METAPLECTIC WHITTAKER CATEGORY AND QUANTUM GROUPS : THE "SMALL" FLE

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Contents

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Introduction

0.1. **Towards the FLE.** The present work lies in within the general paradigm of *quantum geometric Langlands theory.* In the same way as the usual (i.e., non-quantum) geometric Langlands theory, arguably, originates from the geometric Satake equivalence, the quantum geometric Langlands theory originates from the FLE, the *fundamental local equivalence.* In this subsection we will review what the FLE says.

0.1.1. The original FLE is a statement within the theory of D-modules, so it takes place over an algebraically closed field k of characteristic zero. Let G be a reductive group and let $\check{G}$ be its Langlands dual, both considered as groups over k. Let Λ denote the coweight lattice of G, and let $\check{\Lambda}$ be the dual lattice, which is by definition the coweight lattice of $\check{G}$.

We fix a *level* for G, which is by definition, a Weyl group-invariant symmetric bilinear form

$$\kappa : \Lambda \otimes \Lambda \to k,$$

We will assume that resulting (symmetric) bilinear form

$$\mathfrak{t} \underset{k}{\otimes} \mathfrak{t} \to k \tag{0.1}$$

is *non-degenerate*, where $\mathfrak{t} := k \underset{\mathbb{Z}}{\otimes} \Lambda$ is the Lie algebra of the Cartan subgroup $T \subset G$. The non-degeneracy of the pairing (0.1) means that the resulting map

$$\mathfrak{t} \to \mathfrak{t}^* =: \check{\mathfrak{t}}$$

is an isomorphism. Consider the inverse map

$$\check{\mathfrak{t}} \to \mathfrak{t} \simeq \check{\mathfrak{t}}^*,$$

which we can interpret as a (symmetric) bilinear form

$$\check{\mathfrak{t}} \underset{k}{\otimes} \check{\mathfrak{t}} \to k, \tag{0.2}$$

or a symmetric bilinear form

$$\check{\kappa} : \check{\Lambda} \otimes \check{\Lambda} \to k.$$

We will refer to $\check{\kappa}$ as the level *dual* dual to κ.

0.1.2. We turn (0.1) and (0.2) into Ad-invariant symmetric bilinear forms

$$(-,-)_\kappa : \mathfrak{g} \otimes \mathfrak{g} \to k \text{ and } (-,-)_{\check{\kappa}} : \check{\mathfrak{g}} \otimes \check{\mathfrak{g}} \to k,$$

respectively, where the correspondence

$$\mathrm{SymBilin}(\mathfrak{g}, k)^G \Leftrightarrow \mathrm{SymBilin}(\mathfrak{t}, k)^W$$

is given by

$$(-,-) \mapsto (-,-)|_{\mathfrak{t}} + \frac{(-,-)_{\mathrm{Kil}}}{2},$$

where $(-,-)_{\mathrm{Kil}}$ is the Killing form, and similarly for $\check{\mathfrak{g}}$.

0.1.3. The form $(-,-)_\kappa$ gives rise to a Kac-Moody extension

$$0 \to k \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}(\!(t)\!) \to 0$$

and to the category of twisted D-modules $\mathrm{D\text{-}mod}_\kappa(\mathrm{Gr}_G)$ on the affine Grassmannian $\mathrm{Gr}_G = G(\!(t)\!)/G[\![t]\!]$ of G.

We consider two categories associated with the above data:

$$\mathrm{KL}_\kappa(G) := \widehat{\mathfrak{g}}_\kappa\text{-mod}^{G[\![t]\!]} \text{ and } \mathrm{Whit}_\kappa(G) := \mathrm{D\text{-}mod}_\kappa(\mathrm{Gr}_G)^{N(\!(t)\!),\chi}. \tag{0.3}$$

Here the superscript $G[\![t]\!]$ stands for $G[\![t]\!]$-equivariance, i.e., $\mathrm{KL}_\kappa(G)$ is modules over the Harish-Chandra pair $(\widehat{\mathfrak{g}}_\kappa, G[\![t]\!])$. The superscript $N(\!(t)\!), \chi$ stands for equivariance for $N(\!(t)\!)$ against the non-degenerate character χ (see Sect. 6.1 for the detailed definition).

We consider the similar categories for the Langlands dual group

$$\mathrm{KL}_{\check{\kappa}}(\check{G}) \text{ and } \mathrm{Whit}_{\check{\kappa}}(\check{G}).$$

0.1.4. We are now ready to state the FLE. It says that we have a canonical equivalence

$$\mathrm{Whit}_\kappa(G) \simeq \mathrm{KL}_{\check{\kappa}}(\check{G}). \tag{0.4}$$

Symmetrically, we are also supposed to have an equivalence

$$\mathrm{Whit}_{\check{\kappa}}(\check{G}) \simeq \mathrm{KL}_\kappa(G).$$

The equivalence (0.4) is still conjectural, and the present work may be regarded as a step towards its proof.

0.1.5. The categories appearing on the two sides of (0.3) are not mere (DG) categories, but they carry extra structure. Namely, the pair $k((t)) \supset k[[t]]$ that appears in the definition of both sides should be thought of as attached to a point x on a curve X, where t is a local parameter at x.

Each of the categories appearing in (0.3) has a structure of *factorization category*, i.e., it can be more generally attached to a finite collection of points $\underline{x} \subset X$, i.e., a point of the Ran space of X (see Sect. 1.1),

$$\underline{x} \rightsquigarrow \mathcal{C}_{\underline{x}}, \tag{0.5}$$

and we have a system of isomorphisms

$$\mathcal{C}_{\underline{x}_1 \sqcup \underline{x}_2} \simeq \mathcal{C}_{\underline{x}_1} \otimes \mathcal{C}_{\underline{x}_2}, \tag{0.6}$$

whenever $\underline{x}_1$ and $\underline{x}_2$ are *disjoint*.

The additional structure involved in (0.4) is that it is supposed to be an equivalence of factorization categories.

0.2. **Quantum groups perspective.** There is a third (and eventually there will be also a fourth) player in the equivalence (0.4). This third player is the category

$$\mathrm{Rep}_q(\check{G}),$$

which is the category of representations of the "big" (i.e., Lusztig's quantum group). We will now explain how it fits into the picture.

0.2.1. The category $\mathrm{Rep}_q(\check{G})$ is of algebraic nature and can be defined over an arbitrary field of coefficients e (say, also assumed algebraically closed and of characterstic zero). The structure that $\mathrm{Rep}_q(\check{G})$ possesses is that if a *braided monoidal category.*

Here the quantum parameter q is a quadratic form on Λ (which is the weight lattice for $\check{G}$) with values in $\mathsf{e}^\times$.

0.2.2. Assume now that $\mathsf{e} = \mathbb{C} = k$. Take the curve X to be $\mathbb{A}^1$ and $x = 0 \in \mathbb{A}^1$. In this case, for a given DG category $\mathcal{C}$, a braided monoidal structure on it (under appropriate finiteness conditions), via Riemann-Hilbert correspondence gives rise to a factorization category (see Sect. 0.1.5 above) with $\mathcal{C} = \mathcal{C}_{\{0\}}$.

0.2.3. Now, the equivalence established in the series of papers [KL] can be formulated as saying that the factorization category corresponding by the above procedure to $\mathrm{Rep}_q(\check{G})$ identifies with $\mathrm{KL}_{\check{\kappa}}(\check{G})$ for the quantum parameter

$$q = \exp(2\pi i q_\kappa),$$

where q_κ is the quadratic form $\Lambda \to k$ such that the associated bilinear form is κ, i.e.,

$$q_\kappa(\lambda) = \frac{\kappa(\lambda, \lambda)}{2}.$$

In particular, we have an equivalence as plain DG categories

$$\mathrm{KL}_{\check{\kappa}}(\check{G}) \simeq \mathrm{Rep}_q(\check{G}). \tag{0.7}$$

Remark 0.2.4. Note that the equivalence (0.7) does *not* involve Langlands duality: on both sides we are dealing with the same reductive group, in this case $\check{G}$.

0.2.5. Thus, combining, for $\mathsf{e} = \mathbb{C} = k$ we are supposed to have the equivalences

$$\mathrm{Whit}_\kappa(G) \simeq \mathrm{KL}_{\check{\kappa}}(\check{G}) \simeq \mathrm{Rep}_q(\check{G}). \tag{0.8}$$

What we prove in this work is a result that goes a long way towards the composite equivalence

$$\mathrm{Whit}_\kappa(G) \simeq \mathrm{Rep}_q(\check{G}). \tag{0.9}$$

The precise statement of what we actually prove will be explained in the rest of this Introduction. Here let us note the following two of its features:

- The right-hand side of *our* equivalence will not be $\mathrm{Rep}_q(\check{G})$, but rather $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod, the category of modules over the *small* quantum group.
- Our equivalence will be geometric (or motivic) in nature in that it will not be tied to the situation of $\mathsf{e} = \mathbb{C} = k$ and neither will it rely on Riemann-Hilbert. Rather, it applies over any ground field and for an arbitrary sheaf theory (see Sect. 0.8.8).

Let us also add that the equivalence (0.9) had been conjectured by J. Lurie and the first-named author around 2007; it stands at the origin of the FLE.

0.2.6. One could of course try to prove the FLE (0.4) by combining the Kazhdan-Lusztig equivalence (0.7) with the (yet to be established) equivalence (0.9). But this is not how we plan to proceed about proving the FLE. Nor do quantum groups appear in the statement of the main theorem of the present work, Theorem 19.2.5.

So in a sense, the equivalences with the category of modules over the quantum group is just an add-on to the FLE. Yet, it is a very useful add-on: quantum groups are “much more finite-dimensional”, than the other objects involved, so perceiving things from the perspective of quantum groups is convenient in that it really explains “what is going on”. We will resort to this perspective on multiple occasions in this Introduction (notably, in Sect. 0.6) as a guiding principle to some of the key constructions.

0.3. **Let’s prove the FLE: the Jacquet functor.** We will now outline a strategy towards the proof of the FLE. This strategy will not quite work (rather one needs to do more work in order to make it work, and this will be done in a future publication). Yet, this strategy will explain the context for what we will do in the main body of the text.

0.3.1. The inherent difficulty in establishing the equivalence (0.4) is that it involves Langlands duality: each side is a category extracted from the geometry of the corresponding reductive group (i.e., G and $\check{G}$, respectively), while the relationship between these two groups is purely combinatorial (duality on the root data).

So, a reasonable idea to prove (0.4) would be to describe both sides in *combinatorial terms*, i.e., in terms that only involve the root data and the parameter κ, with the hope that the resulting descriptions will match up.

The process of expressing a category associated with the group G in terms that involve just the root data is something quite familiar in representation theory: if $\mathcal{C}_G$ is a category attached to G and $\mathcal{C}_T$ is a similar category attached to the Cartan subgroup T, one seeks to construct a *Jacquet functor*

$$\mathfrak{J} : \mathcal{C}_G \to \mathcal{C}_T,$$

and express $\mathcal{C}_G$ as objects of $\mathcal{C}_T$ equipped with some additional structure.

For example, this is essentially how one proves the geometric Satake equivalence: one constructs the corresponding Jacquet functor

$$\mathrm{Perv}(\mathrm{Gr}_G)^{G[\![t]\!]} \to \mathrm{Perv}(\mathrm{Gr}_T)^{T[\![t]\!]} \simeq \mathrm{Rep}(\check{T})$$

by pull-push along the diagram

$$\begin{tikzcd} \mathrm{Gr}_B \arrow[r] \arrow[d] & \mathrm{Gr}_G \\ \mathrm{Gr}_T. & \end{tikzcd}$$

0.3.2. Let us try to do the same for the two sides of the FLE. First, we notice that for $G = T$, the two categories, i.e.,

$$\mathrm{Whit}_\kappa(T) \simeq \text{D-mod}_\kappa(\mathrm{Gr}_T) \text{ and } \mathrm{KL}_{\check{\kappa}}(\check{T}) \tag{0.10}$$

are indeed equivalent *as factorization categories* in a more or less tautological way.

We now have to figure out what functors to use

$$\mathfrak{J}^{\mathrm{Whit}} : \mathrm{Whit}_\kappa(G) \to \mathrm{Whit}_\kappa(T) \text{ and } \mathfrak{J}^{\mathrm{KL}} : \mathrm{KL}_{\check{\kappa}}(\check{G}) \to \mathrm{KL}_{\check{\kappa}}(\check{T}),$$

so that we will have a chance to express the G-categories in terms of the corresponding T-categories.

0.3.3. Here the quantum group perspective will be instrumental. The functor

$$\mathfrak{J}^{\mathrm{Quant}} : \mathrm{Rep}_q(\check{G}) \to \mathrm{Rep}_q(\check{T})$$

that we want to use, to be denoted $\mathfrak{J}^{\mathrm{Quant}}_!$, is given by

$$M \mapsto \mathrm{C}^{\cdot}(U_q^{\mathrm{Lus}}(\check{N}), M),$$

where $U_q^{\mathrm{Lus}}(\check{N})$ is the positive (i.e., "upper triangular") part of the big (i.e., Lusztig's) quantum group.

The next step is to construct the functors

$$\mathfrak{J}^{\mathrm{Whit}}_! : \mathrm{Whit}_\kappa(G) \to \mathrm{Whit}_\kappa(T) \text{ and } \mathfrak{J}^{\mathrm{KL}}_! : \mathrm{KL}_{\check{\kappa}}(\check{G}) \to \mathrm{KL}_{\check{\kappa}}(\check{T})$$

that under the equivalences (0.8) are supposed to correspond to $\mathfrak{J}^{\mathrm{Quant}}_!$.

0.3.4. The functor

$$\mathfrak{J}^{\mathrm{KL}}_! : \mathrm{KL}_{\check{\kappa}}(\check{G}) \to \mathrm{KL}_{\check{\kappa}}(\check{T})$$

is a version of the functor of semi-infinite cohomology with respect to $\check{\mathfrak{n}}((t))$. More precisely, it is literally that when $\check{\kappa}$ is *positive* or *irrational* and a certain non-trivial modification when $\check{\kappa}$ is *negative rational*.

The precise construction of $\mathfrak{J}^{\mathrm{KL}}_!$ will be given in a subsequent publication, which will deal with the "Kazhdan-Lusztig vs factorization modules" counterpart of our main theorem (the latter deals with "Whittaker vs factorization modules" equivalence).

0.3.5. Let us describe the sought-for functor

$$\mathfrak{J}^{\mathrm{Whit}}_! : \mathrm{Whit}_\kappa(G) \to \mathrm{Whit}_\kappa(T).$$

We will first make a naive attempt. Namely, let

$$\mathfrak{J}^{\mathrm{Whit}}_* : \mathrm{Whit}_\kappa(G) \to \mathrm{Whit}_\kappa(T)$$

be the functor defined by !-pull and *-push along the diagram

$$\begin{tikzcd} \mathrm{Gr}_{B^-} \arrow[r] \arrow[d] & \mathrm{Gr}_G \\ \mathrm{Gr}_T. \end{tikzcd} \tag{0.11}$$

This is a meaningful functor, but it does not produce what we need. Namely, one can show that with respect to the equivalence (0.9), the functor $\mathfrak{J}^{\mathrm{Whit}}_*$ corresponds to the functor

$$\mathfrak{J}^{\mathrm{Quant}}_* : \mathrm{Rep}_q(\check{G}) \to \mathrm{Rep}_q(\check{T}), \quad M \mapsto \mathrm{C}^{\cdot}(U_q^{\mathrm{DK}}(\check{N}), M),$$

where $U_q^{\mathrm{DK}}(\check{N})$ is the De Concini-Kac version of $U_q(\check{N})$ (we note that when κ is irrational, i.e., when q takes values in non-roots of unity, the two versions do coincide).

0.3.6. To obtain the desired functor $\mathfrak{J}_!^{\mathrm{Whit}}$ we proceed as follows, Note that

$$\mathrm{Whit}_\kappa(T) \simeq \text{D-mod}_\kappa(\mathrm{Gr}_T),$$

and as a plain DG category, it is equivalent to the category of Λ-graded vector spaces. For a given $\lambda \in \Lambda$, let us describe the λ-component $(\mathfrak{J}_!^{\mathrm{Whit}})^\lambda$ of $\mathfrak{J}_!^{\mathrm{Whit}}$.

Going back to $\mathfrak{J}_*^{\mathrm{Whit}}$, its λ-component $(\mathfrak{J}_*^{\mathrm{Whit}})^\lambda$ is given by

$$\mathcal{F} \mapsto \mathrm{C}^\cdot(\mathrm{Gr}_G, \mathcal{F} \overset{!}{\otimes} (\mathbf{i}_\lambda^-)_*(\omega_{S^{-,\lambda}})),$$

where

$$N^-((t)) \cdot t^\lambda =: S^{-,\lambda} \overset{\mathbf{i}_\lambda^-}{\hookrightarrow} \mathrm{Gr}_G.$$

Now, $(\mathfrak{J}_!^{\mathrm{Whit}})^\lambda$ is given by

$$\mathcal{F} \mapsto \mathrm{C}^\cdot(\mathrm{Gr}_G, \mathcal{F} \overset{!}{\otimes} (\mathbf{i}_\lambda^-)_!(\omega_{S^{-,\lambda}})).$$

So, the difference between $(\mathfrak{J}_!^{\mathrm{Whit}})^\lambda$ and $(\mathfrak{J}_*^{\mathrm{Whit}})^\lambda$ is that we take $(\mathbf{i}_\lambda^-)_!(\omega_{S^{-,\lambda}})$ instead of $(\mathbf{i}_\lambda^-)_*(\omega_{S^{-,\lambda}})$.

Remark 0.3.7. Let us remark that the functor that appears in the main body of this work is yet a different one, to be denoted $\mathfrak{J}_{!*}^{\mathrm{Whit}}$. Its λ-component $(\mathfrak{J}_{!*}^{\mathrm{Whit}})^\lambda$ is given by

$$\mathcal{F} \mapsto \mathrm{C}^\cdot(\mathrm{Gr}_G, \mathcal{F} \overset{!}{\otimes} \mathrm{IC}^{\lambda+\frac{\infty}{2},-}),$$

where $\mathrm{IC}^{\lambda+\frac{\infty}{2},-}$ is a certain twisted D-module defined in Sect. 13.

Under the equivalence (0.9), the functor $\mathfrak{J}_{!*}^{\mathrm{Whit}}$ corresponds to the functor

$$\mathfrak{J}_{!*}^{\mathrm{Quant}} : \mathrm{Rep}_q(\check{G}) \to \mathrm{Rep}_q(\check{T}), \quad M \mapsto \mathrm{C}^\cdot(\mathfrak{u}_q(\check{N}), M), \tag{0.12}$$

where $\mathfrak{u}_q(\check{N})$ is the positive part of the small quantum group.

0.3.8. Having described the functor $\mathfrak{J}_!^{\mathrm{Whit}}$, let us address the question of whether we can use it in order to express the category $\mathrm{Whit}_\kappa(G)$ in terms of $\mathrm{Whit}_\kappa(T)$. To do so, we will again resort to the quantum group picture.

The functor $\mathfrak{J}_!^{\mathrm{Quant}}$ has an additional structure: it is *lax braided monoidal.* As such, it sends the (monoidal) unit object $\mathsf{e} \in \mathrm{Rep}_q(\check{G})$ to the object that we will denote

$$\Omega_q^{\mathrm{Lus}} \in \mathrm{Rep}_q(\check{T}),$$

which has a structure of $\mathbb{E}_2$-algebra. Moreover, the functor $\mathfrak{J}_!^{\mathrm{Quant}}$ can be upgraded to a functor

$$(\mathfrak{J}_!^{\mathrm{Quant}})^{\mathrm{enh}} : \mathrm{Rep}_q(\check{G}) \to \Omega_q^{\mathrm{Lus}}\text{-mod}_{\mathbb{E}_2}(\mathrm{Rep}_q(\check{T})). \tag{0.13}$$

Similarly, the functors $\mathfrak{J}_!^{\mathrm{Whit}}$ and $\mathfrak{J}_!^{\mathrm{KL}}$ have factorization structures. By applying to the unit, we obtain *factorization algebras*

$$\Omega_\kappa^{\mathrm{Whit,Lus}} \text{ and } \Omega_{\check{\kappa}}^{\mathrm{KL,Lus}} \tag{0.14}$$

in $\mathrm{Whit}_\kappa(T)$ and $\mathrm{KL}_{\check{\kappa}}(\check{T})$, respectively, and the upgrades

$$(\mathfrak{J}_!^{\mathrm{Whit}})^{\mathrm{enh}} : \mathrm{Whit}_\kappa(G) \to \Omega_\kappa^{\mathrm{Whit,Lus}}\text{-FactMod}(\mathrm{Whit}_\kappa(T))_{\mathrm{untl}} \tag{0.15}$$

and

$$(\mathfrak{J}_!^{\mathrm{KL}})^{\mathrm{enh}} : \mathrm{KL}_{\check{\kappa}}(\check{G}) \to \Omega_{\check{\kappa}}^{\mathrm{KL,Lus}}\text{-FactMod}(\mathrm{KL}_{\check{\kappa}}(\check{T}))_{\mathrm{untl}}, \tag{0.16}$$

respectively, where the subscript untl stands for “unital modules”.

One expects the functors (0.15) and (0.16) to be equivalences *if and only if* (0.13) is. However, (0.13) is *not* an equivalence, for the reasons explained in Sect. 0.3.10 below.

0.3.9. In a different world, if the functors (0.15) and (0.16) *were* equivalences, one would complete the proof of (0.4) by proving that under the factorization algebras (0.14) correspond to one another under the equivalence

$$\mathrm{Whit}_\kappa(T) \simeq \mathrm{KL}_{\check{\kappa}}(\check{T})$$

of (0.10).

However, not all is lost in the real world either. In a subsequent publication, we will introduce a device that allows to express the LHS of (0.15) in terms of the RHS (and similarly, for (0.16)) and thereby prove the FLE.

In the case of (0.16), the same device should be able to give a different proof of the Kazhdan-Lusztig equivalence (0.7).

0.3.10. As was mentioned above, the functor (0.13) is *not* an equivalence. Rather, we have an equivalence

$$\mathrm{Rep}_q^{\mathrm{mxd}}(\check{G}) \simeq \Omega_q^{\mathrm{Lus}}\text{-mod}_{\mathbb{E}_2}(\mathrm{Rep}_q(\check{T})), \tag{0.17}$$

where $\mathrm{Rep}_q^{\mathrm{mxd}}(\check{G})$ is a category of modules over the "mixed" quantum group, introduced in [Ga8].

The original functor (0.13) is the composition of (0.17) with the restriction functor

$$\mathrm{Rep}_q^{\mathrm{mxd}}(\check{G}) \to \mathrm{Rep}_q(\check{G}). \tag{0.18}$$

Remark 0.3.11. Recall following [Ga8] that the mixed version $\mathrm{Rep}_q^{\mathrm{mxd}}(\check{G})$ has Lustig's algebra $U_q^{\mathrm{Lus}}(\check{N})$ for the positive part and the De Concini-Kac algebra $U_q^{\mathrm{DK}}(\check{N}^-)$ for the negative part. However, even when q takes in non-roots of unity, the functor (0.18) is not an equivalence (and not even fully faithful unless we restrict to abelian categories).

Namely, in the non-root of unity case, the category $\mathrm{Rep}_q(\check{G})$ consists of modules that are locally finite for the action of the entire quantum group, whereas $\mathrm{Rep}_q^{\mathrm{mxd}}(\check{G})$ is the quantum category $\mathcal{O}$, i.e., we only impose local finiteness for the positive part.

0.3.12. As we just saw, although the functor (0.13) is not an equivalence, it is the composition of a (rather explicit) forgetful functor with an equivalence from another meaningful representation-theoretic category. We have a similar situation for the functors (0.15) and (0.16).

Namely, the functor (0.16) is the composition of the forgetful functor

$$\mathrm{KL}_\kappa(\check{G}) := \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{G}[\![t]\!]} \to \widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{I}},$$

and a functor

$$\widehat{\check{\mathfrak{g}}}_{\check{\kappa}}\text{-mod}^{\check{I}} \to \Omega_{\check{\kappa}}^{\mathrm{KL,Lus}}\text{-FactMod}(\mathrm{KL}_{\check{\kappa}}(\check{T}))_{\mathrm{untl}},$$

which is conjectured to be an equivalence. In the above formula $\check{I} \subset \check{G}[\![t]\!]$ is the Iwahori subgroup. The latter conjectural equivalence is essentially equivalent to the main conjecture of [Ga8], see Conjecture 9.2.2 in *loc.cit.*

Similarly, the functor (0.15) is the composition of the pullback functor

$$\mathrm{Whit}_\kappa(G) := \text{D-mod}_\kappa(\mathrm{Gr}_G)^{N(\!(t)\!),\chi} \to \text{D-mod}_\kappa(\mathrm{Fl}_G)^{N(\!(t)\!),\chi}$$

and a functor

$$\text{D-mod}_\kappa(\mathrm{Fl}_G)^{N(\!(t)\!),\chi} \to \Omega_\kappa^{\mathrm{Whit,Lus}}\text{-FactMod}(\mathrm{Whit}_\kappa(T))_{\mathrm{untl}},$$

which is conjectured to be an equivalence. In the above formula $\mathrm{Fl}_G = G(\!(t)\!)/I$ is the affine flag space. The latter conjectural equivalence should be provable by methods close to those developed in the present work.

0.4. **The present work: the "small" FLE.** Having explained the idea of the proof of the FLE via Jacquet functors, we will now modify the source categories, and explain what it is that we actually prove in this work.

0.4.1. Just as above, we will first place ourselves in the context of quantum groups. From now on we will assume that q takes values in the group of roots of unity.

In this case, following Lusztig, to the datum of $(\check{G}, q)$ one attaches another reductive group H, see Sect. 2.3.6 for the detailed definition. Here we will just say that weight lattice of H, denoted Λ_H is a sublattice of Λ and consists of the kernel of the symmetric bilinear form b associated to q

$$b(\lambda_1, \lambda_2) = q(\lambda_1 + \lambda_2) - q(\lambda_1) - q(\lambda_2),$$

where we use the additive notation for the abelian group $\mathsf{e}^\times$.

In addition, following Lusztig, we have the *quantum Frobenius* map, which we will interpret as a monoidal functor

$$\mathrm{Frob}_q^* : \mathrm{Rep}(H) \to \mathrm{Rep}_q(\check{G}). \tag{0.19}$$

We will use Frob_q^* to regard $\mathrm{Rep}(H)$ as a monoidal category acting on $\mathrm{Rep}_q(\check{G})$,

$$V, M \mapsto \mathrm{Frob}_q^*(V) \otimes M.$$

Given this data, we can consider the *graded Hecke category* of $\mathrm{Rep}_q(\check{G})$ with respect to $\mathrm{Rep}(H)$,

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Rep}_q(\check{G})) := \mathrm{Rep}(T_H) \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}_q(\check{G}).$$

See Sect. 10.3 for the discussion of the formalism of the formation of Hecke categories.

Recall the functor $\mathfrak{J}_{!*}^{\mathrm{Quant}}$ of (0.12). It has the following structure: it intertwines the actions of $\mathrm{Rep}(H)$ on $\mathrm{Rep}_q(\check{G})$ and of $\mathrm{Rep}(T_H)$ on $\mathrm{Rep}_q(\check{T})$ via the restriction functor $\mathrm{Rep}(H) \to \mathrm{Rep}(T_H)$, where T_H is the Cartan subgroup of H, and the latter action is given by the quantum Frobenius for T.

This formally implies that $\mathfrak{J}_{!*}^{\mathrm{Quant}}$ gives rise to a functor

$$(\mathfrak{J}_{!*}^{\mathrm{Quant}})^{\overset{\bullet}{\mathrm{Hecke}}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Rep}_q(\check{G})) \to \mathrm{Rep}_q(\check{T}), \tag{0.20}$$

and further to a functor

$$(\mathfrak{J}_{!*}^{\mathrm{Quant}})^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{enh}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Rep}_q(\check{G})) \to \Omega_q^{\mathrm{small}}\text{-mod}_{\mathbb{E}_2}(\mathrm{Rep}_q(\check{T})), \tag{0.21}$$

where $\Omega_q^{\mathrm{small}}$ is the $\mathbb{E}_2$-algebra in $\mathrm{Rep}_q(\check{T})$ obtained by applying $\mathfrak{J}_{!*}^{\mathrm{Quant}}$ to the unit.

A key observation is that the functor (0.21) is an equivalence (up to renormalization, which we will ignore in this Introduction). We will explain the mechanism of why this equivalence takes place in a short while (see Sect. 0.4.6 below).

0.4.2. We will can perform the same procedure for Whit and KL. In the case of the latter, metaplectic geometric Satake (see Sect. 2.4) defines an action of $\mathrm{Rep}(H)$ on $\mathrm{KL}_{\check{\kappa}}(\check{G})$. This action matches the action of $\mathrm{Rep}(H)$ on $\mathrm{Rep}_q(\check{G})$ via the Kazhdan-Lusztig equivalence.

Furthermore, one can define a functor

$$\mathfrak{J}_{!*}^{\mathrm{KL}} : \mathrm{KL}_{\check{\kappa}}(\check{G}) \to \mathrm{KL}_{\check{\kappa}}(\check{T})$$

with properties mirroring those of $\mathfrak{J}_{!*}^{\mathrm{Quant}}$. In particular, we obtain:

$$(\mathfrak{J}_{!*}^{\mathrm{KL}})^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{enh}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{KL}_{\check{\kappa}}(\check{G})) \to \Omega_{\check{\kappa}}^{\mathrm{KL,small}}\text{-FactMod}(\mathrm{KL}_{\check{\kappa}}(\check{T}))_{\mathrm{untl}}. \tag{0.22}$$

If we use the Kazhdan-Lusztig equivalence (0.7), it would follow formally that the functor (0.22) is an equivalence.

Alternatively, we expect it to be possible to prove that (0.22) is an equivalence directly. Juxtaposing this with the equivalence (0.21), with some extra work, this would provide an alternative construction of the Kazhdan-Lusztig equivalence (0.7).

0.4.3. We now come to the key point of the present work. Using metaplectic geometric Satake we define an action of the same category $\mathrm{Rep}(H)$ also on $\mathrm{Whit}_\kappa(G)$. Hence, we can form the category $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_\kappa(G))$.

Furthermore, in Part V of this work, we construct a functor

$$\mathfrak{J}_{!*}^{\mathrm{Whit}} : \mathrm{Whit}_\kappa(G) \to \mathrm{Whit}_\kappa(T), \tag{0.23}$$

and we show that it also intertwines the $\mathrm{Rep}(H)$ and $\mathrm{Rep}(T_H)$-actions. From here we obtain the functor

$$(\mathfrak{J}_{!*}^{\mathrm{Whit}})^{\overset{\bullet}{\mathrm{Hecke}}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_\kappa(G)) \to \mathrm{Whit}_\kappa(T),$$

and further a functor

$$(\mathfrak{J}_{!*}^{\mathrm{Whit}})^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{enh}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_\kappa(G)) \to \Omega_\kappa^{\mathrm{Whit,small}}\text{-}\mathrm{FactMod}(\mathrm{Whit}_\kappa(T))_{\mathrm{untl}}. \tag{0.24}$$

0.4.4. The main result of this work, Theorem 19.2.5 says that the functor (0.24) is an equivalence.

Furthermore, another of our key results, Theorem 18.4.2, essentially says that under the equivalence

$$\mathrm{Whit}_\kappa(T) \simeq \mathrm{KL}_{\check{\kappa}}(\check{T}),$$

the factorization algebras $\Omega_\kappa^{\mathrm{Whit,small}}$ and $\Omega_{\check{\kappa}}^{\mathrm{KL,small}}$ correspond to one another.

Hence, taking into account the equivalence (0.22), we obtain that our Theorem 19.2.5, combined with Theorem 18.4.2, imply the following equivalence

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_\kappa(G)) \simeq \overset{\bullet}{\mathrm{Hecke}}(\mathrm{KL}_{\check{\kappa}}(\check{G})) \tag{0.25}$$

We refer to the equivalence (0.25) as the "small FLE", for reasons that will be explained in Sect. 0.4.6 right below.

We remark that, on the one hand, (0.25) is a consequence of (0.4). On the other hand, it should not be a far stretch to get the original (0.4) from (0.25), which we plan to do in the future.

0.4.5. Here is what we *do not* prove: the two sides in Theorem 19.2.5 are naturally factorization categories, and one wants the equivalence to preserve this structure.

However, in main body of the text we do not give the definition of factorization categories, so we do not formulate this extension of Theorem 19.2.5. That said, such an extension (in our particular situation) is rather straightforward:

In general, if a functor between factorization categories is an equivalence at a point, this *does not at all* guarantee that it is an equivalence *as factorization categories*. What makes it possible to prove this in our situation is the fact that we can control how our functor (i.e., $(\mathfrak{J}_{!*}^{\mathrm{Whit}})^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{enh}}$) commutes with Verdier duality, see Theorem 18.2.9.

0.4.6. Let us now explain the origin of the terminology "small".

A basic fact established in [ArG] is that we have an equivalence of categories

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Rep}_q(\check{G})) \simeq \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}, \tag{0.26}$$

where the RHS is the (graded version of the) category of representations of the small quantum group.

With respect to this equivalence, the functor $(\mathfrak{J}_{!*}^{\mathrm{Quant}})^{\overset{\bullet}{\mathrm{Hecke}}}$ of (0.20) is simply the functor

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \mathrm{Rep}_q(\mathfrak{T}), \quad M \mapsto \mathrm{C}^\cdot(\mathfrak{u}_q(\check{N}), M).$$

Now, the resulting functor

$$(\mathfrak{J}_{!*}^{\mathrm{Quant}})^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{enh}} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \Omega_q^{\mathrm{small}}\text{-mod}_{\mathbb{E}_2}(\mathrm{Rep}_q(\check{T})) \tag{0.27}$$

is indeed an equivalence for the following reason:

The category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod is (the relative with respect to $\mathrm{Rep}_q(\check{T})$) Drinfeld center of the monoidal category $\mathfrak{u}_q(\check{N})$-mod$(\mathrm{Rep}_q(\check{T}))$, see Sect. 27.2) for what this means. Then the equivalence (0.27) is a combination of the following two statements (see Sect. 29.3 for details):

- A Koszul duality equivalence (up to renormalization) between $\mathfrak{u}_q(\check{N})$-mod$(\mathrm{Rep}_q(\check{T}))$ and $\Omega_q^{\mathrm{small}}$-mod;
- A general fact that for an $\mathbb{E}_2$-algebra A, the Drinefeld center of the monoidal category A-mod identifies with $A\text{-mod}_{\mathbb{E}_2}$.

Remark 0.4.7. At the heart of the equivalence (0.27) was the fact that $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$ is the Drinfeld double of $\mathfrak{u}_q(\check{N})$ relative to $\mathrm{Rep}_q(\check{T})$. This boils down to the fact that $\mathfrak{u}_q(\check{N}^-)$ is the component-wise dual of $\mathfrak{u}_q(\check{N})$.

A similar fact is responsible for the equivalence (0.17). There we have that $U_q^{\mathrm{DK}}(\check{N}^-)$ is the component-wise due of $U_q^{\mathrm{Lus}}(\check{N})$.

0.5. **What do we actually prove?** As was explained above, the goal of this work is to prove the equivalence

$$(\mathfrak{J}^{\mathrm{Whit}}_{!*})^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{enh}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_\kappa(G)) \to \Omega_\kappa^{\mathrm{Whit,small}}\text{-FactMod}(\mathrm{Whit}_\kappa(T))_{\mathrm{untl}} \tag{0.28}$$

of (0.24). However, at a glance, our Theorem 19.2.5 looks a little different. We will now explain the precise statement of what we actually prove.

0.5.1. Perhaps, the main difference is that we actually work in greater generality than the one explained above. Namely, the story of FLE inherently lives in the world of D-modules (in particular, it was tied to the assumption that our ground field k and the field of coefficients e are the same); this is needed in order for categories such as $\mathrm{KL}_\kappa(\check{G})$ to make sense.

However, the assertion that (0.28) is an equivalence is entirely geometric. I.e., it can be formulated within *any* sheaf theory from the list in Sect. 0.8.8. For example, we can work with ℓ-adic sheaves over an arbitrary ground field k (so in this case $\mathsf{e} = \overline{\mathbb{Q}}_\ell$). This is the context in which we prove our Theorem 19.2.5.

0.5.2. That said, we need to explain what replaces the role of the parameter κ.

In the context of a general sheaf theory, we can no longer talk about *twistings* (those only make sense for D-modules). However, we can talk about *gerbes* with respect to to the multiplicative group $\mathsf{e}^\times$ of the field e of coefficients. The assumption that κ was rational gets replaced by the assumption that we take gerbes with respect to the group $\mathsf{e}^{\times,\mathrm{tors}}$ of torsion elements in $\mathsf{e}^\times$, i.e., the group of roots of unity.

Now, instead of the datum of κ, our input is a *geometric metaplectic data* $\mathcal{G}^G$, which is a *factorization gerbe* on the factorization version of the affine Grassmannian. We refer the reader to [GLys], where this theory is developed.

Returning to the context of D-modules, one shows that a datum of κ gives rise to a geometric metaplectic data for the sheaf theory of D-modules.

0.5.3. The second point of difference between (0.28) and the statement of Theorem 19.2.5 involves the fact that the former talks about factorization modules for a factorization algebra on Gr_T, and the latter talks about a factorization algebra on the configuration space instead.

However, this difference is immaterial (we chose the configuration space formulation because of its finite-dimensional nature and the proximity to the language of [BFS]). Indeed, as explained in Sect. 5.5, factorization algebras on Gr_T satisfying a certain support condition (specifically, if they are supported on the connected components of Gr_T corresponding to elements of $(\Lambda^{\mathrm{neg}} - 0) \subset \Lambda)$ can be equivalently though of as factorization algebras on the configuration space.

The factorization algebra on Gr_T that we deal with is the *augmentation ideal* in $\Omega_\kappa^{\mathrm{Whit,small}}$, so factorization modules for it are the same as *unital* factorization modules $\Omega_\kappa^{\mathrm{Whit,small}}$.

0.5.4. The final point of difference is that in order for the equivalence to hold, we need to apply a certain renormalization procedure to the category of factorization modules

$$\Omega_q^{\rm small}\text{-FactMod} \rightsquigarrow \Omega_q^{\rm small}\text{-FactMod}^{\rm ren}.$$

Specifically, we will have enlarge the class of objects that we declare as compact, see Sect. 19.1.

This renormalization procedure follows the pattern of how one obtains IndCoh from QCoh: it does not affect the heart (of the naturally defined t-structure); we have a functor

$$\Omega_q^{\rm small}\text{-FactMod}^{\rm ren} \to \Omega_q^{\rm small}\text{-FactMod}$$

which is t-exact and induces an equivalence on the eventually coconnective parts. So the two categories differ only at the cohomological $-\infty$.

0.5.5. In Part IX of this work we prove that in the Betti situation (i.e., for the ground field being $\mathbb{C}$ and for the sheaf theory being constructible sheaves in classical topology with e-coefficients for any e), we have a canonical equivalence

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\rm baby\text{-}ren} \simeq \Omega_q^{\rm small}\text{-FactMod} \tag{0.29}$$

and

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\rm ren} \simeq \Omega_q^{\rm small}\text{-FactMod}^{\rm ren}, \tag{0.30}$$

where

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\rm baby\text{-}ren} \text{ and } \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\rm ren}$$

are two renormalizations of the (usual version of the) category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod.

The equivalence (0.29) had been previously established in (and was the subject of) the book [BFS]. The proof we give is just a remake in a modern language, where the key new tool we use is Lurie's equivalence between $\mathbb{E}_2$ algebras and factorization algebras. The proof is an expansion of the contents of Sect. 0.4.6 above.

0.5.6. Thus, for $k = \mathbb{C} = \mathsf{e}$, and using Riemann-Hilbert correspondence, we obtain the equivalences

$$\overset{\bullet}{\text{Hecke}}(\text{Whit}_\kappa(G)) \simeq \Omega_q^{\rm small}\text{-FactMod}^{\rm ren} \simeq \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\rm ren}. \tag{0.31}$$

The composite equivalence

$$\overset{\bullet}{\text{Hecke}}(\text{Whit}_\kappa(G)) \simeq \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\rm ren} \tag{0.32}$$

is a "small" version of the equivalence (0.9).

It is the equivalence (0.32) that is responsible for the title of this work.

0.6. **How do we prove it?** Modulo technical nuances that have to do with renormalization, our main result says that a functor denoted

$$\Phi_{\rm Fact}^{\overset{\bullet}{\rm Hecke}} : \overset{\bullet}{\text{Hecke}}(\text{Whit}_q(G)) \to \Omega_q^{\rm small}\text{-FactMod}, \tag{0.33}$$

which is the configuration space version of the functor (0.28), is an equivalence.

Let us explain the main ideas involved in the proof.

0.6.1. Consider the following general paradigm: let

$$\Phi : \mathcal{C}_1 \to \mathcal{C}_2$$

be a functor between (unital) factorization categories. It induces a functor

$$\Phi^{\mathrm{enh}} : \mathcal{C}_1 \to \Omega\text{-ModFact}(\mathcal{C}_2)_{\mathrm{untl}},$$

and we wish to know when the latter is an equivalence.

Such an equivalence is not something we can generally expect. For example, it does hold for Φ being the functor

$$\mathfrak{J}^{\mathrm{Quant}}_{!*} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \mathrm{Rep}_q(\check{T}),$$

but it fails for Φ being the functor

$$\mathfrak{J}^{\mathrm{Quant}}_{!} : \mathrm{Rep}_q(\check{G})\text{-mod} \to \mathrm{Rep}_q(\check{T}).$$

So we will need to prove something very particular about the functor $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$ in order to know that it is an equivalence.

0.6.2. Our proof of the equivalence (0.33) follows a rather standard pattern in representation theory. (the challenge is, rather, to show that this pattern is realized in our situation):

We will define a family of objects in either category that we will call “standard”, indexed by the elements of Λ

$$\mu \mapsto \mathcal{M}^{\mu,!}_{\mathrm{Whit}} \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)) \text{ and } \mathcal{M}^{\mu,!}_{\mathrm{Conf}} \in \Omega^{\mathrm{small}}_q\text{-FactMod}^{\mathrm{ren}}.$$

These objects will be compact and “almost” generate $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$ (resp., $\Omega^{\mathrm{small}}_q$-FactMod$^{\mathrm{ren}}$). Define the co-standard objects $\mathcal{M}^{\mu,*}$ (in either context) by

$$\mathcal{H}om(\mathcal{M}^{\mu',!}, \mathcal{M}^{\mu,*}) = \begin{cases} \mathsf{e} \text{ if } \mu' = \mu \\ 0 \text{ otherwise.} \end{cases} \tag{0.34}$$

A standard argument shows that in order to prove that the functor $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$ is an equivalence, it is sufficient to prove that it sends

$$\mathcal{M}^{\mu,!}_{\mathrm{Whit}} \mapsto \mathcal{M}^{\mu,!}_{\mathrm{Conf}} \text{ and } \mathcal{M}^{\mu,*}_{\mathrm{Whit}} \mapsto \mathcal{M}^{\mu,*}_{\mathrm{Conf}}. \tag{0.35}$$

So the essence of the proof is in defining the corresponding families of objects (on each side) and proving (0.35).

0.6.3. Our guiding principle is again the comparison with the quantum group, in this case, with the category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod.

Namely, we want that the objects $\mathcal{M}^{\mu,!}$ correspond under the equivalences (0.31) to the *baby Verma modules*

$$\mathcal{M}^{\mu,!}_{\mathrm{Quant}} \in \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod},$$

i.e., the induced modules

$$\mathrm{Ind}^{\overset{\bullet}{\mathfrak{u}}_q(\check{G})}_{\overset{\bullet}{\mathfrak{u}}_q(\check{B})}(\mathsf{e}^\mu),$$

where e^μ denotes the one-dimensional module over the quantum Borel with character μ.

Given this, the corresponding objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$ and $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ are easy to guess: they are given by !- and *- extensions, respectively, from the corresponding strata on the configuration space.

0.6.4. The situation with $\mathcal{M}^{\mu,!}_{\mathrm{Whit}}$ and $\mathcal{M}^{\mu,*}_{\mathrm{Whit}}$ is more interesting. If we were dealing with $\mathrm{Whit}_q(G)$ rather than with $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$, we would have a natural collection of standard objects, denoted $W^{\lambda,!}_{\mathrm{Whit}}$, indexed by *dominant* elements of Λ^{pos}. However, the sought-for objects $\mathcal{M}^{\mu,!}_{\mathrm{Whit}} \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$ are *not* obtained from the objects $W^{\lambda,!}_{\mathrm{Whit}}$ by applying the (obvious) functor

$$\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}} : \mathrm{Whit}_q(G) \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)).$$

Indeed, drawing on the equivalence with quantum groups (i.e., (0.9)), the objects $W^{\lambda,!}_{\mathrm{Whit}}$ correspond to the *Weyl* modules over the big quantum group, denoted $W^{\lambda,!}_{\mathrm{Quant}}$, and the latter do *not* restrict to the baby Verma modules over the small quantum group.

However, there exists an *explicit colimit procedure* that allows to express the *dual* baby Verma module $\mathcal{M}^{\mu,*}_{\mathrm{Quant}}$ in terms of the *dual* Weyl modules $W^{\lambda,*}_{\mathrm{Quant}}$ and modules pulled back by the quantum Frobenius (0.19). This procedure fits into the Drinfeld-Plücker paradigm, explained in Sect. 23. (How exactly it can be applied to the quantum groups situation is explained in [Ga8, Sect. 10.3].)

By essentially mimicking the procedure by which one expresses $\mathcal{M}^{\mu,*}_{\mathrm{Quant}}$ via the $W^{\lambda,*}_{\mathrm{Quant}}$'s, or rigorously speaking applying the Drinfeld-Plücker formalism in the situation of $\mathrm{Whit}_q(G)$ (see Sect. 25), we define the sought-for objects $\mathcal{M}^{\mu,*}_{\mathrm{Whit}}$ in terms of the corresponding objects $W^{\lambda,*}_{\mathrm{Whit}}$.

0.6.5. The next task is to show that these objects satisfy

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Whit}}) \simeq \mathcal{M}^{\mu,*}_{\mathrm{Conf}}. \tag{0.36}$$

This is equivalent to showing that the !-fiber $(\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Whit}}))_{\mu'\cdot x}$ of $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Whit}})$ at a point $\mu' \cdot x$ of the configuration space satisfies:

$$(\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Whit}}))_{\mu'\cdot x} = \begin{cases} \mathsf{e} \text{ if } \mu' = \mu \\ 0 \text{ otherwise.} \end{cases} \tag{0.37}$$

This is a non-tautological calculation because it does not reduce to just a calculation of the cohomology of some sheaf: indeed we are dealing with the object $\mathcal{M}^{\mu,*}_{\mathrm{Whit}}$ of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$ and not with just a *sheaf*, which would be an object of $\mathrm{Whit}_q(G)$.

Of course, in order to perform this calculation we need to know something about the functor $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$. This functor is obtained via the Hecke property from the functor

$$\Phi_{\mathrm{Fact}} : \mathrm{Whit}_q(G)) \to \Omega^{\mathrm{small}}_q\text{-FactMod},$$

and when we take the !-fiber at $\mu' \cdot x$ the corresponding functor is

$$\mathcal{F} \mapsto \mathrm{C}^{\cdot}(\mathrm{Gr}_{G,x}, \mathcal{F} \overset{!}{\otimes} \mathrm{IC}^{\mu'+\frac{\infty}{2},-}_x),$$

already mentioned in Remark 0.3.7.

In the above formula, $\mathrm{IC}^{\mu'+\frac{\infty}{2},-}_x$ is the *metaplectic semi-infinite IC sheaf* on the $N(\!(t)\!)$ orbit

$$S^{\mu'} = N(\!(t)\!) \cdot t^{\mu'} \subset \mathrm{Gr}_{G,x}.$$

A significant part of this work (namely, the entire Part IV) is devoted to the construction of this semi-infinite IC sheaf and the discussion of its properties (we should say that this Part closely follows [Ga7], where a non-twisted situation is considered).

On the one hand, we define the semi-infinite IC by the procedure of Goresky-MacPherson extension inside the *semi-infinite category on the Ran version of the affine Grassmannian*, and $\mathrm{IC}^{\mu'+\frac{\infty}{2},-}_x$ is obtained as !-restriction to the fiber over $x \in \mathrm{Ran}_x$.

On the other hand, it turns out to be possible to describe $\mathrm{IC}^{\mu'+\frac{\infty}{2},-}_x$ explicitly, and it turns out that this description can also be phrased in terms of the Drinfeld-Plücker formalism (see Sect. 24.3).

This is what makes the calculation (0.37) feasible: the Drinfeld-Plücker nature of both $\mathcal{M}^{\mu,*}_{\mathrm{Whit}}$ and $\mathrm{IC}_x^{\mu'+\frac{\infty}{2},-}$ so to say cancel each other out. See Sect. 26 for how exactly this happens.

0.6.6. We are now left with the following task: we need to define the family of objects $\mathcal{M}^{\mu,!}_{\mathrm{Whit}}$, so that they satisfy (0.34) and the following property holds:

$$\Phi^{\overset{\bullet}{\mathrm{H}}\mathrm{ecke}}_{\mathrm{Fact}}(\mathcal{M}^{\mu,!}_{\mathrm{Whit}}) \simeq \mathcal{M}^{\mu,!}_{\mathrm{Conf}}. \tag{0.38}$$

To construct the objects $\mathcal{M}^{\mu,!}_{\mathrm{Whit}}$ we once again resort to the equivalence with the quantum group situation.

We know that contragredient duality defines an equivalence

$$(\mathrm{Rep}_q(\check{G})^c)^{\mathrm{op}} \to \mathrm{Rep}_{q^{-1}}(\check{G})^c,$$

which induces a similar equivalence for the small quantum group

$$(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^c)^{\mathrm{op}} \to \overset{\bullet}{\mathfrak{u}}_{q^{-1}}(\check{G})\text{-mod}^c.$$

The latter has the property that it sends $\mathcal{M}^{\mu,*}_{\mathrm{Quant}}$ to $\mathcal{M}^{\mu,!}_{\mathrm{Quant}}$.

Hence, it is naturally to try to define a duality

$$(\mathrm{Whit}_q(G)^c)^{\mathrm{op}} \to \mathrm{Whit}_{q^{-1}}(G)^c, \tag{0.39}$$

which would then give rise to a duality

$$(\overset{\bullet}{\mathrm{H}}\mathrm{ecke}(\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \to \overset{\bullet}{\mathrm{H}}\mathrm{ecke}(\mathrm{Whit}_{q^{-1}}(G))^c, \tag{0.40}$$

and we will define $\mathcal{M}^{\mu,!}_{\mathrm{Whit}}$ as the image of $\mathcal{M}^{\mu,*}_{\mathrm{Whit}}$ under the latter functor.

0.6.7. We define the desired duality functor (0.39) in Sect. 7.2, but this is far from automatic.

The difficulty is that the category $\mathrm{Whit}_q(G)$ is defined by imposing *invariance* with respect to a group *ind*-scheme, while the dual category would naturally be *coinvariants*. So we need to show that the categories of invariants and coinvariants are equivalent to one another. This turns out to be a general phenomenon in the Whittaker situation, as is shown in the striking paper [Ras2]. For completeness we supply a proof of this equivalence in our situation using global methods, see Theorem 7.1.8 and its proof in Sect. 7.4.

Thus, we obtain the duality functor in (0.40) with desired properties, and one can prove (0.34) by essentially mimicking the quantum group situation.

0.6.8. Finally, we need to prove (0.38). At the first glance, this may appear as super hard: we need to prove an analog of (0.37) for the objects $\mathcal{M}^{\mu,!}_{\mathrm{Whit}}$, but for **-fibers* instead of the !-fibers, and we do not really know how to do that: our theory is well-adjusted to computing !-pullbacks and *-direct images, but not *-pullbacks and !-direct images.

What saves the game is that we can show that the functor Φ_{Fact} (and then $\Phi^{\overset{\bullet}{\mathrm{H}}\mathrm{ecke}}_{\mathrm{Fact}}$), composed with the forgetful functor from $\Omega^{\mathrm{small}}_q$-FactMod to just sheaves on the configuration space, intertwines the duality (0.39) on Φ_{Fact} with Verdier duality. This is the assertion of our Theorem 18.2.9 (and its Hecke extension Theorem 22.1.5).

The above results about commutation of the (various versions of the) functor Φ with Verdier duality point to yet another layer in this work:

In order to prove this commutation, we give a *global interpretation* of both the Whittaker category (in Sects 7.3-7.4) and of the functor Φ (in Part VII). In other words, we realize $\mathrm{Whit}_q(G)$ as sheaves on (ind)-algebraic stacks (rather than general prestacks). Eventually this leads to the desired commutation with Verdier duality because we show that the morphisms between algebraic stacks involved in the construction are *proper*.

0.7. **Organization of the text.** This work consists of nine Parts and an Appendix. We will now outline the contents of each Part, in order to help the reader navigating this rather lengthy text.

0.7.1. Part I is preliminaries. It can be skipped on the first pass and returned to when necessary.

In Sect. 1 we recall the definitions of the affine Grassmannian, the loop groups, etc., along with their *factorizable versions*.

In Sect. 2 we summarise, following [GLys], the basic tenets of the *geometric metaplectic theory*. In particular, we explain what a geometric metaplectic datum is (the "q" parameter), the construction of the metaplectic dual H (the recipient of Lusztig's quantum Frobenius), and some relevant facts regarding the metaplectic version of the geometric Satake functor.

In Sect. 3 we discuss the formalism of factorization algebras and modules over them. In particular, we give (one of the possible) definitions of these objects in the context of ∞-categories.

In Sect. 4 we discuss some basics of the geometry of the configuration space (of divisors on X colored by elements of $\Lambda^{\mathrm{neg}} - 0$). In particular, we explain how this configuration space relates to the Ran version of the affine Grassmannian $\mathrm{Gr}_{T,\mathrm{Ran}}$ of T.

In Sect. 5 we talk about about factorization algebras and modules over them *on the configuration space*. In particular, we explain that if a factorization algebra is *perverse*, then the corresponding category of its modules has a t-structure and is a highest weight category.

0.7.2. Part II is devoted to the study of the metaplectic Whittaker category of the affine Grassmannian. The material here is largely parallel to one in [Ga9], where the non-twisted situation is considered.

In Sect. 6 we define the metaplectic Whittaker category $\mathrm{Whit}_{q,x}(G)$ as $(N(\!(t)\!), \chi)$-invariants in the category of (metaplectically twisted) sheaves on the affine Grassmannian, and study its basic properties, such as the t-structure, standard and costandard objects, etc.

In Sect. 7 we give a dual definition of the Whittaker category, denoted $\mathrm{Whit}_{q,x}(G)_{\mathrm{co}}$, equal to $(N(\!(t)\!), \chi)$-coinvariants rather than invariants. We state a (non-trivial) theorem to the effect that some (non-tautological) functor

$$\text{Ps-Id} : \mathrm{Whit}_{q,x}(G)_{\mathrm{co}} \to \mathrm{Whit}_{q,x}(G)$$

is an equivalence. This gives rise to a duality identification

$$\mathrm{Whit}_{q,x}(G)^{\vee} \simeq \mathrm{Whit}_{q^{-1},x}(G). \tag{0.41}$$

We also introduce a global version of the Whittaker category, denoted $\mathrm{Whit}_{q,\mathrm{glob}}(G)$ (using a projective curve X and the stack $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$, which is version of *Drinfeld's compactification*), and we state a theorem that says that the global version is actually equivalent to the local ones, i.e., $\mathrm{Whit}_{q,x}(G)$ and $\mathrm{Whit}_{q,x}(G)_{\mathrm{co}}$. In particular, the duality identification corresponds to the usual Verdier duality on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$.

In Sect. 8 we prove the local-to-global equivalence for the metaplectic Whittaker category

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Whit}_{q,x}(G).$$

In the process of doing so, we introduce the Ran version of the Whittaker category $\mathrm{Whit}_{q,\mathrm{Ran}}(G)$, and a *factorization algebra object*

$$\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}} \in \mathrm{Whit}_{q,\mathrm{Ran}}(G).$$

We also define a functor, crucially used in the sequel:

$$\mathrm{sprd}_{\mathrm{Ran}_x} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Whit}_{q,\mathrm{Ran}_x}(G), \tag{0.42}$$

which, so to say, inserts $\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}$ on points of X other than the marked point x. The functor (0.42) can be further upgraded to a functor

$$\mathrm{Whit}_{q,x}(G) \to \mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}\text{-FactMod}(\mathrm{Whit}_{q,\mathrm{Ran}_x}(G)).$$

0.7.3. In Part III we study the Hecke action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_{q,x}(G)$.

In Sect. 9 we define this Hecke action and prove two crucial results. One says that this action is t-exact (in the t-structure on $\mathrm{Whit}_{q,x}(G)$ introduced in Sect. 6). The other says that we start with an irreducible object $W^{\lambda,!*}_{\mathrm{Whit}} \in (\mathrm{Whit}_{q,x}(G))^{\heartsuit}$ corresponding to the orbit $S^{\lambda,-} \subset \mathrm{Gr}_G$ for λ a *restricted coweight* (see Sect. 9.4 for what this means), for any irreducible $V^\gamma \in \mathrm{Rep}(H)$, the result of the convolution

$$W^{\lambda,!*}_{\mathrm{Whit}} \star V^\gamma$$

is the irreducible object $W^{\lambda+\gamma,!*}_{\mathrm{Whit}} \in (\mathrm{Whit}_{q,x}(G))^{\heartsuit}$.

This is a direct counterpart of Steinberg's theorem in the context of quantum groups: for an irreducible object $W^{\lambda,!*}_{\mathrm{Quant}} \in (\mathrm{Rep}_q(\check{G}))^{\heartsuit}$, the tensor product

$$W^{\lambda,!*}_{\mathrm{Quant}} \otimes \mathrm{Frob}^*_q(V^\gamma)$$

equals $W^{\lambda+\gamma,!*}_{\mathrm{Quant}}$ for λ restricted and any γ.

In Sect. 10 we discuss the general formalism of the formation of the category of Hecke eigen-objects $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$ for a category $\mathbf{C}$ equipped with an action of $\mathrm{Rep}(H)$. In particular, we introduce the notion of such an action being *accessible* (with respect to a given t-structure on $\mathbf{C}$) and show that in this case the resulting t-structure on $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$ is Artinian.

In Sect. 11 we apply the formalism of Sect. 10 to $\mathbf{C} = \mathrm{Whit}_{q,x}(G)$. The resulting category $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ is the LHS in our main theorem. We discuss some basic properties of this category. In particular, we describe the irreducible objects of $(\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)))^{\heartsuit}$.

0.7.4. In Part IV we are concerned with the construction and properties of the metaplectic semi-infinite IC sheaf, denoted $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$.

In Sect. 12 we discuss the metaplectic semi-infinite category $\mathrm{SI}_{q,\mathrm{Ran}}(G)$, which is a full subcategory inside $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}})$, obtained by imposing the equivariance condition with respect to the loop group of N. We restrict our attention to the "non-negative part" of $\mathrm{SI}_{q,\mathrm{Ran}}(G)$, denoted $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$. We define a stratification of $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$ indexed by elements of Λ^{neg}. We also introduce a full subcategory

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$$

that consists of *unital* objects. We show that for every $\lambda \in \Lambda^{\mathrm{neg}}$ the corresponding subcategory

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}$$

is equivalent to the category of (gerbe-twisted) sheaves on the corresponding connected component Conf^λ of the configuration space.

In Sect. 13 we introduce a t-structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$ and define $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ as the minimal extension of the dualizing sheaf on $S^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}$. We describe the !-restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ to

$$\mathrm{Gr}_{G,x} = \{x\} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}_{G,\mathrm{Ran}}.$$

We also discuss the *factorization structure* on $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$, which makes it into a *factorization algebra* on $\mathrm{Gr}_{G,\mathrm{Ran}}$. Finally, we discuss the relationship between $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ and the (gerbe-twisted) IC sheaf on Drinfeld's compactification $\overline{\mathrm{Bun}}_N^{\omega^\rho}$.

In Sect. 14 we establish the (crucial for the sequel) Hecke property of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$.

0.7.5. In Part V we define (various forms of) the Jacquet functor that maps $\mathrm{Whit}_{q,x}(G)$ to the coresponding category for T.

In Sect. 15 we first consider the functor

$$\mathfrak{J}_{!*,\mathrm{Ran}} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}}),$$

defined using the diagram

$$\begin{CD} \mathrm{Gr}_{B^-,\mathrm{Ran}} @>>> \mathrm{Gr}_{G,\mathrm{Ran}} \\ @VVV \\ \mathrm{Gr}_{T,\mathrm{Ran}}, \end{CD}$$

and using $\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}}$ as a kernel.

We precompose this functor with the functor $\mathrm{sprd}_{\mathrm{Ran}_x}$ of (0.42) and obtain a functor

$$\mathfrak{J}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}_x}).$$

We define the factorization algebra $\Omega_q^{\mathrm{Whit}_{!*}}$ on $\mathrm{Gr}_{T,\mathrm{Ran}}$ as $\mathfrak{J}_{!*,\mathrm{Ran}}(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}})$, and we upgrade the functor $\mathfrak{J}_{!*,\mathrm{sprd}}$ to a functor

$$\mathfrak{J}_{!*,\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}\left(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}_x})\right).$$

In Sect. 16 we establish the Hecke property of the functor $\mathfrak{J}_{!*,\mathrm{Fact}}$ and extend it to the functor

$$\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}\left(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}_x})\right).$$

0.7.6. In Part VI we interpret the functor $\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}}$ via the factorization space.

In Sect. 17 we define a particular (in fact, the most basic) factorization algebra $\Omega_q^{\mathrm{small}}$ in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ (here $\mathcal{G}^\Lambda$ is a factorization gerbe on Conf constructed from the geometric metaplectic data $\mathcal{G}^G$). Namely, $\Omega_q^{\mathrm{small}}$, viewed as a sheaf, is *perverse*, and is the Goresky-MacPherson extension of its restriction to the open locus $\overset{\circ}{\mathrm{Conf}} \subset \mathrm{Conf}$ that consists of multiplicity-free divisors, where, in its turn, $\Omega_q^{\mathrm{small}}|_{\overset{\circ}{\mathrm{Conf}}}$ is the *sign local system* (the latter makes sense since the gerbe $\mathcal{G}^\Lambda$ has the property that it is canonically trivial when restricted to $\overset{\circ}{\mathrm{Conf}}$).

In Sect. 18 we use the diagrams

$$\begin{CD} (\mathrm{Gr}_{T,\mathrm{Ran}})^{\mathrm{neg}} @>>> \mathrm{Gr}_{T,\mathrm{Ran}} \\ @VVV \\ \mathrm{Conf} \end{CD}$$

and

$$\begin{CD} ((\mathrm{Gr}_{T,\mathrm{Ran}_x})^{\mathrm{neg}})_{\infty\cdot x} @>>> \mathrm{Gr}_{T,\mathrm{Ran}_x} \\ @VVV \\ \mathrm{Conf}_{\infty\cdot x} \end{CD}$$

to transfer our constructions from (gerbe-twisted) sheaves on $\mathrm{Gr}_{T,\mathrm{Ran}}$ (resp., $\mathrm{Gr}_{T,\mathrm{Ran}_x}$) to those on Conf (resp., $\mathrm{Conf}_{\infty\cdot x}$).

We prove the theorem that the factorization algebra on Conf, corresponding under this procedure to $\Omega_q^{\mathrm{Whit}_{!*}} \in \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}})$, identifies with the factorization algebra $\Omega_q^{\mathrm{small}} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ introduced above.

In particular, the functor $\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}}$ gives rise to a functor

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{small}}\text{-FactMod}.$$

Let $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}$ be the functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}),$$

obtained from $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$ by forgetting the factorization $\Omega_q^{\mathrm{small}}$-module structure. Let Φ be the pre-composition of $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}$ with the tautological functor

$$\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}} : \mathrm{Whit}_{q,x}(G) \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)).$$

We state a key result, Theorem 18.2.9, which says that the functor Φ commutes with Verdier duality. We use this fact to show that the functor $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$ is t-exact and sends irreducible objects in $(\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)))^\heartsuit$ to irreducible objects in $(\Omega_q^{\mathrm{small}}\text{-FactMod})^\heartsuit$.

In Sect. 19 we define the renormalized version $\Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}}$ of the category $\Omega_q^{\mathrm{small}}\text{-FactMod}$ (see Sect. 0.5.4 above), and we state our main result, Theorem 19.2.5, which says that the resulting functor

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}}$$

is an equivalence. In the same section we outline the strategy of the proof of Theorem 19.2.5, which was already mentioned in Sect. 0.6 above.

0.7.7. In Part VII we establish the commutation of Φ with Verdier duality, i.e., Theorem 18.2.9.

In Sect. 20 we give a global interpretation of the functor Φ, in which the geometric object known as the Zastava space plays a prominent role.

In Sect. 21 we prove Theorem 18.2.9 (or rather, its more precise version, which is local on the Zastava space). The key idea is a certain ULA property of the *global* metaplectic IC sheaf on $\overline{\mathrm{Bun}}_{B^-}$.

In Sect. 22 we prove a Hecke extension of Theorem 18.2.9, which says that the functor $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}$ commutes with Verdier duality.

0.7.8. Part VIII is devoted to the realization of the outline of the proof of Theorem 19.2.5 indicated at the end of Sect. 19. I.e., we need to construct the standard and costandard objects in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ and prove their properties.

In Sect. 23 we discuss the general framework of the Drinfeld-Plücker formalism (this formalism was suggested by S. Raskin). We show that the restriction of the metaplectic semin-infinite IC sheaf to the fiber over $x \in \mathrm{Ran}_x$ can be constructed via this formalism.

In Sect. 24 we make a (necessary) digression and discuss the construction of the (dual) baby Verma object in the category of *Iwahori*-equivariant sheaves on the affine Grassmannian. We remark that this object has appeared prominently in the papers [ABBGM], [FG2], [FG3], [Ga8].

In Sect. 25 we construct the sought-for objects $\mathcal{M}^{\mu,*}_{\mathrm{Whit}}$ (using the (dual) baby Verma object in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I$ constructed in Sect. 24). We then construct the objects $\mathcal{M}^{\mu,!}_{\mathrm{Whit}}$ by applying duality. Finally, we verify the orthogonality property (0.34).

In Sect. 26 we prove (0.35). This is where everything comes together in this work. The isomorphism (0.36) uses the full force of the Drinfeld-Plücker formalism. The isomorphism (0.38) is proved using the commutation of the functor $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}$ with Verdier duality.

0.7.9. Part IX is logically disjoint from this rest of this work. Here we prove that in the Betti context we have a canonical equivalence

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \simeq \Omega_q^{\text{small}}\text{-FactMod}$$

of (0.29).

In Sect. 27 we introduce the category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod from the perspective of *higher algebra* (i.e., with using explicit formulas as little as possible).

In Sect. 28 we introduce the two renormalizations of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod, denoted

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})^{\text{ren}}\text{-mod and } \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}.$$

The former is compactly generated by the irreducible objects in $(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^{\heartsuit}$, and the latter by the *baby Verma modules* (hence the notation).

In Sect. 29 we prove the equivalence (0.29). The proof consists of three steps: (i) Koszul duality; (ii) interpretation of the category of $\mathbb{E}_2$-modules over an $\mathbb{E}_2$-algebra as the Drinfeld center of the monoidal category of left modules; (iii) equivalence between $\mathbb{E}_2$-modules over an $\mathbb{E}_2$-algebra and factorization modules over the corresponding factorization algebra.

0.7.10. In the Appendix we introduce a device that we call the Kirillov model that allows to talk about Whittaker categories when the Artin-Schreier sheaf does not exist, for example in the Betti setting (i.e., constructible sheaves in the classical topology for schemes over $\mathbb{C}$).

0.8. **Conventions.**

0.8.1. *Algebraic geometry.* We will work over an algebraically closed field k (of arbitrary characteristic). Our algebro-geometric objects will be schemes, and more generally pre-stacks over k, i.e., (accessible) functors

$$(\text{Sch}^{\text{aff}})^{\text{op}} \to \text{Groupoids}\,.$$

Most of our prestacks will be *locally of finite type*, which means that they are left-Kan-extended from the full subcategory

$$(\text{Sch}^{\text{aff}}_{\text{ft}})^{\text{op}} \subset (\text{Sch}^{\text{aff}})^{\text{op}},$$

consisting of affine schemes of finite type. See, e.g., [Ga9, Sect. 0.5.1] for details.

For the purposes of this work, we will *not* need derived alegebraic geometry. Also, for our purposes it is sufficient to consider classical (and not higher) groupoids.

We denote $\text{pt} := \text{Spec}(k)$.

We let X be a smooth connected curve over k with a marked point $x \in X$. In some places we will need X to be complete; we will explicitly say so when this is the case. We let ω denote the canonical line bundle on X.

0.8.2. *Reductive groups.* We let G be a reductive group over k, with a chosen Borel and Cartan subgroups

$$T \subset B \subset G.$$

We let Λ denote the cocharacter lattice of T. We let $\Lambda^+ \subset \Lambda$ be the sub-monoid of dominant coweights. Let I denote the set of vertices of the Dynkin diagram of G. For each $i \in I$ we let $\alpha_i \in \Lambda$ denote the corresponding coroot. We let $\Lambda^{\text{pos}} \subset \Lambda$ be the submonoid equal to the positive integral span of the elements α_i.

0.8.3. *DG categories.* We let e be an algebraically closed field of characteristic 0, which will serve as our field of coefficients. Our object of study is various e-linear DG categories. We refer the reader to [GR1, Chapter 1, Sect. 10] for a detailed exposition of the theory of DG categories.

Unless specified otherwise, our DG categories will be assumed cocomplete (i.e., closed under infinite direct sums). By default, we will only consider functors between DG categories that commute with infinite direct sums (we call such functors *continuous*).

We let Vect denote the DG category of chain complexes of vector spaces.

For $\mathbf{c}_1, \mathbf{c}_2 \in \mathbf{C}$, we denote by

$$\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) \in \mathrm{Vect}$$

their "internal Hom", i.e.,

$$\mathrm{Hom}_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2) = H^0(\mathcal{H}om_{\mathbf{C}}(\mathbf{c}_1, \mathbf{c}_2)).$$

0.8.4. If $\mathbf{C}$ is a DG category equipped with a t-structure, we let $(\mathbf{C})^{\leq 0}$ (resp., $(\mathbf{C})^{\geq 0}$) denote the full subcategory consisting of *connective* (resp., *coconnective*) objects. We let $(\mathbf{C})^{\heartsuit} = (\mathbf{C})^{\leq 0} \cap (\mathbf{C})^{\geq 0}$ denote the heart of the t-structure.

0.8.5. Recall that if $\mathbf{C}$ is a (complete) DG category, one can talk about the (non-cocomplete) category of its compact objects, denoted $\mathbf{C}^c$. Vice versa, starting with a non-cocomplete category $\mathbf{C}_0$, one can form its ind-completion, denoted $\mathrm{IndCompl}(\mathbf{C}_0)$, which is universal among non-cocomplete categories receiving an exact functor from $\mathbf{C}_0$.

Recall that a DG category is said to be compactly generated if the tautological functor

$$\mathrm{IndCompl}(\mathbf{C}^c) \to \mathbf{C}$$

is an equivalence.

For $\mathbf{C}_0$ as above, we always have $\mathbf{C}_0 \subset \mathrm{IndCompl}(\mathbf{C}_0)^c$; moreover, every object in $\mathrm{IndCompl}(\mathbf{C}_0)^c$ is a retract of one in $\mathbf{C}_0$.

0.8.6. *Limits of DG categories.* One important think about DG categories is that they form an ∞-category. So one needs the full force of higher category theory when forming limits and colimits of DG categories.

Here is one paradigm that appears often in representation theory. Let

$$J \to \mathrm{DGCat}, \quad j \mapsto \mathbf{C}_j$$

be a functor, where J is some index category. Suppose that for every arrow $(j_1 \to j_2) \in J$, the corresponding functor $\mathbf{C}_{j_1} \to \mathbf{C}_{j_2}$ admits a continuous right adjoint. By passing to the right adjoints we obtain another functor

$$J^{\mathrm{op}} \to \mathrm{DGCat}\,.$$

Then for every $j_0 \in J$, the tautological functor

$$\mathrm{ins}_{j_0} : \mathbf{C}_{j_0} \to \underset{j \in J}{\mathrm{colim}}\, \mathbf{C}_j$$

admits a continuous right adjoint. Furthermore, the resulting functor

$$\underset{j \in J}{\mathrm{colim}}\, \mathbf{C}_j \to \underset{j \in I^{\mathrm{op}}}{\mathrm{colim}}\, \mathbf{C}_j,$$

formed by these right adjoints, is an equivalence.

0.8.7. The ∞-category DGCat of DG categories carries a symmetric monoidal structure, Lurie's tensor product

$$\mathbf{C}_1, \mathbf{C}_2 \mapsto \mathbf{C}_1 \otimes \mathbf{C}_2.$$

Hence, one can talk about dualizable DG categories. It is known that compactly generated categories are dualizable. Moreover, for $\mathbf{C}_0$ as in Sect. 0.8.5, we have

$$\mathrm{IndCompl}(\mathbf{C}_0)^\vee \simeq \mathrm{IndCompl}((\mathbf{C}_0)^{\mathrm{op}}).$$

In particular, if $\mathbf{C}$ is compactly generated, we have a canonical equivalence

$$(\mathbf{C}^c)^{\mathrm{op}} \simeq (\mathbf{C}^\vee)^c.$$

0.8.8. *Sheaf theories.* By a sheaf theory we will mean a right-lax symmetric monoidal functor

$$(\mathrm{Sch}^{\mathrm{aff}}_{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{DGCat}, \quad S \mapsto \mathrm{Shv}(S), \quad (S_1 \overset{f}{\to} S_2) \mapsto (\mathrm{Shv}(S_2) \overset{f^!}{\to} \mathrm{Shv}(S_1)), \tag{0.43}$$

$$\mathrm{Shv}(S_1) \otimes \mathrm{Shv}(S_2) \to \mathrm{Shv}(S_1 \otimes S_2). \tag{0.44}$$

Rather than axiomatizing the situation, we will list the examples of sheaf theories that we have in mind:

- For k being of characteristic zero we can take $\mathrm{Shv}(S) = \text{D-mod}(S)$. In this case the field e of coefficients equals k. We refer to this example as the D-module context.
- For k as above, we can take $\mathrm{Shv}(S)$ to be the ind-completion of the full subcategory of $\text{D-mod}(S)$ consisting of holonomic (resp., regular holonomic) D-modules. We refer to this example as holonomic (resp., regular holonomic) context.
- For k being $\mathbb{C}$, we can take $\mathrm{Shv}(S)$ to be the ind-completion of the category of e-constructible sheaves in the classical topology, for any e. We refer to this example as the *Betti context*.
- For any k, we can take $\mathrm{Shv}(S)$ to be the ind-completion of the category of constructible $\overline{\mathbb{Q}}_\ell$-adic sheaves on S. In this case the field e of coefficients is $\overline{\mathbb{Q}}_\ell$. We refer to this example as the *ℓ-adic context*.

Note that in all of the above examples, the functor (0.44) is fully faithful, and in the first example, it is actually an equivalence.

We will refer to the last three examples as the *constructible context*.

0.8.9. *Sheaves on prestacks.* Given a sheaf theory, we apply a procedure of *right Kan extension* to extend it to a functor

$$\mathrm{Shv} : (\mathrm{PreStk}_{\mathrm{lft}})^{\mathrm{op}} \to \mathrm{DGCat}\,. \tag{0.45}$$

Explicitly, for a prestack $\mathcal{Y}$

$$\mathrm{Shv}(\mathcal{Y}) = \lim_{S \to \mathcal{Y}} \mathrm{Shv}(S),$$

where the limit is taken over the category (opposite to that) of affine schemes of finite type mapping to $\mathcal{Y}$. Note that the above limit is formed within DGCat, so it is a higher categorical procedure. In particular, are using the fact that (0.43) is a functor of ∞-categories.

0.8.10. *Functors defined on sheaves.* The functor (0.45) has in fact more functoriality: it actually extends to a functor out of the category of correspondences on prestacks, where we allow to take direct images along ind-schematic maps, see [GR1, Chapter 5, Sect. 2] and [GR2, Chapter 3, Sect. 5].

In the constructible context, for any ind-schematic morphism $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the functor $f^!$ has a left adjoint, denoted $f_!$. In the context of D-modules, this left adjoint is only partially defined.

Similarly, for a map of schemes $f : Y_1 \to Y_2$, the functor $f_* : \mathrm{Shv}(Y_1) \to \mathrm{Shv}(Y_2)$ has a partially defined left adjoint f^*, which is always defined in the constructible context.

Finally, Verdier duality for a ind-scheme Y is an equivalence

$$\mathbb{D}^{\mathrm{Verdier}}(\mathrm{Shv}(Y)^c)^{\mathrm{op}} \to \mathrm{Shv}(Y)^c,$$

uniquely characterized by the requirement that

$$\mathcal{H}om(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}_1), \mathcal{F}_2) = \mathrm{C}^{\cdot}(Y, \mathcal{F}_1 \overset{!}{\otimes} \mathcal{F}_2),$$

where $\mathrm{C}^{\cdot}(Y, -)$ is the functor of cochains, i.e., direct image along $Y \to \mathrm{pt}$.

0.8.11. *Gerbes.* Let $\mathcal{Y}$ be a prestack and let $\mathcal{G}$ be a gerbe on it with respect to the group $\mathsf{e}^{\times,\mathrm{tors}}$ of torsion elements in $\mathsf{e}^{\times}$ of orders prime to $\mathrm{char}(p)$, see [GLys, Sect. 1.3].

The data of a gerbe $\mathcal{G}$ allows to twist the category $\mathrm{Shv}(\mathcal{Y})$ and obtain a new category $\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})$, see [GLys, Sect. 1.7.4]. Moreover, the functor (0.45) extends to a functor

$$(\mathrm{PreStk} + \mathrm{Grb})^{\mathrm{op}} \to \mathrm{DGCat},$$

where the source category consists now of pairs

$$(\mathcal{Y} \in \mathrm{PreStk}_{\mathrm{lft}}, \mathcal{G} \in \mathrm{Grb}(\mathcal{Y})),$$

and where the morphisms $(\mathcal{Y}_1, \mathcal{G}_1) \to (\mathcal{Y}_2, \mathcal{G}_2)$ are maps of prestacks $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ equipped with an identification of the gerbes $f^*(\mathcal{G}_2) \simeq \mathcal{G}_1$.

0.9. **Acknowledgements.**

0.9.1. The first-named author is enormously indebted to J. Lurie: this entire project originated from his original guess as to the existence of the equivalence (0.9).

We are very grateful to S. Raskin for suggesting the idea Drinfeld-Plücker formalism: this made the proof of (0.36) short and conceptual (it was a mess in the previous version).

Special thanks are due to M. Finkelberg: it is from him that we learned the idea that one can describe interesting representation-theoretic categories as factorization modules (the book [BFS] was the first example of this).

The idea of quantum Langlands theory originated in the (mostly unpublished) work of B. Feigin, E. Frenkel and A. Stoyanovsky in the mid-1990’s. Their insight was crucial in informing our formulation of the FLE.

This works inhabits the area known as the geometric Langlands theory, which was put forth in its modern form by A. Beilinson and V. Drinfeld. The influence of their ideas over this work is all-pervasive.

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Part I: Background and preliminaries

In order not to interrupt the flow of exposition in the main body of this work, in this Part we have collected several pieces of mathematical background that we will use.

These include the definitions of the main geometric objects (such as the affine Grassmannian and its various relatives), main tenets of the *geometric metaplectic theory*; factorization algebras and modules, and some discussion of configuration spaces (a.k.a. colored divisors).

1. The geometric objects

In this section we will recall the definition of the key geometric objects that we will work with in this paper: the affine Grassmannian, the loop group, the Hecke stack, etc.

An important feature of these objects is that they have a *factorizable nature*; we will take this perspective when introducing them.

1.1. **The Ran space.** The Ran space is a gadget that allows us to talk about factorization. In this subsection we recall its definition and key structures.

1.1.1. Recall that the Ran space Ran of X assigns to a test scheme S the set of finite non-empty subsets $\mathcal{I} \subset \operatorname{Maps}(S, X)$.

For an element $i \in \mathcal{I}$ we will denote by $\Gamma_i \subset S \times X$ the graph of the corresponding map. Let $\Gamma_{\mathcal{I}}$ denote the (set-theoretic) union of Γ_i over $i \in \mathcal{I}$.

1.1.2. A basic structure that exists on the Ran space is the structure of commutative semi-group: for a finite set J we have the map

$$\operatorname{Ran}^J \to \operatorname{Ran}, \quad (\mathcal{I}_j \subset \operatorname{Maps}(S,X),\, j \in J) \;\mapsto\; (\underset{j\in J}{\cup} \mathcal{I}_j \subset \operatorname{Maps}(S,X)). \tag{1.1}$$

1.1.3. For a finite set J, let

$$(\operatorname{Ran}^J)_{\mathrm{disj}} \subset \operatorname{Ran}^J$$

be the open subfunctor corresonding to the following condition: we allow those J-subsets

$$\mathcal{I}_j \subset \operatorname{Maps}(S,X),\, j \in J$$

such that for every $j_1 \neq j_2$

$$\Gamma_{\mathcal{I}_{j_1}} \cap \Gamma_{\mathcal{I}_{j_2}} = \emptyset.$$

An important feature is that the map

$$(\operatorname{Ran}^J)_{\mathrm{disj}} \to \operatorname{Ran}, \tag{1.2}$$

induced by (1.1), is étale.

1.1.4. By a factorization space over Ran we will mean a prestack $Z_{\operatorname{Ran}} \to \operatorname{Ran}$ equipped with a system of identifications (factorization isomorphisms)

$$Z_{\operatorname{Ran}} \underset{\operatorname{Ran}}{\times} (\operatorname{Ran}^J)_{\mathrm{disj}} \simeq Z^J_{\operatorname{Ran}} \underset{\operatorname{Ran}^J}{\times} (\operatorname{Ran}^J)_{\mathrm{disj}}, \tag{1.3}$$

for every finite set J, that are compatible in a natural sense (see [GLys, Sect. 2.2.1] or Sect. 3.1.2 below).

1.2. **Examples of factorization spaces.** Having introduced the Ran space, we will now move one step closer to the definition of the geometric objects we will be working with.

These objects are obtained by forming *loop spaces*, which have to do with mapping the *multi-disc* parameterized by points of Ran to a given target.

1.2.1. Let $\mathcal{I} \subset \mathrm{Hom}(S,X)$ be as in Sect. 1.1.1. We let $\widehat{\mathcal{D}}_{\mathcal{I}}$ the formal scheme equal to the completion of $S \times X$ along $\Gamma_{\mathcal{I}}$.

Let $\mathcal{D}_{\mathcal{I}}$ denote the affine scheme obtained from $\widehat{\mathcal{D}}_{\mathcal{I}}$ (i.e., the universal recipient of a map from $\mathcal{D}_{\mathcal{I}}$ among affine schemes, see [GLys, Sect. 7.1.2]).

Note that Γ_J is naturally a closed subset of $\mathcal{D}_{\mathcal{I}}$. Denote

$$\overset{\circ}{\mathcal{D}}_{\mathcal{I}} := \mathcal{D}_{\mathcal{I}} - \Gamma_{\mathcal{I}}.$$

It is easy to see that $\overset{\circ}{\mathcal{D}}_{\mathcal{I}}$ is also affine.

1.2.2. Note that for

$$(\mathcal{I}_j, j \in J) \in (\mathrm{Ran}^J)_{\mathrm{disj}}, \quad \mathcal{I} := \underset{j}{\cup} \mathcal{I}_j$$

we have

$$\widehat{\mathcal{D}}_{\mathcal{I}} \simeq \underset{j}{\sqcup} \widehat{\mathcal{D}}_{\mathcal{I}_j}.$$

From here,

$$\mathcal{D}_{\mathcal{I}} \simeq \underset{j}{\sqcup} \mathcal{D}_{\mathcal{I}_j} \text{ and } \overset{\circ}{\mathcal{D}}_{\mathcal{I}} \simeq \underset{j}{\sqcup} \overset{\circ}{\mathcal{D}}_{\mathcal{I}_j}. \tag{1.4}$$

1.2.3. For a prestack Y, define the prestack

$$\mathfrak{L}^+(Y)_{\mathrm{Ran}} \to \mathrm{Ran}$$

by assigning to $\mathcal{I}$ as above the space of maps

$$\widehat{\mathcal{D}}_{\mathcal{I}} \to Y.$$

When Y is affine scheme, maps as above are the same as maps

$$\mathcal{D}_{\mathcal{I}} \to Y,$$

by the universal property of $\mathcal{D}_{\mathcal{I}}$.

1.2.4. Define

$$\mathfrak{L}(Y)_{\mathrm{Ran}} \to \mathrm{Ran}$$

by assigning to $\mathcal{I}$ the space of maps $\overset{\circ}{\mathcal{D}}_{\mathcal{I}} \to Y$.

These spaces have natural factorization structures due to the identifications (1.4).

1.2.5. The above definitions have a variant when instead of an affine scheme Y, we have an affine morphism $Y_X \to X$. In this case we will be looking at maps

$$\mathcal{D}_{\mathcal{I}} \to Y_X \text{ and } \overset{\circ}{\mathcal{D}}_{\mathcal{I}} \to Y_X,$$

respectively, *over* X.

The assignments

$$Y_X \rightsquigarrow \mathfrak{L}^+(X)_{\mathrm{Ran}} \text{ and } Y_X \rightsquigarrow \mathfrak{L}(Y_X)_{\mathrm{Ran}}$$

are functorial in Y_X.

In particular, we obtain that a section $X \to Y_X$ of $Y_X \to X$ gives rise to a map of factorization spaces

$$\mathrm{Ran} \to \mathfrak{L}^+(X)_{\mathrm{Ran}}.$$

1.3. **The affine Grassmannian and other animals.** We will now specialize to the case when the target space Y is an algebraic group G, and thus define the actual geometric objects of interest.

1.3.1. Note that, by functoriality, when the target space is G, the prestacks $\mathfrak{L}^+(G)_{\mathrm{Ran}}$ and $\mathfrak{L}(G)_{\mathrm{Ran}}$ acquire a structure of group-spaces over Ran.

1.3.2. Consider the quotient $\mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Ran}$ (understood in the étale sense). It attaches to $\mathcal{I}$ the datum of a G-bundle on $\mathcal{D}_\mathcal{I}$. It is easy to see, however, that restriction defines an equivalence from G-bundles on $\mathcal{D}_\mathcal{I}$ to those on $\widehat{\mathcal{D}}_\mathcal{I}$.

Note that we can think of $\mathfrak{L}^+(G)\backslash\mathrm{Ran}$ also as $\mathfrak{L}^+(G\backslash\,\mathrm{pt})$.

In particular, from Sect. 1.2.5, we obtain that a G-bundle on X defines a map of factorization spaces

$$\mathrm{Ran} \to \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Ran}.$$

1.3.3. The (Ran version of the) affine Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$ can be defined as the étale sheafification of the quotient

$$\mathfrak{L}(G)_{\mathrm{Ran}}/\mathfrak{L}^+(G)_{\mathrm{Ran}}.$$

Equivalently,

$$\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathrm{Ran}$$

assigns to $\mathcal{I}$ as above the data of pairs $(\mathcal{P}_G, \alpha)$, where:

- $\mathcal{P}^G$ is a G-bundle $\mathcal{P}^G$ on $\mathcal{D}_\mathcal{I}$ (equivalently, on $\widehat{\mathcal{D}}_\mathcal{I}$);
- α is a trivialization of the restriction to $\mathcal{P}_G$ to $\overset{\circ}{\mathcal{D}}_\mathcal{I}$.

1.3.4. A theorem of Beauville and Laszlo says that the affine Grassmannian can be defined using the curve X instead of the disc. Namely, we can take pairs $(\mathcal{P}_G, \alpha)$, where:

- $\mathcal{P}_G$ is a G-bundle $\mathcal{P}^G$ on $S \times X$;
- α is a trivialization of the restriction to $\mathcal{P}_G$ to $S \times X - \Gamma_\mathcal{I}$.

Restriction defines a map from the data above to that in Sect. 1.3.3. The Beauville-Laszlo theorem says that this map is a bijection.

1.3.5. Set

$$\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}} := \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Gr}_{G,\mathrm{Ran}};$$

where the quotient is understood in the étale sense.

The functor of points of $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ consists of triples $(\mathcal{P}'_G, \mathcal{P}_G, \alpha)$, where:

- $\mathcal{P}'_G$ is a G-bundle $\mathcal{P}^G$ on $\mathcal{D}_\mathcal{I}$;
- $\mathcal{P}_G$ is a G-bundle $\mathcal{P}^G$ on $\mathcal{D}_\mathcal{I}$;
- α is an isomorphism $\mathcal{P}'_G|_{\overset{\circ}{\mathcal{D}}_\mathcal{I}} \simeq \mathcal{P}_G|_{\overset{\circ}{\mathcal{D}}_\mathcal{I}}$.

1.3.6. In what follows we will denote by $\overleftarrow{h}_G$, $\overrightarrow{h}_G$ the two maps

$$\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}} \rightrightarrows \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Ran}$$

equal to

$$\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}} = \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Gr}_{G,\mathrm{Ran}} \to \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Ran}$$

$$\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}} = \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Gr}_{G,\mathrm{Ran}} \simeq \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathfrak{L}(G)_{\mathrm{Ran}}/\mathfrak{L}^+(G)_{\mathrm{Ran}} \to \\ \to \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathfrak{L}(G)_{\mathrm{Ran}}/\mathfrak{L}(G)_{\mathrm{Ran}} \simeq \mathfrak{L}^+(G)_{\mathrm{Ran}}\backslash\mathrm{Ran},$$

respectively.

In other words, the map $\overleftarrow{h}_G$ remembers the data of $\mathcal{P}'_G$ and the map $\overrightarrow{h}_G$ remembers the data of $\mathcal{P}_G$.

1.3.7. The prestack $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ carries an involution (swapping the roles of $\mathcal{P}_G$ and $\mathcal{P}'_G$) denoted $\mathbf{inv}^G$, which interchanges $\overleftarrow{h}_G$ and $\overrightarrow{h}_G$.

1.4. **Twist by the canonical bundle.** In this subsection we will take G to be reductive, with a chosen Borel subgroup B and a Cartan subgroup $T \subset B$.

In this subsection we will introduce twisted versions of Gr_G, $\mathfrak{L}(G)$, etc., that have to do with the canonical line bundle on X.

The necessity for such a twist can be explained succinctly as follows: it makes the additive character on the loop group $\mathfrak{L}(N)$ canonical.

1.4.1. For what follows we will choose a square root $\omega^{\otimes \frac{1}{2}}$ of the canonical line bundle ω on X. Let ω^ρ denote the T-bundle on X induced from $\omega^{\otimes \frac{1}{2}}$ by means of the homomorphism

$$2\rho : \mathbb{G}_m \to T.$$

We will denote by the same character ω^ρ the induced B-bundle and G-bundle via

$$T \hookrightarrow B \hookrightarrow G.$$

1.4.2. By Sect. 1.3.2, the datum of ω^ρ defines a map of factorization spaces

$$\mathrm{Ran} \to \mathfrak{L}^+(T)_{\mathrm{Ran}} \backslash \mathrm{Ran},$$

i.e., a $\mathfrak{L}^+(T)_{\mathrm{Ran}}$-torsor over Ran, denoted $\omega^\rho_{\mathrm{Ran}}$, compatible with factorization.

Using the (adjoint) action of $\mathfrak{L}^+(T)_{\mathrm{Ran}}$ on the objects introduced in Sect. 1.3, we obtain their twisted versions, to be denoted

$$\mathfrak{L}^+(G)^{\omega^\rho}_{\mathrm{Ran}},\ \mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}},\ \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}.$$

1.4.3. Note that we can identify $\mathfrak{L}^+(G)^{\omega^\rho}_{\mathrm{Ran}}$ (resp., $\mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}}$) with $\mathfrak{L}^+(G^{\omega^\rho})_{\mathrm{Ran}}$ (resp., $\mathfrak{L}(G^{\omega^\rho})_{\mathrm{Ran}}$), where G^{ω^ρ} is the group-scheme over X obtained by twisting G by means of ω^ρ.

1.4.4. The twisted version $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ of the affine Grassmannian can be explicitly described as follows: it assigns to $\mathcal{J}$ the set of pairs $(\mathcal{P}_G, \alpha)$, where:

- $\mathcal{P}^G$ is a G-bundle $\mathcal{P}^G$ on $\mathcal{D}_\mathcal{J}$ (equivalently, on $\widehat{\mathcal{D}}_\mathcal{J}$);
- α is an identification of the restriction to $\mathcal{P}_G$ to $\overset{\circ}{\mathcal{D}}_\mathcal{J}$ with that of ω^ρ.

Remark 1.4.5. Note that the adjoint action of $\mathfrak{L}^+(G)_{\mathrm{Ran}}$ on $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ is canonically trivialized. Hence, the twist

$$(\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}})^{\omega^\rho}$$

identifies canonically with the non-twisted version $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$.

Remark 1.4.6. Note that in the case when $G = T$, the operation of tensoring a given T-torsor with the T-torsor ω^ρ, gives rise to an identification

$$\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \simeq \mathrm{Gr}_{T,\mathrm{Ran}}.$$

However, we will avoid using it, as it is *incompatible* with gerbes considered in the next section, see Remark 2.1.5.

1.4.7. Since $N \subset B \subset G$ are normalized by T, we can form the twisted forms of the corresponding loop groups

$$\mathfrak{L}^+(B)^{\omega^\rho}_{\mathrm{Ran}},\ \mathfrak{L}(B)^{\omega^\rho}_{\mathrm{Ran}}$$

and

$$\mathfrak{L}^+(N)^{\omega^\rho}_{\mathrm{Ran}},\ \mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}}.$$

Note, however, that since the adjoint action of T on itself is trivial, we have

$$\mathfrak{L}^+(T)^{\omega^\rho}_{\mathrm{Ran}} = \mathfrak{L}^+(T)_{\mathrm{Ran}} \text{ and } \mathfrak{L}(T)^{\omega^\rho}_{\mathrm{Ran}} = \mathfrak{L}(T)_{\mathrm{Ran}}.$$

1.4.8. Consider the special case when $G = PGL_2$, in which case $N \simeq \mathbb{G}_a$, and the adjoint action of $T = \mathbb{G}_m$ is given by dilations. We have:

$$\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \simeq \mathfrak{L}(\mathbb{G}_a)^{\omega}_{\mathrm{Ran}} \simeq \mathfrak{L}(\mathrm{Tot}(\omega))_{\mathrm{Ran}},$$

where $\mathrm{Tot}(\omega)$ as the total space of ω, viewed as a group-scheme over X.

Note that have a canonically defined homomorphism

$$\mathfrak{L}(\mathbb{G}_a)^{\omega}_{\mathrm{Ran}} \to \mathbb{G}_a \times \mathrm{Ran},$$

given by taking the residue.

