THE GEOMETRIC SATAKE EQUIVALENCE FOR INTEGRAL MOTIVES

ROBERT CASS, THIBAUD VAN DEN HOVE AND JAKOB SCHOLBACH

Abstract. We prove the geometric Satake equivalence for mixed Tate motives over the integral motivic cohomology spectrum. This refines previous versions of the geometric Satake equivalence for split groups and power series affine Grassmannians. Our new geometric results include Whitney–Tate stratifications of Beilinson–Drinfeld Grassmannians and cellular decompositions of semi-infinite orbits. With future global applications in mind, we also achieve an equivalence relative to a power of the affine line. Finally, we use our equivalence to give Tannakian constructions of the C-group and a modified form of Vinberg's monoid.

Contents

1. Introduction

1.1. **Motivation and goals.** The *geometric Satake equivalence* roughly says that L^+G-equivariant perverse sheaves on the affine Grassmannian of a reductive group G are equivalent to algebraic representations of the Langlands dual group. It is of fundamental importance, having a wide range of applications such as in the geometric Langlands program and geometric representation theory. It exists in many shapes and forms, using for example complex analytic sheaves [Gin00, MV07], ℓ-adic sheaves [Ric14, Zhu15], D-modules [BD99, XZ22], or parity sheaves [MR18]. The affine Grassmannian also depends on the context: it can be an ind-scheme living over various base fields as in the previous references, it can be upgraded to an ind-scheme over a power of a global curve as in [Gai07, Zhu17b], or it can be a more complicated object living in mixed characteristic such as in [Zhu17a, FS21]. On the other hand, the category of representations of the dual group remains mostly the same, depending only on the coefficients of the chosen cohomology theory. Thus, it is natural to ask whether one can formulate a version using a universal cohomology theory, i.e., motivically.

This question was answered by T. Richarz and one of the authors (J.S.) for mixed Tate motives with rational coefficients in [RS21], and by X. Zhu for numerical motives in [Zhu18]. The goal of this paper is to generalize [RS21] by considering integral coefficients. Aside from addressing the above philosophical question, this has multiple concrete applications. One of our motivations is the ongoing project [RS20, RS21] on a motivic refinement of V. Lafforgue's global Langlands parametrization [Laf18]. Namely, the current work also refines [RS21] by not only considering mixed Tate motives on affine Grassmannians, but more generally on Beilinson–Drinfeld Grassmannians over powers of the affine line. The resulting factorization properties are a key tool in [Laf18]. For an application in modular representation theory, we address a question asked in the recent work of J. Eberhardt with J.S. [ES22, §1.6.1], which provides a step toward deducing the Finkelberg–Mirković conjecture [FM99] from its graded version [AR18]. Our work is also related to Zhu's integral Satake isomorphism [Zhu20], which is controlled by a monoid derived from the Vinberg monoid of the dual group. By restricting to anti-effective motives (Definitions 2.2 and 2.4), we obtain a Tannakian construction of the monoid in loc. cit. with integral rather than ℓ-adic coefficients.

1.2. **Main theorem.** Let S be a base scheme satisfying the conditions in Notation 2.1 and (2.4), such as the spectrum of a finite field, $\operatorname{Spec} \mathbf{Z}$, or more generally a localization of the ring of integers in a number field. Let G be a split reductive group over S and let Gr_G be the affine Grassmannian of G over S. Fixing the smooth curve $X = \mathbf{A}^1_S$, one can define the Beilinson–Drinfeld affine Grassmannian $\mathrm{Gr}_{G,I}$ over X^I, for any nonempty finite set

I, which interpolates between various self-products of $\mathrm{Gr}_G \times_S X$. This is an ind-projective scheme over X^I which admits a left action of a global positive loop group $L_I^+ G$.

We use the category DM of integral motives on S-schemes constructed by Spitzweck [Spi18]. Although the existence of a t-structure on $\mathrm{DM}(Y)$ for an S-scheme Y is still a wide-open question, it does exist on the full subcategory $\mathrm{DTM}(Y) \subset \mathrm{DM}(Y)$ of *stratified Tate motives*. These are motives generated by the Tate twists of the constant sheaf $\mathbf{Z}$ supported on the strata of an admissible stratification of Y in the sense of Definition 2.7.

Building on [RS21], which treats the case $I = \{*\}$, we show the following.

Theorem 1.1 (Theorem 4.21)**.** *The stratification of $\mathrm{Gr}_{G,I}$ in Definition 4.18, which combines the Schubert cells in Gr_G with a stratification of X^I, is admissible, and in particular, Whitney–Tate.*

This allows us to define the abelian category $\mathrm{MTM}(\mathrm{Gr}_{G,I}) \subset \mathrm{DTM}(\mathrm{Gr}_{G,I})$ of *mixed Tate motives*. These are precisely those Tate motives whose (Betti or ℓ-adic) realization is a perverse sheaf. Let $\mathrm{MTM}_{L_I^+ G}(\mathrm{Gr}_{G,I}) \subset \mathrm{MTM}(\mathrm{Gr}_{G,I})$ be the full subcategory of $L_I^+ G$-equivariant motives, cf. Proposition 4.31. In Definition 5.14 we define the *global Satake category* $\mathrm{Sat}^{G,I} \subset \mathrm{MTM}_{L_I^+ G}(\mathrm{Gr}_{G,I})$ as a certain full subcategory. This subcategory is equipped with a convolution product ${}^{\mathrm{p}}\star$ making it into a symmetric monoidal category.

To state our main theorem, let grAb be the category of graded abelian groups. Let $\hat{G}$ be the split reductive group over $\mathbf{Z}$ with root datum dual to that of G. There is a fully faithful embedding $\mathrm{grAb} \to \mathrm{MTM}(S)$, so in particular a graded Hopf algebra over $\mathbf{Z}$ can be viewed as a Hopf algebra in $\mathrm{MTM}(S)$.

Theorem 1.2 (§6)**.** *After fixing a suitable notion of pinning and a particular grading of $\hat{G}$, cf. (6.16), there is a canonical equivalence of symmetric monoidal categories*

$$(\mathrm{Sat}^{G,I}, {}^{\mathrm{p}}\star) \cong (\mathrm{Rep}_{\hat{G}^I}(\mathrm{MTM}(S)), \otimes).$$

Some comments are in order. First, we note that we do not impose any finiteness conditions on our representations. Second, if $I = \{*\}$ then $\mathrm{Sat}^{G,\{*\}} = \mathrm{MTM}_{L_{\{*\}}^+ G}(\mathrm{Gr}_{G,\{*\}})$. If $|I| > 1$, we need some type of universal local acyclicity with respect to X^I. In the absence of a nearby cycles functor for DM with integral coefficients, we take an ad-hoc approach. It turns out that $\mathrm{Sat}^{G,I} \subset \mathrm{MTM}_{L_I^+ G}(\mathrm{Gr}_{G,I})$ is the full subcategory whose compact objects have dense support relative to X^I, cf. Corollary 5.25. This is sufficient because constructibility with respect to our stratification of $\mathrm{Gr}_{G,I}$ is already a strong condition.

The proof of Theorem 1.2 consists of two steps in subsections 6.1 and 6.2. The first step is to construct a Hopf algebra object $H^{G,I} \in \mathrm{MTM}(S)$, for which the category of comodules is equivalent to $\mathrm{Sat}^{G,I}$. This is done via a Tannakian approach, with the Barr–Beck comonadicity theorem as a key ingredient. The second step is to identify the group corresponding to this Hopf algebra, for which we mostly follow [FS21, §VI.11]. The fact that $H^{G,I}$ comes from a graded abelian group, and is thus independent of S, is proved by a geometric argument in Theorem 6.24. It would be interesting to investigate the connection between the Satake category and the base S in the case of non-split groups.

The notation ${}^{\mathrm{p}}\star$ indicates that we take a perverse truncation, which is necessary because we do not impose any flatness conditions on the Satake category. Convolution can be described via the standard construction using multiplication on global loop groups, cf. (4.7). Our construction of the commutativity constraint follows the technique in [MV07] of interpreting the convolution product as a fusion product, cf. Theorem 5.32. This is also the only point where we use a Betti realization functor to deduce a property of our motives, namely, right t-exactness of convolution.

1.3. **Variants.** In Corollary 6.23 we deduce a version of Theorem 1.2 with coefficients Λ in a finite field or $\mathbf{Q}$. This is necessary for the identification of the dual group. Furthermore, if Λ and S both have characteristic p, this provides the first version of geometric Satake which produces the dual group in equal characteristic p to the best of our knowledge. By contrast, one of the authors (R.C.) applied derived categories of étale sheaves with $\mathbf{F}_p$-coefficients on schemes $X/\mathbf{F}_p$ in the context of geometric Satake in [Cas22]. The mod p étale cohomology of X is related to the part of crystalline cohomology on which Frobenius acts by a p-adic unit. There are also non-homotopy invariant phenomena present in [Cas22]. As a rough analogy we expect that for suitable X, anti-effective motives in $\mathrm{DTM}(X)$, cf. Theorem 1.3 below, give a categorical version of the generic Hecke algebra over $\mathbf{F}_p[\mathbf{q}]$, where $\mathbf{q}$ is an indeterminate. Adjoining an inverse $\mathbf{q}^{-1}$ or specializing to $\mathbf{q} = 0$ should correspond to the categories $\mathrm{DTM}(X)$ and $\mathrm{D}(X_{\text{ét}}, \mathbf{F}_p)$, respectively. However, no direct comparison between $\mathrm{DM}(X)$ and $\mathrm{D}(X_{\text{ét}}, \mathbf{F}_p)$ is available because the étale sheafification of $\mathrm{DM}(X)$ is $\mathbf{Z}[p^{-1}]$-linear. We plan to investigate this analogy in future work.

For applications in geometric representation theory, it is sometimes useful to eliminate the extensions between Tate twists. In [ES22], the authors construct a category of *reduced motives* DM_{r} with this property by killing the motivic cohomology of S. In particular, $\mathrm{MTM}_{\mathrm{r}}(S) = \mathrm{grAb}$. There is a reduction functor $\mathrm{DTM} \to \mathrm{DTM}_{\mathrm{r}}$ which commutes with the six operations. Theorem 1.2 is valid for reduced motives as well. The advantage is that representations of $\hat{G}$ in grAb are equivalent to representations of a slightly larger ungraded group in Ab. More precisely, we recover the following variants of the dual group.

Theorem 1.3 (Corollary 6.37, Theorem 6.40)**.** *For $I = \{*\}$, there are canonical equivalences*

$$\mathrm{Sat}_{\mathrm{r}}^{G,\{*\}} \cong \mathrm{Rep}_{\hat{G}}(\mathrm{grAb}) \cong \mathrm{Rep}_{{}^C G}(\mathrm{Ab}),$$

where ${}^C G$ is the C-group from [Zhu20, §1.1]*, which is constructed using the same pinning of $\hat{G}$ as in Theorem 1.2. If $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\},\mathrm{anti}} \subseteq \mathrm{Sat}_{\mathrm{r}}^{G,\{*\}}$ denotes the full subcategory of anti-effective motives (Definitions 2.2 and 2.5), this restricts to to an equivalence*

$$(\mathrm{Sat}_{\mathrm{r}}^{G,\{*\},\mathrm{anti}}, {}^{\mathrm{p}}\star) \cong (\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab}), \otimes),$$

where $V_{\hat{G},\rho_{\mathrm{adj}}}$ is the affine monoid in [Zhu20].

Reduced motives are also a significant tool in our proof of Theorem 1.2 in the non-reduced case, using, e.g., Lemma 2.15. For more discussion on anti-effective motives, see Theorem 6.40 ff. Moreover, passing to Grothendieck rings, we can decategorify the above equivalence and deduce a *generic Satake isomorphism* involving the generic Hecke algebra, cf. Definition 6.43 and Corollary 6.45.

1.4. **Constant terms.** We study the Satake category using *constant term functors*, obtained by hyperbolic localization. Being a standard technique in geometric representation theory, they have been considered in a global setup in [FS21, Corollary VI.3.5]. Namely, for a maximal split torus and Borel $T \subseteq B \subseteq G$ defined over S, there is a functor (5.4) $\mathrm{DM}_{\mathbf{G}_{\mathrm{m}}}(\mathrm{Gr}_{G,I}) \to \mathrm{DM}(\mathrm{Gr}_{T,I})$, for a certain $\mathbf{G}_{\mathrm{m}}$-action on $\mathrm{Gr}_{G,I}$, that can be defined using either left or right adjoints so that it has the favourable properties of both. Moreover, these constant term functors preserve the Satake subcategories. The hardest part of this is showing that they preserve Tate motives, for which we use the following geometric result.

Theorem 1.4 (Theorem 3.32)**.** *The intersection of any Schubert cell with any semi-infinite orbit in Gr_G admits a filtrable decomposition into cellular schemes., i.e., products of copies of $\mathbf{A}^1$ and $\mathbf{G}_{\mathrm{m}}$.*

We remind the reader that the irreducible components of these intersections are called Mirković–Vilonen cycles. The proof of this theorem, which is independent of the rest of the paper, is highly combinatorial and is based on refining previous results of Gaussent–Littelmann [GL05] and Deodhar [Deo85]. In fact, we prove a similar result with T replaced by a rank one Levi subgroup in Theorem 3.44, which is needed to identify the dual group. The restriction to rank one Levi subgroups is an artifact of the proof; we believe it could be removed by delving deeper into the combinatorics, but have not pursued this.

As in [FS21], the constant term functors are used to construct a fiber functor $F^I \colon \mathrm{Sat}^{G,I} \to \mathrm{MTM}(X^I) \cong \mathrm{MTM}(S)$, cf. Definition 5.29. The formula here is a geometric version of the classical Satake transform. The conservativity and t-exactness of this functor (Lemma 5.7, Proposition 5.9) make it an indispensable tool for studying $\mathrm{Sat}^{G,I}$. It is used, for example, in Proposition 5.28, Proposition 5.37, and Theorem 5.41.

Acknowledgements. We thank Esmail Arasteh Rad, Patrick Bieker, Somayeh Habibi, Shane Kelly, Cédric Pépin, Timo Richarz, Markus Spitzweck, Can Yaylali, and Xinwen Zhu for helpful discussions. We also thank Timo Richarz for very useful comments on a preliminary version of this paper. T.v.d.H. thanks the participants of a workshop on geometric Satake organized in Clermont-Ferrand in January 2022. R.C. and J.S. also thank the Excellence Cluster Mathematics Münster for logistical support. R.C. was supported by the National Science Foundation under Award No. 2103200. T.v.d.H. acknowledges support by the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement 101002592), and by the Deutsche Forschungsgemeinschaft (DFG), through the TRR 326 GAUS (project number 444845124).

2. Motivic sheaves

Notation 2.1. Throughout this paper, we fix a connected base scheme S that is smooth of finite type over a Dedekind ring or a field. By convention, a scheme is always supposed to be of finite type over S.

2.1. **Recollections.**

2.1.1. *Motives.* Our main sheaf formalism in this paper is the category of motivic sheaves defined using the Nisnevich topology, with integral coefficients. For a scheme X over a base scheme S as above, the category $\mathrm{DM}(X)$ of *motives over* X (or motivic sheaves on X) was constructed by Spitzweck [Spi18] (building on the works of Ayoub, Bloch, Cisinski, Déglise, Geisser, Levine and Morel–Voevodsky; we refer to op. cit. for further references). Very briefly, there is a motivic ring spectrum $\mathrm{M}\mathbf{Z} \in \mathrm{SH}(\mathrm{Spec}\,\mathbf{Z})$ representing motivic cohomology. Defining $\mathrm{M}\mathbf{Z}_X := \pi^*\mathrm{M}\mathbf{Z}$ for $\pi : X \to \mathrm{Spec}\,\mathbf{Z}$, the category of motives on X is defined as $\mathrm{DM}(X) := \mathrm{Mod}_{\mathrm{M}\mathbf{Z}_X}(\mathrm{SH}(X))$.

The functor $\mathrm{DM}^! : (\mathrm{Sch}_S^{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Pr}_{\mathbf{Z}}^{\mathrm{St}}$ is a Nisnevich (but not an étale, a fortiori not an h-) sheaf. This follows from the corresponding sheaf property of the stable homotopy category SH [Hoy17, Proposition 6.24].

Definition 2.2. For a scheme X, the subcategories

$$\mathrm{DM}(X) \supset \mathrm{DTM}(X) \supset \mathrm{DTM}(X)^{\mathrm{anti}}$$

of *Tate motives* (resp. *anti-effective Tate motives*) are defined to be the presentable subcategories generated by $\mathbf{Z}(n)$ with $n \in \mathbf{Z}$ (resp. $n \leq 0$).

The terminology *anti-effective* reflects the fact that we consider the opposite of the usual notion of effective motives in the literature, e.g., [CD19, Definition 11.1.10]. We will eventually relate representations of the Vinberg monoid with a certain category consisting of anti-effective (stratified) Tate motives, cf. Theorem 6.40.

2.1.2. *Betti realization.* In order to prove that the fusion product preserves the Satake category (Theorem 5.32), we will appeal to the Betti realization functor

$$\rho_{\mathrm{B}} : \mathrm{DM}(X) \to \mathrm{D}(X^{\mathrm{an}})$$

taking values in the derived category of sheaves on the analytic space associated to any scheme X (by convention always of finite type) over $S = \operatorname{Spec} \mathbf{C}$. This functor can be constructed by using Ayoub's Betti realization functor $\mathrm{SH}(X) \to \mathrm{D}(X^{\mathrm{an}})$ [Ayo10, Définition 2.1], and using that for $S = \operatorname{Spec} \mathbf{C}$ the spectrum $\mathrm{M}\mathbf{Z}$ constructed by Spitzweck (cf. Section 2.1.1) is isomorphic to the classical Eilenberg–MacLane spectrum, which is mapped to $\mathbf{Z}$ under the above functor [Lev14, Theorem 5.5]. The restriction of ρ_{B} to the subcategory of constructible motives is compatible with the six functors [Ayo10].

2.1.3. *Reduced motives.* Reduced motives have been introduced in [ES22]. They behave, roughly speaking, like motives, except that the higher motivic cohomology of the base scheme S has been removed. This will allow to exhibit the Tannaka dual of the Satake category as a group associated to a $\mathbf{Z}$-graded Hopf algebra, cf. Theorem 6.20.

The category of *reduced motives* and its full subcategory of *reduced Tate motives* on a scheme X/S is defined as

$$\begin{aligned} \mathrm{DM}_{\mathrm{r}}(X) &:= \mathrm{DM}(X) \otimes_{\mathrm{DTM}(S)} \mathrm{D}(\mathrm{grAb}), \\ \mathrm{DTM}_{\mathrm{r}}(X) &:= \mathrm{DTM}(X) \otimes_{\mathrm{DTM}(S)} \mathrm{D}(\mathrm{grAb}). \end{aligned} \tag{2.1}$$

We will write $\mathrm{DM}_{(\mathrm{r})}(X)$ to denote either $\mathrm{DM}(X)$ or $\mathrm{DM}_{\mathrm{r}}(X)$, and similarly with $\mathrm{DTM}_{(\mathrm{r})}(X)$.

2.1.4. *Functoriality.* The assignment $X \mapsto \mathrm{DM}_{(\mathrm{r})}(X)$ can be organized into a lax symmetric monoidal functor

$$\mathrm{DM}_{(\mathrm{r})} : \mathrm{Corr}(\mathrm{Sch}_S^{\mathrm{ft}}) \to \mathrm{Pr}^{\mathrm{St}}_{\mathbf{Z}}.$$

This functor encodes the existence of *-pullbacks, !-pushforwards (along maps of finite-type S-schemes), the existence of their left adjoints, the existence of the exterior product $\boxtimes$, as well as various projection formulas.

The functor $\mathrm{DTM}(S) \to \mathrm{D}(\mathrm{grAb})$ used in the definition of $\mathrm{DM}_{\mathrm{r}}(X)$ gives rise to a natural transformation, called the *reduction functor*

$$\rho_{\mathrm{r}} : \mathrm{DM} \to \mathrm{DM}_{\mathrm{r}}.$$

This functor is given on objects by $\mathrm{DM}(X) \to \mathrm{DM}_{\mathrm{r}}(X)$. It is compatible with the !- and *-pullback and pushforward functors, $\otimes$ and $\boxtimes$ [ES22]. At least for stratified Tate motives, it is compatible with Verdier duality (Lemma 2.6).

2.1.5. *Motives on prestacks.* The above formalism of (reduced) motives on finite-type S-schemes extends formally (by means of appropriate Kan extensions) to a functor

$$\mathrm{DM}^!_{(\mathrm{r})} : (\mathrm{PreStk}_S)^{\mathrm{op}} \to \mathrm{Pr}^{\mathrm{St}}_{\mathbf{Z}}. \tag{2.2}$$

The source category is the ∞-category of prestacks, i.e., presheaves of anima on the category of affine, but not necessarily finite type S-schemes. The target is the ∞-category of presentable stable $\mathbf{Z}$-linear categories with colimit-preserving functors. This construction is parallel to the one for rational coefficients in [RS20, §2]; Section 2.4 unwinds this definition in the case of equivariant motives. We also refer to [KR21, Cho21] for related constructions of categories of motives on (appropriate) stacks.

2.1.6. *Hyperbolic localization.* The following statement is needed below in order to decompose the fiber functor into weight functors.

