{R}$ -cofree (resp. graded $\mathcal{R}$ -free).

b) Let M be a graded $\mathcal{R}$ -free module. There is a natural isomorphism of DG $\mathcal{A}^{op}$ -modules

$$i^*M \xrightarrow{\sim} i^!(M \otimes_{\mathcal{R}} \mathcal{R}^*).$$

Proof. a) If M is graded $\mathcal{R}$ -free, then obviously $M \otimes_{\mathcal{R}} \mathcal{R}^*$ is graded $\mathcal{R}$ -cofree. Assume that N is graded $\mathcal{R}$ -cofree, i.e. $N^{\mathrm{gr}} = (K \otimes \mathcal{R}^*)^{\mathrm{gr}}$. Then

$$(\operatorname{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, N))^{\mathrm{gr}} = (K \otimes \operatorname{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, \mathcal{R}^*))^{\mathrm{gr}},$$

since $\dim_k \mathcal{R} < \infty$. On the other hand

$$\operatorname{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, \mathcal{R}^*) = \operatorname{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, \operatorname{Hom}_k(\mathcal{R}, k)) = \operatorname{Hom}_k(\mathcal{R}^* \otimes_{\mathcal{R}} \mathcal{R}, k) = \mathcal{R},$$

so $(\operatorname{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, N))^{\mathrm{gr}} = (K \otimes \mathcal{R})^{\mathrm{gr}}$.

b) For an arbitrary DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module M we have a natural (closed degree zero) morphism of DG $\mathcal{A}^{op}$ -modules

$$i^*M \to i^!(M \otimes_{\mathcal{R}} \mathcal{R}^*), \quad m \otimes 1 \mapsto (1 \mapsto m \otimes i),$$

where $i : \mathcal{R} \to k$ is the augmentation map. If M is graded $\mathcal{R}$ -free this map is an isomorphism. □

Proposition 3.10. *Let $\mathcal{R}$ be an artinian DG algebra. Assume that a DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module M satisfies property (P) (resp. property (I)). Then M is graded $\mathcal{R}$ -free (resp. graded $\mathcal{R}$ -cofree).*

Proof. Notice that the collection of graded $\mathcal{R}$ -free objects in $\mathcal{A}^{op}_{\mathcal{R}}$-mod is closed under taking direct sums, direct summands (since the maximal ideal $m \subset \mathcal{R}$ is nilpotent) and direct products (since $\mathcal{R}$ is finite dimensional). Similarly for graded $\mathcal{R}$ -cofree objects since the DG functors in Lemma 3.9 a) preserve direct sums and products. Also notice that for any $A \in \mathcal{A}_{\mathcal{R}}$ the DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module h^A (resp. h^*_A) is graded $\mathcal{R}$ -free (resp. graded $\mathcal{R}$ -cofree). Now the proposition follows since a DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module P (resp. I) with property (P) (resp. property (I)) as a graded module is a direct sum of relatively projective DG modules (resp. a direct product of relatively injective DG modules). $\square$

Corollary 3.11. *Let $\mathcal{R}$ be an artinian DG algebra. Then for any DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module M there exist quasi-isomorphisms $P \to M$ and $M \to I$ such that $P \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$, $I \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$ and P is graded $\mathcal{R}$ -free, I is graded $\mathcal{R}$ -cofree.*

Proof. Indeed, this follows from Theorem 3.4 and Proposition 3.10 above. $\square$

Proposition 3.12. *Let $\mathcal{R}$ be an artinian DG algebra and $S, T \in \mathcal{A}^{op}_{\mathcal{R}}$-mod be graded $\mathcal{R}$ -free (resp. graded $\mathcal{R}$ -cofree).*

a) There is an isomorphism of graded vector spaces $\operatorname{Hom}(S,T) = \operatorname{Hom}(i^*S, i^*T) \otimes \mathcal{R}$, (resp. $\operatorname{Hom}(S,T) = \operatorname{Hom}(i^!S, i^!T) \otimes \mathcal{R}$), which is an isomorphism of algebras if $S = T$. In particular, the map $i^* : \operatorname{Hom}(S,T) \to \operatorname{Hom}(i^*S, i^*T)$ (resp. $i^! : \operatorname{Hom}(S,T) \to \operatorname{Hom}(i^!S, i^!T)$) is surjective.

b) The DG module S has a finite filtration with subquotients isomorphic to i^*S as DG $\mathcal{A}^{op}$ -modules (resp. to $i^!S$ as DG $\mathcal{A}^{op}$ -modules).

c) The DG algebra $\operatorname{End}(S)$ has a finite filtration by DG ideals with subquotients isomorphic to $\operatorname{End}(i^*S)$ (resp. $\operatorname{End}(i^!S)$).

d) If $f \in \operatorname{Hom}(S,T)$ is a closed morphism of degree zero such that i^*f (resp. $i^!f$) is an isomorphism or a homotopy equivalence or a quasi-isomorphism, then f is also such.

Proof. Because of Lemma 3.9 above it suffices to prove the proposition for graded $\mathcal{R}$ -free modules. So assume that S , T are graded $\mathcal{R}$ -free.

a) This holds because $\mathcal{R}$ is finite dimensional.

b) We can refine the filtration of $\mathcal{R}$ by powers of the maximal ideal to get a filtration $F_i\mathcal{R}$ by ideals with 1-dimensional subquotients (and zero differential). Then the filtration $F_iS := S \cdot F_i\mathcal{R}$ satisfies the desired properties.

c) Again the filtration $F_i \operatorname{End}(S) := \operatorname{End}(S) \cdot F_i\mathcal{R}$ has the desired properties.

d) If i^*f is an isomorphism, then f is surjective by the Nakayama lemma for $\mathcal{R}$. Also f is injective since T is graded $\mathcal{R}$ -free.

Assume that i^*f is a homotopy equivalence. Let $C(f) \in \mathcal{A}^{op}_{\mathcal{R}}$-mod be the cone of f . (It is also graded $\mathcal{R}$ -free.) Then $i^*C(f) \in \mathcal{A}^{op}$-mod is the cone $C(i^*f)$ of the morphism i^*f . By assumption the DG algebra $\operatorname{End}(C(i^*f))$ is acyclic. But by part c) the complex $\operatorname{End}(C(f))$ has a finite filtration with subquotients isomorphic to the complex $\operatorname{End}(C(i^*f))$. Hence $\operatorname{End}(C(f))$ is also acyclic, i.e. the DG module $C(f)$ is null-homotopic, i.e. f is a homotopy equivalence.

Assume that i^*f is a quasi-isomorphism. Then in the above notation $C(i^*f)$ is acyclic. Since by part b) $C(f)$ has a finite filtration with subquotients isomorphic to $C(i^*f)$, it is also acyclic. Thus f is a quasi-isomorphism. $\square$

3.4. **More DG functors.** So far we considered DG functors F_*, F^*, $F^!$ between the DG categories $\mathcal{A}^{op}$-mod and $\mathcal{B}^{op}$-mod which came from a DG functor $F:\mathcal{A}\to\mathcal{B}$. We will also need to consider a different type of DG functors.

Example 3.13. *For an artinian DG algebra $\mathcal{R}$ and a small DG category $\mathcal{A}$ we will consider two types of "restriction of scalars" DG functors $\pi_*,\pi_!:\mathcal{A}^{op}_{\mathcal{R}}$-mod $\to\mathcal{R}^{op}$-mod. Namely, for $M\in\mathcal{A}^{op}_{\mathcal{R}}$-mod put*

$$\pi_*M:=\prod_{A\in Ob\mathcal{A}_{\mathcal{R}}}M(A),\quad \pi_!M:=\bigoplus_{A\in Ob\mathcal{A}_{\mathcal{R}}}M(A).$$

We will also consider the two "extension of scalars" functors $\pi^*,\pi^!:\mathcal{R}^{op}$-mod $\to\mathcal{A}^{op}_{\mathcal{R}}$-mod defined by

$$\pi^*(N)(A):=N\otimes\bigoplus_{B\in Ob\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(A,B),\quad \pi^!(N)(A):=\mathrm{Hom}_k(\bigoplus_{B\in Ob\mathcal{A}}\mathrm{Hom}_{\mathcal{A}}(B,A),N)$$

for $A\in Ob\mathcal{A}_{\mathcal{R}}$. Notice that the DG functors (π^*,π_*) and $(\pi_!,\pi^!)$ are adjoint, that is for $M\in\mathcal{A}^{op}_{\mathcal{R}}$-mod and $N\in\mathcal{R}^{op}$-mod there is a functorial isomorphism of complexes

$$\mathrm{Hom}(\pi^*N,M)=\mathrm{Hom}(N,\pi_*M),\quad \mathrm{Hom}(\pi_!M,N)=\mathrm{Hom}(M,\pi^!N).$$

The DG functors $\pi^*,\pi^!$ preserve acyclic DG modules, hence π_* preserves h-injectives and $\pi_!$ preserves h-projectives.

We have the following commutative functorial diagrams

$$\begin{array}{ccc}\mathcal{A}^{op}_{\mathcal{R}}\text{-}mod & \xrightarrow{i^*} & \mathcal{A}^{op}\text{-}mod\\ \pi_!\downarrow & & \pi_!\downarrow\\ \mathcal{R}^{op}\text{-}mod & \xrightarrow{i^*} & DG(k),\end{array}$$

$$\begin{array}{ccc}\mathcal{A}^{op}_{\mathcal{R}}\text{-}mod & \xrightarrow{i^!} & \mathcal{A}^{op}\text{-}mod\\ \pi_*\downarrow & & \pi_*\downarrow\\ \mathcal{R}^{op}\text{-}mod & \xrightarrow{i^!} & DG(k).\end{array}$$

Example 3.14. *Fix $E\in\mathcal{A}^{op}$-mod and put $\mathcal{B}=\mathrm{End}(E)$. Consider the DG functor*

$$\Sigma=\Sigma^E:\mathcal{B}^{op}\text{-}mod\to\mathcal{A}^{op}\text{-}mod$$

defined by $\Sigma(M)=M\otimes_{\mathcal{B}}E$. Clearly, $\Sigma(\mathcal{B})=E$. This DG functor gives rise to the functor

$$\mathbf{L}\Sigma:D(\mathcal{B}^{op})\to D(\mathcal{A}^{op}),\qquad \mathbf{L}\Sigma(M)=M\overset{\mathbf{L}}{\otimes}_{\mathcal{B}}E.$$

3.5. **Pre-triangulated DG categories.** For any DG category $\mathcal{A}$ there exists a DG category $\mathcal{A}^{pre-tr}$ and a canonical full and faithful DG functor $F:\mathcal{A}\to\mathcal{A}^{\text{pre-tr}}$ (see [BK, Dr1]). The homotopy category $\mathrm{Ho}(\mathcal{A}^{\text{pre-tr}})$ is canonically triangulated. The DG category $\mathcal{A}$ is called *pre-triangulated* if the DG functor F is a quasi-equivalence. The DG category $\mathcal{A}^{\text{pre-tr}}$ is pre-triangulated.

Let $\mathcal{B}$ be another DG category and $G:\mathcal{A}\to\mathcal{B}$ be a quasi-equivalence. Then $G^{\text{pre-tr}}:\mathcal{A}^{\text{pre-tr}}\to\mathcal{B}^{\text{pre-tr}}$ is also a quasi-equivalence.

The DG functor F induces a DG isomorphism of DG categories $F_*:(\mathcal{A}^{\text{pre-tr}})^{op}\text{-mod}\to\mathcal{A}^{op}\text{-mod}$. Hence the functors $F_*:D((\mathcal{A}^{\text{pre-tr}})^{op})\to D(\mathcal{A}^{op})$ and $\mathbf{L}F^*:D(\mathcal{A}^{op})\to D((\mathcal{A}^{\text{pre-tr}})^{op})$ are equivalences. We obtain the following corollary.

Corollary 3.15. *Assume that a DG functor $G_1:\mathcal{A}\to\mathcal{B}$ induces a quasi-equivalence $G_1^{\text{pre-tr}}:\mathcal{A}^{\text{pre-tr}}\to\mathcal{B}^{\text{pre-tr}}$. Let $\mathcal{C}$ be another DG category and consider the DG functor $G:=G_1\otimes\mathrm{id}:\mathcal{A}\otimes\mathcal{C}\to\mathcal{B}\otimes\mathcal{C}$. Then the functors $G_*,\mathbf{L}G^*,\mathbf{R}G^!$ between the derived categories $D((\mathcal{A}\otimes\mathcal{C})^{op})$ and $D((\mathcal{B}\otimes\mathcal{C})^{op})$ are equivalences.*

Proof. The DG functor G induces the quasi-equivalence $G^{\text{pre-tr}}:(\mathcal{A}\otimes\mathcal{C})^{\text{pre-tr}}\to(\mathcal{B}\otimes\mathcal{C})^{\text{pre-tr}}$. Hence the corollary follows from the above discussion and Proposition 3.6. □

Example 3.16. *Suppose $\mathcal{B}$ is a pre-triangulated DG category. Let $G_1:\mathcal{A}\hookrightarrow\mathcal{B}$ be an embedding of a full DG subcategory so that the triangulated category $\mathrm{Ho}(\mathcal{B})$ is generated by the collection of objects $G_1(Ob\mathcal{A})$. Then the assumptions of the previous corollary hold.*

3.6. **A few lemmas.**

Lemma 3.17. *Let $\mathcal{R}$, $\mathcal{Q}$ be DG algebras and M be a DG $\mathcal{Q}\otimes\mathcal{R}^{op}$-module.*

a) For any DG modules N, S over the DG algebras $\mathcal{Q}^{op}$ and $\mathcal{R}^{op}$ respectively there is a natural isomorphism of complexes

$$\mathrm{Hom}_{\mathcal{R}^{op}}(N\otimes_{\mathcal{Q}}M,S)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{Q}^{op}}(N,\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)).$$

b) There is a natural quasi-isomorphism of complexes

$$\mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(N\overset{\mathbf{L}}{\otimes}_{\mathcal{Q}}M,S)\xrightarrow{\sim}\mathbf{R}\mathrm{Hom}_{\mathcal{Q}^{op}}(N,\mathbf{R}\mathrm{Hom}_{\mathcal{R}^{op}}(M,S)).$$

Proof. a) Indeed, for $f\in\mathrm{Hom}_{\mathcal{R}^{op}}(N\otimes_{\mathcal{Q}}M,S)$ define $\alpha(f)\in\mathrm{Hom}_{\mathcal{Q}}(N,\mathrm{Hom}_{\mathcal{R}^{op}}(M,S))$ by the formula $\alpha(f)(n)(m)=f(n\otimes m)$. Conversely, for $g\in\mathrm{Hom}_{\mathcal{Q}}(N,\mathrm{Hom}_{\mathcal{R}^{op}}(M,S))$ define $\beta(g)\in\mathrm{Hom}_{\mathcal{R}^{op}}(N\otimes_{\mathcal{Q}}M,S)$ by the formula $\beta(g)(n\otimes m)=g(n)(m)$. Then α and β are mutually inverse isomorphisms of complexes.

b) Choose quasi-isomorphisms $P\to N$ and $S\to I$, where $P\in\mathcal{P}(\mathcal{Q}^{op})$ and $I\in\mathcal{I}(\mathcal{R}^{op})$ and apply a). □

Lemma 3.18. *Let $\mathcal{R}$ be an artinian DG algebra. Then in the DG category $\mathcal{R}^{op}$-mod a direct sum of copies of $\mathcal{R}^*$ is h-injective.*

Proof. Let V be a graded vector space, $M = V \otimes \mathcal{R}^* \in \mathcal{R}^{op}$-mod and C an acyclic DG $\mathcal{R}^{op}$-module. Notice that $M = \operatorname{Hom}_k(\mathcal{R}, V)$ since $\dim \mathcal{R} < \infty$. Hence the complex

$$\operatorname{Hom}_{\mathcal{R}^{op}}(C, M) = \operatorname{Hom}_{\mathcal{R}^{op}}(C, \operatorname{Hom}_k(\mathcal{R}, V)) = \operatorname{Hom}_k(C \otimes_{\mathcal{R}} \mathcal{R}, V) = \operatorname{Hom}_k(C, V)$$

is acyclic. □

Lemma 3.19. *Let $\mathcal{B}$ be a DG algebra, such that $\mathcal{B}^i = 0$ for $i > 0$. Then the category $D(\mathcal{B}^{op})$ has truncation functors: for any DG $\mathcal{B}$-module M there exists a short exact sequence in the abelian category $Z^0(\mathcal{B}$-mod$)$*

$$\tau_{<0}M \to M \to \tau_{\geq 0}M,$$

where $H^i(\tau_{<0}M) = 0$ if $i \geq 0$ and $H^i(\tau_{\geq 0}M) = 0$ for $i < 0$.

Proof. Indeed, put $\tau_{<0}M := \oplus_{i<0}M^i \oplus d(M^{-1})$. □

Lemma 3.20. *Let $\mathcal{B}$ be a DG algebra, s.t. $\mathcal{B}^i = 0$ for $i > 0$ and $\dim \mathcal{B}^i < \infty$ for all i. Let N be a DG $\mathcal{B}$-module with finite dimensional cohomology. Then there exists an h-projective DG $\mathcal{B}$-module P and a quasi-isomorphism $P \to N$, where P in addition satisfies the following conditions*

a) $P^i = 0$ *for* $i >> 0$,

b) $\dim P^i < \infty$ *for all* i.

Proof. First assume that N is concentrated in one degree, say $N^i = 0$ for $i \neq 0$. Consider N as a k-module and put $P_0 := \mathcal{B} \otimes N$. We have a natural surjective map of DG $\mathcal{B}$-modules $\epsilon : P_0 \to N$ which is also surjective on the cohomology. Let $K := \operatorname{Ker} \epsilon$. Then $K^i = 0$ for $i > 0$ and $\dim K^i < \infty$ for all i. Consider K as a DG k-module and put $P_{-1} := \mathcal{B} \otimes K$. Again we have a surjective map of DG $\mathcal{B}$-modules $P_{-1} \to K$ which is surjective and surjective on cohomology. And so on. This way we obtain an exact sequence of DG $\mathcal{B}$-modules

$$\ldots \to P_{-1} \to P_0 \stackrel{\epsilon}{\to} N \to 0,$$

where $P_{-j}^i = 0$ for $i > 0$ and $\dim P_{-j}^i < \infty$ for all j. Let $P := \oplus_j P_{-j}[j]$ be the "total" DG $\mathcal{B}$-module of the complex $\ldots \to P_{-1} \to P_0 \to 0$. Then $\epsilon : P \to N$ is a quasi-isomorphism. Since each DG $\mathcal{B}$-module P_{-j} has the property (P), the module P is h-projective by Remark 3.5a). Also $P^i = 0$ for $i > 0$ and $\dim P^i < \infty$ for all i.

