P-ADIC WHITTAKER PATTERNS

Contents

0.1. Conventions 1
1. Introduction: mixed characteristic Casselman-Shalika formula 3
1.1. The associated function from Frobenius trace 3
1.2. The geometric Casselman-Shalika formula 3
1.3. Related works 4
1.4. Check list 5
2. Drinfeld's Compactification 6
2.1. Generalization: Bundles with divisors 7
3. Character sheaf 8
4. Orbit Intersections: Mirkovic-Vilonen Cycles 10
4.1. Recollection on affine Grassmanian 12
5. Convolution 13
6. Cohomological Computation 14
6.1. The case when $\nu \neq \lambda$ 16
7. Random thoughts 17
8. Some thoughts on 11.1 18
8.1. Is our definition of h bogus 20
8.2. What could potentially work? 20
8.3. Thoughts on the Fourier transform. 22
9. Witt vector attempt 22
9.1. Character sheaf 23
9.2. Equivariance 24
9.3. Proof 25
References 29

0.1. **Conventions.** We will fix p a prime. $l \neq p$.

- Let $\mathcal{O}$ be a complete discrete valuation ring, with fraction field K, residue field k of characteristic p.
- Pftd is the category of affinoid perfectoid spaces.
- k is a complete algebraically closed field of characteristic p, and $|k| = q$. We will sometimes write $* = \operatorname{Spa} k$ for the basepoint.
- $\mathrm{Pftd}_k := \mathrm{Pftd}_{\operatorname{Spd} k}$ is the category of perfectoid spaces over $\operatorname{Spd} k$. We will be taking the valuation topology.
- E is a local field with residue field $\mathbb{F}_q$, char. p, uniformizer p.

Date: December 2, 2023.

- For any $S \in \mathrm{Pftd}_k$ we let X_S denote the relative Fargues–Fontaine curve over S.
- $e \in \mathrm{CAlg}_{\mathbb{Z}_l[\sqrt{q}]}$, i.e. $\overline{\mathbb{Q}_l}$.
- $L \in \mathrm{CAlg}^{l-\mathrm{tors}}_{\mathbb{Z}_l[\sqrt{q}]}$, i.e. $l^i L = 0$ for some $i \geq 1$.

1. Introduction: mixed characteristic Casselman-Shalika formula

Let G be a split connected reductive algebraic group over the finite field $\mathbb{F}_q$. Let

$$\mathrm{Sph}_{G,e}^{\heartsuit} := \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G, e)$$

be the *spherical category* of G, or the category of L^+G equivariant perverse sheaves on Gr_G with coefficients in e. For e a field, this is a *highest weight* category, with standard and costandard objects,

$$j_!(\lambda, e) := \pi_0 j_!^{\lambda} k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle] \text{ and } j_*(\lambda, e) := \pi_0 j_*^{\lambda} k_{\mathrm{Gr}^\lambda}[\langle \lambda, 2\check{\rho}\rangle]$$

If e is of characteristic 0, the category is semisimple, with simple objects

$$\{\mathcal{A}_\lambda := j_{!*}(\lambda, e)\}_{\lambda\in\Lambda_+}$$

By the classical Satake isomorphism, this is isomorphic to

$$\mathrm{Rep}(\widehat{G}, e)$$

algebraic representations of the dual group of G with coefficients in e, [**MV07**]. The reader is welcome to skip from here to the statement of geometric Casselman-Shalika, 1.2.

1.1. The associated function from Frobenius trace.

$$A_\lambda(x) := \mathrm{Tr}(\mathrm{Fr}_q, (\mathcal{A}_\lambda)_x)$$

defined on the set of k points of $\overline{\mathrm{Gr}^\lambda}$, can be viewed as a function of the unramified Hecke algebra [**Gro98**], $\mathcal{H}_G$[1]. The constant term map

$$\mathcal{H}_G \to \mathcal{H}_T, f \mapsto f^B$$

has formula given by

$$f^B(t) := \delta_{B(K)}^{1/2}(t) \int_{N(K)} f(tu)\, du$$

The obvious basis elements $\{f_\lambda\}_{\lambda\in X_{\bullet_+}} \subset \mathcal{H}_G$, defined as indicator functions of double cosets, has a surprisingly simple formula, [NP01], under the constant term map

$$f_\lambda^B(t) = \int_{N(K)} A_\lambda(x\varpi^\nu)\, dx = (-1)^{2\langle\rho,\nu\rangle} q^{\langle\rho,\nu\rangle} m_\lambda(\nu)$$

where ρ is the half sum of positive roots.

1.2. The geometric Casselman-Shalika formula. The equal characteristic *geometric* Casselman -Shalika states

Theorem 1.1. *[FGV01]*8.1.2*

$$H_c^i(S^\mu, j_{!*}(\lambda, e)\Big|_{S^\mu} \otimes_e \chi_\mu^*(\mathcal{L}_\psi)) = \begin{cases} e & \text{if } \lambda = \mu \text{ and } \langle 2\check{\rho}, \lambda\rangle = i \\ 0 & \text{otherwise.}\end{cases}$$

where $\mathcal{L}_\psi$ is pullback of Artin-Schrier sheaf from a nondegenerate character $\psi : N \to \mathbb{G}_a$.

[1] compactly supported functions in $G(K)$ this is bi-equivariant with respect to $G(\mathcal{O})$

This is a geometrization of the classical Casselman-Shalika formula described in 1.1. A baby version without the character is used by Lusztig in giving the weight structure of the Satake category. The first goal of the project is therefore to give a mixed characteristic (of the geometry) version. This will make extensive use of recent of results of Fargues and Scholze, [FS21].

The project's second goal is to set up the foundations of Whittaker category in mixed characteristic, by understanding it as a left module over the spherical Hecke category. This is important in setting up geometric Langlands in the mixed characteristic setting, see 1.3.

By generalizing, suggests a fundamental property of the representation theory of reductive groups over local non-archimedean fields and allows one to import further arithmetic information.

1.3. **Related works.** Beyond its applications in the original paper. [FGV01], the geometric CS formula in equal characteristic has been applied in recent work [Bez+19] to give an *Iwahori-Whittaker model* of the Satake category.

The implication of such a geometric model is twofold. Firstly, it gives a geometric description of the representation category.

$$D^b_{\mathrm{IW}}(\mathrm{Gr}_G, e) \simeq D^b(\mathrm{Rep}_e(\check{G})^\heartsuit)$$

But further shows the derived category is *abelian*, which is much more easy to control.

Secondly, this result fits in the framework of *fundamental local equivalence* (FLE), a program initiated by D. Gaitsgory, [**Gai16**]. The equivalence is present in [**DR20**]*Thm. 3. The Iwahori-Whittaker model is what the Whittaker filtration stabilizes to, see [**Ras16**].

1.4. **Check list.**

(1) Construction of candidate Whittaker category.
   - Compatification, and allowing divisors. We define this in 2.1.
   - "Evaluation" morphism.

(2) Affineness of embedding
$$\mathrm{Bun}_N^{\mathcal{F}_T} \hookrightarrow \overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$$
Affiness guarantee's nice preservation of perversity. This is content of [FGV01]*3.

(3) Constructing the Hecke action, Hk $\circlearrowright$ Whit. This action satisfies: [FGV01]*Thm. 4,
$$\bar{\Psi}_{\varpi}^{x,0} * \mathcal{A}_\lambda \simeq \bar{\Psi}_{\varpi}^{x,\lambda}$$
which is the content of [FGV01]*7. As a formal consequence, we first obtain proof of semi-simplicity, [FGV01]*Thm. 3(1).

(4) [FGV01]*6, one obtains the cleanness property.

(5) The cleanness property is used to deduce the main theorem, [FGV01]*8.

Things we would like to see elaborated:

(1)

2. Drinfeld's Compactification

We make the following constructions, [FGV01]*p15

$$
{}_{\bar{x},\bar{\nu}}\mathrm{Bun}_N^{\mathcal{F}_T} \xhookrightarrow{\text{open}} {}_{\bar{x},\bar{\nu}}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T} \xhookrightarrow{\text{open}} {}_{\bar{x},\bar{\nu}}\overline{\mathrm{Bun}}_N^{\mathcal{F}_T} \xhookrightarrow{\text{open}} {}_{\bar{x},\infty}\overline{\mathrm{Bun}}_N^{\mathcal{F}_T} \tag{1}
$$

and prove the following pull backs,

$$
\begin{tikzcd}
{}^{k}\mathcal{N}_y^{\epsilon} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Sch} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & {}^{k}\tilde{\mathcal{N}}_y^{\epsilon} \arrow[d] \\
\mathrm{Bun}_N^{\mathcal{F}_T} \arrow[r, hook, "\text{open}"] & {}_{y,0}\widetilde{\mathrm{Bun}}_{N,\mu}^{\mathcal{F}_T} \arrow[r] & {}_{y,0}\widetilde{\mathrm{Bun}}_N^{\mathcal{F}_T}
\end{tikzcd} \tag{2}
$$

Via the Tannakian formalism, [FS21]*III, $\mathrm{Bun}_B \in \mathrm{Shv}(\mathrm{Pftd}_k, e)$ has moduli description

$$S \mapsto (\mathcal{F}_{G,S}, \mathcal{F}_{T,S}, \kappa)$$

- $\mathcal{F}_G \in \mathrm{Bun}_G(S)$,
- $\mathcal{F}_T \in \mathrm{Bun}_T(S)$, and
- κ is a collection of injective morphisms

  $$\kappa^{\mathcal{V}} : (\mathcal{V}^U)_{\mathcal{F}_T} \to \mathcal{V}_{\mathcal{F}_G}$$

  satisfying the Plücker relations e.g. as stated in [Ham22, Section 5].

