EQUIVARIANT SATAKE CATEGORY AND KOSTANT-WHITTAKER REDUCTION

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To Victor Ginzburg on his birthday

CONTENTS

ABSTRACT. We explain (following V. Drinfeld) how the $G(\mathbb{C}[[t]])$ equivariant derived
category of the affine Grassmannian can be described in terms of coherent sheaves on
the Langlands dual Lie algebra equivariant with respect to the adjoint action, due to
some old results of V. Ginzburg. The global cohomology functor corresponds under
this identification to restriction to the Kostant slice. We extend this description to
loop rotation equivariant derived category, linking it to Harish-Chandra bimodules for
the Langlands dual Lie algebra, so that the global cohomology functor corresponds
to the quantum Kostant-Whittaker reduction of a Harish-Chandra bimodule. We
derive a conjecture of [11], which identifies the loop-rotation equivariant homology of
the affine Grassmannian with quantized Toda lattice.

1. INTRODUCTION

Let G be a semi-simple algebraic group over an algebraically closed characteristic
zero field k. The fundamental object of the geometric Langlands duality theory is the
so-called *Satake category* $Perv_{\mathbf{G_O}}(\mathrm{Gr})$. The latter is defined as the category of perverse
sheaves on the affine (loop) Grassmannian Gr equivariant with respect to the group of
regular loops $\mathbf{G_O}$.

It turns out that convolution provides $Perv_{\mathbf{G_O}}(\mathrm{Gr})$ with a tensor structure, and the
celebrated geometric Satake isomorphism theorem establishes an equivalence between
$Perv_{\mathbf{G_O}}(\mathrm{Gr})$ and the category of representations of the Langlands dual group $\check{G}$.

By its very definition $Perv_{\mathbf{G_O}}(\mathrm{Gr})$ arises as the heart of the t-structure on a monoidal
triangulated category – the equivariant derived category $D_{\mathbf{G_O}}(\mathrm{Gr})$. It is a natural
question (raised, in particular, by V. Drinfeld) to describe $D_{\mathbf{G_O}}(\mathrm{Gr})$ in terms of the
dual group. Drinfeld has also noticed that at least some form of the answer[1] follows
from the results of V. Ginzburg's preprint [18]. In the present paper we reproduce
this description and extend it to a description of the *loop rotation equivariant* derived
Satake category $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$.

The description of $D_{\mathbf{G_O}}(\mathrm{Gr})$ links it to *conjugation equivariant coherent sheaves* on
the Langlands dual Lie algebra. The additional S^1 (or $\mathbb{G}_m$) equivariance is connected
to quantization of these to Harish-Chandra bimodules (see Theorem 1 for a precise
formulation).

The argument follows the strategy of [18]; it is based on another result of
Ginzburg [17], which reduces the question to computation of the global equivariant

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[1]We do not address Drinfeld's problem to find a more natural derivation of the description, making
compatibility with finer structures transparent. We understand that D. Gaitsgory and J. Lurie have
made a significant progress in this direction.

cohomology of IC sheaves as modules over the global equivariant cohomology algebra $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$. By an explicit calculation we show that $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ is related to the tensor square of the center of the enveloping of the dual Lie algebra $\check{\mathfrak{g}}$, while the global cohomology modules correspond to the bimodules, which describe twisting with a finite dimensional $\check{G}$-modules on the category of *Whittaker modules.* This allows us to relate $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ to Harish-Chandra bimodules, so that the global cohomology is identified with the Kostant-Whittaker reduction.

As an application we prove a conjecture of [11] which identifies the algebra of global equivariant homology of Gr equipped with the convolution algebra structure with the quantized Toda lattice (Theorem 3). Note that the quantized Toda lattice also appears in the apparently related computations by Givental, Kim and others of quantum D-module (quantum cohomology) of the flag variety of G, see e.g. [20].

D. Ben-Zvi and D. Nadler have informed us that they have a more conceptual proof of some of our results, see [8].

2. Notations and statements of the results

2.1. **Notations.** k is the algebraically closed characteristic zero coefficient field. Let G be a semisimple complex algebraic group, $\mathbf{G_O} = G(\mathbb{C}[[t]])$, $\mathbf{G_F} = G(\mathbb{C}((t)))$. The affine Grassmannian $\mathrm{Gr} = \mathrm{Gr}_G = \mathbf{G_F}/\mathbf{G_O}$ carries the category $Perv_{\mathbf{G_O}}(\mathrm{Gr})$ of $\mathbf{G_O}$-equivariant perverse constructible sheaves. It is equipped with the convolution monoidal structure, and is tensor equivalent to the tensor category $Rep(\check{G})$ of representations of the Langlands dual group $\check{G}$ over the field k (see [23], [6], [18]). We denote by $\tilde{S} : Rep(\check{G}) \to Perv_{\mathbf{G_O}}(\mathrm{Gr})$ the geometric Satake isomorphism functor, and by $S : Rep(\check{G}) \to Perv_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ its extension to the monoidal category of $\mathbf{G_O} \rtimes \mathbb{G}_m$-equivariant perverse constructible sheaves. The Lie algebra of $\check{G}$ is denoted by $\check{\mathfrak{g}}$. We choose a Cartan subalgebra $\check{\mathfrak{t}} \subset \check{\mathfrak{g}}$; the corresponding Cartan torus in $\check{G}$ is denoted $\check{T}$. We choose the opposite Borel subalgebras $\check{\mathfrak{b}}_\pm \supset \check{\mathfrak{t}}$ with nilpotent radicals $\check{\mathfrak{n}}_\pm$ and corresponding unipotent subgroups $\check{N}_\pm \subset \check{G}$. The Weyl group of $\check{G}$ is denoted by W.

Let $e, h, f \in \check{\mathfrak{g}}$ be a principal $\mathfrak{sl}_2$ triple such that $f \in \check{\mathfrak{n}}_-$, $e \in \check{\mathfrak{n}}_+$. We have the Kostant slice $e + \mathfrak{z}(f)$ to the principal nilpotent orbit. It is known that $e + \mathfrak{z}(f) \xrightarrow{\sim} \check{\mathfrak{g}}/Ad(\check{G})$, and also $e + \mathfrak{z}(f) \xrightarrow{\sim} (e + \check{\mathfrak{b}}_-)/\check{N}_-$; moreover the $\check{N}_-$ action on $e + \check{\mathfrak{b}}_-$ is free. Let Σ, Υ be the images of $e + \mathfrak{z}(f)$, $e + \check{\mathfrak{b}}_-$ under a $\check{G}$-invariant isomorphism $\check{\mathfrak{g}} \cong \check{\mathfrak{g}}^*$. Thus we have $\Sigma \xrightarrow{\sim} \Upsilon/\check{N}_- = \check{\mathfrak{t}}^*/W$ canonically.

The total space of the tangent bundle of $\check{\mathfrak{t}}^*/W$ is denoted $\mathbf{T}(\check{\mathfrak{t}}^*/W)$.

2.2. **Asymptotic $\check{\mathfrak{g}}$-modules.** Let $U = U(\check{\mathfrak{g}})$ be the enveloping algebra, and let $U_\hbar$ be the "graded enveloping" algebra, i.e. the graded $\mathsf{k}[\hbar]$-algebra generated by $\mathfrak{g}$ with relations $xy - yx = \hbar[x, y]$ for $x, y \in \check{\mathfrak{g}}$ (thus $U_\hbar$ is obtained from U by the standard Rees construction which produces a graded algebra from a filtered one); the adjoint action extends to the action of $\check{G}$ on $U_\hbar$. We define the category $\widetilde{\mathcal{HC}}_\hbar$ of "$\hbar$-Harish-Chandra bimodules" as follows: an object M of this category is a graded $U_\hbar^2 := U_\hbar \otimes_{\mathsf{k}[\hbar]} U_\hbar \simeq U_\hbar \otimes_\mathsf{k} U$-module equipped with an algebraic action ρ of $\check{G}$ such that: (1) the action map $U_\hbar \otimes_{\mathsf{k}[\hbar]} U_\hbar \otimes M \to M$ is $\check{G}$ equivariant, and (2) for $x \in \check{\mathfrak{g}}$ the action of $(x \otimes 1 + 1 \otimes x) \in U_\hbar \otimes_{\mathsf{k}[\hbar]} U_\hbar$ coincides with $\hbar \cdot d\rho(x)$. The functor of restriction from $U_\hbar \otimes U_\hbar$ to $U_\hbar \otimes 1$ is an equivalence between $\widetilde{\mathcal{HC}}_\hbar$ and the category of $\check{G}$-modules equipped with an equivariant $U_\hbar$-action; the same is true for the restriction to $1 \otimes U_\hbar$. We let $\mathcal{HC}_\hbar \subset \widetilde{\mathcal{HC}}_\hbar$ denote the full subcategory of objects which are finitely generated as $U_\hbar \otimes 1$ modules (equivalently, as $1 \otimes U_\hbar$ modules).

Notice that the full subcategory of $\mathcal{HC}_\hbar$ consisting of objects where $\hbar$ acts by zero is identified with the category $Coh^{\check{G}}(\check{\mathfrak{g}}^*)$ of coherent sheaves on $\check{\mathfrak{g}}^*$ equivariant under the coadjoint action; while for $s \in \mathsf{k}$, $s \neq 0$ the subcategory where $\hbar$ acts by s is identified with the category of Harish-Chandra bimodules.

2.3. **Kostant functor $\kappa_\hbar$.** We now proceed to define a functor $\kappa_\hbar : \mathcal{HC}_\hbar \to QCoh^{\mathbb{G}_m}((\check{\mathfrak{t}}^*/W)^2 \times \mathbb{A}^1)$.

Let $\psi : U_\hbar(\check{\mathfrak{n}}_-) \to \mathsf{k}[\hbar]$ be a homomorphism such that $\psi(f_\alpha) = 1$ for any simple root α, and a root generator $f_\alpha \in \check{\mathfrak{n}}_- \subset U_\hbar(\check{\mathfrak{n}}_-)$.

Define $U_\hbar^2(\check{\mathfrak{n}}_-) \subset U_\hbar^2$ by $U_\hbar^2(\check{\mathfrak{n}}_-) = U_\hbar(\check{\mathfrak{n}}_-) \otimes U(\check{\mathfrak{n}}_-)$. We extend ψ to a character $\psi_{(2)} : U_\hbar^2(\check{\mathfrak{n}}_-) = U_\hbar(\check{\mathfrak{n}}_-) \otimes U(\check{\mathfrak{n}}_-) \to \mathsf{k}[\hbar]$ trivial on the second multiple. Note that its restriction to the first copy of $U_\hbar$ is ψ, and its restriction to the second copy is $(-\psi)$.

We set $\kappa_\hbar(M) = (M \overset{\mathrm{L}}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-)_2} (-\psi))^{\check{N}_-}$ where the action of the second copy of $U_\hbar$ is used (though using the first one we get a canonically isomorphic functor). Clearly, $\kappa_\hbar(M)$ is equipped with the action of the Harish-Chandra center $Z(U_\hbar) \otimes_{\mathsf{k}[\hbar]} Z(U_\hbar) = \mathcal{O}((\check{\mathfrak{t}}^*/W) \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1)$, and with the grading (coming from the action of the Cartan element h of the principal $\mathfrak{sl}_2$), so we may view $\kappa_\hbar(M)$ as a $\mathbb{G}_m$-equivariant quasicoherent sheaf on $(\check{\mathfrak{t}}^*/W)^2 \times \mathbb{A}^1$.

If X is a scheme, and $Z \subset X$ is a closed subscheme let $N_X Z$ be the *deformation to the normal cone* to Z, see [15]. It is equipped with a morphism $N_X Z \to X \times \mathbb{A}^1$ (with coordinate $\hbar$ on $\mathbb{A}^1$), and is defined as the relative spectrum of the sheaf of subalgebras in $\mathcal{O}_X[\hbar^{\pm 1}]$ generated by the elements $f\hbar^{-1}$, $f \in \mathcal{O}_X : f|_Z = 0$.

If M is a free $\mathsf{k}[\hbar]$-module, the action of $\mathcal{O}((\check{\mathfrak{t}}^*/W)^2 \times \mathbb{A}^1)$ on $\kappa_\hbar(M)$ extends uniquely to the action of $\mathcal{O}(N_{(\check{\mathfrak{t}}^*/W)^2}\Delta)$ where $\Delta \subset (\check{\mathfrak{t}}^*/W)^2$ is the diagonal. So we can and will view $\kappa_\hbar(M)$ as a $\mathbb{G}_m$-equivariant coherent sheaf on $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta$.

For $V \in Rep(\check{G})$ we define the $\hbar$-Harish-Chandra bimodule $Fr(V)$ by $Fr(V) = U_\hbar \otimes V$ with its natural $\check{G}$-module structure $g(y \otimes v) = Ad(g)(y) \otimes g(v)$, and with the $U_\hbar \otimes U_\hbar$ action specified by $x \otimes u(y \otimes v) = xyu \otimes v + \hbar \cdot xy \otimes u(v)$ for $x, u \in \check{\mathfrak{g}} \subset U_\hbar$. In other words, $Fr(V)$ is obtained by applying the induction (left adjoint to the restriction functor) $Rep(\check{G}) \to \mathcal{HC}_\hbar$ to V. We set $\phi(V) := \kappa_\hbar(Fr(V))$.

Clearly $Fr(V)$ is a projective object of $\mathcal{HC}_\hbar$ for any $V \in Rep(\check{G})$; we call an object of the form $Fr(V)$ a free $\hbar$-Harish-Chandra bimodule. We define the full subcategory $\mathcal{HC}_\hbar^{fr} \subset \mathcal{HC}_\hbar$ to consist of all free objects.

2.4. **Equivariant cohomology of** Gr_G. Note that $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) = H^\bullet_{\mathbb{G}_m}(\mathbf{G_O} \backslash \mathbf{G_F}/\mathbf{G_O})$, whence two morphisms $pr_1^*, pr_2^* : \mathcal{O}(\mathfrak{t}/W) = H^\bullet_{\mathbf{G_O}}(pt) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$. We also have a morphism $pr^* : \mathsf{k}[\hbar] = H^\bullet_{\mathbb{G}_m}(pt) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$.

Theorem 1. *a) Assume G is simply connected. We have a canonical isomorphism $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \cong \mathcal{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)$ where $\Delta \subset (\check{\mathfrak{t}}^*/W)^2$ is the diagonal. Here the projection $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta \to \mathbb{A}^1$ corresponds to the homomorphism $H^\bullet_{\mathbb{G}_m}(pt) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$; and the two projections $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta \to \check{\mathfrak{t}}^*/W = \mathfrak{t}/W$ correspond to the two homomorphisms $H^\bullet_{\mathbf{G_O}}(pt) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$. The isomorphism is specified uniquely by these requirements.*

b) For arbitrary G we have a canonical isomorphism $H^*_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \cong \bigoplus_{\pi_1(G)} \mathcal{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)$.

Remark 1. To simplify the exposition we assume from now on that G is simply connected.

Cohomology of any complex of sheaves on a topological space carries an action of the cohomology algebra of the space; thus we have the functor of equivariant cohomology

$$H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m} : D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) - mod^{gr} = Coh^{\mathbb{G}_m}(N_{(\check{\mathfrak{t}}^*/W)^2}\Delta)$$

where $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ denotes the bounded constructible equivariant derived category, and the grading on $H^*_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}, ?)$ is the one by the cohomology degree.

Theorem 2. *The functor $S : Rep(\check{G}) \to Perv_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ extends to a full imbedding $S_\hbar : \mathcal{HC}_\hbar^{fr} \to D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$, such that*

$$(1) \qquad\qquad\qquad \kappa_\hbar \cong H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m} \circ S_\hbar.$$

Such an extension $S_\hbar$ (for a fixed isomorphism (1)) is unique.

2.5. **Equivariant homology and quantum Toda lattice.** Let $\mathcal{D}_\hbar(\check{G})$ stand for the sheaf of $\hbar$-differential operators on $\check{G}$: its global sections is the smash product of $U_\hbar$ and $\mathcal{O}(\check{G})$. The action of $\check{\mathfrak{n}}_-$ by the left-invariant (resp. right-invariant) vector fields on $\check{G}$ gives rise to the homomorphism l (resp. r): $U_\hbar(\check{\mathfrak{n}}_-) \to \mathcal{D}_\hbar(\check{G})$. Let $I_\psi \subset \mathcal{D}_\hbar(\check{G})$

be the left ideal generated by the $\hbar$-differential operators of the sort $\mathsf{l}(u_1) - \psi(u_1) + \mathsf{r}(u_2) + \psi(u_2)$; $u_1, u_2 \in U_\hbar(\check{\mathfrak{n}}_-)$. We consider the quantum hamiltonian reduction

$$(\mathcal{D}_\hbar(\check{G})/I_\psi)^{\check{N}_- \times \check{N}_-}$$

where the first (resp. second) copy of $\check{N}_-$ acts on $\check{G}$ (and hence on $\mathcal{D}_\hbar(\check{G})$) by the left (resp. right) translations: $(n_1, n_2) \circ g := n_1 g n_2^{-1}$. It is an algebra containing a commutative subalgebra $Z(U_\hbar)$ (via the embedding $Z(U_\hbar) \hookrightarrow \mathcal{D}_\hbar(\check{G})$ as both left- and right-invariant $\hbar$-differential operators). Note that the action of $\check{N}_- \times \check{N}_-$ on the "big Bruhat cell" $C_{w_0} := \check{N}_- \cdot \check{T} \cdot w_0 \cdot \check{N}_- \subset \check{G}$ is free, and hence the quantum hamiltonian reduction of $\mathcal{D}_\hbar(C_{w_0})$ is isomorphic to $\mathcal{D}_\hbar(\check{T})$. This is the classical Kazhdan-Kostant construction of the quantum Toda lattice, see [22]. Thus, the quantum Toda lattice is a certain localization of $(\mathcal{D}_\hbar(\check{G})/I_\psi)^{\check{N}_- \times \check{N}_-}$. In what follows we will call $(\mathcal{D}_\hbar(\check{G})/I_\psi)^{\check{N}_- \times \check{N}_-}$ "the quantized Toda lattice", somewhat abusing the language. The following result was conjectured in [11].

Theorem 3. *The convolution algebra of equivariant homology $H_\bullet^{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ is naturally isomorphic to the quantized Toda lattice $(\mathcal{D}_\hbar(\check{G})/I_\psi)^{\check{N}_- \times \check{N}_-}$. The embedding $Z(U_\hbar) \simeq H_{\mathbf{G_O} \rtimes \mathbb{G}_m}(pt) \hookrightarrow H_\bullet^{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ corresponds to the embedding $Z(U_\hbar) \hookrightarrow (\mathcal{D}_\hbar(\check{G})/I_\psi)^{\check{N}_- \times \check{N}_-}$.*

2.6. **Quasiclassical limit.** Recall that the fiber of $N_X Z$ over $0 \in \mathbb{A}^1$ is the normal cone to Z in X. In particular, the fiber of $N_{(\check{\mathfrak{t}}^*/W)^2}\Delta$ over $0 \in \mathbb{A}^1$ is the total space of the tangent bundle $\mathbf{T}(\check{\mathfrak{t}}^*/W)$. Thus, Theorem 1 implies the canonical isomorphism $H_{\mathbf{G_O}}^\bullet(\mathrm{Gr}) \simeq \mathcal{O}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$. On the other hand, $H_{\mathbf{G_O}}^\bullet(\mathrm{Gr})$ was computed by V. Ginzburg in [18] in terms of the universal centralizer bundle of $\check{\mathfrak{g}}$. The two computations are related as follows.

