CONSTRUCTIBLE SHEAVES ON SCHEMES AND A CATEGORICAL KÜNNETH FORMULA

TAMIR HEMO, TIMO RICHARZ AND JAKOB SCHOLBACH

ABSTRACT. We present a uniform theory of constructible sheaves on arbitrary schemes with coefficients in condensed rings. We use it to prove a Künneth-type equivalence of derived categories of lisse and constructible Weil sheaves on schemes in characteristic $p > 0$ for a wide variety of coefficients, including finite discrete rings, algebraic field extensions $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ and their rings of integers $\mathcal{O}_E$. When combined with results of Xue, this applies to the cohomology of moduli stacks of shtukas over global function fields.

CONTENTS

1. INTRODUCTION

1.1. **Constructible sheaves on schemes.** Initially, étale cohomology of a scheme X only works well for sheaves with torsion coefficients, such as $\mathbb{Z}/n$. In order to have a cohomology theory producing vector spaces over characteristic zero fields, one uses ℓ-adic sheaves [SGA77, Exposé VI] whose cohomology groups are $\mathbb{Q}_\ell$-vector spaces. The name "sheaves" is fully justified with the advent of the proétale topology introduced in [BS15]: such ℓ-adic sheaves are ordinary topos-theoretic sheaves on the site $X_{\text{proét}}$. This is decidedly more conceptual than the classical approach which builds ℓ-adic sheaves by retracing the formula $\mathbb{Q}_\ell = (\lim_n \mathbb{Z}/\ell^n)[\ell^{-1}]$ on the level of sheaves.

For many purposes, it is useful to impose a finiteness condition on sheaves known as constructibility. This is accomplished in [BS15] for adic coefficient rings like $\mathbb{Z}_\ell = \lim \mathbb{Z}/\ell^n$. Their category satisfies proétale descent and compares well to [SGA77, Del80, Eke90]. If the underlying topological space of X is Noetherian, [BS15] also defines a category of constructible $\mathbb{Q}_\ell$-sheaves. However, the finiteness condition on X prevents one from formulating descent on $X_{\text{proét}}$: typical proétale covers $\{U_i \to X\}$ have a profinite set of connected components and so are not topologically Noetherian. Examples show that the category of proétale locally constant sheaves is not well behaved in general. This prevents an obvious generalization of constructible $\mathbb{Q}_\ell$-sheaves to such large schemes.

Here we introduce a notion of lisse and constructible sheaves on arbitrary schemes X with coefficients in an arbitrary condensed (unital, commutative) ring Λ. Recall from [CS] (see also [BH19]) that a condensed ring is a sheaf of rings on the site $*_{\text{proét}}$ of profinite sets. Examples abound, since every T1-topological ring naturally gives rise to a condensed ring via the Yoneda embedding. Let $\Lambda_* = \Gamma(*, \Lambda)$ be the underlying ring.

The pullback of Λ along the canonical map of sites $X_{\text{proét}} \to *_{\text{proét}}$ is a sheaf of rings on $X_{\text{proét}}$. We denote by $D(X, \Lambda)$ the derived category of Λ-sheaves. This is a Λ_*-linear symmetric monoidal closed stable ∞-category (so its homotopy category is a triangulated category). Recall the notion of dualizable objects in symmetric monoidal categories [Lura, Section 4.6.1]. For example, the dualizable objects in the derived category of Λ_*-modules are the perfect complexes, that is, bounded complexes of finite projective Λ_*-modules.

Definition 1.1. Let X be a scheme, and let Λ be a condensed ring.

(1) A sheaf $M \in D(X, \Lambda)$ is called *lisse* if it is dualizable.
(2) A sheaf $M \in D(X, \Lambda)$ is called *constructible* if for every open affine subscheme $U \subset X$ there exists a finite subdivision $U_i \subset U$ into constructible locally closed subschemes such that each $M|_{U_i}$ is lisse.

We denote by $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ the full subcategories of $\mathrm{D}(X, \Lambda)$ of lisse, respectively constructible sheaves. By the setup, these are naturally Λ_*-linear symmetric monoidal stable ∞-categories. This is a technically important prerequisite for stating the categorical Künneth formula, see Theorem 1.12.

Recall that every scheme admits a proétale cover $\{U_i \to X\}$ where the U_i are w-contractible affine schemes. Loosely speaking, these objects form a basis of the topology on $X_{\mathrm{proét}}$. The following key lemma makes the categories of lisse and constructible sheaves manageable:

Lemma 1.2 (Lemma 3.7). *Assume that X is w-contractible affine. Then the global sections functor induces an equivalence of categories*

$$\mathrm{R}\Gamma(X, \text{-})\colon \mathrm{D}_{\mathrm{lis}}(X, \Lambda) \xrightarrow{\cong} \mathrm{Perf}_{\Gamma(X, \Lambda)},$$

where the target is the category of perfect complexes of $\Gamma(X, \Lambda)$-modules up to quasi-isomorphism.

Even for $X = *$, $\Lambda = \mathbb{Z}$, this is noteworthy: the derived category of condensed abelian groups $\mathrm{D}(*, \mathbb{Z})$ contains fully faithfully the category of compactly generated T1-topological abelian groups, and is thus much larger than the category $\mathrm{D}_{\mathrm{lis}}(*, \mathbb{Z}) \cong \mathrm{Perf}_{\mathbb{Z}}$, which consists of bounded complexes of finitely generated free abelian groups equipped with the discrete topology.

Lemma 1.3 (Corollary 3.13). *The functors $U \mapsto \mathrm{D}_{\mathrm{lis}}(U, \Lambda), \mathrm{D}_{\mathrm{cons}}(U, \Lambda)$ are hypersheaves of ∞-categories on $X_{\mathrm{proét}}$.*

Together with Lemma 1.2, this says that $U \mapsto \mathrm{D}_{\mathrm{lis}}(U, \Lambda)$ is the unique hypersheaf on $X_{\mathrm{proét}}$ whose values at the basis of w-contractible affines compute the category $\mathrm{Perf}_{\Gamma(U, \Lambda)}$. It can be used to relate lisse sheaves to the condensed shape (Proposition A.1) which is related to the stratified shape developed in [BGH20].

By the above definition, lisse and constructible sheaves start out life in a derived setting. The natural t-structure on $\mathrm{D}(X, \Lambda)$ restricts to a t-structure on such sheaves only under additional assumptions on X and Λ:

Definition 1.4. A condensed ring Λ is called *t-admissible* if Λ_* is regular coherent (that is, every finitely generated ideal is finitely presented and has finite projective dimension) and, for any extremally disconnected profinite set S, the map $\Lambda_* \to \Gamma(S, \Lambda)$ is flat.

In Section 3.6, we show that Λ is t-admissible if and only if the t-structure on $\mathrm{D}(*, \Lambda)$ restricts a t-structure on $\mathrm{D}_{\mathrm{lis}}(*, \Lambda)$. Examples of t-admissible condensed rings include discrete rings that are regular Noetherian of finite Krull dimension, and all T1-topological rings such that Λ_* is semi-hereditary (=every finitely generated ideal is projective). This covers algebraic field extensions $E \supset \mathbb{Q}_\ell$ and their rings of integers $\mathcal{O}_E$, but also more exotic choices such as the real and complex numbers $\mathbb{R}, \mathbb{C}$ with their Euclidean topology and the ring of adeles $\mathbb{A}_K^T$ prime to some finite set of places T in some number field K. In view of Remark 3.30, the following seems close to optimal:

Theorem 1.5 (Theorem 3.28). *Let Λ be a t-admissible condensed ring. Assume that X has locally a finite number of irreducible components, respectively that locally every constructible subset has so (for example, X locally Noetherian). Then the natural t-structure on $\mathrm{D}(X, \Lambda)$ restricts to a t-structure on $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, respectively on $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$.*

The following result allows for the comparison with the categories of lisse and constructible sheaves as in [Del77, SGA77, Del80, Eke90, BS15, Sta17, GL19]:

Theorem 1.6. *Let X be a scheme, and let Λ be a condensed ring.*

(1) *(Proposition 3.40) For a discrete topological ring Λ, the category $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ is equivalent to the full subcategory of $\mathrm{D}(X_{\mathrm{ét}}, \Lambda)$ of complexes that are étale-locally perfect complexes of Λ_*-modules. Consequently, $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is equivalent to the resulting category of étale constructible sheaves of Λ-modules.*

(2) *(Proposition 3.19) Assume that $\Lambda = \lim \Lambda_n$ is a sequential limit of condensed rings such that the transition maps are surjective with locally nilpotent kernel. Then the natural functors*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \xrightarrow{\cong} \lim \mathrm{D}_{\mathrm{lis}}(X, \Lambda_n), \quad \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \xrightarrow{\cong} \lim \mathrm{D}_{\mathrm{cons}}(X, \Lambda_n),$$

are equivalences.

(3) *(Proposition 3.23) Assume that $T \subset \Lambda_*$ is a multiplicatively closed subset. If X is quasi-compact and quasi-separated (qcqs), then the natural functors*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{Perf}_{T^{-1}\Lambda_*} \to \mathrm{D}_{\mathrm{lis}}\left(X, T^{-1}\Lambda\right), \quad \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{Perf}_{T^{-1}\Lambda_*} \to \mathrm{D}_{\mathrm{cons}}\left(X, T^{-1}\Lambda\right)$$

are fully faithful.

(4) *(Proposition 3.20) Assume that $\Lambda = \operatorname{colim} \Lambda_i$ is a filtered colimit of condensed rings. If X is qcqs, then the natural functors*

$$\operatorname{colim} \mathrm{D}_{\mathrm{lis}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \operatorname{colim} \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$$

are equivalences.

(5) (Theorem 3.26) Assume that X has locally a finite number of irreducible components. Then $M \in \mathrm{D}(X, \Lambda)$ is lisse if and only if M is locally on $X_{\text{proét}}$ isomorphic to $\underline{N} \otimes_{\Lambda_*} \Lambda_X$ for some perfect complex of Λ_*-modules N, where the underline denotes the associated constant sheaf on $X_{\text{proét}}$.

Part (2) applies to adic topological rings $\Lambda = \lim \Lambda/I^n$ such as the ℓ-adic integers $\mathbb{Z}_\ell = \lim \mathbb{Z}/\ell^n$. The key computation for the categories of perfect complexes is [Bha16, Lemma 4.2]. Together with (1), this also shows that $\mathrm{D}_{\text{cons}}(X, \Lambda)$ is equivalent to the full subcategory of $\mathrm{D}(X, \Lambda)$ of I-adically complete objects M such that $M \otimes \Lambda/I$ is étale constructible as in [BS15, Section 6]. For algebraic field extensions $E \supset \mathbb{Q}_\ell$, one easily deduces from (5) that the categories $\mathrm{D}_{\text{cons}}(X, E)$, $\mathrm{D}_{\text{cons}}(X, \mathcal{O}_E)$ agree with the categories defined in [BS15, Definition 6.8.8] whenever X is topologically Noetherian. See, however, Example 3.6 for a lisse sheaf on a profinite set that is not proétale-locally perfect-constant. In conclusion, the above result extends the previously known approaches.

The functor in (3) is not an equivalence for lisse sheaves in general. This relates to the difference between the étale versus the proétale fundamental group, see Remark 3.25 and Theorem 3.45 for a positive result for constructible sheaves. Part (4) gives the comparison of the category of constructible sheaves with coefficients in $\overline{\mathbb{Q}}_\ell = \mathrm{colim}\, E$ or $\overline{\mathbb{Z}}_\ell = \mathrm{colim}\, \mathcal{O}_E$ where the colimits run through finite field extensions $E \supset \mathbb{Q}_\ell$.

Motivated by [Laf18a], we also consider inductive systems of lisse, respectively constructible sheaves. For qcqs schemes X of finite Λ-cohomological dimension (see Section 3.8 for details), we introduce the following notion:

Definition 1.7. A sheaf $M \in \mathrm{D}(X, \Lambda)$ is called *ind-lisse*, respectively *ind-constructible* if it is equivalent to a filtered colimit of lisse, respectively constructible sheaves.

We denote by $\mathrm{D}_{\text{indlis}}(X, \Lambda) \subset \mathrm{D}_{\text{indcons}}(X, \Lambda)$ the resulting full subcategories of $\mathrm{D}(X, \Lambda)$. These categories satisfy étale descent, see Corollary 3.55. Examples of pairs (X, Λ) satisfying the cohomological finiteness assumption include schemes X of finite type over finite and separably closed fields with coefficients Λ being a discrete torsion ring, an algebraic field extension $E \supset \mathbb{Q}_\ell$, or its ring of integers $\mathcal{O}_E$, see Lemma 3.54.

Proposition 1.8 (Proposition 3.50, Corollary 3.51)**.** *Let X be a qcqs scheme of finite Λ-cohomological dimension. Then an object $M \in \mathrm{D}_{\text{indcons}}(X, \Lambda)$ is compact if and only if M is constructible, and likewise for (ind-)lisse sheaves. Consequently, the Ind-completion functor induces equivalences*

$$\mathrm{Ind}\big(\mathrm{D}_{\text{lis}}(X, \Lambda)\big) \xrightarrow{\cong} \mathrm{D}_{\text{indlis}}(X, \Lambda), \quad \mathrm{Ind}\big(\mathrm{D}_{\text{cons}}(X, \Lambda)\big) \xrightarrow{\cong} \mathrm{D}_{\text{indcons}}(X, \Lambda).$$

Combined with Theorem 1.6 (1) and [BS15, Section 6.4], the proposition shows that $\mathrm{D}_{\text{indcons}}(X, \Lambda) \cong \mathrm{D}(X_{\text{ét}}, \Lambda)$ for discrete rings Λ and qcqs schemes X of finite Λ-cohomological dimension.

Remark 1.9. Another motivation for this work is the ℓ-adic realization functor for motives. As explained in [CD16], this functor is in essence a completion functor. It seems an interesting question whether it can be expressed as a scalar extension functor for a yet-to-be-defined category of motives with condensed coefficients.

1.2. **A categorical Künneth formula for Weil sheaves.** Künneth-type formulas relate the (co-)homology of product spaces to the tensor product of the (co-)homology of the single factors. In [GKRV20, Section A.2], this is upgraded to an equivalence of the respective categories of topological sheaves and D-modules on varieties in characteristic 0. In characteristic $p > 0$, such categorical Künneth formulas fail for classical étale sheaves, in fact, already for étale fundamental groups: for example, the natural map $\pi_1(\mathbb{A}^2) \to \pi_1(\mathbb{A}^1) \times \pi_1(\mathbb{A}^1)$ induced from the projections $\mathbb{A}^2 \to \mathbb{A}^1$ of the affine plane onto its coordinate axes is not an isomorphism (not injective) over any algebraically closed field of characteristic $p > 0$, see Example 1.13. This failure of the Künneth formula for π_1 can be rectified by introducing equivariance data under Frobenius morphisms [Dri80, Theorem 2.1]. When adding such equivariance data in a sheaf-theoretic context, one arrives at the notion of Weil sheaves [Del80, Definition 1.1.10]. Before stating the main result in Theorem 1.12, the categorical Künneth formula for Weil sheaves, we explain a site-theoretic approach to such sheaves which slightly differs from [Gei04, Lic05].

Let X be a scheme over a finite field $\mathbb{F}_q$, where q is a p-power. Fix an algebraic closure $\mathbb{F}/\mathbb{F}_q$, and denote by $X_{\mathbb{F}}$ the base change. The partial (q-)Frobenius $\phi_X := \mathrm{Frob}_X \times \mathrm{id}_{\mathbb{F}}$ defines an endomorphism of $X_{\mathbb{F}}$.

Definition 1.10. The *Weil-proétale site* $X_{\text{proét}}^{\text{Weil}}$ is the following site: Objects are pairs (U, φ) consisting of $U \in (X_{\mathbb{F}})_{\text{proét}}$ equipped with an endomorphism $\varphi \colon U \to U$ of $\mathbb{F}$-schemes covering ϕ_X. Morphisms are given by equivariant maps. A family $\{(U_i, \varphi_i) \to (U, \varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\text{proét}}$.

The Weil-proétale site sits in the sequence of sites

$$(X_{\mathbb{F}})_{\text{proét}} \to X_{\text{proét}}^{\text{Weil}} \to X_{\text{proét}}$$

given by the functors $U \leftarrowtail (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrowtail U$. Thus, for any condensed ring Λ, we get pullback functors on derived categories of Λ-sheaves

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}\big(X^{\text{Weil}}, \Lambda\big) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda).$$

In analogy with Definition 1.1, we introduce the categories of lisse and constructible Weil sheaves $\mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda)$ as the full subcategories of $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ that are dualizable, respectively that are Zariski locally on X dualizable along a constructible stratification. These categories are equivalent to the corresponding categories of sheaves on the prestack $X_{\mathbb{F}}/\phi_X$, that is, equivalent to the homotopy fixed points of the induced ϕ_X^*-action:

Proposition 1.11 (Proposition 4.4, Proposition 4.11). *The pullback of sheaves along* $(X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}} \to X^{\mathrm{Weil}}_{\mathrm{pro\acute{e}t}}$ *induces an equivalence of* Λ_*-*linear symmetric monoidal stable* ∞-*categories*

$$\mathrm{D}_{\bullet}(X^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\bullet}(X_{\mathbb{F}}, \Lambda)^{\phi_X^* = \mathrm{id}},$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

Thus, objects in $\mathrm{D}_{\bullet}(X^{\mathrm{Weil}}, \Lambda)$ are pairs (M, α) with $M \in \mathrm{D}_{\bullet}(X_{\mathbb{F}}, \Lambda)$ and $\alpha \colon M \cong \phi_X^* M$. On the abelian level, we recover the classical approach [Del80, Definition 1.1.10]. If Λ is a finite discrete ring, then every Weil descent datum on constructible Λ-sheaves is effective so that $\mathrm{D}_{\mathrm{cons}}(X^{\mathrm{Weil}}, \Lambda) \cong \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$, see Proposition 4.15. However, the categories are not equivalent if $\Lambda = \mathbb{Z}, \mathbb{Z}_{\ell}, \mathbb{Q}_{\ell}$, say. This relates to the difference between continuous representations of Galois groups such as $\hat{\mathbb{Z}}$ versus Weil groups such as $\mathbb{Z}$.

For several $\mathbb{F}_q$-schemes $X_1, \ldots, X_n$, a similar process is carried out for their product $X := X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$ equipped with the partial Frobenii $\phi_{X_i} \colon X_{\mathbb{F}} \to X_{\mathbb{F}}$, see Section 4.2. Generalizing Proposition 1.11, there is an equivalence of Λ_*-linear symmetric monoidal stable ∞-categories

$$\mathrm{D}_{\bullet}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\bullet}(X_{\mathbb{F}}, \Lambda)^{\phi_{X_1}^* = \mathrm{id}, \ldots, \phi_{X_n}^* = \mathrm{id}} \tag{1.1}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$. The category on the left is defined using the Weil-proétale site $(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}})_{\mathrm{pro\acute{e}t}}$ consisting of objects $(U, \varphi_1, \ldots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\mathrm{pro\acute{e}t}}$ and pairwise commuting endomorphisms $\varphi_i \colon U \to U$ covering the partial Frobenii $\phi_{X_i} \colon X_{\mathbb{F}} \to X_{\mathbb{F}}$ for all $i = 1, \ldots, n$. The category on the right is the category of simultaneous homotopy fixed points, see Section 2.3. For constructible Weil sheaves, (1.1) relies on decompositions of partial Frobenius invariant cycles in $X_{\mathbb{F}}$, see Proposition 4.8.

The following result is referred to as the categorical Künneth formula for Weil sheaves (or, derived Drinfeld's lemma):

Theorem 1.12 (Theorem 5.2, Remark 5.3). *Let* $\mathbb{F}_q$ *be a finite field of characteristic* $p > 0$. *Let* $X_1, \ldots, X_n$ *be finite type* $\mathbb{F}_q$-*schemes. Let* Λ *be either a finite discrete ring of prime-to-p-torsion, or an algebraic field extension* $E \supset \mathbb{Q}_{\ell}$, $\ell \neq p$, *or its ring of integers* $\mathcal{O}_E$.

Then the external tensor product of sheaves $(M_1, \ldots, M_n) \mapsto M_1 \boxtimes \ldots \boxtimes M_n$ *induces an equivalence*

$$\mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda). \tag{1.2}$$

Likewise, for the categories of lisse Weil sheaves if, in the case $\Lambda = E$, *one assumes the schemes* $X_1, \ldots, X_n$ *to be geometrically unibranch (for example, normal).*

The tensor product of ∞-categories (see Section 2.1) is formed using the natural Λ_*-linear structures on the categories. We even have an analogous equivalence for the categories of lisse Weil sheaves with coefficients Λ in finite discrete p-torsion rings like $\mathbb{Z}/p^m$, $m \geq 1$, see Theorem 5.2. As the following example shows, the use of Weil sheaves is necessary for the essential surjectivity to hold. This behavior is mentioned in the first arXiv version of [GKRV20, Equation (0.8)] which is one of the main motivations for our work.

Example 1.13 (compare [SGA03, Exposé X, §1, Remarques 1.10]). Let $X_{1,\mathbb{F}} = X_{2,\mathbb{F}} = \mathbb{A}^1_{\mathbb{F}}$ be the affine line so that $X_{\mathbb{F}} = \mathbb{A}^2_{\mathbb{F}}$ with coordinates denoted by x_1 and x_2. Then

$$U := \{t^p - t = x_1 \cdot x_2\} \longrightarrow \mathbb{A}^2_{\mathbb{F}}$$

defines a finite étale cover with Galois group $\mathbb{Z}/p$. Let $M \in \mathrm{D}_{\mathrm{lis}}(\mathbb{A}^2_{\mathbb{F}}, \Lambda)$ be the sheaf in degree 0 associated with some non-trivial character $\mathbb{Z}/p \to \Lambda_*^{\times}$. For $\lambda, \mu \in \mathbb{F}$ not differing by a scalar in $\mathbb{F}_p^{\times}$, the fibers $U|_{\{x_1 = \lambda\}}$, $U|_{\{x_1 = \mu\}}$ are not isomorphic over $\mathbb{A}^1_{\mathbb{F}}$ by Artin-Schreier theory. Hence, $M \not\cong \phi_{X_i}^* M$ and one can show that $M \not\cong M_1 \boxtimes M_2$ for any $M_i \in \mathrm{D}(\mathbb{A}^1_{\mathbb{F}}, \Lambda)$.

If Λ as above is p-torsion free, then the full faithfulness of (1.2) is a direct consequence of the Künneth formula for $X_{i,\mathbb{F}}$, $i = 1, \ldots, n$. For $\Lambda = \mathbb{Z}/p^m$, we use Artin–Schreier theory instead. It would be interesting to see whether the lisse p-torsion case can be extended to constructible sheaves, say, using the mod-p-Riemann–Hilbert correspondence [BL19]. In both cases, the essential surjectivity relies on a variant of Drinfeld's lemma [Dri80, Theorem 2.1] (see also [Laf97, IV.2, Theorem 4], [Lau04, Theorem 8.1.4], [Laf18a, Lemme 8.11], and [Ked19, Theorem 4.2.12], [Hei18, Lemma 6.3], [SW18, Theorem 16.2.4] for expositions) for Weil group representations, see Theorem 5.6.

With a view towards [Laf18a], we consider Weil sheaves whose underlying sheaf is ind-constructible, but where the action of the partial Frobenii do not necessarily preserve the constructible pieces. For finite type $\mathbb{F}_q$-schemes $X_1, \ldots, X_n$ and Λ as in Theorem 1.12, we consider the category of simultaneous homotopy fixed points

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \overset{\mathrm{def}}{=} \mathrm{D}_\bullet(X_{\overline{\mathbb{F}}}, \Lambda)^{\phi_{X_1}^* = \mathrm{id}, \ldots, \phi_{X_n}^* = \mathrm{id}}$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$. Then the external tensor product induces a fully faithful functor

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet\big(X_n^{\mathrm{Weil}}, \Lambda\big) \longrightarrow \mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big). \tag{1.3}$$

Unlike the case of lisse or constructible sheaves, the functor is not essentially surjective as one can add freely actions by the partial Frobenii, see Remark 6.5. However, we can identify a large class of objects in the essential image of (1.3). When combined with the recent results [Xue20c, Theorem 4.2.3], we obtain, for example, that the compactly supported cohomology of moduli stacks of shtukas over global function fields lies in the essential image of (1.3), see Section 6.2 for details.

Remark 1.14. Another motivation for this work is our (T.R. and J.S.) ongoing project aiming for a motivic refinement of [Laf18a]. In this project, we will need a motivic variant of Drinfeld's lemma. Since triangulated categories of motives such as $\mathrm{DM}(X, \mathbb{Q})$ carry t-structures only conditionally, we need a Drinfeld lemma to be a statement about triangulated categories. In conjunction with the conjecture relating Weil-étale motivic cohomology to Weil-étale cohomology [Kah03, Gei04, Lic05], our results suggest to look for a Drinfeld lemma for constructible Weil motives.

2. Prelude on ∞-categories

Throughout this section, Λ denotes a unital, commutative ring. We briefly collect some notation pertaining to ∞-categories from [Lura, Lur09]. As in [Lur09, Section 5.5.3], Pr^{L} denotes the ∞-category of presentable ∞-categories with colimit-preserving functors. It contains the subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories.

2.1. **Monoidal aspects.** The category Pr^{L} carries the Lurie tensor product [Lura, Section 4.8.1]. This tensor product induces one on the full subcategory $\mathrm{Pr}^{\mathrm{St}} \subset \mathrm{Pr}^{\mathrm{L}}$ consisting of stable ∞-categories [Lura, Proposition 4.8.2.18]. For our commutative ring Λ, the ∞-category Mod_Λ of chain complexes of Λ-modules, up to quasi-isomorphism, is a commutative monoid in $\mathrm{Pr}^{\mathrm{St}}$ with respect to this tensor product. This structure includes, in particular, the existence of a functor

$$\mathrm{Mod}_\Lambda \times \mathrm{Mod}_\Lambda \to \mathrm{Mod}_\Lambda$$

which, after passing to the homotopy categories is the classical *derived* tensor product on the unbounded derived category of Λ-modules.

We define $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ to be the category of modules, in $\mathrm{Pr}^{\mathrm{St}}$, over Mod_Λ. This category is denoted $\mathrm{DGCat}_{\mathrm{cont}, \Lambda}$ in [GR17, Chapter 1, Section 10.3], and we will freely use results from there. Noting that modules over Mod_Λ are in particular modules over Sp, the ∞-category of spectra, this can be described as the ∞-category consisting of *stable* presentable ∞-categories together with a Λ-linear structure, and such that functors are continuous and Λ-linear. Therefore $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ carries a symmetric monoidal structure, whose unit is Mod_Λ.

In order to express monoidal properties of ∞-categories consisting, say, of bounded complexes, recall from [Lura, Corollary 4.8.1.4 joint with Lemma 5.3.2.11] or [BZFN10, Proposition 4.4] the symmetric monoidal structure on the ∞-category $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ of idempotent complete stable ∞-categories and exact functors: it is characterized by

$$D_1 \otimes D_2 \overset{\mathrm{def}}{=} \big(\mathrm{Ind}(D_1) \otimes \mathrm{Ind}(D_2)\big)^\omega, \tag{2.1}$$

that is, the compact objects in the Lurie tensor product of the Ind-completions. With respect to these monoidal structures, the Ind-completion functor (taking values in compactly generated presentable ∞-categories with the Lurie tensor product) and the functor forgetting the compact generatedness

$$\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}) \overset{\mathrm{Ind}}{\underset{\cong}{\longrightarrow}} \mathrm{Pr}^{\mathrm{St}}_\omega \longrightarrow \mathrm{Pr}^{\mathrm{St}} \tag{2.2}$$

are both symmetric monoidal [Lura, Lemmas 5.3.2.9, 5.3.2.11].

The subcategory of compact objects in Mod_Λ is given by perfect complexes of Λ-modules [Lura, Proposition 7.2.4.2.]. It is denoted Perf_Λ. Under the equivalence in (2.2), the category $\mathrm{Perf}_\Lambda \in \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ corresponds to Mod_Λ. Moreover, Perf_Λ is a commutative monoid in $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$, so that we can consider its category of modules, denoted as $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda}(\mathrm{Idem})$. This category inherits a symmetric monoidal structure denoted by $D_1 \otimes_{\mathrm{Perf}_\Lambda} D_2$.

Any stable ∞-category D is canonically enriched over the category of spectra Sp. We write $\mathrm{Hom}_D(-,-)$ for the mapping spectrum. Any category in $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ is canonically enriched over Mod_Λ, so that we refer to $\mathrm{Hom}_D(-,-) \in \mathrm{Mod}_\Lambda$ as the mapping complex. For example, for $M, N \in \mathrm{Mod}_\Lambda$, then $\mathrm{Hom}_{\mathrm{Mod}_\Lambda}(M, N)$ is commonly also denoted by $\mathrm{RHom}(M, N)$. Its n-th cohomology is the Hom-group $\mathrm{Hom}(M, N[n])$ in the classical derived category.

2.2. Limits and filtered colimits.
Throughout, we freely use general facts about the forgetful functors

$$\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}) \subset \mathrm{Cat}_\infty^{\mathrm{Ex}} \subset \mathrm{Cat}_\infty$$

between the ∞-categories of idempotent complete stable, respectively stable, respectively arbitrary ∞-categories, together with exact, respectively exact, respectively all functors. Recall that any functor between idempotent complete categories automatically preserves retracts or, equivalently, colimits indexed by Idem.

All three categories have small limits and filtered colimits. As both inclusions preserve these, we will usually not specify in which of the above ∞-categories limits and filtered colimits are formed. See [Lura, Theorem 1.1.4.4, Proposition 1.1.4.6] for the claims concerning the latter functor. Limits in $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ exist since a colimit of a diagram $\mathrm{Idem} \to \lim D_i$ exists as soon as this is true for each D_i. The inclusion $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem}) \subset \mathrm{Cat}_\infty^{\mathrm{Ex}}$ preserves limits, being a right adjoint to the idempotent completion functor. Filtered colimits in $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$ are computed a priori by taking the idempotent completion of the colimit in $\mathrm{Cat}_\infty^{\mathrm{Ex}}$. According to [Lur09, Corollary 4.4.5.21], however, that filtered colimit is already idempotent complete, showing that the former inclusion functor also preserves filtered colimits.

Lemma 2.1. *Let Λ be a ring.*

(1) If $\Lambda = \mathrm{colim}\,\Lambda_i$ is a filtered colimit of rings, then the natural functor

$$\mathrm{colim}\,\mathrm{Perf}_{\Lambda_i} \xrightarrow{\cong} \mathrm{Perf}_\Lambda$$

is an equivalence. Here the transition functors are given by $(\text{-}) \otimes_{\Lambda_i} \Lambda_j$ for $j \geq i$.

(2) Let $\Lambda = \lim_{i \geq 1} \Lambda_i$ be a sequential limit of rings such that all transition maps $\Lambda_{i+1} \to \Lambda_i$ are surjective with locally nilpotent kernel. Then the natural functor

$$\mathrm{Perf}_\Lambda \xrightarrow{\cong} \lim \mathrm{Perf}_{\Lambda_i}, \quad M \mapsto (M \otimes_\Lambda \Lambda_i)_{i \geq 1}$$

is an equivalence. In addition, the functor $\mathrm{Perf}_\Lambda \to \mathrm{Perf}_{\Lambda_1}, M \mapsto M \otimes_\Lambda \Lambda_1$ is conservative.

Proof. In part (1), the full faithfulness follows from standard $\otimes$-Hom-adjunctions using that perfect complexes are dualizable. Since Λ lies in the essential image, the functor is an equivalence given that both categories are in $\mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$.

Part (2) is [Bha16, Lemma 4.2]: for full faithfulness, we note that the functor $M \otimes_\Lambda (\text{-}) \cong \underline{\mathrm{Hom}}_{\mathrm{Mod}_\Lambda}(M^\vee, \text{-})$ commutes with limits so that $M \cong \lim M \otimes_\Lambda \Lambda_i$ for any $M \in \mathrm{Perf}_\Lambda$. For essential surjectivity, we use that a quasi-inverse of the functor is provided by $\{M_i\} \mapsto \lim M_i$, see [Sta17, Tag 0CQG].

For the final statement, it remains to prove that the functor $\mathrm{Perf}_{\Lambda_i} \to \mathrm{Perf}_{\Lambda_1}, M \mapsto M \otimes_{\Lambda_i} \Lambda_1$ is conservative for any $i \geq 1$. So let $M \in \mathrm{Perf}_{\Lambda_i}$ such that $M \otimes_{\Lambda_i} \Lambda_1 \simeq 0$. We choose a bounded complex $(M^a \to \ldots \to M^b)$ of finitely generated projective Λ_i-modules representing M. We need to show that the complex is exact. The derived tensor product $M \otimes_{\Lambda_i} \Lambda_1$ is represented by $(M^a \otimes_{\Lambda_i} \Lambda_1 \to \ldots \to M^b \otimes_{\Lambda_i} \Lambda_1)$. As the complex is exact by assumption, the map $M^{b-1} \otimes_{\Lambda_i} \Lambda_1 \to M^b \otimes_{\Lambda_i} \Lambda_1$ is surjective. Nakayama's lemma shows that $M^{b-1} \to M^b$ is surjective as well, using that $\ker(\Lambda_i \to \Lambda_1)$ is generated by nilpotent elements. By projectivity, this map splits so that $M^{b-1} \simeq M^b \oplus N^{b-1}$. We are reduced to show that the resulting complex $(M^a \to \ldots \to M^{b-2} \to N^{b-1})$ is exact. Continuing by induction on the length $b - a$ implies our claim. $\square$

Write $\mathrm{Cat}_n \subset \mathrm{Cat}_\infty$ for the full subcategory spanned by n-categories [Lur09, Section 2.3.4], and similarly $\mathrm{Cat}_n(\mathrm{Idem}) \subset \mathrm{Cat}_\infty(\mathrm{Idem})$ for the full subcategory consisting of idempotent complete n-categories. Recall that $\mathrm{Cat}_n, \mathrm{Cat}_n(\mathrm{Idem})$ admit limits and filtered colimits and that the above inclusions preserve these. Let us record the following lemma from [BM20, Lemma 3.7] for later use:

Lemma 2.2. *In Cat_n and $\mathrm{Cat}_n(\mathrm{Idem})$, filtered colimits commute with totalizations (=Δ-indexed limits).*

Proof. By [BM20, Lemma 3.7, Example 3.6], this holds true for Cat_n. Now use that $\mathrm{Cat}_n(\mathrm{Idem}) \subset \mathrm{Cat}_n$ preserves limits and filtered colimits.

$\square$

2.3. Fixed points of ∞-categories.
A basic structure in Drinfeld's lemma is the equivariance datum for the partial Frobenii. In this section, we assemble some abstract results where such ∞-categorical constructions are carried out.

Definition 2.3. Let $\phi\colon D \to D$ be an endofunctor in $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$. The category of ϕ-*fixed points* is

$$\mathrm{Fix}(D,\phi) \overset{\mathrm{def}}{=} \lim\left(D \underset{\mathrm{id}_D}{\overset{\phi}{\rightrightarrows}} D\right).$$

Recall that for a symmetric monoidal ∞-category D, a commutative monoid object $\mathrm{L} \in \mathrm{CAlg}(D)$, the forgetful functors $\mathrm{CAlg}(D) \to D$ and $\mathrm{Mod}_\mathrm{L}(D) \to D$ preserve limits [Lura, Corollary 3.2.2.5, Corollary 4.2.3.3]. In particular, if D is in addition Λ-linear, that is, an object in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$, and ϕ is also Λ-linear, then $\mathrm{Fix}(D,\phi)$ admits a natural Λ-linear structure as well.