1.5. **Marked point version.** Factorization structures in this paper will appear as a device to introduce representation-theoretic categories of interest. However, the actual representation theory will occur at (or around) a given point x of the curve X.

In this subsection we will introduce versions of the factorization spaces considered above but *with the marked point* x.

This will eventually allow us to talk about *factorization modules* (at x) for a given *factorization algebra*.

1.5.1. The marked point version of the Ran space, denoted Ran_x is defined as follows.

For a test scheme S, the space $\mathrm{Maps}(S, \mathrm{Ran}_x)$ is the set of finite subsets $\mathcal{I} \subset \mathrm{Maps}(S, X)$ with a distinguished element i_x, which corresponds to the constant map

$$S \to \mathrm{pt} \overset{x}{\to} X. \tag{1.5}$$

We have the natural forgetful map

$$\mathrm{Ran}_x \to \mathrm{Ran}$$

as well as a map

$$\mathrm{Ran} \to \mathrm{Ran}_x,$$

obtained by adding the element i_x.

The semi-group Ran acts on Ran_x by the operation of union of finite sets, i.e., for a finite set J we have a map

$$\mathrm{Ran}^J \times \mathrm{Ran}_x \to \mathrm{Ran}_x \tag{1.6}$$

1.5.2. For a finite set J, let

$$(\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}} \subset \mathrm{Ran}^J \times \mathrm{Ran}_x$$

be equal to the preimage of

$$(\mathrm{Ran}^{J \sqcup *})_{\mathrm{disj}} \subset \mathrm{Ran}^{J \sqcup *}$$

under the forgetful map

$$\mathrm{Ran}^J \times \mathrm{Ran}_x \to \mathrm{Ran}^J \times \mathrm{Ran} = \mathrm{Ran}^{J \sqcup *}.$$

The map

$$(\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}} \to \mathrm{Ran}_x, \tag{1.7}$$

induced by (1.6), is étale.

1.5.3. Let Z_{Ran} be a factorization space over Ran. By a *factorization module space* with respect to Z_{Ran} we will mean a prestack

$$Z_{\mathrm{Ran}_x} \to \mathrm{Ran}_x,$$

equipped with a system of identifications (factorization isomorphisms)

$$Z_{\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} (\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}} \simeq (Z_{\mathrm{Ran}_x} \times Z^J_{\mathrm{Ran}}) \underset{\mathrm{Ran}_x \times \mathrm{Ran}^J}{\times} (\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}}, \tag{1.8}$$

that are compatible in the natural sense.

In what follows we will denote by Z_x the fiber of Z_{Ran_x} over $\{x\} \in \mathrm{Ran}_x$.

1.5.4. *An example.* Let Z_{Ran} be a factorization space over Ran. We can produce from it a factorization module space by setting

$$Z_{\mathrm{Ran}_x} := \mathrm{Ran}_x \underset{\mathrm{Ran}}{\times} Z_{\mathrm{Ran}}.$$

1.5.5. In this way, from the factorization spaces discussed in Sect. 1.3 we obtain their marked point versions, denoted

$$\mathfrak{L}^+(G)_{\mathrm{Ran}_x},\ \mathfrak{L}(G)_{\mathrm{Ran}_x},\ \mathrm{Gr}_{G,\mathrm{Ran}_x}.$$

along with their twists

$$\mathfrak{L}^+(G)^{\omega^\rho}_{\mathrm{Ran}_x},\ \mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}_x},\ \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x},$$

etc.

Remark 1.5.6. We should point out that there are many more examples of factorization module spaces that *do not* arise by the construction of Sect. 1.5.4.

For example, if we take the factorization space $\mathrm{Gr}_{G,\mathrm{Ran}}$, we can create a factorization module space $\mathrm{Fl}_{G,\mathrm{Ran}_x}$ by assigning to $\mathcal{I}$ the set of triples $(\mathcal{P}_G,\alpha,\beta)$, where $(\mathcal{P}_G,\alpha)$ are as in the definition of $\mathrm{Gr}_{G,\mathrm{Ran}_x}$, and β is the datum of reduction of the restriction of $\mathcal{P}_G$ at $S\times x$ to B.

1.5.7. Let $\mathfrak{L}^+(G)_{\mathrm{Ran}_x,\infty\cdot x}$ denote the following factorizatiom module space with respect to $\mathfrak{L}(G)_{\mathrm{Ran}}$. It assigns to $\mathcal{I}$ the datum of a map

$$(\mathcal{D}_{\mathcal{I}} - S\times x)\to G.$$

Note that the fiber $\mathfrak{L}^+(G)_{x,\infty\cdot x}$ of $\mathfrak{L}^+(G)_{\mathrm{Ran}_x,\infty\cdot x}$ over $x\in\mathrm{Ran}_x$ identifies with $\mathfrak{L}(G)_x$.

Note also that restriction to

$$\overset{\circ}{\mathcal{D}}_x \hookrightarrow (\mathcal{D}_{\mathcal{I}} - S\times x)$$

(where $\overset{\circ}{\mathcal{D}}_x$ is the pictured disc corresponding to the constant map (1.5)) defines a map

$$\mathfrak{L}^+(G)_{x,\infty\cdot x}\to\mathfrak{L}(G)_x. \tag{1.9}$$

The inclusion

$$(\mathcal{D}_{\mathcal{I}} - S\times x)\hookrightarrow \overset{\circ}{\mathcal{D}}_{\mathcal{I}}$$

defines a closed embedding

$$\mathfrak{L}^+(G)_{\mathrm{Ran}_x,\infty\cdot x}\to\mathfrak{L}(G)_{\mathrm{Ran}_x}.$$

1.5.8. Consider the double quotient

$$\mathfrak{L}^+(G)\backslash\mathfrak{L}^+(G)_{x,\infty\cdot x}/\mathfrak{L}^+(G). \tag{1.10}$$

On the other hand, we can view it as a closed sunfunctor in $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}_x}$, and as such, as a groupoid acting on $\mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash\mathrm{Ran}_x$.

On the other hand, the map (1.9) gives rise to the following commutative diagram in which both squares are Cartesian:

$$\begin{tikzcd}
\mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash\mathrm{Ran}_x \arrow[d] & \mathfrak{L}^+(G)_{\mathrm{Ran}_x,\infty\cdot x} \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] \arrow[d] & \mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash\mathrm{Ran}_x \arrow[d] \\
\mathfrak{L}^+(G)_x\backslash\,\mathrm{pt} & \mathrm{Hecke}^{\mathrm{loc}}_{G,x} \arrow[l, "\overleftarrow{h}"'] \arrow[r, "\overrightarrow{h}"] & \mathfrak{L}^+(G)_x\backslash\,\mathrm{pt},
\end{tikzcd}$$

where the maps $\mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash\mathrm{Ran}_x\to\mathfrak{L}^+(G)_x\backslash\,\mathrm{pt}$ are given by restricting bundles along

$$\mathcal{D}_x\hookrightarrow\mathcal{D}_{\mathcal{I}}.$$

So we can view (1.10) as incarnating the lift of the action of $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ on $\mathfrak{L}^+(G)_x\backslash\,\mathrm{pt}$ to that on $\mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash\mathrm{Ran}_x$.

1.6. **Technical detour: unital structures.** The discussion of *unitality* for factorization structures will be largely suppressed in this work. However, for technical purposes that have to do with the construction of the semi-infinite IC sheaf, we will need to include a brief discussion.

The idea of "unitality" in the context of spaces over Ran is to account for the following additional structure: we can talk not only about *equality* of two finite subsets of $\mathrm{Hom}(S, X)$, but also about *containment* of one subset in another.

1.6.1. Let

$$(\mathrm{Ran} \times \mathrm{Ran})^{\subset} \subset \mathrm{Ran} \times \mathrm{Ran}$$

be the following subfunctor:

A point $(\mathcal{I}, \mathcal{I}') \in \mathrm{Hom}(S, \mathrm{Ran})$ belongs to $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$ if $\Gamma_{\mathcal{I}}$ is set-theoretically contained in $\Gamma_{\mathcal{I}'}$.

Remark 1.6.2. Note that the substack $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$ as defined above is larger than the substack denoted by the same symbol in [Ga7, 4.1.1]. However, this is largely immaterial for our purposes: the above inclusion induces an equivalence on categories of sheaves as it is surjective in the topology generated by finite surjective maps.

1.6.3. Note that the diagonal map $\Delta_{\mathrm{Ran}} : \mathrm{Ran} \to \mathrm{Ran} \times \mathrm{Ran}$ factors through $(\mathrm{Ran} \times \mathrm{Ran})^{\subset}$.

Let φ_{small} and φ_{big} be the two maps

$$(\mathrm{Ran} \times \mathrm{Ran})^{\subset} \to \mathrm{Ran}$$

that remember $\mathcal{I}$ and $\mathcal{I}'$, respectively.

The following is established in [Ga7, Lemma 4.1.2]:

Lemma 1.6.4.

(a) *For any prestack* $\mathcal{Y} \to \mathrm{Ran}$*, pullback with respect to the base-changed map*

$$\mathrm{id}_{\mathcal{Y}} \times \varphi_{\mathrm{small}} : \mathcal{Y} \underset{\mathrm{Ran}, \varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset} \to \mathcal{Y}$$

induces an equivalence on categories of étale gerbes.

(b) *For any gerbe* $\mathcal{G}$ *on* $\mathcal{Y}$*, the functor of !-pullback along* $\mathrm{id}_{\mathcal{Y}} \times \varphi_{\mathrm{small}}$ *induces a fully faithful functor*

$$\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}) \to \mathrm{Shv}_{\mathcal{G}}(\mathcal{Y} \underset{\mathrm{Ran}, \varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset}).$$

1.6.5. Let Z_{Ran} be a prestack over Ran. By a unital structure on Z_{Ran} we will mean a map

$$\varphi_{\mathrm{big}} : Z_{\mathrm{Ran}} \underset{\mathrm{Ran}, \varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset}) \to Z_{\mathrm{Ran}},$$

which makes the following diagram commute

$$\begin{tikzcd} Z_{\mathrm{Ran}} \underset{\mathrm{Ran}, \varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset}) \arrow[r, "\varphi_{\mathrm{big}}"] \arrow[d] & Z_{\mathrm{Ran}} \arrow[d] \\ (\mathrm{Ran} \times \mathrm{Ran})^{\subset} \arrow[r, "\varphi_{\mathrm{big}}"] & \mathrm{Ran}. \end{tikzcd}$$

We also require that φ_{big} be associative in a natural sense, and that the composite

$$Z_{\mathrm{Ran}} \overset{\mathrm{id} \times \Delta_{\mathrm{Ran}}}{\longrightarrow} Z_{\mathrm{Ran}} \underset{\mathrm{Ran}, \varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset}) \overset{\varphi_{\mathrm{big}}}{\longrightarrow} Z_{\mathrm{Ran}}$$

be the identity map.

1.6.6. From Lemma 1.6.4(a) we obtain:

Corollary 1.6.7. *For a gerbe $\mathcal{G}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ we have a canonical isomorphism*

$$\varphi_{\mathrm{big}}^*(\mathcal{G}) \simeq \varphi_{\mathrm{small}}^*(\mathcal{G}),$$

uniquely characterized by the requirement that the composite

$$\mathcal{G} \simeq \Delta_{\mathrm{Ran}}^* \circ \varphi_{\mathrm{big}}^*(\mathcal{G}) \simeq \Delta_{\mathrm{Ran}}^* \circ \varphi_{\mathrm{small}}^*(\mathcal{G}) \simeq \mathcal{G}$$

is the identity map.

1.6.8. If Z_{Ran} has a factorization structure over Ran, then $Z_{\mathrm{Ran}} \underset{\mathrm{Ran},\varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset}$, viewed as mapping to Ran via

$$Z_{\mathrm{Ran}} \underset{\mathrm{Ran},\varphi_{\mathrm{small}}}{\times} (\mathrm{Ran} \times \mathrm{Ran})^{\subset} \to (\mathrm{Ran} \times \mathrm{Ran})^{\subset} \xrightarrow{\varphi_{\mathrm{big}}} \mathrm{Ran}$$

also has a natural factorization structure.

This, we can talk about a unital structure being compatible with factorization.

1.6.9. Note that $\mathrm{Gr}_{G,\mathrm{Ran}}$ (or its variant $\mathrm{Gr}_{G,\mathrm{Ran}}^{\omega^\rho}$) provides an example of a factorization space with a unital structure. Indeed the map φ_{big} is defined as follows:

Let us interpret the affine Grassmannian as in Sect. 1.3.4. Then φ_{big} sends $(\mathcal{I}, \mathcal{I}', \mathcal{P}_G, \alpha)$ to $(\mathcal{I}', \mathcal{P}_G, \alpha')$, where α' is the restriction of α along

$$(S \times X - \Gamma_{\mathcal{I}}) \hookrightarrow (S \times X - \Gamma_{\mathcal{I}'}).$$

2. Geometric metaplectic data

The object of study of this work is *metaplectically twisted* sheaves on geometries attached to the loop group $\mathfrak{L}(G)$. In this section we explain what data goes into defining such a twist.

We will also discuss the phenomenon of *metaplectic Langlands duality.*

2.1. **Definition of the geometric metaplectic data.** In this subsection we recall, following [GLys], the definition of geometric metaplectic data.

2.1.1. Given a factorization space over Ran, it makes sense to talk about $\mathsf{e}^{\times,\mathrm{tors}}$-gerbes over it compatible with factorization (see [GLys, Sect. 2.2.4] or Sect. 3.1.3 below).

By a *geometric metaplectic data* for G over X we will mean a factorization gerbe over $\mathrm{Gr}_{G,\mathrm{Ran}}$. In what follows we will denote such a gerbe by $\mathcal{G}^G$. We denote the ∞-groupoid formed by them by

$$\mathrm{FactGe}_G\,.$$

The operation of tensor product defines on FactGe_G a structure of connective spectrum (i.e., commutative monoid in spaces).

2.1.2. Consider now the group-objects

$$\mathfrak{L}^+(G)_{\mathrm{Ran}} \hookrightarrow \mathfrak{L}(G)_{\mathrm{Ran}}.$$

We can talk about *multiplicative* factorization gerbes on $\mathfrak{L}(G)_{\mathrm{Ran}}$ (resp., $\mathfrak{L}^+(G)_{\mathrm{Ran}}$), i.e., factorization gerbes compatible with the group structure. According to [GLys, Proposition 7.3.5], the map

from the space of

–*Multiplicative factorization gerbes on* $\mathfrak{L}(G)_{\mathrm{Ran}}$ *equipped with a trivialization of the restriction to* $\mathfrak{L}^+(G)_{\mathrm{Ran}}$ *(as a multiplicative factorization gerbe)*

to the space of

–*Factorization gerbes on* $\mathrm{Gr}_{G,\mathrm{Ran}}$,

given by descent along $\mathfrak{L}(G)_{\mathrm{Ran}} \to \mathrm{Gr}_{G,\mathrm{Ran}}$, is an equivalence.

Thus, given a geometric metaplectic data, we obtain a multiplicative factorization gerbe on $\mathfrak{L}(G)_{\mathrm{Ran}}$, also denoted $\mathcal{G}^G$, equipped with a trivialization of its restriction to $\mathfrak{L}^+(G)_{\mathrm{Ran}}$.

By construction, the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ is twisted-equivariant with respect to the action of $\mathfrak{L}(G)_{\mathrm{Ran}}$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ against the multiplicative gerbe $\mathcal{G}^G$ on $\mathfrak{L}(G)_{\mathrm{Ran}}$.

2.1.3. Since the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ is $\mathfrak{L}^+(G)_{\mathrm{Ran}}$-equivariant, it descends to a factorization gerbe on $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ that we will denote by $\mathcal{G}^{G,G,\mathrm{ratio}}$.

Note that the involution on $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ from Sect. 1.3.7 turns $\mathcal{G}^{G,G,\mathrm{ratio}}$ to $(\mathcal{G}^{G,G,\mathrm{ratio}})^{-1}$.

2.1.4. Recall the twisted version of $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ of $\mathrm{Gr}_{G,\mathrm{Ran}}$ from Sect. 1.4.4. We have a projection

$$\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \to \mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}\,.$$

Pulling back $\mathcal{G}^{G,G,\mathrm{ratio}}$ with respect to this projection gives rise to a factorization gerbe on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$. We will denote it by the same character $\mathcal{G}^G$.

Remark 2.1.5. There should be no danger of confusing $\mathcal{G}^G$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ and $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$, as they live on different spaces. See, however, Remark 1.4.6.

2.1.6. Assume for a moment that $G=T$ is a torus. Here is explicit description of the fibers of the gerbe $\mathcal{G}^T$ on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$:

For a point $x\in X$ and the point $t^\lambda\in \mathrm{Gr}^{\omega^\rho}_{T,x}$, the fiber $\mathcal{G}^T|_{t^\lambda}$ identifies with

$$\mathcal{G}^T_{\lambda\cdot x}\otimes (\omega_x^{\otimes\frac{1}{2}})^{b(\lambda,2\rho)}, \tag{2.1}$$

where:

- $\mathcal{G}^T_{\lambda\cdot x}$ is the fiber of the gerbe $\mathcal{G}^T$ on $\mathrm{Gr}_{T,x}$ at the point $t^\lambda\in \mathrm{Gr}_{T,x}$;
- $\omega_x^{\otimes\frac{1}{2}}$ is the fiber of $\omega^{\otimes\frac{1}{2}}$ ar $x\in X$;
- $b:\Lambda\times\Lambda\to \mathsf{e}^\times(-1)$ is the symmetric bilinear form associated to $\mathcal{G}^T$, see Sect. 2.2.2 below.

In other words, the passage $\mathrm{Gr}_{T,x}\rightsquigarrow \mathrm{Gr}^{\omega^\rho}_{T,x}$ results in the additional factor isomorphic to $(\omega_x^{\otimes\frac{1}{2}})^{b(\lambda,2\rho)}$.

2.1.7. The gerbe $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ can be also seen as obtained from the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}_{G,\mathrm{Ran}}$ by applying the twisting construction using the $\mathfrak{L}^+(T)_{\mathrm{Ran}}$-action by means of the $\mathfrak{L}^+(T)_{\mathrm{Ran}}$-torsor $\omega^\rho_{\mathrm{Ran}}$, see Sect. 1.4.2.

This twisting construction produces also a multiplicative factorization gerbe (still denoted $\mathcal{G}^G$) on $\mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}}$, equipped with the trivialization of its restriction to $\mathfrak{L}^+(G)^{\omega^\rho}_{\mathrm{Ran}}$.

The gerbe $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ is twisted-equivariant with respect to the action of $\mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}}$ on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ against the multiplicative gerbe $\mathcal{G}^G$ on $\mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}}$.

2.1.8. Assume for a moment that X is complete. I this case we can consider the algebraic stack Bun_G classifying G-bundles on X.

Consider the projection

$$\mathrm{Gr}_{G,\mathrm{Ran}}\to \mathrm{Bun}_G\,.$$

According to [GLys, Sect. 2.3.5], *any* gerbe on $\mathrm{Gr}_{G,\mathrm{Ran}}$ uniquely descends to a gerbe on Bun_G. We will denote by the same character $\mathcal{G}^G$ the resulting gerbe on Bun_G.

Remark 2.1.9. Note that we also have a map

$$\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}\to \mathrm{Bun}_G\,.$$

The pullback of $\mathcal{G}^G$ along this map differs from the gerbe we denoted $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ by tensoring by $\mathcal{G}^G|_{\omega^\rho}$.

2.2. **The case of tori.** In this subsection we will take $G=T$ to be a torus. We will analyze some explicit combinatorial and geometric objects attached to a geometric metaplectic data for T.

2.2.1. According to [GLys, Sect. 4.2], to a geometric metaplectic data $\mathcal{G}^T$ for T one attaches a quadratic form q on Λ with values in $\mathsf{e}^{\times,\mathrm{tors}}(-1)$.

According to [GLys, Sect. 4.2.10], the space of geometric metaplectic data for which q is trivial is canonically isomorphic to the space of gerbes on X with respect to the group

$$\mathrm{Hom}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}) \simeq \check{T}(\mathsf{e})^{\mathrm{tors}},$$

where $\check{T}$ is the Langlands dual torus of T, thought of as an algebraic group over e.

Thus, we have a fiber sequence of spectra

$$0 \to \mathrm{Ge}(X, \check{T}(\mathsf{e})^{\mathrm{tors}}) \to \mathrm{FactGe}_T \to \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \to 0.$$

2.2.2. Let

$$b : \Lambda \otimes \Lambda \to \mathsf{e}^{\times,\mathrm{tors}}(-1)$$

denote the symmetric bilinear form associated with q.

Let $\Lambda^\sharp \subset \Lambda$ denote the kernel of b. Let

$$T^\sharp \to T$$

be an isogenous torus whose lattice of coweights equals $\Lambda^\sharp$. Let T_H denote the Langlands dual torus of $T^\sharp$, thought of as an algebraic group over e.

2.2.3. The restriction of q to $\Lambda^\sharp$ is a *linear* map

$$\Lambda^\sharp \to \pm 1 \subset \mathsf{e}^{\times,\mathrm{tors}}(-1).$$

We can view it as an element of order 2, denoted ϵ, in T_H.

2.2.4. Let $\mathcal{G}^{T^\sharp}$ be the geometric metaplectic data for $T^\sharp$ obtained from $\mathcal{G}^T$ by pulling back along

$$\mathrm{Gr}_{T^\sharp,\mathrm{Ran}} \to \mathrm{Gr}_{T,\mathrm{Ran}}.$$

Note that since $T^\sharp$ is commutative, the factorization space $\mathrm{Gr}_{T^\sharp,\mathrm{Ran}}$ carries a group structure over Ran. Hence, we can talk about (factorization) gerbes on $\mathrm{Gr}_{T^\sharp,\mathrm{Ran}}$ equipped with a multiplicative structure.

The following is established in [GLys, Proposition 4.3.2 and Sect. 4.5]:

Proposition 2.2.5.

(a) *The factorization gerbe $\mathcal{G}^{T^\sharp}$ on $\mathrm{Gr}_{T^\sharp,\mathrm{Ran}}$ carries a uniquely defined multiplicative structure.*

(b) *To $\mathcal{G}^{T^\sharp}$ one can canonically attach a geometric metaplectic data $\mathcal{G}_0^{T^\sharp}$ for $T^\sharp$ with a vanishing quadratic form, such that $\mathcal{G}^{T^\sharp}$ and $\mathcal{G}_0^{T^\sharp}$ are isomorphic as multiplicative structure (without the factorization structure).*

Remark 2.2.6. The discrepancy between the factorization structures on $\mathcal{G}^{T^\sharp}$ and $\mathcal{G}_0^{T^\sharp}$ is controlled by the element ϵ of Sect. 2.2.3, see [GLys, Sects. 4.5 and 4.6].

2.2.7. Let $\mathcal{G}^T$ and $\mathcal{G}_0^{T^\sharp}$ be as in Proposition 2.2.5. According to Sect. 2.2.1, to $\mathcal{G}_0^{T^\sharp}$ we can canonically attach a $T_H(\mathsf{e})^{\mathrm{tors}}$-gerbe on X, denoted $\mathcal{G}_{T_H}$.

2.3. **The metaplectic dual datum.** Let $\mathcal{G}^G$ be a geometric metaplectic data. Following [GLys, Sect. 6.3], we will now attach to it a *metaplectic Langlands dual datum*, which is triple $(H, \mathcal{G}_H, \epsilon)$, as explained below.

2.3.1. Consider the diagram

$$\begin{array}{ccc}\mathrm{Gr}_{B,\mathrm{Ran}} & \longrightarrow & \mathrm{Gr}_{G,\mathrm{Ran}} \\ \downarrow & & \\ \mathrm{Gr}_{T,\mathrm{Ran}}. & & \end{array}$$

It is easy to see that any (factorization) gerbe on $\mathrm{Gr}_{B,\mathrm{Ran}}$ comes as pullback from a uniquely defined (factorization) gerbe on $\mathrm{Gr}_{T,\mathrm{Ran}}$. Thus, restricting $\mathcal{G}^G$ to $\mathrm{Gr}_{B,\mathrm{Ran}}$, we obtain a geometric metaplectic data $\mathcal{G}^T$ for T. This operation defines a map of (commutative monoids in) spaces:

$$\mathrm{FactGe}_G \to \mathrm{FactGe}_T\,. \tag{2.2}$$

Remark 2.3.2. Our convention here is sightky different from one in [GLys, Sect. 5] by a certain sign gerbe, which will be irrelevant for the purposes of this work.

2.3.3. Consider the resulting quadratic form q on Λ, see Sect. 2.2.1. One shows (see [GLys, Sect. 3.3]) that q is Weyl group invariant and *restricted*, i.e., belongs to the subset

$$\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}})^W_{\mathrm{restr}} \subset \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}),$$

introduced in [GLys, Sect. 3.2.2]. Let us recall what this means:

A W-invariant quadratic form q on Λ is restricted if for any coroot α and any $\lambda \in \Lambda$, we have

$$b(\alpha,\lambda) = q(\alpha)^{\langle \lambda,\check{\alpha}\rangle}. \tag{2.3}$$

According to [GLys, Corollary 3.3.5], we have a commutative diagram of fiber sequence of spectra

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}}) & \longrightarrow & \mathrm{FactGe}_G & \longrightarrow & \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Ge}(X, \check{T}(\mathsf{e})^{\mathrm{tors}}) & \longrightarrow & \mathrm{FactGe}_T & \longrightarrow & \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) & \longrightarrow & 0, \end{array} \tag{2.4}$$

where $Z_{\check{G}}$ is the center of the Langlands dual $\check{G}$ of G. Note that the group $Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}}$ can be also thought as

$$\mathrm{Hom}(\pi_{1,\mathrm{alg}}(G), \mathsf{e}^{\times,\mathrm{tors}}(-1)),$$

where $\pi_{1,\mathrm{alg}}(G)$ is the algebraic fundamental group of G.

2.3.4. Let T_H be the torus associated to $\mathcal{G}^T$, see Sect. 2.2.2. The first ingredient in the triple $(H, \mathcal{G}_{Z_H}, \epsilon)$, namely H, is a reductive group over the field of coefficients e with maximal torus T_H. We will now specify its root datum.

As was just mentioned, the weight lattice of H equals $\Lambda^\sharp$; we will sometimes denote it also by Λ_H. In particular, we have an inclusion

$$\Lambda_H \subset \Lambda,$$

which is a rational isomorphism.

The set of roots (resp., positive roots, simple roots) of H is in bijection with those of G. For a coroot α of G let $q_\alpha \in \mathsf{e}^\times(-1)$ denote the element $q(\alpha)$. Let ℓ_α denote the order of q_α.

Definition 2.3.5. *We will say that a geometric metaplectic datum is* non-degenerate *of $\ell_\alpha \neq 1$ for all α.*

If α is a simple root α_i for $i \in I$ we will simply write q_i instead of q_{α_i}. By W-invariance, the above non-degeneracy condition is equivalent to the non-triviality of all q_i.

2.3.6. Set
$$\alpha_H = \ell_\alpha \cdot \alpha \in \Lambda \text{ and } \check{\alpha}_H = \frac{\check{\alpha}}{\ell_\alpha} \in \mathbb{Q} \underset{\mathbb{Z}}{\otimes} \check{\Lambda}.$$

According to [GLys, Sect. 6.1], we have:
$$\alpha_H \in \Lambda_H \text{ and } \check{\alpha}_H \in \check{\Lambda}_H,$$
and the quadruple
$$(\Lambda_H, \check{\Lambda}_H, \{\alpha_H\}, \{\check{\alpha}_H\})$$
forms a root datum of a reductive group. This is the sought-for group H.

2.3.7. The second component in the triple $(H, \mathcal{G}_{Z_H}, \epsilon)$ is $\mathcal{G}_{Z_H}$. This is a gerbe on X with respect to $Z_H(\mathsf{e})^{\text{tors}}$, where Z_H is the center of H:

One shows (see [GLys, Sect. 6.2]) that the $T_H(\mathsf{e})^{\text{tors}}$-gerbe $\mathcal{G}_{T_H}$ of Sect. 2.2.7 is induced from a canonically defined $Z_H(\mathsf{e})^{\text{tors}}$-gerbe $\mathcal{G}_{Z_H}$ along the inclusion $Z_H \hookrightarrow T_H$.

2.3.8. In the rest of this paper we will choose a trivialization of the fiber $\mathcal{G}_{Z_H,x}$ of $\mathcal{G}_{Z_H}$ at the chosen point $x \in X$. This choice is made in order to unburden the notation.

The trivialization of $\mathcal{G}_{Z_H,x}$ induces a trivialization of the $T_H(\mathsf{e})^{\text{tors}}$-gerbe $\mathcal{G}_{T_H,x} := \mathcal{G}_{T_H}|_x$.

2.3.9. The last component in the triple $(H, \mathcal{G}_{Z_H}, \epsilon)$ is the element $\epsilon \in T_H$ from Sect. 2.2.3. One shows that this element actually belongs to $Z_H \subset T_H$.

That said, the above element ϵ will not play any role in the present work.

2.4. **Metaplectic geometric Satake.** Recall that the (usual) geometric Satake, in its weak form, is a monoidal functor
$$\operatorname{Rep}(\check{G}) \to \operatorname{Sph}_x(G) := \operatorname{Shv}(\operatorname{Gr}_{G,x})^{\mathfrak{L}^+(G)_x}.$$

In this subsection we will recall, following [GLys, Sect. 9], its metaplectic counterpart.

2.4.1. We fix a point $x \in X$. We define the metaplectic spherical category, denoted $\operatorname{Sph}_{q,x}(G)$ to be
$$\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_{G,x})^{\mathfrak{L}^+(G)_x}.$$

We regard it as equipped with a monoidal structure, given by convolution.

2.4.2. The category $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_{G,x})^{\mathfrak{L}^+(G)_x}$ carries the (perverse) t-structure, and the convolution functor
$$\operatorname{Sph}_{q,x}(G) \otimes \operatorname{Sph}_{q,x}(G) \to \operatorname{Sph}_{q,x}(G)$$
is t-exact.

2.4.3. Let $\operatorname{Rep}(H)$ be the DG category of representations of H, defined, e.g., as the category of quasi-coherent sheaves on the algebraic stack (over e) BH.

According to [GLys, Sect. 9.2], there exists a canonically defined monoidal functor
$$\operatorname{Sat}_{q,G} : \operatorname{Rep}(H) \to \operatorname{Sph}_{q,x}(G)$$
(here we are using the trivialization of the gerbe $\mathcal{G}_{Z_H,x}$, see Sect. 2.3.8).

In what follows we will discuss some of properties of the functor $\operatorname{Sat}_{q,G}$ that we will use in the future.

2.4.4. Let us first take $G = T$ to be a torus. Consider the corresponding torus $T^\sharp$. Since the $\mathfrak{L}^+(T^\sharp)_x$-action on $\mathrm{Gr}_{T^\sharp,x}$ is trivial, we obtain a canonically defined functor

$$\mathrm{Shv}(\mathrm{Gr}_{T^\sharp,x}) \to \mathrm{Shv}(\mathrm{Gr}_{T^\sharp,x})^{\mathfrak{L}^+(T^\sharp)_x}. \tag{2.5}$$

The functor $\mathrm{Sat}_{q,T}$ is the composition

$$\mathrm{Rep}(T_H) \simeq \mathrm{Shv}(\mathrm{Gr}_{T^\sharp,x}) \overset{(2.5)}{\longrightarrow} \mathrm{Shv}(\mathrm{Gr}_{T^\sharp,x})^{\mathfrak{L}^+(T^\sharp)_x} \simeq$$
$$\simeq \mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp,x})^{\mathfrak{L}^+(T^\sharp)_x} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,x})^{\mathfrak{L}^+(T)_x} =: \mathrm{Sph}_{q,x}(T),$$

where:

- The equivalence $\mathrm{Shv}(\mathrm{Gr}_{T^\sharp,x})^{\mathfrak{L}^+(T^\sharp)_x} \simeq \mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp,x})^{\mathfrak{L}^+(T^\sharp)_x}$ is given by the trivialization of $\mathcal{G}_{T_H,x}$;
- The functor $\mathrm{Shv}_{\mathcal{G}^{T^\sharp}}(\mathrm{Gr}_{T^\sharp,x})^{\mathfrak{L}^+(T^\sharp)_x} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,x})^{\mathfrak{L}^+(T)_x}$ is given by direct image along $\mathfrak{L}^+(T^\sharp)_x\backslash\mathrm{Gr}_{T^\sharp,x} \to \mathfrak{L}^+(T)_x\backslash\mathrm{Gr}_{T,x}$.

2.4.5. Let now G be general.

Due to the trivialization of $\mathcal{G}_{T_H,x}$, the gerbe $\mathcal{G}^G$ is trivialized when restricted to the orbits

$$S^\gamma = \mathfrak{L}(N)_x \cdot t^\gamma \subset \mathrm{Gr}_{G,x}, \quad \gamma \in \Lambda^\sharp.$$

Hence, for $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$ it makes sense to consider

$$\Gamma(S^\gamma, \mathcal{F}|_{S^\gamma}) \in \mathrm{Vect}\,.$$

The compatibility of metaplectic geometric Satake with Jacquet functors (see [GLys, Sect. 9.4.3]) implies:

$$\Gamma(S^\gamma, \mathrm{Sat}_{q,G}(V))[-\langle\gamma, 2\check{\rho}\rangle] \simeq (\mathrm{Res}^H_{T_H}(V))(\gamma), \quad V \in \mathrm{Rep}(H), \tag{2.6}$$

where (γ) means weight component γ.

2.4.6. The trivialization of $\mathcal{G}^G|_{S^\gamma}$ in particular implies that the fiber of $\mathcal{G}^G$ at t^γ is trivialized. Hence, the object

$$\mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G} \in \mathrm{Sph}_{q,x}(G)$$

is well-defined.

It follows from (2.6) that we have a *canonical* identification

$$\mathrm{Sat}_{q,G}(V^\gamma) \simeq \mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}, \tag{2.7}$$

where V^γ is the irreducible representation H of highest weight γ *with a trivialized highest weight line.*

In particular, the functor $\mathrm{Sat}_{q,G}$ is t-exact.

Remark 2.4.7. The functor $\mathrm{Sat}_{q,G}$ is *not* an equivalence, but it *does* induce an equivalence of monoidal abelian categories

$$(\mathrm{Rep}(H))^\heartsuit \to (\mathrm{Sph}_{q,x}(G))^\heartsuit.$$

Remark 2.4.8. Note that the isomorphisms (2.6) and (2.7) give rise to canonical trvializations of the *lowest weight lines* in each V^γ:

$$\mathsf{e} \simeq \Gamma(S^{w_0(\gamma)}, \mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G})[\langle\gamma, 2\check{\rho}\rangle] \simeq V^\gamma(w_0(\gamma)).$$

This system of trivializations corresponds to a canonically defined representative

$$\mathsf{w}_0 \in \mathrm{Norm}_H(T_W)$$

of the longest element of the Weyl group $w_0 \in W_H$. The element w_0 is characterized by the property that it makes the diagrams

$$\begin{tikzcd} \mathsf{e} \arrow[r, "\simeq"] \arrow[d, "\mathrm{id}"'] & V^\gamma(\gamma) \arrow[d, "\mathsf{w}_0"] \\ \mathsf{e} \arrow[r, "\simeq"] & V^\gamma(w_0(\gamma)) \end{tikzcd}$$

commute.

2.5. **Metaplectic geometric Satake and Verdier duality.** One of the crucial steps in the proof of our main theorem depends on a Verdier duality manipulation. In order to do this we will need to study how metaplectic geometric Satake interacts with Verdier duality, and this is the subject of the present subsection.

2.5.1. We note that inversion on $\mathfrak{L}(G)_x$ (or, which is equivalent, the involution on $\mathrm{Hecke}^{\mathrm{loc}}_{G,\mathrm{Ran}}$ from Sect. 1.3.7), defines an equivalence

$$\mathrm{inv}^G : \mathrm{Sph}_{q,x}(G) \to \mathrm{Sph}_{q^{-1},x}(G),$$

which *reverses* the monoidal structures.

Consider the Verdier duality functor

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Sph}_{q,x}(G)^c)^{\mathrm{op}} \to \mathrm{Sph}_{q^{-1},x}(G)^c.$$

It follows from the definitions that the composite

$$\mathbb{D}^{\mathrm{Verdier}} \circ \mathrm{inv}^G : (\mathrm{Sph}_{q,x}(G)^c)^{\mathrm{op}} \to \mathrm{Sph}_{q,x}(G)^c$$

is the functor of *monoidal dualization* on $\mathrm{Sph}_{q,x}(G)^c$.

Remark 2.5.2. Note that the functor $\mathbb{D}^{\mathrm{Verdier}} \circ \mathrm{inv}^G$ sends $\mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}$ to $\mathrm{IC}_{q^{-1},\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G}$. Combined with (2.7), this implies a *canonical* identification

$$(V^\gamma)^* \simeq V^{-w_0(\gamma)}. \tag{2.8}$$

In particular, (2.8) implies that each V^γ has a canonically trivialized lowest weight line. However, it is easy to see that this is the same trivialization as the one specified in Remark 2.4.8.

2.5.3. It follows from the constructions that if $(H, \mathcal{G}_{Z_H}, \epsilon)$ is the metaplectic dual datum for $\mathcal{G}^G$, then the one corresponding to $(\mathcal{G}^G)^{-1}$ is given by $(H, (\mathcal{G}_{Z_H})^{-1}, \epsilon)$.

In particular, a trivialization of the gerbe $\mathcal{G}_{Z_H,x}$ (see Sect. 2.3.8) induces a trivialization of the corresponding gerbe arising from $\mathcal{G}^{-1}$.

In particular, we obtain a geometric Satake functor

$$\mathrm{Sat}_{q^{-1},G} : \mathrm{Rep}(H) \to \mathrm{Sph}_{q^{-1},x}(G).$$

2.5.4. We normalize the Cartan involution τ^H on a reductive group H with chosen Cartan and Borel subgroups $T_H \subset B_H$ so that it acts as inversion on T_H and swaps B_H and B_H^-. We have a commutative diagram

$$\begin{tikzcd} T_H \arrow[r] \arrow[d, "\tau^{T_H}"'] & H \arrow[d, "\tau^H"] \\ T_H \arrow[r] & H, \end{tikzcd} \tag{2.9}$$

where, according to the above conventions, τ^{T_H} is inversion on T_H.

We will denote by the same symbol τ^H the corresponding involution on $\operatorname{Rep}(H)$. We have the corresponding commutative diagram

$$
\begin{tikzcd}
\operatorname{Rep}(H) \arrow[r, "\operatorname{Res}^H_{T_H}"] \arrow[d, "\tau^H"'] & \operatorname{Rep}(T_H) \arrow[d, "\tau^{T_H}"] \\
\operatorname{Rep}(H) \arrow[r, "\operatorname{Res}^H_{T_H}"] & \operatorname{Rep}(T_H).
\end{tikzcd}
\tag{2.10}
$$

Note that we have a *canonical* identification

$$\tau^H(V^\gamma) \simeq (V^\gamma)^*. \tag{2.11}$$

Indeed, both representations are irreducible and have trivialized *lowest* weight lines.

2.5.5. Combining (2.11) with (2.7), we obtain that the following diagram of monoidal functors canonically commutes

$$
\begin{tikzcd}
(\operatorname{Rep}(H)^c)^{\mathrm{op}} \arrow[r, "(\operatorname{Sat}_{q,G})^{\mathrm{op}}"] \arrow[d, "\tau^H \circ \mathbb{D}^{\mathrm{lin}}"'] & (\operatorname{Sph}_{q,x}(G)^c)^{\mathrm{op}} \arrow[d, "\mathbb{D}^{\mathrm{Verdier}}"] \\
\operatorname{Rep}(H)^c \arrow[r, "\operatorname{Sat}_{q,G}"] & (\operatorname{Sph}_{q^{-1},x}(G))^c,
\end{tikzcd}
\tag{2.12}
$$

where $\mathbb{D}^{\mathrm{lin}}$ is the (usual) dualization functor

$$(\operatorname{Rep}(H)^c)^{\mathrm{op}} \to \operatorname{Rep}(H)^c.$$

Juxtaposing (2.12) with Sect. 2.5.1, we obtain the following commutative diagram of monoidal functors:

$$
\begin{tikzcd}
\operatorname{Rep}(H) \arrow[r, "\operatorname{Sat}_{q,G}"] \arrow[d, "\tau^H"'] & \operatorname{Sph}_{q,x}(G) \arrow[d, "\operatorname{inv}^G"] \\
\operatorname{Rep}(H) \arrow[r, "\operatorname{Sat}_{q,G}"] & \operatorname{Sph}_{q^{-1},x}(G).
\end{tikzcd}
\tag{2.13}
$$

3. Factorization algebras and modules

Our main theorem compares the twisted Whittaker category on the affine Grassmannian with the category of *factorization modules* over a cartain *factorization algebra*.

In this section we will recall the definition of these objects in the context of factorization spaces over the Ran space.

3.1. **Factorization algebras.** In this subsection we will recall the definition of factorization algebras (on factorization spaces over the Ran space).

3.1.1. Let $Z_{\mathrm{Ran}} \to \mathrm{Ran}$ be a factorization space over Ran, and let $\mathcal{G}$ be a factorization gerbe on Z_{Ran}. By a *factorization algebra* in $\operatorname{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}})$ we will mean an object $\mathcal{A} \in \operatorname{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}})$ equipped with a *homotopy compatible* system of isomorphisms

$$\mathcal{A}|_{Z_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}} \simeq \mathcal{A}^{\boxtimes J}|_{Z^J_{\mathrm{Ran}} \underset{\mathrm{Ran}^J}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}},$$

where the two spaces are identified by (1.3).

The expression "homotopy compatible" can be formalized is several different (but equivalent) ways. Below we discuss one of the possibilities (which is very close to one from [Ras1, Sect. Sect. 6]). We start with spelling out the details in the definition of the notion of *factorization space*.

3.1.2. First, consider the assignment

$$J \rightsquigarrow (\mathrm{Ran}^J)_{\mathrm{disj}} \tag{3.1}$$

as a functor

$$\mathrm{fSet}^{\mathrm{surj}} \to \mathrm{PreStk},$$

where $\mathrm{fSet}^{\mathrm{surj}}$ is the category of finite non-empty sets and surjective maps.

The operation of disjoint union makes $\mathrm{fSet}^{\mathrm{surj}}$ into a symmetric monoidal category. The functor (3.1) has a natural op-lax symmetric monoidal structure, which means that we have the natural maps

$$(\mathrm{Ran}^{J_1 \sqcup J_2})_{\mathrm{disj}} \to (\mathrm{Ran}^{J_1})_{\mathrm{disj}} \times (\mathrm{Ran}^{J_2})_{\mathrm{disj}},$$

etc.

A factorization space over Ran is an op-lax symmetric monoidal functor

$$\mathrm{fSet}^{\mathrm{surj}} \to \mathrm{PreStk}, \quad J \mapsto Z_J, \tag{3.2}$$

equipped with a natural transformation to the functor (3.1), such that the following two requirements hold:

- The map
$$Z_J \to (\mathrm{Ran}^J)_{\mathrm{disj}} \underset{\mathrm{Ran}}{\times} Z_*,$$
induced by the map $I \to *$, is an isomorphism.
- The map
$$Z_J \to (\mathrm{Ran}^J)_{\mathrm{disj}} \underset{\mathrm{Ran}^J}{\times} (Z_*)^J,$$
induced by the op-lax symmetric monoidal structure, is an isomorphism

The relation of this definition to the naive one in Sect. 1.1.4 is that

$$Z_* := Z_{\mathrm{Ran}},\ Z_I := (\mathrm{Ran}^J)_{\mathrm{disj}} \underset{\mathrm{Ran}^J}{\times} (Z_{\mathrm{Ran}})^J.$$

3.1.3. Replacing the symmetric monoidal category PreStk by that of $\mathrm{PreStk} + \mathrm{Grb}$ consisting pairs $(\mathcal{Y}, \mathcal{G})$, where $\mathcal{Y}$ is a prestack and $\mathcal{G}$ is a gerbe on $\mathcal{Y}$, we obtain the notion of factorization gerbe over a factorization space.

3.1.4. Let

$$I \mapsto (Z_I, \mathcal{G}_I) \tag{3.3}$$

be a factorization gerbe on a factorization space.

Composing with the (lax symmetric monoidal) functor

$$\mathrm{Shv} : (\mathrm{PreStk} + \mathrm{Grb})^{\mathrm{op}} \to \infty\text{-Cat}, \quad (\mathcal{Y}, \mathcal{G}) \mapsto \mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}), \quad (\mathcal{Y}_0 \xrightarrow{f} \mathcal{Y}_1) \mapsto f^!, \tag{3.4}$$

we obtain a lax symmetric monoidal functor

$$(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \infty\text{-Cat}, \quad I \mapsto \mathrm{Shv}_{\mathcal{G}_I}(Z_I). \tag{3.5}$$

We can view (3.5) as a Cartesian fibration

$$\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{fSet}^{\mathrm{surj}}}) \to \mathrm{fSet}^{\mathrm{surj}}, \tag{3.6}$$

where $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{fSet}^{\mathrm{surj}}})$ is equipped with a symmetric monoidal structure and (3.6) is a symmetric monoidal functor.

A factorization algebra in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}})$ is by definition a symmetric monoidal section of (3.6), which is Cartesian as a section of ∞-categories (i.e., sends arrows in $\mathrm{fSet}^{\mathrm{surj}}$ to arrows in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{fSet}^{\mathrm{surj}}})$ that are Cartesian with respect to (3.6)).

Remark 3.1.5. Above we gave a definition of factorization algebras *in* $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}})$, where $(Z_{\mathrm{Ran}}, \mathcal{G})$ has a factorization structure. However, for many purposes it is convenient to give a more general definition– that of factorization algebra inside a general *factorization category*. The latter will not appear explicitly in this work.

3.1.6. *Example.* Take $Z_{\mathrm{Ran}} = \mathrm{Ran}$, with its tautological structure of factorization space. Then

$$\omega_{\mathrm{Ran}} \in \mathrm{Shv}(\mathrm{Ran})$$

acquires a structure of factorization algebra.

3.2. **Functoriality of factorization algebras.** In this subsection we will study functoriality properties of factorization algebras with respect to maps of factorization spaces.

3.2.1. Let $f : Z^1_{\mathrm{Ran}} \to Z^2_{\mathrm{Ran}}$ be a map of factorization spaces. Let $\mathcal{G}^2$ be a factorization gerbe on Z^2_{Ran}, and let $\mathcal{G}^1$ be its pullback to Z^1_{Ran}, equipped with its natural factorization structure.

It is clear that the pullback functor

$$f^! : \mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}}) \to \mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}})$$

induces a functor on the corresponding categories of factorization algebras

$$f^! : \mathrm{FactAlg}(\mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}})) \to \mathrm{FactAlg}(\mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}})).$$

3.2.2. A basic example of this situation is when $Z^1_{\mathrm{Ran}} = Z_{\mathrm{Ran}}$ is an arbitrary factorization space and $Z^2_{\mathrm{Ran}} = \mathrm{Ran}$. Taking the factorization algebra $\omega_{\mathrm{Ran}} \in \mathrm{Shv}(\mathrm{Ran})$, we obtain that

$$\omega_{Z_{\mathrm{Ran}}} \in \mathrm{Shv}(Z_{\mathrm{Ran}})$$

has a natural structure of factorization algebra.

3.2.3. Let $f : Z^1_{\mathrm{Ran}} \to Z^2_{\mathrm{Ran}}$ be as before, but let us assume that f is *ind-schematic*. Then the pushforward functor

$$f_* : \mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}}) \to \mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}})$$

induces a functor on the corresponding categories of factorization algebras

$$f_* : \mathrm{FactAlg}(\mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}})) \to \mathrm{FactAlg}(\mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}})).$$

3.2.4. Let $Z^2_{\mathrm{Ran}} = Z_{\mathrm{Ran}}$ is an arbitrary factorization space and $Z^1_{\mathrm{Ran}} = \mathrm{Ran}$. Let

$$\mathrm{unit} : \mathrm{Ran} \to Z_{\mathrm{Ran}}$$

be a section of the tautological projection; assume that it is schematic as a morphism of prestacks.

We obtain that

$$\mathrm{unit}_*(\omega_{\mathrm{Ran}}) \in \mathrm{Shv}(Z_{\mathrm{Ran}})$$

has a natural structure of factorization algebra.

3.3. **Factorization modules.** We now come to a definition crucial for this work: that of factorization module over a given factorization algebra.

3.3.1. Let Z_{Ran} be a facorization space over Ran, and let $Z_{\mathrm{Ran}_x} \to \mathrm{Ran}_x$ be a factorization module space. Let $\mathcal{G}$ be a factorization gerbe on Z_{Ran}. Assume being given a gerbe $\mathcal{G}$ over Z_{Ran_x}, equipped with a factorization structure with respect to the gerbe $\mathcal{G}$ on Z_{Ran}.

Let $\mathcal{A}$ be a factorization algebra in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}})$. By a *factorization module* module in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x})$ with respect to $\mathcal{A}$ we will mean an object $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x})$ equipped with a homotopy compatible system of isomorphisms

$$\mathcal{F}|_{Z_{\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} (\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}}} \simeq (\mathcal{F} \boxtimes \mathcal{A}^{\boxtimes J})|_{(\mathrm{Ran}_x \times Z^J_{\mathrm{Ran}}) \underset{\mathrm{Ran}^J \times \mathrm{Ran}_x}{\times} ((\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}})},$$

where two spaces are identified by (1.8).

Below we give one of the possible formulations of the expression "homotopy coherence" in this context.

3.3.2. Let $\mathrm{fSet}_*^{\mathrm{surj}}$ be the category of pointed finite sets and surjective maps. We will view it as a module category over the monoidal category $\mathrm{fSet}^{\mathrm{surj}}$ under the operation of disjoint union.

We consider the functor

$$\mathrm{fSet}_*^{\mathrm{surj}} \to \mathrm{PreStk}, \quad J \mapsto (\mathrm{Ran}_*^J)_{\mathrm{disj}} := (\mathrm{Ran}^J)_{\mathrm{disj}} \underset{\mathrm{Ran}}{\times} \mathrm{Ran}_x, \tag{3.7}$$

where the map $(\mathrm{Ran}^J)_{\mathrm{disj}} \to \mathrm{Ran}$ corresponds to the element $* \in J$.

When we regard PreStk as a module category over itself, the functor (3.7) has a structure of op-lax compatibility with actions, with respect to the op-lax monoidal functor (3.1) and the above module structure on $\mathrm{fSet}_*^{\mathrm{surj}}$ over $\mathrm{fSet}^{\mathrm{surj}}$.

3.3.3. Given Z_{Ran}, thought of as a functor (3.2), a factorization module space over Z_{Ran} is a functor

$$\mathrm{fSet}_*^{\mathrm{surj}} \to \mathrm{PreStk}, \quad J \mapsto \widetilde{Z}_J,$$

equipped with a functor of op-lax compatibility with actions, and a natural transformation to (3.7), such that the following requirements hold:

- The map
$$\widetilde{Z}_J \to (\mathrm{Ran}_*^J)_{\mathrm{disj}} \underset{\mathrm{Ran}_x}{\times} \widetilde{Z}_*,$$
induced by $J \to *$, is an isomorphism.
- The map
$$\widetilde{Z}_J \to (\mathrm{Ran}_*^J)_{\mathrm{disj}} \underset{\mathrm{Ran}^{J-*}\times \mathrm{Ran}_x}{\times} (Z_*^{J-*} \times \widetilde{Z}_*),$$
induced by the structure of op-lax compatibility with actions, is an isomorphism.

The relation of this definition to the naive one is that

$$\widetilde{Z}_* := Z_{\mathrm{Ran}_x}.$$

3.3.4. Given a factorization gerbe on Z_{Ran}, we define a factorization structure on a gerbe on Z_{Ran_x}, following the recipe of Sect. 3.1.3.

3.3.5. Given a factorization algebra $\mathcal{A} \in \mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}})$, we define the notion of factorization module for it in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x})$, along the lines of Sect. 3.1.4:

Namely, composing with the functor (3.4), from Z_{Ran_x} we create a functor

$$(\mathrm{fSet}_*^{\mathrm{surj}})^{\mathrm{op}} \to \infty\text{-Cat}, \quad J \mapsto \mathrm{Shv}_{\mathcal{G}^G}(\widetilde{Z}_J),$$

which we turn into Cartesian fibration

$$\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{fSet}_*^{\mathrm{surj}}}) \to \mathrm{fSet}_*^{\mathrm{surj}}, \tag{3.8}$$

so that the category $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{fSet}_*^{\mathrm{surj}}})$ is equipped with a monoidal action of $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{fSet}^{\mathrm{surj}}})$, and the functor (3.8) is compatible with the actions with respect to the (symmetric) monoidal functor (3.6).

When we view $\mathcal{A}$ as a (symmetric) monoidal section of (3.6), a factorization module for $\mathcal{A}$ in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x})$ is a Cartesian section of (3.8), compatible with the actions.

3.3.6. We denote the category of factorization $\mathcal{A}$-modules in $\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x})$ by $\mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x}))$, or simply $\mathcal{A}$-FactMod when no confusion is likely to occur. We let $\mathbf{oblv}_{\mathrm{Fact}}$ denote the forgetful functor

$$\mathcal{A}\text{-FactMod} \to \mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x}).$$

Remark 3.3.7. Let $X' := X - x$, and let Ran' denote the Ran space of X. Note that in the definition of a factorization module space for a given factorization space Z_{Ran}, only

$$Z_{\mathrm{Ran}'} := \mathrm{Ran}' \underset{\mathrm{Ran}}{\times} Z_{\mathrm{Ran}}$$

plays a role. Indeed, for $J \in \mathrm{fSet}_*^{\mathrm{surj}}$ and $J' \in \mathrm{fSet}^{\mathrm{surj}}$, we have

$$(\mathrm{Ran}_*^{J'\sqcup J})_{\mathrm{disj}} \simeq \mathrm{Ran}'^{J'} \underset{\mathrm{Ran}^{J'}}{\times} (\mathrm{Ran}_*^{J'\sqcup J})_{\mathrm{disj}}.$$

The same remark applies to factorization gerbes and factorization algebras.

3.3.8. The first (non-zero) example of a factorization module is the so-called vacuum module: take

$$Z_{\mathrm{Ran}_x} := \mathrm{Ran}_x \underset{\mathrm{Ran}}{\times} Z_{\mathrm{Ran}}$$

and let $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}}(Z_{\mathrm{Ran}_x})$ be the pulllback of $\mathcal{A}$ itself under the forgetful map

$$Z_{\mathrm{Ran}_x} \to \mathrm{Ran}.$$

This incarnates the principle that "a commutative algebra is naturally a left module over itself".

3.3.9. *Modules for the unit.* We will now describe a particular (albeit tautological) example of construction of factorization modules. This construction will play an important role in the sequel, as it can *generate* other constructions using functoriality (see Sect. 3.4 below).

Let Z be an arbitary prestack with a gerbe $\mathcal{G}$ on it. We can regard

$$\mathrm{Ran}_x \times Z,$$

equipped with its tautological projection to Ran_x as a factorization module space with respect to the factorization space equal to Ran itself.

The pullback $\mathcal{G}|_{\mathrm{Ran}_x \times \mathcal{G}}$ has a natural factorization structure with respect to the (necessarily) trivial factorization gerbe on Ran.

Then the pullback functor

$$\mathrm{Shv}_{\mathcal{G}}(Z) \to \mathrm{Shv}_{\mathcal{G}}(\mathrm{Ran}_x \times Z)$$

naturally lifts to a functor

$$\mathrm{Shv}_{\mathcal{G}}(Z) \to \omega_{\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}}(\mathrm{Ran}_x \times Z)),$$

where we regard ω_{Ran} as a factorization algebra in Shv(Ran), see Sect. 3.2.2.

3.3.10. Here is an example of a situation where we can describe the category of factorization modules explicitly.

Let $Z_{\mathrm{Ran}'}$ be a factorization space over Ran' (see Remark 3.3.7), equipped with a factorization gerbe $\mathcal{G}'$. Let Z_x be an arbitrary prestack with a gerbe $\mathcal{G}_x$.

We define Z_{Ran_x} as follows: for an affine test scheme S and $* \in \mathcal{I} \subset \mathrm{Hom}(S, X)$, denote $\mathcal{I}' := \mathcal{I} - *$, and set

$$S \underset{\mathrm{Ran}_x}{\times} Z_{\mathrm{Ran}_x} := (S \underset{\mathcal{I}',\mathrm{Ran}}{\times} Z_{\mathrm{Ran}'}) \times Z_x.$$

We have the projections

$$Z_x \leftarrow Z_{\mathrm{Ran}_x} \to Z_{\mathrm{Ran}'},$$

and we define the gerbe $\mathcal{G}$ on Z_{Ran_x} as the tensor product of the pullbacks of $\mathcal{G}_x$ and $\mathcal{G}'$, respectively. The gerbe $\mathcal{G}$ acquires a natural structure of factorization with respect to $\mathcal{G}'$ (see Remark 3.3.7).

Let $\mathcal{A}'$ be a factorization algebra in $\mathrm{Shv}_{\mathcal{G}'}(Z_{\mathrm{Ran}'})$. By unwinding the definitions, we obtain that the functor of restriction along $Z_x \to Z_{\mathrm{Ran}_x}$ defines an equivalence

$$\mathcal{A}'\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}'}(Z_{\mathrm{Ran}_x})) \to \mathrm{Shv}_{\mathcal{G}_x}(Z_x)$$

is an equivalence.

3.4. **Functoriality properties of factorization modules.** We will now study the functoriality of the category of factorization modules under the change of factorization (module) space.

3.4.1. Let

$$
\begin{tikzcd}
Z^{1,2}_{\mathrm{Ran}} \arrow[r, "f"] \arrow[d, "g"'] & Z^1_{\mathrm{Ran}} \\
Z^2_{\mathrm{Ran}}
\end{tikzcd}
\tag{3.9}
$$

be a *correspondence* between factorization spaces, where the morphism g is ind-schematic. Let $\mathcal{G}^1$ and $\mathcal{G}^2$ be factorization gerbes on Z^1 and Z^2, respectively, equipped with an isomorphism $\mathcal{G}^1|_{Z^{1,2}} \simeq \mathcal{G}^2|_{Z^{1,2}}$ as factorization gerbes.

Then according to Sects. 3.2.1 and 3.2.3, given a factorization algebra $\mathcal{A}^1 \in \mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}})$, the object

$$\mathcal{A}^2 := g_* \circ f^!(\mathcal{A}^1) \in \mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}})$$

acquires a structure of factorization algebra in $\mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}})$.

Let now

$$
\begin{tikzcd}
Z^{1,2}_{\mathrm{Ran}_x} \arrow[r, "f"] \arrow[d, "g"'] & Z^1_{\mathrm{Ran}_x} \\
Z^2_{\mathrm{Ran}_x}
\end{tikzcd}
\tag{3.10}
$$

be a diagram of factorization module spaces for the factorization spaces appearing in (3.9). We obtain that the functor

$$g_* \circ f^! : \mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}_x}) \to \mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}_x}) \tag{3.11}$$

gives rise to a functor

$$g_* \circ f^! : \mathcal{A}^1\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}_x})) \to \mathcal{A}^2\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}_x})). \tag{3.12}$$

3.4.2. The functors $g_* \circ f^!$ inherit the usual properties of functors defined by correspondences. For example, if g is ind-proper, then $g^!$ is the right adjoint of $g_* =: g_!$. Similarly, if f is étale, then $f^! =: f^*$ is the left adjoint of f_*.

3.4.3. In addition, the fact that the map (1.7) is étale has the following consequences:

Suppose that f is étale and g is proper, and suppose that $\mathcal{F} \in \mathcal{A}^2\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}_x}))$ is such that the (partially defined) left adjoint

$$f_! \circ g^* : \mathrm{Shv}_{\mathcal{G}^2}(Z^2_{\mathrm{Ran}_x}) \to \mathrm{Shv}_{\mathcal{G}^1}(Z^1_{\mathrm{Ran}_x})$$

of (3.11) is defined on $\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{F})$.

Then the (partially defined) left adjoint $f_! \circ g^*$ of (3.12) is defined on $\mathcal{F}$, and we have

$$\mathbf{oblv}_{\mathrm{Fact}} \circ (f_! \circ g^*) \simeq (f_! \circ g^*) \circ \mathbf{oblv}_{\mathrm{Fact}}.$$

4. Configuration spaces

In the previous section we discussed factorization spaces and factorization algebras (and modules over them). However, factorization spaces over Ran are prestacks that are not even ind-schemes (such as Ran itself), and sheaves on them may be unwieldy.

In this section we will introduce another paradigm for factorization: the underlying geometry will be the (pointed) configuration space, which has the advantage of being a scheme (resp., ind-scheme).

We will see also that the affine Grassmannian for Gr_T has a closed subfunctor essentially isomorphic to Conf. This will allow us to transfer the information between the two contexts.

4.1. **Configuration space as the spaces of colored divisors.** In this subsection we introduce the configuration space.

4.1.1. Let Conf be the scheme that classifies the data of $(\Lambda^{\mathrm{neg}}-0)$-valued divisors on X, i.e., expressions of the form

$$D = \sum_k \lambda_k \cdot x_k, \tag{4.1}$$

where:

- The index k runs over some finite set;
- The points $x_k \in X$ are pairwise distinct;
- All λ_k are in $\Lambda^{\mathrm{neg}} - 0$.

4.1.2. We have:

$$\mathrm{Conf} = \bigsqcup_{\lambda \in \Lambda^{\mathrm{neg}}-0} \mathrm{Conf}^\lambda,$$

where where λ is the total degree (i.e., for a point (4.1) its total degree is $\sum_k \lambda_k$).

Each Conf^λ is isomorphic to X^λ, where for

$$\lambda = \sum_i n_i \cdot (-\alpha_i),\ \ \alpha_i \text{ are the simple coroots},\ \ n_i \in \mathbb{Z}^{\geq 0}$$

we have

$$X^\lambda = \prod_i X^{(n_i)}.$$

Remark 4.1.3. Note that if G is semi-simple and simply connected, then Conf can also be interpreted as the moduli space of non-zero homomorphisms from the monoid $\check{\Lambda}^+$ to the scheme of effective divisors

$$\mathrm{Div}^{\mathrm{eff}}(X) \simeq \underset{n\geq 0}{\sqcup} X^{(n)}.$$

4.1.4. Let

$$\overset{\circ}{\mathrm{Conf}} \subset \mathrm{Conf}$$

be the open subscheme corresponding to the condition that in (4.1) every λ_k is a negative simple coroot.

We have

$$\overset{\circ}{\mathrm{Conf}} = \bigsqcup_{\lambda \in \Lambda^{\mathrm{neg}}} \overset{\circ}{\mathrm{Conf}}{}^\lambda,$$

where each $\overset{\circ}{\mathrm{Conf}}{}^\lambda$ is isomorphic to the open subscheme

$$\overset{\circ}{X}{}^\lambda \subset X^\lambda,$$

obtained by removing the diagonal divisor.

4.1.5. The scheme Conf has a natural structure of commutative semigroup: for a finite non-empty set I we have the map

$$\mathrm{Conf}^I \to \mathrm{Conf}, \tag{4.2}$$

given by the addition of operation on $(\Lambda^{\mathrm{neg}} - 0)$-valued divisors.

4.1.6. We will denote by

$$(\mathrm{Conf}^I)_{\mathrm{disj}} \subset \mathrm{Conf}^I$$

the open subscheme given by the following condition:

The corresponding configurations $\Sigma\, \lambda^i_k \cdot x^i_k$ must have disjoint support, i.e., $x^i_k \neq x^{i'}_{k'}$ for all k, k' for every pair of indices $i \neq i'$.

Note that the map (4.2), restricted to $(\mathrm{Conf}^I)_{\mathrm{disj}}$, is étale.

4.2. **Configurations with a marked point.** In this subsection we introduce a version of Conf, where at a marked point x, we allow the value of our divisor to be any element of Λ. The resulting space $\mathrm{Conf}_{\infty\cdot x}$ will no longer be a scheme, but it will be an ind-scheme.

4.2.1. Fix a point $x \in X$. Let $\mathrm{Conf}_{\infty\cdot x}$ denote the ind-scheme classifying the data of Λ-colored divisors on X of the form

$$D = \lambda_x \cdot x + \sum_k \lambda_k \cdot x_k, \tag{4.3}$$

where:

- The index k runs over some finite set;
- The points $x_k \in X$ are pair-wise distinct as well as distinct from x;
- $\lambda_k \in \Lambda^{\mathrm{neg}} - 0$ and λ_x is an arbitrary element of Λ.

4.2.2. One can explicitly write down $\mathrm{Conf}_{\infty\cdot x}$ as follows. It equals the colimit

$$\mathrm{Conf}_{\infty\cdot x} = \underset{\mu\in\Lambda}{\underrightarrow{\mathrm{colim}}}\, \mathrm{Conf}_{\leq\mu\cdot x}, \tag{4.4}$$

where $\mathrm{Conf}_{\leq\mu\cdot x}$ is the space of those configurations (4.3) for which $\lambda_x \leq \mu$ in the standard order relation (i.e., $\mu - \lambda_x \in \Lambda^{\mathrm{pos}}$).

Each $\mathrm{Conf}_{\leq\mu\cdot x}$ is a scheme. Explicitly, it is the disjoint union

$$\mathrm{Conf}_{\leq\mu\cdot x} = \bigsqcup_{\lambda\in\mu+\Lambda^{\mathrm{neg}}} (\mathrm{Conf}_{\leq\mu\cdot x})^{\lambda},$$

where λ is the total degree (i.e., for a point (4.3) its total degree is $\lambda_x + \sum_k \lambda_k$).

For every fixed λ, we have

$$(\mathrm{Conf}_{\leq\mu\cdot x})^{\lambda} \simeq X^{\lambda-\mu}. \tag{4.5}$$

In terms of the identifications (4.5), the transition maps

$$X^{\lambda-\mu_1} \to X^{\lambda-\mu_2}$$

in forming the colimit (4.4) are given by adding the divisor $(\mu_1 - \mu_2)\cdot x$.

4.2.3. Note that Conf can be also thought of as a closed subscheme of $\mathrm{Conf}_{\infty\cdot x}$. Namely, it identifies with $\mathrm{Conf}_{x,\leq 0}$, with the connected component

$$(\mathrm{Conf}_{x,\leq 0})^0 \simeq \mathrm{pt}$$

removed.

4.2.4. The indscheme $\mathrm{Conf}_{\infty\cdot x}$ has a natural structure of module over Conf.

For a finite set I, we denote by

$$(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}} \subset \mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x}$$

the open ind-subscheme given by the following condition:

The corresponding two configurations $\Sigma\, \lambda^i_k \cdot x^i_k$ and $\lambda_x \cdot x + \Sigma\, \mu_j \cdot y_j$ must have disjoint support, i.e., $x^i_k \neq x^{i'}_{k'}$ for all k, k' every pair of indices $i \neq i'$ *and* $y_j \neq x^i_k \neq x$ for all i, j, k.

Note that the action map

$$(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}} \to \mathrm{Conf}_{\infty\cdot x} \tag{4.6}$$

is étale.

4.3. **Sheaves on configuration spaces.** In this subsection we introduce factorization gerbes and the corresponding categories of sheaves on configuration spaces.

4.3.1. Note that for a gerbe $\mathcal{G}^\Lambda$ on Conf, one can talk about a *factorization structure* on it. This means a system of isomorphisms

$$\mathcal{G}^\Lambda|_{(\mathrm{Conf}^I)_{\mathrm{disj}}} \simeq (\mathcal{G}^\Lambda)^{\boxtimes I}|_{(\mathrm{Conf}^I)_{\mathrm{disj}}}$$

for every finite set I that are compatible in the evident sense.

Given a factorization gerbe $\mathcal{G}^\Lambda$ on Conf, one can talk about a factorization structure on a gerbe $\mathcal{G}^\Lambda$ on $\mathrm{Conf}_{\infty\cdot x}$. By definition, this means a compatible system of isomorphisms

$$\mathcal{G}^\Lambda|_{(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}}} \simeq (\mathcal{G}^\Lambda)^{\boxtimes I} \boxtimes \mathcal{G}^\Lambda|_{(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}}}. \tag{4.7}$$

4.3.2. For the duration of this section we fix such a pair of factorization gerbes Λ^Λ on Conf and $\mathrm{Conf}_{\infty\cdot x}$.

We will consider the corresponding categories of sheaves

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}) \text{ and } \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

4.3.3. Being a category of sheaves on a scheme (resp., ind-scheme), the category $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ (resp., $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$) is compactly generated.

For any ind-scheme (ind-algebraic stack) $\mathcal{Y}$ with a gerbe $\mathcal{G}$ on it, let

$$\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})$$

denote the full subcategory consisting of objects $\mathcal{F}$ that satisfy the following:

- The support of $\mathcal{F}$ is a *scheme* (resp., *algebraic stack*), to be denoted $\mathcal{Y}'$;
- The restriction of $\mathcal{F}$ to every quasi-compact open subscheme (resp., substack) $\overset{\circ}{\mathcal{Y}} \subset \mathcal{Y}'$ belongs to $\mathrm{Shv}_{\mathcal{G}}(\overset{\circ}{\mathcal{Y}})$.

It is clear that

$$\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})^c \subset \mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})^{\mathrm{loc.c}}.$$

Another feature of this subcategory is that we have a well-defined Verdier duality equivalence

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Shv}_{\mathcal{G}^{-1}}(\mathcal{Y})^{\mathrm{loc.c}}.$$

Consider the corresponding full subcategory

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}).$$

Note that it consists objects that are compact when restricted to every connected component Conf^λ of Conf. Similarly, consider the full subcategory

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

Verdier duality defines equivalences

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf})^{\mathrm{loc.c}}$$

and

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}}.$$

4.4. **Translation action on colored divisors.** In this subsection we introduce a piece of structure, crucial for the rest of the paper: on action of a sublattice on $\mathrm{Conf}_{\infty\cdot x}$ by adding divisors supported at the point x.

4.4.1. Let $\Lambda^\sharp \subset \Lambda$ be a sublattice. We will consider $\Lambda^\sharp$ as a discrete scheme acting on $\mathrm{Conf}_{\infty\cdot x}$ by adding the corresponding divisor at x:

$$\gamma, D \mapsto \mathrm{Tr}^\gamma(D) := D + \gamma \cdot x.$$

We will make the following assumption: the gerbe $\mathcal{G}^\Lambda$ on $\mathrm{Conf}_{\infty\cdot x}$ is *equivariant* with respect to this action, in a way compatible with the factorization structure with respect to the given gerbe $\mathcal{G}^\Lambda$ on Conf.

This means that we are given a compatible system of identifications

$$\mathrm{Tr}^\gamma(\mathcal{G}^\Lambda) \simeq \mathcal{G}^\Lambda \tag{4.8}$$

that make the following diagrams commute:

$$\begin{tikzcd}
\mathrm{Tr}^\gamma(\mathcal{G}^\Lambda)|_{(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}}} \arrow[r] \arrow[d] & (\mathcal{G}^\Lambda)^{\boxtimes I} \boxtimes \mathrm{Tr}^\gamma(\mathcal{G}^\Lambda)|_{(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}}} \arrow[d] \\
\mathcal{G}^\Lambda|_{(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}}} \arrow[r] & (\mathcal{G}^\Lambda)^{\boxtimes I} \boxtimes \mathcal{G}^\Lambda|_{(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}}},
\end{tikzcd}$$

where the top horizontal isomorphism is induced by (4.7) via the commutative diagram

$$\begin{tikzcd}
(\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}} \arrow[r, "\mathrm{id} \times \mathrm{Tr}^\gamma"] \arrow[d, "(4.6)"'] & (\mathrm{Conf}^I \times \mathrm{Conf}_{\infty\cdot x})_{\mathrm{disj}} \arrow[d, "(4.6)"] \\
\mathrm{Conf}_{\infty\cdot x} \arrow[r, "\mathrm{Tr}^\gamma"] & \mathrm{Conf}_{\infty\cdot x}\,.
\end{tikzcd}$$

4.4.2. The maps (4.8) induce functors

$$\mathrm{Tr}^\gamma : \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

We will regard this collection of functors as an action of the (symmetric) monoidal category $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$, where T_H is the torus whose lattice of characters is $\Lambda^\sharp$.

By definition, for $\gamma \in \Lambda^\sharp$, the object $\mathsf{e}^\gamma \in \mathrm{Rep}(T_H)$ acts as Tr^γ.

4.5. **Isogenies.** In this subsection we will study the behavior of the category $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$ under the change of the lattice Λ by an isogenous one. This will be handy in the future, as it will be convenient for us to replace the given group G by another one in the same isogeny class.

4.5.1. Let is be given a short exact sequence of lattices

$$0 \to \Lambda \to \widetilde{\Lambda} \to \Lambda_0 \to 0. \tag{4.9}$$

We let $\widetilde{\Lambda}^{\mathrm{neg}} \subset \widetilde{\Lambda}$ be the image of Λ^{neg} under the above map.

Denote by $\widetilde{\mathrm{Conf}}_{\infty\cdot x}$ the corresponding configuration space with a marked point. The map $\Lambda \to \widetilde{\Lambda}$ defines a closed embedding

$$i : \mathrm{Conf}_{\infty\cdot x} \to \widetilde{\mathrm{Conf}}_{\infty\cdot x}. \tag{4.10}$$

4.5.2. Let $\mathcal{G}^{\widetilde{\Lambda}}$ be a gerbe on $\widetilde{\mathrm{Conf}}_{\infty\cdot x}$ equipped with a factorization structure with respect to the given factorization gerbe $\mathcal{G}^\Lambda$ on Conf.

Let us assume being given an identification

$$\mathcal{G}^{\widetilde{\Lambda}}|_{\mathrm{Conf}_{\infty\cdot x}} \simeq \mathcal{G}^\Lambda$$

as gerbes on $\mathrm{Conf}_{\infty\cdot x}$ equipped with a factorization structure with respect to the factorization gerbe $\mathcal{G}^\Lambda$ on Conf.

Then the map i of (4.10) gives rise to a functor

$$i_* : \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^{\widetilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x}). \tag{4.11}$$

4.5.3. Let us now be in the situation of Sect. 4.4 for Λ and $\widetilde{\Lambda}$, and assume that that we have a commutative diagram

$$\begin{tikzcd} \widetilde{\Lambda}^\sharp \arrow[r] & \widetilde{\Lambda} \\ \Lambda^\sharp \arrow[r] \arrow[u] & \Lambda. \arrow[u] \end{tikzcd} \tag{4.12}$$

We obtain an action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$ and an action of $\mathrm{Rep}(T_{\widetilde{H}})$ on $\mathrm{Shv}_{\mathcal{G}^{\widetilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x})$, which are intertwined by the functor i_* of (4.11) and the restriction functor

$$\mathrm{Rep}(T_H) \to \mathrm{Rep}(T_{\widetilde{H}}).$$

In particular, we obtain a functor

$$\mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^{\widetilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x}). \tag{4.13}$$

4.5.4. Assume now that (4.12) is a *push-out* diagram.

The following results easily from the definitions:

Lemma 4.5.5. *Under the above assumptions, the functor* (4.13) *is an equivalence.*

4.5.6. Note that the assumption in Sect. 4.5.4 implies that we have a short exact sequence of lattices

$$0 \to \Lambda^\sharp \to \widetilde{\Lambda}^\sharp \to \Lambda_0 \to 0. \tag{4.14}$$

A choice of a splitting of (4.14) defines an equivalence

$$\mathrm{Rep}(T_H) \otimes \mathrm{Rep}(T_0) \simeq \mathrm{Rep}(T_{\widetilde{H}}),$$

as $\mathrm{Rep}(T_H)$-module categories, where T_0 is a torus with weight lattice Λ_0.

Combining with Lemma 4.5.5, we obtain:

Corollary 4.5.7. *A choice of a splitting of* (4.14) *defines an equivalence*

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \otimes \mathrm{Rep}(T_0) \to \mathrm{Shv}_{\mathcal{G}^{\widetilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x}).$$

4.6. **Configuration space via the affine Grassmannian.** Consider the Ran Grassmannian $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$ of T. We will now describe certain closed subfunctors

$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \subset (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}} \subset \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}.$$

It will turn out that the prestack $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}$ is essentially equivalent to the configuration space Conf.

4.6.1. Consider the simply connected cover G_{sc} of (the derived group of) G. Let T_{sc} denote the Cartan group of G_{sc}. Note that the map

$$\mathrm{Gr}^{\omega^\rho}_{T_{\mathrm{sc}},\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \tag{4.15}$$

is a closed embedding.

For the definition of $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}$ and $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}$ we stipulate that the equal equals the images of

$$(\mathrm{Gr}^{\omega^\rho}_{T_{\mathrm{sc}},\mathrm{Ran}})^{\mathrm{neg}} \subset (\mathrm{Gr}^{\omega^\rho}_{T_{\mathrm{sc}},\mathrm{Ran}})^{\text{non-pos}} \subset \mathrm{Gr}^{\omega^\rho}_{T_{\mathrm{sc}},\mathrm{Ran}}$$

along (4.15), so for the definition of $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}$ and $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}$ we will assume that $G = G_{\mathrm{sc}}$.

4.6.2. An S-point $(J,\mathcal{P}_T,\alpha)$ of $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$ belongs to $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}$ if the following condition hold:

- *Regularity*: for every *dominant* weight $\check{\lambda}\in\check{\Lambda}^+$, the meromorphic map of line bundles on $S\times X$ (resp., $\mathcal{D}_J$)
$$\check{\lambda}(\mathcal{P}_T)\to\check{\lambda}(\mathcal{P}^0_T),$$
induced by α, is regular.

An S-point $(\mathcal{J},\mathcal{P}_T,\alpha)$ as above $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}$ if moreover the following holds:

- *Non-redundancy*: for every point $s\in S$ and every element $j\in\mathcal{J}$ there exists at least one $\check{\lambda}\in\check{\Lambda}^+$, for which the above map of line bundles has a zero at the point of X corresponding to $s\to S\overset{j}{\to}X$.

Note that $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}$ and $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}$ have a natural structure of factorization spaces over Ran.

4.6.3. Evaluation on fundamental weights defines a map of prestacks
$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}\to\mathrm{Conf}\tag{4.16}$$
(see Remark 4.1.3).

The following is obtained as [Ga4, Lemma 8.1.4]:

Lemma 4.6.4. *The map* (4.16) *induces an isomorphism* of the sheafifications *in the topology generated by finite surjective maps.*

4.6.5. As a corollary, we obtain that the map (4.16) induces an isomorphisms on spaces of gerbes. In particular, we obtain that for a geometric metaplectic data for T, the factorization gerbe $\mathcal{G}^T|_{(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}}$ on $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}$ is the pullback of a uniquely defined factorization gerbe, deboted $\mathcal{G}^\Lambda$ on Conf.

Furthermore, Lemma 4.6.4 implies that pullback defines an equivalence
$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})\to\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}).\tag{4.17}$$

4.6.6. We define the closed subfunctors
$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{neg}}_{\infty\cdot x}\subset(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{non-pos}}_{\infty\cdot x}\subset\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}$$
as follows:

A point $(J,\mathcal{P}_T,\alpha)$ of $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}$ belongs to $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{neg}}_{\infty\cdot x}$ if there exists another T-bundle $\mathcal{P}'_T$ on $S\times X$ and an isomorphism $\mathcal{P}_T|_{S\times(X-x)}\simeq\mathcal{P}'_T|_{S\times(X-x)}$, such that the resulting point $(J,\mathcal{P}'_T,\alpha')$ of $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}$ belongs to
$$\mathrm{Ran}_x\underset{\mathrm{Ran}}{\times}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}\subset\mathrm{Ran}_x\underset{\mathrm{Ran}}{\times}\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}.$$

Replacing $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}$ by $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}$ we obtain the definition of $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{non-pos}}_{\infty\cdot x}$.

4.6.7. As in (4.16) we have a canonically defined map
$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{neg}}_{\infty\cdot x}\to\mathrm{Conf}_{\infty\cdot x},\tag{4.18}$$
and a counterpart of Lemma 4.6.4 holds.

Hence, stating with a geometric metaplectic data for T, we obtain that the corresponding gerbe
$$\mathcal{G}^T|_{(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{neg}}_{\infty\cdot x}},$$
viewed as a equipped with a factorization structure with respect to $\mathcal{G}^T|_{(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}}$, is the pullback of a uniquely defined gerbe $\mathcal{G}^\Lambda$ on $\mathrm{Conf}_{\infty\cdot x}$ equipped with a factorization structure with respect to the facorization gerbe $\mathcal{G}^\Lambda$ on Conf.

Furthermore, pullback with respect to (4.18) defines an equivalence
$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})\to\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{neg}}_{\infty\cdot x}).\tag{4.19}$$

5. Factorization algebras and modules on configuration spaces

In this section we will finally define factorization algebras and modules over them on the configuration spaces.

We will see that they model a certain subcategory of factorization algebras (resp., modules over them) on Gr_T.

5.1. **Factorization algebras on configuration spaces.** In this subsection we define the notion of factorization algebra on Conf.

5.1.1. Let $\mathcal{G}^\Lambda$ be a factorization gerbe on Conf. A factorization algebra in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ is an object $\mathcal{A} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ equipped with a *homotopy-compatible* system of identifications

$$\mathcal{A}|_{(\mathrm{Conf}^I)_{\mathrm{disj}}} \simeq (\mathcal{A})^{\boxtimes I}|_{(\mathrm{Conf}^I)_{\mathrm{disj}}} \tag{5.1}$$

for finite non-empty sets I.

Below we explain one of the possible ways to formalize the phrase "homotopy-coherent" in this context. We will follow the the same idea as in Sects. 3.1.2-3.1.4.

5.1.2. The assignment

$$I \mapsto (\mathrm{Conf}^I)_{\mathrm{disj}}$$

has a structure of op-lax symmetric monoidal functor

$$\mathrm{fSet}^{\mathrm{surj}} \to \mathrm{Sch}\,.$$

A factorization gerbe on Conf is a lift of the above functor to a functor with values in the symmetric monoidal category

$$\mathrm{Sch} + \mathrm{Grb}\,.$$

Composing with (3.4), we obtain that the assignment

$$I \mapsto \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{\boxtimes I}}((\mathrm{Conf}^I)_{\mathrm{disj}})$$

has a structure of lax monoidal functor

$$(\mathrm{fSet}^{\mathrm{surj}})^{\mathrm{op}} \to \infty\text{-}\mathrm{Cat}\,.$$

We interpret this functor as a Cartesian fibration

$$\mathrm{Shv}_{\mathcal{G}}((\mathcal{G}^\Lambda)^{\mathrm{fSet}^{\mathrm{surj}}}) \to \mathrm{fSet}^{\mathrm{surj}} \tag{5.2}$$

of symmetric monoidal categories.

A factorization algebra in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ is a symmetric monoidal Cartesian section of (5.2).

5.1.3. Proceeding as in Sect. 3.3, given a factorization algebra $\mathcal{A} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$, we define the categories of factorization modules with respect to it in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$. We denote this category

$$\mathcal{A}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})),$$

or simply $\mathcal{A}$-FactMod if no confusion is likely to occur.

We have a tautological conservative forgetful functor

$$\mathbf{oblv}_{\mathrm{Fact}} : \mathcal{A}\text{-}\mathrm{FactMod} \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

5.2. **Change of lattice and isogenies.** In this subsection we will remark that the material of Sects. 4.4 and 4.5 carries over to categories of factorization modules.

5.2.1. Let us be in the situation of Sect. 4.4. As in Sect. 3.4.1, it follows that the action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$ gives rise to one on $\mathcal{A}$-FactMod.

For $\gamma \in \Lambda^\sharp$, we will denote by the same symbol Tr^γ the corresponding translation endo-functor of $\mathcal{A}$-FactMod.

5.2.2. Let us now be in the situation of Sect. 4.5. As in Sect. 3.4.1, it follows that the functor i_* of Sect. 4.11 induces a functor

$$i_* : \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})) \to \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x}))$$

that intertwines the actions of $\mathrm{Rep}(T_H)$ and $\mathrm{Rep}(T_{\widetilde{H}})$, respectively.

In particular, we obtain a functor

$$\mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})) \to \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x})). \tag{5.3}$$

We claim:

Proposition 5.2.3. *Under the assumption of Sect. 4.5.4, the functor* (5.3) *is an equivalence.*

Proof. It follows as in Sect. 3.4.1 that the functor Sect. 4.13 and its right adjoint induce an adjoint pair of functors

$$\mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})) \rightleftarrows \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x})).$$

We wish to show that, under the assumption of Sect. 4.5.4, these functors are mutually inverse. I.e., we need to show that the unit and the counit of this adjunction are isomorphisms. For the latter, it is sufficient to show that the natural transformations become isomorphisms after applying the (conservative) forgeful functors

$$\mathbf{oblv}_{\mathrm{Fact}} : \mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})) \to \mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

and

$$\mathbf{oblv}_{\mathrm{Fact}} : \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x})) \to \mathrm{Shv}_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x}),$$

respectively.

Now fact that the resulting natural transformations are isomorphisms follows from Lemma 4.5.5. □

Corollary 5.2.4. *A choice of a splitting of* (4.14) *defines an equivalence*

$$\mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})) \otimes \mathrm{Rep}(T_0) \to \mathcal{A}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^{\tilde{\Lambda}}}(\widetilde{\mathrm{Conf}}_{\infty\cdot x})).$$

5.3. **Structure of the category of factorization modules on the configuration space.** In this subsection we fix a factorization algebra $\mathcal{A} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$.

5.3.1. For $\mu \in \Lambda$ we let $\mathcal{A}\text{-FactMod}_{\leq\mu}$ be the category of factorization $\mathcal{A}$-modules in the category $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\leq\mu\cdot x})$, or which is the same, the preimage of

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\leq\mu\cdot x}) \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

under the functor $\mathbf{oblv}_{\mathrm{Fact}}$.

Let ι_μ denote the closed embedding

$$\mathrm{Conf}_{\leq\mu\cdot x} \hookrightarrow \mathrm{Conf}.$$

As in Sect. 3.4.2, the adjoint pair

$$(\iota_\mu)_! : \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\leq\mu\cdot x}) \rightleftarrows \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) : (\iota_\mu)^!$$

induces a pair of adjoint functors

$$(\iota_\mu)_! : \mathcal{A}\text{-FactMod}_{\leq\mu} \rightleftarrows \mathcal{A}\text{-FactMod} : (\iota_\mu)^!$$

both of which commute with the forgetful functor $\mathbf{oblv}_{\mathrm{Fact}}$.

Since the unit of the adjunction

$$\mathrm{Id} \to (\iota_\mu)^! \circ (\iota_\mu)_!$$

is an isomorphism on $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\leq\mu\cdot x})$, the conservativity of $\mathbf{oblv}_{\mathrm{Fact}}$ implies that it is also an isomorphism on $\mathcal{A}\text{-FactMod}_{\leq\mu}$.

Hence, the functor

$$(\iota_\mu)_! : \mathcal{A}\text{-FactMod}_{\leq \mu} \to \mathcal{A}\text{-FactMod} \tag{5.4}$$

is fully faithful.

We will often identify $\mathcal{A}\text{-FactMod}_{\leq \mu}$ with its essential image in $\mathcal{A}\text{-FactMod}$. We have

$$\mathcal{A}\text{-FactMod}_{\leq \mu_1} \subset \mathcal{A}\text{-FactMod}_{\leq \mu_2} \text{ for } \mu_1 \leq \mu_2.$$

5.3.2. The presentation (4.4) implies that the functors $(\iota_\mu)_!$ define an equivalence

$$\underset{\mu\in\Lambda}{\underset{\longrightarrow}{\operatorname{colim}}}\, \operatorname{Shv}_{\mathcal{G}^\Lambda}(\operatorname{Conf}_{\leq \mu\cdot x}) \to \operatorname{Shv}_{\mathcal{G}^\Lambda}(\operatorname{Conf}_{\infty\cdot x}),$$

(see Sect. 0.8.6 for the general paradigm).

In particular, the map

$$\underset{\mu\in\Lambda}{\underset{\longrightarrow}{\operatorname{colim}}}\, (\iota_\mu)_! \circ (\iota_\mu)^! \to \operatorname{Id} \tag{5.5}$$

is an isomorphism on $\operatorname{Shv}_{\mathcal{G}^\Lambda}(\operatorname{Conf}_{\infty\cdot x})$. By the conservativity of $\mathbf{oblv}_{\mathrm{Fact}}$, we obtain that (5.5) is an isomorphism also in $\mathcal{A}\text{-FactMod}$. Hence, we obtain that the functors (5.4) also define an equivalence

$$\underset{\mu\in\Lambda}{\underset{\longrightarrow}{\operatorname{colim}}}\, \mathcal{A}\text{-FactMod}_{\leq \mu} \to \mathcal{A}\text{-FactMod}\,.$$

5.3.3. Let

$$\operatorname{Conf}_{=\mu\cdot x} \subset \operatorname{Conf}_{\leq \mu\cdot x}$$

be the open subscheme consisting of points (4.3) for which $\lambda_x = \mu$. Note that the above open embedding, to be denoted by $\jmath_\mu$, is affine.

We can consider the corresponding category $\mathcal{A}\text{-FactMod}_{=\mu}$, along with the pair of adjoint functors

$$(\jmath_\mu)^* : \mathcal{A}\text{-FactMod}_{\leq \mu} \rightleftarrows \mathcal{A}\text{-FactMod}_{=\mu} : (\jmath_\mu)_*, \tag{5.6}$$

commuting with the forgetful functor $\mathbf{oblv}_{\mathrm{Fact}}$ and with $(\jmath_\mu)_*$ being fully faithful.

Note also that the essential image of $(\jmath_\mu)_*$ is the right orthogonal to the full subcategory of $\mathcal{A}\text{-FactMod}_{\leq \mu}$ generated by $\mathcal{A}\text{-FactMod}_{\leq \mu'}$ for $\mu' < \mu$.

5.3.4. We have the following assertion:

Lemma 5.3.5. *The functor of stalk at the (unique) point*

$$\mu\cdot x \in (\operatorname{Conf}_{=\mu\cdot x})^\mu \subset \operatorname{Conf}_{=\mu\cdot x}$$

defines a t-exact equivalence from $\mathcal{A}\text{-FactMod}_{=\mu}$ to the category $\operatorname{Vect}_{\mathcal{G}^\Lambda_{\mu\cdot x}}$ (i.e., the category of vector spaces twisted by the fiber of the gerbe $\mathcal{G}^\Lambda$ at the point $\mu\cdot x$).

Proof. Follows, as in Sect. 3.3.10, from the fact that the map (4.6) defines an isomorphism from

$$(\operatorname{Conf}\times\operatorname{Conf}_{\infty\cdot x})_{\mathrm{disj}} \cap \operatorname{Conf}\times\{\mu\cdot x\}$$

to $\operatorname{Conf}_{=\mu\cdot x}$. $\square$

5.3.6. Assume now that $\mathcal{A} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ is *holonomic* if our theory is that of D-modules (the condition is vacuous for other choices of sheaf theory).

Then Lemma 5.3.5 implies that the functor

$$(\jmath_\mu)_! : \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{=\mu\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\leq\mu\cdot x}),$$

left adjoint to $(\jmath_\mu)^! = (\jmath_\mu)^*$, is well-defined on the essential image on the functor $\mathbf{oblv}_{\mathrm{Fact}}$.

Hence, as in Sect. 3.4.3, we obtain that the functor $(\jmath_\mu)^! = (\jmath_\mu)^*$ in (5.6) also admits a *left adjoint*, to be denoted $(\jmath_\mu)_!$, which commutes with the forgetful functor $\mathbf{oblv}_{\mathrm{Fact}}$.

The existence of $(\jmath_\mu)_!$ and its commutation with $\mathbf{oblv}_{\mathrm{Fact}}$ implies that the functor $(\iota_\mu)_! =: (\iota_\mu)_*$ of (5.4) admits also a left adjoint, to be denoted $(\iota_\mu)^*$, which also commutes with $\mathbf{oblv}_{\mathrm{Fact}}$.

5.3.7. Choose a trivialization of the gerbe $\mathcal{G}^\Lambda_{\mu\cdot x}$, thereby identifying $\mathcal{A}\text{-FactMod}_{=\mu}$ with Vect.

Let $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ (resp., $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$) denote the object of $\mathcal{A}$-FactMod equal to $(\iota_\mu)_* \circ (\jmath_\mu)_*$ (resp., $(\iota_\mu)_! \circ (\jmath_\mu)_!$) applied to

$$\mathsf{e} \in \mathrm{Vect} \simeq \mathcal{A}\text{-FactMod}_{=\mu}\,.$$

We will call $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ (resp., $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$) the *co-standard* (resp., *standard*) object.

5.3.8. Note that the objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$ form a set of compact generators of the category $\mathcal{A}$-FactMod. Indeed, the functor $\mathcal{H}om_{\mathcal{A}\text{-FactMod}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}, -)$ identifies with the functor

$$\mathcal{A}\text{-FactMod} \xrightarrow{\iota^!_\mu} \mathcal{A}\text{-FactMod}_{\leq\mu} \xrightarrow{\jmath^*_\mu} \mathcal{A}\text{-FactMod}_{=\mu} \simeq \mathrm{Vect}_{\mathcal{G}^\Lambda_{\mu\cdot x}} \simeq \mathrm{Vect}\,.$$

Note that above functor identifies with

$$\mathcal{A}\text{-FactMod} \xrightarrow{\mathbf{oblv}_{\mathrm{Fact}}} \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \to \mathrm{Vect}_{\mathcal{G}^\Lambda_{\mu\cdot x}} \simeq \mathrm{Vect}, \tag{5.7}$$

where the middle arrow is the functor of !-fiber at $\mu \cdot x \in \mathrm{Conf}_{\infty\cdot x}$.

In particular, we obtain:

$$\mathcal{H}om_{\mathcal{A}\text{-FactMod}}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) = \begin{cases} \mathsf{e} \text{ if } \mu' = \mu \\ 0 \text{ otherwise.} \end{cases} \tag{5.8}$$

Note also that the objects $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ *co-generate* $\mathcal{A}$-FactMod.

5.3.9. Let us be in the situation of Sect. 4.5. It is clear that for $\gamma \in \Lambda^\sharp$, we have

$$\mathrm{Tr}^\gamma(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}) \simeq \mathcal{M}^{\mu+\gamma,!}_{\mathrm{Conf}} \text{ and } \mathrm{Tr}^\gamma(\mathcal{M}^{\mu,*}_{\mathrm{Conf}}) \simeq \mathcal{M}^{\mu+\gamma,*}_{\mathrm{Conf}}.$$

5.3.10. Assume now that $\mathcal{A}$, viewed as an object of $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$, belongs to $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})^{\mathrm{loc.c}}$.

Let

$$\mathcal{A}\text{-FactMod}^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

be the full subcategory consisting of objects whose image under $\mathbf{oblv}_{\mathrm{Fact}}$ belongs to the subcategory $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$.

Evidently, the objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$ and $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ belong to $\mathcal{A}\text{-FactMod}^{\mathrm{loc.c}}$.