The variety $(\check{\mathfrak{g}}^*)^{reg}$ of regular elements in $\check{\mathfrak{g}}^*$ carries a sheaf of commutative Lie algebras $\mathfrak{z} \subset \check{\mathfrak{g}} \otimes \mathcal{O}$ whose fiber at a point $\xi \in (\check{\mathfrak{g}}^*)^{reg}$ is the stabilizer of ξ. We claim a canonical isomorphism $\mathfrak{z} \cong pr^*(\mathcal{T}^*)$ where $pr : (\check{\mathfrak{g}}^*)^{reg} \to \check{\mathfrak{t}}^*/W$ is the projection to the spectrum of invariant polynomials, and $\mathcal{T}^*$ stands for the cotangent sheaf. Indeed, the fiber of $pr^*(\mathcal{T}^*)$ at a point $\xi \in \check{\mathfrak{g}}^*$ is dual to the cokernel of the map $\check{\mathfrak{g}} \to \check{\mathfrak{g}}^*$, $x \mapsto coad(x)(\xi)$; thus it is canonically isomorphic to the kernel of the dual map (which happens to coincide with the original map), which is exactly the fiber of $\mathfrak{z}$ at ξ.

In view of this identification, one should compare Lemma 9 in subsection 4.7 below with Ginzburg's description in [18] of $\mathbf{G_O}$-equivariant Intersection Cohomology of a $\mathbf{G_O}$-orbit in Gr as a $\mathfrak{z}$-module.

We now proceed to define the *Kostant functor* $\kappa : Coh^{\check{G} \times \mathbb{G}_m}(\check{\mathfrak{g}}^*) \to Coh^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$, $\mathbb{G}_m$-equivariant coherent sheaves on the tangent bundle to $\check{\mathfrak{t}}^*/W$.

If $\mathcal{F} \in Coh^{\check{G}}(\check{\mathfrak{g}}^*)$ is equipped with an equivariant structure, then $\mathcal{F}|_{(\check{\mathfrak{g}}^*)^{reg}}$ carries an action of $\mathfrak{z}$; thus by the previous paragraph it defines a coherent sheaf on the total space of the pull-back of the tangent bundle under pr. Restricting this sheaf to the preimage of Σ we get a coherent sheaf on the tangent bundle to $\Sigma = \check{\mathfrak{t}}^*/W$ which we denote by $\overline{\kappa}(\mathcal{F})$. Notice that $\overline{\kappa}(\mathcal{F}) = (\mathcal{F}|_\Upsilon)^{\check{N}_-}$ (where we do not distinguish between a coherent sheaf on an affine variety and the module of its global sections). An obvious

modification of this definition yields a functor $\kappa : Coh^{\check{G} \times \mathbb{G}_m}(\check{\mathfrak{g}}^*) \to Coh^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$ (where the action of $\mathbb{G}_m$ on $\check{\mathfrak{t}}^*/W$ is the natural one).

Define the full subcategory $Coh_{fr}^{\check{G} \times \mathbb{G}_m}(\check{\mathfrak{g}}^*) \subset Coh^{\check{G} \times \mathbb{G}_m}(\check{\mathfrak{g}}^*)$ to consist of all objects of the form $V \otimes \mathcal{O}_{\check{\mathfrak{g}}^*}$, for $V \in Rep(\check{G} \times \mathbb{G}_m)$.

Recall that $\tilde{S} : Rep(\check{G}) \to Perv_{\mathbf{G_O}}(\mathrm{Gr})$ is the composition of S with the forgetful functor $Perv_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \to Perv_{\mathbf{G_O}}(\mathrm{Gr})$. We have the functor of $\mathbf{G_O}$-equivariant cohomology

$$H^\bullet_{\mathbf{G_O}} : D_{\mathbf{G_O}}(\mathrm{Gr}) \to H^\bullet_{\mathbf{G_O}}(\mathrm{Gr}) - mod^{gr} = Coh^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$$

Theorem 4. *The functor $\tilde{S}$ extends to a full imbedding $\tilde{S}_{qc} : Coh_{fr}^{\check{G} \times \mathbb{G}_m}(\check{\mathfrak{g}}^*) \to D_{\mathbf{G_O}}(\mathrm{Gr})$, such that there exists an isomorphism*

$$(2) \qquad\qquad\qquad \kappa \cong H^\bullet_{\mathbf{G_O}} \circ \tilde{S}_{qc}.$$

Such an extension $\tilde{S}_{qc}$ (for a fixed isomorphism (2)) is unique.

2.7. **Equivalences.** To a differential graded algebra A one can associate the triangulated category $D(A)$ of differential graded modules localized by quasi-isomorphism; and a full triangulated subcategory $D_{perf}(A) \subset D(A)$ of perfect complexes. Thus $D_{perf}(A)$ is the full subcategory in the latter category consisting of perfect complexes (i.e. generated by the free module under cones and direct summands). Given an algebraic group H acting on a dg-algebra A, we can consider equivariant dg-modules and localize them by quasi-isomorphisms, arriving at the equivariant version $D^H_{perf}(A)$.

We now consider the "differential-graded versions" $\mathrm{Sym}^{[]}(\check{\mathfrak{g}})$, $U_\hbar^{[]}$ of the graded algebras $\mathrm{Sym}(\check{\mathfrak{g}})$, $U_\hbar(\check{\mathfrak{g}})$. By definition $\mathrm{Sym}^{[]}(\check{\mathfrak{g}})$, $U_\hbar^{[]}$ are differential graded algebras with zero differential, which as algebras are isomorphic to $\mathrm{Sym}(\check{\mathfrak{g}})$, $U_\hbar(\check{\mathfrak{g}})$ respectively. The cohomological grading is defined so that elements of $\check{\mathfrak{g}}$ and $\hbar$ have degree two. Recall from section 2.2 that an asymptotic Harish-Chandra bimodule $M \in \mathcal{HC}_\hbar$ is nothing but a $\check{G}$-equivariant $U_\hbar^{[]}$-module. Using this identification we can transfer tensor product of asymptotic Harish-Chandra bimodules to a monoidal structure on the category of $\check{G}$-equivariant $U_\hbar^{[]}$-modules. It gives rise to a monoidal structure on $D^{\check{G}}_{perf}(U_\hbar^{[]})$. Similarly, we define a monoidal structure on $D^{\check{G}}_{perf}(\mathrm{Sym}^{[]}(\check{\mathfrak{g}}))$.

Theorem 5. *There exist canonical equivalences of monoidal triangulated categories $D^{\check{G}}_{perf}(U_\hbar^{[]}) \cong D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$, $D^{\check{G}}_{perf}(\mathrm{Sym}^{[]}(\check{\mathfrak{g}})) \cong D_{\mathbf{G_O}}(\mathrm{Gr})$.*

The following statement is an immediate consequence of the Theorem, which has the (psychological) advantage of bypassing the notion of a dg-algebra.

Corollary 1. *a) The derived graded Harish-Chandra bimodule category is Koszul dual to a graded version of the loop-rotation equivariant Satake category, i.e. (cf. [7], [2], [10]) there exists a functor $\Phi : D^b(\mathcal{HC}_\hbar) \to D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$, such that*
   i) $\Phi(M(1)) \cong \Phi(M)[1]$ in a natural way.
   ii) For M_1, $M_2 \in D^b(\mathcal{HC}_\hbar)$, Φ induces an isomorphism

$$\sum_{n,m} Hom(M_1, M_2(n)[m]) \cong \sum_k Hom(\Phi(M_1), \Phi(M_2)[k])$$

*iii) The image of Φ generates the target category as a triangulated category.
Moreover, Φ carries a natural monoidal structure, and $\Phi|_{\mathcal{HC}_\hbar^{fr}}$ coincides with $S_\hbar$.*

b) The derived category of graded $\check{G}$-equivariant $\mathrm{Sym}(\check{\mathfrak{g}})$ modules is Koszul dual to a graded version of the Satake category, i.e. there exists a functor $\Phi' : D^b(Coh^{\check{G}\times\mathbb{G}_m}(\check{\mathfrak{g}})) \to D_{\mathbf{G_O}}(\mathrm{Gr})$, satisfying the properties similar to those listed in part (a). This functor has a natural monoidal structure.

3. Topology

3.1. **Proof of Theorem 1.** a) First we construct the morphism α : $\mathcal{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta) \to H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$. Recall that $H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr}) = H^\bullet_{\mathbb{G}_m}(\mathbf{G_O}\backslash\mathbf{G_F}/\mathbf{G_O})$, whence two morphisms pr_1^*, pr_2^* : $\mathcal{O}(\mathfrak{t}/W) = H^\bullet_{\mathbf{G_O}}(pt) \to H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$. We also have a morphism pr^* : $\mathsf{k}[\hbar] = H^\bullet_{\mathbb{G}_m}(pt) \to H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$. Since $H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})|_{\hbar=0} = H^\bullet_{\mathbf{G_O}}(\mathrm{Gr})$, it follows that $pr_1^*|_{\hbar=0} = pr_2^*|_{\hbar=0}$. Hence the morphism (pr_1^*, pr_2^*, pr^*) : $\mathcal{O}(\mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1) \to H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$ factors through the desired morphism $\mathcal{O}(\mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1) \to \mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta) \xrightarrow{\alpha} H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$.

Next we prove that α is an embedding. It suffices to prove that the localized morphism

$$\alpha_{loc} : \mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta)\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)}\mathrm{Frac}(\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1)) \to H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)}\mathrm{Frac}(\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1))$$

is an embedding. Note that the RHS is the localized equivariant cohomology $H^\bullet_{T\times\mathbb{G}_m}(\mathrm{Gr})_{loc}$, which embeds into the inverse limit of the localized equivariant cohomology $H^\bullet_{T\times\mathbb{G}_m}(\mathrm{Gr}_\lambda)_{loc}$ of the $\mathbf{G_O}$-orbit closures $\mathrm{Gr}_\lambda \subset \mathrm{Gr}$. By the Localization Theorem for torus-equivariant cohomology, the latter is $\prod_\lambda \mathrm{Frac}(\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1))$ (cf. 3.2), and α_{loc} is an embedding.

Finally, it remains to check that the graded dimensions of $H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$, and of $\mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta)$ coincide. Here the grading of $\mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta)$ comes from the natural $\mathbb{G}_m$-actions on $\mathfrak{t}$ and $\mathbb{A}^1$. To this end note that the graded dimension of $H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$ coincides with that of $H^\bullet_{\mathbf{G_O}}(pt) \otimes H^\bullet_{\mathbb{G}_m}(pt) \otimes H^\bullet(\mathrm{Gr})$, that is $\mathsf{k}[x_1, \ldots, x_r, y_1, \ldots, y_r, \hbar]$. Here r is the rank of G; the degree of $\hbar$ is 2; the degrees of x's are twice the exponents of $\check{\mathfrak{g}}$; the degrees of y's are twice the exponents of $\check{\mathfrak{g}}$ minus 2.

Now to compute the graded dimension of $\mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta)$ we use that $\mathfrak{t}/W$ is isomorphic to the vector space Σ. More generally, for vector spaces V, V' we have an isomorphism $\beta : V \times V' \times \mathbb{A}^1 \xrightarrow{\sim} N_{V\times V'}V$. In effect, the map $\gamma : V \times V' \times \mathbb{A}^1 \to V \times V' \times \mathbb{A}^1$, $(v, v', a) \mapsto (v, av', a)$, factors through the desired isomorphism: $V \times V' \times \mathbb{A}^1 \xrightarrow{\beta} N_{V\times V'}V \to V \times V' \times \mathbb{A}^1$. We derive an isomorphism $N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta \cong \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1$ which lowers the $\mathbb{G}_m$-weights in the second copy of $\mathfrak{t}/W$ by 1, whence the desired formula for the graded dimension of $\mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta)$.

This completes the proof of the part a) of the theorem.

b) Let $\tilde{G}$ stand for the simply connected cover of G. Then the Grassmannian Gr_G is a union of connected components numbered by the characters $\chi \in \pi_1(G)$, and each connected component is isomorphic to $\mathrm{Gr}_{\tilde{G}}$. Moreover, the isomorphisms of various connected components are $T \times \mathbb{G}_m$-equivariant. Hence $H^\bullet_{T\times\mathbb{G}_m}(\mathrm{Gr}_G)$ is equal

to $\prod_{\pi_1(G)} H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_G)$. Finally, $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G) = H^\bullet_{G \times \mathbb{G}_m}(\mathrm{Gr}_G)$ coincides with the W-invariants in $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_G)$. $\square$

3.2. **Canonical filtration on** $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F})$**.** For a $\mathbf{G_O} \rtimes \mathbb{G}_m$-equivariant perverse sheaf $\mathcal{F}$ on Gr_G we will define a canonical filtration on $H^\bullet_{T \times \mathbb{G}_m}(\mathcal{F}) = H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathcal{F}) \otimes_{\mathcal{O}(\mathfrak{t}/W)} \mathcal{O}(\mathfrak{t})$. Note that if π is the projection $\mathfrak{t} \to \mathfrak{t}/W$, and $(\pi, \mathrm{Id}, \mathrm{Id})$ is the projection $\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1 \to \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1$, then $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathcal{F}) \otimes_{\mathcal{O}(\mathfrak{t}/W)} \mathcal{O}(\mathfrak{t}) = (\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathcal{F})$. For $\lambda \in X_*(T)$ we denote by λ the corresponding T-fixed point of Gr_G. We denote by $\mathfrak{T}_\lambda$ the semiinfinite $N_-(\mathbf{F})$-orbit through λ. We denote by $\overline{\mathfrak{T}}_\lambda$ the closure of $\mathfrak{T}_\lambda$, that is the union of $\mathfrak{T}_\mu$ over $\mu \geq \lambda$. We filter $H^\bullet_{T \times \mathbb{G}_m}(\mathcal{F})$ by the images of $r_\lambda : H^\bullet_{\overline{\mathfrak{T}}_\lambda, T \times \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}) \to H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F})$ (cohomology with supports). The associated graded of this filtration is $\bigoplus_\lambda H^\bullet_{\mathfrak{T}_\lambda, T \times \mathbb{G}_m}(\mathrm{Gr}_n \mathcal{F})$. Since λ is the only $T \times \mathbb{G}_m$-fixed point of $\mathfrak{T}_\lambda$, we have $H^\bullet_{\mathfrak{T}_\lambda, T \times \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}) = H^\bullet_{T \times \mathbb{G}_m}(\lambda) \otimes \mathrm{j}_\lambda^* \iota_\lambda^! \mathcal{F}$ where ι_λ is the locally closed embedding of $\mathfrak{T}_\lambda$ into Gr_G, and j_λ is the embedding of λ into $\mathfrak{T}_\lambda$.

Now recall that $\mathcal{F} \mapsto \mathrm{j}_\lambda^* \iota_\lambda^! \mathcal{F}$ is the λ-weight component of the Mirković-Vilonen fiber functor on the category of $\mathbf{G_O} \rtimes \mathbb{G}_m$-equivariant perverse sheaves on Gr_G. In other words, if V is a $\check{G}$-module, and $\mathcal{F} = S(V)$, then $\mathcal{F} \mapsto \mathrm{j}_\lambda^* \iota_\lambda^! \mathcal{F} = {}_\lambda V$ where ${}_\lambda V$ is the λ-weight component of V.

Furthermore, we claim that the $\mathcal{O}(\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\lambda)$ is canonically isomorphic to $(\mathrm{Id}, \pi, Id)_* \mathcal{O}(\Gamma_\lambda)$ where $\Gamma_\lambda \subset \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1$ is given by the equation $\Gamma_\lambda = \{(x_1, x_2, a) : x_2 = x_1 + a\lambda\}$. In effect, let p stand for the projection from the affine flag variety Fl_G to the affine Grassmannian Gr_G. Let $\tilde{\lambda}$ be a $T \times \mathbb{G}_m$-fixed point of Fl_G such that p projects $\tilde{\lambda}$ isomorphically onto λ. Let $\mathbf{I}$ stand for the Iwahori subgroup of $\mathbf{G_O}$. We have $H^\bullet_{T \times \mathbb{G}_m}(\mathsf{Fl}_G) = H^\bullet_{\mathbb{G}_m}(\mathbf{I} \backslash \mathbf{G_F} / \mathbf{I})$, and so $H^\bullet_{T \times \mathbb{G}_m}(\tilde{\lambda})$ is a module over $\mathcal{O}(\mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1)$. Clearly, the $\mathcal{O}(\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\lambda)$ is isomorphic to the direct image of the $\mathcal{O}(\mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\tilde{\lambda})$ under the projection $\mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1 \to \mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1$. So it suffices to check that the $\mathcal{O}(\mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\tilde{\lambda})$ is isomorphic to $\mathcal{O}(\Gamma_\lambda)$ after localization along $\mathfrak{t} \times \mathbb{A}^1$.

The set of T-fixed points in Fl_G is canonically identified with the extended affine Weyl group W_{aff} of G, and we choose $\tilde{\lambda}$ so that it coincides with $\lambda \in W_{aff}$. Then the preimage T_λ of $\tilde{\lambda} \in \mathsf{Fl}_G$ in $\mathbf{G_F}$ is homotopically equivalent to T, and the action of $T \times T \times \mathbb{G}_m$ on T_λ is homotopically equivalent to $(t_1, t_2, z)(t) = t_1 t t_2^{-1} \lambda(z)$. We conclude that the $\mathcal{O}(\mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\tilde{\lambda}) = H^\bullet_{\mathbb{G}_m}(T \backslash T_\lambda / T)$ is isomorphic to $\mathcal{O}(\Gamma_\lambda)$.

We have proved the following

Lemma 1. *For $V \in Rep(\check{G})$, the $\mathcal{O}(\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$ has a canonical filtration with the associated graded $\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes {}_\lambda V$. In particular, $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$ is flat as an $\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)$-module.*

$\square$

3.3. **Levi-equivariant cohomology.** Let $T \subset L \subset G$ be a Levi subgroup. We denote by P_L (resp. P_L^-) the parabolic subgroup generated by L and the positive (resp. negative) Borel subgroup B (resp. B_-). We denote by $W_L \subset W$ the Weyl group of

L. We denote by π_L the projection from $\mathfrak{t}/W_L$ to $\mathfrak{t}/W$. We denote by X_L^+ the set of highest weights of irreducible $\check{L}$-modules, where $\check{L} \subset \check{G}$ stands for the Langlands dual Levi subgroup. We have a natural projection from X_L^+ to the lattice $X^*(Z(\check{L}))$ of characters of the center $Z(\check{L})$ of $\check{L}$. The set of $P_L^-(\mathbf{F})$-orbits in Gr_G is numbered by $X^*(Z(\check{L}))$. For $\lambda \in X^*(Z(\check{L}))$ we will denote the corresponding orbit by $_L\mathfrak{T}_\lambda$, and its closure by $_L\overline{\mathfrak{T}}_\lambda$. The locally closed embedding of $\mathfrak{T}_\lambda$ into Gr_G is denoted by $\imath_\lambda$. For an $\check{L}$-module V we denote by $S_L(V)$ the corresponding $L(\mathbf{O}) \rtimes \mathbb{G}_m$-equivariant perverse sheaf on Gr_L.

