RESEARCH STATEMENT

MILTON LIN

Introduction

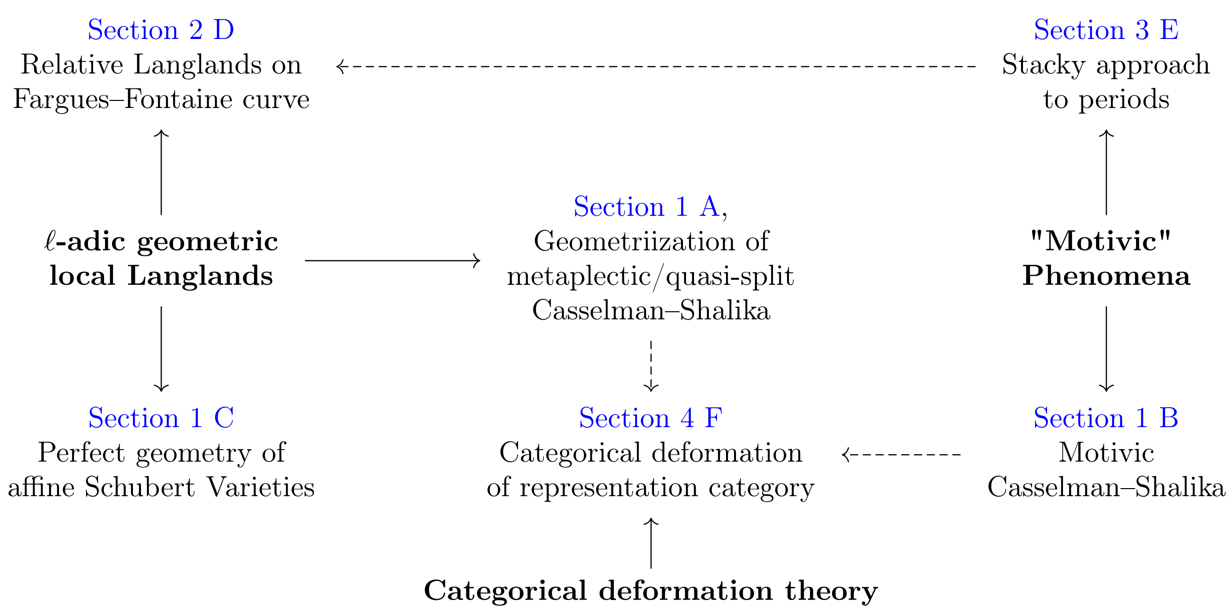
My areas of interest in mathematics are:

(1) The **Langlands correspondence**, which is now a huge web of conjectures, from special values of L functions to conformal field theory. My particular interest is in various incarnations of the **Casselman–Shalika formula**, A, and **relative aspects of the ℓ-adic geometric local Langlands correspondence**, D.

(2) **Stable homotopy theory**, through the lens of **categorical deformation theory**, F.

(3) **Motivic phenomena**, where I hope to explore the motivic version of Casselman–Shalika formula, B, and a stacky approach to periods, E.

Majority of the research presented here originates from my study of the Casselman–Shalika formula in the mixed characteristic setting, as outlined in Section 1. These areas of research are interconnected, as shown in the following diagram.



The priority of research is listed in the following order[1],

A=D=F>B>E> C.

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[1]The alphabet links to the goal rather than the section.

Notations. Theorems stated have full proof written by either me or joint with collaborators, *unless* it is annotated with: ***conjecture*** – no proofs have been written down but is believed to hold, or ***in progress*** – where we have partial progress. We use freely the language of higher categories and higher algebra, [Lur09], [Lur18].

