NOTES ON LOCAL LANGLANDS FOR GL_n

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ABSTRACT. These are some extremely rough notes for a learning seminar at Harvard, in the fall semester of 2017. These notes are expanded from the first several talks given by the author. Because of absence of experts among both speakers and audience, there are probably mistakes in the notes that we are unaware of (thus use with caution). I would also appreciate if crucial misunderstandings could be pointed out to me via emails.

CONTENTS

1. Weil-Deligne representations

In this section, we let K be a fixed non-archimedean local field with residue characteristic p. Let k be its residue field, with $q = \#k = p^f$ elements. Fix a uniformizer $\varpi_K \in \mathcal{O}_K$. Finally let $G_K = G(K^s/K)$ be the Galois group of a fixed separable closure K^s over K.

1.1. Weil-Deligne representations.

1.1.1. *The Weil group W_K.*

By definition we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I_K & \longrightarrow & W_K & \longrightarrow & \mathrm{Frob}^{\mathbb{Z}} & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & I_K & \longrightarrow & G_K & \longrightarrow & G_k & \longrightarrow & 0 \end{array}$$

here W_K is topologized such that $I_K \subset W_K$ is open with its usual topology and $\mathrm{Frob}^{\mathbb{Z}} \cong \mathbb{Z}$ has the discrete topology. In particular this is *not* the subspace topology induced from G_K.

Remark. In general, if K is either a local or Global field, the definition of W_K associated to K involves a triple $(W_K, \iota, \{r_E\})$ where W_K is a topological group, $\iota : W_K \to G_K$ a continuous homomorphism with dense image, and $r_E : C_E \xrightarrow{\sim} W_E^{ab}$ for each E/F finite where $C_E = E^\times$ or $E^\times \backslash \mathbb{A}_E^\times$ respectively in the local and global case, which satisfy certain natural conditions. One can show that W_K exists and is unique. In the non-archimedean local case this agrees with our definition, with ι the natural inclusion, and r_E the isomorphism from local class field theory.

1.1.2. *The tame character.*

Fix a compatible system of primitive roots of unity $\{\zeta_n\}_{p \neq n}$ in K^s. We have the maximal unramified extension $K^{nr} = \underset{p\neq n}{\cup} K(\zeta_n)$ and the maximal tamely ramified extension $K^t = \underset{p\neq n}{\cup} K^{nr}(\varpi_K^{1/n})$ over K. The wild inertia $P_K = G(K^s/K^t)$ is the unique pro-p Sylow subgroup of I_K, and the tame character (depending on the choice of $\{\zeta_n\}$) is the following isomorphism:

$$t : I_K/P_K \xrightarrow{\sim} \prod_{l \neq p} \mathbb{Z}_l, \qquad \sigma \mapsto t(\sigma) \mod n \quad \text{where } \frac{\sigma(\varpi_K^{1/n})}{\varpi_K^{1/n}} = \zeta_n^{t(\sigma)}.$$

Moreover, for $l \neq p$, let $K^{t,l} = K^{nr}(\varpi_K^{1/l^\infty})$ be the l-part of the maximal tamely ramified extension, then we have (*isomorphism as abelian groups*)

$$t_l : I_{K,l}/P_K = G(K^{t,l}/K^{nr}) \xrightarrow{\sim} \mathbb{Z}_l$$

which agrees with the projection of the tame character onto $\mathbb{Z}_l$.

G_K (or W_K) acts on I_K/P_K by conjugation (since P_K is the unique pro-p Sylow subgroup). On each $\mathbb{Z}_l$ factor, the action is given by the l-adic cyclotomic character $\chi = \chi_l$:

$$I_{K,l}/P_K \xrightarrow{\sim} \mathbb{Z}_l(1)$$

where G_K acts on $I_{K,l}/P_K$ by conjugation. In other words, for any $\sigma \in G_K, \tau \in I_{K,l}$, we have

$$t_l(\sigma\tau\sigma^{-1}) = \chi(\sigma) t_l(\tau).$$

1.1.3. *Weil-Deligne representations.*

Let L be a field of characteristic 0 and V a finite dimensional vector space over L.

Definition. A Weil-Deligne representation of W_K on V is a pair (ρ, N) where

(1) ρ is a continuous representation of W_K on V with respect to the *discrete topology* on L, or equivalently, I_K has finite image;
(2) $N \in \mathrm{End}(V)$ is an endomorphism such that for any $\sigma \in W_K$,
$$\sigma N \sigma^{-1} = q^{-v_K(\sigma)} N$$
where $q = \#k$ and v_K is determined by $\sigma \mapsto \mathrm{Frob}_K^{v_K(\sigma)} \in W_K/I_K \xrightarrow{\sim} \mathrm{Frob}_K^{\mathbb{Z}}$.

Definition. Let $\mathfrak{wd} = \mathfrak{wd}_L(W_K)$ be the category of Weil-Deligne representations, where morphisms are morphisms of representations compatible with the endomorphisms N.

1.1.4. *Basic properties.*

Let $(\rho, N) : W_K \to \mathrm{GL}(V)$ be a Weil-Deligne representation as above. In particular, for a lift of Frobenius $\varphi \in W_K$, we have
$$\varphi N \varphi^{-1} = q^{-1} N.$$

We record the following properties:

(1) N is necessarily nilpotent: the characteristic polynomial
$$f(X) = \det(I_n \cdot X - N) = \det(I_n \cdot X - q^{-1}N) = q^{-n} f(qX).$$
Hence $f(X) = X^n$, which forces N to be nilpotent.
(2) N is clearly *not* Galois equivariant, in fact, N induces a Galois equivariant map
$$N(-1) : V \to V(-1)$$
where $V(-1)$ denotes (as a formal symbol) the twist of V by the unramified character $\epsilon : G_K \to L^\times$, $\varphi \mapsto q$ where φ is a lift of Frobenius. If $L/\mathbb{Q}_l$ for $l \neq p$, then (-1) agrees with the usual notation for Tate twist.
(3) Let $r : W_K \to \mathrm{GL}(V)$ be a continuous representation of W_K with respect to the discrete topology on V, then we define the Weil-Deligne representation $\mathrm{Sp}_m(r)$ of W_K on V^m by
$$\mathrm{Sp}_m(r) := r \oplus r(1) \oplus ... \oplus r(m-1)$$
where $r(i)$ is the representation r twisted by χ^{-1}, and then define $N \in \mathrm{End}(V^m)$ by sending each $r(i)$ isomorphically to $r(i+1)$ for $i \leq m-2$ and $N = 0$ on $r(m-1)$. Alternatively,
$$N(-1) : \mathrm{Sp}_m(r) = r \oplus ... \oplus r(m-1) \longrightarrow \mathrm{Sp}_m(r)(-1) = r(-1) \oplus ... \oplus r(m-2)$$
is identity on $r \oplus ... \oplus r(m-2)$ and 0 on $r(m-1)$.
(4) By induction on the minimum of m such that $N^m = 0$, we can show that every Weil-Deligne representation (ρ, N) where ρ is semisimple is isomorphic to a direct sum of Weil-Deligne representations of the form $\mathrm{Sp}_m(r)$, where r is an irreducible representation of W_K with $r(I_K)$ finite.

1.1.5. *Remarks on semi-simple representations.*

For this discussion we let V be a vector space over an arbitrary field k. Let $H < G$ be a subgroup and $\rho : G \to \mathrm{GL}(V)$ a representation of G. We want to know the relation between semi-simplicity of ρ and semi-simplicity of $\rho|_H$.

Lemma 1.1.

(1). If V is finite dimensional and $H \lhd G$ is a normal subgroup, then ρ is semi-simple implies that $\rho|_H$ is semi-simple.

(2). If $H < G$ is of finite index r, where r is invertible in the coefficient field k, then ρ is semi-simple if and only if $\rho|_H$ is semi-simple.

Proof.

(1). Without loss of generality assume that ρ is irreducible. Take $W = \oplus_i W_i$ where the $\{W_i\}$'s are distinct H-subrepresentations of V, then W is a G-subrepresentation of V by normality of H. Since V is finite dimensional, we know that $W \neq 0$, so $W = V$ and $\rho|_H = \oplus W_i$ is semi-simple.

(2). Let $G = \bigsqcup_{i=1,\dots,r} Hg_i$.

(a). First suppose ρ is irreducible. Any $v \neq 0$ generates V as a simple $k[G]$-module, so $\{g_i v\}$ generates V as an $k[H]$-module. Now let $W \subset V$ be a maximal $k[H]$-submodule (which exists since V is finitely generated). Now each V/g_iW is a simple $k[H]$-module, and we have natural $k[H]$-linear map $V \to \oplus V/g_iW$, whose kernel is a proper $k[G]$-submodule of V so equals 0. Therefore, V is a semi-simple $k[H]$-module since $V \hookrightarrow \oplus V/g_iW$.

(b). The "if" part is Maschke's theorem. Suppose $\rho|_H$ is semi-simple, let $W \subset V$ be a G-subrepresentation, therefore a semi-simple H-subrepresentation. Let $\pi : V \to W$ be an H-linear projection, and average π we get a G-linear projection:

$$\pi_G : V \to W \quad \text{by } v \mapsto \frac{1}{r} \sum_{i=1,\dots,r} g_i^{-1} \pi(g_i \cdot v)$$

hence the claim. (The point here is that even though g_i might not commute with h, $g_i^{-1} \circ \pi \circ g_i$ commutes with h, so π_G is H-linear, then also G-linear). □

1.1.6. *φ-semisimple representations.*

Lemma 1.2. *Let ρ be a representation of W_K on V where $\rho(I_K)$ is finite. Suppose for simplicity that L is algebraically closed. Then TFAE:*

1. *$\rho(\sigma)$ is semisimple for all $\sigma \in W_K$;*
2. *$\rho(\varphi)$ is semisimple for (any) lift of Frobenius φ;*
3. *ρ is a semisimple representation.*

Sketch. Note that char $L = 0$. Since $\operatorname{im} \rho(I_K)$ is finite, we claim that for any $\sigma \in W_K$ there exists some $m \in \mathbb{Z}_{>0}$ such that $\rho(\sigma^m)$ lies in the center of $\rho(W_K)$. To prove the claim, pick $\tau_1, ..., \tau_r \in I_K$ which generates $\rho(I_K)$ and let $\tau_0 = \varphi$ be a lift of Frobenius. For each τ_i, consider $\rho(\tau_i^n \sigma \tau_i^{-n})$ for all $n \in \mathbb{Z}_{>0}$, by the finiteness of $\rho(I_K)$ we know that there exists some m_i such that $\rho(\tau_i^{m_i})$ commutes with $\rho(\sigma)$, then put $m = m_0 m_1 ... m_r$. Therefore, for any $\sigma \in W_K$ there exists $n \geq 0, n' > 0$ such that $\rho(\sigma^{n'}) = \rho(\varphi^n)$, hence (1) and (2) are equivalent since L has characteristic 0 (so a power of an linear operator T is semisimple if and only if T is semisimple – note that this fails in general in positive characteristic). Now $\rho(\varphi)$ generates a subgroup of finite index in $\rho(W_K)$, and L has characteristic 0, so by part (2) of Lemma 1.1,

$\rho(W_K)$ acts semisimply on V if and only if $\langle\rho(\varphi)\rangle$ acts semisimply, if and only if $\rho(\varphi)$ is semisimple. $\square$

Remark. Note that we have assumed that L is algebraically closed in the lemma above. This is because in our convention, a linear operator is called semisimple if it is "potentially" diagonalizable, namely if it is diagonalizable after a base change. For a general L we just have to replace the statement in part (3) by "potentially semi-simple".

Definition. A Weil-Deligne representation $(\rho, N) \in \mathfrak{wd}_K$ is φ-semisimple (Frobenius-semisimple) if ρ is semisimple, or equivalently, if $\rho(\varphi)$ is semisimple.

Remark.

(1) Let $(\rho, N) \in \mathfrak{wd}$ be a Weil-Deligne representation. If ρ is irreducible then N is necessarily 0, since $\ker N$ is stable under W_K.
(2) The φ-semisimple representations (ρ, N) are in general not the semisimple objects in the category $\mathfrak{wd}$. The semi-simple objects in $\mathfrak{wd}$ are direct sums of simple objects, namely they are φ-semisimple representations with $N = 0$.

Remark (Frobenius-semi-simplification)**.**
Let (ρ, N) be a Weil-Deligne representation of W_K, let ρ^{ss} the semi-simplification of ρ, then (ρ^{ss}, N) is a Weil-Deligne representation, and called the φ-semi-simplification of (ρ, N).

1.1.7. *Tensor.*
Let (ρ_1, N_1) and (ρ_2, N_2) be two Weil-Deligne representations, then their tensor product is

$$(\rho_1 \otimes \rho_2, N_1 \otimes id + id \otimes N_2),$$

and similarly, we have $\mathrm{Hom}(\rho_1, \rho_2) = (r, N)$ given by

$$\begin{cases} \Big(r(\tau)\varphi\Big)v_1 = \rho_2(\tau) \cdot \varphi\Big(\rho_1(\tau)^{-1}v_1\Big) \\ (N\varphi)v_1 = N_2 \cdot \varphi(v_1) - \varphi(N_1 \cdot v_1) \end{cases}$$

1.1.8. *Weil-Deligne group scheme.*
Consider W_K as a (constant) group scheme over $\mathbb{Q}$, obtained as limit $\lim_{I' \lhd I_K} W_K/I'$ as I' runs over all open normal subgroups of I_K. This is an infinite type group scheme over $\mathbb{Q}$. Let

$$\widetilde{W}_K := \mathbb{G}_a \rtimes W_K$$

where the action of W_K on $\mathbb{G}_a$ is given by $wxw^{-1} = |w|_K\, x$.
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1.1.9. *Jacobson-Morozov.*
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1.1.10. *ϵ-factors of Tate and Deligne-Langlands.*
We need to define L- and ϵ- conductors for a Weil-Deligne representation (ρ, N), along the way we also define the conductor $c((\rho, N))$. First we do this for representations of W_K. Recall that for us, a representation of W_K means it is continuous with respect to the discrete topology on the target. If we fix an isomorphism $\overline{L} = \overline{\mathbb{Q}}_l \xrightarrow{\sim} \mathbb{C}$, these correspond to Artin representations of W_K.