5.4. **The t-structure on factorization modules.** We retain the assumptions of Sect. 5.3.6. Let assume, in addition, that $\mathcal{A}$, when viewed as an object of $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$, belongs to $(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}))^\heartsuit$.

We will show that in this case, the category $\mathcal{A}$-FactMod has a well-behaved t-structure and the abelian category $(\mathcal{A}\text{-FactMod})^\heartsuit$ is a highest weight category.

5.4.1. We define a t-structure on the category $\mathcal{A}$-FactMod by declaring an object $\mathcal{F}$ to be coconnective if

$$\mathrm{Hom}_{\mathcal{A}\text{-FactMod}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}[k], \mathcal{F}) = 0$$

for all $\mu \in \Lambda$ and $k > 0$.

By factorization and the assumption on $\mathcal{A}$, we obtain that this condition is equivalent to

$$\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{F}) \in (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))^{\geq 0}.$$

We claim:

Proposition 5.4.2.

(a) *The functor* $\mathbf{oblv}_{\mathrm{Fact}}$ *is t-exact.*

(b) *The objects* $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ *belong to* $(\mathcal{A}\text{-FactMod})^\heartsuit$.

Proof. The fact that the functor $\mathbf{oblv}_{\mathrm{Fact}}$ is left t-exact has been noted above. Since $(\mathcal{A}\text{-FactMod})^{\leq 0}$ is generated under colimits by the objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$, in order to show that $\mathbf{oblv}_{\mathrm{Fact}}$ is right t-exact, it suffices to show that $\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}) \in (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))^{\geq 0}$. However, this follows from the fact that the open embedding $\jmath_\mu$ is affine.

Thus point (a) of the proposition has been proved. In order to prove point (b), since the functor $\mathbf{oblv}_{\mathrm{Fact}}$ is conservative, it suffices to show that

$$\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}),\ \mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Conf}}) \in (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))^{\heartsuit}.$$

This fact for $\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}) \in (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))^{\heartsuit}$ has been noted above. The corresponding fact for $\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Conf}})$ also follows from the fact that the open embedding $\jmath_\mu$ is affine.

□

5.4.3. By [Lur, Theorem 1.3.3.2], we have a canonically defined t-exact functor

$$D^+\left((\mathcal{A}\text{-FactMod})^\heartsuit\right) \to \mathcal{A}\text{-FactMod}\,. \tag{5.9}$$

We claim:

Proposition 5.4.4. *The functor* (5.9) *is an equivalence onto the eventually coconnective part.*

Before we give a proof, let us note the following:

The functor (5.9) induces a *bijection*

$$\mathrm{Ext}^i_{(\mathcal{A}\text{-FactMod})^\heartsuit}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) \to H^i\left(\mathcal{H}om_{\mathcal{A}\text{-FactMod}}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}})\right)$$

for $i = 0, 1$ and and an injection

$$\mathrm{Ext}^2_{(\mathcal{A}\text{-FactMod})^\heartsuit}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) \to H^2\left(\mathcal{H}om_{\mathcal{A}\text{-FactMod}}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}})\right).$$

In particular, it follows from (5.8) that

$$\mathrm{Ext}^2_{(\mathcal{A}\text{-FactMod})^\heartsuit}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) = 0.$$

Hence, we obtain that $(\mathcal{A}\text{-FactMod})^\heartsuit$ has a structure of *highest weight category*.

Proof of Proposition 5.4.4. It is enough to show that the functor (5.4.4) is fully faithful. Since the objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$ (resp., $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$) generate (resp., co-generate) $\mathcal{A}$-FactMod, it is sufficient to show that the map

$$\mathrm{Ext}^i_{(\mathcal{A}\text{-FactMod})^\heartsuit}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) \to H^i\left(\mathcal{H}om_{\mathcal{A}\text{-FactMod}}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}})\right)$$

is an isomorphism for all i.

However, this follows from (5.8) and the fact that in a highest weight category

$$\mathrm{Ext}^i_{(\mathcal{A}\text{-FactMod})^\heartsuit}(\mathcal{M}^{\mu',!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}}) = 0, \text{ for } i \geq 1.$$

□

5.4.5. Consider the canonical map

$$\mathcal{M}^{\mu,!}_{\mathrm{Conf}} \to \mathcal{M}^{\mu,*}_{\mathrm{Conf}}.$$

Let $\mathcal{M}^{\mu,!*}_{\mathrm{Conf}}$ denote its image, viewed as an object in $(\mathcal{A}\text{-FactMod})^{\heartsuit}$.

We obtain that the objects $\mathcal{M}^{\mu,!*}_{\mathrm{Conf}}$ are the irreducibles in the abelian category $(\mathcal{A}\text{-FactMod})^{\heartsuit}$.

5.4.6. Let us be in the situation of Sect. 4.5. It is clear that for $\gamma \in \Lambda^\sharp$, we have

$$\mathrm{Tr}^\gamma(\mathcal{M}^{\mu,!*}_{\mathrm{Conf}}) \simeq \mathcal{M}^{\mu+\gamma,!*}_{\mathrm{Conf}}.$$

5.5. **Comparison with the affine Grassmannian.** Recall the setting of Sect. 4.6. In this subsection we will use it to compare factorization algebras in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$ and those in $\mathrm{Shv}\,\mathcal{G}^\Lambda(\mathrm{Conf})$, as well as modules over them.

5.5.1. First off, the equivalence (4.17) implies that pullback along (4.16) defines an equivalence

$$\mathrm{FactAlg}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})) \simeq \mathrm{FactAlg}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}})), \quad \mathcal{A}_{\mathrm{Conf}} \mapsto \mathcal{A}_{\mathrm{Gr}_T}.$$

5.5.2. Let $\mathcal{A}_{\mathrm{Gr}_T}$ be a factorization algebra in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$, and let $\mathcal{A}_{\mathrm{Conf}}$ be the corresponding factorization algebra in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$.

We obtain that pullback with respect to (4.18) defines an equivalence

$$\mathcal{A}_{\mathrm{Conf}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})) \to \mathcal{A}_{\mathrm{Gr}_T}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathrm{neg}}_{\infty\cdot x})). \tag{5.10}$$

Remark 5.5.3. Let $\mathcal{A}$ be a factorization algebra in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$ supported on $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}$ (resp., $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}$).

It follows automatically from factorization, that any object in $\mathcal{A}_{\mathrm{Gr}_T}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))$ is supported on $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathrm{neg}}_{\infty\cdot x}$ (resp., $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{non-pos}}_{\infty\cdot x}$).

5.5.4. *Hecke action.* Let $\Lambda^\sharp$ be as in Sect. 2.2.2. Recall that we have an action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$. It is easy to see that this action preserves the subcategory

$$\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathrm{neg}}_{\infty\cdot x}) \subset \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}).$$

Let $\mathcal{A}_{\mathrm{Gr}_T}$ be a factorization algebra in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$ supported on $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}$. By Sect. 3.4.1, the above action gives rise to an action of $\mathrm{Rep}(T_H)$ on $\mathcal{A}_{\mathrm{Gr}_T}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathrm{neg}}_{\infty\cdot x}))$.

It follows from the constructions that the equivalence (5.10) intertwines this action and the action of $\mathrm{Rep}(T_H)$ on $\mathcal{A}_{\mathrm{Conf}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))$ from Sect. 5.2.1.

Part II: The metaplectic Whittaker category of the affine Grassmannian

The goal of this work is to establish an equivalence of two categories: the (Hecke version of the) metaplectic Whittaker category of the affine Grassmannian and a certain category of factorization modules. In this Part we initiate the study of the former of these categories.

6. The metaplectic Whittaker category

In this section we introduce the metaplectic Whittaker category of the affine Grassmannian, and study its basic properties.

6.1. **Definition of the metaplectic Whittaker category.** In this subsection we introduce the metaplectic Whittaker category of the affine Grassmannian, denoted $\mathrm{Whit}_{q,x}(G)$. The definition involves infinite-dimensional algebro-geometric objects, and we will rewrite it as a limit of categories of finite-dimensional nature.

6.1.1. We start with a geometric metaplectic data for the group G, i.e., a factorization gerbe $\mathcal{G}^G$ on the affine Grassmannian $\mathrm{Gr}_{G,\mathrm{Ran}}$.

We consider the twisted version $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$, see Sect. 1.4.4, and its fiber $\mathrm{Gr}^{\omega^\rho}_{G,x}$ over $\{x\} \in \mathrm{Ran}$. We denote by the same symbol $\mathcal{G}^G$ the resulting gerbe on $\mathrm{Gr}^{\omega^\rho}_{G,x}$.

The indscheme $\mathrm{Gr}^{\omega^\rho}_{G,x}$ is acted on by the ω^ρ-twist of the loop group at x, denoted $\mathfrak{L}(G)^{\omega^\rho}_x$. This is the group of automorphisms of the G-bundle ω^ρ on the formal punctured disc around x.

Recall (see Sect. 2.1.7) that $\mathfrak{L}(G)^{\omega^\rho}_x$ carries a canonically defined multiplicative gerbe (also denoted $\mathcal{G}^G$), so that the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$ is twisted-equivariant against the gerbe $\mathcal{G}^G$ on $\mathfrak{L}(G)^{\omega^\rho}_x$ with respect to the above action.

6.1.2. Consider now the subgroup $\mathfrak{L}(N)^{\omega^\rho}_x \subset \mathfrak{L}(G)^{\omega^\rho}_x$. Due to the fact that $\mathfrak{L}(N)^{\omega^\rho}_x$ is ind-pro-unipotent, any gerbe over $\mathfrak{L}(N)^{\omega^\rho}_x$ is trivial (and the trivialization is uniquely fixed by its value at the origin). In particular, any multiplicative gerbe on $\mathfrak{L}(N)^{\omega^\rho}_x$ admits a canonical trivialization compatible with the multiplicative structure.

Thus, the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$ is *equivariant* with respect to $\mathfrak{L}(N)^{\omega^\rho}_x$.

6.1.3. Note that the group

$$\mathfrak{L}(N)^{\omega^\rho}_x/[\mathfrak{L}(N)^{\omega^\rho}_x, \mathfrak{L}(N)^{\omega^\rho}_x]$$

identifies with $(\mathfrak{L}(\mathbb{G}_a)^{\omega}_x)^I$, where $\mathfrak{L}(\mathbb{G}_a)^{\omega}_x$ is the group indscheme of meromorphic differentials on the formal punctured disc around x, and I is the set of vertices of the Dynkin diagram of G.

The residue map defines a homomorphism $\mathfrak{L}(\mathbb{G}_a)^{\omega}_x \to \mathbb{G}_a$, see Sect. 1.4.8, Let χ_N denote the pullback of the Artin-Schreier sheaf χ under the map

$$\mathfrak{L}(N)^{\omega^\rho}_x \to \mathfrak{L}(N)^{\omega^\rho}_x/[\mathfrak{L}(N)^{\omega^\rho}_x, \mathfrak{L}(N)^{\omega^\rho}_x] \simeq (\mathfrak{L}(\mathbb{G}_a)^{\omega}_x)^I \to (\mathbb{G}_a)^I \xrightarrow{\mathrm{sum}} \mathbb{G}_a.$$

We define $\mathrm{Whit}_{q,x}(G)$ to be the category of $(\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N)$-equivariant objects in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$, i.e.,

$$\mathrm{Whit}_{q,x}(G) := \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x, \chi_N}.$$

6.1.4. Let us rewrite this definition in more detail. In particular, we will see that the forgetful functor

$$\mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \tag{6.1}$$

is fully faithful.

First, let us write $\mathfrak{L}(N)^{\omega^\rho}_x$ as a union of its group subschemes N_k, $k = 1, 2, ...$ By definition, we have

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \simeq \lim_k \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N}, \tag{6.2}$$

where the limit is taken with respect to the forgetful functors

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_{k'},\chi_N} \to \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N}, \quad k' \geq k.$$

For the fully faithfulness of (6.1) it would suffice to see that each of the forgetful functors

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \tag{6.3}$$

is fully faithful, and so the limit (6.2) amounts to the intersection of the corresponding nested family of the subcategories $\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N}$.

Thus, from now on we fix a particular index k.

6.1.5. Next, we write $\mathrm{Gr}^{\omega^\rho}_{G,x}$ as a union of an increasing family of its closed subschemes

$$\mathrm{Gr}^{\omega^\rho}_{G,x} \simeq \bigcup_j Y_j.$$

With no restriction of generality, we can assume that all Y_j are N_k-invariant. We have

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \simeq \lim_j \mathrm{Shv}_{\mathcal{G}^G}(Y_j), \tag{6.4}$$

where the limit is taken with respect to the !-restriction functors

$$\mathrm{Shv}_{\mathcal{G}^G}(Y_{j'}) \to \mathrm{Shv}_{\mathcal{G}^G}(Y_j), \quad j' \geq j.$$

Thus, from (6.4) we obtain

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N} \simeq \lim_j \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_k,\chi_N}. \tag{6.5}$$

Thus, for the fully faithfulness of (6.3), it would suffice to show that each of the functors

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_k,\chi_N} \to \mathrm{Shv}_{\mathcal{G}^G}(Y_j) \tag{6.6}$$

is fully faithful.

Thus, from now on, the index j will also be fixed.

6.1.6. The group indscheme N_k can be written as a limit of finite-dimensional unipotent groups $N_{k,l}$, $l = 1, 2, ...$ With no restriction of generality, we can assume that the action of N_k on Y_j factors through $N_{k,l}$ for every l.

By definition, we have

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_k,\chi_N} = \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_{k,l},\chi_N}$$

for *any* such l, where we notice that the forgetful functors

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_{k,l},\chi_N} \to \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_{k,l'},\chi_N}, \quad l' \geq l$$

are equivalences, because

$$\ker(N_{k,l'} \to N_{k,l})$$

are unipotent. Since the groups $N_{k,l}$ are themselves unipotent, the forgetful functors

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_{k,l},\chi_N} \to \mathrm{Shv}_{\mathcal{G}^G}(Y_j) \tag{6.7}$$

are fully faithful, and hence so is (6.6).

6.1.7. We note that the forgetful functors (6.7) admit right adjoints, denoted $\mathrm{Av}_*^{N_{k,l},\chi_N}$. Denote the resulting forgetful functor to (6.6) by $\mathrm{Av}_*^{N_k,\chi_N}$.

These right adjoints make each of the diagrams

$$\begin{tikzcd} \mathrm{Shv}_{\mathcal{G}^G}(Y_j) \arrow[r, "\mathrm{Av}_*^{N_k,\chi_N}"] & \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_k,\chi_N} \\ \mathrm{Shv}_{\mathcal{G}^G}(Y_{j'}) \arrow[u] \arrow[r, "\mathrm{Av}_*^{N_k,\chi_N}"] & \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_{j'})\right)^{N_k,\chi_N} \arrow[u] \end{tikzcd}$$

with $j' \geq j$ commutative, e.g., because the corresponding diagram of left adjoints

$$\begin{tikzcd} \mathrm{Shv}_{\mathcal{G}^G}(Y_j) \arrow[d] & \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_j)\right)^{N_k,\chi_N} \arrow[l] \arrow[d] \\ \mathrm{Shv}_{\mathcal{G}^G}(Y_{j'}) & \left(\mathrm{Shv}_{\mathcal{G}^G}(Y_{j'})\right)^{N_k,\chi_N} \arrow[l] \end{tikzcd}$$

is commutative (here the vertical arrows are given by direct image with respect to the closed embedding $Y_j \hookrightarrow Y_{j'}$).

Thus, the above right adjoints combine to a right adjoint, also denoted $\mathrm{Av}_*^{N_k,\chi_N}$ to (6.3).

6.1.8. Using the fully faithful embedding (6.3), we can view $\mathrm{Av}_*^{N_k,\chi_N}$ as an endo-functor of

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho}).$$

Being a composition of a right adjoint followed by a left adjoint, it has a natural structure of a comonad; moreover the co-multiplication map

$$\mathrm{Av}_*^{N_k,\chi_N} \to \mathrm{Av}_*^{N_k,\chi_N} \circ \mathrm{Av}_*^{N_k,\chi_N}$$

is an isomorphism.

The subcategory

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho})\right)^{N_k,\chi_N} \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho})$$

consists of those objects $\mathcal{F}$, for which the counit map

$$\mathrm{Av}_*^{N_k,\chi_N}(\mathcal{F}) \to \mathcal{F} \tag{6.8}$$

is an isomorphism. The subcategory

$$\mathrm{Whit}_{q,x}(G) \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,x}^{\omega^\rho}) \tag{6.9}$$

consists of those objects $\mathcal{F}$, for which the maps (6.8) are isomorphisms for all k.

6.1.9. The inclusion (6.9) admits a *discontinuous* right adjoint given by

$$\mathrm{Av}_*^{\mathfrak{L}(N)_x^{\omega^\rho},\chi_N} := \lim_k \mathrm{Av}_*^{N_k,\chi_N}.$$

6.2. **Structure of the metaplectic Whittaker category.** In this subsection we will study the basic structural properties of $\mathrm{Whit}_{q,x}(G)$, namely, its stratification indexed by the elements of Λ^+, and the corresponding standard and costandard objects.

6.2.1. In addition to right adjoint given by $\mathrm{Av}_*^{N_k,\chi_N}$, the functor (6.1) admits a *partially defined* left adjoint, denoted $\mathrm{Av}_!^{N_k,\chi_N}$, given by !-averaging. This partially defined adjoint is always defined in the context of ℓ-adic sheaves. In the context of D-modules, it is defined on holonomic objects.

If $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ is an object on which all $\mathrm{Av}_!^{N_k,\chi_N}$ are defined, then the partially left adjoint to the inclusion (6.9) is defined on $\mathcal{F}$ and is given by

$$\mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\mathcal{F}) := \operatorname*{colim}_k \mathrm{Av}_!^{N_k,\chi_N}(\mathcal{F}).$$

6.2.2. Pick a uniformizer t at x and for $\lambda \in \Lambda$ let t^λ denote the corresponding point of $\mathrm{Gr}^{\omega^\rho}_{G,x}$. Choose a trivialization of the fiber of $\mathcal{G}^G$ at t^λ. Let $\delta_{t^\lambda,\mathrm{Gr}}$ denote the resulting point of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$.

Denote

$$W^{\lambda,!} := \mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})[-\langle \lambda, 2\check{\rho}\rangle] \in \mathrm{Whit}_{q,x}(G).$$

By construction, the objects $W^{\lambda,!}$ are compact in $\mathrm{Whit}_{q,x}(G)$. We will shortly see that these objects generate $\mathrm{Whit}_{q,x}(G)$ (and they vanish unless λ is dominant).

6.2.3. For $\lambda \in \Lambda$, let S^λ be the corresponding $\mathfrak{L}(N)^{\omega^\rho}_x$-orbit on $\mathrm{Gr}^{\omega^\rho}_{G,x}$, i.e.,

$$S^\lambda = \mathfrak{L}(N)^{\omega^\rho}_x \cdot t^\lambda \cdot \mathfrak{L}^+(G)^{\omega^\rho}_x / \mathfrak{L}^+(G)^{\omega^\rho}_x, \tag{6.10}$$

and let $\overline{S}^\lambda$ denote its closure. Denote by $\bar{\mathbf{i}}_\lambda$ the closed embedding

$$\overline{S}^\lambda \to \mathrm{Gr}^{\omega^\rho}_{G,x},$$

and by $\mathbf{i}_\lambda$ the locally closed embedding

$$S^\lambda \to \mathrm{Gr}^{\omega^\rho}_{G,x}.$$

6.2.4. Recall the context of Sect. 0.8.6. In particular, if Y is a prestack equipped with a gerbe $\mathcal{G}$ and written as

$$Y = \lim_\alpha Y_\alpha, \quad (\alpha \to \alpha') \mapsto Y_\alpha \overset{f_{\alpha,\alpha'}}{\longrightarrow} Y_{\alpha'}$$

so that

$$\mathrm{Shv}_{\mathcal{G}}(Y) \simeq \lim_\alpha \mathrm{Shv}_{\mathcal{G}}(Y_\alpha),$$

where the transition functors are $f^!_{\alpha,\alpha'}$, and if the functors $(f_{\alpha,\alpha'})_!$ are well-defined, then we also have

$$\mathrm{Shv}_{\mathcal{G}}(Y) \simeq \operatorname*{colim}_\alpha \mathrm{Shv}_{\mathcal{G}}(Y_\alpha),$$

where the transition functors are now $(f_{\alpha,\alpha'})_!$.

6.2.5. We have

$$\mathrm{Gr}^{\omega^\rho}_{G,x} \simeq \operatorname*{colim}_{\lambda\in\Lambda} \overline{S}^\lambda,$$

where Λ is regarded as a poset with the standard order relation.

Thus, we obtain

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \simeq \lim_{\lambda\in\Lambda} \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda)$$

with respect to the pullback functors, *and also*

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \simeq \operatorname*{colim}_{\lambda\in\Lambda} \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda),$$

with respect to the pushforward functors.

For each pair of indices $\lambda \leq \lambda'$, we have the following commutative diagrams

$$\begin{tikzcd}
\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda'}) \arrow[d] & (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda'}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \arrow[l] \arrow[d] \\
\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}) & (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \arrow[l]
\end{tikzcd}$$

(where the left vertical arrows are given by pullback) and

$$
\begin{tikzcd}
\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda'}) & (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda'}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \arrow[l] \\
\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}) \arrow[u] & (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \arrow[l] \arrow[u]
\end{tikzcd}
$$

(where the left vertical arrows are given by pushforward).

From here we obtain the presentations

$$\mathrm{Whit}_{q,x}(G) \simeq \lim_{\lambda\in\Lambda} (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N},$$

with respect to pullbacks, and

$$\mathrm{Whit}_{q,x}(G) \simeq \underset{\lambda\in\Lambda}{\mathrm{colim}}\, (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N},$$

with respect to pushforwards.

Moreover, since each of the pushforward functors

$$(\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \to (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda'}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}$$

is fully faithful, so are the resulting pushforward functors

$$(\bar{\mathbf{i}}_\lambda)_! : (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \to \mathrm{Whit}_{q,x}(G).$$

We denote the essential image of the latter functor by $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$. By construction, an object of $\mathrm{Whit}_{q,x}(G)$ belongs to $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ if it is supported on $\overline{S}^\lambda$. We have:

$$\lambda \leq \lambda' \;\Rightarrow\; \mathrm{Whit}_{q,x}(G)_{\leq\lambda} \subset \mathrm{Whit}_{q,x}(G)_{\leq\lambda'},$$

and

$$W^{\lambda,!} \in \mathrm{Whit}_{q,x}(G)_{\leq\lambda}.$$

6.2.6. Let $\mathbf{j}_\lambda$ denote the open embedding

$$S^\lambda \hookrightarrow \overline{S}^\lambda.$$

The adjoint pair

$$(\mathbf{j}_\lambda)^* : \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}) \rightleftarrows \mathrm{Shv}_{\mathcal{G}^G}(S^{\lambda}) : (\mathbf{j}_\lambda)_* \tag{6.11}$$

gives rise to an adjunction

$$
\begin{aligned}
(\mathbf{j}_\lambda)^* : \mathrm{Whit}_{q,x}(G)_{\leq\lambda} = \\
= (\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \rightleftarrows (\mathrm{Shv}_{\mathcal{G}^G}(S^{\lambda}))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} =: \mathrm{Whit}_{q,x}(G)_{=\lambda} : (\mathbf{j}_\lambda)_*
\end{aligned} \tag{6.12}
$$

such that make both circuits in the diagram

$$
\begin{tikzcd}
\mathrm{Whit}_{q,x}(G)_{\leq\lambda} \arrow[r, shift left, "\mathbf{j}_\lambda^*"] \arrow[d] & \mathrm{Whit}_{q,x}(G)_{=\lambda} \arrow[l, shift left, "(\mathbf{j}_\lambda)_*"] \arrow[d] \\
\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^{\lambda}) \arrow[r, shift left, "\mathbf{j}_\lambda^*"] & \mathrm{Shv}_{\mathcal{G}^G}(S^{\lambda}) \arrow[l, shift left, "(\mathbf{j}_\lambda)_*"]
\end{tikzcd}
$$

commute.

In particular, since the co-unit of the adjunction

$$\mathbf{j}_\lambda^* \circ (\mathbf{j}_\lambda)_* \to \mathrm{Id}$$

is an isomorphism for (6.11), it is such also for (6.12). In particular, we obtain that the functor

$$(\mathbf{j}_\lambda)_* : \mathrm{Whit}_{q,x}(G)_{=\lambda} \to \mathrm{Whit}_{q,x}(G)_{\leq\lambda}$$

is fully faithful.

The essential image of $\mathrm{Whit}_{q,x}(G)_{=\lambda}$ in $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ is the *right orthogonal* of the full subcategory

$$\mathrm{Whit}_{q,x}(G)_{<\lambda} \subset \mathrm{Whit}_{q,x}(G)_{\leq\lambda}$$

generated by the essential images of $\mathrm{Whit}_{q,x}(G)_{\leq\lambda'}$ with $\lambda' < \lambda$.

6.2.7. Set

$$\overset{\circ}{W}{}^{\lambda} := \mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})[-\langle\lambda,2\check{\rho}\rangle] \in \mathrm{Whit}_{q,x}(G)_{=\lambda}.$$

Clearly,

$$W^{\lambda,!} \simeq (\mathbf{i}_\lambda)_!(\overset{\circ}{W}{}^{\lambda}) \simeq (\bar{\mathbf{i}}_\lambda)_* \circ (\mathbf{j}_\lambda)_!(\overset{\circ}{W}{}^{\lambda}),$$

where $(\mathbf{i}_\lambda)_!$ (resp., $(\mathbf{j}_\lambda)_!$) denotes the (partially defined) left adjoint of $\mathbf{i}_\lambda^!$ (resp., $\mathbf{j}_\lambda^!$).

We denote by

$$W^{\lambda,*} \in \mathrm{Whit}_{q,x}(G)_{\leq\lambda}$$

the object $(\mathbf{i}_\lambda)_*(\overset{\circ}{W}{}^{\lambda})$.

Almost by definition, we have:

$$\mathrm{Maps}_{\mathrm{Whit}_{q,x}(G)}(W^{\lambda,!}, W^{\lambda',*}) = \begin{cases} \mathsf{e} \text{ if } \lambda = \lambda' \\ 0 \text{ otherwise.} \end{cases} \tag{6.13}$$

6.2.8. We claim:

Proposition 6.2.9.
(a) *The category* $\mathrm{Whit}_{q,x}(G)_{=\lambda}$ *is zero unless* λ *is dominant.*
(b) *For* λ *dominant, the category* $\mathrm{Whit}_{q,x}(G)_{=\lambda}$ *is non-canonically equivalent to* Vect*, with the generator given by* $\overset{\circ}{W}{}^{\lambda}$.

From here we easily obtain:

Corollary 6.2.10.
(a) *The objects* $W^{\lambda,!}$ *and* $W^{\lambda,*}$ *with* λ *dominant generate* $\mathrm{Whit}_{q,x}(G)$.
(b) *The canonical map* $W^{\lambda,!} \to W^{\lambda,*}$ *is an isomorphism for any* λ *that is minimal in* Λ^+ *with respect to the standard order relation (in particular, for* $\lambda = 0$*).*
(c) *The category* $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$ *is zero unless* λ *is dominant.*

Proof of Proposition 6.2.9. Consider the functor

$$\mathrm{Whit}_{q,x}(G)_{=\lambda} \to \mathrm{Vect}, \tag{6.14}$$

equal to the composition of the forgetful functor

$$\mathrm{Whit}_{q,x}(G)_{=\lambda} \to \mathrm{Shv}_{\mathcal{G}^G}(S^\lambda),$$

followed by the functor of taking the !-fiber at t^λ.

We claim that the functor (6.14) is conservative. Indeed, if for some $\mathcal{F} \in (\mathrm{Shv}_{\mathcal{G}^G}(S^\lambda))^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}$ its fiber at t^λ is zero, then the restriction of $\mathcal{F}$ to the N_k-orbit of t^λ is zero for all k. However,

$$S^\lambda = \underset{k}{\cup}\, N_k \cdot t^\lambda,$$

and hence $\mathcal{F} = 0$.

The functor (6.14) admits a left adjoint that sends the generator $\mathsf{e} \in \mathrm{Vect}$ to

$$\mathrm{Av}_!^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}}) \simeq \underset{k}{\mathrm{colim}}\, \mathrm{Av}_!^{N_k,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}}).$$

We claim that the resulting monad on Vect is the identity if λ is dominant and zero otherwise.

Indeed, note that $\mathrm{Av}_!^{N_k,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})$ identifies with $\chi_N|_{N_k\cdot t^\lambda}$ if χ_N, restricted to $\mathrm{Stab}_{N_k}(t^\lambda)$, is trivial, and zero otherwise. Hence, its !-fiber at t^λ of $\mathrm{Av}_!^{N_k,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})$ is e if χ_N restricted to $\mathrm{Stab}_{N_k}(t^\lambda)$ is trivial, and zero otherwise. Now, the dominance condition on λ is equivalent to the fact that χ_N restricted to $\mathrm{Stab}_{\mathfrak{L}(N)^{\omega^\rho}_x}(t^\lambda)$ is trivial.

□

6.3. **The t-structure.** In this subsection we will show how to endow $\mathrm{Whit}_{q,x}(G)$ with a t-structure. We note, however, that this t-structure "has nothing to do" with the t-structure on the ambient category $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$.

6.3.1. We introduce a t-structure on $\mathrm{Whit}_{q,x}(G)$ by declaring that an object $\mathcal{F}$ is coconnective if

$$\mathrm{Hom}_{\mathrm{Whit}_{q,x}(G)}(W^{\lambda,!}[k],\mathcal{F})=0$$

for all λ and $k>0$.

In Sect. 7.4.3 we will prove:

Proposition 6.3.2.

(a) *The subcategories* $\mathrm{Whit}_{q,x}(G)_{\leq\lambda}\subset\mathrm{Whit}_{q,x}(G)$ *are compatible with the t-structure.*

(b) *The objects* $W^{\lambda,!}$ *and* $W^{\lambda',*}$ *belong to the heart of the t-structure and are of finite length.*

Remark 6.3.3. It follows formally from Proposition 6.3.2(b) (see Proposition 5.4.4) that the functor

$$D^+(\mathrm{Whit}_{q,x}(G))\to\mathrm{Whit}_{q,x}(G)$$

is an equivalence onto the eventually connective part.

6.3.4. Let $W^{\lambda,!*}$ denote the image of the canonical map $W^{\lambda,!}\to W^{\lambda,*}$.

Corollary 6.3.5.

(a) *The irreducibles in* $(\mathrm{Whit}_{q,x}(G))^\heartsuit$ *are the objects* $W^{\lambda,!*}$.

(b) *The objects* $W^{\lambda,!*}$ *are compact in* $\mathrm{Whit}_{q,x}(G)$ *and they generate* $\mathrm{Whit}_{q,x}(G)$.

Proof of Corrollary 6.3.5. Let L be an object in $(\mathrm{Whit}_{q,x}(G))^\heartsuit$. By the definition of the t-structure, it admits a non-zero map $W^{\lambda,!}\to L$ for some $\lambda\in\Lambda^+$. If L is irreducible, then the above map is surjective. In particular, $L\in\mathrm{Whit}_{q,x}(G)_{\leq\lambda}$, and $(\mathbf{j}_\lambda)^*\circ(\bar{\mathbf{i}}_\lambda)^!(L)$ admits a non-zero map to $\mathbf{j}^*_\lambda(W^{\lambda,!})$. Hence, L admits a non-zero map to $W^{\lambda,*}$. It follows from Corrollary 6.2.10 that L equals $W^{\lambda,!*}$.

Vice versa, let L be an irreducile submodule of $W^{\lambda,!*}$. By the above, it is of the form $W^{\lambda',!*}$ for some λ'. In particular, we obtain a non-zero map

$$W^{\lambda',!}\twoheadrightarrow L\hookrightarrow W^{\lambda,!*}\hookrightarrow W^{\lambda,!},$$

which by Corrollary 6.2.10 implies that $\lambda'=\lambda$.

To prove that $W^{\lambda,!*}$ are compact in $\mathrm{Whit}_{q,x}(G)$ we argue by induction on $\lambda\in\Lambda^+$ with respect to the standard order relation. The base of the induction is provided by Corrollary 6.2.10(b).

Suppose the statement is true for $\lambda'<\lambda$. It is enough to show that $\ker(W^{\lambda,!}\to W^{\lambda,!*})$ is compact. We note, however, that the above object belongs to $\mathrm{Whit}_{q,x}(G)_{<\lambda}$ and is of finite length (by Proposition 6.3.2(b)), and thus is a finite extension of objects $W^{\lambda',!*}$ for $\lambda'<\lambda$. This implies the assertion.

□

Corollary 6.3.6. *The objects* $W^{\lambda,*}$ *are compact in* $\mathrm{Whit}_{q,x}(G)$.

Proof. Follows by combining Proposition 6.3.2(b) and Corollary 6.3.5(b). □

Remark 6.3.7. Note that objects lying in the image of the forgetful functor

$$\mathrm{Whit}_{q,x}(G)\to\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$$

is *infinitely connective*, i.e., it sends all objects of $\mathrm{Whit}_{q,x}(G)$ to objects of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ that have zero cohomologies with respect to the natural t-structure on that category.

To show this, it suffices to show that the generators of $\mathrm{Whit}_{q,x}(G)$, i.e., the objects $W^{\lambda,!}$, map to infinitely connective objects in. This follows from the fact that $\mathrm{Av}_!^{N_k,\chi_N}(\delta_{t^\lambda,\mathrm{Gr}})$ lives in cohomological degrees $\leq -\dim(N_k \cdot t^\lambda)$.

6.3.8. *A digression: properties of t-structures.* We shall say that a t-structure on a compactly generated category $\mathbf{C}$ is *compactly generated* if:

- $(\mathbf{C})^{\leq 0}$ is generated under colimits by $\mathbf{C}^c \cap (\mathbf{C})^{\leq 0}$.

We shall say that a t-structure on a compactly generated category $\mathbf{C}$ is *coherent* if, moreover,

- Compact objects in $\mathbf{C}$ are cohomologically bounded;
- The subcategory $\mathbf{C}^c \subset \mathbf{C}$ is preserved by the truncation functors.

We shall say that a t-structure is *Noetherian* if, in addition:

- The subcategory $\mathbf{C}^c \cap (\mathbf{C})^\heartsuit \subset (\mathbf{C})^\heartsuit$ is stable under subquotients (in particular is abelian).

Finally, we will that a t-structure is *Artinian* if, moreover:

- Each object of $\mathbf{C}^c \cap (\mathbf{C})^\heartsuit \subset (\mathbf{C})^\heartsuit$ has finite length.

It is easy to see that a t-structure on $\mathbf{C}$ is Artinian if and only if irreducible objects in $(\mathbf{C})^\heartsuit$ are compact and they generate $\mathbf{C}$.

6.3.9. With the above definitions, we obtain that Corollary 6.3.5 implies the following:

Corollary 6.3.10. *The t-structure on* $\mathrm{Whit}_{q,x}(G)$ *is Artinian.*

7. The dual and the global definitions of the metaplectic Whittaker category

The goal of this section is two-fold: we will show that the metaplectic Whittaker category is essentially self-dual (as a DG category), and that one can define it alternatively using global geometry.

Both these facts will be used in the proof of the main theorem, which compares $\mathrm{Whit}_{q,x}(G)$ with a certain category of factorization modules.

7.1. **The definition as *coinvariants*.** In the previous section we defined $\mathrm{Whit}_{q,x}(G)$ as the category of $(\mathfrak{L}(N)_x^{\omega^\rho}, \chi_N)$-invariant objects in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$. We will now introduce another category, denoted $\mathrm{Whit}_{q,x}(G)_{\mathrm{co}}$, by taking $\mathfrak{L}(N)_x^{\omega^\rho}$-coinvariant objects in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$.

7.1.1. We define $\mathrm{Whit}_{q,x}(G)_{\mathrm{co}}$ to be the quotient DG category of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ by the full subcategory generated by objects

$$\mathrm{Fib}(\mathrm{Av}_*^{N_k,\chi_N}(\mathcal{F}) \to \mathcal{F}), \quad \forall \mathcal{F} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}), \forall k.$$

I.e., for a test DG category $\mathbf{C}$, the datum of a functor $\mathrm{Whit}_{q,x}(G)_{\mathrm{co}} \to \mathbf{C}$ is equivalent to the datum of a functor $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathbf{C}$ that maps all morphisms (6.8) to isomorphisms.

7.1.2. We can give a similar definition for every fixed subgroup N_k; denote the resulting category of coinvariants by

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)_{N_k,\chi_N}.$$

Note that since the co-monad $\mathrm{Av}_*^{N_k,\chi_N}$ is idempotent, the averaging functor

$$\mathrm{Av}_*^{N_k,\chi_N} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N}$$

defines a functor

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)_{N_k,\chi_N} \to \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{N_k\chi_N}. \tag{7.1}$$

Proposition 7.1.3. *The functor* (7.1) *is an equivalence.*

Proof. This is a formal assertion, valid for any idempotent co-monad acting on a DG category. □

7.1.4. Consider now the colmit DG category

$$\underset{k}{\operatorname{colim}} \left(\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N},$$

where for $k \leq k'$ the transition functor

$$\left(\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N} \to \left(\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})\right)^{N_{k'},\chi_N}$$

is given by $\operatorname{Av}_*^{N_{k'},\chi_N}$. (Compare this with the limit in (6.2), which was taken with respect to the transition functors being the forgetful functors.)

From Proposition 7.1.3 we obtain that we have a canonical equivalence

$$\underset{k}{\operatorname{colim}} \left(\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N} \simeq \operatorname{Whit}_{q,x}(G)_{\mathrm{co}}. \tag{7.2}$$

7.1.5. For a pair of indices $k \leq k'$, let $\ell_{k,k'}$ denote the *-fiber of the dualizing sheaf on $N_{k'}/N_k$ at the origin (ignoring the Galois action, this is just $\mathsf{e}[2\dim(N_{k'}/N_k)]$).

In addition to the tautological natural transformation

$$\operatorname{Av}_*^{N_{k'},\chi_N} \to \operatorname{Av}_*^{N_k,\chi_N}.$$

We have a natural transformation

$$\operatorname{Av}_*^{N_k,\chi_N} \to \ell_{k,k'} \otimes \operatorname{Av}_*^{N_{k'},\chi_N}. \tag{7.3}$$

With no restriction of generality we can assume that our set of indices has a smallest element $k = 0$ such that $N_0 = \mathfrak{L}^+(N)^{\omega^\rho}_x$. Consider the functor

$$\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x}) \to \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})$$

defined as

$$\underset{k}{\operatorname{colim}}\, \ell_{0,k} \otimes \operatorname{Av}_*^{N_k,\chi_N}, \tag{7.4}$$

where the transition maps are given by (7.3) and the isomorphisms

$$\ell_{0,k} \circ \ell_{k,k'} \simeq \ell_{0,k'}.$$

It is clear that the image of (7.4) belongs to $\left(\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})\right)^{N_k,\chi_N}$ for every k. Moreover, it maps all the morphisms of the form (6.8) to isomorphisms. Hence, (7.4) gives rise to a functor

$$\text{Ps-Id} : \operatorname{Whit}_{q,x}(G)_{\mathrm{co}} \to \operatorname{Whit}_q(\operatorname{Gr}^{\omega^\rho}_{G,x}) \tag{7.5}$$

7.1.6. For $\lambda \in \Lambda$, let $W^{\lambda,*}_{\mathrm{co}} \in \operatorname{Whit}_{q,x}(G)_{\mathrm{co}}$ denote the image of $\delta_{t^\lambda,\mathrm{Gr}}[\langle \lambda, 2\check{\rho}\rangle] \in \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x})$ under the projection

$$\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}^{\omega^\rho}_{G,x}) \to \operatorname{Whit}_{q,x}(G)_{\mathrm{co}}.$$

It follows from the definitions that for λ dominant

$$\text{Ps-Id}(W^{\lambda,*}_{\mathrm{co}}) \simeq W^{\lambda,*}; \tag{7.6}$$

(the shift by twice $\langle \lambda, 2\check{\rho}\rangle$ appears since this integer equals $\dim(\mathfrak{L}^+(N)^{\omega^\rho}/\operatorname{Stab}_{\mathfrak{L}(N)^{\omega^\rho}}(t^\lambda)))$.

Also, as in the proof of Proposition 6.2.9, it is easy to see that if λ is non-dominant, then $W^{\lambda,*}_{\mathrm{co}} = 0$.

7.1.7. In Sect. 7.4.6 we will prove:

Theorem 7.1.8. *The functor* (7.5) *is an equivalence.*

Corollary 7.1.9. *The category* $\operatorname{Whit}_{q,x}(G)_{\mathrm{co}}$ *is compactly generated.*

7.2. **Duality for the Whittaker category.** In this subsection we will show that the metaplectic Whittaker category is (essentially) self-dual as a DG category. This is not tautological, as the definition of $\mathrm{Whit}_{q,x}(G)$ involved taking invariants with respect to a group ind-scheme, and this operation is in general not self-dual.

7.2.1. Being compactly generated, the category $\mathrm{Whit}_{q,x}(G)_{\mathrm{co}}$ is dualizable. We will now construct a canonical equivalence[1]

$$(\mathrm{Whit}_{q,x}(G)_{\mathrm{co}})^{\vee} \simeq \mathrm{Whit}_{q^{-1},x}(G), \tag{7.7}$$

where q^{-1} indicates that we are taking the inverse geometric metaplectic data.

7.2.2. In order to construct (7.7), we need to identify the categories $\mathrm{Whit}_{q^{-1},x}(G)$ and

$$\mathrm{Funct}(\mathrm{Whit}_{q,x}(G)_{\mathrm{co}}, \mathrm{Vect}),$$

where the latter is, by definition, the full subcategory of

$$\mathrm{Funct}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}), \mathrm{Vect})$$

that consists of those functors that map all morphisms (6.8) to isomorphisms.

Verdier duality defines an equivalence

$$(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))^{\vee} \simeq \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x}),$$

i.e., an equivalence

$$\mathrm{Funct}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}), \mathrm{Vect}) \simeq \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x}). \tag{7.8}$$

Under this equivalence, precomposition with $\mathrm{Av}_*^{N_k,\chi_N}$ on the left-hand side of (7.8) goes over to the functor $\mathrm{Av}_*^{N_k,-\chi_N}$ on the right-hand side. Thus defines the sought-for equivalence (7.7).

7.2.3. Combining with the equivalence (7.5), we thus obtain an equivalence

$$(\mathrm{Whit}_{q,x}(G))^{\vee} \simeq \mathrm{Whit}_{q^{-1},x}(G). \tag{7.9}$$

In particular, we obtain an equivalence

$$((\mathrm{Whit}_{q,x}(G))^c)^{\mathrm{op}} \to (\mathrm{Whit}_{q^{-1},x}(G))^c \tag{7.10}$$

that we denote by $\mathbb{D}^{\mathrm{Verdier}}$.

We note that by construction, the equivalences (7.9) and (7.10) are *involutive*.

7.2.4. We claim:

Proposition 7.2.5. $\mathbb{D}^{\mathrm{Verdier}}(W^{\lambda,!}) \simeq W^{\lambda,*}$.

By the involutivity of $\mathbb{D}^{\mathrm{Verdier}}$ we then obtain:

Corollary 7.2.6. *We have* $\mathbb{D}^{\mathrm{Verdier}}(W^{\lambda,*}) \simeq W^{\lambda,!}$ *and* $\mathbb{D}^{\mathrm{Verdier}}(W^{\lambda,!*}) \simeq W^{\lambda,!*}$.

Corollary 7.2.7. *A compact object* $\mathcal{F} \in \mathrm{Whit}_{q,x}(G)$ *is connective with respect to the t-structure if and only if* $\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \in \mathrm{Whit}_{q^{-1},x}(G)$ *is coconnective.*

Proof of Proposition 7.2.5. Taking into account (7.6), we need to show that the functor $\mathrm{Whit}_{q,x}(G) \to \mathrm{Vect}$, given by

$$\mathcal{F} \mapsto \mathcal{M}aps_{\mathrm{Whit}_{q,x}(G)}(W^{\lambda,!}, \mathcal{F}), \tag{7.11}$$

identifies with the functor

$$\mathcal{F} \mapsto \langle W^{\lambda,*}_{\mathrm{co}}, \mathcal{F} \rangle, \tag{7.12}$$

where $\langle -, - \rangle$ denotes the canonical pairing

$$\mathrm{Whit}_{q,x}(G) \otimes \mathrm{Whit}_{q^{-1},\mathrm{co}}(G) \to \mathrm{Vect}\,.$$

[1] Up to replacing the Artin-Schreier sheaf by its inverse.

The functor (7.12) is by definition

$$\mathcal{F} \mapsto \langle \mathcal{F}, \delta_{t^\lambda,\mathrm{Gr}} \rangle [\langle \lambda, 2\check{\rho} \rangle],$$

where $\langle -,- \rangle$ now denotes the canonical pairing

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \otimes \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathrm{Vect}\,.$$

However, again by definition, $\langle \mathcal{F}, \delta_{t^\lambda,\mathrm{Gr}} \rangle$ is given by taking the !-fiber of $\mathcal{F}$ at t^λ. Now, by the definition of $W^{\lambda,!}$, the expression in (7.11) is also given by taking the !-fiber of $\mathcal{F}$ at t^λ, shifted by $[\langle \lambda, 2\check{\rho} \rangle]$.

□

7.3. **The global definition.** In this subsection we will explore a different way to define the metaplectic Whittaker category, where instead of the affine Grassmannian we will use a "global" algebro-geometric object. The advantage of this approach is that it provides a finite-dimensional model for $\mathrm{Whit}_{q,x}(G)$.

7.3.1. In this subsection we take X to be complete. Let $(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}$ be the version of Drinfeld's compactification introduced in [Ga9, Sect. 4.1]. Namely, $(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}$ classifies the data of a G-bundle $\mathcal{P}_G$ on X and that of injective maps of coherent sheaves

$$\kappa^{\check{\lambda}} : (\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\infty \cdot x), \quad \check{\lambda} \in \check{\Lambda}^+ \tag{7.13}$$

(here $\mathcal{V}^{\check{\lambda}}$ denotes the Weyl module of highest weight $\check{\lambda}$), such that the maps $\kappa^{\check{\lambda}}$ satisfy the Plücker relations, i.e., they define a reduction of $\mathcal{P}_G$ to B at the generic point of X.

Remark 7.3.2. When the derived group of G is not simply connected, in addition to the Plücker relations one imposes another closed condition, restricting the possible defect of the maps (7.13), see [Sch, Sect. 7]. However, for the purposes of defining the global Whittaker category, the difference is material, as the objects satisfying the Whittaker condition will be automatically supported on the closed substack in question.

7.3.3. For $\lambda \in \Lambda$, let

$$(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\leq \lambda\cdot x} \overset{\bar{\mathbf{i}}_\lambda}{\hookrightarrow} (\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}$$

be the closed substack where we require that for every $\check{\lambda} \in \check{\Lambda}^+$, the corresponding map (7.13) has a pole of order $\leq \langle \check{\lambda}, \lambda \rangle$.

We denote by

$$(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{=\lambda\cdot x} \overset{\mathbf{j}_\lambda}{\hookrightarrow} (\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\leq \lambda\cdot x}$$

the open substack, where we require that for for every $\check{\lambda} \in \check{\Lambda}^+$, the corresponding map (7.13) has a pole of order equal $\langle \check{\lambda}, \lambda \rangle$, *and is non-vanishing* at other points of X.

Set

$$\mathbf{i}_\lambda = \bar{\mathbf{i}}_\lambda \circ \mathbf{j}_\lambda.$$

We note that the strata $\overline{\mathrm{Bun}}^{\omega^\rho}_{N,=\lambda'\cdot x}$ with $\lambda' \leq \lambda$ *do not* cover all of $(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\leq \lambda\cdot x}$. Namely, they miss all the points for which the maps (7.13) have zeroes on $X - x$.

Note that the stack $\overline{\mathrm{Bun}}^{\omega^\rho}_{N,=0\cdot x}$ identifies with

$$\mathrm{Bun}^{\omega^\rho}_N := \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \{\omega^\rho\}.$$

7.3.4. According to Sect. 2.1.8, the geometric metaplectic data $\mathcal{G}^G$ descends to a gerbe, also denoted $\mathcal{G}^G$, on Bun_G. We will denote by the same symbol $\mathcal{G}^G$ the gerbe on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$ equal to the ratio of the pullback of $\mathcal{G}^G$ along the forgetful map

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x} \to \mathrm{Bun}_G$$

and the fiber of $\mathcal{P}^G$ at the point $\omega^\rho \in \mathrm{Bun}_G$.

Note that wit this definition, the restriction of $\mathcal{G}^G$ to the locally closed substack

$$\mathrm{Bun}_N^{\omega^\rho} \hookrightarrow (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$$

is canonically trivialized.

7.3.5. Inside $\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})$ one singles out a full subcategory, denoted $\mathrm{Whit}_{q,\mathrm{glob}}(G)$, by imposing the condition of equivariance with respect to a certain unipotent groupoid. We refer the reader to [Ga9, Sects. 4.4-4.7], where this equivariance condition is written out in detail. We note that in Remark 8.2.5 below we will give another (but of course equivalent) way to characterize this subcategory.

The embedding

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})$$

is compatible with the (perverse) t-structure on $\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})$, and admits a continuous right adjoint, denoted $\mathrm{Av}_*^{N_{\mathrm{glob}},\chi_N}$, see [Ga9, Corollary 4.7.4].

7.3.6. Replacing $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$ by $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}$ or $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}$, one has the similarly defined full subcategories

$$\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda} \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \text{ and } \mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda} \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}).$$

The adjunctions

$$(\bar{\mathbf{i}}_\lambda)_! : \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \rightleftarrows \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) : \bar{\mathbf{i}}_\lambda^!$$

and

$$\mathbf{j}_\lambda^* : \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \rightleftarrows \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}) : (\mathbf{j}_\lambda)_*$$

give rise to commutative diagrams

$$\begin{tikzcd}
\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda} \arrow[r, shift left, "(\bar{\mathbf{i}}_\lambda)_!"] \arrow[d] & \mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda} \arrow[l, shift left, "\bar{\mathbf{i}}_\lambda^!"] \arrow[d] \\
\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \arrow[r, shift left, "(\bar{\mathbf{i}}_\lambda)_!"] & \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \arrow[l, shift left, "\bar{\mathbf{i}}_\lambda^!"]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda} \arrow[r, shift left, "\mathbf{j}_\lambda^*"] \arrow[d] & \mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda} \arrow[l, shift left, "(\mathbf{j}_\lambda)_*"] \arrow[d] \\
\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \arrow[r, shift left, "\mathbf{j}_\lambda^*"] & \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}). \arrow[l, shift left, "(\mathbf{j}_\lambda)_*"]
\end{tikzcd}$$

Furthermore, the diagrams

$$\begin{tikzcd}
\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda} \arrow[r, shift left, "(\bar{\mathbf{i}}_\lambda)_!"] & \mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda} \arrow[l, shift left, "\bar{\mathbf{i}}_\lambda^!"] \\
\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \arrow[r, shift left, "(\bar{\mathbf{i}}_\lambda)_!"] \arrow[u, "\mathrm{Av}_*^{N_{\mathrm{glob}},\chi_N}"] & \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \arrow[l, shift left, "\bar{\mathbf{i}}_\lambda^!"] \arrow[u, "\mathrm{Av}_*^{N_{\mathrm{glob}},\chi_N}"']
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda} \arrow[r, shift left, "\mathbf{j}_\lambda^*"] & \mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda} \arrow[l, shift left, "(\mathbf{j}_\lambda)_*"] \\
\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}) \arrow[u, "\mathrm{Av}_*^{N_{\mathrm{glob}},\chi_N}"] \arrow[r, shift left, "\mathbf{j}_\lambda^*"] & \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}). \arrow[u, "\mathrm{Av}_*^{N_{\mathrm{glob}},\chi_N}"'] \arrow[l, shift left, "(\mathbf{j}_\lambda)_*"]
\end{tikzcd}$$

commute as well.

The partially defined functor

$$(\mathbf{j}_\lambda)_! : \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}),$$

left adjoint to $\mathbf{j}_\lambda^*$ is defined on $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda}$, and makes the diagram

$$\begin{tikzcd}
\mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda} \arrow[r, "(\mathbf{j}_\lambda)_!"] \arrow[d] & \mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda} \arrow[d] \\
\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}) \arrow[r, "(\mathbf{j}_\lambda)_!"] & \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x})
\end{tikzcd}$$

commute.

Finally, we have:

Lemma 7.3.7.

(a) *The category* $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda}$ *is zero unless* λ *is dominant.*

(b) *For* λ *dominant, the category* $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda}$ *is non-canonically equivalent to* Vect*; its generator is locally constant (as an object of* $\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x})$*);*

(c) *Every object of* $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda}$*, whose restriction to* $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}$ *vanishes, is supported on* $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{<\lambda\cdot x} := \bigcup\limits_{\lambda'<\lambda} \overline{\mathrm{Bun}}_{N,\leq\lambda'\cdot x}^{\omega^\rho}$.

Corollary 7.3.8. *For every* $\lambda\in\Lambda^+$ *there exists a* quasi-compact *open substack* $U\subset(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\leq\lambda\cdot x}$*, such that every object of* $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{\leq\lambda}$ *is a* clean extension *of its restriction to* U.

7.3.9. For λ dominant, pick a generator of $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda}$ that is perverse on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\lambda\cdot x}$, and let $W^{\lambda,!}_{\mathrm{glob}}\in\mathrm{Whit}_{q,\mathrm{glob}}(G)$ (resp., $W^{\lambda,*}_{\mathrm{glob}}\in\mathrm{Whit}_{q,\mathrm{glob}}(G)$) be obtained by applying to it the functor $(\mathbf{i}_\lambda)_! := (\bar{\mathbf{i}}_\lambda)_!\circ(\mathbf{j}_\lambda)_!$ (resp., $(\mathbf{i}_\lambda)_* := (\bar{\mathbf{i}}_\lambda)_!\circ(\mathbf{j}_\lambda)_*$).

Let also $W^{\lambda,!*}_{\mathrm{glob}}$ be the Goresky-MacPherson extension of the above object in $\mathrm{Whit}_{q,\mathrm{glob}}(G)_{=\lambda}$. The objects $W^{\lambda,!*}_{\mathrm{glob}}$ are the irreducibles in $(W^{\lambda,!}_{\mathrm{glob}})^\heartsuit$.

It follows from Lemma 7.3.7 that the objects $W^{\lambda,!}_{\mathrm{glob}}$ generate $\mathrm{Whit}_{q,\mathrm{glob}}(G)$. Since the open embedding $(\mathbf{j}_\lambda)_*$ is affine (see [FGV, Sect. Prop. 3.3.1]), we have

$$W^{\lambda,!}_{\mathrm{glob}}, W^{\lambda,*}_{\mathrm{glob}} \in (\mathrm{Whit}_{q,\mathrm{glob}}(G))^\heartsuit.$$

They are of finite length and compact as objects of $\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})$ by Corrollary 7.3.7.

Furthermore, an object $\mathcal{F}\in\mathrm{Whit}_{q,\mathrm{glob}}(G)$ is coconnective if and only if

$$\mathrm{Hom}_{\mathrm{Whit}_{q,\mathrm{glob}}(G)}(W^{\lambda,!}_{\mathrm{glob}}[k],\mathcal{F}) = 0$$

for all $\lambda\in\Lambda^+$ and $k>0$.

7.3.10. It follows from Corrollary 7.3.7 that Verdier duality for $\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})^{\mathrm{loc.c}}$ (see Sect. 4.3.3) defines an equivalence[2]

$$(\mathrm{Whit}_{q,\mathrm{glob}}(G))^\vee \simeq \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G). \tag{7.14}$$

Denote the resulting equivalence

$$((\mathrm{Whit}_{q,\mathrm{glob}}(G))^c)^{\mathrm{op}} \to (\mathrm{Whit}_{q^{-1},\mathrm{glob}}(G))^c \tag{7.15}$$

by $\mathbb{D}^{\mathrm{Verdier}}$.

We have

$$\mathbb{D}^{\mathrm{Verdier}}(W^{\lambda,!}_{\mathrm{glob}}) \simeq W^{\lambda,*}_{\mathrm{glob}} \text{ and } \mathbb{D}^{\mathrm{Verdier}}(W^{\lambda,*}_{\mathrm{glob}}) \simeq W^{\lambda,!}_{\mathrm{glob}}.$$

7.4. **The local vs global equivalence.** In this subsection we will state a theorem to the effect that the global Whittaker category $\mathrm{Whit}_{q,\mathrm{glob}}(G)$ is equivalent to the local Whittaker category $\mathrm{Whit}_{q,x}(G)$ defined earlier.

7.4.1. We have a natural projection

$$\pi_x : \mathrm{Gr}^{\omega^\rho}_{G,x} \to (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}.$$

Note that, according to our conventions, the gerbe on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$ that we denoted $\mathcal{G}^G$ pulls back to the gerbe $\mathcal{G}^G$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$. Consider the corresponding pullback functor

$$\pi_x^! : \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}).$$

According to [Ga9, Theorem 5.1.4(a)], the functor $\pi_x^!$ sends $\mathrm{Whit}_{q,\mathrm{glob}}(G)$ to

$$\mathrm{Whit}_{q,x}(G) \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}).$$

In Sect. 8.3 we will prove:

Theorem 7.4.2. *The resulting functor*

$$\pi_x^! : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Whit}_{q,x}(G)$$

is an equivalence. Moreover, after applying the cohomological shift by

$$d_g := \dim(\mathrm{Bun}_N^{\omega^\rho}) = (g-1)(d - \langle 2\check{\rho}, 2\rho\rangle), \quad d = \dim(\mathfrak{n}),$$

it is t-exact and sends standards (resp., costandards) to standards (resp., costandards).

7.4.3. Some remarks are in order.

First off, it is easy to see from the definitions that $\pi_x^!$, shifted cohomologically by d_g sends $W^{\mu,*}_{\mathrm{glob}}$ to $W^{\mu,*}$. Since the latter objects generate $\mathrm{Whit}_{q,x}(G)$, in order to prove that $\pi_x^!$ is an equivalence, it suffices to show that it is fully faithful. The proof of fully faithfulness will be given in Sect. 8.3.1.

Second, we have a tautological map

$$W^{\mu,!} \to \pi_x^!(W^{\mu,!}_{\mathrm{glob}})[d_g] \tag{7.16}$$

If we assume that $\pi_x^!$ is fully faithful, we obtain that the map (7.16) induces an isomorphism on maps into any $W^{\mu',*}$. Since the latter objects *co-generate* $\mathrm{Whit}_{q,x}(G)$, we obtain that (7.16) is an isomorphism.

Since the t-structures on both $\mathrm{Whit}_{q,x}(G)$ and $W^{\mu,*}_{\mathrm{glob}}$ are characterized in terms of the objects $W^{\mu,!}$ and $W^{\mu,!}_{\mathrm{glob}}$, respectively, we obtain that the functor

$$\pi_x^![d_g]$$

is t-exact.

This implies the assertion of Proposition 6.3.2.

[2] Up to replacing the Artin-Schreier sheaf by its inverse.

7.4.4. Since the morphism π_x is ind-schematic, we have a well-defined functor

$$(\pi_x)_* : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty \cdot x}).$$

Consider the composite

$$\mathrm{Av}^{N_{\mathrm{glob}},\chi_N}_* \circ (\pi_x)_* : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathrm{Whit}_{q,\mathrm{glob}}(G).$$

It is not difficult to show (see [Ga9, Lemma 5.3.3]) that the above functor factors through a functor

$$\mathrm{Whit}_{q,x}(G)_{\mathrm{co}} \to \mathrm{Whit}_{q,\mathrm{glob}}(G); \tag{7.17}$$

moreover, the latter is the functor dual to

$$\pi_x^! : \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G) \to \mathrm{Whit}_{q^{-1},x}(G)$$

in terms of the identifications

$$\mathrm{Whit}_{q,\mathrm{glob}}(G)^\vee \simeq \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G) \text{ and } \mathrm{Whit}_{q,x}(G)^\vee \simeq \mathrm{Whit}_{q^{-1},\mathrm{co}}(G).$$

By a slight abuse of notation, we will denote the functor appearing in (7.17) by the same character $\mathrm{Av}^{N_{\mathrm{glob}},\chi_N}_* \circ (\pi_x)_*$. From Theorem 7.4.2 we obtain:

Corollary 7.4.5. *The functor* (7.17) *is an equivalence of categories.*

7.4.6. Consider now the composite functor

$$\mathrm{Whit}_{q,x}(G)_{\mathrm{co}} \overset{\mathrm{Av}^{N_{\mathrm{glob}},\chi_N}_* \circ (\pi_x)_*}{\longrightarrow} \mathrm{Whit}_{q,\mathrm{glob}}(G) \xrightarrow{\pi_x^!} \mathrm{Whit}_{q,x}(G). \tag{7.18}$$

It follows from the construction (see [Ga9, Corollary 5.4.5]) that the functor (7.18) identifies with the functor

$$\text{Ps-Id} \otimes \ell_g,$$

where Ps-Id is as in (7.5), where ℓ_g is line equal to the !-fiber of the constant sheaf on $\mathrm{Bun}^{\omega^\rho}_N$ (at any k-point).

Thus, we obtain that Theorem 7.4.2 (combined with Corrollary 7.4.5) implies Theorem 7.1.8.

Remark 7.4.7. It follows by unwinding the constructions that the equivalence of Theorem 7.4.2 intertwines the duality equivalences

$$(\mathrm{Whit}_{q,\mathrm{glob}}(G))^\vee \simeq \mathrm{Whit}_{q^{-1},\mathrm{glob}}$$

of (7.14) and

$$(\mathrm{Whit}_{q,x}(G))^\vee \simeq \mathrm{Whit}_{q^{-1},x}(G)$$

of (7.9), up to tensoring by ℓ_g.

8. The Ran version and proof of Theorem 7.4.2

This section is ostensibly devoted to the proof of Theorem 7.4.2. However, in the process, we will introduce another player–the Ran version of the Whittaker category.

It will play a crucial role in the sequel as it will provide one of the ingredients for the construction of the *factorization enhancement* of the Jacquet functor.

8.1. **The Ran version of the semi-infinite orbit.** In this subsection we will state a general fully-faithfulness result that allows to compare categories of sheaves on $(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty \cdot x}$ and (various versions of) the affine Grassmannian.

8.1.1. Consider the Ran Grassmannian $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ and its version $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}$, see Sect. 1.5.5. We let

$$(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x} \subset \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}$$

be the closed subfunctor given by the following condition:

A point $(\mathcal{I}, \mathcal{P}_G, \alpha)$ belongs to $(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}$ if for every dominant weight $\check{\lambda}$, the composite meromorphic map

$$(\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \xrightarrow{\alpha} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G}(\infty\cdot x) \tag{8.1}$$

is *regular on* $X - x$, where:

- $\mathcal{P}'_G$ denotes the G-bundle induced from the T-bundle ω^ρ;
- The map $(\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G}$ corresponds to the highest weight vector in $\mathcal{V}^\lambda$.

Note that we have a Cartesian square

$$\begin{tikzcd} \mathrm{Gr}^{\omega^\rho}_{G,x} \arrow[r] \arrow[d] & (\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x} \arrow[d] \\ \mathrm{pt} \arrow[r] & \mathrm{Ran}_x, \end{tikzcd}$$

where $\mathrm{pt} \to \mathrm{Ran}_x$ corresponds to the point $\{x\}$.

8.1.2. Let us denote by π_{Ran_x} the natural forgetful map

$$(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x} \to (\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}.$$

Note the composite

$$\mathrm{Gr}^{\omega^\rho}_{G,x} \hookrightarrow (\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x} \xrightarrow{\pi_{\mathrm{Ran}_x}} (\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}$$

is the map that we have earlier denoted by π_x.

8.1.3. Note that the pullback of the gerbe $\mathcal{G}^G$ on $(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}$ identifies with the gerbe $\mathcal{G}^G$ on $(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}$. Hence, we have a well-defined functor

$$\pi^!_{\mathrm{Ran}_x} : \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}). \tag{8.2}$$

We claim:

Theorem 8.1.4. *The functor* (8.2) *is fully faithful.*

We omit the proof of this theorem as it repeats verbatim the proof of [Ga7, Theorem 3.4.4].

8.2. **The Ran version of the metaplectic Whittaker category.** In this subsection we will introduce another key player–the Ran version of the Whittaker category. It will play a technical role in the proof of Theorem 7.4.2, and also a central role in the construction of the functor in the main theorem.

8.2.1. Recall the group ind-schemes over Ran denoted

$$\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \subset \mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}},$$

see Sects. 1.4.3-1.4.7. Note that as in the case of $\mathfrak{L}(N)^{\omega^\rho}_x$ we have a canonically defined homomorphism $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \to \mathbb{A}^1$. We denote by the same character χ_N the pullback of the Artin-Schreier sheaf χ to $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}}$.

As in the case of $\mathfrak{L}(N)^{\omega^\rho}_x$, the restriction of the multiplicative gerbe $\mathcal{G}^G$ along

$$\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \to \mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Ran}}$$

admits a unique trivialization, normalized by the requirement that it is the tautological one on the unit section.

8.2.2. Let $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x}$ denote the pullback of $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}}$ along the map $\mathrm{Ran}_x \to \mathrm{Ran}$. Note that $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x}$ acts on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}$, preserving $(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}$. Hence, it makes sense to talk about the categories

$$\mathrm{Whit}_{q,\mathrm{Ran}_x}(G) := \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x},\chi_N}$$

and

$$\mathrm{Whit}_{q,\mathrm{Ran}_x}(G)^{\leq 0}_{\infty\cdot x} := \left(\mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x},\chi_N},$$

the latter being a full subcategory of the former consisting of objects that are supported on $(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x} \subset \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}$.

8.2.3. One shows (see [Ga9, Corollary 6.2.2]):

Proposition 8.2.4. *The pullback functor* $\pi^!_{\mathrm{Ran}_x}$ *sends*

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x})$$

to

$$\mathrm{Whit}_{q,\mathrm{Ran}_x}(G)^{\leq 0}_{\infty\cdot x} \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}).$$

Remark 8.2.5. One can show (see [Ga9, Theorem 5.1.4(b)]) that

$$\begin{array}{ccc} \mathrm{Whit}_{q,\mathrm{Ran}_x}(G)^{\leq 0}_{\infty\cdot x} & \longrightarrow & \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}) \\ \uparrow & & \uparrow \\ \mathrm{Whit}_{q,\mathrm{glob}}(G) & \longrightarrow & \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}) \end{array}$$

is a pullback diagram, i.e., full subcategory $\mathrm{Whit}_{q,\mathrm{glob}}(G) \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x})$ consists exactly of those objects that satisfy the Whittaker equivariance condition when pulled back to $(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}$.

8.2.6. Note now that there is a tautological map

$$\mathrm{unit} : \mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x},$$

whose image belongs to $(\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}$: namely, a G-bundle trivialized away from x can be thought of as trivialized away from a finite set of points containing $\{x\}$.

We have the following crucial result, whose proof repeats verbatim the proof of [Ga9, Theorem 6.2.5].

Theorem 8.2.7. *The functor*

$$\mathrm{unit}^! : \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x})$$

defines an equivalence from $\mathrm{Whit}_{q,\mathrm{Ran}_x}(G)^{\leq 0}_{\infty\cdot x}$ *to*

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N}.$$

We note that in the statement of Theorem 8.2.7, the target category is a version of $\mathrm{Whit}_{q,x}(G)$, where instead of $\mathrm{Gr}^{\omega^\rho}_{G,x}$ we take $\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x}$, with $\mathfrak{L}(N)^{\omega^\rho}_x$ acting trivially on Ran_x.

8.2.8. In what follows we will denote the functor

$$\mathrm{Whit}_{q,x}(G) = \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \to \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \to$$
$$\to \left(\mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}_x},\chi_N} \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})$$

by $\mathrm{sprd}_{\mathrm{Ran}_x}$, where in the above formula, the second arrow is !-pullback along the projection

$$\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{G,x},$$

and the third arrow is the equivalence inverse to one in Theorem 8.2.7.

Remark 8.2.9. The functor $\mathrm{sprd}_{\mathrm{Ran}_x}$ encodes the *unital structure* on $\mathrm{Whit}_{q,x}(G)$, viewed as a factorization category.

8.3. **Proof of Theorem 7.4.2.** In this subsection we will combine Theorems 8.1.4 and 8.2.7 to prove Theorem 7.4.2.

8.3.1. According to Sect. 7.4.3, it suffices to show that the functor

$$(\pi_x)^! : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Whit}_{q,x}(G)$$

is fully faithful. We factor the above functor as a composite

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \overset{\pi^!_{\mathrm{Ran}_x}}{\longrightarrow} \mathrm{Whit}_{q,\mathrm{Ran}_x}(G)^{\leq 0}_{\infty\cdot x} \overset{\mathrm{unit}^!}{\longrightarrow} \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x,\chi_N} \to \mathrm{Whit}_{q,x}(G),$$

where the last arrow is restriction along

$$\{x\} \times \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x}.$$

8.3.2. According to Theorem 8.1.4, the first arrow is fully faithful and, according to Theorem 8.2.7, the second arrow is an equivalence. Hence, the functor of pullback along $\pi_{\mathrm{Ran}_x} \circ \mathrm{unit}$ is fully faithful when restricted to $\mathrm{Whit}_{q,\mathrm{glob}}(G)$.

8.3.3. Note, however, that the map $\pi_{\mathrm{Ran}_x} \circ \mathrm{unit}$ factors as

$$\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{G,x} \overset{\pi_x}{\longrightarrow} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}.$$

Hence, $\pi^!_x$, restricted to $\mathrm{Whit}_{q,\mathrm{glob}}(G)$ is a retract of a fully faithful functor, and therefore is itself fully faithful.

□[Theorem 7.4.2]

8.4. **The non-marked case.** For future reference, we will discuss variants of the constructions in Sects. 8.1-8.2 *without* the marked point x.

8.4.1. We define the closed subfunctor

$$\overline{S}^0_{\mathrm{Ran}} \subset \mathrm{Gr}_{G,\mathrm{Ran}}$$

as follows:

A point $(\mathcal{I}, \mathcal{P}_G, \alpha)$ belongs to $\overline{S}^0_{\mathrm{Ran}}$ if for every dominant weight $\check{\lambda}$, the corresponding meromorphic map

$$(\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \overset{\alpha}{\longrightarrow} \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \tag{8.3}$$

is regular on all of X (we retain the notations used in (8.3)).

8.4.2. We define the categories

$$\mathrm{Whit}_{q,\mathrm{Ran}}(G) := \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}},\chi_N}$$

and

$$\mathrm{Whit}_{q,\mathrm{Ran}}(G)^{\leq 0} := \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}},\chi_N}.$$

8.4.3. We have the tautological section

$$\mathrm{unit} : \mathrm{Ran} \to \overline{S}^0_{\mathrm{Ran}},$$

and a counterpart of Theorem 8.2.7 says that the functor

$$\mathrm{unit}^! : \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}}) \to \mathrm{Shv}(\mathrm{Ran})$$

induces an equivalence from

$$\mathrm{Whit}_{q,\mathrm{Ran}}(G)^{\leq 0} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}}) \tag{8.4}$$

to Shv(Ran).

8.4.4. We let

$$\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}} \in \mathrm{Whit}_{q,\mathrm{Ran}}(G)^{\leq 0}$$

denote the object equal to the image of $\omega_{\mathrm{Ran}} \in \mathrm{Shv}(\mathrm{Ran})$ under the equivalence inverse to the above equivalence

$$\mathrm{Whit}_{q,\mathrm{Ran}}(G)^{\leq 0} \to \mathrm{Whit}_{q,\mathrm{Ran}}(G)^{\leq 0}.$$

Sometimes, by a slight abuse of notation, we will regard $\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}$ as an object just of $\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})$ or $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$.

Note that the restriction

$$\mathrm{Vac}_{\mathrm{Whit},x} := \mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}|_{\mathrm{Gr}^{\omega^\rho}_{G,x}} \in \mathrm{Whit}_{q,x}(G)$$

identifies with the object

$$W^{0,*} \simeq W^{0,!*} \simeq W^{0,!}.$$

8.4.5. The following results from the equivalences (8.4) and Theorem 8.2.7:

Theorem 8.4.6.
(a) *The object* $\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}$ *has a structure of factorization algebra in* $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$*, uniquely characterized by the requirement that the induced factorization algebra structure on*

$$\mathrm{unit}^!(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}) \in \mathrm{Shv}(\mathrm{Ran})$$

corresponds to the tautological one on ω_{Ran} *(see Sect. 3.2.2) with respect to the identification*

$$\mathrm{unit}^!(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}) \simeq \omega_{\mathrm{Ran}}.$$

(b) *The functor*

$$\mathrm{sprd}_{\mathrm{Ran}_x} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})$$

lifts to a functor

$$\mathrm{sprd}_{\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})),$$

uniquely characterized by the requirement that the composite functor

$$\mathrm{unit}^! \circ \mathrm{sprd}_{\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \omega_{\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x}))$$

identifies with the composite

$$\mathrm{Whit}_{q,x}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})) \to \omega_{\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x})),$$

where the second arrow is the functor of Sect. 3.3.9.

Part III: Hecke action and Hecke eigen-objects

The main theorem of this paper describes not the category $\mathrm{Whit}_{q,x}(G)$ itself, but rather its *de-equivariantization* with respect to the Hecke action. In this Part we will introduce and study the resulting category, denoted $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$.

9. Hecke action on the metaplectic Whittaker category

In this section we study the Hecke action of the category of representations of the group H (see Sect. 2.3.6) on $\mathrm{Whit}_{q,x}(G)$. This is a structure needed to define $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$.

9.1. **Definition of the Hecke action on the Whittaker category.** In this subsection we define the Hecke action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_{q,x}(G)$.

9.1.1. Note that

$$\mathrm{Sph}_q(G) := \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,x})\right)^{\mathfrak{L}^+(G)_x}$$

identifies *canonically* with

$$\left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}^+(G)^{\omega^\rho}_x}.$$

Therefore, we obtain a (right) action of $\mathrm{Sph}_q(G)$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ by convolutions (on the right), which preserves the action of $\mathfrak{L}(G)^{\omega^\rho}$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ by left translations.

Convention: *For the duration of this section, in order to unburden the notation, we will omit the superscript ω^ρ and the subscript x, so $\mathrm{Gr}^{\omega^\rho}_{G,x}$ will be denoted simply by Gr_G.*

9.1.2. The action of $\mathrm{Sph}_q(G)$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$ commutes with the functors $\mathrm{Av}_*^{N_k,\chi_N}$. Thus, we obtain that the action of $\mathrm{Sph}_q(G)$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$ preserves the full subcategory

$$\mathrm{Whit}_q(G) \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G).$$

Thus, we obtain a monoidal right action of $\mathrm{Sph}_q(G)$ on $\mathrm{Whit}_q(G)$.

9.1.3. Combining with metaplectic geometric Satake (see Sect. 2.3.8), we obtain a monoidal action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_q(G)$. We refer to it as the *Hecke action*, and will denote it by

$$\mathcal{F}, V \mapsto \mathcal{F} \star \mathrm{Sat}_{q,G}(V).$$

9.2. **Hecke action and duality.** In this subsection we will study the interaction of the Hecke action and self-duality of $\mathrm{Whit}_q(G)$.

9.2.1. Since the action of $\mathrm{Sph}_q(G)$ on on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$ commutes with the functors $\mathrm{Av}_*^{N_k,\chi_N}$, we obtain a canonically defined action of $\mathrm{Sph}_q(G)$ also on the category $\mathrm{Whit}_q(G)_{\mathrm{co}}$.

9.2.2. By construction, the functor

$$\mathrm{Ps\text{-}Id} : \mathrm{Whit}_q(G)_{\mathrm{co}} \to \mathrm{Whit}_q(G)$$

intertwines the actions of $\mathrm{Sph}_q(G)$ on both sides.

In particular, we obtain that $\mathrm{Whit}_q(G)_{\mathrm{co}}$ is a $\mathrm{Rep}(H)$-module category, and the functor Ps-Id is a map of such.

9.2.3. Since the convolution action of $\mathrm{Sph}_q(G)$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$ is given by a *proper pushforward*, we have a commutative diagram of actions

$$\begin{tikzcd}
(\mathrm{Sph}_q(G)^c)^{\mathrm{op}} \otimes ((\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \arrow[r] \arrow[d, "\mathbb{D}^{\mathrm{Verdier}} \otimes \mathbb{D}^{\mathrm{Verdier}}"'] & ((\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \arrow[d, "\mathbb{D}^{\mathrm{Verdier}}"] \\
\mathrm{Sph}_{q^{-1}}(G)^c \otimes (\mathrm{Whit}_{q^{-1},\mathrm{co}}(G))^c \arrow[r] & (\mathrm{Whit}_{q^{-1},\mathrm{co}}(G))^c,
\end{tikzcd}$$

where the equivalence

$$\mathbb{D}^{\mathrm{Verdier}} : ((\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \simeq (\mathrm{Whit}_{q^{-1},\mathrm{co}}(G))^c$$

is that induced by (7.7).

9.2.4. Combining with Sect. 2.5, we obtain a commutative diagram of actions

$$\begin{tikzcd}
(\mathrm{Rep}(H)^c)^{\mathrm{op}} \otimes ((\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \arrow[r] \arrow[d, "(\tau^H \circ \mathbb{D}^{\mathrm{lin}}) \otimes \mathbb{D}^{\mathrm{Verdier}}"'] & ((\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \arrow[d] \\
\mathrm{Rep}(H)^c \otimes (\mathrm{Whit}_{q^{-1}}(G))^c \arrow[r] & (\mathrm{Whit}_{q^{-1}}(G))^c.
\end{tikzcd} \tag{9.1}$$

where we remind that τ^H denoted the Cartan involution on H, see Sect. 2.5.4. In the above diagram, we denoted by

$$\mathbb{D}^{\mathrm{Verdier}} : ((\mathrm{Whit}_q(G))^c)^{\mathrm{op}} \simeq (\mathrm{Whit}_{q^{-1}}(G))^c$$

the identification of (7.10).

9.3. **Hecke action and the t-structure.** In this subsection we will study how the Hecke action interacts with the t-structure on $\mathrm{Whit}_q(G)$, which was introduced in Sect. 6.3.

9.3.1. We claim:

Proposition 9.3.2. *For $V \in \mathrm{Rep}(H)^{\heartsuit}$, the corresponding Hecke functor*

$$\mathcal{F} \mapsto \mathcal{F} \star \mathrm{Sat}_{q,G}(V) \tag{9.2}$$

is t-exact.

Before we prove this proposition, having future needs in mind, we will perform a certain elementary but crucial calculation.

9.3.3. *Stalks of the convolution, 1-st approximation.* In this subsection we will give an explicit expression for the cohomology of the !-fiber at t^μ of $W^{\lambda,*} \star \mathrm{Sat}_{q,G}(V)$ for $V \in \mathrm{Rep}(H)$.

First off, we note that the fiber in question vanishes unless $\mu \in \Lambda^+$ (by Proposition 6.2.9(a)), so henceforth we will assume that μ is dominant.

Consider the ind-subscheme

$$S^{\mu-\lambda} = \mathfrak{L}(N) \cdot t^{\mu-\lambda} \subset \mathrm{Gr}_G.$$

Let χ_N^λ be the character sheaf on $\mathfrak{L}(N)$ obtained from χ_N by pullback with respect to the automorphism Ad_{t^λ}. Since μ was assumed dominant, χ_N^λ descends to a well-defined object of $\mathrm{Shv}(S^{\mu-\lambda})$, which we denote by the same character χ_N^λ.

Since $\mathfrak{L}(N)$ is ind-pro-unipotent, the restriction of $\mathcal{G}^G$ to $\mathfrak{L}(N)$ is canonically trivialized, and the restriction of $\mathcal{G}^G$ to $S^{\mu-\lambda}$ admits a unique (up to a non-canonical isomorphism) $\mathfrak{L}(N)$-equivariant trivialization. Due to this trivialization, we can regard $\mathrm{Sat}_{q,G}(V)|_{S^{\mu-\lambda}}$ as an object of the non-twisted category $\mathrm{Shv}(S^{\mu-\lambda})$.

By unwinding the definitions, we obtain:

$$(W^{\lambda,*} \star \mathrm{Sat}_{q,G}(V))_{t^\mu} \simeq H(S^{\mu-\lambda}, \mathrm{Sat}_{q,G}(V)|_{S^{\mu-\lambda}} \otimes \chi_N^\lambda)[\langle \lambda, 2\check{\rho}\rangle]. \tag{9.3}$$

9.3.4. *Stalks of the convolution, 2nd approximation.* We now claim that the expression in (9.3) lives in the cohomological degrees $\geq -\langle \mu, 2\check{\rho}\rangle$. I.e., we claim that

$$H(S^{\mu-\lambda}, \mathrm{Sat}_{q,G}(V)|_{S^{\mu-\lambda}} \otimes \chi_N^\lambda) \tag{9.4}$$

lives in cohomological degrees $\geq \langle \lambda - \mu, 2\check{\rho}\rangle$.

We stratify $S^{\mu-\lambda}$ by the intersections $S^{\mu-\lambda} \cap \mathrm{Gr}_G^\nu$ with $\nu \in \Lambda^+$. So, it sufficient to show that each

$$H(S^{\mu-\lambda} \cap \mathrm{Gr}_G^\nu, \mathrm{Sat}_{q,G}(V)|_{S^{\mu-\lambda}\cap \mathrm{Gr}_G^\nu} \otimes \chi_N^\lambda) \tag{9.5}$$

lives in cohomological degrees $\geq \langle \lambda - \mu, 2\check{\rho}\rangle$.

By perversity, $\mathrm{Sat}_{q,G}(V)|_{\mathrm{Gr}_G^\nu}$ lives in non-negative *perverse* cohomological degrees, and it is lisse due to $\mathfrak{L}^+(G)$-equivariance. Hence, it is the Verdier dual of an object that lives in the *usual* cohomological degrees $\leq -\langle \nu, 2\check{\rho}\rangle$ (we recall that $\langle \nu, 2\check{\rho}\rangle = \dim(\mathrm{Gr}_G^\nu)$).

Therefore, $\mathrm{Sat}_{q,G}(V)|_{S^{\mu-\lambda}\cap \mathrm{Gr}_G^\nu}$ is the Verdier dual of an object that lives in the *usual* cohomological degrees $\leq -\langle \nu, 2\check{\rho}\rangle$.

Now the required cohomological estimate follows from the fact that

$$\dim(S^{\mu-\lambda} \cap \mathrm{Gr}_G^\nu) \leq \langle \nu + \mu - \lambda, \check{\rho}\rangle.$$

9.3.5. *Proof of Proposition 9.3.2.* It suffices to consider the case when V is finite-dimensional. Note that both the left and right adjoints of the functor (9.2) identify with

$$\mathcal{F} \mapsto \mathcal{F} \star \mathrm{Sat}_{q,G}(V^*),$$

where V^* is the dual representation of V. So, it suffices to show that (9.2) is left t-exact.

By the definition of the t-structure on $\mathrm{Whit}_q(G)$, its subcategory of connective objects is generated under colimits by the objects $W^{\mu,!}$. Taking into account (6.13), we obtain that the subcategory of coconnective objects in $\mathrm{Whit}_q(G)$ is *co-generated* under limits by the objects $W^{\lambda,*}$.

Thus, it suffices to show that for every λ and μ, the object

$$\mathcal{H}om_{\mathrm{Whit}_q(G)}(W^{\mu,!}, W^{\lambda,*} \star \mathrm{Sat}_{q,G}(V)) \in \mathrm{Vect}$$

lives in cohomological degrees ≥ 0. In other words, we have to show that that the !-fiber at t^μ of $W^{\lambda,*}\star\mathrm{Sat}_{q,G}(V)$ lives in the cohomological degrees $\geq -\langle \mu, 2\check{\rho}\rangle$. However, the latter has been established in Sect. 9.3.4.

$\square$[Proposition 9.3.2]

9.4. **Restricted coweights.** In this subsection we will make the analysis of the action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_q(G)$ even more explicit.

Namely, we will show that for certain coweights λ (called *restricted*), the image of the corresponding $W^{\lambda,!*}$ under $-\star \mathrm{Sat}_{q,G}(V)$ for $V \in \mathrm{Irrep}(H)$ stays irreducible.

This is a counterpart of Steinberg's theorem in the context of quantum groups.

9.4.1. We shall say that a coweight $\mu \in \Lambda^+$ is *restricted* if for every vertex i of the Dynkin diagram we have

$$\langle \mu, \check{\alpha}_i\rangle < \mathrm{ord}(q_i),$$

where q_i is as in Sect. 2.3.4.

First, we note:

Lemma 9.4.2. *Assume that the derived group of H is simply connected. Then any element $\lambda \in \Lambda^+$ can be written as $\mu + \gamma$ with $\mu \in \Lambda^+$ restricted and $\gamma \in (\Lambda^\sharp)^+$.*

9.4.3. We are now ready to state the main result of this section:

Theorem 9.4.4. *Suppose that $\mu \in \Lambda^+$ is restricted. Then for an irreducible $V^\gamma \in \mathrm{Rep}(\check{G})$ with highest weight $\gamma \in (\Lambda^\sharp)^+$, we have*
$$W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma) \simeq W^{\mu+\gamma,!*}.$$

This theorem was proved in [Lys, Sect. 7]. We include the proof for completeness.

9.4.5. *Proof of Theorem 9.4.4, Step 0.* It is easy to see that $W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma)$ is supported on $\overline{S}^{\mu+\gamma}$ and
$$W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma)|_{S^{\mu+\gamma}} \simeq W^{\mu+\gamma,!*}|_{S^{\mu+\gamma}}.$$

In particular, we have the maps
$$W^{\mu+\gamma,!} \to W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma) \text{ and } W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma) \to W^{\mu+\gamma,*}, \tag{9.6}$$
whose composition (is a non-zero scalar multiple of) the canonical map $W^{\mu+\gamma,!} \to W^{\mu+\gamma,*}$.

Thus, we have to show that the maps in (9.6) are surjective and injective, respectively. We will show the former, as the latter would follow by duality (see Sect. 9.2).

The surjectivity of the map $W^{\mu+\gamma,!} \to W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma)$ is equivalent to the assertion that there are no non-zero maps
$$W^{\mu,!*} \star \mathrm{Sat}_{q,G}(V^\gamma) \to W^{\lambda,!*}, \quad \lambda \neq \mu + \gamma.$$

Using the t-exactness of the convolution, it suffices to show that if there exists a nonzero Hom
$$W^{\mu,!} \star \mathrm{Sat}_{q,G}(V^\gamma) \to W^{\lambda,*}.$$
then $\lambda = \mu + \gamma$.

9.4.6. *Proof of Theorem 9.4.4, Step 1.* By adjunction, we have to show that if there exists a nonzero Hom
$$W^{\mu,!} \to W^{\lambda,*} \star \mathrm{Sat}_{q,G}(V^\gamma),$$
then $\lambda - \mu = -w_0(\gamma)$.

I.e., we have to show that *if* the expression
$$(W^{\lambda,*} \star \mathrm{Sat}_{q,G}(V^\gamma))_{t^\mu}$$
has cohomology in degree $-\langle \mu, 2\check{\rho}\rangle$, *then* $\mu - \lambda = w_0(\gamma)$.

By Sect. 9.3.4, we need to analyze when
$$H^{\langle \lambda-\mu, 2\check{\rho}\rangle}(S^{\mu-\lambda} \cap \mathrm{Gr}^\gamma_G, \mathrm{Sat}_{q,G}(V^\gamma)|_{S^{\mu-\lambda}\cap \mathrm{Gr}^\gamma_G} \otimes \chi^\lambda_N) \tag{9.7}$$
is non-zero (we note that the strata with $\nu \neq \gamma$ do not contribute to this cohomology as $\mathrm{Sat}_{q,G}(V^\gamma)|_{\mathrm{Gr}^\nu}$ will sit in strictly positive perverse cohomological degrees).

Note also that the condition that $\mu - \lambda = w_0(\gamma)$ implies that $w_0(\gamma)$ is the smallest element of Λ, for which the intersection $S^{\mu-\lambda} \cap \mathrm{Gr}^\gamma_G$ is non-empty, and in the latter case it consists of one point.

9.4.7. *Stalks of the convolution, bottom cohomology.* Note that for $\gamma \in \Lambda^\sharp$, the gerbe $\mathcal{G}^G|_{\mathrm{Gr}^\gamma_G}$ admits a unique (up to a non-canonical isomorphism) $\mathfrak{L}^+(G)$-equivariant trivialization.

Thus, we obtain that over the intersection
$$S^{\mu-\lambda} \cap \mathrm{Gr}^\gamma_G,$$
the gerbe $\mathcal{G}^G$ admits two *different* trivializations. Hence, their ratio is given by a local system that we temporarily denote by Ψ_q.

Thus, the expression in (9.7) is non-zero if and only if the local system
$$\chi^\lambda_N \otimes \Psi_q$$
on $S^{\mu-\lambda} \cap \mathrm{Gr}^\gamma_G$ is *trivial* on some irreducible component of of (the top) dimension $\langle \mu - \lambda + \gamma, \check{\rho}\rangle$.

9.4.8. *Proof of Theorem 9.4.4, Step 2: Reduction to an intersection of semi-infinite orbits.* Let $S^{-,w_0(\gamma)}$ denote the $\mathfrak{L}(N^-)$-orbit of $t^{w_0(\gamma)}$. It is known (see, e.g., [BFGM, Sect. 6]) that the inclusions

$$S^{\mu-\lambda} \cap \mathrm{Gr}^{\gamma}_G \hookleftarrow S^{\mu-\lambda} \cap \mathfrak{L}^+(N^-) \cdot t^{w_0(\gamma)} \hookrightarrow S^{\mu-\lambda} \cap S^{-,w_0(\gamma)}$$

induce bijections on the sets of irreducible components of (the top) dimension $\langle \mu-\lambda+\gamma, \check{\rho}\rangle$.

The restriction of $\mathcal{G}^G$ to $S^{-,w_0(\gamma)}$ also acquires a non-canonical trivialization. Hence, the discrepancy between the two trivializations over $S^{\mu-\lambda}\cap S^{-,w_0(\gamma)}$ is given by a local system that we also temporarily denote by Ψ_q. Its further restriction to

$$S^{\mu-\lambda} \cap \mathfrak{L}^+(N^-) \cdot t^{w_0(\gamma)}$$

identifies (non-canonically) with the restriction of the local system on $S^{\mu-\lambda} \cap \mathrm{Gr}^{\gamma}_G$ that we had earlier denoted by Ψ_q.

Thus, it suffices to show that if μ is restricted and $\mu - \lambda \neq w_0(\gamma)$, then the resulting local system

$$\chi^{\lambda}_N \otimes \Psi_q$$

on $S^{\mu-\lambda}\cap S^{-,w_0(\gamma)}$ is *non-trivial* on every irreducible component of (the top) dimension $\langle \mu-\lambda+\gamma, \check{\rho}\rangle$.

Translating by t^{λ}, we obtain that the required statement follows from the next result, proved in [Lys, Sect. 6]:

Theorem 9.4.9. *For a restricted dominant coweight μ and any $\nu \neq \mu$, the local system*

$$\chi^{0}_N \otimes \Psi_q$$

on $S^{\mu}\cap S^{-,\nu}$ is non-trivial on every irreducible component of (the top) dimension $\langle \mu-\nu, \check{\rho}\rangle$.

□[Theorem 9.4.4]

9.5. **Proof of Theorem 9.4.9.** For the sake of completeness, we will now reproduce a sketch of the proof of Theorem 9.4.9.

We note that, in addition to the proof of Theorem 9.4.4, we will use Theorem 9.4.9 one more time, for the analysis of the Hecke action on $\mathrm{Shv}(\mathrm{Gr}_G)^I$, where I is the Iwahori subgroup of $\mathfrak{L}^+(G)$.

9.5.1. Consider the action of the torus $T \subset \mathfrak{L}^+(T)$ on Gr_G. Since it stabilizes the points t^{μ} and normalizes $\mathfrak{L}(N)$, it acts on each S^{μ} and $S^{-,\nu}$. By [GLys, Sect. 7.4.2], the $\mathfrak{L}(N)$-equivariant trivialization of $\mathcal{G}^G|_{S^{\eta}}$ is T-twisted equivariant against the Kummer local system on T corresponding to the character

$$b(\mu,-) : \Lambda \to \mathsf{e}^*(-1).$$

Similarly, the $\mathfrak{L}(N^-)$-equivariant trivialization of $\mathcal{G}^G|_{S^{-,\nu}}$ is T-twisted equivariant against the Kummer local system on T corresponding to the character

$$b(\nu,-) : \Lambda \to \mathsf{e}^*(-1).$$

Hence, the local system Ψ_q on $S^{\mu}\cap S^{-,\nu}$ is T-twisted equivariant against the Kummer local system on T corresponding to the character

$$b(\mu-\nu,-) : \Lambda \to \mathsf{e}^*(-1).$$

9.5.2. Note now that the local system χ^0_N on S^{μ} is the pullback of the Artin-Schreier sheaf along a map $S^{\mu} \to \mathbb{G}_a$ that is $\mathbb{G}_m$-equivariant, where $\mathbb{G}_m$ acts on S^{μ} via the cocharacter ρ.

In particular, the push-forward of Ψ_q along the resulting map $S^{\mu}\cap S^{-,\nu} \to \mathbb{G}_a$ is twisted $\mathbb{G}_m$-equivariant against the Kummer local system on $\mathbb{G}_m$ corresponding to

$$b(\mu-\nu,\rho) \in \mathsf{e}^*(-1).$$

In particular, we obtain that the local system $\chi^0_N\otimes\Psi_q$ can be trivial on a given irreducible component of $S^{\mu}\cap S^{-,\nu}$ of (the top) dimension $\langle \mu-\nu, \check{\rho}\rangle$ only if both χ^0_N and Ψ_q are trivial on that component.

In particular, this can only happen if $\mu-\nu \in \Lambda^{\sharp}$.

9.5.3. We now recall that the union of sets of irreducible components of $S^\lambda \cap S^{-,0}$ over $\lambda \in \Lambda$ has a structure of crystal.

For a vertex i of the Dynkin diagram, let ϕ_i be the corresponding function (measuring the power of the lowering operator needed to kill the given element). On the one hand, it is shown in [Lys, Sect. 6]

On the one hand, it is shown in [Lys, Prop. 6.1.7] that if K is an irreducible component on which Φ_q is *trivial*, we have

$$\phi_i(K) \in \mathbb{Z}^{\geq 0} \cdot \mathrm{ord}(q_i).$$

Remark 9.5.4. In fact, it follows from [Lys, Sect. 6] that the set of such irreducible components also has a structure of crystal, where instead of e_i and f_i operators we take their $\mathrm{ord}(q_i)$-powers.