How consider the general case. Let $H^s(N) = 0$ and $H^i(N) = 0$ for all $i < s$. Replacing N by $\tau_{\geq s}N$ (Lemma 3.19) we may and will assume that $N^i = 0$ for $i < s$. Then $M := (\operatorname{Ker} d_N) \cap N^s$ is a DG $\mathcal{B}$-submodule of N which is not zero. If the embedding $M \hookrightarrow N$ is a quasi-isomorphism, then we may replace N by M and so we are done by the previous argument. Otherwise we have a short exact sequence of DG $\mathcal{B}$-modules

$$o \to M \to N \to N/M \to 0$$

with $\dim H(M), \dim H(N/M) < \dim H(N)$. By the induction on $\dim H(N)$ we may assume that the lemma holds for M and N/M . But then it also holds for N . □

Corollary 3.21. *Let $\mathcal{B}$ be a DG algebra, s.t. $\mathcal{B}^i = 0$ for $i > 0$, $\dim \mathcal{B}^i < \infty$ for all i and the algebra $H^0(\mathcal{B})$ is local. Let N be a DG $\mathcal{B}$ -module with finite dimensional cohomology. Then N is quasi-isomorphic to a finite dimensional DG $\mathcal{B}$ -module.*

Proof. By Lemma 3.20 there exists a bounded above and locally finite DG $\mathcal{B}$ -module P which is quasi-isomorphic to N . It remains to apply the appropriate truncation functor to P (Lemma 3.19). □

Corollary 3.22. *Let $\mathcal{B}$ be an augmented DG algebra, s.t. $\mathcal{B}^i = 0$ for $i > 0$, $\dim \mathcal{B}^i < \infty$ for all i and the algebra $H^0(\mathcal{B})$ is local. Denote by $\langle k \rangle \subset D(\mathcal{B})$ the triangulated envelope of the DG $\mathcal{B}$ -module k . Let N be a DG $\mathcal{B}$ -module with finite dimensional cohomology. Then $N \in \langle k \rangle$.*

Proof. By the previous corollary we may assume that N is finite dimensional. But then an easy applying of the Nakayama lemma for $H^0(\mathcal{B})$ shows that N has a filtration by DG $\mathcal{B}$ -modules with subquotients isomorphic to k . □

Lemma 3.23. *Let $\mathcal{B}$ and $\mathcal{C}$ be DG algebras. Consider the DG algebra $\mathcal{B} \otimes \mathcal{C}$ and a homomorphism of DG algebras $F : \mathcal{B} \to \mathcal{B} \otimes \mathcal{C}$, $F(b) = b \otimes 1$. Let N be an h-projective (resp. h-injective) DG $\mathcal{B} \otimes \mathcal{C}$ -module. Then the DG $\mathcal{B}$ -module $F_* N$ is also h-projective (resp. h-injective).*

Proof. The assertions follow from the fact that the DG functor $F_* : \mathcal{B} \otimes \mathcal{C}\text{-mod} \to \mathcal{B}\text{-mod}$ has a left adjoint DG functor F^* (resp. right adjoint DG functor $F^!$) which preserves acyclic DG modules. Indeed,

$$F^*(M) = \mathcal{C} \otimes_k M, \qquad F^!(M) = \mathrm{Hom}_k(\mathcal{C}, M).$$

□

Part 2. Deformation functors

4. The homotopy deformation and co-deformation pseudo-functors

Denote by **Gpd** the 2-category of groupoids.

Let $\mathcal{E}$ be a category and $F, G : \mathcal{E} \to \mathbf{Gpd}$ two pseudo-functors. A morphism $\epsilon : F \to G$ is called full and faithful (resp. an equivalence) if for every $X \in Ob\mathcal{E}$ the functor $\epsilon_X : F(X) \to G(X)$ is full and faithful (resp. an equivalence). We call F and G equivalent if there exists an equivalence $F \to G$.

It the rest of this paper we will usually denote by $\mathcal{A}$ a fixed DG category and by E a DG $\mathcal{A}^{op}$ -module.

Let us define the homotopy deformation pseudo-functor $\operatorname{Def}^{\mathrm{h}}(E) : \mathrm{dgart} \to \mathbf{Gpd}$. This functor describes "infinitesimal" (i.e. along artinian DG algebras) deformations of E in the homotopy category of DG $\mathcal{A}^{op}$ -modules.

Definition 4.1. *Let $\mathcal{R}$ be an artinian DG algebra. An object in the groupoid $\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ is a pair (S,σ) , where $S\in\mathcal{A}^{op}_{\mathcal{R}}$-mod and $\sigma : i^*S \to E$ is an isomorphism of DG $\mathcal{A}^{op}$ -modules such that the following holds: there exists an isomorphism of graded $\mathcal{A}^{op}_{\mathcal{R}}$ -modules $\eta : (E\otimes\mathcal{R})^{\mathrm{gr}} \to S^{\mathrm{gr}}$ so that the composition*

$$E = i^*(E\otimes\mathcal{R}) \stackrel{i^*(\eta)}{\to} i^*S \stackrel{\sigma}{\to} E$$

is the identity.

Given objects $(S,\sigma),(S',\sigma')\in\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ a map $f:(S,\sigma)\to(S',\sigma')$ is an isomorphism $f:S\to S'$ such that $\sigma'\cdot i^*f=\sigma$. An allowable homotopy between maps f,g is a homotopy $h:f\to g$ such that $i^*(h)=0$. We define morphisms in $\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ to be classes of maps modulo allowable homotopies.

Note that a homomorphism of artinian DG algebras $\phi:\mathcal{R}\to\mathcal{Q}$ induces the functor $\phi^*:\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(E)\to\operatorname{Def}^{\mathrm{h}}_{\mathcal{Q}}(E)$. This defines the pseudo-functor

$$\operatorname{Def}^{h}(E) : \mathrm{dgart} \to \mathbf{Gpd}.$$

We refer to objects of $\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ as homotopy $\mathcal{R}$ -deformations of E .

The term "homotopy" in the above definition is used to distinguish the pseudo-functor $\operatorname{Def}^{\mathrm{h}}$ from the pseudo-functor Def of *derived deformations* (Definition 10.1). It may be justified by the fact that $\operatorname{Def}^{\mathrm{h}}(E)$ depends (up to equivalence) only on the isomorphism class of E in $\operatorname{Ho}(\mathcal{A}^{op}\text{-mod})$ (Corollary 8.4 a)).

Example 4.2. *We call $(p^*E,\mathrm{id})\in\operatorname{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ the trivial $\mathcal{R}$ -deformation of E .*

Definition 4.3. *Denote by $\operatorname{Def}^{\mathrm{h}}_{+}(E)$, $\operatorname{Def}^{\mathrm{h}}_{-}(E)$, $\operatorname{Def}^{\mathrm{h}}_{0}(E)$, $\operatorname{Def}^{\mathrm{h}}_{\mathrm{cl}}(E)$ the restrictions of the pseudo-functor $\operatorname{Def}^{\mathrm{h}}(E)$ to subcategories dgart_+ , dgart_- , art , cart respectively.*

Let us give an alternative description of the same deformation problem. We will define the homotopy *co-deformation* pseudo-functor $\operatorname{coDef}^{\mathrm{h}}(E)$ and show that it is equivalent to $\operatorname{Def}^{\mathrm{h}}(E)$. The point is that in practice one should use $\operatorname{Def}^{\mathrm{h}}(E)$ for a h-projective E and $\operatorname{coDef}^{\mathrm{h}}(E)$ for a h-injective E (see Section 11).

For an artinian DG algebra $\mathcal{R}$ recall the $\mathcal{R}^{op}$ -module $\mathcal{R}^*=\operatorname{Hom}_k(\mathcal{R},k)$.

Definition 4.4. *Let $\mathcal{R}$ be an artinian DG algebra. An object in the groupoid $\operatorname{coDef}^{\mathrm{h}}_{\mathcal{R}}(E)$ is a pair (T,τ) , where T is a DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module and $\tau:E\to i^!T$ is an isomorphism of DG $\mathcal{A}^{op}$ -modules so that the following holds: there exists an isomorphism of graded $\mathcal{A}^{op}_{\mathcal{R}}$ -modules $\delta:T^{\mathrm{gr}}\to(E\otimes\mathcal{R}^*)^{\mathrm{gr}}$ such that the composition*

$$E \stackrel{\tau}{\to} i^!T \stackrel{i^!(\delta)}{\to} i^!(E\otimes\mathcal{R}^*) = E$$

is the identity.

Given objects (T,τ) *and* $(T',\tau')\in \mathrm{coDef}^h_{\mathcal{R}}(E)$ *a map* $g:(T,\tau)\to(T',\tau')$ *is an isomorphism* $f:T\to T'$ *such that* $i^! f\cdot\tau=\tau'$ *. An allowable homotopy between maps* f,g *is a homotopy* $h:f\to g$ *such that* $i^!(h)=0$ *. We define morphisms in* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(E)$ *to be classes of maps modulo allowable homotopies.*

Note that a homomorphism of DG algebras $\phi:\mathcal{R}\to\mathcal{Q}$ *induces the functor* $\phi^!:\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(E)\to \mathrm{coDef}^{\mathrm{h}}_{\mathcal{Q}}(E)$ *. This defines the pseudo-functor*

$$\mathrm{coDef}^{\mathrm{h}}(E):\mathrm{dgart}\to\mathbf{Gpd}.$$

We refer to objects of $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(E)$ as homotopy $\mathcal{R}$ -co-deformations of E .

Example 4.5. *For example we can take* $T=E\otimes\mathcal{R}^*$ *with the differential* $d_{E,R^*}:=d_E\otimes 1+1\otimes d_{R^*}$ *(and* $\tau=\mathrm{id}$ *). This we consider as the trivial* $\mathcal{R}$ *-co-deformation of* E *.*

Definition 4.6. *Denote by* $\mathrm{coDef}^{\mathrm{h}}_{+}(E)$ *,* $\mathrm{coDef}^{\mathrm{h}}_{-}(E)$ *,* $\mathrm{coDef}^{\mathrm{h}}_{0}(E)$ *,* $\mathrm{coDef}^{\mathrm{h}}_{\mathrm{cl}}(E)$ *the restrictions of the pseudo-functor* $\mathrm{coDef}^{\mathrm{h}}(E)$ *to subcategories* dgart_+ *,* dgart_- *,* art *,* cart *respectively.*

Proposition 4.7. *There exists a natural equivalence of pseudo-functors*

$$\delta=\delta^E:\mathrm{Def}^{\mathrm{h}}(E)\to\mathrm{coDef}^{\mathrm{h}}(E).$$

Proof. We use Lemma 3.9 above. Namely, let S be an $\mathcal{R}$ -deformation of E . Then $S\otimes_{\mathcal{R}}\mathcal{R}^*$ is an $\mathcal{R}$ -co-deformation of E . Conversely, given an $\mathcal{R}$ -co-deformation T of E the DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module $\mathrm{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*,T)$ is an $\mathcal{R}$ -deformation of E . This defines mutually inverse equivalences $\delta_{\mathcal{R}}$ and $\delta_{\mathcal{R}}^{-1}$ between the groupoids $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ and $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(E)$, which extend to morphisms between pseudo-functors $\mathrm{Def}^{\mathrm{h}}(E)$ and $\mathrm{coDef}^{\mathrm{h}}(E)$. Let us be a little more explicit.

Let $\phi:\mathcal{R}\to\mathcal{Q}$ be a homomorphism of artinian DG algebras and $S\in\mathrm{Def}^{\mathrm{h}}(E)$. Then

$$\delta_{\mathcal{Q}}\cdot\phi^*(S)=S\otimes_{\mathcal{R}}Q\otimes_{\mathcal{Q}}\mathcal{Q}^*=S\otimes_{\mathcal{R}}\mathcal{Q}^*,\qquad \phi^!\cdot\delta_{\mathcal{R}}(S)=\mathrm{Hom}_{\mathcal{R}^{op}}(\mathcal{Q},S\otimes_{\mathcal{R}}\mathcal{R}^*).$$

The isomorphism α_ϕ of these DG $\mathcal{A}^{op}_{\mathcal{Q}}$ -modules is defined by $\alpha_\phi(s\otimes f)(q)(r):=sf(q\phi(r))$ for $s\in S$, $f\in\mathcal{Q}^*$, $q\in\mathcal{Q}$, $r\in\mathcal{R}$. Given another homomorphism $\psi:\mathcal{Q}\to\mathcal{Q}'$ of DG algebras one checks the cocycle condition $\alpha_{\psi\phi}=\psi^!(\alpha_\phi)\cdot\alpha_\psi$ (under the natural isomorphisms $(\psi\phi)^*=\psi^*\phi^*$, $(\psi\phi)^!=\psi^!\phi^!$). □

5. Maurer-Cartan pseudo-functor

Definition 5.1. *For a DG algebra* $\mathcal{C}$ *with the differential* d *consider the (inhomogeneous) quadratic map*

$$Q:\mathcal{C}^1\to\mathcal{C}^2;\quad Q(\alpha)=d\alpha+\alpha^2.$$

We denote by $MC(\mathcal{C})$ *the (usual) Maurer-Cartan cone*

$$MC(\mathcal{C})=\{\alpha\in\mathcal{C}^1|Q(\alpha)=0\}.$$

Note that $\alpha \in MC(\mathcal{C})$ is equivalent to the operator $d+\alpha : \mathcal{C} \to \mathcal{C}$ having square zero. Thus the set $MC(\mathcal{C})$ describes the space of "internal" deformations of the differential in the complex $\mathcal{C}$.

Definition 5.2. *Let $\mathcal{B}$ be a DG algebra with the differential d and a nilpotent DG ideal $\mathcal{I} \subset \mathcal{B}$. We define the Maurer-Cartan groupoid $\mathcal{MC}(\mathcal{B},\mathcal{I})$ as follows. The set of objects of $\mathcal{MC}(\mathcal{B},\mathcal{I})$ is the cone $MC(\mathcal{I})$. Maps between objects are defined by means of the gauge group $G(\mathcal{B},\mathcal{I}) := 1+\mathcal{I}^0$ ($\mathcal{I}^0$ is the degree zero component of $\mathcal{I}$) acting on $\mathcal{MC}(\mathcal{B},\mathcal{I})$ by the formula*

$$g : \alpha \mapsto g\alpha g^{-1} + gd(g^{-1}),$$

where $g \in G(\mathcal{B},\mathcal{I})$, $\alpha \in MC(\mathcal{I})$. (This comes from the conjugation action on the space of differentials $g : d+\alpha \mapsto g(d+\alpha)g^{-1}$.) So if $g(\alpha) = \beta$, we call g a map from α to β . Denote by $G(\alpha,\beta)$ the collection of such maps. We define the set $\mathrm{Hom}(\alpha,\beta)$ in the category $\mathcal{MC}(\mathcal{B},\mathcal{I})$ to consist of homotopy classes of maps, where the homotopy relation is defined as follows. There is an action of the group $\mathcal{I}^{-1}$ on the set $G(\alpha,\beta)$:

$$h : g \mapsto g + d(h) + \beta h + h\alpha,$$

for $h \in \mathcal{I}^{-1}, g \in G(\alpha,\beta)$. We call two maps homotopic, if they lie in the same $\mathcal{I}^{-1}$ -orbit.

To make the category $\mathcal{MC}(\mathcal{B},\mathcal{I})$ well defined we need to prove a lemma.

Lemma 5.3. *Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in MC(\mathcal{I})$ and $g_1 \in G(\alpha_1,\alpha_2)$, $g_1, g_3 \in G(\alpha_2,\alpha_3)$, $g_4 \in G(\alpha_3,\alpha_4)$. If g_2 and g_3 are homotopic, then so are $g_2 g_1$ and $g_3 g_1$ (resp. $g_4 g_2$ and $g_4 g_3$).*

Proof. Omit. □

Let $\mathcal{C}$ be another DG algebra with a nilpotent DG ideal $\mathcal{J} \subset \mathcal{C}$. A homomorphism of DG algebras $\psi : \mathcal{B} \to \mathcal{C}$ such that $\psi(\mathcal{I}) \subset \mathcal{J}$ induces the functor

$$\psi^* : \mathcal{MC}(\mathcal{B},\mathcal{I}) \to \mathcal{MC}(\mathcal{C},\mathcal{J}).$$

Definition 5.4. *Let $\mathcal{B}$ be a DG algebra and $\mathcal{R}$ be an artinian DG algebra with the maximal ideal $m \subset \mathcal{R}$. Denote by $\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$ the Maurer-Cartan groupoid $\mathcal{MC}(\mathcal{B}\otimes\mathcal{R}, \mathcal{B}\otimes m)$. A homomorphism of artinian DG algebras $\phi : \mathcal{R} \to \mathcal{Q}$ induces the functor $\phi^* : \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) \to \mathcal{MC}_{\mathcal{Q}}(\mathcal{B})$. Thus we obtain the Maurer-Cartan pseudo-functor*

$$\mathcal{MC}(\mathcal{B}) : \mathrm{dgart} \to \mathbf{Gpd}.$$

We denote by $\mathcal{MC}_+(\mathcal{B})$, $\mathcal{MC}_-(\mathcal{B})$, $\mathcal{MC}_0(\mathcal{B})$, $\mathcal{MC}_{\mathrm{cl}}(\mathcal{B})$ the restrictions of the pseudo-functor $\mathcal{MC}(\mathcal{B})$ to subcategories dgart_+ , dgart_- , art , cart .

Remark 5.5. *A homomorphism of DG algebras $\psi : \mathcal{C} \to \mathcal{B}$ induces a morphism of pseudo-functors*

$$\psi^* : \mathcal{MC}(\mathcal{C}) \to \mathcal{MC}(\mathcal{B}).$$

6. Description of pseudo-functors $\mathrm{Def}^{\mathrm{h}}(E)$ and $\mathrm{coDef}^{\mathrm{h}}(E)$

We are going to give a description of the pseudo-functor $\mathrm{Def}^{\mathrm{h}}$ and hence also of the pseudo-functor $\mathrm{coDef}^{\mathrm{h}}$ via the Maurer-Cartan pseudo-functor $\mathcal{MC}$.