Fix $\varpi = \varpi_K$. First recall (from Tate's thesis) that for χ a quasi-character $\chi : K^\times \to \mathbb{C}$ Tate defines a local L-factor

$$L(\chi) = \begin{cases} (1-\chi(\varpi))^{-1} & \text{if } \chi \text{ is unramified,} \\ 1 & \text{if } \chi \text{ is ramified.} \end{cases}$$

For $\omega_s = \|\cdot\|^s$, $s \in \mathbb{C}$, $L(\chi\omega_s)$ is meromorphic in s. Let ψ be a nontrivial additive character on K, and dx a Haar measure on K. Denote by d^*x the Haar measure $\|x\|^{-1}dx$ on $K^\times$. Finally, let $\chi^\vee := \chi^{-1}\omega_1 = \chi^{-1}\|\cdot\|$ be the twisted dual of χ. Tate considers a Fourier transform $\hat{f}(y) = \displaystyle\int f(x)\psi(xy)dx$ and showed that

$$\frac{\int \hat{f}\chi^\vee(x)d^*x}{L(\chi^\vee)} = \epsilon(\chi,\psi,dx)\cdot \frac{\int f\chi(x)d^*x}{L(\chi)}$$

for a non-vanishing holomorphic function $\epsilon(\chi,\psi,dx)$ of χ, called the ϵ-factor, which depends on ψ and dx. Tate also shows that, if we let $n(\psi)$ be the largest integer such that $\psi(\varpi^{-n}\mathcal{O}_K) = 1$ and let $c(\chi)$ be the conductor of χ, and let $c \in K^\times$ be an element with valuation $n(\psi)+c(\chi)$, then

$$\epsilon(\chi,\psi,dx) = \begin{cases} \|c\|^{-1}\chi(c)\cdot \mathrm{Vol}_{dx}(\mathcal{O}_K) & \text{if } \chi \text{ is unramified,} \\ \int_{c^{-1}\mathcal{O}_K^\times}\chi^{-1}(x)\psi(x)dx & \text{if } \chi \text{ is ramified.} \end{cases}$$

Namely he provides an explicit formula for the ϵ-factors of characters.

For a higher dimensional representation, (according to [10]) there is only an existence theorem due to Langlands (and Deligne gives a short proof)

Theorem 1.3. *There exists a unique function ϵ which associates a number $\epsilon(\rho,\psi,dx) \in \mathbb{C}^\times$ to each choice of local field K, a representation ρ of W_K, a nontrivial additive character ψ and a Haar measure dx of K, such that*

(1) *For characters χ, $\epsilon(\chi,\psi,dx)$ agrees with above;*
(2) *$\epsilon(\cdot,\psi,dx)$ is multiplicative on short exact sequences of W_K representations, so in particular descends to a map on the Grothendieck group*

$$\epsilon(\cdot,\psi,dx) : \mathrm{R}(W_K) := \mathrm{K}_0(\mathrm{Rep}_{W_K}) \longrightarrow \mathbb{C}^\times.$$

(3) *ϵ is inductive in degree 0 in the following sense: for every field extension $E/L/K$, and additive Haar measure ν_E, μ_L, we have*

$$\epsilon(\mathrm{ind}_{E/L}[r],\ \psi\circ\mathrm{Tr}_{L/K},\ \mu_L) = \epsilon([r],\ \psi\circ\mathrm{Tr}_{E/K},\ \mu_E)$$

for all $[r] \in \mathrm{R}(W_K)$ of degree 0.

Remark. Let $c(\rho)$ be the Artin conductor of the representation (ρ, V) of W_K, which by definition equals

$$c(\rho) := c(\chi_\rho) = \sum \frac{1}{[G_0 : G_i]}(\dim V - \dim V^{G_i})$$

where χ_ρ is the character of this representation and G_i are the i^{th} ramification subgroups of I_K. Then one can show that

$$\epsilon(\rho\cdot\omega_s,\psi,dx) = (\#k)^{-\left(c(\rho)+n(\psi)\dim V\right)\cdot s}\cdot\epsilon(\rho,\psi,dx)$$

1.1.11. *Conductors, L- and ϵ- factors.*

Now we are ready to define the corresponding invariants for a Weil-Deligne representation.

Definition. Let $\rho = (\rho', N)$ be a Weil-Deligne representation of W_K. Let ψ be an additive character on K and dx an additive Haar measure on K. Let $q = \#k$ be the size of the residue field.

(1) We define the conductor $c(\rho) = c(\rho', N)$ to be
$$c(\rho) := c(\rho') + \dim V^{I_K} - \dim(\ker N)^{I_K}$$
(2) Let Φ be (any) geometric Frobenius. Define the L-factor to be
$$L(\rho, s) := \det\Big(1 - q^{-s}\Phi\Big|(\ker N)^{I_K}\Big)^{-1}$$
(3) Define the ϵ-factor to be (where we have suppressed the notation on dx)
$$\epsilon(\rho, s, \psi) := q^{-\big(c(\rho)+n(\psi)\dim V\big)\cdot s} \cdot \det\Big(-\Phi\Big|V^{I_K}/(\ker N)^{I_K}\Big) \cdot \epsilon(\rho', \psi, dx)$$

In particular, we single out the following trivial corollary from the definition.

Corollary 1.4.
$$\epsilon(\rho, s, \psi) := q^{-\big(c(\rho)+n(\psi)\dim V\big)\cdot s} \cdot \epsilon(\rho, 0, \psi)$$

Remark. The definition of $\epsilon(\rho, s, \psi)$ is slightly different from [10], namely Tate defined the epsilon-factor to be $q^{n(\psi)\dim V\cdot s} \cdot \epsilon(\rho, s, \psi)$ in our terminology, which does not seem to be consistent with the Remark in the previous page. Our convention agrees with [11].

1.2. l-adic representations.

1.2.1. *l-adic Galois representations.*

Now let $L/\mathbb{Q}_l$ be a finite extension with $l \neq p$. Let V be a finite dimensional vector space over L. Let $\{\zeta_n\}_{p\nmid n}$ be a fixed compatible system of roots of unity and $t_l : I_K/P_L \to \mathbb{Z}_l$ the l-adic tame character defined as above. Recall Grothendieck's quasi-unipotent monodromy theorem:

Lemma 1.5. *Let $\rho : G_K \to \mathrm{GL}(V)$ be an l-adic representation (i.e., ρ is continuous with respect to the l-adic topology on V) of G_K, then*

(1) *There exists an open subgroup $I'_K \subset I_K$ (corresponding to a finite extension K'/K) and a uniquely determined nilpotent operator $N \in \mathrm{End}(V)$ such that for all $\sigma \in I'_K$,*
$$\rho(\sigma) = \exp(t_l(\sigma)N).$$
(2) *Moreover, for any $\tau = \sigma\varphi^n \in W_K$, where $\sigma \in I_K$ and φ a (fixed) lift of Frobenius,*
$$\rho(\tau)N\rho(\tau^{-1}) = q^{-n}N.$$

Sketch. By continuity G_K stabilizes a $\mathcal{O}_L$ lattice $\Lambda \subset V$, so reducing mod l we get $\overline{\rho} : G_K \to \mathrm{Aut}(\Lambda/l)$. The kernel $\ker\overline{\rho} = G_{K''} \subset G_K$ is a subgroup of finite index, namely K''/K is finite. Since the kernel of $\mathrm{GL}(\Lambda)$ reducing mod l is pro-l, we know that $\rho|_{I_{K''}}$ factors through $t_l : I_{K''} \to \mathbb{Z}_l(1)$. Let $\sigma \in I_{K''}$ be an element such that $t_l(\sigma)$ topologically generates $\mathbb{Z}_l(1)$ (as an abelian group). Note that conjugation by $G_{K''}$ on $t_l(I_{K''})$ is the cyclotomic character, namely $t_l(\varphi\sigma\varphi^{-1}) = \chi(\varphi)t_l(\sigma)$, this forces the eigenvalues of $\rho(\sigma)$ to be roots of unity. Therefore, there exists K'/K finite extension such that $\rho(I'_K)$ acts unipotently. Now replace σ by a new topological generator in $I_{K'}$ if necessary, still call it σ, then define $N = \log(\rho(\sigma))$, which is well defined by unipotence. □

Corollary 1.6. *Let $\rho : G_K \to \mathrm{GL}(V)$ be an l-adic representation (i.e., ρ is continuous with respect to the l-adic topology on V) of G_K. If ρ is semi-simple, then $\rho(I_K)$ is finite.*

Proof. By part (1) of Lemma 1.1, $\rho|_{I_K}$ is semisimple. Further restrict ρ to $I_{K'} \lhd I_K$ as in the proof of 1.5, so $\rho|_{I'_K}$ is semisimple and unipotent, therefore it is identity. □

1.2.2. *The functor* $\mathrm{WD}_{\zeta,\varphi}$.

Let $r : W_K \to \mathrm{GL}_n(L)$ be a continuous representation with respect to the discrete topology on L, where $L/\mathbb{Q}_p$. Then the eigenvalues of $r(\sigma)$ (in $\overline{\mathbb{Q}}_l$) are l-adic units for all $\sigma \in W_K$ if and only if they are l-adic units for $r(\varphi)$, or $r(\tau)$ for any $\tau \notin I_K$, if and only if $\det \rho(\varphi) \in \mathcal{O}_L^\times$ and the characteristic polynomial of φ lives in $\mathcal{O}_L[X]$. In which case, r is called *l-integral* or *l-bounded*.

Corollary 1.7. *Fix $\zeta = \{\zeta_n\}$ and φ as above. Then there is an equivalence of categories*

$$\mathrm{WD} = \mathrm{WD}_{\zeta,\varphi} : \mathrm{Rep}_L(G_K) \to \mathfrak{wd}_L^{l\text{-int}}(W_K)$$

from the category of l-adic G_K representations on L-vector spaces to the category of l-integral Weil-Deligne representations of W_K on L-vector spaces. The functor WD is given by

$$\mathrm{WD}(\rho) = (\rho', N), \quad \rho'(\sigma\varphi^n) := \rho(\sigma\varphi^n)\exp(-t_l(\sigma)N).$$

Remark.

(1) $\mathrm{WD}_{\zeta',\varphi'}$ and $\mathrm{WD}_{\zeta,\varphi}$ are naturally isomorphic.
(2) φ-semisimple representations in $\mathfrak{wd}_L^{l\text{-int}}(W_K)$ correspond to l-adic representations ρ where $\rho(\varphi)$ is a semisimple operator.
(3) Let $\mathrm{WD}(\rho) = (\rho', N)$, then ρ is an irreducible l-adic representation if and only if $N = 0$ and ρ' is irreducible.

Now we have the following corollary of the two corollaries above:

Corollary 1.8. *Retain notations from above, $\rho : G_K \to \mathrm{GL}(V)$ corresponds to $\mathrm{WD}(\rho) = (\rho', N)$, then the following are equivalent:*

(1) *ρ is semisimple after a possible base change of L;*
(2) *ρ is φ-semisimple (i.e., φ acts semi-simply) and $\rho(I_K)$ is finite;*
(3) *ρ' is φ-semisimple and $N = 0$.*

Proof. It is clear that (1) $\Leftrightarrow$ (3) $\Rightarrow$ (2). Finally, (2) $\Rightarrow$ (3) by the last assertion of the discussion on "basic properties" immediately after the definition of Weil-Deligne representations, or alternatively (2) $\Rightarrow$ (1) as follows: (2) implies that after replacing L by $\overline{L}$, $\langle\rho(\varphi)\rangle$ is a semisimple representation and $\langle\rho(\varphi)\rangle < \rho(G_K)$ has finite index, therefore we can apply 1.1. □

1.2.3. *Weil-Deligne inertia types and Galois inertia types.*

Definition. Let V be a finite dimensional L-vector space as before.

(1) A Weil-Deligne inertia type (sometimes also referred to as an inertia type) is a representation
$$\tau : I_K \to \mathrm{GL}(V)$$
such that there exists a φ-semisimple Weil-Deligne representation $(\rho', N) : W_K \to \mathrm{GL}(V)$ with $\tau = \rho'|_{I_K}$. In particular, τ has finite image and carries no information about the monodromy operator N.

(2) A Galois inertia type is a continuous φ-semisimple representation
$$\xi : I_K \to \mathrm{GL}(V)$$
which extends to a continuous (l-adic) representation $\rho : W_K \to \mathrm{GL}(V)$.

In particular, a Galois inertia type restricts to a Weil-Deligne inertia type, by taking $\rho'|_{I_K}$ where $(\rho', N) = \mathrm{WD}(\rho)$, for some extension ρ of ξ. We describe this more directly in terms of a classification of Galois inertia types.

Both inertia types can be thought of as the following data. n-dimensional (φ-semisimple) Weil-Deligne representations are of the form $\oplus_i \mathrm{Sp}_{m_i}(r_i)$ for irreducible representations r_i of W_K of dimension n_i such that $\sum_i n_i m_i = n$.

Lemma 1.9. *Let $(r, N) = \bigoplus_{i=1}^{d} \mathrm{Sp}_{m_i}(r_i)$ and $(r', N') = \bigoplus_{i=1}^{d'} \mathrm{Sp}_{m'_i}(r'_i)$ be two Weil-Deligne representations of W_K of dimension n, as above (so all the r_i and r'_i are irreducible). Let ρ and ρ' be the corresponding l-adic representations of W_K given by WD^{-1}.*

- *Let $\tau = r|_{I_K}$ and $\tau' = r'|_{I_K}$ be the two Weil-Deligne inertia types;*
- *Let $\xi = \rho|_{I_K}$ and $\xi' = \rho'|_{I_K}$ be the two corresponding Galois inertia types.*

Then

(1) *$\tau \cong \tau'$ as I_K representations if and only if the collection of all $\{r_i\}$'s with repetition agrees with $\{r'_i\}$'s up to twisting by unramified characters (on the collection of $\{r_i\}'s$ for example).*

(2) *$\xi \cong \xi'$ if and only if $d = d'$, and up to permutation of indices, $m_i = m'_i$, $r_i \cong r'_i \otimes \eta_i$ for unramified characters η_i of W_K.*

Proof. The proof follows immediately from Lemma 4.3. □

Summary (of the correspondence).

Under the correspondence and the corresponding φ-$s.s.$ part,
$$\begin{array}{ccc} \mathrm{Rep}_L(G_K) & \longleftrightarrow & \mathfrak{wd}_L^{l\text{-int}}(W_K) \\ \mathrm{Rep}_L^{\varphi\text{-}ss}(G_K) & \longleftrightarrow & \mathfrak{wd}_L^{\varphi\text{-}ss, l\text{-int}}(W_K) \end{array}$$
we have
$$\mathrm{Rep}_L^{ss}(G_K) = \{\rho \in \mathrm{Rep}_L^{\varphi\text{-}ss}(G_K) : \rho(I_K) < \infty\} \longleftrightarrow \{(\rho', N) \in \mathfrak{wd}_L^{\varphi\text{-}ss, l\text{-int}}(W_K) : N = 0\}.$$

2. Representations of GL_n

2.1. **Admissible representations of** GL_n**.**

2.1.1. *General definitions.*

Let $\mathbb{G}$ be a connected reductive group over K, and let $G = \mathbb{G}(K)$. Let V be a $\mathbb{C}$-vector space (in what follows we could replace $\mathbb{C}$ with any algebraically closed characteristic 0 field, in fact we fix an isomorphism $\iota : \overline{L} = \overline{\mathbb{Q}}_l \cong \mathbb{C}$ and no longer distinguish these coefficient fields in the notes).

