

THE LANGUAGE OF SCHEMES (EGA I)

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SUMMARY

- §1. Affine schemes.
- §2. Preschemes and morphisms of preschemes.
- §3. Products of preschemes.
- §4. Subpreschemes and immersion morphisms.
- §5. Reduced preschemes; separation condition.
- §6. Finiteness conditions.
- §7. Rational maps.
- §8. Chevalley schemes.
- §9. Supplement on quasi-coherent sheaves.
- §10. Formal schemes.

Sections §§1–8 do little more than develop a language, which will be used in what follows. It should be noted, however, that, in accordance with the general spirit of this treatise, §§7–8 will be used less than the others, and in a less essential way; we have speak of Chevalley’s schemes only to make the link with the language of Chevalley [CC] and Nagata [Nag58]. Section §9 gives

definitions and results concerning quasi-coherent sheaves, some of which are no longer limited to a translation into a “geometric” language of known notions of commutative algebra, but are already of a global nature; they will be indispensable, in the following chapters, for the global study of morphisms. Finally, §10 introduces a generalization of the notion of schemes, which will be used as an intermediary in Chapter III to formulate and prove, in a convenient way, the fundamental results of the cohomological study of the proper morphisms; moreover, it should be noted that the notion of formal schemes seems indispensable in expressing certain facts about the “theory of modules” (classification problems of algebraic varieties). The results of §10 will not be used before §3 of Chapter III, and it is recommended to omit their reading until then.

§1. AFFINE SCHEMES

1.1. The prime spectrum of a ring.

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(1.1.1). *Notation.* Let A be a (commutative) ring, and M an A -module. In this chapter and the following, we will constantly use the following notation:

- $\text{Spec}(A)$ = set of prime ideals of A , also called the *prime spectrum* of A ; for $x \in X = \text{Spec}(A)$, it will often be convenient to write \mathfrak{j}_x instead of x . For $\text{Spec}(A)$ to be *empty*, it is necessary and sufficient for the ring A to be 0.
- $A_x = A_{\mathfrak{j}_x}$ = (local) ring of fractions $S^{-1}A$, where $S = A - \mathfrak{j}_x$.
- $\mathfrak{m}_x = \mathfrak{j}_x A_{\mathfrak{j}_x}$ = maximal ideal of A_x .
- $k(x) = A_x / \mathfrak{m}_x$ = residue field of A_x , canonically isomorphic to the field of fractions of the integral ring A / \mathfrak{j}_x , with which we identify it.
- $f(x)$ = class of f mod. \mathfrak{j}_x in $A / \mathfrak{j}_x \subset k(x)$, for $f \in A$ and $x \in X$. We also say that $f(x)$ is the value of f at a point $x \in \text{Spec}(A)$; the equations $f(x) = 0$ and $f \in \mathfrak{j}_x$ are equivalent.
- $M_x = M \otimes_A A_x$ = module of fractions with denominators in $A - \mathfrak{j}_x$.
- $\tau(E)$ = radical of the ideal of A generated by a subset E of A .
- $V(E)$ = set of $x \in X$ such that $E \subset \mathfrak{j}_x$ (or the set of $x \in X$ such that $f(x) = 0$ for all $f \in E$), for $E \subset A$. So we have

$$(1.1.1.1) \quad \tau(E) = \bigcap_{x \in V(E)} \mathfrak{j}_x.$$

- $V(f) = V(\{f\})$ for $f \in A$.
- $D(f) = X - V(f)$ = set of $x \in X$ where $f(x) \neq 0$.

Proposition (1.1.2). — We have the following properties:

- (i) $V(0) = X$, $V(1) = \emptyset$.
- (ii) The relation $E \subset E'$ implies $V(E) \supset V(E')$.
- (iii) For each family (E_λ) of subsets of A , $V(\bigcup_\lambda E_\lambda) = V(\sum_\lambda E_\lambda) = \bigcap_\lambda V(E_\lambda)$.
- (iv) $V(E E') = V(E) \cup V(E')$.
- (v) $V(E) = V(\tau(E))$.

Proof. The properties (i), (ii), (iii) are trivial, and (v) follows from (ii) and from equation (1.1.1.1). It is evident that $V(E E') \supset V(E) \cup V(E')$; conversely, if $x \notin V(E)$ and $x \notin V(E')$, then there exists $f \in E$ and $f' \in E'$ such that $f(x) \neq 0$ and $f'(x) \neq 0$ in $k(x)$, hence $f(x)f'(x) \neq 0$, i.e., $x \notin V(E E')$, which proves (iv). \square

Proposition (1.1.2) shows, among other things, that sets of the form $V(E)$ (where E varies over the subsets of A) are the *closed sets* of a topology on X , which we will call the *spectral topology*¹; unless expressly stated otherwise, we always assume that $X = \text{Spec}(A)$ is equipped with the spectral topology.

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¹The introduction of this topology in algebraic geometry is due to Zariski. So this topology is usually called the “Zariski topology” on X .

(1.1.3). For each subset Y of X , we denote by $j(Y)$ the set of $f \in A$ such that $f(y) = 0$ for all $y \in Y$; equivalently, $j(Y)$ is the intersection of the prime ideals j_y for $y \in Y$. It is clear that the relation $Y \subset Y'$ implies that $j(Y) \supset j(Y')$ and that we have

$$(1.1.3.1) \quad j\left(\bigcup_{\lambda} Y_{\lambda}\right) = \bigcap_{\lambda} j(Y_{\lambda})$$

for each family (Y_{λ}) of subsets of X . Finally we have

$$(1.1.3.2) \quad j(\{x\}) = j_x.$$

Proposition (1.1.4). —

- (i) For each subset E of A , we have $j(V(E)) = \tau(E)$.
- (ii) For each subset Y of X , $V(j(Y)) = \overline{Y}$, the closure of Y in X .

Proof. (i) is an immediate consequence of the definitions and (1.1.1.1); on the other hand, $V(j(Y))$ is closed and contains Y ; conversely, if $Y \subset V(E)$, we have $f(y) = 0$ for $f \in E$ and all $y \in Y$, so $E \subset j(Y)$, $V(E) \supset V(j(Y))$, which proves (ii). \square

Corollary (1.1.5). — The closed subsets of $X = \text{Spec}(A)$ and the ideals of A equal to their radicals (in other words, those that are the intersection of prime ideals) correspond bijectively by the inclusion-reversing maps $Y \mapsto j(Y)$, $\mathfrak{a} \mapsto V(\mathfrak{a})$; the union $Y_1 \cup Y_2$ of two closed subsets corresponds to $j(Y_1) \cap j(Y_2)$, and the intersection of any family (Y_{λ}) of closed subsets corresponds to the radical of the sum of the $j(Y_{\lambda})$.

Corollary (1.1.6). — If A is a Noetherian ring, $X = \text{Spec}(A)$ is a Noetherian space.

Note that the converse of this corollary is false, as shown by any non-Noetherian integral ring with a single prime ideal $\neq \{0\}$ (for example a nondiscrete valuation ring of rank 1).

As an example of ring A whose spectrum is not a Noetherian space, one can consider the ring $\mathcal{C}(Y)$ of continuous real functions on an infinite compact space Y ; we know that, as a set, Y corresponds to the set of maximal ideals of A , and it is easy to see that the topology induced on Y by that of $X = \text{Spec}(A)$ is the original topology of Y . Since Y is not a Noetherian space, the same is true for X .

Corollary (1.1.7). — For each $x \in X$, the closure of $\{x\}$ is the set of $y \in X$ such that $j_x \subset j_y$. For $\{x\}$ to be closed, it is necessary and sufficient that j_x is maximal.

Corollary (1.1.8). — The space $X = \text{Spec}(A)$ is a Kolmogoroff space.

Proof. If x and y are two distinct points of X , we have either $j_x \not\subset j_y$ or $j_y \not\subset j_x$, so one of the points x , y does not belong to the closure of the other. \square

(1.1.9). According to Proposition (1.1.2, iv), for two elements f, g of A , we have

$$(1.1.9.1) \quad D(fg) = D(f) \cap D(g).$$

Note also that the equality $D(f) = D(g)$ means, according to Proposition (1.1.4, i) and Proposition (1.1.2, v), that $\tau(f) = \tau(g)$, or that the minimal prime ideals containing (f) and (g) are the same; in particular, it is also the case when $f = ug$, where u is invertible.

Proposition (1.1.10). —

- (i) When f ranges over A , the sets $D(f)$ forms a basis for the topology of X .
- (ii) For every $f \in A$, $D(f)$ is quasi-compact. In particular, $X = D(1)$ is quasi-compact.

Proof.

- (i) Let U be an open set in X ; by definition, we have $U = X - V(E)$ where E is a subset of A , and $V(E) = \bigcap_{f \in E} V(f)$, hence $U = \bigcup_{f \in E} D(f)$.
- (ii) By (i), it suffices to prove that, if $(f_{\lambda})_{\lambda \in L}$ is a family of elements of A such that $D(f) \subset \bigcup_{\lambda \in L} D(f_{\lambda})$, then there exists a finite subset J of L such that $D(f) \subset \bigcup_{\lambda \in J} D(f_{\lambda})$. Let \mathfrak{a} be the ideal of A generated by the f_{λ} ; we have, by hypothesis, that $V(f) \supset V(\mathfrak{a})$, so $\tau(f) \subset \tau(\mathfrak{a})$; since $f \in \tau(f)$, there exists an integer $n \geq 0$ such that $f^n \in \mathfrak{a}$. But then f^n belongs to the ideal \mathfrak{b} generated by the finite subfamily $(f_{\lambda})_{\lambda \in J}$, and we have $V(f) = V(f^n) \supset V(\mathfrak{b}) = \bigcap_{\lambda \in J} V(f_{\lambda})$, that is to say, $D(f) \supset \bigcup_{\lambda \in J} D(f_{\lambda})$.

□

Proposition (1.1.11). — For each ideal \mathfrak{a} of A , $\text{Spec}(A/\mathfrak{a})$ is canonically identified with the closed subspace $V(\mathfrak{a})$ of $\text{Spec}(A)$.

Proof. We know there is a canonical bijective correspondence (respecting the inclusion order structure) between ideals (resp. prime ideals) of A/\mathfrak{a} and ideals (resp. prime ideals) of A containing \mathfrak{a} . □

Recall that the set \mathfrak{N} of nilpotent elements of A (the *nilradical* of A) is an ideal equal to $\tau(0)$, the intersection of all the prime ideals of A (0, 1.1.1).

Corollary (1.1.12). — The topological spaces $\text{Spec}(A)$ and $\text{Spec}(A/\mathfrak{N})$ are canonically homeomorphic.

Proposition (1.1.13). — For $X = \text{Spec}(A)$ to be irreducible (0, 2.1.1), it is necessary and sufficient that the ring A/\mathfrak{N} is integral (or, equivalently, that the ideal \mathfrak{N} is prime).

Proof. By virtue of Corollary (1.1.12), we can restrict to the case where $\mathfrak{N} = 0$. If X is reducible, then there exist two distinct closed subsets Y_1 and Y_2 of X such that $X = Y_1 \cup Y_2$, so $j(X) = j(Y_1) \cap j(Y_2) = 0$, since the ideals $j(Y_1)$ and $j(Y_2)$ are distinct from (0) (1.1.5); so A is not integral. Conversely, if there are elements $f \neq 0, g \neq 0$ of A such that $fg = 0$, we have $V(f) \neq X, V(g) \neq X$ (since the intersection of all the prime ideals of A is (0)), and $X = V(fg) = V(f) \cup V(g)$. □

Corollary (1.1.14). —

- (i) In the bijective correspondence between closed subsets of $X = \text{Spec}(A)$ and ideals of A equal to their radicals, the irreducible closed subsets of X correspond to the prime ideals of A . In particular, the irreducible components of X correspond to the minimal prime ideals of A .
- (ii) The map $x \mapsto \overline{\{x\}}$ establishes a bijective correspondence between X and the set of closed irreducible subsets of X (in other words, all closed irreducible subsets of X admit exactly one generic point).

Proof. (i) follows immediately from (1.1.13) and (1.1.11); and for proving (ii), we can, by (1.1.11), restrict to the case where X is irreducible; then, according to Proposition (1.1.13), there exists a smaller prime ideal \mathfrak{N} in A , which corresponds to the generic point of X ; in addition, X admits at most one generic point since it is a Kolmogoroff space ((1.1.8) and (0, 2.1.3)). □

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Proposition (1.1.15). — If \mathfrak{J} is an ideal in A containing the radical $\mathfrak{N}(A)$, the only neighborhood of $V(\mathfrak{J})$ in $X = \text{Spec}(A)$ is the whole space X .

Proof. Each maximal ideal of A belongs, by definition, to $V(\mathfrak{J})$. As each ideal $\mathfrak{a} \neq A$ of A is contained in a maximal ideal, we have $V(\mathfrak{a}) \cap V(\mathfrak{J}) \neq \emptyset$, whence the proposition. □

1.2. Functorial properties of prime spectra of rings.

(1.2.1). Let A, A' be two rings, and

$$\phi : A' \longrightarrow A$$

a homomorphism of rings. For each prime ideal $x = j_x \in \text{Spec}(A) = X$, the ring $A'/\phi^{-1}(j_x)$ is canonically isomorphic to a subring of A/j_x , and so it is integral, or, in other words, $\phi^{-1}(j_x)$ is a prime ideal of A' ; we denote it by ${}^a\phi(x)$, and we have thus defined a map

$${}^a\phi : X = \text{Spec}(A) \longrightarrow X' = \text{Spec}(A')$$

(also denoted $\text{Spec}(\phi)$), that we call the map *associated* to the homomorphism ϕ . We denote by ϕ^x the injective homomorphism from $A'/\phi^{-1}(j_x)$ to A/j_x induced by ϕ by passing to quotients, as well as its canonical extension to a monomorphism of fields

$$\phi^x : k({}^a\phi(x)) \longrightarrow k(x);$$

for each $f' \in A'$, we therefore have, by definition,

$$(1.2.1.1) \quad \phi^x(f'({}^a\phi(x))) = (\phi(f'))(x) \quad (x \in X).$$

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Proposition (1.2.2). —

(i) For each subset E' of A' , we have

$$(1.2.2.1) \quad {}^a\phi^{-1}(V(E')) = V(\phi(E')),$$

and in particular, for each $f' \in A'$,

$$(1.2.2.2) \quad {}^a\phi^{-1}(D(f')) = D(\phi(f')).$$

(ii) For each ideal \mathfrak{a} of A , we have

$$(1.2.2.3) \quad \overline{{}^a\phi(V(\mathfrak{a}))} = V(\phi^{-1}(\mathfrak{a})).$$

Proof. The relation ${}^a\phi(x) \in V(E')$ is, by definition, equivalent to $E' \subset \phi^{-1}(j_x)$, so $\phi(E') \subset j_x$, and finally $x \in V(\phi(E'))$, hence (i). To prove (ii), we can suppose that \mathfrak{a} is equal to its radical, since $V(\mathfrak{r}(\mathfrak{a})) = V(\mathfrak{a})$ (1.1.2, v) and $\phi^{-1}(\mathfrak{r}(\mathfrak{a})) = \mathfrak{r}(\phi^{-1}(\mathfrak{a}))$; the relation $f' \in \mathfrak{a}'$ is, by definition, equivalent to $f'(x') = 0$ for each $x \in {}^a\phi(Y)$, so, by Equation (1.2.1.1), it is also equivalent to $\phi(f')(x) = 0$ for each $x \in Y$, or to $\phi(f') \in j(Y) = \mathfrak{a}$, since \mathfrak{a} is equal to its radical; hence (ii). \square

Corollary (1.2.3). — The map ${}^a\phi$ is continuous.

We remark that, if A'' is a third ring, and ϕ' a homomorphism $A'' \rightarrow A'$, then we have ${}^a(\phi' \circ \phi) = {}^a\phi \circ {}^a\phi'$; this result, with Corollary (1.2.3), says that $\text{Spec}(A)$ is a *contravariant functor* in A , from the category of rings to that of topological spaces.

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Corollary (1.2.4). — Suppose that ϕ is such that every $f \in A$ can be written as $f = h\phi(f')$, where h is invertible in A (which, in particular, is the case when ϕ is surjective). Then ${}^a\phi$ is a homeomorphism from X to ${}^a\phi(X)$.

Proof. We show that for each subset $E \subset A$, there exists a subset E' of A' such that $V(E) = V(\phi(E'))$; according to the (T_0) axiom (1.1.8) and the formula (1.2.2.1), this implies first of all that ${}^a\phi$ is injective, and then, by (1.2.2.1), that ${}^a\phi$ is a homeomorphism. But it suffices, for each $f \in E$, to take $f' \in A'$ such that $h\phi(f') = f$ with h invertible in A ; the set E' of these elements f' is exactly what we are searching for. \square

(1.2.5). In particular, when ϕ is the canonical homomorphism from A to a ring quotient A/\mathfrak{a} , we again get (1.1.12), and ${}^a\phi$ is the canonical injection of $V(\mathfrak{a})$, identified with $\text{Spec}(A/\mathfrak{a})$, into $X = \text{Spec}(A)$.

Another particular case of (1.2.4):

Corollary (1.2.6). — If S is a multiplicative subset of A , the spectrum $\text{Spec}(S^{-1}A)$ is canonically identified (with its topology) with the subspace of $X = \text{Spec}(A)$ consisting of the x such that $j_x \cap S = \emptyset$.

Proof. We know by (0, 1.2.6) that the prime ideals of $S^{-1}A$ are the ideals $S^{-1}j_x$ such that $j_x \cap S = \emptyset$, and that we have $j_x = (i_A^S)^{-1}(S^{-1}j_x)$. It then suffices to apply Corollary (1.2.4) to the i_A^S . \square

Corollary (1.2.7). — For ${}^a\phi(X)$ to be dense in X' , it is necessary and sufficient for each element of the kernel $\text{Ker } \phi$ to be nilpotent.

Proof. Applying Equation (1.2.2.3) to the ideal $\mathfrak{a} = (0)$, we have $\widetilde{{}^a\phi(X)} = V(\text{Ker } \phi)$, and for $V(\text{Ker } \phi) = X'$ to hold, it is necessary and sufficient for $\text{Ker } \phi$ to be contained in all the prime ideals of A' , or, equivalently, in the nilradical \mathfrak{r}' of A' . \square

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1.3. Sheaf associated to a module.

(1.3.1). Let A be a commutative ring, M an A -module, f an element of A , and S_f the multiplicative set consisting of the f^n , where $n \geq 0$. Recall that we set $A_f = S_f^{-1}A$, $M_f = S_f^{-1}M$. If S'_f is the saturated multiplicative subset of A consisting of the $g \in A$ which divide an element of S_f , we know that A_f and M_f are canonically identified with $S'_f{}^{-1}A$ and $S'_f{}^{-1}M$ (0, 1.4.3).

Lemma (1.3.2). — The following conditions are equivalent:

- (a) $g \in S'_f$;
- (b) $S'_g \subset S'_f$;
- (c) $f \in \mathfrak{r}(g)$;
- (d) $\mathfrak{r}(f) \subset \mathfrak{r}(g)$;

- (e) $V(g) \subset V(f)$;
- (f) $D(f) \subset D(g)$.

Proof. This follows immediately from the definitions and (1.1.5). \square

(1.3.3). If $D(f) = D(g)$, then Lemma (1.3.2, b) shows that $M_f = M_g$. More generally, if $D(f) \supset D(g)$, then $S'_f \subset S'_g$, and we know (0, 1.4.1) that there exists a canonical functorial homomorphism

$$\rho_{g,f} : M_f \longrightarrow M_g,$$

and if $D(f) \supset D(g) \supset D(h)$, we have (0, 1.4.4)

$$(1.3.3.1) \quad \rho_{h,g} \circ \rho_{g,f} = \rho_{h,f}.$$

When f ranges over the elements of $A - \mathfrak{j}_x$ (for a given x in $X = \text{Spec}(A)$), the sets S'_f constitute an increasing filtered set of subsets of $A - \mathfrak{j}_x$, since for elements f and g of $A - \mathfrak{j}_x$, S'_f and S'_g are contained in S'_{fg} ; since the union of the S'_f over $f \in A - \mathfrak{j}_x$ is $A - \mathfrak{j}_x$, we conclude (0, 1.4.5) that the A_x -module M_x is canonically identified with the inductive limit $\varinjlim M_f$, relative to the family of homomorphisms $(\rho_{g,f})$. We denote by

$$\rho_x^f : M_f \longrightarrow M_x$$

the canonical homomorphism for $f \in A - \mathfrak{j}_x$ (or, equivalently, $x \in D(f)$).

Definition (1.3.4). — We define the structure sheaf of the prime spectrum $X = \text{Spec}(A)$ (resp. the sheaf associated to the A -module M), denoted by \tilde{A} or \mathcal{O}_X (resp. \tilde{M}) as the sheaf of rings (resp. the \tilde{A} -module) associated to the presheaf $D(f) \mapsto A_f$ (resp. $D(f) \mapsto M_f$), defined on the basis \mathfrak{B} of X consisting of the $D(f)$ for $f \in A$ ((1.1.10), (0, 3.2.1), and (0, 3.5.6)).

We saw (0, 3.2.4) that the stalk \tilde{A}_x (resp. \tilde{M}_x) can be identified with the ring A_x (resp. the A_x -module M_x); we denote by

$$\theta_f : A_f \longrightarrow \Gamma(D(f), \tilde{A})$$

$$(\text{resp. } \theta_f : M_f \longrightarrow \Gamma(D(f), \tilde{M})),$$

the canonical map, so that, for all $x \in D(f)$ and all $\xi \in M_f$, we have

$$(1.3.4.1) \quad (\theta_f(\xi))_x = \rho_x^f(\xi).$$

Proposition (1.3.5). — \tilde{M} is an exact functor, covariant in M , from the category of A -modules to the category of \tilde{A} -modules.

Proof. Indeed, let M, N be two A -modules, and u a homomorphism $M \rightarrow N$; for each $f \in A$, u corresponds canonically to a homomorphism u_f from the A_f -module M_f to the A_f -module N_f , and the diagram (for $D(g) \subset D(f)$)

$$\begin{array}{ccc} M_f & \xrightarrow{u_f} & N_f \\ \rho_{g,f} \downarrow & & \downarrow \rho_{g,f} \\ M_g & \xrightarrow{u_g} & N_g \end{array}$$

is commutative (1.4.1); these homomorphisms then define a homomorphism of \tilde{A} -modules $\tilde{u} : \tilde{M} \rightarrow \tilde{N}$ (0, 3.2.3). In addition, for each $x \in X$, \tilde{u}_x is the inductive limit of the u_f for $x \in D(f)$ ($f \in A$), and as a result (0, 1.4.5), if we canonically identify \tilde{M}_x and \tilde{N}_x with M_x and N_x respectively, then \tilde{u}_x is identified with the homomorphism u_x canonically induced by u . If P is a third A -module, v a homomorphism $N \rightarrow P$, and $w = v \circ u$, it is immediate that $w_x = v_x \circ u_x$, so $\tilde{w} = \tilde{v} \circ \tilde{u}$. We have therefore clearly defined a covariant (in M) functor \tilde{M} , from the category of A -modules to that of \tilde{A} -modules. This functor is exact, since, for each $x \in X$, M_x is an exact functor in M (0, 1.3.2); in addition, we have $\text{Supp}(M) = \text{Supp}(\tilde{M})$, by definition ((0, 1.7.1) and (0, 3.1.6)). \square

Proposition (1.3.6). — For each $f \in A$, the open subset $D(f) \subset X$ is canonically identified with the prime spectrum $\text{Spec}(A_f)$, and the sheaf \widetilde{M}_f associated to the A_f -module M_f is canonically identified with the restriction $\widetilde{M}|_{D(f)}$.

Proof. The first assertion is a particular case of (1.2.6). In addition, if $g \in A$ is such that $D(g) \subset D(f)$, then M_g is canonically identified with the module of fractions of M_f whose denominators are the powers of the canonical image of g in A_f (0, 1.4.6). The canonical identification of \widetilde{M}_f with $\widetilde{M}|_{D(f)}$ then follows from the definitions. \square

Theorem (1.3.7). — For each A -module M and each $f \in A$, the homomorphism

$$\theta_f : M_f \longrightarrow \Gamma(D(f), \widetilde{M})$$

is bijective (in other words, the presheaf $D(f) \mapsto M_f$ is a sheaf). In particular, M can be identified with $\Gamma(X, \widetilde{M})$ via θ_1 .

Proof. We note that, if $M = A$, then θ_f is a homomorphism of structure rings; Theorem (1.3.7) then implies that, if we identify the rings A_f and $\Gamma(D(f), \widetilde{A})$ via θ_f , the homomorphism $\theta_f : M_f \rightarrow \Gamma(D(f), \widetilde{M})$ is an isomorphism of modules.

We show first that θ_f is injective. Indeed, if $\xi \in M_f$ is such that $\theta_f(\xi) = 0$, then, for each prime ideal \mathfrak{p} of A_f , there exists $h \notin \mathfrak{p}$ such that $h\xi = 0$; as the annihilator of ξ is not contained in any prime ideal of A_f , each A_f integral, and so $\xi = 0$.

It remains to show that θ_f is surjective; we can restrict to the case where $f = 1$ (the general case then following by “localizing”, using (1.3.6)). Now let s be a section of \widetilde{M} over X ; according to (1.3.4) and (1.1.10, ii), there exists a finite cover $(D(f_i))_{i \in I}$ of X ($f_i \in A$) such that, for each $i \in I$, the restriction $s_i = s|_{D(f_i)}$ is of the form $\theta_{f_i}(\xi_i)$, where $\xi_i \in M_{f_i}$. If i, j are indices of I , and if the restrictions of s_i and s_j to $D(f_i) \cap D(f_j) = D(f_i f_j)$ are equal, then it follows, by definition of M , that

$$(1.3.7.1) \quad \rho_{f_i f_j, f_i}(\xi_i) = \rho_{f_i f_j, f_j}(\xi_j).$$

By definition, we can write, for each $i \in I$, $\xi_i = z_i / f_i^{n_i}$, where $z_i \in M$, and, since I is finite, by multiplying each z_i by a power of f_i , we can assume that all the n_i are equal to one single n . Then, by definition, (1.3.7.1) implies that there exists an integer $m_{ij} \geq 0$ such that $(f_i f_j)^{m_{ij}}(f_j^n z_i - f_i^n z_j) = 0$, and we can moreover suppose that the m_{ij} are equal to the one single m ; then replacing the z_i by $f_i^m z_i$, it remains to prove the case where $m = 0$, in other words, the case where we have

$$(1.3.7.2) \quad f_j^n z_i = f_i^n z_j$$

for any i, j . We have $D(f_i^n) = D(f_i)$, and since the $D(f_i)$ form a cover of X , the ideal generated by the f_i^n is A ; in other words, there exist elements $g_i \in A$ such that $\sum_i g_i f_i^n = 1$. Then consider the element $z = \sum_i g_i z_i$ of M ; in (1.3.7.2), we have $f_i^n z = \sum_j g_j f_i^n z_j = (\sum_j g_j f_j^n) z_i = z_i$, where, by definition, $\xi_i = z/1$ in M_{f_i} . We conclude that the s_i are the restrictions to $D(f_i)$ of $\theta_1(z)$, which proves that $s = \theta_1(z)$ and finishes the proof. \square

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Corollary (1.3.8). — Let M and N be A -modules; the canonical homomorphism $u \mapsto \widetilde{u}$ from $\text{Hom}_A(M, N)$ to $\text{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ is bijective. In particular, the equations $M = 0$ and $\widetilde{M} = 0$ are equivalent.

Proof. Consider the canonical homomorphism $v \mapsto \Gamma(v)$ from $\text{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$ to $\text{Hom}_{\Gamma(\widetilde{A})}(\Gamma(\widetilde{M}), \Gamma(\widetilde{N}))$; the latter module is canonically identified with $\text{Hom}_A(M, N)$, by Theorem (1.3.7). It remains to show that $u \mapsto \widetilde{u}$ and $v \mapsto \Gamma(v)$ are inverses of each other; it is evident that $\Gamma(\widetilde{u}) = u$ by definition of \widetilde{u} ; on the other hand, if we let $u = \Gamma(v)$ for $v \in \text{Hom}_{\widetilde{A}}(\widetilde{M}, \widetilde{N})$, then the map $w : \Gamma(D(f), \widetilde{M}) \rightarrow \Gamma(D(f), \widetilde{N})$ canonically induced from v is such that the diagram

$$\begin{array}{ccc} M & \xrightarrow{u} & N \\ \rho_{f,1} \downarrow & & \downarrow \rho_{f,1} \\ M_f & \xrightarrow{w} & N_f \end{array}$$

is commutative; so we necessarily have that $w = u_f$ for all $f \in A$ (0, 1.2.4), which shows that $\Gamma(\widetilde{v}) = v$. \square

Corollary (1.3.9). —

- (i) Let u be a homomorphism from an A -module M to an A -module N ; then the sheaves associated to $\text{Ker } u$, $\text{Im } u$, and $\text{Coker } u$, are $\text{Ker } \widetilde{u}$, $\text{Im } \widetilde{u}$, and $\text{Coker } \widetilde{u}$ (respectively). In particular, for \widetilde{u} to be injective (resp. surjective, bijective), it is necessary and sufficient for u to be so too.
- (ii) If M is an inductive limit (resp. direct sum) of a family of A -modules (M_λ) , then \widetilde{M} is the inductive limit (resp. direct sum) of the family (\widetilde{M}_λ) , via a canonical isomorphism.

Proof.

- (i) It suffices to apply the fact that \widetilde{M} is an exact functor in M (1.3.5) to the two exact sequences of A -modules

$$\begin{aligned} 0 &\longrightarrow \text{Ker } u \longrightarrow M \longrightarrow \text{Im } u \longrightarrow 0, \\ 0 &\longrightarrow \text{Im } u \longrightarrow N \longrightarrow \text{Coker } u \longrightarrow 0. \end{aligned}$$

The second claim then follows from Theorem (1.3.7).

- (ii) Let $(M_\lambda, g_{\mu\lambda})$ be an inductive system of A -modules, with inductive limit M , and let g_λ be the canonical homomorphism $M_\lambda \rightarrow M$. Since we have $\widetilde{g_{\nu\mu}} \circ \widetilde{g_{\mu\lambda}} = \widetilde{g_{\nu\lambda}}$ and $\widetilde{g_\lambda} = \widetilde{g_\mu} \circ \widetilde{g_{\mu\lambda}}$ for $\lambda \leq \mu \leq \nu$, it follows that $(\widetilde{M}_\lambda, \widetilde{g_{\mu\lambda}})$ is an inductive system of sheaves on X , and if we denote by h_λ the canonical homomorphism $\widetilde{M}_\lambda \rightarrow \varinjlim \widetilde{M}_\lambda$, then there is a unique homomorphism $v : \varinjlim \widetilde{M}_\lambda \rightarrow \widetilde{M}$ such that $v \circ h_\lambda = \widetilde{g_\lambda}$. To see that v is bijective, it suffices to check, for each $x \in X$, that v_x is a bijection from $(\varinjlim \widetilde{M}_\lambda)_x$ to \widetilde{M}_x ; but $\widetilde{M}_x = M_x$, and

$$(\varinjlim \widetilde{M}_\lambda)_x = \varinjlim (\widetilde{M}_\lambda)_x = \varinjlim (M_\lambda)_x = M_x \quad (0, 1.3.3).$$

Conversely, it follows from the definitions that $(\widetilde{g_\lambda})_x$ and $(h_\lambda)_x$ are both equal to the canonical map from $(M_\lambda)_x$ to M_x ; since $(\widetilde{g_\lambda})_x = v_x \circ (h_\lambda)_x$, v_x is the identity. Finally, if M is the direct sum of two A -modules N and P , it is immediate that $\widetilde{M} = \widetilde{N} \oplus \widetilde{P}$; each direct sum being the inductive limit of finite direct sums, the claims of (ii) are thus proved. \square

We note that Corollary (1.3.8) proves that the sheaves isomorphic to the associated sheaves of A -modules form an *abelian category* (T, I, 1.4).

We also note that Corollary (1.3.9) implies that, if M is an A -module of *finite type* (that is to say, there exists a surjective homomorphism $A^n \rightarrow M$) then there exists a surjective homomorphism $\widetilde{A}^n \rightarrow \widetilde{M}$, or, in other words, the \widetilde{A} -module \widetilde{M} is *generated by a finite family of sections over X* (0, 5.1.1), and vice versa.

(1.3.10). If N is a submodule of an A -module M , the canonical injection $j : N \rightarrow M$ gives, by (1.3.9), an injective homomorphism $\widetilde{N} \rightarrow \widetilde{M}$, which allows us to canonically identify \widetilde{N} with an \widetilde{A} -submodule of \widetilde{M} ; we will always assume that we have made this identification. If N and P are submodules of M , then we have

$$(1.3.10.1) \quad (N + P)^\sim = \widetilde{N} + \widetilde{P},$$

$$(1.3.10.2) \quad (N \cap P)^\sim = \widetilde{N} \cap \widetilde{P},$$

since $N + P$ and $N \cap P$ are the image of the canonical homomorphism $N \oplus P \rightarrow M$ and the kernel of the canonical homomorphism $M \rightarrow (M/N) \oplus (M/P)$ (respectively), and it suffices to apply (1.3.9).

We conclude from (1.3.10.1) and (1.3.10.2) that, if $\widetilde{N} = \widetilde{P}$, then we have $N = P$.

Corollary (1.3.11). — *On the category of sheaves isomorphic to the associated sheaves of A -modules, the functor Γ is exact.*

Proof. Let $\widetilde{M} \xrightarrow{\widetilde{u}} \widetilde{N} \xrightarrow{\widetilde{v}} \widetilde{P}$ be an exact sequence corresponding to two homomorphisms $u : M \rightarrow N$ and $v : N \rightarrow P$ of A -modules. If $Q = \text{Im } u$ and $R = \text{Ker } v$, we have $\widetilde{Q} = \text{Im } \widetilde{u} = \text{Ker } \widetilde{v} = \widetilde{R}$ (Corollary (1.3.9)), hence $Q = R$. \square

Corollary (1.3.12). — *Let M and N be A -modules.*

- (i) *The sheaf associated to $M \otimes_A N$ is canonically identified with $\tilde{M} \otimes_{\tilde{A}} \tilde{N}$.*
- (ii) *If, in addition, M admits a finite presentation, then the sheaf associated to $\text{Hom}_A(M, N)$ is canonically identified with $\mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N})$.*

Proof.

- (i) The sheaf $\mathcal{F} = \tilde{M} \otimes_{\tilde{A}} \tilde{N}$ is associated to the presheaf

$$U \longmapsto \mathcal{F}(U) = \Gamma(U, \tilde{M}) \otimes_{\Gamma(U, \tilde{A})} \Gamma(U, \tilde{N}),$$

with U varying over the basis (1.1.10, i) of X consisting of the $D(f)$, where $f \in A$. We know that $\mathcal{F}(D(f))$ is canonically identified with $M_f \otimes_{A_f} N_f$, by (1.3.7) and (1.3.6). Moreover, we know that the A_f -module $M_f \otimes_{A_f} N_f$ is canonically isomorphic to $(M \otimes_A N)_f$ (0, 1.3.4), which is itself canonically isomorphic to $\Gamma(D(f), (M \otimes_A N)^\sim)$ (Theorem (1.3.7) and Proposition (1.3.6)). In addition, we see immediately that the canonical isomorphisms

$$\mathcal{F}(D(f)) \simeq \Gamma(D(f), (M \otimes_A N)^\sim)$$

thus obtained satisfy the compatibility conditions with respect to the restriction operations (0, 1.4.2), so they define a canonical functorial isomorphism

$$\tilde{M} \otimes_{\tilde{A}} \tilde{N} \simeq (M \otimes_A N)^\sim.$$

- (ii) The sheaf $\mathcal{G} = \mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N})$ is associated to the presheaf

$$U \longmapsto \mathcal{G}(U) = \text{Hom}_{\tilde{A}|U}(\tilde{M}|U, \tilde{N}|U),$$

with U varying over the basis of X consisting of the $D(f)$. We know that $\mathcal{G}(D(f))$ is canonically identified with $\text{Hom}_{A_f}(M_f, N_f)$ (Proposition (1.3.6) and Corollary (1.3.8)), which, according to the hypotheses on M , is canonically identified with $(\text{Hom}_A(M, N))_f$ (0, 1.3.5). Finally, $(\text{Hom}_A(M, N))_f$ is canonically identified with $\Gamma(D(f), (\text{Hom}_A(M, N))^\sim)$ (Proposition (1.3.6) and Theorem (1.3.7)), and the canonical isomorphisms $\mathcal{G}(D(f)) \simeq \Gamma(D(f), (\text{Hom}_A(M, N))^\sim)$ thus obtained are compatible with the restriction operations (0, 1.4.2); they thus define a canonical isomorphism $\mathcal{H}om_{\tilde{A}}(\tilde{M}, \tilde{N}) \simeq (\text{Hom}_A(M, N))^\sim$.

□

(1.3.13). Now let B be a (commutative) A -algebra; this can be understood by saying that B is an A -module such that we have some given element $e \in B$ and an A -homomorphism $\phi : B \otimes_A B \rightarrow B$, so that (a) the diagrams

$$\begin{array}{ccc} B \otimes_A B \otimes_A B & \xrightarrow{\phi \otimes 1} & B \otimes_A B \\ 1 \otimes \phi \downarrow & & \downarrow \phi \\ B \otimes_A B & \xrightarrow{\phi} & B \end{array} \quad \begin{array}{ccc} B \otimes_A B & \xrightarrow{\sigma} & B \otimes_A B \\ & \searrow \phi & \swarrow \phi \\ & B & \end{array}$$

(σ being the canonical symmetry map) are commutative; and (b) $\phi(e \otimes x) = \phi(x \otimes e) = x$. By Corollary (1.3.12), the homomorphism $\tilde{\phi} : \tilde{B} \otimes_{\tilde{A}} \tilde{B} \rightarrow \tilde{B}$ of \tilde{A} -modules satisfies the analogous conditions, and so it defines an \tilde{A} -algebra structure on \tilde{B} . In a similar way, the data of a B -module N is the same as the data of an A -module N and an A -homomorphism $\psi : B \otimes_A N \rightarrow N$ such that the diagram

$$\begin{array}{ccc} B \otimes_A B \otimes_A N & \xrightarrow{\phi \otimes 1} & B \otimes_A N \\ 1 \otimes \psi \downarrow & & \downarrow \psi \\ B \otimes_A N & \xrightarrow{\psi} & N \end{array}$$

is commutative and $\psi(e \otimes n) = n$; the homomorphism $\tilde{\psi} : \tilde{B} \otimes_{\tilde{A}} \tilde{N} \rightarrow \tilde{N}$ satisfies the analogous condition, and so defines a \tilde{B} -module structure on \tilde{N} .

In a similar way, we see that if $u : B \rightarrow B'$ (resp. $v : N \rightarrow N'$) is a homomorphism of A -algebras (resp. of B -modules), then \tilde{u} (resp. \tilde{v}) is a homomorphism of \tilde{A} -algebras (resp. of \tilde{B} -modules), and $\text{Ker } \tilde{u}$ is a \tilde{B} -ideal (resp. $\text{Ker } \tilde{v}$, $\text{Coker } \tilde{v}$, and $\text{Im } \tilde{v}$ are \tilde{B} -modules). If N is a B -module, then \tilde{N} is a \tilde{B} -module of finite type if and only if N is a B -module of finite type (0, 5.2.3).

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If M, N are B -modules, then the \tilde{B} -module $\tilde{M} \otimes_{\tilde{B}} \tilde{N}$ is canonically identified with $(M \otimes_B N)^\sim$; similarly $\mathcal{H}om_{\tilde{B}}(\tilde{M}, \tilde{N})$ is canonically identified with $(\text{Hom}_B(M, N))^\sim$ whenever M admits a finite presentation; the proofs are similar to those for Corollary (1.3.12).

If \mathfrak{J} is an ideal of B , and N is a B -module, then we have $(\mathfrak{J}N)^\sim = \tilde{\mathfrak{J}} \cdot \tilde{N}$.

Finally, if B is an A -algebra graded by the A -submodules B_n ($n \in \mathbb{Z}$), then the \tilde{A} -algebra \tilde{B} , the direct sum of the \tilde{A} -modules \tilde{B}_n (1.3.9), is graded by these \tilde{A} -submodules, the axiom of graded algebras saying that the image of the homomorphism $B_m \otimes B_n \rightarrow B$ is contained in B_{m+n} . Similarly, if M is a B -module graded by the submodules M_n , then \tilde{M} is a \tilde{B} -module graded by the \tilde{M}_n .

(1.3.14). If B is an A -algebra, and M a submodule of B , then the \tilde{A} -subalgebra of \tilde{B} generated by \tilde{M} (0, 4.1.3) is the \tilde{A} -subalgebra \tilde{C} , where we denote by C the subalgebra of B generated by M . Indeed, C is the direct sum of the submodules of B which are the images of the homomorphisms $\bigotimes^n M \rightarrow B$ ($n \geq 0$), so it suffices to apply (1.3.9) and (1.3.12).

1.4. Quasi-coherent sheaves over a prime spectrum.

Theorem (1.4.1). — *Let X be the prime spectrum of a ring A , V a quasi-compact open subset of X , and \mathcal{F} an $(\mathcal{O}_X|V)$ -module. The following four conditions are equivalent.*

- (a) *There exists an A -module M such that \mathcal{F} is isomorphic to $\tilde{M}|V$.*
- (b) *There exists a finite open cover (V_i) of V by sets of the form $D(f_i)$ ($f_i \in A$) contained in V , such that, for each i , $\mathcal{F}|V_i$ is isomorphic to a sheaf of the form \tilde{M}_i , where M_i is an A_{f_i} -module.*
- (c) *The sheaf \mathcal{F} is quasi-coherent (0, 5.1.3).*
- (d) *The two following properties are satisfied:*
 - (d1) *For each $f \in A$ such that $D(f) \subset V$, and for each section $s \in \Gamma(D(f), \mathcal{F})$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section of \mathcal{F} over V .*
 - (d2) *For each $f \in A$ such that $D(f) \subset V$ and for each section $t \in \Gamma(V, \mathcal{F})$ such that the restriction of t to $D(f)$ is 0, there exists an integer $n \geq 0$ such that $f^n t = 0$.*

(In the statement of the conditions (d1) and (d2), we have tacitly identified A and $\Gamma(\tilde{A})$ using Theorem (1.3.7)).

Proof. The fact that (a) implies (b) is an immediate consequence of Proposition (1.3.6) and the fact that the $D(f_i)$ form a basis for the topology of X (1.1.10). As each A -module is isomorphic to the cokernel of a homomorphism of the form $A^{(I)} \rightarrow A^{(J)}$, (1.3.6) implies that each sheaf associated to an A -module is quasi-coherent; so (b) implies (c). Conversely, if \mathcal{F} is quasi-coherent, each $x \in V$ has a neighborhood of the form $D(f) \subset V$ such that $\mathcal{F}|D(f)$ is isomorphic to the cokernel of a homomorphism $\tilde{A}_f^{(I)} \rightarrow \tilde{A}_f^{(J)}$, so also to the sheaf \tilde{N} associated to the module N , the cokernel of the corresponding homomorphism $A_f^{(I)} \rightarrow A_f^{(J)}$ (Corollaries (1.3.8) and (1.3.9)); since V is quasi-compact, it is clear that (c) implies (b).

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To prove that (b) implies (d1) and (d2), we first assume that $V = D(g)$ for some $g \in A$, and that \mathcal{F} is isomorphic to the sheaf \tilde{N} associated to an A_g -module N ; by replacing X with V and A with A_g (1.3.6), we can reduce to the case where $g = 1$. Then $\Gamma(D(f), \tilde{N})$ and N_f are canonically identified with one another (Proposition (1.3.6) and Theorem (1.3.7)), so a section $s \in \Gamma(D(f), \tilde{N})$ is identified with an element of the form z/f^n , where $z \in N$; the section $f^n s$ is identified with the element $z/1$ of N_f and, as a result, is the restriction to $D(f)$ of the section of \tilde{N} over X that is identified with the element $z \in N$; hence (d1) in this case. Similarly, $t \in \Gamma(X, \tilde{N})$ is identified with an element $z' \in N$, the restriction of t to $D(f)$ is identified with the image $z'/1$ of z' in N_f , and to say that this image is zero means that there exists some $n \geq 0$ such that $f^n z' = 0$ in N , or, equivalently, $f^n t = 0$.

To finish the proof, that (b) implies (d1) and (d2), it suffices to establish the following lemma.

Lemma (1.4.1.1). — Suppose that V is the finite union of sets of the form $D(g_i)$, and that all of the sheaves $\mathcal{F}|_{D(g_i)}$ and $\mathcal{F}|_{(D(g_i) \cap D(g_j))} = \mathcal{F}|_{D(g_i g_j)}$ satisfy (d1) and (d2); then \mathcal{F} has the following two properties:

- (d'1) For each $f \in A$ and for each section $s \in \Gamma(D(f) \cap V, \mathcal{F})$, there exists an integer $n \geq 0$ such that $f^n s$ extends to a section of \mathcal{F} over V .
- (d'2) For each $f \in A$ and for each section $t \in \Gamma(V, \mathcal{F})$ such that the restriction of t to $D(f) \cap V$ is 0, there exists an integer $n \geq 0$ such that $f^n t = 0$.

We first prove (d'2): since $D(f) \cap D(g_i) = D(f g_i)$, there exists, for each i , an integer n_i such that the restriction of $(f g_i)^{n_i} t$ to $D(g_i)$ is zero: since the image of g_i in A_{g_i} is invertible, the restriction of $f^{n_i} t$ to $D(g_i)$ is also zero; taking n to be the largest of the n_i , we have proved (d'2).

To show (d'1), we apply (d1) to the sheaf $\mathcal{F}|_{D(g_i)}$: there exists an integer $n_i \geq 0$ and a section s'_i of \mathcal{F} over $D(g_i)$ extending the restriction of $(f g_i)^{n_i} s$ to $D(f g_i)$; since the image of g_i in A_{g_i} is invertible, there is a section s_i of \mathcal{F} over $D(g_i)$ such that $s'_i = g_i^{n_i} s_i$, and s_i extends the restriction of $f^{n_i} s$ to $D(f g_i)$; in addition we can suppose that all the n_i are equal to a single integer n . By construction, the restriction of $s_i - s_j$ to $D(f) \cap D(g_i) \cap D(g_j) = D(f g_i g_j)$ is zero; by (d2) applied to the sheaf $\mathcal{F}|_{D(g_i g_j)}$, there exists an integer $m_{ij} \geq 0$ such that the restriction to $D(g_i g_j)$ of $(f g_i g_j)^{m_{ij}} (s_i - s_j)$ is zero; since the image of $g_i g_j$ in $A_{g_i g_j}$ is invertible, the restriction of $f^{m_{ij}} (s_i - s_j)$ to $D(g_i g_j)$ is zero. We can then assume that all the m_{ij} are equal to a single integer m , and so there exists a section $s' \in \Gamma(V, \mathcal{F})$ extending the $f^m s_i$; as a result, this section extends $f^{n+m} s$, hence we have proved (d'1).

It remains to show that (d1) and (d2) imply (a). We first show that (d1) and (d2) imply that these conditions are satisfied for each sheaf $\mathcal{F}|_{D(g)}$, where $g \in A$ is such that $D(g) \subset V$. It is evident for (d1); on the other hand, if $t \in \Gamma(D(g), \mathcal{F})$ is such that its restriction to $D(f) \subset D(g)$ is zero, there exists, by (d1), an integer $m \geq 0$ such that $g^m t$ extends to a section s of \mathcal{F} over V ; applying (d2), we see that there exists an integer $n \geq 0$ such that $f^n g^m t = 0$, and as the image of g in A_g is invertible, $f^n t = 0$.

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That being so, since V is quasi-compact, Lemma (1.4.1.1) proves that the conditions (d'1) and (d'2) are satisfied. Consider then the A -module $M = \Gamma(V, \mathcal{F})$, and define a homomorphism of \tilde{A} -modules $u : \tilde{M} \rightarrow j_*(\mathcal{F})$, where j is the canonical injection $V \rightarrow X$. Since the $D(f)$ form a basis for the topology of X , it suffices, for each $f \in A$, to define a homomorphism $u_f : M_f \rightarrow \Gamma(D(f), j_*(\mathcal{F})) = \Gamma(D(f) \cap V, \mathcal{F})$, with the usual compatibility conditions (0, 3.2.5). Since the canonical image of f in A_f is invertible, the restriction homomorphism $M = \Gamma(V, \mathcal{F}) \rightarrow \Gamma(D(f) \cap V, \mathcal{F})$ factors as $M \rightarrow M_f \xrightarrow{u_f} \Gamma(D(f) \cap V, \mathcal{F})$ (0, 1.2.4), and the verification of these compatibility conditions for $D(g) \subset D(f)$ is immediate. This being so, we show that the condition (d'1) (resp. (d'2)) implies that each of the u_f are surjective (resp. injective), which proves that u is bijective, and as a result that \mathcal{F} is the restriction to V of an \tilde{A} -module isomorphic to \tilde{M} . If $s \in \Gamma(D(f) \cap V, \mathcal{F})$, there exists, by (d'1), an integer $n \geq 0$ such that $f^n s$ extends to a section $z \in M$; we then have $u_f(z/f^n) = s$, so u_f is surjective. Similarly, if $z \in M$ is such that $u_f(z/1) = 0$, this means that the restriction to $D(f) \cap V$ of the section z is zero; according to (d'2), there exists an integer $n \geq 0$ such that $f^n z = 0$, hence $z/1 = 0$ in M_f , and so u_f is injective. \square

Corollary (1.4.2). — Each quasi-coherent sheaf over a quasi-compact open subset of X is induced by a quasi-coherent sheaf on X .

Corollary (1.4.3). — Every quasi-coherent \mathcal{O}_X -algebra over $X = \text{Spec}(A)$ is isomorphic to an \mathcal{O}_X -algebra of the form \tilde{B} , where B is an A -algebra over A ; every quasi-coherent \tilde{B} -module is isomorphic to a \tilde{B} -module of the form \tilde{N} , where N is a B -module.

Proof. Indeed, a quasi-coherent \mathcal{O}_X -algebra is a quasi-coherent \mathcal{O}_X -module, and therefore of the form \tilde{B} , where B is an A -module; the fact that B is an A -algebra follows from the characterization of the structure of an \mathcal{O}_X -algebra using the homomorphism $\tilde{B} \otimes_{\tilde{A}} \tilde{B} \rightarrow \tilde{B}$ of \tilde{A} -modules, as well as Corollary (1.3.12). If \mathcal{G} is a quasi-coherent \tilde{B} -module, it suffices to show, in a similar way, that it is also a quasi-coherent \tilde{A} -module to conclude the proof; since the question is local, we can, by restricting to an open subset of X of the form $D(f)$, assume that \mathcal{G} is the cokernel of a homomorphism $\tilde{B}^{(I)} \rightarrow \tilde{B}^{(I)}$

of \tilde{B} -modules (and *a fortiori* of \tilde{A} -modules); the proposition then follows from Corollaries (1.3.8) and (1.3.9). \square

1.5. Coherent sheaves over a prime spectrum.

Theorem (1.5.1). — *Let A be a Noetherian ring, $X = \text{Spec}(A)$ its prime spectrum, V an open subset of X , and \mathcal{F} an $(\mathcal{O}_X|V)$ -module. The following conditions are equivalent.*

- (a) \mathcal{F} is coherent.
- (b) \mathcal{F} is of finite type and quasi-coherent.
- (c) There exists an A -module M of finite type such that \mathcal{F} is isomorphic to the sheaf $\tilde{M}|V$.

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Proof. (a) trivially implies (b). To see that (b) implies (c), note that, since V is quasi-compact (0, 2.2.3), we have previously seen that \mathcal{F} is isomorphic to a sheaf $\tilde{N}|V$, where N is an A -module (1.4.1). We have $N = \varinjlim M_\lambda$, where M_λ run over the set of A -submodules of N of finite type, hence (1.3.9) $\mathcal{F} = \tilde{N}|V = \varinjlim \tilde{M}_\lambda|V$; but since \mathcal{F} is of finite type, and V is quasi-compact, there exists an index λ such that $\mathcal{F} = \tilde{M}_\lambda|V$ (0, 5.2.3).

Finally, we show that (c) implies (a). It is clear that \mathcal{F} is then of finite type ((1.3.6) and (1.3.9)); in addition, the question being local, we can restrict to the case where $V = D(f)$, $f \in A$. Since A_f is Noetherian, we see that it suffices to prove that the kernel of a homomorphism $\tilde{A}^n \rightarrow \tilde{M}$, where M is an A -module, is of finite type. But such a homomorphism is of the form \tilde{u} , where u is a homomorphism $A^n \rightarrow M$ (1.3.8), and if $P = \text{Ker } u$ then we have $\tilde{P} = \text{Ker } \tilde{u}$ (1.3.9). Since A is Noetherian, P is of finite type, which finishes the proof. \square

Corollary (1.5.2). — *Under the hypotheses of (1.5.1), the sheaf \mathcal{O}_X is a quasi-coherent sheaf of rings.*

Corollary (1.5.3). — *Under the hypotheses of (1.5.1), every coherent sheaf over an open subset of X is induced by a coherent sheaf on X .*

Corollary (1.5.4). — *Under the hypotheses of (1.5.1), every quasi-coherent \mathcal{O}_X -module \mathcal{F} is the inductive limit of the coherent \mathcal{O}_X -submodules of \mathcal{F} .*

Proof. Indeed, $\mathcal{F} = \tilde{M}$, where M is an A -module, and M is the inductive limit of its submodules of finite type; we conclude the proof by appealing to (1.3.9) and (1.5.1). \square

1.6. Functorial properties of quasi-coherent sheaves over a prime spectrum.

(1.6.1). Let A, A' be rings,

$$\phi : A' \longrightarrow A$$

a homomorphism, and

$${}^a\phi : X = \text{Spec}(A) \longrightarrow X' = \text{Spec}(A')$$

the continuous map associated to ϕ (1.2.1). We will define a canonical homomorphism

$$\tilde{\phi} : \mathcal{O}_{X'} \longrightarrow {}^a\phi_*(\mathcal{O}_X)$$

of sheaves of rings. For each $f' \in A'$, we put $f = \phi(f')$; we have ${}^a\phi^{-1}(D(f')) = D(f)$ (1.2.2.2). The rings $\Gamma(D(f'), \tilde{A}')$ and $\Gamma(D(f), \tilde{A})$ are identified with $A'_{f'}$ and A_f (respectively) ((1.3.6) and (1.3.7)). The homomorphism ϕ canonically defines a homomorphism $\phi_{f'} : A'_{f'} \rightarrow A_f$ (0, 1.5.1), in other words, we have a homomorphism of rings

$$\Gamma(D(f), \tilde{A}) \longrightarrow \Gamma({}^a\phi^{-1}(D(f')), \tilde{A}) = \Gamma(D(f'), {}^a\phi_*(\tilde{A})).$$

In addition, these homomorphisms satisfy the usual compatibility conditions: for $D(f') \supset D(g')$, I | 94 the diagram

$$\begin{array}{ccc} \Gamma(D(f'), \tilde{A}') & \longrightarrow & \Gamma(D(f'), {}^a\phi_*(\tilde{A})) \\ \downarrow & & \downarrow \\ \Gamma(D(g'), \tilde{A}') & \longrightarrow & \Gamma(D(g'), {}^a\phi_*(\tilde{A})) \end{array}$$

is commutative (0, 1.5.1); we have thus defined a homomorphism of $\mathcal{O}_{X'}$ -algebras, as the $D(f')$ form a basis for the topology of X' (0, 3.2.3). The pair $\Phi = ({}^a\phi, \tilde{\phi})$ is thus a *morphism* of ringed spaces

$$\Phi : (X, \mathcal{O}_X) \longrightarrow (X', \mathcal{O}_{X'}),$$

(0, 4.1.1).

We also note that, if we put $x' = {}^a\phi(x)$, then the homomorphism $\tilde{\phi}_x^\#$ (0, 3.7.1) is exactly the homomorphism

$$\phi_x : A'_{x'} \longrightarrow A_x$$

canonically induced by $\phi : A' \rightarrow A$ (0, 1.5.1). Indeed, each $z' \in A'_{x'}$ can be written as g'/f' , where f', g' are in A' and $f' \notin \mathfrak{j}_{x'}$; $D(f')$ is then a neighborhood of x' in X' , and the homomorphism $\Gamma(D(f'), \tilde{A}') \rightarrow \Gamma({}^a\phi^{-1}(D(f')), \tilde{A})$ induced by $\tilde{\phi}$ is exactly $\phi_{f'}$; by considering the section $s' \in \Gamma(D(f'), \tilde{A}')$ corresponding to $g'/f' \in A'_{f'}$, we obtain $\tilde{\phi}_x^\#(z') = \phi(g')/\phi(f')$ in A_x .

Example (1.6.2). — Let S be a multiplicative subset of A , and ϕ the canonical homomorphism $A \rightarrow S^{-1}A$; we have already seen (1.2.6) that ${}^a\phi$ is a *homeomorphism* from $Y = \text{Spec}(S^{-1}A)$ to the subspace of $X = \text{Spec}(A)$ consisting of the x such that $\mathfrak{j}_x \cap S = \emptyset$. In addition, for each x in this subspace, which is thus of the form ${}^a\phi(y)$ with $y \in Y$, the homomorphism $\tilde{\phi}_y^\# : \mathcal{O}_x \rightarrow \mathcal{O}_y$ is *bijective* (0, 1.2.6); in other words, \mathcal{O}_Y is identified with the sheaf on Y induced by \mathcal{O}_X .

Proposition (1.6.3). — *For every A -module M , there exists a canonical functorial isomorphism from the $\mathcal{O}_{X'}$ -module $(M_{[\phi]})^\sim$ to the direct image $\Phi_*(\tilde{M})$.*

Proof. For purposes of abbreviation, we write $M' = M_{[\phi]}$, and for each $f' \in A'$, we put $f = \phi(f')$. The modules of sections $\Gamma(D(f'), \tilde{M}')$ and $\Gamma(D(f), \tilde{M})$ are identified, respectively, with the modules $M'_{f'}$ and M_f (over $A'_{f'}$ and A_f , respectively); in addition, the $A'_{f'}$ -module $(M_f)_{[\phi_{f'}]}$ is canonically isomorphic to $M'_{f'}$ (0, 1.5.2). We thus have a functorial isomorphism of $\Gamma(D(f'), \tilde{A}')$ -modules: $\Gamma(D(f'), \tilde{M}') \simeq \Gamma({}^a\phi^{-1}(D(f')), \tilde{M})_{[\phi_{f'}]}$ and these isomorphisms satisfy the usual compatibility conditions with the restrictions (0, 1.5.6), thus defining the desired functorial isomorphism. We note that, in a precise way, if $u : M_1 \rightarrow M_2$ is a homomorphism of A -modules, it can be considered as a homomorphism $(M_1)_{[\phi]} \rightarrow (M_2)_{[\phi]}$ of A' -modules; if we denote this homomorphism by $u_{[\phi]}$, then $\Phi_*(\tilde{u})$ is identified with $(u_{[\phi]})^\sim$. \square

This proof also shows that, for each A -algebra B , the canonical functorial isomorphism $(B_{[\phi]})^\sim \simeq \Phi_*(\tilde{B})$ is an isomorphism of $\mathcal{O}_{X'}$ -algebras; if M is a B -module, the canonical functorial isomorphism $(M_{[\phi]})^\sim \simeq \Phi_*(\tilde{M})$ is an isomorphism of $\Phi_*(\tilde{B})$ -modules. I | 95

Corollary (1.6.4). — *The direct image functor Φ_* is exact on the category of quasi-coherent \mathcal{O}_X -modules.*

Proof. Indeed, it is clear that $M_{[\phi]}$ is an exact functor in M and \tilde{M}' is an exact functor in M' (1.3.5). \square

Proposition (1.6.5). — *Let N' be an A' -module, and N the A -module $N' \otimes_{A'} A_{[\phi]}$; then there exists a canonical functorial isomorphism from the \mathcal{O}_X -module $\Phi^*(\tilde{N}')$ to \tilde{N} .*

Proof. We first remark that $j : z' \mapsto z' \otimes 1$ is an A' -homomorphism from N' to $N_{[\phi]}$: indeed, by definition, for $f' \in A'$, we have $(f'z') \otimes 1 = z' \otimes \phi(f') = \phi(f')(z' \otimes 1)$. We have (1.3.8) a homomorphism $\tilde{j} : \tilde{N}' \rightarrow (N_{[\phi]})^\sim$ of $\mathcal{O}_{X'}$ -modules, and, thanks to (1.6.3), we can consider \tilde{j} as mapping \tilde{N}' to $\Phi_*(\tilde{N})$. There canonically corresponds to this homomorphism \tilde{j} a homomorphism $h = \tilde{j}^\#$ from $\Phi^*(\tilde{N}')$ to \tilde{N} (0, 4.4.3); we will see that, for each stalk, h_x is *bijective*. Put $x' = {}^a\phi(x)$ and let $f' \in A'$ be such that $x' \in D(f')$; let $f = \phi(f')$. The ring $\Gamma(D(f), \tilde{A})$ is identified with A_f , the modules $\Gamma(D(f), \tilde{N})$ and $\Gamma(D(f'), \tilde{N}')$ with N_f and $N'_{f'}$ (respectively); let $s \in \Gamma(D(f'), \tilde{N}')$, identified with n'/f'^p ($n' \in N'$), and s be its image under \tilde{j} in $\Gamma(D(f), \tilde{N})$; s is identified with $(n' \otimes 1)/f^p$. On the other hand, let $t \in \Gamma(D(f), \tilde{A})$, identified with g/f^q ($g \in A$); then, by definition, we have $h_x(s'_x \otimes t_x) = t_x \cdot s_x$ (0, 4.4.3). But we can canonically identify N_f with $N'_{f'} \otimes_{A'_{f'}} (A_f)_{[\phi_{f'}]}$ (0, 1.5.4);

s then corresponds to the element $(n'/f'^p) \otimes 1$, and the section $y \mapsto t_y \cdot s_y$ with $(n'/f'^p) \otimes (g/f^q)$. The compatibility diagram of (0, 1.5.6) shows that h_x is exactly the canonical isomorphism

$$(1.6.5.1) \quad N'_{x'} \otimes_{A'_{x'}} (A_x)_{[\phi_{x'}]} \simeq N_x = (N' \otimes_{A'} A_{[\phi]})_x.$$

In addition, let $v : N'_1 \rightarrow N'_2$ be a homomorphism of A' -modules; since $\tilde{v}_{x'} = v_{x'}$ for each $x' \in X'$, it follows immediately from the above that $\Phi^*(\tilde{v})$ is canonically identified with $(v \otimes 1)^\sim$, which finishes the proof of (1.6.5). \square

If B' is an A' -algebra, the canonical isomorphism from $\Phi^*(\tilde{B}')$ to $(B' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of \mathcal{O}_X -algebras; if, in addition, N' is a B' -module, then the canonical isomorphism from $\Phi^*(\tilde{N}')$ to $(N' \otimes_{A'} A_{[\phi]})^\sim$ is an isomorphism of $\Phi^*(\tilde{B}')$ -modules.

Corollary (1.6.6). — *The sections of $\Phi^*(\tilde{N}')$, the canonical images of the sections s' , where s' varies over the A' -module $\Gamma(\tilde{N}')$, generate the A -module $\Gamma(\Phi^*(N'))$.*

Proof. Indeed, these images are identified with the elements $z' \otimes 1$ of N , when we identify N' and N with $\Gamma(\tilde{N}')$ and $\Gamma(\tilde{N})$ (respectively) (1.3.7), and z' varies over N' . \square

(1.6.7). In the proof of (1.6.5), we had proved in passing that the canonical map (0, 4.4.3.2) $\rho : \tilde{N}' \rightarrow \Phi_*(\Phi^*(\tilde{N}'))$ is exactly the homomorphism \tilde{j} , where $j : N' \rightarrow N' \otimes_{A'} A_{[\phi]}$ is the homomorphism $z' \mapsto z' \otimes 1$. Similarly, the canonical map (0, 4.4.3.3) $\sigma : \Phi^*(\Phi_*(\tilde{M})) \rightarrow \tilde{M}$ is exactly \tilde{p} , where $p : M_{[\phi]} \otimes_{A'} A_{[\phi]} \rightarrow M$ is the canonical homomorphism, which sends each tensor product $z \otimes a$ ($z \in M, a \in A$) to $a \cdot z$; this follows immediately from the definitions ((0, 3.7.1), (0, 4.4.3), and (1.3.7)).

We conclude ((0, 4.4.3) and (0, 3.5.4.4)) that if $v : N' \rightarrow M_{[\phi]}$ is an A' -homomorphism, then $\tilde{v}^\# = (v \otimes 1)^\sim$.

(1.6.8). Let N'_1 and N'_2 be A' -modules, and assume N'_1 admits a *finite presentation*; it then follows from (1.6.7) and (1.3.12, ii) that the canonical homomorphism (0, 4.4.6)

$$\Phi^*(\mathcal{H}om_{\tilde{A}'}(\tilde{N}'_1, \tilde{N}'_2)) \longrightarrow \mathcal{H}om_{\tilde{A}}(\Phi^*(\tilde{N}'_1), \Phi^*(\tilde{N}'_2))$$

is exactly $\tilde{\gamma}$, where γ denotes the canonical homomorphism of A -modules $\text{Hom}_{A'}(N'_1, N'_2) \otimes_{A'} A \rightarrow \text{Hom}_A(N'_1 \otimes_{A'} A, N'_2 \otimes_{A'} A)$.

(1.6.9). Let \mathfrak{J}' be an ideal of A' , and M an A -module; since, by definition, $\tilde{\mathfrak{J}}'\tilde{M}$ is the image of the canonical homomorphism $\Phi^*(\tilde{\mathfrak{J}}') \otimes_{\tilde{A}} \tilde{M} \rightarrow \tilde{M}$, it follows from Proposition (1.6.5) and Corollary (1.3.12, i) that $\tilde{\mathfrak{J}}'\tilde{M}$ canonically identifies with $(\mathfrak{J}'M)^\sim$; in particular, $\Phi^*(\tilde{\mathfrak{J}}')\tilde{A}$ is identified with $(\mathfrak{J}'A)^\sim$, and, taking the right exactness of the functor Φ^* into account, the \tilde{A} -algebra $\Phi^*((A'/\mathfrak{J}')^\sim)$ is identified with $(A/\mathfrak{J}'A)^\sim$.

(1.6.10). Let A'' be a third ring, ϕ' a homomorphism $A'' \rightarrow A'$, and write $\phi'' = \phi \circ \phi'$. It follows immediately from the definitions that ${}^a\phi'' = ({}^a\phi') \circ ({}^a\phi)$, and $\tilde{\phi}'' = \tilde{\phi} \circ \tilde{\phi}'$ (0, 1.5.7). We conclude that $\Phi'' = \Phi' \circ \Phi$; in other words, $(\text{Spec}(A), \tilde{A})$ is a *functor* from the category of rings to that of ringed spaces.

1.7. Characterization of morphisms of affine schemes.

Definition (1.7.1). — We say that a ringed space (X, \mathcal{O}_X) is an *affine scheme* if it is isomorphic to a ringed space of the form $(\text{Spec}(A), \tilde{A})$, where A is a ring; we then say that $\Gamma(X, \mathcal{O}_X)$, which is canonically identified with the ring A (1.3.7), is the ring of the affine scheme (X, \mathcal{O}_X) , and we denote it by $A(X)$ when there is no chance of confusion.

By abuse of language, when we speak of an *affine scheme* $\text{Spec}(A)$; it will always be the ringed space $(\text{Spec}(A), \tilde{A})$.

(1.7.2). Let A and B be rings, and (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) the affine schemes corresponding to the prime spectra $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$. We have seen (1.6.1) that each ring homomorphism $\phi : B \rightarrow A$ corresponds to a morphism $\Phi = ({}^a\phi, \tilde{\phi}) = \text{Spec}(\phi) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$. We note that ϕ is entirely determined by Φ , since we have, by definition, $\phi = \Gamma(\tilde{\phi}) : \Gamma(\tilde{B}) \rightarrow \Gamma({}^a\phi_*(\tilde{A})) = \Gamma(\tilde{A})$.

Theorem (1.7.3). — ² Let $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ be affine schemes. For a morphism of ringed spaces $(\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ to be of the form $(^a\phi, \tilde{\phi})$, where ϕ is a homomorphism of rings $A(Y) \rightarrow A(X)$, it is necessary and sufficient that, for each $x \in X$, $\theta_x^\#$ is a local homomorphism: $\mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$.

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Proof. Let $A = A(X)$, $B = A(Y)$. The condition is necessary, since we saw (1.6.1) that $\tilde{\phi}_x^\#$ is the homomorphism from $B_{\phi(x)}$ to A_x canonically induced by ϕ , and, by definition, of $^a\phi(x) = \phi^{-1}(j_x)$, this homomorphism is local.

We now prove that the condition is sufficient. By definition, θ is a homomorphism $\mathcal{O}_Y \rightarrow \psi_*(\mathcal{O}_X)$, and we canonically obtain a ring homomorphism

$$\phi = \Gamma(\theta) : B = \Gamma(Y, \mathcal{O}_Y) \longrightarrow \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X) = A.$$

The hypotheses on $\theta_x^\#$ mean that this homomorphism induces, by passing to quotients, a monomorphism θ^x from the residue field $k(\psi(x))$ to the residue field $k(x)$, such that, for each section $f \in \Gamma(Y, \mathcal{O}_Y) = B$, we have $\theta^x(f(\psi(x))) = \phi(f)(x)$. The relation $f(\psi(x)) = 0$ is therefore equivalent to $\phi(f)(x) = 0$, which means that $j_{\psi(x)} = j_{^a\phi(x)}$, and we now write $\psi(x) = ^a\phi(x)$ for each $x \in X$, or $\psi = ^a\phi$. We also know that the diagram

$$\begin{array}{ccc} B = \Gamma(Y, \mathcal{O}_Y) & \xrightarrow{\phi} & \Gamma(X, \mathcal{O}_X) = A \\ \downarrow & & \downarrow \\ B_{\psi(x)} & \xrightarrow{\theta_x^\#} & A_x \end{array}$$

is commutative (0, 3.7.2), which means that $\theta_x^\#$ is equal to the homomorphism $\phi_x : B_{\psi(x)} \rightarrow A_x$ canonically induced by ϕ (0, 1.5.1). As the data of the $\theta_x^\#$ completely characterize $\theta^\#$, and as a result θ (0, 3.7.1), we conclude that we have $\theta = \tilde{\phi}$, by the definition of $\tilde{\phi}$ (1.6.1). \square

We say that a morphism (ψ, θ) of ringed spaces satisfying the condition of (1.7.3) is a *morphism of affine schemes*.