Proposition 6.1. *Let $\mathcal{A}$ be a DG category and $E \in \mathcal{A}^{op}$-mod. Denote by $\mathcal{B}$ the DG algebra* $\mathrm{End}(E)$. *Then there exists an equivalence of pseudo-functors $\theta = \theta^E : \mathcal{MC}(\mathcal{B}) \to \mathrm{Def}^{\mathrm{h}}(E)$. (Hence also $\mathcal{MC}(\mathcal{B})$ and $\mathrm{coDef}^{\mathrm{h}}(E)$ are equivalent.)*

Proof. Fix an artinian DG algebra $\mathcal{R}$ with the maximal ideal m. Let us define an equivalence of groupoids

$$\theta_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E).$$

Denote by $S_0 = p^*E \in \mathcal{A}^{op}_{\mathcal{R}}$-mod the trivial $\mathcal{R}$-deformation of E with the differential $d_{E,\mathcal{R}} = d_E \otimes 1 + 1 \otimes d_{\mathcal{R}}$. There is a natural isomorphism of DG algebras $\mathrm{End}(S_0) = \mathcal{B} \otimes \mathcal{R}$.

Let $\alpha \in \mathcal{MC}(\mathcal{B} \otimes m) = \mathcal{MC}_{\mathcal{R}}(\mathcal{B})$. Then in particular $\alpha \in \mathrm{End}^1(S_0)$. Hence $d_\alpha := d_{E,\mathcal{R}} + \alpha$ is an endomorphism of degree 1 of the graded module S_0^{gr}. The Maurer-Cartan condition on α is equivalent to $d_\alpha^2 = 0$. Thus we obtain an object $S_\alpha \in \mathcal{A}^{op}_{\mathcal{R}}$-mod. Clearly $i^*S_\alpha = E$, so that

$$\theta_{\mathcal{R}}(\alpha) := (S_\alpha, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E).$$

One checks directly that this map on objects extends naturally to a functor $\theta_{\mathcal{R}} : \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) \to \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$. Indeed, maps between Maurer-Cartan objects induce isomorphisms of the corresponding deformations; also homotopies between such maps become allowable homotopies between the corresponding isomorphisms.

It is clear that the functors $\theta_{\mathcal{R}}$ are compatible with the functors ϕ^* induced by morphisms of DG algebras $\phi : \mathcal{R} \to \mathcal{Q}$. So we obtain a morphism of pseudo-functors

$$\theta : \mathcal{MC}(\mathcal{B}) \to \mathrm{Def}^{\mathrm{h}}(E).$$

It suffices to prove that $\theta_{\mathcal{R}}$ is an equivalence for each $\mathcal{R}$.

Surjective. Let $(T, \tau) \in \mathrm{Def}^{h}_{\mathcal{R}}(E)$. We may and will assume that $T^{gr} = S_0^{gr}$ and $\tau = \mathrm{id}$. Then $\alpha_T := d_T - d_{\mathcal{R},E} \in \mathrm{End}^1(S_0) = (\mathcal{B} \otimes \mathcal{R})^1$ is an element in $MC(\mathcal{B} \otimes \mathcal{R})$. Since $i^*\alpha_T = 0$ it follows that $\alpha_T \in \mathcal{MC}_{\mathcal{R}}(\mathcal{B})$. Thus $(T, \tau) = \theta_{\mathcal{R}}(\alpha_T)$.

Full. Let $\alpha, \beta \in \mathcal{MC}_{\mathcal{R}}(\mathcal{B})$. An isomorphism between the corresponding objects $\theta_{\mathcal{R}}(\alpha)$ and $\theta_{\mathcal{R}}(\beta)$ is defined by an element $f \in \mathrm{End}(S_0) = (\mathcal{B} \otimes \mathcal{R})$ of degree zero. The condition $i^*f = \mathrm{id}_Z$ means that $f \in 1 + (\mathcal{B} \otimes m)^0$. Thus $f \in G(\alpha, \beta)$.

Faithful. Let $\alpha, \beta \in \mathcal{MC}_{\mathcal{R}}(\mathcal{B})$ and $f, g \in G(\alpha, \beta)$. One checks directly that f and g are homotopic (i.e. define the same morphism in $\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$) if and only if there exists an allowable homotopy between $\theta_{\mathcal{R}}(f)$ and $\theta_{\mathcal{R}}(g)$. This proves the proposition. $\square$

Corollary 6.2. *For $E \in \mathcal{A}^{op}$-mod the pseudo-functors* $\mathrm{Def}^{\mathrm{h}}(E)$ *and* $\mathrm{coDef}^{\mathrm{h}}(E)$ *depend (up to equivalence) only on the DG algebra* $\mathrm{End}(E)$.

We will prove a stronger result in Corollary 8.2 below.

Example 6.3. *Let* $E \in \mathcal{A}^{op}$*-mod and denote* $\mathcal{B} = \mathrm{End}(E)$ *. Consider* $\mathcal{B}$ *as a (free) right* $\mathcal{B}$ *-module, i.e.* $\mathcal{B} \in \mathcal{B}^{op}$*-mod . Then* $\mathrm{Def}^h(\mathcal{B}) \simeq \mathrm{Def}^h(E)$ *(* $\simeq \mathrm{coDef}^h(\mathcal{B}) \simeq \mathrm{coDef}^h(E)$ *) because* $\mathrm{End}(\mathcal{B}) = \mathrm{End}(E) = \mathcal{B}$ *. We will describe this equivalence directly in Section 9 below.*

7. Obstruction Theory

It is convenient to describe the obstruction theory for our (equivalent) deformation pseudo-functors Def^h and coDef^h using the Maurer-Cartan pseudo-functor $\mathcal{MC}(\mathcal{B})$ for a fixed DG algebra $\mathcal{B}$.

Let $\mathcal{R}$ be an artinian DG algebra with a maximal ideal m , such that $m^{n+1} = 0$. Put $I = m^n$, $\overline{\mathcal{R}} = \mathcal{R}/I$ and $\pi : \mathcal{R} \to \overline{\mathcal{R}}$ the projection morphism. We have $mI = Im = 0$.

Note that the kernel of the homomorphism $1 \otimes \pi : \mathcal{B} \otimes \mathcal{R} \to \mathcal{B} \otimes \overline{\mathcal{R}}$ is the (DG) ideal $\mathcal{B} \otimes I$. The next proposition describes the obstruction theory for lifting objects and morphisms along the functor

$$\pi^* : \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) \to \mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B}).$$

It is close to [GM]. Note however a difference in part 3) and part 4) since we do not assume that out DG algebras live in nonnegative dimensions (and of course we work with DG algebras and not with DG Lie algebras).

Proposition 7.1. *1). There exists a map* $o_2 : Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B}) \to H^2(\mathcal{B} \otimes I)$ *such that* $\alpha \in Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B})$ *is in the image of* π^* *if and only if* $o_2(\alpha) = 0$ *. Furthermore if* $\alpha, \beta \in Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B})$ *are isomorphic, then* $o_2(\alpha) = 0$ *if and only if* $o_2(\beta) = 0$ *.*

2). Let $\xi \in Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B})$ *. Assume that the fiber* $(\pi^*)^{-1}(\xi)$ *is not empty. Then there exists a simply transitive action of the group* $Z^1(\mathcal{B} \otimes I)$ *on the set* $Ob(\pi^*)^{-1}(\xi)$ *. Moreover the composition of the difference map*

$$Ob(\pi^*)^{-1}(\xi) \times Ob(\pi^*)^{-1}(\xi) \to Z^1(\mathcal{B} \otimes I)$$

with the projection

$$Z^1(\mathcal{B} \otimes I) \to H^1(\mathcal{B} \otimes I)$$

which we denote by

$$o_1 : Ob(\pi^*)^{-1}(\xi) \times Ob(\pi^*)^{-1}(\xi) \to H^1(\mathcal{B} \otimes I)$$

has the following property: for $\alpha, \beta \in Ob(\pi^*)^{-1}(\xi)$ *there exists a morphism* $\gamma : \alpha \to \beta$ *s.t.* $\pi^*(\gamma) = \mathrm{id}_\xi$ *if and only if* $o_1(\alpha, \beta) = 0$ *.*

3). Let $\tilde{\alpha}, \tilde{\beta} \in Ob\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$ *be isomorphic objects and let* $f : \alpha \to \beta$ *be a morphism from* $\alpha = \pi^*(\tilde{\alpha})$ *to* $\beta = \pi^*(\tilde{\beta})$ *. Then there is a transitive action of the group* $H^0(\mathcal{B} \otimes I)$ *on the set* $(\pi^*)^{-1}(f)$ *of morphisms* $\tilde{f} : \tilde{\alpha} \to \tilde{\beta}$ *such that* $\pi^*(\tilde{f}) = f$ *.*

4). In the notation of 3) suppose that the fiber $(\pi^*)^{-1}(f)$ *is non-empty. Then the kernel of the above action coincides with the kernel of the map*

$$H^0(\mathcal{B}\otimes I)\to H^0(\mathcal{B}\otimes m, d^{\alpha,\beta}), \tag{7.1}$$

where $d^{\alpha,\beta}$ *is a differential on the graded vector space* $\mathcal{B}\otimes m$ *given by the formula*

$$d^{\alpha,\beta}(x)=dx+\beta x-(-1)^{\bar{x}}x\alpha.$$

In particular the difference map

$$o_0:(\pi^*)^{-1}(f)\times(\pi^*)^{-1}(f)\to \operatorname{Im}(H^0(\mathcal{B}\otimes I)\to H^0(\mathcal{B}\otimes m, d^{\alpha,\beta}))$$

has the property: if $\tilde{f},\tilde{f}'\in(\pi^*)^{-1}(f)$ *, then* $\tilde{f}=\tilde{f}'$ *if and only if* $o_0(\tilde{f},\tilde{f}')=0$ *.*

Proof. 1) Let $\alpha\in Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B})=MC(\mathcal{B}\otimes(m/I))$. Choose $\tilde{\alpha}\in(\mathcal{B}\otimes m)^1$ such that $\pi(\tilde{\alpha})=\alpha$. Consider the element

$$Q(\tilde{\alpha})=d\tilde{\alpha}+\tilde{\alpha}^2\in(\mathcal{B}\otimes m)^2.$$

Since $Q(\alpha)=0$ we have $Q(\tilde{\alpha})\in(\mathcal{B}\otimes I)^2$. We claim that $dQ(\tilde{\alpha})=0$. Indeed,

$$dQ(\tilde{\alpha})=d(\tilde{\alpha}^2)=d(\tilde{\alpha})\tilde{\alpha}-\tilde{\alpha}d(\tilde{\alpha}).$$

We have $d(\tilde{\alpha})\equiv\tilde{\alpha}^2(mod(\mathcal{B}\otimes I))$. Hence $dQ(\tilde{\alpha})=-\tilde{\alpha}^3+\tilde{\alpha}^3=0$ (since $I\cdot m=0$).

Furthermore suppose that $\tilde{\alpha}'\in(\mathcal{B}\otimes m)^1$ is another lift of α , i.e. $\tilde{\alpha}'-\tilde{\alpha}\in(\mathcal{B}\otimes I)^1$. Then

$$Q(\tilde{\alpha}')-Q(\tilde{\alpha})=d(\tilde{\alpha}'-\tilde{\alpha})+(\tilde{\alpha}'-\tilde{\alpha})(\tilde{\alpha}'+\tilde{\alpha})=d(\tilde{\alpha}'-\tilde{\alpha}).$$

Thus the cohomology class of the cocycle $Q(\tilde{\alpha})$ is independent of the lift $\tilde{\alpha}$. We denote this class by $o_2(\alpha)\in H^2(\mathcal{B}\otimes I)$.

If $\alpha=\pi^*(\tilde{\alpha})$ for some $\tilde{\alpha}\in Ob\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$, then clearly $o_2(\alpha)=0$. Conversely, suppose $o_2(\alpha)=0$ and let $\tilde{\alpha}$ be as above. Then $dQ(\tilde{\alpha})=d\tau$ for some $\tau\in(\mathcal{B}\otimes I)^1$. Put $\tilde{\alpha}'=\tilde{\alpha}-\tau$. Then

$$Q(\tilde{\alpha}')=d\tilde{\alpha}-d\tau+\tilde{\alpha}^2-\tilde{\alpha}\tau-\tau\tilde{\alpha}+\tau^2=Q(\tilde{\alpha})-d\tau=0.$$

Let us prove the last assertion in 1). Assume that $\pi^*(\tilde{\alpha})=\alpha$ and $\beta=g(\alpha)$ for some $g\in 1+(\mathcal{B}\otimes m/I)^0$. Choose a lift $\tilde{g}\in 1+(\mathcal{B}\otimes m)^0$ of g and put $\tilde{\beta}:=\tilde{g}(\tilde{\alpha})$. Then $\pi^*(\tilde{\beta})=\beta$. This proves 1).

2). Let $\alpha\in Ob(\pi^*)^{-1}(\xi)$ and $\eta\in Z^1(\mathcal{B}\otimes I)$. Then

$$Q(\alpha+\eta)=d\alpha+d\eta+\alpha^2+\alpha\eta+\eta\alpha+\eta^2=Q(\alpha)+d\eta=0.$$

So $\alpha+\eta\in Ob(\pi^*)^{-1}(\xi)$. This defines the action of the group $Z^1(\mathcal{B}\otimes I)$ on the set $Ob(\pi^*)^{-1}(\xi)$.

Let $\alpha,\beta\in Ob(\pi^*)^{-1}(\xi)$. Then $\alpha-\beta\in(\mathcal{B}\otimes I)^1$ and

$$d(\alpha-\beta)=d\alpha-d\beta+\beta(\alpha-\beta)+(\alpha-\beta)\beta+(\alpha-\beta)^2=Q(\alpha)-Q(\beta)=0.$$

Thus $Z^1(\mathcal{B}\otimes I)$ acts simply transitively on $Ob(\pi^*)^{-1}(\xi)$. Now let $o_1(\alpha,\beta)\in H^1(\mathcal{B}\otimes I)$ be the cohomology class of $\alpha-\beta$. We claim that there exists a morphism $\gamma:\alpha\to\beta$ covering id_ξ if and only if $o_1(\alpha,\beta)=0$.

Indeed, let γ be such a morphism. Then by definition the morphisms $\pi^*(\gamma)$ and id_ξ are homotopic. That is there exists $h\in(\mathcal{B}\otimes(m/I))^{-1}$ such that

$$\mathrm{id}_\xi=\pi^*(\gamma)+d(h)+\xi h+h\xi.$$

Choose a lifting $\tilde{h}\in(\mathcal{B}\otimes m)^{-1}$ on h and replace the morphism γ by the homotopical one

$$\delta=\gamma+d(\tilde{h})+\beta\tilde{h}+\tilde{h}\alpha.$$

Thus $\delta=1+u$, where $u\in(\mathcal{B}\otimes I)^0$. But then

$$\beta=\delta\alpha\delta^{-1}+\delta d(\delta^{-1})=\alpha-du,$$

so that $o_1(\alpha,\beta)=0$.

Conversely, let $\alpha-\beta=du$ for some $u\in(\mathcal{B}\otimes I)^0$. Then $\delta=1+u$ is a morphism from α to β and $\pi^*(\delta)=\mathrm{id}_\xi$. This proves 2).

3). Let us define the action of the group $Z^0(\mathcal{B}\otimes I)$ on the set $(\pi^*)^{-1}(f)$. Let $\tilde{f}:\tilde{\alpha}\to\tilde{\beta}$ be a lift of f , and $v\in Z^0(\mathcal{B}\otimes I)$. Then $\tilde{f}+v$ also belongs to $(\pi^*)^{-1}(f)$. If $v=du$ for $u\in(\mathcal{B}\otimes I)^{-1}$, then

$$\tilde{f}+v=\tilde{f}+du+\tilde{\beta}u+u\tilde{\alpha}$$

and hence morphisms $\tilde{f}$ and $\tilde{f}+v$ are homotopic. This induces the action of $H^0(\mathcal{B}\otimes I)$ on the set $(\pi^*)^{-1}(f)$.

To show that this action is transitive let $\tilde{f}':\tilde{\alpha}\to\tilde{\beta}$ be another morphism in $(\pi^*)^{-1}(f)$. This means by definition that there exists $h\in(\mathcal{B}\otimes(m/I))^{-1}$ such that

$$f=\pi^*(\tilde{f}')+dh+\beta h+h\alpha.$$

Choose a lifting $\tilde{h}\in(\mathcal{B}\otimes m)^{-1}$ of h and replace $\tilde{f}'$ by the homotopical morphism

$$\tilde{g}=\tilde{f}'+d\tilde{h}+\tilde{\beta}\tilde{h}+\tilde{h}\tilde{\alpha}.$$

Then $\tilde{g}=\tilde{f}+v$ for $v\in(\mathcal{B}\otimes I)^0$. Since $\tilde{f},\tilde{g}:\tilde{\alpha}\to\tilde{\beta}$ we must have that $v\in Z^0(\mathcal{B}\otimes I)$. This shows the transitivity and proves 3).

4). Suppose that for some $v\in Z^0(\mathcal{B}\otimes I)$ and for some $\tilde{f}\in(\pi^*)^{-1}(f)$ we have that $\tilde{f}+v=\tilde{f}$. This means, by definition, that there exists an element $h\in(\mathcal{B}\otimes m)^{-1}$ such that $d^{\alpha,\beta}(h)=v$. In other words, the class $[v]\in H^0(\mathcal{B}\otimes I)$ lies in the kernel of the map (7.1). This proves 4). $\square$

8. Invariance theorem and its implications

Theorem 8.1. *Let $\phi : \mathcal{B} \to \mathcal{C}$ be a quasi-isomorphism of DG algebras. Then the induced morphism of pseudo-functors*

$$\phi^* : \mathcal{MC}(\mathcal{B}) \to \mathcal{MC}(\mathcal{C})$$

is an equivalence.

Proof. The proof is almost the same as that of Theorem 2.4 in [GM]. We present it for reader's convenience and also because of the slight difference in language: in [GM] they work with DG Lie algebras as opposed to DG algebras.