Definition. Let (π, V) be a representation of G,

(1) π is smooth if the stabilizer of each $v \in V$ is open in G.
(2) π is admissible if it is smooth, and for every open subgroup $H \subset G$, V^H is finite dimensional.

Let $\mathrm{Rep}_{\mathrm{G}}^{\infty}$ be the category of smooth representations, and $\mathrm{Rep}_{\mathrm{G}}^{\mathrm{adm}}$ the category of admissible representations.

Remark. Note that π is smooth if and only if $V = \cup_{\mathtt{K}} V^{\mathtt{K}}$ where $\mathtt{K}$ runs through all compact open subgroups of G. It is admissible if in addition every $V^{\mathtt{K}}$ is finite dimensional.

Remark. Let $\mathtt{K}$ be any compact open subgroup of G, and let $\mathcal{E}(\mathtt{K})$ be the set of equivalence classes of smooth irreducible representations (note that smoothness implies that they are necessarily finite dimensional). The restriction $V|_{\mathtt{K}}$ is clearly semisimple (since every vector $v \in V$ is contained in a (finite dimensional) irreducible representation of $\mathtt{K}$ by the smoothness assumption). For each $\sigma = (\sigma, \Sigma) \in \mathcal{E}(\mathtt{K})$, let V_σ be the σ-isotypic component of V, namely the minimal $\mathtt{K}$-invariant subspace of V affording a representation of $\mathtt{K}$ of the given class σ, then $V = \bigoplus_{\sigma \in \mathcal{E}(\mathtt{K})} V_\sigma$, and each $V_\sigma|_{\mathtt{K}}$ is a direct sum of the representation Σ of $\mathtt{K}$. (π, V) is admissible precisely when each V_σ is finite dimensional, in which case we have

$$V|_{\mathtt{K}} = \oplus V_\sigma = \bigoplus_{\sigma \in \mathcal{E}(\mathtt{K})} \Sigma^{m_\sigma}$$

where $m_\sigma \in \mathbb{Z}_{\geq 0}$ is the multiplicity of σ in $V|_{\mathtt{K}}$.

Definition. Let $(\pi, V) \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$, let V^* be the linear dual of V (in the category of vector spaces), it receives an action of G (as usual) by $(g\psi)(v) = \psi(g^{-1}v)$. We define $V^\vee$ by

$$V^\vee = \widetilde{V} := \{\psi \in V^* : \text{each stabilizer of } \psi \text{ is open}\}.$$

$V^\vee = \widetilde{V}$ is called the contragradient of the representation V, also denoted by $\pi^\vee$ or $\widetilde{\pi}$, and is smooth by definition.

Remark. In general, the natural morphism $V \to V^{\vee\vee}$ is not an isomorphism. In some sense admissibility is defined so that the problem goes away:

(1) π is admissible if and only if $\pi^\vee$ is admissible.
(2) π is admissible if and only if $V \xrightarrow{\sim} V^{\vee\vee}$ is an isomorphism.
(3) π is irreducible if and only if $\pi^\vee$ is irreducible.
(4) (Gelfand-Kazhdan). When $G = \mathrm{GL}_n(K)$ and $\pi \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$ and irreducible, $\pi^\vee$ is isomorphism to the representation $g \mapsto \pi({}^t g^{-1})$, hence the name contragradient.

2.1.2. *Hecke algebras.*

Definition. Let G be as above, normalize the Haar measure on G so that $\mu(\mathtt{K}_0) = 1$ where $\mathtt{K}_0 \subset G$ is the maximal compact subgroup.

(1) The Hecke algebra $\mathcal{H}(G) = \{\text{locally constant } f : G \to \mathbb{C} \text{ with compact support}\}$ is an associative algebra under convolution (after choice of the Haar measure):

$$(f_1 * f_2)(h) = \int_G f_1(hg^{-1}) f_2(g) dg = \int_G f_1(g) f_2(g^{-1}h) dg$$

(2) Let $\mathtt{K} \subset G$ be a compact open subgroup, we define $\mathcal{H}(G/\!\!/\mathtt{K})$ to be the subalgebra consisting of $f \in \mathcal{H}(G)$ which are left and right invariant under $\mathtt{K}$.
(3) More generally, let $Z \subset Z(G)$ be a closed subspace of the center and $\chi : Z \to \mathbb{C}$ a character of Z. We have the following variant of the double-quotient Hecke algebra:

$$\mathcal{H}_\chi(G) =: \{\text{loc. const. } f : G \to \mathbb{C} \text{ cpt supp. mod } Z, \text{ s.t. } f(zg) = \chi(z)^{-1} f(g) \forall z \in Z\}.$$

The multiplication structure is $(f_1 * f_2)(h) = \displaystyle\int_{G/Z} f_1(x) f_2(x^{-1}h) gx$.

Remark.

(1) $\mathcal{H}(G/\!\!/\mathtt{K}) = \{\phi : \mathtt{K}\backslash G/\mathtt{K} \to \mathbb{C} \text{ with finite support}\}$. In other words $\mathcal{H}(G, \mathtt{K})$ consists of $\phi \in \mathcal{H}(G)$ such that $\phi(kgk') = \phi(g)$ for all $k, k' \in \mathtt{K}$ whose support consists of a finite union of double cosets $\mathtt{K}g\mathtt{K}$
(2) $\mathcal{H}(G/\!\!/\mathtt{K})$ has a two-sided unit, given by $e_{\mathtt{K}} := \frac{1}{\mathrm{Vol}(\mathtt{K})} \cdot 1_{\mathtt{K}}$, the normalized characteristic function on $\mathtt{K}$.
(3) However $\mathcal{H}(G)$ is not a unital algebra. $\mathcal{H}(G) = \bigcup \mathcal{H}(G/\!\!/\mathtt{K})$ is the union of subalgebras $\mathcal{H}(G/\!\!/\mathtt{K})$, but with different units $e_{\mathtt{K}}$ for each $\mathtt{K}$.

2.1.3. *G action on $\mathcal{H}(G)$.*

There is a natural left (resp. right) action of G on $\mathcal{H}(G)$, [1] which we now describe.

- The left action of G on $\mathcal{H}(G)$, denoted by λ_g, is given by

$$\lambda_g f(z) = f(g^{-1}z) \quad \forall z \in G$$

  It is clear that for any $f \in \mathcal{H}(G)$, $\lambda_g f \in \mathcal{H}(G)$ and this indeed defines an action.
- Likewise, the right action ρ_g is given by

$$\rho_g f(z) = f(zg) \quad \forall z \in G.$$

Remark. Part of the reason why we define $\mathcal{H}(G) := \mathcal{C}_c^\infty(G, \mathbb{C})$ is to make both actions of G on $\mathcal{H}(G)$ defined above a smooth representation of G.

[1] not to be confused with what we call *Hecke actions* in next subsection.

2.1.4. *Hecke modules.*

Let (π, V) be a smooth representation, $\mathcal{H}(G)$ acts on V by

$$f \star v := \pi(f)(v) = \int_G f(g)\pi(g)v\, dg.$$

Again this is a finite sum by the remark above.

Since $V = \bigcup_{\mathtt{K}} V^{\mathtt{K}}$, the Hecke action makes V a non-degenerate $\mathcal{H}(G)$ module, namely $\mathcal{H}(G) \cdot V = V$ (for example, $\frac{1}{\mu(\mathtt{K})} 1_{\mathtt{K}} \star v = \pi(e_{\mathtt{K}})v = v$).

Lemma 2.1. *There is a natural functor from*

$$\mathrm{Rep}_{\mathrm{G}}^{\infty} \to \{\text{non deg. } \mathcal{H}(G)\text{-modules}\}$$

which induces an equivalence of categories.

Before we proceed, we fix some notations and record some simple observations.

(1) For any compact open subgroup $\mathtt{K} \subset G$, let $e_{g\mathtt{K}} := \frac{1}{\mu(\mathtt{K})} 1_{g\mathtt{K}}$ be the normalized characteristic function on $g\mathtt{K}$. Note that when $g = \mathrm{id}$, this agrees with our notation for $e_{\mathtt{K}}$.
(2) It is straightforward to check that, for $f \in \mathcal{H}(G)$, $e_{\mathtt{K}} * f = f$ iff and only if $f(kg) = f(g)$ for all $k \in \mathtt{K}$. In particular, for any compact open subgroup $\mathtt{K}' \subset \mathtt{K}$, we have $e_{\mathtt{K}} = e_{\mathtt{K}'} * e_{\mathtt{K}}$.
(3) Again let $\mathtt{K}' \subset \mathtt{K}$ as above, and $g \in G$ any element. We claim that
$$e_{\mathtt{gK}} = e_{g\mathtt{K}'} * e_{\mathtt{K}}.$$
Since by definition $e_{g\mathtt{K}'} * e_{\mathtt{K}}(z) = \displaystyle\int_G e_{g\mathtt{K}'}(x) * e_{\mathtt{K}}(x^{-1}z)dx$, which is non-zero if and only if $z \in x\mathtt{K}$ where $x \in g\mathtt{K}'$, namely if only if $z \in g\mathtt{K}$. The value of $e_{g\mathtt{K}'} * e_{\mathtt{K}}(z)$ for such z is precisely $\dfrac{1}{\mu(\mathtt{K})}\dfrac{1}{\mathtt{K}'} \cdot \mu(g\mathtt{K}' \cap g\mathtt{K}) = \dfrac{1}{\mu(\mathtt{K})}$. It is important here that the second characteristic function is supported on a *subgroup* $\mathtt{K}$ (not some translate $g\mathtt{K}$ of it).
(4) In the functor described above, a subrepresentation of (π, V) corresponds to a submodule under the action of $\mathcal{H}(G)$.

Proof of Lemma 2.1. We have already constructed the functor from $\mathrm{Rep}_{\mathrm{G}}^{\infty}$ to non-degenerate Hecke modules. One checks that this functor is fully faithful by unwinding definition. Now we construct a functor in the other direction. Let M be a non-degenerate Hecke module. First we claim that for any $m \in M$, there exists some compact open subsgroup $\mathtt{K} \subset G$ such that $e_{\mathtt{K}} \star m = m$. To see this, write $m = \sum f_i \star m_i$ for a finite sum with $f_i \in \mathcal{H}(G)$ and $m_i \in M$, by the non-degenerate assumption. Now pick $\mathtt{K}$ small enough so that both left and right actions of $\mathtt{K} \subset G$ on $\mathcal{H}(G)$ stabilize each f_i [see subsection 2.1.3], in other words $f_i \in \mathcal{H}(G /\!\!/ \mathtt{K})$. Then $e_{\mathtt{K}} \star m = e_{\mathtt{K}} \star (\sum f_i \star m) = \sum \Big((e_{\mathtt{K}} * f_i) \star m_i \Big) = m$.

Pick such a compact open subgroup $\mathtt{K}$, then we define

$$g \cdot m = \pi(g)m := e_{\mathtt{gK}} \star m.$$

First this is well-defined, namely does not depend on the choice of $\mathtt{K}$ (as long as it is small enough such that $e_{\mathtt{K}} \star m = m$). This is straightforward: suppose $\mathtt{K}' \subset \mathtt{K}$, then

$$e_{g\mathtt{K}'} \star m = e_{g\mathtt{K}'} \star (e_{\mathtt{K}} \star m) = (e_{g\mathtt{K}'} * e_{\mathtt{K}}) \star m = e_{g\mathtt{K}} \star m$$

where the last equality is observation (3) above.

Then we need to show that this defines an action, which is in fact a bit tricky (but then the smoothness is straightforward). Let $g_1, g_2 \in G$ and $m \in M$, we will choose $\mathtt{K}$ and $\mathtt{K}'$ such that

$$e_{g_1 g_2 \mathtt{K}} \star m = e_{g_1 \mathtt{K}'} \star e_{g_2 \mathtt{K}} \star m = (e_{g_1 \mathtt{K}'} * e_{g_2 \mathtt{K}}) \star m.$$

The choice of $\mathtt{K}$ is as before, so that $e_{\mathtt{K}} \star m = m$. Now let $m' := g_2 \cdot m$, we claim that there exists $\mathtt{K}'$ small enough such that $e_{g_2 \mathtt{K}} \in \mathcal{H}(G /\!\!/ \mathtt{K}')$ and such that $e_{\mathtt{K}'} \star m' = m'$. To see this, observe that $e_{g_2 \mathtt{K}} \in \mathcal{H}(G /\!\!/ \mathtt{K}')$ means $\mathtt{K}' g_2 \mathtt{K} = g_2 \mathtt{K}$, for which we first take $\mathtt{K}' = g_2 \mathtt{K} g_2^{-1} \cap \mathtt{K}$, [2] and then shrink it if necessary to stabilize m'. From this we can check directly that

$$e_{g_1 \mathtt{K}'} * e_{g_2 \mathtt{K}} = e_{g_1 g_2 \mathtt{K}},$$

which implies that, indeed, $e_{g_1 g_2 \mathtt{K}} \star m = (e_{g_1 \mathtt{K}'} * e_{g_2 \mathtt{K}}) \star m$, we may replace $\mathtt{K}$ and $\mathtt{K}'$ by their intersection, to conclude $(g_1 g_2) \cdot m = g_1 \cdot (g_2 \cdot m)$.

$\square$

Remark. Another (more concise) way for the same construction is to consider the natural map of $\mathcal{H}(G)$-modules

$$\mathcal{H}(G) \otimes_{\mathcal{H}(G)} M \to M$$

where we regard $\mathcal{H}(G)$ with a right action on itself. This map is in fact an isomorphism (despite the fact that $\mathcal{H}(G)$ does not contain a unit) – this follows from the non-degenerate assumption. The get a G-action on M via the isomorphism $\mathcal{H}(G) \otimes M \cong M$, by letting G act on the left on $\mathcal{H}(G)$. Note that this precisely corresponds to $g \cdot m = g \cdot (e_{\mathtt{K}} \otimes m) = e_{g\mathtt{K}} \otimes m$.

2.1.5.

Now fix $\mathtt{K}$, then the functor above restricts to a functor from $\mathrm{Rep}_{\mathrm{G}}^{\mathrm{adm}}$ to the category of finite dimensional $\mathcal{H}(G /\!\!/ \mathtt{K})$ modules. In these notes the term *irreducible modules* are used interchangeably with *simple modules.*

Lemma 2.2 (Casselman)**.** *Fix a compact open subgroup $\mathtt{K} \subset G$.*

(1) *The functor*

$$\{\pi \in \mathrm{Rep}_G^{f.l.\mathrm{adm}} : \forall W \text{irred. subquot}, W^{\mathtt{K}} \neq 0\} \xrightarrow{\sim} \mathcal{H}(G /\!\!/ \mathtt{K})\text{-}\mathfrak{mod}^{f.d}$$

given by $V \mapsto V^{\mathtt{K}}$, is an equivalence from the category of finite length admissible representations such that all irreducible sub-quotients have nonzero vector fixed by $\mathtt{K}$, to the category of finite dimensional $\mathcal{H}(G /\!\!/ \mathtt{K})$ modules.

