# THE LANGLANDS CORRESPONDENCE FOR GLOBAL FUNCTION FIELDS

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LECTURE NOTES BY TONY FENG

## Contents

Disclaimer

This document consists of (unofficial) lecture notes I took from a course offered by Zhiwei Yun at Stanford in the winter quarter of 2015, entitled "The Langlands Correspondence for Global Function Fields." I found this class quite challenging to follow, and my misunderstandings will doubtless be reflected in these notes. These misunderstandings are obviously my (and not the lecturer's) fault.

## 1. Introduction

1.1. **Setup.** The goal of the course is to understand V. Lafforgue's work on the Langlands correspondence for $\mathrm{GL}_n$ over global function field. First let's try to state a rough version of the result. To do so, we need to introduce some notation.

Throughout, $F$ will denote the function field of a (smooth, projective, and geometrically connected) curve $X/(k=\mathbb{F}_q)$. Fix a split, connected, reductive group $G/k$. In this course the main example is $G=\mathrm{GL}_n$, but it is important to note htat Lafforgue's work applies to *general* $G$.

We denote by $|X|$ the set of closed points of $X$. For any $x\in X$ denote by $\mathscr{O}_x$ the *completed local ring* of $X$ at $x$. After choosing a uniformizer $\varpi_x$ for $\mathscr{O}_{X,x}$, we can identify this with $k_x[[\varpi_x]]$, where $k_x$ is the residue field at $x$. We let $F_x=\mathrm{Frac}(\mathscr{O}_x)$, which is (non-canonically) isomorphic to $k_x((\varpi_x))$. It is useful to think of $\operatorname{Spec}\mathscr{O}_x$ as a formal disk about $x$.

We recall the ring of *adeles* associated to $F$:

$$\mathbb{A}_F=\prod_{x\in|X|}{}'(F_x,\mathscr{O}_X)=\{(a_x)_{x\in|X|}\mid a_x\in\mathscr{O}_x \text{ for almost all } x\}.$$

Note that $\mathbb{A}_F$ is a $k$-algebra, so we can consider the locally compact topological group $G(\mathbb{A}_F)$.

*Definition* 1.1.1. We define

$$\mathscr{A}_G=\left\{\begin{matrix}\text{“cuspidal, smooth, automorphic”}\\ \text{representations of } G(\mathbb{A}_F)\end{matrix}\right\}/\text{isomorphism}.$$

Choosing some prime $\ell\neq p=\mathrm{ch}\,k$, we set

$$\mathscr{G}_{\widehat{G}}=\left\{\begin{matrix}\text{continuous homomorphisms}\\ \rho:\mathrm{Gal}(\overline{F}/F)\to\widehat{G}(\overline{\mathbb{Q}_\ell})\end{matrix}\right\}/\text{conjugacy}.$$

Here $\widehat{G}$ to be the Langlands dual of $G$, defined over $\mathbb{Z}$. It is determined by the combinatorial data involved in the classification of reductive groups.

The Langlands correspondence says roughly that *There is a map $\mathscr{A}_G\to\mathscr{G}_{\widehat{G}}$ which is finite-to-one*. In particular, to an automorphic representation of $G$ we should be able to associated a Galois representation valued in the Langlands dual group of $G$.

We will spend the rest of the section explaining the precise meaning of the ingredients.

1.2. **The Langlands dual group.** Split, connected reductive groups over $k$ are classified combinatorially by the datum of a four-tuple $(P,\Delta,P^\vee,\Delta^\vee)$ where $P,P^\vee$ are lattices and $\Delta\subset P,\Delta^\vee\subset P^\vee$ are root systems. The classification associates to $G$ the character group $P=X^\bullet(T)$ with its roots $\Delta$, and the co-character group $P^\vee=X_\bullet(T)$ with its coroots $\Delta^\vee$.

Then $\widehat{G}$ is the group determined by the involution

$$(P,\Delta,P^\vee,\Delta^\vee)\longleftrightarrow(P^\vee,\Delta^\vee,P,\Delta).$$

*Example* 1.2.1. Some examples of Langlands dual groups:

- If $G=\mathrm{GL}_n$, then $\widehat{G}\cong\mathrm{GL}_n$.
- If $G=\mathrm{SL}_n$, $\widehat{G}\cong\mathrm{PGL}_n$.
- If $G=\mathrm{Sp}_{2n}$, then $\widehat{G}\cong\mathrm{SO}_{2n+1}$.

A general property is that simply connected groups go to adjoint groups (and vice versa, of course). In all examples except the last one, the Dynkin diagram is unchanged.

1.3. **Automorphic representations.** The Langlands philosophy says roughly that there should be a map $\mathscr{A}_G \to \mathscr{G}_{\widehat{G}}$. That is, to an "automorphic representation" one should be able to attach a "Galois representation." Moreover, this map should be finite-to-one.

What Lafforgue proves is a little more precise. Fix a "level" $K \subset G(\mathbb{A})$ of the form

$$K = \prod_x K_x$$

where $K_x \subset G(\mathscr{O}_x)$ is a congruence subgroup for each $x$, and $K_x = G(\mathscr{O}_x)$ for almost all $x$. That is, $K_x$ is specified by a finite number of "congruence conditions." We choose a (finite) "bad subset" $S \subset |X|$ containing all $x \in |X|$ such that $K_x \subsetneq G(\mathscr{O}_x)$.

We consider, as a first approximation, the space of *all* functions

$$C(G(F)\backslash G(\mathbb{A}_F)/K) := \mathrm{Fun}(G(F)\backslash G(\mathbb{A}_F)/K, \overline{\mathbb{Q}_\ell}).$$

*Remark* 1.3.1. Note that since $K$ is compact open, $G(F)\backslash G(\mathbb{A}_F)/K$ is discrete, so the notion of coninuity is trivial.

*Remark* 1.3.2. As far as the representation theory is concerned, the theory is the same as long as one uses an algebraically closed field of characteristic 0 as the target field.

This space of all functions is "too big" for at least two reasons, so we have to modify it.

The first and more trivial reason is that it has many "components." One can see this even for $G = \mathrm{GL}_1 = \mathbb{G}_m$. Then one is considering $F^\times\backslash\mathbb{A}^\times/K$, which by the obvious quotient map admits a surjection to $F^\times\backslash\mathbb{A}^\times/\prod \mathscr{O}_x^\times$.

$$\begin{array}{c} F^\times\backslash\mathbb{A}^\times/K \\ \downarrow \\ F^\times\backslash\mathbb{A}^\times/\prod \mathscr{O}_x^\times \end{array}$$

The space $F^\times\backslash\mathbb{A}^\times/\prod \mathscr{O}_x^\times$ is isomorphic to $F^\times\backslash\bigoplus_{x\in|X|}(F_x^\times/\mathscr{O}_x^\times)$, which is the *class group* $F^\times\backslash\,\mathrm{Div}(X) \cong \mathrm{Cl}(X)$. This has a "degree map" $\mathrm{Cl}(X) \xrightarrow{\deg} \mathbb{Z}$.

$$\begin{tikzcd} F^\times\backslash\mathbb{A}^\times/K \arrow[d] & & & \\ F^\times\backslash\mathbb{A}^\times/\prod \mathscr{O}_x^\times \arrow[r] & F^\times\backslash\bigoplus_{x\in|X|}(F_x^\times/\mathscr{O}_x^\times) \arrow[r, equal] & \mathrm{Cl}(X) \arrow[r, "\deg"] & \mathbb{Z} \end{tikzcd}$$

The fiber over each degree $d \in \mathbb{Z}$ represents a different "component." So this is a trivial source of infinitely many components in our original space.

For $\mathrm{GL}_n$, the same issue arises via the determinant map $G \to \mathrm{GL}_1$. In fact, this issue will arise whenever $G$ is not semisimple (i.e. whenever $G$ has a torus in its center, or equivalently whenever $G$ admits a non-trivial map to $\mathbb{G}_m$).

Here is how we can correct this problem. If $G = \mathrm{GL}_n$, the inclusion of the center $Z(G)$ is a map $\mathbb{G}_m \to \mathrm{GL}_n$. Choose $a \in \mathbb{A}^\times$ such that $\deg a = 1$. Then consider instead the double coset space

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/Ka^{\mathbb{Z}}$$

where we abuse notation by identifying $a \in \mathrm{GL}_1(\mathbb{A})$ with the diagonal element of $\mathrm{GL}_n(\mathbb{A})$. Under $\deg\circ\det$, $a \in \mathrm{GL}_n(\mathbb{A})$ maps to $n$, so $\deg\circ\det$ maps the double coset space to $\mathbb{Z}/n$, a *finite* set.

In dealing with general $G$, one replaces $a$ with the choice of a *lattice* in $Z^0(G)$ (the connected component of the center $Z(G)$). To elaborate, the degree map takes $Z^0(\mathbb{A})$ to $X_\bullet(Z^0)$. Since the latter is just a free abelian group of finite rank, one can choose a section of this map and call its image $\Xi$, and consider the double coset

$$G(F)\backslash G(\mathbb{A})/K\cdot\Xi.$$

This has finitely many "components" in the sense we have been discussing.

1.4. **Cuspidal representations.** The function space $C(G(F)\backslash G(\mathbb{A})/K\cdot\Xi)$ is still infinite-dimensional, and we prefer to work with finite-dimensional representations, so we introduce another refinement. The idea is to study "cusp forms," which form a subset

$$C_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K\cdot\Xi) \subset C(G(F)\backslash G(\mathbb{A})/K\cdot\Xi).$$

Crucially, it turns out that this space is *finite-dimensional.* In fact, something much stronger is true: there is a finite subset

$$\Sigma_{K,\Xi} \subset G(F)\backslash G(\mathbb{A})/K\cdot\Xi$$

such that any $f \in C_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K\cdot K\cdot\Xi)$ vanishes outside $\Sigma_{K,\Xi}$. In other words, the support of functions in this space is *uniformly* bounded. We will see later why this is so.

The space $C_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K\cdot\Xi)$ admits an action of the *Hecke algebra*

$$\mathscr{H}_{K,\Xi} = \underbrace{C_c(K\cdot\Xi\backslash G(\mathbb{A})/K\cdot\Xi)}_{\text{compactly (=finitely) supported}}.$$

by convolution. The multiplication within the Hecke algebra is also via convolution. Note that $G(\mathbb{A})$ acts on $C_c(G(F)\backslash G(\mathbb{A}))$ by right translation; the Hecke action is a "linearized version" of this action.

*Definition* 1.4.1. We define

$$\mathscr{A}_{G,K,\Xi} \subset \mathscr{A}_G$$

to be the subset of isomorphism classes of simple $H_{K,\Xi}$-subalgebras of $C_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K\cdot\Xi)$.

1.5. **Lafforgue's work.** We can now explain Lafforgue's results. Lafforgue constructs a commutative algebra

$$\mathscr{B} \subset \mathrm{End}_{\mathbb{Q}_\ell}(C_{\mathrm{cusp}}(G(F)\backslash G(\mathbb{A})/K\cdot\Xi))$$

with the following properties:

(1) The action of $\mathscr{B}$ on $C_{\mathrm{cusp}}$ commutes with the action of the Hecke algebra $\mathscr{H} := \mathscr{H}_{K,\Xi}$,
(2) $\mathscr{B}$ is an artinian $\overline{\mathbb{Q}_\ell}$-algebra, and there is a canonical map $\mathrm{Spec}\,\mathscr{B} \to \mathscr{G}_{\widehat{G}}$. Thus $\mathscr{B}$ is designed to "access" Galois representations.

(3) There is a compatibility with the "Satake parameters," which we will explain later. This implies that there is a "generalized eigenspace decomposition"

$$C_{\text{cusp}} = \bigoplus_{\nu\colon \mathscr{B}\to\overline{Q_\ell}} C_{\text{cusp}}(\nu)$$

such that each summand is preserved by the action of the Hecke algebra. Note that $\nu$ can be interpreted as a closed point of Spec $\mathscr{B}$. Then the map Spec $\mathscr{B} \to \mathscr{G}_{\widehat{G}}$ associates to $\nu \in$ Spec $\mathscr{B}$ a representation $\rho_\nu\colon \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ such that any $\mathscr{H}$-submodule of $C_{\text{cusp}}(\nu)$ is sent to $\rho_\nu \in \mathscr{G}_{\widehat{G}}$. This is the desired map

$$\mathscr{A}_{G,K,\Xi} \to \mathscr{G}_{\widehat{G}}.$$

Since we have defined $\mathscr{A}_{G,K,\Xi}$ to consist of isomorphism *classes* of simple $\mathscr{H}$-modules, it is possible a priori that an isomorphism class appears in multiple eigenspaces, say associated to $\nu_1$ and $\nu_2$, such that $\rho_{\nu_1} \not\cong \rho_{\nu_2}$.

In fact, this *does* happen, but not for $\mathrm{GL}_n$. So in general, one doesn't get a map - just a correspondence. The reason underlying this ambiguity is that there can be two homomorphisms $\rho_1, \rho_2\colon \Gamma \to \widehat{G}$ such that for *each* $\gamma \in \Gamma$ the elements $\rho_1(\gamma)$ and $\rho_2(\gamma)$ are conjugate, but $\rho_1$ and $\rho_2$ are not "globally" conjugate. Since the Hecke algebra is "local," it can't detect this sort of phenomena.

1.6. **What is known.** The case $G = \mathrm{GL}_n$ was settled completely by L. Lafforgue (2002). In this case the statements simplify somewhat: there is a canonical *bijection* between

$$\mathscr{A}_n = \left\{ \begin{smallmatrix} \text{cuspidal automorphic} \\ \text{reps. of } \mathrm{GL}_n(\mathbb{A}) \end{smallmatrix} \right\} /\text{iso} \longleftrightarrow \mathscr{G}_n = \left\{ \begin{smallmatrix} \text{irreducible, cont. reps.} \\ \rho\colon W(\overline{F}/F) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}) \\ \text{ramified at fin. many places} \end{smallmatrix} \right\} /\text{iso}.$$

In particular, the left hand side is independent of $\ell$, so the right hand side must be as well. This shows that $\mathscr{G}_{n,\ell} \longleftrightarrow \mathscr{G}_{n,\ell'}$ if $\ell \neq \ell'$.

In fact, every $\rho \in \mathscr{G}_{n,\ell}$ "comes from geometry" in the sense that there is a variety $Y/F$ such that $\rho$ appears in $H^i_{\text{ét}}(Y_{\overline{F}}, \overline{Q_\ell})$.

The other direction, "Galois to automorphic," is still unclear for general $G$, and beyond the methods we will discuss. Indeed, V. Lafforgue's decomposition

$$C_{\text{cusp}} = \bigoplus_{\nu\colon B_\ell \to \overline{\mathbb{Q}_\ell}} C_{\text{cusp}}(\nu)$$

depends on $\ell$ already, while we expect that there should be a decomposition that is *independent* of $\ell$. So there is much more work to be done!

## 2. Cusp Forms and Cuspidal Representations

We keep the notation from before: $G/k$ is a split reductive group and $F = k(X)$ is a global function field.

We conider the quotient

$$[G] := G(F)\backslash G(\mathbb{A}).$$

This admits an action of $G(\mathbb{A})$ by right translation. Then

$$C([G]) := \mathrm{Fun}([G], \overline{\mathbb{Q}_\ell}).$$

has an induced action of $G(\mathbb{A})$. The study of automorphic forms concerns the decomposition of $C([G])$ into irreducible representations for $G(\mathbb{A})$.

As we mentioned last time, this space is "too big" to get a reasonable answer. For instance, it is certainly not a direct sum of its irreducible representations - it is more like a "direct integral." Therefore, one can only hope to parametrize the irreducible subrepresentations by *continuous* parameters, and that introduces functional-analytic issues. In this section, however, we will focus on a special class of subrepresentations for which we can ignore these issues.

### 2.1. Parabolic subgroups.

If $G = \mathrm{GL}_n$, the "basic parabolic subgroups" are those which are "block-upper-triangular":

$$P = \left\{ \begin{pmatrix} \boxed{\mathrm{GL}_{n_1}} & * & * \\ 0 & \boxed{\mathrm{GL}_{n_2}} & * \\ 0 & 0 & \ddots \end{pmatrix} \right\}.$$

In general, a subgroup $Q < \mathrm{GL}_n$ defined over $F$ is *parabolic* if it can be conjugated to a basic one.

A more invariant definition is that if we view $\mathrm{GL}_n = \mathrm{Aut}_F(V)$, where $\dim_F V = n$, then a parabolic subgroup is the stabilizer of a (not necessarily maximal) flag

$$V^\bullet : \quad V = V^0 \supsetneq V^1 \supsetneq V^2 \supsetneq \ldots \supsetneq V^d = 0.$$

We let

$$P(V^\bullet) = \{g \in \mathrm{Aut}(V) \mid g V^i = V^i\}$$

be the associated parabolic, so $V^\bullet \mapsto P(V^\bullet)$ is a bijection between flags in $V$ and parabolic subgroups in $\mathrm{Aut}(V)$.

For any parabolic subgroup, there is an exact sequence

$$1 \to N_P \to P \to L_P \to 1$$

where $N_P$ is the called the *unipotent radical* of $P$ and $L_P$ is called the *Levi quotient*. If $P = P(V^\bullet)$, then

$$N_P = \{g \in \mathrm{Aut}(V) \mid (g - I)V^i \subset V^{i+1}\}$$

and

$$L_P = \prod_i \mathrm{Aut}(V^i/V^{i+1}).$$

In concrete terms, if $P$ is a basic parabolic then $N_P$ is the subgroup which is the identity along the "block diagonal" and $L_P$ are the "block diagonal" matrices. From this one sees that the exact sequence is *split*.

$$N_P = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & \ddots \end{pmatrix} \right\} \qquad L_P = \left\{ \begin{pmatrix} \boxed{\mathrm{GL}_{n_1}} & 0 & 0 \\ 0 & \boxed{\mathrm{GL}_{n_2}} & 0 \\ 0 & 0 & \ddots \end{pmatrix} \right\}.$$

2.2. **Cusp forms.** Let $f \in C([G])$. In analogy to the classical theory of modular forms, we want to define the notion of "constant term" of $f$, so that cusp forms are those with vanishing constant term. If $P < G$ is parabolic (and defined over $F$), we consider the diagram

$$\begin{tikzcd} & {[P] = P(F)\backslash P(\mathbb{A})} \arrow[dl] \arrow[dr] & \\ {[G] = G(F)\backslash G(\mathbb{A})} & & {[L_P] = L_P(F)\backslash L_P(\mathbb{A})} \end{tikzcd}$$

Then for any $f \in [G]$, we can pull it back (i.e. restrict) to $[P]$, and then "integrate along fibers" to obtain an element of $[L_P]$.

*Definition* 2.2.1. We define the "Constant Term" map

$$CT_P^G \colon C^\infty([G]) \to C^\infty([L_P])$$

explicitly as

$$f \mapsto \left( \underbrace{\ell}_{\text{in } L_P(\mathbb{A})} \mapsto \int_{N_P(F)\backslash N_P(\mathbb{A})} f(n\ell)\, dn \right)$$

There are some issues involved in making precise what we mean here. First, we are implicitly choosing a splitting of the exact sequence in order to view $\ell \in L_P(\mathbb{A})$ as an element of $[G]$.

Second, we should explain the measure $dn$. It is induced from a Haar measure on $N_P(\mathbb{A})$. Concretely, if $K_N \subset N(\mathbb{A})$ is compact open, then the double coset space $N(F)\backslash N(\mathbb{A})/K_N$ is *finite*. If, for instance, $G = \mathrm{GL}_2$ so $N \cong \mathbb{A}^1$, the "level 1" choice $K_N = \prod \mathscr{O}_x$ leads to

$$N(F)\backslash N(\mathbb{A})/K_N = F\backslash \mathbb{A}/\prod_{x\in|X|} \mathscr{O}_x.$$

We know that this is finite from the theory of algebraic curves (it describes the obstruction to approximating Laurent tails using global rational functions; by Riemann-Roch, we know that only Laurent tails of bounded degree can contribute, and there are finitely many of those since we are over a finite field). In general, $N(F)\backslash N(\mathbb{A})/K_N$ will be a successive extension of copies of this by $\mathbb{G}_a$. For $h \colon N(F)\backslash N(\mathbb{A})/K_N \to \overline{\mathbb{Q}_\ell}$, we can define

$$\int_{[N]} h\, dn = \sum_{x\in[N]/K_N} h(x) \cdot \# \mathrm{Stab}_{K_N}(x).$$

The normalization here makes the result independent of the choice of $K_N$. Since any locally constant function $f\colon N_P(F)\backslash N_P(\mathbb{A}) \to \overline{\mathbb{Q}_\ell}$ will be invariant by *some* compact open subgroup $K_N$ (depending on $f$), so we can always make this definition.

*Exercise* 2.2.2. Use the translation from classical modular forms to automorphic forms on $\mathrm{GL}_2$ to compare the classical notion of cuspidality with the general one we just defined.

*Definition* 2.2.3. We say that $f \in C^\infty([G])$ is *cuspidal* if for all parabolic subgroups $P \subsetneq G$ defined over $F$,

$$CT_P^G(f) = 0.$$

*Remark* 2.2.4. In fact, it suffices to check this for maximal parabolics because there is a "transitivity of constant term." Also, it suffices to check one parabolic within each $G(F)$-conjugacy class.

*Exercise* 2.2.5. Prove these statements.

*Example* 2.2.6. For $G = \mathrm{GL}_n$, this means that we only need to check $CT_{P_m}^G(f)$ where

$$P_m = \left\{ \begin{pmatrix} \boxed{\mathrm{GL}_m} & * \\ 0 & \boxed{\mathrm{GL}_{n-m}} \end{pmatrix} \right\}.$$

*Exercise* 2.2.7. Check that the subspace $C_{\mathrm{cusp}}([G]) \subset C^\infty([G])$ is invariant under $G(\mathbb{A})$.

*Definition* 2.2.8. A *cuspidal automorphic representation* of $G(\mathbb{A})$ is a subquotient of $C_{\mathrm{cusp}}([G])$.

This is correct only if $Z(G)$ doesn't have a split torus over $F$. Since we assumed that $G$ was split, that is equivalent to $G$ being semisimple. This doesn't apply to $\mathrm{GL}_n$, so we have to do something extra: choose $a \in \mathbb{A}^\times$ with $\deg a \neq 0$, and define similarly $C_{\mathrm{cusp}}([G]/a^{\mathbb{Z}})$. Its subquotients are then the cuspidal automorphic representations.

*Remark* 2.2.9. Brian Conrad points out that this is the same as considering the subquotients with a fixed finite order central character.

2.3. **The main result.** Let $K$ be a level (i.e. a compact open subgroup of $G(\mathbb{A})$). For $\mathrm{GL}_n$, we consider

$$C_{\mathrm{cusp}}([G]/Ka^{\mathbb{Z}}) = C_{\mathrm{cusp}}([G]/a^{\mathbb{Z}})^K.$$

Notice that this space admits an action of the Hecke algebra $\mathscr{H}_K = C_c(a^{\mathbb{Z}}K\backslash G(\mathbb{A})/a^{\mathbb{Z}}K)$ (the compactly supported bi-invariant functions).

The main goal of this section is to prove the following theorems.

**Theorem 2.3.1.** *$C_{\mathrm{cusp}}([G]/a^{\mathbb{Z}} \cdot K)$ is finite-dimensional, and all of its elements have uniformly finite support.*

By this we mean that the size of the support can be bounded *independently of the function*. Using this, we can prove the semi-simplicity of the cuspidal space with level $a^{\mathbb{Z}} \cdot K$.

**Theorem 2.3.2.** *$C_{\mathrm{cusp}}([G]/a^{\mathbb{Z}} \cdot K)$ decomposes into a direct sum of irreducible $\mathscr{H}_K$-modules.*

*Proof of Theorem 2.3.2 assuming Theorem 2.3.1.* It suffices to construct a positive definite $\mathscr{H}_K$-invariant Hermitian form on $C_{\mathrm{cusp}}([G]/a^{\mathbb{Z}} \cdot K)$. (Then by the usual argument, the orthogonal complement of a subspace is invariant under the Hecke algebra and one wins by induction.) Choosing an isomorphism $\overline{\mathbb{Q}_\ell} \cong \mathbb{C}$, an obvious form is

$$\langle f_1, f_2 \rangle = \int_{[G]/a^{\mathbb{Z}} \cdot K} f_1 \overline{f_2}.$$

By Theorem 2.3.1, we know that $f_1, f_2$ have finite support so the integral is well-defined, and it is obviously positive definite.

It remains to check the compatibility with the Hecke action.

*Exercise* 2.3.3. Check that if $h \in \mathscr{H}_K$, then we have

$$\langle h * f_1, f_2 \rangle = \langle f_1, h^* * f_2 \rangle$$

where $h^*(x) = \overline{h(x^{-1})}$.

□

The proof of Theorem 2.3.1 requires more preparation.

## 2.4. **Weil's Uniformization Theorem.**

We specialize $G = \mathrm{GL}_n$ and $K = \prod_{x \in |X|} \mathrm{GL}_n(\mathscr{O}_x)$.

**Theorem 2.4.1** (Weil). *There is a canonical isomorphism of groupoids*

$$\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}) / \prod_{x \in |X|} \mathrm{GL}_n(\mathscr{O}_x) \cong \mathrm{Bun}_n(k) := \left\{ \begin{smallmatrix} \textit{vector bundles of} \\ \textit{rank } n \textit{ over } X \end{smallmatrix} \right\}.$$

*Remark* 2.4.2. At this point $\mathrm{Bun}_n(k)$ is purely notation, but we will eventually see that there exists a moduli stack $\mathrm{Bun}_n$ of rank $n$ vector bundles on $X$, whose $k$-points are precisely $\mathrm{Bun}_n(k)$ as we defined above.

Recall that a groupoid can be thought of as a set, with a group attached to each element. On the left, the set is the double coset space and the group is the stabilizer, and on the right the group is the automorphism group.

*Definition* 2.4.3. For $\mathscr{E} \to X$ a vector bundle of rank $n$, a *full level structure* of $\mathscr{E}$ at $x$ is an isomorphism

$$\alpha_x : \mathscr{E}|_{\mathrm{Spec}\, \mathscr{O}_x} \cong \mathscr{O}_x^{\oplus n} \quad (\text{as } \mathscr{O}_x\text{-modules})$$

and a *generic trivialization* of $\mathscr{E}$ is an isomoprhism

$$\tau : \mathscr{E}|_{\mathrm{Spec}\, F} \cong F^n.$$

Weil's theorem can be proved by finding an appropriate interpretation of $\mathrm{GL}_n(\mathbb{A})$ as parametrizing the data of rank $n$ vector bundles on $X$ plus some extra structure, and comparing the quotient map on one side with the forgetful map on the other.

Our candidate for comparison with $\mathrm{GL}_n(\mathbb{A})$ is the set of vector bundles $\mathscr{E}$ of rank $n$ over $X$ endowed with full level structure $\alpha_x$ at all $x \in |X|$ and a generic trivialization $\tau$:

$$\Sigma := \left\{ (\mathscr{E} \to X, \{\alpha_x\}_{x \in |X|}, \tau) : \begin{smallmatrix} \mathscr{E} \to X = \text{vec. bundle. rank } n \\ \{\alpha_x\} = \text{full level structure} \\ \tau = \text{generic trivialization} \end{smallmatrix} \right\}.$$

By forgetting the trivializations, maps to the groupoid of all vector bundles, as a torsor for

$$\mathrm{GL}_n(F) \times \prod_{x \in |X|} \mathrm{GL}_n(\mathscr{O}_x).$$

Therefore, we seek a $\mathrm{GL}_n(F) \times \prod_{x\in|X|} \mathrm{GL}_n(\mathscr{O}_x)$-equivariant map from $\Sigma$ to $\mathrm{GL}_n(\mathbb{A})$.

$$\begin{tikzcd}
\Sigma \arrow[r, "?"] \arrow[d] & \mathrm{GL}_n(\mathbb{A}) \arrow[d] \\
\mathrm{Bun}_n(k) \arrow[r, dotted] & \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\prod_x \mathrm{GL}_n(\mathscr{O}_x).
\end{tikzcd}$$

Given $(\mathscr{E}, \alpha_x, \tau)$ we can consider $\mathscr{E}|_{F_x}$ (thought of as the restriction to the punctured formal neighborhood of $x$). Then we have *two* trivalizations of $\mathscr{E}|_{F_x}$: one coming from the generic trivialization $\tau$, and the other coming from the local trivialization $\alpha_x$. The transition map then defines an element $g_x \in \mathrm{GL}_n(F)_x$, defined by the composition

$$F_x^n \xrightarrow{\alpha_x^{-1}} \mathscr{E}|_{\mathrm{Spec}\ F_x} \xrightarrow{\tau\otimes \mathrm{Id}_{F_x}} F_x^n.$$

*Exercise* 2.4.4. Check that for almost all $x$, $g_x$ lies in $\mathrm{GL}_n(\mathscr{O}_x)$.

This defines a map $\Sigma \to \mathrm{GL}_n(\mathbb{A})$, and moreover $\mathrm{GL}_n(\mathscr{O}_n)$ acts by pre-composing with the inverse and $\mathrm{GL}_n(F)$ acts by post-composing, so the map is equivariant and descends to a map

$$\mathrm{Bun}_n(k) \to \mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\prod_x \mathrm{GL}_n(\mathscr{O}_x).$$

It remains to see that this is a bijection. Let's try to define an inverse, which is a variant of the operation of recovering the vector bundle from its transition functions. Let's think about how these "transition" functions arose. We started with the data of a rational trivialization of $\mathscr{E}$ and local trivializations around each point $x$. When the transition function in the punctured disk Spec $F_x$ is actually an element of $\mathrm{GL}_n(\mathscr{O}_x)$, that says that the rational trivialization of $\mathscr{E}|_{\mathrm{Spec}\ F}$ extends to $x$. So given a collection $(g_x)_{x\in|X|} \in \mathrm{GL}_n(\mathbb{A})$, $\mathscr{E}$ should be a "modification" of the trivial bundle $\mathscr{O}_X^n$ at the points $x$ where $g_x \notin \mathrm{GL}_n(\mathscr{O}_x)$.

*Example* 2.4.5. Let's first consider the case of line bundles to see how things work out. The generic trivialization $\tau$ corresponds to a rational section of $\mathscr{O}_X$. At the points where $g_x \notin \mathscr{O}_x^\times$, we may assume that $g_x$ is a power $\varpi_x^{n_x}$ of the local uniformizer $\varpi_x \in \mathscr{O}_x$. This says that the rational section has order $-n_x$ at $x$, so the line bundle determining these transition functions is $\mathscr{O}(-\sum n_x x)$.

This suggests the general procedure. First suppose that $g_x$ has entries in $\mathscr{O}_x$ for all $x$. Let $\Lambda_x = g_x \cdot \mathscr{O}_x^n$, which will be a sublattice of $\mathscr{O}_x^n$ (and equal to it for all but finitely many $x$). Defining $Q_x := \mathscr{O}_x^n/\Lambda_x$, we set

$$\mathscr{E} = \ker\left(\mathscr{O}_x^n \to \bigoplus_{x\in X} Q_x\right).$$

Now in general, pick $m_x$ to be the least integer such that $\Lambda_x = g_x \cdot \mathscr{O}_x^n \subset \varpi_x^{-m_x} \mathscr{O}_x^n$, define $Q_x := \varpi_x^{-m_x} \mathscr{O}_x^n / \Lambda_x$, and then set

$$\mathscr{E} = \ker\left( \mathscr{O}_x^n(\sum m_x x) \to \bigoplus_{x \in X} Q_x \right)$$

*Exercise* 2.4.6. Check that this indeed gives a well-defined inverse.

**Variants.** We can add level structure into this correspondence. Let $K \cong \prod K_x \subset \mathrm{GL}_n(\mathscr{O}_x)$ where all but finitely many $K_x$ are precisely $\mathrm{GL}_n(\mathscr{O}_x)$.

*Example* 2.4.7. We could take $K_x := \ker(\mathrm{GL}_n(\mathscr{O}_x) \to \mathrm{GL}_n(\mathscr{O}_x/\varpi_x^{d_x}))$. We denote such a $K = \prod K_x$ by $K = K_D$, where $D = \sum d_x x$ is an effective divisor. Then the analogue of Weil's theorem is:

$$\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A}) / K_D \cong \left\{ (V, \alpha) \mid {}^{V \text{ rank } n \text{ vec. bundle over } X}_{\alpha \colon V|_D \cong \mathscr{O}_D^n} \right\}.$$

This generalizes to arbitrary split group.

**Theorem 2.4.8** (Uniformization)**.** *For a general algebraic group $G$ split over $k$, we have a canonical bijection*

$$G(F) \backslash G(\mathbb{A}) / \prod_x G(\mathscr{O}_x) \cong \{\textit{principal } G\textit{-bundles over } X\}.$$

*Remark* 2.4.9. We have *not* assumed that $G$ is reductive here, and indeed we will shortly apply it to non-reductive groups.

*Example* 2.4.10. For $G = \mathrm{Sp}_{2n}$, a $G$-torsor over $X$ is the same as the data of a vector bundle $V$ over $X$ of rank $n$ and a symplectic form $\omega \colon V \otimes_{\mathscr{O}_x} V \to \mathscr{O}_X$ (a perfecting alternating form).

In the most general case, $G$ may be defined only over $F$, and not over $k$. However, it is possible to choose an integral model $\mathscr{G}/X$ ("spreading out"), so that $\mathscr{G}(\mathscr{O}_X)$ makes sense. Then one has a correspondence

$$G(F) \backslash G(\mathbb{A}) / \prod_x \mathscr{G}(\mathscr{O}_x) \hookrightarrow \{\mathscr{G}\text{-torsor over } X\}.$$

The problem is that this is not surjective in general. Indeed, only $G$-torsors that are trivial at the generic point lie in the image of this map. But *unlike* vector bundles, $G$-torsors are not Zariski-locally trivial, and hence may not be trivial at the generic point.

*Remark* 2.4.11. It's not really obvious why this issue *doesn't* arise in the case where $G$ is a split, connected reductive group defined over $k$. A proof of this non-trivial fact, supplied via private communication by Brian Conrad, is reproduced below.

Let $Z$ be the (split!) maximal central torus of $G$, $G'$ the semisimple derived group, and $\widetilde{G'} \twoheadrightarrow G'$ its (split!) simply connected central cover, so we have a central isogeny

$$1 \to \mu \to \widetilde{G'} \times Z \to G \to 1. \tag{2.4.1}$$

If $T$ is a split maximal $k$-torus in $\widetilde{G'}$, then $\mu$ lies inside $T' := T \times Z$, so we can make a central pushout of (2.4.1) along $\mu \to T'$ to get a central extension

$$1 \to T' \to E \to G \to 1 \tag{2.4.2}$$

where $T'$ is a split $k$-torus and $E$ is split with simply connected derived group. So far this has nothing to do with finite fields. By centrality of (2.4.2), we have an exact sequence of pointed sets

$$1 \to H^1(X,E) \to H^1(X,G) \to H^2(X,T) \to 1$$

whose final term is $\mathrm{Br}(X) \otimes X_*(T)$. But as $X$ is a smooth projective curve over a finite field, $\mathrm{Br}(X)$ vanishes class field theory for $F$ ♠♠♠ TONY: [I know how to prove this, but not by class field theory?!] so every $G$-torsor on $X$ is a pushout of an $E$-torsor. Hence, it suffices to show $H^1(F,E) = 1$.

But $E'$ is simply connected by design, and the quotient $E/E'$ is a torus that is split, so $H^1(F, E/E') = 1$ and hence $H^1(F,E') \to H^1(F,E)$ is surjective. Thus, it is enough to prove the vanishing of $H^1(F,-)$ on connected semisimple $F$-groups that are *simply connected* (even handling the split case is enough for us, e.g. $\mathrm{SL}_n$, $\mathrm{Sp}_{2n}$, $E_8$, $G_2$, etc.). This vanishing is a deep theorem of Harder (maybe he only handled the split case, but it is true in general; I have never read the proof by Harder, which is written in German, but my vague recollection is that it somehow uses automorphic methods).

2.5. **Cusp forms on $\mathrm{GL}_n$.** Let's try to understand where cusp forms fit into this picture for $G = \mathrm{GL}_n$. Let $\mathrm{Bun}_n(k)$ be the groupoid of rank $n$ vector bundles over $X$. By Weil's Theorem 2.4.1, we may interpret cusp forms as certain functions

$$f \colon \mathrm{Bun}_n(k) \to \overline{\mathbb{Q}_\ell}.$$

Let $P_m$ be the maximal parabolic consisting of upper-block-triangular matrices with block sizes $(m, n-m)$ (see Example 2.2.6). Then, using Weil's theorem, we have a correspondence

$$\begin{tikzcd}
 & P_m(F)\backslash P_m(\mathbb{A})/\prod_x P_m(\mathscr{O}_x) \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_n(k) & & L_n(F)\backslash L_n(\mathbb{A})/\prod_x L_n(\mathscr{O}_x)
\end{tikzcd}$$

By the extension of Weil's theorem for split $G/k$ (Theorem 2.4.8), we have

$$P_m(F)\backslash P_m(\mathbb{A})/\prod_x P_m(\mathscr{O}_x) \longleftrightarrow \left\{(V' \subset V) \mid {}_{V' \subset V \text{ rank } m \text{ sub-bundle}}^{V = \text{vec. bun. rank } n}\right\}.$$

and $L_n(F)\backslash L_n(\mathbb{A})/L_n(\mathscr{O}_x)$ can be identified with $\mathrm{Bun}_m(k) \times \mathrm{Bun}_{n-m}(k)$. On elements, the correspondence is

$$\begin{tikzcd}
 & (V' \subset V) \arrow[dl, maps to] \arrow[dr, maps to] & \\
V & & (V', V/V')
\end{tikzcd}$$

Then the "constant term" map

$$CT_{P_m}^{Q_n} \colon C(\mathrm{Bun}_n(k)) \to C(\mathrm{Bun}_m(k) \times \mathrm{Bun}_{n-m}(k))$$

is defined on elements by

$$f \mapsto \left[ (V', V'') \mapsto \sum_{[V] \in \mathrm{Ext}^1(V'', V')} \frac{1}{\# \mathrm{Hom}(V'', V')} f(V) \right] \tag{2.5.1}$$

Note that $\mathrm{Hom}(V'', V')$ is a finite-dimensional $k$-vector space, so it has a finite size. By definition, $f$ is cuspidal if $CT_{P_m}^{\mathrm{GL}_n}(f) = 0$ for all $m = 1, 2, \ldots, n$.

*Exercise* 2.5.1. Check that the formula above is compatible with our general definition of the constant term map.

Recall that we wanted to show that any cusp form $f$ has uniformly finite support. How might we show this? Suppose that *any* extension of $V''$ by $V'$ splits. Then the sum in formula (2.5.1) involves only one term, so if $f$ is cuspidal then $f$ *must* vanish on $V' \oplus V''$. Therefore, we are interested in studying how often this occurs.

2.6. **Semistability of vector bundles.** So when does it happen that *any* extension of $V''$ by $V'$ splits? This is tautologically equivalent to $\mathrm{Ext}^1(V'', V') = 0$, which occurs if and only if $\mathrm{Hom}(V', V'' \otimes \omega_X) = 0$ by a form of Serre duality.

*Exercise* 2.6.1. Think to the case of line bundles: intuitively, a high-degree bundle can't map to a low-degree bundle (Indeed, $\mathrm{Hom}(\mathscr{L}_1, \mathscr{L}_2) \cong \mathscr{L}_1^\vee \otimes \mathscr{L}_2$).

Vector bundles are a bit more subtle; so we require a digression on semi-stability.

*Definition* 2.6.2. Let $V$ be a vector bundle. We define the *slope* of $V$ to be

$$\mu(V) := \frac{\deg V}{\operatorname{rank} V}.$$

*Definition* 2.6.3. $V$ is *semistable* if $\mu(V') \subset \mu(V)$ for all sub-bundles $V' \subset V$.

*Remark* 2.6.4. Recall that a sub-bundle is an inclusion $V' \hookrightarrow V$ such that the quotient is *also* a vector bundle (i.e. the inclusion map has constant rank). This is tronger than requiring $V' \hookrightarrow V$ to be an injective map of sheaves.

*Example* 2.6.5. On $X = \mathbb{P}^1$, we have

$$\mu(\mathscr{O}(a) \oplus \mathscr{O}(b)) = \frac{a+b}{2}$$

and hence $\mathscr{O}(a) \oplus \mathscr{O}(b)$ is not semistable if $a > \frac{a+b}{2}$. Therefore, $\mathscr{O}(a) \oplus \mathscr{O}(b)$ is semistable if and only if $a = b$. We easily see that in general, $\mathscr{O}(a_1) \oplus \ldots \oplus \mathscr{O}(a_n)$ is semistable if and only if $a_1 = a_2 = \ldots = a_n$.

Any vector bundle $V$ has a canonical filtration, called the *Harder-Narasimhan* filtration,

$$0 \subset V_1 \subsetneq V_2 \subset \ldots \subsetneq V_m = V$$

having the property that $V_i / V_{i-1}$ is semistable with slope $\mu_i$, and

$$\mu_1 > \mu_2 > \ldots > \mu_{m-1}.$$

Indeed, to construct this we may choose $V_1$ to be the sub-bundle with the large slope. We then choose $V_2$ to be the pre-image under the natural quotient of the sub-bundle of $V/V_1$ with largest slope, etc.

Therefore, for any vector bundle we have a "slope invariant"

$$V \mapsto (\mu_{\max}(V) := \mu_1 > \mu_2 > \ldots > \mu_{\min}(V))$$

assigning to any vector bundles the tuple of slopes determined by its Harder-Narasimhan filtration.

**Lemma 2.6.6.** *If* $\mu_{\min}(V') > \mu_{\max}(V'')$, *then* $\mathrm{Hom}(V', V'') = 0$.

Since $\mathrm{Ext}^1(V', V'') \cong \mathrm{Hom}(V', V'' \otimes \omega_X)$ by Serre duality, this is a useful criterion for determining when $\mathrm{Ext}^1(V', V'')$ is trivial.