The natural maps $G \leftarrow P \to M$ induce morphisms of v-stacks

$$\mathrm{Bun}_G \leftarrow \mathrm{Bun}_P \to \mathrm{Bun}_M \tag{3}$$

by precomposition.

Definition 2.1. For $\mathcal{F}_T \in \mathrm{Bun}_T(*)$, let $\mathrm{Bun}_N^{\mathcal{F}_T}$ the pullback

$$
\begin{tikzcd}
\mathrm{Bun}_N^{\mathcal{F}_T} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Bun}_N \arrow[d] \\
\mathrm{pt} \arrow[r, "\mathcal{F}_T"] & \mathrm{Bun}_T
\end{tikzcd}
$$

Definition 2.2. Let $\overline{\mathrm{Bun}}_N^{\mathcal{F}_T}$ denote the v-stack[2]

$$S \mapsto (\mathcal{F}_{G,S}, \overline{\kappa})$$

- $\mathcal{F}_G \in \mathrm{Bun}_G(S)$

[2]Not sure why this is so yet

- $\overline{\kappa}$ consists of the collection of
$$\left\{\kappa^{\mathcal{V}} : (\mathcal{V}^U)_{\mathcal{F}_T} \to \mathcal{V}_{\mathcal{F}_G}\right\}_{\mathcal{V}\in \mathrm{Rep}(G)}$$
except now $\overline{\kappa}$ is a map of $\mathcal{O}_{X_S}$-modules such that
  - each $\overline{\kappa}^{\mathcal{V}}$ is fiberwise injective (in the sense of [AL21]*2.3) and
  - the usual Plücker relations are satisfied, as [Ham22, Definition 5.6].

2.1. **Generalization: Bundles with divisors.**

Proposition 2.3.
$$_{\bar{x},\bar{\nu}}Bun_N^{\mathcal{F}_T} \simeq Bun_N^{\mathcal{F}_T}$$

Denote $\bar{\nu}' \geq \bar{\nu}$ if $\bar{\nu}' - \bar{\nu} \in \mathbb{N}_+\check{\Phi}^+$.

Definition 2.4.
$$_{\bar{x},\infty}\overline{\mathrm{Bun}_N}^{\mathcal{F}_T} := \varinjlim {}_{\bar{x},\bar{\nu}}\overline{\mathrm{Bun}_N}^{\mathcal{F}_T}$$

3. Character sheaf

Lemma 3.1. *There is an isomorphism*

$$Bun_N \cong [*/\underline{N(E)}]$$

where $\underline{N(E)}$ denotes the constant pro-étale sheaf associated with the locally profinite group $N(E)$.

Proof. We prove this by induction on N.

First, suppose $N \cong \mathbb{G}_a$. By the Tannakian formalism, the data of a $\mathbb{G}_a$-bundle on X_S is the same as a short exact sequence

$$0 \to \mathcal{O}_{X_S} \to \mathcal{V} \to \mathcal{O}_{X_S} \to 0$$

of vector bundles on X_S. In other words, it is determined by an element of

$$\mathrm{Ext}^1_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}, \mathcal{O}_{X_S}) = H^1(X_S, \mathcal{O}_{X_S}).$$

By [FS21, Proposition II.2.5] the pro-étale sheafification of the functor $S \mapsto H^1(X_S, \mathcal{O}_{X_S})$ vanishes so pro-étale locally, the only $\mathbb{G}_a$-bundle is

$$0 \to \mathcal{O}_{X_S} \to \mathcal{O}_{X_S} \oplus \mathcal{O}_{X_S} \to \mathcal{O}_{X_S} \to 0$$

up to isomorphism. An endomorphism of this $\mathbb{G}_a$-bundle is a morphism of short exact sequences which induces identities on the ends, which can be represented as a matrix $\begin{pmatrix} \mathrm{id} & \alpha \\ 0 & \mathrm{id} \end{pmatrix}$ where

$$\alpha \in \mathrm{End}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}) = \mathrm{Hom}_{\mathcal{O}_{X_S}}(\mathcal{O}_{X_S}, \mathcal{O}_{X_S}) = H^0(X_S, \mathcal{O}_{X_S})$$

which is pro-étale locally $\underline{E}(S)$. Therefore the natural map

$$[*/\underline{E}] \to \mathrm{Bun}_{\mathbb{G}_a}$$

given by inclusion of the trivial bundle is an isomorphism of stacks.

Now suppose $\dim N > 1$, so that there is a nontrivial unipotent subgroup N' of N such that $N'/N \cong \mathbb{G}_a$, **[Spr98]**. This induces a sequence of maps

$$\begin{tikzcd} \mathrm{Bun}_{N'} \arrow[r, "\sim"] \arrow[d] & B\underline{N'(E)} \arrow[d] \\ \mathrm{Bun}_N \arrow[r] \arrow[d] & B\underline{N(E)} \arrow[d] \\ \mathrm{Bun}_{\mathbb{G}_a} \arrow[r, "\sim"] & B\underline{E} \end{tikzcd}$$

Both vertical sequences are fibre sequences; therefore, the middle horizontal map is an isomorphism. □

Recall from [FS21, p. III.3] that there is a Beauville–Laszlo uniformization map

$$\mathrm{Gr}_G \to \mathrm{Bun}_G$$

which is a surjective morphism of v-stacks.

We can use this to construct a map

$$h : LN \to LN/L^+N = \mathrm{Gr}_G \to \mathrm{Bun}_N \xrightarrow{\sim} B\underline{N(E)} \to B\underline{E}$$

where the last map is induced by

$$N \to N/[N,N] \cong \bigoplus_{\text{simple roots}} \mathbb{G}_a \xrightarrow{+} \mathbb{G}_a$$

But $[*/\underline{E}]$ is the moduli stack of pro-étale $\underline{E}$-torsors on the Fargues–Fontaine curve, so any representation $\rho : E \to \mathrm{GL}_n(\mathbb{Q}_\ell)$ corresponds to an ℓ-adic local system on $B\underline{E}$ of rank $\dim \rho$.

Definition 3.2. Fix a non-trivial character $\psi : E \to \overline{\mathbb{Q}}_\ell^\times$. We let $\mathcal{L}_\psi$ denote the ℓ-adic local system on $B\underline{E}$ corresponding to ψ.

We can then pull this back to obtain an ℓ-adic local system $h^*\mathcal{L}_\psi$ on LN.

4. Orbit Intersections: Mirkovic-Vilonen Cycles

To compute the Hecke action, we need to understand the intersection of semi-infinite orbits [Fre+98, p. 7]. These played a dominant role in the first complete proof of geometric Langlands [**MV07**]. Over $\mathbb{C}$, the statement has already appeared in the work of [**Lus82**]. In mixed characteristic, this was discussed [Zhu17, p. 2.2]. Let us recall the semi-infinite orbits in the p-adic setting from [FS21, p. VI.3]. [Ham22, p. 4.2]. To make the first cohomolgoical computation, we follow the argument of Ngô-Polo [NP01, p. 5].

Definition 4.1. Let $\Omega_\mu := \{\mu \in X_\bullet : \lambda^+ \leq \mu\}$, where λ^+ is the unique dominant W-translate of λ. [3]

For (possible) future use, we consider the *Beilinson Drinfeld Grassmanian*, which we recall in 4.1. For convenience, we omit the base stack of divisors Div^I. In this section, G is a split reductive group over K, a p-adic field. [4] We thus fix a split reductive model over $\mathcal{O}_K$.

Definition 4.2. Let I be a finite set. For $\nu_\bullet := (\nu_i)_{i\in I} \in (X_\bullet)^I$. The *semi-infinite obrit* associated to $\nu_\bullet$ is the small v-sheaf $S_G^{\nu_\bullet} \in \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_p}, v)_{/\mathrm{Div}^I}$ given by the pullback

$$\begin{tikzcd}
S_G^{\nu_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_B^I \arrow[d] \\
\mathrm{Gr}_T^{\nu_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_T^I \arrow[d] \\
\mathrm{pt} \arrow[r, "\nu_\bullet"] & (X_\bullet)^I
\end{tikzcd}$$

Definition 4.3. For $\lambda \in X_{\bullet,+}^I$, we let $\mathrm{Gr}_G^{\lambda_\bullet}$ be the locally closed subfunctor of Gr_G^I.

Definition 4.4. Let

$$\begin{tikzcd}
\mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \arrow[d] \\
\mathrm{Hck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \arrow[r, hook] & \mathrm{Hck}_{G,\mathrm{Div}^1_{\mathcal{Y}}}
\end{tikzcd}$$

be the inclusion of open cells, [FS21, p. IV.7.5], and denote

$$\mathcal{A}_\mu := j_{\mu!}\Lambda[d_\mu]$$

as the IC sheaves.[5]

To set the stage, we recall the Satake isomorphism in the mixed characteristic setting

Theorem 4.5. *[FS21, p. I.6.3] For a finite index I,*

$$Sat_G^I \simeq Rep_\Lambda({}^L G^I)$$

[3]Alternatively, this is $\lambda + \mathbb{Z}\Phi^\vee \cap \mathrm{Conv}(W\lambda)$

[4]One can always base change when necessary.