Lemma 2. *For* $V \in Rep(\check{G})$*, the* $\mathcal{O}(\mathfrak{t}/W_L \times \mathfrak{t}/W \times \mathbb{A}^1)$*-module* $(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V))$ *carries a canonical filtration* $F_L^\bullet$ *such that the associated graded is equipped with a canonical isomorphism*

$$^{\mathrm{top}}\Xi_L : \ gr(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) \xrightarrow{\sim} (\mathrm{Id}, \pi_L, \mathrm{Id})_* H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, S_L(V|_{\check{L}})).$$

Proof : We have $(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) = H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) = H_{L \times \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V))$. The canonical filtration in question is filtration by the images of r_λ : $H_{_L\overline{\mathfrak{T}}_\lambda, L \times \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) \to H_{L \times \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V))$ (cohomology with supports; here $\lambda \in X^*(Z(\check{L}))$). The associated graded of this filtration is $\bigoplus_{\lambda \in X^*(Z(\check{L}))} H_{_L\mathfrak{T}_\lambda, L \times \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V))$. Let $\mathfrak{p}^\lambda$: $_L\mathfrak{T}_\lambda \to \mathrm{Gr}_L$ denote the natural $L(\mathbf{O}) \rtimes \mathbb{G}_m$-equivariant projection. Then we have $H_{_L\mathfrak{T}_\lambda, L \times \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) = H_{L \times \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, \mathfrak{p}_*^\lambda \imath_\lambda^! S(V))$. However, according to [6], we have a canonical isomorphism $\bigoplus_{\lambda \in X^*(Z(\check{L}))} \mathfrak{p}_*^\lambda \imath_\lambda^! S(V) = S_L(V|_{\check{L}})$. The lemma is proved. $\square$

3.4. **Transitivity for a pair of Levi subgroups.** We have a canonical isomorphism $^{\mathrm{top}}\Xi_L$: $gr(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) \xrightarrow{\sim} (\mathrm{Id}, \pi_L, \mathrm{Id})_* H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, S_L(V|_{\check{L}}))$. In the RHS we have the restriction of $\mathcal{O}(\mathfrak{t}/W_L \times \mathfrak{t}/W_L \times \mathbb{A}^1)$-module $H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, S_L(V|_{\check{L}}))$ to $\mathcal{O}(\mathfrak{t}/W_L \times \mathfrak{t}/W \times \mathbb{A}^1)$. To save a bit of notation in what follows we will write simply

$$^{\mathrm{top}}\Xi_L : \ gr(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) \xrightarrow{\sim} H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, S_L(V|_{\check{L}})).$$

If $\check{T} \subset \check{L}' \subset \check{L}$ is another Levi subgroup, then we denote by $\pi_{L'}^L$ the projection from $\mathfrak{t}/W_{L'}$ to $\mathfrak{t}/W_L$. Note that the filtration $F_{L'}^\bullet$ on $(\pi_{L'}, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) = (\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^*(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V))$ is a refinement of the filtration $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* F_L^\bullet$, and hence induces a canonical filtration $F_{L'}^{L\bullet}$ on $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* gr_{F_L^\bullet}(\pi_L, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V))$. The isomorphism $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* \ ^{\mathrm{top}}\Xi_L$ carries the filtration $F_{L'}^{L\bullet}$ to the filtration $F_{L'}^\bullet$ on $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, S_L(V|_{\check{L}}))$. We have a canonical isomorphism $^{\mathrm{top}}\Xi_{L'}^L$: $gr_{F_{L'}^\bullet}(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* H_{L(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_L, S_L(V|_{\check{L}})) \xrightarrow{\sim} H_{L'(\mathbf{O}) \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_{L'}, S_{L'}(V|_{\check{L}'}))$. We consider the composition

$$(3) \qquad gr_{F_{L'}^\bullet}(\pi_{L'}, \mathrm{Id}, \mathrm{Id})^* H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathrm{Gr}_G, S(V)) \xrightarrow{gr_{F_{L'}^{L\bullet}}(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* \ ^{\mathrm{top}}\Xi_L}$$

$$gr_{F^\bullet_{L'}}(\pi^L_{L'}, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{L(\mathbf{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_L, S_L(V|_{\check{L}})) \xrightarrow{\mathrm{top}\Xi^L_{L'}} H^\bullet_{L'(\mathbf{O}) \rtimes \mathbb{G}_m}(\mathrm{Gr}_{L'}, S_{L'}(V|_{\check{L'}}))$$

Then we have

Lemma 3. $\mathrm{top}\Xi^L_{L'} \circ gr_{F^{L\bullet}_{L'}}(\pi^L_{L'}, \mathrm{Id}, \mathrm{Id})^*\, \mathrm{top}\Xi_L =\, \mathrm{top}\Xi_{L'}.$

$\square$

3.5. **Tensor structure on equivariant cohomology.** A $\mathbf{G_O} \rtimes \mathbb{G}_m$-equivariant sheaf $\mathcal{F}$ on Gr_G will be viewed as a sheaf on the stack $\mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m$. Given two such sheaves $\mathcal{F}_1, \mathcal{F}_2$ we define $\mathcal{F}_1 \boxtimes_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathcal{F}_2$ as the descent of $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ from

$$(\mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m) \times (\mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m)$$

to $\mathbf{G_O} \rtimes \mathbb{G}_m \backslash (\mathbf{G_F} \rtimes \mathbb{G}_m \times_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathbf{G_F} \rtimes \mathbb{G}_m) / \mathbf{G_O} \rtimes \mathbb{G}_m$. Clearly,

$$H^\bullet(\mathbf{G_O} \rtimes \mathbb{G}_m \backslash (\mathbf{G_F} \rtimes \mathbb{G}_m \times_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathbf{G_F} \rtimes \mathbb{G}_m) / \mathbf{G_O} \rtimes \mathbb{G}_m, \mathcal{F}_1 \boxtimes_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathcal{F}_2) =$$

$$H^\bullet(\mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m, \mathcal{F}_1) \otimes_{H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(pt)} H^\bullet(\mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m, \mathcal{F}_2) =$$

$$H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_1) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_2) =: H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_1) \star H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_2).$$

The multiplication in $\mathbf{G_F} \rtimes \mathbb{G}_m$ gives rise to the map

$$m: \mathbf{G_O} \rtimes \mathbb{G}_m \backslash (\mathbf{G_F} \rtimes \mathbb{G}_m \times_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathbf{G_F} \rtimes \mathbb{G}_m) / \mathbf{G_O} \rtimes \mathbb{G}_m \to \mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m.$$

The convolution $\mathcal{F}_1 * \mathcal{F}_2$ is defined as $m_*(\mathcal{F}_1 \boxtimes_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathcal{F}_2)$. Hence

$$H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_1 * \mathcal{F}_2) = H^\bullet(\mathbf{G_O} \rtimes \mathbb{G}_m \backslash \mathbf{G_F} \rtimes \mathbb{G}_m / \mathbf{G_O} \rtimes \mathbb{G}_m, \mathcal{F}_1 * \mathcal{F}_2) =$$

$$H^\bullet(\mathbf{G_O} \rtimes \mathbb{G}_m \backslash (\mathbf{G_F} \rtimes \mathbb{G}_m \times_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathbf{G_F} \rtimes \mathbb{G}_m) / \mathbf{G_O} \rtimes \mathbb{G}_m, \mathcal{F}_1 \boxtimes_{\mathbf{G_O} \rtimes \mathbb{G}_m} \mathcal{F}_2) =$$

$$H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_1) \star H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, \mathcal{F}_2).$$

Now for $V_1, V_2 \in Rep(\check{G})$, and $\mathcal{F}_1 = S(V_1)$, $\mathcal{F}_2 = S(V_2)$ we have a canonical isomorphism $S(V_1 \otimes V_2) \xrightarrow{\sim} S(V_1) * S(V_2)$, and thus

$$\mathrm{top}\omega_{V_1, V_2}: H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_1 \otimes V_2)) \xrightarrow{\sim} H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_1)) \star H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_2)).$$

According to Lemma 1 (cf. also Lemma 2), we have a canonical isomorphism

$$\mathrm{top}\Xi_V =\, \mathrm{top}\Xi_{T,V}: gr(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \xrightarrow{\sim} (\mathrm{Id}, \pi, \mathrm{Id})_* H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_T, S_T(V|_{\check{T}})).$$

In the RHS we have the restriction of $\mathcal{O}(\mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1)$-module $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_T, S_T(V|_{\check{T}}))$ to $\mathcal{O}(\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1)$. To save a bit of notation in what follows we will write simply $\mathrm{top}\Xi_V: gr(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \xrightarrow{\sim} H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_T, S_T(V|_{\check{T}}))$. It follows that after tensoring with $\mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$ (over the first and third factors in $\mathcal{O}(\mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1)$) we have a canonical isomorphism

$$\mathrm{top}\Xi^{\mathrm{gen}}_V: H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) =$$

$$= gr(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_T, S_T(V|_{\check{T}})) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) = \bigoplus_\lambda \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)\right) \otimes {}_\lambda V$$

Now we have a canonical isomorphism $\mathcal{O}(\Gamma_\mu) \star \mathcal{O}(\Gamma_\nu) := \mathcal{O}(\Gamma_\mu) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathcal{O}(\Gamma_\nu) = \mathcal{O}(\Gamma_{\mu+\nu})$. Hence we get a canonical isomorphism

$$\mathrm{top}\Xi^{\mathrm{gen}}_{V_1} \star \mathrm{top}\Xi^{\mathrm{gen}}_{V_2}: (H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_1)) \star H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_2))) \otimes_{\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) \xrightarrow{\sim}$$

$$\overset{\sim}{\longrightarrow} \bigoplus_{\mu+\nu=\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1)} \mathsf{k}(\mathfrak{t}\times\mathbb{A}^1)\right) \otimes\ {}_\mu V_1 \otimes\ {}_\nu V_2 =$$

$$= \bigoplus_{\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1)} \mathsf{k}(\mathfrak{t}\times\mathbb{A}^1)\right) \otimes\ {}_\lambda(V_1 \otimes V_2)$$

We want to compare it with ${}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_1\otimes V_2}$:

$$H^\bullet_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr}_G, S(V_1\otimes V_2))\otimes_{\mathcal{O}(\mathfrak{t}/W\times\mathbb{A}^1)}\mathsf{k}(\mathfrak{t}\times\mathbb{A}^1)\overset{\sim}{\longrightarrow}\bigoplus_{\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\mathfrak{t}\times\mathbb{A}^1)} \mathsf{k}(\mathfrak{t}\times\mathbb{A}^1)\right)\otimes {}_\lambda(V_1\otimes V_2)$$

Proposition 1. ${}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_1\otimes V_2} = ({}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_1} \star {}^{\mathrm{top}}\Xi^{\mathrm{gen}}_{V_2}) \circ {}^{\mathrm{top}}\omega_{V_1,V_2}.$

Proof : The equality readily reduces to the following compatibility. Let $\Phi_{MV} = \bigoplus_\lambda \Phi^\lambda_{MV} : \mathcal{F} \mapsto \bigoplus_\lambda \mathsf{j}^*_\lambda\imath^!_\lambda\mathcal{F}$ be the Mirković-Vilonen fiber functor on the Satake category $Perv_{\mathbf{G_O}}(\mathrm{Gr})$ (notations of 3.2, see [24]). The (proof of) Lemma 1 provides a canonical isomorphism

$$(4) \qquad\qquad gr(H^\bullet_{T\times\mathbb{G}_m}(Gr_G,\mathcal{F})) \cong \bigoplus \Phi^\lambda_{MV} \otimes \mathcal{O}(\Gamma_\lambda).$$

We have to check that this isomorphism is compatible with the tensor structure, i.e. for $\mathcal{F}, \mathcal{G} \in Perv_{\mathbf{G_O}}(\mathrm{Gr})$ we have to check coincidence of the two embeddings from $\Phi_{MV}(\mathcal{F}) \otimes \Phi_{MV}(\mathcal{G})$ in $gr(H^\bullet_{T\times\mathbb{G}_m}(Gr_G,\mathcal{F}*\mathcal{G}))$, where the first one comes from the isomorphisms (4) for $\mathcal{F}$, $\mathcal{G}$ and tensor structure on the functor $gr(H^\bullet_{T\times\mathbb{G}_m})$, and the second one comes from the tensor structure on the functor Φ_{MV} and isomorphism (4) for $\mathcal{F}*\mathcal{G}$.

To check the equality, we recall a "filtration" in the I-equivariant derived category on a $\mathbf{G_O}$ equivariant perverse sheaf $\mathcal{F}$, which induces the above filtration on $H^\bullet_{T\times\mathbb{G}_m}(Gr_G,\mathcal{F})$. Here and below by a "filtration" on an object X of a triangulated category we mean a collection of object $X_0 = 0,\dots,X_n = X$ and distinguished triangles $X_i \to X_{i+1} \to Y_i$; the objects Y_i will be called the "subquotients" of the filtration.

Let $\mathcal{C}, \mathcal{D}$ be the equivariant constructible derived category with respect to the natural action of $\mathbf{I}\rtimes\mathbb{G}_m$ on Gr_G and on Fl_G respectively. Thus convolution $*_I$ provides $\mathcal{D}$ with a monoidal structure, and $\mathcal{C}$ with an action of the monoidal category $\mathcal{D}$.

Recall the *Wakimoto sheaves* $J_\lambda \in \mathcal{D}$, characterized by the following properties: $J_{\lambda+\mu} \cong J_\lambda *_I J_\mu$, while for a dominant weight $\lambda \in \Lambda^+$ the sheaf J_λ is the $*$-extension of the constant perverse sheaf from the Iwahori orbit corresponding to λ, see, e.g. [1].

Recall that p stands for the projection $\mathsf{Fl}_G \to \mathrm{Gr}_G$. We set $J^{\mathrm{Gr}}_\lambda = p_*(J_\lambda)$. It is not hard to show that for $\mathcal{F} \in D_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$, $J^{\mathrm{Gr}}_\lambda * \mathcal{F} \cong J_\lambda *_I \mathcal{F}$ canonically. Also J^{Gr}_λ can be characterized by $\mathsf{j}^*_\mu\imath^!_\mu J^{\mathrm{Gr}}_\lambda = \mathsf{k}^{\delta_{\lambda,\mu}}$ (cf. [1]).

Fix $\mathcal{F} \in Perv_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr})$, and choose a coweight λ deep inside the dominant chamber. Then one shows that $j^!_\nu(J_\lambda * \mathcal{F}) \cong \mathsf{k}[\dim Gr_\nu] \otimes \mathsf{j}^*_{\nu-\lambda}\imath^!_{\nu-\lambda}\mathcal{F}$ for all ν. Thus one can consider the Cousin "filtration" on $J_\lambda * \mathcal{F}$ with subquotients $j_{\nu*}j^!_\nu(J_\lambda * \mathcal{F})$ and apply the functor $J_{-\lambda}*_I$ to it, thereby obtaining a "filtration" on $\mathcal{F}$ with "subquotients" $\Phi^\mu_{MV}(\mathcal{F}) \otimes J^{\mathrm{Gr}}_\mu$. It is clear that this "filtration" induces the above filtration on $H_{T\times\mathbb{G}_m}(\mathcal{F})$.

Let now $\mathcal{F}, \mathcal{G}$ be a pair of objects of $Perv_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$. The above "filtration" on $\mathcal{F}$ induces a "filtration" on $\mathcal{F} * \mathcal{G}$ with "subquotients"

$$\Phi_{MV}^{\mu}(\mathcal{F}) \otimes J_{\mu}^{\mathrm{Gr}} * \mathcal{G} = \Phi_{MV}^{\mu}(\mathcal{F}) \otimes J_{\mu} *_I \mathcal{G}.$$

Using the "filtration" on $\mathcal{G}$ with "subquotients" $\Phi_{MV}^{\nu}(\mathcal{G}) \otimes J_{\nu}^{\mathrm{Gr}}$ we get a "filtration" on $\mathcal{F} * \mathcal{G}$ with "subquotients"

$$\Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \otimes J_{\mu} *_I J_{\nu}^{\mathrm{Gr}} = \Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \otimes J_{\mu+\nu}^{\mathrm{Gr}}.$$

Comparing it with the "filtration" on $\mathcal{F} * \mathcal{G}$ with "subquotients" $\Phi_{MV}^{\eta}(\mathcal{F} * \mathcal{G})$ we get an isomorphism $\Phi_{MV}(\mathcal{F} * \mathcal{G}) \cong \Phi_{MV}(\mathcal{F}) \otimes \Phi_{MV}(\mathcal{G})$. It is not hard to see that this isomorphism coincides with any of the standard definitions of tensor structure on Φ_{MV}; in fact, a close description of the tensor structure appears in [12], Theorem 3.2.8.

Now we see that the isomorphism

$$gr\, H_{T \times \mathbb{G}_m}^{\bullet}(\mathcal{F} * \mathcal{G}) \cong gr\, H_{T \times \mathbb{G}_m}^{\bullet}(\mathcal{F}) \star \, gr\, H_{T \times \mathbb{G}_m}^{\bullet}(\mathcal{G})$$

breaks as a direct sum of maps

$$\left(\Phi_{MV}^{\mu}(\mathcal{F}) \otimes H_{T \times \mathbb{G}_m}^{\bullet}(J_{\mu}^{\mathrm{Gr}})\right) \star \left(\Phi_{MV}^{\nu}(\mathcal{G}) \otimes H_{T \times \mathbb{G}_m}^{\bullet}(J_{\nu}^{\mathrm{Gr}})\right) \to \Phi_{MV}^{\mu+\nu}(\mathcal{F} * \mathcal{G}) \otimes H_{T \times \mathbb{G}_m}^{\bullet}(J_{\mu+\nu}^{\mathrm{Gr}}),$$

coming from the map $\Phi_{MV}^{\mu}(\mathcal{F}) \otimes \Phi_{MV}^{\nu}(\mathcal{G}) \to \Phi_{MV}^{\mu+\nu}(\mathcal{F} * \mathcal{G})$ induced by the tensor structure on Φ_{MV}, and the natural isomorphism

$$\mathcal{O}(\Gamma_{\mu}) \star \mathcal{O}(\Gamma_{\nu}) = H_{T \times \mathbb{G}_m}^{\bullet}(J_{\mu}^{\mathrm{Gr}}) \star H_{T \times \mathbb{G}_m}^{\bullet}(J_{\nu}^{\mathrm{Gr}}) \cong H_{T \times \mathbb{G}_m}^{\bullet}(J_{\mu+\nu}^{\mathrm{Gr}}) = \mathcal{O}(\Gamma_{\mu+\nu}).$$

The claim follows. $\square$

4. Algebra

4.1. **Some properties of the Kostant functor $\kappa_{\hbar}$.** The following properties of the Kostant functor will play an important role in the proof of the main results.