Corollary (1.7.4). — If (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are affine schemes, there exists a canonical isomorphism from the set of morphisms of affine schemes $\text{Hom}((X, \mathcal{O}_X), (Y, \mathcal{O}_Y))$ to the set of ring homomorphisms from B to A , where $A = \Gamma(\mathcal{O}_X)$ and $B = \Gamma(\mathcal{O}_Y)$.

Furthermore, we can say that the functors $(\text{Spec}(A), \tilde{A})$ in A and $\Gamma(X, \mathcal{O}_X)$ in (X, \mathcal{O}_X) define an equivalence between the category of commutative rings and the opposite category of affine schemes (T, I, 1.2).

Corollary (1.7.5). — If $\phi : B \rightarrow A$ is surjective, then the corresponding morphism $(^a\phi, \tilde{\phi})$ is a monomorphism of ringed spaces (cf. (4.1.7)).

Proof. Indeed, we know that $^a\phi$ is injective (1.2.5), and, since ϕ is surjective, for each $x \in X$, $\phi_x^\# : B_{\phi(x)} \rightarrow A_x$, which is induced by ϕ by passing to rings of fractions, is also surjective (0, 1.5.1); hence the conclusion (0, 4.1.1). \square

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1.8. Morphisms from locally ringed spaces to affine schemes. Due to a remark by J. Tate, the statements of Theorem (1.7.3) and Proposition (2.2.4) can be generalized as follows:³

Proposition (1.8.1). — Let (S, \mathcal{O}_S) be an affine scheme, and (X, \mathcal{O}_X) a locally ringed space. Then there is a canonical bijection from the set of ring homomorphisms $\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(X, \mathcal{O}_X)$ to the set of morphisms of ringed spaces $(\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$ such that, for each $x \in X$, $\theta_x^\#$ is a local homomorphism $\mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$.

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²[Trans.] See (1.8) and the footnote there.

³[Trans.] The following section (I.1.8) was added in the errata of EGA II, hence the temporary change in page numbers, which refer to EGA II.

Proof. We note first that if (X, \mathcal{O}_X) and (S, \mathcal{O}_S) are any two ringed spaces, then a morphism (ψ, θ) from (X, \mathcal{O}_X) to (S, \mathcal{O}_S) canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(X, \mathcal{O}_X)$, hence a first map

$$(1.8.1.1) \quad \rho : \text{Hom}((X, \mathcal{O}_X), (S, \mathcal{O}_S)) \longrightarrow \text{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)).$$

Conversely, under the stated hypotheses, we set $A = \Gamma(S, \mathcal{O}_S)$, and consider a ring homomorphism $\phi : A \rightarrow \Gamma(X, \mathcal{O}_X)$. For each $x \in X$, it is clear that the set of the $f \in A$ such that $\phi(f)(x) = 0$ is a *prime ideal* of A , since $\mathcal{O}_x/\mathfrak{m}_x = k(x)$ is a field; it is therefore an element of $S = \text{Spec}(A)$, which we denote by ${}^a\phi(x)$. In addition, for each $f \in A$, we have, by definition (0, 5.5.2), that ${}^a\phi^{-1}(D(f)) = X_f$, which proves that ${}^a\phi$ is a *continuous map* $X \rightarrow S$. We then define a homomorphism

$$\tilde{\phi} : \mathcal{O}_S \longrightarrow {}^a\phi_*(\mathcal{O}_X)$$

of \mathcal{O}_S -modules; for each $f \in A$, we have $\Gamma(D(f), \mathcal{O}_S) = A_f$ (1.3.6); for each $s \in A$, we associate to $s/f \in A_f$ the element $(\phi(s)|X_f)(\phi(f)|X_f)^{-1}$ of $\Gamma(X_f, \mathcal{O}_X) = \Gamma(D(f), {}^a\phi_*(\mathcal{O}_X))$, and we immediately see (by passing from $D(f)$ to $D(fg)$) that this is a well-defined homomorphism of \mathcal{O}_S -modules, hence a morphism $({}^a\phi, \tilde{\phi})$ of ringed spaces. In addition, with the same notation, and setting $y = {}^a\phi(x)$ for brevity, we immediately see (0, 3.7.1) that we have $\tilde{\phi}_x^\#(s_y/f_y) = (\phi(s)_x)(\phi(f)_x)^{-1}$; since the relation $s_y \in \mathfrak{m}_y$ is, by definition, equivalent to $\phi(s)_x \in \mathfrak{m}_x$, we see that $\tilde{\phi}_x^\#$ is a *local homomorphism* $\mathcal{O}_y \rightarrow \mathcal{O}_x$, and we have thus defined a second map $\sigma : \text{Hom}(\Gamma(S, \mathcal{O}_S), \Gamma(X, \mathcal{O}_X)) \rightarrow \mathfrak{L}$, where \mathfrak{L} is the set of the morphisms $(\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$ such that $\theta_x^\#$ is local for each $x \in X$. It remains to prove that σ and ρ (restricted to \mathfrak{L}) are inverses of each other; the definition of $\tilde{\phi}$ immediately shows that $\Gamma(\tilde{\phi}) = \phi$, and, as a result, that $\rho \circ \sigma$ is the identity. To see that $\sigma \circ \rho$ is the identity, start with a morphism $(\psi, \theta) \in \mathfrak{L}$ and let $\phi = \Gamma(\theta)$; the hypotheses on $\theta_x^\#$ mean that this morphism induces, by passing to quotients, a monomorphism $\theta^x : k(\psi(x)) \rightarrow k(x)$ such that for each section $f \in A = \Gamma(S, \mathcal{O}_S)$, we have $\theta^x(f(\psi(x))) = \phi(f)(x)$; the equation $f(\phi(x)) = 0$ is therefore equivalent to $\phi(f)(x) = 0$, which proves that ${}^a\phi = \psi$. On the other hand, the definitions imply that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \Gamma(X, \mathcal{O}_X) \\ \downarrow & & \downarrow \\ A_{\psi(x)} & \xrightarrow{\theta_x^\#} & \mathcal{O}_x \end{array}$$

is commutative, and it is the same for the analogous diagram where $\theta_x^\#$ is replaced by $\tilde{\phi}_x^\#$, hence $\tilde{\phi}_x^\# = \theta_x^\#$ (0, 1.2.4), and, as a result, $\tilde{\phi} = \theta$. \square

(1.8.2). When (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are *locally ringed spaces*, we will consider the morphisms $(\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ such that, for each $x \in X$, $\theta_x^\#$ is a *local homomorphism* $\mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$. Henceforth when we speak of a *morphism of locally ringed spaces*, it will always be a morphism like the above; with this definition of morphisms, it is clear that the locally ringed spaces form a *category*; for any two objects X and Y of this category, $\text{Hom}(X, Y)$ thus denotes the set of morphisms of locally ringed spaces from X to Y (the set denoted \mathfrak{L} in (1.8.1)); when we consider the set of *morphisms of ringed spaces* from X to Y , we will denote it by $\text{Hom}_{\text{rs}}(X, Y)$ to avoid any confusion. The map (1.8.1.1) is then written as

$$(1.8.2.1) \quad \rho : \text{Hom}_{\text{rs}}(X, Y) \longrightarrow \text{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

and its restriction

$$(1.8.2.2) \quad \rho' : \text{Hom}(X, Y) \longrightarrow \text{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X))$$

is a *functorial map* in X and Y on the category of locally ringed spaces.

Corollary (1.8.3). — *Let (Y, \mathcal{O}_Y) be a locally ringed space. For Y to be an affine scheme, it is necessary and sufficient that, for each locally ringed space (X, \mathcal{O}_X) , the map (1.8.2.2) be bijective.*

Proof. Proposition (1.8.1) shows that the condition is necessary. Conversely, if we suppose that the condition is satisfied, and if we put $A = \Gamma(Y, \mathcal{O}_Y)$, then it follows from the hypotheses and from (1.8.1) that the functors $X \mapsto \text{Hom}(X, Y)$ and $X \mapsto \text{Hom}(X, \text{Spec}(A))$, from the category of locally

ringed spaces to that of sets, are *isomorphic*. We know that this implies the existence of a canonical isomorphism $X \rightarrow \text{Spec}(A)$ (cf. 0, 8). \square

(1.8.4). Let $S = \text{Spec}(A)$ be an affine scheme; denote by (S', A') the ringed space whose underlying space is a point and the structure sheaf A' is the (necessarily simple) sheaf on S' defined by the ring A . Let $\pi : S \rightarrow S'$ be the unique map from S to S' ; on the other hand, we note that, for each open subset U of S , we have a canonical map $\Gamma(S', A') = \Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(U, \mathcal{O}_S)$ which thus defines a π -morphism $\iota : A' \rightarrow \mathcal{O}_S$ of sheaves of rings. We have thus canonically defined a *morphism of ringed spaces* $i = (\pi, \iota) : (S, \mathcal{O}_S) \rightarrow (S', A')$. For each A -module M , we denote by M' the simple sheaf on S' defined by M , which is evidently an A' -module. It is clear that $i_*(\tilde{M}) = M'$ (1.3.7).

Lemma (1.8.5). — *With the notation of (1.8.4), for each A -module M , the canonical functorial \mathcal{O}_S -homomorphism (0, 4.3.3)*

$$(1.8.5.1) \quad i^*(i_*(\tilde{M})) \longrightarrow \tilde{M}$$

is an isomorphism.

Proof. Indeed, the two parts of (1.8.5.1) are right exact (the functor $M \mapsto i_*(\tilde{M})$ evidently being exact) and commute with direct sums; by considering M as the cokernel of a homomorphism $A^{(I)} \rightarrow A^{(J)}$, we can reduce to proving the lemma for the case where $M = A$, and it is evident in this case. \square

Corollary (1.8.6). — *Let (X, \mathcal{O}_X) be a ringed space, and $u : X \rightarrow S$ a morphism of ringed spaces. For each A -module M , we have (with the notation of (1.8.4)) a canonical functorial isomorphism of \mathcal{O}_X -modules* II | 220

$$(1.8.6.1) \quad u^*(\tilde{M}) \simeq u^*(i^*(M')).$$

Corollary (1.8.7). — *Under the hypotheses of (1.8.6), we have, for each A -module M and each \mathcal{O}_X -module \mathcal{F} , a canonical isomorphism, functorial in M and \mathcal{F} ,*

$$(1.8.7.1) \quad \text{Hom}_{\mathcal{O}_S}(\tilde{M}, u_*(\mathcal{F})) \simeq \text{Hom}_A(M, \Gamma(X, \mathcal{F})).$$

Proof. We have, according to (0, 4.4.3) and Lemma (1.8.5), a canonical isomorphism of bifunctors

$$\text{Hom}_{\mathcal{O}_S}(\tilde{M}, u_*(\mathcal{F})) \simeq \text{Hom}_{A'}(M', i_*(u_*(\mathcal{F})))$$

and it is clear that the right hand side is exactly $\text{Hom}_A(M, \Gamma(X, \mathcal{F}))$. We note that the canonical homomorphism (1.8.7.1) sends each \mathcal{O}_S -homomorphism $h : \tilde{M} \rightarrow u_*(\mathcal{F})$ (in other words, each u -morphism $\tilde{M} \rightarrow \mathcal{F}$) to the A -homomorphism $\Gamma(h) : M \rightarrow \Gamma(S, u_*(\mathcal{F})) = \Gamma(X, \mathcal{F})$. \square

(1.8.8). With the notation of (1.8.4), it is clear (0, 4.1.1) that each morphism of ringed spaces $(\psi, \theta) : X \rightarrow S'$ is equivalent to the data of a ring homomorphism $A \rightarrow \Gamma(X, \mathcal{O}_X)$. We can thus interpret Proposition (1.8.1) as defining a canonical bijection $\text{Hom}(X, S) \simeq \text{Hom}(X, S')$ (where we understand that the right-hand side is the collection of morphisms of ringed spaces, since in general A is not a local ring). More generally, if X and Y are locally ringed spaces, and if (Y', A') is the ringed space whose underlying space is a point and whose sheaf of rings A' is the simple sheaf defined by the ring $\Gamma(Y, \mathcal{O}_Y)$, we can interpret (1.8.2.1) as a map

$$(1.8.8.1) \quad \rho : \text{Hom}_{\text{rs}}(X, Y) \longrightarrow \text{Hom}(X, Y').$$

The result of Corollary (1.8.3) is interpreted by saying that affine schemes are characterized among locally ringed spaces as those for which the restriction of ρ to $\text{Hom}(X, Y)$:

$$(1.8.8.2) \quad \rho' : \text{Hom}(X, Y) \longrightarrow \text{Hom}(X, Y')$$

is *bijective* for every locally ringed space X . In the following chapter, we generalize this definition, which allows us to associate to *any* ringed space Z (and not only to a ringed space whose underlying space is a point) a locally ringed space which we will call $\text{Spec}(Z)$; this will be the starting point for a “relative” theory of preschemes over any ringed space, extending the results of Chapter I.

(1.8.9). We can consider the pairs (X, \mathcal{F}) consisting of a locally ringed space X and an \mathcal{O}_X -module \mathcal{F} as forming a category, a *morphism* in this category being a pair (u, h) consisting of a morphism of locally ringed spaces $u : X \rightarrow Y$ and a u -morphism $h : \mathcal{G} \rightarrow \mathcal{F}$ of modules; these morphisms (for II | 221

(X, \mathcal{F}) and (Y, \mathcal{G}) fixed) form a set which we denote by $\text{Hom}((X, \mathcal{F}), (Y, \mathcal{G}))$; the map $(u, h) \mapsto (\rho'(u), \Gamma(h))$ is a canonical map

$$(1.8.9.1) \quad \text{Hom}((X, \mathcal{F}), (Y, \mathcal{G})) \longrightarrow \text{Hom}((\Gamma(Y, \mathcal{O}_Y), \Gamma(Y, \mathcal{G})), (\Gamma(X, \mathcal{O}_X), \Gamma(X, \mathcal{F})))$$

functorial in (X, \mathcal{F}) and (Y, \mathcal{G}) , the right-hand side being the set of di-homomorphisms corresponding to the rings and modules considered (0, 1.0.2).

Corollary (1.8.10). — *Let Y be a locally ringed space, and \mathcal{G} an \mathcal{O}_Y -module. For Y to be an affine scheme and \mathcal{G} to be a quasi-coherent \mathcal{O}_Y -module, it is necessary and sufficient that, for each pair (X, \mathcal{F}) consisting of a locally ringed space X and an \mathcal{O}_X -module \mathcal{F} , the canonical map (1.8.9.1) be bijective.*

We leave the proof, which is modelled on that of (1.8.3), using (1.8.1) and (1.8.7), to the reader.

Remark (1.8.11). — The statements (1.7.3), (1.7.4), and (2.2.4) are particular cases of (1.8.1), as well as the definition in (1.6.1); similarly, (2.2.5) follows from (1.8.7). Corollary (1.8.7) also implies (1.6.3) (and, as a result, (1.6.4)) as a particular case, since if X is an affine scheme and $\Gamma(X, \mathcal{F}) = N$, then the functors $M \mapsto \text{Hom}_{\mathcal{O}_S}(\tilde{M}, u_*(\tilde{N}))$ and $M \mapsto \text{Hom}_{\mathcal{O}_S}(\tilde{M}, (N_{[\phi]})^\sim)$ (where $\phi : A \rightarrow \Gamma(X, \mathcal{O}_X)$ corresponds to u) are isomorphic, by Corollaries (1.8.7) and (1.3.8). Finally, (1.6.5) (and, as a result, (1.6.6)) follow from (1.8.6), and the fact that, for each $f \in A$, the A_f -modules $N' \otimes_{A'} A_f$ and $(N' \otimes_{A'} A)_f$ (with the notation of (1.6.5)) are canonically isomorphic.

§2. PRESCHMES AND MORPHISMS OF PRESCHMES

2.1. Definition of preschemes.

(2.1.1). Given a ringed space (X, \mathcal{O}_X) , we say that an open subset V of X is an *affine open subset* if the ringed space $(V, \mathcal{O}_X|_V)$ is an affine scheme (1.7.1).

Definition (2.1.2). — We define a *prescheme* to be a ringed space (X, \mathcal{O}_X) such that every point of X admits an affine open neighborhood.

Proposition (2.1.3). — *If (X, \mathcal{O}_X) is a prescheme, then its affine open subsets form a basis for the topology of X .*

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Proof. If V is an arbitrary open neighborhood of $x \in X$, then there exists by hypothesis an open neighborhood W of x such that $(W, \mathcal{O}_X|_W)$ is an affine scheme; we write A to mean its ring. In the space W , $V \cap W$ is an open neighborhood of x ; so there exists some $f \in A$ such that $D(f)$ is an open neighborhood of x contained inside $V \cap W$ (1.1.10, i). The ringed space $(D(f), \mathcal{O}_X|_{D(f)})$ is thus an affine scheme, isomorphic to A_f (1.3.6), whence the proposition. \square

Proposition (2.1.4). — *The underlying space of a prescheme is a Kolmogoroff space.*

Proof. If x and y are two distinct points of a prescheme X , then it is clear that there exists an open neighborhood of one of these points that does not contain the other if x and y are not in the same affine open subset; and if they are in the same affine open subset, this is a result of (1.1.8). \square

Proposition (2.1.5). — *If (X, \mathcal{O}_X) is a prescheme, then every closed irreducible subset of X admits exactly one generic point, and the map $x \mapsto \overline{\{x\}}$ is thus a bijection of X onto its set of closed irreducible subsets.*

Proof. If Y is a closed irreducible subset of X and $y \in Y$, and if U is an affine open neighborhood of y in X , then $U \cap Y$ is dense in Y , and also irreducible ((0, 2.1.1) and (0, 2.1.4)); thus, by Corollary (1.1.14), $U \cap Y$ is the closure in U of a point x , and so $Y \subset \overline{U}$ is the closure of x in X . The uniqueness of the generic point of X is a result of Proposition (2.1.4) and of (0, 2.1.3). \square

(2.1.6). If Y is a closed irreducible subset of X , and y its generic point, then the local ring \mathcal{O}_y (also written $\mathcal{O}_{X/Y}$) is called the *local ring of X along Y* , or the *local ring of Y in X* .

If X itself is irreducible and x its generic point then we say that \mathcal{O}_x is the *ring of rational functions on X* (cf. §7).

Proposition (2.1.7). — *If (X, \mathcal{O}_X) is a prescheme, then the ringed space $(U, \mathcal{O}_X|_U)$ is a prescheme for every open subset U .*

Proof. This follows directly from Definition (2.1.2) and Proposition (2.1.3). \square

We say that $(U, \mathcal{O}_X|_U)$ is the prescheme *induced* on U by (X, \mathcal{O}_X) , or the *restriction* of (X, \mathcal{O}_X) to U .

(2.1.8). We say that a prescheme (X, \mathcal{O}_X) is *irreducible* (resp. *connected*) if the underlying space X is irreducible (resp. connected). We say that a prescheme is *integral* if it is *irreducible and reduced* (cf. (5.1.4)). We say that a prescheme (X, \mathcal{O}_X) is *locally integral* if every $x \in X$ admits an open neighborhood U such that the prescheme induced on U by (X, \mathcal{O}_X) is integral.

2.2. Morphisms of preschemes.

Definition (2.2.1). — Given two preschemes, (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) , we define a morphism (of preschemes) from (X, \mathcal{O}_X) to (Y, \mathcal{O}_Y) to be a morphism of ringed spaces (ψ, θ) such that, for all $x \in X$, θ_x^\sharp is a local homomorphism $\mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$.

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By passing to quotients, the map $\mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$ gives us a monomorphism $\theta^x : k(\psi(x)) \rightarrow k(x)$, which lets us consider $k(x)$ as an *extension* of the field $k(\psi(x))$.

(2.2.2). The composition (ψ'', θ'') of two morphisms (ψ, θ) , (ψ', θ') of preschemes is also a morphism of preschemes, since it is given by the formula $\theta''^\sharp = \theta^\sharp \circ \psi'^*(\theta'^\sharp)$ (0, 3.5.5). From this we conclude that preschemes form a *category*; using the usual notation, we will write $\text{Hom}(X, Y)$ to mean the set of morphisms from a prescheme X to a prescheme Y .

Example (2.2.3). — If U is an open subset of X , then the canonical injection (0, 4.1.2) of the induced prescheme $(U, \mathcal{O}_X|_U)$ into (X, \mathcal{O}_X) is a morphism of preschemes; it is further a *monomorphism* of ringed spaces (and *a fortiori* a monomorphism of preschemes), which follows rapidly from (0, 4.1.1).

Proposition (2.2.4). — ⁴ Let (X, \mathcal{O}_X) be a prescheme, and (S, \mathcal{O}_S) an affine scheme associated to a ring A . Then there exists a canonical bijective correspondence between morphisms of preschemes from (X, \mathcal{O}_X) to (S, \mathcal{O}_S) and ring homomorphisms from A to $\Gamma(X, \mathcal{O}_X)$.

Proof. First note that, if (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) are two arbitrary ringed spaces, a morphism (ψ, θ) from (X, \mathcal{O}_X) to (Y, \mathcal{O}_Y) canonically defines a ring homomorphism $\Gamma(\theta) : \Gamma(Y, \mathcal{O}_Y) \rightarrow \Gamma(Y, \psi_*(\mathcal{O}_X)) = \Gamma(X, \mathcal{O}_X)$. In the case that we consider, everything boils down to showing that any homomorphism $\phi : A \rightarrow \Gamma(X, \mathcal{O}_X)$ is of the form $\Gamma(\theta)$ for exactly one θ . Now, by hypothesis, there is a covering (V_α) of X by affine open subsets; by composing ϕ with the restriction homomorphism $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(V_\alpha, \mathcal{O}_X|_{V_\alpha})$, we obtain a homomorphism $\phi_\alpha : A \rightarrow \Gamma(V_\alpha, \mathcal{O}_X|_{V_\alpha})$ that corresponds to a unique morphism $(\psi_\alpha, \theta_\alpha)$ from the prescheme $(V_\alpha, \mathcal{O}_X|_{V_\alpha})$ to (S, \mathcal{O}_S) , by Theorem (1.7.3). Furthermore, for each pair of indices (α, β) , each point of $V_\alpha \cap V_\beta$ admits an affine open neighborhood W contained inside $V_\alpha \cap V_\beta$ (2.1.3); it is clear that, by composing ϕ_α and ϕ_β with the restriction homomorphisms to W , we obtain the same homomorphism $\Gamma(S, \mathcal{O}_S) \rightarrow \Gamma(W, \mathcal{O}_X|_W)$, so, with the equation $(\theta_\alpha^\sharp)_x = (\phi_\alpha)_x$ for all $x \in V_\alpha$ and all α (1.6.1), the restriction to W of the morphisms $(\psi_\alpha, \theta_\alpha)$ and $(\psi_\beta, \theta_\beta)$ coincide. From this we conclude that there is a morphism $(\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$ of ringed spaces, and only one such that its restriction to each V_α is $(\psi_\alpha, \theta_\alpha)$, and it is clear that this morphism is a morphism of preschemes and such that $\Gamma(\theta) = \phi$.

Let $u : A \rightarrow \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism, and $v = (\psi, \theta)$ the corresponding morphism $(X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$. For each $f \in A$, we have that

$$(2.2.4.1) \quad \psi^{-1}(D(f)) = X_{u(f)}$$

with the notation of (0, 5.5.2) relative to the locally free sheaf \mathcal{O}_X . In fact, it suffices to verify this formula when X itself is affine, and then this is nothing but (1.2.2.2). \square

Proposition (2.2.5). — Under the hypotheses of Proposition (2.2.4), let $\phi : A \rightarrow \Gamma(X, \mathcal{O}_X)$ be a ring homomorphism, $f : (X, \mathcal{O}_X) \rightarrow (S, \mathcal{O}_S)$ the corresponding morphism of preschemes, \mathcal{G} (resp. \mathcal{F}) an \mathcal{O}_X -module (resp. \mathcal{O}_S -module), and $M = \Gamma(S, \mathcal{F})$. Then there exists a canonical bijective correspondence between f -morphisms $\mathcal{F} \rightarrow \mathcal{G}$ (0, 4.4.1) and A -homomorphisms $M \rightarrow (\Gamma(X, \mathcal{G}))_{[\phi]}$.

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Proof. Reasoning as in Proposition (2.2.4), we reduce to the case where X is affine, and the proposition then follows from Proposition (1.6.3) and from Corollary (1.3.8). \square

⁴[Trans.] See (1.8) and the footnote there.

(2.2.6). We say that a morphism of preschemes $(\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ is *open* (resp. *closed*) if, for all open subsets U of X (resp. all closed subsets F of X), $\psi(U)$ is open (resp. $\psi(F)$ is closed) in Y . We say that (ψ, θ) is *dominant* if $\psi(X)$ is dense in Y , and *surjective* if ψ is surjective. We note that these conditions rely only on the continuous map ψ .

Proposition (2.2.7). — *Let*

$$f = (\psi, \theta) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

and

$$g = (\psi', \theta') : (Y, \mathcal{O}_Y) \longrightarrow (Z, \mathcal{O}_Z)$$

be morphisms of preschemes.

- (i) *If f and g are both open (resp. closed, dominant, surjective), then so is $g \circ f$.*
- (ii) *If f is surjective and $g \circ f$ closed, then g is closed.*
- (iii) *If $g \circ f$ is surjective, then g is surjective.*

Proof. Claims (i) and (iii) are evident. Write $g \circ f = (\psi'', \theta'')$. If F is closed in Y then $\psi^{-1}(F)$ is closed in X , so $\psi''(\psi^{-1}(F))$ is closed in Z ; but since ψ is surjective, $\psi(\psi^{-1}(F)) = F$, so $\psi''(\psi^{-1}(F)) = \psi'(F)$, which proves (ii). \square

Proposition (2.2.8). — *Let $f = (\psi, \theta)$ be a morphism $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$, and (U_α) an open cover of Y . For f to be open (resp. closed, surjective, dominant), it is necessary and sufficient for its restriction to each induced prescheme $(\psi^{-1}(U_\alpha), \mathcal{O}_X|_{\psi^{-1}(U_\alpha)})$, considered as a morphism of preschemes from this induced prescheme to the induced prescheme $(U_\alpha, \mathcal{O}_Y|_{U_\alpha})$ to be open (resp. closed, surjective, dominant).*

Proof. The proposition follows immediately from the definitions, taking into account the fact that a subset F of Y is closed (resp. open, dense) in Y if and only if each of the $F \cap U_\alpha$ are closed (resp. open, dense) in U_α . \square

(2.2.9). Let (X, \mathcal{O}_X) and (Y, \mathcal{O}_Y) be two preschemes; suppose that X (resp. Y) has a finite number of irreducible components X_i (resp. Y_i) ($1 \leq i \leq n$); let ξ_i (resp. η_i) be the generic point of X_i (resp. Y_i) (2.1.5). We say that a morphism

$$f = (\psi, \theta) : (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$$

is *birational* if, for all i , $\psi^{-1}(\eta_i) = \{\xi_i\}$ and $\theta_{\xi_i}^\# : \mathcal{O}_{\eta_i} \rightarrow \mathcal{O}_{\xi_i}$ is an isomorphism. It is clear that a birational morphism is dominant (0, 2.1.8), and thus it is surjective if it is also closed.

Notation (2.2.10). — In all that follows, when there is no risk of confusion, we *suppress* the structure sheaf (resp. the morphism of structure sheaves) from the notation of a prescheme (resp. morphism of preschemes). If U is an open subset of the underlying space X of a prescheme, then whenever we speak of U as a prescheme we always mean the induced prescheme on U .

2.3. Gluing preschemes.

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(2.3.1). It follows from Definition (2.1.2) that every ringed space obtained by *gluing* preschemes (0, 4.1.7) is again a prescheme. In particular, although every prescheme admits, by definition, a cover by affine open sets, we see that every prescheme can actually be obtained by *gluing affine schemes*.

Example (2.3.2). — Let K be a field, $B = K[s]$ and $C = K[t]$ polynomial rings in one indeterminate over K , and define $X_1 = \text{Spec}(B)$ and $X_2 = \text{Spec}(C)$, which are isomorphic affine schemes. In X_1 (resp. X_2), let U_{12} (resp. U_{21}) be the affine open $D(s)$ (resp. $D(t)$) where the ring B_s (resp. C_t) is formed of rational fractions of the form $f(s)/s^m$ (resp. $g(t)/t^n$) with $f \in B$ (resp. $g \in C$). Let u_{12} be the isomorphism of preschemes $U_{21} \rightarrow U_{12}$ corresponding (2.2.4) to the isomorphism from B_s to C_t that, to $f(s)/s^m$, associates the rational fraction $f(1/t)/(1/t^m)$. We can glue X_1 and X_2 along U_{12} and U_{21} by using u_{12} , because there is clearly no gluing condition. We later show that the prescheme X obtained in this manner is a particular case of a general method of construction (II, 2.4.3). Here we show only that X is *not an affine scheme*; this will follow from the fact that the ring $\Gamma(X, \mathcal{O}_X)$ is *isomorphic* to K , and so its spectrum reduces to a point. Indeed, a section of \mathcal{O}_X over X has a restriction over X_1 (resp. X_2), identified with an affine open of X , that is a polynomial $f(s)$ (resp. $g(t)$), and it follows from the definitions that we should have $g(t) = f(1/t)$, which is only possible if $f = g \in K$.

2.4. Local schemes.

(2.4.1). We say that an affine scheme is a *local scheme* if it is the affine scheme associated to a local ring A ; there then exists, in $X = \text{Spec}(A)$, a single closed point $a \in X$, and for all other $b \in X$ we have that $a \in \overline{\{b\}}$ (1.1.7).

For all preschemes Y and points $y \in Y$, the local scheme $\text{Spec}(\mathcal{O}_y)$ is called the *local scheme of Y at the point y* . Let V be an affine open subset of Y containing y , and B the ring of the affine scheme V ; then \mathcal{O}_y is canonically identified with B_y (1.3.4), and the canonical homomorphism $B \rightarrow B_y$ thus corresponds (1.6.1) to a morphism of preschemes $\text{Spec}(\mathcal{O}_y) \rightarrow V$. If we compose this morphism with the canonical injection $V \rightarrow Y$, then we obtain a morphism $\text{Spec}(\mathcal{O}_y) \rightarrow Y$ which is *independent* of the affine open subset V (containing y) that we chose: indeed, if V' is some other affine open subset containing y , then there exists a third affine open subset W that contains y and is such that $W \subset V \cap V'$ (2.1.3); we can thus assume that $V \subset V'$, and then if B' is the ring of V' , so everything relies on remarking that the diagram

$$\begin{array}{ccc} B' & \xrightarrow{\quad} & B \\ & \searrow & \swarrow \\ & \mathcal{O}_y & \end{array}$$

is commutative (0, 1.5.1). The morphism

$$\text{Spec}(\mathcal{O}_y) \longrightarrow Y$$

thus defined is said to be *canonical*.

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Proposition (2.4.2). — Let (Y, \mathcal{O}_Y) be a prescheme; for all $y \in Y$, let (ψ, θ) be the canonical morphism $(\text{Spec}(\mathcal{O}_y), \tilde{\mathcal{O}}_y) \rightarrow (Y, \mathcal{O}_Y)$. Then ψ is a homeomorphism from $\text{Spec}(\mathcal{O}_y)$ to the subspace S_y of Y given by the z such that $y \in \overline{\{z\}}$ (or, equivalently, the generalizations of y (0, 2.1.2)); furthermore, if $z = \psi(\mathfrak{p})$, then $\theta_z^\# : \mathcal{O}_z \rightarrow (\mathcal{O}_y)_{\mathfrak{p}}$ is an isomorphism; (ψ, θ) is thus a monomorphism of ringed spaces.

Proof. Since the unique closed point a of $\text{Spec}(\mathcal{O}_y)$ is contained in the closure of any point of this space, and since $\psi(a) = y$, the image of $\text{Spec}(\mathcal{O}_y)$ under the continuous map ψ is contained in S_y . Since S_y is contained in every affine open containing y , one can consider just the case where Y is an affine scheme; but then this proposition follows from (1.6.2). \square

We see (2.1.5) that there is a bijective correspondence between $\text{Spec}(\mathcal{O}_y)$ and the set of closed irreducible subsets of Y containing y .

Corollary (2.4.3). — For $y \in Y$ to be the generic point of an irreducible component of Y , it is necessary and sufficient for the only prime ideal of the local ring \mathcal{O}_y to be its maximal ideal (in other words, for \mathcal{O}_y to be of dimension zero).

Proposition (2.4.4). — Let (X, \mathcal{O}_X) be a local scheme of some ring A , a its unique closed point, and (Y, \mathcal{O}_Y) a prescheme. Every morphism $u = (\psi, \theta) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ then factors uniquely as $X \rightarrow \text{Spec}(\mathcal{O}_{\psi(a)}) \rightarrow Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a local homomorphism $\mathcal{O}_{\psi(a)} \rightarrow A$. This establishes a canonical bijective correspondence between the set of morphisms $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ and the set of local homomorphisms $\mathcal{O}_y \rightarrow A$ for $(y \in Y)$.

Indeed, for all $x \in X$, we have that $a \in \overline{\{x\}}$, so $\psi(a) \in \overline{\{\psi(x)\}}$, which shows that $\psi(X)$ is contained in every affine open subset that contains $\psi(a)$. So it suffices to consider the case where (Y, \mathcal{O}_Y) is an affine scheme of ring B , and then we have that $u = ({}^a\phi, \tilde{\phi})$, where $\phi \in \text{Hom}(B, A)$ (1.7.3). Further, we have that $\phi^{-1}(\mathfrak{j}_a) = \mathfrak{j}_{\psi(a)}$, and hence that the image under ϕ of any element of $B - \mathfrak{j}_{\psi(a)}$ is invertible in the local ring A ; the factorization in the result follows from the universal property of the ring of fractions (0, 1.2.4). Conversely, to each local homomorphism $\mathcal{O}_y \rightarrow A$ there is a unique corresponding morphism $(\psi, \theta) : X \rightarrow \text{Spec}(\mathcal{O}_y)$ such that $\psi(a) = y$ (1.7.3), and, by composing with the canonical morphism $\text{Spec}(\mathcal{O}_y) \rightarrow Y$, we obtain a morphism $X \rightarrow Y$, which proves the proposition.

(2.4.5). The affine schemes whose ring is a field K have an underlying space that is just a point. If A is a local ring with maximal ideal \mathfrak{m} , then each local homomorphism $A \rightarrow K$ has kernel equal to \mathfrak{m} , and so factors as $A \rightarrow A/\mathfrak{m} \rightarrow K$, where the second arrow is a monomorphism. The morphisms $\text{Spec}(K) \rightarrow \text{Spec}(A)$ thus correspond bijectively to monomorphisms of fields $A/\mathfrak{m} \rightarrow K$.

Let (Y, \mathcal{O}_Y) be a prescheme; for each $y \in Y$ and each ideal \mathfrak{a}_y of \mathcal{O}_y , the canonical homomorphism $\mathcal{O}_y \rightarrow \mathcal{O}_y/\mathfrak{a}_y$ defines a morphism $\text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \rightarrow \text{Spec}(\mathcal{O}_y)$; if we compose this with the canonical morphism $\text{Spec}(\mathcal{O}_y) \rightarrow Y$, then we obtain a morphism $\text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \rightarrow Y$, again said to be *canonical*. For $\mathfrak{a}_y = \mathfrak{m}_y$, this says that $\mathcal{O}_y/\mathfrak{a}_y = k(y)$, and so Proposition (2.4.4) says that:

Corollary (2.4.6). — Let (X, \mathcal{O}_X) be a local scheme whose ring K is a field, ξ the unique point of X , and (Y, \mathcal{O}_Y) a prescheme. Then each morphism $u : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ factors uniquely as $X \rightarrow \text{Spec}(k(\psi(\xi))) \rightarrow Y$, where the second arrow denotes the canonical morphism, and the first corresponds to a monomorphism $k(\psi(\xi)) \rightarrow K$. This establishes a canonical bijective correspondence between the set of morphisms $(X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$ and the set of monomorphisms $k(y) \rightarrow K$ (for $y \in Y$).

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Corollary (2.4.7). — For all $y \in Y$, every canonical morphism $\text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \rightarrow Y$ is a monomorphism of ringed spaces.

Proof. We have already seen this when $\mathfrak{a}_y = 0$ (2.4.2), and it suffices to apply Corollary (1.7.5). \square

Remark. — 2.4.8 Let X be a local scheme, and a its unique closed point. Since every affine open subset containing a is necessarily equal to the whole of X , every *invertible* \mathcal{O}_X -module (0, 5.4.1) is necessarily *isomorphic* to \mathcal{O}_X (or, as we say, again, *trivial*). This property does not hold in general for an arbitrary affine scheme $\text{Spec}(A)$; we will see in Chapter V that if A is a normal ring then this is true when A is a unique factorisation domain.

2.5. Preschemes over a prescheme.

Definition (2.5.1). — Given a prescheme S , we say that the data of a prescheme X and a morphism of preschemes $\phi : X \rightarrow S$ defines a prescheme X *over the prescheme* S , or an S -prescheme; we say that S is the *base prescheme* of the S -prescheme X . The morphism ϕ is called the *structure morphism* of the S -prescheme X . When S is an affine scheme of ring A , we also say that X endowed with ϕ is a *prescheme over the ring* A (or an A -prescheme).

It follows from (2.2.4) that the data of a prescheme over a ring A is equivalent to the data of a prescheme (X, \mathcal{O}_X) whose structure sheaf \mathcal{O}_X is a sheaf of A -algebras. An arbitrary prescheme can always be considered as a \mathbf{Z} -prescheme in a unique way.

If $\phi : X \rightarrow S$ is the structure morphism of an S -prescheme X , we say that a point $x \in X$ is *over a point* $s \in S$ if $\phi(x) = s$. We say that X *dominates* S if ϕ is a dominant morphism (2.2.6).

(2.5.2). Let X and Y be S -preschemes; we say that a morphism of preschemes $u : X \rightarrow Y$ is a *morphism of preschemes over* S (or an S -morphism) if the diagram

$$\begin{array}{ccc} X & \xrightarrow{u} & Y \\ & \searrow & \swarrow \\ & S & \end{array}$$

(where the diagonal arrows are the structure morphisms) is commutative: this ensures that, for all $s \in S$ and $x \in X$ over s , $u(x)$ also lies over s .

It follows immediately from this definition that the composition of any two S -morphisms is an S -morphism; S -preschemes thus form a *category*.

We denote by $\text{Hom}_S(X, Y)$ the set of S -morphisms from an S -prescheme X to an S -prescheme Y ; the identity morphism of an S -prescheme X is denoted by 1_X .

When S is an affine scheme of ring A , we will also say A -morphism instead of S -morphism.

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(2.5.3). If X is an S -prescheme, and $v : X' \rightarrow X$ a morphism of preschemes, then the composition $X' \rightarrow X \rightarrow S$ endows X' with the structure of an S -prescheme; in particular, every prescheme induced by an open set U of X can be considered as an S -prescheme by the canonical injection.

If $u : X \rightarrow Y$ is an S -morphism of S -preschemes, then the restriction of u to any prescheme induced by an open subset U of X is also an S -morphism $U \rightarrow Y$. Conversely, let (U_α) be an open cover of X , and for each α let $u_\alpha : U_\alpha \rightarrow Y$ be an S -morphism; if, for all pairs of indices (α, β) , the restrictions of u_α and u_β to $U_\alpha \cap U_\beta$ agree, then there exists an S -morphism $X \rightarrow Y$, and exactly one such that the restriction to each U_α is u_α .

If $u : X \rightarrow Y$ is an S -morphism such that $u(X) \subset V$, where V is an open subset of Y , then u , considered as a morphism from X to V , is also an S -morphism.

(2.5.4). Let $S' \rightarrow S$ be a morphism of preschemes; for all S' -preschemes, the composition $X \rightarrow S' \rightarrow S$ endows X with the structure of an S -prescheme. Conversely, suppose that S' is the induced prescheme of an open subset of S ; let X be an S -prescheme and suppose that the structure morphism $f : X \rightarrow S$ is such that $f(X) \subset S'$; then we can consider X as an S' -prescheme. In this latter case, if Y is another S -prescheme whose structure morphism sends the underlying space to S' , then every S -morphism from X to Y is also an S' -morphism.

(2.5.5). If X is an S -prescheme, with structure morphism $\phi : X \rightarrow S$, we define an S -section of X to be an S -morphism from S to X , that is to say a morphism of preschemes $\psi : S \rightarrow X$ such that $\phi \circ \psi$ is the identity on S . We denote by $\Gamma(X/S)$ the set of S -sections of X .

§3. PRODUCTS OF PRESCHEMES

3.1. Sums of preschemes. Let (X_α) be any family of preschemes; let X be a topological space which is the *sum* of the underlying spaces X_α ; X is then the union of pairwise disjoint open subspaces X'_α , and for each α there is a homomorphism ϕ_α from X_α to X'_α . If we equip each of the X'_α with the sheaf $(\phi_\alpha)_*(\mathcal{O}_{X_\alpha})$, it is clear that X becomes a prescheme, which we call the *sum* of the family of preschemes (X_α) and which we denote $\bigsqcup_\alpha X_\alpha$. If Y is a prescheme, then the map $f \mapsto (f \circ \phi_\alpha)$ is a *bijection* from the set $\text{Hom}(X, Y)$ to the product set $\prod_\alpha \text{Hom}(X_\alpha, Y)$. In particular, if the X_α are S -preschemes, with structure morphisms ψ_α , then X is an S -prescheme by the unique morphism $\psi : X \rightarrow S$ such that $\psi \circ \phi_\alpha = \psi_\alpha$ for each α . The sum of two preschemes X and Y is denoted by $X \sqcup Y$. It is immediate that, if $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$, then $X \sqcup Y$ is canonically identified with $\text{Spec}(A \times B)$.

3.2. Products of preschemes.

Definition (3.2.1). — Given S -preschemes X and Y , we say that a triple (Z, p_1, p_2) , consisting of an S -prescheme Z , and S -morphisms $p_1 : Z \rightarrow X$ and $p_2 : Z \rightarrow Y$, is a *product* of the S -preschemes X and Y , if, for each S -prescheme T , the map $f \mapsto (p_1 \circ f, p_2 \circ f)$ is a bijection from the set of S -morphisms from T to Z , to the set of pairs consisting of an S -morphism $T \rightarrow X$ and an S -morphism $T \rightarrow Y$ (in other words, a bijection

$$\text{Hom}_S(T, Z) \simeq \text{Hom}_S(T, X) \times \text{Hom}_S(T, Y).$$

This is the general notion of a *product* of two objects in a category, applied to the category of S -preschemes (T, I, 1.1); in particular, a product of two S -preschemes is *unique* up to a unique S -isomorphism. Because of this uniqueness, most of the time we will denote a product of two S -preschemes X and Y by $X \times_S Y$ (or simply $X \times Y$, when there is no chance of confusion), with the morphisms p_1 and p_2 (the *canonical projections* of $X \times_S Y$ to X and to Y , respectively) being suppressed in the notation. If $g : T \rightarrow X$ and $h : T \rightarrow Y$ are S -morphisms, we denote by $(g, h)_S$, or simply (g, h) , the S -morphism $f : T \rightarrow X \times_S Y$ such that $p_1 \circ f = g$, $p_2 \circ f = h$. If X' and Y' are two S -preschemes, p'_1 and p'_2 the canonical projections of $X' \times_S Y'$ (assumed to exist), and $u : X' \rightarrow X$ and $v : Y' \rightarrow Y$ S -morphisms, then we write $u \times_S v$ (or simply $u \times v$) for the S -morphism $(u \circ p'_1, v \circ p'_2)_S$ from $X' \times_S Y'$ to $X \times_S Y$.

When S is an affine scheme given by some ring A , we often replace S by A in the above notation.

Proposition (3.2.2). — Let X , Y , and S be affine schemes, given by rings B , C , and A (respectively). Let $Z = \text{Spec}(B \otimes_A C)$, and let p_1 and p_2 be the S -morphisms corresponding (2.2.4) to the canonical A -homomorphisms $u : b \mapsto b \otimes 1$ and $v : c \mapsto 1 \otimes c$ (respectively) from B and C to $B \otimes_A C$; then (Z, p_1, p_2) is a product of X and Y .

Proof. According to (2.2.4), it suffices to check that, if, to each A -homomorphism $f : B \otimes_A C \rightarrow L$ (where L is an A -algebra), we associate the pair $(f \circ u, f \circ v)$, then this defines a bijection $\text{Hom}_A(B \otimes_A C, L) \simeq \text{Hom}_A(B, L) \times \text{Hom}_A(C, L)$,⁵ which follows immediately from the definitions and the fact that $b \otimes c = (b \otimes 1)(1 \otimes c)$. \square

Corollary (3.2.3). — *Let T be an affine scheme given by some ring D , and $\alpha = ({}^a\rho, \tilde{\rho})$ (resp. $\beta = ({}^a\sigma, \tilde{\sigma})$) an S -morphism $T \rightarrow X$ (resp. $T \rightarrow Y$), where ρ (resp. σ) is an A -homomorphism from B (resp. C) to D ; then $(\alpha, \beta)_S = ({}^a\tau, \tilde{\tau})$, where τ is the homomorphism $B \otimes_A C \rightarrow D$ such that $\tau(b \otimes c) = \rho(b)\sigma(c)$.*

Proposition (3.2.4). — *Let $f : S' \rightarrow S$ be a monomorphism of preschemes (T, I, 1.1), and let X and Y be S' -preschemes, also considered as S -preschemes via f . Every product of the S -preschemes X and Y is then a product of the S' -preschemes X and Y , and vice versa.*

Proof. Let $\phi : X \rightarrow S'$ and $\psi : Y \rightarrow S'$ be the structure morphisms. If T is an S -prescheme, and $u : T \rightarrow X$ and $v : T \rightarrow Y$ are S -morphisms, then we have, by definition, that $f \circ \phi \circ u = f \circ \psi \circ v = \theta$, the structure morphism of T ; the hypotheses on f imply that $\phi \circ u = \psi \circ v = \theta'$, and so we can consider T as an S' -prescheme with structure morphism θ' , and u and v as S' -morphisms. The conclusion of the proposition follows immediately, taking (3.2.1) into account. \square

Corollary (3.2.5). — *Let X and Y be S -preschemes, with structure morphisms $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$, and let S' be an open subset of S such that $\phi(X) \subset S'$ and $\psi(Y) \subset S'$. Every product of the S -preschemes X and Y is then also a product of the S' -preschemes X and Y , and conversely.*

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It suffices to apply (3.2.4) to the canonical injection $S' \rightarrow S$.

Theorem (3.2.6). — *Given S -preschemes X and Y , there exists a product $X \times_S Y$.*

The proof proceeds in several steps.

Lemma (3.2.6.1). — *Let (Z, p, q) be a product of X and Y , and U and V open subsets of X and Y , respectively. If we let $W = p^{-1}(U) \cap q^{-1}(V)$, then the triple consisting of W and the restrictions of p and q to W (considered as the morphisms $W \rightarrow U$ and $W \rightarrow V$, respectively) is a product of U and V .*

Indeed, if T is an S -prescheme, then we can identify the S -morphisms $T \rightarrow W$ with the S -morphisms $T \rightarrow Z$ mapping T to W . Then, if $g : T \rightarrow U$ and $h : T \rightarrow V$ are any two S -morphisms, we can consider them as S -morphisms from T to X and to Y , respectively, and, by hypothesis, there is then a unique S -morphism $f : T \rightarrow Z$ such that $g = p \circ f$ and $h = q \circ f$. Since $p(f(Y)) \subset U$, $q(f(T)) \subset V$, we have

$$f(T) \subset p^{-1}(U) \cap q^{-1}(V) = W,$$

hence our claim.

Lemma (3.2.6.2). — *Let Z be an S -prescheme, $p : Z \rightarrow X$ and $q : Z \rightarrow Y$ both S -morphisms, (U_α) an open cover of X , and (V_λ) an open cover of Y . Suppose that, for each pair (α, λ) , the S -prescheme $W_{\alpha\lambda} = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ and the restrictions of p and q to $W_{\alpha\lambda}$ form a product of U_α and V_λ . Then (Z, p, q) is a product of X and Y .*

We first show that, if f_1 and f_2 are S -morphisms $T \rightarrow Z$, then the equations $p \circ f_1 = p \circ f_2$ and $q \circ f_1 = q \circ f_2$ imply that $f_1 = f_2$. Indeed, Z is the union of the $W_{\alpha\lambda}$, so the $f_1^{-1}(W_{\alpha\lambda})$ form an open cover of T , and similarly for $f_2^{-1}(W_{\alpha\lambda})$. In addition, we have

$$f_1^{-1}(W_{\alpha\lambda}) = f_1^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)) = f_2^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)) = f_2^{-1}(W_{\alpha\lambda})$$

by hypothesis, and it thus reduces to noting that the the restrictions of f_1 and f_2 to $f_1^{-1}(W_{\alpha\lambda}) = f_2^{-1}(W_{\alpha\lambda})$ are identical for each pair of indices. But since these restrictions can be considered as S -morphisms from $f_1^{-1}(W_{\alpha\lambda})$ to $W_{\alpha\lambda}$, our claim follows from the hypotheses and Definition (3.2.1).

Suppose now that we are given S -morphisms $g : T \rightarrow X$ and $h : T \rightarrow Y$. Let $T_{\alpha\lambda} = g^{-1}(U_\alpha) \cap h^{-1}(V_\lambda)$; then the $T_{\alpha\lambda}$ form an open cover of T . By hypothesis, there exists an S -morphism $f_{\alpha\lambda}$ such that $p \circ f_{\alpha\lambda}$ and $q \circ f_{\alpha\lambda}$ are the restrictions of g and h to $T_{\alpha\lambda}$ (respectively). Now, we will show that

⁵The notation Hom_A denotes here the set of homomorphisms of A -algebras.

the restrictions of $f_{\alpha\lambda}$ and $f_{\beta\mu}$ to $T_{\alpha\lambda} \cap T_{\beta\mu}$ coincide, which will finish the proof of Lemma (3.2.6.2). The images of $T_{\alpha\lambda} \cap T_{\beta\mu}$ under $f_{\alpha\lambda}$ and $f_{\beta\mu}$ are contained in $W_{\alpha\lambda} \cap W_{\beta\mu}$ by definition. Since

$$W_{\alpha\lambda} \cap W_{\beta\mu} = p^{-1}(U_\alpha \cap U_\beta) \cap q^{-1}(V_\lambda \cap V_\mu),$$

it follows from Lemma (3.2.6.1) that $W_{\alpha\lambda} \cap W_{\beta\mu}$ and the restrictions to this prescheme of p and q form a product of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$. Since $p \circ f_{\alpha\lambda}$ and $p \circ f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$ and similarly for $q \circ f_{\alpha\lambda}$ and $q \circ f_{\beta\mu}$, we see that $f_{\alpha\lambda}$ and $f_{\beta\mu}$ coincide on $T_{\alpha\lambda} \cap T_{\beta\mu}$.

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Lemma (3.2.6.3). — *Let (U_α) be an open cover of X , (V_λ) an open cover of Y , and suppose that, for each pair (α, λ) , there exists a product of U_α and V_λ ; then there exists a product of X and Y .*

Applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$, we see that there exists a product of S -preschemes induced, respectively, by X and Y on these open sets; in addition, the uniqueness of the product shows that, if we set $i = (\alpha, \lambda)$ and $j = (\beta, \mu)$, then there is a canonical isomorphism h_{ij} (resp. h_{ji}) from this product to an S -prescheme W_{ij} (resp. W_{ji}) induced by $U_\alpha \times_S V_\lambda$ (resp. $U_\beta \times_S V_\mu$) on an open set; then $f_{ij} = h_{ij} \circ h_{ji}^{-1}$ is an isomorphism from W_{ji} to W_{ij} . In addition, for a third pair $k = (\gamma, \nu)$, we have $f_{ik} = f_{ij} \circ f_{jk}$ on $W_{ki} \cap W_{kj}$, by applying Lemma (3.2.6.1) to the open sets $U_\alpha \cap U_\beta \cap U_\gamma$ and $V_\lambda \cap V_\mu \cap V_\nu$ in U_β and V_μ , respectively. It follows that we have a prescheme Z , an open cover (Z_i) of the underlying space of Z , and, for each i , an isomorphism g_i from the induced prescheme Z_i to the prescheme $U_\alpha \times_S V_\lambda$, so that, for each pair (i, j) , we have $f_{ij} = g_i \circ g_j^{-1}$ (2.3.1); in addition, we have $g_i(Z_i \cap Z_j) = W_{ij}$. If p_i, q_i , and θ_i are the projections and the structure morphism of the S -prescheme $U_\alpha \times_S V_\lambda$ (respectively), we immediately see that $p_i \circ g_i = p_j \circ g_j$ on $Z_i \cap Z_j$, and similarly for the two other morphisms. We can thus define the morphisms of preschemes $p : Z \rightarrow X$ (resp. $q : Z \rightarrow Y, \theta : Z \rightarrow S$) by the condition that p (resp. q, θ) coincide with $p_i \circ g_i$ (resp. $q_i \circ g_i, \theta_i \circ g_i$) on each of the Z_i ; Z , equipped with θ , is then an S -prescheme. We now show that $Z'_i = p^{-1}(U_\alpha) \cap q^{-1}(V_\lambda)$ is equal to Z_i . For each index $j = (\beta, \mu)$, we have $Z_j \cap Z'_i = g_j^{-1}(p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda))$. We have, by Lemma (3.2.6.1),

$$p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = p_j^{-1}(U_\alpha \cap U_\beta) \cap q_j^{-1}(V_\lambda \cap V_\mu);$$

with the restrictions of p_j and q_j to $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda)$ defining, on this S -prescheme, the structure of a product of $U_\alpha \cap U_\beta$ and $V_\lambda \cap V_\mu$; but the uniqueness of the product then implies that $p_j^{-1}(U_\alpha) \cap q_j^{-1}(V_\lambda) = W_{ji}$. As a result, we have $Z_j \cap Z'_i = Z_j \cap Z_i$ for each j , hence $Z'_i = Z_i$. We then deduce from Lemma (3.2.6.2) that (Z, p, q) is a product of X and Y .

Lemma (3.2.6.4). — *Let $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$ be the structure morphisms of X and Y , (S_i) an open cover of S , and let $X_i = \phi^{-1}(S_i)$, $Y_i = \psi^{-1}(S_i)$. If each of the products $X_i \times_S Y_i$ exists, then $X \times_S Y$ exists.*

According to Lemma (3.2.6.3), everything follows from proving that the products $X_i \times_S Y_i$ exists for any i and j . Set $X_{ij} = X_i \cap X_j = \phi^{-1}(S_i \cap S_j)$, $Y_{ij} = Y_i \cap Y_j = \psi^{-1}(S_i \cap S_j)$; by Lemma (3.2.6.1), the product $Z_{ij} = X_{ij} \times_S Y_{ij}$ exists. We now note that, if T is an S -prescheme, and if $g : T \rightarrow X_i$ and $h : T \rightarrow Y_j$ are S -morphisms, then we necessarily have that $\phi(g(T)) = \psi(h(T)) \subset S_i \cap S_j$ by the definition of an S -morphism, and thus that $g(T) \subset X_{ij}$ and $h(T) \subset Y_{ij}$; it is then immediate that Z_{ij} is the product of X_i and Y_j .

(3.2.6.5). We can now complete the proof of Theorem (3.2.6). If S is an affine scheme, then there are covers (U_α) and (V_λ) of X and Y (respectively) consisting of affine open subsets; since $U_\alpha \times_S V_\lambda$ exists, by (3.2.2), $X \times_S Y$ exists similarly, by Lemma (3.2.6.3). If S is any prescheme, then there is a cover (S_i) of S consisting of affine open subsets. If $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structure morphisms, and if we set $X_i = \phi^{-1}(S_i)$ and $Y_i = \psi^{-1}(S_i)$, then the products $X_i \times_{S_i} Y_i$ exist, by the above; but then the products $X_i \times_S Y_i$ also exist (3.2.5), therefore $X \times_S Y$ exists similarly, by Lemma (3.2.6.4). \square

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Corollary (3.2.7). — *Let $Z = X \times_S Y$ be the product of two S -preschemes, p and q the projections from Z to X and to Y (respectively), and ϕ (resp. ψ) the structure morphism of X (resp. Y). Let S' be an open subset of S , and U (resp. V) an open subset of X (resp. Y) contained in $\phi^{-1}(S')$ (resp. $\psi^{-1}(S')$). Then the product*

$U \times_{S'} V$ is canonically identified with the prescheme induced on Z by $p^{-1}(U) \cap q^{-1}(V)$ (considered as an S' -prescheme). In addition, if $f : T \rightarrow X$ and $g : T \rightarrow Y$ are S -morphisms such that $f(T) \subset U$ and $g(T) \subset V$, then the S' -morphism $(f, g)_{S'}$ can be identified with the restriction of $(f, g)_S$ to $p^{-1}(U) \cap q^{-1}(V)$.

Proof. This follows from Corollary (3.2.5) and Lemma (3.2.6.1). \square

(3.2.8). Let (X_α) and (Y_λ) be families of S -preschemes, and X (resp. Y) the sum of the family (X_α) (resp. (Y_λ)) (3.1). Then $X \times_S Y$ can be identified with the sum of the family $(X_\alpha \times_S Y_\lambda)$; this follows immediately from Lemma (3.2.6.3).

(3.2.9).⁶ It follows from (1.8.1) that we can state (3.2.2) in the following manner: $Z = \text{Spec}(B \otimes_A C)$ is not only a product of $X = \text{Spec}(B)$ and $Y = \text{Spec}(C)$ in the category of S -preschemes, but also in the category of locally ringed spaces over S (with a definition of S -morphisms modelled on that of (2.5.2)). The proof of (3.2.6) also proves that, for any two S -preschemes X and Y , the prescheme $X \times_S Y$ is not only the product of X and Y in the category of S -preschemes, but also in the category of locally ringed spaces over the prescheme S .

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3.3. Formal properties of the product; change of the base prescheme.

(3.3.1). The reader will notice that all the properties stated in this section, except (3.3.13) and (3.3.15), are true without modification in any category, whenever the products involved in the statements exist (since it is clear that the notions of an S -object and of an S -morphism can be defined exactly as in (2.5) for any object S of the category).

(3.3.2). First of all, $X \times_S Y$ is a *covariant bifunctor* in X and Y on the category of S -preschemes: it suffices in fact to note that the diagram

$$\begin{array}{ccccc} X \times Y & \xrightarrow{f \times 1} & X' \times Y & \xrightarrow{f' \times 1} & X'' \times Y \\ \downarrow & & \downarrow & & \downarrow \\ X & \xrightarrow{f} & X' & \xrightarrow{f'} & X'' \end{array}$$

is commutative.

Proposition (3.3.3). — For each S -prescheme X , the first (resp. second) projection from $X \times_S S$ (resp. $S \times_S X$) is a functorial isomorphism from $X \times_S S$ (resp. $S \times_S X$) to X , whose inverse isomorphism is $(1_X, \phi)_S$ (resp. $(\phi, 1_X)_S$), where we denote by ϕ the structure morphism $X \rightarrow S$; we can therefore write, up to a canonical isomorphism,

$$X \times_S S = S \times_S X = X.$$

Proof. It suffices to prove that the triple $(X, 1_X, \phi)$ is a product of X and S . If T is an S -prescheme, then the only S -morphism from T to S is necessarily the structure morphism $\psi : T \rightarrow S$. If f is an S -morphism from T to X , we necessarily have $\psi = \phi \circ f$, hence our claim. \square

Corollary (3.3.4). — Let X and Y be S -preschemes, with structure morphisms $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$. If we canonically identify X with $X \times_S S$, and Y with $S \times_S Y$, then the projections $X \times_S Y \rightarrow X$ and $X \times_S Y \rightarrow Y$ are identified with $1_X \times \psi$ and $\phi \times 1_Y$ (respectively).

The proof is immediate and is left to the reader.

(3.3.5). We can define, in a manner similar to (3.2), the product of a finite number n of S -preschemes, and the existence of these products follows from (3.2.6) by induction on n , and by noting that $(X_1 \times_S X_2 \times_S \cdots \times_S X_{n-1}) \times_S X_n$ satisfies the definition of a product. The uniqueness of the product implies, as in any category, its *commutativity* and *associativity* properties. If, for example, p_1, p_2 , and p_3 denote the projections from $X_1 \times_S X_2 \times_S X_3$, and if we identify this prescheme with $(X_1 \times_S X_2) \times_S X_3$, then the projection to $X_1 \times_S X_2$ is identified with $(p_1, p_2)_S$.

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⁶[Trans.] (3.2.9) is from the errata of EGA II, on page 221, whence the change in page numbering.

(3.3.6). Let S and S' be preschemes, and $\phi : S' \rightarrow S$ a morphism, which lets us consider S' as an S -prescheme. For each S -prescheme X , consider the product $X \times_S S'$, and let p and π' be the projections to X and to S' (respectively). Equipped with π' , this product is an S' -prescheme; when we consider it as such, we denote it by $X_{(S')}$ or $X_{(\phi)}$, and we say that this is the prescheme obtained by *base change* (or *a change of base*) from S to S' by means of the morphism ϕ , or the *inverse image* of X by ϕ . We note that, if π is the structure morphism of X , and θ the structure morphism of $X \times_S S'$, considered as an S -prescheme, then the diagram

$$\begin{array}{ccc} X & \xleftarrow{p} & X_{(S')} \\ \pi \downarrow & \theta \swarrow & \downarrow \pi' \\ S & \xleftarrow{\phi} & S' \end{array}$$

is commutative.

(3.3.7). With the notation of (3.3.6), for each S -morphism $f : X \rightarrow Y$, we denote by $f_{(S')}$ the S' -morphism $f \times_S 1 : X_{(S')} \rightarrow Y_{(S')}$, and we say that $f_{(S')}$ is the *base change* (or *inverse image*) of f by ϕ . Therefore $X_{(S')}$ is a *covariant functor* in X , from the category of S -preschemes to that of S' -preschemes.

(3.3.8). The prescheme $X_{(S')}$ can be considered as a solution to a *universal mapping problem*: each S' -prescheme T is also an S -prescheme via ϕ ; each S -morphism $g : T \rightarrow X$ is then uniquely written as $g = p \circ f$, where f is an S' -morphism $T \rightarrow X_{(S')}$, as follows from the definition of the product applied to the S -morphisms f and $\psi : T \rightarrow S'$ (the structure morphism of T).

Proposition (3.3.9). — (“Transitivity of base change”). *Let S'' be a prescheme, and $\phi' : S'' \rightarrow S$ a morphism. For each S -prescheme X , there exists a canonical functorial isomorphism from the S'' -prescheme $(X_{(\phi)})_{(\phi')}$ to the S'' -prescheme $X_{(\phi \circ \phi')}$.*

Proof. Let T be a S'' -prescheme, ψ its structure morphism, and g an S -morphism from T to X (T being considered as an S -prescheme with structure morphism $\phi \circ \phi' \circ \psi$). Since T is also an S' -prescheme with structure morphism $\phi' \circ \psi$, we can write $g = p \circ g'$, where g' is an S' -morphism $T \rightarrow X_{(\phi)}$, and then $g' = p' \circ g''$, where g'' is an S'' -morphism $T \rightarrow (X_{(\phi)})_{(\phi')}$:

$$\begin{array}{ccccc} X & \xleftarrow{p} & X_{(\phi)} & \xleftarrow{p'} & (X_{(\phi)})_{(\phi')} \\ \pi \downarrow & & \pi' \downarrow & & \downarrow \pi'' \\ S & \xleftarrow{\phi} & S & \xleftarrow{\phi'} & S'' \end{array}$$

So the result follows by the uniqueness of the solution to a universal mapping problem. \square I | 110

This result can be written as the equality (up to a canonical isomorphism) $(X_{(S')})_{(S'')} = X_{(S'')}$ (if there is no chance of confusion), or also as

$$(3.3.9.1) \quad (X \times_S S') \times_{S'} S'' = X \times_S S'';$$

the functorial nature of the isomorphism defined in (3.3.9) can similarly be expressed by the transitivity formula for base change morphisms

$$(3.3.9.2) \quad (f_{(S')})_{(S'')} = f_{(S'')}$$

for each S -morphism $f : X \rightarrow Y$.

Corollary (3.3.10). — *If X and Y are S -preschemes, then there exists a canonical functorial isomorphism from the S' -prescheme $X_{(S')} \times_{S'} Y_{(S')}$ to the S' -prescheme $(X \times_S Y)_{(S')}$.*

Proof. We have, up to canonical isomorphism,

$$(X \times_S S') \times_{S'} (Y \times_S S') = X \times_S (Y \times_S S') = (X \times_S Y) \times_S S'$$

according to (3.3.9.1) and the associativity of products of S -preschemes. \square

The functorial nature of the isomorphism defined in Corollary (3.3.10) can be expressed by the formula

$$(3.3.10.1) \quad (u_{(S')}, v_{(S')})_{S'} = ((u, v)_S)_{(S')}$$

for each pair of S -morphisms $u : T \rightarrow X, v : T \rightarrow Y$.

In other words, the base change functor $X_{(S')}$ commutes with products; it also commutes with sums (3.2.8).

Corollary (3.3.11). — *Let Y be an S -prescheme, and $f : X \rightarrow Y$ a morphism which makes X a Y -prescheme (and, as a result, also an S -prescheme). The prescheme $X_{(S')}$ is then identified with the product $X \times_Y Y_{(S')}$, the projection $X \times_Y Y_{(S')} \rightarrow Y_{(S')}$ being identified with $f_{(S')}$.*

Proof. Let $\psi : Y \rightarrow S$ be the structure morphism of Y ; we have the commutative diagram

$$\begin{array}{ccccc} S' & \longleftarrow & Y_{(S')} & \xleftarrow{f_{(S')}} & X_{(S')} \\ \downarrow & & \downarrow & & \downarrow \\ S & \xleftarrow{\psi} & Y & \xleftarrow{f} & X. \end{array}$$

We have that $Y_{(S')}$ is identified with $S'_{(\psi)}$, and $X_{(S')}$ with $S'_{(\psi \circ f)}$; taking (3.3.9) and (3.3.4) into account, we thus deduce the corollary. \square

(3.3.12). Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be S -morphisms which are monomorphisms of preschemes (I, 1.1); then $f \times_S g$ is a monomorphism. Indeed, if p and q are the projections of $X \times_S Y$, p' and q' the projections of $X' \times_S Y'$, and u and v both S -morphisms $T \rightarrow X \times_S Y$, then the equation $(f \times_S g) \circ u = (f \times_S g) \circ v$ implies that $p' \circ (f \times_S g) \circ u = p' \circ (f \times_S g) \circ v$, or, in other words, that $f \circ p \circ u = f \circ p \circ v$, and since f is a monomorphism, $p \circ u = p \circ v$; using the fact that g is a monomorphism, we similarly obtain $q \circ u = q \circ v$, hence $u = v$.

It follows that, for each base change $S' \rightarrow S$,

$$f_{(S')} : X_{(S')} \longrightarrow Y_{(S')}$$

is a monomorphism.

(3.3.13). Let S and S' be affine schemes of rings A and A' (respectively); a morphism $S' \rightarrow S$ then corresponds to a ring homomorphism $A \rightarrow A'$. If X is an S -prescheme, we denote by $X_{(A')}$ or $X \otimes_A A'$ the S' -prescheme $X_{(S')}$; when X is also affine of ring B , $X_{(A')}$ is affine of ring $B_{(A')} = B \otimes_A A'$ obtained by extension of scalars from the A -algebra B to A' .

(3.3.14). With the notation of (3.3.6), for each S -morphism $f : S' \rightarrow X$, we have that $f' = (f, 1_{S'})_S$ is an S' -morphism $S' \rightarrow X' = X_{(S')}$ such that $p \circ f' = f$, $\pi' \circ f' = 1_{S'}$, or, in other words, an S' -section of X' ; conversely, if f' is such an S' -section, then $f = p \circ f'$ is an S -morphism $S' \rightarrow X$. We thus define a canonical bijective correspondence

$$\mathrm{Hom}_S(S', X) \simeq \mathrm{Hom}_{S'}(S', X').$$

We say that f' is the *graph morphism* of f , and we denote it by Γ_f .

(3.3.15). Given a prescheme X , which we can always consider as a \mathbf{Z} -prescheme, it follows, in particular, from (3.3.14) that the X -sections of $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate) correspond bijectively with morphisms $\mathbf{Z}[T] \rightarrow X$. We will show that these X -sections also correspond bijectively with sections of the structure sheaf \mathcal{O}_X over X . Indeed, let (U_α) be a cover of X by affine open subsets; let $u : X \rightarrow X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ be an X -morphism, and let u_α be its restriction to U_α ; if A_α is the ring of the affine scheme U_α , then $U_\alpha \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ is an affine scheme of ring $A_\alpha[T]$ (3.2.2), and u_α canonically corresponds to an A_α -homomorphism $A_\alpha[T] \rightarrow A_\alpha$ (1.7.3). Now, since such a homomorphism is completely determined by the data of the image of T in A_α , let $s_\alpha \in A_\alpha = \Gamma(U_\alpha, \mathcal{O}_X)$, and if we suppose that the restrictions of u_α and u_β to an affine open subset $V \subset U_\alpha \cap U_\beta$ coincide, then we see immediately that s_α and s_β coincide on V ; thus the family (s_α) consists of the restrictions to U_α of a section s of \mathcal{O}_X over X ; conversely, it is clear that such a section defines a family (u_α) of morphisms which are the restrictions to U_α of an X -morphism $X \rightarrow X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$. This result is generalized in (II, 1.7.12).

3.4. Points of a prescheme with values in a prescheme; geometric points.

(3.4.1). Let X be a prescheme; for each prescheme T , we then denote by $X(T)$ the set $\text{Hom}(T, X)$ of morphisms $T \rightarrow X$, and the elements of this set are called *the points of X with values in T* . If we associate to each morphism $f : T \rightarrow T'$ the map $u' \mapsto u' \circ f$ from $X(T')$ to $X(T)$, we see, for fixed X , that $X(T)$ is a *contravariant functor in T* , from the category of preschemes to that of sets. In addition, each morphism of preschemes $g : X \rightarrow Y$ defines a functorial homomorphism $X(T) \rightarrow Y(T)$, which sends $v \in X(T)$ to $g \circ v$.

(3.4.2). Given sets P, Q , and R , and maps $\phi : P \rightarrow R$ and $\psi : Q \rightarrow R$, we define the *fibre product of P and Q over R* (relative to ϕ and ψ) as the subset of the product set $P \times Q$ consisting of the pairs (p, q) such that $\phi(p) = \psi(q)$; we denote it by $P \times_R Q$. Definition (3.2.1) of the product of S -preschemes can be interpreted, with the notation of (3.4.1), via the formula

$$(3.4.2.1) \quad (X \times_S Y)(T) = X(T) \times_{S(T)} Y(T).$$

the maps $X(T) \rightarrow S(T)$ and $Y(T) \rightarrow S(T)$ corresponding to the structure morphisms $X \rightarrow S$ and $Y \rightarrow S$.