9.5.5. On the other hand, it is known (see [FGV, Sect. 7.3]) that under the bijection

$$S^\mu \cap S^{-,\nu} \simeq S^{\mu-\nu} \cap S^{-,0}$$

given by the action of $t^{-\nu}$, the set of irreducible components of $S^\mu \cap S^{-,\nu}$ on which χ_N^0 is trivial corresponds to the subset of irreducible components K of $S^{\mu-\nu} \cap S^{-,0}$ for which

$$\phi_i(K) \leq \check{\alpha}_i(\mu), \ \forall \ i.$$

9.5.6. Combining, we obtain that for an irreducible component of $S^\mu \cap S^{-,\nu}$, denoted K, on which both Ψ_q and χ_N^0 are trivial, we have:

$$\phi_i(K) \in \mathbb{Z}^{\geq 0} \cdot \mathrm{ord}(q_i) \text{ and } \phi_i(K) \leq \check{\alpha}_i(\mu) < \mathrm{ord}(q_i),$$

where the latter inequality is due to the fact that μ is restricted.

Hence, we obtain that $\phi_i(K) = 0$ for all i, which forces $\mu - \nu = 0$.

□[Theorem 9.4.9]

10. Hecke eigen-objects

In this section we will study the general paradigm of forming Hecke categories: given a category $\mathbf{C}$ with an action of $\mathrm{Rep}(H)$, we will define a new category $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$ and study its properties.

10.1. **Tensor products over** $\mathrm{Rep}(H)$**: a reminder.** Recall that if $\mathbf{C}$ and $\mathbf{D}$ are DG categories that are right and left modules, respectively, over a monoidal DG category $\mathbf{A}$, we can form the tensor product

$$\mathbf{C} \underset{\mathbf{A}}{\otimes} \mathbf{D},$$

which is another DG category.

In this subsection we discuss some general features of this operation, when $\mathbf{A}$ is the category $\mathrm{Rep}(H)$ of representations of an algebraic group H.

10.1.1. Let H be an algebraic group, and $\mathbf{C}$ and $\mathbf{D}$ categories with an action of the monoidal category $\mathrm{Rep}(H)$.

Consider the tensor product

$$\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}. \tag{10.1}$$

By definition, the category (10.1) comes equipped with a functor

$$\Phi_{\mathrm{univ}} : \mathbf{C} \otimes \mathbf{D} \to \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D},$$

universal among functors

$$\Phi : \mathbf{C} \otimes \mathbf{D} \to \mathbf{E}$$

equipped with functorial isomorphisms

$$\Phi(\mathbf{c} \star V, \mathbf{d}) \overset{\alpha_V}{\simeq} \Phi(\mathbf{c}, V \star \mathbf{d}), \quad \mathbf{c} \in \mathbf{C}, \mathbf{d} \in \mathbf{D}, V \in \mathrm{Rep}(H),$$

compatible with associativity in the sense that for $V_1, V_2 \in \mathrm{Rep}(H)$, the diagrams

$$\begin{tikzcd}
\Phi(\mathbf{c} \star (V_1 \otimes V_2), \mathbf{d}) \arrow[r, "\alpha_{V_1 \otimes V_2}"] \arrow[d, "\sim"'] & \Phi(\mathbf{c}, (V_1 \otimes V_2) \star \mathbf{d}) \arrow[d, "\sim"] \\
\Phi((\mathbf{c} \star V_1) \star V_2, \mathbf{d}) \arrow[d, "\alpha_{V_2}"'] & \Phi(\mathbf{c}, V_1 \star (V_2 \star \mathbf{d})) \\
\Phi(\mathbf{c} \star V_1, V_2 \star \mathbf{d}) \arrow[r, "\mathrm{id}"] & \Phi(\mathbf{c} \star V_1, V_2 \star \mathbf{d}) \arrow[u, "\alpha_{V_1}"']
\end{tikzcd}$$

should commute, along with a homotopy-coherent system of compatibilties for multi-fold tensor products.

10.1.2. According to [GR1, Chapter1, Proposition 9.4.8], the functor Φ_{univ} admits a (continuous) conservative right adjoint, denoted

$$\Psi_{\mathrm{univ}} : \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D} \to \mathbf{C} \otimes \mathbf{D}$$

that realizes $\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}$ as the category consisting of objects $\mathbf{e} \in \mathbf{C} \otimes \mathbf{D}$, equipped with a system of isomorphisms

$$((\mathrm{Id} \star V) \otimes \mathrm{Id})(\mathbf{e}) \overset{\beta_V}{\simeq} (\mathrm{Id} \otimes (V \star \mathrm{Id}))(\mathbf{e}), \quad V \in \mathrm{Rep}(H)$$

compatible with associativity in the sense that for $V_1, V_2 \in \mathrm{Rep}(H)$, the diagrams

$$\begin{tikzcd}
((\mathrm{Id} \star (V_1 \otimes V_2)) \otimes \mathrm{Id})(\mathbf{e}) \arrow[r, "\beta_{V_1 \otimes V_2}"] \arrow[d, "\sim"'] & (\mathrm{Id} \otimes ((V_1 \otimes V_2) \star \mathrm{Id}))(\mathbf{e}) \arrow[d, "\sim"] \\
((\mathrm{Id} \star V_2) \otimes \mathrm{Id}) \circ ((\mathrm{Id} \star V_1) \otimes \mathrm{Id})(\mathbf{e}) \arrow[d, "\beta_{V_1}"'] & (\mathrm{Id} \otimes (V_1 \star \mathrm{Id})) \circ (\mathrm{Id} \otimes (V_2 \star \mathrm{Id}))(\mathbf{e}) \\
((\mathrm{Id} \star V_2) \otimes \mathrm{Id}) \circ (\mathrm{Id} \otimes (V_1 \star \mathrm{Id}))(\mathbf{e}) \arrow[r, "\sim"] & (\mathrm{Id} \otimes (V_1 \star \mathrm{Id})) \circ ((\mathrm{Id} \star V_2) \otimes \mathrm{Id})\ (\mathbf{e}) \arrow[u, "\beta_{V_2}"']
\end{tikzcd}$$

should commute, along with a homotopy-coherent system of compatibilties for multi-fold tensor products.

10.1.3. Thus, we obtain that $\mathbf{C} \otimes \mathbf{D}$ and $\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}$ are related by a pair of (continuous) adjoint functors

$$\Phi_{\mathrm{univ}} : \mathbf{C} \otimes \mathbf{D} \rightleftarrows \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D} : \Psi_{\mathrm{univ}} \tag{10.2}$$

with the right adjoint being conservative.

Hence, by the Barr-Beck-Lurie theorem, the category $\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}$ identifies with the category

$$\mathrm{Reg}(H)\text{-mod}(\mathbf{C} \otimes \mathbf{D}),$$

where $\mathrm{Reg}(H)$ is the monad on $\mathbf{C} \otimes \mathbf{D}$, given by the action of the associative algebra object

$$\mathrm{Reg}(H) \in \mathrm{Rep}(H) \otimes \mathrm{Rep}(H),$$

the "regular representation" of H.

10.1.4. Assume now that $\mathbf{C}$ and $\mathbf{D}$ are compactly generated. In this case $\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}$ is also compactly generated, with the set of compact generators provided by

$$\Phi_{\mathrm{univ}}(\mathbf{c} \otimes \mathbf{d}), \quad \mathbf{c} \in \mathbf{C}^c,\ \mathbf{d} \in \mathbf{D}^c.$$

10.1.5. Consider $\mathbf{C}^\vee$ and $\mathbf{D}^\vee$ equipped with the natural $\operatorname{Rep}(H)$-actions, and consider the corresponding functors

$$\Phi_{\mathrm{univ}} : \mathbf{D}^\vee \otimes \mathbf{C}^\vee \rightleftarrows \mathbf{D}^\vee \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{C}^\vee : \Psi_{\mathrm{univ}}. \tag{10.3}$$

It follows from [GR1, Prop. 9.4.8] that we have a canonical identification

$$\mathbf{D}^\vee \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{C}^\vee \simeq (\mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{D})^\vee$$

so that the functors in (10.3) identify with the duals of those in (10.2).

In particular, if we denote by $x \mapsto x^\vee$ the corresponding equivalences

$$(\mathbf{C}^c)^{\mathrm{op}} \to (\mathbf{C}^\vee)^c, \quad (\mathbf{D}^c)^{\mathrm{op}} \to (\mathbf{D}^\vee)^c \text{ and } ((\mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{D})^c)^{\mathrm{op}} \simeq (\mathbf{D}^\vee \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{C}^\vee)^c,$$

we have

$$(\Phi_{\mathrm{univ}}(\mathbf{c} \otimes \mathbf{d}))^\vee \simeq \Phi_{\mathrm{univ}}(\mathbf{c}^\vee \otimes \mathbf{d}^\vee).$$

Remark 10.1.6. The above discussion applies to $\operatorname{Rep}(H)$ replaced by any *rigid* symmetric monoidal category $\mathbf{A}$. The role of the regular representation is played by the image of the unit object under the functor

$$\mathbf{A} \to \mathbf{A} \otimes \mathbf{A},$$

right adjoint adjoint to the tensor product functor $\mathbf{A} \otimes \mathbf{A} \to \mathbf{A}$.

10.1.7. Assume now that $\mathbf{C}$ and $\mathbf{D}$ are each equipped with a t-structure, such that the connective subcategories are generated by compact objects.

Assume also that the action functors

$$\mathbf{C} \otimes \operatorname{Rep}(H) \to \mathbf{C}, \quad \operatorname{Rep}(H) \otimes \mathbf{D} \to \mathbf{D}$$

are t-exact.

Then $\mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{D}$ also acquires a t-structure, with both functors

$$\Phi_{\mathrm{univ}} : \mathbf{C} \otimes \mathbf{D} \rightleftarrows \mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{D} : \Psi_{\mathrm{univ}}$$

being t-exact.

10.2. **The paradigm of Hecke eigen-objects: a reminder.** In this section we introduce a key definition: that of the category of Hecke eigen-objects arising from a category equipped with an action of $\operatorname{Rep}(H)$.

10.2.1. We apply the discussion in Sect. 10.1 to the case when $\mathbf{D} = \mathrm{Vect}$ with the action of $\operatorname{Rep}(H)$ on $\mathbf{D}$ given by the forgetful functor

$$\operatorname{Res}^H : \operatorname{Rep}(H) \to \mathrm{Vect},$$

i.e., the functor that sends a representation V to its underlying vector space $\underline{V}$.

We will refer to the resulting category

$$\mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathrm{Vect}$$

as the category of Hecke eigen-objects in $\mathbf{C}$, and denote it also by $\operatorname{Hecke}(\mathbf{C})$.

10.2.2. By definition, the functor

$$\Phi_{\mathrm{univ}} : \mathbf{C} \to \operatorname{Hecke}(\mathbf{C})$$

is universal among functors $\Phi : \mathbf{D} \to \mathbf{E}$ equipped with a system of isomorphisms

$$\Phi(\mathbf{c} \star V) \overset{\alpha_V}{\simeq} \underline{V} \otimes \Phi(\mathbf{c}), \quad V \in \operatorname{Rep}(H), \ \mathbf{c} \in \mathbf{C},$$

compatible with tensor products of representations.

10.2.3. The monad on $\mathbf{C}$, corresponding to the pair of adjoint functors

$$\mathbf{C} \rightleftarrows \mathrm{Hecke}(\mathbf{C})$$

is given by the action of

$$(\mathrm{Id} \otimes \mathrm{Res}^H)(\mathrm{Reg}(H)) \in \mathrm{Rep}(H),$$

i.e., we think of $\mathrm{Reg}(H)$ as a representation of one copy of H, rather than $H \times H$.

In what follows we will use the notation

$$\mathbf{ind}_{\mathrm{Hecke}} := \Phi_{\mathrm{univ}} \text{ and } \mathbf{oblv}_{\mathrm{Hecke}} := \Psi_{\mathrm{univ}}.$$

10.2.4. By Sect. 10.1.2, we can think of $\mathrm{Hecke}(\mathbf{C})$ as the category of objects $\mathbf{c} \in \mathbf{C}$, equipped with a a system of isomorphisms

$$\mathbf{c} \star V \overset{\beta_V}{\simeq} \underline{V} \otimes \mathbf{c}, \tag{10.4}$$

compatible with tensor products of representations.

The latter interpretation is the source of the name "Hecke eigen-objects".

10.2.5. The category

$$\mathrm{Hecke}(\mathbf{C}) = \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Vect}$$

has a natural structure of category acted on by H.

Explicitly, when we think of objects $\mathrm{Hecke}(\mathbf{C})$ as in (10.4), the action action of a point $h \in H$ on such an object is given by modifying the isomorphisms β_V via the action of h on $\underline{V}$.

10.2.6. For a category $\widetilde{\mathbf{C}}$ equipped with an action of H, let $\widetilde{\mathbf{C}}^H$ denote the corresponding category of H-equivariant objects. By [Ga5, Theorem 2.2.2], the category $\widetilde{\mathbf{C}}^H$ is equipped with a natural action of $\mathrm{Rep}(H)$, and the assignments

$$\mathbf{C} \mapsto \mathrm{Hecke}(\mathbf{C}) \text{ and } \widetilde{\mathbf{C}} \mapsto (\widetilde{\mathbf{C}})^H$$

define mutually inverse equivalences between the $(\infty, 2)$-categories

$$\mathrm{Rep}(H)\text{-}\mathbf{mod} \text{ and } H\text{-}\mathbf{mod}.$$

In particular, the category $\mathbf{C}$ can be recovered from $\mathrm{Hecke}(\mathbf{C})$ as the category of H-equivariant objects, i.e.,

$$\mathbf{C} \simeq (\mathrm{Hecke}(\mathbf{C}))^H.$$

The latter point of view allows us to think of $\mathrm{Hecke}(\mathbf{C})$ as a "de-equivariantization" of $\mathbf{C}$.

10.2.7. In particular, we can think of the functor

$$\mathbf{ind}_{\mathrm{Hecke}} : \mathbf{C} \to \mathrm{Hecke}(\mathbf{C})$$

as

$$\mathrm{Res}^H : (\mathrm{Hecke}(\mathbf{C}))^H \to \mathrm{Hecke}(\mathbf{C}),$$

and of

$$\mathbf{oblv}_{\mathrm{Hecke}} : \mathrm{Hecke}(\mathbf{C}) \to \mathbf{C}$$

as

$$\mathrm{coInd}^H : \mathrm{Hecke}(\mathbf{C}) \to (\mathrm{Hecke}(\mathbf{C}))^H.$$

Here, for a category $\widetilde{\mathbf{C}}$ acted on by H, we denote by

$$\mathrm{Res}^H : \widetilde{\mathbf{C}}^H \rightleftarrows \widetilde{\mathbf{C}} : \mathrm{coInd}^H$$

the corresponding adjoint pair of functors.

10.2.8. *Example.* Let H be a torus with the weight lattice Λ_H. Then the datum of a category equipped with an action of $\operatorname{Rep}(H)$ is the same as that of a category equipped with an action of Λ_H.

Let A be a Λ_H-graded algebra. Then the we can take as $\mathbf{C}$ the category $\overset{\bullet}{A}$-mod of Λ_H-graded A-modules. The action of Λ_H on $\overset{\bullet}{A}$-mod is given by shifting the grading.

The corresponding category $\operatorname{Hecke}(\mathbf{C})$ can be identified with the category A-mod. The action of H on A defines an H-action on A-mod, and the corresponding equivariant category $A\text{-mod}^H$ can be identified with $\overset{\bullet}{A}$-mod.

The functor $\mathbf{ind}_{\mathrm{Hecke}} \simeq \operatorname{Res}^H$ is the forgetful functor

$$\overset{\bullet}{A}\text{-mod} \to A\text{-mod},$$

and $\mathbf{oblv}_{\mathrm{Hecke}} \simeq \operatorname{coInd}^H$ is its right adjoint, given by averaging along Λ_H.

10.3. **Graded Hecke eigen-objects.** From now on, until the end of this section, we let H be a reductive group and $T_H \subset H$ be its Cartan subgroup.

In this subsection we will discuss a version of the construction of Sect. 10.2, where instead of the "absolute" de-equivariantization, we perform a partial one, relative to T_H.

10.3.1. We will apply the framework of Sect. 10.1 to the case when $\mathbf{D} = \operatorname{Rep}(T_H)$, where T_H is a torus mapping to H. We denote the corresponding category

$$\mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \operatorname{Rep}(T_H)$$

by $\operatorname{H\overset{\bullet}{e}cke}(\mathbf{C})$.

We denote the resulting pair of adjoint functors by

$$\mathbf{ind}_{\operatorname{H\overset{\bullet}{e}cke}} : \mathbf{C} \otimes \operatorname{Rep}(T_H) \rightleftarrows \operatorname{H\overset{\bullet}{e}cke}(\mathbf{C}) : \mathbf{oblv}_{\operatorname{H\overset{\bullet}{e}cke}}.$$

The corresponding monad on $\mathbf{C} \otimes \operatorname{Rep}(T_H)$ identifies with the action of

$$(\operatorname{Id} \otimes \operatorname{Res}^H_{T_H})(\operatorname{Reg}(H)) \in \operatorname{Rep}(H) \otimes \operatorname{Rep}(T_H).$$

10.3.2. By construction, $\operatorname{H\overset{\bullet}{e}cke}(\mathbf{C})$ is acted on by $\operatorname{Rep}(T_H)$. If we take its Hecke category with respect to T_H, we recover $\operatorname{Hecke}(\mathbf{C})$. In particular, we obtain that we can recover $\operatorname{H\overset{\bullet}{e}cke}(\mathbf{C})$ as

$$\operatorname{H\overset{\bullet}{e}cke}(\mathbf{C}) \simeq (\operatorname{Hecke}(\mathbf{C}))^{T_H}.$$

In particular, we have a pair of adjoint functors

$$\operatorname{Res}^{T_H} : \operatorname{H\overset{\bullet}{e}cke}(\mathbf{C}) \rightleftarrows \operatorname{Hecke}(\mathbf{C}) : \operatorname{coInd}^{T_H}.$$

The composite

$$\mathbf{oblv}_{\mathrm{Hecke}} \circ \operatorname{Res}^{T_H} : \operatorname{H\overset{\bullet}{e}cke}(\mathbf{C}) \to \mathbf{C}$$

is the forgetful functor

$$\operatorname{H\overset{\bullet}{e}cke}(\mathbf{C}) \overset{\mathbf{oblv}_{\operatorname{H\overset{\bullet}{e}cke}}}{\longrightarrow} \mathbf{C} \otimes \operatorname{Rep}(T_H) \overset{\operatorname{Id} \otimes \operatorname{Res}^{T_H}}{\longrightarrow} \mathbf{C}.$$

10.3.3. By definition, the functor $\mathbf{ind}_{\operatorname{H\overset{\bullet}{e}cke}}$ is a universal recipient among functors

$$\Phi : \mathbf{C} \to \mathbf{E},$$

where $\mathbf{E}$ is a $\operatorname{Rep}(T_H)$-module category, equipped with a system of identifications

$$\Phi(\mathbf{c} \star V) \overset{\alpha_V}{\simeq} \Phi(\mathbf{c}) \star \operatorname{Res}^H_{T_H}(V),$$

compatible with the with tensor products of representations.

10.3.4. By Sect. 10.1.2, we can think of $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$ as the category of objects $\overset{\bullet}{\mathbf{c}} \in \mathrm{Rep}(T_H)\otimes\mathbf{C}$, equipped with a a system of isomorphisms

$$\overset{\bullet}{\mathbf{c}} \star V \overset{\beta_V}{\simeq} \mathrm{Res}^H_{T_H}(V) \otimes \overset{\bullet}{\mathbf{c}} \tag{10.5}$$

(where $\mathrm{Rep}(T_H)$ acts on $\mathrm{Rep}(T_H) \otimes \mathbf{C}$ via the 1st factor), compatible with tensor products of representations.

For this reason we will refer to $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$ as graded (with respect to lattice of weights of T_H) Hecke eigen-objets of $\mathbf{C}$.

10.4. **The relative Hecke category.** For future use we will introduce yet another variant of the Hecke category.

10.4.1. Let $\mathbf{C}$ be acted by $\mathrm{Rep}(H)$ (on the right) and by $\mathrm{Rep}(T_H)$ (on the left) so that these two actions commute. In other words, we have an action of $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$ on $\mathbf{C}$.

We introduce the category $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{C})$ as

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{C}) := \mathbf{C} \underset{\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)}{\otimes} \mathrm{Rep}(T_H),$$

where the functors

$$\mathrm{Rep}(H) \to \mathrm{Rep}(T_H) \leftarrow \mathrm{Rep}(T_H)$$

are restriction and identity, respectively.

10.4.2. By Sect. 10.1.2, we can think of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{C})$ as the category of objects $\mathbf{c} \in \mathbf{C}$, equipped with a system of isomorphisms

$$\mathbf{c} \underset{H}{\star} V \simeq \mathrm{Res}^H_{T_H}(V) \star \mathbf{c}, \tag{10.6}$$

compatible with tensor products of representations.

10.4.3. We will denote by $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}}$ the functor

$$\mathbf{C} \overset{\mathrm{Id}\otimes\mathrm{unit}}{\longrightarrow} \mathbf{C} \otimes \mathrm{Rep}(T_H) \overset{\Phi_{\mathrm{univ}}}{\longrightarrow} \mathbf{C} \underset{\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)}{\otimes} \mathrm{Rep}(T_H) =: \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{C}),$$

and by

$$\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}} : \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{C}) \to \mathbf{C}.$$

its right adjoint.

When we think of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{C})$ as objects $\mathbf{c} \in \mathbf{C}$ equipped with a system of isomorphisms (10.6), the functor $\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}}$ remembers the data of $\mathbf{c}$.

10.5. **Duality for the Hecke category.** In this short subsection we will study how the formation of the Hecke category interacts with duality on categories acted on by $\mathrm{Rep}(H)$.

10.5.1. Let $\mathbf{C}$ be again a compactly generated category, acted on by $\mathrm{Rep}(H)$. Let us consider $\mathbf{C}^\vee$ as acted on by $\mathrm{Rep}(H)$ by the formula

$$\mathbf{c}^\vee \star V = (\mathbf{c} \star \tau^H(V^*))^\vee, \quad \mathbf{c} \in \mathbf{C}^c, V \in \mathrm{Rep}(H),$$

where τ^H is the Cartan involution on H, see Sect. 2.5.4.

According to Sect. 10.1.5, we have a canonical identification

$$\mathrm{Hecke}(\mathbf{C})^\vee \simeq \mathrm{Hecke}(\mathbf{C}^\vee),$$

so that the diagram

$$\begin{tikzcd}
(\mathbf{C}^c)^{\mathrm{op}} \arrow[r, "\mathbf{ind}_{\mathrm{Hecke}}"] \arrow[d] & (\mathrm{Hecke}(\mathbf{C})^c)^{\mathrm{op}} \arrow[d] \\
(\mathbf{C}^\vee)^c \arrow[r, "\mathbf{ind}_{\mathrm{Hecke}}"] & \mathrm{Hecke}(\mathbf{C}^\vee)^c
\end{tikzcd}$$

commutes.

10.5.2. Similarly, we define a duality

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\vee \simeq \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}^\vee)$$

by making the following diagram commute:

$$\begin{tikzcd}
((\mathbf{C} \otimes \mathrm{Rep}(T_H))^c)^{\mathrm{op}} \arrow[r, "\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}"] \arrow[d] & (\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^c)^{\mathrm{op}} \arrow[d] \\
(\mathbf{C}^\vee \otimes \mathrm{Rep}(T_H)^c \arrow[r, "\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}"] & \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}^\vee)^c,
\end{tikzcd}$$

where the left vertical arrow sends

$$(\mathbf{c} \otimes V) \mapsto \mathbf{c}^\vee \otimes \tau^{T_H}(V^*).$$

10.6. **Irreducible objects in the Hecke category.** In this subsection we will assume that $\mathbf{C}$ is equipped with a t-structure satisfying the assumption of Sect. 10.1.7.

Note that in this case, according to Sect. 10.1.7, the category $\mathrm{Hecke}(\mathbf{C})$ acquires a t-structure, in which both functors $\mathbf{ind}_{\mathrm{Hecke}}$ and $\mathbf{oblv}_{\mathrm{Hecke}}$ are t-exact.

Our goal in this subsection is to give an explicit description of the irreducible objects in $\mathrm{Hecke}(\mathbf{C})$. We will be able to do so under an additional assumption on the action of $\mathrm{Rep}(H)$ on $\mathbf{C}$, namely, when this action is *accessible*.

10.6.1. We shall say that an irreducible $\mathbf{c} \in \mathbf{C}^\heartsuit$ is *restricted* for the Hecke action if for every $V \in \mathrm{Irrep}(H)$, the object $\mathbf{c} \star V \in \mathbf{C}^\heartsuit$ is irreducible.

For example, Theorem 9.4.4 says that if $\lambda \in \Lambda$ is restricted, then the object $W^{\lambda,!*} \in \mathrm{Whit}_{q,x}(G)$ is a restricted irreducible.

10.6.2. We are going to prove:

Proposition 10.6.3. *Let $\mathbf{c} \in \mathbf{C}^\heartsuit$ be restricted. Then $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}) \in \mathrm{Hecke}(\mathbf{C})^\heartsuit$ is irreducible.*

Proof. Let $\mathbf{c}' \in \mathrm{Hecke}(\mathbf{C})^\heartsuit$ be equipped with a non-zero map $\mathbf{c}' \to \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c})$; let us show that this map is a surjection.

We have a surjection

$$\mathbf{ind}_{\mathrm{Hecke}} \circ \mathbf{oblv}_{\mathrm{Hecke}}(\mathbf{c}') \to \mathbf{c}',$$

so we can assume that $\mathbf{c}'$ is of the form $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_1)$ for some $\mathbf{c}_1 \in \mathbf{C}^\heartsuit$. Hence, the map in question comes from a map in $\mathbf{C}$

$$\mathbf{c}_1 \to \mathbf{oblv}_{\mathrm{Hecke}} \circ \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}) \simeq \underset{V \in \mathrm{Irrep}(H)}{\oplus} (\mathbf{c} \star V) \otimes \underline{V}^*. \tag{10.7}$$

Let $V \in \mathrm{Irrep}(H)$ be such that the component $\mathbf{c}_1 \to (\mathbf{c} \star V) \otimes \underline{V}^*$ of the map (10.7) is non-zero. Replacing $\mathbf{c}_1$ by the preimage of $(\mathbf{c} \star V) \otimes \underline{V}^*$ under (10.7), and using the assumption on $\mathbf{c}$, we can assume that $\mathbf{c}_1$ is isomorphic to $\mathbf{c} \star V$ and (10.7) corresponds to an element $\xi \in \underline{V}^*$.

Hence, the original map $\mathbf{c}' \to \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c})$ identifies with

$$\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c} \star V) \simeq \underline{V} \otimes \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}) \overset{\xi \otimes \mathrm{id}}{\longrightarrow} \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}),$$

and hence is manifestly a surjection.

□

10.6.4. We shall say that the action of $\mathrm{Rep}(H)$ on $\mathbf{C}$ is *accessible* if every irreducible object of $\mathbf{C}^\heartsuit$ is of the form $\mathbf{c} \star V$ for $\mathbf{c} \in \mathbf{C}^\heartsuit$ restricted and $V \in \mathrm{Irrep}(H)$.

For example, Lemma 9.4.2 says that if the derived group of H is simply-connected, then the action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_{q,x}(G)$ is accessible.

10.6.5. From Proposition 10.6.3 we obtain:

Corollary 10.6.6. *Assume that the t-structure on* $\mathbf{C}$ *is Artinian and that the action of* $\mathrm{Rep}(H)$ *is accessible. Then:*

(a) *Every irreducible object of* $\mathrm{Hecke}(\mathbf{C})^\heartsuit$ *is of the form* $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c})$ *for a restricted* $\mathbf{c} \in \mathbf{C}^\heartsuit$*.*

(b) *If for two such objects we have* $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_1) \simeq \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_2)$*, then* $\mathbf{c}_1 \simeq \mathbf{c}_2 \star V$ *for a 1-dimensional representation* V *of* H*.*

Proof. Let $\mathbf{c}'$ be an irreducible object of $\mathrm{Hecke}(\mathbf{C})^\heartsuit$. There exists an object $\mathbf{c}_1 \in \mathbf{C}^\heartsuit$ equipped with a non-zero map $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_1) \to \mathbf{c}'$. By Artinianness, we can assume that $\mathbf{c}_1$ is irreducible. Write $\mathbf{c}_1 \simeq \mathbf{c} \star V$ for $\mathbf{c}$ restricted.

Thus, we obtain a non-zero map

$$\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c} \star V) \simeq \underline{V} \otimes \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}) \to \mathbf{c}',$$

from which we deduce the existence of a non-zero map

$$\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}) \to \mathbf{c}'.$$

However, by Proposition 10.6.3, $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c})$ is irreducible, and hence $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}) \simeq \mathbf{c}'$. This proves point (a).

For point (b), let us be given a non-zero map

$$\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_1) \to \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_2)$$

for $\mathbf{c}_1, \mathbf{c}_2$ as in Proposition 10.6.3. Then we obtain a non-zero map in $\mathbf{C}$

$$\mathbf{c}_1 \to \underset{V \in \mathrm{Irrep}(H)}{\oplus} (\mathbf{c}_2 \star V) \otimes \underline{V}^*.$$

Hence, we obtain a non-zero map

$$\mathbf{c}_1 \to \mathbf{c}_2 \star V$$

for some irreducible V. By the assumption on $\mathbf{c}_2$, the latter map is an isomorphism.

Symmetrically, we have: $\mathbf{c}_2 \simeq \mathbf{c}_1 \star W$ for some $W \in \mathrm{Irrep}(H)$. Hence,

$$\mathbf{c}_1 \simeq \mathbf{c}_1 \star (V \otimes W).$$

From here we obtain that V and W are 1-dimensional.

□

Corollary 10.6.7. *Assume that the t-structure on* $\mathbf{C}$ *is Artinian and that the action of* $\mathrm{Rep}(H)$ *is accessible. Then the t-structure on* $\mathrm{Hecke}(\mathbf{C})^\heartsuit$ *is Artinian.*

Proof. It suffices to show that for an irreducible $\mathbf{c}_1 \in \mathbf{C}^\heartsuit$, the object $\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_1) \in \mathrm{Hecke}(\mathbf{C})^\heartsuit$ has finite length.

Write $\mathbf{c}_1 = \mathbf{c} \star V$ for $\mathbf{c}$ as in Proposition 10.6.3. Then

$$\mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}_1) \simeq \underline{V} \otimes \mathbf{ind}_{\mathrm{Hecke}}(\mathbf{c}),$$

and the assertion follows.

□

10.7. **Irreducible objects in the graded version.** In this subsection we retain the assumptions of Sect. 10.6. We will will adapt the results of *loc.cit.* to describe irreducibles in the graded Hecke category $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$.

10.7.1. First off note that if H is a torus, any irreducible object in $\mathbf{C}$ is restricted for this action: indeed the irreducible objects of $\mathrm{Rep}(H)$ are 1-dimensional and act by invertible functors.

In particular, an action of $\mathrm{Rep}(H)$ is automatically accessible.

10.7.2. Let us be in the context of Sect. 10.3. Applying Proposition 10.6.3 and Corollaries 10.6.7 and 10.6.6 to the torus T_H, we obtain:

Corollary 10.7.3.
(a) *The forgetful functor*

$$\mathrm{Res}^{T_H} : \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\heartsuit \to \mathrm{Hecke}(\mathbf{C})^\heartsuit$$

sends irreducibles to irreducibles.

(b) *Every irreducible of* $\mathrm{Hecke}(\mathbf{C})^\heartsuit$ *is of the form* $\mathrm{Res}^{T_H}(\mathbf{c})$ *for some irreducible* $\mathbf{c} \in \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\heartsuit$.

(c) *If for two irreducibles* $\mathbf{c}_1, \mathbf{c}_2 \in \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\heartsuit$, *we have* $\mathrm{Res}^{T_H}(\mathbf{c}_1) \simeq \mathrm{Res}^{T_H}(\mathbf{c}_2)$, *then* $\mathbf{c}_1$ *and* $\mathbf{c}_2$ *differ by a translation by an element of* Λ_H.

Combining with Proposition 10.6.3 and Corollaries 10.6.7 and 10.6.6, we obtain:

Corollary 10.7.4.

(a) *For every restricted irreducible* $\mathbf{c} \in \mathbf{C}^\heartsuit$ *and every* $\gamma \in \Lambda_H$, *the object* $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(\mathbf{c} \otimes \mathsf{e}^\gamma) \in \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\heartsuit$ *is irreducible.*

(b) *Suppose that the t-structure on* $\mathbf{C}$ *is Artinian and that the action of* $\mathrm{Rep}(H)$ *on* $\mathbf{C}$ *is accessible. Then:*

(i) *The t-structure on* $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\heartsuit$ *is Artinian;*

(ii) *Every irreducible object of* $\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^\heartsuit$ *is of the form* $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(\mathbf{c} \otimes \mathsf{e}^\gamma)$ *for some* $\mathbf{c} \in \mathbf{C}^\heartsuit$ *as in Proposition 10.6.3 and* $\gamma \in \Lambda_H$.

(iii) *Two irreducible objects* $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(\mathbf{c}_1 \otimes \mathsf{e}^{\gamma_1})$ *and* $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(\mathbf{c}_2 \otimes \mathsf{e}^{\gamma_2})$ *are isomorphic if and only if* $\gamma_1 - \gamma_2$ *extends to a character (to be denoted* γ*) of* H, *and* $\mathbf{c}_2 \simeq \mathbf{c}_1 \star \mathsf{e}^\gamma$, *where* e^γ *denotes the corresponding one-dimensional representation of* H.

11. The category of Hecke eigen-objects in the Whittaker category

In this section we will finally define and study the main character of this paper, the category of graded Hecke eigensheaves in the metaplectic Whittaker category.

11.1. **Definition.** In this subsection we introduce the category of graded Hecke eigensheaves in the metaplectic Whittaker category,

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)),$$

taken with respect to the $\mathrm{Rep}(H)$ action on $\mathrm{Whit}_{q,x}(G)$, which was defined in Sect. 9.1.

We will also consider the non-graded version $\mathrm{Hecke}(\mathrm{Whit}_{q,x}(G))$.

11.1.1. By Sect. 10.1.3, we have a pair of adjoint functors

$$\mathbf{ind}_{\overset{\bullet}{\text{Hecke}}} : \text{Whit}_{q,x}(G) \otimes \text{Rep}(T_H) \rightleftarrows \overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G)) : \mathbf{oblv}_{\overset{\bullet}{\text{Hecke}}}.$$

The corresponding monad on $\text{Whit}_{q,x}(G)$ is given by the action of the object

$$\text{Sat}_{q,G} \otimes \text{Res}^H_{T_H}(\text{Reg}(\check{G})) \in \text{Sph}_{q,x}(G) \otimes \text{Rep}(T_H).$$

In particular, the category $\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G))$ is compactly generated by the essential image of $(\text{Whit}_{q,x}(G) \otimes \text{Rep}(T_H))^c$ under the functor $\mathbf{ind}_{\overset{\bullet}{\text{Hecke}}}$.

11.1.2. By Sect. 10.1.2, we can rewrite $\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G))$ as the category consisting of

$$\mathcal{F} \in \text{Whit}_{q,x}(G) \otimes \text{Rep}(T_H),$$

equipped with a system of identifications

$$\mathcal{F} \star \text{Sat}_{q,G}(V) \overset{\beta_V}{\simeq} \text{Res}^H_{T_H}(V) \otimes \mathcal{F},$$

(where $\text{Rep}(T_H)$ acts on $\text{Whit}_{q,x}(G) \otimes \text{Rep}(T_H)$ via the 2nd factor), compatible with tensor products of representations.

11.1.3. Taking into account Sect. 9.2.4, the recipe of Sect. 10.5.2 defines an identification

$$\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G))^\vee \simeq \overset{\bullet}{\text{Hecke}}(\text{Whit}_{q^{-1},x}(G)), \tag{11.1}$$

i.e., an equivalence

$$\mathbb{D}^{\text{Verdier}} : ((\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G)))^c)^{\text{op}} \simeq (\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q^{-1},x}(G)))^c, \tag{11.2}$$

which makes the following diagram commute

$$\begin{tikzcd}
((\text{Whit}_{q,x}(G) \otimes \text{Rep}(T_H))^c)^{\text{op}} \arrow[r, "(\mathbf{ind}_{\overset{\bullet}{\text{Hecke}}})^{\text{op}}"] \arrow[d, "\mathbb{D}^{\text{Verdier}} \otimes (\tau^{T_H} \circ \mathbb{D}^{\text{lin}})"'] & ((\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G)))^c)^{\text{op}} \arrow[d, "\mathbb{D}^{\text{Verdier}}"] \\
(\text{Whit}_{q^{-1},x}(G) \otimes \text{Rep}(T_H))^c \arrow[r, "\mathbf{ind}_{\overset{\bullet}{\text{Hecke}}}"] & (\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q^{-1},x}(G)))^c.
\end{tikzcd}$$

11.2. **Behavior with respect to isogenies.** As a convenient technical tool, we will study the behavior of the categores $\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G))$ and $\text{Hecke}(\text{Whit}_{q,x}(G))$ under isogenies $G \to \widetilde{G}$.

11.2.1. Let us be given a short exact sequence of reductive groups

$$1 \to G \to \widetilde{G} \to T_0 \to 1,$$

where T_0 is a torus. Consider the corresponding short exact sequence of tori

$$0 \to T \to \widetilde{T} \to T_0 \to 0$$

and lattices

$$0 \to \Lambda \to \widetilde{\Lambda} \to \Lambda_0 \to 0.$$

Let us be given a geometric metaplectic data $\mathcal{G}^{\widetilde{G}}$ for $\widetilde{G}$, whose restriction to G gives $\mathcal{G}^G$. We obtain a map of lattices

$$\Lambda^\sharp \to \widetilde{\Lambda}^\sharp$$

and a map of reductive groups

$$\widetilde{H} \to H.$$

11.2.2. We will say that the isogeny is *strictly compatible with the geometric metaplectic data* if the diagram

$$\begin{CD} \widetilde{\Lambda}^\sharp @>>> \widetilde{\Lambda} \\ @AAA @AAA \\ \Lambda^\sharp @>>> \Lambda \end{CD}$$

is a push-out square, cf. Sect. 4.5.4.

In particular, in this case we obtain a short exact sequence of lattices

$$0 \to \Lambda^\sharp \to \widetilde{\Lambda}^\sharp \to \Lambda_0 \to 0, \tag{11.3}$$

and an isogeny of metaplectic duals

$$1 \to \check{T}_0 \to \widetilde{H} \to H \to 1.$$

11.2.3. Note that the image of the resulting map

$$\mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}$$

is a union of some connected components.

In particular, direct image defines a fully faithful functor

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}), \tag{11.4}$$

and in particular a fully faithful monoidal functor

$$\mathrm{Sph}_{q,x}(G) \to \mathrm{Sph}_{q,x}(\widetilde{G}).$$

We have a commutative diagram

$$\begin{CD} \mathrm{Rep}(H) @>>> \mathrm{Rep}(\widetilde{H}) \\ @V{\mathrm{Sat}_{q,G}}VV @VV{\mathrm{Sat}_{q,\widetilde{G}}}V \\ \mathrm{Sph}_{q,x}(G) @>>> \mathrm{Sph}_{q,x}(\widetilde{G}). \end{CD}$$

11.2.4. Consider the resulting functor

$$\mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})) \to \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})), \tag{11.5}$$

where Hecke on the left-hand (resp., right-hand) side is taken with respect to action of $\mathrm{Rep}(H)$ (resp., $\mathrm{Rep}(\widetilde{H})$).

We have a commutative diagram

$$\begin{CD} \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) @>{(11.4)}>> \mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}) \\ @V{\mathbf{ind}_{\mathrm{Hecke}}}VV @VV{\mathbf{ind}_{\mathrm{Hecke}}}V \\ \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})) @>{(11.5)}>> \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})). \end{CD}$$

We claim:

Proposition 11.2.5.

(a) *The functor* (11.5) *is fully faithful.*

(b) *If the isogeny is stricttly compatible with the geometric metaplectic data (see Sect. 11.2.2), then* (11.5) *is an equivalence.*

Proof. To prove point (a), it suffices to show that for $\mathcal{F}_0, \mathcal{F}_1 \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$, the map

$$\mathrm{Maps}_{\mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))}(\mathbf{ind}_{\mathrm{Hecke}}(\mathcal{F}_0), \mathbf{ind}_{\mathrm{Hecke}}(\mathcal{F}_1)) \to$$
$$\to \mathrm{Maps}_{\mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}))}(\mathbf{ind}_{\mathrm{Hecke}}(\mathcal{F}_0), \mathbf{ind}_{\mathrm{Hecke}}(\mathcal{F}_1))$$

is an isomorphism, where in the left-hand side $\mathbf{ind}_{\mathrm{Hecke}}$ denotes the functor

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})),$$

and in the right-hand side, it denotes the functor

$$\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}) \to \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})).$$

By adjunction, this is equivalent to showing that the map

$$\mathrm{Maps}_{\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})}\left(\mathcal{F}_0, \underset{V\in \mathrm{Irrep}(H)}{\oplus} \mathcal{F}_1 \star \mathrm{Sat}_{q,G}(V)\otimes \underline{V}\right) \to$$
$$\mathrm{Maps}_{\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})}\left(\mathcal{F}_0, \underset{V\in \mathrm{Irrep}(\widetilde{H})}{\oplus} \mathcal{F}_1 \star \mathrm{Sat}_{q,G}(V)\otimes \underline{V}\right)$$

is an isomorphism.

The required isomorphism follows from the fact that for $\mathcal{F}$ supported on $\mathrm{Gr}^{\omega^\rho}_{G,x} \subset \mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}$ and $V \in \mathrm{Irrep}(\widetilde{H})$, the object $\mathcal{F} \star \mathrm{Sat}_{q,G}(V) \in \mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})$ is still supported on $\mathrm{Gr}^{\omega^\rho}_{G,x}$ if and only if

$$V \in \mathrm{Irrep}(H) \subset \mathrm{Irrep}(\widetilde{H}).$$

For point (b) we note that the condition in Sect. 11.2.2 implies that for every $0 \neq \mathcal{F}_1 \in \mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})$ there exists $V \in \mathrm{Irrep}(\widetilde{H})$ so that $\mathcal{F}_1 \star \mathrm{Sat}_{q,G}(V)$ is non-zero when restricted to $\mathrm{Gr}^{\omega^\rho}_{G,x} \subset \mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}$.

To prove that (11.5) is an equivalence, it suffices to show that for every $\mathcal{F}' \in \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}))$ there exists $\mathcal{F} \in \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}))$ equipped with a non-zero map

$$\mathbf{ind}_{\mathrm{Hecke}}(\mathcal{F}) \to \mathcal{F}'.$$

Choose $V \in \mathrm{Irrep}(\widetilde{H})$ so that $\mathbf{oblv}_{\mathrm{Hecke}}(\mathcal{F}') \star \mathrm{Sat}_{q,G}(V)$ is non-zero when restricted to $\mathrm{Gr}^{\omega^\rho}_{G,x}$. Let $\mathcal{F}$ be the resulting object of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$. By construction, we have a non-zero map

$$\mathcal{F} \star \mathrm{Sat}_{q,G}(V^*) \to \mathbf{oblv}_{\mathrm{Hecke}}(\mathcal{F}'),$$

and hence a non-zero map

$$\underline{V}^* \otimes \mathbf{ind}_{\mathrm{Hecke}}(\mathcal{F}) \to \mathcal{F}',$$

as desired.

□

From Proposition 11.2.5, we obtain:

Corollary 11.2.6. *Assume that the isogeny is strictly compatible with the geometric metaplectic data. Then:*

(a) *The functors* (11.4) *and* $\mathrm{Rep}(T_H) \to \mathrm{Rep}(T_{\widetilde{H}})$ *induce an equivalence of* $\mathrm{Rep}(T_H)$*-module categories:*

$$\mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})) \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})),$$

where $\overset{\bullet}{\mathrm{Hecke}}$ *on the left-hand (resp., right-hand) side is taken with respect to* $\mathrm{Rep}(H)$ *and* $\mathrm{Rep}(T_H)$ *(resp.,* $\mathrm{Rep}(\widetilde{H})$ *and* $\mathrm{Rep}(T_{\widetilde{H}})$*).*

(b) *A choice of a splitting of* (11.3) *defines an equivalence*

$$\mathrm{Rep}(\check{T}_0) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})) \simeq \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^{\widetilde{G}}}(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x})).$$

11.2.7. Imposing the Whittaker condition, from (11.4), we obtain a fully faithful functor

$$\mathrm{Whit}_{q,x}(G) \to \mathrm{Whit}_q(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}). \tag{11.6}$$

From Proposition 11.2.5 and Corrollary 11.2.6 we obtain:

Corollary 11.2.8. *Assume that the isogeny is strictly compatible with the geometric metaplectic data. Then:*

(a) *The functor*

$$\mathrm{Hecke}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Hecke}(\mathrm{Whit}_q(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}))$$

is an equivalence.

(b) *The functor*

$$\mathrm{Rep}(T_{\widetilde{H}}) \underset{\mathrm{Rep}(T_H)}{\otimes} \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}))$$

is an equivalence.

(c) *A choice of a splitting of* (11.3) *defines an equivalence*

$$\mathrm{Rep}(\check{T}_0) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \simeq \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(\mathrm{Gr}^{\omega^\rho}_{\widetilde{G},x}))$$

is an equivalence.

11.3. **The t-structure and the description of irreducibles.** In this subsection we will study the behavior of the t-structure on the category $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(\widetilde{G}))$. In particular, we will obtain an explicit description of irreducibles.

11.3.1. According to Sect. 10.1.7 and Proposition 9.3.2, the category $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(\widetilde{G}))$ acquires a t-structure in which both functors

$$\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}} : \mathrm{Whit}_{q,x}(G) \otimes \mathrm{Rep}(T_H) \rightleftarrows \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) : \mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}}$$

are t-exact.

11.3.2. We claim:

Proposition 11.3.3.

(a) *The t-structure on* $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ *is Artinian.*

(b) *There is a natural bijection between irreducibles of* $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^\heartsuit$ *and elements of* Λ.

In the course of the proof of Proposition 11.3.3 we will give an explicit description of the irreducibles of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^\heartsuit$, which will also be useful later.

11.3.4. *Proof of Proposition 11.3.3, first case.* Let us first assume that the pair $(G, \mathcal{G}^G)$ is such that the derived group of H is simply connected. In this case, by Lemma 9.4.2 and Theorem 9.4.9, the action of $\mathrm{Rep}(H)$ on $\mathrm{Whit}_{q,x}(G)$ is accessible.

Hence the assertion of the proposition follows from Corrollary 10.7.4(b).

11.3.5. *Proof of Proposition 11.3.3, reduction step.* Suppose now that we are given an isogeny

$$G \to \widetilde{G}$$

strictly compatible with the geometric metaplectic data. From Corrollary 11.2.8(b,c), we obtain that the assertion of Proposition 11.3.3 for $(\widetilde{G}, \mathcal{G}^{\widetilde{G}})$ implies that for $(G, \mathcal{G}^G)$.

Hence, in order to prove Proposition 11.3.3, it suffices to show the following:

Proposition 11.3.6. *Given* $(G, \mathcal{G}^G)$*, there exists a pair* $(\widetilde{G}, \mathcal{G}^{\widetilde{G}})$ *and an isogeny* $G \to \widetilde{G}$*, strictly compatible with the geometric metaplectic data, such the derived group of* $\widetilde{H}$ *is simply connected.*

The proof of Proposition 11.3.6 is given in Sect. 11.4 below.

$\square$[Proposition 11.3.3]

11.3.7. For $\lambda \in \Lambda$, let $\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$ denote the corresponding irreducible in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$. By construction, for $\gamma \in \Lambda^\sharp$, we have:

$$\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}} \otimes \mathsf{e}^\gamma \simeq \mathcal{M}^{\lambda+\gamma,!*}_{\mathrm{Whit}},$$

where $\otimes$ stands for the action of $\mathrm{Rep}(T_H)$ on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$. Moreover, for λ restricted, we have

$$\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}} := \mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,!*}).$$

We claim:

Corollary 11.3.8. *For $\lambda \in \Lambda^+$, the object $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,*}) \in (\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)))^\heartsuit$ receives a non-zero map from the irreducible $\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$, and the Jordan-Holder constituents of the quotient are of the form $\mathcal{M}^{\lambda',!*}_{\mathrm{Whit}}$ for $\lambda' \leq \lambda$.*

Proof. It is enough to prove the assertion for $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,!*})$ instead of $\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,*})$. Again, we can assume that the derived group of H is simply-connected. Write $\lambda = \lambda_1 + \gamma$ with λ_1 restricted and $\gamma \in \Lambda^{\sharp,+}$. Then

$$\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(W^{\lambda,!*}) \simeq \mathcal{M}^{\lambda_1,!*}_{\mathrm{Whit}} \otimes \mathrm{Res}^H_{T_H}(V^\gamma),$$

and the assertion follows. □

11.3.9. Recall the duality functor

$$\mathbb{D}^{\mathrm{Verdier}} : (\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^c)^{\mathrm{op}} \to \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q^{-1},x}(G))^c.$$

By Corollary 7.2.6 and the construction of the irreducibles $\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}$, we have

$$\mathbb{D}^{\mathrm{Verdier}}(\mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}) \simeq \mathcal{M}^{\lambda,!*}_{\mathrm{Whit}}.$$

From here and Proposition 11.3.3 we obtain:

Corollary 11.3.10. *An object $\mathcal{F} \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^c$ is connective/coconnective if and only if $\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})$ is coconnective/connective*

11.4. **Proof of Proposition 11.3.6.** The proof of Proposition 11.3.6 will amount to a manipulation with lattices and root data.

11.4.1. We first choose an isogeny

$$1 \to \check{T}_0 \to \widetilde{H} \twoheadrightarrow H \to 1$$

so that the derived group of $\widetilde{H}$ is simply-connected. Let $\widetilde{\Lambda}^\sharp$ denote the weight lattice of $\widetilde{H}$.

We define the lattice $\widetilde{\Lambda}$ to be the push-out

$$\widetilde{\Lambda}^\sharp \underset{\Lambda^\sharp}{\sqcup} \Lambda.$$

By construction, the map

$$\widetilde{\Lambda}^\sharp \to \widetilde{\Lambda}$$

is a rational equivalence.

11.4.2. We now construct a root datum for which $\widetilde{\Lambda}$ is the coweight lattice. For a coroot α of G, we let $\widetilde{\alpha} \in \widetilde{\Lambda}$ be the image of α under the natural embedding $\Lambda \to \widetilde{\Lambda}$.

The corresponding root $\widetilde{\check{\alpha}}$ is constructed as follows:

$$\widetilde{\check{\alpha}} = \ell_\alpha \cdot \widetilde{\check{\alpha}}_H,$$

where $\widetilde{\check{\alpha}}_H$ is the corresponding coroot of $\widetilde{H}$, and ℓ_α is as in Sect. 2.3.4.

A priori, $\widetilde{\check{\alpha}}$ is defined as an element of

$$\widetilde{\check{\Lambda}}^\sharp \subset \widetilde{\check{\Lambda}}^\sharp \underset{\mathbb{Z}}{\otimes} \mathbb{Q} \simeq \widetilde{\check{\Lambda}} \underset{\mathbb{Z}}{\otimes} \mathbb{Q}.$$

However, it is easy to see that it in fact belongs to $\widetilde{\check{\Lambda}}$.

It follows from Sect. 2.3.6 that the elements

$$\{\widetilde{\alpha} \in \widetilde{\Lambda}, \widetilde{\check{\alpha}} \in \widetilde{\check{\Lambda}}\}$$

indeed form a root system so that

$$\Lambda \to \widetilde{\Lambda}$$

is an isogeny.

11.4.3. Let $\widetilde{G}$ be the corresponding reductive group over k. By construction, we have a short exact sequence

$$1 \to G \to \widetilde{G} \to T_0 \to 1,$$

where T_0 is the torus dual to $\check{T}_0$.

11.4.4. It remains to show that, on a Zariski neighborhood of the point $x \in X$, there exists a geometric metaplectic data $\mathcal{G}^{\widetilde{G}}$ for $\widetilde{G}$ such that the map $G \to \widetilde{G}$ is strictly compatible with the geometric metaplectic data.

Note, however, that by [GLys, Sect. 3.3], on *affine curves*, geometric metaplectic data are classified by their associated quadratic forms. Hence, it remains to show that the quadratic form q on Λ can be extended to an element

$$\widetilde{q} \in \mathrm{Quad}(\widetilde{\Lambda}, \mathsf{e}^\times(-1))^W_{\mathrm{restr}}$$

(see [GLys, Sect. 3.2.2] or Sect. 27.1.1 for the notation), in such a way that kernel of the associated symmetric bilinear form $\widetilde{b}$ equals $\widetilde{\Lambda}^\sharp \subset \widetilde{\Lambda}$.

11.4.5. Note that the restriction of the quadratic form q to $\Lambda^\sharp$ is such that the associated symmetric bilinear form vanishes. Hence $q^\sharp := q|_{\Lambda^\sharp}$ is a linear map

$$\Lambda^\sharp \to \pm 1 \subset \mathsf{e}^\times.$$

We extend the above map in an arbitrary way to a map

$$\widetilde{q}^\sharp : \widetilde{\Lambda}^\sharp \to \pm 1.$$

We define a quadratic form $\widetilde{q}$ on $\widetilde{\Lambda}$ by the formula

$$\widetilde{q}(\lambda + \widetilde{\lambda}^\sharp) = q(\lambda) + \widetilde{q}^\sharp(\widetilde{\lambda}^\sharp), \quad \lambda \in \Lambda,\ \widetilde{\lambda}^\sharp \in \widetilde{\Lambda}^\sharp.$$

It is easy to see that $\widetilde{q}$, constructed above, indeed belongs to

$$\mathrm{Quad}(\widetilde{\Lambda}, \mathsf{e}^\times(-1))^W_{\mathrm{restr}} \subset \mathrm{Quad}(\widetilde{\Lambda}, \mathsf{e}^\times(-1)),$$

as required.

Part IV: The metaplectic semi-infinite IC sheaf

Having studied the Whittaker category on the affine Grassmannian and its de-equivariantization, i.e., the Hecke category, we move on to the next step: we want to relate it to the right-hand side of our main theorem, which is the category of factorization modules over some factorization algebra on (an object closely related to) the affine Grassmannian for the Cartan subgroup T.

The passage between Gr_G to Gr_T can justifiably be called a *Jacquet functor* as it involves taking cohomology along $\mathfrak{L}(N)$-orbits. However, there is a caveat: this is not just cohomology, but rather it is taken against a non-trivial kernel, The kernel is metaplectic semi-infinite IC sheaf, denoted $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$, is the object of study in this Part.

12. The metaplectic semi-infinite category of the affine Grassmannian

The metaplectic semi-infinite IC sheaf is constructed by the usual procedure of *intermediate extension* inside a certain DG category equipped with a t-structure.

The goal of the present section is to introduce this DG category: this is the (unital version of) the metaplectic semi-infinite category on $\mathrm{Gr}_{G,\mathrm{Ran}}$, denoted $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$.

12.1. **The semi-infinite category.** In this subsection we will define the metaplectic semi-infinite category, first at a fixed point $x \in X$, denoted $\mathrm{SI}_{q,x}(G)$, and then its Ran version, denoted $\mathrm{SI}_{q,\mathrm{Ran}}(G)$.

12.1.1. We define the metaplectic semi-infinite category of the affine Grassmannian, denoted $\mathrm{SI}_{q,x}(G)$ as the full subcategory in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})$ that consists of $\mathfrak{L}(N)^{\omega^\rho}_x$-equivariant objects, i.e.,

$$\mathrm{SI}_{q,x}(G) := \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})\right)^{\mathfrak{L}(N)^{\omega^\rho}_x}.$$

The difference between $\mathrm{Whit}_{q,x}(G)$ and $\mathrm{SI}_{q,x}(G)$ is that instead of the non-degenerate character we use the trivial one.

12.1.2. Much of the discussion pertaining to the definition of $\mathrm{Whit}_{q,x}(G)$ applies to $\mathrm{SI}_{q,x}(G)$. In particular, we have the full subcategories

$$\mathrm{SI}_{q,x}(G)_{=\mu} \subset \mathrm{SI}_{q,x}(G)_{\leq\mu} \subset \mathrm{SI}_{q,x}(G)$$

and the corresponding adjoint functors.

However, instead of Proposition 6.2.9, we have the following assertion (with the same proof):

Lemma 12.1.3. *The category $\mathrm{SI}_{q,x}(G)_{=\mu}$ is (non-canonically) equivalent to* Vect *for* any $\mu \in \Lambda$*, via the functor of !-fiber at the point $t^\mu \in \mathrm{Gr}^{\omega^\rho}_{G,x}$. For $\mu = 0$ this equivalence is canonical.*

Remark 12.1.4. We note, however, that although the standard objects (i.e., the !-extensions of the generators of each $\mathrm{SI}_{q,x}(G)_{=\mu}$) are compact, the corresponding co-standard objects (i.e., the *-extensions) are no longer such. This contrasts with the case of $\mathrm{Whit}_{q,x}(G)$, see Corrollary 6.3.6.

12.1.5. Let $\mathrm{SI}_{q,\mathrm{Ran}}(G)$ be the Ran space version of the semi-infinite category, i.e.,

$$\mathrm{SI}_{q,\mathrm{Ran}}(G) := \left(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})\right)^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}}}.$$

12.1.6. Let $\overline{S}^0_{\mathrm{Ran}} \subset \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ be the corresponding closed subfunctor, see Sect. 8.4.1. We let

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)$$

be the full subcategory that consists of objects supported on $\overline{S}^0_{\mathrm{Ran}}$.

12.2. **Stratifications.** In this subsection we introduce a stratification of $\overline{S}^0_{\mathrm{Ran}}$ by locally closed subfunctors, according to the order of degeneracy of the Drinfeld structure. This stratification will give rise to a stratification of the category $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$.

12.2.1. Recall the space of colored divisors denoted Conf, see Sect. 4.1.1. Let $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$ be the prestack over Conf that classifies triples $(D, \mathcal{P}_G, \alpha)$, where

- $D = \sum\limits_k \mu_k \cdot x_k$ is a point of Conf;
- $\mathcal{P}_G$ is a G-bundle;
- α is an identification of $\mathcal{P}_G$ with ω^ρ away from $\{x_k\}$.

In a similar way we define the group (ind)-schemes

$$\mathfrak{L}^+(G)^{\omega^\rho}_{\mathrm{Conf}} \subset \mathfrak{L}(G)^{\omega^\rho}_{\mathrm{Conf}} \text{ and } \mathfrak{L}^+(N)^{\omega^\rho}_{\mathrm{Conf}} \subset \mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Conf}}$$

over Conf.

12.2.2. Let $\overline{S}^{\mathrm{Conf}}_{\mathrm{Conf}}$ be the closed subfunctor of $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$ consisting of points $(D, \mathcal{P}_G, \alpha)$ as above, for which for every $\check{\lambda} \in \check{\Lambda}^+$, the composite map

$$(\omega^{\frac{1}{2}})^{\langle \check{\lambda}, 2\rho \rangle}(\sum_k - \langle \check{\lambda}, \mu_k \rangle \cdot x_k) \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}'_G} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \tag{12.1}$$

which is a priori defined on $X - \{x_k\}$, extends to a regular map on all of X.

Let

$$S^{\mathrm{Conf}}_{\mathrm{Conf}} \overset{\mathbf{j}_{\mathrm{Conf}}}{\hookrightarrow} \overline{S}^{\mathrm{Conf}}_{\mathrm{Conf}}$$

be the open subfunctor, where we require the composite map (12.1) to have no zeroes on X.

Let p^{Conf} (resp., $\overline{p}^{\mathrm{Conf}}$) denote the projection $S^{\mathrm{Conf}}_{\mathrm{Conf}} \to \mathrm{Conf}$ (resp., $\overline{S}^{\mathrm{Conf}}_{\mathrm{Conf}} \to \mathrm{Conf}$).

12.2.3. We can also think of

$$S^{\mathrm{Conf}}_{\mathrm{Conf}} \subset \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$$

as follows:

The embedding $T \to G$ gives rise to a map $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Conf}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$. In addition, the projection

$$\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Conf}} \to \mathrm{Conf}$$

has a canonical section. Composing, we obtain a section

$$\mathrm{Conf} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}.$$

Then $S^{\mathrm{Conf}}_{\mathrm{Conf}}$ is the orbit of the group $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Conf}}$ acting on the above section.

12.2.4. Let

$$(\mathrm{Conf} \times \mathrm{Ran})^{\subset} \subset \mathrm{Ran} \times \mathrm{Conf}$$

be the ind-closed subfunctor corresponding to the following condition:

An S-point of

$$(\mathcal{I} \subset \mathrm{Hom}(S, X); D \in \mathrm{Hom}(S, \mathrm{Conf}))$$

belongs to $(\mathrm{Conf} \times \mathrm{Ran})^{\subset}$ if and only if the support of the divisor D is *set-theoretically* contained in the union of the graphs of the maps $i : S \to X, i \in \mathcal{J}$ (cf. [Ga7, Sect. 1.3.2]).

Note that we have a canonical identification

$$\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Conf} \times \mathrm{Ran})^{\subset} \simeq \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Conf} \times \mathrm{Ran})^{\subset}. \tag{12.2}$$

Let $\mathrm{pr}_{\mathrm{Ran}}$ denote the projection $(\mathrm{Conf} \times \mathrm{Ran})^{\subset} \to \mathrm{Conf}$.

12.2.5. Denote

$$\overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}} := (\mathrm{Conf}\times\mathrm{Ran})^{\subset} \underset{\mathrm{Conf}}{\times} \overline{S}^{\mathrm{Conf}}_{\mathrm{Conf}}.$$

Denote by $\overline{p}^{\mathrm{Conf}}_{\mathrm{Ran}}$ the projection

$$\overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}} \to (\mathrm{Conf}\times\mathrm{Ran})^{\subset}.$$

Note that the identification (12.2) realizes $\overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}}$ as a closed subfunctor in

$$(\mathrm{Conf}\times\mathrm{Ran})^{\subset} \underset{\mathrm{Ran}}{\times} \overline{S}^{0}_{\mathrm{Ran}}.$$

Let $\overline{\mathbf{i}}^{\mathrm{Conf}}_{\mathrm{Ran}}$ denote the composite map

$$\overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}} \hookrightarrow (\mathrm{Conf}\times\mathrm{Ran})^{\subset} \underset{\mathrm{Ran}}{\times} \overline{S}^{0}_{\mathrm{Ran}} \to \overline{S}^{0}_{\mathrm{Ran}}.$$

Note that the map $\overline{\mathbf{i}}^{\mathrm{Conf}}_{\mathrm{Ran}}$ is proper.

12.2.6. Denote

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} := (\mathrm{Conf}\times\mathrm{Ran})^{\subset} \underset{\mathrm{Conf}}{\times} S^{\mathrm{Conf}}_{\mathrm{Conf}}.$$

Denote by $p^{\mathrm{Conf}}_{\mathrm{Ran}}$ the projection

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} \to (\mathrm{Conf}\times\mathrm{Ran})^{\subset}.$$

Denote by $\mathbf{j}^{\mathrm{Conf}}_{\mathrm{Ran}}$ the open embedding

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} \hookrightarrow \overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}}.$$

12.2.7. For $\lambda\in\Lambda^{\mathrm{neg}}-0$, denote

$$(\mathrm{Conf}^{\lambda}\times\mathrm{Ran})^{\subset} := (\mathrm{Conf}\times\mathrm{Ran})^{\subset} \underset{\mathrm{Conf}}{\times} \mathrm{Conf}^{\lambda}.$$

Let $\mathrm{pr}^{\lambda}_{\mathrm{Ran}}$ denote the restriction of the map $\mathrm{pr}_{\mathrm{Ran}}$ to $(\mathrm{Conf}^{\lambda}\times\mathrm{Ran})^{\subset}$. Denote also

$$\overline{S}^{\lambda}_{\mathrm{Ran}} := (\mathrm{Conf}^{\lambda}\times\mathrm{Ran})^{\subset} \underset{(\mathrm{Conf}\times\mathrm{Ran})^{\subset}}{\times} \overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}}$$

and

$$S^{\lambda}_{\mathrm{Ran}} := (\mathrm{Conf}^{\lambda}\times\mathrm{Ran})^{\subset} \underset{(\mathrm{Conf}\times\mathrm{Ran})^{\subset}}{\times} S^{\mathrm{Conf}}_{\mathrm{Ran}}.$$

Denote by $\mathbf{j}^{\lambda}_{\mathrm{Ran}}$ the resulting map

$$S^{\lambda}_{\mathrm{Ran}} \to \overline{S}^{\lambda}_{\mathrm{Ran}}$$

and by $\overline{\mathbf{i}}^{\lambda}_{\mathrm{Ran}}$ the corresponding map

$$\overline{S}^{\lambda}_{\mathrm{Ran}} \to \overline{S}^{\mathrm{Conf}}_{\mathrm{Ran}} \xrightarrow{\overline{\mathbf{i}}_{\mathrm{Ran}}} \overline{S}^{0}_{\mathrm{Ran}}.$$

Denote

$$\mathbf{i}^{\lambda}_{\mathrm{Ran}} := \overline{\mathbf{i}}^{\lambda}_{\mathrm{Ran}} \circ \mathbf{j}^{\lambda}_{\mathrm{Ran}}.$$

Denote by $\overline{p}^{\lambda}_{\mathrm{Ran}}$ (resp., $p^{\lambda}_{\mathrm{Ran}}$) the restriction of $\overline{p}_{\mathrm{Ran}}$ (resp., p_{Ran}) to $\overline{S}^{\lambda}_{\mathrm{Ran}}$ (resp., $S^{\lambda}_{\mathrm{Ran}}$).

12.2.8. We extend the above definitions to formally include the case of $\lambda=0$, in which case we set

$$\mathrm{Conf}^{0} = \mathrm{pt}.$$

We let

$$S^{0}_{\mathrm{Ran}} \overset{\mathbf{j}_{\mathrm{Ran}}}{\hookrightarrow} \overline{S}^{0}_{\mathrm{Ran}}$$

be the open subfunctor, where we require that the map (8.3) have no zeroes.

We have:

$$\overline{\mathbf{i}}^{0}_{\mathrm{Ran}} = \mathrm{id},\ \ \mathbf{i}^{0}_{\mathrm{Ran}} = \mathbf{j}^{0}_{\mathrm{Ran}} = \mathbf{j}_{\mathrm{Ran}},\ \ (\mathrm{Conf}^{0}\times\mathrm{Ran})^{\subset} = \mathrm{Ran},$$

p^{0}_{Ran} (resp., $\overline{p}^{0}_{\mathrm{Ran}}$) is the map $S^{0}_{\mathrm{Ran}}\to\mathrm{Ran}$ (resp., $\overline{S}^{0}_{\mathrm{Ran}}\to\mathrm{Ran}$), and $\mathrm{pr}^{0}_{\mathrm{Ran}}$ is the projection $\mathrm{Ran}\to\mathrm{pt}$.

12.2.9. The following results easily from the definitions:

Lemma 12.2.10. *The map*

$$\mathbf{i}^\lambda_{\mathrm{Ran}} : S^\lambda_{\mathrm{Ran}} \to \overline{S}^0_{\mathrm{Ran}}$$

is a locally closed embedding. Every field-valued point of $\overline{S}^0_{\mathrm{Ran}}$ *belongs to the image of exactly one such map.*

12.3. **Stratification of the category.** The strata S^λ_{Ran} of $\overline{S}^0_{\mathrm{Ran}}$ give rise to a *recollement* pattern on $\mathrm{SI}_{q,\mathrm{Ran}}(G)$.

12.3.1. Let $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq\lambda}$ denote the full subcategory of $\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda_{\mathrm{Ran}})$ given by the condition of equivariance with respect to the pullback of $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}}$ to $(\mathrm{Conf}^\lambda \times \mathrm{Ran})^{\subset}$.

Let $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}$ be the corresponding full subcategory of $\mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{\mathrm{Ran}})$.

12.3.2. The maps $\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}}$ and $\mathbf{j}^\lambda_{\mathrm{Ran}}$ define pairs of mutually adjoint functors

$$(\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})_! = (\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})_* : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq\lambda} \rightleftarrows \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0} : (\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})^!;$$

$$(\mathbf{j}^\lambda_{\mathrm{Ran}})^* = (\mathbf{j}^\lambda_{\mathrm{Ran}})^! : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq\lambda} \rightleftarrows \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda} : (\mathbf{j}^\lambda_{\mathrm{Ran}})_*.$$

12.3.3. In addition, as in [Ga7, Corollary 1.4.5], one shows that the partially defined left adjoint $(\mathbf{i}^\lambda_{\mathrm{Ran}})^*$ of

$$(\mathbf{i}^\lambda_{\mathrm{Ran}})_* := (\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})_* \circ (\mathbf{j}^\lambda_{\mathrm{Ran}})_*$$

is defined on

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}}),$$

giving rise to an adjoint pair

$$(\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})^* : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0} \rightleftarrows \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq\lambda} : (\mathbf{i}^\lambda_{\mathrm{Ran}})_*.$$

12.3.4. Also, the partially defined left adjoint $(\mathbf{i}^\lambda_{\mathrm{Ran}})_!$ of $(\mathbf{i}^\lambda_{\mathrm{Ran}})^!$ is defined on

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda} \subset \mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{\mathrm{Ran}}),$$

giving rise to an adjoint pair

$$(\mathbf{i}^\lambda_{\mathrm{Ran}})_! : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda} \rightleftarrows \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0} : (\mathbf{i}^\lambda_{\mathrm{Ran}})^!.$$

12.4. **Description of the category on a stratum.** In this subsection we will describe explicitly the category $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}$. It will turn out to be equivalent to the category of (gerbe-twisted) sheaves on $(\mathrm{Conf}^\lambda \times \mathrm{Ran})^{\subset}$.

12.4.1. We first observe:

Proposition 12.4.2. *The pullback of the gerbe* $\mathcal{G}^G$ *along the map*

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} \to \overline{S}^0_{\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \tag{12.3}$$

identifies canonically with the pullback of the gerbe $\mathcal{G}^\Lambda$ *on* Conf *of Sect. 4.6.5 along the map*

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} \overset{p_{\mathrm{Ran}}}{\longrightarrow} (\mathrm{Conf} \times \mathrm{Ran})^{\subset} \overset{\mathrm{pr}_{\mathrm{Ran}}}{\longrightarrow} \mathrm{Conf}. \tag{12.4}$$

The proof will be essentially a diagram chase, modulo the additional structure on the gerbe $\mathcal{G}^G$ specified in Sect. 1.6.6.

Proof. By construction, the map (12.3) factors as

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}.$$

By definition, the correspondence between $\mathcal{G}^G$ and $\mathcal{G}^T$ is such that their pullbacks to $\mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}}$ along the maps

$$\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \leftarrow \mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$$

are identified, see Sect. 2.3.1.

Hence, it suffices to show that pullback of $\mathcal{G}^T$ along

$$S^{\mathrm{Conf}}_{\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{B,\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \tag{12.5}$$

identifies with the pullback of $\mathcal{G}^\Lambda$ along the map (12.4).

In order to do so, we can replace $S^{\mathrm{Conf}}_{\mathrm{Ran}}$ by

$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Conf}}{\times} S^{\mathrm{Conf}}_{\mathrm{Ran}},$$

(see Sect. 4.6.1 for the notation), which identifies with

$$S^{\mathrm{Conf}}_{\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}{\times} \mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}},$$

where the map $\mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \to \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$ is φ_{small}, see Sect. 1.6.9 for the notation.

With respect to this identification, the composition

$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Conf}}{\times} S^{\mathrm{Conf}}_{\mathrm{Ran}} \to S^{\mathrm{Conf}}_{\mathrm{Ran}} \overset{(12.4)}{\longrightarrow} \mathrm{Conf}$$

identifies with

$$S^{\mathrm{Conf}}_{\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}{\times} \mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \to (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \to \mathrm{Conf}\,.$$

Hence, the pullback of $\mathcal{G}^\Lambda$ along this map identifies with the pullback of $\mathcal{G}^T$ along the map

$$S^{\mathrm{Conf}}_{\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}{\times} \mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \to \mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \overset{\varphi_{\mathrm{small}}}{\longrightarrow} \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}.$$

The composition

$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Conf}}{\times} S^{\mathrm{Conf}}_{\mathrm{Ran}} \to S^{\mathrm{Conf}}_{\mathrm{Ran}} \overset{(12.5)}{\longrightarrow} \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}$$

identifies with

$$S^{\mathrm{Conf}}_{\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \underset{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}{\times} \mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \to \mathrm{Gr}^{\omega^\rho}_{T,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}} \overset{\varphi_{\mathrm{big}}}{\longrightarrow} \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}.$$

Hence, the required isomorphism follows from Corrollary 1.6.7.

□

12.4.3. From Proposition 12.4.2, we obtain that we have a canonically defined pullback functor

$$p^!_{\mathrm{Ran}} : \mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}\times\mathrm{Ran})^{\subset}) \to \mathrm{Shv}_{\mathcal{G}^G}(S^{\mathrm{Conf}}_{\mathrm{Ran}}),$$

and for every individual $\lambda \in \Lambda^{\mathrm{neg}}$, a functor

$$(p^\lambda_{\mathrm{Ran}})^! : \mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset}) \to \mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{\mathrm{Ran}}),$$

As in Lemma 12.1.3 (see also [Ga7, Lemma 1.4.8], we have:

Lemma 12.4.4. *For every $\lambda \in \Lambda^{\mathrm{neg}}$, functor $(p^\lambda_{\mathrm{Ran}})^!$ induces an equivalence*

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset}) \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}.$$

12.5. **The unital subcategory.** As we just saw in Lemma 12.4.4, the category $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}$ is equivalent to that of (gerbe-twisted) sheaves on $(\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset}$. This category is too large for our needs: it contains the “junk” directions that have to do with Ran, and we would like to cut those down.

A device to do so is the *unitality structure*.

12.5.1. Let us recall the setting of Sect. 1.6. Note that according to Lemma 1.6.4, the pullback functor

$$\varphi_{\mathrm{small}} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}}) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}})$$

is fully faithful.

We define the category $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}})_{\mathrm{untl}}$ to consist of objects $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$ equipped with an isomorphism

$$\varphi^!_{\mathrm{small}}(\mathcal{F}) \simeq \varphi^!_{\mathrm{big}}(\mathcal{F})$$

and an identification of the composite map

$$\mathcal{F} \simeq \Delta_{\mathrm{Ran}}! \circ \varphi^!_{\mathrm{small}}(\mathcal{F}) \simeq \Delta_{\mathrm{Ran}}! \circ \varphi^!_{\mathrm{big}}(\mathcal{F}) \simeq \mathcal{F}$$

with the identity map on $\mathcal{F}$.

Note that due to the fully faithfulness result quoted above, the forgetful functor

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})_{\mathrm{untl}} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$$

is fully faithful.

12.5.2. Define

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)_{\mathrm{untl}} := \mathrm{SI}_{q,\mathrm{Ran}}(G) \cap \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})_{\mathrm{untl}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}}).$$

12.5.3. Along with $\mathrm{Gr}^{\omega^\rho}_{G,(\mathrm{Ran}\times\mathrm{Ran})^{\subset}}$, we can consider the prestacks

$$\overline{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}},\ \overline{S}^{\mathrm{Conf}}_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}},\ S^{\mathrm{Conf}}_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}},$$

etc.

Proceeding as above, we define the full subcategories

$$\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}})_{\mathrm{untl}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}}),$$

$$\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}})_{\mathrm{untl}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}})$$

and

$$\mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}})_{\mathrm{untl}} \subset \mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{(\mathrm{Ran}\times\mathrm{Ran})^{\subset}}),$$

as well as

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0},$$
$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda}_{\mathrm{untl}} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda},$$
$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{= \lambda}_{\mathrm{untl}} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{= \lambda}.$$

It is clear that the functors $(\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})^!$, $(\mathbf{j}^\lambda_{\mathrm{Ran}})^!$ (and hence also $(\mathbf{i}^\lambda_{\mathrm{Ran}})^!$) as well as $(\bar{\mathbf{i}}^\lambda_{\mathrm{Ran}})_*$, $(\mathbf{j}^\lambda_{\mathrm{Ran}})_*$ (and hence also $(\mathbf{i}^\lambda_{\mathrm{Ran}})_*$) preserve the corresponding unital subcategories.

In addition, as in [Ga7, Proposition 4.2.2], we have:

Proposition 12.5.4. *The functors $(\mathbf{i}^\lambda_{\mathrm{Ran}})_!$ and $(\mathbf{i}^\lambda_{\mathrm{Ran}})^*$ also preserve the unital subcategories.*

12.5.5. By a similar token we define the full subcategory

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}\times\mathrm{Ran})^{\subset})_{\mathrm{untl}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}\times\mathrm{Ran})^{\subset}).$$

It follows from Lemma 12.4.4 that the functor $(p^\lambda_{\mathrm{Ran}})^!$ induces an equivalence

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset})_{\mathrm{untl}} \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}. \tag{12.6}$$

12.5.6. The following is proved in [Ga7, Proposition 4.2.7]:

Proposition 12.5.7. *Pullback with respect to*

$$\mathrm{pr}^\lambda_{\mathrm{Ran}} : (\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset} \to \mathrm{Conf}^\lambda$$

defines an equivalencce

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}^\lambda) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda\times\mathrm{Ran})^{\subset})_{\mathrm{untl}}.$$

12.5.8. Combining, we obtain the following explicit description of the category $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}$:

Corollary 12.5.9. *For every $\lambda \in \Lambda^{\mathrm{neg}}$, pullback with respect to $\mathrm{pr}^{\lambda}_{\mathrm{Ran}} \circ p^{\lambda}_{\mathrm{Ran}}$ defines an equivalence*

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}^\lambda) \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}.$$

12.5.10. *Example.* For $\lambda = 0$ the above corollary says that the functor

$$\mathsf{e} \mapsto \omega_{S^0_{\mathrm{Ran}}}$$

defines an equivalence

$$\mathrm{Vect} \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=0}_{\mathrm{untl}}.$$

13. The metaplectic semi-infinite IC sheaf

In this section we finally construct the main object of study in this Part: the metaplectic semi-infinite IC sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$.

13.1. **The t-structure on the semi-infinite category.** In this subsection we will introduce a t-structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$, and as a result we will define the metaplectic semi-infinite IC sheaf $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$.

13.1.1. We define a t-structure on the category $\mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda \times \mathrm{Ran})^{\subset})_{\mathrm{untl}}$ via the equivalence of Proposition 12.5.7:

$$(\mathrm{pr}^{\lambda}_{\mathrm{Ran}})^! : \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}^\lambda) \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda \times \mathrm{Ran})^{\subset})_{\mathrm{untl}},$$

i.e., we transfer the (perverse) t-structure from $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}^\lambda)$.

13.1.2. We define a t-structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}$ via the equivalence of (12.6), *up to a cohomological shift by $\langle \lambda, 2\check{\rho} \rangle$.*

Namely, we declare on object of $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}$ to be connective/coconnective if and only if its image in $\mathrm{Shv}_{\mathcal{G}^\Lambda}((\mathrm{Conf}^\lambda \times \mathrm{Ran})^{\subset})_{\mathrm{untl}}$, shifted cohomologically by $[\langle \lambda, 2\check{\rho} \rangle]$, is connective/coconnective.

We apply a similar procedure to define a t-structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=0}_{\mathrm{untl}}$ via its identification with Vect.

13.1.3. We define a structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$ by declaring an object to be coconnective if and only if its image under $(\mathbf{i}^{\lambda}_{\mathrm{Ran}})^!$ is coconnective in $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}$ for every λ.

13.1.4. By construction, the functors

$$(\mathbf{i}^{\lambda}_{\mathrm{Ran}})_! : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}} \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$$

are right t-exact, and the functors

$$(\mathbf{i}^{\lambda}_{\mathrm{Ran}})_* : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}} \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$$

are left t-exact. From here, we obtain that the functors

$$(\mathbf{i}^{\lambda}_{\mathrm{Ran}})^* : \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}} \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}$$

are right t-exact.

Moreover, as in [Ga7, Lemma 2.1.8], we have:

Lemma 13.1.5. *An object $\mathcal{F} \in \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$ is connective if and only if $(\mathbf{i}^{\lambda}_{\mathrm{Ran}})^*(\mathcal{F}) \in \mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}}$ is connective for every λ.*

13.1.6. We let

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}} \in (\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}})^{\heartsuit} \subset \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$$

be the object equal to the image of the resulting map

$$H^0((\mathbf{j}_{\mathrm{Ran}})_!(\omega_{S^0_{\mathrm{Ran}}})) \to H^0((\mathbf{j}_{\mathrm{Ran}})_*(\omega_{S^0_{\mathrm{Ran}}})),$$

where H^0 refers to the t-structure on $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$ introduced in Sect. 13.1.3 above.

Remark 13.1.7. For the purposes of this paper we will largely forget the unitality property of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$, and regard it simply as an object of $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}$.

We emphasize, however, that we needed the unitality device in order to have a well-behaved t-structure.

13.1.8. For future use we record:

Lemma 13.1.9. *The !-restriction of* $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ *to* $S^{\lambda}_{\mathrm{Ran}}$ *is zero unless* $\lambda \in \Lambda^{\sharp}$.

The proof will be given in Sect. 14.1.5.

13.1.10. In the rest of this section we will discuss various properties of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$:

–We will explicitly describe its restriction to

$$\mathrm{Gr}^{\omega^{\rho}}_{G,x} \subset \mathrm{Gr}_{G,\mathrm{Ran}};$$

–We will endow it with a *factorization structure*;

–We will describe its relationship with the *global* metaplectic semi-infinite IC sheaf.

13.2. **Description of the fiber.** The definition of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ might appear as very abstract, and the result object may seem hardly calculable. This turns out to be not so.

In this subsection we will describe explicitly its restriction to $\mathrm{Gr}^{\omega^{\rho}}_{G,x} \subset \mathrm{Gr}^{\omega^{\rho}}_{G,\mathrm{Ran}}$, i.e., the fiber of $\mathrm{Gr}^{\omega^{\rho}}_{G,\mathrm{Ran}}$ over the point $\{x\} \in \mathrm{Ran}$.

13.2.1. For an element $\gamma \in \Lambda^{\sharp}$ consider the corresponding point

$$t^{\gamma} \in S^{\gamma} \subset \mathrm{Gr}^{\omega^{\rho}}_{G,x}.$$

The trivialization of the gerbe $\mathcal{G}_{T_H,x}$ in Sect. 2.3.8 gives rise to trivialization of $\mathcal{G}^G|_{t^{\gamma}}$. Hence, $\delta_{t^{\gamma},\mathrm{Gr}}$ makes sense as an object of $\mathrm{Shv}_{\mathcal{G}}(\mathrm{Gr}^{\omega^{\rho}}_{G,x})$.

13.2.2. Assume now that γ is dominant. Let V^{γ} denote the corresponding irreducible object in $\mathrm{Rep}(H)$ (see our conventions in Sect. 2.4.5). It follows from (2.7) that the !-fiber of $\mathrm{Sat}_{q,G}(V^{\gamma})$ at t^{γ} identifies canonically with $\mathsf{e}[-\langle \gamma, 2\check{\rho}\rangle]$.

Hence, we obtain a canonically defined map

$$\delta_{t^{\gamma},\mathrm{Gr}}[-\langle \gamma, 2\check{\rho}\rangle] \to \mathrm{Sat}_{q,G}(V^{\gamma}). \tag{13.1}$$

By adjunction, we obtain a map

$$\delta_{1,\mathrm{Gr}} \to \delta_{t^{-\gamma},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma})[\langle \gamma, 2\check{\rho}\rangle]. \tag{13.2}$$

13.2.3. We consider $(\Lambda^{\sharp})^+$ as a poset with respect to the order relation that

$$\gamma_2 \preceq \gamma_1 \;\Leftrightarrow\; \gamma_2 - \gamma_1 \in (\Lambda^{\sharp})^+.$$

Note that this is a *different* order relation than the standard one denoted $\leq$. Note also that, taken with respect to $\preceq$, the set $(\Lambda^{\sharp})^+$ is *filtered*.

13.2.4. For $\gamma_2 \succeq \gamma_1$ with $\gamma_2 - \gamma_1 = \gamma \in (\Lambda^\sharp)^+$ we define the map

$$\delta_{t^{-\gamma_1},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma_1})[\langle \gamma_1, 2\check{\rho}\rangle] \to \delta_{t^{-\gamma_2},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma_2})[\langle \gamma_2, 2\check{\rho}\rangle]$$

to be the composite

$$\delta_{t^{-\gamma_1},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma_1})[\langle \gamma_1, 2\check{\rho}\rangle] \simeq \delta_{t^{-\gamma_1},\mathrm{Gr}} \star \delta_{1,\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma_1})[\langle \gamma_1, 2\check{\rho}\rangle] \overset{(13.2)}{\longrightarrow}$$
$$\to \delta_{t^{-\gamma_1},\mathrm{Gr}} \star \delta_{t^{-\gamma},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma}) \star \mathrm{Sat}_{q,G}(V^{\gamma_1})[\langle \gamma_2, 2\check{\rho}\rangle] \simeq \delta_{t^{-\gamma_2},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma}) \star \mathrm{Sat}_{q,G}(V^{\gamma_1})[\langle \gamma_2, 2\check{\rho}\rangle] \to$$
$$\to \delta_{t^{-\gamma_2},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^{\gamma_2})[\langle \gamma_2, 2\check{\rho}\rangle] \tag{13.3}$$

where the last arrow is induced by the Plücker map

$$V^\gamma \otimes V^{\gamma_1} \to V^{\gamma_2},$$

normalized so that it is the *idenitity* map on the (trivialized) highest weight lines.

13.2.5. As in [Ga6, Sect. 2.3], one shows that the assingment

$$\gamma \rightsquigarrow \delta_{t^{-\gamma},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^\gamma)[\langle \gamma, 2\check{\rho}\rangle]$$

together with the above transition maps define a functor of ∞-categories

$$((\Lambda^\sharp)^+, \preceq) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}).$$

Define:

$$'\mathrm{IC}^{\frac{\infty}{2}}_{q,x} := \underset{\gamma \in ((\Lambda^\sharp)^+, \preceq)}{\mathrm{colim}}\, \delta_{t^{-\gamma},\mathrm{Gr}} \star \mathrm{Sat}_{q,G}(V^\gamma)[\langle \gamma, 2\check{\rho}\rangle] \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}). \tag{13.4}$$

As in [Ga6, Proposition 2.3.7], one shows:

Proposition 13.2.6.

(a) *The object* $'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ *belongs to* $\mathrm{SI}_{q,x}(G)$.

(b) *The object* $'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ *is supported on* $\overline{S}^0 \subset \mathrm{Gr}^{\omega^\rho}_{G,x}$.

(c) *There is a canonical identification* $(\mathbf{j}^0)^!('\mathrm{IC}^{\frac{\infty}{2}}_{q,x}) \simeq \omega_{S^0}$.

13.2.7. A metaplectic analog of the isomorphism of [Ga7, Corollary 2.7.7] reads:

Theorem 13.2.8. *There exists a canonical isomorphism between*

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,x} := \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{\mathrm{Gr}^{\omega^\rho}_{G,x}}$$

and $'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$.

13.2.9. In what follows we will need the following property of $\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$:

Proposition 13.2.10. *The !-restriction of* $\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ *to a stratum* S^λ *with* $\lambda \neq 0$ *is of the form*

$$\omega_{S^\lambda}[-\langle \lambda, 2\check{\rho}\rangle] \otimes \mathsf{K}_{x,\lambda},$$

where K_λ *is an object of* $\mathrm{Shv}_{\mathcal{G}^\Lambda_{\lambda\cdot x}}(\mathrm{pt})$ *that lives in cohomological degrees* ≥ 2.

Proof. Consider the restriction of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ to the stratum

$$X \underset{\mathrm{Conf}^\lambda}{\times} S^\lambda_{\mathrm{Ran}},$$

where $X \to \mathrm{Conf}^\lambda$ is the diagonal map $x \mapsto \lambda \cdot x$.

This restriction is the pullback of an object

$$\mathsf{K}_{X,\lambda}[-\langle \lambda, 2\check{\rho}\rangle] \in \mathrm{Shv}_{\mathcal{G}^\Lambda|_X}(X),$$

and $\mathsf{K}_{x,\lambda}$ is the !-fiber of $\mathsf{K}_{X,\lambda}$ at x. A priori, $\mathsf{K}_{X,\lambda}$ lives in cohomological degrees > 0.

Now, the expression of $\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ given by (13.4) implies that $\mathsf{K}_{x,\lambda}$ would be (non-canonically) the same for *another choice* of a curve X and a point x on it. So for the purposes of proving the cohomological estimate, we can assume that $X = \mathbb{A}^1$ with a choice of geometric metaplectic datum which is translation-invariant with the same quadratic form q.

Since the assignment $X \rightsquigarrow \mathsf{K}_{X,\lambda}$ respects automorphisms of the situation, we obtain that $\mathsf{K}_{X,\lambda}$ is translation-invariant, and in particular lisse. This implies that the !-fiber $\mathsf{K}_{X,\lambda}$ at x lives in degrees > 1, as required.

□

13.3. **Factorization.** We will now specify a key structure possessed by $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$: that of factorization algebra. This structure will be used to define a factorization structure on the Jacquet functor

$$\mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_T).$$

13.3.1. Parallel to the factorization structure on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ (see Sect. 1.3.3), we also have a factorization on $\overline{S}^0_{\mathrm{Ran}}$ and $\overline{S}_{\mathrm{Ran}}$:

$$\overline{S}^0_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}} \simeq (\overline{S}^0_{\mathrm{Ran}})^J \underset{\mathrm{Ran}^J}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}. \tag{13.5}$$

$$S^0_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}} \simeq (S^0_{\mathrm{Ran}})^J \underset{\mathrm{Ran}^J}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}. \tag{13.6}$$

13.3.2. The following assertion is an extension of [Ga7, Sect. 4.6]:

Theorem 13.3.3. *The object $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}} \in \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})$ has a structure of factorization algebra (see Sect. 3.1.1), uniquely characterized by the requirement that the induced structure of factorization algebra on $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{S^0_{\mathrm{Ran}}} \in \mathrm{Shv}_{\mathcal{G}^G}(S^0_{\mathrm{Ran}})$ corresponds to the tautological one on $\omega_{S^0_{\mathrm{Ran}}}$ (see Sect. 3.2.2) with respect to the identification*

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{S^0_{\mathrm{Ran}}} \simeq \omega_{S^0_{\mathrm{Ran}}}$$

Remark 13.3.4. In fact, a slightly stronger assertion is true: for an individual finite set J, there exists factorization isomorphism

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{\overline{S}^0_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}} \simeq (\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}})^{\boxtimes J}|_{(\overline{S}^0_{\mathrm{Ran}})^J \underset{\mathrm{Ran}^J}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}}, \tag{13.7}$$

is *uniquely characterized* by the requirement that the induced isomorphism

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{S^0_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}} \simeq (\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}})^{\boxtimes J}|_{(S^0_{\mathrm{Ran}})^J \underset{\mathrm{Ran}^J}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}}$$

corresponds to the tautological one under on

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{S^0_{\mathrm{Ran}}} \simeq \omega_{S^0_{\mathrm{Ran}}}.$$

Remark 13.3.5. It is in the proof of Theorem 13.3.3 that the unitality property/structure on $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ plays the most essential role, see [Ga7, Sect. 4.6]: one shows that both sides in (13.7) are Goresky-MacPherson extensions of their respective restrictions to $S^0_{\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Ran}^J)_{\mathrm{disj}}$ for the appropriately defined t-structure.

Remark 13.3.6. The unitality structure is also helpful to combat difficulties of homotopy-theoretic nature: it allows us to consider all the objects involved in (13.7) as living inside appropriately defined *abelian categories*; hence the *homotopy coherence* of factorization isomorphisms is automatic.

13.4. **Comparison with the global metaplectic IC sheaf.** In this subsection we will show how to express $\mathrm{IC}_{q,\mathrm{Ran}}$ in terms of finite-dimensional algebraic geometry (specifically, in terms of gerbe-twisted sheaves on $\overline{\mathrm{Bun}}_N^{\omega^\rho}$).

This description will be subsequently used in order to establish one of the key properties of the Jacquet functor, namely, its commutation with Verdier duality.

13.4.1. Consider the algebraic stack $\overline{\mathrm{Bun}}_N^{\omega^\rho}$. As in the case of

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho}),$$

one singles out a full subcategory denoted

$$\mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq 0} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho}), \tag{13.8}$$

by imposing equivariance with respect to a certain unipotent groupoid, see [Ga9, Sects. 4.4-4.7]. The subcategory (13.8) is compatible with the (perverse) t-structure on $\mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho})$.

We will give an explicit description of this subcategory below, see Sect. 13.4.4.

13.4.2. For $\lambda \in \Lambda^{\mathrm{neg}}$, let

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{\leq\lambda} \overset{\bar{\mathbf{i}}^\lambda_{\mathrm{glob}}}{\hookrightarrow} \overline{\mathrm{Bun}}_N^{\omega^\rho}$$

be the closed substack corresponding to the locus where the generalized B structure has total defect at least $-\lambda$. Let

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{=\lambda} \overset{\mathbf{j}^\lambda_{\mathrm{glob}}}{\hookrightarrow} (\overline{\mathrm{Bun}}_N^{\omega^\rho})^{\leq\lambda}$$

denote the open substack, where we the generalized B structure has total defect exactly λ. Denote

$$\mathbf{i}^\lambda_{\mathrm{glob}} = \bar{\mathbf{i}}^\lambda_{\mathrm{glob}} \circ \mathbf{j}^\lambda_{\mathrm{glob}}.$$

In particular, for $\lambda = 0$, we have $\bar{\mathbf{i}}^0_{\mathrm{glob}} = \mathrm{id}$, and

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{\leq 0} = \overline{\mathrm{Bun}}_N^{\omega^\rho} \text{ and } (\overline{\mathrm{Bun}}_N^{\omega^\rho})^{=0} = \mathrm{Bun}_N^{\omega^\rho},$$

and the map $\mathbf{j}^0_{\mathrm{glob}} = \mathbf{i}^0_{\mathrm{glob}}$ is the open embedding

$$\mathrm{Bun}_N^{\omega^\rho} \overset{\mathbf{j}_{\mathrm{glob}}}{\hookrightarrow} \overline{\mathrm{Bun}}_N^{\omega^\rho}.$$

For $\lambda \in \Lambda^{\mathrm{neg}} - 0$ we have a canonical isomorphism

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{=\lambda} \simeq \mathrm{Bun}_B \underset{\mathrm{Bun}_T}{\times} \mathrm{Conf}^\lambda,$$

where

$$\mathrm{AJ}^{\omega^\rho} : \mathrm{Conf} \to \mathrm{Bun}_T, \quad \sum_k \lambda_k \cdot x_k \mapsto \omega^\rho(\sum_k \lambda_k \cdot x_k). \tag{13.9}$$

13.4.3. We have the corresponding full subcategories

$$\mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq\lambda} \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})^{\leq\lambda})$$

and

$$\mathrm{SI}_{q,\mathrm{glob}}(G)^{=\lambda} \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})^{=\lambda}),$$

and the corresponding pairs of adjoint functors

$$(\mathbf{j}^\lambda_{\mathrm{glob}})_! : \mathrm{SI}_{q,\mathrm{glob}}(G)^{=\lambda} \rightleftarrows \mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq\lambda} : (\mathbf{j}^\lambda_{\mathrm{glob}})^!,$$

$$(\mathbf{j}^\lambda_{\mathrm{glob}})^* = (\mathbf{j}^\lambda_{\mathrm{glob}})^! : \mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq\lambda} \rightleftarrows \mathrm{SI}_{q,\mathrm{glob}}(G)^{=\lambda} : (\mathbf{j}^\lambda_{\mathrm{glob}})_*;$$

$$(\bar{\mathbf{i}}^\lambda_{\mathrm{glob}})_! = (\bar{\mathbf{i}}^\lambda_{\mathrm{glob}})_* : \mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq\lambda} \rightleftarrows \mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq 0} : (\bar{\mathbf{i}}^\lambda_{\mathrm{glob}})^!,$$

$$(\bar{\mathbf{i}}^\lambda_{\mathrm{glob}})^* : \mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq 0} \rightleftarrows \mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq\lambda} : (\bar{\mathbf{i}}^\lambda_{\mathrm{glob}})_*.$$

13.4.4. The full subcategory $\mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq 0} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho})$ is characterized as follows:

An object $\mathcal{F} \in \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho})$ belongs to $\mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq 0}$ if and only if each $(\mathbf{i}^\lambda_{\mathrm{glob}})^!(\mathcal{F})$ (or, equivalently $(\mathbf{i}^\lambda_{\mathrm{glob}})^*(\mathcal{F})$) belongs to the corresponding full subcategory

$$\mathrm{SI}_{q,\mathrm{glob}}(G)^{=\lambda} \subset \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})^{=\lambda}).$$

In its turn, pullback along

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})^{=\lambda} \to \mathrm{Conf}^\lambda$$

is fully faithful, and its essential image is exactly $\mathrm{SI}_{q,\mathrm{glob}}(G)^{=\lambda}$.