1. Mixed Characteristic Geometry and the Casselman–Shalika formula

Let G be a connected reductive group over a nonarchimedian local field with residue characteristic $p \neq \ell$, and $\Lambda := \overline{\mathbb{Q}}_\ell$. In [FS24], Fargues and Scholze have formulated the geometric Langlands conjecture for the Fargues Fontaine curve, similar to the function field correspondence of Beilinson-Drinfeld, see [AG15], [Gai14]. We let $\mathrm{LS}_{\hat{G}}$, be the moduli stack of L-parameters of $\hat{G}$, Bun_G the moduli stack of G-bundles on the Fargues–Fontaine curve. They constructed two actions

$$\begin{tikzcd} & \mathrm{IndPerf}(\mathrm{LS}_{\hat{G}}) \arrow[dl, phantom, "\circlearrowleft"] \arrow[dr, phantom, "\circlearrowright"] & \\ \mathrm{IndCoh}^{qc}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}}) & & D_{\mathrm{lis}}(\mathrm{Bun}_G) \end{tikzcd}$$

The left hand is induced from the natural tensor structure, while the right hand is the so-called *spectral action* [NY19]. For a fixed choice of Whittaker datum, [FS24, I.10.2] conjectured a $\mathrm{IndPerf}(\mathrm{LS}_{\hat{G}})$-linear equivalence

$$\mathbb{L}_G : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \simeq \mathrm{IndCoh}^{\mathrm{qc}}_{\mathrm{Nilp}}(\mathrm{LS}_{\hat{G}})$$

which can be thought of as a generalization of the classical local Langlands.

One fundamental aspect of the program is to understand the *Whittaker-Fourier coefficient functor*,

$$\mathrm{coeff} : D_{\mathrm{lis}}(\mathrm{Bun}_G, \Lambda) \to D(\Lambda)$$

and its various properties, see [FR22, Ch. 5] for definitions and theorems. In the classical setting, this corresponds to finding the Fourier coefficients of automorphic functions, see [Gel75, Ch.3].

The first fundamental result in this context is the *global Casselman-Shalika formula*, as proven in [FGV01], which we aim to replicate in the mixed characteristic setting. In joint work with Ashwin Iyengar (American Mathematical Society) and Konrad Zou (Bonn University) [ILZ24], we proved a variation of this problem: the *geometric Casselman–Shalika formula* over the Witt vector affine Grassmannian Gr_G, analogous to the geometrization carried out in [NP01] for equi-characteristic local fields.

Theorem 1.1. *[ILZ24] The geometric Casselman–Shalika formula holds over the Witt vector Grassmanian.*

At the same time, we have established basic properties of averaging functors (cf. [FR22, Ch. 7], [BBM21]), yielding Iwahori–Satake equivalence, [Bez+19] without using nearby cycles.

Theorem 1.2 (I.-Lin-Z., in progress)**.** *The Iwahori-Whittaker category is equivalent to the spherical Hecke category in mixed characteristics.* [2]

As of writing, D. Hansen, L. Hamman, and L. Mann have announced significant advancements to the global version of the formula. Building on all these progresses, I have developed a strong interest in exploring the geometrization of the Casselman–Shalika formula for covering groups [GGW18].

[2] In the set up of [Bez+19], the equivalence was applied to modular representation theory, as the Iwahori-Whittaker category has a much simpler categorical structure.

Research Goal A. Geometrization of Metaplectic/Quasi-split Casselman–Shalika. *We propose two explorations of the Casselman–Shalika formula over equal-characteristic local fields. First, a geometric metaplectic Casselman–Shalika formula, building on the works of Gaitsgory and Lysenko [GL22], McNamara [McN16], and Brubaker et al. [Bru+20; Bru+24]. Second, a geometric Casselman–Shalika formula for quasi-split groups, following [GK20].*

In a different direction, inspired by my joint project in Chow groups, [Has+24], I hope to extend this framework to the context of motivic sheaves. For nice schemes X, Levine showed that t-structures exist on a nice subcategory of category of motives, $\mathrm{DTM}(X) \hookrightarrow \mathrm{DM}(X)$, referred as *mixed Tate movies*. This was extended to schemes with cellular Whitney-Tate stratification by Soergel and Wendt, and to prestacks in [RS20], which led to a series of work in applying the theory of motives to the geometric Langlands.

Research Goal B. Motivic Whittaker categories. *Define a Whittaker category in motivic setting, obtaining a similar equivalence at [FGV01]. To begin, we can prove the same statement in [NP01] in the category of mixed Tate motives, in which all the relevant objects are well defined. The difficulty, however, is that the original argument requires characteristic* 0 *coefficients, whilst in positive characteristic, there are problems of extensions.*

Lastly, in [ILZ24], we have studied the smoothness of perfect Schubert subvarieties, $\mathrm{Gr}_{G,\leq\mu}$, for minuscule and quasi-minuscule μ. In the equal characteristic case over $\mathbb{C}((t))$, this was first examined by Evens and Mirkovic and later extended by Haines [HR20], with applications to classifying Shimura varieties with good or semi-stable reductions.

