1 The natural numbers

Week 1, Monday, August 28th, Last updated: 01/09/23, dmy.
Reading: [5, Ch.2-3]

We assume the notion of *set*, 2, and take it as a primitive notion to mean a "collection of distinct objects."

Learning Objectives

Next eight lectures:

- To construct the objects:

$$\mathbb{N}, \quad \mathbb{Z}, \quad \mathbb{Q}, \quad \mathbb{R}$$

  and define the notion of *sets*, 2.

- To prove properties and reason with these objects. In the process, you will learn various proof techniques. Most importantly, *proof by induction* and *proof by contradiction*.

This lecture:

- how to define the natural numbers, $\mathbb{N}$, and appreciate the role of *definitions*.

- how to apply induction. In particular, we would see that even proving statements as associativity of natural numbers is nontrivial!

Pedagogy

1. $\mathbb{N}$ is presented differently in distinct foundations, such as ZFC or type theory. Our presentation is to be *agnostic* of the foundation. From a working mathematician point of view, it *does not matter*, how the natural numbers are constructed, as long as they obey the properties of the axioms, 1.1.

2. We take the point of view that in mathematics, there are various type of objects. Among all objects studied, some happened to be *sets*. Some presentation of mathematics[a] will regard all objects as sets.

The various types of mathematics are more or less equivalent in our context.

[a] such as ZFC

Why should we delve into the foundations? Two reasons:

1. Foundational language is how many mathematicians do new mathematics. One defines new axioms and explores the possibilities.

2. How can we even discuss mathematics without having a rigorous understanding of our objects?

Discussion

A *natural (counting) number*[a], as we conceived informally is an element of

$$\mathbb{N} := \{0, 1, 2, \ldots\}$$

What is ambiguous about this?

- What does "$\cdots$" mean? How are we sure that the list does not cycle back?

- How does one perform operations?

- What *exactly* is a natural number? What happens if I say

$$\{0, A, AA, AAA, AAAA, \ldots\}$$

  are the numbers?

We will answer these questions over the course.

[a] It does not matter if we regard 0 as a natural number or not. This is a convention.

Forget about the natural numbers we love and know. If one were to define the *numbers*, one might conclude that the numbers are about a concept.

Axioms 1.1. The *Peano Axioms*: [1] Guiseppe Peano, 1858-1932.

1. 0 is a natural number.
$$0 \in \mathbb{N}$$

2. if n is a natural number then we have a natural number, called the *successor of* n, denoted $S(n)$.
$$\forall n \in \mathbb{N}, S(n) \in \mathbb{N}$$

[1] In 1900, Peano met Russell in the mathematical congress. The methods laid the foundation of *Principa Mathematica*

3. 0 is not the successor of any natural number.

$$\forall n \in \mathbb{N}, S(n) \neq 0$$

4. If $S(n) = S(m)$ then $n = m$.

$$\forall n, m \in \mathbb{N}, S(n) = S(m) \Rightarrow n = m$$

5. Principle of induction. Let $P(n)$ be any *property* on the natural number n. Suppose that

   a. $P(0)$ is true.
   b. When ever $P(n)$ is true, so is $P(S(n))$.

   Then $P(n)$ is true for all n natural numbers.

Discussion

What could be meant by a *property?* The principle of induction is in fact an *axiom schema*, consisting of a collection of axioms.

- "n is a prime".
- "$n^2 + 1 = 3$".

We have not yet shown that any collection of object would satisfy the axioms. This will be a topic of later lectures. So we will assume this for know.

Axiom 1.2. There exists a set $\mathbb{N}$, whose elements are the *natural numbers*, for which 1.1 are satisfied.

There can be many such systems, but they are all equivalent for doing mathematics.

Discussion

With only up to axiom 4: This can be *not* so satisfying. What have we done? We said we have a collection of objects that satisfy some concept F="natural numbers". But how do we know, Julius Ceasar does not belong to this concept?

Definition 1.3. We define the following natural numbers:

$$1 := S(0), 2 := S(1) = S(S(0)), 3 := S(2) = S(S(S(0)))$$

$$4 := S(3), 5 := S(4)$$

Intuitively, we want to continue the above process and say that whatever created iteratively by the above process are the *natural numbers*.

Discussion

- Give a set that satisfies axioms 1 and 2 but not 3.
- Give a set that satisfies axioms 1,2 and 3 but not 4.
- Give a set satisfying axioms 1,2,3 and 4, but not 5.

$$\{n/2 : n \in \mathbb{N}\} = \{0, 0.5, 1, 1.5, 2, 2.5, \cdots\}$$

Proposition 1.4. 1 is not 0.

Proof. Use axiom 3. □

Proposition 1.5. 3 is not equal to 0.

Proof. $3 = S(2)$ by definition, 1.3. If $S(2) = 0$, then we have a contradiction with Axiom 2, 1.1. □

1.1 Addition

Definition 1.6. (Left) Addition. Let $m \in \mathbb{N}$.

$$0 + m := m$$

Suppose, by induction, we have defined $n + m$. Then we define

$$S(n) + m := S(n + m)$$

In the context of 1.13, for each n, our function is $f_n := S : \mathbb{N} \to \mathbb{N}$ is $a_{S(n)} := S(a_n)$ with $a_0 = m$.