Because of these facts, we will usually not specify where the limit above is formed. Note that all functors

$$\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}) \overset{\mathrm{Ind}}{\underset{\cong}{\longrightarrow}} \mathrm{Pr}^{\mathrm{St}}_\omega \overset{(*)}{\longrightarrow} \mathrm{Pr}^{\mathrm{St}} \longrightarrow \mathrm{Pr}^{\mathrm{L}} \longrightarrow \widehat{\mathrm{Cat}_\infty} \tag{2.3}$$

except for the forgetful functor marked $(*)$ preserve limits, see [Lura, Corollary 4.2.3.3] and [Lur09, Proposition 5.5.3.13] for the rightmost two functors. To give a concrete example of that failure in our situation, note that $\mathrm{Fix}(D,\mathrm{id}_D) = \mathrm{Fun}(B\mathbb{Z},D)$, that is, objects are pairs (M,α) consisting of some $M \in D$ and some automorphism $\alpha\colon M \cong M$. Now consider $D = \mathrm{Vect}^{\mathrm{fd}}_\Lambda$, the (abelian) category of finite-dimensional vector spaces over a field Λ. The natural functor

$$\mathrm{Ind}\left(\lim\left(\mathrm{Vect}^{\mathrm{fd}}_\Lambda \rightrightarrows \mathrm{Vect}^{\mathrm{fd}}_\Lambda\right)\right) \to \lim\left(\mathrm{Ind}\left(\mathrm{Vect}^{\mathrm{fd}}_\Lambda\right) \rightrightarrows \mathrm{Ind}\left(\mathrm{Vect}^{\mathrm{fd}}_\Lambda\right)\right) = \lim\left(\mathrm{Vect}_\Lambda \rightrightarrows \mathrm{Vect}_\Lambda\right)$$

is fully faithful, but *not* essentially surjective: given an automorphism α of an infinite-dimensional vector space M, there need not be a filtration $M = \bigcup M_i$ by finite-dimensional subspaces M_i that is compatible with α.

Fixed point categories inherit t-structures as follows:

Lemma 2.4. *Let* $\phi\colon D \to D$ *be a functor in* $\mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$. *Suppose* D *carries a t-structure such that* ϕ *is t-exact. Then* $\mathrm{Fix}(D,\phi)$ *carries a unique t-structure such that the evaluation functor is t-exact. There is a natural equivalence*

$$\mathrm{Fix}(D^\heartsuit,\phi) \overset{\cong}{\longrightarrow} \mathrm{Fix}(D,\phi)^\heartsuit.$$

Proof. Let us abbreviate $\widetilde{D} := \mathrm{Fix}(D,\phi)$. For $\bullet$ being either "≤ 0" or "≥ 0", we put $\widetilde{D}^\bullet := \mathrm{Fix}(D^\bullet,\phi)$, which is a (non-stable) ∞-category. This is clearly the only choice for a t-structure making ev a t-exact functor. It satisfies the claim about the hearts of the t-structure by definition.

We need to show that it is a t-structure. Being a limit of full subcategories, the categories $\widetilde{D}^\bullet$ are full subcategories of $\widetilde{D}$. Since ϕ, being t-exact, commutes with $\tau_D^{\leq 0}$ and $\tau_D^{\geq 0}$, these two functors also yield truncation functors for $\widetilde{D}$. For $M \in \widetilde{D}^{\leq 0}$, $N \in \widetilde{D}^{\geq 1}$ (we use cohomological conventions), we have

$$\mathrm{Hom}_{\widetilde{D}}\left(M,N\right) = \lim\left(\mathrm{Hom}_D(M,N) \rightrightarrows \mathrm{Hom}_D(M,N)\right),$$

where on the right hand side M, N denote the underlying objects in D. Since $M \in D^{\leq 0}$, $N \in D^{\geq 1}$, we have $\mathrm{H}^i \mathrm{Hom}_D(M,N) = 0$ for $i = -1, 0$. Thus, $\mathrm{H}^0 \mathrm{Hom}_{\widetilde{D}}(M,N) = 0$ as well. $\qquad\square$

Definition 2.3 can be generalized as follows: Let $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ be a diagram. For example, for $n = 1$, this amounts to giving $D = \varphi(*) \in \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ and an equivalence $\phi = \varphi(1)\colon D \to D$. For $n = 2$, such a datum corresponds to giving D, equivalences $\phi_1, \phi_2\colon D \overset{\cong}{\to} D$ together with an equivalence $\phi_1 \circ \phi_2 \overset{\cong}{\to} \phi_2 \circ \phi_1$. So we define the ∞-category of *simultaneous fixed points* as

$$\mathrm{Fix}(D,\phi_1,\ldots,\phi_n) \overset{\mathrm{def}}{=} \lim\varphi \in \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem}).$$

Remark 2.5. The statement of Lemma 2.4 carries over verbatim assuming that D has a t-structure and all ϕ_i are t-exact, noting that $B\mathbb{Z}^n = (S^1)^n$ is a finite simplicial set.

Lemma 2.6. *Let* $\varphi\colon B\mathbb{Z}^n \to \mathrm{Cat}^{\mathrm{Ex}}_\infty(\mathrm{Idem})$ *be a diagram. Denote* $D = \varphi(*)$ *and* $\phi_i = \varphi(e_i)$ *for the i-th standard vector* $e_i \in \mathbb{Z}^n$. *The functor*

$$\mathrm{Fix}(D,\phi_1,\ldots,\phi_n) \to \mathrm{Fix}(\mathrm{Ind}(D),\phi_1,\ldots,\phi_n)$$

induced from the inclusion $D \subset \mathrm{Ind}(D)$ *is fully faithful and takes values in compact objects. In particular, it yields a fully faithful functor*

$$\mathrm{Ind}(\mathrm{Fix}(D,\phi_1,\ldots,\phi_n)) \to \mathrm{Fix}(\mathrm{Ind}(D),\phi_1,\ldots,\phi_n).$$

Proof. Let $M \in \mathrm{Fix}(D,\phi_1,\ldots,\phi_n)$ and denote its underlying object in D by the same symbol. For every $N \in \mathrm{Fix}(\mathrm{Ind}(D),\phi_1,\ldots,\phi_n)$, we have a limit diagram of mapping complexes

$$\mathrm{Hom}_{\mathrm{Fix}(\mathrm{Ind}(D))}(M,N) \cong \mathrm{Fix}(\mathrm{Hom}_{\mathrm{Ind}(D)}(M,N),\phi_1,\ldots,\phi_n).$$

Since filtered colimits commute with finite limits in Ani [Lur09, Proposition 5.3.3.3.], we see that M is compact in $\mathrm{Fix}(\mathrm{Ind}(D))$ because M is so in $\mathrm{Ind}(D)$. $\qquad\square$

Lemma 2.7. *Let* $\varphi_i \colon B\mathbb{Z} \to \mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Lambda}(\mathrm{Idem})$, $i = 1,\ldots,n$ *be given. Denote* $D_i = \varphi_i(*)$, $\phi_i = \varphi_i(1)$ *and* $\widetilde{D}_i = \mathrm{Ind}(D_i)$. *Then there is a canonical equivalence*

$$\mathrm{Fix}\big(\widetilde{D}_1, \phi_1\big) \otimes_{\mathrm{Mod}_\Lambda} \ldots \otimes_{\mathrm{Mod}_\Lambda} \mathrm{Fix}\big(\widetilde{D}_n, \phi_n\big) \overset{\cong}{\to} \mathrm{Fix}\left(\widetilde{D}_1 \otimes_{\mathrm{Mod}_\Lambda} \ldots \otimes_{\mathrm{Mod}_\Lambda} \widetilde{D}_n, \phi_1, \ldots, \phi_n\right).$$

Proof. Any compactly generated category in $\mathrm{Pr}^{\mathrm{St}}_\Lambda$ is dualizable [Lurb, Remark D.7.7.6 (1)] so that tensoring with it preserves limits. $\square$

3. Lisse and constructible sheaves

In this section, we develop the basics of a theory of lisse (French for smooth) and constructible sheaves on schemes, with coefficients in a condensed (unital, commutative) ring Λ.

3.1. **Definitions and basic facts.** For a scheme X, we denote by $X_{\mathrm{proét}}$ its proétale site introduced in [BS15, §4]. Thus, $X_{\mathrm{proét}}$ is the category of weakly étale schemes over X with covers given by families $\{U_i \to U\}$ of maps in $X_{\mathrm{proét}}$ that are fpqc-coverings, that is, any open affine in U is mapped onto by an open affine in $\sqcup_i U_i$. Being a major difference with étale site $X_{\mathrm{ét}}$, any scheme X admits a proétale cover by w-contractible affine schemes, that is, by affine schemes U such that any weakly étale surjection $V \to U$ splits.

Also, for any profinite set S, we denote by $S_{\mathrm{proét}}$ the category of profinite sets over S with covers given by finite families of jointly surjective maps. The w-contractible objects in $S_{\mathrm{proét}}$ are the extremally disconnected profinite sets. We use this in the special case where $S = *$ is the singleton: For any scheme X, there is the map of sites

$$p_X \colon X_{\mathrm{proét}} \longrightarrow *_{\mathrm{proét}}$$

given by the limit-preserving functor (in the opposite direction) $S = \lim S_i \mapsto \lim X^{S_i} =: X \times S$.

Throughout this section, Λ is a condensed (unital, commutative) ring Λ, that is, a sheaf of rings on $*_{\mathrm{proét}}$, see [CS]. We write $\Lambda_X := p_X^{-1}\Lambda$ for the corresponding sheaf of rings on $X_{\mathrm{proét}}$. The condensed rings considered in this paper arise as follows:

Example 3.1. Every T1-topological ring Λ induces sheaf of rings, also denoted by Λ, on $*_{\mathrm{proét}}$:

$$S \longmapsto \mathrm{Maps}_{\mathrm{cont}}(S, \Lambda),$$

that is, continuous maps from the profinite set S to Λ. In fact, we have a functor from the category of T1-topological rings into the category of condensed rings. This functor is fully faithful when restricted to those Λ whose underlying topological space is compactly generated.

Of major interest to us is the case of *totally disconnected* topological rings Λ. In this case, for any scheme or profinite set X, the sheaf Λ_X is nothing but the sheaf of rings on $X_{\mathrm{proét}}$ given by

$$U \longmapsto \mathrm{Maps}_{\mathrm{cont}}(U, \Lambda),$$

that is, continuous maps from the underlying topological space of U into Λ: any continuous map $U \to \Lambda$ from a quasi-compact and quasi-separated $U \in X_{\mathrm{proét}}$ factors uniquely through $U \to \pi_0 U$. In [BS15, Lemma 4.2.12], this sheaf is denoted $\mathcal{F}_\Lambda$.

Examples of such totally disconnected topological rings Λ include all discrete rings, adic rings with finitely generated ideals of definition [Sta17, Tag 05GG], algebraic field extensions $E \supset \mathbb{Q}_\ell$ equipped with the colimit topology, their open subrings of integers $\mathcal{O}_E \subset E$, but also the ring of adeles $\mathbb{A}^T_K$ prime to some finite set of places T containing all Archimedean places in some number field K.

For any scheme X, we denote by $\mathrm{D}(X, \Lambda)$ the derived ∞-category of the abelian category of sheaves of Λ_X-modules on $X_{\mathrm{proét}}$. This is a presentable stable ∞-category which is symmetric monoidal and closed. The monoidal structure is denoted by $- \otimes_{\Lambda_X} -$ and the inner homomorphisms by $\underline{\mathrm{Hom}}_{\Lambda_X}(-, -)$, so that

$$\mathrm{Hom}_{\Lambda_X}(-, -) = \mathrm{Hom}_{\mathrm{D}(X,\Lambda)}(-, -) = \mathrm{R}\Gamma\big(X, \underline{\mathrm{Hom}}_{\Lambda_X}(-, -)\big)$$

is the mapping complex.

For any $n \in \mathbb{Z}$, the truncations $\tau^{\geq n}$, $\tau^{\leq n}$ in the standard t-structure on $\mathrm{D}(X, \Lambda)$ induce (in cohomological notation) adjunctions

$$\tau^{\geq n} \colon \mathrm{D}(X, \Lambda) \leftrightarrows \mathrm{D}^{\geq n}(X, \Lambda) : \mathrm{incl}, \quad \mathrm{incl} \colon \mathrm{D}^{\leq n}(X, \Lambda) \leftrightarrows \mathrm{D}(X, \Lambda) : \tau^{\leq n},$$

where incl is the inclusion of the respective subcategories. As $X_{\mathrm{proét}}$ is locally weakly contractible [BS15, Proposition 4.2.8], the t-structure on $\mathrm{D}(X, \Lambda)$ is left-complete (equivalently, Postnikov towers converge in the associated hypercompleted ∞-topos) and the category $\mathrm{D}(X, \Lambda)$ is compactly generated, see [BS15, Proposition 3.2.3]. A family of compact generators is given by the objects $\Lambda_X[U] \in \mathrm{D}(X, \Lambda)$, for $U \in X_{\mathrm{proét}}$ w-contractible affine, corepresenting the functor $\mathrm{R}\Gamma(U, -)$. We remark that $\mathrm{D}(X, \Lambda)$ is equivalent to the category of Λ_X-modules on the hypercompleted ∞-topos associated with $X_{\mathrm{proét}}$ by [Lurb, Theorem 2.1.2.2, Definition 2.1.0.1]. Similarly, this discussion applies to any profinite set S (in fact, this is just a special case) in place of X and the category $\mathrm{D}(S, \Lambda)$.

If $f\colon Y \to X$ is any morphism of schemes, then $\Lambda_Y = f^{-1}\Lambda_X$. It is formal to check that the ordinary pullback, respectively pushforward of sheaves induces an adjunction

$$f^* = f^{-1}\colon \mathrm{D}(X,\Lambda) \leftrightarrows \mathrm{D}(Y,\Lambda) : f_*,$$

where f^* is exact, t-exact and symmetric monoidal. Similarly, if $\Lambda \to \Lambda'$ is a morphism of condensed rings, then the forgetful functor $\mathrm{D}(X,\Lambda') \to \mathrm{D}(X,\Lambda)$ admits a symmetric monoidal left adjoint

$$\mathrm{D}(X,\Lambda) \to \mathrm{D}(X,\Lambda'), \quad M \mapsto M \otimes_{\Lambda_X} \Lambda'_X.$$

Let us denote by $\Gamma(X,\Lambda)$ the (underived) global sections of Λ_X viewed as a ring. Then there is the functor

$$\mathrm{Mod}_{\Gamma(X,\Lambda)} \to \mathrm{D}(X,\Lambda), \quad M \mapsto M_X, \tag{3.1}$$

characterized in $\mathrm{Pr}^{\mathrm{St}}$ as the colimit-preserving extension of $\Gamma(X,\Lambda) \mapsto \Lambda_X$. Explicitly, M_X is the hypersheaf associated with the presheaf on $X_{\mathrm{pro\acute{e}t}}$ given by $U \mapsto M \otimes_{\Gamma(X,\Lambda)} \mathrm{R}\Gamma(U,\Lambda)$. This functor is symmetric monoidal and makes $\mathrm{D}(X,\Lambda)$ a commutative algebra object in $\mathrm{Pr}^{\mathrm{St}}_{\Gamma(X,\Lambda)}$. The sections of M_X on w-contractible qcqs $U \in X_{\mathrm{pro\acute{e}t}}$ are computed as

$$\mathrm{R}\Gamma(U, M_X) \cong M \otimes_{\Gamma(X,\Lambda)} \Gamma(U,\Lambda). \tag{3.2}$$

In particular, if $f\colon Y \to X$ lies in $X_{\mathrm{pro\acute{e}t}}$, then $f^*M_X \cong \big(M \otimes_{\Gamma(X,\Lambda)} \Gamma(Y,\Lambda)\big)_Y$.

Remark 3.2. Here is an equivalent way of defining the functor $M \mapsto M_X$ in (3.1): For any scheme or profinite set X, there is a natural map $\underline{\Gamma(X,\Lambda)} \to \Lambda_X$ of sheaf of rings on $X_{\mathrm{pro\acute{e}t}}$ where $\underline{\Gamma(X,\Lambda)}$ denotes the constant sheaf associated with the discrete ring $\Gamma(X,\Lambda)$. Then the functor (3.1) is equivalent to the functor

$$M \longmapsto \underline{M} \otimes_{\underline{\Gamma(X,\Lambda)}} \Lambda_X,$$

where $\underline{M}$ denotes the constant sheaf. As an example, let $X = *$, $\Lambda = \mathbb{Q}_\ell$ (see Example 3.1) and $M \in \mathrm{Mod}^{\heartsuit}_{\mathbb{Q}_\ell}$ a $\mathbb{Q}_\ell$-vector space. Then, loosely speaking, the above functor equips M with the relatively discrete topology. More precisely, writing $M = \mathrm{colim}\,\mathbb{Q}_\ell^I$ as an increasing union of finite-dimensional vector spaces, we take the product topology on $\mathbb{Q}_\ell^I$ and the colimit topology on M.

Recall that a subset of a qcqs topological space is called *constructible* if it is a finite Boolean combination of quasi-compact open subsets. Also, recall the notion of dualizable objects in symmetric monoidal categories [Lura, Definition 4.6.1.1, Remark 4.6.1.12].

Definition 3.3. Let X be a scheme or a profinite set, and Λ a condensed ring.

    (1) A sheaf $M \in \mathrm{D}(X,\Lambda)$ is called *lisse* if it is dualizable.

    (2) A sheaf $M \in \mathrm{D}(X,\Lambda)$ is called *constructible* if, for any open affine $U \subset X$, there exists a finite subdivision of U into constructible locally closed subschemes $U_i \subset U$ such that $M|_{U_i}$ is lisse.

If X is qcqs (=quasi-separated and quasi-compact) and $M \in \mathrm{D}(X,\Lambda)$ constructible, then there is a finite subdivision of X into constructible locally closed subschemes $X_i \subset X$ such that $M|_{X_i}$ is lisse. The argument is purely topological and the same as in [Sta17, Tag 095E].

The full subcategories of $\mathrm{D}(X,\Lambda)$ of lisse, respectively constructible Λ-sheaves are denoted by

$$\mathrm{D}_{\mathrm{lis}}(X,\Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X,\Lambda).$$

Both categories are naturally commutative algebra objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty,\Gamma(X,\Lambda)}(\mathrm{Idem})$, that is, idempotent complete stable $\Gamma(X,\Lambda)$-linear symmetric monoidal ∞-categories.

If $f\colon Y \to X$ is any map of schemes, then the pullback $f^*\colon \mathrm{D}(X,\Lambda) \to \mathrm{D}(Y,\Lambda)$ preserves lisse, respectively constructible sheaves and hence induces functors

$$f^*\colon \mathrm{D}_{\mathrm{lis}}(X,\Lambda) \to \mathrm{D}_{\mathrm{lis}}(Y,\Lambda), \quad f^*\colon \mathrm{D}_{\mathrm{cons}}(X,\Lambda) \to \mathrm{D}_{\mathrm{cons}}(Y,\Lambda).$$

For lisse sheaves, this follows from the monoidality of f^*. For constructible sheaves, one additionally reduces to the case of affine schemes so that f induces a spectral map on the underlying topological spaces, and thus is continuous in the constructible topology, see [Sta17, Tag 0A2S].

If $\Lambda \to \Lambda'$ is a map of condensed rings, then the functor $(\text{-}) \otimes_{\Lambda_X} \Lambda'_X$ preserves lisse, respectively constructible sheaves and hence induces functors

$$\mathrm{D}_{\mathrm{lis}}(X,\Lambda) \to \mathrm{D}_{\mathrm{lis}}(X,\Lambda'), \quad \mathrm{D}_{\mathrm{cons}}(X,\Lambda) \to \mathrm{D}_{\mathrm{cons}}(X,\Lambda').$$

For any constructible closed immersion $i\colon Z \hookrightarrow X$ with open complement $j\colon U \hookrightarrow X$, we have adjunctions

$$j_!\colon \mathrm{D}(U,\Lambda) \leftrightarrows \mathrm{D}(X,\Lambda) : j^*, \quad i_*\colon \mathrm{D}(Z,\Lambda) \leftrightarrows \mathrm{D}(X,\Lambda) : i^!, \tag{3.3}$$

fitting in fiber sequences $j_!j^* \to \mathrm{id} \to i_*i^*$ and $i_*i^! \to \mathrm{id} \to j_*j^*$, see [BS15, §6.1]. The functors $j_*, j_!, i_*$ are fully faithful and satisfy the usual formulas

$$i^*i_* \simeq j^*j_* \simeq j^*j_! \simeq \mathrm{id}, \quad j^*i_* \simeq i^*j_! \simeq 0. \tag{3.4}$$

Lemma 3.4. *In the above situation, the functors $i_*, j_!$ are t-exact and preserve the full subcategories of constructible sheaves.*

Proof. By [BS15, Lemma 6.2.1 (1) + (2)], the functors $i_*, j_!$ induce equivalences onto the full subcategory of $D(X_{\text{proét}}, \Lambda)$ spanned by objects supported on Z, respectively U. Their inverses are given by i^*, j^* which are clearly t-exact, hence so are the functors $i_*, j_!$. Using the formulas (3.4) it is clear that $i_*, j_!$ preserve constructibility. $\square$

We use the following terminology throughout:

Definition 3.5. A sheaf $M \in D(X, \Lambda)$ is called *(perfect-)constant* if $M \simeq \underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_X$ for some (perfect) complex of Λ_*-modules N, where $\Lambda_* = \Gamma(*, \Lambda)$ is the underlying ring. It is called *(pro-)étale-locally (perfect-)constant* if it is so locally on $X_{(\text{pro-})\text{ét}}$.

Any proétale-locally perfect-constant sheaf is lisse. The converse holds for discrete coefficient rings Λ (Corollary 3.43) in which case $\underline{\Lambda_*} \cong \Lambda_X$. It also holds for schemes having locally finitely many irreducible components, see Theorem 3.26. The following example of a lisse sheaf, which is based on [BS15, Example 6.6.12], shows however that lisse sheaves on profinite sets do not have such a simple description:

Example 3.6. Let $S = \hat{\mathbb{Z}} = \lim_m \mathbb{Z}/m$ be the profinite completion of the integers viewed as a profinite set. Take $\Lambda = \hat{\mathbb{Z}}$ viewed as a profinite ring. Then the endomorphisms of Λ_S in $D(S, \Lambda)^{\heartsuit}$ are computed as

$$H^0(S, \Lambda) = \text{Maps}_{\text{cont}}(S, \hat{\mathbb{Z}}) = \text{Maps}_{\text{cont}}(\hat{\mathbb{Z}}, \hat{\mathbb{Z}}).$$

The constant map $f \equiv s \in \hat{\mathbb{Z}}$ corresponds to the endomorphism of Λ_S given by multiplication with the scalar s. By contrast, if $f: \Lambda_S \to \Lambda_S$ corresponds to the identity in $\text{Maps}_{\text{cont}}(\hat{\mathbb{Z}}, \hat{\mathbb{Z}})$, then its stalk f_s at $s \in S$ is multiplication with s viewed as element in $\hat{\mathbb{Z}}$. The complex $(\Lambda_S \xrightarrow{f} \Lambda_S) \in D_{\text{lis}}(S, \Lambda)$ is étale-locally perfect-constant after each reduction modulo $m \neq 0$. However, the complex is not proétale-locally perfect-constant as any cover $\{S_i \to S\}$ has a member $S_i \to S$ whose image is infinite.

3.2. Lisse sheaves on w-contractible schemes.
Recall the functor $\text{Mod}_{\Gamma(X,\Lambda)} \to D(X, \Lambda), M \mapsto M_X$ from (3.1).

Lemma 3.7. *Let X be a w-contractible qcqs scheme, or an extremally disconnected profinite set.*

(1) The functor (3.1) *induces an adjunction*

$$(\text{-})_X : \text{Mod}_{\Gamma(X,\Lambda)} \rightleftarrows D(X, \Lambda) : R\Gamma(X, \text{-}).$$

(2) Both adjoints are colimit-preserving and symmetric monoidal. In addition, $(\text{-})_X$ is fully faithful and $R\Gamma(X, \text{-})$ is t-exact.

(3) The adjunction induces an equivalence on dualizable objects:

$$(\text{-})_X : \text{Perf}_{\Gamma(X,\Lambda)} \cong D_{\text{lis}}(X, \Lambda) : R\Gamma(X, \text{-})$$

Proof. For (1), we note that the spectra-valued functor $R\Gamma(X, \text{-}): D(X, \Lambda) \to \text{Sp}$ is right adjoint to the unique colimit-preserving functor $\text{Sp} \to D(X, \Lambda)$ mapping the sphere spectrum to Λ_X. Since $\text{Mod}_{R\Gamma(X,\Lambda)}$ is the category of modules over the monad associated to the adjunction, this induces an adjunction

$$\text{Mod}_{R\Gamma(X,\Lambda)} \rightleftarrows D(X, \Lambda).$$

Since X is w-contractible qcqs, respectively extremally disconneced, it is w-contractible coherent in the topos-theoretic sense so that $R\Gamma(X, \text{-})$ is (co-)limit-preserving and t-exact. In particular, $R\Gamma(X, \Lambda) = \Gamma(X, \Lambda)$.

For (2), we observe that the unit of the adjunction $\text{id} \to R\Gamma(X, \text{-}) \circ (\text{-})_X$ is an equivalence by (3.2) so that $(\text{-})_X$ is fully faithful. Clearly, $(\text{-})_X$ is also colimit-preserving and symmetric monoidal. Also, $R\Gamma(X, \text{-})$ is colimit-preserving and t-exact as noted in (1). For $M, N \in D(X, \Lambda)$, their tensor product $M \otimes_{\Lambda_X} N$ is the ∞-sheafification of the presheaf $U \mapsto R\Gamma(U, M) \otimes_{R\Gamma(U,\Lambda)} R\Gamma(U, N)$. Again, since X is w-contractible coherent, its global sections are equivalent to $R\Gamma(X, M) \otimes_{\Gamma(X,\Lambda)} R\Gamma(X, N)$.

For (3), we note that the adjunction restricts to an adjunction on dualizable objects by monoidality of both functors. The counit of this adjunction $(\text{-})_X \circ R\Gamma(X, \text{-}) \to \text{id}$ is an equivalence if and only if the functor

$$R\Gamma(X, \text{-}): D_{\text{lis}}(X, \Lambda) \to \text{Perf}_{\Gamma(X,\Lambda)}$$

is fully faithful, see [Lur09, Proposition 5.2.7.4]. For $M, N \in D_{\text{lis}}(X, \Lambda)$, this follows from $\underline{\text{Hom}}_{\Lambda_X}(M, N) \cong N \otimes_{\Lambda_X} M^{\vee}$ upon applying the symmetric monoidal functor $R\Gamma(X, \text{-})$. $\square$

The following example, communicated to us by Peter Scholze, shows that the functor $(\text{-})_X$ is not t-exact in general.

Example 3.8. Let $X = \beta\mathbb{N}$ be the Stone-Čech compactification of the natural numbers viewed as an extremally disconnected profinite set. Let $\Lambda = \mathbb{Q}_\ell$ viewed as a condensed ring, see Example 3.1. The map $\mathbb{N} \to \mathbb{Q}_\ell$, $n \mapsto \ell^n$ uniquely extends to a continuous map $f \colon \beta\mathbb{N} \to \mathbb{Q}_\ell$ by the universal property of β, that is, $f \in \mathrm{Maps}_{\mathrm{cont}}(\beta\mathbb{N}, \mathbb{Q}_\ell) = \Gamma(X, \Lambda)$. One checks that the complex $0 \to \Gamma(X, \Lambda) \xrightarrow{f} \Gamma(X, \Lambda)$ is exact. However, the induced complex on the level of sheaves is not exact because $f|_{\partial X} = 0$, where $\partial X = \beta\mathbb{N} \backslash \mathbb{N}$ denotes the boundary.

If X is a qcqs scheme, then its underlying topological space is spectral [Sta17, Tag 094L]. Thus, the set of connected components $\pi_0 X$ endowed with the quotient topology is a profinite space. Any map $S \to \pi_0 X$ of profinite sets can be written as profinite $\pi_0 X$-sets $S = \lim S_i$ such that each $S_i \to \pi_0 X$ is the base change of a map of finite sets, see [BS15, Proof of Lemma 2.2.8]. If we equip the topological space $X \times_{\pi_0 X} S_i \to X$ with the sheaf of rings given by the pullback of the structure sheaf on X, then it is representable by an object of the Zariski site X_{Zar}. The induced transition maps $X \times_{\pi_0 X} S_j \to X \times_{\pi_0 X} S_i$, $j \geq i$ are affine so that the limit

$$X \times_{\pi_0 X} S \overset{\mathrm{def}}{=} \lim X \times_{\pi_0 X} S_i$$

exists in the category of X-schemes. The functor $S \mapsto X \times_{\pi_0 X} S$ from profinite $\pi_0 X$-sets to X-schemes is limit-preserving and induces a map of sites

$$\pi_X \colon X_{\mathrm{proét}} \longrightarrow (\pi_0 X)_{\mathrm{proét}}, \tag{3.5}$$

factorizing $p_X \colon X_{\mathrm{proét}} \to *_{\mathrm{proét}}$.

Proposition 3.9. *Let X be a w-contractible affine scheme.*

(1) The functor

$$\pi_X^* \colon \mathrm{D}(\pi_0 X, \Lambda) \longrightarrow \mathrm{D}(X, \Lambda)$$

is fully faithful and commutes with the formation of inner homomorphisms.

(2) A Λ-sheaf $M \in \mathrm{D}(X, \Lambda)$ lies in the essential image of π_X^* if and only if for all maps $U \to V$ in $X_{\mathrm{proét}}$ between w-contractible affine schemes inducing isomorphisms $\pi_0 U \cong \pi_0 V$, the map

$$\mathrm{R}\Gamma(V, M) \xrightarrow{\cong} \mathrm{R}\Gamma(U, M)$$

is an equivalence.

(3) The functor π_X^* induces an equivalence

$$\mathrm{D}_{\mathrm{lis}}(\pi_0 X, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda).$$

Proof. We adjust the argument given in [BS15, Lemma 4.2.13] for the abelian categories. Abbreviate $\pi = \pi_X$, $\mathrm{D}(X) = \mathrm{D}(X, \Lambda)$ and $\mathrm{D}(\pi_0 X) = \mathrm{D}(\pi_0 X, \Lambda)$.

For (1), we show that the natural map $\mathrm{id} \to \pi_* \pi^*$ is an equivalence which formally implies the full faithfulness. Any continuous $\pi_0 X$-map $U \to S$ with affine $U \in X_{\mathrm{proét}}$, $S \in (\pi_0 X)_{\mathrm{proét}}$ factors uniquely through $U \to \pi_0 U$, since any profinite set is totally disconnected. Hence, if $M \in \mathrm{D}(\pi_0 X)$, then $\pi^* M$ is the sheafification of the presheaf $U \mapsto \mathrm{R}\Gamma(\pi_0 U, M)$. In particular, if U is also w-contractible, then we have an equivalence

$$\mathrm{R}\Gamma(U, \pi^* M) \cong \mathrm{R}\Gamma(\pi_0 U, M). \tag{3.6}$$

In this case, $\pi_0 U$ is extremally disconnected by [BS15, Lemma 2.4.8]. We apply these observations to show that the map $M \to \pi_* \pi^* M$ is an equivalence as follows. By evaluating at any extremally disconnected $S \in (\pi_0 X)_{\mathrm{proét}}$ it suffices to show that the map

$$\mathrm{R}\Gamma(S, M) \longrightarrow \mathrm{R}\Gamma(X \times_{\pi_0 X} S, \pi^* M)$$

is an equivalence. As $X \times_{\pi_0 X} S \to X$ is an pro-(Zariski open) pro-finite map with $\pi_0(X \times_{\pi_0 X} S) \cong S$ (by construction) we see that $X \times_{\pi_0 X} S$ is w-contractible affine as well: affine is clear; $X \to \pi_0 X$ has a section $s \colon \pi_0 X \to X$ given by the closed points in X by w-locality [BS15, Lemma 2.1.4] so that

$$s \times_{\pi_0 X} \mathrm{id} \colon S = \pi_0 X \times_{\pi_0 X} S \to X \times_{\pi_0 X} S$$

identifies S with the closed points in $X \times_{\pi_0 X} S$; finally, $X \times_{\pi_0 X} S \to X$ induces an isomorphism on local rings which are therefore strictly Henselian at all closed points. This shows that $X \times_{\pi_0 X} S$ is w-strictly local and hence w-contractible by [BS15, Lemma 2.4.8] using that its set of connected components is S (which is extremally disconnected). This implies $M \cong \pi_* \pi^* M$. The preservation of inner homomorphisms is immediate from the full faithfulness using (3.6).

For (2), if $M \in \mathrm{D}(X)$ is equivalent to the π^*-pullback of some object in $D(\pi_0 X)$, then it satisfies the desired condition by (3.6). Conversely, assume that M is localizing for maps $U \to V$ in $X_{\mathrm{proét}}$ of w-contractible affine schemes inducing an isomorphism on π_0. We claim that the map $\pi^* \pi_* M \to M$ is an equivalence. Indeed, evaluating at some w-contractible affine $U \in X_{\mathrm{proét}}$ gives the map

$$\mathrm{R}\Gamma(X \times_{\pi_0 X} \pi_0 U, M) \longrightarrow \mathrm{R}\Gamma(U, M) \tag{3.7}$$

induced from the canonical map $U \to X \times_{\pi_0 X} \pi_0 U$ over $\pi_0 U$. One argues as in (1) above to see that $X \times_{\pi_0 X} \pi_0 U$ is w-contractible affine with space of components $\pi_0 U$. Thus, (3.7) is an isomorphism by our assumption on M. We conclude $M \cong \pi^* \pi_* M$.

For (3), we note $\Lambda_X \cong \pi^* \Lambda_{\pi_0 X}$ so that $\Gamma(X, \Lambda) = \Gamma(\pi_0 X, \Lambda)$ by (1). By (3.6) the diagram

$$
\begin{array}{ccc}
\mathrm{D_{lis}}(\pi_0 X) & \xrightarrow{\ \pi^* \ } & \mathrm{D_{lis}}(X) \\[4pt]
\cong \uparrow {\scriptstyle (-)_{\pi_0 X}} & & \cong \uparrow {\scriptstyle (-)_X} \\[4pt]
\mathrm{Perf}_{\Gamma(\pi_0 X, \Lambda)} & =\!=\!= & \mathrm{Perf}_{\Gamma(X, \Lambda)}.
\end{array}
$$

commutes up to equivalence. The vertical functors are equivalences by Lemma 3.7 (3). $\qquad \square$

The following corollary shows that lisse sheaves extend proétale locally to small neighborhoods:

Corollary 3.10. *Let A be a ring Henselian along an ideal I. Let $i \colon Z := \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A =: X$ be the closed immersion induced by the quotient map $A \to A/I$.*

(1) The map i induces an isomorphism $\pi_0 Z \cong \pi_0 X$.

(2) The affine scheme X is w-contractible if and only if Z is w-contractible. In this case, there is an equivalence

$$
i^* \colon \mathrm{D_{lis}}(X, \Lambda) \xrightarrow{\ \cong \ } \mathrm{D_{lis}}(Z, \Lambda).
$$

Proof. For (1), the following argument was explained to us by Kęstutis Česnavičius: Being profinite sets the map $\pi_0 Z \to \pi_0 X$ is obtained as a limit over (finer and finer) finite subdivisions of Z and X into clopen (=closed and open) subsets. By the unique lifting of idempotents along the quotient map $A \to A/I$ [Sta17, Tag 09XI], these finite subdivisions match, so do the limits.