13.4.5. Consider the map

$$\pi_{\mathrm{Ran}} : \overline{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}_N^{\omega^\rho}.$$

Note that it follows from Theorem 8.1.4 that the pullback functor

$$\pi^!_{\mathrm{Ran}} : \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho}) \to \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})$$

is fully faithful.

The following is a metaplectic analog of [Ga7, Corollary 3.5.7 and Theorem 4.3.2]:

Theorem 13.4.6. *The functor*

$$\pi^!_{\mathrm{Ran}} : \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho}) \to \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})$$

induces an equivalence between

$$\mathrm{SI}_{q,\mathrm{glob}}(G)^{\leq 0} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho})$$

and

$$\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}}).$$

Moreover, after applying the cohomological shift by

$$d_g := \dim(\mathrm{Bun}_N^{\omega^\rho}) = (g-1)(d - \langle 2\check{\rho}, 2\rho\rangle), \quad d = \dim(\mathfrak{n}),$$

it is t-exact.

13.4.7. Let $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} \in (\mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathrm{Bun}}_N^{\omega^\rho}))^\heartsuit$ be the IC-extension (in the category of $\mathcal{G}^G$-twisted sheaves) of the constant perverse sheaf on $\mathrm{Bun}_N^{\omega^\rho}$. From Theorem 13.4.6 we obtain:

Corollary 13.4.8. *We have a unique isomorphism* $(\pi_{\mathrm{Ran}})^!(\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}})[d_g] \simeq \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ *extending the tautological identification over* S^0_{Ran}.

14. Torus equivariance and the Hecke property of the semi-infinite IC sheaf

The goal of this section is two-fold. First, we will introduce a twisting construction, which will allow us to put $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ on twisted versions of $\overline{S}^0_{\mathrm{Ran}}$. Second, we will study the Hecke eigen-property of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$, which is a crucial ingredient for the Hecke enhancement of the Jacquet functor.

14.1. **Adding T-equivariance.** In this subsection we will show that $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ has a natural structure of equivariance with respect to the group $\mathfrak{L}^+(T)_{\mathrm{Ran}}$.

14.1.1. The constructions in Sect. 12 all have a variant when we replace the *condition* of equivariance with respect to $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}}$ by the *structure* of equivariance with respect to $\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \cdot \mathfrak{L}^+(T)_{\mathrm{Ran}}$.

Thus, we obtain a full subcategory

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} := \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \cdot \mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}},$$

along with its variant

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} := \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \cdot \mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$$

as well as

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} := \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda_{\mathrm{Ran}})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \cdot \mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset \mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^\lambda_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$$

and

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{= \lambda})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} = \mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{\mathrm{Ran}})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \cdot \mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset \mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}.$$

14.1.2. For future use we note:

Lemma 14.1.3. *The category* $(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$ *is zero unless* $\lambda \in \Lambda^\sharp$.

Proof. The proof follows by analyzing the stabilizers:

By factorization, we may consider the case of a single point $x \in X$, i.e., we have to show that the category $(\mathrm{SI}_{q,x}(G)_{=\lambda})^{\mathfrak{L}^+(T)_x}$ is zero unless $\lambda \in \Lambda^\sharp$.

Consider the (unique) T-fixed point $t^\lambda \in S^\lambda \subset \mathrm{Gr}^{\omega^\rho}_{G,x}$. Restriction defines an equivalence

$$\mathrm{SI}_{q,x}(G)_{=\lambda} \to \mathrm{Shv}_{\mathcal{G}^G}(\{t^\lambda\}),$$

and hence an equivalence

$$(\mathrm{SI}_{q,x}(G)_{=\lambda})^{\mathfrak{L}^+(T)_x} \simeq \mathrm{Shv}_{\mathcal{G}^G}(\{t^\lambda\})^{\mathfrak{L}^+(T)_x}.$$

The $\mathfrak{L}^+(T)_x$-equivariance structure on the gerbe $\mathcal{G}^G|_{\{t^\lambda\}}$ corresponds to a character sheaf on $\mathfrak{L}^+(T)_x$, and we have to show that this character sheaf is non-trivial if $\lambda \notin \Lambda^\sharp$.

Now, by [GLys, Sect. 7.4], the above character sheaf is described as follows: it is the pullback along $\mathfrak{L}^+(T)_x \to T$ of the character sheaf on T arising by Kummer theory from the map

$$b(\lambda, -) : \Lambda \to \mathsf{e}(-1).$$

The triviality of the latter means that $\lambda \in \Lambda^\sharp$.

□

14.1.4. We also have the corresponding unital variants:

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset (\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}},$$

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset (\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq \lambda})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}},$$

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{= \lambda}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} \subset (\mathrm{SI}_{q,\mathrm{Ran}}(G)^{= \lambda})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}.$$

Note that the corresponding variant of Corrollary 12.5.9 says that pullback with respect to $\mathrm{pr}^\lambda_{\mathrm{Ran}} \circ p^\lambda_{\mathrm{Ran}}$ defines an equivalence

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}^\lambda)^{\mathfrak{L}^+(T)_{\mathrm{Conf}^\lambda}} \to (\mathrm{SI}_{q,\mathrm{Ran}}(G)^{=\lambda}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}},$$

where $\mathfrak{L}^+(T)_{\mathrm{Conf}^\lambda}$ acts trivially on its base Conf^λ, but the equivariance structure for the gerbe $\mathcal{G}^\Lambda$ is non-trivial.

In particular, the recipe of Sect. 13.1 defines a t-structure on $(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$ so that the forgetful functor

$$(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}} \to \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$$

is t-exact, and its restriction to the hearts is fully faithful.

14.1.5. Thus, we obtain that $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ naturally upgrades to an object of $(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$.

In particular, we note that the assertion of Lemma 13.1.9 follows from Lemma 14.1.3.

14.1.6. Note that the colimit (13.4) is naturally an object of

$$(\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)_x} := \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{\mathfrak{L}(N)^{\omega^\rho}_x \cdot \mathfrak{L}^+(T)_x}.$$

Thus, by Theorem 13.2.8, we obtain that (13.4) gives a description of

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,x} := \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}|_{\mathrm{Gr}^{\omega^\rho}_{G,x}}$$

as an object of $(\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)_x}$.

14.1.7. Consider the category $\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$. Thinking of it as

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash\overline{S}^0_{\mathrm{Ran}}),$$

we can talk about factorization algebras in $\mathrm{Shv}_{\mathcal{G}^G}(\overline{S}^0_{\mathrm{Ran}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$.

The next result is be proved along with Theorem 13.3.3:

Theorem 14.1.8. *There exists a factorization structure on* $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ *thought of as an object of* $(\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}})^{\mathfrak{L}^+(T)_{\mathrm{Ran}}}$*, uniquely characterized by the requirement that it gives rise to the factorization structure on* $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$*, thought of as an object of* $\mathrm{SI}_{q,\mathrm{Ran}}(G)^{\leq 0}_{\mathrm{untl}}$*, specified by Theorem 13.3.3.*

14.2. **Hecke action on the semi-infinite category.** In this subsection we will explore a key structure possessed by $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$, namely, that it is a Hecke eigensheaf.

14.2.1. Let us perform the base change for all the objects involved with respect to the forgetful map

$$\mathrm{Ran}_x \to \mathrm{Ran}.$$

Consider the resulting category

$$(\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}} = \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}(N)^{\omega^\rho}_{\mathrm{Ran}} \cdot \mathfrak{L}^+(T)_{\mathrm{Ran}_x}},$$

and the object

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x} \in (\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}},$$

equal to the !-pullback of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ along the map $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$.

Note that $(\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}$ is acted on naturally on the right by

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}^+(G)_{\mathrm{Ran}_x}},$$

and on the left by

$$\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}.$$

14.2.2. Recall the groupoid $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ acting on $\mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash\mathrm{Ran}_x$, see Sect. 1.5.8. Consider the resulting monoidal functor

$$\mathrm{Sph}_{q,x}(G) := \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{\mathfrak{L}^+(G)^{\omega^\rho}_x} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}^+(G)_{\mathrm{Ran}_x}}. \tag{14.1}$$

Composing with the geometric Satake functor

$$\mathrm{Sat}_{q,G} : \mathrm{Rep}(H) \to \mathrm{Sph}_{q,x}(G)$$

we obtain that $(\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}$ is acted on by $\mathrm{Rep}(H)$ (on the right).

14.2.3. Similarly, we obtain a monoidal functor

$$\mathrm{Sph}_{q,x}(T) := \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,x})^{\mathfrak{L}^+(T)_x} \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}. \tag{14.2}$$

In what follows we will replace the usual geometric Satake functor for T

$$\mathrm{Sat}_{q,T} : \mathrm{Rep}(T_H) \to \mathrm{Sph}_{q,x}(T),$$

by its variant that takes into account a cohomological shift:

$$\mathrm{Sat}'_{q,T} : \mathrm{Rep}(T_H) \to \mathrm{Sph}_{q,x}(T),$$

where

$$\mathrm{Sat}'_{q,T}(\mathsf{e}^\lambda) = \mathrm{Sat}(\mathsf{e}^\lambda)[-\langle \lambda, 2\check{\rho}\rangle].$$

Composing, with (14.2), we obtain that $(\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}$ is acted on by $\mathrm{Rep}(T_H)$ (on the left), in a way commuting with the above $\mathrm{Rep}(H)$-action.

14.2.4. Thus, we find ourselves in the paradigm of Sect. 10.4, and it makes sense to consider the corresponding category

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}((\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}).$$

We have the following result, which is a metaplectic version of [Ga7, Theorem 5.1.8]:

Theorem 14.2.5. *The object* $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x} \in (\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}$ *naturally lifts to object of the category*

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}((\mathrm{SI}_{q,\mathrm{Ran}_x}(G))^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}).$$

14.2.6. Restricting to the fiber over $x \in \mathrm{Ran}_x$, we obtain that the object

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,x} \in (\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)_x}$$

lifts to an object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)_x})$.

Remark 14.2.7. In Sect. 23.5 we will show how this structure follows from the identification of $\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ with ${}'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ via the *Drinfeld-Plücker formalism*.

14.3. **Hecke structure and factorization.** For what follows we will need to complement Theorem 14.2.5 by the following statement.

14.3.1. The factorization isomorphisms (13.7) (viewed as taking place in the $\mathfrak{L}^+(T)$-equivariant category) give rise to factorization isomorphisms

$$\begin{aligned}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}|_{\overline{S}^0_{\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} (\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}}} \simeq \\ \simeq ((\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}})^{\boxtimes J} \boxtimes \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x})|_{((\overline{S}^0_{\mathrm{Ran}})^J \times \overline{S}^0_{\mathrm{Ran}_x}) \underset{\mathrm{Ran}^J \times \mathrm{Ran}_x}{\times} (\mathrm{Ran}^J \times \mathrm{Ran}_x)_{\mathrm{disj}}}.\end{aligned} \tag{14.3}$$

I.e., $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$ is naturally an object of the category

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}). \tag{14.4}$$

14.3.2. Recall now (see Sect. 3.4.1) that the actions of $\mathrm{Sph}_{q,x}(G)$ and $\mathrm{Sph}_{q,x}(T)$ on

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}}$$

give rise to ones on (14.4).

Composing with the metaplectic geometric Satake functors for G and T, we obtain that (14.4) acquires a structure of module category over $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$.

14.3.3. The following comes along with the construction of the Hecke structure on $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$:

Theorem 14.3.4. *The relative Hecke structure on* $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$ *given by Theorem 14.2.5 is compatible in the natural sense with the factorization isomorphisms* (13.7) *and* (14.3). *I.e.,*

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x} \in \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}})$$

lifts to an object of

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}\left(\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})^{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}})\right).$$

14.4. **A T-twisting construction.** In this subsection we will perform a twisting construction that will allow us to consider variants of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ on spaces that are fibered over a base with typical fiber being $\overline{S}^0_{\mathrm{Ran}}$. This construction will be used in the definition of the Jacquet functor, and also for the local-to-global comparison.

14.4.1. Let $\mathcal{Y}$ be a prestack equipped with a map to $\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash\mathrm{Ran}$. We let

$${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$$

denote the fiber product

$$\mathcal{Y} \underset{\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash\mathrm{Ran}}{\times} \left(\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}\right).$$

We will use a similar notation for ${}_{\mathcal{Y}}\overline{S}^0$, etc.

14.4.2. Pulling back with respect to the forgetful map

$${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}},$$

from $\mathcal{G}^G$, we obtain a gerbe on ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$, which we denote by ${}_{\mathcal{Y}}\mathcal{G}^G$.

Further, pulling back $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$, we obtain an object

$${}_{\mathcal{Y}}\mathrm{IC}^{\frac{\infty}{2}}_q \in \mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\overline{S}^0).$$

In the sequel we will need the following two particular cases of this construction.

14.4.3. Take

$$\mathcal{Y} = \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}},$$

equipped with its tautological map to $\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash\mathrm{Ran}$,

$$(\mathcal{I}, \mathcal{P}_G, \alpha) \mapsto (\mathcal{I}, \mathcal{P}_G).$$

Note that we have a canonical identification

$$ {}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G \simeq \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}. \tag{14.5}$$

The multiplicativity property of the gerbe $\mathcal{G}^{G,G,\mathrm{ratio}}$ (see Sect. 2.1.3) implies that the resulting gerbe ${}_{\mathcal{Y}}\mathcal{G}^G$ on ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$ goes over under the identification (14.5) to the gerbe on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ equal to

$$\mathcal{G}^{G,T,\mathrm{ratio}} := (\mathcal{G}^T)^{-1} \boxtimes \mathcal{G}^G.$$

Thus, we obtain that

$${}_{\mathcal{Y}}\mathrm{IC}^{\frac{\infty}{2}}_q \in \mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\overline{S}^0)$$

can be thought of as an object, to be denoted

$${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}).$$

From we obtain:

Corollary 14.4.4. *The object* ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ *has a natural structure of factorization algebra in*

$$\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}).$$

14.4.5. Let us now take

$$\mathcal{Y} = \mathrm{Bun}_T \times \mathrm{Ran},$$

equipped with its natural map to $\mathfrak{L}^+(T)_{\mathrm{Ran}}\backslash\mathrm{Ran}$. Consider the corresponding object

$${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}} := {}_{\mathrm{Bun}_T \times \mathrm{Ran}}\mathrm{IC}^{\frac{\infty}{2}}_q \in \mathrm{Shv}_{{}_{\mathrm{Bun}_T \times \mathrm{Ran}}\mathcal{G}^G}({}_{\mathrm{Bun}_T \times \mathrm{Ran}}\overline{S}^0).$$

Note that the prestack ${}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}} := {}_{\mathrm{Bun}_T \times \mathrm{Ran}}\overline{S}^0$ admits a natural map

$$\pi_{\mathrm{Ran}} : {}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}_B,$$

so that we have a Cartesian square

$$\begin{tikzcd}
\overline{S}^0_{\mathrm{Ran}} \arrow[r] \arrow[d, "\pi_{\mathrm{Ran}}"'] & {}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}} \arrow[d, "\pi_{\mathrm{Ran}}"] \\
\overline{\mathrm{Bun}}^{\omega^\rho}_N \arrow[r] \arrow[d] & \overline{\mathrm{Bun}}_B \arrow[d, "\overline{\mathfrak{q}}"] \\
\mathrm{pt} \arrow[r, "\omega^\rho"] & \mathrm{Bun}_T .
\end{tikzcd}$$

14.4.6. Let $\mathcal{G}^{G,T,\mathrm{ratio}}$ denote the gerbe on $\overline{\mathrm{Bun}}_B$ equal to

$$\mathcal{G}^G \otimes (\mathcal{G}^T)^{-1},$$

where $\mathcal{G}^G$ is the pullback of the same-named gerbe on Bun_G (see Sect. 7.3.4) along the map

$$\overline{\mathfrak{p}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_G$$

and $\mathcal{G}^T$ is the pullback of the same-named gerbe on Bun_T (see Sect. 7.3.4) along the map

$$\overline{\mathfrak{q}} : \overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T .$$

Note that we have a canonical identification of gerbes on ${}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}}$

$${}_{\mathrm{Bun}_T \times \mathrm{Ran}}\mathcal{G}^G \simeq (\pi_{\mathrm{Ran}})^*(\mathcal{G}^{G,T,\mathrm{ratio}}). \tag{14.6}$$

14.4.7. Note also that we have a canonical trivialization of the restriction of $\mathcal{G}^{G,T,\mathrm{ratio}}$ along

$$\mathrm{Bun}_B \overset{\mathbf{j}_{\mathrm{glob}}}{\hookrightarrow} \overline{\mathrm{Bun}}_B.$$

This trivialization is compatible with the trivializations of the restriction of both sides of (14.6) to ${}_{\mathrm{Bun}_T}S^0_{\mathrm{Ran}} \subset {}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}}$.

We let ${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}$ denote the IC extension (in the category of $\mathcal{G}^{G,T,\mathrm{ratio}}$-twisted sheaves on $\overline{\mathrm{Bun}}_B$) of the constant perverse sheaf on Bun_B (the latter makes sense due to the above trivialization of $\mathcal{G}^{G,T,\mathrm{ratio}}|_{\mathrm{Bun}_B}$).

The following is a metaplectic analog of [Ga7, Theorem 6.3.2]:

Theorem 14.4.8. *There is a unique isomorphism*

$$(\pi_{\mathrm{Ran}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}})[d_g + \dim(\mathrm{Bun}_T) + \deg] \simeq {}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$$

extending the tautological identification over ${}_{\mathrm{Bun}_T}S^0_{\mathrm{Ran}}$*, where the value of* $\deg$ *equals* $\langle\lambda, 2\check{\rho}\rangle$ *over the connected component* Bun^λ_T *of* Bun_T*.*

Remark 14.4.9. We normalize the bijection

$$\pi_0(\mathrm{Bun}_T) \simeq \Lambda$$

so that the map $\mathrm{Gr}_{T,x} \to \mathrm{Bun}_T$ sends

$$\mathrm{Gr}^\lambda_{T,x} \to \mathrm{Bun}^\lambda_T,$$

where $\mathrm{Gr}^\lambda_{T,x}$ is the connected component of $\mathrm{Gr}_{T,x}$ that contains the point t^λ.

14.5. **Hecke property in the twisted context.** In this subsection we will show how the Hecke property of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$ translates to a Hecke property of its twisted version ${}_{\mathcal{Y}}\mathrm{IC}^{\frac{\infty}{2}}_q$ constructed in Sect. 14.4. This is a necessary ingredient for establishing the Hecke property of the Jacquet functor.

14.5.1. Let us base change the discussion in Sect. 14.4 along the map $\mathrm{Ran}_x \to \mathrm{Ran}$. I.e., let us assume having a prestack $\mathcal{Y}$, equipped with a map

$$\mathcal{Y} \to \mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x.$$

Consider the corresponding prestack

$${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G := \mathcal{Y} \underset{\mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x}{\times} \left(\mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}\right).$$

14.5.2. Recall again the groupoid $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ acting on $\mathfrak{L}^+(G)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x$, see Sect. 1.5.8.

By construction, this action lifts to one on ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$. Hence, we obtain a monoidal action of $\mathrm{Sph}_{q,x}(G)$ on $\mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G)$.

Composing with the geometric Satake functor $\mathrm{Sat}_{q,G}$, we obtain an action of $\mathrm{Rep}(H)$ on $\mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G)$.

14.5.3. Consider now the action of the groupoid $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ acting on $\mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x$. Assume that we are given a lift of this action to one on $\mathcal{Y}$.

I.e., we assume being given a prestack ${}_{\mathcal{Y}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ equipped with maps

$$\mathcal{Y} \xleftarrow{\overleftarrow{h}_T} {}_{\mathcal{Y}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \xrightarrow{\overrightarrow{h}_T} \mathcal{Y}$$

and a map

$${}_{\mathcal{Y}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \to \mathrm{Hecke}^{\mathrm{loc}}_{T,x}$$

that make both square in the diagram

$$\begin{array}{ccccc}
\mathcal{Y} & \xleftarrow{\overleftarrow{h}_T} & {}_{\mathcal{Y}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} & \xrightarrow{\overrightarrow{h}_T} & \mathcal{Y} \\
\downarrow & & \downarrow & & \downarrow \\
\mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x & \xleftarrow{\overleftarrow{h}_T} & \mathrm{Hecke}^{\mathrm{loc}}_{T,x} & \xrightarrow{\overrightarrow{h}_T} & \mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x,
\end{array}$$

Cartesian.

14.5.4. Under the above circumstances, the above action of $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ on $\mathcal{Y}$ can be further lifted to an action on ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$ (given by leaving the G-bundle $\mathcal{P}_G$ intact).

In other words, we have a prestack ${}_{{}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G}\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ equipped with maps

$${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G \leftarrow {}_{{}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \to {}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$$

and a map

$${}_{{}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \to {}_{\mathcal{Y}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$$

that make both squares in the diagram

$$\begin{tikzcd}
{}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho} \arrow[d] & {}_{\mathcal{Y}\mathrm{Gr}_G^{\omega^\rho}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \arrow[l, "\overleftarrow{h}_T"'] \arrow[r, "\overrightarrow{h}_T"] \arrow[d] & {}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho} \arrow[d] \\
\mathcal{Y} & {}_{\mathcal{Y}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \arrow[l, "\overleftarrow{h}_T"'] \arrow[r, "\overrightarrow{h}_T"] & \mathcal{Y}
\end{tikzcd}$$

Cartesian.

14.5.5. From here we obtain that the monoidal category $\mathrm{Sph}_{q,x}(T)$ acts on the left $\mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho})$. Moreover, this action commutes with the right action of $\mathrm{Sph}_{q,x}(G)$.

Composing with the geometric Satake functor $\mathrm{Sat}'_{q,T}$, we obtain an action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho})$, which commutes with the $\mathrm{Rep}(H)$-action defined above.

14.5.6. Thus, we obtain that $\mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho})$ can be viewed as a category equipped with an action of $\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)$, and we find ourselves in the context of Sect. 10.4.

From Theorem 14.2.5 we obtain:

Corollary 14.5.7. *The object*

$$ {}_{\mathcal{Y}}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x} \in \mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho}) $$

lifts to object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{{}_{\mathcal{Y}}\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho}))$.

14.6. **Hecke action over** Gr_T**.** We will now consider an example of the situation described in Sect. 14.5 (another example will be described in Sect. 14.7 below). The construction in this subsection has a direct import on the Hecke property of the Jacquet functor considered in the next Part.

14.6.1. Let us be in the context of Sect. 14.4.3, but with a marked point x. I.e., we take

$$ \mathcal{Y} := \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}, $$

together with its natural map to $\mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x$.

The action of $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ on $\mathfrak{L}^+(T)_{\mathrm{Ran}_x}\backslash \mathrm{Ran}_x$ naturally lifts to a *right* action on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}$. Using the inversion involution, we obtain a left action of $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}$. Hence, we find ourselves in the context of Sect. 14.5.3.

14.6.2. Recall the identification (14.5):

$$ {}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho} \simeq \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}. $$

This identification intertwines the left action of $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ on ${}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho}$ of Sect. 14.5.2 and the natural right action of $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}$ via the second factor.

In addition, the above identification intertwines the left action of $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ on ${}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho}$ of Sect. 14.5.4 and the natural right action of $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}$ via the first factor, *precomposed* with the inversion involution on $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ (note that inversion turns a left action into a right action).

Hence, we can view $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})$ as a module over $\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)$, *where we precompose* the action of $\mathrm{Rep}(T_H)$ with the Cartan involution τ^{T_H}, see Sect. 2.5.4.

14.6.3. From Corrollary 14.5.7, combined with Sect. 2.5.5, we obtain:

Corollary 14.6.4. *The object*

$$_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})$$

lifts to an object of

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})).$$

14.6.5. We will now need to complement of statement of Corrollary 14.6.4 to take into account the factorization structure.

By Corrollary 14.4.4 we can regard $_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ as a factorization algebra in the category

$$\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$$

and $_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$ as an object of

$$_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}\text{-FactMod}\left(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})\right). \tag{14.7}$$

As in Sect. 14.3.2, we can regard (14.7) as a module category over $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$. From Theorem 14.3.4 we obtain:

Corollary 14.6.6. *The relative Hecke structure on* $_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$ *given by Corrollary 14.6.4 is compatible with the factorization structure in a natural sense. I.e.,* $_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$ *naturally lifts to an object of the category*

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}\left(_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}))\right).$$

14.7. **Global Hecke property.** In this subsection we will consider another example of the paradigm of Sect. 14.5, which contains a global aspect, incarnated by the stack Bun_T of T-bundles on a complete curve X.

The material in this subsection is needed for the local-to-global comparison for the (Hecke version of the) Jacquet functor, which is, in turn, used to establish the commutation of the (Hecke version of the) local Jacquet functor with Verdier duality.

14.7.1. Consider the ind-algebraic stack $(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}$. By construction, it is equipped with a pair of maps

$$\mathrm{pt}/\mathfrak{L}^+(G)_x \leftarrow (\overline{\mathrm{Bun}}_B)_{\infty\cdot x} \to \mathrm{pt}/\mathfrak{L}^+(T)_x.$$

The action on $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ on $\mathrm{pt}/\mathfrak{L}^+(T)_x$ naturally lifts to a (left) action on $(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}$: we modify the T-bundle, while leaving the G-bundle intact.

Similarly, the right action of $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ on $\mathrm{pt}/\mathfrak{L}^+(G)_x$ naturally lifts to an action on $(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}$: we modify the G-bundle, while leaving the T-bundle intact.

Thus, the category $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$ acquires a left action of $\mathrm{Sph}_{q,x}(T)$ and a commuting right action of $\mathrm{Sph}_{q,x}(G)$.

Remark 14.7.2. Concretely, the action of $\mathrm{Sph}_{q,x}(G)$ on $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$ is given by the formula

$$\mathcal{F}, \mathcal{S}_G \mapsto (\overleftarrow{h}_G)_*\left((\overrightarrow{h}_G)^!(\mathcal{F}) \overset{!}{\otimes} \mathcal{S}_G|_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}\,\mathrm{Hecke}^{\mathrm{loc}}_{G,x}}\right), \quad \mathcal{F} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x}),\ \mathcal{S}_G \in \mathrm{Sph}_{q,x}(G)$$

for the maps in the following diagram

$$\begin{tikzcd}
(\overline{\mathrm{Bun}}_B)_{\infty\cdot x} \arrow[d] & {}_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}}\mathrm{Hecke}^{\mathrm{loc}}_{G,x} \arrow[l, "\overleftarrow{h}_G"'] \arrow[r, "\overrightarrow{h}_G"] \arrow[d] & (\overline{\mathrm{Bun}}_B)_{\infty\cdot x} \arrow[d] \\
\mathrm{pt}/\mathfrak{L}^+(G)_x & \mathrm{Hecke}^{\mathrm{loc}}_{G,x} \arrow[l, "\overleftarrow{h}_G"'] \arrow[r, "\overrightarrow{h}_G"] & \mathrm{pt}/\mathfrak{L}^+(G)_x.
\end{tikzcd}$$

Similarly, the action of $\mathrm{Sph}_{q,x}(T)$ on $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$ is given by the formula

$$\mathcal{F},\mathcal{S}_T\mapsto(\overrightarrow{h}_T)_*\left((\overleftarrow{h}_T)^!(\mathcal{F})\overset{!}{\otimes}\mathcal{S}_T|_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}\mathrm{Hecke}^{\mathrm{loc}}_{T,x}}\right),\quad \mathcal{F}\in\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x}),\ \mathcal{S}_T\in\mathrm{Sph}_{q,x}(T)$$

for the maps in the following diagram

$$\begin{tikzcd}
(\overline{\mathrm{Bun}}_B)_{\infty\cdot x} \arrow[d] & {}_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}}\mathrm{Hecke}^{\mathrm{loc}}_{T,x} \arrow[l, "\overleftarrow{h}_T"'] \arrow[r, "\overrightarrow{h}_T"] \arrow[d] & (\overline{\mathrm{Bun}}_B)_{\infty\cdot x} \arrow[d] \\
\mathrm{pt}/\mathfrak{L}^+(T)_x & \mathrm{Hecke}^{\mathrm{loc}}_{T,x} \arrow[l, "\overleftarrow{h}_T"'] \arrow[r, "\overrightarrow{h}_T"] & \mathrm{pt}/\mathfrak{L}^+(T)_x.
\end{tikzcd}$$

14.7.3. Applying geometric Satake, we obtain an action of $\mathrm{Rep}(H)$ and a commuting action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$.

Warning: Here, unlike the local version, we use the usual geometric Satake functor $\mathrm{Sat}_{q,T}$ (and not the cohomologically shifted version $\mathrm{Sat}'_{q,T}$, see Sect. 14.2.3).

The following is a metaplectic version of [BG, Theorem 3.1.4]:

Theorem 14.7.4. *The object* ${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}\in\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$ *naturally lifts to an object of* $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x}))$.

Remark 14.7.5. In [BG], the Hecke property of $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$ was established with respect to objects $V\in\mathrm{Rep}(\check{G})$ that *lie in the abelian category* $(\mathrm{Rep}(\check{G}))^{\heartsuit}$. This is, however, sufficient because $\mathrm{Rep}(\check{G})$ is the derived category if its heart. The same remark applies in the metaplectic case.

14.7.6. Let us return to the setting of Sect. 14.5 with

$$\mathcal{Y}:=\mathrm{Bun}_T\times\mathrm{Ran}_x.$$

Consider the resulting prestack

$${}_{\mathrm{Bun}_T}\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}:={}_{\mathrm{Bun}_T\times\mathrm{Ran}_x}\mathrm{Gr}^{\omega^\rho}_G.$$

and the corresponding closed sub-prestack

$$({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}\subset{}_{\mathrm{Bun}_T}\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x},$$

see Sect. 8.1.1.

Note that the actions of the groupoids $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ and $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ preserve this sub-prestack. In particular, we can consider $\mathrm{Shv}_{\mathrm{Bun}_T\times\mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})$ as equipped with a right action of $\mathrm{Sph}_{q,x}(G)$ and a commuting left action of $\mathrm{Sph}_{q,x}(T)$.

In particular, we obtain that $\mathrm{Shv}_{\mathrm{Bun}_T\times\mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})$ is a module over $\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)$. It follows from Corrollary 14.5.7 that the resulting object

$${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}\in\mathrm{Shv}_{\mathrm{Bun}_T\times\mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})$$

naturally lifts to an object of

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathrm{Bun}_T\times\mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})).$$

14.7.7. Note that the projection

$$\pi_{\mathrm{Ran}_x} : ({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x} \to (\overline{\mathrm{Bun}}_B)_{\infty\cdot x}$$

intertwines the actions of $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$ and $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ on $({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}$ and $(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}$.

Hence, the pullback functor

$$(\pi_{\mathrm{Ran}_x})^! : \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x}) \to \mathrm{Shv}_{\mathrm{Bun}_T \times \mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})$$

gives rise to a functor

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})) \to \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathrm{Bun}_T \times \mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})).$$

14.7.8. The following is a metaplectic version of [Ga7, Theorem 6.3.5]:

Theorem 14.7.9. *The isomorphism*

$$(\pi_{\mathrm{Ran}_x})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}})[d_g + \dim(\mathrm{Bun}_T) + \deg] \simeq {}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$$

of Theorem 14.4.8 lifts to an isomorphism of objects in $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathrm{Bun}_T \times \mathrm{Ran}\,\mathcal{G}}(({}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x}))$.

14.8. **Local vs global Hecke property.** In this subsection we will study the compatibility of the constructions in Sects. 14.6 and 14.7, respectively.

14.8.1. Recall the closed subfunctor

$${}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}\overline{S}^0 \subset {}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}\mathrm{Gr}^{\omega^\rho}_G \simeq \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}.$$

We have a Cartesian square

$$\begin{tikzcd}
{}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}\overline{S}^0 \arrow[r] \arrow[d] & {}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}} \arrow[d] \\
\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \arrow[r] & \mathrm{Bun}_T .
\end{tikzcd}$$

Composing with the map $\pi_{\mathrm{Ran}} : {}_{\mathrm{Bun}_T}\overline{S}^0_{\mathrm{Ran}} \to \overline{\mathrm{Bun}}_B$, we obtain a map

$$\pi_{\mathrm{Gr}_T} : {}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}\overline{S}^0 \to \overline{\mathrm{Bun}}_B$$

so that the diagram

$$\begin{tikzcd}
{}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}\overline{S}^0 \arrow[r, "\pi_{\mathrm{Gr}_T}"] \arrow[d] & \overline{\mathrm{Bun}}_B \arrow[d, "\mathfrak{q}"] \\
\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \arrow[r] & \mathrm{Bun}_T
\end{tikzcd} \tag{14.8}$$

commutes.

Note that the pullback of the gerbe $\mathcal{G}^{G,T,\mathrm{ratio}}$ along π_{Gr_T} goes over to the restriction of the gerbe denoted $\mathcal{G}^{G,T,\mathrm{ratio}}$ on $\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$.

Unwinding the constructions, from Theorem 14.4.8 we obtain:

Corollary 14.8.2. *There exists canonical isomorphism in* $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}({}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}}\overline{S}^0)$

$$(\pi_{\mathrm{Gr}_T})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}})[d_g + \dim(\mathrm{Bun}_T) + \deg] \simeq {}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}},$$

where the value of $\deg$ *equals* $\langle \lambda, 2\check{\rho} \rangle$ *over the connected component* Bun_T^λ *of* Bun_T.

14.8.3. Consider now the closed subfunctor

$$_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}}(\overline{S}^0)_{\infty\cdot x} \subset {}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}}\mathrm{Gr}^{\omega^\rho}_G \simeq \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}.$$

Similarly, to the above, we have a map

$$\pi_{\mathrm{Gr}_T} : {}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}}(\overline{S}^0)_{\infty\cdot x} \to (\overline{\mathrm{Bun}}_B)_{\infty\cdot x},$$

compatible with the gerbes, and with the actions of the groupoids $\mathrm{Hecke}^{\mathrm{loc}}_{G,x}$ and $\mathrm{Hecke}^{\mathrm{loc}}_{T,x}$.

In particular, the functor of pullback

$$(\pi_{\mathrm{Gr}_T})^! : \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}({}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}}(\overline{S}^0)_{\infty\cdot x})$$

gives rise to a functor

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})) \to \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}({}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}}(\overline{S}^0)_{\infty\cdot x}).$$

From Theorem 14.7.9 we obtain:

Corollary 14.8.4. *The isomorphism*

$$(\pi_{\mathrm{Gr}_T})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}})[d_g + \dim(\mathrm{Bun}_T) + \deg] \simeq {}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}_x}$$

of Corrollary 14.8.2 lifts to an isomorphism of objects of

$$\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}({}_{\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}}(\overline{S}^0)_{\infty\cdot x}).$$

Part V: The Jacquet functor

In this Part we begin the process of relating the two categories involved in our main theorem: the Hecke category of $\mathrm{Whit}_{q,x}(G)$ and a certain category of factorization modules. The relationship will be realized by a functor from the former to the latter, the main ingredient of which is (one of the versions of) the Jacquet functor

$$\mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{R,\mathrm{Ran}_x}).$$

15. Construction of the Jacquet functor

The goal of this section is to construct the Jacquet functor

$$\mathfrak{J}_{!*,\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})),$$

where $\Omega_q^{\mathrm{Whit}_{!*}}$-mod is a certain factorization algebra in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$, and

$$\Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))$$

denotes the category of factorization modules over it in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$.

In the next section, we will upgrade the functor $\mathfrak{J}_{!*,\mathrm{Fact}}$ to a functor

$$\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

15.1. **The bare version of the Jacquet functor.** The definition of the functor $\mathfrak{J}_{!*,\mathrm{Fact}}$ will proceed in stages. In this subsection we will define the functor

$$\mathfrak{J}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}),$$

which we should think of as the composition of $\mathfrak{J}_{!*,\mathrm{Fact}}$ and the forgetful functor

$$\Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}).$$

15.1.1. Recall the object

$${}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}).$$

We will consider its counterpart denoted

$${}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}),$$

in which we replace $N \rightsquigarrow N^-$ and $\mathcal{G}^G \rightsquigarrow (\mathcal{G}^G)^{-1}$.

15.1.2. We define the most basic version of the Jacquet functor

$$\mathfrak{J}_{!*,\mathrm{Ran}} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$$

as

$$\mathcal{F} \mapsto (p_T)_*({}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} (p_G)^!(\mathcal{F})),$$

where p_T and p_G are the projections

$$\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \leftarrow \mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}, \tag{15.1}$$

respectively.

Note that the functor $(p_T)_*$ makes sense since the tensor product

$${}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} (p_G)^!(\mathcal{F})$$

belongs to the category

$$\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}).$$

In other words, the functor $\mathfrak{J}_{!*}$ is defined using the correspondence (15.1) with kernel ${}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}_x}$.

Remark 15.1.3. Let us explain the origin of the name "Jacquet functor". Suppose that instead of $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{Ran}}$ we use

$$(\mathbf{j}_{\mathrm{Ran}})_*(\omega_{S^0_{\mathrm{Ran}}}) \in \mathrm{SI}_{q,\mathrm{Ran}}(G)^{\mathfrak{L}^+(T)_{\mathrm{Ran}}},$$

along with its variants.

Denote the resulting functor

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$$

by $\mathfrak{J}_{*,\mathrm{Ran}}$.

Then it is easy to see that $\mathfrak{J}_{*,\mathrm{Ran}}$ is given by

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}) \overset{!\text{-pullback}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{B^-,\mathrm{Ran}}) \overset{*\text{-pushforward}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}).$$

15.1.4. Along with $\mathfrak{J}_{!*,\mathrm{Ran}}$, we will consider its variants

$$\mathfrak{J}_{!*,\mathrm{Ran}_x} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$$

and

$$\mathfrak{J}_{!*,x} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,x}) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,x}),$$

defined using the objects

$${}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}_x} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}) \text{ and } {}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x} \in \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}(\mathrm{Gr}^{\omega^\rho}_{T,x} \times \mathrm{Gr}^{\omega^\rho}_{G,x}),$$

respectively, obtained as pullbacks of ${}_{\mathrm{Gr}^{\omega^\rho}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q,\mathrm{Ran}}$.

15.1.5. Recall now the functor

$$\mathrm{sprd}_{\mathrm{Ran}_x} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}),$$

see Sect. 8.2.8.

We define the functor

$$\mathfrak{J}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$$

as the composite

$$\mathfrak{J}_{!*,\mathrm{Ran}_x} \circ \mathrm{sprd}_{\mathrm{Ran}_x}.$$

15.2. **The factorization algebra $\Omega^{\mathrm{Whit}_{!*}}_q$.** In this subsection we define a factorization algebra

$$\Omega^{\mathrm{Whit}_{!*}}_q \in \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}}).$$

It will be essentially equivalent to the factorization algebra on Conf used to define the category of factorization modules that appears in the right-hand side in our main theorem.

15.2.1. Recall that the object

$$\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}).$$

Recall that, according to Theorem 8.4.6(a), $\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}$ has a structure of factorization algebra in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$

Recall also that, according to Corrollary 14.4.4, ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}}$ has a structure of factorization algebra in $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$.

Hence, by Sect. 3.2.1, we obtain that

$${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} (p_G)^!(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}})$$

acquires a natural structure of factorization algebra in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}})$.

15.2.2. Set

$$\Omega_q^{\mathrm{Whit}_{!*}} := (p_T)_*({}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} (p_G)^!(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}})).$$

By Sect. 3.2.3, we obtain that $\Omega_q^{\mathrm{Whit}_{!*}}$ acquires a natural structure of factorization algebra in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$.

Remark 15.2.3. In Sect. 18.4 we will a very explicit description of $\Omega_q^{\mathrm{Whit}_{!*}}$.

15.3. **Adding the factorization structure.** In this subsection we will upgrade $\mathfrak{J}_{!*,\mathrm{sprd}}$ to a functor

$$\mathfrak{J}_{!*,\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

15.3.1. Recall that Theorem 8.4.6(b) says that the functor $\mathrm{sprd}_{\mathrm{Ran}_x}$ canonically lifts to a functor

$$\mathrm{sprd}_{\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})). \tag{15.2}$$

Consider ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}_x}$ as an object of

$${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}}\text{-FactMod}\left(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})\right).$$

Using Sect. 3.4.1, we obtain that the functor

$$\mathcal{F} \mapsto {}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} p_G^! \circ \mathrm{sprd}_{\mathrm{Ran}_x}(\mathcal{F})$$

upgrades to a functor

$$\begin{aligned}(15.3)\quad & \mathrm{Whit}_{q,x}(G) \to \\ & \to ({}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} (p_G)^!(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}))\text{-FactMod}\left(\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})\right).\end{aligned}$$

15.3.2. Composing (15.3) with the functor $(p_T)_*$ (see Sect. 3.4.1), we obtain that the functor $\mathfrak{J}_{!*,\mathrm{sprd}}$ upgrades to the sought-for functor

$$\mathfrak{J}_{!*,\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

16. Hecke enhancement of the Jacquet functor

The goal of this section is to perform the key construction of this paper, namely, to extend the functor

$$\mathfrak{J}_{!*,\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))$$

constructed in the previous section to a functor

$$\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

16.1. **Extension of the bare version of the functor.** In this subsection, as a warm-up, we will extend the functor

$$\mathfrak{J}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$$

to a functor

$$\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}).$$

16.1.1. By Sect. 10.3.3, the construction of the sought-for functor $\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{sprd}}$ is equivalent to the following:

Theorem-Construction 16.1.2. *The functor*

$$\mathfrak{J}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$$

intertwines the $\mathrm{Rep}(H)$*-action on* $\mathrm{Whit}_{q,x}(G)$ *and the* $\mathrm{Rep}(T_H)$*-action on* $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$.

The rest of this subsection is devoted to the proof of this theorem.

16.1.3. First, we note that the functor

$$\mathrm{sprd}_{\mathrm{Ran}_x} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})$$

intertwines the actions of $\mathrm{Sph}_{q,x}(G)$ on $\mathrm{Whit}_{q,x}(G)$ and $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})$.

Hence, Theorem 16.1.2 follows from the next more general result:

Theorem-Construction 16.1.4. *The functor*

$$\mathfrak{J}_{!*,\mathrm{Ran}_x} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x}) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$$

intertwines the $\mathrm{Rep}(H)$*-action on* $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})$ *and the* $\mathrm{Rep}(T_H)$*-action on* $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$.

We will see that the proof of Theorem 16.1.4 amounts to no more than diagram chase once we know Corrollary 14.6.4.

16.1.5. The proof of Theorem 16.1.4 will fit into the following general paradigm:

Let $\mathbf{C}$ be a category equipped with an action of $\mathrm{Rep}(H)$, and let $\mathbf{D}$ be a category equipped with an action of $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$. Let $\mathbf{E}$ be a category, equipped with an action of $\mathrm{Rep}(T_H)$.

Let us be given a functor

$$\Psi : \mathbf{C} \otimes \mathbf{D} \to \mathbf{E},$$

equipped with a factorization

$$\mathbf{C} \otimes \mathbf{D} \to \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D} \xrightarrow{\widetilde{\Psi}} \mathbf{E}. \tag{16.1}$$

Note that $\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}$ is acted on by $\mathrm{Rep}(T_H)$. Assume that $\widetilde{\Psi}$ intertwines this action and the given one on $\mathbf{E}$.

Let $\mathbf{d} \in \mathbf{D}$ be an object equipped with a structure of object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D})$. By unwinding the definitions, in this case we obtain that the functor

$$\Phi := \Psi_{\mathbf{d}} : \mathbf{C} \to \mathbf{E}, \quad \mathbf{c} \mapsto \Psi(\mathbf{c} \otimes \mathbf{d})$$

intertwines the $\mathrm{Rep}(H)$-action on $\mathbf{C}$ and the $\mathrm{Rep}(T_H)$-action on $\mathbf{E}$.

16.1.6. We apply the above paradigm as follows. We take

$$\mathbf{C} := \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x}),\ \mathbf{E} := \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}),$$

and

$$\mathbf{D} := \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}).$$

We will now supply the above categories with the required pieces of structure. First off, the functor Ψ is the functor

$$\mathcal{F}, \mathcal{F}' \mapsto (p_T)_* \left(\mathcal{F}' \overset{!}{\otimes} (p_G)^!(\mathcal{F}) \right). \tag{16.2}$$

16.1.7. The action of $\mathrm{Rep}(H)$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})$ (resp., the action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$) is given by composing the $\mathrm{Sph}_{q,x}(G)$-action on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})$ (resp., the action of $\mathrm{Sph}_{q,x}(T)$ on $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$) and the geometric Satake functor $\mathrm{Sat}_{q,G}$ (resp., $\mathrm{Sat}'_{q,T}$).

The action of $\mathrm{Rep}(T_H)$ on $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})$ is also given by $\mathrm{Sat}'_{q,T}$.

Now, the action of $\mathrm{Rep}(H)$ on $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})$ is given by *composing*

$$\mathrm{Sat}_{q,G} : \mathrm{Rep}(H) \to \mathrm{Sph}_{q,x}(G)$$

with the inversion anti-homomorphism

$$\mathrm{inv}^G : \mathrm{Sph}_{q,x}(G) \to \mathrm{Sph}_{q^{-1},x}(G),$$

see Sect. 2.5.1. Note that by Sect. 2.5.5, the latter is the same as the action of $\mathrm{Sph}_{q^{-1},x}(G)$, *precomposed* with the Cartan involution τ^H on $\mathrm{Rep}(H)$.

The factorization (16.1) follows from the formula

$$(p_T)_*\left(\mathcal{F}'\overset{!}{\otimes}(p_G)^!(\mathcal{F}\star\mathcal{S}_G)\right)\simeq(p_T)_*\left((\mathcal{F}'\star\mathrm{inv}^G(\mathcal{S}_G))\overset{!}{\otimes}\mathcal{F}\right),\quad \mathcal{S}_G\in\mathrm{Sph}_{q,x}(G).$$

The fact that the resulting functor $\widetilde{\Psi}$ is compatible with the actions of $\mathrm{Rep}(T_H)$ follows from the formula

$$(p_T)_*\left((\mathcal{F}'\star\mathcal{S}_T)\overset{!}{\otimes}(p_G)^!(\mathcal{F})\right)\simeq\left((p_T)_*\left(\mathcal{F}'\overset{!}{\otimes}\mathcal{F}\right)\right)\star\mathcal{S}_T,\quad \mathcal{S}_T\in\mathrm{Sph}_{q,x}(T).$$

16.1.8. Finally, we take

$$\mathbf{d}:={}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}}\in\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}\underset{\mathrm{Ran}_x}{\times}\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x}).$$

The structure on $\mathbf{d}$ of an object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D})$ is provided by Corrollary 14.6.4:

The Cartan involution on T_H (appearing in Corrollary 14.6.4), the Cartan involution on H (involved in the action of $\mathrm{Rep}(H)$ on $\mathbf{D}$, see above), and the swap of B and B^- (involved in the passage ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{Ran}}\rightsquigarrow{}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}}$) cancel each other out.

16.2. **Hecke structure on the factorizble version.** In this subsection we will upgrade the construction of the previous subsection to obtain the functor

$$\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact}}:\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))\to\Omega_q^{\mathrm{Whit}_{!*}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

16.2.1. By Sect. 3.4.1, the category $\Omega_q^{\mathrm{Whit}_{!*}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))$ carries a (right) monoidal action of $\mathrm{Sph}_{q,x}(T)$, and hence an action of $\mathrm{Rep}(T_H)$.

By Sect. 10.3.3, the construction of the sought-for functor $\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact}}$ is equivalent to the following:

Theorem-Construction 16.2.2. *The functor*

$$\mathfrak{J}_{!*,\mathrm{Fact}}:\mathrm{Whit}_{q,x}(G)\to\Omega_q^{\mathrm{Whit}_{!*}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))$$

intertwines the $\mathrm{Rep}(H)$*-action on* $\mathrm{Whit}_{q,x}(G)$ *and the* $\mathrm{Rep}(T_H)$*-action on*

$$\Omega_q^{\mathrm{Whit}_{!*}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

The rest of this subsection is devoted to the proof of this theorem.

16.2.3. First, by Sect. 3.4.1, the category $\mathrm{Vac}_{\mathrm{Whit,Ran}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x}))$ carries a (right) monoidal action of $\mathrm{Sph}_{q,x}(G)$. From the construction of the functor

$$\mathrm{sprd}_{\mathrm{Fact}}:\mathrm{Whit}_{q,x}(G)\to\mathrm{Vac}_{\mathrm{Whit,Ran}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x}))$$

in Theorem 8.4.6(b), we obtain that it intertwines the actions of $\mathrm{Sph}_{q,x}(G)$ on the two sides.

Hence, Theorem 16.2.2 follows from the next more general result:

Theorem-Construction 16.2.4. *The functor*

$$\mathrm{Vac}_{\mathrm{Whit,Ran}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x}))\to\Omega_q^{\mathrm{Whit}_{!*}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))\tag{16.3}$$

induced by $\mathfrak{J}_{!*,\mathrm{Ran}_x}$ *intertwines the* $\mathrm{Rep}(H)$*-action on* $\mathrm{Vac}_{\mathrm{Whit,Ran}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x}))$ *and the* $\mathrm{Rep}(T_H)$*-action on* $\Omega_q^{\mathrm{Whit}_{!*}}\text{-}\mathrm{FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x}))$.

16.2.5. We will prove Theorem 16.2.4 by applying the paradigm of Sect. 16.1.5.

We take

$$\mathbf{C} = \mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Ran}_x})),$$

$$\mathbf{E} := \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}_x})),$$

$$\mathbf{D} := \left({}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Ran}} \overset{!}{\otimes} (p_G)^!(\mathrm{Vac}_{\mathrm{Whit},\mathrm{Ran}})\right)\text{-FactMod}(\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x} \underset{\mathrm{Ran}_x}{\times} \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}_x})).$$

We take Ψ to be the functor, induced by the functor (16.2).

16.2.6. Now the proof of Theorem 16.2.4 follows verbatim that of Theorem 16.1.4, using Corrollary 14.6.6 as an additional input.

Part VI: Interpretation via configuration spaces

In the previous Part, we constructed the functor

$$\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})).$$

This functor is *almost* an equivalence: we will need to apply a certain renormalization procedure to the right-hand side in order to turn it into one. However, in order to make sense of this renormalization procedure, we will need to interpret the above category of factorization modules in terms of the configuration space, following the procedure of Sect. 5.5. The passage $\mathrm{Gr}_T \rightsquigarrow \mathrm{Conf}$ will have the additional advantage of placing us in the context of finite-dimensional algebraic geometry, thereby making the category of factorization modules more accessible to calculations (see Sect. 5.3).

Once we have reinterpreted the Jacquet functor as taking place in the category of factorization modules over the configuration space, we will be able to state our main theorem.

17. Factorization algebra $\Omega_q^{\mathrm{small}}$

In this section we put ourselves in the context of Sect. 5.1. We will describe a particular factorization algebra, denoted $\Omega_q^{\mathrm{small}}$, in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ that exists under some additional conditions on the geometric metaplectic data $\mathcal{G}^T$.

The right-hand side in our main theorem will be the (renormalized version of the) category of factorization modules over $\Omega_q^{\mathrm{small}}$.

17.1. **Rigitification of geometric metaplectic data for T.** In this subsection we start with a geometric metaplectic data $\mathcal{G}^T$ for the torus T and describe the additional condition that allows to define the factorization algebra $\Omega_q^{\mathrm{small}}$.

17.1.1. Recall (see Sect. 4.6.5) that the geometric metaplectic data $\mathcal{G}^T$ for T gives rise to a factorization gerbe, denoted $\mathcal{G}^\Lambda$, on Conf. In particular, for each element $\lambda \in \Lambda$, we obtain a gerbe $\mathcal{G}^\lambda$ on X equal to the restriction of $\mathcal{G}^\Lambda$ along

$$X \overset{x\mapsto \lambda\cdot x}{\longrightarrow} X^\lambda.$$

Let $\mathrm{FactGe}_T^{\mathrm{rigid}}$ denote the space consisting of pairs: a geometric metaplectic data $\mathcal{G}^T$ and a *trivialization* of $\mathcal{G}^{-\alpha_i}$ for every simple coroot α_i. We have a fiber sequence of *connective* spectra

$$\prod_i \mathrm{Tors}(X, \mathsf{e}^{\times,\mathrm{tors}}) \to \mathrm{FactGe}_T^{\mathrm{rigid}} \to \mathrm{FactGe}_T,$$

which fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}}) \arrow[r] \arrow[d] & \mathrm{FactGe}_T^{\mathrm{rigid}} \arrow[r] \arrow[d] & \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \arrow[r] \arrow[d, "\mathrm{id}"] & 0 \\
0 \arrow[r] & \mathrm{Ge}(X, \check{T}(\mathsf{e})^{\mathrm{tors}}) \arrow[r] & \mathrm{FactGe}_T \arrow[r] & \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \arrow[r] & 0.
\end{tikzcd}$$

Remark 17.1.2. Note that, in particular, that if G is semi-simple simply-connected, the connective spectrum $\mathrm{FactGe}_T^{\mathrm{rigid}}$ is discrete, i.e., it maps isomorphically to $\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))$.

17.1.3. Note that for an object $\mathcal{G}^T \in \mathrm{FactGe}_T^{\mathrm{rigid}}$, the data of trivialization for the negative simple coroots uniquely extends to a trivialization of the restriction of $\mathcal{G}^\Lambda$

$$\overset{\circ}{\mathrm{Conf}} \subset \mathrm{Conf},$$

in a way compatible with the factorization structure on $\mathcal{G}^\Lambda$.

17.1.4. Let us show that if the geometric metaplectic data $\mathcal{G}^T$ for T arises from a geometric metaplectic data $\mathcal{G}^G$, then the above condition is automatically satisfied.

First, we note that by [GLys, Sect. 5.1], this question reduces to the case of $G = SL_2$. Next, by [GLys, Proposition 3.1.9 and Theorem 3.2.6] every factorization gerbe on Gr_{SL_2} is canonically of the form

$$(\det_{SL_2})^a, \quad a \in \mathsf{e}^\times(-1).$$

The resulting gerbe on $X = X^{-\alpha_i}$ equals $\mathcal{L}^a$ (see [GLys, Sect. 1.4.2] for the notation), where $\mathcal{L}$ is the line bundle on X that sends $x \in X$ to the line

$$\mathrm{rel.\,det.}(\omega^{\otimes \frac{1}{2}}(x), \omega^{\otimes \frac{1}{2}}) \otimes \mathrm{rel.\,det.}(\omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes -\frac{1}{2}}).$$

17.1.5. We claim that $\mathcal{L}$ is canonically trivial. Indeed:

$$\mathrm{rel.\,det.}(\omega^{\otimes \frac{1}{2}}(x), \omega^{\otimes \frac{1}{2}}) \simeq \omega_x^{\otimes -\frac{1}{2}}$$

and

$$\mathrm{rel.\,det.}(\omega^{\otimes -\frac{1}{2}}, \omega^{\otimes -\frac{1}{2}}(-x)) \simeq \omega_x^{\otimes -\frac{1}{2}}$$

as required.

17.1.6. Thus we obtain that the map (2.2) canonically lifts to a map

$$\mathrm{FactGe}_G \to \mathrm{FactGe}_T^{\mathrm{rigid}}, \tag{17.1}$$

which fits into the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}}) \arrow[r] \arrow[d, "\mathrm{id}"'] & \mathrm{FactGe}_G \arrow[r] \arrow[d] & \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathrm{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}}) \arrow[r] & \mathrm{FactGe}_T^{\mathrm{rigid}} \arrow[r] & \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \arrow[r] & 0,
\end{tikzcd}$$

cf. (2.4). In particular, we obtain that the map (17.1) induces an *isomorphism*

$$\mathrm{FactGe}_G \to \mathrm{FactGe}_T^{\mathrm{rigid}} \underset{\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))}{\times} \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}}. \tag{17.2}$$

17.2. **Construction of $\Omega_q^{\mathrm{small}}$ as a sheaf.** In this subsection we will define the (gerbe-twisted) perverse sheaf underlying the factorization algebra $\Omega_q^{\mathrm{small}}$.

17.2.1. As our initial datum, for each vertex of the Dynkin diagram i, let us choose an e-line denoted $\mathsf{f}^{i,\mathrm{fact}}$.

In Sect. 18.4.5 we will see the geometric meaning of these lines (they appear as *geometric Gauss sums*).

17.2.2. Since Conf is a scheme locally of finite type, we have a well-defined full subcategory of twisted *perverse* sheaves

$$\mathrm{Perv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}) \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}).$$

We are going to define an object

$$\Omega_q^{\mathrm{small}} \in \mathrm{Perv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}).$$

We first define its restriction to $\overset{\circ}{\mathrm{Conf}}$,

$$\overset{\circ}{\Omega}{}_q^{\mathrm{small}} \in \mathrm{Perv}_{\mathcal{G}^\Lambda}(\overset{\circ}{\mathrm{Conf}}).$$

17.2.3. By Sect. 17.1.3, the gerbe $\mathcal{G}^\Lambda$ is trivialized over $\overset{\circ}{\mathrm{Conf}}$, so

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\overset{\circ}{\mathrm{Conf}}) \simeq \mathrm{Shv}(\overset{\circ}{\mathrm{Conf}}) \tag{17.3}$$

and

$$\mathrm{Perv}_{\mathcal{G}^\Lambda}(\overset{\circ}{\mathrm{Conf}}) \simeq \mathrm{Perv}(\overset{\circ}{\mathrm{Conf}}).$$

So, we can regard the sought-for twisted perverse sheaf $\overset{\circ}{\Omega}{}^{\mathrm{small}}_q$ as an object of $\mathrm{Perv}(\overset{\circ}{\mathrm{Conf}})$.

17.2.4. Recall that

$$\overset{\circ}{\mathrm{Conf}} = \underset{\lambda\in\Lambda^{\mathrm{neg}}}{\sqcup} \overset{\circ}{\mathrm{Conf}}{}^\lambda \simeq \underset{\lambda\in\Lambda^{\mathrm{neg}}}{\sqcup} \overset{\circ}{X}{}^\lambda,$$

where $\overset{\circ}{X}{}^\lambda$ is obtained from

$$X^\lambda = \prod_i X^{(n_i)} \text{ if } \lambda = \underset{i}{\Sigma}\, n_i\cdot(-\alpha_i)$$

by removing the diagonal divisor.

For a fixed λ and each i we consider the n_i-th *anti-symmetric power* of the constant sheaf $(\mathsf{f}^{i,\mathrm{fact}})_X$ on X, as a local system on

$$\overset{\circ}{X}{}^{(n_i)} = X^{(n_i)} - \mathrm{Diag}\,.$$

Denote it by $(\mathsf{f}^{i,\mathrm{fact}})_X^{(n_i),\mathrm{sign}}$. Note that up to a trivialization of $\mathsf{f}^{i,\mathrm{fact}}$, this is just the sign local system on $X^{(n_i)} - \mathrm{Diag}$.

17.2.5. Consider the perverse sheaf

$$(\mathsf{f}^{i,\mathrm{fact}})_X^{(n_i),\mathrm{sign}}[n_i] \in \mathrm{Perv}(\overset{\circ}{X}{}^{(n_i)}).$$

We set

$$\overset{\circ}{\Omega}{}^{\mathrm{small}}_q \in \mathrm{Perv}(\overset{\circ}{\mathrm{Conf}}) = \mathrm{Perv}(\overset{\circ}{X}{}^\lambda)$$

to be the external product

$$\Big(\underset{i}{\boxtimes}\, (\mathsf{f}_{i,\mathrm{fact}})_X^{(n_i),\mathrm{sign}}[n_i]\Big)\,|_{\overset{\circ}{X}{}^\lambda}.$$

17.2.6. Finally, we define $\Omega^{\mathrm{small}}_q$ to be the Goresky-MacPherson extension of $\overset{\circ}{\Omega}{}^{\mathrm{small}}_q$ *in the category of twsited perverse sheaves* $\mathrm{Perv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$.

17.3. **Factorization structure on $\Omega^{\mathrm{small}}_q$.** In this subsection we endow $\Omega^{\mathrm{small}}_q$ with a structure of factorization algebra.

17.3.1. We have to construct a system of isomorphisms

$$\Omega^{\mathrm{small}}_q|_{(\mathrm{Conf}^J)_{\mathrm{disj}}} \simeq (\Omega^{\mathrm{small}}_q)^{\boxtimes J}|_{(\mathrm{Conf}^J)_{\mathrm{disj}}}, \tag{17.4}$$

satisfying a homotopy coherent system of compatibilities.

17.3.2. Note that both sides of (17.4) are *perverse sheaves* that are Goresky-MacPherson extensions of their respective restrictions to

$$(\overset{\circ}{\mathrm{Conf}})^J_{\mathrm{disj}} := (\overset{\circ}{\mathrm{Conf}})^J \cap (\mathrm{Conf}^J)_{\mathrm{disj}} \subset \mathrm{Conf}^J\,.$$

This is due to the fact that the addition map

$$\mathrm{Conf}^J \to \mathrm{Conf}$$

is étale when restricted to $(\mathrm{Conf}^J)_{\mathrm{disj}}$.

17.3.3. Hence, instead of (17.4), it suffices to construct the corresponding isomorphisms

$$\overset{\circ}{\Omega}{}^{\text{small}}_q|_{(\overset{\circ}{\text{Conf}}{}^J)_{\text{disj}}} \simeq (\overset{\circ}{\Omega}{}^{\text{small}}_q)^{\boxtimes J}|_{(\overset{\circ}{\text{Conf}}{}^J)_{\text{disj}}}. \tag{17.5}$$

However, the latter results directly from the construction.

18. Jacquet functor to the configuration space

In this section we will perform one of the main constructions in this paper: we will use the functor

$$\overset{\bullet}{\mathfrak{J}}{}^{\text{Hecke}}_{!*,\text{Fact}} : \text{Whit}_{q,x}(G) \to \Omega_q^{\text{Whit}_{!*}}\text{-FactMod}(\text{Shv}_{\mathcal{G}^T}(\text{Gr}^{\omega^\rho}_{T,\text{Ran}_x}))$$

constructed earlier into a functor to produce a functor

$$\overset{\bullet}{\Phi}{}^{\text{Hecke}}_{\text{Fact}} : \text{Whit}_{q,x}(G) \to \Omega_q^{\text{small}}\text{-FactMod}(\text{Shv}_{\mathcal{G}^\Lambda}(\text{Conf}_{\infty\cdot x})).$$

18.1. **Support of the Jacquet functor.** As a warm-up for what follows, we will show that the support of objects in the image of the functor $\mathfrak{J}_{!*,\text{sprd}}$ is contained in the *non-positive* part of $\text{Gr}^{\omega^\rho}_{T,\text{Ran}_x}$, see Sect. 4.6.

18.1.1. We will presently prove:

Proposition 18.1.2.
(a) *The support of the object*

$$\Omega_q^{\text{Whit}_{!*}} \in \text{Shv}_{\mathcal{G}^T}(\text{Gr}^{\omega^\rho}_{T,\text{Ran}})$$

is contained in $(\text{Gr}^{\omega^\rho}_{T,\text{Ran}})^{\text{non-pos}}$.
(b) *The unit map*

$$\text{unit}_*(\omega_{\text{Ran}}) \to \Omega_q^{\text{Whit}_{!*}} \tag{18.1}$$

gives rise to an isomorphism

$$\omega_{\text{Ran}} \to \text{unit}^!(\Omega_q^{\text{Whit}_{!*}}).$$

18.1.3. *Proof of Proposition 18.1.2(a).* By factorization, the assertion of point (a) of the proposition is equivalent to the following: the support of the object

$$\mathfrak{J}_{!*,x}(\text{Vac}_{\text{Whit},x}) \in \text{Shv}_{\mathcal{G}^T}(\text{Gr}^{\omega^\rho}_{T,x})$$

is contained in the union of the connected components of $\text{Gr}^{\omega^\rho}_{T,x}$ corresponding to the elements of Λ^{neg}.

To prove this, it suffices to show that if

$$\Big(\{t^\lambda\} \times \text{supp}(W^{0,*})\Big) \cap \text{supp}({}_{\text{Gr}_T}\text{IC}^{\frac{\infty}{2},-}_{q^{-1},x}) \neq \emptyset,$$

then $\lambda \in \Lambda^{\text{neg}}$.

The intersection

$$(p_T)^{-1}(\{t^\lambda\}) \cap \text{supp}({}_{\text{Gr}_T}\text{IC}^{\frac{\infty}{2},-}_{q^{-1},x})$$

equals $\overline{S}^{-,\lambda}$, viewed as a subset of $\text{Gr}^{\omega^\rho}_{G,x}$. This is while

$$\text{supp}(W^{0,*}) = \overline{S}^0 \subset \text{Gr}^{\omega^\rho}_{G,x}.$$

Now, the assertion follows from the fact that

$$\overline{S}^{-,\lambda} \cap \overline{S}^0 \neq \emptyset \;\Rightarrow\; \lambda \in \Lambda^{\text{neg}}.$$

18.1.4. *Proof of Proposition 18.1.2(b).* To prove point (b) we note that the restriction of ${}_{\mathrm{Gr}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x}$ to

$$\{1\} \times \mathrm{Gr}^{\omega^\rho}_{G,x} \subset \mathrm{Gr}^{\omega^\rho}_{T,x} \times \mathrm{Gr}^{\omega^\rho}_{G,x}$$

identifies with $\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x}$. Hence, we need to show that the map

$$\mathsf{e} \to H(\mathrm{Gr}^{\omega^\rho}_{G,x}, \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x} \overset{!}{\otimes} W^{0,*}),$$

induced by (18.1), is an isomorphism.

We recall that $\mathrm{supp}(\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x}) = \overline{S}^{-,0}$, while $\overline{S}^{-,0} \cap \overline{S}^0 = \{1\}$. Hence,

$$\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x} \overset{!}{\otimes} W^{0,*} \simeq \delta_{1,\mathrm{Gr}},$$

and the assertion follows.

18.1.5. As an immediate corollary of Proposition 18.1.2, we obtain:

Corollary 18.1.6. *The support of the objects in the essential image of the functor*

$$\mathfrak{J}_{!*,\mathrm{sprd}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})$$

is contained in $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{non-pos}}_{\infty\cdot x}$.

Thus, we can consider the functors $\mathfrak{J}_{!*,\mathrm{Fact}}$ and $\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}}$ as taking values in

$$\Omega^{\mathrm{Whit}_{!*}}_q\text{-mod-FactMod}\left(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x})\right).$$

18.2. **Jacquet functor as mapping to configuration spaces.** In this subsection we will finally construct the functors

$$\Phi_{\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to \Omega^{\mathrm{small}}_q\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))$$

and

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega^{\mathrm{small}}_q\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}))$$

18.2.1. Recall the factorization subspace

$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \hookrightarrow (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}, \tag{18.2}$$

see Sect. 4.6.

Let

$$\Omega^{\mathrm{Whit}_{!*}}_{q,\mathrm{red}} \in \mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}})$$

denote the factorization algebra obtained from $\Omega^{\mathrm{Whit}_{!*}}_q$ by applying the functor of !-pullback with respect to (18.2), (see Sect. 3.2.1).

18.2.2. Consider the closed embedding

$$(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\mathrm{neg}}_{\infty\cdot x} \hookrightarrow (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}_x})^{\text{non-pos}}_{\infty\cdot x}, \tag{18.3}$$

see Sect. 4.6.6.

Let $\mathfrak{J}_{!*,\mathrm{Fact},\mathrm{red}}$ (resp., $\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact},\mathrm{red}}$) be the functor obtained from $\mathfrak{J}_{!*,\mathrm{Fact}}$ (resp., $\mathfrak{J}^{\overset{\bullet}{\mathrm{Hecke}}}_{!*,\mathrm{Fact}}$) by composing with the functor

$$\Omega^{\mathrm{Whit}_{!*}}_q\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x})) \to \Omega^{\mathrm{Whit}_{!*}}_{q,\mathrm{red}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}_{\infty\cdot x})), \tag{18.4}$$

given by !-pullback along (18.3) (see Sect. 3.4.1).

Remark 18.2.3. Although we will not use this in the present work, one can show that the functor (18.4) is *almost* an equivalence. Namely, inside

$$\Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x}))$$

one singles out a *full subcategory* (one that contains the essential image of the functors $\mathfrak{J}_{!*,\mathrm{Fact}}$ and $\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact}}$) that consists of *unital* factorization modules, to be denoted

$$\Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x}))_{\mathrm{untl}}.$$

Now, the functor (18.4) defines an equivalence from this subcategory to

$$\Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x}))_{\mathrm{untl}} \to \Omega_{q,\mathrm{red}}^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{neg}}_{\infty\cdot x})).$$

This is due to the fact that the objects $\Omega_q^{\mathrm{Whit}_{!*}}$ and $\Omega_{q,\mathrm{red}}^{\mathrm{Whit}_{!*}}$, both viewed as factorization algebras in $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})$, are related by the mutually inverse procedures of “adding the unit” and “passage to the augmentation ideal”.

18.2.4. We now apply the equivalence of Sect. 5.5.1. Let

$$\Omega_{q,\mathrm{Conf}}^{\mathrm{Whit}_{!*}} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$$

be the factorization algebra that corresponds to

$$\Omega_{q,\mathrm{red}}^{\mathrm{Whit}_{!*}} \in \mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}})$$

under the above equivalence.

Consider the resulting equivalence of (5.10)

$$\Omega_{q,\mathrm{red}}^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x})) \simeq \Omega_{q,\mathrm{Conf}}^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})). \tag{18.5}$$

Let

$$\mathfrak{J}_{!*,\mathrm{Fact},\mathrm{Conf}} : \mathrm{Whit}_{q,x}(G) \to \Omega_{q,\mathrm{Conf}}^{\mathrm{Whit}_{!*}}\text{-FactMod}$$

and

$$\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact},\mathrm{Conf}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_{q,\mathrm{Conf}}^{\mathrm{Whit}_{!*}}\text{-FactMod}$$

denote the functors obtained from $\mathfrak{J}_{!*,\mathrm{Fact},\mathrm{red}}$ and $\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact},\mathrm{red}}$, respectively, by composing with the equivalence (18.5).

18.2.5. We will denote by

$$'\Omega_q^{\mathrm{small}}$$

the factorization algebra in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ obtained from $\Omega_{q,\mathrm{Conf}}^{\mathrm{Whit}_{!*}}$ by applying the cohomological shift $[\langle\lambda,2\check{\rho}\rangle]$ on the connected component Conf^λ of Conf.

The functor

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}), \tag{18.6}$$

given by cohomological shift $[\langle\mu,2\check{\rho}\rangle]$ on the connected component $\mathrm{Conf}^\mu_{\infty\cdot x}$ of $\mathrm{Conf}_{\infty\cdot x}$ defines an equivalence

$$\Omega_{q,\mathrm{Conf}}^{\mathrm{Whit}_{!*}}\text{-FactMod} \to {'\Omega_q^{\mathrm{small}}}\text{-FactMod}.$$

18.2.6. Let

$$\Phi_{\mathrm{Fact}} : \mathrm{Whit}_{q,x}(G) \to {'\Omega_q^{\mathrm{small}}}\text{-FactMod}$$

and

$$\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to {'\Omega_q^{\mathrm{small}}}\text{-FactMod}$$

be the functors obtained from $\mathfrak{J}_{!*,\mathrm{Fact},\mathrm{Conf}}$ and $\overset{\bullet}{\mathfrak{J}}{}^{\mathrm{Hecke}}_{!*,\mathrm{Fact},\mathrm{Conf}}$, respectively, by composing with (18.6).

18.2.7. Let

$$\Phi : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \text{ and } \Phi^{\overset{\bullet}{\mathrm{Hecke}}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

denote the functors obtained from Φ_{Fact} and $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$, respectively, by composing with the forgetful functor

$$\mathbf{oblv}_{\mathrm{Fact}} : \Omega^{\mathrm{Whit}_{!*}}_{q,\mathrm{Conf}}\text{-FactMod} \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

18.2.8. Recall the subcategory

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}),$$

see Sect. 4.3.3. Recall also the Verdier duality equivalence

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}},$$

see Sect. 4.3.3.

A key technical assertion, which we will prove in Sect. 20.4 is the following:

Theorem 18.2.9. *The functor* Φ *sends compact objects in* $\mathrm{Whit}_{q,x}(G)$ *to* $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}}$, *and the diagram*

$$\begin{array}{ccc} (\mathrm{Whit}_q(G)^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}^{\mathrm{Verdier}}} & \mathrm{Whit}_{q^{-1}}(G)^c \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} & \xrightarrow{\mathbb{D}^{\mathrm{Verdier}}} & \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}}, \end{array}$$

commutes, where the upper horizontal arrow is the equivalence (7.10).

Remark 18.2.10. A curious aspect of Theorem 18.2.9 is that we will use global methods to prove it. In fact, it is this theorem that was the reason to introduce the global version of the Whittaker category, $\mathrm{Whit}_{q,\mathrm{glob}}(G)$.

18.3. **Explicit description of the functor Φ.** We will now make a pause and describe explicitly the functor Φ and the sheaf $'\Omega^{\mathrm{small}}_q$.

18.3.1. Recall the context of Sect. 14.4. Take

$$\mathcal{Y} := (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}.$$

The corresponding twist ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$ of $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ identifies with

$$\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \simeq \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}}.$$

Since the map

$${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G \simeq \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}} \underset{\mathrm{Conf}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$$

is a base-change of the map $(\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \to \mathrm{Conf}$, it induces an equivalence between spaces of gerbes and gerbe-twisted sheaves.

Let $\mathcal{G}^{G,T,\mathrm{ratio}}$ be the gerbe on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$ whose pullback to ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$ gives ${}_{\mathcal{Y}}\mathcal{G}^G$. Let $\mathcal{G}^\Lambda$ be the pullback of the same-named gerbe along $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}} \to \mathrm{Conf}$. Note that

$$\mathcal{G}^{G,T,\mathrm{ratio}} \simeq \mathcal{G}^G \otimes (\mathcal{G}^\Lambda)^{-1},$$

where $\mathcal{G}^G$ is the gerbe on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$ whose further pullback to ${}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G$ is the pullback of the same-named gerbe on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}$ along

$$(18.7) \qquad {}_{\mathcal{Y}}\mathrm{Gr}^{\omega^\rho}_G \simeq \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}} \underset{\mathrm{Ran}}{\times} (\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\mathrm{neg}} \to \mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Ran}}.$$

18.3.2. Consider the object

$$_{\mathcal{Y}}\mathrm{IC}_{q^{-1}}^{\frac{\infty}{2},-} \in \mathrm{Shv}_{({}_{\mathcal{Y}}\mathcal{G}^G)^{-1}}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho}).$$

Let

$$\mathrm{IC}_{q^{-1},\mathrm{Conf}}^{\frac{\infty}{2},-} \in \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}_{G,\mathrm{Conf}}^{\omega^\rho})$$

be the object that corresponds to it under the equivalence

$$\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}_{G,\mathrm{Conf}}^{\omega^\rho}) \to \mathrm{Shv}_{({}_{\mathcal{Y}}\mathcal{G})^{-1}}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho}).$$

18.3.3. Let

$$\mathrm{Vac}_{\mathrm{Whit,Conf}} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Conf}}^{\omega^\rho})$$

be the object that corresponds under the equivalence

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Conf}}^{\omega^\rho}) \to \mathrm{Shv}_{\mathcal{G}^G}({}_{\mathcal{Y}}\mathrm{Gr}_G^{\omega^\rho})$$

to the pullback of $\mathrm{Vac}_{\mathrm{Whit,Ran}}$ along (18.7).

18.3.4. Note that the tensor product

$$\mathrm{Vac}_{\mathrm{Whit,Conf}} \overset{!}{\otimes} \mathrm{IC}_{q^{-1},\mathrm{Conf}}^{\frac{\infty}{2},-} \tag{18.8}$$

is naturally an object of

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Gr}_{G,\mathrm{Conf}}^{\omega^\rho}).$$

Unwinding the constructions, we obtain that

$$'\Omega_q^{\mathrm{small}} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$$

is the direct image of (18.8) along the projection

$$\mathrm{Gr}_{G,\mathrm{Conf}}^{\omega^\rho} \to \mathrm{Conf},$$

cohomologically shifted by $[\langle\lambda, 2\check{\rho}\rangle]$ on the connected component Conf^λ of Conf.

18.3.5. Let us now modify the above discussion by taking

$$\mathcal{Y} := (\mathrm{Gr}_{T,\mathrm{Ran}_x}^{\omega^\rho})_{\infty\cdot x}^{\mathrm{neg}}.$$

Consider the corresponding version of the affine Grassmannian $\mathrm{Gr}_{G,\mathrm{Conf}_{\infty\cdot x}}^{\omega^\rho}$, so that

$$\mathrm{Gr}_{G,\mathrm{Conf}_{\infty\cdot x}}^{\omega^\rho} \underset{\mathrm{Conf}_{\infty\cdot x}}{\times} (\mathrm{Gr}_{T,\mathrm{Ran}_x}^{\omega^\rho})_{\infty\cdot x}^{\mathrm{neg}} \simeq \mathrm{Gr}_{G,\mathrm{Ran}_x}^{\omega^\rho} \underset{\mathrm{Ran}_x}{\times} (\mathrm{Gr}_{T,\mathrm{Ran}_x}^{\omega^\rho})_{\infty\cdot x}^{\mathrm{neg}}.$$

18.3.6. Applying the same procedure as above, we obtain an object

$$\mathrm{IC}_{q^{-1},\mathrm{Conf}_{\infty\cdot x}}^{\frac{\infty}{2},-} \in \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\mathrm{Gr}_{G,\mathrm{Conf}_{\infty\cdot x}}^{\omega^\rho}),$$

and a functor

$$\mathrm{sprd}_{\mathrm{Conf}_{\infty\cdot x}} : \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_{G,\mathrm{Conf}_{\infty\cdot x}}^{\omega^\rho}).$$

18.3.7. For $\mathcal{F} \in \mathrm{Whit}_{q,x}(G)$, the tensor product

$$\mathrm{sprd}_{\mathrm{Conf}_{\infty\cdot x}}(\mathcal{F}) \overset{!}{\otimes} \mathrm{IC}_{q^{-1},\mathrm{Conf}_{\infty\cdot x}}^{\frac{\infty}{2},-} \tag{18.9}$$

is naturally an object of $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Gr}_{G,\mathrm{Conf}_{\infty\cdot x}}^{\omega^\rho})$.

Then for $\mathcal{F}$ as above the object

$$\Phi(\mathcal{F}) \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

is the direct image of (18.9) along the projection

$$\mathrm{Gr}_{G,\mathrm{Conf}_{\infty\cdot x}}^{\omega^\rho} \to \mathrm{Conf}_{\infty\cdot x},$$

cohomologically shifted by $[\langle\lambda, 2\check{\rho}\rangle]$ on the connected component $\mathrm{Conf}_{\infty\cdot x}^\lambda$ of $\mathrm{Conf}_{\infty\cdot x}$.

18.4. **Identification of factorization algebras.** We now come to the first crucial computational results of this paper (which makes everything work).

18.4.1. Recall the factorization algebra $\Omega_q^{\mathrm{small}}$ introduced in Sect. 17. We claim:

Theorem 18.4.2. *There exists a canonical isomorphism of factorization algebras*

$$'\Omega_q^{\mathrm{small}} \simeq \Omega_q^{\mathrm{small}}$$

for the choice of the lines $\mathsf{f}^{i,\mathrm{fact}}$ *specified in Sect. 18.4.5.*

The assertion of Theorem 18.4.2 naturally splits into two parts:

Proposition 18.4.3. *For the choice of the lines* $\mathsf{f}^{i,\mathrm{fact}}$ *specified in Sect. 18.4.5, we have a canonical isomorphism of factorization algebras in* $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\overset{\circ}{\mathrm{Conf}})$

$$'\Omega_q^{\mathrm{small}}|_{\overset{\circ}{\mathrm{Conf}}} \simeq \Omega_q^{\mathrm{small}}|_{\overset{\circ}{\mathrm{Conf}}}.$$

Theorem 18.4.4. *The object* $'\Omega_q^{\mathrm{small}} \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$ *is a perverse sheaf, which is the Goresky-MacPherson extension of its restriction to* $\overset{\circ}{\mathrm{Conf}}$.

We will prove Proposition 18.4.3 in the rest of this subsection. The proof of Theorem 18.4.4 will be given in Sect. 18.7.

18.4.5. Recall that Kummer theory attaches to an element $c \in \mathsf{e}^\times(-1)$ a character sheaf on $\mathbb{G}_m$; to be denoted Ψ_c. When $c \neq 0$, the extension of Ψ_c along $\mathbb{G}_m \to \mathbb{A}^1$ is *clean*; by a slight abuse of notation we will denote it by the same character Ψ_c.

Recall also that a geometric metaplectic datum defines for each vertex of the Dynkin diagram an element $q_i \in \mathsf{e}^\times(-1)$. Recall also that we impose a non-degeneracy condition on the geomtetric metaplectic datum, which says that all q_i are non-trivial, see Definition 2.3.5.

Finally, recall that for our definition of the Whittaker category, we *chose* an Artin-Schreier character sheaf χ on $\mathbb{G}_a$.

We let $\mathsf{f}^{i,\mathrm{fact}}$ be the line (a "Gauss sum")

$$H^1(\mathbb{A}^1, \Psi_{q_i} \otimes \chi).$$

Note this cohomology is a geomtric Gauss sum, in particular, $H^0 = H^2 = 0$.

By factorization, in order to prove Proposition 18.4.3, it suffices to perform the calculation on each individual $\mathrm{Conf}^{-\alpha_i} \simeq X$. To simplify the notation, we will perform the calculation at a fixed point $x \in X$.

18.4.6. For $\lambda \in \Lambda$, consider the action of the element $t^\lambda \in \mathfrak{L}(T)_x$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$. It induces a functor

$$\mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{\mathfrak{L}^+(T)_x} \to \mathrm{Shv}_{(\mathcal{G}^G)^{-1}\otimes \mathcal{G}^\Lambda_{\lambda\cdot x}}(\mathrm{Gr}^{\omega^\rho}_{G,x})^{\mathfrak{L}^+(T)_x}.$$

Note that $t^\lambda \star \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x}$ identifies with the restriction of $\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}}$ to the fiber over the point

$$\lambda \cdot x \in \mathrm{Conf}\,.$$

Set

$$\mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x} = t^\lambda \star \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},x}[\langle \lambda, 2\check{\rho}\rangle] \in \mathrm{Shv}_{(\mathcal{G}^G)^{-1}\otimes \mathcal{G}^\Lambda_{\lambda\cdot x}}(\mathrm{Gr}^{\omega^\rho}_{G,x}).$$

Note that the restriction of the gerbe $(\mathcal{G}^G)^{-1} \otimes \mathcal{G}^\Lambda_{\lambda\cdot x}$ to $S^{-,\lambda} \subset \mathrm{Gr}^{\omega^\rho}_{G,x}$ is canonically trivialized, and in terms of this trivialization, we have

$$\mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x}|_{S^{-,\lambda}} \simeq \omega_{S^{-,\lambda}}[\langle \lambda, 2\check{\rho}\rangle].$$

18.4.7. Thus, we have to show that

$$H(\mathrm{Gr}^{\omega^\rho}_{G,x}, W^{0,*} \overset{!}{\otimes} \mathrm{IC}^{-\alpha_i+\frac{\infty}{2},-}_{q^{-1},x}) \simeq \Gamma(\mathbb{A}^1, \Psi_{q_i} \otimes \chi). \tag{18.10}$$

Note that the tensor product $W^{0,*} \overset{!}{\otimes} \mathrm{IC}^{-\alpha_i+\frac{\infty}{2},-}_{q^{-1},x}$ is an *untwisted* sheaf on $\mathrm{Gr}^{\omega^\rho}_{G,x}$ due to the trivialization of the restriction of $\mathcal{G}^\Lambda$ to the point $-\alpha_i \cdot x \in \mathrm{Conf}$, given by Sect. 17.1.4.

18.4.8. First, by Corrollary 6.2.10(b), the object $W^{0,*}$ is the *-extension of its restriction to S^0.

The object $\mathrm{IC}^{-\alpha_i+\frac{\infty}{2},-}_{q^{-1},x}$ is supported on $\overline{S}^{-,-\alpha_i}$. Note now that the residue map defines an isomorphism

$$S^0 \cap \overline{S}^{-,-\alpha_i} \to \mathbb{A}^1. \tag{18.11}$$

We will show that we have a canonical isomorphism

$$(W^{0,*} \overset{!}{\otimes} \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}})|_{S^0\cap\overline{S}^{-,-\alpha_i}} \simeq \Psi_{q_i} \otimes \chi \tag{18.12}$$

taking place in $\mathrm{Shv}(\mathbb{A}^1)$.

18.4.9. Note that the open subset

$$S^0 \cap S^{-,-\alpha_i} \subset S^0 \cap \overline{S}^{-,-\alpha_i} \tag{18.13}$$

corresponds to

$$\mathbb{A}^1 - 0 \subset \mathbb{A}^1,$$

while $\{0\} \subset \mathbb{A}^1$ corresponds to

$$S^0 \cap S^{-,0} \subset S^0 \cap \overline{S}^{-,-\alpha_i}. \tag{18.14}$$

We claim that the object

$$(W^{0,*} \overset{!}{\otimes} \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}})|_{S^0\cap\overline{S}^{-,-\alpha_i}}$$

is the *-extension of its own restriction along the open embedding (18.13). I.e., we claim that its !-restriction along (18.14) vanishes. However, this follows from Lemma 13.1.9 and the assumption that q_i is non-trivial.

Hence, it suffices to establish an isomorphism

$$(W^{0,*} \overset{!}{\otimes} \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}})|_{S^0\cap S^{-,-\alpha_i}} \simeq \Psi_{q_i} \otimes \chi \tag{18.15}$$

taking place in $\mathrm{Shv}(\mathbb{A}^1 - 0)$.

18.4.10. Recall that the gerbe $\mathcal{G}^G|_{S^0}$ is trivialized, and in terms of this trivialization, the object $W^{0,*}|_{S^0}$ corresponds to ω_{S^0} tensored by the pullback of χ. Hence, in terms of *this* trivialization,

$$W^{0,*}|_{S^0\cap S^{-,-\alpha_i}} \simeq \chi \otimes \omega_{\mathbb{A}^1-0}.$$

Recall also that the gerbe $(\mathcal{G}^G)^{-1} \otimes \mathcal{G}^T|_{S^{-,-\alpha_i}}$ is also canonically trivialized, and in terms of *this* trivialization, $\mathrm{IC}^{-\alpha_i+\frac{\infty}{2},-}_{q^{-1},x}|_{S^{-,-\alpha_i}}$ corresponds to $\omega_{S^{-,-\alpha_i}}[-2]$. Hence, in terms of *this* trivialization,

$$\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}}|_{S^0_{\mathrm{Conf}^{-\alpha_i}}\cap S^{-,-\alpha_i}_{\mathrm{Conf}^{-\alpha_i}}} \simeq \mathsf{e}_{\mathbb{A}^1-0}.$$

Hence, in order to establish (18.15), we need to show that the resulting trivialization of

$$\mathcal{G}^G|_{S^0\cap S^{-,-\alpha_i}} \otimes ((\mathcal{G}^G)^{-1} \otimes \mathcal{G}^T)|_{S^0\cap S^{-,-\alpha_i}} \simeq \mathcal{G}^T|_{S^0\cap S^{-,-\alpha_i}}$$

differs from the trialization of $\mathcal{G}^T|_{S^{-,-\alpha_i}}$ of Sect. 17.1.4 by the local system equal to the pullback of Ψ_{q_i} along the residue map

$$S^0 \cap S^{-,-\alpha_i} \to \mathbb{A}^1 - 0.$$

This is a calculation performed in the next subsection.

18.5. **Calculation of the discrepancy.**

18.5.1. As in Sect. 17.1.4, the calculation reduces to the case of $G = SL_2$, in which case the gerbe $\mathcal{G}^G$ is canonically of the form $\det^a_{SL_2}$ for $a \in \mathsf{e}^\times(-1)$.

The corresponding quadratic form q takes the value a on the (unique) coroot α_i, i.e., $q_i = a$.

The line bundle $\det_{SL_2}$ admits a canonical trivialization when restricted to S^0 and also, by Sect. 17.1.5, to $S^{-,-\alpha_i}$. We need to show that the discrepancy of these two trivializations, viewed as a function,

$$S^0 \cap S^{-,-\alpha_i} \to \mathbb{G}_m$$

equals the residue map.

18.5.2. A point of $S^0 \cap S^{-,-\alpha_i}$ is a rank 2-bundle $\mathcal{M}$ on X that fits into a diagram

$$\begin{tikzcd} & \omega^{\otimes \frac{1}{2}}(x) & \\ \omega^{\otimes \frac{1}{2}} \arrow[r] & \mathcal{M} \arrow[u] \arrow[r] & \omega^{\otimes -\frac{1}{2}} \\ & \omega^{\otimes -\frac{1}{2}}(-x) \arrow[u] & \end{tikzcd}$$

where the row and the column are exact sequences. Such an $\mathcal{M}$ is uniquely determined by the choice of a line

$$\ell \subset \omega_x^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x)_x$$

that projects isomorphically to both factors

$$\omega^{\otimes -\frac{1}{2}}(-x)_x \leftarrow \ell \to \omega_x^{\otimes \frac{1}{2}}. \tag{18.16}$$

The resulting isomorphism

$$\omega^{\otimes -\frac{1}{2}}(-x)_x \to \omega_x^{\otimes \frac{1}{2}} \tag{18.17}$$

can be regarded as a non-zero element of $\omega^{\otimes \frac{1}{2}}(x)_x$, where $\omega^{\otimes \frac{1}{2}}(x)_x \simeq \mathbb{A}^1$ by the residue map. It is easy to see that the thus constructed map $S^0 \cap S^{-,-\alpha_i} \to \mathbb{A}^1 - 0$ is the residue map of (18.11).

18.5.3. The two embeddings

$$\omega^{\otimes \frac{1}{2}}(x) \oplus \omega^{\otimes -\frac{1}{2}}(-x) \hookleftarrow \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x) \hookrightarrow \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}$$

induce the identifications

$$\begin{aligned} &\mathrm{rel.}\det(\omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \simeq \\ &\qquad \simeq \mathrm{rel.}\det(\omega^{\otimes \frac{1}{2}}(x) \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \omega_x^{\otimes \frac{1}{2}} \simeq \omega_x^{\otimes \frac{1}{2}}, \end{aligned} \tag{18.18}$$

(where the last isomorphism comes from Sect. 17.1.5) and

$$\begin{aligned} &\mathrm{rel.}\det(\omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \simeq \\ &\qquad \simeq \mathrm{rel.}\det(\omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}, \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \omega_x^{\otimes \frac{1}{2}} \simeq \omega_x^{\otimes \frac{1}{2}}. \end{aligned} \tag{18.19}$$

However, by the construction of the isomorphism in Sect. 17.1.5, it follows that the above two isomorphisms

$$\mathrm{rel.}\det(\omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes \frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \simeq \omega_x^{\otimes \frac{1}{2}}$$

coincide.

18.5.4. The fiber of $\det_{SL_2}$ at $\mathcal{M}$ is given by

$$\det.\mathrm{rel.}(\mathcal{M}, \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}).$$

We have a canonical short exact sequence

$$0 \to \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x) \to \mathcal{M} \to \ell \otimes \omega_x^{\otimes -1} \to 0,$$

where the line $\ell \otimes \omega_x^{\otimes -1}$ is regarded as a skyscraper sheaf at x. Hence, the fiber of $\det_{SL_2}$ at $\mathcal{M}$ can be further identified with

$$\det.\mathrm{rel.}(\omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \ell \otimes \omega_x^{\otimes -1},$$

and its two trivializations are given by

$$\det.\mathrm{rel.}(\omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \ell \otimes \omega_x^{\otimes -1} \overset{\leftarrow \text{in } (18.16)}{\simeq}$$
$$\simeq \det.\mathrm{rel.}(\omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \omega^{\otimes -\frac{1}{2}}(-x)_x \otimes \omega_x^{\otimes -1} \overset{(18.18)}{\simeq} \omega_x^{\otimes\frac{1}{2}} \otimes \omega_x^{\otimes -\frac{1}{2}} \simeq k$$

and

$$\det.\mathrm{rel.}(\omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \ell \otimes \omega_x^{\otimes -1} \overset{\to \text{in } (18.16)}{\simeq}$$
$$\simeq \det.\mathrm{rel.}(\omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}(-x), \omega^{\otimes\frac{1}{2}} \oplus \omega^{\otimes -\frac{1}{2}}) \otimes \omega_x^{\otimes\frac{1}{2}} \otimes \omega_x^{\otimes -1} \overset{(18.19)}{\simeq} \omega_x^{\otimes\frac{1}{2}} \otimes \omega_x^{\otimes -\frac{1}{2}} \simeq k,$$

respectively.

Hence, the discrepancy between the two is given by (18.17), as required.

18.6. **Properties of the functor Φ and its variants.** By Theorem 18.4.2, we can consider Φ_{Fact} and $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}$ as taking values in the category $\Omega_q^{\mathrm{small}}$-FactMod. In this subsection we will study some basic properties of this functor.

