GEOMETRIC CASSELMAN–SHALIKA IN MIXED CHARACTERISTIC

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Contents

1. Introduction

The goal of this article is to prove the following theorem.

Theorem 1.1. *If λ is a dominant coweight and ν and μ are coweights such that $\mu+\nu$ are dominant, then*

$$H^i_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\lambda|_{\mathrm{MV}_{\lambda,\nu}} \otimes (h^{\lambda,\nu}_\mu)^*(\mathcal{L}_\psi)) = \begin{cases} \mathrm{Hom}_{\mathrm{Rep}(\widehat{G})}(V^\lambda \otimes V^\mu, V^{\mu+\nu}) & i = (2\rho,\nu) \text{ and } \mu \in X_*(T)_+ \\ 0 & \text{otherwise} \end{cases}$$

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Remark 1.2. When $\mu = 0$, we recover [NP01, Thm. 3.2]

$$H^i_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\nu|_{\mathrm{MV}_{\lambda,\nu}} \otimes (h_0^{\lambda,\nu})^*(\mathcal{L}_\psi)) = \begin{cases} \overline{\mathbb{Q}}_\ell\langle \rho, \lambda\rangle & \text{if } i = (2\rho,\nu) \text{ and } \nu = \lambda \\ 0 & \text{otherwise} \end{cases}$$

2. Notation

Fix a finite extension $F/\mathbb{Q}_p$ with ring of integers $\mathcal{O} \subset F$, uniformizer $\varpi \in \mathcal{O}$, and residue field $k = \mathcal{O}/\varpi$. Write $q = |k|$. If R is a perfect k-algebra, write

$$W_{\mathcal{O}}(R) = W(R) \otimes_{W(k)} \mathcal{O}$$

where $W(-)$ denotes the p-typical Witt vectors. We also define the truncated Witt vectors

$$W_{\mathcal{O},h}(R) = W_{\mathcal{O}}(R) \otimes_{W(k)} \mathcal{O}/\varpi^n.$$

We also fix some notation for the reductive group.

- Let G be a split reductive group over F.
- Fix a maximal torus T and a Borel B containing it, and let N denote its unipotent radical.
- Let $\bar{G}, \bar{B}, \bar{T}, \bar{N}$ denote the special fibers over k.
- Let Φ denote the set of all roots, and let Φ_+ denote the set of positive roots corresponding to B.
- Let $X_*(T) = \mathrm{Hom}(\mathbb{G}_m, T)$ denote the lattice of cocharacters, and let $X_*(T)^+$ denote the cone of dominant cocharacters corresponding to B.
- Write $\leq$ for the usual Bruhat order with respect to the positive roots.
- If $\nu \in X_*(T)$, we write $\varpi^\nu := \nu(\varpi)$.

3. Witt vector affine Grassmannian

Definition 3.1 ([Zhu17, Section 1])**.**

- If $\mathcal{X}$ is an affine scheme over $\mathcal{O}$, let $L^+\mathcal{X} \in \mathrm{AlgSpc}_k^{\mathrm{pf}}$ denote the positive loop space. As a consequence of [Gre61], we have

$$L^+\mathcal{X} \simeq \varprojlim_h L^h\mathcal{X}$$

where $L^h\mathcal{X}$ is the perfection of the prestack $L^h_p\mathcal{X} \in \mathrm{Shv}(\mathrm{Aff}_k)$, whose R points are $\mathcal{X}(W_{\mathcal{O},h}(R))$.

- if $X \in \mathrm{Aff}_F$, let LX denote the loop space whose R points, for a perfect k-scheme R, are

$$LX(R) = X(W_{\mathcal{O}}(R)[1/\varpi]).$$

The functor LX is represented by an ind perfect scheme.

- If H is any smooth affine group scheme over $\mathcal{O}$, we write

$$\mathrm{Gr}_H = LH/L^+H$$

for the Witt vector affine Grassmannian for H, where we take the quotient in the étale topology.

Recall that Gr_G can be written as the colimit of perfections of projective varieties, called *(affine) Schubert varieties*:

$$\mathrm{Gr}_G = \mathrm{colim}_{\lambda \in X_*(T)^+} \mathrm{Gr}_{\leq \lambda}$$

and that the Schubert varieties are the closure of their maximal Schubert cells:

$$\mathrm{Gr}_{\leq \lambda} = \overline{\mathrm{Gr}_\lambda} = \bigcup_{\lambda' \leq \lambda} \mathrm{Gr}_{\lambda'},$$

where $\mathrm{Gr}_\lambda \subset \mathrm{Gr}_G$ is locally closed, and such that on k-points we get

$$\mathrm{Gr}_\lambda(k) = G(\mathcal{O})\lambda(\varpi)G(\mathcal{O}),$$

in accordance with the Cartan decomposition. By definition there is a left action of LG on Gr_G. This restricts to an action of L^+G on $\mathrm{Gr}_{\leq \lambda}$.

Lemma 3.2. *The action of L^+G on $\mathrm{Gr}_{\leq \lambda}$ factors through L^hG for h large enough.*

Proof. This is explained in the proof of [Zhu17, Proposition 1.23]. □

For $\lambda \in X_*(T)^+$ we let $\mathcal{A}_\lambda$ denote the intersection cohomology sheaf on $\mathrm{Gr}_{\leq \lambda}$, which is defined as the intermediate extension of the constant sheaf $\overline{\mathbb{Q}}_\ell$ on Gr_λ to all of $\mathrm{Gr}_{\leq \lambda}$. We have

$$\mathcal{A}_\lambda \in P_{L^+G}(\mathrm{Gr}_G).$$

Its restriction is

$$\mathcal{A}_\lambda|_{\mathrm{Gr}_\lambda} = \overline{\mathbb{Q}}_\ell[(2\rho, \mu)].$$

The inclusion $N \hookrightarrow G$ functorially induces an inclusion $\mathrm{Gr}_N \hookrightarrow \mathrm{Gr}_G$. The Iwasawa decomposition gives us the following alternative stratification of Gr_G.

Definition 3.3 ([Zhu17, somewhere])**.** The *semi-infinite orbit* of a cocharacter $\nu \in X_*(T)$ is

$$S_\nu = \varpi^\lambda \mathrm{Gr}_N \subset \mathrm{Gr}_G .$$

Definition 3.4. Let

$$\mathrm{MV}_{\lambda,\nu} := \mathrm{Gr}_{\leq \lambda} \cap S_\nu,$$

where “MV” is short for “Mirkovic–Vilonen”. In the literature a *Mirkovic–Vilonen cycle* is typically an irreducible component of $\mathrm{MV}_{\lambda,\nu}$, but we use MV to denote the whole intersection.

3.5. **Character sheaf.** Fix, once and for all, an additive character

$$\psi : F \to F/\mathcal{O} \to \overline{\mathbb{Q}}_\ell^\times$$

such that $\psi(p^{-1}\mathcal{O}) \neq 1$. Choosing conductor zero will simplify the rest of the arguments, but does not amount to any real loss of generality in Theorem 1.1.

In order to geometrize the additive character and consider Whittaker sheaves, we first consider the natural map

$$h : LN \to LN/[LN, LN] \xrightarrow{\sim} \prod_{\alpha\in\Phi_+} L\mathbb{G}_a \xrightarrow{+} L\mathbb{G}_a \to L\mathbb{G}_a/L^+\mathbb{G}_a. \tag{1}$$

This has a natural descent to S_ν.

Lemma 3.6. *If $\mu \in X_\bullet(T)$ is a character such that $\mu+\nu$ is dominant, then h induces a map*

$$h^\nu_\mu : S_\nu \to L\mathbb{G}_a/L^+\mathbb{G}_a.$$

explicitly given by

$$\begin{aligned}(\varpi^\nu LN(R))/L^+N(R) &\to L\mathbb{G}_a(R)/L^+\mathbb{G}_a(R)\\ \varpi^\nu n \mod L^+N(R) &\mapsto h(\mathrm{ad}(\varpi^{\mu+\nu})(n)).\end{aligned}$$

Proof. Note that $S_\nu = \varpi^\nu \operatorname{Gr}_N = (\varpi^\nu LN)/L^+N$. But this is the étale sheafification of the naïve quotient of presheaves. So for R a perfect k-algebra we define

$$\begin{aligned}(\varpi^\nu LN(R))/L^+N(R) &\to L\mathbb{G}_a(R)/L^+\mathbb{G}_a(R)\\ \varpi^\nu n \mod L^+N(R) &\mapsto h(\mathrm{ad}(\varpi^{\mu+\nu})(n)).\end{aligned}$$

To see that this is well-defined, suppose $\varpi^\nu nL^+N(R) = \varpi^\nu mL^+N(R)$. Then $n^{-1}m \in L^+N(R)$, but $\mu+\nu$ is dominant so $\mathrm{ad}(\varpi^{\mu+\nu})(n^{-1}m) \in L^+N(R)$, which maps to $L^+\mathbb{G}_a(R)$ under the group homomorphism h. This is clearly functorial and extends to a morphism of presheaves, which we then sheafify. ☐

We will turn the nontrivial additive character

$$\psi : F \to F/\mathcal{O} \to \overline{\mathbb{Q}}_\ell$$

into a character sheaf (i.e. a multiplicative rank 1 étale local system) on

$$\operatorname{Gr}_{\mathbb{G}_a} := L\mathbb{G}_a/L^+\mathbb{G}_a$$

(whose k points are exactly $F/\mathcal{O}$) and pull it back along h^ν_μ. However, $\operatorname{Gr}_{\mathbb{G}_a}$ is a group ind-scheme, and a geometric version of ψ on $\operatorname{Gr}_{\mathbb{G}_a}$ would have to be supported everywhere. To formalize this, one would have to define the category of étale sheaves on $\operatorname{Gr}_{\mathbb{G}_a}$ as a *limit* of sheaves on finite pieces of the ind-scheme, as opposed to Definition 14.3, which is defined by taking a colimit. We want to avoid making the limit definition.

Remark 3.7. In the existing proofs of geometric Casselman–Shalika in equal characteristic, the character sheaf is induced from residue map h, [FGV01], [FR22],

$$h : \operatorname{Bun}_N^\Omega \to \mathbb{G}_a$$

which ends with the residue map $L\mathbb{G}_a \xrightarrow{\sum c_i t^i \mapsto c_{-1}} \mathbb{G}_a$. In mixed characteristic, this cannot work because ψ does not factor through any finite subgroup of $F/\mathcal{O}$.

But Lemma 3.9 below saves us from this predicament.

Definition 3.8. If H is a smooth affine group scheme over $\mathcal{O}$ and $s \in \mathbb{Z}$, we let $L^{\geq s}H$ denote the image of L^+H under the isomorphism

$$LH \xrightarrow{\cdot \varpi^s} LH.$$

For $s > 0$ it's clear that the natural embedding $L^+H \to LH$ factors through $L^{\geq -s}H$, so we can form the quotient

$$L^{\geq s}H/L^+H,$$

which is isomorphic to L^sH.

Lemma 3.9. *If λ is a dominant coweight and and ν is a coweight, there is a factorization*

$$\begin{tikzcd} \mathrm{MV}_{\lambda,\nu} \arrow[r, dashed, "h_\mu^{\lambda,\nu}"] \arrow[d, hook] & L\mathbb{G}_a^{\geq -s}/L^+\mathbb{G}_a \arrow[d, hook] \\ S_\nu \arrow[r, "h_\mu^\nu"'] & L\mathbb{G}_a/L^+\mathbb{G}_a \end{tikzcd}$$

where $s > 0$ is some large enough positive integer.

Proof. Note $\mathrm{MV}_{\lambda,\nu}$ is a subscheme of $\mathrm{Gr}_{\leq\lambda}$, which is the perfection of a projective variety over k, by the results of [BS17], and is therefore quasi-compact over k. So the morphism to the ind-scheme

$$L\mathbb{G}_a/L^+\mathbb{G}_a = \mathrm{colim}_s L\mathbb{G}_a^{\geq -s}/L^+\mathbb{G}_a$$

must factor through one of the $L\mathbb{G}_a^{\geq -s}/L^+\mathbb{G}_a$. [1]

□

Lemma 3.10. *The quotient $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ is represented by a pfp perfect group scheme and its k-points are naturally identified with $\varpi^{-s}\mathcal{O}/\mathcal{O}$.*

Proof. We exhibit an isomorphism $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \xrightarrow{\sim} L^s\mathbb{G}_a$. If R is a perfect k-algebra, we can define an isomorphism of group-valued presheaves

$$L^{\geq -s}\mathbb{G}_a(R)/L^+\mathbb{G}_a(R) \to L^s\mathbb{G}_a(R)$$
$$\sum_{i=-s}^{-1}[r_i]\varpi^i \mapsto \sum_{i=0}^{s-1}[r_{i-s}]\varpi^i$$

and then take the sheafification. We conclude by noting that $L^s\mathbb{G}_a$ is the perfection of the finite type group scheme $L_p^s\mathbb{G}_a$ whose k-points are $\mathcal{O}/\varpi^s\mathcal{O}$. □

[1]Note that $\mathrm{Aff}_k \hookrightarrow \mathrm{Ind\,Sch}_k^{\mathrm{str}} \hookrightarrow \mathrm{Ind\,Sch}_k$, embeds as compact objects, hence, mapping out of an affine scheme factors through a finite stage of an indscheme. Any quasicompact scheme X is given by a finite cover of affine schemes, which implies the same property holds for quasicompact schemes.

Theorem 3.11 (Lusztig, [Lus06]). *For each group homomorphism*

$$\chi : \varpi^{-s}\mathcal{O}/\mathcal{O} \to \overline{\mathbb{Q}}_\ell^\times$$

there is a unique rank 1 $\overline{\mathbb{Q}}_\ell$-local system $\mathcal{L}_\psi$ on $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ [Ashwin: this sheaf is really defined on the base change to $\overline{k}$, but we need to change everything at some point to reflect the fact that we're working over the algebraic closure and doing a bunch of descent] *such that*

(1) *$a^*\mathcal{L}_\psi \cong \mathcal{L}_\psi \boxtimes \mathcal{L}_\psi$, where a is the addition map on $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$, and*

(2) *the value of the trace of Frobenius at the stalk of $\mathcal{L}_\psi$ at $g \in G(k)$ is $\chi(g)$.*

Remark 3.12. Lusztig did not consider perfections of finite type group schemes in his result, but since the étale site is insensitive to perfection and $L^{\geq -s}\mathbb{G}_a/\mathbb{G}_a$ is a pfp perfect group scheme, the theorem applies to $L^{\geq -s}\mathbb{G}_a/\mathbb{G}_a$ without any further work. See [DW23, Theorem 2.9] for another account of this. [Konrad: Is it clear that it is a pfp perfect group scheme? it seems to use the fact that $L^s\mathbb{G}_a$ is the perfection of a finite type group scheme, and while it is true that $L^s\mathbb{G}_a$ is the perfection of a finite type scheme, and that the multiplication map is of pfp, it is not clear to me that there is a suitable choice of deperfection that makes $L^s\mathbb{G}_a$ come from a finite type group scheme] [Ashwin: Look at Zhu 1.1.1, he defines $L^s\mathbb{G}_a$ as the perfection of something of finite type (just the naive Witt vector scheme)]

Moreover, if $t > s$ there is an inclusion

$$\iota : L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \hookrightarrow L^{\geq -t}\mathbb{G}_a/L^+\mathbb{G}_a$$

and it is easy to check that $\iota^*\mathcal{L}_\psi = \mathcal{L}_\psi$.