Proposition 2.3. *Let X be an (ind-)scheme with an action of $\mathbf{G}_{\mathrm{m}}$ that is respecting some ind-presentation $X = \operatorname{colim} X_i$. We also assume the $\mathbf{G}_{\mathrm{m}}$-action is étale-locally linearizable. Consider the fixed points X^0, the attractors X^+ and repellers X^- of this action:*

$$\begin{tikzcd}[column sep=small] & X^{\pm} = \underline{\mathrm{Hom}}_{\mathbf{G}_{\mathrm{m}}}(\mathbf{A}^1_{\pm}, X) \arrow[dl, "q^{\pm}"'] \arrow[dr, "p^{\pm}"] & \\ X^0 = \underline{\mathrm{Hom}}_{\mathbf{G}_{\mathrm{m}}}(S, X) & & X. \end{tikzcd}$$

Here $\mathbf{A}^1_\pm$ *is* $\mathbf{A}^1$ *with the* $\mathbf{G}_\mathrm{m}$*-action given by* $\lambda \cdot t := \lambda^{\pm 1} t$*. These functors are representable by (ind-)schemes.*

There is a natural transformation

$$h : q^-_* p^{-!} \to q^+_! p^{+*}.$$

The restriction of this natural transformation to the full subcategory $\mathrm{DM}(X)^{\mathbf{G}_\mathrm{m}-\mathrm{mono}} \subset \mathrm{DM}(X)$ *of* $\mathbf{G}_\mathrm{m}$-monodromic motives*, i.e., the subcategory generated under finite colimits by the image of the forgetful functor* $\mathrm{DM}(\mathrm{coeq}(\mathbf{G}_\mathrm{m} \times X \underset{a}{\overset{p}{\rightrightarrows}} X)) \to \mathrm{DM}(X)$ *is an equivalence.*

Proof. In the context of étale torsion sheaves on algebraic spaces the statement above is Richarz' version of hyperbolic localization [Ric19]. The proof in loc. cit. only uses the *- and !-functors, localization, and homotopy invariance for étale torsion sheaves, and can be repeated verbatim for motives on schemes. (Another proof in the context of D-modules (over schemes in characteristic 0) due to Drinfeld–Gaitsgory [DG14] that again only uses these formal properties of a sheaf context can also be adapted verbatim to motives).

The statement for $\mathbf{G}_\mathrm{m}$-monodromic motives on an ind-scheme X as above follows since all four functors appearing in the natural transformation h are colimit-preserving and $\mathrm{DM}(\mathrm{coeq}(\mathbf{G}_\mathrm{m} \times X \rightrightarrows X)) = \mathrm{colim}_i\, \mathrm{DM}(\mathrm{coeq}(\mathbf{G}_\mathrm{m} \times X_i \rightrightarrows X_i))$. □

2.2. **Stratified Tate motives.** In the sequel, we will be using standard terminology about stratified (ind-)schemes, as in [RS20, §3].

Definition 2.4. ([SW18, §4], [RS20, 3.1.11]) Let $\iota : X^\dagger = \bigsqcup_{w \in W} X^w \to X$ be a stratified ind-scheme. We say M is a *stratified Tate motive* if $\iota^* M \in \mathrm{DTM}(X^\dagger)$. The stratification ι is called *Whitney–Tate* if $\iota^* \iota_* \mathbf{Z} \in \mathrm{DTM}(X^\dagger)$. (Equivalently, $\iota^{v*} \iota^w_* \mathbf{Z} \in \mathrm{DTM}(X^v)$ for all the strata $X^w \overset{\iota^w}{\to} X \overset{\iota^v}{\leftarrow} X^v$.) We similarly define an *anti-effective stratified Tate motive* and an *anti-effective Whitney–Tate stratification* by replacing $\mathrm{DTM}(X^\dagger)$ with $\mathrm{DTM}(X^\dagger)^{\mathrm{anti}}$ everywhere.

If we have an (anti-effective) Whitney–Tate stratification, we denote by $\mathrm{DTM}(X, X^\dagger)^{(\mathrm{anti})} \subset \mathrm{DM}(X)$ the full subcategory of motives such that $\iota^* M \in \mathrm{DTM}(X^\dagger)^{(\mathrm{anti})}$. This category is called the category of *(anti-effective) stratified Tate motives*, and is also denoted by $\mathrm{DTM}(X)^{(\mathrm{anti})}$ if the choice of $X^\dagger$ is clear from the context.

A Whitney–Tate stratification is called *universally Whitney–Tate* if for any scheme $Y \to S$, the natural map

$$p^* \iota_* \mathbf{Z} \to (\mathrm{id}_Y \times \iota)_* p^{\dagger *} \mathbf{Z} \tag{2.3}$$

resulting from the following cartesian diagram is an isomorphism:

$$\begin{tikzcd} Y \times_S X^\dagger \arrow[r, "\mathrm{id} \times \iota"] \arrow[d, "p^\dagger"] & Y \times_S X \arrow[d, "p"] \\ X^\dagger \arrow[r, "\iota"] & X. \end{tikzcd}$$

Remark 2.5. Having an (anti-effective) Whitney–Tate stratification ensures that ι^*, $\iota^!$, $\iota_!$ and ι_* preserve the categories of (anti-effective) stratified Tate motives, cf. [SW18, §4].

Any stratification such that the closures $\overline{X^w}$ are smooth over S is Whitney–Tate. This follows from relative purity [Wil17, Remark 4.7].

If X is universally Whitney–Tate, then the product stratification $Y \times_S X^\dagger \to Y \times_S X$ is Whitney–Tate for any scheme Y/S.

Lemma 2.6. *For a Whitney–Tate stratified ind-scheme* X*, the reduction functor* $\rho_\mathrm{r} : \mathrm{DTM}(X, X^\dagger) \to \mathrm{DTM}_\mathrm{r}(X, X^\dagger)$ *is compatible with the internal* Hom*-functor. In particular, if Verdier duality preserves Tate motives (e.g.,* $X^\dagger$ *is smooth),* ρ_r *is compatible with Verdier duality.*

Proof. The functor $\iota^!$ is conservative and satisfies $\iota^! \underline{\mathrm{Hom}}(M, N) = \underline{\mathrm{Hom}}(\iota^* M, \iota^! N)$. We can therefore assume X consists of a single stratum. In this case we conclude using $\underline{\mathrm{Hom}}(\mathbf{Z}(k), N) = N(-k)$. □

Definition 2.7. A stratification $\iota : X^\dagger \to X$ of an ind-scheme X is called *admissible* if a) it is a Whitney–Tate stratification, b) the structural map $\pi^\dagger : X^\dagger \to S$ is smooth, and c) $\pi^\dagger_* \mathbf{Z} \in \mathrm{DTM}(S)^{\geq 0}$, the smallest subcategory of $\mathrm{DTM}(S)$ containing $\mathbf{Z}(n)$, $n \in \mathbf{Z}$, stable under extensions, coproducts and shifts $[k]$ for $k < 0$.

Remark 2.8. The admissibility of the stratification enforces in particular that the motives of the individual strata X^w are Tate motives. The condition that we only allow negative shifts will be used in order to construct the motivic t-structure.

A Whitney–Tate stratification for which $X^w = \mathbf{G}_\mathrm{m}^{n_w} \times_S \mathbf{A}^{m_w}$ is admissible. Such stratifications are called *strongly cellular*.

More generally, if each X^w is smooth and admits (in its own right) a strongly cellular Whitney–Tate stratification, then the stratification of X by the X^w is admissible. We call these stratifications *cellular*, as in [RS20, Def 3.1.5].

The following lemma will be applied in the context of the Beilinson–Drinfeld Grassmannian (Theorem 4.21) to $X = \mathbf{A}^n$ and U being the open subscheme consisting of points in $\mathbf{A}^n$ whose coordinates are pairwise distinct. The scheme U is not cellular (e.g. for $n = 3$, U has no $\mathbf{F}_2$-points), but the following lemma implies it satisfies part c) of Definition 2.7.

Lemma 2.9. *Let*

$$D := \bigcup_{i \in I} D_i \xrightarrow{i} X \leftarrow U := X \setminus D$$

be the inclusion of a strict normal crossings divisor (i.e., the $D_J := \bigcap_{j \in J} D_j$ for all $J \subset I$ are smooth over S, including $D_\emptyset = X$) and its complement. Suppose for each J, the structural map $\pi_J : D_J \to S$ has the property $\pi_{J*}\mathbf{Z} \in \mathrm{DTM}(S)^{\geq 0}$. Then $f_*\mathbf{Z} \in \mathrm{DTM}(S)^{\geq 0}$, for $f : U := X \setminus D \to S$.

Proof. Let $D^n = \bigsqcup_{J \subset I, |J| = n} D_J \xrightarrow{\pi^{(n)}} S$ be the disjoint union of the n-fold intersections of the individual divisors, so that $D^0 = X$. By localization, the object $f_*f^!\mathbf{Z}$ is the homotopy colimit of a diagram of the form

$$\ldots \to \pi^{(2)}_* \pi^{(2)!}\mathbf{Z} \to \pi^{(1)}_* \pi^{(1)!}\mathbf{Z} \to \pi^{(0)}_* \pi^{(0)!}\mathbf{Z}.$$

Applying relative purity to the smooth maps $\pi^{(n)}$ and f gives the claim. ☐

The following lemma allows to zig-zag between reduced motives on the Hecke prestack over $\operatorname{Spec}\mathbf{Q}$ and over $\operatorname{Spec}\mathbf{F}_p$ (cf. Lemma 4.28).

Lemma 2.10. *Consider a cartesian diagram*

$$\begin{tikzcd} X'^{\dagger} \arrow[r, "\iota'"] \arrow[d, "f^{\dagger}"] & X' \arrow[r, "\pi'"] \arrow[d] & S' \arrow[d, "f"] \\ X^{\dagger} \arrow[r, "\iota"] & X \arrow[r, "\pi"] & S, \end{tikzcd}$$

in which ι determines a universally admissibly stratified (ind-)scheme and S' is an S-scheme such that $f^*\pi_*\iota_*\mathbf{Z}_{X^\dagger} \overset{\cong}{\to} \pi'_*\iota'_*f^{\dagger *}\mathbf{Z}_{X^\dagger}$. Then $f^* : \mathrm{DTM}_{\mathrm{r}}(X) \to \mathrm{DTM}_{\mathrm{r}}(X \times_S S')(:= \mathrm{DTM}(X \times_S S') \otimes_{\mathrm{DTM}(S')} \mathrm{D}(\mathrm{grAb}))$ is an equivalence.

Proof. This is the content of [ES22, Proposition 4.25] if the stratification is cellular (as opposed to just admissible). As in loc. cit., using the universality, one reduces to the case where X is a single stratum. We consider the monad $T = \pi_*\pi^*$ associated to the adjunction $\pi^* : \mathrm{DTM}_{\mathrm{r}}(S) \rightleftarrows \mathrm{DTM}_{\mathrm{r}}(X) : \pi_*$. By definition of DTM, π_* is conservative and colimit-preserving, so that the Barr–Beck–Lurie theorem implies that $\mathrm{DTM}_{\mathrm{r}}(X) = \mathrm{Alg}_T(\mathrm{DTM}_{\mathrm{r}}(S))$. In order to establish the equivalence, we first observe that the claim holds true for S in place of X by the definition in (2.1). By our assumption, f^* commutes with π_*, so that f^* maps the monad $\pi_*\pi^*$ to the monad $\pi'_*\pi'^*$. ☐

2.3. **t-structures.** In this subsection, we summarize some basic properties related to motivic t-structures. Throughout, let X be an ind-scheme with an admissible stratification $X^\dagger = \bigsqcup_{w \in W} X_w$. The construction works parallelly for reduced and regular motives, except that in the latter case we always (have to) assume (in addition to our running assumption in Notation 2.1) that S satisfies the *Beilinson–Soulé vanishing* condition

$$\mathrm{H}^n(S, \mathbf{Q}(q)) = 0 \quad \text{for } n < 0, q \in \mathbf{Z}. \tag{2.4}$$

This is satisfied, for example, if $S = \operatorname{Spec}\mathbf{Q}$, $\operatorname{Spec}\mathbf{Z}$ or $\operatorname{Spec}\mathbf{F}_p$. Recall that $\mathrm{DTM}_{(\mathrm{r})}$ denotes either DTM or $\mathrm{DTM}_{\mathrm{r}}$.

Lemma 2.11. *In the above situation, the category $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)^{\mathrm{c}}$ of compact objects in $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)$ carries a t-structure whose heart is the abelian category generated by direct summands and extensions by the objects*

$$\mathbf{Z}_{X_w}/m(k)[\dim_S X^w], k \in \mathbf{Z}, m \geq 0. \tag{2.5}$$

The same statement holds for $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)^{\mathrm{anti}}$ with $k \leq 0$ instead.

Proof. The existence of the t-structure is [Spi16, Theorem 9.10]. To see the description of the generators (which is certainly also well-known), we may replace $X^\dagger$ by a connected component thereof. The condition in Definition 2.7 ensures that $\mathrm{Hom}_{\mathrm{DM}_{(\mathrm{r})}(X^\dagger)}(\mathbf{Z}, \mathbf{Z}(k)[n]) = \mathrm{Hom}_{\mathrm{DTM}_{(\mathrm{r})}(S)}(\mathbf{Z}, \pi^\dagger_*\mathbf{Z}(k)[n]) = 0$ for $n < 0$. Here we use that the Beilinson–Soulé vanishing condition with $\mathbf{Q}$-coefficients in (2.4) implies a similar vanishing for $\mathbf{Z}$-coefficients [Spi16, Lemma 3.4]. (For reduced motives, this vanishing is trivial since $\mathrm{DTM}_{\mathrm{r}}(S) = \mathrm{D}(\mathrm{grAb})$.) Therefore, the category $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)^{\mathrm{c}}$ satisfies the conditions in [Lev93, Definition 1.1, Equation (1.5)] with $\mathbf{Q}$ replaced by $\mathbf{Z}$ throughout. The argument in loc. cit. goes through, with the minor amendment that the triangulated subcategory of $\mathrm{DTM}(X^\dagger)$ generated by $\mathbf{Z}(c)$, for a *fixed* c, is equivalent to the $\mathrm{Perf}_{\mathbf{Z}}$ (the subcategory of the derived category of abelian groups consisting of perfect complexes). The heart of the usual t-structure, which is simply the category of abelian groups, is generated by (split) extensions by $\mathbf{Z}/m$ ($m \in \mathbf{Z}$). The resulting filtration in Theorem 1.4.(iii) in op. cit. then gives our claim about the heart of the t-structure. ☐

The t-structure on $\mathrm{DTM}(X^\dagger)^{\mathrm{c}}$ yields formally a t-structure on its Ind-completion $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)$ such that the subcategories $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)^{\le n}$, $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)^{\ge n}$ admit and the truncation functors $\tau^{\ge n}, \tau^{\le n}$ and ${}^{\mathrm{p}}\mathrm{H}^n$ preserve filtered colimits.

Lemma 2.12. *Let X be a smooth S-scheme. Then, for unstratified Tate motives,*

$$\mathrm{DTM}_{(\mathrm{r})}(X)^{\mathrm{anti}} = \{M \in \mathrm{DTM}_{(\mathrm{r})}(X) \,:\, \mathrm{Maps}_{\mathrm{DTM}_{(\mathrm{r})}(X)}(\mathbf{Z}(p), M) = 0 \text{ for all } p \ge 1\},$$
$$\mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}} = \{M \in \mathrm{MTM}_{(\mathrm{r})}(X) \,:\, \mathrm{Hom}_{\mathrm{MTM}_{(\mathrm{r})}(X)}(\mathbf{Z}(p)[\dim X], M) = 0 \text{ for all } p \ge 1\}.$$

Moreover, $\mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}} \subset \mathrm{MTM}_{(\mathrm{r})}(X)$ is stable under subquotients.

Proof. To show "$\subset$", it suffices to see $\mathrm{Maps}_{\mathrm{DTM}_{(\mathrm{r})}(X)}(\mathbf{Z}(p), M) = 0$ or equivalently $\mathrm{Hom}_{\mathrm{DTM}_{(\mathrm{r})}(X)}(\mathbf{Z}(p), M[s]) = 0$ for all $s \in \mathbf{Z}$. This group is given by $\mathrm{H}^s(X, \mathbf{Z}(n-p))$, which vanishes if $p \ge 1$ and $n \le 0$ because it is a higher Chow group of codimension $n-p$ cycles. For reduced motives, this vanishing still holds by [ES22, (3.4)].

To show "$\supset$", let us write $\mathcal{C}$ for the right hand category. The inclusion $i\colon \mathrm{DTM}_{(\mathrm{r})}(X)^{\mathrm{anti}} \to \mathcal{C}$ admits a right adjoint R. In order to show $iR = \mathrm{id}$, it suffices to show $\mathrm{Maps}(\mathbf{Z}(e), iRM) \to \mathrm{Maps}(\mathbf{Z}(e), M)$ is an equivalence for all $M \in \mathcal{C}$ and $e \in \mathbf{Z}$. For $e \ge 1$ this is immediate because the cofiber of $iRM \to M$ is in $\mathcal{C}$. For $e \le 0$, by adjunction and $Ri = \mathrm{id}$, we have equivalences $\mathrm{Maps}(\mathbf{Z}(e), iRM) = \mathrm{Maps}(\mathbf{Z}(e), RM) = \mathrm{Maps}(\mathbf{Z}(e), M)$.

The proof for $\mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$ is analogous. This also implies that $\mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$ is stable under subquotients. Indeed, if $N \to M$ is an injection in $\mathrm{MTM}_{(\mathrm{r})}(X)$, so is $\mathrm{Hom}(\mathbf{Z}(p)[\dim X], N) \to \mathrm{Hom}(\mathbf{Z}(p)[\dim X], M)$. Stability under quotients then follows as well. $\square$

Remark 2.13. Under our assumptions on $X, X^\dagger$ above, the category $\mathrm{DTM}_{(\mathrm{r})}(X)$ and its subcategory of compact objects carries a t-structure glued from the t-structure on the strata, i.e., on $\mathrm{DTM}_{(\mathrm{r})}(X^\dagger)$. (Again, for regular motives, we need to assume that S satisfies the Beilinson–Soulé vanishing condition). The heart of this t-structure is denoted by $\mathrm{MTM}_{(\mathrm{r})}(X, X^\dagger)$ or just $\mathrm{MTM}_{(\mathrm{r})}(X)$ if $X^\dagger$ is clear from the context. Its objects are called *mixed Tate motives*. Again, there is an obvious variant for anti-effective Whitney–Tate stratifications. The heart of the t-structure on $\mathrm{DTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$ is denoted $\mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$.

2.3.1. *Conservativity results.* The t-structure can be used to prove that various realization(-like) functors are conservative. Lemma 2.15 will be used to prove that the fusion product preserves mixed Tate motives.

Lemma 2.14. *Let $(X, X^\dagger)$ be an ind-scheme over $S = \operatorname{Spec} \mathbf{Z}$ with an admissible stratification. Then the restriction to the generic fiber gives a conservative functor*

$$\mathrm{DTM}(X, X^\dagger)^{\mathrm{c}} \xrightarrow{\eta^*} \mathrm{DTM}(X_{\mathbf{Q}}, X^\dagger_{\mathbf{Q}})^{\mathrm{c}}.$$

Proof. The restriction functors $\iota^!$ and $\iota^!_{\mathbf{Q}}$ to the strata are conservative by localization, and commute with η^*. Thus, we may replace X by a single stratum, with structural map $\pi : X \to S$. By definition of Tate motives, the functor $\pi_* : \mathrm{DTM}(X) \to \mathrm{DTM}(S)$ is conservative. It also commutes with η^*, so that we may replace X by S.

Suppose $M \in \mathrm{DTM}(S)$ is such that $\eta^* M = 0$. Every compact object in $\mathrm{DTM}(S)$ is a finite iterated extension of objects in $\mathrm{MTM}(S)$ and the functor η^* is t-exact. We may therefore assume $M \in \mathrm{MTM}(S)$. By Lemma 2.11, M admits a finite filtration (inside $\mathrm{MTM}(S)$) whose associated graded pieces are, up to a Tate twist, of the form $\bigoplus_i \mathbf{Z}/m_i$ ($m_i \in \mathbf{Z}$). Again using the t-exactness of η^*, we may replace M by these graded pieces, in which case our claim is obvious. $\square$

Lemma 2.15. *Suppose S satisfies the Beilinson–Soulé vanishing and let X/S be an ind-scheme with an admissible stratification $\iota : X^\dagger \to X$. Then the reduction and Betti realization functors*

$$\rho_{\mathrm{r}}\colon \mathrm{DTM}(X)^{\mathrm{c}} \to \mathrm{DTM}_{\mathrm{r}}(X)$$
$$\rho_{\mathrm{B}}\colon \mathrm{DTM}(X)^{\mathrm{c}} \to \mathrm{D}(X^{\mathrm{an}}) \quad (\text{for } S = \operatorname{Spec} \mathbf{Z}, \mathbf{Q})$$

are t-exact and conservative. Thus some $M \in \mathrm{DTM}(X)^{\mathrm{c}}$ lies in positive (resp. negative) t-degrees if and only if $\rho_{\mathrm{r}}(M)$ or $\rho_{\mathrm{B}}(M)$ has the corresponding property. In particular, some $M \in \mathrm{DTM}(X)^{\mathrm{c}}$ lies in $\mathrm{MTM}(X)$ iff its Betti realization is a perverse sheaf on X^{an}.