[5]The typical analysis of such sheaves on Hck stack pullsback further to the Demazure resolution.

Proposition 4.6. *[Ham22, p. 4.4] For all finite index sets I, the followin diagram commutes*

$$\begin{tikzcd} Sat_G^I \arrow[r, "CT[\deg]"] \arrow[d, "F_G^I"] & Sat_T^I \arrow[d, "F_T^I"] \\ Rep_\Lambda({}^LG) \arrow[r, "res_T^I"] & Rep_\Lambda^I({}^LT) \end{tikzcd}$$

where

- *CT is the constant term functor.*
- *F_G^I, F_I^T are due to Tannakian equivalence [FS21, Thm 1.6.3].*

Proposition 4.7. *Let $\lambda \in X_{\bullet,+}$. Let $x \to Div^1$ be a geometric point.*

$$H_c^k({}_xS^\nu \cap \overline{{}_x\operatorname{Gr}^\lambda}, \mathcal{A}_\lambda)$$

vanishes unless $k = \langle 2\rho, \nu\rangle$, in which case, it is isomorphic to $V^\lambda(\nu)^\vee$.

Proof. Let us consider the following diagram

$$\begin{tikzcd} \mathrm{pt} & S^\lambda \arrow[l, "p"'] \arrow[r, hook, "q"] & \mathrm{Gr} \\ & S^\lambda \cap \overline{\mathrm{Gr}^\mu} \arrow[ul, "p'"] \arrow[u, hook] \arrow[r, hook, "q'"] & \overline{\mathrm{Gr}^\mu} \arrow[u, hook] \\ & & \mathrm{Gr}^\mu \arrow[u, hook] \end{tikzcd}$$

Let $\mathcal{S}_{V^\lambda}$ be the sheaf corresponding to highest weight representation V^λ, as 4.5. Then by applying 4.6,

$$\begin{aligned} H_c^k({}_xS^\nu \cap \overline{{}_x\operatorname{Gr}^\lambda}, \mathcal{A}_\lambda) &= (p')_!(q')^*(\mathcal{A}_\lambda) \\ &\simeq p_!q^*(\mathcal{S}_{V^\lambda}) \\ &= H_c^{-\langle 2\rho,\nu\rangle}(S^\nu, \mathcal{S}_{V^\lambda}) \\ &\simeq V^\lambda(\nu)^\vee \end{aligned}$$

□

4.0.1. *Properties of orbit intersection.*

Proposition 4.8. *[**BR18**], [**She22**] Let $\lambda, \nu \in X_\bullet$ with λ dominant, $x \to Div^1$ be a geometric point.*

(1) Nonemptiness.

$${}_xS^\nu \cap \overline{{}_x\operatorname{Gr}^\lambda} \neq \emptyset \Leftrightarrow \nu \in \Omega_\lambda$$

(2) *Dimension.*

$$ {}_xS^\nu \cap {}_x\operatorname{Gr}^{\leq\nu} $$

is equidimensional of rank $\langle\rho,\nu+\lambda\rangle$.

(3) *Containment property.*

$$ \bigsqcup_{\nu\in\Omega_\lambda} {}_xS^\nu \cap \overline{{}_x\operatorname{Gr}^\lambda} \xrightarrow{\simeq} {}_x\operatorname{Gr}^{\leq\nu} $$

of underlying topological spaces.

4.1. **Recollection on affine Grassmanian.** We will consider the B_{dR}^+ affine Grassmanian. The local definition can be specialized from the global definition. We include the latter when we need to describe the Hecke action.

Let $S \in \mathrm{Pftd}_{\mathbb{F}_q}$. Recall in [FS21, p. II], we could construct curves

$$ \mathcal{Y}_S, Y_S := \mathcal{Y}_S \backslash V(\pi) \text{ and } X_S = Y_S/\varphi^{\mathbb{Z}} $$

We can define the following stacks of divisors on such curves.

Definition 4.9. We have the following small v-sheaves $\mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_q}, v)$

$$ \mathrm{Div}^1_{\mathcal{Y}} := \mathrm{Spd}(\mathcal{O}_K) $$

$$ \mathrm{Div}^1_X := \mathrm{Div}^1 := \mathrm{Spd}\, K/\varphi^{\mathbb{Z}} $$

where Div^1 is the *mirror curve* [6] For a finite set I with $|I| = d$, we will denote

$$ \mathrm{Div}^I_{\mathcal{Y}} := (\mathrm{Div}^1_{\mathcal{Y}})^d $$

Definition 4.10. Let I be a finite set.

$$ \operatorname{Gr}^I_{G,\mathrm{Div}^1_{\mathcal{Y}}} \to \mathrm{Div}^I_{\mathcal{Y}} $$

$$ \operatorname{Gr}^I_{G,\mathrm{Div}^1} \to \mathrm{Div}^I $$

be the *Beilinson-Drinfeld* Grassmanian [FS21, p. VI.1.8]. This is a small v-sheaf. Unless stated otherwise, will omit the Div^I. For $S \to \mathrm{Div}^d_{\mathcal{Y}}$ we denote

$$ \operatorname{Gr}_{G,S} := \operatorname{Gr}_G \times_{\mathrm{Div}^d_{\mathcal{Y}}} S $$

[6] Its S points are the degree 1 Cartier divisors on X_S, where one has $\pi_1(\mathrm{Div}^1) = W_K$.

5. Convolution

Recall, def. **??**.

Definition 5.1 (Twisted product)**.** If H is an algebraic group and X is an L^+H-space, then the *twisted product*

$$\begin{array}{c} \mathrm{Gr}_H \tilde{\times} X := LH \times^{L^+H} X \\ \downarrow \\ \mathrm{Gr}_H \end{array}$$

forms a new fiber bundle with fibers X.

There is a moduli description

$$\mathrm{Gr}_G \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_G = \left\{ \mathcal{E}_1 \overset{\beta_1}{\dashrightarrow} \cdots \overset{\beta_{n-1}}{\dashrightarrow} \mathcal{E}_n \overset{\beta_n}{\dashrightarrow} \mathcal{E}^0 \right\}$$

Recall that we have a fiber sequence $N \to B \to T$ which functorially induces

$$\mathrm{Gr}_N \to \mathrm{Gr}_B \to \mathrm{Gr}_T\,.$$

But $\mathrm{Gr}_T = \bigsqcup_{\nu \in X_*(T)} \mathrm{Gr}_T^{\nu}$ and so we let

$$S_\nu := \mathrm{Gr}_B \times_{\mathrm{Gr}_T} \mathrm{Gr}_T^{\nu}$$

Note that the restriction of the L^+G-torsor $LG \to \mathrm{Gr}_G$ over S_ν has a canonical reduction as a L^+N-torsor given by

$$LN \to S_\nu, \quad n \mapsto n \cdot t^\lambda \mod L^+G.$$

So if we take $H = N$, for $\nu_\bullet = (\nu_1, \ldots, \nu_m)$ any tuple in $X_*(T)$ we can form the twisted product

$$S_{\nu_\bullet} = S_{\nu_1} \tilde{\times} \cdots \tilde{\times} S_{\nu_m}$$

Definition 5.2 (multiplication map)**.** Let

$$m : \mathrm{Gr}_G \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_G \to \mathrm{Gr}_G$$
$$(\mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_n) \mapsto (\mathcal{E}_n, \beta_1 \cdots \beta_n)$$

be the projection on to the nth component.

Proposition 5.3.

$$\begin{tikzcd} S_{\nu_\bullet} \arrow[r, "\simeq"] \arrow[d, hook] & S_{\nu_1} \times S_{\nu_1+\nu_2} \times \cdots \times S_{|\nu_\bullet|} \arrow[d, hook] \\ \mathrm{Gr}\, wt \times \cdots wt \times \mathrm{Gr} \arrow[r] & \mathrm{Gr} \times \cdots \times \mathrm{Gr} \simeq \mathrm{Gr}^n \end{tikzcd}$$

Recall the map Def. **??**.

Definition 5.4. For $\sigma \in X_*(T)$, let $h_\sigma := h \circ \mathrm{ad}(t^\sigma)$, where $\mathrm{ad}(t^\sigma) : LN \to LN$ is the adjoint action.

6. Cohomological Computation

Recall the construction of h, **??**.

Theorem 6.1 ([NP01, p. 3.1])**.** *For $\lambda \in X_{\bullet,+}$*

$$R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h^*\mathcal{L}) = \begin{cases} \bar{\mathbb{Q}}_l(\langle \rho, \lambda \rangle) & \nu = \lambda \\ 0 & \nu \neq \lambda \end{cases}$$

Proof. The case when $\nu = \lambda$ follows from the fact that h is trivial on $\mathrm{MV}_{\lambda,\lambda}$, so that $h^*\mathcal{L}$ is constant, and we are reduced the case in Prop. **??**. □

As we do not have the splitting as [NP01, p. 9.1], we will follow [Zhu17] to construct a splitting.

