Derived smooth induction with applications

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Abstract

In natural characteristic, smooth induction from an open subgroup does not always give an exact functor. In this article we initiate a study of the right derived functors, and we give applications to the non-existence of projective representations and duality.

1 Introduction

Let G be a profinite group, and let k be a field of characteristic p. The category of smooth G-representations on k-vector spaces $\mathrm{Mod}_k(G)$ has nonzero projective objects if and only if p has finite exponent in the pro-order $|G|$. See [CK23, Thm. 3.1] for example (or [DK23, Rk. 2.20, p. 20] for a less precise result). In this paper we study the question about non-existence of projectives for *locally* profinite groups. More precisely for p-adic Lie groups G. We approach the problem via the right derived functors of smooth induction Ind_K^G from a compact open subgroup K. This is the *right* adjoint to the restriction functor and, in contrast to compact induction, the functor Ind_K^G is provably not exact in general for non-compact G.

As a sample result, suppose G is a p-adic Lie group with a non-discrete center. We show in Proposition 4.2 that the category $\mathrm{Mod}_k(G)$ has no nonzero projective objects.

For general p-adic reductive groups (with no restriction on the center) we prove the following result, which has been a folklore expectation for some time:

Theorem 1.1. *Let $G = \mathbf{G}(\mathfrak{F})$ for a nontrivial connected reductive group $\mathbf{G}$ defined over a finite extension $\mathfrak{F}/\mathbb{Q}_p$. Then $\mathrm{Mod}_k(G)$ has no nonzero projective objects.*

We deduce 1.1 from the vanishing of $R^d\,\mathrm{Ind}_K^G(k)$ for certain principal congruence subgroups K. More precisely, we fix a special vertex x_0 in the Bruhat-Tits building of G and consider the associated group scheme $\mathbf{G}_{x_0}^\circ$ over $\mathfrak{O}$ (the valuation ring in $\mathfrak{F}$). The congruence subgroup

$$K_m := \ker\left(\mathbf{G}_{x_0}^\circ(\mathfrak{O}) \longrightarrow \mathbf{G}_{x_0}^\circ(\mathfrak{O}/\pi^m\mathfrak{O})\right)$$

is a uniform pro-p group for $m \in e\mathbb{N}$ (with the extra assumption that $m > e$ if $p = 2$). Here π is a choice of uniformizer in $\mathfrak{F}$, and e denotes the ramification index of $\mathfrak{F}/\mathbb{Q}_p$.

The most technical part of our paper is finding the precise vanishing range for $R^i\,\mathrm{Ind}_{K_m}^G(k)$. This range is given by the number

$$i_0 := \dim_{\mathbb{Q}_p}(G/P_{\min})$$

where $P_{\min}$ denotes the group of $\mathfrak{F}$-points of a minimal parabolic $\mathfrak{F}$-subgroup of $\mathbf{G}$. We have:

Theorem 1.2. $R^i\,\mathrm{Ind}_{K_m}^G(k) = 0$ *if and only if* $i > i_0$.

This answers a question in [Sor] about the higher smooth duals of the compactly induced representation $\mathrm{ind}_{K_m}^G(k)$.

We introduce the functor $\underline{\mathrm{Ind}}$ by taking the union of Ind_K^G as K varies. This takes a smooth G-representation to a smooth $G \times G$-representation. At the derived level this gives a functor

$$R\underline{\mathrm{Ind}} : D(G) \longrightarrow D(G \times G)$$

where $D(G) := D(\mathrm{Mod}_k(G))$. In [SS] we studied the smooth duality functor $R\underline{\mathrm{Hom}}(-, k)$ on this category. The complex $R\underline{\mathrm{Ind}}(k)$ in some sense represents $R\underline{\mathrm{Hom}}(-, k)$ on the subcategory $D(G)^c$ of compact objects. The precise statement is the following: For all compact $V^\bullet$ there is an isomorphism

$$\tau_{V^\bullet} : R\underline{\mathrm{Hom}}(V^\bullet, k) \xrightarrow{\sim} R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V^\bullet, R\underline{\mathrm{Ind}}(k))$$

in $D(G)$. Here, if V is a smooth G-representation and W is a smooth $G \times G$-representation, we let $\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V, W)$ denote the space of k-linear maps $V \longrightarrow W$ which are G-equivariant on the right and smooth on the left (referring to the two G-factors in $G \times G$ acting on W).

When G is a p-adic reductive group, as in Theorems 1.1 and 1.2, we show that $R\underline{\mathrm{Ind}}(k)$ only has cohomology in degree zero:

Theorem 1.3. *Keep the group $G = \mathbf{G}(\mathfrak{F})$ as in Theorem 1.1. There is an isomorphism*

$$\mathcal{C}^\infty(G, k)[0] \xrightarrow{\sim} R\underline{\mathrm{Ind}}(k)$$

in $D(G \times G)$. (Here $\mathcal{C}^\infty(G, k)[0]$ is the space of k-valued functions on G, which are smooth on both sides, viewed as a complex concentrated in degree zero.)

2 The derived functors of smooth induction

For now G denotes an arbitrary locally profinite group, and k is any field. Let Mod_k and $\mathrm{Mod}_k(G)$ be the category of k-vector spaces and of smooth G-representations on k-vector spaces, respectively. Fix a compact open subgroup $K \subseteq G$ and consider the restriction functor $\mathrm{res}_K^G : \mathrm{Mod}_k(G) \longrightarrow \mathrm{Mod}_k(K)$. The compact induction functor ind_K^G is an exact left adjoint. The full smooth induction functor Ind_K^G is a right adjoint of res_K^G, but it is not exact in general when the characteristic of k divides the pro-order of G. The purpose of this paper is to understand the derived functors $R^i \mathrm{Ind}_K^G$ better in that case.

Starting with an object V from $\mathrm{Mod}_k(K)$ we will follow the convention in [Vig96, Ch. I, Sect. 5] and realize $\mathrm{Ind}_K^G(V)$ as the space of all smooth functions $f : G \longrightarrow V$ satisfying the transformation property $f(\kappa x) = \kappa f(x)$ for $\kappa \in K$. Thus in this article our convention is that G acts by right translations.

Definition 2.1. *For an open subgroup $U \subset G$ the K-action on G/U gives rise to the following:*

i. Let $K \curvearrowright G/U$ denote the groupoid with objects the elements $x \in G/U$ and morphisms $\mathrm{Hom}(x, y) = \{\kappa \in K : \kappa x = y\}$ for $x, y \in G/U$;

ii. The representation V gives a functor $F_V : K \curvearrowright G/U \longrightarrow \mathrm{Mod}_k$ sending $x \mapsto V^{K \cap xUx^{-1}}$, and if $\kappa x = y$ the k-linear map associated with κ is

$$\begin{aligned} F_V(\kappa) : V^{K \cap xUx^{-1}} &\xrightarrow{\sim} V^{K \cap yUy^{-1}} \\ v &\longmapsto \kappa v. \end{aligned}$$

We can think of the U-invariants $\mathrm{Ind}_K^G(V)^U$ as the limit of F_V.

Lemma 2.2. $\mathrm{Ind}_K^G(V)^U \simeq \varprojlim_{x\in G/U} V^{K\cap xUx^{-1}}$.

Proof. The space $\mathrm{Ind}_K^G(V)^U$ consists of all K-equivariant functions $f : G/U \longrightarrow V$. For such an f, as $x \in G/U$ varies the vectors $f(x) \in V^{K\cap xUx^{-1}}$ are compatible via the isomorphisms $F_V(\kappa)$. Vice versa, a compatible tuple of vectors arise from a unique U-invariant function. $\square$

Remark 2.3. The point of this categorical description of $\mathrm{Ind}_K^G(V)^U$ is to avoid having to pick double coset representatives R for $K\backslash G/U$. With such a choice R one can of course describe the U-invariants in simpler terms as just a product $\prod_{x\in R} V^{K\cap xUx^{-1}}$. However, as U varies the transition maps become more cumbersome to work with.

There is a formula for $R^i\,\mathrm{Ind}_K^G(V)$ of the same nature.

Proposition 2.4. $R^i\,\mathrm{Ind}_K^G(V) \simeq \varinjlim_U \varprojlim_{x\in G/U} H^i(K\cap xUx^{-1}, V)$.

Proof. Let $V \longrightarrow J^\bullet$ be an injective resolution of V in $\mathrm{Mod}_k(K)$. This remains injective upon restriction to an open subgroup since compact induction is exact. Therefore we have

$$\begin{aligned} R^i\,\mathrm{Ind}_K^G(V) &\simeq h^i(\mathrm{Ind}_K^G(J^\bullet)) \\ &\simeq \varinjlim_U \varprojlim_{x\in G/U} h^i((J^\bullet)^{K\cap xUx^{-1}}) \\ &\simeq \varinjlim_U \varprojlim_{x\in G/U} H^i(K\cap xUx^{-1}, V). \end{aligned}$$

In the second isomorphism we moved h^i inside $\varinjlim_U$ and $\varprojlim_{x\in G/U}$, which is justified by the fact that Mod_k satisfies AB5 and AB4* (filtered colimits and products are exact). Recall from Remark 2.3 that $\varprojlim_{x\in G/U}$ can be identified with a product $\prod_{x\in R}$. $\square$

Remark 2.5. For a fixed U the limit $\varprojlim_{x\in G/U} H^i(K\cap xUx^{-1}, V)$ coincides with the groupoid cohomology of $K \curvearrowright G/U$ as described in [Ron, Df. 6] for example. It is the limit of the functor F_V^i sending $x \mapsto H^i(K\cap xUx^{-1}, V)$. Concretely, an element of this limit is a function

$$c : G/U \longrightarrow \bigoplus_{x\in G/U} H^i(K\cap xUx^{-1}, V)$$

with the following properties:

i. $c_x := c(x) \in H^i(K\cap xUx^{-1}, V)$ for all $x \in G/U$;

ii. If $\kappa x = y$ then $c_x \mapsto c_y$ via the isomorphism

$$\kappa_* = F_V^i(\kappa) : H^i(K\cap xUx^{-1}, V) \xrightarrow{\sim} H^i(K\cap yUy^{-1}, V).$$

With this description we can make the transition maps in the colimit $\varinjlim_U$ explicit. Let $U' \subset U$ be an open subgroup. Then the transition map is

$$\begin{aligned} t^i_{U,U'} : \varprojlim_{x\in G/U} H^i(K\cap xUx^{-1}, V) &\longrightarrow \varprojlim_{x'\in G/U'} H^i(K\cap x'U'x'^{-1}, V) \\ c &\longmapsto \left(\mathrm{res}^{K\cap x'Ux'^{-1}}_{K\cap x'U'x'^{-1}}\, c_{x'U}\right)_{x'\in G/U'}. \end{aligned}$$

This above formula for $t^i_{U,U'}$ will play a crucial role throughout this paper.

3 The connection to higher smooth duality

To motivate the ensuing discussion we establish a relation between the functors $R^i \operatorname{Ind}_K^G$ and the higher smooth duality functors introduced in [Koh], and recast in [SS].

Lemma 3.1. *For V in $\operatorname{Mod}_k(K)$ and W in $\operatorname{Mod}_k(G)$ there are functorial isomorphisms*

$$\operatorname{Ind}_K^G \underline{\operatorname{Hom}}(V, W|_K) \simeq \underline{\operatorname{Hom}}(\operatorname{ind}_K^G V, W).$$

(Here $\underline{\operatorname{Hom}}$ denotes the smooth k-linear maps, as defined in [SS, Sect. 1] for example.)