Proof. Let $\rho = \rho_{\mathrm{B}}$ or ρ_{r}. The functor ρ commutes with ι^* and $\iota^!$. These two functors are conservative by localization. Since the t-structure on $\mathrm{DTM}(X)$ is obtained by glueing the ones on the strata $\mathrm{DTM}(X^w)$, we may replace X by a single stratum. In this case the t-exactness of ρ is obvious from the definitions. Using the t-exactness, the conservativity of ρ then follows as in the proof of Lemma 2.14, using that $\rho_{\mathrm{B}}(\mathbf{Z}/m(n)) = \mathbf{Z}/m \ne 0$ (for $|m| \ne 1$), respectively $\rho_{\mathrm{r}}(\mathbf{Z}/m(n)) = \mathbf{Z}/m(n) \ne 0$. $\square$

Definition 2.16. Let $w \in W$ be a stratum, and let $X \overset{\iota^w}{\leftarrow} X^w \overset{\pi_w}{\to} S$ be the obvious maps. Finally, pick $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$. (In the case of non-reduced motives, we assume S satisfies the Beilinson–Soulé vanishing so this makes sense.) We call

$$\mathrm{IC}_{w,L} := \iota^w_{!*}(\pi_w^* L[\dim X_w]) \in \mathrm{MTM}_{(\mathrm{r})}(X, X^\dagger) \tag{2.6}$$

the *(reduced) intersection motive*. Here $\iota^w_{!*}$ is the intermediate extension functor defined as in [BBD82, Définition 1.4.22], namely $\iota^w_{!*}M := \mathrm{im}({}^{\mathrm{p}}\mathrm{H}^0\iota^w_! M \to {}^{\mathrm{p}}\mathrm{H}^0\iota^w_* M)$.

Remark 2.17. The reduction functor ρ_{r} is t-exact and commutes with $\iota_!$, ι_*, and π_w^*, and therefore respects intersection motives, i.e., sends the unreduced $\mathrm{IC}_{w,L}$ to the reduced version. Similarly, $\rho_{\mathrm{B}}(\mathrm{IC}_{w,\mathbf{Z}(n)})$ is the classical intersection complex $\mathrm{IC}_{\overline{X}_w} \in \mathrm{Perv}(X^{\mathrm{an}})$.

Lemma 2.18. *The compact objects in $\mathrm{MTM}_{(\mathrm{r})}(X)$ are generated, by means of extensions and retracts, by the intersection motives $\mathrm{IC}_{w,L}$ for $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$ as in Definition 2.16.*

Proof. According to Lemma 2.11, $\mathrm{MTM}_{(\mathrm{r})}(X_w)^{\mathrm{c}}$ is generated under extensions and retracts by objects of the form $\pi_w^* L[\dim X_w]$ for $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$. The statement then reduces to generalities about hearts of glued t-structures, cf. [Ach21, Theorem 3.4.2, Lemma 3.4.3]. □

Lemma 2.19. *Let X be an ind-scheme with an admissible stratification. Let $j\colon U \to X$ be the inclusion of a union of open strata and let $i\colon Z \to X$ be the complement. Let $A, B \in \mathrm{MTM}_{(\mathrm{r})}(X)$ be such that A has no quotients supported on Z and B has no subobjects supported on Z. Then there is a natural isomorphism*

$$\mathrm{Hom}(A, j_{!*}j^*B) \cong \mathrm{Hom}(A, B).$$

Proof. By localization there is an exact sequence

$$0 \to j_{!*}j^*B \to {}^{\mathrm{p}}j_*j^*B \to i_*{}^{\mathrm{p}}\mathrm{H}^1(i^! j_{!*}j^*B) \to 0.$$

We have $\mathrm{Hom}(A, i_*{}^{\mathrm{p}}\mathrm{H}^1(i^! j_{!*}j^*B)) = 0$ by the assumption on A, so $\mathrm{Hom}(A, j_{!*}j^*B) \cong \mathrm{Hom}(A, {}^{\mathrm{p}}j_*j^*B)$. By the assumption on B, there is also an exact sequence

$$0 \to B \to {}^{\mathrm{p}}j_*j^*B \to i_*{}^{\mathrm{p}}\mathrm{H}^1(i^!B) \to 0.$$

Hence $\mathrm{Hom}(A, B) \cong \mathrm{Hom}(A, {}^{\mathrm{p}}j_*j^*B)$. □

2.4. **Equivariant motives.**

2.4.1. *Basic definitions and averaging functors.* The functor in (2.2) gives a category $\mathrm{DM}_{(\mathrm{r})}(Y)$ of (reduced) motives on any prestack Y over S, and a !-pullback functor between such categories, for *any* map of prestacks. An important example of a prestack is a quotient prestack

$$G \setminus X := \mathrm{colim}(\dots G \times_S G \times_S X \overset{\to}{\underset{\to}{\to}} G \times_S X \rightrightarrows X),$$

where X is any prestack acted upon by a group prestack G. (An example coming up below is the quotient LG/L^+G of the loop group, which is an ind-scheme, by the positive loop group, which is a group scheme, although not of finite type.) For such quotients, the definition gives

$$\mathrm{DM}_{(\mathrm{r})}(G \setminus X) = \lim \left(\mathrm{DM}_{(\mathrm{r})}(X) \rightrightarrows \mathrm{DM}_{(\mathrm{r})}(G \times_S X) \overset{\to}{\underset{\to}{\to}} \mathrm{DM}_{(\mathrm{r})}(G \times_S G \times_S X) \to \dots \right), \tag{2.7},$$

where the limit is formed using !-pullback (along the various action and projection maps).

Remark 2.20. (Functoriality for equivariant motives) Suppose $f : X \to Y$ is a G-equivariant map of prestacks, and write $\overline{f} : G \setminus X \to G \setminus Y$ for the induced map. If $f^!$ admits a left adjoint $f_!$, then the adjoint functor theorem (cf. [RS20, Lemma 2.2.9]) guarantees the existence of a left adjoint, denoted $\overline{f}_!$, of $\overline{f}^!$. If G is a pro-smooth group scheme over S, then $\overline{f}_!$ commutes (via the forgetful functors, i.e., !-pullbacks along $X \to G \setminus X$ etc.) with $f_!$.

This construction of adjoints can be iterated. For example, if f is a proper schematic map, there are adjoints $(\overline{f}^*, \overline{f}_!, \overline{f}^!)$ between the categories $\mathrm{DM}_{(\mathrm{r})}(G \setminus X)$ and $\mathrm{DM}_{(\mathrm{r})}(G \setminus Y)$. Again, if G is pro-smooth, then these functors can be computed as f^* etc. on the level of the underlying motives.

Lemma 2.21. *Let a smooth algebraic group G act on a scheme X.*

1. *The forgetful functor $u^! : \mathrm{DM}_{(\mathrm{r})}(G \setminus X) \to \mathrm{DM}_{(\mathrm{r})}(X)$ admits a left adjoint* $\mathrm{coav} := \mathrm{coav}_G$ *and a right adjoint* $\mathrm{av} := \mathrm{av}_G$*, called (co)averaging functors.*
2. *The composite $u^! \mathrm{av}_G$ can be computed as $a_* p^* = a_* p^!(-d)[-2d]$, where $d := \dim G/S$ and $G \times X \underset{p}{\overset{a}{\rightrightarrows}} X$ are the action and projection map. Likewise, $u^! \mathrm{coav}_G = a_! p^!$.*

(3) Verdier duality (denoted by D) exchanges averaging and coaveraging functors in the sense that there is a natural isomorphism of functors $\mathrm{D}_{X/G}\operatorname{coav} = \operatorname{av}\mathrm{D}_X$.
(4) The reduction functor ρ_{r} commutes with av_G and coav_G (for DM, respectively DM_{r}).
(5) If $f : Y \to X$ is a map of G-schemes, then $u^!$ and av commute with $f^!$. If f is smooth, then coav also commutes with $f^!$.
(6) Given another such pair (G', X'), there is an isomorphism

$$\operatorname{coav}_G(-) \boxtimes \operatorname{coav}_{G'}(-) \xrightarrow{\cong} \operatorname{coav}_{G\times_S G'}(- \boxtimes -).$$

The same holds for the averaging functors if $S = \operatorname{Spec} k$ is a field of characteristic zero.

Proof. (1): This follows from the adjoint functor theorem, but also from the following explicit description, which proves (2). We describe equivariant motives using the limit description as in (2.7), using the isomorphism $X \cong G \setminus (G \times X)$. The following diagram displays the low degrees of the simplicial diagrams whose colimits are shown in the bottom line:

$$\begin{tikzcd}
X \arrow[r, shift left, "e"] \arrow[d, equal] & G \times (G \times X) \arrow[l, shift left, "p_X"] \arrow[r, "\mathrm{id}_G \times a"] \arrow[d, shift right, "a_G \times \mathrm{id}_X"'] \arrow[d, shift left, "p_{G\times X}"] & G \times X \arrow[d, shift right, "a"'] \arrow[d, shift left, "p_X"] \\
X \arrow[r, shift left, "e"] \arrow[d, equal] & G \times X \arrow[l, shift left, "p_X"] \arrow[r, "a"] \arrow[d] & X \arrow[d] \\
X \arrow[r, shift left] & G \setminus (G \times X) \arrow[l, shift left, "\cong"] \arrow[r, "G\setminus a"] & G \setminus X.
\end{tikzcd}$$

The left horizontal bottom maps are isomorphisms of prestacks. Under this equivalence, the functor $u^!$ is induced levelwise by !-pullback along the right horizontal maps, i.e., $(\mathrm{id}_{G^n} \times a)^!$ in degree $n \geq 0$. These functors admit a left (resp. a right) adjoint, namely $(\mathrm{id}_{G^n} \times a)_!$ (resp. $(\mathrm{id}_{G^n} \times a)_*(-d)[-2d]$), which both commute with !-pullback along the vertical maps in the right and middle diagram since the squares are cartesian and G is smooth. Thus, they assemble to the asserted adjoints of $u^!$.

Under the isomorphism $X \cong G \setminus (G \times X)$, the composite $u^! \operatorname{av}_G$ is given by $a^! a_*(-d)[-2d]$. In terms of motives on X, this means we have to evaluate

$$e^! a^! a_* p^! (-d)[-2d],$$

which is isomorphic to $a_* p^*$. Likewise, $u^! \operatorname{coav}_G$ corresponds to the endofunctor $a^! a_!$ on $\mathrm{DM}(G \times X)$, and $e^!(a^! a_!)p^! = a_! p^!$.

(3): This follows from (2) since for any map $f : Y \to Z$ of finite-type S schemes, we have $\mathrm{D} f_! = f_* \mathrm{D}$, and for any smooth map (such as $f = a$ or p) we have $f^* \mathrm{D}_Z = \underline{\mathrm{Hom}}(f^* -, f^* \omega_Z) = \underline{\mathrm{Hom}}(f^! -, \omega_Y) = \mathrm{D}_Y f^!$, as consequences of the projection formula, resp. the projection formula for $f_\sharp$ vs. $\otimes$ and relative purity [CD19, Theorem 2.4.50, §1.1.33].

(4): By definition, $u^!$ commutes with ρ_{r}, so there is a natural map $\rho_{\mathrm{r}} \operatorname{av}_{\mathrm{DM}} \to \operatorname{av}_{\mathrm{DM}_{\mathrm{r}}} \rho_{\mathrm{r}}$ (and analogously for coav). To check it is an isomorphism it suffices to append the conservative functor $u^!$, so the claim follows from (2) since ρ_{r} commutes with *-functors.

(5): This is similarly reduced to the observation that $a_! p^!$ (resp. $a_* p^*$) commutes with $f^!$ as asserted, by base-change and relative purity.

(6): By the definition of $\boxtimes$ on prestacks of the form X/G [RS21, Appendix A], the forgetful functor is compatible with exterior products. This gives the map as displayed. In order to check it is an isomorphism, we apply $(u \times u')^! = u^! \boxtimes u'^!$, so we need to prove

$$a_* p^* - \boxtimes a'_* p'^* - = (a \times a')_* (p \times p')^* (- \boxtimes -).$$

This holds by [JY21, Theorem 2.4.6] (for *-pushforwards, this needs the assumption on S, which is used to apply resolution of singularities). The argument for coav is similar but only uses the compatibility of $\boxtimes$ with !-pullbacks along smooth maps and !-pushforwards (i.e., the projection formula). $\square$

2.4.2. *Equivariant Tate motives.*

Definition 2.22. For a stratified (ind-)scheme X with an action of a group scheme G (not necessarily of finite type over S), the category $\mathrm{DTM}_{(\mathrm{r})}(G \setminus X)$ of *equivariant (reduced) Tate motives* on the prestack quotient $G \setminus X$ is defined as the full subcategory of $\mathrm{DM}_{(\mathrm{r})}(G \setminus X)$ whose underlying object in $\mathrm{DM}_{(\mathrm{r})}(X)$ is in $\mathrm{DTM}_{(\mathrm{r})}(X)$, cf. [RS20, §3.1]. Note that this is a slight abuse of notation in that it not only depends on the prestack $G \setminus X$, but rather on X and G.

Lemma 2.23. *Suppose $X = \operatorname{colim} X_i$ is an admissibly stratified (ind-)scheme with an action of a pro-algebraic group $G = \lim G_i$ such that the G-action preserves X_i and factors over G_i. Also assume each G_i is cellular and*

$\ker(G \to G_i)$ *is split pro-unipotent* [RS20, Definition A.4.5]. *Then* $\mathrm{DTM}_{(\mathrm{r})}(G \setminus X)$ *(Definition 2.22) carries a (unique) t-structure such that the forgetful functor to* $\mathrm{DTM}_{(\mathrm{r})}(X)$ *is t-exact. Its heart is denoted by* $\mathrm{MTM}_{(\mathrm{r})}(G\setminus X)$.

The intersection motives $\mathrm{IC}_{w,L}$ *(Definition 2.16) are naturally objects in this abelian category.*

Proof. The proof of [RS20, Proposition 3.2.15] showing the existence of this t-structure carries over. Using the notation of Remark 2.20, the object $\mathrm{im}({}^{\mathrm{p}}\mathrm{H}^0\bar{\iota}_{w!}M \to {}^{\mathrm{p}}\mathrm{H}^0\bar{\iota}_{w*}M) \in \mathrm{MTM}_{(\mathrm{r}),G}(X)$ maps to $\mathrm{IC}_{w,L}$ under the forgetful functor. ☐

2.5. **Rational and modular coefficients.** In order to compute the dual group of the Satake category, we will also need to work with rational and $\mathbf{F}_p$-coefficients. For a commutative ring Λ and a prestack X, the category $\mathrm{DM}_{(\mathrm{r})}(X,\Lambda)$ of *(reduced) motives on X with Λ-coefficients* is defined as $\mathrm{DM}_{(\mathrm{r})}(X) \otimes_{\mathrm{D}(\mathrm{Mod}_{\mathbf{Z}})} \mathrm{D}(\mathrm{Mod}_\Lambda)$. By construction, $\mathrm{DM}(X,\mathbf{Q})$ is the category of Beilinson motives [CD19, §14]. The existence and properties of the six functors holds without any change, as does the definition and properties of (stratified) Tate motives. Thus, the category $\mathrm{DTM}(X,\mathbf{Q})$ of stratified Tate motives with rational coefficients is exactly the one considered in [RS20, RS21].

Lemma 2.24. *Let X/S be an admissibly stratified ind-scheme. In the case of unreduced motives, also assume that S satisfies the Beilinson–Soulé vanishing condition (2.4). Then the category* $\mathrm{DTM}_{(\mathrm{r})}(X,\mathbf{Q})$ *carries a t-structure such that the natural functor* $\mathrm{DTM}_{(\mathrm{r})}(X) \to \mathrm{DTM}_{(\mathrm{r})}(X,\mathbf{Q})$ *is t-exact. The forgetful functor* $\mathrm{DTM}_{(\mathrm{r})}(X,\mathbf{Q}) \to \mathrm{DTM}_{(\mathrm{r})}(X)$ *is t-exact and fully faithful. Its essential image is the full subcategory consisting of objects $\mathcal{F}$ such that $n \cdot \mathrm{id}_{\mathcal{F}}$ is an isomorphism for all nonzero integers n. For $S = \operatorname{Spec} \mathbf{F}_q$, the reduction functor* $\rho_{\mathrm{r}} : \mathrm{DTM}(X,\mathbf{Q}) \to \mathrm{DTM}_{\mathrm{r}}(X,\mathbf{Q})$ *is an equivalence.*

Proof. The properties of the t-structure hold by (the proof of) Lemma 2.11. The asserted full faithfulness holds by $\mathbf{Q}\otimes\mathbf{Q} = \mathbf{Q} \in \mathrm{DM}_{(\mathrm{r})}(X)$. The final statement is [ES22, Proposition 5.3]. ☐

For $\mathbf{F}_p$-coefficients, the proof of Lemma 2.11 does not immediately apply. We therefore adopt a slightly different approach.

Definition and Lemma 2.25. For X/S as in Lemma 2.24, we define $\mathrm{MTM}_{(\mathrm{r})}(X,\mathbf{F}_p) \subset \mathrm{MTM}_{(\mathrm{r})}(X)$ to be the full subcategory consisting of the objects $\mathcal{F}$ such that $p \cdot \mathrm{id}_{\mathcal{F}} = 0$. This is an abelian subcategory.

3. Affine Grassmannians

3.1. **Definitions and basic Whitney–Tateness properties.** Throughout this paper, G denotes the base change to S of a split reductive group over $\mathbf{Z}$ (all reductive groups are assumed to be connected). We fix a split maximal torus and a Borel $T \subset B \subset G$, also defined over $\mathbf{Z}$. By a parabolic subgroup of G, we mean a subgroup $P \subset G$ containing B associated to a subset of the simple roots. In order to give some constructions uniformly, we let $\mathcal{G}$ be a smooth affine group scheme over S. The loop group (resp. positive loop group) is the functor $L\mathcal{G} : \mathrm{AffSch}_S^{\mathrm{op}} \to \mathrm{Set}$, $\operatorname{Spec} R \mapsto \mathcal{G}(R(\!(t)\!))$ (resp. $L^+\mathcal{G}(R) = \mathcal{G}(R[\![t]\!])$). The affine Grassmannian of $\mathcal{G}$ is the étale sheafification of the presheaf quotient

$$\mathrm{Gr}_{\mathcal{G}} := (L\mathcal{G}/L^+\mathcal{G})_{\text{ét}}. \tag{3.1}$$

The above choices determine a standard apartment of G with origin 0 and a standard alcove $\mathbf{a}$. For any facet $\mathbf{f}$ in the closure of $\mathbf{a}$, there is an associated parahoric subgroup $\mathcal{P} \subset LG$. If $\mathbf{f} = 0$ then $\mathcal{P} = L^+G$, and if $\mathbf{f} = \mathbf{a}$ we write $\mathcal{I}$ for the corresponding Iwahori subgroup.

Definition and Lemma 3.1. For a parahoric subgroup $\mathcal{P} \subset LG$, the Zariski, Nisnevich, and étale sheafifications of the quotient $LG/\mathcal{P}$ agree. We denote this common quotient by $\mathrm{Fl}_{\mathcal{P}}$. The sheaf $\mathrm{Fl}_{\mathcal{P}}$ is represented by an ind-projective scheme over S. As usual, we denote the *affine Grassmannian* by $\mathrm{Gr}_G := \mathrm{Fl}_{L^+G}$, and we denote the *affine flag variety* by $\mathrm{Fl} := \mathrm{Fl}_{\mathcal{I}}$.

Proof. Because sheafification commutes with base change we may assume $S = \mathbf{Z}$. Then the agreement of the different sheafifications of $LG/\mathcal{P}$ follows from [Fal03, Def. 5 ff.] (see also [dCHL18]). The representability of $\mathrm{Fl}_{\mathcal{P}}$ and Gr_P is a consequence of [PZ13, Corollary 11.7] and [HR18b, Corollary 3.11 (i)] (see also [HR18a, Lemma 2.1 ff.] for further discussion about ind-projectivity). ☐

The following four statements will be used to prove that averaging the dualizing sheaf on Gr_B yields a Tate motive, cf. the proof of Proposition 6.2.

Proposition 3.2. *Let $\mu \in X_*(T)^+$. The stabilizer of $t(\mu)\cdot e \subset \mathrm{Gr}_G(\mathbf{Z})$ is represented by a subgroup $\mathcal{P}_\mu \subset L^+G$ which is an extension of a split reductive $\mathbf{Z}$-group by a split pro-unipotent $\mathbf{Z}$-group in the sense of* [RS20, Definition A.4.5]. *The étale sheaf-theoretic image*

$$\mathrm{Gr}_G^\mu := L^+G \cdot t(\mu) \cdot e \subset \mathrm{Gr}_G^{\leq \mu}$$

agrees with $(L^+G/\mathcal{P}_\mu)_{\text{ét}}$. Using a superscript n to denote jet groups, this quotient agrees with $(L^nG/\mathcal{P}_\mu^n)_{\mathrm{Zar}}$ for $n \gg 0$. For such n, $\mathcal{P}_\mu^n$ is an extension of a split reductive $\mathbf{Z}$-group by a split unipotent $\mathbf{Z}$-group.