*Proof.* Note that the slope of any semi-stable bundle can only increase under a quotient, by the addivity of rank and degree in the exact sequence

$$0 \to V' \to V \to V'' \to 0.$$

Therefore, a semi-stable vector bundle cannot map to one of lower slope. The result follows by applying this observation repeatedly with respect to Harder-Naransimhan filtrations for $V'$ and $V''$.

□

Let $g$ be the genus of $X$, and suppose that in the slope sequence of the Harder-Narasimhan filtration of a vector bundle $V$ on $X$ there is a gap with size $> 2g-2$, i.e. there exists $i$ such that

$$\mu_i - \mu_{i+1} > 2g - 2.$$

That means there exists $V' \subset V$ such that $V/V' = V''$ and $\mu_{\min}(V') = \mu_i > \mu_{\max}(V'') + 2g - 2$, so

$$\mathrm{Hom}(V', V'' \otimes \omega_X) = 0 \implies \mathrm{Ext}^1(V'', V') = 0.$$

Therefore, $V$ is the *unique* extension of $V''$ by $V'$. Thus, $f$ has to vanish on $V' \oplus V'' \cong V$. In conclusion, any cusp form must vanish on a vector bundle with a "big gap" (i.e. $> 2g-2$) in its slope sequence.

2.7. **Uniformly finite support of cusp forms.** Let $\mathrm{Bun}_n^d(k)$ be the set of vector bundles of rank $n$ and degree $d$ on $X$. In here, we have the subset $\mathrm{Bun}_n^{d,\mathrm{gaps} \le 2g-2}$ consisting of vector bundles of rank $n$, degree $d$, and having all gaps $\le 2g-2$ in their slope sequences.

**Lemma 2.7.1.** $\mathrm{Bun}_n^{d,gaps \le 2g-2}$ *is a finite set.*

*Sketch of Proof.* The *HN-polygon* of a vector bundle is the lower convex hull of the (rank, degree) points of the elements of the filtration. (So the slope invariants are the slopes between successive vertices.) We claim that there are only finitely many possible HN polygons for a vector with given degree and rank, and bounded gaps. This is easy to show: for instance, the maximum slope is bounded above, because the minimum slope can be bounded in terms of the maximum slope and the rank. Anyway, we will assume this point and go on.

Next, for a given HN polygon, we claim that there are finitely many vector bundles with a given HN polygon. The key is to establish "boundedness" of semistable bundles of given rank and degree (and then use the fact that each graded piece $V_i/V_{i+1}$ of the associated graded of the Harder-Narasimhan filtration is semistable, by definition). This fact comes from the construction of a coarse moduli space classifying bundles of given rank and degree: one constructs a line bundle $\mathscr{L}$ with very negative degree and large rank, and proves that any semistable bundle is a quotient of $\mathscr{L}^N$ for some uniform $N \gg 0$.

The upshot is that there is a coarse moduli space of semistable vector bundles with a given HN polygon which is *finite type* over $k$, and since $k$ is a finite field that implies that it has only finitely many $k$-points. $\square$

The conclusion is that any cusp form $f \in \mathrm{Bun}_n(k)$ is supported on $\coprod_{d\in\mathbb{Z}} \mathrm{Bun}_n^{d,\mathrm{gaps}\le 2g-2}(k)$. This is still possibly infinite, but we already encountered this issue in the first lecture, when we discussed how one needs to modify the representation space in the presence of non-trivial central tori. Choose $a \in \mathbb{A}^\times$ which has image 1 under the degree map

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\prod_x \mathrm{GL}_n(\mathscr{O}_x) \xrightarrow{\deg} \mathbb{Z}.$$

Then we get a map

$$\mathrm{GL}_n(F)\backslash \mathrm{GL}_n(\mathbb{A})/\prod_x \mathrm{GL}_n(\mathscr{O}_x)a^{\mathbb{Z}} \to \mathbb{Z}/n\mathbb{Z}$$

and the left hand side is equivalent to $\mathrm{Bun}_n(k)$ modulo a certain equivalence relation, which has the following meaning. We can interpret $\mathrm{Div}(a)$ as a divisor on $X$, corresponding to a line bundle $\mathscr{L} = \mathscr{O}_X(\mathrm{Div}(a)) \in \mathrm{Pic}^1(X)$. Then the equivalence relation is $V \sim V'$ if $V' \cong V \otimes \mathscr{L}$.

By the preceding discussion, if a cusp form $f$ is invariant under this equivalence relation (which is the right notion of cusp form for $\mathrm{GL}_n$), then it is supported on the set $\coprod_{d\in\mathbb{Z}} \mathrm{Bun}_n^{d,\mathrm{gaps}\le 2g-2}(k)$ *modulo* the equivalence relation, which is finite.

## 3. The Hecke Algebra

Recall that the (global) Hecke algebra is defined to be $\mathscr{H}_K := C_c(K\backslash G(\mathbb{A})/K)$. We now consider its action (by convolution) on $C(G(F)\backslash G(\mathbb{A})/K)$.

If $K = \prod_x K_x$, then $\mathscr{H}_K \cong \bigotimes_{x\in|X|} \mathscr{H}_{K_x}$ where $\mathscr{H}_{K_x} = C_c(K_x\backslash G(F_x)/K_x)$.

*Remark* 3.0.2. This tensor product is the coproduct in the category of algebras, which more concretely mean that almost all factors should be the identity (the identity function of the identity double coset).

By assumption, $K_x = G(\mathscr{O}_x)$ for almost all $x$, and we denote $\mathscr{H}_x := \mathscr{H}_{G(\mathscr{O}_x)}$.

### 3.1. **The local Hecke algebra.**

Now we study the ring $\mathscr{H}_x$. We have a bijection

$$G(\mathscr{O}_x)\backslash G(F_x)/G(\mathscr{O}_x) \longleftrightarrow G(F_x)\backslash(G(F_x)/G(\mathscr{O}_x) \times G(F_x)/G(\mathscr{O}_x))$$

sending $[g] \mapsto (1,[g])$. We give an alternative interpretation of the objects here. For $G = \mathrm{GL}_n$, we have $\mathrm{GL}_n(F_x)$ acting transitively on

$$\mathrm{Lat}_x := \{\Lambda \subset F_x^n \mid \Lambda = \text{ free } \mathscr{O}_x\text{-module of rank } n\}.$$

The stabilizers are all conjugate to $\mathrm{GL}_n(\mathscr{O}_x)$, as a "base point" is the standard lattice $\Lambda = \mathscr{O}_x^n$. Thus, we may identify $\mathrm{Lat}_x$ as the homogeneous space $\mathrm{GL}_n(F_x)/\mathrm{GL}_n(\mathscr{O}_x)$. Let's use this to interpret the double coset space as

$$\mathrm{GL}_n(F_x)\backslash \mathrm{Lat}_x \times \mathrm{Lat}_x$$

which one can think of as parametrizing "all relative positions of two lattices."

An easy consequence of the structure theorem for modules over a DVR says that if $\Lambda, \Lambda'$ are two lattices of rank $n$ over a DVR $\mathscr{O}_x$ with uniformizer $\varpi_x$, then there exists an $\mathscr{O}_x$-basis $(e_1,\dots,e_n)$ of $\Lambda$ such that $\Lambda' = \langle \varpi^{d_1}e_1,\dots \varpi^{d_n}e_n\rangle$. Thus, from a pair of lattices we obtain an *unordered* set of $n$ integers, and this completely classifies the "relative position" of $\Lambda$ and $\Lambda'$.

In conclusion, we have identified

$$G(F_x)\backslash \mathrm{Lat}_x \times \mathrm{Lat}_x \longleftrightarrow \mathbb{Z}^n/S_n.$$

Thus we can interpret the local Hecke algebra $\mathscr{H}_x = C(G(\mathscr{O}_x)\backslash G(F_x)/G(\mathscr{O}_x))$ as the set of functions $f\colon \mathrm{Lat}_x \times \mathrm{Lat}_x \to \mathbb{Z}$ which are invariant under $G(F_x)$ and supported on finitely many $G(F_x)$-orbits (this is the translation of the compactly supported condition).

*Exercise* 3.1.1. Check that in these terms, the multiplicative structure is given by

$$(f_1 * f_2)(\Lambda,\Lambda') = \sum_{\Lambda''\in \mathrm{Lat}_x} f_1(\Lambda,\Lambda'')f_2(\Lambda'',\Lambda').$$

Note that this is well-defined because $f_1, f_2$ are compactly supported.

**Lemma 3.1.2.** *$H_x(\mathbb{Z})$ is commutative.*

*Proof.* The key ingredient is "Gelfand's trick," which is similar in a way to Weyl's unitary trick in that it works by introducing auxiliary structure (in Weyl's case, the data of a hermitian form). We equip $F_x^n$ with a symmetric bilinear form $(\cdot,\cdot)$ such that $\mathscr{O}_x^n$ is self-dual.

*Remark* 3.1.3. What does this mean? If $\Lambda \subset F_x^n$ is a lattice, we define the *dual lattice* to be

$$\Lambda^\vee = \{y \in F_x^n : (y, \Lambda) \subset \mathscr{O}_x\}.$$

For instance, we can take the form to be the "standard" one.

Then there is an involution $\sigma \colon \mathrm{Lat}_x \times \mathrm{Lat}_x \to \mathrm{Lat}_x \times \mathrm{Lat}_x$ sending $(\Lambda, \Lambda') \mapsto ((\Lambda')^\vee, \Lambda^\vee)$.

*Exercise* 3.1.4. Check that $(f_1 * f_2) \circ \sigma = (f_2 \circ \sigma) * (f_1 \circ \sigma)$.

On the other hand, you can check by hand that $\sigma$ does not change the relative position of lattices. Therefore, the action of $\sigma$ on $\mathscr{H}_x$ is trivial - but putting this into the identity from Exercise 3.1.4 shows that $f_1$ and $f_2$ commute. $\square$

3.2. **Geometic interpretation of the Hecke action.** What is a "geometric" interpretation of the Hecke action? We say that two vector bundles $V$ and $V'$ "differ only at $x$" if $V|_{X-x} \cong V'|_{X-x}$. In this case, their "diffference" is measured by a pair of lattices $(\Lambda, \Lambda')$ as follows. Choose a trivialization $V|_{\mathrm{Spec}\, F_x} \cong V'|_{\mathrm{Spec}\, F_x} \cong F_x^n$ (since $V$ and $V'$ are isomorphic away from $x$). The restriction of $V$ and $V'$ to Spec $\mathscr{O}_x$ gives a pair of lattices $(\Lambda, \Lambda')$ that can be identified as lying in a common $F_x^n$ by the preceding trivializations. We then define, for $f \in \mathscr{H}_x$,

$$f(V, V') = f(\Lambda, \Lambda').$$

Now we can describe the Hecke action. For $\varphi \in C(\mathrm{Bun}_n(k)) = C(G(F)\backslash G(\mathbb{A})/\prod_x G(\mathscr{O}_x))$ and $f \in \mathscr{H}_x$, we set

$$(f * \varphi)(V) = \sum_{V, V' \text{ differing only at } x} f((V, V'))\varphi(V').$$

3.3. **A presentation for the Hecke algebra.** For $i = 0, 1, \ldots, n$ we denote by $O_i \subset \mathrm{Lat}_n \times \mathrm{Lat}_n$ the subset

$$O_i = \{(\Lambda, \Lambda') \mid \varpi\Lambda \subset \Lambda' \underbrace{\subset}_{\text{colength i}} \Lambda\}$$

*Remark* 3.3.1. Here we are considering pairs where $\Lambda/\Lambda'$ is a $k = \mathscr{O}_x/\varpi_x$-vector space of dimension $i$, so the length is the same as the dimension over $k$. Later, we will consider pairs where the quotient is *not* a $k$-vector space (i.e. $\varpi$ does not act trivially), where it only makes sense to consider the length.

*Exercise* 3.3.2. Show that $O_i$ is the $\mathrm{GL}_n(\mathbb{F}_x)$-orbit of $(\mathscr{O}_x^n, (\varpi\mathscr{O}_x)^i \oplus \mathscr{O}_x^{n-i})$.

By the exercise, $\lambda_i := 1_{O_i} \in \mathscr{H}_x$. The element $\lambda_0 = 1_{\mathrm{diag}(\mathrm{Lat}_n)}$ is the unit of $\mathscr{H}_x$. Also, $O_n = \{(\Lambda, \varpi\Lambda)\}$ so $\lambda_n$ is invertible, with inverse $1_{O_{-n}}$ where $O_{-n} = \{(\varpi\Lambda, \Lambda)\}$. So we have an inclusion $\mathbb{Z}[e_1, \ldots, e_n^{\pm}] \hookrightarrow \mathscr{H}_x$ sending $e_i \mapsto \lambda_i$.

**Theorem 3.3.3.** *This map is a ring isomorphism.*

*Proof.* Define a subalgebra

$$\mathscr{H}_x^+ \subset \mathscr{H}_x$$

as follows. We have a subset

$$(\mathrm{Lat}_n \times \mathrm{Lat}_n)^+ = \{(\Lambda, \Lambda') \mid \Lambda \supset \Lambda'\} \subset (\mathrm{Lat}_n \times \mathrm{Lat}_n)$$

and $\mathscr{H}_x^+$ is defined as the functions supported on $(\mathrm{Lat}_n \times \mathrm{Lat}_n)^+$.

*Exercise* 3.3.4. Check that $\mathscr{H}_x^+$ is a subring.

We will show that
$$\mathbb{Z}[e_1,\dots,e_n]\hookrightarrow\mathscr{H}_n^+$$
sending $e_i\mapsto\lambda_i$ is actually an isomorphism. From this the result follows easily, as any pair of lattices can be brought to an element of $(\mathrm{Lat}_n\times\mathrm{Lat}_n)^+$ by multiplying the second lattice by a sufficiently high power of $\varpi$, which corresponds to inverting $\lambda_n$. To elaborate, note that $\lambda_n^k=\mathbf{1}\{(\varpi^k\Lambda,\Lambda)\}$. Thus if $f$ is any lattice function, then
$$f*\lambda_n^k(\Lambda,\Lambda')=\sum_{\Lambda''}f(\Lambda,\Lambda'')\mathbf{1}\{(\Lambda''=\varpi^k\Lambda')\}=f(\Lambda,\varpi^k\Lambda')$$
and since $f$ has compact support, for large enough $k$ we can guarantee that $\varpi^k\Lambda'\subset\Lambda$ for all of the finitely many pairs $(\Lambda,\Lambda')$ on which $f$ doesn't vanish.

We put a grading on $\mathscr{H}_n^+=\bigoplus_{d\geq 0}\mathscr{H}_n^{+,d}$ where $\mathscr{H}_n^{+,d}$ are the functions supported on the set
$$(\mathrm{Lat}_n\times\mathrm{Lat}_n)^{+,d}=\{(\Lambda,\Lambda')\mid\Lambda\underbrace{\supset}_{\text{length }d}\Lambda'\}$$
Now $\mathbb{Z}[e_1,\dots,e_n]$ also has a grading with $\deg e_i=i$, compatible with the grading on $\mathscr{H}_n^+$. So we only need to show that
$$\phi_d\colon\mathbb{Z}[e_1,\dots,e_n]_d\cong\mathscr{H}_n^{+,d}.$$
The left hand side is a free $\mathbb{Z}$-module of finite rank. In fact, the rank is equal to $p_{\leq n}(d)$, the number of partitions of $d$ with all parts $\leq n$.

On the right hand side, we are considering functions on the set of pairs $(\Lambda,\Lambda')$ such that $\Lambda\supset_d\Lambda'$, up to the action of $\mathrm{GL}_2(\mathbb{F}_x)$. This is precisely the data of the quotient $\Lambda/\Lambda'$, aa torsion $\mathscr{O}_x$ module of length $d$ generated by at most $n$ elements, so by the classification theorem for finitely generated modules over a DVR we have
$$\Lambda/\Lambda'\cong\bigoplus_{i=1}^n\mathscr{O}_x/\varpi_x^{d_i}\qquad\sum_i d_i=d$$
where some of the $d_i$ may be 0. It is easily checked that $\mathrm{GL}_n(F_x)$-orbits on $(\mathrm{Lat}_n\times\mathrm{Lat}_n)^{+,d}$ are in bijection with partitions of $d$ with at most $n$ parts, via the map sending $(\Lambda,\Lambda')$ to the indecomposable factors of $\Lambda/\Lambda'$. The number of orbits is the same as $p_{\leq n}(d)$, by the usual trick of associating the "conjugate" partition (obtained by flipping the diagram representation) which sends partitions with at most $n$ parts bijectively to partitions with parts of size at most $n$.

Since we are considering a map of $\mathbb{Z}$-modules of the same finite rank, we only need to show that $\phi_d$ is *surjective*. Start with a "Jordan type"
$$\bigoplus_i\mathscr{O}_x/\varpi^{d_i}\mathscr{O}_x\text{ such that }\underline{d}=(d_1\geq d_2\geq\ldots\geq d_n\geq 0),\quad\sum d_i=d.$$
Then we get a basis element $1_{\underline{d}}\in\mathscr{H}_n^{+,d}$ which is the function supported on this orbit. We will be done if we can find a monomial $e_1^{k_1}\dots e_n^{k_n}$ whose image under $\phi_d$ is $1_{\underline{d}}+\sum_{\underline{d'}\prec\underline{d}}\alpha_{\underline{d'}}1_{\underline{d'}}$ for some partial order $\prec$, so that the matrix of the map looks "upper-triangular" with ones along the diagonal.

Here we choose the partial order where the smaller partitions are "more generic," e.g. $\underline{d} = (d,0,\dots,0)$ is biggest and $\underline{d} = (1,1,\dots,1)$ is smallest. More precisely, we say that $(d_1,\dots,d_n) \prec (d'_1,\dots,d'_n)$ (with tuples in decreasing order) if

$$\begin{aligned} d_1 &\le d'_1 \\ d_1+d_2 &\le d'_1+d'_2 \\ \vdots &\le \vdots \\ d_1+\dots+d_n &\le d'_1+\dots+d'_n \end{aligned}$$

Another way of saying this is that looking at the affine variety $N_d$ of $d\times d$ nilpotent matrices over $k_x$, the orbits are classified by Jordan types and each partition gives an orbit $O_{\underline{d}}$; we say that $\underline{d} \prec \underline{d}'$ if $\mathscr{O}_d \subset \overline{\mathscr{O}_{d'}}$.

*Example* 3.3.5. For example, every orbit is in the closure of

$$\begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \dots & \ddots & 0 & 1 \\ 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

which corresponds to $(d,0,\dots,0)$, so this is the largest element of the poset. The zero matrix corresponds to the partition $(1,\dots,1)$ and lies is in the closure of every orbit, hence is the smallest element of the poset.

*Example* 3.3.6. If $\underline{d} = (6,4,1)$ then we take the "conjugate partition" $(1,1,2,2,2,3)$ and

$$\phi_d(e_1^2e_2^3e_3^1) = \phi_d(e_1)^2\phi_d(e_2)^3\phi_d(e_3).$$

Evaluating this function at $(\Lambda,\Lambda')$ gives

$$\#\{\Lambda \supset^3 \Lambda_1 \supset^2 \Lambda_2 \supset^2 \Lambda_3 \supset^2 \Lambda_4 \supset^1 \Lambda_5 \supset^1 \Lambda_6 = \Lambda'\}.$$

where the successive quotients have lengths specified by the partition, and the quotients factor through the residue field (i.e. $\varpi_x$ acts trivially on successive quotients).

*Exercise* 3.3.7. Show that if $\Lambda/\Lambda'$ has Jordan type $\underline{d}$, then there is a unique such chain. [Hint: $\Lambda_1$ must correspond to $\ker\varpi$, $\Lambda_2$ to $\ker\varpi^2$, etc.]

*Exercise* 3.3.8. Show that if such a chain exists, then the Jordan type of $\Lambda/\Lambda'$ is $\prec \underline{d}$.

These two claims establish what we wanted.

□

## 4. The Satake Isomorphism

4.1. **The classical Satake isomorphism.** Let $G/k$ be a split reductive group and $\mathscr{H}_{G,x}$ its local Hecke algebra at a closed point $x \in X$. For $G = \mathrm{GL}_n$, we proved a presentation

$$\mathscr{H}_{G,x} \cong \mathbb{Z}[e_1, \dots, e_n^{\pm}]$$

as graded rings, where $\deg e_i = i$. This is not quite phrased in a way conducive to generalization. It is better to view $\mathbb{Z}[e_1, \dots, e_n^{\pm}] \cong \mathbb{Z}[x_1^{\pm}, \dots, x_n^{\pm}]^{S_n}$ sending $e_i$ to the $i$th elementary symmetric polynomial in $x_1, \dots, x_n$. That is the presentation that is easier to generalize.

**Theorem 4.1.1** (Satake)**.** *Let $F_x$ be a local field. Then $\mathscr{H}_{G,x} \cong \mathbb{Z}[X_\bullet(T)]^W$ where $T < G$ is a (split) maximal torus, $X_\bullet(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ is the cocharacter group of $T$, and $W$ is the Weyl group of $G$.*

*Example* 4.1.2. For $G = \mathrm{GL}_n$, one can take $T$ to be the diagonal entries and then the cocharacter lattice consists of maps to each diagonal entry. The Weyl group is $N_G(T)/T \cong S_n$, acting by permutation.

Satake actually constructed an inverse map - see the exercises. Showing surjectivity is similar to what we did: impose a partial order, etc.

Let $H/\mathbb{C}$ be a reductive group. We can consider the category

$$\mathbf{Rep}(H) = \{\text{finite dimensional algebraic representations of } H\}.$$

Here "algebraic representation" just means that it is given by an algebraic morphism $H \to \mathrm{GL}_n$. By Weyl's unitary trick, this category is semisimple (since every representation is determined by its restriction to a maximal compact).

We can form the Grothendieck group

$$R(H) := K_0(\mathbf{Rep}(H)),$$

which by the above remarks is the free abelian group generated by isomorphism classes of irreducible representations. This has a natural ring structure, induced by the tensor product.

4.2. **The Langlands dual group.** Choose a maximal torus $T_H < H$. Then we have a map $R(H) \to R(T_H)$ induced by restriction, the latter being the free $\mathbb{Z}$-module with basis $X^\bullet(T_H)$ where $X^\bullet(T_H) = \mathrm{Hom}(T_H, \mathbb{G}_m)$ is the *character lattice*. Therefore, $R(T_H)$ can also be interpreted as the group ring of the character lattice. So we have a ring homomorphism

$$R(H) \to R(T_H) = \mathbb{Z}[X^\bullet(T_H)].$$

But since the character of any representation is invariant under conjugation, the image lies in $\mathbb{Z}[X^\bullet(T_H)]^W$.

**Theorem 4.2.1** (Classical)**.** *This map induces an isomorphism*

$$R(H) \cong \mathbb{Z}[X^\bullet(T_H)]^W.$$

Comparing this with the Satake isomorphism motivates the Langlands dual group. We want to construct a reductive group $\widehat{G}/\mathbb{C}$ with maximal torus $\widehat{T}$ such that

$$X^\bullet(\widehat{T}) = X_\bullet(T)$$

and $W(\widehat{G}, \widehat{T}) = W(G,T) =: W$. Then we would have a map

$$R(\widehat{G}) \xrightarrow{\sim} \mathbb{Z}[X^\bullet(\widehat{T})]^W \cong Z[X_\bullet(T)]^W \xleftarrow{\sim} \mathscr{H}_{G,x}.$$

Putting the two isomorphisms together, we obtain $\mathscr{H}_{G,x} \cong R(\widehat{G})$.

Now let's relate this back to automorphic forms. We have a (self-adjoint) action of the local Hecke algebra $H_{G,x}$ on $C_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K)$ where $K = \prod_x K_x$ with $K_x = G(\mathscr{O}_x)$. Therefore, the space $C_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K)$ decomposes into a direct sum of eigenspaces under $\mathscr{H}_{G,x}$. The eigenspaces are indexed by eigenvalues, which are ring homomorphisms $\chi \colon \mathscr{H}_{G,x} \to \overline{\mathbb{Q}_\ell}$.

$$C_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K) \cong \bigoplus_{\nu \colon \mathscr{H}_{G,x} \to \overline{\mathbb{Q}_\ell}} C_{\text{cusp}}(\nu).$$

Now, the eigenvalues may be viewed as elements of MaxSpec($\mathscr{H}_{G,x} \otimes_{\mathbb{Z}} \overline{\mathbb{Q}_\ell}$) (since the Hecke algebra is of finite type over $\mathbb{Z}$). By the Satake isomorphism, this is the same as MaxSpec($R(\widehat{G}) \otimes \overline{\mathbb{Q}_\ell}$). So we get a decomposition

$$C_{\text{cusp}}(G(F)\backslash G(\mathbb{A})/K) \cong \bigoplus_{\nu \in \operatorname{MaxSpec}(R(\widehat{G}) \otimes \overline{\mathbb{Q}_\ell})} C_{\text{cusp}}(\nu).$$

Now we assume that $\widehat{G}$ is defined over $\overline{\mathbb{Q}_\ell}$ (we put it over $\mathbb{C}$ before, but that doesn't matter). There is a map $\widehat{G} \to \operatorname{Spec}(R(\widehat{G}) \otimes \overline{\mathbb{Q}_\ell})$ as follows: the image of $g \in \widehat{G}$ is the maximal ideal taking a representation $[V] \mapsto \operatorname{Tr}(g \mid V)$.

*Exercise* 4.2.2. Check that this is a ring homomorphism.

Since this is evidently conjugation-invariant, it descends to a map

$$\widehat{G}//\widehat{G} \to \operatorname{Spec}(R(\widehat{G})_{\overline{\mathbb{Q}_\ell}})$$

where $\widehat{G}//\widehat{G} = \operatorname{Spec}(\mathscr{O}(\widehat{G})^{\widehat{G}})$, the action being conjugation.

**Theorem 4.2.3** (Chevalley)**.** *This map is an isomorphism, i.e.*

$$\mathscr{O}(\widehat{G})^{\widehat{G}} \cong R(\widehat{G})_{\overline{\mathbb{Q}_\ell}}.$$

The map is actually easier to describe on the level of rings, as it takes a representation $[V]$ to its character $\chi_V$.

**Corollary 4.2.4.** *The eigenvalues of $\mathscr{H}_{g,x}$ are in bijection with $\overline{\mathbb{Q}_\ell}$-points of $\widehat{G}//\widehat{G}$, which are (by definition) in bijection with $\widehat{G}(\overline{\mathbb{Q}_\ell})^{ss}/conj$.*

*Example* 4.2.5. If $G = \mathrm{GL}_n$, then the semisimple elements are precisely the diagonalizable elements, and diagonalizable elements up to permutation are just classified by the eigenvalues.

4.3. **An overview of Lafforgue's work.** In the case $G = \mathrm{GL}_n$, the dual group is $\widehat{G} = \mathrm{GL}_n / \overline{\mathbb{Q}_\ell}$. Then we showed that

$$C_{\mathrm{cusp}}(\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A})/K \cdot a^{\mathbb{Z}})$$

is a finite-dimensional space, which has an action of the Hecke algebra $\mathscr{H}_K$. In particular, for each "good" $x \in |X|$ (i.e. $K_x = \mathrm{GL}_n(\mathscr{O}_x)$), we have an action of $\mathscr{H}_{G,x}$ via its inclusion into $\mathscr{H}_K$.

V. Lafforgue's breakthrough is based on constructing a commutative ring $\mathscr{B}$ acting on $C_{\mathrm{cusp}}$ (for fixed level $K$), which induces a decomposition

$$C_{\mathrm{cusp}}(\mathrm{GL}_n(F) \backslash \mathrm{GL}_n(\mathbb{A})/K \cdot a^{\mathbb{Z}}) = \bigoplus_{\nu \in \mathrm{MaxSpec}(B)} C_{\mathrm{cusp}}(\nu).$$

V. Lafforgue then constructs a map from $\mathrm{MaxSpec}(\mathscr{B})$ to $\mathscr{G}_{\widehat{G}}$ (recall that this is the set of maps $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$ up to conjugacy), which is that predicted by the Langlands correspondence.

Part of this correspondence specifies a "compatibility with the Hecke action" in the following sense. Suppose $\nu \in \mathrm{MaxSpec}(\mathscr{B})$ maps back to $\nu_x \in \mathrm{MaxSpec}\mathscr{H}_{G,x}$ and $\rho_\nu \in \mathscr{G}_{\widehat{G}}$. Then $\rho_\nu(\mathrm{Frob}_x)^{ss} \in \widehat{G}^{ss}/\mathrm{conj}$. On the other hand, we also have a map $\mathrm{MaxSpec}(\mathscr{H}_{G,x}) \to \mathrm{MaxSpec}(R(\widehat{G})^{\widehat{G}} = \widehat{G}^{ss}/\mathrm{conj}$, and the compatibility says that the diagram commutes:

$$\begin{tikzcd}
\nu \arrow[r] \arrow[d] & \nu_x \in \mathrm{MaxSpec}(\mathscr{H}_{G,x}) \arrow[d] \\
\rho_\nu \arrow[r] & \rho_\nu(\mathrm{Frob}_x)^{ss} \in \widehat{G}^{ss}/\mathrm{conj}
\end{tikzcd}$$

*Remark* 4.3.1. Note that the commutativity of $\mathscr{G}_{G,x}$, which acts both through $\mathscr{B}$ and $\mathscr{H}_K$, is necessary for these actions to be compatible.

Let's restrict our attention to the case $G = \mathrm{GL}_n$ to see more explicitly what this says. Denote $\mathscr{H}_x = \mathscr{H}_{\mathrm{GL}_n,x}$. Then associated to the $\nu$ eigenspace is a homomorphism $\nu_x : \mathscr{H}_x \to \overline{\mathbb{Q}_\ell}$, and we saw that $\mathscr{H}_x \cong \overline{\mathbb{Q}_\ell}[e_1, e_2, \ldots, e_n^{\pm}]$, so $\nu_x$ is specified by the data of $\{a_{i,x}\}$ such that $e_i \mapsto a_{i,x} \in \overline{\mathbb{Q}_\ell}$.

Also associated to $\nu$ is a representation $\rho_\nu \in \mathscr{G}_{\mathrm{GL}_n}$. Then element $\mathrm{Frob}_x$ maps to $\rho(\mathrm{Frob}_x) = \sigma_x = \sigma_x^{ss}\sigma_x^u$. The semi-simple part is well-defined up to conjugacy, i.e. the data of the characteristic polynomial of $\sigma_x^{ss}$, and the compatibility with the Satake parameters says that this characteristic polynomial is precisely

$$T^n - a_{1,x}T^{n-1} + a_{2,x}T^{n-2} + \ldots + (-1)^n a_{n,x}.$$

## 5. Moduli of Vector Bundles

### 5.1. Construction of $\mathrm{Bun}_n^d$.

Fix $n \geq 1, d \in \mathbb{Z}$. We have a moduli functor $\mathbf{Sch}/k \to \mathbf{Groupoids}$ sending

$$S \mapsto \left\{ \substack{\text{rank } n \text{ vec. bun. over } X \times_k S \\ \text{with fiberwise degree } d} \right\}.$$

**Theorem 5.1.1.** *This functor is represented by an algebraic stack* $\mathrm{Bun}_n^d$*, which is locally of finite type.*

*Proof.* This is Laumon-Moret-Bailly, Theorem 4.6.2.1. It is relatively easy to show that it is a stack - that is just a question about gluing vector bundles. The algebraicity is the hard part: you need to construct a morphism from a scheme of finite type. The idea is to use the construction of Quot schemes. If you twist a vector bundle by a high enough power of an ample bundle, then it will be generated by global sections, hence can be presented as a quotient of $\mathscr{O}^N$. One needs to have a uniformity result saying, and then one can classify this as quotients $\mathscr{O}^N \twoheadrightarrow \mathscr{V}(n)$. □

*Remark* 5.1.2. The automorphism groups of $\mathscr{V} \in \mathrm{Bun}_n^d$ are not finite over $k$ in general. For instance, they trivially contain $\mathbb{G}_m$. Even after modding out by this action, they may not be finite (for instance, the automorphisms of the trivial bundle are all of $\mathrm{GL}_n$). So $\mathrm{Bun}_n^d$ is *not* a Deligne-Mumford stack.

*Example* 5.1.3. $\mathrm{Bun}_n^d$ is only locally of finite type (not necessarily globally quasicompact). If $X = \mathbb{P}^1$,

$$\mathrm{Bun}_n^d/\mathrm{iso} = \{\mathscr{O}(d_1) \oplus \ldots \oplus \mathscr{O}(d_n) \mid \sum d_i = d\} \longleftrightarrow \mathbb{Z}^n/S_n.$$

So even if $k$ is a finite point, there are infinitely many points on $\mathrm{Bun}_n^d$, hence it is not of finite type.

One can put a partial order on the points via the Zariski topology on $\mathrm{Bun}_n^d(k)$. As an example, we consider the case $d = 0$ and $n = 2$. Since every vector bundle on $\mathbb{P}^1$ is a sum of line bundles, we have

$$\mathrm{Bun}_n^0(k) = \{\mathscr{O}^2, \mathscr{O}(1) + \mathscr{O}(-1), \mathscr{O}(2) \oplus \mathscr{O}(-2), \ldots \mathscr{O}(n) \oplus \mathscr{O}(-n), \ldots\}.$$

The trivial bundle is not only generic, it is open. Down the sequence, the points get "more and more closed." What do the automorphism groups look like? We have $\mathrm{Aut}(\mathscr{O}^2) = \mathrm{GL}_2$, while $\mathrm{Aut}(\mathscr{O}(1) \oplus \mathscr{O}(-1))$ is upper triangular since there are no morphisms from $\mathscr{O}(1)$ to $\mathscr{O}(-1)$, but the morphisms in the other direction form an $\mathscr{O}(2)$, so the automorphism group has dimension 5 in total. It is easy to see that the automorphism groups get larger and larger.

$\mathrm{Bun}_n^d$ is not of finite type, as for instance it has an infinite stratification by Harder-Naransimhan polynomials. In general,

$$\mathrm{Bun}_n^d = \bigcup_{\mathscr{P} \in \text{HN polygon}} \mathrm{Bun}_n^{\mathscr{P}}$$

and $\mathrm{Bun}_n^{\mathscr{P}}$ is a locally closed substack of $\mathrm{Bun}_n^d$ of *finite type* over $k$. Under specialization $\eta \rightsquigarrow s$, $\mathscr{P}_s$ lies above $\mathscr{P}_\eta$ (the HN polygon goes up under specialization).

*Exercise* 5.1.4. Check this for $\mathbb{P}^1$. Each of the bundles has a different HN polygon, which looks like a sequencing of triangles of increasing height.

The point is that given a polygon, we get a fixed lower bound on the slope of a filtration, which gives a uniform bound in terms of some Quot scheme.

**Theorem 5.1.5.** $\mathrm{Bun}_n^d$ *is smooth over* $k$.

*Proof.* One calculates that the obstruction to infinitesimally deforming $\mathscr{V}$ lies in $H^2(X, \underline{\mathrm{End}}(\mathscr{V}))$, which vanishes because it's the second cohomology group on a curve.

One needs to make some local calculations to convince oneself that this space really does control the deformations of the curve. In these calculations, one sees that the tangent complex at $\mathscr{V}$ is given by $R\Gamma(X, \underline{\mathrm{End}}(\mathscr{V}))[1]$. This has cohomology in two degrees: in degree $-1$, it is $H^0(X, \underline{\mathrm{End}}(\mathscr{V})) = \mathrm{End}(\mathscr{V}) = \mathrm{Lie}(\mathrm{Aut}(\mathscr{V}))$, and in degree 0 it is $H^1(X, \underline{\mathrm{End}}(\mathscr{V}))$ which is the tangent space at $\mathscr{V}$. $\square$

For general $G$, $\mathrm{Bun}_G$ is an algebraic stack locally of finite type, smooth over $k$. If $\mathscr{E} \in \mathrm{Bun}_G$, we can consider $H^*(X, \mathrm{Ad}(\mathscr{E}))$ where $\mathrm{Ad}(\mathscr{E}) = \mathscr{E} \times \mathfrak{g}/G$ is the vector bundle over $X$ of rank $\dim \mathfrak{g}$ associated to the principal $G$-bundle $\mathscr{E}$. The obstructions to infinitesimal deformations will be an $H^2$ group, which again vanishes for dimension reasons as $X$ is a curve, verifying smoothness. As before, one can calculate that the relevant $H^1$ is the tangent space, and the $H^0$ is the Lie algebra of the automorphism group.

5.2. **Local Hecke correspondences.** Let $\mathrm{Bun}_n = \coprod_{d \in \mathbb{Z}} \mathrm{Bun}_n^d$. Fix $x \in X(k)$. The basic construction is a correspondence

$$\begin{tikzcd} & \mathscr{H}_x^{(i)} \arrow[dl, "p"'] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

where

$$\mathscr{H}_x^{(i)} = \{(\mathscr{V} \supset_i \mathscr{V}') \mid \mathscr{V}(-x) \subset \mathscr{V}' \subset \mathscr{V}\}.$$

Note that the condition says that $V, V'$ are the same away from $x$ and $\mathfrak{m}_x$ acts trivially on the quotient $\mathscr{V}/\mathscr{V}'$. Now, $p^{-1}(\mathscr{V})$ classifies sub-bundles of $\mathscr{V}$ such that $\mathscr{V}_x/\mathscr{V}'_x$ is an $i$-dimensional quotient of $\mathscr{V} \otimes k_x$, so the fibers are $\mathrm{Gr}(i, n)$. That shows that $\mathscr{H}_x^{(i)}$ is also an algebraic stack.

This can be generalized in several ways. For instance, one can allow "deeper" modifications at a point, by dropping the requirement $\mathscr{V}' \supset \mathscr{V}(-x)$. This condition can be rephrased as saying that $\varpi_x$ acting trivially on the quotient, so to generalize it, fix $\underline{d} = (d_1 \geq d_2 \geq \ldots \geq d_n \geq 0)$ and consider

$$\mathscr{H}_x^{\underline{d}} = \left\{ (\mathscr{V} \supset \mathscr{V}') \mid {\scriptstyle \varpi_x \text{ acts on } \mathscr{V}/\mathscr{V}' \atop \text{with Jordan type } \prec \underline{d}} \right\}$$

with the same partial order on Jordan types that we defined earlier. This means that

$$\mathscr{V}/\mathscr{V}' \cong \bigoplus_i \mathscr{O}_x / \varpi_x^{d_i} \mathscr{O}_x$$

and $\underline{d} \succ \underline{d}'$ if

$$\begin{aligned} d_1 &\le d_1' \\ d_1 + d_2 &\le d_1' + d_2' \\ \vdots &\le \vdots \\ d_1 + \ldots + d_n &\le d_1' + \ldots + d_n' \end{aligned}$$

*Example* 5.2.1. A single Jordan block corresponds to $\underline{d} = (d, 0, \ldots, 0)$ is the most generic (biggest in the partial order). In this case, $\mathscr{H}_x^{\underline{d}} = \{(V \supset_d \mathscr{V}')\}$, with no extra conditions on the action of $\varpi$.

Even more generally, we do not need to restrict ourselves to considering sub-bundles. Let $\underline{d} = (d_1 \ge d_2 \ge \ldots \ge d_n)$ where we do *not* require that $d_n \ge 0$. For some $N \gg 0$, $\underline{d} + N \ge 0$ and we define

$$\mathscr{H}_x^{\underline{d}} = \left\{ (\mathscr{V}, \mathscr{V}') \mid \substack{\mathscr{V}' \subset \mathscr{V}(Nx) \text{ vec. bun. on } X \times_k S \\ \mathscr{V}(Nx)/\mathscr{V}' \text{ has Jordan type } \prec \underline{d} + N} \right\}.$$

You can prove that this is an algebraic stack by studying the fibers over $\mathrm{Bun}_n$. Identifying $\mathscr{V}_x \cong \mathscr{O}_x^n$, the fiber over $\mathscr{V}$ is

$$p^{-1}(\mathscr{V}) = \left\{ \Lambda' \subset (\varpi_x^{-N} \mathscr{O}_x)^n \mid \mathscr{O}_x^n / \Lambda' \text{ Jordan type } \prec \underline{d} + N \right\} =: \mathrm{Gr}_{\prec d}.$$

More precisely, we also pick $M \gg 0$ and consider

$$\left\{ \Lambda' \mid (\varpi_x^M \mathscr{O}_x)^n \subset \Lambda' \subset_{\prec \underline{d} + N} (\varpi_x^{-N} \mathscr{O}_x)^n \right\} =: \mathrm{Gr}_{\prec \underline{d}}^{(M,N)}.$$

This is a projective scheme over $k_x$, and it is "independent of $M, N$" in the sense that the closed embedding

$$\mathrm{Gr}_{\prec \underline{d}}^{(M,N)} \hookrightarrow \mathrm{Gr}_{\prec \underline{d}}^{(M',N')}$$

existing if $(M, N) \le (M', N')$ is a bijection on field-valued points. That implies that there is a well-defined reduced structure obtained by picking any large enough $M, N$.

*Example* 5.2.2. If $n = 1$, $\underline{d} = (0)$, then there is only one underlying point $\Lambda' = \mathscr{O}_x$. If $M = N = 1$, then we seek to classify

$$\{\varpi_x \mathscr{O}_x \subset \Lambda' \subset \varpi_x^{-1} \mathscr{O}_x \mid \varpi_x^{-1} \mathscr{O}_x \supset^1 \Lambda'\}.$$

Identifying $\mathscr{O}_x \cong k_x[[\varpi_x]]$, $\Lambda'/\varpi_x \mathscr{O}_x$ is a line in the quotient $k[[\varpi_x]]/\varpi_x^2$. Thus it can be viewed as a point of $\mathbb{P}^1$. The fact that this line is invariant under multiplication by $\varpi_x$ implies that it must be equal to the span of $\varpi_x$.

However, the scheme is non-reduced in this case (it turns out to be the first-order neighborhood of the point). To see this, note that we can describe the line as the span of $a + b\varpi_x$ for some $(a, b) \neq 0, 0$. The condition that this line is stable under multiplication by $\varpi_x$ implies that $\varpi_x(a + b\varpi_x) = a\varpi_x \in \mathrm{Span}(a + b\varpi_x)$, i.e. the matrix

$$\begin{pmatrix} a+b & \\ & a \end{pmatrix} \text{ is singular.}$$

That cuts out $a^2 = 0$, so the subscheme in question is $\mathrm{Proj}\, k[a, b]/a^2 = 0$, which is a double point.