Lemma 4. *a) The functors κ, $\kappa_{\hbar}$ are exact.*
b) The functors $\kappa|_{Coh_{fr}^{\check{G} \times \mathbb{G}_m}(\check{\mathfrak{g}}^*)}$, $\kappa_{\hbar}|_{\mathcal{HC}_{\hbar}^{fr}}$ are full embeddings.

Proof is essentially due to B. Kostant, cf. [21].

a) For the exactness of κ, note that the functor of restriction $\mathcal{F} \mapsto \mathcal{F}|_{\Upsilon}$ is exact on $Coh^{\check{G}}(\check{\mathfrak{g}}^*)$, and then $\mathcal{F}|_{\Upsilon}$ is an $\check{N}_-$-equivariant coherent sheaf on Υ. Recall that $\check{N}_-$ acts on Υ freely, and $\Upsilon/\check{N}_- \simeq \Sigma$. Hence the functor of invariants $\mathcal{G} \mapsto \mathcal{G}^{\check{N}_-}$ is exact on $Coh^{\check{N}_-}(\Upsilon)$. The exactness of κ follows.

For the exactness of $\kappa_{\hbar}$, we will prove that both the functors of $-\psi$-coinvariants, and $\check{N}_-$-invariants are exact, and hence $\kappa_{\hbar}$ is exact as their composition. It is enough to check it on the positively graded $\hbar$-Harish-Chandra bimodules. Then it is enough to check the exactness on the subcategory of $\hbar$-Harish-Chandra bimodules with grading degrees between 0 and n for a fixed $n \gg 0$. Thus it suffices to check the exactness on the subcategory of $\hbar$-Harish-Chandra bimodules with nilpotent action of $\hbar$, and then it suffices to consider the subcategory of $\hbar$-Harish-Chandra bimodules with the *trivial* action of $\hbar$. However, an $\hbar$-Harish-Chandra bimodule M with the trivial action of $\hbar$ is nothing else than a $\check{G}$-equivariant coherent sheaf on $\check{\mathfrak{g}}^*$, and $M \overset{\mathrm{L}}{\otimes}_{U_{\hbar}(\check{\mathfrak{n}}_-)_2}(-\psi) = M|_{\Upsilon}$. In

particular, the functor of $-\psi$-coinvariants is exact according to the previous paragraph. For the same reason, $M \mapsto (M \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-)_2} (-\psi))^{\check{N}_-} = (M|_\Upsilon)^{\check{N}_-}$ is exact. This completes the proof of a).

b) $\kappa|_{Coh^{\check{G}\times\mathbb{G}m}_{fr}(\check{\mathfrak{g}}^*)}$ is fully faithful since the complement to $(\check{\mathfrak{g}}^*)^{reg}$ in $\check{\mathfrak{g}}^*$ has codimension 2, and the centralizer of a generic regular element is connected.

To prove that $\kappa_\hbar|_{\mathcal{HC}^{fr}_\hbar}$ is fully faithful, we consider free $\hbar$-Harish-Chandra bimodules $M_1 = U_\hbar \otimes V_1$, $M_2 = U_\hbar \otimes V_2$, and the following commutative diagram:

(5)

$$
\begin{array}{ccccc}
\mathrm{Hom}(M_1, M_2) & \overset{\epsilon}{\longrightarrow} & \mathrm{Hom}(\kappa_\hbar M_1, \kappa_\hbar M_2) & \overset{\delta}{\longrightarrow} & \mathrm{Hom}(\kappa_\hbar M_1, \kappa_\hbar M_2)/\hbar \\
\downarrow{\scriptstyle\beta} & & & & \downarrow{\scriptstyle\gamma} \\
\end{array}
$$
$$
\mathrm{Hom}(M_1/\hbar, M_2/\hbar) \overset{\alpha}{\longrightarrow} \mathrm{Hom}(\kappa(M_1/\hbar), \kappa(M_2/\hbar)) \overset{\sim}{\longrightarrow} \mathrm{Hom}((\kappa_\hbar M_1)/\hbar, (\kappa_\hbar M_2)/\hbar)
$$

We have just proved that α is an isomorphism. Moreover, β is surjective since $\mathrm{Hom}(U_\hbar \otimes V_1, U_\hbar \otimes V_2) = \mathrm{Hom}_{\check{G}}(V_1 \otimes V_2^*, U_\hbar)$, and all the $\check{G}$-modules in question are semisimple. It follows that γ is surjective. On the other hand, γ is injective since $\kappa_\hbar M_1$, $\kappa_\hbar M_2$ are free over $\mathsf{k}[\hbar]$. Now that γ is proved to be an isomorphism, the composition $\delta \circ \epsilon$ must be surjective. Hence ϵ is surjective by Nakayama Lemma. It remains to prove that ϵ is injective. Since $\kappa_\hbar$ is exact, it is enough to prove that $\kappa_\hbar M \neq 0$ for a nonzero subobject M of a free $\hbar$-Harish-Chandra bimodule M_2. We consider a nonzero subobject $M/\hbar \subset M_2/\hbar$ of a free $\mathcal{O}(\check{\mathfrak{g}}^*)$-module $M_2/\hbar$. It suffices to prove that $\kappa M \neq 0$. However, the support of any nonzero section of a free $\mathcal{O}(\check{\mathfrak{g}}^*)$-module is the whole of $\check{\mathfrak{g}}^*$, hence its restriction to Υ is nonzero.

The lemma is proved. $\square$

4.2. **De-symmetrized Kostant functor $\varkappa_\hbar$.** We denote by π the projection $\check{\mathfrak{t}}^* \to \check{\mathfrak{t}}^*/W$, and we denote by $(\pi, \mathrm{Id}, \mathrm{Id})$ the projection $\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1 \to \check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1$. For $V \in Rep(\check{G})$ we are going to describe $(\pi, \mathrm{Id}, \mathrm{Id})^*\phi(V) \in Coh(\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1)$.

To this end we consider the universal Verma module $\mathcal{M}_\hbar(-\rho) = U_\hbar \otimes_{U_\hbar(\check{\mathfrak{b}})} \mathsf{k}[\hbar][\check{\mathfrak{t}}](-\rho)$, and $\mathsf{k}[\hbar][\check{\mathfrak{t}}](-\rho)$ is a $U_\hbar(\check{\mathfrak{b}})$-module which factors through the $U_\hbar(\check{\mathfrak{t}}) = \mathsf{k}[\hbar][\check{\mathfrak{t}}]$-module where $t \in \check{\mathfrak{t}}$ acts by multiplication by $t - \hbar\rho(t)$ (recall that ρ is the halfsum of positive roots of $\check{\mathfrak{g}}$). For an $\hbar$-Harish-Chandra bimodule M we set $\varkappa_\hbar(M) := \mathcal{M}_\hbar(-\rho) \overset{L}{\otimes}_{U_\hbar} M \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-)_2} \psi = \mathsf{k}[\hbar][\check{\mathfrak{t}}](-\rho) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{b}})_1} M \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-)_2} \psi$. This is an $\mathcal{O}(\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1)$-module: the action of $\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) = U_\hbar(\check{\mathfrak{t}})$ comes from the fact that $U_\hbar(\check{\mathfrak{t}})$ normalizes $U_\hbar(\check{\mathfrak{b}})$, and the action of $\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$ is the action of the center $Z(U_\hbar)$ of the second copy of $U_\hbar$, as before.

For $V \in Rep(\check{G})$ we set $\varphi(V) := \varkappa_\hbar(Fr(V))$.

Lemma 5. *For $V \in Rep(\check{G})$ we have a canonical isomorphism $\varphi(V) \simeq (\pi, \mathrm{Id}, \mathrm{Id})^*\phi(V)$.*

Proof : We denote by $\mathcal{W}^-_\hbar := U_\hbar \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$ the Whittaker $U_\hbar$-module. By a theorem of Kostant, $\mathrm{End}_{U_\hbar}(\mathcal{W}^-_\hbar) = Z(U_\hbar)$, and the category $\mathcal{A}$ of $(U_\hbar(\check{\mathfrak{n}}_-), \psi)$-integrable $U_\hbar$-modules is equivalent to the category of $Z(U_\hbar)$-modules (here a $U_\hbar(\check{\mathfrak{n}}_-)$-module

L is called $(U_\hbar(\check{\mathfrak{n}}_-), \psi)$-integrable if the action of $\check{\mathfrak{n}}_-$ on $L \otimes (-\psi)$ is locally nilpotent). Namely, a $(U_\hbar(\check{\mathfrak{n}}_-), \psi)$-integrable $U_\hbar$-module L goes to the $Z(U_\hbar)$-module $Hom_{U_\hbar}(\mathcal{W}_\hbar^-, L)$. Conversely, a $Z(U_\hbar)$-module R goes to $\mathcal{W}_\hbar^- \otimes_{Z(U_\hbar)} R$. In particular, $\mathcal{W}_\hbar^-$ goes to the free module $Z(U_\hbar)$.

For an $\hbar$-Harish-Chandra bimodule M we will construct a canonical isomorphism $\kappa_\hbar(M) \simeq Hom_{U_\hbar}(\mathcal{W}_\hbar^-, M \otimes_{U_\hbar} \mathcal{W}_\hbar^-)$. In effect, $L \mapsto M \otimes_{U_\hbar} L$ is a right-exact endofunctor of the category of $(U_\hbar(\check{\mathfrak{n}}_-), \psi)$-integrable $U_\hbar$-modules. Under Kostant's equivalence, this endofunctor goes to the convolution with the $Z(U_\hbar)$-bimodule X which corresponds by Kostant to our endofunctor applied to $\mathcal{W}_\hbar^-$. In other words, $X = Hom_{U_\hbar}(\mathcal{W}_\hbar^-, M \otimes_{U_\hbar} \mathcal{W}_\hbar^-)$. We have a tautological isomorphism $X \otimes_{Z(U_\hbar)} \mathcal{W}_\hbar^- \overset{\sim}{\longrightarrow} M \otimes_{U_\hbar} \mathcal{W}_\hbar^-$. This yields the desired isomorphism $X \overset{\sim}{\longrightarrow} \kappa_\hbar(M)$. In particular, for a free $\hbar$-Harish-Chandra bimodule $M = Fr(V)$, we obtain $\phi(V) \otimes_{Z(U_\hbar)} \mathcal{W}_\hbar^- \overset{\sim}{\longrightarrow} V \otimes_{\mathsf{k}} \mathcal{W}_\hbar^-$.

Now let us compute $(\mathcal{M}_\hbar(-\rho) \otimes V \otimes \mathcal{W}_\hbar^-) \otimes_{U_\hbar} \mathsf{k}[\hbar] = (\mathcal{M}_\hbar(-\rho) \otimes (V \otimes \mathcal{W}_\hbar^-)) \otimes_{U_\hbar} \mathsf{k}[\hbar] \overset{\sim}{\longrightarrow} (\mathcal{M}_\hbar(-\rho) \otimes (\mathcal{W}_\hbar^- \otimes_{Z(U_\hbar)} \phi(V))) \otimes_{U_\hbar} \mathsf{k}[\hbar] \overset{\sim}{\longrightarrow} (\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V)$. The last isomorphism arises from $(\mathcal{M}_\hbar(-\rho) \otimes \mathcal{W}_\hbar^-) \otimes_{U_\hbar} \mathsf{k}[\hbar] \overset{\sim}{\longrightarrow} U_\hbar(\check{\mathfrak{t}}) = \mathcal{O}(\check{\mathfrak{t}}^*)$, since $\mathcal{M}_\hbar(-\rho) = U_\hbar \otimes_{U_\hbar(\check{\mathfrak{b}})} \mathsf{k}[\hbar][\check{\mathfrak{t}}](-\rho)$, and $\mathcal{W}_\hbar^- = U_\hbar \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$.

On the other hand, $(\mathcal{M}_\hbar(-\rho) \otimes V \otimes \mathcal{W}_\hbar^-) \otimes_{U_\hbar} \mathsf{k}[\hbar] = ((\mathcal{M}_\hbar(-\rho) \otimes V) \otimes \mathcal{W}_\hbar^-) \otimes_{U_\hbar} \mathsf{k}[\hbar] \overset{\sim}{\longrightarrow} (\mathcal{M}_\hbar(-\rho) \otimes V) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-)_2} \psi \overset{\sim}{\longrightarrow} \mathsf{k}[\hbar][\check{\mathfrak{t}}](-\rho) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{b}})_1} (U_\hbar \otimes V) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-)_2} \psi = \varphi(V)$.

This completes the proof of the lemma. $\square$

4.3. **Canonical filtration on** $\varphi(V)$. For $V \in Rep(\check{G})$ we have $\varphi(V) = (\mathcal{M}_\hbar(-\rho) \otimes V) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$. Note that $\mathcal{M}_\hbar(-\rho) \otimes V$ has a canonical filtration with associated graded $\bigoplus_\lambda \mathcal{M}_\hbar(\lambda - \rho) \otimes {}_\lambda V$, where λ is a weight of $\check{\mathfrak{t}}$, and ${}_\lambda V$ is the corresponding weight space of V; furthermore, $\mathcal{M}_\hbar(\lambda - \rho) = U_\hbar \otimes_{U_\hbar(\check{\mathfrak{b}})} \mathsf{k}[\hbar][\check{\mathfrak{t}}](\lambda - \rho)$, and $\mathsf{k}[\hbar][\check{\mathfrak{t}}](\lambda - \rho)$ is a $U_\hbar(\check{\mathfrak{b}})$-module which factors through the $U_\hbar(\check{\mathfrak{t}}) = \mathsf{k}[\hbar][\check{\mathfrak{t}}]$-module where $t \in \check{\mathfrak{t}}$ acts by multiplication by $t + \hbar\lambda(t) - \hbar\rho(t)$.

It follows that $\varphi(V)$ has a canonical filtration with associated graded $\bigoplus_\lambda (\mathcal{M}_\hbar(\lambda - \rho) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi) \otimes {}_\lambda V$. Note that $\mathcal{M}_\hbar(\lambda - \rho) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$ is a $\mathcal{O}(\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1)$-module since $\mathcal{M}_\hbar(\lambda - \rho)$ is a $U_\hbar(\check{\mathfrak{t}}) - Z(U_\hbar)$-bimodule. To describe $\mathcal{M}_\hbar(\lambda - \rho) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$ as a coherent sheaf on $\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1$, we denote by $(\mathrm{Id}, \pi, \mathrm{Id})$ the projection from $\check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \times \mathbb{A}^1$ to $\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1$, and we denote by $\Gamma_\lambda \subset \check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \times \mathbb{A}^1$ the subscheme defined by the equations $\Gamma_\lambda = \{(t_1, t_2, a) : t_2 = t_1 + a\lambda\}$. Then $\mathcal{M}_\hbar(\lambda - \rho) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi = (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda)$.

We have proved the following

Lemma 6. *For $V \in Rep(\check{G})$, the $\mathcal{O}(\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1)$-module $\varphi(V)$ has a canonical filtration with associated graded $\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes {}_\lambda V$. In particular, $\varphi(V)$ is flat as an $\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$-module.*

$\square$

4.4. **Whittaker modules for Levi subalgebras.** Let $\check{T} \subset \check{L} \subset \check{G}$ be a Levi subgroup with the Lie algebra $\check{\mathfrak{t}} \subset \check{\mathfrak{l}} \subset \check{\mathfrak{g}}$. We denote by $\check{\mathfrak{p}}_L$ (resp. $\check{\mathfrak{p}}_L^-$) the parabolic subalgebra generated by $\check{\mathfrak{l}}$ and the positive (resp. negative) Borel subalgebra $\check{\mathfrak{b}}$ (resp. $\check{\mathfrak{b}}_-$). We denote by π_L the projection from $\check{\mathfrak{t}}^*/W_L$ to $\check{\mathfrak{t}}^*/W$.

Lemma 7. *For $V \in Rep(\check{G})$, the $\mathcal{O}(\check{\mathfrak{t}}^*/W_L \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$-module $(\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V)$ carries a canonical filtration $F_L^\bullet$ such that the associated graded is equipped with a canonical isomorphism $\,^{\mathrm{alg}}\Xi_L : \, gr(\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V) \xrightarrow{\sim} (\mathrm{Id}, \pi_L, \mathrm{Id})_* \phi_L(V|_{\check{L}})$.*

Proof : We have $\check{\mathfrak{l}} = [\check{\mathfrak{l}}, \check{\mathfrak{l}}] \oplus \mathfrak{z}_{\check{\mathfrak{l}}}$ where $\mathfrak{z}_{\check{\mathfrak{l}}}$ stands for the center of $\check{\mathfrak{l}}$. We consider the nilpotent subalgebra $\check{\mathfrak{n}}_-^L = \check{\mathfrak{l}} \cap \check{\mathfrak{n}}_-$, and a nondegenerate homomorphism $\psi_L : U_\hbar(\check{\mathfrak{n}}_-^L) \to \mathsf{k}[\hbar]$ such that $\psi(f_\alpha) = 1$ for any simple root α of $\check{\mathfrak{l}}$. We define the Whittaker $U_\hbar([\check{\mathfrak{l}}, \check{\mathfrak{l}}])$-module $\mathcal{W}_L^-$ as $U_\hbar([\check{\mathfrak{l}}, \check{\mathfrak{l}}]) \otimes_{U_\hbar(\check{\mathfrak{n}}_-^L)} \psi_L$. We define a free $U_\hbar(\mathfrak{z}_{\check{\mathfrak{l}}}) = \mathsf{k}[\hbar][\mathfrak{z}_{\check{\mathfrak{l}}}]$-module $\mathfrak{z}_{\check{\mathfrak{l}}}(-\rho + \rho_L)$ as $\mathsf{k}[\hbar][\mathfrak{z}_{\check{\mathfrak{l}}}]$ where $t \in \mathfrak{z}_{\check{\mathfrak{l}}}$ acts by multiplication by $t - \hbar(\rho - \rho_L)(t)$ (here ρ_L is the halfsum of positive roots of $\check{\mathfrak{l}}$). We define a $U_\hbar(\check{\mathfrak{l}})$-module $\mathcal{W}_L^-(-\rho + \rho_L)$ as $\mathcal{W}_L^- \otimes_{\mathsf{k}[\hbar]} \mathfrak{z}_{\check{\mathfrak{l}}}(-\rho + \rho_L)$. The projection $\check{\mathfrak{p}}_L \to \check{\mathfrak{l}}$ gives rise to the homomorphism $U_\hbar(\check{\mathfrak{p}}_L) \to U_\hbar(\check{\mathfrak{l}})$, and thus we can consider $\mathcal{W}_L^-(-\rho + \rho_L)$ as a $U_\hbar(\check{\mathfrak{p}}_L)$-module. Finally, we define the *Verma-Whittaker $U_\hbar$*-module $\mathcal{MW}_L(-\rho + \rho_L)$ as $U_\hbar \otimes_{U_\hbar(\check{\mathfrak{p}}_L)} \mathcal{W}_L^-(-\rho + \rho_L)$. Note that the center $Z(U_\hbar(\check{\mathfrak{l}})) = \mathcal{O}(\check{\mathfrak{t}}^*/W_L \times \mathbb{A}^1)$ acts by endomorphisms of $\mathcal{W}_L^-(-\rho + \rho_L)$, and hence of $\mathcal{MW}_L(-\rho + \rho_L)$.