(3.4.3). If we are given a prescheme S and we consider only the S -preschemes and S -morphisms, then we will denote by $X(T)_S$ the set $\text{Hom}_S(T, X)$ of S -morphisms $T \rightarrow X$, and suppress the subscript S when there is no chance of confusion; we say that the elements of $X(T)_S$ are the *points* (or *S -points*, when there is a possibility of confusion) of the *S -prescheme X with values in the S -prescheme T* . In particular, an *S -section* of X is none other than a *point of X with values in S* . The formula (3.4.2.1) can then be written as

$$(3.4.3.1) \quad (X \times_S Y)(T)_S = X(T)_S \times_{Y(T)_S} Y(T)_S;$$

more generally, if Z is an S -prescheme, and X, Y , and T are Z -preschemes (thus *ipso facto* S -preschemes), then we have

$$(3.4.3.2) \quad (X \times_Z Y)(T)_S = X(T)_S \times_{Y(T)_S} Y(T)_S.$$

We note that, to show that a triple (W, r, s) consisting of an S -prescheme W and S -morphisms $r : W \rightarrow X$ and $s : W \rightarrow Y$ is a product of X and Y (over Z), it suffices, by definition, to check that, for each S -prescheme T , the diagram

$$\begin{array}{ccc} W(T)_S & \xrightarrow{r'} & X(T)_S \\ s' \downarrow & & \downarrow \phi' \\ Y(T)_S & \xrightarrow{\psi'} & Z(T)_S \end{array}$$

makes $W(T)_S$ the fibre product of $X(T)_S$ and $Y(T)_S$ over $Z(T)_S$, where r' and s' correspond to r and s , and ϕ' and ψ' to the structure morphisms $\phi : X \rightarrow Z$ and $\psi : Y \rightarrow Z$.

(3.4.4). When T (resp. S) in the above is an affine scheme of ring B (resp. A), we replace T (resp. S) by B (resp. A) in the above notation, and we then call the elements of $X(B)$ the *points of X with values in the ring B* , and the elements of $X(B)_A$ the *points of the A -prescheme X with values in the A -algebra B* . We note that $X(B)$ and $X(B)_A$ are *covariant* functors in B . We similarly write $X(T)_A$ for the set of points of the A -prescheme X with values in the A -prescheme T .

(3.4.5). Consider, in particular, the case where T is of the form $\text{Spec}(A)$, where A is a *local* ring; the elements of $X(A)$ then correspond bijectively to *local* homomorphisms $\mathcal{O}_x \rightarrow A$ for $x \in X$ (2.2.4); we say that the point x of the underlying space of X is the *location*⁷ of the point of X with values in A to which it corresponds.

More specifically, we define the *geometric points* of a prescheme S to be the *points of X with values in a field K* : the data of such a point is equivalent to the data of its location x in the underlying subspace of X , and of an *extension* K of $k(x)$; K will be called the *field of values* of the corresponding geometric point, and we say that this geometric point is *located at x* . We also define a map $X(K) \rightarrow X$, sending a geometric point with values in K to its location.

⁷[Trans.] We also say that the geometric point lies over this x .

If $S' = \text{Spec}(K)$ is an S -prescheme (in other words, if K is considered as an extension of the residue field $k(s)$, where $s \in S$), and if X is an S -prescheme, then an element of $X(K)_S$, or, as we say, a *geometric point of X lying over s with values in K* , consists of the data of a $k(s)$ -monomorphism from the residue field $k(x)$ to K , where x is a point of X lying over s (therefore $k(x)$ is an extension of $k(s)$).

In particular, if $S = \text{Spec}(K) = \{\xi\}$, then the geometric points of X with values in K can be identified with the points $x \in X$ such that $k(x) = K$; we say that these latter points are the K -rational points of the K -prescheme X ; if K' is an extension of K , then the geometric points of X with values in K' bijectively correspond to the K' -rational points of $X' = X_{(K')}$ (3.3.14).

Lemma (3.4.6). — Let X_i ($1 \leq i \leq n$) be S -preschemes, s a point of S , and x_i ($1 \leq i \leq n$) points of X_i lying over s . Then there exists an extension K of $k(s)$ and a geometric point of the product $Y = X_1 \times_S X_2 \times_S \cdots \times_S X_n$, with values in K , whose projections to the X_i are localized at the x_i .

Proof. There exist $k(s)$ -monomorphisms $k(x_i) \rightarrow K$, all in the same extension K of $k(s)$ (Bourbaki, Alg., chap. V, §4, prop. 2). The compositions $k(s) \rightarrow k(x_i) \rightarrow K$ are all identical, and so the morphisms $\text{Spec}(K) \rightarrow X_i$ corresponding to the $k(x_i) \rightarrow K$ are all S -morphisms, and we thus conclude that they define a unique morphism $\text{Spec}(K) \rightarrow Y$. If y is the corresponding point of Y , it is clear that its projection in each of the X_i is x_i . \square

Proposition (3.4.7). — Let X_i ($1 \leq i \leq n$) be S -preschemes, and, for each index i , let x_i be a point of X_i . For there to exist a point y of $Y = X_1 \times_S X_2 \times \cdots \times_S X_n$ whose image is x_i under the i th projection for each $1 \leq i \leq n$, it is necessary and sufficient that the x_i all lie above the same point s of S .

Proof. The condition is evidently necessary; Lemma (3.4.6) proves that it is sufficient. \square

In other words, if we denote by (X) the underlying set of X , we see that we have a canonical surjective function $(X \times_S Y) \rightarrow (X) \times_{(S)} (Y)$; we must point out that this function is *not injective* in general; in other words, there can exist multiple distinct points z in $X \times_S Y$ that have the same projections $x \in X$ and $y \in Y$; we have already seen this when S , X , and Y are prime spectra of fields k , K , and K' (respectively), since the tensor product $K \otimes_k K'$ has, in general, multiple distinct prime ideals (cf. (3.4.9)).

Corollary (3.4.8). — Let $f : X \rightarrow Y$ be an S -morphism, and $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ the S' -morphism induced by f by an extension $S' \rightarrow S$ of the base prescheme. Let p (resp. q) be the projection $X_{(S')} \rightarrow X$ (resp. $Y_{(S')} \rightarrow Y$); for every subset M of X , we have

$$q^{-1}(f(M)) = f_{(S')}(p^{-1}(M)).$$

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Proof. Indeed (3.3.11), $X_{(S')}$ can be identified with the product $X \times_Y Y_{(S')}$ thanks to the commutative diagram

$$\begin{array}{ccc} X & \xleftarrow{p} & X_{(S')} \\ f \downarrow & & \downarrow f_{(S')} \\ Y & \xleftarrow{q} & Y_{(S')} \end{array}$$

By (3.4.7), the equation $q(y') = f(x)$ for $x \in M$ and $y' \in Y_{(S')}$ is equivalent to the existence of some $x' \in X_{(S')}$ such that $p(x') = x$ and $f_{(S')}(x') = y'$, whence the corollary. \square

Lemma (3.4.6) can be made clearer in the following manner:

Proposition (3.4.9). — Let X and Y be S -preschemes, x a point of X , and y a point of Y , with both x and y lying above the same point $s \in S$. The set of points of $X \times_S Y$ with projections x and y is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ considered as extensions of $k(s)$ (Bourbaki, Alg., chap. VIII, §8, prop. 2).

Proof. Let p (resp. q) be the projection from $X \times_S Y$ to X (resp. Y), and E the subspace $p^{-1}(x) \cap q^{-1}(y)$ of the underlying space of $X \times_S Y$. First, note that the morphisms $\text{Spec}(k(x)) \rightarrow S$ and $\text{Spec}(k(y)) \rightarrow$

S factor as $\text{Spec}(k(x)) \rightarrow \text{Spec}(k(s)) \rightarrow S$ and $\text{Spec}(k(y)) \rightarrow \text{Spec}(k(s)) \rightarrow S$; since $\text{Spec}(k(s)) \rightarrow S$ is a monomorphism (2.4.7), it follows from (3.2.4) that we have

$$P = \text{Spec}(k(x)) \times_S \text{Spec}(k(y)) = \text{Spec}(k(x)) \times_{\text{Spec}(k(s))} \text{Spec}(k(y)) = \text{Spec}(k(x) \otimes_{k(s)} k(y)).$$

We will define two maps, $\alpha : P_0 \rightarrow E$ and $\beta : E \rightarrow P_0$, inverse to one another (where P_0 denotes the underlying set of the prescheme P). If $i : \text{Spec}(k(x)) \rightarrow X$ and $j : \text{Spec}(k(y)) \rightarrow Y$ are the canonical morphisms (2.4.5), we take α to be the map of underlying spaces corresponding to the morphism $i \times_S j$. On the other hand, every $z \in E$ defines, by hypothesis, two $k(s)$ -monomorphisms, $k(x) \rightarrow k(z)$ and $k(y) \rightarrow k(z)$, and thus a $k(s)$ -monomorphism $k(x) \otimes_{k(s)} k(y) \rightarrow k(z)$, and thus a morphism $\text{Spec}(k(z)) \rightarrow P$; $\beta(z)$ will be the image of z in P_0 under this morphism. The verification of the fact that $\alpha \circ \beta$ and $\beta \circ \alpha$ are the identity maps follows from (2.4.5) and the definition of the product (3.2.1). Finally, we know that P_0 is in bijective correspondence with the set of types of extensions (?) composed of $k(x)$ and $k(y)$ (Bourbaki, *Alg.*, chap. VIII, §8, prop. 1). \square

3.5. Surjections and injections.

(3.5.1). In a general sense, consider a property **P** of morphisms of preschemes, and the following two propositions:

- (i) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are S -morphisms that have property **P**, then $f \times_S g$ also has property **P**.
- (ii) If $f : X \rightarrow Y$ is an S -morphism that has property **P**, then every S' -morphism $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$, induced by f by an extension of the base prescheme, also has property **P**.

Since $f_{(S')} = f \times_S 1_{S'}$, we see that, if, for every prescheme X , the identity 1_X has property **P**, then (i) implies (ii); on the other hand, since $f \times_S g$ is the composite morphism

$$X \times_S Y \xrightarrow{f \times 1_Y} X' \times_S Y \xrightarrow{1_{X'} \times g} X' \times_S Y',$$

we see that, if the composition of two morphisms has property **P**, then so does the product $f \times_S g$, and so (ii) implies (i).

A first application of this remark is

Proposition (3.5.2). —

- (i) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are surjective S -morphisms, then $f \times_S g$ is surjective.
- (ii) If $f : X \rightarrow Y$ is a surjective S -morphism, then $f_{(S')}$ is surjective for every extension S' of the base prescheme.

Proof. The composition of any two surjections being a surjection, it suffices to prove (ii); but this proposition follows immediately from (3.4.8) applied to $M = X$. \square

Proposition (3.5.3). — For a morphism $f : X \rightarrow Y$ to be surjective, it is necessary and sufficient that, for every field K and every morphism $\text{Spec}(K) \rightarrow Y$, there exist an extension K' of K and a morphism $\text{Spec}(K') \rightarrow X$ that make the following diagram commute:

$$\begin{array}{ccc} X & \longleftarrow & \text{Spec}(K') \\ f \downarrow & & \downarrow \\ Y & \longleftarrow & \text{Spec}(K). \end{array}$$

Proof. The condition is sufficient because, for all $y \in Y$, it suffices to apply it to a morphism $\text{Spec}(K) \rightarrow Y$ corresponding to a monomorphism $k(y) \rightarrow K$, with K being an extension of $k(y)$ (2.4.6). Conversely, suppose that f is surjective, and let $y \in Y$ be the image of the unique point of $\text{Spec}(K)$; there exists some $x \in X$ such that $f(x) = y$; we will consider the corresponding monomorphism $k(y) \rightarrow k(x)$ (2.2.1); it then suffices to take K' to be the extension of $k(y)$ such that there exist $k(y)$ -monomorphisms from $k(x)$ and K to K' (Bourbaki, *Alg.*, chap. V, Â§4, prop. 2); the morphism $\text{Spec}(K') \rightarrow X$ corresponding to $k(x) \rightarrow K'$ is exactly that for which we are searching. \square

With the language introduced in (3.4.5), we can say that every geometric point of Y with values in K comes from a geometric point of X with values in an extension of K .

Definition (3.5.4). — We say that a morphism $f : X \rightarrow Y$ of preschemes is *universally injective*, or a *radicial morphism*, if, for every field K , the corresponding map $X(K) \rightarrow Y(K)$ is injective.

It follows also from the definitions that every *monomorphism of preschemes* (T, 1.1) is radicial.

(3.5.5). For a morphism $f : X \rightarrow Y$ to be radicial, it suffices that the condition of Definition (3.5.4) hold for every *algebraically closed* field. In fact, if K is an arbitrary field, and K' an algebraically-closed extension of K , then the diagram

$$\begin{array}{ccc} X(K) & \xrightarrow{\alpha} & Y(K) \\ \phi \downarrow & & \downarrow \phi' \\ X(K') & \xrightarrow{\alpha'} & Y(K') \end{array}$$

commutes, where ϕ and ϕ' come from the morphism $\text{Spec}(K') \rightarrow \text{Spec}(K)$, and α and α' corresponding to f . However, ϕ is injective, and so too is α' , by hypothesis; hence α is necessarily injective.

Proposition (3.5.6). — Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms of preschemes.

- (i) If f and g are radicial, then so is $g \circ f$.
- (ii) Conversely, if $g \circ f$ is radicial, then so is f .

Proof. Taking into account Definition (3.5.4), the proposition reduces to the corresponding claims for the maps $X(K) \rightarrow Y(K) \rightarrow Z(K)$, and these claims are evident. \square

Proposition (3.5.7). —

- (i) If the S -morphisms $f : X \rightarrow X'$ and $g : X' \rightarrow Y$ are radicial, then so is $f \times_S g$.
- (ii) If the S -morphism $f : X \rightarrow Y$ is radicial, then so is $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ for every extension $S' \rightarrow S$ of the base prescheme.

Proof. Given (3.5.1), it suffices to prove (i). We have seen (3.4.2.1) that

$$\begin{aligned} (X \times_S Y)(K) &= X(K) \times_{S(K)} Y(K), \\ (X' \times_S Y')(K) &= X'(K) \times_{S(K)} Y'(K), \end{aligned}$$

with the map $(X \times_S Y)(K) \rightarrow (X' \times_S Y')(K)$ corresponding to $f \times_S g$ thus being identified with $(u, v) \rightarrow (f \circ u, g \circ v)$, and the proposition then follows. \square

Proposition (3.5.8). — For a morphism $f = (\psi, \theta) : X \rightarrow Y$ to be radicial, it is necessary and sufficient for ψ to be injective and for the monomorphism $\theta^x : k(\psi(x)) \rightarrow k(x)$ to make $k(x)$ a radicial extension of $k(\psi(x))$ for every $x \in X$.

Proof. We suppose that f is radicial and first show that the equation $\psi(x_1) = \psi(x_2) = y$ necessarily implies that $x_1 = x_2$. Indeed, there exists a field K , and an extension of $k(y)$, along with $k(y)$ -monomorphisms $k(x_1) \rightarrow K$ and $k(x_2) \rightarrow K$ (Bourbaki, Alg., chap. V, Âg4, prop. 2); the corresponding morphisms $u_1 : \text{Spec}(K) \rightarrow X$ and $u_2 : \text{Spec}(K) \rightarrow X$ are then such that $f \circ u_1 = f \circ u_2$, and so $u_1 = u_2$ by hypothesis, and this implies, in particular, that $x_1 = x_2$. We now consider $k(x)$ as the extension of $k(\psi(x))$ by means of θ^x : if $k(x)$ is not a radicial algebraically-closed extension, then there exist two distinct $k(\psi(x))$ -monomorphisms from $k(x)$ to an algebraically-closed extension K of $k(\psi(x))$, and the two corresponding morphisms $\text{Spec}(K) \rightarrow X$ would contradict the hypothesis. Conversely, taking (2.4.6) into account, it is immediate that the conditions stated are sufficient for f to be radicial. \square

Corollary (3.5.9). — If A is a ring, and S is a multiplicative set of A , then the canonical morphism $\text{Spec}(S^{-1}A) \rightarrow \text{Spec}(A)$ is radicial.

Proof. Indeed, this morphism is a monomorphism (1.6.2). \square

Corollary (3.5.10). — Let $f : X \rightarrow Y$ be a radicial morphism, $g : Y' \rightarrow Y$ a morphism, and $X' = X_{(Y')} = X \times_Y Y'$. Then the radicial morphism $f_{(Y')}$ (3.5.7, ii) is a bijection from the underlying space of X to $g^{-1}(f(X))$; further, for every field K , the set $X'(K)$ can be identified with the subset of $Y'(K)$ given by the inverse image of the map $Y'(K) \rightarrow Y(K)$ (corresponding to g) from the subset $X(K)$ of $Y(K)$.

Proof. The first claim follows from (3.5.8) and (3.4.8); the second, from the commutativity of the following diagram:

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$$\begin{array}{ccc} X'(K) & \longrightarrow & Y'(K) \\ \downarrow & & \downarrow \\ X(K) & \longrightarrow & Y(K) \end{array}$$

□

Remark (3.5.11). — We say that a morphism $f = (\psi, \theta)$ of preschemes is *injective* if the map ψ is injective. For a morphism $f = (\psi, \theta) : X \rightarrow Y$ to be *radicial*, it is necessary and sufficient that, for every morphism $Y' \rightarrow Y$, the morphism $f_{(Y')} : X_{(Y')} \rightarrow Y'$ be injective (which justifies the terminology of a *universally injective* morphism). In fact, the condition is necessary by (3.5.7, ii) and (3.5.8). Conversely, the condition implies that ψ is injective; if, for some $x \in X$, the monomorphism $\theta^x : k(\psi(x)) \rightarrow k(x)$ were not radicial, then there would be an extension K of $k(\psi(x))$, and two distinct morphisms $\text{Spec}(K) \rightarrow X$ corresponding to the same morphism $\text{Spec}(K) \rightarrow Y$ (3.5.8). But then, setting $Y' = \text{Spec}(K)$, there would be two distinct Y' -sections of $X_{(Y')}$ (3.3.14), which contradicts the hypothesis that $f_{(Y')}$ is injective.

3.6. Fibres.

Proposition (3.6.1). — Let $f : X \rightarrow Y$ be a morphism, y a point of Y , and \mathfrak{a}_y an ideal of definition for \mathcal{O}_y for the \mathfrak{m}_y -preadic topology. Then the projection $p : X \times_Y \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \rightarrow X$ is a homeomorphism from the underlying space of the prescheme $X \times_Y \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y)$ to the fibre $f^{-1}(y)$ equipped with the topology induced from that of the underlying space of X .

Proof. Since $\text{Spec}(\mathcal{O}_y/\mathfrak{a}_y) \rightarrow Y$ is radicial ((3.5.4) and (2.4.7)), since $\text{Spec}(\mathcal{O}_y/\mathfrak{a}_y)$ is a single point, and since the ideal $\mathfrak{m}_y/\mathfrak{a}_y$ is nilpotent by hypothesis (1.1.12), we already know ((3.5.10) and (3.3.4)) that p identifies, as sets, the underlying space of $X \times_Y \text{Spec}(\mathcal{O}_y/\mathfrak{a}_y)$ with $f^{-1}(y)$; everything reduces to proving that p is a homeomorphism. By (3.2.7), the question is local on X and Y , and so we can suppose that $X = \text{Spec}(B)$ and $Y = \text{Spec}(A)$, with B being an A -algebra. The morphism p then corresponds to the homomorphism $1 \otimes \phi : B \rightarrow B \otimes_A A'$, where $A' = A_y/\mathfrak{a}_y$ and ϕ is the canonical map from A to A' . Then every element of $B \otimes_A A'$ can be written as

$$\sum_i b_i \otimes \phi(a_i) / \phi(s) = \left(\sum_i (a_i b_i \otimes 1) \right) (1 \otimes \phi(s))^{-1},$$

where $s \notin \mathfrak{j}_y$, and Proposition (1.2.4) applies. □

(3.6.2). Throughout the rest of this treatise, whenever we consider a fibre $f^{-1}(y)$ of a morphism as having the structure of a $k(y)$ -prescheme, it will always be the prescheme obtained by transporting the structure of $X \times_Y \text{Spec}(k(y))$ by the projection to X . We will also write this (latter) product as $X \times_Y k(y)$, or $X \otimes_{\mathcal{O}_Y} k(y)$; more generally, if B is an \mathcal{O}_Y -algebra, we will denote by $X \times_Y B$ or $X \otimes_{\mathcal{O}_Y} B$ the product $X \times_Y \text{Spec}(B)$.

With the preceding convention, it follows from (3.5.10) that the points of X with values in an extension K of $k(y)$ are identified with the points of $f^{-1}(y)$ with values in K .

(3.6.3). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms, and $h = g \circ f$ their composition; for all $z \in Z$, the fibre $h^{-1}(z)$ is a prescheme isomorphic to

$$X \times_Z \text{Spec}(k(z)) = (X \times_Y Y) \times_Z \text{Spec}(k(z)) = X \times_Y g^{-1}(z).$$

In particular, if U is an open subset of X , then the prescheme induced on $U \cap f^{-1}(y)$ by the prescheme $f^{-1}(y)$ is isomorphic to $f_U^{-1}(y)$ (f_U being the restriction of f to U),

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Proposition (3.6.4). — (Transitivity of fibres) Let $f : X \rightarrow Y$ and $g : Y' \rightarrow Y$ be morphisms; let $X' = X \times_Y Y' = X_{(Y')}$ and $f' = f_{(Y')} : X' \rightarrow Y'$. For every $y' \in Y'$, if we let $y = g(y')$, then the prescheme $f'^{-1}(y')$ is isomorphic to $f^{-1}(y) \otimes_{k(y)} k(y')$.

Proof. Indeed, it suffices to remark that the two preschemes $(X \otimes_Y k(y)) \otimes_{k(y)} k(y')$ and $(X \times_Y Y') \otimes_{Y'} k(y')$ are both canonically isomorphic to $X \times_Y \text{Spec}(k(y'))$ by (3.3.9.1). \square

In particular, if V is an open neighborhood of y in Y , and we denote by f_V the restriction of f to the induced prescheme on $f^{-1}(V)$, then the preschemes $f^{-1}(y)$ and $f_V^{-1}(y)$ are canonically identified.

Proposition (3.6.5). — *Let $f : X \rightarrow Y$ be a morphism, y a point of Y , Z the local prescheme $\text{Spec}(\mathcal{O}_y)$, and $p = (\psi, \theta)$ the projection $X \times_Y Z \rightarrow X$; then p is a homeomorphism from the underlying space of $X \times_Y Z$ to the subspace $f^{-1}(Z)$ of X (when the underlying space of Z is identified with a subspace of Y , cf. (2.4.2)), and, for all $t \in X \times_Y Z$, letting $z = \psi(t)$, we have that $\theta_t^\#$ is an isomorphism from \mathcal{O}_x to \mathcal{O}_t .*

Proof. Since Z (identified as a subspace of Y) is contained inside every affine open containing y (2.4.2), we can, as in (3.6.1), reduce to the case where $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine schemes, with A being a B -algebra. Then $X \times_Y Z$ is the prime spectrum of $A \otimes_B B_y$, and this ring is canonically identified with $S^{-1}A$, where S is the image of $B - \mathfrak{p}_y$ in A (0, 1.5.2); since p then corresponds to the canonical homomorphism $A \rightarrow S^{-1}A$, the proposition follows from (1.6.2). \square

3.7. Application: reduction of a prescheme mod. \mathfrak{J} . *This section, which makes use of notions and results from Chapter I and Chapter II, will not be used in what follows in this treatise, and is only intended for readers familiar with classical algebraic geometry.*

(3.7.1). Let A be a ring, X an A -prescheme, and \mathfrak{J} an ideal of A ; then $X_0 = X \otimes_A (A/\mathfrak{J})$ is an (A/\mathfrak{J}) -prescheme, which we sometimes say is induced from X by *reduction mod. \mathfrak{J}* .

. This terminology is used foremost when A is a *local ring* and \mathfrak{J} its maximal ideal, in such a way that X_0 is a prescheme over the residue field $k = A/\mathfrak{J}$ of A .

When A is also integral, with field of fractions K , we can consider the K -prescheme $X' = X \otimes_A K$. By an abuse of language which we will not use, it has been said, up until now, that X_0 is *induced by X' by reduction mod. \mathfrak{J}* . In the case where this language was used, A was a local ring of dimension 1 (most often a discrete valuation ring) and it was implied (be it more or less explicitly) that the given K -prescheme X' was a closed subprescheme of a K -prescheme P' (in fact, a projective space of the form (?) \mathbf{P}_K^r , cf. (II, 4.1.1)), itself of the form $P' = P \otimes_A K$, where P is a given A -prescheme (in fact, the A -scheme \mathbf{P}_A^r , with the notation of (II, 4.1.1)). In our language, the definition of X_0 in terms of X' is formulated as follows:

We consider the affine scheme $Y = \text{Spec}(A)$, formed of two points, the unique closed point $y = \mathfrak{J}$ and the generic point (0) , the singleton set U of the generic point being thus an open $U = \text{Spec}(K)$ in Y . If X is an A -prescheme (or, in other words, a Y -prescheme), then $X \otimes_A K = X'$ is exactly the prescheme induced by X on $\psi^{-1}(U)$, denoting by ψ the structure morphism $X \rightarrow Y$. In particular, if ϕ is the structure morphism $P \rightarrow Y$, then a closed subprescheme X' of $P' = \phi^{-1}(U)$ is a (locally closed) subprescheme of P . If P is Noetherian (for example, if A is Noetherian and P is of finite type over A), then there exists a smaller closed subprescheme $X = \overline{X'}$ of G that through which X' factors (9.5.10), and X' is the prescheme induced by X on the open $\phi^{-1}(U) \cap X$, and so is isomorphic to $X \otimes_A K$ (9.5.10). *The immersion of X' into $P' = P \otimes_A K$ thus lets us canonically consider X' as being of the form $X' = X \otimes_A K$, where X is an A -prescheme.* We can then consider the reduced mod. \mathfrak{J} prescheme $X_0 = X \otimes_A k$, which is exactly the fibre $\psi^{-1}(y)$ of the closed point y . Up until now, lacking the adequate terminology, we had avoided explicitly introducing the A -prescheme X . One ought to, however, note that all the claims normally made about the “reduced mod. \mathfrak{J} ” prescheme X_0 should be seen as consequences of more complicated claims about X itself, and cannot be satisfactorily formulated or understood except by interpreting them as such. It seems also that any hypotheses made on X_0 always reduce to hypotheses on X itself (independent of the prior data of an immersion of X' in \mathbf{P}_K^r), which lets us give more intrinsic statements.

(3.7.3). Lastly, we draw attention to a very particular fact, which has undoubtedly contributed to slowing the conceptual clarification of the situation envisaged here: if A is a discrete valuation ring, and if X is *proper* over A (which is indeed the case if X is a closed subprescheme of some \mathbf{P}_A^r , cf. (II, 5.5.4)), then the points of X with values in A and the points of X' with values in k are in bijective correspondence (II, 7.3.8). This is why we often believe that facts about X' have been proved, when

in reality we have proved facts about X , and these remain valuable (in this form) whenever we no longer suppose that the base local ring is of dimension 1.

§4. SUBPRESCHEMES AND IMMERSION MORPHISMS

4.1. Subpreschemes.

(4.1.1). As the notion of a quasi-coherent sheaf (0, 5.1.3) is local, a quasi-coherent \mathcal{O}_X -module \mathcal{F} over a prescheme X can be defined by the following condition: for each affine open V of X , $\mathcal{F}|_V$ is isomorphic to the sheaf associated to a $\Gamma(V, \mathcal{O}_X)$ -module (1.4.1). It is clear that, over a prescheme X , the structure sheaf \mathcal{O}_X is quasi-coherent, and that the kernels, cokernels, and images of homomorphisms of quasi-coherent \mathcal{O}_X -modules, as well as inductive limits and direct sums of quasi-coherent \mathcal{O}_X -modules, are also quasi-coherent (Theorem (1.3.7) and Corollary (1.3.9)).

Proposition (4.1.2). — *Let X be a prescheme, and \mathcal{I} a quasi-coherent sheaf of ideals of \mathcal{O}_X . Then the support Y of the sheaf $\mathcal{O}_X/\mathcal{I}$ is closed, and if we denote by \mathcal{O}_Y the restriction of $\mathcal{O}_X/\mathcal{I}$ to Y , then (Y, \mathcal{O}_Y) is a prescheme.*

Proof. It evidently suffices (2.1.3) to consider the case where X is an affine scheme, and to show that, in this case, Y is closed in X , and is an affine scheme. Indeed, if $X = \text{Spec}(A)$, then we have $\mathcal{O}_X = \tilde{A}$ and $\mathcal{I} = \tilde{\mathfrak{I}}$, where \mathfrak{I} is an ideal of A (1.4.1); Y is then equal to the closed subset $V(\mathfrak{I})$ of X and can be identified with the prime spectrum of the ring $B = A/\mathfrak{I}$ (1.1.11); in addition, if ϕ is the canonical homomorphism $A \rightarrow B = A/\mathfrak{I}$, then the direct image ${}^a\phi_*(\tilde{B})$ is canonically identified with the sheaf $\tilde{A}/\tilde{\mathfrak{I}} = \mathcal{O}_X/\mathcal{I}$ (Proposition (1.6.3) and Corollary (1.3.9)), which finishes the proof. \square

We say that (Y, \mathcal{O}_Y) is the *subprescheme* of (X, \mathcal{O}_X) defined by the sheaf of ideals \mathcal{I} ; this is a particular case of the more general notion of a *subprescheme*:

Definition (4.1.3). — We say that a ringed space (Y, \mathcal{O}_Y) is a subprescheme of a prescheme (X, \mathcal{O}_X) if:

- 1st. Y is a locally closed subspace of X ;
- 2nd. if U denotes the largest open subset of X containing Y such that Y is closed in U (equivalently, the complement in X of the boundary of Y with respect to \bar{Y}), then (Y, \mathcal{O}_Y) is a subprescheme of $(U, \mathcal{O}_X|_U)$ defined by a quasi-coherent sheaf of ideals of $\mathcal{O}_X|_U$.

We say that the subprescheme (Y, \mathcal{O}_Y) of (X, \mathcal{O}_X) is closed if Y is closed in X (in which case $U = X$).

It follows immediately from this definition and Proposition (4.1.2) that the closed subpreschemes of X are in canonical *bijective correspondence* with the quasi-coherent sheaves of ideals \mathcal{I} of \mathcal{O}_X , since if two such sheaves \mathcal{I} and \mathcal{I}' have the same (closed) support Y , and if the restrictions of $\mathcal{O}_X/\mathcal{I}$ and $\mathcal{O}_X/\mathcal{I}'$ to Y are identical, then we have $\mathcal{I}' = \mathcal{I}$.

(4.1.4). Let (Y, \mathcal{O}_Y) be a subprescheme of X , U the largest open subset of X such that Y is closed (and thus contained) in U , and V an open subset of X contained in U ; then $V \cap Y$ is closed in V . In addition, if Y is defined by the quasi-coherent sheaf of ideals \mathcal{I} of $\mathcal{O}_X|_U$, then $\mathcal{I}|_V$ is a quasi-coherent sheaf of ideals of $\mathcal{O}_X|_V$, and it is immediate that the prescheme induced by Y on $Y \cap V$ is the closed subprescheme of V defined by the sheaf of ideals $\mathcal{I}|_V$. Conversely:

Proposition (4.1.5). — *Let (Y, \mathcal{O}_Y) be a ringed space such that Y is a subspace of X , and there exists a cover (V_α) of Y by open subsets of X such that, for each α , $Y \cap V_\alpha$ is closed in V_α , and the ringed space $(Y \cap V_\alpha, \mathcal{O}_Y|(Y \cap V_\alpha))$ is a closed subprescheme of the prescheme induced on V_α by X . Then (Y, \mathcal{O}_Y) is a subprescheme of X .*

Proof. The hypotheses imply that Y is locally closed in X and that the largest open U in which Y is closed (and thus contained) contains all the V_α ; we can thus reduce to the case where $U = X$ and Y is closed in X . We then define a quasi-coherent sheaf of ideals \mathcal{I} of \mathcal{O}_X by taking $\mathcal{I}|_{V_\alpha}$ to be the sheaf of ideals of $\mathcal{O}_X|_{V_\alpha}$ which define the closed subprescheme $(Y \cap V_\alpha, \mathcal{O}_Y|(Y \cap V_\alpha))$, and, for each open subset W of X not intersecting Y , $\mathcal{I}|_W = \mathcal{O}_X|_W$. We see immediately, by Definition (4.1.3) and (4.1.4), that there exists a unique sheaf of ideals \mathcal{I} satisfying these conditions, and that it defines the closed subprescheme (Y, \mathcal{O}_Y) . \square

In particular, the *induced* (by X) prescheme on an *open subset* of X is a *subprescheme* of X .

Proposition (4.1.6). — *A subprescheme (resp. a closed subprescheme) of a subprescheme (resp. closed subprescheme) of X is canonically identified with a subprescheme (resp. closed subprescheme) of X .* I | 121

Proof. Since a locally closed subset of a locally closed subspace of X is a locally closed subspace of X , it is clear (4.1.5) that the question is local and that we can thus suppose that X is affine; the proposition then follows from the canonical identification of A/\mathfrak{J}' with $(A/\mathfrak{J})/(\mathfrak{J}'/\mathfrak{J})$, where A is some ring, and \mathfrak{J} and \mathfrak{J}' are ideals of A such that $\mathfrak{J} \subset \mathfrak{J}'$. \square

In what follows, we will always make use of the above identification.

(4.1.7). Let Y be a subprescheme of a prescheme X , and denote by ψ the canonical injection $Y \rightarrow X$ of the *underlying subspaces*; we know that the inverse image $\psi^*(\mathcal{O}_X)$ is the restriction $\mathcal{O}_X|_Y$ (0, 3.7.1). If, for each $y \in Y$, we denote by ω_y the canonical homomorphism $(\mathcal{O}_X)_y \rightarrow (\mathcal{O}_Y)_y$, then these homomorphisms are the restrictions to stalks of a *surjective* homomorphism ω of sheaves of rings $\mathcal{O}_X|_Y \rightarrow \mathcal{O}_Y$: indeed, it suffices to check locally on Y , that is to say, we can suppose that X is affine and that the subprescheme Y is closed; if in this case \mathcal{I} is the sheaf of ideals in \mathcal{O}_X which defines Y , then the ω_y are none other than the restriction to stalks of the homomorphism $\mathcal{O}_X|_Y \rightarrow (\mathcal{O}_X/\mathcal{I})|_Y$. We have thus defined a *monomorphism of ringed spaces* (0, 4.1.1) $j = (\psi, \omega^\flat)$ which is evidently a morphism $Y \rightarrow X$ of preschemes (2.2.1), and we call this the *canonical injection morphism*.

If $f : X \rightarrow Z$ is a morphism, we then say that the composite morphism $Y \xrightarrow{j} X \xrightarrow{f} Z$ is the *restriction* of f to the subprescheme Y .

(4.1.8). Conforming to the general definitions (T, I, 1.1), we will say that a morphism of preschemes $f : Z \rightarrow X$ is *majoré*⁸ by the injection morphism $j : Y \rightarrow X$ of a subprescheme Y of X if f factors as $Z \xrightarrow{g} Y \xrightarrow{j} X$, where g is a morphism of preschemes; further, g is necessarily *unique* since j is a monomorphism.

Proposition (4.1.9). — *For a morphism $f : Z \rightarrow X$ to be factor through an injection morphism $j : Y \rightarrow X$, it is necessary and sufficient that $f(Z) \subset Y$ and, for all $z \in Z$, letting $y = f(z)$, that the homomorphism $(\mathcal{O}_X)_y \rightarrow \mathcal{O}_z$ corresponding to f factor as $(\mathcal{O}_X)_y \rightarrow (\mathcal{O}_Y)_y \rightarrow \mathcal{O}_z$ (or equivalently, for the kernel of $(\mathcal{O}_X)_y \rightarrow \mathcal{O}_z$ to contain the kernel of $(\mathcal{O}_X)_y \rightarrow (\mathcal{O}_Y)_y$).*

Proof. The conditions are evidently necessary. To see that they are sufficient, we can reduce to the case where Y is a *closed* subprescheme of X , by replacing X by an open U such that Y is closed in U (4.1.3) if necessary; Y is then defined by a quasi-coherent sheaf of ideals \mathcal{I} of \mathcal{O}_X . Let $f = (\psi, \theta)$, and let \mathcal{J} be the sheaf of ideals of $\psi^*(\mathcal{O}_X)$, kernel of $\theta^\sharp : \psi^*(\mathcal{O}_X) \rightarrow \mathcal{O}_Z$; considering the properties of the functor ψ^* (0, 3.7.2), the hypotheses imply that, for each $z \in Z$, we have $(\psi^*(\mathcal{I}))_z \subset \mathcal{J}_z$, and, as a result, that $\psi^*(\mathcal{I}) \subset \mathcal{J}$. Thus θ^\sharp factors as

$$\psi^*(\mathcal{O}_X) \longrightarrow \psi^*(\mathcal{O}_X)/\psi^*(\mathcal{I}) = \psi^*(\mathcal{O}_X/\mathcal{I}) \xrightarrow{\omega} \mathcal{O}_Z,$$

the first arrow being the canonical homomorphism. Let ψ' be the continuous map $Z \rightarrow Y$ coinciding with ψ ; it is clear that we have $\psi'^*(\mathcal{O}_Y) = \psi^*(\mathcal{O}_X/\mathcal{I})$; on the other hand, ω is evidently a local homomorphism, so $g = (\psi', \omega^\flat)$ is a morphism of preschemes $Z \rightarrow Y$ (2.2.1), which, according to the above, is such that $f = j \circ g$, hence the proposition. \square

Corollary (4.1.10). — *For an injection morphism $Z \rightarrow X$ to be factor through the injection morphism $Y \rightarrow X$, it is necessary and sufficient for Z to be a subprescheme of Y .* I | 122

We then write $Z \leqslant Y$, and this condition is evidently an *ordering* on the set of subpreschemes of X .

⁸[Trans.] There doesn't seem to be an English equivalent of this, except for 'bounded above', which doesn't make much sense in this context. We would normally just say that ' f factors through j ', but to avoid having to entirely restructure the often-lengthy sentences in the original, we sometimes (but as little as we can) use 'majoré'.

4.2. Immersion morphisms.

Definition (4.2.1). — We say that a morphism $f : Y \rightarrow X$ is an immersion (resp. a closed immersion, an open immersion) if it factors as $Y \xrightarrow{g} Z \xrightarrow{j} X$, where g is an isomorphism, Z is a subprescheme of X (resp. a closed subprescheme, a subprescheme induced by an open set), and j is the injection morphism.

The subprescheme Z and the isomorphism g are then determined in a *unique* way, since if Z' is a second subprescheme of X , j' the injection $Z' \rightarrow X$, and g' an isomorphism $Y \rightarrow Z'$ such that $j \circ g = j' \circ g'$, then we have $j' = j \circ g \circ g'^{-1}$, hence $Z' \leq Z$ (4.1.10), and we can similarly show that $Z \leq Z'$, hence $Z' = Z$, and, since j is a monomorphism of preschemes, $g' = g$.

We say that $f = j \circ g$ is the *canonical factorization* of the immersion f , and the subprescheme Z and the isomorphism g are those *associated* to f .

It is clear that an immersion is a *monomorphism* of preschemes (4.1.7) and *a fortiori* a *radicial* morphism (3.5.4).

Proposition (4.2.2). —

- (a) For a morphism $f = (\psi, \theta) : Y \rightarrow X$ to be an open immersion, it is necessary and sufficient for ψ to be a homeomorphism from Y to some open subset of X , and, for all $y \in Y$, that the homomorphism $\theta_y^\# : \mathcal{O}_{\psi(y)} \rightarrow \mathcal{O}_y$ be bijective.
- (b) For a morphism $f = (\psi, \theta) : Y \rightarrow X$ to be an immersion (resp. a closed immersion), it is necessary and sufficient for ψ to be a homeomorphism from Y to some locally closed (resp. closed) subset of X , and, for all $y \in Y$, that the homomorphism $\theta_y^\# : \mathcal{O}_{\psi(y)} \rightarrow \mathcal{O}_y$ be surjective.

Proof.

- (a) The conditions are evidently necessary. Conversely, if they are satisfied, then it is clear that $\theta^\#$ is an isomorphism from \mathcal{O}_Y to $\psi^*(\mathcal{O}_X)$, and $\psi^*(\mathcal{O}_X)$ is the sheaf induced by “transport of structure” via ψ^{-1} from $\mathcal{O}_X|_{\psi(Y)}$; hence the conclusion.
- (b) The conditions are evidently necessary—we prove that they are sufficient. Consider first the particular case where we suppose that X is an affine scheme, and that $Z = \psi(Y)$ is *closed* in X . We then know (0, 3.4.6) that $\psi_*(\mathcal{O}_Y)$ has support equal to Z , and that, denoting its restriction to Z by \mathcal{O}'_Z , the ringed space (Z, \mathcal{O}'_Z) is induced from (Y, \mathcal{O}_Y) by transport of structure via the homeomorphism ψ considered as a map from Y to Z . Let us now show that $f_*(\mathcal{O}_Y) = \psi_*(\mathcal{O}_Y)$ is a *quasi-coherent* \mathcal{O}_X -module. Indeed, for all $x \notin Z$, $\psi_*(\mathcal{O}_Y)$ restricted to a suitable neighborhood of x is zero. On the contrary, if $z \in Z$, then we have $z = \psi(y)$ for some well-defined $y \in Y$; let V be an affine open neighborhood of y in Y ; $\psi(V)$ is then open in Z , and so equal to the intersection of Z with an open subset U of X , and the restriction of U to $\psi_*(\mathcal{O}_Y)$ is identical to the restriction of U to the direct image $(\psi_V)_*(\mathcal{O}_Y|_V)$, where ψ_V is the restriction of ψ to V . The restriction of the morphism (ψ, θ) to $(V, \mathcal{O}_Y|_V)$ is a morphism from this aforementioned prescheme to (X, \mathcal{O}_X) , and, as a result, is of the form $(^a\phi, \tilde{\phi})$, where ϕ is the homomorphism from the ring $A = \Gamma(X, \mathcal{O}_X)$ to the ring $\Gamma(V, \mathcal{O}_Y)$ (1.7.3); we conclude that $(\psi_V)_*(\mathcal{O}_Y|_V)$ is a quasi-coherent \mathcal{O}_X -module (1.6.3), which proves our assertion, due to the local nature of quasi-coherent sheaves. In addition, the hypothesis that ψ is a homeomorphism implies (0, 3.4.5) that, for all $y \in Y$, ψ_y is an isomorphism $(\psi^*(\mathcal{O}_Y))_{\psi(y)} \rightarrow \mathcal{O}_y$; since the diagram

$$\begin{array}{ccc} \mathcal{O}_{\psi(y)} & \xrightarrow{\theta_{\psi(y)}} & (\psi^*(\mathcal{O}_Y))_{\psi(y)} \\ \psi_y \circ \rho_{\psi(y)} \downarrow & & \downarrow \psi_y \\ (\psi^*(\mathcal{O}_X))_y & \xrightarrow{\theta_y^\#} & \mathcal{O}_y \end{array}$$

is commutative, and the vertical arrows are the isomorphisms (0, 3.7.2), the hypothesis that $\theta_y^\#$ is surjective implies that $\theta_{\psi(y)}$ is surjective as well. Since the support of $\psi_*(\mathcal{O}_Y)$ is $Z = \psi(Y)$, θ is a *surjective* homomorphism from $\mathcal{O}_X = \tilde{A}$ to the quasi-coherent \mathcal{O}_X -module $f_*(\mathcal{O}_Y)$. As

a result, there exists a unique isomorphism ω from a sheaf quotient $\tilde{A}/\tilde{\mathfrak{J}}$ (with \mathfrak{J} being an ideal of A) to $f_*(\mathcal{O}_Y)$ which, when composed with the canonical homomorphism $\tilde{A} \rightarrow \tilde{A}/\tilde{\mathfrak{J}}$, gives θ (1.3.8); if \mathcal{O}_Z denotes the restriction of $\tilde{A}/\tilde{\mathfrak{J}}$ to Z , then (Z, \mathcal{O}_Z) is a subprescheme of (X, \mathcal{O}_X) , and f factors through the canonical injection of this subprescheme into X and the isomorphism (ψ_0, ω_0) , where ψ_0 is ψ considered as a map from Y to Z , and ω_0 the restriction of ω to \mathcal{O}_Z .

We now pass to the general case. Let U be an affine open subset of X such that $U \cap \psi(Y)$ is closed in U and nonempty. By restricting f to the prescheme induced by Y on the open subset $\psi^{-1}(U)$, and by considering it as a morphism from this prescheme to the prescheme induced by X on U , we reduce to the first case; the restriction of f to $\psi^{-1}(U)$ is thus a closed immersion $\psi^{-1}(U) \rightarrow U$, canonically factoring as $j_U \circ g_U$, where g_U is an isomorphism from the prescheme $\psi^{-1}(U)$ to a subprescheme Z_U of U , and j_U is the canonical injection $Z_U \rightarrow U$. Let V be a second affine open subset of X such that $V \subset U$; since the restriction Z'_V of Z_U to V is a subprescheme of the prescheme V , the restriction of f to $\psi^{-1}(V)$ factors as $j'_V \circ g'_V$, where j'_V is the canonical injection $Z'_V \rightarrow V$ and g'_V is an isomorphism from $\psi^{-1}(V)$ to Z'_V . By the uniqueness of the canonical factorization of an immersion (4.2.1), we necessarily have that $Z'_V = Z_V$ and $g'_V = g_V$. We conclude (4.1.5) that there is a subprescheme Z of X whose underlying space is $\psi(Y)$, and whose restriction to each $U \cap \psi(Y)$ is Z_U ; the g_U are then the restrictions to $\psi^{-1}(U)$ of an isomorphism $g : Y \rightarrow Z$ such that $f = j \circ g$, where j is the canonical injection $Z \rightarrow X$.

□

Corollary (4.2.3). — *Let X be an affine scheme. For a morphism $f = (\psi, \theta) : Y \rightarrow X$ to be a closed immersion, it is necessary and sufficient for Y to be an affine scheme, and the homomorphism $\Gamma(\psi) : \Gamma(\mathcal{O}_X) \rightarrow \Gamma(\mathcal{O}_Y)$ to be surjective.*

Corollary (4.2.4). —

- (a) *Let f be a morphism $Y \rightarrow X$, and (V_λ) a cover of $f(Y)$ by open subsets of X . For f to be an immersion (resp. an open immersion), it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be an immersion (resp. an open immersion) into V_λ .*
- (b) *Let f be a morphism $Y \rightarrow X$, and (V_λ) an open cover of X . For f to be a closed immersion, it is necessary and sufficient for its restriction to each of the induced preschemes $f^{-1}(V_\lambda)$ to be a closed immersion into V_λ .*

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Proof. Let $f = (\psi, \theta)$; in the case (a), $\theta_y^\#$ is surjective (resp. bijective) for all $y \in Y$, and in the case (b), $\theta_y^\#$ is surjective for all $y \in Y$; it thus suffices to check, in case (a), that ψ is a homeomorphism from Y to a locally closed (resp. open) subset of X , and, in case (b), that ψ is a homeomorphism from Y to a closed subset of X . Now ψ is evidently injective, and sends each neighborhood of y in Y to a neighborhood of $\psi(y)$ in $\psi(Y)$ for all $y \in Y$, by virtue of the hypothesis; in case (a), $\psi(Y) \cap V_\lambda$ is locally closed (resp. open) in V_λ , so $\psi(Y)$ is locally closed (resp. open) in the union of the V_λ , and *a fortiori* in X ; in case (b), $\psi(Y) \cap V_\lambda$ is closed in V_λ , so $\psi(Y)$ is closed in X since $X = \bigcup_\lambda V_\lambda$. □

Proposition (4.2.5). — *The composition of any two immersions (resp. of two open immersions, of two closed immersions) is an immersion (resp. an open immersion, a closed immersion).*

Proof. This follows easily from (4.1.6). □

4.3. Products of immersions.

Proposition (4.3.1). — *Let $\alpha : X' \rightarrow X$, $\beta : Y' \rightarrow Y$ be two S -morphisms; if α and β are immersions (resp. open immersions, closed immersions), then $\alpha \times_S \beta$ is an immersion (resp. an open immersion, a closed immersion). In addition, if α (resp. β) identifies X' (resp. Y') with a subprescheme X'' (resp. Y'') of X (resp. Y), then $\alpha \times_S \beta$ identifies the underlying space of $X' \times_S Y'$ with the subspace $p^{-1}(X'') \cap q^{-1}(Y'')$ of the underlying space of $X \times_S Y$, where p and q denote the projections from $X \times_S Y$ to X and Y respectively.*

Proof. According to Definition (4.2.1), we can restrict to the case where X' and Y' are subpreschemes, and α and β the injection morphisms. The proposition has already been proven for the subpreschemes induced by open sets (3.2.7); since every subprescheme is a closed subprescheme

of a prescheme induced by an open set (4.1.3), we can reduce to the case where X' and Y' are *closed* subpreschemes.

Let us first show that we can assume S to be *affine*. Let (S_λ) be a cover of S by affine open sets; if ϕ and ψ are the structure morphisms of X and Y , then let $X_\lambda = \phi^{-1}(S_\lambda)$ and $Y_\lambda = \psi^{-1}(S_\lambda)$. The restriction X'_λ (resp. Y'_λ) of X' (resp. Y') to $X_\lambda \cap X'$ (resp. $Y_\lambda \cap Y'$) is a closed subprescheme of X_λ (resp. Y_λ), the preschemes $X_\lambda, Y_\lambda, X'_\lambda$, and Y'_λ can be considered as S_λ -preschemes, and the products $X_\lambda \times_S Y_\lambda$ and $X_\lambda \times_{S_\lambda} Y_\lambda$ (resp. $X'_\lambda \times_S Y'_\lambda$ and $X'_\lambda \times_{S_\lambda} Y'_\lambda$) are identical (3.2.5). If the proposition is true when S is affine, then the restriction of $\alpha \times_S \beta$ to each of the $X'_\lambda \times_S Y'_\lambda$ is thus an immersion (3.2.7). Since the product $X'_\lambda \times_S Y'_\mu$ (resp. $X_\lambda \times_S Y_\mu$) can be identified with $(X'_\lambda \cap X'_\mu) \times_S (Y'_\lambda \cap Y'_\mu)$ (resp. $(X_\lambda \cap X_\mu) \times_S (Y_\lambda \cap Y_\mu)$) (3.2.6.4), the restriction of $\alpha \times_S \beta$ to each of the $X'_\lambda \times_S Y'_\mu$ is also an immersion; the same is true for $\alpha \times_S \beta$ by (4.2.4). I | 125

Next, we show that we can assume X and Y to be *affine*. Indeed, let (U_i) (resp. (V_j)) be a cover of X (resp. Y) by affine open sets, and let X'_i (resp. Y'_j) be the restriction of X' (resp. Y') to $X' \cap U_i$ (resp. $Y' \cap V_j$), which is a closed subprescheme of U_i (resp. V_j); $U_i \times_S V_j$ can be identified with the restriction of $X \times_S Y$ to $p^{-1}(U_i) \cap q^{-1}(V_j)$ (3.2.7); similarly, if p' and q' are the projections from $X' \times_S Y'$, then $X'_i \times_S Y'_j$ can be identified with the restriction of $X' \times_S Y'$ to $p'^{-1}(X'_i) \cap q'^{-1}(Y'_j)$. Set $\gamma = \alpha \times_S \beta$; we have, by definition, $p \circ \gamma = \alpha \circ p'$ and $q \circ \gamma = \beta \circ q'$; since $X'_i = \alpha^{-1}(U_i)$ and $Y'_j = \beta^{-1}(V_j)$, we also have $p'^{-1}(X'_i) = \gamma^{-1}(p^{-1}(U_i))$ and $q'^{-1}(Y'_j) = \gamma^{-1}(q^{-1}(V_j))$, hence

$$p'^{-1}(X'_i) \cap q'^{-1}(Y'_j) = \gamma^{-1}(p^{-1}(U_i) \cap q^{-1}(V_j)) = \gamma^{-1}(U_i \times_S V_j),$$

and we then conclude as in the previous part of the proof.

So suppose X, Y , and S are affine, and let B, C , and A be their respective rings. Then B and C are A -algebras, and X' and Y' are affine schemes whose rings are quotient algebras B' and C' of B and C respectively. In addition, we have $\alpha = ({}^a\rho, \tilde{\rho})$ and $\beta = ({}^a\sigma, \tilde{\sigma})$, where ρ and σ are (respectively) the canonical homomorphisms $B \rightarrow B'$ and $C \rightarrow C'$ (1.7.3). With that in mind, we know that $X \times_S Y$ (resp. $X' \times_S Y'$) is an affine scheme with ring $B \otimes_A C$ (resp. $B' \otimes_A C'$), and $\alpha \times_S \beta = ({}^a\tau, \tilde{\tau})$, where τ is the homomorphism $\rho \otimes \sigma$ from $B \otimes_A C$ to $B' \otimes_A C'$ (Proposition (3.2.2) and Corollary (3.2.3)); since this homomorphism is surjective, $\alpha \times_S \beta$ is an immersion. In addition, if \mathfrak{b} (resp. \mathfrak{c}) is the kernel of ρ (resp. σ), then the kernel of τ is $u(\mathfrak{b}) + v(\mathfrak{c})$, where u (resp. v) is the homomorphism $b \mapsto b \otimes 1$ (resp. $c \mapsto 1 \otimes c$). Since $p = ({}^au, \tilde{u})$ and $q = ({}^av, \tilde{v})$, this kernel corresponds, in the prime spectrum of $B \otimes_A C$, to the closed set $p^{-1}(X') \cap q^{-1}(Y')$ ((1.2.2.1) and Proposition (1.1.2, iii)), which finishes the proof. □

Corollary (4.3.2). — *If $f : X \rightarrow Y$ is an immersion (resp. an open immersion, a closed immersion) and an S -morphism, then $f_{(S')}$ is an immersion (resp. an open immersion, a closed immersion) for every extension $S' \rightarrow S$ of the base prescheme.*

4.4. Inverse images of a subprescheme.

Proposition (4.4.1). — *Let $f : X \rightarrow Y$ be a morphism, Y' a subprescheme (resp. a closed subprescheme, a prescheme induced by an open set) of Y , and $j : Y' \rightarrow Y$ the injection morphism. Then the projection $p : X \times_Y Y' \rightarrow X$ is an immersion (resp. a closed immersion, an open immersion); the underlying space of the subprescheme of X associated to p is $f^{-1}(Y')$; in addition, if j' is the injection morphism of this subprescheme into X , then for a morphism $h : Z \rightarrow X$ to be such that $f \circ h : Z \rightarrow Y$ factors through j , it is necessary and sufficient for h to factor through j' .*

Proof. Since $p = 1_X \times_Y j$ (3.3.4), the first claim follows from Proposition (4.3.1); the second is a particular case of Corollary (3.5.10) (after swapping the roles of X and Y'). Finally, if we have $f \circ h = j \circ h'$, where h' is a morphism $Z \rightarrow Y'$, then it follows from the definition of the product that we have $h = p \circ u$, where u is a morphism $Z \rightarrow X \times_Y Y'$, whence the last claim. □

We say that the subprescheme of X thus defined is the *inverse image* of the subprescheme Y' of Y under the morphism f , terminology which is consistent with that introduced more generally in (3.3.6). When we speak of $f^{-1}(Y')$ as a subprescheme of X , this will always be the subprescheme we mean. I | 126

When the preschemes $f^{-1}(Y')$ and X are identical, j' is the identity and each morphism $h : Z \rightarrow X$ thus factors through j' , so the morphism $f : X \rightarrow Y$ factors as $X \xrightarrow{g} Y' \xrightarrow{j} Y$.

When y is a closed point of Y and $Y' = \text{Spec}(k(y))$ is the smallest closed subprescheme of Y having $\{y\}$ as its underlying space (4.1.9), the closed subprescheme $f^{-1}(Y')$ is canonically isomorphic to the fibre $f^{-1}(y)$ defined in (3.6.2), and we will use this identification in all that follows.

Corollary (4.4.2). — Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms, and $h = g \circ f$ their composition. For each subprescheme Z' of Z , the subpreschemes $f^{-1}(g^{-1}(Z'))$ and $h^{-1}(Z')$ of X are identical.

Proof. This follows from the existence of the canonical isomorphism $X \times_Y (Y \times_Z Z') \simeq X \times_Z Z'$ (3.3.9.1). \square

Corollary (4.4.3). — Let X' and X'' be subpreschemes of X , and $j' : X' \rightarrow X$, and $j'' : X'' \rightarrow X$ their injection morphisms; then $j'^{-1}(X'')$ and $j''^{-1}(X')$ are both equal to the greatest lower bound $\inf(X', X'')$ of X' and X'' for the ordering \leq on subpreschemes, and this is canonically isomorphic to $X' \times_X X''$.

Proof. This follows immediately from Proposition (4.4.1) and Corollary (4.1.10). \square

Corollary (4.4.4). — Let $f : X \rightarrow Y$ be a morphism, and Y' and Y'' subpreschemes of Y ; then we have $f^{-1}(\inf(Y', Y'')) = \inf(f^{-1}(Y'), f^{-1}(Y''))$.

Proof. This follows from the existence of the canonical isomorphism between $(X \times_Y Y') \times_X (X \times_Y Y'')$ and $X \times_Y (Y' \times_Y Y'')$ (3.3.9.1). \square

Proposition (4.4.5). — Let $f : X \rightarrow Y$ be a morphism, and Y' a closed subprescheme of Y defined by a quasi-coherent sheaf of ideals \mathcal{K} of \mathcal{O}_Y (4.1.3); the closed subprescheme $f^{-1}(Y')$ of X is then defined by the quasi-coherent sheaf of ideals $f^*(\mathcal{K})\mathcal{O}_X$ of \mathcal{O}_X .

Proof. The statement is evidently local on X and Y ; it thus suffices to note that if A is a B -algebra and \mathfrak{K} an ideal of B , then we have $A \otimes_B (B/\mathfrak{K}) = A/\mathfrak{K}A$, and to then apply (1.6.9). \square

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Corollary (4.4.6). — Let X' be a closed subprescheme of X defined by a quasi-coherent sheaf of ideals \mathcal{J} of \mathcal{O}_X , and i the injection $X' \rightarrow X$; for the restriction $f \circ i$ of f to X' to factor through the injection $j : Y' \rightarrow Y$ (in other words, for it to factor as $j \circ g$, with g a morphism $X' \rightarrow Y'$), it is necessary and sufficient that $f^*(\mathcal{K}) \subset \mathcal{J}$.

Proof. It suffices to apply Proposition (4.4.1) to i , taking Proposition (4.4.5) into account. \square

4.5. Local immersions and local isomorphisms.

Definition (4.5.1). — Let $f : X \rightarrow Y$ be a morphism of preschemes. We say that f is a local immersion at a point $x \in X$ if there exists an open neighborhood U of x in X and an open neighborhood V of $f(x)$ in Y such that the restriction of f to the induced prescheme U is a closed immersion of U into the induced prescheme V . We say that f is a local immersion if f is a local immersion at each point of X .

Definition (4.5.2). — We say that a morphism $f : X \rightarrow Y$ is a local isomorphism at a point $x \in X$ if there exists an open neighborhood U of x in X such that the restriction of f to the induced prescheme U is an open immersion of U into Y . We say that f is a local isomorphism if f is a local isomorphism at each point of X .

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(4.5.3). An immersion (resp. a closed immersion) $f : X \rightarrow Y$ can be characterized as a local immersion such that f is a homeomorphism from the underlying space of X to a subset (resp. a closed subset) of Y . An open immersion f can be characterized as an *injective* local isomorphism.

Proposition (4.5.4). — Let X be an irreducible prescheme, and $f : X \rightarrow Y$ a dominant injective morphism. If f is a local immersion, then f is an immersion, and $f(X)$ is open in Y .

Proof. Let $x \in X$, and let U be an open neighborhood of x , and V an open neighborhood of $f(x)$ in Y such that the restriction of f to U is a closed immersion into V ; since U is dense in X , $f(U)$ is dense in Y by hypothesis, so $f(U) = V$, and f is a homeomorphism from U to V ; the hypothesis that f is injective implies that $f^{-1}(V) = U$, hence the proposition. \square

Proposition (4.5.5). —

- (i) *The composition of any two local immersions (resp. of two local isomorphisms) is a local immersion (resp. a local isomorphism).*
- (ii) *Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be two S -morphisms. If f and g are local immersions (resp. local isomorphisms), then so too is $f \times_S g$.*
- (iii) *If an S -morphism f is a local immersion (resp. a local isomorphism), then so too is $f_{(S')}$ for every extension $S' \rightarrow S$ of the base prescheme.*

Proof. According to (3.5.1), it suffices to prove (i) and (ii).

(i) follows immediately from the transitivity of closed (resp. open) immersions (4.2.5) and from the fact that if f is a homeomorphism from X to a closed subset of Y , then for every open $U \subset X$, $f(U)$ is open in $f(X)$, so there exists an open subset V of Y such that $f(U) = V \cap f(X)$, and, as a result, $f(U)$ is closed in V .

To prove (ii), let p and q be the projections from $X \times_X Y$, and p' and q' the projections from $X' \times_S Y'$. There exists, by hypothesis, open neighborhoods U, U', V , and V' of $x = p(z), x' = p'(z'), y = q(z)$, and $y' = q'(z')$ (respectively), such that the restrictions of f and g to U and V (respectively) are closed (resp. open) immersions into U' and V' (respectively). Since the underlying spaces of $U \times_S V$ and $U' \times_S V'$ can be identified with the open neighborhoods $p^{-1}(U) \cap q^{-1}(V)$ and $p'^{-1}(U') \cap q'^{-1}(V')$ of z and z' (respectively) (3.2.7), the proposition follows from Proposition (4.3.1). \square

§5. REDUCED PRESCHMES; SEPARATION CONDITION

5.1. Reduced preschemes.

Proposition (5.1.1). — *Let (X, \mathcal{O}_X) be a prescheme, and \mathcal{B} a quasi-coherent \mathcal{O}_X -algebra. Then there exists a unique quasi-coherent \mathcal{O}_X -module \mathcal{N} whose stalk \mathcal{N}_x at any $x \in X$ is the nilradical of the ring \mathcal{B}_x . When X is affine, and, consequently, $\mathcal{B} = \tilde{B}$, where B is an algebra over $A(X)$, then we have $\mathcal{N} = \tilde{\mathfrak{N}}$, where \mathfrak{N} is the nilradical of B .*

Proof. The statement is local, so it suffices to show the latter claim. We know that $\tilde{\mathfrak{N}}$ is a quasi-coherent \mathcal{O}_X -module (1.4.1), and that its stalk at a point $x \in X$ is the ideal \mathfrak{N}_x of the ring of fractions B_x ; it remains to prove that the nilradical of B_x is contained in \mathfrak{N}_x , the converse inclusion being evident. Let z/s be an element of the nilradical of B_x , with $z \in B$, and $s \notin \mathfrak{j}_x$; by hypothesis, there exists an integer k such that $(z/s)^k = 0$, which implies that there exists some $t \notin \mathfrak{j}_x$ such that $tz^k = 0$. We conclude that $(tz)^k = 0$, and, as a result, that $z/s = (tz)/(ts) \in \mathfrak{N}_x$. \square

We say that the quasi-coherent \mathcal{O}_X -module \mathcal{N} thus defined is the *nilradical* of the \mathcal{O}_X -algebra \mathcal{B} ; in particular, we denote by \mathcal{N}_X the nilradical of \mathcal{O}_X .

Corollary (5.1.2). — *Let X be a prescheme; the closed subscheme of X defined by the sheaf of ideals \mathcal{N}_X is the only reduced subscheme (0, 4.1.4) of X that has X as its underlying space; it is also the smallest subscheme of X that has X as its underlying space.*

Proof. Since the structure sheaf of the closed subscheme of Y defined by \mathcal{N}_X is $\mathcal{O}_X / \mathcal{N}_X$, it is immediate that Y is reduced and has X as its underlying space, because $\mathcal{N}_x \neq \mathcal{O}_x$ for any $x \in X$. To show the other claims, note that a subscheme Z of X that has X as its underlying space is defined by a sheaf of ideals \mathcal{I} (4.1.3) such that $\mathcal{I}_x \neq \mathcal{O}_x$ for any $x \in X$. We can restrict to the case where X is affine, say $X = \text{Spec}(A)$ and $\mathcal{I} = \tilde{\mathfrak{I}}$, where \mathfrak{I} is an ideal of A ; then, for every $x \in X$, we have $\mathfrak{I}_x \subset \mathfrak{j}_x$, and so \mathfrak{I} is contained in every prime ideal of A , and so also in their intersection \mathfrak{N} , the nilradical of A . This proves that Y is the small subscheme of X that has X as its underlying space (4.1.9); furthermore, if Z is distinct from Y , we necessarily have $\mathcal{I}_x \neq \mathcal{N}_x$ for at least one $x \in X$, and so (5.1.1) Z is not reduced. \square

Definition (5.1.3). — We define the reduced prescheme associated to a prescheme X , denoted by X_{red} , to be the unique reduced subscheme of X that has X as its underlying space.

Saying that a prescheme X is reduced thus implies that $X = X_{\text{red}}$.

Proposition (5.1.4). — *For the prime spectrum of a ring A to be a reduced (resp. integral) prescheme (2.1.7), it is necessary and sufficient for A to be a reduced (resp. integral) ring.*

Proof. Indeed, it follows immediately from (5.1.1) that the condition $\mathcal{N} = (0)$ is necessary and sufficient for $X = \operatorname{Spec}(A)$ to be reduced; the claim corresponding to integral rings is then a consequence of (1.1.13). \square

Since every ring of fractions $\neq \{0\}$ of an integral ring is integral, it follows from (5.1.4) that, for every *locally integral* prescheme X , \mathcal{O}_x is an integral ring for every $x \in X$. The converse is true whenever the underlying space of X is *locally Noetherian*: indeed, X is then reduced, and if U is an affine open subset of X , which is a Noetherian space, then U has only a finite number of irreducible components, and so its ring A has only a finite number of minimal prime ideals (1.1.14). If two of the irreducible components of U had a common point x , then \mathcal{O}_x would have at least two distinct minimal prime ideals, and would thus not be integral; the components of U are thus open subsets that are pairwise disjoint, and each of them is thus integral.

(5.1.5). Let $f = (\psi, \theta) : X \rightarrow Y$ be a morphism of preschemes; the homomorphism $\theta_x^\sharp : \mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$ sends each nilpotent element of $\mathcal{O}_{\psi(x)}$ to a nilpotent element of \mathcal{O}_x ; by passing to the quotients, θ_x^\sharp induces a homomorphism $\omega : \psi^*(\mathcal{O}_Y/\mathcal{N}_Y) \rightarrow \mathcal{O}_X/\mathcal{N}_X$. I | 129

$$\omega : \psi^*(\mathcal{O}_Y/\mathcal{N}_Y) \rightarrow \mathcal{O}_X/\mathcal{N}_X.$$

It is clear that, for every $x \in X$, $\omega_x : \mathcal{O}_{\psi(x)}/\mathcal{N}_{\psi(x)} \rightarrow \mathcal{O}_x/\mathcal{N}_x$ is a local homomorphism, and so (ψ, ω^\flat) is a morphism of preschemes $X_{\text{red}} \rightarrow Y_{\text{red}}$, which we denote by f_{red} , and call the *reduced* morphism associated to f . It is immediate that, for morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$, we have $(g \circ f)_{\text{red}} = g_{\text{red}} \circ f_{\text{red}}$, and so we have defined X_{red} as a *functor, covariant* in X .

The preceding definition shows that the diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y \end{array}$$

is commutative, where the vertical arrows are the injection morphisms; in other words, $X_{\text{red}} \rightarrow X$ is a *functorial* morphism. We note in particular that, if X is reduced, then every morphism $f : X \rightarrow Y$ factors as $X \xrightarrow{f_{\text{red}}} Y_{\text{red}} \rightarrow Y$; in other words, f factors through the injection morphism $Y_{\text{red}} \rightarrow Y$.

Proposition (5.1.6). — *Let $f : X \rightarrow Y$ be a morphism; if f is surjective (resp. radicial, an immersion, a closed immersion, an open immersion, a local immersion, a local isomorphism), then so too is f_{red} . Conversely, if f_{red} is surjective (resp. radicial), then so too is f .*

Proof. The proposition is trivial if f is surjective; if f is radicial, then the proposition follows from the fact that, for every $x \in X$, the field $k(x)$ is the same for the preschemes X and X_{red} (3.5.8). Finally, if $f = (\psi, \theta)$ is an immersion, a closed immersion, or a local immersion (resp. an open immersion, or a local isomorphism), then the proposition follows from the fact that, if θ_x^\sharp is surjective (resp. bijective), then so too is the homomorphism obtained by passing to the quotients by the nilradicals $\mathcal{O}_{\psi(x)}$ and \mathcal{O}_x ((5.1.2) and (4.2.2)) (cf. (5.5.12)). \square

Proposition (5.1.7). — *If X and Y are S -preschemes, then the preschemes $X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}}$ and $X_{\text{red}} \times_S Y_{\text{red}}$ are identical, and canonically identified with a subscheme of $X \times_S Y$ that has the same underlying subspace as the two aforementioned products.*

Proof. The canonical identification of $X_{\text{red}} \times_S Y_{\text{red}}$ with a subscheme of $X \times_S Y$ that has the same underlying space follows from (4.3.1). Furthermore, if ϕ and ψ are the structure morphisms $X_{\text{red}} \rightarrow S$ and $Y_{\text{red}} \rightarrow S$ (respectively), then they factor through S_{red} (5.1.5), and since $S_{\text{red}} \rightarrow S$ is a monomorphism, the first claim of the proposition follows from (3.2.4). \square

Corollary (5.1.8). — *The preschemes $(X \times_S Y)_{\text{red}}$ and $(X_{\text{red}} \times_{S_{\text{red}}} Y_{\text{red}})_{\text{red}}$ are canonically identified with one another.*

Proof. This follows from (5.1.2) and (5.1.7). \square

We note that, even if X and Y are reduced preschemes, $X \times_S Y$ might not be reduced, because the tensor product of two reduced algebras can have nilpotent elements.