Definition 6.2. If Z is an affine scheme over E and $r \geq 0$ is an integer, the *truncated loop space* of level r is

$$L^r Z = Z(B^+_{\mathrm{Div}_X}/\mathcal{I}^r_S).$$

More precisely, for $S = \mathrm{Spa}(R, R^+) \to \mathrm{Div}_X$ denote by $(R^\sharp, R^{\sharp+})$ the corresponding untilt, and let ξ denote a generator of $\ker(\theta : W_{\mathcal{O}_E}(R^+) \to R^{\sharp+})$. Then

$$L^r Z(R, R^+) \simeq Z(B^+_{\mathrm{dR}}(R^\sharp)/\xi^r).$$

For $r \geq 0$ is then a natural quotient map

$$L^+ Z \to L^r Z$$

Definition 6.3. Let $\mathrm{MV}_{\nu,\mu} := S_\nu \cap \mathrm{Gr}_{\leq\mu}$.

Definition 6.4. For $r \in \mathbb{N}_{\geq 0} \cup \{\infty\}$, we will consider the following bundles

$$\begin{tikzcd} (\mathrm{MV}_{\nu,\mu})^{(r)} \arrow[r] \arrow[d, "p_r"] \arrow[dr, phantom, "\lrcorner"] & S_\nu^{(r)} := L^r N \times_{L^+N} LN \arrow[d] \\ \mathrm{MV}_{\nu,\mu} \arrow[r, hook] & S_\nu \end{tikzcd}$$

where by convention we set $L^\infty N := L^+ N$.

Lemma 6.5. *For $r \geq 0$, the action of L^+N on $\mathrm{MV}^{(r)}_{\nu,\mu}$ factors through $L^{r'}N$ for some $r' > 0$.*

Proof. Working pro-étale locally, this reduces to the fact that the L^+G-action on $\mathrm{Gr}_{\leq\mu}$ factors through L^rG for some $r > 0$ which depends on μ. TODO: NEED TO EXTEND THIS TO THE ACTION ON $\mathrm{MV}^{(r)}_{\nu,\mu}$, BUT THIS MIGHT BE IMMEDIATE BECAUSE IT'S AN L^r-TORSOR. Should actually probably just use the moduli description for this via bundles and the action via changing the trivialization. □

By the lemma we can choose integers $r_1, \ldots, r_m \geq 0$ such that $r_m = 0$ and such that the action of L^+N on $\mathrm{MV}^{(r_i)}_{\nu_i,\mu_i}$ factors through $L^{r_{i-1}}N$.

Lemma 6.6. *There is an* $\prod_i L^{r_i}U$ *torsor*

$$\begin{tikzcd} \prod_{i=1}^n (\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)} \arrow[d, "q_\bullet"] \\ \mathrm{MV}_{\nu_\bullet,\mu_\bullet} \end{tikzcd}$$

such that

$$q_\bullet^* IC_{\mu_\bullet} \cong p_\bullet^* \left(IC_{\mu_1} \boxtimes \cdots \boxtimes IC_{\mu_n} \right)$$

where $p_\bullet$ *is the map*

$$\begin{tikzcd} \prod_{i=1}^n (\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)} \arrow[d, "p_\bullet"] \\ \prod (\mathrm{MV}_{\nu_i,\mu_i}) \end{tikzcd}$$

Proof. For simplicity, first suppose $m = 2$.

There is an L^+N-torsor $LN \to S_{\nu_i}$. Since $\mathrm{MV}_{\nu_i,\mu_i}$ is an L^+N-invariant subspace, this restricts to an L^+N-torsor $\mathrm{MV}^{(\infty)}_{\nu_i,\mu_i} \to \mathrm{MV}_{\nu_i,\mu_i}$. Since the action of L^+N on $\mathrm{MV}_{\nu_2,\mu_2}$ factors through the quotient map $L^+N \to L^rN$, we get a commuting diagram

$$\begin{tikzcd} & \mathrm{MV}^{(\infty)}_{\nu_1,\mu_1} \times \mathrm{MV}_{\nu_2,\mu_2} \arrow[d, "p_\infty"] \arrow[ddl, "p"'] \arrow[ddr, "q"] & \\ & \mathrm{MV}^{(r_1)}_{\nu_1,\mu_1} \times \mathrm{MV}_{\nu_2,\mu_2} \arrow[dl, "p_r \times \mathrm{id}"] \arrow[dr, "q_r"'] & \\ \mathrm{MV}_{\nu_1,\mu_1} \times \mathrm{MV}_{\nu_2,\mu_2} & & \mathrm{MV}_{\nu_1,\mu_1} \tilde{\times} \mathrm{MV}_{\nu_2,\mu_2} \end{tikzcd}$$

in which q is an L^+N-torsor and q_r is an L^rN-torsor. The morphism p_∞ is just the quotient by $\ker(L^+N \to L^rN)$ in the first slot and the identity in the second. The point now is that there is a unique perverse sheaf $\mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}$ on $\mathrm{MV}_{\nu_1,\mu_1} \tilde{\times} \mathrm{MV}_{\nu_1,\mu_2}$ satisfying

$$p^*(\mathrm{IC}_{\mu_1} \boxtimes \mathrm{IC}_{\mu_2}) \cong q^*(\mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}).$$

There is also a unique perverse sheaf $\mathcal{L}$ satisfying

$$q_r^* \mathcal{L} \cong p_r^*(\mathrm{IC}_{\mu_1} \boxtimes \mathrm{IC}_{\mu_2})$$

But pulling back by p_∞ gives $q^*\mathcal{L} \cong p^*(\mathrm{IC}_{\mu_1} \boxtimes \mathrm{IC}_{\mu_2})$ so we must have $\mathcal{L} \cong \mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}$ by uniqueness.

For $m > 2$, the same argument above gives an $L^{r_{m-1}}N$-torsor

$$\mathrm{MV}^{(r_{m-1})}_{\nu_{n-1},\mu_{m-1}} \to \mathrm{MV}_{\nu_{m-1},\mu_{m-1}} \tilde{\times} \mathrm{MV}_{\nu_m,\mu_m}$$

Then one can continue inductively, with $\mathrm{MV}^{(r_{m-1})}_{\nu_{m-1},\mu_{m-1}} \tilde{\times} \mathrm{MV}_{\nu_m,\mu_m}$ (with its natural $L^{r_{m-2}}N$-action) playing the role of $\mathrm{MV}_{\nu_m,\mu_m}$. ∎

$$\begin{tikzcd}
\prod_{i=1}^n (\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)} \arrow[d, "q_\bullet"] \arrow[drrr, "h_\bullet"] & & & \\
\mathrm{MV}_{\nu_\bullet,\mu_\bullet} \arrow[rr] & & S_{|\nu_\bullet|} \arrow[r, "h"] & B\underline{N(E)}
\end{tikzcd}$$

Lemma 6.7. *If $\mu_\bullet \subset M$ is a tuple of nonzero quasi-minuscule coweights and $(\nu_1, \ldots, \nu_n)$ is a tuple of coweights, then*

$$R\Gamma_c(\mathrm{MV}_{\nu_1,\mu_1}\, wt \times \cdots wt \times \mathrm{MV}_{\nu_n,\mu_n}, IC_{\mu_\bullet} \otimes h_\bullet^* \mathcal{L}_\psi) \simeq \bigotimes_{i=1}^n R\Gamma_c(\mathrm{MV}^{(r_i)}_{\nu_i,\mu_i}, p_i^* IC_{\mu_i} \otimes h^*_{\sigma_i} \mathcal{L}_\psi)$$

Proof. $\mathrm{IC}_{\mu_\bullet}$ splits as $\boxtimes_{i=1}^n p_i^* \mathrm{IC}_{\mu_i}$ over $\prod_{i=1}^n \mathrm{MV}^{(r)}_{\nu_i,\mu_i}$ by Lem. 9.10. As $*$-pullback is symmetric monoidal we have

$$\begin{aligned}
&R\Gamma_c\left(\mathrm{MV}_{\nu_1,\mu_1} \tilde{\times} \cdots \tilde{\times}\, \mathrm{MV}_{\nu_m,\mu_m}, \mathrm{IC}_{\mu_\bullet} \otimes h_\bullet^* \mathcal{L}_\psi\right) \\
&\simeq R\Gamma_c\left(\prod_{i=1}^n (\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)}, \boxtimes_{i=1}^n \mathrm{pr}_i^* \mathrm{IC}_{\mu_i} \otimes \boxtimes_{i=1}^n (h_\sigma \circ q_i)^* \mathcal{L}_\psi\right) \\
&\simeq \bigotimes_{i=1}^n R\Gamma_c((\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)}, \mathrm{pr}_i^* \mathrm{IC}_{\mu_i} \otimes (h_\sigma \circ q_i)^* \mathcal{L}_\psi)[2 \dim N \cdot r_i]
\end{aligned}$$

□

6.1. **The case when $\nu \neq \lambda$.** Using the computation in 9.11, we are thus reduced to the case when each partial sums of $\nu_\bullet$ are non dominant.

The following is a geometric version of the PRV conjecture, and follows from the geometric Satake equivalence in this context.

Lemma 6.8. *There exists a sequence of quasi-minuscule coweights $\mu_\bullet = (\mu_1, \ldots, \mu_m)$ such that $V^\lambda_{\mu_\bullet} \neq 0$ in the decomposition*

$$IC_{\mu_1} \star \cdots \star IC_{\mu_n} = \bigoplus_{\substack{\xi \in X_*(T)_+ \\ \xi \leq \mu_1 + \cdots + \mu_n}} IC_\xi \otimes V^\xi_{\mu_\bullet}.$$

Recall that our goal is to show that

$$R\Gamma_c(S_\lambda, \mathrm{IC}_\lambda \otimes h^* \mathcal{L}_\psi) = 0.$$

By the above direct sum decomposition, it suffices to show the following.