We claim that for $V \in Rep(\check{G})$, we have a canonical isomorphism

$$(\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V)[n_L] \cong (\mathcal{MW}_L(-\rho + \rho_L) \otimes V \otimes \mathcal{W}_{\bar{\hbar}}^-) \overset{L}{\otimes}_{U_\hbar} \mathsf{k}[\hbar]$$

(the LHS is homologically shifted to the degree $-n_L$, that is negative dimension of $\check{\mathfrak{n}}_-^L$). In effect, arguing like in the proof of Lemma 5, we only have to check that $(\mathcal{MW}_L(-\rho + \rho_L) \otimes \mathcal{W}_{\bar{\hbar}}^-) \overset{L}{\otimes}_{U_\hbar} \mathsf{k}[\hbar] \xrightarrow{\sim} Z(U_\hbar(\check{\mathfrak{l}}))[n_L] = \mathcal{O}(\check{\mathfrak{t}}^*/W_L \times \mathbb{A}^1)[n_L]$. To this end we note that $(\mathcal{MW}_L(-\rho + \rho_L) \otimes \mathcal{W}_{\bar{\hbar}}^-) \overset{L}{\otimes}_{U_\hbar} \mathsf{k}[\hbar] \xrightarrow{\sim} \mathcal{W}_L^-(-\rho + \rho_L) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{p}}_L)} (\mathcal{W}_{\bar{\hbar}}^-|_{U_\hbar(\check{\mathfrak{p}}_L)}) = \mathcal{W}_L^-(-\rho + \rho_L) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{p}}_L)} (U_\hbar(\check{\mathfrak{p}}_L) \otimes_{U_\hbar(\check{\mathfrak{n}}_-^L)} \psi_L) \xrightarrow{\sim} \mathcal{W}_L^-(-\rho + \rho_L) \overset{L}{\otimes}_{U_\hbar(\check{\mathfrak{n}}_-^L)} \psi_L \xrightarrow{\sim} Z(U_\hbar(\check{\mathfrak{l}}))[n_L]$.

Moreover, it follows that for an $\check{L}$-module W we have a canonical isomorphism of $\mathcal{O}(\mathfrak{t}/W_L \times \mathfrak{t}/W \times \mathbb{A}^1)$-modules $([U_\hbar \otimes_{U_\hbar(\check{\mathfrak{p}}_L)} (\mathcal{W}_L^-(-\rho + \rho_L) \otimes W)] \otimes \mathcal{W}_{\bar{\hbar}}^-) \overset{L}{\otimes}_{U_\hbar} \mathsf{k}[\hbar] \xrightarrow{\sim} (\mathrm{Id}, \pi_L, \mathrm{Id})_* \phi_L(W)$. Now it remains to notice that for a $\check{G}$-module V the $U_\hbar$-module $\mathcal{MW}_L(-\rho + \rho_L) \otimes V$ has a canonical filtration with associated graded $U_\hbar \otimes_{U_\hbar(\check{\mathfrak{p}}_L)} (\mathcal{W}_L^-(-\rho + \rho_L) \otimes V|_{\check{L}})$. This completes the proof of the lemma. $\square$

4.5. **Transitivity for a pair of Levi subgroups.** We have a canonical isomorphism $\,^{\mathrm{alg}}\Xi_L : \, gr(\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V) \xrightarrow{\sim} (\mathrm{Id}, \pi_L, \mathrm{Id})_* \phi_L(V|_{\check{L}})$. In the RHS we have the restriction of $\mathcal{O}(\check{\mathfrak{t}}^*/W_L \times \check{\mathfrak{t}}^*/W_L \times \mathbb{A}^1)$-module $\phi_L(V|_{\check{L}})$ to $\mathcal{O}(\check{\mathfrak{t}}^*/W_L \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$. To save a bit of notation in what follows we will write simply $\,^{\mathrm{alg}}\Xi_L : \, gr(\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V) \xrightarrow{\sim} \phi_L(V|_{\check{L}})$.

If $\check{T} \subset \check{L}' \subset \check{L}$ is another Levi subgroup, then we denote by $\pi_{L'}^L$ the projection from $\check{\mathfrak{t}}^*/W_{L'}$ to $\check{\mathfrak{t}}^*/W_L$. Note that the filtration $F_{L'}^\bullet$ on $(\pi_{L'}, \mathrm{Id}, \mathrm{Id})^* \phi(V) = (\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* (\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V)$ is a refinement of the filtration $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* F_L^\bullet$, and hence induces a canonical filtration $F_{L'}^{L\bullet}$ on $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* gr_{F_L^\bullet}(\pi_L, \mathrm{Id}, \mathrm{Id})^* \phi(V)$. The isomorphism $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* \,^{\mathrm{alg}}\Xi_L$ carries the filtration $F_{L'}^{L\bullet}$ to the filtration $F_{L'}^\bullet$ on $(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* \phi_L(V|_{\check{L}})$. We have a canonical isomorphism $\,^{\mathrm{alg}}\Xi_{L'}^L : \, gr_{F_{L'}^\bullet}(\pi_{L'}^L, \mathrm{Id}, \mathrm{Id})^* \phi_L(V|_{\check{L}}) \xrightarrow{\sim} \phi_{L'}(V|_{\check{L}'})$. We consider the

composition
(6)

$$gr_{F^{\bullet}_{L'}}(\pi_{L'}, \mathrm{Id}, \mathrm{Id})^* \phi(V) \xrightarrow{gr_{F^{L\bullet}_{L'}}(\pi^L_{L'}, \mathrm{Id}, \mathrm{Id})^* {}^{\mathrm{alg}}\Xi_L} gr_{F^{\bullet}_{L'}}(\pi^L_{L'}, \mathrm{Id}, \mathrm{Id})^* \phi_L(V|_{\check{L}}) \xrightarrow{{}^{\mathrm{alg}}\Xi^L_{L'}} \phi_{L'}(V|_{\check{L}'})$$

Then we have

Lemma 8. ${}^{\mathrm{alg}}\Xi^L_{L'} \circ gr_{F^{L\bullet}_{L'}}(\pi^L_{L'}, \mathrm{Id}, \mathrm{Id})^* {}^{\mathrm{alg}}\Xi_L = {}^{\mathrm{alg}}\Xi_{L'}$.

$\square$

4.6. **Tensor structure on Kostant functor.** Recall that according to the proof of Lemma 5, for $V \in Rep(\check{G})$, we have a canonical isomorphism $\phi(V) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathcal{W}^-_{\hbar} \xrightarrow{\sim} V \otimes \mathcal{W}^-_{\hbar}$. Thus, for $V_1, V_2 \in Rep(\check{G})$, we have $\phi(V_1) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \phi(V_2) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathcal{W}^-_{\hbar} \xrightarrow{\sim} \phi(V_1) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} (V_2 \otimes \mathcal{W}^-_{\hbar}) \xrightarrow{\sim} (V_1 \otimes V_2) \otimes \mathcal{W}^-_{\hbar} \xleftarrow{\sim} \phi(V_1 \otimes V_2) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathcal{W}^-_{\hbar}$. Composing (and inverting) these isomorphisms we obtain $\phi(V_1 \otimes V_2) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathcal{W}^-_{\hbar} \xrightarrow{\sim} \phi(V_1) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \phi(V_2) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathcal{W}^-_{\hbar}$, and thus

$${}^{\mathrm{alg}}\omega_{V_1, V_2} : \phi(V_1 \otimes V_2) \xrightarrow{\sim} \phi(V_1) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \phi(V_2) =: \phi(V_1) \star \phi(V_2).$$

According to Lemma 6 (cf. also Lemma 7), we have a canonical isomorphism ${}^{\mathrm{alg}}\Xi_V = {}^{\mathrm{alg}}\Xi_{T,V} : gr(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \xrightarrow{\sim} (\mathrm{Id}, \pi, \mathrm{Id})_* \phi_T(V|_{\check{T}})$. In the RHS we have the restriction of $\mathcal{O}(\check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \times \mathbb{A}^1)$-module $\phi_T(V|_{\check{T}})$ to $\mathcal{O}(\check{\mathfrak{t}}^* \times (\check{\mathfrak{t}}^*/W) \times \mathbb{A}^1)$. To save a bit of notation in what follows we will write simply ${}^{\mathrm{alg}}\Xi_V : gr(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \xrightarrow{\sim} \phi_T(V|_{\check{T}})$. It follows that after tensoring with $\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$ (over the first and third factors in $\mathcal{O}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$) we have a canonical isomorphism

$${}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_V : \phi(V) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) = gr(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \xrightarrow{\sim}$$

$$\xrightarrow{\sim} \phi_T(V|_{\check{T}}) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) = \bigoplus_{\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \right) \otimes {}_\lambda V$$

Now we have a canonical isomorphism $\mathcal{O}(\Gamma_\mu) \star \mathcal{O}(\Gamma_\nu) := \mathcal{O}(\Gamma_\mu) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathcal{O}(\Gamma_\nu) = \mathcal{O}(\Gamma_{\mu+\nu})$. Hence we get a canonical isomorphism ${}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_1} \star {}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_2}$:

$$(\phi(V_1) \star \phi(V_2)) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \xrightarrow{\sim} \bigoplus_{\mu+\nu=\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \right) \otimes {}_\mu V_1 \otimes {}_\nu V_2 =$$

$$= \bigoplus_{\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \right) \otimes {}_\lambda(V_1 \otimes V_2)$$

We want to compare it with

$${}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_1 \otimes V_2} : \phi(V_1 \otimes V_2) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \xrightarrow{\sim} \bigoplus_{\lambda} \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \right) \otimes {}_\lambda(V_1 \otimes V_2)$$

Proposition 2. ${}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_1 \otimes V_2} = ({}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_1} \star {}^{\mathrm{alg}}\Xi^{\mathrm{gen}}_{V_2}) \circ {}^{\mathrm{alg}}\omega_{V_1, V_2}$.

Proof : We consider the generic universal Verma module $\mathcal{M}_\hbar^{\mathrm{gen}}(-\rho) = U_\hbar \otimes_{U_\hbar(\check{\mathfrak{b}})} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)(-\rho)$, and $\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)(-\rho)$ is a $U_\hbar(\check{\mathfrak{b}})$-module which factors through the $U_\hbar(\check{\mathfrak{t}}) = \mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$-module where $t \in \check{\mathfrak{t}}$ acts by multiplication by $t - \hbar\rho(t)$. It is well known that $\mathrm{End}_{U_\hbar}(\mathcal{M}_\hbar^{\mathrm{gen}}(-\rho)) = \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$, and the category $\mathcal{B}$ of $U_\hbar(\check{\mathfrak{n}})$-integrable $U_\hbar \otimes \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$-modules is semisimple, and any simple object is isomorphic to $\mathcal{M}_\hbar^{\mathrm{gen}}(-\rho)$. In particular, $\mathcal{B}$ is equivalent to the category of $\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$-modules.

For $V \in Rep(\check{G})$ we put $\varphi^{\mathrm{gen}}(V) := \varphi(V) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) = gr\varphi(V) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)$. This is the restriction of a $\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \otimes \mathcal{O}(\check{\mathfrak{t}}^*)$-module to $\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \otimes \mathcal{O}(\check{\mathfrak{t}}^*/W)$, but we will view it as a $\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \otimes \mathcal{O}(\check{\mathfrak{t}}^*)$-module.

Arguing like in the proof of Lemma 5, we obtain a canonical isomorphism $\varphi^{\mathrm{gen}}(V) \otimes_{\mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathcal{M}_\hbar^{\mathrm{gen}}(-\rho) \xrightarrow{\sim} V \otimes \mathcal{M}_\hbar^{\mathrm{gen}}(-\rho)$. This gives rise to the tensor structure on the functor φ^{gen}:

$$\varphi^{\mathrm{gen}}(V_1 \otimes V_2) \xrightarrow{\sim} \varphi^{\mathrm{gen}}(V_1) \star \varphi^{\mathrm{gen}}(V_2).$$

Clearly, the identification $^{\mathrm{alg}}\Xi_V^{\mathrm{gen}} : \varphi^{\mathrm{gen}}(V) \xrightarrow{\sim} \bigoplus_\lambda \left(\mathcal{O}(\Gamma_\lambda) \otimes_{\mathcal{O}(\check{\mathfrak{t}}^* \times \mathbb{A}^1)} \mathsf{k}(\check{\mathfrak{t}}^* \times \mathbb{A}^1) \right) \otimes {}_\lambda V$ commutes with the obvious tensor structure in the RHS.

On the other hand, arguing like in the proof of Lemma 5, we obtain a canonical isomorphism $\varphi^{\mathrm{gen}}(V) \xrightarrow{\sim} (\mathcal{M}_\hbar^{\mathrm{gen}}(-\rho) \otimes V \otimes \mathcal{W}_\hbar^-) \otimes_{U_\hbar} \mathsf{k}[\hbar]$ which implies that the tensor structures on ϕ and φ^{gen} are compatible as well. This completes the proof of the proposition. $\square$

4.7. **Quasiclassical limit of** $\phi(V)$**.** For $V \in Rep(\check{G})$, Lemma 6 implies that the $\mathcal{O}(\check{\mathfrak{t}}^*/W)$-bimodule $\phi(V)$ is supported at the diagonal $\Delta \subset \check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W$. It follows that the action of $\mathcal{O}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$ on $\phi(V)$ actually extends to the action of $\mathcal{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)$. As we know from 2.6, $\mathcal{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)/\hbar \simeq \mathcal{O}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$ is the universal centralizer.

Lemma 9. *For $V \in Rep(\check{G})$, the $\mathcal{O}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$-module $\phi(V)|_{\hbar=0}$ is canonically isomorphic to the module $\mathcal{O}(\Sigma) \otimes V$ over the universal centralizer.*

Proof : Consider the $\mathcal{O}(\check{\mathfrak{g}}^*)^{\check{G}}$-module $Pol(\check{\mathfrak{g}}^*, \check{\mathfrak{g}})^{\check{G}}$ of $\check{G}$-invariant polynomial maps from $\check{\mathfrak{g}}^*$ to $\check{\mathfrak{g}}$; this is a vector bundle over $Spec\mathcal{O}(\check{\mathfrak{g}}^*)^{\check{G}} = \Sigma$. Given a polynomial $P \in \mathcal{O}(\check{\mathfrak{g}}^*)^{\check{G}}$, its differential dP defines a section of this vector bundle. For a central element $z \in Z(U(\check{\mathfrak{g}})) = \mathcal{O}(\check{\mathfrak{g}}^*)^{\check{G}}$ we denote the corresponding section by σ_z. If z runs through a set of generators of $Z(U(\check{\mathfrak{g}}))$, the corresponding sections σ_z form a basis of the universal centralizer bundle, and identify it with the cotangent bundle $\mathbf{T}^*(\Sigma)$.

Thus it suffices to check the following statement about the free $\hbar$-Harish-Chandra bimodule $U_\hbar \otimes V$. Let $z^{(1)}$ (resp. $z^{(2)}$) stand for the left (resp. right) action of z in $U_\hbar \otimes V$. Then the action of $\frac{z^{(1)} - z^{(2)}}{\hbar}|_{\hbar=0}$ on $(U_\hbar \otimes V)|_{\hbar=0} = \mathcal{O}(\check{\mathfrak{g}}^*) \otimes V$ coincides with the action of $\sigma_z \in \mathcal{O}(\check{\mathfrak{g}}^*) \otimes \check{\mathfrak{g}}$.

In effect, if $v \in V$, and $z = \sum_{\underline{i}} a_{\underline{i}} x_{i_1} \ldots x_{i_k}$ where $x_{i_l} \in \check{\mathfrak{g}}$, and $\tilde{y} \in U_\hbar$ is a lift of $y \in \mathcal{O}(\check{\mathfrak{g}}^*) = U_\hbar|_{\hbar=0}$, then $\frac{z(\tilde{y} \otimes v) - (\tilde{y} \otimes v)z}{\hbar}|_{\hbar=0} = \sum_{i_l \in \underline{i}} a_{\underline{i}} x_{i_1} \ldots \widehat{x_{i_l}} \ldots x_{i_k} y \otimes x_{i_l}(v) = \sigma_z(y \otimes v)$. The lemma is proved. $\square$

5. Rank 1

5.1. **Equivariant cohomology for $G = PGL(2)$.** Let us describe $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$ in the case $G = PGL(2)$. Then $\check{G}$ is isomorphic to $SL(2)$. For a positive integer n let V_n be the $n+1$-dimensional irreducible $SL(2)$-module. Let Gr_n be the closure of the n-dimensional $\mathbf{G_O}$-orbit in Gr_G. It is known that Gr_n is rationally smooth, so we are interested in $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_n)$ as a module over $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$. The non-equivariant cohomology $H^\bullet(\mathrm{Gr}_n)$ is a cyclic $H^\bullet(\mathrm{Gr})$-module (see e.g. [19]). Hence, by graded Nakayama lemma, $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_n)$ is a cyclic module over $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$. Recall that $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \cong \mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta) \oplus \mathcal{O}(N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta)$, and so $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_n)$ is the structure sheaf of a subscheme $A_n \subset N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta$ of a copy of $N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta$ specified by the parity of n.

Now we describe the subscheme A_n. Let $P_n = \{n\omega, (n-2)\omega, \ldots, (2-n)\omega, -n\omega\}$ be the set of weights of $\check{G}$-module V_n. We have $P_n \subset \mathfrak{t} = \check{\mathfrak{t}}^*$. For $i = -n, -n+2, \ldots, n-2, n$, let $\Gamma_i \subset \mathfrak{t} \times \mathfrak{t} \times \mathbb{A}^1$ be a subscheme defined by the equations $\Gamma_i = \{(x_1, x_2, a) : x_2 = x_1 + ia\omega\}$. Let $\Gamma(n)$ stand for the subscheme defined by the product of the above equations (over $i = -n, -n+2, \ldots, n-2, n$). Recall that π stands for the projection $\mathfrak{t} \to \mathfrak{t}/W$, and consider the subscheme $(\pi, \pi, \mathrm{Id})(\Gamma(n)) \subset \mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1$. Finally, we can formulate

Lemma 10. A_n *is the proper preimage of* $(\pi, \pi, \mathrm{Id})(\Gamma(n))$ *in* $N_{\mathfrak{t}/W \times \mathfrak{t}/W}\Delta$.