3.13. **Convolution.** For those familiar with the twisted product construct is free to skip this section.

Definition 3.14 (Twisted product). If H is an algebraic group and X is an L^+H-space, then the *twisted product*

$$\begin{tikzcd} \mathrm{Gr}_H \tilde{\times} X := LH \times^{L^+H} X \arrow[d] \\ \mathrm{Gr}_H \end{tikzcd}$$

forms a new fiber bundle with fibers X.

There is a moduli description

$$\mathrm{Gr}_G \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_G = \left\{ \mathcal{E}_1 \overset{\beta_1}{\dashrightarrow} \cdots \overset{\beta_{n-1}}{\dashrightarrow} \mathcal{E}_n \overset{\beta_n}{\dashrightarrow} \mathcal{E}^0 \right\}$$

Recall that we have a fiber sequence $N \to B \to T$ which functorially induces

$$\mathrm{Gr}_N \to \mathrm{Gr}_B \to \mathrm{Gr}_T .$$

But $\mathrm{Gr}_T = \bigsqcup_{\nu \in X_*(T)} \mathrm{Gr}_T^\nu$ and so we let

$$S_\nu := \mathrm{Gr}_B \times_{\mathrm{Gr}_T} \mathrm{Gr}_T^\nu$$

Note that the restriction of the L^+G-torsor $LG \to \mathrm{Gr}_G$ over S_ν has a canonical reduction as a L^+N-torsor given by

$$LN \to S_\nu, \quad n \mapsto n \cdot t^\lambda \mod L^+G.$$

So if we take $H = N$, for $\nu_\bullet = (\nu_1, \dots, \nu_m)$ any tuple in $X_*(T)$ we can form the twisted product

$$S_{\nu_\bullet} = S_{\nu_1} \tilde{\times} \cdots \tilde{\times} S_{\nu_m}$$

Definition 3.15. Let

$$m : \mathrm{Gr}_G \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_G \to \mathrm{Gr}_G$$

$$(\mathcal{E}_1 \dashrightarrow \cdots \dashrightarrow \mathcal{E}_n) \mapsto (\mathcal{E}_n, \beta_1 \cdots \beta_n)$$

be the projection on to the nth component.

Definition 3.16. Let $\mathcal{A}_1, \mathcal{A}_2 \in P_{L^+G}(\mathrm{Gr})$, we define the convolution product

$$\mathcal{A}_1 \star \mathcal{A}_2 := m_!(\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2)$$

where $\mathcal{A}_1 \widetilde{\boxtimes} \mathcal{A}_2$ is the unique sheaf of $\mathcal{A}_1 \boxtimes \mathcal{A}_2$, [Zhu17, p. 2.2].

Proposition 3.17.

$$\begin{tikzcd}
S_{\nu_\bullet} \arrow[r, "\simeq"] \arrow[d, hook] & S_{\sigma_1} \times S_{\sigma_2} \times \cdots \times S_{\sigma_n} \arrow[d, hook] \\
\mathrm{Gr} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr} \arrow[r] & \mathrm{Gr} \times \cdots \times \mathrm{Gr} \simeq \mathrm{Gr}^n
\end{tikzcd}$$

where $\sigma_i = \sum_{k=1}^{i} \nu_k$

4. Orbit Intersections: Mirkovic-Vilonen Cycles

To compute the Hecke action, we need to understand the intersection of semi-infinite orbits [Fre+98, p. 7]. These played a dominant role in the first complete proof of geometric Langlands [MV07]. Over $\mathbb{C}$, the statement has already appeared in the work of [**Lus82**]. In mixed characteristic, this was discussed [Zhu17, p. 2.2]. Let us recall the semi-infinite orbits in the p-adic setting from [FS21, p. VI.3]. [Ham22, p. 4.2]. To make the first cohomolgoical computation, we follow the argument of Ngô-Polo [NP01, p. 5].

Definition 4.1. Let $\Omega_\mu := \{\lambda \in X_\bullet : \lambda^+ \leq \mu\}$, where λ^+ is the unique dominant W-translate of λ. [2]

For (possible) future use, we consider the *Beilinson Drinfeld Grassmanian*, which we recall in 4.9. For convenience, we omit the base stack of divisors Div^I. In this section, G is a split reductive group over K, a p-adic field. [3] We thus fix a split reductive model over $\mathcal{O}_K$.

[2]Alternatively, this is $\lambda + \mathbb{Z}\Phi^\vee \cap \mathrm{Conv}(W\lambda)$

[3]One can always base change when necessary.

Definition 4.2. Let I be a finite set. For $\nu_\bullet := (\nu_i)_{i\in I} \in (X_\bullet)^I$. The *semi-infinite obrit* associated to $\nu_\bullet$ is the small v-sheaf $S_G^{\nu_\bullet} \in \mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_p}, v)_{/\mathrm{Div}^I}$ given by the pullback

$$\begin{tikzcd} S_G^{\nu_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_B^I \arrow[d] \\ \mathrm{Gr}_T^{\nu_\bullet} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Gr}_T^I \arrow[d] \\ \mathrm{pt} \arrow[r, "\nu_\bullet"] & (X_\bullet)^I \end{tikzcd}$$

Definition 4.3. For $\lambda \in X_{\bullet,+}^I$, we let $\mathrm{Gr}_G^{\lambda_\bullet}$ be the locally closed subfunctor of Gr_G^I.

Definition 4.4. Let

$$\begin{tikzcd} \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \arrow[d] \\ \mathrm{Hck}_{G,\mathrm{Div}^1_{\mathcal{Y}},\mu} \arrow[r, hook] & \mathrm{Hck}_{G,\mathrm{Div}^1_{\mathcal{Y}}} \end{tikzcd}$$

be the inclusion of open cells, [FS21, p. IV.7.5], and denote

$$\mathcal{A}_\mu := j_{\mu !}\Lambda[d_\mu]$$

as the IC sheaves.[4]

To set the stage, we recall the Satake isomorphism in the mixed characteristic setting

Theorem 4.5. *[FS21, p. I.6.3] For a finite index I,*

$$Sat_G^I \simeq \mathrm{Rep}_\Lambda({}^L G^I)$$

Proposition 4.6. *[Ham22, p. 4.4] For all finite index sets I, the followin diagram commutes*

$$\begin{tikzcd} Sat_G^I \arrow[r, "CT{[\deg]}"] \arrow[d, "F_G^I"] & Sat_T^I \arrow[d, "F_T^I"] \\ \mathrm{Rep}_\Lambda({}^L G) \arrow[r, "res_T^I"] & \mathrm{Rep}_\Lambda^I({}^L T) \end{tikzcd}$$

where

- *CT is the constant term functor.*
- *F_G^I, F_I^T are due to Tannakian equivalence [FS21, Thm 1.6.3].*

Proposition 4.7. *Let $\lambda \in X_{\bullet,+}$. Let $x \to Div^1$ be a geometric point.*

$$H_c^k({}_xS^\nu \cap \overline{{}_x\,\mathrm{Gr}^\lambda}, \mathcal{A}_\lambda)$$

vanishes unless $k = \langle 2\rho, \nu\rangle$, in which case, it is isomorphic to $V^\lambda(\nu)^\vee$.

[4]The typical analysis of such sheaves on Hck stack pullsback further to the Demazure resolution.

Proof. Let us consider the following diagram

$$\begin{tikzcd}
\mathrm{pt} & S^{\lambda} \arrow[l, "p"'] \arrow[r, hook, "q"] & \mathrm{Gr} \\
& S^{\lambda} \cap \overline{\mathrm{Gr}^{\mu}} \arrow[ul, "p'"] \arrow[u, hook] \arrow[r, hook, "q'"] & \overline{\mathrm{Gr}^{\mu}} \arrow[u, hook] \\
& & \mathrm{Gr}^{\mu} \arrow[u, hook]
\end{tikzcd}$$

Let $\mathcal{S}_{V^\lambda}$ be the sheaf corresponding to highest weight representation V^λ, as 4.5. Then by applying 4.6,

$$\begin{aligned}
H_c^k({}_xS^\nu \cap \overline{{}_x\,\mathrm{Gr}^\lambda}, \mathcal{A}_\lambda) &= (p')_!(q')^*(\mathcal{A}_\lambda) \\
&\simeq p_!q^*(\mathcal{S}_{V^\lambda}) \\
&= H_c^{-\langle 2\rho,\nu\rangle}(S^\nu, \mathcal{S}_{V^\lambda}) \\
&\simeq V^\lambda(\nu)^\vee
\end{aligned}$$

□

4.7.1. *Properties of orbit intersection.*

Proposition 4.8. *[***BR18***], [***She22***] Let $\lambda, \nu \in X_\bullet$ with λ dominant, $x \to Div^1$ be a geometric point.*

(1) Nonemptiness.

$${}_xS^\nu \cap \overline{{}_x\,\mathrm{Gr}^\lambda} = {}_xS^\nu \cap {}_x\,\mathrm{Gr}^{\leq\lambda} \neq \emptyset \Leftrightarrow \nu \in \Omega_\lambda$$

(2) Dimension.

$${}_xS^\nu \cap {}_x\,\mathrm{Gr}^{\leq\nu}$$

is equidimensional of rank $\langle \rho, \nu + \lambda\rangle$.

(3) Containment property.

$$\bigsqcup_{\nu\in\Omega_\lambda} {}_xS^\nu \cap \overline{{}_x\,\mathrm{Gr}^\lambda} \xrightarrow{\simeq} {}_x\,\mathrm{Gr}^{\leq\nu}$$

of underlying topological spaces.

4.9. **Recollection on affine Grassmanian.** We will consider the B_{dR}^+ affine Grassmanian. The local definition can be specialized from the global definition. We include the latter when we need to describe the Hecke action.

Let $S \in \mathrm{Pftd}_{\mathbb{F}_q}$. Recall in [FS21, p. II], we could construct curves

$$\mathcal{Y}_S, Y_S := \mathcal{Y}_S \backslash V(\pi) \text{ and } X_S = Y_S/\varphi^{\mathbb{Z}}$$

We can define the following stacks of divisors on such curves.

Definition 4.10. We have the following small v-sheaves $\mathrm{Shv}(\mathrm{Pftd}_{\mathbb{F}_q}, v)$

$$\mathrm{Div}^1_{\mathcal{Y}} := \mathrm{Spd}(\mathcal{O}_K)$$

$$\mathrm{Div}^1_X := \mathrm{Div}^1 := \mathrm{Spd}\, K/\varphi^{\mathbb{Z}}$$

where Div^1 is the *mirror curve* [5] For a finite set I with $|I| = d$, we will denote

$$\mathrm{Div}^I_{\mathcal{Y}} := (\mathrm{Div}^1_{\mathcal{Y}})^d$$

Definition 4.11. Let I be a finite set.

$$\mathrm{Gr}^I_{G,\mathrm{Div}^1_{\mathcal{Y}}} \to \mathrm{Div}^I_{\mathcal{Y}}$$

$$\mathrm{Gr}^I_{G,\mathrm{Div}^1} \to \mathrm{Div}^I$$

be the *Beilinson-Drinfeld* Grassmanian [FS21, p. VI.1.8]. This is a small v-sheaf. Unless stated otherwise, will omit the Div^I. For $S \to \mathrm{Div}^d_{\mathcal{Y}}$ we denote

$$\mathrm{Gr}_{G,S} := \mathrm{Gr}_G \times_{\mathrm{Div}^d_{\mathcal{Y}}} S$$

[5] Its S points are the degree 1 Cartier divisors on X_S, where one has $\pi_1(\mathrm{Div}^1) = W_K$.

5. Non-dominant case

In this section, we verify Theorem 1.1 when $\mu \in X_*(T) \setminus X_*(T)_+$.

By Lemma 3.2 the L^+G-action on $\mathrm{Gr}_{\leq\lambda}$ factors through L^hG for some large enough $h > 0$. Therefore, the L^+N-action on $\mathrm{MV}_{\lambda,\mu}$ factors through L^hN as well. A direct computation shows that the map $h_\mu|_{L^+N} : L^+N \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ also factors as

$$h_\mu|_{L^+N} : L^+N \to L^hN \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$$

for large enough h.

Proposition 5.1. *Choose s such that $h_\mu|_{L^+N}$ and $h_\mu^{\lambda,\nu}$ both factor through $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \to L\mathbb{G}_a/L^+\mathbb{G}_a$. Then the following diagram commutes:*

$$\begin{tikzcd}
L^+N \times \mathrm{MV}_{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[d] & \mathrm{MV}_{\lambda,\nu} \arrow[d, equal] \\
L^hN \times \mathrm{MV}_{\lambda,\nu} \arrow[r, "\mathrm{act}"] \arrow[d, "h_\mu \times h_\mu^{\lambda,\nu}"'] & \mathrm{MV}_{\lambda,\nu} \arrow[d, "h_\mu^{\lambda,\nu}"] \\
L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \times L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "a"'] \arrow[d] & L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a \arrow[d] \\
L\mathbb{G}_a/L^+\mathbb{G}_a \times L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "a"] & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

Proof. This is a diagram chase. □

Corollary 5.2. *If μ is non-dominant, $\mu+\nu$ is dominant, and λ is dominant, then*

$$R\Gamma_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) = 0.$$

Proof. By Proposition 5.1 and the fact that $\mathcal{A}_\lambda$ is L^+G-equivariant,

$$\begin{aligned}
\mathrm{act}^*(\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) &= \mathrm{act}^*\mathcal{A}_\lambda \otimes \mathrm{act}^*(h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu \times h_\mu^{\lambda,\nu})^* a^*\mathcal{L}_\psi \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu \times h_\mu^{\lambda,\nu})^* (\mathcal{L}_\psi \boxtimes \mathcal{L}_\psi) \\
&= (\overline{\mathbb{Q}}_\ell \boxtimes \mathcal{A}_\lambda) \otimes (h_\mu^*\mathcal{L}_\psi \boxtimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi) \\
&= h_\mu^*\mathcal{L}_\psi \boxtimes (\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi),
\end{aligned}$$

so $\mathcal{A}_\lambda \otimes (h_\mu^{\lambda,\nu})^*\mathcal{L}_\psi$ is $(L^hN, h_\mu^*\mathcal{L}_\psi)$-equivariant.