Lemma 6.9. *The inclusion of the direct factor*

$$R\Gamma_c(S_\nu, IC_\nu \otimes h^* \mathcal{L}_\psi) \otimes V^\nu_{\mu_\bullet} \to R\Gamma_c(S_\nu, IC_{\mu_1} \star \cdots \star IC_{\mu_n} \otimes h^* \mathcal{L}_\psi)$$

is a quasi-isomorphism.

7. Random thoughts

Definition 7.1. The additive character on LN is $LN \to LN/L^+N \to \mathrm{Bun}_N \cong B\underline{N(E)} \to B\underline{E}$.

Definition 7.2. The $\mathrm{Bun}_N^{\mathcal{F}_T}$ are basically the $\widetilde{\mathcal{M}}_b$ charts attached to unramified elements in the Kottwitz set. (I have to check this from Linus work, but I am pretty sure). A remark on its cohomology:

In particular they are cohomologically contractible in the sense that there is a point $i\colon * \subset \widetilde{\mathcal{M}}_b$ such that $R\Gamma(\widetilde{\mathcal{M}}_b, A) \cong i^*A$ (note that $\mathcal{D}(*, \Lambda) = \mathcal{D}(\Lambda)$).

Definition 7.3. Affineness of the embedding into the Compactification does not help, we will need to prove t-exactness results by hand.

I am confused about cleanness of the extensions, this should only work in characterstic 0. One can probably already see why this is important classically, but I don't know where (pray to god that this does not use the decomposition theorem, hope that we can just use the corresponding fact for the Satake category).

Remark 7.4. I am pretty sure the simply connectedness assumption on $[G, G]$ is not needed by the way. The point is classically this ensures that the Beaville-Laszlo map is surjective (it guarantees that any G-bundle becomes trivializable after removing a point from the curve). However, for the Fargues-Fontaine curve this is not needed.

8. Some thoughts on 11.1

This is regarding [NP01, p. 11.1]

Proposition 8.1. *If $\sigma \notin X_{\bullet,+}$ we have that*

$$R\Gamma_c(S_\nu, \mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi) = 0$$

Proof. this is classical argument

(1) $\mathcal{A}_\lambda \otimes h_\sigma^* \mathcal{L}_\psi$ is $(\mathbb{G}_a, \mathcal{L}_\psi)$s equivariant. We have a $\mathbb{G}_a$ action on S_ν inducing the following commutative diagram.

$$\begin{tikzcd}
& & LU \times \mathrm{MV}^{\lambda,\nu} \arrow[d, hook] \\
\mathbb{G}_a \times \mathrm{MV}^{\lambda,\nu} \arrow[r, hook, "u_\alpha"] \arrow[dd, "\mathrm{id}\times h_\sigma"] & L^+U \times \mathrm{MV}^{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[ur, hook] \arrow[d, "\mathrm{id}\times h_\sigma^\nu"] & \mathrm{MV}^{\lambda,\nu} \arrow[d] \\
& L^+U \times \mathbb{G}_a \arrow[d, "h_\sigma\times\mathrm{id}"] \arrow[r, phantom, "(1)"] & \mathbb{G}_a \arrow[d] \\
\mathbb{G}_a \times \mathbb{G}_a \arrow[r] & \mathbb{G}_a \times \mathbb{G}_a \arrow[r] & \mathbb{G}_a
\end{tikzcd} \tag{4}$$

where $u_\alpha : \mathbb{G}_a \hookrightarrow L^+U$ root group embedding $u_\alpha : \mathbb{G}_a \to L^+G$, twisted by $t^{-\langle\alpha,\sigma\rangle-1}$.[7] Thus we have

$$\begin{aligned}
\mathrm{act}^*\left(\mathcal{A}_\lambda \otimes h_\sigma^*\mathcal{L}_\psi\right) &\simeq \mathrm{act}^*\mathcal{A}_\lambda \otimes \mathrm{act}^* h_\sigma^*\mathcal{L}_\psi \\
&\simeq \mathrm{act}^*\mathcal{A}_\lambda \otimes (\mathrm{id}\times h_\sigma)^* a^*\mathcal{L}_\psi \\
&\simeq \left(\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda\right) \otimes \left(\mathcal{L}_\psi \boxtimes h_\sigma^*\mathcal{L}_\psi\right) \\
&\simeq \left(\overline{\mathbb{Q}}_\ell \otimes \mathcal{L}_\psi\right) \boxtimes \left(\mathcal{A}_\lambda \otimes h_\sigma^*\mathcal{L}_\psi\right)
\end{aligned}$$

Where we used that the box tensor product satisfies

$$(A \otimes B) \boxtimes (C \otimes D) \simeq (A \boxtimes C) \otimes (B \boxtimes D)$$

and that $\mathbb{G}_a$ acts equivariant on $(S_\nu, \mathcal{A}_\lambda)$.

(2) $(\mathbb{G}_a, \mathcal{L}_\psi)$ equivariant sheaves have vanishing cohomology.

□

In [FGV01, p42], they made an alternative argument. This lemma is explained using the following argument:

Proposition 8.2. *Suppose the following two conditions are satisfied.*

- $\mathcal{A}_\lambda \otimes \left(\chi_\mu^\nu\right)^* \mathcal{L}_\psi$ *is* (L^+N, χ_μ) *-equivariant.*

[7] This ensures a scaling of back to -1 after adjoint action.

- *χ_μ is nontrivial for μ dominant.*

Then the cohomology vanishes.

When we ponder about diagram (4) the remaining two questions are:

(1) was the embedding of u_α every necessary?

(2) Does this depend on the fact that L^+N is unipotent?

(3) Would we not be able to replace $\mathbb{G}_a$ with $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ as 9.2.

Proof. Let $S_\nu^\lambda = \mathrm{Gr}^\lambda \cap S_\nu$, let us have maps $S_\nu^\lambda \xrightarrow{i} S_\nu \xrightarrow{h} */\underline{E} \xrightarrow{p} *$. Then by projection formula $R\Gamma(S_\nu, A_\lambda \otimes h^*\mathcal{L}_\psi) = p_!(h_!A_\lambda \otimes \mathcal{L}_\psi)$. Identifying $D(*/\underline{E})$ with smooth E-representations, $h_!A_\lambda$ has the trivial action (since we have $A_\lambda = i_!1$ and clearly the constant sheaf corresponds to the trivial representation), and $\mathcal{L}_\psi$ is a non-trivial character. We we have to check that $p_!\mathcal{L}_\psi = 0$, hopefully easy? (the problem: it is ok for group cohomology, but $p_!$ is not quite group cohomology...) $\square$

8.1. **Is our definition of h bogus.** What is wrong with the h map? Suppose we want to copy and paste the classical argument, first from *our definition* of rank 1-local system, $\mathcal{L} \in D(B\underline{E})$, it satisfies the character condition $a^*\mathcal{L} \simeq \mathcal{L} \boxtimes \mathcal{L}$.

$$\begin{tikzcd}
\mathbb{G}_a \times LN \arrow[r, hook] \arrow[d, "\mathrm{id}\times h_\sigma"] & LN \times LN \arrow[r, "\mathrm{act}"] \arrow[d, phantom, "(1)"] & LN \arrow[d, "h_\sigma"] \\
\mathbb{G}_a \times LN/LN^+ \arrow[rr, "\mathrm{act}"] \arrow[d] & (2) & LN/LN^+ \arrow[d] \\
\mathbb{G}_a \times \mathrm{Bun}_N \arrow[rr, "\mathrm{triv!}"] \arrow[d] & (3) & \mathrm{Bun}_N \arrow[d] \\
B\underline{E} \times B\underline{E} \arrow[rr, "a"] & & B\underline{E}
\end{tikzcd}$$

Ideally: the above diagram should commute/ Then as is the classical case , $\mathcal{A}_\lambda \otimes h_\sigma^*\mathcal{L}_\psi$ is $(\mathbb{G}_a, \mathcal{L}_\psi)$-equivariant.
However, the action of $\mathbb{G}_a \hookrightarrow LN \circlearrowright \mathrm{Gr}_N$ is the one induced on points given by

$$(A, (\mathcal{E}, \varepsilon)) \mapsto (\mathcal{E}, A\varepsilon)$$

Use Gr_N's (global) moduli problem: $\mathcal{E} \in N\mathrm{Tors}(X)$ and trivalization $\varepsilon : \mathcal{E} \simeq \mathcal{E}^0\Big|_{X-x}$ which has a canonical forgetful map (BL uiformization)

$$\mathrm{Gr}_N \to \mathrm{Bun}_N$$

Thus, the action of $\mathbb{G}_a$ becomes trivial after quotienting out to Bun_N.

(1) Would the bottom square *commute*? What is the map

$$\mathbb{G}_a \times \mathrm{Bun}_N \to B\underline{E} \times B\underline{E}?$$

Is this (BC, id)? [8] However, the bottom box doesn't look like it would commute!