Proof. For any representation X in $\operatorname{Mod}_k(G)$ we have functorial isomorphisms

$$\begin{aligned}
\operatorname{Hom}_{\operatorname{Mod}_k(G)}(X, \operatorname{Ind}_K^G \underline{\operatorname{Hom}}(V, W|_K)) &\simeq \operatorname{Hom}_{\operatorname{Mod}_k(K)}(X|_K, \underline{\operatorname{Hom}}(V, W|_K)) \\
&\simeq \operatorname{Hom}_{\operatorname{Mod}_k(K)}(X|_K \otimes_k V, W|_K) \\
&\simeq \operatorname{Hom}_{\operatorname{Mod}_k(G)}(\operatorname{ind}_K^G(X|_K \otimes_k V), W) \\
&\simeq \operatorname{Hom}_{\operatorname{Mod}_k(G)}(X \otimes_k \operatorname{ind}_K^G V, W) \\
&\simeq \operatorname{Hom}_{\operatorname{Mod}_k(G)}(X, \underline{\operatorname{Hom}}(\operatorname{ind}_K^G V, W)).
\end{aligned}$$

The fourth isomorphism follows from [Vig96, p. 40]; part d) just prior to Section 5.3. The others use standard adjunction properties. The claim then follows from the Yoneda lemma. □

This gives the following spectral sequence (with V and W as above).

Proposition 3.2. $E_2^{i,j} = R^i \operatorname{Ind}_K^G \underline{\operatorname{Ext}}^j(V, W|_K) \Longrightarrow \underline{\operatorname{Ext}}^{i+j}(\operatorname{ind}_K^G V, W)$.

Proof. Note that the functor $\underline{\operatorname{Hom}}(V, -)$ preserves injective objects since $(-) \otimes_k V$ is an exact left adjoint. So does $(-)|_K$ as observed in the proof of Proposition 2.4. The Grothendieck spectral sequence for Ind_K^G composed with $\underline{\operatorname{Hom}}(V, (-)|_K)$ takes the stated form by 3.1. □

We emphasize the special case $W = k$ below.

Corollary 3.3. *Suppose V is a finite-dimensional object of $\operatorname{Mod}_k(K)$ and let $V^* = \operatorname{Hom}_k(V, k)$ denote its contragredient. Then there is an isomorphism of G-representations*

$$R^i \operatorname{Ind}_K^G(V^*) \simeq \underline{\operatorname{Ext}}^i(\operatorname{ind}_K^G V, k).$$

Proof. When $W = k$ the spectral sequence in Proposition 3.2 becomes

$$E_2^{i,j} = R^i \operatorname{Ind}_K^G \underline{\operatorname{Ext}}^j(V, k) \Longrightarrow \underline{\operatorname{Ext}}^{i+j}(\operatorname{ind}_K^G V, k).$$

When V is finite-dimensional $\underline{\operatorname{Hom}}(V, -) = \operatorname{Hom}_k(V, -)$ is exact, so $\underline{\operatorname{Ext}}^j(V, k) = 0$ for $j > 0$ and $\underline{\operatorname{Ext}}^0(V, k) = V^*$. In this case the spectral sequence degenerates into the isomorphisms in 3.3, and we are done. □

4 The top-dimensional derived functor

In this section we take G to be a p-adic Lie group of dimension $d = \dim_{\mathbb{Q}_p}(G)$, and we assume $\mathrm{char}(k) = p$.

Remark 4.1. In part i of Proposition 6.2 we will show that, for any V in $\mathrm{Mod}_k(K)$,

$$R^i \operatorname{Ind}_K^G(V) = 0 \quad \forall i > d.$$

A natural question is whether it is possible to compute the top-dimensional derived functors $R^d \operatorname{Ind}_K^G(V)$. For the trivial representation $V = k$ we have the following.

Proposition 4.2. *Assume G has a non-discrete center. Then:*

i. $R^d \operatorname{Ind}_K^G(k) = 0$ for all compact open subgroups $K \subset G$;

ii. The category $\mathrm{Mod}_k(G)$ has no nonzero projective objects.

Proof. For the proof of part one let $U \subset K$ be any open Poincaré subgroup, and let c be as in Remark 2.5 with $V = k$. We must find an open subgroup $U' \subset U$ such that $t^d_{U,U'}(c) = 0$. The corestriction map

$$\mathrm{cor}^{K\cap x'Ux'^{-1}}_{K\cap x'U'x'^{-1}} : H^d(K \cap x'U'x'^{-1}, k) \longrightarrow H^d(K \cap x'Ux'^{-1}, k)$$

is known to be an isomorphism (of one-dimensional spaces) for all U'. Its composition with the restriction map $\mathrm{res}^{K\cap x'Ux'^{-1}}_{K\cap x'U'x'^{-1}}$ is multiplication by the index. Thus $\mathrm{res}^{K\cap x'Ux'^{-1}}_{K\cap x'U'x'^{-1}} = 0$ if this index is > 1. To summarize, we must find $U' \subsetneq U$ such that we have strict inclusions

$$K \cap gU'g^{-1} \subsetneq K \cap gUg^{-1}$$

for all $g \in G$. Let Z denote the center of G. Intersecting both sides above with Z shows it is enough to pick a U' such that $Z \cap U' \subsetneq Z \cap U$. For example, for the right-hand side we get

$$Z \cap (K \cap gUg^{-1}) = K \cap g(Z \cap U)g^{-1} = K \cap (Z \cap U) = Z \cap U.$$

Since we are assuming Z is non-discrete $Z \cap U$ contains a non-identity element z say. Pick an open neighborhood $z(Z \cap U') \subset Z \cap U$ not containing the identity. Any such U' works.

For part two let W be an arbitrary object of $\mathrm{Mod}_k(G)$ and let V be an object of $\mathrm{Mod}_k(K)$ as before. Note that Ind_K^G takes injective objects to injective objects (since restriction is an exact left adjoint) and we therefore have a Grothendieck spectral sequence of the form

$$E_2^{j,i} = \mathrm{Ext}^j_{\mathrm{Mod}_k(G)}(W, R^i \operatorname{Ind}_K^G(V)) \Longrightarrow \mathrm{Ext}^{i+j}_{\mathrm{Mod}_k(K)}(W, V)$$

coming from Frobenius reciprocity. See [Vig96, p. 42] for example. If $\mathrm{Hom}_{\mathrm{Mod}_k(G)}(W, -)$ is exact we find that $E_2^{j,i} = 0$ for all $j > 0$, from which we deduce an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_k(G)}(W, R^i \operatorname{Ind}_K^G(V)) \simeq \mathrm{Ext}^i_{\mathrm{Mod}_k(K)}(W, V).$$

We specialize to the case $V = k$ and $i = d$. As we have just shown, the left-hand side vanishes in this case. For part two K only plays an auxiliary role and we may take it to be Poincaré. We infer that

$$\mathrm{Hom}_k(H^0(K, W), k) \simeq \mathrm{Ext}^d_{\mathrm{Mod}_k(K)}(W, k) = 0 \Longrightarrow H^0(K, W) = 0 \Longrightarrow W = 0$$

by duality for Poincaré groups. See the review in [SS, Sect. 1] for instance. □

Remark 4.3. Both parts of the previous Proposition clearly fail if G is discrete (as $d = 0$ and smooth G-representations are the same as abstract $k[G]$-modules). We do not know whether the Proposition holds if we only assume G itself is non-discrete.

Also, a natural question in the context of Proposition 4.2 is whether every homotopically projective complex of objects in $\mathrm{Mod}_k(G)$ is necessarily contractible. An affirmative answer would vastly generalize part ii of Proposition 4.2.

Theorem 1.1 in the introduction gives a supplement to Proposition 4.2 for p-adic reductive groups G.

5 The case of p-adic reductive groups

5.1 Notation, conventions, and background

In this article $\mathbb{N} = \{1, 2, 3, \dots\}$ denotes the set of all positive integers.

We let $\mathfrak{F}/\mathbb{Q}_p$ be a finite extension with valuation ring $\mathfrak{O}$, and we choose a uniformizer π. Take $\mathrm{val}_{\mathfrak{F}}$ to be the valuation on $\mathfrak{F}$ satisfying $\mathrm{val}_{\mathfrak{F}}(\pi) = 1$. As usual $q = p^f$ is the cardinality of the residue field $\mathfrak{O}/\pi\mathfrak{O}$, and $e = e(\mathfrak{F}/\mathbb{Q}_p)$ denotes the ramification index.

In this section $\mathbf{G}$ is an arbitrary connected reductive group defined over $\mathfrak{F}$, and $G = \mathbf{G}(\mathfrak{F})$. We choose a maximal $\mathfrak{F}$-split subtorus $\mathbf{S} \subset \mathbf{G}$ and let $\mathbf{Z}$ denote its centralizer. Following our earlier convention, $S = \mathbf{S}(\mathfrak{F})$ and $Z = \mathbf{Z}(\mathfrak{F})$ denote the p-adic Lie groups of $\mathfrak{F}$-rational points.

Let Φ denote the roots of $\mathbf{G}$ relative to $\mathbf{S}$, and select a subset of positive roots Φ^+ once and for all (then $\Phi^- = -\Phi^+$ is the set of negative roots). The root system may not be reduced, and we let Φ_{red} be the subset of reduced roots ($\alpha \in \Phi$ such that $\frac{1}{2}\alpha \notin \Phi$). By Φ^+_{red} and Φ^-_{red} we mean the subsets of positive and negative reduced roots respectively.

Furthermore, we pick a special vertex x_0 in the apartment associated with $\mathbf{S}$, and consider the Bruhat-Tits group scheme $\mathbf{G}^\circ_{x_0}$ over $\mathfrak{O}$. The special subgroup $K_0 = \mathbf{G}^\circ_{x_0}(\mathfrak{O})$ and its principal congruence subgroups

$$K_m = \ker\left(\mathbf{G}^\circ_{x_0}(\mathfrak{O}) \longrightarrow \mathbf{G}^\circ_{x_0}(\mathfrak{O}/\pi^m\mathfrak{O})\right)$$

play a pivotal role. The argument proving [OS19, Cor. 7.8] applies verbatim and shows K_m is a uniform pro-p group if $m \in e\mathbb{N}$ and $m > e$ if $p = 2$. We give the proof below and compute the lower p-series for such K_m.

Let $\mathbf{A}$ be the maximal $\mathfrak{F}$-split subtorus of the center of $\mathbf{G}$. We can arrange for x_0 to be the origin of the apartment $X_*(\mathbf{S})/X_*(\mathbf{A}) \otimes \mathbb{R}$. In this case the action of $z \in Z$ on the apartment is translation by the image of $\nu(z)$ where

$$\nu : Z \longrightarrow X_*(\mathbf{S}) \otimes \mathbb{R}$$

is the homomorphism for which

$$\langle \nu(z), \chi|\mathbf{S}\rangle = -\mathrm{val}_{\mathfrak{F}}\chi(z)$$

for all $\chi \in X^*(\mathbf{Z})$. We refer to [SSt, Sect. I.1] for more details. Later on we consider the monoid Z^+ of all $z \in Z$ such that $\langle \nu(z), \alpha\rangle \leq 0$ for all $\alpha \in \Phi^+$.