Proof. Let $\mathbf{f} = 0$ be the origin in the apartment of G. By lifting μ to the extended affine Weyl group we can form the translate $\mu\mathbf{f}$. Then $\mathcal{P}_\mu \subset L^+G$ is constructed explicitly in [RS20, Lemma 4.3.7] as the parahoric associated to $\mathbf{f} \cup \mu\mathbf{f}$. It has the stated form by the proof of [RS20, Lemma 4.2.7], see also [RS20, Remark 4.2.8]. Then $\mathcal{P}_\mu^0 \subset G$ is generated by the root subgroups U_α such that $\langle \alpha, \mu \rangle \leq 0$ by the discussion following [Zhu17b, Corollary 2.1.11]. By [RS20, Lemma 4.3.7], $\mathrm{Gr}_G^\mu \cong (L^+G/\mathcal{P}_\mu)_{\text{ét}} \cong (L^nG/\mathcal{P}_\mu^n)_{\text{ét}}$ for $n \gg 0$. It remains to show that $L^nG \to (L^nG/\mathcal{P}_\mu^n)_{\text{ét}}$ admits sections Zariski-locally. Fix such an n and let $L^{>0}G = \ker(L^nG \xrightarrow{t \mapsto 0} G)$. Then $(L^nG/L^{>0}G)_{\text{ét}} \cong G$ and we have the following diagram, where the square is cartesian.

$$\begin{tikzcd} L^nG \arrow[r, "g"] & (L^nG/L^{>0}G \cap \mathcal{P}_\mu^n)_{\text{ét}} \arrow[r, "f"] \arrow[d] & \mathrm{Gr}_G^\mu \arrow[d] \\ & G \arrow[r] & (G/\mathcal{P}_\mu^0)_{\text{ét}} \end{tikzcd}$$

The left vertical map is the quotient by $L^{>0}G$. The bottom horizontal map can be trivialized Zariski-locally over a covering of $(G/\mathcal{P}_\mu^0)_{\text{ét}}$ by affine spaces, cf. [Jan87, 1.10(5)]. By the Quillen–Suslin theorem [Qui76, Sus76], every finitely generated projective module over $\mathbf{Z}[x_1, \ldots, x_m]$ is free. Thus the right vertical map, which is an affine bundle, also trivializes over each of the affine cells in $(G/\mathcal{P}_\mu^0)_{\text{ét}}$. It follows that Gr_G^μ is covered by affine cells over which f is a trivial $\mathcal{P}_\mu^0$-torsor. The map g is also an affine bundle, so by using the Quillen–Suslin theorem again, g admits a section over the image of any of our sections of f. This gives enough sections of the composition $f \circ g$ Zariski-locally. $\square$

Lemma 3.3. *For any parabolic subgroup $P \subset G$, Gr_P is represented by an ind-scheme of ind-finite type over S. The natural morphism $\mathrm{Gr}_P \to \mathrm{Gr}_G$ identifies Gr_P with the attracting locus of a $\mathbf{G}_\mathrm{m}$-action on Gr_G, and it restricts to a locally closed embedding on connected components of Gr_P. If k is a field, then $\mathrm{Gr}_P(k) \to \mathrm{Gr}_G(k)$ is a bijection.*

Proof. Choose a cocharacter $\eta \in X_*(T)^+$ which is orthogonal to the simple roots associated to P, but not orthogonal to any other simple root. Then the action of $\mathbf{G}_\mathrm{m}$ on G via conjugation by η extends to a $\mathbf{G}_\mathrm{m}$-action on Gr_G. By [HR18b, Theorem 3.17], Gr_P identifies with the the attractor $(\mathrm{Gr}_G)^+$ for this $\mathbf{G}_\mathrm{m}$-action. The representability of Gr_P then follows from [HR18b, Theorem 2.1 (iii)]. To show that we have a locally closed embedding, first note that there exists a closed embedding $G \to \mathbf{GL}_n$ for some n. Then the proof of [HR18b, Lemma 3.16] shows the $\mathbf{G}_\mathrm{m}$-action on Gr_G is Zariski-locally linearizable. Thus, we may write $\mathrm{Gr}_G = \operatorname{colim} X_i$, where each X_i is projective over S and $\mathbf{G}_\mathrm{m}$-stable, and there is a $\mathbf{G}_\mathrm{m}$-equivariant Zariski cover $U_i \to X_i$ which is affine. Then $X^+ = \operatorname{colim}(X_i)^+$, and by [Ric19, Lemma 1.11] we have $(U_i)^+ = (U_i)^0 \times_{(X_i)^0} (X_i)^+$. Since $(U_i)^+$ is representable by a closed subscheme of U_i [Ric19, Lemma 1.9], we conclude that $\mathrm{Gr}_P \to \mathrm{Gr}_G$ is Zariski-locally on Gr_P a locally closed immersion. The final claim about points over a field k will then imply it is a locally closed immersion on each connected component. For this, we note that $\mathrm{Gr}_G(k) = LG(k)/L^+G(k)$ and $\mathrm{Gr}_P(k) = LP(k)/L^+P(k)$. It follows that $\mathrm{Gr}_P(k) \to \mathrm{Gr}_G(k)$ is injective, and it is surjective by the Iwasawa decomposition of $G(k(\!(t)\!))$. $\square$

By [Ric19, Corollary 1.12], there are bijections $\pi_0(\mathrm{Gr}_B) \cong \pi_0(\mathrm{Gr}_T) \cong X_*(T)$. The resulting connected components of Gr_B are denoted by S_ν for $\nu \in X_*(T)$, and called the *semi-infinite orbits*. If B^- is the opposite Borel subgroup, we denote by T_ν the corresponding connected component of Gr_{B^-}. Now fix $\mu \in X_*(T)^+$ and $\nu \in X_*(T)$ such that $S_\nu \cap \mathrm{Gr}_G^\mu \neq \emptyset$. Let $n \gg 0$ be such that the action of L^+G on Gr_G^μ factors through L^nG. Let $\mathcal{P}_\mu^n \subset L^nG$ be the stabilizer of $t(\mu)$ as in Proposition 3.2.

Lemma 3.4. *The $\mathcal{P}_\mu^n$-torsor $L^nG \to \mathrm{Gr}_G^\mu$ is trivial over $S_\nu \cap \mathrm{Gr}_G^\mu$.*

Proof. Fix a regular dominant cocharacter, so that that S_ν is a connected component of the attracting locus of the resulting $\mathbf{G}_\mathrm{m}$-action on Gr_G. Let $\mathcal{P}_\mu^0 \subset G$ be the mod t reduction of $\mathcal{P}_\mu$. The attractors for the $\mathbf{G}_\mathrm{m}$-action on $G/\mathcal{P}_\mu^0$ are affine cells isomorphic to the orbits $Uw\mathcal{P}_\mu^0/\mathcal{P}_\mu^0$, where $U \subset B$ is the unipotent radical and w belongs to the Weyl group, cf. [NP01, Lemme 6.2]. Let $q\colon \mathrm{Gr}_G^\mu \to G/\mathcal{P}_\mu^0$ be the reduction mod t map. Then $S_\nu \cap \mathrm{Gr}_G^\mu$ is contained in $q^{-1}(Uw\mathcal{P}_\mu^0/\mathcal{P}_\mu^0)$ for some w. By the proof of Proposition 3.2, it suffices to show that the $\mathcal{P}_0^\mu$-torsor $G \to G/\mathcal{P}_\mu^0$ is trivial over $Uw\mathcal{P}_\mu^0/\mathcal{P}_\mu^0$. As in the proof of [NP01, Lemme 6.2], there is a vector subgroup $U_w \subset U$ [1] which maps isomorphically onto $U_w = Uw\mathcal{P}_\mu^0/\mathcal{P}_\mu^0$. The map $U_w \times \mathcal{P}_\mu^0 \to G$, $(u,p) \mapsto uwp$ then induces a trivialization of $G \to G/\mathcal{P}_\mu^0$ over $Uw\mathcal{P}_\mu^0/\mathcal{P}_\mu^0$. $\square$

Proposition 3.5. *Let $a\colon L^nG \times (S_\nu \cap \mathrm{Gr}_G^\mu) \to \mathrm{Gr}_G^\mu$ be the action map and let $t^\mu\colon S \to \mathrm{Gr}_G^\mu$ be the point corresponding to μ. Then $a^{-1}(t^\mu) \cong \mathcal{P}_\mu^n \times (S_\nu \cap \mathrm{Gr}_G^\mu)$.*

Proof. The projection $a^{-1}(t^\nu) \to S_\nu \cap \mathrm{Gr}_G^\mu$ is a $\mathcal{P}_\mu^n$-torsor. Let $r\colon S_\nu \cap \mathrm{Gr}_G^\mu \to L^nG$ be a section, as per Lemma 3.4. Then there is an isomorphism $\mathcal{P}_\mu^n \times (S_\nu \cap \mathrm{Gr}_G^\mu) \to a^{-1}(t^\mu)$, $(p,x) \mapsto (r(x) \cdot p \cdot r(x)^{-1}, x)$. $\square$

[1] It is $U_w = wU_\mu^+ w^{-1} \cap U$, where U_μ^+ is the unipotent radical of the parabolic opposite to $\mathcal{P}_\mu^0$.

The next proposition is an extension (but not strictly speaking a corollary) of the Whitney–Tateness of partial affine flag varieties [RS20, Theorem 5.1.1]. It will be used in order to show the Whitney–Tateness of the Beilinson–Drinfeld Grassmannian. The following lemma serves to show the anti-effectivity.

Lemma 3.6. *Let X and Y be ind-schemes, each having a Whitney–Tate stratification by affine spaces. Let $\pi\colon X \to Y$ be a smooth map which sends strata onto strata, and such that for each stratum $X_w \subset X$, the induced map on strata $X_w \to \pi(X_w)$ is a relative affine space. Then the functors $\pi_!$ and $\pi^*\pi_!$ preserve anti-effective stratified Tate motives.*

Proof. By excision and base change, it suffices to consider a motive $\iota_{w!}\mathbf{Z}$, where $\iota_w\colon X_w \to X$ is a stratum. Then the lemma follows from the fact that the structure map $f\colon \mathbf{A}^n_S \to S$ satisfies $f_!(\mathbf{Z}) \cong \mathbf{Z}(-n)[-2n]$. ☐

Proposition 3.7. *For any parahoric subgroups $\mathcal{P}$, $\mathcal{P}' \subset LG$, the stratification of the partial affine flag variety $\mathrm{Fl}_{\mathcal{P}}$ by $\mathcal{P}'$-orbits is anti-effective universally Whitney–Tate.*

Proof. This follows by revisiting the proof in [RS20, Theorem 5.1.1]: Beginning with the case $\mathcal{P} = \mathcal{P}' = \mathcal{I}$ is the Iwahori subgroup, let $\iota : \mathrm{Fl}^\dagger \to \mathrm{Fl}$ be the stratification map and, for any S-ind-scheme Y, let $\iota' : \mathrm{Fl}^\dagger \times_S Y \to \mathrm{Fl} \times_S Y$ be the product stratification. For an element w of the extended affine Weyl group, one shows by induction on the length $l(w)$ that $\iota'^!\iota'_{w!}\mathbf{Z} \in \mathrm{DTM}(\mathrm{Fl}^\dagger \times Y)$: this is clear if $l(w) = 0$. Inductively, for $w = vs$ for a simple reflection s and an element with $l(v) = l(w) - 1$, there is a cartesian diagram, where $\mathcal{P}_s$ is the parahoric subgroup associated to s and the map π arises from the inclusion $\mathcal{I} \subset \mathcal{P}_s$:

$$\begin{tikzcd}
(\mathrm{Fl}^v \sqcup \mathrm{Fl}^w) \times Y \arrow[r] \arrow[dr, "\pi^\dagger"'] & \pi^{-1}(\mathrm{Fl}^v_{\mathcal{P}_s} \times Y) \arrow[r] \arrow[d, "\tilde{\pi}"] & \mathrm{Fl} \times Y \arrow[d, "\pi"] \\
& \mathrm{Fl}^v_{\mathcal{P}_s} \times Y \arrow[r] & \mathrm{Fl}_{\mathcal{P}_s} \times Y.
\end{tikzcd}$$

The map $\pi^\dagger$ is isomorphic to the disjoint union of $\mathrm{id}_{\mathrm{Fl}^v}$ and the projection $p : \mathbf{A}^1_{\mathrm{Fl}^v_{\mathcal{P}_s}} \to \mathrm{Fl}^v_{\mathcal{P}_s}$. More generally, the map π is smooth and proper, and the induced map from each stratum of $\mathrm{Fl} \times Y$ onto its image in $\mathrm{Fl}_{\mathcal{P}_s} \times Y$ is either an isomorphism or an affine space of relative dimension one. Applying Verdier duality D to the localization sequence [RS20, (5.1.2)] and noting that $\omega_{\mathbf{A}^n} = \mathbf{Z}(n)[2n]$ gives a fiber sequence

$$\iota'_{v*}\mathbf{Z}(-1)[-2] \to \pi^!\pi_!\iota'_{v*}\mathbf{Z}(-1)[-2] \to \iota'_{w*}\mathbf{Z}. \tag{3.2}$$

Thus, the fact $p_!\mathbf{Z} = \mathbf{Z}(-1)[-2]$ shows that we have a Whitney-Tate stratification, and the condition of (2.3) being an isomorphism on the summand corresponding to v implies the same for the summand of w.

The Whitney–Tateness for general $\mathcal{P}, \mathcal{P}' \subset LG$ is then treated identically as in loc. cit.

We prove the anti-effectivity of the stratification using the same reduction steps, as follows.

First case: $\mathcal{I} = \mathcal{P} = \mathcal{P}'$. We apply Lemma 3.6 to π (we may assume $Y = S$), and conclude the claim using (3.2).

Second case: $\mathcal{I} = \mathcal{P}' \subset \mathcal{P}$. By [RS20, Lemma 4.3.13], the projection $\mathrm{Fl} \to \mathrm{Fl}_{\mathcal{P}}$ satisfies the hypotheses of Lemma 3.6, where both ind-schemes are stratified by $\mathcal{I}$-orbits. For a stratum $\iota_w\colon \mathrm{Fl}^w_{\mathcal{P}} \to \mathrm{Fl}_{\mathcal{P}}$, let $s\colon \mathrm{Fl}^w_{\mathcal{P}} \to \mathrm{Fl}$ be a section which includes $\mathrm{Fl}^w_{\mathcal{P}}$ as a stratum of Fl. Then $\iota_{w*}\mathbf{Z} = \pi_*s_*\mathbf{Z}$, so we conclude by the first case and Lemma 3.6.

Third case: $\mathcal{P}$, $\mathcal{P}'$ arbitrary. Let $\iota\colon \mathrm{Fl}^\dagger_{\mathcal{P}} \to \mathrm{Fl}_{\mathcal{P}}$ be the stratification by $\mathcal{P}'$-orbits, and let ι' be the stratification of $\mathrm{Fl}^\dagger_{\mathcal{P}}$ by $\mathcal{I}$-orbits. To check if $\iota^*\iota_*\mathbf{Z}$ is anti-effective, we apply Lemma 2.12. By localization, the condition $\mathrm{Maps}(\mathbf{Z}(p), \iota^*\iota_*\mathbf{Z}) = 0$ is equivalent to $\mathrm{Maps}(\iota'_!\iota'^*\mathbf{Z}(p), \iota^*\iota_*\mathbf{Z}) = \mathrm{Maps}(\iota'^*\mathbf{Z}(p), \iota'^!\iota^*\iota_*\mathbf{Z}) = 0$. Since the motive $\mathbf{Z}$ on $\mathrm{Fl}^\dagger_{\mathcal{P}'}$ is anti-effective with respect to ι', we conclude by the second case.

☐

Lemma 3.8. *Consider some schematic map $f : X' \to X''$ of (ind-)schemes over S, some $M \in \mathrm{DM}(X')$ and some stratified Tate motive $N \in \mathrm{DTM}(\mathrm{Fl}_{\mathcal{P}})$, where the stratification on $\mathrm{Fl}_{\mathcal{P}}$ is by any $\mathcal{P}'$-orbits. Then the following natural maps are isomorphisms:*

$$f_*M \boxtimes N \to (f \times \mathrm{id})_*(M \boxtimes N),$$
$$(f \times \mathrm{id})^!(M \boxtimes N) \to f^!M \boxtimes N.$$

Remark 3.9. Resolution of singularities implies that *-pushforward functors are compatible with exterior products: for a field k, and two maps $X' \xrightarrow{f} X''$, $Y' \xrightarrow{g} Y''$, the natural map

$$f_*M \boxtimes g_*N \to (f \times g)_*(M \boxtimes N)$$

is an isomorphism for any $M \in \mathrm{DM}(X')$, $N \in \mathrm{DM}(Y')$ if k is of characteristic 0 or if $\mathrm{char}\, k$ is invertible in the ring of coefficients [JY21, Theorem 2.4.6]. Since below we are interested in motives with integral coefficients, and work over $\mathrm{Spec}\, \mathbf{Z}$, we need to supply a more specific argument.

Proof. We may refine our stratification and replace $\mathcal{P}'$ by the Iwahori subgroup $\mathcal{I}$. The proof of Proposition 3.7 implies that (cf. [RS20, Proposition 5.2.2]) $\mathrm{DTM}(\mathrm{Fl}_{\mathcal{P}})$ is generated (under colimits) by $\pi_! \mathrm{DTM}(\mathrm{Fl})$, where $\pi : \mathrm{Fl} \to \mathrm{Fl}_{\mathcal{P}}$ is the quotient map. This map is proper, so the projection formula and the fact that *-pushforwards along schematic maps (as well as any !-pullback) preserve colimits reduce the claim for $\mathcal{P}$ to the one for $\mathcal{I}$. In this case, again by loc. cit., the category $\mathrm{DTM}(\mathrm{Fl})$ is the smallest presentable subcategory containing $\tau_* \mathrm{DTM}(S)$, where $\tau : S \to \mathrm{Fl}$ is the closed embedding of a base point, and stable under $\pi_s^* \pi_{s*}$, where $\pi_s : \mathrm{Fl} \to \mathrm{Fl}_{\mathcal{P}_s}$ is as in the proof above. Now, $\boxtimes$ commutes with τ_* and also with π_s^* and π_{s*}, since this map is smooth and proper. $\square$

The following corollary will be used in order to show that Beilinson–Drinfeld affine Grassmannians are Whitney–Tate stratified.

Corollary 3.10. *Let X be any stratified Whitney–Tate (ind-)scheme X. Then the product stratification on $X \times_S \mathrm{Fl}_{\mathcal{P}}$ (i.e., strata are products of X_w times $\mathcal{P}'$-orbits, for an arbitrary fixed parahoric subgroup $\mathcal{P}'$) is again Whitney–Tate.*

Proof. Abbreviate $\mathrm{Fl} := \mathrm{Fl}_{\mathcal{P}}$ and write ι_X and ι_{Fl} for the stratification maps. It suffices to have an isomorphism $(\iota_X \times \iota_{\mathrm{Fl}})_* \mathbf{Z} = \iota_{X*} \mathbf{Z} \boxtimes \iota_{\mathrm{Fl}*} \mathbf{Z}$, since in any case $*$-pullbacks commute with exterior products. We have

$$(\iota_X \times \iota_{\mathrm{Fl}})_* \mathbf{Z} = (\iota_X \times \mathrm{id})_* (\mathrm{id} \times \iota_{\mathrm{Fl}})_* \mathbf{Z}.$$

Since Fl is universally Whitney–Tate, $(\mathrm{id} \times \iota_{\mathrm{Fl}})_* \mathbf{Z} = p^* \iota_{\mathrm{Fl}*} \mathbf{Z}$, where $p : X^\dagger \times \mathrm{Fl} \to \mathrm{Fl}$ is the projection. This motive can also be written as $\mathbf{Z}_{X^\dagger} \boxtimes \iota_{\mathrm{Fl}*} \mathbf{Z}$. Applying $(\iota_X \times \mathrm{id})_*$ to it gives, by Lemma 3.8, $\iota_{X*} \mathbf{Z} \boxtimes \iota_{\mathrm{Fl}*} \mathbf{Z}$. $\square$

3.2. **Semi-infinite orbits over an algebraically closed field.** In this section we assume that $S = \mathrm{Spec}\, k$ is the spectrum of an algebraically closed field and that G is simple and simply connected. See Lemma 3.31 for how to generalize the argument to general reductive groups. Let $\mathcal{I} \subset L^+G$ be the Iwahori subgroup. The $\mathcal{I}$-orbits in Gr_G are parametrized by $X_*(T)$, see [RS20, 4.2.12]. For $\lambda, \nu \in X_*(T)$, let $\mathrm{Gr}_G^{\mathcal{I}\lambda} \subset \mathrm{Gr}_G$ be the corresponding $\mathcal{I}$-orbit, and consider the semi-infinite orbit S_ν. View the intersection $S_\nu \cap \mathrm{Gr}_G^{\mathcal{I}\lambda}$ as a reduced subscheme of $\mathrm{Gr}_G^{\mathcal{I}\lambda}$.

In this section we show that $S_\nu \cap \mathrm{Gr}_G^{\mathcal{I}\lambda}$ has Tate cohomology. In the case of the L^+G-orbits Gr_G^{λ}, we will prove the stronger result that $S_\nu \cap \mathrm{Gr}_G^{\lambda}$ has a filtrable cellular decomposition over $\mathbf{Z}$ for any split G in Theorem 3.32, and a similar result for semi-infinite orbits with respect to a rank one parabolic subgroup in Theorem 3.44. The latter result will be used to identify the Tannakian group of the Satake category, cf. Theorem 6.27, and more precisely Proposition 6.33. Our purpose in this section is to give a short proof that the constant term functors for B preserve Tate motives over an algebraically closed field without delving into the detailed combinatorics in Section 3.3.1.

Recall the following definition from [BB76, Definition 2].

Definition 3.11. A decomposition of a scheme X into locally closed subschemes $(X_\alpha)_{\alpha \in A}$ is called *filtrable*, if there exists a finite decreasing sequence $X = X_0 \supset X_1 \supset \ldots \supset X_m = \varnothing$ of closed subschemes of X, such that for each $j = 1, \ldots, m$, the complement $X_{j-1} \setminus X_j$ is one of the X_α's.

In particular, every stratification is filtrable. While not every filtrable decomposition is a stratification, it is enough for the purposes of inductively applying localization.

Proposition 3.12. *If $p \colon S_\nu \cap \mathrm{Gr}_G^{\mathcal{I}\lambda} \to S$ is the structure morphism, $p_!(\mathbf{Z}) \in \mathrm{DTM}(S)$.*

Proof. Pick a regular cocharacter $\mathbf{G}_{\mathrm{m}} \to T$, so that the reduced locus of Gr_T is the set of fixed points for the resulting $\mathbf{G}_{\mathrm{m}}$-action on Gr_G. Because G is simple and simply connected, by [Zhu17b, Theorem 2.5.3] we may identify Gr_G with the flag variety of an affine Kac–Moody group. Choose a Bott–Samelson resolution $m \colon X \to \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}}$ such that m is an isomorphism over $\mathrm{Gr}_G^{\mathcal{I}\lambda}$. Details about the construction of X can be found in [JMW14, §4].