If μ is not dominant, pick a simple root α such that $\langle \alpha, \mu \rangle < 0$ and let $u_\alpha : \mathbb{G}_a \to N$ denote the inclusion of the root subgroup. Then the composition

$$L^+\mathbb{G}_a \hookrightarrow L\mathbb{G}_a \xrightarrow{u_\alpha} LN \xrightarrow{\mathrm{ad}\,\varpi^\mu} LN \to LN/[LN, LN] \xrightarrow{+} L\mathbb{G}_a$$

is just the multiplication by $\varpi^{\langle\alpha,\mu\rangle}$ map. Therefore, $h_\mu|_{L^+N}$ is non-trivial. This implies that $h_\mu^*\mathcal{L}_\psi$ is also nontrivial. To see why, note that the local system $h_\mu^*\mathcal{L}_\psi$ corresponds to the character

$$\pi_1^{\text{ét}}(L^h N) \to \pi_1^{\text{ét}}(L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a) \twoheadrightarrow \varpi^{-s}\mathcal{O}/\mathcal{O} \to \overline{\mathbb{Q}}_\ell^\times$$

and the first map is surjective since the morphism $L^h N \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ has connected geometric fibers, so this character is nontrivial. We conclude by applying Proposition 5.3. □

Proposition 5.3. *Suppose Z is a pfp perfect group scheme over k with an action*

$$\mathrm{act} : G \times Z \to Z$$

of a pfp perfect group scheme G defined over k. If $\mathcal{L}$ is a non-trivial rank 1 local system on G and $\mathcal{F} \in \mathrm{Shv}(Z)$ is $(G, \mathcal{L})$-equivariant, i.e.

$$\mathrm{act}^* \mathcal{F} \simeq \mathcal{L} \boxtimes \mathcal{F}$$

then

$$R\Gamma_c(Z, \mathcal{F}) = 0.$$

Proof. The proof of [Ngô00, Lemma 3.3] goes through verbatim, for G a connected commutative algebraic group replacing $\mathbb{G}_a$, and noting that the statement depends only on the étale topology, which is insensitive to perfection. □

6. Dominant case: equal cocharacters

Now we treat the case of a dominant twist of h_μ. From now on suppose $\mu \in X_*(T)_+$.

In this section we treat the case where $\nu = \lambda$ in $S_\nu \cap \mathrm{Gr}_{\leq\lambda}$. Since $V^{\lambda+\mu}$ appears with multiplicity one inside of $V^\lambda \otimes V^\mu$, we want to show

$$R\Gamma_c(\mathrm{MV}_{\lambda,\lambda}, \mathcal{A}_\lambda|_{\mathrm{MV}_{\lambda,\lambda}} \otimes (h_\mu^{\lambda,\lambda})^*(\mathcal{L}_\psi)) = \overline{\mathbb{Q}}_\ell[2(\rho,\lambda)]$$

Lemma 6.1. *Suppose $\lambda \in X_*(T)_+$ and $w \in W$. Then*

$$h_\mu^{\lambda,w\lambda} : \mathrm{MV}_{\lambda,w\lambda} \to L^{\geq -s}\mathbb{G}_a / L^+\mathbb{G}_a$$

factors through the identity section $\operatorname{Spec} k \to L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$.

Proof. First note that [Zhu17, Corollary 2.8] implies that

$$S_{w\lambda} \cap \mathrm{Gr}_{\leq\lambda'} \neq \varnothing \text{ if and only if } w\lambda \in \Omega(\lambda').$$

If $\lambda' < \lambda$ we cannot have $w\lambda \in \Omega(\lambda')$. [Milton: to check] So since

$$\mathrm{MV}_{\lambda,w\lambda} = \bigcup_{\lambda' \leq \lambda} S_{w\lambda} \cap \mathrm{Gr}_{\lambda'}$$

we see that $\mathrm{MV}_{\lambda,w\lambda} = S_{w\lambda} \cap \mathrm{Gr}_\lambda$.

Since $S_{w\lambda} \subset \mathrm{Gr}_G$ is locally closed, its intersection with $\mathrm{Gr}_{\leq\lambda}$ is again locally closed. But [BS17] shows that $\mathrm{Gr}_{\leq\lambda}$ is the perfection of a projective k-variety, so $\mathrm{MV}_{\lambda,w\lambda}$ is the perfection of a quasi-projective reduced k-scheme. As explained in Lemma 3.10, $L^{\geq -s}\mathbb{G}_a/L^+\mathbb{G}_a$ is the perfection of a finite type k-group scheme. So as explained in the proof of [Zhu17, Proposition A.17], $h_\mu^{\lambda,w\lambda}$ is the perfection of a morphism

$$h : X \to Y$$

of finite type weakly normal (and in particular reduced) k-schemes such that

$$h^{\mathrm{perf}} = h_\mu^{\lambda,w\lambda}$$

$$X(k) = X^{\mathrm{perf}}(k), Y(k) = Y^{\mathrm{perf}}(k)$$

since perfection does not affect k-points. Now since k is algebraically closed, to prove that $X \to Y$ is the trivial map, it suffices to check that

$$h_\mu^{\lambda,w\lambda}(k) : N(F)\varpi^{w\lambda}/G(\mathcal{O}) \cap G(\mathcal{O})\varpi^\lambda/G(\mathcal{O}) \to \varpi^{-s}\mathcal{O}/\mathcal{O}$$

sends every element to $0 \in \varpi^{-s}\mathcal{O}/\mathcal{O}$.

By [Sat63, p.44][6],

$$N(F)\varpi^{w\lambda}/G(\mathcal{O}) \cap G(\mathcal{O})\varpi^\lambda/G(\mathcal{O}) = N(\mathcal{O})\varpi^{w\lambda}/G(\mathcal{O}) = [\varpi^{w\lambda}(\varpi^{-w\lambda}N(\mathcal{O})\varpi^{w\lambda})]/G(\mathcal{O}),$$

and then the conclusion is clear from the definition of h (see Lemma 3.6). ☐

Corollary 6.2. *Let λ be quasiminuscule. $\mu \in X_\bullet(T)_+$*

$$R\Gamma_c(\mathrm{MV}_{\lambda,w\lambda}, \mathcal{A}_\lambda \otimes (h_\mu^{\lambda,w\lambda})^*(\mathcal{L}_\psi)) \simeq R\Gamma_c(\mathrm{MV}_{\lambda,w\lambda}, \mathcal{A}_\lambda) = \overline{\mathbb{Q}}_\ell[2\langle\rho,\lambda\rangle]$$

[6]Satake's paper assumes F is a finite extension of $\mathbb{Q}_p$, but the same proof works when F is finite and totally ramified over $F_0 = W(k)$, where k is an algebraically closed field of characteristic p.

Proof. By Lemma 6.1, $(h_\mu^{\lambda,w\lambda})^*\mathcal{L}_\psi = \overline{\mathbb{Q}}_\ell$, which implies the first equality.

Note that by [Zhu17, Prop 2.7], the cohomology is concentrated in one degree, $\langle 2\rho, \lambda\rangle$. The number of irreducible components of $\mathrm{MV}_{\lambda,w\lambda}$ is equal to the dimension of the weight space $w\lambda$ in the highest weight representation V_λ, [Zhu17, Prop. 2.8]. These irreducible components form a basis of the cohomology, [Zhu17, Prop 2.9]. We observe that the the weight space of $w\lambda$ in V^λ is 1-dimensional. $\square$

7. Dominant case: unequal cocharacters

We still assume $\mu \in X_*(T)_+$, but now we assume $\nu \neq \lambda$ in $S_\nu \cap \mathrm{Gr}_{\leq\lambda}$. Our goal is to show:

$$R\Gamma_c(\mathrm{MV}_{\lambda,\nu}, \mathcal{A}_\nu\big|_{\mathrm{MV}_{\lambda,\nu}} \otimes (h_0^{\lambda,\nu})^*\mathcal{L}_\psi) = 0. \tag{2}$$

We will follow the strategy of [NP01]: we will construct a fibration and reduce the problem to studying the geometry of $\mathrm{MV}_{\lambda,\nu}$ for λ quasi-minuscule.

First, we use Zhu's geometric version of the PRV conjecture:

Lemma 7.1 ([Zhu17, Lemma 2.16])**.** *Given $\lambda \in X_\bullet(T)_+$, there exists a sequence of quasi-minuscule cocharacters $\lambda_\bullet = (\lambda_1, \dots, \lambda_m)$ such that $W^\lambda_{\lambda_\bullet} \neq 0$ in the decomposition*

$$\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_m} = \bigoplus_{\substack{\xi \in X_*(T)_+,\\ \xi \leq |\lambda_\bullet|}} \mathcal{A}_\xi \otimes W^\xi_{\lambda_\bullet}.$$

in the Satake category $P_{L^+G}(\mathrm{Gr}_G)$. Here, the dimension of $W^\xi_{\lambda_\bullet}$ is equal to the multiplicity of $\mathcal{A}_\xi$ in the convolution.

Fix a sequence $\lambda_\bullet = (\lambda_1, \dots, \lambda_m)$ as in Lemma 7.1. The decomposition in the statement of the lemma gives rise to an isomorphism

$$\begin{aligned}&R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi))\\ &= \bigoplus_{\substack{\xi \in X_*(T)_+,\\ \xi \leq |\lambda_\bullet|}} R\Gamma_c(\mathrm{MV}_{\xi,\nu}, \mathcal{A}_\xi \otimes (h_0^{\xi,\nu})^*(\mathcal{L}_\psi)) \otimes W^\xi_{\lambda_\bullet}\end{aligned}$$

So we have proven Equation 2 if we can show that the direct factor map

$$\begin{aligned}&R\Gamma_c(\mathrm{MV}_{\nu,\nu}, \mathcal{A}_\nu \otimes (h_0^{\nu,\nu})^*(\mathcal{L}_\psi)) \otimes W^\nu_{\lambda_\bullet}\\ &\to R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi))\end{aligned}$$

is a quasi-isomorphism.

But by Corollary 6.2, the left hand side is isomorphic $\overline{\mathbb{Q}}_\ell[\langle 2\rho, \nu\rangle]\,(-\langle \rho, \nu\rangle)$, so it suffices to show that

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) = W^\nu_{\lambda_\bullet}[\langle 2\rho, \nu\rangle]\,(-\langle \rho, \nu\rangle)\,.$$

Proposition 8.1 shows that we can break down the cohomology.

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) = \bigoplus_{|\nu_\bullet| = \nu} \bigotimes_{i=1}^m R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes h^{\lambda_i,\nu_i}_{\sigma_{i-1}}\mathcal{L}_\psi)$$

Now fix an n-tuple $\nu_\bullet = (\nu_1, \dots, \nu_n)$ such that $|\nu_\bullet| = \nu$. We may make the following two assumptions.

- Every σ_i is dominant. If not, then some σ_i is non-dominant. In this case
$$R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes (h^{\lambda_i,\nu_i}_{\sigma_i})^*\mathcal{L}_\psi) = 0$$
by Corollary 5.2, so the whole tensor product vanishes as well.
- Either $\nu_i = w\lambda_i$ for some $w \in W$, or $\nu_i = 0$. Recall from [NP01, p. 1.1], that as λ_i are minuscule, we have $\Omega(\lambda_i) = W\lambda_i \cup \{0\}$. We may thus suppose $\nu_i \in \Omega(\lambda_i) = W\lambda_i \cup \{0\}$, for otherwise $\mathrm{MV}_{\lambda_i,\nu_i} = \varnothing$ by Proposition 4.8.

We now split into cases based on whether $\nu_i = w\lambda_i$ or $\nu_i = 0$.

7.1.1. *Weyl orbit.* If $\nu_i = w\lambda_i$ for some $w \in W$,
$$R\Gamma_c(\mathrm{MV}_{\lambda_i,w\lambda_i}, \mathcal{A}_{\lambda_i} \otimes h^*_{\sigma_i}\mathcal{L}_\psi) = \bar{\mathbb{Q}}_\ell[\langle 2\rho, w\lambda_i\rangle][(\rho, w\lambda_i)]$$
by Corollary 6.2.

If $\nu_i = 0$, we will use the computation in **??**. Combining these two, we deduce that
$$H^i_c(\mathrm{MV}_{\lambda_\bullet,\nu_\bullet}, \mathcal{A}_{\lambda_\bullet} \otimes h^*\mathcal{L}_\psi) = \begin{cases} 0 & i \neq 2\langle\rho,\nu\rangle \\ |\,\{\text{dominant } \lambda_\bullet \text{ paths from } 0 \text{ to } \nu\}\,| & i = 2\langle\rho,\nu\rangle \end{cases}$$

8. Breaking down the convolution

Our goal for this section is to prove that the cohomology

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)).$$

breaks down as follows

Proposition 8.1. *Let $\sigma_i = \nu_1 + \cdots + \nu_i$ for $i = 1, \ldots, m$, then*

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) = \bigoplus_{|\nu_\bullet|=\nu} \bigotimes_{i=1}^{m} R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes h_{\sigma_{i-1}}^{\lambda_i,\nu_i}\mathcal{L}_\psi)$$

where $\nu_\bullet$ runs over all n-tuples of elements of $X_*(T)$ summing to ν.

Recall that

$$\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n} = m_!(\mathcal{A}_{\lambda_1} \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} \mathcal{A}_{\lambda_n})$$

So the projection formula gives us

$$R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) = R\Gamma_c(\bigcup_{|\nu_\bullet|=\nu} \widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}, \mathcal{A}_{\lambda_1} \widetilde{\boxtimes} \cdots \widetilde{\boxtimes} \mathcal{A}_{\lambda_n} \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)),$$

[Milton: this is not so clear.] noting that

$$m^{-1}(\mathrm{MV}_{|\lambda_\bullet|,\nu}) = \bigcup_{|\nu_\bullet|=\nu} \widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}$$

where

$$\widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet} = \mathrm{MV}_{\lambda_1,\nu_1} \mathbin{\widetilde{\times}} \cdots \mathrm{MV}_{\lambda_n,\nu_n}.$$

Recall that the *right* multiplication action of L^+G on LG makes $LG \to \mathrm{Gr}$ a right L^+G-torsor, and this canonically descends to an L^+N-torsor

$$\begin{aligned} \varpi^\nu LN &\to S_\nu \\ \varpi^\nu n &\mapsto \varpi^\nu n \mod L^+G. \end{aligned}$$

This map is L^+N-equivariant from the left.