Proposition (5.1.9). — *Let X be a prescheme, and \mathcal{I} a quasi-coherent sheaf of ideals of \mathcal{O}_X such that $\mathcal{I}^n = 0$ for some integer $n > 0$. Let X_0 be the closed subscheme $(X, \mathcal{O}_X/\mathcal{I})$ of X ; for X to be an affine scheme, it is necessary and sufficient for X_0 to be an affine scheme.*

The condition is clearly necessary, so we will show that it is sufficient. If we set $X_k = (X, \mathcal{O}_X/\mathcal{I}^{k+1})$, it is enough to prove by induction on k that X_k is affine, and so we are led to consider the base case, where $\mathcal{I}^2 = 0$. We set

$$\begin{aligned} A &= \Gamma(X, \mathcal{O}_X) \\ A_0 &= \Gamma(X_0, \mathcal{O}_{X_0}) = \Gamma(X, \mathcal{O}_X/\mathcal{I}). \end{aligned}$$

The canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{I}$ induces a homomorphism of rings $\phi : A \rightarrow A_0$. We will see below that ϕ is *surjective*, which implies that

$$(5.1.9.1) \quad 0 \longrightarrow \Gamma(X, \mathcal{I}) \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X, \mathcal{O}_X/\mathcal{I}) \longrightarrow 0$$

is an *exact* sequence. We now prove, assuming that this is true, the proposition. Note that $\mathfrak{K} = \Gamma(X, \mathcal{I})$ is an ideal whose square is zero in A , and thus a module over $A_0 = A/\mathfrak{K}$. By hypothesis, we have $X_0 = \text{Spec}(A)$, and, since the underlying spaces of X_0 and X are identical, $\mathfrak{K} = \Gamma(X_0, \mathcal{I})$; Additionally, since $\mathcal{I}^2 = 0$, \mathcal{I} is a quasi-coherent $(\mathcal{O}_X/\mathcal{I})$ -module, so we have $\mathcal{I} \cong \tilde{\mathfrak{K}}$ and $\mathfrak{K}_x = \mathcal{I}_x$ for all $x \in X_0$ (1.4.1). With this in mind, let $X' = \text{Spec}(A)$, and consider the morphism $f = (\psi, \theta) : X \rightarrow X'$ of preschemes that corresponds to the identity map $A \rightarrow \Gamma(X, \mathcal{O}_X)$ (2.2.4). For every affine open subset V of X , the diagram

$$\begin{array}{ccc} A & \longrightarrow & \Gamma(V, \mathcal{O}_X|V) \\ \downarrow & & \downarrow \\ A_0 = A/\mathfrak{K} & \longrightarrow & \Gamma(V, \mathcal{O}_{X_0}|V) \end{array}$$

commutes, whence the diagram

$$\begin{array}{ccc} X' & \xleftarrow{f} & X \\ j' \uparrow & & \uparrow j \\ X'_0 & \xleftarrow{f_0} & X_0 \end{array}$$

also commutes (X'_0 being the closed subscheme of X' defined by the quasi-coherent sheaf of ideals $\tilde{\mathfrak{K}}$, and j and j' being the canonical injection morphisms). But since X_0 is affine, f_0 is an isomorphism, and since the underlying continuous maps of j and j' are identity maps, we see straight away that $\psi : X \rightarrow X'$ is a homeomorphism. Furthermore, the equation $\mathfrak{K}_x = \mathcal{I}_x$ shows that the restriction of $\theta^\# : \psi^*(\mathcal{O}_{X'}) \rightarrow \mathcal{O}_X$ is an *isomorphism* from $\psi^*(\tilde{\mathfrak{K}})$ to \mathcal{I} ; additionally, by passing to the quotients, $\theta^\#$ gives an *isomorphism* $\psi^*(\mathcal{O}_X/\tilde{\mathfrak{K}}) \rightarrow \mathcal{O}_X/\mathcal{I}$, because f_0 is an isomorphism; we thus immediately conclude, by the 5 lemma (M, I, 1.1), that $\theta^\#$ is itself an isomorphism, and thus that f is an *isomorphism*, and thus that X is affine. So everything reduces to proving the exactitude of (5.1.9.1), which will follow from showing that $H^1(X, \mathcal{I}) = 0$. But $H^1(X, \mathcal{I}) = H^1(X_0, \mathcal{I})$, and we have seen that \mathcal{I} is a quasi coherent \mathcal{O}_{X_0} -module. Our proof will thus follow from

Lemma (5.1.9.2). — *If Y is an affine scheme, and \mathcal{F} a quasi-coherent \mathcal{O}_Y -module, then $H^1(Y, \mathcal{F}) = 0$.*

Proof. This lemma will be proven in Chapter III, Â§1, as a consequence of the more general theorem that $H^i(Y, \mathcal{F}) = 0$ for all $i > 0$. To give an independent proof, note that $H^1(Y, \mathcal{F})$ can be identified with the module $\text{Ext}_{\mathcal{O}_Y}^1(Y; \mathcal{O}_Y, \mathcal{F})$ of extensions classes of the \mathcal{O}_Y -module \mathcal{O}_Y by the \mathcal{O}_Y -module \mathcal{F} (T, 4.2.3); so everything reduces to proving that such an extension \mathcal{G} is trivial. But, for all $y \in Y$, there is a neighbourhood V of y in Y such that $\mathcal{G}|V$ is isomorphic to $\mathcal{F}|V \oplus \mathcal{O}_Y|V$ (0, 5.4.9); from this we conclude that \mathcal{G} is a *quasi-coherent* \mathcal{O}_Y -module. If A is the ring of Y , then we have $\mathcal{F} = \tilde{M}$ and $\mathcal{G} = \tilde{N}$, where M and N are A -modules, and, by hypothesis, N is an extension of the A -module A by the A -module M (1.3.11). Since this extension is necessarily trivial, the lemma is proven, and thus so too is (5.1.9). \square

Corollary (5.1.10). — *Let X be a prescheme such that \mathcal{N}_X is nilpotent. For X to be an affine scheme, it is necessary and sufficient for X_{red} to be an affine scheme.*

5.2. Existence of a subprescheme with a given underlying space.

Proposition (5.2.1). — *For every locally closed subspace Y of the underlying space of a prescheme X , there exists exactly one reduced subprescheme of X that has Y as its underlying space.*

Proof. The uniqueness follows from (5.1.2), so it remains only to show the existence of the prescheme in question.

If X is affine, given by some ring A , and Y closed in X , then the proposition is immediate: $j(Y)$ is the largest ideal $\mathfrak{a} \subset A$ such that $V(\mathfrak{a}) = Y$, and it is equal to its radical (1.1.4, i), so $A/j(Y)$ is a reduced ring.

In the general case, for every affine open $U \subset X$ such that $U \cap Y$ is closed in U , consider the closed subprescheme Y_U of U defined by the sheaf of ideals associated to the ideal $j(U \cap Y)$ of $A(U)$, which is reduced. We can show that, if V is an affine open subset of X contained in U , then Y_V is induced by Y_U on $V \cap Y$; but this induced prescheme is a closed subprescheme (of V) which is reduced and has $V \cap Y$ as its underlying space; the uniqueness of Y_V thus implies our claim. \square

Proposition (5.2.2). — *Let X be a reduced subprescheme of a prescheme Y ; if Z is the closed reduced subprescheme of Y that has \bar{X} as its underlying space, then X is a subprescheme induced on an open subset of Z .*

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Proof. There is indeed an open subset U of Y such that $X = U \cap \bar{X}$; since, by (5.2.2), X is a reduced subprescheme of Z , the subprescheme X is induced by Z on the open subspace X by uniqueness (5.2.1). \square

Corollary (5.2.4). — *Let $f : X \rightarrow Y$ be a morphism, and X' (resp. Y') a closed subprescheme of X (resp. Y) defined by a quasi-coherent sheaf of ideals \mathcal{J} (resp. \mathcal{K}) of \mathcal{O}_X (resp. \mathcal{O}_Y). Suppose that X' is reduced, and that $f(X') \subset Y'$. Then $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{J}$.*

Proof. Since, by (5.2.2), the restriction of f to X' factors as $X' \rightarrow Y' \rightarrow Y$, it suffices to apply (4.4.6). \square

5.3. Diagonal; graph of a morphism.

(5.3.1). Let X be an S -prescheme; we define the *diagonal morphism* of X in $X \times_S X$, denoted by $\Delta_{X|S}$, or Δ_X , or even Δ if no confusion may arise, to be the S -morphism $(1_X, 1_X)_S$, or, in other words, the unique S -morphism Δ_X such that

$$(5.3.1.1) \quad p_1 \circ \Delta_X = p_2 \circ \Delta_X = 1_X,$$

where p_1 and p_2 are the projections of $X \times_S X$ (Definition (3.2.1)). If $f : T \rightarrow X$ and $g : T \rightarrow Y$ are S -morphisms, we immediately have that

$$(5.3.1.2) \quad (f, g)_S = (f \times_S g) \circ \Delta_{T|S}.$$

The reader will note that the preceding definition and the results stated in (5.3.1) to (5.3.8) are true in any category, provided that the products used within exist in the category.

Proposition (5.3.2). — *Let X and Y be S -preschemes; if we make the canonical identification between $(X \times Y) \times (X \times Y)$ and $(X \times X) \times (Y \times Y)$, then the morphism $\Delta_{X \times Y}$ is identified with $\Delta_X \times \Delta_Y$.*

Proof. Indeed, if $p_1 : X \times X \rightarrow X$ and $q_1 : Y \times Y \rightarrow Y$ are the projections onto the first component, then the projection onto the first component $(X \times Y) \times (X \times Y) \rightarrow X \times Y$ is identified with $p_1 \times q_1$, and we have

$$(p_1 \times q_1) \circ (\Delta_X \times \Delta_Y) = (p_1 \circ \Delta_X) \times (q_1 \circ \Delta_Y) = 1_{X \times Y}$$

and we can argue similarly for the projections onto the second component. \square

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Corollary (5.3.4). — *For every extension $S' \rightarrow S$ of the base prescheme, $\Delta_{X_{S'}}$ is canonically identified with $(\Delta_X)_{(S')}$.*

Proof. It suffices to remark that $(X \times_S X)_{(S')}$ is canonically identified with $X_{(S')} \times_{S'} X_{(S')}$ (3.3.10). \square

Proposition (5.3.5). — *Let X and Y be S -preschemes, and $\phi : S \rightarrow T$ a morphism of preschemes, which lets us consider every S -prescheme as a T -prescheme. Let $f : X \rightarrow S$ and $g : Y \rightarrow S$ be the structure morphisms, p and q the projections of $X \times_S Y$, and $\pi = f \circ p = g \circ q$ the structure morphism $X \times_S Y \rightarrow S$. Then the diagram*

$$(5.3.5.1) \quad \begin{array}{ccc} X \times_S Y & \xrightarrow{(p,q)_T} & X \times_T Y \\ \pi \downarrow & & \downarrow f \times_T g \\ S & \xrightarrow{\Delta_{S|T}} & S \times_T S \end{array}$$

commutes, and identifies $X \times_S Y$ with the product of the $(S \times_T S)$ -preschemes S and $X \times_T Y$, and the projections with π and $(p, q)_T$. I | 133

Proof. By (3.4.3), we are led to proving the corresponding proposition in the category of sets, replacing X, Y , and S by $X(Z)_T, Y(Z)_T$, and $S(Z)_T$ (respectively), with Z being an arbitrary T -prescheme. But, in the category of sets, the proof is immediate and left to the reader. \square

Corollary (5.3.6). — *The morphism $(p, q)_T$ can be identified (letting $P = S \times_T S$) with $1_{X \times_T Y} \times_P \Delta_S$.*

Proof. This follows from (5.3.5) and (3.3.4). \square

Corollary (5.3.7). — *If $f : X \rightarrow Y$ is an S -morphism, then the diagram*

$$\begin{array}{ccc} X & \xrightarrow{(1_X, f)_S} & X \times_S Y \\ f \downarrow & & \downarrow f \times_S 1_Y \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y \end{array}$$

commutes, and identifies X with the product of the $(Y \times_S Y)$ -preschemes Y and $X \times_S Y$.

Proof. It suffices to apply (5.3.5), replacing S by Y , and T by S , and noting that $X \times_Y Y = X$ (3.3.3). \square

Proposition (5.3.8). — *For $f : X \rightarrow Y$ to be a monomorphism of preschemes, it is necessary and sufficient for $\Delta_{X|Y}$ to be an isomorphism from X to $X \times_Y X$.*

Proof. Indeed, to say that f is a monomorphism implies that, for every Y -prescheme Z , the corresponding map $f' : X(Z)_Y \rightarrow Y(Z)_Y$ is an injection, and, since $Y(Z)_Y$ consists of a single element, this implies that $X(Z)_Y$ consists of a single element as well. But this can also be expressed by saying that $X(Z)_Y \times X(Z)_Y$ is canonically isomorphic to $X(Z)_Y$; the former is exactly the set $(X \times_Y X)(Z)_Y$ (3.4.3.1), which implies that $\Delta_{X|Y}$ is an isomorphism. \square

Proposition (5.3.9). — *The diagonal morphism Δ_X is an immersion from X to $X \times_S X$.*

Proof. Indeed, since the continuous maps p_1 and Δ_X from the underlying spaces are such that $p_1 \circ \Delta_X$ is the identity, Δ_X is a homeomorphism from X to $\Delta_X(X)$. Similarly, the composite homomorphism $\mathcal{O}_X \rightarrow \mathcal{O}_{\Delta_X(X)} \rightarrow \mathcal{O}_X$ (composed of the homomorphisms corresponding to p_1 and Δ_X) is the identity, which means that the homomorphism corresponding to Δ_X is surjective; the proposition thus follows from (4.2.2). \square

We say that the subprescheme of $X \times_S X$ associated to the immersion Δ_X (4.2.1) is the *diagonal* of $X \times_S X$.

Corollary (5.3.10). — *Under the hypotheses of (5.3.5), $(p, q)_T$ is an immersion.*

Proof. This follows from (5.3.6) and (4.3.1). \square

We say (under the hypotheses of (5.3.5)) that $(p, q)_T$ is the *canonical immersion* of $X \times_S Y$ into $X \times_T Y$.

Corollary (5.3.11). — Let X and Y be S -preschemes, and $f : X \rightarrow Y$ an S -morphism; then the graph morphism $\Gamma_f = (1_X, f)_S$ of f (3.3.14) is an immersion of X into $X \times_S Y$.

Proof. This is a particular case of Corollary (5.3.10), where we replace S by Y , and T by S (cf. (5.3.7)). \square

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The subprescheme of $X \times_S Y$ associated to the immersion Γ_f (4.2.1) is called *the graph* of the morphism f ; the subpreschemes of $X \times_S Y$ that are graphs of morphisms $X \rightarrow Y$ are characterised by the property that the restriction to such a subprescheme G of the projection $p_1 : X \times_S Y \rightarrow X$ is an isomorphism g from G to X : G is the graph of the morphism $p_2 \circ g^{-1}$, where p_2 is the projection $X \times_S Y \rightarrow Y$.

When we take, in particular, $X = S$, then the S -morphisms $S \rightarrow Y$ (which are exactly the S -sections of Y (2.5.5)) are equal to their graph morphisms; the subpreschemes of Y that are the graphs of S -sections (in other words, those that are isomorphic to S by the restriction of the structure morphism $Y \rightarrow S$) are then also called the *images* of these sections, or, by an abuse of language, the S -sections of Y .

Corollary (5.3.12). — With the hypotheses and notation of (5.3.11), for every morphism $g : S' \rightarrow S$, let f' be the inverse image of f under g (3.3.7); then $\Gamma_{f'}$ is the inverse image of Γ_f under g .

Proof. This is a particular case of (3.3.10.1). \square

Corollary (5.3.13). — Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms; if $g \circ f$ is an immersion (resp. a local immersion), then so too is f .

Proof. Indeed, f factors as $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$. Furthermore, p_2 can be identified with $(g \circ f) \times_Z 1_Y$ (3.3.4); if $g \circ f$ is an immersion (resp. a local immersion), then so too is p_2 ((4.3.1) and (4.5.5)), and since Γ_f is an immersion (5.3.11), we are done, by (4.2.4) (resp. (4.5.5)). \square

Corollary (5.3.14). — Let $j : X \rightarrow Y$ and $g : Y \rightarrow Z$ be S -morphisms. If j is an immersion (resp. a local immersion), then so too is $(j, g)_S$.

Proof. Indeed, if $p : Y \times_S Z \rightarrow Y$ is the projection onto the first component, then we have $j = p \circ (j, g)_S$, and it suffices to apply (5.3.13). \square

Proposition (5.3.15). — If $f : X \rightarrow Y$ is an S -morphism, then the diagram

$$(5.3.15.1) \quad \begin{array}{ccc} X & \xrightarrow{\Delta_X} & X \times_S X \\ f \downarrow & & \downarrow f \times_S f \\ Y & \xrightarrow{\Delta_Y} & Y \times_S Y \end{array}$$

commutes (in other words, Δ_X is a functorial morphism in the category of preschemes).

Proof. The proof is immediate and left to the reader. \square

Corollary (5.3.16). — If X is a subprescheme of Y , then the diagonal $\Delta_X(X)$ can be identified with a subprescheme of $\Delta_Y(Y)$, and the underlying space can be identified with

$$\Delta_Y(Y) \cap p_1^{-1}(X) = \Delta_Y \cap p_2^{-1}(X)$$

(p_1 and p_2 being the projections of $Y \times_S Y$).

Proof. Applying (5.3.15) to the injection morphism $f : X \rightarrow Y$, we see that $f \times_S f$ is an immersion that identifies the underlying space of $X \times_S X$ with the subspace $p_1^{-1}(X) \cap p_2^{-1}(X)$ of $Y \times_S Y$ (4.3.1); further, if $z \in \Delta_Y(Y) \cap p_1^{-1}(X)$, then we have $z = \Delta_Y(y)$ and $y = p_1(z) \in X$, so $y = f(y)$, and $z = \Delta_Y(f(y))$ belongs to $\Delta_X(X)$ by the commutativity of (5.3.15.1). \square

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Corollary (5.3.17). — Let $f_1 : Y \rightarrow X$ and $f_2 : Y \rightarrow X$ be S -morphisms, and y a point of Y such that $f_1(y) = f_2(y) = x$, and such that the homomorphisms $k(x) \rightarrow k(y)$ corresponding to f_1 and f_2 are identical. Then, if $f = (f_1, f_2)_S$, the point $f(y)$ belongs to the diagonal $\Delta_{X|S}(X)$.

Proof. The two homomorphisms $k(x) \rightarrow k(y)$ corresponding to f_i ($i = 1, 2$) define two S -morphisms $g_i : \text{Spec}(k(y)) \rightarrow \text{Spec}(k(x))$ such that the diagrams

$$\begin{array}{ccc} \text{Spec}(k(y)) & \xrightarrow{g_i} & \text{Spec}(k(x)) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{f_i} & X \end{array}$$

commute. The diagram

$$\begin{array}{ccc} \text{Spec}(k(y)) & \xrightarrow{(g_1, g_2)_S} & \text{Spec}(k(x)) \times_S \text{Spec}(k(x)) \\ \downarrow & & \downarrow \\ Y & \xrightarrow{(f_1, f_2)_S} & X \times_S X \end{array}$$

thus also commutes. But it follows from the equality $g_1 = g_2$ that the image under $(g_1, g_2)_S$ of the unique point of $\text{Spec}(k(y))$ belongs to the diagonal of $\text{Spec}(k(x)) \times_S \text{Spec}(k(x))$; the conclusion then follows from (5.3.15). \square

5.4. Separated morphisms and separated preschemes.

Definition (5.4.1). — We say that a morphism of preschemes $f : X \rightarrow Y$ is *separated* if the diagonal morphism $X \rightarrow X \times_Y X$ is a *closed* immersion; we then also say that X is a *separated prescheme* over Y , or a Y -*scheme*. We say that a prescheme X is separated if it is separated over $\text{Spec}(\mathbf{Z})$; we then also say that X is a *scheme*⁹ (cf. (5.5.7)).

By (5.3.9), for X to be separated over Y , it is necessary and sufficient for $\Delta_X(X)$ to be a *closed* subspace of the underlying space of $X \times_Y X$.

Proposition (5.4.2). — Let $S \rightarrow T$ be a separated morphism. If X and Y are S -preschemes, then the canonical immersion $X \times_S Y \rightarrow X \times_T Y$ (5.3.10) is closed.

Proof. Indeed, if we refer to the diagram in (5.3.5.1), we see that $(p, q)_T$ can be considered as being obtained from $\Delta_{S|T}$ by the extension $f \times_T g : X \times_T Y \rightarrow S \times_T S$ of the base prescheme $S \times_T S$; the proposition then follows from (4.3.2). \square

Corollary (5.4.3). — Let Y be an S -scheme, and $f : X \rightarrow Y$ an S -morphism. Then the graph morphism $\Gamma_f : X \rightarrow X \times_S Y$ (5.3.11) is a closed immersion.

Proof. This is a particular case of (5.4.2), where we replace S by Y , and T by S . \square

Corollary (5.4.4). — Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms, with g separated. If $g \circ f$ is a closed immersion, then so too is f .

Proof. The proof using (5.4.3) is the same as that of (5.3.13) using (5.3.11). \square

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Corollary (5.4.5). — Let Z be an S -scheme, and $j : X \rightarrow Y$ and $g : X \rightarrow Z$ S -morphisms. If j is a closed immersion, then so too is $(j, g)_S : X \rightarrow Y \times_S Z$.

Proof. The proof using (5.4.4) is the same as that of (5.3.14) using (5.3.13). \square

Corollary (5.4.6). — If X is an S -scheme, then every S -section of X (2.5.5) is a closed immersion.

Proof. If $\phi : X \rightarrow S$ is the structure morphism, and $\psi : S \rightarrow X$ an S -section of X , it suffices to apply (5.4.5) to $\phi \circ \psi = 1_S$. \square

Corollary (5.4.7). — Let X be an integral prescheme with generic point s , and X an S -scheme. If two S -sections f and g are such that $f(s) = g(s)$, then $f = g$.

⁹[Trans.] We repeat here the warning given at the very start of this translation: the early versions of the EGA use *prescheme* to mean is now usually called a *scheme*, and *scheme* for what is now usually called a *separated scheme*. Grothendieck himself later said that the more modern terminology was preferable, but we have decided to keep this translation ‘historically accurate’ by using the older nomenclature.

Proof. Indeed, if $x = f(s) = g(s)$, then the homomorphisms $k(x) \rightarrow k(s)$ corresponding to f and g are necessarily identical. If $h = (f, g)_S$, we thus deduce (5.3.17) that $h(s)$ belongs to the diagonal $Z = \Delta_X(X)$; but since $S = \overline{\{s\}}$, and since Z is closed by hypothesis, we have $h(S) \subset Z$. It then follows from (5.2.2) that h factors as $S \rightarrow Z \rightarrow X \times_S X$, and we thus conclude that $f = g$, by definition of the diagonal. \square

Remark (5.4.8). — If we suppose, conversely, that the conclusion of (5.4.3) is true when $f = 1_Y$, then we can conclude that Y is separated over S ; similarly, if we suppose that the conclusion of (5.4.5) applies to the two morphisms $Y \xrightarrow{\Delta_Y} Y \times_Z Y \xrightarrow{p_1} Y$, then we can conclude that Δ_Y is a closed immersion, and thus that Y is separated over Z ; finally, if we assume that the conclusion of (5.4.6) is true for the Y section Δ_Y of the Y -prescheme $Y \times_S Y \rightarrow Y$, then this implies that Y is separated over S .

5.5. Separation criteria.

Proposition (5.5.1). —

- (i) Every monomorphism of preschemes (and, in particular, every immersion) is a separated morphism.
- (ii) The composition of any two separated morphisms is separated.
- (iii) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are separated S -morphisms, then $f \times_S g$ is separated.
- (iv) If $f : X \rightarrow Y$ is a separated S -morphism, then the S' -morphism $f_{(S')}$ is separated for every extension $S' \rightarrow S$ of the base prescheme.
- (v) If the composition $g \circ f$ is separated, then f is separated.
- (vi) For a morphism f to be separated, it is necessary and sufficient for f_{red} (5.1.5) to be separated.

Proof. Note that (i) is an immediate consequence of (5.3.8). If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are morphisms, then the diagram

$$(5.5.1.1) \quad \begin{array}{ccc} X & \xrightarrow{\Delta_{X|Z}} & X \times_Z X \\ & \searrow \Delta_{X|Y} & \nearrow j \\ & X \times_Y X & \end{array}$$

commutes (where j denotes the canonical immersion (5.3.10)), as can be immediately verified. If f and g are separated, then $\Delta_{X|Y}$ is a closed immersion by definition, and j is a closed immersion by (5.4.2), whence $\Delta_{X|Z}$ is a closed immersion by (4.2.4), which proves (ii). Given (i) and (ii), (iii) and (iv) are equivalent (3.5.1), so it suffices to prove (iv). But $X_{(S')} \times_{Y_{(S')}} X_{(S')}$ is canonically identified with $(X \times_Y X) \times_Y Y_{(S')}$ by (3.3.11) and (3.3.9.1), and we immediately see that the diagonal morphism $\Delta_{X_{(S')}}$ can then be identified with $\Delta_X \times_Y 1_{Y_{(S')}}$; the proposition then follows from (4.3.1).

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To prove (v), consider, as in (5.3.13), the factorisation $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$ of f , noting that $p_2 = (g \circ f) \times_Z 1_Y$; the hypothesis (that $g \circ f$ is separated) implies that g_2 is separated, by (iii) and (i), and, since Γ_f is an immersion, Γ_f is separated, by (i), whence f is separated, by (ii). Finally, to prove (vi), recall that the preschemes $X_{\text{red}} \times_{Y_{\text{red}}} X_{\text{red}}$ and $X_{\text{red}} \times_Y X_{\text{red}}$ are canonically identified with one another (5.1.7); if we denote by j the injection $X_{\text{red}} \rightarrow X$, then the diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{\Delta_{X_{\text{red}}}} & X_{\text{red}} \times_Y X_{\text{red}} \\ j \downarrow & & \downarrow j \times_Y j \\ X & \xrightarrow{\Delta_X} & X \times_Y X \end{array}$$

commutes (5.3.15), and the proposition follows from the fact that the vertical arrows are homeomorphisms of the underlying spaces (4.3.1). \square

Corollary (5.5.2). — If $f : X \rightarrow Y$ is separated, then the restriction of f to any subprescheme of X is separated.

Proof. This follows from (5.5.1, i and ii). \square

Corollary (5.5.3). — *If X and Y are S -preschemes such that Y is separated over S , then $X \times_S Y$ is separated over X .*

Proof. This is a particular case of (5.5.1, iv). \square

Proposition (5.5.4). — *Let X be a prescheme, and assume that its underlying space is a finite union of closed subsets X_k ($1 \leq k \leq n$); for each k , consider the reduced subprescheme of X that has X_k as its underlying space (5.2.1), and denote this again by X_k . Let $f : X \rightarrow Y$ be a morphism, and, for each k , let Y_k be a closed subset of Y such that $f(X_k) \subset Y_k$; we again denote by Y_k the reduced subprescheme of Y that has Y_k as its underlying space, so that the restriction $X_k \rightarrow Y$ of f to X_k factors as $X_k \xrightarrow{f_k} Y_k \rightarrow Y$ (5.2.2). For f to be separated, it is necessary and sufficient for all the f_k to be separated.*

Proof. The necessity follows from (5.5.1, i, ii, and v). Conversely, if the condition of the statement is satisfied, then each of the restrictions $X_k \rightarrow Y$ of f is separated (5.5.1, (i) and (ii)); if p_1 and p_2 are the projections of $X \times_Y X$, then the subspace $\Delta_{X_k}(X_k)$ can be identified with the subspace $\Delta_X(X) \cap p_1^{-1}(X_k)$ of the underlying space of $X \times_Y X$ (5.3.16); these subspaces are closed in $X \times_Y X$, and thus so too is their union $\Delta_X(X)$. \square

Suppose, in particular, that the X_k are the irreducible components of X ; then we can suppose that the Y_k are the irreducible components of Y (0, 2.1.5); Proposition (5.5.4) then, in this case, leads to the idea of separation in the case of integral preschemes (2.1.7).

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Proposition (5.5.5). — *Let (Y_λ) be an open cover of a prescheme Y ; for a morphism $f : X \rightarrow Y$ to be separated, it is necessary and sufficient for each of its restrictions $f^{-1}(Y_\lambda) \rightarrow Y_\lambda$ to be separated.*

Proof. If we set $X_\lambda = f^{-1}(Y_\lambda)$, everything reduces, by taking (4.2.4, b) and the identification of the products $X_\lambda \times_Y X_\lambda$ and $X_\lambda \times_{Y_\lambda} X_\lambda$ into account, to proving that the $X_\lambda \times_Y X_\lambda$ form a cover of $X \times_Y X$. But if we set $Y_{\lambda\mu} = Y_\lambda \cap Y_\mu$ and $X_{\lambda\mu} = X_\lambda \cap X_\mu = f^{-1}(Y_{\lambda\mu})$, then $X_\lambda \times_Y X_\mu$ can be identified with the product $X_{\lambda\mu} \times_{Y_{\lambda\mu}} X_{\lambda\mu}$ (3.2.6.4), and so also with $X_{\lambda\mu} \times_Y X_{\lambda\mu}$ (3.2.5), and finally with an open subset of $X_\lambda \times_Y X_\lambda$, which proves our claim (3.2.7). \square

Proposition (5.5.4) allows us, by taking an affine open cover of Y , to restrict our study of separated morphisms to just those that take values in affine schemes.

Proposition (5.5.6). — *Let Y be an affine scheme, X a prescheme, and (U_α) a cover of X by affine open subsets. For a morphism $f : X \rightarrow Y$ to be separated, it is necessary and sufficient for $U_\alpha \cap U_\beta$ to be, for every pair of indices (α, β) , an affine open subset, and for the ring $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ to be generated by the union of the canonical images of the rings $\Gamma(U_\alpha, \mathcal{O}_X)$ and $\Gamma(U_\beta, \mathcal{O}_X)$.*

Proof. The $U_\alpha \times_Y U_\beta$ form an open cover of $X \times_Y X$ (3.2.7); denoting the projections of $X \times_Y X$ by p and q , we have

$$\begin{aligned} \Delta_X^{-1}(U_\alpha \times_Y U_\beta) &= \Delta_X^{-1}(p^{-1}(U_\alpha) \cap q^{-1}(U_\beta)) \\ &= \Delta_X^{-1}(p^{-1}(U_\alpha)) \cap \Delta_X^{-1}(q^{-1}(U_\beta)) = U_\alpha \cap U_\beta; \end{aligned}$$

so everything reduces to proving that the restriction of Δ_X to $U_\alpha \cap U_\beta$ is a closed immersion into $U_\alpha \times_Y U_\beta$. But this restriction is exactly $(j_\alpha, j_\beta)_Y$, where j_α (resp. j_β) denotes the injection morphism from $U_\alpha \cap U_\beta$ to U_α (resp. U_β), as follows from the definitions. Since $U_\alpha \times_Y U_\beta$ is an affine scheme whose ring is canonically isomorphic to $\Gamma(U_\alpha, \mathcal{O}_X) \otimes_{\Gamma(Y, \mathcal{O}_Y)} \Gamma(U_\beta, \mathcal{O}_X)$ (3.2.2), we see that $U_\alpha \cap U_\beta$ must be an affine scheme, and that the map $h_\alpha \otimes h_\beta \mapsto h_\alpha h_\beta$ from the ring $A(U_\alpha \times_Y U_\beta)$ to $\Gamma(U_\alpha \cap U_\beta, \mathcal{O}_X)$ must be surjective (4.2.3), which finishes the proof. \square

Corollary (5.5.7). — *An affine scheme is separated (and is thus a scheme, which justifies the terminology of (5.4.1)).*

Corollary (5.5.8). — *Let Y be an affine scheme; for $f : X \rightarrow Y$ to be a separated morphism, it is necessary and sufficient for X to be separated (in other words, for X to be a scheme).*

Proof. Indeed, we see that the criteria of (5.5.6) do not depend on f . \square

Corollary (5.5.9). — For a morphism $f : X \rightarrow Y$ to be separated, it is necessary and sufficient for the induced prescheme $f^{-1}(U)$ to be separated, for every open subset of U on which Y induces a separated prescheme, and it is sufficient for it to be the case for every affine open subset $U \subset Y$.

Proof. The necessity of the condition follows from (5.5.4) and (5.5.1, ii); the sufficiency follows from (5.5.4) and (5.5.8), taking into account the existence of affine open covers of Y . \square

In particular, if X and Y are affine schemes, then every morphism $X \rightarrow Y$ is separated.

Proposition (5.5.10). — Let Y be a scheme, and $f : X \rightarrow Y$ a morphism. For every affine open subset U of X , and every affine open subset V of Y , $U \cap f^{-1}(V)$ is affine.

Proof. Let p_1 and p_2 be the projections of $X \times_{\mathbf{Z}} Y$; the subspace $U \cap f^{-1}(V)$ is the image of $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ under p_1 . But $p_1^{-1}(U) \cap p_2^{-1}(V)$ can be identified with the underlying space of the prescheme $U \times_{\mathbf{Z}} V$ (3.2.7), and is thus an affine scheme (3.2.2); since $\Gamma_f(X)$ is closed in $X \times_{\mathbf{Z}} Y$ (5.4.3), $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ is closed in $U \times_{\mathbf{Z}} V$, and so the prescheme induced by the subprescheme of $X \times_{\mathbf{Z}} Y$ associated to Γ_f (4.2.1) on the open subset $\Gamma_f(X) \cap p_1^{-1}(U) \cap p_2^{-1}(V)$ of its underlying space is a closed subprescheme of an affine scheme, and thus an affine scheme (4.2.3). The proposition then follows from the fact that Γ_f is an immersion. \square

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Example (5.5.11). — The prescheme from Example (2.3.2) (“the projective line over a field K ”) is separated, because, for the cover (X_1, X_2) of X by affine open subsets, $X_1 \cap X_2 = U_{12}$ is affine, and $\Gamma(U_{12}, \mathcal{O}_X)$, the ring of rational fractions of the form $f(s)/s^m$ with $f \in K[s]$, is generated by $K[s]$ and $1/s$, so the conditions of (5.5.6) are satisfied.

With the same choice of X_1, X_2, U_{12} , and U_{21} as in Example (2.3.2), now take u_{12} to be the isomorphism which sends $f(s)$ to $f(t)$; we now obtain, by gluing, a non-separated integral prescheme X , because the first condition of (5.5.6) is satisfied, but not the second. It is immediate here that $\Gamma(X, \mathcal{O}_X) \rightarrow \Gamma(X_1, \mathcal{O}_X) = K[s]$ is an isomorphism; the inverse isomorphism defines a morphism $f : X \rightarrow \text{Spec}(K[s])$ that is surjective, and for every $y \in \text{Spec}(K[s])$ such that $j_y \neq (0)$, $f^{-1}(y)$ consists of a single point, but for $j_y = (0)$, $f^{-1}(y)$ consists of two distinct points (we say that X is the “affine line over K with the point 0 doubled”).

We can also give examples where neither of the two conditions of (5.5.6) are satisfied. First note that, in the prime spectrum Y of the ring $A = K[s, t]$ of polynomials in two indeterminates over a field K , the open subset U given by the union of $D(s)$ and $D(t)$ is not an affine open subset. Indeed, if z is a section of \mathcal{O}_Y over U , there exist two integers $m, n \geq 0$ such that $s^m z$ and $t^n z$ are the restrictions of polynomials in s and t to U (1.4.1), which is clearly possible only if the section z extends to a section over the whole of Y , identified with a polynomial in s and t . If U were an affine open subset, then the injection morphism $U \rightarrow Y$ would be an isomorphism (1.7.3), which is a contradiction, since $U \neq Y$.

With the above in mind, take two affine schemes Y_1 and Y_2 , prime spectra of the rings $A_1 = K[s_1, t_1]$ and $A_2 = K[s_2, t_2]$ (respectively); take $U_{12} = D(s_1) \cup D(t_1)$ and $U_{21} = D(s_2) \cup D(t_2)$, and take u_{12} to be the restriction of an isomorphism $Y_2 \rightarrow Y_1$ to U_{21} corresponding to the isomorphism of rings that sends $f(s_1, t_1)$ to $f(s_2, t_2)$; we then have an example where the conditions of (5.5.6) are not satisfied (the integral prescheme thus obtained is called “the affine plane over K with the point 0 doubled”).

Remark (5.5.12). — Given some property P of morphisms of preschemes, consider the following propositions.

- (i) Every closed immersion has property P .
- (ii) The composition of any two morphisms that both have property P also has property P .
- (iii) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are S -morphisms that have property P , then $f \times_S g$ has property P .
- (iv) If $f : X \rightarrow Y$ is an S -morphism that has property P , then every S' -morphism $f_{(S')}$ obtained by an extension $S' \rightarrow S$ of the base prescheme also has property P .
- (v) If the composition $g \circ f$ of two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ has property P , and g is separated, then f has property P .
- (vi) If a morphism $f : X \rightarrow Y$ has property P , then so too does f_{red} (5.1.5).

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If we suppose that (i) and (ii) are both true, then (iii) and (iv) are *equivalent*, and (v) and (vi) are *consequences* of (i), (ii), and (iii).

The first claim has already been shown (3.5.1). Consider the factorisation (5.3.13) $X \xrightarrow{\Gamma_f} X \times_Z Y \xrightarrow{p_2} Y$ of f ; the equation $p_2 = (g \circ f) \times_Z 1_Y$ shows that, if $g \circ f$ has property \mathbf{P} , then so too does p_2 , by (iii); if g is separated, then Γ_f is a closed immersion (5.4.3), and so also has property \mathbf{P} , by (i); finally, by (ii), f has property \mathbf{P} .

Finally, consider the commutative diagram

$$\begin{array}{ccc} X_{\text{red}} & \xrightarrow{f_{\text{red}}} & Y_{\text{red}} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y, \end{array}$$

where the vertical arrows are the closed immersions (5.1.5), and thus have property \mathbf{P} , by (i). The hypothesis that f has property \mathbf{P} implies, by (ii), that $X_{\text{red}} \xrightarrow{f_{\text{red}}} Y_{\text{red}} \rightarrow Y$ has property \mathbf{P} ; finally, since a closed immersion is separated (5.5.1, i), f_{red} has property \mathbf{P} , by (v).

Note that, if we consider the propositions

(i') Every immersion has property \mathbf{P} ;

(v') If $g \circ f$ has property \mathbf{P} , then so too does f ;

then the above arguments show that (v') is a consequence of (i'), (ii), and (iii).

(5.5.13). Note that (v) and (vi) are again consequences of (i), (iii), and

(ii') If $j : X \rightarrow Y$ is a closed immersion, and $g : Y \rightarrow Z$ is a morphism that has property \mathbf{P} , then $g \circ j$ has property \mathbf{P} .

Similarly, (v') is a consequence of (i'), (iii), and

(ii'') If $j : X \rightarrow Y$ is an immersion, and $g : Y \rightarrow Z$ is a morphism that has property \mathbf{P} , then $g \circ j$ has property \mathbf{P} .

This follows immediately from the arguments of (5.5.12).

§6. FINITENESS CONDITIONS

6.1. Noetherian and locally Noetherian preschemes.

Definition (6.1.1). — We say that a prescheme X is Noetherian (resp. locally Noetherian) if it is a finite union (resp. union) of affine open V_α in such a way that the ring of the of the induced scheme on each of the V_α is Noetherian.

It follows immediately from (1.5.2) that, if X is locally Noetherian, then the structure sheaf \mathcal{O}_X is a *coherent sheaf of rings*, the question being a local one. Every *quasi-coherent* \mathcal{O}_X -submodule (resp. quasi-coherent quotient \mathcal{O}_X -module) of a *coherent* \mathcal{O}_X -module \mathcal{F} is *coherent*, as the question is once again a local one, and it suffices to apply (1.5.1), (1.4.1), and (1.3.10), combined with the fact that a submodule (resp. quotient module) of a module of finite type over a Noetherian ring is of finite type. In particular, every *quasi-coherent sheaf of ideals* of \mathcal{O}_X is *coherent*.

If a prescheme X is a finite union (resp. union) of open subsets W_λ in such a way that the preschemes induced on the W_λ are Noetherian (resp. locally Noetherian), it is clear that X is Noetherian (resp. locally Noetherian).

Proposition (6.1.2). — For a prescheme X to be Noetherian, it is necessary and sufficient for it to be locally Noetherian and have a quasi-compact underlying space. The underlying space itself is then also Noetherian.

Proof. The first claim follows immediately from the definitions and (1.1.10, ii). The second follows from (1.1.6) and the fact that every space that is a finite union of Noetherian subspaces is itself Noetherian (0, 2.2.3). \square

Proposition (6.1.3). — Let X be an affine scheme given by a ring A . The following conditions are equivalent: (a) X is Noetherian; (b) X is locally Noetherian; (c) A is Noetherian.

Proof. The equivalence between (a) and (b) follows from (6.1.2) and the fact that the underlying space of every affine scheme is quasi-compact (1.1.10); it is furthermore clear that (c) implies (a). To see that (a) implies (c), we remark that there is a finite cover (V_i) of X by affine open subsets such that the ring A_i of the prescheme induced on V_i is Noetherian. So let (\mathfrak{a}_n) be an increasing sequence of ideals of A ; by a canonical bijective correspondence, there is a corresponding sequence $(\tilde{\mathfrak{a}}_n)$ of sheaves of ideals in $\tilde{A} = \mathcal{O}_X$; to see that the sequence (\mathfrak{a}_n) is stable (?), it suffices to prove that the sequence $(\tilde{\mathfrak{a}}_n)$ is. But the restriction $\tilde{\mathfrak{a}}_n|_{V_i}$ is a quasi-coherent sheaf of ideals in $\mathcal{O}_X|_{V_i}$, being the inverse image of $\tilde{\mathfrak{a}}_n$ under the canonical injection $V_i \rightarrow X$ (0, 5.1.4); $\tilde{\mathfrak{a}}_n|_{V_i}$ is thus of the form $\tilde{\mathfrak{a}}_{ni}$, where \mathfrak{a}_{ni} is an ideal of A_i (1.3.7). Since A_i is Noetherian, the sequence (\mathfrak{a}_{ni}) is stable for all i , whence the proposition. \square

We note that the above argument proves also that if X is a Noetherian prescheme, then every increasing sequence of coherent sheaves of ideals of \mathcal{O}_X is stable (?).

Proposition (6.1.4). — *Every subprescheme of a Noetherian (resp. locally Noetherian) prescheme is Noetherian (resp. locally Noetherian).*

Proof. It suffices to give a proof for a Noetherian prescheme X ; further, by definition (6.1.1), we can also restrict to the case where X is an affine scheme. Since every subprescheme of X is a closed subprescheme of a prescheme induced on an open subset (4.1.3), we can restrict to the case of a subprescheme Y , either closed or induced on an open subset of X . The proof in the case where Y is closed is immediate, since if A is the ring of X , we know that Y is an affine scheme given by the ring A/\mathfrak{J} , where \mathfrak{J} is an ideal of A (4.2.3); since A is Noetherian (6.1.3), so too is A/\mathfrak{J} .

Now suppose that Y is open in X ; the underlying space of Y is Noetherian (6.1.2), hence quasi-compact, and thus a finite union of open subsets $D(f_i)$ ($f_i \in A$); everything reduces to showing the proposition in the case where $Y = D(f)$ with $f \in A$. But then Y is an affine scheme whose ring is isomorphic to A_f (1.3.6); since A is Noetherian (6.1.3), so too is A_f . \square

(6.1.5). We note that the product of two Noetherian S -preschemes is not necessarily Noetherian, even if the preschemes are affine, since the tensor product of two Noetherian algebras is not necessarily a Noetherian ring (cf. (6.3.8)).

Proposition (6.1.6). — *If X is a Noetherian prescheme, the nilradical \mathcal{N}_X of \mathcal{O}_X is nilpotent.*

Proof. We can in fact cover X with a finite number of affine open subsets U_i , and it suffices to prove that there exists whole numbers n_i such that $(\mathcal{N}_X|_{U_i})^{n_i} = 0$; if n is the largest of the n_i , then we will have $\mathcal{N}_X^n = 0$. We can thus restrict to the case where $X = \text{Spec}(A)$ is affine, with A a Noetherian ring; by (5.1.1) and (1.3.13), it suffices to observe that the nilradical of A is nilpotent ([Sam53, p. 127, cor. 4]). \square

Corollary (6.1.7). — *Let X be a Noetherian prescheme; for X to be an affine scheme, it is necessary and sufficient that X_{red} be affine.*

Proof. This follows from (6.1.6) and (5.1.10). \square

Lemma (6.1.8). — *Let X be a topological space, x a point of X , and U an open neighbourhood of x having only a finite number of irreducible components. Then there exists a neighbourhood V of x such that every open neighbourhood of x contained in V is connected.*

Proof. Let U_i ($1 \leq i \leq m$) be the irreducible components of U not containing x ; the complement (in U) of the union of the U_i is an open neighbourhood V of x inside U , and thus so too in X ; it is also, incidentally, the complement (in X) of the union of the irreducible components of X that do not contain x (0, 2.1.6). So let W be an open neighbourhood of x contained in V . The irreducible components of W are the intersections of W with the irreducible components of U (0, 2.1.6), so these components contain x ; since they are connected, so too is W . \square

Corollary (6.1.9). — *A locally Noetherian topological space is locally connected (which implies, amongst other things, that its connected components are open).*

Proposition (6.1.10). — *Let X be a locally Noetherian topological space. The following conditions are equivalent.*

- (a) *The irreducible components of X are open.*

- (b) *The irreducible components of X are exactly its connected components.*
- (c) *The connected components of X are irreducible.*
- (d) *Two distinct irreducible components of X have an empty intersection.*

Finally, if X is a prescheme, then these conditions are also equivalent to

- (e) *For every $x \in X$, $\text{Spec}(\mathcal{O}_x)$ is irreducible (or, in other words, the nilradical of \mathcal{O}_x is prime).*

Proof. It is immediate that (a) implies (b), because an irreducible space is connected, and (a) implies that the irreducible components of X are the sets that are both open and closed. It is trivial that (b) implies (c); conversely, a closed set F containing a connected component C of X , with C distinct from F , cannot be irreducible, because not being connected means that F is the union of two disjoint nonempty sets that are both open and closed in F , and thus closed in X ; as a result, (c) implies (b). We immediately conclude from this that (c) implies (d), since two distinct connected components have no points in common.

We have not yet used the fact that X is locally Noetherian. Suppose now that this is indeed the case, and we will show that (d) implies (a): by (0, 2.1.6), we can restrict ourselves to the case where the space X is Noetherian, and so has only a finite number of irreducible components. Since they are closed and pairwise disjoint, they are open.

Finally, the equivalence between (d) and (e) holds true even without the assumption that the underlying space of the prescheme X is locally Noetherian. We can in fact restrict ourselves to the case where $X = \text{Spec}(A)$ is affine, by (0, 2.1.6); to say that x is contained in only one single irreducible component of X is to say that \mathfrak{j}_x contains only one single minimal ideal of A (1.1.14), which is equivalent to saying that $\mathfrak{j}_x \mathcal{O}_x$ contains only one single minimal ideal of \mathcal{O}_x , whence the conclusion. \square

Corollary (6.1.11). — *Let X be a locally Noetherian space. For X to be irreducible, it is necessary and sufficient that X be connected and nonempty, and that any two distinct irreducible components of X have an empty intersection. If X is a prescheme, this latter condition is equivalent to asking that $\text{Spec}(\mathcal{O}_x)$ be irreducible for all $x \in X$.*

Proof. The second claim has already been shown in (6.1.10); the only thing thus remaining to show is that the conditions in the first claim are sufficient. But by (6.1.10), these conditions imply that the irreducible components of X are exactly its connected components, and since X is connected and nonempty, it is irreducible. \square

Corollary (6.1.12). — *Let X be a locally Noetherian prescheme. For X to be integral, it is necessary and sufficient that X be connected and that \mathcal{O}_x be integral for all $x \in X$.*

Proposition (6.1.13). — *Let X be a locally Noetherian prescheme, and let $x \in X$ be a point such that the nilradical \mathcal{N}_x of \mathcal{O}_x is prime (resp. such that \mathcal{O}_x is reduced, resp. integral); then there exists an open neighbourhood U of x that is irreducible (resp. reduced, resp. integral).*

Proof. It suffices to consider two cases: where \mathcal{N}_x is prime, and where $\mathcal{N}_x = 0$; the third hypotheses is a combination of the first two. If \mathcal{N}_x is prime, then x belongs to only one single irreducible component Y of X (6.1.10); the union of the irreducible components of X that do not contain x is closed (the set of these components being locally finite), and the complement U of this union is thus open and contained in Y , and thus irreducible (0, 2.1.6). If $\mathcal{N}_x = 0$, we also have $\mathcal{N}_y = 0$ for any y in a neighbourhood of x , because \mathcal{N} is quasi-coherent (5.1.1), and thus coherent, since X is locally Noetherian, and the conclusion then follows from (0, 5.2.2). \square

6.2. Artinian preschemes.

Definition (6.2.1). — We say that a prescheme is *Artinian* if it is affine, and given by an Artinian ring.

Proposition (6.2.2). — *Given a prescheme X , the following conditions are equivalent:*

- (a) *X is an Artinian scheme;*
- (b) *X is Noetherian and its underlying space is discrete;*
- (c) *X is Noetherian and the points of its underlying space are closed (the T_1 condition).*

When any of the above hold, the underlying space of X is finite, and the ring A of X is the direct sum of local (Artinian) rings of points of X .

Proof. We know that (a) implies the last claim ([SZ60, p. 205, th. 3]), so every prime ideal of A is thus maximal and is the inverse image of a maximal ideal of one of the local components of A , and so the space X is finite and discrete; (a) thus implies (b), and (b) clearly implies (c). To see that (c) implies (a), we first show that X is then finite; we can indeed restrict to the case where X is affine, and we know that a Noetherian ring whose prime ideals are all maximal is Artinian ([SZ60, p. 203]), whence our claim. The underlying space X is then discrete, the topological sum of a finite number of points x_i , and the local rings $\mathcal{O}_{x_i} = A_i$ are Artinian; it is clear that X is isomorphic to the prime spectrum affine scheme of the ring A (the direct sum of the A_i) (1.7.3). \square

6.3. Morphisms of finite type.

Definition (6.3.1). — We say that a morphism $f : X \rightarrow Y$ is of *finite type* if Y is the union of a family (V_α) of affine open subsets having the following property:

- (P) $f^{-1}(V_\alpha)$ is a finite union of affine open subsets $U_{\alpha i}$ that are such that each ring $A(U_{\alpha i})$ is an algebra of finite type over $A(V_\alpha)$.

We then say that X is a prescheme of finite type over Y , or a Y -prescheme of finite type.

Proposition (6.3.2). — If $f : X \rightarrow Y$ is a morphism of finite type, then every affine open subset W of Y satisfies property (P) of (6.3.1).

We first show

Lemma (6.3.2.1). — If $T \subset Y$ is an affine open subset, satisfying property (P), then, for every $g \in A(T)$, $D(g)$ also satisfies property (P).

Proof. By hypothesis, $f^{-1}(T)$ is a finite union of affine open subsets Z_j , that are such that $A(Z_j)$ is an algebra of finite type over $A(T)$; let $\phi_j : A(T) \rightarrow A(Z_j)$ be the homomorphism of rings corresponding to the restriction of f to Z_j (2.2.4), and set $g_j = \phi_j(g)$; we then have $f^{-1}(D(g)) \cap Z_j = D(g_j)$ (1.2.2.2). But $A(D(g_j)) = A(Z_j)_{g_j} = A(Z_j)[1/g_j]$ is of finite type over $A(Z_j)$, and a fortiori over $A(T)$ by the hypothesis, and so also over $A(D(g)) = A(T)[1/g]$, which proves the lemma. \square

Proof. With the above lemma, since W is quasi-compact (1.1.10), there exists a finite covering of W by sets of the form $D(g_i) \subset W$, where each g_i belongs to a ring $A(V_{\alpha(i)})$. Each $D(g_i)$, being quasi-compact, is a finite union of sets $D(h_{ik})$, where $h_{ik} \in A(W)$; if $\phi_i : A(W) \rightarrow A(D(g_i))$ is the canonical map, then we have $D(h_{ik}) = D(\phi_i(h_{ik}))$ by (1.2.2.2). By (6.3.2.1), each of the $f^{-1}(D(h_{ik}))$ admits a finite covering by affine open subsets U_{ijk} , that are such that the $A(U_{ijk})$ are algebras of finite type over $A(D(h_{ik})) = A(W)[1/h_{ik}]$, whence the proposition. \square

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We can thus say that the notion of a prescheme of finite type over Y is *local on Y* .

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Proposition (6.3.3). — Let X and Y be affine schemes; for X to be of finite type over Y , it is necessary and sufficient that $A(X)$ be an algebra of finite type over $A(Y)$.

Proof. Since the condition clearly suffices, we show that it is necessary. Set $A = A(Y)$ and $B = A(X)$; by (6.3.2), there exists a finite affine open cover (V_i) of X such that each of the rings $A(V_i)$ is an A -algebra of finite type. Further, since the V_i are quasi-compact, we can cover each of them with a finite number of open subsets of the form $D(g_{ij}) \subset V_i$, where $g_{ij} \in B$; if ϕ_i is a homomorphism $B \rightarrow A(V_i)$ that corresponds to the canonical injection $V_i \rightarrow X$, then we have $B_{g_{ij}} = (A(V_i))_{\phi_i(g_{ij})} = A(V_i)[1/\phi_i(g_{ij})]$, so $B_{g_{ij}}$ is an A -algebra of finite type. We can thus restrict to the case where $V_i = D(g_i)$ with $g_i \in B$. By hypothesis, there exists a finite subset F_i of B and an integer $n_i \geq 0$ such that B_{g_i} is the algebra generated over A by the elements $b_i/g_i^{n_i}$, where the b_i run over all of F_i . Since there are only finitely many of the g_i , we can assume that all the n_i are equal to the same integer n . Further, since the $D(g_i)$ form a cover of X , the ideal generated in B by the g_i is equal to B , or, in other words, there exist $h_i \in B$ such that $\sum_i h_i g_i = 1$. So let F be the finite subset of B given by the union of the F_i , the set of the g_i , and the set of the h_i ; we will show that the subring $B' = A[F]$ of B is equal to B . By hypothesis, for every $b \in B$ and every i , the canonical image of b in B_{g_i} is of the

form $b'_i/g_i^{m_i}$, where $b'_i \in B'$; by multiplying the b'_i by suitable powers of the g_i , we can again assume that all the m_i are equal to the same integer m . By the definition of the ring of fractions, there is thus an integer N (dependant on b) such that $N \geq m$ and $g_i^N b \in B'$ for all i ; but, in the ring B' , the g_i^N generate the ideal B' , because the g_i do (and the h_i belong to B'); there are thus $c_i \in B'$ such that $\sum_i c_i g_i^N = 1$, whence $b = \sum_i c_i g_i^N b \in B'$, Q.E.D. \square

Proposition (6.3.4). —

- (i) Every closed immersion is of finite type.
- (ii) The composition of any two morphisms of finite type is of finite type.
- (iii) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are S -morphisms of finite type, then $f \times_S g$ is of finite type.
- (iv) If $f : X \rightarrow Y$ is an S -morphism of finite type, then $f_{(S')}$ is of finite type for any extension $g : S' \rightarrow S$ of the base prescheme.
- (v) If the composition $g \circ f$ of two morphisms is of finite type, with g separated, then f is of finite type.
- (vi) If a morphism f is of finite type, then f_{red} is of finite type.

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iv).

To show (i), we can restrict to the case of a canonical injection $X \rightarrow Y$, with X being a closed subprescheme of Y ; further (6.3.2), we can assume that Y is affine, in which case X is also affine (4.2.3) and its ring is isomorphic to a quotient ring A/\mathfrak{J} , where A is the ring of Y and \mathfrak{J} is an ideal of A ; since A/\mathfrak{J} is of finite type over A , the conclusion follows.

Now we show (ii). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms of finite type, and let U be an affine open subset of Z ; g^{-1} admits a finite covering by affine open subsets V_i that are such that each $A(V_i)$ is an algebra of finite type over $A(U)$ (6.3.2); similarly, each of the f^{-1} admits a finite cover by affine open subsets W_{ij} that are such that each $A(W_{ij})$ is an algebra of finite type over $A(V_i)$, and so also an algebra of finite type over $A(U)$, whence the conclusion.

Finally, to show (iv), we can restrict to the case where $S = Y$; then $f_{(S')}$ is also equal to $f_{Y_{(S')}}$, where we consider f as a Y -morphism, and the base extension is $Y_{(S')} \rightarrow Y$ (3.3.9). So let p and q be the projections $X_{(S')} \rightarrow X$ and $X_{(S')} \rightarrow S'$. Let V be an affine open subset of S ; $f^{-1}(V)$ is a finite union of affine open subsets W_i , each of which is such that $A(W_i)$ is an algebra of finite type over $A(V)$ (6.3.2). Let V' be an affine open subset of S' contained in $g^{-1}(V)$; since $f \circ p = g \circ q$, $q^{-1}(V')$ is contained in the union of the $p^{-1}(W_i)$; on the other hand, the intersection $p^{-1}(W_i) \cap q^{-1}(V')$ can be identified with the product $W_i \times_V V'$ (3.2.7), which is an affine scheme whose ring is isomorphic to $A(W_i) \otimes_{A(V)} A(V')$ (3.2.2); this ring is, by hypothesis, an algebra of finite type over $A(V')$, which proves the proposition. \square

Corollary (6.3.5). — Let $f : X \rightarrow Y$ be an immersion morphism. If the underlying space of Y (resp. X) is locally Noetherian (resp. Noetherian), then f is of finite type.

Proof. We can always assume that Y is affine (6.3.2); if the underlying space of Y is locally Noetherian, then we can further assume that it is Noetherian, and then the underlying space of X , which is a subspace, is also Noetherian. In other words, we can assume that Y is affine and that the underlying space of X is Noetherian; there then exists a covering of X by a finite number of affine open subsets $D(g_i) \subset Y$, where $g_i \in A(Y)$, that are such that the $X \cap D(g_i)$ are closed in $D(g_i)$ (and thus affine schemes (4.2.3)), because X is locally closed in Y (4.1.3). Then $A(X \cap D(g_i))$ is an algebra of finite type over $A(D(g_i))$, by (6.3.4, i) and (6.3.3), and $A(D(g_i)) = A(Y)_{g_i} = A(Y)[1/g_i]$ is of finite type over $A(Y)$, which finishes the proof. \square

Corollary (6.3.6). — Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be morphisms. If $g \circ f$ is of finite type, with either X Noetherian or $X \times_Z Y$ locally Noetherian, then f is of finite type.

Proof. This follows immediately from the proof of (5.5.12) and from (6.3.5) applied to the immersion morphism Γ_f . \square

Proposition (6.3.7). — Let $f : X \rightarrow Y$ be a morphism of finite type; if Y is Noetherian (resp. locally Noetherian), then X is Noetherian (resp. locally Noetherian).

Proof. We can restrict to proving the proposition for when Y is Noetherian. Then Y is a finite union of affine open subsets V_i that are such that the $A(V_i)$ are Noetherian rings. By (6.3.2), each of the

$f^{-1}(V_i)$ is a finite union of affine open subsets W_{ij} that are such that the $A(W_{ij})$ are algebras of finite type over $A(V_i)$, and thus Noetherian rings; this proves that X is Noetherian. \square

Corollary (6.3.8). — *Let X be a prescheme of finite type over S . For every base extension $S' \rightarrow S$ with S' Noetherian (resp. locally Noetherian), $X_{(S')}$ is Noetherian (resp. locally Noetherian).*

Proof. This follows from (6.3.7), since $X_{(S')}$ is of finite type over S' by (6.3.4, iv). \square

We can also say that, for a product $X \times_S Y$ of S -preschemes, if one of the factors X or Y is of finite type over S and the other is Noetherian (resp. locally Noetherian), then $X \times_S Y$ is Noetherian (resp. locally Noetherian).

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Corollary (6.3.9). — *Let X be a prescheme of finite type over a locally Noetherian prescheme S . Then every S -morphism $f : X \rightarrow Y$ is of finite type.*

Proof. In fact, we can assume that S is Noetherian; if $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structure morphisms, then we have $\phi = \psi \circ f$, and X is Noetherian by (6.3.7); f is thus of finite type by (6.3.6). \square

Proposition (6.3.10). — *Let $f : X \rightarrow Y$ be a morphism of finite type. For f to be surjective, it is necessary and sufficient that, for every algebraically closed field Ω , the map $X(\Omega) \rightarrow Y(\Omega)$ that corresponds to f (3.4.1) be surjective.*

Proof. The condition suffices, as we can see by considering, for all $y \in Y$, an algebraically closed extension Ω of $k(y)$, and the commutative diagram

$$\begin{array}{ccc} X & & \\ \downarrow f & \swarrow & \text{Spec}(\Omega) \\ Y & \nwarrow & \end{array}$$

(cf. (3.5.3)). Conversely, suppose that f is surjective, and let $g : \{\zeta\} = \text{Spec}(\Omega) \rightarrow Y$ be a morphism, where Ω is an algebraically closed field. If we consider the diagram

$$\begin{array}{ccc} X & \longleftarrow & X_{(\Omega)} \\ \downarrow f & & \downarrow f_{(\Omega)} \\ Y & \longleftarrow & \text{Spec}(\Omega), \end{array}$$

then it suffices to show that there exists a rational point over Ω in $X_{(\Omega)}$ ((3.3.14), (3.4.3), and (3.4.4)). Since f is surjective, $X_{(\Omega)}$ is nonempty (3.5.10), and since f is of finite type, so too is $f_{(\Omega)}$ (6.4.3, iv); thus $X_{(\Omega)}$ contains a nonempty affine open subset Z such that $A(Z)$ is a non-null algebra of finite type over Ω . By Hilbert's Nullstellensatz [Zar47], there exists an Ω -homomorphism $A(Z) \rightarrow \Omega$, and thus a section of $X_{(\Omega)}$ over $\text{Spec}(\Omega)$, which proves the proposition. \square

6.4. Algebraic preschemes.

Definition (6.4.1). — Given a field K , we define an *algebraic K -prescheme* to be a prescheme X of finite type over K ; K is called the *base field* of X . If in addition X is a scheme (or if X is a K -scheme, which is equivalent (5.5.8)), we say that X is an *algebraic K -scheme*.

Every algebraic K -prescheme is Noetherian (6.3.7).

Proposition (6.4.2). — *Let X be an algebraic K -prescheme. For a point $x \in X$ to be closed, it is necessary and sufficient that $k(x)$ be an algebraic extension of K of finite degree.*

Proof. We can assume that X is affine, with the ring A of X being a K -algebra of finite type. Indeed, the affine open subsets U of X such that $A(U)$ is a K -algebra of finite type form a finite cover of X (6.3.1). The closed points of X are thus the points such that \mathfrak{j}_x is a maximal ideal of A , or in

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other words, such that A/j_x is a field (necessarily equal to $k(x)$). Since A/j_x is a K -algebra of finite type, we see that if x is closed, then $k(x)$ is a field that is an algebra of finite type over K , and so necessarily a K -algebra of finite rank [Zar47]. Conversely, if $k(x)$ is of finite rank over K , then so is $A/j_x \subset k(x)$, and since every integral ring that is also a K -algebra of finite rank is a field, we have that $A/j_x = k(x)$, and hence x is closed. \square

Corollary (6.4.3). — *Let K be an algebraically-closed field, and X an algebraic K -prescheme; the closed points of X are then the rational points over K (3.4.4) and can be canonically identified with the points of X with values in K .*

Proposition (6.4.4). — *Let X be an algebraic prescheme over a field K . The following properties are equivalent.*

- (a) X is Artinian.
- (b) The underlying space of X is discrete.
- (c) The underlying space of X has only a finite number of closed points.
- (c') The underlying space of X is finite.
- (d) The points of X are closed.
- (e) X is isomorphic to $\text{Spec}(A)$, where A is a K -algebra of finite rank.

Proof. Since X is Noetherian, it follows from (6.2.2) that the conditions (a), (b), and (d) are equivalent, and imply (c) and (c'); it is also clear that (e) implies (a). It remains to see that (c) implies (d) and (e); we can restrict to the case where X is affine. Then $A(X)$ is a K -algebra of finite type (6.3.3), and thus a Jacobson ring ([CC, p. 3-11 and 3-12]), in which there are, by hypothesis, only a finite number of maximal ideals. Since a finite intersection of prime ideals can only be a prime ideal if it is equal to one of the prime ideals being intersected, every prime ideal of $A(X)$ is thus maximal, whence (d). Further, we then know (6.2.2) that $A(X)$ is an Artinian K -algebra of finite type, and so necessarily of finite rank [Zar47]. \square

(6.4.5). When the conditions of (6.4.4) are satisfied, we say that X is a scheme *finite over K* (cf. (II, 6.1.1)), or a *finite K -scheme*, of rank $[A : K]$, which we also denote by $\text{rg}_K(X)$; if X and Y are finite K -schemes, we have

$$(6.4.5.1) \quad \text{rg}_K(X \sqcup Y) = \text{rg}_K(X) + \text{rg}_K(Y),$$

$$(6.4.5.2) \quad \text{rg}_K(X \times_K Y) = \text{rg}_K(X) \text{rg}_K(Y),$$

as a result of (3.2.2).

Corollary (6.4.6). — *Let X be a finite K -scheme. For every extension K' of K , $X \otimes_K K'$ is a finite K' -scheme, and its rank over K' is equal to the rank of X over K .*

Proof. If $A = A(X)$, then we have $[A \otimes_K K' : K'] = [A : K]$. \square

Corollary (6.4.7). — *Let X be a scheme finite over a field K ; we let $n = \sum_{x \in X} [k(x) : K]_S$ (we recall that if K' is an extension of K , then $[K' : K]_S$ is the separable rank of K' over k , the rank of the largest algebraic separable extension of K contained in K'); then for every algebraically closed extension Ω of K , the underlying space of $X \otimes_K \Omega$ has exactly n points, which can be identified with the points of X with values in Ω .*

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Proof. We can clearly restrict to the case where the ring $A = A(X)$ is local (6.2.2); let \mathfrak{m} be its maximal ideal, and $L = A/\mathfrak{m}$ its residue field, an algebraic extension of K . The points of X with values in Ω then correspond, bijectively, to the Ω -sections of $X \otimes_K \Omega$ ((3.4.1) and (3.3.14)), and also to the K -homomorphisms from L to Ω (1.7.3), whence the proposition (Bourbaki, *Alg.*, chap. V, Â§7, n° 5, prop. 8), taking (6.4.3) into account. \square

(6.4.8). The number n defined in (6.4.7) is called the *separable rank* of A (or of X) over K , or also the *geometric number of points* of X ; it is equal to the number of elements of $X(\Omega)_K$. It follows immediately from this definition that, for every extension K' of K , $X \otimes_K K'$ has the same geometric number of points as X . If we denote this number by $n(X)$, it is clear that, if X and Y are two schemes, finite over K , then

$$(6.4.8.1) \quad n(X \sqcup Y) = n(X) + n(Y).$$

Under the same hypotheses, we also have

$$(6.4.8.2) \quad n(X \times_K Y) = n(X)n(Y)$$

because of the interpretation of $n(X)$ as the number of elements of $X(\Omega)_K$ and Equation (3.4.3.1).

Proposition (6.4.9). — *Let K be a field, X and Y algebraic K -preschemes, $f : X \rightarrow Y$ a K -morphism, and Ω an algebraically closed extension of K of infinite transcendence degree over K . For f to be surjective, it is necessary and sufficient that the map $X(\Omega)_K \rightarrow Y(\Omega)_K$ that corresponds to f (3.4.1) be surjective.*

Proof. The necessity follows from (6.3.10), noting that f is necessarily of finite type (6.3.9). To see that the condition is sufficient, we argue as in (6.3.10), noting that, for every $y \in Y$, $k(y)$ is an extension of K of finite type, and so is K -isomorphic to a subfield of Ω . \square

Remark (6.4.10). — We will see in chapter IV that the conclusion of (6.4.9) still holds without the hypothesis on the transcendence degree of Ω over K .

Proposition (6.4.11). — *If $f : X \rightarrow Y$ is a morphism of finite type, then, for every $y \in Y$, the fibre $f^{-1}(y)$ is an algebraic prescheme over the residue field $k(y)$, and for every $x \in f^{-1}(y)$, $k(x)$ is an extension of $k(y)$ of finite type.*

Proof. Since $f^{-1}(y) = X \otimes_Y k(y)$ (6.3.6), the proposition follows from (6.3.4, iv) and (6.3.3). \square

Proposition (6.4.12). — *Let $f : X \rightarrow Y$ and $g : Y' \rightarrow Y$ be morphisms; set $X' = X \times_Y Y'$, and let $f' = f_{(Y')} : X' \rightarrow Y'$. Let $y' \in Y'$ and set $y = g(y')$; if the fibre $f^{-1}(y)$ is a finite algebraic scheme over $k(y)$, then the fibre $f'^{-1}(y')$ is a finite algebraic scheme over $k(y')$, and has the same rank and geometric number of points as $f^{-1}(y)$ does.*

Proof. Taking into account the transitivity of fibres (3.6.5), this follows immediately from (6.4.6) and (6.4.8). \square

(6.4.13). Proposition (6.4.11) shows that the morphisms of finite type that correspond, intuitively, to the “algebraic families of algebraic varieties”, with the points of Y playing the role of “parameters”, which gives these morphisms a “geometric” meaning. The morphisms which are not of finite type will show up in the following mostly in questions of “changing the base prescheme”, by localisation or completion, for example.

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6.5. Local determination of a morphism.

Proposition (6.5.1). — *Let X and Y be S -preschemes, with Y of finite type over S ; let $x \in X$ and $y \in Y$ lie over the same point $s \in S$.*

- (i) *If two S -morphisms $f = (\psi, \theta)$ and $f' = (\psi', \theta')$ from X to Y are such that $\psi(x) = \psi'(x) = y$, and the (local) \mathcal{O}_S -homomorphisms θ_x^\sharp and θ'_x^\sharp from \mathcal{O}_y to \mathcal{O}_x are identical, then f and f' agree on an open neighbourhood of x .*
- (ii) *Suppose further that S is locally Noetherian. For every local \mathcal{O}_S -homomorphism $\phi : \mathcal{O}_y \rightarrow \mathcal{O}_x$, there exists an open neighbourhood U of x in X , and an S -morphism $f = (\psi, \theta)$ from U to Y such that $\psi(x) = y$ and $\theta_x^\sharp = \phi$.*

Proof.