18.6.1. Recall (see Sect. 5.4) that sense $\Omega_q^{\mathrm{small}}$ is perverse, the category $\Omega_q^{\mathrm{small}}$-FactMod has a t-structure for which the functor $\mathbf{oblv}_{\mathrm{Fact}}$ is t-exact. Recall also the irreducible objects

$$\mathcal{M}^{\lambda,!*}_{\mathrm{Fact}} \in (\Omega_q^{\mathrm{small}}\text{-FactMod})^{\heartsuit}.$$

Here is the key result, proved in Sect. 18.7 below:

Theorem 18.6.2. *Let $\lambda \in \Lambda^+$ be restricted. Then*

$$\Phi(W^{\lambda,!*}) \simeq \mathbf{oblv}(\mathcal{M}^{\lambda,!*}_{\mathrm{Fact}}).$$

The above theorem has a slew of consequences pertaining to the properties of the functors Φ, Φ_{Fact} and $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}$.

Corollary 18.6.3. *Let $\lambda \in \Lambda^+$ be restricted. Then*

$$\Phi_{\mathrm{Fact}}(W^{\lambda,!*}) \simeq \mathcal{M}^{\lambda,!*}_{\mathrm{Fact}}.$$

Proof. Follows from the fact that if $'\mathcal{M}^{\lambda,!*}_{\mathrm{Fact}} \in \Omega_q^{\mathrm{small}}$-FactMod is such that

$$\mathbf{oblv}('\mathcal{M}^{\lambda,!*}_{\mathrm{Fact}}) \simeq \mathbf{oblv}(\mathcal{M}^{\lambda,!*}_{\mathrm{Fact}}),$$

then $'\mathcal{M}^{\lambda,!*}_{\mathrm{Fact}} \simeq \mathcal{M}^{\lambda,!*}_{\mathrm{Fact}}$. □

Corollary 18.6.4. *For any $\mu \in \Lambda$,*

$$\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}}) \simeq \mathcal{M}^{\mu,!*}_{\mathrm{Fact}}.$$

Proof. As in Sect. 11.3.5 and using Sect. 5.2.2, we can reduce the assertion of the proposition to the case when H is such that its derived group is simply-connected. In this case we can write $\mu = \lambda + \gamma$, where $\lambda \in \Lambda^+$ is restricted and $\gamma \in \Lambda^\sharp$. We have:

$$\Phi^{\overset{\bullet}{\text{Hecke}}}_{\text{Fact}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\text{Whit}}) \simeq \Phi^{\overset{\bullet}{\text{Hecke}}}_{\text{Fact}}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,!*}_{\text{Whit}}) \otimes \mathsf{e}^\gamma) \simeq \Phi^{\overset{\bullet}{\text{Hecke}}}_{\text{Fact}}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,!*}_{\text{Whit}})) \star \delta_{t^\gamma} \simeq$$
$$\simeq \Phi^{\overset{\bullet}{\text{Hecke}}}_{\text{Fact}}(\mathbf{ind}_{\overset{\bullet}{\text{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,!*}_{\text{Whit}})) \star \delta_{t^\gamma} \simeq \Phi_{\text{Fact}}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,!*}_{\text{Whit}}) \star \delta_{t^\gamma} \simeq \mathcal{M}^{\lambda,!*}_{\text{Fact}} \star \delta_{t^\gamma} \simeq \mathcal{M}^{\mu,!*}_{\text{Fact}}.$$

□

Corollary 18.6.5.

(a) *The functor* $\Phi^{\overset{\bullet}{\text{Hecke}}}_{\text{Fact}}$ *is t-exact.*

(b) *The functor* Φ_{Fact} *is t-exact.*

(c) *The functor* Φ *is t-exact.*

Proof. Point (a) is immediate from Corrollary 18.6.4 using the fact that the t-structure on the category $\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G))$ is Artinian (the latter by Proposition 11.3.3(a)).

Point (b) follows from point (a) since

$$\Phi_{\text{Fact}} \simeq \Phi^{\overset{\bullet}{\text{Hecke}}}_{\text{Fact}} \circ \mathbf{ind}_{\overset{\bullet}{\text{Hecke}}},$$

where $\mathbf{ind}_{\overset{\bullet}{\text{Hecke}}}$ is also t-exact.

Point (c) is logically equivalent to point (b).

□

Remark 18.6.6. We will give an alternative proof of Corrollary 18.6.5(c) in Sect. 18.7, in the course of the proof of Theorem 18.6.2.

18.7. **Computation of stalks and proofs of Theorems 18.4.4 and 18.6.2.** In this subsection we will assume Theorem 18.2.9 and deduce from it Theorems 18.4.4 and 18.6.2.

18.7.1. In order to prove Theorem 18.4.4, using the Verdier autoduality statement established in Theorem 18.2.9 and factorization, it suffices to show the following:

For $\lambda \in \Lambda^{\text{neg}}$ with $\lambda \neq -\alpha_i$, the !-fiber of $\Phi(W^0)$ at the point $\lambda \cdot x \in \text{Conf}$ lives in cohomological degrees ≥ 2.

We will now derive a general expression for the !-fiber, denoted

$$\Phi(\mathcal{F})_{\lambda\cdot x} \in \text{Shv}_{\mathcal{G}^G_{\lambda\cdot x}}(\text{pt}) =: \text{Vect}_{\mathcal{G}^G_{\lambda\cdot x}}$$

of $\Phi(\mathcal{F})$ at $\lambda \cdot x \in \text{Conf}_{\infty\cdot x}$ for $\lambda \in \Lambda$ and $\mathcal{F} \in \text{Whit}_q(G)$. By factorization, this would give an answer to the !-fiber of $\Phi(\mathcal{F})$ at any other point of $\text{Conf}_{\infty\cdot x}$.

We will show that for any $\mathcal{F} \in (\text{Whit}_q(G))^\heartsuit$, the object $\Phi(\mathcal{F})_{\lambda\cdot x}$ lives in cohomological degrees ≥ 0. Taking into account Theorem 18.2.9 and Corollary 7.2.6, this will imply that the functor Φ is t-exact (see Remark 18.6.6 above).

Furthermore, we will show that for μ restricted, $\Phi(W^{\mu,!*})_{\lambda\cdot x}$ lives in cohomological degrees ≥ 1 for $\lambda \neq \mu$. Using Theorem 18.2.9 and Corollary 7.2.6, this will imply Theorem 18.6.2.

18.7.2. Recall the notation

$$\text{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x} \in \text{Shv}_{(\mathcal{G}^G)^{-1}\otimes\mathcal{G}^\Lambda_{\lambda\cdot x}}(\text{Gr}^{\omega^\rho}_{G,\text{Ran}}),$$

see Sect. 18.4.6.

Unwinding the definitions, we obtain:

$$\Phi(\mathcal{F})_{\lambda\cdot x} \simeq \Gamma(\text{Gr}^{\omega^\rho}_{G,x}, \mathcal{F} \overset{!}{\otimes} \text{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x}) \in \text{Vect}_{\mathcal{G}^G_{\lambda\cdot x}}. \tag{18.20}$$

18.7.3. We now claim:

Proposition 18.7.4. *For $\mathcal{F} \in (\mathrm{Whit}_q(G))^\heartsuit$, the cohomologies $H^i(\mathrm{Gr}^{\omega^\rho}_{G,x}, \mathcal{F} \overset{!}{\otimes} \mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x})$ satisfy:*
(a) $H^i = 0$ *for* $i < 0$.
(b) H^0 *identifies with* $H^0(S^{-,\lambda}, \mathcal{F}|_{S^{-,\lambda}}[\langle \lambda, 2\check{\rho}\rangle])$.
(c) H^1 *injects into* $H^1(S^{-,\lambda}, \mathcal{F}|_{S^{-,\lambda}}[\langle \lambda, 2\check{\rho}\rangle])$.

Proof. Note that the support of $\mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x}|_{S^{-,\lambda}}$ is $\overline{S}^{-,\lambda}$, which is the union of $S^{-,\lambda'}$ for $\lambda' \in \lambda + \Lambda^{\mathrm{pos}}$. Using the Cousin decomposition, it suffices to show that for any $\lambda' \in \lambda + (\Lambda^{\mathrm{pos}} - 0)$, the cohomologies

$$H^i(S^{-,\lambda'}, \mathcal{F}|_{S^{-,\lambda'}} \overset{!}{\otimes} \mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x}|_{S^{-,\lambda'}})$$

vanish in degrees $\leq 1 + \langle \lambda, 2\check{\rho}\rangle$ and that the cohomolgies

$$H^i(S^{-,\lambda}, \mathcal{F}|_{S^{-,\lambda}} \overset{!}{\otimes} \mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x}|_{S^{-,\lambda}}) \tag{18.21}$$

vanish in degrees $< \langle \lambda, 2\check{\rho}\rangle$.

Note that by Proposition 13.2.10, the restriction $\mathrm{IC}^{\lambda+\frac{\infty}{2},-}_{q^{-1},x}|_{S^{-,\lambda'}}$ is isomorphic to

$$\omega_{S^{-,\lambda'}}[\langle \lambda', 2\check{\rho}\rangle] \otimes \mathsf{K}_{\lambda,\lambda'},$$

where $\mathsf{K}_{\lambda,\lambda'}$ is an object of the category $\mathrm{Shv}_{\mathcal{G}^\Lambda_{\lambda\cdot x} \otimes \mathcal{G}^\Lambda_{-\lambda'\cdot x}}(\mathrm{pt})$ that lives in cohomological degrees > 1.

Hence, it suffices to show that the cohomologies

$$H^i(S^{-,\lambda'}, \mathcal{F}|_{S^{-,\lambda'}}) \in \mathrm{Shv}_{\mathcal{G}^\Lambda_{\lambda'\cdot x}}(\mathrm{pt})$$

vanish in degrees $< \langle \lambda', 2\check{\rho}\rangle$.

Applying the Cousin decomposition with respect to the orbits S^μ, it suffices to show that the cohomologies

$$H^i(S^\mu \cap S^{-,\lambda'}, \mathcal{F}|_{S^\mu \cap S^{-,\lambda'}})$$

vanish in degrees $< \langle \lambda', 2\check{\rho}\rangle$.

By the definition of the t-structure on $\mathrm{Whit}_q(G)$, the restriction $\mathcal{F}|_{S^\mu}$ is isomorphic to

$$\chi_N \otimes \omega_{S^\mu}[-\langle \mu, 2\check{\rho}\rangle] \otimes \mathsf{K}'_\mu,$$

where K'_μ is object of the category $\mathrm{Shv}_{\mathcal{G}^\Lambda_{\mu\cdot x}}(\mathrm{pt})$ that lives in cohomological degrees ≥ 0.

The discrepancy of the identifications

$$\mathcal{G}^\Lambda_{\mu\cdot x} \simeq \mathcal{G}^\Lambda|_{S^\mu \cap S^{-,\lambda'}} \simeq \mathcal{G}^\Lambda_{\lambda'\cdot x}$$

is given by a local system in $\mathrm{Shv}_{\mathcal{G}^\Lambda_{\lambda'\cdot x} \otimes (\mathcal{G}^\Lambda_{\mu\cdot x})^{-1}}(S^\mu \cap S^{-,\lambda'})$. Hence, up to tensoring by *lisse* sheaves, the restriction $\mathcal{F}|_{S^\mu \cap S^{-,\lambda'}}$ identifies with

$$\omega_{S^\mu \cap S^{-,\lambda'}}[-\langle \mu, 2\check{\rho}\rangle] \otimes \mathsf{K}'_\mu,$$

and hence lives in perverse cohomological degrees

$$\geq \langle \mu, 2\check{\rho}\rangle - \dim(S^\mu \cap S^{-,\lambda'}).$$

Hence, its cohomologies live in degrees

$$\geq \langle \mu, 2\check{\rho}\rangle - \dim(S^\mu \cap S^{-,\lambda'}) - \dim(S^\mu \cap S^{-,\lambda'}) =$$
$$= \langle \mu, 2\check{\rho}\rangle - 2\dim(S^\mu \cap S^{-,\lambda'}) = \langle \mu, 2\check{\rho}\rangle - \langle \mu - \lambda', 2\check{\rho}\rangle = \langle \lambda', 2\check{\rho}\rangle,$$

as required.

The cohomologies (18.21) are analyzed similarly.

□

18.7.5. We are now ready to prove the cohomological estimates stated in Sect. 18.7.1. The fact that for any $\mathcal{F} \in (\mathrm{Whit}_q(G))^{\heartsuit}$, the object $\Phi(\mathcal{F})_{\lambda\cdot x}$ lives in cohomological degrees ≥ 0 follows from Proposition 18.7.4(a).

Let us take $\mathcal{F} = W^{\mu,!*}$ with μ restricted. Applying Proposition 18.7.4(b), we need to show that

$$H^0(S^\mu \cap S^{-,\lambda}, \chi_N|_{S^\mu \cap S^{-,\lambda}} \otimes \Psi_q \otimes \omega_{S^\mu \cap S^{-,\lambda}}[\langle \lambda - \mu, 2\check{\rho}\rangle]) = 0 \text{ if } \lambda \neq \mu$$

where Ψ_q is the local system on $S^\mu \cap S^{-,\lambda}$ from Sect. 9.4.8. However, this is the statement of Theorem 9.4.9.

18.7.6. Finally, let us take $\mathcal{F} = W^0$. Applying Proposition 18.7.4(c), for the proof of Theorem 18.4.4 it remains to show that the *sup-bottom* cohomology

$$H^1(S^0 \cap S^{-,\lambda}, \chi_N|_{S^0 \cap S^{-,\lambda}} \otimes \Psi_q \otimes \omega_{S^0 \cap S^{-,\lambda}}[\langle \lambda, 2\check{\rho}\rangle])$$

vanishes.

This vanishing was established in [Lys, Theorem 1.1.5] under the (mild) assumption that the order of each q_i is large enough. In a subsequent publication we will give a proof in the general case (assuming $q_i \neq 1$).

19. Statement of the main theorem

In this section we will finally formulate our main theorem, which compares $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ with a certain modification of the category $\Omega_q^{\mathrm{small}}$-FactMod.

19.1. **Renormalization of $\Omega_q^{\mathrm{small}}$-FactMod.** In this subsection we will introduce a *renormalized version* of $\Omega_q^{\mathrm{small}}$-FactMod, denoted $\Omega_q^{\mathrm{small}}$-FactMod$^{\mathrm{ren}}$. It is this modified version that will end up being equivalent to $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$.

The nature of this modification is that we *declare a larger class of objects as compact*; it mimics the procedure that produces IndCoh from QCoh, see [Ga3, Sect. 1].

19.1.1. We define $\Omega_q^{\mathrm{small}}$-FactMod$^{\mathrm{ren}}$ to be the ind-completion of the non-cocomplete full subcategory of $\Omega_q^{\mathrm{small}}$-FactMod that consists of objects that are finite extensions of (shifts of) the objects $\mathcal{M}^{\mu,!*}_{\mathrm{Conf}}$.

Ind-extension of the tautological inclusion defines a functor

$$\text{un-ren} : \Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}} \to \Omega_q^{\mathrm{small}}\text{-FactMod}. \tag{19.1}$$

Remark 19.1.2. Note that the functor un-ren is *not fully faithful*, even though it is such when restricted to the full subcategory of compact objects.

19.1.3. Recall that the category $\Omega_q^{\mathrm{small}}$-FactMod is compactly generated by the objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$ for $\mu \in \Lambda$. We will need the following result:

Proposition 19.1.4. *The objects $\mathcal{M}^{\mu,!}_{\mathrm{Conf}}$ and $\mathcal{M}^{\mu,*}_{\mathrm{Conf}}$ have finite length.*

The proof will be given in Sect. 19.3.3.

Remark 19.1.5. When $k = \mathbb{C}$, an alternative proof of Proposition 19.1.4 will be given in Sect. 29.2.6.

19.1.6. By Proposition 19.1.4, the category of compact objects in $\Omega_q^{\mathrm{small}}$-FactMod can be thought as a full subcategory in the category of compact objects in $\Omega_q^{\mathrm{small}}$-FactMod$^{\mathrm{ren}}$.

Therefore, the procedure of ind-extension defines a fully faithful functor

$$\mathrm{ren} : \Omega_q^{\mathrm{small}}\text{-FactMod} \to \Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}},$$

which is easily seen to be the left adjoint of the functor un-ren of (19.1).

19.1.7. As in [Ga3, Sect. 1.2] we have:

Proposition 19.1.8. *The category* $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$ *has a t-structure, uniquely characterized by the property that an object is connective if and only if its image under the functor* un-ren *is connective. Moreover, the functor* un-ren *has the following properties with respect to this t-structure:*

(a) *It is t-exact;*

(b) *It induces an equivalence*

$$(\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}})^{\geq n} \to (\Omega_q^{\text{small}}\text{-FactMod})^{\geq n}$$

for any n;

(c) *It induces an equivalence of the hearts.*

Corollary 19.1.9. *The kernel of the functor* un-ren *consists of infinitely coconnective objects, i.e.,*

$$\bigcap_n (\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}})^{\leq -n}.$$

Remark 19.1.10. Note that there are two possible meaning for the notation $\mathcal{M}^{\mu,!}_{\text{Conf}}$ and $\mathcal{M}^{\mu,*}_{\text{Conf}}$ and $\mathcal{M}^{\mu,!*}_{\text{Conf}}$. On the one hand, we can view them as objects of the original category $\Omega_q^{\text{small}}\text{-FactMod}$.

On the other hand, we can view them (using Proposition 19.1.4 for $\mathcal{M}^{\mu,!}_{\text{Conf}}$ and $\mathcal{M}^{\mu,*}_{\text{Conf}}$) as (compact) objects in $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$.

We will not distinguish the two usages notationally, but will explicitly mention which one we mean if a confusion is likely to occur.

We note that the functor un-ren sends $\mathcal{M}^{\mu,!}_{\text{Conf}}$ (resp., $\mathcal{M}^{\mu,*}_{\text{Conf}}$, $\mathcal{M}^{\mu,!*}_{\text{Conf}}$), viewed as an object of $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$, to $\mathcal{M}^{\mu,!}_{\text{Conf}}$ (resp., $\mathcal{M}^{\mu,*}_{\text{Conf}}$, $\mathcal{M}^{\mu,!*}_{\text{Conf}}$), viewed as an object of $\Omega_q^{\text{small}}\text{-FactMod}$.

The functor ren sends $\mathcal{M}^{\mu,!}_{\text{Conf}}$, viewed as an object of $\Omega_q^{\text{small}}\text{-FactMod}$, to $\mathcal{M}^{\mu,!}_{\text{Conf}}$, viewed as an object of $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$.

However, the functor ren *does not* send $\mathcal{M}^{\mu,*}_{\text{Conf}}$ (resp., $\mathcal{M}^{\mu,!*}_{\text{Conf}}$), viewed as an object of $\Omega_q^{\text{small}}\text{-FactMod}$, to $\mathcal{M}^{\mu,!}_{\text{Conf}}$, viewed as an object of $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$. This is due to the fact that $\mathcal{M}^{\mu,*}_{\text{Conf}}$ and $\mathcal{M}^{\mu,!*}_{\text{Conf}}$ are *not* compact in $\Omega_q^{\text{small}}\text{-FactMod}$. In fact, the images of these objects under ren do not lie in the heart of $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$.

19.1.11. Recall (see Sect. 6.3.8) what it means for a t-structure on a DG category to be Artinian.

We obtain that the t-structure on $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$ is Artinian. This is the main point of difference between $\Omega_q^{\text{small}}\text{-FactMod}$ and $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$.

Remark 19.1.12. Since introducing the configuration space may seem a bit artificial, let us remark that one define the renormalization $\Omega_q^{\text{small}}\text{-FactMod}^{\text{ren}}$ purely in terms of the factorization algebra $\Omega_q^{\text{Whit}_{!*}} \in \text{Shv}_{\mathcal{G}^T}(\text{Gr}^{\omega}_{T,\text{Ran}})$.

Namely, recall (see Remark 18.2.3) that the category $\Omega_q^{\text{small}}\text{-FactMod}$ is equivalent to

$$\Omega_q^{\text{Whit}_{!*}}\text{-FactMod}(\text{Shv}_{\mathcal{G}^T}((\text{Gr}^{\omega^\rho}_{T,\text{Ran}})^{\text{non-pos}}_{\infty\cdot x}))_{\text{untl}}.$$

We introduce the t-structure on this category by declaring an object co-connective if its !-restriction to $\{x\} \subset \text{Ran}_x$, viewed as an object of

$$\text{Shv}_{\mathcal{G}^T}(\text{Gr}^{\omega^\rho}_{T,x}),$$

is coconnective with respect to the following t-structure:

For $\mu \in \Lambda$, on the connected component of $\text{Gr}^{\omega^\rho}_{T,x} \simeq \Lambda$, we shift the standard t-structure by $[-\langle \mu, 2\check{\rho}\rangle]$.

Having defined the t-structure, we can define the sought-for renormalization:

The renormalized category is the ind-completion of objects that are finite extensions of (shifts of) irreducible objects in

$$\left(\Omega_q^{\mathrm{Whit}_{!*}}\text{-FactMod}(\mathrm{Shv}_{\mathcal{G}^T}((\mathrm{Gr}^{\omega^\rho}_{T,\mathrm{Ran}})^{\text{non-pos}}_{\infty\cdot x}))_{\mathrm{untl}}\right)^{\heartsuit}.$$

19.2. **Statement of the theorem.** In this subsection we define a renormalized version of the functor $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}$, which will allow us to state our main result, Theorem 19.2.5.

19.2.1. Note that Corrollary 18.6.4 implies that the restriction of the functor $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}$ to

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))^c \subset \mathrm{Whit}_{q,x}(G)$$

takes values in

$$(\Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}})^c \subset \Omega_q^{\mathrm{small}}\text{-FactMod}.$$

Hence, ind-extension defines a functor

$$\overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}}$$

so that

$$\text{un-ren} \circ \overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}} \simeq \overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}.$$

19.2.2. By construction, the functor $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}}$ preserves compactness. This is the main difference between $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}}$ and $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}}$.

19.2.3. By construction, we have:

$$\overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}}) \simeq \mathcal{M}^{\mu,!*}_{\mathrm{Fact}}, \tag{19.2}$$

as objects of $\Omega_q^{\mathrm{small}}$-FactMod$^{\mathrm{ren}}$.

Since the t-structure on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G))$ is Artinian, we obtain that the functor $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}}$ is t-exact.

19.2.4. The main result of this work is the following:

Theorem 19.2.5. *The functor* $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke,ren}}_{\mathrm{Fact}}$ *is an equivalence.*

19.3. **Outline of the proof.** In this subsection we outline the main steps involved in the proof of Theorem 19.2.5.

19.3.1. First, by Propositions 11.3.6 and 11.2.5, we can assume that the derived group of H is simply-connected.

19.3.2. We will introduce objects

$$\mu \rightsquigarrow \overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}} \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$$

with the following properties:

- (i)
$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) = \begin{cases} \mathsf{e} \text{ if } \mu' = \mu \\ 0 \text{ otherwise.} \end{cases}$$
- (ii) The unique (up to a scalar) map $\overset{\bullet}{\mathcal{M}}{}^{\mu,!} \to \overset{\bullet}{\mathcal{M}}{}^{\mu,*}$ factors through the irreducible object $\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}}$, and the fiber (resp., cofiber) of the map from $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}$ to it (resp., from it to $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$) has a finite filtration with subquotients being irreducibles $\overset{\bullet}{\mathcal{M}}{}^{\mu',!*}_{\mathrm{Whit}}$ with $\mu' < \mu$.

- (iii)
$$\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}) \simeq \overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Conf}} \text{ and } \Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \simeq \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Conf}}$$
as objects of $\Omega_q^{\mathrm{small}}$-FactMod.

Let us show how having a collection of objects with the above properties implies Theorem 19.2.5.

19.3.3. First, property (ii) above implies that the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}$ and $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}$ are compact.

Let us note that this, combined with property (iii) and (19.2), implies the assertion of Proposition 19.1.4.

19.3.4. Next we note that the isomorphisms of property (iii) above imply that we also have
$$\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}) \simeq \overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Conf}} \text{ and } \Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \simeq \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Conf}}$$
as objects of $\Omega_q^{\mathrm{small}}$-FactMod$^{\mathrm{ren}}$, see Remark 19.1.10.

Proof of Theorem 19.2.5. In view of (19.2), it only remains to show that the functor $\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}$ is fully faithful. Since the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}$ compactly generate the category $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$, it suffices to show that the functor $\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}$ induces an isomorphism
$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!*}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}}) \to \mathcal{H}om_{\Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}}}(\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}), \Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}))$$
for all $\mu, \mu' \in \Lambda$.

However, by a standard argument with the 5-lemma, it follows from (ii) above that it is sufficient to show that the functor $\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}$ induces an isomorphism
$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \to \mathcal{H}om_{\Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}}}(\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}), \Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}))$$
for all $\mu, \mu' \in \Lambda$.

The latter is equivalent to showing that the functor $\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}}}$ induces an isomorphism
$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \to \mathcal{H}om_{\Omega_q^{\mathrm{small}}\text{-FactMod}}(\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}), \Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}))$$
for all $\mu, \mu' \in \Lambda$.

In view of properties (i) and (iii), and (5.8), both sides vanish for $\mu \neq \mu'$. So, it remains to consider the case of $\mu = \mu'$. We have to show that the resulting map
$$\mathsf{e} \simeq \mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \to \mathcal{H}om_{\Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Fact}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Fact}}) \simeq \mathsf{e}$$
is non-zero.

We argue by contradiction. If this map where zero, by property (ii) and the t-exactness of $\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}$, this would imply that $\Phi_{\mathrm{Fact}}^{\overset{\bullet}{\mathrm{Hecke}},\mathrm{ren}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}}) = 0$, which would contradict (19.2).

$\square$

Part VII: Zastava spaces and global interpretation of the Jacquet functor

The goal of this part is to prove Theorem 18.2.9, and its Hecke extension, Theorem 22.1.5. A salient feature of the proof is that it uses global methods, i.e., we will be working with a complete curve X.

20. Zastava spaces

In this section we will rewrite the Jacquet functor Φ in terms of the global version of the Whittaker category, $\mathrm{Whit}_{q,x}(G)$. The link between the local geometry (which involves $\mathfrak{L}(N)$-orbits on Gr_G) and the global one (which involves $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$) is provided by the *Zastava spaces*.

20.1. **Zastava spaces: local definition.** In this subsection we will interpret the spaces involved in the construction of the functor Φ as *Zastava spaces*.

20.1.1. The (completed) Zastava space, denoted $\overline{\mathcal{Z}}$ is defined to be

$$\overline{S}^0_{\mathrm{Conf}} \cap \overline{S}^{-,\mathrm{Conf}}_{\mathrm{Conf}},$$

where the intersection is taking place in $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$.

Let $\mathsf{v}_{\mathrm{Conf}}$ denote the projection $\overline{\mathcal{Z}} \to \mathrm{Conf}$. The properness of the affine Grassmannian implies that the map $\mathsf{v}_{\mathrm{Conf}}$ is ind-proper. However, we will soon see that $\overline{\mathcal{Z}}$ is actually a *scheme*, so that the map $\mathsf{v}_{\mathrm{Conf}}$ is actually a proper map of schemes.

20.1.2. Let $\mathcal{Z} \subset \overline{\mathcal{Z}}$ be the open subfunctor equal to

$$\overline{S}^0_{\mathrm{Conf}} \cap S^{-,\mathrm{Conf}}_{\mathrm{Conf}}.$$

Let $\overline{\overset{\circ}{\mathcal{Z}}} \subset \overline{\mathcal{Z}}$ be the open subfunctor equal to

$$S^0_{\mathrm{Conf}} \cap \overline{S}^{-,\mathrm{Conf}}_{\mathrm{Conf}}.$$

Finally, let

$$\overset{\circ}{\mathcal{Z}} := \mathcal{Z} \cap \overline{\overset{\circ}{\mathcal{Z}}} \subset \overline{\mathcal{Z}}.$$

20.1.3. We introduce the polar version of the Zastava spaces as follows:

$$\overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x} := (\overline{S}^0_{\mathrm{Conf}_{\infty\cdot x}})_{\infty\cdot x} \cap (\overline{S}^{-,\mathrm{Conf}_{\infty\cdot x}}_{\mathrm{Conf}_{\infty\cdot x}})_{\infty\cdot x},$$

where the intersection is taking place in $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}_{\infty\cdot x}}$.

Let $\mathsf{v}_{\mathrm{Conf}_{\infty\cdot x}}$ denote the projection $\overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x} \to \mathrm{Conf}_{\infty\cdot x}$. The properness of the affine Grassmannian implies that the map $\mathsf{v}_{\mathrm{Conf}_{\infty\cdot x}}$ is ind-proper.

20.1.4. We introduce the closed subspace $\overline{\mathcal{Z}}_{\infty\cdot x} \subset \overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x}$ to be

$$\overline{\mathcal{Z}}_{\infty\cdot x} := (\overline{S}^0_{\mathrm{Conf}_{\infty\cdot x}})_{\infty\cdot x} \cap (\overline{S}^{-,\mathrm{Conf}_{\infty\cdot x}}_{\mathrm{Conf}_{\infty\cdot x}}).$$

Remark 20.1.5. Note that the space $\overline{\mathcal{Z}}_{\infty\cdot x}$ was used in the definition of the functor Φ, see Sect. 18.3.7. Indeed, the support of objects of the form

$$\mathrm{sprd}_{\mathrm{Conf}_{\infty\cdot x}}(\mathcal{F}) \overset{!}{\otimes} \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}_{\infty\cdot x}}$$

is contained in $\overline{\mathcal{Z}}_{\infty\cdot x}$.

20.2. **Global interpretation of the Zastava spaces.** In this subsection we let the curve X be complete. We will reinterpret the various versions of the Zastava space in terms of moduli spaces of bundles on X.

20.2.1. Consider the algebraic stacks

$$\overline{\mathrm{Bun}}_N^{\omega^\rho} \xrightarrow{\overline{\mathfrak{p}}} \mathrm{Bun}_G \xleftarrow{\overline{\mathfrak{p}}^-} \overline{\mathrm{Bun}}_{B^-}.$$

Consider the open substack

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-})^{\mathrm{gen}} \subset \overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-}$$

corresponding to the condition that the (generic) N-reduction and the (generic) B^--reduction of a given G-bundles are *transversal at the generic point of* X.

The maps

$$\pi : \overline{S}^0_{\mathrm{Conf}} \to \overline{\mathrm{Bun}}_N$$

and

$$\pi^-_{\mathrm{Conf}} : \overline{S}^{-,\mathrm{Conf}}_{\mathrm{Conf}} \to \overline{\mathrm{Bun}}_{B^-}$$

(see Sect. 14.8.1) define a map

$$\overline{\mathcal{Z}} \to \overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-},$$

and it is easy to see that its image lands in $(\overline{\mathrm{Bun}}_N \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-})^{\mathrm{gen}}$.

We claim:

Proposition 20.2.2. *The map*

$$\overline{\mathcal{Z}} \to (\overline{\mathrm{Bun}}_N \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-})^{\mathrm{gen}}$$

is an isomorphism. Under this identification, the subspaces

$$\mathcal{Z} \subset \overline{\mathcal{Z}} \supset \overline{\overset{\circ}{\mathcal{Z}}}$$

correspond to

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \mathrm{Bun}_{B^-})^{\mathrm{gen}} \subset (\overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-})^{\mathrm{gen}} \supset (\mathrm{Bun}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-})^{\mathrm{gen}},$$

respectively.

Proof. Proceeding as in [Sch, Sect. 7], we can assume that the derived group of G is simply-connected. In this case, we will explicitly construct an inverse map.

An S-point of $(\overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-})^{\mathrm{gen}}$ consists of a G-bundle $\mathcal{P}_G$, equipped with a Plücker data for N and B^-

$$(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\}, \{\kappa^{-,\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\}),$$

where:

- $(\mathcal{P}_G, \{\kappa^{\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\})$ is as in the definition of $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$, i.e., this is a data of a G-bundle on X, equipped with a generalized reduction (a.k.a. Drinfeld structure) to N^{ω^ρ};
- $\{\kappa^{-,\check{\lambda}}, \check{\lambda} \in \check{\Lambda}^+\}$ is a generalized reduction of $\mathcal{P}_G$ to B^-, i.e., these are maps

$$'\mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \to \check{\lambda}(\mathcal{P}_T)$$

  for *some* T-bundle $\mathcal{P}_T$ on X that satisfy the Plucker relations (here $'\mathcal{V}^{\check{\lambda}}$ denotes the dual Weyl module with highest weight $\check{\lambda}$);
- The above generalized reductions of $\mathcal{P}_G$ to N^{ω^ρ} and B^- are mutually transversal away from a closed subset $S \times X$ that is *finite over* S.

The transversality condition means that the composite maps

$$(\omega^{\frac{1}{2}})^{\langle\check{\lambda},2\rho\rangle} \to \mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \to {}'\mathcal{V}^{\check{\lambda}}_{\mathcal{P}_G} \to \check{\lambda}(\mathcal{P}_T), \check{\lambda} \in \check{\Lambda}^+ \tag{20.1}$$

are isomorphisms away from a closed subset $S \times X$ that is *finite over* S.

The assumption that the derived group of G is simply connected implies (see Remark 4.1.3) that the system of maps (20.1) is equivalent to a datum of an S-point D of Conf.

By construction, over the open subset

$$S \times X - D \subset S \times X,$$

the maps $\kappa^{\check{\lambda}}$ and the maps $\kappa^{-,\check{\lambda}}$ are *bundle maps*, hence the (generalized) N-reduction and the (generalized) B^--reduction of $\mathcal{P}_G|_{S\times X}$ are *genuine* and mutually transversal. Hence, over this open subset these two reductions uniquely determine a trivialization of $\mathcal{P}_G$.

I.e., we obtain an S-point of $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$ that projects to the S-point of Conf, given by D. By construction, the above S-point of $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$ actually belongs to the subfunctor $\overline{\mathcal{Z}}$, as desired.

□

As an immediate corollary of Proposition 20.2.2, we obtain:

Corollary 20.2.3. *The prestack $\overline{\mathcal{Z}}$ is a scheme.*

Proof. Indeed, by construction, $\overline{\mathcal{Z}}$ is an ind-scheme, but Proposition 20.2.2 implies that it is also an algebraic stack.

□

20.2.4. We will also need a variant of the above picture with pole points. Consider the maps

$$(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \xrightarrow{\bar{\mathfrak{p}}} \mathrm{Bun}_G \xleftarrow{\bar{\mathfrak{p}}^-} (\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}.$$

We consider the corresponding open subfunctors

$$\left((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} (\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}\right)^{\mathrm{gen}} \subset (\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} (\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}$$

and

$$\left((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-}\right)^{\mathrm{gen}} \subset (\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-}.$$

We have the naturally defined maps

$$\overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x} \to \left((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} (\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}\right)^{\mathrm{gen}} \tag{20.2}$$

and

$$\overline{\mathcal{Z}}_{\infty\cdot x} \to \left((\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-}\right)^{\mathrm{gen}}. \tag{20.3}$$

As in Proposition 20.2.2, one shows:

Proposition 20.2.5. *The maps* (20.2) *and* (20.3) *are isomorphisms.*

20.2.6. Let $\mathcal{G}^G$ denote the gerbe on $\overline{\mathcal{Z}}$ (resp., $\overline{\mathcal{Z}}_{\infty\cdot x}$, $\overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x}$) obtained by restriction from the same-named gerbe on $\mathrm{Gr}^{\omega^\rho}_{G,\mathrm{Conf}}$. Let $\mathcal{G}^\Lambda$ denote the gerbe on $\overline{\mathcal{Z}}$ (resp., $\overline{\mathcal{Z}}_{\infty\cdot x}$, $\overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x}$) obtained as a pullback from the same-named gerbe on Conf (resp., $\mathrm{Conf}_{\infty\cdot x}$). Denote

$$\mathcal{G}^{G,T,\mathrm{ratio}} := \mathcal{G}^G \otimes (\mathcal{G}^T)^{-1}.$$

We obtain that under the identification of Proposition 20.2.2 (resp., (20.2) and (20.3)), $\mathcal{G}^G$ corresponds to the pullback of the same-named gerbe on $\overline{\mathrm{Bun}}^{\omega^\rho}_N$ (resp., $(\overline{\mathrm{Bun}}^{\omega^\rho}_N)_{\infty\cdot x}$), and $\mathcal{G}^{G,T,\mathrm{ratio}}$ corresponds to the pullback of the same-named gerbe on $\overline{\mathrm{Bun}}_{B^-}$ (resp., $(\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}$).

20.3. **Global interpretation of the functor Φ.** In this subsection we will use Proposition 20.2.5 to give a global interpretation of the functor Φ.

20.3.1. Consider the Cartesian square

$$\begin{tikzcd} \overline{\mathrm{Bun}}_N^{\omega^\rho} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-} \arrow[r, "'\overline{\mathsf{p}}"] \arrow[d, "'\overline{\mathsf{p}}^-"'] & \overline{\mathrm{Bun}}_{B^-} \arrow[d, "\overline{\mathsf{p}}^-"] \\ \overline{\mathrm{Bun}}_N^{\omega^\rho} \arrow[r, "\overline{\mathsf{p}}"] & \mathrm{Bun}_G . \end{tikzcd}$$

By a slight abuse of notation let us denote by the same symbols $'\overline{\mathsf{p}}^-$ and $'\overline{\mathsf{p}}$ the resulting maps

$$\overline{\mathrm{Bun}}_N^{\omega^\rho} \overset{'\overline{\mathsf{p}}^-}{\longleftarrow} \mathcal{Z} \overset{'\overline{\mathsf{p}}}{\longrightarrow} \overline{\mathrm{Bun}}_{B^-}$$

arising from the identification of Proposition 20.2.2.

We will use a similar notation also for the spaces $\overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x}$ and $\overline{\mathcal{Z}}_{\infty\cdot x}$.

20.3.2. Define the functor

$$\Phi_{\mathrm{glob}} : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

to be the composition

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \overset{('\overline{\mathsf{p}}^-)^!}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathcal{Z}}_{\infty\cdot x}) \overset{-\overset{!}{\otimes}('\overline{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)]}{\longrightarrow}$$
$$\to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\overline{\mathcal{Z}}_{\infty\cdot x}) \overset{(\mathsf{v}_{\mathrm{Conf}})_*}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}). \tag{20.4}$$

20.3.3. Recall the equivalence

$$\pi_x^! : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Whit}_q(G).$$

We claim:

Proposition 20.3.4. *The functor*

$$\Phi \circ \pi_x^![d_g] : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

identifies canonically with Φ_{glob}.

This proposition follows by applying $\mathsf{v}_{\mathrm{Conf}_x}$ from the following more precise result:

Proposition 20.3.5. *The composite functor*

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \overset{('\overline{\mathsf{p}}^-)^!}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathcal{Z}}_{\infty\cdot x}) \overset{-\overset{!}{\otimes}('\overline{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)]}{\longrightarrow}$$
$$\to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\overline{\mathcal{Z}}_{\infty\cdot x}) \tag{20.5}$$

identifies canonically with the composition of $\pi_x^![d_g]$ *and the functor*

$$\mathcal{F} \mapsto \mathrm{sprd}_{\mathrm{Conf}_{\infty\cdot x}}(\mathcal{F}) \overset{!}{\otimes} \mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}_{\infty\cdot x}}[\deg], \quad \mathrm{Whit}_{q,x}(G) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\overline{\mathcal{Z}}_{\infty\cdot x}). \tag{20.6}$$

Proof. First, we note that by Corrollary 14.8.2 (or, rather, its version with N replaced by N^-), the object

$$\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{Conf}_{\infty\cdot x}}|_{\overline{\mathcal{Z}}_{\infty\cdot x}}[\deg + d_g] \in \mathrm{Shv}_{(\mathcal{G}^G)^{-1}\otimes\mathcal{G}^\Lambda}(\overline{\mathcal{Z}}_{\infty\cdot x})$$

identifies with

$$('\overline{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)].$$

Hence, it remains to show that the composition

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \overset{\pi_x^!}{\longrightarrow} \mathrm{Whit}_q(G) \overset{\mathrm{sprd}_{\mathrm{Conf}_{\infty\cdot x}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Conf}_{\infty\cdot x}})_{\infty\cdot x}) \overset{!\text{-restriction}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathcal{Z}}_{\infty\cdot x})$$

identifies with the functor

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \overset{('\bar{\mathsf{p}}^-)^!}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}(\overline{\mathcal{Z}}_{\infty\cdot x}).$$

In fact, we claim that the functor

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \xrightarrow{\pi_x^!} \mathrm{Whit}_q(G) \overset{\mathrm{sprd}_{\mathrm{Conf}_{\infty\cdot x}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Conf}_{\infty\cdot x}})_{\infty\cdot x})$$

identifies canonically with

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \overset{\pi^!_{\mathrm{Conf}_{\infty\cdot x}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Conf}_{\infty\cdot x}})_{\infty\cdot x}).$$

For this, it suffices to show that the composition

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \xrightarrow{\pi_x^!} \mathrm{Whit}_q(G) \overset{\mathrm{sprd}_{\mathrm{Ran}_x}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Ran}_x})_{\infty\cdot x})$$

identifies canonically with

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}) \overset{\pi^!_{\mathrm{Ran}_x}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Conf}_{\mathrm{Ran}_x}})_{\infty\cdot x}).$$

The latter follows from Theorem 8.2.7 by composing both sides with the functor

$$\mathrm{unit}^! : \mathrm{Shv}_{\mathcal{G}^G}((\overline{S}^0_{\mathrm{Conf}_{\mathrm{Ran}_x}})_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x}).$$

Indeed, both sides are given by the forgetful functor $\mathrm{Whit}_{q,\mathrm{glob}}(G) \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})$, followed by pullback along

$$\mathrm{Ran}_x \times \mathrm{Gr}^{\omega^\rho}_{G,x} \to \mathrm{Gr}^{\omega^\rho}_{G,x} \xrightarrow{\pi_x} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}.$$

□

20.3.6. Let us observe that Proposition 20.3.4 immediately implies the first assertion Theorem 18.2.9, namely that the functor Φ sends compact objects in $\mathrm{Whit}_{q,x}(G)$ to locally compact ones in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$:

Indeed, over each connected component $\mathrm{Conf}^\lambda_{\infty\cdot x}$ of $\mathrm{Conf}_{\infty\cdot x}$, all the functors in the composition (20.4) preserve compactness.

20.4. **A sharpened version of Theorem 18.2.9.** In this subsection we will formulate a sharpened version of Theorem 18.2.9, and deduce from it the original statement.

20.4.1. Let

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}} \subset \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathcal{Z}_{\infty\cdot x}).$$

be the subcategory of Sect. 4.3.3; it consists of objects whose restriction to the preimage of every individual $\mathrm{Conf}^\lambda_{\infty\cdot x} \subset \mathrm{Conf}_{\infty\cdot x}$ is compact.

Since $\mathcal{Z}_{\infty\cdot x}$ is an ind-scheme, we have a well-defined Verdier duality functor

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}}. \tag{20.7}$$

We are going to prove the following result:

Theorem 20.4.2. *The following diagram of functors commutes*

$$\begin{tikzcd}
(\mathrm{Whit}_q(G)^c)^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{Verdier}}"] \arrow[d, "(20.6)"'] & \mathrm{Whit}_{q^{-1}}(G)^c \arrow[d, "(20.6)"'] \\
(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{Verdier}}"] & \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}}
\end{tikzcd}$$

commutes.

Note that combined with the properness of the map $\mathsf{v}_{\mathrm{Conf}_x}$, the assertion of Theorem 20.4.2 implies that of Theorem 18.2.9.

20.4.3. Note that statement of Theorem 20.4.2 is *local* (i.e., does not appeal to a global curve X). However, we will use global methods to prove it. Namely, we will use the interpretation of the functor (20.6) as (20.5) in order to prove it.

Note that, according to Remark 7.4.7, the functor $\pi_x^![d_g]$ intertwines the duality (7.10) with the Verdier duality functor

$$(\mathrm{Whit}_{q,\mathrm{glob}}(G)^c)^{\mathrm{op}} \to \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G)^c$$

induced by Verdier duality

$$\mathbb{D}^{\mathrm{Verdier}} : (\mathrm{Shv}_{\mathcal{G}^G}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x})^{\mathrm{loc.c}}.$$

Thus, using Proposition 20.3.5, we obtain that Theorem 20.4.2 is equivalent to the following one:

Theorem 20.4.4. *The following diagram of functors commutes*

$$\begin{tikzcd}
(\mathrm{Whit}_{q,\mathrm{glob}}(G)^c)^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{Verdier}}"] \arrow[d, "(20.5)"'] & \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G)^c \arrow[d, "(20.5)"'] \\
(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{Verdier}}"] & \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathcal{Z}_{\infty\cdot x})^{\mathrm{loc.c}}
\end{tikzcd}$$

commutes.

We will prove Theorem 20.4.4 in the next section, using the notion of ULA (universal local acyclicity).

20.4.5. For future reference we record the following corollary of Theorem 20.4.4 (which, given Proposition 20.3.4, is equivalent to Theorem 18.2.9):

Corollary 20.4.6. *The diagram*

$$\begin{tikzcd}
(\mathrm{Whit}_{q,\mathrm{glob}}(G)^c)^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{Verdier}}"] \arrow[d, "\Phi_{\mathrm{glob}}"'] & \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G)^c \arrow[d, "\Phi_{\mathrm{glob}}"] \\
(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} \arrow[r, "\mathbb{D}^{\mathrm{Verdier}}"] & \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}}.
\end{tikzcd}$$

21. Proof of the local Verdier duality theorem

The goal of this section is to prove Theorem 20.4.4. The main ingredient in the proof is a certain local acyclicity (a.k.a. ULA) property of

$$_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}} \in \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\overline{\mathrm{Bun}}_{B^-})$$

with respect to the projection

$$\overline{\mathsf{p}}^- : \overline{\mathrm{Bun}}_{B^-} \to \mathrm{Bun}_G .$$

21.1. **Construction of the natural transformation.** In this subsection we will describe a framework that leads to the construction of a natural transformation in Theorem 20.4.4.

21.1.1. Let us be given a Cartesian diagram of algebraic stacks

$$\begin{tikzcd}
\mathcal{Y}_1 \underset{\mathcal{Y}}{\times} \mathcal{Y}_2 \arrow[r, "'f"] \arrow[d, "'f^-"'] & \mathcal{Y}_2 \arrow[d, "f^-"] \\
\mathcal{Y}_1 \arrow[r, "f"] & \mathcal{Y},
\end{tikzcd}$$

where $\mathcal{Y}$ is smooth of dimension d. Note that for $\mathcal{F}_i \in \mathrm{Shv}_{\mathcal{G}_i}(\mathcal{Y}_i)^{\mathrm{loc.c}}$ there exists a canonically defined map in $\mathrm{Shv}_{\mathcal{G}_1\otimes\mathcal{G}_2}(\mathcal{Y}_1 \underset{\mathcal{Y}}{\times} \mathcal{Y}_2)$

$$('f^-)^*(\mathcal{F}_1) \overset{*}{\otimes} ('f)^*(\mathcal{F}_2)[-d] \to ('f^-)^!(\mathcal{F}_1) \overset{!}{\otimes} ('f)^!(\mathcal{F}_2)[d], \tag{21.1}$$

see [BG, Sect. 5.1].

In particular, we obtain that there exists a natural transformation

$$\mathbb{D}^{\mathrm{Verdier}}(('f^-)^!(\mathcal{F}_1) \overset{!}{\otimes} ('f)^!(\mathcal{F}_2)[d]) \to ('f^-)^!(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}_1)) \overset{!}{\otimes} ('f)^!(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}_2))[d]. \tag{21.2}$$

21.1.2. We apply the above paradigm to

$$\begin{CD} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-} @>{'\bar{\mathsf{p}}}>> \overline{\mathrm{Bun}}_{B^-} \\ @V{'\bar{\mathsf{p}}^-}VV @VV{\bar{\mathsf{p}}^-}V \\ (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x} @>{\bar{\mathsf{p}}}>> \mathrm{Bun}_G, \end{CD} \tag{21.3}$$

where we take $\mathcal{F}_1 = \mathcal{F} \in \mathrm{Whit}_{q,\mathrm{glob}}(G)$ and $\mathcal{F}_2 = {}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}}$.

We compose the two functors in (21.2) with restriction along the open embedding

$$\overline{\mathcal{Z}}_{\infty\cdot x} \hookrightarrow (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-}.$$

We will prove the following more precise version of Theorem 20.4.2:

Theorem 21.1.3. *The natural transformation*

$$\mathbb{D}^{\mathrm{Verdier}}\left(('\bar{\mathsf{p}}^-)^!(\mathcal{F}) \overset{!}{\otimes} ('\bar{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)]\right) \to \tag{21.4}$$
$$\to ('\bar{\mathsf{p}}^-)^!(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \overset{!}{\otimes} ('\bar{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)],$$

arising from (21.2) *is an isomorphism in* $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\overline{\mathcal{Z}}_{\infty\cdot x})$ *for any* $\mathcal{F} \in \mathrm{Whit}_{q,\mathrm{glob}}(G)^c$.

21.2. Proof of Theorem 21.1.3, Step 1.

21.2.1. With no restriction of generality, we can assume that $\mathcal{F} \in \mathrm{Whit}_{q,\mathrm{glob}}(G)$ is perverse and irreducible. Hence it is of the form $W^{\mu,!*}_{\mathrm{glob}}$ for some $\mu \in \Lambda^+$, see Sect. 7.3.9.

Then $\mathcal{F}$ is the Goresky-MacPherson extension of its restriction to the locally closed substack $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu\cdot x}$. Let $\overline{\mathcal{Z}}_{=\mu\cdot x}$ be the corresponding locally closed substack of $\overline{\mathcal{Z}}_{\infty\cdot x}$.

The key step in the proof of Theorem 21.1.3 is the following:

(*) *The object*

$$('\bar{\mathsf{p}}^-)^!(\mathcal{F}) \overset{!}{\otimes} ('\bar{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)] \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\overline{\mathcal{Z}}_{\infty\cdot x}) \tag{21.5}$$

is perverse and is isomorphic to the Goresky-MacPherson extension of its restriction to $\overline{\mathcal{Z}}_{=\mu\cdot x}$.

In the rest of this subsection we will show how (*) implies the assertion of Theorem 21.1.3.

21.2.2. By (*), we know that both sides in (21.4) are the Goresky-MacPherson extensions of their respective restrictions to $\overline{\mathcal{Z}}_{=\mu\cdot x}$. Hence, it is enough to show that the corresponding map in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\overline{\mathcal{Z}}_{=\mu\cdot x})$ is an isomorphism. Thus we can replace the diagram (21.3) by

$$\begin{CD} (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu\cdot x} \underset{\mathrm{Bun}_G}{\times} \overline{\mathrm{Bun}}_{B^-} @>{'\bar{\mathsf{p}}}>> \overline{\mathrm{Bun}}_{B^-} \\ @V{'\bar{\mathsf{p}}^-}VV @VV{\bar{\mathsf{p}}^-}V \\ (\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu\cdot x} @>{\bar{\mathsf{p}}}>> \mathrm{Bun}_G, \end{CD}$$

Since $\mathcal{F}|_{(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu\cdot x}}$ is lisse and $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu\cdot x}$ is smooth, the statement about the isomorphism is equivalent to one when $\mathcal{F}$ is replaced by the constant/dualizing sheaf.

Thus, we have to show that the map

$$\mathbb{D}^{\mathrm{Verdier}}\left(('\overline{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)-d_{g,\mu}]\right)\to$$
$$\to ('\overline{\mathsf{p}}^-)^!(\mathbb{D}^{\mathrm{Verdier}}({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}}))[\dim(\mathrm{Bun}_G)-d_{g,\mu}]$$

is an isomorphism, where $d_{g,\mu}=\dim((\overline{\mathrm{Bun}}_N^{\omega^\rho})_{=\mu\cdot x})$.

21.2.3. By (the metaplectic analog of) [Ga6, Proposition 3.6.5], the object

$$('\overline{\mathsf{p}})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}})[\dim(\mathrm{Bun}_G)-d_{g,\mu}]$$

is perverse and is isomorphic to the Goresky-MacPherson extension of its restriction to

$$\mathcal{Z}_{=\mu\cdot x}:=\overline{\mathcal{Z}}_{=\mu\cdot x}\underset{\overline{\mathrm{Bun}}_{B^-}}{\times}\mathrm{Bun}_{B^-}.$$

Hence, it is enough to show that the map

$$\mathbb{D}^{\mathrm{Verdier}}\left(('\mathsf{p})^!({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}}|_{\mathrm{Bun}_{B^-}})[\dim(\mathrm{Bun}_G)-d_{g,\mu}]\right)\to$$
$$\to ('\mathsf{p}^-)^!(\mathbb{D}^{\mathrm{Verdier}}({}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}}|_{\mathrm{Bun}_{B^-}}))[\dim(\mathrm{Bun}_G)-d_{g,\mu}] \quad (21.6)$$

is an isomorphism in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathcal{Z}_{=\mu\cdot x})$.

Note now that the object ${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q^{-1},\mathrm{glob}}|_{\mathrm{Bun}_{B^-}}$ is the constant sheaf (up to a cohomological shift). Now the desired assertion follows from the next result:

Lemma 21.2.4. *In the setting of Sect. 21.1.1, assume that $\mathcal{Y}_1$, $\mathcal{Y}_2$ and $\mathcal{Y}$ are equidimensional and smooth, and let*

$$\mathcal{Y}'\hookrightarrow\mathcal{Y}_1\underset{\mathcal{Y}}{\times}\mathcal{Y}_2$$

be a smooth open substack of dimension $\dim(\mathcal{Y}_1)+\dim(\mathcal{Y}_2)-\dim(\mathcal{Y})$. *Then for $\mathcal{F}_1$ and $\mathcal{F}_2$ lisse, the restriction of the map* (21.2) *to $\mathcal{Y}'$ is an isomorphism.*

21.3. **Proof of Theorem 21.1.3, Step 2.** In this subsection we will show how a ULA property of the global $\mathrm{IC}^{\frac{\infty}{2}}$ implies statement (*) used in the previous subsection.

21.3.1. Let us be in the situation of Sect. 21.1.1. Let $\overset{\circ}{\mathcal{Y}}_1\subset\mathcal{Y}_1$ be a locally closed substack. Assume that $\mathcal{F}_1\in\mathrm{Shv}_{\mathcal{G}_1}(\mathcal{Y}_1)$ is perverse and is isomorphic to the Goresky-MacPherson extension of its restriction to $\overset{\circ}{\mathcal{Y}}_1$.

Recall the notion of *ULA object* for $\mathcal{F}\in\mathrm{Shv}_{\mathcal{G}}(\mathcal{Y}')$ relative to a morphism $\mathcal{Y}'\to\mathcal{Y}$, see [BG, Sect. 5.1]. A key observation that we will use in the proof of Theorem 21.1.3 is the following:

Lemma 21.3.2. *Assume that $\mathcal{F}_2\in\mathrm{Perv}_{\mathcal{G}_2}(\mathcal{Y}_2)$ is ULA with respect to f^-. Then the object*

$$('f^-)^!(\mathcal{F}_1)\overset{!}{\otimes}('f)^!(\mathcal{F}_2)[d]$$

is perverse and is isomorphic to the Goresky-MacPherson extension of its restriction to $\overset{\circ}{\mathcal{Y}}_1\underset{\mathcal{Y}}{\times}\mathcal{Y}_2$.

The proof of property (*) will consist of reducing to the situation in which we can apply Lemma 21.3.2.

21.3.3. We now proceed with the proof of (*). For $\nu \in \Lambda$, denote

$$\overline{\mathcal{Z}}^{\nu}_{\infty\cdot x} := \overline{\mathcal{Z}}_{\infty\cdot x} \underset{\mathrm{Conf}_{\infty\cdot x}}{\times} \mathrm{Conf}^{\nu}_{\infty\cdot x}.$$

Let $\mathcal{F}^{\nu}$ denote the restriction of (21.5) to $\overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}$.

For an element $\nu' \in \Lambda^{\mathrm{neg}}$, consider the factorization isomorphism

$$(\overline{\mathcal{Z}}^{\nu'} \times \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}) \underset{\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}} \simeq \overline{\mathcal{Z}}^{\nu'+\nu}_{\infty\cdot x} \underset{\mathrm{Conf}^{\nu'+\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}}$$

and consider the open substack of the LHS equal to

$$(\overset{\circ}{\mathcal{Z}}{}^{\nu'} \times \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}) \underset{\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}}. \tag{21.7}$$

We will consider the following two maps from (21.7) to $\overline{\mathcal{Z}}_{\infty\cdot x}$. One, denoted $f_1^{\nu'}$, is the projection on the second factor

$$(\overset{\circ}{\mathcal{Z}}{}^{\nu'} \times \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}) \underset{\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}} \to \overset{\circ}{\mathcal{Z}}{}^{\nu'} \times \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x} \to \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x} \tag{21.8}$$

and the other, denoted $f_2^{\nu'}$, the composite

$$\begin{aligned}(\overset{\circ}{\mathcal{Z}}{}^{\nu'} \times \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}) &\underset{\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}} \hookrightarrow \\ &\to (\overline{\mathcal{Z}}^{\nu'} \times \overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}) \underset{\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}} \simeq \\ &\simeq \overline{\mathcal{Z}}^{\nu'+\nu}_{\infty\cdot x} \underset{\mathrm{Conf}^{\nu'+\nu}_{\infty\cdot x}}{\times} \left(\mathrm{Conf}^{\nu'} \times \mathrm{Conf}^{\nu}_{\infty\cdot x}\right)_{\mathrm{disj}} \to \overline{\mathcal{Z}}^{\nu'+\nu}_{\infty\cdot x}.\end{aligned} \tag{21.9}$$

Both these maps are smooth.

21.3.4. Fix an element $\lambda \in \Lambda^{\mathrm{pos}}$. We will consider an open substack in (21.7), to be denoted $\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}$, that consists of points satisfying the following conditions:

(i) We require that for the point of $\overline{\mathcal{Z}}_{\infty\cdot x}$ (obtained by either (21.8) or (21.9)), the generalized B^--reduction has total order of degeneracy is $\leq \lambda$.

(ii) We require that $-\nu' - \nu - \lambda$ be "deep enough" in the dominant chamber, in the sense that

$$\langle -\nu' - \nu - \lambda, \check{\alpha}_i \rangle > d$$

for some fixed integer d (specified in Theorem 21.3.7 below).

It is easy to see that the union of all λ's and ν' of the images of $\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}$ under the maps $f_1^{\nu'}$ cover $\overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}$.

21.3.5. Hence, in order to prove (*), it suffices to show that for all ν', the object

$$(f_1^{\nu'})^!(\mathcal{F}^{\nu})|_{\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}}, \tag{21.10}$$

shifted cohomologically by $[\langle \nu', 2\check{\rho}\rangle] = [-\dim(\overset{\circ}{\mathcal{Z}}{}^{\nu'})]$, is perverse and is isomorphic to the Goresky-MacPherson extension of its restriction to

$$\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}_{=\mu\cdot x} := \mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda} \underset{f_1^{\nu'},\overline{\mathcal{Z}}^{\nu}_{\infty\cdot x}}{\times} \overline{\mathcal{Z}}^{\nu}_{\mu\cdot x}.$$

21.3.6. Note now that the locally closed substack $\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}_{=\mu\cdot x} \subset \mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}$ also equals

$$\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda} \underset{f_2^{\nu'},\overline{\mathcal{Z}}^{\nu'+\nu}_{\infty\cdot x}}{\times} \overline{\mathcal{Z}}^{\nu'+\nu}_{\mu\cdot x}.$$

Now, by repeating the argument of [Ga6, Sect. 3.9], one shows that there exists an isomorphism

$$(f_2^{\nu'})^!(\mathcal{F}^{\nu}) \simeq \mathcal{E} \otimes (f_1^{\nu'})^!(\mathcal{F}^{\nu}),$$

where $\mathcal{E}$ is *lisse sheaf*, pulled back from the $\overset{\circ}{\mathcal{Z}}{}^{\nu'}$ factor, and placed in perverse cohomological degree $-\langle \nu', 2\check{\rho}\rangle$.

Hence, in order to prove the desired property of (21.10), it suffices to establish the same property of

$$(f_2^{\nu'})^!(\mathcal{F}^{\nu})|_{\mathcal{Z}^{\mathrm{Fact},\nu',\leq\lambda}}.$$

Now, the required assertion follows from Lemma 21.3.2 and the following result of [Camp]:

Theorem 21.3.7. *There exists an integer d that only depends on the genus of X with the following property: for any $\lambda \in \Lambda^{\mathrm{pos}}$ and $\mu \in \Lambda$, satisfying $\langle \mu - \lambda, \check{\alpha}_i\rangle > d$, the restriction of ${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2},-}_{q,\mathrm{glob}}$ to the open locus of $\overline{\mathrm{Bun}}_{B^-}$ consisting of generalized B^--reductions of total order of degeneracy $\leq \lambda$ and degree μ is ULA with respect to the projection*

$$\overline{\mathsf{p}}^- : \overline{\mathrm{Bun}}_{B^-} \to \mathrm{Bun}_G.$$

22. Hecke enhancement of the Verdier duality theorem

Recall the functor

$$\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}} = \mathbf{oblv}_{\mathrm{Fact}} \circ \overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q,x}(G)) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

For the proof of the main theorem (more precisely, for property (iii) for $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}$ in Sect. 19.3.2), we will need an extension of Theorem 18.2.9 for the functor $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}$. This is the subject of the present section.

More precisely, we will state and prove Theorem 22.1.5, which will be used in the proof of Corrollary 26.1.5.

22.1. **Hecke enhancement and duality.** In this subsection we will state Theorem 22.1.5, which expresses the commutation property of the functor $\overset{\bullet}{\Phi}{}^{\mathrm{Hecke}}$ with Verdier duality.

22.1.1. Note that Theorem 16.1.2 supplies a system of functorial isomorphisms

$$\Phi(\mathcal{F} \star \mathrm{Sat}_{q,G}(V)) \simeq \mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V)) \star \Phi(\mathcal{F}), \quad \mathcal{F} \in \mathrm{Whit}_q(G),\ V \in \mathrm{Rep}(H), \tag{22.1}$$

compatible with tensor products of objects $V \in \mathrm{Rep}(H)$.

In addition, Theorem 18.2.9 establishes functorial isomorphisms

$$\mathbb{D}^{\mathrm{Verdier}}(\Phi(\mathcal{F} \star \mathrm{Sat}_{q,G}(V))) \simeq \Phi\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F} \star \mathrm{Sat}_{q,G}(V))\right) \simeq \Phi\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \star \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,G}(V))\right) \simeq$$
$$\simeq \Phi\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \star \mathrm{Sat}_{q^{-1},G}(\tau^H(V^*))\right)$$

and

$$\mathbb{D}^{\mathrm{Verdier}}\left(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V)) \star \Phi(\mathcal{F})\right) \simeq \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V))) \star \mathbb{D}^{\mathrm{Verdier}}(\Phi(\mathcal{F})) \simeq$$
$$\simeq \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V))) \star \Phi(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \simeq \mathrm{Sat}_{q^{-1},T}(\tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*))) \star \Phi(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})).$$

where we assume that both $\mathcal{F}$ and V are compact.

22.1.2. Thus, on the one hand, applying $\mathbb{D}^{\mathrm{Verdier}}$ to both sides of (22.1), we obtain a system of functorial isomorphisms

$$\Phi\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \star \mathrm{Sat}_{q^{-1},G}(\tau^H(V^*))\right) \simeq \mathrm{Sat}_{q^{-1},T}(\tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*))) \star \Phi(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})), \tag{22.2}$$

compatible with tensor products of objects $V \in \mathrm{Rep}(H)$.

On the other hand, applying Theorem 16.1.2 to the functor

$$\Phi : \mathrm{Whit}_{q^{-1}}(G) \to \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x}),$$

we obtain a system of identifications

$$\Phi\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \star \mathrm{Sat}_{q^{-1},G}(\tau^H(V^*))\right) \simeq \mathrm{Sat}_{q^{-1},T}(\mathrm{Res}^H_{T_H}(\tau^H(V^*))) \star \Phi(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})), \tag{22.3}$$

compatible with tensor products of objects $V \in \mathrm{Rep}(H)$.

We claim:

Theorem 22.1.3. *The identifications* (22.2) *and* (22.3) *are compatible via the canonical isomorphism*

$$\tau^{T_H} \circ \mathrm{Res}^H_{T_H} \simeq \mathrm{Res}^H_{T_H} \circ \tau^H.$$

These identifications satisfy a homotopy-coherent system of comptatibilities for tensor products of the objects $V \in \mathrm{Rep}(H)^c$.

22.1.4. As a formal consequence of Theorem 22.1.3 we obtain:

Theorem 22.1.5. *We have a commutative diagram*

$$\begin{array}{ccc}
(\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))^c)^{\mathrm{op}} & \xrightarrow{\mathbb{D}^{\mathrm{Verdier}}} & \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q^{-1}}(G)) \\
{\scriptstyle \Phi^{\overset{\bullet}{\mathrm{Hecke}}}}\big\downarrow & & \big\downarrow{\scriptstyle \Phi^{\overset{\bullet}{\mathrm{Hecke}}}} \\
(\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}})^{\mathrm{op}} & \xrightarrow{\mathbb{D}^{\mathrm{Verdier}}} & \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x})^{\mathrm{loc.c}},
\end{array}$$

where the upper horizontal arrow is the equivalence (11.2).

22.1.6. The rest of this subsection is devoted to the proof of Theorem 22.1.3. As the isomorphism of Theorem 18.2.9 was proved by global methods, we will have to resort to global methods to prove Theorem 22.1.3. Thus, for the rest of this section the curve X will be complete.

22.2. **Hecke structure on the functor Φ_{glob}.** Recall the functor

$$\Phi_{\mathrm{glob}} : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Shv}_{\mathcal{G}^\lambda}(\mathrm{Conf}_{\infty\cdot x}),$$

see Sect. 20.3.2. As a first step towards the proof of Theorem 22.1.3, we will replace it by a statement that involves Φ_{glob} instead of Φ.

22.2.1. The Hecke action on $(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x}$ makes $\mathrm{Whit}_{q,\mathrm{glob}}(G)$ into a category acted on by $\mathrm{Sph}_{q,x}(G)$ on the right (see Remark 14.7.2). In particular, we obtain a $\mathrm{Rep}(H)$ on $\mathrm{Whit}_{q,\mathrm{glob}}(G)$ via $\mathrm{Sat}_{q,G}$.

We claim that an analog of Theorem 16.1.2 holds in this situation:

Theorem-Construction 22.2.2. *The functor Φ_{glob} intertwines the $\mathrm{Rep}(H)$-action on $\mathrm{Whit}_{q,x}(G)$ and the $\mathrm{Rep}(T_H)$-action on $\mathrm{Shv}_{\mathcal{G}^\lambda}(\mathrm{Conf}_{\infty\cdot x})$ given by translation functors (see Sect. 4.4.2).*

Proof. We employ the paradigm of Sect. 16.1.5. We take $\mathbf{C} := \mathrm{Whit}_{q,\mathrm{glob}}(G)$, $\mathbf{E} := \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$ and

$$\mathbf{D} := \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}).$$

We take the functor

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \otimes \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}) \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

to be

$$\mathcal{F}, \mathcal{F}' \mapsto (\mathsf{v}_{\mathrm{Conf}_{\infty\cdot x}})_* \left(('\overline{\mathsf{p}}^-)^!(\mathcal{F}) \overset{!}{\otimes} ('\overline{\mathsf{p}})^!(\mathcal{F}')[\dim(\mathrm{Bun}_G)] \right), \tag{22.4}$$

where $'\overline{\mathsf{p}}^-$ and $'\overline{\mathsf{p}}$ denote the two projections

$$(\overline{\mathrm{Bun}}_N^{\omega^\rho})_{\infty\cdot x} \xleftarrow{'\overline{\mathsf{p}}^-} \overline{\mathcal{Z}}_{\infty\cdot x,\infty\cdot x} \xrightarrow{'\overline{\mathsf{p}}} (\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}.$$

The Hecke actions for G and T on $(\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}$ make $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$ into a module category for $\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)$ (see Remark 14.7.2).

We take $\mathbf{d} \in \overset{\bullet}{\mathrm{Hecke}}(\mathbf{D})$ to be the object

$$_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}} \in \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$$

of Theorem 14.7.4.

One shows that the required compatibilities hold as in the local case. This gives the functor Φ_{glob} the required structure.

□

22.2.3. Recall now that according to Proposition 20.3.4, we have a canonical isomorphism

$$\Phi_{\mathrm{glob}} \simeq \Phi \circ \pi_x^![d_g].$$

Since the functor

$$\pi_x^![d_g] : \mathrm{Whit}_{q,\mathrm{glob}}(G) \to \mathrm{Whit}_q(G)$$

commutes with Hecke actions, the structure on of commutation with the action of $\mathrm{Rep}(H)$ on Φ, provided by Theorem 16.1.2 induces one on Φ_{glob}.

However, if follows from the constructions that this structure on Φ_{glob} equals one constructed in Sect. 22.2.1 above.

22.2.4. Since $\pi_x^![d_g]$ is an equivalence that commutes with duality, we obtain that Theorem 22.1.3 is equivalent to the corresponding statement for Φ_{glob}:

Theorem 22.2.5. *The following diagram commutes for $\mathcal{F} \in \mathrm{Whit}_{q,\mathrm{glob}}(G)^c$ and $V \in \mathrm{Rep}(H)^c$.*

$$\begin{tikzcd}
\mathbb{D}^{\mathrm{Verdier}}(\Phi_{\mathrm{glob}}(\mathcal{F}\star\mathrm{Sat}_{q,G}(V))) \arrow[r, "\textit{Thm. 22.2.2}"] \arrow[d, "\textit{Cor. 20.4.6}"'] & \mathbb{D}^{\mathrm{Verdier}}\left(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V))\star\Phi_{\mathrm{glob}}(\mathcal{F})\right) \arrow[d] \\
\Phi_{\mathrm{glob}}\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}\star\mathrm{Sat}_{q,G}(V))\right) \arrow[d] & \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V)))\star\mathbb{D}^{\mathrm{Verdier}}(\Phi_{\mathrm{glob}}(\mathcal{F})) \arrow[d, "\textit{Cor. 20.4.6}"] \\
\Phi_{\mathrm{glob}}\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})\star\mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,G}(V))\right) \arrow[dd] & \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V)))\star\Phi_{\mathrm{glob}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \arrow[d] \\
& \mathrm{Sat}_{q^{-1},T}(\tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*)))\star\Phi_{\mathrm{glob}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})) \arrow[d] \\
\Phi_{\mathrm{glob}}\left(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})\star\mathrm{Sat}_{q^{-1},G}(\tau^H(V^*))\right) \arrow[r, "\textit{Thm. 22.2.2}"] & \mathrm{Sat}_{q^{-1},T}(\mathrm{Res}^H_{T_H}(\tau^H(V^*)))\star\Phi_{\mathrm{glob}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F})).
\end{tikzcd}$$

The commutation identifications satisfy a homotopy-coherent system of comptatibilities for tensor products of the objects $V \in \mathrm{Rep}(H)^c$.

The rest of this section is devoted to Theorem 22.2.5.

22.3. **A framework for commutation of Hecke structure with duality.** In this subsection we will describe a general categorical framework for the proof of Theorem 22.2.5.

22.3.1. Let us be in the paradigm of Sect. 16.1.5. Assume that all categories involved are compactly generated.

Let us consider $\mathbf{C}^\vee$ and $\mathbf{D}^\vee$ equipped with actions of $\mathrm{Rep}(H)$ given on compact objects by the formula

$$\mathbf{c}^\vee \star V = (\mathbf{c} \star \tau^H(V^*))^\vee,\ V \star \mathbf{d}^\vee = (\tau^H(V^*) \star \mathbf{d})^\vee,$$

and let us equip $\mathbf{D}^\vee$ with an action of $\mathrm{Rep}(T_H)$ by the formula

$$\mathbf{d}^\vee \star W = (\mathbf{d} \star \tau^H(W^*))^\vee.$$

22.3.2. Assume that the functor $\Psi : \mathbf{C} \otimes \mathbf{D} \to \mathbf{E}$ preserves compactness, and let Υ denote the resulting functor

$$\mathbf{C}^\vee \otimes \mathbf{D}^\vee \to \mathbf{E}^\vee, \quad \Upsilon(\mathbf{c}^\vee \otimes \mathbf{d}^\vee) := (\Psi(\mathbf{c} \otimes \mathbf{d}))^\vee.$$

Identifying

$$\mathbf{C}^\vee \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}^\vee \simeq (\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D})^\vee, \quad \mathbf{c}^\vee \otimes \mathbf{d}^\vee \mapsto (\mathbf{c} \otimes \mathbf{d})^\vee,$$

we obtain that the functor Υ is also equipped with a factorization

$$\mathbf{C}^\vee \otimes \mathbf{D}^\vee \to \mathbf{C}^\vee \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}^\vee \xrightarrow{\widetilde{\Upsilon}} \mathbf{E}$$

for some canonically defined functor $\widetilde{\Upsilon}$.

Let now $\mathbf{d}$ be an object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D})$. (We are not assuming that $\mathbf{d}$ is compact in $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D})$, and a fortiori not in $\mathbf{D}$.)

22.3.3. Recall that to *any* object $\mathbf{f}$ in a compactly generated category $\mathbf{F}$ one can attach its dual $\mathbf{f}^\vee \in \mathbf{F}^\vee$, characterized uniquely by the property that

$$\mathcal{H}om_{\mathbf{F}^\vee}(\mathbf{f}_1^\vee, \mathbf{f}^\vee) = \mathcal{H}om_{\mathbf{F}}(\mathbf{f}, \mathbf{f}_1), \quad \mathbf{f}_1 \in \mathbf{F}^c.$$

Explicitly, if $\mathbf{f} = \underset{k}{\mathrm{colim}}\, \mathbf{f}_k$, then

$$\mathbf{f}^\vee \simeq \lim_k \mathbf{f}_k^\vee.$$

Let $\Psi : \mathbf{F} \to \mathbf{E}$ be a continuous functor that preserves compactness. Consider the corresponding functor

$$\Upsilon : \mathbf{F}^\vee \to \mathbf{E}^\vee, \quad \Upsilon(\mathbf{f}_1^\vee) := (\Psi(\mathbf{f}_1))^\vee, \quad \mathbf{f}_1 \in \mathbf{F}^c.$$

Then we have a natural map

$$\Upsilon(\mathbf{f}^\vee) \to \Psi(\mathbf{f})^\vee. \tag{22.5}$$

This map is an isomorphism if Ψ also commutes with *limits*.

22.3.4. Let $\mathbf{d}^\vee$ be the corresponding object of $\mathbf{D}^\vee$. Note that for any finite-dimensional $V \in \mathrm{Rep}(H)$ (resp., $W \in \mathrm{Rep}(T_H)$), the canonical maps

$$V \star \mathbf{d}^\vee \to (\tau^H(V^*) \star \mathbf{d})^\vee \text{ and } \mathbf{d}^\vee \star W = (\mathbf{d} \star \tau^{T_H}(W^*))^\vee$$

coming from (22.5) are isomorphisms (indeed, the functors $V \star -$ and $- \star W$ admit left adjoints and hence commute with limits).

Thus, we obtain that the object $\mathbf{d}^\vee \in \mathbf{D}^\vee$ is a system of isomorphisms

$$V \star \mathbf{d}^\vee \simeq \mathbf{d}^\vee \star \tau^{T_H}(\mathrm{Res}^H_{T_H}(\tau^H(V))).$$

We identify

$$\tau^{T_H}(\mathrm{Res}^H_{T_H}(\tau^H(V))) \simeq \mathrm{Res}^H_{T_H}(V).$$

This identification defines a lift of $\mathbf{d}^\vee$ to an object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}^\vee)$.

22.3.5. Consider the resulting functors

$$\Psi_{\mathbf{d}} : \mathbf{C} \to \mathbf{E} \text{ and } \Upsilon_{\mathbf{d}^\vee} : \mathbf{C}^\vee \to \mathbf{E}^\vee.$$

From (22.5), for a compact $\mathbf{c} \in \mathbf{C}$ we obtain a naturally defined map

$$\Upsilon_{\mathbf{d}^\vee}(\mathbf{c}^\vee) \to (\Psi_{\mathbf{d}}(\mathbf{c}))^\vee. \tag{22.6}$$

Assume that these maps are isomorphisms.

22.3.6. By Sect. 16.1.5, we have the isomorphisms

$$\Psi_{\mathbf{d}}(\mathbf{c} \star V) \simeq \mathrm{Res}^H_{T_H}(V) \star \Psi_{\mathbf{d}}(\mathbf{c}),$$

from which by duality we obtain the isomorphisms

$$(\Psi_{\mathbf{d}}(\mathbf{c} \star V))^\vee \simeq \tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*)) \star (\Psi_{\mathbf{d}}(\mathbf{c}))^\vee. \tag{22.7}$$

We also have the isomorphisms

$$\Upsilon_{\mathbf{d}^\vee}(\mathbf{c}^\vee \star V) \simeq \mathrm{Res}^H_{T_H}(V) \star \Upsilon_{\mathbf{d}^\vee}(\mathbf{c}^\vee). \tag{22.8}$$

Unwinding the constructions, we obtain that the following diagrams are commutative:

$$\begin{tikzcd}
(\Psi_{\mathbf{d}}(\mathbf{c} \star V))^\vee \arrow[r, "(22.7)"] & \tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*)) \star (\Psi_{\mathbf{d}}(\mathbf{c}))^\vee \\
\Upsilon_{\mathbf{d}^\vee}((\mathbf{c} \star V)^\vee) \arrow[u, "(22.6)"] & \tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*)) \star \Upsilon_{\mathbf{d}^\vee}(\mathbf{c}^\vee) \arrow[u, "(22.6)"'] \\
\Upsilon_{\mathbf{d}^\vee}(\mathbf{c}^\vee \star \tau^H(V^*)) \arrow[u, "\sim"] \arrow[r, "(22.8)"] & \mathrm{Res}^H_{T_H}(\tau^H(V^*)) \star \Upsilon_{\mathbf{d}^\vee}(\mathbf{c}^\vee), \arrow[u, "\sim"']
\end{tikzcd} \tag{22.9}$$

where lower right vertical arrow comes from the identification

$$\tau^{T_H} \circ \mathrm{Res}^H_{T_H} \simeq \mathrm{Res}^H_{T_H} \circ \tau^H.$$

Moreover, the commutation identifications satisfy a homotopy-coherent system of comptatibilities for tensor products of the objects $V \in \mathrm{Rep}(H)^c$.

22.4. **Proof of Theorem 22.2.5.** In this subsection we will prove Theorem 22.2.5 by showing that it fits into the paradigm of Sect. 22.3 above.

22.4.1. We take $\mathbf{C}$ to be the category $\mathrm{Whit}_{q,\mathrm{glob}}(G)$, and $\mathbf{E}$ to be $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$.

Recall that in Sect. 22.2.1, the category $\mathbf{D}$ was taken to be $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$. However, here we will have to somewhat modify this choice.

22.4.2. Note that the algebraic stack $\overline{\mathrm{Bun}}_{B^-}$ is disconnected, and its individual connected components are *not* quasi-compact. So, the question of compact generation of $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}(\overline{\mathrm{Bun}}_{B^-})$ may be non-trivial. We will skirt this problem as follows:

Consider the full (but non-cocomplete) subcategory of $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$ generated by T-Hecke translates of the direct summands of ${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}}$ (these direct summands correspond to the different connected components of $\overline{\mathrm{Bun}}_{B^-}$). We let $\mathbf{D}$ (to be henceforth denoted $\mathbf{D}_q$) be the ind-completion of this category. By construction, we have a tautological functor

$$\mathbf{D}_q \to \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}).$$

Note that compact objects of $\mathbf{D}_q$ map to *locally compact* objects of $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$. In particular, Verdier duality on $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$ is well-defined for these objects.

This allows to identify $\mathbf{D}_q^\vee$ with the category $\mathbf{D}_{q^{-1}}$ so that we have a commutative diagram

$$\begin{array}{ccc} (\mathbf{D}_q^c)^{\mathrm{op}} & \longrightarrow & \left(\left(\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})\right)^{\mathrm{loc.c}}\right)^{\mathrm{op}} \\ \downarrow & & \downarrow \\ \mathbf{D}_{q^{-1}}^c & \longrightarrow & \left(\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})\right)^{\mathrm{loc.c}}. \end{array}$$

22.4.3. We take the functor Ψ to be the composite of

$$\mathrm{Whit}_{q,\mathrm{glob}}(G) \otimes \mathbf{D}_q \to \mathrm{Whit}_{q,\mathrm{glob}}(G) \otimes \mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$$

with the functor (22.4). Let us denote this functor by Ψ_q.

By Theorem 21.1.3, the resulting functor

$$\Upsilon : \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G) \otimes \mathbf{D}^\vee \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

from Sect. 22.3.2 identifies with $\Psi_{q^{-1}}$, i.e., with

$$\mathrm{Whit}_{q^{-1},\mathrm{glob}}(G) \otimes \mathbf{D}_{q^{-1}} \simeq \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G) \otimes \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x}) \to \\ \overset{(22.4)}{\longrightarrow} \mathrm{Shv}_{(\mathcal{G}^\Lambda)^{-1}}(\mathrm{Conf}_{\infty\cdot x}).$$

22.4.4. Consider the functor

$$\widetilde{\Psi}_q := \widetilde{\Psi} : \mathrm{Whit}_{q,\mathrm{glob}}(G) \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}_q \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}).$$

We claim that the resulting functor

$$\widetilde{\Upsilon} : \mathrm{Whit}_{q^{-1},\mathrm{glob}}(G) \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D}_{q^{-1}} \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

from Sect. 22.3.2 identifies with $\widetilde{\Psi}_{q^{-1}}$.

This statement amounts to the fact that for $\mathcal{F} \in \mathrm{Whit}_{q,\mathrm{glob}}(G)^c$, $\mathcal{F}' \in \mathbf{D}_q$ and $V \in \mathrm{Rep}(H)^c$, the isomorphisms

$$\Psi_q(\mathcal{F} \star V, \mathcal{F}') \simeq \Psi_q(\mathcal{F}, V \star \mathcal{F}')$$

and

$$\Psi_{q^{-1}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \star \tau^H(V^*), \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}')) \simeq \Psi_{q^{-1}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}), \tau^H(V^*) \star \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}'))$$

make the following diagram is commutative (in a way compatible with tensor products of objects V):

$$\begin{array}{ccc} \mathbb{D}^{\mathrm{Verdier}}(\Psi_q(\mathcal{F} \star V, \mathcal{F}')) & \longrightarrow & \mathbb{D}^{\mathrm{Verdier}}(\Psi_q(\mathcal{F}, V \star \mathcal{F}')) \\ \downarrow & & \downarrow \\ \Psi_{q^{-1}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}) \star \tau^H(V^*), \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}')) & \longrightarrow & \Psi_{q^{-1}}(\mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}), \tau^H(V^*) \star \mathbb{D}^{\mathrm{Verdier}}(\mathcal{F}')). \end{array}$$

This follows from the fact that the natural transformations (21.2) involved in the construction of the isomophism

$$\Upsilon \simeq \Psi_{q^{-1}}$$

commute with proper pushforwards.

Similarly, the data of commutation with the action of $\mathrm{Rep}(T_H)$ on the functor $\widetilde{\Upsilon}$ that arises from one on $\widetilde{\Psi}_q$ agrees with the corresponding data on $\widetilde{\Psi}_{q^{-1}}$.

22.4.5. We take the object $\mathbf{d} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}_q)$ (to be henceforth denoted $\mathbf{d}_q$) to be

$$ {}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}} . $$

Note that although $\mathbf{d}_q$, viewed as an object of $\mathbf{D}_q$, is not compact (indeed, it is spread over all connected components on $\overline{\mathrm{Bun}}_{B^-}$), that its image in $\mathrm{Shv}_{(\mathcal{G}^{G,T,\mathrm{ratio}})^{-1}}((\overline{\mathrm{Bun}}_{B^-})_{\infty\cdot x})$ is locally compact.

The corresponding object $\mathbf{d}_q^\vee \in \mathbf{D}_q^\vee$ identifies with $\mathbf{d}_{q^{-1}}$, i.e.,

$$ {}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} . $$

Recall now that according to Sect. 22.3.4, the Hecke structure on $\mathbf{d}_q$ gives rise to one on $\mathbf{d}_q^\vee \in \mathbf{D}_q^\vee$; i.e., it lifts to an object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}_q^\vee)$. We have the following key assertion:

Theorem 22.4.6. *Under the identifications $\mathbf{D}_q^\vee \simeq \mathbf{D}_{q^{-1}}$ and $\mathbf{d}_q^\vee \simeq \mathbf{d}_{q^{-1}}$, the structure on $\mathbf{d}_q^\vee$ of object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}_q^\vee)$ coincides with the structure on $\mathbf{d}_{q^{-1}}$ of object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}_{q^{-1}})$.*

22.4.7. Assuming for a moment Theorem 22.4.6 we complete the proof of the desired global version of Theorem 22.1.3 by invoking the system of commutative diagrams (22.9).

22.5. **Hecke structure on the global IC sheaf and Verdier duality.** This rest of this section is devoted to the proof of Theorem 22.4.6. In order to unburden the notation we will write $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}$ instead of ${}_{\mathrm{Bun}_T}\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}$. We will also switch from B^- back to B.

In this subsection we will explain what Theorem 22.4.6 says in "down-to-earth" terms.

22.5.1. Let us write down what Theorem 22.4.6 says in concrete terms. According to Theorem 14.7.4, for $V \in \mathrm{Rep}(H)$ we have canonical isomorphisms

$$ \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} \star \mathrm{Sat}_{q,G}(V) \simeq \mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V)) \star \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} $$

and

$$ \mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}} \star \mathrm{Sat}_{q^{-1},G}(V) \simeq \mathrm{Sat}_{q^{-1},T}(\mathrm{Res}^H_{T_H}(V)) \star \mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}} . $$

The claim is that for $V \in \mathrm{Rep}(H)^c$, the following diagram commutes:

(22.10)
$$ \begin{tikzcd} \mathbb{D}^{\mathrm{Verdier}}(\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} \star \mathrm{Sat}_{q,G}(V)) \arrow[r] \arrow[dd] & \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V)) \star \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}) \arrow[d] \\ & \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,T}(\mathrm{Res}^H_{T_H}(V))) \star \mathbb{D}^{\mathrm{Verdier}}(\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}})) \arrow[d] \\ \mathbb{D}^{\mathrm{Verdier}}(\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}) \star \mathbb{D}^{\mathrm{Verdier}}(\mathrm{Sat}_{q,G}(V))) \arrow[d] & \mathrm{Sat}_{q^{-1},T}(\tau^{T_H}(\mathrm{Res}^H_{T_H}(V^*))) \star \mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}} \arrow[d] \\ \mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}} \star \mathrm{Sat}_{q^{-1},G}(\tau^H(V^*)) \arrow[r] & \mathrm{Sat}_{q^{-1},T}(\mathrm{Res}^H_{T_H}(\tau^H(V^*))) \star \mathrm{IC}^{\frac{\infty}{2}}_{q^{-1},\mathrm{glob}}, \end{tikzcd} $$

where the lower right vertical arrow is given by

$$ \tau^{T_H} \circ \mathrm{Res}^H_{T_H} \simeq \mathrm{Res}^H_{T_H} \circ \tau^H. $$

Moreover, the data of commutation is compatible with tensor products of the objects V.

22.5.2. Some simplifying remarks are in order:

(i) As in Remark 14.7.5 it is sufficient to establish the commutativity of the diagrams (22.10) for $V \in (\operatorname{Rep}(H))^{\heartsuit}$.

(ii) For V in the abelian category, all the objects involved in (22.10) lie in the heart of the perverse t-structure on $\operatorname{Shv}_{\mathcal{G}^{G,T,\text{ratio}}}((\overline{\operatorname{Bun}}_B)_{\infty\cdot x})$. Hence, once we check the commutation for individual objects V, the higher compatibilities would follow.

22.6. **Digression: gluing different components of $\overline{\operatorname{Bun}}_B$ together.** Note that $\operatorname{IC}^{\frac{\infty}{2}}_{q,\text{glob}}$ is not an irreducible perverse sheaf for the simple reason that it is supported on all the different connected components of $\overline{\operatorname{Bun}}_B$.

In this subsection we will introduce a geometric device that allows to "sew together" the various components of $\overline{\operatorname{Bun}}_B$. More precisely, we will define a category such that, when regarded as an object in it, $\operatorname{IC}^{\frac{\infty}{2}}_{q,\text{glob}}$ will be irreducible.

22.6.1. Fix another point $x \neq y \in X$. Let

$$(\overline{\operatorname{Bun}}_B)_{\infty\cdot x,\text{good at } y} \subset (\overline{\operatorname{Bun}}_B)_{\infty\cdot x}$$

be the open sub-functor, where we require that our generalized B-reduction be non-degenerate at y. Restriction to the formal disc around y defines a map

$$(\overline{\operatorname{Bun}}_B)_{\infty\cdot x,\text{good at } y} \to \operatorname{pt}/\mathfrak{L}^+(B)_y.$$

Consider the corresponding Hecke groupoid

$$\begin{tikzcd}
(\overline{\operatorname{Bun}}_B)_{\infty\cdot x,\text{good at } y} \arrow[d] & {}_{(\overline{\operatorname{Bun}}_B)_{\infty\cdot x,\text{good at } y}}\operatorname{Hecke}^{\text{loc}}_{B,y} \arrow[l, "\overleftarrow{h}_B"'] \arrow[r, "\overrightarrow{h}_B"] \arrow[d] & (\overline{\operatorname{Bun}}_B)_{\infty\cdot x,\text{good at } y} \arrow[d] \\
\operatorname{pt}/\mathfrak{L}^+(B)_y & \operatorname{Hecke}^{\text{loc}}_{B,y} \arrow[l] \arrow[r] & \operatorname{pt}/\mathfrak{L}^+(B)_y,
\end{tikzcd}$$

where in this diagram both squares are Cartesian.

The pullbacks of the gerbe $\mathcal{G}^{G,T,\text{ratio}}$ with respect to $\overleftarrow{h}_B$ and $\overrightarrow{h}_B$ are naturally identified.

22.6.2. The natural projection

$$\operatorname{Hecke}^{\text{loc}}_{B,y} \to \operatorname{Hecke}^{\text{loc}}_{T,y} \to \Lambda$$

defines a decomposition of $\operatorname{Hecke}^{\text{loc}}_{B,y}$ into connected components, indexed by the elements of Λ; denote them by $\operatorname{Hecke}^{\text{loc},\lambda}_{B,y}$. Denote

$$\operatorname{Hecke}^{\text{loc},+}_{B,y} = \underset{\lambda\in\Lambda^+}{\sqcup} \operatorname{Hecke}^{\text{loc},\lambda}_{B,y}$$

.

For $\lambda \in \Lambda^+$ let

$$\operatorname{Hecke}^{\text{loc},\lambda,\text{restr}}_{B,y} \subset \operatorname{Hecke}^{\text{loc},\lambda}_{B,y}$$

be the subfunctor

$$\mathfrak{L}^+(N)_y\backslash(\mathfrak{L}^+(N)_y\cdot t^\lambda)/\mathfrak{L}^+(N)_y \subset \mathfrak{L}^+(N)_y\backslash(\mathfrak{L}(N)_y\cdot t^\lambda)/\mathfrak{L}^+(N)_y.$$

Then the map

$$\operatorname{Hecke}^{\text{loc},\lambda,\text{restr}}_{B,y} \overset{\overrightarrow{h}^{\lambda,\text{restr}}_B}{\longrightarrow} \operatorname{pt}/\mathfrak{L}^+(B)_y$$

is an isomorphism and the map

$$\operatorname{pt}/\mathfrak{L}^+(B)_y \overset{\overleftarrow{h}^{\lambda,\text{restr}}_B}{\longleftarrow} \operatorname{Hecke}^{\text{loc},\lambda,\text{restr}}_{B,y}$$

is a fibration into affine spaces of dimensions $\langle\lambda, 2\check{\rho}\rangle$.

22.6.3. Consider the corresponding substacks

$$_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}}\mathrm{Hecke}^{\mathrm{loc},\lambda,\mathrm{restr}}_{B,y} \subset {}_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}}\mathrm{Hecke}^{\mathrm{loc}}_{B,y}.$$

The resulting map

$$_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}}\mathrm{Hecke}^{\mathrm{loc},\lambda,\mathrm{restr}}_{B,y} \overset{\overrightarrow{h}^{\lambda,\mathrm{restr}}_B}{\longrightarrow} (\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}$$

is an isomorphism, and the map

$$(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y} \overset{\overleftarrow{h}^{\lambda,\mathrm{restr}}_B}{\longleftarrow} {}_{(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}}\mathrm{Hecke}^{\mathrm{loc},\lambda,\mathrm{restr}}_{B,y}$$

is a fibration into affine spaces of dimensions $\langle\lambda, 2\check{\rho}\rangle$.

22.6.4. The (ind)-algebraic stack $(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}$ splits into connected components

$$(\overline{\mathrm{Bun}}_B)^{\lambda}_{\infty\cdot x,\mathrm{good\ at}\ y}, \quad \lambda\in\Lambda.$$

The above maps $\overrightarrow{h}^{\lambda,\mathrm{restr}}_B, \overleftarrow{h}^{\lambda,\mathrm{restr}}_B$ define a system of maps

$$m^{\lambda_2,\lambda_1} : (\overline{\mathrm{Bun}}_B)^{\lambda_2}_{\infty\cdot x,\mathrm{good\ at}\ y} \to (\overline{\mathrm{Bun}}_B)^{\lambda_1}_{\infty\cdot x,\mathrm{good\ at}\ y}, \quad \lambda_2-\lambda_1\in\Lambda^+.$$

22.6.5. We can view the assignment

$$\lambda\mapsto (\overline{\mathrm{Bun}}_B)^{\lambda}_{\infty\cdot x,\mathrm{good\ at}\ y}$$

as a functor from (the opposite of) Λ viewed as a poset

$$\lambda_1\preceq\lambda_2 \Leftrightarrow \lambda_2-\lambda_1\in\Lambda^+$$

to the category of (ind)-algebraic stacks.

Consider the functor

$$(\Lambda,\preceq)\to \mathrm{DGCat}$$

that sends

$$\lambda\mapsto \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)^{\lambda}_{\infty\cdot x,\mathrm{good\ at}\ y})$$

and $\lambda_1\preceq\lambda_2$ to the functor $(m^{\lambda_2,\lambda_1})^![\langle\lambda_1-\lambda_2, 2\check{\rho}\rangle]$.

22.6.6. Define

$$\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y})^{\mathrm{Hecke}_{T,y}} := \lim_{(\Lambda,\preceq)^{\mathrm{op}}} \mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)^{\lambda}_{\infty\cdot x,\mathrm{good\ at}\ y}). \tag{22.11}$$

Informally, objects of this category are objects $\mathcal{F}\in\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y})$ equipped with a homotopy-compatible system of identifications

$$(m^{\lambda_2,\lambda_1})^!(\mathcal{F}^{\lambda_1})[\langle\lambda_1-\lambda_2,2\check{\rho}\rangle]\simeq\mathcal{F}^{\lambda_2},$$

where

$$\mathcal{F}^{\lambda} := \mathcal{F}|_{(\overline{\mathrm{Bun}}_B)^{\lambda}_{\infty\cdot x,\mathrm{good\ at}\ y}}.$$

22.6.7. By a slight abuse of notation let us continue to denote by $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}$ its restriction along the open embedding

$$(\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y}\hookrightarrow(\overline{\mathrm{Bun}}_B)_{\infty\cdot x}.$$

It is clear that $\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}$ naturally lifts to an object of $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\mathrm{good\ at}\ y})^{\mathrm{Hecke}_{T,y}}$.