Definition 8.2. Let $r \in \mathbb{N}_{\geq 0} \cup \{\infty\}$. We can form L^rN-torsors over S^ν and $\mathrm{MV}_{\lambda,\nu}$ using the following pullback diagram:

$$\begin{tikzcd} \mathrm{MV}_{\lambda,\nu}^{(r)} \arrow[r] \arrow[d, "p_r"] & S_\nu^{(r)} := \varpi^\nu LN \times^{L^+N} L^rN \arrow[d] \\ \mathrm{MV}_{\lambda,\nu} \arrow[r, hook] & S_\nu \end{tikzcd}$$

We adopt the convention $L^\infty N := L^+N$. Note that $S_\nu^{(0)} = S_\nu$ and $S_\nu^{(\infty)} = \varpi^\nu LN$.

Lemma 8.3. *For $r \geq 0$, the left action of L^+N on $\mathrm{MV}_{\lambda,\nu}^{(r)}$ factors through $L^{r'}N$ for some $r' > 0$.*

Proof. If we write

$$\mathrm{MV}^{(r)}_{\lambda,\nu} = \mathrm{MV}_{\lambda,\nu} \times_{S_\nu} (\varpi^\nu LN \times^{L^+N} L^rN),$$

then the left L^+N-action is just the diagonal action, which descends to the fiber product. Thus, it suffices to check individually on each component that L^+N factors through some L^fN, $f \in \mathbb{N}$.

[Ashwin: make this work]

For the first factor, the left action of L^+G on $\mathrm{Gr}_{\leq\lambda}$ factors through $L^{r'}G$ for some $r > 0$ (which depends on λ), so the left L^+N-action on $\mathrm{MV}_{\lambda,\nu}$ factors through $L^{r'}N$ as well.

For the second factor, note that an arbitrary element of $\varpi^\nu LN \times^{L^+N} L^rN$ is of the form $(\varpi^\nu n, LN^{(r)})$ for some $n \in LN$. We want to show that there exists some large enough $r'' > r'$ such that if $h \in LN^{(r'')}$ then

$$(h\varpi^\nu n, LN^{(r)}) \sim (\varpi^\nu n, LN^{(r)})$$

Since $\varpi^\nu nL^+G \in \mathrm{MV}_{\lambda,\nu}$, if $h \in LN$, then h fixes $\varpi^\nu nL^+G \in \mathrm{MV}_{\lambda,\nu}$, so

$$h\varpi^\nu n = \varpi^\nu ng$$

for some $g \in L^+G$. In fact $g \in LN$, since $g = \mathrm{ad}((\varpi^\nu n)^{-1})(h)$, so $g \in L^+N = LN \cap L^+G$. Then

$$(h\varpi^\nu n, LN^{(r)}) = (\varpi^\nu ng, LN^{(r)}) \sim (\varpi^\nu n, gLN^{(r)}),$$

so we are done if $g \in LN^{(r)}$.

Since $\varpi^\nu nL^+G \in \mathrm{Gr}_{\leq\lambda}$, there exists some $x, g' \in L^+G$ and some dominant $\lambda' \leq \lambda$ such that $\varpi^\nu n = x\varpi^{\lambda'}g'$. Thus

$$n = \varpi^{-\nu}x\varpi^{\lambda'}g'.$$

We conclude by proving two facts:

(1) For any cocharacter ν and any $r' \in \mathbb{N}$, there exists $s \in \mathbb{N}$ so that

$$\mathrm{ad}(\varpi^\nu)(LN^{(s)}) \subseteq LN^{(r')}.$$

(2) For any $x \in L^+G$, $r' \in \mathbb{N}$, there exists $s \in \mathbb{N}$ so that

$$\mathrm{ad}(x)(LN^{(s)}) \subseteq LG^{(r')}.$$

Indeed by 1 and 2 we can show that there exists s such that $g \in LG^{(r)}$. As $LG^{(r)} \cap L^+N$, from the diagram

$$\begin{tikzcd} LN^{(r)} \arrow[r, hook] \arrow[d] & LG^{(r)} \arrow[d] \\ L^+N \arrow[r, hook] \arrow[d] & L^+G \arrow[d] \\ L^rN \arrow[r, hook] & L^rG \end{tikzcd}$$

we have $g \in LN^{(r)}$. $\square$

Lemma 8.4. *For any cocharacter ν and any $r' \in \mathbb{N}$, there exists $s \in \mathbb{N}$ so that*

$$\mathrm{ad}(\varpi^{\nu})(LN^{(s)}) \subseteq LN^{(r')}.$$

Proof. The case for GL_n is clear. The general case follows from embedding into GL_n and the diagram :

$$\begin{tikzcd}
L^+N^{(s)} \arrow[r, hook] \arrow[d] & LU^{(s)} \arrow[d] \\
L^+N \arrow[r, hook] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & L^+U \arrow[d] \\
L^+G \arrow[r, hook] \arrow[d, two heads] & L^+\mathrm{GL}_n \arrow[d, two heads] \\
L^{r'}G \arrow[r, hook] & L^{r}\mathrm{GL}_n
\end{tikzcd}$$

and the fact that being unipotent for an element is an intrinsic property. □

Now pick $\nu_1, \dots, \nu_m$ such that $\nu_1 + \cdots + \nu_m = \nu$.

By the lemma we can choose integers $r_1, \dots, r_m \geq 0$ such that $r_m = 0$ and such that the action of L^+N on $\prod_{k=i}^{m} \mathrm{MV}_{\lambda_k,\nu_k}^{(r_k)}$ factors through $L^{r_{i-1}}N$ for $i = 2, \dots, m$.

Lemma 8.5. *There are two $\prod_i L^{r_i}N$ torsors $p_\bullet = \prod p_i$ and $q_\bullet$*

$$\begin{tikzcd}
& \prod_{i=1}^{m} (\mathrm{MV}_{\lambda_i,\nu_i})^{(r_i)} \arrow[dl, "p_\bullet"'] \arrow[dr, "q_\bullet"] & \\
\prod_{i=1}^{m} \mathrm{MV}_{\lambda_i,\nu_i} & & \widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}
\end{tikzcd}$$

such that

$$q_\bullet^* \mathcal{A}_{\lambda_\bullet} \cong p_1^* \mathcal{A}_{\lambda_1} \boxtimes \cdots \boxtimes p_m^* \mathcal{A}_{\lambda_m}.$$

Proof. The torsor $p_\bullet$ is just the product of each individual $L^{r_i}N$-torsor $\mathrm{MV}_{\lambda_i,\nu_i}^{(r_i)} \to \mathrm{MV}_{\lambda_i,\nu_i}$. If $m = 1$ there is nothing to do, so suppose $m > 1$. Since the L^+N-action on $\mathrm{MV}_{\lambda_m,\nu_m}$ factors through $L^{(r_{m-1})}N$, we can form the diagram

$$\begin{tikzcd}
& \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}}^{(\infty)} \times \mathrm{MV}_{\lambda_m,\nu_m} \arrow[d, "p_\infty"'] \arrow[ddl, "p"'] \arrow[ddr, "q"] & \\
& \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}}^{(r_{m-1})} \times \mathrm{MV}_{\lambda_m,\nu_m} \arrow[dl, "p_{r_{m-1}} \times \mathrm{id}"] \arrow[dr, "q_{r_{m-1}}"'] & \\
\mathrm{MV}_{\lambda_{m-1},\nu_{m-1}} \times \mathrm{MV}_{\lambda_m,\nu_m} & & \mathrm{MV}_{\lambda_{m-1},\nu_{m-1}} \tilde{\times} \mathrm{MV}_{\lambda_m,\nu_m}
\end{tikzcd}$$

in which q is an L^+N-torsor and q_r is an $L^{r_{m-1}}N$-torsor. The morphism p_∞ is just the pushout along the morphism $L^+N \to L^rN$ in the first slot and the identity in the second. The point now is that there is a unique perverse sheaf $\mathcal{A}_{\lambda_{m-1}}\tilde{\boxtimes}\mathcal{A}_{\lambda_m}$ on $\mathrm{MV}_{\lambda_{m-1},\nu_{m-1}} \tilde{\times} \mathrm{MV}_{\lambda_m,\nu_m}$ satisfying

$$p^*(\mathcal{A}_{\lambda_{m-1}} \boxtimes \mathcal{A}_{\lambda_m}) \cong q^*(\mathcal{A}_{\lambda_{m-1}}\tilde{\boxtimes}\mathcal{A}_{\lambda_m}).$$

There is also a unique perverse sheaf $\mathcal{L}$ satisfying

$$q^*_{r_{m-1}}\mathcal{L} \cong p^*_{r_{m-1}}(\mathcal{A}_{\lambda_{m-1}} \boxtimes \mathcal{A}_{\lambda_m})$$

But pulling back by p_∞ gives $q^*\mathcal{L} \cong p^*(\mathcal{A}_{\lambda_{m-1}} \boxtimes \mathcal{A}_{\lambda_m})$ so we must have $\mathcal{L} \cong \mathcal{A}_{\lambda_{m-1}}\tilde{\boxtimes}\mathcal{A}_{\lambda_m}$ by uniqueness.

If $m > 2$, one can repeat the same process as above inductively. For example, first replace $\mathrm{MV}_{\lambda_m,\nu_m}$ with $\mathrm{MV}^{(r_{m-1})}_{\lambda_{m-1},\nu_{m-1}} \times \mathrm{MV}_{\lambda_m,\nu_m}$ and run the same argument. □

Lemma 8.6. *The following diagram commutes:*

$$\begin{tikzcd}
\prod_{i=1}^m \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i} \arrow[r, "q_\bullet"] \arrow[d, "p_\bullet"'] & \widetilde{\prod}_{i=1}^m \mathrm{MV}_{\lambda_i,\nu_i} \arrow[d, "m"] \\
\prod_{i=1}^m \mathrm{MV}_{\lambda_i,\nu_i} \arrow[d, "\prod_{i=1}^m h^{\lambda_i,\nu_i}_{\sigma_{i-1}}"'] & \mathrm{MV}_{|\lambda_\bullet|,\nu} \arrow[d, "h_0^{|\lambda_\bullet|,\nu}"] \\
\prod_{i=1}^m L\mathbb{G}_a/L^+\mathbb{G}_a \arrow[r, "+"] & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

As a direct consequence,

$$(h_0^{|\lambda_\bullet|,\nu} \circ m \circ q_\bullet)^*\mathcal{L}_\psi \simeq (h_0^{\lambda_1,\nu_1} \circ p_1)^*\mathcal{L}_\psi \boxtimes (h^{\lambda_2,\nu_2}_{\sigma_1} \circ p_2) \boxtimes \cdots \boxtimes (h^{\lambda_m,\nu_m}_{\sigma_{m-1}} \circ p_m)^*\mathcal{L}_\psi.$$

Proof. The following diagram commutes

$$\begin{tikzcd}
\prod_{i=1} \mathrm{MV}^{(r_i)}_{\lambda_i,\nu_i} \arrow[d] & & \\
\widetilde{\prod}_{i=1} \mathrm{MV}_{\lambda_i,v_i} \arrow[r, "\simeq"] & S_{\nu_\bullet} \cap \mathrm{Gr}_{\leq\mu_\bullet} \arrow[r, hook] \arrow[d] & S_{\nu_\bullet} \arrow[d, "m"] \\
& S_{\sigma_n} \cap \mathrm{Gr}_{\leq|\mu_\bullet|} \arrow[dr] & S_\nu \arrow[d, "h^\nu"] \\
& & L\mathbb{G}_a/L^+\mathbb{G}_a
\end{tikzcd}$$

where the map m is defined as the composition of the identification in Proposition 3.17 and the projection:

$$S_{\nu_\bullet} \xrightarrow{\simeq} S_{\sigma_1} \times \cdots \times S_{\sigma_n} \longrightarrow S_{\sigma_n} = S_\nu$$

One can check that a general element

$$(\varpi^{\nu_1} x_1, \ldots, \varpi^{\nu_n} x_n) \in \prod_{i=1} \mathrm{MV}^{(r_i)}_{\lambda_i, \nu_i}$$

which, since ϖ^ν normalizes LN, can also be written as

$$(y_1 \varpi^{\nu_1}, \ldots, y_n \varpi^{\nu_n}) \in \prod_{i=1}^{n} \mathrm{MV}^{(r_i)}_{\lambda_i, \nu_i}$$

where

$$y_i = \mathrm{ad}(\varpi^{\nu_i}) x_i \in LN \quad i = 1, \ldots, n,$$

thus maps to

$$\mathrm{ad}(\varpi^{\sigma_1}) x_1 \cdots \mathrm{ad}(\varpi^{\sigma_n}) x_n \varpi^{\sigma_n} \in S_\nu$$

under the composition. Thus, the right hand side computes as

$$h^\nu(\mathrm{ad}(\varpi^{\sigma_1}) x_1 \cdots \mathrm{ad}(\varpi^{\sigma_n}) x_n \varpi^\nu) = \sum_{i=1}^{m} h_{\sigma_i}(x_i) = \sum_{i=1}^{m} h^{\nu_i}_{\sigma_{i-1}}(y_i \varpi^{\nu_i} L^+ G) = \sum_{i=1}^{m} (h^{\lambda_i, \nu_i}_{\sigma_{i-1}} \circ p_i)(y_i \varpi^{\nu_i}).$$

□

8.6.1. *Proof of Proposition 8.1.*

Proof. In contrast to the proof of [NP01, p31], which just passes to the convolution Grassmannian, we need to further resolve by using the $\prod L^{r_i} N$-torsors constructed in Lemma 8.5. This yields a diagram

$$\begin{tikzcd}
\bigcup_{|\nu_\bullet| = \nu} \prod_{i=1}^{m} \mathrm{MV}^{(r_i)}_{\lambda_i, \nu_i} \arrow[d, "q_\bullet"] & \\
m^{-1}(\mathrm{MV}_{|\lambda_\bullet|, \nu}) = \bigcup_{|\nu_\bullet| = \nu} \widetilde{\mathrm{MV}}_{\lambda_\bullet, \nu_\bullet} \arrow[r, hook] \arrow[d] & \mathrm{Gr}_{\leq \lambda_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{Gr}_{\leq \lambda_n} \arrow[d, "m"] \\
\mathrm{MV}_{|\lambda_\bullet|, \nu} \arrow[r, hook] & \mathrm{Gr}_{\leq |\lambda_\bullet|}
\end{tikzcd}$$

where the first map m, as Definition 3.15,

$$\widetilde{\mathrm{MV}}_{\lambda_\bullet, \nu_\bullet} := \mathrm{MV}_{\lambda_1, \nu_1} \widetilde{\times} \cdots \widetilde{\times} \mathrm{MV}_{\lambda_n, \nu_n}$$