For each $\alpha \in \Phi$ we have a root subgroup $\mathcal{U}_\alpha$ of $\mathbf{G}$ with $\mathfrak{F}$-rational points U_α; the latter is normalized by Z. According to [KP23, Thm. 9.6.5] the root datum of $\mathbf{G}$ carries a valuation attached to x_0. By [KP23, Df. 6.1.2] this gives rise to a descending filtration of U_α by subgroups $(U_{\alpha,r})_{r\in\mathbb{R}}$ satisfying

$$zU_{\alpha,r}z^{-1} = U_{\alpha,r-\langle\nu(z),\alpha\rangle} \qquad \text{for any } z \in Z.\text{[1]} \tag{1}$$

This filtration needs to be modified. For any concave function f on $\Phi \cup \{0\}$ [KP23, Df. 7.3.3] introduces the subgroup

$$U_{\alpha,f} := U_{\alpha,x_0,f} := U_{\alpha,f(\alpha)} \cdot U_{2\alpha,f(2\alpha)}$$

of U_α. For any real number $r \geq 0$ the constant function f_r with value r is concave, and we abbreviate $\tilde{U}_{\alpha,r} := U_{\alpha,f_r}$.[2]

Before [KP23, Lem. 9.8.1] a descending filtration $(Z_r)_{r\geq 0}$ of Z is constructed. We point out that this filtration depends on the connected reductive group $\mathbf{Z}$ and not the ambient group $\mathbf{G}$. Since the Bruhat-Tits building of Z is a single point ([KP23, Prop. 9.3.9]), namely x_0, it follows from [KP23, Prop. 13.2.5, part (2)] that each Z_r is a normal subgroup of Z.

Using these filtrations [KP23, Def. 7.3.3] then introduces a descending filtration $(\mathcal{P}_{x_0,r})_{r\geq 0}$ of $K_0 = \mathbf{G}^\circ_{x_0}(\mathfrak{O})$. The crucial property of these filtrations is the Iwahori factorization ([KP23, Prop. 13.2.5, part (3)]): For any $r > 0$ the multiplication map defines a homeomorphism

$$\prod_{\alpha\in\Phi^-_{\mathrm{red}}} \tilde{U}_{\alpha,r} \times Z_r \times \prod_{\alpha\in\Phi^+_{\mathrm{red}}} \tilde{U}_{\alpha,r} \xrightarrow{\sim} \mathcal{P}_{x_0,r} \ . \tag{2}$$

(The factors in each product are ordered in a fixed but arbitrary way.)

Lemma 5.1. *For any $m > 0$ we have $K_m = \mathcal{P}_{x_0,m}$.*

Proof. By [KP23, Prop. 9.8.3] there is, for any $r \geq 0$, a smooth affine $\mathfrak{O}$-group scheme of finite type $\mathcal{G}_{x_0,r}$ such that $\mathcal{G}_{x_0,r}(\mathfrak{O}) = \mathcal{P}_{x_0,r}$. It comes by descend from the maximal unramified extension $\mathfrak{F}^{\mathrm{ur}}$ of $\mathfrak{F}$. By [KP23, Prop. 8.5.16, Df. A.5.12] and passing to $\mathrm{Gal}(\mathfrak{F}^{\mathrm{ur}}/\mathfrak{F})$-invariants we have

$$\mathcal{G}_{x_0,r+m}(\mathfrak{O}) = \ker\big(\mathcal{G}_{x_0,r}(\mathfrak{O}) \longrightarrow \mathcal{G}_{x_0,r}(\mathfrak{O}/\pi^m\mathfrak{O})\big).$$

Now take $r = 0$ and observe that $\mathcal{G}_{x_0,0} = \mathbf{G}^\circ_{x_0}$ since $K_0 = \mathcal{P}_{x_0,0}$. $\square$

As a consequence we obtain from (2) the Iwahori factorization

$$\prod_{\alpha\in\Phi^-_{\mathrm{red}}} \tilde{U}_{\alpha,m} \times Z_m \times \prod_{\alpha\in\Phi^+_{\mathrm{red}}} \tilde{U}_{\alpha,m} \xrightarrow{\sim} K_m \qquad \text{for any } m > 0. \tag{3}$$

Remark 5.2. For any $m > 0$ and $\alpha \in \Phi_{\mathrm{red}}$ the filtration subgroups $\tilde{U}_{\alpha,m}$ and Z_m are themselves principal congruence subgroups. For Z_m this is a special case of Lemma 5.1 since Z_m is the analog of $\mathcal{P}_{x_0,m}$ for the connected reductive $\mathfrak{F}$-group $\mathbf{Z}$.

[1]The ν in [KP23, Df. 6.1.2 V6] is the composite of our ν and the canonical splitting of the natural monomorphism $X_*(\mathbf{S}') \otimes \mathbb{R} \hookrightarrow X_*(\mathbf{S}) \otimes \mathbb{R}$ where $\mathbf{S}'$ is the maximal subtorus of $\mathbf{S}$ contained in the derived subgroup of $\mathbf{G}$ ([KP23, 4.1.4]). This canonical splitting exists since the difference between $\mathbf{S}$ and $\mathbf{S}'$, up to isogeny, comes from the center of $\mathbf{G}$. Hence our number $\langle\nu(z),\alpha\rangle$, for a root $\alpha \in \Phi$, coincides with $\alpha(\nu(z))$ in [KP23, Df. 6.1.2 V6]. Note that [KP23] contains a sign mistake as explained in [Kal, 2nd item on p. 23].

[2][KP23, Sect. 13.2] replaces f_r again by r in the notation, which we find too confusing. For simplicity we drop the point x_0, which we fixed once and for all, from the notation.

To see that $\tilde{U}_{\alpha,m}$ is a principal congruence subgroup we argue as follows. By [KP23, Prop. 2.11.9, Prop. 9.8.3] there is, for $r \geq 0$, a unique closed subgroup scheme $\mathcal{U}_{\alpha,x_0,r} \subset \mathcal{G}_{x_0,r}$ such that $\mathcal{U}_{\alpha,x_0,r}(\mathfrak{O}) = \tilde{U}_{\alpha,r}$. Because $\mathcal{U}_{\alpha,x_0,r}$ is closed the map $\mathcal{U}_{\alpha,x_0,r}(\mathfrak{O}/\pi^m\mathfrak{O}) \hookrightarrow \mathcal{G}_{x_0,r}(\mathfrak{O}/\pi^m\mathfrak{O})$ is still an inclusion and therefore

$$\begin{aligned}\tilde{U}_{\alpha,m} &= \mathcal{P}_{x_0,m} \cap \tilde{U}_{\alpha,0} = K_m \cap \tilde{U}_{\alpha,0} \\ &= \ker\big(\mathbf{G}^\circ_{x_0}(\mathfrak{O}) \longrightarrow \mathbf{G}^\circ_{x_0}(\mathfrak{O}/\pi^m\mathfrak{O})\big) \cap \tilde{U}_{\alpha,0} \\ &= \ker\big(\mathcal{U}_{\alpha,x_0,0}(\mathfrak{O}) \longrightarrow \mathbf{G}^\circ_{x_0}(\mathfrak{O}/\pi^m\mathfrak{O})\big) \\ &= \ker\big(\mathcal{U}_{\alpha,x_0,0}(\mathfrak{O}) \longrightarrow \mathcal{U}_{\alpha,x_0,0}(\mathfrak{O}/\pi^m\mathfrak{O})\big).\end{aligned}$$

The first equality follows from [KP23, Prop. 13.2.5, part (3)], applied to $\mathcal{P}_{x_0,m}$, after comparing the components associated with the root α. The second equality is Lemma 5.1.

5.2 Some uniform subgroups and their cohomology

We begin with the following variant of [OS19, Cor. 7.7]. Let $\mathcal{G} = \mathrm{Spec}(A)$ be a smooth affine $\mathfrak{O}$-group scheme, and $\widehat{\mathcal{G}} = \mathrm{Spf}(\widehat{A}^{\mathfrak{p}})$ be its formal completion in the unit section. Upon choosing an isomorphism $\widehat{A}^{\mathfrak{p}} \simeq \mathfrak{O}[\![X_1, \ldots, X_\delta]\!]$ we get a homeomorphism $\xi : \widehat{\mathcal{G}}(\mathfrak{O}) \xrightarrow{\sim} (\pi\mathfrak{O})^\delta$. We let

$$\widehat{\mathcal{G}}_m(\mathfrak{O}) := \xi^{-1}\big((\pi^m\mathfrak{O})^\delta\big) = \ker\big(\mathcal{G}(\mathfrak{O}) \longrightarrow \mathcal{G}(\mathfrak{O}/\pi^m\mathfrak{O})\big).$$

With this notation we have the following slight generalization of [OS19, Cor. 7.7].

Lemma 5.3. *Let $m \in e\mathbb{N}$, and assume $m > e$ if $p = 2$. Then the congruence subgroup $\widehat{\mathcal{G}}_m(\mathfrak{O})$ is a uniform pro-p group. Furthermore*

$$\widehat{\mathcal{G}}_{m+e}(\mathfrak{O}) = \widehat{\mathcal{G}}_m(\mathfrak{O})^p = \{g^p : g \in \widehat{\mathcal{G}}_m(\mathfrak{O})\}.$$

(This gives the lower p-series for $\widehat{\mathcal{G}}_m(\mathfrak{O})$ by iteration.)

Proof. The fact that $\widehat{\mathcal{G}}_m(\mathfrak{O})$ is uniform is precisely the content of [OS19, Cor. 7.7]. Here we compute its lower p-series. By considering $\mathcal{G}_0 = \mathrm{Res}_{\mathfrak{O}/\mathbb{Z}_p}\mathcal{G}$, and noting that $\widehat{\mathcal{G}}_{0,j}(\mathbb{Z}_p) = \widehat{\mathcal{G}}_m(\mathfrak{O})$ if $m = ej$, we may (and will) assume $\mathfrak{O} = \mathbb{Z}_p$.

We first deal with the case $p > 2$. As explained in [OS19, 7.2.1] the group $\widehat{\mathcal{G}}(\mathbb{Z}_p)$ is a standard group in the sense of [DdSMS, Df. 8.22]. The (last paragraph of the) proof of [DdSMS, Thm. 8.31] shows that $\widehat{\mathcal{G}}(\mathbb{Z}_p)^{p^{m-1}} = \widehat{\mathcal{G}}_m(\mathbb{Z}_p)$. (See also part (iii) of [DdSMS, Thm. 3.6] which gives the lower p-series of a uniform group.) Raising both sides to the p^{th} power gives the result.

For $p = 2$ we work with $\widehat{\mathcal{G}}_2(\mathbb{Z}_2)$ which again is standard for the same reason. The proof of [DdSMS, Thm. 8.31] (with $\varepsilon = 1$) also shows that $\widehat{\mathcal{G}}_2(\mathbb{Z}_2)^{2^{m-2}} = \widehat{\mathcal{G}}_m(\mathbb{Z}_2)$ for $m \geq 2$. Taking squares gives the result. $\square$

In particular K_m as well as $\tilde{U}_{\alpha,m}$ and Z_m (by Remark 5.2) all are uniform pro-p groups if $m \in e\mathbb{N}$ and $m > e$ if $p = 2$; moreover

$$K_m^p = K_{m+e}, \ Z_m^p = Z_{m+e}, \ \text{and} \quad \tilde{U}_{\alpha,m}^p = \tilde{U}_{\alpha,m+e} \ . \tag{4}$$

Next we will see that in certain situations intersections of uniform subgroups are uniform. In the following we put $\wp := p$ if $p > 2$, and $\wp := 4$ if $p = 2$.