Fix an artinian DG algebra $\mathcal{R}$ with the maximal ideal $m \subset \mathcal{R}$, such that $m^{n+1} = 0$. We prove that

$$\phi^* : \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) \to \mathcal{MC}_{\mathcal{R}}(\mathcal{C})$$

is an equivalence by induction on n. If $n = o$, then both groupoids contain one object and one morphism, so are equivalent. Let $n > 0$. Put $I = m^n$ with the projection $\pi : \mathcal{R} \to \mathcal{R}/I = \overline{\mathcal{R}}$. We have the commutative functorial diagram

$$\begin{array}{ccc} \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) & \xrightarrow{\phi^*} & \mathcal{MC}_{\mathcal{R}}(\mathcal{C}) \\ \pi^* \downarrow & & \downarrow \pi^* \\ \mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{B}) & \xrightarrow{\phi^*} & \mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{C}). \end{array}$$

By induction we may assume that the bottom functor is an equivalence. To prove the same about the top one we need to analyze the fibers of the functor π^*. This has been done by the obstruction theory.

We will prove that the functor

$$\phi^* : \mathcal{MC}_{\mathcal{R}}(\mathcal{B}) \to \mathcal{MC}_{\mathcal{R}}(\mathcal{C})$$

is surjective on the isomorphism classes of objects, is full and is faithful.

Surjective on isomorphism classes. Let $\beta \in Ob\mathcal{MC}_{\mathcal{R}}(\mathcal{C})$. Then $\pi^*\beta \in Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{C})$. By the induction hypothesis there exists $\alpha' \in Ob\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{C})$ and an isomorphism $g : \phi^*\alpha' \to \pi^*\beta$. Now

$$H^2(\phi)o_2(\alpha') = o_2(\phi^*\alpha') = o_2(\pi^*\beta) = 0.$$

Hence $o_2(\alpha') = 0$, so there exists $\tilde{\alpha} \in Ob\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$ such that $\pi^*\tilde{\alpha} = \alpha'$, and hence

$$\phi^*\pi^*\tilde{\alpha} = \pi^*\phi^*\tilde{\alpha} = \phi^*\alpha'.$$

Choose a lift $\tilde{g} \in 1 + (\mathcal{C} \otimes m)^0$ of g and put $\tilde{\beta} = \tilde{g}^{-1}(\beta)$. Then

$$\pi^*(\tilde{\beta}) = \pi^*(\tilde{g}^{-1}(\beta)) = g^{-1}\pi^*\beta = \phi^*\alpha'.$$

The obstruction to the existence of an isomorphism $\phi^*\tilde{\alpha} \to \tilde{\beta}$ covering $\mathrm{id}_{\pi^*(\alpha')}$ is an element $o_1(\phi^*(\tilde{\alpha}), \tilde{\beta}) \in H^1(\mathcal{C} \otimes I)$. Since $H^1(\phi)$ is surjective there exists a cocycle $u \in Z^1(\mathcal{B} \otimes I)$ such that $H^1(\phi)[u] = o_1(\phi^*(\tilde{\alpha}), \tilde{\beta})$. Put $\alpha = \tilde{\alpha} - u \in Ob\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$. Then

$$\begin{aligned} o_1(\phi^*\alpha, \tilde{\beta}) &= o_1(\phi^*\alpha, \phi^*\tilde{\alpha}) + o_1(\phi^*\tilde{\alpha}, \tilde{\beta}) \\ &= H^1(\phi)o_1(\alpha, \tilde{\alpha}) + o_1(\phi^*\tilde{\alpha}, \tilde{\beta}) \\ &= -H^1(\phi)[u] + o_1(\phi^*\tilde{\alpha}, \beta) = 0 \end{aligned}$$

This proves the surjectivity of ϕ^* on isomorphism classes.

Full. Let $f : \phi^*\alpha_1 \to \phi^*\alpha_2$ be a morphism in $\mathcal{MC}_{\mathcal{R}}(\mathcal{C})$. Then π^*f is a morphism in $\mathcal{MC}_{\overline{\mathcal{R}}}(\mathcal{C})$:

$$\pi^*(f) : \phi^*\pi^*\alpha_1 \to \phi^*\pi^*\alpha_2.$$

By induction hypothesis there exists $g : \pi^*\alpha_1 \to \pi^*\alpha_2$ such that $\phi^*(g) = \pi^*(f)$. Let $\tilde{g} \in 1 + (\mathcal{C} \otimes m)^0$ be any lift of g . Then $\pi^*(\tilde{g}\alpha_1) = \pi^*\alpha_2$. The obstruction to the existence of a morphism $\gamma : \tilde{g}\alpha_1 \to \alpha_2$ covering $\mathrm{id}_{\pi^*\alpha_2}$ is an element $o_1(\tilde{g}\alpha_1, \alpha_2) \in H^1(\mathcal{B} \otimes I)$. By assumption $H^1(\phi)$ is an isomorphism and we know that

$$H^1(\phi)(o_1(\tilde{g}\alpha_1, \alpha_2)) = o_1(\phi^*\tilde{g}\alpha_1, \phi^*\alpha_2) = 0,$$

since the morphism $f \cdot (\phi^*\tilde{g})^{-1}$ is covering the identity morphism $\mathrm{id}_{\pi^*\phi^*\alpha_2}$. Thus $o_1(\tilde{g}\alpha_1, \alpha_2) = 0$ and γ exists. Then $\gamma \cdot \tilde{g} : \alpha_1 \to \alpha_2$ is covering $g : \pi^*\alpha_1 \to \pi^*\alpha_2$. Hence both morphisms $\phi^*(\gamma \cdot \tilde{g})$ and f are covering $\pi^*(f)$. The obstruction to their equality is an element $o_0(\phi^*(\gamma \cdot \tilde{g}), f) \in \mathrm{Im}(H^0(\mathcal{C} \otimes I) \to H^0(\mathcal{C} \otimes m))$. Let $v \in H^0(\mathcal{C} \otimes I)$ be a representative of this element and $u \in Z^0(\mathcal{B} \otimes I)$ be a representative of the inverse image of v under $H^0(\phi)$. Then $\phi^*(\gamma \cdot \tilde{g} + u) = f$.

Faithful. Let $\gamma_1, \gamma_2 : \alpha_1 \to \alpha_2$ be morphisms in $\mathcal{MC}_{\mathcal{R}}(\mathcal{B})$ with $\phi^*\gamma_1 = \phi^*\gamma_2$. Then $\phi^*\pi^*\gamma_1 = \phi^*\pi^*\gamma_2$. By the induction hypothesis $\pi^*\gamma_1 = \pi^*\gamma_2$, so the obstruction $o_0(\gamma_1, \gamma_2) \in \mathrm{Im}(H^0(\mathcal{B} \otimes I) \to H^0(\mathcal{B} \otimes m, d^{\alpha_1,\alpha_2}))$ is defined. Now the image of $o_0(\gamma_1, \gamma_2)$ under the map

$$\mathrm{Im}(H^0(\mathcal{B} \otimes I) \to H^0(\mathcal{B} \otimes m, d^{\alpha_1,\alpha_2})) \to \mathrm{Im}(H^0(\mathcal{C} \otimes I) \to H^0(\mathcal{C} \otimes m, d^{\phi^*\alpha_1,\phi^*\alpha_2})) \tag{8.1}$$

equals to $o_0(\phi^*\gamma_1, \phi^*\gamma_2) = 0$. So it remains to prove that the map (8.1) is an isomorphism. Clearly, it is sufficient to prove that the morphism of complexes

$$\phi_{\mathcal{R}}^{\alpha_1,\alpha_2} : (\mathcal{B} \otimes m, d^{\alpha_1,\alpha_2})) \to (\mathcal{C} \otimes m, d^{\phi^*\alpha_1,\phi^*\alpha_2}))$$

is a quasi-isomorphism. Note that these complexes have finite filtrations by subcomplexes $\mathcal{B} \otimes m^i$ and $\mathcal{C} \otimes m^i$ respectively. The morphism $\phi_{\mathcal{R}}^{\alpha_1,\alpha_2}$ is compatible with these filtrations and induces quasi-isomorphisms on the subquotients. Hence $\phi_{\mathcal{R}}^{\alpha_1,\alpha_2}$ is a quasi-isomorphism. This proves the theorem. □

Corollary 8.2. *The homotopy (co-) deformation pseudo-functor of* $E \in \mathcal{A}^{op}$*-mod depends (up to equivalence) only on the quasi-isomorphism class of the DG algebra* $\mathrm{End}(E)$.

Proof. This follows from Theorem 8.1 and Proposition 6.1. □

The next proposition provides two examples of this situation. It was communicated to us by Bernhard Keller.

Proposition 8.3. *(Keller) a) Assume that* $E' \in \mathcal{A}^{op}$*-mod is homotopy equivalent to* E *. Then the DG algebras* $\mathrm{End}(E)$ *and* $\mathrm{End}(E')$ *are canonically quasi-isomorphic.*

b) Let $P \in \mathcal{P}(\mathcal{A}^{op})$ *and* $I \in \mathcal{I}(\mathcal{A}^{op})$ *be quasi-isomorphic. Then the DG algebras* $\mathrm{End}(P)$ *and* $\mathrm{End}(I)$ *are canonically quasi-isomorphic.*

Proof. a) Let $g : E \to E'$ be a homotopy equivalence. Consider its cone $C(g) \in \mathcal{A}^{op}$-mod . Let $\mathcal{C} \subset \mathrm{End}(C(g))$ be the DG subalgebra consisting of endomorphisms which leave E' stable. There are natural projections $p : \mathcal{C} \to \mathrm{End}(E')$ and $q : \mathcal{C} \to \mathrm{End}(E)$. We claim that p and q are quasi-isomorphisms. Indeed, $\mathrm{Ker}(p)$ (resp. $\mathrm{Ker}(q)$) is the complex $\mathrm{Hom}(E[1], C(g))$ (resp. $\mathrm{Hom}(C(g), E')$). These complexes are acyclic, since g is a homotopy equivalence.

b) The proof is similar. Let $f : P \to I$ be a quasi-isomorphism. Then the cone $C(f)$ is acyclic. We consider the DG subalgebra $\mathcal{D} \subset \mathrm{End}(C(f))$ which leaves I stable. Then $\mathcal{D}$ is quasi-isomorphic to $\mathrm{End}(I)$ and $\mathrm{End}(P)$ because the complexes $\mathrm{Hom}(P[1], C(f))$ and $\mathrm{Hom}(C(f), I)$ are acyclic. □

Corollary 8.4. *a) If DG* $\mathcal{A}^{op}$ *-modules* E *and* E' *are homotopy equivalent then the pseudo-functors* $\mathrm{Def}^{\mathrm{h}}(E)$ *,* $\mathrm{coDef}^{\mathrm{h}}(E)$ *,* $\mathrm{Def}^{\mathrm{h}}(E')$ *,* $\mathrm{coDef}^{\mathrm{h}}(E')$ *are canonically equivalent.*

b) Let $P \to I$ *be a quasi-isomorphism between* $P \in \mathcal{P}(\mathcal{A}^{op})$ *and* $I \in \mathcal{I}(\mathcal{A}^{op})$ *. Then the pseudo-functors* $\mathrm{Def}^{\mathrm{h}}(P)$ *,* $\mathrm{coDef}^{\mathrm{h}}(P)$ *,* $\mathrm{Def}^{\mathrm{h}}(I)$ *,* $\mathrm{coDef}^{\mathrm{h}}(I)$ *are canonically equivalent.*

Proof. Indeed, this follows from Proposition 8.3 and Corollary 8.2. □

Actually, one can prove a more precise statement.

Proposition 8.5. *Fix an artinian DG algebra* $\mathcal{R}$ *.*

a) Let $g : E \to E'$ *be a homotopy equivalence of DG* $\mathcal{A}^{op}$ *-modules. Assume that* $(V, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ *and* $(V', \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E')$ *are objects that correspond to each other via the equivalence* $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E) \simeq \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E')$ *of Corollary 8.4. Then there exists a homotopy equivalence* $\tilde{g} : V \to V'$ *which extends* g *, i.e.* $i^*\tilde{g} = g$ *. Similarly for the objects of* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}$ *with* $i^!$ *instead of* i^* *.*

b) Let $f : P \to I$ *be a quasi-isomorphism with* $P \in \mathcal{P}(\mathcal{A}^{op})$ *,* $I \in \mathcal{I}(\mathcal{A}^{op})$ *. Assume that* $(S, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ *and* $(T, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(I)$ *are objects that correspond to each other via the equivalence* $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P) \simeq \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(I)$ *of Corollary 8.4. Then there exists a quasi-isomorphism* $\tilde{f} : S \to T$ *which extends* f *, i.e.* $i^*\tilde{f} = f$ *. Similarly for the objects of* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}$ *with* $i^!$ *instead of* i^* *.*

Proof. a) Consider the DG algebra

$$\mathcal{C} \subset \mathrm{End}(C(g))$$

as in the proof of Proposition 8.3. We proved there that the natural projections $\mathrm{End}(E) \leftarrow \mathcal{C} \rightarrow \mathrm{End}(E')$ are quasi-isomorphisms. Hence the induced functors between groupoids $\mathcal{MC}_{\mathcal{R}}(\mathrm{End}(E)) \leftarrow \mathcal{MC}_{\mathcal{R}}(\mathcal{C}) \rightarrow \mathcal{MC}_{\mathcal{R}}(\mathrm{End}(E'))$ are equivalences by Theorem 8.1. Using Proposition 6.1 we may and will assume that deformations (V, id), (V', id) correspond to elements $\alpha_E \in \mathcal{MC}_{\mathcal{R}}(\mathrm{End}(E))$, $\alpha_{E'} \in \mathcal{MC}_{\mathcal{R}}(\mathrm{End}(E'))$ which come from the same element $\alpha \in \mathcal{MC}_{\mathcal{R}}(\mathcal{C})$.

Consider the DG modules $E \otimes \mathcal{R}$, $E' \otimes \mathcal{R}$ with the differentials $d_E \otimes 1 + 1 \otimes d_{\mathcal{R}}$ and $d_{E'} \otimes 1 + 1 \otimes d_{\mathcal{R}}$ respectively and the morphism $g \otimes 1 : E \otimes \mathcal{R} \rightarrow E' \otimes \mathcal{R}$. Then

$$\mathcal{C} \otimes \mathcal{R} = \begin{pmatrix} \mathrm{End}(E' \otimes \mathcal{R}) & \mathrm{Hom}(E[1] \otimes \mathcal{R}, E' \otimes \mathcal{R}) \\ 0 & \mathrm{End}(E \otimes \mathcal{R}) \end{pmatrix} \subset \mathrm{End}(C(g \otimes 1)),$$

and

$$\alpha = \begin{pmatrix} \alpha_{E'} & t \\ 0 & \alpha_E \end{pmatrix}.$$

Recall that the differential in the DG module $C(g \otimes 1)$ is of the form $(d_{E'} \otimes 1, d_E[1] \otimes 1 + g[1] \otimes 1)$. The element α defines a new differential d_α on $C(g \otimes 1)$ which is $(d_{E'} \otimes 1 + \alpha_{E'}, (d_E[1] \otimes 1 + \alpha_E) + (g[1] \otimes 1 + t))$. The fact that $d_\alpha^2 = 0$ implies that $\tilde{g} := g \otimes 1 + t[-1] : V \rightarrow V'$ is a closed morphism of degree zero and hence the DG module $C(g \otimes 1)$ with the differential d_α is the cone $C(\tilde{g})$ of this morphism.

Clearly, $i^* \tilde{g} = g$ and it remains to prove that $\tilde{g}$ is a homotopy equivalence. This in turn is equivalent to the acyclicity of the DG algebra $\mathrm{End}(C(\tilde{g}))$. But recall that the differential in $\mathrm{End}(C(\tilde{g}))$ is an " $\mathcal{R}$-deformation" of the differential in the DG algebra $\mathrm{End}(C(g))$ which is acyclic, since g is a homotopy equivalence. Therefore $\mathrm{End}(C(\tilde{g}))$ is also acyclic. This proves the first statement in a). The last statement follows by the equivalence of groupoids $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}} \simeq \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}$ (Proposition 4.7).

The proof of b) is similar: exactly in the same way we construct a closed morphism of degree zero $\tilde{f} : S \rightarrow T$ which extends f. Then $\tilde{f}$ is a quasi-isomorphism, because f is such. $\square$

Corollary 8.6. *Fix an artinian DG algebra* $\mathcal{R}$.

a) Let $g : E \rightarrow E'$ *be a homotopy equivalence as in Proposition 8.5a). Let* $(V, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E)$ *and* $(V', \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E')$ *be objects corresponding to each other under the equivalence* $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E) \simeq \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(E')$. *Then* $i^*V = \mathbf{L}i^*V$ *if and only if* $i^*V' = \mathbf{L}i^*V'$. *Similarly for the objects of* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}$ *with* $i^!$ *and* $\mathbf{R}i^!$ *instead of* i^* *and* $\mathbf{L}i^*$.

b) Let $f : P \rightarrow I$ *be a quasi-isomorphism as in Proposition 8.5b). Let* $(S, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ *and* $(T, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(I)$ *be objects which correspond to each other under the equivalence* $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P) \simeq \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(I)$. *Then* $i^*S = \mathbf{L}i^*S$ *if and only if* $i^*T = \mathbf{L}i^*T$. *Similarly for the objects of* $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}$ *with* $i^!$ *and* $\mathbf{R}i^!$ *instead of* i^* *and* $\mathbf{L}i^*$.