Recall from Lemma 8.5

$$
\begin{tikzcd}
& \prod_{i=1}^{m} (\mathrm{MV}_{\lambda_i,\nu_i})^{(r_i)} \arrow[dl, "p_\bullet"'] \arrow[dr, "q_\bullet"] & \\
\prod_{i=1}^{m} \mathrm{MV}_{\lambda_i,\nu_i} & & \widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}
\end{tikzcd}
$$

that we have a unique sheaf $\mathcal{A}_{\lambda_\bullet} \otimes \mathcal{L}_\psi$ on $\widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}$ such that

$$q_\bullet^* \left(\mathcal{A}_{\lambda_\bullet} \otimes \mathcal{L}_\psi\right) \cong \left(p_1^* \mathcal{A}_{\lambda_1} \otimes h_0^* \mathcal{L}_\psi\right) \boxtimes \cdots \boxtimes \left(p_m^* \mathcal{A}_{\lambda_n} \otimes h_{\sigma_{n-1}}^* \mathcal{L}_\psi\right)$$

and that [Milton: I'm not totally sure why this is true]

$$m_!(\mathcal{A}_{\lambda_\bullet} \otimes \mathcal{L}_\psi) = (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)$$

where by Lemma 8.5, each component splits as a *direct product* in the second resolution.

$$
\begin{aligned}
R\Gamma_c(\widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}, \mathcal{A}_{\lambda_\bullet} \otimes h_\bullet^* \mathcal{L}_\psi) &\simeq R\Gamma_c\left(\prod_{i=1}^{m} \mathrm{MV}_{\lambda_i,\nu_i}^{(r_i)}, q_\bullet^* \mathcal{A}_{\lambda_\bullet}\right)\left[2 \dim N \cdot \sum_{i=1}^{n} r_i\right] \\
&\simeq \bigotimes_{i=1}^{n} \left(R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}^{(r_i)}, p_i^* \mathcal{A}_{\lambda_i} \otimes h_{\sigma_i}^* \mathcal{L}_\psi)[2 \dim N \cdot r_i]\right) \\
&\simeq \bigotimes_{i=1}^{n} R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes (h_{\sigma_i}^{\lambda_i,\nu_i})^* \mathcal{L}_\psi)
\end{aligned}
$$

Now we have

$$
\begin{aligned}
R\Gamma_c(\mathrm{MV}_{|\lambda_\bullet|,\nu}, (\mathcal{A}_{\lambda_1} \star \cdots \star \mathcal{A}_{\lambda_n}) \otimes (h_0^{|\lambda_\bullet|,\nu})^*(\mathcal{L}_\psi)) &= \bigoplus_{|\nu_\bullet|=\nu} R\Gamma_c(\widetilde{\mathrm{MV}}_{\lambda_\bullet,\nu_\bullet}, \mathcal{A}_{\lambda_\bullet} \otimes h_\bullet^* \mathcal{L}_\psi) \\
&\simeq \bigoplus_{|\nu_\bullet|=\nu} \bigotimes_{i=1}^{n} R\Gamma_c(\mathrm{MV}_{\lambda_i,\nu_i}, \mathcal{A}_{\lambda_i} \otimes (h_{\sigma_i}^{\lambda_i,\nu_i})^* \mathcal{L}_\psi)
\end{aligned}
$$

□

9. Zero orbit

Let us begin with some notations

Definition 9.1.

$$\Delta_{\lambda^\vee} := \{\alpha \in \Phi : \alpha = w\lambda^\vee\}$$

denote the set of simple roots Weyl-conjugate to $\lambda^\vee$. If $\sigma \in X_*(T)$ we let

$$\Delta^\sigma_{\lambda^\vee} := \{\alpha \in \Delta_{\lambda^\vee} : \langle \alpha, \sigma \rangle < 0\}.$$

Definition 9.2. Let $P_\lambda := \langle T, U_\alpha : \langle \alpha, \lambda \rangle \leq 0 \rangle \in \mathrm{Grp}\,\mathrm{Sch}_k$ denote the parabolic subgroup of G generated by T and the root subgroups.

Example 9.3. Note P_λ always contains the opposite of the standard Borel. For GL_2 or GL_3 this containment is an equality. For GL_4, you also have to throw in the root subgroup for the positive simple root $e_2 - e_3$.

This section is devoted to proving the following result.

Theorem 9.4.

$$R\Gamma_c(\mathrm{MV}_{\lambda,0}, \mathcal{A}_\lambda \otimes h^*_\sigma \mathcal{L}_\psi) = \bar{\mathbb{Q}}_\ell^{|\Delta^\sigma_{\lambda^\vee}|}$$

Proof. By Equation 5, □

9.5. **Resolution of singularity.** To prove this we will use a resolution of $\mathrm{MV}_{\lambda,0}$ pulled back from Zhu's resolution

$$\pi : \widetilde{\mathrm{Gr}}_{\leq\lambda} \to \mathrm{Gr}_{\leq\lambda}$$

defined in [Zhu17, Lemma 2.12]. We briefly recall the construction, that closely imitates the Moy-Prasad filtration, [CI21, p. 2.4].

Definition 9.6. Given $r \in [0,1]$, consider the parahoric groups scheme scheme $\mathcal{G}_r \in \mathrm{Grp}\,\mathrm{Sch}_{\mathcal{O}}$ such that

$$\mathcal{G}_r(\mathcal{O}) = \langle T(\mathcal{O}), \varpi^{\lceil \langle r\lambda, \alpha \rangle \rceil} U_\alpha(\mathcal{O}) : \alpha \in \Phi \rangle.$$

Define $Q_r := L^+\mathcal{G}_r \in \mathrm{Ind}\,\mathrm{Sch}_k$, as described in Definition 3.1: k-algebra points R are given by

$$Q_r(R) := \mathcal{G}_r(W(R))$$

This is representable by an affine group scheme.

Example 9.7. Let $\lambda = (1,-1)$. Observe that $\langle \lambda, \lambda^\vee \rangle = 2$.

$$Q_0 = \begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}, \quad Q_{1/4} = \begin{pmatrix} \mathcal{O} & \varpi\mathcal{O} \\ \mathcal{O} & \mathcal{O} \end{pmatrix}$$

$$Q_{1/2} = \begin{pmatrix} \mathcal{O} & \varpi\mathcal{O} \\ \varpi^{-1}\mathcal{O} & \mathcal{O} \end{pmatrix} \quad Q_{\frac{3}{4}} = \begin{pmatrix} \mathcal{O} & \varpi^2\mathcal{O} \\ \varpi^{-1}\mathcal{O} & \mathcal{O} \end{pmatrix} Q_1 = \begin{pmatrix} \mathcal{O} & \varpi^2\mathcal{O} \\ \varpi^{-2}\mathcal{O} & \mathcal{O} \end{pmatrix}$$

This is pictured via the following inclusion relation

$$\begin{tikzcd}
G/P_\lambda & & & \left(\mathbb{P}^1\right)^{\mathrm{pf}} \simeq Q_{\frac{1}{2}}/Q_{\frac{3}{4}} & & & \\
& Q_0 \arrow[ul] & & & Q_{\frac{1}{2}} \arrow[ul] & & Q_1 \\
& & Q_{\frac{1}{4}} \arrow[ul] \arrow[urr] & & & Q_{\frac{3}{4}} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

where the left diagonal sequences are quotients.

The resolution of singularities is then given by taking

$$\begin{aligned}
\pi : \widetilde{\mathrm{Gr}}_{\leq\lambda} := Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{2}}/Q_{\frac{3}{4}} &\to \mathrm{Gr}_{\leq\lambda} \\
(g, g') &\mapsto gg'\varpi^\lambda.
\end{aligned}$$

Proposition 9.8.

$$\begin{aligned}
Q_{\frac{1}{2}}/Q_{\frac{3}{4}} &\simeq (\mathbb{P}^1)^{\mathrm{pf}} \\
Q_0/Q_{\frac{1}{4}} &\simeq (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}},
\end{aligned}$$

so the map $\pi : \widetilde{\mathrm{Gr}}_{\leq\lambda} \to \mathrm{Gr}_{\leq\lambda}$ *is a* $\mathbb{P}^{1,\mathrm{pf}}$*-fibration over* $(\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}$*.*

Proof. For $Q_0/Q_{\frac{1}{4}}$: we use the following exact sequence

$$Q_0 \cap Q_{\frac{1}{2}}/Q_0 \cap Q_1 \longrightarrow Q_0/Q_0 \cap Q_1 \longrightarrow G/P_\lambda$$

and that

$$Q_0 \cap Q_{\frac{1}{2}} = Q_{\frac{1}{4}}$$

For $Q_{\frac{1}{2}}/Q_{\frac{3}{4}}$ a sketch is suggested in Example 10.14. ☐

There is a relative Bruhat decomposition [Ashwin: justify this]

$$Q_{\frac{1}{2}}/Q_{\frac{3}{4}} = Q_{\frac{1}{4}}Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \sqcup Q_{\frac{1}{4}}s_{1,\theta}Q_{\frac{3}{4}}/Q_{\frac{3}{4}}$$

which corresponds, under the isomorphism in Proposition 9.8, to

$$\mathbb{P}^{1,\mathrm{pf}} = \mathbb{A}^{1,\mathrm{pf}} \sqcup *.$$

As stated in Zhu, this gives rise to a decomposition

$$\begin{tikzcd}
Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{4}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \arrow[r, equal] \arrow[d, hook] & \pi^{-1}(\mathrm{Gr}_\lambda) \arrow[r, "\simeq"] \arrow[d, hook] & \mathrm{Gr}_\lambda \arrow[d, hook] \\
Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{2}}/Q_{\frac{3}{4}} \arrow[r, equal] & \widetilde{\mathrm{Gr}}_{\leq\lambda} \arrow[r, "\pi"'] & \mathrm{Gr}_{\leq\lambda} \\
Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{4}} s_{1,\mu^\vee} Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \arrow[r, equal] \arrow[u, hook] & (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}} \arrow[r] \arrow[u, hook] & \mathrm{Gr}_0 \arrow[u, hook]
\end{tikzcd} \tag{3}$$

We have the following identification

Proposition 9.9. *For each $w \in W$, define Q_w as the pullback*

$$\begin{tikzcd}
Q_w \times^{Q^{\frac{1}{4}}} Q_{\frac{1}{4}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Q_0 \times^{Q_{\frac{1}{4}}} Q_{\frac{1}{4}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \arrow[d] \\
NwP_\lambda/P_\lambda \arrow[r] & Q_0 \times^{Q_{\frac{1}{4}}} pt \simeq G/P_\lambda
\end{tikzcd}$$

Then

(1) if $w\lambda \in \check{\Phi}_-$:the zero section is a cycle,

$$Q_w \times^{Q^{\frac{1}{4}}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \xrightarrow{\simeq} S_{w\lambda} \cap \mathrm{Gr}_\lambda$$
$$(g, g') \mapsto gg'\varpi^\lambda$$

(2) if $w\lambda \in \check{\Phi}_+$: the total space of the restricted bundle is cycle,

$$Q_w \times^{Q^{\frac{1}{4}}} Q_{\frac{1}{4}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}} \xrightarrow{\simeq} S_{w\lambda} \cap \mathrm{Gr}_\lambda$$

Proof. (1). Note that $S_{w\lambda} \cap \mathrm{Gr}_\lambda = L^+N\varpi^{w\lambda}L^+G/L^+G$, which was explained in the end of Lemma 6.1. [Milton: This argument seems fishy] We can show the map is surjective on k-points. Pick any lift

$$(nw, 1) \quad n \in L^+N$$

Then this is sent to

$$nw\varpi^\lambda L^+G = n\varpi^{w\lambda}wL^+G = n\varpi^{w\lambda}L^+G$$

using that $\mathrm{ad}(w)\varpi^\lambda = \varpi^{w\lambda}$. We also know that it is injective since π restricts to an to an isomorphism on Gr_λ. □

Corollary 9.10. *The complement of $\mathcal{L}_w^\times \hookrightarrow \mathcal{L}_w$ is given by the zero section: $NwP/P \to Q_w \times^{Q^{\frac{1}{4}}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}}$*

Proof. We use the equality $S_0 \cap \mathrm{Gr}_\lambda = \mathrm{Gr}_\lambda \setminus \bigcup_{w\lambda} S_{\lambda w} \cap \mathrm{Gr}_\lambda$, i.e. given by taking away the pieces which contains $Q_0 \times^{Q^{\frac{1}{4}}} Q_{\frac{3}{4}}/Q_{\frac{3}{4}}$. □

The restriction

$$\mathring{\phi} : \mathrm{Gr}_\lambda \xrightarrow{\sim} \pi^{-1}(\mathrm{Gr}_\lambda) \to (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}$$

is the natural $(\mathbb{A}^1)^{\mathrm{pf}}$-bundle described in the discussion following [Zhu17, Proposition 1.23]. After restriction to S_0 (which contains Gr_0) we obtain

$$\begin{tikzcd}
\pi^{-1}(S_0 \cap \mathrm{Gr}_\lambda) \arrow[r, "\simeq"] \arrow[d] & S_0 \cap \mathrm{Gr}_\lambda \arrow[d] \\
\pi^{-1}(S_0 \cap \mathrm{Gr}_{\leq\lambda}) \arrow[r, "\pi"'] & S_0 \cap \mathrm{Gr}_{\leq\lambda} \\
(\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}} \arrow[r] \arrow[u, hook] & \mathrm{Gr}_0 \arrow[u, hook]
\end{tikzcd} \tag{4}$$

Let

$$(\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_{-} := (\bigcup_{w : w\lambda^\vee < 0} \bar{N} w \bar{P}_\lambda / \bar{P}_\lambda)^{\mathrm{pf}}.$$

Then

$$\mathring{\phi}(\pi^{-1}(S_0 \cap \mathrm{Gr}_\lambda)) \subset (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_{-}$$

and the map $\pi^{-1}(S_0 \cap \mathrm{Gr}_\lambda) \to (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_{-}$ is exactly the $(\mathbb{G}_m)^{\mathrm{pf}}$-bundle obtained by taking the complement of the zero section from the $(\mathbb{A}^1)^{\mathrm{pf}}$-bundle $\mathring{\phi}^{-1}((\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_{-}) \to (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_{-}$. To simplify the notation a bit, we make the following definition.