Lemma 5.4. *Let $(A, \|\cdot\|)$ be a finite dimensional normed $\mathbb{Q}_p$-algebra whose norm $\|\cdot\|$ is submultiplicative and let $A_0 := \{a \in A : \|a\| \leq \wp\}$. Fix a closed subgroup $\Gamma \leq 1 + A_0$. Let K, K' be uniform open subgroups of Γ, and $a \in A^\times$ an arbitrary unit. If $a^{-1}K'a \cap K$ is open in K then it is uniform.*

Proof. First of all note that A_0 is a open pro-p subgroup of the additive group A, and $1+A_0$ is an open pro-p subgroup of the multiplicative group $A^\times$. It is well known (cf. [DdSMS, Prop. 6.22 and Cor. 6.25]) that the usual exponential and logarithm power series induce homeomorphisms

$$\exp : A_0 \longrightarrow 1 + A_0 \quad \text{and} \quad \log : 1 + A_0 \longrightarrow A_0 ,$$

which are inverse to each other.

We recall that a pro-p group H is uniform if and only if the following conditions are satisfied ([DdSMS, Thm. 4.5]):

(1) H is (topologically) finitely generated;

(2) H is torsion-free;

(3) H is powerful – which means $H/\overline{H^\wp}$ is abelian.

Here H^n, for $n \in \mathbb{N}$, is a priori the subgroup generated by the set of n-powers. However, when H is uniform $H^\wp$ is the same as the set of all $\wp$-powers by [DdSMS, Lemma 3.4] ; the same result shows that $H^\wp$ is open.

Obviously the intersection is torsion-free since K is. Since $a^{-1}K'a \cap K$ is open in K by assumption it is finitely generated by [DdSMS, Prop. 1.7] . It remains to show it is powerful, so choose two elements $x, y \in a^{-1}K'a \cap K$ arbitrarily. It suffices to show the commutator $[x, y]$ is a $\wp$-power from $a^{-1}K'a \cap K$. Since K and $a^{-1}K'a \cong K'$ are uniform we can write

$$[x, y] = \kappa^\wp = (a^{-1}\kappa'a)^\wp$$

for suitable $\kappa \in K$ and $\kappa' \in K'$. We just need to argue that $\kappa = a^{-1}\kappa'a$. Taking log above, using [DdSMS, Cor. 6.25(iii)], yields $\log[x, y] = \wp \log \kappa$. Since $a^{-1}\kappa'a$ may not lie in $1 + A_0$ we have to note that nevertheless $\log(a^{-1}\kappa'a)$ makes sense. Indeed the series

$$\log(a^{-1}\kappa'a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}(a^{-1}\kappa'a - 1)^n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} a^{-1}(\kappa' - 1)^n a = a^{-1}\log(\kappa')a$$

still converges. Having checked this, taking log as above we deduce $\log[x, y] = \wp \log(a^{-1}\kappa'a)$. Comparing the two expressions for $\log[x, y] \in \wp A_0$ and dividing by $\wp$ (in A) we infer the equality $\log(\kappa) = \log(a^{-1}\kappa'a)$ in A_0. Finally take exp on both sides, again noting that

$$\exp\big(\log(a^{-1}\kappa'a)\big) = \exp\big(a^{-1}\log(\kappa')a\big) = a^{-1}\exp\big(\log(\kappa')\big)a = a^{-1}\kappa'a .$$

This gives the equality $\kappa = a^{-1}\kappa'a$ as desired. We conclude $a^{-1}K'a \cap K$ that is uniform. $\square$

Example 5.5. *Our main example is the matrix algebra $A = M_N(\mathbb{Q}_p)$ with the norm $\|a\| = \max_{i,j}|a_{ij}|$. The multiplicative group $1 + A_0$ is the congruence subgroup $\ker(\mathrm{GL}_N(\mathbb{Z}_p) \to \mathrm{GL}_N(\mathbb{Z}_p/\wp\mathbb{Z}_p))$. Then Lemma 5.4 implies the following: If K and K' are uniform closed subgroups of $\ker(\mathrm{GL}_N(\mathbb{Z}_p) \to \mathrm{GL}_N(\mathbb{Z}_p/\wp\mathbb{Z}_p))$ then so is $a^{-1}K'a \cap K$ for any $a \in \mathrm{GL}_N(\mathbb{Q}_p)$ such that $a^{-1}K'a \cap K$ is open in K.*

Proposition 5.6. *Let $m, n \in e\mathbb{N}$ assuming $m, n > e$ if $p = 2$; we then have:*

a. $K_m \cap gK_n g^{-1}$ *is uniform for all* $g \in G$*;*

b. $\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1}$ *is uniform for all* $\alpha \in \Phi$ *and* $z \in Z$*,*

Proof. As we noted before (4) all groups of which we take intersections are uniform. The assertion then follows from the above Example by using a faithful representation of the Weil restriction to $\mathbb{Z}_p$ of $\mathbf{G}^{\circ}_{x_0}$ into some GL_N. $\square$

Throughout we fix a field k of characteristic p. A uniform pro-p group U admits a canonical p-valuation defined by the lower p-series. Thus U becomes equi-p-valued, and by [Laz, Ch. V, Prop. 2.5.7.1] the cup product gives an isomorphism of graded k-algebras

$$\bigwedge H^1(U,k) \xrightarrow{\sim} H^*(U,k).$$

Moreover $H^1(U,k) = \mathrm{Hom}_{\mathbb{F}_p}(U/U^p, k)$ is the dual Frattini quotient, which we will henceforth denote U^*_Φ. (Here the subscript Φ is standard notation for Frattini quotients, and should not be confused with the root system.) Our exterior products $\bigwedge = \bigwedge_k$ are always over k.

Proposition 5.7. *Consider arbitrary $m, n \in e\mathbb{N}$, both assumed to be $> e$ if $p = 2$, and $z \in Z$. Then $H^i(K_m \cap zK_n z^{-1}, k)$ decomposes as a direct sum*

$$\bigoplus \Big[\bigotimes_{\alpha \in \Phi^-_{\mathrm{red}}} \bigwedge^{a_\alpha} (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \otimes \bigwedge^{b} (Z_{\max\{m,n\}})^*_\Phi \otimes \bigotimes_{\alpha \in \Phi^+_{\mathrm{red}}} \bigwedge^{c_\alpha} (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \Big]$$

where $\{a_\alpha\}_{\alpha \in \Phi^-_{\mathrm{red}}}$, b, $\{c_\alpha\}_{\alpha \in \Phi^+_{\mathrm{red}}}$ in the sum $\bigoplus$ range over non-negative integers with sum i.

Proof. Conjugating the Iwahori factorization of K_n by z, using that Z_n is normal in Z, and intersecting with K_m we find that

$$\Big(\prod_{\alpha \in \Phi^-_{\mathrm{red}}} \tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1} \Big) \times Z_{\max\{m,n\}} \times \Big(\prod_{\alpha \in \Phi^+_{\mathrm{red}}} \tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1} \Big) \xrightarrow{\sim} K_m \cap zK_n z^{-1}. \tag{5}$$

By the same argument as in the proof of [OS19, Cor. 7.11], which uses the uniformity of all groups involved, quotienting out p-powers induces an isomorphism on the level of Frattini quotients. Taking k-linear duals and forming the exterior algebra gives an isomorphism of graded k-algebras

$$\begin{aligned}
&\bigwedge (K_m \cap zK_n z^{-1})^*_\Phi \xrightarrow{\simeq} \\
&\bigwedge \Big[\bigoplus_{\alpha \in \Phi^-_{\mathrm{red}}} (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \oplus (Z_{\max\{m,n\}})^*_\Phi \oplus \bigoplus_{\alpha \in \Phi^+_{\mathrm{red}}} (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \Big] \simeq \\
&\bigotimes_{\alpha \in \Phi^-_{\mathrm{red}}} \bigwedge (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \otimes \bigwedge (Z_{\max\{m,n\}})^*_\Phi \otimes \bigotimes_{\alpha \in \Phi^+_{\mathrm{red}}} \bigwedge (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \ .
\end{aligned}$$

Comparing the degree i graded pieces gives the result. $\square$

5.3 On the vanishing of certain restriction maps

We fix an $m \in e\mathbb{N}$. If $p = 2$ we always assume $m > e$. For $n, n' \in e\mathbb{N}$ such that $n \leq n'$ we have restriction maps

$$\mathrm{res}^i_{n,n'}(g) : H^i(K_m \cap gK_ng^{-1}, k) \longrightarrow H^i(K_m \cap gK_{n'}g^{-1}, k).$$

When $g = z \in Z$, this map is compatible with the decomposition in Prop. 5.7 in the obvious sense. (In general restriction commutes with cup products in group cohomology.)

Lemma 5.8. *Suppose $m \leq n < n'$ all lie in $e\mathbb{N}$ (and are $> e$ if $p = 2$). Then $\mathrm{res}^i_{n,n'}(g) = 0$ for all $g \in G$ and $i > i_0 := \dim_{\mathbb{Q}_p}(G/P_{min})$.*

Proof. By the Cartan decomposition $G = K_0Z^+K_0$, as described in [KP23, Thm. 5.2.1, part (1)] for example, we may write $g = hzh'$ for some $z \in Z^+$ and $h, h' \in K_0$. Since K_m and K_n are both normal in K_0 we note that

$$K_m \cap gK_ng^{-1} = h(K_m \cap zK_nz^{-1})h^{-1}.$$

Therefore $x \mapsto hx$ induces isomorphisms h_* on cohomology which fit in the following commutative diagram, where the horizontal maps are the restriction maps:

$$\begin{tikzcd}
H^i(K_m \cap gK_ng^{-1}, k) \arrow[r] & H^i(K_m \cap gK_{n'}g^{-1}, k) \\
H^i(K_m \cap zK_nz^{-1}, k) \arrow[r] \arrow[u, "h_*"] & H^i(K_m \cap zK_{n'}z^{-1}, k). \arrow[u, "h_*"']
\end{tikzcd}$$

Fix an $i > i_0$. Our claim that $\mathrm{res}^i_{n,n'}(g) = 0$ is therefore equivalent to $\mathrm{res}^i_{n,n'}(z) = 0$, which we now proceed to show using the decomposition in Proposition 5.7.