Claim: embed $\mathbb{G}_a \hookrightarrow L\mathbb{G}_a$ via $a \mapsto a\xi^{-1}$. This defines an action of $\mathbb{G}_a$ on $L\mathbb{G}_a/L^+\mathbb{G}_a$, as $ab\xi^{-2}\varepsilon = ab\xi^{-1}\varepsilon$ once you mod out the action of $L^+\mathbb{G}_a$. Then (this is complete speculation)

$$\begin{tikzcd}
\mathbb{G}_a \times L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r] \arrow[d] & L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[d] \\
BE \times BE \arrow[r] & BE
\end{tikzcd}$$

commutes.

8.2. **What could potentially work?** Consider the Lang map ,

$$\begin{tikzcd}
\underline{E} \arrow[r] & L\mathbb{G}_a \arrow[d, "x \mapsto \mathrm{Fr}(x)-x"] \\
& L\mathbb{G}_a
\end{tikzcd}$$

[8] The torsor induced from the fundamental exact sequence of p-adic Hodge theory.

This induces a map $L\mathbb{G}_a \to B\underline{E}$, which induces

$$\mathrm{LS}(B\underline{E}) \to \mathrm{LS}(L\mathbb{G}_a)$$

This allows us to pullback sheaf $\psi \in \mathrm{LS}(B\underline{E})$ to $\mathcal{L}_\psi \in \mathrm{LS}(L\mathbb{G}_a)$.
If rather we defined $h : LN \to L\mathbb{G}_a$, Then we would have to modify our diagram from the classical proof to

$$\begin{tikzcd} L\mathbb{G}_a \times LN \arrow[r] \arrow[d] & LN \arrow[d] \\ L\mathbb{G}_a \times L\mathbb{G}_a \arrow[r] & L\mathbb{G}_a \end{tikzcd}$$

However, this diagram wouldn't commute due to the fact that the original diagram commutes precisely due to our choice of $\sigma \notin X_{\bullet,+}$.

(1) Suppose we could get $L\mathbb{G}_a, \psi$ equivariant sheaf

$$\mathrm{act}^* \mathcal{F} \simeq \mathcal{F} \boxtimes \mathcal{L}_\psi$$

In which case the difficulty is in (2). Does an analogue of [Ngô00, Lem3.3] holds? That $(L\mathbb{G}_a, \mathcal{L}_\psi)$-equviaraint sheaves have vanishing of cohomology.

8.3. **Thoughts on the Fourier transform.** Here's a thought. I think we don't need to work "absolutely", and instead can just work over $\operatorname{Spd} C$ or $\operatorname{Spd} E$ or something, but whatever.

If we use the Anschütz–Le Bras formalism, we get the following diagram in the quasi-minuscule case. (We need to first ensure that $\mathrm{MV} = \mathrm{MV}_{\lambda,\nu}$ is a very nice stack in E-vector spaces. They're affine in the minuscule case. I don't know how hard this is in the quasi-minuscule case. Hopefully not hard, since it should be an affine bundle over an affine space.)

$$\begin{tikzcd} & \mathrm{MV}^\vee \times \mathrm{MV} \arrow[dl, "\pi^\vee"'] \arrow[dr, "\pi"] \arrow[rr, "\alpha"] & & B\underline{E} \\ \mathrm{MV}^\vee & & \mathrm{MV} & \end{tikzcd}$$

Given the nondegenerate character $\psi : E \to \overline{\mathbb{Q}}_\ell^\times$, we get a character sheaf $\mathcal{L}_\psi$ on $B\underline{E}$. We can form the Fourier transform

$$\mathcal{F}(A_\lambda) := \pi^\vee_!(\pi^* A_\lambda \otimes \alpha^* \mathcal{L}_\psi)$$

Taking the stalk of the Fourier transform at a point corresponds to evaluating it at a point. Pick a point $y : * \to \mathrm{MV}^\vee$. The choice of a point corresponds, in the usual Fourier transform over $\mathbb{R}$, to choosing some additive character to integrate against. For instance, if $\mathrm{MV} = \mathbb{A}^1$, then choosing the point "1" corresponds to taking the Fourier coefficient corresponding to the additive character ψ. Choosing another (nonzero) point corresponds to twisting ψ first and then

Then we can look at the fiber $\{y\} \times \mathrm{MV} \hookrightarrow \mathrm{MV}^\vee \times \mathrm{MV}$. So we can look at the composite map

$$\mathrm{MV} = \{y\} \times \mathrm{MV} \hookrightarrow \mathrm{MV}^\vee \times \mathrm{MV} \xrightarrow{\sim} \mathrm{MV} \times \mathrm{MV} \xrightarrow{m} \mathrm{MV} \xrightarrow{+} \mathbb{A}^1 \xrightarrow{\mathrm{BC}(\mathcal{O}(1))} B\underline{E}.$$

But this is the same as

$$\mathrm{MV} \xrightarrow{+} \mathbb{A}^1 \xrightarrow{\mathrm{BC}(O(1))} B\underline{E}.$$

But the base change formula means that

$$p^* \pi^\vee_!(\pi^* A_\lambda \otimes \alpha^* \mathcal{L}_\psi) \cong R\Gamma_c(\mathrm{MV}, A_\lambda \otimes)$$

9. Witt vector attempt

Let $L^+\mathcal{X}$ denote the positive loop space if $\mathcal{X}$ is an affine scheme over $\mathcal{O}$, and let LX denote the loop space if X is an affine scheme over F. Let G denote a connected reductive group scheme over $\mathcal{O}$, and let Gr denote the Witt vector affine Grassmannian for G. Let $\mathrm{Gr}^{\leq \lambda}$ and S^ν denote the usual affine Schubert varieties and semi-infinite orbits. We also let

$$\mathrm{MV}_{\lambda,\nu} := \mathrm{Gr}_{\leq \lambda} \cap S_\nu,$$

where "MV" is short for "Mirkovic–Vilonen".

First we define

$$h : LN \to LN/[LN, LN] \xrightarrow{\sim} \prod_\alpha L\mathbb{G}_a \xrightarrow{+} L\mathbb{G}_a \to L\mathbb{G}_a / L^+\mathbb{G}_a$$

If μ is a coweight, we twist h and define

$$h_\mu : LN \xrightarrow{\mathrm{ad}(\varpi^\sigma)} LN \xrightarrow{h} L\mathbb{G}_a/L^+\mathbb{G}_a.$$

Lemma 9.1. *If ν and μ are two coweights such that $\mu+\nu$ is dominant, then the map*

$$\begin{aligned} h^\nu_\mu : S_\nu &\to L\mathbb{G}_a/L^+\mathbb{G}_a \\ n\cdot\varpi^\nu &\mapsto h_\mu(n). \end{aligned}$$

is well-defined.

Proof. If $n_1\varpi^\nu L^+G = n_2\varpi^\nu L^+G$ then $\mathrm{ad}(\varpi^{-\nu})(n_1n_2^{-1}) \in L^+G$. But then

$$h_\mu(n_1n_2^{-1}) = h_\mu(\mathrm{ad}(\varpi^\nu)\mathrm{ad}(\varpi^{-\nu})(n_1n_2^{-1})) = h(\mathrm{ad}(\varpi^{\mu+\nu})\mathrm{ad}(\varpi^{-\nu})(n_1n_2^{-1}))$$

But $\mu+\nu$ is dominant, so $\mathrm{ad}(\varpi^{\mu+\nu})$ preserves L^+G, so we conclude by noting that h is trivial on L^+G. ☐

In the existing proofs of geometric Casselman–Shalika in equal characteristic, the definition of h ends with the residue map $L\mathbb{G}_a \to \mathbb{G}_a$ instead of the projection $L\mathbb{G}_a \to L\mathbb{G}_a/L^+\mathbb{G}_a$. In mixed characteristic this cannot work because additive characters of $\mathbb{Q}_p/\mathbb{Z}_p$ don't factor through $\frac{1}{p}\mathbb{Z}_p/\mathbb{Z}_p$; in fact they don't factor through *any* proper subgroup of $\mathbb{Q}_p/\mathbb{Z}_p$. However, since we only care about the cohomology of finite dimensional subspaces of Gr, once we restrict there, the map h *does* factor through a proper subgroup.

For any $s \in \mathbb{Z}$ there is a multiplication map $L^+\mathbb{G}_a \xrightarrow{\cdot p^s} L\mathbb{G}_a$, and we denote its image by $L^{\geq s}\mathbb{G}_a$.

Lemma 9.2. *If λ is a dominant coweight and and ν is a coweight, there is a factorization*

$$\begin{tikzcd} \mathrm{MV}_{\lambda,\nu} \arrow[r, dashed, "h^{\lambda,\nu}_\mu"] \arrow[d, hook] & L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[d, hook] \\ S_\nu \arrow[r, "h^\nu_\mu"'] & L\mathbb{G}_a/L^+\mathbb{G}_a \end{tikzcd}$$

where $s > 0$ is some positive integer.

9.1. **Character sheaf.** Since all of our geometric spaces are defined over $\mathbb{F}_p$, there is a natural Artin–Schreier–Witt sequence

$$0 \to \underline{\frac{1}{p^s}\mathbb{Z}_p/\mathbb{Z}_p} \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \xrightarrow{\mathrm{Frob}-\mathrm{id}} L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \to 0$$

The restricted character $\psi|_{\frac{1}{p^s}\mathbb{Z}_p/\mathbb{Z}_p}$ gives rise to a rank 1 local system on $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$, which we abusively denote $\mathcal{L}_\psi$ for simplicity.