- (i) Since the question is local on S , X , and Y , we can assume that S , X , and Y are affine, given by rings A , B , and C (respectively), and with f and f' of the form $(^a\phi, \tilde{\phi})$ and $(^a\phi', \tilde{\phi}')$ (respectively), where ϕ and ϕ' are A -homomorphisms from C to B such that $\phi^{-1}(j_x) = \phi'^{-1}(j_x) = j_y$, and the homomorphisms ϕ_x and ϕ'_x from C_y to B_x , induced by ϕ and ϕ' , are identical; we can further suppose that C is an A -algebra of finite type. Let c_i ($1 \leq i \leq n$) be the generators of the A -algebra C , and set $b_i = \phi(c_i)$ and $b'_i = \phi'(c_i)$; by hypothesis, we have $b_i/1 = b'_i/1$ in the ring of fractions B_x ($1 \leq i \leq n$). This implies that there exist elements $s_i \in B - j_x$ such that $s_i(b_i - b'_i) = 0$ for $1 \leq i \leq n$, and we can clearly assume that all the s_i are equal to a single element $g \in B - j_x$. From this, we conclude that we have $b_i/1 = b'_i/1$ for $1 \leq i \leq n$ in the ring of fractions B_g ; if i_g is the canonical homomorphism $B \rightarrow B_g$, we then have $i_g \circ \phi = i_g \circ \phi'$; so the restrictions of f and f' to $D(g)$ are identical.

- (ii) We can restrict to the situation as in (i), and further assume that the ring A is Noetherian. Let c_i ($1 \leq i \leq n$) be the generators of the A -algebra C , and let $\alpha : A[X_1, \dots, X_n] \rightarrow C$ be the homomorphism of polynomial algebras that sends X_i to c_i for $1 \leq i \leq n$. Also let i_y be the canonical homomorphism $C \rightarrow C_y$, and consider the composite homomorphism

$$\beta : A[X_1, \dots, X_n] \xrightarrow{\alpha} C \xrightarrow{i_y} C_y \xrightarrow{\phi} B_x.$$

We denote by \mathfrak{a} the kernel of β ; since A is Noetherian, so too is $A[X_1, \dots, X_n]$, and so \mathfrak{a} admits a finite system of generators $Q_j(X_1, \dots, X_n)$ ($1 \leq j \leq m$). Furthermore, each of the elements $\phi(i_y(c_i))$ can be written in the form b_i/s_i , where $b_i \in B$ and $s_i \notin \mathfrak{j}_x$; we can further assume that all of the s_i are equal to a single element $g \in B - \mathfrak{j}_x$. With this, by hypothesis, we have $Q_j(b_1/g, \dots, b_n/g) = 0$ in B_x ; set

$$Q_j(X_1/T, \dots, X_n/T) = P_j(X_1, \dots, X_n, T)/T^{k_j}$$

where P_j is homogeneous of degree k_j . Then let $d_j = P_j(b_1, \dots, b_n, g) \in B$. By hypothesis, we have $t_j d_j = 0$ for some $t_j \in B - \mathfrak{j}_x$ ($1 \leq j \leq m$), and we can clearly assume that all the t_j are equal to a single element $h \in B - \mathfrak{j}_x$; from this we conclude that $P_j(hb_1, \dots, hb_n, hg) = 0$ for $1 \leq j \leq m$. With this, consider the homomorphism ρ from $A[X_1, \dots, X_n]$ to the ring of fractions B_{hg} which sends X_i to hb_i/hg ($1 \leq i \leq n$); the image of \mathfrak{a} under this homomorphism is 0, and is *a fortiori* the same as the image of the kernel $\alpha^{-1}(0)$ under ρ . So ρ factors as $A[X_1, \dots, X_n] \xrightarrow{\alpha} C \xrightarrow{\gamma} B_{hg}$, with $\gamma(c_i) = hb_i/hg$, and it is clear that, if i_x is the canonical homomorphism $B_{hg} \rightarrow B_x$, then the diagram

$$(6.5.1.1) \quad \begin{array}{ccc} C & \xrightarrow{\gamma} & B_{hg} \\ i_y \downarrow & & \downarrow i_x \\ C_y & \xrightarrow{\phi} & B_x \end{array}$$

is commutative; we thus have $\phi = \gamma_x$, and since ϕ is a local homomorphism, ${}^a\gamma(x) = y$; $f = ({}^a\gamma, \tilde{\gamma})$ is thus an S -morphism from the neighbourhood $D(hg)$ of x to Y as claimed in the proposition. □

Corollary (6.5.2). — Under the hypotheses of (6.5.1, ii), if, further, X is of finite type over S , then we can assume that the morphism f is of finite type.

Proof. This follows from Corollary (6.3.6). □

Corollary (6.5.3). — Suppose that the hypotheses of Proposition (6.5.1, ii), and suppose further that Y is integral, and that ϕ is an injective homomorphism. Then we can assume that $f = ({}^a\gamma, \tilde{\gamma})$, where γ is injective.

Proof. Indeed, we can assume C to be integral (5.1.4), hence i_y injective; it then follows from the diagram (6.5.1.1) that γ is injective. □

Proposition (6.5.4). — Let $f = (\psi, \theta) : X \rightarrow Y$ be a morphism of finite type, x a point of X , and $y = \psi(x)$.

- (i) For f to be a local immersion at the point x (4.5.1), it is necessary and sufficient that $\theta_x^\sharp : \mathcal{O}_y \rightarrow \mathcal{O}_x$ be surjective.
- (ii) Assume further that Y is locally Noetherian. For f to be a local isomorphism at the point x (4.5.2), it is necessary and sufficient that θ_x^\sharp be an isomorphism.

Proof.

- (ii) By (6.5.1), there exists an open neighbourhood V of Y and a morphism $g : V \rightarrow X$ such that $g \circ f$ (resp. $f \circ g$) is defined and agrees with the identity on a neighbourhood of x (resp. y), whence we can easily see that f is a local isomorphism.

- (i) Since the question is local on X and Y , we can assume that X and Y are affine, given by rings A and B (respectively); we have $f = ({}^a\phi, \tilde{\phi})$, where ϕ is a homomorphism of rings $B \rightarrow A$ that makes A a B -algebra of finite type; we have $\phi^{-1}(\mathfrak{j}_x) = \mathfrak{j}_y$, and the homomorphism $\phi_x : B_y \rightarrow A_x$ induced by ϕ is *surjective*. Let (t_i) ($1 \leq i \leq n$) be a system of generators of the B -algebra A ; the hypothesis on ϕ_x implies that there exist $b_i \in B$ and some $c \in B - \mathfrak{j}_x$ such that, in the ring of fractions A_x , we have $t_i/1 = \phi(b_i)/\phi(c)$ for $1 \leq i \leq n$. Then (1.3.3) there exists some $a \in A - \mathfrak{j}_x$ such that, if we let $g = a\phi(c)$, we also have $t_i/1 = a\phi(b_i)/g$ in the ring of fractions A_g . With this, there exists, by hypothesis, a polynomial $Q(X_1, \dots, X_n)$, with coefficients in the ring $\phi(B)$, such that $a = Q(t_1, \dots, t_n)$; let $Q(X_1/T, \dots, X_n/T) = P(X_1, \dots, X_n, T)/T^m$, where P is homogeneous of degree m . In the ring A_g , we have

$$a/1 = a^m P(\phi(b_1), \dots, \phi(b_n), \phi(c))/g^m = a^m \phi(d)/g^m$$

where $d \in B$. Since, in A_g , $g/1 = (a/1)(\phi(c)/1)$ is invertible by definition, so too are $a/1$ and $\phi(c)/1$, and we can thus write $a/1 = (\phi(d)/1)(\phi(c)/1)^{-m}$. From this we conclude that $\phi(d)/1$ is also invertible in A_g . So let $h = cd$; since $\phi(h)/1$ is invertible in A_g , the composite homomorphism $B \xrightarrow{\phi} A \rightarrow A_g$ factors as $B \rightarrow B_h \xrightarrow{\gamma} A_g$ (0, 1.2.4). We will show that γ is *surjective*; it suffices to show that the image of B_h in A_g contains the $t_i/1$ and $(g/1)^{-1}$. But we have $(g/1)^{-1} = (\phi(c)/1)^{m-1}(\phi(d)/1)^{-1} = \gamma(c^m/h)$, and $a/1 = \gamma(d^{m+1}/h^m)$, so $(a\phi(b_i))/1 = \gamma(b_i d^{m+1}/h^m)$, and since $t_i/1 = (a\phi(b_i)/1)(g/1)^{-1}$, our claim is proved. The choice of h implies that $\psi(D(g)) \subset D(h)$, and we also know that the restriction of f to $D(g)$ is equal to $({}^a\gamma, \tilde{\gamma})$; since γ is surjective, this restriction is a closed immersion of $D(g)$ into $D(h)$ (4.2.3). □

Corollary (6.5.5). — Let $f = (\psi, \theta) : X \rightarrow Y$ be a morphism of finite type. Assume that X is irreducible, and denote by x its generic point, and let $y = \psi(x)$.

- (i) For f to be a local immersion at any point of X , it is necessary and sufficient that $\theta_x^\# : \mathcal{O}_y \rightarrow \mathcal{O}_x$ be surjective.
- (ii) Assume further that Y is irreducible and locally Noetherian. For f to be a local isomorphism at any point of X , it is necessary and sufficient that y be the generic point of Y (or, equivalently (0, 2.1.4), that f be a dominant morphism) and that $\theta_x^\#$ be an isomorphism (in other words, that f be birational (2.2.9)).

Proof. It is clear that (i) follows from (6.5.4, i), taking into account the fact that every nonempty open subset of X contains x ; similarly, (ii) follows from (6.5.4, ii). □

6.6. Quasi-compact morphisms and morphisms locally of finite type.

Definition (6.6.1). — We say that a morphism $f : X \rightarrow Y$ is *quasi-compact* if the inverse image of any quasi-compact open subset of Y under f is quasi-compact.

Let \mathfrak{B} be a base of the topology of Y consisting of quasi-compact open subsets (for example, affine open subsets); for f to be quasi-compact, it is necessary and sufficient that the inverse image of every set of \mathfrak{B} under f be quasi-compact (or, equivalently, a finite union of affine open subsets), because every quasi-compact open subset of Y is a finite union of sets of \mathfrak{B} . For example, if X is *quasi-compact* and Y *affine*, then every morphism $f : X \rightarrow Y$ is quasi-compact: indeed, X is a finite union of affine open subsets U_i , and for every affine open subset V of Y , $U_i \cap f^{-1}(V)$ is affine (5.5.10), and so quasi-compact.

If $f : X \rightarrow Y$ is a quasi-compact morphism, it is clear that, for every open subset V of Y , the restriction of f to $f^{-1}(V)$ is a quasi-compact morphism $f^{-1}(V) \rightarrow V$. Conversely, if (U_α) is an open cover of Y , and $f : X \rightarrow Y$ a morphism such that the restrictions $f^{-1}(U_\alpha) \rightarrow U_\alpha$ are quasi-compact, then f is quasi-compact.

Definition (6.6.2). — We say that a morphism $f : X \rightarrow Y$ is *locally of finite type* if, for every $x \in X$, there exists an open neighbourhood U of x and an open neighbourhood $V \supset f(U)$ of y such that the restriction of f to U is a morphism of finite type from U to V . We then also say that X is a prescheme I | 153

locally of finite type over Y , or a Y -prescheme locally of finite type.

It follows immediately from (6.3.2) that, if f is locally of finite type, then, for every open subset W of Y , the restriction of f to $f^{-1}(W)$ is a morphism $f^{-1}(W) \rightarrow W$ that is locally of finite type.

If Y is locally Noetherian and X locally of finite type over Y , then X is locally Noetherian thanks to (6.3.7).

Proposition (6.6.3). — *For a morphism $f : X \rightarrow Y$ to be of finite type, it is necessary and sufficient that it be quasi-compact and locally of finite type.*

Proof. The necessity of the conditions is immediate, given (6.3.1) and the remark following (6.6.1). Conversely, suppose that the conditions are satisfied, and let U be an affine open subset of Y , given by some ring A ; for all $x \in f^{-1}(U)$, there is, by hypothesis, a neighbourhood $V(x) \subset f^{-1}(U)$ of x , and a neighbourhood $W(x) \subset U$ of $y = f(x)$ containing $f(V(x))$, and such that the restriction of f to $V(x)$ is a morphism $V(x) \rightarrow W(x)$ of finite type. Replacing $W(x)$ with a neighbourhood $W_1(x) \subset W(x)$ of x of the form $D(g)$ (with $g \in A$), and $V(x)$ with $V(x) \cap f^{-1}(W_1(x))$, we can assume that $W(x)$ is of the form $D(g)$, and thus of finite type over U (because its ring can be written as $A[1/g]$); so $V(x)$ is of finite type over U . Further, $f^{-1}(U)$ is quasi-compact by hypothesis, and so the finite union of open subsets $V(x_i)$, which finishes the proof. \square

Proposition (6.6.4). —

- (i) *An immersion $X \rightarrow Y$ is quasi-compact if it is closed, or if the underlying space of Y is locally Noetherian, or if the underlying space of X is Noetherian.*
- (ii) *The composition of any two quasi-compact morphisms is quasi-compact.*
- (iii) *If $f : X \rightarrow Y$ is a quasi-compact S -morphism, then so too is $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ for any extension $g : S \rightarrow S'$ of the base prescheme.*
- (iv) *If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are two quasi-compact S -morphisms, then $f \times_S g$ is quasi-compact.*
- (v) *If the composition of any two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ is quasi-compact, and if either g is separated or the underlying space of X is locally Noetherian, then f is quasi-compact.*
- (vi) *For a morphism f to be quasi-compact, it is necessary and sufficient that f_{red} be quasi-compact.*

Proof. We note that (vi) is evident because the property of being quasi-compact, for a morphism, depends only on the corresponding continuous map of underlying spaces. We will similarly prove the part of (v) corresponding to the case where the underlying space of X is locally Noetherian. Set $h = g \circ f$, and let U be a quasi-compact open subset of Y ; $g(U)$ is quasi-compact (but not necessarily open) in Z , and so contained in a finite union of quasi-compact open subsets V_j (2.1.3), and $f^{-1}(U)$ is thus contained in the union of the $h^{-1}(V_j)$, which are quasi-compact subspaces of X , and thus Noetherian subspaces. We thus conclude (0, 2.2.3) that $f^{-1}(U)$ is a Noetherian space, and *a fortiori* quasi-compact.

To prove the other claims, it suffices to prove (i), (ii), and (iii) (5.5.12). But (ii) is evident, and (i) follows from (6.3.5) whenever the space Y is locally Noetherian or the space X is Noetherian, and is evident for a closed immersion. To show (iii), we can restrict to the case where $S = Y$ (3.3.11); let $f' = f_{(S')}$, and let U' be a quasi-compact open subset of S' . For every $s' \in U'$, let T be an affine open neighbourhood of $g(s')$ in S , and let W be an affine open neighbourhood of s' contained in $U' \cap g^{-1}(T)$; it will suffice to show that $f'^{-1}(W)$ is quasi-compact; in other words, we can restrict to showing that, when S and S' are affine, the underlying space of $X \times_S S'$ is quasi-compact. But since X is then, by hypothesis, a finite union of affine open subsets V_j , $X \times_S S'$ is a union of the underlying spaces of the affine schemes $V_j \times_S S'$ ((3.2.2) and (3.2.7)), which proves the proposition. \square

We note also that, if $X = X' \sqcup X''$ is the sum of two preschemes, a morphism $f : X \rightarrow Y$ is quasi-compact if and only if its restrictions to both X' and X'' are quasi-compact.

Proposition (6.6.5). — *Let $f : X \rightarrow Y$ be a quasi-compact morphism. For f to be dominant, it is necessary and sufficient that, for every generic point y of an irreducible component of Y , $f^{-1}(y)$ contain the generic point of an irreducible component of X .*

Proof. It is immediate that the condition is sufficient (even without assuming that f is quasi-compact). To see that it is necessary, consider an affine open neighbourhood U of y ; $f^{-1}(U)$ is quasi-compact,

and so a *finite* union of affine open subsets V_i , and the hypothesis that f be dominant implies that y belongs to the closure in U of one of the $f(V_i)$. We can clearly assume X and Y to be reduced; since the closure in X of an irreducible component of V_i is an irreducible component on X (0, 2.1.6), we can replace X by V_i , and Y by the closed reduced subscheme of U that has $\overline{f(V_i)} \cap U$ as its underlying space (5.2.1), and we are thus led to proving the proposition when $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine and reduced. Since f is dominant, B is a subring of A (1.2.7), and the proposition then follows from the fact that every minimal prime ideal of B is the intersection of B with a minimal prime ideal of A (0, 1.5.8). \square

Proposition (6.6.6). —

- (i) Every local immersion is locally of finite type.
- (ii) If two morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are locally of finite type, then so too is $g \circ f$.
- (iii) If $f : X \rightarrow Y$ is an S -morphism locally of finite type, then $f_{(S')} : X_{(S')} \rightarrow Y_{(S')}$ is locally of finite type for any extension $S' \rightarrow S$ of the base prescheme.
- (iv) If $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ are S -morphisms locally of finite type, then $f \times_S g$ is locally of finite type.
- (v) If the composition $g \circ f$ of two morphisms is locally of finite type, then f is locally of finite type.
- (vi) If a morphism f is locally of finite type, then so too is f_{red} .

Proof. By (5.5.12), it suffices to prove (i), (ii), and (iii). If $j : X \rightarrow Y$ is a local immersion then, for every $x \in X$, there is an open neighbourhood V of $j(x)$ in Y and an open neighbourhood U of x in X such that the restriction of j to U is a closed immersion $U \rightarrow V$ (4.5.1), and so this restriction is of finite type. To prove (ii), consider a point $x \in X$; by hypothesis, there is an open neighbourhood W of $g(f(x))$ and an open neighbourhood V of $f(x)$ such that $g(V) \subset W$ and such that V is of finite type over W ; furthermore, $f^{-1}(V)$ is locally of finite type over V (6.6.2), so there is an open neighbourhood U of x that is contained in $f^{-1}(V)$ and of is finite type over V ; thus we have $g(f(U)) \subset W$, and that U is of finite type over W (6.3.4, ii). Finally, to prove (iii), we can restrict to the case where $Y = S$ (3.3.11); for every $x' \in X' = X_{(S')}$, let x be the image of x' in X , s the image of x in S , T an open neighbourhood of s , T' the inverse image of T in S' , and U an open neighbourhood of x that is of finite type over T and whose image is contained in T ; then $U \times_S T' = U \times_T T'$ is an open neighbourhood of x' (3.2.7) that is of finite type over T' (6.3.4, iv). \square

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Corollary (6.6.7). — Let X and Y be S -preschemes that are locally of finite type over S . If S is locally Noetherian, then $X \times_S Y$ is locally Noetherian.

Proof. Indeed, X being locally of finite type over S means that it is locally Noetherian, and that $X \times_S Y$ is locally of finite type over X , and so $X \times_S Y$ is also locally Noetherian. \square

Remark (6.6.8). — Proposition (6.3.10) and its proof extend immediately to the case where we suppose only that the morphism f is locally of finite type. Similarly, propositions (6.4.2) and (6.4.9) hold true when we suppose only that the preschemes X and Y in the claim are locally of finite type over the field K .

§7. RATIONAL MAPS

7.1. Rational maps and rational functions.

(7.1.1). Let X and Y be preschemes, U and V dense open subsets of X , and f (resp. g) a morphism from U (resp. V) to Y ; we say that f and g are *equivalent* if they agree on a dense open subset of $U \cap V$. Since a finite intersection of dense open subsets of X is a dense open subset of X , it is clear that this relation is an *equivalence relation*.

Definition (7.1.2). — Given preschemes X and Y , we define a rational map from X to Y to be an equivalence class of morphisms from a dense open subset of X to a dense open subset of Y , under the equivalence relation defined in (7.1.1). If X and Y are S -preschemes, we say that such a class is a rational S -map if there exists a representative of the class that is also an S -morphism. We define a rational S -section of X to be any rational S -map from S to X . We define a rational

function on a prescheme X to be any rational X -section on the X -prescheme $X \otimes_{\mathbf{Z}} \mathbf{Z}[T]$ (where T is an indeterminate).

By an abuse of language, whenever we are discussing only S -preschemes, we will say “rational map” instead of “rational S -map” if no confusion may arise.

Let f be a rational map from X to Y , and U an open subset of X ; if f_1 and f_2 are two morphisms belonging to the class of f , defined (respectively) on dense open subsets V and W of X , then the restrictions $f_1|_{(U \cap V)}$ and $f_2|_{(U \cap W)}$ agree on $U \cap V \cap W$, which is dense in U ; the class f of morphisms thus defines a rational map from U to Y , called the *restriction of f to U* , and denoted by $f|_U$.

If, to every S -morphism $f : X \rightarrow Y$, we take the corresponding rational S -map to which f belongs, we obtain a canonical map from $\text{Hom}_S(X, Y)$ to the set of rational S -maps from X to Y . We denote by $\Gamma_{\text{rat}}(X/Y)$ the set of rational Y -sections on X , and we thus have a canonical map $\Gamma(X/Y) \rightarrow \Gamma_{\text{rat}}(X/Y)$. It is also clear that, if X and Y are S -preschemes, then the set of rational S -maps from X to Y is canonically identified with $\Gamma_{\text{rat}}((X \times_S Y)/X)$ (3.3.14).

(7.1.3). It also follows from (7.1.2) and (3.3.14) that the rational functions on X are canonically identified with *equivalence classes of sections of the structure sheaf \mathcal{O}_X* over dense open subsets of X , where two such sections are equivalent if they agree on some dense open subset of X contained inside the intersection of the subsets on which they are defined. In particular, it follows that the rational functions on X form a *ring* $R(X)$.

(7.1.4). When X is an *irreducible* prescheme, every nonempty open subset of X is dense in X ; so we can say that the nonempty open subsets of X are the *open neighbourhoods of the generic point x* of X . To say that two morphisms from nonempty open subsets of X to Y are equivalent thus means, in this case, that they have the *same germ* at the point x . In other words, the rational maps (resp. rational S -maps) $X \rightarrow Y$ are identified with the *germs of morphisms* (resp. *S -morphisms*) from nonempty open subsets of X to Y at the generic point x of X . In particular:

Proposition (7.1.5). — *If X is an irreducible prescheme, then the ring $R(X)$ of rational maps on X is canonically identified with the local ring \mathcal{O}_x of the generic point x of X . It is a local ring of dimension 0, and thus a local Artinian ring when X is Noetherian; it is a field when X is integral, and, when X is further an affine scheme, it is identified with the field of fractions of $A(X)$.*

Proof. Given the above, and the identification of rational functions with sections of \mathcal{O}_X over a dense open subset of X , the first claim is nothing but the definition of the fibre of a sheaf above a point. For the other claims, we can reduce to the case where X is affine, given by some ring A ; then \mathfrak{j}_x is the nilradical of A , and \mathcal{O}_x is thus of dimension 0; if A is integral, then $\mathfrak{j}_x = (0)$, and \mathcal{O}_x is thus the field of fractions of A . Finally, if A is Noetherian, we know ([Sam53, p. 127, cor. 4]) that \mathfrak{j}_x is nilpotent, and $\mathcal{O}_x = A_x$ Artinian. \square

If X is *integral*, then the ring \mathcal{O}_z is integral for all $z \in X$; every affine open subset U containing z also contains x , and $R(U)$, being equal to the field of fractions of $A(U)$, is identified with $R(X)$; we thus conclude that $R(X)$ can also be identified with the *field of fractions of \mathcal{O}_z* : the canonical identification of \mathcal{O}_z to a subring of $R(X)$ consists of associating, to every germ of a section $s \in \mathcal{O}_z$, the unique rational function on X , class of a section of \mathcal{O}_X , (necessarily defined on a dense open subset of X) having s as its germ at the point z .

(7.1.6). Now suppose that X has a *finite* number of irreducible components X_i ($1 \leq i \leq n$) (which will be the case whenever the underlying space of X is *Noetherian*); let X'_i be the open subset of X given by the complement of the $X_j \cap X_i$ for $j \neq i$ inside X_i ; X'_i is irreducible, its generic point x_i is the generic point of X_i , and the X'_i are pairwise disjoint, with their union being dense in X (0, 2.1.6). For every dense open subset U of X , $U_i = U \cap X'_i$ is a nonempty dense open subset of X'_i , with the U_i being pairwise disjoint, and so $U' = \bigcup_i U'_i$ is dense in X . Giving a morphism from U' to Y consists of giving (arbitrarily) a morphism from each of the U_i to Y .

Thus:

Proposition (7.1.7). — *Let X and Y be two preschemes (resp. S -preschemes) such that X has a finite number of irreducible components X_i , with generic points x_i ($1 \leq i \leq n$). If R_i is the set of germs of morphisms (resp.*

S -morphisms) from open subsets of X to Y at the point x_i , then the set of rational maps (resp. rational S -maps) from X to Y can be identified with the product of the R_i ($1 \leq i \leq n$).

Corollary (7.1.8). — Let X be a Noetherian prescheme. The ring of rational functions on X is an Artinian ring, whose local components are the rings \mathcal{O}_{x_i} of the generic points x_i of the irreducible components of X .

Corollary (7.1.9). — Let A be a Noetherian ring, and $X = \text{Spec}(A)$. If Q is the complement of the union of the minimal prime ideals of A , then the ring of rational functions on X can be canonically identified with the ring of fractions $Q^{-1}A$.

This will follow from the following lemma:

Lemma (7.1.9.1). — For an element $f \in A$ to be such that $D(f)$ is dense in X , it is necessary and sufficient that $f \in Q$; every dense open subset of X contains an open subset of the form $D(f)$, where $f \in Q$.

Proof. To show (7.1.9.1), we again denote by X_i ($1 \leq i \leq n$) the irreducible components of X ; if $D(f)$ is dense in X then $D(f) \cap X_i \neq \emptyset$ for $1 \leq i \leq n$, and vice-versa; but this means that $f \notin \mathfrak{p}_i$ for $1 \leq i \leq n$, where we set $\mathfrak{p}_i = \mathfrak{j}(X_i)$, and since the \mathfrak{p}_i are the minimal prime ideals of A (1.1.14), the conditions $f \notin \mathfrak{p}_i$ ($1 \leq i \leq n$) are equivalent to $f \in Q$, whence the first claim of the lemma. For the other claim, if U is a dense open subset of X , the complement of U is a set of the form $V(\mathfrak{a})$, where \mathfrak{a} is an ideal which is not contained in any of the \mathfrak{p}_i ; it is thus not contained in their union ([Nor53, p. 13]), and there thus exists some $f \in \mathfrak{a}$ belonging to Q ; whence $D(f) \subset U$, which finishes the proof. \square

(7.1.10). Suppose again that X is irreducible, with generic point x . Since every nonempty open subset U of X contains x , and thus also contains every $z \in X$ such that $x \in \overline{\{z\}}$, every morphism $U \rightarrow Y$ can be composed with the canonical morphism $\text{Spec}(\mathcal{O}_x) \rightarrow X$ (2.4.1); and any two morphisms into Y from two nonempty open subsets of X which agree on a nonempty open subset of X give, by composition, the same morphism $\text{Spec}(\mathcal{O}_x) \rightarrow Y$. In other words, to every rational map from X to Y there is a corresponding well-defined morphism $\text{Spec}(\mathcal{O}_x) \rightarrow Y$.

Proposition (7.1.11). — Let X and Y be two S -preschemes; suppose that X is irreducible with generic point x , and that Y is of finite type over S . Any two rational S -maps from X to Y that correspond to the same S -morphism $\text{Spec}(\mathcal{O}_x) \rightarrow Y$ are then identical. If we further suppose S to be locally Noetherian, then every S -morphism from $\text{Spec}(\mathcal{O}_x)$ to Y corresponds to exactly one rational S -map from X to Y .

Proof. Taking into account that every nonempty subset of X is dense in X , this follows from (6.5.1). \square

Corollary (7.1.12). — Suppose that S is locally Noetherian, and that the other hypotheses of (7.1.11) are satisfied. The rational S -maps from X to Y can then be identified with points of the S -prescheme Y , with values in the S -prescheme $\text{Spec}(\mathcal{O}_x)$.

Proof. This is nothing but (7.1.11), with the terminology introduced in (3.4.1). \square

Corollary (7.1.13). — Suppose that the conditions of (7.1.12) are satisfied. Let s be the image of x in S . The data of a rational S -map from X to Y is equivalent to the data of a point y of Y over s along with a local \mathcal{O}_s -homomorphism $\mathcal{O}_y \rightarrow \mathcal{O}_x = R(X)$.

Proof. This follows from (7.1.11) and (2.4.4). \square

In particular:

Corollary (7.1.14). — Under the conditions of (7.1.12), rational S -maps from X to Y depend only (for any given Y) on the S -prescheme $\text{Spec}(\mathcal{O}_x)$, and, in particular, remain the same whenever X is replaced by $\text{Spec}(\mathcal{O}_z)$, for any $z \in X$.

Proof. Since $z \in \overline{\{x\}}$, x is the generic point of $Z = \text{Spec}(\mathcal{O}_z)$, and $\mathcal{O}_{X,x} = \mathcal{O}_{Z,z}$. \square

When X is integral, $R(X) = \mathcal{O}_x = k(x)$ is a field (7.1.5); the preceding corollaries then specialize to the following:

Corollary (7.1.15). — Suppose that the conditions of (7.1.12) are satisfied, and further that X is integral. Let s be the image of x in S . Then rational S -maps from X to Y can be identified with the geometric points of $Y \otimes_S k(s)$ with values in the extension $R(X)$ of $k(s)$, or, in other words, every such map is equivalent to the data of a point $y \in Y$ above s along with a $k(s)$ -monomorphism from $k(y)$ to $k(x) = R(X)$.

Proof. The points of Y above s are identified with the points of $Y \otimes_S k(s)$ (3.6.3), and the local \mathcal{O}_s -homomorphisms $\mathcal{O}_y \rightarrow R(X)$ with the $k(s)$ -monomorphisms $k(y) \rightarrow R(X)$. \square

More precisely:

Corollary (7.1.16). — Let k be a field, and X and Y two algebraic preschemes over k (6.4.1); suppose further that X is integral. Then the rational k -maps from X to Y can be identified with the geometric points of Y with values in the extension $R(X)$ of k (3.4.4).

7.2. Domain of definition of a rational map.

(7.2.1). Let X and Y be preschemes, and f rational map from X to Y . We say that f is *defined at a point* $x \in X$ if there exists a dense open subset U of X that contains x , and a morphism $U \rightarrow Y$ belonging to the equivalence class of f . The set of points $x \in X$ where f is defined is called the *domain of definition* of f ; it is clear that it is an open dense subset of X .

Proposition (7.2.2). — Let X and Y be S -preschemes such that X is reduced and Y is separated over S . Let f be a rational S -map from X to Y , with domain of definition U_0 . Then there exists exactly one S -morphism $U_0 \rightarrow Y$ belonging to the class of f . I | 159

Since, for every morphism $U \rightarrow Y$ belonging to the class of f , we necessarily have $U \subset U_0$, it is clear that the proposition will be a consequence of the following:

Lemma (7.2.2.1). — Under the hypotheses of (7.2.2), let U_1 and U_2 be two dense open subsets of X , and $f_i : U_i \rightarrow Y$ ($i = 1, 2$) two S -morphisms such that there exists an open subset $V \subset U_1 \cap U_2$, dense in X , and on which f_1 and f_2 agree. Then f_1 and f_2 agree on $U_1 \cap U_2$.

Proof. We can clearly restrict to the case where $X = U_1 = U_2$. Since X (and thus V) is reduced, X is the smallest closed subprescheme of X containing V (5.2.2). Let $g = (f_1, f_2)_S : X \rightarrow Y \times_S Y$; since, by hypothesis, the diagonal $T = \Delta_Y(Y)$ is a closed subprescheme of $Y \times_S Y$, $Z = g^{-1}(T)$ is a closed subprescheme of X (4.4.1). If $h : V \rightarrow Y$ is the common restriction of f_1 and f_2 to V , then the restriction of g to V is $g' = (h, h)_S$, which factors as $g' = \Delta_Y \circ h$; since $\Delta_Y^{-1}(T) = Y$, we have that $g'^{-1}(T) = V$, and so Z is a closed subprescheme of X inducing V , thus containing V , which implies that $Z = X$. From the equation $g^{-1}(T) = X$, we deduce (4.4.1) that g factors as $\Delta_Y \circ f$, where f is a morphism $X \rightarrow Y$, which implies, by the definition of the diagonal morphism, that $f_1 = f_2 = f$. \square

It is clear that the morphism $U_0 \rightarrow Y$ defined in (7.2.2) is the unique morphism of the class f that *cannot be extended* to a morphism from an open subset of X that strictly contains U_0 . Under the hypotheses of (7.2.2), we can thus *identify* the rational maps from X to Y with the *non-extendible* (to strictly larger open subsets) morphisms from dense open subsets of X to Y . With this identification, Proposition (7.2.2) implies:

Corollary (7.2.3). — With the hypotheses from (7.2.2) on X and Y , let U be a dense open subset of X . Then there exists a canonical bijective correspondence between S -morphisms from U to Y and rational S -maps from X to Y that are defined at all points of U .

Proof. By (7.2.2), for every S -morphism f from U to Y , there exists exactly one rational S -map \bar{f} from X to Y which extends f . \square

Corollary (7.2.4). — Let S be a scheme, X a reduced S -prescheme, Y an S -scheme, and $f : U \rightarrow Y$ an S -morphism from a dense open subset U of X to Y . If \bar{f} is the rational \mathbf{Z} -map from X to Y that extends f , then \bar{f} is an S -morphism (and thus the rational S -map from X to Y that extends f).

Proof. Indeed, if $\phi : X \rightarrow S$ and $\psi : Y \rightarrow S$ are the structure morphisms, U_0 the domain of definition of \bar{f} , and j the injection $U_0 \rightarrow X$, then it suffices to show that $\psi \circ \bar{f} = \phi \circ j$, but this follows from (7.2.2.1), since f is an S -morphism. \square

Corollary (7.2.5). — Let X and Y be two S -preschemes; suppose that X is reduced, and that X and Y are separated over S . Let $p : Y \rightarrow X$ be an S -morphism (making Y an X -prescheme), U a dense open subset of X , and f a U -section of Y ; then the rational map \bar{f} from X to Y extending f is a rational X -section of Y .

Proof. We have to show that $p \circ \bar{f}$ is the identity on the domain of definition of \bar{f} ; since X is separated over S , this again follows from (7.2.2.1). I | 160 \square

Corollary (7.2.6). — Let X be a reduced prescheme, and U a dense open subset of X . Then there is a canonical bijective correspondence between sections of \mathcal{O}_X over U and rational functions on X defined at every point of U .

Proof. Taking (7.2.3), (7.1.2), and (7.1.3) into account, it suffices to note that the X -prescheme $X \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ is separated over X (5.5.1, iv). \square

Corollary (7.2.7). — Let Y be a reduced prescheme, $f : X \rightarrow Y$ a separated morphism, U a dense open subset of Y , $g : U \rightarrow f^{-1}(U)$ a U -section of $f^{-1}(U)$, and Z the reduced subprescheme of X that has $\overline{g(U)}$ as its underlying space (5.2.1). For g to be the restriction of a Y -section of X (in other words (7.2.5), for the rational map from Y to X extending g to be defined everywhere), it is necessary and sufficient for the restriction of f to Z to be an isomorphism from Z to Y .

Proof. The restriction of f to $f^{-1}(U)$ is a separated morphism (5.5.1, i), so g is a closed immersion (5.4.6), and so $g(U) = Z \cap f^{-1}(U)$, and the subprescheme induced by Z on the open subset $g(U)$ of Z is identical to the closed subprescheme of $f^{-1}(U)$ associated to g (5.2.1). It is then clear that the stated condition is sufficient, because, if satisfied, and if $f_Z : Z \rightarrow Y$ is the restriction of f to Z , and $\bar{g} : Y \rightarrow Z$ is the inverse isomorphism, then \bar{g} extends g . Conversely, if g is the restriction to U of a Y -section h of X , then h is a closed immersion (5.4.6), and so $h(Y)$ is closed, and, since it is contained in Z , is equal to Z , and it follows from (5.2.1) that h is necessarily an isomorphism from Y to the closed subprescheme Z of X . \square

(7.2.8). Let X and Y be two S -preschemes, with X reduced, and Y separated over S . Let f be a rational S -map from X to Y , and let x be a point of X ; we can compose f with the canonical S -morphism $\text{Spec}(\mathcal{O}_x) \rightarrow X$ (2.4.1) provided that the intersection of $\text{Spec}(\mathcal{O}_x)$ with the domain of definition of f is dense in $\text{Spec}(\mathcal{O}_x)$ (identified with the set of $z \in X$ such that $x \in \overline{\{z\}}$ (2.4.2)). This will happen in the follow cases:

- 1st. X is *irreducible* (and thus *integral*), because then the generic point ξ of X is the generic point of $\text{Spec}(\mathcal{O}_x)$; since the domain of definition U of f contains ξ , $U \cap \text{Spec}(\mathcal{O}_x)$ contains ξ , and so is dense in $\text{Spec}(\mathcal{O}_x)$.
- 2nd. X is *locally Noetherian*; our claim then follows from:

Lemma (7.2.8.1). — Let X be a prescheme whose underlying space is locally Noetherian, and x a point of X . The irreducible components of $\text{Spec}(\mathcal{O}_x)$ are the intersections of $\text{Spec}(\mathcal{O}_x)$ with the irreducible components of X containing x . For an open subset $U \subset X$ to be such that $U \cap \text{Spec}(\mathcal{O}_x)$ is dense in $\text{Spec}(\mathcal{O}_x)$, it is necessary and sufficient for it to have a nonempty intersection with the irreducible components of X that contain x (which will be the case whenever U is dense in X).

Proof. It suffices to show just the first claim, since the second then follows. Since $\text{Spec}(\mathcal{O}_x)$ is contained in every affine open subset U that contains x , and since the irreducible components of U that contain x are the intersections of U with the irreducible components of X containing x (0, 2.1.6), we can suppose that X is affine, given by some ring A . Since the prime ideals of A_x correspond bijectively to the prime ideals of A that are contained in \mathfrak{j}_x (2.1.6), the minimal prime ideals of A_x correspond to the minimal prime ideals of A that are contained in \mathfrak{j}_x , hence the lemma. I | 161 \square

With this in mind, suppose that we are in one of the two cases mentioned in (7.2.8). If U is the domain of definition of the rational S -map f , then we denote by f' the rational map from $\text{Spec}(\mathcal{O}_x)$ to Y which agrees (taking (2.4.2) into account) with f on $U \cap \text{Spec}(\mathcal{O}_x)$; we say that this rational map is *induced by* f .

Proposition (7.2.9). — Let S be a locally Noetherian prescheme, X a reduced S -prescheme, and Y an S -scheme of finite type. Suppose further that X is either irreducible or locally Noetherian. Then let f be a

rational S -map from X to Y , and x a point of X . For f to be defined at a point x , it is necessary and sufficient for the rational map f' from $\text{Spec}(\mathcal{O}_x)$ to Y , induced by f (7.2.8), to be a morphism.

Proof. The condition clearly being necessary (since $\text{Spec}(\mathcal{O}_x)$ is contained in every open subset containing x), we show that it is sufficient. By (6.5.1), there exists an open neighbourhood U of x in X , and an S -morphism g from U to Y that induces f' on $\text{Spec}(\mathcal{O}_x)$. If X is irreducible, then U is dense in X , and, by (7.2.3), we can suppose that g is a rational S -map. Further, the generic point of X belongs to both $\text{Spec}(\mathcal{O}_x)$ and the domain of definition of f , and so s and g agree at this point, and thus on a nonempty open subset of X (6.5.1). But since f and g are rational S -maps, they are identical (7.2.3), and so f is defined at x .

If we now suppose that X is locally Noetherian, then we can suppose that U is Noetherian; then there are only a finite number of irreducible components X_i of X that contain x (7.2.8.1), and we can suppose that they are the only ones that have a nonempty intersection with U , by replacing, if needed, U with a smaller open subset (since there are only a finite number of irreducible components of X that have a nonempty intersection with U , because U is Noetherian). We then have, as above, that f and g agree on a nonempty open subset of each of the X_i . Taking into account the fact that each of the X_i is contained in \bar{U} , we consider the morphism f_1 , defined on a dense open subset of $U \cup (X - \bar{U})$, equal to g on U , and to f on the intersection of $X - \bar{U}$ with the domain of definition of f . Since $U \cup (X - \bar{U})$ is dense in X , f_1 and f agree on a dense open subset of X , and since f is a rational map, f is an extension of f_1 (7.2.3), and is thus defined at the point x . \square

7.3. Sheaf of rational functions.

(7.3.1). Let X be a prescheme. For every open subset $U \subset X$, we denote by $R(U)$ the ring of rational functions on U (7.1.3); this is a $\Gamma(U, \mathcal{O}_X)$ -algebra. Further, if $V \subset U$ is a second open subset of X , then every section of \mathcal{O}_X over a dense (in X) open subset of V gives, by restriction to V , a section over a dense (in X) open subset of V , and if two sections agree on a dense (in X) open subset of U , then their restrictions to V agree on a dense (in X) open subset of V . We can thus define a di-homomorphism of algebras $\rho_{V,U} : R(U) \rightarrow R(V)$, and it is clear that, if $U \supset V \supset W$ are open subsets of X , then we have $\rho_{W,U} = \rho_{W,V} \circ \rho_{V,U}$; the $R(U)$ thus define a *presheaf* of algebras on X .

Definition (7.3.2). — We define the sheaf of rational functions on a prescheme X , denoted by $\mathcal{R}(X)$, to be the \mathcal{O}_X -algebra associated to the presheaf defined by the $R(U)$. I | 162

For every prescheme X and open subset $U \subset X$, it is clear that the induced sheaf $\mathcal{R}(X)|_U$ is exactly $\mathcal{R}(U)$.

Proposition (7.3.3). — Let X be a prescheme such that the family (X_λ) of its irreducible components is locally finite (which is the case whenever the underlying space of X is locally Noetherian). Then the \mathcal{O}_X -module $\mathcal{R}(X)$ is quasi-coherent, and for every open subset U of X that has a nonempty intersection with only finitely many of the components X_λ , $R(U)$ is equal to $\Gamma(U, \mathcal{R}(X))$, and can be canonically identified with the direct sum of the local rings of the generic points of the X_λ such that $U \cap X_\lambda \neq \emptyset$.

Proof. We can evidently restrict to the case where X has only a finite number of irreducible components X_i , with generic points x_i ($1 \leq i \leq n$). The fact that $R(U)$ can be canonically identified with the direct sum of the $\mathcal{O}_{X_i} = R(X_i)$ such that $U \cap X_i \neq \emptyset$ then follows from (7.1.7). We will show that the presheaf $U \rightarrow R(U)$ satisfies the sheaf axioms, which will prove that $R(U) = \Gamma(U, \mathcal{R}(X))$. Indeed, it satisfies (F1) by what has already been discussed. To see that it satisfies (F2), consider a cover of an open subset U of X by open subsets $V_\alpha \subset U$; if the $s_\alpha \in R(V_\alpha)$ are such that the restrictions of s_α and s_β to $V_\alpha \cap V_\beta$ agree for every pair of indices, then we can conclude that, for every index i such that $U \cap X_i \neq \emptyset$, the components in $R(X_i)$ of all the s_α such that $V_\alpha \cap X_i \neq \emptyset$ are all the same; denoting this component by t_i , it is clear that the element of $R(U)$ that has the t_i as its components has s_α as its restriction to each V_α . Finally, to see that $\mathcal{R}(X)$ is quasi-coherent, we can restrict to the case where $X = \text{Spec}(A)$ is affine; by taking U to be an affine open subset of the form $D(f)$, where $f \in A$, it follows from the above and from Definition (1.3.4) that we have $\mathcal{R}(X) = \tilde{M}$, where M is the direct sum of the A -modules A_{x_i} . \square

Corollary (7.3.4). — Let X be a reduced prescheme that has only a finite number of irreducible components, and let X_i ($1 \leq i \leq n$) be the closed reduced preschemes of X that have the irreducible components of X as

their underlying spaces (5.2.1). If h_i is the canonical injection $X_i \rightarrow X$, then $\mathcal{R}(X)$ is the direct sum of the \mathcal{O}_X -algebras $(h_i)_*(\mathcal{R}(X_i))$.

Corollary (7.3.5). — If X is irreducible, then every quasi-coherent $\mathcal{R}(X)$ -module \mathcal{F} is a simple sheaf.

Proof. It suffices to show that every $x \in X$ admits a neighbourhood U such that $\mathcal{F}|_U$ is a simple sheaf (0, 3.6.2); in other words, we are led to considering the case where X is affine; we can further suppose that \mathcal{F} is the cokernel of a homomorphism $(\mathcal{R}(X))^I \rightarrow (\mathcal{R}(X))^J$ (0, 5.1.3), and everything then follows from showing that $\mathcal{R}(X)$ is a simple sheaf; but this is evident, because $\Gamma(U, \mathcal{R}(X)) = R(X)$ for every nonempty open subset U , where U contains the generic point of X . \square

Corollary (7.3.6). — If X is irreducible, then, for every quasi-coherent \mathcal{O}_X -module \mathcal{F} , $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}(X)$ is a simple sheaf; if, further, X is reduced (and thus integral), then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}(X)$ is isomorphic to a sheaf of the form $(\mathcal{R}(X))^{(I)}$.

Proof. The second claim follows from the fact that $R(X)$ is a field. \square

Proposition (7.3.7). — Suppose that the prescheme X is locally integral or locally Noetherian. Then $\mathcal{R}(X)$ is a quasi-coherent \mathcal{O}_X -algebra; if, further, X is reduced (which will be the case whenever X is locally integral), then the canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{R}(X)$ is injective. I | 163

Proof. The question being local, the first claim follows from (7.3.3); the second follows from (7.2.3). \square

(7.3.8).¹⁰ Let X and Y be two integral preschemes, which implies that $\mathcal{R}(X)$ (resp. $\mathcal{R}(Y)$) is a quasi-coherent \mathcal{O}_X -module (resp. \mathcal{O}_Y -module) (7.3.3). Let $f : X \rightarrow Y$ be a dominant morphism; then there exists a canonical homomorphism of \mathcal{O}_X -modules

$$(7.3.8.1) \quad \tau : f^*(\mathcal{R}(Y)) \longrightarrow \mathcal{R}(X).$$

Proof. Suppose first that $X = \text{Spec}(A)$ and $Y = \text{Spec}(B)$ are affine, given by integral rings A and B , with f thus corresponding to an injective homomorphism $B \rightarrow A$ which extends to a monomorphism $L \rightarrow K$ from the field of fractions L of B to the field of fractions K of A . The homomorphism (7.3.8.1) then corresponds to the canonical homomorphism $L \otimes_B A \rightarrow K$ (1.6.5). In the general case, for each pair of nonempty affine open sets $U \subset X$ and $V \subset Y$ such that $f(U) \subset V$, we define, as above, a homomorphism $\tau_{U,V}$ and we immediately have that, if $U' \subset U$, $V' \subset V$, $f(U') \subset V'$, then $\tau_{U,V}$ extends $\tau_{U',V'}$, and hence our assertion. If x and y are the generic points of X and Y respectively, then we have $f(x) = y$,

$$(f^*(\mathcal{R}(Y)))_x = \mathcal{O}_y \otimes_{\mathcal{O}_y} \mathcal{O}_x = \mathcal{O}_x$$

(0, 4.3.1) and τ_x is thus an isomorphism. \square

7.4. Torsion sheaves and torsion-free sheaves.

(7.4.1). Let X be an integral scheme. For every \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $\mathcal{O}_X \rightarrow \mathcal{R}(X)$ defines, by tensoring, a homomorphism (again said to be *canonical*) $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}(X)$, which, on each fibre, is exactly the homomorphism $z \rightarrow z \otimes 1$ from \mathcal{F}_x to $\mathcal{F}_x \otimes_{\mathcal{O}_x} R(X)$. The kernel \mathcal{T} of this homomorphism is a \mathcal{O}_X -submodule of \mathcal{F} , called the *torsion sheaf* of \mathcal{F} ; it is quasi-coherent if \mathcal{F} is quasi-coherent ((4.1.1) and (7.3.6)). We say that \mathcal{F} is *torsion free* if $\mathcal{T} = 0$, and that \mathcal{F} is a *torsion sheaf* if $\mathcal{T} = \mathcal{F}$. For every \mathcal{O}_X -module \mathcal{F} , \mathcal{F}/\mathcal{T} is torsion free. We deduce from (7.3.5) that:

Proposition (7.4.2). — If X is an integral prescheme, then every torsion-free quasi-coherent \mathcal{O}_X -module \mathcal{F} is isomorphic to a subsheaf \mathcal{G} of a simple sheaf of the form $(\mathcal{R}(X))^{(I)}$, generated (as a $\mathcal{R}(X)$ -module) by \mathcal{G} .

The cardinality of I is called the *rank* of \mathcal{F} ; for every nonempty affine open subset U of X , the rank of \mathcal{F} is equal to the rank of $\Gamma(U, \mathcal{F})$ as a $\Gamma(U, \mathcal{O}_X)$ -module, as we see by considering the generic point of X , contained in U . In particular:

Corollary (7.4.3). — On an integral prescheme X , every torsion-free quasi-coherent \mathcal{O}_X -module of rank 1 (in particular, every invertible \mathcal{O}_X -module) is isomorphic to a \mathcal{O}_X -submodule of $\mathcal{R}(X)$, and vice versa.

¹⁰[Trans.] This paragraph was changed entirely in the Errata of EGA II.

Corollary (7.4.4). — Let X be an integral prescheme, \mathcal{L} and \mathcal{L}' torsion-free \mathcal{O}_X -modules, and f (resp. f') a section of \mathcal{L} (resp. \mathcal{L}') over X . In order to have $f \otimes f' = 0$, it is necessary and sufficient for one of the sections f and f' to be zero.

Proof. Let x be the generic point of X ; we have, by hypothesis, that $(f \otimes f')_x = f_x \otimes f'_x = 0$. Since \mathcal{L}_x and \mathcal{L}'_x can be identified with \mathcal{O}_x -submodules of the field \mathcal{O}_x , the above equation leads to $f_x = 0$ or $f'_x = 0$, and thus $f = 0$ or $f' = 0$, since \mathcal{L} and \mathcal{L}' are torsion free (7.3.5). \square

Proposition (7.4.5). — Let X and Y be integral preschemes, and $f : X \rightarrow Y$ a dominant morphism. For every torsion-free quasi-coherent \mathcal{O}_X -module \mathcal{F} , $f_*(\mathcal{F})$ is a torsion-free \mathcal{O}_Y -module.

Proof. Since f_* is left exact (0, 4.2.1), it suffices, by (7.4.2), to prove the proposition in the case where $\mathcal{F} = (\mathcal{R}(X))^{(I)}$. But every nonempty open subset U of Y contains the generic point of Y , so $f^{-1}(U)$ contains the generic point of X (0, 2.1.5), so we have that $\Gamma(U, f_*(\mathcal{F})) = \Gamma(f^{-1}(U), \mathcal{F}) = (R(X))^{(I)}$; in other words, $f_*(\mathcal{F})$ is the simple sheaf with fibre $(R(X))^{(I)}$, considered as a $\mathcal{R}(Y)$ -module, and it is clearly torsion free. \square

Proposition (7.4.6). — Let X be an integral prescheme, and x its generic point. For every quasi-coherent \mathcal{O}_X -module \mathcal{F} of finite type, the following conditions are equivalent: (a) \mathcal{F} is a torsion sheaf; (b) $\mathcal{F}_x = 0$; (c) $\text{Supp}(\mathcal{F}) \neq X$.

Proof. By (7.3.5) and (7.4.1), the equations $\mathcal{F}_x = 0$ and $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{R}(X) = 0$ are equivalent, so (a) and (b) are equivalent; then $\text{Supp}(\mathcal{F})$ is closed in X (0, 5.2.2), and since every nonempty open subset of X contains x , (b) and (c) are equivalent. \square

(7.4.7). We generalise (by an abuse of language) the definitions of (7.4.1) to the case where X is a *reduced* prescheme having only a *finite* number of irreducible components; it then follows from (7.3.4) that the equivalence between a) and c) in (7.4.6) still holds true for such a prescheme.

§8. CHEVALLEY SCHEMES

8.1. Allied local rings. For each local ring A , we denote by $\mathfrak{m}(A)$ the maximal ideal of A .

Lemma (8.1.1). — Let A and B be two local rings such that $A \subset B$; Then the following conditions are equivalent.

- (i) $\mathfrak{m}(B) \cap A = \mathfrak{m}(A)$.
- (ii) $\mathfrak{m}(A) \subset \mathfrak{m}(B)$.
- (iii) 1 is not an element of the ideal of B generated by $\mathfrak{m}(A)$.

Proof. It is evident that (i) implies (ii), and (ii) implies (iii); lastly, if (iii) is true, then $\mathfrak{m}(B) \cap A$ contains $\mathfrak{m}(A)$, and does not contain 1, and is thus equal to $\mathfrak{m}(A)$.

When the equivalent conditions of (8.1.1) are satisfied, we say that B *dominates* A ; this is equivalent to saying that the injection $A \rightarrow B$ is a *local* homomorphism. It is clear that, in the set of local subrings of a ring R , the relation given by domination is an order. \square

(8.1.2). Now consider a *field* R . For all subrings A of R , we denote by $L(A)$ the set of local rings $A_{\mathfrak{p}}$, where \mathfrak{p} ranges over the prime spectrum of A ; such local rings are identified with the subrings of R containing A . Since $\mathfrak{p} = (\mathfrak{p}A_{\mathfrak{p}}) \cap A$, the map $\mathfrak{p} \mapsto A_{\mathfrak{p}}$ from $\text{Spec}(A)$ to $L(A)$ is bijective.

Lemma (8.1.3). — Let R be a field, and A a subring of R . For a local subring M of R to dominate a ring $A_{\mathfrak{p}} \in L(A)$, it is necessary and sufficient that $A \subset M$; the local ring $A_{\mathfrak{p}}$ dominated by M is then unique, and corresponds to $\mathfrak{p} = \mathfrak{m}(M) \cap A$.

Proof. If M dominates $A_{\mathfrak{p}}$, then $\mathfrak{m}(M) \cap A_{\mathfrak{p}} = \mathfrak{p}A_{\mathfrak{p}}$, by (8.1.1), whence the uniqueness of \mathfrak{p} ; on the other hand, if $A \subset M$, then $\mathfrak{m}M \cap A = \mathfrak{p}$ is prime in A , and since $A - \mathfrak{p} \subset M$, we have that $A_{\mathfrak{p}} \subset M$ and $\mathfrak{p}A_{\mathfrak{p}} \subset \mathfrak{m}(M)$, so M dominates $A_{\mathfrak{p}}$. \square

Lemma (8.1.4). — Let R be a field, M and N local subrings of R , and P the subring of R generated by $M \cup N$. Then the following conditions are equivalent.

- (i) There exists a prime ideal \mathfrak{p} of P such that $\mathfrak{m}(M) = \mathfrak{p} \cap M$ and $\mathfrak{m}(N) = \mathfrak{p} \cap N$.
- (ii) The ideal \mathfrak{a} generated in P by $\mathfrak{m}(M) \cup \mathfrak{m}(N)$ is distinct from P .

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(iii) *There exists a local subring Q of R simultaneously dominating both M and N .*

Proof. It is clear that (i) implies (ii); conversely, if $\mathfrak{a} \neq P$, then \mathfrak{a} is contained in a maximal ideal \mathfrak{n} of P , and since $1 \notin \mathfrak{n}$, $\mathfrak{n} \cap M$ contains $\mathfrak{m}(M)$ and is distinct from M , so $\mathfrak{n} \cap M = \mathfrak{m}(M)$, and similarly $\mathfrak{n} \cap N = \mathfrak{m}(N)$. It is clear that, if Q dominates both M and N , then $P \subset Q$ and $\mathfrak{m}(M) = \mathfrak{m}(Q) \cap M = (\mathfrak{m}(Q) \cap P) \cap M$, and $\mathfrak{m}(N) = (\mathfrak{m}(Q) \cap P) \cap N$, so (iii) implies (i); the converse is evident when we take $Q = P_{\mathfrak{p}}$. \square

When the conditions of (8.1.4) are satisfied, we say, with C. Chevalley, that the local rings M and N are *allied*.

Proposition (8.1.5). — *Let A and B be subrings of a field R , and C the subring of R generated by $A \cup B$. Then the following conditions are equivalent.*

- (i) *For every local ring Q containing A and B , we have that $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{m}(Q) \cap A$ and $\mathfrak{q} = \mathfrak{m}(Q) \cap B$.*
- (ii) *For all prime ideals \mathfrak{r} of C , we have that $A_{\mathfrak{p}} = B_{\mathfrak{q}}$, where $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$.*
- (iii) *If $M \in L(A)$ and $N \in L(B)$ are allied, then they are identical.*
- (iv) *$L(A) \cap L(B) = L(C)$.*

Proof. Lemmas (8.1.3) and (8.1.4) prove that (i) and (iii) are equivalent; it is clear that (i) implies (ii) by taking $Q = C_{\mathfrak{r}}$; conversely, (ii) implies (i), because if Q contains $A \cup B$ then it contains C , and if $\mathfrak{r} = \mathfrak{m}(Q) \cap C$, then $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$, by (8.1.3). It is immediate that (iv) implies (i), because if Q contains $A \cup B$ then it dominates a local ring $C_{\mathfrak{r}} \in L(C)$ by (8.1.3); by hypothesis we have that $C_{\mathfrak{r}} \in L(A) \cap L(B)$, and (8.1.1) and (8.1.3) prove that $C_{\mathfrak{r}} = A_{\mathfrak{p}} = B_{\mathfrak{q}}$. We prove finally that (iii) implies (iv). Let $Q \in L(C)$; Q dominates some $M \in L(A)$ and some $N \in L(B)$ (8.1.3), so M and N , being allied, are identical by hypothesis. As we then have that $C \subset M$, we know that M dominates some $Q' \in L(C)$ (8.1.3), so Q dominates Q' , whence necessarily (8.1.3) $Q = Q' = M$, so $Q \in L(A) \cap L(B)$. Conversely, if $Q \in L(A) \cap L(B)$, then $C \subset Q$, so (8.1.3) Q dominates some $Q'' \in L(C) \subset L(A) \cap L(B)$; Q and Q'' , being allied, are identical, so $Q'' = Q \in L(C)$, which completes the proof. \square

8.2. Local rings of an integral scheme.

(8.2.1). Let X be an *integral* prescheme, and R its field of rational functions, identical to the local ring of the generic point a of X ; for all $x \in X$, we know that \mathcal{O}_x can be canonically identified with a subring of R (7.1.5), and for every rational function $f \in R$, the domain of definition $\delta(f)$ of f is the open set of $x \in X$ such that $f \in \mathcal{O}_x$. It thus follows, from (7.2.6), that, for every open $U \subset X$, we have

$$(8.2.1.1) \quad \Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x.$$

Proposition (8.2.2). — *Let X be an integral prescheme, and R its field of rational fractions. For X to be a scheme, it is necessary and sufficient for the relation “ \mathcal{O}_x and \mathcal{O}_y are allied” (8.1.4), for points x and y of X , to imply that $x = y$.*

Proof. We suppose that this condition is satisfied, and aim to show that X is separated. Let U and V be two distinct affine open subsets of X , given by rings A and B (respectively), identified with subrings of R ; U (resp. V) is thus identified (8.1.2) with $L(A)$ (resp. $L(B)$), and the hypotheses tell us (8.1.5) that C is the subring of R generated by $A \cup B$, and $W = U \cap V$ is identified with $L(A) \cap L(B) = L(C)$. Furthermore, we know ([CC], p. 5-03, 4 bis) that every subring E of R is equal to the intersection of the local rings belonging to $L(E)$; C is thus identified with the intersection of the rings \mathcal{O}_z for $z \in W$, or, equivalently (8.2.1.1), with $\Gamma(W, \mathcal{O}_X)$. So consider the subprescheme induced by X on W ; to the identity morphism $\phi : C \rightarrow \Gamma(W, \mathcal{O}_X)$ there corresponds (2.2.4) a morphism $\Phi = (\psi, \theta) : W \rightarrow \text{Spec}(C)$; we will see that Φ is an *isomorphism* of preschemes, whence W is an *affine* open subset. The identification of W with $L(C) = \text{Spec}(C)$ shows that ψ is *bijective*. On the other hand, for all $x \in W$, θ_x^\sharp is the injection $C_{\mathfrak{r}} \rightarrow \mathcal{O}_x$, where $\mathfrak{r} = \mathfrak{m}_x \cap C$, and, by definition, $C_{\mathfrak{r}}$ is identified with \mathcal{O}_x , so θ_x^\sharp is *bijective*. It thus remains to show that ψ is a *homeomorphism*, or, in other words, that for every closed subset $F \subset W$, $\psi(F)$ is closed in $\text{Spec}(C)$. But F is the intersection

of W with a closed subspace of U of the form $V(\mathfrak{a})$, where \mathfrak{a} is an ideal of A ; we will show that $\psi(F) = V(\mathfrak{a}C)$, which proves our claim. In fact, the prime ideals of C containing $\mathfrak{a}C$ are the prime ideals of C containing \mathfrak{a} , and so are the ideals of the form $\psi(x) = \mathfrak{m}_x \cap C$, where $\mathfrak{a} \subset \mathfrak{m}_x$ and $x \in W$; since $\mathfrak{a} \subset \mathfrak{m}_x$ is equivalent to $x \in V(\mathfrak{a}) = W \cap F$ for $x \in U$, we do indeed have that $\psi(F) = V(\mathfrak{a}C)$.

It follows that X is separated, because $U \cap V$ is affine and its ring C is generated by the union $A \cup B$ of the rings of U and V (5.5.6).

Conversely, suppose that X is separated, and let x and y be points of X such that \mathcal{O}_x and \mathcal{O}_y are allied. Let U (resp. V) be an affine open subset containing x (resp. y), of ring A (resp. B); we then know that $U \cap V$ is affine and that its ring C is generated by $A \cup B$ (5.5.6). If $\mathfrak{p} = \mathfrak{m}_x \cap A$ and $\mathfrak{q} = \mathfrak{m}_y \cap B$, then $A_{\mathfrak{p}} = \mathcal{O}_x$ and $B_{\mathfrak{q}} = \mathcal{O}_y$, and since $A_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are allied, there exists a prime ideal \mathfrak{r} of C such that $\mathfrak{p} = \mathfrak{r} \cap A$ and $\mathfrak{q} = \mathfrak{r} \cap B$ (8.1.4). But then there exists a point $z \in U \cap V$ such that $\mathfrak{r} = \mathfrak{m}_z \cap C$, since $U \cap V$ is affine, and so evidently $x = z$ and $y = z$, whence $x = y$. \square

Corollary (8.2.3). — *Let X be an integral scheme, and x and y points of X . In order for $x \in \overline{\{y\}}$, it is necessary and sufficient for $\mathcal{O}_x \subset \mathcal{O}_y$, or, equivalently, for every rational function defined at x to also be defined at y .*

Proof. The condition is evidently necessary because the domain of definition $\delta(f)$ of a rational function $f \in R$ is open; we now show that it is sufficient. If $\mathcal{O}_x \subset \mathcal{O}_y$, then there exists a prime ideal \mathfrak{p} of \mathcal{O}_x such that \mathcal{O}_y dominates $(\mathcal{O}_x)_{\mathfrak{p}}$ (8.1.3); but (2.4.2) there exists some $z \in X$ such that $x \in \overline{\{z\}}$ and $\mathcal{O}_z = (\mathcal{O}_x)_{\mathfrak{p}}$; since \mathcal{O}_z and \mathcal{O}_y are allied, we have that $z = y$ by (8.2.2), whence the corollary. \square

Corollary (8.2.4). — *If X is an integral scheme then the map $x \rightarrow \mathcal{O}_x$ is injective; equivalently, if x and y are two distinct points of X , then there exists a rational function defined at one of these points but not the other.*

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Proof. This follows from (8.2.3) and the axiom (T_0) (2.1.4). \square

Corollary (8.2.5). — *Let X be an integral scheme whose underlying space is Noetherian; letting f range over the field R of rational functions on X , the sets $\delta(f)$ generate the topology of X .*

In fact, every closed subset of X is thus a finite union of irreducible closed subsets, or, in other words, of the form $\overline{\{y\}}$ (2.1.5). But, if $x \notin \overline{\{y\}}$, then there exists a rational function f defined at x but not at y (8.2.3), or, equivalently, we have that $x \in \delta(f)$ and that $\delta(f)$ is not contained in $\overline{\{y\}}$. The complement of $\overline{\{y\}}$ is thus a union of sets of the form $\delta(f)$, and, by virtue of the first remark, every open subset of X is the union of finite intersections of open sets of the form $\delta(f)$.

(8.2.6). Corollary (8.2.5) shows that the topology of X is entirely characterised by the data of the local rings $(\mathcal{O}_x)_{x \in X}$ that have R as their field of fractions. It is equivalent to say that the closed subsets of X are defined in the following manner: given a finite subset $\{x_1, \dots, x_n\}$ of X , consider the set of $y \in X$ such that $\mathcal{O}_y \subset \mathcal{O}_{x_i}$ for at least one index i , and these sets (over all choices of $\{x_1, \dots, x_n\}$) are the closed subsets of X . Further, once the topology on X is known, the structure sheaf \mathcal{O}_X is also determined by the family of the \mathcal{O}_x , since $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x$, by (8.2.1.1). The family $(\mathcal{O}_X)_{x \in X}$ thus completely determines the prescheme X when X is an integral scheme whose underlying space is Noetherian.

Proposition (8.2.7). — *Let X and Y be integral schemes, $f : X \rightarrow Y$ a dominant morphism (2.2.6), and K (resp. L) the field of rational functions on X (resp. Y). Then L can be identified with a subfield of K , and, for all $x \in X$, $\mathcal{O}_{f(x)}$ is the unique local ring of Y dominated by \mathcal{O}_x .*

Proof. If $f = (\psi, \theta)$ and a is the generic point of X , then $\psi(a)$ is the generic point of Y (0, 2.1.5); θ_a^\sharp is then a monomorphism of fields, from $L = \mathcal{O}_{\psi(a)}$ to $K = \mathcal{O}_a$. Since every nonempty affine open subset U of Y contains $\psi(a)$, it follows from (2.2.4) that the homomorphism $\Gamma(U, \mathcal{O}_Y) \rightarrow \Gamma(\psi^{-1}(U), \mathcal{O}_X)$ corresponding to f is the restriction of θ_a^\sharp to $\Gamma(U, \mathcal{O}_Y)$. So, for every $x \in X$, θ_x^\sharp is the restriction to $\mathcal{O}_{\psi(a)}$ of θ_a^\sharp , and is thus a monomorphism. We also know that θ_x^\sharp is a local homomorphism, so, if we identify L with a subfield of K by θ_a^\sharp , $\mathcal{O}_{\psi(x)}$ is dominated by \mathcal{O}_x (8.1.1); it is also the only local ring of Y dominated by \mathcal{O}_x , since two local rings of Y that are allied are identical (8.2.2). \square

Proposition (8.2.8). — Let X be an irreducible prescheme, $f : X \rightarrow Y$ a local immersion (resp. local isomorphism), and suppose further that f is separated. Then f is an immersion (resp. an open immersion).

Proof. Let $f = (\psi, \theta)$; it suffices, in both cases, to prove that ψ is a homeomorphism from X to $\psi(X)$ (4.5.3). Replacing f by f_{red} ((5.1.6) and (5.5.1, vi)), we can assume that X and Y are reduced. If Y' is the closed reduced subscheme of Y that has $\overline{\psi(X)}$ as its underlying space, then f factors as $X \xrightarrow{f'} Y' \xrightarrow{j} Y$, where j is the canonical injection (5.2.2). It follows from (5.5.1, v) that f' is again a separated morphism; further, f' is again a local immersion (resp. a local isomorphism), because, since the condition is local on X and Y , we can restrict to the case where f is a closed immersion (resp. open immersion), and our claim then follows immediately from (4.2.2).

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We can thus suppose that f is a dominant morphism, which leads to the fact that Y is, itself, irreducible (0, 2.1.5), and so X and Y are both integral. Further, the condition being local on Y , we can suppose that Y is an affine scheme; since f is separated, X is a scheme (5.5.1, ii), and we are finally at the hypotheses of Proposition (8.2.7). Then, for all $x \in X$, $\theta_x^\#$ is injective; but the hypothesis that f is a local immersion implies that $\theta_x^\#$ is surjective (4.2.2), so $\theta_x^\#$ is bijective, or, equivalently (with the identification of Proposition (8.2.7)) we have that $\mathcal{O}_{\psi(x)} = \mathcal{O}_x$. This implies, by Corollary (8.2.4), that ψ is an injective map, which already proves the proposition when f is a local isomorphism (4.5.3). When we suppose that f is only a local immersion, for all $x \in X$ there exists an open neighborhood U of x in X and an open neighborhood V of $\psi(x)$ in Y such that the restriction of ψ to U is a homeomorphism from U to a closed subset of V . But U is dense in X , so $\psi(U)$ is dense in Y and a fortiori in V , which proves that $\psi(U) = V$; since ψ is injective, $\psi^{-1}(V) = U$ and this proves that ψ is a homeomorphism from X to $\psi(X)$. \square

8.3. Chevalley schemes.

(8.3.1). Let X be a Noetherian integral scheme, and R its field of rational functions; we denote by X' the set of local subrings $\mathcal{O}_x \subset R$, where x ranges over all points of X . The set X' satisfies the following three conditions.

- (Sch. 1) For all $M \in X'$, R is the field of fractions of M .
- (Sch. 2) There exists a finite set of Noetherian subrings A_i of R such that $X' = \bigcup_i L(A_i)$, and, for all pairs of indices i, j , the subring A_{ij} of R generated by $A_i \cup A_j$ is an algebra of finite type over A_i .
- (Sch. 3) Any two elements M and N of X' that are allied are identical.

We have seen in (8.2.1) that (Sch. 1) is satisfied, and (Sch. 3) follows from (8.2.2). To show (Sch. 2), it suffices to cover X by a finite number of affine open subsets U_i whose rings are Noetherian, and to take $A_i = \Gamma(U_i, \mathcal{O}_X)$; the hypothesis that X is a scheme implies that $U_i \cap U_j$ is affine, and also that $\Gamma(U_i \cap U_j, \mathcal{O}_X) = A_{ij}$ (5.5.6); further, since the space U_i is Noetherian, the immersion $U_i \cap U_j \rightarrow U_i$ is of finite type (6.3.5), so A_{ij} is an A_i -algebra of finite type (6.3.3).

(8.3.2). The structures whose axioms are (Sch. 1), (Sch. 2), and (Sch. 3) generalise “schemes”, in the sense of C. Chevalley, who additionally supposes that R is an extension of finite type of a field K , and that the A_i are K -algebras of finite type (which renders a part of (Sch. 2) useless) [CC]. Conversely, if we have such a structure on a set X' , then we can associate to it an integral scheme X by using the remarks from (8.2.6): the underlying space of X is equal to X' endowed with the topology defined in (8.2.6), and with the sheaf \mathcal{O}_X such that $\Gamma(U, \mathcal{O}_X) = \bigcap_{x \in U} \mathcal{O}_x$ for all open $U \subset X$, with the evident definition of restriction homomorphisms. We leave to the reader the task of verifying that we thus obtain an integral scheme, whose local rings are the elements of X' ; we will not use this result in what follows.