Consider $\mathrm{res}^i_{n,n'}(z)$ restricted to the piece of $H^i(K_m \cap zK_nz^{-1}, k)$ indexed by $\{a_\alpha\}_{\alpha \in \Phi^-_{\mathrm{red}}}$, b, $\{c_\alpha\}_{\alpha \in \Phi^+_{\mathrm{red}}}$. Observe that

- $\bigwedge^b (Z_n)^*_\Phi \longrightarrow \bigwedge^b (Z_{n'})^*_\Phi$ vanishes for $b > 0$ since $Z_{n'} \subseteq Z_{n+e} = (Z_n)^p$ by (4).
- When $\alpha \in \Phi^+_{\mathrm{red}}$ we have $\langle \nu(z), \alpha \rangle \leq 0$. In particular

  $$z\tilde{U}_{\alpha,n}z^{-1} = U_{\alpha,n-\langle \nu(z),\alpha\rangle} \cdot U_{2\alpha,n-\langle \nu(z),2\alpha\rangle} \subseteq \tilde{U}_{\alpha,n-\langle \nu(z),\alpha\rangle} \subseteq \tilde{U}_{\alpha,n} \subseteq \tilde{U}_{\alpha,m},$$

  so that

  $$\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1} = z\tilde{U}_{\alpha,n}z^{-1}.$$

  For such α the map in question is

  $$\bigwedge^{c_\alpha} (z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi \longrightarrow \bigwedge^{c_\alpha} (z\tilde{U}_{\alpha,n'}z^{-1})^*_\Phi$$

  which vanishes for $c_\alpha > 0$ since

  $$z\tilde{U}_{\alpha,n'}z^{-1} \subseteq z\tilde{U}_{\alpha,n+e}z^{-1} = (z\tilde{U}_{\alpha,n}z^{-1})^p,$$

  again by (4).

If $\mathrm{res}^i_{n,n'}(z) \neq 0$ there must be a piece of cohomology with $b = 0$ and all $c_\alpha = 0$ where $\mathrm{res}^i_{n,n'}(z)$ is nonzero. Since $\sum_{\alpha\in\Phi^-_{\mathrm{red}}} a_\alpha = i$ and all $a_\alpha \leq \dim_{\mathbb{Q}_p} U_\alpha (= \dim_{\mathbb{Q}_p}(\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1}))$ we conclude that

$$i \leq \sum_{\alpha\in\Phi^-_{\mathrm{red}}} \dim_{\mathbb{Q}_p} U_\alpha = \dim_{\mathbb{Q}_p}(G/P_{\min}) = i_0.$$

See [Bor, Sect. 21.11] for the first equality. □

Next we show the above bound i_0 is sharp, as made precise in the result below.

Lemma 5.9. *Given $m \leq n \leq n'$ in $e\mathbb{N}$ (all $> e$ if $p = 2$) there exists $z \in Z^+$ satisfying*

$$-\langle \nu(z), \alpha\rangle \leq m - n', \quad \forall \alpha \in \Phi^-.$$

For such z, $\mathrm{res}^i_{n,n'}(z) \neq 0$ for all $i \leq i_0$.

Proof. For the existence part, take $z := \mu(\pi) \in S \subseteq Z$ where $\mu \in X_*(\mathbf{S})^+$ is a dominant cocharacter satisfying $\langle \mu, \alpha\rangle \leq m - n'$ for all $\alpha \in \Phi^-$. We note that $X^*(\mathbf{Z}) \to X^*(\mathbf{S})$ has finite cokernel, so some multiple of α extends to an $\mathfrak{F}$-rational character of $\mathbf{Z}$, and therefore $-\langle \nu(\mu(\pi)), \alpha\rangle = \mathrm{val}_{\mathfrak{F}}\alpha(\mu(\pi)) = \langle \mu, \alpha\rangle$ by the defining property of ν. We see that $z \in S \cap Z^+$.

For the non-vanishing part, we first work out the case $i = i_0$. Consider the contribution to $H^{i_0}(K_m \cap zK_nz^{-1}, k)$ with indices $a_\alpha = \dim_{\mathbb{Q}_p} U_\alpha$ for all $\alpha \in \Phi^-_{\mathrm{red}}$ (and therefore $b = 0$ and $c_\alpha = 0$ for $\alpha \in \Phi^+_{\mathrm{red}}$) which is the line

$$\mathcal{L}_{z,n} := \bigotimes_{\alpha\in\Phi^-_{\mathrm{red}}} \bigwedge^{\mathrm{top}} (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_\Phi.$$

We are assuming $z \in Z^+$ satisfies the inequalities $-\langle \nu(z), \alpha\rangle \leq m - n'$ for *all* $\alpha \in \Phi^-$ (even the non-reduced roots). Under this assumption we have

$$U_{\alpha,m} \subseteq zU_{\alpha,n'}z^{-1} \subseteq zU_{\alpha,n}z^{-1}$$

since $m \geq n' - \langle \nu(z), \alpha\rangle$, and similarly for 2α if it is a root. As a result

$$\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1} = \tilde{U}_{\alpha,m} = \tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n'}z^{-1}$$

and consequently $\mathrm{res}^{i_0}_{n,n'}(z)$ maps $\mathcal{L}_{z,n}$ isomorphically to the line $\mathcal{L}_{z,n'}$ in $H^{i_0}(K_m \cap zK_{n'}z^{-1}, k)$. In particular $\mathrm{res}^{i_0}_{n,n'}(z)$ is nonzero on $\mathcal{L}_{z,n}$ for all such z.

For $i \leq i_0$ we generalize the argument from the previous paragraph as follows. Once and for all we write $i = \sum_{\alpha\in\Phi^-_{\mathrm{red}}} a_\alpha$ for a choice of integers $0 \leq a_\alpha \leq \dim_{\mathbb{Q}_p} U_\alpha$. One way of doing this is to list the roots, $\Phi^-_{\mathrm{red}} = \{\alpha_1, \alpha_2, \ldots\}$. Then let q be the largest index for which

$$\dim_{\mathbb{Q}_p} U_{\alpha_1} + \cdots + \dim_{\mathbb{Q}_p} U_{\alpha_q} \leq i.$$

By convention $q := 0$ if $i < \dim_{\mathbb{Q}_p} \mathcal{U}_{\alpha_1}$. Now let $a_{\alpha_j} := \dim_{\mathbb{Q}_p} U_{\alpha_j}$ for $j \leq q$. If $i < i_0$ we let

$$a_{\alpha_{q+1}} := i - (\dim_{\mathbb{Q}_p} U_{\alpha_1} + \cdots + \dim_{\mathbb{Q}_p} U_{\alpha_q}).$$

If there are any remaining roots α_j with $j > q + 1$ we declare that $a_{\alpha_j} := 0$.

Having chosen this expansion $i = \sum_{\alpha \in \Phi^-_{\mathrm{red}}} a_\alpha$, we introduce the following auxiliary subspace of $H^i(K_m \cap zK_nz^{-1}, k)$:

$$\mathcal{V}_{z,n} := \bigotimes_{\alpha \in \Phi^-_{\mathrm{red}}} \bigwedge^{a_\alpha} (\tilde{U}_{\alpha,m} \cap z\tilde{U}_{\alpha,n}z^{-1})^*_{\Phi}.$$

The restriction map $\mathrm{res}^i_{n,n'}(z)$ restricts to an isomorphism $\mathcal{V}_{z,n} \xrightarrow{\sim} \mathcal{V}_{z,n'}$ for z as above. In particular $\mathrm{res}^i_{n,n'}(z) \neq 0$ as claimed. □

5.4 The proof of Theorem 1.2

We can now prove our main result for p-adic reductive groups, which we recall here:

Theorem 5.10. *Fix an $m \in e\mathbb{N}$ ($> e$ if $p = 2$) and let $i_0 = \dim_{\mathbb{Q}_p}(G/P_{min})$ as above. Then:*

(a) $R^i \operatorname{Ind}^G_{K_m}(k) = 0$ *for all* $i > i_0$*;*

(b) $R^i \operatorname{Ind}^G_{K_m}(k) \neq 0$ *for all* $i \leq i_0$*.*

Proof. To show the vanishing in part (a) suppose $i > i_0$. Let $n \in (e\mathbb{N})_{\geq m}$ be arbitrary and consider any

$$c \in \varprojlim_{g \in G/K_n} H^i(K_m \cap gK_ng^{-1}, k)$$

as in Remark 2.5. It suffices to show $t^i_{K_n,K_{n'}}(c) = 0$ for all $n' \in (e\mathbb{N})_{>n}$, which follows from Lemma 5.8 since

$$t^i_{K_n,K_{n'}}(c)_{gK_{n'}} = \mathrm{res}^i_{n,n'}(g)(c_{gK_n}) = 0$$

for all $g \in G$. (We have used the formula for the transition maps $t^i_{K_n,K_{n'}}$ given in 2.5.)

For the non-vanishing in part (b) now suppose $i \leq i_0$ and pick some $n \in (e\mathbb{N})_{\geq m}$ once and for all. We will construct a nonzero class c which survives all the transition maps. That is, such that $t^i_{K_n,K_{n'}}(c) \neq 0$ for all $n' \in (e\mathbb{N})_{>n}$.

The class c will be prescribed on a set of representatives $\mathcal{X}$ for Z^+/Z^0. (For comparison Z^0 is shorthand notation for the subgroup denoted by $Z(\mathfrak{F})^0$ in [KP23, Df. 2.6.23].) We recall the Cartan decomposition $G = \bigsqcup_{z \in \mathcal{X}} K_0zK_0$.

For every $z \in \mathcal{X}$ we once and for all select a nonzero vector $v_z \in \mathcal{V}_{z,n}$ (recall the notation introduced in the proof of Lemma 5.9). Corresponding to these data there is a unique groupoid cohomology class

$$c : G/K_n \longrightarrow \bigoplus_{g \in G/K_n} H^i(K_m \cap gK_ng^{-1}, k)$$

with the following properties:

- c is K_m-equivariant (in the sense described in Remark 2.5);
- c is supported on $K_m\mathcal{X}K_n/K_n$;
- $c_{zK_n} = v_z$ for all $z \in \mathcal{X}$.

The uniqueness of c is clear. For its existence let $g \in K_m \mathcal{X} K_n$ and write $g = hzh'$ accordingly. Thus $h \in K_m$, $h' \in K_n$, and $z \in \mathcal{X}$. By the Cartan decomposition the factor z is uniquely determined, and the left coset $h(K_m \cap zK_n z^{-1})$ is independent of the factorization of g. Now let $c_{gK_n} = h_*(v_z)$ which is therefore well-defined.

To see this c has the desired property let $n' \in (e\mathbb{N})_{>n}$. Our task is to verify the tuple $t^i_{K_n,K_{n'}}(c)$ has at least one nonzero component. Take $z \in \mathcal{X}$ to be an element such that $-\langle \nu(z), \alpha \rangle \leq m - n'$ for all $\alpha \in \Phi^-$. Then by (the proof of) Lemma 5.9 we know that $\mathrm{res}^i_{n,n'}(z)$ restricts to an isomorphism $\mathcal{V}_{z,n} \xrightarrow{\sim} \mathcal{V}_{z,n'}$. In particular

$$t^i_{K_n,K_{n'}}(c)_{zK_{n'}} = \mathrm{res}^i_{n,n'}(z)(c_{zK_n}) = \mathrm{res}^i_{n,n'}(z)(v_z) \neq 0$$

which finishes the proof. □

We finish this section by emphasizing an application to duality, which essentially reproves and strengthens one of the main results from [Sor].

Corollary 5.11. *Let $m \in e\mathbb{N}$ ($m > e$ if $p = 2$). Then* $\underline{\mathrm{Ext}}^i(\mathrm{ind}^G_{K_m} k, k) = 0 \Longleftrightarrow i > i_0$.