9.2. **Equivariance.** Note that the L^+G-action on $\mathrm{Gr}^{\leq\lambda}$ factors through L^hG for some large enough $h > 0$. Therefore, the L^+N-action on $\mathrm{MV}_{\lambda,\mu}$ factors through L^hN as well. Note that the map $h_\mu : L^+N \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ also factors as $h_\mu : L^+N \to L^hN \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ for large enough h.

Proposition 9.3. *Choose s such that $h_\mu|_{L^+N}$ and $h_\mu^{\lambda,\nu}$ both factor through $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \to L\mathbb{G}_a/L^+\mathbb{G}_a$. Then the following diagram commutes:*

$$\begin{tikzcd}
L^+N \times \mathrm{MV}_{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[d] & \mathrm{MV}_{\lambda,\nu} \arrow[d] \\
L^hN \times \mathrm{MV}_{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[d, "h_\mu \times h_\mu^{\lambda,\nu}"'] & \mathrm{MV}_{\lambda,\nu} \arrow[d, "h_\mu^{\lambda,\nu}"] \\
L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \times L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "a"'] \arrow[d] & L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[d] \\
L\mathbb{G}_a/L^+\mathbb{G}_a \times L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "a"] & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

Proof. An element $(n, n' \cdot p^\nu)$ gets sent to $(nn' \cdot p^\nu)$ gets sent to $h_\mu(nn')$. In the other direction $(n, n' \cdot p^\nu)$ gets sent to $(h_\mu(n), h_\mu(n'))$ gets sent to $h_\mu(n) + h_\mu(n') = h_\mu(nn')$. $\square$

Let $\mathcal{A}_\lambda \in P_{L^+G}(\mathrm{Gr})$ denote the sheaf corresponding to the highest weight representation V_λ via Zhu's geometric Satake equivalence.

Corollary 9.4. *If μ is non-dominant, $\mu + \nu$ is dominant, and λ is dominant, then*

$$R\Gamma_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) = 0.$$

Proof. By Proposition 9.3 we prove equivariance with respect to the middle square.

$$\begin{aligned}
\mathrm{act}^*(\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) &= \mathrm{act}^*\mathcal{A}_\lambda \otimes \mathrm{act}^*(h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu \times h_\mu^{\lambda,\nu})^* a^*\mathcal{L}_\psi \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu \times h_\mu^{\lambda,\nu})^*(\mathcal{L}_\psi \boxtimes \mathcal{L}_\psi) \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu^*\mathcal{L}_\psi \boxtimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) \\
&= h_\mu^*\mathcal{L}_\psi \boxtimes (\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi)
\end{aligned}$$

So $\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi$ is (L^+N, h_μ)-equivariant. If μ is non-dominant then $h_\mu|_{L^+N}$ is non-trivial, so $h_\mu^*\mathcal{L}_\psi$ is non-trivial. Finally we conclude by Prop. 9.5, which holds verbatim for any pfp perfect group scheme acting on a a pfp perfect scheme, so the result follows. $\square$

Proposition 9.5. *Let $Z \in \mathrm{Sch}^{\mathit{fin\,typ}}$ with an action of*

$$a : G \times Z \to Z$$

Let $\mathcal{L} \in \mathrm{Shv}^{\mathit{loc\,free}, r=1}(G)$. Let $\mathcal{F} \in \mathrm{Shv}(Z)$ be $(G, \mathcal{L})$ equivariant, i.e.

$$a^*\mathcal{F} \simeq \mathcal{L} \boxtimes \mathcal{F}$$

Than $\pi_! \mathcal{F} \simeq 0$.

Proof.
Consider the diagram

$$\begin{tikzcd} \mathbb{G}_a \times Z \arrow[r, "\mathrm{id}\times a"] \arrow[d] & \mathbb{G}_a \times Z \arrow[d] \\ \mathbb{G}_a \arrow[r] & \mathbb{G}_a \end{tikzcd}$$

we obtain

$$(\mathrm{id} \times a)^* (k \boxtimes \mathcal{F}) \simeq \mathcal{L} \boxtimes \mathcal{F}$$

or by adjunction

$$k \boxtimes \mathcal{F} \simeq (\mathrm{id} \times a)_* \mathcal{L} \boxtimes \mathcal{F}$$

suppose $\pi_! \mathcal{F} \in \mathrm{Shv}(k) \simeq \mathrm{Mod}_k$ were non zero. This means there exists $i : z \hookrightarrow Z$, such that[9]

$$\pi_! i^* \mathcal{F} \not\simeq 0$$

In otherwords, we'd have

$$(\mathrm{id} \times i)^* (k \boxtimes \mathcal{F}) \simeq (\mathrm{id} \times i)^* (\mathrm{id} \times a)_* (\mathcal{L} \boxtimes \mathcal{F})$$

yields

$$k \otimes \pi_! i^* \mathcal{F} \simeq \mathcal{L} \otimes \pi_! i^* \mathcal{F}$$

in $\mathrm{Shv}(k)$, and as both $k, \mathcal{L}$ are irreducible sheaves we have $k \simeq \mathcal{L}$.[10] □

9.3. **Proof.**

Theorem 9.6. *If λ is a dominant coweight and ν and μ are coweights such that $\mu + \nu$ are dominant, then the cohomology*

$$H^i_c(\mathrm{MV}_{\lambda,\nu}, A_\lambda|_{\mathrm{MV}_{\lambda,\nu}} \otimes (h^{\lambda,\nu}_\mu)^*(\mathcal{L}_\psi))$$

vanishes unless $i = (2\rho, \nu)$ and μ is dominant, in which case it is canonically isomorphic to $\mathrm{Hom}_{\widehat{G}}(V^\lambda \otimes V^\nu, V^{\mu+\nu})$.

Note that Corollary 9.4 implies the vanishing part when μ is non-dominant, so it remains to treat the dominant case. For this, we mimic the strategy of [NP01]; in particular, we exploit the fact that the geometry of the $\mathrm{MV}_{\lambda,\nu}$ becomes simpler when λ is quasi-minuscule. Luckily, we have the following geometric version of the PRV conjecture:

Lemma 9.7 ([Zhu17, Lemma 2.16])**.** *There exists a sequence of quasi-minuscule coweights $\lambda_\bullet = (\lambda_1, \dots, \lambda_m)$ such that $V^\lambda_{\lambda_\bullet} \neq 0$ in the decomposition*

$$\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_m} = \bigoplus_{\substack{\xi \in X_*(T)_+, \\ \xi \leq \lambda_1 + \cdots + \lambda_n}} \mathcal{A}_\xi \otimes V^\xi_{\lambda_\bullet}.$$

[9]Indeed, for topological spaces, if $\mathcal{F}$ is bdd below complex of sheaves $\pi_! i^* \mathcal{F} \simeq \varinjlim_{Z \in U} H^k(U, \mathcal{F}_U)$. In our setting $\mathcal{F}$ is quasicoherent sheaf, this implies that $\pi_! i^* \mathcal{F} \simeq \varinjlim_D M_D \simeq M_z$ where we localize $M := \Gamma(Z, \mathcal{F})$.

[10]For instance, use semisimplicity representation category.

Pick such a sequence $\lambda_1, \dots, \lambda_m$. Recall that the *right* multiplication action of L^+G on LG makes $LG \to \mathrm{Gr}$ an L^+G-torsor, and this canonically descends to an L^+N-torsor

$$\begin{aligned} LN &\to S^\nu \\ n &\mapsto p^\nu n \mod L^+G. \end{aligned}$$

Definition 9.8. Let $r \in \mathbb{N}_{\geq 0} \cup \{\infty\}$. Via pushout we can form the following L^rN-torsors over S^ν and $\mathrm{MV}_{\lambda,\nu}$:

$$\begin{tikzcd} \mathrm{MV}^{(r)}_{\lambda,\nu} \arrow[r] \arrow[d, "p_r"] & S^{(r)}_\nu := LN \times^{L^+N} L^rN \arrow[d] \\ \mathrm{MV}_{\lambda,\nu} \arrow[r, hook] & S_\nu \end{tikzcd}$$

We adopt the convention $L^\infty N := L^+N$. Note that $S^{(0)}_\nu = S_\nu$ and $S^{(\infty)}_\nu = LN$.

Lemma 9.9. *For $r \geq 0$, the left action of L^+N on $\mathrm{MV}^{(r)}_{\lambda,\nu}$ factors through $L^{r'}N$ for some $r' > 0$.*

Proof. First note that the left action of L^+G on $\mathrm{Gr}_{\leq\lambda}$ factors through L^rG for some $r > 0$ (which depends on λ). This implies that the left L^+N-action on $\mathrm{MV}^{(0)}_{\leq\lambda,\nu} = \mathrm{MV}_{\leq\lambda,\nu}$ factors through L^rN as well.

The space $\mathrm{MV}^{(r)}_{\lambda,nu}$ acquires an action of L^+N as follows.

TODO: NEED TO EXTEND THIS TO THE ACTION ON $\mathrm{MV}^{(r)}_{\nu,\mu}$, BUT THIS MIGHT BE IMMEDIATE BECAUSE IT'S AN L^rN-TORSOR. Should actually probably just use the moduli description for this via bundles and the action via changing the trivialization. □

Now pick $\nu_1, \dots, \nu_m$ such that $\nu_1 + \cdots + \nu_m = \nu$.