Research Goal C. Geometry of general perfect Schubert variety *Prove the results of Pappas and Zou, [PZ24] in perfect geometry. Associated to an absolutely special vertex in the Bruhats-Tits building of G, we have an associated group scheme $\mathcal{G}$ over $\mathcal{O}$.*

Conjecture 1.3. The smooth locus of $\mathrm{Gr}_{\mathcal{G},\leq\mathfrak{s}}$ is $\mathrm{Gr}_{\mathcal{G},\mathfrak{s}}$ in perfect geometry, in the sense of [Zhu17].

2. Relative langlands on the Fargues Fontaine curve

Joint with Yuta Takaya (Tokyo University), we explicitly compare the period sheaves on $\mathcal{A}$-side (automorphic) and L-sheaves on $\mathcal{B}$-side (Galois) under the relative Langlands conjectures of Ben-Zvi-Sakellaridis-Venkatesh. [BSV]. In *op. cit.*, one interprets Langlands correspondence as a form of arithmetic 4d quantum field theory, which has genesis in the Kapustin-Witten interpretation, [KW07], and the Knots and Primes promoted by B. Mazur, M. Kim, [Kim16]. The Fargues-Fontaine curve should be a *global object* of dimension 2 under the dictionary.

The theory suggests a duality extending that Langlands dual group: (G, X) with $(\hat{G}, \hat{X})$ of hyperspherical varieties. We considered the *Iwasawa-Tate case*: $G = \mathbb{G}_{m,F}$ and $X = \mathbb{A}_F$ with dual pair $\hat{G} = \mathbb{G}_{m,\Lambda}$ and $\hat{X} = \mathbb{A}_\Lambda$. We constructed two maps

$$\pi : \mathrm{Bun}_G^X \to \mathrm{Bun}_G, \quad \hat{\pi} : \mathrm{LS}_{\hat{G}}^{\hat{X}} \to \mathrm{LS}_{\hat{G}}$$

yielding the *period sheaf*, $\mathcal{P}_X := \pi_!\Lambda$, and *L-sheaf*, $\mathcal{L}_{\hat{X}} := \pi_*\omega_{\mathrm{Loc}_{\hat{G}}^{\hat{X}}}$. Bun_G has a Hardar-Narasimhan straification by locally closed substacks Bun_G^b indexed by the Kottwitz set $B(G)$. In our case, $G = \mathbb{G}_m$, $\mathrm{Bun}_{\mathbb{G}_m}$ is stratified by $\mathrm{Bun}_{\mathbb{G}_m}^n$ for $n \in \mathbb{Z} = B(T)$. The following theorem explicitly computes the period sheaf.

Proposition 2.1 (Lin-T.)**.** *Let $n > 0$. Let $\mathcal{BC}(n)$ be the Banach-Colmez space of the line bundle $\mathcal{O}(n)$. The relative stack $\mathrm{Bun}_G^{X,n} := \pi^{-1}(\mathrm{Bun}_G^n)$ is a $\mathcal{BC}(n)$-torsor over Bun_G^n. $\mathcal{P}_X^n := \mathcal{P}_X|_{\mathrm{Bun}_G^n} \simeq \Lambda[-2n]$.*

On the other hand, we expect that one can study the spectral side using derived Fourier vector bundles (cf. [FW24, Ch. 6]).

Conjecture 2.2. [Lin-T., in progress] Under the geometric local Langlands correspondence, $\mathbb{L}_G$, (appropriately normalized) $\mathcal{P}_X$ is sent to[3] $\mathcal{L}_{\hat{X}}$.