Proof. Due to Corollary 3.3 this is just a restatement of Theorem 5.10. □

5.5 The proof of Theorem 1.1

By assumption $\mathbf{G}$ is nontrivial and connected. Hence the group Z has positive $\mathbb{Q}_p$-dimension. To show $\mathrm{Mod}_k(G)$ has no nonzero projective objects, the proof of Proposition 4.2 applies, noting that $R^d\,\mathrm{Ind}^G_{K_m}(k) = 0$. Indeed, by (3) for instance,

$$d = \sum_{\alpha \in \Phi^-_{\mathrm{red}}} \dim_{\mathbb{Q}_p} U_\alpha + \dim_{\mathbb{Q}_p} Z + \sum_{\alpha \in \Phi^+_{\mathrm{red}}} \dim_{\mathbb{Q}_p} U_\alpha = 2i_0 + \dim_{\mathbb{Q}_p} Z > i_0 \ .$$

We immediately deduce from Theorem 5.10 that $R^d\,\mathrm{Ind}^G_{K_m}(k) = 0$.

Remark 5.12. There is a quicker route to Theorem 1.1 by directly showing $R^d\,\mathrm{Ind}^G_{K_m}(k) = 0$. This amounts to verifying the condition appearing in the proof of Proposition 4.2. Indeed, for all large enough n there is an $n' > n$ (for instance $n' = n + 1$) such that we have a *strict* inclusion

$$K_m \cap gK_{n'}g^{-1} \subsetneq K_m \cap gK_n g^{-1}$$

for all $g \in G$. Indeed, by the Cartan decomposition it suffices to check this for $g = z \in Z^+$. For such z the strict inclusion follows from the factorization (5) by noting that $Z_{n'} \subsetneq Z_n$.

6 The derived functor $R\underline{\mathrm{Ind}}$

6.1 Preliminary remarks

We return to the general setup and let G be an arbitrary p-adic Lie group of $\mathbb{Q}_p$-dimension d, and k is still a field of characteristic p. Recall that $\mathrm{Mod}_k(G)$ is the abelian category of smooth G-representations on k-vector spaces. We let $D(G)$ denote its (unbounded) derived category.

We remind the reader that our convention is that G acts on $\mathrm{Ind}^G_K(V)$ by right translations. Thus, for any V in $\mathrm{Mod}_k(K)$, the space $\mathrm{Ind}^G_K(V)$ consists of all functions $f : G \longrightarrow V$ such that

- $f(\kappa g) = \kappa f(g)$ for any $g \in G$ and $\kappa \in K$;
- $f(gu) = f(g)$ for any $g \in G$ and $u \in U$ for some open $U \leq G$ (depending on f).

The fact that Ind_K^G is right adjoint to restriction is a consequence of Frobenius reciprocity (see [Vig96, I.5.7(i)] for example) which is the isomorphism

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Mod}_k(K)}(W, V) &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(W, \mathrm{Ind}_K^G(V)) \\ A &\longmapsto B(v)(g) := A(gv) \end{aligned} \tag{6}$$

for V in $\mathrm{Mod}_k(K)$ and W in $\mathrm{Mod}_k(G)$. The functor Ind_K^G is left exact and preserves injective objects (see [Vig96, I.5.9]). If $K' \leq K$ then $\mathrm{Ind}_K^G(V) \subseteq \mathrm{Ind}_{K'}^G(V)$. We may therefore take the union over K and introduce the left exact functor

$$\begin{aligned} \underline{\mathrm{Ind}} : \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(G) \\ V &\longmapsto \varinjlim_K \mathrm{Ind}_K^G(V) \ . \end{aligned}$$

Later we will endow $\underline{\mathrm{Ind}}(V)$ with a smooth $G \times G$-action, provided V is in $\mathrm{Mod}(G)$. We are interested in the derived functors $R^i\underline{\mathrm{Ind}}$ and their relation to $R^i \mathrm{Ind}_K^G$.

Lemma 6.1. $R^i\underline{\mathrm{Ind}}(V) = \varinjlim_K R^i \mathrm{Ind}_K^G(V)$.

Proof. This is immediate from the exactness of inductive limits and the fact that any injective object in $\mathrm{Mod}_k(G)$ remains injective in any $\mathrm{Mod}_k(K)$. (Indeed, the left adjoint of restriction is compact induction, which is exact since $k[G]$ is flat over $k[K]$.) □

As a preliminary observation, we prove that $R^i\underline{\mathrm{Ind}}$ vanishes for $i > d$:

Proposition 6.2. *The following holds:*

i. For any compact open subgroup $K \subseteq G$, the functor Ind_K^G has cohomological dimension at most d.

ii. The functor $\underline{\mathrm{Ind}}$ has cohomological dimension at most d.

Proof. This is a consequence of Proposition 2.4, but we prefer to include the following direct argument (which also gives an alternate proof of the derived Mackey decomposition 2.4).

i. Consider any V in $\mathrm{Mod}_k(K)$ and let $U \subseteq G$ be a compact open subgroup. We choose a set $R \subseteq G$ of representatives for the double cosets $K\backslash G/U$. The Mackey decomposition (see [Vig96, I.5.5] for example) is a natural isomorphism

$$\begin{aligned} \mathrm{Ind}_K^G(V)^U &\xrightarrow{\cong} \prod_{x \in R} V^{K \cap xUx^{-1}} \\ f &\longmapsto (f(x))_{x \in R} \ . \end{aligned}$$

Let $V \xrightarrow{\mathrm{qis}} \mathcal{J}^\bullet$ be a choice of injective resolution in $\mathrm{Mod}_k(K)$. On the one hand it remains an injective resolution in any $\mathrm{Mod}_k(K \cap xUx^{-1})$. Hence the complex $\prod_{x \in R}(\mathcal{J}^\bullet)^{K \cap xUx^{-1}}$ computes $\prod_{x \in R} H^*(K \cap xUx^{-1}, V)$. On the other hand the complex $\mathrm{Ind}_K^G(\mathcal{J}^\bullet)$ computes $R^* \mathrm{Ind}_K^G(V)$

and is a complex of injective objects in $\mathrm{Mod}_k(G)$. Therefore the composed functor spectral sequence for the functors $(-)^U$ and $\mathrm{Ind}_K^G(-)$ exists and reads

$$E_2^{r,s} = H^r(U, R^s \mathrm{Ind}_K^G(V)) \Longrightarrow \prod_{x \in R} H^{r+s}(K \cap xUx^{-1}, V) \ .$$

By passing to the limit with respect to U it degenerates into isomorphisms

$$R^s \mathrm{Ind}_K^G(V) \cong \varinjlim_U \prod_{x \in R} H^s(K \cap xUx^{-1}, V) \ . \tag{7}$$

For this note that profinite group cohomology commutes with inductive limits and that, for any M in $\mathrm{Mod}_k(K)$, we have

$$\varinjlim_U H^r(U, M) = \begin{cases} M & \text{if } r = 0, \\ 0 & \text{otherwise.} \end{cases}$$

In the limit (7) we may take U to run over Poincaré subgroups of G. However, with U the open subgroup $K \cap xUx^{-1}$ is also a Poincaré group of dimension d. Therefore all the cohomology groups on the right-hand side of (7) vanish for $s > d$.

ii. Because of Lemma 6.1 this follows from i. $\square$

Remark 6.3. If G is compact then the functors Ind_K^G and $\underline{\mathrm{Ind}}$ are exact.

Proof. For compact G we have $\mathrm{Ind}_K^G = \mathrm{ind}_K^G =$ compact induction, which is exact. $\square$

As a consequence of Prop. 6.2.ii, $\underline{\mathrm{Ind}}$ has finite cohomological dimension, and we therefore have (by [Har, Cor. I.5.3(γ)]) the total derived functor between the unbounded derived categories:

$$R\underline{\mathrm{Ind}} : D(G) \longrightarrow D(G) \ .$$

This functor has more structure, as we now explain. In the following we use the convention that for a $G \times G$-action we write G_ℓ, resp. G_r, if we refer to the action of the left, resp. right, factor in the product $G \times G$.

Lemma 6.4. *For any two representations V and V' in $\mathrm{Mod}_k(G)$ we have*

i. For $f \in \mathrm{Ind}_K^G(V')$ and $(g_1, g_2) \in G \times G$ the function

$$^{(g_1,g_2)}f(g) := g_1 f(g_1^{-1} g g_2)$$

lies in $\mathrm{Ind}_{g_1 K g_1^{-1}}^G(V')$;

ii. this defines a smooth $G \times G$-action on $\underline{\mathrm{Ind}}(V')$; more precisely, $\mathrm{Ind}_K^G(V') = \underline{\mathrm{Ind}}(V')^{K_\ell}$;

iii. the earlier G-action on $\underline{\mathrm{Ind}}(V')$ is the G_r-action;

iv. the adjunction isomorphism

$$\underline{\mathrm{Hom}}(V, V') \xrightarrow{\cong} \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V, \underline{\mathrm{Ind}}(V')) := \varinjlim_K \mathrm{Hom}_{\mathrm{Mod}_k(G_r)}(V, \underline{\mathrm{Ind}}(V'))^{K_\ell} \tag{8}$$

obtained by passing to the inductive limit with respect to K in (6) is G-equivariant where G acts on the target through the G_ℓ-action on $\underline{\mathrm{Ind}}(V')$;

v. if the representation V is finitely generated then

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V, \underline{\mathrm{Ind}}(V')) = \mathrm{Hom}_{\mathrm{Mod}_k(G_r)}(V, \underline{\mathrm{Ind}}(V')) \ .$$

Proof. i. Suppose f is fixed by right translation by U. Then ${}^{(g_1,g_2)}f$ is fixed by right translation by $g_2 U g_2^{-1}$. Furthermore, for $\kappa \in K$, we compute

$${}^{(g_1,g_2)}f((g_1\kappa g_1^{-1})g) = g_1 f(g_1^{-1}(g_1\kappa g_1^{-1})gg_2) = g_1\kappa f(g_1^{-1}gg_2) = (g_1\kappa g_1^{-1}) \cdot {}^{(g_1,g_2)}f(g) \ .$$

Thus ${}^{(g_1,g_2)}f \in \underline{\mathrm{Ind}}(V')^{g_1 K g_1^{-1} \times g_2 U g_2^{-1}}$. Now ii and iii are obvious.

iv. For a fixed K we have by (6) and ii:

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Mod}_k(K)}(V, V') &\xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Mod}_k(G)}(V, \mathrm{Ind}_K^G(V')) \\ &= \mathrm{Hom}_{\mathrm{Mod}_k(G)}(V, \underline{\mathrm{Ind}}(V')^{K_\ell}) \\ &= \mathrm{Hom}_{\mathrm{Mod}_k(G_r)}(V, \underline{\mathrm{Ind}}(V'))^{K_\ell}\end{aligned}$$

which in the limit with respect to K gives rise to the isomorphism (8). A straightforward computation shows its equivariance. (We refer to [SS, p. 32] for the definition of the smooth linear maps $\underline{\mathrm{Hom}}(V, V')$.)

v. Under the finiteness assumption, the image of any G-homomorphism from V into $\underline{\mathrm{Ind}}(V')$ lies in $\mathrm{Ind}_K^G(V')$ for some open $K \leq G$ (which depends on the homomorphism). $\square$

Hence we actually have a left exact functor $\underline{\mathrm{Ind}} : \mathrm{Mod}_k(G) \longrightarrow \mathrm{Mod}_k(G \times G)$ which derives to a functor $R\underline{\mathrm{Ind}} : D(G) \longrightarrow D(G \times G)$ computable by homotopically injective resolutions. Our next goal is to lift the adjunction (8) to the derived level, using [KS, Thm. 14.4.8]. For this we first have to discuss its right-hand side in more detail.