If $\underline{d} \prec \underline{d}'$, then we have an obvious embedding $\mathrm{Gr}_{\prec \underline{d}} \hookrightarrow \mathrm{Gr}_{\prec \underline{d}'}$. Denoting $|\underline{d}| = \sum d_i$, we can form

$$\mathrm{Gr}_n^d = \varinjlim_{|\underline{d}|=d} \mathrm{Gr}_{\prec \underline{d}}$$

and

$$\mathrm{Gr}_n^d(k_x) = \mathrm{Lat}_n^d = \{\Lambda' \subset F_x^n \mid [\mathscr{O}_x^n : \Lambda'] = d\}$$

Here the index is defined in the only reasonable way: $\Lambda'$ is not necessarily a sublattice of $\mathscr{O}_x^n$, but they are both sublattices of some common lattices, and one transfers the notion of index via this common lattice. This is a component of the *affine Grassmannian* for $\mathrm{GL}_n$. The full affine Grassmannian for $\mathrm{GL}_n$ is

$$\mathrm{Gr}_n = \coprod_{d \in \mathbb{Z}} \mathrm{Gr}_n^d,$$

which is an inductive limit of reduced projective schemes over $k_x$.

There is a more intrinsic construction of the affine Grassmannian, which can also be stated in greater generality. Let $G$ a group scheme over $k_x$. We can consider the functor

$$S \mapsto G(\mathscr{O}_S((t)))/G(\mathscr{O}_S[[t]])$$

from affine schemes over $k_x$ to sets. After sheafifying this in the faithfully flat topology, one obtains a functor Gr. Modulo nilpotents, this agrees with the previous definition. However, in practice one never works with this formulation.

5.3. **Global Hecke correspondences.** We just defined a "local" Hecke correspondence

$$\begin{tikzcd} & \mathscr{H}_x^{\prec \underline{d}} \arrow[dl] \arrow[dr] & \\ \mathrm{Bun}_n & & \mathrm{Bun}_n \end{tikzcd}$$

where $\mathscr{H}_x^{\prec \underline{d}}$ consists of pairs $(V, V')$ where $V$ differs from $V'$ at $x$, with relative position $\prec \underline{d}$.

We then defined

$$\mathscr{H}_x^d = \varinjlim_{|\underline{d}|=d} \mathscr{H}_x^{\prec \underline{d}}$$

which is a "global" (on $\mathrm{Bun}_n$) analogue of the Grassmannian $\mathrm{Gr}_n^d(k_x)$. In particular, the fiber over the trivial bundle $\mathscr{O}_x^n$ is precisely $\mathrm{Gr}_n^d(k_x)$.

**Globalizing.** Next we want to set up a *global* (on $X$) version of this problem. This should yield a correspondence $\mathscr{H}^{\prec \underline{d}} \to X$ whose fiber over $x$ is precisely $\mathscr{H}_x^{\prec \underline{d}}$.

So let's set up the following moduli problem. Define a functor $\mathscr{H}^{\prec \underline{d}}$ to be, roughly speaking, the groupoid of pairs of bundles which are modifications of each other along a *section* of $S \times X \to S$ (we had previous considered modifications at a single point of $x$). More formally, $\mathscr{H}^{\prec \underline{d}}(S)$ is the groupoid

$$\mathscr{H}^{\prec \underline{d}}(S) = \{(\xi : S \to X, \mathscr{V}, \mathscr{V}') \mid (*)\}$$

where

- $\xi \colon S \to X$ is a map such that $\Gamma_\xi \colon S \hookrightarrow X \times S$ (the graph of $\xi$) is a divisor, and
- $\mathscr{V}, \mathscr{V}'$ are vector bundles over $X \times S$ with an isomorphism $\tau \colon \mathscr{V}|_{X\times S-\Gamma_\xi} \cong \mathscr{V}'|_{X\times S-\Gamma_\xi}$ and satisfying the conditions:
  (1) $\mathscr{V}(-d_1\Gamma_\xi) \subset \mathscr{V}' \subset \mathscr{V}(-d_n\Gamma_\xi)$,
  (2) $\bigwedge^2 \mathscr{V}(-(d_1+d_2)\Gamma_\xi) \subset \bigwedge^2 \mathscr{V}' \subset \bigwedge^2 \mathscr{V}(-(d_{n-1}+d_n)\Gamma_\xi)$
  (3) etc.

  Here we think of $\tau$ as inducing a rational map $\mathscr{V}' \dashrightarrow \mathscr{V}$, which induces a rational map on the exterior powers (so that the conditions actually make sense).

*Remark* 5.3.1. In this moduli problem $\mathscr{V}'$ is a sub-bundle of $\mathscr{V}(-d_n\Gamma_\xi)$ - think of $\mathscr{V}$ as part of the data, and $\mathscr{V}'$ as a choice of sub-bundle (so the data of the inclusion is given).

The condition is precisely the globalization of the condition that $(\Lambda, \Lambda') = (\mathscr{V}_x, \mathscr{V}'_x)$ lie in $\mathrm{Gr}_n^{\prec \underline{d}}$, i.e. the elementary divisors of $\varpi$ on $\Lambda/\Lambda'$ are $(\varpi^{d_1}, \dots, \varpi^{d_n})$.

So we can again form a direct limit $\mathscr{H}^e := \varinjlim_{|\underline{d}|=e} \mathscr{H}^{\underline{d}}$, which is concretely described as

$$\mathscr{H}^e(S) = \left\{ (\xi, \mathscr{V}, \mathscr{V}') \mid \tau \colon \mathscr{V}|_{X\times S-\Gamma_\xi} \cong \mathscr{V}'|_{X\times S-\Gamma_\xi}, \deg \mathscr{V} - \deg \mathscr{V}' = e \right\}.$$

**Another variant.** We can modify the bundle at several sections rather than one. That's the correspondence obtained by stringing together two of these Hecke correspondences:

$$\begin{tikzcd}
& & {}^2\mathscr{H} \arrow[dl] \arrow[dr] & & \\
& \mathscr{H} \arrow[dl] \arrow[dr] & & \mathscr{H} \arrow[dl] \arrow[dr] & \\
\mathrm{Bun}_n & & \mathrm{Bun}_n & & \mathrm{Bun}_n
\end{tikzcd}$$

Here ${}^2\mathscr{H}$ parametrizes

$$\{(\xi_1, \xi_2, \mathscr{V}_1, \mathscr{V}_2, \mathscr{V}_3 \mid (\mathscr{V}_1)_{X\times S-\Gamma_{\xi_1}} \to (\mathscr{V}_2)|_{X\times S-\Gamma_{\xi_2}} \to (\mathscr{V}_3)|_{X\times S-\Gamma_{\xi_3}}\}.$$

This maps to the two Hecke correspondences from before, by forgetting the first or third bundles. If you forget the middle bundle, then you get a map to the tuples parametrizing vector bundles modified along *two* sections.

## 6. Moduli of Shtukas

### 6.1. **The definition of Shtuka.**

*Definition* 6.1.1. Define the following moduli problem Shtuka: $\mathbf{Sch}/k \to \mathbf{Grpd}$. To a scheme $S/k$ we associate the tuples

$$(\xi_0, \xi_\infty, \alpha\colon \mathscr{V}_0 \hookrightarrow \mathscr{V}_1 \hookleftarrow {}^\tau\mathscr{V}_0 \colon \beta)$$

where:

- $\xi_0, \xi_\infty$ are maps $S \to X$,
- $\mathscr{V}_0, \mathscr{V}_1$ are vector bundles over $X \times S$,
- ${}^\tau\mathscr{V}_0 = (\mathrm{id}_X \times \mathrm{Frob}_{S/k})^* \mathscr{V}_0$.
- $\mathscr{V}_1/\alpha(\mathscr{V}_0)$ is supported scheme-theoretically along $\Gamma_{\xi_0}$ and is locally free of rank 1 as an $\mathscr{O}_S$-module (since $\Gamma_{\xi_0} \cong S$), and similarly for $\mathscr{V}_1/\beta({}^\tau\mathscr{V}_0)$.

Intuitively, $\alpha \circ \beta^{-1}$ defines a "rational isomorphism" between $\mathscr{V}_0$ and ${}^\tau\mathscr{V}_0$ with zeros and poles along $\Gamma_{\xi_0}$ and $\Gamma_{\xi_\infty}$.

This is similar to the Hecke correspondences that we defined in the last section. Consider, in the notation that we used before, the moduli stack

$${}^2\mathscr{H}^{(0,\dots,0,-1),(1,0,\dots,0)} = \{(\xi_1, \xi_2, V_0 \hookrightarrow V_1 \hookleftarrow V_2) \mid (*)\}$$

where the technical conditions amount to saying that $V_1/V_0$ supported on $\Gamma_{\xi_1}$, and $V_1/V_2$ is supported on $\Gamma_{\xi_2}$. More precisely, $V_1/V_0$ restricts to a degree 1 line bundle on $\Gamma_{\xi_1}$ This maps to $X^2$ by forgetting the bundle data, and to $\mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n$ by forgetting the bundle inclusions and $\xi_i$.

$$\begin{tikzcd} {}^2\mathscr{H}^{(0,\dots,0,-1),(1,0,\dots,0)} \arrow[r] \arrow[d] & X^2 \\ \mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n & \end{tikzcd}$$

The base space $\mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n$ in turn admits a map from $\mathrm{Bun}_n \times \mathrm{Bun}_n$ via

$$(\mathscr{V}_0, \mathscr{V}_1) \mapsto (\mathscr{V}_0, \mathscr{V}_1, \mathrm{Frob}_{\mathrm{Bun}_n/k}(\mathscr{V}_0)).$$

**Proposition 6.1.2.** *We have the pullback diagram*

$$\begin{tikzcd} \mathrm{Shtuka} \arrow[r] \arrow[d] & {}^2\mathscr{H}^{(0,\dots,0,-1),(1,0,\dots,0)} \arrow[d] \\ \mathrm{Bun}_n \times \mathrm{Bun}_n \arrow[r] & \mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n . \end{tikzcd}$$

*Proof.* The only content here is that $\mathrm{Frob}_{\mathrm{Bun}_n/k}(\mathscr{V}_0) \cong (\mathrm{Id}_X \times \mathrm{Frob}_{S/k})^*.\mathscr{V}_0$ This is actually tautological, after one establishes a non-trivial definition of Frobenius for stacks. We can define $\tau\colon \mathscr{X}(S) \to \mathscr{X}(S)$ induced by the (relative) $\mathrm{Frob}_{S/k}$. We then have to check that this coincides with the the usual definition when $\mathscr{X}$ is the stack represented by a scheme. $\square$

One could more generally try to define

$$\begin{tikzcd} \mathrm{Shtuka}^{\underline{d}} \arrow[r] \arrow[d] & {}^2\mathscr{H}^{\underline{d},\underline{d}'} \arrow[d] \\ \mathrm{Bun}_n \times \mathrm{Bun}_n \arrow[r] & \mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n . \end{tikzcd}$$

However, we claim that $\mathrm{Shtuka}^{\underline{d},\underline{d}'}$ is empty unless $|\underline{d}+\underline{d}'| =: \sum d_i + d_i' = 0$. Indeed, $|\underline{d}|+|\underline{d}'|$ is the *difference* of the degrees of $\mathscr{V}$ and ${}^{\tau}\mathscr{V}$, but that is 0 because the relative Frobenius doesn't change the degree.

6.2. **A special case.** In this case we undertake an extensive study of a special case of the construction where $\underline{d} = \underline{d}' = (0,\dots,0)$, i.e.

$$\mathrm{Shtuka}^{(0,\dots,0)}(S) = \{(\xi \colon S \to X, \mathscr{V} \cong {}^{\tau}\mathscr{V})\}.$$

Now the isomorphism is honest over all of $X \times S$, so $\xi$ is extraneous. To focus on the data that we're interested in, define

$$\boxed{\mathrm{Shtuka}_{\emptyset}(S) = \{\mathscr{V} \text{ vector bundle on } X \times S, \alpha \colon \mathscr{V} \cong^{\tau} \mathscr{V}\}.}$$

So $\mathrm{Shtuka}^{(0,\dots,0)}(S)$ is just $\mathrm{Shtuka}_{\emptyset}^{(0,\dots,0)}(S) \times \mathrm{Hom}(S,X)$. If $S = \mathrm{Spec}\, \overline{k}$, then

$$\mathrm{Shtuka}_{\emptyset}(\overline{k}) = \{\mathscr{V} \text{ on } X_{\overline{k}} \text{ plus descent datum for } \mathrm{Gal}(\overline{k}/k)\}.$$

This is just the same as a vector bundle on $X$.

*Remark* 6.2.1. This equivalence here is slightly non-trivial, and depends on $k$ being a *finite* field. Indeed, we know that $\mathscr{V}$ will be defined over some finite extension $k'/k$, say of degree $n$. Then we are given $\alpha \colon \mathscr{V} \cong {}^{\tau}\mathscr{V}$. However, the cocycle condition that we need is that $\alpha^n \colon \mathscr{V} \cong {}^{\tau^n}\mathscr{V} = \mathscr{V}$ is not any automorphism but the *identity* automorphism.

Over a finite field one can arrange this by exponentiating $\alpha$, since $\mathrm{GL}_n(k)$ is finite. In more general circumstances, the equivalence simply need not hold.

6.3. **Crystals.** We first study a simpler problem that looks like a "shtuka over a point." Suppose $S/k = \mathbb{F}_q$ is an affine scheme.

*Definition* 6.3.1. A *unit root $F$-crystal* over $S$ is a pair $(M,\varphi)$, where $M$ is a vector bundle over $S$ and $\varphi \colon M \cong \mathrm{Frob}_{S/k}^* M = \mathscr{O}_S \otimes_{(\mathscr{O}_S,\mathrm{Frob})} M$.

As $\mathrm{Shtuka}_{\emptyset}(S)$ parametrizes vector bundles over $X \times S$ and an isomorphism with the Frobenius twist over $S$, we can think of unit root $F$-crystals as the fiber of such a datum over a closed point $x \in |X|$.

*Remark* 6.3.2. The pair $(M,\varphi)$ was traditionally denoted $(M,F)$, which explains the terminology.

The $F$-unit root crystals form a $k$-linear tensor category, as there is an obvious notion of sum and tensor product.

*Remark* 6.3.3. Since the category consists of $\mathcal{O}_S$-modules, one might think that it should even be an $\mathcal{O}_S$-linear tensor category. However, multiplication by $f \in \mathcal{O}_S$ only induces a commutative diagram

$$\begin{tikzcd} M \arrow[r] \arrow[d, "\varphi_M"'] & N \arrow[d, "\varphi_N"'] \\ \mathrm{Frob}_{S/k}^* M \arrow[r, "\mathrm{Frob}_{S/k}^* \varphi"'] & \mathrm{Frob}_{S/k}^* N \end{tikzcd}$$

when $f^q = f$, which is equivalent to $f \in k$.

In particular, the "obvious" map $M \xrightarrow{\iota} \mathcal{O}_S \otimes_{\mathcal{O}_S} M$ by $m \mapsto 1 \otimes m$ is not $(\mathcal{O}_S, \mathrm{Frob})$-linear, as $fm \mapsto 1 \otimes fm = f^q \otimes m$. However, the map $\varphi \colon M \to \mathcal{O}_S \otimes_{\mathcal{O}_S} M$ is $\mathcal{O}_S$-linear by definition.

**Theorem 6.3.4** (Katz)**.** *There is an equivalence of categories*

$$\left\{ \begin{matrix} \text{unit root} \\ F\text{-crystals} \end{matrix} \right\} \longleftrightarrow \left\{ \begin{matrix} \text{étale } \mathbb{F}_q\text{-local systems} \\ \text{over } S \text{ (finite rank)} \end{matrix} \right\}$$

*Remark* 6.3.5. The "éspace étale" of an étale $\mathbb{F}_q$-local system can be viewed as an "$\mathbb{F}_q$-vector bundle" over $S$ (that is, the fibers are $\mathbb{F}_q$ vector spaces), with the local system being its sheaf of sections.

*Proof.* We construct a natural map

$$\left\{ \begin{matrix} \text{unit root} \\ F\text{-crystals} \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{étale } \mathbb{F}_q\text{-local systems} \\ \text{over } S \text{ (finite rank)} \end{matrix} \right\}.$$

Let $(M, \varphi)$ be a unit root $F$-crystal and let $E$ be the total space of $M$. We can view $E$ as an additive group scheme over $S$. The relative Frobenius $\mathrm{Frob}_{E/S}$ gives a map $E \to \mathrm{Frob}_{S/k}^* E$.

$$\begin{tikzcd} E \arrow[r, "\mathrm{Frob}_{E/S}"] \arrow[dr] & \mathrm{Frob}_{S/k}^* E \arrow[r] \arrow[d] & E \arrow[d] \\ & S \arrow[r, "\mathrm{Frob}_{S/k}"'] & S \end{tikzcd}$$

On the other hand, $\varphi$ gives another map $E \to \mathrm{Frob}_{S/k}^* E$. We set $G(M, \varphi)$ to be the equalizer of the diagram

$$E \rightrightarrows \mathrm{Frob}_{S/k}^* E$$

i.e.

$$G(M, \varphi) := \ker(M \xrightarrow{\varphi - \mathrm{Frob}_{E/S}} \mathrm{Frob}^* M) \subset M.$$

*Example* 6.3.6. If $S = \mathrm{Spec}\, A$ and $M = A^n$, then $\varphi$ is specified by an $n \times n$ matrix $(a_{ij})$. So $\varphi - \mathrm{Frob}_{E/S}$ sends

$$(x_1, \ldots, x_n) \mapsto (\sum a_{1i} x_i - x_1^q, \sum a_{2i} x_i - x_2^q, \ldots).$$

Over $\overline{k}$ the kernel has size $q^n$, and $G(M, \varphi)$ is a finite étale group scheme over $\overline{k}$.

In general, one sees by a local calculation that $G(M, \varphi)$ is finite over $S$ with order $q^n$, where $n = \operatorname{rank} M$: it is of the form

$$\mathcal{O}_S[x_1, \ldots, x_n]/(x_1^q - \text{linear terms}, x_2^q - \text{linear terms}, \ldots).$$

The Jacobian criterion shows that $G(M,\varphi)$ is étale over $S$.

Here's a more geometric reformulation of the étaleness. We can realize $G(M,\varphi)$ from the following pullback diagram

$$\begin{tikzcd} G(M,\varphi) \arrow[r] \arrow[d] & E \arrow[d, "(\mathrm{Id},\varphi)"] \\ E \arrow[r, "(\mathrm{Id},\mathrm{Frob}_{E/S})"'] & E \times_S \mathrm{Frob}^*_{S/k} E \end{tikzcd}$$

Then one checks that the graph of $\varphi$ is transverse to the graph of $\mathrm{Frob}_{E/S}$. Of course, this is essentially the same calculation as we performed above, but this geometric viewpoint of intersecting graphs with correspondences is a useful theme that we'll encounter repeatedly.

We claim that the quasi-inverse functor is $G \mapsto G \otimes_{\mathbb{F}_q} \mathscr{O}_S$. This is an étale vector bundle, so by descent of vector bundles it is even a Zariski vector bundle. The natural inclusion $G(M,\varphi) \hookrightarrow M$ induces a map $G(M,\varphi) \otimes_{\mathbb{F}_q} \mathscr{O}_S \to M$. One has to check that

$$(G(M,\varphi) \otimes_{\mathbb{F}_q} \mathscr{O}_S, \mathrm{Id} \otimes \mathrm{Frob}_{S/k}) \to (M,\varphi)$$

is an isomorphism (it is then easy to check that it's a left adjoint). For this, we can pass to geometric points and therefore assume that $S = \mathrm{Spec}\, \overline{k}$. As we saw above, if $\mathrm{rank}\, M = n$ then $G(M,\varphi)$ has order $q^n$ over $\overline{k}$, hence corresponds to an $n$-dimensional $\mathbb{F}_q$-vector space. Therefore, $G(M,\varphi) \otimes_{\mathbb{F}_q} \mathscr{O}_S$ is an $n$-dimensional vector space over $\overline{k}$, so it suffices to show that it *injects* into $M$.

Pick a basis $(x_1,\ldots,x_n)$ for $G(M,\varphi)$ over $k$. It suffices to show that it remains independent over $\overline{k}$. If not, then we may choose a linear relation

$$\sum a_i x_i = 0 \quad a_i \in \overline{k}$$

and assume without loss of generality that $a_1 = 1$. Applying $\varphi^{-1} \circ \mathrm{Frob}_{E/S}$ to this relation, we obtain that

$$x_1 + \sum \varphi^{-1}(a_i) x_i = 0.$$

We can then subtract these to obtain a smaller independence relation, contradicting the minimality unless $\varphi(a_i) = a_i$ for each $i$. But then that contradicts the choice of $x_1,\ldots,x_n$ as independent over $k$.

For the other inverse, we have

$$G \rightsquigarrow (G \otimes_{\mathbb{F}_q} \mathscr{O}_S, \mathrm{Id} \otimes \mathrm{Frob}_{S/k}) = (M,\varphi) \rightsquigarrow G(M,\varphi) \supset G$$

but this inclusion must be an equality because $G(M,\varphi)$ and $G$ both have order $q^n$ over $\overline{k}$. $\square$

6.4. **A global version.** We now apply the theory just discussed to study the $\mathrm{Shtuka}_\emptyset$. The result is essentially that $\mathrm{Shtuka}_\emptyset \cong \mathrm{Bun}$, with the latter interpreted as a stack as

$$\mathrm{Bun} = \coprod_{\mathscr{F} \in \mathrm{Vec}(X)/\mathrm{iso}} [\mathrm{Spec}\, k / \mathrm{Aut}(\mathscr{F})].$$

**Theorem 6.4.1.** *Let $X/k = \mathbb{F}_q$ be a projective scheme and $S/k$ any scheme. Then there is an equivalence of categories*

$$\left\{(\mathscr{F},\varphi) \mid {}^{\mathscr{F}\in Coh(X\times_k S),\, flat}_{\varphi:\mathscr{F}\cong(\mathrm{Id}_X\times\mathrm{Frob}_{S/k})^*\mathscr{F}}\right\} \cong Maps(S, \coprod_{\mathscr{F}\in Coh(X)/iso} [\mathrm{Spec}\ k/\mathrm{Aut}(\mathscr{F})]).$$

*Sketch of Proof.* When $X = \mathrm{Spec}\ k$, the left hand side becomes the category of unit root $F$-crystals over $S$, and the right hand side becomes $\mathrm{Map}(S, \coprod_n [\mathrm{pt}/\mathrm{GL}_n(k)])$. By definition, the latter is equivalent to giving an étale $k$-local system on $S$.

We want to somehow reduce to this case. Now, coherent sheaves on $X$ are the same as finitely generated graded $R$-modules, where $R$ is the homogeneous coordinate ring of $X$, modulo negligible modules. Similarly, coherent sheaves on $X \times_k S$ which are flat over $S$ are equivalent to finitely generated graded $R \otimes \mathscr{O}_S$-modules which are flat over $S$, modulo negligible modules. By the previous theorem, this latter is equivalent to "graded $R$-modules in étale local systems over $S$." (In other words, each graded piece comprises a local system over $S$ - this is the key of the projectivity hypothesis!) But one can check that this is precisely the right hand side of the theorem. □

If $X$ be a projective curve, then the theorem says that

$$\mathrm{Shtuka}_{\emptyset}(S) = \{(\mathscr{F},\varphi) \mid {}^{\mathscr{F}=\text{vector bundle of rank } n/X\times_k S}_{\varphi:\mathscr{F}\cong^{\tau}\mathscr{F}}\} = \mathrm{Map}(S, \mathrm{Bun}_n(k)).$$

Here $\mathrm{Bun}_n = \coprod_{\mathscr{F}} [\mathrm{Spec}\ k/\mathrm{Aut}(\mathscr{F})]$ as $\mathscr{F}$ varies over the isomorphism classes of vector bundles on $X$ (the automorphisms being considered as the constant group scheme over $k$), so in particular

$$\mathrm{Shtuka}_{\emptyset}(\overline{k}) = \mathrm{Bun}_n(k)$$

is the groupoid of rank $n$ vector bundles on $X$.

6.5. **Cohomology of Shtukas.** Now we can discuss the cohomology of Shtukas. For instance,

$$H^0_c(\mathrm{Shtuka}_{\emptyset,\overline{k}}; \overline{\mathbb{Q}_\ell}) \cong C_c(\mathrm{Bun}_n(k) = G(F)\backslash G(\mathbb{A})/K; \overline{\mathbb{Q}_\ell})$$

This gives a re-interpretation of classical automorphic forms as a cohomology group of the moduli stack of Shtukas.

The idea of Lafforgue is to study operators on the cohomology coming from more complicated Shtukas.

## 7. Classification of Generic Fibers

### 7.1. **Drinfeld's equivalence.**
For example, consider the diagram from before:

$$\begin{tikzcd}
\mathrm{Shtuka}^{(-1,0,\dots,0),(0,\dots,0,1)} \arrow[r] \arrow[d] & \mathscr{H}_k^{(-1,0,\dots,0),(0,\dots,0,1)} \arrow[r] \arrow[d] & X^2 \\
\mathrm{Bun}_n \times \mathrm{Bun}_n \arrow[r] & \mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n &
\end{tikzcd}$$

Then $\mathrm{Shtuka}(\overline{k}) = \{\mathscr{E} \hookrightarrow_1 \mathscr{E}' \hookleftarrow_1 {}^{\tau}\mathscr{E} := \mathscr{E} \otimes_{\overline{k},\mathrm{Frob}_{S/k}} \overline{k}\}$. If we pull this data back to the generic point of $X$, then the vector bundles become vector spaces over $F$, and we get $\mathscr{E} \otimes F \cong \mathscr{E}' \otimes F$. So the data becomes that of $M = \mathscr{E} \otimes F$, an $n$-dimensional $F \otimes \overline{k}$-vector space together with $\varphi : M \cong M \otimes_{\overline{k},\mathrm{Frob}_{S/k}} \overline{k}$. This almost looks like the datum of a unit root F-crystal, except the ground field is much bigger (it is a global function field rather than a finite field). So we should try to emulate what we did with the crystals.

*Definition* 7.1.1. This motivates the definition of the category

$$\mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob}) = \left\{ (V,\varphi) \mid {\scriptstyle V = \text{fin. dim. } F \otimes \overline{k} \text{ vec. space} \atop \varphi : V \cong V \otimes_{\overline{k},\mathrm{Frob}} \overline{k}} \right\}.$$

This is evidently an $F$-linear abelian category.

**Theorem 7.1.2** (Drinfeld)**.** $\mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob})$ *is semisimple, and there is a bijection*

$$\left\{ \textit{simple objects in } \mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob}) \right\} / \textit{iso} \longleftrightarrow \varinjlim_{E/F \textit{ finite separable}} \mathrm{Div}^0(E;\mathbb{Q}).$$

*Remark* 7.1.3. $\mathrm{Div}(E)$ is the free abelian group on all the valuations on $E$, i.e. the divisor group of the projective curve corresponding to $E$, and $\mathrm{Div}(E;\mathbb{Q}) := \mathrm{Div}(E) \otimes \mathbb{Q}$. Then principal divisors $\mathrm{Div}^0(E;\mathbb{Q})$ is the subgroup of degree 0 divisors tensored with $\mathbb{Q}$. By the short exact sequence

$$0 \to E^\times \otimes \mathbb{Q} \to \mathrm{Div}^0(E) \otimes \mathbb{Q} \to \mathrm{Cl}(E) \otimes \mathbb{Q} \to 0$$

and the fact that the class group of $E$ is *finite*, we see that $\mathrm{Div}^0(E) \otimes \mathbb{Q} \cong E^\times$. The theorem is true more generally replacing $\mathrm{Div}^0(E;\mathbb{Q})$ with $E^\times \otimes \mathbb{Q}$, but the geometric interpretation will be the useful one for our applications.

Here we regard $E \hookrightarrow E'$ as being induced by $Y' \xrightarrow{f} Y$, and the transition maps being $f^* : \mathrm{Div}^0(E;\mathbb{Q}) \to \mathrm{Div}^0(E';\mathbb{Q})$. We are *not* regarding the fields as lying in a fixed separable closure; if we did, then we would have to consider the fields up to Galois action, and the limit would be viewed as

$$\varinjlim_{E/F} E^\times \otimes_{\mathbb{Z}} \mathbb{Q} \cong (F^s)^\times \otimes_{\mathbb{Z}} \overline{\mathbb{Q}} / \mathrm{Gal}(F^s/F).$$

The full theorem actually predicts something more precise. Let $(V,\varphi)$ be a simple object in $\mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob})$. Then $\mathrm{End}(V,\varphi)$ is a division algebra over $F$ with center $E$, so $E/F$ is separable (Koethe's theorem), and $E$ acts on $(V,\varphi)$. If $(V,\varphi) \longleftrightarrow a \in \mathrm{Div}^0(E)_{\mathbb{Q}}$ under the theorem, then we can write

$$a = \sum_{y \in |E|} v_y(a) \cdot y, \quad v_y(a) \in \mathbb{Q}$$

where $\sum v_y(a)[k(y):k] = 0$ by definition of being degree 0. The full theorem predicts that if $N$ is the LCM of the denominators of $\{v_y(a)[k(y):k]\}$ (with the convention that the denominator of 0 is 1) then $\dim_E \mathrm{End}(V,\varphi) = N^2$ and $\dim_E V = N$.

*Proof.* Let $(V,\varphi)$ be a simple object over $F \otimes \overline{k}$. We want to associate to $(V,\varphi)$ a value in $(F^s)^\times \otimes \mathbb{Q}/\mathrm{Gal}(F^s/F)$. The only natural such choice is something which would be roughly speaking an "eigenvalue" of $\varphi$ on $V$. Since $\varphi$ is given by a finite amount of data, the pair $(V,\varphi)$ is actually defined over a *finite* extension $k_n/k$ of degree $n$. So $(V,\varphi) \cong (V_n,\varphi) \otimes_{k_n} \overline{k}$.

Since $\varphi$ is a $\sigma$-linear automorphism of $V_n/F \otimes k_n$, $\varphi^n$ is an $F \otimes k_n$-*linear* automorphism of $V_n$. Then it makes sense to talk about the eigenvalues of $\varphi^n$ in $\overline{F}^\times$. To recover the eigenvalues of $\varphi$, we can take the $n$th roots of the eigenvalues of $\varphi^n$. This is ambiguous up to $n$th roots of unity, but we can extract $n$th roots *canonically in* $\overline{F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ (as this removes torsion).

How well-defined is this? If we replace $n$ by $mn$, then we consider the eigenvalues of $\varphi^{mn}$ instead of $\varphi^n$, which just has the effect of raising all the eigenvalues to the $m$th power, and that difference is undone when we take the $mn$th root.

So we've associated to $(V,\varphi)$ a subset of $\overline{F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, which is evidently invariant under $\mathrm{Gal}(\overline{F}/F \otimes k_n)$. But since we killed roots of unity by tensoring with $\mathbb{Q}$, the Galois group of the finite cyclotomic Galois extension $F \otimes k_n/F$ also preserves the subset. Also, we have

$$(F^s)^\times \otimes_{\mathbb{Z}} \mathbb{Q} \cong \overline{F}^\times \otimes_{\mathbb{Z}} \mathbb{Q}$$

because any element becomes separable after raising to a sufficiently high $p$th power. So this shows that our subset is even a $\mathrm{Gal}(F^s/F)$-orbit.

Now why is this subset a *single* orbit? If it consists of at least two orbits, then $\mathrm{Spec}(\varphi^n)$ (the analytic spectrum, i.e. the eigenvalues!) contains at least two orbits. That means that the Jordan decomposition for $(V,\varphi^n) \otimes \overline{F}$ descends to a direct sum decomposition for $(V_n,\varphi^n)$. But each direct summand is $\varphi$-stable, as the idempotent projection onto each factor can be expressed as a polynomial in $\varphi$, which violates simplicity.

Let $\lambda \in (F^s)^\times \otimes \mathbb{Q}/\mathrm{Gal}(F^s/F)$. Let $M(\lambda)$ be the subcategory

$$M(\lambda) = \{(V,\varphi) \mid \mathrm{Spec}(\varphi^n)^{1/n} = \text{orbit of } \lambda)\}.$$

We can check that if $\lambda \not\sim \lambda'$, then there is no simple extension between objects in $M(\lambda)$ and $M(\lambda')$ (for the same eigenspace splitting reasons as before), so we have a splitting of categories

$$\mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob}) \cong \bigoplus M(\lambda).$$

We now need to show that each $M(\lambda)$ is semisimple and has only 1 simple object (up to isomorphism). Let $E/F$ be a finite extension, chosen to be minimal among finite extensions such that $\lambda \in E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ (so $E$ doesn't necessarily contain $\lambda$; it only needs to contain a root of unity multiple of $\lambda$). For each $n = 1,2,\dots$ and $b \in E^\times$ with $b \equiv \lambda^n$ in $E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, let

$$M(\lambda)_{n,b} := \left\{ (V_n, \varphi) \mid \substack{V_n / E\otimes k_n \\ \varphi = \sigma\text{-linear aut. of } V \\ \varphi^n = b\cdot \mathrm{Id}} \right\}$$

Every object of $M(\lambda)_{n,b}$ gives an object of $M(\lambda)$ by tensoring with $\overline{k}$. In fact, we claim that

$$M(\lambda) = \varinjlim_{n,b} M(\lambda)_{n,b}.$$

To see this, suppose that $(V,\varphi)$ is an object of $M(\lambda)$. Since $\varphi$ involves only a finite amount of data, any object of $M(\lambda)$ certainly defined over some $E \otimes k_n$. The only subtlety is why we may assume that $\varphi^n$ is actually *semisimple* (referring to the definition of $M(\lambda)_{n,b}$). If we decompose $\varphi = \varphi_{ss} + \varphi_u$ into the semisimple and unipotent parts, then since we are in characteristic $p$ we can raise to some large $p$th power to kill off the unipotent part.

Strictly speaking, one also has to check that if all homomorphisms are also obtained in the direct limit. But any homomorphism is again specified by a finite amount of data, which will be defined over a finite field extension, and the homomorphism groups must stabilize as they are ultimately finite-dimensional.

So we only need to show that $M(\lambda)_{n,b}$ is semisimple with 1 simple object. But by the preceding discussion, $M(\lambda)_{n,b}$ is the category of modules over $(E\otimes k_n)\langle\tau\rangle/(\tau^n - b)$, where $\tau \cdot x = \mathrm{Frob}(x)\tau$ for $x \in k_n$ and $\tau$ commutes with $E$. The proof will then be completed by the following standard exercise in the theory of central simple algebras.

*Exercise* 7.1.4. Show that $(E\otimes k_n)\langle\tau\rangle/(\tau^n - b)$ is a central simple algebra over $E$ (just find some field extension splitting this as a matrix algebra).

□

*Remark* 7.1.5. For a given $(V,\varphi)$, the field $E = \mathrm{End}(V,\varphi)$ can be viewed as analogous to "complex multiplication" for $(V,\varphi)$.

7.2. **The Dieudonné-Manin classification.** There is a local version of Drinfeld's theorem, going by the name of the *Dieudonné-Manin classification*, which describes a local version of the category $\mathbf{Mod}(F\otimes\overline{k}, 1\otimes \mathrm{Frob})$. Let $K$ be a local function field with residue field $k$. Let $L = \widehat{K^{ur}}$. Then we define

$$\mathrm{IsoCrystal}(L) = \left\{ (V,\varphi) \mid \substack{V = \text{fin.-dim. } / L \\ \varphi : V \cong V \otimes_{\overline{k},\mathrm{Frob}} \overline{k}} \right\}.$$

**Theorem 7.2.1** (Dieudonné-Manin classification)**.** *IsoCrystal*$(L)$ *is a semisimple $K$-linear category, whose simple objects (up to isomorphism) are in bijection with* $\mathbb{Q}$. *If* $\frac{a}{b} \longleftrightarrow (V_{a/b},\varphi)$, *with* $b > 0$, *then* $\dim_L V_{a/b} = b$ *and* $\mathrm{End}(V_{a/b},\varphi)$ *is a central simple algebra over $K$ with invariant* $-a/b \in \mathbb{Q}/\mathbb{Z}$.

This local version of the theorem informs the global version via a "local-global compatibility" between Drinfeld's theorem and the Dieudonné-Manin classification. Given $(V,\varphi) \in \mathbf{Mod}(F\otimes\overline{k}, 1\otimes\mathrm{Frob})$ which is actually defined over a finite extension $E/F$, for each $y \in |E|$ we can consider

$$(V_y, \varphi_y) := (V, \varphi^{[k(y):k]}) \otimes_E \widehat{E_y^{ur}} \in \mathrm{IsoCrystal}(\widehat{E_y^{ur}}).$$

This will be a direct sum of simple isocrystals, *each* of slope $v_y(a)[k(y):k]$. Such an object, a direct sum of isomorphic simples, is called *isoclinic*. To summarize, the following diagram commutes:

$$\begin{tikzcd}
\text{simple} \in \mathbf{Mod}(F\otimes \overline{k}, 1\otimes \mathrm{Frob}) \arrow[r, "\text{Drinfeld}"'] \arrow[d, "\text{localize}"'] & (F^s)^\times \otimes \mathbb{Q}/\operatorname{Gal}(F^s/F) \arrow[d, "\text{valuation}"'] \\
\text{isoclinic} \in \text{IsoCrystal}(L) \arrow[r, "\text{Dieudonné-Manin}"'] & \mathbb{Q}
\end{tikzcd}$$

This compatibility generalizes the local-global compatibility from class field theory. If $(V,\varphi)$ is a simple object of $\mathbf{Mod}(F\otimes\overline{k}, 1\otimes\mathrm{Frob})$ then $D=\operatorname{End}(V,\varphi)$ will be a central simple algebra over $E$. For any $y\in|X|$ the algebra $D_y := D\otimes_E E_y$ has local invariant $-v_y(a)[k(y):k]\in\mathbb{Q}/\mathbb{Z}$.

On the other hand, by Drinfeld's equivalence $(V,\varphi)$ corresponds to an element of $\operatorname{Div}^0(E,\mathbb{Q})$. We have a map $\operatorname{Div}^0(E,\mathbb{Q})\to\operatorname{Div}^0(E,\mathbb{Q}/\mathbb{Z})\cong\operatorname{Br}(E)$ sending $a\in\operatorname{Div}^0(E,\mathbb{Q})$ to the central simple algebra $D_a$ over $E$, which by class field theory has rank equal to the square of the LCM of the denominators of the local invariants. This CSA is precisely $D=\operatorname{End}(V,\varphi)$.

$$\begin{tikzcd}
\mathbf{Mod}(F\otimes \overline{k}, 1\otimes \mathrm{Frob}) \arrow[r, "\text{Drinfeld}"'] \arrow[d, "\text{localize}"'] \arrow[rr, bend left, "(V,\varphi)\mapsto\operatorname{End}(V,\varphi)"] & (F^s)^\times \otimes \mathbb{Q}/\operatorname{Gal}(F^s/F) \arrow[r] \arrow[d, "\text{valuation}"'] & \operatorname{Br}(E) \arrow[d, "\text{CFT}"] \\
\text{IsoCrystal}(L) \arrow[r, "\text{Dieudonné-Manin}"'] & \mathbb{Q} \arrow[r] & \mathbb{Q}/\mathbb{Z} = \operatorname{Br}(E_y)
\end{tikzcd}$$

7.3. **Generic fibers of shtukas.** Let $\overline{x},\overline{y}\in X(\overline{k})$ lying over closed points $x,y\in|X|$. Assume $x\neq y$. We are interested in

$$\mathrm{Shtuka}_{\overline{x},\overline{y}}(\overline{k}) = \{(\mathscr{E},\mathscr{E}',\alpha\colon \mathscr{E}\overset{\overline{x}}{\underset{1}{\hookrightarrow}}\mathscr{E}'\overset{\overline{y}}{\underset{1}{\hookleftarrow}}{}^\tau\mathscr{E}=(1\otimes\mathrm{Frob})^*\mathscr{E}:\beta\}.$$

The bundle $\mathscr{E}$' is actually determined by the rest of the data, so this is just the data of a rational map $\mathscr{E}\dashrightarrow{}^\tau\mathscr{E}$ with a simple zero at $\overline{x}$ and a simple pole at $\overline{y}$. By taking the generic fiber, we get an object $(V,\varphi)$ of $\mathbf{Mod}(F\otimes\overline{k}, 1\otimes\mathrm{Frob})$. We want to ponder the question of what possible $(V,\varphi)$ appear as the image of this functor.

Since we just showed that the latter category is semisimple, we know that

$$(V,\varphi)=\bigoplus_{\lambda\in F^s\otimes_{\mathbb{Z}}\mathbb{Q}/\sim}(V_\lambda,\varphi_\lambda)^{m_\lambda}.$$

What can we say about the $\lambda$ and $m_\lambda$?

**Theorem 7.3.1.** *If $(V,\varphi)$ is the generic fiber of $\{\mathscr{E}\dashrightarrow{}^\tau\mathscr{E}\}$, then*

$$(V,\varphi)=(F\otimes\overline{k},1\otimes\mathrm{Frob})^r\oplus(V_\lambda,\varphi_\lambda),$$

*i.e. there is only one non-trivial component, which appears with multiplicity* $1$. *Moreover, if $E$ is the minimal field such that $\lambda \in E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$, then* $\mathrm{Div}(\lambda) \in \mathrm{Div}^0(E,\mathbb{Q})$ *must be of the form*

$$\frac{1}{m}\left(\frac{1}{[k(\widetilde{x}):k]}\widetilde{x} - \frac{1}{[k(\widetilde{y}):k]}\widetilde{y}\right)$$

*where $\widetilde{x},\widetilde{y}$ are points of $|E|$ lying over $x, y \in |F|$ respectively, and $r + m[E:F] = n$.*

*Remark* 7.3.2. The minimality is needed here to give $V$ the structure of an $E$-vector space. But what is that structure anyway? Here is an intrinsic definition of (the minimal) $E$ attached to a simple $(V,\varphi)$: if $(V_n,\varphi_n)$ is a model of $(V,\varphi)$ over $F \otimes_k k_n$, then $\mathrm{End}(V,\varphi)$ is a division algebra with center $E$.