Definition 9.11. Let

$$\begin{aligned} \mathcal{L}^\times &:= S_0 \cap \mathrm{Gr}_\lambda \\ \mathcal{L} &:= \mathring{\phi}^{-1}((\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}_{-}) \end{aligned}$$

For $w \in W$ such that $w\lambda^\vee \leq 0$ we also write $\mathcal{L}_w = \mathcal{L}|_{(\bar{N}w\bar{P}_\lambda/\bar{P}_\lambda)^{\mathrm{pf}}}$ and $\mathcal{L}_w^\times = \mathcal{L}|^\times_{(\bar{N}w\bar{P}_\lambda/\bar{P}_\lambda)^{\mathrm{pf}}}$

By abuse of notation, we will identify $S_0 \cap \mathrm{Gr}_\lambda$ with its isomorphic preimage under π.

We attach a useful picture to have in mind:

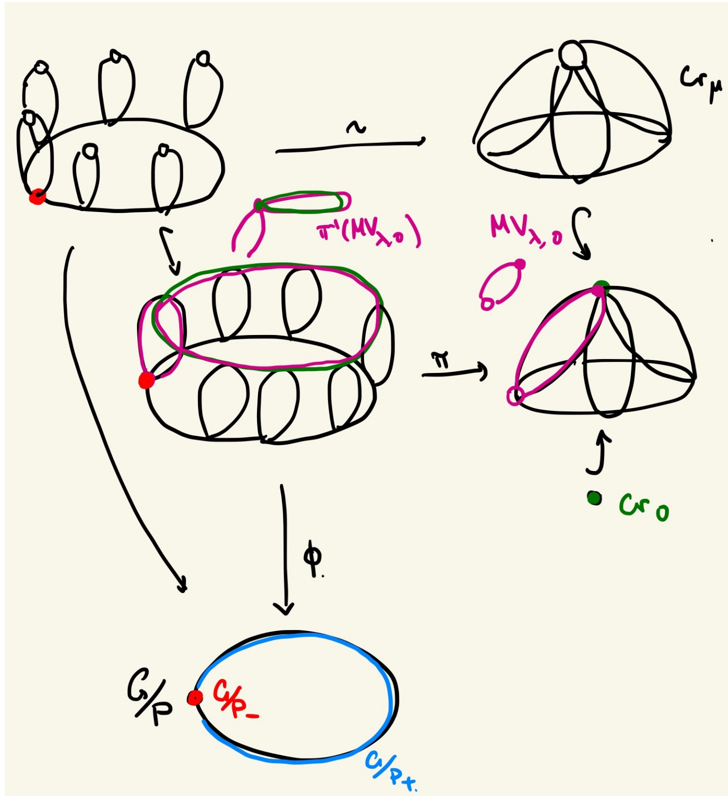

FIGURE 1. Resolution of $\pi : \widetilde{\mathrm{Gr}}_{\leq\lambda} \to \mathrm{Gr}_{\leq\lambda}$, which is also regarded as a $(\mathbb{P}^1)^{\mathrm{pf}}$-bundle over $\bar{G}/\bar{P}_\lambda$.

Lemma 9.12. *Let $d = \langle 2\rho, \lambda\rangle$. With the notation as above,*

$$\pi_*\pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi[d] \simeq (\mathcal{A}_\mu \otimes (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \oplus \mathcal{C}$$

where $\mathcal{C}$ is a complex of $\overline{\mathbb{Q}}_\ell$-vector spaces supported on Gr_0 *satisfying*

$$H^i(\mathcal{C}) = \begin{cases} H^{i+d}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) & i \geq 0 \\ H^{i+d-2}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) & i < 0 \end{cases}$$

Proof. As in [Zhu17, Section 2.2.2] we use the decomposition theorem to obtain

$$\pi_*\overline{\mathbb{Q}}_\ell[d] = \mathcal{A}_\lambda \oplus \mathcal{C}$$

with $\mathcal{C}$ having the desired cohomology. Then the projection formula gives

$$\begin{aligned} \pi_*\pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi[d] &\simeq \pi_*(\overline{\mathbb{Q}}_\ell[d] \otimes \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \\ &\simeq \pi_*\overline{\mathbb{Q}}_\ell[d] \otimes (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi \\ &\simeq \big(\mathcal{A}_\lambda \otimes (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi\big) \oplus \big(\mathcal{C} \otimes i^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi\big) \\ &\simeq \big(\mathcal{A}_\lambda \otimes (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi\big) \oplus \mathcal{C}. \end{aligned}$$

□

Since π is proper, we obtain

$$R\Gamma_c(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi)[d] = R\Gamma_c(\mathrm{MV}_{\lambda,0}, \mathcal{A}_\lambda \otimes (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \oplus \mathcal{C} \tag{5}$$

10. Proof 1

To compute the left-hand side we consider the open-closed decomposition

$$\mathcal{L}^\times \xhookrightarrow{j} \pi^{-1}(\mathrm{MV}_{\lambda,0}) \xhookleftarrow{i} \pi^{-1}(\mathrm{Gr}_0) = (\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}}$$

This induces a long exact sequence

$$\cdots \to H^i_c(\mathcal{L}^\times, (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) \to H^i_c(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) \to H^i(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) \to \tag{6}$$

Note $\pi^*(h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi$ restricts to the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $\pi^{-1}(\mathrm{Gr}_0)$ since the map factors as

$$\begin{tikzcd}
\pi^{-1}(\mathrm{MV}_{\lambda,0}) \arrow[r] & \mathrm{MV}_{\lambda,0} \\
(\bar{G}/\bar{P}_\lambda)^{\mathrm{pf}} \simeq \pi^{-1}(\mathrm{Gr}_0) \arrow[u, hook] \arrow[r] & \mathrm{Gr}_0 = \mathrm{pt} \arrow[u, hook]
\end{tikzcd}$$

Lemma 10.1. *Let $w \in W$, then*

$$\langle \rho, w\lambda^\vee \rangle = -1$$

if and only if $w\lambda^\vee$ is a simple root.

Proposition 10.2. *We have*

$$\dim H^{i+d}_c(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) = \begin{cases} \dim H^{i+d}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) & \text{if } i > 0 \\ \dim H^{i+d-2}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) & \text{if } i < 0 \\ |\Delta^\sigma_{\lambda^\vee}| + |\Delta_{\lambda^\vee}| & \text{if } i = 0 \end{cases}$$

Proof. Let us first recall the dimension of all objects of interest, [Zhu17, Corollary 2.8],

Total space	dimension
$\mathcal{L}^\times$	$d/2$
$\mathcal{L}^\times_w$	$\langle \rho, w\lambda \rangle + \frac{d}{2} + 1$

and note that the corresponding base

Base space	dimension
G/P_λ	$d/2 - 1$
NwP_λ/P_λ	$\langle \rho, w\lambda \rangle + \frac{d}{2}$

and hat $\langle \rho, w\lambda \rangle \le -1$ with equality if and only if $-w\lambda^\vee$ is a simple root.

First suppose $i > 0$. As, $\dim \mathcal{L}^\times = d/2$,

$$H^{i+d}_c(\mathcal{L}^\times, (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) = H^{i+d+1}_c(\mathcal{L}^\times, (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) = 0$$

and Equation 6 yields the desired equality.

Next suppose $i = 0$. Using the fact that $\dim \mathcal{L}^\times \le d/2$ again, we see that [Mil80, p220]

$$H^{d+1}_c(\mathcal{L}^\times, (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) = 0.$$

The cohomology of $\bar{G}/\bar{P}_\lambda$ is concentrated in even degrees, so since $d = 2\langle \rho, w\lambda^\vee\rangle \in 2\mathbb{Z}$,

$$H^{d-1}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) = 0.$$

Thus Equation 6 reduces to

$$0 \to H_c^d(\mathcal{L}^\times, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^d(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H^d(\bar{G}/\bar{P}_\lambda, \bar{\mathbb{Q}}_l) \to 0$$

We know $\dim H_c^d(\bar{G}/\bar{P}_\lambda, \bar{\mathbb{Q}}_l) = |\Delta_{\lambda^\vee}|$, so we need to understand the first term in the sequence. Choose an order-preserving injection $\alpha : W \to \mathbb{N}$. This induces a filtration on $\mathcal{L}^\times$ by closed subspaces such that the successive complements are exactly the $\mathcal{L}_w^\times$. This gives rise to a spectral sequence (see e.g. [Mil80, Remark III.1.30])

$$E_1^{p,q} = \bigoplus_{\alpha(w)=p} H_c^{p+q}(\mathcal{L}_w^\times, j_p^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \Rightarrow H_c^d(\mathcal{L}^\times, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi), \quad j_p : \mathcal{L}_{w_p}^\times \hookrightarrow \mathcal{L}^\times$$

So it remains to show that

$$\dim H_c^d\left(\mathcal{L}_w^\times, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi\right) = \begin{cases} 1 & \langle -w\lambda^\vee, \sigma\rangle > 0 \text{ and } w\lambda^\vee \text{ is a simple root} \\ 0 & \text{otherwise} \end{cases}$$

We know that

$$\dim \mathcal{L}_w^\times = \langle \rho, w\lambda + \lambda\rangle + 1 = \langle \rho, w\lambda\rangle + \frac{d}{2} + 1.$$

But $\langle \rho, w\lambda\rangle \leq -1$ with equality if and only if $-w\lambda^\vee$ is a simple root. So if $-w\lambda^\vee$ is not simple, the degree d cohomology of any ℓ-adic sheaf on $\mathcal{L}_w^\times$ vanishes. So assume $-w\lambda^\vee$ is a simple root so that $\dim \mathcal{L}_w^\times = \frac{d}{2}$.

- If $\langle -w\lambda^\vee, \sigma\rangle > 0$ then the map $h_\sigma^{\lambda,0}$ is trivial by Proposition 10.7, so $j_w^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi = \overline{\mathbb{Q}}_\ell$. By Poincaré duality (e.g. [Mil80, Thm 11.2]), and the fact that $\mathcal{L}_w^\times$ (being a $\mathbb{G}_\mathfrak{m}^{\mathrm{pf}}$-fibration over the perfection of an affine space) is connected,

$$H_c^d(\mathcal{L}_w^\times, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathcal{L}_w^\times, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell.$$

- Suppose $\langle -w\lambda^\vee, \sigma\rangle = 0$. The open-closed decomposition

$$\mathcal{L}_w^\times \hookrightarrow \mathcal{L}_w \hookleftarrow (\bar{N}w\bar{P}_\lambda/\bar{P}_\lambda)^{\mathrm{pf}}$$

gives rise to a long exact sequence

$$\cdots \to H_c^i(\mathcal{L}_w^\times, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^i(\mathcal{L}_w, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^i(\mathcal{L}_w \backslash \mathcal{L}_w^\times, ??) \simeq H^i(NwP/P, ??) \to \cdots$$

[Milton: ?? should be constant? Something similar is defined before lemma 9.5, but probably not the same.] [Ashwin: probably, but I'm a little confused as to how the map $(G/P)_w \to \mathcal{L}_w$ is actually defined in the context of Zhu; in Ngo-Polo they define it as the zero section of the line bundle, can we do the same thing? Presumably...]

Finally, suppose $i < 0$ We use the Gysin sequence to obtain that in

$$\cdots \to H^{i-2}((G/P_\lambda)_-) \to H^i((G/P_\lambda)_-) \to H^i(\mathcal{L}^\times) \to H^{i-1}((G/P_\lambda)_-) \to H^{i+1}((G/P_\lambda)_-) \to \cdots$$

where we used the identification $H^i(\mathcal{L}) \simeq H^{i-2}((G/P)_-)$. [Milton: I am not sure we can do this, is this some chern class argument: we need some chern class theory in characteristic p?] Now we split into two cases:

(1) When i is odd: we have $H^{d+i}(G/P) = 0$ at Equation 6

$$\operatorname{coker}(H^{d+i-1}(G/P) \to H^{d+i}(\mathcal{L}^\times)) \simeq H^{d+i}\left(\pi^{-1}(\mathrm{MV}_{\lambda,0})\right)$$

but then by the diagram **??** the map this factors through the restriction map

$$H^{d+i-1}(G/P) \to H^{d+i-1}((G/P)_-)$$

since these are affinely stratified space, as [Hai]. Thus,

$$H^{d+i}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) = 0 = H^{d+i-2}(G/P)$$

(2) When i is even: we have the short exact sequence

$$0 \to H^{i+d-2}(\mathcal{L}^\times) \to H^{i+d-2}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) \to H^{i+d-2}(G/P) \to H^{i+d-1}(\mathcal{L}^\times) \to H^{i+d-1}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) \to 0$$

and

$$0 \to H^{i+d-4}((G/P)_-) \to H^{i+d-2}((G/P)_-) \to H^{i+d-2}(\mathcal{L}^\times) \to 0$$

and

$$0 \to H^{i+d-1}(\mathcal{L}^\times) \to H^{i+d}((G/P)_-) \to H^{i+d+2}((G/P)_-) \to 0$$

The Euler characteristic of this sequence is 0. Now we apply the result of [NP01] as follows: we have diagram **??** and hence the same long exact sequence, as above, the result of [NP01, p. 8], thus gives us

$$\dim H^{i+d-2}(\pi^{-1}(\mathrm{MV}_{\lambda,0})) = \dim H^{i+d-2}(G/P)$$

which is what we wanted

□

Lemma 10.3. *Vanishing global sections. Let $f : X \to Y$ be a proper map. If*

$$R\Gamma_c(X_y, i_y^*\mathcal{F}) \simeq 0 \qquad X_y := f^{-1}(y) \quad y \in Y$$

Then

$$R\Gamma_c(X, \mathcal{F}) \simeq 0$$

Proof. Indeed, $R\Gamma_c(X, \mathcal{F}) \simeq \pi_! f_! \mathcal{F}$, where $\pi : X \to \mathrm{pt}$ is the canonical map to the point. Then as $(f_!\mathcal{F})_y \simeq R\Gamma_c(X_y, \mathcal{F})$ by smooth base change, $f_!\mathcal{F} \simeq 0$. Hence, $\pi_! f_! \mathcal{F} \simeq \pi_! 0 \simeq 0$. □

Lemma 10.4. *Let $\mathcal{F} \in Mod_{\mathcal{O}_E}^{rank\ 1}$ and*

$$\begin{tikzcd} F \arrow[r, "i_b"] & E \arrow[d] \\ & B \end{tikzcd}$$

an affine fibration. If

(1) $i_b^*\mathcal{F}$ is the constant sheaf for all b, Then

$$H_c^r(B, R^{2n}\pi_!\mathcal{F}) \simeq H_c^{r+2n}(E, \mathcal{F})$$

(2)

Proof. By the Leray spectral sequence, to do this, one follows the argument [Mil08, p. 12.7] we have

$$E_2^{r,s} := H_c^r(B, R^s\pi_!\mathcal{F}) \Rightarrow H_c^{r+s}(E, \mathcal{F})$$

Suppose that $i^*\mathcal{F}$ is:

(1) The constant sheaf. For all $s \neq 2n$,

$$(R^s\pi_!\mathcal{F})_b \simeq H^s(F_b, i_b^*\mathcal{F}) \simeq 0$$

where F_b is the fiber. Thus, the spectral sequence collapses and we have

$$H_c^r(B, R^{2n}\pi_!\mathcal{F}) \simeq H_c^{r+2n}(E, \mathcal{F})$$

(2) The nontrivial local system. Then

$$R^s\pi_!\mathcal{F}_b \simeq 0$$

for all s. Thus, we have that

$$H^n(E, \mathcal{F}) \simeq 0$$

for all n.