We begin by recalling some general nonsense about the adjunction between tensor products and the Hom-functor which for three k-vector spaces V_1, V_2, and V_3 is given by the linear isomorphism

$$\begin{aligned}\mathrm{Hom}_k(V_1 \otimes_k V_2, V_3) &\xrightarrow{\cong} \mathrm{Hom}_k(V_1, \mathrm{Hom}_k(V_2, V_3)) \\ A &\longmapsto \lambda_A(v_1)(v_2) := A(v_1 \otimes v_2) \ .\end{aligned} \tag{9}$$

Suppose that all three vector spaces carry a left G-action. Then $\mathrm{Hom}_k(V_1 \otimes_k V_2, V_3)$ and $\mathrm{Hom}_k(V_1, \mathrm{Hom}_k(V_2, V_3))$ are equipped with the $G \times G \times G$-action defined by

$${}^{(g_1,g_2,g_3)}A(v_1 \otimes v_2) := g_3 A(g_1^{-1}v_1 \otimes g_2^{-1}v_2) \quad \text{and} \quad {}^{(g_1,g_2,g_3)}\lambda(v_1)(v_2) := g_3(\lambda(g_1^{-1}v_1)(g_2^{-1}v_2)),$$

respectively. The above adjunction is equivariant for these two actions. If we restrict to the diagonal G-action, then the above adjunction induces the adjunction isomorphism

$$\mathrm{Hom}_{k[G]}(V_1 \otimes_k V_2, V_3) \xrightarrow{\cong} \mathrm{Hom}_{k[G]}(V_1, \mathrm{Hom}_k(V_2, V_3)) \ .$$

If the G-action on the V_i is smooth then this can also be written as an isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_k(G)}(V_1 \otimes_k V_2, V_3) \cong \mathrm{Hom}_{\mathrm{Mod}_k(G)}(V_1, \underline{\mathrm{Hom}}(V_2, V_3)) \ . \tag{10}$$

In the next paragraph we discuss a variant of this.

Now suppose that in the adjunction (9) the vector spaces V_1 and V_2 carry a G-action whereas V_3 carries a $G \times G$-action. Then $V_1 \otimes_k V_2$ carries a $G \times G$-action as well. Moreover, $\mathrm{Hom}_{k[G_r]}(V_2, V_3)$ still carries a G-action through the G_ℓ-action on V_3. The above adjunction induces the adjunction isomorphism

$$\mathrm{Hom}_{k[G \times G]}(V_1 \otimes_k V_2, V_3) \xrightarrow{\cong} \mathrm{Hom}_{k[G_\ell]}(V_1, \mathrm{Hom}_{k[G_r]}(V_2, V_3)) \ .$$

Suppose in addition that the actions on the V_i are smooth. Then the G_ℓ-action on

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_2, V_3) := \varinjlim_K \mathrm{Hom}_{\mathrm{Mod}_k(G_r)}(V_2, V_3)^{K_\ell} \ ,$$

where K runs over the compact open subgroups, is smooth. Hence we may rewrite the latter isomorphism as

$$\mathrm{Hom}_{\mathrm{Mod}_k(G \times G)}(V_1 \otimes_k V_2, V_3) \cong \mathrm{Hom}_{\mathrm{Mod}_k(G_\ell)}(V_1, \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_2, V_3)) \ .$$

This works as well with V_1 and V_2 interchanged:

Remark 6.5. We could have defined the initial adjunction for vector spaces analogously by the linear isomorphism

$$\begin{aligned} \mathrm{Hom}_k(V_1 \otimes_k V_2, V_3) &\xrightarrow{\cong} \mathrm{Hom}_k(V_2, \mathrm{Hom}_k(V_1, V_3)) \\ A &\longmapsto \mu_A(v_2)(v_1) := A(v_1 \otimes v_2) \ . \end{aligned}$$

This leads to the isomorphism

$$\mathrm{Hom}_{\mathrm{Mod}_k(G \times G)}(V_1 \otimes_k V_2, V_3) \cong \mathrm{Hom}_{\mathrm{Mod}_k(G_r)}(V_2, \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_\ell)}(V_1, V_3)) \ .$$

We are now in a position to apply [KS, Thm. 14.4.8]. For our present context we conclude that the functor

$$\begin{aligned} \mathrm{Mod}_k(G)^{op} \times \mathrm{Mod}_k(G \times G) &\longrightarrow \mathrm{Mod}_k(G) \\ (V, V') &\longmapsto \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V, V') \end{aligned}$$

has the left adjoint functor

$$\begin{aligned} \mathrm{Mod}_k(G) \times \mathrm{Mod}_k(G) &\longrightarrow \mathrm{Mod}_k(G \times G) \\ (V, V') &\longmapsto V \otimes_k V' \end{aligned}$$

This shows that the adjointness requirements in [KS, Thm. 14.4.8] are satisfied, so that we have the total derived functor

$$R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)} : D(G)^{op} \times D(G \times G) \longrightarrow D(G)$$

satisfying (14.4.6) in [KS]. Namely, for $V_1^\bullet$, $V_2^\bullet$ in $D(G)$ and $V_3^\bullet$ in $D(G \times G)$ we have isomorphisms

$$\begin{aligned} R\,\mathrm{Hom}_{\mathrm{Mod}_k(G \times G)}(V_1^\bullet \otimes_k V_2^\bullet, V_3^\bullet) &\cong R\,\mathrm{Hom}_{\mathrm{Mod}_k(G_\ell)}(V_1^\bullet, R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_2^\bullet, V_3^\bullet)) \\ &\cong R\,\mathrm{Hom}_{\mathrm{Mod}_k(G_\ell)}(V_2^\bullet, R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_1^\bullet, V_3^\bullet)). \end{aligned}$$

The derived functor is computable via homotopically injective resolutions, see part (ii) of [KS, Thm. 14.4.8]:

$$R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V^\bullet, V'^\bullet) = \mathrm{tot}_\Pi \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V^\bullet, J'^\bullet) \tag{11}$$

where $V'^\bullet \xrightarrow{\simeq} J'^\bullet$ is a homotopically injective resolution in $\mathrm{Mod}_k(G \times G)$. Here tot_Π is the direct product totalization of the double complex. We emphasize that Π refers to the direct product in $\mathrm{Mod}_k(G)$, in other words the smooth vectors of the direct product of abstract $k[G]$-modules.

6.2 Duality on compact objects

Next we relate the duality functor $R\underline{\mathrm{Hom}}(-, k)$ from [SS] to the object $R\underline{\mathrm{Ind}}(k)$. In this section we are primarily interested in the restriction of the duality functor to $D(G)^c$ (the subcategory of compact objects). Our main result here is Corollary 6.9 below.

First, let $V_1^\bullet$ and $V_2^\bullet$ be any two objects of $D(G)$ and fix a homotopically injective resolution $V_2^\bullet \xrightarrow{\simeq} J^\bullet$. Then, by [SS, Prop. 3.1]:

$$R\underline{\mathrm{Hom}}(V_1^\bullet, V_2^\bullet) = \underline{\mathrm{Hom}}^\bullet(V_1^\bullet, J^\bullet) \quad \text{and} \quad R\underline{\mathrm{Ind}}(V_2^\bullet) = \underline{\mathrm{Ind}}(J^\bullet)\ . \tag{12}$$

By Lemma 6.4.iv. we have the G-equivariant adjunction isomorphism of actual complexes

$$\underline{\mathrm{Hom}}^\bullet(V_1^\bullet, J^\bullet) \xrightarrow{\cong} \mathrm{tot}_\Pi \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_1^\bullet, \underline{\mathrm{Ind}}(J^\bullet))\ . \tag{13}$$

Using (12), (13), and (11) we now consider the G-equivariant map

$$\begin{aligned}
\tau_{V_1^\bullet, V_2^\bullet} : R\underline{\mathrm{Hom}}(V_1^\bullet, V_2^\bullet) &= \underline{\mathrm{Hom}}^\bullet(V_1^\bullet, J^\bullet) \\
&\cong \mathrm{tot}_\Pi \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_1^\bullet, \underline{\mathrm{Ind}}(J^\bullet)) \\
&\to \mathrm{tot}_\Pi \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_1^\bullet, J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)}) \\
&= R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_1^\bullet, \underline{\mathrm{Ind}}(J^\bullet)) \\
&= R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(V_1^\bullet, R\underline{\mathrm{Ind}}(V_2^\bullet))
\end{aligned}$$

where $\underline{\mathrm{Ind}}(J^\bullet) \xrightarrow{\simeq} J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)}$ is a homotopically injective resolution in $\mathrm{Mod}_k(G \times G)$.

Proposition 6.6. *Let $U \subseteq G$ be a torsionfree pro-p open subgroup. Viewing $\mathbf{X}_U := \mathrm{ind}_U^G(k)$ as a complex (concentrated in degree zero) the above map $\tau_{\mathbf{X}_U, V_2^\bullet}$ is a quasi-isomorphism for any $V_2^\bullet$.*

Proof. We have to show that the map

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(\mathbf{X}_U, \underline{\mathrm{Ind}}(J^\bullet)) \longrightarrow \underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(\mathbf{X}_U, J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)})$$

is a quasi-isomorphism. Frobenius reciprocity implies that

$$\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(\mathbf{X}_U, -) = \varinjlim_K \mathrm{Hom}_{\mathrm{Mod}_k(G_r)}(\mathbf{X}_U, -)^{K \times \{1\}} = \varinjlim_K (-)^{K \times U} = (-)^{\{1\} \times U}\ . \tag{14}$$

Hence the map in question is the map $\underline{\mathrm{Ind}}(J^\bullet)^{\{1\} \times U} \longrightarrow (J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)})^{\{1\} \times U}$. Of course we have the isomorphism $H^*(U, \underline{\mathrm{Ind}}(J^\bullet)) \xrightarrow{\cong} H^*(U, J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)})$ where, for simplicity, we write simply U instead of $\{1\} \times U$. We adopt this convention for the rest of this proof. Hence it is enough to verify the following:

(a) $H^*(U, \underline{\mathrm{Ind}}(J^\bullet)) = h^*(\underline{\mathrm{Ind}}(J^\bullet)^U)$;

(b) $H^*(U, J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)}) = h^*((J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)})^U)$.

We first note ([Lur, 1.3.5], [ScSc, 3.1]) that we may assume that all our homotopically injective resolutions are even semi-injective, i.e., in addition each of their terms is an injective object.

- *Part (a)*

Each J^m, for $m \in \mathbb{Z}$, is injective in $\mathrm{Mod}_k(G)$ and hence in $\mathrm{Mod}_k(K)$. Then any $\mathrm{Ind}_K^G(J^m)$ is injective in $\mathrm{Mod}_k(G)$ and hence in $\mathrm{Mod}_k(U)$. Since the cohomology of U commutes with inductive limits it follows that the complex $\underline{\mathrm{Ind}}(J^\bullet)$ consists of $H^0(U,-)$-acyclic objects. The functor $H^0(U,-)$ has finite cohomological dimension. Therefore a) holds by [Har, Cor. I.5.3(γ)] (and its proof).