For (2), we note that by [BS15, Theorem 1.8, Lemma 2.2.9] an affine scheme is w-contractible if and only if it is w-strictly local and its profinite set of connected components is extremally disconnected. Since A is Henselian along I, the ring A is w-strictly local if and only if A/I is w-strictly local, see [BS15, Lemma 2.2.13]. So the first statement in (2) follows from (1). The second then follows, again using (1), from Proposition 3.9 (3). $\qquad \square$

3.3. **Hyperdescent.** In this subsection, let X be a scheme and Λ a condensed ring.

Lemma 3.11. *The property of Λ-sheaves of being lisse, respectively constructible is local on $X_{\mathrm{proét}}$.*

Proof. It is enough to prove the following: if X is affine and $j \colon U \to X$ a w-contractible affine cover, then $M \in \mathrm{D}(X, \Lambda)$ is lisse, respectively constructible if and only if $j^* M$ is so. Since $j^* \colon \mathrm{D}(X, \Lambda) \to \mathrm{D}(U, \Lambda)$ is monoidal, conservative and commutes with inner homomorphisms, the statement for the property "lisse" follows.

Now assume that $j^* M$ is constructible. Since the 1-topos of $X_{\mathrm{proét}}$ is generated by pro-étale affine objects [BS15, Lemma 4.2.4], we can assume that $U = \lim_i U_i \to X$ is a cofiltered limit of affine schemes $U_i \in X_{\mathrm{ét}}$. If the stratification on U witnessing the constructibility of $j^* M$ arises by pullback from X, then we are done using the case of lisse sheaves above. Following [BS15, Lemmas 6.3.10, 6.3.13] we reduce to this situation in several steps.

First, each constructible subset of U arises by pullback from some U_i. So the stratification witnessing the constructibility of $j^* M$ arises by pullback from some U_i. We reduce to the case where $U = U_i \to X$ is an étale cover.

Next, stratifying X by constructible locally closed subschemes $X_i \subset X$ such that the base change $U \times_X X_i \to X_i$ is finite étale [Sta17, Tag 03S0] we may assume that $U \to X$ is finite étale (after replacing X by some X_i, and possibly a Zariski localization to preserve the affineness of X).

Now writing $X = \lim_i X_i$ as a cofiltered inverse limit of finite type $\mathbb{Z}$-schemes the map $U \to X$ arises as the base change of some finite étale map $U_i \to X_i$. The connected components of X_i are open and closed. After possibly replacing X by a finite clopen cover we may assume that X_i is connected. Likewise, we may assume that U_i is connected. Then we may replace $U_i \to X_i$ by its Galois closure and assume that $U_i \to X_i$ is a finite Galois cover with group $G = \operatorname{Aut}(U_i/X_i)$. Hence, we reduced to the case where $j \colon U \to X$ is a G-torsor under some finite constant X-group scheme G.

Finally, using the G-action on U one easily constructs a finite subdivision of U into G-equivariant constructible locally closed $U_i \subset U$ such that $j^* M|_{U_i}$ is lisse. Clearly, these strata arise by pullback along the G-torsor $U \to X$. This implies the constructibility of M (again using the case of lisse sheaves above). $\qquad \square$

Corollary 3.12. *If $j \colon U \to X$ is quasi-compact étale (respectively, finite étale), then $j_! \colon \mathrm{D}(U, \Lambda) \to \mathrm{D}(X, \Lambda)$ preserves the subcategories of constructible sheaves (respectively, lisse sheaves).*

Proof. As in the proof of Lemma 3.11, one reduces to the finite étale case and further to the case of a G-torsor $j \colon U \to X$ under some finite constant X-group scheme G. Then $U \times_X U \cong G \times X$ which implies $j^* j_! M \cong \oplus_{g \in G} M$ for any $\mathrm{D}(X, \Lambda)$. The corollary follows. $\qquad \square$

Corollary 3.13. *The functors $U \mapsto \mathrm{D_{cons}}(U, \Lambda)$, $\mathrm{D_{lis}}(U, \Lambda)$ are hypersheaves of ∞-categories on $X_{\mathrm{pro\acute{e}t}}$.*

Proof. We only spell out the constructible case, the one for lisse sheaves is identical. We first check that $U \mapsto \mathrm{D_{cons}}(U, \Lambda)$ is a sheaf on $X_{\mathrm{pro\acute{e}t}}$. Given an object $U \in X_{\mathrm{pro\acute{e}t}}$ and an étale cover $\{U_i \to U\}$ we denote by $\mathcal{U}$ the covering sieve generated by the maps $\{U_i \to U\}$. By [Lurb, Remark 2.1.0.5], we know that the functor $U \mapsto \mathrm{D}(U, \Lambda)$ is a hypersheaf on $X_{\mathrm{pro\acute{e}t}}$. Thus, we have an equivalence

$$\mathrm{D}(U, \Lambda) \xrightarrow{\cong} \lim_{V \in \mathcal{U}} \mathrm{D}(V, \Lambda).$$

As constructibility is preserved by pullback, we have inclusions of full subcategories

$$\mathrm{D_{cons}}(U, \Lambda) \subset \lim_{V \in \mathcal{U}} \mathrm{D_{cons}}(V, \Lambda) \subset \mathrm{D}(U, \Lambda).$$

The essential image of the limit consists of objects $M \in \mathrm{D}(U, \Lambda)$ such that $M|_V$ is constructible for every $V \in \mathcal{U}$. In particular, $M|_{U_i}$ is constructible for every U_i in the cover $\{U_i \to U\}$. Hence, $M \in \mathrm{D_{cons}}(U, \Lambda)$ by Lemma 3.11.

Given the sheaf property, being a hypersheaf can be checked locally, we reduce to the case where X is affine. In this case, the Grothendieck topology $X_{\mathrm{pro\acute{e}t}}$ is finitary in the sense of [Lurb, Section A.3.2]. So by [Lurb, Proposition A.5.7.2] it is enough to show that for every hypercover $U_\bullet \to U$ with $U \in X_{\mathrm{pro\acute{e}t}}$ qcqs, the natural functor

$$\mathrm{D_{cons}}(U, \Lambda) \longrightarrow \mathrm{Tot}\big(\mathrm{D_{cons}}(U_\bullet, \Lambda)\big) := \lim_{[n] \in \Delta} \big(\mathrm{D_{cons}}(U_n, \Lambda)\big).$$

is an equivalence. Since $U \mapsto \mathrm{D}(U, \Lambda)$ is a hypersheaf, it satisfies descent. As constructibility is preserved by pullback, we have inclusions of full subcategories

$$\mathrm{D_{cons}}(U, \Lambda) \subset \mathrm{Tot}\big(\mathrm{D_{cons}}(U_\bullet, \Lambda)\big) \subset \mathrm{D}(U, \Lambda).$$

The totalization is the full subcategory of objects $M \in \mathrm{D}(U, \Lambda)$ such that $M|_{U_0}$ is constructible. Hence, $M \in \mathrm{D_{cons}}(U, \Lambda)$ by Lemma 3.11. $\qquad\square$

Corollary 3.14. *Let X be a scheme, and let $M \in \mathrm{D}(X, \Lambda)$. Then M is lisse if and only if, for every map of w-contractible affines $V \to U$ in $X_{\mathrm{pro\acute{e}t}}$, the natural map*

$$\mathrm{R\Gamma}(U, M) \otimes_{\Gamma(U, \Lambda)} \Gamma(V, \Lambda) \xrightarrow{\cong} \mathrm{R\Gamma}(V, M)$$

is an equivalence and the complex is perfect.

Proof. Combine Lemma 3.7 (3) with Corollary 3.13. $\qquad\square$

In order to compare our definition of, say, constructible $\bar{\mathbb{Q}}_\ell$-sheaves to the classical one in terms of E-sheaves for finite field extensions $E \supset \mathbb{Q}_\ell$, it is necessary to control hyperdescent not only for $\mathrm{D_{lis}}$, but for appropriate colimits of such categories. To do this, we filter the $\mathrm{D_{lis}}$-categories according to the amplitude of objects:

Definition 3.15. For an integer $n \geq 0$, we write $\mathrm{D_{lis}^{\{-n,n\}}}(X, \Lambda)$ for the full subcategory of $\mathrm{D_{lis}}(X, \Lambda)$ of objects M such that M and its dual $M^\vee$ lie in degrees $[-n, n]$ with respect to the t-structure on $\mathrm{D}(X, \Lambda)$.

The purpose of introducing this subcategory is to have a $(2n + 1)$-category: $M \cong \tau^{\leq n}\tau^{\geq -n}M$ for each such object. For example, the category $\mathrm{D_{lis}^{\{0,0\}}}(X, \Lambda)$ is the full subcategory of dualizable objects in $\mathrm{D}(X, \Lambda)^\heartsuit$, that is, those Λ-sheaves M that are locally on $X_{\mathrm{pro\acute{e}t}}$ isomorphic to $\underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_X$ for some finite projective $\Gamma(X, \Lambda)$-module N.

Lemma 3.16. *Assume that X is qcqs. Then*

$$\mathrm{D_{lis}}(X, \Lambda) = \bigcup_{n \geq 0} \mathrm{D_{lis}^{\{-n,n\}}}(X, \Lambda)$$

as full subcategories of $\mathrm{D_{lis}}(X, \Lambda)$.

Proof. The condition of being in the subcategory $\mathrm{D_{lis}^{\{-n,n\}}}(X, \Lambda)$ can be checked proétale locally: the restriction functors are monoidal, conservative and preserve the t-structure. So we may assume that X is a w-contractible qcqs scheme. Then, under the equivalence $\mathrm{D_{lis}}(X, \Lambda) \cong \mathrm{Perf}_{\Gamma(X, \Lambda)}$ (see Lemma 3.7 (3)), an object lies in the subcategory $\mathrm{D_{lis}^{\{-n,n\}}}(X, \Lambda)$ if and only if it is represented by a bounded complex of finitely generated projective $\Gamma(X, \Lambda)$-modules that is concentrated in degrees $[-n, n]$. Hence, the lemma follows from the corresponding filtration $\mathrm{Perf}_{\Gamma(X, \Lambda)} = \bigcup_{n \geq 0} \mathrm{Perf}_{\Gamma(X, \Lambda)}^{\{-n,n\}}$ on perfect modules. $\qquad\square$

Corollary 3.17. *Every constructible Λ-sheaf on a qcqs scheme is bounded.*

Proof. By an induction on the finite number of strata witnessing the constructibility, using the conservativity and t-exactness of the pair of functors (j^*, i^*) in the notation of (3.3), one reduces to the case of lisse sheaves. So we are done by Lemma 3.16. $\qquad\square$

Lemma 3.18. *Let $n \geq 0$ be an integer.*

(1) For any map $f \colon Y \to X$ of schemes the pullback functor f^* restricts to a functor

$$f^* \colon D_{\mathrm{lis}}^{\{-n,n\}}(X,\Lambda) \to D_{\mathrm{lis}}^{\{-n,n\}}(Y,\Lambda).$$

(2) For any map of condensed rings $\Lambda \to \Lambda'$ the base change functor $(\text{-}) \otimes_{\Lambda_X} \Lambda'_X$ restricts to a functor

$$(\text{-}) \otimes_{\Lambda_X} \Lambda'_X \colon D_{\mathrm{lis}}^{\{-n,n\}}(X,\Lambda) \to D_{\mathrm{lis}}^{\{-n,n\}}(X,\Lambda').$$

(3) The functor $X \mapsto D_{\mathrm{lis}}^{\{-n,n\}}(X,\Lambda)$ satisfies hyperdescent on $X_{\mathrm{pro\acute{e}t}}$.

Proof. Part (1) is clear since f^* is t-exact and, being monoidal, preserves duals. For (2) we use that $\otimes$ is right t-exact in general. On the other hand,

$$M \mapsto M \otimes_{\Lambda_X} \Lambda'_X = (M^\vee)^\vee \otimes_{\Lambda_X} \Lambda'_X = \underline{\mathrm{Hom}}_{\Lambda_X}(M^\vee, \Lambda'_X)$$

is also left t-exact. Part (3) is immediate from Corollary 3.13 and (1), using that the condition of lying in the subcategory $D_{\mathrm{lis}}^{\{-n,n\}}(X,\Lambda)$ can be checked proétale locally. $\qquad\qquad\square$

3.4. **Change of coefficients.** We show that the category of lisse and constructible sheaves behaves well under certain sequential limits and filtered colimits in the condensed coefficients Λ. Throughout, let X be a scheme.

3.4.1. *Sequential limits.* In this subsection, let $\Lambda = \lim_{i \geq 1} \Lambda_i$ be a sequential limit of condensed rings such that all transition maps $\Lambda_{i+1} \to \Lambda_i$ are surjective with locally nilpotent kernel. The last condition means that, for all profinite sets S, all elements of the kernel of $\Gamma(S, \Lambda_{i+1}) \to \Gamma(S, \Lambda_i)$ are nilpotent. We note that Λ_X identifies via the natural map with

$$\Lambda_X \xrightarrow{\cong} \lim \Lambda_{i,X} \cong \mathrm{Rlim}\,\Lambda_{i,X},$$

where we use that sequential limits of surjections are exact in a replete topos, see [BS15, Proposition 3.1.10]. In the following all limits will be derived unless mentioned otherwise. Also recall the generalities about limits of stable (idempotent complete) ∞-categories from Section 2.2.

Proposition 3.19. *The natural functors*

$$D_{\mathrm{lis}}(X,\Lambda) \xrightarrow{\cong} \lim D_{\mathrm{lis}}(X,\Lambda_i), \quad D_{\mathrm{cons}}(X,\Lambda) \xrightarrow{\cong} \lim D_{\mathrm{cons}}(X,\Lambda_i)$$

are equivalences. Both limits are formed using $(\text{-}) \otimes_{\Lambda_j} \Lambda_i$ for $j \geq i$. An inverse functor is given by $\{M_i\} \mapsto \lim M_i$.

Proof. We start with the categories of lisse sheaves. Both functors $X \mapsto D_{\mathrm{lis}}(X,\Lambda), \lim D_{\mathrm{lis}}(X,\Lambda_i)$ are hypersheaves on $X_{\mathrm{pro\acute{e}t}}$ by Corollary 3.13. So we reduce to the case where X is w-contractible and affine. Using Lemma 3.7 (3), we get a commutative (up to equivalence) diagram:

$$
\begin{array}{ccc}
D_{\mathrm{lis}}(X,\Lambda) & \longrightarrow & \lim D_{\mathrm{lis}}(X,\Lambda_i) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{Perf}_{\Gamma(X,\Lambda)} & \longrightarrow & \lim \mathrm{Perf}_{\Gamma(X,\Lambda_i)}
\end{array}
$$

Since X is w-contractible and affine, all transition maps $\Gamma(X,\Lambda_{i+1}) \to \Gamma(X,\Lambda_i)$ are surjective with locally nilpotent kernel. Thus, the lower horizontal functor is an equivalence by Lemma 2.1 (2).

As for constructible sheaves, we claim that the map

$$M \xrightarrow{\cong} \lim (M \otimes_{\Lambda_X} \Lambda_{i,X})$$

is an equivalence for any $M \in D_{\mathrm{cons}}(X,\Lambda)$: the functor ι^* commutes with limits for any constructible locally closed immersion $\iota \colon Z \to X$ by [BS15, Corollary 6.1.5]. Using the localization sequences (3.3) we may assume that M is lisse where our claim is already proven. This formally implies the full faithfulness: for any $N \in D_{\mathrm{cons}}(X,\Lambda)$, $M \in D(X,\Lambda)$,

$$\mathrm{Hom}_{\Lambda_X}(M,N) \cong \lim \mathrm{Hom}_{\Lambda_X}(M, N \otimes_{\Lambda_X} \Lambda_{i,X}) \cong \lim \mathrm{Hom}_{\Lambda_{i,X}}(M \otimes_{\Lambda_X} \Lambda_{i,X}, N \otimes_{\Lambda_X} \Lambda_{i,X}).$$

For essential surjectivity, it is enough to show that, for any $\{M_i\} \in \lim D_{\mathrm{cons}}(X,\Lambda_i)$ and fixed $j \geq 1$, the limit $\lim M_i \in D(X_{\mathrm{pro\acute{e}t}}, \Lambda)$ is constructible and the natural map

$$(\lim M_i) \otimes_{\Lambda_X} \Lambda_{j,X} \longrightarrow M_j \tag{3.8}$$

is an equivalence. As before, we reduce to the case where M_j is lisse. We claim that $\lim M_i$ is lisse as well. Evaluating at any w-contractible affine $U \in X_{\mathrm{pro\acute{e}t}}$ gives

$$(\lim \mathrm{R}\Gamma(U,M_i)) \otimes_{\Gamma(U,\Lambda)} \Gamma(U,\Lambda_j) \longrightarrow \mathrm{R}\Gamma(U,M_j) \tag{3.9}$$

14

Since $\mathrm{R}\Gamma(U, M_j)$, and hence $\mathrm{R}\Gamma(U, M_1)$, is perfect by Lemma 3.7 (3), this map is an equivalence and $\lim \mathrm{R}\Gamma(U, M_i) = \mathrm{R}\Gamma(U, \lim M_i)$ is perfect by [Sta17, Tag 0CQG]. Since U was arbitrary, (3.8) is an equivalence. To see that $\lim M_i$ is lisse, it is remains (Corollary 3.14) to show that the natural map

$$(\mathrm{R}\Gamma(U, \lim M_i)) \otimes_{\Gamma(U, \Lambda)} \Gamma(V, \Lambda) \longrightarrow \mathrm{R}\Gamma(V, \lim M_i)$$

is an equivalence for any map of w-contractible affines $V \to U$ in $X_{\mathrm{pro\acute{e}t}}$. Using the conservativity of (-) $\otimes_{\Gamma(V,\Lambda)}$ $\Gamma(V, \Lambda_j)$ on perfect complexes proven in Lemma 2.1 (2), this follows from (3.9). $\qquad \square$

3.4.2. *Filtered colimits.* In this subsection, we assume that $\Lambda = \mathrm{colim}\, \Lambda_i$ is a filtered colimit of condensed rings Λ_i. Recall the generalities about filtered colimits of stable (idempotent complete) ∞-categories from Section 2.2.

Proposition 3.20. *If X is qcqs, then the natural functors*

$$\mathrm{colim}\, \mathrm{D}_{\mathrm{lis}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \mathrm{colim}\, \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda),$$

are equivalences. Both filtered colimits are formed using (-) $\otimes_{\Lambda_i} \Lambda_j$ *for $j \geq i$.*

Proof. For lisse sheaves, it suffices by Lemma 3.16 to show that the functor

$$\mathrm{colim}_i\, \mathrm{D}_{\mathrm{lis}}^{\{-n,n\}}(X, \Lambda_i) \longrightarrow \mathrm{D}_{\mathrm{lis}}^{\{-n,n\}}(X, \Lambda)$$

is an equivalence for any fixed $n \geq 0$. Both sides satisfy hyperdescent on $X_{\mathrm{pro\acute{e}t}}$ (Lemma 3.18 (3) using Lemma 2.2), so we may assume that X is w-contractible qcqs. In this case we have $\mathrm{D}_{\mathrm{lis}}^{\{-n,n\}}(X, \Lambda) \cong \mathrm{Perf}_{\Gamma(X,\Lambda)}^{\{-n,n\}}$ by Lemma 3.7 (3), see also the proof of Lemma 3.16. Since X is qcqs, we have a presentation $\Gamma(X, \Lambda) = \mathrm{colim}\, \Gamma(X, \Lambda_i)$ as a filtered colimit of rings. We conclude using Lemma 2.1 (1).

As for constructible sheaves we note that for any constructible locally closed immersion $\iota : Z \to X$ and $M \in \mathrm{D}(Z_{\mathrm{pro\acute{e}t}}, \Lambda)$, $N \in \mathrm{D}(X_{\mathrm{pro\acute{e}t}}, \Lambda)$ we have

$$\iota_!(M \otimes_{\Lambda_Z} \iota^* N) \cong \iota_! M \otimes_{\Lambda_X} N \tag{3.10}$$

by [BS15, Lemma 6.2.3 (3)]. Applying this with $N = \Lambda_i$ and using standard arguments involving the fiber sequence $j_! j^* \to \mathrm{id} \to i_* i^*$ in the notation of (3.3) the essential surjectivity follows from the case of lisse sheaves. For full faithfulness, it suffices to show (after using standard $\otimes$-Hom-adjunctions) that, for any $M, N \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda_i)$, the natural map

$$\mathrm{colim}_{j \geq i}\, \mathrm{Hom}_{\Lambda_{X,i}}\left(M, N \otimes_{\Lambda_{X,i}} \Lambda_{X,j}\right) \longrightarrow \mathrm{Hom}_{\Lambda_{X,i}}\left(M, \mathrm{colim}_{j \geq i}\, N \otimes_{\Lambda_{X,i}} \Lambda_{X,j}\right)$$

is an equivalence. By Lemma 3.21 below, it is enough to show that there exists an integer $n \geq 0$ such that $N \otimes_{\Lambda_{i,X}} \Lambda_{j,X} \in \mathrm{D}^{\geq -n}(X, \Lambda_i)$ for all $j \geq i$. Using that X is qcqs, we can perform an induction on the number of strata of a stratification witnessing the constructibility of N. Applying (3.10) to $\iota_! \iota^* N \otimes_{\Lambda_{i,X}} \Lambda_{j,X}$ and using the t-exactness of $\iota_!$ (Lemma 3.4), we may then assume that N is lisse. Then it is lies in $\mathrm{D}_{\mathrm{lis}}^{\{-n,n\}}(X, \Lambda_i)$ for $n \gg 0$, so that $N \otimes_{\Lambda_{i,X}} \Lambda_{j,X}$ lies in the same subcategory as well, see Lemma 3.18 (2). $\qquad \square$

In the proof, we used the following general lemma. An analogous result for étale sheaves is proven in [BS15, Lemma 6.3.14]:

Lemma 3.21. *For any fixed integer $n \in \mathbb{Z}$, the following functors commute with filtered colimits with terms in $\mathrm{D}^{\geq n}(X, \Lambda)$:*

(1) $f_* : \mathrm{D}(X, \Lambda) \to \mathrm{D}(Y, \Lambda)$ for any qcqs map $f : X \to Y$;

(2) $\underline{\mathrm{Hom}}_{\Lambda_X}(M, \text{-}) : \mathrm{D}(X, \Lambda) \to \mathrm{D}(X, \Lambda)$ for any qcqs scheme X and $M \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$.

In particular, under the conditions in (2), the functor

$$\mathrm{Hom}_{\Lambda_X}(M, \text{-}) = \mathrm{R}\Gamma\left(X, \underline{\mathrm{Hom}}_{\Lambda_X}(M, \text{-})\right) : \mathrm{D}(X, \Lambda) \to \mathrm{Mod}_{\Gamma(X,\Lambda)}$$

commutes with such colimits as well.

Proof. For the final assertion, we apply (1) to the map of sites $f = p_X : X_{\mathrm{pro\acute{e}t}} \to *_{\mathrm{pro\acute{e}t}}$. Then $\mathrm{R}\Gamma(X, \text{-})$ is the composition of the functors

$$\mathrm{D}(X, \Lambda) \xrightarrow{f_*} \mathrm{D}(*, \Lambda) \xrightarrow{\mathrm{R}\Gamma(*, \text{-})} \mathrm{Mod}_{\Gamma(X,\Lambda)},$$

and hence commutes with filtered colimits with terms in $\mathrm{D}^{\geq n}(X, \Lambda)$ as well. So using (2), we see that $\mathrm{Hom}_{\Lambda_X}(M, \text{-})$ commutes with such filtered colimits as well. Here we use that any constructible sheaf on a qcqs scheme is bounded (see Corollary 3.17), so that the functor $\underline{\mathrm{Hom}}_{\Lambda_X}(M, \text{-})$ maps $\mathrm{D}^{\geq n}(X, \Lambda)$ into $\mathrm{D}^{\geq m}(X, \Lambda)$ for some $m \leq n$.

For (1), let $N = \mathrm{colim}\, N_j$ be a filtered colimit of some sheaves $N_j \in \mathrm{D}^{\geq n}(X, \Lambda)$. It is enough to show that the natural map

$$\mathrm{colim}_j\, \mathrm{H}^p \circ f_*(N_j) \to \mathrm{H}^p \circ f_*(N) \tag{3.11}$$

is an equivalence in $D(Y,\Lambda)^{\heartsuit}$ for any $p \in \mathbb{Z}$, $H^p := \tau^{\leq p} \circ \tau^{\geq p}$. As filtered colimits are t-exact we can write $N = \operatorname{colim} N_j = \operatorname{colim}_{m,j} \tau^{\leq m} N_j$. By left exactness of f_* only the terms $\tau^{\leq p} N_j, \tau^{\leq p} N$ contribute to $H^p \circ f_*$ and we may assume $N_j, N \in D^{[n,p]}(X,\Lambda)$. An induction on the length $p - n$ reduces us further to the case where N_j, N are in a single t-degree. So after possibly renumbering we may assume $N_j, N \in D(X,\Lambda)^{\heartsuit}$ embedded in degree 0. Evaluating (3.11) at any $V \in Y_{\text{proét}}$ w-contractible affine it is enough to show that

$$\operatorname{colim}_j H^p(X \times_Y V, N_j) = \operatorname{colim}_j \Gamma(V, H^p \circ f_*(N_j)) \to \Gamma(V, H^p \circ f_*(N)) = H^p(X \times_Y V, N)$$

is an isomorphism. By our assumption on f, the base change $X \times_Y V$ is qcqs. It remains to show that for any qcqs scheme X the cohomology functor $H^p(X,\text{-}) : D(X,\Lambda)^{\heartsuit} \to \operatorname{Mod}^{\heartsuit}_{\Gamma(X,\Lambda)}$ commutes with filtered colimits. Choosing a hypercover $U_\bullet \to X$ in $X_{\text{proét}}$ by w-contractible affine schemes, this can be computed as the p-th cohomology of the complex

$$\ldots \to 0 \to \Gamma(U_0, \text{-}) \to \Gamma(U_1, \text{-}) \to \Gamma(U_2, \text{-}) \to \ldots .$$

As each $\Gamma(U_i, \text{-})$, $i \geq 0$ commutes with filtered colimits so does $H^p(X,\text{-})$.

For (2), let X be qcqs and $M \in D_{\text{cons}}(X,\Lambda)$. We claim that the functor $\underline{\operatorname{Hom}}_{\Lambda_X}(M,\text{-})$ commutes with filtered colimits with terms in $D^{\geq 0}(X_{\text{proét}},\Lambda)$. If M is lisse (=dualizable), then $\underline{\operatorname{Hom}}_{\Lambda_X}(M,\text{-}) = (\text{-}) \otimes_{\Lambda_X} M^\vee$ commutes with all colimits. In general, by an induction on the finite number of strata in X witnessing the constructibility of M, we reduce to the case $M = \iota_! \iota^* M$ where $\iota^* M$ is lisse for some constructible locally closed immersion $\iota : Z \hookrightarrow X$. Using standard adjunctions we compute

$$\underline{\operatorname{Hom}}_{\Lambda_X}(\iota_! \iota^* M, \text{-}) = \iota_* \underline{\operatorname{Hom}}_{\Lambda_Z}(\iota^* M, \iota^!(\text{-})) = \iota_* (\iota^!(\text{-}) \otimes_{\Lambda_Z} (\iota^* M)^\vee).$$

Note that $\iota^!$ is left t-exact (as the right adjoint of the t-exact functor $\iota_!$), so preserves the subcategory $D^{\geq 0}(X,\Lambda)$. In light of (1) applied to ι_*, it remains to show that $\iota^!$ commutes with the desired colimits. If ι is an open immersion, then the t-exact functor $\iota^! = \iota^*$ commutes with all colimits. We reduce to the case where $\iota = i : Z \hookrightarrow X$ is a constructible closed immersion with open complement $j : U \hookrightarrow X$. Then the fiber sequence $i_* i^! \to \operatorname{id} \to j_* j^*$ and (1) applied to j_* shows that $i^!$ commutes with the desired colimits as well. $\square$

Remark 3.22. The condition that the filtered colimit is formed using objects in $D^{\geq n}(X,\Lambda)$ can not in general be dropped in Lemma 3.21 (see however Lemma 3.53 for a positive result in this direction): Assume that $R\Gamma(X,\Lambda)$ is concentrated in infinitely many degrees. For example, $R\Gamma(\operatorname{Spec}(\mathbb{R}), \mathbb{Z}/2)$ computes the group cohomology of $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) = \mathbb{Z}/2$ on the trivial module $\mathbb{Z}/2$ which is equal to $\mathbb{Z}/2$ in all even positive degrees. Since $D(X,\Lambda)$ is left-complete [BS15, Proposition 3.3.3], the natural map $\oplus_{n \geq 0} \Lambda[n] \to \prod_{n \geq 0} \Lambda[n]$ is an equivalence. If $H^0(X,\text{-})$ would commute with infinite direct sums, then we could compute

$$\bigoplus_{n \geq 0} H^0(X, \Lambda[n]) = H^0\left(X, \bigoplus_{n \geq 0} \Lambda[n]\right) = H^0\left(X, \prod_{n \geq 0} \Lambda[n]\right) = \prod_{n \geq 0} H^0(X, \Lambda[n]),$$

which is a contradiction.

3.4.3. *Localizations.* In this subsection, let Λ be a condensed ring and $T \subset \Gamma(*,\Lambda)$ a multiplicatively closed subset. Then the localization $T^{-1}\Lambda = \operatorname{colim}_{t \in T} \Lambda$ viewed as a filtered colimit of sheaves defines a condensed ring. Since $*$-pullbacks commute with colimits, we have $T^{-1}\Lambda_X = \operatorname{colim}_{t \in T} \Lambda_X$. Its values on any qcqs $U \in X_{\text{proét}}$ are computed as

$$\Gamma(U, T^{-1}\Lambda) = \operatorname{colim}_{t \in T} \Gamma(U, \Lambda) = T^{-1}\Gamma(U, \Lambda).$$

The second equality is clear, and the first equality is an instance of (3.11). Let $\Lambda_* := \Gamma(*, \Lambda)$, and denote by $T^{-1}\Lambda_*$ its localization.

Proposition 3.23. *If X is qcqs, then the natural functors*

$$D_{\text{lis}}(X,\Lambda) \otimes_{\operatorname{Perf}_{\Lambda_*}} \operatorname{Perf}_{T^{-1}\Lambda_*} \to D_{\text{lis}}(X, T^{-1}\Lambda), \quad D_{\text{cons}}(X,\Lambda) \otimes_{\operatorname{Perf}_{\Lambda_*}} \operatorname{Perf}_{T^{-1}\Lambda_*} \to D_{\text{cons}}(X, T^{-1}\Lambda),$$

induced by $M \otimes_{\Lambda_*} T^{-1}\Lambda_* \mapsto M \otimes_{\Lambda_X} T^{-1}\Lambda_X$ are fully faithful.

Proof. For $M \in D_{\text{cons}}(X,\Lambda)$, we denote $T^{-1}M = T^{-1}\Lambda_X \otimes_{\Lambda_X} M$. Using that $\operatorname{Perf}_{T^{-1}\Lambda_*}$ is generated under finite colimits by $T^{-1}\Lambda_*$, it is enough to show that the natural map

$$\operatorname{Hom}_{\Lambda_X}(M, N) \otimes_{\Lambda_*} T^{-1}\Lambda_* \to \operatorname{Hom}_{\Lambda_X}(M, T^{-1}N) = \operatorname{Hom}_{T^{-1}\Lambda_X}(T^{-1}M, T^{-1}N)$$

is an equivalence for any $M, N \in D_{\text{cons}}(X,\Lambda)$. This follows from Lemma 3.21. $\square$

The functor from $T1$-topological abelian to condensed abelian groups does not commute with filtered colimits in general. However, the following lemma shows, for example, that $\mathbb{Q}_\ell = \operatorname{colim}_{\ell \times} \mathbb{Z}_\ell$ and that $\bar{\mathbb{Q}}_\ell = \operatorname{colim}_{E/\mathbb{Q}_\ell \text{ finite}} E$ (writing each E as a filtered colimit of $\mathcal{O}_E$'s) holds as condensed rings:

Lemma 3.24. *Let $\Lambda = \operatorname{colim} \Lambda_i$ be a countable filtered colimit of quasi-compact Hausdorff topological abelian groups with injective transition maps. Then the induced map of condensed abelian groups $\operatorname{colim}_i \Lambda_i \to \Lambda$ is an isomorphism.*

Proof. First off, we note that filtered colimits exist in the category of topological abelian groups (or topological rings) and are formed by taking the colimit in the category of abelian groups (or rings) equipped with its colimit topology. It is enough to show that the map $\operatorname{colim} \Gamma(S, \Lambda_i) \to \Gamma(S, \Lambda)$ is an isomorphism for any profinite set S. Injectivity is clear. For surjectivity, we claim that every continuous map $S \to \Lambda$ factors through some Λ_i. As injections between quasi-compact Hausdorff spaces are closed embeddings, this follows from [BS15, Lemma 4.3.7]. $\qquad\square$

Remark 3.25. The functor between the categories of lisse sheaves in Proposition 3.23 is not an equivalence in general as the source category does not satisfy descent, compare with the discussion above [BS15, Lemma 7.4.7] (see, however, Theorem 3.45 for a positive result for constructible sheaves). More precisely, the failure of essential surjectivity accounts for the difference between the étale and proétale fundamental group. For example, let $X = \mathbb{P}^1/0 \sim \infty$ be the nodal curve over some algebraically closed field. Its proétale fundamental group (with respect to the choice of some geometric point) is $\pi_1^{\mathrm{proét}}(X) = \mathbb{Z}$ equipped with the discrete topology, whereas the étale fundamental group is its profinite completion $\pi_1^{\mathrm{ét}}(X) = \widehat{\mathbb{Z}}$. So the category $\mathrm{D}_{\mathrm{lis}}^{\{0,0\}}(X, \mathbb{Q}_\ell)$ is the category of (continuous) representations of $\mathbb{Z}$ on finite-dimensional $\mathbb{Q}_\ell$-vector spaces, whereas the source category corresponds to the strict full subcategory of those representations stabilizing a $\mathbb{Z}_\ell$-lattice.

3.5. **Local constancy of lisse sheaves.** Recall from Definition 3.5 that a sheaf $M \in \mathrm{D}(X, \Lambda)$ is called proétale-locally perfect-constant if M is locally on $X_{\mathrm{proét}}$ isomorphic to $\underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_X$ for some $N \in \mathrm{Perf}_{\Lambda_*}$, where $\Lambda_* = \Gamma(*, \Lambda)$ is the underlying ring.

Theorem 3.26. *Let Λ be a condensed ring. Let X be a scheme that has locally a finite number of irreducible components. Then $M \in \mathrm{D}(X, \Lambda)$ is lisse if and only if M is proétale-locally perfect-constant.*

Proof. Let M be lisse (the other direction is clear). After a Zariski localization, we reduce to the case where X is affine and connected with finitely many irreducible components. As any two points of X can be joined by a finite zig-zag of specializations, the pullback of M to any geometric point is perfect-constant (Lemma 3.7) with the same value $N \in \mathrm{Perf}_{\Lambda_*}$. Let $U \in X_{\mathrm{proét}}$ be any w-contractible affine cover. We claim that there exists an isomorphism $M|_U \simeq \underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_U$, implying the theorem.