(2) *If $V \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$ is irreducible and $V^{\mathtt{K}} \neq 0$, then $V^{\mathtt{K}}$ is an irreducible $\mathcal{H}(G /\!\!/ \mathtt{K})$ module. Moreover, there is a bijection between*
   (a) *equivalence classes of irreducible smooth representations (π, V) with $V^{\mathtt{K}} \neq 0$*
   (b) *isomorphism classes of irreducible $\mathcal{H}(G /\!\!/ \mathtt{K})$-modules.*

Remark. It is straightforward to show that, if $V \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$ is irreducible, then $V^{\mathtt{K}}$ is either or an irreducible $\mathcal{H}(G)$-module. The converse, however, is a nontrivial statement.

As a corollary, we know that

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[2] This is for example how one shows $\mathcal{H}(G) = \cup_{\mathtt{K}} \mathcal{H}(G /\!\!/ \mathtt{K})$, or equivalently, both left and right actions of G on $\mathcal{H}(G)$ are smooth

Corollary 2.3. *If $V \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$ is irreducible and $\mathcal{H}(G/\!\!/\mathtt{K})$ is commutative, then either $V^{\mathtt{K}} = 0$ or $V^{\mathtt{K}}$ is 1-dimensional.*

2.1.6. *Spherical Hecke algebra.*

In our setup $G = \mathrm{GL}_n(K)$, if $C = \mathrm{GL}_n(\mathcal{O}_K)$ is the maximal compact open subgroup of G, then $\mathcal{H}(G/\!\!/C)$ is commutative, in fact, we have

$$\mathcal{H}(G/\!\!/C) \xrightarrow{\sim} \mathbb{C}[t_1^{\pm 1}, ..., t_n^{\pm 1}]^{\mathfrak{S}_n}.$$

More generally, for $G = \mathbb{G}(K)$ where $\mathbb{G}$ is split reductive (with fixed $T \subset B \subset G$) and $C \subset G$ hyperspecial, the (classical) Satake isomorphism states that

$$\mathcal{H}(G/\!\!/C) \xrightarrow{\sim} (\mathcal{O}_{\widehat{T}})^W,$$

where $\widehat{T} = \operatorname{Spec} \mathbb{C}[\Lambda]$ with Λ the coweight lattice and W is the Weyl group. In particular, the spherical Hecke algebra is commutative.

Remark. In the GL_n case, the commutativity of $\mathcal{H}$ can be seen from Gelfand's trick. Observe that

$$C\backslash G/C \cong G\backslash \Big(G/C \times G/C\Big)$$

by $[g] \mapsto (1, [g])$. Then observe that G/C parametrizes lattices in K^n. Namely we can identify

$$C\backslash G/C \longleftrightarrow G\backslash(\mathcal{L}_n \times \mathcal{L}_n) = \{\text{relative position of lattices}\}$$

where we use $\mathcal{L}_n$ to denote the set of lattices in K^n. From this perspective, we have

$$(\phi * \psi)(\Lambda, \Lambda') = \sum_{\Lambda''} \phi(\Lambda, \Lambda'')\psi(\Lambda'', \Lambda')$$

Note that the the sum is finite because of compact support. Gelfand's idea was that the relative position of (Λ, Λ') is the same as $((\Lambda')^\vee, \Lambda^\vee)$, [3] therefore we have

$$\begin{aligned}
(\phi * \psi)(\Lambda, \Lambda') &= \sum_{\Lambda''} \phi(\Lambda, \Lambda'') \cdot \psi(\Lambda'', \Lambda') \\
&= \sum_{\Lambda''} \phi((\Lambda'')^\vee, \Lambda^\vee) \cdot \psi((\Lambda')^\vee, (\Lambda'')^\vee) = \sum_{\Lambda''} \phi(\Lambda'', \Lambda^\vee) \cdot \psi((\Lambda')^\vee, \Lambda'') \\
&= (\psi * \phi)((\Lambda')^\vee, \Lambda^\vee) = (\psi * \phi)(\Lambda, \Lambda')
\end{aligned}$$

This proof is almost tautological.

Corollary-Definition. Let (π, V) be an irreducible admissible representation of G, $C = \mathrm{GL}_n(\mathcal{O}_K)$ the maximal compact subgroup of G, then V^C is either 0 or 1-dimensional. V is called unramified if $V^\pi \neq 0$, i.e., V^π is 1-dimensional.

2.1.7. *Schur's lemma.*

Lemma 2.4. *Let $(\pi, V) \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$ be a smooth irreducible representation of G, then every G-endomorphism of V is scalar.*

Unlike the case when V is finite dimensional, Schur's lemma is a non-trivial statement. In the case when $\pi \in \mathrm{Rep}_{\mathrm{G}}^{\mathrm{adm}}$ is irreducible and admissible [4], we can use Lemma 2.2 to get ourselves back to dealing with finite dimensional modules – namely by choosing $C \subset G$ small enough so that $V^C \neq 0$, we know that the only $\mathcal{H}(G/\!\!/C)$-linear endomorphisms of V^C are scalars.

[3] after introducing a non-degenerate symmetric bilinear form on K^n

[4] Note that we cannot actually apply Theorem 2.5 since its proof relies Schur's lemma.

Sketch. In our setup $G = \mathbb{G}(K)$, note that G has a countable basis and that $\mathcal{H}(G/\!\!/\mathtt{K})$ has countable dimension over $\mathbb{C}$ for all compact open subgroup group $\mathtt{K} \subset G$, thus $\mathcal{H}(G)$ has countable dimension over $\mathbb{C}$. Let $v \in V$ be a nonzero vector in V, since (π, V) is irreducible, we have a surjective map $\mathcal{H}(G) \to V$ given by $f \mapsto f \star v$, so V has countable dimension. Now consider the ring G-endomorphisms $\mathrm{End}_G(V)$, again by irreducibility of (π, V), the map $\mathrm{End}_G(V) \to V$ given by $\mu \mapsto \mu(v)$ is injective. Therefore, $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{C}$ of countable dimension, but any division algebra D of countable dimension over an uncountable algebraically closed field F is just the field itself (proof: suppose $D \neq F$, there exists some $x \in D \backslash F$ then x is transcendental over F since $F = \overline{F}$, but there are uncountably many elements $\{\frac{1}{x-\alpha} : \alpha \in F\}$ which can be seen to be linearly independent. QED), therefore the lemma follows. □

2.1.8. *Smooth irreducible representations.*

Theorem 2.5. *Let (π, V) be a smooth irreducible representation of G, then π is admissible.*

The proof uses Jacquet's functor to embed π into parabolic induction of admissible (supercuspidal) representations. We omit the proof and refer readers to Jacquet's treatment or [Bernstein-Zelevinsky].

Lemma 2.6. *Every smooth irreducible representation of $G = \mathrm{GL}_n(K)$ is either 1-dimensional or infinite dimensional.*

Proof. This is due to the structure of G. Suppose $\pi : G \to \mathrm{GL}_n(\mathbb{C})$ is smooth, then $\ker(\pi) \lhd G$ is open, hence contains $1 + \varpi_K^m \mathrm{GL}_n(\mathcal{O}_K)$ for some m and therefore $1+\varpi_K^m N$, the upper-triangular matrices with diagonal entries equal to 1 and the rest in $\varpi^m \mathcal{O}_K$. By conjugation we know that the unipotent upper triangular matrices $U \subset \ker \pi$. Finally by normality we know that $\mathrm{SL}_n(K) \subset \ker \pi$. We have now proven the lemma, and a bit more – any such finite dimensional (hence dimension 1) irreducible representation π factors through the determinant map, so it is of the form $\chi \circ \det$ for some quasi-character χ of $K^\times$. □

2.1.9. *Harish-Chandra characters.* [5]

Recall that, given a smooth representation (π, V), the Hecke algebra $\mathcal{H}(G)$ acts on V by

$$\pi(f)(v) = \int_G f(g)\pi(g) v dg$$

where $f \in \mathcal{H}(G) = \mathcal{C}_c^\infty(G)$. Under the equivalence from smooth representations of G to non-degenerate $\mathcal{H}(G)$-modules, admissible representations correspond to the $\mathcal{H}(G)$-modules where $\pi(f)$ is a finite rank operator for all f. This allows us to define the trace of $\pi(f)$, and hence the distribution character

$$\Theta_\pi : \mathcal{H}(G) \to \mathbb{C}$$

given by $f \mapsto \Theta_\pi(f) := \mathrm{Tr}\, \pi(f)$.

Lemma 2.7. *If $\{\pi_1, ..., \pi_n\}$ are pair-wise non-isomorphic irreducible admissible representations of G, then $\{\Theta_{\pi_1}, ..., \Theta_{\pi_n}\}$ are linearly independent.*

[5] This part of the discussion was omitted in my talk due to limit of time

Remark. A useful way to compute the distribution character Θ_π for (π, V) is as follows. Pick a countable set of basis $\mathtt{K}_m$ for G (for example, $\mathtt{K}_i = 1 + \varpi_K^i \mathtt{K}_0$ where $\mathtt{K}_0 = \mathrm{GL}_2(\mathcal{O}_K)$). Let $V_i := V^{\mathtt{K}_i}$, so we have $V_0 \subset V_1 \subset \cdots \subset V = \cup_i V_i$. For each i, and each $g \in G$, consider the following composition

$$\pi(e_{\mathtt{K}_i}) \circ \pi(g) \circ \pi(e_{\mathtt{K}_i}) : V_i \longrightarrow V_i.$$

Denote the trace of this operator by $\xi_{\pi,i}(g)$. For $f \in \mathcal{H}(G)$, there exists some i (depending on f) such that $f \in \mathcal{H}(G /\!\!/ \mathtt{K}_i)$, then for each $m \geq i$, we have

$$\Theta_\pi(f) = \int_G \xi_{\pi,m}(g) f(g) dg.$$

where the right hand side does not depend on m for m large enough.

We state a deep theorem of Harish-Chandra (In fact this is a theorem of Harish-Chandra when K has characteristic 0 and of Lemaire when K has positive characteristic), which says that, Θ_π is represented by the Harish-Chandra character χ_π, which is locally constant on the regular semi-simple elements $\mathrm{G}^{r.s.}$. Let us first set up the notation:

(1) Let $\mathrm{G}^{r.s.}$ be the set of regular semi-simple elements in $\mathrm{GL}_n(K)$. (Recall that an element $g \in \mathrm{GL}_n(K)$ is regular semi-simple if the characteristic polynomial $P_g(X) \in k[X]$ of g has distinct roots over $\overline{K}$.)
(2) Let $\mathrm{G}^{r.e.} \subset \mathrm{G}^{r.s.}$ to be the subset of regular elliptic elements (Recall that an element $g \in \mathrm{G}^{r.s.}$ is regular elliptic if in addition its characteristic polynomial P_g is irreducible).

Theorem 2.8 (Harish-Chandra)**.** *There is a locally integrable function χ_π on G, which is locally constant on the regular semi-simple set $\mathrm{G}^{r.s.}$ and 0 on the complement of $\mathrm{G}^{r.s.}$, such that*

$$\Theta_\pi(f) = \int_G f(g)\chi_\pi(g) dg$$

for all $f \in \mathcal{H}(G)$.

Remark. χ_π can be viewed as a character $\chi_\pi : \widetilde{G} \to \mathbb{C}$ where $\widetilde{G} = \mathrm{G}^{r.s.}/\sim$ is the set of conjugacy classes of regular semi-simple elements in G.

Remark. We record a definition which is useful in discussing the Jacquet-Langlands correspondence.

(1) A smooth admissible representation (π, V) of $\mathrm{GL}_n(K)$ is elliptic if its Harish-Chandra character χ_π is not identically 0 on $\mathrm{G}^{r.e.}$.
(2) Every essentially square integrable representation is elliptic.

2.2. **Supercuspidal representations.**

2.2.1. *Induction and compact induction.*

We work with a slightly more general setup, consider a locally compact (possibly totally disconnected) topological group G and a closed subgroup $H < G$, and their smooth complex representations.

Definition. Now we define two functors $\mathrm{Ind}, \mathrm{ind} : \mathrm{Rep}_H^\infty \to \mathrm{Rep}_{\mathrm{G}}^\infty$

(1) $\mathrm{Ind}_H^G(W) := \{f : G \to W \mid \text{such that}$

$$f(hx) = hf(x), \text{for all } h \in H, x \in G; \text{and } f \text{ is uniformally locally const}\},$$

where the last condition means that there is a open subgroup $C \subset G$ (ind. of x) such that $f(xC) = f(x)$ for all x. G acts on $\mathrm{Ind}(W)$ by right translation.

(2) $\mathrm{ind}_H^G \subset \mathrm{Ind}_H^G$ is a subfunctor

$$\mathrm{ind}(W) = \{f \in \mathrm{Ind}(W) | f \text{ is compactly supported mod } H\}$$

Remark.

(1) If G/H is compact, then ind = Ind.
(2) Suppose G/H is compact, then ind = Ind sends admissible representations of H to admissible representations of G.
(3) Ind is exact (and so is Res the restriction functor); if $H < G$ is in addition open, then ind is also exact.

Consider restriction functor $\mathrm{Res} : \mathrm{Rep}_G^\infty \to \mathrm{Rep}_H^\infty$.

Lemma 2.9.

(1) *(Frobenius reciprocity)*[6] *We have adjunction*

$$\mathrm{Res} \dashv \mathrm{Ind}\,.$$

(2) *Further assume $H < G$ is open (so G/H is discrete), then*

$$\mathrm{ind} \dashv \mathrm{Res}\,.$$

Remark.

(1) The functors ind, Res, Ind correspond to $f_!, f^*, f_*$ in a sense which could be made precise. [e.g., Ngo's notes]. This is helpful to remember the adjunctions above.
(2) Suppose $H < G$ is open, then ind could be interpreted as coInd, namely

$$\mathrm{ind}(W) = W \otimes_{\mathbb{C}[H]} \mathbb{C}[G].$$

Now consider a quotient $\psi : G \to M = G/N$ for some (closed) normal subgroup N. We still have a "Res" functor:

$$\mathrm{Res} = \psi^* : \mathrm{Rep}_M^\infty \to \mathrm{Rep}_G^\infty$$

by composing with ψ (this is sometimes called the inflation functor). The left adjoint of Res is taking the N-coinvariant subspace, and the right adjoint of Res is taking N-invariant.