7.4. **Newton and Hodge polygons.** We begin by examining the local structure, which will involve Newton and Hodge polygons. Then for any $(V,\varphi) \in \mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob})$ we let $E = \mathrm{End}(V,\varphi)$ and for a place $\widetilde{u}$ of $E$ lying over $u$ of $F$, we let $L_{\widetilde{u}} = \widehat{F_{\widetilde{u}}} \cong \overline{k(\widetilde{u})}((t))$. We then define

$$(V_u,\varphi_u) := (V,\varphi) \otimes_{E\otimes\overline{k}} L_{\widetilde{u}}.$$

*Remark* 7.4.1. What if we instead considered

$$(V'_u,\varphi'_u) := (V,\varphi)\otimes_{F\otimes_k\overline{k}} L_u \in \mathrm{IsoCrystal}(L_u)?$$

If $\mathrm{End}(V,\varphi) = E$ then

$$(V,\varphi)\otimes_{F\otimes_k\overline{k}} L_u \cong (V,\varphi)\otimes_{E\otimes\overline{k}}(E\otimes\overline{k})\otimes_{F\otimes_k\overline{k}} L_u$$

But $(E\otimes\overline{k})\otimes_{F\otimes_k\overline{k}} L_u$ breaks into a product of local fields, namely the completions of $E$ at places above $u$. So we see that

$$(V'_u,\varphi'_u) = \bigoplus_{\widetilde{u}|u}(V_{\widetilde{u}},\varphi_{\widetilde{u}}).$$

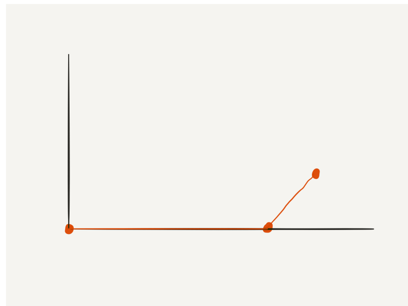
*The Newton polygon.* The Dieudonné-Manin classification gives a Newton polygon $NP(V,\varphi)$ for every $(V,\varphi) \in \mathrm{IsoCrystal}(L)$ as follows. Each simple summand is associated to an invariant $m/n \in \mathbb{Q}$ by the Dieudonné-Manin classification, and the polygon for that simple is just a segment of slope $m/n$. In general, the Newton polygon for a semisimple module has a slope for each simple summand, put in increasing order.

*The Hodge polygon.* If $\Lambda$ is an $\mathscr{O}_L$-lattice in $V$, then we also have a *Hodge polygon* $HP(\Lambda, V,\varphi)$ associated to $\Lambda$, which measures the relative position of $\Lambda$ and $\varphi(\Lambda)$. It basically returns the analogue of the Jordan type partition from before: if $\varphi:\Lambda\to\Lambda$, then

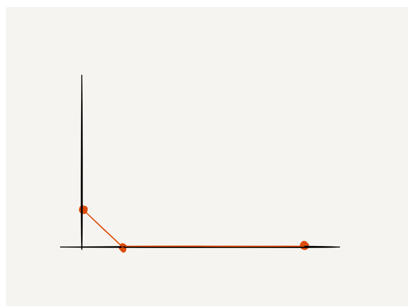
$$\Lambda/\varphi(\Lambda) \cong \bigoplus_{d_1\le\ldots\le d_n} \mathscr{O}_L/\varpi^{d_i}$$

and the slopes of the $HP(\Lambda, V,\varphi)$ are the $d_i$'s.

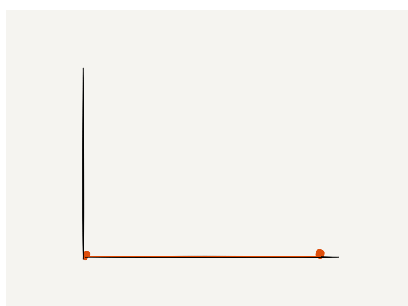
*Example* 7.4.2. Suppose $\Lambda$ has rank $n$ and $\varphi(\Lambda) \subset^1 \Lambda$ has length 1. Then the Hodge polygon is the convex hull of $(0,0),(n-1,0),(n,1)$.



If $\varphi(\Lambda) \supset^1 \lambda$, then the Hodge polygon is the convex hull of $(0,1),(1,0),(n,0)$.



If $\varphi(\Lambda) = \Lambda$, then we just get a straight line with slope 0.



If $u \neq x,y$, then $(V,\varphi)$ has a model $(V_n,\varphi_n)$ over a finite extension, and $(V_n,\varphi_n)$ has an $\mathscr{O}_{L_u}$-model, i.e. there exists a lattice $\Lambda_n \subset V_n$ such that $\varphi_n : \Lambda_u \cong {}^\tau\Lambda_u = \Lambda_u \otimes_{\overline{k},\sigma} \overline{k}$. In this case, $(V_n,\varphi_n) \cong (L_u, 1\otimes\sigma)^n$ is isoclinic with slope 0. So we know the Hodge polygons at all $u$: it is flat if $u \neq x,y$, which is usually the case, but it can also be one of the two exceptional examples if $u = x$ or $u = x$). This will allow us to get constraints on the Newton polygon from the following fact.

**Theorem 7.4.3** (Mazur)**.** *For any $(V,\varphi) \in$ IsoCrystal$(L)$, the Newton polygon lies above the Hodge polygon with the same endpoints.*

If the Hodge polygon is flat (e.g. at all $u \neq x,y$), then Mazur's theorem constrains the Newton polygon to be the same. Therefore, under the Dieudonné-Manin classifications the slope must be 0 at $u$, hence also for any $\widetilde{u} \in E$ lying over $u$. Then local-global compatibility implies that the slope can be read off from the coefficient of the divisor. That is, if $(V_\lambda, \varphi_\lambda)$ appears as a summand of $(V, \varphi)$ then

$$\mathrm{Div}(\lambda) = \sum_{\widetilde{u} \in |E|} v_{\widetilde{u}}(\lambda) \widetilde{u}$$

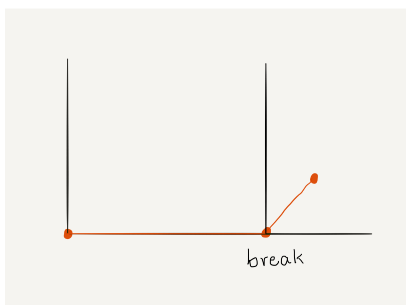
and $(V_{\widetilde{u}}, \varphi_{\widetilde{u}})$ is isoclinic with slope $v_{\widetilde{u}}(\lambda)[k(\widetilde{u}) : k]$. So putting this together, we see that $v_{\widetilde{u}}(\lambda) = 0$ if $\widetilde{u}$ does not lie over $x$ or $y$, i.e. $(V_\lambda, \varphi_\lambda)$ can only appear in $(V, \varphi)$ if it becomes a direct sum of trivial isocrystals after localizing at $u$.

*Definition* 7.4.4. We say that a *breaking point* of a Newton polygon is a vertex where the slope changes.

A result of Katz says that if the NP and HP meet at a breaking point, then the isocrystal splits in a corresponding way. More precisely:

**Theorem 7.4.5** (Katz)**.** *If $NP(V, \varphi)$ meets $HP(\Lambda, V, \varphi)$ at a breaking point, then there exists a canonical decomposition $M = M_1 \oplus M_2$ (and hence $V = V_1 \oplus V_2$) such that each $V_i$ is stable under $\varphi$, and $NP(V_1, \varphi)$ is the first half of $NP(V, \varphi)$, and $HP(M_1, \varphi)$ is the first half of $HP(M, \varphi)$.*

*Example* 7.4.6. We only need a special case of this result. Suppose that the Hodge polygon looks like ___/ with endpoint $(n, 1)$.




The Newton polygon must start off flat, and break at some $(m, 0)$. This implies that $(V, \varphi)$ can be decomposed as $V = V_1 \oplus V_2$ where $NP(V_1, \varphi)$ and $HP(V_1, \varphi)$ are both flat, and $(V_2, \varphi)$ is a simple isocrystal over $L_u$ with slope $1/m$, and $(V_1, \varphi) = (L_u, \mathrm{Frob})^{n-m}$.

We have basically given the proof of Drinfeld's Theorem 7.1.2, but we collect together the arguments for the sake of clarity.

*Proof of Theorem 7.3.1.* For a closed point $u$, we denote by $d(u)$ the degree $[k(u) : k]$. Suppose

$$(V, \varphi) = \bigoplus_i (V_i, \varphi_i)$$

is a decomposition into simple objects, with $(V_i, \varphi_i) \longleftrightarrow (E_i, a_i)$ under the classification in Theorem 7.1.2. For $u \in |X|$, we can tensor with $L_u$ to get $(V_u, \varphi_u) \in \mathrm{IsoCrystal}(L_u)$, where $\varphi_u = \varphi^{d(u)}$.

By local-global compatibility, we may understand the local modules $(V_u, \varphi_u)$ in terms of the divisor which is the pre-image of $u$: the slopes of $NP(V_u, \varphi_u)$ are

$$\{v_{\widetilde{u}}(a_i)\cdot d(\widetilde{u}) \mid \widetilde{u} \in E_i \text{ above } u\}.$$

We know that $(V, \varphi)$ comes from a *vector bundle*, not just a vector space over the generic point, which essentially means that there is an *integral* structure, so it makes sense to look at the Hodge polygon. Recall that we started out with $\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^{\tau}\mathscr{E}$. We can view $\mathscr{E}_u \subset V_u$ as a lattice, and the Hodge polygon measures relative position of $(\mathscr{E}_u, \varphi_u(\mathscr{E}_u))$ (where $\varphi$ is the rational map $\mathscr{E} \to \mathscr{E}$).

There are two distinct cases: $u \notin \{x, y\}$ or $u \in \{x, y\}$. If $u \neq x, y$ then (as we saw in the discussion above) the Hodge polygon is flat. Therefore, $v_{\widetilde{u}}(a_i) = 0$ for all $\widetilde{u} \in |E_i|$ above $u$.

On the other hand, if $u = x$ then the Hodge polygon ___/ starts off flat and then moves up to $(n, 1)$ at the last step, because there is a zero with colength 1. If $u = y$, then the Hodge polygon looks like ___ as there is a pole with colength 1.

So if $u = x$, then there exists a unique $(E_1, a_1)$ and unique $\widetilde{x} \in |E_1|$ above $x$ such that $v_{\widetilde{x}}(a_1) \neq 0$. If $u = y$, then there exists a unique $(E_1', a_1')$ and unique $\widetilde{y} \in |E_1'|$ above $y$ such that $v_{\widetilde{y}}(a_1') \neq 0$. Furthermore, since the total degree has to add up to 0 for each simple object, the points $\widetilde{x}, \widetilde{y}$ must be in the same direct summand, i.e. $E_1 = E_1'$. As there are no divisors on any other curve, the other summands must be trivial (by Drinfeld's classification of the simple objects, only the trivial one has all vanishing local invariants).

In conclusion, $(M, \varphi)$ must decompose as a single non-trivial simple plus several copies of the trivial simple. To describe it completely, it suffices to describe the divisor of the non-trivial simple $(E_1, a_1)$. We know that this is $v_{\widetilde{u}}(a_1)\cdot d(\widetilde{u})$. On the other hand, it is the slope of the non-flat "irreducible piece" of the Newton polygon, which by the discussion of Example 7.4.6 is $1/m$ for some $m$. However, there is a slight subtlety here in that this is with respect to $(E_1)_{\widetilde{u}}$, which relative to $F_x$ is scaled by $\frac{1}{d(\widetilde{u})}$.

By the condition that the divisor must have degree 0, the component supported at $\widetilde{y}$ must be $-\frac{\widetilde{y}}{m\, d(\widetilde{y})}$. So we are finding the non-trivial summand to be classified by $(E, a)$ where $a$ is the divisor $\frac{1}{m}\left(\frac{\widetilde{x}}{d(\widetilde{x})} - \frac{\widetilde{y}}{d(\widetilde{y})}\right) \in \mathrm{Div}^0(E, \mathbb{Q})$.

Finally, by class field theory we understand a global CSA in terms of its local invariants: the LCM of the denominators of $v_{\widetilde{u}}(a_i)d(\widetilde{u})$ for $\widetilde{u} \in |E|$, which in our notation above is $m$, is precisely $\mathrm{rank}_{E\otimes \overline{k}} V_i$. This establishes the equality

$$n = r + m[E : F].$$

□

*Example* 7.4.7. Let's write out all the possibilities for $n = 2$ (the generic behavior of rank 2 shtukas). We have $2 = r + m[E : F]$, where $m > 0$ (since the Newton polygon is not flat, there must be a non-trivial summand). There is a small number of cases to consider:

(1) $E = F$, $m = 1, r = 1$. Then $(V, \varphi)$ is isomorphic to the direct sum of the trivial module and a one-dimensional vector space over $F \otimes \overline{k}$. What is this non-trivial module? Under Drinfeld's classification, it corresponds to $a = \frac{x}{d(x)} - \frac{y}{d(y)}$, where

$d(x)$ and $d(y)$ are the degrees of the rational points $x$ and $y$. This is $\mathrm{Div}(f)$ for some $f \in F$, as $\mathrm{Div}^0(F)_{\mathbb{Q}} \cong F^\times \otimes_{\mathbb{Z}} \mathbb{Q}$. Then $\varphi\colon F \otimes \overline{k} \to F \otimes \overline{k}$ sends $1 \mapsto f \otimes 1$.

(2) $E = F$, $m = 2$, $r = 0$. Then $a = \frac{1}{2}\left(\frac{x}{d(x)} - \frac{y}{d(y)}\right)$ so $(V, \varphi)$ is a simple, rank 2 module over $F \otimes \overline{k}$. From the classification, we know that $\mathrm{End}(V, \varphi) = D$ is a quaternion algebra over $F$ ramified exactly at $x$ and $y$, and $\mathrm{Aut}(V, \varphi) = D^\times$. Thus, this case can be thought of as an analog of *super-singular elliptic curves*. There are two places whose localizations have Newton polygons _/ and _.

(3) $[E : F] = 2$, $m = 1$, $r = 0$. Then $(V, \varphi)$ is a 1-dimensional space over $E \otimes \overline{k}$. There exists $f \in E^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ with $\mathrm{Div}(f) = a = \frac{\tilde{x}}{d(\tilde{x})} - \frac{\tilde{y}}{d(\tilde{y})}$ and $(V, \varphi) = (E \otimes \overline{k}, 1 \mapsto f \otimes 1)$. This can be thought of as an analog of *complex multiplication*.

## 8. Uniformization of Shtukas

8.1. **The Uniformization Theorem.** We apply the preceding results to prove a uniformization theorem for shtukas analogous to Weil's Theorem 2.4.1.

Let's return to conidering

$$_{\overline{x},\overline{y}}\,\mathrm{Shtuka}(\overline{k}) = \left\{ \alpha \colon \mathscr{E} \underset{\overline{x}}{\overset{1}{\hookrightarrow}} \mathscr{E}' \underset{\overline{y}}{\overset{1}{\hookleftarrow}} {}^{\tau}\mathscr{E} \colon \beta \right\}.$$

We know that if we take the generic fiber of any such $\alpha \colon \mathscr{E} \underset{1}{\overset{\overline{x}}{\hookrightarrow}} \mathscr{E}' \underset{1}{\overset{\overline{y}}{\hookleftarrow}} {}^{\tau}\mathscr{E} \colon \beta$, we will obtain $(V,\varphi) \in \mathbf{Mod}(F \otimes \overline{k}, 1 \otimes \mathrm{Frob})$. Fix $(V,\varphi)$ a possible such generic fiber. Then we can studied the "rigidified Shtuka" consisting of bundles with generic fiber isomorphic to $(V,\varphi)$:

$$\boxed{_{\overline{x},\overline{y}}\widetilde{\mathrm{Shtuka}}(\overline{k})_{(V,\varphi)} = \left\{ \begin{matrix} \alpha \colon \mathscr{E} \underset{\overline{x}}{\overset{1}{\hookrightarrow}} 1 \mathscr{E}' \underset{\overline{y}}{\overset{1}{\hookleftarrow}} {}^{\tau}\mathscr{E} \colon \beta \\ \psi \colon (\mathscr{E} \otimes F, \beta \circ \alpha^{-1}) \cong (V,\varphi) \end{matrix} \right\}.}$$

Now the rigidified shtuka $\overline{x},\overline{y}\widetilde{\mathrm{Shtuka}}(\overline{k})_{(V,\varphi)}$ receives an action of $\mathrm{Aut}(V,\varphi)$, and in fact is an $\mathrm{Aut}(V,\varphi)$-torsor over its image in $_{\overline{x},\overline{y}}\,\mathrm{Shtuka}(\overline{k})$ (in the groupoid sense). Thus we have

$$\mathrm{Shtuka}_{\underline{x},\underline{y}}(\overline{k}) = \coprod_{(V,\varphi)/\cong} {}_{\overline{x},\overline{y}}\widetilde{\mathrm{Shtuka}}(\overline{k})_{(V,\varphi)} / \mathrm{Aut}(V,\varphi)$$

**Construction of a restricted product.** We now give an "adelic description" of the moduli space of shtukas.

*Exercise* 8.1.1. Show that $\widetilde{\mathrm{Shtuka}}(\overline{k})_{(V,\varphi)} \neq \emptyset$. You can check this by hand; you just have to construct an appropriate global shtuka from a given generic fiber.

By the exercise, we may pick a basepoint $(\mathscr{E}_0 \hookrightarrow \mathscr{E}_0' \hookleftarrow {}^{\tau}\mathscr{E}_0)$. Any other point $(\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^{\tau}\mathscr{E})$ sharing the generic fiber $(V,\varphi)$ is obtained by modifying this basepoint at finitely many $u \in |X|$.

Picking a basis of $\mathscr{E}_0'$ gives a trivialization of the vector bundles on the complement of finitely many points. This gives, for every $u \neq x, y$, a $M_u^0 \subset V_u$ such that $\varphi_u(\Lambda_u^0) = \Lambda_u^0$. Set

$$\mathscr{M}_u = \{\Lambda_u \subset V_u / \widehat{F_u^{ur}} \mid (*)\}$$

where the condition (*) is that

- for all $u \neq x, y$, we have $\varphi_u(\Lambda_u) = \Lambda_u$
- if $u = x$ then $\varphi_x(\Lambda_x) \subset \Lambda_x$ has colength 1, and
- if $u = y$ then $\varphi_y(\Lambda_y) \supset \Lambda_y$ has colength 1.

Then, using the lattices $M_u^0$ we may define the restricted direct product

$$\prod_{u \in |X|}{}' (\mathscr{M}_u, M_u^0)$$

which consists of collections $(\Lambda_u \in \mathscr{M}_u)$ for each $u$, such that almost all $\Lambda_u$ are equal to $M_u^0$.

*Remark* 8.1.2. We don't need a basepoint to talk about $\mathscr{M}_u$, but to form the restricted direct product and define the map we do need to pick a lattice in each $\mathscr{M}_u$, which requires a global basepoint. However, the map is sort of independent of basepoint.

This is analogous to how we used the trivial bundle as a basepoint in Weil's Theorem 2.4.1, but we could have used any other vector bundle together with full level structure and generic trivialization.

This discussion basically proves:

**Theorem 8.1.3.** *There is a canonical isomorphism*

$$_{\overline{x},\overline{y}}\widetilde{\mathrm{Shtuka}}(\overline{k})_{(V,\varphi)} \cong \prod{}' (\mathscr{M}_u, M_u^0)$$

Let's digest this into a form more reminiscent of Weil's Uniformization Theorem. First suppose $u \neq x, y$. Then we saw that $(V_u, \varphi_u) \cong (\widehat{F_u^{ur}}, \mathrm{Frob}_{V/\overline{k}})^n$ (because the Newton polygon is flat at $u$). To ease the notation, denote $\sigma = 1 \otimes \mathrm{Frob}$. We claim that

$$\mathscr{M}_u \cong \mathrm{GL}_n(F_u)/\mathrm{GL}_n(\mathscr{O}_u).$$

The leftwards map is $g \mapsto (g(\widehat{\mathscr{O}_u^{ur}})^n, \sigma)$. That this map is well-defined is clear (the original lattice $(\widehat{\mathscr{O}_u^{ur}})^n$ was stable under Frobenius, so its image is as well), and the content of the isomorphism is that *every* lattice in $\mathscr{M}_u$ comes from this construction. So why is this the case?

Set $L = \widehat{F_u^{ur}}$. Then $\mathrm{Frob}_u$ acts on $\mathrm{GL}_n(L)/\mathrm{GL}_n(\mathscr{O}_L)$, which is the space of all lattices in $L$. The lattices in $\mathscr{M}_u$ are those which are fixed points under this action:

$$\mathscr{M}_u = (\mathrm{GL}_n(L)/\mathrm{GL}_n(\mathscr{O}_L))^{\mathrm{Frob}_u}.$$

There is a certainly a map to here from $\mathrm{GL}_n(L^\sigma) = \mathrm{GL}_n(F_u)$. The surjectivity amounts to vanishing of some Galois cohomology group, which can be checked.

There is also a more concrete way to see this. By stratifying the space of lattices according to their relative positions with respect to the standard lattice, we get a stratification of by affine spacse over $\overline{k}$ with the standard action of Frobenius, i.e. we can realize $(\mathrm{GL}_n(L)/\mathrm{GL}_n(\mathscr{O}_L))$ as $\bigcup \overline{k}^N$ with the standard action of Frobenius, and taking fixed points one gets $\bigcup k^N$, which is $\mathrm{GL}_n(F_u)/\mathrm{GL}_n(\mathscr{O}_u)$.

In summary, if $u \neq x, y$ then

$$\mathscr{M}_u \cong \mathrm{GL}_n(F_n)/\mathrm{GL}_n(\mathscr{O}_u)$$

upon choosing a trivialization $(V_u, \varphi_u) \cong ((\widehat{F_u^{ur}})^n, \sigma)$.

Now suppose that $u = x$. Then the Newton polygon has vertices $(0,0), (n-\ell, 0), (n, 1)$ and the Hodge polygon has vertices $(0,0), (n-1, 0), (n, 1)$. There is a breaking point at $(n-\ell, 0)$, which induces a decomposition $V_x = V_{x,1} \oplus V_{x,2}$ compatible with $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Lambda_i = \Lambda \cap V_{x,i}$. Applying the preceding discussion to $V_{x,1}$, we see that

$$\mathscr{M}_x = \mathrm{GL}_{n-\ell}(F_x)/\mathrm{GL}_{n-\ell}(\mathscr{O}_x) \times \mathscr{N}_x,$$

with the first factor parametrizing choices for $\Lambda_1$, and

$$\mathscr{N}_x = \{\Lambda_2 \subset V_{x,2} \mid \varphi(\Lambda_2) \subset^1 \Lambda_2\}.$$

It remains to understand $\mathcal{N}_x$. Note that there is an action of $\varphi$ on $\mathcal{N}_x$ just by replacing $\Lambda_2$ with $\varphi(\Lambda_2)$. Therefore $\mathcal{N}_x$ has an action of the group $\varphi^{\mathbb{Z}} \cong \mathbb{Z}$.

**Proposition 8.1.4.** *$\mathcal{N}_x$ is a torsor for $\varphi_x^{\mathbb{Z}}$.*

*Proof.* Fixing $\varphi\colon V \cong V$, suppose $M$ and $M'$ are lattices in $V$ such that $\varphi(M)$ has colength 1 in $M$ and $\varphi(M')$ has colength 1 in $M'$. We want to show that $M' = \varphi^n M$ for some $n$.

**Lemma 8.1.5.** *There exists some $n$ such that $M' \subset \varphi^n M$.*

*Proof.* Left as exercise. $\square$

By choosing a maximal such $n$ and replacing $M$ by $\varphi^n M$, we may assume that $M' \subset M$ but $M' \not\subset \varphi^{-1} M$. Consider the reduction

$$\overline{\varphi}\colon M/M' \to M/M'.$$

This factors through $(\varphi(M)+M')/M'$:

$$\begin{tikzcd} M/M' \arrow[rr, "\overline{\varphi}"] \arrow[dr] & & M/M' \\ & (\varphi(M)+M')/M' \arrow[ur, hook] & \end{tikzcd}$$

Since $\varphi(M)$ has colength 1 in $M$, $(\varphi(M)+M')/M'$ has colength at most 1 in $M/M'$. However, since $\varphi(M) \not\subset M'$, the equality case cannot occur. That means that $\overline{\varphi}$ is an isomorphism, but on the other hand it must be nilpotent by switching the roles of $M$ and $M'$ in Lemma 8.1.5. $\square$

*Remark* 8.1.6. By Drinfeld's theorem we know that $\mathrm{Aut}(V_{x,2}, \varphi_{x,2}) = D^\times$ (a central simple algebra with center $E_{\widetilde{x}}$ and invariant $1/m$). The action of $D^\times$ on $\mathcal{N}_x$ factors through $\mathbb{Z}$ via the valuation.

The theory for $u = y$ is similar, except using $\varphi(\Lambda_2) \supset^1 \Lambda_2$. Putting these discussions together, we obtain:

**Theorem 8.1.7** (Uniformization Theorem for Shtukas)**.** *There is a canonical isomorphism*

$$\begin{aligned} {}_{x,y}\,\mathrm{Shtuka}(\overline{k})_{(V,\varphi)} \cong \mathrm{GL}_n(\mathbb{A}^{x,y})/\mathrm{GL}_n(\mathscr{O}^{x,y}) \\ \times \mathrm{GL}_{n-\ell}(F_x)/\mathrm{GL}_{n-\ell}(\mathscr{O}_x) \times (\varphi_x^{\mathbb{Z}} - \mathit{torsor}) \\ \times \mathrm{GL}_{n-\ell'}(F_y)/\mathrm{GL}_{n-\ell'}(\mathscr{O}_y) \times (\varphi_y^{\mathbb{Z}} - \mathit{torsor}) \end{aligned}$$

What is the action of $\mathrm{Aut}(V,\varphi)$? We can view $\mathrm{Aut}(V,\varphi)$ as the $F$-points of a reductive group over $F$, which is a product of factors $\mathrm{GL}_2(D)$ where $D$ is a (not necessarily central) division algebra over $F$. This is an inner form of a Levi subgroup in $\mathrm{GL}_n$, as it preserves each summand. We can embed this into $\mathrm{Aut}(V_u, \varphi_u)$, which can similaly be viewed as the $F_u$-points of a (potentially bigger) reductive group over $F_u$, which also preserves each summand. When $u = x$ we will have $\mathrm{Aut}(V_x, \varphi_x) \cong \mathrm{GL}_{n-\ell}(F_x) \times D_x^\times$ and similarly for $u = y$.

8.2. **More symmetries on** $\text{Shtuka}(\overline{k})$**.** The uniformization theorem allows us to see many symmetries of the moduli stack of shtukas.

**Hecke operators.** Suppose $u \in X(k)$ and $u \neq x, y$. There is an action of the local Hecke algebra $\mathscr{H}_u = C_c(G(\mathscr{O}_u)\backslash G(F_u)/G(\mathscr{O}_u))$ on $C(\text{Shtuka}(\overline{k})) = C(G(F)\backslash G(F_u)/G(\mathscr{O}_u))$ by convolution (on the right, as we quotient out by automorphisms on the left). By Cartan decomposition, all double cosets are of the form

$$G(\mathscr{O}_u)\begin{pmatrix} \varpi^{d_1} & & \\ & \varpi^{d_2} & \\ & & \varpi^{d_n} \end{pmatrix} G(\mathscr{O}_u)$$

for integers $d_1, \ldots, d_n$, so we may think of an element of $\mathscr{H}_u$ as a function on finitely many tuples of integers up to permuttion. How does this act on $C(\text{Shtuka}(\overline{k}))$? Let $\underline{d} = (d_1, \ldots, d_n)$, and suppose for simplicity that all entries are $\geq 0$. Then we have a correspondence

$$\begin{tikzcd} & C_{u,\underline{d}} \arrow[dl, "c_1"'] \arrow[dr, "c_2"] & \\ \text{Shtuka}_{x,y} & & \text{Shtuka}_{x,y} \end{tikzcd}$$

where $C_{u,\underline{d}}$ is the set of pairs of diagrams $(\mathscr{E} \overset{1}{\underset{x}{\hookrightarrow}} \mathscr{E}' \overset{1}{\underset{y}{\hookleftarrow}} {}^\tau\mathscr{E})$ and $(\mathscr{F} \overset{1}{\underset{x}{\hookrightarrow}} \mathscr{F}' \overset{1}{\underset{y}{\hookleftarrow}} {}^\tau\mathscr{F})$ together with a map between diagrams

$$\begin{tikzcd} \mathscr{E} \arrow[r, hook, "x"'] \arrow[d, "\alpha"] & \mathscr{E}' \arrow[d, "\alpha'"] & {}^\tau\mathscr{E} \arrow[l, hook', "y"'] \arrow[d, "{}^\tau\alpha"] \\ \mathscr{F} \arrow[r, hook, "x"'] & \mathscr{F}' & {}^\tau\mathscr{F} \arrow[l, hook', "y"'] \end{tikzcd}$$

such that $\text{coker}(\alpha)$ is supported at $u$ with Jordan type $\underline{d}$. Let's be careful about what this means. There is a canonical isomorphism $\text{coker}(\alpha) \cong \text{coker}({}^\tau\alpha)$. Viewing $\text{coker}(\alpha)$ as a torsion coherent sheaf on $\{u\} \times S$ equipped with an $F$-unit root crystal structure, we know that $\text{coker}(\alpha)$ descends to an étale $k$-local system on $X$ supported at $\{u\}$, and then we can talk about its Jordan type over $k$.

Having constructed the correspondence, we get the Hecke correspondence $f \mapsto c_{2!}c_1^* f$ where $c_1$ sends the diagram to the top row and $c_2$ sends it to the bottom row. We claim that the fibers of $c_1, c_2$ are discrete - more precisely, we claim that $c_1, c_2$ are finite étale maps with discrete fibers. Indeed, the fiber of $c_1$ over a point is $G(\mathscr{O}_u)\varpi^{\underline{d}}G(\mathscr{O}_u)/G(\mathscr{O}_u)$ and the fiber over $C_2$ is $G(\mathscr{O}_u)\varpi^{-\underline{d}}G(\mathscr{O}_u)/G(\mathscr{O}_u)$.

*Special points.* We now consider what happens when $u = x$ or $u = y$. The uniformization furnished a description of $\text{Shtuka}_{x,y}$ as $(\ldots) \times \mathscr{N}_x \times \mathscr{N}_y$ and the actions of $\varphi_x^{\mathbb{Z}}$ and $\varphi_y^{\mathbb{Z}}$ commute with the $\text{Aut}(V, \varphi)$-action. Therefore, $\text{Shtuka}_{\underline{x},\underline{y}}(\overline{k})$ admits an action of $\varphi_x^{\mathbb{Z}}, \varphi_y^{\mathbb{Z}}$.

**Partial Frobenius.** Given

$$\alpha\colon \mathscr{E} \overset{1}{\underset{\underline{x}}{\hookrightarrow}} \mathscr{E}' \overset{1}{\underset{\underline{y}}{\hookleftarrow}} {}^{\tau}\mathscr{E}\colon \beta$$

view $\mathscr{E}'$ as an enlargement of $\mathscr{E}$. Then we can form

$$\begin{tikzcd}
& {}^{\tau}\mathscr{E} \arrow[dl, hook, "\overline{y}"'] \arrow[dr, hook, "\tau(\overline{x})"] & \\
\mathscr{E}' \arrow[dr, hook, "\tau(\overline{x})"'] & & {}^{\tau}\mathscr{E}' \arrow[dl, hook', "\overline{y}"] \\
& \mathscr{E}'' &
\end{tikzcd}$$

by enlarging along $\overline{y}$ and then along $\tau(\overline{x})$. Then we can forget the upper half of the diagram to end up with

$$\begin{tikzcd}
\mathscr{E}' \arrow[dr, hook, "\tau(\overline{x})"'] & & {}^{\tau}\mathscr{E}' \arrow[dl, hook', "\overline{y}"] \\
& \mathscr{E}'' &
\end{tikzcd}$$

This defines a map $\Phi_1\colon \mathrm{Shtuka}_{\overline{x},\overline{y}} \to \mathrm{Shtuka}_{\overline{x},\overline{y}}$, such that the diagram commutes:

$$\begin{tikzcd}
\mathrm{Shtuka}_{\overline{x},\overline{y}} \arrow[r, "\Phi_1"] \arrow[d] & \mathrm{Shtuka}_{\overline{x},\overline{y}} \arrow[d] \\
(X\times X)-\Delta \arrow[r, "\mathrm{Frob},1"'] & (X\times X)-\Delta
\end{tikzcd}$$

*Remark* 8.2.1. This is called "partial Frobenius" because it only performs Frobenius on one of the factors. The relation to Frobenius is that $\Phi_1\circ\Phi_2=\Phi_2\circ\Phi_1=\mathrm{Frob}_{S/k}$.

If we repeatedly apply the partial Frobenius, then we obtain a sequence of maps

$$\mathrm{Shtuka}_{\overline{x},\overline{y}} \to \mathrm{Shtuka}_{\tau(\overline{x}),\overline{y}} \to \ldots \to \mathrm{Shtuka}_{\tau^{d(x)}(\overline{x}),\overline{y})} = \mathrm{Shtuka}_{\overline{x},\overline{y}}$$

because $\mathrm{Frob}^{d(x)}$ fixes $x$. Thus, $\Phi_1^{d(x)}$ defines an *automorphism* of $\mathrm{Shtuka}_{\overline{x},\overline{y}}$. In fact, we claim that the $\varphi_x$-action is induced from $\Phi_1^{d(x)}$ and the $\varphi_y$-action is induced from $\Phi_2^{d(y)}$.

*Exercise* 8.2.2. Prove this. [Hint: Proposition 8.1.4.]

## 9. More on Shtukas

9.1. **Some geometric properties.** In this section we abbreviate $\mathrm{Shtuka} = \mathrm{Shtuka}^{(0,\ldots,0,-1),(1,0,\ldots,0)}$ for ease of notation. Our goal is to prove the following theorem.

**Theorem 9.1.1.** Shtuka *is a Deligne-Mumford stack locally of finite type, and the map* $\mathrm{Shtuka} \to X \times X$ *is smooth of relative dimension* $2(n-1)$.

We break the proof up into a couple of steps.

9.1.1. *Locally finite type.*

**Proposition 9.1.2.** Shtuka *is a Deligne-Mumford stack locally of finite type.*

*Proof.* For the first point, recall the diagram

$$\begin{tikzcd} \mathrm{Shtuka} \arrow[r] \arrow[d] & {}^2\mathscr{H}^{(0,\ldots,0,-1),(1,0,\ldots,0)} \arrow[d] \\ \mathrm{Bun}_n \times \mathrm{Bun}_n \arrow[r] & \mathrm{Bun}_n \times \mathrm{Bun}_n \times \mathrm{Bun}_n . \end{tikzcd}$$

We know that $\mathrm{Bun}_n$ is locally of finite-type, and we can identify a finite-type piece by bounding the HN-polygon. Namely, if $\mu$ is a HN-polygon then $\mathrm{Bun}_n^{\prec\mu}$, parametrizing vector bundles with HN polygon bounded by $\mu$, is finite type. Using that each stratum is defined over the ground field, hence preserved by Frobenius, we get a diagram

$$\begin{tikzcd} \mathrm{Shtuka}^{\prec\mu} \arrow[r] \arrow[d] & ({}^2\mathscr{H}^{(0,\ldots,0,-1),(1,0,\ldots,0)})^{\prec\mu} \arrow[d] \\ \mathrm{Bun}_n^{\prec\mu} \times \mathrm{Bun}_n^{\prec\mu} \arrow[r] & \mathrm{Bun}_n^{\prec\mu} \times \mathrm{Bun}_n^{\prec\mu} \times \mathrm{Bun}_n^{\prec\mu} . \end{tikzcd}$$

It will certainly suffice to show that $\mathrm{Shtuka}^{\prec\mu}$ is a finite type DM stack. Now let's *rigidify*. Let $D \subset X$ be a finite subscheme, and add level structure:

$$\mathrm{Bun}_D^{\prec\mu} = \left\{ (\mathscr{E}, \phi) \mid {\textstyle \substack{HN(\mathscr{E}) \prec\mu \\ \phi : \mathscr{E}|_D \cong \mathscr{O}_D^n}} \right\}.$$

If $D$ is "sufficiently thick" then $\mathrm{Bun}_D^{\prec\mu}$ will actually a quasiprojective scheme. (This is analogous to how adding enough level structure to modular curves makes them into finite moduli spaces.) We have a map $\mathrm{Bun}_D^{\prec\mu} \to \mathrm{Bun}^{\prec\mu}$ which is in fact a $\mathrm{GL}_n(\mathscr{O}_D)$-torsor.

Now construct the pullback diagram

$$\begin{tikzcd} \mathrm{Shtuka}_D^{\prec\mu} \arrow[r] \arrow[d] & ({}^2\mathscr{H}_D^{(0,\ldots,0,-1),(1,0,\ldots,0)})^{\prec\mu} \arrow[d] \\ (\mathrm{Bun}_n^{\prec\mu})_D \times (\mathrm{Bun}_n^{\prec\mu})_D \arrow[r] & (\mathrm{Bun}_n^{\prec\mu})_D \times (\mathrm{Bun}_n^{\prec\mu})_D \times (\mathrm{Bun}_n^{\prec\mu})_D . \end{tikzcd}$$

*Remark* 9.1.3. We have to be careful in the definition of $\mathscr{H}_D^{\prec\mu}$ to specify only those bundles that are modified *away* from $D$.

Then $\mathrm{Shtuka}_D^{\prec\mu} \to \mathrm{Shtuka}^{\prec\mu}$ is finite étale, with fibers $\mathrm{GL}_n(\mathscr{O}_D)$-torsors. Indeed, $\mathrm{Shtuka}_D^{\prec\mu} = \{\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^\tau\mathscr{E} \mid *\}$ where we require

$$\begin{tikzcd} \mathscr{E}|_D \arrow[r, "\cong"] \arrow[d, "\cong"'] & \mathscr{O}_D^n \arrow[d, "\cong"'] \\ {}^\tau\mathscr{E}|_D \arrow[r, "\cong"] & \mathscr{O}_D^n \end{tikzcd}$$

to commute (i.e. the two trivializations are compatible). So it's enough to show that $\mathrm{Shtuka}_D^{\prec\mu}$ is a quasiprojective scheme, which we've already discussed. $\square$

9.1.2. *Smoothness.*

**Theorem 9.1.4.** $\mathrm{Shtuka} \to X \times X$ *is smooth of relative dimension* $2(n-1)$. *Moreover, one has a diagram*

$$\begin{tikzcd} & U \arrow[dl, "\acute{e}t"'] \arrow[dr, "\acute{e}t"] & \\ \mathrm{Shtuka} \arrow[dr] & & (\mathbb{P}^{n-1} \times X) \times ((\mathbb{P}^{n-1})^\vee \times X) \arrow[dl] \\ & X^2 & \end{tikzcd}$$

*where $U$ is a scheme.*

*Proof.* Let $B = \mathrm{Bun}_n$. We "pretend" that this is a scheme of finite type over $k$ (it is really a stack locally of finite type, but the question is local anyway). We have the usual diagram

$$\begin{tikzcd} \mathrm{Shtuka}_{x,y} \arrow[r] \arrow[d] & {}^2\mathscr{H}_{x,y} = \{\mathscr{E} \overset{1}{\underset{x}{\hookrightarrow}} \mathscr{E}' \overset{1}{\underset{y}{\hookleftarrow}} \mathscr{E}'' \mid (*)\} \arrow[d, "{(h,h'')}"'] \\ B \arrow[r] & B \times B \end{tikzcd}$$

What is the fiber of $h'' \colon {}^2\mathscr{H} \to B$? Given $\mathscr{E}''$, we first need to choose an embedding $\mathscr{E}'' \hookrightarrow \mathscr{E}'$. As $\mathscr{E}'$ modifies $\mathscr{E}''$ at one point $y$, this amounts to choosing a *line* in $\varpi^{-1}\mathscr{E}''_y/\mathscr{E}''_y$, i.e. a point of $\mathbb{P}^{n-1}$.

$$\begin{tikzcd} {}^2\mathscr{H}_{x,y} = \{\mathscr{E} \overset{1}{\underset{x}{\hookrightarrow}} \mathscr{E}' \overset{1}{\underset{y}{\hookleftarrow}} \mathscr{E}'' \mid (*)\} \arrow[d] \\ \{\mathscr{E}' \overset{1}{\underset{y}{\hookleftarrow}} \mathscr{E}'' \mid (*)\} \arrow[d, "\text{fiber} \cong \mathbb{P}^{n-1}"] \\ B = \{\mathscr{E}''\} \end{tikzcd}$$

Now let's globalize this discussion by dropping our fixed choices of points $x, y$. To emphasize this, we represent variable modified points by $u, v$. We have a diagram

$$\begin{tikzcd}
\text{Shtuka} \arrow[r] \arrow[d] & {}^2\mathscr{H} = \{\mathscr{E} \overset{1}{\hookrightarrow} \mathscr{E}' \overset{1}{\hookleftarrow} \mathscr{E}'' \mid (*)\} \arrow[d, "(h,h'')"'] \\
X^2 \times B \arrow[r] & X^2 \times B \times B
\end{tikzcd}$$

We can factorize

$$\begin{tikzcd}
{}^2\mathscr{H} = \{\mathscr{E} \underset{u}{\overset{1}{\hookrightarrow}} \mathscr{E}' \underset{v}{\overset{1}{\hookleftarrow}} \mathscr{E}'' \mid (*)\} \arrow[d] \\
Y = \{\mathscr{E}' \underset{v}{\overset{1}{\hookleftarrow}} \mathscr{E}'' \mid (*)\} \arrow[d, "\text{fiber} \cong \mathbb{P}^{n-1}"] \\
X \times B = \{(v, \mathscr{E}'')\}
\end{tikzcd}$$

Then the globalization of the local case is that *intermediate space* $Y = \{\mathscr{E}' \underset{y}{\hookleftarrow} \mathscr{E}'' \mid (*)\}$ *can be thought of as isomorphic to the projectivization of the universal bundle over* $X \times B$.