First, assume that X is irreducible, and fix a geometric generic point $\eta \to X$. Let $U_\eta := U \times_X \eta$ be the base change, and consider the commutative diagram of sites

$$
\begin{array}{ccccc}
\eta_{\mathrm{proét}} & \longleftarrow & (U_\eta)_{\mathrm{proét}} & \xrightarrow[\cong]{\ \pi_{U_\eta}\ } & (\pi_0 U_\eta)_{\mathrm{proét}} \\
\downarrow & & \downarrow & & \Big\updownarrow \\
X_{\mathrm{proét}} & \longleftarrow & U_{\mathrm{proét}} & \xrightarrow{\ \pi_U\ } & (\pi_0 U)_{\mathrm{proét}}
\end{array}
\tag{3.12}
$$

Here π_{U_η} is an equivalence because η is a geometric point and $U_\eta \to \eta$ is proétale. Further, the map $\pi_0 U_\eta \to \pi_0 U$ is surjective and admits a splitting: the map on topological spaces $|U_\eta| \to |U| \times_{|X|} |\eta|$ is surjective by [Sta17, Tag 03H4], and hence induces a surjection on connected components. So we need to see that the image of every connected component of $|U|$ under the map $|U| \to |X|$ contains the unique generic point of the irreducible space $|X|$. This is true because the map $|U| \to |X|$ is generalizing and connected components are closed under generalizations. Now as U is w-contractible affine, so $\pi_0 U$ is extremally disconnected profinite, there exists a section to the surjection $\pi_0 U_\eta \to \pi_0 U$.

To finish the argument in the irreducible case, we apply $\mathrm{D}_{\mathrm{lis}}(\text{-}, \Lambda)$ to the diagram (3.12) and observe that π_U induces an equivalence by Proposition 3.9 (3). More concretely, any isomorphism $M|_\eta \simeq \underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_\eta$ induces an isomorphism over U_η, and hence over U by using the section.

Next, if $X = \cup X_i$ is the union of finitely many irreducible components, then we denote $U_i = X_i \times_X U$. It follows from the irreducible case that there exist isomorphisms $\varphi_i \colon M|_{U_i} \simeq \underline{N} \otimes_{\Lambda_*} \Lambda_{U_i}$. As U is w-contractible affine, so are the closed subschemes $U_i \subset U$ by [BS15, Lemma 2.2.15]. We denote by $\tilde{U}_i \to U$ their Henselizations. Using Corollary 3.10 (2), the isomorphisms φ_i uniquely extend to $\tilde{\varphi}_i \colon M|_{\tilde{U}_i} \simeq \underline{N} \otimes_{\Lambda_*} \Lambda_{\tilde{U}_i}$. As there are only finitely many irreducible components, the disjoint union $\sqcup \tilde{U}_i \to U$ is a cover in $X_{\mathrm{proét}}$, and hence admits a section because U is w-contractible affine. Therefore, we can pullback the isomorphism $\sqcup \tilde{\varphi}_i$ along the section to obtain an isomorphism $M|_U \simeq \underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_U$ as desired. $\qquad\square$

3.6. **t-Structures.** The definition of lisse and constructible sheaves is well-adapted to the derived setting. The natural t-structure on the category of all sheaves only restricts to a t-structure on the categories of lisse and constructible sheaves under additional assumptions on the scheme X and the condensed ring of coefficients Λ. Let us denote by

$$
\mathrm{D}_{\mathrm{lis}}^{\geq 0}(X, \Lambda) := \mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cap \mathrm{D}^{\geq 0}(X, \Lambda), \quad \mathrm{D}_{\mathrm{cons}}^{\geq 0}(X, \Lambda) := \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \cap \mathrm{D}^{\geq 0}(X, \Lambda)
\tag{3.13}
$$

the full subcategories of $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, respectively $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$, and likewise for the subcategories in cohomological degrees ≤ 0. Following [Gla89, Chapter 6, Section 2], we say that a (unital, commutative) ring is *regular coherent* if every finitely generated ideal is finitely presented (that is, the ring is coherent [Sta17, Tag 05CU]) and has finite projective dimension.

Definition 3.27. A condensed ring Λ is called *t-admissible* if the underlying ring $\Lambda_* = \Gamma(*, \Lambda)$ is regular coherent and $\Lambda_* \to \Gamma(S, \Lambda)$ is flat for any extremally disconnected profinite set S.

We show in Proposition 3.34 that the flatness condition is automatic for a discrete topological ring Λ (viewed as a condensed ring as in Example 3.1). Thus, Λ is t-admissible if and only if Λ_* is regular coherent. For example, this holds if Λ_* is regular Noetherian of finite Krull dimension, see Lemma 3.33 (4). Further examples of t-admissible condensed rings include all T1-topological rings such that Λ_* is semi-hereditary, see Lemma 3.36. This covers algebraic field extensions of $E \supset \mathbb{Q}_\ell$ for some prime ℓ and their rings of integers $\mathcal{O}_E$, but also more exotic choices such as the real and complex numbers $\mathbb{R}$, $\mathbb{C}$ with their Euclidean topology and the ring of adeles $\mathbb{A}_K$ for some number field K, see Corollary 3.37.

Theorem 3.28. *Let Λ be a condensed ring. Then Λ is t-admissible if and only if the natural t-structure on $\mathrm{D}(*, \Lambda)$ restricts to a t-structure on $\mathrm{D}_{\mathrm{lis}}(*, \Lambda)$. In this case, assume that X has Zariski locally a finite number of irreducible components, respectively that locally every constructible subset has so. Then the categories in (3.13) define a t-structure on $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$, respectively $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$, and the following hold:*
   (1) The heart $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)^{\heartsuit}$ is the full subcategory of $\mathrm{D}(X, \Lambda)^{\heartsuit}$ of sheaves M that are locally on $X_{\mathrm{proét}}$ isomorphic to $\underline{N} \otimes_{\Lambda_*} \Lambda_X$ for some finitely presented Λ_*-module N.
   (2) The heart $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)^{\heartsuit}$ is the full subcategory of $\mathrm{D}(X, \Lambda)^{\heartsuit}$ of sheaves M such that, for any open affine $U \subset X$, there exists a finite subdivision of U into constructible locally closed subschemes $U_i \subset U$ such that $M|_{U_i} \in \mathrm{D}_{\mathrm{lis}}(X, \Lambda)^{\heartsuit}$.

Corollary 3.29. *Let Λ be a t-admissible condensed ring. Let X be a qcqs scheme having locally a finite number of irreducible components. Then $M \in \mathrm{D}(X, \Lambda)$ is lisse if and only if M is bounded and each cohomology sheaf is locally on $X_{\mathrm{proét}}$ isomorphic to $\underline{N} \otimes_{\Lambda_*} \Lambda_X$ for some finitely presented Λ_*-module N.*

Proof. If M is lisse, then M is bounded (as X is qcqs, see Lemma 3.16) and each cohomology sheaf $\mathrm{H}^p(M)$, $p \in \mathbb{Z}$ is lisse, using the t-admissibility of Λ (Theorem 3.28). The converse follows from an easy induction on the length using that $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ is stable. $\square$

Remark 3.30. Some finiteness assumption on X is necessary in order to have a t-structure on $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ such that the inclusion into $\mathrm{D}(X, \Lambda)$ is t-exact. As a concrete example take $X = \beta\mathbb{N}$, $\Lambda = \mathbb{Q}_\ell$ and $f \in \Gamma(X, \Lambda)$ as in Example 3.8. Let K be the kernel of $f: \Lambda_X \to \Lambda_X$ formed in $\mathrm{D}(X, \Lambda)^{\heartsuit}$. Then $\mathrm{R}\Gamma(X, K) = \Gamma(X, K) = 0$, but $K \neq 0$ as its stalks at the boundary $\partial X = \beta\mathbb{N}\backslash\mathbb{N}$ are non-zero. When combined with the equivalence $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Perf}_{\Gamma(X, \Lambda)}$ from Lemma 3.7 (3), this shows that K is not lisse. In view of Corollary 3.14, the failure is explained by the lack of the depicted base change property in this corollary. As a warning, let us point out that $\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Perf}_{\Gamma(X, \Lambda)}$ inherits the t-structure from $\mathrm{Mod}_{\Gamma(X, \Lambda)}$ because $\Gamma(X, \Lambda) = \mathrm{Maps}_{\mathrm{cont}}(\beta\mathbb{N}, \mathbb{Q}_\ell)$ is regular coherent, see Theorem 3.38. However, if one equips $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ with this t-structure, the inclusion into $\mathrm{D}(X, \Lambda)$ will not be t-exact.

The proof of Theorem 3.28 relies on the following key characterization of regular coherent rings. We first provide a well-known auxiliary lemma:

Lemma 3.31. *A ring Λ is coherent if and only if the subcategory $\mathrm{Mod}_{\Lambda}^{\heartsuit, \mathrm{fp}} \subset \mathrm{Mod}_{\Lambda}^{\heartsuit}$ (of the abelian category of Λ-modules) spanned by the finitely presented Λ-modules is abelian.*

Proof. If Λ is coherent, then $\mathrm{Mod}_{\Lambda}^{\heartsuit, \mathrm{fp}}$ is abelian by [Sta17, Tag 05CW]. Conversely, assume that $\mathrm{Mod}_{\Lambda}^{\heartsuit, \mathrm{fp}}$ is abelian. If $I \subset \Lambda$ is a finitely generated ideal, then $\Lambda \to \Lambda/I$ is a map of finitely presented Λ-modules, and hence $I = \ker(\Lambda \to \Lambda/I)$ is finitely presented as well. Here we used that the inclusion $\mathrm{Mod}_{\Lambda}^{\heartsuit, \mathrm{fp}} \subset \mathrm{Mod}_{\Lambda}^{\heartsuit}$ is left-exact. This shows that Λ is coherent. $\square$

Proposition 3.32. *For any ring Λ, the following are equivalent:*
   (1) The natural t-structure on Mod_{Λ} restricts to a t-structure on Perf_{Λ}.
   (2) The ring Λ is coherent and every finitely presented Λ-module has finite Tor dimension.
   (3) The ring Λ is coherent and every finitely generated ideal has finite Tor dimension.
   (4) The ring Λ is regular coherent.

Proof. Assume that (1) holds. Then there are inclusions
$$\mathrm{Mod}_{\Lambda}^{\heartsuit, \mathrm{fp}} \subset \mathrm{Perf}_{\Lambda}^{\heartsuit} \subset \left(\mathrm{Mod}_{\Lambda}^{\heartsuit}\right)^{\omega} \tag{3.14}$$

of full subcategories of $\mathrm{Mod}_\Lambda^\heartsuit$ of finitely presented Λ-modules at the left and of compact objects at the right: The first inclusion means that every finitely presented Λ-module is of the form $\mathrm{H}^0(M)$ for some $M \in \mathrm{Perf}_\Lambda$ which is clear. For the second inclusion, we note that any perfect complex is compact in Mod_Λ [Sta17, Tag 07LT] and that the inclusion $\mathrm{Mod}_\Lambda^\heartsuit \subset \mathrm{Mod}_\Lambda$ is full and preserves filtered colimits. It is well-known that the categories at the left and at the right in (3.14) agree. Thus, both inclusions are equalities. Being the heart of a t-structure, $\mathrm{Mod}_\Lambda^{\heartsuit,\mathrm{fp}}$ is abelian, so that Λ is coherent (Lemma 3.31). The inclusion $M \in \mathrm{Mod}_\Lambda^{\heartsuit,\mathrm{fp}} = \mathrm{Perf}_\Lambda^\heartsuit \subset \mathrm{Perf}_\Lambda$ shows that every finitely presented Λ-module M is perfect. By [Sta17, Tag 0658], this is equivalent to M being pseudo-coherent (or, almost perfect) and of finite Tor dimension. This implies (2).

Conversely, assume that (2) holds. Then $\mathrm{Mod}_\Lambda^{\heartsuit,\mathrm{fp}}$ is abelian (Lemma 3.31) so that every Λ-module of the form $\mathrm{H}^0(M)$, $M \in \mathrm{Perf}_\Lambda$ is finitely presented. Also, every finitely presented Λ-module is pseudo-coherent by [Sta17, Tag 0EWZ] and, hence perfect since it has finite Tor dimension [Sta17, Tag 0658]. So Perf_Λ is stable under the truncation functors $\tau^{\leq n}$, $\tau^{\geq n}$ for all $n \in \mathbb{Z}$. This implies (1) since the other properties of a t-structure are inherited from Mod_Λ.

It is clear that (2) implies (3). So assume that (3) holds. Let M be a finitely presented Λ-module. We need to show that there exists an integer $n > 0$ (possibly depending on M) such that $\mathrm{H}^p(N \otimes_\Lambda M) = 0$ for all $p > n$ and $N \in \mathrm{Mod}_\Lambda^\heartsuit$. The argument is similar to the proof of [Sta17, Tag 00HD]: As M is finitely presented, there is some $m \geq 1$ and an exact sequence $0 \to M' \to \Lambda^m \to M \to 0$. Then M' is finitely presented as well because Λ is coherent (so $\mathrm{Mod}_\Lambda^{\heartsuit,\mathrm{fp}}$ is abelian). We reduce to the case where $M \subset R^m$ is a submodule. If $m = 1$, then M is a finitely generated ideal and we are done. If $m \geq 2$, then there is an exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

where $M' = M \cap (R \oplus 0^{m-1}) \subset R$ and $M'' \subset R^{m-1}$ are submodules. By induction, there are finitely many finitely generated ideals in R whose Tor dimension bound the Tor dimension of M. This implies (2).

It remains to prove the equivalence of (3) and (4). If Λ is as in (3), then every finitely generated ideal admits a finite resolution by finite projective modules, using the equivalent characterization (1). Thus, (3) implies (4). Conversely, any finite projective resolution is K-flat. So ideals admitting such resolutions are of finite Tor dimension, proving (4) implies (3). $\qquad\square$

Proof of Theorem 3.28. First, assume that Λ is t-admissible and that X has locally a finite number of irreducible components, respectively every constructible subset has so. We show that the categories $\mathrm{D}_{\mathrm{lis}}(X,\Lambda)$, respectively $\mathrm{D}_{\mathrm{cons}}(X,\Lambda)$ are closed under the truncation functors $\tau^{\leq 0}$, and hence inherit the t-structure from $\mathrm{D}(X,\Lambda)$. Since restriction commutes with truncation functors, we reduce to the case of $\mathrm{D}_{\mathrm{lis}}(X,\Lambda)$ with X being affine and connected with finitely many irreducible components. So pick $M \in \mathrm{D}_{\mathrm{lis}}(X,\Lambda)$. We need to show that $\tau^{\leq 0}M$ is lisse as well. For any w-contractible affine cover $U \in X_{\mathrm{proét}}$, there is an isomorphism $M|_U \simeq \underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_U$ for some $N \in \mathrm{Perf}_{\Lambda_*}$, see Theorem 3.26. We compute

$$(\tau^{\leq 0}M)|_U \cong \tau^{\leq 0}M|_U \simeq \tau^{\leq 0}\left(\underline{N} \otimes_{\underline{\Lambda_*}} \Lambda_U\right) \xrightarrow{\cong} \underline{\tau^{\leq 0}N} \otimes_{\underline{\Lambda_*}} \Lambda_U,$$

where the last map is checked to be an isomorphism by evaluating at w-contractible affines $V \in U_{\mathrm{proét}}$ and using the flatness of $\Lambda_* \to \Gamma(V,\Lambda) = \Gamma(\pi_0 V,\Lambda)$. Note that $\mathrm{R}\Gamma(V,\text{-})$ is t-exact by Lemma 3.7 (1), that $\pi_0(V)$ is extremally disconnected [BS15, Lemma 2.4.8], and that Λ is assumed to be t-admissible. Further, since Λ_* is regular coherent, Proposition 3.32 shows that $\tau^{\leq 0}N \in \mathrm{Perf}_{\Lambda_*}$. So $\tau^{\leq 0}M$ is proétale-locally perfect-constant, thus lisse. Also, the description of the hearts in (1) and (2) follows immediately from (3.14).

It remains to show that t-admissibility is necessary in order to have the restricted t-structure on lisse sheaves on the point. So assume that the natural t-structure on $\mathrm{D}(*,\Lambda)$ restricts to a t-structure on $\mathrm{D}_{\mathrm{lis}}(*,\Lambda)$. In particular, the latter category is closed under truncation in $\mathrm{D}(*,\Lambda)$. As $\mathrm{R}\Gamma(*,\text{-})$ is t-exact, we see that $\mathrm{Perf}_{\Lambda_*}$ is closed under truncation in Mod_{Λ_*}. By the equivalent characterization in Proposition 3.32, the ring Λ_* is regular coherent. Similarly, using the t-exactness of $\mathrm{R}\Gamma(S,\text{-})$ for any extremally disconnected profinite set S, we see that the functor $\mathrm{Perf}_{\Lambda_*} \to \mathrm{Mod}_{\Gamma(S,\Lambda)}$, $N \mapsto N \otimes_{\Lambda_*} \Gamma(S,\Lambda)$ is t-exact. We claim that $\mathrm{Tor}_1^{\Lambda_*}(\Gamma(S,\Lambda),\Lambda_*/I) = 0$ for all finitely generated ideals $I \subset \Lambda_*$ which implies flatness of $\Lambda_* \to \Gamma(S,\Lambda)$ by [Sta17, Tag 00M5]. Indeed, as Λ_* is regular coherent, we see $\Lambda_*/I \in \mathrm{Perf}_{\Lambda_*}$ when placed in cohomological degree 0, say. By assumption, $(\Lambda_*/I) \otimes_{\Lambda_*} \Gamma(S,\Lambda)$ is concentrated in degree 0 so that we get the desired vanishing. We conclude that Λ is t-admissible which finishes the proof. $\qquad\square$

We now exhibit examples of t-admissible condensed rings. Throughout, we freely use the equivalent characterizations of regular coherent rings in Proposition 3.32. Recall that a *semi-hereditary* ring is one where every finitely generated ideal is projective.

Lemma 3.33. *There holds:*

(1) Filtered colimits of regular coherent rings with flat transition maps are regular coherent.

(2) *Localizations of regular coherent rings are regular coherent.*

(3) *Finite products of regular coherent rings are regular coherent.*

(4) *Regular Noetherian rings of finite Krull dimension and semi-hereditary rings are regular coherent.*

(5) *Localizations and arbitrary products of semi-hereditary rings are semi-hereditary.*

Proof. Clearly, semi-hereditary rings are regular coherent: every finitely generated projective module is finitely presented. The remainder of part (4) follows from [Sta17, Tags 00OE, 0CXE]. For (5), we use [Oli83, Section 4.2, Proposition] for products. The statement about localizations is readily checked. Also, part (3) is easy and left to the reader. For (1), we observe that every finitely generated ideal $I \subset \Lambda = \operatorname{colim} \Lambda_i$ is of the form $I_j \otimes_{\Lambda_j} \Lambda$ for some finitely generated ideal $I_j \subset \Lambda_j$, using the flatness of the transition maps. Thus, if I_j is finitely presented and of finite Tor dimension as a Λ_j-module, so is I as a Λ-module. Similarly, in (2), given a finitely generated ideal in an localization $I \subset T^{-1}\Lambda$, there exists a finitely generated ideal $J \subset \Lambda$ such that $I = T^{-1}J$. So, if J is finitely presented and of finite Tor dimension, so is I. $\qquad\square$

Now let Λ be a condensed ring. Recall that if Λ is associated with a topological ring, we have $\Gamma(S, \Lambda) = \operatorname{Maps_{cont}}(S, \Lambda)$ for any profinite set S, see Example 3.1. For discrete rings, the situation is as nice as possible.

Proposition 3.34. *For a discrete condensed ring Λ, the map $\Lambda_* \to \Gamma(S, \Lambda)$ is flat for any profinite set S. Thus, Λ is t-admissible if and only if Λ_* is regular coherent.*

Proof. Write $S = \lim S_i$ as a cofiltered limit of finite sets. As Λ is discrete, we have

$$\Gamma(S, \Lambda) = \operatorname{colim} \Gamma(S_i, \Lambda) = \operatorname{colim} \Lambda_*^{S_i}$$

which is flat, being a filtered colimit of free, hence flat Λ_*-modules. $\qquad\square$

Remark 3.35. Combining the above proof with Lemma 3.33 (1) and (3) shows that, more generally, $\Gamma(S, \Lambda)$ is regular coherent for any profinite set S provided Λ_* is so.

The above shows that for any $\ell > 0$, lisse sheaves with $\mathbb{Z}/\ell^2$-coefficients are not closed under truncation, which bars the attempt to construct the t-structure for $\mathbb{Z}_\ell$-coefficients by a direct limit argument. We do easily get the t-structure for $\Lambda = \mathbb{Z}_\ell$ using the following method, though.

Lemma 3.36. *Let Λ be the condensed ring associated with a T1-topological ring. If the underlying ring Λ_* is semi-hereditary, then Λ is t-admissible.*

Proof. Lemma 3.33 (4) shows that Λ_* is regular coherent. We claim that $\Lambda_* \to \Gamma(S, \Lambda)$ is flat for any profinite set S. Since Λ is associated with a topological ring, the natural map $\Gamma(S, \Lambda) \to \prod_{s \in S} \Lambda_*$, $f \mapsto (f(s))_{s \in S}$ is injective. So $\Gamma(S, \Lambda)$ is a torsionless Λ_*-module in the sense of Bass [Lam99, Definition 4.64], and hence flat by a theorem of Chase [Lam99, Theorem 4.67]. $\qquad\square$

Corollary 3.37. *The condensed rings associated with the following T1-topological rings are t-admissible:*

(1) *All T1-topological Prüfer domains, for example, algebraic field extensions $E \supset \mathbb{Q}_\ell$ and their rings of integers $\mathcal{O}_E$ for any prime ℓ or the real and complex numbers $\mathbb{R}$, $\mathbb{C}$ with their Euclidean topology.*

(2) *The adeles $\mathbb{A}_K$, the finite adeles $\mathbb{A}_{K,f}$ and the profinite completion $\widehat{\mathcal{O}}_K$ for any finite field extension $K \supset \mathbb{Q}$ with ring of integers $\mathcal{O}_K$.*

Proof. In view of Lemma 3.36, it suffices to check that all aforementioned rings are semi-hereditary. Clearly, Prüfer domains (for example, fields, Bézout domains or Dedekind domains) are semi-hereditary which implies (1). As localizations and products of semi-hereditary rings are semi-heredity by Lemma 3.33 (5), part (2) follows from the formulas $\widehat{\mathcal{O}}_K = \prod_{\mathfrak{p}} \widehat{\mathcal{O}}_{K,\mathfrak{p}}$ and $\mathbb{A}_{K,f} = \widehat{\mathcal{O}}_K[T^{-1}]$, $T = K \setminus \{0\}$ and $\mathbb{A}_K \simeq \mathbb{A}_{K,f} \times \mathbb{R}^r \times \mathbb{C}^s$ for some $r, s \geq 0$. $\qquad\square$

Finally, let us mention the following result due to Brookshear [Bro78], De Marco [DM83], Neville [Nev90] and Vechtomov [Vec96] on the structure of rings of continuous functions. Its corollary below is used in [HS] (see Theorem 3.45) to compute the category of constructible $\mathbb{Q}_\ell$-sheaves:

Theorem 3.38. *Let Λ be one of the following topological rings:*

(1) *The real numbers $\mathbb{R}$ or the complex numbers $\mathbb{C}$ with their Euclidean topology*

(2) *An algebraic field extension $E \supset \mathbb{Q}_\ell$ for some prime ℓ, or its ring of integers $\mathcal{O}_E$.*

(3) *The adeles $\mathbb{A}_K$, the finite adeles $\mathbb{A}_{K,f}$ or the profinite completion $\widehat{\mathcal{O}}_K$ for some finite field extension $K \supset \mathbb{Q}$ with ring of integers $\mathcal{O}_K$.*

Then, for any extremally disconnected profinite set S, the ring $\Gamma(S, \Lambda) = \operatorname{Maps_{cont}}(S, \Lambda)$ is semi-hereditary. In particular, it is regular coherent.

Proof. For (1), let Λ denote either $\mathbb{R}$ or $\mathbb{C}$. If $U \subset S$ is open, then its topological closure $\bar{U}$ is again open [Sta17, Tag 08YI]. In particular, S is a Λ-basically disconnected Tychonoff ($=T_{3\frac{1}{2}}$) space or, equivalently, every finitely generated ideal in $\Gamma(S, \Lambda)$ is principal and projective, see [Gla89, Theorems 8.4.3, 8.4.4] for $\Lambda = \mathbb{R}$ and [Vec96, Theorem 12.12] for $\Lambda = \mathbb{C}$.

Part (2) reduces to the case of finite field extensions $E \supset \mathbb{Q}_\ell$. It is enough to show that every finitely generated ideal in $\Gamma(S, \mathcal{O}_E)$ is principal and projective.

First, we show that every ideal generated by two elements $f, g \in \Gamma(S, \mathcal{O}_E)$ is principal. Let us denote by $|\text{-}| : E \to \mathbb{R}_{\geq 0}$ the normalized valuation so that $\mathcal{O}_E$ is the subring of elements $x \in E$ with $|x| \leq 1$. Let $U = \{s \in S \mid |f(s)| > |g(s)|\}$ and $V = \{s \in S \mid |f(s)| < |g(s)|\}$ which are open disjoint sets in S. As S is extremally disconnected, their topological closures $\bar{U}, \bar{V}$ are clopen ($=$closed and open) and still disjoint [Sta17, Tag 08YK]. The characteristic function $e_{\bar{U}}$ on $\bar{U}$ defines an idempotent in $\Gamma(S, \mathcal{O}_E)$. We claim that the ideal (f, g) is the principal ideal generated by $h := e_{\bar{U}} f + (1 - e_{\bar{U}}) g \in \Gamma(S, \mathcal{O}_E)$. It is clear that $h \in (f, g)$. Conversely, we note that $|f(s)|, |g(s)| \leq |h(s)|$ for all $s \in S$ by construction. Using the equivalent characterizations in [Vec96, Theorem 12.2], we obtain functions $a, b \in \Gamma(S, E)$ with $f = a \cdot h$ and $g = b \cdot h$. Comparing valuations, we necessarily have $|a(s)|, |b(s)| \leq 1$ for all $s \in S$, that is, $a, b \in \Gamma(S, \mathcal{O}_E)$. So $(f, g) = (h)$ as ideals in $\Gamma(S, \mathcal{O}_E)$.

It remains to show that every principal ideal is projective. More generally, the proof of [Gla89, Theorem 8.4.4] shows that this holds for any $T1$-topological ring Λ without zero divisors: For any $f \in \Gamma(S, \Lambda)$, there is a short exact sequence

$$0 \longrightarrow \mathrm{Ann}(f) \longrightarrow \Gamma(S, \Lambda) \xrightarrow{f\times} (f) \longrightarrow 0, \tag{3.15}$$

where $\mathrm{Ann}(f) = \{g \mid g \cdot f = 0\}$ is the annihilator ideal of f. We show that (3.15) splits so that (f) is projective, being a direct summand of $\Gamma(S, \Lambda)$. Let $U = \{s \in S \mid f(s) \neq 0\}$. As Λ is $T1$, this subset is open and its closure $\bar{U} \subset S$ is clopen. The characteristic function $e_{S \setminus \bar{U}}$ on $S \setminus \bar{U}$ defines an idempotent in $\Gamma(S, \Lambda)$. We claim that $\mathrm{Ann}(f) = (e_{S \setminus \bar{U}})$ which will imply that (3.15) splits. Clearly, $e_{S \setminus \bar{U}} \cdot f = 0$ by construction of $S \setminus \bar{U}$. Conversely, if $g \cdot f = 0$ for some $g \in \Gamma(S, \Lambda)$, then $g|_U = 0$ because Λ is without zero divisors. As $g^{-1}(0)$ is closed in S, we still have $g|_{\bar{U}} = 0$. Hence, $g = g \cdot e_{S \setminus \bar{U}}$, that is, $g \in (S \setminus \bar{U})$.

For (3), we note

$$\Gamma(S, \widehat{\mathcal{O}}_K) = \prod_{\mathfrak{p}} \Gamma(S, \widehat{\mathcal{O}}_{K_\mathfrak{p}}),$$

where $\mathfrak{p}$ runs through the places of K. As products of semi-hereditary rings are semi-hereditary [Oli83, Section 4.2, Proposition], so is $\Gamma(S, \widehat{\mathcal{O}}_K)$. Next, writing $\mathbb{A}_{K,\mathrm{f}} = \widehat{\mathcal{O}}_K[T^{-1}]$, $T = K \setminus \{0\}$ as a localization and using Lemma 3.24, we see that $\Gamma(S, \mathbb{A}_{K,\mathrm{f}}) = \Gamma(S, \widehat{\mathcal{O}}_K)[T^{-1}]$ is semi-hereditary. Finally, $\mathbb{A}_K \simeq \mathbb{A}_{K,\mathrm{f}} \times \mathbb{R}^r \times \mathbb{C}^s$ for some $r, s \geq 0$ implies the result in this case as well. $\square$

Corollary 3.39. *Let S be an extremally disconnected profinite set. Let $E \supset \mathbb{Q}_\ell$ and $\mathbb{A}_{K,\mathrm{f}}$ be as in Theorem 3.38. Then every finite projective module over $\Gamma(S, E)$, respectively over $\Gamma(S, \mathbb{A}_{K,\mathrm{f}})$ extends to a finite projective module over $\Gamma(S, \mathcal{O}_E)$, respectively over $\Gamma(S, \widehat{\mathcal{O}}_K)$.*

Proof. All rings are semi-hereditary by Theorem 3.38. Every finite projective module M over a semi-hereditary ring R is a direct sum of finitely generated (hence projective) ideals [Lam99, Theorem 2.29]. Hence, it is enough to show that every finitely generated ideal extends. This is clear from $\Gamma(S, E) = \Gamma(S, \mathcal{O}_E)[\ell^{-1}]$ and $\Gamma(S, \mathbb{A}_{K,\mathrm{f}}) = \Gamma(S, \widehat{\mathcal{O}}_K)[T^{-1}]$, $T = K \setminus \{0\}$ by multiplying with a common denominator of the continuous functions generating the ideal. $\square$

3.7. **Comparison results.** We compare our definition with classical definitions for discrete rings, adic rings and their localizations. The upshot is that we recover [BS15, Definition 6.3.1, Definition 6.5.1] for discrete and adic rings, and [BS15, Definition 6.8.8] for algebraic extensions of $\mathbb{Q}_\ell$. In Section 3.7.3, we give some examples and make the connection to more classical approaches [Del80, (1.1)], [ILO14, Exposé XIII, §4], [Sta17, Tag 0F4M], see also [KW01, II.5, Appendix A].

3.7.1. *Discrete rings.* In this subsection, let Λ be the condensed ring associated with a *discrete* topological ring, also denoted by Λ, see Example 3.1. Then, for any scheme X, the sheaf $\Lambda_X = \underline{\Lambda}$ is the constant sheaf of rings on $X_{\mathrm{proét}}$ associated with Λ. The morphism onto the étale site $\nu : X_{\mathrm{proét}} \to X_{\mathrm{ét}}$ induces a pullback functor

$$\nu^* : \mathrm{D}(X_{\mathrm{ét}}, \Lambda) \to \mathrm{D}(X, \Lambda).$$

Recall from [CD16, Remark 6.3.27] that an object $M \in \mathrm{D}(X_{\mathrm{ét}}, \Lambda)$ is dualizable if and only if there exists a covering $\{U_i \to X\}$ in $X_{\mathrm{ét}}$ such that each restriction $M|_{U_i}$ is constant with perfect values. Let us temporarily denote by

$$\mathrm{D}_{\mathrm{lis}}(X_{\mathrm{ét}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_{\mathrm{ét}}, \Lambda)$$

the full subcategories of $\mathrm{D}(X_{\mathrm{ét}}, \Lambda)$ of objects which are lisse ($=$dualizable in $\mathrm{D}(X_{\mathrm{ét}}, \Lambda)$, by definition), respectively Zariski locally lisse along a finite subdivision into constructible locally closed subschemes.

Proposition 3.40. *The functor ν^* induces equivalences*

$$\mathrm{D}_{\mathrm{lis}}(X_{\text{ét}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X, \Lambda).$$

Remark 3.41. If X is qcqs and Λ Noetherian, then $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda) = \mathrm{D}_{\mathrm{ctf}}(X_{\text{ét}}, \Lambda)$ coincides with the full subcategory of $\mathrm{D}(X_{\text{ét}}, \Lambda)$ of constructible Λ-sheaves of finite Tor-dimension. This can be deduced from the characterization in [Sta17, Tag 03TT].

We need some preparation before giving the proof of Proposition 3.40.

Lemma 3.42. *Recall that Λ is discrete. Let $S = \lim S_i$ be a profinite set. Then the natural functor*

$$\operatorname{colim} \mathrm{Perf}_{\Gamma(S_i, \Lambda)} \xrightarrow{\cong} \mathrm{Perf}_{\Gamma(S, \Lambda)},$$

is an equivalence. Here the transition functors are given by $(\text{-}) \otimes_{\Gamma(S_i, \Lambda)} \Gamma(S_j, \Lambda)$ for $j \geq i$.

Proof. Any continuous map $S \to \Lambda$ factors through some $S \to S_i$ because S is quasi-compact and Λ discrete. Hence, $\Gamma(S, \Lambda) = \operatorname{colim} \Gamma(S_i, \Lambda)$ is a filtered colimit. So we are done by Lemma 2.1 (1). $\qquad\square$

For $N \in \mathrm{Mod}_\Lambda$, recall that $\underline{N}$ denotes the associated constant sheaf of Λ-modules on $X_{\mathrm{proét}}$. If $N \in \mathrm{Perf}_\Lambda$, then $\mathrm{R}\Gamma(X, \underline{N}) \cong \mathrm{R}\Gamma(X, \Lambda) \otimes_{\Lambda_*} N$ shows that

$$\underline{N} \cong (N \otimes_{\Lambda_*} \Gamma(X, \Lambda))_X \in \mathrm{D}(X, \Lambda). \tag{3.16}$$

Corollary 3.43. *An object $M \in \mathrm{D}(X, \Lambda)$ is lisse if and only if there exists a covering $\{U_i \to X\}$ in $X_{\mathrm{proét}}$ such that each restriction $M|_{U_i}$ is constant with perfect values.*

Proof. Let us assume that M is lisse (the other direction is clear). By Lemma 3.11, we may assume X to be w-contractible and affine. In this case, $\pi_0 X = \lim (\pi_0 X)_i$ is a profinite set. Then Lemma 3.42 together with Lemma 3.7 (3) give equivalences

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Perf}_{\Gamma(X, \Lambda)} = \mathrm{Perf}_{\Gamma(\pi_0 X, \Lambda)} \cong \operatorname{colim} \mathrm{Perf}_{\Gamma((\pi_0 X)_i, \Lambda)},$$

using that $\Gamma(X, \Lambda) = \Gamma(\pi_0 X, \Lambda)$. In down to earth terms, we find some $N \in \mathrm{Perf}_{\Gamma((\pi_0 X)_i, \Lambda)}$ together with an isomorphism

$$\left(N \otimes_{\Gamma((\pi_0 X)_i, \Lambda)} \Gamma(X, \Lambda) \right)_X \cong M.$$

The fibers of the projection $X \to (\pi_0 X)_i$ induce a finite subdivision of X into clopen subsets. After Zariski localizing on X we may therefore assume that $(\pi_0 X)_i = *$. It follows from (3.16) that $\underline{N} \cong M$ is constant. $\qquad\square$

Proof of Proposition 3.40. Since $\nu^* \colon \mathrm{D}(X_{\text{ét}}, \Lambda) \to \mathrm{D}(X, \Lambda)$ is monoidal, it induces functors

$$\mathrm{D}_{\mathrm{lis}}(X_{\text{ét}}, \Lambda) \to \mathrm{D}_{\mathrm{lis}}(X, \Lambda), \quad \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda) \to \mathrm{D}_{\mathrm{cons}}(X, \Lambda), \tag{3.17}$$

compatible with étale localization on X. By étale descent (Zariski descent is enough), we may assume X to be affine. The functor ν^* is fully faithful when restricted to bounded below complexes by [BS15, Corollary 5.1.6]. Since X is quasi-compact, any object in $\mathrm{D}_{\mathrm{lis}}(X_{\text{ét}}, \Lambda)$ and hence in $\mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \Lambda)$ is bounded. This implies the full faithfulness of the functors in (3.17).