Proof. This follows immediately from Proposition 8.5. $\square$

Proposition 8.7. *Let* $F:\mathcal{A}\to\mathcal{C}$ *be a DG functor which induces an equivalence of derived categories* $\mathbf{L}F^*: D(\mathcal{A}^{op})\to D(\mathcal{C}^{op})$ *. (For example, this is the case if* F *induces a quasi-equivalence* $F^{\text{pre-tr}}:\mathcal{A}^{\text{pre-tr}}\to\mathcal{C}^{\text{pre-tr}}$ *(Corollary 3.15)).*

a) Let $P\in\mathcal{P}(\mathcal{A}^{op})$ *. Then the map of DG algebras* $F^*:\mathrm{End}(P)\to\mathrm{End}(F^*(P))$ *is a quasi-isomorphism. Hence the deformation pseudo-functors* $\mathrm{Def}^{\mathrm{h}}$ *and* $\mathrm{coDef}^{\mathrm{h}}$ *of* P *and* $F^*(P)$ *are equivalent.*

b) Let $I\in\mathcal{I}(\mathcal{A}^{op})$ *. Then the map of DG algebras* $F^!:\mathrm{End}(I)\to\mathrm{End}(F^!(I))$ *is a quasi-isomorphism. Hence the deformation pseudo-functors* $\mathrm{Def}^{\mathrm{h}}$ *and* $\mathrm{coDef}^{\mathrm{h}}$ *of* I *and* $F^!(I)$ *are equivalent.*

Proof. a) By Lemma 3.6 we have $F^*(P)\in\mathcal{P}(\mathcal{C}^{op})$. Hence the assertion follows from Theorems 3.1 and 8.1.

b) The functor $\mathbf{R}F^!: D(\mathcal{A}^{op})\to D(\mathcal{C}^{op})$ is also an equivalence because of adjunctions $(F_*,\mathbf{R}F^!),(\mathbf{L}F^*,F_*)$. Also $F^!(I)\in\mathcal{I}(\mathcal{C}^{op})$ (Lemma 3.6). Hence the assertion follows from Theorems 3.1 and 8.1. □

9. Direct relation between pseudo-functors $\mathrm{Def}^{\mathrm{h}}(F)$ and $\mathrm{Def}^{\mathrm{h}}(\mathcal{B})$ ($\mathrm{coDef}^{\mathrm{h}}(F)$ and $\mathrm{coDef}^{\mathrm{h}}(\mathcal{B})$)

9.1. **DG functor** Σ **.** Let $F\in\mathcal{A}^{op}\text{-mod}$ and put $\mathcal{B}=\mathrm{End}(F)$. Recall the DG functor from Example 3.14

$$\Sigma=\Sigma^F:\mathcal{B}^{op}\text{-mod}\to\mathcal{A}^{op}\text{-mod},\quad \Sigma(M)=M\otimes_{\mathcal{B}}F.$$

For each artinian DG algebra $\mathcal{R}$ we obtain the corresponding DG functor

$$\Sigma_{\mathcal{R}}:(\mathcal{B}\otimes\mathcal{R})^{op}\text{-mod}\to\mathcal{A}^{op}_{\mathcal{R}}\text{-mod},\quad \Sigma_{\mathcal{R}}(M)=M\otimes_{\mathcal{B}}F.$$

Lemma 9.1. *The DG functors* $\Sigma_{\mathcal{R}}$ *have the following properties.*

a) If a DG $(\mathcal{B}\otimes\mathcal{R})^{op}$ *-module* M *is graded* $\mathcal{R}$ *-free (resp. graded* $\mathcal{R}$ *-cofree), then so is the DG* $\mathcal{A}^{op}_{\mathcal{R}}$ *-module* $\Sigma_{\mathcal{R}}(M)$ *.*

b) Let $\phi:\mathcal{R}\to\mathcal{Q}$ *be a homomorphism of artinian DG algebras. Then there are natural isomorphisms of DG functors*

$$\Sigma_{\mathcal{Q}}\cdot\phi^*=\phi^*\cdot\Sigma_{\mathcal{R}},\quad \Sigma_{\mathcal{R}}\cdot\phi_*=\phi_*\cdot\Sigma_{\mathcal{Q}}.$$

In particular,

$$\Sigma\cdot i^*=i^*\cdot\Sigma_{\mathcal{R}}.$$

c) There is a natural isomorphism of DG functors

$$\Sigma_{\mathcal{Q}}\cdot\phi^!=\phi^!\cdot\Sigma_{\mathcal{R}}$$

on the full DG subcategory of DG $(\mathcal{B}\otimes\mathcal{R})^{op}$ *-modules* M *such that* $M^{\mathrm{gr}}\simeq M_1^{\mathrm{gr}}\otimes M_2^{\mathrm{gr}}$ *for a* $\mathcal{B}^{op}$ *-module* M_1 *and an* $\mathcal{R}^{op}$ *-module* M_2 *. (This subcategory includes in particular graded*

$\mathcal{R}$ -cofree modules.) Therefore

$$\Sigma \cdot i^! = i^! \cdot \Sigma_{\mathcal{R}}$$

on this subcategory.

d) For a graded $\mathcal{R}$ -free DG $(\mathcal{B}\otimes\mathcal{R})^{op}$ -module M there is a functorial isomorphism

$$\Sigma_{\mathcal{R}}(M\otimes_{\mathcal{R}}\mathcal{R}^*)=\Sigma_{\mathcal{R}}(M)\otimes_{\mathcal{R}}\mathcal{R}^*$$

Proof. The only nontrivial assertion is c). For any DG $(\mathcal{B}\otimes\mathcal{R})^{op}$ -module M there is a natural closed morphism of degree zero of DG $\mathcal{A}^{op}_{\mathcal{Q}}$ -modules

$$\gamma_M:\mathrm{Hom}_{\mathcal{R}^{op}}(Q,M)\otimes_{\mathcal{B}}F\to\mathrm{Hom}_{\mathcal{R}^{op}}(Q,M\otimes_{\mathcal{B}}F),\quad \gamma(g\otimes f)(q)=(-1)^{\bar{f}\bar{q}}g(q)\otimes f.$$

Since $\mathcal{Q}$ is a finite $\mathcal{R}^{op}$ -module γ_M is an isomorphism if $M^{\mathrm{gr}}\simeq M_1^{\mathrm{gr}}\otimes M_2^{\mathrm{gr}}$ for a $\mathcal{B}^{op}$ -module M_1 and an $\mathcal{R}^{op}$ -module M_2 . □

Proposition 9.2. *a) For each artinian DG algebra $\mathcal{R}$ the DG functor $\Sigma_{\mathcal{R}}$ induces functors between groupoids*

$$\mathrm{Def}^{\mathrm{h}}(\Sigma_{\mathcal{R}}):\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B})\to\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(F),$$
$$\mathrm{coDef}^{\mathrm{h}}(\Sigma_{\mathcal{R}}):\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B})\to\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(F),$$

b) The collection of DG functors $\{\Sigma_{\mathcal{R}}\}_{\mathcal{R}}$ defines morphisms of pseudo-functors

$$\mathrm{Def}^{\mathrm{h}}(\Sigma):\mathrm{Def}^{\mathrm{h}}(\mathcal{B})\to\mathrm{Def}^{\mathrm{h}}(F),$$

$$\mathrm{coDef}^{\mathrm{h}}(\Sigma):\mathrm{Def}^{\mathrm{h}}(\mathcal{B})\to\mathrm{Def}^{\mathrm{h}}(F).$$

c) The morphism $\mathrm{Def}^{\mathrm{h}}(\Sigma)$ is compatible with the equivalence θ of Proposition 6.1. That is the functorial diagram

$$\begin{array}{ccc}\mathcal{MC}(\mathcal{B}) & = & \mathcal{MC}(\mathcal{B})\\ \theta^{\mathcal{B}}\downarrow & & \downarrow\theta^{F}\\ \mathrm{Def}^{\mathrm{h}}(\mathcal{B}) & \stackrel{\mathrm{Def}^{\mathrm{h}}(\Sigma)}{\to} & \mathrm{Def}^{\mathrm{h}}(F)\end{array}$$

is commutative.

d) The morphisms $\mathrm{Def}^{\mathrm{h}}(\Sigma)$ and $\mathrm{coDef}^{\mathrm{h}}(\Sigma)$ are compatible with the equivalence δ of Proposition 4.7. That is the functorial diagram

$$\begin{array}{ccc}\mathrm{Def}^{\mathrm{h}}(\mathcal{B}) & \stackrel{\mathrm{Def}^{\mathrm{h}}(\Sigma)}{\to} & \mathrm{Def}^{\mathrm{h}}(F)\\ \delta^{\mathcal{B}}\downarrow & & \downarrow\delta^{F}\\ \mathrm{coDef}^{\mathrm{h}}(\mathcal{B}) & \stackrel{\mathrm{coDef}^{\mathrm{h}}(\Sigma)}{\to} & \mathrm{coDef}^{\mathrm{h}}(F)\end{array}$$

is commutative.

e) The morphisms $\mathrm{Def}^{\mathrm{h}}(\Sigma)$ and $\mathrm{coDef}^{\mathrm{h}}(\Sigma)$ are equivalences, i.e. for each $\mathcal{R}$ the functors $\mathrm{Def}^{\mathrm{h}}(\Sigma_{\mathcal{R}})$ and $\mathrm{coDef}^{\mathrm{h}}(\Sigma_{\mathcal{R}})$ are equivalences.

Proof. a) and b) follow from parts a),b),c) of Lemma 9.1; c) is obvious; d) follows from part d) of Lemma 9.1; e) follows from c) and d). □

9.2. **DG functor** ψ^***.** Let $\psi : \mathcal{C} \to \mathcal{B}$ be a homomorphism of DG algebras. Recall the corresponding DG functor

$$\psi^* : \mathcal{C}^{op}\text{-mod} \to \mathcal{B}^{op}\text{-mod}, \quad \psi^*(M) = M \otimes_{\mathcal{C}} \mathcal{B}.$$

For each artinian DG algebra $\mathcal{R}$ we obtain a similar DG functor

$$\psi^*_{\mathcal{R}} : (\mathcal{C} \otimes \mathcal{R})^{op}\text{-mod} \to (\mathcal{B} \otimes \mathcal{R})^{op}\text{-mod}, \quad \psi^*(M) = M \otimes_{\mathcal{C}} \mathcal{B}.$$

The next lemma and proposition are complete analogues of Lemma 9.1 and Proposition 9.2.

Lemma 9.3. *The DG functors* $\psi^*_{\mathcal{R}}$ *have the following properties.*

a) If a DG $(\mathcal{C} \otimes \mathcal{R})^{op}$ *-module* M *is graded* $\mathcal{R}$ *-free (resp. graded* $\mathcal{R}$ *-cofree), then so is the DG* $(\mathcal{B} \otimes \mathcal{R})^{op}$ *-module* $\psi^*_{\mathcal{R}}(M)$ *.*

b) Let $\phi : \mathcal{R} \to \mathcal{Q}$ *be a homomorphism of artinian DG algebras. Then there are natural isomorphisms of DG functors*

$$\psi^*_{\mathcal{Q}} \cdot \phi^* = \phi^* \cdot \psi^*_{\mathcal{R}}, \quad \psi^*_{\mathcal{R}} \cdot \phi_* = \phi_* \cdot \psi^*_{\mathcal{Q}}.$$

In particular,

$$\psi^* \cdot i^* = i^* \cdot \psi^*_{\mathcal{R}}.$$

c) There is a natural isomorphism of DG functors

$$\psi^*_{\mathcal{Q}} \cdot \phi^! = \phi^! \cdot \psi^*_{\mathcal{R}}$$

on the full DG subcategory of DG $(\mathcal{C} \otimes \mathcal{R})^{op}$ *-modules* M *such that* $M^{\mathrm{gr}} \simeq M_1^{\mathrm{gr}} \otimes M_2^{\mathrm{gr}}$ *for a* $\mathcal{C}^{op}$ *-module* M_1 *and an* $\mathcal{R}^{op}$ *-module* M_2 *. (This subcategory includes in particular graded* $\mathcal{R}$ *-cofree modules.) Therefore*

$$\psi^* \cdot i^! = i^! \cdot \psi^*_{\mathcal{R}}$$

on this subcategory.

d) For a graded $\mathcal{R}$ *-free DG* $(\mathcal{C} \otimes \mathcal{R})^{op}$ *-module* M *there is a functorial isomorphism*

$$\psi^*_{\mathcal{R}}(M \otimes_{\mathcal{R}} \mathcal{R}^*) = \psi^*_{\mathcal{R}}(M) \otimes_{\mathcal{R}} \mathcal{R}^*$$

Proof. As in Lemma 9.1, the only nontrivial assertion is c). For any DG $(\mathcal{C} \otimes \mathcal{R})^{op}$ -module M there is a natural closed morphism of degree zero of DG $A^{op}_{\mathcal{Q}}$ -modules

$$\eta_M : \mathrm{Hom}_{\mathcal{R}^{op}}(Q, M) \otimes_{\mathcal{C}} \mathcal{B} \to \mathrm{Hom}_{\mathcal{R}^{op}}(Q, M \otimes_{\mathcal{C}} \mathcal{B}), \quad \gamma(g \otimes f)(q) = (-1)^{\bar{f}\bar{q}} g(q) \otimes f.$$

Since $\mathcal{Q}$ is a finite $\mathcal{R}^{op}$ -module η_M is an isomorphism if $M^{\mathrm{gr}} \simeq M_1^{\mathrm{gr}} \otimes M_2^{\mathrm{gr}}$ for a $\mathcal{B}^{op}$ -module M_1 and an $\mathcal{R}^{op}$ -module M_2 . □

Proposition 9.4. *a) For each artinian DG algebra* $\mathcal{R}$ *the DG functor* $\psi^*_{\mathcal{R}}$ *induces functors between groupoids*

$$\mathrm{Def}^{\mathrm{h}}(\psi^*_{\mathcal{R}}) : \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{C}) \to \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}),$$
$$\mathrm{coDef}^{\mathrm{h}}(\psi^*_{\mathcal{R}}) : \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{C}) \to \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(\mathcal{B}),$$

b) The collection of DG functors $\{\psi^*_{\mathcal{R}}\}_{\mathcal{R}}$ *defines morphisms*

$$\mathrm{Def}^{\mathrm{h}}(\psi^*) : \mathrm{Def}^{\mathrm{h}}(\mathcal{C}) \to \mathrm{Def}^{\mathrm{h}}(\mathcal{B}),$$

$$\mathrm{coDef}^{\mathrm{h}}(\psi^*) : \mathrm{Def}^{\mathrm{h}}(\mathcal{C}) \to \mathrm{Def}^{\mathrm{h}}(\mathcal{B}).$$

c) The morphism $\mathrm{Def}^{\mathrm{h}}(\psi^*)$ *is compatible with the equivalence* θ *of Proposition 6.1. That is the functorial diagram*

$$\begin{array}{ccc} \mathcal{MC}(\mathcal{C}) & \stackrel{\psi^*}{\to} & \mathcal{MC}(\mathcal{B}) \\ \theta^{\mathcal{C}} \downarrow & & \downarrow \theta^{\mathcal{B}} \\ \mathrm{Def}^{\mathrm{h}}(\mathcal{C}) & \stackrel{\mathrm{Def}^{\mathrm{h}}(\psi^*)}{\to} & \mathrm{Def}^{\mathrm{h}}(\mathcal{B}) \end{array}$$

is commutative.

d) The morphisms $\mathrm{Def}^{\mathrm{h}}(\psi^*)$ *and* $\mathrm{coDef}^{\mathrm{h}}(\psi^*)$ *are compatible with the equivalence* δ *of Proposition 4.7. That is the functorial diagram*

$$\begin{array}{ccc} \mathrm{Def}^{\mathrm{h}}(\mathcal{C}) & \stackrel{\mathrm{Def}^{\mathrm{h}}(\psi^*)}{\to} & \mathrm{Def}^{\mathrm{h}}(\mathcal{B}) \\ \delta^{\mathcal{C}} \downarrow & & \downarrow \delta^{\mathcal{B}} \\ \mathrm{coDef}^{\mathrm{h}}(\mathcal{C}) & \stackrel{\mathrm{coDef}^{\mathrm{h}}(\psi^*)}{\to} & \mathrm{coDef}^{\mathrm{h}}(\mathcal{B}) \end{array}$$

is commutative.

e) Assume that ψ *is a quasi-isomorphism. Then the morphisms* $\mathrm{Def}^{\mathrm{h}}(\psi^*)$ *and* $\mathrm{coDef}^{\mathrm{h}}(\psi^*)$ *are equivalences, i.e. for each* $\mathcal{R}$ *the functors* $\mathrm{Def}^{\mathrm{h}}(\psi^*_{\mathcal{R}})$ *and* $\mathrm{coDef}^{\mathrm{h}}(\psi^*_{\mathcal{R}})$ *are equivalences.*

Proof. a) and b) follow from parts a),b),c) of Lemma 9.3; c) is obvious; d) follows from part d) of Lemma 9.3; e) follows from c),d) and Theorem 8.1. $\square$

Later we will be especially interested in the following example.

Lemma 9.5. *(Keller). a) Assume that the DG algebra* $\mathcal{B}$ *satisfies the following conditions:* $H^i(\mathcal{B}) = 0$ *for* $i < 0$ *,* $H^0(\mathcal{B}) = k$ *(resp.* $H^0(\mathcal{B}) = k$ *). Then there exists a DG subalgebra* $\mathcal{C} \subset \mathcal{B}$ *with the properties:* $\mathcal{C}^i = 0$ *for* $i < 0$ *,* $\mathcal{C}^0 = k$ *, and the embedding* $\psi : \mathcal{C} \hookrightarrow \mathcal{B}$ *is a quasi-isomorphism (resp. the induced map* $H^i(\psi) : H^i(\mathcal{C}) \to H^i(\mathcal{B})$ *is an isomorphism for* $i \geq 0$ *).*

Proof. Indeed, put $\mathcal{C}^0 = k$, $\mathcal{C}^1 = K \oplus L$, where $d(K) = 0$ and K projects isomorphically to $H^1(\mathcal{B})$, and $d : L \stackrel{\sim}{\to} d(\mathcal{B}^1) \subset \mathcal{B}^2$. Then take $\mathcal{C}^i = \mathcal{B}^i$ for $i \geq 2$ and $\mathcal{C}^i = 0$ for $i < 0$. $\square$

10. The derived deformation and co-deformation pseudo-functors

10.1. **The pseudo-functor** $\mathrm{Def}(E)$ **.** Fix a DG category $\mathcal{A}$ and an object $E \in \mathcal{A}^{op}$-mod . We are going to define a pseudo-functor $\mathrm{Def}(E)$ from the category dgart to the category **Gpd** of groupoids. This pseudo-functor assigns to a DG algebra $\mathcal{R}$ the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ of $\mathcal{R}$ -deformations of E in the *derived* category $D(\mathcal{A}^{op})$.

Definition 10.1. *Fix an artinian DG algebra $\mathcal{R}$. An object of the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is a pair (S,σ), where $S\in D(\mathcal{A}^{op}_{\mathcal{R}})$ and σ is an isomorphism (in $D(\mathcal{A}^{op})$)*

$$\sigma : \mathbf{L}i^*S \to E.$$

A morphism $f:(S,\sigma)\to(T,\tau)$ between two $\mathcal{R}$-deformations of E is an isomorphism (in $D(\mathcal{A}^{op}_{\mathcal{R}})$) $f:S\to T$, such that

$$\tau\cdot \mathbf{L}i^*(f)=\sigma.$$

This defines the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$. A homomorphism of artinian DG algebras $\phi:\mathcal{R}\to\mathcal{Q}$ induces the functor

$$\mathbf{L}\phi^* : \mathrm{Def}_{\mathcal{R}}(E)\to \mathrm{Def}_{\mathcal{Q}}(E).$$

Thus we obtain a pseudo-functor

$$\mathrm{Def}(E):\mathrm{dgart}\to \mathbf{Gpd}.$$

We call $\mathrm{Def}(E)$ the pseudo-functor of derived deformations of E.