22.6.8. The key observation now is that for any $\gamma_1, \gamma_2 \in \Lambda^\sharp$, we have

$$\mathrm{Hom}_{\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\text{good at } y})^{\mathrm{Hecke}_{T,y}}}(\mathsf{e}^{\gamma_1} \star \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}, \mathsf{e}^{\gamma_2} \star \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}) = \begin{cases} \mathsf{e} \text{ if } \gamma_1 = \gamma_2 \text{ and} \\ 0 \text{ otherwise.} \end{cases} \tag{22.12}$$

We emphasize that in the above formula, we are taking $\mathrm{Hom}(-,-)$, i.e., $H^0(\mathcal{H}om(-,-))$.

Remark 22.6.9. The isomorphism (22.12) takes place for Hom taken in the category (22.11), but not in $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x})$, because in the latter each connected component would contribute its own factor of e. This was the reason for introducing the category (22.11).

22.6.10. We will use (22.12) as follows:

First off, it follows from the definitions that the G- and T- Hecke actions at x lift naturally to actions on the category (22.11).

Now, from (22.12) and Theorem 14.7.4 we obtain

(A) There exists a *monoidal* functor ${}_q\mathrm{Res}^H_{T_H} : \mathrm{Rep}(H)^\heartsuit \to \mathrm{Rep}(T_H)^\heartsuit$, uniquely characterized by the the system of isomorphisms

$$\mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} \star \mathrm{Sat}_{q,G}(V) \simeq \mathrm{Sat}_{q,T}({}_q\mathrm{Res}^H_{T_H}(V)) \star \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}, \quad V \in \mathrm{Rep}(H)^\heartsuit, \tag{22.13}$$

taking place in $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\text{good at } y})^{\mathrm{Hecke}_{T,y}}$. Indeed, for $W \in \mathrm{Rep}(T_H)^\heartsuit$ we have

$$\mathrm{Hom}(W, {}_q\mathrm{Res}^H_{T_H}(V)) :=$$
$$= \mathrm{Hom}_{\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\text{good at } y})^{\mathrm{Hecke}_y}}(\mathrm{Sat}_{q,T}(W) \star \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}}, \mathrm{IC}^{\frac{\infty}{2}}_{q,\mathrm{glob}} \star \mathrm{Sat}_{q,G}(V)).$$

(B) There exists an isomorphism between monoidal functors ${}_q\mathrm{Res}^H_{T_H} \simeq \mathrm{Res}^H_{T_H}$.

22.7. **Proof Theorem 22.4.6.** In this subsection we will finally prove Theorem 22.4.6.

22.7.1. It is easy to see that Verdier duality is well-defined on objects of (22.11) that are locally compact as objects of $\mathrm{Shv}_{\mathcal{G}^{G,T,\mathrm{ratio}}}((\overline{\mathrm{Bun}}_B)_{\infty\cdot x,\text{good at } y})$.

This implies that the objects appearing in the diagram (22.10) can be considered as objects in (22.11). Hence, it is sufficient to establish the commutativity of the diagram (22.10) in this context.

22.7.2. We prove the required equality as follows:

Applying Verdier duality to (22.13), we obtain *an* isomorphism

$${}_q\mathrm{Res}^H_{T_H} \simeq {}_{q^{-1}}\mathrm{Res}^H_{T_H}$$

as monoidal functors $\mathrm{Rep}(H) \to \mathrm{Rep}(T_H)$.

We need to show that the composite isomorphism

$$\mathrm{Res}^H_{T_H} \simeq {}_q\mathrm{Res}^H_{T_H} \simeq {}_{q^{-1}}\mathrm{Res}^H_{T_H} \simeq \mathrm{Res}^H_{T_H}$$

is the identity map.

22.7.3. A priori, the above composite map is given by an element $t \in T_H$, and we need to see that $t = 1$. For that it is sufficient to see that t acts as identity on the highest weight lines for each $V = V^\gamma$.

However, the latter is easy to see from the constructions.

Part VIII: Baby Verma objects

The goal of this Part is to carry out the program indicated in Sect. 19.3.2, i.e., to construct the "(dual) baby Verma" objects $\overset{\bullet}{\mathcal{M}}^{\mu,!}_{\mathrm{Whit}}$ and $\overset{\bullet}{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}}$, and establish their properties.

The term "baby Verma" is due to the fact that under the equivalence with the category of modules over the small quantum group, the objects $\overset{\bullet}{\mathcal{M}}^{\mu,!}_{\mathrm{Whit}}$ (resp., $\overset{\bullet}{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}}$) correspond to baby Verma (resp., dual baby Verma) mdoules, see Remark 27.4.2.

23. The B-Hecke category and the Drinfeld-Plücker formalism

The construction of the objects $\overset{\bullet}{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}}$ is based on the *Drinfeld-Plücker formalism*[3], which is the subject of the present section.

23.1. **The B-Hecke category.** Recall the setting of Sects. 10.2 and 10.3. In this subsection we will need to complement that discussion by introducing yet another version of the Hecke category, this time relative to the Borel subgroup $B_H \subset H$.

23.1.1. Let $\mathbf{C}$ be a category acted on by $\mathrm{Rep}(H)$. We define the category $B\text{-Hecke}(\mathbf{C})$ to be

$$\mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(B_H),$$

where B_H is the Borel subgroup of H, see Sect. 10.1.

Tautologically, we can rewrite

$$B\text{-Hecke}(\mathbf{C}) \simeq (\mathrm{Hecke}(\mathbf{C}))^{B_H},$$

where we view $\mathrm{Hecke}(\mathbf{C})$ as acted on by H, see Sect. 10.2.6.

In what follows we will assume that $\mathbf{C}$ is compactly generated, in which case $B\text{-Hecke}(\mathbf{C})$ is also compactly generated.

23.1.2. The pair of adjoint functors

$$\mathrm{Res}^{B_H}_{T_H} : \mathrm{Rep}(B_H) \rightleftarrows \mathrm{Rep}(T_H) : \mathrm{coInd}^{B_H}_{T_H}$$

gives rise to the (same named) functors

$$\mathrm{Res}^{B_H}_{T_H} : B\text{-Hecke}(\mathbf{C}) \rightleftarrows \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) : \mathrm{coInd}^{B_H}_{T_H}.$$

Being the left adjoint of a continuous functor, the functor $\mathrm{Res}^{B_H}_{T_H}$ preserves compactness. However, we claim that more is true:

Lemma 23.1.3. *If* $\mathbf{c} \in B\text{-Hecke}(\mathbf{C})$ *is such that* $\mathrm{Res}^{B_H}_{T_H}(\mathbf{c}) \in \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})$ *is compact, then* $\mathbf{c}$ *is compact.*

Proof. This is a general phenomenon: for a category $\mathbf{D}$ acted on by an algebraic group H' (in out case $\mathbf{D} = \mathrm{Hecke}(\mathbf{C})$ and $H' = B_H$), if an object $\mathbf{d} \in \mathbf{D}^{H'}$ is such that the underlying object $\mathrm{Res}^{H'}(\mathbf{d}) \in \mathbf{D}$ is compact, then $\mathbf{d}$ itself is compact.

□

[3] Both the name "Drinfeld-Plücker" and the mathematical idea belong to S. Raskin.

23.1.4. We now consider the functor

$$\mathrm{Res}^H_{B_H} : \mathrm{Rep}(H) \to \mathrm{Rep}(B_H),$$

and its right and left adjoints, denoted $\mathrm{coInd}^H_{B_H}$ and $\mathrm{Ind}^H_{B_H}$, respectively. The functor $\mathrm{Res}^H_{B_H}$ induces the (same named) functor:

$$\mathrm{Res}^H_{B_H} : \mathbf{C} \simeq \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(H) \to \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(B_H) \simeq B\text{-Hecke}(\mathbf{C})$$

and its right and left adjoints

$$\mathrm{coInd}^H_{B_H}, \mathrm{Ind}^H_{B_H} : B\text{-Hecke}(\mathbf{C}) \to \mathbf{C}. \tag{23.1}$$

Being left adjoints of continuous functors, the functors $\mathrm{Res}^H_{B_H}$ and $\mathrm{Ind}^H_{B_H}$ preserve compactness.

23.1.5. Recall also that Serre duality for H/B_H implies that we have a canonical isomorphism

$$\mathrm{coInd}^H_{B_H}(-) \simeq \mathrm{Ind}^H_{B_H}(\otimes k^{-2\rho_H})[-d], \tag{23.2}$$

(here $d = \dim(H/B_H)$) as $\mathrm{Rep}(H)$-linear functors $\mathrm{Rep}(B_H) \to \mathrm{Rep}(H)$.

This implies a similar relationship between the functors (23.1). In particular, we obtain that the functor $\mathrm{coInd}^H_{B_H}$ also preserves compactness.

23.1.6. Let us consider $\mathbf{C}^\vee$ as a category acted on by $\mathrm{Rep}(H)$ as in Sect. 10.5.

Then we obtain a canonical identification

$$B\text{-Hecke}(\mathbf{C})^\vee \simeq B^-\text{-Hecke}(\mathbf{C}^\vee), \tag{23.3}$$

or equivalently

$$(B\text{-Hecke}(\mathbf{C})^c)^{\mathrm{op}} \simeq B^-\text{-Hecke}(\mathbf{C}^\vee)^c, \quad \mathbf{c} \mapsto \mathbf{c}^\vee, \tag{23.4}$$

for which the diagram

$$\begin{CD} ((\mathbf{C} \otimes \mathrm{Rep}(B_H))^c)^{\mathrm{op}} @>>> (B\text{-Hecke}(\mathbf{C})^c)^{\mathrm{op}} \\ @VVV @VV{(23.4)}V \\ ((\mathbf{C}^\vee \otimes \mathrm{Rep}(B_H^-))^c)^{\mathrm{op}} @>>> B^-\text{-Hecke}(\mathbf{C}^\vee)^c, \end{CD} \tag{23.5}$$

where the left vertical arrow is the tensor product of

$$(\mathbf{C}^c)^{\mathrm{op}} \to (\mathbf{C}^\vee)^c, \quad \mathbf{c} \mapsto \mathbf{c}^\vee,$$

and the functor

$$(\mathrm{Rep}(B_H)^c)^{\mathrm{op}} \to \mathrm{Rep}(B_H^-)^c, \quad V \mapsto \tau^H(V^*),$$

where we use τ^H as an isomorphism $B_H \to B_H^-$.

23.1.7. Note that from (23.2) we obtain

$$(\mathrm{coInd}^H_{B_H}(\mathbf{c}))^\vee \simeq \mathrm{coInd}^H_{B_H^-}(\mathbf{c}^\vee \otimes \mathsf{e}^{-2\rho_H})[d], \quad \mathbf{c} \in \mathrm{Rep}(B_H)^c. \tag{23.6}$$

Note also, that we have a canonical identification

$$(\mathrm{Res}^{B_H}_{T_H}(\mathbf{c}))^\vee \simeq \mathrm{Res}^{B_H^-}_{T_H}(\mathbf{c}^\vee), \quad \mathbf{c} \in \mathrm{Rep}(B_H)^c$$

where we use the identification

$$(\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C})^c)^{\mathrm{op}} \to \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}^\vee)^c$$

as in Sect. 10.5.2 (i.e., we combine the usual duality for T_H with τ^{T_H}).

23.2. **Behavior of the t-structure.** In this subsection we will study the behavior of the t-structure on B-Hecke$(\mathbf{C})$.

23.2.1. Assume that $\mathbf{C}$ is equipped with a t-structure so that the action of $\operatorname{Rep}(H)$ on $\mathbf{C}$ is given by t-exact functors. Then, according to Sect. 10.1.7, the category $B\text{-Hecke}(\mathbf{C})$ also acquires a t-structure.

By construction, the functors

$$\operatorname{Res}^H_{B_H} : \mathbf{C} \to B\text{-Hecke}(\mathbf{C}) \text{ and } \operatorname{Res}^{B_H}_{T_H} : B\text{-Hecke}(\mathbf{C}) \to \overset{\bullet}{\text{Hecke}}(\mathbf{C})$$

are t-exact.

23.2.2. By adjunction, the functor $\operatorname{Ind}^H_{B_H} : B\text{-Hecke}(\mathbf{C}) \to \mathbf{C}$ is right t-exact, while the functor $\operatorname{coInd}^H_{B_H} : B\text{-Hecke}(\mathbf{C}) \to \mathbf{C}$ is left t-exact.

Note, however, that it follows from (23.2) that the right cohomological amplitude of $\operatorname{coInd}^H_{B_H}$ (resp., left cohomological amplitude of $\operatorname{Ind}^H_{B_H}$) is bounded by d.

23.2.3. In what follows we will assume that the t-structure on $\mathbf{C}$ is compactly generated (see Sect. 6.3.8 for what this means). We are going to prove the following analog of Serre's theorem on coherent sheaves on the projective space.

Note that for $\mathbf{c} \in B\text{-Hecke}(\mathbf{C})$ and γ we have a canonically defined map

$$\operatorname{Res}^H_{B_H}(\operatorname{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^{-\gamma})) \otimes \mathsf{e}^{\gamma} \to \mathbf{c}. \tag{23.7}$$

We have:

Proposition 23.2.4. *Let* $\mathbf{c}$ *be an object of* $B\text{-Hecke}(\mathbf{C})^c \cap (B\text{-Hecke}(\mathbf{C}))^{\leq 0}$. *Then for all* γ *deep enough in the dominant chamber (i.e.,* $\gamma \in \gamma_0 + \Lambda^+_H$ *for some fixed* γ_0*) we have:*

(a) *The object* $\operatorname{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^{-\gamma})$ *is connective.*

(b) *The cofiber of the map* (23.7) *belongs to* $(B\text{-Hecke}(\mathbf{C}))^{<0}$.

Proof. We can find $\mathbf{c}' \in \mathbf{C}^c \cap (\mathbf{C})^{\leq 0}$ and $V \in \operatorname{Rep}(B_H)^c \cap (\operatorname{Rep}(B_H))^{\leq 0}$ together with a map

$$\operatorname{Res}^H_{B_H}(\mathbf{c}') \otimes V =: \mathbf{c}_1 \to \mathbf{c}$$

whose cofiber belongs to $(B\text{-Hecke}(\mathbf{C}))^{<0}$. Let $\mathbf{c}_2$ denote the fiber of the above map.

It is easy to see that both points of the proposition hold for $\mathbf{c}_1$. From here we obtain that point (a) for $\mathbf{c}_2$ implies point (b) for $\mathbf{c}$.

We will prove point (a) by descending induction. Namely, we claim that $\operatorname{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^{-\gamma})$ belongs to $(B\text{-Hecke}(\mathbf{C}))^{\leq i}$ for γ deep enough in the dominant chamber. The assertion for $i > d$ follows from Sect. 23.2.2. The induction step follows the fiber sequence

$$\mathbf{c}_2 \to \mathbf{c}_1 \to \mathbf{c}$$

and the fact that the assertion holds for $\mathbf{c}_1$. □

23.2.5. Let us assume that the t-structure on $\mathbf{C}$ in Artinian and the action of $\operatorname{Rep}(H)$ is accessible (see Sect. 10.6.4 for what this means).

Recall (see Corrollary 10.7.4) that in this case the t-structure on $\overset{\bullet}{\text{Hecke}}(\mathbf{C})$ is also Artinian. From here, combining with Lemma 23.1.3 we obtain:

Corollary 23.2.6. *Under the above circumstances, the t-structure on* $B\text{-Hecke}(\mathbf{C})$ *is Artinian.*

23.3. **Drinfeld-Plücker formalism.** In this subsection we will finally introduce the Drinfeld-Plücker formalism.

23.3.1. Let now $\mathbf{C}$ be as in Sect. 10.4, i.e., it is acted on by $\operatorname{Rep}(H) \otimes \operatorname{Rep}(T_H)$. Define

$$B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C}) := \mathbf{C} \underset{\operatorname{Rep}(H)\otimes \operatorname{Rep}(T_H)}{\otimes} \operatorname{Rep}(B_H),$$

where $\operatorname{Rep}(T_H) \to \operatorname{Rep}(B_H)$ is the functor of restriction along the *projection* $B_H \to T_H$.

We have the following diagram of categories

$$\begin{array}{ccc} B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C}) & \xrightarrow{\operatorname{Res}^{B_H}_{T_H}} & \overset{\bullet}{\text{Hecke}}_{\mathrm{rel}}(\mathbf{C}) \\ {\scriptstyle \mathbf{oblv}_{\mathrm{rel}}}\big\downarrow & & \big\downarrow{\scriptstyle \mathbf{oblv}_{\mathrm{rel}}} \\ B\text{-Hecke}(\mathbf{C}) & \xrightarrow{\operatorname{Res}^{B_H}_{T_H}} & \overset{\bullet}{\text{Hecke}}(\mathbf{C}), \end{array}$$

where the vertical arrows are the functors

$$\mathbf{C} \underset{\operatorname{Rep}(H)\otimes \operatorname{Rep}(T_H)}{\otimes} \mathbf{D} \to \mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{D},$$

right adjoint to the projections

$$\mathbf{C} \underset{\operatorname{Rep}(H)}{\otimes} \mathbf{D} \to \mathbf{C} \underset{\operatorname{Rep}(H)\otimes \operatorname{Rep}(T_H)}{\otimes} \mathbf{D},$$

for $\mathbf{D} = \operatorname{Rep}(B_H)$ and $\mathbf{D} = \operatorname{Rep}(T_H)$.

Let $\mathbf{oblv}_{B\text{-Hecke}_{\mathrm{rel}}}$ denote the forgetful functor

$$B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C}) \xrightarrow{\operatorname{Res}^{B_H}_{T_H}} \overset{\bullet}{\text{Hecke}}_{\mathrm{rel}}(\mathbf{C}) \xrightarrow{\mathbf{oblv}_{\overset{\bullet}{\text{Hecke}}_{\mathrm{rel}}}} \mathbf{C}.$$

23.3.2. Consider the base affine space $\overline{H/N_H}$ for the group H. This is an affine scheme acted on by $H \times T_H$. We consider the algebra of regular functions $\operatorname{Fun}(\overline{H/N_H})$ on $\overline{H/N_H}$ as an algebra object inside the monoidal category $\operatorname{Rep}(H) \otimes \operatorname{Rep}(T_H)$.

For $\mathbf{C}$ as in Sect. 23.3.1, define

$$\operatorname{DrPl}(\mathbf{C}) := \operatorname{Fun}(\overline{H/N_H})\text{-mod}(\mathbf{C}) \simeq \mathbf{C} \underset{\operatorname{Rep}(H)\otimes \operatorname{Rep}(T_H)}{\otimes} \operatorname{Fun}(\overline{H/N_H})\text{-mod}(\operatorname{Rep}(H) \otimes \operatorname{Rep}(T_H))$$

23.3.3. Explicitly, we can think about an object of $\operatorname{DrPl}(\mathbf{C})$ as follows: this is an object $\mathbf{c} \in \mathbf{C}$ endowed with a system of maps

$$\mathbf{c} \star (V^\gamma)^* \to \mathsf{e}^{-\gamma} \star \mathbf{c}, \quad \gamma \in \Lambda^+_H \tag{23.8}$$

that satisfy a homotopy-coherent system of compatibilities, starting from the commutative diagram

$$\begin{array}{ccc} \mathbf{c} \star ((V^{\gamma_1})^* \otimes (V^{\gamma_2})^*) & \longrightarrow & \mathbf{c} \star (V^{\gamma_1+\gamma_2})^* \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow \\ (\mathbf{c} \star (V^{\gamma_1})^*) \star (V^{\gamma_2})^* & & \mathsf{e}^{-\gamma_1-\gamma_2} \star \mathbf{c} \\ \big\downarrow & & \big\downarrow{\scriptstyle\sim} \\ (\mathsf{e}^{-\gamma_1} \star \mathbf{c}) \star (V^{\gamma_2})^* & \longrightarrow & (\mathsf{e}^{-\gamma_1} \otimes \mathsf{e}^{-\gamma_2}) \star \mathbf{c} \\ {\scriptstyle\sim}\big\downarrow & & \big\downarrow{\scriptstyle\sim} \\ \mathsf{e}^{-\gamma_1} \star (\mathbf{c} \star (V^{\gamma_2})^*) & \longrightarrow & \mathsf{e}^{-\gamma_1} \star (\mathsf{e}^{-\gamma_2} \star \mathbf{c}). \end{array} \tag{23.9}$$

Remark 23.3.4. In the above commutative diagram, the upper horizontal arrow comes from the Plücker map

$$(V^{\gamma_1})^* \otimes (V^{\gamma_2})^* \to (V^{\gamma_1+\gamma_2})^* \tag{23.10}$$

dual to the map

$$V^{\gamma_1+\gamma_2} \to V^{\gamma_1} \otimes V^{\gamma_2}$$

which induces the *identity* map on the *trivialized* highest weight lines.

Equivalently, by definition

$$V^\gamma := \mathrm{Ind}_{B_H}^H(\mathsf{e}^\gamma) \text{ and } (V^\gamma)^* \simeq \mathrm{coInd}_{B_H}^H(\mathsf{e}^{-\gamma}),$$

and the map (23.10) is the canonical map

$$\mathrm{coInd}_{B_H}^H(\mathsf{e}^{-\gamma_1}) \otimes \mathrm{coInd}_{B_H}^H(\mathsf{e}^{-\gamma_2}) \to \mathrm{coInd}_{B_H}^H(\mathsf{e}^{-\gamma_1-\gamma_2}).$$

23.3.5. Let j denote the open embedding

$$H\backslash(H/N_H)/T_H \hookrightarrow H\backslash(\overline{H/N_H})/T_H.$$

The pair of adjoint functors

$$\mathsf{j}^* : \mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H) \rightleftarrows \mathrm{QCoh}(H\backslash(H/N_H)/T_H) : \mathsf{j}_* \tag{23.11}$$

induces an adjoint pair

$$\mathsf{j}^* : \mathbf{C} \underset{\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)}{\otimes} \mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H) \rightleftarrows \mathbf{C} \underset{\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)}{\otimes} \mathrm{QCoh}(H\backslash(H/N_H)/T_H) : \mathsf{j}_*.$$

We identify

$$H\backslash(H/N_H)/T_H \simeq \mathrm{pt}/B_H$$

and

$$\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H) \simeq \mathrm{Fun}(\overline{H/N_H})\text{-mod}(\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)).$$

Hence, we obtain an adjunction

$$\mathsf{j}^* : \mathrm{DrPl}(\mathbf{C}) \rightleftarrows B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathbf{C}) : \mathsf{j}_*. \tag{23.12}$$

Since the co-unit of the adjunction

$$\mathsf{j}^* \circ \mathsf{j}_* \to \mathrm{Id}$$

is an isomorphism in (23.11), the same is true for (23.12). I.e., the functor j_* in (23.12) is fully faithful.

23.3.6. The composite functor

$$\mathrm{DrPl}(\mathbf{C}) \xrightarrow{\mathsf{j}^*} B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathbf{C}) \overset{\mathbf{oblv}_{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}}}{\longrightarrow} \mathbf{C}$$

can be explicitly described as follows (see [Ga6, Proposition 6.2.4]):

If we think of an object of $\mathrm{DrPl}(\mathbf{C})$ as in Sect. 23.3.3, then the resulting object of $\mathbf{C}$ identifies with

$$\underset{\gamma\in\Lambda_H^+}{\mathrm{colim}}\, \mathsf{e}^{-\gamma} \star \mathbf{c} \star V^\gamma, \tag{23.13}$$

where we regard Λ_H^+ as a (filtered!) poset with respect to

$$\gamma_1 \preceq \gamma_2 \;\Leftrightarrow\; \gamma_2 - \gamma_1 =: \gamma \in \Lambda_H^+,$$

and the transition maps are given by

$$\mathsf{e}^{-\gamma_1} \star \mathbf{c} \star V^{\gamma_1} \to \mathsf{e}^{-\gamma_1} \star \mathbf{c} \star ((V^\gamma)^* \otimes V^\gamma \otimes V^{\gamma_1}) \simeq (\mathsf{e}^{-\gamma_1} \star \mathbf{c} \star (V^\gamma)^*) \star (V^\gamma \otimes V^{\gamma_1}) \to$$
$$\to (\mathsf{e}^{-\gamma_1} \star \mathsf{e}^{-\gamma} \star \mathbf{c}) \star (V^\gamma \otimes V^{\gamma_1}) \to \mathsf{e}^{-\gamma_1-\gamma} \star \mathbf{c} \star V^{\gamma_1+\gamma}.$$

23.4. **The relative vs non-relative case.** In this subsection we will discuss some variants of the construction in Sect. 23.3.

23.4.1. Let now $\mathbf{C}_0$ be a category equipped just with an action of $\mathrm{Rep}(H)$. Set

$$\mathbf{C} := \mathrm{Rep}(T_H) \otimes \mathbf{C}_0,$$

so that

$$B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \simeq B\text{-Hecke}(\mathbf{C}_0).$$

Note that the functor

$$\mathbf{oblv}_{B\text{-Hecke}_{\mathrm{rel}}} : B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \to \mathrm{Rep}(T_H) \otimes \mathbf{C}_0$$

identifies with the composite

$$B\text{-Hecke}(\mathbf{C}_0) \overset{\mathrm{Rep}^H_{B_H}}{\longrightarrow} \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}_0) \overset{\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}}}{\longrightarrow} \mathrm{Rep}(T_H) \otimes \mathbf{C}_0$$

.

23.4.2. Consider the adjunction

$$\mathsf{j}^* : \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \rightleftarrows B\text{-Hecke}(\mathbf{C}_0) : \mathsf{j}_*.$$

We can think of objects of $\mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0)$ as follows. These are families of objects

$$\{\mathbf{c}^\gamma \in \mathbf{C}, \gamma \in \Lambda_H\}$$

equipped with a system of maps

$$\mathbf{c}^{\gamma_1} \star (V^\gamma)^* \to \mathbf{c}^{\gamma_1 - \gamma}, \quad \gamma \in \Lambda^+_H, \tag{23.14}$$

satisfying an appropriate system of compatibilities.

In terms of this description, the corresponding functor

$$\mathbf{oblv}_{B\text{-Hecke}_{\mathrm{rel}}} \circ \mathsf{j}^* : \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \to \mathrm{Rep}(T_H) \otimes \mathbf{C}_0$$

sends a system $\{\mathbf{c}^\gamma\}$ as above to an object

$$\{\widetilde{\mathbf{c}}^\gamma\} \in \mathrm{Rep}(T_H) \otimes \mathbf{C}_0$$

with

$$\widetilde{\mathbf{c}}^{\gamma'} \simeq \underset{\gamma \in \Lambda^+_H}{\mathrm{colim}}\, \mathbf{c}^{-\gamma+\gamma'} \star V^\gamma, \tag{23.15}$$

23.4.3. The functor

$$\mathsf{j}_* : B\text{-Hecke}(\mathbf{C}_0) \to \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0)$$

can be described as follows. It sends an object $\mathbf{c} \in B\text{-Hecke}(\mathbf{C}_0)$ to the system $\{\mathbf{c}^\lambda\}$ with

$$\mathbf{c}^\gamma = \mathrm{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^\gamma),$$

where the maps (23.14) are given by

$$\mathrm{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^{\gamma_1}) \star (V^\gamma)^* \simeq \mathrm{coInd}^H_{B_H}\left(\mathbf{c} \otimes \mathsf{e}^{\gamma_1} \otimes \mathrm{Res}^H_{B_H}((V^\gamma)^*)\right) \to$$
$$\to \mathrm{coInd}^H_{B_H}\left(\mathbf{c} \otimes \mathsf{e}^{\gamma_1} \otimes \mathsf{e}^{-\gamma}\right) = \mathrm{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^{\gamma_1 - \gamma}).$$

23.4.4. Note that by adjunction we obtain that for any

$$\{\mathbf{c}^\gamma\} \in \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0)$$

and $\mathbf{c} := \mathsf{j}^*(\{\mathbf{c}^\gamma\}) \in B\text{-Hecke}(\mathbf{C})$, there exists a canonical map

$$\mathbf{c}^\gamma \to \mathrm{coInd}^H_{B_H}(\mathbf{c} \otimes \mathsf{e}^\gamma). \tag{23.16}$$

23.4.5. Recall the duality (23.3). For a compact object $\mathbf{c} \in B\text{-Hecke}(\mathbf{C}_0)$, consider its dual $\mathbf{c}^\vee \in B^-\text{-Hecke}(\mathbf{C}_0^\vee)$, and consider the corresponding object

$$\mathsf{j}_*(\mathbf{c}^\vee) \in \mathrm{DrPl}^-(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0).$$

From (23.6) we obtain that $\mathsf{j}_*(\mathbf{c}^\vee)$ is given by the system $\{\mathbf{c}^\gamma\}$ with

$$\mathbf{c}^\gamma := (\mathrm{coInd}_{B_H}^H(\mathbf{c} \otimes \mathsf{e}^{\gamma+2\rho_H}))^\vee[-d].$$

23.4.6. Let $\mathbf{C}$ be again a category acted on by $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$, and take $\mathbf{C}_0 := \mathbf{C}$, where we disregard the $\mathrm{Rep}(T_H)$-action. Consider the right adjoint of the action functor

$$\mathbf{C} \to \mathrm{Rep}(T_H) \otimes \mathbf{C}_0.$$

This is a functor of $\mathrm{Rep}(T_H)$-module categories.

Hence, it induces functors

$$\mathrm{DrPl}(\mathbf{C}) \to \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \tag{23.17}$$

$$B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C}) \to B\text{-Hecke}(\mathbf{C}_0) \tag{23.18}$$

that make the diagrams

$$\begin{tikzcd}
\mathrm{DrPl}(\mathbf{C}) \arrow[r] \arrow[d, "\mathsf{j}^*"'] & \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \arrow[d, "\mathsf{j}^*"] \\
B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C}) \arrow[r] & B\text{-Hecke}(\mathbf{C}_0)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathrm{DrPl}(\mathbf{C}) \arrow[r] & \mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathbf{C}_0) \\
B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C}) \arrow[r] \arrow[u, "\mathsf{j}_*"] & B\text{-Hecke}(\mathbf{C}_0) \arrow[u, "\mathsf{j}_*"']
\end{tikzcd}$$

commute.

Note that the functor (23.18) is the same as the functor that we denoted $\mathbf{oblv}_{\mathrm{rel}}$ in Sect. 23.3.1.

The functor (23.17) sends an object $\mathbf{c}$ as in Sect. 23.3.3 to the system $\{\mathbf{c}^\gamma\}$ with

$$\mathbf{c}^\gamma := \mathsf{e}^\gamma \star \mathbf{c}.$$

23.4.7. Applying (23.15), we obtain that the functor

$$\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}} \circ \mathrm{Res}^{B_H}_{T_H} \circ \mathbf{oblv}_{\mathrm{rel}} \circ \mathsf{j}^* : \mathrm{DrPl}(\mathbf{C}) \to \mathrm{Rep}(T_H) \otimes \mathbf{C}$$

sends $\mathbf{c} \in \mathrm{DrPl}(\mathbf{C})$ to the object $\{\widetilde{\mathbf{c}}^\gamma\} \in \mathrm{Rep}(T_H) \otimes \mathbf{C}$ with

$$\widetilde{\mathbf{c}}^{\gamma'} \simeq \underset{\gamma \in \Lambda_H^+}{\mathrm{colim}}\, \mathsf{e}^{-\gamma+\gamma'} \star \mathbf{c} \star V^\gamma. \tag{23.19}$$

23.5. **The semi-infinite IC sheaf via the Drinfeld-Plücker formalism.** We will now show that the object ${}'\mathrm{IC}^{\frac{\infty}{2}}_{q,x} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$ defined in Sect. 13.2 can be obtained via the Drinfeld Plücker formalism.

We will change the notation slightly and denoted the object ${}'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$, when viewed as equipped with the relative Hecke structure (that latter thanks to Theorem 14.2.5) by

$${}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)).$$

From now on we will omit the superscript ω^ρ and the subscript x. So we will write Gr_G instead of $\mathrm{Gr}^{\omega^\rho}_{G,x}$, and $\mathfrak{L}(N)$ instead of $\mathfrak{L}(N)^{\omega^\rho}_x$, etc.

23.5.1. Consider the category $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$. We regard it as acted on from the right by $\mathrm{Rep}(H)$ (via $\mathrm{Sat}_{q,G}$) and on the left by $\mathrm{Rep}(T_H)$ (via $\mathrm{Sat}'_{q,T}$, see Sect. 14.2.3).

We claim that the object $\delta_{1,\mathrm{Gr}} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$ naturally upgrades to an object

$$(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)} \in \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}).$$

The corresponding maps (23.8)

$$\mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{-w_0(\gamma)}} \simeq \delta_{1,\mathrm{Gr}} \star \mathrm{Sat}((V^\gamma)^*) \to \mathsf{e}^{-\gamma} \star \delta_{1,\mathrm{Gr}} \simeq \delta_{t^{-\gamma},\mathrm{Gr}}[\langle \gamma, 2\check{\rho}\rangle]$$

are obtained from the maps (13.1) by adjunction.

The higher compatibilities for these maps are explained in [Ga6, Sect. 2.7].

23.5.2. Consider the resulting object

$$\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)}) \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}).$$

We claim:

Proposition 23.5.3. *The object*

$$\mathrm{Res}^{B_H}_{T_H}(\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)})) \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)})$$

identifies canonically with $'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x}$.

Proof. The fact that

$$\mathbf{oblv}_{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}}(\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)})) \simeq {}'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$$

as plain objects of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$ follows from Sect. 23.3.6.

The fact that the isomorphism respects the graded Hecke structure follows from the construction of the latter on $'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$, see [Ga7, Theorem 5.1.8].

□

23.5.4. From now on we will denote

$$'\widetilde{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x} := \mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)}) \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}),$$

so that the identification of Proposition 23.5.3 says that

$$\mathrm{Res}^{B_H}_{T_H}({}'\widetilde{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x}) \simeq {}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x}.$$

23.5.5. Recall now that $'\mathrm{IC}^{\frac{\infty}{2}}_{q,x}$ was actually an object of the full subcategory

$$((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)} := \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)} \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}.$$

Since the above embeddings are fully faithful, we obtain that $'\widetilde{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x}$ automatically belongs to

$$B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)}) \subset B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}).$$

We will now show that when viewed as such, the object $'\widetilde{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x}$ can also be obtained by applying j^* to an object in $\mathrm{DrPl}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)})$.

23.5.6. Namely, consider the object

$$(\mathbf{i}_0)_!(\omega_{S^0}) \in (\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)}.$$

We claim that it naturally upgrades to an object of

$$((\mathbf{i}_0)_!(\omega_{S^0}))^{\mathrm{DrPl}} \in \mathrm{DrPl}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)}),$$

so that the resulting object $\mathsf{j}^*(((\mathbf{i}_0)_!(\omega_{S^0}))^{\mathrm{DrPl}}) \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)})$ goes over under the embedding

$$B\text{-}\mathrm{Hecke}_{\mathrm{rel}}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)}) \hookrightarrow B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)})$$

to $\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)}) := {}'\widetilde{\mathrm{IC}}^{\frac{\infty}{2}}_{q,x}$.

23.5.7. In order to construct the Drinfeld-Plücker structure on $(\mathbf{i}_0)_!(\omega_{S^0})$, we consider the (partially defined) left adjoint $\mathrm{Av}_!^{\mathfrak{L}(N)}$ to the embedding

$$(\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)} \hookrightarrow \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}.$$

Note that for $\gamma \in \Lambda^\sharp$, we have a canonical identification

$$\mathrm{Av}_!^{\mathfrak{L}(N)}(\delta_{t^\gamma,\mathrm{Gr}}) \simeq (\mathbf{i}_\gamma)_!(\omega_{S^\gamma}).$$

Since Hecke convolutions (for G and for T) are given by proper pushforwards, level-wise application of $\mathrm{Av}_!^{\mathfrak{L}(N)}$ induces a (partially defined) left adjoint to

$$\mathrm{DrPl}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)}) \hookrightarrow \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)})$$

and to

$$B\text{-}\mathrm{Hecke}_{\mathrm{rel}}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)}) \hookrightarrow B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}).$$

Moreover, these functors are intertwined by the corresponding functors j^*, j_*, etc.

23.5.8. Applying $\mathrm{Av}_!^{\mathfrak{L}(N)}$ to $(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)}$ we obtain the desired object $((\mathbf{i}_0)_!(\omega_{S^0}))^{\mathrm{DrPl}}$. Moreover,

$$\mathrm{Av}_!^{\mathfrak{L}(N)}(\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)})) \simeq \mathsf{j}^*(((\mathbf{i}_0)_!(\omega_{S^0}))^{\mathrm{DrPl}}). \tag{23.20}$$

However, since $\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)})$ already belongs to $B\text{-}\mathrm{Hecke}_{\mathrm{rel}}((\mathrm{SI}_{q,x}(G))^{\mathfrak{L}^+(T)})$, we obtain that the left-hand side in (23.20) identifies with $\mathsf{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},\mathfrak{L}^+(T)})$, as desired.

24. The dual baby Verma object in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I$

In order to construct the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$, we will need to make a detour and discuss the category of Iwahori-equivariant sheaves on Gr_G. We will construct a particular “dual baby Verma” object

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}} \in B\text{-}\mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I),$$

and study its properties.

Results of this section are of independent interest as the object $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}$ (and its descendents $\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}$, $\mathcal{F}^{\frac{\infty}{2}}$) are quite ubiquitous in this branch of representation theory, see e.g., [ABBGM], [FG2], [FG3], [Ga8].

In the next section, we apply a simple manipulation to $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}$ and produce from it the sought-for objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$.

24.1. **The Iwahori-equivariant category.** In this subsection we recollect some facts pertaining to the behavior of the category of Iwahori-equivariant sheaves on Gr_G.

24.1.1. Let $I \subset \mathfrak{L}^+(G)$ be the Iwahori subgroup. We consider the category

$$\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^I.$$

In what follows we will use a slightly renormalized version of the category, namely the ind-completion of $(\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^I)^{\text{loc.c}}$. We denote it $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^{I,\text{ren}}$. The category $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^I$ carries a t-structure, for which the tautological functor

$$\text{un-ren} : \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^{I,\text{ren}} \to \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^I \tag{24.1}$$

is t-exact.

The advantage of $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^{I,\text{ren}}$ is that the t-structure on it is Artinian (see Sect. 6.3.8 for what this means).

24.1.2. The category $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^{I,\text{ren}}$ is acted on from the right by $\operatorname{Sph}_{q,x}(G)$ by convolutions.

Consider the affine flag space

$$\operatorname{Fl}_G := \mathfrak{L}(G)/I.$$

Convolution on the left defines an action of the monoidal category $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I$ on $\operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Gr}_G)^{I,\text{ren}}$. This action commutes with the above right action of $\operatorname{Sph}_{q,x}(G)$.

To avoid notational confusion, henceforth, we will denote these two convolution functors by

$$\underset{\mathfrak{L}^+(G)}{\star} - \;\text{ and }\; - \underset{I}{\star},$$

respectively.

24.1.3. We claim that there exists a canonically defined monoidal functor

$$\operatorname{Rep}(T_H) \to \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I, \tag{24.2}$$

Namely, for an element $\gamma \in \Lambda^\sharp = \Lambda_H \subset \Lambda$ consider the corresponding orbit

$$I \cdot t^\gamma \cdot I/I \subset \operatorname{Fl}_G.$$

Our choice of trivialization of $\mathcal{G}_{T_H,x}$ defines an I-equivariant trivialization of the restriction $\mathcal{G}^G|_{I\cdot t^\gamma \cdot I/I}$. Let

$$j_{\gamma,!}, j_{\gamma,*} \in \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I$$

denote the corresponding standard (resp., costandard) objects.

Proposition 24.1.4. *We have canonical isomorphisms*

$$j_{\gamma,!} \underset{I}{\star} j_{\gamma,*} \simeq \delta_{1,\operatorname{Fl}} \simeq j_{\gamma,*} \underset{I}{\star} j_{\gamma,!}.$$

Proof. First, we notice that the functor of convolution

$$j_{\gamma,!} \underset{I}{\star} - : \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I \to \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I$$

is the left adjoint of

$$j_{-\gamma,*} \underset{I}{\star} - : \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I \to \operatorname{Shv}_{\mathcal{G}^G}(\operatorname{Fl}_G)^I.$$

Now the assertion follows from the fact that these functors are self-equivalences, hence "adjoint" is the same as "inverse".

□

24.1.5. The sought-for functor (24.2) is uniquely determined by the condition that it sends

$$\mathsf{e}^\gamma \mapsto j_{\gamma,!} \text{ for } \gamma \in \Lambda_H^+.$$

From Proposition 24.1.4 it follows that (24.2) sends

$$\mathsf{e}^{-\gamma} \mapsto j_{-\gamma,*} \text{ for } \gamma \in \Lambda_H^+.$$

For general $\gamma \in \Lambda_H$, let us denote by J_γ the image of e^γ under (24.2).

24.1.6. Thus, we obtain that the category $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$ is acted on by $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$.

Remark 24.1.7. Note, however, that while the action of $(\mathrm{Rep}(H))^\heartsuit$ on $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$ is given by t-exact functors, this is *not* the case for $(\mathrm{Rep}(T_H))^\heartsuit$. So, while the categories

$$B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}) \text{ and } \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

carry well-behaved t-structures, the relative versions

$$B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}) \text{ and } \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

do not.

24.2. **Construction of the dual baby Verma object.** In this subsection we define the main player in this section–the dual baby Verma object in $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$.

24.2.1. Consider the object $\delta_{1,\mathrm{Gr}} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$. We claim that it naturally upgrades to an object

$$(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},I} \in \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

The corresponding system of maps (23.8) is given by

$$\mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{-w_0(\gamma)}} \to j_{-\gamma,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \tag{24.3}$$

obtained by adjunction from the maps

$$j_{\gamma,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \to \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{\gamma}}, \tag{24.4}$$

the latter being maps corresponding to the open embeddings

$$I \cdot t^\gamma \mathfrak{L}^+(G)/\mathfrak{L}^+(G) \hookrightarrow \mathfrak{L}^+(G) \cdot t^\gamma \mathfrak{L}^+(G)/\mathfrak{L}^+(G) = \mathrm{Gr}_G^\gamma.$$

One easily checks that the maps (24.3) satisfy the compatibility expressed by diagram (23.9); namely one checks the commutativity of the corresponding diagram for the maps (24.4):

$$\begin{tikzcd}
j_{\gamma_1,!} \underset{I}{\star} j_{\gamma_2,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \arrow[r] \arrow[d, "\sim"'] & j_{\gamma_1,!} \underset{I}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{\gamma_2}} \arrow[d, "\sim"] \\
j_{\gamma_1+\gamma_2,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \arrow[r] \arrow[d] & j_{\gamma_1,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{\gamma_2}} \arrow[d, "\sim"] \\
\mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{\gamma_1+\gamma_2}} \arrow[r] & \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{\gamma_1}} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{\gamma_2}} .
\end{tikzcd}$$

The higher compatibilities hold automatically, as the objects involved in (24.4) belong to the heart of the t-structure.

24.2.2. Let $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}$ denote the object of $B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$ equal to

$$\mathrm{j}^*((\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},I}).$$

We will also consider several objects obtained from $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}$ by applying the various forgetful functors:

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{\mathrm{rel}} := \mathrm{Res}^{B_H}_{T_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}) \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}});$$

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}} := \mathbf{oblv}_{\mathrm{rel}}(\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}) \in B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}});$$

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}} := \mathrm{Res}^{B_H}_{T_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2}}) \simeq \mathbf{oblv}_{\mathrm{rel}}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{\mathrm{rel}}) \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

Remark 24.2.3. We can also consider the object

$$\mathcal{F}^{\frac{\infty}{2}} := \mathrm{Res}^{B_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2}}) \simeq \mathrm{Res}^{T_H}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}) \in \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

But this object will not play a prominent role in this paper.

24.2.4. Recall now (see Remark 24.1.7) that the categories

$$B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}) \text{ and } \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

each carries a well-behaved t-structure, and the restriction functor $\mathrm{Rep}_{T_H}^{B_H}$ t-exact.

We claim:

Proposition 24.2.5. *The object $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}$ (resp., $\overset{\bullet}{\mathcal{F}}^{\frac{\infty}{2}}$) belongs to $(B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^{\heartsuit}[d]$ (resp., $(\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^{\heartsuit}[d]$).*

Proof. It suffices to prove the assertion for $\overset{\bullet}{\mathcal{F}}^{\frac{\infty}{2}}$. Applying (23.19), and using the fact that the poset $(\Lambda_H, \preceq)$ is filtered, it suffices to see that for a fixed γ' and cofinal set of γ's, the objects

$$j_{-\gamma+\gamma',*} \underset{I}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{\gamma}_G}$$

belong to $(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^{\heartsuit}[d]$.

We claim that this happens as soon as $-\gamma+\gamma' =: \gamma_0$ is dominant regular. Indeed, convolution with $\mathrm{Sph}_{q,x}(G)$ is t-exact, so it suffices to see that the objects

$$j_{-\gamma_0,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}}$$

belong to $(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^{\heartsuit}[d]$ for γ_0 dominant regular. Indeed, in this case, the map

$$I \cdot t^{-\gamma_0} \cdot I/I \to I \cdot t^{-\gamma} \cdot \mathfrak{L}^+(G)/\mathfrak{L}^+(G)$$

is a fibration into affine spaces of dimension d, while the inclusion

$$I \cdot t^{-\gamma_0} \cdot \mathfrak{L}^+(G)/\mathfrak{L}^+(G) \hookrightarrow \mathrm{Gr}_G$$

is affine.

□

24.3. **Relation to the semi-infinite IC sheaf.** In this subsection we will see that the object

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}} \in B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

introduced above, is closely related to the semi-infinite IC sheaf

$$'\widetilde{\mathrm{IC}}^{\frac{\infty}{2}}_{q,x} \in B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}).$$

A similar relationship will hold for their descendants, in particular for

$$\overset{\bullet}{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}) \text{ and } '\overset{\bullet}{\mathrm{IC}}^{\frac{\infty}{2}}_{q,x} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

24.3.1. We recall (see, e.g., [Ga6, Proposition 5.2.2]) that the (partially defined) functor

$$\mathrm{Av}_!^{\mathfrak{L}(N)} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)} =: \mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)}.$$

restricted to

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$$

defines an equivalence

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I \to \mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)}.$$

The inverse equivalence is given by $\mathrm{Av}_*^{\overset{\circ}{I}}$, where $\overset{\circ}{I}$ is the unipotent radical of I.

This equivalence is compatible with the right action of $\mathrm{Rep}(H)$ and with the left action of $\mathrm{Rep}(T_H)$. Hence, it induces equivalences

$$\mathrm{Av}_!^{\mathfrak{L}(N)} : \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I) \to \mathrm{DrPl}(\mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)})$$

and

$$\mathrm{Av}_!^{\mathfrak{L}(N)} : B\text{-Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I) \to B\text{-Hecke}_{\mathrm{rel}}(\mathrm{DrPl}(\mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)}),$$

which intertwine the functors j^* and j_*.

24.3.2. By a slight abuse of notation, let us denote by the same symbol $(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},I}$ the image of (what was previously denoted) $(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},I}$ under the functor

$$\mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}} \to \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I)$$

induced by the functor un-ren of (24.1). We will use a similar convention for $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}$.

We claim:

Proposition 24.3.3. *The functor* $\mathrm{Av}_!^{\mathfrak{L}(N)}$ *sends*

$$(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},I} \in \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I)$$

to

$$((\mathbf{i}_0)_!(\omega_{S^0}))^{\mathrm{DrPl}} \in \mathrm{DrPl}(\mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)}).$$

Proof. The proof follows from the fact that the objects $j_{-\gamma,*}$ for $\gamma \in \Lambda_H^+$ equipped with the maps

$$\mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{-w_0(\gamma)}} \to j_{-\gamma,*}$$

can be obtained from the objects $\mathrm{Av}_!^{\mathfrak{L}(N)}(\delta_{t^\gamma,\mathrm{Gr}})[\langle -\gamma, 2\check{\rho}\rangle]$ equipped with the maps

$$\mathrm{Av}_!^{\mathfrak{L}(N)}(\delta_{1,\mathrm{Gr}}) \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{-w_0(\gamma)}} \to \mathrm{Av}_!^{\mathfrak{L}(N)}(\delta_{t^{-\gamma},\mathrm{Gr}})[\langle -\gamma, 2\check{\rho}\rangle]$$

by applying the functor $\mathrm{Av}_*^{\overset{\circ}{I}}$.

□

Corollary 24.3.4. *The functor* $\mathrm{Av}_!^{\mathfrak{L}(N)}$ *sends*

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}} \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I) \mapsto {}'\widetilde{\mathrm{IC}}^{\frac{\infty}{2}}_{q,x} \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{DrPl}(\mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)})$$

and

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{\mathrm{rel}} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I) \mapsto {}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{DrPl}(\mathrm{SI}_{q,x}(G)^{\mathfrak{L}^+(T)}).$$

24.4. **A twist by w_0.** We will now perform a (rather elementary) manipulation with $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}}$–a twist by the longest element of the Weyl group. This way, we will define the object $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}_{\mathrm{rel}}$ that we are actually after.

24.4.1. Consider the object

$$j_{w_0,*} \in \mathrm{Shv}(G/B)^B \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G)^I.$$

Define

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2},w_0} := j_{w_0,*} \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2}}[-d] \in B\text{-}\mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}});$$

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},w_0} := j_{w_0,*} \underset{I}{\star} \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}[-d] \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}),$$

where $d = \dim(G/B)$.

24.4.2. Note that by Sect. 23.4, the object $\widetilde{\mathcal{F}}^{\frac{\infty}{2},w_0}$ can be thought of as obtained by applying the functor j^* to an object of

$$\mathrm{DrPl}(\mathrm{Rep}(T_H) \otimes \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

with components

$$j_{w_0,*} \underset{I}{\star} J_{\gamma,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}}[-d]. \tag{24.5}$$

Note that for $\gamma \in \Lambda_H^+$, we have

$$j_{w_0,*} \underset{I}{\star} J_{-\gamma,!} \simeq j_{w_0,*} \underset{I}{\star} j_{-\gamma,*} \simeq j_{-w_0(\gamma),*} \underset{I}{\star} j_{w_0,*}.$$

Hence, the above object identifies with

$$j_{-w_0(\gamma),*} \underset{I}{\star} \delta_{1,\mathrm{Gr}}, \tag{24.6}$$

where we have used the fact that $j_{w_0,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \simeq \delta_{1,\mathrm{Gr}}[d]$.

24.4.3. We claim:

Proposition 24.4.4. *The object* $\widetilde{\mathcal{F}}^{\frac{\infty}{2},w_0}$ *belongs to* $(B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^\heartsuit$*; the same is true for* $\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},w_0}$.

Proof. As in the proof of Proposition 24.2.5, it suffices to show that the objects

$$j_{w_0,*} \underset{I}{\star} J_{-\gamma,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}[-d]$$

belong to $(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^\heartsuit$ for $\gamma \in \Lambda^+$. Using (24.6), we rewrite the above object as

$$j_{-w_0(\gamma),*} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \star \mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}.$$

Since the functor $- \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}$ is t-exact, it suffices to show that

$$j_{\gamma,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \in (\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^\heartsuit$$

for $\gamma \in \Lambda^+_H$.

The latter follows from the fact that the projection

$$I \cdot t^\gamma \cdot I/I \to I \cdot t^\gamma \cdot \mathfrak{L}^+(G)/\mathfrak{L}^+(G)$$

is an isomorphism, while the inclusion

$$I \cdot t^\gamma \cdot \mathfrak{L}^+(G)/\mathfrak{L}^+(G) \hookrightarrow \mathrm{Gr}_G$$

is affine.

□

24.4.5. *Digression.* Note that we have a *canonical* equivalence

$$\mathrm{Rep}(B_H) \to \mathrm{Rep}(B_H^-)$$

as $\mathrm{Rep}(H)$-module categories.

Indeed, choose of a representative w_0' of $w_0 \in W_H$. Then conjugation by w_0' defines the required functor. Now, as two such choices differ by an element in B_H, the corresponding functors are canonically identified.

Similarly, the action of w_0 defines a canonical self-equivalence of $\mathrm{Rep}(T_H)$ as a $\mathrm{Rep}(H)$-module category.

In particular, for $\mathbf{C}$ acted on by $\mathrm{Rep}(H)$, we have canonical equivalences

$$B\text{-Hecke}(\mathbf{C}) \overset{w_0}{\to} B^-\text{-Hecke}(\mathbf{C}),$$

and

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \overset{w_0}{\to} \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}),$$

that make the diagram

$$\begin{tikzcd} B\text{-Hecke}(\mathbf{C}) \arrow[r, "w_0"] \arrow[d, "\mathrm{Res}^{B_H}_{T_H}"'] & B^-\text{-Hecke}(\mathbf{C}) \arrow[d, "\mathrm{Res}^{B_H^-}_{T_H}"] \\ \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \arrow[r, "w_0"] & \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \end{tikzcd}$$

commute.

24.4.6. Define the objects

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} := w_0(\widetilde{\mathcal{F}}^{\frac{\infty}{2},w_0}) \in B^-\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

and

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-} := w_0(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},w_0}) \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

Note that we have

$$\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-} \simeq \mathrm{Res}^{B_H^-}_{T_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}).$$

For completeness, define also

$$\mathcal{F}^{\frac{\infty}{2},-} : \mathrm{Res}^{B_H^-}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}) \simeq \mathrm{Res}^{T_H}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-}) \in \mathrm{Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

24.4.7. By construction, the object $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}$ is obtained by applying the functor j^* to an object of $\mathrm{DrPl}^-(\mathrm{Rep}(T_H) \otimes \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$ that corresponds, in terms of Sect. 23.4.2, to the system

$$\gamma \mapsto j_{w_0,*} \underset{I}{\star} J_{w_0(\gamma),!} \underset{I}{\star} \delta_{1,\mathrm{Gr}}[-d].$$

Note that for $\gamma \in \Lambda_H^+$, the above object identifies with

$$j_{\gamma,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}}. \tag{24.7}$$

Hence, according to (23.19), the object

$$\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}} \circ \mathrm{Res}^{B_H^-}_{T_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}) \in \mathrm{Rep}(T_H) \otimes \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$$

is given by

$$\gamma' \rightsquigarrow \underset{\gamma}{\mathrm{colim}}\, j_{\gamma+\gamma',*} \underset{I}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}_G^{-w_0(\gamma)}}, \tag{24.8}$$

where the colimit runs over the set $\gamma \in (-\gamma' + \Lambda_H^+) \cap \Lambda_H^+$.

24.5. **Finiteness properties of $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}$.** In this subsection we will state a crucial finiteness property of the object $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}$ constructed above. It will instrumental in establishing the required properties of the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$.

24.5.1. Recall, following (23.16), that according to Sect. 24.4.7, for $\gamma \in \Lambda_H^+$, there exists a canonical map

$$j_{\gamma,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \to \mathrm{coInd}^H_{B_H^-}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^\gamma). \tag{24.9}$$

We are going to prove:

Theorem 24.5.2.
(a) *The object $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \in B^-\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$ is compact.*
(b) *The maps* (24.9) *(with $\gamma \in \Lambda_H^+$) are isomorphisms.*

As a special case of point (b) of the theorem we obtain:

Corollary 24.5.3. *The map $\delta_{1,\mathrm{Gr}} \to \mathrm{coInd}^H_{B_H^-}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-})$ is an isomorphism.*

We can now amplify point (b) of Theorem 24.5.2 as follows:

Corollary 24.5.4. *For* any $\gamma \in \Lambda_H$ *we have*

$$\mathrm{coInd}^H_{B_H^-}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^\gamma) \simeq j_{w_0,*} \underset{I}{\star} J_{w_0(\gamma)} \underset{I}{\star} \delta_{1,\mathrm{Gr}}[-d].$$

Proof. By construction, for any $\gamma \in \Lambda_H$, we have

$$J_\gamma \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2}} \simeq \widetilde{\mathcal{F}}^{\frac{\infty}{2}} \otimes \mathsf{e}^\gamma.$$

Hence,

$$j_{w_0,*} \underset{I}{\star} J_\gamma \underset{I}{\star} j_{w_0,!} \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \simeq \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^{w_0(\gamma)}.$$

This implies the required assertion by applying $\mathrm{coInd}^H_{B^-_H}$ to both sides and using Corollary 24.5.3. □

Remark 24.5.5. Note that for $\gamma \in \Lambda^+_H$, the object

$$j_{w_0,*} \underset{I}{\star} J_{w_0(-\gamma)} \underset{I}{\star} \delta_{1,\mathrm{Gr}}[-d]$$

that appears in Corrollary 24.5.4 identifies with

$$j_{w_0,*} \underset{I}{\star} j_{w_0(-\gamma),!} \underset{I}{\star} \delta_{1,\mathrm{Gr}}[-d].$$

Note also that if γ is regular, then the latter object identifies with

$$j_{-\gamma\cdot w_0,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}}[-d],$$

where we note that

$$j_{-\gamma\cdot w_0,!} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \in (\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})^\heartsuit.$$

24.5.6. The rest of this section is essentially devoted to the proof of Theorem 24.5.2. We will give two proofs: the first one by mimicking certain arguments from the paper [ABBGM]. And the second one, which actually explains "what is going on" via a metaplectic version of (some aspects of) the Arkhipov-Bezrukavnikov theory.

24.6. **First proof of Theorem 24.5.2.**

24.6.1. In Sect. 24.7 we will prove:

Theorem 24.6.2. *The action of* $\mathrm{Rep}(H)$ *on* $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$ *is accessible.*

In the rest of this subsection we will show how Theorem 24.6.2 implies Theorem 24.5.2.

24.6.3. First, we have the following assertion, whose proof is a verbatim repetition of the proof of the argument in [ABBGM, Proposition 3.2.6]:

Proposition 24.6.4. *The map*

$$\mathrm{Res}^H_{B^-_H}(j_{\gamma,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}}) \to \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^\gamma \tag{24.10}$$

arising by adjunction from (24.9)*, is surjective, as a map of objects in* $(B^-\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^\heartsuit$*, for any* γ *which is regular.*

24.6.5. Note that by Corollary 23.2.6, from Proposition 24.6.4 we immediately obtain that $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}$ is compact. This is point (a) of Theorem 24.5.2.

To prove point (b), by the argument in Corollary 24.5.4, it is enough to prove the assertion for some γ that is sufficiently dominant. Fix a dominant *regular* γ_0. We will show that the map (24.9) is an isomorphism for $\gamma_0 + \gamma$ for all γ that are sufficiently dominant.

24.6.6. Consider the map (24.10) for our γ_0. By Propositions 23.2.4 and 24.6.4, for all γ that are sufficiently dominant, the map

$$j_{\gamma_0,*} \underset{I}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G} \simeq (j_{\gamma_0,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}}) \underset{\mathfrak{L}^+(G)}{\star} \mathrm{Sat}_{q,G}(V^\gamma) \simeq \mathrm{coInd}^H_{B^-_H}(\mathrm{Res}^H_{B^-_H}(j_{\gamma_0,*} \underset{I}{\star} \delta_{1,\mathrm{Gr}}) \otimes \mathsf{e}^\gamma) \to$$
$$\to \mathrm{coInd}^H_{B^-_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^{\gamma_0} \otimes \mathsf{e}^\gamma) \simeq \mathrm{coInd}^H_{B^-_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^{\gamma_0+\gamma})$$

is surjective.

However, it is easy to see that the latter map factors as

$$j_{\gamma_0,*} \underset{I}{\star} \mathrm{Sat}_{q,G}(V^\gamma) \to j_{\gamma_0+\gamma,*} \to \mathrm{coInd}^H_{B^-_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^{\gamma_0+\gamma}),$$

where:

- The first arrow is the map given by applying
$$j_{\gamma_0,*} \underset{I}{\star} j_{w_0,*} \underset{I}{\star} -[-d]$$
to the map (24.3);
- The second arrow is the map (24.9) for $\gamma_0 + \gamma$.

Thus, we obtain that the map

$$j_{\gamma_0+\gamma,*} \to \mathrm{coInd}^H_{B^-_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^{\gamma_0+\gamma}) \tag{24.11}$$

above is surjective, as a map of objects in $(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})^\heartsuit$. Hence, it remains to show that the map (24.11) is injective (as a map of objects in $(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})^\heartsuit$).

24.6.7. We will show that the map (24.9) is injective for any $\gamma \in \Lambda^+_H$.

Note that the map

$$\mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G} \to j_{\gamma,*}$$

identifies $\mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}$ with the *socle* of $j_{\gamma,*}$. Hence, it is sufficient show that the composite

$$\mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G} \to j_{\gamma,*} \to \mathrm{coInd}^H_{B^-_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^\gamma)$$

is injective (equivalently, non-zero). Equivalently, we have to show that the map

$$\mathrm{Res}^H_{B^-_H}(\mathrm{IC}_{q,\overline{\mathrm{Gr}}^\gamma_G}) \to \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^\gamma \tag{24.12}$$

is non-zero.

After passing from $\widetilde{\mathcal{F}}^{\frac{\infty}{2},-}$ back to $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}$, the latter map fits into the following paradigm.

24.6.8. We start from an object $\mathbf{c}^{\mathrm{DrPl}} \in \mathrm{DrPl}(\mathbf{C})$ with the underlying object of $\mathbf{C}$ denoted by $\mathbf{c}$. Denote

$$\mathbf{c}' := \mathbf{oblv}_{\mathrm{rel}} \circ \mathsf{j}^*(\mathbf{c}^{\mathrm{DrPl}}) \in \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Rep}(B_H).$$

Then, according to (23.16), we have a canonical map

$$\mathrm{Res}^H_{B_H}(\mathbf{c}) \to \mathbf{c}'. \tag{24.13}$$

The map (24.12) equals the composite

$$\mathrm{Res}^H_{B_H}(\mathbf{c} \star (V^\gamma)^*) \to \mathbf{c}' \otimes \mathrm{Res}^H_{B_H}((V^\gamma)^*) \to \mathbf{c}' \otimes \mathsf{e}^{-\gamma}. \tag{24.14}$$

We claim that this map is non-zero under the following circumstances:

- (i) $\mathbf{C}$ is equipped with a t-structure such that $(\mathrm{Rep}(H))^\heartsuit$ acts by t-exact functors;
- (ii) $\mathsf{e}^{-\gamma} \star \mathbf{c} \in (\mathbf{C})^\heartsuit$ for γ *sufficiently dominant*;
- (iii) $\mathbf{c}$ is compact
- (iv) The maps (23.8) are non-zero.

24.6.9. Indeed, the map (24.14) comes by adjunction from the map

$$\mathrm{Res}^H_{B_H}(\mathbf{c}) \to \mathbf{c}' \to \mathbf{c}' \otimes \mathsf{e}^{-\gamma} \otimes \mathrm{Res}^H_{B_H}(V^\gamma). \tag{24.15}$$

So it is sufficient to show that the latter map is non-zero. By (23.19), and assumptions (i) and (ii) on the object $\mathbf{c}$,

$$\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}} \circ \mathrm{Res}^{B_H}_{T_H}(\mathbf{c}') \in (\mathrm{Rep}(T_H) \otimes \mathbf{C})^{\heartsuit}.$$

Hence, $\mathbf{c}' \in (B\text{-Hecke}(\mathbf{C}))^{\heartsuit}$ and the second arrow in (24.15) is injective. Hence, it remains to show that the first arrow in (24.15), i.e., map (24.13), is non-zero.

24.6.10. To prove the latter, it suffices to show that the induced map

$$\mathbf{ind}_{\overset{\bullet}{\mathrm{Hecke}}}(\mathbf{c}) \simeq \mathrm{Res}^H_{T_H}(\mathbf{c}) \simeq \mathrm{Res}^{B_H}_{T_H} \circ \mathrm{Res}^H_{B_H}(\mathbf{c}) \to \mathrm{Res}^{B_H}_{T_H}(\mathbf{c}')$$

is non-zero, i.e., the map

$$\mathsf{e} \otimes \mathbf{c} \to \mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}} \circ \mathrm{Res}^{B_H}_{T_H}(\mathbf{c}')$$

is non-zero.

Applying (23.19), this equivalent to the fact that the map

$$\mathbf{c} \to \underset{\gamma \in \Lambda^+_H}{\mathrm{colim}}\, \mathsf{e}^{-\gamma} \star \mathbf{c} \star V^\gamma$$

is non-zero in $\mathrm{Rep}(T_H) \otimes \mathbf{C}$.

Since $\mathbf{c}$ is compact and $(\Lambda^+_H, \preceq)$ is filtered, it suffices to show that the individual maps

$$\mathbf{c} \to \mathsf{e}^{-\gamma} \star \mathbf{c} \star V^\gamma$$

are non-zero. However, these maps are obtained by adjunction from the maps (23.8).

24.7. **Proof of Theorem 24.6.2.** The proof will be an adaptation of the argument of [ABBGM, Theorem 1.3.5].

24.7.1. Recall that I-orbits on Gr_G are in bijection with cosets $W^{\mathrm{aff,ext}}/W$. Note that for an element $\widetilde{w} = w \cdot \lambda$ of $W^{\mathrm{aff,ext}}$, the orbit

$$I \cdot \widetilde{w} \cdot \mathfrak{L}^+(G)/\mathfrak{L}^+(G)$$

carries an I-equivariant $\mathcal{G}^G$-twisted sheaf if and only if $\lambda \in \Lambda^+_H$.

For $\gamma \in \Lambda^+_H$, let

$$\mathrm{IC}_{q,w\cdot\gamma} \in (\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^{\heartsuit}$$

denote the corresponding irreducible object.

We claim:

Theorem 24.7.2. *Let $\gamma \in \Lambda^+_H$ and $w \in W$ be a pair of elements such that*

$$\langle \gamma, \check{\alpha}_i \rangle = \begin{cases} 0, & \text{if } w(\alpha_i) \in \Lambda^{\mathrm{pos}}; \\ \mathrm{ord}(q_i), & \text{if } w(\alpha_i) \in \Lambda^{\mathrm{neg}}. \end{cases}$$

Then the object $\mathrm{IC}_{q,w\cdot\gamma} \in (\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}))^{\heartsuit}$ is restricted. In fact,

$$\mathrm{IC}_{q,w\cdot\gamma} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\mathrm{Gr}^{\gamma'}_G} \simeq \mathrm{IC}_{q,w\cdot(\gamma+\gamma')}.$$

This theorem implies Theorem 24.6.2 in the same way as [ABBGM, Theorem 1.3.5(1)] implies [ABBGM, Theorem 1.3.5(2)] using the assumption that the derived group of H is simply-connected.

24.7.3. To prove Theorem 24.7.2 we repeat the argument in [ABBGM, Sect. 2.1]. The only difference is that instead of [FGV, Theorem 7.1.7] we use Theorem 9.4.9, applied to the weight λ defined by the formula

$$\langle \lambda, \check{\alpha}_i \rangle = \begin{cases} 0, & \text{if } w(\alpha_i) \in \Lambda^{\mathrm{pos}}; \\ \mathrm{ord}(q_i) - 1, & \text{if } w(\alpha_i) \in \Lambda^{\mathrm{neg}}. \end{cases}$$

24.8. **Metaplectic Arkhipov-Bezrukavnikov theory.** In this section we will make preparations for another proof of Theorem 24.5.2 by introducing a metaplectic analog of (some aspects of) the theory developed in [AB].

24.8.1. Consider the following example of a category equipped with an action of $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$:

$$\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H).$$

The self-duality of $\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H)$ as a module over $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$ implies that for any $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$-module category $\mathbf{C}$, we have:

$$\mathrm{DrPl}(\mathbf{C}) \simeq \mathrm{Funct}_{\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)}(\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H), \mathbf{C}). \tag{24.16}$$

In particular, taking the identity functor on $\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H)$, we obtain that the object

$$\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H} \in \mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H)$$

admits a canonical lift to an object

$$(\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H})^{\mathrm{DrPl}} \in \mathrm{DrPl}(\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H)).$$

Under the equivalence (24.16), for a functor

$$\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H) \to \mathbf{C}$$

the corresponding object of $\mathrm{DrPl}(\mathbf{C})$ is the image of $(\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H})^{\mathrm{DrPl}}$ under this functor.

24.8.2. Next consider

$$\mathrm{Rep}(B_H) \simeq \mathrm{QCoh}(H\backslash(H/N_H)/T_H)$$

as a category equipped with an action of $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$.

The self-duality of $\mathrm{Rep}(B_H)$ as a module over $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$ implies that for any $\mathbf{C}$, we have:

$$B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathbf{C}) \simeq \mathrm{Funct}_{\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)}(\mathrm{Rep}(B_H), \mathbf{C}). \tag{24.17}$$

In particular, the object $\mathsf{e} \in \mathrm{Rep}(B_H)$ admits a canonical lift to an object

$$\mathsf{e}^{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}} \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Rep}(B_H)),$$

which corresponds to the identity functor on $\mathrm{Rep}(B_H)$.

Under the equivalence (24.17), for a functor $\mathrm{Rep}(B_H) \to \mathbf{C}$, the corresponding object of $B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathbf{C})$ is the image of $\mathsf{e}^{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}}$ under the above functor.

24.8.3. One can describe the object $\mathsf{e}^{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}}$ explicitly. Namely, we identify

$$B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathrm{Rep}(B_H)) \simeq \mathrm{QCoh}((H/N_H)/\mathrm{Ad}(B_H)),$$

and $\mathsf{e}^{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}}$ is the image of the structure sheaf along the following composition of closed embeddings

$$\mathrm{pt}/B_H \to T_H/\mathrm{Ad}(B_H) \hookrightarrow (H/N_H)/\mathrm{Ad}(B_H).$$

In particular, the object

$$\mathsf{e}^{B\text{-}\mathrm{Hecke}} := \mathbf{oblv}_{\mathrm{rel}}(\mathsf{e}^{B\text{-}\mathrm{Hecke}_{\mathrm{rel}}}) \in B\text{-}\mathrm{Hecke}(\mathrm{Rep}(B_H))$$

with respect to the identification

$$B\text{-}\mathrm{Hecke}(\mathrm{Rep}(B_H)) \simeq \mathrm{QCoh}(B_H\backslash H/B_H)$$

is the image of the structure sheaf along the map

$$\mathsf{i} : \mathrm{pt}/B_H \simeq B_H\backslash B_H/B_H \hookrightarrow B_H\backslash H/B_H.$$

From here it follows that given an object $\mathbf{c} \in B\text{-}\mathrm{Hecke}_{\mathrm{rel}}(\mathbf{C})$, the resulting functor $\mathrm{Rep}(B_H) \to \mathbf{C}$ sends

$$\mathsf{e} \in \mathrm{Rep}(B_H) \rightsquigarrow \mathrm{coInd}^H_{B_H}(\mathbf{oblv}_{\mathrm{rel}}(\mathbf{c})). \tag{24.18}$$

24.8.4. Under the identifications (24.16) and (24.17), the functor $\mathsf{j}^* : \mathrm{DrPl}(\mathbf{C}) \to B\text{-Hecke}_{\mathrm{rel}}(\mathbf{C})$ corresponds to precomposition with

$$\mathsf{j}_* : \mathrm{QCoh}(H\backslash(H/N_H)/T_H) \to \mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H).$$

24.8.5. Take $\mathbf{C} = \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$ and

$$(\delta_{1,\mathrm{Gr}})^{\mathrm{DrPl},I} \in \mathrm{DrPl}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

Consider the corresponding functor, denoted

$$\widetilde{\mathrm{AB}} : \mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}. \tag{24.19}$$

By the above, we obtain that

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}}_{\mathrm{rel}} \simeq \widetilde{\mathrm{AB}} \circ \mathsf{j}_*(\mathsf{e}^{B\text{-Hecke}_{\mathrm{rel}}}),$$

and hence

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}} \simeq \widetilde{\mathrm{AB}} \circ \mathsf{j}_*((\mathsf{i})_*(\mathsf{e})). \tag{24.20}$$

24.8.6. The following is the metaplectic analog of the result of [AB, Theorem 3.1.4]:

Theorem 24.8.7. *The functor* $\widetilde{\mathrm{AB}}$ *of* (24.19) *factors through the localization*

$$\mathsf{j}^* : \mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H) \to \mathrm{QCoh}(H\backslash(H/N_H)/T_H).$$

Remark 24.8.8. Another way to formulate Theorem 24.8.7 is that the functor (24.19) sends $\ker(\mathsf{j}^*)$ to zero.

24.8.9. Theorem 24.8.7 can be proved by repeating verbatim the proof in [AB] of the usual (i.e., nin-metaplectic) version. Alternatively, in the next subsection we will see that Theorem 24.8.7 is logically equivalent to Theorem 24.5.2.

24.9. **Second proof of Theorem 24.5.2.** We will now show that Theorems 24.8.7 and 24.5.2 tautologically imply one another.

24.9.1. First, let us assume Theorem 24.8.7.

Let

$$\mathrm{AB} : \mathrm{Rep}(B_H) \simeq \mathrm{QCoh}(H\backslash(H/N_H)/T_H) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}} \tag{24.21}$$

denote the resulting functor. Note, however, that since $\mathsf{j}^* \circ \mathsf{j}_* \simeq \mathrm{Id}$, we have

$$\mathrm{AB} \simeq \widetilde{\mathrm{AB}} \circ \mathsf{j}_*. \tag{24.22}$$

24.9.2. Point (a) of Theorem 24.5.2 is equivalent to the assertion that $\widetilde{\mathcal{F}}^{\frac{\infty}{2}} \in B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$ is compact.

The functor $\widetilde{\mathrm{AB}}$ sends the generator of $\mathrm{QCoh}(H\backslash(\overline{H/N_H})/T_H)$, viewed as a $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$-module category, i.e., $\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H}$, to a compact object of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}$, i.e., $\delta_{1,\mathrm{Gr}}$. Hence, $\widetilde{\mathrm{AB}}$ sends compacts to compacts.

The latter formally implies that the functor AB also sends compacts to compacts. Hence, AB admits a continuous right adjoint, which automatically also respects the action of $\mathrm{Rep}(H) \otimes \mathrm{Rep}(T_H)$. The adjoint pair

$$\mathrm{AB} : \mathrm{QCoh}(H\backslash(H/N_H)/T_H) \rightleftarrows \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}} : \mathrm{AB}^R$$

induces the (same-named) adjoint pair

$$B\text{-Hecke}(\mathrm{QCoh}(H\backslash(H/N_H)/T_H)) \rightleftarrows B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}}).$$

In particular, the functor

$$\mathrm{AB} : B\text{-Hecke}(\mathrm{Rep}(B_H)) \to B\text{-Hecke}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}})$$

admits a continuous right adjoint, and hence sends compacts to compacts. In particular, the image of $(\mathsf{i})_*(\mathsf{e})$ is compact. However,

$$\mathrm{AB}\circ(\mathsf{i})_*(\mathsf{e}) \overset{(24.22)}{\simeq} \widetilde{\mathrm{AB}}\circ \mathsf{j}_*\circ(\mathsf{i})_*(\mathsf{e}) \overset{(24.20)}{\simeq} \widetilde{\mathcal{F}}^{\frac{\infty}{2}},$$

whence the latter is compact, as required.

24.9.3. Point (b) of Theorem 24.5.2 is equivalent to the assertion that the maps

$$J_\gamma \underset{I}{\star} \delta_{1,\mathrm{Gr}} \to \mathrm{coInd}^H_{B_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2}})$$

that arise from (23.16) are isomorphisms.

To prove this, by applying the functor AB, it suffices to show that the corresponding maps

$$\mathsf{e}^\gamma \to \mathrm{coInd}^H_{B_H}((\mathsf{i})_*(\mathsf{e})\otimes \mathsf{e}^\gamma)$$

are isomorphisms in $\mathrm{Rep}(B_H)$. Note that

$$\mathrm{coInd}^H_{B_H}: B\text{-Hecke}(\mathrm{Rep}(B_H)) \to \mathrm{Rep}(B_H)$$

identifies with the direct image functor along

$$B_H\backslash H/B_H \to \mathrm{pt}/B_H.$$

This makes the assertion obvious.

24.9.4. Vice versa, let us assume Theorem 24.5.2(b) and deduce Theorem 24.8.7. We need to show that the natural transformation

$$\widetilde{\mathrm{AB}} \to \widetilde{\mathrm{AB}}\circ \mathsf{j}_*\circ \mathsf{j}^*, \tag{24.23}$$

induced by the unit of the adjunction $\mathrm{Id}\to \mathsf{j}_*\circ\mathsf{j}^*$ is an isomorphism.

Since both functors respect the action of $\mathrm{Rep}(H)\otimes\mathrm{Rep}(T_H)$, it suffices to show that the natural transformation (24.23) induces an isomorphism

$$\widetilde{\mathrm{AB}}(\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H}) \to \widetilde{\mathrm{AB}}\circ \mathsf{j}_*\circ \mathsf{j}^*(\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H}).$$

However, it is easy to see that we have a commutative diagram

$$\begin{tikzcd}
\widetilde{\mathrm{AB}}(\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H}) \arrow[r] \arrow[dd, "\sim"'] & \widetilde{\mathrm{AB}}\circ \mathsf{j}_*\circ \mathsf{j}^*(\mathcal{O}_{H\backslash(\overline{H/N_H})/T_H}) \arrow[d, "\sim"] \\
& \widetilde{\mathrm{AB}}\circ \mathsf{j}_*(\mathcal{O}_{H\backslash(H/N_H)/T_H}) \arrow[d, "\sim"', "(24.20)\text{ and }(24.18)"] \\
\delta_{1,\mathrm{Gr}} \arrow[r, "(24.9)"] & \mathrm{coInd}^H_{B_H}(\widetilde{\mathcal{F}}^{\frac{\infty}{2}}).
\end{tikzcd}$$

Now, the bottom arrow is the above diagram is an isomorphism by Corrollary 24.5.3.

25. Baby Verma objects in the Whittaker category

In this section we will realize a part of the program indicated in Sect. 19.3.2: we will construct the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$ and $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}$ and verify proprties (i) and (ii).

25.1. **Construction of dual baby Verma objects in the Whittaker category.** In this subsection we will construct the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$.

25.1.1. Consider the category

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G)^{\mathfrak{L}(N),\chi_N} \subset \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G),$$

defined in the same way as

$$\mathrm{Whit}_q(G) = \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N),\chi_N},$$

but with Gr_G replaced by Fl_G.

25.1.2. Note now that we have a well-defined convolution functor

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G) \otimes \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G), \quad \mathcal{F}_1, \mathcal{F}_2 \to \mathcal{F}_1 \underset{I}{\star} \mathcal{F}_2, \tag{25.1}$$

which respects the action of $\mathrm{Sph}_{q,x}(G)$ on the right, and the convolution action of $\mathrm{Shv}_{\mathcal{G}^G}(\mathfrak{L}(G))$ on the left.

In particular, the above functor induces a functor

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G)^{\mathfrak{L}(N),\chi_N} \otimes \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{I,\mathrm{ren}} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N),\chi_N} =: \mathrm{Whit}_q(G), \tag{25.2}$$

which respects the action of $\mathrm{Sph}_{q,x}(G)$ on the right.

We notice:

Lemma 25.1.3. *The functor*

$$- \underset{I}{\star} \delta_{1,\mathrm{Gr}} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G) \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)$$

identifies with direct image along $\mathrm{Fl}_G \to \mathrm{Gr}_G$.

25.1.4. For $\lambda \in \Lambda^+$, denote

$$S^\lambda_{\mathrm{Fl}} = \mathfrak{L}(N) \cdot t^\lambda \cdot I/I \subset \mathrm{Fl}_G .$$

As in the case of the affine Grassmannian, the functor of taking the fiber at $t^\lambda \in \mathrm{Fl}_G$ defines an equivalence

$$\mathrm{Shv}_{\mathcal{G}^G}(S^\lambda_{\mathrm{Fl}})^{\mathfrak{L}(N),\chi_N} \to \mathrm{Vect}\,. \tag{25.3}$$

Let

$$W^{\lambda,*}_{\mathrm{Fl}} \in \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Fl}_G)^{\mathfrak{L}(N),\chi_N}$$

be the *-extension of the image of $\mathsf{e}[-\langle\lambda, 2\check{\rho}\rangle] \in \mathrm{Vect}$ under the equivalence (25.3).

25.1.5. Denote

$$\widetilde{\mathcal{M}}^{\lambda,*}_{\mathrm{Whit}} := W^{\lambda,*}_{\mathrm{Fl}} \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \in B^- \text{-Hecke}(\mathrm{Whit}_q(G)).$$

Denote

$$\overset{\bullet}{\mathcal{M}}{}^{\lambda,*}_{\mathrm{Whit}} := \mathrm{Res}^{B^-_H}_{T_H}(\mathcal{M}^\lambda_{\mathrm{Whit}}) \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)).$$

25.1.6. We observe:

Lemma 25.1.7. *For $\gamma \in \Lambda^+_H$ we have an isomorphism*

$$\widetilde{\mathcal{M}}^{\lambda,*}_{\mathrm{Whit}} \otimes \mathsf{e}^\gamma \simeq \widetilde{\mathcal{M}}^{\lambda+\gamma,*}_{\mathrm{Whit}}.$$

Proof. By the construction of $\widetilde{\mathcal{F}}^{\frac{\infty}{2}}$, we have

$$J_\gamma \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2}} \simeq \widetilde{\mathcal{F}}^{\frac{\infty}{2}} \otimes \mathsf{e}^\gamma.$$

Hence,

$$j_{w_0,*} \underset{I}{\star} J_\gamma \underset{I}{\star} j_{w_0,!} \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \simeq \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^{w_0(\gamma)}.$$

Hence, for $\gamma \in \Lambda^+_H$, we obtain

$$j_{\gamma,*} \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \simeq \widetilde{\mathcal{F}}^{\frac{\infty}{2},-} \otimes \mathsf{e}^\gamma.$$

From here we obtain that $\widetilde{\mathcal{M}}^{\lambda,*}_{\mathrm{Whit}} \otimes \mathsf{e}^\gamma$ identifies with

$$W^{\lambda,*}_{\mathrm{Fl}} \underset{I}{\star} j_{\gamma,*} \underset{I}{\star} \widetilde{\mathcal{F}}^{\frac{\infty}{2},-}.$$

Finally, we notice that for $\gamma \in \Lambda^+_H$, we have:

$$W^{\lambda,*}_{\mathrm{Fl}} \underset{I}{\star} j_{\gamma,*} \simeq W^{\lambda+\gamma,*}_{\mathrm{Fl}}, \tag{25.4}$$

which implies the assertion of the lemma.

□

Let now μ be an arbitrary element of Λ. Write

$$\mu = \lambda - \gamma, \quad \lambda \in \Lambda^+_H. \tag{25.5}$$

Define:

$$\widetilde{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}} := \widetilde{\mathcal{M}}^{\lambda,*}_{\mathrm{Whit}} \otimes \mathsf{e}^{-\gamma}.$$

Note that Lemma 25.1.7 implies that this definition is independent of the choice of a presentation of μ as in Sect. 25.5. Moreover, for any μ and γ we have:

$$\widetilde{\mathcal{M}}^{\mu+\gamma,*}_{\mathrm{Whit}} \simeq \widetilde{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}} \otimes \mathsf{e}^{\gamma}. \tag{25.6}$$

Define:

$$\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}} := \mathrm{Res}^{B^-_H}_{T_H}(\widetilde{\mathcal{M}}^{\mu}_{\mathrm{Whit}}) \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)).$$

We also have:

$$\overset{\bullet}{\mathcal{M}}{}^{\mu+\gamma,*}_{\mathrm{Whit}} \simeq \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}} \otimes \mathsf{e}^{\gamma}.$$

25.1.8. For completeness, define Define:

$$\mathcal{M}^{\mu,*}_{\mathrm{Whit}} := \mathrm{Res}^{T_H}(\overset{\bullet}{\mathcal{M}}{}^{\mu}_{\mathrm{Whit}}) \in \mathrm{Hecke}(\mathrm{Whit}_q(G)).$$

However, these objects will not be used in this paper.

25.2. **Properties of dual baby Verma objects in the Whittaker category.** In this subsection we investigate some basic properties of the objects $\widetilde{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}}$ constructed in the previous subsection.

25.2.1. First, we claim:

Lemma 25.2.2. *The object*

$$\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \in \mathrm{Rep}(T_H) \otimes \mathrm{Whit}_q(G)$$

is given by

$$\gamma' \rightsquigarrow \underset{\gamma}{\mathrm{colim}}\, W^{\mu+\gamma+\gamma',*} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G},$$

where the colimit runs over the set $\gamma \in (-\mu - \gamma' + \Lambda^+_H) \cap \Lambda^+_H$.

Proof. By (25.6), with no restriction of generality we can assume that $\mu = \lambda \in \Lambda^+$.

By (24.8), the object

$$\mathbf{oblv}_{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,*}_{\mathrm{Whit}})$$

is given by

$$\gamma' \rightsquigarrow \underset{\gamma}{\mathrm{colim}}\, W^{\lambda,*}_{\mathrm{Fl}} \underset{I}{\star} j_{\gamma+\gamma',*} \underset{I}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G},$$

where the colimit runs over the set $\gamma \in (-\gamma' + \Lambda^+_H) \cap \Lambda^+_H$.

Using (25.4), we have:

$$W^{\lambda,*}_{\mathrm{Fl}} \underset{I}{\star} j_{\gamma+\gamma',*} \underset{I}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G} \simeq W^{\lambda+\gamma+\gamma',*}_{\mathrm{Fl}} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G}.$$

Finally, we notice that by Lemma 25.1.3 for any $\lambda' \in \Lambda^+$

$$W^{\lambda',*}_{\mathrm{Fl}} \underset{I}{\star} \delta_{1,\mathrm{Gr}} \simeq W^{\lambda',*}.$$

□

Corollary 25.2.3. *The objects* $\widetilde{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}}$ *belong to* $(B^- \text{-Hecke}(\mathrm{Whit}_q(G)))^{\heartsuit}$.

Proof. It is sufficient to show that

$$\mathbf{oblv}_{\overset{\bullet}{\text{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,*}_{\text{Whit}}) \in \text{Whit}_q(G)$$

belongs to $(\text{Whit}_q(G))^\heartsuit$ for $\lambda \in \Lambda^+$.

Since the poset Λ^+_H is filtered, it suffices to show that each term in the colimit in Lemma 25.2.2 belongs to $(\text{Whit}_q(G))^\heartsuit$. Now the assertion follows from Proposition 9.3.2.

□

Next, we claim:

Lemma 25.2.4. *For $\lambda \in \Lambda^+$, we have*

$$\text{coInd}^H_{B^-_H}(\widetilde{\mathcal{M}}^{\lambda,*}_{\text{Whit}}) \simeq W^{\lambda,*}.$$

Proof. Follows immediately from Corrollary 24.5.3 using Lemma 25.1.3.

□

25.2.5. For $\lambda \in \Lambda^+$ consider now the map

$$\text{Res}^H_{B^-_H}(W^{\lambda,*}) \to \widetilde{\mathcal{M}}^{\lambda,*}_{\text{Whit}} \tag{25.7}$$

arising by adjunction from the isomorphism of Lemma 25.2.4.

Proposition 25.2.6. *For $\lambda \in \Lambda^+$ and $\gamma \in \Lambda_H$ consider the map*

$$\text{Res}^H_{B^-_H}(W^{\lambda+\gamma,*}) \otimes \mathsf{e}^{-\gamma} \to \widetilde{\mathcal{M}}^{\lambda+\gamma,*}_{\text{Whit}}.$$

If γ is deep enough in the dominant chamber, this map has the following properties:
(a) *It is surjective (in the abelian category $(B^-\text{-Hecke}(\text{Whit}_q(G)))^\heartsuit$).*
(b) *Its kernel admits a finite left resolution each of whose terms admits a filtration with subquotients of the form*

$$\text{Res}^H_{B^-_H}(W^{\lambda'+\gamma',*}) \otimes \mathsf{e}^{-\gamma'}$$

for $\lambda' \in \lambda - (\Lambda^{\text{pos}} - 0)$, $\gamma' \in \Lambda^+_H$ and $\lambda' + \gamma' \in \Lambda^+$.

Proof. Recall the functor

$$\text{AB} : \text{Rep}(B_H) \to \text{Shv}_{\mathcal{G}^G}(\text{Gr}_G)^{I,\text{ren}}$$

of (24.21). We will denote by the same character the resulting functor

$$\text{QCoh}(B_H\backslash H/B_H) \to B\text{-Hecke}(\text{Shv}_{\mathcal{G}^G}(\text{Gr}_G)^{I,\text{ren}}).$$

Recall that

$$\widetilde{\mathcal{F}}^{\frac{\infty}{2}} \simeq \text{AB}((\mathsf{i})_*(\mathsf{e})).$$

We note now that for $\gamma \in \Lambda_H$ deep enough in the dominant chamber, the object

$$(\mathsf{i})_*(\mathsf{e}) \in \text{QCoh}(B_H\backslash H/B_H) \simeq \text{Rep}(B_H) \underset{\text{Rep}(H)}{\otimes} \text{Rep}(B_H)$$

admits a finite left resolution whose initial term is

$$\mathsf{e}^\gamma \otimes \mathsf{e}^{-\gamma},$$

and each of the other terms admits a filtration with terms

$$\mathsf{e}^{\gamma'+\gamma_0} \otimes \mathsf{e}^{-\gamma'}, \quad \gamma_0 \in \Lambda^{\text{neg}}_H - 0, \quad \gamma' + \gamma_0 \in \Lambda^+_H.$$

Applying to this resolution the functor AB term-wise, then convolving with $j_{w_0,*}[-d]$ on the left and with $W^{\lambda,*}_{\text{Fl}}$ on the right, and finally applying the functor

$$w_0 : B\text{-Hecke}(\text{Whit}_q(G)) \to B^-\text{-Hecke}(\text{Whit}_q(G)),$$

we obtain that $\widetilde{\mathcal{M}}^{\lambda+\gamma,*}_{\text{Whit}}$ admits a left resolution of the form specified in the proposition.

□

25.3. **Jordan-Holder series of dual baby Verma modules.**

25.3.1. First, we claim:

Lemma 25.3.2. *The objects* $\widetilde{\mathcal{M}}^{\mu,*}_{\mathrm{Whit}} \in B^-\text{-Hecke}(\mathrm{Whit}_q(G))$ *are compact.*

Proof. It is enough to show that the object $\widetilde{\mathcal{M}}^{\mu+\gamma,*}_{\mathrm{Whit}} \in B^-\text{-Hecke}(\mathrm{Whit}_q(G))$ is compact for *some* γ. By Proposition 25.2.6, we can choose γ large enough so that $\widetilde{\mathcal{M}}^{\mu+\gamma,*}_{\mathrm{Whit}}$ admits a surjection from $\mathrm{Res}^H_{B_H}(W^{\mu+\gamma,*})$. Hence, $\widetilde{\mathcal{M}}^{\mu+\gamma,*}_{\mathrm{Whit}}$ is compact as the t-structure on $B^-\text{-Hecke}(\mathrm{Whit}_q(G))$ is Artinian. □

Corollary 25.3.3. *The objects* $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}} \in B^-\text{-Hecke}(\mathrm{Whit}_q(G))$ *are compact.*

25.3.4. Since the t-structure on $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$, is Artinian, Corrollary 25.3.2 implies that the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}} \in (\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)))^{\heartsuit}$ have finite length.

We claim:

Proposition 25.3.5. *There exists a non-zero map* $\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}} \to \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$, *such that the Jordan-Holder constituents of the quotient are of the form*

$$\overset{\bullet}{\mathcal{M}}{}^{\mu',!*}_{\mathrm{Whit}}, \quad \mu' \in \mu - (\Lambda^{\mathrm{pos}} - 0).$$

Proof. We can replace the initial μ by any $\mu + \gamma$ for $\gamma \in \Lambda_H$. Let γ be as in Proposition 25.2.6. Applying this proposition, we can assume that there exists a surjection

$$\mathrm{Res}^H_{T_H}(W^{\mu,*}) \twoheadrightarrow \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}, \tag{25.8}$$

whose kernel admits a left resolution each of which terms admits a filtration with subquotients of the form

$$\mathrm{Res}^H_{T_H}(W^{\mu'+\gamma',*}) \otimes \mathsf{e}^{-\gamma'}, \quad \mu' \in \mu - (\Lambda^{\mathrm{pos}} - 0), \quad \gamma' \in \Lambda^+_H. \tag{25.9}$$

Recall the (injective) map

$$\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}} \to \mathrm{Res}^H_{T_H}(W^{\mu,*}) \tag{25.10}$$

(see Corrollary 11.3.8). Composing, we obtain a map

$$\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}} \to \mathrm{Res}^H_{T_H}(W^{\mu,*}) \to \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}.$$

We claim that this composite in non-zero. Indeed, if it were zero, the image of (25.10) would hit the kernel of (25.8). However, the Jordan-Holder constituents of the latter are among those of (25.9), and by Corrollary 11.3.8 those are of the form

$$\overset{\bullet}{\mathcal{M}}{}^{\mu',!*}_{\mathrm{Whit}}, \quad \mu' \in \mu - (\Lambda^{\mathrm{pos}} - 0).$$

Finally, the fact that the cokernel of the map $\overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\mathrm{Whit}} \to \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$ has constituents of the form specified in the proposition follows by applying Corrollary 11.3.8 again. □

25.4. **The (actual) baby Verma objects in the Whittaker category.** In this subsection we will finally define the objects $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}} \in \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))$.