We first show the essential surjectivity for lisse sheaves. Corollary 3.43 shows that any $M \in \mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ is perfect-locally constant on $X_{\mathrm{proét}}$. Clearly, any constant sheaf in $\mathrm{D}(X, \Lambda)$ arises as ν^*-pullback from $\mathrm{D}(X_{\text{ét}}, \Lambda)$, that is, it is classical in the terminology of [BS15, §5]. Therefore, M is locally on $X_{\mathrm{proét}}$ classical, and hence classical itself by [BS15, Lemma 5.1.4] which holds for general bounded (below) complexes with the same proof, see also [BS15, above Remark 5.1.7]. Finally, by [BS15, Lemma 6.3.13] the sheaf M necessarily arises as pullback of a locally on $X_{\text{ét}}$ constant sheaf with perfect values, that is, from an object in $\mathrm{D}_{\mathrm{lis}}(X_{\text{ét}}, \Lambda)$.

It remains to show the essential surjectivity on constructible sheaves. If $\iota \colon Z \hookrightarrow X$ is a constructible locally closed subset, the functors $\iota^*, \iota_!$ commute with ν^*, see [BS15, Lemma 6.2.3 (4)] for $\iota_!$. Using the full faithfulness of ν^* on bounded complexes, we reduce by induction on the number of constructible locally closed strata to the case of lisse sheaves. $\qquad\square$

3.7.2. *Adic rings.* In this subsection, let Λ be the condensed ring associated with a Noetherian ring, also denoted by Λ, complete for the topology defined by an ideal $I \subset \Lambda$, see Example 3.1. Then each quotient Λ/I^i is discrete and $\Lambda = \lim_{i \geq 1} \Lambda/I^i$ as condensed rings so that Section 3.4.1 applies. In the following, all limits are derived unless mentioned otherwise.

An object $M \in \mathrm{D}(X, \Lambda)$ is called *derived I-complete* if the natural map $M \to \lim(M \otimes_{\Lambda_X} (\Lambda/I^i)_X)$ is an equivalence. Let us temporarily denote by

$$\mathrm{D}_{\mathrm{lis}}(X, \widehat{\Lambda}) \subset \mathrm{D}_{\mathrm{cons}}(X, \widehat{\Lambda})$$

the full subcategories of $\mathrm{D}(X, \Lambda)$ of objects M which are derived I-complete and such that its reduction $M \otimes_{\Lambda_X} (\Lambda/I)_X$ is lisse, respectively constructible in $\mathrm{D}(X, \Lambda/I)$. By Proposition 3.40 applied to the discrete ring Λ/I, the homotopy category of $\mathrm{D}_{\mathrm{cons}}(X, \widehat{\Lambda})$ agrees with the category defined in [BS15, Definition 6.5.1].

Proposition 3.44. *There are equalities*

$$\mathrm{D}_{\mathrm{lis}}(X,\Lambda) = \mathrm{D}_{\mathrm{lis}}(X,\widehat{\Lambda}), \quad \mathrm{D}_{\mathrm{cons}}(X,\Lambda) = \mathrm{D}_{\mathrm{cons}}(X,\widehat{\Lambda}).$$

as full subcategories of $\mathrm{D}(X,\Lambda)$.

Proof. First off, for each $M \in \mathrm{D}_{\mathrm{lis}}(X,\widehat{\Lambda})$ we have $M \otimes_{\Lambda_X} (\Lambda/I^i)_X \in \mathrm{D}_{\mathrm{lis}}(X,\Lambda/I^i)$ for all $i \geq 1$ and likewise for constructible sheaves. This easily follows from the proof of Proposition 3.19. An alternative argument is as follows: Using Proposition 3.40 for Λ/I, one sees by induction on i that each reduction $M \otimes_{\Lambda_X} (\Lambda/I^i)_X$ lies in the essential image of the functor $\nu^* \colon \mathrm{D}(X_{\text{ét}}, \Lambda/I^i) \to \mathrm{D}(X_{\mathrm{proét}}, \Lambda/I^i)$ and is étale locally constant. Passing to a suitable étale covering of X, our claim for lisse sheaves reduces to the corresponding statement for Λ/I^i-modules and the nilpotent ideal I/I^i, see [Sta17, Tag 07LU]. For constructible sheaves M, it follows that any stratification witnessing the constructibility of $M \otimes_{\Lambda_X} (\Lambda/I)_X$ induces such a stratification for $M \otimes_{\Lambda_X} (\Lambda/I^i)_X$.

In view of Proposition 3.19, it suffices to show that the natural functors

$$\mathrm{D}_{\mathrm{lis}}(X,\widehat{\Lambda}) \to \lim \mathrm{D}_{\mathrm{lis}}(X,\Lambda/I^i), \quad \mathrm{D}_{\mathrm{cons}}(X,\widehat{\Lambda}) \to \lim \mathrm{D}_{\mathrm{cons}}(X,\Lambda/I^i)$$

are equivalences. The full faithfulness is proven in [BS15, Lemma 3.5.7 (2)], and the essential surjectivity is immediate from the definition. $\qquad\square$

3.7.3. *Classical approaches and some examples.* Fix a prime ℓ and denote by $\mathbb{Z}_\ell = \lim \mathbb{Z}/\ell^n$ the ring of ℓ-adic integers viewed as a profinite topological ring. Equip $\mathbb{Q}_\ell = \mathbb{Z}_\ell[\ell^{-1}]$ with its (usual) colimit topology so that $\mathbb{Z}_\ell \subset \mathbb{Q}_\ell$ is an open subring.

For a scheme X, let use denote by $\mathrm{D}_{\mathrm{ctf}}(X_{\text{ét}}, \mathbb{Z}/\ell^n)$ the category of constructible étale $\mathbb{Z}/\ell^n$-sheaves of finite Tor-dimension. It is classical to consider the limit

$$\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Z}_\ell) = \lim \mathrm{D}_{\mathrm{ctf}}(X_{\text{ét}}, \mathbb{Z}/\ell^n),$$

that is, the category of compatible systems of such objects. From here one usually passes to $\mathbb{Q}_\ell$-coefficients by inverting ℓ:

$$\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Q}_\ell) = \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Z}_\ell) \otimes_{\mathrm{Perf}_{\mathbb{Z}_\ell}} \mathrm{Perf}_{\mathbb{Q}_\ell}$$

Note that this tensor product agrees with the idempotent completion of the localization $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Z}_\ell)[\ell^{-1}]$. The following lemma gives the comparison with [ILO14, Exposé XIII, §4], [Sta17, Tag 0F4M] and [BS15, Definition 6.8.8]:

Theorem 3.45. *For any qcqs scheme X, there are natural equivalences*

$$\mathrm{D}_{\mathrm{cons}}(X,\mathbb{Z}/\ell^n) \cong \mathrm{D}_{\mathrm{ctf}}(X,\mathbb{Z}/\ell^n), \quad \mathrm{D}_{\mathrm{cons}}(X,\mathbb{Z}_\ell) \cong \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Z}_\ell), \quad \mathrm{D}_{\mathrm{cons}}(X,\mathbb{Q}_\ell) \cong \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Q}_\ell).$$

Proof. By Proposition 3.40 and Remark 3.41, the pullback of sheaves along $X_{\mathrm{proét}} \to X_{\text{ét}}$ induces equivalences $\mathrm{D}_{\mathrm{ctf}}(X,\mathbb{Z}/\ell^n) = \mathrm{D}_{\mathrm{cons}}(X_{\text{ét}}, \mathbb{Z}/\ell^n) \cong \mathrm{D}_{\mathrm{cons}}(X,\mathbb{Z}/\ell^n)$. Passing to the limits, we obtain an equivalence $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Z}_\ell) \cong \mathrm{D}_{\mathrm{cons}}(X,\mathbb{Z}_\ell)$ by Proposition 3.19. Finally, we have a fully faithful embedding $\mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathbb{Q}_\ell) \hookrightarrow \mathrm{D}_{\mathrm{cons}}(X,\mathbb{Q}_\ell)$ by Proposition 3.23 which is essentially surjective by [HS]. The latter uses Corollary 3.39. Alternatively, if X is topologically Noetherian, one can use Corollary 3.29 to see that the category $\mathrm{D}_{\mathrm{cons}}(X,\mathbb{Q}_\ell)$ agrees with the category defined in [BS15, Definition 6.8.8]. In this case, the result also follows from [BS15, Proposition 6.8.14]. $\qquad\square$

Remark 3.46. Likewise, one can show the following formula for any qcqs scheme X with coefficients in the finite adeles $\mathbb{A}_{\mathbb{Q},\mathrm{f}}$:

$$\mathrm{D}_{\mathrm{cons}}(X,\mathbb{A}_{\mathbb{Q},\mathrm{f}}) \cong \big(\lim \mathrm{D}_{\mathrm{ctf}}(X,\mathbb{Z}/n) \big) \otimes_{\mathrm{Perf}_{\widehat{\mathbb{Z}}}} \mathrm{Perf}_{\mathbb{A}_{\mathbb{Q},\mathrm{f}}}$$

If X is of finite type over a finite or separably closed field, then the homotopy category of $\mathrm{D}_{\mathrm{cons}}(X,\mathbb{Z}_\ell)$ is the 2-limit of the categories $\mathrm{D}_{\mathrm{ctf}}(X,\mathbb{Z}/\ell^n)$ considered in [Del80, Equation (1.1.2)], see also [KW01, Section II.6]. For the relation with [Eke90], the reader is referred to [BS15, Section 5.5], in view of Section 3.7.2. Also, $\mathrm{D}_{\mathrm{cons}}(X,\mathbb{Z}_\ell)$ is equivalent to the ∞-category defined in [GL19, Definition 4.3.10] when X is quasi-projective over an algebraically closed field. We leave the details to the reader.

The same reasoning as in Theorem 3.45 applies to obtain, more generally, natural equivalences

$$\mathrm{D}_{\mathrm{cons}}(X,\mathcal{O}_E/\ell^n) \cong \mathrm{D}_{\mathrm{ctf}}(X,\mathcal{O}_E/\ell^n), \quad \mathrm{D}_{\mathrm{cons}}(X,\mathcal{O}_E) \cong \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathcal{O}_E), \quad \mathrm{D}_{\mathrm{cons}}(X,E) \cong \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,E),$$

where $E \supset \mathbb{Q}_\ell$ is a finite field extension with ring of integers $\mathcal{O}_E$. For coefficients, say in $\bar{\mathbb{Q}}_\ell$ or $\bar{\mathbb{Z}}_\ell$, we recover the classical approach [Del80, (1.1.3)], [KW01, II.5, Appendix A] of passing to the colimit over its finite subextensions:

Lemma 3.47. *For any qcqs scheme X, there are natural equivalences*

$$\operatorname*{colim}_{E/\mathbb{Q}_\ell \text{ finite}} \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,\mathcal{O}_E) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X,\bar{\mathbb{Z}}_\ell), \quad \operatorname*{colim}_{E/\mathbb{Q}_\ell \text{ finite}} \mathrm{D}_{\mathrm{c}}^{\mathrm{b}}(X,E) \xrightarrow{\cong} \mathrm{D}_{\mathrm{cons}}(X,\bar{\mathbb{Q}}_\ell),$$

where $\bar{\mathbb{Z}}_\ell = \operatorname{colim} \mathcal{O}_E$ and $\bar{\mathbb{Q}}_\ell = \operatorname{colim} E$ are equipped with the colimit topology which is used to form the target categories.

Proof. This immediate from the above discussion using Proposition 3.20. □

Remark 3.48 (The 6 functors)**.** Under the usual finiteness, excellency and ℓ-coprimality assumptions on the schemes, one obtains a 6 functor formalism along the usual lines for the categories of constructible sheaves. In light of Theorem 3.45, the reader is referred to the treatment in [BS15, Section 6.7].

3.8. **Ind-lisse and ind-constructible sheaves.** Let X be a qcqs scheme and Λ a condensed ring. In this section, we impose the following finiteness assumption on the Λ-cohomological dimension: there exists an integer $d_X \geq 0$ such that for all $p > d_X$,

$$\mathrm{H}^p(U, N) = 0, \tag{3.18}$$

for all pro-étale affines $U = \lim U_i \in X_{\mathrm{proét}}$ and all sheaves $N \in \mathrm{D}(X, \Lambda)^\heartsuit$ of the form $N = \mathrm{H}^0(M)$ for some $M \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$. Examples include finite type schemes X over finite or separably closed fields with coefficients Λ being either a discrete torsion ring, an algebraic extension $E/\mathbb{Q}_\ell$ or its ring of integers $\mathcal{O}_E$, see Lemma 3.54.

Recall from Section 3.1 that the category $\mathrm{D}(X, \Lambda)$ admits small (co-)limits.

Definition 3.49. A sheaf $M \in \mathrm{D}(X, \Lambda)$ is called *ind-lisse*, respectively *ind-constructible* if it is equivalent to a filtered colimit of lisse, respectively constructible Λ-sheaves.

The full subcategories of $\mathrm{D}(X, \Lambda)$ of ind-lisse, respectively ind-constructible sheaves are denoted by

$$\mathrm{D}_{\mathrm{indlis}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Lambda).$$

Both categories are naturally commutative algebra objects in $\mathrm{Pr}^{\mathrm{St}}_{\Gamma(X,\Lambda)}$, that is, $\Gamma(X, \Lambda)$-linear symmetric monoidal stable presentable ∞-categories, see Corollary 3.51 for the properties stable and presentable. Recall the notion of compact objects [Lur09, Section 5.3.4].

Proposition 3.50. *An object $M \in \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ is compact if and only if M is constructible, and likewise for (ind-)lisse sheaves. Consequently, passing to compact objects induces equalities*

$$\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)^\omega = \mathrm{D}_{\mathrm{cons}}(X, \Lambda), \quad \mathrm{D}_{\mathrm{indlis}}(X, \Lambda)^\omega = \mathrm{D}_{\mathrm{lis}}(X, \Lambda).$$

Before giving the proof, let us point out the following corollary which, for example, gives the comparison with the presentable categories of ℓ-adic sheaves considered in [GL19, §4].

Corollary 3.51. *The inclusion $\mathrm{D}_{\mathrm{cons}}(X, \Lambda) \subset \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ extends to a colimit-preserving equivalence*

$$\mathrm{Ind}\big(\mathrm{D}_{\mathrm{cons}}(X, \Lambda)\big) \xrightarrow{\cong} \mathrm{D}_{\mathrm{indcons}}(X, \Lambda),$$

and likewise for (ind-)lisse sheaves.

Proof. By [Lur09, Proposition 5.3.5.11], the functor is fully faithful because all objects of $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ are compact in $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$. The essential surjectivity is immediate from the definition. □

Remark 3.52. If Λ is a discrete ring, then Corollary 3.51 together with Proposition 3.40 shows that $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda) \cong \mathrm{D}(X_{\mathrm{ét}}, \Lambda)$, see also [BS15, Proposition 6.4.8]. In particular, the natural t-structure on $\mathrm{D}(X, \Lambda)$ restricts to a t-structure on $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ even if Λ is not t-admissible.

The proof of Proposition 3.50 builds on the following lemma that crucially relies on assumption (3.18), see Remark 3.22:

Lemma 3.53. *The following functors commute with filtered colimits with terms in $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$:*

(1) $f_* \colon \mathrm{D}(X, \Lambda) \to \mathrm{D}(Y, \Lambda)$ for map $f \colon X \to Y$ between qcqs schemes satisfying (3.18);
(2) $\underline{\mathrm{Hom}}_{\Lambda_X}(M, \text{-}) \colon \mathrm{D}(X, \Lambda) \to \mathrm{D}(X, \Lambda)$ for any qcqs X scheme satisfying (3.18) and $M \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$.
In particular, under the conditions in (2), the functor

$$\mathrm{Hom}_{\Lambda_X}(M, \text{-}) = \mathrm{R}\Gamma\big(X, \underline{\mathrm{Hom}}_{\Lambda_X}(M, \text{-})\big) \colon \mathrm{D}(X, \Lambda) \to \mathrm{Mod}_{\Gamma(X,\Lambda)}$$

commutes with such colimits as well.

Proof. Assuming that part (1) holds, the rest is proven analogously to Lemma 3.21. For (1), let $\{M_i\}$ be a filtered system of constructible Λ-sheaves M_i on X. We need to show that the map

$$\mathrm{colim}\,\mathrm{H}^p \circ f_*(M_i) \to \mathrm{H}^p \circ f_*(\mathrm{colim}\,M_i),$$

is an equivalence in $\mathrm{D}(Y, \Lambda)^\heartsuit$ for every $p \in \mathbb{Z}$, $\mathrm{H}^p := \tau^{\leq p} \circ \tau^{\geq p}$. By Lemma 3.21, this holds true if all M_i lie in $\mathrm{D}^{\geq n}(X, \Lambda)$ for some $n \in \mathbb{Z}$. So it is enough to show that there exists $n \in \mathbb{Z}$ such that the map $\mathrm{H}^p \circ f_*(\tau^{\geq n}M) \to \mathrm{H}^p \circ f_*(M)$ is an isomorphism for all $M \in \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$. Evaluating at any w-contractible pro-étale affine $V \in Y_{\mathrm{proét}}$, we obtain the map on proétale cohomology groups

$$\mathrm{H}^p\big(X \times_Y V, \tau^{\geq n}M\big) \to \mathrm{H}^p\big(X \times_Y V, M\big)$$

Using the left-completeness of $\mathrm{D}(X, \Lambda)$ [BS15, Proposition 3.3.3], we are reduced to showing: there exists some integer $d_X \geq 0$ such that $\mathrm{H}^p(U, N) = 0$ for all $p > d_X$, all qcqs $U \in X_{\mathrm{pro\acute{e}t}}$ that admit an open cover by pro-étale affines and all $N \in \mathrm{D}(X, \Lambda)^\heartsuit$ of the form $N = \mathrm{H}^0(M)$ for some $M \in \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$. Since $\mathrm{H}^p(U, \text{-})$ commutes with filtered colimits in $\mathrm{D}(X, \Lambda)^\heartsuit$ (Lemma 3.21), we may assume that $N = \mathrm{H}^0(M)$ for some $M \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$. By an induction on the finite number pro-étale affines covering U, passing through the case of separated U first, we may assume that $U = \lim U_i$ is pro-étale affine. So the desired integer d_X exists by our assumption (3.18). $\qquad\square$

Proof of Proposition 3.50. We only treat (ind-)constructible sheaves as the argument for (ind-)lisse sheaves is completely analogous.

To show that $M \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is compact in $\mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$, we need to show that the natural map

$$\operatorname{colim} \operatorname{Hom}_{\Lambda_X}(M, N_j) \to \operatorname{Hom}_{\Lambda_X}(M, \operatorname{colim} N_j)$$

is an equivalence for every filtered system $\{N_j\}$ of constructible Λ-sheaves. This follows from Lemma 3.53.

Conversely, let $M = \operatorname{colim} M_i \in \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$ be a compact object. Then the identity $\mathrm{id}_M \colon M \to M$ factors through some M_i, presenting M as a direct summand of M_i. As $\mathrm{D}_{\mathrm{cons}}(X, \Lambda)$ is idempotent complete, we see that M is constructible.

$\qquad\square$

Lemma 3.54. *The pair* (X, Λ) *satisfies* (3.18) *in each of the following cases:*

(1) *The scheme* X *is of finite type over a finite or separably closed field and* Λ *is either a discrete torsion ring, an algebraic field extension* $E \supset \mathbb{Q}_\ell$ *or its ring of integers* $\mathcal{O}_E$.
(2) *The scheme* X *is a qcqs scheme in characteristic* $p > 0$ *and* Λ *is either a discrete p-torsion ring, an algebraic field extension* $E \supset \mathbb{Q}_p$ *or its ring of integers* $\mathcal{O}_E$.

Proof. Let $U = \lim U_i \in X_{\mathrm{pro\acute{e}t}}$ be pro-étale affine, and let $N = \mathrm{H}^0(M)$ for some $M \in \mathrm{D}_{\mathrm{cons}}(X, \Lambda)$. The case where Λ is an algebraic extension $E \supset \mathbb{Q}_\ell$ or its ring of integers $\mathcal{O}_E$ are reduced to the case of finite extensions (Lemma 3.47) and further to the case of finite discrete torsion rings (Theorem 3.45). It remains to treat the case where Λ is a discrete torsion ring. Then $\mathrm{H}^p(U, N) = \operatorname{colim} \mathrm{H}^p(U_i, N)$ using Proposition 3.40 and the continuity of the étale site [Sta17, Tag 03Q4]. So part (1) follows from Artin vanishing (see [Sta17, Tag 0F0V]), noting that the abelian sheaf underlying N is torsion and that separably closed fields (respectively, finite fields) have cohomological dimension 0 (respectively, 1). For (2), we claim that $\mathrm{H}^i(X, N) = 0$ for all $i > 2$, affine schemes $X = \operatorname{Spec} R$ in characteristic $p > 0$ and p-torsion abelian sheaves N. Using dévissage arguments [Sta17, Tags 09Z4, 03SA], it suffices to consider a constructible $\mathbb{F}_p$-sheaf N. By topological invariance of the étale site, we may assume R to be perfect. The category $\mathrm{D}_{\mathrm{cons}}(X, \mathbb{F}_p)$ embeds fully faithfully and t-exact into the derived category of modules over $R[F]$ defined in [BL19, Notation 2.1.5], see Theorem 12.1.5 there. This functor sends the constant sheaf $\mathbb{F}_p$ to R, which has a length two resolution by projective $R[F]$-modules, see [BL19, Section 3]. This shows the claim. $\qquad\square$

Corollary 3.55. *Let* $U \to X$ *be a quasi-compact étale surjection with Čech nerve* $U_\bullet$. *Then the natural functor*

$$\mathrm{D}_{\mathrm{indcons}}(X, \Lambda) \xrightarrow{\cong} \operatorname{Tot}\big(\mathrm{D}_{\mathrm{indcons}}(U_\bullet, \Lambda)\big)$$

is an equivalence. If $U \to X$ *is finite étale, then the same holds for ind-lisse sheaves.*

Proof. By the descent equivalence $\operatorname{Tot}\big(\mathrm{D}(U_\bullet, \Lambda)\big) = \mathrm{D}(X, \Lambda)$, there are full inclusions

$$\mathrm{D}_{\mathrm{indcons}}(X, \Lambda) \subset \operatorname{Tot}\big(\mathrm{D}_{\mathrm{indcons}}(U_\bullet, \Lambda)\big) \subset \mathrm{D}(X, \Lambda).$$

It remains to show: if $M \in \mathrm{D}(X, \Lambda)$ such that $M|_U$ is ind-constructible, then M is so. We denote $j_\bullet \colon U_\bullet \to X$. For every $M \in \mathrm{D}(X, \Lambda)$, we have a canonical equivalence $|(j_\bullet)_! \circ j_\bullet^* M| \xrightarrow{\sim} M$. Since $U \to X$ is finitely presented étale, the same is true for every $j_n \colon U_n \to X$. In particular, each functor $(j_n)_!$ preserves the constructible subcategories (Corollary 3.12). So, if $j_n^* M$ can be written as a filtered colimit of constructible objects, then the same is true for $(j_n)_! j_n^* M$. This shows $M \in \mathrm{D}_{\mathrm{indcons}}(X, \Lambda)$.

Finally, if $U \to X$ is finite étale, then each functor $(j_n)_!$ preserves the lisse subcategories (Corollary 3.12), and we can proceed as before. $\qquad\square$

3.9. **External tensor products.** For schemes $X_1, \ldots, X_n$ over a fixed base scheme S (for example, the spectrum of a field) and a condensed ring Λ, we denote the external product in the usual way:

$$\boxtimes \colon \mathrm{D}(X_1, \Lambda) \times \ldots \times \mathrm{D}(X_n, \Lambda) \longrightarrow \mathrm{D}\left(X_1 \times_S \ldots \times_S X_n, \Lambda\right),$$

$$(M_1, \ldots, M_n) \longmapsto M_1 \boxtimes \ldots \boxtimes M_n := p_1^*(M_1) \otimes_{\Lambda_X} \ldots \otimes_{\Lambda_X} p_n^*(M_n).$$

Here $p_i \colon X_1 \times_S \ldots \times_S X_n \to X_i$ are the projections. This functor induces the functor

$$\boxtimes \colon \mathrm{D}(X_1, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}(X_n, \Lambda) \to \mathrm{D}\left(X_1 \times_S \ldots \times_S X_n, \Lambda\right), \tag{3.19}$$

in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$ for $\Lambda_* = \Gamma(*, \Lambda)$ the underlying ring. Here we regard $\mathrm{D}(X_i, \Lambda)$ as objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$ via the natural functor $\mathrm{Mod}_{\Lambda_*} \to \mathrm{D}(*, \Lambda) \to \mathrm{D}(X_i, \Lambda)$, $\Lambda_* \mapsto \Lambda_{X_i}$, see (3.1) and Section 2.1 for tensor products of ∞-categories. The external tensor product of constructible sheaves is again constructible, and hence induces a functor

$$\boxtimes \colon \mathrm{D}_{\mathrm{cons}}(X_1, \Lambda) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}(X_n, \Lambda) \to \mathrm{D}_{\mathrm{cons}}\left(X_1 \times_S \ldots \times_S X_n, \Lambda\right), \tag{3.20}$$

in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda_*}(\mathrm{Idem})$ and likewise for the categories of ind-constructible, respectively (ind-)lisse sheaves.

4. Weil sheaves

In this section, we introduce the categories

$$\mathrm{D}_{\mathrm{lis}}\left(X^{\mathrm{Weil}}, \Lambda\right) \subset \mathrm{D}_{\mathrm{cons}}\left(X^{\mathrm{Weil}}, \Lambda\right) \subset \mathrm{D}\left(X^{\mathrm{Weil}}, \Lambda\right)$$

consisting of lisse, respectively constructible, respectively all Weil sheaves. These are the categories featuring in the categorical Künneth formula (Theorem 1.12).

Throughout this section, X is a scheme over a finite field $\mathbb{F}_q$ of characteristic $p > 0$. Unless the contrary is mentioned, we impose no conditions on X. Moreover, Λ is a condensed ring. We fix an algebraic closure $\mathbb{F}$ of $\mathbb{F}_q$, and denote by $X_{\mathbb{F}} := X \times_{\mathbb{F}_q} \mathrm{Spec}\,\mathbb{F}$ the base change. Denote by ϕ_X (respectively $\phi_{\mathbb{F}}$) the endomorphism of $X_{\mathbb{F}}$ that is the q-Frobenius on X (respectively $\mathrm{Spec}\,\mathbb{F}$) and the identity on the other factor.

Let $\mathrm{D}(X_{\mathbb{F}}, \Lambda)$ be the category of Λ-sheaves introduced in Section 3, and $\mathrm{D}_{\mathrm{lis}}(X_{\mathbb{F}}, \Lambda) \subset \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)$ its full subcategories of lisse and constructible sheaves. These categories are objects in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda_*}(\mathrm{Idem})$, that is, Λ_*-linear stable idempotent complete symmetric monoidal ∞-categories where $\Lambda_* = \Gamma(*, \Lambda)$ is the underlying ring.

4.1. The Weil-proétale site.
The Weil-étale topology for schemes over finite field is introduced in [Lic05], see also [Gei04]. Our approach for the proétale topology is slightly different:

Definition 4.1. The *Weil-proétale site* of X, denoted by $X^{\mathrm{Weil}}_{\mathrm{proét}}$, is the following site: Objects in $X^{\mathrm{Weil}}_{\mathrm{proét}}$ are pairs (U, φ) consisting of $U \in (X_{\mathbb{F}})_{\mathrm{proét}}$ equipped with an endomorphism $\varphi \colon U \to U$ of $\mathbb{F}$-schemes such that the map $U \to X_{\mathbb{F}}$ intertwines φ and ϕ_X. Morphisms in $X^{\mathrm{Weil}}_{\mathrm{proét}}$ are given by equivariant maps, and a family $\{(U_i, \varphi_i) \to (U, \varphi)\}$ of morphisms is a cover if the family $\{U_i \to U\}$ is a cover in $(X_{\mathbb{F}})_{\mathrm{proét}}$.

Note that $X^{\mathrm{Weil}}_{\mathrm{proét}}$ admits small limits formed componentwise as $\lim(U_i, \varphi_i) = (\lim U_i, \lim \varphi_i)$. In particular, there are limit-preserving maps of sites

$$(X_{\mathbb{F}})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}} \to X_{\mathrm{proét}} \tag{4.1}$$

given by the functors (in the opposite direction) $U \leftarrow\!\!\shortmid (U, \varphi)$ and $(U_{\mathbb{F}}, \phi_U) \leftarrow\!\!\shortmid U$. We denote by $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ the unbounded derived category of sheaves of Λ_X-modules on $X^{\mathrm{Weil}}_{\mathrm{proét}}$. The maps of sites (4.1) induce functors

$$\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}(X_{\mathbb{F}}, \Lambda), \tag{4.2}$$

whose composition is the usual pullback functor along $X_{\mathbb{F}} \to X$.

Remark 4.2. The functor $\mathrm{D}(X, \Lambda) \to \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is not an equivalence in general. This relates to the difference between continuous representations Galois versus Weil groups. See, however, Proposition 4.15 for filtered colimits of finite discrete rings Λ.

We have the following basic functoriality: Let $j \colon U \to X$ be a weakly étale morphism and consider the corresponding object $(U_{\mathbb{F}}, \phi_U)$ of $X^{\mathrm{Weil}}_{\mathrm{proét}}$. Then the slice site $(X^{\mathrm{Weil}}_{\mathrm{proét}})_{/(U_{\mathbb{F}}, \phi_U)}$ is equivalent to $U^{\mathrm{Weil}}_{\mathrm{proét}}$. This gives a functor $(X_{\mathrm{proét}})^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}}_{\Lambda}$, $U \mapsto \mathrm{D}(U^{\mathrm{Weil}}, \Lambda)$ which is a hypercomplete sheaf of Λ_*-linear presentable stable categories.

Also, we obtain an adjunction

$$j_! \colon \mathrm{D}(U^{\mathrm{Weil}}, \Lambda) \rightleftarrows \mathrm{D}(X^{\mathrm{Weil}}, \Lambda) \colon j^*$$

that is compatible with the $((j_{\mathbb{F}})_!, (j_{\mathbb{F}})^*)$-adunction under (4.2). The category $\mathrm{D}(X^{\mathrm{Weil}}, \Lambda)$ is equivalent to the category of ϕ_X-equivariant sheaves on $X_{\mathbb{F}}$, as we will now explain.

For each $i \geq 0$, consider the object $(X_i, \Phi_i) \in X^{\mathrm{Weil}}_{\mathrm{proét}}$ with $X_i = \mathbb{Z}^{i+1} \times X_{\mathbb{F}}$ the countably disjoint union of $X_{\mathbb{F}}$, the map $X_i \to X_{\mathbb{F}}$ given by projection and the endomorphism $\Phi_i \colon X_i \to X_i$ given by $(\underline{n}, x) \mapsto (\underline{n} - (1, \ldots, 1), \phi_X(x))$ on sections. The inclusion $\mathbb{Z}^i \to \mathbb{Z}^{i+1}$, $\underline{n} \mapsto (0, \underline{n})$ induces a map of schemes $X_{i-1} \to X_i$ where $X_{-1} := X_{\mathbb{F}}$. By pullback, we get a limit-preserving map of sites

$$(X_{i-1})_{\mathrm{proét}} \to X^{\mathrm{Weil}}_{\mathrm{proét}}/(X_i, \Phi_i) \tag{4.3}$$

Lemma 4.3. *For each $i \geq 0$, the map (4.3) induces an equivalence on the associated 1-topoi.*

Proof. As universal homeomorphisms induce equivalences on proétale 1-topoi [BS15, Lemma 5.4.2], we may assume that X is perfect. In this case, the sites (4.3) are equivalent because ϕ_X is an isomorphism. Explicitly, an inverse is given by sending an object $U \in (X_{i-1})_{\text{proét}}$ to the object $V = \bigsqcup_{\underline{n} \in \mathbb{Z}^{i+1}} V_{\underline{n}}$, $V_{\underline{n}} \to \{\underline{n}\} \times X_{\mathbb{F}}$ defined by

$$V_{\underline{n}} = U_{(n_2 - n_1, \ldots, n_{i+1} - n_1)} \times_{X_{\mathbb{F}}, \phi_X^{n_1}} X_{\mathbb{F}},$$

and with endomorphism $\varphi \colon V \to V$ defined by the maps $V_{\underline{n}} = V_{\underline{n} - (1, \ldots, 1)} \times_{X_{\mathbb{F}}, \phi_X} X_{\mathbb{F}} \to V_{\underline{n} - (1, \ldots, 1)}$. $\qquad\square$

Weil sheaves admit the following presentation as the ϕ_X^*-fixed points of $D(X_{\mathbb{F}}, \Lambda)$, see Definition 2.3:

Proposition 4.4. *The last functor in* (4.2) *induces an equivalence*

$$D(X^{\text{Weil}}, \Lambda) \cong \lim \left(D(X_{\mathbb{F}}, \Lambda) \overset{\phi_X^*}{\underset{\text{id}}{\rightrightarrows}} D(X_{\mathbb{F}}, \Lambda) \right). \tag{4.4}$$

Remark 4.5. Objects in (4.4) are pairs (M, α) where $M \in D(X_{\mathbb{F}}, \Lambda)$ and α is an isomorphism $M \cong \phi_X^* M$. Note that the composition $\phi_X \circ \phi_{\mathbb{F}}$ is the absolute q-Frobenius. In particular, it induces the identity on proétale topoi, see [BS15, Lemma 5.4.2]. Therefore, replacing ϕ_X^* by $\phi_{\mathbb{F}}^*$ in (4.4) yields an equivalent category.

Proof of Proposition 4.4. The structural morphism $(X_0, \Phi_0) \to (X_{\mathbb{F}}, \phi_X)$ is a cover in $X_{\text{proét}}^{\text{Weil}}$. Its Čech nerve has objects $(X_i, \Phi_i) \in X_{\text{proét}}^{\text{Weil}}$, $i \geq 0$ as above. By descent, there is an equivalence

$$D(X^{\text{Weil}}, \Lambda) \overset{\cong}{\longrightarrow} \text{Tot} \left(D((X_{\text{proét}}^{\text{Weil}})_{/(X_\bullet, \Phi_\bullet)}, \Lambda) \right). \tag{4.5}$$

Under Lemma 4.3, the cosimplicial 1-topos associated with $(X_{\text{proét}}^{\text{Weil}})_{/(X_\bullet, \Phi_\bullet)}$ is equivalent to the cosimplicial 1-topos associated with the action of ϕ_X^* on $(X_{\mathbb{F}})_{\text{proét}}$. The equivalence (4.5) then becomes

$$D(X^{\text{Weil}}, \Lambda) \overset{\cong}{\longrightarrow} \lim_{B\mathbb{Z}} D(X_{\mathbb{F}}, \Lambda),$$

for the diagram $B\mathbb{Z} \to \text{Pr}_\Lambda^{\text{St}}$ corresponding to the endomorphism ϕ_X^* of $D(X_{\mathbb{F}}, \Lambda)$. That is, $D(X^{\text{Weil}}, \Lambda)$ is equivalent to the homotopy fixed points of $D(X_{\mathbb{F}}, \Lambda)$ with respect to the action of ϕ_X^*, which is our claim. $\qquad\square$

4.2. Weil sheaves on products.

The discussion of the previous section generalizes to products of schemes as follows. Let $X_1, \ldots, X_n$ be schemes over $\mathbb{F}_q$, and denote by $X := X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$ their product. For every $1 \leq i \leq n$, we have a morphism $\phi_{X_i} \colon X_{i,\mathbb{F}} \to X_{i,\mathbb{F}}$ as in the previous section. We use the notation ϕ_{X_i} to also denote the corresponding map on $X_{\mathbb{F}} = X_{1,\mathbb{F}} \times_{\mathbb{F}} \ldots \times_{\mathbb{F}} X_{n,\mathbb{F}}$ which is ϕ_{X_i} on the i-th factor and the identity on the other factors.