□

To extend to arbitrary sheaf we need to understand how h_σ restricts onto each pieces of the bundle $\pi : \widetilde{\mathrm{Gr}_\lambda}\Big|_{MV_{\lambda,0}} \to \mathrm{MV}_{\lambda,0}$.

Lemma 10.5. *Each $y \in S_0 \cap \mathrm{Gr}_\lambda$ can be written in the form*

$$y = nwN_{\lambda^\vee}(\varpi x)\varpi^\lambda L^+G$$

for $n \in L^+N$ and $x \in L^+\mathbb{G}_a \setminus \varpi L^+\mathbb{G}_a$ and $w \in W$ where $w\lambda^\vee < 0$. Here $N_{\lambda^\vee} : L\mathbb{G}_a \to LG$ denotes the inclusion of the root subgroup for the root $\lambda^\vee$.

Proof. An element of $S_0 \cap \mathrm{Gr}_\lambda$ can be written as $g\varpi^\lambda L^+G$ for some $g \in L^+G$. Its image under the reduction map $\mathrm{Gr}_\lambda \to \bar{G}/\bar{P}_\lambda$ lands in

$$(\bar{G}/\bar{P}_\lambda)_- = \bigsqcup_{w : w\lambda < 0} \bar{N}w\bar{P}_\lambda/\bar{P}_\lambda,$$

so we may write $\bar{g}\bar{P}_\lambda = \bar{n}w\bar{P}_\lambda$ for some arbitrary lift $n \in L^+N$, and therefore $g = nwp$ for some element $p \in L^+G$ which maps to $\bar{P}_\lambda$ modulo ϖ. We claim we can write

$$p = N_{\lambda^\vee}(\varpi x)\widetilde{p}$$

for some $\widetilde{p} \in L^+G$ and $x \in L^+\mathbb{G}_a \setminus \varpi L^+\mathbb{G}_a$ [Ashwin: hmm but I suppose x could be very divisible by ϖ? or maybe we can just choose p so that it isn't] such that $\operatorname{ad}\varpi^{-\lambda}(\widetilde{p}) \in L^+G$. [Ashwin: prove this claim for any group other than GL_n lol. wait actually maybe you can prove it for GL_n and then use an embedding to get the result?]. Therefore

$$\begin{aligned} g\varpi^\lambda &= nwN_{\lambda^\vee}(\varpi x)\widetilde{p}\varpi^\lambda L^+G \\ &= nwN_{\lambda^\vee}(\varpi x)\varpi^\lambda(\operatorname{ad}\varpi^{-\lambda}(\widetilde{p}))L^+G \\ &= nwN_{\lambda^\vee}(\varpi x)\varpi^\lambda L^+G \end{aligned}$$

as desired. □

Lemma 10.6. *The map*

$$\begin{aligned} \mathcal{L}_w^\times &\to (\bar{N}w\bar{P}_\lambda/P_\lambda)^{\mathrm{pf}} \times \mathbb{G}_m^{\mathrm{pf}} \\ nwN_{\lambda^\vee}(\varpi x)\varpi^\lambda L^+G &\mapsto (\bar{n}w\bar{P}_\lambda, \bar{x}) \end{aligned}$$

is an isomorphism.

Proof. The map is well defined. Case of GL_n. Suppose that

$$g = n_1 w N_{\lambda^\vee}(wx_1)L^+G = n_2 w N_{\lambda^\vee}(wx_2)L^+G$$

Write $n_2^{-1}n_1 = \prod_{i=1}^r N_{\alpha_i}(y_i)$ in terms of its root subgroups...

Then as

$$[N_{\lambda^\vee}, N_\alpha] = 0$$

□

Proposition 10.7. *The restriction of $h_\sigma^{\lambda,0}$ to $\mathcal{L}_w$ is*

(1) trivial when $\langle -w\lambda^\vee, \sigma\rangle \neq 0$, [7] *and*

(2) the identity map fibers when $\langle -w\lambda^\vee, \sigma\rangle = 0$. In otherwords, we have the following

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r] & \mathcal{L}_w \arrow[r] \arrow[d] & S_0 \cap \mathrm{Gr}_\lambda \arrow[r, "h_\sigma^{\lambda,0}"] & L^{\geq -1}\mathbb{G}_a/L\mathbb{G}_a \simeq \mathbb{G}_a \\ & \bar{N}w\bar{P}_\lambda/\bar{P}_\lambda & & \end{tikzcd}$$

In particular, the integer s chosen in Lemma 3.6 can be taken to be 1 for $\mathrm{MV}_{\lambda,0}$.

Proof. We follow [NP01, Lemme 8.5]. Using Lemma 10.5, we may write every element $y \in S_0 \cap \mathrm{Gr}_\lambda$ as

$$y = nwN_{\lambda^\vee}(\varpi x)\varpi^\lambda L^+G = nN_{w\lambda^\vee}(\varpi x)\varpi^{w\lambda}L^+G.$$
$$n \in L^+N, x \in L^+\mathbb{G}_a \backslash \varpi L^+\mathbb{G}_a$$

Now let $t := -\varpi x \in L^+\mathbb{G}_a$ and $\alpha = w\lambda^\vee$. As argued in [Ste16, pp. 17-20], the Steinberg relations hold for any Chevalley group base changed to any field: for any root $\beta \in \Phi$, and invertible $s \in L\mathbb{G}_a$,

$$s^{\beta^\vee} w_\beta = N_\beta(s)N_{-\beta}(-s^{-1})N_\beta(s)$$

Now as

$$N_\alpha(t)w_\alpha^{-1} \in L^+G$$

we deduce that

$$nN_\alpha(-t)t^{\alpha^\vee}L^+G = nN_{-\alpha}(\varpi^{-1}x^{-1})L^+G$$

Since σ is dominant and $n \in L^+N$ we have $h_\sigma(n) = 0$, so

$$h_\sigma^{\lambda,0}(y) = h_\sigma(N_{-\alpha}(\varpi^{-1}x^{-1})) = h(N_{-\alpha}(\varpi^{\langle -\alpha,\sigma\rangle -1}x^{-1})) = \begin{cases} 0 & \text{if } \alpha \notin \Delta \text{ or } \langle -\alpha, \sigma\rangle > 0 \\ \varpi^{-1}x^{-1} & \langle -\alpha,\sigma\rangle = 0 \end{cases}$$

[7] note that this is always > 0.

Indeed, if $-\alpha$ is not a simple root the map h kills its root subgroup. If $\langle -\alpha, \sigma\rangle > 0$, then $\varpi^{\langle -\alpha,\sigma\rangle -1}x^{-1} \in L^+\mathbb{G}_a$, and h is trivial on L^+G. □

Corollary 10.8. *For any $\sigma \in X_*(T)_+$*

$$R\Gamma_c(S_0 \cap \mathrm{Gr}_\lambda, (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) = \overline{\mathbb{Q}}_\ell^{|\Delta^\sigma_{\lambda^\vee}|}$$

where $\Delta^\sigma_{\lambda^\vee} = \{\alpha \in \Delta_{\lambda^\vee} : \langle \alpha, \sigma\rangle > 0\}$.

Proof. We have a stratification

$$S_0 \cap \mathrm{Gr}_\lambda = \bigcup_{w : w\lambda < 0} \phi^{-1}(\bar{N}w\bar{P}_\lambda/\bar{P}_\lambda).$$

The spectral sequence for compactly supported cohomology of the induced filtration by closed subspaces gives a spectral sequence

$$E_1^{p,q} = \bigoplus_w H_c^{p+q}(\phi^{-1}(\bar{N}w\bar{P}_\lambda/\bar{P}_\lambda), (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi) \Longrightarrow H_c^{p+q}(S_0 \cap \mathrm{Gr}_\lambda, (h^{\lambda,0}_\sigma)^*\mathcal{L}_\psi).$$

But since the cohomology of each stratum is concentrated in one degree, this spectral sequence degenerates at the E_1-page.

But note that $R\Gamma_c$ □

The above corollary shows: we can break up $S_0 \cap \mathrm{Gr}_\mu$ into disjoint pieces.

(1) There are $|\Delta^\sigma_{\lambda^\vee}|$ pieces for which $h^{\lambda,0}_\sigma$ is trivial.
(2) The others are nontrivial for which the monodromic argument implies we have vanishing cohomology. [Milton: Not sure what this should be, see [FGV01, p. 7.1.8] for a similar looking statement.]

Lemma 10.9. *Basis of Schubert cohomology.*

Lemma 10.10.

$$H^*(\mathcal{L}^\times) \simeq \begin{cases} \operatorname{coker}(H^{*-2}((G/P_\lambda)_-) \to H^*((G/P_\lambda)_-)) & \text{if } * \text{ is even} \\ \ker\left(H^{*-1}((G/P_\lambda)_-) \to H^{*+1}((G/P_\lambda)_-)\right) & \text{if } * \text{ is odd} \end{cases}$$

by substituting

Note that the connecting maps here are explicitly given by the Pieri or Chevellay formula.

Proposition 10.11.

$$S_\nu \cap \mathrm{Gr}_{\leq\lambda} = S_\nu \cap \mathrm{Gr}_\lambda = \begin{cases} \phi_-^{-1}\left(\bar{U}w\bar{P}_\lambda/\bar{P}_\lambda\right) & \text{if } \nu = w\lambda \in \Phi_+^\vee \\ \bar{U}w\bar{P}_\lambda/\bar{P}_\lambda & \text{if } \nu = w\lambda \in \Phi_-^\vee \\ \emptyset & \text{otherwise} \end{cases}$$

Thus we have

$$\mathrm{MV}_{\lambda,0} = \pi\left(\phi^{-1}\left(\bigcup_{w\lambda\in\Phi_-^\vee} UwP_\lambda/P_\lambda\right) \setminus \bigcup_{w\lambda\in\Phi_-^\vee}(S_{w\lambda}\cap \mathrm{Gr}_{\leq\lambda})\right)$$

Proof. The first equality comes from Proposition 4.8 [Milton: How is this true?] The other equality is [Zhu17, p26]. We explain the argument here. Consider the stratification of $\mathrm{Gr} = \bigsqcup S_\nu$, intersected with $\mathrm{Gr}_{\leq\lambda}$. □

10.12. **Notes for the resolution.**

Remark 10.13. The filtration by Zhu is *different* to that given by Moy-Prasad. We recall the latter briefly .

- Let $x_\alpha : \mathbb{G}_a(K) \to U_\alpha \hookrightarrow G(K)$ be root subgroups from the Chevalley system.
- $v : K \to \mathbb{Z}$ we can define a canonical filtration on the root subgroups
$$U_{\alpha,r} := 1 \cup \{x_\beta(f) \,:\, f \in K, v(f) \geq r\}$$

Once we fix a point $\mathbf{x} \in \mathcal{A}_{T,\breve{K}}$, we can define

$$\breve{P}_{\mathbf{x}} = P_{\mathbf{x}}(\mathcal{O})$$

with a collection of subgroups

$$\breve{P}_{\mathbf{x}}^r \hookrightarrow \breve{P}_{\mathbf{x}}$$

Example 10.14. We denote $\mathcal{G}_a$ the parahoric $\mathcal{O}$-scheme. Suppose that the following map

$$\mathcal{G}_0(R) \xrightarrow{\operatorname{ad} g} \mathcal{G}_{1/2}(R) \quad g := \begin{pmatrix} \varpi & \\ & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & \varpi b \\ \varpi^{-1}c & d \end{pmatrix}$$

is well defined for each $\mathcal{O}$-algebra R.

we deduce that

$$Q_{\frac{1}{2}}/Q_{\frac{3}{4}} \simeq Q_0/Q_{\frac{1}{4}} \simeq \mathrm{GL}_2/B_- \simeq \mathbb{P}^1$$

More generally, in the GL_n case we have

$$Q_{\frac{1}{2}}/Q_{\frac{3}{4}} \simeq \begin{pmatrix} \mathcal{O} & \cdots & \mathcal{O} \\ \mathcal{O} & \ddots & \mathcal{O} \\ \mathcal{O} & \cdots \mathcal{O} & \mathcal{O} \end{pmatrix} \Big/ \begin{pmatrix} \mathcal{O} & \cdots & \varpi\mathcal{O} \\ \mathcal{O} & \ddots & \mathcal{O} \\ \mathcal{O} & \cdots \mathcal{O} & \mathcal{O} \end{pmatrix} \simeq \mathbb{P}^1$$

[Milton: This argument is bugged: $Q_{1/2}$]

11. Alternative argument

Proposition 11.1. *We have*

$$\dim H_c^{i+d}(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) = \begin{cases} \dim H^{i+d}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) & \text{if } i > 0 \\ \dim H^{i+d-2}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) & \text{if } i < 0 \\ |\Delta^\sigma_{\lambda^\vee}| + |\Delta_{\lambda^\vee}| & \text{if } i = 0 \end{cases}$$

We have the following open-closed decomposition:

$$\mathcal{L} \longrightarrow \pi^{-1}(\mathrm{MV}_{\lambda,0}) \longleftarrow \phi^{-1}((\bar{G}/\bar{P}_\lambda)_+) \cap \pi^{-1}(\mathrm{MV}_{\lambda,0}) \simeq (\bar{G}/\bar{P}_\lambda)_+$$

inducing long exact sequence

$$\cdots \to H_c^i(\mathcal{L}, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^i(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^i((\bar{G}/\bar{P}_\lambda)_+, \overline{\mathbb{Q}}_\ell) \to \cdots \tag{7}$$

Let us first recall the dimension of all objects of interest,

Lemma 11.2. *[Zhu17, Corollary 2.8],*

Total space	*dimension*
$\mathcal{L}$	$d/2$
$\mathcal{L}_w, w\lambda^\vee \in \Phi_-$	$\langle \rho, w\lambda \rangle + \frac{d}{2} + 1 \leq \frac{d}{2}$

where $\langle \rho, w\lambda\rangle \leq -1$ with equality if and only if $-w\lambda^\vee$ is a simple root. and note that the corresponding base