25.4.1. Recall the duality operation of (23.4):

$$(B^- \text{-Hecke}(\text{Whit}_q(G))^c)^{\text{op}} \to B\text{-Hecke}(\text{Whit}_{q^{-1}}(G))^c.$$

Applying this functor to

$$\widetilde{\mathcal{M}}^{\mu,*}_{\text{Whit}} \in B^- \text{-Hecke}(\text{Whit}_q(G))^c,$$

and up to replacing q^{-1} by q, we obtain an object that we will denote

$$\widetilde{\mathcal{M}}^{\mu,!}_{\text{Whit}} \in B\text{-Hecke}(\text{Whit}_q(G))^c.$$

25.4.2. Denote

$$\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\text{Whit}} := \text{Res}^{B_H}_{T_H}(\widetilde{\mathcal{M}}^{\mu,!}_{\text{Whit}}) \in \overset{\bullet}{\text{Hecke}}(\text{Whit}_q(G))^c.$$

By (23.5), the object $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\text{Whit}}$ is obtained from $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\text{Whit}}$ by the duality functor

$$(\overset{\bullet}{\text{Hecke}}(\text{Whit}_q(G))^c)^{\text{op}} \to \overset{\bullet}{\text{Hecke}}(\text{Whit}_{q^{-1}}(G))^c. \tag{25.11}$$

25.4.3. One can deduce many of the properties of $\widetilde{\mathcal{M}}^{\mu,!}_{\text{Whit}}$ (resp., $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\text{Whit}}$) from those of $\widetilde{\mathcal{M}}^{\mu,*}_{\text{Whit}}$ (resp., $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\text{Whit}}$) by duality. We will need the following few:

Corollary 25.4.4. *There exists a non-zero map $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\text{Whit}} \to \overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\text{Whit}}$, such that the Jordan-Holder constituents of the kernel are of the form*

$$\overset{\bullet}{\mathcal{M}}{}^{\mu',!*}_{\text{Whit}}, \quad \mu' \in \mu - (\Lambda^{\text{pos}} - 0).$$

Corollary 25.4.5. *For every $\mu \in \Lambda$ one can find $\gamma \in \Lambda_H$ sufficiently deep in the dominant chamber, such that there exists a map*

$$\widetilde{\mathcal{M}}^{\mu+\gamma,!}_{\text{Whit}} \to \text{Res}^H_{B_H}(W^{\mu+\gamma,!}) \otimes \mathsf{e}^{-\gamma}$$

with the the following properties:

(a) *It is injective.*

(b) *Its cokernel admits a finite right resolution each of whose terms admits a filtration with subquotients of the form*

$$\text{Res}^H_{B_H}(W^{\mu'+\gamma',!}) \otimes \mathsf{e}^{-\gamma'}$$

for $\mu' \in \mu - (\Lambda^{\text{pos}} - 0)$, $\gamma' \in \Lambda^+_H$ and $\mu' + \gamma' \in \Lambda^+$.

Corollary 25.4.6. *For $\lambda \in \Lambda^+$, we have*

$$\text{Ind}^H_{B_H}(\overset{\bullet}{\mathcal{M}}{}^{\lambda,!}_{\text{Whit}}) \simeq W^{\lambda,!}.$$

25.5. **Orthogonality.** In this subsection we will show that the $\overset{\bullet}{\mathcal{M}}{}^{\lambda,!}_{\text{Whit}}$, $\overset{\bullet}{\mathcal{M}}{}^{\lambda,*}_{\text{Whit}}$ satisfy properties (i) and (ii) from Sect. 19.3.2.

25.5.1. First, by combining Proposition 25.3.5 and Corrollary 25.4.4, we obtain that there exist maps

$$\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\text{Whit}} \twoheadrightarrow \overset{\bullet}{\mathcal{M}}{}^{\mu,!*}_{\text{Whit}} \hookrightarrow \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\text{Whit}}$$

whose kernel/cokernel have Jordan-Holder constituents of the form

$$\overset{\bullet}{\mathcal{M}}{}^{\mu',!*}_{\text{Whit}}, \quad \mu' \in \mu - (\Lambda^{\text{pos}} - 0).$$

We now claim:

Theorem 25.5.2. *We have:*

$$\mathcal{H}om_{\overset{\bullet}{\text{Hecke}}(\text{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\text{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\text{Whit}}) = \begin{cases} \mathsf{e} \text{ if } \mu' = \mu \\ 0 \text{ otherwise.} \end{cases}$$

The rest of this subsection is devoted to the proof of this theorem.

25.5.3. *Reduction step 1.* First, we claim that it is sufficient to consider the case when

$$\mu' \notin \mu + (\Lambda^{\mathrm{pos}} - 0).$$

Indeed, $\mu' \in \mu + (\Lambda^{\mathrm{pos}} - 0)$, applying the duality functor (25.11), we obtain:

$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu',!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \simeq \mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q^{-1}}(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}, \overset{\bullet}{\mathcal{M}}{}^{\mu',*}_{\mathrm{Whit}}),$$

and now $\mu \notin \mu' + (\Lambda^{\mathrm{pos}} - 0)$.

25.5.4. *Reduction step 2.* Let us choose $\gamma \in \Lambda^+_H$ deep enough in the dominant chamber, so that $\overset{\bullet}{\mathcal{M}}{}^{\mu'+\gamma,!}_{\mathrm{Whit}}$ admits a resolution as in Corrollary 25.4.5. Replacing

$$\mu' \rightsquigarrow \mu' + \gamma, \ \ \mu \rightsquigarrow \mu + \gamma,$$

we obtain that it is sufficient to show that for $\lambda' \in \Lambda^+$ and $\lambda' \notin \lambda + (\Lambda^{\mathrm{pos}} - 0)$, we have:

$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\mathrm{Res}^H_{T_H}(W^{\lambda',!}), \overset{\bullet}{\mathcal{M}}{}^{\lambda,*}_{\mathrm{Whit}}) = \begin{cases} \mathsf{e} \text{ if } \lambda' = \lambda \\ 0 \text{ otherwise.} \end{cases}$$

25.5.5. Using Lemma 25.2.2, we rewrite

$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\mathrm{Res}^H_{T_H}(W^{\lambda',!}), \overset{\bullet}{\mathcal{M}}{}^{\lambda,*}_{\mathrm{Whit}})$$

as

$$\underset{\gamma \in \Lambda^+_H}{\mathrm{colim}}\ \mathcal{H}om_{\mathrm{Whit}_q(G)}(W^{\lambda',!}, W^{\lambda+\gamma,*} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G}).$$

It is therefore sufficient to show that for $\lambda' \in \Lambda^+$ and $\lambda' \notin \lambda + (\Lambda^{\mathrm{pos}} - 0)$, we have:

$$\mathcal{H}om_{\mathrm{Whit}_q(G)}(W^{\lambda',!}, W^{\lambda+\gamma,*} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G}) = \begin{cases} \mathsf{e} \text{ if } \lambda' = \lambda \\ 0 \text{ otherwise.} \end{cases} \tag{25.12}$$

25.5.6. We rewrite

$$\mathcal{H}om_{\mathrm{Whit}_q(G)}(W^{\lambda',!}, W^{\lambda+\gamma,*} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{-w_0(\gamma)}_G}) \simeq \mathcal{H}om_{\mathrm{Whit}_q(G)}(W^{\lambda',!} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{\gamma}_G}, W^{\lambda+\gamma,*}).$$

Note that

$$W^{\lambda',!} \underset{\mathfrak{L}^+(G)}{\star} \mathrm{IC}_{q,\overline{\mathrm{Gr}}^{\gamma}_G}$$

is supported on $\overline{S}^{\lambda'+\gamma}$, and that its restriction to $S^{\lambda'+\gamma}$ is the generator of $\mathrm{Whit}_q(G)_{=\lambda'+\gamma}$. This implies (25.12):

Indeed, the case $\lambda' = \lambda$ is obvious. For $\lambda' \neq \lambda$, the condition that $\lambda' \notin \lambda + (\Lambda^{\mathrm{pos}} - 0)$ implies that

$$\overline{S}^{\lambda'+\gamma} \cap S^{\lambda+\gamma} = \emptyset.$$

26. Calculation of stalks

The goal of this section is to complete the program indicated in Sect. 19.3.2 by proving property (iii) in *loc.cit.*

26.1. **Statement of the result.** In this subsection we will state precisely the calculation that we will perform.

26.1.1. Fix an element $\mu \in \Lambda$. In order to simply the notation, we will trivialize the fiber of $\mathcal{G}^\Lambda$ at the point $\mu \cdot x \in \mathrm{Conf}_{\infty\cdot x}$. Note that this fiber identifies canonically with the fiber of $\mathcal{G}^G$ at $t^\mu \in \mathfrak{L}(G)^{\omega^\rho}_x$.

Due to this trivialization, we have a *well-defined* functor

$$\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \to \mathrm{Vect}, \tag{26.1}$$

given by taking the !-fiber at $\mu \cdot x$. In particular, we a *well-defined* objects

$$\mathcal{M}^{\mu,!}_{\mathrm{Conf}}, \mathcal{M}^{\mu,*}_{\mathrm{Conf}} \in \Omega^{\mathrm{small}}_q\text{-FactMod}\,.$$

The trivialization of the fiber of $\mathcal{G}^G$ at $t^\mu \in \mathfrak{L}(G)^{\omega^\rho}_x$ gives rise to an identification

$$\mathrm{Whit}_q(G)_{=\mu} \simeq \mathrm{Vect},$$

and hence to a pair of *well-defined* objects

$$W^{\mu,!}, W^{\mu,*} \in \mathrm{Whit}_q(G).$$

We normalize $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$ so that

$$\mathbf{coind}^H_{B^-_H(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}})} \simeq W^{\mu,*}.$$

26.1.2. The main result of the present section is the following theorem:

Theorem 26.1.3. *The functor*

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}} : \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)) \to \Omega^{\mathrm{small}}_q\text{-FactMod}$$

sends $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$ *to* $\mathcal{M}^{\mu,*}_{\mathrm{Conf}} \in \Omega^{\mathrm{small}}_q$-FactMod.

26.1.4. Before we proceed further, let us show how Theorem 26.1.3 completes the outline in Sect. 19.3.2. In fact, it remains to prove the following:

Corollary 26.1.5. *The functor* $\Phi^{\overset{\bullet}{\mathrm{Hecke}}}_{\mathrm{Fact}}$ *sends* $\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}$ *to* $\mathcal{M}^{\mu,!}_{\mathrm{Conf}} \in \Omega^{\mathrm{small}}_q$-FactMod.

Proof. Note that if an object $\mathcal{F} \in \Omega^{\mathrm{small}}_q$-FactMod is equipped with an isomorphism

$$\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{F}) \simeq \mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}),$$

then this isomorphism lifts uniquely to an isomorphism

$$\mathcal{F} \simeq \mathcal{M}^{\mu,!}_{\mathrm{Conf}}.$$

Hence, in order to prove the corollary, it suffices establish an isomorphism

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}) \simeq \mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,!}_{\mathrm{Conf}}). \tag{26.2}$$

We start with the isomorphism

$$\Phi^{\overset{\bullet}{\mathrm{Hecke}}}(\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}) \simeq \mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Conf}}),$$

provided by Theorem 26.1.3. Now (26.2) follows by applying Theorem 22.1.5.

$\square$

26.1.6. The rest of this section is devoted to the proof of Theorem 26.1.3. We will deduce it from the following result

Theorem 26.1.7. *The functor*

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)) \overset{\Phi^{\overset{\bullet}{\mathrm{Hecke}}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x}) \overset{!\text{-fiber at } \mu\cdot x}{\longrightarrow} \mathrm{Vect} \tag{26.3}$$

identifies canonically with

$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}, -).$$

Let us see how Theorem 26.1.7 implies Theorem 26.1.3:

Proof of Theorem 26.1.3. The object $\mathcal{M}^{\mu,*}_{\mathrm{Conf}} \in \Omega^{\mathrm{small}}_q$-FactMod is uniquely characterized by the property that the underlying object

$$\mathbf{oblv}_{\mathrm{Fact}}(\mathcal{M}^{\mu,*}_{\mathrm{Conf}}) \in \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$$

has !-fiber e at $\mu \cdot x$ and has a zero !-fiber at $\mu' \cdot x$ for $\mu' \neq \mu$.

Now the assertion follows from Theorem 25.5.2.

□

26.1.8. Thus, the rest of this section is devoted to the proof of Theorem 26.1.7. We should remark, however, that the proof will essentially be a formal manipulation, given the relationship between $'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}}_{q,x}$ and $\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2}}_{\mathrm{rel}}$ explained in Sect. 24.3.

26.2. **Framework for the proof of Theorem 26.1.7.** In this subsection we will explain a general categorical framework in which Theorem 26.1.7 will be proved.

26.2.1. Let $\mathbf{C}$ and $\mathbf{D}$ be module categories over $\mathrm{Rep}(H)$, and let us be given a pairing

$$\widetilde{\Psi} : \mathbf{C} \underset{\mathrm{Rep}(H)}{\otimes} \mathbf{D} \to \mathbf{E}.$$

We claim that the datum of Ψ gives rise to the datum of a pairing

$$\overset{\bullet}{\Psi} : \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathbf{D}) \to \mathbf{E}.$$

Indeed, assume for simplicity that $\mathbf{D}$ is compactly generated. Then we can interpret $\widetilde{\Psi}$ as a $\mathrm{Rep}(H)$-linear functor

$$\mathbf{C} \to \mathbf{D}^\vee \otimes \mathbf{E}.$$

The latter gives rise to a functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \to \overset{\bullet}{\mathrm{Hecke}}(\mathbf{D}^\vee) \otimes \mathbf{E}.$$

Now, identifying

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{D}^\vee) \simeq \overset{\bullet}{\mathrm{Hecke}}(\mathbf{D})^\vee$$

as in Sect. 10.5.2, we thus obtain a functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \to \overset{\bullet}{\mathrm{Hecke}}(\mathbf{D})^\vee \otimes \mathbf{E},$$

hence the desired pairing $\overset{\bullet}{\Psi}$.

26.2.2. Let us now be in the context of Sect. 16.1.5. It is easy to see that the functor

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \otimes \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}) \to \mathbf{E}$$

constructed in *loc. cit.* identifies with

$$\overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \otimes \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathbf{D}) \overset{\mathrm{Id} \otimes \mathbf{oblv}_{\mathrm{rel}}}{\longrightarrow} \overset{\bullet}{\mathrm{Hecke}}(\mathbf{C}) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathbf{D}) \overset{\overset{\bullet}{\Psi}}{\longrightarrow} \mathbf{E}.$$

26.3. **Applying the framework: the left-hand side.**

26.3.1. Let Ψ denote the pairing

$$\mathrm{Whit}_q(G)\otimes \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)} \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)\otimes \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G) \overset{-\overset{!}{\otimes}-}{\longrightarrow} \\ \to \mathrm{Shv}(\mathrm{Gr}_G) \overset{\Gamma(\mathrm{Gr}_G,-)}{\longrightarrow} \mathrm{Vect}\,. \tag{26.4}$$

By construction, it factors via

$$\widetilde{\Psi}: \mathrm{Whit}_q(G) \underset{\mathrm{Rep}(H)}{\otimes} \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)} \to \mathrm{Vect},$$

where $\mathrm{Rep}(H)$ acts on $\mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}$ by

$$V,\mathcal{F}\mapsto \mathcal{F}\underset{\mathfrak{L}^+(G)}{\star} \mathrm{inv}^G(\mathrm{Sat}_{q,G})(V) \simeq \mathcal{F}\underset{\mathfrak{L}^+(G)}{\star} \mathrm{Sat}_{q^{-1},G}(\tau^H(V)).$$

26.3.2. Consider the corresponding functor

$$\overset{\bullet}{\Psi}: \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))\otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}) \to \mathrm{Vect}\,.$$

Recall the object

$$'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2}} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}),$$

and consider the corresponding object.

$$'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-} \in \overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}).$$

Due to the trivialization in Sect. 26.1.1, the translate

$$t^\mu\cdot {}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-}$$

makes sense as on object of $\overset{\bullet}{\mathrm{Hecke}}_{\mathrm{rel}}(\mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)})$.

Unwinding the definitions and using Sect. 26.2.1 above, we obtain that the functor (26.3) identifies with the functor

$$\overset{\bullet}{\Psi}(-,\mathbf{oblv}_{\mathrm{rel}}(t^\mu\cdot {}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-}))[\langle\mu,2\check{\rho}\rangle]. \tag{26.5}$$

26.3.3. Note that since objects of $\mathrm{Whit}_q(G)$ are $\mathfrak{L}^+(N)$-equivariant, the pairing Ψ of (26.4) is isomorphic to its precomposition with the endo-functor

$$\mathrm{Whit}_q(G)\otimes \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)} \overset{\mathrm{Id}\otimes \mathrm{Av}_*^{\mathfrak{L}^+(N)}}{\longrightarrow} \mathrm{Whit}_q(G)\otimes \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{\mathfrak{L}^+(T)}.$$

Hence, the same is true for the functors $\widetilde{\Psi}$ and $\overset{\bullet}{\Psi}$. Therefore, we can rewrite the functor in (26.5) as

$$\overset{\bullet}{\Psi}(-,\mathrm{Av}_*^{\mathfrak{L}^+(N)}(\mathbf{oblv}_{\mathrm{rel}}(t^\mu\cdot {}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-})))[\langle\mu,2\check{\rho}\rangle]. \tag{26.6}$$

26.4. **Applying the framework: the right-hand side.**

26.4.1. By Sect. 26.2.1, the pairing

$$\Upsilon: \mathrm{Whit}_q(G)\otimes \mathrm{Whit}_{q^{-1}}(G)\to \mathrm{Vect}$$

arising from (7.9), gives rise to a pairing

$$\overset{\bullet}{\Upsilon}: \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))\otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_{q^{-1}}(G))\to \mathrm{Vect}\,.$$

By definition, the functor

$$\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}},-): \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))\to \mathrm{Vect}$$

is given by

$$\overset{\bullet}{\Upsilon}(-,\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}),$$

26.4.2. Unwinding the definitions of $\overset{\bullet}{\mathcal{M}}{}^{\mu,*}_{\mathrm{Whit}}$ and of Υ we obtain:

Lemma 26.4.3. *The pairings*

$$\mathrm{Whit}_q(G) \otimes \mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{I,\mathrm{ren}} \to \mathrm{Vect}$$

given by

$$\mathcal{F}, \mathcal{F}' \mapsto \Psi(\mathcal{F}, t^\mu \cdot \mathcal{F}')[\langle \mu, 2\check{\rho}\rangle]$$

and

$$\mathcal{F}, \mathcal{F}' \mapsto \Upsilon(\mathcal{F}, W^{\mu,*}_{\mathrm{Fl}} \underset{I}{\star} \mathcal{F}')$$

are canonically isomorphic.

Corollary 26.4.4. *For $\mu \in \Lambda$, the pairings*

$$\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G)) \otimes \overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{(\mathcal{G}^G)^{-1}}(\mathrm{Gr}_G)^{I,\mathrm{ren}}) \to \mathrm{Vect}$$

given by

$$\mathcal{F}, \mathcal{F}' \mapsto \overset{\bullet}{\Psi}(\mathcal{F}, t^\mu \cdot \mathcal{F}')[\langle \mu, 2\check{\rho}\rangle]$$

and

$$\mathcal{F}, \mathcal{F}' \mapsto \overset{\bullet}{\Upsilon}(\mathcal{F}, W^{\mu,*}_{\mathrm{Fl}} \underset{I}{\star} \mathcal{F}')$$

are canonically isomorphic.

26.4.5. Thus, we obtain that the functor $\mathcal{H}om_{\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Whit}_q(G))}(\overset{\bullet}{\mathcal{M}}{}^{\mu,!}_{\mathrm{Whit}}, -)$ can be identified with

$$\overset{\bullet}{\Psi}(-, t^\mu \cdot \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-})[\langle \mu, 2\check{\rho}\rangle]. \tag{26.7}$$

Using once again $\mathfrak{L}^+(N)$-equivariance of objects of $\mathrm{Whit}_q(G)$, we obtain that the latter expression cab be rewritten as

$$\overset{\bullet}{\Psi}(-, \mathrm{Av}_*^{\mathfrak{L}^+(N)}(t^\mu \cdot \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-}))[\langle \mu, 2\check{\rho}\rangle]. \tag{26.8}$$

26.5. **Conclusion of proof of Theorem 26.1.7.**

26.5.1. Comparing (26.6) and (26.8), we obtain that Theorem 26.1.7 follows from the next assertion:

Proposition 26.5.2. *For $\mu \in \Lambda^+$, there exists a canonical isomorphism in $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G))$:*

$$\mathrm{Av}_*^{\mathfrak{L}^+(N)}(t^\mu \cdot \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-}) \simeq \mathrm{Av}_*^{\mathfrak{L}^+(N)}(\mathbf{oblv}_{\mathrm{rel}}(t^\mu \cdot {}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-}))).$$

The rest of this subsection is devoted to the proof of Proposition 26.5.2.

26.5.3. Recall (see Sect. 24.3.1) that the functor $\mathrm{Av}_!^{\mathfrak{L}(N)}$ defines an equivalence

$$\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}.$$

Note that we have a commutative diagram of functors

$$\begin{tikzcd}
\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)} \arrow[r, "w_0\cdot"] & \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N^-)\cdot\mathfrak{L}^+(T)} \\
\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I \arrow[u, "\mathrm{Av}_!^{\mathfrak{L}(N)}"] \arrow[r, "j_{w_0,!}\underset{I}{\star}-[d]"] & \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I. \arrow[u, "\mathrm{Av}_!^{\mathfrak{L}(N^-)}"']
\end{tikzcd} \tag{26.9}$$

In particular, we obtain that the functor

$$\mathrm{Av}_!^{\mathfrak{L}(N^-)} : \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^I \to \mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N^-)\cdot\mathfrak{L}^+(T)} \tag{26.10}$$

is also an equivalence.

26.5.4. It follows formally that the right adjoint of the functor (26.10) is given by $\mathrm{Av}_*^{\overset{\circ}{I}}$, where $\overset{\circ}{I}$ is the unipotent radical of I. Since (26.10) is an equivalence, we obtain that $\mathrm{Av}_*^{\overset{\circ}{I}}$ defines an inverse equivalence.

From the decomposition

$$\overset{\circ}{I} = \overset{\circ}{I}{}^+ \cdot \overset{\circ}{I}{}^0 \cdot \overset{\circ}{I}{}^-,$$

where

$$\overset{\circ}{I}{}^+ := \overset{\circ}{I} \cap \mathfrak{L}(N) = \mathfrak{L}^+(N),\ \overset{\circ}{I}{}^0 := \overset{\circ}{I} \cap \mathfrak{L}(T),\ \overset{\circ}{I}{}^{:} = \overset{\circ}{I} \cap \mathfrak{L}(N^-),$$

it follows that when applied to objects of $\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G)^{\mathfrak{L}(N)\cdot\mathfrak{L}^+(T)}$, the natural transformation

$$\mathrm{Av}_*^{\overset{\circ}{I}} \to \mathrm{Av}_*^{\mathfrak{L}^+(N)}$$

is an isomorphism.

26.5.5. It follows from Proposition 24.3.3 and the commutative diagram (26.9) that

$$\mathrm{Av}_!^{\mathfrak{L}(N^-)}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-}) \simeq \mathbf{oblv}_{\mathrm{rel}}({}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-}).$$

Hence, we obtain an isomorphism

$$\mathrm{Av}_*^{\mathfrak{L}^+(N)}(\mathbf{oblv}_{\mathrm{rel}}({}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-}))) \simeq \overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-} \tag{26.11}$$

as objects of $\overset{\bullet}{\mathrm{Hecke}}(\mathrm{Shv}_{\mathcal{G}^G}(\mathrm{Gr}_G))$.

We claim that this implies the isomorphism stated in Proposition 26.5.2.

26.5.6. Indeed, since μ is dominant, we have

$$\mathfrak{L}^+(N) \subset \mathrm{Ad}_{t^{-\mu}}(\mathfrak{L}^+(N)).$$

Hence, (26.11) implies

$$\mathrm{Av}_*^{\mathrm{Ad}_{t^{-\mu}}(\mathfrak{L}^+(N))}(\mathbf{oblv}_{\mathrm{rel}}({}'\overset{\bullet}{\mathrm{IC}}{}^{\frac{\infty}{2},-}))) \simeq \mathrm{Av}_*^{\mathrm{Ad}_{t^{-\mu}}(\mathfrak{L}^+(N))}(\overset{\bullet}{\mathcal{F}}{}^{\frac{\infty}{2},-}).$$

Translating both sides by t^μ we arrive at the isomorphism of Proposition 26.5.2.

□

Part IX: Relation to quantum groups

This Part is disjoint from the rest of this work. Here we will establish an equivalence between the category $\Omega_q^{\mathrm{small}}$-FactMod$^{\mathrm{ren}}$ and a (renormalized) version of the category of modules over the small quantum group.

The contents of this Part can be thought of recasting the modern language the equivalence of categories, which was the subject of the work [BFS].

27. Modules over the small quantum group

In this section we define the category of modules over the small quantum group, as a e-linear category. It will be introduced as *relative Drinfeld center* of the category of modules over the *positive part*, denoted $\mathfrak{u}_q(\check{N})$.

27.1. **The small quantum group: the positive part.** In this subsection we will introduce the positive part of the small quantum group, $\mathfrak{u}_q(\check{N})$.

27.1.1. We start with a datum of a bilinear form

$$b' : \Lambda \otimes \Lambda \to \mathsf{e}^{\times,\mathrm{tors}}.$$

Let q be the associated quadratic form $\Lambda \to \mathsf{e}^{\times,\mathrm{tors}}$. We will assume that q is *non-degenerate*, i.e., $q(\alpha) \neq 1$ for all coroots α.

We will also assume that the quadratic form q belongs to the subset

$$\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}})^W_{\mathrm{restr}} \subset \mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}),$$

see Sect. 2.3.3.

27.1.2. Starting from this data, we will eventually define the category of modules over the *Langlands dual* small quantum group, to be denoted $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod.

Remark 27.1.3. The dot $\bullet$ over $\mathfrak{u}_q$ is meant to emphasize that we will be dealing with the category of Λ-*graded* modules.

Remark 27.1.4. The fact that our quantum group corresponds to the Langlands dual group manifests itself in that its lattice of *weights* is the lattice Λ of coweights of G.

27.1.5. Consider the category $\mathrm{Rep}(\check{T}) \simeq \mathrm{Vect}^\Lambda$, where Vect^Λ is the category of Λ-graded vector spaces. For $\lambda \in \Lambda$ we let

$$\mathsf{e}^\lambda \in \mathrm{Vect}^\lambda \subset \mathrm{Vect}^\Lambda$$

denote the vector space e placed in the graded component λ.

We consider Vect^Λ as endowed with the standard monoidal structure. Now, the data of b' defines a new braiding on Vect^Λ. Denote the resulting braided monoidal category Vect_q^Λ.

27.1.6. Choose *some* 1-dimensional objects

$$\mathsf{e}^{i,\mathrm{quant}} \in \mathrm{Vect}^{\alpha_i} \subset \mathrm{Vect}^\Lambda.$$

I.e., each $\mathsf{e}^{i,\mathrm{quant}}$ is *non-canonically* isomorphic to e^{α_i}.

Let $U_q(\check{N})^{\mathrm{free}}$ be the free associative algebra in $\mathrm{Vect}^{\Lambda^{\mathrm{pos}}} \subset \mathrm{Vect}^\Lambda$ on

$$\underset{i}{\oplus}\, \mathsf{e}^{i,\mathrm{quant}} \in \mathrm{Vect}^\Lambda.$$

I.e., to specify a map from $U_q(\check{N})^{\mathrm{free}}$ to an associative algebra A in Vect^Λ is equivalent to specifying maps

$$\mathsf{e}^{i,\mathrm{quant}} \to A.$$

Let e_i denote the tautological map $\mathsf{e}^{i,\mathrm{quant}} \to U_q(\check{N})^{\mathrm{free}}$.

We endow $U_q(\check{N})^{\text{free}}$ with a Hopf algebra structure in $\text{Vect}_q^{\Lambda^{\text{pos}}}$ by letting the comultiplication

$$U_q(\check{N})^{\text{free}} \to U_q(\check{N})^{\text{free}} \otimes U_q(\check{N})^{\text{free}}$$

correspond to the maps

$$\mathsf{e}_{i,\text{quant}} \overset{e_i \otimes \text{unit} + \text{unit} \otimes e_i}{\longrightarrow} U_q(\check{N})^{\text{free}} \otimes U_q(\check{N})^{\text{free}}.$$

27.1.7. Let $U_q(\check{N})^{\text{co-free}}$ denote the co-free graded co-associative co-algebra in $\text{Vect}^{\Lambda^{\text{pos}}}$ on the co-generators $\mathsf{e}^{i,\text{quant}}$. I.e., to specify a map from a co-associative co-algebra A in $\text{Vect}^{\Lambda^{\text{pos}}}$ to $U_q(\check{N})^{\text{co-free}}$ is equivalent to specifying maps

$$A \to \mathsf{e}^{i,\text{quant}}.$$

Let e_i^* denote the tautological map $U_q(\check{N})^{\text{co-free}} \to \mathsf{e}_{i,\text{quant}}$.

We endow $U_q(\check{N})^{\text{co-free}}$ with a Hopf algebra structure in Vect_q^{Λ} by letting the multiplication

$$U_q(\check{N})^{\text{co-free}} \otimes U_q(\check{N})^{\text{co-free}} \to U_q(\check{N})^{\text{co-free}}$$

correspond to the maps

$$U_q(\check{N})^{\text{co-free}} \otimes U_q(\check{N})^{\text{co-free}} \overset{e_i^* \otimes \text{aug} + \text{aug} \otimes e_i^*}{\longrightarrow} \mathsf{e}_{i,\text{quant}}.$$

27.1.8. We define a map of Hopf algebras

$$U_q(\check{N})^{\text{free}} \to U_q(\check{N})^{\text{co-free}} \tag{27.1}$$

to correspond to the projections

$$U_q(\check{N})^{\text{free}} \to \mathsf{e}_{i,\text{quant}}$$

onto the α_i components.

27.1.9. We define $\mathfrak{u}_q(\check{N}^+)$ to be the image of the map (27.1). The key fact that uses the non-degeneracy assumption on q is that $\mathfrak{u}_q(\check{N}^+)$ is *finite-dimensional.*

Moreover, it can be explicitly described (as an algebra) as a quotient of $U_q(\check{N})^{\text{free}}$ by the quantum Serre relations and the relations

$$(e_i)^{\text{ord}(q_i)} = 1.$$

27.2. **Digression: the notion of relative Drinfeld center.** In this subsection we recollect the general framework for defining the notion of relative Drinfeld center.

27.2.1. Recall that if $\mathbf{A}$ is a category, it makes sense to talk about an action of a monoidal category $\mathbf{O}$ on $\mathbf{A}$. The category of such pairs $(\mathbf{O}, \mathbf{A})$ itself forms a symmetric monoidal category under the operation of Cartesian product.

Consider the category of associative algebra objects in the above category. If $(\mathbf{O}, \mathbf{A})$ is such an algebra object, the forgetful functor $(\mathbf{O}, \mathbf{A}) \mapsto \mathbf{A}$ endows $\mathbf{A}$ with a structure of monoidal category, and the forgetful functor $(\mathbf{O}, \mathbf{A}) \mapsto \mathbf{O}$ endows $\mathbf{O}$ with a structure of associative algebra in the category of monoidal categories. In other words $\mathbf{O}$ acquires a structure of *braided monoidal category.* In this case we shall say that $\mathbf{O}$ acts on $\mathbf{A}$ (sometimes, for emphasis, we shall say that $\mathbf{O}$ acts on $\mathbf{A}$ *on the left*).

Given $\mathbf{A}$, there exists a universal braided monoidal category that acts on $\mathbf{A}$ in the above sense. It is called the *Drinfeld center* of $\mathbf{A}$, and is denoted $Z_{\text{Dr}}(\mathbf{A})$.

The objects of $Z_{\text{Dr}}(\mathbf{A})$ are $z \in \mathbf{A}$, equipped with a family of isomorphisms

$$R_{z,\mathbf{a}} : z \otimes \mathbf{a} \simeq \mathbf{a} \otimes z,$$

compatible with tensor products of the $\mathbf{a}$'s.

27.2.2. Similar definitions apply for *right* actions of monoidal categories. In this way we obtain the notion of *right* action of a braided monoidal category on a monoidal category.

Thus, we can talk about triples $(\mathbf{O}, \mathbf{A}, \mathbf{O}')$, where $\mathbf{A}$ is a monoidal category, and $\mathbf{O}$ and $\mathbf{O}'$ are braided monoidal categories, acting compatibly on the left and right on $\mathbf{A}$, respectively.

Given $\mathbf{A}$ equipped with an action of $\mathbf{O}'$ on the right, there exists a universal braided monoidal category $\mathbf{O}$ that acts on $\mathbf{A}$ (on the left) in a way compatible with the right action of $\mathbf{O}'$. It is called the *relative (to* $\mathbf{O}'$*) Drinfeld center* of $\mathbf{A}$, and it is denoted $Z_{\mathrm{Dr},\mathbf{O}'}(\mathbf{A})$.

Objects of $Z_{\mathrm{Dr},\mathbf{O}'}(\mathbf{A})$ are $z \in \mathbf{A}$, equipped with a family of isomorphisms

$$R_{z,\mathbf{a}} : z \otimes \mathbf{a} \simeq \mathbf{a} \otimes z,$$

compatible with tensor products of the $\mathbf{a}$'s, and compatible with the action of $\mathbf{O}'$ in the sense that for $\mathbf{o}' \in \mathbf{O}'$ the map

$$R_{z,\mathbf{o}'} : z \otimes \mathbf{o}' \simeq \mathbf{o}' \otimes z,$$

agrees with the one induced by the right action of $\mathbf{O}'$ on $\mathbf{A}$.

We have a natural forgetful functor

$$Z_{\mathrm{Dr},\mathbf{O}'}(\mathbf{A}) \to \mathbf{A}.$$

Remark 27.2.3. Note that unless the braided monoidal structure on $\mathbf{O}'$ is *symmetric*, there is no naturally defined homomorphism from $\mathbf{O}'$ to $Z_{\mathrm{Dr},\mathbf{O}'}(\mathbf{A})$.

27.2.4. Let $\mathbf{O}$ be a braided monoidal category, and let A be a Hopf algebra in $\mathbf{O}$. In this case, the category

$$\mathbf{A} := A\text{-mod}$$

of A-modules in $\mathbf{O}$ acquires a natural monoidal structure, compatible with the forgetful monoidal functor

$$\mathbf{oblv}_A : A\text{-mod} \to \mathbf{O}.$$

We also note that tensoring on the right defines a *right* action of $\mathbf{O}$ on A-mod. Thus, we can talk about the braided monoidal category

$$Z_{\mathrm{Dr},\mathbf{O}}(A\text{-mod}).$$

Note that the monoidal forgetful functor

$$Z_{\mathrm{Dr},\mathbf{O}}(A\text{-mod}) \to A\text{-mod} \to \mathbf{O} \tag{27.2}$$

is *not* compatible with the braided structures.

27.2.5. Suppose now that A is dualizable as an object of $\mathbf{O}$. In this case, the forgetful functor

$$Z_{\mathrm{Dr},\mathbf{O}}(A\text{-mod}) \to A\text{-mod} \tag{27.3}$$

admits both a left and right adjoints.

The composition of the left adjoint to (27.3) with the forgetful functor (27.2) identifies with

$$M \mapsto \mathbf{oblv}_A(M) \otimes A^\vee,$$

The composition of the right adjoint to (27.3) with the forgetful functor (27.2) identifies with

$$M \mapsto \mathbf{oblv}_A(M) \otimes A.$$

27.3. **The category of modules over the small quantum group.** In this subsection we will finally introduce the category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod of modules over the (Langlands dual) small quantum group.

27.3.1. We apply the discussion in Sect. 27.2.4 to the braided monoidal category Vect_q^Λ and the Hopf algebra $\mathfrak{u}_q(\check{N}^+)$ *in* Vect_q^Λ.

We introduce the (braided monoidal) category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod to be

$$Z_{\mathrm{Dr},\mathrm{Vect}_q^\Lambda}(\mathfrak{u}_q(\check{N}^+)\text{-mod}).$$

We emphasize that in the above formula, and elsewhere, $\mathfrak{u}_q(\check{N}^+)$-mod denotes the category of $\mathfrak{u}_q(\check{N}^+)$-modules *in* Vect_q^Λ.

27.3.2. Let $\mathbf{oblv}_{\mathfrak{u}_q(\check{G})}$ denote the (conservative) forgetful functor

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \mathfrak{u}_q(\check{N}^+)\text{-mod} \to \mathrm{Vect}_q^\Lambda, \tag{27.4}$$

i.e., the functor (27.2).

This functor admits a left adjoint, which we denote by

$$\mathbf{ind}_{\mathfrak{u}_q(\check{G})} : \mathrm{Vect}_q^\Lambda \rightleftarrows \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}.$$

The resulting monad $\mathbf{oblv}_{\mathfrak{u}_q(\check{G})} \circ \mathbf{ind}_{\mathfrak{u}_q(\check{G})}$ is t-exact with respect to the (obvious) t-structure on Vect_q^Λ. This implies that $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod acquires a t-structure, uniquely characterized by the requirement that the forgetful functor $\mathbf{oblv}_{\mathfrak{u}_q(\check{G})}$ is t-exact. Moreover, the functor $\mathbf{ind}_{\mathfrak{u}_q(\check{G})}$ is also t-exact.

27.3.3. Denote

$$\mathfrak{u}_q(\check{G})^\mu := \mathbf{ind}_{\mathfrak{u}_q(\check{G})}(\mathsf{e}^\mu),$$

where we remind that $\mathsf{e}^\mu \in \mathrm{Vect}^\Lambda$ is the vector space e placed in degree μ.

The objects $\mathfrak{u}_q(\check{G})^\mu$ form a set of compact generators of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod; they are projective as objects of $(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^\heartsuit$.

From here we obtain that the canonically defined functor

$$D^+((\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^\heartsuit) \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$$

extends to an equivalence

$$D((\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^\heartsuit) \simeq \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}.$$

27.4. **Standard and costandard objects.**

27.4.1. The forgetful functor

$$\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \mathfrak{u}_q(\check{N}^+)\text{-mod}$$

also admits a t-exact left adjoint, denoted $\mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}$, see Sect. 27.2.5.

For $\mu \in \Lambda$, consider e^μ as an object of $\mathfrak{u}_q(\check{N}^+)$-mod, where the action of $\mathfrak{u}_q(\check{N}^+)$ is trivial. Set

$$\mathcal{M}^{\mu,!}_{\mathrm{quant}} := \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}(\mathsf{e}^\mu).$$

We call it the *standard object* of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod of highest weight μ. It belongs to the heart of the t-structure.

Remark 27.4.2. The objects $\mathcal{M}^{\mu,!}_{\mathrm{quant}}$ are sometimes called the "baby Verma modules".

27.4.3. Note that each $\mathfrak{u}_q(\check{G})^\mu$ has a finite filtration with subquotients $\mathcal{M}^{\mu+\lambda,!}_{\mathrm{quant}}$ with $\lambda \in \Lambda^{\mathrm{pos}}$.

27.4.4. It is well-known that $(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^\heartsuit$ has a structure of highest weight category, in which $\mathcal{M}^{\mu,!}_{\text{quant}}$ are the standard objects. In particular, for every μ there exists a *co-standard* object

$$\mathcal{M}^{\mu,*}_{\text{quant}} \in (\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^\heartsuit,$$

uniquely characterized by the requirement that

$$\mathcal{H}om_{\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}}(\mathcal{M}^{\mu',!}_{\text{quant}}, \mathcal{M}^{\mu,*}_{\text{quant}}) = \begin{cases} \mathsf{e} \text{ if } \mu' = \mu \\ 0 \text{ otherwise.} \end{cases} \tag{27.5}$$

27.4.5. We let $\mathcal{M}^{\mu,!*}_{\text{quant}}$ denote the image of the canonical map

$$\mathcal{M}^{\mu,!}_{\text{quant}} \to \mathcal{M}^{\mu,*}_{\text{quant}}.$$

The objects $\mathcal{M}^{\mu,!*}_{\text{quant}}$ are the irreducibles of $(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^\heartsuit$.

28. Renormalization for quantum groups

Recall (see Sect. 19.1) that we modified the category of factorization modules in order to obtained a category that eventually turned to be equivalent to $\overset{\bullet}{\text{Hecke}}(\text{Whit}_{q,x}(G))$. In this section we will apply a similar renormalization procedure to $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod.

In fact, we will define two different renormalizations of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod: one will be equivalent to the original category of factorization modules, and the other two its renormalized version.

28.1. **The "obvious" renormalization.** Along with $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod we will consider its renormalized version $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$, endowed with a pair of adjoint functors

$$\text{ren} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \rightleftarrows \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}} : \text{un-ren}\,.$$

The material here is parallel to Sect. 19.1.

28.1.1. The category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$ is defined as the the ind-completion of the full (but not cocomplete) subcategory of

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{fin.dim}} \subset \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$$

that consists of that go to compact objects in Vect_q^Λ under the forgetful functor $\mathbf{oblv}_{\mathfrak{u}_q(\check{G})}$ of (27.4).

Note that $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{fin.dim}}$ can also be characterized as consisting of finite extensions of (shifts of) irreducible objects.

Ind-extension of the tautological embedding defines a functor

$$\text{un-ren} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}.$$

28.1.2. Since the objects $\mathfrak{u}_q(\check{G})^\mu$ have finite length, we have

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^c \subset \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{fin.dim}}.$$

Ind-extension of this embedding defines a fully faithful functor

$$\text{ren} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}},$$

which is a left adjoint to un-ren.

28.1.3. We have the following assertion parallel to Proposition 19.1.8:

Proposition 28.1.4. *The category* $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$ *has a t-structure, uniquely characterized by the property that an object is connective if and only if its image under the functor* un-ren *is connective. Moreover, the functor* un-ren *has the following properties with respect to this t-structure:*

(a) *It is t-exact;*

(b) *It induces an equivalence*

$$(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}})^{\geq n} \to (\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^{\geq n}$$

for any n;

(c) *It induces an equivalence of the hearts.*

Corollary 28.1.5. *The kernel of the functor* un-ren *consists of infinitely coconnective objects, i.e.,*

$$\bigcap_n (\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}})^{\leq -n}.$$

Remark 28.1.6. We will use the notation $\mathcal{M}^{\mu,!}_{\text{quant}}$, $\mathcal{M}^{\mu,*}_{\text{quant}}$ and $\mathcal{M}^{\mu,!*}_{\text{quant}}$ for the corresponding objects of *either* $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod *or* $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$, see Remark 19.1.10.

28.1.7. It follows from the construction that the t-structure on $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$ is Artinian (see Sect. 6.3.8 for what this means).

28.2. **A different renormalization.** We will now introduce a different category, denoted

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}},$$

which will be sandwiched between $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod and $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$.

28.2.1. Consider the category $\mathfrak{u}_q(\check{N}^+)$-mod.

We define its renormalized version, denoted $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$, by the same procedure as above. I.e., $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$ is the ind-completion of the full (but not cocomplete) subcategory

$$(\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{fin.dim}} \subset \mathfrak{u}_q(\check{N}^+)\text{-mod}$$

consisting of objects that map to compact objects under the forgetful functor

$$\mathfrak{u}_q(\check{N}^+)\text{-mod} \to \text{Vect}_q^\Lambda .$$

We have a pair of adjoint functors

$$\text{ren} : \mathfrak{u}_q(\check{N}^+)\text{-mod} \rightleftarrows \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} : \text{un-ren}\,.$$

28.2.2. The subcategory $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{fin.dim}} \subset \mathfrak{u}_q(\check{N}^+)$-mod is preserved by the tensor product operation. Hence, it inherits a monoidal structure.

Ind-extending, we obtain that $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$ acquires a monoidal structure, which can be uniquely characterized by the requirement that the functor un-ren be monoidal.

Furthermore, we have a right action of (the braided monoidal category) Vect_q^Λ on $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$, so that the functor un-ren is compatible with the actions.

28.2.3. We define $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ to be (the relative to Vect_q^Λ) Drinfeld center of $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$, i.e.,

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} := Z_{\text{Dr},\text{Vect}_q^\Lambda}(\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}).$$

28.2.4. Note now that the monoidal operation

$$\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \otimes \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \to \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$$

factors as

$$\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \otimes \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \to \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \otimes \mathfrak{u}_q(\check{N}^+)\text{-mod} \to \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$$

and also

$$\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \otimes \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \to \mathfrak{u}_q(\check{N}^+)\text{-mod} \otimes \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \to \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}.$$

From here, it is easy to see that $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ is endowed by a pair of adjoint functors

$$\mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} : \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \rightleftarrows \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} : \mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}$$

and also a pair of adjoint functors

$$\text{ren}' : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \rightleftarrows \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} : \text{un-ren}'$$

that make all the circuits in the diagram

$$\begin{tikzcd}
\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}} \arrow[r, shift left, "\mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}"] \arrow[d, shift left, "\text{un-ren}"] & \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \arrow[l, shift left, "\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}"] \arrow[d, shift left, "\text{un-ren}'"] \\
\mathfrak{u}_q(\check{N}^+)\text{-mod} \arrow[u, shift left, "\text{ren}"] \arrow[r, shift left, "\mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}"] & \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \arrow[l, shift left, "\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}"] \arrow[u, shift left, "\text{ren}'"]
\end{tikzcd}$$

commute.

28.2.5. The above commutative diagram implies in particular that the monad $\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} \circ \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}$ acting on $\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$ is t-exact. Since the forgetful functor

$$\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \to \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{ren}}$$

is conservative, we obtain that $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ acquires a t-structure, uniquely characterized by the condition that $\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}$ is t-exact.

In particular, we obtain that the functor

$$\text{un-ren}' : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$$

is t-exact.

28.2.6. Thus, we obtain that $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ can also be obtained as a renormalization of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$.

Namely, $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ identifies with the ind-completion of the full subcategory

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby}} \subset \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$$

that consists of objects that finite extensions of (shifts of) standard (i.e., baby Verma) objects.

The functor

$$\text{un-ren}' : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$$

is the ind-extension of the above tautological embedding.

The functor

$$\text{ren}' : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$$

is the ind-extension of the embedding

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^c \subset \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby}},$$

the latter due to the fact that each $\mathfrak{u}_q(\check{G})^\mu$ has a filtration by standards, see Sect. 27.4.3.

By [FG1, Sect. 23], the functor $\text{un-ren}' : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$ induces an equivalence

$$(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}})^{\geq n} \to (\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod})^{\geq n}$$

for any n, and thus also an equivalence of the hearts.

Remark 28.2.7. We will use the notation $\mathcal{M}^{\mu,!}_{\text{quant}}$, $\mathcal{M}^{\mu,*}_{\text{quant}}$ and $\mathcal{M}^{\mu,!*}_{\text{quant}}$ for the corresponding objects of *either* $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$ *or* $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$, see Remark 19.1.10.

Remark 28.2.8. The definition of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ can also be rephrased as follows:

The monad $\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} \circ \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)}$ acting on $\mathfrak{u}_q(\check{N}^+)\text{-mod}$ preserves the subcategories

$$\mathfrak{u}_q(\check{N}^+)\text{-mod}^c \subset \mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{fin.dim}} \subset \mathfrak{u}_q(\check{N}^+)\text{-mod}.$$

We have:

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^c \simeq (\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} \circ \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)})\text{-mod}(\mathfrak{u}_q(\check{N}^+)\text{-mod}^c)$$

and

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby}} \simeq (\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} \circ \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)})\text{-mod}(\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{fin.dim}}),$$

where the functor ren' is induced by the embedding

$$(\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} \circ \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)})\text{-mod}(\mathfrak{u}_q(\check{N}^+)\text{-mod}^c) \hookrightarrow$$
$$\hookrightarrow (\mathbf{oblv}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)} \circ \mathbf{ind}^{\mathfrak{u}_q(\check{G})}_{\mathfrak{u}_q(\check{N}^+)})\text{-mod}(\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\text{fin.dim}}).$$

28.3. **Relationship between the two renormalizations.**

28.3.1. Note now that the category of compact objects in $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ can be thought of as a subcategory of the category of compact objects in $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$. Hence, we obtain a fully faithful functor

$$\text{ren}'' : \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \to \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$$

that admits a continuous right adjoint, denoted $\text{un-ren}''$.

The composition

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod} \xrightarrow{\text{ren}'} \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \xrightarrow{\text{ren}''} \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$$

identifies with the functor ren, and the composition

$$\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}} \xrightarrow{\text{un-ren}''} \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}} \xrightarrow{\text{un-ren}'} \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$$

identifies with the functor ren.

As in [AG, Corollary 4.4.3], we obtain that the functor $\text{un-ren}''$ is t-exact.

28.3.2. Thus, we can think of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$ as a renormalization of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$.

Namely, $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{ren}}$ is the ind-completion of the full subcategory of $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ consisting of objects that are finite extensions of (shifts of) the the objects $\mathcal{M}^{\mu,!*}_{\text{quant}}$, *where the latter are viewed as objects in* $(\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}})^{\heartsuit}$.

29. Quantum groups vs factorization modules equivalence

In this section we will take the ground field to be $\mathbb{C}$, the curve X to be $\mathbb{A}^1$, and the sheaf theory to be that of constructible sheaves in the classical topology. We will relate the category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}$ (or rather its renormalized version) to the category of factorizations modules over Ω^{small}_q.

29.1. **Matching the parameters.** In this subsection we will explain how to match the parameters needed to define the quantum group with those needed to define Ω^{small}_q.

29.1.1. Recall (see Sect. 27.1.1) that in order to define the category of modules over the quantum group, we started with a bilinear form

$$b' : \Lambda \otimes \Lambda \to \mathsf{e}^{\times,\mathrm{tors}},$$

such that the corresponding quadratic form belongs to $\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}})^W_{\mathrm{restr}}$.

Since the ground field is $\mathbb{C}$, the Tate twist is canonically trivialized, so we can regard b' as a map

$$b' : \Lambda \otimes \Lambda \to \mathsf{e}^{\times,\mathrm{tors}}(-1). \tag{29.1}$$

We claim that the datum of (29.1) gives rise to a geometric metaplectic datum for T equipped with a rigidification, see Sect. 17.1.1 [4].

29.1.2. First, we claim that a bilinear form (29.1) (without the extra condition on q) gives rise to a geometric metaplectic datum $\mathcal{G}^T$ for T.

Recall the description of factorization gerbes for tori, given in [GLys, Sect. 4.1.3]. Namely, to specify a factorization gerbe $\mathcal{G}^T$, we need to specify for every finite set J and a map $\lambda_J : J \to \Lambda$ a gerbe $\mathcal{G}^T_{\lambda^J}$ on X^J, along with the compatibilities of [GLys, Equations (4.3) and (4.4)].

For j, denote $\lambda_j = \lambda_J(j)$. For un *unordered* pair of elements $j_1 \neq j_2$, let Δ_{j_1,j_2} denote the corresponding diagonal divisor in X^J. We set

$$\mathcal{G}^T_{\lambda^J} := \left(\underset{j\in J}{\boxtimes} \omega^{q(\lambda_j)} \right) \bigotimes \left(\underset{j_1\neq j_2/\Sigma_2}{\otimes} \mathcal{O}(-\Delta_{j_1,j_2})^{b(\lambda_{j_1},\lambda_{j_2})} \right). \tag{29.2}$$

The isomorphisms of [GLys, Equations (4.4)] are automatic. In order to construct the isomorphisms of [GLys, Equations (4.3)], to simplify the notation we will consider the case $J = \{1, 2\}$. Thus, we need to construct an isomorphism of gerbes

$$\omega^{q(\lambda_1)} \otimes \omega^{q(\lambda_2)} \otimes (\mathcal{O}(-\Delta)|_\Delta)^{b(\lambda_{j_1},\lambda_{j_2})} \simeq \omega^{q(\lambda_1+\lambda_2)}. \tag{29.3}$$

29.1.3. Let us note that for every element $c \in \mathsf{e}^\times(-1)$ we have a well-defined $\mathsf{e}^\times$-torsor, denoted $(-1)^c$, constructed as follows:

To c we associate the corresponding Kummer sheaf Ψ_c on $\mathbb{G}_m$. We set $(-1)^c$ to be equal to the fiber of Ψ_c at $(-1) \in \mathbb{G}_m$.

29.1.4. To define (29.3) let us first choose an ordering, namely $(1, 2)$ on $\{1, 2\}$. This ordering identifies the line bundle $\mathcal{O}(-\Delta)|_\Delta$ with ω.

We let (29.3) be the tautological isomorphism coming from the identity

$$q(\lambda_1) \cdot q(\lambda_2) \cdot b(\lambda_{j_1}, \lambda_{j_2}) = q(\lambda_1 + \lambda_2),$$

tensored with the line

$$(-1)^{b'(\lambda_1,\lambda_2)}.$$

Note that this is the only place in the construction where we use the data of a bilinear form b', as opposed to that of a quadratic form q.

29.1.5. Let us now show that the isomorphism (29.3) is canonically independent of the choice of the ordering. Indeed, the swap of two factors multiplies the identification

$$\mathcal{O}(-\Delta)|_\Delta \simeq \omega$$

by (-1).

The required isomorphism follows now from

$$(-1)^{b(\lambda_1,\lambda_2)} \otimes (-1)^{b'(\lambda_1,\lambda_2)} \simeq (-1)^{b'(\lambda_2,\lambda_1)}.$$

[4] The discussion in the rest of this subsection applies to a general sheaf theory of a general ground field.

29.1.6. The above construction defines a map

$$\mathrm{Bilin}(\Lambda, \mathsf{e}^{\times}(-1)) \to \mathrm{FactGe}_T\,. \tag{29.4}$$

In fact, one can describe FactGe_T entirely in terms of this map. Namely, by unwinding the constructions, one obtains that FactGe_T, viewed as a connective spectrum, is represented by the push-out along $B^2(\mathrm{Hom}(\Lambda, \pm 1))$ of

$$\mathrm{Ge}(X, \check{T}(\mathsf{e})^{\mathrm{tors}})$$

and the connective spectrum represented by the complex

$$\mathrm{Quad}(\Lambda, \pm 1) \to \mathrm{Alt}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)), \tag{29.5}$$

where:

- The map $\mathrm{Alt}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))$ is
$$\mathrm{Alt}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \overset{x\to x^2}{\longrightarrow} \mathrm{Alt}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \hookrightarrow \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))$$
- The map $\mathrm{Quad}(\Lambda, \pm 1) \to \mathrm{Alt}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))$ is
$$\mathrm{Quad}(\Lambda, \pm 1) \to \mathrm{SymBilin}(\Lambda, \pm 1) = \mathrm{Alt}(\Lambda, \pm 1) \hookrightarrow \mathrm{Alt}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \tag{29.6}$$
- The map from $B^2(\mathrm{Hom}(\Lambda, \pm 1))$ to the spectrum represented by the complex (29.5) is given by the identification of $\mathrm{Hom}(\Lambda, \pm 1)$ with the kernel of (29.6);
- The map $B^2(\mathrm{Hom}(\Lambda, \pm 1)) \to \mathrm{Ge}(X, \check{T}(\mathsf{e})^{\mathrm{tors}})$ is
$$B^2(\mathrm{Hom}(\Lambda, \pm 1)) \to B^2(\mathrm{Hom}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}})) = B^2(\check{T}(\mathsf{e})^{\mathrm{tors}}) \to \mathrm{Ge}(X, \check{T}(\mathsf{e})^{\mathrm{tors}}).$$

Remark 29.1.7. Note also that the complex (29.5) admits a natural quasi-isomorphic embedding into the complex maps quasi-isomorphically to the complex

$$\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)), \tag{29.7}$$

in which the map

$$\mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1)) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))$$

is

$$b'' \mapsto b'; \quad b''(\lambda, \mu) = b'(\lambda, \mu) - b'(\mu, \lambda).$$

This complex is acyclic in degree -1. Its H^0 identifies with $\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))$ and H^{-2} with $\mathrm{Hom}(\Lambda, \pm 1)$.

Note that the assignment

$$b' \mapsto \mathrm{Rep}_q(T) \tag{29.8}$$

is a bijection between $\mathrm{Bilin}(\Lambda, \mathsf{e}^{\times}$ and the set of braided structures on the monoidal category $\mathrm{Rep}(T)$.

The meaning of the complex (29.7) is the following (up to replacing $\mathsf{e}^{\times,\mathrm{tors}}(-1)$ by $\mathsf{e}^{\times}$):

It follows from [Del, P.S.] that the assignment (29.8) extends to an equivalence between the pushout-out along $B^2(\mathrm{Hom}(\Lambda, \pm 1))$ of $B^2(\mathrm{Hom}(\Lambda, \mathsf{e}^{\times}))$ with the connective spectrum represented by the complex

$$\mathrm{Quad}(\Lambda, \mathsf{e}^{\times}) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times}) \to \mathrm{Bilin}(\Lambda, \mathsf{e}^{\times}), \tag{29.9}$$

and the groupoid of braided monoidal categories whose monoid of isomorphism classes of objects is identified with Λ.

29.1.8. Let us now restore the condition that q belong to $\mathrm{Quad}(\Lambda, \mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}}$. Let us show that the resulting factorization gerbe on Conf is equipped with a rigidification (see Sect. 17.1.1).

Indeed, according to formula (2.1), we need construct a trivialization of the gerbe

$$\omega^{q(-\alpha_i)} \otimes (\omega^{\otimes \frac{1}{2}})^{b(-\alpha_i, 2\rho)}.$$

For this, it suffices to show that

$$q(-\alpha_i)^2 \cdot b(-\alpha_i, 2\rho) = 1,$$

but this follows from condition (2.3).

29.1.9. The above construction defines a map

$$\begin{aligned}(29.10)\quad \operatorname{Bilin}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1)) \underset{\operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))}{\times} \operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}} \to \\ \to \operatorname{FactGe}^{\mathrm{rigid}}_T \underset{\operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))}{\times} \operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}}\end{aligned}$$

(while the latter is isomorphic to FactGe_G, see (17.2)).

As in Sect. 29.1.6 above, one can describe the space

$$\operatorname{FactGe}^{\mathrm{rigid}}_T \underset{\operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))}{\times} \operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}}$$

entirely in terms of the map (29.10). Namely, this space identifies with the push-out along $B^2(\operatorname{Hom}(\pi_{1,\mathrm{alg}}(G),\pm 1))$ of $\operatorname{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}})$ and the connective spectrum represented by the complex

$$\begin{aligned}(29.11)\quad \ker(\operatorname{Quad}(\Lambda,\pm 1)\to \prod_i \pm 1)\to \\ \to \operatorname{Alt}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1)) \to \operatorname{Bilin}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1)) \underset{\operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))}{\times} \operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}},\end{aligned}$$

where:

- The map $\operatorname{Quad}(\Lambda,\pm 1)\to \prod_i \pm 1$ is given by evaluating a quadratic form on the simple negative coroots;
- The map from $B^2(\operatorname{Hom}(\pi_{1,\mathrm{alg}}(G),\pm 1))$ to the connective spectrum represented by the complex (29.11) is given by identifying H^{-2} of this complex with
$$\ker(\operatorname{Hom}(\Lambda,\pm 1)\to \prod_i \pm 1)\simeq \operatorname{Hom}(\pi_{1,\mathrm{alg}}(G),\pm 1);$$
- The map $B^2(\operatorname{Hom}(\pi_{1,\mathrm{alg}}(G),\pm 1))\to \operatorname{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}})$ is
$$B^2(\operatorname{Hom}(\pi_{1,\mathrm{alg}}(G),\pm 1))\to B^2(\operatorname{Hom}(\pi_{1,\mathrm{alg}}(G),\mathsf{e}^{\times,\mathrm{tors}})) = B^2(Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}})\to \operatorname{Ge}(X, Z_{\check{G}}(\mathsf{e})^{\mathrm{tors}}).$$

Remark 29.1.10. Recall (see Remark 17.1.2) that if G is semi-simple simply-connected, the connective spectrum $\operatorname{FactGe}^{\mathrm{rigid}}_T \underset{\operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))}{\times} \operatorname{Quad}(\Lambda,\mathsf{e}^{\times,\mathrm{tors}}(-1))^W_{\mathrm{restr}}$ is discrete.

This matches the fact that in this case the complex (29.11) is acyclic off degree 0. Indeed, in this case we have an isomorphism

$$\ker(\operatorname{Quad}(\Lambda,\pm 1)\to \prod_i \pm 1)\oplus \operatorname{Hom}(\Lambda,\pm 1)\to \operatorname{Quad}(\Lambda,\pm 1),$$

while $\operatorname{Hom}(\Lambda,\pm 1)$ identifies with the kernel of the map (29.6).

In particular, for G semi-simple simply-connected, the complex (29.9) is canonically isomorphic to a direct sum

$$\operatorname{Quad}(\Lambda,\mathsf{e}^{\times})\oplus \operatorname{Hom}(\Lambda,\pm 1)[-2].$$

29.2. **Statement of the result.** In this subsection we will state the main theorem of this Part that establishes an equivalence between the categories $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$ and $\Omega^{\mathrm{small}}_q$-FactMod.

29.2.1. We start with a form b' as in Sect. 29.1.1. Choose the lines $\mathsf{e}^{i,\mathrm{quant}}$, see Sect. 27.1.6. Let $\mathsf{f}^{i,\mathrm{fact}}$ be the dual lines.

To the data $(b',\{\mathsf{e}^{i,\mathrm{quant}}\})$ be we associate the category $\overset{\bullet}{\mathfrak{u}}_q(\check{G})$-mod, and its renormalized version $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$.

To the data of b' we associate a geometric metaplectic data $\mathcal{G}^T$ for T (see Sect. 29.1.1), and to the data $(\mathcal{G}^T,\{\mathsf{f}^{i,\mathrm{fact}}\})$ we attach the category $\Omega^{\mathrm{small}}_q$-FactMod, see Sect. 17.

29.2.2. The goal of this section is to prove the following:

Theorem 29.2.3. *There exists a canonical equivalence*

$$\Omega_q^{\mathrm{small}}\text{-FactMod} \simeq \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\mathrm{baby\text{-}ren}},$$

which is t-exact and maps standards to standards.

We note that the assertion of Theorem 29.2.3 at the level of the hearts of the corresponding categories is the main result of the book [BFS], specifically Theorem 17.1 in Part III of *loc. cit.*

29.2.4. As a formal corollary of Theorem 29.2.3 we obtain:

Corollary 29.2.5. *There exists a canonical equivalence*

$$\Omega_q^{\mathrm{small}}\text{-FactMod}^{\mathrm{ren}} \simeq \overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\mathrm{ren}},$$

which is t-exact and maps standards to standards.

The rest of this section is devoted to the proof of Theorem 29.2.5[5].

29.2.6. Note also that Theorem 29.2.3 gives a proof of Proposition 19.1.4 when $k = \mathbb{C}$.

29.3. **Koszul duality for Hopf algebras.** In this section we will perform the first step in the proof of Theorem 29.2.3: it consists of passing from a Hopf algebra in Vect_q^Λ (such as $\mathfrak{u}_q(\check{N}^+)$) to its Koszul dual $\mathbb{E}_2$-algebra.

29.3.1. Let A be a Hopf algebra in Vect_q^Λ such that its augmentation ideal is contained in $\mathrm{Vect}_q^{\Lambda^{\mathrm{pos}}-0}$, and each graded component is finite-dimensional.

Consider the category A-mod, and let $A\text{-mod}^{\mathrm{ren}}$ denote its renormalized version defined as in Sect. 28.2.1.

By the definition of $A\text{-mod}^{\mathrm{ren}}$, the functor of trivial action

$$\mathbf{triv}_A : \mathrm{Vect}_q^\Lambda \to A\text{-mod}^{\mathrm{ren}},$$

admits a *continious* right adjoint.

This right adjoint is also conservative, because by the condition on A, the essential image of the functor $\mathbf{triv}_A$ generates $A\text{-mod}^{\mathrm{ren}}$ under colimits (indeed, the essential image of $(\mathrm{Vect}_q^\Lambda)^{\mathrm{fin.dim}}$ along $\mathbf{triv}_A$ generates $A\text{-mod}^{\mathrm{fin.dim}}$ under finite colimits).

29.3.2. The monad

$$\mathbf{inv}_A \circ \mathbf{triv}_A$$

acting on Vect_q^Λ commutes with right multiplication. Hence, it is given by an associative algebra in Vect_q^Λ, to be denoted Inv_A. The underlying object of Vect_q^Λ identifies with $\mathbf{inv}_A \circ \mathbf{triv}_A(\mathsf{e})$, i.e.,

$$\lambda \mapsto \mathcal{H}om_{A\text{-mod}}(\mathbf{triv}_A(\mathsf{e}^0), \mathbf{triv}_A(\mathsf{e}^\lambda)).$$

Tautologically, the functor $\mathbf{inv}_A$ upgrades to a functor

$$\mathbf{inv}_A^{\mathrm{enh}} : A\text{-mod}^{\mathrm{ren}} \to \mathrm{Inv}_A\text{-mod}. \tag{29.12}$$

By the Barr-Beck-Lurie theorem, the above functor (29.12) is an equivalence.

[5]The proof of Theorem 29.2.3 given below is the result of discussions between the first-named author and J. Lurie. However, the responsibility for any shortcomings that may result from its publication lie with D.G.

29.3.3. Let now A be a Hopf algebra in Vect_q^Λ. This structure is equivalent to giving A-mod (or $A\text{-mod}^{\mathrm{ren}}$) a structure of monoidal category, for which the forgetful functor

$$A\text{-mod}^{\mathrm{ren}} \to \mathrm{Vect}_q^\Lambda$$

is monoidal, in a way compatible with the right action of the braided monoidal category Vect_q^Λ (see Sect. 27.2.4). The unit in $A\text{-mod}^{\mathrm{ren}}$ is given by $\mathbf{triv}_A(k)$.

Hence, the equivalence of (29.12) induces a monoidal structure on Inv_A-mod, for which the unit object is

$$\mathrm{Inv}_A \in \mathrm{Inv}_A\text{-mod}.$$

Such a structure is equivalent to a structure on Inv_A of $\mathbb{E}_2$-algebra in Vect_q^Λ, so that (29.12).

29.3.4. Thus, we obtain an equivalence of the corresponding (relative to Vect_q^Λ) Drinfeld centers

$$Z_{\mathrm{Dr},\mathrm{Vect}_q^\Lambda}(A\text{-mod}^{\mathrm{ren}}) \to Z_{\mathrm{Dr},\mathrm{Vect}_q^\Lambda}(\mathrm{Inv}_A\text{-mod}). \tag{29.13}$$

29.3.5. By [Fra, Proposition 4.36] we have:

$$Z_{\mathrm{Dr},\mathrm{Vect}_q^\Lambda}(\mathrm{Inv}_A\text{-mod}) \simeq \mathrm{Inv}_A\text{-mod}_{\mathbb{E}_2},$$

where the latter denotes the category of $\mathbb{E}_2$-modules over the $\mathbb{E}_2$-algebra Inv_A in the braided monoidal category Vect_q^Λ.

29.3.6. To summarize, we obtain an equivalence

$$Z_{\mathrm{Dr},\mathrm{Vect}_q^\Lambda}(A\text{-mod}^{\mathrm{ren}}) \simeq \mathrm{Inv}_A\text{-mod}_{\mathbb{E}_2}. \tag{29.14}$$

29.4. **Factorization algebras vs $\mathbb{E}_2$-algebras.** We will now perform a crucial step in the transition between $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\mathrm{ren}}$ and $\Omega_q^{\mathrm{small}}$-FactMod: we will relate (a certain kind of) $\mathbb{E}_2$-algebras in Vect_q^Λ and factorization algebras in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$.

29.4.1. Recall that in this section the curve X is taken to be $\mathbb{A}^1$ (with $x \in X$ being $0 \in \mathbb{A}^1$).

According to [Lur], to a braided monoidal category $\mathbf{O}$ one can attach a *factorization category* over the Ran space of $\mathbb{A}^1$, denoted $\mathrm{Fact}(\mathbf{O})$.

Futhermore, if B is an $\mathbb{E}_2$-algebra in $\mathbf{O}$, then to it there corresponds a *factorization algebra* Ω_B in $\mathrm{Fact}(\mathbf{O})$, and we have an equivalence between the category of $\mathbb{E}_2$-modules with respect to B in $\mathbf{O}$ and factorization Ω_B-modules in $\mathrm{Fact}(\mathbf{O})$, i.e.,

$$\Omega_B\text{-FactMod} \simeq B\text{-mod}_{\mathbb{E}_2}. \tag{29.15}$$

29.4.2. We take $\mathbf{O} = \mathrm{Vect}_q^\Lambda$. In this case $\mathrm{Fact}(\mathbf{O})$ identifies with $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}})$.

Let now B be a (non-unital) $\mathbb{E}_2$-algebra Vect_q^Λ, which is contained in ${\mathrm{Vect}_q^\Lambda}^{\mathrm{neg}-0}$. Then Ω_B, viewed as an object of $\mathrm{Shv}_{\mathcal{G}^T}(\mathrm{Gr}_{T,\mathrm{Ran}})$, is supported on $(\mathrm{Gr}_{T,\mathrm{Ran}})^{\mathrm{neg}}$. Hence, by Sect. 5.5.1, we can think of Ω_B as a $\mathcal{G}^\Lambda$-twisted factorization algebra on Conf.

Furthermore, according to (5.10), we can think of factorization Ω_B-modules on $\mathrm{Gr}_{T,\mathrm{Ran}}$ as factorization Ω_B-modules on $\mathrm{Conf}_{\infty\cdot x}$.

Hence, (29.15) becomes an equivalence

$$\Omega_B\text{-FactMod} \simeq B\text{-mod}_{\mathbb{E}_2}, \tag{29.16}$$

where Ω_B-FactMod denotes the category of factorization Ω_B-modules in $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty\cdot x})$.

Let us write how certain functors on one side of the equivalence (29.16) translate to the other side.

29.4.3. Since our curve X is $\mathbb{A}^1$, the canonical line bundle ω on X is trivialized. In addition, we have a canonical generator for the diagonal divisor $\Delta \subset X \times X$. In particular, by formulas (29.2) and (2.1), the fiber of $\mathcal{G}^\Lambda$ at any point $\mu \cdot x \in \mathrm{Conf}_{\infty \cdot x}$ admits a canonical trivialization.

Remark 29.4.4. It is easy to see that the gerbe $\mathcal{G}^\Lambda$ on all of $\mathrm{Conf}_{\infty \cdot x}$ (and on Conf) admits a canonical trivialization. However, these trivializations are *incompatible* with factorization.

The above trivialization of $\mathcal{G}^\Lambda$ on Conf is also incompatible with the trivialization of $\mathcal{G}^\Lambda$ on $\overset{\circ}{\mathrm{Conf}}$ of Sect. 17.1.3. The discrepancy of these two trivialization is given by a *non-trivial* local system on $\overset{\circ}{\mathrm{Conf}}(X, \Lambda^{\mathrm{neg}})$. This is the braiding local system of [BFS, Part III, Sect. 3.1]. As a result, the twisted perverse sheaf $\overset{\circ}{\Omega}{}^{\mathrm{small}}_q$, viewed as a plain perverse sheaf on $\overset{\circ}{\mathrm{Conf}}$ (via the above trivialization of $\mathcal{G}^\Lambda$ specific to $\mathbb{A}^1$) is not just the sign local system, but has a monodromy that depends on q.

29.4.5. First off, for $\mu \in \Lambda$, the functor

$$B\text{-mod}_{\mathbb{E}_2} \to \mathrm{Vect}_q^\Lambda \to \mathrm{Vect}_q^\mu \simeq \mathrm{Vect},$$

(where the second arrow is the projection on the μ-component) corresponds in terms of the equivalence (29.16) to the composite:

$$\Omega_B\text{-FactMod} \overset{\mathbf{oblv}_{\mathrm{Fact}}}{\longrightarrow} \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty \cdot x}) \overset{!\text{-fiber at } \mu\cdot x}{\longrightarrow} \mathcal{G}^\lambda|_{\mu \cdot x} \simeq \mathrm{Vect}\,.$$

29.4.6. Consider now the functor $\Omega_B\text{-FactMod} \to \mathrm{Vect}$ equal to the composite:

- The forgetful functor $\Omega_B\text{-FactMod} \to \mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf}_{\infty \cdot x})$;
- The functor of !-restriction to the subspace of *non-negative real configurations*

$$\mathrm{Conf}_{\infty \cdot x}^{\mathbb{R}^{\geq 0}} \subset \mathrm{Conf}_{\infty \cdot x};$$

- The functor of *-fiber at $\mu \cdot x \in \mathrm{Conf}_{\infty \cdot x}^{\mathbb{R}^{\geq 0}}$.
- The identification $\mathcal{G}^\lambda|_{\mu \cdot x} \simeq \mathrm{Vect}$.

The corresponding functor $B\text{-mod}_{\mathbb{E}_2} \to \mathrm{Vect}$ is the composite

$$B\text{-mod}_{\mathbb{E}_2} \to B\text{-mod} \overset{\mathbf{e} \underset{B}{\otimes} -}{\longrightarrow} \mathrm{Vect}_q^\Lambda \to \mathrm{Vect}_q^\mu \simeq \mathrm{Vect}\,.$$

29.5. **Hopf algebras vs factorization algebras.** In this subsection we will supply some explicit information on the factorization algebra Ω_B corresponding via (29.15) to the augmentation ideal in Inv_A, where A is a Hopf algebra as in Sect. 29.3.1.

29.5.1. Let B be the augmentation ideal in Inv_A. Composing (29.16) with (29.14), we obtain an equivalence

$$Z_{\mathrm{Dr}, \mathrm{Vect}_q^\Lambda}(A\text{-mod}^{\mathrm{ren}}) \simeq \Omega_B\text{-FactMod}\,. \tag{29.17}$$

29.5.2. Let us see how the obvious forgetful functors on one side of the equivalence (29.17) look on the other side.

The functor of !-fiber at the point $\mu \cdot x \in \mathrm{Conf}_{\infty \cdot x}$ on $\Omega_B\text{-FactMod}$ (i.e., the composite described in Sect. 29.4.5) corresponds to the composite

$$Z_{\mathrm{Dr}, \mathrm{Vect}_q^\Lambda}(A\text{-mod}^{\mathrm{ren}}) \to A\text{-mod}^{\mathrm{ren}} \overset{\mathbf{inv}_A}{\to} \mathrm{Vect}_q^\Lambda \to \mathrm{Vect}_q^\mu \simeq \mathrm{Vect},$$

where the first arrow in the forgetful functor and the third arrow is the projection onto the μ-component.

Consider now the composite functor $\Omega_B\text{-FactMod} \to \mathrm{Vect}$, described in Sect. 29.4.6. It corresponds to the forgetful functor

$$Z_{\mathrm{Dr}, \mathrm{Vect}_q^\Lambda}(A\text{-mod}^{\mathrm{ren}}) \to \mathrm{Vect}_q^\Lambda \to \mathrm{Vect}_q^\mu \simeq \mathrm{Vect},$$

where the second arrow is the projection onto the μ-component.

29.5.3. We have the following additional two properties of the assignment $A \rightsquigarrow \Omega_B$:

Proposition 29.5.4.

(a) *For $\mu \in \Lambda^{\mathrm{neg}}$, the vector space equal to the *-fiber at $\mu \cdot x$ of Ω_B identifies with the $-\mu$-component of* $\mathrm{Inv}_{A^\vee}$*, where $A^\vee$ is the component-wise linear dual of A, viewed as a Hopf algebra in* $\mathrm{Vect}^\Lambda_{q^{-1}}$*.*

(b) *For $\mu \in \Lambda^{\mathrm{neg}}$, the vector space equal to the *-fiber at $\mu \cdot x$ of the !-restriction of Ω_B to* $\mathrm{Conf}^{\mathbb{R}} \subset \mathrm{Conf}$ *identifies with the vector space* dual *to the $-\mu$-component of A.*

29.5.5. We now claim:

Corollary 29.5.6.

(a) *If A is concentrated in cohomological degrees* ≥ 0 *(resp.,* ≤ 0*), with respect to the obvious t-structure on* Vect^Λ_q*, then Ω_B, viewed as an object of* $\mathrm{Shv}_{\mathcal{G}^\Lambda}(\mathrm{Conf})$*, is concentrated in perverse cohomological degrees* ≥ 0 *(resp.,* ≤ 0*).*

(a') *If A is concentrated in cohomological degree* 0*, then Ω_B is perverse.*

(b) *If A_1 and A_2 are both concentrated in cohomological degree* 0 *and $A_1 \to A_2$ is a surjective (resp., injective) map of Hopf algebras, then the induced map $\Omega_{B_2} \to \Omega_{B_1}$ is injective (resp., surjective).*

(c) *Under the assumption of* (a')*, the equivalence* (29.17) *is t-exact.*

Proof. For a fixed $\lambda \in \Lambda^{\mathrm{neg}}$ consider the full subcategory of $\mathrm{Shv}(X^\lambda)$ that consists of complexes locally constant along the diagonal stratification. Then the functor that sends an object to the *-fiber at $\mu \cdot x$ of its !-restriction to $X^\lambda_{\mathbb{R}}$ is conservative and t-exact (in the perverse t-structure).

This implies points (a) and (b) in view of Proposition 29.5.4(b). Point (a') is a particular case of (a).

For point (c) we consider the full subcategory of $\mathrm{Shv}(X^\lambda)$ that consists of complexes locally constant along the stratification given by diagonals and incidence with x. Then on this subcategory, the functor that sends an object to the *-fiber at $\mu \cdot x$ of its !-restriction to $X^\lambda_{\mathbb{R}_{\geq 0}}$ is t-exact (in the perverse t-structure).

□

29.6. **The case of quantum groups.** We will now combine the contents of Sects. 29.3-29.5, and complete the proof of Theorem 29.2.3.

29.6.1. We apply the above discussion to the Hopf algebra $A = \mathfrak{u}_q(\check{N}^+)$. First, we claim that the resulting factorization algebra $\mathrm{Conf}\ \Omega_B$ on Conf identifies with $\Omega_q^{\mathrm{small}}$.

Using Corrollary 29.5.6(b), it suffices to show that the (twisted) sheaf $\Omega_{A'}$ on Conf corresponding to $A' = U_q(\check{N})^{\mathrm{free}}$ (resp., $A' = U_q(\check{N})^{\text{co-free}}$) is given by extension by * (resp., !) of $\overset{\circ}{\Omega}{}_q^{\mathrm{small}}$ along the embedding

$$\overset{\circ}{\mathrm{Conf}} \hookrightarrow \mathrm{Conf}\,. \tag{29.18}$$

We will prove the assertion regarding $U_q(\check{N})^{\mathrm{free}}$; the one about $U_q(\check{N})^{\text{co-free}}$ follows similarly by applying Proposition 29.5.4(a).

29.6.2. We have to show that !-fibers of Ω_B for $B = \mathrm{Inv}_A$ with $A = U_q(\check{N})^{\mathrm{free}}$ are zero on the complement to (29.18). By factorization, this is equivalent to showing that !-fibers of Ω_B are zero at points $\mu \cdot x$ for μ not being a negative simple root.

By Sect. 29.4.5, we need to show that $\mathbf{inv}_A(\mathsf{e})$ lives only in degrees that are negative simple roots. However, this follows from the fact that A is free as an associative algebra on the generators in degrees equal to simple roots.

29.6.3. Thus, the equivalence (29.16) translates in our case to the equivalence

$$\Omega_q^{\mathrm{small}}\text{-FactMod} \simeq Z_{\mathrm{Dr},\mathrm{Vect}_q^\Lambda}(\mathfrak{u}_q(\check{N}^+)\text{-mod}^{\mathrm{ren}}),$$

where the right-hand side is by definition $\overset{\bullet}{\mathfrak{u}}_q(\check{G})\text{-mod}^{\text{baby-ren}}$.

This establishes the equivalence of categories claimed in Theorem 29.2.3. The t-exactness property of the above equivalence follows from Corrollary 29.5.6(c). The fact that this equivalence maps standards to standards follows from the construction.

Appendix A. Whittaker vs Kirillov models

In this Appendix we will explain a device that replaces the Whittaker model when the Artin-Schreier sheaf does not exist, e.g., in the ℓ-adic context when the ground field has characteristic zero, or in the Betti context. This device is called the Kirillov model.

A.1. **The context.**

A.1.1. Let $\mathcal{Y}$ be an algebraic stack of finite type over the ground field k. Let first $\mathcal{Y}$ be equipped with an action of $\mathbb{G}_a$. If k is of characteristic zero and we are working with D-modules over if k is of positive characteristic and we are working with ℓ-adic sheaves, we have a well-defined *full subcategory*

$$\mathrm{Whit}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$$

that consists of objects that are $\mathbb{G}_a$-invariant against a chosen Artin-Schreier sheaf χ on $\mathbb{G}_a$ (in the context of D-modules, χ is the exponential D-module).

I.e., this is the category whose objects are those $\mathcal{F} \in \mathrm{Shv}(\mathcal{Y})$ for which there *exists* an isomorphism

$$\mathrm{act}^*(\mathcal{F}) \simeq \chi \boxtimes \mathcal{F},$$

whose further *-pullback along

$$\{0\} \times \mathcal{Y} \xrightarrow{i_0} \mathbb{G}_a \times \mathcal{Y}$$

is the identity map

$$\mathcal{F} \simeq i_0^* \circ \mathrm{act}^*(\mathcal{F}) \simeq i_0^* \circ (\chi \boxtimes \mathcal{F}) \simeq \mathcal{F}.$$

A.1.2. The subcategory $\mathrm{Whit}(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})$ has some favorable properties:

(1) It is compatible with the t-structure (i.e., is preserved by the truncation functors);
(2) For a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the functors $f^!$ and f_* send the categories $\mathrm{Whit}(\mathcal{Y}_1)$ and $\mathrm{Whit}(\mathcal{Y}_2)$ to one another. Furthermore, the partially defined left adjoints of these functors, i.e., the functors $f_!$ and f^* also send $\mathrm{Whit}(\mathcal{Y}_i)$ to $\mathrm{Whit}(\mathcal{Y}_j)$ on those objects on which they are defined;
(3) Verdier duality (which is defined on $\mathrm{Shv}(\mathcal{Y})^{\mathrm{loc.c.}}$, see Sect. 4.3.3) sends $\mathrm{Whit}(\mathcal{Y})$ to a similarly defined character for the opposite choice of the Artin-Schreier sheaf.

A.1.3. Assume now that we are given an extension of the action of $\mathbb{G}_a$ on $\mathcal{Y}$ to an action of the semi-direct product

$$\mathbb{G}_m \ltimes \mathbb{G}_a.$$

In this case we will be able to define another category, denoted $\mathrm{Kir}_*(\mathcal{Y})$. This category $\mathrm{Kir}_*(\mathcal{Y})$ will be defined in an arbitrary sheaf-theoretic context (in that it does not require the existence of the Artin-Schreier sheaf).

In the context when Artin-Schreier is defined, we will have a canonical equivalence

$$\mathrm{Kir}_*(\mathcal{Y}) \simeq \mathrm{Whit}(\mathcal{Y}).$$

In addition, in the context of constructible sheaves or holonomic D-modules, we will be able to define another category, denoted $\mathrm{Kir}(\mathcal{Y})_!$, and we will have also an equivalence

$$\mathrm{Kir}_!(\mathcal{Y}) \simeq \mathrm{Kir}_*(\mathcal{Y}).$$

Furthermore, restricting to locally compact objects, the Verdier duality functor on $\mathrm{Shv}(\mathcal{Y})^{\mathrm{loc.c}}$ induces an equivalence

$$(\mathrm{Kir}_!(\mathcal{Y})^{\mathrm{loc.c}})^{\mathrm{op}} \to \mathrm{Kir}_*(\mathcal{Y})^{\mathrm{loc.c}}.$$

Remark A.1.4. That said, the category $\mathrm{Kir}_*(\mathcal{Y})$ (or $\mathrm{Kir}_!(\mathcal{Y})$) does not enjoy the favorable properties of $\mathrm{Whit}(\mathcal{Y})$ mentioned in Sect. A.1.2.

Most importantly, for a map $f : \mathcal{Y}_1 \to \mathcal{Y}_2$, the functors $f^!$ and f_* do send $\mathrm{Kir}_*(\mathcal{Y}_1)$ and $\mathrm{Kir}_*(\mathcal{Y}_2)$ to one another, but the functors $f_!$ and f^* do not. The situation with $\mathrm{Kir}_!(\mathcal{Y})$ is the opposite.

A.1.5. For the purposes of this work, we take the stacks $\mathcal{Y}$ to be the following ones:

Recall the subschemes $Y_j \subset \mathrm{Gr}^{\omega^\rho}_{G,x}$ and the subgroups $N_k \subset \mathfrak{L}(N)^{\omega^\rho}_x$ of Sect. 6.1, so that the action of N_k on Y_j factors through some finite-dimensional quotient $N_{k,l}$.

Consider the action of $\mathbb{G}_m$ on $\mathfrak{L}(N)^{\omega^\rho}_x$ obtained from the adjoint action of $T_{\mathrm{ad}} \subset \mathfrak{L}^+(T)_x$ on $\mathfrak{L}(N)^{\omega^\rho}_x$ and the cocharacter $\rho : \mathbb{G}_m \to T_{\mathrm{ad}}$.

With no restriction of generality, we can assume that the subgroup N_k is preserved by this action, as well as the kernel of the projection to $N_{k,l}$. Finally, we can assume that the restriction of the canonical homomorphism $\mathfrak{L}(N)^{\omega^\rho}_x \to \mathbb{G}_a$ to N_k factors through $N_{k,l}$. Let $N'_{k,l}$ be the kernel of the resulting homomorphism $N_{k,l} \to \mathbb{G}_a$.

Set

$$\mathcal{Y} := N'_{k,l} \backslash Y_j.$$

By construction, $\mathcal{Y}$ carries a residual action of $\mathbb{G}_a$. In addition, we note that the action of $T \subset \mathfrak{L}^+(T)_x$ on $\mathrm{Gr}^{\omega^\rho}_{G,x}$ also factors through T_{ad}. So we obtain a well-defined action of $\mathbb{G}_m$ on Y_j via ρ, and hence on $\mathcal{Y}$.

It is easy to, however, that the above $\mathbb{G}_a$- and $\mathbb{G}_m$-actions on $\mathcal{Y}$ combine to an action of the semi-direct product $\mathbb{G}_m \ltimes \mathbb{G}_a$.

A.2. **Definition of the Kirillov model.**

A.2.1. Consider the category $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$ and its full subcategory $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m \ltimes \mathbb{G}_a}$. The forgetful functor

$$\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m \ltimes \mathbb{G}_a} \hookrightarrow \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \tag{A.1}$$

admits a right adjoint, denoted $\mathrm{Av}^{\mathbb{G}_a}_*$, which makes the following diagram commute:

$$\begin{tikzcd} \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \arrow[r, "\mathrm{Av}^{\mathbb{G}_a}_*"] \arrow[d] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m \ltimes \mathbb{G}_a} \arrow[d] \\ \mathrm{Shv}(\mathcal{Y}) \arrow[r, "\mathrm{Av}^{\mathbb{G}_a}_*"] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_a}, \end{tikzcd}$$

where the vertical arrows are the forgetful functors.

We let $\mathrm{Kir}_*(\mathcal{Y})$ be the full subcategory of $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$ equal to the kernel of the functor $\mathrm{Av}^{\mathbb{G}_a}_*$.

A.2.2. Note that the embedding

$$\mathrm{Kir}_*(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$$

admits a *left* adjoint, given by

$$\mathcal{F} \mapsto \mathrm{coFib}(\mathrm{Av}^{\mathbb{G}_a}_*(\mathcal{F}) \to \mathcal{F}). \tag{A.2}$$

A.2.3. Similarly, in the constructible situation, the functor (A.1) admits a left adjoint, denoted $\mathrm{Av}^{\mathbb{G}_a}_!$, which makes the following diagram commute:

$$\begin{tikzcd} \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \arrow[r, "\mathrm{Av}^{\mathbb{G}_a}_!"] \arrow[d] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m \ltimes \mathbb{G}_a} \arrow[d] \\ \mathrm{Shv}(\mathcal{Y}) \arrow[r, "\mathrm{Av}^{\mathbb{G}_a}_!"] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_a}. \end{tikzcd}$$

We let $\mathrm{Kir}_!(\mathcal{Y})$ be the full subcategory of $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$ equal to the kernel of the functor $\mathrm{Av}^{\mathbb{G}_a}_!$.

A.2.4. Note that now the embedding

$$\mathrm{Kir}_!(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$$

admits a *right* adjoint, given by

$$\mathcal{F} \mapsto \mathrm{Fib}(\mathcal{F} \to \mathrm{Av}^{\mathbb{G}_a}_!(\mathcal{F})). \tag{A.3}$$

A.2.5. It is clear that the Verdier duality functor

$$\Big((\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m})^{\mathrm{loc.c}}\Big)^{\mathrm{op}} \to (\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m})^{\mathrm{loc.c}}$$

sends

$$\mathrm{Kir}_!(\mathcal{Y})^{\mathrm{loc.c}} := \mathrm{Kir}_!(\mathcal{Y}) \cap (\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m})^{\mathrm{loc.c}} \subset (\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m})^{\mathrm{loc.c}}$$

to

$$\mathrm{Kir}_*(\mathcal{Y})^{\mathrm{loc.c}} := \mathrm{Kir}_*(\mathcal{Y}) \cap (\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m})^{\mathrm{loc.c}} \subset (\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m})^{\mathrm{loc.c}}$$

and vice versa.

A.3. **The Whittaker vs Kirillov equivalence.**

A.3.1. Let us be again in the situation when the Artin-Schreier sheaf is defined. Consider the functor

$$\mathrm{Whit}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \tag{A.4}$$

equal to

$$\mathrm{Whit}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\mathrm{Av}_*^{\mathbb{G}_m}} \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m},$$

where the first arrow is the forgetful functor.

It is easy to see that the image of (A.4) belongs to $\mathrm{Kir}_!(\mathcal{Y}) \subset \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$: this follows from the fact that the diagram

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{Y}) \arrow[r, "\mathrm{Av}_*^{\mathbb{G}_m}"] \arrow[d, "\mathrm{Av}_*^{\mathbb{G}_a}"'] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \arrow[d, "\mathrm{Av}_*^{\mathbb{G}_a}"] \\
\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_a} \arrow[r, "\mathrm{Av}_*^{\mathbb{G}_m}"] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}
\end{tikzcd}$$

commutes.

Hence, we obtain a functor

$$\mathrm{Whit}(\mathcal{Y}) \to \mathrm{Kir}_*(\mathcal{Y}) \tag{A.5}$$

We claim:

Proposition A.3.2.

(a) *The functor* (A.5) *is an equivalence.*

(b) *The (partially defined) functor* $\mathrm{Av}_!^{\mathbb{G}_a,\chi}$ *left adjoint to the embedding* $\mathrm{Whit}(\mathcal{Y}) \to \mathrm{Shv}(\mathcal{Y})$ *is defined and maps isomorphically to* $\mathrm{Av}_*^{\mathbb{G}_a,\chi}[2]$ *on the essential image of the forgetful functor* $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \to \mathrm{Shv}(\mathcal{Y})$.

(c) *The resulting functor*

$$\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \to \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\mathrm{Av}_!^{\mathbb{G}_a,\chi}} \mathrm{Whit}(\mathcal{Y})$$

factors as

$$\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \xrightarrow{(A.2)} \mathrm{Kir}_*(\mathcal{Y}) \to \mathrm{Whit}(\mathcal{Y}),$$

and the resulting functor $\mathrm{Kir}_*(\mathcal{Y}) \to \mathrm{Whit}(\mathcal{Y})$ *is the inverse of* (A.5).

Proof. The proof follows from the Fourier-Deligne transform picture: we interpret the *-convolution action of $\mathrm{Shv}(\mathbb{G}_m)$ on $\mathrm{Shv}(\mathcal{Y})$ as an action of the monoidal category $\mathrm{Shv}(\mathbb{A}^1)$ with respect to the pointwise $\overset{!}{\otimes}$ tensor product.

□

A.3.3. In the constructible situation we have a similarly defined equivalence

$$\mathrm{Whit}(\mathcal{Y}) \to \mathrm{Kir}_!(\mathcal{Y}) \tag{A.6}$$

using the dual functors, i.e.,

$$\mathrm{Whit}(\mathcal{Y}) \hookrightarrow \mathrm{Shv}(\mathcal{Y}) \xrightarrow{\mathrm{Av}_!^{\mathbb{G}_m}} \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}.$$

The inverse functor makes the following diagram commutative

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \arrow[r, "(A.3)"] \arrow[d] & \mathrm{Kir}_!(\mathcal{Y}) \arrow[d] \\
\mathrm{Shv}(\mathcal{Y}) \arrow[r, "\mathrm{Av}_*^{\mathbb{G}_a,\chi}"] & \mathrm{Whit}(\mathcal{Y}).
\end{tikzcd}$$

A.4. **The *-Kirillov vs !-Kirillov equivalence.**

A.4.1. Let us first be in the constructible situation when the Artin-Schreier sheaf is defined. Combining the equivalences (A.5) and (A.6), we obtain an equivalence

$$\mathrm{Kir}_*(\mathcal{Y}) \simeq \mathrm{Kir}_!(\mathcal{Y}). \tag{A.7}$$

Let us describe the corresponding functors

$$\mathrm{Kir}_*(\mathcal{Y}) \leftrightarrow \mathrm{Kir}_!(\mathcal{Y})$$

explicitly.

A.4.2. Note that for any $\mathcal{G} \in \mathrm{Shv}(\mathbb{G}_a)^{\mathbb{G}_m}$, we have the well-defined endo-functors of $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$

$$\mathcal{F} \mapsto \mathcal{G} \overset{*}{\star} \mathcal{F} \text{ and } \mathcal{F} \mapsto \mathcal{G} \overset{!}{\star} \mathcal{F}$$

intertwined by the forgetful functor $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \to \mathrm{Shv}(\mathcal{Y})$ with the same-named endo-functors of $\mathrm{Shv}(\mathcal{Y})$.

A.4.3. By unwinding the definitions, we obtain that the resulting functor

$$\mathrm{Kir}_*(\mathcal{Y}) \to \mathrm{Kir}_!(\mathcal{Y})$$

fits into the commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \arrow[r, "j_*(\mathsf{e})[3] \overset{!}{\star} -"] \arrow[d, "(A.2)"'] & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \\
\mathrm{Kir}_*(\mathcal{Y}) \arrow[r] & \mathrm{Kir}_!(\mathcal{Y}). \arrow[u]
\end{tikzcd}$$

The functor

$$\mathrm{Kir}_*(\mathcal{Y}) \leftarrow \mathrm{Kir}_!(\mathcal{Y})$$

fits into the commutative diagram

$$\begin{tikzcd}
\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} & \mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m} \arrow[l, "j_!(\mathsf{e})[-1] \overset{*}{\star} -"'] \arrow[d, "(A.3)"] \\
\mathrm{Kir}_*(\mathcal{Y}) \arrow[u] & \mathrm{Kir}_!(\mathcal{Y}). \arrow[l]
\end{tikzcd}$$

In the above formulas j denotes the open embedding

$$\mathbb{G}_a - 0 \hookrightarrow \mathbb{G}_a,$$

and $\mathsf{e} \in \mathrm{Shv}(\mathbb{G}_a - 0)$ stands for the constant sheaf on $\mathbb{G}_a - 0$.

A.4.4. Let us now be in the constructible situation, but where the Artin-Schreier sheaf is not necessarily defined. Note that the endo-functors of $\mathrm{Shv}(\mathcal{Y})^{\mathbb{G}_m}$ given by $j_*(\mathsf{e})[3] \overset{!}{\star} -$ and $j_!(\mathsf{e})[3] \overset{*}{\star} -$ respectively, define a pair of mutually adjoint functors

$$\mathrm{Kir}_*(\mathcal{Y}) \rightleftarrows \mathrm{Kir}_!(\mathcal{Y}) \tag{A.8}$$

We claim:

Proposition A.4.5. *The functors* (A.8) *are mutually inverse equivalences.*

Proof. By Lefschetz principle, we can reduce to the situation when $k = \mathbb{C}$ and the sheaf theory is that of constructible sheaves in the classical topology with coefficients in $\mathsf{e} = \mathbb{C}$. In the latter case, we can apply Riemann-Hilbert and thus embed our situation into that of holonomic D-modules. Now the assertion follows from Sect. A.4.3 above.

□

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