§9. SUPPLEMENT ON QUASI-COHERENT SHEAVES

9.1. Tensor product of quasi-coherent sheaves.

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Proposition (9.1.1). — Let X be a prescheme (resp. a locally Noetherian prescheme). Let \mathcal{F} and \mathcal{G} be quasi-coherent (resp. coherent) \mathcal{O}_X -modules; then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is quasi-coherent (resp. coherent); it is further of finite type if both \mathcal{F} and \mathcal{G} are of finite type. If \mathcal{F} admits a finite presentation and if \mathcal{G} is quasi-coherent (resp. coherent), then $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ is quasi-coherent (resp. coherent).

Proof. Being a local proposition, we can suppose that X is affine (resp. Noetherian affine); further, if \mathcal{F} is coherent, then we can assume that it is the cokernel of a homomorphism $\mathcal{O}_X^m \rightarrow \mathcal{O}_X^n$. The claims pertaining to quasi-coherent sheaves then follow from Corollaries (1.3.12) and (1.3.9); the claims pertaining to coherent sheaves follow from Theorem (1.5.1) and from the fact that if M and N are modules of finite type over a Noetherian ring A then $M \otimes_A N$ and $\text{Hom}_A(M, N)$ are both A -modules of finite type. \square

Definition (9.1.2). — Let X and Y be S -preschemes, p and q the projections of $X \times_S Y$, and \mathcal{F} (resp. \mathcal{G}) a quasi-coherent \mathcal{O}_X -module (resp. quasi-coherent \mathcal{O}_Y -module). We define the tensor product of \mathcal{F} and \mathcal{G} over \mathcal{O}_S (or over S), denoted by $\mathcal{F} \otimes_{\mathcal{O}_S} \mathcal{G}$ (or $\mathcal{F} \otimes_S \mathcal{G}$) to be the tensor product $p^*(\mathcal{F}) \otimes_{\mathcal{O}_{X \times_S Y}} q^*(\mathcal{G})$ over the prescheme $X \times_S Y$.

If X_i ($1 \leq i \leq n$) are S -preschemes, and \mathcal{F}_i ($1 \leq i \leq n$) are quasi-coherent \mathcal{O}_{X_i} -modules, then we similarly define the tensor product $\mathcal{F}_1 \otimes_S \mathcal{F}_2 \otimes_S \cdots \otimes_S \mathcal{F}_n$ over the prescheme $Z = X_1 \times_S X_2 \times_S \cdots \times_S X_n$; it is a quasi-coherent \mathcal{O}_Z -module by virtue of (9.1.1) and (0, 5.1.4); it is further coherent if all the \mathcal{F}_i are coherent and Z is locally Noetherian, by virtue of (9.1.1), (0, 5.3.11), and (6.1.1).

Note that, if we take $X = Y = S$, then Definition (9.1.2) gives us back the tensor product of \mathcal{O}_S -modules. Furthermore, since $q^*(\mathcal{O}_Y) = \mathcal{O}_{X \times_S Y}$ (0, 4.3.4), the product $\mathcal{F} \otimes_S \mathcal{O}_Y$ is canonically identified with $p^*(\mathcal{F})$, and, in the same way, $\mathcal{O}_X \otimes_S \mathcal{G}$ is canonically identified with $q^*(\mathcal{G})$. In particular, if we take $Y = S$ and denote by f the structure morphism $X \rightarrow Y$, then we have that $\mathcal{O}_X \otimes_Y \mathcal{G} = f^*(\mathcal{G})$: the ordinary tensor product and the inverse image thus appear as particular cases of the general tensor product.

Definition (9.1.2) leads immediately to the fact that, for fixed X and Y , $\mathcal{F} \otimes_S \mathcal{G}$ is a right-exact additive covariant bifunctor in \mathcal{F} and \mathcal{G} .

Proposition (9.1.3). — Let S , X , and Y be affine schemes of rings A , B , and C (respectively), with B and C being A -algebras. Let M (resp. N) be a B -module (resp. C -module), and $\mathcal{F} = \tilde{M}$ (resp. $\mathcal{G} = \tilde{N}$) the associated quasi-coherent sheaf; then $\mathcal{F} \otimes_S \mathcal{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$ -module $M \otimes_A N$.

Proof. According to Proposition (1.6.5), $\mathcal{F} \otimes_S \mathcal{G}$ is canonically isomorphic to the sheaf associated to the $(B \otimes_A C)$ -module

$$(M \otimes_B (B \otimes_A C)) \otimes_{B \otimes_A C} ((B \otimes_A C) \otimes_C N)$$

and, by the canonical isomorphisms between tensor products, this latter module is isomorphic to

$$M \otimes_B (B \otimes_A C) \otimes_C N = (M \otimes_B B) \otimes_A (C \otimes_C N) = M \otimes_A N.$$

\square

Proposition (9.1.4). — Let $f : T \rightarrow X$ and $g : T \rightarrow Y$ be S -morphisms, and \mathcal{F} (resp. \mathcal{G}) a quasi-coherent \mathcal{O}_X -module (resp. quasi-coherent \mathcal{O}_Y -module). Then

$$(f, g)_S^*(\mathcal{F} \otimes_S \mathcal{G}) = f^*(\mathcal{F}) \otimes_{\mathcal{O}_T} g^*(\mathcal{G}).$$

Proof. If p, q are the projections of $X \times_S Y$, then the formula follows from the equalities $(f, g)_S^* \circ p^* = f^*$ and $(f, g)_S^* \circ q^* = g^*$ (0, 3.5.5), and the fact that the inverse image of a tensor product of algebraic sheaves is the tensor product of their inverse images (0, 4.3.3). \square

Corollary (9.1.5). — Let $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ be S -morphisms, and \mathcal{F}' (resp. \mathcal{G}') a quasi-coherent $\mathcal{O}_{X'}$ -module (resp. quasi-coherent $\mathcal{O}_{Y'}$ -module). Then

$$(f, g)_S^*(\mathcal{F}' \otimes_S \mathcal{G}') = f^*(\mathcal{F}') \otimes_S g^*(\mathcal{G}')$$

Proof. This follows from (9.1.4) and the fact that $f \times_S g = (f \circ p, g \circ q)_S$, where p and q are the projections of $X \times_S Y$. \square

Corollary (9.1.6). — Let X, Y , and Z be S -preschemes, and \mathcal{F} (resp. \mathcal{G}, \mathcal{H}) a quasi-coherent \mathcal{O}_X -module (resp. quasi-coherent \mathcal{O}_Y -module, quasi-coherent \mathcal{O}_Z -module); then the sheaf $\mathcal{F} \otimes_S \mathcal{G} \otimes_S \mathcal{H}$ is the inverse image of $(\mathcal{F} \otimes_S \mathcal{G}) \otimes_S \mathcal{H}$ by the canonical isomorphism from $X \times_S Y \times_S Z$ to $(X \times_S Y) \times_S Z$.

Proof. This isomorphism is given by $(p_1, p_2)_S \times_S p_3$, where p_1 , p_2 , and p_3 are the projections of $X \times_S Y \times_S Z$.

Similarly, the inverse image of $\mathcal{G} \otimes_S \mathcal{F}$ under the canonical isomorphism from $X \times_S Y$ to $Y \times_S X$ is $\mathcal{F} \otimes_S \mathcal{G}$. \square

Corollary (9.1.7). — *If X is an S -prescheme, then every quasi-coherent \mathcal{O}_X -module \mathcal{F} is the inverse image of $\mathcal{F} \otimes_S \mathcal{O}_S$ by the canonical isomorphism from X to $X \times_S S$ (3.3.3).*

Proof. This isomorphism is $(1_X, \phi)_S$, where ϕ is the structure morphism $X \rightarrow S$, and the corollary follows from (9.1.4) and the fact that $\phi^*(\mathcal{O}_S) = \mathcal{O}_X$. \square

(9.1.8). Let X be an S -prescheme, \mathcal{F} a quasi-coherent \mathcal{O}_X -module, and $\phi : S' \rightarrow S$ a morphism; we denote by $\mathcal{F}_{(\phi)}$ or $\mathcal{F}_{(S')}$ the quasi-coherent sheaf $\mathcal{F} \otimes_S \mathcal{O}_{S'}$ over $X \times_S S' = X_{(\phi)} = X_{(S')}$; so $\mathcal{F}_{(S')} = p^*(\mathcal{F})$, where p is the projection $X_{(S')} \rightarrow X$.

Proposition (9.1.9). — *Let $\phi'' : S'' \rightarrow S'$ be a morphism. For every quasi-coherent \mathcal{O}_X -module \mathcal{F} on the S -prescheme X , $(\mathcal{F}_{(\phi)})_{(\phi')}$ is the inverse image of $\mathcal{F}_{(\phi \circ \phi')}$ by the canonical isomorphism $(X_{(\phi)})_{(\phi')} \simeq X_{(\phi \circ \phi')}$ (3.3.9).*

Proof. This follows immediately from the definitions and from (3.3.9), and is written

$$(9.1.9.1) \quad (\mathcal{F} \otimes_S \mathcal{O}_{S'}) \otimes_{S'} \mathcal{O}_{S''} = \mathcal{F} \otimes_S \mathcal{O}_{S''}.$$

\square

Proposition (9.1.10). — *Let Y be an S -prescheme, and $f : X \rightarrow Y$ an S -morphism. For every quasi-coherent \mathcal{O}_Y -module \mathcal{G} and every morphism $S' \rightarrow S$, we have that $(f_{(S')})^*(\mathcal{G}_{(S')}) = (f^*(\mathcal{G}))_{(S')}$.*

Proof. This follows immediately from the commutativity of the diagram

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$$\begin{array}{ccc} X_{(S')} & \xrightarrow{f_{(S')}} & Y_{(S')} \\ \downarrow & & \downarrow \\ X & \xrightarrow{f} & Y. \end{array}$$

\square

Corollary (9.1.11). — *Let X and Y be S -preschemes, and \mathcal{F} (resp. \mathcal{G}) a quasi-coherent \mathcal{O}_X -module (resp. quasi-coherent \mathcal{O}_Y -module). Then the inverse image of the sheaf $(\mathcal{F}_{(S')}) \otimes_{(S')} (\mathcal{G}_{(S')})$ by the canonical isomorphism $(X \times_S Y)_{(S')} \simeq (X_{(S')}) \times_{S'} (Y_{(S')})$ (3.3.10) is equal to $(\mathcal{F} \otimes_S \mathcal{G})_{(S')}$.*

Proof. If p and q are the projections of $X \times_S Y$, then the isomorphism in question is nothing but $(p_{(S')}, q_{(S')})_{S'}$; the corollary then follows from Propositions (9.1.4) and (9.1.10). \square

Proposition (9.1.12). — *With the notation from Definition (9.1.2), let z be a point of $X \times_S Y$, and let $x = p(z)$, and $y = q(z)$; the stalk $(\mathcal{F} \otimes_S \mathcal{G})_z$ is isomorphic to $(\mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z) \otimes_{\mathcal{O}_z} (\mathcal{G}_y \otimes_{\mathcal{O}_y} \mathcal{O}_z) = \mathcal{F}_x \otimes_{\mathcal{O}_x} \mathcal{O}_z \otimes_{\mathcal{O}_y} \mathcal{G}_y$.*

Proof. Since we can reduce to the affine case, the proposition follows from Equation (1.6.5.1). \square

Corollary (9.1.13). — *If \mathcal{F} and \mathcal{G} are of finite type, then*

$$\text{Supp}(\mathcal{F} \otimes_S \mathcal{G}) = p^{-1}(\text{Supp}(\mathcal{F})) \cap q^{-1}(\text{Supp}(\mathcal{G})).$$

Proof. Since $p^*(\mathcal{F})$ and $q^*(\mathcal{G})$ are both of finite type over $\mathcal{O}_{X \times_S Y}$, we reduce, by Proposition (9.1.12) and by (0, 1.7.5), to the case where $\mathcal{G} = \mathcal{O}_Y$, that is, it remains to prove the following equation:

$$(9.1.13.1) \quad \text{Supp}(p^*(\mathcal{F})) = p^{-1}(\text{Supp}(\mathcal{F})).$$

The same reasoning as in (0, 1.7.5) leads us to prove that, for all $z \in X \times_S Y$, we have $\mathcal{O}_z / \mathfrak{m}_x \mathcal{O}_z \neq 0$ (with $x = p(z)$), which follows from the fact that the homomorphism $\mathcal{O}_x \rightarrow \mathcal{O}_z$ is *local*, by hypothesis. \square

We leave it to the reader to extend the results in this section to the more general case of an arbitrary (but finite) number of factors, instead of just two.

9.2. Direct image of a quasi-coherent sheaf.

Proposition (9.2.1). — *Let $f : X \rightarrow Y$ be a morphism of preschemes. We suppose that there exists a cover (Y_α) of Y by affine opens having the following property: every $f^{-1}(Y_\alpha)$ admits a finite cover $(X_{\alpha i})$ by affine opens that are contained in $f^{-1}(Y_\alpha)$ and that are such that every intersection $X_{\alpha i} \cap X_{\alpha j}$ is itself a finite union of affine opens. With these hypotheses, for every quasi-coherent \mathcal{O}_X -module \mathcal{F} , $f_*(\mathcal{F})$ is a quasi-coherent \mathcal{O}_Y -module.*

Proof. Since this is a local condition on Y , we can assume that Y is equal to one of the Y_α , and thus omit the indices α .

- (a) First, suppose that the $X_i \cap X_j$ are themselves affine opens. We set $\mathcal{F}_i = \mathcal{F}|_{X_i}$ and $\mathcal{F}_{ij} = \mathcal{F}|_{(X_i \cap X_j)}$, and let \mathcal{F}'_i and \mathcal{F}'_{ij} be the images of \mathcal{F}_i and \mathcal{F}_{ij} (respectively) by the restriction of f to X_i and to $X_i \cap X_j$ (respectively); we know that the \mathcal{F}'_i and \mathcal{F}'_{ij} are quasi-coherent (1.6.3). Set $\mathcal{G} = \bigoplus_i \mathcal{F}'_i$ and $\mathcal{H} = \bigoplus_{i,j} \mathcal{F}'_{ij}$; \mathcal{G} and \mathcal{H} are quasi-coherent \mathcal{O}_Y -modules; we will define a homomorphism $u : \mathcal{G} \rightarrow \mathcal{H}$ such that $f_*(\mathcal{F})$ is the kernel of u ; it will follow from this that $f_*(\mathcal{F})$ is quasi-coherent (1.3.9). It suffices to define u as a homomorphism of presheaves; taking into account the definitions of \mathcal{G} and \mathcal{H} , it thus suffices, for every open subset $W \subset Y$, to define a homomorphism

$$u_W : \bigoplus_i \Gamma(f^{-1}(W) \cap X_i, \mathcal{F}) \longrightarrow \bigoplus_{i,j} \Gamma(f^{-1}(W) \cap X_i \cap X_j, \mathcal{F})$$

that satisfies the usual compatibility conditions when we let W vary. If, for every section $s_i \in \Gamma(f^{-1}(W) \cap X_i, \mathcal{F})$, we denote by $s_{i|j}$ its restriction to $f^{-1}(W) \cap X_i \cap X_j$, then we set

$$u_W((s_i)) = (s_{i|j} - s_{j|i})$$

and the compatibility conditions are clearly satisfied. To prove that the kernel \mathcal{R} of u is $f_*(\mathcal{F})$, we define a homomorphism from $f_*(\mathcal{F})$ to \mathcal{R} by sending each section $s \in \Gamma(f^{-1}(W), \mathcal{F})$ to the family (s_i) , where s_i is the restriction of s to $f^{-1}(W) \cap X_i$; axioms (F1) and (F2) of sheaves (G, II, 1.1) tell us that this homomorphism is *bijective*, which finishes the proof in this case.

- (b) In the general case, the same reasoning applies once we have established that the \mathcal{F}_{ij} are quasi-coherent. But, by hypothesis, $X_i \cap X_j$ is a finite union of affine opens X_{ijk} ; and since the X_{ijk} are affine opens in a scheme, the intersection of any two of them is again an affine open (5.5.6). We are thus led to the first case, and so we have proved Proposition (9.2.1). \square

Corollary (9.2.2). — *The conclusion of Proposition (9.2.1) holds true in each of the following cases:*

- (a) f is separated and quasi-compact;
- (b) f is separated and of finite type;
- (c) f is quasi-compact, and the underlying space of X is locally Noetherian.

Proof. In case (a), the $X_{\alpha i} \cap X_{\alpha j}$ are affine (5.5.6). Case (b) is a particular example of case (a) (6.6.3). Finally, in case (c), we can reduce to the case where Y is affine and the underlying space of X is Noetherian; then X admits a finite cover of affine opens (X_i) , and the $X_i \cap X_j$, being quasi-compact, are finite unions of affine opens (2.1.3). \square

9.3. Extension of sections of quasi-coherent sheaves.

Theorem (9.3.1). — *Let X be a prescheme whose underlying space is Noetherian, or a scheme whose underlying space is quasi-compact. Let \mathcal{L} be an invertible \mathcal{O}_X -module (0, 5.4.1), f a section of \mathcal{L} over X , X_f the open set of $x \in X$ such that $f(x) \neq 0$ (0, 5.5.1), and \mathcal{F} a quasi-coherent \mathcal{O}_X -module.*

- (i) *If $s \in \Gamma(X, \mathcal{F})$ is such that $s|_{X_f} = 0$, then there exists an integer $n > 0$ such that $s \otimes f^{\otimes n} = 0$.*
- (ii) *For every section $s \in \Gamma(X_f, \mathcal{F})$, there exists an integer $n > 0$ such that $s \otimes f^{\otimes n}$ extends to a section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ over X .*

Proof.

- (i) Since the underlying space of X is quasi-compact, and thus the union of finitely-many affine opens U_i with $\mathcal{L}|_{U_i}$ isomorphic to $\mathcal{O}_X|_{U_i}$, we can reduce to the case where X is affine and $\mathcal{L} = \mathcal{O}_X$. In this case, f can be identified with an element of $A(X)$, and we have that $X_f = D(f)$; s can be identified with an element of an $A(X)$ -module M , and $s|_{X_f}$ to the corresponding element of M_f , and the result is then trivial, recalling the definition of a module of fractions.
- (ii) Again, X is a finite union of affine opens U_i ($1 \leq i \leq r$) such that $\mathcal{L}|_{U_i} \cong \mathcal{O}_X|_{U_i}$, and, for every i , $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ can be identified (by the aforementioned isomorphism) with $(f|(U_i \cap X_f))^n(s|(U_i \cap X_f))$. We then know (1.4.1) that there exists an integer $n > 0$ such that, for all i , $(s \otimes f^{\otimes n})|(U_i \cap X_f)$ extends to a section s_i of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$ over U_i . Let $s_{i|j}$ be the restriction of s_i to $U_i \cap U_j$; by definition we have that $s_{i|j} - s_{j|i} = 0$ on $X_f \cap U_i \cap U_j$. But, if X is a Noetherian space, then $U_i \cap U_j$ is quasi-compact; if X is a scheme, then $U_i \cap U_j$ is an affine open (5.5.6), and so again quasi-compact. By virtue of (i), there thus exists an integer m (independent of i and j) such that $(s_{i|j} - s_{j|i}) \otimes f^{\otimes m} = 0$. It immediately follows that there exists a section s' of $\mathcal{F} \otimes \mathcal{L}^{\otimes(n+m)}$ over X that restricts to $s_i \otimes f^{\otimes m}$ over each U_i , and restricts to $s \otimes f^{\otimes(n+m)}$ over X_f .

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□

The following corollaries give an interpretation of Theorem (9.3.1) in a more algebraic language:

Corollary (9.3.2). — *With the hypotheses of (9.3.1), consider the graded ring $A_* = \Gamma_*(\mathcal{L})$ and the graded A_* -module $M_* = \Gamma_*(\mathcal{L}, \mathcal{F})$ (0, 5.4.6). If $f \in A_n$, where $n \in \mathbf{Z}$, then there is a canonical isomorphism $\Gamma(X_f, \mathcal{F}) \simeq ((M_*)_f)_0$ (the subgroup of the module of fractions $(M_*)_f$ consisting of elements of degree 0).*

Corollary (9.3.3). — *Suppose that the hypotheses of (9.3.1) are satisfied, and suppose further that $\mathcal{L} = \mathcal{O}_X$. Then, setting $A = \Gamma(X, \mathcal{O}_X)$ and $M = \Gamma(X, \mathcal{F})$, the A_f -module $\Gamma(X_f, \mathcal{F})$ is canonically isomorphic to M_f .*

Proposition (9.3.4). — *Let X be a Noetherian prescheme, \mathcal{F} a coherent \mathcal{O}_X -module, and \mathcal{J} a coherent sheaf of ideals in \mathcal{O}_X , such that the support of \mathcal{F} is contained in that of $\mathcal{O}_X/\mathcal{J}$. Then there exists an integer $n > 0$ such that $\mathcal{J}^n \mathcal{F} = 0$.*

Proof. Since X is a union of finitely-many affine opens whose rings are Noetherian, we can suppose that X is affine, given by some Noetherian ring A ; then $\mathcal{F} = \tilde{M}$, where $M = \Gamma(X, \mathcal{F})$ is an A -module of finite type, and $\mathcal{J} = \tilde{\mathfrak{J}}$, where $\mathfrak{J} = \Gamma(X, \mathcal{J})$ is an ideal of A ((1.4.1) and (1.5.1)). Since A is Noetherian, \mathfrak{J} admits a finite system of generators f_i ($1 \leq i \leq m$). By hypothesis, every section of \mathcal{F} over X is zero on each of the $D(f_i)$; if s_j ($1 \leq j \leq q$) are sections of \mathcal{F} that generate M , then there exists an integer h , independent of i and j , such that $f_i^h s_j = 0$ (1.4.1), whence $f_i^h s = 0$ for all $s \in M$. We thus conclude that, if $n = mh$, then $\mathfrak{J}^n M = 0$, and so the corresponding \mathcal{O}_X -module $\mathcal{J}^n \mathcal{F} = \tilde{\mathfrak{J}^n M}$ (1.3.13) is zero. □

Corollary (9.3.5). — *With the hypotheses of (9.3.4), there exists a closed subprescheme Y of X , whose underlying space is the support of $\mathcal{O}_X/\mathcal{J}$, such that, if $j : Y \rightarrow X$ is the canonical injection, then $\mathcal{F} = j_*(j^*(\mathcal{F}))$.*

Proof. First, note that the supports of $\mathcal{O}_X/\mathcal{J}$ and $\mathcal{O}_X/\mathcal{J}^n$ are the same, since, if $\mathcal{J}_x = \mathcal{O}_x$, then $\mathcal{J}_x^n = \mathcal{O}_x$, and we also have that $\mathcal{J}_x^n \subset \mathcal{J}_x$ for all $x \in X$. We can, thanks to (9.3.4), thus suppose that $\mathcal{J} \mathcal{F} = 0$; we can then take Y to be the closed subprescheme of X defined by \mathcal{J} , and since \mathcal{F} is then an $(\mathcal{O}_X/\mathcal{J})$ -module, the conclusion follows immediately. □

9.4. Extension of quasi-coherent sheaves.

(9.4.1). Let X be a topological space, \mathcal{F} a sheaf of sets (resp. of groups, of rings) on X , U an open subset of X , $\psi : U \rightarrow X$ the canonical injection, and \mathcal{G} a subsheaf of $\mathcal{F}|_U = \psi^*(\mathcal{F})$. Since ψ_* is left exact, $\psi_*(\mathcal{G})$ is a subsheaf of $\psi_*(\psi^*(\mathcal{F}))$; we denote by ρ the canonical homomorphism $\mathcal{F} \rightarrow \psi_*(\psi^*(\mathcal{F}))$ (0, 3.5.3), and we denote by $\overline{\mathcal{G}}$ the subsheaf $\rho^{-1}(\psi_*(\mathcal{G}))$ of \mathcal{F} . It follows immediately from the definitions that, for every open subset V of X , $\Gamma(V, \overline{\mathcal{G}})$ consists of sections $s \in \Gamma(V, \mathcal{F})$

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whose restriction to $V \cap U$ is a section of \mathcal{G} over $V \cap U$. We thus have that $\overline{\mathcal{G}}|U = \psi^*(\overline{\mathcal{G}}) = \mathcal{G}$, and that $\overline{\mathcal{G}}$ is the *largest* subsheaf of \mathcal{F} that restricts to \mathcal{G} over U ; we say that $\overline{\mathcal{G}}$ is the *canonical extension* of the subsheaf \mathcal{G} of $\mathcal{F}|U$ to a subsheaf of \mathcal{F} .

Proposition (9.4.2). — *Let X be a prescheme, and U an open subset of X such that the canonical injection $j : U \rightarrow X$ is a quasi-compact morphism (which will be the case for all U if the underlying space of X is locally Noetherian (6.6.4, i)). Then:*

- (i) *for every quasi-coherent $(\mathcal{O}_X|U)$ -module \mathcal{G} , $j_*(\mathcal{G})$ is a quasi-coherent \mathcal{O}_X -module, and $j_*(\mathcal{G})|U = j^*(j_*(\mathcal{G})) = \mathcal{G}$;*
- (ii) *for every quasi-coherent \mathcal{O}_X -module \mathcal{F} and every quasi-coherent $(\mathcal{O}_X|U)$ -submodule \mathcal{G} , the canonical extension $\overline{\mathcal{G}}$ of \mathcal{G} (9.4.1) is a quasi-coherent \mathcal{O}_X -submodule of \mathcal{F} .*

Proof. If $j = (\psi, \theta)$ (ψ being the injection $U \rightarrow X$ of underlying spaces), then, by definition, we have that $j_*(\mathcal{G}) = \psi_*(\mathcal{G})$ for every $(\mathcal{O}_X|U)$ -module \mathcal{G} , and, further, that $j^*(\mathcal{H}) = \psi^*(\mathcal{H}) = \mathcal{H}|U$ for every \mathcal{O}_X -module \mathcal{H} , by definition of the prescheme induced over an open subset. So (i) is thus a particular case of ((9.2.2, a)); for the same reason, $j_*(j^*(\mathcal{F}))$ is quasi-coherent, and since $\overline{\mathcal{G}}$ is the inverse image of $j_*(\mathcal{G})$ by the homomorphism $\rho : \mathcal{F} \rightarrow j_*(j^*(\mathcal{F}))$, (ii) follows from (4.1.1). \square

Note that the hypothesis that the morphism $j : U \rightarrow X$ is quasi-compact holds whenever the open subset U is *quasi-compact* and X is a *scheme*: indeed, U is then a union of finitely-many affine opens U_i , and, for every affine open V of X , $V \cap U_i$ is an affine open (5.5.6), and thus quasi-compact.

Corollary (9.4.3). — *Let X be a prescheme, and U a quasi-compact open subset of X such that the injection morphism $j : U \rightarrow X$ is quasi-compact. Suppose as well that every quasi-coherent \mathcal{O}_X -module is the inductive limit of its quasi-coherent \mathcal{O}_X -submodules of finite type (which will be the case if X is an affine scheme). Then let \mathcal{F} be a quasi-coherent \mathcal{O}_X -module, and \mathcal{G} a quasi-coherent $(\mathcal{O}_X|U)$ -submodule of $\mathcal{F}|U$ of finite type. Then there exists a quasi-coherent \mathcal{O}_X -submodule \mathcal{G}' of \mathcal{F} of finite type such that $\mathcal{G}'|U = \mathcal{G}$.*

Proof. We have $\mathcal{G} = \overline{\mathcal{G}}|U$, and $\overline{\mathcal{G}}$ is quasi-coherent, from (9.4.2), so the inductive limit of its quasi-coherent \mathcal{O}_X -submodules \mathcal{H}_λ of finite type. It follows that \mathcal{G} is the inductive limit of the $\mathcal{H}_\lambda|U$, and thus equal to one of the $\mathcal{H}_\lambda|U$, since it is of finite type (0, 5.2.3). \square

Remark (9.4.4). — Suppose that for every affine open $U \subset X$, the injection morphism $U \rightarrow X$ is quasi-compact. Then, if the conclusion of (9.4.3) holds for every affine open U and for every quasi-coherent $(\mathcal{O}_X|U)$ -submodule \mathcal{G} of $\mathcal{F}|U$ of finite type, it follows that \mathcal{F} is the inductive limit of its quasi-coherent \mathcal{O}_X -submodules of finite type. Indeed, for every affine open $U \subset X$, we have that $\mathcal{F}|U = \tilde{M}$, where M is an $A(U)$ -module, and since the latter is the inductive limit of its quasi-coherent submodules of finite type, $\mathcal{F}|U$ is the inductive limit of its $(\mathcal{O}_X|U)$ -submodules of finite type (1.3.9). But, by hypothesis, each of these submodules is induced on U by a quasi-coherent \mathcal{O}_X -submodule $\mathcal{G}_{\lambda,U}$ of \mathcal{F} of finite type. The finite sums of the $\mathcal{G}_{\lambda,U}$ are again quasi-coherent \mathcal{O}_X -modules of finite type, because this is a local property, and the case where X is affine was covered in (1.3.10); it is clear then that \mathcal{F} is the inductive limit of these finite sums, whence our claim.

Corollary (9.4.5). — *Under the hypotheses of Corollary (9.4.3), for every quasi-coherent $(\mathcal{O}_X|U)$ -module \mathcal{G} of finite type, there exists a quasi-coherent \mathcal{O}_X -module \mathcal{G}' of finite type such that $\mathcal{G}'|U = \mathcal{G}$.*

Proof. Since $\mathcal{F} = j_*(\mathcal{G})$ is quasi-coherent (9.4.2) and $\mathcal{F}|U = \mathcal{G}$, it suffices to apply Corollary (9.4.3) to \mathcal{F} . \square

Lemma (9.4.6). — *Let X be a prescheme, L a well-ordered set, $(V_\lambda)_{\lambda \in L}$ a cover of X by affine opens, and U an open subset of X ; for all $\lambda \in L$, we set $W_\lambda = \bigcup_{\mu < \lambda} V_\mu$. Suppose that: (1) for every $\lambda \in L$, $V_\lambda \cap W_\lambda$ is quasi-compact; and (2) the immersion morphism $U \rightarrow X$ is quasi-compact. Then, for every quasi-coherent \mathcal{O}_X -module \mathcal{F} and every quasi-coherent $(\mathcal{O}_X|U)$ -submodule \mathcal{G} of $\mathcal{F}|U$ of finite type, there exists a quasi-coherent \mathcal{O}_X -submodule \mathcal{G}' of \mathcal{F} of finite type such that $\mathcal{G}'|U = \mathcal{G}$.*

Proof. Let $U_\lambda = U \cup W_\lambda$; we will define a family (\mathcal{G}'_λ) by induction, where \mathcal{G}'_λ is a quasi-coherent $(\mathcal{O}_X|U_\lambda)$ -submodule of $\mathcal{F}|U_\lambda$ of finite type, such that $\mathcal{G}'_\lambda|U_\mu = \mathcal{G}'_\mu$ for $\mu < \lambda$ and $\mathcal{G}'_\lambda|U = \mathcal{G}$. The unique \mathcal{O}_X -submodule \mathcal{G}' of \mathcal{F} such that $\mathcal{G}'|U_\lambda = \mathcal{G}'_\lambda$ for all $\lambda \in L$ (0, 3.3.1) will then give us what we want. So suppose that the \mathcal{G}'_μ are defined and have the preceding properties for $\mu < \lambda$;

if λ does not have a predecessor then we take \mathcal{G}'_λ to be the unique $(\mathcal{O}_X|U_\lambda)$ -submodule of $\mathcal{F}|U_\lambda$ such that $\mathcal{G}'_\lambda|U_\mu = \mathcal{G}'_\mu$ for all $\mu < \lambda$, which is allowed since the U_μ with $\mu < \lambda$ then form a cover of U_λ . If, conversely, $\lambda = \mu + 1$, then $U_\lambda = U_\mu \cup V_\mu$, and it suffices to define a quasi-coherent $(\mathcal{O}_X|V_\mu)$ -submodule \mathcal{G}''_μ of $\mathcal{F}|V_\mu$ of finite type such that

$$\mathcal{G}''_\mu|(U_\mu \cap V_\mu) = \mathcal{G}'_\mu|(U_\mu \cap V_\mu);$$

and then to take \mathcal{G}'_λ to be the $(\mathcal{O}_X|U_\lambda)$ -submodule of $\mathcal{F}|U_\lambda$ such that $\mathcal{G}'_\lambda|U_\mu = \mathcal{G}'_\mu$ and $\mathcal{G}'_\lambda|V_\mu = \mathcal{G}''_\mu$ (0, 3.3.1). But, since V_μ is affine, the existence of \mathcal{G}''_μ is guaranteed by (9.4.3) as soon as we show that $U_\mu \cap V_\mu$ is quasi-compact; but $U_\mu \cap V_\mu$ is the union of $U \cap V_\mu$ and $W_\mu \cap V_\mu$, which are both quasi-compact by virtue of the hypotheses. \square

Theorem (9.4.7). — *Let X be a prescheme, and U an open subset of X . Suppose that one of the following conditions is verified:*

- (a) *the underlying space of X is locally Noetherian;*
- (b) *X is a quasi-compact scheme and U is a quasi-compact open.*

Then, for every quasi-coherent \mathcal{O}_X -module \mathcal{F} and every quasi-coherent $(\mathcal{O}_X|U)$ -submodule \mathcal{G} of $\mathcal{F}|U$ of finite type, there exists a quasi-coherent \mathcal{O}_X -submodule \mathcal{G}' of \mathcal{F} of finite type such that $\mathcal{G}'|U = \mathcal{G}$.

Proof. Let $(V_\lambda)_{\lambda \in L}$ be a cover of X by affine opens, with L assumed to be finite in case (b); since L is equipped with the structure of a well-ordered set, it suffices to check that the conditions of (9.4.6) are satisfied. It is clear in the case of (a), since the spaces V_λ are Noetherian. For case (b), the $V_\lambda \cap V_\mu$ are affine (5.5.6), and thus quasi-compact, and, since L is finite, $V_\lambda \cap W_\lambda$ is quasi-compact. Whence the theorem. \square

Corollary (9.4.8). — *Under the hypotheses of (9.4.7), for every quasi-coherent $(\mathcal{O}_X|U)$ -module \mathcal{G} of finite type, there exists a quasi-coherent \mathcal{O}_X -module \mathcal{G}' of finite type such that $\mathcal{G}'|U = \mathcal{G}$.*

Proof. It suffices to apply (9.4.7) to $\mathcal{F} = j_*(\mathcal{G})$, which is quasi-coherent (9.4.2) and such that $\mathcal{F}|U = \mathcal{G}$. \square

Corollary (9.4.9). — *Let X be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent \mathcal{O}_X -module is the inductive limit of its quasi-coherent \mathcal{O}_X -submodules of finite type.*

Proof. This follows from Theorem (9.4.7) and Remark (9.4.4). \square

Corollary (9.4.10). — *Under the hypotheses of (9.4.9), if a quasi-coherent \mathcal{O}_X -module \mathcal{F} is such that every quasi-coherent \mathcal{O}_X -submodule of finite type of \mathcal{F} is generated by its sections over X , then \mathcal{F} is generated by its sections over X .*

Proof. Let U be an affine open neighborhood of a point $x \in X$, and let s be a section of \mathcal{F} over U ; the \mathcal{O}_X -submodule \mathcal{G} of $\mathcal{F}|U$ generated by s is quasi-coherent and of finite type, so there exists a quasi-coherent \mathcal{O}_X -submodule \mathcal{G}' of \mathcal{F} of finite type such that $\mathcal{G}'|U = \mathcal{G}$ (9.4.7). By hypothesis, there thus exists a finite number of sections t_i of \mathcal{G}' over X and of sections a_i of \mathcal{O}_X over a neighborhood $V \subset U$ of x such that $s|V = \sum_i a_i(t_i|V)$, which proves the corollary. \square

9.5. Closed image of a prescheme; closure of a subprescheme.

Proposition (9.5.1). — *Let $f : X \rightarrow Y$ be a morphism of preschemes such that $f_*(\mathcal{O}_X)$ is a quasi-coherent \mathcal{O}_Y -module (which will be the case if f is quasi-compact and, in addition, either f is separated or X is locally Noetherian (9.2.2)). Then there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \rightarrow Y$ (or, equivalently (4.4.1), such that the subprescheme $f^{-1}(Y')$ of X is identical to X).*

More precisely:

Corollary (9.5.2). — *Under the conditions of (9.5.1), let $f = (\psi, \theta)$, and let \mathcal{J} be the (quasi-coherent) kernel of the homomorphism $\theta : \mathcal{O}_Y \rightarrow f_*(\mathcal{O}_X)$. Then the closed subprescheme Y' of Y defined by \mathcal{J} satisfies the conditions of (9.5.1).*

Proof. Since the functor ψ^* is exact, the canonical factorization $\theta : \mathcal{O}_Y \rightarrow \mathcal{O}_Y / \mathcal{J} \xrightarrow{\theta'} \psi_*(\mathcal{O}_X)$ gives (0, 3.5.4.3) a factorization $\theta^\# : \psi^*(\mathcal{O}_Y) \rightarrow \psi^*(\mathcal{O}_Y) / \psi^*(\mathcal{J}) \xrightarrow{\theta'^\#} \mathcal{O}_X$; since $\theta^\#$ is a local homomorphism for every $x \in X$, the same is true of $\theta'^\#$; if we denote by ψ_0 the continuous map ψ considered as a map from X to Y' , and by θ_0 the restriction $\theta'|_{Y'} : (\mathcal{O}_Y / \mathcal{J})|_{Y'} \rightarrow \psi_*(\mathcal{O}_X)|_{Y'} = (\psi_0)_*(\mathcal{O}_X)$, then we see that $f_0 = (\psi_0, \theta_0)$ is a morphism of preschemes $X \rightarrow Y'$ (2.2.1) such that $f = j \circ f_0$. Now, if Y'' is a second closed subprescheme of Y , defined by a quasi-coherent sheaf of ideals \mathcal{J}' of \mathcal{O}_Y , such that f factors through the injection $j' : Y'' \rightarrow Y$, then we should immediately have that $\psi(X) \subset Y''$, and thus that $Y' \subset Y''$, since Y'' is closed. Furthermore, for all $y \in Y''$, θ should factor as $\mathcal{O}_y \rightarrow \mathcal{O}_y / \mathcal{J}'_y \rightarrow (\psi_*(\mathcal{O}_X))_y$, which, by definition, leads to $\mathcal{J}'_y \subset \mathcal{J}_y$, and thus X' is a closed subprescheme of Y'' (4.1.10). \square

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Definition (9.5.3). — Whenever there exists a smaller subprescheme Y' of Y such that f factors through the canonical injection $j : Y' \rightarrow Y$, we say that Y' is the *closed image* prescheme of X under the morphism f .

Proposition (9.5.4). — If $f_*(\mathcal{O}_X)$ is a quasi-coherent \mathcal{O}_Y -module, then the underlying space of the closed image of X under f is the closure $\overline{f(X)}$ in Y .

Proof. As the support of $f_*(\mathcal{O}_X)$ is contained in $\overline{f(X)}$, we have (with the notation of (9.5.2)) $\mathcal{J}_y = \mathcal{O}_y$ for $y \notin \overline{f(X)}$, thus the support of $\mathcal{O}_Y / \mathcal{J}$ is contained in $\overline{f(X)}$. In addition, this support is closed and contains $f(X)$: indeed, if $y \in f(X)$, the unit element of the ring $(\psi_*(\mathcal{O}_X))_y$ is not zero, being the germ at y of the section

$$1 \in \Gamma(X, \mathcal{O}_X) = \Gamma(Y, \psi_*(\mathcal{O}_X));$$

since it is the image under θ of the unit element of \mathcal{O}_y , the latter does not belong to \mathcal{J}_y , hence $\mathcal{O}_y / \mathcal{J}_y \neq 0$; this finishes the proof. \square

Proposition (9.5.5). — (Transitivity of closed images). Let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be two morphisms of preschemes; we suppose that the closed image Y' of X under f exists, and that, if g' is the restriction of g to Y' , then the closed image Z' of Y' under g' exists. Then the closed image of X under $g \circ f$ exists and is equal to Z' .

Proof. It suffices (9.5.1) to show that Z' is the smallest closed subprescheme Z_1 of Z such that the closed subprescheme $(g \circ f)^{-1}(Z_1)$ of X (equal to $f^{-1}(g^{-1}(Z_1))$ by Corollary (4.4.2)) is equal to X ; it is equivalent to say that Z' is the smallest closed subprescheme of Z such that f factors through the injection $g^{-1}(Z_1) \rightarrow Y$ (4.4.1). By virtue of the existence of the closed image Y' , every Z_1 with this property is such that $g^{-1}(Z_1)$ factors through Y' , which is equivalent to saying that $j^{-1}(g^{-1}(Z_1)) = g'^{-1}(Z_1) = Y'$, denoting by j the injection $Y' \rightarrow Y$. By the definition of Z' , we indeed conclude that Z' is the smallest closed subprescheme of Z satisfying the preceding condition. \square

Corollary (9.5.6). — Let $f : X \rightarrow Y$ be an S -morphism such that Y is the closed image of X under f . Let Z be an S -scheme; if two S -morphisms g_1, g_2 from Y to Z are such that $g_1 \circ f = g_2 \circ f$, then $g_1 = g_2$.

Proof. Let $h = (g_1, g_2)_S : Y \rightarrow Z \times_S Z$; since the diagonal $T = \Delta_Z(Z)$ is a closed subprescheme of $Z \times_S Z$, $Y' = h^{-1}(T)$ is a closed subprescheme of Y (4.4.1). Let $u = g_1 \circ f = g_2 \circ f$; we then have, by definition of the product, $h' = h \circ f = (u, u)_S$, so $h \circ f = \Delta_Z \circ u$; since $\Delta_Z^{-1}(T) = Z$, we have $h'^{-1}(T) = u^{-1}(Z) = X$, so $f^{-1}(Y') = X$. From this, we conclude (4.4.1) that the f factors through the canonical injection $Y' \rightarrow Y$, so $Y' = Y$ by hypothesis; it then follows (4.4.1) that h factors as $\Delta_Z \circ v$, where v is a morphism $Y \rightarrow Z$, which implies that $g_1 = g_2 = v$. \square

Remark (9.5.7). — If X and Y are S -schemes, Proposition (9.5.6) implies that, when Y is the closed image of X under f , f is an *epimorphism* in the category of S -schemes (T, 1.1). We will show in Chapter V that, conversely, if the closed image Y' of X under f exists and if f is an epimorphism of S -schemes, then we necessarily have that $Y' = Y$.

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Proposition (9.5.8). — Suppose that the hypotheses of (9.5.1) are satisfied, and let Y' be the closed image of X under f . For every open V of Y , let $f_V : f^{-1}(V) \rightarrow V$ be the restriction of f ; then the closed image of $f^{-1}(V)$ under f_V in V exists and is equal to the prescheme induced by Y' on the open $V \cap Y'$ of Y' (in other words, to the subprescheme $\inf(V, Y')$ of Y (4.4.3)).

Proof. Let $X' = f^{-1}(V)$; since the direct image of $\mathcal{O}_{X'}$ by f_V is exactly the restriction of $f_*(\mathcal{O}_X)$ to V , it is clear that the kernel \mathcal{J}' of the homomorphism $\mathcal{O}_V \rightarrow (f_V)_*(\mathcal{O}_{X'})$ is the restriction of \mathcal{J} to V , from which the proposition quickly follows. \square

We will see that this result can be understood as saying that taking the closed image commutes with an extension $Y_1 \rightarrow Y$ of the base prescheme, which is an *open immersion*. We will see in Chapter IV that it is the same for an extension $Y_1 \rightarrow Y$ which is a *flat* morphism, provided that f is separated and quasi-compact.

Proposition (9.5.9). — *Let $f : X \rightarrow Y$ be a morphism such that the closed image Y' of X under f exists.*

- (i) *If X is reduced, then so too is Y' .*
- (ii) *If the hypotheses of Proposition (9.5.1) are satisfied and X is irreducible (resp. integral), then so too is Y' .*

Proof. By hypothesis, the morphism f factors as $X \xrightarrow{g} Y' \xrightarrow{j} Y$, where j is the canonical injection.

Since X is reduced, g factors as $X \xrightarrow{h} Y'_{\text{red}} \xrightarrow{j'} Y'$, where j' is the canonical injection (5.2.2), and it then follows from the definition of Y' that $Y'_{\text{red}} = Y'$. If, moreover, the conditions of Proposition (9.5.1) are satisfied, then it follows from (9.5.4) that $f(X)$ is dense in Y' ; if X is irreducible, then so is Y' (0, 2.1.5). The claim about integral preschemes follows from the conjunction of the two others. \square

Proposition (9.5.10). — *Let Y be a subprescheme of a prescheme X , such that the canonical injection $i : Y \rightarrow X$ is a quasi-compact morphism. Then there exists a smaller closed subprescheme \bar{Y} of X containing Y ; its underlying space is the closure of that of Y ; the latter is open in its closure, and the prescheme Y is induced on this open by \bar{Y} .*

Proof. It suffices to apply Proposition (9.5.1) to the injection j , which is separated (5.5.1) and quasi-compact by hypothesis; (9.5.1) thus proves the existence of \bar{Y} , and (9.5.4) shows that its underlying space is the closure of Y in X ; since Y is locally closed in X , it is open in \bar{Y} , and the last claim comes from (9.5.8) applied to an open subset V of X such that Y is closed in V . \square

With the above notation, if the injection $V \rightarrow X$ is quasi-compact, and if \mathcal{J} is the quasi-coherent sheaf of ideals of $\mathcal{O}_X|_V$ defining the closed subprescheme Y of V , it follows from Proposition (9.5.1) that the quasi-coherent sheaf of ideals of \mathcal{O}_X defining \bar{Y} is the canonical extension (9.4.1) $\bar{\mathcal{J}}$ of \mathcal{J} , because it is clearly the largest quasi-coherent subsheaf of ideals of \mathcal{O}_X inducing \mathcal{J} on V .

Corollary (9.5.11). — *Under the hypotheses of Proposition (9.5.10), every section of $\bar{\mathcal{O}}_{\bar{Y}}$ over an open V of \bar{Y} that is zero on $V \cap Y$ is zero.* I | 179

Proof. By Proposition (9.5.8), we can reduce to the case where $V = \bar{Y}$. If we take into account that the sections of $\bar{\mathcal{O}}_{\bar{Y}}$ over \bar{Y} canonically correspond to the \bar{Y} -sections of $\bar{Y} \otimes_{\mathbb{Z}} \mathbb{Z}[T]$ (3.3.15) and that the latter is separated over \bar{Y} , then the corollary appears as a specific case of (9.5.6). \square

When there exists a smaller closed subprescheme Y' of X containing a subprescheme Y of X , we say that Y' is the *closure* of Y in X , when there is little cause for confusion.

9.6. Quasi-coherent sheaves of algebras; change of structure sheaf.

Proposition (9.6.1). — *Let X be a prescheme, and \mathcal{B} a quasi-coherent \mathcal{O}_X -algebra (0, 5.1.3). For a \mathcal{B} -module \mathcal{F} to be quasi-coherent (on the ringed space (X, \mathcal{B})) it is necessary and sufficient that \mathcal{F} be a quasi-coherent \mathcal{O}_X -module.*

Proof. Since the question is local, we can assume X to be affine, given by some ring A , and thus $\mathcal{B} = \tilde{B}$, where B is an A -algebra (1.4.3). If \mathcal{F} is quasi-coherent on the ringed space (X, \mathcal{B}) then we can also assume that \mathcal{F} is the cokernel of a \mathcal{B} -homomorphism $\mathcal{B}^{(I)} \rightarrow \mathcal{B}^{(J)}$; since this homomorphism is also an \mathcal{O}_X -homomorphism of \mathcal{O}_X -modules, and $\mathcal{B}^{(I)}$ and $\mathcal{B}^{(J)}$ are quasi-coherent \mathcal{O}_X -modules (1.3.9, ii), \mathcal{F} is also a quasi-coherent \mathcal{O}_X -module (1.3.9, i).

Conversely, if \mathcal{F} is a quasi-coherent \mathcal{O}_X -module, then $\mathcal{F} = \tilde{M}$, where M is a B -module (1.4.3); M is isomorphic to the cokernel of a B -homomorphism $B^{(I)} \rightarrow B^{(J)}$, so \mathcal{F} is a \mathcal{B} -module isomorphic to the cokernel of the corresponding homomorphism $\mathcal{B}^{(I)} \rightarrow \mathcal{B}^{(J)}$ (1.3.13), which finishes the proof. \square

In particular, if \mathcal{F} and \mathcal{G} are two quasi-coherent \mathcal{B} -modules, then $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{G}$ is a quasi-coherent \mathcal{B} -module; similarly for $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ whenever we further suppose that \mathcal{F} admits a finite presentation (1.3.13).

(9.6.2). Given a prescheme X , we say that a quasi-coherent \mathcal{O}_X -algebra \mathcal{B} is of *finite type* if, for all $x \in X$, there exists an open *affine* neighborhood U of x such that $\Gamma(U, \mathcal{B}) = B$ is an algebra of finite type over $\Gamma(U, \mathcal{O}_X) = A$. We then have that $\mathcal{B}|_U = \tilde{B}$ and, for all $f \in A$, the induced $(\mathcal{O}_X|_U D(f))$ -algebra $\mathcal{B}|_U D(f)$ is of finite type, because it is isomorphic to $(B_f)^\sim$, and $B_f = B \otimes_A A_f$ is clearly an algebra of finite type over A_f . Since the $D(f)$ form a basis for the topology of U , we thus conclude that if \mathcal{B} is a quasi-coherent \mathcal{O}_X -algebra of finite type then, for every open V of X , $\mathcal{B}|_V$ is a quasi-coherent $(\mathcal{O}_X|_V)$ -algebra of finite type.

Proposition (9.6.3). — *Let X be a locally Noetherian prescheme. Then every quasi-coherent \mathcal{O}_X -algebra \mathcal{B} of finite type is a coherent sheaf of rings (0, 5.3.7).*

Proof. We can once again restrict to the case where X is an affine scheme given by a Noetherian ring A , and where $\mathcal{B} = \tilde{B}$, with B being an A -algebra of finite type; B is then a Noetherian ring. With this, it remains to prove that the kernel \mathcal{N} of a \mathcal{B} -homomorphism $\mathcal{B}^m \rightarrow \mathcal{B}$ is a \mathcal{B} -module of finite type; but it is isomorphic (as a \mathcal{B} -module) to \tilde{N} , where N is the kernel of the corresponding homomorphism of B -modules $B^m \rightarrow B$ (1.3.13). Since B is Noetherian, the submodule N of B^m is a B -module of finite type, so there exists a homomorphism $B^p \rightarrow B^m$ with image N ; since the sequence $B^p \rightarrow B^m \rightarrow B$ is exact, so too is the corresponding sequence $\mathcal{B}^p \rightarrow \mathcal{B}^m \rightarrow \mathcal{B}$ (1.3.5), and since \mathcal{N} is the image of $\mathcal{B}^p \rightarrow \mathcal{B}^m$ (1.3.9, i), this proves the proposition. \square

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Corollary (9.6.4). — *Under the hypotheses of (9.6.3), for a \mathcal{B} -module \mathcal{F} to be coherent, it is necessary and sufficient that it be a quasi-coherent \mathcal{O}_X -module and a \mathcal{B} -module of finite type. If this is the case, and if \mathcal{G} is a \mathcal{B} -submodule or a quotient module of \mathcal{F} , then in order for \mathcal{G} to be a coherent \mathcal{B} -module, it is necessary and sufficient that it be a quasi-coherent \mathcal{O}_X -module.*

Proof. Taking (9.6.1) into account, the conditions on \mathcal{F} are clearly necessary; we will show that they are sufficient. We can restrict to the case where X is affine, given by some Noetherian ring A , where $\mathcal{B} = \tilde{B}$, with B an A -algebra of finite type, where $\mathcal{F} = \tilde{M}$, with M a B -module, and where there exists a surjective \mathcal{B} -homomorphism $\mathcal{B}^m \rightarrow \mathcal{F} \rightarrow 0$. We then have the corresponding exact sequence $B^m \rightarrow M \rightarrow 0$, so M is a B -module of finite type; further, the kernel P of the homomorphism $B^m \rightarrow M$ is then a B -module of finite type, since B is Noetherian. We thus conclude (1.3.13) that \mathcal{F} is the cokernel of a \mathcal{B} -homomorphism $\mathcal{B}^m \rightarrow \mathcal{B}^n$, and is thus coherent, since \mathcal{B} is a coherent sheaf of rings (0, 5.3.4). The same reasoning shows that a quasi-coherent \mathcal{B} -submodule (resp. a quotient \mathcal{B} -module) of \mathcal{F} is of finite type, from whence the second part of the corollary. \square

Proposition (9.6.5). — *Let X be a quasi-compact scheme, or a prescheme whose underlying space is Noetherian. For all quasi-compact \mathcal{O}_X -algebras \mathcal{B} of finite type, there exists a quasi-coherent \mathcal{O}_X -submodule \mathcal{E} of \mathcal{B} of finite type such that \mathcal{E} generates (0, 4.1.3) the \mathcal{O}_X -algebra \mathcal{B} .*

Proof. In fact, by hypothesis, there exists a finite cover (U_i) of X consisting of affine opens such that $\Gamma(U_i, \mathcal{B}) = B_i$ is an algebra of finite type over $\Gamma(U_i, \mathcal{O}_X) = A_i$; let E_i be a A_i -submodule of B_i of finite type that generates the A_i -algebra B_i ; thanks to (9.4.7), there exists a \mathcal{O}_X -submodule \mathcal{E}_i of \mathcal{B} , quasi-coherent and of finite type, such that $\mathcal{E}_i|_{U_i} = \tilde{E}_i$. It is clear that the sum \mathcal{E} of the \mathcal{E}_i is the desired object. \square

Proposition (9.6.6). — *Let X be a prescheme whose underlying space is locally Noetherian, or a quasi-compact scheme. Then every quasi-coherent \mathcal{O}_X -algebra \mathcal{B} is the inductive limit of its quasi-coherent \mathcal{O}_X -subalgebras of finite type.*

Proof. In fact, it follows from (9.4.9) that \mathcal{B} is the inductive limit (as an \mathcal{O}_X -module) of its quasi-coherent \mathcal{O}_X -submodules of finite type; the latter generating quasi-coherent \mathcal{O}_X -subalgebras of \mathcal{B} of finite type (1.3.14), and so \mathcal{B} is a fortiori their inductive limit. \square

§10. FORMAL SCHEMES

10.1. Formal affine schemes.

(10.1.1). Let A be an *admissible* topological ring (0, 7.1.2); for each ideal of definition \mathfrak{J} of A , $\mathrm{Spec}(A/\mathfrak{J})$ can be identified with the closed subspace $V(\mathfrak{J})$ of $\mathrm{Spec}(A)$ (1.1.11), the set of *open* prime ideals of A ; this topological space does not depend on the ideal of definition \mathfrak{J} considered; we denote this topological space by \mathfrak{X} . Let (\mathfrak{J}_λ) be a fundamental system of neighborhoods of 0 in A , consisting of ideals of definition, and for each λ , let \mathcal{O}_λ be the structure sheaf of $\mathrm{Spec}(A/\mathfrak{J}_\lambda)$; this sheaf is induced on \mathfrak{X} by $\tilde{A}/\tilde{\mathfrak{J}}_\lambda$ (which is zero outside of \mathfrak{X}). For $\mathfrak{J}_\mu \subset \mathfrak{J}_\lambda$, the canonical homomorphism $A/\mathfrak{J}_\mu \rightarrow A/\mathfrak{J}_\lambda$ thus defines a homomorphism $u_{\lambda\mu} : \mathcal{O}_\mu \rightarrow \mathcal{O}_\lambda$ of sheaves of rings (1.6.1), and (\mathcal{O}_λ) is a *projective system of sheaves of rings* for these homomorphisms. Since the topology of \mathfrak{X} admits a basis consisting of quasi-compact open subsets, we can associate to each \mathcal{O}_λ a *pseudo-discrete sheaf of topological rings* (0, 3.8.1) which have \mathcal{O}_λ as the underlying sheaf of rings (without topologies), and that we denote also by \mathcal{O}_λ ; and the \mathcal{O}_λ give again a *projective system of sheaves of topological rings* (0, 3.8.2). We denote by $\mathcal{O}_{\mathfrak{X}}$ the *sheaf of topological rings* on \mathfrak{X} , the projective limit of the system (\mathcal{O}_λ) ; for each *quasi-compact* open subset U of \mathfrak{X} , $\Gamma(U, \mathcal{O}_{\mathfrak{X}})$ is a topological ring, the projective limit of the system of *discrete* rings $\Gamma(U, \mathcal{O}_\lambda)$ (0, 3.2.6).

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Definition (10.1.2). — Given an admissible topological ring A , we define the formal spectrum of A , denoted by $\mathrm{Spf}(A)$, to be the closed subspace \mathfrak{X} of $\mathrm{Spec}(A)$ consisting of the open prime ideals of A . We say that a topologically ringed space is a formal affine scheme if it is isomorphic to a formal spectrum $\mathrm{Spf}(A) = \mathfrak{X}$ equipped with a sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$ which is the projective limit of sheaves of pseudo-discrete topological rings $(\tilde{A}/\tilde{\mathfrak{J}}_\lambda)|_{\mathfrak{X}}$, where \mathfrak{J}_λ varies over the filtered set of ideals of definition of A .

When we speak of a *formal spectrum* $\mathfrak{X} = \mathrm{Spf}(A)$ as a formal affine scheme, it will always be as the topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ where $\mathcal{O}_{\mathfrak{X}}$ is defined as above.

We note that every *affine scheme* $X = \mathrm{Spec}(A)$ can be considered as a formal affine scheme in only one way, by considering A as a discrete topological ring: the topological rings $\Gamma(U, \mathcal{O}_X)$ are then discrete whenever U is quasi-compact (but not, in general, when U is an arbitrary open subset of X).

Proposition (10.1.3). — If $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an admissible ring, then $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is topologically isomorphic to A .

Proof. Indeed, since \mathfrak{X} is closed in $\mathrm{Spec}(A)$, it is quasi-compact, and so $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is topologically isomorphic to the projective limit of the discrete rings $\Gamma(\mathfrak{X}, \mathcal{O}_\lambda)$; but $\Gamma(\mathfrak{X}, \mathcal{O}_\lambda)$ is isomorphic to A/\mathfrak{J}_λ (1.3.7); since A is separated and complete, it is topologically isomorphic to $\varprojlim A/\mathfrak{J}_\lambda$ (0, 7.2.1), whence the proposition. \square

Proposition (10.1.4). — Let A be an admissible ring, $\mathfrak{X} = \mathrm{Spf}(A)$, and, for every $f \in A$, let $\mathfrak{D}(f) = D(f) \cap \mathfrak{X}$; then the topologically ringed space $(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}}|_{\mathfrak{D}(f)})$ is isomorphic to the formal affine spectrum $\mathrm{Spf}(A_{\{f\}})$ (0, 7.6.15).

Proof. For every ideal of definition \mathfrak{J} of A , the discrete ring $S_f^{-1}A/S_f^{-1}\mathfrak{J}$ is canonically identified with $A_{\{f\}}/\mathfrak{J}_{\{f\}}$ (0, 7.6.9), so, by (1.2.5) and (1.2.6), the topological space $\mathrm{Spf}(A_{\{f\}})$ is canonically identified with $\mathfrak{D}(f)$. Further, for every quasi-compact open subset U of \mathfrak{X} contained in $\mathfrak{D}(f)$, $\Gamma(U, \mathcal{O}_\lambda)$ can be identified with the module of sections of the structure sheaf of $\mathrm{Spec}(S_f^{-1}A/S_f^{-1}\mathfrak{J}_\lambda)$ over U (1.3.6), so, setting $\mathfrak{Y} = \mathrm{Spf}(A_{\{f\}})$, $\Gamma(U, \mathcal{O}_{\mathfrak{X}})$ can be identified with the module of sections $\Gamma(U, \mathcal{O}_{\mathfrak{Y}})$, which proves the proposition. \square

(10.1.5). As a sheaf of rings *without topology*, the structure sheaf $\mathcal{O}_{\mathfrak{X}}$ of $\mathrm{Spf}(A)$ admits, for every $x \in \mathfrak{X}$, a fibre which, by (10.1.4), can be identified with the inductive limit $\varinjlim A_{\{f\}}$ for the $f \notin \mathfrak{j}_x$. Then, by (0, 7.6.17) and (0, 7.6.18):

Proposition (10.1.6). — For every $x \in \mathfrak{X} = \mathrm{Spf}(A)$, the fibre \mathcal{O}_x is a local ring whose residue field is isomorphic to $k(x) = A_x/\mathfrak{j}_x A_x$. If, further, A is adic and Noetherian, then \mathcal{O}_x is a Noetherian ring.

Since $k(x)$ is not reduced at 0, we conclude from this that the *support* of the ring of sheaves $\mathcal{O}_{\mathfrak{X}}$ is equal to \mathfrak{X} .

10.2. Morphisms of formal affine schemes.

(10.2.1). Let A, B be two admissible rings, and let $\phi : B \rightarrow A$ be a *continuous* morphism. The continuous map ${}^a\phi : \operatorname{Spec}(A) \rightarrow \operatorname{Spec}(B)$ (1.2.1) then maps $\mathfrak{X} = \operatorname{Spf}(A)$ to $\mathfrak{Y} = \operatorname{Spf}(B)$, since the inverse image under ϕ of an open prime ideal of A is an open prime ideal of B . Conversely, for all $g \in B$, ϕ defines a continuous homomorphism $\Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{Y}}) \rightarrow \Gamma(\mathfrak{D}(\phi(g)), \mathcal{O}_{\mathfrak{X}})$ according to (10.1.4), (10.1.3), and (0, 7.6.7); since these homomorphisms satisfy the compatibility conditions for the restrictions corresponding to the change from g to a multiple of g , and since $\mathfrak{D}(\phi(g)) = {}^a\phi^{-1}(\mathfrak{D}(g))$, they define a *continuous* homomorphism of sheaves of topological rings $\mathcal{O}_{\mathfrak{Y}} \rightarrow {}^a\phi_*(\mathcal{O}_{\mathfrak{X}})$ (0, 3.2.5) that we denote by $\tilde{\phi}$; we have thus defined a morphism $\Phi = ({}^a\phi, \tilde{\phi})$ of topologically ringed spaces $\mathfrak{X} \rightarrow \mathfrak{Y}$. We note that, as a homomorphism of sheaves without topology, $\tilde{\phi}$ defines a homomorphism $\tilde{\phi}_x^\# : \mathcal{O}_{\phi(x)} \rightarrow \mathcal{O}_x$ on the stalks, for all $x \in \mathfrak{X}$.

Proposition (10.2.2). — *Let A and B be admissible topological rings, and let $\mathfrak{X} = \operatorname{Spf}(A)$ and $\mathfrak{Y} = \operatorname{Spf}(B)$. For a morphism $u = (\psi, \theta) : \mathfrak{X} \rightarrow \mathfrak{Y}$ of topologically ringed spaces to be of the form $({}^a\phi, \tilde{\phi})$, where ϕ is a continuous ring homomorphism $B \rightarrow A$, it is necessary and sufficient that $\theta_x^\#$ be a local homomorphism $\mathcal{O}_{\phi(x)} \rightarrow \mathcal{O}_x$ for all $x \in \mathfrak{X}$.*

Proof. The condition is necessary: let $\mathfrak{p} = \mathfrak{j}_x \in \operatorname{Spf}(A)$, and let $\mathfrak{q} = \phi^{-1}(\mathfrak{j}_x)$; if $g \notin \mathfrak{q}$, then we have $\phi(g) \notin \mathfrak{p}$, and it is immediate that the homomorphism $B_{\{g\}} \rightarrow A_{\{\phi(g)\}}$ induced by ϕ (0, 7.6.7) sends $\mathfrak{q}_{\{g\}}$ to a subset of $\mathfrak{p}_{\{\phi(g)\}}$; by passing to the inductive limit, we see (taking (10.1.5) and (0, 7.6.17) into account) that $\tilde{\phi}_x^\#$ is a local homomorphism.

Conversely, let (ψ, θ) be a morphism satisfying the condition of the proposition; by (10.1.3), θ defines a continuous ring homomorphism

$$\phi = \Gamma(\theta) : B = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \longrightarrow \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = A.$$

By virtue of the hypothesis on θ , for the section $\phi(g)$ of $\mathcal{O}_{\mathfrak{X}}$ over \mathfrak{X} to be an invertible germ at the point x , it is necessary and sufficient that g be an invertible germ at the point $\psi(x)$. But, by (0, 7.6.17), the sections of $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{\mathfrak{Y}}$) over \mathfrak{X} (resp. \mathfrak{Y}) that have a non-invertible germ at the point x (resp. $\psi(x)$) are exactly the elements of \mathfrak{j}_x (resp. $\mathfrak{j}_{\psi(x)}$); the above remark thus shows that ${}^a\phi = \psi$. Finally, I | 183

for all $g \in B$ the diagram

$$\begin{array}{ccc} B = \Gamma(\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) & \xrightarrow{\phi} & \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) = A \\ \downarrow & & \downarrow \\ B_{\{g\}} = \Gamma(\mathfrak{D}(g), \mathcal{O}_{\mathfrak{Y}}) & \xrightarrow{\Gamma(\theta_{\mathfrak{D}(g)})} & \Gamma(\mathfrak{D}(\phi(g)), \mathcal{O}_{\mathfrak{X}}) = A_{\{\phi(g)\}} \end{array}$$

is commutative; by the universal property of completed rings of fractions (0, 7.6.6), $\theta_{\mathfrak{D}(g)}$ is equal to $\tilde{\phi}_{\mathfrak{D}(g)}$ for all $g \in B$, and so (0, 3.2.5) we have $\theta = \tilde{\phi}$. \square

We say that a morphism (ψ, θ) of topologically ringed spaces satisfying the condition of Proposition (10.2.2) is a *morphism of formal affine schemes*. We can say that the functors $\operatorname{Spf}(A)$ in A and $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ in \mathfrak{X} define an *equivalence* between the category of admissible rings and the opposite category of formal affine schemes (T, I, 1.2).

(10.2.3). As a particular case of (10.2.2), note that, for $f \in A$, the canonical injection of the formal affine scheme induced by \mathfrak{X} on $\mathfrak{D}(f)$ corresponds to the continuous canonical homomorphism $A \rightarrow A_{\{f\}}$. Under the hypotheses of Proposition (10.2.2), let h be an element of B , and g an element of A that is a multiple of $\phi(h)$; we then have $\psi(\mathfrak{D}(g)) \subset \mathfrak{D}(h)$; the restriction of u to $\mathfrak{D}(g)$, considered as a morphism from $\mathfrak{D}(g)$ to $\mathfrak{D}(h)$, is the unique morphism v making the diagram

$$\begin{array}{ccc} \mathfrak{D}(g) & \xrightarrow{v} & \mathfrak{D}(h) \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{u} & \mathfrak{Y} \end{array}$$

commutate.

This morphism corresponds to the unique continuous homomorphism $\phi' : B_{\{h\}} \rightarrow A_{\{g\}}$ (0, 7.6.7) making the diagram

$$\begin{array}{ccc} A & \xleftarrow{\phi} & B \\ \downarrow & & \downarrow \\ A_{\{g\}} & \xleftarrow{\phi'} & B_{\{h\}} \end{array}$$

commutate.

10.3. Ideals of definition for a formal affine scheme.

(10.3.1). Let A be an admissible ring, \mathfrak{J} an open ideal of A , and \mathfrak{X} the formal affine scheme $\mathrm{Spf}(A)$. Let (\mathfrak{J}_λ) be the set of those ideals of definition for A that are contained in \mathfrak{J} ; then $\tilde{\mathfrak{J}}/\tilde{\mathfrak{J}}_\lambda$ is a sheaf of ideals of $\tilde{A}/\tilde{\mathfrak{J}}_\lambda$. Denote by \mathfrak{J}^Δ the projective limit of the sheaves on \mathfrak{X} induced by $\tilde{\mathfrak{J}}/\tilde{\mathfrak{J}}_\lambda$, which is identified with a *sheaf of ideals* of $\mathcal{O}_{\mathfrak{X}}$ (0, 3.2.6). For every $f \in A$, $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta)$ is the projective limit of the $S_f^{-1}\mathfrak{J}/S_f^{-1}\mathfrak{J}_\lambda$, or, in other words, can be identified with the open ideal $\mathfrak{J}_{\{f\}}$ of the ring $A_{\{f\}}$ (0, 7.6.9), and, in particular, $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$; we conclude (the $\mathfrak{D}(f)$ forming a basis for the topology of \mathfrak{X}) that

$$(10.3.1.1) \quad \mathfrak{J}^\Delta|_{\mathfrak{D}(f)} = (\mathfrak{J}_{\{f\}})^\Delta.$$

(10.3.2). With the notation of (10.3.1), for all $f \in A$, the canonical map from $A_{\{f\}} = \Gamma(\mathfrak{D}(f), \mathcal{O}_{\mathfrak{X}})$ to $\Gamma(\mathfrak{D}(f), (\tilde{A}/\tilde{\mathfrak{J}})|_{\mathfrak{X}}) = S_f^{-1}A/S_f^{-1}\mathfrak{J}$ is *surjective* and has $\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta) = \mathfrak{J}_{\{f\}}$ as its kernel (0, 7.6.9); these maps thus define a *surjective* continuous homomorphism, said to be *canonical*, from the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$ to the sheaf of discrete rings $(\tilde{A}/\tilde{\mathfrak{J}})|_{\mathfrak{X}}$, whose kernel is \mathfrak{J}^Δ ; this homomorphism is none other than $\tilde{\phi}$ (10.2.1), where ϕ is the continuous homomorphism $A \rightarrow A/\mathfrak{J}$; the morphism $({}^a\phi, \tilde{\phi}) : \mathrm{Spec}(A/\mathfrak{J}) \rightarrow \mathfrak{X}$ of formal affine schemes (where ${}^a\phi$ is the identity homeomorphism from \mathfrak{X} to itself) is also said to be *canonical*. We thus have, according to the above, a *canonical isomorphism*

$$(10.3.2.1) \quad \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^\Delta \simeq (\tilde{A}/\tilde{\mathfrak{J}})|_{\mathfrak{X}}.$$

It is clear (since $\Gamma(\mathfrak{X}, \mathfrak{J}^\Delta) = \mathfrak{J}$) that the map $\mathfrak{J} \rightarrow \mathfrak{J}^\Delta$ is *strictly increasing*; according to the above, for $\mathfrak{J} \subset \mathfrak{J}'$, the sheaf $\mathfrak{J}'^\Delta/\mathfrak{J}^\Delta$ is canonically isomorphic to $\tilde{\mathfrak{J}}'/\tilde{\mathfrak{J}} = (\mathfrak{J}'/\mathfrak{J})^\sim$.