Base space	*dimension*
G/P_λ	$d/2 - 1$
$NwP_\lambda/P_\lambda, w\lambda^\vee \in \Phi_-$	$\langle \rho, w\lambda \rangle + \frac{d}{2}$
$NwP_\lambda/P_\lambda, w\lambda^\vee \in \Phi_+$	$\langle \rho, w\lambda \rangle + \frac{d}{2} - 1 \geq \frac{d}{2}$

11.3. **Case of $i > 0$.** As, $\dim \mathcal{L} = d/2$, from Lemma 11.2,

$$H_c^{i+d}(\mathcal{L}, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) = H_c^{i+d+1}(\mathcal{L}, (h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) = 0$$

and Equation 7 yields the desired equality. Here we used that

$$H_c^{i+d}((\bar{G}/\bar{P}_\lambda)_+, \overline{\mathbb{Q}}_\ell) = H_c^{i+d}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell)$$

whenever $i > 0$, which follows from the fact that if $w\lambda < 0$, then

$$\dim \bar{N} w \bar{P}_\lambda/\bar{P}_\lambda = \langle \rho, w\lambda \rangle + \frac{d}{2} \leq \frac{d}{2} - 1.$$

11.4. **Case of $i = 0$.** As $\dim \mathcal{L} \leq d/2$ again, we see that [Mil80, p220]

$$H_c^{d+1}(\mathcal{L}, \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) = 0.$$

Note $\pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi$ restricts to the constant sheaf $\overline{\mathbb{Q}}_\ell$ on $(\bar{G}/\bar{P}_\lambda)_+ \subset \pi^{-1}(\mathrm{Gr}_0)$ since the map factors as

$$\begin{tikzcd}
\pi^{-1}(\mathrm{MV}_{\lambda,0}) \arrow[r, "\pi"] & \mathrm{MV}_{\lambda,0} \\
\pi^{-1}(\mathrm{Gr}_0) \arrow[r] \arrow[u, hook] & \mathrm{Gr}_0 = \mathrm{pt} \arrow[u, hook]
\end{tikzcd}$$

The cohomology of $(\bar{G}/\bar{P}_\lambda)_+$ is concentrated in even degrees, [Milton: why is this true?] so since $d = 2\langle \rho, w\lambda^\vee\rangle \in 2\mathbb{Z}$,

$$H^{d-1}(\bar{G}/\bar{P}_\lambda, \overline{\mathbb{Q}}_\ell) = 0.$$

Thus Equation 7 reduces to

$$0 \to H_c^d(\mathcal{L}, \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^d(\pi^{-1}(\mathrm{MV}_{\lambda,0}), \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \to H_c^d((\bar{G}/\bar{P}_\lambda)_+, \overline{\mathbb{Q}}_\ell) \to 0$$

We know $\dim H_c^d((\bar{G}/\bar{P}_\lambda)_+, \overline{\mathbb{Q}}_\ell)$, which is $|\Delta_{\lambda^\vee}|$.

We have a filtration on $\mathcal{L}$ by closed subspaces such that the successive complements are exactly the $\mathcal{L}_w$. This gives rise to a spectral sequence (see e.g. [Mil80, Remark III.1.30])

$$E_1^{p,q} = \bigoplus_{\alpha(w)=p} H_c^{p+q}(\mathcal{L}_{w_p}, j_{w_p}^*\pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi) \Rightarrow H_c^{p+q}(\mathcal{L}, \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi), \quad j_{w_p} : \mathcal{L}_{w_p} \hookrightarrow \mathcal{L}$$

Proposition 11.5. *Suppose $w\lambda < 0$.*

$$\dim H_c^d\left(\mathcal{L}_w, j_w^*\pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi\right) = \begin{cases} 1 & \langle -w\lambda^\vee, \sigma\rangle > 0 \textit{ and } w\lambda^\vee \textit{ is a simple root} \\ 0 & \textit{otherwise} \end{cases}$$

Proof.

- If $\langle -w\lambda^\vee, \sigma\rangle > 0$ then the map $h_\sigma^{\lambda,0}$ is trivial by Proposition 10.7, so

$$j_w^*\pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi = \overline{\mathbb{Q}}_\ell$$

  - $-w\lambda^\vee$ is not simple: then $H_c^d(\mathcal{L}_w, \bar{\mathbb{Q}}_l)$ vanishes, as $\dim \mathcal{L}_w < \frac{d}{2}$.
  - $-w\lambda^\vee$ is simple: then $\dim \mathcal{L}_w = \frac{d}{2}$. By Poincaré duality (e.g. [Mil80, Thm 11.2]), and the fact that $\mathcal{L}_w$ (being a $(\mathbb{A}^1)^{\mathrm{pf}}$-fibration over the perfection of an affine space) is connected,

$$H_c^d(\mathcal{L}_w, \overline{\mathbb{Q}}_\ell) \simeq H^0(\mathcal{L}_w, \overline{\mathbb{Q}}_\ell) = \overline{\mathbb{Q}}_\ell.$$

- Suppose $\langle -w\lambda^\vee, \sigma\rangle = 0$. By Proposition 10.7, we know the map $h_\sigma^{\lambda,0} \circ j_w$ induces the identity map.

$$\begin{tikzcd}
\mathbb{A}^1 \arrow[r, hook, "i_b"] \arrow[d] \arrow[rrrr, bend left, "\mathrm{id}"] \arrow[dr, phantom, "\lrcorner"] & \mathcal{L}_w \arrow[r] \arrow[d] & S_0 \cap \mathrm{Gr}_\lambda \arrow[r] & \mathrm{Gr}_{\mathbb{G}_a} \arrow[r] & L^{\geq -1}\mathbb{G}_a/L\mathbb{G}_a \simeq \mathbb{G}_a \\
\{b\} \arrow[r, hook] & \bar{N}w\bar{P}_\lambda/\bar{P}_\lambda
\end{tikzcd}$$

  $i_b^* j_w^* (h_\sigma^{\lambda,0})^* \mathcal{L}_\psi$, has trivial cohomology, on the fibers of the affine bundle $\mathcal{L}_w$. Hence, by Lemma 10.3, $j_w^* \pi^* (h_\sigma^{\lambda,0})^* \mathcal{L}_\psi$ vanishes.

□

11.6. **Case of $i < 0$.** Set $\mathcal{F}_\sigma := \pi^*(h_\sigma^{\lambda,0})^*\mathcal{L}_\psi$.

$H^{i+d}((G/P)_+, \overline{\mathbb{Q}}_\ell) = 0$ for $i < 0$. Thus, we are reduced to showing that

$$\dim H^{i+d}(\mathcal{L}, \mathcal{F}_\sigma) = \dim H_c^{i+d-2}((G/P)_-, \bar{\mathbb{Q}}_l)$$

As components of spectral sequence coincide, the dimension of what the spectral sequence converge to are equal.

$$\begin{tikzcd}
H_c^{i+d-2}(C_w, \mathcal{F}_{\sigma,w}) \arrow[r, Rightarrow] \arrow[d, phantom, "\simeq"] & H_c^{i+d-2}((G/P)_-, \bar{\mathbb{Q}}_l) \arrow[d, phantom, "?"] \\
H_c^{i+d}(\mathcal{L}_w, \mathcal{F}_\sigma) \arrow[r, Rightarrow] & H_c^{i+d}(\mathcal{L}, \mathcal{F}_\sigma)
\end{tikzcd}$$

where

$$\mathcal{F}_{\sigma,w} = \begin{cases} \bar{\mathbb{Q}}_l & \langle w\lambda^\vee, \sigma\rangle > 0 \text{ and is simple} \\ 0 & \text{otherwsie} \end{cases}$$

We prove the equivalences on the left vertical side:

- $\langle w\lambda^\vee, \sigma\rangle > 0$ and is simple: we know the restriction of $\mathcal{F}_\sigma$ to $\mathcal{L}_w$ is the constant sheaf, by Proposition 10.7. We do note require that $\mathcal{L}_w$ is trivial bundle - we can use the Čech to cohomology spectral sequence [Stacks, 03OU], [Mil80, III, Thm. 2.17]: for any étale covering $\{U_i \to U\}_{i\in I}$ of $U \in \mathrm{Aff}_{\bar{k}}$, there is a spectral sequence

  $$E_2^{p,q} := \check{H}^p(U, \mathcal{H}^q(\mathcal{F})) \Rightarrow H^{p+q}(U, \mathcal{F})$$

  We are thus reduce to computing cohomology of a trivial affine bundle

  $$U \times \mathbb{A}^1 \to U$$

  By Poincaré duality, we deduce that

  $$R\Gamma_c(U) \xrightarrow{\simeq} R\Gamma_c(U \times \mathbb{A}^1) \simeq R\Gamma_c(U) \otimes R\Gamma_c(\mathbb{A}^1)$$

  Thus

  $$H_c^{i+d}(\mathcal{L}_w, \mathcal{F}_\sigma) \simeq H_c^{i+d-2}(C_w, \bar{\mathbb{Q}}_l)$$

- Otherwise: By the Leray spectral sequence, [Mil08, p. 12.7],

$$E_2^{rs} := H_c^r(C_w, R^s\phi_!\mathcal{F}_\sigma) \Rightarrow H_c^{r+s}(\mathcal{L}_w, \mathcal{F}_\sigma)$$
$$(R^s\phi_!\mathcal{F})_b \simeq H_c^s(\mathbb{A}^1, i_b^*\mathcal{F}_\sigma) \simeq 0$$

  for all s, as $i_b^*\mathcal{F}_\sigma$ is a nontrivial local system from Proposition 10.7.

12. Recovering classical Casselman Shalika

The proof follows that explained [Fre+98, p. 5.4]. Let $\widehat{G}$ denote the Tannakian dual group.

Theorem 12.1. *Let $\gamma \in \widehat{G}$. There exists a unique*

$$W_\gamma \in Fct(G(K), \bar{\mathbb{Q}}_l)$$

satisfying the following property.

- $W_\gamma(gh) = W_\gamma(h)$.
- $W_\gamma(ug) = \Psi^{-1}(u) W_\gamma(g)$.

Further for $\lambda \in X_\bullet$,

$$W_\gamma(\varpi^\lambda) = q^{-(\rho,\mu)}\, Tr(\gamma, V(\lambda))$$

These are the *Whittaker functions* which induces a map

$$s_\gamma : \mathrm{Fct}(G/K)^{N,\psi} \to \bar{\mathbb{Q}}_{l,\psi}$$
$$\phi \mapsto \int_{N\backslash G} W_\gamma \cdot \phi$$

in $\mathrm{Mod}_{\mathrm{cHk}(G,K)}$.

Let us recall the basic properties of function dictionary.

Proposition 12.2. *Let $\mathcal{F} \in \mathrm{Shv}^b_{cstr}(X, \tau_{\text{ét}})$*

13. Appendix:cohomology for stratified spaces

Let X be a scheme, whose underlying space is a locally stratified by spaces,

- $X = \bigcup_{w\in\Delta} C_w$, where C_w are a locally closed.

We will given an order

$$\sigma : \Delta \to \mathbb{Z}$$
$$\alpha < \beta \Rightarrow \sigma(\alpha) < \sigma(\beta)$$

Proposition 13.1. *Suppose we have a stratification $T_0 \hookrightarrow \cdots T_n = X$, we have on a spectral sequence*

$$E_1^{p,q} = H^{p+q}(X, gr^p\mathcal{F}) \simeq H^{p+q}(T_p \backslash T_{p-1}, i_p^*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

Proof. By recollement is $\mathcal{F} \in \mathrm{Shv}(X)$, we get a filtered complex, for each p, we have an adjunction

$$T_p \xrightarrow{i_p} T_p \xleftarrow{j_p} T_p \backslash T_{p-1}$$
$$i_{p*}i_p^!\mathcal{F} \to \mathcal{F} \to j_{p*}j_p^*\mathcal{F} =: \mathrm{gr}^p\mathcal{F}$$

by spectral sequence for filtered complex, □

14. Appendix:perfect geometry

We have the following categories

$$
\begin{tikzcd}
\mathrm{Aff}_k^{\mathrm{pf}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Aff}_k \arrow[d, hook] \\
\mathrm{Sch}_k^{\mathrm{pf}} \arrow[r, hook] \arrow[d, hook] & \mathrm{Sch}_k \arrow[d, hook] \\
\mathrm{AlgSpc}_k^{\mathrm{pf}} \arrow[r, hook] \arrow[d, hook] & \mathrm{AlgSpc}_k^{\mathrm{pf}} \arrow[d, hook] \\
\mathrm{Stk}_k^{\mathrm{pf}} := \mathrm{Shv}(\mathrm{Aff}_k^{\mathrm{pf}}, \tau) \arrow[r, hook] \arrow[d] & \mathrm{Stk}_k := \mathrm{Shv}(\mathrm{Aff}_k, \tau) \arrow[d] \arrow[l, bend left] \\
\mathrm{PShv}(\mathrm{Aff}_k) \arrow[r] & \mathrm{PShv}(\mathrm{Aff}_k) \arrow[l, bend left, "\mathrm{res}"]
\end{tikzcd}
\tag{8}
$$

where the last functor corresponds to the restriction of sheaves from $i : \mathrm{Aff}_k^{\mathrm{pf}} \hookrightarrow \mathrm{Aff}_k$.

Proposition 14.1. *Let $X \in AlgSpc_k$, there is an equivalence of sites,* [8]

$$(X, \tau_{\text{ét}}) \underset{\varepsilon_*}{\overset{\varepsilon^*}{\rightleftarrows}} \left(X^{\mathrm{pf}}, \tau_{\text{ét}}\right)$$

Our main geometric object of interest is the affine Grassmanian and this an ind-scheme, [CW24]. These are of the form

$$X = \varinjlim X_i, \text{ where } X_i \in \mathrm{Stk}_k^{\mathrm{Art,lft}} \text{ with closed immersions } t_{ij} : X_i \to X_j \text{ as transitions.} \tag{9}$$

Note that we can construct the category

$$\mathrm{Shv} : \mathrm{Stk}_k^{\mathrm{pf}} \to \mathrm{DGCat}$$

Proposition 14.2. *sheaves on ind-schemes of ind-finite types satisfies*

(1) f^* is defined.

Our geometric objects

Definition 14.3. Let $X = \varinjlim X_i$ be of form described Equation 9

$$\mathrm{Shv}^!(X) := \varinjlim_{t^!} \mathrm{Shv}(X_i)$$

where the colimit takes place in DGCat.

Theorem 14.4. *[RS21, Thm. 2.6]* $\mathrm{Shv}^!$ *restricts to a six functor formalism.*

[8]The maps written in topological setting

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