Similarly, there is a $\mathbb{P}^n$-bundle over $Z$ whose fiber is $\mathscr{E}'$ over a given point. Then ${}^2\mathscr{H}$ is isomorphic to the projectivization of *this* $\mathbb{P}^n$-bundle on $Y$.

This discussion shows there is a natural diagram

$$\begin{tikzcd}
& W \arrow[dl, "\text{ét}"'] \arrow[dr, "\text{ét}"] & \\
{}^2\mathscr{H} \arrow[dr] & & (\mathbb{P}^{n-1} \times X) \times ((\mathbb{P}^{n-1})^\vee \times X) \times B \arrow[dl] \\
& X^2 \times B &
\end{tikzcd}$$

(The point is that after étale pullback, the projective bundles are trivialized.) We form $U$ by pulling this back via the map Shtuka $\to {}^2\mathscr{H}$, which fights into a commutative diagram:

$$\begin{tikzcd}
U \arrow[rr] \arrow[dr] \arrow[dd] & & W \arrow[dr, "\text{ét}"] \arrow[dd] & \\
& \text{Shtuka} \arrow[rr] \arrow[dl] & & {}^2\mathscr{H} \arrow[dl] \\
B \arrow[rr, "{(1,\mathrm{Frob}_{S/k})}"'] & & B \times B &
\end{tikzcd}$$

Since pullbacks of surjective étale maps are surjective étale, $U \to$ Shtuka is surjective étale. Note that since the bottom square is cartesian, as discussed earler, and the top is cartesian by definition, the front square is also cartesian.

We need to check that the map $u\colon U \to (\mathbb{P}^{n-1} \times X) \times ((\mathbb{P}^{n-1})^{\vee} \times X) =: Z$ is étale. Abstracting the situation a bit, we claim that in general if the map $W \to Z \times B$ is étale, and $B$ is smooth, and we have the diagram defining $U$:

$$\begin{tikzcd} U \arrow[r] \arrow[dd] & W \arrow[d] \\ & Z \times B \arrow[d, "?,\mathrm{Id}"] \\ B \arrow[r, "\mathrm{Id},\mathrm{Frob}"'] & B \times B \end{tikzcd}$$

then $U \to Z$ is étale. This is just some transversality property. Affine locally, let $Z = \mathrm{Spec}(R)$ and $B_R := Z \times B$. The hypothesis is that $W \to B_R$ is étale. Let's calculate morphism on tangent spaces corresponding to the diagram above:

$$\begin{tikzcd} & T_{W/R} \arrow[d, "(?,1)"'] \\ T_{B_R/R} \arrow[r, "(1,0)"'] & T_{B_R/R} \oplus T_{B_R/R} \end{tikzcd}$$

It is transparent that the tangent spaces are transverse.

*Remark* 9.1.5. Notice that we only used that $B$ is smooth.

□

9.2. **More general constructions of shtukas.** Let $I$ be a finite set. If $G = \mathrm{GL}_n$, we consider a function

$$\lambda\colon I \to \mathbb{Z}^n/S_n$$

*Remark* 9.2.1. More generally, we should replace $\mathbb{Z}^n/S_n$ by $(X_\bullet(T))^{\mathrm{dom}}$, which is in bijection with Weyl group orbits on cocharacters of $T$. In these terms, $\mathbb{Z}^n/S_n$ is in bijection with $(\mathbb{Z}^n)^{\mathrm{dom}} = \{(d_1 \geq d_2 \geq \ldots \geq d_n)\}$.

We will define a Hecke correspondence $\mathscr{H}^I_\lambda$ generalizing the one from before, which will fit into a diagram

$$\begin{tikzcd} \mathscr{H}^I_\lambda \arrow[r] \arrow[d] & X^I \\ \mathrm{Bun} \times \mathrm{Bun} & \end{tikzcd}$$

*Definition* 9.2.2. Assume for simplicity that $\lambda_i \geq 0$ for all $i$ (i.e. $\lambda_i$ maps to the subset of $(\mathbb{Z}^n)^{\mathrm{dom}}$ where all integers are non-negative). Then $\mathscr{H}^I_\lambda(S)$ is the data of

- maps $(x_i\colon S \to X)_{i \in I}$,
- an inclusion of rank $n$ vector bundles $\mathscr{E} \hookrightarrow \mathscr{E}'$ on $X \times S$, such that $\mathscr{E}'/\mathscr{E}$ is supported on $\bigcup_i \Gamma(x_i)$ and its restriction to $\Gamma(x_i)$ is a flat $\mathscr{O}_S$-module for each $i$, satisfying:

- for all $\overline{s} \in S$,
$$(\mathscr{E}'/\mathscr{E})_{\overline{s}} = \bigoplus_{\substack{\overline{x} \in x_i(\overline{s}) \\ \text{for some } i}} (\mathscr{E}'/\mathscr{E})_{\overline{x}}$$
and $(\mathscr{E}'/\mathscr{E})_{\overline{x}}$ has Jordan type
$$\prec \Big( \sum_{x_i(\overline{s})=\overline{x}} d_1^i \geq \sum_{x_i(\overline{s})=x} d_2^i \geq \ldots \sum_{x_i(\overline{s})=x} d_n^i \Big).$$

It is clear how to extend the definition when not all $\lambda_i$ are non-negative: instead of demanding that $\mathscr{E} \to \mathscr{E}'$ be an inclusion, we simply bound its kernel.

This is the generalization you would get from "following your nose," except that it is not necessarily obvious what constraints to put on the modification of vector bundles when the graphs meet. When the graphs don't meet (i.e. there is only one contributing $i$ in the third condition), we get the usual thing. When they do meet, then we "add" the Jordan type restrictions for the different components

*Example* 9.2.3. Let $G = \mathrm{GL}_2$, $I = \{1,2\}$, and $\lambda$ be defined by $\lambda_1 = (1 \geq 0)$ and $\lambda_2 = (0 \geq -1)$. Then
$$\mathscr{H}^I_\lambda(\overline{k}) = \{x_1, x_2, \mathscr{E} \dashrightarrow \mathscr{E}' \mid (*)\}$$
where the rational map has a zero at $x_1$ and a pole at $x_2$. This admits a map to $X^2$ by forgetting everything except $(x_1, x_2)$. Let's also think about the moduli problem
$$\widetilde{\mathscr{H}^I_\lambda}(\overline{k}) = \{x_1, x_2, \mathscr{E} \overset{1}{\underset{x_1}{\hookrightarrow}} \mathscr{F} \overset{1}{\underset{x_2}{\hookleftarrow}} \mathscr{E}' \mid (*)\}$$
which admits an obvious map to $\mathscr{H}^I_\lambda(\overline{k})$ by forgetting $\mathscr{F}$.

Over the complement of the diagonal, i.e. when $x_1 \neq x_2$, for every $\{\mathscr{E} \dashrightarrow \mathscr{E}'\}$ we can *canonically* insert an $\mathscr{F}$ such that $\mathscr{E} \hookrightarrow_{x_1} \mathscr{F} \hookleftarrow_{x_2} \mathscr{E}'$. (The vector bundles are identified generically, so we can view then as lying in a common $F^n$, and then just take their sum.) Thus, $\mathscr{H}^I_\lambda(\overline{k})$ and $\widetilde{\mathscr{H}^I_\lambda}(\overline{k})$ are isomorphic away from the diagonal.

The case $x_1 = x_2$ is more interesting. The fiber of $\widetilde{\mathscr{H}^I_\lambda}(\overline{k})$ over $x_1 = x_2$ has a closed stratum where $\mathscr{E} \cong \mathscr{E}'$, and an open stratum where $\mathscr{E} \ncong \mathscr{E}'$. Fixing $\mathscr{E}$, the fiber is obtained by choosing an embedding $\mathscr{E} \hookrightarrow \mathscr{F}$, and then a sub-bundle $\mathscr{F} \hookleftarrow \mathscr{E}'$. Therefore, the fiber is set-theoretically $\mathbb{P}^1 \times \mathbb{P}^1$, and in fact it is scheme-theoretically a $\mathbb{P}^1$-bundle over $\mathbb{P}^1$. The closed stratum $\mathscr{E} \cong \mathscr{E}'$ allows us to choose $\mathscr{F}$ freely, so we get a $\mathbb{P}^1$.

If $\mathscr{E}$ and $\mathscr{E}'$ are not isomorphic, then again $\mathscr{F}$ is uniquely determined as their sum, so again $\widetilde{\mathscr{H}^I_\lambda} \to \mathscr{H}^I_\lambda$ is an ismorphism over the open stratum. However, the closed stratum on $\mathscr{H}^I_\lambda$ is a point whose fiber in $\widetilde{\mathscr{H}^I_\lambda}$ is a $\mathbb{P}^1$.

In conclusion, the fiber of $\widetilde{\mathscr{H}^I_\lambda}$ over $x_1 = x_2$ is a smooth $\mathbb{P}^1$-bundle over $\mathbb{P}^1$, and the fiber of $\mathscr{H}^I_\lambda(\overline{k})$ over $x_1 = x_2$ is the quadric cone obtained by contracting a $\mathbb{P}^1$ (which is the exceptional divisor in its blowup). So the augmented moduli problem $\widetilde{\mathscr{H}^I_\lambda}$ is like a resolution of the singularity of $\mathscr{H}^I_\lambda$ at a point of the special locus $x_1 = x_2$.

9.3. **Refinements of shtukas.** Let $I, \lambda$ be as before. Let $I = I_1 \sqcup I_2 \ldots \sqcup I_r$ be a composition (i.e. ordered partition) of $I$ and $\lambda_i = \lambda|_{I_i}$. Then we define the space $\mathscr{H}_\lambda^{I_1,\ldots,I_r}$ to be the fibered product

$$\mathscr{H}_{\lambda_1}^{I_1} \times_{\mathrm{Bun}} \mathscr{H}_{\lambda_2}^{I_2} \times_{\mathrm{Bun}} \ldots \times_{\mathrm{Bun}} \mathscr{H}_{\lambda_r}^{I_r}.$$

More concretely, this is $\{\mathscr{E}_0 \dashrightarrow \mathscr{E}_1 \dashrightarrow \ldots \dashrightarrow \mathscr{E}_r\}$ where the chain has modification bounded by $\lambda_1, \lambda_2$, etc. This maps by $p_1, \ldots, p_r$ to $\mathrm{Bun}_n \times \ldots \times \mathrm{Bun}_n$.

*Remark* 9.3.1. Later, when we study the cohomology of the moduli space of shtukas, we will see that the maps (induced by refinement)

$$\mathscr{H}_\lambda^{I_1,\ldots,I_r} \to \mathscr{H}_\lambda^{I'_1,\ldots} \ldots \to \mathscr{H}_\lambda^I$$

are "stratified small maps."

Consider the pullback diagram

$$\begin{tikzcd}
\mathscr{H}_\lambda^{I_1,\ldots,I_r}|_{X^I - \Delta_{I_1,\ldots,I_r}} \arrow[r] \arrow[d] & \mathscr{H}_\lambda^I \arrow[d] \\
X^I - \Delta_{I_1,\ldots,I_r} \arrow[r] & X^I
\end{tikzcd}$$

where $X^I - \Delta_{I_1,\ldots,I_r} = \{(x_i) \mid \{x_j\}_{j\in I_1}, \ldots \{x_j\}_{j \in I_r} \text{ disjoint}\}$ is the complement of the "large diagonal." This parametrizes pullbacks $\mathscr{E} \dashrightarrow \mathscr{E}_1 \dashrightarrow \mathscr{E}_2 \dashrightarrow \ldots \to \mathscr{E}_n$ where the successive modifications occur at *disjoint* sets of points. Therefore, these modifications can be considered "independently" - this is called the *factorization property*.

What happens if we restrict to the diagonal? Heres one way to think about the diagonal. Any map of index sets $\varphi \colon I \twoheadrightarrow J$ induces $\Delta_\varphi \colon X^J \hookrightarrow X^I$, via $(x_j) \mapsto (x_{\varphi(i)})_{i \in I}$, i.e. $x_j$ is put into the $i$th coordinate if $\varphi(i) = j$.

In particular, if $I \twoheadrightarrow \{1\}$ collapses the index set, then the corresponding map $\Delta_\varphi \colon X \hookrightarrow X^I$ has image the "small" diagonal $(x, x, \ldots, x)$.

More generally, suppose that $I = I_1 \sqcup \ldots \sqcup I_r \twoheadrightarrow J = J_1 \sqcup \ldots \sqcup J_r$ with $I_i = \varphi^{-1}(J_i)$. Then for any $\lambda \colon I \to (\mathbb{Z}^n)^{\mathrm{dom}}$ we get $\varphi_* \lambda \colon J \to (\mathbb{Z}^n)^{\mathrm{dom}}$ by adding the values along fibers of $\varphi$. This induces a cartesian diagram

$$\begin{tikzcd}
\mathscr{H}_{\varphi_*\lambda}^{J_1,\ldots,J_r} \arrow[r] \arrow[d] & \mathscr{H}_\lambda^{I_1,\ldots,I_r} \arrow[d] \\
X^J \arrow[r, hook, "\Delta_\varphi"] & X^I
\end{tikzcd}$$

Now we define the corresponding notion of shtuka.

*Definition* 9.3.2. Let $I = I_1 \sqcup I_2 \sqcup \ldots \sqcup I_r$ and $\lambda \colon I \to (\mathbb{Z}^n)^{\mathrm{dom}}$. Then we define $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}$ by the pullback diagram

$$\begin{tikzcd} \mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r} \arrow[r] \arrow[d] & \mathscr{H}_\lambda^{I_1,\ldots,I_r} \arrow[d, "(p_0, p_r)"'] \\ \mathrm{Bun} \arrow[r, "(\mathrm{Id},\mathrm{Frob})"'] & \mathrm{Bun}_n \times \mathrm{Bun}_n \end{tikzcd}$$

In concrete terms, we think of $\mathscr{H}_\lambda^{I_1,\ldots,I_r}$ as parametrizing chains

$$\{\mathscr{E}_0 \dashrightarrow \mathscr{E}_1 \dashrightarrow \ldots \dashrightarrow \mathscr{E}_r\}$$

such that the successive modifications are bounded by $\lambda|_{I_1}, \lambda|_{I_2}$, etc.; then $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}$ parametrizes such chains with $\mathscr{E}_r = {}^\tau\mathscr{E}_1$.

9.4. **Partial Frobenius.** We can generalize the partial Frobenius map from earlier to a map $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r} \to \mathrm{Shtuka}_\lambda^{I_2,\ldots,I_r,I_1}$ as follows. We have a diagram

$$\begin{tikzcd} \mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r} \arrow[r] \arrow[d, "(p_0,\ldots,p_{r-1})"'] & X^I \\ \mathrm{Bun} \times \ldots \times \mathrm{Bun} & \end{tikzcd}$$

On points, $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}(S) \to \mathrm{Shtuka}_\lambda^{I_2,\ldots,I_r,I_1}(S)$ is defined by sending

$$(\mathscr{E}_0 \dashrightarrow \mathscr{E}_1 \dashrightarrow \ldots \dashrightarrow \mathscr{E}_r = {}^\tau\mathscr{E}_0) \mapsto (\mathscr{E}_1 \dashrightarrow \mathscr{E}_2 \dashrightarrow \ldots \dashrightarrow \mathscr{E}_r = {}^\tau\mathscr{E}_0 \dashrightarrow {}^\tau\mathscr{E}_1)$$

where the last has index set ${}^\tau I_1 = ({}^\tau x_j)_{j \in I_1}$, and the bounds are the same. This is called partial Frobenius because we are basically applying Frobenius only over the factor $X^{I_1}$:

$$\begin{tikzcd} \mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r} \arrow[r, "\mathrm{Frob}_{I_1}"'] \arrow[d] & \mathrm{Shtuka}_\lambda^{I_2,\ldots,I_r,I_1} \arrow[d] \\ X^I \arrow[r, "\mathrm{Frob}_{I_1}"'] & X^I \end{tikzcd}$$

the bottom map sending $(x_1, \ldots, x_r) \mapsto ({}^\tau x_1, y_2, \ldots, x_r)$. Composing all the partial Frobenius maps gives the usual (full) Frobenius:

$$\mathrm{Frob}_{I_r} \circ \ldots \circ \mathrm{Frob}_{I_1} = \mathrm{Frob}_{\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}/k}.$$

9.5. **Local structure.** We now want to discuss a local model for $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}$, for instance in order to understand its singularities.

*Definition* 9.5.1. We have a map $\mathrm{Spec}\, k \to \mathrm{Bun}_G$ corresponding to the trivial $G$-bundle. The fiber is called the *Beilinson-Drinfeld Grassmannian*:

$$\begin{tikzcd} \mathrm{Gr}_\lambda^{I_1,\ldots,I_n} \arrow[r, hook] \arrow[d] & \mathscr{H}_\lambda^{I_1,\ldots,I_n} \arrow[d, "p_n"'] \\ \mathrm{Spec}\, k \arrow[r, hook] & \mathrm{Bun}_G \end{tikzcd}$$

*Example* 9.5.2. If $I=\{1,\dots,m\}$, then

$$\mathrm{Gr}^I_\lambda=\{\mathscr{E}_0 \dashrightarrow \mathscr{E}_2\cong\mathscr{O}_X^n \mid {}^{\text{modification at } x_1,\dots,x_m}_{\text{bounded by } \lambda_i \text{ at } x_i}\}.$$

*Exercise* 9.5.3. Show that for any reductive group $G$,

$$\dim \mathrm{Gr}_\lambda=\langle 2\rho,\lambda\rangle$$

where $\rho=\frac{1}{2}\sum_{\alpha\in\Phi^+}\rho$ is the usual half sum of positive roots.

*Remark* 9.5.4. Although we are keeping $\mathrm{GL}_n$ in mind as our main example, we want the discussion to apply to general reductive groups.

What does it mean to "modify" a $G$-bundle when $G\neq\mathrm{GL}_n$? For general reductive groups $G$, there is an equivalence of categories between principal $G$-bundles over $X$ and tensor-functors $\mathbf{Rep}(G)\to\mathbf{Vect}(X)$, sending a principal $G$-bundle $\mathscr{E}$ to the functor

$$V\mapsto V_{\mathscr{E}}:=V\times_G\mathscr{E}:=V\times\mathscr{E}/G.$$

To modify the trivial $G$-bundle, we need to give, for each $V\in\mathbf{Rep}(G)$, a natural (rational) map $\mathscr{E}_V \dashrightarrow V\otimes_k\mathscr{O}_X$ such that $\varphi_V\otimes\varphi_W=\varphi_{V\otimes W}$, etc. The points at which $\mathscr{E}$ is a "modification" of $G\times X$ are those at which this rational map fails to be an isomorphism.

For the classical groups, one only needs to supply a modification for the standard representation, as this "generates" the category $\mathbf{Rep}(G)$. For instance, for $G=\mathrm{Sp}_{2n}$, to modify the trivial $G$-bundle we need a map $\mathscr{E}\dashrightarrow\mathscr{O}_X^{2n}$ such that the standard symplectic form on $\mathscr{O}_X^{2n}$ extends to a symplectic form $\bigwedge^2\mathscr{E}\to\mathscr{O}_X$.

The Beilinson-Drinfeld Grassmannian should be viewed as a "local (on $\mathrm{Bun}_G$) version" of $\mathscr{H}_\lambda^{I_1,\dots,I_n}$. There is an obvious forgetful map

$$\begin{tikzcd}\mathrm{Gr}_\lambda^{I_1,\dots,I_n} \arrow[d, "\pi"] \\ X^n\end{tikzcd}$$

It has the following nice factorization property: for each tuple of *distinct* geometric points $(x_1,\dots,x_n)\in X^I$, we have

$$\pi^{-1}(x_1,\dots,x_n)=\prod_{i=1}^n \mathrm{Gr}_{x_i,\lambda_i}$$

because we can *independently* specify the modifications at the different $x_i$.

We have a natural inclusion $\mathrm{Gr}_{x,\lambda}\subset\mathrm{Lat}_{n,x}=\{\mathscr{O}_x\text{-lattices in } F_x^n\}$. The latter is what we called the affine Grassmannian Gr. The subscheme $\mathrm{Gr}_{x,\lambda}$ consists of the subset of lattices $\Lambda$ such that the relative position of $\Lambda$ "in" $\mathscr{O}_x^n$ is $\prec\lambda$.

*Local structure.* Since all the $F_x$ "look the same" (independent of the geometric point $x$) $\mathrm{Gr}_{x,\lambda}$ should look like some kind of affine Grassmannian bundle over $X$.

For concreteness, suppose $I=\{i\}$. Then we want to make precise the statement that $\mathrm{Gr}_\lambda^I$ "looks like"

$$X\times\mathrm{Gr}_\lambda:=X\times\{\text{lattices } \Lambda\subset k((t))^n \text{ with rel. pos. } \prec\lambda \text{ w.r.t } k[[t]]^n\}$$

where $k = \overline{\mathbb{F}_q}$.

Let

$$\mathrm{Jet}_N(X)(k) = \{(x,\alpha) \mid x \in X(k), \alpha\colon \mathscr{O}_{X,x}/\mathfrak{m}_x^{N+1} \cong k[t]/k^{N+1}\}.$$

This admits an obvious map to $X$, so consider the diagram

$$\begin{tikzcd} & \mathrm{Gr}_\lambda^I \arrow[d] \\ \mathrm{Jet}_N(X)(k) \arrow[r] & X \end{tikzcd}$$

The map $\mathrm{Jet}_N(X)(k) \to X$ is an $\mathrm{Aut}(k[t]/t^{N+1})$-torsor. Such an automorphism can be described concretely: it is determined by the image of $t$, which can be any $a_0 + a_1 t + \ldots$ such that $a_0$ is invertible.

We claim that if $N$ is large enough, we can trivialize the pullback fibration over $\mathrm{Jet}_N(X)(k)$:

$$\begin{tikzcd} \mathrm{Gr}_\lambda \times \mathrm{Jet}_N(X) \arrow[r] \arrow[d] & \mathrm{Gr}_\lambda^I \arrow[d] \\ \mathrm{Jet}_N(X)(k) \arrow[r] & X \end{tikzcd}$$

Why? For $\lambda \geq 0$,

$$\mathrm{Gr}_{x,\lambda} = \{\Lambda \overset{\prec\lambda}{\hookrightarrow} \mathscr{O}_x^n\}.$$

But since $|\lambda|$ is bounded,

$$\{\Lambda \overset{\prec\lambda}{\hookrightarrow} \mathscr{O}_x^n\} = \{(\mathfrak{m}_x^{N+1}\mathscr{O}_x)^n \subset \Lambda \subset^{\prec\lambda} \mathscr{O}_x^n\} \text{ for } N \gg 0.$$

But that just describes an $\mathscr{O}_x/\mathfrak{m}_x^{N+1}$-submodule of $(\mathscr{O}_x/\mathfrak{m}_x^{N+1})^n$ bounded by $\lambda$. Since the Jet space has a *built-in* "trivialization" $\mathscr{O}_x/\mathfrak{m}_x^{N+1} \cong k[t]/t^{N+1}$, this problem is completely independent of $x$. Since $\mathrm{Jet}_N(X)$ is evidently étale over $X$ for any $N$, we obtain the desired *étale local* factorization.

This same argument works in general. $\mathrm{Gr}_\lambda^{I_1,\ldots,I_r} \cong \prod_{i=1}^r \mathrm{Gr}_{\lambda_i}^{I_i}$. Unfortunately, $\mathrm{Gr}_\lambda^I$ doesn't decompose étale locally if $|I| \geq 2$, as you'll get funny stuff over the diagonal. But if each $|I_i| = 1$, then one has étale locally

$$\mathrm{Gr}_\lambda^{I_1,\ldots,I_r} \sim (\prod \mathrm{Gr}_{\lambda_i}) \times X^I.$$

In fact, the globalization of these statements is true:

**Proposition 9.5.5.** *Let $I = I_1 \sqcup \ldots \sqcup I_r$. We have étale locally,*

$$\begin{tikzcd} \mathscr{H}_\lambda^{I_1,\ldots,I_r} \arrow[rr, squiggly, no head] \arrow[dr] & & \mathrm{Gr}_\lambda^{I_1,\ldots,I_r} \times \mathrm{Bun}_G \arrow[dl] \\ & \mathrm{Bun}_G & \end{tikzcd}$$

*and*

$$\begin{tikzcd}
\mathrm{Shtuka}_\lambda^{I_1,\dots,I_r} \arrow[rr, squiggly, dash] \arrow[dr] & & \mathrm{Gr}_\lambda^{I_1,\dots,I_r} \arrow[dl] \\
& X^I &
\end{tikzcd}$$

*meaning that there exist schemes $W$ and $U$ and commutative diagrams*

$$\begin{tikzcd}
& W \arrow[dl, "\text{ét surj}"'] \arrow[dr, "\text{ét}"] & \\
\mathscr{H}_\lambda^{I_1,\dots,I_r} \arrow[dr] & & \mathrm{Gr}_\lambda^{I_1,\dots,I_r} \times \mathrm{Bun}_G \arrow[dl] \\
& \mathrm{Bun}_G &
\end{tikzcd}$$

*and*

$$\begin{tikzcd}
& U \arrow[dl, "\text{ét surj}"'] \arrow[dr, "\text{ét}"] & \\
\mathrm{Shtuka}_\lambda^{I_1,\dots,I_r} \arrow[dr] & & \mathrm{Gr}_\lambda^{I_1,\dots,I_r} \arrow[dl] \\
& X^I &
\end{tikzcd}$$

We won't give the proof. It is similar to the special case that we discussed before for $G = \mathrm{GL}_n$, where $I_1 = \{1\}$ and $I_2 = \{2\}$, $\lambda_1 = (1, 0, \dots, 0)$ and $\lambda_2 = (0, \dots, 0, -1)$, in which case one gets $\mathrm{Gr}_{\lambda_1} \cong \mathbb{P}^{n-1}$ and $\mathrm{Gr}_{\lambda_2} \cong (\mathbb{P}^{n-1})^\vee$ and $\mathrm{Gr}_\lambda^{I_1,I_2} \sim (\mathbb{P}^{n-1} \times X) \times ((\mathbb{P}^{n-1})^\vee \times X)$.

## 10. Intersection Cohomology

10.1. **Intersection Cohomology sheaves.** We previously defined a projective variety $\mathrm{Gr}_\lambda$ over $k$, which was possibly singular. There is a canonically defined complex of $\mathbb{Q}_\ell$ sheaves $IC_\lambda$ on $\mathrm{Gr}_\lambda$, called *intersection cohomology sheaves*. There is much to say about intersection cohomology sheaves, but since we don't have much time to spend on them we will content ourselves with a bare bones definition. To describe these sheaves, we need the fact that the subscheme

$$\mathrm{Gr}_\lambda^0 = \{\Lambda \text{ rel. pos. exactly} = \lambda\} \subset \mathrm{Gr}_X,$$

is open and smooth. The complex of intersection cohomology sheaves has (and in fact, is characterized by) the following properties:

- $IC_\lambda|_{\mathrm{Gr}_\lambda^0} \cong \mathbb{Q}_\ell[\dim \mathrm{Gr}_\lambda^0]$ (i.e. the constant sheaf $\mathbb{Q}_\ell$ in degree $-\dim \mathrm{Gr}_\lambda^0$).
- The complex is self-dual under Verdier duality, i.e. $\mathbb{D}(IC_\lambda) \cong IC_\lambda$.
- The complex is indecomposable.
- There is a "boundedness condition" for the inclusion $\iota_\mu \colon \mathrm{Gr}_\mu^0 \hookrightarrow \mathrm{Gr}_\lambda$ (induced by $\mu \prec \lambda$): the complex $\iota_\mu^* IC_\lambda$ lies in degree $\leq -\dim \mathrm{Gr}_\mu^0 - 1$ if $\mu \neq \lambda$.

10.2. **Example computations.** We're going to use a more "practical" definition of the intersection cohomology sheaves, illustrated via examples.

*Example* 10.2.1. If $\lambda = (2, 0, \ldots, 0)$ then what does $IC_\lambda$ look like? First off,

$$\mathrm{Gr}_\lambda = \{\Lambda \subset \mathscr{O}^n \mid \mathscr{O}^n/\Lambda \text{ has length } 2\}.$$

This has two strata, one open and one closed. The open stratum $\mathrm{Gr}_\lambda^0$ consists of $\Lambda$ such that $\mathscr{O}^n/\Lambda \cong k[t]/t^2$. The closed stratum consists of $\Lambda$ where $\mathscr{O}^n/\Lambda \cong k^2$. Let's call this $\mathrm{Gr}_{\omega_2} = \mathrm{Gr}_{\omega_2}^0$ where $\omega_2 = (1, 1, 0, \ldots, 0)$. So we have a decomposition into two strata, both of which are (separately) smooth.

We can calculate $IC_\lambda$ by constructing a resolution of $\mathrm{Gr}_\lambda$. We already mentioned this last time in the $\mathrm{GL}_2$ case: when $n = 2$, $\mathrm{Gr}_\lambda$ looks like a singular quadric cone, and the resolution is the blowup. We resolved the singularity by a moduli space of lattices *specifying an additional intermediate lattice*. If $\omega_1 = (1, 0, \ldots, 0)$ then we have a map

$$\begin{tikzcd}
\mathrm{Gr}_{\omega_1,\omega_1} = \{\Lambda \overset{1}{\hookrightarrow} \Lambda' \overset{1}{\hookrightarrow} \mathscr{O}^n\} \arrow[r, equal] \arrow[d] & \mathrm{Gr}_\lambda^0 \cup \{\mathbb{P}_k^{n-1} \text{ fibration}/\mathrm{Gr}_{\omega_2}\} \arrow[d] \\
\mathrm{Gr}_\lambda = \{\Lambda \overset{2}{\hookrightarrow} \mathscr{O}^n\} \arrow[r, equal] & \mathrm{Gr}_\lambda^0 \cup \mathrm{Gr}_{\omega_2}
\end{tikzcd}$$

since over $\Lambda \in \mathrm{Gr}_\lambda^0$, specifying $\Lambda'$ corresponds to specifying a $\varpi$-stable line in the quotient $\varpi^{-1}\Lambda/\Lambda$, but there is a unique such choice. On the other hand, over $\mathrm{Gr}_{\omega_2}$ we can specify it freely.

*Theorem* 10.2.2. *If $f \colon X \to Y$ is a birational projective morphism, then*

$$Rf_* IC_X = IC_Y \oplus (\ldots).$$

This *deep* fact is a special case of a general Decomposition Theorem. We won't discuss the proof.

We can use the theorem if we can identify the intersection cohomology resolution upstairs. But that is smooth, because it is clearly a $\mathbb{P}^{n-1}$ fibration over $\mathbb{P}^{n-1}$ (in the case $n=2$ which we studied before, it is $\mathbb{P}(\mathscr{O}\oplus\mathscr{O}(-2))$), and we know that on a smooth scheme the intersection cohomology is just the constant sheaf, in this case $\mathbb{Q}_\ell[2(n-1)]$.

So by the special case of the decomposition theorem, we know that

$$R\pi_*\mathbb{Q}_\ell[2(n-1)] = IC_\lambda \oplus (?).$$

The (?) must be supported on the closed stratum, since $R\pi_*\mathbb{Q}_\ell[2(n-1)]$ and $IC_\lambda$ agree on the open stratum by the definition of the intersection cohomology sheaf (there is "no room" for anything else).

To detect the mystery summand, we take the stalk of $R\pi_*\mathbb{Q}_\ell[2(n-1)]$ at $s\in \mathrm{Gr}_{\omega_2}$. Using a form of proper base change (which is actually *easier* for $\ell$-adic sheaves) we obtain

$$H^*(\mathbb{P}^{n-1})[2(n-1)] = i_s^* IC_\lambda \oplus i_s^*(?).$$

The first summand on the right hand side is non-zero, and has degree $\le -\dim \mathrm{Gr}_{\omega_2} - 1 = -2(n-2)-1$ (the fourth property of IC sheaves above), while the left hand side is supported in degree $-2(n-1), -2(n-2), \ldots, -2, 0$. Thus we see that $i_s^* IC_\lambda$ "takes up" the part supported in degree $-2(n-1)$, and the rest must be from (?).

In the special case $n=2$, we see that (?) is a skyscraper sheaf of dimension 1 supported at $s$. Interesting! This prompts the speculation: is $(?) = \mathbb{Q}_\ell[\dim \mathrm{Gr}_{\omega_2}] = IC_{\omega_2}$ on $\mathrm{Gr}_{\omega_2}$? The answer is yes, by a form of equivariance that we will discuss later. So the conclusion is that

$$\boxed{R\pi_* IC_{\mathrm{Gr}_{\omega_1,\omega_1}} \cong IC_\lambda \oplus IC_{\omega_2}}$$

where we are really abusing notation by identifying $IC_{\omega_2}$ with its pushforward via the natural inclusion.

In particular, we've found that $IC_\lambda|_{\mathrm{Gr}_{\omega_2}} \cong \mathbb{Q}_\ell[\dim \mathrm{Gr}_{\omega_2}]$ as well. Therefore,

$$\boxed{IC_\lambda \cong \mathbb{Q}_\ell[2(n-1)]}.$$

When $n=2$, this is a reflection of the fact that the cone, while not smooth, is *rationally smooth.*

This is a general phenomenon:

**Theorem 10.2.3.** *The map* $\mathrm{Gr}_{\lambda,\mu} \to \mathrm{Gr}_{\lambda+\mu}$ *sending*

$$\{\Lambda \overset{\prec\lambda}{\dashrightarrow} \Lambda' \overset{\prec\mu}{\dashrightarrow} \mathscr{O}^n\} \to \{\Lambda \overset{\prec\lambda+\mu}{\dashrightarrow} \mathscr{O}^n\}$$

*is a "partial resolution" in the sense that:*

$$\boxed{R\pi_{\lambda,\mu *} IC_{\lambda,\mu} \cong \bigoplus_{\nu\le\lambda+\mu} IC_\nu^{m_\nu}}$$

*where the embedding* $\mathrm{Gr}_\nu \hookrightarrow \mathrm{Gr}_{\lambda+\mu}$ *is induced by* $\nu \prec \lambda+\mu$.

*Example* 10.2.4. Let $G = \mathrm{GL}_n$ and $\lambda = (1,0,\dots,0,-1)$. We view this as the highest root of $\widehat{\mathrm{GL}}_n$ (the Langlands dual). Let's try to compute the IC sheaf of $\mathrm{Gr}_\lambda$. For $n=2$,

$$\mathrm{Gr}_{(1,-1)} = \{\Lambda \subset F^2 \mid (\varpi\mathscr{O})^2 \subset \Lambda \subset (\varpi^{-1}\mathscr{O})^2, [\mathscr{O}^2 : \Lambda] = 1\}.$$

As before, we compute by "resolving" into steps

$$(1,0,\dots,0,-1) = \underbrace{(1,0,\dots,0)}_{\omega_1} + \underbrace{(0,\dots,0,-1)}_{\omega_2}.$$

Then we have

$$\begin{tikzcd} \widetilde{X} = \mathrm{Gr}_{\omega_1,\omega_2} \arrow[r, equal] \arrow[d, "\pi"'] & \{\Lambda_0 \overset{1}{\hookrightarrow} \Lambda_1 \overset{1}{\hookleftarrow} \mathscr{O}^2 \mid (*)\} \arrow[d, "\pi"'] \\ X = \mathrm{Gr}_\lambda \arrow[r, equal] & \{\Lambda_0 \mid (*)\} \end{tikzcd}$$

When $\Lambda_0$ and $\mathscr{O}^2$ are not isomorphic, $\Lambda_1$ is uniquely determined. If $\Lambda_0 \cong \mathscr{O}^2$, we can choose $\Lambda_1$ arbitrarily. Therefore, over the open stratum where $\Lambda_0 \not\cong \mathscr{O}^2$, the map is an *isomorphism*. Over the point $\Lambda_0 \cong \mathscr{O}^2$, we have a $\mathbb{P}^{n-1}$ in the fiber.

In the case $n=2$, $X$ is birational to $\mathbb{P}^1 \times \mathbb{P}^1$ and we see that $\dim \widetilde{X} = \dim X = 2(n-1)$.

Let $\delta$ be the skyscraper sheaf at pt, the distinguished point of $X$ corresponding to the trivial bundle $\mathscr{O}^n$.

*Proposition* 10.2.5. *$R\pi_*\mathbb{Q}_\ell[2(n-1)]$ is a direct sum of shifts of $IC_X$ and $\delta$.*

This comes from a general theorem that the right derived functors of the pushforward must be a direct sum of simple perverse sheaves. In this case, there is an equivariance constraint that forces these two possibilities. To elaborate, there is an action of $\mathrm{GL}_n(\mathscr{O})$ on $X$ and $\widetilde{X}$ induced by the action on the standard lattice $\mathscr{O}^n$, which changes $\Lambda_0$ and $\Lambda_1$ but not the relative positions. The strata we just described are orbits of $\mathrm{GL}_n(\mathscr{O})$. [We regard $\mathrm{GL}_n(\mathscr{O}) = \varprojlim \mathrm{GL}_n(\mathscr{O}/\varpi^i)$, but here the action factors through $\mathrm{GL}_n(\mathscr{O}/\varpi^2)$.] On then checks that $\overline{IC}|_X$ and $\delta$ are the only equivariant sheaves. This is non-trivial (by the way, we should be over an algebraically closed base field for it to be true) - a subtle point is to rule out local systems, which comes from the fact that the action has connected stabilizers. ♠♠♠ TONY: [???]

So we get that

$$R\pi_*\mathbb{Q}_\ell([2(n-1)]) = \left(\bigoplus_i IC_X[a_i]\right) \oplus \left(\bigoplus_i \delta[b_i]\right).$$

Taking the stalk over $X^0$ (the smooth open stratum), the definition of intersection cohomology sheaves says that $IC_X|_{X^0} \cong \mathbb{Q}_\ell[2(n-1)]$. Therefore, we see that only $IC_X$ appears, with no shift and multiplicity 1.

Next let's try taking the stalk at pt:

$$i^*_{\mathrm{pt}} R\pi_*\mathbb{Q}_\ell[2(n-1)] \cong H^{*+2(n-1)}(\mathbb{P}^{n-1})$$

which is supported in degrees $-2(n-1), -2(n-1)+2, \dots, -2, 0$.

What else can we use? Note that $R\pi_*\mathbb{Q}_\ell[2(n-1)]$ is self-dual under Verdier duality, as $\mathbb{Q}_\ell[2(n-1)]$ was self-dual and then we pushed it forward under a proper map, which preserves the self-duality. As Verdier duality *negates* degrees, if $b_i$ appears then so does $-b_i$. But the $b_i$'s must be non-positive, by our observations concerning the cohomology of projective space, so $b_i = 0$. Then we also see that the multiplicity is at most 1, as that is the case for the top cohomology of projective space.

So we have narrowed down to two possibilities. Either

$$R\pi_*\mathbb{Q}_\ell[2(n-1)] = IC_X \oplus \delta \text{ or } IC_X.$$

However, recall that $i_{\mathrm{pt}}^* IC_X$ lies in degree $< 0$. Therefore, it cannot capture the top cohomology which we saw lies in degree 0, so $\delta$ must appear!

Therefore, $\iota_{\mathrm{pt}}^* IC_X$ has dimension 1 in degrees $-2(n-1), -2(n-1)+2, \ldots, -2$. The sheaf appearing in degree $-2(n-1)$ is the constant sheaf on $X$, shifted by $2(n-1)$. The other terms measure the singularity of $X$ at pt. More precisely,

$$i_{\mathrm{pt}}^* IC_X \cong H^{<2n}(U - \mathrm{pt})$$

where $U$ is some neighborhood of pt. One can view $U - \mathrm{pt}$ as being homotopy equivalent to a real manifold of real dimension $4(n-1)-1$ (the $-1$ coming from shrinking the $\mathbb{C}^*$-bundle to an $S^1$ bundle), so this is basically saying that the intersection cohomology sheaf captures "half" of the local cohomology.

In the case $n = 2$, we get that $IC_X \cong \mathbb{Q}_\ell[2]$, which is the same as it would be if $X$ were smooth, so the singularity is *not detected* by cohomology with rational coefficients - thus it is "rationally smooth."

10.3. **Semi-small maps.** The fact that $R\pi_* IC_{\lambda_1,\lambda_2}$ is a direct sum of $IC$ sheaves follows from the "stratified semi-smallness" of the map $\pi\colon \mathrm{Gr}_{\lambda_1,\lambda_2} \to \mathrm{Gr}_{\lambda_1+\lambda_2}$, which is a notion of morphism that plays well with *perverse sheaves* (a category of sheaves in which the IC sheaves are naturally viewed).

*Definition* 10.3.1. If $\pi\colon X \to Y$ is proper and generically finite, and $X$ is smooth, then $\pi$ is *semi-small* if for all $d$,

$$\mathrm{codim}_Y\{y \in Y \mid \dim f^{-1}(y) \geq d\} \geq 2d.$$

*Example* 10.3.2. Let $X$ be a surface and $\pi\colon \widetilde{X} \to X$ the blowup along a finite collection of points. Then the blown up points are the only ones whose fibers is positive-dimensional, and they have codimension 2 in $X$, so $\pi$ is semi-small.

However, if you blow up a threefold at a point, then the fiber is 2-dimensional but the points have codimension $3 < 4$, so the blowup map is *not* semi-small.

*Exercise* 10.3.3. Prove that $X \to Y$ is semi-small if and only if $\dim(X \times_Y X) = \dim X$.

**Theorem 10.3.4.** *Suppose $\pi\colon X \to Y$ is semi-small. Then*

$$R\pi_*\mathbb{Q}_\ell[\dim X] = \bigoplus \textit{(IC sheaves)}.$$

*Remark* 10.3.5. Another way of formulating the right hand side is that the IC sheaves are precisely the simple perverse sheaves.