By the lemma we can choose integers $r_1, \dots, r_m \geq 0$ such that $r_m = 0$ and such that the action of L^+N on $\mathrm{MV}^{(r_i)}_{\nu_i,\mu_i}$ factors through $L^{r_{i-1}}N$.

Lemma 9.10. *There is an $\prod_i L^{r_i}N$ torsor*

$$\prod_{i=1}^n (\mathrm{MV}_{\lambda_i,\nu_i})^{(r_i)} \xrightarrow{q_\bullet} \mathrm{MV}_{\nu_\bullet,\mu_\bullet}$$

such that

$$q_\bullet^* \mathcal{A}_{\mu_\bullet} \cong p_\bullet^* \left(\mathcal{A}_{\mu_1} \boxtimes \cdots \boxtimes \mathcal{A}_{\mu_m}\right)$$

where $p_\bullet$ is the map

$$\prod_{i=1}^n (\mathrm{MV}_{\lambda_i,\nu_i})^{(r_i)} \xrightarrow{p_\bullet} \prod \mathrm{MV}_{\lambda_i,\nu_i}$$

Proof. For simplicity, first suppose $m = 2$.

$$\begin{tikzcd}
& \mathrm{MV}^{(\infty)}_{\nu_1,\mu_1} \times \mathrm{MV}_{\nu_2,\mu_2} \arrow[dl, "p"'] \arrow[d, "p_\infty"'] \arrow[dr, "q"] & \\
& \mathrm{MV}^{(r_1)}_{\nu_1,\mu_1} \times \mathrm{MV}_{\nu_2,\mu_2} \arrow[dl, "p_r \times \mathrm{id}"] \arrow[dr, "q_r"'] & \\
\mathrm{MV}_{\nu_1,\mu_1} \times \mathrm{MV}_{\nu_2,\mu_2} & & \mathrm{MV}_{\nu_1,\mu_1} \tilde{\times} \mathrm{MV}_{\nu_2,\mu_2}
\end{tikzcd}$$

in which q is an L^+N-torsor and q_r is an L^rN-torsor. The morphism p_∞ is just the quotient by $\ker(L^+N \to L^rN)$ in the first slot and the identity in the second. The point now is that there is a unique perverse sheaf $\mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}$ on $\mathrm{MV}_{\nu_1,\mu_1} \tilde{\times} \mathrm{MV}_{\nu_1,\mu_2}$ satisfying

$$p^*(\mathrm{IC}_{\mu_1} \boxtimes \mathrm{IC}_{\mu_2}) \cong q^*(\mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}).$$

There is also a unique perverse sheaf $\mathcal{L}$ satisfying

$$q_r^* \mathcal{L} \cong p_r^*(\mathrm{IC}_{\mu_1} \boxtimes \mathrm{IC}_{\mu_2})$$

But pulling back by p_∞ gives $q^*\mathcal{L} \cong p^*(\mathrm{IC}_{\mu_1} \boxtimes \mathrm{IC}_{\mu_2})$ so we must have $\mathcal{L} \cong \mathrm{IC}_{\mu_1} \tilde{\boxtimes} \mathrm{IC}_{\mu_2}$ by uniqueness.

For $m > 2$, the same argument above gives an $L^{r_{m-1}}N$-torsor

$$\mathrm{MV}^{(r_{m-1})}_{\nu_{n-1},\mu_{m-1}} \to \mathrm{MV}_{\nu_{m-1},\mu_{m-1}} \tilde{\times} \mathrm{MV}_{\nu_m,\mu_m}$$

Then one can continue inductively, with $\mathrm{MV}^{(r_{m-1})}_{\nu_{m-1},\mu_{m-1}} \tilde{\times} \mathrm{MV}_{\nu_m,\mu_m}$ (with its natural $L^{r_{m-2}}N$-action) playing the role of $\mathrm{MV}_{\nu_m,\mu_m}$. $\square$

$$\begin{tikzcd}
\prod_{i=1}^n (\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)} \arrow[d, "q_\bullet"] \arrow[drr, "h_\bullet"] & & \\
\mathrm{MV}_{\nu_\bullet,\mu_\bullet} \arrow[r] & S_{|\nu_\bullet|} \arrow[r, "h"] & B\underline{N(E)}
\end{tikzcd}$$

Lemma 9.11. *If $\mu_\bullet \subset M$ is a tuple of nonzero quasi-minuscule coweights and $(\nu_1, \dots, \nu_n)$ is a tuple of coweights, then*

$$R\Gamma_c(\mathrm{MV}_{\nu_1,\mu_1}\, wt \times \cdots wt \times \mathrm{MV}_{\nu_n,\mu_n}, IC_{\mu_\bullet} \otimes h_\bullet^* \mathcal{L}_\psi) \simeq \bigotimes_{i=1}^n R\Gamma_c(\mathrm{MV}^{(r_i)}_{\nu_i,\mu_i}, p_i^* IC_{\mu_i} \otimes h^*_{\sigma_i} \mathcal{L}_\psi)$$

Proof. $\mathrm{IC}_{\mu_\bullet}$ splits as $\boxtimes_{i=1}^n p_i^* \mathrm{IC}_{\mu_i}$ over $\prod_{i=1}^n \mathrm{MV}^{(r)}_{\nu_i,\mu_i}$ by Lem. 9.10. As $*$-pullback is symmetric monoidal we have

$$
\begin{aligned}
&R\Gamma_c\left(\mathrm{MV}_{\nu_1,\mu_1}\,\tilde{\times}\cdots\tilde{\times}\,\mathrm{MV}_{\nu_m,\mu_m},\mathrm{IC}_{\mu_\bullet}\otimes h_\bullet^*\mathcal{L}_\psi\right)\\
&\simeq R\Gamma_c\left(\prod_{i=1}^n(\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)},\boxtimes_{i=1}^n\mathrm{pr}_i^*\mathrm{IC}_{\mu_i}\otimes\boxtimes_{i=1}^n(h_\sigma\circ q_i)^*\mathcal{L}_\psi\right)\\
&\simeq\bigotimes_{i=1}^n R\Gamma_c((\mathrm{MV}_{\nu_i,\mu_i})^{(r_i)},\mathrm{pr}_i^*\mathrm{IC}_{\mu_i}\otimes(h_\sigma\circ q_i)^*\mathcal{L}_\psi)[2\dim N\cdot r_i]
\end{aligned}
$$

$\square$

References

[AL21] Anschütz, Johannes and Le Bras, Arthur-César. "A Fourier Transform for Banach-Colmez spaces". In: *arXiv e-prints*, arXiv:2111.11116 (Nov. 2021), arXiv:2111.11116. arXiv: 2111.11116 [math.AG] (cit. on p. 7).

[Bez+19] Bezrukavnikov, Roman et al. "An Iwahori-Whittaker model for the Satake category". In: *J. Éc. polytech. Math.* 6 (2019), pp. 707–735. ISSN: 2429-7100,2270-518X. URL: https://doi.org/10.5802/jep.104 (cit. on p. 4).

[FGV01] Frenkel, E., Gaitsgory, D., and Vilonen, K. "Whittaker patterns in the geometry of moduli spaces of bundles on curves". In: *Ann. of Math. (2)* 153.3 (2001), pp. 699–748. ISSN: 0003-486X,1939-8980. URL: https://doi.org/10.2307/2661366 (cit. on pp. 3–6, 18).

[Fre+98] Frenkel, E. et al. "Geometric realization of Whittaker functions and the Langlands conjecture". In: *J. Amer. Math. Soc.* 11.2 (1998), pp. 451–484. ISSN: 0894-0347,1088-6834. URL: https://doi.org/10.1090/S0894-0347-98-00260-4 (cit. on p. 10).

[FS21] Fargues, Laurent and Scholze, Peter. "Geometrization of the local Langlands correspondence". In: *arXiv e-prints*, arXiv:2102.13459 (Feb. 2021), arXiv:2102.13459. arXiv: 2102.13459 [math.RT] (cit. on pp. 4, 6, 8, 10–12).

[Ham22] Hamann, Linus. "Geometric Eisenstein Series, Intertwining Operators, and Shin's Averaging Formula". In: *arXiv e-prints*, arXiv:2209.08175 (Sept. 2022), arXiv:2209.08175. arXiv: 2209.08175 [math.NT] (cit. on pp. 6, 7, 10, 11).

[Ngô00] Ngô, Bao Châu. "Preuve d'une conjecture de Frenkel-Gaitsgory-Kazhdan-Vilonen pour les groupes linéaires généraux". In: *Israel J. Math.* 120 (2000), pp. 259–270. ISSN: 0021-2172,1565-8511. URL: https://doi.org/10.1007/s11856-000-1279-5 (cit. on p. 21).

[NP01] Ngô, B. C. and Polo, P. "Résolutions de Demazure affines et formule de Casselman-Shalika géométrique". In: *J. Algebraic Geom.* 10.3 (2001), pp. 515–547. ISSN: 1056-3911,1534-7486 (cit. on pp. 3, 10, 14, 18, 25).

[Zhu17] Zhu, Xinwen. "Affine Grassmannians and the geometric Satake in mixed characteristic". In: *Ann. of Math. (2)* 185.2 (2017), pp. 403–492. ISSN: 0003-486X,1939-8980. URL: https://doi.org/10.4007/annals.2017.185.2.2 (cit. on pp. 10, 14, 25).