We define the site $(X_1^{\text{Weil}} \times \ldots \times X_n^{\text{Weil}})_{\text{proét}}$ whose underlying category consists of tuples $(U, \varphi_1, \ldots, \varphi_n)$ with $U \in (X_{\mathbb{F}})_{\text{proét}}$ and pairwise commuting endomomorphisms $\varphi_i \colon U \to U$ such that the following diagram commutes

$$
\begin{array}{ccc}
U & \overset{\varphi_i}{\longrightarrow} & U \\
\downarrow & & \downarrow \\
X_{\mathbb{F}} & \overset{\phi_{X_i}}{\longrightarrow} & X_{\mathbb{F}},
\end{array}
$$

for all $1 \leq i \leq n$. As before, we denote by $D(X_1^{\text{Weil}} \times \ldots \times X_n^{\text{Weil}}, \Lambda)$ the corresponding derived category of Λ-sheaves.

Using a similar reasoning as in the previous section, we can identify this category of sheaves with the homotopy fixed points

$$D(X_1^{\text{Weil}} \times \ldots \times X_n^{\text{Weil}}, \Lambda) \overset{\cong}{\longrightarrow} \text{Fix}(D(X_{\mathbb{F}}, \Lambda), \phi_{X_1}^*, \ldots, \phi_{X_n}^*) \tag{4.6}$$

of the commuting family of the functors $\phi_{X_i}^*$, see Remark 2.5. Explicitly, for $n = 2$, this is the homotopy limit of the diagram

$$
\begin{array}{ccc}
D(X_{\mathbb{F}}, \Lambda) & \overset{\phi_{X_1}^*}{\underset{\text{id}}{\longrightarrow}} & D(X_{\mathbb{F}}, \Lambda) \\
{\scriptstyle \text{id}} \downarrow \downarrow {\scriptstyle \phi_{X_2}^*} & & {\scriptstyle \text{id}} \downarrow \downarrow {\scriptstyle \phi_{X_2}^*} \\
D(X_{\mathbb{F}}, \Lambda) & \overset{\phi_{X_1}^*}{\underset{\text{id}}{\longrightarrow}} & D(X_{\mathbb{F}}, \Lambda).
\end{array}
$$

Roughly speaking, objects in the category $D(X_1^{\text{Weil}} \times \ldots \times X_n^{\text{Weil}}, \Lambda)$ are given by tuples $(M, \alpha_1, \ldots, \alpha_n)$ with $M \in D(X_{\mathbb{F}}, \Lambda)$ and with pairwise commuting equivalences $\alpha_i \colon M \cong \phi_{X_i}^* M$. That is, equipped with a collection of equivalences $\phi_{X_j}^*(\alpha_i) \circ \alpha_j \simeq \phi_{X_i}^*(\alpha_j) \circ \alpha_i$ for all i, j satisfying higher coherence conditions.

4.3. **Partial-Frobenius stability.** For schemes $X_1, \ldots, X_n$ over $\mathbb{F}_q$, we denote by $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$ their product together with the partial Frobenii $\phi_{X_i} \colon X \to X$, $1 \leq i \leq n$. To give a reasonable definition of lisse and constructible Weil sheaves, we need to understand the relation between partial-Frobenius invariant constructible subsets in X and constructible subsets in the single factors X_i:

Definition 4.6. A subset $Z \subset X$ is called *partial-Frobenius invariant* if $\phi_{X_i}(Z) = Z$ for all $1 \leq i \leq n$.

The composition $\phi_{X_1} \circ \cdots \circ \phi_{X_n}$ is the absolute q-Frobenius on X and thus induces the identity on the topological space underlying X. Therefore, in order to check that $Z \subset X$ is partial-Frobenius invariant, it suffices that, for any fixed i, the subset Z is ϕ_{X_j}-invariant for all $j \neq i$. This remark, which also applies to $X_{\mathbb{F}} = X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n \times_{\mathbb{F}_q} \operatorname{Spec} \mathbb{F}$, will be used below without further comment.

We first investigate the case of two factors with one being a separably closed field. This eventually rests on Drinfeld's descent result [Dri87, Proposition 1.1] for coherent sheaves:

Lemma 4.7. *Let X be a qcqs $\mathbb{F}_q$-scheme, and let $k/\mathbb{F}_q$ be a separably closed field. Denote by $p \colon X_k \to X$ the projection. Then $Z \mapsto p^{-1}(Z)$ induces a bijection*

$$\{\text{constructible subsets in } X\} \leftrightarrow \{\text{partial-Frobenius invariant, constructible subsets in } X_k\}.$$

Proof. The injectivity is clear because p is surjective. It remains to check the surjectivity. Without loss of generality we may assume that k is algebraically closed, and replace ϕ_X by ϕ_k which is an automorphism. Given that $Z \mapsto p^{-1}(Z)$ is compatible with passing to complements, unions and localizations on X, we are reduced to proving the bijection for constructible closed subsets Z and for X affine over $\mathbb{F}_q$. By Noetherian approximation (Lemma 4.9), we reduce further to the case where X is of finite type over $\mathbb{F}_q$ and still affine. Now we choose a locally closed embedding $X \to \mathbb{P}^n_{\mathbb{F}_q}$ into projective space. A closed subset $Z' \subset X_k$ is ϕ_k-invariant if and only if its closure inside $\mathbb{P}^n_k$ is so. Hence, it is enough to consider the case where $X = \mathbb{P}^n_{\mathbb{F}_q}$ is the projective space. Let Z' be a closed ϕ_k-invariant subset of X_k. When viewed as a reduced subscheme, the isomorphism ϕ_k restricts to an isomorphism of Z'. In particular, $\mathcal{O}_{Z'}$ is a coherent $\mathcal{O}_{X_k}$-module equipped with an isomorphism $\mathcal{O}_{Z'} \cong \phi_k^* \mathcal{O}_{Z'}$. Hence, Drinfeld's descent result [Dri87, Proposition 1.1] (see also [Ked19, Section 4.2] for a recent exposition) yields $Z' = Z_k$ for a unique closed subscheme $Z \subset X$. $\qquad\square$

The following proposition generalizes the results [Lau04, Lemma 9.2.1] and [Laf18a, Lemme 8.12] in the case of curves.

Proposition 4.8. *Let $X_1, \ldots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Then any partial-Frobenius invariant constructible closed subset $Z \subset X$ is a finite set-theoretic union of subsets of the form $Z_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} Z_n$, for appropriate constructible closed subschemes $Z_i \subset X_i$.*

In particular, any partial-Frobenius invariant constructible open subscheme $U \subset X$ is a finite union of constructible open subschemes of the form $U_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} U_n$, for appropriate constructible open subschemes $U_i \subset X_i$.

Proof. By induction, we may assume $n = 2$. By Noetherian approximation (Lemma 4.9), we reduce to the case where both X_1, X_2 are of finite type over $\mathbb{F}_q$. In the following, all products are formed over $\mathbb{F}_q$, and locally closed subschemes are equipped with their reduced subscheme structure. Let $Z \subset X_1 \times X_2$ be a partial-Frobenius invariant closed subscheme. The complement $U = X_1 \times X_2 \setminus Z$ is also partial-Frobenius invariant.

In the proof, we can replace X_1 (and likewise X_2) by a stratification in the following sense: Suppose $X_1 = A' \sqcup A''$ is a set-theoretic stratification into a closed subset A' with open complement A''. Once we know $Z \cap A' \times X_2 = \bigcup_j Z'_{1j} \times Z'_{2j}$ and $Z \cap A'' \times X_2 = \bigcup_j Z''_{1j} \times Z''_{2j}$ for appropriate closed subschemes $Z'_{1j} \subset A'$, $Z''_{1j} \subset A''$ and $Z'_{2j}, Z''_{2j} \subset X_2$, we have the set-theoretic equality

$$Z = \bigcup_j Z'_{1i} \times Z'_{2j} \cup \bigcup_j \overline{Z''_{1j}} \times Z''_{2j},$$

where $\overline{Z''_{1j}} \subset X_1$ denotes the scheme-theoretic closure. Here we note that taking scheme-theoretic closures commutes with products because the projections $X_1 \times X_2 \to X_i$ are flat, and that the topological space underlying the scheme-theoretic closure agrees with the topological closure because all schemes involved are of finite type.

The proof is now by Noetherian induction on X_2, the case $X_2 = \emptyset$ being clear (or, if the reader prefers the case where X_2 is zero dimensional reduces to Lemma 4.7). In the induction step, we may assume, using the above stratification argument, that both X_i are irreducible with generic point η_i. We let $\overline{\eta}_i$ be a geometric generic point over η_i, and denote by $p_i \colon X_1 \times X_2 \to X_i$ the two projections. Both p_i are faithfully flat of finite type and in particular open, so that $p_i(U)$ is open in X_i. We have a set-theoretic equality

$$Z = \big((X_1 \setminus p_1(U)) \times X_2\big) \cup \big(X_1 \times (X_2 \setminus p_2(U))\big) \cup \big(Z \cap p_1(U) \times p_2(U)\big).$$

Once we know $Z \cap p_1(U) \times p_2(U) = \bigcup_j Z_{1j} \times Z_{2j}$ for appropriate closed $Z_{ij} \subset p_i(U)$, we are done. We can therefore replace X_i by $p_i(U)$ and assume that both $p_i \colon U \to X_i$ are surjective.

The base change $U \times_{X_2} \overline{\eta}_2$ is a $\phi_{\overline{\eta}_2}$-invariant subset of $X_1 \times \overline{\eta}_2$. By Lemma 4.7, it is thus of the form $U_1 \times \overline{\eta}_2$ for some open subset $U_1 \subset X_1$. There is an inclusion (of open subschemes of $X_1 \times \eta_2$): $U \times_{X_2} \eta_2 \subset U_1 \times \eta_2$. It becomes a set-theoretic equality, and therefore an isomorphism of schemes, after base change along $\overline{\eta}_2 \to \eta_2$. By faithfully flat descent, this implies that the two mentioned subsets of $X_1 \times \eta_2$ agree. We claim $U_1 = X_1$. Since the projection $U \to X_2$ is surjective, in particular its image contains η_2, so that U_1 is a non-empty subset, and therefore open dense in the irreducible scheme X_1. Let $x_1 \in X_1$ be a point. Since the projection $U \to X_1$ is surjective, $U \cap (\{x_1\} \times X_2)$ is a non-empty open subscheme of $\{x_1\} \times X_2$. So it contains a point lying over (x_1, η_2). We conclude $X_1 \times \eta_2 \subset U$.

We claim that there is a non-empty open subset $A_2 \subset X_2$ such that

$$X_1 \times A_2 \subset U \text{ or, equivalently, } X_1 \times (X_2 \setminus A_2) \supset X_1 \times X_2 \setminus U.$$

The underlying topological space of $V = X_1 \times X_2 \setminus U$ is Noetherian and thus has finitely many irreducible components V_j. The closure of the projection $\overline{p_2(V_j)} \subset X_2$ does not contain η_2, since $X_1 \times \eta_2 \subset U$. Thus, $A_2 := \bigcap_j X_1 \setminus \overline{p_2(V_j)}$ satisfies our requirements.

Now we continue by Noetherian induction applied to the stratification $X_2 = A_2 \sqcup (X_2 \setminus A_2)$: We have $Z \cap X_1 \times A_2 = \emptyset$, so that we may replace X_2 by the proper closed subscheme $X_2 \setminus A_2$. Hence, the proposition follows by Noetherian induction. $\qquad\square$

The following lemma on Noetherian approximation of partial Frobenius invariant subsets is needed for the reduction to finite type schemes:

Lemma 4.9. *Let $X_1, \ldots, X_n$ be qcqs $\mathbb{F}_q$-schemes, and denote $X = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Let $X_i = \lim_j X_{ij}$ be a cofiltered limit of finite type $\mathbb{F}_q$-schemes with affine transition maps, and write $X = \lim_j X_j$, $X_j := X_{1j} \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_{nj}$ (see [Sta17, Tag 01ZA] for the existence of such presentations). Let $Z \subset X$ be a constructible closed subset. Then the intersection*

$$Z' = \bigcap_{i=1}^n \bigcap_{m \in \mathbb{Z}} \phi_{X_i}^m(Z)$$

is partial Frobenius invariant, constructible closed and there exists an index j and a partial Frobenius invariant closed subset $Z_j' \subset X_j$ such that $Z' = Z_j' \times_{X_j} X$ as sets.

We note that each ϕ_{X_i} induces a homeomorphism on the underlying topological space of X so that Z' is well-defined. This lemma applies, in particular, to partial Frobenius invariant constructible closed subsets $Z \subset X$ in which case we have $Z = Z'$.

Proof. As Z is constructible, there exists an index j and a constructible closed subscheme $Z_j \subset X_j$ such that $Z = Z_j \times_{X_j} X$ as sets. We put $Z_j' = \cap_{i=1}^n \cap_{m \in \mathbb{Z}} \phi_{X_{ij}}^m(Z_j)$. As X_j is of finite type over $\mathbb{F}_q$, the subset Z_j' is still constructible closed. As partial Frobenii induce bijections on the underlying topological spaces, one checks that $\phi_{X_{ij}}^m(Z_j) \times_{X_j} X = \phi_{X_i}^m(Z)$ as sets for all $m \in \mathbb{Z}$. Thus, $Z' = Z_j' \times_{X_j} X$ which, also, is constructible closed because $X \to X_j$ is affine. $\qquad\square$

4.4. **Lisse and constructible Weil sheaves.** In this subsection, we define the subcategories of lisse and constructible Weil sheaves and establish a presentation similar to (4.4). Let $X_1, \ldots, X_n$ be schemes over $\mathbb{F}_q$, and denote $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$. Let Λ be a condensed ring.

Definition 4.10. Let $M \in D\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$.

(1) The Weil sheaf M is called *lisse* if it is dualizable. (Here dualizability refers to the symmetric monoidal structure on $D\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$, given by the derived tensor product of Λ-sheaves on the Weil-proétale topos.)

(2) The Weil sheaf M is called *constructible* if for any open affine $U_i \subset X_i$ there exists a finite subdivision into constructible locally closed subschemes $U_{ij} \subseteq U_i$ such that each restriction $M|_{U_{1j}^{\mathrm{Weil}} \times \ldots \times U_{nj}^{\mathrm{Weil}}} \in D\big(U_{1j}^{\mathrm{Weil}} \times \ldots \times U_{nj}^{\mathrm{Weil}}, \Lambda\big)$ is lisse.

The full subcategories of $D\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ consisting of lisse, respectively constructible Weil sheaves are denoted by

$$D_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \subset D_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big).$$

Both categories are idempotent complete stable $\Gamma(X, \Lambda)$-linear symmetric monoidal ∞-categories.

From the presentation (4.6), we get that a Weil sheaf M is lisse if and only if the underlying object $M_{\mathbb{F}} \in D(X_{\mathbb{F}}, \Lambda)$ is lisse. So (4.6) restricts to an equivalence

$$D_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \cong \mathrm{Fix}\big(D_{\mathrm{lis}}\big(X_{\mathbb{F}}, \Lambda\big), \phi_{X_1}^*, \ldots, \phi_{X_n}^*\big). \tag{4.7}$$

The same is true for constructible Weil sheaves by the following proposition:

Proposition 4.11. *A Weil sheaf $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ is constructible if and only if the underlying sheaf $M_{\mathbb{F}} \in \mathrm{D}(X_{\mathbb{F}}, \Lambda)$ is constructible. Consequently, (4.6) restricts to an equivalence*

$$\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \cong \mathrm{Fix}\big(\mathrm{D}_{\mathrm{cons}}\big(X_{\mathbb{F}}, \Lambda\big), \phi_{X_1}^*, \ldots, \phi_{X_n}^*\big). \tag{4.8}$$

Proof. Clearly, if M is constructible, so is $M_{\mathbb{F}}$ by Definition 4.10. Let $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ such that $M_{\mathbb{F}}$ is constructible. We may assume that all X_i are affine. We claim that there is a finite subdivision $X_{\mathbb{F}} = \sqcup X_\alpha$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_\alpha}$ is lisse and such that each X_α is partial Frobenius invariant.

Assuming the claim we finish the argument as follows. By Proposition 4.8, any open stratum $U = X_{j_0} \subset X_{\mathbb{F}}$ is a finite union of subsets of the form $U_{1,\mathbb{F}} \times_{\mathbb{F}} \ldots \times_{\mathbb{F}} U_{n,\mathbb{F}}$ and the restriction of M to each of them is lisse. In particular, the complement $X_{\mathbb{F}} \backslash U$ is defined over $\mathbb{F}_q$ and arises as a finite union of schemes of the form $X' = X_1' \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n'$ for suitable qcqs schemes X_i' over $\mathbb{F}_q$. Intersecting each $X_{\mathbb{F}}'$ with the remaining strata $\sqcup_{j \neq j_0} X_j$, we conclude by induction on the number of strata.

It remains to prove the claim. We start with any finite subdivision $X_{\mathbb{F}} = \sqcup X_j'$ into constructible locally closed subsets such that $M_{\mathbb{F}}|_{X_j'}$ is lisse. Pick an open stratum X_{j_0}', and set

$$X_{j_0} = \bigcup_{i=1}^{n} \bigcup_{m \in \mathbb{Z}} \phi_{X_i}^m(X_{j_0}'). \tag{4.9}$$

This is a constructible open subset of $X_{\mathbb{F}}$ by Lemma 4.9 applied to its closed complement. Furthermore, $M_{\mathbb{F}}|_{X_{j_0}}$ is lisse by its partial Frobenius equivariance, noting that $\phi_{X_i}^*$ induces equivalences on proétale topoi to treat the negative powers in (4.9). As before, $X_{\mathbb{F}} \backslash X_{j_0}$ is defined over $\mathbb{F}_q$. So replacing X_j', $j \neq j_0$ by $X_j' \cap (X_{\mathbb{F}} \backslash X_{j_0})$, the claim follows by induction on the number of strata. $\square$

In the case of a single factor $X = X_1$, the preceding discussion implies

$$\mathrm{D}_\bullet\big(X^{\mathrm{Weil}}, \Lambda\big) \cong \lim \left(\mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \overset{\phi_X^*}{\underset{\mathrm{id}}{\rightrightarrows}} \mathrm{D}_\bullet(X_{\mathbb{F}}, \Lambda) \right), \tag{4.10}$$

for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$.

4.5. **Relation with the Weil groupoid.** In this subsection, we relate lisse Weil sheaves with representations of the Weil groupoid. Throughout, we work with étale fundamental groups as opposed to their proétale variants in order to have Drinfeld's lemma available, see Section 5.2. The two concepts differ in general, but agree for geometrically unibranch (for example, normal) Noetherian schemes, see [BS15, Lemma 7.4.10].

For a Noetherian scheme X, let $\pi_1(X)$ be the *étale fundamental groupoid* of X as defined in [SGA03, Exposé V, §7 and §9]. Its objects are geometric points of X, and its morphisms are isomorphisms of fiber functors on the finite étale site of X. This is an essentially small category. The automorphism group in $\pi_1(X)$ at a geometric point $x \to X$ is profinite. It is denoted $\pi_1(X, x)$ and called the *étale fundamental group* of (X, x). If X is connected, then the natural map $B\pi_1(X, x) \to \pi_1(X)$ is an equivalence for any $x \to X$. If X is the disjoint sum of schemes X_i, $i \in I$, then $\pi_1(X)$ is the disjoint sum of the $\pi_1(X_i)$, $i \in I$. In this case, if $x \to X$ factors through X_i, then $\pi_1(X, x) = \pi_1(X_i, x)$.

Definition 4.12. Let $X_1, \ldots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and write $X = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. The *Frobenius-Weil groupoid* is the stacky quotient

$$\mathrm{FWeil}(X) = \pi_1(X_{\mathbb{F}})/\langle \phi_{X_1}^{\mathbb{Z}}, \ldots, \phi_{X_n}^{\mathbb{Z}} \rangle, \tag{4.11}$$

where we use that the partial Frobenii ϕ_{X_i} induce automorphisms on the finite étale site of $X_{\mathbb{F}}$.

For $n = 1$, we denote $\mathrm{FWeil}(X) = \mathrm{Weil}(X)$. Even if X is connected, its base change $X_{\mathbb{F}}$ might be disconnected in which case the action of ϕ_X permutes some connected components. Therefore, fixing a geometric point of $X_{\mathbb{F}}$ is inconvenient, and the reason for us to work with fundmental groupoids as opposed to fundmental groups. The automorphism groups in $\mathrm{Weil}(X)$ carry the structure of locally profinite groups: indeed, if X is connected, then $\mathrm{Weil}(X)$ is, for any choice of a geometric point $x \to X_{\mathbb{F}}$, equivalent to the classifying space of the Weil group $\mathrm{Weil}(X, x)$ from [Del80, Définition 1.1.10]. Recall that this group sits in an exact sequence of topological groups

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{Weil}(X, x) \to \mathrm{Weil}(\mathbb{F}/\mathbb{F}_q) \simeq \mathbb{Z}, \tag{4.12}$$

where $\pi_1(X_{\mathbb{F}}, x)$ carries its profinite topology and $\mathbb{Z}$ the discrete topology. The topology on the morphisms groups in $\mathrm{Weil}(X)$ obtained in this way is independent from the choice of $x \to X_{\mathbb{F}}$. The image of $\mathrm{Weil}(X, x) \to \mathbb{Z}$ is the subgroup $m\mathbb{Z}$ where m is the degree of the largest finite subfield in $\Gamma(X, \mathcal{O}_X)$. In particular, we have $m = 1$ if $X_{\mathbb{F}}$ is connected. Let us add that if $x \to X_{\mathbb{F}}$ is fixed under ϕ_X, then the action of ϕ_X on $\pi_1(X_{\mathbb{F}}, x)$ corresponds by virtue of the formula $\phi_X^* = (\phi_{\mathbb{F}}^*)^{-1}$ to the action of the geometric Frobenius, that is, the inverse of the q-Frobenius in $\mathrm{Weil}(\mathbb{F}/\mathbb{F}_q)$.

Likewise, for every $n \geq 1$, the stabilizers of the Frobenius-Weil groupoid are related to the partial Frobenius-Weil groups introduced in [Dri87, Proposition 6.1] and [Laf18a, Remarque 8.18]. In particular, there is an exact sequence

$$1 \to \pi_1(X_{\mathbb{F}}, x) \to \mathrm{FWeil}(X, x) \to \mathbb{Z}^n,$$

for each geometric point $x \to X_{\mathbb{F}}$. This gives $\mathrm{FWeil}(X)$ the structure of a locally profinite groupoid.

Let Λ be either of the following coherent topological rings: a coherent discrete ring, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for some prime ℓ, or its ring of integers $\mathcal{O}_E \supset \mathbb{Z}_\ell$. For a topological groupoid W, we will denote by $\mathrm{Rep}_\Lambda(W)$ the category of continuous representations of W with values in finitely presented Λ-modules and by $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W) \subset \mathrm{Rep}_\Lambda(W)$ its full subcategory of representations on finite projective Λ-modules. Here finitely presented Λ-modules M carry the quotient topology induced from the choice of any surjection $\Lambda^n \to M$, $n \geq 0$ and the product topology on Λ^n.

Lemma 4.13. *In the situation above, the category $\mathrm{Rep}_\Lambda(W)$ is Λ_*-linear and abelian. In particular, its full subcategory $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ is Λ_*-linear and additive.*

Proof. Let W_{disc} be the discrete groupoid underlying W, and denote by $\mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ the category of W_{disc}-representations on finitely presented Λ-modules. Evidently, this category is Λ_*-linear and abelian since Λ is coherent (Lemma 3.31). We claim that $\mathrm{Rep}_\Lambda(W) \subset \mathrm{Rep}_\Lambda(W_{\mathrm{disc}})$ is a Λ_*-linear full abelian subcategory. If Λ is discrete (and coherent), then every finitely presented Λ-module carries the discrete topology and the claim is immediate, see also [Sta17, Tag 0A2H]. For $\Lambda = E, \mathcal{O}_E$, one checks that every map of finitely presented Λ-modules is continuous, every surjective map is a topological quotient and every injective map is a closed embedding. For the latter, we use that every finitely presented Λ-module can be written as a countable filtered colimit of compact Hausdorff spaces along injections, and that every injection of compact Hausdorff spaces is a closed embedding. This implies the claim. $\square$

We apply this for W being either of the locally profinite groupoids $\pi_1(X)$, $\pi_1(X_{\mathbb{F}})$ or $\mathrm{FWeil}(X)$. Note that restricting representations along $\pi_1(X_{\mathbb{F}}) \to \mathrm{FWeil}(X)$ induces an equivalence of Λ_*-linear abelian categories

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \cong \mathrm{Fix}\big(\mathrm{Rep}_\Lambda\big(\pi_1(X_{\mathbb{F}})\big), \phi_{X_1}, \ldots, \phi_{X_n}\big), \tag{4.13}$$

and similarly for the Λ_*-linear additive category $\mathrm{Rep}_\Lambda^{\mathrm{fp}}\big(\mathrm{FWeil}(X)\big)$.

Lemma 4.14. *In the situation above, there is a natural functor*

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \to \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)^\heartsuit, \tag{4.14}$$

that is fully faithful. Moreover, the following properties hold if Λ is either finite discrete or $\Lambda = \mathcal{O}_E$ for $E \supset \mathbb{Q}_\ell$ finite:

(1) An object M lies in the essential image of (4.14) if and only if its underlying sheaf $M_{\mathbb{F}}$ is locally on $(X_{\mathbb{F}})_{\mathrm{proét}}$ isomorphic to $\underline{N} \otimes_{\Lambda_*} \Lambda_{X_{\mathbb{F}}}$ for some finitely presented Λ_*-module N.

(2) The functor (4.14) restricts to an equivalence of Λ_*-linear additive categories

$$\mathrm{Rep}_\Lambda^{\mathrm{fp}}\big(\mathrm{FWeil}(X)\big) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}^{\{0,0\}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big).$$

(3) If Λ is t-admissible, then (4.14) restricts to an equivalence of Λ_*-linear abelian categories

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \xrightarrow{\cong} \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)^\heartsuit.$$

If all X_i, $i = 1, \ldots, n$ are geometrically unibranch, then (1), (2) and (3) hold for general coherent topological rings Λ as above.

Proof. There is a canonical equivalence of topological groupoids $\pi_1(X_{\mathbb{F}}) \cong \widehat{\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})}$ with the profinite completion of the proétale fundamental groupoid, see [BS15, Lemma 7.4.3]. It follows from [BS15, Lemmas 7.4.5, 7.4.7] that restricting representations along $\pi_1^{\mathrm{proét}}(X_{\mathbb{F}}) \to \pi_1(X_{\mathbb{F}})$ induces full embeddings

$$\mathrm{Rep}_\Lambda\big(\pi_1(X_{\mathbb{F}})\big) \hookrightarrow \mathrm{Rep}_\Lambda\big(\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})\big) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^\heartsuit, \tag{4.15}$$

that are compatible with the action of ϕ_{X_i} for all $i = 1, \ldots, n$. So we obtain the fully faithful functor (4.14) by passing to fixed points, see (4.13), (4.7) and Lemma 2.4 (see also Remark 2.5).

Part (1) describes the essential image of $\mathrm{Rep}_\Lambda\big(\pi_1^{\mathrm{proét}}(X_{\mathbb{F}})\big) \hookrightarrow \mathrm{D}(X_{\mathbb{F}}, \Lambda)^\heartsuit$. So if Λ is finite discrete or profinite, then the first functor in (4.15) is an equivalence, and we are done. Part (2) is immediate from (1), noting that an object in the essential image of (4.15) is lisse if and only if its underlying module is finite projective. Likewise, part (3) is immediate from (1), using Corollary 3.29. Here we need to exclude rings like $\Lambda = \mathbb{Z}/\ell^2$ in order to have a t-structure on lisse sheaves.

Finally, if all X_i are geometrically unibranch, so is $X_{\mathbb{F}}$ which follows from the characterization [Sta17, Tag 0BQ4]. In this case, we get $\pi_1(X_{\mathbb{F}}) \cong \pi_1^{\mathrm{proét}}(X_{\mathbb{F}})$ by [BS15, Lemma 7.4.10]. This finishes the proof. $\square$

4.6. **Weil-étale versus étale sheaves.** We end this section with the following description of Weil sheaves with (ind-)finite coefficients. Note that such simplification in terms of ordinary sheaves is not possible for $\Lambda = \mathbb{Z}, \mathbb{Z}_\ell, \mathbb{Q}_\ell$, say.

Proposition 4.15. *Let X be a qcqs $\mathbb{F}_q$-scheme. Let Λ be a finite discrete ring or a filtered colimit of such rings. Then the natural functors*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \to \mathrm{D}_{\mathrm{lis}}\big(X^{\mathrm{Weil}}, \Lambda\big), \quad \mathrm{D}_{\mathrm{cons}}(X, \Lambda) \to \mathrm{D}_{\mathrm{cons}}\big(X^{\mathrm{Weil}}, \Lambda\big),$$

are equivalences.

Proof. Throughout, we repeatedly use that filtered colimits commute with finite limits in Cat_∞. Using Proposition 3.20, we may assume that Λ is finite discrete. By the comparison result Proposition 3.40, we can identify the categories $\mathrm{D}_\bullet(X, \Lambda)$, respectively $\mathrm{D}_\bullet(X_\mathbb{F}, \Lambda)$ for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$ with full subcategories of the derived category of étale Λ-sheaves $\mathrm{D}(X_{\mathrm{ét}}, \Lambda)$, respectively $\mathrm{D}(X_{\mathbb{F}, \mathrm{ét}}, \Lambda)$. Write $X = \lim X_i$ as a cofiltered limit of finite type $\mathbb{F}_q$-schemes X_i with affine transition maps [Sta17, Tag 01ZA]. Using the continuity of étale sites [Sta17, Tag 03Q4], there are natural equivalences

$$\mathrm{colim}\,\mathrm{D}_\bullet(X_i, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X, \Lambda), \quad \mathrm{colim}\,\mathrm{D}_\bullet(X_i^{\mathrm{Weil}}, \Lambda) \xrightarrow{\cong} \mathrm{D}_\bullet(X^{\mathrm{Weil}}, \Lambda) \tag{4.16}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Hence, we can assume that X is finite type over $\mathbb{F}_q$.

To show full faithfulness, we claim more generally that the natural map

$$\mathrm{D}(X_{\mathrm{ét}}, \Lambda) \to \lim \left(\mathrm{D}(X_{\mathbb{F}, \mathrm{ét}}, \Lambda) \underset{\mathrm{id}}{\overset{\phi_X^*}{\rightrightarrows}} \mathrm{D}(X_{\mathbb{F}, \mathrm{ét}}, \Lambda) \right) =: \mathrm{D}\big(X_{\mathrm{ét}}^{\mathrm{Weil}}, \Lambda\big)$$

is fully faithful. As Λ is torsion, this is immediate from [Gei04, Corollary 5.2] applied to the inner homomorphisms between sheaves. Let us add that this induces fully faithful functors

$$\mathrm{D}^+(X_{\mathrm{ét}}, \Lambda) \to \mathrm{D}^+\big(X_{\mathrm{ét}}^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big) \tag{4.17}$$

on bounded below objects, see [BS15, Proposition 5.2.6 (1)].

It remains to prove essential surjectivity. Using a stratification as in Definition 4.10, it is enough to consider the lisse case. Pick $M \in \mathrm{D}_{\mathrm{lis}}(X^{\mathrm{Weil}}, \Lambda)$. It is enough to show that M lies is in the essential image of (4.17), noting that the functor detects dualizability. As M is bounded, this will follow from showing that for every $j \in \mathbb{Z}$, the cohomology sheaf $\mathrm{H}^j(M) \in \mathrm{D}(X^{\mathrm{Weil}}, \Lambda)^\heartsuit$ is in the essential image of (4.17).

Fix $j \in \mathbb{Z}$. As M is lisse, the underlying sheaf $\mathrm{H}^j(M)_\mathbb{F} \in \mathrm{D}(X_\mathbb{F}, \Lambda)^\heartsuit$ is proétale-locally constant and valued in finitely presented Λ-modules. By Lemma 4.14 (1), it comes from a representation of $\mathrm{Weil}(X)$. Restriction of representations along $\mathrm{Weil}(X) \to \pi_1(X)$ fits into a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Rep}_\Lambda\big(\pi_1(X)\big) & \xrightarrow{\;\cong\;} & \mathrm{Rep}_\Lambda\big(\mathrm{Weil}(X)\big) \\
\downarrow & & \downarrow \\
\mathrm{D}(X_{\mathrm{ét}}, \Lambda)^\heartsuit & \longrightarrow & \mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big)^\heartsuit,
\end{array}
$$

where the upper horizontal arrow is an equivalence since Λ is finite. In particular, the object $\mathrm{H}^j(M)$ is in the essential image of the fully faithful functor (4.17). $\qquad\square$

5. The categorical Künneth formula

We continue with the notation of Section 4. In particular, $\mathbb{F}_q$ denotes a finite field of characteristic $p > 0$. Recall from Section 2.1 the tensor product of Λ_*-linear idempotent complete stable ∞-categories. The external tensor product of sheaves $(M_1, \dots, M_n) \mapsto M_1 \boxtimes \dots \boxtimes M_n$ as in (3.19) induces a functor

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \dots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet\big(X_n^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \times \dots \times X_n^{\mathrm{Weil}}, \Lambda\big), \tag{5.1}$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Throughout, we consider the following situation, see also Remark 5.3 for the compatibility of (5.1) with certain (co-)limits in the schemes X_i and coefficients Λ:

Situation 5.1. The schemes $X_1, \dots, X_n$ are of finite type over $\mathbb{F}_q$, and Λ is the condensed ring associated with one of the following topological rings:

(a) a finite discrete ring of prime-to-p-torsion;
(b) the ring of integers $\mathcal{O}_E$ of a finite field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$;
(c) a finite field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$;
(d) a finite discrete p-torsion ring that is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$.

Theorem 5.2. *In Situation 5.1, the functor (5.1) is an equivalence in each of the following cases:*

(1) $\bullet$ = cons *and* Λ *is as in* (a), (b) *or* (c);

(2) $\bullet$ = lis *and* Λ *is as in* (a), (b), (d) *or as in* (c) *if all* X_i, $i = 1, \ldots, n$ *are geometrically unibranch (for example, normal).*

In the p-torsion free cases (a), (b) and (c), the full faithfulness is a direct consequence of the Künneth formula applied to the $X_{i,\mathbb{F}}$. In the p-torsion case (d), we use Artin–Schreier theory instead. It would be interesting to see whether this part can be extended to constructible sheaves using the mod-p-Riemann–Hilbert correspondence as in, say, [BL19]. In all cases, the essential surjectivity relies on a variant of Drinfeld's lemma for Weil group representations.

Before turning to the proof of Theorem 5.2, let us point out the following compatibility of the functor (5.1) with (co-)limits. This can be used to extend Theorem 5.2, for example, to condensed rings like $\Lambda = \bar{\mathbb{Z}}_\ell, \bar{\mathbb{Q}}_\ell$, $\ell \neq p$ in cases (b) and (c), or general qcqs $\mathbb{F}_q$-schemes X_i and finite discrete rings like $\mathbb{Z}/m$ for any integer $m \geq 1$ in cases (a) and (d):

Remark 5.3 (Compatibility of (5.1) with certain (co-)limits)**.** Throughout, we repeatedly use that filtered colimits commute with finite limits and both commute with tensor products in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda_*}(\mathrm{Idem})$, see also Section 2.