The Decomposition Theorem says that the derived pushfoward is a direct sum of shifted simple perverse sheaves (and this is already a very deep result), and this theorem is saying that there are *no shifts*. So it is a refinement of the decomposition theorem in this case.

This applies in our situation because the relevant maps of Beilinson-Drinfeld grassmannians, e.g. $\mathrm{Gr}_{\lambda,\mu} \to \mathrm{Gr}_{\lambda+\mu}$, are "stratified semi-small."

*Definition* 10.3.6. A map $\pi \colon X \to Y$ is called *stratified semi-small* if $\pi$ is proper and generically finite, and we can stratify $X = \bigcup_\alpha X_\alpha$ and $Y = \bigcup_\beta Y_\beta$ (with each stratum irreducible) such that for all $\alpha, \beta$ and all $y \in Y_\beta$,

$$\dim(f^{-1}(y) \cap X_\alpha) \leq \frac{\dim X_\alpha - \dim Y_\beta}{2}.$$

*Remark* 10.3.7. A semi-small map is not stratified semi-small with the trivial stratifications; instead one should stratify by dimensions of the fibers.

**Theorem 10.3.8.** *The conclusion of Theorem 10.3.4 applies if $f$ is stratified semi-small.*

## 11. Geometric Satake

11.1. **The Geometric Satake Equivalence.** For now, we work over an *algebraically closed* field $k$. Write $\mathrm{Gr} = \varinjlim_\lambda \mathrm{Gr}_\lambda$. From $\lambda$ we get an intersection cohomology sheaf $IC_\lambda$ on $\mathrm{Gr}_\lambda$ extended by 0 to Gr.

We consider a category $\mathscr{P}(\mathrm{Gr})$, which is defined as the subcategory of the derived category of complexes of sheaves on Gr whose objects are finite successive extensions of $IC_\lambda$, i.e. $\mathscr{F} \in \mathscr{P}(\mathrm{Gr})$ if there exist short exact sequences

$$\begin{aligned} 0 \to \mathscr{F}^1 \to \mathscr{F} \to IC_{\lambda_1} \to 0 \\ 0 \to \mathscr{F}^2 \to \mathscr{F}^1 \to IC_{\lambda_2} \to 0 \\ \vdots \\ 0 \to \mathscr{F}^r \to IC_{\lambda_r} \to 0 \end{aligned}$$

Note that we are *not* allowing shifts and twists. This is an abelian category, and it turns out that all objects are in fact direct sums of $IC_\lambda$, so it is quite concrete. Another way to describe $\mathscr{P}(\mathrm{Gr})$, if it helps, is that it is the abelian category of perverse sheaves generated by the $IC_\lambda$.

**The fusion product.** There is a convolution product on sheaves in Gr. If $\lambda_1, \lambda_2 \in X_\bullet(T)^{\mathrm{dom}}$, we have a map

$$\begin{tikzcd} \mathrm{Gr}_{\lambda_1,\lambda_2} \arrow[r, equal] \arrow[d] & \{\Lambda_0 \overset{\prec\lambda_1}{\dashrightarrow} \Lambda_1 \overset{\prec\lambda_2}{\dashrightarrow} \mathscr{O}^n\} \arrow[d, "\text{proper, birational}"'] \\ \mathrm{Gr}_{\lambda_1+\lambda_2} \arrow[r, equal] & \{\Lambda_0 \overset{\prec\lambda_1+\lambda_2}{\dashrightarrow} \mathscr{O}^n\} \end{tikzcd}$$

This is an *isomorphism* over the open stratum where the relative position is exactly $\lambda_1 + \lambda_2$. If $IC_{\lambda_1,\lambda_2}$ is the intersection cohomology complex of $\mathrm{Gr}_{\lambda_1,\lambda_2}$ then by Theorem 10.2.3 we have

$$R\pi_* IC_{\lambda_1,\lambda_2} = IC_{\lambda_1+\lambda_2} \oplus \bigoplus_{\mu \prec \lambda_1+\lambda_2} IC_\mu^{m_\mu(\lambda_1,\lambda_2)}.$$

We then define

$$IC_{\lambda_1} * IC_{\lambda_2} := R\pi_* IC_{\lambda_1,\lambda_2}.$$

It's not easy to see that this is actually associative, etc. without a more extensive discussion of perverse sheaves, which we don't want to have.

There is another perspective which makes the associativity clearer. Consider the diagram

$$\begin{tikzcd} \mathrm{Gr}_{\lambda_1,\lambda_2} \arrow[r, hook] & \mathrm{Gr}^{(2)} \arrow[r, equal] \arrow[d, "\pi"'] & \{\Lambda_0 \dashrightarrow \Lambda_1 \dashrightarrow \mathscr{O}^n\} \arrow[r, equal] \arrow[d, "\pi"'] & \mathrm{Gr} \times \mathrm{Gr} \\ & \mathrm{Gr} \arrow[r, equal] & \{\Lambda_0\} & \end{tikzcd}$$

Let $\mathscr{F}_1, \mathscr{F}_2 \in \mathscr{P}(\mathrm{Gr})$; we want to define a "convolution product" by $\mathscr{F}_1, \mathscr{F}_2 \mapsto \pi_!(\mathscr{F}_1 \boxtimes \mathscr{F}_2)$. Unfortunately the isomorphism $\mathrm{Gr}^{(2)} \cong \mathrm{Gr} \times \mathrm{Gr}$ really only occurs at "infinite level," where $\Lambda_0$ and $\Lambda_1$ really do become "independent."

There is a way of making this work "at finite level," i.e. on finite type schemes. The idea is to create a "twisted product" $\mathrm{Gr}_{\lambda_1,\lambda_2} \to \mathrm{Gr}_{\lambda_2}$ which is a fibration with fibers isomorphic to $\mathrm{Gr}_{\lambda_1}$. One can create a twisted version of the exterior tensor product on the twisted product, $\mathscr{F}_1 \widetilde{\boxtimes} \mathscr{F}_2$ on $\mathrm{Gr}^{(2)}$, and we define

$$\mathscr{F}_1 * \mathscr{F}_2 = R\pi_*(\mathscr{F}_1 \widetilde{\boxtimes} \mathscr{F}_2)$$

Associativity is now automatic from that for $\widetilde{\boxtimes}$. The unit is $\delta$, the skyscraper sheaf at $pt = \mathrm{Gr}_0$.

**Theorem 11.1.1** (Geometric Satake Equivalence)**.** *We have the following properties of* $\mathscr{P}(\mathrm{Gr})$.

(1) $\mathscr{P}(\mathrm{Gr})$ *is a semisimple abelian category, and the convolution product* $*\colon \mathscr{P}(\mathrm{Gr}) \times \mathscr{P}(\mathrm{Gr}) \to \mathscr{P}(\mathrm{Gr})$ *makes* $\mathscr{P}(\mathrm{Gr})$ *a tensor category (a category equipped with a bifunctor satisfyng associativity, commutativity, ...).*

(2) *There is a tensor-equivalence of categories* $\mathscr{S} : \mathscr{P}(\mathrm{Gr}) \to \mathrm{Rep}(\widehat{G})$*, such that the following diagram commutes:*

$$\begin{tikzcd} \mathscr{P} \arrow[rr, "\mathscr{S}"] \arrow[dr, "H(G{,}-)"'] & & \mathrm{Rep}(\widehat{G}) \\ & \mathbf{Vec} \arrow[ur, "forget"'] & \end{tikzcd}$$

*Remark* 11.1.2. This implies that every representation of $\mathrm{Rep}(\widehat{G})$ has a natural grading (coming from the grading on cohomology). This may be surprising at first, but remember that $\widehat{G}$ is not just a reductive group; it comes with a *split torus.* The map $\mathbb{G}_m \xrightarrow{2\rho} \widehat{T} \subset \widehat{G}$ gives any $\widehat{G}$-representation an action of $\mathbb{G}_m$, i.e. a $\mathbb{Z}$-grading, and this is the same as the grading coming from cohomology.

Let's highlight the concrete meaning of this being an equivalence of tensor categories. Let $\pi$ denote the map

$$\begin{tikzcd} \mathrm{Gr}_{\lambda_1,\lambda_2} \arrow[d, "\pi"'] \\ \mathrm{Gr}_{\lambda_1+\lambda_2} \end{tikzcd}$$

so that $R\pi_* IC_{\lambda_1,\lambda_2} = IC_{\lambda_1} * IC_{\lambda_2}$ (by definition). Then

$$\mathscr{S}(IC_{\lambda_1} * IC_{\lambda_2}) \cong V_{\lambda_1} \otimes V_{\lambda_2}.$$

Now,

$$IC_{\lambda_1} * IC_{\lambda_2} = \bigoplus_{\mu \prec \lambda_1+\lambda_2} IC_\mu \otimes M^\mu_{\lambda_1,\lambda_2}$$

where $M^{\mu}_{\lambda_1,\lambda_2} = \mathrm{Hom}_{\widehat{G}}(V_\mu, V_{\lambda_1} \otimes V_{\lambda_2})$ regarded as a trivial representation (i.e. just to put the multiplicities in), which corresponds to the "Clebsch-Gordon" decomposition

$$V_{\lambda_1} \otimes V_{\lambda_2} = \bigoplus_{\mu \prec \lambda_1+\lambda_2} V_\mu \otimes M^{\mu}_{\lambda_1,\lambda_2}.$$

*Example* 11.1.3. Let $\widehat{G} = \mathrm{GL}(V)$, $\lambda = (2,0,\ldots,0) = (1,0,\ldots,0)+(1,0,\ldots,0)$. Then we computed in Example 10.2.1 that

$$IC_{(1,0,\ldots,0)} * IC_{(1,0,\ldots,0)} \cong IC_{(2,0,\ldots,0)} \oplus IC_{(1,0,\ldots,0)}.$$

Under the geometric Satake correspondence, $IC_{(2,0,\ldots,0,)} \longleftrightarrow \mathrm{Sym}^2 V$ and $IC_{(1,1,0,\ldots,0)} \longleftrightarrow \bigwedge^2 V$, so our computation in Example 10.2.1 just reflects the classical decomposition

$$V \otimes V = \mathrm{Sym}^2 V \oplus \bigwedge^2 V.$$

*Example* 11.1.4. Let $\widehat{G} = \mathrm{GL}(V)$, $\lambda = (1,0,\ldots,0,-1) = (1,0,\ldots,0)+(0,\ldots,0,-1)$. Then we computed in Example 10.2.4 that

$$IC_{(1,0,\ldots,0)} * IC_{(0,\ldots,0,-1)} = IC_{(1,0,\ldots,0,-1)} \oplus IC_{(0,0,\ldots,0)}.$$

Under geometric Satake, this recovers the classical decomposition

$$V \otimes V^* \cong \mathrm{End}^0(V) \oplus 1$$

11.2. **A mixed version.** Previously, we conidered the category

$$\mathscr{P}(\mathrm{Gr}) = \left\{ \bigoplus_{\lambda \in X_\bullet(T)^{\mathrm{dom}}} IC_\lambda^{\oplus m_\lambda} \right\}.$$

over $k$ and algebraically closed field.

Now we want to work over $k = \mathbb{F}_q$, and view Gr as a direct limit of projective schemes over $k$., and develop a "mixed" version of Geometric Satake. Working over a finite $k$ adds a whole new slew of étale sheaves, as we can twist by any non-trivial character coming from the ground field.

*Example* 11.2.1. After making a choice of $q^{1/2}$, we can define $\overline{\mathbb{Q}_\ell}(1/2)$ to be the étale sheaf on Spec $k$ corresponding to the character $\mathrm{Gal}(\overline{k}/k) \to \overline{\mathbb{Q}_\ell}^\times$ sending Frob $\mapsto q^{-1/2}$. (The $q^{1/2}$ is for normalization purposes, which will be made clearer later.)

*Definition* 11.2.2. Given $\mathscr{F}$ on $\mathrm{Gr} \xrightarrow{\pi} \mathrm{Spec}\, k$, we define

$$\mathscr{F}(n/2) := \mathscr{F} \otimes (\pi^* \overline{\mathbb{Q}_\ell}(1/2)^{\otimes n}).$$

We are now prepared the analogue of the category $\mathscr{P}(\mathrm{Gr})$.

*Definition* 11.2.3. We define $\mathscr{P}_{\mathrm{mix}}(\mathrm{Gr})$ to be the semisimple subcategory of perverse sheaves generated by $IC_\lambda(n/2)^{m(\lambda,n)}$ over all $n, \lambda$:

$$\mathscr{P}_{\mathrm{mix}}(\mathrm{Gr}) := \left\{ \bigoplus_{\lambda, n \in \mathbb{Z}} IC_\lambda(n/2)^{m(\lambda,n)} \right\}.$$

*Remark* 11.2.4. The category $\mathscr{P}_{\text{mix}}(\text{Gr})$ is semisimple *by definition.* If it helps, $\mathscr{P}_{\text{mix}}(\text{Gr})$ is the full subcategory of "perverse Weil sheaves on Gr" but not including non-trivial extensions.

it is a theorem that $\mathscr{P}_{\text{mix}}(\text{Gr})$ is stable under the convolution product. In fact, there is a convolution product on perverse Weil sheaves in general. In our case, it turns out that

$$IC_\lambda * IC_\mu = \bigoplus_\nu IC_\nu^{\bigoplus m_\nu(\lambda,\mu)}.$$

The striking features is that *no twists are necessary!* It's easy to show that this is true after semisimplification, but the fact that it's true on the nose is difficult. This shows that

$$\mathscr{P}_0(\text{Gr}) := \left\{ \bigoplus IC_\lambda \right\} \subset P_{\text{mix}}(\text{Gr})$$

is also stable under $*$.

We have a natural (tensor)-equivalence $\mathscr{P}_0(\text{Gr}) \cong \mathscr{P}(\text{Gr}_{\overline{k}})$. (This is clear on objects, and it follows for endomorphisms by semisimplicity). By Geometric Satake, both are equivalent to $\text{Rep}(\widehat{G})$.

The link between geometric and classical Satake is via the *Grothendieck group*, which we now consider. If $v$ denotes the class of $\overline{\mathbb{Q}_\ell}(-1/2)$ in the Grothendieck group $K_0(\mathscr{P}_{\text{mix}}(\text{Gr}))$, then

$$K_0(\mathscr{P}_{\text{mix}}(\text{Gr})) \cong K_0(P_0(\text{Gr})) \otimes_{\mathbb{Z}} \mathbb{Z}[v, v^{-1}] \underbrace{\cong}_{\text{geom. Satake}} R(\widehat{G}) \otimes_{\mathbb{Z}} \mathbb{Z}[v, v^{-1}]$$

where $R(\widehat{G})$ is the Grothendieck group of $\text{Rep}(\widehat{G})$.

11.3. **The function-sheaf correspondence.** Let $\mathscr{H} = C_c(G(\mathscr{O})\backslash G(F)/G(\mathscr{O}); \mathbb{Z})$ where $F = k((\varpi))$ (this is an integral version of the local Hecke algebra). Then we have a "sheaf-to-function" map

$$\begin{array}{c} K_0(P_{\text{mix}}(\text{Gr})) \\ \downarrow \\ \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \end{array}$$

Let's remind you of how this correspondence works. If $X/k$ is a scheme and $\mathscr{F}$ is a constructible étale sheaf on $X$ with $\overline{\mathbb{Q}_\ell}$-coefficients (stalks are $\overline{\mathbb{Q}_\ell}$-vector spaces), then we define $f_{\mathscr{F}} : X(k) \to \overline{\mathbb{Q}_\ell}$ by

$$x \mapsto \text{Tr}(\text{Frob}_x)|_{\mathscr{F}_{\overline{x}}}.$$

More generally, if $\mathscr{F}$ is a *complex* of sheaves then

$$\boxed{f_{\mathscr{F}}(x) = \sum_{i \in \mathbb{Z}} (-1)^i \, \text{Tr}(\text{Frob}_x)|_{H^i \mathscr{F}_{\overline{x}}}}.$$

This gives a functor $\mathbf{Shv}(X) \to \mathbf{Fun}(X(k))$ sending

$$\begin{aligned} \mathscr{F} &\mapsto f_{\mathscr{F}} \\ \pi^* &\mapsto \pi^* \\ \pi_! &\mapsto \text{summation along fibers} \\ \otimes &\mapsto \text{pointwise multiplication.} \end{aligned}$$

Moreover, as the functor is additive it factors through $K_0(\mathbf{Shv}(X))$.

The function-sheaf correspondence induces a map

$$K_0(\mathscr{P}_{\text{mix}}(\text{Gr})) \to \text{Fun}_c(\text{Gr}(k) = G(F)/G(\mathscr{O}), \mathbb{Z}[q^{\pm 1/2}]).$$

This is because the eigenvalues of Frobenius are all half-integral powers of $q$, by design. But the objects $\mathscr{P}_{\text{mix}}(Gr)$ are sums of twists of IC sheaves, which are locally constant along each $G(\mathscr{O})$-orbit under left multiplication (as these are all smooth). Therefore, the image of $K_0(\mathscr{P}_{\text{mix}}(\text{Gr}))$ lies in

$$\text{Fun}_c(G(\mathscr{O})\backslash G(F)/G(\mathscr{O}), \mathbb{Z}[q^{\pm 1/2}]) = \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}].$$

This discussion has shown:

**Proposition 11.3.1.** *The function-sheaf correspondence factors through the isomorphism*

$$\begin{tikzcd} K_0(\mathscr{P}_{mix}(\text{Gr})) \arrow[d, "\cong"'] \arrow[dd, bend right=60, "\textit{function-sheaf}"'] \\ \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[v^{\pm 1/2}] \arrow[d, "v \mapsto q^{-1/2}"] \\ \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \end{tikzcd}$$

11.4. **Classical and Geometric Satake.** We can now state the compatibility between geometric and classical Satake.

**Theorem 11.4.1.** *The following diagram commutes.*

$$\begin{tikzcd} K_0(P_{mix}(\text{Gr})) \arrow[r, "\textit{geometric Satake}", "\sim"'] \arrow[d, "\sim"', "\textit{function-sheaf}"] & R(\widehat{G}) \arrow[d, equal] \\ \mathscr{H} \otimes_{\mathbb{Z}} \mathbb{Z}[q^{\pm 1/2}] \arrow[r, "\textit{classical Satake}", "\sim"'] & R(\widehat{G}) \end{tikzcd}$$

*Proof.* The theorem is a bit of a cheat, as we have to use a different normalization for the classical Satake isomorphism than we did before.

Let $1_\lambda$ denote the characteristic function of $G(\mathscr{O})\varpi^\lambda G(\mathscr{O}) \in \mathscr{H}$. Then $\{1_\lambda\}$ is a $\mathbb{Z}$-basis of $\mathscr{H}$, and

$$1_\lambda \cdot 1_\mu = \sum_{\nu \prec \lambda+\mu} (\text{universal polynomial in } q) 1_\nu$$

where the "universal polynomials" depend only on $\lambda, \mu, \nu$. In terms of our preferred bases, the diagram is as follows

$$\begin{tikzcd}
{[IC_\lambda]} \arrow[r] \arrow[d, mapsto] & {[V_\lambda]} \arrow[d, mapsto] \\
C_\lambda = (-q^{1/2})^{\langle 2\rho, \lambda\rangle} 1_\lambda + (\text{lower-order}) & {[V_\lambda]}
\end{tikzcd}$$

(The factor $(-q^{1/2})^{\langle 2\rho,\lambda\rangle}$ appears because of the shift involved in defining $IC_\lambda$ in order to make it Verdier self-dual; note that $\dim \mathrm{Gr}_\lambda = \langle 2\rho, \lambda\rangle$.)

We had previously set up the (classical) Satake isomorphism to send $1_\lambda$+(lower-order) to $[V_\lambda]$. However, as mentioned at the beginning of the proof, we are going to re-normalize it to make the diagram commute, so that the end conclusion is somewhat tautological. $\square$

*Example* 11.4.2. Consider $G = \mathrm{GL}_n$. If

$$\omega_i = (\underbrace{1,\dots,1}_{i \text{ ones}}, 0, \dots, 0)$$

then $\mathrm{Gr}_{\omega_i} \cong \mathrm{Gr}(i,n)$, and $C_{\omega_i} = (-q^{1/2})^{i(n-i)} 1_{\omega_i} \longleftrightarrow [\bigwedge^i \mathbb{C}^n]$.

*Example* 11.4.3. There are bijections.

$$\begin{tikzcd}
\left\{ \text{characters } \mathscr{H} \to \overline{\mathbb{Q}_\ell} \right\} \arrow[r, leftrightarrow] \arrow[d, leftrightarrow] & \left\{ \text{s.s. conj. classes in } \widehat{G}(\overline{\mathbb{Q}_\ell}) \right\} \\
\left\{ \begin{smallmatrix} \text{irr. rep'n. of } G(F) \\ \text{with } G(\mathscr{O})\text{-inv't vector} \end{smallmatrix} \right\} /\text{iso} &
\end{tikzcd}$$

By composing these bijection, we should be able to associate a unique conjugacy class in $\widehat{G}(\overline{\mathbb{Q}_\ell})$ to any representation in the bottom set. What conjugacy class does the trivialization representation of $G(F)$ correspond to?

You might guess the trivial, but because of the normalization we have chosen, this is not the case. The answer turns out to be the conjugacy class of $2\rho(-q^{1/2}) \in \widehat{G}(\overline{\mathbb{Q}_\ell})$ viewing $2\rho$ as a cocharacter $\mathbb{G}_m \to \widehat{G}$.

*Exercise* 11.4.4. Check this.

The representation corresponding to $C_\lambda$ acts on the $G(\mathscr{O})$-invariant vector by $\mathrm{Tr}(g)|_{V_\lambda}$, which can be interepreted more geometrically as $\mathrm{Tr}(\mathrm{Frob})|_{H^*(IC_\lambda))}$.

## 12. Lafforgue's Excursion Operators

**12.1. Sheaves on moduli of shtukas.** Consider the map

$$\begin{tikzcd} \mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r} \arrow[d] \\ X^I \end{tikzcd}$$

where $I = I_1 \cup \ldots \cup I_r$. Recall that by Proposition 9.5.5, this map looks étale locally like $\mathrm{Gr}_{\lambda}^{I_1,\dots,I_r} \to X^I$.

*Definition* 12.1.1. We set $\mathscr{F}_{\lambda}^{I_1,\dots,I_r}$ to be the IC-sheaf on $\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r}$.

There are a few technical issues here. $\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r}$ is only étale-locally isomorphic to the Beilinson-Drinfeld grassmannian; in fact, it is of infinite type. We're going to brush these concerns under the rug and pretend we are working with honest finite type schemes and constructible sheaves. You can just imagine that by bounding $\lambda$, we can exhaust $\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r}$ by finite-type strata. Viewing

$$\mathrm{Shtuka}^{I_1,\dots,I_r} = \varinjlim_{\lambda} \mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r}$$

we can then define $\mathscr{F}_{\lambda}^{I_1,\dots,I_r} = \varinjlim_{\lambda} IC_{\lambda}^{I_1,\dots,I_r}$.

*Remark* 12.1.2. Although we defined $\lambda$ as a map from $I$ to $X_{\bullet}(T)^{\mathrm{dom}}$, one should think of $\lambda$ as indexing irreducible representations of $\widehat{G}$. Namly, if $\lambda$ sending $i \mapsto \lambda_i$, then let $V_{\lambda_i} \in \mathrm{Rep}(\widehat{G})$ be the irreducible representation with highest weight $\lambda_i$. Then the datum of $\lambda$ corresponds to an irreducible representation of $\widehat{G}^I$, namely $\boxtimes V_{\lambda_i}$.

More generally, if $W \in \mathrm{Rep}(\widehat{G}^I)$ and $W = \bigoplus W_j$ (with each summand irreducible) then we define

$$\mathscr{F}_W^{I_1,I_2,\dots I_r} = \bigoplus \mathscr{F}_{W_j}^{I_1,\dots,I_r}.$$

This is a complex of sheaves on $\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r}$ for $\lambda \gg 0$. But recall that we have a map $\mathrm{Shtuka}^{I_1,\dots,I_r} \xrightarrow{p} X^I$, so we can derive the proper pushforward

$$Rp_! \mathscr{F}_W^{I_1,\dots,I_r} = \varinjlim_{\mu} Rp_!^{\prec\mu} \left( \mathscr{F}_W^{I_1,\dots,I_r} |_{\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r,\prec\mu}} \right)$$

As $\mathrm{Bun}_G^{\prec\mu}$ is finite type (since the HN polygon is bounded), and $\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r,\prec\mu}$ is defined by the pullback diagram

$$\begin{tikzcd} \mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r,\prec\mu} \arrow[r, hook] \arrow[d] & \mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r} \arrow[d] \\ \mathrm{Bun}_G^{\prec\mu} \arrow[r, hook] & \mathrm{Bun}_G \end{tikzcd}$$

we have that $\mathrm{Shtuka}_{\lambda}^{I_1,\dots,I_r,\prec\mu}$ is of finite type.

*Example* 12.1.3. If $I=\emptyset$, then what is $\mathscr{F}^{\emptyset}$? Recall that we found $\mathrm{Shtuka}^{\emptyset}(\overline{k})\cong \mathrm{Bun}_G(k)$:

$$\begin{tikzcd} \mathrm{Shtuka}^{\emptyset} \arrow[r, "\cong"] \arrow[d] & \mathrm{Bun}_G(k) \\ \mathrm{Spec}\, k & \end{tikzcd}$$

Then $R^0 p_! \mathscr{F}^{\emptyset} = H^0_c(\mathrm{Bun}_G(k)) = C_c(\mathrm{Bun}_G(k))$. This is infinite-dimensional, which is bad. What we would like is to cut out the cuspidal part $C_{\mathrm{cusp}}(\mathrm{Bun}_G(k))\subset C_c(\mathrm{Bun}_G(k))$. In the general case, we want to analogously cut out something that corresponds to the "cuspidal part" in hopes that the limit will stabilize.

Lafforgue hacks around this problem by cutting out instead a "Hecke finite part." In this case, they are the same. However, it is not clear in general if this is true.

*Definition* 12.1.4. For $W\in \mathrm{Rep}(\widehat{G}^I)$ we define the "shifted' (by $\#I$) perverse sheaf" $\mathscr{S}_W^{I_1,\ldots,I_r}$ on $\mathrm{Shtuka}^{I_1,\ldots,I_r}$ as follows. Let

$$W=\bigoplus_{\lambda\in X_\bullet(T)^{\mathrm{dom}}} (\boxtimes_{i\in I} V_{\lambda_i})^{m_\lambda}$$

be the decomposition into irreducibles. Then we set

$$\boxed{\mathscr{S}_W^{I_1,\ldots,I_r} := \bigoplus_{\lambda} \left(\mathscr{S}^{I_1,\ldots,I_r}_{\boxtimes V_{\lambda_i}}\right)^{m_\lambda} := \bigoplus_{\lambda} (\mathscr{F}_\lambda^{I_1,\ldots,I_r}[-\#I])^{m_\lambda}.}$$

This is not a perverse sheaf on $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}$. However, if we view $\mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}$ a family over $X^I$, then we have precisely shifted by the dimension of the base to make this sheaf perverse along *fibers*.

If $p\colon \mathrm{Shtuka}_\lambda^{I_1,\ldots,I_r}$ is the natural projection, then we define

$$\boxed{\mathscr{H}_W^{I_1,\ldots,I_r} := Rp_!\mathscr{S}_W^{I_1,\ldots,I_r}.}$$

This definition relies on the result that the category is semisimple, but it can be given a more intrinsic definition, which however we usually won't work with.

**A Miracle.** We have a map

$$\begin{tikzcd} \mathrm{Shtuka}^{I_1,\ldots,I_r} \arrow[d, "\pi"'] \\ \mathrm{Shtuka}^I \end{tikzcd}$$

obtained by forgetting all of the intermediate bundles. One might ask, what is the relationship between $\mathscr{S}_\lambda^{I_1,\ldots,I_r}$ and $\mathscr{S}_\lambda^I$? The miraculous answer is:

**Theorem 12.1.5.** *For all* $W\in\mathrm{Rep}(\widehat{G}^I)$, *we have*

$$R\pi_!\mathscr{S}_W^{I_1,\ldots,I_r} = \mathscr{S}_W^I.$$

12.2. **Digression on small maps.** Theorem 12.1.5 is not proved by defining a map in either direction, but by using a general fact that if $\pi\colon X \to Y$ is a birational, proper, "small" map and $X$ is smooth and irreducible, then

$$R\pi_*(\overline{\mathbb{Q}_\ell})_X[\dim X] \cong IC_Y.$$

*Definition* 12.2.1. We say that $\pi\colon X \to Y$ is *small* if

$$\operatorname{codim}_y\{y \in Y \mid \dim \pi^{-1}(y) \geq d\} \geq 2d+1 \text{ for any } d > 0.$$

This is a bit stronger than semi-small. A map of three-folds induced by contracting a curve to a point (e.g. a blow-down) is small, but the same is not true after replacing "three" by "two."

The proof of the theorem proceeds by checking that $R\pi_*\overline{\mathbb{Q}_\ell}_X[\dim X]$ satisfies the axioms of the IC sheaves. You might worry how this could be sufficient- where would a map come from? Well, you can write down a map over open sets where you just have a constant sheaf, and extending over the rest using the estimates involved in the definition of IC sheaf.

We want to get a similar result for replacing the constant sheaf by the constant sheaf upstairs. To do this, we need a stronger condition than smallness, which is "stratified smallness" (which just extends the inequality in stratified semi-smallness by one).

The map

$$\begin{tikzcd} \mathrm{Shtuka}^{I_1,\dots,I_r} \arrow[d, "\pi"'] \\ \mathrm{Shtuka}^I \end{tikzcd}$$

is in fact stratified small. The reason that stratified semi-smallness came up was because the morphism

$$\begin{tikzcd} \mathrm{Gr}_{\lambda_1,\lambda_2} \arrow[d] \\ \mathrm{Gr}_{\lambda_1+\lambda_2} \end{tikzcd}$$

(obtained by forgetting the intermediate bundles) was stratified semi-small, and the extra parameter of the Shtuka (which is like a global version of the Grassmannian over the entire curve) gives that extra dimension in the inequality.

**Theorem 12.2.2** (Miracle Theorem)**.** *If $f\colon X \to Y$ is stratified small, then $R\pi_* IC_X \cong IC_Y$.*

In particular, we have a stratified small map

$$\begin{tikzcd} \mathrm{Shtuka}^{I_1,\dots,I_r} \arrow[d, "p"'] \\ X^I \end{tikzcd}$$

and the miraculous Theorem 12.1.5 tells us that $\mathscr{H}_W^{I_1,\dots,I_r} := Rp_!\mathscr{S}_W^{I_1,\dots,I_r}$ is *independent* of the decomposition of $I$; we have a *canonical* isomorphism

$$\mathscr{H}_W^{I_1,\dots,I_r} \cong \mathscr{H}_W^I.$$

*Remark* 12.2.3. In practice, we think of this as follows: if we let $\eta_I \in X^I$ be the generic point, then

$$\mathscr{H}_W^{I_1,\dots,I_r} = \varinjlim_{\mu} (Rp_!^{\prec\mu} \mathscr{S}_W^{I_1,\dots,I_r})_{\overline{\eta_I}}.$$

12.3. **Partial Frobenius.** If the sheaf $\mathscr{H}_W^{I_1,\dots,I_r}$ doesn't end up depending on the decomposition $I_1,\dots,I_r$, then what was the point of discussing the decompositions anyway? The answer is that we need them in order to define the partial Frobenius. If we define

$$\mathrm{Shtuka}^{\{1\},\{2\}} = \left\{ \mathscr{E} \xhookrightarrow[x_1]{1} \mathscr{E}' \xhookleftarrow[x_2]{1} {}^\tau\mathscr{E} \right\}$$

and

$$\mathrm{Shtuka}^{\{2\},\{1\}} = \left\{ \mathscr{F} \xhookleftarrow[x_2]{1} \mathscr{F}' \xhookrightarrow[x_1]{1} {}^\tau\mathscr{F} \right\}$$

then we get a map $\mathrm{Shtuka}^{\{1\},\{2\}} \to \mathrm{Shtuka}^{\{2\},\{1\}}$ corresponding to

$$(\mathscr{E} \xhookrightarrow[x_1]{1} \mathscr{E}' \xhookleftarrow[x_2]{1} {}^\tau\mathscr{E}) \mapsto (\mathscr{E}' \xhookleftarrow[x_2]{1} {}^\tau\mathscr{E} \xhookrightarrow[\tau(x_1)]{1} {}^\tau\mathscr{E}')$$

The reason that this is called partial Frobenius is because over $X \times X$, this map is $(\mathrm{Frob}_X, \mathrm{Id})$. Typically (i.e. if $x_1 \neq x_2$) there is a unique choice of $\mathscr{E}'$, so we can forget it and remember only $\mathscr{E}$. However, over the diagonal $x_1 = x_2$, it is ill-defined if you forget intermediate bundle.

If we were to forget the decompsition, then we would have something like

$$\mathrm{Shtuka}^{\{1,2\}} = \left\{ \mathscr{E} \overset{1,-1}{\underset{x_1,x_2}{\dashrightarrow}} {}^\tau\mathscr{E} \right\}$$

but this just doesn't make sense when $x_1 = x_2$. The condition in that diagonal case should be ${}^\tau\mathscr{E}(-x_1) \hookrightarrow \mathscr{E} \hookrightarrow {}^\tau\mathscr{E}(x_1)$. But since there are many possibilities for the intermediate bundle in such a case, it's not clear how to define the partial Frobenius. So that is why it is important to remember the decompositions.

Let $I$ be an index set and $J \subset I$ any subset. Consider $F_J : X^I \to X^I$ sending $(x_i) \mapsto (x_i')$ where

$$x_i' = \begin{cases} \mathrm{Frob}(x_i) & i \in J \\ x_i & i \notin J \end{cases}$$

Then there's a canonical isomorphism

$$\varphi_J : F_J^* \mathscr{H}_W^I \cong \mathscr{H}_W^I$$

coming from the partial Frobenius on

$$\begin{tikzcd} \mathrm{Shtuka}_W^I \arrow[r, "F_J"] \arrow[d] & \mathrm{Shtuka}_W^I \arrow[d] \\ X^I \arrow[r, "F_J"] & X^I \end{tikzcd}$$

Why is it an isomorphism? Note that $\mathscr{H}_W^I \cong \mathscr{H}_W^{J,I\setminus J}$ (by the Miracle Theorem 12.2.2), and we can pull back the partial Frobenius to

$$\begin{tikzcd}
\mathrm{Shtuka}_W^{J,I\setminus J} \arrow[r, "F_J"] \arrow[d] & \mathrm{Shtuka}_W^{I\setminus J,J} \arrow[d] \\
X^I \arrow[r, "F_J"] & X^I
\end{tikzcd}$$

which is cartesian up to a radicial map (the usual theory of the Frobenius morphism). This gives a definition of $\varphi_J$, and the Miracle Theorem implies that it is an isomorphism.

This is compatible in the following sense: if we decompose $J = J_1 \cup J_2$, then the following diagram

$$\begin{tikzcd}
F_{J_2}^* F_{J_1}^* \mathscr{H}_W^I \arrow[r, "\cong", "F_{J_1}^*"'] \arrow[d, equal] & F_{J_2}^* \mathscr{H}_W^I \arrow[r, "\cong", "F_{J_2}^*"'] & \mathscr{H}_W^I \\
F_J^* \mathscr{H}_W^I \arrow[urr, bend right, "\cong", "F_J^*"'] & &
\end{tikzcd}$$

12.4. **Construction of excursion operators.** Now let's "pretend" that $\mathscr{H}_W^I$ was a constructible complex (i.e. complex of constructible sheaves) on $X^I$. We know that it isn't really, as it's too "infinite type," but we will later cut out a "Hecke finite" component. Notably, $\mathscr{H}_W^I$ is equipped with partial Frobenius.

**Theorem 12.4.1** (Drinfeld)**.** *Suppose $\mathscr{K}$ is a constructible sheaf on some open dense subset $\Omega \subset X^I$, equipped with maps $\varphi_i \colon F_{\{i\}}^* \mathscr{K}|_{F_{\{i\}}^{-1}(\Omega)\cap\Omega} \cong \mathscr{K}|_{F_{\{i\}}^{-1}(\Omega)\cap\Omega}$ such that*

*(1) the $\varphi_i$ commute,*
*(2) $\prod_{i\in I} \varphi_i$ is the canonical isomorphism $F_\Omega^* \mathscr{K} \cong \mathscr{K}$.*

*Then there exists an open dense subset $U \subset X$ such that $K$ can be extended to a local system on $U^I$.*

*Moreover, if $\eta$ is the generic point of $\Omega$ then the monodromy representation $\pi_1(U^I, \Delta(\overline{\eta}))$ (the basepoint being $\overline{\eta} \to X \xrightarrow{\Delta} X^I$) acting on $\mathscr{K}_{\Delta(\overline{\eta})}$ factors through $\pi_1(U,\overline{\eta})^I$, i.e. the fundamental group of the product acts through the product of the fundamental groups.*

What does this mean geometrically? The first part says that the locus where the sheaf is not a local system must look something like a disjoint union of coordinate axes. This is easy to see. For simplicity take $|I| = 2$. Then the locus where $K$ fails to be a local system is some curve in $X \times X$, but applying partial Frobenius gives another such curve, so it must be the case that *partial Frobenius preserves the curve*, and that forces the curve to be of the desired form. We'll postpone the rest of the proof to the next section.

*Definition* 12.4.2. We set

$$^0\mathscr{H}_W^I := R^0 p_! \mathscr{S}_W^I.$$

*Example* 12.4.3. If $I = \emptyset$, then

$$^0\mathscr{H}^\emptyset = C_c(G(F)\backslash G(\mathbb{A})/G(\mathscr{O}), \overline{\mathbb{Q}_\ell})$$

is the space of automorphic forms.

For $W \in \mathrm{Rep}(\widehat{G}^I)$ we associate $\Delta^* W \in \mathrm{Rep}(\widehat{G})$ (e.g. this takes an external tensor product to an internal tensor product). Suppose we have the data of $\widehat{G}$-equivariant maps $x, \xi$:

$$\overline{\mathbb{Q}_\ell} \xrightarrow{x} \Delta^* W \xrightarrow{\xi} \overline{\mathbb{Q}_\ell}.$$

*Example* 12.4.4. If $W = V \boxtimes V^*$ then we can take

$$\overline{\mathbb{Q}_\ell} \xrightarrow{x=\text{co-ev}} V \otimes V^* \xrightarrow{\xi=\text{ev}} \overline{\mathbb{Q}_\ell}.$$

*Example* 12.4.5. $\mathrm{Shtuka}^{\{1\}}_{\lambda=0} = \mathrm{Shtuka}^{\{1\}}_{W=\mathrm{triv}}$ is just $\mathrm{Shtuka}^{\emptyset} \times X$, so

$$\mathscr{H}^{\{1\}}_{\mathrm{triv}} = Rp_! \mathbb{Q}_\ell = {}^0\mathscr{H}^{\emptyset} \otimes \mathbb{Q}_\ell \text{ on } X.$$

One can think of $\lambda = 0$ interchangeably with the trivial representation of $G$. (See remark 12.1.2.)

Let $[G] = G(F)\backslash G(\mathbb{A})/\prod_x G(\mathscr{O}_x)$. We have compositions

$$C_c([G]) \xrightarrow{\sim} {}^0\mathscr{H}^{\emptyset} \xrightarrow{\sim} {}^0\mathscr{H}^{\{1\}}_{\mathrm{triv}}|_{\overline{\eta}} \xrightarrow{x} {}^0\mathscr{H}^{\{1\}}_{\Delta^* W}|_{\overline{\eta}} \xrightarrow{\sim} {}^0\mathscr{H}^I_W|_{\Delta(\overline{\eta})}.$$

A priori ${}^0\mathscr{H}^I_W$ is just a local system on some open subset $\Omega \subset X^I$, but Drinfeld's theorem implies that it extends to a local system on $U^I$ for some open $U \subset X$, and the action of $\pi_1(U^I, \Delta(\overline{\eta}))$ on ${}^0\mathscr{H}^I_W|_{\Delta(\overline{\eta})}$ factors through $\pi_1(U, \overline{\eta})^I$. This implies that given the data of $(\gamma_i)_{i\in I} \subset \mathrm{Gal}(F^s/F)$, we get an operator

$$(\gamma_i)_{i\in I} \colon {}^0\mathscr{H}^I_W|_{\Delta(\overline{\eta})} \to {}^0\mathscr{H}^I_W|_{\Delta(\overline{\eta})}.$$

*Definition* 12.4.6. With the data given above, consider the diagram

$$\begin{tikzcd}
C_c([G]) \arrow[r, "\cong"] \arrow[d, dotted, "\mathscr{S}_{I,W,x,\xi,(\gamma_i)_{i\in I}}"'] & {}^0\mathscr{H}^{\emptyset} \arrow[r, "\cong"] & {}^0\mathscr{H}^{\{1\}}_{\mathrm{triv}}|_{\overline{\eta}} \arrow[r, "x"] & {}^0\mathscr{H}^{\{1\}}_{\Delta^* W}|_{\overline{\eta}} \arrow[r, "\cong"] & {}^0\mathscr{H}^I_W|_{\Delta(\overline{\eta})} \arrow[d, "(\gamma_i)_{i\in I}"'] \\
C_c([G]) & {}^0\mathscr{H}^{\emptyset} \arrow[l, "\cong"] & {}^0\mathscr{H}^{\{1\}}_{\mathrm{triv}}|_{\overline{\eta}} \arrow[l, "\cong"] & {}^0\mathscr{H}^{\{1\}}_{\Delta^* W}|_{\overline{\eta}} \arrow[l, "\xi"] & {}^0\mathscr{H}^I_W|_{\Delta(\overline{\eta})} \arrow[l, "\cong"]
\end{tikzcd}$$

The big composition $\mathscr{S}_{I,W,x,\xi,(\gamma_i)_{i\in I}} \colon C_c([G]) \to C_c([G])$ is the *excursion operator* associated to the data $I, W, x, \xi, (\gamma_i)_{i\in I}$.