- *Part (b)*

This is a similar argument since in the subsequent Lemma 6.7 we will show that any term of the complex $J^\bullet_{\underline{\mathrm{Ind}}(J^\bullet)}$ being injective in $\mathrm{Mod}_k(G \times G)$ is $H^0(U,-)$-acyclic. $\square$

At the end of the previous proof we alluded to:

Lemma 6.7. *Let $U \subseteq G$ be an open subgroup. Then any injective object V in $\mathrm{Mod}_k(G \times G)$ is $H^0(\{1\} \times U, -)$-acyclic.*

Proof. First we allow an arbitrary object V in $\mathrm{Mod}_k(G\times G)$. For any compact open subgroup $K \subset G$ we have the Hochschild-Serre spectral sequence (where we write U instead of $\{1\} \times U$ on the E_2-page):

$$E_2^{rs} = H^r(K, H^s(U, V)) \Longrightarrow H^{r+s}(K \times U, V) \ .$$

It is functorial with respect to the restriction to a smaller compact open subgroup $K' \subseteq K$ (see [NSW, II.4 Ex. 3]). Hence we may pass to the limit with respect to smaller $K' \subseteq K$ in this spectral sequence. As in the proof of Prop. 6.2.i the limit spectral sequence degenerates into isomorphisms

$$H^s(U, V) \cong \varinjlim_{K'} H^s(K' \times U, V) \ .$$

Now, if V is injective in $\mathrm{Mod}_k(G \times G)$ then it is injective in each $\mathrm{Mod}_k(K' \times U)$ so that the above right-hand side vanishes for $s > 0$. Therefore so does $H^s(U, V)$. $\square$

We easily deduce the following result.

Corollary 6.8. *Let $U \subseteq G$ be a torsionfree pro-p open subgroup; we then have:*

i. The functors $R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(\mathbf{X}_U, -)$ and $RH^0(\{1\} \times U, -)$ from $D(G \times G)$ to $D(G)$ are naturally isomorphic;

ii. the functors $R\underline{\mathrm{Hom}}(\mathbf{X}_U, -)$ and $RH^0(\{1\} \times U, R\underline{\mathrm{Ind}}(-))$ from $D(G)$ to $D(G)$ are naturally isomorphic;

iii. if G is compact, then the functors $R\underline{\mathrm{Hom}}(\mathbf{X}_U, -)$ and $RH^0(\{1\} \times U, \underline{\mathrm{Ind}}(-))$ from $D(G)$ to $D(G)$ are naturally isomorphic.

Proof. i. This follows from (14) and the above Lemma 6.7. ii. Combine i. and Prop. 6.6. iii. If G is compact then, by Remark 6.3, the functor $\underline{\mathrm{Ind}}$ is exact so that $R\underline{\mathrm{Ind}} = \underline{\mathrm{Ind}}$. Now apply part ii. $\square$

We now specialize the map $\tau_{V_1^\bullet, V_2^\bullet}$ to the case where $V_2^\bullet$ is the complex with the trivial representation k in degree zero, and obtain a natural transformation

$$\tau_- := \tau_{-,k} : R\underline{\mathrm{Hom}}(-,k) \longrightarrow R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(-, R\underline{\mathrm{Ind}}(k)) \tag{15}$$

between exact functors from $D(G)$ to $D(G)$.

Recall from [DGA, Rk. 10] that the full subcategory $D(G)^c$ of all compact objects in $D(G)$ is the smallest strictly full triangulated subcategory closed under direct summands which contains $\mathbf{X}_U$ for some (or equivalently any) open torsionfree pro-p subgroup $U \subseteq G$.

Corollary 6.9. *τ_- restricted to $D(G)^c$ is a natural isomorphism.*

Proof. The full subcategory of all objects $V^\bullet$ in $D(G)$ for which $\tau_{V^\bullet}$ is an isomorphism is a strictly full triangulated subcategory closed under direct summands which contains $\mathbf{X}_U$ by Prop. 6.6. $\square$

6.3 The complex $R\underline{\mathrm{Ind}}(k)$ for reductive groups

In this section we again focus on p-adic reductive groups, and we put ourselves in the setup from Section 5. Thus $G = \mathbf{G}(\mathfrak{F})$ is the group of $\mathfrak{F}$-rational points of a connected reductive group $\mathbf{G}$ defined over some finite extension $\mathfrak{F}/\mathbb{Q}_p$. The goal in this section is to establish the following vanishing result.

Theorem 6.10. *$R^i\underline{\mathrm{Ind}}(k) = 0$ for all $i > 0$.*

The proof requires some preparation. We start with the following observation.

Lemma 6.11. *Let $U' \subset U$ be two uniform pro-p groups; if $U' \subset U^p$, then the restriction map*

$$H^i(U,k) \xrightarrow{res} H^i(U',k)$$

is the zero map for all $i > 0$.

Proof. For $i = 1$ the map in question is the natural map $\mathrm{Hom}_k(U/U^p, k) \to \mathrm{Hom}_k(U'/U'^p, k)$ which is the zero map by assumption. For uniform pro-p groups, and a coefficient field k of characteristic p, the cohomology is generated under the cup product in degree 1 (one can reduce to the case $k = \mathbb{F}_p$ which is [Laz, Prop. 2.5.7.1, p. 567]). Since restriction maps commute with cup products the assertion follows. $\square$

To apply Lemma 6.11 the key input is the following.

Lemma 6.12. *Let $n \in e\mathbb{N}$ (and assume $n > e$ if $p = 2$). Then, for all $g \in G$ we have*

$$(K_n \cap gK_ng^{-1})^p = K_{n+e} \cap gK_{n+e}g^{-1}.$$

Proof. By the Cartan decomposition $G = K_0 Z^+ K_0$ we may assume that $g = z \in Z^+$, noting that K_n and K_{n+e} are both normal subgroups of K_0. For any $z \in Z$ we have, by (5) in the proof of Proposition 5.7, the homeomorphism

$$\Big(\prod_{\alpha\in\Phi^-_{\mathrm{red}}} \tilde{U}_{\alpha,n} \cap z\tilde{U}_{\alpha,n}z^{-1}\Big) \times Z_n \times \Big(\prod_{\alpha\in\Phi^+_{\mathrm{red}}} \tilde{U}_{\alpha,n} \cap z\tilde{U}_{\alpha,n}z^{-1}\Big) \xrightarrow{\sim} K_n \cap zK_nz^{-1}.$$

In that proof we also argued that by dividing out by the subgroups of p-powers (even if $p = 2$) we get an isomorphism. Hence the above map restricted to p-powers must still be a homeomorphism, i.e.,

$$(K_n \cap zK_nz^{-1})^p = \prod_{\alpha\in\Phi^-_{\mathrm{red}}} (\tilde{U}_{\alpha,n} \cap z\tilde{U}_{\alpha,n}z^{-1})^p \times Z_n^p \times \prod_{\alpha\in\Phi^+_{\mathrm{red}}} (\tilde{U}_{\alpha,n} \cap z\tilde{U}_{\alpha,n}z^{-1})^p \ .$$

Using (1) we compute, now for $z \in Z^+$,

$$\begin{aligned} z\tilde{U}_{\alpha,n}z^{-1} &= zU_{\alpha,n}z^{-1} \cdot zU_{2\alpha,n}z^{-1} \\ &= U_{\alpha,n-\langle\nu(z),\alpha\rangle} \cdot U_{2\alpha,n-\langle\nu(z),2\alpha\rangle} \\ &\begin{cases} \supseteq U_{\alpha,n} \cdot U_{2\alpha,n} = \tilde{U}_{\alpha,n} & \text{if } \alpha \in \Phi^-, \\ \subseteq U_{\alpha,n} \cdot U_{2\alpha,n} = \tilde{U}_{\alpha,n} & \text{if } \alpha \in \Phi^+ \end{cases} \end{aligned}$$

and therefore

$$\tilde{U}_{\alpha,n} \cap z\tilde{U}_{\alpha,n}z^{-1} = \begin{cases} \tilde{U}_{\alpha,n} & \text{if } \alpha \in \Phi^-, \\ z\tilde{U}_{\alpha,n}z^{-1} & \text{if } \alpha \in \Phi^+. \end{cases}$$

Using (4) it follows that

$$(\tilde{U}_{\alpha,n} \cap z\tilde{U}_{\alpha,n}z^{-1})^p = \tilde{U}^p_{\alpha,n} \cap z\tilde{U}^p_{\alpha,n}z^{-1} = \tilde{U}_{\alpha,n+e} \cap z\tilde{U}_{\alpha,n+e}z^{-1}$$

as well as $Z_n^p = Z_{n+e}$. We conclude that

$$\begin{aligned} (K_n \cap zK_nz^{-1})^p &= \prod_{\alpha\in\Phi^-_{\mathrm{red}}} (\tilde{U}_{\alpha,n+e} \cap z\tilde{U}_{\alpha,n+e}z^{-1}) \times Z_{n+e} \times \prod_{\alpha\in\Phi^+_{\mathrm{red}}} (\tilde{U}_{\alpha,n+e} \cap z\tilde{U}_{\alpha,n+e}z^{-1}) \\ &= K_{n+e} \cap zK_{n+e}z^{-1} \end{aligned}$$

as desired. $\square$

We can now prove our vanishing result for $R^i\underline{\mathrm{Ind}}(k)$.

Proof. (Theorem 6.10.) Upon passing to the diagonal colimit $K = U$ when combining Proposition 2.4 and Lemma 6.1 we see that

$$\begin{aligned} R^i\underline{\mathrm{Ind}}(k) &\simeq \varinjlim_K \varinjlim_U \varprojlim_{x\in G/U} H^i(K \cap xUx^{-1}, k) \\ &\simeq \varinjlim_U \varprojlim_{x\in G/U} H^i(U \cap xUx^{-1}, k) \\ &\simeq \varinjlim_{n\in e\mathbb{N}} \varprojlim_{x\in G/K_n} H^i(K_n \cap xK_nx^{-1}, k). \end{aligned}$$

Now it is obvious from Lemmas 6.11 and 6.12 that the transition maps

$$\begin{aligned} \varprojlim_{x\in G/K_n} H^i(K_n \cap xK_nx^{-1}, k) &\longrightarrow \varprojlim_{x'\in G/K_{n'}} H^i(K_{n'} \cap x'K_{n'}x'^{-1}, k) \\ c &\longmapsto \big(\mathrm{res}^{K_n\cap x'K_nx'^{-1}}_{K_{n'}\cap x'K_{n'}x'^{-1}} c_{x'K_n}\big)_{x'\in G/K_{n'}} \end{aligned}$$

are all zero for $i > 0$ and $n' = n + e$. Therefore $R^i\underline{\mathrm{Ind}}(k) = 0$ for $i > 0$. $\square$

Since the complex $R\underline{\mathrm{Ind}}(k)$ is concentrated in non-negative degrees, and has zero cohomology in positive degrees by 6.10, there is a quasi-isomorphism

$$\underline{\mathrm{Ind}}(k)[0] \xrightarrow{\text{qis}} R\underline{\mathrm{Ind}}(k) \tag{16}$$

for p-adic reductive groups G. Note that $\underline{\mathrm{Ind}}(k)$ is simply the space $\mathcal{C}^\infty(G,k)$ of smooth vectors in the $G \times G$-representation on the space of all k-valued functions on G.

Corollary 6.13. *The functors $R\underline{\mathrm{Hom}}(-,k)$ and $R\underline{\mathrm{Hom}}_{\mathrm{Mod}_k(G_r)}(-,\mathcal{C}^\infty(G,k))$ restricted to $D(G)^c$ are isomorphic.*

Proof. Combine Corollary 6.9 with (16). ☐

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