Let $f = m|_{m^{-1}(S_\nu \cap \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}})}$ and let $i \colon (S_\nu \cap \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}}) \setminus (S_\nu \cap \mathrm{Gr}_G^{\mathcal{I}\lambda}) \to S_\nu \cap \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}}$ be the closed immersion. By applying localization to $f_! \mathbf{Z}$ we have an exact triangle

$$p_! \mathbf{Z} \to p_! f_! \mathbf{Z} \to p_! i_! i^* f_! \mathbf{Z}.$$

We will prove the middle and right terms lie in $\mathrm{DTM}(S)$.

Note that the variety X is smooth, projective, and it has a $\mathbf{G}_{\mathrm{m}}$-action such that m is $\mathbf{G}_{\mathrm{m}}$-equivariant. As X embeds equivariantly into a product of affine flag varieties, this $\mathbf{G}_{\mathrm{m}}$-action has isolated fixed points. Since k is algebraically closed, the attractors for this action of $\mathbf{G}_{\mathrm{m}}$ as in [BB73] then give a decomposition of X into affine spaces. By the existence of T-equivariant ample line bundles on affine flag varieties, cf. [Zhu17b, §1.5], X also embeds $\mathbf{G}_{\mathrm{m}}$-equivariantly into some projective space. By [BB76, Th. 3], this implies the Białynicki-Birula decomposition of X is a filtrable decomposition into affine cells. Since $S_\nu \cap \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}}$ is an attractor, the fiber $m^{-1}(S_\nu \cap \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}})$ is a union of attractors. By repeatedly applying localization and noting that affine spaces have Tate cohomology it follows that $p_! \circ f_!(\mathbf{Z}) \in \mathrm{DTM}(S)$.

The proof of [JMW14, Prop. 4.11] shows that $m_!(\mathbf{Z})$ is isomorphic to a composition of $*$-pullbacks and !-pushforwards along stratified maps between Kac-moody flag varieties which are stratified by (affine) $\mathcal{I}$-orbits. In particular, $m_!(\mathbf{Z}) \in \mathrm{DTM}(\overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}})$, where $\overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}}$ is stratified by $\mathcal{I}$-orbits. By proper base change, the restriction of $f_!(\mathbf{Z})$ to each $S_\nu \cap \mathrm{Gr}_G^{\mathcal{I}\lambda'} \subset \overline{\mathrm{Gr}_G^{\mathcal{I}\lambda}}$ is Tate. Thus $p_! i_! i^* f_! \mathbf{Z} \in \mathrm{DTM}(S)$ by induction on λ and localization with respect to the stratification by intersections of S_ν with $\mathcal{I}$-orbits. $\square$

3.3. **Intersections of Schubert cells and semi-infinite orbits.** In the following subsection, we prove that the intersections of the Schubert cells and semi-infinite orbits admit a cellular decomposition, following [GL05]. This will later allow us to show the constant term functor associated to the Borel preserves Tate motives. Over an algebraically closed field, this can also be shown using Proposition 3.12. The proof in this section, while much longer and much more combinatorial, works over any base. Moreover, it will allow us to prove a more general cellularity result, which implies the preservation of Tate motives by constant term functors associated to higher Levi subgroups. For the rest of this section, we will work over $S = \operatorname{Spec} \mathbf{Z}$ for simplicity, the general case then follows by base change.

Notation 3.13. Consider $T \subseteq B \subseteq G$ as before. Recall that if $\mathrm{ev} : L^+G \to G$ is the evaluation at 0, then the Iwahori $\mathcal{I} \subseteq L^+G \subseteq LG$ is the inverse image $\mathrm{ev}^{-1}(B)$. Similarly, we define $\mathcal{I}^- := \mathrm{ev}^{-1}(B^-)$ and $\mathcal{T} := \mathrm{ev}^{-1}(T)$. Let $N := N(T)$ be the normalizer of T in G, which gives us the finite Weyl group $W = N/T$ of G. We also have the affine Weyl group $W^{\mathfrak{a}}$, which agrees with the extended affine Weyl group $N(\mathbf{Z}(\!(t)\!))/T(\mathbf{Z}[\![t]\!])$ when G is simply connected. We denote by Φ be the roots of G, and $\Phi^+ \subset \Phi$ the positive roots with respect to B. Similarly, we denote by $R^+ \subseteq R$ the (positive) affine roots. Finally, we fix a Chevalley system of G. In particular, for any root $\beta \in \Phi$, we get an isomorphism $x_\beta \colon \mathbf{G}_{\mathrm{a}} \to U_\beta$, where U_β is the root group associated to β.

3.3.1. *Review of combinatorial galleries.* For the rest of this section, we will assume G is semisimple and simply connected, unless mentioned otherwise (which will be the case for the main theorems in this section). Under this assumption, a weaker version of what we want was already proved by Gaussent–Littelmann in [GL05] for complex groups. However, their arguments work almost verbatim over any algebraically closed field, and we will also use this in the sequel. Our aim will be to generalize (and strengthen) their work to groups over $\operatorname{Spec} \mathbf{Z}$. We first recall the most important notation and terminology from [GL05], and refer to loc. cit. for details. By the following remark, we can work over an arbitrary algebraically closed field k for now.

Remark 3.14. For any field k, the natural maps $N(\mathbf{Z}(\!(t)\!))/T(\mathbf{Z}(\!(t)\!)) \xrightarrow{\cong} N(k(\!(t)\!))/T(k(\!(t)\!))$ and $T(\mathbf{Z}(\!(t)\!))/T(\mathbf{Z}[\![t]\!]) \xrightarrow{\cong} T(k(\!(t)\!))/T(k[\![t]\!])$ are isomorphisms. In particular, there is a natural isomorphism $N(\mathbf{Z}(\!(t)\!))/T(\mathbf{Z}[\![t]\!]) \xrightarrow{\cong} N(k(\!(t)\!))/T(k[\![t]\!])$, i.e., an isomorphism between the (extended) affine Weyl groups of G and G_k. Using this, we get an isomorphism of apartments $\mathcal{A}(G,T) \cong \mathcal{A}(G_k, T_k)$, equivariant for the actions of the affine Weyl groups of G and G_k. This isomorphism is moreover compatible with the identifications $\mathcal{A}(G,T) \cong X_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$ and $\mathcal{A}(G_k, T_k) \cong X_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$, if we choose the canonical Chevalley valuations on $G \otimes_{\mathbf{Z}} \mathbf{Z}[\![t]\!]$ and $G_k \otimes_k k[\![t]\!]$ as basepoints of the apartments.

Consider the group $X_*(T)$ of cocharacters of G, and let $\mathcal{A}_G := X_*(T) \otimes_{\mathbf{Z}} \mathbf{R}$. The affine Weyl group $W^{\mathfrak{a}}$ acts on $\mathcal{A}_G$ by affine reflections. The reflection hyperplanes (also called *walls*) in $\mathcal{A}_G$ for this action are all of the form $\mathbf{H}_{\beta,m} = \{a \in \mathcal{A}_G \mid \langle a, \beta \rangle = m\}$, for some positive root $\beta \in \Phi^+$ and $m \in \mathbf{Z}$. Let $s_{\beta,m} \in W^{\mathfrak{a}}$ denote the corresponding affine reflection, and $\mathbf{H}^+_{\beta,m} = \{a \in \mathcal{A}_G \mid \langle a, \beta \rangle \geq m\}$ and $\mathbf{H}^-_{\beta,m} = \{a \in \mathcal{A}_G \mid \langle a, \beta \rangle \leq m\}$ the associated closed half-spaces.

Remark 3.15. We use the Tits convention for the action of $W^{\mathfrak{a}}$ on $\mathcal{A}_G$, i.e., we let $t \in T(k(\!(t)\!))/T(k[\![t]\!]) = X_*(T)$ act via the translation by $-t$. The advantage of this convention is that if we let $W^{\mathfrak{a}}$ act on the set R of affine roots as usual, via $w(\alpha)(x) := \alpha(w^{-1}x)$ for $\alpha \in R$, $w \in W^{\mathfrak{a}}$ and $x \in \mathcal{A}_G$, then $wU_\alpha w^{-1} = U_{w(\alpha)}$. This will be used later on.

Let $\mathbf{H}^{\mathfrak{a}} := \bigcup_{\beta \in \Phi^+, m \in \mathbf{Z}} \mathbf{H}_{\beta,m}$ denote the union of the reflection hyperplanes. Then the connected components of $\mathcal{A}_G \setminus \mathbf{H}^{\mathfrak{a}}$ are called *open alcoves*, and their closures simply *alcoves*. More generally, a *face* of $\mathcal{A}_G$ is a subset F that can be obtained by intersecting closed affine half-spaces and reflection hyperplanes, one for each $\beta \in \Phi^+$ and $m \in \mathbf{Z}$. One example of an alcove is the *fundamental alcove* $\Delta_f = \{a \in \mathcal{A}_G \mid 0 \leq \langle a, \beta \rangle \leq 1, \forall \beta \in \Phi^+\}$.

It is well-known that $W^{\mathfrak{a}}$ is generated by the affine reflections $S^{\mathfrak{a}}$, consisting of those $s_{\beta,m} \in W^{\mathfrak{a}}$ such that Δ_f admits a face lying in $\mathbf{H}_{\beta,m}$. For a face F contained in Δ_f, we define the *type* $S^{\mathfrak{a}}(F)$ as the subset of $S^{\mathfrak{a}}$ consisting of those $s_{\beta,m} \in S^{\mathfrak{a}}$ such that F is contained in the hyperplane $\mathbf{H}_{\beta,m}$. In particular, we have $S^{\mathfrak{a}}(0) = S := \{s_{\beta,0} \mid \beta \in \Phi^+\}$ and $S^{\mathfrak{a}}(\Delta_f) = \varnothing$. Since $W^{\mathfrak{a}}$ acts simply transitively on the set of alcoves, we can translate any face to a face of Δ_f using this action, and use this to define the type of arbitrary faces in $\mathcal{A}_G$. Moreover, to any subset $t \subset S^{\mathfrak{a}}$, we can associate a parahoric subgroup $\mathcal{P}_t := \bigcup_{w \in W^{\mathfrak{a}}_t} \mathcal{I} w \mathcal{I} \subseteq LG$, where $W^{\mathfrak{a}}_t$ is the subgroup of $W^{\mathfrak{a}}$ generated by t. We call this the *standard parahoric of type* t. For example, we have $\mathcal{P}_\varnothing = \mathcal{I}$, and $\mathcal{P}_S = L^+G$. Conversely, any parahoric subgroup $\mathcal{P}$ containing $\mathcal{I}$ arises uniquely in this way; we denote the associated subgroup of $W^{\mathfrak{a}}$ by $W^{\mathfrak{a}}_{\mathcal{P}}$.

Now, consider the isomorphisms $x_\beta : \mathbf{G}_a \to U_\beta$ arising from the Chevalley system. Let v denote the canonical valuation on $k(\!(t)\!)$, and define, for any $r \in \mathbf{R}$, the subgroup

$$U_{\beta,r} := 1 \cup \{x_\beta(f) \mid f \in k(\!(t)\!), v(f) \geq r\} \subseteq G(k(\!(t)\!)).$$

Letting $\ell_\beta(\Omega) := -\inf_{x\in\Omega}\langle x, \beta\rangle$ for any $\varnothing \neq \Omega \subseteq \mathcal{A}_G$, we can define the subgroups $U_\Omega := \langle U_{\beta,\ell_\beta(\Omega)} \mid \beta \in \Phi\rangle$ of $G(k(\!(t)\!))$, and use this to define the affine building of G.

Definition 3.16. The *affine building* $\mathscr{J}^{\mathfrak{a}}$ of G is the quotient $G(k(\!(t)\!)) \times \mathcal{A}_G/\sim$, where two pairs (g, x) and (h, y) are equivalent if there is some $n \in N(k(\!(t)\!))$ such that $nx = y$ and $g^{-1}hn \in U_x$.

There is an $N(k(\!(t)\!))$-equivariant injection $\mathcal{A}_G \hookrightarrow \mathscr{J}^{\mathfrak{a}}: x \mapsto (1, x)$. An *apartment* of $\mathscr{J}^{\mathfrak{a}}$ is a subset of the form $g\mathcal{A}_G$, for some $g \in G(k(\!(t)\!))$. In particular, $\mathcal{A}_G$ is an apartment, called the *standard* apartment.

Finally, we define the faces in $\mathscr{J}^{\mathfrak{a}}$ as the $G(k(\!(t)\!))$-translates of the faces in $\mathcal{A}_G$. The type of a face in $\mathscr{J}^{\mathfrak{a}}$ is defined similarly, by translating to a face in $\mathcal{A}_G$; this is a well-defined notion.

The following definition is a central topic in [GL05].

Definition 3.17. A *gallery* in $\mathscr{J}^{\mathfrak{a}}$ is a sequence of faces

$$\gamma = (\Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \ldots \supset \Gamma'_p \subset \Gamma_p \supset \Gamma'_{p+1})$$

in $\mathscr{J}^{\mathfrak{a}}$, such that

- Γ'_0 and Γ'_1, called the *source* and *target* of γ, are vertices,
- the Γ_j's are all faces of the same dimension, and
- for $1 \leq j \leq p$, the face Γ'_j is a codimension one face of both Γ_{j-1} and Γ_j.

The *gallery of types* of such a gallery γ is the sequence of types of the faces of γ:

$$t_\gamma = (t'_0 \supset t_0 \subset t'_1 \supset \ldots t'_p \subset t_p \supset t'_{p+1}),$$

where the t'_j and t_j are the types of Γ'_j and Γ_j respectively.

We will be especially interested in galleries of a more combinatorial nature, depending on a fixed minimal gallery. To define these, we need the following extension of cocharacters.

Remark 3.18. If $G_{\mathrm{adj}} = G/Z_G$ is the adjoint quotient of G and $T_{\mathrm{adj}} \subseteq G_{\mathrm{adj}}$ the adjoint torus, then we have natural injections $X_*(T) \subseteq X_*(T_{\mathrm{adj}}) \subseteq X_*(T) \otimes_{\mathbf{Z}} \mathrm{R}$. The lattice $X_*(T_{\mathrm{adj}}) \subseteq X_*(T) \otimes_{\mathbf{Z}} \mathrm{R}$ consists exactly of those vertices that are a face of $\mathcal{A}_G$. In what follows, we will often consider all elements of $X_*(T_{\mathrm{adj}}) \subseteq X_*(T) \otimes_{\mathbf{Z}} \mathrm{R}$ instead of just the cocharacters, this will help us when applying the results of this section to non-simply connected groups. Note that notions such as dominance and regularity extend to $X_*(T_{\mathrm{adj}})$.

For $\mu \in X_*(T_{\mathrm{adj}})$, let $\mathbf{H}_\mu = \bigcap_{\langle\mu,\alpha\rangle=0} \mathbf{H}_{\alpha,0}$ be the intersection of those reflection hyperplanes corresponding to the roots orthogonal to μ (so that $\mathbf{H}_\mu = \mathcal{A}_G$ for regular μ).

Definition 3.19. Fix some $\mu \in X_*(T_{\mathrm{adj}})$, let F_f be the face corresponding to $0 \in \mathcal{A}_G$, and F_μ the face corresponding to $\mu \in \mathcal{A}_G$; note that both are vertices. A gallery $\gamma = (\Gamma'_0 \subset \Gamma_0 \supset \Gamma'_1 \subset \ldots \supset \Gamma'_p \subset \Gamma_p \supset \Gamma'_{p+1})$ contained in $\mathcal{A}_G$ is said to *join 0 with* μ if its source is F_f, its target F_μ, and if the dimension of the large faces Γ_j is equal to the dimension of $\mathbf{H}_\mu$.

In fact, we are interested in those galleries that are minimal in a precise sense. While one can define what it means for an arbitrary gallery in $\mathscr{J}^{\mathfrak{a}}$ to be minimal, we will content ourselves to give an equivalent definition for galleries joining 0 with some dominant $\mu \in X_*(T_{\mathrm{adj}})$, cf. [GL05, Lemma 4]. We say that a reflection hyperplane $\mathbf{H}$ *separates* a subset Ω and a face F of $\mathcal{A}_G$, if Ω lies in a closed half-space defined by $\mathbf{H}$, and F is contained in the opposite open half-space. For two faces E, F in $\mathcal{A}_G$, let $\mathcal{M}_{\mathcal{A}_G}(E, F)$ be the set of such hyperplanes separating E and F. It is known that this set is finite.

Definition 3.20. Let $\gamma_\mu = (F_f \subset \Gamma_0 \supset \Gamma'_1 \subset \ldots \supset \Gamma'_p \subset \Gamma_p \supset F_\mu)$ be a gallery joining 0 with μ. For each $0 \leq j \leq p$, let $\mathcal{H}_j$ be the set of reflection hyperplanes $\mathbf{H}$ such that $\Gamma'_j \subset \mathbf{H}$ and $\Gamma_j \not\subseteq \mathbf{H}$. We say γ_μ is *minimal* if all the faces of γ_μ are contained in $\mathbf{H}_\mu$, and if there is a disjoint union $\bigsqcup_{0\leq j<p} \mathcal{H}_j = \mathcal{M}_{\mathcal{A}_G}(F_f, F_\mu)$.

Let us fix a minimal gallery γ_μ joining 0 with μ, with associated gallery of types $t_{\gamma_\mu} = (S \supset t_0 \subset t'_1 \supset \ldots \supset t'_p \subset t_p \supset t_\mu)$. We denote by $\Gamma(\gamma_\mu)$ the set of all galleries of type t_{γ_μ} and of source F_f contained in the standard apartment $\mathcal{A}_G$. Such galleries are called *combinatorial of type* t_{γ_μ}. One can describe $\Gamma(\gamma_\mu)$ quite explicitly; recall that for some type t, we defined W_t as the subgroup of $W^{\mathfrak{a}}$ generated by t. For simplicity, we will also write $W_\mu := W_{t_\mu}$, and similarly $W_i := W_{t_i}$ and $W'_i := W_{t'_i}$. Then, by [GL05, Proposition 2], there is a bijection

$$W \times^{W_0} W'_1 \times^{W_1} \ldots \times^{W_{p-1}} W'_p/W_p \to \Gamma(\gamma_\mu),$$

sending an equivalence class $[\delta_0, \delta_1, \ldots, \delta_p]$ to the combinatorial gallery of type t_{γ_μ} given by $(F_f \subset \Sigma_0 \supset \Sigma'_1 \subset \ldots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu)$, where $\Sigma_j = \delta_0\delta_1 \ldots \delta_j F_{t_j}$.

Most important for us will be the subset of *positively folded combinatorial galleries.* Before we can explain their definition, note that we can (and will) assume that for any $[\delta_0, \delta_1, \ldots, \delta_p] \in W \times^{W_0} W'_1 \times^{W_1} \ldots \times^{W_{p-1}} W'_p/W_p$, each $\delta_j \in W'_j$ is the minimal length representative of its class in W'_j/W_j. Moreover, the minimal gallery γ_μ is represented by $[1, \tau_1^{\min}, \ldots, \tau_p^{\min}]$, where each $\tau_j^{\min} \in W'_j$ is the minimal length representative of the longest class in W'_j/W_j. The positively folded galleries are now defined as in [GL05, Definition 16].

Definition 3.21. Let $\delta = (F_f \subset \Sigma_0 \supset \Sigma'_1 \subset \ldots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu)$ be a combinatorial gallery in $\Gamma(\gamma_\mu)$ corresponding to $[\delta_0, \delta_1, \ldots, \delta_p]$, where each δ_j is a minimal representative of its class in W'_j/W_j.

(1) If $j \geq 1$ and $\delta_j \neq \tau_j^{\min}$, we say δ is *folded* around Σ'_j.

Now, consider for each $j \geq 1$, the combinatorial galleries

$$\gamma^{j-1} = [\delta_0, \ldots, \delta_{j-1}, \tau_j^{\min}, \ldots, \tau_p^{\min}] = (F_f \subset \ldots \subset \Sigma_{j-1} \supset \Sigma'_j \subset \Omega_j \supset \Omega'_{j+1} \subset \ldots)$$

and

$$\gamma^{j} = [\delta_0, \ldots, \delta_{j}, \tau_{j+1}^{\min}, \ldots, \tau_p^{\min}] = (F_f \subset \ldots \subset \Sigma_{j-1} \supset \Sigma'_j \subset \Sigma_j \supset \Sigma'_{j+1} \subset \ldots).$$

Then, by [GL05, Lemma 5], there exist positive roots $\beta_1, \ldots, \beta_q$ and integers $m_1, \ldots, m_q$ such that the small face Σ'_j is contained in $\bigcap_{i=1}^q \mathbf{H}_{\beta_i, m_i}$, and where $\Sigma_j = s_{\beta_q, m_q} \ldots s_{\beta_1, m_1}(\Omega_j)$.

(2) If δ is folded around Σ'_j, we say this folding is *positive* if $\Sigma_j \subset \bigcap_{i=1}^q \mathbf{H}^+_{\beta_i, m_i}$.

(3) The combinatorial gallery δ is *positively folded*, if all of its folds are positive.

We denote the subset of $\Gamma(\gamma_\mu)$ consisting of the positively folded combinatorial galleries by $\Gamma^+(\gamma_\mu)$.

As an example, γ_μ does not have any folds, so that it is automatically positively folded. We will also need the following definition.