(1) *Filtered colimits in* Λ. First off, extension of scalars along any map of condensed rings $\Lambda \to \Lambda'$ induces a commutative diagram in $\mathrm{Cat}^{\mathrm{Ex}}_{\infty, \Lambda_*}(\mathrm{Idem})$:

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_\bullet\big(X_n^{\mathrm{Weil}}, \Lambda\big) \longrightarrow \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$$

$$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}}, \Lambda'\big) \otimes_{\mathrm{Perf}_{\Lambda'_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda'_*}} \mathrm{D}_\bullet\big(X_n^{\mathrm{Weil}}, \Lambda'\big) \longrightarrow \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \ldots \times X_n^{\mathrm{Weil}}, \Lambda')$$

It follows from Proposition 3.20 that both sides of (5.1) are compatible with filtered colimits in Λ.

(2) *Finite products in* Λ. Let $\Lambda = \prod \Lambda_i$ be a finite product of condensed rings. For any scheme X, the natural map $\mathrm{D}_\bullet(X, \Lambda) \to \prod \mathrm{D}_\bullet(X, \Lambda_i)$ is an equivalence for $\bullet \in \{\varnothing, \mathrm{lis}, \mathrm{cons}\}$, and likewise for Weil sheaves if X is defined over $\mathbb{F}_q$. As $\Lambda_* = \prod \Lambda_{i,*}$, we see that (5.1) is compatible finite products in the coefficients.

(3) *Limits in* X_i *for discrete* Λ. Assume that Λ is finite discrete, see Situation 5.1 (a), (d). Let $X_1, \ldots, X_n$ be qcqs $\mathbb{F}_q$-schemes. Write each X_i as a cofiltered limit $X_i = \lim X_{ij}$ of finite type $\mathbb{F}_q$-schemes X_{ij} with affine transition maps [Sta17, Tag 01ZA]. As Λ is finite discrete, we can use the continuity of étale sites as in (4.16) to show that the natural map

$$\mathrm{colim}_j \mathrm{D}_\bullet\big(X_{1j}^{\mathrm{Weil}} \times \ldots \times X_{nj}^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big),$$

is an equivalence for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Thus, (5.1) is compatible with cofiltered limits of finite type $\mathbb{F}_q$-schemes with affine transition maps.

5.1. **Full faithfulness.** In this section, we prove that the functor (5.1) is fully faithful under the conditions of Theorem 5.2. We first consider the p-torsion free cases:

Proposition 5.4. *Let* $X_1, \ldots, X_n$ *and* Λ *be as in Situation* 5.1 (a), (b) *or* (c). *Then the functor* (5.1) *is fully faithful for* $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.

Proof. For constructible sheaves on $X_{i,\mathbb{F}}$ (as opposed to X_i^{Weil}), this interpretation of the Künneth formula appears already in [GKRV20, Section A.2]. Throughout, we drop Λ from the notation. It is enough to verify that for all $M_i, N_i \in \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}})$ the natural map

$$\bigotimes_{i=1}^n \mathrm{Hom}_{\mathrm{D}(X_i^{\mathrm{Weil}})}(M_i, N_i) \to \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}})}(M_1 \boxtimes \ldots \boxtimes M_n, N_1 \boxtimes \ldots \boxtimes N_n) \qquad (5.2)$$

is an equivalence. As (5.2) is functorial in the objects and compatible with shifts, it suffices, by Definition 4.10, to consider the case where M_i, $i = 1, \ldots, n$ is the extension by zero of a lisse Weil Λ-sheaf on some locally closed subscheme $Z_i \subset X_i$. Using the adjunction

$$(\iota_i)_! : \mathrm{D}_{\mathrm{cons}}(Z_i^{\mathrm{Weil}}) \rightleftarrows \mathrm{D}_{\mathrm{cons}}(X_i^{\mathrm{Weil}}) : (\iota_i)^!,$$

and the dualizability of lisse sheaves, we reduce to the case $M_i = \Lambda_{X_i}$, $i = 1, \ldots, n$. That is, (5.2) becomes a map of cohomology complexes. By Proposition 4.4, we have

$$\mathrm{R}\Gamma\big(X_i^{\mathrm{Weil}}, N_i\big) = \lim \left(\mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i) \xrightarrow{\phi_{X_i}^* - \mathrm{id}} \mathrm{R}\Gamma(X_{i,\mathbb{F}}, N_i) \right). \qquad (5.3)$$

33

A similar computation holds for the mapping complexes in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}})$, see (4.6). Such finite limits commute with the tensor product in Mod_Λ. Thus, (5.3) reduces to the Künneth formula

$$\mathrm{R}\Gamma\big(X_{1,\mathbb{F}}, N_1\big) \otimes \ldots \otimes \mathrm{R}\Gamma\big(X_{n,\mathbb{F}}, N_n\big) \xrightarrow{\cong} \mathrm{R}\Gamma\big(X_{1,\mathbb{F}} \times_\mathbb{F} \ldots \times_\mathbb{F} X_{n,\mathbb{F}}, N_1 \boxtimes \ldots \boxtimes N_n\big),$$

where we use that the X_i are of finite type and the coprimality assumptions on Λ, see [Sta17, Tag 0F1P]. $\qquad\square$

Next, we consider the p-torsion case:

Proposition 5.5. *Let $X_1, \ldots, X_n$ and Λ be as in Situation 5.1 (d). Then the functor (5.1) is fully faithful for $\bullet = \mathrm{lis}$.*

Proof. As in the proof of Proposition 5.4, we need to show that the map

$$\bigotimes_{i=1}^n \mathrm{R}\Gamma\big(X_i^{\mathrm{Weil}}, N_i\big) \to \mathrm{R}\Gamma\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, N_1 \boxtimes \ldots \boxtimes N_n\big) \tag{5.4}$$

is an equivalence for any $N_i \in \mathrm{D}_{\mathrm{lis}}(X_i^{\mathrm{Weil}})$. Using Zariski descent for both sides, we may assume that each X_i is affine. As Λ is finite discrete (see also the discussion around (4.16)), the invariance of the étale site under perfection reduces us to the case where each X_i is perfect. The proof now proceeds by several reduction steps: 1) reduce to $N_i = \Lambda_{X_i}$; 2) reduce to $\Lambda = \mathbb{Z}/p$; 3) reduce to $q = p$ being a prime. The last step 4) is then an easy computation.

Step 1): *We may assume $N_i = \Lambda_{X_i}$.* In order to show (5.4) is a quasi-isomorphism, it suffices to show this after applying $\tau^{\leq r}$ for arbitrary r. By shifting the bounded (Corollary 3.17) complexes N_i appropriately, we may assume $r = 0$. Note that $\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, N_i) \cong \mathrm{R}\Gamma(X_i, N_i)$, see Proposition 4.15. By right exactness of the tensor product, we have $\tau^{\leq 0}\big(\bigotimes_i \mathrm{R}\Gamma(X_i, N_i)\big) = \bigotimes_i \tau^{\leq 0} \mathrm{R}\Gamma(X_i, N_i)$. By Proposition 3.40 and the discussion preceding it, there is an étale covering $U_i \to X_i$ such that $N_i|_{U_i}$ is perfect-constant. Let $U_{i,\bullet}$ be the Čech nerve of this covering. By étale descent, we have

$$\mathrm{R}\Gamma(X_i, N_i) = \lim_{[j] \in \Delta} \mathrm{R}\Gamma(U_{i,j}, N_i).$$

For each $r \in \mathbb{Z}$, there is some j_r such that

$$\tau^{\leq r} \lim_{[j] \in \Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \lim_{[j] \in \Delta, j \leq j_r} \tau^{\leq r} \mathrm{R}\Gamma(U_{i,j}, N_i).$$

This can be seen from the spectral sequence (note that it is concentrated in degrees $j \geq 0$ and degrees $j' \geq r$ for some r, since the complexes N_i are bounded from below)

$$\mathrm{H}^{j'}(U_{i,j}, N_i) \Rightarrow \mathrm{H}^{j'+j} \lim_{j \in \Delta} \mathrm{R}\Gamma(U_{i,j}, N_i) = \mathrm{H}^{j'+j}(X_i, N_i).$$

As the tensor product in (5.4) commutes with *finite* limits, we may thus assume that each N_i is perfect-constant. Another dévissage reduces us to the case $N_i = \Lambda_{X_i}$, the constant sheaf itself.

Step 2): *We may assume $\Lambda = \mathbb{Z}/p$.* By assumption, Λ is flat over $\mathbb{Z}/p^m$ for some $m \geq 1$. We immediately reduce to $\Lambda = \mathbb{Z}/p^m$. For any perfect affine scheme $X = \mathrm{Spec}\, R$ in characteristic $p > 0$, we claim that $\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \cong \mathrm{R}\Gamma(X, \mathbb{Z}/p^r)$. Assuming the claim, we finish the reduction step by tensoring (5.4) with the short exact sequence of $\mathbb{Z}/p^m$-modules $0 \to \mathbb{Z}/p^{m-1} \to \mathbb{Z}/p^m \to \mathbb{Z}/p \to 0$, using that finite limits commutes with tensor products. It remains to prove the claim. The Artin–Schreier–Witt exact sequence of sheaves on $X_{\text{ét}}$ yields

$$\mathrm{R}\Gamma(X, \mathbb{Z}/p^m) = [W_m(R) \xrightarrow{F - \mathrm{id}} W_m(R)].$$

Now we use that $W_m(R) \otimes_{\mathbb{Z}/p^m} \mathbb{Z}/p^r \xrightarrow{\cong} W_r(R)$ compatibly with F, which holds since R is perfect. This shows the claim, and we have accomplished Step 2).

Step 3): *We may assume q is prime.* Recall that $q = p^r$ is a prime power. In order to reduce to the case $r = 1$, let $X_i' := X_i$, but now regarded as a scheme over $\mathbb{F}_p$. We have $X_{i,\mathbb{F}}' = \bigsqcup_{i=1}^r X_{i,\mathbb{F}}$. The Galois group $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ is generated by the p-Frobenius, which acts by permuting the components in this disjoint union. Thus, we have $\mathrm{D}((X_i')^{\mathrm{Weil}}) = \mathrm{D}(X_i^{\mathrm{Weil}})$. The same reasoning also applies to several factors X_i^{Weil}, so we may assume our ground field to be $\mathbb{F}_p$.

Step 4): Set $R := \bigotimes_{i, \mathbb{F}_p} R_i$, $R_\mathbb{F} := R \otimes_{\mathbb{F}_p} \mathbb{F}$. We write ϕ_i for the p-Frobenius on R_i and also for any map on a tensor product involving R_i, by taking the identity on the remaining tensor factors. By Artin–Schreier theory, we have

$$\mathrm{R}\Gamma(X_i^{\mathrm{Weil}}, \mathbb{Z}/p) \overset{4.15}{=} \mathrm{R}\Gamma(X_i, \mathbb{Z}/p) = [R_i \xrightarrow{\phi_i - \mathrm{id}} R_i],$$

$$\mathrm{R}\Gamma(X_{1,\mathbb{F}} \times_\mathbb{F} \cdots \times_\mathbb{F} X_{n,\mathbb{F}}, \mathbb{Z}/p) = [R_\mathbb{F} \xrightarrow{\phi - \mathrm{id}} R_\mathbb{F}],$$

where ϕ is the absolute p-Frobenius of $R_\mathbb{F}$. Thus, the right hand side in (5.4) is the homotopy orbits of the action of $\mathbb{Z}^{n+1}$ on $R_\mathbb{F}$, whose basis vectors act as $\phi_1, \ldots, \phi_n$ and ϕ. Note that ϕ is the composite $\phi_\mathbb{F} \circ \phi_1 \circ \cdots \circ \phi_n$, where $\phi_\mathbb{F}$ is the Frobenius on $\mathbb{F}$. Thus, the previously mentioned $\mathbb{Z}^{n+1}$-action on $R_\mathbb{F}$ is equivalent to the one where the

basis vectors act as $\phi_1, \ldots, \phi_n$ and $\phi_{\mathbb{F}}$. We conclude our claim by using that $[R_{\mathbb{F}} \overset{\mathrm{id}-\phi_{\mathbb{F}}}{\to} R_{\mathbb{F}}]$ is quasi-isomorphic to $R[0]$.   □

5.2. Drinfeld's lemma.
The essential surjectivity in Theorem 5.2 is based on the following variant of Drinfeld's lemma. Its formulation is close to [Lau04, Theorem 8.1.4], and in this form is a slight extension of [Laf18a, Lemme 8.2] for $\mathbb{Z}_\ell$-coefficients and [Xue20b, Lemma 3.3.2] for $\mathbb{Q}_\ell$-coefficients. We will drop the coefficient ring Λ from the notation whenever convenient.

Let $X_1, \ldots, X_n$ be Noetherian schemes over $\mathbb{F}_q$, and denote $X = X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Recall the Frobenius–Weil groupoid $\mathrm{FWeil}(X)$, see Definition 4.12. The projections $X_{\mathbb{F}} \to X_{i,\mathbb{F}}$ onto the single factors induce a continuous map of locally profinite groupoids

$$\mu \colon \mathrm{FWeil}(X) \to \mathrm{Weil}(X_1) \times \ldots \times \mathrm{Weil}(X_n). \tag{5.5}$$

Theorem 5.6 (Version of Drinfelds's lemma). *Let Λ be as in Situation 5.1. Restriction along the map (5.5) induces an equivalence*

$$\mathrm{Rep}_\Lambda\big(\mathrm{Weil}(X_1) \times \ldots \times \mathrm{Weil}(X_n)\big) \overset{\cong}{\to} \mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big), \tag{5.6}$$

between the abelian categories of continuous representations on finitely presented Λ-modules.

Proof. For all objects $x \in \mathrm{FWeil}(X)$, that is, all geometric points $x \to X_{\mathbb{F}}$, passing to the automorphism groups induces a commutative diagram of locally profinite groups

$$
\begin{array}{ccccccccc}
1 & \longrightarrow & \pi_1(X_{\mathbb{F}}, x) & \longrightarrow & \mathrm{FWeil}(X, x) & \longrightarrow & \mathbb{Z}^n & . \\
 & & \downarrow & & \downarrow{\scriptstyle \mu_x} & & \| & \\
1 & \longrightarrow & \prod_{i=1}^n \pi_1(X_{i,\mathbb{F}}, x) & \longrightarrow & \prod_{i=1}^n \mathrm{Weil}(X_i, x) & \longrightarrow & \mathbb{Z}^n &
\end{array}
$$

The left vertical arrow is surjective [Sta17, Tags 0BN6, 0385]. Thus μ_x is surjective as well and hence (5.6) is fully faithful. For essential surjectivity, it remains to show that any continuous representation $\mathrm{FWeil}(X, x) \to \mathrm{GL}(M)$ on a finitely presented Λ-module M factors through μ_x. The key input is Drinfeld's lemma: it implies that μ_x induces an isomorphism on profinite completions. Therefore, it is enough to apply Lemma 5.7 below with $H := \mathrm{FWeil}(X, x) \to \mathrm{Weil}(X_1) \times \ldots \times \mathrm{Weil}(X_n) =: G$ and $K := \pi_1(X_{\mathbb{F}}, x)$. This completes the proof of (5.6).   □

The following lemma formalizes a few arguments from [Xue20b, §3.2.3], and we reproduce the proof for the convenience of the reader:

Lemma 5.7 (Drinfeld, Xue). *Let Λ be as in Situation 5.1. Let $\mu \colon H \to G$ be a continuous surjection of locally profinite groups that induces an isomorphism on profinite completions. Assume that there exists a compact open normal subgroup $K \subset H$ containing $\ker \mu$ such that H/K is finitely generated and injects into its profinite completion. Then μ induces an equivalence*

$$\mathrm{Rep}_\Lambda(G) \cong \mathrm{Rep}_\Lambda(H)$$

between their categories of continuous representations on finitely presented Λ-modules.

Proof. The case where Λ is finite discrete is obvious, and hence so is the case $\Lambda = \mathcal{O}_E$ for some finite field extension $E \supset \mathbb{Q}_\ell$. The case $\Lambda = E$ is reduced to $\Lambda = \mathbb{Q}_\ell$. As μ is surjective, it remains to show that every continuous representation $\rho \colon H \to \mathrm{GL}(M)$ on a finite-dimensional $\mathbb{Q}_\ell$-vector space factors through G, that is, $\ker \mu \subset \ker \rho$. One shows the following properties:

(1) The group $\ker \mu$ is the intersection over all open subgroups in K which are normal in H.
(2) The group $\ker \rho \cap K$ is a closed normal subgroup in H such that $K/\ker \rho \cap K \cong \rho(K)$ is topologically finitely generated.

These properties imply $\ker \mu \subset \ker \rho \cap K$ as follows: For a finite group L, let $U_L := \cap \ker(K \to L)$ where the intersection is over all continuous morphisms $K \to L$ that are trivial on $\ker \rho \cap K$. Because of the topologically finite generatedness in (2), this is a finite intersection so that U_L is open in K. Also, it is normal in H, and hence $\ker \mu \subset U_L$ by (1). On the other hand, it is evident that $\ker \rho \cap K = \cap_L U_L$ because K is profinite.

For the proof of (1) observe that $\ker \mu$ agrees with the kernel of $H \to H^\wedge \cong G^\wedge$ by our assumption on the profinite completions. Using $\ker \mu \subset K$ and the injection $H/K \to (H/K)^\wedge$ implies (1).

For (2) it is evident that $\ker \rho \cap K$ is a closed normal subgroup in H. Since K is compact, its image $\rho(K)$ is a closed subgroup of the ℓ-adic Lie group $\mathrm{GL}(M)$, hence an ℓ-adic Lie group itself. The final assertion follows from [Ser64, théorème 2].   □

For the overall goal of proving essential surjectivity in Theorem 5.2, we need to investigate how representations of product groups factorize into external tensor products of representations. In view of Lemma 4.13 and its proof, it is enough to consider representations of abstract groups, disregarding the topology. This is done in the next section.

5.3. Factorizing representations. In this subsection, let Λ be a Dedekind domain [Sta17, Tag 034X]. Thus, any submodule N of a finite projective Λ-module M is again finite projective.

Given any group W, we write $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ for the category of W-representations on finite projective Λ-modules. As in [CR06, Sections 73.8, 75], we say that such a W-representation M is *fp-simple* if any subrepresentation $0 \neq N \subset M$ has maximal rank. By induction on the rank, every non-zero representation in $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ admits a non-zero fp-simple subrepresentation. The proof of the following lemma is left to the reader, see also [CR06, Theorem 75.6]:

Lemma 5.8. *A representation $M \in \mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ is fp-simple if and only if $M \otimes_\Lambda \mathrm{Frac}(\Lambda)$ is fp-simple (hence, simple).*

The following proposition will serve in the proof of Theorem 5.2 using Theorem 5.6, where we will need to decompose representations of a product of Weil groups into decompositions of the individual Weil groups.

Proposition 5.9. *Let $W = W_1 \times W_2$ be a product of two groups. Let $M \in \mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$ be fp-simple. Fix W_1-subrepresentation $M_1 \subset M$ that is fp-simple. Consider the W_2-representation $M_2 := \mathrm{Hom}_{W_1}(M_1, M)$ and the associated evaluation map*

$$\mathrm{ev} \colon M_1 \boxtimes M_2 \to M.$$

(1) If Λ is an algebraically closed field, then ev is an isomorphism and M_2 is simple.
(2) If Λ is a perfect field, then ev is a split surjection and M_2 is semi-simple.
(3) If Λ is a Dedekind domain of Krull dimension 1 with perfect fraction field, then there is a short exact sequence

$$0 \to M \oplus \ker \mathrm{ev} \to M_1 \boxtimes M_2 \to T \to 0, \tag{5.7}$$

where T is Λ-torsion.

Proof. Note that ev is a map in $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(W)$. Its image has maximal rank by the fp-simplicity of M. Thus, if Λ is a field, then it is surjective.

In case (1), we claim that ev is an isomorphism. The following argument was explained to us by Jean-François Dat: For injectivity, observe that $M_1 \boxtimes M_2 = M_1^{\oplus \dim M_2}$ as W_1-representations. Hence, if the kernel of ev is non-trivial, then it contains M_1 as an irreducible constituent. Therefore, it suffices to prove that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is injective. Since Λ is algebraically closed, we have $\mathrm{End}_{W_1}(M_1) = \Lambda$ by Schur's lemma. Hence, the composition

$$M_2 = \mathrm{Hom}_{W_1}(\mathrm{End}_{W_1}(M_1), M_2) \cong \mathrm{Hom}_{W_1}(M_1, M_1 \boxtimes M_2) \to \mathrm{Hom}_{W_1}(M_1, M) = M_2$$

is the identity. This shows that $\mathrm{Hom}_{W_1}(M_1, \mathrm{ev})$ is an isomorphism.

In case (2), we claim that $M_1 \boxtimes M_2$ is semi-simple, and hence that M appears as a direct summand. Using [Bou12, Section 13.4 Corollaire] applied to the group algebras it is enough to show that M_1 and M_2 are absolutely semi-simple. Since Λ is perfect, any finite-dimensional representation is semi-simple if and only if it is absolutely semi-simple, see [Bou12, Section 13.1]. Hence, it remains to check that $M_{2,\bar\Lambda} = M_2 \otimes_\Lambda \bar\Lambda$ is semi-simple where $\bar\Lambda/\Lambda$ is an algebraic closure. The module $M_{2,\bar\Lambda} = \mathrm{Hom}_{W_1}(M_{1,\bar\Lambda}, M_{\bar\Lambda})$ splits as a direct sum according to the simple constituents $\bar M_1 \subset M_{1,\bar\Lambda}$ and $\bar M \subset M_{\bar\Lambda}$. Finally, each $\bar M_2 = \mathrm{Hom}_{W_1}(\bar M_1, \bar M)$ is either simple or vanishes: If there exists a non-zero W_1-equivariant map $\bar M_1 \to \bar M$, then it must be injective by the simplicity of $\bar M_1$. As $\bar\Lambda$ is algebraically closed, the proof of (1) shows that $\bar M \cong \bar M_1 \boxtimes \bar M_2$ so that $\bar M_2$ must be simple because $\bar M$ is so. This shows that M_2 is absolutely semi-simple as well.

In case (3), abbreviate $\Lambda' := \mathrm{Frac}\,\Lambda$, $M' := M \otimes_\Lambda \Lambda'$ and so on. We will repeatedly use that $(\text{-}) \otimes_\Lambda \Lambda'$ preserves and detects fp-simplicity of representations, see Lemma 5.8. By (2), the evaluation map $\mathrm{ev}' := \mathrm{ev} \otimes \Lambda'$ admits a Λ'-linear section $\tilde i \colon M' \to (M_1 \boxtimes M_2)'$. As M' is finitely presented, there is some $0 \neq \lambda \in \Lambda$ such that $\lambda \tilde i$ arises by scalar extension of a map $i \colon M \to M_1 \boxtimes M_2$. By construction, the map $i \oplus \mathrm{incl} \colon M \oplus \ker(\mathrm{ev}) \to M_1 \boxtimes M_2$ is an isomorphism after tensoring with Λ'. So its cokernel is Λ-torsion, and it is injective as both modules at the left are projective (hence Λ-torsion free). This finishes the proof of the proposition. $\square$

5.4. Essential surjectivity. In this section, we prove the essential surjectivity asserted in Theorem 5.2. Throughout, we freely use the fully faithfulness proven in Proposition 5.4 and Proposition 5.5.

Recall that $X_1, \ldots, X_n$ are finite type $\mathbb{F}_q$-schemes, and write $X := X_1 \times_{\mathbb{F}_q} \ldots \times_{\mathbb{F}_q} X_n$. Let Λ be either a finite discrete ring, a finite field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$ or its ring of integers $\mathcal{O}_E$. Note that this covers all cases from Situation 5.1.

First, we show that it suffices to prove containment in the essential image étale locally:

Lemma 5.10. *Let $U_i \to X_i$ be quasi-compact étale surjections for $i = 1, \ldots, n$. Then the following properties hold:*

(1) An object $M \in \mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$ belongs to the full subcategory

$$\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big)$$

if and only if its restriction $M|_{U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}}$ *belongs to the full subcategory*

$$\mathrm{D}_{\mathrm{cons}}\big(U_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(U_n^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}\big(U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}, \Lambda\big).$$

(2) Assume that all $U_i \to X_i$ are finite étale. Then (1) holds for the categories of lisse sheaves.

Proof. The only if direction in part (1) is clear. Conversely, assume that $M|_{U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}}$ lies in the essential image of the external tensor product. By étale descent, we have an equivalence:

$$\mathrm{D}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \xrightarrow{\cong} \mathrm{Tot}\big(\mathrm{D}\big(U_{1,\bullet}^{\mathrm{Weil}} \times \ldots \times U_{n,\bullet}^{\mathrm{Weil}}, \Lambda\big)\big).$$

In particular, we get an equivalence $|(j_\bullet)_! \circ j_\bullet^* M| \xrightarrow{\sim} M$ where $j_\bullet := j_{1,\bullet} \times \ldots \times j_{n,\bullet}$ with $j_{i,\bullet} \colon U_{i,\bullet} \to X_i$ for $i = 1, \ldots, n$. For each $m \geq 0$, the object $j_m^* M$ lies in

$$\mathrm{D}_{\mathrm{cons}}\big(U_{1,m}^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(U_{n,m}^{\mathrm{Weil}}, \Lambda\big).$$

It follows from Corollary 3.12 that these subcategories are preserved under $(j_m)_!$. So we see

$$(j_m)_! j_m^*(M) \in \mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big)$$

for all $m \geq 0$. For every $m \geq 0$, let M_m denote the realization of the m-th skeleton of the simplicial object $(j_\bullet)_! \circ j_\bullet^* M$ so that we have a natural equivalence $\mathrm{colim}\, M_m \xrightarrow{\cong} M$ in $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$. We claim that M is a retract of some M_m, and hence lies in $\mathrm{D}_{\mathrm{cons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{cons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big)$ by idempotent completeness. To prove the claim, note that the sheaf $M_{\mathbb{F}} \in \mathrm{D}_{\mathrm{cons}}(X_{\mathbb{F}}, \Lambda)$ underlying M is compact in the category of ind-constructible sheaves $\mathrm{D}_{\mathrm{indcons}}(X_{\mathbb{F}}, \Lambda)$, see Proposition 3.50 which is applicable in view of Lemma 3.54. As taking partial Frobenius fixed points is a finite limit, so commutes with filtered colimits, we see that the natural map of mapping complexes

$$\mathrm{colim}\, \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)}(M, M_m) \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)}(M, \mathrm{colim}\, M_m)$$

is an equivalence. In particular, the inverse equivalence $M \xrightarrow{\cong} \mathrm{colim}\, M_m$ factors through some M_m, presenting M as a retract of M_m. This proves the claim, and hence (1).

For (2), note that if $U_i \to X_i$ are finite étale, then the functors $(j_m)_!$ preserve the lisse categories by Corollary 3.12. In particular, for every $m \geq 0$ the object $(j_m)_! j_m^*(M)$ is lisse and so is M_m. We conclude using compactness as before. $\square$

We will also need the following variant of Lemma 5.10. Using Lemma 2.6 and Proposition 3.50, the fully faithful functor (5.1) uniquely extends to a fully faithful functor

$$\mathrm{Ind}\big(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}\big(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda)\big) \to \mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \qquad (5.8)$$

for $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$.

Lemma 5.11. *The statements (1) and (2) of Lemma 5.10 hold for the functor (5.8) with $\bullet \in \{\mathrm{lis}, \mathrm{cons}\}$. Namely, to check that an object lies in the essential image of (5.8), one can pass to a quasi-compact étale cover if $\bullet = \mathrm{cons}$, and to a finite étale cover if $\bullet = \mathrm{lis}$.*

Proof. This is immediate from the proof of Lemma 5.10: Arguing as above and using étale descent for ind-constructible, respectively ind-lisse sheaves (Corollary 3.55), we see that $M \cong \mathrm{colim}\, M_m$ with

$$M_m \in \mathrm{Ind}\big(\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}\big(\mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda)\big)$$

for all $m \geq 0$ and $\bullet = \mathrm{cons}$, respectively $\bullet = \mathrm{lis}$. As the essential image of (5.8) is closed under colimits, M lies in the corresponding subcategory as well. $\square$

Now we have enough tools to prove the categorical Künneth formula alias derived Drinfeld's lemma:

Proof of Theorem 5.2. In view of Proposition 5.4 and Proposition 5.5, it remains to show the essential surjectivity of the external tensor product functor on Weil sheaves (5.1) under the assumptions in Theorem 5.2. Part (1), the case of constructible sheaves, is reduced to part (2), the case of lisse sheaves, by taking a stratification as in Definition 4.10 (2) and using the full faithfulness already proven. Here we note that by refining the stratification witnessing the constructibility if necessary, we can even assume all strata to be smooth, so in particular geometrically unibranch. Hence, it remains to prove part (2), that is, the essential surjectivity of the fully faithful functor

$$\boxtimes \colon \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}\big(X_n^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big), \qquad (5.9)$$

when either Λ is finite discrete as in cases (a), (d) in Theorem 5.2 (2), or $\Lambda = \mathcal{O}_E$ for a finite field extension $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ as in case (b), or $\Lambda = E$ and the X_i are geometrically unibranch as in the remaining case (c). In fact, the latter two cases are easier to handle due to the presence of natural t-structures on the categories of lisse sheaves

(Theorem 3.28, compare also Remark 3.35). So we will distinguish two cases below: 1) $\Lambda = \mathcal{O}_E$, or $\Lambda = E$ and all X_i geometrically unibranch; 2) Λ is finite discrete.

Now pick $M \in \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$. Corollary 3.17 implies that M is bounded in the standard t-structure on $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$. So M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$. As M is lisse, Lemma 4.14 (1) shows in both cases 1) and 2) that each $\mathrm{H}^j(M)$ comes from a continuous representation on a finitely presented Λ-module in

$$\mathrm{Rep}_\Lambda\big(\mathrm{FWeil}(X)\big) \overset{5.6}{\cong} \mathrm{Rep}_\Lambda(W), \tag{5.10}$$

where we denote $W := W_1 \times \ldots \times W_n$ with $W_i := \mathrm{Weil}(X_i)$.

Throughout, we repeatedly use that the functor (5.9) is fully faithful, commutes with finite (co-)limits and shifts, and that its essential image is closed under retracts (as the source category is idempotent complete, by definition) and contains all perfect-constant sheaves.

Case 1): *Assume* $\Lambda = \mathcal{O}_E$, *or* $\Lambda = E$ *and all* X_i *geometrically unibranch.* In this case, we have a t-structure on lisse Weil sheaves so that each $\mathrm{H}^j(M)$ belongs to $\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit$. By induction on the length of M, using the full faithfulness of (5.9), we reduce to the case where M is abelian, that is, a continuous W-representation on a finitely presented Λ-module. The external tensor product induces a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Rep}_\Lambda(W_1) \times \ldots \times \mathrm{Rep}_\Lambda(W_n) & \overset{\boxtimes}{\longrightarrow} & \mathrm{Rep}_\Lambda(W) \\
\downarrow{\scriptstyle\cong} & & \downarrow{\scriptstyle\cong} \\
\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)^\heartsuit \times \ldots \times \mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit & \overset{\boxtimes}{\longrightarrow} & \mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)^\heartsuit,
\end{array}
$$

where the vertical equivalences are given by Lemma 4.14. Note that M splits into a direct sum $M_{\mathrm{tor}} \oplus M_{\mathrm{fp}}$ where the finitely presented Λ-module underlying M_{tor} is Λ-torsion and M_{fp} is projective. So we can treat either case separately. Using that the essential image of (5.9) is closed under extensions (by fully faithfulness) and retracts, the finite projective case is reduced to the fp-simple case and, by Proposition 5.9, to the finite torsion case. Note that the W_i-representations constructed in, say (5.7), are obtained from M_{fp} by taking subquotients and tensor products, so are automatically continuous. Next, as the Λ-module underlying M_{tor} is finite torsion, the Λ-sheaf M_{tor} is perfect-constant along some finite étale cover. So we conclude by Lemma 5.10 (2).

Case 2): *Assume* Λ *is finite discrete as above.* In a nutshell, the argument is similar to the last step in case 1), but a little more involved due to the absence of natural t-structures on the categories of lisse sheaves in general, see Theorem 3.28 and Remark 3.35. More precisely, in the special case, where Λ is a finite field, the argument of case 1) applies, but not so if $\Lambda = \mathbb{Z}/\ell^2$, say. So, instead, we extend (5.9) by passing to Ind-completions to a commutative diagram

$$
\begin{array}{ccc}
\mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Perf}_{\Lambda_*}} \ldots \otimes_{\mathrm{Perf}_{\Lambda_*}} \mathrm{D}_{\mathrm{lis}}\big(X_n^{\mathrm{Weil}}, \Lambda\big) & \overset{\boxtimes}{\longrightarrow} & \mathrm{D}_{\mathrm{lis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \\
\downarrow & & \downarrow \\
\mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)\big) & \overset{\mathrm{Ind}(\boxtimes)}{\longrightarrow} & \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)\big),
\end{array}
$$

of full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$, see the discussion around (5.8). Note that the fully faithful embedding (5.8) factors through $\mathrm{Ind}(\boxtimes)$. Both vertical arrows are the inclusion of the subcategories of compact objects by idempotent completeness of the involved categories and (2.1). Thus, if M lies in the essential image of $\mathrm{Ind}(\boxtimes)$, then it is a retract of a finite colimit of objects in the essential image of $\boxtimes$, so lies itself in this essential image. As M is a successive extension of its cohomology sheaves $\mathrm{H}^j(M)$, it suffices to show

$$\mathrm{H}^j(M) \in \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(X_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(X_n^{\mathrm{Weil}}, \Lambda)\big),$$

for all $j \in \mathbb{Z}$. So fix j and denote $N := \mathrm{H}^j(M)$ viewed as a continuous W-representation on a finitely presented Λ-module. As Λ is finite, N comes from a continuous representation of $\pi_1(X_1) \times \ldots \pi_1(X_n)$ on which some open subgroup acts trivially. Hence, there exist finite étale surjections $U_i \to X_i$ such that the subgroup $\pi_1(U_1) \times \ldots \times \pi_1(U_n)$ acts trivially on N. In particular, $N|_{U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}}$ is constant, and hence lies in the essential image of the functor

$$\mathrm{Mod}_R \cong \mathrm{Ind}\big(\mathrm{Perf}_R\big) \to \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}, \Lambda)\big),$$

where $R := \Gamma(\pi_0(U_1) \times \ldots \times \pi_0(U_n), \Lambda)$. As the sets $\pi_0(U_i)$ are finite discrete, each $R_i := \Gamma(\pi_0(U_i), \Lambda)$ is a finite free Λ_*-algebra, and we have $R \cong R_1 \otimes_{\Lambda_*} \ldots \otimes_{\Lambda_*} R_n$. Thus, the external tensor product induces a commutative

diagram

$$\mathrm{Mod}_{R_1} \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Mod}_{R_n} \xrightarrow{\ \cong\ } \mathrm{Mod}_R$$

$$\mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}}, \Lambda)\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(U_n^{\mathrm{Weil}}, \Lambda)\big) \xrightarrow{\mathrm{Ind}(\boxtimes)} \mathrm{Ind}\big(\mathrm{D}_{\mathrm{lis}}(U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}, \Lambda)\big),$$

where the upper horizontal arrow is an equivalence. So $N|_{U_1^{\mathrm{Weil}} \times \ldots \times U_n^{\mathrm{Weil}}}$ lies in the essential image of $\mathrm{Ind}(\boxtimes)$, and we conclude by Lemma 5.11 applied to the finite étale covers $U_i \to X_i$ and $\bullet = \mathrm{lis}$. $\qquad\square$

6. Ind-constructible Weil sheaves

In this section, we introduce the full subcategories

$$\mathrm{D}_{\mathrm{indlis}}\big(X^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}_{\mathrm{indcons}}\big(X^{\mathrm{Weil}}, \Lambda\big)$$

of $\mathrm{D}\big(X^{\mathrm{Weil}}, \Lambda\big)$ consisting of ind-objects of lisse, respectively constructible sheaves equipped with partial Frobenius action. That is, the partial Frobenius only preserves the ind-system of objects, but not necessarily each member. We will define analogous categories for a product of schemes. Similarly to the lisse, respectively constructible case, there is a fully faithful functor

$$\mathrm{D}_{\mathrm{indcons}}\big(X_1^{\mathrm{Weil}}, \Lambda\big) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_{\mathrm{indcons}}\big(X_n^{\mathrm{Weil}}, \Lambda\big) \to \mathrm{D}_{\mathrm{indcons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big),$$

which, however, will not be an equivalence in general, see Remark 6.5. Nevertheless, we can identify a class of objects that lie in the essential image and that include many cases of interest such as the shtuka cohomology studied in [Laf18a, LZ19, Xue20b, Xue20c].