*Example* 12.4.7. Let $I = \{1,2\}$, $W = V \boxtimes V^*$, and $(\gamma_1, \gamma_2) = (\widetilde{\mathrm{Frob}_x}, \mathrm{Id})$, where $\widetilde{\mathrm{Frob}_x}$ is a lift of Frobenius (which depends on a choice of embedding $\mathrm{Gal}(F_x^s/F_x) \hookrightarrow \mathrm{Gal}(F^s/F)$). Then we can take

$$\overline{\mathbb{Q}_\ell} \xrightarrow{x=\text{co-eval}} \Delta^* W = V \otimes V^* \xrightarrow{\xi=\text{eval}} \overline{\mathbb{Q}_\ell}$$

which induces the excursion operator $\mathscr{S}_{\{1,2\}, V\boxtimes V^*, \text{co-ev}, \text{ev}, (\widetilde{\mathrm{Frob}_v}, \mathrm{Id})}$.

We will see later:

*Theorem* 12.4.8 (Lafforgue). *$\mathscr{S}_{\{1,2\}, V\boxtimes V^*, co\text{-}ev, ev, (\widetilde{\mathrm{Frob}_v}, \mathrm{Id})}$ coincides with the Hecke operator $h_{V,x} \in C_c(G(\mathscr{O}_x\backslash G(F_x)/G(\mathscr{O}_x), \overline{\mathbb{Q}_\ell})$ corresponding to $[V] \in R(\widehat{G})$ under the Satake correspondence.*

12.5. **Proof of Drinfeld's Theorem.** We want to prove the second part of the theorem stated last time: given a local system $\mathscr{L}$ on $\Omega \subset X^n$, equipped with an isomorphism $F_i^*\mathscr{L}_{\eta_n} \cong \mathscr{L}|_{\eta_n}$ ($\eta_n \in X^n$ the generic point) such that these isomorphisms commute and the composition is the tautological isomorphism $F_\Omega^*\mathscr{L} \cong \mathscr{L}$, then the monodromy representation factors as

$$\begin{tikzcd} \mathrm{Gal}(F_n^s/F_n) \cong \pi_1(\eta_n, \overline{\eta}_n) \arrow[r, two heads] \arrow[d, "\text{projection to each factor}"'] & \pi_1(\Omega, \eta_n) \arrow[r] & \mathrm{GL}(\mathscr{L}_{\overline{\eta}_n}) \\ \pi_1(\eta, \overline{\eta})^n \arrow[urr, dotted] & & \end{tikzcd}$$

We already argued that $\mathscr{L}$ can be viewed as local system on $U^n \subset X^n$ where $U \subset X$ is open and dense, as the symmetry under partial Frobenius forces the bad set to be a union of coordinate hyperplanes.

*Reduction to finite covers.* Let $\mathbf{f\acute{e}t}(U^n/\mathrm{pFrob})$ be the category consisting of finite étale morphisms $Y \to U^n$ plus the data of commuting isomorphisms $\varphi_i \colon F_i^*Y \cong Y$ over $U^n$ and such that $\prod_{i=1}^n \varphi_i$ is the tautological isomorphism $F_{U^n}^* Y \cong Y$, where these are defined by

$$\begin{tikzcd} Y & F_i^*Y \arrow[l, "\varphi_i"', "\cong"] \arrow[r] \arrow[d] & Y \arrow[d] \\ & U^n \arrow[r] & U^n \end{tikzcd}$$

and

$$\begin{tikzcd} Y & F_{U^n}^*Y \arrow[l, "\text{taut}"', "\cong"] \arrow[r, "F_Y"'] \arrow[d, "\pi"'] & Y \arrow[d, "\pi"'] \\ & U^n \arrow[r, "\mathrm{Frob}_{U^n}"'] & U^n \end{tikzcd}$$

This forms a Galois category. Given a geometric generic point $\overline{\eta}_n \in X^n$, we get a fiber functor

$$\omega_{\overline{\eta}_n} \colon \mathbf{f\acute{e}t}(U^n/\mathrm{pFrob}) \to \{\text{finite sets}\}$$

and we define

$$\pi_1(U^n/\text{partial Frob}, \overline{\eta}_n) := \mathrm{Aut}(\omega_{\overline{\eta}_n}).$$

We can reduce to the case of torsion sheaves by considering $E = \varprojlim_m \mathscr{O}_E/\varpi_E^m$, which shows how to determine the local system at finite level:

$$\begin{tikzcd} \pi_1(U^n) \arrow[r] \arrow[dd] & \mathrm{GL}_n(\mathscr{O}_E) \arrow[dr] & \\ & & \mathrm{GL}_n(\mathscr{O}_E/\varpi_E^m) \\ \pi_1(U)^n \arrow[r, dotted, "?"] & \mathrm{GL}_n(\mathscr{O}_E) \arrow[ur] & \end{tikzcd}$$

Therefore, a reformulation of Drinfeld's theorem is then

$$\pi_1(U^n/\mathrm{pFrob}, \overline{\eta}_n) \cong \pi_1(U, \overline{\eta})^n.$$

This is the statement that we prove.

*Proof.* Recall "Frobenius descent" from Theorem 6.4.1. If $X$ is a projective variety over $k = \mathbb{F}_q$ and $S/k$ is a scheme, then we can consider the category of coherent sheaves $\mathscr{F}$ on $X \times_k S$ equipped with an isomorphism

$$(\mathrm{Id}_X \times \mathrm{Frob}_S)^* \mathscr{F} \cong \mathscr{F}.$$

This category is equivalent $\mathbf{Maps}(S, \mathbf{Coh}(X))$, with $\mathbf{Coh}(X)$ a groupoid with finite automorphisms, viewed as a discrete stack over $K$:

$$\coprod_{\mathscr{F} \in \mathbf{Coh}(X)} [\mathrm{Spec}\ k/\mathrm{Aut}(\mathscr{F})]$$

*Example* 12.5.1. When $X = \mathrm{Spec}\ k$, the left hand side becomes vector bundles $V \to X$ equipped with an isomorphism $\mathrm{Frob}_S^* V \cong V$ (i.e. unit root $F$-crystals), and the right hand side becomes étale $k = \mathbb{F}_q$-local systems over $S$, which recovers Katz' Theorem6.3.4. Indeed, to give a map from a connected $S$ to

$$\mathbf{Coh}(\mathrm{Spec}\ k) = \coprod_{n \geq 0} [\mathrm{Spec}\ k/\mathrm{GL}_n(k)]$$

is, by definition, the same as picking an $n$ and a $\mathrm{GL}_n$-torsor over $k$, which is just the data of an $n$-dimensional étale $k = \mathbb{F}_q$-local system over $S$.

Let $X$ be a complete curve. Then

$$\mathbf{f\acute{e}t}(X \times S/\mathrm{pFrob}) \cong \left\{ \begin{matrix} \text{finite étale } \mathscr{O}_{X\times S}\text{-algebras } \mathscr{A} \\ \text{equipped with partial Frobenius} \end{matrix} \right\}.$$

The right hand side is the full subcategory of étale objects in $\mathbf{Coh}(X)$, hence equivalent to $\mathbf{Map}(S, \mathbf{f\acute{e}t}(X) \subset \mathbf{Coh}(X))$. But this in turn is the same as

$$\mathbf{Map}(S, \text{ finite discrete (i.e. continuous action) } \pi_1(X, \overline{x})\text{-sets}).$$

When $S$ is connected, with base point $\overline{s}$, we can further identify the latter with $\pi_1(S, \overline{s}) \times \pi_1(X, \overline{x})$-sets. To see this, note that an object of the category of finite discrete (i.e. continuous action) $\pi_1(X, \overline{x})$-sets is a set $\Sigma$ equipped with a (continuous) $\pi_1(X, \overline{x})$-action. Then to give a map from a connected $S$ (with basepoint), you have to pick a single such object, and give a map

$$\pi_1(S, \overline{s}) \to \mathrm{Aut}_{\pi_1(X, \overline{x})}(\Sigma).$$

That's the same as a finite discrete (i.e. continuous) $\pi_1(S, \overline{s}) \times \pi_1(X, \overline{s})$-set.

By applying this with $S = X^{n-1}, X^{n-2}, \ldots$ we see that

$$\pi_1(X^n/\mathrm{pFrob}) \cong \pi_1(X/\mathrm{pFrob})^n.$$

In the general case, we have to handle $U \times S$ (where we are thinking of $S$ as $U^{n-1}$). We omit the argument. □

12.6. **The Weil group.** Let $X$ be a smooth projective curve over $k = \mathbb{F}_q$. Then we have an exact sequence

$$1 \to \pi_1(X_{\overline{k}}) \to \pi_1(X) \to \mathrm{Gal}(\overline{k}/k) \to 1.$$

*Definition* 12.6.1. We define the *Weil group* $W(X)$ of $X$ to be the pullback of $\mathrm{Frob}_k^{\mathbb{Z}} \hookrightarrow \mathrm{Gal}(\overline{k}/k)$:

$$\begin{tikzcd} 1 \arrow[r] & \pi_1(X_{\overline{k}}) \arrow[r] \arrow[d] & W(X) \arrow[r] \arrow[d] & \mathrm{Frob}_k^{\mathbb{Z}} \arrow[r] \arrow[d, hook] & 1 \\ 1 \arrow[r] & \pi_1(X_{\overline{k}}) \arrow[r] & \pi_1(X) \arrow[r] & \mathrm{Gal}(\overline{k}/k) \arrow[r] & 1 \end{tikzcd}$$

If $K = k(X)$, then we define $W(K)$ similarly as a subgroup of $\mathrm{Gal}(K^s/K) = \mathrm{Gal}(\overline{K}/K^{\mathrm{perf}})$ whos image in $\mathrm{Gal}(\overline{k}/k)$ is an integer power of Frobenius.

More generally, suppose $K_1, K_2, \ldots, K_n$ are function fields over $k$ such that $k$ is algebraically closed in each $K_i$. Then set $K = \mathrm{Frac}(K_1 \otimes_k K_2 \otimes_k \ldots \otimes_k K_n)$, and define

$$W(K_1, \ldots, K_n) := \{g \in \mathrm{Aut}_{\overline{k}}(\overline{K}) \mid g|_{K_i^{\mathrm{perf}}} \text{ is an integral power of } \mathrm{Frob}_{K_i^{\mathrm{perf}}}\}.$$

This admit a homomorphism to $\mathbb{Z}^n$. What's the kernel? It consists of those automorphisms acting by the identity on $K_i^{\mathrm{perf}}$ and also $\overline{k}$, i.e. the "geometric Galois group" $\mathrm{Aut}(\overline{K}/\overline{k}K^{\mathrm{perf}}) \cong \mathrm{Gal}(K^s/\overline{k}K)$.

We have a map

$$\begin{tikzcd} W(K) \arrow[r, hook] \arrow[d] & W(K_1, \ldots, K_n) \arrow[d] \\ \mathbb{Z} \arrow[r, "\text{diagonal}"'] & \mathbb{Z}^n \end{tikzcd}$$

as every element of $W(K)$ acts by an integral power of Frob on all of $K_i^{\mathrm{perf}}$. However, after profinite completion we get a *surjection* onto the subgroup lying over $\Delta(\widehat{\mathbb{Z}})$. This follows from essentially the same argument as in Drinfeld's theorem.

*Exercise* 12.6.2. Prove this.

## 13. Compatibilty with Hecke Operators

13.1. **A special excursion operator.** We have now defined excursion operators $\mathscr{S}_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ parametrized by the data of

- an index set $I$,
- a representation $W\in \operatorname{Rep}(\widehat{G}^I)$,
- maps $\operatorname{triv}\xrightarrow{x} W|_{\widehat{G}}\xrightarrow{\xi}\operatorname{triv}$,
- $\gamma_i\in\Gamma:=\operatorname{Gal}(F^s/F)$ for each $i\in I$.

In this section we will focus on proving a theorem, promised earlier, relating Hecke operators to the excursion operator for the specific data

- $I=\{1,2\}$,
- $W=V\boxtimes V^*$,
- the maps $x,\xi$ are
$$\operatorname{triv}\xrightarrow{\text{coev}}\underbrace{V\otimes V^*}_{\cong\operatorname{End}(V)}\xrightarrow{\text{ev}}\operatorname{triv}$$
- $(\gamma_1,\gamma_2)=(\widetilde{\operatorname{Frob}_v},1)$ where $v\in X(k)$ (the rationality is just for the sake of simplicity) and $\widetilde{\operatorname{Frob}}_v$ is a lift of $\operatorname{Frob}_v\in\operatorname{Gal}(k_v^s/k_v)$.

$$\begin{array}{ccc}\widetilde{\operatorname{Frob}_v}\in & \Gamma_v:=\operatorname{Gal}(F_v^s/F_v)\\ \big\downarrow & \big\downarrow\\ \operatorname{Frob}_v\in & \operatorname{Gal}(k_v^s/k_v)\end{array}$$

**Hecke operators.** Let $[G]=G(F)\backslash G(\mathbb{A}_F)/\prod_x G(\mathscr{O}_x)$. Recall that the integral version of the (local) Hecke algebra $\mathscr{H}_v=\operatorname{Fun}_c(G(\mathscr{O}_v)\backslash G(F_v)/G(\mathscr{O}_v),\mathbb{Z})$ acts on $C_c([G])$ by convolution on the right, and we denote the action map by $T$.

There is an element $h_{V,v}\in\mathscr{H}_v\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm1/2}]$, determined by the property that its image in $R(\widehat{G}\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm1/2}])$ under the Satake correspondence is $[V]$:

$$\begin{array}{ccc}h_{V,v}\in & \mathscr{H}_v\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm1/2}]\\ \big\downarrow & \big\downarrow{\scriptstyle\text{Satake}}\\ [V]\in & R(\widehat{G})\otimes_{\mathbb{Z}}\mathbb{Z}[q^{\pm1/2}].\end{array}$$

This involves a choice of $q^{1/2}$, but since that is built into the ring $Z[q^{\pm1/2}]$, we can regard $h_{V,v}$ as lying *canonically* in $\operatorname{Fun}_c(G(\mathscr{O}_v)\backslash G(F_v)/G(\mathscr{O}_v),\mathbb{Z}[q^{\pm1/2}])$.

*Definition* 13.1.1. The element $T(h_{V,v})\in\operatorname{End}(C_c([G]))$ is the *Hecke operator at $v$* associated with $V$.

**Theorem 13.1.2.** *We have an equality of operators*
$$\mathscr{S}_{\{1,2\},V\boxtimes V^*,coev,ev,(\widetilde{\operatorname{Frob}}_v,1)}=T(h_{V,v})\in\operatorname{End}(C_{\text{cusp}}([G]).$$

*Proof.* We focus on the special case $G=\operatorname{GL}_n$ and $V$ the standard representation, and $v\in X(k)$. There is a technical point here, which is that we are still pretending the sheaves are constructible. This issue underlies the difference between working with the cusp forms and all compactly supported functions. So we are brushing a finiteness problem under the rug for now.

Let's recall the construction of the excursion operator with these specific choices.

(1) Recalling that $\mathscr{H}^{\{*\}}_{\mathrm{triv}}$ is the constant sheaf $C_c([G])\otimes\overline{\mathbb{Q}}_\ell$ on $X$ ♠♠♠ TONY: [we have been sloppy about $\mathbb{Q}_\ell$ vs its algebraic closure...], we obtain trivially an isomorphism

$$C_c([G]) \xrightarrow{\sim} {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v.$$

(2) There is a map ${}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \hookrightarrow {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)}$ induced by coevaluation. This factors through ${}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v$, simply because $\operatorname{Spec} k \xrightarrow{v,v} \Delta(V)$ factors through $\operatorname{Spec} k \xrightarrow{v} X \xrightarrow{\Delta} \Delta(X)$, which maps isomorphically to its image, and restricting to the diagonal essentially just changes the internal tensor product to the external tensor product.

In conclusion, we have constructed

$$\begin{tikzcd}
C_c([G]) \cong {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \arrow[r] \arrow[dr, "\mathrm{coev}"'] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \\
& {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[u, "\cong"']
\end{tikzcd}$$

(3) We then apply partial Frobenius map $F_1$ (we would ordinarily need to exponentiate this to the degree of $v$, but because we assumed that $v$ was rational the exponent is 1) to the same thing, and then surject to ${}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v$ via the map induced by ev.

$$\begin{tikzcd}
C_c([G]) \cong {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \arrow[r] \arrow[dr, "\mathrm{coev}"'] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r, "F_1"] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r] \arrow[d, "\cong"'] & {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \cong C_c([G]) \\
& {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[u, "\cong"'] & {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[ur, "\mathrm{ev}"'] &
\end{tikzcd}$$

What's the geometric interpretation here? The stalk ${}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v$ is the compactly supported cohomology $H^0_c(\mathrm{Shtuka}^{\uparrow\downarrow}_v, IC)$ where $\mathrm{Shtuka}^{\uparrow\downarrow}_v$ parametrizes modifications of the form $\{\mathscr{E} \underset{v}{\overset{1}{\hookrightarrow}} \mathscr{E}' \underset{v}{\overset{1}{\hookleftarrow}} {}^\tau\mathscr{E}\}$. The space $\mathrm{Shtuka}^{\uparrow\downarrow}_v$ is smooth of dimension $2(n-1)$, as it is (étale) locally over $\mathrm{Bun}_n$ just a product of two projective spaces. So in this case the IC sheaf is $\overline{\mathbb{Q}}_\ell[2(n-1)](n-1)$ - a constant sheaf of the right dimension, plus a Tate twist that makes it pure of weight 0. In particular, we see that

$$H^0_c(\mathrm{Shtuka}^{\uparrow\downarrow}_v, IC) \cong H^{2(n-1)}_c(\mathrm{Shtuka}^{\uparrow\downarrow}_v)(n-1).$$

♠♠♠ TONY: [to check that I understand the weight thing: if we had used $Rp_*$ instead of $Rp_!$, then we would be twisting by $1-n$ instead of $n-1$?]

For formal reasons, ${}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v$ *also* calculates the compactly supported cohomology of a *different* shtuka $\mathrm{Shtuka}_v^{\downarrow\uparrow}$:

$${}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \cong H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\downarrow\uparrow})(n-1)$$

where $\mathrm{Shtuka}_v^{\downarrow\uparrow}$ parametrizes $\{\mathscr{F} \overset{1}{\underset{v}{\hookleftarrow}} \mathscr{F}' \overset{1}{\underset{v}{\hookrightarrow}} \mathscr{F}''\}$. The fact that $H_c^*(\mathrm{Shtuka}_v^{\uparrow\downarrow}) \cong H_c^*(\mathrm{Shtuka}_v^{\downarrow\uparrow})$ is highly nontrivial, a consequence of the commutativity of convolution product in $\mathscr{P}(\mathrm{Gr})$ (the perverse sheaves on the affine Grassmannian).

So we choose to identify

$$\begin{tikzcd}
C_c([G]) \cong {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \arrow[r] \arrow[dr, "\mathrm{coev}"] \arrow[ddr, dotted, "???"'] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r, "F_1"] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r] \arrow[d, "\cong"'] & {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \cong C_c([G]) \\
 & {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[u, "\cong"'] \arrow[d, equal] & {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[ur, "\mathrm{ev}"] \arrow[d, equal] & \\
 & H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\downarrow\uparrow}) \arrow[r, "F^*_{\{1\}}"'] & H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\uparrow\downarrow}) \arrow[uur, dotted, "???"'] &
\end{tikzcd}$$

where the map $F_{\{1\}}\colon \mathrm{Shtuka}_v^{\uparrow\downarrow} \to \mathrm{Shtuka}_v^{\downarrow\uparrow}$ is the "partial Frobenius,"

$$(\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^\tau\mathscr{E}) \mapsto (\mathscr{E}' \hookleftarrow {}^\tau\mathscr{E} \hookrightarrow {}^\tau\mathscr{E}').$$

To identify the outer maps ???, we have to really unravel the geometric Satake correspondence. We'll just say the answer.

We can consider the closed substack $Z^{\downarrow\uparrow}$ of $\mathrm{Shtuka}_v^{\downarrow\uparrow} = \{\mathscr{F} \overset{1}{\underset{v}{\hookleftarrow}} \mathscr{F}' \overset{1}{\underset{v}{\hookrightarrow}} \mathscr{F}\}$ parametrizing

$$Z^{\downarrow\uparrow} = \{\mathscr{F} \overset{1}{\underset{v}{\hookleftarrow}} \mathscr{F}' \overset{1}{\underset{v}{\hookrightarrow}} {}^\tau\mathscr{F} \mid \mathscr{F} = {}^\tau\mathscr{F}\}$$

(where the equality makes sense by identifying $\mathscr{F}$ and ${}^\tau\mathscr{F}$ in a common bundle using $\mathscr{F}'$). Forgetting $\mathscr{F}'$ gives a map

$$Z^{\downarrow\uparrow} \to \mathrm{Bun}_G(k) = [G],$$

whose fiber over $\mathscr{F}$ is $\mathbb{P}(\mathscr{F}_v/\varpi_v\mathscr{F}_v)$. ♠♠♠ TONY: [since we need to choose a hyperplane, it seems to me that this is an element of the dual projective space, unless we are using Grothendieck's convention] Thus, $Z^{\downarrow\uparrow} \cong \coprod_{[G]} \mathbb{P}^{n-1}$. This gives an isomorphism $C_c([G]) \cong A_{n-1}(Z^{\downarrow\uparrow})$, where $A_\bullet$ is the Chow ring, which then admits a cycle class map to

$$A_{n-1}(Z^{\downarrow\uparrow}) \xrightarrow{cl} H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\downarrow\uparrow}).$$

The relevant map is the composition:

$$\begin{tikzcd}
C_c([G]) \cong {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \arrow[r] \arrow[dr, "\mathrm{coev}"'] \arrow[dd, "\cong"] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r, "F_1"] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r] \arrow[d, "\cong"'] & {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \cong C_c([G]) \\
& {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[u, "\cong"'] \arrow[d, equal] & {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[ur, "\mathrm{ev}"] \arrow[d, equal] & \\
A_{n-1}(Z^{\downarrow\uparrow}) \arrow[r, "\mathrm{cl}"'] & H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\downarrow\uparrow}) \arrow[r, "F_{\{1\}}^*"'] & H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\uparrow\downarrow}) &
\end{tikzcd}$$

A similar story holds for $\mathrm{Shtuka}_v^{\uparrow\downarrow} \supset Z^{\uparrow\downarrow} = \coprod_{[G]} \mathbb{P}(\varpi^{-1}\mathscr{E}_v/\mathscr{E}_v)^\vee$, and we have a restriction map $H_c^*(\mathrm{Shtuka}_v^{\uparrow\downarrow}) \xrightarrow{\mathrm{res}} H_c^{2(n-1)}(Z^{\uparrow\downarrow})(n-1)$.

$$\begin{tikzcd}
C_c([G]) \cong {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \arrow[r] \arrow[dr, "\mathrm{coev}"'] \arrow[dd, "\cong"] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r, "F_1"] & {}^0\mathscr{H}^{\{1,2\}}_{V\boxtimes V^*}|_{\Delta(v)} \arrow[r] \arrow[d, "\cong"'] & {}^0\mathscr{H}^{\{*\}}_{\mathrm{triv}}|_v \cong C_c([G]) \arrow[dd, "\cong"'] \\
& {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[u, "\cong"'] \arrow[d, equal] & {}^0\mathscr{H}^{\{*\}}_{V\otimes V^*}|_v \arrow[ur, "\mathrm{ev}"] \arrow[d, equal] & \\
A_{n-1}(Z^{\downarrow\uparrow}) \arrow[r, "\mathrm{cl}"'] & H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\downarrow\uparrow}) \arrow[r, "F_{\{1\}}^*"'] & H_c^{2(n-1)}(\mathrm{Shtuka}_v^{\uparrow\downarrow}) \arrow[r, "\mathrm{res}"'] & H_c^{2(n-1)}(Z^{\uparrow\downarrow})(n-1).
\end{tikzcd}$$

The bottom row can be clarified by writing its *adjoint* in terms of (Borel-Moore) homology:

$$H_0^{BM}(Z^{\downarrow\uparrow}) \xleftarrow{\cap [Z^{\downarrow\uparrow}]} H_{2(n-1)}^{BM}(\mathrm{Shtuka}_v^{\uparrow\downarrow}) \xleftarrow{F_{\{1\}}} H_{2(n-1)}^{BM}(\mathrm{Shtuka}_v^{\uparrow\downarrow}) \leftarrow H_{2(n-1)}^{BM}(Z^{\uparrow\downarrow})$$

Let's try to use this to describe the excursion operator more explicitly. Given $\mathscr{E} \in [G]$, tracing through the composition on homology and then the identification with $H_c([G])$ should send $[\mathbb{P}(\mathscr{E}_v)^\vee]$ to a function on $[G]$. What function? The one taking

$$\boxed{\mathscr{F} \in [G] \mapsto \langle F_{\{1\}}(\mathbb{P}(\mathscr{E}_v)^\vee) \cdot \mathbb{P}(\mathscr{F}_v) \rangle_{\mathrm{Shtuka}_v^{\downarrow\uparrow}}.}$$

To compute the right hand side, we claim that $F_{\{1\}}|_{\mathbb{P}(\mathscr{E}_v)^\vee}$ is locally an immersion, and $F_{\{1\}}(\mathbb{P}(\mathscr{E})^\vee)$ is transverse to $\mathbb{P}(\mathscr{F}_v)$. This is just a tangent space calculation, which we presently sketch.

What is the tangent space to $\mathrm{Shtuka}_v^{\downarrow\uparrow}$ at $(\mathscr{F} \hookleftarrow \mathscr{F}' \hookrightarrow {}^\tau\mathscr{F})$? Recall the diagram

$$\begin{tikzcd}
\mathrm{Shtuka}_v^{\downarrow\uparrow} \arrow[r] \arrow[d] & \mathscr{H}_v^{\downarrow\uparrow} \arrow[d] \\
\mathrm{Bun} \times \mathrm{Bun} \arrow[r] & \mathrm{Bun} \times \mathrm{Bun} \times \mathrm{Bun}
\end{tikzcd}$$

which identifies the tangent space of $\mathrm{Shtuka}_v^{\downarrow\uparrow}$ at $(\mathscr{F} \hookleftarrow \mathscr{F}' \hookrightarrow {}^\tau\mathscr{F})$ with the relative tangent space of the third projection map $\mathscr{H}_v^{\downarrow\uparrow} \xrightarrow{p_3} \mathrm{Bun}_G$, sending $\{\mathscr{F} \hookleftarrow \mathscr{F}' \hookrightarrow \mathscr{F}''\} \mapsto \mathscr{F}''$.

The fiber is determined by first choosing $\mathscr{F}'$ of colength 1, then $\mathscr{F}$ containing it. This shows that there is a *filtration* on the tangent space, coming from a short exact sequence

$$0 \to T(\underbrace{\mathscr{H}^{\uparrow} \xrightarrow{p_1} \mathrm{Bun}}_{(\mathscr{F}\hookrightarrow\mathscr{F}')\mapsto\mathscr{F}'}) \to T(\mathscr{H}^{\downarrow\uparrow} \xrightarrow{p_3} \mathrm{Bun}) \to T(\underbrace{\mathscr{H}^{\downarrow} \xrightarrow{p_1} \mathrm{Bun}}_{(\mathscr{F}'\hookleftarrow\mathscr{F}'')\mapsto\mathscr{F}''}) \to 0$$

We just have to check two things. First, $dF_{\{1\}}|_{T\mathbb{P}(\mathscr{E}_v)^\vee}$ has image equal to the subspace above. That's obvious, because $T\mathbb{P}(\mathscr{E}_v)^\vee$ is the relative tangent space to $(\mathscr{E} \hookrightarrow \mathscr{E}') \mapsto \mathscr{E}$ at $\mathscr{E}$, after applying partial Frobenius this becomes $(\mathscr{E}' \hookleftarrow {}^\tau\mathscr{E}) \mapsto {}^\tau\mathscr{E}$. Second, it is easy to check that $T\mathbb{P}(\mathscr{F}_v)$ maps isomorphically to the quotient. These two combine to show the claim.

*Exercise* 13.1.3. Write out the details.

Since we now have two natural representatives of the relevant classes intersecting transversely, it is enough to count the *naïve intersection number*. That is, how many modifications $\{\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^\tau\mathscr{E}\} = \mathbb{P}(\mathscr{E}_v)^\vee$ intersect $Z^{\downarrow\uparrow}$ after applying partial Frobenius?

Under partial Frobenius we have

$$(\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^\tau\mathscr{E}) \mapsto (\mathscr{E}' \hookleftarrow {}^\tau\mathscr{E} = \mathscr{E} \hookrightarrow {}^\tau\mathscr{E}')$$

which lies in $Z^{\downarrow\uparrow}$ if and only if $\mathscr{E}' = {}^\tau\mathscr{E}'$.

In conclusion, $(\mathscr{E} \hookrightarrow \mathscr{E}' \hookleftarrow {}^\tau\mathscr{E})$ contributes to the intersection if and only if both $\mathscr{E}, \mathscr{E}' \in \mathrm{Bun}_G(k)$ (i.e. are defined over $k$) *and* $\mathscr{E} \hookrightarrow \mathscr{E}'$ is also defined over $k$ (because we are keeping track of the maps as well in all of this). Thus, we have *finally* computed that

$$\boxed{\langle F_{\{1\}}(\mathbb{P}(\mathscr{E}_v)^\vee) \cdot \mathbb{P}(\mathscr{F}_v)\rangle = \#\{\mathscr{E} \overset{1}{\underset{v}{\hookrightarrow}} \mathscr{F} \text{ defined over } k\}.}$$

To prove the theorem, we compare this to the matrix coefficient of the relevant Hecke operator. Since $G = \mathrm{GL}_n$ and $V$ is the standard representation, $h_{V,v}$ is the indicator function of the double coset

$$G(\mathscr{O}_v)\begin{pmatrix} \varpi_v & & \\ & 1 & \\ & & \ddots \end{pmatrix} G(\mathscr{O}_v).$$

Therefore, $T(h_{V,v})(1_{\mathscr{F}})$ (applying to the characteristic function of the bundle $\mathscr{F}$) sends $\mathscr{E}$ to $\#\{\mathscr{E} \overset{1}{\underset{v}{\hookrightarrow}} \mathscr{F}$ defined over $k\}$. So we've manually checked that the two functions agree. Actually, we have found something even better than that: not only do their values agree, but the *underlying sets being counted are the same.*

□

*Remark* 13.1.4. There is a more general version. We have an action of $\mathscr{S}_{\{1,2\}, V\boxtimes V^*, \mathrm{coev}, \mathrm{ev}, \{\widetilde{\mathrm{Frob}_v}, 1\}}$ on the sheaf ${}^0\mathscr{H}^I_W$. However, and this is a difference from the $I = \emptyset$ case, the Hecke operator $T(h_{V,v})$ doesn't act on the whole sheaf but on ${}^0\mathscr{H}^I_W|_{(X-v)^I}$. That is, we only get an action away from points not involved in the modification. The general theorem is then that the two operators agree when they are both defined.

In particular, we see that Hecke the action of $T(h_{V,v})$ can be extended to an endomorphism of ${}^0\mathscr{H}^I_W$ on $X^I$, via the excursion operator.

*Remark* 13.1.5. There was one (important!) step we skipped, which was the proof of the geometric interpretation of the coevaluation map Geometric Satake. It is perhaps most instructive to illustrate through an example. Recall from earlier

$$\begin{tikzcd}
\mathrm{Gr}^{\downarrow\uparrow} = \mathrm{Gr}_{(1,0,\ldots,0),(0,\ldots,0,-1)} \arrow[r, equal] \arrow[d] & \{\Lambda \overset{1}{\hookleftarrow} \Lambda' \overset{1}{\hookrightarrow} \mathscr{O}^{\oplus n}\} \arrow[d] & \mathbb{P}^{n-1} \arrow[l, hook'] \arrow[d] \\
\mathrm{Gr}_{(1,0,\ldots,0,-1)} \arrow[r, equal] & \{\Lambda \dashrightarrow \mathscr{O}^{\oplus n}\} & \{\Lambda = \mathscr{O}^{\oplus n}\} = \mathrm{pt} \arrow[l, hook']
\end{tikzcd}$$

Under Geometric Satake, the decomposition of representations $V \otimes V^* = \mathrm{End}^0(V) \oplus \mathrm{triv}$ corresponds to the decomposition of perverse sheaves $R\pi_*\mathbb{Q}_\ell[2(n-1)] \cong IC \oplus \delta$. The coevaluation map corresponds to the inclusion $\mathrm{triv} \hookrightarrow V \otimes V^*$, hece of $\delta \hookrightarrow R\pi_*\mathbb{Q}_\ell 2(n-1)$. So why does that corresopnd to the inclusion of the fundamental class? Taking talks of this decomposition at pt gives

$$H^*(\mathbb{P}^{n-1}) \cong H^{<2(n-1)}(\mathbb{P}^{n-1}) \oplus H^{2(n-1)}(\mathbb{P}^{n-1}),$$

so we really see that the trivial summand corresponds to the top class of $H^*(\mathbb{P}^{n-1})$.

13.2. **The Eichler-Shimura relation.** For $V \in \mathrm{Rep}(\widehat{G})$, we get an action of a partial Frobenius operator $F_{\{0\}}$ on $\mathscr{H}^{I\cup\{0\}}_{W\boxtimes V}$. If we also pick $v \in X(k)$, we get an action of an excursion operator $\mathscr{S}_{\{1,2\},V\boxtimes V^*,\mathrm{coev},\mathrm{ev},(\widetilde{\mathrm{Frob}_v},1)} =: \mathscr{S}_{V,v}$. The Eichler-Shimura relation is a polynomial equation satisfied by $F_{\{0\}}$ with coefficients being of the form $\mathscr{S}_{\bigwedge^* V,v}$.

If $d = \dim V$, then it takes the explicit form

$$\boxed{(F_{\{0\}})^d - F_{\{0\}}^{d-1} \circ \mathscr{S}_{V,v} + F_{\{0\}}^{d-2}\mathscr{S}_{\bigwedge^2 V,v} \pm \ldots \pm \mathscr{S}_{\bigwedge^d V,v} = 0.}$$

According to the theorem we just proved, these coefficients are really Hecke operators. However, it is difficult to show that directly.

*Example* 13.2.1. In the case of modular curves (with good reduction), we have

$$\begin{tikzcd}
\mathscr{X} \arrow[d] & X_{\overline{\mathbb{F}_p}} \arrow[d] \\
\mathrm{Spec}\,\mathbb{Z} & \mathrm{Spec}\,\overline{\mathbb{F}}_p
\end{tikzcd}$$

Frobenius acts on $X_{\overline{\mathbb{F}_p}}$, hence on its cohomology $H^*(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell) \cong H^*(\mathscr{X}_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$. In addition, the integral model $\mathscr{X}$ admits an action of Hecke operators $T_p$, inducing an action on $H^*(\mathscr{X}_{\overline{\mathbb{Q}_p}}, \mathbb{Q}_\ell)$ and thus also on $H^*(X_{\overline{\mathbb{F}_p}}, \mathbb{Q}_\ell)$. However, the Hecke operators *don't* directly act on the special fibers. Anyway, one has the relation

$$\boxed{\mathrm{Frob}_p^2 - T_p\,\mathrm{Frob}_p + p = 0 \in \mathrm{End}(H^*(X_{\overline{\mathbb{F}_p}})).}$$

This is the classical Eichler-Shimura relation.

In the function field case, one doesn't have modular curves. The analogue of a modular curve should be a relative curve over $X$, but the problem is that no moduli stack of shtukas is a relative curve. For instance, consider the "nonexistent moduli stack" $\mathrm{Shtuka}_{(1,0)}^{\{1\}} \to X$, which parametrizes

$$\{\mathscr{E} \overset{1}{\hookrightarrow} {}^{\tau}\mathscr{E}\}.$$

The space $\mathrm{Shtuka}_{(1,0)}^{\{1\}}$ is the empty set, because the degrees of $\mathscr{E}$ and ${}^{\tau}\mathscr{E}$ must be equal. One rectifies this by punting the extra degree to a point "$\infty$" sitting at infinity:

$$\begin{tikzcd} \mathrm{Shtuka}_{V\boxtimes V^*}^{\{1,2\}}|_{X\times\{\infty\}} \arrow[r, equal] & \mathrm{Shtuka}^{\uparrow\downarrow} = \{\mathscr{E} \underset{x}{\overset{1}{\hookrightarrow}} \mathscr{E}' \underset{\infty}{\overset{1}{\hookleftarrow}} {}^{\tau}\mathscr{E}\} \arrow[d, "\text{rel. dim. } 2"'] \\ & X \end{tikzcd}$$

In this case, the Eichler-Shimura relation should be as follows. The cohomology group $H^*(\mathrm{Shtuka}_{V\boxtimes V^*}^{\{1,2\}}|_{(v,\infty)})$ admits an action of partial Frobenius $(F_{\{1\}})^{\deg v}$ as well as a "Hecke operator" $T(h_{V,v})$. Analogously to Example 13.2.1, the Hecke algebra really acts through $\mathscr{S}_{V,v}$, as $T(h_{V,v})$ was not defined on a special fiber. The Eichler-Shimura relation is then

$$(F_{\{1\}}^{\deg v})^2 - \mathscr{S}_{V,v} \circ F_{\{1\}}^{\deg v} + \mathscr{S}_{\bigwedge^2 V,,v} = 0 \in \mathrm{End}(H_c^*(\mathrm{Shtuka})).$$

## 14. Applictions to Galois Representations

14.1. **Lafforgue's algebra.** Recall that we constructed excursion operators

$$\mathscr{S}_{I,W,x,\xi,(\gamma_i)_{i\in I}} \in \mathrm{End}({}^0\mathscr{H}^\emptyset) = \mathrm{End}(C_{\mathrm{cusp}}[G])$$

which were parametrized by the data of

- an index set $I$,
- a representation $W \in \mathrm{Rep}(\widehat{G}^I)$,
- maps $\mathrm{triv} \xrightarrow{x} W|_{\widehat{G}} \xrightarrow{\xi} \mathrm{triv}$,
- $\gamma_i \in \Gamma := \mathrm{Gal}(F^s/F)$.

*Definition* 14.1.1. Let $\mathscr{B} \subset \mathrm{End}(C_{\mathrm{cusp}}[G])$ be the subalgebra generated by these excursion operators.

**Proposition 14.1.2.** *The algebra $\mathscr{B}$ is commutative.*

*Proof.* Recall from Example 12.4.5 that $\mathrm{Shtuka} \times X \cong \mathrm{Shtuka}^{\{1\}}_{\mathrm{triv}}$, i.e. $\mathrm{Shtuka}^\emptyset$ is the fiber of the moduli of one-point modification shtukas on $X$, but where the modification is *trivial*. The excursion operator was actually realized at the level of the "larger" moduli stacks

$$\begin{tikzcd} \mathrm{Shtuka}^{\{1\}}_{\mathrm{triv}} \arrow[d] & \mathrm{Shtuka}^I_W \arrow[d] \\ X \arrow[r] & X^I \end{tikzcd}$$

Namely, we defined $\mathscr{S}_{\dots}$ by considering the action of the fundamental group on the cohomology of $\mathrm{Shtuka}^I_W$, and then restricting to the diagonal to get an action on the cohomology of $\mathrm{Shtuka}^{\{1\}}_{\mathrm{triv}}$, and finally taking the stalk at a point to get an action on the cohomology of $\mathrm{Shtuka}^\emptyset$.

Suppose that we have two excursion operators, coming from two such setups

$$\begin{tikzcd} \mathrm{Shtuka}^{\{1\}}_{\mathrm{triv}} \arrow[d] & \mathrm{Shtuka}^I_W \arrow[d] \\ X \arrow[r] & X^I \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Shtuka}^{\{1\}}_{\mathrm{triv}} \arrow[d] & \mathrm{Shtuka}^J_W \arrow[d] \\ X \arrow[r] & X^J \end{tikzcd}$$

Then we can splice these together to obtain, for instance,

$$\begin{tikzcd} \mathrm{Shtuka}^{\{1\}}_{\mathrm{triv}} \arrow[r] \arrow[d] & \mathrm{Shtuka}^I_W \arrow[r] \arrow[d] & \mathrm{Shtuka}^I_W \times \mathrm{Shtuka}^J_{W'} \arrow[d] \\ X \arrow[r] & X^I \arrow[r] & X^I \times X^J \end{tikzcd}$$

This says that

$$\mathscr{S}_{I,W_j,x,\xi,(\gamma_i)_{i\in I}} \circ \mathscr{S}_{J,W',x',\xi',(\gamma_j)_{j\in J}} = \mathscr{S}_{I\sqcup J,W\boxtimes W',x\otimes x',\xi\otimes\xi',(\gamma_i,\gamma_j)}.$$

*Exercise* 14.1.3. Check, by writing down a large commutative diagram, that this equality holds (it is obvious that there will be an equality for some permutation of the daa on the right hand side, but why is this the right one?).

This looks good for showing commutativity, but we are not done quite yet: we need to show that we can swap the order in the decomposition of the index set $I \sqcup J, W \boxtimes W', x \otimes x', \xi \otimes \xi', (\gamma_i, \gamma_j)$ without changing the operator $\mathscr{S}_{I\sqcup J, W\boxtimes W', x\otimes x', \xi\otimes\xi', (\gamma_i,\gamma_j)}$, or in other words that

$$\mathscr{S}_{I\sqcup J, W\boxtimes W', x\otimes x', \xi\otimes x', (\gamma_i,\gamma_j)} = \mathscr{S}_{J\sqcup I, W'\boxtimes W, x'\otimes x, \xi'\otimes\xi, (\gamma_j,\gamma_i)}$$

More generally any map $\varphi \colon I \to J$ induces $\widehat{G}^J \xrightarrow{\Delta_\varphi} \widehat{G}^I$. The composite diagonal map can be factored as

$$\widehat{G} \xrightarrow{\Delta} \widehat{G}^J \xrightarrow{\Delta_\varphi} \widehat{G}^I$$

Then the general claim is

$$\mathscr{S}_{I,W,x,\xi,(\gamma_{\varphi(i)})_{i\in I}} = \mathscr{S}_{I,\Delta_\varphi^* W,x,\xi,(\gamma_j)_{j\in J}}$$

This follows from the fact that these operators are identified by the identification $\mathscr{H}_W^I|_{X^J} \cong \mathscr{H}^J_{\Delta_\varphi^* W}$.