Remark 10.2. *A quasi-isomorphism $\phi:\mathcal{R}\to\mathcal{Q}$ of artinian DG algebras induces an equivalence of groupoids*

$$\mathbf{L}\phi^* : \mathrm{Def}_{\mathcal{R}}(E)\to \mathrm{Def}_{\mathcal{Q}}(E).$$

Indeed, $\mathbf{L}\phi^*: D(\mathcal{A}^{op}_{\mathcal{R}})\to D(\mathcal{A}^{op}_{\mathcal{Q}})$ is an equivalence of categories (Proposition 3.7) which commutes with the functor $\mathbf{L}i^*$.

Remark 10.3. *A quasi-isomorphism $\delta:E_1\to E_2$ of DG $\mathcal{A}^{op}$-modules induces an equivalence of pseudo-functors*

$$\delta_*:\mathrm{Def}(E_1)\to\mathrm{Def}(E_2)$$

by the formula $\delta_*(S,\sigma)=(S,\delta\cdot\sigma)$.

Proposition 10.4. *Let $F:\mathcal{A}\to\mathcal{A}'$ be a DG functor which induces a quasi-equivalence $F^{\text{pre-tr}}:\mathcal{A}^{\text{pre-tr}}\to\mathcal{A}'^{\text{pre-tr}}$ (this happens for example if F is a quasi-equivalence). Then for any $E\in D(\mathcal{A}^{op})$ the deformation pseudo-functors $\mathrm{Def}(E)$ and $\mathrm{Def}(\mathbf{L}F^*(E))$ are canonically equivalent. (Hence also $\mathrm{Def}(F_*(E'))$ and $\mathrm{Def}(E')$ are equivalent for any $E'\in D(\mathcal{A}'^0)$).*

Proof. For any artinian DG algebra $\mathcal{R}$ the functor F induces a commutative functorial diagram

$$\begin{array}{ccc} D(\mathcal{A}^{op}_{\mathcal{R}}) & \stackrel{\mathbf{L}(F\otimes \mathrm{id})^*}{\longrightarrow} & D(\mathcal{A}'^0_{\mathcal{R}}) \\ \downarrow \mathbf{L}i^* & & \downarrow \mathbf{L}i^* \\ D(\mathcal{A}^{op}) & \stackrel{\mathbf{L}F^*}{\longrightarrow} & D(\mathcal{A}'^0) \end{array}$$

where $\mathbf{L}F^*$ and $\mathbf{L}(F\otimes\mathrm{id})^*$ are equivalences by Corollary 3.15. The horizontal arrows define a functor $F^*_{\mathcal{R}}:\mathrm{Def}_{\mathcal{R}}(E)\to\mathrm{Def}_{\mathcal{R}}(\mathbf{L}F^*(E))$. Moreover these functors are compatible with the functors $\mathbf{L}\phi^*:\mathrm{Def}_{\mathcal{R}}\to\mathrm{Def}_{\mathcal{Q}}$ induced by morphisms $\phi:\mathcal{R}\to\mathcal{Q}$ of artinian DG algebras. So we get the morphism $F^*:\mathrm{Def}(E)\to\mathrm{Def}(\mathbf{L}F^*(E))$ of pseudo-functors. It is clear that for each $\mathcal{R}$ the functor $F^*_{\mathcal{R}}$ is an equivalence. Thus F^* is also such. $\square$

Example 10.5. *Suppose that* $\mathcal{A}'$ *is a pre-triangulated DG category (so that the homotopy category* $\mathrm{Ho}(\mathcal{A}')$ *is triangulated). Let* $F:\mathcal{A}\hookrightarrow\mathcal{A}'$ *be an embedding of a full DG subcategory so that the triangulated category* $\mathrm{Ho}(\mathcal{A}')$ *is generated by the collection of objects* $F(Ob\mathcal{A})$ *. Then the assumption of the previous proposition holds.*

Remark 10.6. *In the definition of the pseudo-functor* $\mathrm{Def}(E)$ *we could work with the homotopy category of h-projective DG modules instead of the derived category. Indeed, the functors* i^* *and* ϕ^* *preserve h-projective DG modules.*

Definition 10.7. *Denote by* $\mathrm{Def}_+(E)$, $\mathrm{Def}_-(E)$, $\mathrm{Def}_0(E)$, $\mathrm{Def}_{\mathrm{cl}}(E)$ *the restrictions of the pseudo-functor* $\mathrm{Def}(E)$ *to subcategories* dgart_+ , dgart_- , art , cart *respectively.*

10.2. **The pseudo-functor** $\mathrm{coDef}(E)$ **.** Now we define the pseudo-functor $\mathrm{coDef}(E)$ of *derived co-deformations* in a similar way replacing everywhere the functors $(\cdot)^*$ by $(\cdot)^!$.

Definition 10.8. *Fix an artinian DG algebra* $\mathcal{R}$ *. An object of the groupoid* $\mathrm{coDef}_{\mathcal{R}}(E)$ *is a pair* (S,σ) *, where* $S\in D(\mathcal{A}^{op}_{\mathcal{R}})$ *and* σ *is an isomorphism (in* $D(\mathcal{A}^{op})$ *)*

$$\sigma:E\to\mathbf{R}i^!S.$$

A morphism $f:(S,\sigma)\to(T,\tau)$ *between two* $\mathcal{R}$ *-deformations of* E *is an isomorphism (in* $D(\mathcal{A}^{op}_{\mathcal{R}})$ *)* $f:S\to T$ *, such that*

$$\mathbf{R}i^!(f)\cdot\sigma=\tau.$$

This defines the groupoid $\mathrm{coDef}_{\mathcal{R}}(E)$ *. A homomorphism of artinian DG algebras* $\phi:\mathcal{R}\to\mathcal{Q}$ *induces the functor*

$$\mathbf{R}\phi^!:\mathrm{coDef}_{\mathcal{R}}(E)\to\mathrm{coDef}_{\mathcal{Q}}(E).$$

Thus we obtain a pseudo-functor

$$\mathrm{coDef}(E):\mathrm{dgart}\to\mathbf{Gpd}.$$

We call $\mathrm{coDef}(E)$ the functor of derived co-deformations of E .

Remark 10.9. *A quasi-isomorphism* $\phi:\mathcal{R}\to\mathcal{Q}$ *of artinian DG algebras induces an equivalence of groupoids*

$$\mathbf{R}\phi^!:\mathrm{coDef}_{\mathcal{R}}(E)\to\mathrm{coDef}_{\mathcal{Q}}(E).$$

Indeed, $\mathbf{R}\phi^!:D(\mathcal{A}^{op}_{\mathcal{R}})\to D(\mathcal{A}^{op}_{\mathcal{Q}})$ *is an equivalence of categories (Proposition 3.7) which commutes with the functor* $\mathbf{R}i^!$.

Remark 10.10. *A quasi-isomorphism* $\delta:E_1\to E_2$ *of* $\mathcal{A}$ *-DG-modules induces an equivalence of pseudo-functors*

$$\delta^*:\mathrm{coDef}(E_2)\to\mathrm{coDef}(E_1)$$

by the formula $\delta^*(S,\sigma)=(S,\sigma\cdot\delta)$.

Proposition 10.11. *Let* $F:\mathcal{A}\to\mathcal{A}'$ *be a DG functor as in Proposition 10.4 above. Consider the induced equivalence of derived categories* $\mathbf{R}F^!:D(\mathcal{A}^{op})\to D(\mathcal{A}'^0)$ *(Corollary 3.15). Then for any* $E\in D(\mathcal{A}^{op})$ *the deformation pseudo-functors* $\mathrm{coDef}(E)$ *and* $\mathrm{coDef}(\mathbf{R}F^!(E))$ *are canonically equivalent. (Hence also* $\mathrm{coDef}(F_*(E'))$ *and* $\mathrm{coDef}(E')$ *are equivalent for any* $E'\in D(\mathcal{A}'^0)$ *).*

Proof. For any artinian DG algebra $\mathcal{R}$ the functor F induces a commutative functorial diagram

$$\begin{array}{ccc} D(\mathcal{A}^{op}_{\mathcal{R}}) & \stackrel{\mathbf{R}((F\otimes \mathrm{id})^!)}{\longrightarrow} & D(\mathcal{A}'^0_{\mathcal{R}}) \\ \downarrow \mathbf{R}i^! & & \downarrow \mathbf{R}i^! \\ D(\mathcal{A}^{op}) & \stackrel{\mathbf{R}F^!}{\longrightarrow} & D(\mathcal{A}'^0), \end{array}$$

where $\mathbf{R}(F\otimes\mathrm{id})^!$ is an equivalence by Corollary 3.15. The horizontal arrows define a functor $F^!_{\mathcal{R}}:\mathrm{coDef}_{\mathcal{R}}(E)\to\mathrm{coDef}_{\mathcal{R}}(\mathbf{R}F^!(E))$. Moreover these functors are compatible with the functors $\mathbf{R}\phi^!:\mathrm{coDef}_{\mathcal{R}}\to\mathrm{coDef}_{\mathcal{Q}}$ induced by morphisms $\phi:\mathcal{R}\to\mathcal{Q}$ of artinian DG algebras. So we get the morphism $F^!:\mathrm{coDef}(E)\to\mathrm{coDef}(\mathbf{R}F^!(E))$. It is clear that for each $\mathcal{R}$ the functor $F^!_{\mathcal{R}}$ is an equivalence. Thus $F^!$ is also such. □

Example 10.12. *Let* $F:\mathcal{A}'\to\mathcal{A}$ *be as in Example 10.5 above. Then the assumption of the previous proposition holds.*

Remark 10.13. *In the definition of the pseudo-functor* $\mathrm{coDef}(E)$ *we could work with the homotopy category of h-injective DG modules instead of the derived category. Indeed, the functors* $i^!$ *and* $\phi^!$ *preserve h-injective DG modules.*

Definition 10.14. *Denote by* $\mathrm{coDef}_+(E)$, $\mathrm{coDef}_-(E)$, $\mathrm{coDef}_0(E)$, $\mathrm{coDef}_{\mathrm{cl}}(E)$ *the restrictions of the pseudo-functor* $\mathrm{coDef}(E)$ *to subcategories* dgart_+ , dgart_- , art , cart *respectively.*

Remark 10.15. *The pseudo-functors* $\mathrm{Def}(E)$ *and* $\mathrm{coDef}(E)$ *are not always equivalent (unlike their homotopy counterparts* $\mathrm{Def}^{\mathrm{h}}(E)$ *and* $\mathrm{coDef}^{\mathrm{h}}(E)$ *). In fact we expect that pseudo-functors* Def *and* coDef *are the "right ones" only in case they can be expressed in terms of the pseudo-functors* $\mathrm{Def}^{\mathrm{h}}$ *and* $\mathrm{coDef}^{\mathrm{h}}$ *respectively. (See the next section).*

11. Relation between pseudo-functors Def and Def^h (resp. coDef and coDef^h)

The ideal scheme that should relate these deformation pseudo-functors is the following. Let $\mathcal{A}$ be a DG category, $E\in\mathcal{A}^{op}$-mod . Choose quasi-isomorphisms $P\to E$ and $E\to I$, where $P\in\mathcal{P}(\mathcal{A}^{op})$ and $I\in\mathcal{I}(\mathcal{A}^{op})$. Then there should exist natural equivalences

$$\mathrm{Def}(E)\simeq\mathrm{Def}^h(P),\qquad \mathrm{coDef}(E)\simeq\mathrm{coDef}^h(I).$$

Unfortunately, this does not always work.

Example 11.1. *Let $\mathcal{A}$ be just a graded algebra $A=k[t]$, i.e. $\mathcal{A}$ contains a single object with the endomorphism algebra $k[t]$, $\deg(t)=1$ (the differential is zero). Take the artinian DG algebra $\mathcal{R}$ to be $\mathcal{R}=k[\epsilon]/(\epsilon^2)$, $\deg(\epsilon)=0$. Let $E=A$ and consider a DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module $M=E\otimes\mathcal{R}$ with the differential d_M which is the multiplication by $t\otimes\epsilon$. Clearly, M defines an object in $\mathrm{Def}^h_{\mathcal{R}}(E)$ which is not isomorphic to the trivial deformation. However, one can check (Proposition 11.18) that $\mathbf{L}i^*M$ is not quasi-isomorphic to E (although $i^*M=E$), thus M does not define an object in $\mathrm{Def}_{\mathcal{R}}(E)$. This fact and the next proposition show that the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$ is connected (contains only the trivial deformation), so it is not the "right" one.*

Proposition 11.2. *Assume that* $\mathrm{Ext}^{-1}(E,E)=0$ *.*

1) Fix a quasi-isomorphism $P\to E$, $P\in\mathcal{P}(\mathcal{A}^{op})$. Let $\mathcal{R}$ be an artinian DG algebra and $(S,\mathrm{id})\in\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$. The following conditions are equivalent:

a) $S\in\mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$,

b) $i^*S=\mathbf{L}i^*S$,

c) (S,id) defines an object in the groupoid $\mathrm{Def}_{\mathcal{R}}(E)$.

The pseudo-functor $\mathrm{Def}(E)$ is equivalent to the full pseudo-subfunctor of $\mathrm{Def}^{\mathrm{h}}(P)$ consisting of objects $(S,\mathrm{id})\in\mathrm{Def}^{\mathrm{h}}(P)$, where S satisfies a) (or b)) above.

2) Fix a quasi-isomorphism $E\to I$ with $I\in\mathcal{I}(\mathcal{A}^{op})$. Let $\mathcal{R}$ be an artinian DG algebra and $(T,\mathrm{id})\in\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$. The following conditions are equivalent:

a') $T\in\mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$,

b') $i^!T=\mathbf{R}i^!T$,

c') (T,id) defines an object in the groupoid $\mathrm{coDef}_{\mathcal{R}}(E)$.

The pseudo-functor $\mathrm{coDef}(E)$ is equivalent to the full pseudo-subfunctor of $\mathrm{coDef}^{\mathrm{h}}(I)$ consisting of objects $(T,\mathrm{id})\in\mathrm{coDef}^{\mathrm{h}}(I)$, where T satisfies a') (or b')) above.

Proof. 1) It is clear that a) implies b) and b) implies c). We will prove that c) implies a). We may and will replace the pseudo-functor $\mathrm{Def}(E)$ by an equivalent pseudo-functor $\mathrm{Def}(P)$ (Remark 10.3).

Since (S,id) defines an object in $\mathrm{Def}_{\mathcal{R}}(P)$ there exists a quasi-isomorphism $g:\tilde{S}\to S$ where $\tilde{S}$ has property (P) (hence $\tilde{S}\in\mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$), such that $i^*g:i^*\tilde{S}\to i^*S=P$ is also a quasi-isomorphism. Denote $Z=i^*\tilde{S}$. Then $Z\in\mathcal{P}(\mathcal{A}^{op})$ and hence i^*g is a homotopy equivalence. Since both $\tilde{S}$ and S are graded $\mathcal{R}$ -free, the map g is also a homotopy equivalence (Proposition 3.12d)). Thus $S\in\mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$.

Let us prove the last assertion in 1).

Fix an object $(\overline{S},\tau)\in\mathrm{Def}_{\mathcal{R}}(P)$. Replacing $(\overline{S},\tau)$ by an isomorphic object we may and will assume that $\overline{S}$ satisfies property (P). In particular, $\overline{S}\in\mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$ and $\overline{S}$ is graded $\mathcal{R}$ -free. This implies that $(\overline{S},\mathrm{id})\in\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(W)$ where $W=i^*\overline{S}$. We have $W\in\mathcal{P}(\mathcal{A}^{op})$. The quasi-isomorphism $\tau:W\to P$ is therefore a homotopy equivalence. By Corollary 8.4a) and

Proposition 8.5a) there exists an object $(S', \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ and a homotopy equivalence $\tau' : \overline{S} \to S'$ such that $i^*(\tau') = \tau$. This shows that $(\overline{S}, \tau)$ is isomorphic (in $\mathrm{Def}_{\mathcal{R}}(P)$) to an object $(S', \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$, where $S' \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$.

Let $(S, \mathrm{id}), (S', \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ be two objects such that $S, S' \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$. Consider the obvious map

$$\delta : \mathrm{Hom}_{\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)}((S, \mathrm{id}), (S', \mathrm{id})) \to \mathrm{Hom}_{\mathrm{Def}_{\mathcal{R}}(P)}((S, \mathrm{id}), (S', \mathrm{id})).$$

It suffices to show that δ is bijective.

Let $f : (S, \mathrm{id}) \to (S', \mathrm{id})$ be an isomorphism in $\mathrm{Def}_{\mathcal{R}}(P)$. Since $S, S' \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$ and $P \in \mathcal{P}(\mathcal{A}^{op})$ this isomorphism f is a homotopy equivalence $f : S \to S'$ such that $i^* f$ is homotopic to id_P. Let $h : i^* f \to \mathrm{id}$ be a homotopy. Since S , S' are graded $\mathcal{R}$ -free the map $i^* : \mathrm{Hom}(S, S') \to \mathrm{Hom}(P, P)$ is surjective (Proposition 3.12a)). Choose a lift $\tilde{h} : S \to S'[1]$ of h and replace f by $\tilde{f} = f - d\tilde{h}$. Then $i^* \tilde{f} = id$. Since S and S' are graded $\mathcal{R}$ -free $\tilde{f}$ is an isomorphism (Proposition 3.12d)). This shows that δ is surjective.

Let $g_1, g_2 : S \to S'$ be two isomorphisms (in $\mathcal{A}^{op}_{\mathcal{R}}$-mod) such that $i^* g_1 = i^* g_2 = \mathrm{id}_P$. That is g_1, g_2 represent morphisms in $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$. Assume that $\delta(g_1) = \delta(g_2)$, i.e. there exists a homotopy $s : g_1 \to g_2$. Then $d(i^* s) = i^*(ds) = 0$. Since by our assumption $H^{-1} \mathrm{Hom}(P, P) = 0$ there exists $t \in \mathrm{Hom}^{-2}(P, P)$ with $dt = i^* s$. Choose a lift $\tilde{t} \in \mathrm{Hom}^{-2}(S, S')$ of t. Then $\tilde{s} := s - d\tilde{t}$ is an allowable homotopy between g_1 and g_2. This proves that δ is injective and finishes the proof of 1).