Definition 3.22. Let $\delta = (F_f \subset \Sigma_0 \supset \Sigma'_1 \subset \ldots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu) \in \Gamma^+(\gamma_\mu)$ be a positively folded combinatorial gallery. A *load-bearing wall* for δ at Σ_j is a reflection hyperplane $\mathbf{H}$ such that $\Sigma'_j \subset \mathbf{H}$ and $\Sigma_j \nsubseteq \mathbf{H}$, and for which there exist an integer n_β for each positive root $\beta \in \Phi^+$ such that $\mathbf{H}$ separates Σ_j from $\bigcap_{\beta \in \Phi^+} \mathbf{H}^-_{\beta, n_\beta}$.

As δ was assumed positively folded, it follows from the definitions that any folding hyperplane is a load-bearing wall.

Finally, to each $\delta = [\delta_0, \ldots, \delta_p] \in \Gamma^+(\gamma_\mu)$, we will need to attach two sets of indices. For any affine root α of G, we denote by $\mathcal{U}_\alpha$ the corresponding root subgroup of LG.

Definition 3.23.

(1) For any parahorics $\mathcal{Q} \subseteq \mathcal{P}$ and $w \in W_{\mathcal{P}}/W_{\mathcal{Q}}$, we define the subsets of affine roots $R^+(w) := \{\alpha > 0 \mid \mathcal{U}_{w^{-1}(\alpha)} \nsubseteq \mathcal{Q}\}$ and $R^-(w) := \{\alpha < 0 \mid w(\alpha < 0), \mathcal{U}_\alpha \subseteq \mathcal{P}, \mathcal{U}_\alpha \nsubseteq \mathcal{Q}\}$.

(2) Using the same notation as above, we define $\mathcal{U}^+(w) := \prod_{\eta \in R^+(w)} \mathcal{U}_\eta$ and $\mathcal{U}^-(w) := \prod_{\theta \in R^-(w)} \mathcal{U}_\theta$.

(3) Let $\delta = [\delta_0, \ldots, \delta_p] \in \Gamma^+(\gamma_\mu)$ be a positively folded combinatorial gallery of type t_{γ_μ}. For any j, let $\mathcal{P}_j$ and $\mathcal{Q}_j$ be the parahoric subgroups containing $\mathcal{I}$ of type t'_j and t_j respectively. Consider the set of walls in $\mathcal{A}_G$ that contain $\mathcal{F}_{\mathcal{P}_j}$, but not $\delta_j F_{\mathcal{Q}_j}$. If we index this set by I_j, then I_j can be decomposed in as $I_j = I_j^+ \sqcup I_j^-$, such that $R^+(\delta_j) = \{\alpha_i \mid i \in I_j^+\}$, where α_i is the positive root corresponding to the wall $\mathbf{H}_i$, and there is a similar description for $R^-(\delta_j)$; cf. [GL05, §10]. Then, we define $J_{-\infty}(\delta) \subseteq \bigsqcup_{j=0}^p I_j$ as the subset corresponding to those walls that are load-bearing. We also define $J^\pm_{-\infty}(\delta) := J_{-\infty}(\delta) \cap (\bigsqcup_{i=0}^p I_j^\pm)$.

3.3.2. *Cellular stratifications of Bott–Samelson schemes.* In this subsection, we will apply the methods from [GL05], and explain how to generalize them to more general bases. By Remark 3.14, we can use notions that depend only on the standard apartment (instead of the whole affine building) over any base, and independently of the base. Examples of this are the standard apartment $\mathcal{A}_G = \mathcal{A}(G, T)$, and combinatorial galleries of a fixed type, possibly minimal or positively folded.

Fix some dominant $\mu \in X_*(T_{\mathrm{adj}})$, a minimal gallery γ_μ in $\mathcal{A}_G$ joining 0 with μ, and let

$$t_{\gamma_\mu} = (t'_0 \supset t_0 \subset t'_1 \supset \ldots \supset t_{j-1} \subset t'_j \supset t_j \subset \ldots t'_p \supset t_p \subset t_\mu)$$

be its gallery of types. For $0 \leq j \leq p$, let $\mathcal{P}_j \subseteq LG$ (resp. $\mathcal{Q}_j$, resp. $\mathcal{P}_\mu$) be the parahoric subgroup of type t'_j (resp. t_j, resp. t_μ) containing $\mathcal{I}$. Note that if μ is a dominant cocharacter, then $\mathcal{P}_\mu = L^+G$. In general, $\mathcal{P}_\mu$ is exactly the parahoric subgroup such that the sheaf quotient $LG/\mathcal{P}_\mu$ from Definition and Lemma 3.1 is LG-equivariantly universally homeomorphic to the connected component of $\mathrm{Gr}_{G_{\mathrm{adj}}}$ corresponding to the image of μ under $X_*(T_{\mathrm{adj}}) \to \pi_1(G_{\mathrm{adj}}) \cong \pi_0(\mathrm{Gr}_{G_{\mathrm{adj}}})$. This will allow us to reduce the combinatorics needed to the case of simply connected groups, cf. Lemma 3.31. For simplicity, we will denote the L^+G-orbit in $LG/\mathcal{P}_\mu$ corresponding to μ by Gr_G^μ, and its closure by $\mathrm{Gr}_G^{\leq \mu}$, but we emphasize that these are only subschemes of Gr_G when μ is an actual cocharacter of G, as opposed to a cocharacter of G_{adj}.

Definition 3.24. The *Bott–Samelson* scheme $\Sigma(\gamma_\mu)$ is the contracted product

$$\mathcal{P}_0 \times^{\mathcal{Q}_0} \mathcal{P}_1 \times^{\mathcal{Q}_1} \ldots \times^{\mathcal{Q}_{p-1}} \mathcal{P}_p/\mathcal{Q}_p.$$

Proposition 3.25. *The multiplication morphism (defined since $\mathcal{Q}_p \subseteq \mathcal{P}_\mu$)*

$$\psi : \Sigma(\gamma_\mu) \to \mathrm{Gr}_G^{\leq\mu}$$

is an isomorphism over the open subscheme Gr_G^μ.

Proof. Denote the restricted morphism by $\phi\colon \Sigma^\circ(\gamma_\mu) \to \mathrm{Gr}_G^\mu$, i.e., $\Sigma^\circ(\gamma_\mu) = \psi^{-1}(\mathrm{Gr}_G^\mu)$. Over algebraically closed fields, the proposition follows from [GL05, Lemma 10] (while they only show the restricted morphism is bijective, the Bott–Samelson resolutions are always birational, so that ϕ is an isomorphism by Zariski's Main Theorem [Mum99, III, §9, (I)] as the target is smooth and therefore normal). Let us show how this implies the statement over $\operatorname{Spec}\mathbf{Z}$.

Note that $\Sigma(\gamma_\mu)$ is an iterated sequence of Zariski-locally trivial fibrations with smooth fibers, and in particular smooth itself. By [Sta22, Tag 039D] and the result over algebraically closed fields, ϕ is flat. As its geometric fibers are isomorphisms, ϕ is hence smooth, and even étale. Bijectivity of ϕ also follows from the similar statement over algebraically closed fields, and one readily checks that ϕ is even universally bijective. We conclude by [Sta22, Tag 02LC] that ϕ is an isomorphism. □

Since these Bott-Samelson schemes are smooth projective, they are much better behaved with respect to $\mathbf{G}_\mathrm{m}$-actions. So we will first study the decomposition on $\Sigma(\gamma_\mu)$ induced by a certain action, and later restrict this decomposition to Gr_G^μ. Recall the notion of filtrable decompositions from Definition 3.11.

Definition and Lemma 3.26. Consider some regular anti-dominant cocharacter $\lambda \in X_*(T)$, and the induced $\mathbf{G}_\mathrm{m}$-action on $\Sigma(\gamma_\mu)$ via left-multiplication on the first factor. Then the connected components of the attractor locus are indexed by $\Gamma(\gamma_\mu)$, and we denote them by C_δ. These C_δ form a filtrable decomposition of $\Sigma(\gamma_\mu)$.

Proof. The attractor locus of $\Sigma(\gamma_\mu)$ is representable and smooth by [Ric19, Theorem 1.8 (iii)]. Since the Białynicki-Birula decomposition from [BB73, Theorem 4.4] agrees with the attractor locus when working over an algebraically closed base field, [Ric19, Corollary 1.16] tells us that the geometric fibres of the attractor locus are the schemes $C(\delta)$ from [GL05, Proposition 6]; in particular, the connected components of these geometric fibres are indexed by $\Gamma(\gamma_\mu)$. By smoothness, the same holds over $\operatorname{Spec}\mathbf{Z}$. The fact that the C_δ induce a decomposition of $\Sigma(\gamma_\mu)$ also follows from the claim over the geometric points from [GL05, Proposition 6].

To show that $\Sigma(\gamma_\mu) = \bigsqcup_{\delta\in\Gamma(\gamma_\mu)} C_\delta$ is a filtrable decomposition into locally closed subschemes, we can use the proof of [BB76, Theorem 3] (additionally using [Dri18, Lemma 1.4.9]), as soon as $\Sigma(\gamma_\mu)$ admits a $\mathbf{G}_\mathrm{m}$-equivariant embedding into some projective space with linear $\mathbf{G}_\mathrm{m}$-action. For GL_n, this is classical, while for general G this follows by choosing a faithful representation of G. □

The $C(\delta)$ from [GL05, Proposition 6] are affine spaces. In fact, the C_δ are already affine spaces over $\mathbf{Z}$, but we omit details as we will not need this.

By Proposition 3.25 and Definition and Lemma 3.26, we get an induced filtrable decomposition of $\mathrm{Gr}_G^\mu = \bigsqcup_{\delta\in\Gamma(\gamma_\mu)} X_\delta$, with $X_\delta := C_\delta \cap \mathrm{Gr}_G^\mu$. This is related to the semi-infinite orbits S_ν as follows:

Proposition 3.27. *For some $\nu \in X_*(T_\mathrm{adj})$, let $\Gamma(\gamma_\mu,\nu)$ denote those galleries in $\Gamma(\gamma_\mu)$ with target F_ν. Then we have a filtrable decomposition $S_\nu \cap \mathrm{Gr}_G^\mu = \bigsqcup_{\tau\in\Gamma(\gamma_\mu,\nu)} X_\tau$.*

Proof. The fact that $S_\nu \cap \mathrm{Gr}_G^\mu = \bigsqcup_{\tau\in\Gamma(\gamma_\mu,\nu)} X_\tau$ follows from the similar result over the geometric points of $\operatorname{Spec}\mathbf{Z}$, as in [GL05, Theorem 3]. The fact that this decomposition is filtrable then follows from Definition and Lemma 3.26, as Gr_G^μ is open in $\Sigma(\gamma_\mu)$. □

Lemma 3.28. *If $X_\delta \neq \varnothing$, then $\delta \in \Gamma^+(\gamma_\mu)$, i.e., δ is positively folded.*

Proof. Again, this follows from [GL05, Lemma 11] applied to the geometric points of $\operatorname{Spec}\mathbf{Z}$. □

In the proof of our main result, Theorem 3.32, the existence of a cellular decomposition will be reduced to considering galleries with only three faces. The following proposition is a step towards this case. We consider a *triple gallery of types* (t_{j-1}, t'_j, t_j) for some $0 \leq j \leq p$, and the parahoric subgroups $\mathcal{P}$ and $\mathcal{Q}$ containing $\mathcal{I}$ corresponding to t'_j and t_j respectively. Moreover, we let w be (the shortest representative of) some element in $W_\mathcal{P}/W_\mathcal{Q}$, and $\tau^{\min}$ the shortest representative of the longest class in $W_\mathcal{P}/W_\mathcal{Q}$. Recall also the groups $\mathcal{U}^+(w)$ and $\mathcal{U}^-(w)$ from Definition 3.23.

Proposition 3.29. *The intersection $\mathcal{U}^+(w)w\mathcal{U}^-(w) \cap \mathcal{U}^+(\tau^{\min})\tau^{\min}$ admits a stratification into products of $\mathbf{A}^1$'s and $\mathbf{G}_\mathrm{m}$'s.*

Proof. As in [GL05, Proposition 9], we are reduced to showing that for some $v \in W_{\mathcal{P}}^{\mathfrak{a}}$, the intersection $\mathcal{I}v^{-1} \cap \mathcal{I}^- \mathcal{Q}/\mathcal{Q}$ (inside $\mathcal{P}/\mathcal{Q}$) admits a stratification as in the statement of the proposition. This is shown using methods of Deodhar (although in the infinite-dimensional setting), and we postpone this to Corollary 3.41. $\square$

Modulo Corollary 3.41 alluded to in the previous proof, we now have all the ingredients to deduce the main result of this section. We begin with the simpler case where μ is regular. Then, for any $\delta = [\delta_0, \ldots, \delta_p] \in \Gamma(\gamma_\mu)$ and any $1 \leq i \leq p$, δ_i is either trivial, or a simple reflection [GL05, p.80]. There is a unique possibility for this simple reflection, and we denote the corresponding affine root by α_i.

Proposition 3.30. *Assume μ is regular, and let $\delta \in \Gamma^+(\gamma_\mu)$. Then $X_\delta \cong \mathbf{A}^k \times \mathbf{G}_{\mathrm{m}}^l$ for some $k, l \geq 0$.*

Proof. Since $\Sigma(\gamma_\mu)$ was defined as an iterated Zariski-locally trivial $\mathcal{P}_i/\mathcal{Q}_i$-fibration for some parahoric subgroups $\mathcal{Q}_i \subset \mathcal{P}_i$, the Bruhat stratification for each such quotient shows that $\Sigma(\gamma_\mu)$ is stratified by affine spaces, indexed by $W \times^{W_0} W_1' \times^{W_1} \ldots \times^{W_{p-1}} W_p'/W_p$, where the locally closed strata can be described via root groups. (We note that, although this index set is naturally in bijection with $\Gamma(\gamma_\mu)$, the resulting decomposition does *not* agree with the one from Definition and Lemma 3.26.)

Then, [GL05, Proposition 10], tells us that over any geometric point of $\operatorname{Spec} \mathbf{Z}$ and any $\delta = [\delta_0, \delta_1, \ldots, \delta_p] \in \Gamma(\gamma_\mu)$ (with each δ_i a minimal representative of its class in W_i'/W_i), C_δ is exactly the locally closed subscheme of $\Sigma(\gamma_\mu)$ given by

$$\delta_0 \cdot \prod_{\beta<0, \delta_0(\beta)<0} \mathcal{U}_\beta \cdot \prod_{i=1}^{p} \mathcal{U}_{\alpha_i}^{\bullet} \cdot \delta_i$$

where α_i is as above, $\mathcal{U}_{\alpha_i}^{\bullet}$ is defined as $\mathcal{U}_{\alpha_i}$ (resp. $\mathcal{U}_{-\alpha_i}^{\times}$, resp. $\{0\}$) when $i \in J_{-\infty}^+(\delta)$ (resp. $i \in J_{-\infty}^-(\delta)$, resp. $i \notin J_{-\infty}(\delta)$), and $J_{-\infty}(\delta) = J_{-\infty}^+(\delta) \sqcup J_{-\infty}^-(\delta)$ is as in Definition 3.23. In particular, the same description in terms of (punctured) root groups can be given over $\operatorname{Spec} \mathbf{Z}$, showing the proposition. $\square$

In general, X_δ will not be as simple, and we have to stratify it further using Proposition 3.29. Before we state our main result, let us compare Schubert cells in different affine Grassmannians. This will allow us to remove the assumption that G is semisimple or simply connected. In particular, we omit this assumption for the lemma. We will denote by G_{adj} the adjoint quotient of G, and by G_{sc} the simply connected cover of G_{adj}.

Lemma 3.31. *Let $\mu \in X_*(T)^+$ be a dominant cocharacter of G, and denote the induced dominant cocharacter of G_{adj} the same way. Then there exist universal homeomorphisms $\mathrm{Gr}_G^{\leq \mu} \to \mathrm{Gr}_{G_{\mathrm{adj}}}^{\leq \mu} \leftarrow \mathrm{Gr}_{G_{\mathrm{sc}}}^{\leq \mu}$, which are equivariant for the LG-, respectively the LG_{sc}-action. Moreover, these morphisms restrict to isomorphisms on the open Schubert cells.*

Proof. First, consider the morphism $\mathrm{Gr}_G \to \mathrm{Gr}_{G_{\mathrm{adj}}}$ induced by the quotient $G \to G_{\mathrm{adj}}$. It is clearly LG-equivariant, and restricts to a morphism $\mathrm{Gr}_G^{\leq \mu} \to \mathrm{Gr}_{G_{\mathrm{adj}}}^{\leq \mu}$, as both subschemes are defined as orbit closures. This latter morphism is proper, and a universal homeomorphism when restricted to certain geometric fibers of $\operatorname{Spec} \mathbf{Z}$ by [HR22, Proposition 3.5]. In particular, $\mathrm{Gr}_G^{\leq \mu} \to \mathrm{Gr}_{G_{\mathrm{adj}}}^{\leq \mu}$ is a universal homeomorphism, and it restricts to an isomorphism over $\mathrm{Gr}_G^{\mu} \to \mathrm{Gr}_{G_{\mathrm{adj}}}^{\mu}$, as both source and target are smooth over $\operatorname{Spec} \mathbf{Z}$.

Next, we note that LG_{sc} acts on $\mathrm{Gr}_{G_{\mathrm{adj}}}$ via the natural morphism $LG_{\mathrm{sc}} \to LG_{\mathrm{adj}}$. This realizes any connected component of $\mathrm{Gr}_{G_{\mathrm{adj}}}$ as a quotient, up to universal homeomorphism, of LG_{sc} by a hyperspecial parahoric subgroup; this follows as in the previous paragraph for the neutral connected component, and in general by conjugating L^+G_{sc} by a suitable element of $LG_{\mathrm{adj}}(\mathbf{Z})$. In particular, this identification is LG_{sc}-equivariant. Consequently, it restricts to a universal homeomorphism $\mathrm{Gr}_{G_{\mathrm{sc}}}^{\leq \mu} \to \mathrm{Gr}_{G_{\mathrm{adj}}}^{\leq \mu}$, and to an isomorphism $\mathrm{Gr}_{G_{\mathrm{sc}}}^{\mu} \to \mathrm{Gr}_{G_{\mathrm{adj}}}^{\mu}$ by a similar argument as above. $\square$

For the following theorem, we again omit the assumption that G is semisimple or simply connected.

Theorem 3.32. *Let μ and ν be cocharacters of G, with μ dominant. Then the intersection $\mathrm{Gr}_G^{\mu} \cap S_\nu$ admits a filtrable decomposition into cellular schemes.*

Proof. By Lemma 3.31, we may assume G simply connected, so that the results of this subsection apply. By Proposition 3.27, it is enough to prove that for each $\delta \in \Gamma(\gamma_\mu)$, the intersection $X_\delta = C_\delta \cap \mathrm{Gr}_G^{\mu}$ is cellular. Moreover, by Lemma 3.28, we may assume that $\delta \in \Gamma^+(\gamma_\mu)$.

Now, consider the immersion $C_\delta \to \Sigma(\gamma_\mu)$. As in the regular case, we need to determine the preimage X_δ of Gr_G^{μ} under this map. Over algebraically closed fields, this is done in [GL05, Theorem 4], by first identifying the points of $\Sigma(\gamma_\mu)$ with certain galleries in $\mathcal{J}^{\mathfrak{a}}$, and showing that the galleries corresponding to points in Gr_G^{μ} are exactly the minimal galleries. Then, splitting up δ into galleries of triples, the description of $\Sigma(\gamma_\mu)$ as an iterated Zariski-locally trivial fibration and Proposition 3.29 are used to decompose X_δ into locally closed subsets, which are iterated products of (punctured) vector bundles. By induction, each such iterated bundle is already trivial, i.e., a product of $\mathbf{A}^1$'s and $\mathbf{G}_{\mathrm{m}}$'s. Finally, these locally closed subsets can be described via root groups, their zero sections, and the complements of zero sections, using [GL05, Proposition 8 and Lemma 13]. As the result is independent of the choice of algebraically closed field, we can use the same method to deduce the result over $\operatorname{Spec}\mathbf{Z}$.