Proof : Since $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}$ is a flat $\mathcal{O}(\mathfrak{t}/W \times \mathbb{A}^1)$-module, it suffices to identify A_n with $(\pi, \pi, \mathrm{Id})(\Gamma(n))$ generically over $\mathfrak{t}/W \times \mathbb{A}^1$, or else to identify $(\pi, \mathrm{Id}, \mathrm{Id})^{-1}(A_n)$ with $(\mathrm{Id}, \pi, \mathrm{Id})(\Gamma(n))$ generically over $\mathfrak{t} \times \mathbb{A}^1$. This was done in Lemma 1. $\square$

5.2. **Generic splitting of the canonical filtration on equivariant cohomology for $G = PGL(2)$.** Recall the canonical filtration on $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_n)$ (see 3.2). We will compare the identification (see Lemma 1) of the associated graded with $\bigoplus_{i=-n}^{n}(\mathrm{Id}, \pi, \mathrm{Id})_*\mathcal{O}(\Gamma_i) \otimes\ {}_iV_n$ with the identification (see Lemma 10) $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_n) \cong \mathcal{O}\left((\pi, \mathrm{Id}, \mathrm{Id})^{-1}(A_n)\right)$. To this end we recall some basic facts about the cohomology of Gr_n. For $i = -n, -n+2, \ldots, n-2, n$, let $v_i \in H^{i+n}(\mathrm{Gr}_n)$ stand for the (Poincaré dual of the) fundamental class of $\mathrm{Gr}_n \cap \overline{\mathfrak{T}}_i$ (an irreducible subvariety of Gr_n of dimension $\frac{n-i}{2}$). The action of $e, h, f \in \mathfrak{sl}_2$ on $H^\bullet(\mathrm{Gr}_n)$ in this basis is given by

$$hv_i = iv_i, \ ev_{i-2} = \frac{n+i}{2}v_i, \ fv_{i+2} = \frac{n-i}{2}v_i$$

(recall that e is defined as the multiplication by the first Chern class of the determinant line bundle).

The canonical filtration $0 = F^{n+2} \subset F^n \subset F^{n-2} \subset \ldots \subset F^{2-n} \subset F^{-n} = H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_n)$ is given by $F^i = \mathrm{Im}\left(r_i : H^\bullet_{\overline{\mathfrak{T}}_i, T \times \mathbb{G}_m}(\mathrm{Gr}_n) \to H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_n)\right)$. On the other hand, the proof of Lemma 10 shows that $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}_n)$ is generated by the (Poincaré dual of the) fundamental class $\tilde{v}_{-n}$ of $\mathrm{Gr}_n = \mathrm{Gr}_n \cap \overline{\mathfrak{T}}_{-n}$. Recall that j_i stands for the embedding of the T-fixed point i into $\mathfrak{T}_i$, while $\imath_i$ stands for the locally closed embedding of $\mathfrak{T}_i$ into Gr. Let us denote by $\bar{\imath}_i$ the closed embedding of $\overline{\mathfrak{T}}_i$ into Gr. The image of $\bar{\imath}_i^*\tilde{v}_{-n}$ in the nonequivariant cohomology $H^\bullet(\mathrm{Gr}_n \cap \overline{\mathfrak{T}}_i)$ is the fundamental

class v_i of $\mathrm{Gr}_n \cap \overline{\mathfrak{T}}_i$. We can further restrict it to $\mathrm{Gr}_n \cap \mathfrak{T}_i$, and then to i to get $\mathrm{j}_i^* \imath_i^* \tilde{v}_{-n}$. To compare it with $\mathrm{j}_i^* \imath_i^! \tilde{v}_{-n}$ we consider a transversal slice $\mathrm{Gr}_n \cap \mathfrak{S}_i$ to $\mathrm{Gr}_n \cap \overline{\mathfrak{T}}_i$ in Gr_n where $\mathfrak{S}_i$ is the $N(\mathbf{F})$-orbit through the point i.

It is known that $\mathrm{Gr}_n \cap \mathfrak{S}_i$ is isomorphic to a vector space $\mathbb{A}^{\frac{n+i}{2}}$ with the origin at i, and the action of $T \times \mathbb{G}_m$ is linear with weights $x + (i-1)\hbar, x + (i-2)\hbar, \ldots, x + \frac{i-n}{2}\hbar$. It follows that $\imath_i^! \tilde{v}_{-n} = (x + (i-1)\hbar)(x + (i-2)\hbar)\ldots(x + \frac{i-n}{2}\hbar)\imath_i^* \tilde{v}_{-n}$. We conclude that the generator $\tilde{v}_i$ of F^i whose class in the nonequivariant cohomology $H^\bullet(\mathrm{Gr}_n)$ is equal to v_i is given by

$$(7) \qquad \tilde{v}_i = (x + (i-1)\hbar)(x + (i-2)\hbar)\ldots(x + \frac{i-n}{2}\hbar)\tilde{v}_{-n}$$

5.3. Kostant functor for $\check{G} = SL(2)$.
Now we consider the group $\check{G} = SL(2)$ with the Lie algebra $\check{\mathfrak{g}} = \mathfrak{sl}_2$ and Cartan subalgebra $\check{\mathfrak{t}} \subset \check{\mathfrak{g}}$.

Lemma 11. *The $\mathcal{O}(\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$-module $\phi(V_n) = \kappa_\hbar(U_\hbar \otimes V_n)$ is isomorphic to $\mathcal{O}(A_n)$.*

Proof: According to Lemma 9, the restriction of $\phi(V_n)$ to $\hbar = 0$ is isomorphic to the $\mathcal{O}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$-module $V_n \otimes \mathcal{O}(\check{\mathfrak{t}}^*/W)$, that is V_n viewed as a module over the universal centralizer. Further restricting it to $0 \in \check{\mathfrak{t}}^*/W$ we obtain V_n viewed as a module over the centralizer of the regular nilpotent $e \in \mathfrak{sl}_2$. Clearly, V_n is a cyclic $\mathsf{k}[e]$-module. By the graded Nakayama Lemma, $\phi(V_n)$ is a cyclic $\mathcal{O}(N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta)$-module as well, hence $\phi(V_n)$ is isomorphic to the structure sheaf of a subscheme $B_n \subset N_{\check{\mathfrak{t}}^*/W \times \check{\mathfrak{t}}^*/W}\Delta$. We have to check $B_n = A_n$.

According to Lemmas 5, 6, $\phi(V_n)$ is a flat $\mathcal{O}(\check{\mathfrak{t}}^*/W \times \mathbb{A}^1)$-module, so it suffices to identify B_n and A_n generically over $\check{\mathfrak{t}}^*/W \times \mathbb{A}^1$. Moreover, it suffices to identify $(\pi, \pi, \mathrm{Id})^{-1}(B_n)$ with $\Gamma(n) \subset \check{\mathfrak{t}}^* \times \check{\mathfrak{t}}^* \times \mathbb{A}^1$. This was done in Lemma 6. This completes the proof of the lemma. $\square$

5.4. Generic splitting of the canonical filtration on Kostant functor for $\check{G} = SL(2)$.
Recall the canonical filtration on $\varphi(V_n)$ (see 4.3). We will compare the identification (see Lemma 6) of the associated graded with $\bigoplus_{i=-n}^{n}(Id, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_i) \otimes {}_iV_n$ with the identification (see Lemma 11) $\varphi(V_n) \cong \mathcal{O}\left((\pi, \mathrm{Id}, \mathrm{Id})^{-1}(A_n)\right)$. To this end we recall some basic facts about $U_\hbar(\mathfrak{sl}_2)$-modules. First, V_n is a free $\mathsf{k}[\hbar]$-module with a basis $\{v_n, v_{n-2}, \ldots, v_{2-n}, v_{-n}\}$. The action of $e, h, f \in \mathfrak{sl}_2$ is given by

$$hv_i = i\hbar v_i, \ ev_{i-2} = \frac{n+i}{2}\hbar v_i, \ fv_{i+2} = \frac{n-i}{2}\hbar v_i$$

Second, $\mathcal{M}_\hbar(-1)$ is a free $\mathsf{k}[\hbar, x]$-module with a basis $\{m_{-1}, m_{-3}, m_{-5}, \ldots\}$. The action of e, h, f is given by

$$hm_i = (x + i\hbar)m_i, \ em_i = \frac{-i-1}{2}\hbar(x - \frac{i+1}{2}\hbar)m_{i+2}, \ fm_i = m_{i-2}$$

We have a canonical filtration $0 = F^{n+2} \subset F^n \subset F^{n-2} \subset \ldots \subset F^{2-n} \subset F^{-n} = \mathcal{M}_\hbar(-1) \otimes_{\mathsf{k}[\hbar]} V_n$ by $U_\hbar$-submodules such that $F^i/F^{i+2} = \mathcal{M}_\hbar(i-1) \otimes {}_iV_n$ (notations of 4.3). Recall that ${}_iV_n$ is spanned by v_i. There is a unique vector $s_i \in (\mathcal{M}_\hbar(-1) \otimes V_n) \otimes_{\mathsf{k}[\hbar,x]} \mathsf{k}(\hbar, x)$ such that $es_i = 0$, and $s_i \equiv m_{-1} \otimes v_i$ modulo $U_\hbar\langle s_{i+2}, s_{i+4}, \ldots, s_n\rangle$.

Then $F^i = U_\hbar\langle s_i, s_{i+2}, \ldots, s_n\rangle \cap (\mathcal{M}_\hbar(-1) \otimes V_n)$. The image $\bar{s}_i$ of this vector in the ψ-coinvariants $U_\hbar\langle s_i\rangle \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi = \mathcal{M}_\hbar(i-1) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$ is the generator of $(\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_i) \otimes {}_i V_n$. Note that the space of ψ-coinvariants is just the quotient modulo the image of $f - 1$.

On the other hand, the proof of Lemma 11 shows that $\varphi(V_n) = (\mathcal{M}_\hbar(-1) \otimes V_n) \otimes_{U_\hbar(\check{\mathfrak{n}}_-)} \psi$ is generated by the image $\overline{m_{-1} \otimes v_n}$ of $m_{-1} \otimes v_{-n}$ in the space of ψ-coinvariants. Thus we have to express $\overline{m_{-1} \otimes v_{-n}}$ in terms of $\bar{s}_{-n}, \ldots, \bar{s}_n$.

Lemma 12.

$$\overline{m_{-1} \otimes v_{-n}} = \sum_{i=-n}^{n} (x + (i-1)\hbar)^{-1}(x + (i-2)\hbar)^{-1} \ldots \left(x + \frac{i-n}{2}\hbar\right)^{-1} \bar{s}_i$$

Proof: Recall that the space of ψ-coinvariants is just the quotient modulo the image of $f - 1$. This means we have to prove the following equality in $(\mathcal{M}_\hbar(-1) \otimes V_n) \otimes_{\mathsf{k}[\hbar,x]} \mathsf{k}(\hbar, x)$:

$$m_{-1} \otimes v_{-n} = \sum_{i=-n}^{n} (x + (i-1)\hbar)^{-1}(x + (i-2)\hbar)^{-1} \ldots \left(x + \frac{i-n}{2}\hbar\right)^{-1} f^{\frac{n+i}{2}} s_i$$

For $l = 0, 1, \ldots, n$ we introduce a new vector s_i^l such that $es_i^l = 0$, and $s_i^l \equiv e^l(m_{-1} \otimes v_{i-2l})$ modulo $U_\hbar\langle s_{i+2}, s_{i+4}, \ldots, s_n\rangle$ (evidently, s_i^l is proportional to $s_i = s_i^0$ when $i - 2l \geq -n$; otherwise, s_i^l is not defined). Consider the following collection of equalities:

$$e^l(m_{-1} \otimes v_{-n}) = \sum_{i=-n+2l}^{n} \hbar^l (x + (i-1)\hbar)^{-1}(x+(i-2)\hbar)^{-1} \ldots \left(x + \frac{i-n+2l}{2}\hbar\right)^{-1} f^{\frac{n-2l+i}{2}} s_i^l$$

Then the n-th equality is obvious, while the $l+1$-st equality is equivalent to the l-th one by applying e to both sides. Thus the desired equality (for $l = 0$) follows by descending induction in l. $\square$

6. Topology vs Algebra

6.1. Comparison of Kostant functor with equivariant cohomology.
Recall that $\mathfrak{t}$ is identified with $\check{\mathfrak{t}}^*$.

Theorem 6. *a) For $V \in Rep(\check{G})$ there is a unique isomorphism of $\mathcal{O}(\mathfrak{t}/W \times \mathfrak{t}/W \times \mathbb{A}^1)$-modules $\eta_V : H^\bullet_{\mathbf{G}_\mathbf{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \simeq \phi(V)$ such that $(\pi, \mathrm{Id}, \mathrm{Id})^* \eta_V$ preserves the canonical filtrations and induces the identity isomorphism of the associated graded $gr(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G}_\mathbf{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) = (\mathrm{Id}, \pi, \mathrm{Id})_* \bigoplus_\lambda \mathcal{O}(\Gamma_\lambda) \otimes {}_\lambda V = gr(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V)$.*
b) For $V_1, V_2 \in Rep(\check{G})$ the composition

$$H^\bullet_{\mathbf{G}_\mathbf{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_1)) \star H^\bullet_{\mathbf{G}_\mathbf{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_2)) \xrightarrow{\text{top}\omega^{-1}_{V_1, V_2}} H^\bullet_{\mathbf{G}_\mathbf{O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V_1 \otimes V_2))$$

$$\xrightarrow{\eta_{V_1 \otimes V_2}} \phi(V_1 \otimes V_2) \xrightarrow{\text{alg}\omega_{V_1, V_2}} \phi(V_1) \star \phi(V_2)$$

equals $\eta_{V_1} * \eta_{V_2}$ (notations of 3.5 and 4.6).

Proof : a) We have

$$(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) =$$

$$gr(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) =$$

$$\left(\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes \ _\lambda V \right) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) =$$

$$gr(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) = (\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1).$$

Thus we have to identify $(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$ with the natural W-action on it, and $(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V)$ with the natural W-action on it, as two $\mathcal{O}(\mathfrak{t} \times \mathfrak{t}/W \times \mathbb{A}^1)$-submodules of $(\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes \ _\lambda V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$. First we will show that the W-action on $(\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes \ _\lambda V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$ arising from its identification with $(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$ coincides with the W-action arising from the identification with $(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$.

Let α be a simple root, $s_\alpha \in W$ the corresponding simple reflection, and $(e_\alpha, h_\alpha, f_\alpha)$ the corresponding $\mathfrak{sl}_2$-triple in $\check{\mathfrak{g}}$. Let $v \in \ _\lambda V$ be a vector such that $f_\alpha v = 0$. Then $h_\alpha v = \lambda(h_\alpha) v$, and $\lambda(h_\alpha)$ is a nonpositive integer. We consider the following vector in $(\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes \ _\lambda V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1) \ni p_\alpha(v) :=$

$$\sum_{k=0}^{-\lambda(h_\alpha)} (h_\alpha + (2k-1+\lambda(h_\alpha))\hbar)^{-1} (h_\alpha + (2k-2+\lambda(h_\alpha))\hbar)^{-1} \ldots (h_\alpha + (k+\lambda(h_\alpha))\hbar)^{-1} 1_{\lambda + k\alpha} \otimes e_\alpha^k v$$

where 1_μ is the constant function 1 on $(\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\mu$.

It follows from the computation in rank 1 (Lemma 12), and the transitivity equation (8) of subsection 4.5 (applied to the case where $\check{L}$ is a subminimal Levi containing just one positive root α, and $\check{L}' = \check{T}$) that $p_\alpha(v)$ is s_α-invariant (with respect to the action arising from the identification with $(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$), and $M_\alpha := \sum_\lambda^{v \in \ _\lambda V^{f_\alpha}} \mathcal{O}(\mathfrak{t} \times \mathbb{A}^1) p_\alpha(v)$ contains $(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V)$.

On the other hand, it follows from the computation in rank 1 (equation (7) of subsection 5.2), and the transitivity equation (3) of subsection 3.4 (applied to the case where $\check{L}$ is a subminimal Levi containing just one positive root α, and $\check{L}' = \check{T}$) that $p_\alpha(v)$ is s_α-invariant (with respect to the action arising from the identification with $(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V)) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$), and $M_\alpha = \sum_\lambda^{v \in \ _\lambda V^{f_\alpha}} \mathcal{O}(\mathfrak{t} \times \mathbb{A}^1) p_\alpha(v)$ contains $(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$. It follows that the two W-actions on $(\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes \ _\lambda V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$ coincide. In particular, we have defined unambiguously a W-action on $(\bigoplus_\lambda (\mathrm{Id}, \pi, \mathrm{Id})_* \mathcal{O}(\Gamma_\lambda) \otimes \ _\lambda V) \otimes_{\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)} \mathsf{k}(\mathfrak{t} \times \mathbb{A}^1)$.

Now we claim that $(\pi, \mathrm{Id}, \mathrm{Id})^* \phi(V) = \bigcap_{w \in W}^{\alpha \text{ simple}} w(M_\alpha) = (\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$. In effect, note that if $(\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\nu \cap (\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\mu$ has codimension 1 in $(\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\nu$ (and in $(\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\mu$ as well), then necessarily $\mu = \nu + k\beta$ for certain integer k, and certain positive root β (not necessarily a simple root). Let us choose $w \in W$ such that $\alpha := w^{-1}(\beta)$ is a simple root. Then $w^{-1}(\mu) = w^{-1}(\nu) + k\alpha$. Since we know that any section in M_α extends through the generic point of $(\mathrm{Id}, \pi, \mathrm{Id})\Gamma_{w^{-1}(\mu)} \cap (\mathrm{Id}, \pi, \mathrm{Id})\Gamma_{w^{-1}(\mu)+k\alpha}$, we conclude that any section in $w(M_\alpha)$

extends through the generic point of $(\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\mu \cap (\mathrm{Id}, \pi, \mathrm{Id})\Gamma_\nu$. It follows that any section in $\bigcap_{w \in W}^{\alpha \, \mathrm{simple}} w(M_\alpha)$ is regular off a codimension 2 subvariety. Since both $(\pi, \mathrm{Id}, \mathrm{Id})^*\phi(V)$ and $(\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$ are flat $\mathcal{O}(\mathfrak{t} \times \mathbb{A}^1)$-modules coinciding with $\bigcap_{w \in W}^{\alpha \, \mathrm{simple}} w(M_\alpha)$ generically, we conclude that $(\pi, \mathrm{Id}, \mathrm{Id})^*\phi(V) = \bigcap_{w \in W}^{\alpha \, \mathrm{simple}} w(M_\alpha) = (\pi, \mathrm{Id}, \mathrm{Id})^* H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}_G, S(V))$. This completes the proof of the a).

b) follows by comparing Propositions 1 and 2. $\square$

6.2. Cohomology is fully faithful.
The following property of the cohomology functor will play an important role in the proof of the main results. Let $\mathcal{JC}$ (resp. $\widetilde{\mathcal{JC}}$) denote the full subcategory of semisimple complexes in $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ (resp. $D_{\mathbf{G_O}}(\mathrm{Gr})$).