I believe the following convention differs from some existing ones:

Definition. Let $\psi : G \to M = G/N$ as above, we define

$$\psi_! : \mathrm{Rep}_G^\infty \to \mathrm{Rep}_M^\infty$$

by $\psi_! V = V/[N-1]V$ the space of N-coinvariants of V.

Then as expected, we have the following adjunction in this setup: $\psi_! \dashv \psi^*$.

[6] This is called "Dual Frobenius reciprocity" in Cartier's survey.

2.2.2. *Parabolic induction and the Jacquet functor I.*

Let $P < G = \mathrm{GL}_n(K)$ be a parabolic subgroup,[7] associated to P we have unipotent radical U_P and Levi quotient L_P which fits into:

$$1 \to U_P \to P \to L_P \to 1.$$

We have $G = \mathrm{GL}_n(K)$, so parabolic subgroups are conjugates of the standard parabolics

$$P_{\underline{n}} = \begin{pmatrix} \boxed{\mathrm{GL}_{n_1}} & * & * \\ & \boxed{\mathrm{GL}_{n_2}} & * \\ & & \cdots \end{pmatrix}, \quad \underline{n} = (n_1, ..., n_r),$$

where $\underline{n}$ is an ordered partition of n. Inside $P_{\underline{n}}$ we have

$$U_{\underline{n}} = \begin{pmatrix} \boxed{1_{n_1}} & * & * \\ & \boxed{1_{n_2}} & * \\ & & \cdots \end{pmatrix}, \qquad L_{\underline{n}} = \begin{pmatrix} \boxed{\mathrm{GL}_{n_1}} & & \\ & \ddots & \\ & & \boxed{\mathrm{GL}_{n_r}} \end{pmatrix}.$$

Now for any standard parabolic P consider $P < G$ and $\varphi : P \twoheadrightarrow L = P/U$, by the previous discussion we have functors

$$\mathrm{ind} = \mathrm{Ind} : \mathrm{Rep}_P^\infty \to \mathrm{Rep}_G^\infty, \qquad \varphi_! : \mathrm{Rep}_P^\infty \to \mathrm{Rep}_L^\infty$$

and adjunction

$$\mathrm{Res}_P^G \dashv \mathrm{ind}_P^G = \mathrm{Ind}_P^G, \qquad \varphi_! \dashv \varphi^*$$

Now we define the (naive/un-normalized) parabolic induction $\widetilde{I}_P^G$ and $\widetilde{J}_P^G$ as follows:

Definition.

(1) $\widetilde{I}_P^G : \mathrm{Rep}_L^\infty \to \mathrm{Rep}_G^\infty$ is defined by

$$\widetilde{I}_P^G := \mathrm{ind}\,_P^G \circ \varphi^*$$

(2) $\widetilde{J}_P^G : \mathrm{Rep}_G^\infty \to \mathrm{Rep}_L^\infty$ is defined by

$$\widetilde{J}_P^G = \varphi_! \circ \mathrm{Res}_P^G$$

Remark. We have the following properties

(1) $\widetilde{J} \dashv \widetilde{I}$;
(2) Both functors are exact;
(3) Both functors send finite length representations to finite length representations;
(4) Both functors send admissible representations to admissible representations;
(5) The properties above still hold after twisting by a character.

(1) is a formal consequence of Res $\dashv$ Ind and $\varphi_! \dashv \varphi^*$:

$$\mathrm{Hom}_L(\widetilde{J}\sigma_G, \sigma_L) \cong \mathrm{Hom}_P(\mathrm{Res}_P^G(\sigma_G), \varphi^*(\sigma_L)) \cong \mathrm{Hom}_G(\sigma_G, \widetilde{I}\sigma_L).$$

[7] To be more precise, we should take a parabolic K-subgroup $\mathbb{P} \subset \mathbb{G}$ and then take its K-points

2.2.3. *Parabolic induction and the Jacquet functor II.*

The naive Jacquet functor and parabolic induction do not send unitary representations to unitary representations (since parabolics are in general not uni-modular). Let δ be the modular character on P, namely given by

$$(L_x)^*\mu_P^r = \delta(x)\mu_P^r$$

where L_x is the *left* translation by x and μ^r is (any) right invariant Haar measure on P.

Example.

Consider $B \subset \mathrm{GL}_2(K)$ the standard Borel consisting of upper triangular matrices $\left\{\begin{pmatrix} x & y \\ & z \end{pmatrix}\right\} \subset \mathrm{GL}_2$, then we have the left invariant measure μ^l and right invariant measure μ^r:

$$d\mu^l = \frac{1}{|x|^2 \cdot |z|} dx\ dy\ dz, \quad d\mu^r = \frac{1}{|x| \cdot |z|^2} dx\ dy\ dz.$$

By definition, $d\mu^r(\gamma g) = \delta(\gamma) d\mu^r(g)$, therefore it is easy to compute

$$d\mu^r\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\begin{pmatrix} x & y \\ & z \end{pmatrix}\right) = \frac{d(ax)\wedge d(ay+bz)\wedge d(dz)}{|ax|\cdot|dz|^2} = \frac{|a|}{|d|}\cdot d\mu^r\left(\begin{pmatrix} x & y \\ & z \end{pmatrix}\right),$$

so $\delta\left(\begin{pmatrix} a & b \\ & d \end{pmatrix}\right) = |a \cdot d^{-1}|$.

More generally, for $L_{\underline{n}} \subset P_{\underline{n}}$ where we consider the Levi as a subgroup, we have modular character $\delta_{\underline{n}}$ on $P_{\underline{n}}$, then $\delta_{\underline{n}}|_{L_{\underline{n}}}$ can be described by

$$\delta_{\underline{n}}(x) = |\det(\mathrm{ad}_{U_{\underline{n}}}(x))|, \qquad \text{for all } x \in L_{\underline{n}}.$$

Here $\mathrm{ad}_{U_{\underline{n}}}$ denotes the representation of $L_{\underline{n}}$ by acting on $U_{\underline{n}}$ by conjugation.

This discussion leads to the definition of the normalized Jacquet functor J and parabolic induction I.

Definition.

$$J_P^G := \delta^{\frac{-1}{2}} \cdot \widetilde{J}_P^G, \qquad I_P^G := \widetilde{I}_P^G \cdot \delta^{\frac{1}{2}}$$

Remark. By definition, we have for all (π_L, W) a smooth representation of $L_{\underline{n}} = \mathrm{GL}_{\underline{n}}$, given by

$$I_{P_{\underline{n}}}^G(W) = \left\{ f : \mathrm{GL}_n(K) \to W \;\middle|\; f \text{ is smooth}, \right.$$
$$\left. f(uhg) = \delta_{\underline{n}}^{1/2}(h)\pi_L(h)f(g), \quad \text{for all } u \in U_{\underline{n}}, h \in \mathrm{GL}_{\underline{n}}, g \in G \right\}$$

2.2.4. *Supercuspidal representations.*

Definition. Let $(\pi, V) \in \mathrm{Rep}_{\mathrm{G}}^{\mathrm{adm}}$ be an irreducible admissible representation, (π, V) is called supercuspidal if for any proper parabolic $P < G$, $J_P^G(\pi) = 0$.

Supercuspidal representations form building blocks of admissible representations.

Lemma 2.10. *Let $\pi \in \mathrm{Rep}_{\mathrm{G}}^{\infty}$ be an irreducible (hence admissible) representation, then either π is supercuspidal, or π is a sub-representation of $I_P^G(\sigma)$ where $P < G$ is a (proper) parabolic subgroup, and σ an irreducible supercuspidal representation of L.*

Proof. Suppose π is not supercuspidal, then there exists a minimal parabolic $P < G$ such that $J_P^G(\pi) \neq 0$, $J_P^G\pi$ is of finite length, and let σ be an irreducible quotient. Then by adjunction we have

$$0 \neq \mathrm{Hom}_L(J_P^G\pi, \sigma) = \mathrm{Hom}_G(\pi, I_P^G\sigma)$$

which means π is isomorphic to a sub of $I_P^G\sigma$. By construction one can show that σ is supercuspidal. $\square$

The following description [8] (I believe it is due to Harish-Chandra) of supercuspidal representations are helpful. Let $G^0 = \{g \in \mathrm{GL}_n(K) \mid \det g \in \mathcal{O}_K^\times\}$, and let $Z = Z(G)$ be the center of G, then ZG^0 has finite index in G (this works for all reductive p-adic groups, and for GL_n the index is n). The idea is that one can usually study representations of G by restricting it to G^0, then take care of the center separately. We will give an application of this idea later in the notes.

Lemma 2.11. *Let (π, V) be an irreducible admissible representation of G, then TFAE*

1. *π is supercuspidal;*
2. *Every matrix coefficient $C_{v,\lambda} : g \mapsto \lambda(\pi(g)v)$ from $G \to \mathbb{C}$ is compactly supported on G^0;*
3. *Every matrix coefficient is compactly supported mod center $Z \subset G$.*

Moreover, if in addition $\pi^\vee$ is also irreducible, then $(1) - (3)$ is equivalent to

4. *There exists one matrix coefficient which is compactly supported modulo center.*

Sketch. Add sketch in second draft $\square$

The following lemma will be useful for us:

Lemma 2.12. *Let $\mathtt{K}_0$ be a compact open subgroup of $\mathrm{GL}_n(K)$. There are only finitely many isomorphism classes of irreducible supercuspidal representations of G° with a $\mathtt{K}_0$ fixed vector.*

Proof. This is the reformulation of the corollary of Proposition 21 in Section 3 of Bernstein's Harvard notes, see [1]. $\square$

Remark. The following description of parabolic induction from $L_{\underline{n}}$ to G will be a special case of the Bernstein-Zelevinsky classification.

- Let (π_L, W) be an admissible representation of $L_{\underline{n}} = \mathrm{GL}_{n_1} \times ... \times \mathrm{GL}_{n_r}$ given by
$$\pi_{\underline{n}} = \pi_1 \otimes ... \otimes \pi_r, \quad W = V_1 \otimes ... \otimes V_r$$
where each (π_i, V_i) is an irreducible supercuspidal representation of GL_{n_i}. Then the representation
$$\pi_1 \times ... \times \pi_r := I_{P_{\underline{n}}}^G(\pi_L)$$
is an admissible representation of G as discussed above. It is irreducible if and only if for any π_i, π_j where $i \neq j$, $\pi_i \neq |\det| \cdot \pi_j$.
- In the case of GL_2, this is saying that $\chi_1 \times \chi_2$ is irreducible if and only if $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$ – these are the principal series for GL_2.

[8] I think the condition (2), or equivalently (3), might be how supercuspidality is defined originally.

2.3. **Statements of the Local Langlands correspondence.**

So far we have not yet introduced L- and ϵ-factors of Galois representations and admissible representations, or the notion of (essentially-) square integrable, (essentially-) tempered, and generic representations. Though I believe it is already time to state the precise local Langlands correspondence (LLC) at this stage.

2.3.1. *The statement of LLC.*

Theorem 2.13. *There exists a unique collection of bijections of sets*

$$\mathrm{rec}_n : \left\{\begin{matrix}\text{(Equivalent classes of)}\\ \text{irreducible admissible}\\ \text{representations of } \mathrm{GL}_n(K)\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{(Equivalent classes of)}\\ n\text{-dimensional Frobenius-semisimple}\\ \text{Weil-Deligne representations of } W_K\end{matrix}\right\}$$

satisfying the following list of properties:

(1) For every irreducible quasi-character χ of $K^\times$,
$$\mathrm{rec}_1(\chi) = \chi \circ \mathrm{Art}_K^{-1},$$
where $\mathrm{Art}_K : K^\times \xrightarrow{\sim} W_K^{ab}$.

(2) For every pair π_1, π_2 of irreducible admissible representations of GL_{n_1} and GL_{n_2},
$$L(\pi_1 \times \pi_2, s) = L(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s)$$
$$\epsilon(\pi_1 \times \pi_2, s) = \epsilon(\mathrm{rec}_{n_1}(\pi_1) \otimes \mathrm{rec}_{n_2}(\pi_2), s)$$
where the L- and ϵ- factors are defined in Section 3.6.

(3) For every χ quasi-character of $K^\times$ and every irreducible $\pi \in \mathrm{Rep}^{\mathrm{adm}}_{\mathrm{GL}_n(K)}$, we have
$$\mathrm{rec}_n(\pi\chi) = \mathrm{rec}_n(\pi) \otimes \mathrm{rec}_1(\chi)$$

(4) For every irreducible $\pi \in \mathrm{Rep}^{\mathrm{adm}}_{\mathrm{GL}_n(K)}$ with central character ω_π,
$$\det \circ\, \mathrm{rec}_n(\pi) = \mathrm{rec}_1(\omega_\pi).$$

(5) For every irreducible $\pi \in \mathrm{Rep}^{\mathrm{adm}}_{\mathrm{GL}_n(K)}$, we have
$$\mathrm{rec}_n(\pi^\vee) = \mathrm{rec}_n(\pi)^\vee$$

Remark. There are two important invariants preserved by the local Langlands correspondence rec_n

- Conductor: $c(\pi) = c(\mathrm{rec}_n(\pi))$, which follows from the compatibility with ϵ-factors.
- Depth: $d(\pi) = d(\mathrm{rec}_n(\pi))$, introduced by Moy-Prasad.

Remark. By work of Henniart, it suffices to prove that there exists a collection of maps rec_n satisfying $(1)-(5)$ above (on supercuspidal representations). Henniart's theorems asserts that, once rec_n exists, it is necessarily a collection of bijections and will be unique. Henniart's strategy is roughly as follows:

(i) By Bernstein-Zelevinsky, which will be discussed in section 3, we can reduce to showing that there exists a unique collection of bijections rec_n from
$$\{\text{cls. of supercuspidal reps of } \mathrm{GL}_n\} \to \{\text{cls. of n-dim irreducible WD-reps of } W_K\}$$

(ii) By the property of ϵ-factors (and considering rec_n for all n at once), reduce the statement above to existence of bijections $\{\mathrm{rec}_n\}$.

(iii) By Henniart's *numerical local Langlands correspondence*, the surjectivity of the maps rec_n follow from injectivity.

(iv) Use the properties of L-factors to show that, such a collection $\{\mathrm{rec}_n\}$ will necessarily be injecctive.

Hence it suffices to show existence of the maps $\{\mathrm{rec}_n\}$ on irreducible supercuspidal representations of GL_n satisfying $(1)-(5)$ as in the theorem. This is the statement proven by Harris-Taylor, and later by Henniart, when $K/\mathbb{Q}_p$ is a local number field. It seems that Scholze's approach bypasses the numerical local Langlands correspondence.