As we used an induction on the iterated Zariski-locally trivial fibration $\Sigma(\gamma_\mu)$, and as each induction step used an actual stratification by Proposition 3.29, the resulting cellular decomposition of X_δ is a stratification as well. □

Example 3.33. Let us work out what happens for the (non-simply connected) group PGL_2, whose simply connected cover is SL_2. Considering the natural identification $X_*(\mathrm{PGL}_2) \cong \mathbf{Z}$, let $\mu \in \mathbf{Z}_{\geq 0}$ be a dominant cocharacter of PGL_2, which lives in the standard apartment $\mathcal{A}_{\mathrm{SL}_2} = X_*(\mathrm{SL}_2) \otimes_{\mathbf{Z}} \mathbf{R} \cong \mathbf{R}$ of SL_2. Note that μ is (induced by) a cocharacter of SL_2 exactly when μ is even. Consider the unique minimal gallery

$$\gamma_\mu := (\{0\} \subset [0,1] \supset \{1\} \subset \ldots \supset \{\mu-1\} \subset [\mu-1,\mu] \supset \{\mu\})$$

joining 0 with μ. Noting that the affine Weyl group of SL_2 is generated by two reflections $s_0 : x \mapsto -x$ and $s_1 : x \mapsto 2-x$, the gallery of types of γ_μ is given by

$$\left(\{s_0\} \supset \varnothing \subset \{s_1\} \supset \ldots \subset \{s_{\mu-1(\mathrm{mod}\,2)}\} \supset \varnothing \subset \{s_{\mu(\mathrm{mod}\,2)}\}\right),$$

and there is a bijection

$$\Gamma(\gamma_\mu) \cong \langle s_0 \rangle \times \langle s_1 \rangle \times \ldots \times \langle s_{\mu-2(\mathrm{mod}\,2)} \rangle \times \langle s_{\mu-1(\mathrm{mod}\,2)} \rangle.$$

As the face corresponding to the empty type is the fundamental alcove, we see that under this bijection, γ_μ corresponds to $(1, s_1, s_0, \ldots)$, which is just a straight path in $\mathcal{A}_{\mathrm{SL}_2}$ from $\{0\}$ to $\{\mu\}$. The other combinatorial galleries $(\delta_0, \delta_1, \delta_2, \ldots)$ can be described as follows: they begin at 0, and if $\delta_0 = 1$, then they start in the positive direction (towards μ), otherwise they start in the opposite direction. After this, if $\delta_i = 1$, the path turns around, so that the $i-1$th and ith large faces in δ agree, otherwise the path continues in the same direction. However, at the points where the path turns around, there is a fold, which is positive exactly when the path is going into the negative direction, and turns to the positive direction. In particular, there is a unique positively folded combinatorial gallery in $\Gamma^+(\gamma_\mu)$, with source 0 and target ν, where $\nu \in X_*(T_{\mathrm{adj}})$ corresponds to an integer congruent to μ modulo 2, such that $-\mu \leq \nu \leq \mu$. Finally, we note that in this case, the index set $J_{-\infty}$ consists of those indices for which a path in $\mathcal{A}_{\mathrm{SL}_2}$ moves in the positive direction, and $J_{-\infty}^-$ those indices which correspond to a fold (necessarily positive). So, using the proof of Proposition 3.30, we see that $\mathrm{Gr}_G^\mu \cap S_{-\mu} = \operatorname{Spec}\mathbf{Z}$, that $\mathrm{Gr}_G^\mu \cap S_\mu \cong \mathbf{A}^\mu$, and that $\mathrm{Gr}_G^\mu \cap S_\nu \cong \mathbf{G}_{\mathrm{m}} \times \mathbf{A}^{\frac{\mu+\nu}{2}-1}$ when $-\mu < \nu < \mu$.

3.3.3. *Generalizing the Deodhar decomposition.* To finish the proof of Proposition 3.29, we will follow the method of Deodhar as in [Deo85]. Our goal will be to generalize this result to arbitrary bases, to the infinite-dimensional setting, and to more general parahoric subgroups. So let $\mathcal{Q}$ be as in Proposition 3.29. Then there is a Bruhat decomposition

$$LG/\mathcal{Q} = \bigsqcup_{w \in W^{\mathfrak{a}}/W_{\mathcal{Q}}} \mathcal{I}w \cdot \mathcal{Q},$$

where $W_{\mathcal{Q}}$ is the subgroup of $W^{\mathfrak{a}}$ generated by the type of $\mathcal{Q}$. Fix some $y \in W^{\mathfrak{a}}$, which we can assume is a minimal length representative of its class in $W^{\mathfrak{a}}/W_{\mathcal{Q}}$, together with a reduced expression $y = s_1 \cdot \ldots \cdot s_k$, where each s_i is a simple reflection (this can include the simple reflections not appearing in the finite Weyl group W). Let α_i be the (not necessarily distinct) simple roots corresponding to s_i. Denoting ${}^gA := gAg^{-1}$ for any g and A, we define $U_j := U^+ \cap {}^{s_j\cdot\ldots\cdot s_k}U^-$ and $U^j := U^+ \cap {}^{s_k\cdot\ldots\cdot s_j}U^+$, where U^+ (resp. U^-) denotes the pro-unipotent subgroup of LG corresponding to the positive (resp. negative) affine roots.

Lemma 3.34. *(1) U_j is the subgroup of U^+ generated by 1-parameter subgroups corresponding to the roots*

$$\{\alpha \in R^+ \mid s_k \ldots s_j(\alpha) \in R^-\} = \{\alpha_j, s_j(\alpha_{j+1}), \ldots, s_j \ldots s_{k-1}(\alpha_k)\}.$$

In particular, $U_j \cong \mathbf{A}^{k-j+1}$.

(2) $U_1 \supseteq^{s_1} U_2 \supseteq^{s_1 s_2} U_3 \supseteq \ldots \supseteq^{s_1\cdots s_k} U_{k+1} = 0$.

(3) U^j is the subgroup of U^+ generated by 1-parameter subgroups corresponding to the roots $\{\alpha \in R^+ \mid s_j \ldots s_k(\alpha) \in R^+\}$.

(4) $U^1 \subseteq U^2 \subseteq \ldots U^{k+1} = U^+$.

(5) For any j, we have $U^+ = U_j \cdot {}^{s_j\cdots s_k} U^j$, with a uniqueness of expression.

Proof. (1) is clear, using that $wU_\alpha w^{-1} = U_{w(\alpha)}$ for any $w \in W^{\mathfrak{a}}$ and $\alpha \in R$ (cf. Remark 3.15). (3) is similar to (1). (2) is again clear, with (4) being similar, and (5) follows from (1) and (3). □

Using the fixed isomorphisms $x_\alpha \colon \mathbf{G}_{\mathrm{a}} \to U_\alpha$ for all $\alpha \in R$, denote $s_\alpha := x_\alpha(1) \cdot x_{-\alpha}(-1) \cdot x_\alpha(1)$. The following lemma is the infinite-dimensional analogue of [Deo85, Lemma 2.1], and can be proven in a similar way.

Lemma 3.35. *Let $\alpha \in R^+$. For a ring R and $t \in R^\times$, there exists $h \in \mathcal{T}(R)$ such that*

$$x_\alpha(t^{-1}) \cdot s_\alpha \cdot x_\alpha(t) = h \cdot s_\alpha \cdot x_\alpha(-t^{-1}) \cdot s_\alpha^{-1}.$$

The following notion will be used to parametrize certain cells in $\mathcal{I}y \cdot \mathcal{Q}/\mathcal{Q}$.

Definition 3.36. A *subexpression* is a sequence $\underline{\sigma} = (\sigma_0, \dots, \sigma_k)$ of elements of $W^{\mathfrak{a}}/W_{\mathcal{Q}}$ such that, denoting by σ_i^m the unique minimal length representative of σ_i (cf. [Ric13, Lemma 1.6]), the following holds:

(1) $\sigma_0^m = \mathrm{id}$, and
(2) $(\sigma_{j-1}^m)^{-1}\sigma_j^m \in \{\mathrm{id}, s_j\}$ for each $1 \le j \le k$.

We denote the set of such subexpressions by $\mathcal{S}$. An element $\underline{\sigma} \in \mathcal{S}$ is called a *distinguished subexpression* if it satisfies the following additional condition:

(3) $\sigma_j^m \le \sigma_{j-1}^m s_j$ for each $1 \le j \le k$.

We denote the set of all distinguished subexpressions by $\mathcal{D}$.

If $x \in X$ is a point of a scheme, we will usually identify it with the map $\operatorname{Spec} k(x) \to X$.

Lemma 3.37. *For a point $u_1 \in U_1$ and $0 \le j \le k$, let $\sigma_j \in W^{\mathfrak{a}}/W_{\mathcal{Q}}$ be the unique element such that $u_1 s_1 \dots s_j \in \mathcal{I}^- \sigma_j \mathcal{Q}$. Then $\underline{\sigma} = (\sigma_0, \dots, \sigma_k)$ is a distinguished subexpression.*

Proof. First, we note that for any $w \in W^{\mathfrak{a}}$ and simple s, we have $w\mathcal{Q}s \subset \mathcal{I}^- ws\mathcal{Q} \cup \mathcal{I}^- w\mathcal{Q}$. Moreover, if $w\mathcal{Q}s \cap \mathcal{I}^- w\mathcal{Q} \neq \varnothing$, then $l(ws) \ge l(w)$; this variant of the (T3) condition for Tits systems can be proven in a similar way, see e.g. [Bor91, 21.15] for the finite-dimensional case.

Clearly, $\sigma_0^m = \mathrm{id}$, so consider $1 \le j \le k$. Then by induction, $u_1 s_1 \dots s_{j-1} s_j \in \mathcal{I}^- \sigma_{j-1}^m \mathcal{Q} s_j$. So, by the previous paragraph, we have either $u_1 s_1 \dots s_j \in \mathcal{I}^- \sigma_{j-1}^m s_j \mathcal{Q}$ or $u_1 s_1 \dots s_j \in \mathcal{I}^- \sigma_{j-1}^m \mathcal{Q}$, the latter of which can only happen when $l(\sigma_{j-1}^m s_j) \ge l(\sigma_{j-1}^m)$. The conditions for $\underline{\sigma}$ being a distinguished subexpression are now clearly fulfilled. $\square$

In particular, this gives a set-theoretic map $\eta \colon U_1 \to \mathcal{D}$.

Lemma 3.38. *For each $\underline{\sigma} \in \mathcal{D}$ and $1 \le j \le k$, define $\Omega(\underline{\sigma}, j) \subseteq \mathbf{A}^1$ as*

$$\Omega(\underline{\sigma}, j) = \begin{cases} \mathbf{A}^1 & \text{if } \sigma_{j-1}^m > \sigma_j^m \\ \operatorname{Spec} \mathbf{Z} & \text{if } \sigma_{j-1}^m < \sigma_j^m \\ \mathbf{G}_{\mathrm{m}} & \text{if } \sigma_{j-1}^m = \sigma_j^m. \end{cases} \tag{3.3}$$

Then for each j there is a locally closed immersion $f_j \colon \Omega(\underline{\sigma}, j) \times U_{j+1} \to U_j$, such that for any ring R, any $t \in \Omega(\underline{\sigma}, j)(R)$ and any $u_{j+1} \in U_{j+1}(R)$, there exist $b_j^- \in \mathcal{I}^-(R)$ and $v_{j+1} \in U^{j+1}(R)$ such that

$$\sigma_{j-1}^m f_j(t, u_{j+1}) s_j \cdot \ldots \cdot s_k = b_j^- \sigma_j^m u_{j+1} s_{j+1} \cdot \ldots \cdot s_k \cdot v_{j+1}. \tag{3.4}$$

Proof. If $\sigma_{j-1}^m > \sigma_j^m$, then $\sigma_j^m = \sigma_{j-1}^m s_j$, and $\Omega(\underline{\sigma}, j) = \mathbf{A}^1$. Denoting the simple root corresponding to the simple reflection s_j by α_j, we have $\sigma_j^m(\alpha_j) \in R^+$, and we define $f_j : \mathbf{A}^1 \times U_{j+1} \to U_j$ on R-valued points by $f_j(t, u_{j+1}) = x_{\alpha_j}(t) \cdot s_j u_{j+1} s_j^{-1}$. Then f_j is an isomorphism, and the conditions of the proposition are satisfied.

If $\sigma_{j-1}^m < \sigma_j^m$, then again $\sigma_j^m = \sigma_{j-1}^m s_j$, but this time we have $\Omega(\underline{\sigma}, j) = \operatorname{Spec} \mathbf{Z}$ (the zero section of $\mathbf{A}^1$). So now we define $f_j(0, u_{j+1}) = s_j u_{j+1} s_j^{-1}$ (again on R-valued points). This is clearly an isomorphism onto the closed subscheme ${}^{s_j}U_{j+1} \subseteq U_j$, and again (3.4) is satisfied.

Finally, assume $\sigma_{j-1}^m = \sigma_j^m$, in which case we have $\Omega(\underline{\sigma}, j) = \mathbf{G}_{\mathrm{m}}$. Moreover, $l(\sigma_{j-1}^m s_j) \ge l(\sigma_{j-1}^m)$ and hence $\sigma_{j-1}^m(\alpha_j) \in R^+$. For any ring R, any $t \in R^\times = \mathbf{G}_{\mathrm{m}}(R)$, and any $u_{j+1} \in U_{j+1}(R)$, Lemma 3.34 (5) gives unique $\tilde{u}_{j+1} \in U_{j+1}(R)$ and $v_{j+1} \in U^{j+1}(R)$ such that

$$x_{\alpha_j}(t) \cdot u_{j+1} = \tilde{u}_{j+1} \cdot s_{j+1} \dots s_k v_{j+1}^{-1} s_k^{-1} \dots s_{j+1}^{-1}. \tag{3.5}$$

Then we define $f_j : \mathbf{G}_{\mathrm{m}} \times U_{j+1} \to U_j$ as $f_j(t, u_{j+1}) := x_{\alpha_j}(t^{-1}) s_j \tilde{u}_{j+1} s_j^{-1} \in U_j(R)$. Note that f_j is injective on R-valued points, hence a monomorphism, and that f_j is a group homomorphism. Moreover, the image of f_j is the open $U_j \setminus {}^{s_j}U_{j+1}$, and f_j is an isomorphism onto its image: consider $x_{\alpha_j}(d) \cdot s_j \cdot \tilde{u}_{j+1} s_j^{-1}$, for some ring R, some $d \in R^\times$ and $\tilde{u}_{j+1} \in U_{j+1}(R)$. Then by (3.5), we have

$$x_{\alpha_j}(-d^{-1}) \cdot \tilde{u}_{j+1} = u_{j+1} s_{j+1} \dots s_k v_{j+1} \cdot s_k^{-1} \dots s_{j+1}^{-1}$$

for some $u_{j+1} \in U_{j+1}(R)$, so that $f_j(d^{-1}, u_{j+1}) = x_{\alpha_j}(d) \cdot s_j \tilde{u}_{j+1} \cdot s_j^{-1}$. We are left to check (3.4). For this, note that

$$\sigma_{j-1}^m f_j(t, u_{j+1}) \cdot s_j \dots s_k = \sigma_{j-1}^m \cdot x_{\alpha_j}(t^{-1}) \cdot s_j \tilde{u}_{j+1} s_{j+1} \dots s_k.$$

By (3.5) and Lemma 3.35 we get

$$\begin{aligned} \sigma_{j-1}^m f_j(t, u_{j+1}) \cdot s_j \dots s_k &= \sigma_{j-1}^m x_{\alpha_j}(t^{-1}) \cdot s_j \cdot x_{\alpha_j}(t) u_{j+1} s_{j+1} \dots s_k \cdot v_{j+1} \\ &= \sigma_{j-1}^m h s_j x_{\alpha_j}(-t^{-1}) \cdot s_j^{-1} (\sigma_{j-1}^m)^{-1} \sigma_{j-1}^m u_{j+1} s_{j+1} \dots s_k \cdot v_{j+1}, \end{aligned}$$

for some $h \in \mathcal{T}(R)$. But then we are done: $\sigma_{j-1}^m h s_j x_{\alpha_j}(-t^{-1}) s_j^{-1} (\sigma_{j-1}^m)^{-1} \in \mathcal{I}^-(R)$ as $\sigma_{j-1}^m(\alpha_j) \in R^+$, and because $\sigma_{j-1}^m = \sigma_j^m$ in this case. $\square$

The following proposition is the infinite-dimensional analogue of [Deo85, Theorem 1.1], although allowing more general parahorics than just the minimal ones, and a more general base.

Proposition 3.39. *There is a decomposition* $\mathcal{I}y \cdot \mathcal{Q} = \bigsqcup_{\underline{\sigma} \in \mathcal{D}} \mathbf{A}^{m(\underline{\sigma})} \times \mathbf{G}_{\mathrm{m}}^{n(\underline{\sigma})}$. *Moreover, for each* $\underline{\sigma} \in \mathcal{D}$, *there is an* $x \in W^{\mathfrak{a}}/W_{\mathcal{Q}}$ *such that the associated* $\mathbf{A}^{m(\underline{\sigma})} \times \mathbf{G}_{\mathrm{m}}^{n(\underline{\sigma})}$ *is contained in* $\mathcal{I}^{-}x \cdot \mathcal{Q}$.

Proof. Using that the Bruhat cell $\mathcal{I}y \cdot \mathcal{Q}$ can be identified with U_1 (cf. e.g. the proof of [RS20, Proposition 4.3.9]), let $D_{\underline{\sigma}}$ be the preimage of $\underline{\sigma}$ under the map $\eta \colon U_1 \to \mathcal{D}$ defined by Lemma 3.37. It is clear that we have a disjoint union $\mathcal{I}y \cdot \mathcal{Q} = \bigcup_{\mathcal{D}} D_{\underline{\sigma}}$, so fix some $\underline{\sigma} = (\sigma_0, \dots, \sigma_k) \in \mathcal{D}$.

For $1 \leq j \leq k+1$, we define locally closed subschemes $A_j \subseteq U_j$ inductively as follows: first, we define $A_{k+1} := U_{k+1} = 0$, and for $j \leq k$, we let $A_j := f_j(\Omega(\underline{\sigma}, j) \times A_{j+1})$, where f_j is the map constructed in Lemma 3.38. It is clear that A_1 is isomorphic to $\mathbf{A}^{m(\underline{\sigma})} \times \mathbf{G}_{\mathrm{m}}^{n(\underline{\sigma})}$, where $m(\underline{\sigma})$ and $n(\underline{\sigma})$ can be determined explicitly by (3.3). So we are left to show that $A_1 = D_{\underline{\sigma}}$.

For a point $u_1 \in A_1$, let k be its residue field. Then we can inductively find points $u_j : \operatorname{Spec} k \to A_j$ and $t_j : \operatorname{Spec} k \to \Omega(\underline{\sigma}, j)$ such that $u_j = f_j(t_j, u_{j+1})$. By Lemma 3.38 one then sees that $u_1 s_1 \dots s_j \in \mathcal{I}^{-}\sigma_j \cdot \mathcal{Q}$ for each j, so that $u_1 \in D_{\underline{\sigma}}$.

Conversely, let $u_1 \in D_{\underline{\sigma}}$. We want to inductively find $u_j \in U_j$ and $t_j \in \Omega(\underline{\sigma}, j)$, such that $u_j = f_j(t_j, u_{j+1})$ for each j. Recall that the image of $f_j : \Omega(\underline{\sigma}, j) \times U_{j+1} \to U_j$ is U_j if $\sigma_{j-1} > \sigma_j$, ${}^{s_j}U_{j+1}$ if $\sigma_{j-1} < \sigma_j$, and $U_j \setminus {}^{s_j}U_{j+1}$ if $\sigma_{j+1} = \sigma_j$. Now, $u_1 \in D_{\underline{\sigma}}$ implies that $u_1 \in \operatorname{im}(f_1)$. So we can find unique t_1 and u_2 such that $f_1(t_1, u_2) = u_1$. Similarly, by some case distinctions, one can prove that each such $u_2 \in \operatorname{im}(f_2)$, as otherwise $u_1 \notin D_{\underline{\sigma}}$, and hence inductively find u_j and t_j as desired. Moreover, we have $u_{k+1} \in A_{k+1}$, so that by definition of A_j, we get $u_j \in A_j$ by induction. This shows that $u_1 \in A_1$, so that $A_1 = D_{\underline{\sigma}}$, which concludes the first part of the proposition.

For the second part, one easily checks that $D_{\underline{\sigma}} \subset \mathcal{I}^{-}\sigma_k \cdot \mathcal{Q}$, using the definition of the map $\eta \colon U_1 \to \mathcal{D}$. ☐

Remark 3.40. Unfortunately, the decompositions obtained by Deodhar in the finite-dimensional setting fail to satisfy the closure relations, cf. [Dud08] for a counterexample. In particular, we don't expect the decomposition from Proposition 3.39 to be a stratification. Fortunately, we can refine the decomposition into a stratification that still satisfies the required property. This finally finishes the proof of Proposition 3.29, and hence the proof of Theorem 3.32.

Corollary 3.41. *For any* $x, y \in W^{\mathfrak{a}}/W_{\mathcal{Q}}$, *the intersection* $(\mathcal{I}y \cap \mathcal{I}^{-}x) \cdot \mathcal{Q}$ *inside* $LG/\mathcal{Q}$ *admits a stratification into products of* $\mathbf{A}^1$*'s and* $\mathbf{G}_{\mathrm{m}}$*'s.*

Proof. We keep the notation of Proposition 3.39. Consider first the coset $U_1 = \mathcal{I}x \cdot \mathcal{Q} \subseteq LG/\mathcal{Q}$. The proofs of Lemma 3.38 and Proposition 3.39 realize U_1 as a k-fold product $\prod_{i=1}^{k} \mathbf{A}_i^1$ of affine lines, together with a decomposition $\mathbf{A}_i^1 = (\mathbf{G}_{\mathrm{m}})_i \sqcup (\operatorname{Spec} \mathbf{Z})_i$ of these affine lines, such that for any $\underline{\sigma} \in \mathcal{D}$, the component $D_{\underline{\sigma}}$ is isomorphic to a product of $\mathbf{A}^1$'s, $\mathbf{G}_{\mathrm{m}}$'s and $\operatorname{Spec} \mathbf{Z}$'s as above, one for each $1 \leq i \leq k$. In particular, the intersection $X := (\mathcal{I}y \cap \mathcal{I}^{-}x) \cdot \mathcal{Q}$ is the disjoint union of such products inside the affine space $U_1 \cong \mathbf{A}^k$.