Applying this above to swap the order of the decomposition of the index set, we obtain

$$\mathscr{S}_{I\sqcup J, W\boxtimes W', x\otimes x', \xi\otimes x', (\gamma_i,\gamma_j)} = \mathscr{S}_{J\sqcup I, W'\boxtimes W, x'\otimes x, \xi'\otimes\xi, (\gamma_j,\gamma_i)}$$

as desired. $\square$

Now, $\mathscr{B}$ acts on $C_{\text{cusp}}$ and $\mathscr{B}$ is finite-dimensional, as $C_{\text{cusp}}$ is finite-dimensional. So we can decompose

$$C_{\text{cusp}} = \bigoplus_{\sigma \colon \mathscr{B} \to \overline{\mathbb{Q}_\ell}^\times} C_{\text{cusp}}(\sigma).$$

As $\mathscr{B}$ contains the Hecke algebra $\mathscr{H} = \bigotimes_{v\in|X|} \mathscr{H}_v$, this decomposition is potentially *finer* than decomposition into isotypic $\mathscr{H}$-modules (it is indeed finer in some groups $G \neq \mathrm{GL}(n)$), as the Hecke algebra is local but $\mathscr{B}$ carries global data.

14.2. **Lafforgue's map.** Our next goal is to study applications to Galois representations, and specifically to define the map

$$\operatorname{Spec} \mathscr{B} \to \{\rho \colon \Gamma := \mathrm{Gal}(F^s/F) \to \widehat{G}\}.$$

The construction of the excursion operators furnish a $\mathscr{B}$-valued function

$$(I, W, x, \xi) \mapsto f_{I,W,x,\xi} \in \mathrm{Fun}(\Gamma^I, \mathscr{B}) \tag{14.2.1}$$

where $f_{I,W,x,\xi}$ takes

$$(\gamma_1, \dots, \gamma_n) \mapsto \mathscr{S}_{I,\dots,(\gamma_i)}.$$

To warm up for the upcoming consruction, consider $\widehat{G} \times \widehat{G}$ acting on $\mathscr{O}(\widehat{G})$ by left and right translation. By an algebraic version of the Peter-Weyl Theorem,

$$\mathscr{O}(\widehat{G}) \cong \bigoplus_{V\in \mathrm{Irrep}(\widehat{G})} V \boxtimes V^*$$

where as a representation of $\widehat{G}\times\widehat{G}$, the left $\widehat{G}$ acts on the first factor of the tensor product, and the right $\widehat{G}$ on the second. If we restrict this action to the diagonal, then we obtain the representation $\bigoplus_V \mathrm{End}(V, V)$.

More generally, for an index set $I$ we have a Peter-Weyl decomposition

$$\mathscr{O}(\widehat{G}^I) = \bigoplus_{(V_i)_{i\in I}} \underbrace{(\boxtimes_{i\in I} V_i)}_{W} \boxtimes \underbrace{\left(\boxtimes_{i\in I} V_i^*\right)}_{W^*}.$$

The group $\widehat{G}^I \times \widehat{G}^I$ acts on $\widehat{G}^I$ by left and right translation, respectively, no the factors with corresponding indices. We can restrict this to an action of $\widehat{G} \times \widehat{G}$ on $\widehat{G}^I$ via the diagonal embedding

$$\widehat{G} \times \widehat{G} \xrightarrow{\Delta,\Delta} \widehat{G}^I \times \widehat{G}^I.$$

Under this action, we have a decomposition

$$\mathscr{O}(\widehat{G}\backslash \widehat{G}^I/\widehat{G}) = \bigoplus_{W \in \mathrm{Irrep}(\widehat{G}^I)} W^{\widehat{G}} \otimes (W^*)^{\widehat{G}}$$

Now we can view $x \in W^{\widehat{G}}$ and $\xi \in (W^*)^{\widehat{G}}$, and thus $x \otimes \xi \in W^{\widehat{G}} \otimes (W^*)^{\widehat{G}}$. As $x, \xi$ run over bases of $W^{\widehat{G}}$ and $(W^*)^{\widehat{G}}$, $x \otimes \xi$ runs over a basis of $W^{\widehat{G}} \otimes (W^*)^{\widehat{G}}$.

Thus the data of $(I, W, x, \xi)$ is equivalent to the data of an element of $\mathscr{O}(\widehat{G}\backslash \widehat{G}^I/\widehat{G})$, which satisfies some structural compatibility. By (14.2.1), we get a map

$$\theta_I : \mathscr{O}(\widehat{G}\backslash \widehat{G}^I/\widehat{G}) \to C(\Gamma^I, \mathscr{B})$$

In fact, this is even a *ring homomorphism*, where the ring structure on $C(\Gamma^I, \mathscr{B})$ come from pointwise multiplication. Moreover, it is easy to check that the image of $\theta_I$ lies in $C(\Gamma\backslash\Gamma^I/\Gamma, \mathscr{B})$.

*Exercise* 14.2.1. Check these claims.

Let's think about what this double coset looks like. Label $I = \{0, 1, \dots, n\}$. In $\widehat{G}\backslash \widehat{G}^I/\widehat{G}$ we can change the first coordinate to be the identity, which shows that

$$\widehat{G}\backslash \widehat{G}^I/\widehat{G} \xleftarrow{\cong} \widehat{G}^{I-\{0\}}/\widehat{G}$$

sending $(1, g_1, g_2, \dots, g_n) \leftarrow (g_1, \dots, g_n)$, where the quotient on the right hand side is by the action of simultaneous conjugation on all the entries.

If $I$ is finite, then by Exercise 14.2.1 the map $\theta_I$ can be viewed as a ring homomorphism to the invariant subspace

$$\theta_I : \mathscr{O}(\widehat{G}^I)^{\widehat{G}} \to \mathrm{Fun}(\Gamma^I, \mathscr{B})^{\Gamma}$$

where the action in both cases is by simultaneous conjugation. Composing with any $\sigma_i : \mathscr{B} \to \overline{\mathbb{Q}_\ell}$ gives

$$\boxed{\sigma \circ \theta_I : \mathscr{O}(\widehat{G}^I)^{\widehat{G}} \to \mathrm{Fun}(\Gamma^I, \overline{\mathbb{Q}_\ell})^{\Gamma}.}$$

This satisfies some compatibility relations, which fit into the framework of *pseudo-representation* which we presently discuss.

14.3. **Pseudo-representations.**

*Definition* 14.3.1. Let $H$ be an algebraic group over an algebraically closed, characteristic 0 field $E$. A *pseudo-representation* of $\Gamma$ in $H$ consists of the data of a ring homomorphism

$$\theta_I \colon \mathscr{O}(H^I)^H \to C(\Gamma^I)^\Gamma \text{ for all non-empty, finite index sets } I$$

satisfying

(1) For every non-empty finite set $I$, a ring homomorphism $\theta_I \colon \mathscr{O}(H^I)^H \to C(\Gamma^I)^\Gamma$, where the actions in both cases are by simultaneous conjugation on the source.
(2) For all $\varphi \colon I \to J$, a commutative diagram

$$\begin{tikzcd} \mathscr{O}(H^I)^H \arrow[r, "\theta_I"] \arrow[d, "\Delta_\varphi^*"'] & C(\Gamma^I)^\Gamma \arrow[d, "\Delta_\varphi^*"'] \\ \mathscr{O}(H^J)^H \arrow[r, "\theta_J"] & C(\Gamma^J)^J. \end{tikzcd}$$

(3) Compatibility with the group (i.e. Hopf algebra) structure: if $m^*$ denotes the multiplication map

$$\begin{tikzcd} \mathscr{O}(H) \arrow[d, "m^*"'] \\ \mathscr{O}(H \times H) \end{tikzcd}$$

then the diagram commutes

$$\begin{tikzcd} \mathscr{O}(H^n)^H \arrow[r, "\theta_n"] \arrow[d, "1\otimes\ldots\otimes m^*"'] & C(\Gamma^n)^\Gamma \arrow[d, "1\otimes\ldots\otimes m^*"] \\ \mathscr{O}(H^{n+1})^H \arrow[r, "\theta_{n+1}"] & C(\Gamma^{n+1})^\Gamma \end{tikzcd}$$

More concretely, this says that for all $f \colon H^n \to \mathbb{Q}_\ell$ invariant under $H$-conjugation, the function $f' \colon H^{n+1} \to \overline{\mathbb{Q}_\ell}$ taking

$$(h_1, \ldots, h_n, h_{n+1}) \mapsto f(h_1, \ldots, h_n h_{n+1})$$

satisfies $\theta_{n+1}(f')(\gamma_1, \ldots, \gamma_{n+1}) = \theta_n(f)(\gamma_1, \ldots, \gamma_n\gamma_{n+1})$.

*Example* 14.3.2. It is easy to go from a representation to a pseudo-representation by the following recipe: If $\rho \colon \Gamma \to H$ is a group homomorphism, then define

$$\theta_n(f)(\gamma_1, \ldots, \gamma_n) := f(\rho(\gamma_1), \ldots, \rho(\gamma_n)).$$

Going in the other direction is a major theorem.

**Theorem 14.3.3.** *Let $H$ be a reductive group and $\{\theta_I\}_I$ be a pseudo-representation of $\Gamma$ in $H$. Then there exists a homomorphism $\rho \colon \Gamma \to H(E)$ for some finite $E/\overline{\mathbb{Q}_\ell}$ such that for all $\gamma_1, \ldots, \gamma_n \in \Gamma$ and $f \in \mathscr{O}(H^n)^H$,*

$$(\theta_n f)(\gamma_1, \ldots, \gamma_n) = f(\rho(\gamma_1), \ldots, \rho(\gamma_n)).$$

*Moreover, if we require the Zariski closure of $\rho(\Gamma)$ to be reductive (which is always possible), then $\rho$ is unique up to $H$-conjugacy.*

*Example* 14.3.4. Let $\chi \in \mathscr{O}(H)^H$ correspond to the trace. Given a pseudo-representation of $\Gamma$ in $H$, $\theta_1(\chi)$ describes the character of a representation of $\Gamma$ corresponding to an honest representation $\Gamma \to H$.

This is almost as good as the full data of the representation. If $H = \mathrm{GL}_n$, for instance, this *already* determines the representation up to semisimplification!

This reflects the general property that the data involved in a pseudo-representation actually "overspecifies" the representation - the point is not to construct the representation, but to show that it doesn't "contradict itself."

*Example* 14.3.5. If $H = \mathrm{GL}_r$, one can define the notion of a "pseudo-representation of $\Gamma$ into $\mathrm{Mat}_r$ by similar data of

$$\theta_I \colon \mathscr{O}(\mathrm{Mat}_r^I)^{\mathrm{GL}_r} \to C(\Gamma^I)^{\Gamma}.$$

The proof of this theorem shows that there is a semigroup homomorphism $\rho \colon \Gamma \to \mathrm{Mat}_r = \mathrm{End}(V)$ (with $\mathrm{Mat}_r$ regarded as a multiplicative semigroup). This has the property that $\rho(1)^2 = \rho(1)$, and the corresponding map $\rho' \colon \Gamma \to \mathrm{GL}(\rho(1)V)$ is an honest representation.

*Remark* 14.3.6. We could have relaxed the definition a little. Suppose that one is given the data of $\theta_I \colon \mathscr{O}(H^I)^H \to C(\Gamma^I)$ (so not taking the conjugation-invariant subspace on the target) plus the analogs of the compatibility conditions, then it will be *automatic* that $\theta_I$ lands in $C(\Gamma^I)^{\Gamma}$.

To illustrate why, let's just consider $\theta_1 \colon \mathscr{O}(H)^H \to C(\Gamma)$, which takes a class function on $H$ to some function on $\Gamma$. We want to show that its image is also a class function, equivalently

$$\theta_1 f(\gamma_1\gamma_2) = \theta_1 f(\gamma_2\gamma_1). \tag{14.3.1}$$

This follows from the commutative triangle:

$$\begin{tikzcd} & \mathscr{O}(H)^H \arrow[dl, "m^*"'] \arrow[dr, "m^*"] & \\ \mathscr{O}(H^2)^H \arrow[rr, "(12)"] & & \mathscr{O}(H^2)^H \end{tikzcd}$$

The compatibility condition demands that after applying $\theta_1$, we get a commutative triangle

$$\begin{tikzcd} & \theta f \arrow[dl] \arrow[dr] & \\ m^*(\theta f) \arrow[rr, "(12)"] & & m^*(\theta f) \end{tikzcd}$$

But that is precisely (14.3.1).

14.4. **Classical pseudo-representations.** To begin the construction of representations from pseudo-representations, let's relate our notion of pseudo-representation to a more classical one.

*Definition* 14.4.1. A *classical pseudorepresentation of* $\Gamma$ of dimension $d$ is a class function

$$\tau \colon \Gamma \to \overline{\mathbb{Q}_\ell}$$

satisfying

$$\sum_{\sigma \in S_{d+1}} (-1)^{\sigma} \prod_{\sigma = \underbrace{c_1 c_2 \dots c_r}_{\text{cycle decomp.}}} \tau(\gamma_{c_i}) = 0$$

where if $c_i = (1, 2, \dots, \ell)$ then $\gamma_{c_i} = \gamma_1 \gamma_2 \dots \gamma_\ell$. The class function assumption implies that this is well defined.

*Example* 14.4.2. If $d = 1$, then the condition is $\tau(\gamma_1\gamma_2) = \tau(\gamma_1)\tau(\gamma_2)$, which corresponds to an actual 1-dimensional representation.

*Example* 14.4.3. If $d = 2$, then the condition is

$$\begin{aligned} &\tau(\gamma_1)\tau(\gamma_2)\tau(\gamma_3) \\ &\quad -\tau(\gamma_1\gamma_2)\tau(\gamma_3) - \tau(\gamma_2\gamma_3)\tau(\gamma_1) - \tau(\gamma_3\gamma_1)\tau(\gamma_2) \\ &\qquad\qquad + \tau(\gamma_1\gamma_2\gamma_3) + \tau(\gamma_1\gamma_3\gamma_2) = 0. \end{aligned}$$

This seems much more opaque, but it does in fact correspond to a genuine 2-dimensionl representation.

We want to relate the two definitions of pseudo-representations. That may be surprising, since there seem to be so many more functions and relations involved in our definition of pseudo-representation than classical pseudo-representation. Roughly speaking, we'll show that all the functions $\theta_I$ "come from" one function, and all the axioms "come from" one equation.

*Preparations.* We consider the case of $H = \mathrm{GL}(V)$ over a field $E$. We have

$$\mathscr{O}(\mathrm{End}(V)^n) = \mathrm{Sym}(\mathrm{End}(V)x_1 \oplus \dots \oplus \mathrm{End}(V)x_n).$$

This admits a surjection

$$\bigoplus_{\underline{m}} (\mathrm{End}(V)^{\otimes \underline{m}})^{\mathrm{GL}(V)} x^{\underline{m}} \to \mathrm{Sym}(\mathrm{End}(V)x_1 \oplus \dots \oplus \mathrm{End}(V)x_n).$$

**Schur-Weyl duality.** Now, $V^{\otimes n}$ admits a left action of $S_n$ and a right action of $\mathrm{GL}(V)$, which induces corresponding actions on $\mathrm{End}(V^{\otimes n})$. Then *Schur-Weyl* duality implies that there exists a surjection

$$E[S_m] \twoheadrightarrow \mathrm{End}(V^{\otimes m})^H,$$

denoted $\sigma \in S_m \mapsto T_\sigma$. Let $\dim_E V = r$. The kernel is generated by the element

$$\lambda_{r+1} := \begin{cases} \sum_{\sigma \in S_{r+1}} (-1)^\sigma \sigma & S_{r+1} \hookrightarrow S_m, \\ 0 & m \le r. \end{cases}$$

It is easy to see that $\lambda_{r+1}$ lies in the kernel, as its image lies in the endomorphisms of $\underbrace{(\bigwedge^{r+1} V)}_{=0} \otimes V^{\otimes (m-r-1)}$.

We now describe the elements $T_\sigma \in \mathrm{End}(V^{\otimes m})^{\mathrm{GL}_r} \hookrightarrow \mathscr{O}(\mathrm{Mat}_r^{\oplus m})^{\mathrm{GL}_r}$. The image of $T_\sigma$ is the multilinear, $\mathrm{GL}_r$-invariant form

$$T_\sigma \colon \underbrace{\mathrm{Mat}_r \times \ldots \times \mathrm{Mat}_r}_{m} \to E$$

defined by

$$(X_1, \ldots, X_m) \mapsto \mathrm{Tr}(\sigma \circ (X_1 \otimes \ldots \otimes X_m)|_{V^{\otimes m}}) = \prod_{c \in \sigma} \mathrm{Tr}(X_c|_V)$$

For instance, if the cycle $c = (123)$ is in $\sigma$'s cycle decomposition, then $X_c = X_1X_2X_3$.

*Example* 14.4.4. The element $\lambda_{r+1}$, viewed as multilinear map $\underbrace{\mathrm{Mat}_r \times \ldots \times \mathrm{Mat}_r}_{r+1} \to E$, sends

$$(X_1, \ldots, X_m) \mapsto \sum_{\sigma \in S_{r+1}} (-1)^\sigma \prod_{c \in \sigma} \mathrm{tr}(X_c|_V).$$

*Classical pseudo-representation to pseudo-representation.* Suppose $\tau \colon \Gamma \to E$ is a classical pseudo-representation of $\dim \le r$. We want to build a ring homomorphism

$$\theta_I \colon \mathscr{O}(\mathrm{Mat}_r^I)^{\mathrm{GL}_r} \to C(\Gamma^I)^\Gamma.$$

For $I = \{1, \ldots, n\}$, we have defined a surjection

$$\bigoplus_{\underline{m}=(m_1,\ldots,m_n)} \mathrm{End}(V^{\otimes m})^{\mathrm{GL}_r} x^{\underline{m}} \twoheadrightarrow \mathscr{O}(\mathrm{Mat}_r^I)^{\mathrm{GL}_r}.$$

For $\sigma \in S_m$, the image of $T_\sigma x^{\underline{m}}$ is obtained by replacing the $x$'s by matrices appropriately. For example, $T_\sigma x_1^2 x_2 x_3$ $(n = 3, m = 4)$ should go to the function $(X_1, X_2, X_3) \mapsto T_\sigma(X_1, X_2, X_3, X_4)$.

This may be expressed more succintly as follows. If $\varphi = \underline{m}$ as a function $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ (up to the action of $S_m$), then $T_\sigma \mapsto \Delta_\varphi^* T_\sigma$, where

$$\Delta_\varphi^* T_\sigma(X_1, \ldots, X_n) = T_\sigma(X_{\varphi(1)}, \ldots, X_{\varphi(m)}).$$

This implies that $\mathscr{O}(\mathrm{Mat}_r^n)^{\mathrm{GL}_r}$ is linearly spanned by $\Delta_\varphi^* T_\sigma$ where $\varphi \colon \{1, \ldots, m\} \to \{1, \ldots, n\}$, or in other words by the functions

$$X_1, \ldots, X_n \mapsto \mathrm{Tr}(X_{i_1} \ldots X_{i_m}).$$

We've now produced nice generators for $\mathscr{O}(\mathrm{Mat}_r^I)^{\mathrm{GL}_r}$, but we haven't yet specified the map

$$\mathscr{O}(\mathrm{Mat}_r^I)^{\mathrm{GL}_r} \to C(\Gamma^I)^\Gamma.$$

However, the natural choice is now clear. The domain is generated by expressions of the form $\Delta_\varphi^* T_\sigma$, and we send this to function $\Delta_\varphi^* \tau_\sigma$, where the definition of $\tau_\sigma$ is the same as $T_\sigma$ except replacing the trace by the class function $\tau$.

We record the general ingredient we used:

**Theorem 14.4.5** (Procesi)**.** *The map $R = E[t_c \colon c \text{ a cycle}] \to \mathscr{O}(\mathrm{Mat}_r^\infty)^{\mathrm{GL}_r}$ sending $t_c$ to the function*

$$(X_1, x_2, \ldots) \mapsto \mathrm{Tr}(X_c)$$

*is surjective, and the kernel $I$ is generated by*

$$\lambda_{r+1}(M_1,\dots,M_{r+1}) = \sum_{\sigma\in S_{r+1}} (-1)^\sigma \prod_{c\in\sigma} t_c(M_1,\dots,M_{r+1}).$$

**Aside: more about Schur-Weyl duality**. Let $V$ be a vector space of dimension $d$. We have a decomposition of $S_n$-representations

$$V^{\otimes n} = \bigoplus_{\rho\in \mathrm{Irrep}(S_n)} \rho \boxtimes V_\rho$$

(so far this is always true) where $V_\rho$ is 0 or irreducible as a $\mathrm{GL}(V)$-representation.

There is a bijection between $\mathrm{Irrep}(S_n)$ and partitions of $n$, in which the trivial representation goes to $n$, the standard goes to $1+(n-1)$, and the sign goes to $1+1+\ldots+1$. (This is possibly off by conjugation of the partition.)

Now, in the above decomposition not every irreducible representation appears. In fact, the only $\lambda$ that appear are those having at most $d$ parts: $\lambda = (\lambda_1 \geq \ldots \geq \lambda_d \geq 0)$. (By padding with 0 if necessary, we can assume that there are exactly $d$ parts.) If $\rho$ is the representation corresponding to the partition $\lambda$, then Schur-Weyl duality says that $V_\rho$ is the representation of $\mathrm{GL}(V)$ with highest weight $\lambda$.

*Example* 14.4.6. From this we can see that the sign representation doesn't appear in $V^{\otimes(d+1)}$. If it did, then the multiplicity space would be $\bigwedge^{d+1} V = 0$, i.e. the corresponding summand "would be" $\mathrm{sgn}\otimes \bigwedge^{d+1} V \subset V^{\otimes d+1}$.

14.5. **GIT reformulation of pseudo-representations.** Another way to formulate the data of a pseudo-representations in terms of functions

$$\xi_I \colon \Gamma^I/\Gamma \to H^I//H \quad \text{(the GIT quotient)}$$

satisfying:

- compatibility with respect to maps $I \xrightarrow{\varphi} J$ in the sense that the diagram

$$\begin{tikzcd} \Gamma^I/\Gamma \arrow[r, "\xi_I"] & H^I//H \\ \Gamma^J/\Gamma \arrow[u, "\Delta_\varphi"] \arrow[r, "\xi_J"] & H^J//H \arrow[u, "\Delta_\varphi"'] \end{tikzcd}$$

  commutes, and
- compatible with multiplication.

If $I = \{1,\dots,n\}$, then we denote $\xi_I$ by

$$(\gamma_1,\dots,\gamma_n) \mapsto [h_1,\dots,h_n].$$

A consequence of taking the GIT quotient is that the elements $h_1,\dots,h_n$ are not necessarily well-defined up to $H$-conjugacy, but they will be once we impose a semisimplicity requirement.

*Example* 14.5.1. If $I=\{1,\dots,m\}\hookrightarrow J=\{1,\dots,n\}$ then the compatibility with index transfer says

$$\begin{tikzcd} (\gamma_1,\dots,\gamma_n) \arrow[r] \arrow[d] & {[h_1,\dots,h_n]} \arrow[d] \\ (\gamma_1,\dots,\gamma_m) \arrow[r] & {[h_1,\dots,h_m]} \end{tikzcd}$$

commutes, and the compatibility with multiplication is that

$$\begin{tikzcd} (\gamma_1,\dots,\gamma_n,\gamma_{n+1}) \arrow[r] \arrow[d] & {[h_1,\dots,h_n,h_{n+1}]} \arrow[d] \\ (\gamma_1,\dots,\gamma_n\gamma_{n+1}) \arrow[r] & {[h_1,\dots,h_nh_{n+1}]} \end{tikzcd}$$

commutes. Even though the elements weren't well-defined. this is well-defined on the GIT quotient

*Example* 14.5.2. If $\Gamma=F_n=\langle s_1,\dots,s_n\rangle$, then

$$\xi_n:\Gamma^n\to H^n//H$$

is an assigment $(s_1,\dots,s_n)\mapsto[h_1,\dots,h_n]$. Taking $n=1$ already gives $\Gamma\to H$, reflecting the highly overspecified nature of the data.

For $n=1$, and $h\in H$, Richardson's Theorem (stated below) implies:

- $H\cdot h$ is closed if and only if $h$ is semisimple,
- $\overline{H\cdot h}\ni\{1\}$ if and only if $h$ is unipotent.

*Definition* 14.5.3. For general $n$ and $h_1,\dots,h_n\in H$ we define $A(h_1,\dots,h_n)=\langle h_1,\dots,h_n\rangle$, the subgroup generated by $h_1,\dots,h_n$.

- We say that $(h_1,\dots,h_n)$ is *semisimple* if $A(h_1,\dots,h_n)$ is reductive, and
- *unipotent* if $A(h_1,\dots,h_n)$ is unipotent.

*Example* 14.5.4. For $H=\mathrm{SL}_2$,

$$\left(\begin{pmatrix}1&1\\&1\end{pmatrix},\begin{pmatrix}1&\\1&1\end{pmatrix}\right).$$

is semisimple, as these two elements generate the whole group.

**Theorem 14.5.5** (Richardson)**.** *Let $H/E$ be a reductive group over an algebraically closed field. Then:*

*(1) $(h_1,\dots,h_n)\in H^n$ is semisimple if and only if $H\cdot(h_1,\dots,h_n)\subset H^n$ is closed.*
*(2) If $(h_1,\dots,h_n)\in H^n$ we can write $h_i=h_i'h_i''$ such that $(h_1',\dots,h_n')$ is unipotent and $(h_1'',\dots,h_n'')$ is semisimple, and $H\cdot(h_1'',\dots,h_n'')$ is the unique closed orbit in $\overline{H\cdot(h_1,h_2,\dots,h_n)}$.*
*(3) There is a bijection*

$$H^n//H(E)\longleftrightarrow \textit{semisimple } n\textit{-tuples in } H(E).$$

*Remark* 14.5.6. The assumption of an algebraically closed field is actually only needed for the last assertion.

*Example* 14.5.7. Let $H = \mathrm{SL}_2$, $h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, and $h_2 = \begin{pmatrix} a & 1 \\ 0 & a^{-1} \end{pmatrix}$. Then $\langle h_1, h_2 \rangle = B = U \cdot T$. The decomposition corresponding to (2) above is

$$h_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$h_2 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}$$

14.6. **Pseudo-representations to representations.** For every $n$ and $(\gamma_1, \ldots, \gamma_n) \in \Gamma^n$, we have

$$\xi_n(\gamma_1, \ldots, \gamma_n) = [h_1, \ldots, h_n]$$

where $(h_1, \ldots, h_n)$ a semisimple $n$-tuple in $H$, well-defined up to conjugacy.

We now *fix a specific* $n$ and $(\gamma_1, \ldots, \gamma_n)$ and $(h_1, \ldots, h_n)$ such that

(1) $\overline{\langle h_1, \ldots, h_n \rangle}$ has maximal dimension among all such constructions,
(2) among the tuples achieving (1), the centralizer $C(h_1, \ldots, h_n)$ of $\langle h_1, \ldots, h_n \rangle$ is minimal (in dimension and then number of components). Note that because $\overline{\langle h_1, \ldots, h_n \rangle}$ is reductive by the semisimplicity hypothesis, this centralizer is reductive as well.

We now define the desired representation

$$\rho : \Gamma \to H$$

as follows. For $\gamma \in \Gamma$, we set $\rho(\gamma)$ to be the *unique* $h$ such that

$$(\gamma_1, \ldots, \gamma_n, \gamma) \xrightarrow{\xi_{n+1}} [h_1, \ldots, h_n, h]$$

and the tuple $(h_1, \ldots, h_n, h)$ is semisimple. We have to justify the existence and uniqueness.

*Existence.* We want to show that there exists $h$ such that $(h_1, \ldots, h_n, h) \in \xi_{n+1}(\gamma_1, \ldots, \gamma_n, \gamma)$ is semisimple. We can certainly pick *some* semisimple tuple

$$(h_1', \ldots, h_n', h') \in \xi_{n+1}(\gamma_1, \ldots, \gamma_n, \gamma).$$

By compatibility with index transfer applied to the diagram

$$\begin{tikzcd} (\gamma_1, \ldots, \gamma_n, \gamma) \arrow[r] \arrow[d] & {[h_1', \ldots, h_n', h']} \arrow[d] \\ (\gamma_1, \ldots, \gamma_n) \arrow[r] & {[h_1', \ldots, h_n']} \end{tikzcd}$$

we see that $(h_1', \ldots, h_n') \in \xi_n(\gamma_1, \ldots, \gamma_n)$.

We know that

$$(\gamma_1, \ldots, \gamma_n) \xrightarrow{\xi_n} [h_1, \ldots, h_n],$$

so $[h_1, \ldots, h_n] = [h_1', \ldots, h_n']$ in the GIT quotient. We claim that $(h_1', \ldots, h_n')$ is semisimple. If not, we may arrange that $(h_1, \ldots, h_n) = (h_1', \ldots, h_n')^{ss}$ (for any two things with the same image in the GIT quotient, the semisimple one can be moved into the semisimple part of the other). That implies that $\overline{\langle h_1, \ldots, h_n \rangle}$ is a Levi subgroup of $\overline{\langle h_1', \ldots, h_n' \rangle}$, but the first maximality condition forces this to be an equality. Therefore, $(h_1',\dots,h_n')$ is semisimple as well, so by conjugating we may assume that $(h_1,\dots,h_n)=(h_1',\dots,h_n')$, and hence take $h=h'$.

*Uniqueness.* A priori, $h$ is well-defined up to $H$-conjugacy. However, the existence argument shows that $(h_1,\dots,h_n,h)$ is a semisimple tuple, and that furthermore $(h_1,\dots,h_n)$ is semisimple. Therefore, $h$ is really well-defined up to $H$-conjugacy *keeping* $(h_1,\dots,h_n)$ as the first $n$ coordinates; i.e. up to conjugacy by $C(h_1,\dots,h_n)$. So it suffices to show that $h$ commutes with $C(h_1,\dots,h_n)$. But if this were not the case then we would have $C(h_1,\dots,h_n,h)<C(h_1,\dots,h_n)$, contradicting minimality in the definition of $(h_1,\dots,h_n)$.

So this $h$ is well-defined, giving a map $\rho:\Gamma\to H$. However, there are some further conditions to check.

We need to show that $\rho$ is actually a group homomorphism:

$$\rho(\gamma\delta)=\rho(\gamma)\rho(\delta).$$

The same argument as we made for existence shows that there exist $\gamma',\delta'$ such that

$$(h_1,\dots,h_n,h',h'')\in\xi_{n+2}(h_1,\dots,h_n,\gamma,\delta).$$

We have a compatibility diagram

$$\begin{tikzcd}
(\gamma_1,\dots,\gamma_n,\gamma,\delta) \arrow[r, "\xi_{n+2}"] \arrow[d] & {[h_1,\dots,h_n,h',h'']} \arrow[d] \\
(\gamma_1,\dots,\gamma_n,\gamma) \arrow[r, "\xi_{n+1}"] & {[h_1,\dots,h_n,h']}
\end{tikzcd}$$

Again by similar arguments as in existence, we see that $h'=\rho(\gamma)$ and $h''=\rho(\delta)$.

By multiplication compatibility, we also have a diagram

$$\begin{tikzcd}
(\gamma_1,\dots,\gamma_n,\gamma,\delta) \arrow[r, "\xi_{n+2}"] \arrow[d] & {[h_1,\dots,h_n,\rho(\gamma),\rho(\delta)]} \arrow[d] \\
(\gamma_1,\dots,\gamma_n,\gamma\delta) \arrow[r, "\xi_{n+1}"] & {[h_1,\dots,h_n,\rho(\gamma)\rho(\delta)]}
\end{tikzcd}$$

hence $[h_1,\dots,h_n,\rho(\gamma)\rho(\delta)]=[h_1,\dots,h_n,\rho(\gamma\delta)]$. Furthermore $\rho(\gamma)\rho(\delta)$ is semisimple by the same argument used in the proof of existence: otherwise $\langle h_1,\dots,h_n,\rho(\gamma\delta)\rangle$ would be a Levi subgroup of $\langle\langle h_1,\dots,h_n,\rho(\gamma)\rho(\delta)\rangle$, but the former is already maximal because $\langle h_1,\dots,h_n\rangle$ is. Finally, Then the argument used in the proof of uniqueness shows that $\rho(\gamma\delta)=\rho(\gamma)\rho(\delta)$.

*Remark* 14.6.1. We should actually have set up the original problem in a continuous setting, i.e. started with a "continuous pseudo-representation"

$$\theta_n:\mathscr{O}(H^n)^H\to\mathrm{Cont}(\Gamma^n,E)^\Gamma$$

where $\Gamma$ has some topological group structure. Then one can show that the resuting $\rho:\Gamma\to H(E')$ is continuous (where $E'/E$ is a finite extension).

## 15. Conclusion

15.1. **Summary.** We discussed how Lafforgue constructed a commutative subring $\mathscr{B} \subset \mathrm{End}(C_{\mathrm{cusp}}([G]))$. This ring "contains" all Hecke operators, i.e. we have a map

$$\bigotimes_{x \in |X|} \mathscr{H}_x \to \mathscr{B}. \tag{15.1.1}$$

We wanted to construct a map

$$\mathrm{Spec}\ \mathscr{B} \to \{\text{continuous semisimple } \rho \colon \Gamma \to \widehat{G}(\overline{\mathbb{Q}_\ell})\}/\sim$$

as follows. The construction of the excursion operators gives a collection of maps

$$\theta_I \mathscr{O}(\widehat{G}^I)^{\widehat{G}} \to \mathrm{Fun}(\Gamma^I, \mathscr{B})^{\Gamma} \xrightarrow{\sigma \colon \mathscr{B} \mapsto \overline{\mathbb{Q}_\ell}} \mathrm{Fun}(\Gamma^I)^{\Gamma}, \overline{\mathbb{Q}_\ell}$$

and it turns out that these fit together to define a pseudo-representation $\rho_\gamma \colon \Gamma \to \widehat{G}$. We then showed that these pseudo-representations come from an honest representation of $\Gamma$ into $\widehat{G}$, giving (15.1.1).

Thus, we have a decomposition

$$C_{\mathrm{cusp}} = \bigoplus_{\sigma \colon \mathscr{B} \to \overline{\mathbb{Q}_\ell}} C_{\mathrm{cusp}}(\sigma)$$

and for each $\sigma$ a representation $\rho_\sigma \colon \pi_1(X) \to \widehat{G}$ (the unramifiedness come from the well-definedness of the excursion operators, independent of the choice of lift of Frobenius) satisfying the Hecke compatibility condition that for all $\pi \subset C_{\mathrm{cusp}}(\sigma)$ and all $x \in |X|$, the Satake parameter of $\pi_x$ is conjugate to $\rho_\sigma(\mathrm{Frob}_x)$.

This completes the proof of the main result, modulo finiteness concerns that we ignored, which we now briefly address.

15.2. **Some finiteness issues.** A key point that we brushed under the rug is that the map

$$\begin{array}{c} \mathrm{Shtuka}^I \\ \big\downarrow \\ X^I \end{array}$$

is not of finite type. However, the shtukas have a certain "recursive" structure that allows one to analyze them "at infinity."

*Example* 15.2.1. Recall that for $G = \mathrm{GL}_2$, we denoted by $\mathrm{Shtuka}^{\uparrow\downarrow}_{r=2}$ the moduli stack of modifications of rank $r = 2$ vector bundles of the form $\{\mathscr{E} \overset{1}{\hookrightarrow} \mathscr{E}' \overset{1}{\hookleftarrow} {}^{\tau}\mathscr{E}\}$.

Suppose we have a short exact sequence of such modifications (i.e. a diagram as below with short exact *columns*)

$$\begin{tikzcd} \mathscr{L} \arrow[r] \arrow[d] & \mathscr{L}' \arrow[d] & {}^\tau\mathscr{L} \arrow[l] \arrow[d] \\ \mathscr{E} \arrow[r] \arrow[d] & \mathscr{E}' \arrow[d] & {}^\tau\mathscr{E} \arrow[l] \arrow[d] \\ \mathscr{M} \arrow[r] & \mathscr{M}' & {}^\tau\mathscr{M} \arrow[l] \end{tikzcd}$$

There are two cases: the modification of $\mathscr{E}'$ can occur in $\mathscr{L}'$, or $\mathscr{M}'$. Suppose it occurs in $\mathscr{L}'$. Then, by chasing through the diagram, it's not hard to show that $\mathscr{M} = \mathscr{M}' = {}^\tau\mathscr{M}$ and $(\mathscr{L} \hookrightarrow \mathscr{L}' \hookleftarrow {}^\tau\mathscr{L})$ is in $\mathrm{Shtuka}^{\uparrow\downarrow}_{r=1}$. Similarly, if the modification occurs in $\mathscr{M}'$, then $\mathscr{L} = \mathscr{L}' = {}^\tau\mathscr{L}$.

This gives two substacks of $\mathrm{Shtuka}^{\uparrow\downarrow}_{r=2}$. Let's think about what each of these looks like. Obviously, the choices of $\mathscr{L}$ and $\mathscr{M}$ are parametrized by $\mathrm{Shtuka}^{\uparrow\downarrow}_{r=1} \times \mathrm{Pic}(k)$. Fix $\mathscr{L} \in \mathrm{Shtuka}^{\uparrow\downarrow}_{r=1}$ and $\mathscr{M} \in \mathrm{Pic}(k)$. Let $C_{\mathscr{L},\mathscr{M}}$ parametrize extensions of $\mathscr{M}$ by $\mathscr{L}$. Then we have a map

$$i\colon C_{\mathscr{L},\mathscr{M}} \hookrightarrow \mathrm{Shtuka}^{\uparrow\downarrow}_{r=2}.$$

**Fact:** Fix a geometric point $(x,y) \in X^2$ with $y \notin \mathrm{Frob}^{\mathbb{Z}}(x)$. Then $C_{\mathscr{L},\mathscr{M}}|_{(x,y)} \cong \mathbb{A}^1$, and $i$ is a closed embedding.

Thus, in $\mathrm{Shtuka}^{\uparrow\downarrow}_{r=2}|_{(x,y)}$ there are two families of curves isomorphic to $\mathbb{A}^1$, and *these families are themselves parametrized by* $\mathrm{Shtuka}^{\uparrow\downarrow}_{R=1}|_{(x,y)} \times \mathrm{Pic}(k)$.

15.3. **Cutting out the cuspidal forms.** Let $\overline{\eta_2} \in X^2$ be a geometric generic point. Then we have actions of $\bigotimes_{x\in|X|}\mathscr{H}_x$ and $\mathrm{Gal}(\overline{\eta_x}/\eta_x)$ on $\mathscr{H} := \varinjlim H^2_c(\mathrm{Shtuka}^{\uparrow\downarrow}_{\overline{\eta_x}}, E)$. We can define the "Hecke-finite" space

$$\mathscr{H}^{Hf} := \bigcup \begin{matrix}\text{finite-type } \mathscr{O}_E\text{-submodule of } \mathscr{H} \\ \text{stable under Hecke operators}\end{matrix}.$$

This admits both Hecke and Galois actions. We want there to be many finite-type $\mathscr{O}_E$-submodules of $\mathscr{H}^{Hf}$ which are moreover stable under partial Frobenius, so that the construction of the excursion operators goes through.

We can prove this using the Eichler-Shimura relation. Indeed, Let $M$ be a finite type $\mathscr{O}_E$-module, Hecke stable. Then Eichler-Shimura tells us that

$$\sum_{i,n} F^n_{\{i\}}(M) = \sum_{n\le N,i} F^n_{\{i\}}(M)$$

and the latter is of finite type. Then we can apply Drinfeld's theorem and argue as before.

We also want to know that this "Hecke finite" space is interesting, and in fact it agrees with the cuspidal subspace.

**Proposition 15.3.1.** *We have*

$$\mathscr{H}^{Hf} := C_c([G])^{Hf} = C_{\mathrm{cusp}}([G]).$$

*Proof.* The containment $\mathscr{H}^{Hf} \supset C_{\text{cusp}}([G])$ is clear: cuspidal forms are Hecke finite as they span only a finite dimensional space.

To prove the containment $\mathscr{H}^{Hf} \subset C_{\text{cusp}}([G])$, suppose $f \in C_c([G], E)$ is such that $\mathscr{H}_x \cdot f$ is finite-dimensional for some $x \in |X|$. Then we want to show that $f$ is a cusp form.

First, let's see if we can see intuitively why this should be true. Think to the classical case of modular curves. Suppose you have a function $f$ with non-zero constant term in the Fourier expansion at $\infty$. The Hecke operator $T_p$ will move a point in the fundamental domain to somewhere else, and compositions of $T_p$ will actually take it arbitrarily high. Then the space spanned will not be finite-dimensional.

Now let's try to make a rigorous argument. Suppose $f$ is not cuspidal, so $CT_P(f) \neq 0$ for some proper parabolic $P < G$. Let $M$ be the associated Levi of $P$, so $CT_P(f) \in C_c([M])$.

There is a map $\mathscr{H}_{G,x} \to \mathscr{H}_{M,x}$, generalizing the Satake transformation $\mathscr{H}_x \to \mathbb{C}[X_\bullet(T)]^W$ when $M = T$, which we denote by $h \mapsto h_P$. This is compatible with the constant term map $CT_P$ in the sense that

$$CT_P(h \cdot f) = h_P CT_P(f).$$

Let's see why this implies that $\{h \cdot f : h \in \mathscr{H}\}$ cannot span a finite-dimensional space for $G = \mathrm{SL}_2$, $T = \mathbb{G}_m$. In this case, $h_B$ is a function $[T] = \mathrm{Pic}(\mathrm{Spec}\ k) = \mathbb{Z}$. If $f$ has non-zero constant term, then $CT_P(h^n \cdot f) = h_B^N CT_P(f)$ is non-zero. But choosing $h_B = t + t^{-1}$, for example, we see that $h_B^N CT_P(f)$ is supported on an arbitrarily wide interval as $N \to \infty$, so these cannot all lie in a finite-dimensional space. $\square$