2.3.2. *Finer correspondence.*

Note that the properties included in the theorem above are needed to guarantee uniqueness of the correspondence (hence part of the theorem). The following ones can be regarded as extra properties (and they provide finer information on rec_n):

Theorem 2.14. *Under the local Langlands correspondence, namely* rec_n *as in the theorem above, we have*

(1) *The irreducible representation* $\pi \in \mathrm{Rep}_{\mathrm{G}}^{\mathrm{adm}}$ *is supercuspidal if and only if* $\mathrm{rec}_n(\pi)$ *is irreducible.*
(2) π *is essentially square integrable if and only if* $\mathrm{rec}_n(\pi)$ *is indecomposable.*
(3) π *is generic if and only if* $L(\mathrm{ad}\circ\mathrm{rec}_n(\pi), s)$ *has no pole at* $s = 1$.

2.3.3. *Inertial correspondence.*

We state special cases of the theory of ($\mathrm{GL}_n(\mathcal{O}_K)$-) types in the subsequent section. While we cannot expect a correspondence between $\mathrm{GL}_n(\mathcal{O}_K)$-representations and inertia representations, there is a close link between them.

Proposition 2.15.

(1) *Let* π *and* π' *be two generic representations of* G, *if* $\mathrm{r}_n(\pi)|_{I_K} \cong \mathrm{r}_n(\pi')|_{I_K}$, *where* r_n *denotes the* l*–adic representation of* W_K *associated to the Weil-Deligne representation* rec_n, *then*
$$\pi|_{\mathrm{GL}_n(\mathcal{O}_K)} \cong \pi'|_{\mathrm{GL}_n(\mathcal{O}_K)}$$
(2) (Paskunas.) *Let* τ *be a Weil-Deligne inertia type of* I_K *which extends to an* n*-dimensional irreducible Frobenius semi-simple representation of* W_K, *then there exists a unique (up to isomorphism) smooth irreducible representation* $\sigma = \sigma_\tau$ *of* $\mathrm{GL}_n(\mathcal{O}_K)$, *such that for any smooth irreducible infinite dimensional representation* π *of* $\mathrm{GL}_n(K)$,
$$\pi|_{\mathtt{K}} \text{ contains } \sigma_\tau \quad \Leftrightarrow \quad \mathrm{rec}_n(\pi)|_{I_K} \cong \tau.$$

3. Bernstein-Zelevinsky (and other) classifications

3.1. Classification of admissible representations of GL_n.

3.1.1. *The segment* $\Delta(\pi, m)$.

Let $(\pi, V) \in \mathrm{Rep}_{\mathrm{G}}^{\mathrm{adm}}$, for any $s \in \mathbb{C}$ let $\pi(s)$ be the twist of π by the character $|\det|^s$, namely $\pi(s)(g) = |\det(g)|^s \pi(g)$. Note that if π is irreducible supercuspidal, then so is $\pi(s)$.

We give a partial order on the set of isomorphism classes of irreducible supercuspidal representations of $\mathrm{GL}_n(K)$: namely $\pi \leq \pi'$ if and only if $\pi' = \pi(s)$ for some integer $s \geq 0$. Call this poset $\mathcal{A}_n^{sc}$. A finite interval in $\mathcal{A}_n^{sc}$ is called a Zelevinsky segment. To be more precise:

Definition. A Zelevinsky segment [9] starting from π of length m and degree mn is an ordered finite collection of m irreducible supercuspidal representations of $\mathrm{GL}_n(K)$:

$$\Delta(\pi, m) = [\pi, \pi(1), ..., \pi(m-1)].$$

Definition. Let $\Delta_1 = (\pi_1, m_1)$ and $\Delta_2 = (\pi_2, m_2)$ be two Zelevinsky segments in $\mathcal{A}_{n_1}^{sc}$ and $\mathcal{A}_{n_2}^{sc}$ respectively.
(1). Δ_1, Δ_2 are linked if $\Delta_1 \nsubseteq \Delta_2$, $\Delta_2 \nsubseteq \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a Zelevinsky segment. In particular, this could only happen when $n_1 = n_2$.
(2). Δ_1 precedes Δ_2 if (a). Δ_1 and Δ_2 are linked, and (b). $\pi_1 < \pi_2$.

Remark. In particular, if $n_1 \neq n_2$, then Δ_1 and Δ_2 are not linked.

Definition. Let $\Delta = \Delta(\pi, m)$ be a (Zelevinsky) segment, of length m and degree mn, we define an admissible representation of GL_{mn} by

$$\pi(\Delta) := \pi_1 \times ... \times \pi_m,$$

where $\pi_1 \times ... \times \pi_m = I_{P_{(n,n,...)}}^{\mathrm{GL}_{mn}}(\otimes \pi_i)$ as in the last remark in 2.2.4.

3.1.2. *The* $(*)$ *condition.*

Let $\Delta_1, ..., \Delta_r$ be an ordered tuple of r Zelevinsky segments of length m_i, with $\Delta_i \subset \mathcal{A}_{n_i}^{sc}$. We introduce the following $(*)$ condition:

$$(*) \qquad \text{For each } i < j,\ \Delta_i \text{ does not precede } \Delta_j.$$

A $(*)$-permutation of $\Delta_1, \cdots, \Delta_r$ is a permutation σ on Δ_i such that $\Delta_{\sigma(1)}, \cdots, \Delta_{\sigma(r)}$ still satisfies the $(*)$ condition.

Note that

(1) By definition, when $n_i \neq n_j$ the condition is empty;
(2) The $(*)$ condition allows Δ_j to precede Δ_i when $i < j$.

Example. Let $\Delta = [\pi, \pi(1)]$, $\Delta' = [\pi(2), \pi(3)]$, $\Delta'' = [\pi(5), \pi(6)]$, then (Δ, Δ') does not satisfy $(*)$, while (Δ', Δ), or (Δ, Δ''), or the single segment $\Delta \cup \Delta' = [\pi, ..., \pi(3)]$ all satisfy $(*)$.

[9] Sometimes referred to as a Bernstein-Zelevinsky segment in literature. In these notes, we use segments or Zelevinsky segments.

3.1.3. *Statements of the classification theorem.*

Theorem 3.1 (Bernstein-Zelevinsky)**.**

(1) *For each Zelevinsky segment Δ of length m, the representation $\pi(\Delta)$ has length 2^{m-1}.*
(2) *Let Δ be a Zelevinsky segment of finite length, then $\pi(\Delta)$ has a unique irreducible sub-representation $Z(\Delta)$ and a unique irreducible quotient-representation $Q(\Delta)$.*
(3) *Let $\Delta_1, ..., \Delta_r$ be Zelevinsky segments satisfying the $(*)$ condition, then the representation*
$$Q(\Delta_1) \times \cdots \times Q(\Delta_r) \in \mathrm{Rep}^{\mathrm{adm}}(\mathrm{GL}_{\sum n_i m_i})$$
admits a unique irreducible quotient $Q(\Delta_1, ..., \Delta_r)$. Moreover, $Q(\Delta_1, ..., \Delta_r)$ is independent of the order of Δ_i under $(*)$-permutations.
(4) *Any irreducible representation (π, V) of $G = \mathrm{GL}_n(K)$ is isomorphic to one of the form*
$$\pi \cong Q(\Delta_1, ..., \Delta_r),$$
where $\Delta_i = (\pi_i, m_i)$ with *$n = \sum n_i m_i$, for a unique collection of segments $\Delta_1, ...\Delta_r$ up to $(*)$-conjugation.*
(5) *As in (2) above, let $\Delta_1, ..., \Delta_r$ be segments satisfying the $(*)$ condition, then*
$$Q(\Delta_1) \times \cdots \times Q(\Delta_2)$$
is irreducible (hence equal to $Q(\Delta_1, ..., \Delta_r)$) if and only if no two of the segments Δ_i and Δ_j are linked. As a special case, we recover he remark in 2.2.4.
(6) *For $\Delta = \Delta(\pi, m) = [\pi, ..., \pi(m-1)]$, define*
$$\Delta^\vee = [\pi(m-1)^\vee, ..., \pi^\vee] = \Delta(\pi^\vee(1-m), m),$$
then
$$Q(\Delta_1^\vee, ..., \Delta_r^\vee) = Q(\Delta_1, ..., \Delta_r)^\vee.$$
(7) *A similar assertion of $(3)-(6)$ holds for $Z(\Delta_1) \times \cdots \times Z(\Delta_r)$, which admits a unique irreducible sub-representation $Z(\Delta_1, ..., \Delta_r)$.*

Remark.
Given any tuple $\Delta_1, ..., \Delta_r$ of Zelevinsky segments, there exist some permutation of the tuple so it satisfies $(*)$, so the set of isomorphic classes of admissible representations of $\mathrm{GL}_n(K)$ is in bijection with the unordered tuples of $\Delta_1, ..., \Delta_r$ where $n = \sum m_i n_i$.

Remark.
One can give the direct sum $\mathcal{R}_K = \oplus_{n \geq 0} \mathcal{R}_n(K)$ of Grothendieck groups of admissible representations of each $\mathrm{GL}_n(K)$ admits an multiplicative structure by
$$([\pi_1], [\pi_2]) \mapsto [\pi_1 \times \pi_2],$$
which makes $\mathcal{R}_K$ a graded commutative ring. And the map $Q(\Delta_1, ..., \Delta_r) \mapsto Z(\Delta_1, ..., \Delta_r)$ defines an involution on $\mathcal{R}_K$.

3.1.4. *Supercuspidal support.*

Definition. The supercuspidal support of $\pi \cong Q(\Delta_1, ..., \Delta_r)$ as in 3.1 is the (unordered) collection of supercuspidal representations

$$\mathrm{Supp}(\pi) = \{ \pi_i(j) \}_{i=1,...,r, j=1,...,m_i-1}$$

Remark (Zelevinsky)**.** The supercuspidal support of π are is the unique unordered tuple of supercuspidal representations $\sigma_1, ..., \sigma_s$ such that $Q(\Delta_1, ..., \Delta_r)$ and $Z(\Delta_1, ..., \Delta_r)$ are subquotients of $\sigma_1 \times \cdots \times \sigma_s$.

3.1.5. *Steinberg representations.*

Consider the (quasi-) character $|\cdot|^{\frac{1-n}{2}} : K^\times \to \mathbb{C}^\times$, and a Zelevinsky segment

$$\Delta = \Delta(|\cdot|^{\frac{1-n}{2}}, n) = [\, |\cdot|^{\frac{1-n}{2}}, |\cdot|^{\frac{3-n}{2}}, \cdots, |\cdot|^{\frac{n-1}{2}} \,].$$

It is straightforward to compute the modulus character δ_B on the standard Borel B restricted to $T = L_B = \mathrm{GL}_{(1,...,1)}(K)$. Let $x = \mathrm{Diag}(x_1, ..., x_n)$ be the diagonal matrix with entries $x_1, ..., x_n$, then

$$\delta(x) = |\det(\mathrm{ad}_U(x))| = |x_1^{n-1} \cdot x_3^{n-3} \cdots x_n^{1-n}|.$$

In other words, the representation π_T (in this case a character) $|\cdot|^{\frac{1-n}{2}} \otimes ... \otimes |\cdot|^{\frac{n-1}{2}}$ on the diagonal torus T equals $\delta^{\frac{-1}{2}}$. By definition, $\pi(\Delta)$ has the following simple description

$$\pi(\Delta) = |\cdot|^{\frac{1-n}{2}} \times ... \times |\cdot|^{\frac{n-1}{2}} = \{\text{smooth functions } f : B\backslash G \to \mathbb{C}\}.$$

Therefore, the unique irreducible sub-representation $Z(\Delta)$ has to be the trivial representation triv, which corresponds to constant functions on $B\backslash G$.

The unique quotient $Q(\Delta) =: \mathrm{St}_n$ is defined to be the (standard) Steinberg representation of $\mathrm{GL}_n(K)$. It is selfdual, namely, $\mathrm{St}_n^\vee = \mathrm{St}_n$.

Remark. When $n = 2$ and $\Delta = [\, |\cdot|^{\frac{-1}{2}}, |\cdot|^{\frac{1}{2}}]$, part (1) of theorem 3.1 asserts that $\pi(\Delta)$ has length 2. Therefore, we have

$$0 \to \mathrm{triv} \to \pi(\Delta) \to \mathrm{St}_2 \to 0$$

Remark. Alternatively, we may consider $\Delta_1 = [\, |\cdot|^{\frac{1}{2}}]$ and $\Delta_2 = |\cdot|^{\frac{-1}{2}}$, so (Δ_1, Δ_2) satisfies the $(*)$-condition, and $\mathrm{St}_2 = Z(\Delta_1, \Delta_2)$, $\mathrm{triv} = Q(\Delta_1, \Delta_2)$.

To compare, we have short exact sequences:

$$0 \to \mathrm{triv} \to |\cdot|^{\frac{-1}{2}} \times |\cdot|^{\frac{1}{2}} \to \mathrm{St}_2 \to 0$$

$$0 \to \mathrm{St}_2 \to |\cdot|^{\frac{1}{2}} \times |\cdot|^{\frac{-1}{2}} \to \mathrm{triv} \to 0$$

3.1.6. *The local Langlands correspondence for* GL_2.

The case for $n = 2$ is particularly clear. On the GL_2 side, either π is supercuspidal, or it is a quotient of some parabolic induction (of either a single segment of length 2, or two segments both of length 1). Therefore, we have the following possibilities

(1) π is a supercuspidal representation.
(2) $\pi = Q(\chi_1, \chi_2)$ where $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$, so in fact $\pi = \chi_1 \times \chi_2$, these are called *principal series* representations.
(3) $\pi = Q(\chi|\cdot|^{\frac{1}{2}}, \chi|\cdot|^{\frac{-1}{2}})$, this is a 1-dimensional representation $\mathrm{triv} \otimes \chi$.
(4) $\pi = Q(\Delta)$ where $\Delta = [\chi|\cdot|^{\frac{-1}{2}}, \chi|\cdot|^{\frac{1}{2}}]$, which equals $\mathrm{St}_2(\chi) := \mathrm{St}_2 \otimes \chi$. These are called the *Stenberg representation twisted by* χ.

Their corresponding Weil-Deligne representations are

(1) For π supercuspidal, $\mathrm{rec}_2(\pi)$ is irreducible, so
$$\mathrm{rec}_2(\pi) = (\rho, 0)$$
where ρ is irreducible.

(2) For principal series $\pi = \chi_1 \times \chi_2$, $\chi_1/\chi_2 \neq |\;|^{\pm 1}$
$$\mathrm{rec}_2(\pi) = (\chi_1 \oplus \chi_2, 0).$$ [10]

(3) For $\pi = \mathrm{triv} \otimes \chi$,
$$\mathrm{rec}_2(\pi) = (\chi \oplus \chi(1), 0)$$
where $\chi(1)$ is the "Tate twist" of χ.