(10.3.3). The hypotheses and notation being the same as in (10.3.1), we say that a sheaf of ideals \mathcal{J} of $\mathcal{O}_{\mathfrak{X}}$ is a *sheaf of ideals of definition* for \mathfrak{X} (or an *ideal sheaf of definition* for \mathfrak{X}) if, for all $x \in \mathfrak{X}$, there exists an open neighborhood of x of the form $\mathfrak{D}(f)$, where $f \in A$, such that $\mathcal{J}|_{\mathfrak{D}(f)}$ is of the form \mathfrak{H}^Δ , where \mathfrak{H} is an ideal of definition for $A_{\{f\}}$.

Proposition (10.3.4). — *For all $f \in A$, each sheaf of ideals of definition for \mathfrak{X} induces a sheaf of ideals of definition for $\mathfrak{D}(f)$.*

Proof. This follows from (10.3.1.1). □

Proposition (10.3.5). — *If A is an admissible ring, then every sheaf of ideals of definition for $\mathfrak{X} = \mathrm{Spf}(A)$ is of the form \mathfrak{J}^Δ , where \mathfrak{J} is a uniquely determined ideal of definition for A .*

Proof. Let \mathcal{J} be a sheaf of ideals of definition of \mathfrak{X} ; by hypothesis, and since \mathfrak{X} is quasi-compact, there are finitely-many elements $f_i \in A$ such that the $\mathfrak{D}(f_i)$ cover \mathfrak{X} and such that $\mathcal{J}|_{\mathfrak{D}(f_i)} = \mathfrak{H}_i^\Delta$, where \mathfrak{H}_i is an ideal of definition for $A_{\{f_i\}}$. For each i , there exists an open ideal \mathfrak{K}_i of A such that $(\mathfrak{K}_i)_{\{f_i\}} = \mathfrak{H}_i$ (0, 7.6.9); let \mathfrak{K} be an ideal of definition for A containing all the \mathfrak{K}_i . The canonical image of $\mathcal{J}/\mathfrak{K}^\Delta$ in the structure sheaf $(A/\mathfrak{K})^\sim$ of $\mathrm{Spec}(A/\mathfrak{K})$ (10.3.2) is thus such that its restriction to $\mathfrak{D}(f_i)$ is equal to its restriction to $(\mathfrak{K}_i/\mathfrak{K})^\sim$; we conclude that this canonical image is a *quasi-coherent* sheaf on $\mathrm{Spec}(A/\mathfrak{K})$, and so is of the form $(\mathfrak{J}/\mathfrak{K})^\sim$, where \mathfrak{J} is an ideal of definition for A containing \mathfrak{K} (1.4.1), whence $\mathcal{J} = \mathfrak{J}^\Delta$ (10.3.2); in addition, since for each i there exists an integer n_i such that $\mathfrak{H}_i^{n_i} \subset \mathfrak{K}_{\{f_i\}}$, we will have, by setting n to be the largest of the n_i , that $(\mathcal{J}/\mathfrak{K}^\Delta)^n = 0$, and, as a result (10.3.2), that $((\mathfrak{J}/\mathfrak{K})^\sim)^n = 0$, whence finally that $(\mathfrak{J}/\mathfrak{K})^n = 0$ (1.3.13), which proves that \mathfrak{J} is an ideal of definition for A (0, 7.1.4). □

Proposition (10.3.6). — *Let A be an adic ring, and \mathfrak{J} an ideal of definition for A such that $\mathfrak{J}/\mathfrak{J}^2$ is an (A/\mathfrak{J}) -module of finite type. For any integer $n > 0$, we then have $(\mathfrak{J}^\Delta)^n = (\mathfrak{J}^n)^\Delta$.*

Proof. For all $f \in A$, we have (since \mathfrak{J}^n is an open ideal)

$$(\Gamma(\mathfrak{D}(f), \mathfrak{J}^\Delta))^n = (\mathfrak{J}_{\{f\}})^\Delta = (\mathfrak{J}^n)_{\{f\}} = \Gamma(\mathfrak{D}(f^n), (\mathfrak{J}^n)^\Delta)$$

by (10.3.1.1) and (0, 7.6.12). The result then follows from the fact that $(\mathfrak{J}^\Delta)^n$ is associated to the presheaf $U \mapsto (\Gamma(U, \mathfrak{J}^\Delta))^n$ (0, 4.1.6), since the $\mathfrak{D}(f)$ form a basis for the topology of \mathfrak{X} . \square I | 185

(10.3.7). We say that a family (\mathcal{J}_λ) of sheaves of ideals of definition for \mathfrak{X} is a *fundamental system of sheaves of ideals of definition* if each sheaf of ideals of definition for \mathfrak{X} contains one of the \mathcal{J}_λ ; since $\mathcal{J}_\lambda = \mathfrak{J}_\lambda^\Delta$, it is equivalent to say that the \mathfrak{J}_λ form a *fundamental system of neighborhoods of 0* in A . Let (f_α) be a family of elements of A such that the $\mathfrak{D}(f_\alpha)$ cover \mathfrak{X} . If (\mathcal{J}_λ) is a filtered decreasing family of sheaves of ideals of $\mathcal{O}_\mathfrak{X}$ such that, for each α , the family $(\mathcal{J}_\lambda|_{\mathfrak{D}(f_\alpha)})$ is a fundamental system of sheaves of ideals of definition for $\mathfrak{D}(f_\alpha)$, then (\mathcal{J}_λ) is a fundamental system of sheaves of ideals of definition for \mathfrak{X} . Indeed, for each sheaf of ideals of definition \mathcal{J} for \mathfrak{X} , there is a finite cover of \mathfrak{X} by $\mathfrak{D}(f_i)$ such that, for each i , $\mathcal{J}_{\lambda_i}|_{\mathfrak{D}(f_i)}$ is a sheaf of ideals of definition for $\mathfrak{D}(f_i)$ that is contained in $\mathcal{J}|_{\mathfrak{D}(f_i)}$. If μ is an index such that $\mathcal{J}_\mu \subset \mathcal{J}_{\lambda_i}$ for all i , then it follows from (10.3.3) that \mathcal{J}_μ is a sheaf of ideals of definition for \mathfrak{X} , evidently contained in \mathcal{J} , whence our claim.

10.4. Formal preschemes and morphisms of formal preschemes.

(10.4.1). Given a topologically ringed space \mathfrak{X} , we say that an open $U \subset \mathfrak{X}$ is a *formal affine open* (resp. a *formal adic affine open*, resp. a *formal Noetherian affine open*) if the topologically ringed space induced on U by \mathfrak{X} is a formal affine scheme (resp. a scheme whose ring is adic, resp. adic and Noetherian).

Definition (10.4.2). — A *formal prescheme* is a topologically ringed space \mathfrak{X} which admits a formal affine open neighborhood for each point. We say that the formal prescheme \mathfrak{X} is *adic* (resp. *locally Noetherian*) if each point of \mathfrak{X} admits a formal adic (resp. Noetherian) open neighborhood. We say that \mathfrak{X} is *Noetherian* if it is locally Noetherian and its underlying space is quasi-compact (and hence Noetherian).

Proposition (10.4.3). — *If \mathfrak{X} is a formal prescheme (resp. a locally Noetherian formal prescheme), then the formal affine (resp. Noetherian affine) open sets form a basis for the topology of \mathfrak{X} .*

Proof. This follows from Definition (10.4.2) and Proposition (10.1.4) by taking into account that, if A is an adic Noetherian ring, then so too is $A_{\{f\}}$ for all $f \in A$ (0, 7.6.11). \square

Corollary (10.4.4). — *If \mathfrak{X} is a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme), then the topologically ringed space induced on each open set of \mathfrak{X} is also a formal prescheme (resp. a locally Noetherian formal prescheme, resp. a Noetherian formal prescheme).*

Definition (10.4.5). — Given two formal preschemes \mathfrak{X} and \mathfrak{Y} , a morphism (of formal preschemes) from \mathfrak{X} to \mathfrak{Y} is a morphism (ψ, θ) of topologically ringed spaces such that, for all $x \in \mathfrak{X}$, θ_x^\sharp is a local homomorphism $\mathcal{O}_{\psi(x)} \rightarrow \mathcal{O}_x$.

It is immediate that the composition of any two morphisms of formal preschemes is again a morphism of formal preschemes; the formal preschemes thus form a *category*, and we denote by $\text{Hom}(\mathfrak{X}, \mathfrak{Y})$ the set of morphisms from a formal prescheme \mathfrak{X} to a formal prescheme \mathfrak{Y} . I | 186

If U is an open subset of \mathfrak{X} , then the canonical injection into \mathfrak{X} of the formal prescheme induced on U by \mathfrak{X} is a morphism of formal preschemes (and in fact a *monomorphism* of topologically ringed spaces (0, 4.1.1)).

Proposition (10.4.6). — *Let \mathfrak{X} be a formal prescheme, and $\mathfrak{S} = \text{Spf}(A)$ a formal affine scheme. There exists a canonical bijective equivalence between the morphisms from the formal prescheme \mathfrak{X} to the formal prescheme \mathfrak{S} and the continuous homomorphisms from the ring A to the topological ring $\Gamma(\mathfrak{X}, \mathcal{O}_\mathfrak{X})$.*

Proof. The proof is similar to that of (2.2.4), by replacing “homomorphism” with “continuous homomorphism”, “affine open” with “formal affine open”, and by using Proposition (10.2.2) instead of Theorem (1.7.3); we leave the details to the reader. \square

(10.4.7). Given a formal prescheme \mathfrak{S} , we say that the data of a formal prescheme \mathfrak{X} and a morphism $\phi : \mathfrak{X} \rightarrow \mathfrak{S}$ defines a formal prescheme \mathfrak{X} over \mathfrak{S} or an *formal \mathfrak{S} -prescheme*, ϕ being called the *structure morphism* of the \mathfrak{S} -prescheme \mathfrak{X} . If $\mathfrak{S} = \text{Spf}(A)$, where A is an admissible ring, then we also say that the formal \mathfrak{S} -prescheme \mathfrak{X} is a *formal A -prescheme* or a formal prescheme over A . An arbitrary formal prescheme can be considered as a formal prescheme over \mathbb{Z} (equipped with the discrete topology).

If \mathfrak{X} and \mathfrak{Y} are formal \mathfrak{S} -preschemes, then we say that a morphism $u : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a \mathfrak{S} -*morphism* if the diagram

$$\begin{array}{ccc} \mathfrak{X} & \xrightarrow{u} & \mathfrak{Y} \\ & \searrow & \swarrow \\ & \mathfrak{S} & \end{array}$$

(where the downwards arrows are the structure morphisms) is commutative. With this definition, the formal \mathfrak{S} -preschemes (for some fixed \mathfrak{S}) form a *category*. We denote by $\text{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ the set of \mathfrak{S} -morphisms from a formal \mathfrak{S} -prescheme \mathfrak{X} to a formal \mathfrak{S} -prescheme \mathfrak{Y} . When $\mathfrak{S} = \text{Spf}(A)$, we sometimes say *A-morphism* instead of \mathfrak{S} -morphism.

(10.4.8). Since every affine scheme can be considered as a formal affine scheme (10.1.2), every (usual) prescheme can be considered as a formal prescheme. In addition, it follows from Definition (10.4.5) that, for the *usual* preschemes, the morphisms (resp. S -morphisms) of *formal* preschemes coincide with the morphisms (resp. S -morphisms) defined in §2.

10.5. Sheaves of ideals of definition for formal preschemes.

(10.5.1). Let \mathfrak{X} be a formal prescheme; we say that an $\mathcal{O}_{\mathfrak{X}}$ -ideal \mathcal{J} is a *sheaf of ideals of definition* (or an *ideal sheaf of definition*) for \mathfrak{X} if every $x \in \mathfrak{X}$ has a formal affine open neighborhood U such that $\mathcal{J}|_U$ is a sheaf of ideals of definition for the formal affine scheme induced on U by \mathfrak{X} (10.3.3); by (10.3.1.1) and Proposition (10.4.3), for each open $V \subset \mathfrak{X}$, $\mathcal{J}|_V$ is then a sheaf of ideals of definition for the formal prescheme induced on V by \mathfrak{X} .

We say that a family (\mathcal{J}_{λ}) of sheaves of ideals of definition for \mathfrak{X} is a *fundamental system of sheaves of ideals of definition* if there exists a cover (U_{α}) of \mathfrak{X} by formal affine open sets such that, for each α , the family of the $\mathcal{J}_{\lambda}|_{U_{\alpha}}$ is a fundamental system of sheaves of ideals of definition (10.3.6) for the formal affine scheme induced on U_{α} by \mathfrak{X} . It follows from the last remark of (10.3.7) that, when \mathfrak{X} is a formal affine scheme, this definition coincides with the definition given in (10.3.7). For an open subset V of \mathfrak{X} , the restrictions $\mathcal{J}_{\lambda}|_V$ then form a fundamental system of sheaves of ideals of definition for the formal prescheme induced on V , according to (10.3.1.1). If \mathfrak{X} is a *locally Noetherian* formal prescheme, and \mathcal{J} is a sheaf of ideals of definition for \mathfrak{X} , then it follows from Proposition (10.3.6) that the powers \mathcal{J}^n form a fundamental system of sheaves of ideals of definition for \mathfrak{X} .

(10.5.2). Let \mathfrak{X} be a formal prescheme, and \mathcal{J} a sheaf of ideals of definition for \mathfrak{X} . Then the ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$ is a (usual) *prescheme*, which is affine (resp. locally Noetherian, resp. Noetherian) when \mathfrak{X} is a formal affine scheme (resp. a locally Noetherian formal scheme, resp. a Noetherian formal scheme); we can reduce to the affine case, and then the proposition has already been proved in (10.3.2). In addition, if $\theta : \mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{J}$ is the canonical homomorphism, then $u = (1_{\mathfrak{X}}, \theta)$ is a *morphism* (said to be *canonical*) of formal preschemes $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, because, again, this was proved in the affine case (10.3.2), to which it is immediately reduced.

Proposition (10.5.3). — *Let \mathfrak{X} be a formal prescheme, and (\mathcal{J}_{λ}) a fundamental system of sheaves of ideals of definition for \mathfrak{X} . Then the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$ is the projective limit of the pseudo-discrete sheaves of rings $(0, 3.8.1) \mathcal{O}_{\mathfrak{X}}/\mathcal{J}_{\lambda}$.*

Proof. Since the topology of \mathfrak{X} admits a basis of formal quasi-compact affine open sets (10.4.3), we reduce to the affine case, where the proposition is a consequence of Proposition (10.3.5), (10.3.2), and Definition (10.1.1). \square

It is not true that any formal prescheme admits a sheaf of ideals of definition. However:

Proposition (10.5.4). — *Let \mathfrak{X} be a locally Noetherian formal prescheme. There exists a largest sheaf of ideals of definition \mathcal{T} for \mathfrak{X} ; this is the unique sheaf of ideals of definition \mathcal{J} such that the prescheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$*

is reduced. If \mathcal{J} is a sheaf of ideals of definition for \mathfrak{X} , then \mathcal{T} is the inverse image under $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{J}$ of the nilradical of $\mathcal{O}_{\mathfrak{X}}/\mathcal{J}$.

Proof. Suppose first that $\mathfrak{X} = \mathrm{Spf}(A)$, where A is an adic Noetherian ring. The existence and the properties of \mathcal{T} follow immediately from Propositions (10.3.5) and (5.1.1), taking into account the existence and the properties of the largest ideal of definition for A ((0, 7.1.6) and (0, 7.1.7)).

ErrII

To prove the existence and the properties of \mathcal{T} in the general case, it suffices to show that, if $U \supset V$ are two Noetherian formal affine open subsets of X , then the largest sheaf of ideals of definition \mathcal{T}_U for U induces the largest sheaf of ideals of definition \mathcal{T}_V for V ; but as $(V, (\mathcal{O}_{\mathfrak{X}}|V)/(\mathcal{T}_U|V))$ is reduced, this follows from the above. \square

We denote by $\mathfrak{X}_{\mathrm{red}}$ the (usual) reduced prescheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{T})$.

Corollary (10.5.5). — *Let \mathfrak{X} be a locally Noetherian formal prescheme, and \mathcal{T} the largest sheaf of ideals of definition for \mathfrak{X} ; for each open subset V of \mathfrak{X} , $\mathcal{T}|V$ is the largest sheaf of ideals of definition for the formal prescheme induced on V by \mathfrak{X} .*

Proposition (10.5.6). — *Let \mathfrak{X} and \mathfrak{Y} be formal preschemes, \mathcal{J} (resp. \mathcal{K}) be a sheaf of ideals of definition for \mathfrak{X} (resp. \mathfrak{Y}), and $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism of formal preschemes.*

- (i) *If $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$, then there exists a unique morphism $f' : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$ of usual preschemes making the diagram*

$$(10.5.6.1) \quad \begin{array}{ccc} (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}) & \xrightarrow{f} & (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}) \\ \uparrow & & \uparrow \\ (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) & \xrightarrow{f'} & (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K}) \end{array}$$

commutate, where the vertical arrows are the canonical morphisms.

- (ii) *Suppose that $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ are formal affine schemes, $\mathcal{J} = \mathfrak{J}^\Delta$ and $\mathcal{K} = \mathfrak{K}^\Delta$, where \mathfrak{J} (resp. \mathfrak{K}) is an ideal of definition for A (resp. B), and $f = ({}^a\phi, \tilde{\phi})$, where $\phi : B \rightarrow A$ is a continuous homomorphism; for $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ to hold, it is necessary and sufficient that $\phi(\mathfrak{K}) \subset \mathfrak{J}$, and, in this case, f' is then the morphism $({}^a\phi', \tilde{\phi}')$, where $\phi' : B/\mathfrak{K} \rightarrow A/\mathfrak{J}$ is the homomorphism induced from ϕ by passing to quotients.*

Proof.

- (i) If $f = (\psi, \theta)$, then the hypotheses imply that the image under $\theta^\sharp : \psi^*(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \mathcal{O}_{\mathfrak{X}}$ of the sheaf of ideals $\psi^*(\mathcal{K})$ of $\psi^*(\mathcal{O}_{\mathfrak{Y}})$ is contained in \mathcal{J} (0, 4.3.5). By passing to quotients, we thus obtain from θ^\sharp a homomorphism of sheaves of rings

$$\omega : \psi^*(\mathcal{O}_{\mathfrak{Y}}/\mathcal{K}) = \psi^*(\mathcal{O}_{\mathfrak{Y}})/\psi^*(\mathcal{K}) \longrightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{J};$$

furthermore, since, for all $x \in \mathfrak{X}$, θ_x^\sharp is a local homomorphism, so too is ω_x . The morphism of ringed spaces (ψ, ω^\flat) is thus (2.2.1) the unique morphism f' of ringed spaces whose existence was claimed.

- (ii) The canonical functorial correspondence between morphisms of formal affine schemes and continuous homomorphisms of rings (10.2.2) shows that, in the case considered, the relation $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ implies that we have $f' = ({}^a\phi', \tilde{\phi}')$, where $\phi' : B/\mathfrak{K} \rightarrow A/\mathfrak{J}$ is the unique homomorphism making the diagram

$$(10.5.6.2) \quad \begin{array}{ccc} B & \xrightarrow{\phi} & A \\ \downarrow & & \downarrow \\ B/\mathfrak{K} & \xrightarrow{\phi'} & A/\mathfrak{J} \end{array}$$

commutate. The existence of ϕ' thus implies that $\phi(\mathfrak{K}) \subset \mathfrak{J}$. Conversely, if this condition is satisfied, then, denoting by ϕ' the unique homomorphism making the diagram (10.5.6.2) commute and setting $f' = ({}^a\phi', \tilde{\phi}')$, it is clear that the diagram (10.5.6.1) is commutative; considering the homomorphisms ${}^a\phi^*(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \mathcal{O}_{\mathfrak{X}}$ and ${}^a\phi'^*(\mathcal{O}_{\mathfrak{Y}}/\mathcal{K}) \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{J}$

corresponding to f and f' respectively then leads to the fact that this implies the relation $f^*(\mathcal{H})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{I}$.

□

It is clear that the correspondence $f \mapsto f'$ defined above is *functorial*.

10.6. Formal preschemes as inductive limits of preschemes.

(10.6.1). Let \mathfrak{X} be a formal prescheme, and (\mathcal{I}_λ) a fundamental system of sheaves of ideals of definition for \mathfrak{X} ; for each λ , let f_λ be the canonical morphism $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda) \rightarrow \mathfrak{X}$ (10.5.2); for $\mathcal{I}_\mu \subset \mathcal{I}_\lambda$, the canonical morphism $\mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\mu \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda$ defines a canonical morphism $f_{\mu\lambda} : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda) \rightarrow (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\mu)$ of (usual) preschemes such that $f_\lambda = f_\mu \circ f_{\mu\lambda}$. The preschemes $X_\lambda = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda)$ and the morphisms $f_{\mu\lambda}$ thus form (by (10.4.8)) an *inductive system* in the category of formal preschemes.

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Proposition (10.6.2). — *With the notation of (10.6.1), the formal prescheme \mathfrak{X} and the morphisms f_λ form an inductive limit (T, I, 1.8) of the system $(X_\lambda, f_{\mu\lambda})$ in the category of formal preschemes.*

Proof. Let \mathfrak{Y} be a formal prescheme, and, for each index λ , let

$$g_\lambda = (\psi_\lambda, \theta_\lambda) : X_\lambda \longrightarrow \mathfrak{Y}$$

be a morphism such that $g_\lambda = g_\mu \circ f_{\mu\lambda}$ for $\mathcal{I}_\mu \subset \mathcal{I}_\lambda$. This latter condition and the definition of the X_λ imply first of all that the ψ_λ are identical to a single continuous map $\psi : \mathfrak{X} \rightarrow \mathfrak{Y}$ of the underlying spaces; in addition, the homomorphisms $\theta_\lambda^\# : \psi^*(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \mathcal{O}_{X_\lambda} = \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda$ form a *projective system* of homomorphisms of sheaves of rings. By passing to the projective limit, there is an induced homomorphism $\omega : \psi^*(\mathcal{O}_{\mathfrak{Y}}) \rightarrow \varprojlim \mathcal{O}_{\mathfrak{X}}/\mathcal{I}_\lambda = \mathcal{O}_{\mathfrak{X}}$, and it is clear that the morphism $g = (\psi, \omega^\flat)$ of *ringed spaces* is the *unique* morphism making the diagrams

$$(10.6.2.1) \quad \begin{array}{ccc} X_\lambda & \xrightarrow{g_\lambda} & \mathfrak{Y} \\ & \searrow f_\lambda & \nearrow g \\ & \mathfrak{X} & \end{array}$$

commutative. It remains to prove that g is a morphism of *formal preschemes*; the question is local on \mathfrak{X} and \mathfrak{Y} , so we can assume that $\mathfrak{X} = \text{Spf}(A)$ and $\mathfrak{Y} = \text{Spf}(B)$, with A and B admissible rings, and with $\mathcal{I}_\lambda = \mathfrak{I}_\lambda^\Delta$, where (\mathfrak{I}_λ) is a fundamental system of ideals of definition for A (10.3.5); since $A = \varprojlim A/\mathfrak{I}_\lambda$, the existence of a morphism g of formal affine schemes making the diagrams (10.6.2.1) commute then follows from the bijective correspondence (10.2.2) between morphisms of formal affine schemes and continuous ring homomorphisms, and from the definition of the projective limit. But the uniqueness of g as a morphism of ringed spaces shows that it coincides with the morphism in the beginning of the proof. □

The following proposition establishes, under certain additional conditions, the existence of the inductive limit of a given inductive system of (usual) preschemes in the category of formal preschemes:

Proposition (10.6.3). — *Let \mathfrak{X} be a topological space, and (\mathcal{O}_i, u_{ji}) a projective system of sheaves of rings on \mathfrak{X} , with \mathbf{N} for its set of indices. Let \mathcal{I}_i be the kernel of $u_{0i} : \mathcal{O}_i \rightarrow \mathcal{O}_0$. Suppose that:*

- (a) *the ringed space $(\mathfrak{X}, \mathcal{O}_i)$ is a prescheme X_i ;*
- (b) *for all $x \in \mathfrak{X}$ and all i , there exists an open neighborhood U_i of x in \mathfrak{X} such that the restriction $\mathcal{I}_i|_{U_i}$ is nilpotent; and*
- (c) *the homomorphisms u_{ji} are surjective.*

ErrII

Let $\mathcal{O}_{\mathfrak{X}}$ be the sheaf of topological rings given by the projective limit of the pseudo-discrete sheaves of rings \mathcal{O}_i , and let $u_i : \mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_i$ be the canonical homomorphism. Then the topologically ringed space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a formal prescheme; the homomorphisms u_i are surjective; their kernels $\mathcal{I}^{(i)}$ form a fundamental system of sheaves of ideals of definition for \mathfrak{X} , and $\mathcal{I}^{(0)}$ is the projective limit of the sheaves of ideals \mathcal{I}_i .

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Proof. We first note that, on each stalk, u_{ji} is a surjective homomorphism and *a fortiori* a local homomorphism; thus $v_{ij} = (1_{\mathfrak{X}}, u_{ji})$ is a morphism of preschemes $X_j \rightarrow X_i$ ($i \geq j$) (2.2.1). Suppose first that each X_i is an affine scheme of ring A_i . There exists a ring homomorphism $\phi_{ji} : A_i \rightarrow A_j$ such that $u_{ji} = \widetilde{\phi_{ji}}$ (1.7.3); as a result (1.6.3), the sheaf \mathcal{O}_j is a quasi-coherent \mathcal{O}_i -module over X_i (for the external law defined by u_{ji}), associated to A_j considered as an A_i -module by means of ϕ_{ji} . For all $f \in A_i$, let $f' = \phi_{ji}(f)$; by hypothesis, the open sets $D(f)$ and $D(f')$ are identical in \mathfrak{X} , and the homomorphism from $\Gamma(D(f), \mathcal{O}_i) = (A_i)_f$ to $\Gamma(D(f), \mathcal{O}_j) = (A_j)_{f'}$ corresponding to u_{ji} is exactly $(\phi_{ji})_f$ (1.6.1). But when we consider A_j as an A_i -module, $(A_j)_{f'}$ is the $(A_i)_f$ -module $(A_j)_f$, so we also have $u_{ji} = \widetilde{\phi_{ji}}$, where ϕ_{ji} is now considered as a homomorphism of A_i -modules. Then, since u_{ji} is surjective, we conclude that ϕ_{ji} is also surjective (1.3.9), and if \mathfrak{J}_{ji} is the kernel of ϕ_{ji} , then the kernel of u_{ji} is a quasi-coherent \mathcal{O}_i -module equal to $\widetilde{\mathfrak{J}_{ji}}$. In particular, we have $\mathcal{J}_i = \widetilde{\mathfrak{J}_i}$, where \mathfrak{J}_i is the kernel of $\phi_{0i} : A_i \rightarrow A_0$. Hypothesis (b) implies that \mathcal{J}_i is *nilpotent*: indeed, since \mathfrak{X} is quasi-compact, we can cover \mathfrak{X} by a finite number of open sets U_k such that $(\mathcal{J}_i|_{U_k})^{n_k} = 0$, and, by setting n to be the largest of the n_k , we have $\mathcal{J}_i^n = 0$. We thus conclude that \mathfrak{J}_i is nilpotent (1.3.13). Then the ring $A = \varprojlim A_i$ is admissible (0, 7.2.2), the canonical homomorphism $\phi_i : A \rightarrow A_i$ is surjective, and its kernel $\mathfrak{J}^{(i)}$ is equal to the projective limit of the \mathfrak{J}_{ik} for $k \geq i$; the $\mathfrak{J}^{(i)}$ form a fundamental system of neighborhoods of 0 in A . The claims of Proposition (10.6.3) follow in this case from (10.1.1) and (10.3.2), with $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ being $\mathrm{Spf}(A)$.

Again, in this particular case, we note that, if $f = (f_i)$ is an element of the projective limit $A = \varprojlim A_i$, then all the open sets $D(f_i)$ (which are affine open sets in X_i) can be identified with the open subset $\mathfrak{D}(f)$ of \mathfrak{X} , and the prescheme induced on $\mathfrak{D}(f)$ by X_i thus being identified with the affine scheme $\mathrm{Spec}((A_i)_{f_i})$.

In the general case, we remark first that, for every quasi-compact open subset U of \mathfrak{X} , each of the $\mathcal{J}_i|_U$ is nilpotent, as shown by the above reasoning. We will show that, for every $x \in \mathfrak{X}$, there exists an open neighborhood U of x in \mathfrak{X} which is an *affine open set* for all the X_i . Indeed, we take U to be an affine open set for X_0 , and observe that $\mathcal{O}_{X_0} = \mathcal{O}_{X_i}/\mathcal{J}_i$. Since, by the above, $\mathcal{J}_i|_U$ is nilpotent, U is also an affine open set for each X_i , by Proposition (5.1.9). This being so, for each U satisfying the preceding conditions, the study of the affine case as above shows that $(U, \mathcal{O}_X|_U)$ is a formal prescheme whose $\mathcal{J}^{(i)}|_U$ form a fundamental system of sheaves of ideals of definition, and $\mathcal{J}^{(0)}|_U$ is the projective limit of the $\mathcal{J}_i|_U$; whence the conclusion. \square

Corollary (10.6.4). — Suppose that, for $i \geq j$, the kernel of u_{ji} is \mathcal{J}_i^{j+1} and that $\mathcal{J}_1/\mathcal{J}_1^2$ is of finite type over $\mathcal{O}_0 = \mathcal{O}_1/\mathcal{J}_1$. Then \mathfrak{X} is an adic formal prescheme, and if $\mathcal{J}^{(n)}$ is the kernel of $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_n$, then $\mathcal{J}^{(n)} = \mathcal{J}^{n+1}$ and $\mathcal{J}/\mathcal{J}^2$ is isomorphic to \mathcal{J}_1 . If, in addition, X_0 is locally Noetherian (resp. Noetherian), then \mathfrak{X} is locally Noetherian (resp. Noetherian). I | 191

Proof. Since the underlying spaces of \mathfrak{X} and X_0 are the same, the question is local, and we can suppose that all the X_i are affine; taking into account the fact that $\mathcal{J}_{ij} = \widetilde{\mathfrak{J}_{ji}}$ (with the notation of Proposition (10.3.6)), we can immediately reduce the problem to the corresponding claims of Proposition (0, 7.2.7) and Corollary (0, 7.2.8), by noting that $\mathfrak{J}_1/\mathfrak{J}_1^2$ is then an A_0 -module of finite type (1.3.9). \square

In particular, every locally Noetherian formal prescheme \mathfrak{X} is the inductive limit of a sequence (X_n) of locally Noetherian (usual) preschemes satisfying the conditions of Proposition (10.3.6) and Corollary (10.6.4): it suffices to consider a sheaf of ideals of definition \mathcal{J} for \mathfrak{X} (10.5.4) and to take $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ ((10.5.1) and Proposition (10.6.2)).

Corollary (10.6.5). — Let A be an admissible ring. For the formal affine scheme $\mathfrak{X} = \mathrm{Spf}(A)$ to be Noetherian, it is necessary and sufficient for A to be adic and Noetherian.

Proof. The condition is evidently sufficient. Conversely, suppose that \mathfrak{X} is Noetherian, and let \mathfrak{J} be an ideal of definition for A , and $\mathcal{J} = \widetilde{\mathfrak{J}^\Delta}$ the corresponding sheaf of ideals of definition for \mathfrak{X} . The (usual) preschemes $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ are then affine and Noetherian, so the rings $A_n = A/\mathfrak{J}^{n+1}$ are Noetherian (6.1.3), whence we conclude that $\mathfrak{J}/\mathfrak{J}^2$ is an A/\mathfrak{J} -module of finite type. Since the \mathcal{J}^n form a fundamental system of sheaves of ideals of definition for \mathfrak{X} (10.5.1), we

have $\mathcal{O}_{\mathfrak{X}} = \varprojlim \mathcal{O}_{\mathfrak{X}} / \mathcal{J}^n$ (10.5.3); we thus conclude (10.1.3) that A is topologically isomorphic to $\varprojlim A / \mathfrak{J}^n$, which is adic and Noetherian (0, 7.2.8). \square

Remark (10.6.6). — With the notation of Proposition (10.6.3), let \mathcal{F}_i be an \mathcal{O}_i -module, and suppose we are given, for $i \geq j$, a v_{ij} -morphism $\theta_{ji} : \mathcal{F}_i \rightarrow \mathcal{F}_j$ such that $\theta_{kj} \circ \theta_{ji} = \theta_{ki}$ for $k \leq j \leq i$. Since the underlying continuous map of v_{ij} is the identity, θ_{ji} is a homomorphism of sheaves of abelian groups to the space \mathfrak{X} ; in addition, if \mathcal{F} is the projective limit of the projective system (\mathcal{F}_i) of sheaves of abelian groups, then the fact that the θ_{ji} are v_{ij} -morphisms lets us define an $\mathcal{O}_{\mathfrak{X}}$ -module structure on \mathcal{F} by passing to the projective limit; when equipped with this structure, we say that \mathcal{F} is the *projective limit* (with respect to the θ_{ji}) of the system of \mathcal{O}_i -modules (\mathcal{F}_i) . In the particular case where $v_{ij}^*(\mathcal{F}_i) = \mathcal{F}_j$ and θ_{ji} is the *identity*, we say that \mathcal{F} is the projective limit of a system (\mathcal{F}_i) such that $v_{ij}^*(\mathcal{F}_i) = \mathcal{F}_j$ for $j \leq i$ (without mentioning the θ_{ji}).

(10.6.7). Let \mathfrak{X} and \mathfrak{Y} be formal preschemes, \mathcal{J} (resp. \mathcal{K}) a sheaf of ideals of definition for \mathfrak{X} (resp. \mathfrak{Y}), and $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. We then have, for every integer $n > 0$, that $f^*(\mathcal{K}^n)\mathcal{O}_{\mathfrak{X}} = (f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}})^n \subset \mathcal{J}^n$; f thus induces (10.5.6) a morphism of (usual) preschemes $f_n : X_n \rightarrow Y_n$ by setting $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{J}^{n+1})$ and $Y_n = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}} / \mathcal{K}^{n+1})$, and it immediately follows from the definitions that the diagrams

$$(10.6.7.1) \quad \begin{array}{ccc} X_m & \xrightarrow{f_m} & Y_m \\ \downarrow & & \downarrow \\ X_n & \xrightarrow{f_n} & Y_n \end{array}$$

commute for $m \leq n$; in other words, (f_n) is an *inductive system* of morphisms.

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(10.6.8). Conversely, let (X_n) (resp. (Y_n)) be an inductive system of (usual) preschemes satisfying conditions (b) and (c) of Proposition (10.6.3), and let \mathfrak{X} (resp. \mathfrak{Y}) be its inductive limit. By definition of the inductive limit, each sequence (f_n) of morphisms $X_n \rightarrow Y_n$ forms an inductive system that admits an *inductive limit* $f : \mathfrak{X} \rightarrow \mathfrak{Y}$, which is the unique morphism of formal preschemes that makes the diagrams

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ \downarrow & & \downarrow \\ \mathfrak{X} & \xrightarrow{f} & \mathfrak{Y} \end{array}$$

commutate.

Proposition (10.6.9). — Let \mathfrak{X} and \mathfrak{Y} be locally Noetherian formal preschemes, and \mathcal{J} (resp. \mathcal{K}) be a sheaf of ideals of definition for \mathfrak{X} (resp. \mathfrak{Y}); the map $f \mapsto (f_n)$ defined in (10.6.7) is a bijection from the set of morphisms $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ such that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ to the set of sequences (f_n) of morphisms that make the diagrams (10.6.7.1) commute.

Proof. If f is the inductive limit of this sequence, then it remains to show that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$. The statement, being local on \mathfrak{X} and \mathfrak{Y} , can be reduced to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ are affine, with A and B adic Noetherian rings, and with $\mathcal{J} = \mathfrak{J}^\Delta$ and $\mathcal{K} = \mathfrak{K}^\Delta$, where \mathfrak{J} (resp. \mathfrak{K}) is an ideal of definition for A (resp. B). We then have that $X_n = \mathrm{Spec}(A_n)$ and $Y_n = \mathrm{Spec}(B_n)$, with $A_n = A / \mathfrak{J}^{n+1}$ and $B_n = B / \mathfrak{K}^{n+1}$, by Proposition (10.3.6) and (10.3.2); $f_n = ({}^a\phi_n, \tilde{\phi}_n)$, where the homomorphisms $\phi_n : B_n \rightarrow A_n$ forms a projective system, thus $f = ({}^a\phi, \tilde{\phi})$, and so $f = ({}^a\phi, \tilde{\phi})$, where $\phi = \varprojlim \phi_n$. The commutativity of the diagram (10.6.7.1) for $m = 0$ then gives the condition $\phi_n(\mathfrak{K} / \mathfrak{K}^{n+1}) \subset \mathfrak{J} / \mathfrak{J}^{n+1}$ for all n , so, by passing to the projective limit, we have $\phi(\mathfrak{K}) \subset \mathfrak{J}$, which implies that $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{J}$ (10.5.6, ii). \square

Corollary (10.6.10). — Let \mathfrak{X} and \mathfrak{Y} be locally Noetherian formal preschemes, and \mathcal{T} the largest sheaf of ideals of definition for \mathfrak{X} (10.5.4).

- (i) For every sheaf of ideals of definition \mathcal{K} for \mathfrak{Y} and every morphism $f : \mathfrak{X} \rightarrow \mathfrak{Y}$, we have $f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{T}$.

- (ii) *There is a canonical bijective correspondence between $\text{Hom}(\mathfrak{X}, \mathfrak{Y})$ and the set of sequences (f_n) of morphisms making the diagrams (10.6.7.1) commute, where $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{I}^{n+1})$ and $Y_n = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}} / \mathcal{K}^{n+1})$.*

Proof. (ii) follows immediately from (i) and Proposition (10.6.9). To prove (i), we can reduce to the case where $\mathfrak{X} = \text{Spf}(A)$ and $\mathfrak{Y} = \text{Spf}(B)$, with A and B Noetherian, and with $\mathcal{I} = \mathfrak{I}^\Delta$ and $\mathcal{K} = \mathfrak{K}^\Delta$, where \mathfrak{I} is the largest ideal of definition for A and \mathfrak{K} is an ideal of definition for B . Let $f = ({}^a\phi, \tilde{\phi})$, where $\phi : B \rightarrow A$ is a continuous homomorphism; since the elements of \mathfrak{K} are topologically nilpotent (0, 7.1.4, ii), so too are those of $\phi(\mathfrak{K})$, and so $\phi(\mathfrak{K}) \subset \mathfrak{I}$, since \mathfrak{I} is the set of topologically nilpotent elements of A (0, 7.1.6); hence, by Proposition (10.5.6, ii), we are done. \square

Corollary (10.6.11). — *Let $\mathfrak{S}, \mathfrak{X}, \mathfrak{Y}$ be locally Noetherian formal preschemes, and $f : \mathfrak{X} \rightarrow \mathfrak{S}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{S}$ the morphisms that make \mathfrak{X} and \mathfrak{Y} formal \mathfrak{S} -preschemes. Let \mathcal{J} (resp. \mathcal{K}, \mathcal{L}) be a sheaf of ideals of definition for \mathfrak{S} (resp. $\mathfrak{X}, \mathfrak{Y}$), and suppose that $f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{K}$ and $g^*(\mathcal{J})\mathcal{O}_{\mathfrak{Y}} = \mathcal{L}$; set $S_n = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}} / \mathcal{J}^{n+1})$, $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{K}^{n+1})$, and $Y_n = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}} / \mathcal{L}^{n+1})$. Then there exists a canonical bijective correspondence between $\text{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y})$ and the set of sequences (u_n) of S_n -morphisms $u_n : X_n \rightarrow Y_n$ making the diagrams (10.6.7.1) commute.*

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Proof. For each \mathfrak{S} -morphism $u : \mathfrak{X} \rightarrow \mathfrak{Y}$, we have by definition that $f = g \circ u$, whence

$$u^*(\mathcal{L})\mathcal{O}_{\mathfrak{X}} = u^*(g^*(\mathcal{J})\mathcal{O}_{\mathfrak{Y}})\mathcal{O}_{\mathfrak{X}} = f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}} \subset \mathcal{K},$$

and the corollary then follows from Proposition (10.6.9). \square

We note that, for $m \leq n$, the data of a morphism $f_n : X_n \rightarrow Y_n$ determines a unique morphism $f_m : X_m \rightarrow Y_m$ making the diagram (10.6.7.1) commutative, since we immediately see that we can reduce to the affine case; we have thus defined a map $\phi_{mn} : \text{Hom}_{S_n}(X_n, Y_n) \rightarrow \text{Hom}_{S_m}(X_m, Y_m)$, and the $\text{Hom}_{S_n}(X_n, Y_n)$ form, with the ϕ_{mn} , a projective system of sets; Corollary (10.6.11) then says that there is a canonical bijection

$$\text{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \simeq \varprojlim_n \text{Hom}_{S_n}(X_n, Y_n).$$

10.7. Products of formal preschemes.

(10.7.1). Let \mathfrak{S} be a formal prescheme; formal \mathfrak{S} -preschemes form a category, and we can define a notion of a *product* of formal \mathfrak{S} -preschemes.

Proposition (10.7.2). — *Let $\mathfrak{X} = \text{Spf}(B)$ and $\mathfrak{Y} = \text{Spf}(C)$ be formal affine schemes over a formal affine scheme $\mathfrak{S} = \text{Spf}(A)$. Let $\mathfrak{Z} = \text{Spf}(B \hat{\otimes}_A C)$, and let p_1 and p_2 be the \mathfrak{S} -morphisms corresponding (10.2.2) to the canonical (continuous) A -homomorphisms $\rho : B \rightarrow B \hat{\otimes}_A C$ and $\sigma : C \rightarrow B \hat{\otimes}_A C$; then (\mathfrak{Z}, p_1, p_2) is a product of the formal affine \mathfrak{S} -schemes \mathfrak{X} and \mathfrak{Y} .*

Proof. By Proposition (10.4.6), it suffices to check that, if we associate, to each continuous A -homomorphism $\phi : B \hat{\otimes}_A C \rightarrow D$ (where D is an admissible ring which is a topological A -algebra), the pair $(\phi \circ \rho, \phi \circ \sigma)$, then this defines a bijection

$$\text{Hom}_A(B \hat{\otimes}_A C, D) \simeq \text{Hom}_A(B, D) \times \text{Hom}_A(C, D),$$

which is exactly the universal property of the completed tensor product (0, 7.7.6). \square

Proposition (10.7.3). — *Given formal \mathfrak{S} -preschemes \mathfrak{X} and \mathfrak{Y} , their product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ exists.*

Proof. The proof is similar to that of Theorem (3.2.6), replacing affine schemes (resp. affine open sets) by formal affine schemes (resp. formal affine open sets), and replacing Proposition (3.2.2) by Proposition (10.7.2). \square

All the formal properties of the product of preschemes ((3.2.7) and (3.2.8), (3.3.1) and (3.3.12)) hold true without modification for the product of formal preschemes.

(10.7.4). Let $\mathfrak{S}, \mathfrak{X}$, and \mathfrak{Y} be formal preschemes, and let $f : \mathfrak{X} \rightarrow \mathfrak{S}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{S}$ be morphisms. Suppose that there exist, in $\mathfrak{S}, \mathfrak{X}$, and \mathfrak{Y} respectively, three fundamental systems of sheaves of ideals of definitions (\mathcal{J}_λ) , (\mathcal{K}_λ) , and (\mathcal{L}_λ) , all having the same set I of indices, and such that $f^*(\mathcal{J}_\lambda)\mathcal{O}_{\mathfrak{X}} \subset \mathcal{K}_\lambda$ and $g^*(\mathcal{J}_\lambda)\mathcal{O}_{\mathfrak{Y}} \subset \mathcal{L}_\lambda$ for all λ . Set $S_\lambda = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}} / \mathcal{J}_\lambda)$, $X_\lambda = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{K}_\lambda)$, and

$Y_\lambda = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{L}_\lambda)$; for $\mathcal{J}_\mu \subset \mathcal{J}_\lambda$, $\mathcal{K}_\mu \subset \mathcal{K}_\lambda$, and $\mathcal{L}_\mu \subset \mathcal{L}_\lambda$, note that S_λ (resp. X_λ, Y_λ) is a closed subprescheme of S_μ (resp. X_μ, Y_μ) that has the *same* underlying space (10.6.1). Since $S_\lambda \rightarrow S_\mu$ is a monomorphism of preschemes, we see that the products $X_\lambda \times_{S_\lambda} Y_\lambda$ and $X_\lambda \times_{S_\mu} Y_\lambda$ are identical (3.2.4), since $X_\lambda \times_{S_\mu} Y_\lambda$ can be identified with a closed subprescheme of $X_\mu \times_{S_\mu} Y_\mu$ that has the *same* underlying space (4.3.1). With this in mind, the product $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is the *inductive limit* of the usual preschemes $X_\lambda \times_{S_\lambda} Y_\lambda$: indeed, as we see in Proposition (10.6.2), we can reduce to the case where $\mathfrak{S}, \mathfrak{X}$, and \mathfrak{Y} are formal affine schemes. Taking into account both Proposition (10.5.6, ii) and the hypotheses on the fundamental systems of sheaves of ideals of definition for $\mathfrak{S}, \mathfrak{X}$, and \mathfrak{Y} , we immediately see that our claim follows from the definition of the completed tensor product of two algebras (0, 7.7.1).

Furthermore, let \mathfrak{Z} be a formal \mathfrak{S} -prescheme, (\mathcal{M}_λ) a fundamental system of ideals of definition for \mathfrak{Z} having I for its set of indices, and let $u : \mathfrak{Z} \rightarrow \mathfrak{X}$ and $v : \mathfrak{Z} \rightarrow \mathfrak{Y}$ be \mathfrak{S} -morphisms such that $u^*(\mathcal{K}_\lambda)\mathcal{O}_{\mathfrak{Z}} \subset \mathcal{M}_\lambda$ and $v^*(\mathcal{L}_\lambda)\mathcal{O}_{\mathfrak{Z}} \subset \mathcal{M}_\lambda$ for all λ . If we set $Z_\lambda = (\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}}/\mathcal{M}_\lambda)$, and if $u_\lambda : Z_\lambda \rightarrow X_\lambda$ and $v_\lambda : Z_\lambda \rightarrow Y_\lambda$ are the S_λ -morphisms corresponding to u and v (10.5.6), then we immediately have that $(u, v)_{\mathfrak{S}}$ is the inductive limit of the S_λ -morphisms $(u_\lambda, v_\lambda)_{S_\lambda}$.

The ideas of this section apply, in particular, to the case where $\mathfrak{S}, \mathfrak{X}$, and \mathfrak{Y} are locally Noetherian, taking the systems consisting of the powers of a sheaf of ideals of definition (10.5.1) as the fundamental systems of sheaves of ideals of definition. However, we note that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is not necessarily locally Noetherian (see however (10.13.5)).

10.8. Formal completion of a prescheme along a closed subset.

(10.8.1). Let X be a *locally Noetherian* (usual) prescheme, and X' a closed subset of the underlying space of X ; we denote by Φ the set of *coherent* sheaves of ideals \mathcal{J} of \mathcal{O}_X such that the support of $\mathcal{O}_X/\mathcal{J}$ is X' . The set Φ is nonempty ((5.2.1), (4.1.4), (6.1.1)); we order it by the relation \supset .

Lemma (10.8.2). — *The ordered set Φ is filtered; if X is Noetherian, then, for all $\mathcal{J}_0 \in \Phi$, the set of powers \mathcal{J}_0^n ($n > 0$) is cofinal in Φ .*

Proof. If \mathcal{J}_1 and \mathcal{J}_2 are in Φ , and if we set $\mathcal{J} = \mathcal{J}_1 \cap \mathcal{J}_2$, then \mathcal{J} is coherent since \mathcal{O}_X is coherent ((6.1.1) and (0, 5.3.4)), and we have $\mathcal{J}_x = (\mathcal{J}_1)_x \cap (\mathcal{J}_2)_x$ for all $x \in X$, whence $\mathcal{J}_x = \mathcal{O}_x$ for $x \notin X'$, and $\mathcal{J}_x \neq \mathcal{O}_x$ for $x \in X'$, which proves that $\mathcal{J} \in \Phi$. On the other hand, if X is Noetherian and if \mathcal{J}_0 and \mathcal{J} are in Φ , then there exists an integer $n > 0$ such that $\mathcal{J}_0^n(\mathcal{O}_X/\mathcal{J}) = 0$ (9.3.4), which implies that $\mathcal{J}_0^n \subset \mathcal{J}$. \square

(10.8.3). Now let \mathcal{F} be a *coherent* \mathcal{O}_X -module; for all $\mathcal{J} \in \Phi$, we have that $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J})$ is a coherent \mathcal{O}_X -module (9.1.1) with support contained in X' , and we will usually identify it with its restriction to X' . When \mathcal{J} varies over Φ , these sheaves form a *projective system* of sheaves of abelian groups.

Definition (10.8.4). — Given a closed subset X' of a locally Noetherian prescheme X and a coherent \mathcal{O}_X -module \mathcal{F} , we define the *completion of \mathcal{F} along X'* , denoted by $\mathcal{F}_{/X'}$ (or $\widehat{\mathcal{F}}$ when there is little chance of confusion), to be the restriction to X' of the sheaf $\varprojlim_{\Phi} (\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}))$; we say that its sections over X' are the *formal sections of \mathcal{F} along X'* . \square

It is immediate that, for every open $U \subset X$, we have $(\mathcal{F}|_U)_{/(U \cap X')} = (\mathcal{F}_{/X'})|_{(U \cap X')}$.

By passing to the projective limit, it is clear that $(\mathcal{O}_X)_{/X'}$ is a sheaf of rings, and that $\mathcal{F}_{/X'}$ can be considered as an $(\mathcal{O}_X)_{/X'}$ -module. In addition, since there exists a basis for the topology of X' consisting of quasi-compact open sets, we can consider $(\mathcal{O}_X)_{/X'}$ (resp. $\mathcal{F}_{/X'}$) as a *sheaf of topological rings* (resp. of *topological groups*), the projective limit of the *pseudo-discrete* sheaves of rings (resp. groups) $\mathcal{O}_X/\mathcal{F}$ (resp. $\mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{F}) = \mathcal{F}/\mathcal{J}\mathcal{F}$), and, by passing to the projective limit, $\mathcal{F}_{/X'}$ then becomes a *topological $(\mathcal{O}_X)_{/X'}$ -module* ((0, 3.8.1) and (0, 8.2)); recall that, for every *quasi-compact* open $U \subset X$, $\Gamma(U \cap X', (\mathcal{O}_X)_{/X'})$ (resp. $\Gamma(U \cap X', \mathcal{F}_{/X'})$) is then the projective limit of the discrete rings (resp. groups) $\Gamma(U, \mathcal{O}_X/\mathcal{J})$ (resp. $\Gamma(U, \mathcal{F}/\mathcal{J}\mathcal{F})$).

Now, if $u : \mathcal{F} \rightarrow \mathcal{G}$ is a homomorphism of \mathcal{O}_X -modules, then there are canonically induced homomorphisms $u_{\mathcal{J}} : \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J}) \rightarrow \mathcal{G} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J})$ for all $\mathcal{J} \in \Phi$, and these homomorphisms form a projective system. By passing to the projective limit and restricting to X' , these give a continuous $(\mathcal{O}_X)_{/X'}$ -homomorphism $\mathcal{F}_{/X'} \rightarrow \mathcal{G}_{/X'}$, denoted $u_{/X'}$ or \widehat{u} , and called the *completion*

of the homomorphism u along X' . It is clear that, if $v : \mathcal{G} \rightarrow \mathcal{H}$ is a second homomorphism of \mathcal{O}_X -modules, then we have $(v \circ u)_{/X'} = (v_{/X'}) \circ (u_{/X'})$, hence $\mathcal{F}_{/X'}$ is a *covariant additive functor* in \mathcal{F} from the category of coherent \mathcal{O}_X -modules to the category of topological $(\mathcal{O}_X)_{/X'}$ -modules.

Proposition (10.8.5). — *The support of $(\mathcal{O}_X)_{/X'}$ is X' ; the topologically ringed space $(X', (\mathcal{O}_X)_{/X'})$ is a locally Noetherian formal prescheme, and, if $\mathcal{J} \in \Phi$, then $\mathcal{J}_{/X'}$ is a sheaf of ideals of definition for this formal prescheme. If $X = \text{Spec}(A)$ is an affine scheme with Noetherian ring A , $\mathcal{J} = \hat{\mathfrak{J}}$ for some ideal \mathfrak{J} of A , and $X' = V(\mathfrak{J})$, then $(X', (\mathcal{O}_X)_{/X'})$ is canonically identified with $\text{Spf}(\hat{A})$, where \hat{A} is the separated completion of A with respect to the \mathfrak{J} -preadic topology.*

Proof. We can evidently reduce to proving the latter claim. We know (0, 7.3.3) that the separated completion $\hat{\mathfrak{J}}$ of \mathfrak{J} with respect to the \mathfrak{J} -preadic topology can be identified with the ideal $\mathfrak{J}\hat{A}$ of \hat{A} , where \hat{A} is the Noetherian \mathfrak{J} -adic ring such that $\hat{A}/\hat{\mathfrak{J}}^n = A/\mathfrak{J}^n$ (0, 7.2.6). This latter equation shows that the open prime ideals of \hat{A} are the ideals $\hat{\mathfrak{p}} = \mathfrak{p}\hat{\mathfrak{J}}$, where \mathfrak{p} is a prime ideal of A containing \mathfrak{J} , and that we have $\hat{\mathfrak{p}} \cap A = \mathfrak{p}$, and hence $\text{Spf}(\hat{A}) = X'$. Since $\mathcal{O}_X/\mathcal{J}^n = (A/\mathfrak{J}^n)^\sim$, the proposition follows immediately from the definitions. \square

We say that the formal prescheme defined above is the *completion of X along X'* , and we denote it by $X_{/X'}$ or \hat{X} when there is little chance of confusion. When we take $X' = X$, we can set $\mathcal{J} = 0$, and we thus have $X_{/X} = X$.

It is clear that, if U is a subprescheme induced on an open subset of X , then $U_{/(U \cap X')}$ is canonically identified with the formal subprescheme induced on $X_{/X'}$ by the open subset $U \cap X'$ of X' .

Corollary (10.8.6). — *The (usual) prescheme \hat{X}_{red} is the unique reduced subprescheme of X having X' as its underlying space (5.2.1). For \hat{X} to be Noetherian, it is necessary and sufficient for \hat{X}_{red} to be Noetherian, and it suffices that X be Noetherian.*

Proof. Since \hat{X}_{red} is determined locally (10.5.4), we can assume that X is an affine scheme of some Noetherian ring A ; with the notation of Proposition (10.8.5), the ideal \mathfrak{T} of topologically nilpotent elements of \hat{A} is the inverse image under the canonical map $\hat{A} \rightarrow \hat{A}/\hat{\mathfrak{J}} = A/\mathfrak{J}$ of the nilradical of A/\mathfrak{J} (0, 7.1.3), so \hat{A}/\mathfrak{T} is isomorphic to the quotient of A/\mathfrak{J} by its nilradical. The first claim then follows from Propositions (10.5.4) and (5.1.1). If \hat{X}_{red} is Noetherian, then so too is its underlying space X' , and so the $X'_n = \text{Spec}(\mathcal{O}_X/\mathcal{J}^n)$ are Noetherian (6.1.2), and thus so too is \hat{X} (10.6.4); the converse is immediate, by Proposition (6.1.2). \square

(10.8.7). The canonical homomorphisms $\mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J}$ (for $\mathcal{J} \in \Phi$) form a projective system, and give, by passing to the projective limit, a homomorphism of sheaves of rings $\theta : \mathcal{O}_X \rightarrow \psi_*((\mathcal{O}_X)_{/X'}) = \varprojlim_{\Phi} (\mathcal{O}_X/\mathcal{J})$, where ψ denotes the canonical injection $X' \rightarrow X$ of the underlying spaces. We denote by i (or i_X) the morphism (said to be *canonical*)

$$(\psi, \theta) : X_{/X'} \longrightarrow X$$

of ringed spaces.

By taking tensor products, for every coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphisms $\mathcal{O}_X \rightarrow \mathcal{O}_X/\mathcal{J}$ give homomorphisms $\mathcal{F} \rightarrow \mathcal{F} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathcal{J})$ of \mathcal{O}_X -modules which form a projective system, and thus give, by passing to the projective limit, a canonical functorial homomorphism $\gamma : \mathcal{F} \rightarrow \psi_*(\mathcal{F}_{/X'})$ of \mathcal{O}_X -modules.

Proposition (10.8.8). —

- (i) *The functor $\mathcal{F}_{/X'}$ (in \mathcal{F}) is exact.*
- (ii) *The functorial homomorphism $\gamma^\sharp : i^*(\mathcal{F}) \rightarrow \mathcal{F}_{/X'}$ of $(\mathcal{O}_X)_{/X'}$ -modules is an isomorphism.*

Proof.

- (i) It suffices to prove that, if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}'' \rightarrow 0$ is an exact sequence of coherent \mathcal{O}_X -modules, and if U is an affine open subset of X with Noetherian ring A , then the sequence

$$0 \longrightarrow \Gamma(U \cap X', \mathcal{F}'_{/X'}) \longrightarrow \Gamma(U \cap X', \mathcal{F}_{/X'}) \longrightarrow \Gamma(U \cap X', \mathcal{F}''_{/X'}) \longrightarrow 0$$

is exact. We have that $\mathcal{F}|U = \tilde{M}$, $\mathcal{F}'|U = \tilde{M}'$, and $\mathcal{F}''|U = \tilde{M}''$, where M , M' , and M'' are three A -modules of finite type such that the sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is exact ((1.5.1) and (1.3.11)); let $\mathcal{J} \in \Phi$, and let \mathfrak{J} be an ideal of A such that $\mathcal{J}|U = \tilde{\mathfrak{J}}$. We then have

$$\Gamma(U \cap X', \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X / \mathcal{J}^n) = M \otimes_A (A / \mathfrak{J}^n)$$

(3.12); so, by definition of the projective limit, we have

$$\Gamma(U \cap X', \mathcal{F}_{/X'}) = \varprojlim_n (M \otimes_A (A / \mathfrak{J}^n)) = \hat{M},$$

the separated completion of M with respect to the \mathfrak{J} -preadic topology, and similarly

$$\Gamma(U \cap X', \mathcal{F}'_{/X'}) = \hat{M}', \quad \Gamma(U \cap X', \mathcal{F}''_{/X'}) = \hat{M}'';$$

our claim then follows, since A is Noetherian, and since the functor \hat{M} in M is exact on the category of A -modules of finite type (0, 7.3.3).

- (ii) The question is local, so we can assume that we have an exact sequence $\mathcal{O}_X^m \rightarrow \mathcal{O}_X^n \rightarrow \mathcal{F} \rightarrow 0$ (0, 5.3.2); since $\gamma^\#$ is functorial, and the functors $i^*(\mathcal{F})$ and $\mathcal{F}_{/X'}$ are right exact (by (i) and (0, 4.3.1)), we have the commutative diagram

$$(10.8.8.1) \quad \begin{array}{ccccccc} i^*(\mathcal{O}_X^m) & \longrightarrow & i^*(\mathcal{O}_X^n) & \longrightarrow & i^*(\mathcal{F}) & \longrightarrow & 0 \\ \gamma^\# \downarrow & & \gamma^\# \downarrow & & \gamma^\# \downarrow & & \\ (\mathcal{O}_X^m)_{/X'} & \longrightarrow & (\mathcal{O}_X^n)_{/X'} & \longrightarrow & \mathcal{F}_{/X'} & \longrightarrow & 0 \end{array}$$

whose rows are exact. Furthermore, the functors $i^*(\mathcal{F})$ and $\mathcal{F}_{/X'}$ commute with finite direct sums ((0, 3.2.6) and (0, 4.3.2)), and we thus reduce to proving our claim for $\mathcal{F} = \mathcal{O}_X$. We have $i^*(\mathcal{O}_X) = (\mathcal{O}_X)_{/X'} = \mathcal{O}_{\hat{X}}$ (0, 4.3.4), and that $\gamma^\#$ is a homomorphism of $\mathcal{O}_{\hat{X}}$ -modules; so it suffices to check that $\gamma^\#$ sends the unit section of $\mathcal{O}_{\hat{X}}$ over an open subset of X' to itself, which is immediate, and shows, in this case, that $\gamma^\#$ is the identity. \square

Corollary (10.8.9). — *The morphism $i : X_{/X'} \rightarrow X$ of ringed spaces is flat.*

Proof. This follows from (0, 6.7.3) and Proposition (10.8.8, i). \square

Corollary (10.8.10). — *If \mathcal{F} and \mathcal{G} are coherent \mathcal{O}_X -modules, then there exist canonical functorial (in \mathcal{F} and \mathcal{G}) isomorphisms*

$$(10.8.10.1) \quad (\mathcal{F}_{/X'}) \otimes_{(\mathcal{O}_X)_{/X'}} (\mathcal{G}_{/X'}) \simeq (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_{/X'},$$

$$(10.8.10.2) \quad (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G}))_{/X'} \simeq \mathcal{H}om_{(\mathcal{O}_X)_{/X'}}(\mathcal{F}_{/X'}, \mathcal{G}_{/X'}).$$

Proof. This follows from the canonical identification of $i^*(\mathcal{F})$ with $\mathcal{F}_{/X'}$; the existence of the first isomorphism is then a result which holds for all morphisms of ringed spaces (0, 4.3.3.1), and the second is a result which holds for all flat morphisms (0, 6.7.6), by Corollary (10.8.9). \square

Proposition (10.8.11). — *For every coherent \mathcal{O}_X -module \mathcal{F} , the canonical homomorphism $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X', \mathcal{F}_{/X'})$ induced by $\mathcal{F} \rightarrow \mathcal{F}_{/X'}$ has kernel consisting of the zero sections in some neighborhood of X' .*

Proof. It follows from the definition of $\mathcal{F}_{/X'}$ that the canonical image of such a section is zero. Conversely, if $s \in \Gamma(X, \mathcal{F})$ has a zero image in $\Gamma(X', \mathcal{F}_{/X'})$, then it suffices to see that every $x \in X'$ admits a neighborhood in X in which s is zero, and we can thus reduce to the case where $X = \text{Spec}(A)$ is affine, A Noetherian, $X' = V(\mathfrak{J})$ for some ideal \mathfrak{J} of A , and $\mathcal{F} = \tilde{M}$ for some A -module M of finite type. Then $\Gamma(X', \mathcal{F}_{/X'})$ is the separated completion \hat{M} of M for the \mathfrak{J} -preadic topology, and the homomorphism $\Gamma(X, \mathcal{F}) \rightarrow \Gamma(X', \mathcal{F}_{/X'})$ is the canonical homomorphism $M \rightarrow \hat{M}$. We know (0, 7.3.7) that the kernel of this homomorphism is the set of the $z \in M$ killed by an element of $1 + \mathfrak{J}$. So we have $(1 + f)s = 0$ for some $f \in \mathfrak{J}$; for every $x \in X'$ we have $(1_x + f_x)s_x = 0$, and, since $1_x + f_x$ is invertible in \mathcal{O}_x ($\mathfrak{J}_x \mathcal{O}_x$ being contained in the maximal ideal of \mathcal{O}_x), we have $s_x = 0$, which proves the proposition. \square

Corollary (10.8.12). — *The support of $\mathcal{F}_{/X'}$ is equal to $\text{Supp}(\mathcal{F}) \cap X'$.*

Proof. It is clear that $\mathcal{F}_{/X'}$ is an $(\mathcal{O}_X)_{/X'}$ -module of finite type ((10.8.8, ii) and (0, 5.2.4)), so its support is closed (0, 5.2.2) and evidently contained in $\text{Supp}(\mathcal{F}) \cap X'$. To show that it is equal to the latter set, we immediately reduce to proving that the equation $\Gamma(X', \mathcal{F}_{/X'}) = 0$ implies that $\text{Supp}(\mathcal{F}) \cap X' = \emptyset$; this follows from Proposition (10.8.11) and Theorem (1.4.1). \square

Corollary (10.8.13). — *Let $u : \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of coherent \mathcal{O}_X -modules. For $u_{/X'} : \mathcal{F}_{/X'} \rightarrow \mathcal{G}_{/X'}$ to be zero, it is necessary and sufficient for u to be zero on a neighborhood of X' .*

Proof. By Proposition (10.8.8, ii), $u_{/X'}$ can be identified with $i^*(u)$, so, if we consider u as a section over X of the sheaf $\mathcal{H} = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{G})$, then $u_{/X'}$ is the section over X' of $i^*(\mathcal{H}) = \mathcal{H}_{/X'}$ to which it canonically corresponds ((10.8.10.2) and (0, 4.4.6)). It thus suffices to apply Proposition (10.8.11) to the coherent \mathcal{O}_X -module \mathcal{H} . \square

Corollary (10.8.14). — *Let $u : \mathcal{F} \rightarrow \mathcal{G}$ be a homomorphism of coherent \mathcal{O}_X -modules. For $u_{/X'}$ to be a monomorphism (resp. an epimorphism), it is necessary and sufficient for u to be a monomorphism (resp. an epimorphism) on a neighborhood of X' .*

Proof. Let \mathcal{P} and \mathcal{N} be the cokernel and kernel (respectively) of u , so that we have the exact sequence $0 \rightarrow \mathcal{N} \xrightarrow{v} \mathcal{F} \xrightarrow{u} \mathcal{G} \xrightarrow{w} \mathcal{P} \rightarrow 0$, hence (10.8.8, i) the exact sequence

$$0 \longrightarrow \mathcal{N}_{/X'} \xrightarrow{v_{/X'}} \mathcal{F}_{/X'} \xrightarrow{u_{/X'}} \mathcal{G}_{/X'} \xrightarrow{w_{/X'}} \mathcal{P}_{/X'} \longrightarrow 0.$$

If $u_{/X'}$ is a monomorphism (resp. an epimorphism), then we have $v_{/X'} = 0$ (resp. $w_{/X'} = 0$), so there exists a neighborhood of X' on which $v = 0$ (resp. $w = 0$) by Corollary (10.8.13). \square

10.9. Extension of morphisms to completions.

(10.9.1). Let X and Y be locally Noetherian (usual) preschemes, $f : X \rightarrow Y$ a morphism, and X' (resp. Y') a closed subset of the underlying space X (resp. Y) such that $f(X') \subset Y'$. Let \mathcal{J} (resp. \mathcal{K}) be a sheaf of ideals of \mathcal{O}_X (resp. \mathcal{O}_Y) such that the support of $\mathcal{O}_X/\mathcal{J}$ (resp. $\mathcal{O}_Y/\mathcal{K}$) is X' (resp. Y') and $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{J}$; we note that there always exist such sheaves of ideals, since, for example, we can take \mathcal{J} to be the largest sheaf of ideals of \mathcal{O}_X defining a subprescheme of X with underlying space X' (5.2.1), and the hypothesis $f(X') \subset Y'$ implies that $f^*(\mathcal{K})\mathcal{O}_X \subset \mathcal{J}$ (5.2.4). For every integer $n > 0$ we have $f^*(\mathcal{K}^n)\mathcal{O}_X \subset \mathcal{J}^n$ (0, 4.3.5); as a result (4.4.6), if we set $X'_n = (X', \mathcal{O}_X/\mathcal{J}^{n+1})$ and $Y'_n = (Y', \mathcal{O}_Y/\mathcal{K}^{n+1})$, then f induces a morphism $f_n : X'_n \rightarrow Y'_n$, and it is immediate that the f_n form an inductive system. We denote its inductive limit (10.6.8) by $\hat{f} : X_{/X'} \rightarrow Y_{/Y'}$, and we say (by abuse of language) that \hat{f} is the *extension of f to the completions of X and Y along X' and Y'* . It can be checked immediately that this morphism does not depend on the choice of sheaves of ideals \mathcal{J} and \mathcal{K} satisfying the above conditions. It suffices to consider the case where X and Y are Noetherian affine schemes with rings A and B (respectively); then $\mathcal{J} = \tilde{\mathfrak{J}}$ and $\mathcal{K} = \tilde{\mathfrak{K}}$, where \mathfrak{J} (resp. \mathfrak{K}) is an ideal of A (resp. B), f corresponds to a ring homomorphism $\phi : B \rightarrow A$ such that $\phi(\mathfrak{K}) \subset \mathfrak{J}$ ((4.4.6) and (1.7.4)); \hat{f} is then the morphism corresponding (10.2.2) to the continuous homomorphism $\hat{\phi} : \hat{B} \rightarrow \hat{A}$, where \hat{A} (resp. \hat{B}) is the separated completion of A (resp. B) with respect to the \mathfrak{J} -preadic (resp. \mathfrak{K} -preadic) topology (10.6.8); we know that, if we replace \mathcal{J} by another sheaf of ideals $\mathcal{J}' = \tilde{\mathfrak{J}'}$ such that the support of $\mathcal{O}_X/\mathcal{J}'$ is X' , then the \mathfrak{J} -preadic and \mathfrak{J}' -preadic topologies on A are the same (10.82). I | 199

We note that, by definition, the continuous map $X' \rightarrow Y'$ of the underlying spaces of $X_{/X'}$ and $Y_{/Y'}$ corresponding to \hat{f} is exactly the restriction to X' of f .

(10.9.2). It follows immediately from the above definition that the diagram of morphisms of ringed spaces

$$\begin{array}{ccc} \hat{X} & \xrightarrow{\hat{f}} & \hat{Y} \\ i_X \downarrow & & \downarrow i_Y \\ X & \xrightarrow{f} & Y \end{array}$$

commutes, with the vertical arrows being the canonical morphisms (10.8.7).

(10.9.3). Let Z be a third prescheme, $g : Y \rightarrow Z$ a morphism, and Z' a closed subset of Z such that $g(Y') \subset Z'$. If \hat{g} denotes the completion of the morphism g along Y' and Z' , then it immediately follows from (10.9.1) that we have $(g \circ f)^\wedge = \hat{g} \circ \hat{f}$.