Proposition 1.7. For $n \in \mathbb{N}$, $n + 0 = n$.

Proof. Warning: we cannot use the definition 1.6. We will use the principle of induction. What is the *property* here in Axiom 5 of 1.1? The property $P(n)$ is "$0 + n = n$" for each $n \in \mathbb{N}$. We will also have to check the two conditions 5a. and 5b.

a "$P(0)$ *is true.*". People refer to this as the "base case $n = 0$": $0 + 0 = 0$, by 1.6.

b "*If $P(m)$ is true then $P(m+1)$ is true*". The statement "*Suppose $P(m)$ is true*" is often called the "inductive hypothesis". Suppose that $m + 0 = m$. We need to show that $P(S(m))$ *is true*, which is

$$S(m) + 0 = S(m)$$

By def, 1.6,

$$S(m) + 0 = S(m + 0)$$

By hypothesis,

$$S(m + 0) = S(m)$$

By the principle of induction, $P(n)$ is true for all $n \in \mathbb{N}$. □

Such proof format is the typical example for writing inductions, although in practice we will often leave out the italicized part.

Example

[4, 1] Prove by induction

$$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

We observed that we have successfully shown *right* addition with respect to 0 behaves as expected.

Discussion

What should we expect $n + S(m)$ to be?

- Why can't we use 1.6?
- Where would we use 1.7?

Proof is hw.

Proposition 1.8. Prove that for $n, m \in \mathbb{N}$, $n + S(m) = S(n + m)$.

Proof. We induct on n. Base case: $m = 0$.

5b. Suppose $n + S(m) = S(n + m)$. We now prove the statement for

$$S(n) + S(m) = S(S(n) + m)$$

by definition of 1.6,

$$S(n) + S(m) = S(n + S(m))$$

which equals to the right hand side by hypothesis.

□

Proposition 1.9. Addiction is commutative. Prive that for all $n, m \in \mathbb{N}$,

$$n + m = m + n$$

Proof. We prove by induction on n. With m fixed. We leave the base case away. □

Proposition 1.10. Associativity of addition. For all $a, b, c \in \mathbb{N}$, we have

$$(a + b) + c = a + (b + c)$$

Proof. hw. □

Discussion

Can we define "+" on any collection of things? What are examples of operations which are noncommutative and associative? For example, the collection of words?

$$+ : \text{Seq. English words} \times \text{Seq. English words} \to \text{Seq. English words}$$

Of course, this can be a meaningless operation. Let us restrict to the collection of *interpreable* outcomes. Explain why the following are *ambiguous*.

- (Ice) (cream latte)
- (British) (Left) (Waffles on Falkland Islands)
- (Local HS Dropouts) (Cut) (in Half)
-

The latter three are actual news title.

What use is there for addition? We can define the notion of *order* on $\mathbb{N}$. We will see later that this is a *relation* on $\mathbb{N}$.

Definition 1.11. Ordering of $\mathbb{N}$. Let $n, m \in \mathbb{N}$. We write $n \geq m$ or $m \leq n$ iff there is $a \in \mathbb{N}$, such that $n = m + a$.

1.2 Multiplication

Now that we have addition, we are ready to define multiplication as 1.6.

Definition 1.12.

$$0 \cdot m := 0$$

$$S(n) \cdot m := (n \cdot m) + m$$

1.3 Recursive definition

What does the induction axiom bring us? Please ignore the following theorem on first read.

Theorem 1.13. Recursion theorem. Suppose we have for each $n \in \mathbb{N}$,

$$f_n : \mathbb{N} \to \mathbb{N}$$

Let $c \in \mathbb{N}$. Then we can assign a natural number a_n for each $n \in \mathbb{N}$ such that

$$a_0 = c \quad a_{S(n)} = f_n(a_n) \forall n \in \mathbb{N}$$

Discussion

The theorem seems intuitively clear, but there can be pitfalls.

- When defining $a_0 = c$, how are we sure this is *not* redefined after some future steps? This is Axiom 3. of 1.1
- When defining $a_{S(n)}$ how are we sure this is not redefined? This uses Axiom 4. of 1.1.
- One rigorous proof is in [1, p48], but requires more set theory.

Proof. The property $P(n)$ of 1.1 is "{ a_n is well-defined}". Start with $a_0 = c$.

- Inductive hypothesis. Suppose we have defined a_n - meaning that there is only one value!
- We can now define $a_{S(n)} := f_n(a_n)$.

□

1.4 References and additional reading

- Nice lecture notes by Robert.
- Russell's book [2, 1,2] for an informal introduction to cardinals.

2 Naïve Set Theory

Week 1, Wednesday, August 30th
As in the construction of $\mathbb{N}$, we will define a *set* via axioms. Why put a foundation of sets?

- The concept of a set can be used - and is till used in practice - as a practical foundation of mathematics.