The proof of 2) is very similar, but we present it for completeness. Again it is clear that a') implies b') and b') implies c'). We will prove that c') implies a') We may and will replace the functor $\mathrm{coDef}(E)$ by an equivalent functor $\mathrm{coDef}(I)$ (Remark 10.10).

Since (T, id) defines an object in $\mathrm{coDef}_{\mathcal{R}}(I)$, there exists a quasi-isomorphism $g : T \to \tilde{T}$ where $\tilde{T}$ has property (I) (hence $\tilde{T} \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$), such that $i^! g : I = i^! T \to i^! \tilde{T}$ is also a quasi-isomorphism. Denote $K = i^! \tilde{T}$. Then $K \in \mathcal{I}(\mathcal{A}^{op})$ and hence $i^! g$ is a homotopy equivalence. Since both T and $\tilde{T}$ are graded $\mathcal{R}$ -cofree, the map g is also a homotopy equivalence (Proposition 3.12d)). Thus $T \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$.

Let us prove the last assertion in 2).

Fix an object $(\overline{T}, \tau) \in \mathrm{coDef}_{\mathcal{R}}(I)$. Replacing $(\overline{T}, \tau)$ by an isomorphic object we may and will assume that $\overline{T}$ satisfies property (I). In particular, $\overline{T} \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$ and $\overline{T}$ is graded $\mathcal{R}$ -cofree. This implies that $(\overline{T}, \mathrm{id}) \in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(L)$ where $L = i^! \overline{T}$. We have $L \in \mathcal{I}(\mathcal{A}^{op})$ and hence the quasi-isomorphism $\tau : I \to L$ is a homotopy equivalence. By Corollary 8.4a) and Proposition 8.5a) there exist an object $(T', \mathrm{id}) \in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$ and a homotopy equivalence $\tau' : T' \to \overline{T}$ such that $i^! \tau' = \tau$. In particular, $T' \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$. This shows that $(\overline{T}, \tau)$ is isomorphic (in $\mathrm{coDef}_{\mathcal{R}}(I)$) to an object $(T', \mathrm{id}) \in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$ where $T' \in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$.

Let $(T,\mathrm{id}),(T',\mathrm{id})\in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$ be two objects such that $T,T'\in \mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$. Consider the obvious map

$$\delta:\mathrm{Hom}_{\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)}((T,\mathrm{id}),(T',\mathrm{id}))\to \mathrm{Hom}_{\mathrm{coDef}_{\mathcal{R}}(I)}((T,\mathrm{id}),(T',\mathrm{id})).$$

It suffices to show that δ is bijective.

Let $f:(T,\mathrm{id})\to(T',\mathrm{id})$ be an isomorphism in $\mathrm{coDef}_{\mathcal{R}}(I)$. Since $T,T'\in\mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$ and $I\in\mathcal{I}(\mathcal{A}^{op})$ this isomorphism f is a homotopy equivalence $f:T\to T'$ such that $i^!f$ is homotopic to id_I. Let $h:i^!f\to\mathrm{id}$ be a homotopy. Since T, T' are graded $\mathcal{R}$-cofree the map $i^!:\mathrm{Hom}(T,T')\to\mathrm{Hom}(I,I)$ is surjective (Proposition 3.12a)). Choose a lift $\tilde{h}:T\to T'[1]$ of h and replace f by $\tilde{f}=f-d\tilde{h}$. Then $i^!\tilde{f}=id$. Since T and T' are graded $\mathcal{R}$-cofree $\tilde{f}$ is an isomorphism (Proposition 3.12d)). This shows that δ is surjective.

Let $g_1,g_2:T\to T'$ be two isomorphisms (in $\mathcal{A}^{op}_{\mathcal{R}}$-mod) such that $i^!g_1=i^!g_2=\mathrm{id}_I$. That is g_1,g_2 represent morphisms in $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$. Assume that $\delta(g_1)=\delta(g_2)$, i.e. there exists a homotopy $s:g_1\to g_2$. Then $d(i^!s)=i^!(ds)=0$. Since by our assumption $H^{-1}\mathrm{Hom}(I,I)=0$ there exists $t\in\mathrm{Hom}^{-2}(I,I)$ with $dt=i^!s$. Choose a lift $\tilde{t}\in\mathrm{Hom}^{-2}(T,T')$ of t. Then $\tilde{s}:=s-d\tilde{t}$ is an allowable homotopy between g_1 and g_2. This proves that δ is injective. □

Remark 11.3. *In the situation of Proposition 11.2 using Corollary 8.4b) also obtain full and faithful morphisms of pseudo-functors* $\mathrm{Def}(E)$ *,* $\mathrm{coDef}(E)$ *to each of the equivalent pseudo-functors* $\mathrm{Def}^{\mathrm{h}}(P)$ *,* $\mathrm{coDef}^{\mathrm{h}}(P)$ *,* $\mathrm{Def}^{\mathrm{h}}(I)$ *,* $\mathrm{coDef}^{\mathrm{h}}(I)$ *.*

Corollary 11.4. *Assume that* $\mathrm{Ext}^{-1}(E,E)=0$ *. Let* $F\in\mathcal{A}^{op}$*-mod be an h-projective or an h-injective quasi-isomorphic to* E *.*

a) The pseudo-functor $\mathrm{Def}(E)$ *(* $\simeq\mathrm{Def}(F)$ *) is equivalent to the full pseudo-subfunctor of* $\mathrm{Def}^{\mathrm{h}}(F)$ *which consists of objects* (S,id) *such that* $i^*S=\mathbf{L}i^*S$ *.*

b) The pseudo-functor $\mathrm{coDef}(E)$ *(* $\simeq\mathrm{coDef}(F)$*) is equivalent to the full pseudo-subfunctor of* $\mathrm{coDef}^{\mathrm{h}}(F)$ *which consists of objects* (T,id) *such that* $i^!T=\mathbf{R}i^!T$ *.*

Proof. a). In case F is h-projective this is Proposition 11.2 1). Assume that F is h-injective. Choose a quasi-isomorphism $P\to F$ where P is h-projective. Again by Proposition 11.2 1) the assertion holds for P instead of F. But then it also holds for F by Corollary 8.6 b).

b). In case F is h-injective this is Proposition 11.2 2). Assume that F is h-projective. Choose a quasi-isomorphism $F\to I$ where I is h-injective. Then again by Proposition 11.2 2) the assertion holds for I instead of F. But then it also holds for F by Corollary 8.6 b). □

The next theorem provides an example when the pseudo-functors Def_- and $\mathrm{Def}^{\mathrm{h}}_-$ (resp. coDef_- and $\mathrm{coDef}^{\mathrm{h}}_-$) are equivalent.

Definition 11.5. *An object* $M\in\mathcal{A}^{op}$*-mod is called bounded above (resp. below) if there exists* i *such that* $M(A)^j=0$ *for all* $A\in\mathcal{A}$ *and all* $j\geq i$ *(resp.* $j\leq i$ *).*

Theorem 11.6. *Assume that* $\mathrm{Ext}^{-1}(E,E)=0$ *.*

a) Suppose that there exists an h-projective or an h-injective $P\in\mathcal{A}^{op}$*-mod which is bounded above and quasi-isomorphic to* E *. Then the pseudo-functors* $\mathrm{Def}_-(E)$ *and* $\mathrm{Def}^{\mathrm{h}}_-(P)$ *are equivalent.*

b) Suppose that there exists an h-projective or an h-injective $I\in\mathcal{A}^{op}$*-mod which is bounded below and quasi-isomorphic to* E *. Then the pseudo-functors* $\mathrm{coDef}_-(E)$ *and* $\mathrm{coDef}^{\mathrm{h}}_-(I)$ *are equivalent.*

Proof. Fix $\mathcal{R}\in\mathrm{dgart}_-$. In both cases it suffices to show that the embedding of groupoids $\mathrm{Def}_{\mathcal{R}}(E)\simeq\mathrm{Def}_{\mathcal{R}}(P)\subset\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ (resp. $\mathrm{coDef}_{\mathcal{R}}(E)\simeq\mathrm{coDef}_{\mathcal{R}}(I)\subset\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$) in Corollary 11.4 is essentially surjective.

a) It suffices to prove the following lemma.

Lemma 11.7. *Let* $M\in\mathcal{A}^{op}$*-mod be bounded above and* $(S,\mathrm{id})\in\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(M)$ *. The DG* $\mathcal{A}^{op}_{\mathcal{R}}$ *-module* S *is acyclic for the functor* i^* *, i.e.* $\mathbf{L}i^*S=i^*S$ *.*

Indeed, in case $M=P$ the lemma implies that S defines an object in $\mathrm{Def}_{\mathcal{R}}(P)$ (Corollary 11.4 a)).

Proof. Choose a quasi-isomorphism $f:Q\to S$ where $Q\in\mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$. We need to prove that i^*f is a quasi-isomorphism. It suffices to prove that $\pi_!i^*f$ is a quasi-isomorphism (Example 3.13). Recall that $\pi_!i^*=i^*\pi_!$. Thus it suffices to prove that $\pi_!f$ is a homotopy equivalence. Clearly $\pi_!f$ is a quasi-isomorphism. The DG $\mathcal{R}^{op}$ -module $\pi_!Q$ is h-projective (Example 3.13). We claim that the DG $\mathcal{R}^{op}$ -module $\pi_!S$ is also h-projective. Since the direct sum of h-projective DG modules is again h-projective, it suffices to prove that for each object $A\in\mathcal{A}$ the DG $\mathcal{R}^{op}$ -module $S(A)$ is h-projective. Take some object $A\in\mathcal{A}$. We have that $S(A)$ is bounded above and since $\mathcal{R}\in\mathrm{dgart}_-$ this DG $\mathcal{R}^{op}$ -module has an increasing filtration with subquotients being free DG $\mathcal{R}^{op}$ -modules. Thus $S(A)$ satisfies property (P) and hence is h-projective. It follows that the quasi-isomorphism $\pi_!f:\pi_!Q\to\pi_!S$ is a homotopy equivalence. Hence $i^*\pi_!f=\pi_!i^*f$ is also such. □

b) The following lemma implies (by Corollary 11.4 b)) that an object in $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(I)$ is also an object in $\mathrm{coDef}_{\mathcal{R}}(I)$, which proves the theorem. □

Lemma 11.8. *Let* $T\in\mathcal{A}^{op}_{\mathcal{R}}$*-mod be graded cofree and bounded below. Then* T *is acyclic for the functor* $i^!$ *, i.e.* $\mathbf{R}i^!T=i^!T$ *.*

Proof. Denote $N=i^!T\in\mathcal{A}^{op}\text{-mod}$. Choose a quasi-isomorphism $g:T\to J$ where $J\in\mathcal{I}(\mathcal{A}^{op}_{\mathcal{R}})$. We need to prove that $i^!g$ is a quasi-isomorphism. It suffices to show that $\pi_*i^!g$ is a quasi-isomorphism. Recall that $\pi_*i^!=i^!\pi_*$. Thus it suffices to prove that π_*g is a homotopy equivalence. Clearly it is a quasi-isomorphism.

Recall that the DG $\mathcal{R}^{op}$-module $\pi_* J$ is h-injective (Example 3.13) We claim that $\pi_* T$ is also such. Since the direct product of h-injective DG modules is again h-injective, it suffices to prove that for each object $A \in \mathcal{A}$ the DG $\mathcal{R}^{op}$-module $T(A)$ is h-injective. Take some object $A \in \mathcal{A}$. Since $\mathcal{R} \in \mathrm{dgart}_-$ the DG $\mathcal{R}^{op}$-module $T(A)$ has a decreasing filtration

$$G^0 \supset G^1 \supset G^2 \supset ...,$$

with

$$\mathrm{gr}\, T(A) = \oplus_j (T(A))^j \otimes \mathcal{R}^*.$$

A direct sum of shifted copies of the DG $\mathcal{R}^{op}$-module $\mathcal{R}^*$ is h-injective (Lemma 3.18). Thus each $(T(A))^j \otimes \mathcal{R}^*$ is h-injective and hence each quotient $T(A)/G^j$ is h-injective. Also

$$T(A) = \lim_{\leftarrow} T(A)/G^j.$$

Therefore $T(A)$ is h-injective by Remark 3.5.

It follows that $\pi_* g$ is a homotopy equivalence, hence also $i^! \pi_* g$ is such. $\square$

The last theorem allows us to compare the functors Def_- and coDef_- in some important special cases. Namely we have the following corollary.

Corollary 11.9. *Assume that*

a) $\mathrm{Ext}^{-1}(E, E) = 0$ *;*

b) there exists an h-projective or an h-injective $P \in \mathcal{A}^{op}$*-mod which is bounded above and quasi-isomorphic to* E *;*

c) there exists an h-projective or an h-injective $I \in \mathcal{A}^{op}$*-mod which is bounded below and quasi-isomorphic to* E *;*

Then the pseudo-functors $\mathrm{Def}_-(E)$ *and* $\mathrm{coDef}_-(E)$ *are equivalent.*

Proof. We have a quasi-isomorphism $P \to I$. Hence by Proposition 8.3 the DG algebras $\mathrm{End}(P)$ and $\mathrm{End}(I)$ are quasi-isomoprhic. Therefore, in particular, the pseudo-functors $\mathrm{Def}^{\mathrm{h}}_-(P)$ and $\mathrm{coDef}^{\mathrm{h}}_-(I)$ are equivalent (Corollary 8.4b)). It remains to apply the last theorem. $\square$

In practice in order to find the required bounded resolutions one might need to pass to a "smaller" DG category. So it is useful to have the following stronger corollary.

Corollary 11.10. *Let* $F : \mathcal{A} \to \mathcal{A}'$ *be a DG functor which induces a quasi-equivalence* $F^{\text{pre-tr}} : \mathcal{A}^{\text{pre-tr}} \to \mathcal{A}'^{\text{pre-tr}}$. *Consider the corresponding equivalence* $F_* : D(\mathcal{A}'^0) \to D(\mathcal{A}^{op})$ *(Corollary 3.15). Let* $E \in \mathcal{A}'^0$*-mod be such that*

a) $\mathrm{Ext}^{-1}(E, E) = 0$ *;*

b) there exists an h-projective or an h-injective $P \in \mathcal{A}^{op}$*-mod which is bounded above and quasi-isomorphic to* $F_*(E)$ *;*

c) there exists an h-projective or an h-injective $P \in \mathcal{A}^{op}$*-mod which is bounded below and quasi-isomorphic to* $F_*(E)$ *;*

Then the pseudo-functors $\mathrm{Def}_-(E)$ *and* $\mathrm{coDef}_-(E)$ *are equivalent.*

Proof. By the above corollary the pseudo-functors $\mathrm{Def}_-(F_*(E))$ and $\mathrm{coDef}_-(F_*(E))$ are equivalent. By Proposition 10.4 the pseudo-functors $\mathrm{Def}_-(E)$ and $\mathrm{Def}_-(F_*(E))$ are equivalent. Since the functor $\mathbf{R}F^! : D(\mathcal{A}^{op}) \to D(\mathcal{A}'^0)$ is also an equivalence, we conclude that the pseudo-functors $\mathrm{coDef}_-(E)$ and $\mathrm{coDef}_-(F_*(E))$ are equivalent by Proposition 10.11. □

Example 11.11. *If in the above corollary the DG category $\mathcal{A}'$ is pre-triangulated, then one can take for $\mathcal{A}$ a full DG subcategory of $\mathcal{A}'$ such that $\mathrm{Ho}(\mathcal{A}')$ is generated as a triangulated category by the subcategory $\mathrm{Ho}(\mathcal{A})$. One can often choose $\mathcal{A}$ to have one object.*

Example 11.12. *Let $\mathcal{C}$ be a bounded DG algebra, i.e. $\mathcal{C}^i = 0$ for $|i| >> 0$ and also $H^{-1}(\mathcal{C}) = 0$. Then by Theorem 11.6 and Proposition 4.7*

$$\mathrm{coDef}_-(\mathcal{C}) \simeq \mathrm{coDef}^{\mathrm{h}}_-(\mathcal{C}) \simeq \mathrm{Def}^{\mathrm{h}}_-(\mathcal{C}) \simeq \mathrm{Def}_-(\mathcal{C}).$$

The following theorem makes the equivalence of Corollary 11.9 more explicit. Let us first introduce some notation.

For an artinian DG algebra $\mathcal{R}$ consider the DG functors

$$\eta_{\mathcal{R}}, \epsilon_{\mathcal{R}} : \mathcal{A}^{op}_{\mathcal{R}}\text{-mod} \to \mathcal{A}^{op}_{\mathcal{R}}\text{-mod}$$

defined by

$$\epsilon_{\mathcal{R}}(M) = M \otimes_{\mathcal{R}} \mathcal{R}^*, \quad \eta_{\mathcal{R}}(N) = \mathrm{Hom}_{\mathcal{R}^{op}}(\mathcal{R}^*, N).$$

They induce the corresponding functors

$$\mathbf{R}\eta_{\mathcal{R}}, \mathbf{L}\epsilon_{\mathcal{R}} : D(\mathcal{A}^{op}_{\mathcal{R}}) \to D(\mathcal{A}^{op}_{\mathcal{R}}).$$

Theorem 11.13. *Let $E \in \mathcal{A}^{op}$-mod satisfy the assumptions a), b), c) of Corollary 11.9. Fix $\mathcal{R} \in \mathrm{dgart}_-$. Then the following holds.*

1) Let $F \in \mathcal{A}^{op}$-mod be h-projective or h-injective quasi-isomorphic to E.

a) For any $(S, \sigma) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(F)$ we have $i^*S = \mathbf{L}i^*S$.

b) For any $(T, \tau) \in \mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(F)$ we have $i^!T = \mathbf{R}i^!T$.

2) There are natural equivalences of pseudo-functors $\mathrm{Def}^{\mathrm{h}}_-(F) \simeq \mathrm{Def}_-(E)$, $\mathrm{coDef}^{\mathrm{h}}_-(F) \simeq \mathrm{coDef}_-(E)$.