Proposition (10.9.4). — *Let X and Y be locally Noetherian S -preschemes, with Y of finite type over S . Let f and g be S -morphisms from X to Y such that $f(X') \subset Y'$ and $g(X') \subset Y'$. For $\hat{f} = \hat{g}$ to hold, it is necessary and sufficient for f and g to coincide on a neighborhood of X' .*

Proof. The condition is evidently sufficient (even without the finiteness hypothesis on Y). To see that it is necessary, we remark first that the hypothesis $\hat{f} = \hat{g}$ implies that $f(x) = g(x)$ for all $x \in X'$. Also, the question being local, we can assume that X and Y are affine open neighborhoods of x and $y = f(x) = g(x)$ respectively (with Noetherian rings), that S is affine, and that $\Gamma(Y, \mathcal{O}_Y)$ is a $\Gamma(S, \mathcal{O}_S)$ -algebra of finite type (6.3.3). Then f and g correspond to $\Gamma(S, \mathcal{O}_S)$ -homomorphisms ρ and σ (respectively) from $\Gamma(Y, \mathcal{O}_Y)$ to $\Gamma(X, \mathcal{O}_X)$ (1.7.3), and, by hypothesis, the extensions by continuity of these homomorphisms to the separated completion of $\Gamma(Y, \mathcal{O}_Y)$ are the same. We conclude from Proposition (10.8.11) that, for every section $s \in \Gamma(Y, \mathcal{O}_Y)$, the sections $\rho(s)$ and $\sigma(s)$ coincide on a neighborhood (depending on s) of X' ; since $\Gamma(Y, \mathcal{O}_Y)$ is an algebra of finite type over $\Gamma(S, \mathcal{O}_S)$, we have that there exists a neighborhood V of X' such that $\rho(s)$ and $\sigma(s)$ coincide on V for every section $s \in \Gamma(Y, \mathcal{O}_Y)$. If $h \in \Gamma(X, \mathcal{O}_X)$ is such that $D(h)$ is a neighborhood of x contained in V , then we conclude from the above and from Theorem (1.4.1, d) that f and g coincide on $D(h)$. \square

Proposition (10.9.5). — *Under the hypotheses of (10.9.1), for every coherent \mathcal{O}_Y -module \mathcal{G} , there exists a canonical functorial isomorphism of $(\mathcal{O}_X)_{/X'}$ -modules*

$$(f^*(\mathcal{G}))_{/X'} \simeq \hat{f}^*(\mathcal{G}_{/Y'}).$$

Proof. If we canonically identify $(f^*(\mathcal{G}))_{/X'}$ with $i_X^*(f^*(\mathcal{G}))$, and $\hat{f}^*(\mathcal{G}_{/Y'})$ with $\hat{f}^*(i_Y^*(\mathcal{G}))$ (10.8.8), then the proposition follows immediately from the commutativity of the diagram in (10.9.2). \square

(10.9.6). Now let \mathcal{F} be a coherent \mathcal{O}_X -module, and let \mathcal{G} be a coherent \mathcal{O}_Y -module. If $u : \mathcal{G} \rightarrow \mathcal{F}$ is an f -morphism, then it corresponds to an \mathcal{O}_X -homomorphism $u^\sharp : f^*(\mathcal{G}) \rightarrow \mathcal{F}$, thus, by completion, to a continuous $(\mathcal{O}_X)_{/X'}$ -homomorphism $(u^\sharp)_{/X'} : (f^*(\mathcal{G}))_{/X'} \rightarrow \mathcal{F}_{/X'}$, and, by Proposition (10.9.5), there exists a unique \hat{f} -morphism $v : \mathcal{G}_{/Y'} \rightarrow \mathcal{F}_{/X'}$ such that $v^\sharp = (u^\sharp)_{/X'}$. If we consider the triples (\mathcal{F}, X, X') (\mathcal{F} being a coherent \mathcal{O}_X -module, and X' a closed subset of X) as a category, with the morphisms $(\mathcal{F}, X, X') \rightarrow (\mathcal{G}, Y, Y')$ consisting of a morphism of preschemes $f : X \rightarrow Y$ such that $f(X') \subset Y'$ and an f -morphism $u : \mathcal{G} \rightarrow \mathcal{F}$, then we can say that $(X_{/X'}, \mathcal{F}_{/X'})$ is a functor in (\mathcal{F}, X, X') with values in the category of pairs $(\mathfrak{Z}, \mathcal{H})$ consisting of a locally Noetherian formal prescheme \mathfrak{Z} and an $\mathcal{O}_{\mathfrak{Z}}$ -module \mathcal{H} , with the morphisms of the latter category being the pairs consisting of a morphism g of formal preschemes and a g -morphism.

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Proposition (10.9.7). — *Let S , X , and Y be locally Noetherian preschemes, $g : X \rightarrow S$ and $h : Y \rightarrow S$ morphisms, S' a closed subset of S , and X' (resp. Y') a closed subset of X (resp. Y) such that $g(X') \subset S'$ (resp. $h(Y') \subset S'$); let $Z = X \times_S Y$; suppose that Z is locally Noetherian, and let $Z' = p^{-1}(X') \cap q^{-1}(Y')$, where p and q are the projections of $X \times_S Y$. With these conditions, the completion $Z_{/Z'}$ can be identified with the product $(X_{/X'}) \times_{S_{/S'}} (Y_{/Y'})$ of formal $S_{/S'}$ -preschemes, where the structure morphisms are identified with \hat{g} and \hat{h} , and the projections with \hat{p} and \hat{q} .*

Proof. It is immediate that the question is local for S , X , and Y , and we thus reduce to the case where $S = \text{Spec}(A)$, $X = \text{Spec}(B)$, $Y = \text{Spec}(C)$, $S' = V(\mathfrak{J})$, $X' = V(\mathfrak{K})$, and $Y' = V(\mathfrak{L})$, with \mathfrak{J} , \mathfrak{K} , and \mathfrak{L} ideals such that $\phi(\mathfrak{J}) \subset \mathfrak{K}$ and $\psi(\mathfrak{J}) \subset \mathfrak{L}$, where we denote by ϕ and ψ the homomorphisms $A \rightarrow B$ and $A \rightarrow C$ which correspond to g and h (respectively). We know that $Z = \text{Spec}(B \otimes_A C)$ and that $Z' = V(\mathfrak{M})$, where \mathfrak{M} is the ideal $\text{Im}(\mathfrak{K} \otimes_A C) + \text{Im}(B \otimes_A \mathfrak{L})$. The conclusion follows (10.7.2) from the fact that the completed tensor product $(\hat{B} \otimes_{\hat{A}} \hat{C})^\wedge$ (where \hat{A} , \hat{B} , and \hat{C} are, respectively, the separated completions of A , B , and C with respect to the \mathfrak{J} -, \mathfrak{K} -, and \mathfrak{L} -preadic topologies) is the separated completion of the tensor product $B \otimes_A C$ with respect to the \mathfrak{M} -preadic topology (0, 7.7.2). \square

In addition, we note that, if T is a locally Noetherian S -prescheme, $u : T \rightarrow X$ and $v : T \rightarrow Y$ both S -morphisms, and T' a closed subset of T such that $u(T') \subset X'$ and $v(T') \subset Y'$, then the extension to the completion $((u, v)_S)^\wedge$ can be identified with $(\hat{u}, \hat{v})_{S_{/S'}}$.

Corollary (10.9.8). — *Let X and Y be locally Noetherian S -preschemes such that $X \times_S Y$ is locally Noetherian; let S' be a closed subset of S , and X' (resp. Y') a closed subset of X (resp. Y) whose image in S is contained in S' . For every S -morphism $f : X \rightarrow Y$ such that $f(X') \subset Y'$, the graph morphism $\Gamma_{\hat{f}}$ can be identified with the extension $(\Gamma_f)^\wedge$ of the graph morphism of f .*

Corollary (10.9.9). — *Let X and Y be locally Noetherian preschemes, $f : X \rightarrow Y$ a morphism, Y' a closed subset of Y , and $X' = f^{-1}(Y')$. Then the prescheme $X_{/X'}$ can be identified, by the commutative diagram*

$$\begin{array}{ccc} X & \longleftarrow & X_{/X'} \\ f \downarrow & & \downarrow \hat{f} \\ Y & \longleftarrow & Y_{/Y'} \end{array}$$

with the product $X \times_Y (Y_{/Y'})$ of formal preschemes.

Proof. It suffices to apply Proposition (10.9.7), replacing S and S' by Y , and X and X' by X . □

Remark (10.9.10). — If S is the sum $X_1 \sqcup X_2$ (3.1), X' the union $X'_1 \cup X'_2$, where X'_i is a closed subset of X_i ($i = 1, 2$), then we have $X_{/X'} = X_{1/X'_1} \sqcup X_{2/X'_2}$.

10.10. Application to coherent sheaves on formal affine schemes.

(10.10.1). In this paragraph, A denotes an *adic Noetherian ring*, and \mathfrak{J} an ideal of definition for A . Let $X = \text{Spec}(A)$, and $\mathfrak{X} = \text{Spf}(A)$, which can be identified with the closed subset $V(\mathfrak{J})$ of X (10.1.2). In addition, Definitions (10.1.2) and (10.8.4) show that the *formal affine scheme* \mathfrak{X} is identical the completion $X_{/\mathfrak{X}}$ of the affine scheme X along the closed subset \mathfrak{X} of its underlying space. To every coherent \mathcal{O}_X -module \mathcal{F} , corresponds an $\mathcal{O}_{\mathfrak{X}}$ -module of finite type $\mathcal{F}_{/\mathfrak{X}}$, which is a sheaf of topological modules over the sheaf of topological rings $\mathcal{O}_{\mathfrak{X}}$. Every coherent \mathcal{O}_X -module \mathcal{F} is of the form \tilde{M} , where M is an A -module of finite type (1.5.1); we set $(\tilde{M})_{/X} = M^\Delta$. In addition, if $u : M \rightarrow N$ is an A -homomorphism of A -modules of finite type, then it corresponds to a homomorphism $\tilde{u} : \tilde{M} \rightarrow \tilde{N}$, and, as a result, to a continuous homomorphism $\tilde{u}_{/X'} : (\tilde{M})_{/X'} \rightarrow (\tilde{N})_{/X'}$, which we denote by u^Δ . It is immediate that $(v \circ u)^\Delta = v^\Delta \circ u^\Delta$; we have thus defined a *covariant additive functor* M^Δ from the category of A -modules of finite type to the category of $\mathcal{O}_{\mathfrak{X}}$ -modules of finite type. When A is a *discrete ring*, we have $M^\Delta = \tilde{M}$.

Proposition (10.10.2). —

(i) M^Δ is an exact functor in M , and there exists a canonical functorial isomorphism of A -modules $\Gamma(\mathfrak{X}, M^\Delta) \simeq M$.

(ii) If M and N are A -modules of finite type, then there exist canonical functorial isomorphisms

$$(10.10.2.1) \quad (M \otimes_A N)^\Delta \simeq M^\Delta \otimes_{\mathcal{O}_{\mathfrak{X}}} N^\Delta,$$

$$(10.10.2.2) \quad (\text{Hom}_A(M, N))^\Delta \simeq \mathcal{H}\text{om}_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta).$$

(iii) The map $u \mapsto u^\Delta$ is a functorial isomorphism

$$(10.10.2.3) \quad \text{Hom}_A(M, N) \simeq \text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(M^\Delta, N^\Delta).$$

Proof. The exactness of M^Δ follows from the exactness of the functors \tilde{M} (1.3.5) and $\mathcal{F}_{/X'}$ (10.8.8). By definition, $\Gamma(X, M^\Delta)$ is the separated completion of the A -module $\Gamma(X, \tilde{M}) = M$ with respect to the \mathfrak{J} -preadic topology; but, since A is complete and M is of finite type, we know (0, 7.3.6) that M is separated and complete, which proves (i). The isomorphism (10.10.2.1) (resp. (10.10.2.2)) comes from the composition of the isomorphisms (1.3.12, i) and (10.8.10.1) (resp. (1.3.12, ii) and (10.8.10.2)). Finally, since $\text{Hom}_A(M, N)$ is an A -module of finite type, we can apply (i), which identifies $\Gamma(\mathfrak{X}, (\text{Hom}_A(M, N))^\Delta)$ with $\text{Hom}_A(M, N)$, and we can use (10.10.2.2), which proves that the homomorphism (10.10.2.3) is an isomorphism. □

We deduce from Proposition (10.10.2) a series of results analogous to those of Theorem (1.3.7) and Corollary (1.3.12), whose formulation we leave to the reader.

We note that the exactness property of M^Δ , applied to the exact sequence $0 \rightarrow \mathfrak{J} \rightarrow A \rightarrow A/\mathfrak{J} \rightarrow 0$, shows that the sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$ denoted here by \mathfrak{J}^Δ coincides with the one denoted also by \mathfrak{J}^Δ in (10.3.1), by (10.3.2).

Proposition (10.10.3). — *Under the hypotheses of (10.10.1), $\mathcal{O}_{\mathfrak{X}}$ is a coherent sheaf of rings.*

Proof. If $f \in A$, then we know that $A_{\{f\}}$ is an adic Noetherian ring (0, 7.6.11), and, since the question is local, we reduce (10.1.4) to proving that the kernel of the homomorphism $v : \mathcal{O}_{\mathfrak{X}}^n \rightarrow \mathcal{O}_{\mathfrak{X}}$ is an $\mathcal{O}_{\mathfrak{X}}$ -module of finite type. We then have $v = u^\Delta$, where u is an A -homomorphism $A^n \rightarrow A$ (10.10.2); since A is Noetherian, the kernel of u is of finite type, or, equivalently, we have a homomorphism $A^m \xrightarrow{w} A^n$ such that the sequence $A^m \xrightarrow{w} A^n \xrightarrow{u} A$ is exact. We conclude (10.10.2) that the sequence $\mathcal{O}_{\mathfrak{X}}^m \xrightarrow{w^\Delta} \mathcal{O}_{\mathfrak{X}}^n \xrightarrow{v} \mathcal{O}_{\mathfrak{X}}$ is exact, which proves that the kernel of v is of finite type. \square

(10.10.4). With the above notation, set $A_n = A/\mathfrak{J}^{n+1}$, and let S_n be the affine scheme $\text{Spec}(A_n) = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^{n+1})$, with $\mathfrak{J} = \mathfrak{J}^\Delta$ the sheaf of ideals of definition for $\mathcal{O}_{\mathfrak{X}}$ corresponding to the ideal \mathfrak{J} . Let u_{mn} be the morphism of preschemes $X_m \rightarrow X_n$ corresponding to the canonical homomorphism $A_n \rightarrow A_m$ for $m \leq n$; the formal scheme \mathfrak{X} is the inductive limit of the X_n with respect to the u_{mn} (10.6.3).

Proposition (10.10.5). — *Under the hypothesis of (10.10.1), let \mathcal{F} be an $\mathcal{O}_{\mathfrak{X}}$ -module. The following conditions are equivalent:*

- (a) \mathcal{F} is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module;
- (b) \mathcal{F} is isomorphic to the projective limit (10.6.6) of a sequence (\mathcal{F}_n) of coherent \mathcal{O}_{X_n} -modules such that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$; and
- (c) there exists an A -module M of finite type (determined up to canonical isomorphism by Proposition (10.10.2, i)) such that \mathcal{F} is isomorphic to M^Δ .

Proof. We first show that (b) implies (c). We have $\mathcal{F}_n = \widetilde{M_n}$, where M_n is an A_n -module of finite type, and the hypotheses imply that $M_m = M_n \otimes_{A_n} A_m$ for $m \leq n$ (1.6.5); the M_n thus form a projective system for the canonical di-homomorphisms $M_n \rightarrow M_m$ ($m \leq n$), and it follows immediately from the definition of the A_n that this projective system satisfies the conditions of (0, 7.2.9); as a result, its projective limit M is an A -module of finite type such that $M_n = M \otimes_A A_n$ for all n . We deduce that \mathcal{F}_n is induced over X_n by $\widetilde{M} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathfrak{J}^{n+1})$, and so $\mathcal{F} = M^\Delta$ by Definition (10.8.4).

Conversely, (c) implies (b); indeed, if u_n is the immersion morphism $X_n \rightarrow X$, then $u_n^*(\widetilde{M}) = (M \otimes_A A_n)^\sim$ is induced over X_n by $\widetilde{M} \otimes_{\mathcal{O}_X} (\mathcal{O}_X/\mathfrak{J}^{n+1})$, and $M^\Delta = \varprojlim u_n^*(\widetilde{M})$ by Definition (10.8.4); since $u_m = u_n \circ u_{mn}$ for $m \leq n$, the $\mathcal{F}_n = u_n^*(\widetilde{M})$ satisfy the conditions of (b), whence our claim.

We now show that (c) implies (a): indeed, we have, by definition, that $\mathcal{O}_{\mathfrak{X}} = A^\Delta$; since M is the cokernel of a homomorphism $A^m \rightarrow A^n$, it follows from Proposition (10.10.2) that M^Δ is the cokernel of a homomorphism $\mathcal{O}_{\mathfrak{X}}^m \rightarrow \mathcal{O}_{\mathfrak{X}}^n$, and, since the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ is coherent (10.10.3), so too is M^Δ (0, 5.3.4).

Finally, (a) implies (b). Considered as an $\mathcal{O}_{\mathfrak{X}}$ -module, we have that $\mathcal{O}_{X_n} = \mathcal{O}_{\mathfrak{X}}/\mathfrak{J}^{n+1} = A_n^\Delta$; but $\mathcal{F}_n = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module (0, 5.3.5), and, since it is also an \mathcal{O}_{X_n} -module, and \mathfrak{J}^{n+1} is coherent, we conclude that \mathcal{F}_n is a coherent \mathcal{O}_{X_n} -module (0, 5.3.10), and it is immediate that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$ (recalling that the continuous map $X_m \rightarrow X_n$ of the underlying spaces is the identity on \mathfrak{X}). The sheaf $\mathcal{G} = \varprojlim \mathcal{F}_n$ is thus a coherent $\mathcal{O}_{\mathfrak{X}}$ -module, since we have seen that (b) implies (a). The canonical homomorphisms $\mathcal{F} \rightarrow \mathcal{F}_n$ form a projective system, which, by passing to the limit, gives a canonical homomorphism $w : \mathcal{F} \rightarrow \mathcal{G}$, and it remains only to prove that w is bijective. The question is now *local*, so we can reduce to the case where \mathcal{F} is the cokernel of a homomorphism $\mathcal{O}_{\mathfrak{X}}^p \rightarrow \mathcal{O}_{\mathfrak{X}}^q$; since this homomorphism is of the form v^Δ , where v is a homomorphism $A^m \rightarrow A^n$ (10.10.2), \mathcal{F} is isomorphic to M^Δ , where $M = \text{Coker } v$ (10.10.2). We then have, by Proposition (10.10.2), that $\mathcal{F}_n = M^\Delta \otimes_{\mathcal{O}_{\mathfrak{X}}} A_n^\Delta = (M \otimes_A A_n)^\Delta$, and, since the \mathfrak{J} -adic topology on $M \otimes_A A_n$ is discrete, we have $(M \otimes_A A_n)^\Delta = (M \otimes_A A_n)^\sim$ (as an \mathcal{O}_{X_n} -module); we have seen above that $M^\Delta = \varprojlim \mathcal{F}_n$, and w is thus the identity in this case. \square

Corollary (10.10.6). — *If \mathcal{F} satisfies condition (b) of Proposition (10.10.5), then the projective system (\mathcal{F}_n) is isomorphic to the system of the $\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$.*

(10.10.7). Now let A and B be adic Noetherian rings, and $\phi : B \rightarrow A$ a continuous homomorphism; we denote by \mathfrak{J} (resp. \mathfrak{K}) an ideal of definition for A (resp. B) such that $\phi(\mathfrak{K}) \subset \mathfrak{J}$, and we set $X = \text{Spec}(A)$, $Y = \text{Spec}(B)$, $\mathfrak{X} = \text{Spf}(A)$, and $\mathfrak{Y} = \text{Spf}(B)$. Let $f : X \rightarrow Y$ be the morphism of preschemes corresponding to ϕ (1.6.1), and $\hat{f} : \mathfrak{X} \rightarrow \mathfrak{Y}$ its extension to the completions (10.9.1), which is also a morphism of formal preschemes that corresponds to ϕ (10.2.2).

Proposition (10.10.8). — *For every B -module N of finite type, there exists a canonical functorial isomorphism of $\mathcal{O}_{\mathfrak{X}}$ -modules*

$$\hat{f}^*(N^\Delta) \simeq (N \otimes_B A)^\Delta.$$

Proof. Denoting by $i_X : \mathfrak{X} \rightarrow X$ and $i_Y : \mathfrak{Y} \rightarrow Y$ the canonical morphisms, we have (10.8.8), up to canonical functorial isomorphisms, $N^\Delta = i_Y^*(\tilde{N})$ and

$$(N \otimes_B A)^\Delta = i_X^*((N \otimes_B A)^\sim) = i_X^*(f^*(\tilde{N}))$$

(1.6.5); the proposition then follows from the commutativity of the diagram in (10.9.2). \square

Corollary. — *For every ideal \mathfrak{b} of B , we have $\hat{f}^*(\mathfrak{b}^\Delta) \mathcal{O}_{\mathfrak{X}} = (\mathfrak{b}A)^\Delta$.*

Proof. Let j be the canonical injection $\mathfrak{b} \rightarrow B$, to which corresponds the canonical injection $j^\Delta : \mathfrak{b}^\Delta \rightarrow \mathcal{O}_{\mathfrak{Y}}$ of sheaves of $\mathcal{O}_{\mathfrak{Y}}$ -modules; by definition, $\hat{f}^*(\mathfrak{b}^\Delta) \mathcal{O}_{\mathfrak{X}}$ is the image of the homomorphism $\hat{f}^*(j^\Delta) : \hat{f}^*(\mathfrak{b}^\Delta) \rightarrow \mathcal{O}_{\mathfrak{X}} = \hat{f}^*(\mathcal{O}_{\mathfrak{Y}})$; but this homomorphism can be identified with $(j \otimes 1)^\Delta : (\mathfrak{b} \otimes_B A)^\Delta \rightarrow \mathcal{O}_{\mathfrak{X}} = (B \otimes_B A)^\Delta$ by Proposition (10.10.8). Since the image of $j \otimes 1$ is the ideal $\mathfrak{b}A$ of A , the image of $(j \otimes 1)^\Delta$ is thus $(\mathfrak{b}A)^\Delta$, by Proposition (10.10.2), whence the conclusion. \square

10.11. Coherent sheaves on formal preschemes.

Proposition (10.11.1). — *If \mathfrak{X} is a locally Noetherian formal prescheme, then the sheaf of rings $\mathcal{O}_{\mathfrak{X}}$ is coherent, and every sheaf of ideals of definition for \mathfrak{X} is coherent.*

Proof. The question is local, so we can reduce to the case of a Noetherian affine formal scheme, and the proposition then follows from Propositions (10.10.3) and (10.10.5). \square

(10.11.2). Let \mathfrak{X} be a locally Noetherian formal prescheme, \mathcal{I} a sheaf of ideals of definition for \mathfrak{X} , and X_n the locally Noetherian (usual) prescheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$, so that \mathfrak{X} is the inductive limit of the sequence (X_n) with respect to the canonical morphisms $u_{mn} : X_m \rightarrow X_n$ (10.6.3). With this notation:

Theorem (10.11.3). — *For an $\mathcal{O}_{\mathfrak{X}}$ -module \mathcal{F} to be coherent, it is necessary and sufficient for it to be isomorphic to a projective limit of a sequence (\mathcal{F}_n) , where the \mathcal{F}_n are coherent \mathcal{O}_{X_n} -modules such that $u_{mn}^*(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$ (10.6.6). The projective system (\mathcal{F}_n) is then isomorphic to the system of the $u_n^*(\mathcal{F}) = \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$, where u_n is the canonical morphism $X_n \rightarrow \mathfrak{X}$.*

Proof. The question is local, so we can reduce to the case where \mathfrak{X} is a Noetherian affine formal scheme, and the theorem then is a consequence of Proposition (10.10.5) and Corollary (10.10.6). \square

We can thus say that the data of a coherent $\mathcal{O}_{\mathfrak{X}}$ -module is equivalent to the data of a projective system (\mathcal{F}_n) of coherent \mathcal{O}_{X_n} -modules such that $u_{mn}(\mathcal{F}_n) = \mathcal{F}_m$ for $m \leq n$.

Corollary (10.11.4). — *If \mathcal{F} and \mathcal{G} are coherent $\mathcal{O}_{\mathfrak{X}}$ -modules, then we can (with the notation of Theorem (10.11.3)) define a canonical functorial isomorphism*

$$(10.11.4.1) \quad \text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim_n \text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n).$$

Proof. The projective limit on the right-hand side is understood to be taken with respect to the maps $\theta_n \mapsto u_{mn}^*(\theta_n)$ ($m \leq n$) from $\text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$ to $\text{Hom}_{\mathcal{O}_{X_m}}(\mathcal{F}_m, \mathcal{G}_m)$. The homomorphism (10.11.4.1) sends an element $\theta \in \text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ to the sequence $(u_n^*(\theta))$; we see that we can define an inverse homomorphism of the above by sending a projective system $(\theta_n) \in \varprojlim_n \text{Hom}_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$ to its projective limit in $\text{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$, taking into account Theorem (10.11.3). \square

Corollary (10.11.5). — For a homomorphism $\theta : \mathcal{F} \rightarrow \mathcal{G}$ to be surjective, it is necessary and sufficient for the corresponding homomorphism $\theta_0 = u_0^*(\theta) : \mathcal{F}_0 \rightarrow \mathcal{G}_0$ to be surjective.

Proof. The question is local, so we reduce to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ with A an adic Noetherian ring, $\mathcal{F} = M^\Delta$, $\mathcal{G} = N^\Delta$, and $\theta = u^\Delta$, where M and N are A -modules of finite type, and u is a homomorphism $M \rightarrow N$; we then have that $\theta_0 = \tilde{u}_0$, where u_0 is the homomorphism $u \otimes 1 : M \otimes_A A/\mathfrak{J} \rightarrow N \otimes_A A/\mathfrak{J}$; the conclusion follows from the fact θ and u (resp. θ_0 and u_0) are simultaneously surjective ((1.3.9) and (10.10.2)) and that u and u_0 are simultaneously surjective (0, 7.1.14). \square

(10.11.6). Theorem (10.11.3) shows that we can consider every coherent $\mathcal{O}_{\mathfrak{X}}$ -module \mathcal{F} as a *topological $\mathcal{O}_{\mathfrak{X}}$ -module*, considering it as a projective limit of *pseudo-discrete* sheaves of groups \mathcal{F}_n (0, 3.8.1). It then follows from Corollary (10.11.4) that every homomorphism $u : \mathcal{F} \rightarrow \mathcal{G}$ of coherent $\mathcal{O}_{\mathfrak{X}}$ -modules is automatically *continuous* (0, 3.8.2). Furthermore, if \mathcal{H} is a coherent $\mathcal{O}_{\mathfrak{X}}$ -submodule of a coherent $\mathcal{O}_{\mathfrak{X}}$ -module \mathcal{F} , then, for every open $U \subset \mathfrak{X}$, $\Gamma(U, \mathcal{H})$ is a *closed* subgroup of the topological group $\Gamma(U, \mathcal{F})$, since the functor Γ is left exact, and $\Gamma(U, \mathcal{H})$ is the kernel of the homomorphism $\Gamma(U, \mathcal{F}) \rightarrow \Gamma(U, \mathcal{F}/\mathcal{H})$, which is *continuous* by the above, since \mathcal{F}/\mathcal{H} is coherent (0, 5.3.4); our claim follows from the fact that $\Gamma(U, \mathcal{F}/\mathcal{H})$ is a separated topological group. I | 205

Proposition (10.11.7). — Let \mathcal{F} and \mathcal{G} be coherent $\mathcal{O}_{\mathfrak{X}}$ -modules. We can define (with the notation of Theorem (10.11.3)) canonical functorial isomorphisms of topological $\mathcal{O}_{\mathfrak{X}}$ -modules (10.11.6)

$$(10.11.7.1) \quad \mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G} \simeq \varprojlim_n (\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{G}_n),$$

$$(10.11.7.2) \quad \mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \simeq \varprojlim_n \mathcal{H}om_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n).$$

Proof. The existence of the isomorphism (10.11.7.1) follows from the formula

$$\mathcal{F}_n \otimes_{\mathcal{O}_{X_n}} \mathcal{G}_n = (\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}) \otimes_{\mathcal{O}_{X_n}} (\mathcal{G} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}) = (\mathcal{F} \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{G}) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_{X_n}$$

and from Theorem (10.11.3). The isomorphism (10.11.7.2), where both sides are considered as sheaves of modules without topology, follows from the definition of the sections of $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ and $\mathcal{H}om_{\mathcal{O}_{X_n}}(\mathcal{F}_n, \mathcal{G}_n)$, and from the existence of the isomorphism (10.11.4.1), mapping a prescheme induced on an arbitrary Noetherian formal affine open set to \mathfrak{X} . It remains to prove that the isomorphism (10.11.7.2) is bicontinuous over a quasi-compact set, and we can thus reduce to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ with A an adic Noetherian ring, and hence (10.10.5) to the case where $\mathcal{F} = M^\Delta$ and $\mathcal{G} = N^\Delta$, with M and N both A -modules of finite type; taking (10.10.2.1), (10.10.2.3), and Corollary (1.3.12, ii) into account, we reduce to showing that the canonical isomorphism $\mathrm{Hom}_A(M, N) \simeq \varprojlim_n \mathrm{Hom}_{A_n}(M_n, N_n)$ (with $M_n = M \otimes_A A_n$ and $N_n = N \otimes_A A_n$) is continuous, which has already been proved in (0, 7.8.2). \square

(10.11.8). Since $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ is the group of sections of the sheaf of topological groups $\mathcal{H}om_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$, it is equipped with a group topology. If \mathfrak{X} is *Noetherian*, then it follows from (10.11.7.2) that the subgroups $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{I}^n \mathcal{G})$ (for arbitrary n) form a fundamental system of neighborhoods of 0 in this group.

Proposition (10.11.9). — Let \mathfrak{X} be a Noetherian formal prescheme, and \mathcal{F} and \mathcal{G} coherent $\mathcal{O}_{\mathfrak{X}}$ -modules. In the topological group $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$, the surjective (resp. injective, bijective) homomorphisms form an open set.

Proof. By Corollary (10.11.5), the set of surjective homomorphisms in $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ is the inverse image under the continuous map $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{F}_0, \mathcal{G}_0)$ of a subset of the discrete group $\mathrm{Hom}_{\mathcal{O}_{X_0}}(\mathcal{F}_0, \mathcal{G}_0)$, whence the first claim. To show the second, we cover \mathfrak{X} by a finite number of Noetherian formal affine subsets U_i . For $\theta \in \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G})$ to be injective, it is necessary and sufficient for all of the images under the (continuous) restriction maps $\mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}}}(\mathcal{F}, \mathcal{G}) \rightarrow \mathrm{Hom}_{\mathcal{O}_{\mathfrak{X}|U_i}}(\mathcal{F}|U_i, \mathcal{G}|U_i)$ to be injective; we can thus reduce to the affine case, and then this has already been proved in (0, 7.8.3). \square

10.12. Adic morphisms of formal preschemes.

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(10.12.1). Let \mathfrak{X} and \mathfrak{S} be *locally Noetherian* formal preschemes; we say that a morphism $f : \mathfrak{X} \rightarrow \mathfrak{S}$ is *adic* if there exists an ideal of definition \mathcal{J} of \mathfrak{S} such that $\mathcal{K} = f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of \mathfrak{X} ; we then also say that \mathfrak{X} is an *adic \mathfrak{S} -prescheme* (for f). Whenever this is the case, for *every* ideal of definition \mathcal{J}_1 of \mathfrak{S} , $\mathcal{K}_1 = f^*(\mathcal{J}_1)\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of \mathfrak{X} . Indeed, the question being local, we can assume that \mathfrak{X} and \mathfrak{S} are Noetherian and affine; there then exists a whole number n such that $\mathcal{J}^n \subset \mathcal{J}_1$ and $\mathcal{J}_1^n \subset \mathcal{J}$ ((10.3.6) and (0, 7.1.4)), whence $\mathcal{K}^n \subset \mathcal{K}_1$ and $\mathcal{K}_1^n \subset \mathcal{K}$. The first of these relations shows that $\mathcal{K}_1 = \mathfrak{K}_1^\Delta$, where \mathfrak{K}_1 is an open ideal of $A = \Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$, and the second shows that \mathfrak{K}_1 is an ideal of definition of A (0, 7.1.4), whence our claim.

It follows immediately from the above that, if \mathfrak{X} and \mathfrak{Y} are adic \mathfrak{S} -preschemes, then *every* \mathfrak{S} -morphism $u : \mathfrak{X} \rightarrow \mathfrak{Y}$ is *adic*: indeed, if $f : \mathfrak{X} \rightarrow \mathfrak{S}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{S}$ are the structure morphisms, and \mathcal{J} is an ideal of definition of \mathfrak{S} , then we have $f = g \circ u$, and so $u^*(g^*(\mathcal{J})\mathcal{O}_{\mathfrak{Y}})\mathcal{O}_{\mathfrak{X}} = f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of \mathfrak{X} , and, by hypothesis, $g^*(\mathcal{J})\mathcal{O}_{\mathfrak{Y}}$ is an ideal of definition of \mathfrak{Y} .

(10.12.2). In what follows, we suppose that we have some fixed locally Noetherian formal prescheme \mathfrak{S} , and some ideal of definition \mathcal{J} of \mathfrak{S} ; we set $S_n = (\mathfrak{S}, \mathcal{O}_{\mathfrak{S}} / \mathcal{J}^{n+1})$. The (locally Noetherian) adic \mathfrak{S} -preschemes clearly form a *category*. We say that an inductive system (X_n) of locally Noetherian (usual) S_n -preschemes is an *adic inductive (S_n) -system* if the structure morphisms $f_n : X_n \rightarrow S_n$ are such that, for $m \leq n$, the diagrams

$$\begin{array}{ccc} X_n & \longleftarrow & X_m \\ f_n \downarrow & & \downarrow f_m \\ S_n & \longleftarrow & S_m \end{array}$$

commute and *identify* X_m with the product $X_n \times_{S_n} S_m = (X_n)_{(S_m)}$. The adic inductive systems form a *category*: it suffices in fact to define a morphism $(X_n) \rightarrow (Y_n)$ of such systems to be an *inductive system of S_n -morphisms* $u_n : X_n \rightarrow Y_n$ such that u_m is identified with $(u_n)_{(S_m)}$ for $m \leq n$. With this in mind:

Theorem (10.12.3). — *There is a canonical equivalence between the category of adic \mathfrak{S} -preschemes and the category of adic inductive (S_n) -systems.*

The equivalence in question is obtained in the following way: if \mathfrak{X} is an adic \mathfrak{S} -prescheme, and $f : \mathfrak{X} \rightarrow \mathfrak{S}$ is the structure morphism, then $\mathcal{K} = f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition of \mathfrak{X} , and we associate to \mathfrak{X} the inductive system of the $X_n = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}} / \mathcal{K}^{n+1})$, with the structure morphism $f_n : X_n \rightarrow S_n$ corresponding to f (10.5.6). We first show that (X_n) is an *adic inductive system*: if $f = (\psi, \theta)$, then $\psi^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}} = \mathcal{K}$, so $\psi^*(\mathcal{J}^n)\mathcal{O}_{\mathfrak{X}} = \mathcal{K}^n$ for all n , and (by exactness of the functor ψ^*) $\mathcal{K}^{m+1} / \mathcal{K}^{n+1} = \psi^*(\mathcal{J}^{m+1} / \mathcal{J}^{n+1})(\mathcal{O}_{\mathfrak{X}} / \mathcal{K}^{n+1})$ for $m \leq n$; our conclusion thus follows from (4.4.5). Furthermore, it can be immediately verified that a \mathfrak{S} -morphism $u : \mathfrak{X} \rightarrow \mathfrak{Y}$ of adic \mathfrak{S} -preschemes corresponds (with the obvious notation) to an inductive system of S_n -morphisms $u_n : X_n \rightarrow Y_n$ such that u_m is identified with $(u_n)_{(S_m)}$ for $m \leq n$.

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The fact that this equivalence is well defined will follow from the more-precise following proposition.

Proposition (10.12.3.1). — *Let (X_n) be an inductive system of S_n -preschemes; suppose that the structure morphisms $f_n : X_n \rightarrow S_n$ are such that the diagrams in (10.12.2.1) commute and identify X_m with $X_n \times_{S_n} S_m$ for $m \leq n$. Then the inductive system (X_n) satisfies conditions (b) and (c) of (10.6.3); let \mathfrak{X} be the inductive limit, and $f : \mathfrak{X} \rightarrow \mathfrak{S}$ the morphism given by the inductive limit of the inductive system (f_n) . Then, if X_0 is locally Noetherian, \mathfrak{X} is locally Noetherian, and f is an adic morphism.*

Proof. Since the sheaf of ideals of \mathcal{O}_{S_n} that defines the subscheme S_m of S_n is nilpotent, by (4.4.5), so too is the sheaf of ideals of \mathcal{O}_{X_n} that defines the subscheme X_m of X_n , and so the conditions of (10.6.3) are satisfied. The question being local on \mathfrak{X} and \mathfrak{S} , we can assume that $\mathfrak{S} = \text{Spf}(A)$, $\mathcal{J} = \mathfrak{J}^\Delta$ (with A a Noetherian \mathfrak{J} -adic ring), and $X_n = \text{Spec}(B_n)$; if $A_n = A / \mathfrak{J}^{n+1}$, then the hypothesis implies that B_0 is Noetherian, and if we set $\mathfrak{J}_n = \mathfrak{J} / \mathfrak{J}^{n+1}$, then $B_m = B_n / \mathfrak{J}_n^{m+1} B_n$. The kernel of $B_n \rightarrow B_0$ is thus $\mathfrak{K}_n = \mathfrak{J}_n B_n$, and the kernel of $B_n \rightarrow B_m$ is \mathfrak{K}_n^{m+1} for $m \leq n$; further, since A_1 is Noetherian, \mathfrak{J}_1 is of finite type over A_1 , and so $\mathfrak{K}_1 = \mathfrak{K}_1 / \mathfrak{K}_1^2$ is of finite type over B_1 , and *a fortiori* of finite type over

$B_0 = B_1/\mathfrak{K}_1$; the fact that \mathfrak{X} is Noetherian then follows from (10.6.4); if $B = \varprojlim B_n$, then we have $\mathfrak{X} = \mathrm{Spf}(B)$, and, if \mathfrak{K} is the kernel of $B \rightarrow B_0$, then $B_n = B/\mathfrak{K}^{n+1}$. If $\rho_n : A/\mathfrak{J}^{n+1} \rightarrow B/\mathfrak{K}^{n+1}$ is the homomorphism corresponding to f_n , then we have that

$$\mathfrak{K}/\mathfrak{K}^{n+1} = (B/\mathfrak{K}^{n+1})\rho_n(\mathfrak{J}/\mathfrak{J}^{n+1})$$

since the homomorphism $\rho : A \rightarrow B$ corresponding to f is equal to $\varprojlim \rho_n$, and that the ideal $\mathfrak{J}B$ of B is dense in \mathfrak{K} , and, since every ideal of B is closed (0, 7.3.5), we also have that $\mathfrak{K} = \mathfrak{J}B$. If $\mathcal{K} = \mathfrak{K}^\Delta$, the equality $f^*(\mathcal{J})\mathcal{O}_{\mathfrak{X}} = \mathcal{K}$ then follows from (10.10.9), and finishes the proof. \square

(10.12.3.2). The above equivalence gives, for adic \mathfrak{S} -preschemes \mathfrak{X} and \mathfrak{Y} , a canonical bijection

$$\mathrm{Hom}_{\mathfrak{S}}(\mathfrak{X}, \mathfrak{Y}) \simeq \varprojlim_n \mathrm{Hom}_{S_n}(X_n, Y_n)$$

where the projective limit is relative to the maps $u_n \rightarrow (u_n)_{(S_m)}$ for $m \leq n$.

10.13. Morphisms of finite type.

Proposition (10.13.1). — *Let \mathfrak{Y} be a locally Noetherian formal prescheme, \mathcal{K} an ideal of definition of \mathfrak{Y} , and $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ a morphism of formal preschemes. Then the following conditions are equivalent.*

- (a) *X is locally Noetherian, f is an adic morphism (10.12.1), and, if we set $\mathcal{J} = f^*(\mathcal{K})\mathcal{O}_{\mathfrak{X}}$, then the morphism $f_0 : (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J}) \rightarrow (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$ induced by f is of finite type.*
- (b) *\mathfrak{X} is locally Noetherian, and is the inductive limit of an adic inductive (Y_n) -system (X_n) such that the morphism $X_0 \rightarrow Y_0$ is of finite type.*
- (1) *Every point of \mathfrak{Y} has a Noetherian formal affine open neighbourhood V which satisfies the following property:*
 - (Q) *$f^{-1}(V)$ is a finite union of Noetherian formal affine open subsets U_i such that the Noetherian adic ring $\Gamma(U_i, \mathcal{O}_{\mathfrak{X}})$ is topologically isomorphic to the quotient of a formal series algebra, restricted (0, 7.5.1) to $\Gamma(V, \mathcal{O}_{\mathfrak{Y}})$, by an ideal (which is necessarily closed).*

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Proof. It is immediate that (a) implies (b), by (10.12.3). To show that (b) implies (c), we can, since the question is local on \mathfrak{Y} , assume that $\mathfrak{Y} = \mathrm{Spf}(B)$, where B is Noetherian and adic; let $\mathcal{K} = \mathfrak{K}^\Delta$, with \mathfrak{K} an ideal of definition of B . Since, by hypothesis, X_0 is of finite type over Y_0 , X_0 is a finite union of affine open subsets U_i such that the ring A_{i0} of the affine scheme induced by X_0 on U_i is an algebra of finite type over the ring B/\mathfrak{K} of Y_0 (6.3.2). By (5.1.9), U_i is also an affine open subset in each of the Noetherian preschemes X_n , and, if A_{in} is the ring of the affine scheme induced by X_n on U_i , then hypothesis (b) implies, for $m \leq n$, that A_{im} is isomorphic to $A_{in}/\mathfrak{K}^{m+1}A_{in}$. Consequently, the formal prescheme induced by \mathfrak{X} on U_i is isomorphic to $\mathrm{Spf}(A_i)$, where $A_i = \varprojlim_n A_{in}$ (10.6.4); A_i is a $\mathfrak{K}A_i$ -adic ring, and $A_i/\mathfrak{K}A_i$, being isomorphic to A_{i0} , is an algebra of finite type over B/\mathfrak{K} . We thus conclude (0, 7.5.5) that A_i is topologically isomorphic to a quotient of a formal series algebra restricted to B (by a necessarily closed ideal, because such an algebra is Noetherian (0, 7.5.4)).

To show that (c) implies (a), we can restrict to the case where $\mathfrak{X} = \mathrm{Spf}(A)$ is also affine, with A a Noetherian adic ring isomorphic to a quotient of a formal series algebra, restricted to B , by a closed ideal. Then (0, 7.5.5) $A/\mathfrak{K}A$ is an algebra of finite type over B/\mathfrak{K} , and $\mathfrak{K}A = \mathfrak{J}$ is an ideal of definition of A , and so, by (10.10.9), the conditions of (a) are satisfied. \square

We note that, if the conditions of Proposition (10.13.1) are satisfied, then property (a) holds true for any ideal of definition \mathcal{K} of \mathfrak{Y} (by (c)), and so, in property (b), all the f_n are morphisms of finite type.

Corollary (10.13.2). — *If the conditions of (10.13.1) are satisfied, then every Noetherian formal affine open subset V of \mathfrak{Y} has property (Q), and, if \mathfrak{Y} is Noetherian, then so too is \mathfrak{X} .*

Proof. This follows immediately from (10.13.1) and (6.3.2). \square

Definition (10.13.3). — When the equivalent properties (a), (b), and (c) of (10.13.1) are satisfied, we say that the morphism f is of finite type, or that \mathfrak{X} is a formal \mathfrak{Y} -prescheme of finite type, or a formal prescheme of finite type over \mathfrak{Y} .

Corollary (10.13.4). — *Let $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$ be Noetherian formal affine schemes; for \mathfrak{X} to be of finite type over \mathfrak{Y} , it is necessary and sufficient for the Noetherian adic ring A to be isomorphic to the quotient of a formal series algebra, restricted to B , by some closed ideal.*

Proof. With the notation of (10.13.1), if \mathfrak{X} is of finite type over \mathfrak{Y} , then $A/\mathfrak{K}A$ is a (B/\mathfrak{K}) -algebra of finite type by (6.3.3), and $\mathfrak{K}A$ is an ideal of definition of A (10.10.9). We are then done, by (0, 7.5.5). \square

Proposition (10.13.5). —

- (i) *The composition of any two morphisms (of formal preschemes) of finite type is again of finite type.*
- (ii) *Let \mathfrak{X} , \mathfrak{S} , and \mathfrak{S}' be locally Noetherian (resp. Noetherian) formal preschemes, and $f : \mathfrak{X} \rightarrow \mathfrak{S}$ and $\mathfrak{X} \rightarrow \mathfrak{S}'$ morphisms. If f is of finite type, then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is locally Noetherian (resp. Noetherian) and of finite type over \mathfrak{S}' .*
- (iii) *Let \mathfrak{S} be a locally Noetherian formal prescheme, and \mathfrak{X}' and \mathfrak{Y}' formal \mathfrak{S} -preschemes such that $\mathfrak{X}' \times_{\mathfrak{S}} \mathfrak{Y}'$ is locally Noetherian. If \mathfrak{X} and \mathfrak{Y} are locally Noetherian formal \mathfrak{S} -preschemes, and $f : \mathfrak{X} \rightarrow \mathfrak{X}'$ and $g : \mathfrak{Y} \rightarrow \mathfrak{Y}'$ are \mathfrak{S} -morphisms of finite type, then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is locally Noetherian, and $f \times_{\mathfrak{S}} g$ is a \mathfrak{S} -morphism of finite type.*

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Proof. By the formal argument of (3.5.1), (iii) follows from (i) and (ii), so it suffices to prove (i) and (ii).

Let \mathfrak{X} , \mathfrak{Y} , and \mathfrak{Z} be locally Noetherian formal preschemes, and $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{Z}$ morphisms of finite type. If \mathcal{L} is an ideal of definition of \mathfrak{Z} , then $\mathcal{K} = g^*(\mathcal{L})\mathcal{O}_{\mathfrak{Y}}$ is an ideal of definition of \mathfrak{Y} , and $\mathcal{J} = f^*(g^*(\mathcal{L}))\mathcal{O}_{\mathfrak{X}}$ is an ideal of definition for \mathfrak{X} . Let $X_0 = (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{J})$, $Y_0 = (\mathfrak{Y}, \mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$, and $Z_0 = (\mathfrak{Z}, \mathcal{O}_{\mathfrak{Z}}/\mathcal{L})$, and let $f_0 : X_0 \rightarrow Y_0$ and $g_0 : Y_0 \rightarrow Z_0$ be the morphisms corresponding to f and g (respectively). Since, by hypothesis, f_0 and g_0 are of finite type, so too is $g_0 \circ f_0$ (6.3.4), which corresponds to $g \circ f$; thus $g \circ f$ is of finite type, by (10.13.1).

Under the conditions of (ii), \mathfrak{S} (resp. \mathfrak{X} , \mathfrak{S}') is the inductive limit of a sequence (S_n) (resp. (X_n) , (S'_n)) of locally Noetherian preschemes, and we can assume (10.13.1) that $X_m = X_n \times_{S_n} S'_m$ for $m \leq n$. The formal prescheme $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is then the inductive limit of the preschemes $X_n \times_{S_n} S'_n$ (10.7.4), and we have that

$$X_m \times_{S_m} S'_m = (X_n \times_{S_n} S'_m) \times_{S_m} S'_m = (X_n \times_{S_n} S'_n) \times_{S'_n} S'_m.$$

Furthermore, $X_0 \times_{S_0} S'_0$ is locally Noetherian, since X_0 is of finite type over S_0 (6.3.8). We thus conclude (10.12.3.1), first of all, that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is locally Noetherian; then, since $X_0 \times_{S_0} S'_0$ is of finite type over S'_0 (6.3.8), it follows from (10.12.3.1) and (10.13.1) that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{S}'$ is of finite type over \mathfrak{S}' , which proves (ii) (the claim about Noetherian preschemes being an immediate consequence of (6.3.8)). \square

Corollary (10.13.6). — *Under the hypotheses of (10.9.9), if f is a morphism of finite type, then so too is its extension \hat{f} to the completions.*

10.14. Closed subpreschemes of formal preschemes.

Proposition (10.14.1). — *Let \mathfrak{X} be a locally Noetherian formal prescheme, and \mathcal{A} a coherent sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$. If \mathfrak{Y} if the (closed) support of $\mathcal{O}_{\mathfrak{X}}/\mathcal{A}$, then the topologically ringed space $(\mathfrak{Y}, (\mathcal{O}_{\mathfrak{X}}/\mathcal{A})|_{\mathfrak{Y}})$ is a locally Noetherian formal prescheme that is Noetherian if \mathfrak{X} is.*

Proof. Note that $\mathcal{O}_{\mathfrak{X}}/\mathcal{A}$ is coherent by (10.10.3) and (0, 5.3.4), so its support \mathfrak{Y} is closed (0, 5.2.2). Let \mathcal{J} be an ideal of definition of \mathfrak{X} , and let $X_n = (\mathfrak{X}/\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$; the sheaf of rings $\mathcal{O}_{\mathfrak{X}}/\mathcal{A}$ is the projective limit of the sheaves $\mathcal{O}_{\mathfrak{X}}/(\mathcal{A} + \mathcal{J}^{n+1}) = (\mathcal{O}_{\mathfrak{X}}/\mathcal{A}) \otimes_{\mathcal{O}_{\mathfrak{X}}} (\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ (10.11.3), all of which have support \mathfrak{Y} . The sheaf $(\mathcal{A} + \mathcal{J}^{n+1})/\mathcal{J}^{n+1}$ is a coherent $\mathcal{O}_{\mathfrak{X}}$ -module, since \mathcal{J}^{n+1} is coherent, and so $(\mathcal{A} + \mathcal{J}^{n+1})/\mathcal{J}^{n+1}$ is also a coherent $(\mathcal{O}_{\mathfrak{X}}/\mathcal{J}^{n+1})$ -module (0, 5.3.10); if Y_n is the closed subprescheme of X_n defined by this sheaf of ideals, it is immediate that $(\mathfrak{Y}, (\mathcal{O}_{\mathfrak{X}}/\mathcal{A})|_{\mathfrak{Y}})$ is the formal prescheme given by the inductive limit of the Y_n , and, since the conditions of (10.6.4) are satisfied, this proves that this formal prescheme is locally Noetherian, and further Noetherian if \mathfrak{X} is (since then Y_0 is, by (6.1.4)). \square

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Definition (10.14.2). — We define a closed subprescheme of a formal prescheme \mathfrak{X} to be any formal prescheme of the form $(\mathfrak{Y}, (\mathcal{O}_{\mathfrak{X}}/\mathcal{A})|_{\mathfrak{Y}})$ with \mathcal{A} a coherent ideal of $\mathcal{O}_{\mathfrak{X}}$; we say that this prescheme is the subprescheme defined by \mathcal{A} .

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It is clear that the correspondence thus defined between coherent ideals of $\mathcal{O}_{\mathfrak{X}}$ and closed subpreschemes of \mathfrak{X} is bijective.

The morphism of topologically ringed spaces $j = (\psi, \theta) : \mathfrak{Y} \rightarrow \mathfrak{X}$, where ψ is the injection $\mathfrak{Y} \rightarrow \mathfrak{X}$ and θ^\sharp the canonical homomorphism $\mathcal{O}_{\mathfrak{X}} \rightarrow \mathcal{O}_{\mathfrak{X}}/\mathcal{A}$, is evidently (10.4.5) a morphism of formal preschemes, and we call it the *canonical injection* from \mathfrak{Y} to \mathfrak{X} . Note that, if $\mathfrak{X} = \mathrm{Spf}(A)$, or if A is Noetherian and adic, then $\mathcal{A} = \mathfrak{a}^\Delta$, where \mathfrak{a} is an ideal of A (10.10.5), and it then follows immediately from the above that $\mathfrak{Y} = \mathrm{Spf}(A/\mathfrak{a})$, up to isomorphism, and that j corresponds (10.2.2) to the canonical homomorphism $A \rightarrow A/\mathfrak{a}$.

We say that a morphism $f : \mathfrak{Z} \rightarrow \mathfrak{X}$ of locally Noetherian formal preschemes is a *closed immersion* if it factors as $\mathfrak{Z} \xrightarrow{g} \mathfrak{Y} \xrightarrow{j} \mathfrak{X}$, where g is an isomorphism from \mathfrak{Z} to a closed subprescheme \mathfrak{Y} of \mathfrak{X} , and j is the canonical injection. Since j is a monomorphism of ringed spaces, g and \mathfrak{Y} are necessarily unique.

Proposition (10.14.3). — *A closed immersion is a morphism of finite type.*

Proof. We can immediately restrict to the case where \mathfrak{X} is a formal affine scheme $\mathrm{Spf}(A)$, and $\mathfrak{Y} = \mathrm{Spf}(A/\mathfrak{a})$; the proposition then follows from Proposition (10.13.1, c). \square

Lemma (10.14.4). — *Let $f : \mathfrak{Y} \rightarrow \mathfrak{X}$ be a morphism of locally Noetherian formal preschemes, and let (U_α) be a cover of $f(\mathfrak{Y})$ by Noetherian formal affine open subsets of \mathfrak{X} such that the $f^{-1}(U_\alpha)$ are Noetherian formal affine open subsets of \mathfrak{Y} . For f to be a closed immersion, it is necessary and sufficient for $f(\mathfrak{Y})$ to be a closed subset of \mathfrak{X} and, for all α , for the restriction of f to $f^{-1}(U_\alpha)$ to correspond (10.4.6) to a surjective homomorphism $\Gamma(U_\alpha, \mathcal{O}_{\mathfrak{X}}) \rightarrow \Gamma(f^{-1}(U_\alpha), \mathcal{O}_{\mathfrak{Y}})$.*

Proof. The conditions are clearly necessary. Conversely, if the conditions are satisfied, and if we denote by \mathfrak{a}_α the kernel of $\Gamma(U_\alpha, \mathcal{O}_{\mathfrak{X}}) \rightarrow \Gamma(f^{-1}(U_\alpha), \mathcal{O}_{\mathfrak{Y}})$, then we can define a coherent sheaf of ideals \mathcal{A} of $\mathcal{O}_{\mathfrak{X}}$ by setting $\mathcal{A}|_{U_\alpha} = \mathfrak{a}_\alpha^\Delta$ and taking \mathcal{A} to be zero on the complement of the union of the U_α . Since $f(\mathfrak{Y})$ is closed, and since the support of $\mathfrak{a}_\alpha^\Delta$ is $U_\alpha \cap f(\mathfrak{Y})$, everything reduces to proving that $\mathfrak{a}_\alpha^\Delta$ and $\mathfrak{a}_\beta^\Delta$ induce the same sheaf on any Noetherian formal affine open subset $V \subset U_\alpha \cap U_\beta$. But the restriction to $f^{-1}(U_\alpha)$ of f is a closed immersion of this formal prescheme into U_α , $f^{-1}(V)$ is a Noetherian formal affine open subsets of $f^{-1}(U_\alpha)$, and the restriction of f to $f^{-1}(V)$ is a closed immersion; if \mathfrak{b} is the kernel of the surjective homomorphism $\Gamma(V, \mathcal{O}_{\mathfrak{X}}) \rightarrow \Gamma(f^{-1}(V), \mathcal{O}_{\mathfrak{Y}})$ corresponding to this restriction, then it is immediate (10.10.2) that $\mathfrak{a}_\alpha^\Delta$ induces \mathfrak{b}^Δ on V . The sheaf of ideals \mathcal{A} being thus defined, it is then clear that $f = g \circ j$, where $j : \mathfrak{Z} \rightarrow \mathfrak{X}$ is the canonical injection of the closed subprescheme \mathfrak{Z} of \mathfrak{X} defined by \mathcal{A} , and that g is an isomorphism from \mathfrak{Y} to \mathfrak{Z} . \square

Proposition (10.14.5). —

- (i) *If $f : \mathfrak{Z} \rightarrow \mathfrak{Y}$ and $g : \mathfrak{Y} \rightarrow \mathfrak{X}$ are closed immersions of locally Noetherian formal preschemes, then $g \circ f$ is a closed immersion.*
- (ii) *Let \mathfrak{X} , \mathfrak{Y} , and \mathfrak{S} be locally Noetherian formal preschemes, $f : \mathfrak{X} \rightarrow \mathfrak{S}$ a closed immersion, and $g : \mathfrak{Y} \rightarrow \mathfrak{S}$ a morphism. Then the morphism $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \rightarrow \mathfrak{Y}$ is a closed immersion.*
- (iii) *Let \mathfrak{S} be a locally Noetherian formal prescheme, and \mathfrak{X}' and \mathfrak{Y}' locally Noetherian formal \mathfrak{S} -preschemes such that $\mathfrak{X}' \times_{\mathfrak{S}} \mathfrak{Y}'$ is locally Noetherian. If \mathfrak{X} and \mathfrak{Y} are locally Noetherian \mathfrak{S} -preschemes, and $f : \mathfrak{X} \rightarrow \mathfrak{X}'$ and $g : \mathfrak{Y} \rightarrow \mathfrak{Y}'$ are \mathfrak{S} -morphisms that are closed immersions, then $f \times_{\mathfrak{S}} g$ is a closed immersion.*

Proof. By (3.5.1), it again suffices to prove (i) and (ii).

To prove (i), we can assume that \mathfrak{Y} (resp. \mathfrak{Z}) is a closed subprescheme of \mathfrak{X} (resp. \mathfrak{Y}) defined by a coherent sheaf \mathcal{J} (resp. \mathcal{K}) of ideals of $\mathcal{O}_{\mathfrak{X}}$ (resp. $\mathcal{O}_{\mathfrak{Y}}$); if ψ is the injection $\mathfrak{Y} \rightarrow \mathfrak{X}$ of underlying spaces, then $\psi_*(\mathcal{K})$ is a coherent sheaf of ideals of $\psi_*(\mathcal{O}_{\mathfrak{Y}}) = \mathcal{O}_{\mathfrak{X}}/\mathcal{J}$ (0, 5.3.12), and thus also a coherent $\mathcal{O}_{\mathfrak{X}}$ -module (0, 5.3.10); the kernel \mathcal{K}_1 of $\mathcal{O}_{\mathfrak{X}} \rightarrow (\mathcal{O}_{\mathfrak{X}}/\mathcal{J})/\psi_*(\mathcal{K})$ is thus a coherent sheaf of ideals of $\mathcal{O}_{\mathfrak{X}}$ (0, 5.3.4), and $\mathcal{O}_{\mathfrak{X}}/\mathcal{K}_1$ is isomorphic to $\psi_*(\mathcal{O}_{\mathfrak{Y}}/\mathcal{K})$, which proves that \mathfrak{Z} is an isomorphism to a closed subprescheme of \mathfrak{X} .

To prove (ii), we can immediately restrict to the case where $\mathfrak{S} = \mathrm{Spf}(A)$, $\mathfrak{X} = \mathrm{Spf}(B)$, and $\mathfrak{Y} = \mathrm{Spf}(C)$, with A a Noetherian \mathfrak{J} -adic ring, $B = A/\mathfrak{a}$ (where \mathfrak{a} is an ideal of A), and C a Noetherian topological adic A -algebra. Everything then reduces to proving that the homomorphism

$C \rightarrow C \hat{\otimes}_A (A/\mathfrak{a})$ is *surjective*: but A/\mathfrak{a} is an A -module of finite type, and its topology is the \mathfrak{J} -adic topology; it then follows from (0, 7.7.8) that $C \hat{\otimes}_A (A/\mathfrak{a})$ can be identified with $C \otimes_A (A/\mathfrak{a}) = C/\mathfrak{a}C$, whence our claim. \square

Corollary (10.14.6). — *Under the hypotheses of (10.14.5, ii), let $p : \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \rightarrow \mathfrak{X}$ and $q : \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \rightarrow \mathfrak{Y}$ be the projections, so that the diagram*

$$\begin{array}{ccc} \mathfrak{X} & \xleftarrow{p} & \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y} \\ f \downarrow & & \downarrow q \\ \mathfrak{S} & \xleftarrow{g} & \mathfrak{Y} \end{array}$$

commutes. For every coherent $\mathcal{O}_{\mathfrak{X}}$ module \mathcal{F} , we then have a canonical isomorphism of $\mathcal{O}_{\mathfrak{Y}}$ -modules

$$(10.14.6.1) \quad u : g^* f_*(\mathcal{F}) \simeq q_* p^*(\mathcal{F}).$$

Proof. We know that defining a homomorphism $g^* f_*(\mathcal{F}) \rightarrow q_* p^*(\mathcal{F})$ is equivalent to defining a homomorphism $f_*(\mathcal{F}) \rightarrow g_* q_* p^*(\mathcal{F}) = f_* p_* p^*(\mathcal{F})$ (0, 4.4.3): we take $u = f_*(\rho)$, where ρ is the canonical homomorphism $\mathcal{F} \rightarrow p_* p^*(\mathcal{F})$ (0, 4.4.3). To see that u is an isomorphism, we can immediately restrict to the case where \mathfrak{S} , \mathfrak{X} , and \mathfrak{Y} are formal spectra of Noetherian adic rings A , B , and C (respectively), satisfying the conditions in (10.14.5, ii) above; we then have $\mathcal{F} = M^\Delta$, where M is an (A/\mathfrak{a}) -module of finite type (10.10.5), and the two sides of (10.14.6.1) can then be identified, respectively, by (10.10.8), with $(C \otimes_A M)^\Delta$ and $((C/\mathfrak{a}C) \otimes_{A/\mathfrak{a}} M)^\Delta$, whence the corollary, since $(C/\mathfrak{a}C) \otimes_{A/\mathfrak{a}} M = (C \otimes_A (A/\mathfrak{a})) \otimes_{A/\mathfrak{a}} M$ is canonically identified with $C \otimes_A M$. \square

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Corollary (10.14.7). — *Let X be a locally Noetherian usual prescheme, Y a closed subprescheme of X , j the canonical injection $Y \rightarrow X$, X' a closed subset of X , and $Y' = Y \cap X'$; then $\hat{j} : Y_{/Y'} \rightarrow X_{/X'}$ is a closed immersion, and, for every coherent \mathcal{O}_Y -module \mathcal{F} , we have*

$$\hat{j}_*(\mathcal{F}_{/Y'}) = (j_*(\mathcal{F}))_{/X'}.$$

Proof. Since $Y' = j^{-1}(X')$, it suffices to use (10.9.9) and to apply (10.14.5) and (10.14.6). \square

10.15. Separated formal preschemes.

Definition (10.15.1). — Let \mathfrak{S} be a formal prescheme, \mathfrak{X} a formal \mathfrak{S} -prescheme, and $f : \mathfrak{X} \rightarrow \mathfrak{S}$ the structure morphism. We define the diagonal morphism $\Delta_{\mathfrak{X}|\mathfrak{S}} : \mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ (also denoted by $\Delta_{\mathfrak{X}}$) to be the morphism $(1_{\mathfrak{X}}, 1_{\mathfrak{X}})_{\mathfrak{S}}$. We say that \mathfrak{X} is separated over \mathfrak{S} , or is a formal \mathfrak{S} -scheme, or that f is a separated morphism, if the image of the underlying space of \mathfrak{X} under $\Delta_{\mathfrak{X}}$ is a closed subset of the underlying space of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$. We say that a formal prescheme \mathfrak{X} is separated, or is a formal scheme, if it is separated over \mathbf{Z} .

Proposition (10.15.2). — *Suppose that the formal preschemes \mathfrak{S} and \mathfrak{X} are inductive limits of sequences (S_n) and (X_n) (respectively) of usual preschemes, and that the morphism $f : \mathfrak{X} \rightarrow \mathfrak{S}$ is the inductive limit of a sequence of morphisms $f_n : X_n \rightarrow S_n$. For f to be separated, it is necessary and sufficient for the morphism $f_0 : X_0 \rightarrow S_0$ to be separated.*

Proof. Indeed, $\Delta_{\mathfrak{X}|\mathfrak{S}}$ is then the inductive limit of the sequence of morphisms $\Delta_{X_n|S_n}$ (10.7.4), and the image of the underlying space of \mathfrak{X} (resp. of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ under $\Delta_{\mathfrak{X}|\mathfrak{S}}$) is identical to the image of the underlying space of X_0 (resp. of $X_0 \times_{S_0} X_0$) under $\Delta_{X_0|S_0}$; whence the conclusion. \square

Proposition (10.15.3). — *Suppose that all the formal preschemes (resp. morphisms of formal preschemes) in what follows are inductive limits of sequences of usual preschemes (resp. of morphisms of usual preschemes).*

- (i) *The composition of any two separated morphisms is separated.*
- (ii) *If $f : \mathfrak{X} \rightarrow \mathfrak{X}'$ and $g : \mathfrak{Y} \rightarrow \mathfrak{Y}'$ are separated \mathfrak{S} -morphisms, then $f \times_{\mathfrak{S}} g$ is separated.*
- (iii) *If $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ is a separated \mathfrak{S} -morphism, then the \mathfrak{S}' -morphism $f_{(\mathfrak{S}')} : \mathfrak{X}_{(\mathfrak{S}')} \rightarrow \mathfrak{Y}_{(\mathfrak{S}')}$ is separated for every extension $\mathfrak{S}' \rightarrow \mathfrak{S}$ of the base formal prescheme.*
- (iv) *If the composition $g \circ f$ of two morphisms is separated, then f is separated.*

(In the above, it is implicit that if the same formal prescheme \mathfrak{Z} is mentioned more than once in the same proposition, we consider it as the inductive limit of the *same* sequence (Z_n) of usual preschemes wherever it is mentioned, and the morphisms from \mathfrak{Z} to another formal prescheme (resp. from a formal prescheme to \mathfrak{Z}) as inductive limits of morphisms from Z_n to some usual preschemes (resp. from some usual preschemes to Z_n)).

Proof. With the notation of (10.15.2), we have, in fact, that $(g \circ f)_0 = g_0 \circ f_0$ and $(f \times_{\mathfrak{S}} g)_0 = f_0 \times_{S_0} g_0$; the claims of (10.15.3) are then immediate consequences of (10.15.2) and the corresponding claims in (5.5.1) for usual preschemes. \square

We leave it to the reader to state, for the same type of formal preschemes and morphisms as in (10.15.3), the propositions corresponding to (5.5.5), (5.5.9), and (5.5.10) (by replacing “affine open subset” by “formal affine open subset satisfying condition (b) of (10.6.3)”).

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A similar argument also shows that every Noetherian formal affine scheme is separated, which justifies the terminology.

Proposition (10.15.4). — *Let \mathfrak{S} be a locally Noetherian formal prescheme, and \mathfrak{X} and \mathfrak{Y} locally Noetherian formal \mathfrak{S} -preschemes such that \mathfrak{X} or \mathfrak{Y} is of finite type over \mathfrak{S} (so that $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is locally Noetherian) and such that \mathfrak{Y} is separated over \mathfrak{S} . Let $f : \mathfrak{X} \rightarrow \mathfrak{Y}$ be an \mathfrak{S} -morphism; then the graph morphism $\Gamma_f(1_{\mathfrak{X}}, f)_{\mathfrak{S}} : \mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is a closed immersion.*

Proof. We can assume that \mathfrak{S} is the inductive limit of a sequence (S_n) of locally Noetherian preschemes, \mathfrak{X} (resp. \mathfrak{Y}) the inductive limit of a sequence (X_n) (resp. (Y_n)) of S_n -preschemes, and f the inductive limit of a sequence $(f_n : X_n \rightarrow Y_n)$ of S_n -morphisms; then $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ is the inductive limit of the sequence $(X_n \times_{S_n} Y_n)$, and Γ_f the inductive limit of the sequence (Γ_{f_n}) (10.7.4); by hypothesis, Y_0 is separated over S_0 (10.15.2), so the space $\Gamma_{f_0}(X_0)$ is a closed subspace of $X_0 \times_{S_0} Y_0$; since the underlying spaces of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$ (resp. $\Gamma_f(\mathfrak{X})$) and $X_0 \times_{S_0} Y_0$ (resp. $\Gamma_{f_0}(X_0)$) are identical, we already see that $\Gamma_f(\mathfrak{X})$ is a closed subspace of $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$. Now note that, when (U, V) runs over the set of pairs consisting of a Noetherian formal affine open subset U (resp. V) of \mathfrak{X} (resp. \mathfrak{Y}) such that $f(U) \subset V$, the open subsets $U \times_{\mathfrak{S}} V$ form a cover of $\Gamma_f(\mathfrak{X})$ in $\mathfrak{X} \times_{\mathfrak{S}} \mathfrak{Y}$, and, if $f_U : U \rightarrow V$ is the restriction of f to U , then $\Gamma_{f_U} : U \rightarrow U \times_{\mathfrak{S}} V$ is the restriction of Γ_f to U . If we show that Γ_{f_U} is a closed immersion, then Γ_f will be a closed immersion (10.14.4), or, in other words, we are led to consider the case where $\mathfrak{S} = \text{Spf}(A)$, $\mathfrak{X} = \text{Spf}(B)$, and $\mathfrak{Y} = \text{Spf}(C)$ are affine (with A, B , and C Noetherian adics), with f corresponding to a continuous A -homomorphism $\phi : C \rightarrow B$; then Γ_f corresponds to the unique continuous homomorphism $\omega : B \hat{\otimes}_A C \rightarrow B$ which, when composed with the canonical homomorphisms $B \rightarrow B \hat{\otimes}_A C$ and $C \rightarrow B \hat{\otimes}_A C$, gives the identity and ϕ (respectively). But it is clear that ω is surjective, whence our claim. \square

Corollary (10.15.5). — *Let \mathfrak{S} be a locally Noetherian formal prescheme, and \mathfrak{X} an \mathfrak{S} -prescheme of finite type; for \mathfrak{X} to be separated over \mathfrak{S} , it is necessary and sufficient for the diagonal morphism $\mathfrak{X} \rightarrow \mathfrak{X} \times_{\mathfrak{S}} \mathfrak{X}$ to be a closed immersion.*

Proposition (10.15.6). — *A closed immersion $j : \mathfrak{Y} \rightarrow \mathfrak{X}$ of locally Noetherian formal preschemes is a separated morphism.*

Proof. With the notation of (10.14.2), $j_0 : Y_0 \rightarrow X_0$ is a closed immersion, thus a separated morphism, and so it suffices to apply (10.15.2). \square

Proposition (10.15.7). — *Let X be a locally Noetherian (usual) prescheme, X' a closed subset of X , and $\hat{X} = X_{/X'}$. For \hat{X} to be separated, it is necessary and sufficient that \hat{X}_{red} be separated, and it is sufficient that X be separated.*

Proof. With the notation of (10.8.5), for \hat{X} to be separated, it is necessary and sufficient for X'_0 to be separated (10.15.2), and since $\hat{X}_{\text{red}} = (X'_0)_{\text{red}}$, it is equivalent to ask for \hat{X}_{red} to be separated (5.5.1, vi). \square

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