6.1. **Ind-constructible Weil sheaves.** Let $\mathbb{F}_q$ be a finite field of characteristic $p > 0$, and fix an algebraic closure $\mathbb{F}$. Let $X_1, \ldots, X_n$ be schemes of finite type over $\mathbb{F}_q$. Let Λ be a condensed ring associated with the one of the following topological rings: a discrete coherent torsion ring (for example, a discrete finite ring), an algebraic field extension $E \supset \mathbb{Q}_\ell$, or its ring of integers $\mathcal{O}_E$. We write $X := X_1 \times_{\mathbb{F}_q} \cdots \times_{\mathbb{F}_q} X_n$, and denote by $X_{i,\mathbb{F}} := X_i \times_{\mathbb{F}_q} \mathrm{Spec}\,\mathbb{F}$ and $X_\mathbb{F} := X \times_{\mathbb{F}_q} \mathrm{Spec}\,\mathbb{F}$ the base change. Recall that under these assumptions, by Proposition 3.50 and Lemma 3.54, we have a fully faithful embedding

$$\mathrm{Ind}\big(\mathrm{D}_{\mathrm{cons}}(X_\mathbb{F}, \Lambda)\big) \xrightarrow{\ \cong\ } \mathrm{D}_{\mathrm{indcons}}(X_\mathbb{F}, \Lambda) \subset \mathrm{D}(X_\mathbb{F}, \Lambda), \tag{6.1}$$

and likewise for (ind-)lisse sheaves.

Definition 6.1. An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$ is called *ind-lisse*, respectively *ind-constructible* if the underlying sheaf $M_\mathbb{F} \in \mathrm{D}(X_\mathbb{F}, \Lambda)$ is ind-lisse, respectively ind-constructible in the sense of Definition 3.49.

We denote by

$$\mathrm{D}_{\mathrm{indlis}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \subset \mathrm{D}_{\mathrm{indcons}}\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big)$$

the resulting full subcategories of $\mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$ consisting of ind-lisse, respectively ind-constructible objects. Both categories are naturally commutative algebra objects in $\mathrm{Pr}^{\mathrm{St}}_{\Lambda_*}$ (see the notation from Section 2), that is, presentable stable Λ_*-linear symmetric monoidal ∞-categories where $\Lambda_* := \Gamma(*, \Lambda)$ is the ring underlying Λ. It is immediate from Definition 6.1 that the equivalence (4.6) restricts to an equivalence

$$\mathrm{D}_\bullet\big(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda\big) \cong \mathrm{Fix}\big(\mathrm{D}_\bullet(X_\mathbb{F}, \Lambda), \phi_{X_1}^*, \ldots, \phi_{X_n}^*\big)$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$.

Our goal in this section is to obtain a categorical Künneth formula for the categories of ind-lisse, respectively ind-constructible Weil sheaves. In order to state the result, we need the following terminology. Under our assumptions on Λ, each cohomology sheaf $\mathrm{H}^j(M)$, $j \in \mathbb{Z}$ for $M \in \mathrm{D}_{\mathrm{lis}}(X_\mathbb{F}, \Lambda)$ is naturally a continuous representation of the proétale fundamental groupoid $\pi_1^{\mathrm{proét}}(X_\mathbb{F})$ on a finitely presented Λ-module, see Lemma 4.14. Further, the projections $X_\mathbb{F} \to X_{i,\mathbb{F}}$ induce a full surjective map of topological groupoids

$$\pi_1^{\mathrm{proét}}(X_{1,\mathbb{F}}) \times \ldots \times \pi_1^{\mathrm{proét}}(X_{n,\mathbb{F}}) \to \pi_1^{\mathrm{proét}}(X_\mathbb{F}). \tag{6.2}$$

Definition 6.2. Let $M \in \mathrm{D}(X_\mathbb{F}, \Lambda)$.

(1) The sheaf M is called *split lisse* if it is lisse and the action of $\pi_1^{\mathrm{proét}}(X_\mathbb{F})$ on $\mathrm{H}^j(M)$ factors through (6.2) for all $j \in \mathbb{Z}$.

(2) The sheaf M is called *split constructible* if it is constructible and there exists a finite subdivision into locally closed subschemes $X_i \subseteq X$ such that each restriction $M|_{X_{i,\mathbb{F}}}$ is split lisse.

Definition 6.3. An object $M \in \mathrm{D}(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda)$ is called *ind-(split lisse)*, respectively *ind-(split constructible)* if the underlying object $M_\mathbb{F} \in \mathrm{D}(X_\mathbb{F}, \Lambda)$ is a colimit of split lisse, respectively split constructible objects.

As the category $\mathrm{D}_\bullet(X_\mathbb{F}, \Lambda)$, $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$ is cocomplete, every ind-(split lisse) object is ind-lisse, and likewise, every ind-(split constructible) object is ind-constructible.

Theorem 6.4. *Assume that Λ is either a finite discrete ring of prime-to-p torsion, an algebraic field extension $E \supset \mathbb{Q}_\ell$ for $\ell \neq p$, or its ring of integers $\mathcal{O}_E$. Then the functor induced by the external tensor product*

$$\mathrm{D}_\bullet(X_1^{\mathrm{Weil}}, \Lambda) \otimes_{\mathrm{Mod}_{\Lambda_*}} \ldots \otimes_{\mathrm{Mod}_{\Lambda_*}} \mathrm{D}_\bullet(X_n^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \tag{6.3}$$

is fully faithful for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$. For $\bullet = \mathrm{indlis}$, respectively $\bullet = \mathrm{indcons}$ the essential image contains the ind-(split lisse), respectively ind-(split constructible) objects.

Proof. For full faithfulness, it is enough to consider the case $\bullet = \mathrm{indcons}$. Using Lemma 2.7, it remains to show that the functor

$$\bigotimes_i \mathrm{D}_{\mathrm{indcons}}(X_{i,\mathbb{F}}, \Lambda) \cong \mathrm{Ind}\left(\bigotimes_i \mathrm{D}_{\mathrm{cons}}(X_{i,\mathbb{F}}, \Lambda)\right) \to \mathrm{D}_\bullet(X_\mathbb{F}, \Lambda). \tag{6.4}$$

is fully faithful. In view of (6.1), this is immediate from the Künneth formula for constructible Λ-sheaves as explained in Section 5.1.

To identify objects in the essential image, we note that the fully faithful functors (6.3) and (6.4) induce a Cartesian diagram (see Lemma 2.7):

$$\begin{array}{ccc}
\bigotimes_i \mathrm{D}_\bullet\left(X_i^{\mathrm{Weil}}, \Lambda\right) & \longrightarrow & \mathrm{D}_\bullet(X_1^{\mathrm{Weil}} \times \ldots \times X_n^{\mathrm{Weil}}, \Lambda) \\
\downarrow & & \downarrow \\
\bigotimes_i \mathrm{D}_\bullet(X_{i,\mathbb{F}}, \Lambda) & \longrightarrow & \mathrm{D}_\bullet(X_\mathbb{F}, \Lambda),
\end{array} \tag{6.5}$$

for $\bullet \in \{\mathrm{indlis}, \mathrm{indcons}\}$. Thus, it is enough to show that the object $M_\mathbb{F}$ underlying an split object M lies in the image of the lower horizontal arrow. Since this essential image is closed under colimits, it remains to show it contains the split lisse objects for $\bullet = \mathrm{indlis}$, respectively the split constructible objects for $\bullet = \mathrm{indcons}$.

By the fully faithfulness of (6.4), the split constructible case reduces to the split lisse case, see also the proof of Theorem 5.2 in Section 5.4. So assume $\bullet = \mathrm{indlis}$ and let $M_\mathbb{F} \in \mathrm{D}(X_\mathbb{F}, \Lambda)$ be split lisse. As each cohomology sheaf $\mathrm{H}^j(M_\mathbb{F})$, $j \in \mathbb{Z}$ is at least ind-lisse (see also Remark 3.52), an induction on the length of $M_\mathbb{F}$ reduces us to show that $\mathrm{H}^j(M_\mathbb{F})$ lies in the essential image. By definition, being split lisse implies that the action of $\pi_1^{\mathrm{proét}}(X_\mathbb{F})$ on $\mathrm{H}^j(M_\mathbb{F})$ factors through $\pi_1^{\mathrm{proét}}(X_{1,\mathbb{F}}) \times \ldots \times \pi_1^{\mathrm{proét}}(X_{n,\mathbb{F}})$. Then the arguments of Section 5.4 show that $\mathrm{H}^j(M_\mathbb{F})$ lies is in the essential image of the lower horizontal arrow in (6.5). We leave the details to the reader. $\qquad\square$

Remark 6.5. The functor (6.3) is not essentially surjective in general. To see this, note that the functor $\mathrm{D}_{\mathrm{indcons}}(X^{\mathrm{Weil}}, \Lambda) \to \mathrm{D}_{\mathrm{indcons}}(X_\mathbb{F}, \Lambda)$ admits a left adjoint F that adds a free partial Frobenius action. Explicitly, for an object $M \in \mathrm{D}_{\mathrm{indcons}}(X_\mathbb{F}, \Lambda)$ the object $F(M)$ has underlying sheaf $F(M)_\mathbb{F}$ given by a countable direct sum of copies of M. If M was not originally in the image of the external tensor product (for example, M as in Example 1.13), then $F(M)$ will not be as well. This is, however, the only obstacle for essential surjectivity: as noted in the proof of Theorem 6.4, the diagram (6.5) is Cartesian.

6.2. **Cohomology of shtuka spaces.** Finally, let us mention a key application of Theorem 6.4. Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$. Let $N \subset X$ be a finite subscheme, and denote its complement by $Y = X \backslash N$. Let $E \supset \mathbb{Q}_\ell$, $\ell \neq p$ be an algebraic field extension containing a fixed square root of q. Let $\mathcal{O}_E$ be its ring of integers and denote by k_E the residue field. Let Λ be any of the topological rings $E, \mathcal{O}_E, k_E$. Let G be a split (for simplicity) reductive group over $\mathbb{F}_q$. We denote by $\widehat{G}$ the Langlands dual group of G considered as a split reductive group over Λ.

In the seminal works [Dri80, Laf02] ($G = \mathrm{GL}_n$) and [Laf18b, LZ19] (general reductive G) on the Langlands correspondence over global function fields, the construction of the $\mathrm{Weil}(Y)$-action on automorphic forms of level N is realized using the cohomology sheaves of moduli stacks of shtukas, defined in [Var04] and [Laf18b, Section 2]. As explained in [LZ19, GKRV20, Zhu21], the output of the geometric construction of Lafforgue can be encoded as a natural transformation

$$\mathrm{H}_{N,I}\colon \mathrm{Rep}_\Lambda^{\mathrm{fp}}(\widehat{G}^I) \to \mathrm{Rep}_\Lambda^{\mathrm{cts}}(\mathrm{Weil}(Y)^I), \quad I \in \mathrm{FinSet} \tag{6.6}$$

of functors $\mathrm{FinSet} \to \mathrm{Cat}$ from the category of finite sets to the category of 1-categories. Here the functor $\mathrm{Rep}_\Lambda^{\mathrm{fp}}(\widehat{G}^\bullet)$ assigns to a finite set I the category of algebraic representations of $\widehat{G}^I$ on finite free Λ-modules, and $\mathrm{Rep}_\Lambda^{\mathrm{cts}}(\mathrm{Weil}(Y)^\bullet)$ the category of continuous representations of $\mathrm{Weil}(Y)^I$ in Λ-modules. In both cases, the transition maps are given by restriction of representations.

Let us recall some elements of its construction. For a finite set I, [Var04] and [Laf18b, Section 2] define the ind-algebraic stack $\mathrm{Cht}_{N,I}$ classifying I-legged G-shtukas on X with full level-N-structure. The morphism sending

a G-shtuka to its legs

$$\mathfrak{p}_{N,I} \colon \mathrm{Cht}_{N,I} \to Y^I, \tag{6.7}$$

is locally of finite presentation. For every $W \in \mathrm{Rep}_\Lambda^{\mathrm{fp}}(\widehat{G}^I)$, there is the normalized Satake sheaf $\mathcal{F}_{N,I,W}$ on $\mathrm{Cht}_{N,I}$, see [Laf18b, Définition 2.14]. Base changing to $\mathbb{F}$ and taking compactly supported cohomology, we obtain the object

$$\mathcal{H}_{N,I}(W) \overset{\mathrm{def}}{=} (\mathfrak{p}_{N,I,\mathbb{F}})_!(\mathcal{F}_{N,I,W,\mathbb{F}}) \in \mathrm{D}_{\mathrm{indcons}}(Y_\mathbb{F}^I, \Lambda),$$

see [Laf18b, Définition 4.7] and [Xue20a, Definition 2.5.1]. Under the normalization of the Satake sheaves, the degree 0 cohomology sheaf

$$\mathrm{H}_{N,I}(W) \overset{\mathrm{def}}{=} \mathrm{H}^0(\mathcal{H}_I(W)) \in \mathrm{D}_{\mathrm{indcons}}(Y_\mathbb{F}^I, \Lambda)^\heartsuit$$

corresponds to the middle degree compactly supported intersection cohomology of $\mathrm{Cht}_{N,I}$. Using the symmetries of the moduli stacks of shtukas, the sheaf $\mathrm{H}_{N,I}(W)$ is endowed with a partial Frobenius equivariant structure [Laf02, §6]. So we obtain objects

$$\mathrm{H}_{N,I}(W) \in \mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda)^\heartsuit. \tag{6.8}$$

Next, using the finiteness [Xue20b] and smoothness [Xue20c, Theorem 4.2.3] results, the classical Drinfeld's lemma (Theorem 5.6) applies to give objects $\mathrm{H}_{N,I}(W) \in \mathrm{Rep}_\Lambda^{\mathrm{cts}}(\mathrm{Weil}(Y)^I)$. The construction of the natural transformation (6.6) encodes the functoriality and fusion satisfied by the objects $\{\mathrm{H}_{N,I}(W)\}$ for varying I and W.

However, in order to analyze construction (6.6) further, it is desirable to upgrade the natural transformation of functors (6.6) to the derived level. Namely, to have construction for the complexes $\{\mathcal{H}_I(W)\}_{I,W}$ and not just for their cohomology sheaves, compare with [Zhu21]. Such an upgrade is possible using the derived version of Drinfeld's lemma, as given in the following proposition. A further study of this construction will appear in future work of the first named author (T. H.).

Proposition 6.6. *For $\Lambda \in \{E, \mathcal{O}_E, k_E\}$ and any $W \in \mathrm{Rep}_\Lambda(\widehat{G}^I)$, the shtuka cohomology (6.8) lies in the essential image of the fully faithful functor*

$$\mathrm{D}_{\mathrm{indlis}}(Y^{\mathrm{Weil}}, \Lambda)^{\otimes I} \to \mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda). \tag{6.9}$$

Proof. By [Xue20c, Theorem 4.2.3], the ind-constructible sheaf $\mathrm{H}_{N,I}(W)$ is ind-lisse. By [Xue20b, Proposition 3.2.15], the action of $\mathrm{FWeil}(Y^I)$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\mathrm{Weil}(Y)^I$. In particular, the action of $\pi_1(X_\mathbb{F}^I)$ on $\mathrm{H}_{N,I}(W)$ factors through the product $\pi_1(X_\mathbb{F})^I$. So it is ind-(split lisse) in the sense of Definition 6.3, and we are done by Theorem 6.4. $\square$

Remark 6.7. One can upgrade the above construction in a homotopy coherent way to show that the whole complex $\mathcal{H}_{N,I}(W)$ lies in $\mathrm{D}_{\mathrm{indcons}}((Y^{\mathrm{Weil}})^I, \Lambda)$. If $N \neq \varnothing$ so that $\mathcal{H}_{N,I}(W)$ is known to be bounded, then Proposition 6.6 implies that $\mathcal{H}_{N,I}(W)$ lies in the essential image of (6.9).

APPENDIX A. THE CONDENSED SHAPE OF THE PROÉTALE TOPOS

Let X be a qcqs scheme, and Λ a condensed ring. In this appendix, we explain how the formalism of section Section 3 gives a simple realization of lisse Λ-sheaves on X as representations of the condensed shape associated to $X_{\mathrm{proét}}$ valued in perfect Λ-modules. This is related to the stratified shape developed in [BGH20, BH19, Wol20]. This appendix is not used throughout the manuscript. We include the material as it gives another point of view on the categories of lisse sheaves $\mathrm{D}_{\mathrm{lis}}(X, \Lambda)$ introduced in Section 3.1.

We take the following definition of the condensed shape which is similar to the classical definitions of Artin–Mazur–Friedlander. Denote by $\mathrm{HC}(X)$ the ∞-category of hypercoverings in $X_{\mathrm{proét}}$ whose objects consist of hypercovers $U_\bullet \to X$ in X with U_n qcqs for all $n \geq 0$. For precise definitions, the reader is referred to [DHI] and [Hoy18, §5]. We note that the category $\mathrm{HC}(X)$ is cofiltered by [DHI, Proposition 5.1]. The *condensed shape of $X_{\mathrm{proét}}$* is the condensed animated set

$$\Pi_{\mathrm{cond}}(X) \overset{\mathrm{def}}{=} \lim_{U_\bullet \in \mathrm{HC}(X)} |\pi_0(U_\bullet)|,$$

where the geometric realization and the limit are taken in the ∞-category $\mathrm{Cond}(\mathrm{Ani})$ of condensed anima (also called spaces, Kan complexes, or ∞-groupoids). Here we identify a profinite set with the associated condensed set under the Yoneda embedding.

Let $\mathrm{HC}^w(X) \subset \mathrm{HC}(X)$ denote the full subcategory consisting of hypercovers $U_\bullet$ so that, for every $n \geq 0$, the scheme U_n is w-contractible. Since every hypercover can be refined by one consisting of w-contractible schemes, the inclusion $\mathrm{HC}^w(X) \subset \mathrm{HC}(X)$ is co-initial. In particular, the natural map

$$\Pi_{\mathrm{cond}}(X) \overset{\cong}{\longrightarrow} \lim_{U_\bullet \in \mathrm{HC}^w(X)} |\pi_0(U_\bullet)|.$$

is an equivalence. Since covers of w-contractible qcqs objects split, all the condensed sets in the colimit are actually equivalent. As the category HC^w is cofiltered, we get an equivalence of condensed anima

$$\Pi_{\mathrm{cond}}(X) \xrightarrow{\cong} |\pi_0(U_\bullet)|, \tag{A.1}$$

for every $U_\bullet \in \mathrm{HC}^w(X)$.

For the condensed ring Λ, we define a condensed ∞-category $\mathrm{Perf}_\Lambda^{\mathrm{cond}}$ by the assignment

$$\mathrm{Perf}_\Lambda^{\mathrm{cond}} \colon \{\text{extremally disconnected profinite sets}\}^{op} \to \mathrm{Cat}_\infty^{\mathrm{Ex}}(\mathrm{Idem})$$
$$S \mapsto \mathrm{Perf}_{\Gamma(S,\Lambda)}.$$

Note that $\mathrm{Perf}_\Lambda^{\mathrm{cond}}$, when extended to a hypersheaf on $*_{\mathrm{proét}}$, is simply the hypersheaf $S \mapsto \mathrm{D}_{\mathrm{lis}}(S,\Lambda)$, see Lemma 3.7 and Corollary 3.13.

The category of condensed ∞-categories has a canonical enrichment over Cat_∞. Namely, for every ∞-category K, we have the associated $\underline{K} \in \mathrm{Cond}(\mathrm{Cat}_\infty)$ as the presheaf on the category of extremally disconnected profinite sets

$$S = \lim S_i \mapsto \operatorname*{colim}_i \mathrm{Fun}(S_i, K).$$

Then, for condensed ∞-categories $\mathcal{C}, \mathcal{D}$, the mapping object $\mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D}) \in \mathrm{Cat}_\infty$ is characterized by the existence of natural equivalences

$$\mathrm{Hom}_{\mathrm{Cat}_\infty}\big(K, \mathrm{Fun}^{\mathrm{cts}}(\mathcal{C}, \mathcal{D})\big) \cong \mathrm{Hom}_{\mathrm{Cond}(\mathrm{Cat}_\infty)}\big(\underline{K} \times \mathcal{C}, \mathcal{D}\big)$$

with $K \in \mathrm{Cat}_\infty$. In order to simplify notation, we identify every profinite set S with the associated discrete condensed ∞-category. Then, for any extremally disconnected profinite set S, we have the equivalence

$$\mathrm{Perf}_{\Gamma(S,\Lambda)} \cong \mathrm{Fun}^{\mathrm{cts}}(S, \mathrm{Perf}_\Lambda^{\mathrm{cond}}),$$

using the Yoneda embedding.

Proposition A.1. *Let X be a qcqs scheme and Λ a condensed ring. Then there is a canonical equivalence*

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Fun}^{\mathrm{cts}}\big(\Pi_{\mathrm{cond}}(X), \mathrm{Perf}_\Lambda^{\mathrm{cond}}\big). \tag{A.2}$$

Proof. Let $U_\bullet \in \mathrm{HC}^w(X)$. By descent (Corollary 3.13), we have an equivalence

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Tot}\big(\mathrm{Perf}_{\Gamma(U_\bullet,\Lambda)}\big) \cong \mathrm{Tot}\big(\mathrm{Perf}_{\Gamma(\pi_0(U_\bullet),\Lambda)}\big), \tag{A.3}$$

using $\Gamma(U_n, \Lambda) = \Gamma(\pi_0(U_n), \Lambda)$ for $n \geq 0$. Since each U_n is w-contractible qcqs, each profinite set $\pi_0(U_n)$ is extremally disconnected. We get an equivalence

$$\mathrm{Perf}_{\Gamma(\pi_0(U_\bullet),\Lambda)} \cong \mathrm{Fun}^{\mathrm{cts}}\big(\pi_0(U_\bullet), \mathrm{Perf}_\Lambda^{\mathrm{cond}}\big).$$

Then (A.3) becomes

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Tot}\left(\mathrm{Fun}^{\mathrm{cts}}\big(\pi_0(U_\bullet), \mathrm{Perf}_\Lambda^{\mathrm{cond}}\big)\right) \cong \mathrm{Fun}^{\mathrm{cts}}\big(|\pi_0(U_\bullet)|, \mathrm{Perf}_\Lambda^{\mathrm{cond}}\big).$$

This construction is functorial in $U_\bullet$ and gives a canonical equivalence

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Fun}^{\mathrm{cts}}\big(\Pi_{\mathrm{cond}}(X), \mathrm{Perf}_\Lambda^{\mathrm{cond}}\big).$$

$\square$

The condensed shape can be related to the more familiar versions when one restricts the possible class of rings. First, if Λ is discrete, then we have

$$\mathrm{Perf}_{\Gamma(S,\Lambda)} \cong \operatorname{colim} \mathrm{Perf}_{\Gamma(S_i,\Lambda)} \cong \operatorname{colim} \mathrm{Fun}(S_i, \mathrm{Perf}_\Lambda).$$

for profinite set $S = \lim S_i$, see Lemma 3.42. In this case, (A.2) reduces to

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \mathrm{Fun}(\Pi_{\mathrm{proét}}(X), \mathrm{Perf}_\Lambda),$$

where $\Pi_{\mathrm{proét}}(X)$ denotes the profinite shape of the topos $X_{\mathrm{proét}}$ in the sense of [Lurb, Appendix E], namely, the profinite completion of the shape. Here we use Fun as in [Hoy18] to denote the category of functors between pro-categories. Now consider a condensed ring Λ associated to a Noetherian ring complete with respect to the adic topology for some ideal $I \subset \Lambda$ with Λ/I finite. Then Proposition 3.19 implies

$$\mathrm{D}_{\mathrm{lis}}(X, \Lambda) \cong \lim_n \mathrm{Fun}(\Pi_{\mathrm{proét}}(X), \mathrm{Perf}_{\Lambda/I^n}) \cong \mathrm{Fun}(\Pi_{\mathrm{proét}}(X), \mathrm{Perf}_\Lambda),$$

where on the right hand side both arguments are considered as pro-categories.

References

[BGH20] Clark Barwick, Saul Glasman, and Peter Haine. Exodromy, 2020. arXiv:1807.03281. 2, 41

[BH19] Clark Barwick and Peter Haine. Pyknotic objects, i. basic notions, 2019. arXiv:1904.09966. 1, 41

[Bha16] Bhargav Bhatt. Algebraization and Tannaka duality. *Camb. J. Math.*, 4(4):403–461, 2016. 3, 6

[BL19] Bhargav Bhatt and Jacob Lurie. A Riemann-Hilbert correspondence in positive characteristic. *Camb. J. Math.*, 7(1-2):71–217, 2019. 4, 25, 33

[BM20] Bhargav Bhatt and Akhil Mathew. The arc-topology, 2020. arXiv:1807.04725. 6

[Bou12] N. Bourbaki. *Éléments de mathématique. Algèbre. Chapitre 8. Modules et anneaux semi-simples.* Springer, Berlin, 2012. Second revised edition of the 1958 edition [MR0098114]. 36

[Bro78] J. Glenn Brookshear. On projective prime ideals in $C(X)$. *Proc. Amer. Math. Soc.*, 69(1):203–204, 1978. 20

[BS15] Bhargav Bhatt and Peter Scholze. The pro-étale topology for schemes. *Astérisque*, (369):99–201, 2015. 1, 2, 3, 8, 9, 10, 11, 12, 14, 15, 16, 17, 19, 21, 22, 23, 24, 25, 27, 30, 31, 32

[BZFN10] David Ben-Zvi, John Francis, and David Nadler. Integral transforms and Drinfeld centers in derived algebraic geometry. *J. Amer. Math. Soc.*, 23(4):909–966, 2010. 5

[CD16] Denis-Charles Cisinski and Frédéric Déglise. Étale motives. *Compos. Math.*, 152(3):556–666, 2016. 3, 21

[CR06] Charles W. Curtis and Irving Reiner. *Representation theory of finite groups and associative algebras.* AMS Chelsea Publishing, Providence, RI, 2006. Reprint of the 1962 original. 36

[CS] Dustin Clausen and Peter Scholze. Condensed mathematics. 1, 8

[Del77] P. Deligne. *Cohomologie étale.* Lecture Notes in Mathematics, Vol. 569. Springer-Verlag, Berlin-New York, 1977. Séminaire de Géométrie Algébrique du Bois-Marie SGA 41er2, Avec la collaboration de J. F. Boutot, A. Grothendieck, L. Illusie et J. L. Verdier. 2

[Del80] Pierre Deligne. La conjecture de Weil. II. *Inst. Hautes Études Sci. Publ. Math.*, 52:137–252, 1980. 1, 2, 3, 4, 21, 23, 30

[DHI] Daniel Dugger, Sharon Hollander, and Daniel C. Isaksen. Hypercovers and simplicial presheaves. 136. 41

[DM83] G. De Marco. Projectivity of pure ideals. *Rend. Sem. Mat. Univ. Padova*, 69:289–304, 1983. 20

[Dri80] V. G. Drinfeld. Langlands' conjecture for GL(2) over functional fields. In *Proceedings of the International Congress of Mathematicians (Helsinki, 1978)*, pages 565–574. Acad. Sci. Fennica, Helsinki, 1980. 3, 4, 40

[Dri87] V. G. Drinfeld. Moduli varieties of F-sheaves. *Funktsional. Anal. i Prilozhen.*, 21(2):23–41, 1987. 28, 31

[Eke90] Torsten Ekedahl. On the adic formalism. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 197–218. Birkhäuser Boston, Boston, MA, 1990. 1, 2, 23

[Gei04] Thomas Geisser. Weil-étale cohomology over finite fields. *Math. Ann.*, 330(4):665–692, 2004. 3, 5, 26, 32

[GKRV20] D. Gaitsgory, D. Kazhdan, N. Rozenblyum, and Y. Varshavsky. A toy model for the drinfeld-lafforgue shtuka construction, 2020. arXiv:1908.05420. 3, 4, 33, 40

[GL19] Dennis Gaitsgory and Jacob Lurie. *Weil's conjecture for function fields. Vol. 1*, volume 199 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2019. 2, 23, 24

[Gla89] Sarah Glaz. *Commutative coherent rings*, volume 1371 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1989. 18, 21

[GR17] Dennis Gaitsgory and Nick Rozenblyum. *A study in derived algebraic geometry. Vol. I. Correspondences and duality*, volume 221 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2017. 5

[Hei18] Jochen Heinloth. Langlands parameterization over function fields following V. Lafforgue. *Acta Math. Vietnam.*, 43(1):45–66, 2018. 4

[Hoy18] Marc Hoyois. Higher galois theory. *Journal of Pure and Applied Algebra*, 222(7):1859–1877, 2018. 41, 42

[HS] David Hansen and Peter Scholze. Relative perversity. to appear. 20, 23

[ILO14] Luc Illusie, Yves Laszlo, and Fabrice Orgogozo, editors. *Travaux de Gabber sur l'uniformisation locale et la cohomologie étale des schémas quasi-excellents*. Société Mathématique de France, Paris, 2014. Séminaire à l'École Polytechnique 2006–2008. [Seminar of the Polytechnic School 2006–2008], With the collaboration of Frédéric Déglise, Alban Moreau, Vincent Pilloni, Michel Raynaud, Joël Riou, Benoît Stroh, Michael Temkin and Weizhe Zheng, Astérisque No. 363-364 (2014) (2014). 21, 23

[Kah03] Bruno Kahn. Équivalences rationnelle et numérique sur certaines variétés de type abélien sur un corps fini. *Ann. Sci. École Norm. Sup. (4)*, 36(6):977–1002 (2004), 2003. 5

[Ked19] Kiran S. Kedlaya. *Sheaves, Stacks, and Shtukas*, volume 242 of *in Perfectoid spaces: Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2019. Lectures from the 2017 Arizona Winter School, held in Tucson, AZ, March 11–17, Edited and with a preface by Bryden Cais, With an introduction by Peter Scholze. 4, 28

[KW01] Reinhardt Kiehl and Rainer Weissauer. *Weil conjectures, perverse sheaves and l'adic Fourier transform*, volume 42 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 2001. 21, 23

[Laf97] Laurent Lafforgue. Chtoucas de Drinfeld et conjecture de Ramanujan-Petersson. *Astérisque*, (243):ii+329, 1997. 4

[Laf02] Laurent Lafforgue. Chtoucas de Drinfeld et correspondance de Langlands. *Invent. Math.*, 147(1):1–241, 2002. 40, 41

[Laf18a] Vincent Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *J. Amer. Math. Soc.*, 31(3):719–891, 2018. 3, 4, 5, 28, 31, 35, 39

[Laf18b] Vincent Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *J. Amer. Math. Soc.*, 31(3):719–891, 2018. 40, 41

[Lam99] T. Y. Lam. *Lectures on modules and rings*, volume 189 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999. URL: https://doi.org/10.1007/978-1-4612-0525-8, doi:10.1007/978-1-4612-0525-8. 20, 21

[Lau04] Eike Sören Lau. *On generalised D-shtukas*, volume 369 of *Bonner Mathematische Schriften [Bonn Mathematical Publications]*. Universität Bonn, Mathematisches Institut, Bonn, 2004. Dissertation, Rheinische Friedrich-Wilhelms-Universität Bonn, Bonn, 2004. 4, 28, 35

[Lic05] S. Lichtenbaum. The Weil-étale topology on schemes over finite fields. *Compos. Math.*, 141(3):689–702, 2005. 3, 5, 26

[Lura] Jacob Lurie. Higher Algebra. 1, 5, 6, 7, 9

[Lurb] Jacob Lurie. Spectral Algebraic Geometry. 8, 13, 42

[Lur09]   Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. 5, 6, 7, 10, 24

[LZ19]   Vincent Lafforgue and Xinwen Zhu. Décomposition au-dessus des paramètres de langlands elliptiques, 2019. `arXiv:1811.07976`. 39, 40

[Nev90]   Charles W. Neville. When is $C(X)$ a coherent ring? *Proc. Amer. Math. Soc.*, 110(2):505–508, 1990. 20

[Oli83]   J.-P. Olivier. Going up along absolutely flat morphisms. *J. Pure Appl. Algebra*, 30(1):47–59, 1983. 20, 21

[Ser64]   Jean-Pierre Serre. Groupes analytiques p-adiques. In *Séminaire Bourbaki, 1963/64, Fasc. 2, Exposé 270*, page 10. Secrétariat mathématique, Paris, 1964. 35

[SGA77]   *Cohomologie l-adique et fonctions L*. Lecture Notes in Mathematics, Vol. 589. Springer-Verlag, Berlin, 1977. Séminaire de Géometrie Algébrique du Bois-Marie 1965–1966 (SGA 5), Edité par Luc Illusie. 1, 2

[SGA03]   *Revêtements étales et groupe fondamental (SGA 1)*. Documents Mathématiques (Paris) [Mathematical Documents (Paris)], 3. Société Mathématique de France, Paris, 2003. Séminaire de géométrie algébrique du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Lecture Notes in Math., 224, Springer, Berlin; MR0354651 (50 #7129)]. 4, 30

[Sta17]   Stacks Project Authors. Stacks Project. `http://stacks.math.columbia.edu`, 2017. 2, 6, 8, 9, 11, 12, 15, 17, 18, 19, 20, 21, 22, 23, 25, 29, 31, 32, 33, 34, 35, 36

[SW18]   Peter Scholze and Jared Weinstein. Berkeley lectures on p-adic geometry. Annals of Math Studies, to appear, 2018. 4

[Var04]   Yakov Varshavsky. Moduli spaces of principal F-bundles. *Sel. Math., New Ser.*, 10(1):131–166, 2004. 40

[Vec96]   E. M. Vechtomov. Rings of continuous functions with values in a topological division ring. volume 78, pages 702–753. 1996. Topology, 2. 20, 21

[Wol20]   Sebastian Wolf. The pro-étale topos as a category of pyknotic presheaves, 2020. `arXiv:2012.10502`. 41

[Xue20a]   Cong Xue. Cuspidal cohomology of stacks of shtukas. *Compos. Math.*, 156(6):1079–1151, 2020. 41

[Xue20b]   Cong Xue. Finiteness of cohomology groups of stacks of shtukas as modules over Hecke algebras, and applications. *Épijournal Géom. Algébrique*, 4:Art. 6, 42 pp.–41, 2020. 35, 39, 41

[Xue20c]   Cong Xue. Smoothness of cohomology sheaves of stacks of shtukas, 2020. `arXiv:2012.12833`. 5, 39, 41

[Zhu21]   Xinwen Zhu. Coherent sheaves on the stack of langlands parameters, 2021. `arXiv:2008.02998`. 40, 41

Caltech, Department of Mathematics, 91125 Pasadena, CA, USA
Email address: `themo@caltech.edu`

Technical University of Darmstadt, Department of Mathematics, 64289 Darmstadt, Germany
Email address: `richarz@mathematik.tu-darmstadt.de`

Universität Münster, Mathematisches Institut, 48149 Münster, Germany
Email address: `jakob.scholbach@uni-muenster.de`