Learning Objectives

In this lecture:
- We discuss *set* in detail. We will need this to construct the integers, $\mathbb{Z}$.
- We illustrate what one *can* and *can not* do with sets.

Pedagogy

Again, we don't say what they *are*. This approach is often taken, such as [1].

Discussion

What object can be called a *set*?

A *set* should be
- determined by a *description of the objects* [a] For example, we can consider
$$E := \text{"The set of all even numbers"}$$
$$P := \text{"The set of all primes"}$$
- If x is an object and A is a set, then we can ask whether $x \in A$ or $x \notin A$. *Belonging* is a primitive concept in sets.

[a] this set consists of all objects satisfying this description and *only those objects*.

In this lecture we will discuss some axioms.

Axiom 2.1. If A is a *set* then A is also a *object*.

Axiom 2.2. Axiom of extension. Two sets A, B are equal if and only if ($x \in A \Leftrightarrow x \in B$ for all objects x)

Axiom 2.3. There exist a set $\emptyset$ with no elements. I.e. for any object x, $x \notin \emptyset$.

Proposition 2.4 (Single choice)**.** Let A be nonempty. There exists an object x such that $x \in A$.

Proof. Prove by contradiction. Suppose the statement is false. Then for all objects x, $x \notin A$. By axiom of extension, $A = \emptyset$. $\square$

Discussion

How did we use the axiom of extension?

- The logical argument is often referred to as *proof by contradiction.*
- The last use of extension argument is what some mathematicians would say "trivially true".

Can we make sense of subcollection?

Definition 2.5. Let A, B be sets, we say A is a *subset* of B, denoted

$$A \subseteq B$$

if and only if every element of A is also an element of B.

Example

- $\emptyset \subset \{1\}$. The empty set is subset of everything!
- $\{1,2\} \subset \{1,2,3\}$.

2.1 Ordered pairs

Definition 2.6 (Ordered pair)**.** If x, y are objects, we let (x, y) denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1,2\} = \{2,1\}$

In ordered pairs

- $(1,2) \neq (2,1)$

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

2.2 Comprehension axiom

Definition 2.7. Axiom of Comprehension.

Definition 2.8. *General* comprehension principle. (The paradox leading one). For any property φ, one may form the set of all x with property $\varphi(x)$, we denote this set as

$$\{x|\varphi(x)\}$$

Proposition 2.9. Russell, 1901. The general comprehension principle cannot work.

Proof. Let

$$R := \{x : x \text{ is a set and } x \notin x\}$$

This is a set. Then

$$R \in R \Leftrightarrow R \notin R$$

□

Discussion

How is this different from the axiom of specification?

Discussion

How can it even be the case that $x \in x$, for a set? Can this hold for any set x below?

- $\emptyset$
- The set of all primes.
- The set of natural numbers.

The latter two shows that : this set itself is *not even a number*! Indeed, In Zermelo-Frankel set theory foundations it will be proved that $x \notin x$ for all set x. So the set R in 2.9 is the *set of all sets.*

2.3 References

- A nice introduction to set theory is Saltzman's notes [3].
- [5, 3].
- For the axioms of set theory, [1].

3 Homework for week 1

[2]

In these exercises: our goal is to get familiar with

- manipulating axioms in a definition.
- the notion of the principle of induction.

Problems:

1. Prove 5 is not equal to 2.
2. (*) Prove 1.8.
3. (*) Prove 1.9, assuming 1.8 if necessary.
4. (*) Prove 1.10 assuming 1.8, 1.9 if necessary.
5. (*) $n \in \mathbb{N}$ is *positive* if and only if $n \neq 0$. Prove that if $a, b \in \mathbb{N}$, *a is positive,* then $a + b$ is positive.
6. (***) Let M be a set with 2023 elements. Let N be a positive integer, $0 \leq N \leq 2^{2023}$. Prove that it is possible to color each subset of S so that
   (a) The union of two white subsets is white.
   (b) The union of two black subsets is black.
   (c) There are exactly N white subsets.
7. (**) Integers 1 to n are written ordered in a line. We have the following algorithm:
   - If the first number is k then reverse order of the first k numbers.

   Prove that 1 appears first in the line after a finite number of steps.
8. (**) A finite sequence $(a_i)_{i=1}^n := \{a_1, \dots, a_n\}$ of natural numbers is *bounded*, if there exists some other natural number M, such that $a_i \leq M$ for all $1 \leq i \leq n$. Show that every finite sequence of natural numbers, $a_1, \dots, a_n$, is bounded.

[2] *Due: Week 2, Write the numbering of the three questions to be graded clearly on the top of the page. Each unstarred problem worth 12 points. Each star is an extra 5 points.*

References

[1] Paul R. Halmos, *Naive set theory*, 1961.

[2] Bertrand Russell, *Introduction to mathematical philosophy* (2022).

[3] Maththew Saltzman, *A little set theory (never hurt anybody)* (2019).

[4] Michael Spivak, *Calculus, 4th edition*.

[5] Terence Tao, *Analysis I, 4th edition*, 2022.