3) The functors $\mathbf{L}\epsilon_{\mathcal{R}}$ and $\mathbf{R}\eta_{\mathcal{R}}$ induce mutually inverse equivalences

$$\mathbf{L}\epsilon_{\mathcal{R}} : \mathrm{Def}_{\mathcal{R}}(E) \to \mathrm{coDef}_{\mathcal{R}}(E),$$

$$\mathbf{R}\eta_{\mathcal{R}} : \mathrm{coDef}_{\mathcal{R}}(E) \to \mathrm{Def}_{\mathcal{R}}(E).$$

Proof. 1a). We may and will assume that $\sigma = \mathrm{id}$.

Choose a bounded above h-projective or h-injective $P \in \mathcal{A}^{op}$-mod, which is quasi-isomorphic to E. Then there exists a quasi-isomorphism $P \to F$ (or $F \to P$). The pseudo-functors $\mathrm{Def}^{\mathrm{h}}_-(P)$ and $\mathrm{Def}^{\mathrm{h}}_-(F)$ are equivalent by Corollary 8.4 (a) or b)). By Theorem 11.6 a)

$\mathrm{Def}^{\mathrm{h}}_{-}(P) \simeq \mathrm{Def}_{-}(P)$. Hence by Corollary 11.4 a) for each $(S', \mathrm{id}) \in \mathrm{Def}_{\mathcal{R}}(P)$ we have $i^*S' = \mathbf{L}i^*S'$. Now Corollary 8.6 (a) or b)) implies that $i^*S = \mathbf{L}i^*S$. This proves 1a).

1b). We may and will assume that $\tau = \mathrm{id}$.

The proof is similar to that of 1a). Namely, choose a bounded below h-projective or h-injective $I \in \mathcal{A}^{op}$-mod quasi-isomorphic to E . Then there exists a quasi-isomorphism $F \to I$ (or $I \to F$). The pseudo-functors $\mathrm{coDef}^{\mathrm{h}}_{-}(I)$ and $\mathrm{coDef}^{\mathrm{h}}_{-}(F)$ are equivalent and by Corollary 8.4 (a) or b)). By Theorem 11.6 a) $\mathrm{coDef}^{\mathrm{h}}_{-}(I) \simeq \mathrm{coDef}_{-}(I)$. Hence by Corollary 11.4 b) for each $(T', \mathrm{id}) \in \mathrm{coDef}^{\mathrm{h}}(I)$ we have $i^!T' = \mathbf{R}i^!T'$. Now Corollary 8.6 (a) or b)) implies that $i^!T = \mathbf{R}i^!T$.

2) This follows from 1), Corollary 11.4 a), b).

3) This follows from 2) and the fact that $\epsilon_{\mathcal{R}}$ and $\eta_{\mathcal{R}}$ induce inverse equivalences between $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(F)$ and $\mathrm{coDef}^{\mathrm{h}}_{\mathcal{R}}(F)$ (Proposition 4.7). □

Proposition 11.14. *Let DG algebras $\mathcal{B}$ and $\mathcal{C}$ be quasi-isomorphic and $H^{-1}(\mathcal{B}) = 0$ ($= H^{-1}(\mathcal{C})$). Suppose that the pseudo-functors $\mathrm{Def}(\mathcal{B})$ and $\mathrm{Def}^{\mathrm{h}}(\mathcal{B})$ (resp. $\mathrm{coDef}(\mathcal{B})$ and $\mathrm{coDef}^{\mathrm{h}}(\mathcal{B})$) are equivalent. Then the same is true for $\mathcal{C}$.*

Similar results hold for the pseudo-functors $\mathrm{Def}_{-}, \mathrm{Def}^{\mathrm{h}}_{-}, \mathrm{coDef}_{-}, \ldots$.

Proof. We may and will assume that there exists a morphism of DG algebras $\psi : \mathcal{B} \to \mathcal{C}$ which is a quasi-isomorphism.

By Proposition 8.6 a) the pseudo-functors $\mathrm{Def}^{\mathrm{h}}(\mathcal{B})$ and $\mathrm{Def}^{\mathrm{h}}(\mathcal{C})$ are equivalent.

By Proposition 10.4 the pseudo-functors $\mathrm{Def}(\mathcal{B})$ are $\mathrm{Def}(\mathcal{C})$ are equivalent.

By Proposition 11.2 a) $\mathrm{Def}(\mathcal{B})$ (resp. $\mathrm{Def}(\mathcal{C})$) is a full pseudo-subfunctor of $\mathrm{Def}^{\mathrm{h}}(\mathcal{B})$ (resp. $\mathrm{Def}^{\mathrm{h}}(\mathcal{C})$).

Thus is $\mathrm{Def}(\mathcal{B}) \simeq \mathrm{Def}^{\mathrm{h}}(\mathcal{B})$, then also $\mathrm{Def}(\mathcal{C}) \simeq \mathrm{Def}^{\mathrm{h}}(\mathcal{C})$.

The proof for coDef and $\mathrm{coDef}^{\mathrm{h}}$ is similar using Proposition 8.6 a), Proposition 10.11 and Proposition 11.2 b). □

Corollary 11.15. *Let $\mathcal{B}$ be a DG algebra such that $H^{-1}(\mathcal{B}) = 0$. Assume that $\mathcal{B}$ is quasi-isomorphic to a DG algebra $\mathcal{C}$ such that $\mathcal{C}$ is bounded above (resp. bounded below). Then the pseudo-functors $\mathrm{Def}_{-}(\mathcal{B})$ and $\mathrm{Def}^{\mathrm{h}}_{-}(\mathcal{B})$ are equivalent (resp. $\mathrm{coDef}_{-}(\mathcal{B})$ and $\mathrm{coDef}^{\mathrm{h}}_{-}(\mathcal{B})$ are equivalent).*

Proof. By Theorem 11.6 a) we have that $\mathrm{Def}_{-}(\mathcal{C})$ and $\mathrm{Def}^{\mathrm{h}}_{-}(\mathcal{C})$ are equivalent (resp. $\mathrm{coDef}_{-}(\mathcal{C})$ and $\mathrm{coDef}^{\mathrm{h}}_{-}(\mathcal{C})$ are equivalent). It remains to apply Proposition 11.14. □

11.1. **Relation between pseudo-functors $\mathrm{Def}_{-}(E)$, $\mathrm{coDef}_{-}(E)$ and $\mathrm{Def}_{-}(\mathcal{C})$, $\mathrm{coDef}_{-}(\mathcal{C})$.** The next proposition follows immediately from our previous results.

Proposition 11.16. *Let $\mathcal{A}$ be a DG category and $E \in \mathcal{A}^{op}$-mod . Assume that*

a) $\mathrm{Ext}^{-1}(E, E) = 0$ *;*

b) there exists a bounded above (resp. bounded below) h-projective or h-injective $F \in \mathcal{A}^{op}$*-mod which is quasi-isomorphic to* E *;*

c) there exists a bounded above (resp. bounded below) DG algebra $\mathcal{C}$ *which is quasi-isomorphic to* $\mathrm{End}(F)$ *.*

Then the pseudo-functors $\mathrm{Def}_-(E)$ *and* $\mathrm{Def}_-(\mathcal{C})$ *(resp.* $\mathrm{coDef}_-(E)$ *and* $\mathrm{coDef}_-(\mathcal{C})$ *) are equivalent.*

Proof. Assume that F and $\mathcal{C}$ are bounded above. Then $\mathrm{Def}_-(E) \simeq \mathrm{Def}^{\mathrm{h}}_-(F)$ and $\mathrm{Def}_-(\mathcal{C}) \simeq \mathrm{Def}^{\mathrm{h}}_-(\mathcal{C})$ by Theorem 11.6 a). Also $\mathrm{Def}^{\mathrm{h}}_-(F) \simeq \mathrm{Def}^{\mathrm{h}}_-(\mathcal{C})$ by Proposition 6.1 and Theorem 8.1.

Assume that F and $\mathcal{C}$ are bounded below. Then $\mathrm{coDef}_-(E) \simeq \mathrm{coDef}^{\mathrm{h}}_-(F)$ and $\mathrm{coDef}_-(\mathcal{C}) \simeq \mathrm{coDef}^{\mathrm{h}}_-(\mathcal{C})$ by Theorem 11.6 b). Also $\mathrm{coDef}^{\mathrm{h}}_-(F) \simeq \mathrm{coDef}^{\mathrm{h}}_-(\mathcal{C})$ by Proposition 6.1 and Theorem 8.1. □

Remark 11.17. *The equivalences of pseudo-functors* $\mathrm{Def}^{\mathrm{h}}_-(\mathcal{C}) \simeq \mathrm{Def}^{\mathrm{h}}_-(F)$ *,* $\mathrm{coDef}^{\mathrm{h}}_-(\mathcal{C}) \simeq \mathrm{coDef}^{\mathrm{h}}_-(F)$ *in the proof of last proposition can be made explicit. Put* $\mathcal{B} = \mathrm{End}(F)$ *. Assume, for example, that* $\psi : \mathcal{C} \to \mathcal{B}$ *is a homomorphism of DG algebras which is a quasi-isomorphism. Then the composition of DG functors (Propositions 9.2, 9.4)*

$$\Sigma^F \cdot \psi^* : \mathcal{C}^{op}\text{-}mod \to \mathcal{A}^{op}\text{-}mod$$

induces equivalences of pseudo-functors

$$\mathrm{Def}^{\mathrm{h}}(\Sigma^F \cdot \psi^*) : \mathrm{Def}^{\mathrm{h}}(\mathcal{C}) \simeq \mathrm{Def}^{\mathrm{h}}(F)$$

$$\mathrm{coDef}^{\mathrm{h}}(\Sigma^F \cdot \psi^*) : \mathrm{coDef}^{\mathrm{h}}(\mathcal{C}) \simeq \mathrm{coDef}^{\mathrm{h}}(F)$$

by Propositions 9.2e) and 9.4f).

11.2. **Pseudo-functors** $\mathrm{Def}(E)$ **,** $\mathrm{coDef}(E)$ **are not determined by the DG algebra** $\mathbf{R}\mathrm{Hom}(E,E)$ **.** One might expect that the derived deformation and co-deformation pseudo-functors $\mathrm{Def}_-(E)$, $\mathrm{coDef}_-(E)$ depend only on the (quasi-isomorphism class of the) DG algebra $\mathbf{R}\mathrm{Hom}(E,E)$. This would be an analogue of Theorem 8.1 for the derived deformation theory. Unfortunately this is not true as is shown in the next proposition (even for the "classical" pseudo-functors $\mathrm{Def}_{\mathrm{cl}}$, $\mathrm{coDef}_{\mathrm{cl}}$). This is why all our comparison results for the pseudo-functors Def_- and coDef_- such as Theorems 11.6, 11.13, Corollaries 11.9, 11.15, Proposition 11.16 need some boundedness assumptions.

Consider the DG algebra $A = k[x]$ with the zero differential and $\deg(x) = 1$. Let $\mathcal{A}$ be the DG category with one object whose endomorphism DG algebra is A . Then $\mathcal{A}^{op}$-mod is the DG category of DG modules over the DG algebra $A^{op} = A$. Denote by abuse of notation the unique object of $\mathcal{A}$ also by A and consider the DG $\mathcal{A}^{op}$ -modules $P = h^A$ and $I = h^*_A$. The first one is h-projective and bounded below while the second one is h-injective and bounded above (they are the graded dual of each other). Note that the DG algebras $\mathrm{End}(P)$ and $\mathrm{End}(I)$ are isomorphic:

$$\mathrm{End}(P) = A, \quad \mathrm{End}(I) = A^{**} = A.$$

Let $\mathcal{R} = k[\epsilon]/(\epsilon^2)$ be the (commutative) artinian DG algebra with the zero differential and $\deg(\epsilon) = 0$.

Proposition 11.18. *In the above notation the following holds:*

a) The groupoid $\mathrm{Def}_{\mathcal{R}}(P)$ *is connected.*

b) The groupoid $\mathrm{Def}_{\mathcal{R}}(I)$ *is not connected.*

c) The groupoid $\mathrm{coDef}_{\mathcal{R}}(I)$ *is connected.*

d) The groupoid $\mathrm{coDef}_{\mathcal{R}}(P)$ *is not connected.*

Proof. Let $(S, \mathrm{id}) \in \mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(I)$. Then $S = I \otimes_k \mathcal{R}$ as a graded $(A \otimes \mathcal{R})^{op}$ -module and the differential in S is equal to "multiplication by $\lambda(x \otimes \epsilon)$ " for some $\lambda \in k$. We denote this differential d_λ and the deformation S by S_λ . By Lemma 11.7 each (S_λ, id) is also an object in the groupoid $\mathrm{Def}_{\mathcal{R}}(I)$. Notice that for $\lambda \neq 0$ we have $H(S_\lambda) = k$ and if $\lambda = 0$ then $H(S_\lambda) = A \otimes \mathcal{R}$. This shows for example that (S_1, id) and (S_0, id) are non-isomorphic objects in $\mathrm{Def}_{\mathcal{R}}(I)$ and proves b).

The proof of d) is similar using Lemma 11.8.

Let us prove a). By Proposition 11.2, 1) the groupoid $\mathrm{Def}_{\mathcal{R}}(P)$ is equivalent to the full subcategory of $\mathrm{Def}^{\mathrm{h}}_{\mathcal{R}}(P)$ consisting of objects (S, id) such that $S \in \mathcal{P}(\mathcal{A}^{op}_{\mathcal{R}})$ or, equivalently, $i^*S = \mathbf{L}i^*S$. As in the proof of b) above we have $S = P \otimes \mathcal{R}$ as a graded $(A \otimes \mathcal{R})^{op}$ -module and the differential in S is equal to "multiplication by $\lambda(x \otimes \epsilon)$ " for some $\lambda \in k$. Again we denote the corresponding S by S_λ . It is clear that the trivial homotopy deformation S_0 is h-projective in $\mathcal{A}^{op}_{\mathcal{R}}$-mod , hence it is also an object in $\mathrm{Def}_{\mathcal{R}}(P)$. It remains to prove that for $\lambda \neq 0$ the DG $\mathcal{A}^{op}_{\mathcal{R}}$ -module S_λ is not h-projective. Since the DG functor $\pi_!$ preserves h-projectives (Example 3.13) it suffices to show that S_λ considered as a DG $\mathcal{R}$ -module is not h-projective. We have

$$\pi_! S_\lambda = \bigoplus_{n \geq 0} \mathcal{R}[-n]$$

with the differential $\lambda\epsilon : \mathcal{R}[-n] \to \mathcal{R}[-n-1]$. Consider the DG $\mathcal{R}$ -module

$$N = \bigoplus_{n=-\infty}^{\infty} \mathcal{R}[-n]$$

with the same differential $\lambda\epsilon : \mathcal{R}[-n] \to \mathcal{R}[-n-1]$. Note that N is acyclic (since $\lambda \neq 0$) and the obvious embedding of DG $\mathcal{R}$ -modules $\pi_! S_\lambda \hookrightarrow N$ is not homotopic to zero. Hence $\pi_! S_\lambda$ is not h-projective. This proves a).

The proof of c) is similar using Proposition 11.2, 2) and the DG functor π_* from Example 3.13. □

References

[Be] J. Benabou, Introduction to bicategories, Reports of the Midwest Category Seminar, LNM 47 (1967).

[BK] A. I. Bondal, M. M. Kapranov, Enhanced triangulated categories, *Math. USSR - Sbornik* **70** (1991), No. 1, 93-107.

[De1] P. Deligne, Letter to E. Esnault

[De2] P. Deligne, Letter to L. Breen (1994).

[Dr1] V. Drinfeld, DG quotients of DG categories, arXiv:math.KT/0210114.

[Dr2] V. Drinfeld, Letter to V. Schechtman, September 1988.

[CK1] I. Ciocan-Fontanine, M. M. Kapranov, Derived Quot schemes, math.QA/9906063.

[CK2] I. Ciocan-Fontanine, M. M. Kapranov, Derived Hilbert schemes, math.AG/0005155.

[DW] C. F. Doran, S. Wong, Deformation theory: an historical annotated bibliography, Chapter 2 in book: Deformations of Galois Representations, AMS-IP Studies in Advanced Mathematics Series.

[G1] E. Getzler, A Darboux theorem for Hamiltonian operators in the formal calculus of variations, math.DG/0002164

[G2] E. Getzler, Lie theory for nilpotent L_∞ -algebras, math.AT/0404003.

[GK] V. Ginzburg, M. M. Kapranov, Koszul duality for operads, *Duke Math. Journal* **76** (1994), 203-272.

[GM] W. Goldman, J. Milson, The deformation theory of representations of fundamental groups of compact Kahler manifolds, *Publ. Math. IHES*, **67** (1988), 43-96.

[H] V. Hinich, DG coalgebras as formal stacks, arXiv:math.AG/9812034.

[HMS] D. Husemoller, J. Moore, J. Stasheff, Differential homological algebra and homogeneous spaces, Journal of Algebra 5 (1974), 113-185.

[Ka] M. M. Kapranov, Injective resolutions of BG and derived moduli spaces of local systems, alg-geom/9704009.

[Ke] B. Keller, Deriving DG categories, *Ann. scient. Éc. Norm. Sup.*, 4^e série, t. 27, (1994), 63-102.

[Lau] O. A. Laudal, Noncommutative deformations of modules, Homology, Homotopy and Applications, Vol. 4 (2) (2002) 357-396.

[ELO2] A. Efimov, V. A. Lunts, D. Orlov, Deformation theory of objects in homotopy and derived categories II: pro-representability of the deformation functor (preprint).

[ELO3] A. Efimov, V. A. Lunts, D. Orlov, Deformation theory of objects in homotopy and derived categories III: abelian categories (preprint).

[Lo] W. Lowen, Obstruction theory for objects in abelian and derived categories, math.KT/0407019.

[Q] D. Quillen, Rational homotopy theory, *Annals of Math.* **90** (1969), 205-295.

[SS] M. Schlessinger, J. Stasheff, Deformation theory and rational homotopy type, Univ. of North Carolina preprint, 1979; short version: The Lie algebra structure of tangent cohomology and deformation theory, Journal of Pure and Applied Algebra 38 (1985), 313-322.

Department of Mechanics and Mathematics, Moscow State University, Moscow, Russia and Independent Univeroty of Moscow

E-mail address: `efimov@mccme.ru`

Department of Mathematics, Indiana University, Bloomington, IN 47405, USA

E-mail address: `vlunts@indiana.edu`

Algebra Section, Steklov Mathematical Institute, 8 Gubkina str., Moscow, 119991 Russia

E-mail address: `orlov@mi.ras.ru`