Research Goal D. Relative Langlands on the Fargues Fontaine curve. *Complete Conjecture 2.2 as a first step and then the Hecke case, which classically corresponds to Hecke's integral representation of standard L-function for GL_2. Lastly, one can ask whether on the $\mathcal{B}$-side, the same constructions of [BSV, Ch. 11] works for the p-adic (Emerton-Gee) L-parameter stacks, which potentially give new interpretations to p-adic L-functions.*

3. Stacky approaches and periods

Various cohomology theories – crystalline cohomology, syntomic cohomology, and Dolbeaut cohomology – admit a factorization to the category of stacks over some affine scheme $\operatorname{Spec} R$,

$$\mathrm{Sch}_{\mathbb{Z}}^{\mathrm{sep,ft}} \to \mathrm{Stk}_R \to D(R)$$

$$X \mapsto X^{?} \mapsto \Gamma(X^{?}, \mathcal{O}_{X^{?}})$$

For instance, the *de Rham stack* X^{dR} over $\mathbb{Q}$, is defined by $X^{\mathrm{dR}}(A) := X(A_{\mathrm{red}})$ for any $\mathbb{Q}$-algebra A, see [GR14]. This is often referred to as a *stacky approach* [Dri22] or *transmutation* [Bha23], which allows one to use six functor formalism and geometric techniques. I explored this concept in basic constructions of F. Brown's work on *motivic periods*, [Bro14].

Let $X := \mathbb{P}^1 \setminus \{0, 1, \infty\}$ be the projective space minus three points over $\mathbb{Q}$. The period associated is studied through Chen's de Rham comparison theorem, [Che76], which relates to iterated integrals [Che73] and multiple zeta values [Bro14]. To give a "stacky" perspective of this period one uses unipotent types of stacks, which is defined in Toën, [Toë06], and recently in [MR23]. This is an endofunctor $\mathbf{U}$ on stacks, sending a stack X to its unipotent homotopy type. My first result is:

Theorem 3.1. *(Lin) Unipotent de Rham fundamental group, $\pi_1^{u,dR}(X, x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{dR}))$.*

Research Goal E. A stacky approach to motivic periods. *This research aims to use geometric techniques in the study of periods. As a proof of concept, we will recover de Rham comparison theorem through the Riemann-Hilbert correspondence between the analytic Betti stack and the de Rham stack. $\mathbb{P}^1 \setminus \{0, 1, \infty\}$, is not proper, which requires us to incorporate log structures. I expect to prove:*

Conjecture 3.2. There exists X^{Betti} such that the unipotent Betti group $\pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}), x)$ is isomorphic to $\pi_1(\mathbf{U}(X^{\mathrm{Betti}}))$. A logarithmic Riemann-Hilbert comparison should induce Chen's comparison theorem [Hai01, Thm 3.1]

$$\pi_1^{u,\mathrm{dR}}(X, x) \simeq \pi_1^{u,\mathrm{Betti}}(X(\mathbb{C}), x) \otimes \mathbb{C}$$

By similar techniques of [Bha23], we should recover Haine's theorem: the pro-unipotent completion of de Rham fundamental group admits a mixed Hodge structure. [4] *We hope that such work can spark new techniques and new phenomena, such as those used in p-adic integration theory, [Vol01].*

[3] One has to take into account shearing, twisting and tensoring.

[4] In Brown's approach, he reduced his study of motivic periods to mixed Hodge periods [Bro17, p. 3].

4. Deformation theory and the sphere spectrum $\mathbb{S}$

Let $\mathcal{S}$ denote the ∞-category of ∞-groupoids/anima. The *stabilization* of $\mathcal{S}$ is Sp, the ∞-category of spectra. This is the natural category to study cohomological invariants. Within Sp, lies the universal cohomology theory, $\mathbb{S}$, the *sphere spectrum* By Chevalley's works, connected reductive groups over $\mathbb{C}$ have a canonical split $\mathbb{Z}$-form $G_{\mathbb{Z}}$, see [Con15]. One can analogously ask: *is there a* $\mathbb{S}$*-form for algebraic groups*? A first approximation is the existence of an algebraic category $\mathrm{Rep}_{\mathbb{S}}(G_{\mathbb{S}})$, which *deforms* to $\mathrm{Rep}_{\mathbb{Z}}(G_{\mathbb{Z}})$. To study, we begin with *formal deformation of categories*, which we briefly recall.