(4) For $\pi = \mathrm{St}_2(\chi)$,
$$\mathrm{rec}_2(\pi) = \mathrm{Sp}_2(\chi) = (\chi \oplus \chi(1), N)$$
namely N sends χ to $\chi(1)$, as discussed in remark (3) of 1.1.4.

3.1.7. *The recipe for reducing LLC to the supercuspidal case.*

In order to prove the local Langlands correspondence (Theorem 2.13), it suffices to show that there exists a unique collection of bijections
$$\mathrm{rec}_n : \mathcal{A}_n^{sc} \to \mathcal{G}_n^{\mathrm{irr}}$$
satisfying all conditions in 2.13, where $\mathcal{A}_n^{sc}$ is the (po-)set of isomorphic classes of supercuspidal representations of $\mathrm{GL}_n(K)$ and $\mathcal{G}_n^{\mathrm{irr}}$ is the set of isomorphic classes of irreducible Weil-Deligne W_K representations of dimension n.

This is because, once we have rec on $\mathcal{A}_m^{sc}$ for all m, for an irreducible admissible representation $\pi = Q(\Delta_1, ..., \Delta_r)$ of GL_n with Zelevinsky segments $\Delta_i = \Delta_i(\pi_i, m_i)$ and $n = \sum n_i m_i$, we can define
$$\mathrm{rec}_n(\pi) = \bigoplus_{i=1}^{r} \mathrm{Sp}_{m_i}(\mathrm{rec}_{n_i}(\pi_i)).$$
Note that the underlying representation ρ of $\mathrm{rec}_n(\pi) = (\rho, N)$ is just the direct sum
$$\bigoplus_{1\le i\le r} \bigoplus_{j=0}^{m_i-1} \mathrm{rec}_{n_i}(\pi)(j).$$
One can check that this construction satisfies all requirements $(1) - (5)$ in 2.13.

3.2. **Square integrable, tempered and generic representations.**

We introduce the following representations:

supercuspidals $\subset$ square integrable $\subset$ tempered $\subset$ generic $\subset$ admissible

[10] More accurately $(\mathrm{Art}^{-1}\chi_1 \oplus \mathrm{Art}^{-1}\chi_2, 0)$

3.2.1. *Essentially L^2 and essentially $L^{2+\epsilon}$.*

Definition. Let $G^0 = \{g \in G = \mathrm{GL}_n(K) \,|\, |\det(g)|_K = 1\}$.

(1) For any $r \in \mathbb{R}_{>0}$, an admissible representation (π, V) is essentially L^r if for any $v \in V, \lambda \in V^\vee$, the matrix coefficient $c_{v,\lambda}$ is L^r on G^0, namely
$$\int_{G^0} |c_{v,\lambda}|^r dg$$
exists.
(2) $(\pi, V) \in \mathrm{Rep}_\mathrm{G}^\mathrm{adm}$ is L^r if it is essentially L^r and the central character ω_π is unitary.
(3) $(\pi, V) \in \mathrm{Rep}_\mathrm{G}^\mathrm{adm}$ is essentially square integrable (resp. square integrable) if it is essentially L^2 (resp. L^2).

Remark. For $(\pi, V) \in \mathrm{Rep}_\mathrm{G}^\mathrm{adm}$, if π is essentially L^r, then it is essentially $L^{r'}$ for all $r \geq r$.

Remark (Casselman 80')**.** For any smooth representation $(\pi, V) \in \mathrm{Rep}_\mathrm{G}^\infty$ with central character ω_π, there exists (a unique ?) positive real valued quasi-character χ such that $\omega_{\chi\pi}$ is unitary. This allows us to twist essentially L^r representations to L^r representations.

Theorem 3.2 (Zelevinsky)**.**

(1) *An irreducible admissible representation π of $\mathrm{GL}_n(K)$ is essentially square integrable if and only if it is of the form $Q(\Delta)$ for a (single) Zelevinsky segment Δ.*
(2) *The essentially L^2 representation $Q(\Delta)$ above is square integrable if and only if the central character of $\rho(\frac{m-1}{2})$ is unitary, where $\Delta = \Delta(\sigma, m)$ has length m.*

3.2.2. *Essentially tempered and tempered.*

We need a notion which is slightly weaker than (essentially) square integrable.

Definition. A representation is essentially tempered if it is essentially $L^{2+\epsilon}$ for all $\epsilon > 0$, and tempered if further its central character is unitary.

Theorem 3.3 (Zelevinsky)**.**

An irreducible admissible representation $\pi = Q(\Delta_1, ..., \Delta_r)$ of $\mathrm{GL}_n(K)$ is tempered if and only if each $Q(\Delta_i)$ is square integrable.

Corollary 3.4. *If $\pi = Q(\Delta_1, ..., \Delta_r)$ is tempered, then no two of intervals Δ_i and Δ_j are linked, in particular,*
$$\pi \cong Q(\Delta_1) \times \cdots \times Q(\Delta_r)$$

Proof. Let $\Delta_i = \Delta(\sigma_i, m_i)$, since $\pi = Q(\Delta_1, ..., \Delta_r)$ is tempered, $\sigma_i(\frac{m_i-1}{2})$ has a unitary character. If σ has a unitary character, then $\omega_{\sigma(s)}$ is not unitary for any nonzero $s \in \mathbb{R}$. $\square$

Remark. By the Corollary above, the tempered representations are precisely generic ones such that each $Q(\Delta_i)$ has a unitary central character. It seems (?) difficult to classify essentially tempered representations via Zelevinsky-segments.

3.2.3. *Whittaker models and generic representations.* [11]

Let $\psi : K \to \mathbb{C}^\times$ be a nontrivial additive quasi-character on K as in Subsection 1.1.10, and let $n(\psi)$ be the largest integer such that $\psi(\varpi^{-n}\mathcal{O}_K) = 1$ as before. Let $U_n \subset \mathrm{GL}_n(K)$ be the subgroup consisting of strictly upper triangular matrices. Define a one dimensional representation θ_ψ of U_n by

$$\theta_\psi(u_{ij}) = \psi(u_{12} + \cdots + u_{n-1,n}).$$

Definition. An irreducible admissible representation π of $\mathrm{GL}_n(K)$ is generic if

$$\mathrm{Hom}_{U_n}(\pi|_{U_n}, \theta_\psi) \neq 0.$$

Definition. Let (π, V) be an irreducible admissible generic representation of $G = \mathrm{GL}_n(K)$, the Whittaker model of π with respect to ψ is defined as follows: choose a nonzero element $\lambda \in \mathrm{Hom}_{U_n}(\pi|_{U_n}, \theta_\psi)$, let

$$\mathcal{W}_\psi := \{f : G \to \mathbb{C} \; : \; f(ug) = \theta_\psi(u)g \text{ for all } g \in G, u \in U_n\}.$$

G acts on $\mathtt{W}_\psi$ by right translation. Define the following (injective) G-homomorphism

$$V \longrightarrow \mathcal{W}_\psi \quad \text{by } \; v \mapsto (g \mapsto \lambda(\pi(g)v)).$$

The image $\mathcal{W}_{\pi,\psi}$ (which is an isomorphism copy of (π, V)) is called the Whittaker model of π.

Lemma 3.5.

(1) *The definition of being generic is independent of the choice of ψ.*
(2) *π is generic if and only if $\pi^\vee$ is generic.*
(3) *If $\chi : K^\times \to \mathbb{C}^\times$ is any quasi-character, then π is generic if and only if $\chi\pi$ is generic.*

Theorem 3.6 (Zelevinksy)**.**

An irreducible admissible representation $\pi = Q(\Delta_1, ..., \Delta_r)$ of $\mathrm{GL}_n(K)$ is generic if and only if no two Zelevinsky segments Δ_i and Δ_j are linked. By part (5) of theorem 3.1 if and only if

$$\pi = Q(\Delta_1) \times \cdots \times Q(\Delta_r).$$

In particular, corollary 3.4 can be rephrased as

Every irreducible (essentially) tempered representation (π, V) is generic.

Remark (Shalika)**.** Generic representations are more or less what we care about in global applications – let F be a number field, then all local components π_v of an irreducible admissible cuspidal representations $\pi = \otimes'_v \pi_v$ are generic.

[11] Much more to be added in second draft, especially on Whittaker models and its importance in defining ϵ

3.3. $\mathtt{K}$-spherical representations.

Let $\mathtt{K} \subset G = \mathrm{GL}_n(K)$ be a compact open subgroup, an irreducible admissible representation π is called $\mathtt{K}$-spherical if $\pi^{\mathtt{K}} \neq 0$. It is called spherical if it is $\mathtt{K}_0$-spherical for the maximal compact open $\mathtt{K}_0 = \mathrm{GL}_n(\mathcal{O}_K)$. In this subsection we consider spherical and Iwahori-spherical representations.

3.3.1. *Structure of* $\mathrm{GL}_n(K)$.

Notation.

- Let $B \subset G = \mathrm{GL}_n(K)$ be the standard Borel, $U \subset B$ be the strictly upper triangular matrices and U^- the strictly lower triangular matrices. Let $T \subset B$ be the diagonal torus (the corresponding Levi for B).
- $\mathtt{K}_0 = \mathrm{GL}_n(\mathcal{O}_K)$ is the maximal compact subgroup.
- Let $T_0 := T \cap \mathtt{K}_0 \cong (\mathcal{O}_K^\times)^n$.
- Let $\mathtt{B}_0 \subset \mathtt{K}_0$ be the Iwahori subgroup, which is the preimage of the standard Borel $B(k) \subset \mathrm{GL}_n(k)$ under the reduction map. We will use $\mathrm{Iw} = \mathtt{B}_0$ interchangeably in the notes.
- Let $\mathcal{N} = \mathrm{N}_G(T) \subset G$ be the generalized permutation matrices, and let $\mathcal{N}_{\mathtt{K}} := \mathcal{N} \cap \mathtt{K}$. We identify $\mathcal{N}_{\mathtt{K}}/T_0$ with the Weyl group W.
- Let $W_1 := \mathcal{N}/T_0$, sometimes called the generalized Weyl group.

Recall the following decompositions.

Lemma 3.7. (1)

$$\mathtt{K}_0 = \bigsqcup_{w \in W} \mathtt{B}_0 w \mathtt{B}_0.$$

(2) *The Iwasawa decomposition of* G

$$G = B\mathtt{K}_0.$$

(3) *The Cartan decomposition. Let* $T^- \subset T$ *be the subset containing diagonal matrices* $\mathrm{Diag}(\varpi_K^{r_1}, \varpi_K^{r_2}, ..., \varpi_K^{r_n})$ *with* $r_1 \geq r_2 \geq ... \geq r_n$. *Then*

$$G = \bigsqcup_{\lambda \in T^{-1}} \mathtt{K}_0 \lambda \mathtt{K}_0.$$

(4) *The Bruhat-Tits decomposition. Recall that* $W_1 == \mathcal{N}/T_0$*:*

$$G = \bigsqcup_{w_1 \in W_1} \mathtt{B}_0 w_1 \mathtt{B}_0$$

(5) *The Iwahori decomposition*

$$\mathtt{B}_0 = (\mathtt{B}_0 \cap U^-) \cdot T_0 \cdot (\mathtt{B}_0 \cap U)$$

It would also be useful to write $\mathtt{K}_0 \lambda \mathtt{K}_0$ as disjoint union $\lambda' \mathtt{K}_0$ for certain λ and λ' when one discusses Hecke operators. We will skip this in the first draft.

3.3.2. *Unramified principal series.*

3.3.3. *Unramified representations.*

Every irreducible unramified representation π of $G = \mathrm{GL}_n(K)$ is of the form $Q(\chi_1, ..., \chi_n)$ for some unramified quasi-characters $\chi_1, ..., \chi_n$ of $K^\times$.

Example. Consider the case of $\mathrm{GL}_2(K)$, the theorem implies that unramified irreducible representations of $\mathrm{GL}_2(K)$ are one of the following

(1) $\pi = \chi_1 \times \chi_2$ where $\chi_1/\chi_2 \neq |\cdot|^{\pm 1}$ is a principal series representation with χ_1, χ_2 unramified quasi-characters;
(2) $\pi = Q(\chi|\cdot|^{\frac{1}{2}}, \chi|\cdot|^{-\frac{1}{2}}) = \mathrm{triv} \otimes \chi$ where χ is an unramified quasi-character.

3.3.4. *Iwahori-fixed vectors.*

to be added in second draft

3.4. **Conductor of admissible representations and new-vectors.**

3.5. **Depth 0 representations.** We summarize the discussion in the following diagram. Here we are only concerned with irreducible admissible representations on the automorphic side, and Weil-Deligne representations $\mathrm{rec}_n(\pi) = (\rho, N)$ (or the corresponding l-adic representation $\mathrm{r}_n(\pi)$, so that $\mathrm{rec}_n(\pi) = \mathrm{WD}(\mathrm{r}_n(\pi))$).

$$\pi \text{ unramified} \Leftrightarrow \pi^{\mathtt{K}} \neq 0 \longleftrightarrow \mathrm{r}_n(\pi) \text{ unramified} \Leftrightarrow \rho|_{I_K} = 0 \ \& N = 0$$

$$\pi \text{ Iwahori spherical} \Leftrightarrow \pi^{\mathtt{Iw}} \neq 0 \longleftrightarrow \mathrm{rec}_n(\pi)\ I_K\text{-spherical} \Leftrightarrow \rho|_{I_K} = 0$$

$$\pi \text{ has depth } 0 \longleftrightarrow \mathrm{r}_n(\pi) \text{ tamely ramified}$$

3.5.1. *Conductor and depth.*

3.6. **L-factors and ϵ-factors.**

To be completed in second draft

3.6.1. *ϵ-factors for generic representations.* We postpone the discussion to a future draft, but note the following remarks.

The definition of ϵ-factors on the representation theory side is built out of the supercuspidal ones and based on Bernstein-Zelevinsky classification. For a pair π, π' we will define $\epsilon(\pi \times \pi', s, \psi)$.

3.6.2. *The inductive definition.*

3.6.3. *ϵ-factors and conductors.*

Let π be a smooth irreducible representation of $\mathrm{GL}_n(K)$. We will let $L(\pi, s)$ and $\epsilon(\pi, s, \psi)$ to denote the L- and ϵ-factors when $\pi' : K^\times \to \mathbb{C}^\times$ is the trivial character. Let $q = \#k$ as usual. By a theorem of JPPS (and its correction), we have

$$\epsilon(\pi, s, \psi) = q^{-\left(c(\pi) + n \cdot n(\psi)\right) \cdot s} \cdot \epsilon(\pi, 0, \psi)$$

where $c(\pi)$ is the conductor of π defined in Subsection 3.4. (In particular, the two notions of conductors of π in literature, one defined as in Subsection 3.4, the other by functional equation of $L(\pi, s)$, agree with each other).

3.6.4. *Conductors are preserved by LLC.*

Now we have the following trivial observation.