Lemma 13. *a) The functor $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m} : \mathcal{JC} \to Coh^{\mathbb{G}_m}(N_{(\check{\mathfrak{t}}^*/W)^2}\Delta)$ is a full imbedding.*
b) The functor $H^\bullet_{\mathbf{G_O}} : \widetilde{\mathcal{JC}} \to Coh^{\mathbb{G}_m}(\mathbf{T}(\check{\mathfrak{t}}^*/W))$ is a full imbedding.

Proof is due to V. Ginzburg, see [17]. We prove a), and the proof of b) is identical. For $V_1, V_2 \in Rep(\check{G})$ we have $\mathrm{Ext}^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(S(V_1), S(V_2)) = \mathrm{Ext}^\bullet_{G \times \mathbb{G}_m}(S(V_1), S(V_2))$. Let us denote by Res^T_G the forgetting functor from the $G \times \mathbb{G}_m$-equivariant derived category to the $T \times \mathbb{G}_m$-equivariant derived category. Then the Weyl group W acts naturally on $\mathrm{Ext}^\bullet_{T \times \mathbb{G}_m}(Res^T_G S(V_1), Res^T_G(V_2))$, and $\mathrm{Ext}^\bullet_{G \times \mathbb{G}_m}(S(V_1), S(V_2)) = \mathrm{Ext}^\bullet_{T \times \mathbb{G}_m}(Res^T_G S(V_1), Res^T_G(V_2))^W$. On the other hand, we know by Theorem 6 a) that $H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_{1,2})) \simeq \varphi(V_{1,2})$, and clearly, $\mathrm{Hom}(\phi(V_1), \phi(V_2)) = \mathrm{Hom}(\varphi(V_1), \varphi(V_2))^W$. Thus it suffices to prove that

$$\mathrm{Ext}^\bullet_{T \times \mathbb{G}_m}(Res^T_G S(V_1), Res^T_G(V_2)) \simeq$$

$$\mathrm{Hom}_{\mathcal{O}(\mathfrak{t} \times (\mathfrak{t}/W) \times \mathbb{A}^1)}\left(H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_1)), H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_2))\right) =$$

$$\mathrm{Hom}_{H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr})}\left(H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_1)), H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_2))\right).$$

Following [9], recall the definition of the LHS: we choose finite dimensional approximations P_i to the classifying space of $T \times \mathbb{G}_m$, and we have ind-varieties $P_i \mathrm{Gr}$ fibered over P_i with fibers isomorphic to Gr. We also have semisimple perverse sheaves $P_i S(V_{1,2})$ on $P_i \mathrm{Gr}$, and finally $\mathrm{Ext}^\bullet_{T \times \mathbb{G}_m}(Res^T_G S(V_1), Res^T_G(V_2))$ is defined as $\underset{\to}{\lim} \mathrm{Ext}^\bullet_{P_i \mathrm{Gr}}(P_i S(V_1), P_i S(V_2))$. Since $T \times \mathbb{G}_m$ is a torus, we can choose P_i to be the products of projective spaces (of increasing dimension). We can choose a generic action of $\mathbb{G}_m$ on $P_i \mathrm{Gr}$ (linear along P_i, and via a one-parametric subgroup of $T \times \mathbb{G}_m$ along Gr) such that the corresponding Bialynicki-Birula decomposition of $P_i \mathrm{Gr}$ is cellular. Then we can apply the Theorem of [17] to conclude that $\mathrm{Ext}^\bullet_{P_i \mathrm{Gr}}(P_i S(V_1), P_i S(V_2)) \simeq \mathrm{Hom}_{H^\bullet(P_i \mathrm{Gr})}(H^\bullet(P_i \mathrm{Gr}, S(V_1)), H^\bullet(P_i \mathrm{Gr}, S(V_2)))$. But the limit of the RHS as i grows is $\mathrm{Hom}_{H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr})}\left(H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_1)), H^\bullet_{T \times \mathbb{G}_m}(\mathrm{Gr}, S(V_2))\right)$.

The lemma is proved. $\square$

6.3. Proof of Theorems 2, 4.
The Theorems follow from Theorem 6 in view of Lemmas 4(b), 13.

6.4. **Homology.** The goal of this subsection is to express equivariant cohomology of arbitrary (not necessarily semisimple) equivariant complexes in terms of Harish-Chandra bimodules, and to prove Theorem 3. By Theorem 2 we have a monoidal equivalence $\mathcal{S}$ between the category $\mathcal{HC}_\hbar^{fr}$ of free asymptotic Harish-Chandra bimodules and the category $\mathcal{IC}$ of semisimple complexes in $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$.

A standard argument shows any functor on free Harish-Chandra bimodules is representable by a unique up to a unique isomorphism (not necessarily free) Harish-Chandra bimodule. Thus we have a functor from $\mathfrak{F} : D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \to \widetilde{\mathcal{HC}_\hbar}$ equipped with a functorial isomorphism

$$\mathrm{Hom}(\mathcal{S}(F), \mathcal{F}) \cong \mathrm{Hom}(F, \mathfrak{F}(\mathcal{F}));$$

$\mathfrak{F}$ is defined uniquely up to a unique isomorphism.

Since $\mathcal{S}$ is a full embedding, $\mathfrak{F} \circ \mathcal{S} \cong \mathrm{Id}$ canonically, i.e. $\mathfrak{F}$ restricted to the category of semi-simple complexes is the inverse equivalence to $\mathcal{S}$. It is easy to see from the definition that $\mathfrak{F}$ is a homological functor; thus it actually lands in the category $\mathcal{HC}_\hbar$ of finitely generated Harish-Chandra bimodules, $\mathfrak{F} : D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) \to \mathcal{HC}_\hbar$.

We will say that a functor F between two monoidal categories is quasi-monoidal if a functorial map $F(X) * F(Y) \to F(X * Y)$ is fixed for any objects X, Y of the target category, compatible with the associativity isomorphisms in the two categories in the natural sense.

Proposition 3. *a)* $\mathfrak{F}$ *carries a unique quasi-monoidal structure, whose restriction to* $\mathcal{IC}$ *induces the natural monoidal structure on* $\mathrm{Id}_{\mathcal{HC}_\hbar^{fr}} \cong \mathfrak{F} \circ \mathcal{S}$.

b) We have a natural isomorphism $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m} \cong \kappa_\hbar \circ \mathfrak{F}$ *compatible with the (quasi)monoidal structure.*

Proof a) It is easy to see that

$$\mathfrak{F}(\mathcal{F}) = R\mathrm{Hom}(IC_0, \mathcal{O}(\check{G}) * \mathcal{F}) \cong R\mathrm{Hom}(IC_0, \mathcal{F} * \mathcal{O}(\check{G}))$$

canonically (where the last isomorphism comes from the fact that both sides can be identified with $\bigoplus_\lambda RHom(V_\lambda^* \otimes IC_\lambda, \mathcal{F})$). Then the quasi-monoidal structure comes from the coalgebra structure on $\mathcal{O}(\check{G})$. Uniqueness follows from the fact that for every $\mathcal{F} \in D_{\mathbf{G_O} \rtimes \mathbb{G}_m}$ we can find a free asymptotic Harish-Chandra bimodule V and a surjection $V \to \mathfrak{F}(\mathcal{F})$; by definition of $\mathfrak{F}$ it comes from a map $L = \mathcal{S}(V) \to \mathcal{F}$. Then, in view of functoriality, for $\mathcal{F}_1, \mathcal{F}_2 \in D_{\mathbf{G_O} \rtimes \mathbb{G}_m}$ the map $\mathfrak{F}(\mathcal{F}_1) * \mathfrak{F}(\mathcal{F}_2) \to \mathfrak{F}(\mathcal{F}_1 * \mathcal{F}_2)$ is uniquely determined by the isomorphism $\mathfrak{F}(L_1) * \mathfrak{F}(L_2) \to \mathfrak{F}(L_1 * L_2)$ for $L_i \to \mathcal{F}_i$ ($i = 1, 2$) as above.

b) For $\mathcal{F} \in D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ we can find an exact sequence $V_1 \to V_2 \to \mathfrak{F}(\mathcal{F}) \to 0$, where V_i are free asymptotic Harish-Chandra bimodules. To this sequence there corresponds a sequence of maps in $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ with zero composition: $L_1 \to L_2 \to \mathcal{F}$, where $L_i = \mathcal{S}(V_i)$. We have $\kappa_\hbar \circ \mathfrak{F}(\mathcal{F}) = CoKer(\kappa_\hbar(V_1) \to \kappa_\hbar(V_2)) = CoKer(H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(L_1) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(L_2))$. The latter module maps canonically to $H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathcal{F})$. Thus we defined a map $\kappa_\hbar \circ \mathfrak{F}(\mathcal{F}) \to H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathcal{F})$. A standard argument shows that this map does not depend on the choice of V_1, V_2. We have obtained a natural transformation between the two functors. This transformation is an isomorphism on semi-simple complexes.

Since both functors are homological (where we use exactness of $\kappa_\hbar$, Lemma 4(a)) and semi-simple objects generate the triangulated category $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$, we see that the transformation is an isomorphism. $\square$

We can clearly extend all of the above to Ind-objects. We will be particularly interested in the Ind-object $\mathfrak{D} = \varinjlim \mathfrak{D}_\lambda$, where $\mathfrak{D}_\lambda$ is the dualizing sheaf of the closure of the $\mathbf{G_O}$-orbit Gr_λ. Notice that $H_\bullet^{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) = H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathfrak{D})$, and the convolution algebra structure on homology comes from the structure of an algebra in the monoidal category $D_{\mathbf{G_O} \rtimes \mathbb{G}_m}$ on the object $\mathfrak{D}$ and the monoidal structure on the cohomology functor.

We will now describe the corresponding object in the category of Harish-Chandra bimodules. Recall first the duality $\mathbf{D}$ on $\mathcal{HC}_\hbar^{fr}$, $\mathbf{D} : M \mapsto \mathrm{Hom}_{U_\hbar}(M, U_\hbar)$, where Hom is taken with respect to the right action of $U_\hbar$, and the left action on M, $U_\hbar$ is used to define, respectively, the right and the left action on the Hom module. Let $Forg$ denote the forgetful functor from Harish-Chandra bimodules to vector spaces. The functor $Forg \circ \mathbf{D}$ is represented by the Ind-object, which is readily identified with $U_\hbar \otimes \mathcal{O}(\check{G})$, the module of $\hbar$-differential operators on $\check{G}$. Furthermore, $U_\hbar \otimes \mathcal{O}(\check{G})$ carries another commuting structure of a Harish-Chandra bimodule, and we set $\mathcal{K} = \kappa_\hbar(U_\hbar \otimes \mathcal{O}(\check{G}))$, where this second structure of a Harish-Chandra bimodule is used to compute $\kappa_\hbar$. Thus $\mathcal{K}$ is an Ind-object in the category of Harish-Chandra bimodules; moreover, it is an algebra Ind-object in this monoidal category.

Proposition 4. *We have a canonical isomorphism of algebra Ind-objects:* $\mathfrak{F}(\mathfrak{D}) \cong \mathcal{K}$.

To prove Proposition we need another auxiliary

Lemma 14. *We have a canonical isomorphism* $\mathbb{D} \circ \mathcal{S} \cong \mathcal{S} \circ \mathfrak{C}_{\check{G}} \circ \mathbf{D}$*, where* $\mathbb{D}$*,* $\mathfrak{C}_{\check{G}}$ *denote, respectively, Verdier duality and the functor induced by the canonical outer automorphism of* $\check{G}$ *interchanging conjugacy classes of* g *and* g^{-1}*,* $g \in \check{G}$ *(the Chevalley involution).*

Proof : Recall that a monoidal category $\mathcal{C}$ is rigid if for any object $V \in \mathcal{C}$ there exists another object $V^* \in \mathcal{C}$ and morphisms $\iota : \mathbf{1} \to V \otimes V^*$ and $\tau : V^* \otimes V \to \mathbf{1}$ satisfying a certain compatibility constraint, see [14]. Given V, an object V^* together with morphisms ι, τ is unique up to a unique isomorphism if it exists. Thus for a rigid category $\mathcal{C}$ we have a canonical (up to a canonical isomorphism) functor $\mathcal{C} \mapsto \mathcal{C}^{op}$ sending V to V^*. It is immediate to check that monoidal categories $Perv_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$ and $\mathcal{HC}_\hbar^{fr}$ are rigid categories with duality functors given by $\mathfrak{C}_G \circ \mathbb{D}$ and $\mathbf{D}$ respectively, where the functor $\mathfrak{C}_G$ is induced by the Chevalley involution of G. The equivalence between the two categories intertwines the canonically defined dualities. Also, it is well-known that $\mathcal{S} \circ \mathfrak{C}_{\check{G}} \cong \mathfrak{C}_G \circ \mathcal{S}$ canonically. The Lemma follows. $\square$

Proof of Proposition 4.   The Ind-object $\mathfrak{D}$ represents the functor $\mathcal{F} \mapsto H^\bullet_{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathbb{D}(\mathcal{F}))$. In view of Lemma 14 and Proposition 3(c) we see that the Ind-object $\mathfrak{F}(\mathfrak{D})$ represents the functor $M \mapsto \kappa_\hbar(\mathbf{D}(M))$ on the category of free asymptotic Harish-Chandra modules. It is straighforward to see from the definitions that the Ind-object $\mathcal{K}$ represents the same functor. The isomorphism of functors yields an isomorphism of Harish-Chandra bimodules. Since the isomorphism of functors is compatible with the monoidal structure, the isomorphism of Harish-Chandra bimodules is compatible with the algebra structure. $\square$

Proof of Theorem 3. By Propositions 3 and 4 we have an isomorphism of algebras

$$H_\bullet^{\mathbf{G_O} \rtimes \mathbb{G}_m}(\mathrm{Gr}) = H_{\mathbf{G_O} \rtimes \mathbb{G}_m}^\bullet(\mathfrak{D}) \cong \kappa_\hbar(\mathcal{K}).$$

The latter is by definition the algebra of the quantum Toda lattice. $\square$

6.5. **Formality from purity.** In this section we combine the above Ext computation with a standard argument which allows one to derive formality of the RHom algebra from purity of the Ext spaces. The result is a description of the derived Satake category, including the version equivariant with respect to the loop rotation.

Except for some technical details, this section does not contain original contributions of the authors. We have learned the geometric (respectively, algebraic) ideas exposed here from V. Ginzburg (respectively, L. Positselski) around 1998.

In order to be able to use Frobenius weights we extend the basic setting, and consider $\mathbf{G_O}$, Gr etc. over $\bar{\mathbb{F}}_q$, and the categories of equivariant l-adic sheaves on Gr.

Consider the following general situation. Let R be a finitely localized ring of integers of a number field E, and let $\mathbb{F}_q$ be a finite field quotient of R. Let X_R be a flat scheme over R acted upon by a smooth affine group scheme G_R, such that the set of orbits is finite. We denote by $(X_{\bar{\mathbb{F}}_q}, G_{\bar{\mathbb{F}}_q})$ (resp. $(X_{\bar{E}}, G_{\bar{E}})$) the base change of (X_R, G_R) to a geometric point of R over $\mathbb{F}_q$ (resp. over the generic point). We choose a prime l invertible in R. Let $D_{G_{\bar{\mathbb{F}}_q}}(X_{\bar{\mathbb{F}}_q})$ (resp. $D_{G_{\bar{E}}}(X_{\bar{E}})$) stand for the bounded equivariant constructible derived category of étale $\overline{\mathbb{Q}_l}$-sheaves on $X_{\bar{\mathbb{F}}_q}$ (resp. $X_{\bar{E}}$) (see e.g. [6], 7.4). We choose an isomorphism $\overline{\mathbb{Q}_l} \simeq \mathsf{k}$ (under a technical assumption that k has the same cardinality as $\overline{\mathbb{Q}_l}$), and an embedding $\bar{E} \hookrightarrow \mathbb{C}$, and we denote by $(X_\mathbb{C}, G_\mathbb{C})$ the base change of $(X_{\bar{E}}, G_{\bar{E}})$ to $\mathbb{C}$. Let $D_{G_\mathbb{C}}(X_\mathbb{C})$ (resp. $D_{G_\mathbb{C}}^{top}(X_\mathbb{C}, \overline{\mathbb{Q}_l})$, $D_{G_\mathbb{C}}^{top}(X_\mathbb{C})$) stand for the bounded equivariant constructible derived category of étale $\overline{\mathbb{Q}_l}$-sheaves on $X_\mathbb{C}$ (resp. bounded equivariant constructible derived category of sheaves with $\overline{\mathbb{Q}_l}$-coefficients, resp. k-coefficients, *in the classical topology* of $X_\mathbb{C}$).

Proposition 5. *There exists a localization $R_{(r)}$ of R such that for any point $R_{(r)} \twoheadrightarrow \mathbb{F}_q$, we have the following chain of natural equivalences:*

$$D_{G_{\bar{\mathbb{F}}_q}}(X_{\bar{\mathbb{F}}_q}) \xrightarrow{\alpha} D_{G_{\bar{E}}}(X_{\bar{E}}) \xrightarrow{\beta} D_{G_\mathbb{C}}(X_\mathbb{C}) \xrightarrow{\gamma} D_{G_\mathbb{C}}^{top}(X_\mathbb{C}, \overline{\mathbb{Q}_l}) \xrightarrow{\delta} D_{G_\mathbb{C}}^{top}(X_\mathbb{C})$$

Sketch of proof. The argument is taken from [5], 6.1. The first equivalence α is constructed in [5], 6.1.9 (existence of good models). To justify the finiteness assumptions of *loc. cit.* we note that the set of isomorphism classes of G-equivariant irreducible perverse sheaves on X is finite. Since the equivariant derived categories are not considered in *loc. cit.* we note that according to [6], 7.4, to compare the equivariant Exts between equivariant irreducible perverse sheaves on X it suffices to compare the usual Exts between the lifts of these sheaves to $X \times G^n$ (see the canonical spectral sequence (312) of *loc. cit.*). These are calculated by the Künneth formula.

The second equivalence β is just the base change from $\bar{E}$ to $\mathbb{C}$. The third equivalence γ is the classical M. Artin's comparison theorem of étale and classical cohomology, see [5], 6.1.2(B″). Finally, δ is induced by our isomorphism $\overline{\mathbb{Q}_l} \simeq \mathsf{k}$. $\square$

From now on we will restrict our attention to the equivariant derived categories in our new setting, that is, over $\bar{\mathbb{F}}_q$.

Let X be an algebraic variety over a finite $\mathbb{F}_q$, and let $D(X_{\bar{\mathbb{F}}_q})$ stand for the bounded constructible l-adic derived category of $X_{\bar{\mathbb{F}}_q}$. Let $\mathcal{F}$ be a pure weight zero object of the l-adic derived category of X of geometric origin. The space $\text{Ext}^i(\mathcal{F}_{\bar{\mathbb{F}}_q}, \mathcal{F}_{\bar{\mathbb{F}}_q}) = \bigoplus \text{Ext}^i_j(\mathcal{F}_{\bar{\mathbb{F}}_q}, \mathcal{F}_{\bar{\mathbb{F}}_q})$ carries a canonical grading by Frobenius weights; here the subindex denotes base change to $\bar{\mathbb{F}}_q$, and Ext^i_j is the component of weight j. Recall that by Deligne Theorem [13] $\text{Ext}^i_j = 0$ for $j < i$.