Let $\mathcal{C}$ be a symmetric monoidal category. There is a natural hierarchy of commutativity fitting in the diagram

$$\cdots \to \mathbb{E}_n(\mathcal{C}) \to \mathbb{E}_{n-1}(\mathcal{C}) \to \cdots \to \mathbb{E}_1(\mathcal{C})$$

where $\mathrm{CAlg}(\mathcal{C}) := \mathbb{E}_\infty(\mathcal{C}) := \varprojlim \mathbb{E}_n(\mathcal{C})$ of symmetric algebra objects can be identified with the limit.

Let $R \in \mathbb{E}_n(\mathrm{Sp})$ be an $\mathbb{E}_n$ ring, and consider LMod_R, the derived category of left R-modules, as an $\mathbb{E}_1$ object in $\mathrm{Pr}^{\mathrm{st}}$, the category of presentable stable categories. This defines $\mathrm{RMod}_{\mathrm{LMod}_R}(\mathrm{Pr}^{\mathrm{st}})$, the category of presentable stable (right) R-linear categories. For G a connected reductive group over a field k, $D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$, the bounded derived category of finite dimensional algebraic representations with k coefficients lies in $\mathrm{Pr}_R^{\mathrm{st,cg}}$, the full subcategory spanned by those whose underlying category is compactly generated.

Now let $k = \mathbb{C}$. Let $\mathrm{Art}_k^{(n)}$ denote the category of $\mathbb{E}_n$ *artinian* ring spectrum, for $n \geq 0$ over k, see [Lur18, Ch. 15] for definition. This extends the classical definition of Artinian local ring, in particular, $R \in \mathrm{Art}_k^{(n)}$ admits an augmentation map $\epsilon : R \to k$. One defines the $\mathbb{E}_{n+2}$-formal moduli problem,

$$\mathrm{Cat}\mathcal{D}\mathrm{ef}^{(n)}(\mathcal{C}) : \mathrm{Art}_k^{(n+2)} \to \hat{\mathcal{S}}$$
$$R \mapsto |\,\{\mathcal{C}\} \times_{\mathrm{Pr}_R^{\mathrm{st,cg}}} \mathrm{Pr}_k^{\mathrm{st,cg}}|$$

where $|\quad|$ is the underlying Kan complex of the ∞-category. An object consists of: a $\mathcal{C}_B$ right stable R-linear category, and an equivalence $\mathcal{C}_B \otimes_{\mathrm{LMod}_B} \mathrm{LMod}_k \simeq \mathcal{C}$. Our $\mathbb{E}_4$-moduli problem is when $n = 2$ and $\mathcal{C} = D^b(\mathrm{Rep}_k^{\mathrm{fd}}(G))$. The geometric Casselman–Shalika [FGV01], which is the $\mathbb{E}_2$-algebra equivalence of $\mathcal{C}$ with the *Whittaker sheaves* on the *affine Grassmanian* $\mathrm{Gr}_{\hat{G}}$, describes this moduli problem. Consider moduli of functor of $\mathbb{G}_m$*-gerbes over* X

$$\mathrm{Ge}_{\mathbb{G}_m}(X) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(X, B^2R^\times) \quad R \in \mathrm{Art}_k^{(4)}$$

where $R^\times \subset \Omega^\infty R$ are the invertible elements of the underlying space of R [5] and B^2 is the second deloop. It was stated without proof in [Lur10]

Theorem 4.1 (Lurie). *There is an equivalence of formal* $\mathbb{E}_4$*-moduli problems*

$$\widehat{Ge_{\mathbb{G}_m}}(\mathrm{Gr}_{\hat{G}}) \xrightarrow{\simeq} Cat\mathcal{D}ef^{(2)}(\mathrm{Rep}_k^{fd} G)$$

where $\widehat{\ }$ *is the formal completion of the moduli functor at a base point.*

Research Goal F. Categorical deformations for the sphere spectrum. *We will first document carefully Lurie's theorem, Theorem 4.1. Then, we will explore deformations of representation of Lusztig's small quantum group, see op.cit. Remark 10.12, using recent advances in quantum geometric Langlands. Ultimately, this research aims to contribute to the foundations of categorical deformations.*

[5] is the union of the connected components of invertibles in the $\pi_0 R$ of the 0th space of R and is equivalent to the nth loop space of some space, $R^\times \simeq \Omega^n Z$, $n \geq 4$, hence the deloop $B^2R^\times \in \mathbb{E}_2(\mathcal{S})$

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