Lemma 3.8. *Under the local Langlands correspondence, we have*

$$c(\pi) = c(\mathrm{rec}_n(\pi)).$$

In other words, the conductor is preserved.

Proof. By taking π' to be the trivial character, we have by compatibility of ϵ-factors that $\epsilon(\pi, \psi, s) = \epsilon(\mathrm{rec}_n(\pi), \psi, s)$, in particular, by Corollary 1.4 and the discussion above, we have

$$q^{-\left(c(\pi)+n\cdot n(\psi)\right)\cdot s} = q^{-\left(c(\mathrm{rec}_n(\pi))+n\cdot n(\psi)\right)\cdot s}.$$

$\square$

4. Inertia Langlands correspondence

4.1. Bernstein decomposition.

4.1.1. *Bernstein components and decomposition.*

Definition. Let $L \subset \mathrm{GL}_n(K)$ be a Levi subgroup and $P \subset \mathrm{GL}_n(K)$ a parabolic with Levi factor L. Let ς be an irreducible supercuspidal representation of L. A Bernstein component

$$R^{(L,\varsigma)} = R^{(L,\varsigma)}(\mathrm{GL}_n(K)) \subset \mathrm{Rep}_n^\infty$$

is the full subcategory consisting of π such that every irreducible subquotient of π is a subquotient of $\mathrm{I}_L^{\mathrm{GL}_n}(\varsigma \otimes \eta) := \mathrm{Ind}_P^{\mathrm{GL}_n(K)}(\varsigma \otimes \eta)$, where η is an unramified character of L and $\mathrm{I}_L^{\mathrm{GL}_n} := \mathrm{Ind}_P^{\mathrm{GL}_n(K)} \circ j^*$ is the normalized parabolic induction, with $j : P \to L$ the map to the Levi quotient.

Lemma 4.1. *Two Bernstein components $R^{(L_1,\varsigma_1)}$ and $R^{(L_2,\varsigma_2)}$ are equal if and only if they are inertially equivalent, meaning that there exists an unramified character η of L_2 and $g \in \mathrm{GL}_n(K)$ such that*

$$gL_1g^{-1} = L_2, \qquad {}^g\varsigma_1 = \varsigma_2\eta$$

Definition. Define the Bernstein center of $\mathrm{GL}_n(K)$ to be the set of Bernstein components, namely that

$$B = B(\mathrm{GL}_n(K)) := \{(L,\varsigma)\}/\text{inertial equivalence}.$$

Theorem 4.2 (Bernstein decomposition)**.** *The category Rep_n^∞ of smooth representations of $\mathrm{GL}_n(K)$ decomposes as a direct product*

$$\mathrm{Rep}_n^\infty = \prod_{(L,\varsigma)\in B} \mathrm{R}^{(L,\varsigma)}.$$

Remark. To determine which Bernstein component containing a given irreducible representation $\pi \in \mathcal{A}_n(K)$, it suffices to consider its supercuspidal support.

4.1.2. *Bernstein components for GL_2.*

Example (n = 2). The Bernstein components $R^{(L,\sigma)}(\mathrm{GL}_2(K))$ are one of the following types:

1. If $R^{(L,\sigma)(G)}$ contains an irreducible supercuspidal π, then (L,σ) is inertially equivalent to (G,π), and irreducible representations in the Bernstein component $R^{(G,\pi)}(G)$
$$\{\pi \otimes (\eta \circ \det) : \eta \text{ is unramified}\}$$
2. If $R^{(L,\sigma)(G)}$ contains $\pi = \mathrm{St}(\chi) = \mathrm{St}_G \otimes (\chi \circ \det)$, then $(L,\sigma) \sim (T, \chi \otimes \chi)$, and irreducible representations in $R^{(T,\chi)}(G)$ are
$$\{\mathrm{St}_G \otimes (\chi\eta \circ \det)\} \cup \{\, (\eta_1\chi|\cdot|^{-1/2}) \times (\eta_2\chi|\cdot|^{1/2})\,\}$$
where η, η_1, η_2 range over unramified characters (such that the latter remains a principal series)
3. If $\pi = \pi(\chi_1, \chi_2)$ where $\chi_1\chi_2^{-1}$ is ramified (otherwise we are in the previous case), then the irreducible representations in this Bernstein component are
$$\{\, (\eta_1\chi_1) \times (\eta_2\chi_2)\,\}$$
where η_1, η_2 are unramified.

Remark. Up to multiplying by unramified characters, the supercuspidal support corresponding to each Bernstein component in the example above are:

(1) $R^{(G,\pi)}(G)$ correspondes to $\{\pi\}$
(2) $R^{(T,\chi\otimes\chi)}(G)$ correspondes to $\{\chi,\chi\}$
(3) $R^{(T,\chi_1\otimes\chi_2)}(G)$ correspondes to $\{\chi_1,\chi_2\}$, here $\chi_1\chi_2^{-1}$ is ramified.

4.1.3. *The supercuspidal part of a smooth representation.*

4.1.4. *Proof of the Bernstein decomposition.*

4.2. **Inertial correspondences.**

4.2.1. *Weil-Deligne inertia type.*
First we need a standard lemma in (finite dimensional) representation theory:

Lemma 4.3. *Let $H \lhd G$ be a normal subgroup such that G/H is abelian, and let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible finite dimensional representations over $\mathbb{C}$ (or any algebraically closed field) such that $\rho_1|_H \cong \rho_2|_H$, then there exists a character $\psi : G \to G/H \to \mathbb{C}^\times$ such that $\rho_2 \cong \rho_1 \otimes \chi$.*

Sketch. Consider $\mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ as a vector space which receives an action from G, since $\rho_1|_H \cong \rho_2|_H$, we know that

$$W := \mathrm{Hom}_H(\rho_1,\rho_2) = (\mathrm{Hom}_{\mathbb{C}}(V_1,V_2))^H \neq 0$$

Now $W \subset \mathrm{Hom}_{\mathbb{C}}(V_1, V_2)$ is a subrepresentation of G where H acts trivially, so the abelian quotient G/H acts W, which admits a 1-dimensional sub-representation of G:

$$\psi \hookrightarrow W \subset V_1^\vee \otimes V_2.$$

In particular, we may twist by ψ to obtain

$$\mathrm{triv} \hookrightarrow (V_1 \otimes \psi)^\vee \otimes V_2$$

which implies

$$\mathrm{triv} \hookrightarrow ((V_1 \otimes \psi)^\vee \otimes V_2)^G = \mathrm{Hom}_G(V_1 \otimes \psi, V_2) \neq 0.$$

Therefore, by Schur's lemma, $\rho_1\psi \cong \rho_2$. □

Remark. In fact, we have proven a slightly stronger claim, which is needed below. Namely, in the setup above, if ρ_1 and ρ_2 are both irreducible, and suppose that $\mathrm{Hom}_H(\rho_1,\rho_2) \neq 0$, then the same conclusion holds. Namely in the lemma we *a priori* de not need to assume that $\rho_1|_H \cong \rho_2|_H$.

Lemma 4.4. *Two irreducible admissible representations π and π' of G belong to the same Bernstein component $R^{(L,\sigma)}(G)$ if and only if the underlying W_K representations (of the Weil-Deligne representations) from LLC satisfy*

$$\mathrm{rec}_n(\pi)|_{I_K} \xrightarrow{\sim} \mathrm{rec}_n(\pi')|_{I_K}.$$

In other words, the Bernstein center $B(\mathrm{GL}_n)$ is indexed by **Weil-Deligne inertia types**.

Proof. By Bernstein-Zelevinsky classification,

$$\pi = Q(\Delta_1, ..., \Delta_r)$$

with $\Delta_i = \Delta_i(\pi_i, m_i)$ the Zelevinsky segments, so π belongs to the Bernstein component with

$$L = \prod_{i=1}^{r} \mathrm{GL}_{n_i}(K)^{m_i}, \quad \sigma = \bigotimes_{i=1}^{r} \bigotimes_{j=0}^{m_i-1} \pi_{i,j}$$

where $\pi_{i,j}$ are the supercuspidal representations $\pi_i|\cdot|^j$ of $\mathrm{GL}_{n_i}(K)$. By the local Langlands correspondence,

$$\mathrm{rec}_n(\pi) = \bigoplus_{i=1}^{r} \mathrm{Sp}_{m_i}(\mathrm{rec}_{n_i}(\pi_i)),$$

and the underlying W_K representation restricted to I_K is

$$\mathrm{rec}_n(\pi)|_{I_K} = \bigoplus_{i=1}^{r} \mathrm{rec}_{n_i}(\pi_i)^{m_i}|_{I_K}.$$

Therefore we are reduced to the supercuspidal cases, namely two supercuspidal representations π and π' of $\mathrm{GL}_k(K)$ lie in the same Bernstein component (they differ by a twist of an unramified character η), if and only if $\mathrm{rec}(\pi)|_{I_K} \cong \mathrm{rec}(\pi')|_{I_K}$, but this follows from Lemma 4.3. Note that we have implicitly used the remark after Lemma 4.3 in this last step: suppose that

$$\bigoplus_{i=1}^{r} \mathrm{rec}_{n_i}(\pi_i)^{m_i}|_{I_K} \cong \bigoplus_{i=1}^{r'} \mathrm{rec}_{n_i'}(\pi_i')^{m_i'}|_{I_K},$$

we can (successively) apply the remark to show that $\mathrm{rec}_{n_i}(\pi_i) \otimes \psi_i \cong \mathrm{rec}_{n_i}(\pi_i')$ for some unramified character ψ_i. □

4.2.2. *Galois inertia types and generic representations.*

Let $\mathtt{K} = \mathrm{GL}_n(\mathcal{O}_K)$ be the standard maximal compact open subgroup of $\mathrm{GL}_n(K)$.

Lemma 4.5. *Let π, π' be two generic irreducible admissible representations of $\mathrm{GL}_n(K)$ such that, its corresponding l-adic representations satisfy*

$$\mathrm{r}_n(\pi)|_{I_K} \cong \mathrm{r}_n(\pi')|_{I_K},$$

where $\mathrm{r}_n = \mathrm{WD}^{-1}\mathrm{rec}_n$. Then

$$\pi|_{\mathtt{K}} \cong \pi'|_{\mathtt{K}}.$$

In other words, if two generic representations have the same **Galois inertia type***, then they restrict to isomorphic representations on $\mathtt{K}$.*

Proof. As a consequence of the Iwasawa decomposition $\mathrm{GL}_n(K) = B \cdot \mathrm{GL}_n(\mathcal{O}_K)$ where B is the Borel subgroup consisting of upper triangular matrices, and Mackay's decomposition of restriction of induced representations, we know that for any (standard) parabolic $P \subset \mathrm{GL}_n(K)$ and a representation σ of P,

$$(\mathrm{Ind}_P^{\mathrm{GL}_n(K)} \sigma)|_{\mathtt{K}} = \mathrm{Ind}_{P\cap \mathtt{K}}^{\mathtt{K}}(\sigma|_{P\cap\mathtt{K}}).$$

Since π is generic, by Theorem 3.1, $\pi = Q(\Delta_1) \times \cdots \times Q(\Delta_r)$ and $\mathrm{rec}_n(\pi) = \bigoplus_{i=1}^r \mathrm{rec}_{n_i m_i}(Q(\Delta_i))$ and similarly for π'. Two representations of W_K

$$\bigoplus_{i=1}^{r} \mathrm{Sp}_{m_i}(\rho_i), \quad \bigoplus_{i=1}^{r'} \mathrm{Sp}_{m'_i}(\rho'_i)$$

have the same Galois inertia type if and only if $r = r'$ and up to reordering $m_i = m'_i$ and $\rho_i|_{I_K} \cong \rho'_i|_{I_K}$, by Lemma 1.9.

Now it suffices to prove the lemma under the assumption that $\pi = Q(\Delta(\lambda, m))$ and $\pi' = Q(\Delta(\lambda', \pi))$, where λ and λ' are supercuspidal representations of $\mathrm{GL}_d(K)$. Note that $\lambda = \lambda' \otimes \eta$, where η is an unramified character of $\mathrm{GL}_d(K)$. The character η is of the form $\eta = \chi \circ \det$ where χ is an unramified quasi-character $K^\times \to \mathbb{C}^\times$. Therefore, $\mathrm{rec}_n(\pi) = \mathrm{rec}_n(\pi') \otimes \mathrm{rec}_1(\chi)$, from which it follows that $\pi = \pi'\chi$. □

Remark. The generic condition is needed. For example, for $\mathrm{GL}_2(K)$, triv and St_2 lie in the same Bernstein component, but they have different restrictions to $\mathtt{K} = \mathrm{GL}_n(\mathcal{O}_K)$.

4.3. **Types.**

Finally, we briefly discuss the theory of types.

4.3.1. *Standard definitions.*

Definition. Let σ be a smooth irreducible representation of $\mathtt{K}$, let $s = (L, \varsigma) \in B(\mathrm{GL}_n)$ be an element in the Bernstein center.

(1) σ is a type for (L, ς) if for all smooth irreducible representations $\pi \in \mathrm{Rep}_n^\infty$,
$$\pi|_{\mathtt{K}} \text{ contains } \sigma \Leftrightarrow \pi \in R^{(L,\varsigma)}.$$
(2) σ is a $\mathtt{K}$-type for (L, ς) if for all smooth irreducible representations π,
$$\pi|_{\mathtt{K}} \text{ contains } \sigma \Rightarrow \pi \in R^{(L,\varsigma)}.$$
(3) Let $\pi \in R^{(L,\varsigma)}$ be a smooth irreducible representation in $R^{(L,\varsigma)}$, a minimal $\mathtt{K}$-type for π is a $\mathtt{K}$-type σ for (L, ς) of minimal dimension occurring in $\pi|_{\mathtt{K}}$.

Note that each type is trivially a $\mathtt{K}$-type, and both notions only depend on the pair $(L, \varsigma) \in B(\mathrm{GL}_n)$, while the definition of a minimal $\mathtt{K}$-type for π depends on the particular $\pi \in R^{(L,\varsigma)}$. To wit, there could be many $\mathtt{K}$-types for a given $s = (L, \varsigma)$; for each irreducible $\pi \in R^{(L,\varsigma)}$, a $\mathtt{K}$-type σ may or may not appear in $\pi|_{\mathtt{K}}$. Among those which appear in $\pi|_{\mathtt{K}}$, the minimal dimensional ones are the minimal $\mathtt{K}$-types for π.

Remark. Our definition of type agrees with the one given in [9], which is slightly different from [4]. We refer to latter notion as Bushnell-Kutzko types, which is a pair (J, σ) where J is a compact open subgroup and σ a smooth irreducible representation of J – namely it allows more general compact open subgroups than $\mathtt{K} = \mathrm{GL}_n(\mathcal{O}_K)$.

5. Carayol's program on cohomological realizations

6. Families of local Langlands correspondence

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