Let $\mathcal{E}$ be a graded algebra and $\phi : \mathcal{E} \to \text{Ext}^\bullet(\mathcal{F}_{\bar{\mathbb{F}}_q}, \mathcal{F}_{\bar{\mathbb{F}}_q})^{op}$ be a homomorphism sending a graded component $\mathcal{E}^i$ to $Ext^i_i(\mathcal{F}_{\bar{\mathbb{F}}_q}, \mathcal{F}_{\bar{\mathbb{F}}_q})$.

We will consider the graded algebra $\mathcal{E}$ as a dg-algebra with zero differential.

Proposition 6. *There exists a canonical functor* $\Phi_X : D_{perf}(\mathcal{E}) \to D(X_{\bar{\mathbb{F}}_q})$ *sending the free module to* $\mathcal{F}_{\bar{\mathbb{F}}_q}$ *and inducing the map* ϕ *on Ext groups.*

Sketch of proof. The complex $\mathcal{F}_{\bar{\mathbb{F}}_q}$ is semi-simple, i.e. is isomorphic to $\oplus \text{IC}_i[d_i]$, where IC_i is an irreducible perverse sheaf and $d_i \in \mathbb{Z}$. By Beilinson's Theorem [4] the l-adic derived category contains the derived category of perverse sheaves as a full subcategory, thus $\text{Ext}^\bullet(\mathcal{F}_{\bar{\mathbb{F}}_q}, \mathcal{F}_{\bar{\mathbb{F}}_q})$ coincides with Ext in the category of perverse sheaves. We can assume without loss of generality that $\mathcal{E}$ is finitely presented; thus the map ϕ factors through a map $\phi_{fin} : E \to \text{Ext}^\bullet_{\mathcal{A}}(\mathcal{F}_{\bar{\mathbb{F}}_q}, \mathcal{F}_{\bar{\mathbb{F}}_q})$, where $\mathcal{A}$ is the Serre subcategory in the category of perverse sheaves on $X_{\bar{\mathbb{F}}_q}$ generated by a finite set of irreducible objects, including IC_i. Moreover, the argument of [4] (cf. Examples in *loc. cit.*, 1.2, p.28) shows that all irreducible objects of $\mathcal{A}$ can be assumed to be of geometric origin.

We can identify the abelian category $\mathcal{A}$ with the category of finite length A-modules, where the pro-finite dimensional algebra (algebra in the tensor category of pro-finite dimensional vector spaces) A is defined by $A = \text{End}(\oplus \mathcal{P}_s)$; here s runs over the (finite) set of isomorphism classes of irreducible objects in $\mathcal{A}$, and $\mathcal{P}_s$ is a pro-object in $\mathcal{A}$ which is a projective cover of the corresponding irreducible object L_s, cf. [3]. We fix an isomorphism $Fr^*_q(L_s) \cong L_s$, which induces a pure weight zero Weil structure on L_s (this is possible because L_s has geometric origin, see [5]). Since projective cover of an irreducible object is unique up to a non-unique isomorphism, we can (and will) fix an isomorphism $Fr^*_q(\mathcal{P}_s) \cong \mathcal{P}_s$. Then conjugation with Frobenius is an automorphism of A (which we will also call Frobenius).

By a result of [5], Frobenius acts on $\text{Ext}^1(L_s, L_{s'})$ with positive weights. It follows that Frobenius finite elements are dense in A, and they form a graded subalgebra A^{gr} with finite dimensional graded components, where the grading comes from Frobenius weights. Moreover, components of negative degree in A^{gr} vanish, while A^{gr}_0 is semisimple. Obviously, $\mathcal{A}$ is identified with the category of finite length A^{gr} modules, on which A^{gr}_N acts by zero for $N \gg 0$.

We now consider the object $L = \oplus L_i[d_i] \in D^b(\mathcal{A})$ (where L_i corresponds to IC_i) and a dg-algebra $\mathsf{D} = \text{RHom}_{\mathcal{A}}(L, L)$ (well defined as an object of the category of dg-algebras with inverted quasi-isomorphisms). Recall that we have an equivalence $D^b(\mathcal{A}) \cong D_{perf}(\mathsf{D}^{op})$, $M \mapsto \text{RHom}(L, M)$. We lift L to an object $\tilde{L} = \oplus \tilde{L}_i[d_i](d_i)$

of the derived category of graded A^{gr}-modules, where $\tilde{L}_i$ is the irreducible A-module concentrated in degree zero, and (d) stands for shift of grading by d. Then the algebra $\text{Ext}^\bullet(L, L)$ acquires an additional grading, and D can be chosen to carry also an additional grading compatible with the grading on Ext's. We have a homomorphism $\mathcal{E} \to \bigoplus\limits_i \text{Ext}^i_i(L, L)$, where the lower index denotes the additional grading, and the fact that A^{gr} is positively graded implies that $\text{Ext}^i_j = 0$ for $j < 0$. Thus existence of a canonically defined functor $D_{perf}(\mathcal{E}) \to D^b(\mathcal{A})$ follows from the standard Lemma 15a) below.

We leave it as an exercise to the reader to show that the composed functor $\Phi_X : D_{perf}(\mathcal{E}) \to D^b(\mathcal{A}) \to D(X_{\overline{\mathbb{F}}_q})$ does not depend on the choice of $\mathcal{A}$ up to a canonical isomorphism. $\square$

We will also have to use functoriality properties of the above construction. We spell these out now.

Proposition 7. *a) Let X, Y be algebraic varieties over a finite field $\mathbb{F}_q$, and $F : D(X_{\overline{\mathbb{F}}_q}) \to D(Y_{\overline{\mathbb{F}}_q})$ be a functor satisfying the following conditions*

  (1) *F commutes with the pull back under Frobenius functor, i.e. an isomorphism $F \circ Fr_X \cong Fr_Y \circ F$ is fixed.*
  (2) *F sends pure weight zero Weil complexes to pure weight zero Weil complexes.*
  (3) *Let $\mathcal{A}$ be a finitely generated Serre subcategory in $Perv(X_{\overline{\mathbb{F}}_q})$ invariant under the Frobenius pull-back functor. Then there exists a natural exact functor $F_\mathcal{A} : Com(\mathcal{A}) \to Com(Perv(Y_{\overline{\mathbb{F}}_q}))$, compatible with Frobenius and equipped with a natural isomorphism $\beta_Y \circ F_\mathcal{A} \xrightarrow{\sim} F \circ \beta_X$. Here $\beta_X : Com(\mathcal{A}) \to D(X_{\overline{\mathbb{F}}_q})$, $\beta_Y : Com(Perv(Y_{\overline{\mathbb{F}}_q})) \to D(Y_{\overline{\mathbb{F}}_q})$ are the natural functors.*

Then the construction of Proposition 6 is compatible with F, that is, there is a natural isomorphism $\psi_{X \to Y} : \Phi_Y \xrightarrow{\sim} F \circ \Phi_X$.

b) Assume furthermore that $F_1 : D(Y_{\overline{\mathbb{F}}_q}) \to D(Z_{\overline{\mathbb{F}}_q})$ is a functor satisfying the above conditions. Then $F_1 \circ F$ also satisfies these conditions, and the two isomorphisms $\psi_{X \to Z}$, $F_1 \circ \psi_{X \to Y}$ between the two functors $\Phi_Z \xrightarrow{\sim} F_1 \circ F \circ \Phi_X : D_{perf}(\mathcal{E}) \to D(Z_{\overline{\mathbb{F}}_q})$ coincide.

Proof. As above, we find an abelian subcategory $\mathcal{A}' \subset Perv(Y)$ containing all subquotients of complexes $F_\mathcal{A}(\mathcal{G})$, $\mathcal{G} \in \mathcal{A}$; a complex $\mathcal{C}'$ of pro-objects in $Perv_{mix}(Y)$ quasiisomorphic to $F(\mathcal{C})$, whose terms with forgotten Frobenius action are projective pro-objects in $\mathcal{A}'$. We also have a dg-algebra D' equipped with an additional grading, which acts on $\mathcal{C}'$ in a way compatible with the grading by Frobenius weights, so that the action induces a quasiisomorphism $\mathsf{D}' \to \text{RHom}_{\mathcal{A}'}(F(\mathcal{C}), F(\mathcal{C}))$.

Consider a dg-module $B_{X,Y}$ of $\mathsf{D}' \otimes \mathsf{D}^{op}$-module defined by $B_{X,Y} := \text{Hom}^\bullet(\mathcal{C}', F(\mathcal{C}))$. It is not hard to see that the composed functor $D_{perf}(\mathsf{D}) \cong D^b(\mathcal{A}) \xrightarrow{F} D^b(\mathcal{A}') \cong D_{perf}((\mathsf{D}')^{op})$ arises from this bimodule as described in Lemma 15(b). The action of Frobenius endows $B_{X,Y}$ with an additional grading compatible with the gradings on D, D'. Thus Lemma 15(b) provides the sought for isomorphism between the two functors $D_{perf}(\mathcal{E}) \to D^b(\mathcal{A}')$.

Finally part (b) of the Proposition can be deduced from Lemma 15(c). $\square$

Remark 2. We will apply Proposition 7 when $F = f^*$ or $F = g_*$ for a smooth map $f : Y \to X$ or a proper map $g : X \to Y$. Each of the functors $F = f^*$, $F = g_*$ satisfies the requirements of the Propositions: for f^* this is standard, and for g_* property 3 follows from a construction described in [4], page 41, and 2 follows from [5].

Thus Proposition 7 implies functoriality of the construction of Proposition 6 with respect to proper push-forward and smooth pull backs. Also, Proposition 7(b) implies compatibility of the isomorphism of Proposition 6 with the base change isomorphism for a proper map $X \to Y$ and a smooth map $Y' \to Y$.

Remark 3. A result similar to Proposition 7 holds, with a similar proof, for a functor $F : D(\,^1X_{\overline{\mathbb{F}}_q}) \times \cdots \times D(\,^nX_{\overline{\mathbb{F}}_q}) \to D(Y_{\overline{\mathbb{F}}_q})$. Examples of this situation arise when $Y = {}^1X \times \cdots \times {}^nX$, $F : (\mathcal{F}_1, \ldots, \mathcal{F}_n) \mapsto \mathcal{F}_1 \boxtimes \cdots \boxtimes \mathcal{F}_n$; or in a twisted version of this situation (see below).

Lemma 15. *a) Let* $\mathsf{D} = \bigoplus \mathsf{D}^i_j$ *be a dg-algebra equipped with an additional "inner" grading (denoted by a subindex), which is compatible with the differential (thus we have* $d : \mathsf{D}^i_j \mapsto \mathsf{D}^{i+1}_j$*). Assume that* $H^i_j(\mathsf{D}) = 0$ *for* $j < i$*. Then there is a canonical morphism in the category of dg-algebras with inverted quasiisomorphisms:* $H_{pur} = \bigoplus_i H^i_i(\mathsf{D}) \to \mathsf{D}$*.*

In particular, we have a canonical push-forward functor $D_{perf}(H_{pur}) \to D_{perf}(\mathsf{D})$*.*

b) Let D*,* D' *be dg-algebras satisfying the assumptions of (a). Let* $B \in D(\mathsf{D}' \otimes \mathsf{D}^{op})$ *be such that the forgetful functor* $D(\mathsf{D}' \otimes \mathsf{D}^{op}) \to D(\mathsf{D}')$ *sends* B *to the free rank one module over* D'*; thus* B *defines a homomorphism* $\phi_B : H^\bullet(\mathsf{D}) \to H^\bullet(\mathsf{D}')$*. Let* $\mathcal{E}$*,* $\phi : \mathcal{E} \to \bigoplus_i H^i_i(\mathsf{D})$ *be as above, and consider the functors* $\Phi : D_{perf}(\mathcal{E}) \to D_{perf}(\mathsf{D})$*,* $\Phi' : D_{perf}(\mathcal{E}) \to D_{perf}(\mathsf{D}')$ *arising from* ϕ*,* $\phi_B \circ \phi$ *respectively by the construction of part(a).*

Consider the functor $\Phi_B : D_{perf}(\mathsf{D}) \to D_{perf}(\mathsf{D}')$ *given by* $M \mapsto B \overset{\mathrm{L}}{\otimes}_\mathsf{D} M$*. Assume that* B *carries an additional grading compatible with the gradings on* D*,* D'*. Then we have a natural isomorphism* $\Phi' \cong \Phi_B \circ \Phi$*.*

c) Let D*,* D'*,* D'' *be three dg-algebras as above, and* B*,* B'' *be modules for* $\mathsf{D}' \otimes \mathsf{D}^{op}$*,* $\mathsf{D}'' \otimes (\mathsf{D}')^{op}$ *as above. For a homomorphism* $\mathcal{E} \to \bigoplus H^i_i(\mathsf{D})$ *the two isomorphisms between the two functors* $D_{perf}(\mathcal{E}) \to D_{perf}(\mathsf{D}'')$ *arising from part (b) coincide.*

Proof. We remind the idea of the construction in part (a), and leave (b,c) to the interested reader. Let $\mathsf{D}_i \subset \mathsf{D}$ be the subcomplex of elements of inner degree i. We have a sub dg-algebra $\mathsf{D}_{up} := \bigoplus_i \tau_{\leq i}\mathsf{D}_i$, where we use the standard notation τ for truncation of a complex. Furthermore, D_{up} has a quotient algebra with zero differential $\mathsf{D}_{diag} := \bigoplus_i \tau_{\geq i}\tau_{\leq i}\mathsf{D}_i$. The conditions of part (a) guarantee that the projection homomorphism $\mathsf{D}_{up} \to \mathsf{D}_{diag}$ is a quasi-isomorphism. The composition of the formal inverse to this quasi-isomorphism with the embedding $\mathsf{D}_{up} \hookrightarrow \mathsf{D}$ is the desired morphism. $\square$

6.6. **Proof of Theorem 5.** We construct the first equivalence, the second one is similar. We will construct a monoidal functor $\Psi : D^{\check{G}}_{perf}(U^{[]}_\hbar) \to D_{\mathbf{G}_\mathbf{O} \rtimes \mathbb{G}_m}(\mathrm{Gr})$, whose restriction to the full subcategory $\mathcal{HC}^{fr}_\hbar \subset D^{\check{G}}_{perf}(U^{[]}_\hbar)$ is identified with $\mathcal{S}$ (where the

full embedding sends a $\check{G}$-equivariant graded $U_\hbar$-module to the same module considered
as a dg-module with zero differential). Then Ψ sends a set of generators of the source
triangulated category to generators of the target category, and induces an isomorphism
on Hom's between the generators, hence it is an equivalence.

It suffices to construct a collection of functors $\Psi_\lambda : D^{\check{G}}_{perf}(U^{[]}_\hbar)_{\leq\lambda} \to D_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr}_{\leq\lambda})$,
where λ is a coweight of G, $\mathrm{Gr}_{\leq\lambda}$ is the closure of the corresponding $\mathbf{G_O}$ orbit on
Gr, and $D^{\check{G}}_{perf}(U^{[]}_\hbar)_{\leq\lambda}$ is the full subcategory in $D^{\check{G}}_{perf}(U^{[]}_\hbar)$ generated by the objects
$V \otimes U^{[]}_\hbar$, where V is an irreducible representation of $\check{G}$ with a highest weight $\mu \leq \lambda$.
These functors will be compatible for comparable coweights (i.e. we have isomorphisms
$\Psi_\mu \cong \Psi_\lambda|_{D^{\check{G}}_{perf}(U^{[]}_\hbar)_{\leq\mu}}$ for $\mu \leq \lambda$, satisfying the obvious compatibility for a triple of
coweights $\nu \leq \mu \leq \lambda$).

The action of $\mathbf{G_O} \rtimes \mathbb{G}_m$ on $\mathrm{Gr}_{\leq\lambda}$ factors through a finite dimensional algebraic group
H_λ, and $D_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr}_{\leq\lambda}) \cong D_{H_\lambda}(\mathrm{Gr}_{\leq\lambda})$ naturally. To describe Ψ_λ we need to provide
the following data: for a smooth H_λ-equivariant map $X \to \mathrm{Gr}_{\leq\lambda}$, where H_λ acts on X
freely we need to provide a functor $D^{\check{G}}_{perf}(U^{[]}_\hbar)_{\leq\lambda} \to D(X/H_\lambda)$, compatible with pull-
backs (i.e. a Cartesian section of the category of resolutions of $\mathrm{Gr}_{\leq\lambda}/H_\lambda$, cf. [9], 2.4.3).
In view of Theorem 2 we have a map $\mathrm{End}^\bullet(V_{\leq\lambda} \otimes U^{[]}_\hbar) \overset{\sim}{\longrightarrow} \mathrm{End}^\bullet(I_{\leq\lambda}) \to End(I^X_{\leq\lambda})$; here
$V_{\leq\lambda}$ is the sum of all irreducible $\check{G}$-modules with a highest weight less or equal than
λ, $I_{\leq\lambda} \in D_{\mathbf{G_O}\rtimes\mathbb{G}_m}(\mathrm{Gr}_{\leq\lambda})$ is the sum of IC sheaves of all $\mathbf{G_O}$ orbits in $\mathrm{Gr}_{\leq\lambda}$, and $I^X_{\leq\lambda}$
is the pull-back of $I_{\leq\lambda}$ to X. Thus the required functor is given by Proposition 6 in
view of purity of equivariant Ext's between IC sheaves on the affine Grassmannian, see
e.g. [16]. These functors do indeed form a Cartesian section in view of Proposition 7,
cf. Remark 2. Compatibility between Ψ_λ and Ψ_μ for $\mu \leq \lambda$ is left as an exercise for
the reader.

In view of Proposition 7 (cf. Remark 2), a monoidal structure for the constructed
functor would follow if we show that the functors from $[D^{\check{G}}_{perf}(U^{[]}_\hbar)]^2$, $[D^{\check{G}}_{perf}(U^{[]}_\hbar)]^3$
to the derived category of sheaves on the convolution space, respectively, triple con-
volution space, given, respectively, by $(M_1, M_2) \mapsto \Psi(M_1) \boxtimes^{\mathbf{G_O}\rtimes\mathbb{G}_m} \Psi(M_2)$, and by
$(M_1, M_2, M_3) \mapsto \Psi(M_1) \boxtimes^{\mathbf{G_O}\rtimes\mathbb{G}_m} \Psi(M_2) \boxtimes^{\mathbf{G_O}\rtimes\mathbb{G}_m} \Psi(M_3)$ (where $\boxtimes^{\mathbf{G_O}\rtimes\mathbb{G}_m}$ denotes the
twisted external product on the convolution space), are compatible with the functors
stemming from the construction of Proposition 6. This follows from Remark 3. $\square$

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