Now, we apply the proof of [Sta22, Tag 09Y5], although slightly modified for our purpose. Namely, let $Z \subseteq X$ be an irreducible component. As $X = \bigcup \overline{D_{\underline{\sigma}}}$, where the $\underline{\sigma}$ range over a certain subset of $\mathcal{D}$, we have $Z \subseteq \overline{D_{\underline{\tau}}}$ for some $\underline{\tau} \in \mathcal{D}$. As $D_{\underline{\tau}}$ is locally closed in X, we see that $Z \cap D_{\underline{\tau}}$ contains an open subset of Z, so that $Z \cap D_{\underline{\tau}}$ must contain an open subset $V \neq \varnothing$ of X. Moreover, we can choose V to be a product of $\mathbf{A}_i^1$'s, $(\mathbf{G}_{\mathrm{m}})_i$'s, and $(\operatorname{Spec} \mathbf{Z})_i$'s, one for each i. Now, write $D_{\underline{\tau}} = V \sqcup D_{\underline{\tau}}^{+} \sqcup D_{\underline{\tau}}^{-}$, where $D_{\underline{\tau}}^{+} = (D_{\underline{\tau}} \setminus V) \cap \overline{V}$, and $D_{\underline{\tau}}^{-} = (D_{\underline{\tau}} \setminus V) \cap \overline{V}^c$. On the other hand, for $\underline{\sigma} \neq \underline{\tau}$, let us write $D_{\underline{\sigma}}^{+} = D_{\underline{\sigma}} \cap \overline{V}$ and $D_{\underline{\sigma}}^{-} = D_{\underline{\sigma}} \cap \overline{V}^c$, so that $D_{\underline{\sigma}} = D_{\underline{\sigma}}^{+} \sqcup D_{\underline{\sigma}}^{-}$. Now while these $D_{\underline{\sigma}}^{\pm}$'s (where $\underline{\sigma}$ is allowed to be $\underline{\tau}$ again) are not necessarily a product of $\mathbf{A}_i^1$'s, $(\mathbf{G}_{\mathrm{m}})_i$'s and $(\operatorname{Spec} \mathbf{Z})_i$'s, they are the disjoint union of such products; let us write $D_{\underline{\sigma}}^{\pm, j}$ for the subschemes appearing in this disjoint union. Then

$$X \setminus V = \bigsqcup D_{\underline{\sigma}}^{\pm, j},$$

where all the subschemes appearing are products of $\mathbf{A}_i^1$'s, $(\mathbf{G}_{\mathrm{m}})_i$'s and $(\operatorname{Spec} \mathbf{Z})_i$'s, one for each $1 \leq i \leq k$. Moreover, $\overline{V} = \bigsqcup D_{\underline{\sigma}}^{+, j}$. Finally, we note that $X \setminus V$ is closed in X and strictly smaller, so we may apply noetherian induction to find a stratification of $X \setminus V$ of the desired form. Adding V then gives a cellular stratification of X. ☐

3.3.4. *The case of rank 1 Levi subgroups.* Let us generalize Theorem 3.32, by replacing the semi-infinite orbits by the stratification of Gr_G induced by a more general Levi subgroup M of G. While we suspect the result will hold for arbitrary Levi subgroups, we only prove it for Levi subgroups of semisimple rank one, which will be enough for our purposes.

Let α be any positive simple root of G, with associated parabolic $P \subseteq G$ containing B, and its Levi quotient M. Consider the Borel $Q = \mathrm{im}(B \to M)$ of M, together with the diagram

$$\begin{tikzcd} & & \mathrm{Gr}_B \arrow[dl] \arrow[dr] & & \\ & \mathrm{Gr}_Q \arrow[dl] \arrow[dr] & & \mathrm{Gr}_P \arrow[dl] \arrow[dr] & \\ \mathrm{Gr}_T & & \mathrm{Gr}_M & & \mathrm{Gr}_G, \end{tikzcd}$$

where the square is cartesian, cf. Proposition 4.10. For any cocharacters $\mu, \lambda \in X_*(T)$, where μ is dominant for G and λ dominant for M, we want to understand the intersection of the preimages of Gr_G^μ and Gr_M^λ inside Gr_P, and more specifically the restriction of $\mathrm{Gr}_P \to \mathrm{Gr}_M$ to this intersection. As $\mathrm{Gr}_Q \to \mathrm{Gr}_M$ is a stratification, it makes sense to study the similar restriction of $\mathrm{Gr}_B \to \mathrm{Gr}_Q$ first.

Now, while we would like to use similar combinatorial methods as before, the Levi subgroup M is not semisimple. So we will also need its derived subgroup M_{der}, which is simply connected by [Ste75, Theorem 1 and Remark 2.13] (as we still assume G is simply connected). We can now work with the standard apartment $\mathcal{A}_{M_{\mathrm{der}}}$ of M_{der}; however, we will need to be careful as M_{der} has less cocharacters than M.

Let $\pi : \mathcal{A}_G \to \mathcal{A}_{M_{\mathrm{der}}}$ be the projection given by modding out the kernel of α, viewed as a functional on $\mathcal{A}_G$; cf. also [BT72, §7.6]. In particular, the image of any alcove or face in $\mathcal{A}_G$ under π is an alcove or face in $\mathcal{A}_{M_{\mathrm{der}}}$, and the image of any reflection hyperplane is either a reflection hyperplane, or the whole apartment $\mathcal{A}_{M_{\mathrm{der}}}$. We also fix some dominant $\mu \in X_*(T_{\mathrm{adj}})$, and a minimal gallery γ_μ in $\mathcal{A}_G$ joining 0 with μ.

Definition and Lemma 3.42. For any combinatorial gallery $\delta \in \Gamma(\gamma_\mu)$, there is a natural combinatorial gallery $\pi(\delta)$ in $\mathcal{A}_{M_{\mathrm{der}}}$, which is positively folded if δ is.

Proof. Let $\delta = (F_f \subset \Sigma_0 \supset \Sigma'_1 \subset \ldots \supset \Sigma'_p \subset \Sigma_p \supset F_\nu)$, and consider the naive projection

$$(\pi(F_f) \subseteq \pi(\Sigma_0) \supseteq \pi(\Sigma'_1) \subseteq \ldots \supseteq \pi(\Sigma'_p) \subseteq \pi(\Sigma_p) \supseteq \pi(F_\nu)).$$

While this is not necessarily a gallery in $\mathcal{A}_{M_{\mathrm{der}}}$, we do know that each $\pi(\Sigma'_i)$ is either equal to both $\pi(\Sigma_{i-1})$ and $\pi(\Sigma_i)$, or it is a codimension one subface of both; this uses that $\mathcal{A}_{M_{\mathrm{der}}}$ is one-dimensional. In particular, replacing the triples $\pi(\Sigma_{i-1}) = \pi(\Sigma'_i) = \pi(\Sigma_i)$ in this sequence by $\pi(\Sigma_i)$, we do get a gallery in $\mathcal{A}_{M_{\mathrm{der}}}$, which we denote by $\pi(\delta)$. It is clear that if δ is positively folded, then the same holds for $\pi(\delta)$. Let λ be the dominant cocharacter of M_{adj} corresponding to the length of $\pi(\delta)$. As $\mathrm{M}_{\mathrm{der}}$ has rank 1, there is a unique minimal gallery γ_λ in $\mathcal{A}_{M_{\mathrm{der}}}$ joining 0 with λ. It is then clear that $\pi(\delta)$ is a combinatorial gallery of type t_{γ_λ}. $\square$

For any $\delta \in \Gamma(\gamma_\mu, \nu)$, we would like to say that, using the notation X_δ from Definition and Lemma 3.26, there is a natural map $X_\delta \to X_{\pi(\delta)}$, which agrees with the restriction of the natural projection $\mathrm{Gr}_B \to \mathrm{Gr}_M$. However, here we have to be careful about which connected component $X_\delta \subseteq \mathrm{Gr}_G^\mu \cap S_\nu \subseteq \mathrm{Gr}_P$ will map to in Gr_M. As the semi-infinite orbits of both Gr_G and Gr_M are defined as the fibers of the maps $\mathrm{Gr}_B \to \mathrm{Gr}_T$ and $\mathrm{Gr}_Q \to \mathrm{Gr}_T$ respectively, we see that $X_\delta \subseteq \mathrm{Gr}_P$ will map to the connected component that contains the semi-infinite orbit corresponding to ν. So, for the following proposition, we consider $X_{\pi(\delta)} \subseteq C_{\pi(\delta)}$, but with a natural closed immersion into the connected component of Gr_Q corresponding to ν, via Lemma 3.31. (Note that, technically speaking, this only works if ν is an actual cocharacter; in general we need to look at the adjoint quotient of M.)

Proposition 3.43. *For $\delta \in \Gamma^+(\gamma_\mu, \nu)$ as above, the projection $\mathrm{Gr}_B \to \mathrm{Gr}_Q$ restricts to a morphism $X_\delta \to X_{\pi(\delta)}$, which can be stratified (on the source) into products of geometric vector bundles and punctured geometric vector bundles.*

Proof. Let λ be the cocharacter of M_{adj} corresponding to the length of $\pi(\delta)$, as in the proof of Definition and Lemma 3.42.

Let us first assume μ is regular, in which case λ is regular (i.e., nonzero) as well. In that case, we can describe X_δ and $X_{\pi(\delta)}$ explicitly via root groups, by Proposition 3.30. As the set of roots appearing for $X_{\pi(\delta)}$ is a subset of the roots appearing for X_δ, by definition of the index sets $J^+_{-\infty}$ and $J^-_{-\infty}$, and as the image of a load-bearing wall under the projection π is either a load-bearing wall or the whole apartment $\mathcal{A}_{M_{\mathrm{der}}}$, there is a natural projection $X_\delta \to X_{\pi(\delta)}$, which is the product of (punctured) geometric vector bundles. As the projection $P \to M$ is obtained by modding out similar root groups, we see that $X_\delta \to X_{\pi(\delta)}$ agrees with the restriction of $\mathrm{Gr}_B \to \mathrm{Gr}_Q$.

For the general case, note that by Example 3.33, we do not need to use Proposition 3.29 for the cellular decomposition of $\mathrm{Gr}_M^\lambda \cap S_\nu = X_{\pi(\delta)}$. Then we can argue similarly as in the regular case, using that at any step in the gallery where an $\mathbf{A}^1$ or $\mathbf{G}_{\mathrm{m}}$ appears for $X_{\pi(\delta)}$, every stratum appearing in the Deodhar decomposition at this step of X_δ naturally lives over either $\mathbf{A}^1$ or $\mathbf{G}_{\mathrm{m}}$. $\square$

For the next theorem, we again omit the assumption that G is semisimple or simply connected.

Theorem 3.44. *Let $\mu, \lambda \in X_*(T)$ be two cocharacters, with μ dominant for G, and λ dominant for M. Consider the intersection $\mathrm{Gr}_P^{\mu,\lambda}$ of the preimages of Gr_G^{μ} and Gr_M^{λ} in Gr_P. Then the restriction of $\mathrm{Gr}_P \to \mathrm{Gr}_M$ to $\mathrm{Gr}_P^{\mu,\lambda}$ admits a filtrable decomposition into cellular Gr_M^{λ}-schemes.*

Proof. By Lemma 3.31, we can assume G is semisimple and simply connected. As the morphism $\mathrm{Gr}_P^{\mu,\lambda} \to \mathrm{Gr}_M^{\lambda}$ is L^+M-equivariant, and this action on Gr_M^{λ} is transitive, it is enough to show the theorem over an open subset of Gr_M^{λ}. But then we can use Proposition 3.43 and the open subscheme $\mathrm{Gr}_M^{\lambda} \cap S_\lambda \subseteq \mathrm{Gr}_M^{\lambda}$, where we take the union of all combinatorial galleries $\delta \in \Gamma(\gamma_\mu)$ such that X_δ maps to $\mathrm{Gr}_M^{\lambda} \cap S_\lambda$. The fact that the resulting decomposition is filtrable follows from the similar result as for the torus T, by Theorem 3.32. ☐

4. Beilinson–Drinfeld Grassmannians and convolution

4.1. **Beilinson–Drinfeld Grassmannians.** In this section, we collect basic geometric information about the Beilinson–Drinfeld affine Grassmannians, following [Zhu17b, §3] and [HR18b, §3] in the case of constant group schemes.

Let $\mathcal{G}$ be a smooth affine group scheme over S and let $X := \mathbf{A}^1_S$. Then we have a distinguished S-point $\{0\} \in X(S)$. To define the Beilinson–Drinfeld Grassmannians, we first introduce some notation for working with étale $\mathcal{G}$-torsors. For $\mathrm{Spec}(R) \in \mathrm{AffSch}_S$, let $X_R := X \times_S \mathrm{Spec}(R)$. If $x\colon \mathrm{Spec}(R) \to X$ is a morphism, we denote the graph of x by $\Gamma_x \subset X_R$. We fix a trivial $\mathcal{G}$-torsor $\mathcal{E}_0$ on X, and for any $\mathrm{Spec}(R) \in \mathrm{AffSch}_S$ we also denote its base-change to X_R by $\mathcal{E}_0$.

Note that any $\mathrm{Spec}(R) \in \mathrm{AffSch}_S$ can be viewed as an X-scheme lying over 0 by composing with inclusion $S \xrightarrow{0}$ X. In the following proposition we make this identification. A theorem of Beauville and Laszlo [BL95] implies that $\mathrm{Gr}_{\mathcal{G}}$ (defined in (3.1)) has the following moduli interpretation, cf. [HR18b, Example 3.1 (i)].

Proposition 4.1. *There is a canonical isomorphism of étale sheaves*

$$\mathrm{Gr}_{\mathcal{G}}(R) \cong \left\{(\mathcal{E}, \beta) \,:\, \mathcal{E} \textit{ is a } \mathcal{G}\textit{-torsor on } X_R,\ \beta\colon \mathcal{E}\big|_{X_R - \Gamma_0} \cong \mathcal{E}_0\big|_{X_R-\Gamma_0}\right\}.$$

Let I be a nonempty finite set. For a point $x = (x_i) \in X^I(R)$, let

$$\Gamma_x = \bigcup_{i \in I} \Gamma_{x_i} \subset X_R.$$

Definition 4.2. The Beilinson–Drinfeld Grassmannian for $\mathcal{G}$ over X^I is the functor

$$\mathrm{Gr}_{\mathcal{G},I}(R) = \left\{(x, \mathcal{E}, \beta) \,:\, x \in X^I(R),\ \mathcal{E} \text{ is a } \mathcal{G}\text{-torsor on } X_R,\ \beta\colon \mathcal{E}\big|_{X_R-\Gamma_x} \cong \mathcal{E}_0\big|_{X_R-\Gamma_x}\right\}.$$

Remark 4.3. The definition of $\mathrm{Gr}_{\mathcal{G},I}$ makes sense for general smooth curves X, but the existence of the necessary t-structure on $\mathrm{DTM}(X^I)$ is not known in general. In future work we hope to extend the results in this paper to other curves for which the t-structure is known to exist, such as $X = \mathbf{P}^1$.

Remark 4.4. Our definition of $\mathrm{Gr}_{\mathcal{G},I}$ is the following specialization of [HR18b, Eqn. (3.1)]. In the notation of loc. cit., let $\mathrm{Spec}(\mathcal{O}) = \mathbf{A}^I_S$ and $X = \mathbf{A}^1_{\mathcal{O}}$. If $\mathrm{Spec}(\mathcal{O})$ has affine coordinate functions x_i for $i \in I$ and X has the affine coordinate function t, then let D be the divisor on X defined locally by the ideal $\prod_{i\in I}(t - x_i)$. After identifying pairs $(\mathrm{Spec}(R) \in \mathrm{AffSch}_S, x \in \mathbf{A}^I_S(R))$ with objects in $\mathrm{AffSch}_{\mathrm{Spec}(\mathcal{O})}$, Definition 4.2 agrees with $\mathrm{Gr}_{(X,\mathcal{G}\times X,D)}$ as defined in [HR18b, Eqn. (3.1)].

Lemma 4.5. *If $\mathcal{G} = G$ is a split reductive group, $\mathrm{Gr}_{G,I}$ is represented by an ind-projective scheme over X^I. If $\mathcal{G} = P$ is a standard parabolic subgroup of G, $\mathrm{Gr}_{P,I}$ is represented by an ind-scheme of ind-finite type over X^I.*

Proof. For any closed immersion of group schemes $G \to \mathrm{GL}_n$, the quotient GL_n/G is an affine S-scheme, cf. [Alp14, Corollary 9.7.7]. Hence $\mathrm{Gr}_{G,I}$ is ind-projective by [HR18b, Corollary 3.11 (i)]. By [HR18b, Theorem 2.1 (iii) and Theorem 3.17], this also implies the claim for $\mathrm{Gr}_{P,I}$. ☐

In the special case $I = \{*\}$ is a singleton, Proposition 4.1 implies there is a canonical isomorphism

$$\mathrm{Gr}_{\mathcal{G},\{*\}} \cong \mathrm{Gr}_{\mathcal{G}} \times X. \tag{4.1}$$

For general I, the fiber of $\mathrm{Gr}_{\mathcal{G},I}$ over a point in $(x_i) \in X^I$ depends on the partition of (x_i) into pairwise distinct coordinates. More precisely, let

$$\phi\colon I \twoheadrightarrow J$$

be a surjection of nonempty finite sets. This induces a partition

$$I = \bigcup_{j\in J} I_j, \quad I_j := \phi^{-1}(j).$$

Let

$$X^\phi := \{(x_i) \in X^I \,:\, x_i = x_{i'} \text{ if and only if } \phi(i) = \phi(i')\}. \tag{4.2}$$

This is a locally closed subscheme of X^I. In the special case $\phi = \mathrm{id}$ we write $X^\circ = X^{\mathrm{id}}$, which is the locus with pairwise distinct coordinates. For later use, we also define the open subscheme

$$X^{(\phi)} := \{(x_i) \in X^I \,:\, x_i \neq x_{i'} \text{ if } \phi(i) \neq \phi(i')\} \subset X^I. \tag{4.3}$$

Proposition 4.6. *There is a canonical isomorphism*

$$\mathrm{Gr}_{\mathcal{G},I}\big|_{X^\phi} \cong \prod_{j \in J} \mathrm{Gr}_{\mathcal{G}} \times X^\phi \qquad (4.4).$$

Proof. The arguments in [Zhu17b, 3.1.13] generalize to an arbitrary base. □

We now define the global loop groups. For $x \in X^I(R)$, let $\hat{\Gamma}_x$ be the formal completion of Γ_x in X_R. Locally on S, $\hat{\Gamma}_x$ is the formal spectrum of a topological ring, so by forgetting the topology we can view $\hat{\Gamma}_x$ as an object in AffSch_S, cf. [HR18b, §3.1.1]. Following the discussion in [HR18b, §3.1.1], one can view Γ_x as a Cartier divisor in $\hat{\Gamma}_x$, so $\hat{\Gamma}_x^\circ := \hat{\Gamma}_x - \Gamma_x \in \mathrm{AffSch}_S$.

Definition 4.7. The *global positive loop group* L_I^+G (resp. the *global loop group* L_IG) is the functor

$$L_I^+G(R) = \{(x, g) \,:\, x \in X^I(R),\, g \in G(\hat{\Gamma}_x)\}.$$
$$L_IG(R) = \{(x, g) \,:\, x \in X^I(R),\, g \in G(\hat{\Gamma}_x^\circ)\}.$$

By [HR18b, Lemma 3.2], L_I^+G is represented by a pro-smooth affine group scheme over X^I and L_IG is represented by an ind-affine ind-scheme over S. Both groups satisfy a factorization property as in (4.4). If $I = \{*\}$ is a singleton, there is a canonical isomorphism $L_{\{*\}}^+G \cong L^+G \times X$. The proof of the following lemma was explained to us by T. Richarz.

Lemma 4.8. *The Beilinson–Drinfeld Grassmannian* $\mathrm{Gr}_{G,I}$ *can be identified with the Zariski, Nisnevich, and étale sheafifications of* L_IG/L_I^+G.

Proof. Since sheafification commutes with base change, we may assume $S = \mathrm{Spec}\,\mathbf{Z}$. By [HR18b, Lemma 3.4], we have a right L_I^+G-torsor $L_IG \to \mathrm{Gr}_{G,I}$, which we must show is Zariski-locally trivial. The big open cell in [HR18b, Lemma 3.15] is an open sub-ind-scheme of $\mathrm{Gr}_{G,I}$ over which $L_IG \to \mathrm{Gr}_{G,I}$ admits a section. Under the factorization isomorphism (4.4), it restricts to products of the big open cell $L^{--}G = \ker(L^-G \xrightarrow{t^{-1}\mapsto 0} G)$ in Gr_G, where $L^-G(R) = G(R[t^{-1}])$. By [Fal03, Definition 5 ff.], Gr_G is covered by left translates of $L^{--}G$ by the points in $LT(\mathbf{Z})$ given by evaluation of cocharacters in $X_*(T)$ at t. Fix a surjection $\phi\colon I \to J$. Then $L_IT\big|_{X^\phi} \cong (LT)^J \times X^\phi$. By the previous discussion, it suffices to show that a point in $((LT)^J \times X^\phi)(X^\phi)$ corresponding to a tuple in $X_*(T)^J$ lifts to an X^I-point of L_IT. For this we use the explicit description of $L_IT(X^I)$ in [HR18b, Lemma 3.4]. In the notation of Remark 4.4, $L_IT(X^I)$ is given by the T-points of the complement of $D = \prod_{i \in I}(t - x_i)$ in the completion of $\mathbf{Z}[x_1, \ldots, x_n][t]$ at D. Choosing $|J|$ of the coordinates $x_1, \ldots, x_n$ to represent the distinct J coordinates over X^ϕ, it follows that the lift we need exists, and is already defined before passing to the completion. □

By Lemma 4.8, the group L_IG acts on $\mathrm{Gr}_{G,I}$ on the left. We denote the result of the action of $g \in L_IG(R)$ on $(x, \mathcal{E}, \beta) \in \mathrm{Gr}_{G,I}(R)$ by $(x, g\mathcal{E}, g\beta)$. Here $g\mathcal{E}$ is obtained by gluing $\mathcal{E}\big|_{X_R - \Gamma_x}$ to $\mathcal{E}\big|_{\hat{\Gamma}_x}$ along $\beta^{-1}\big|_{\hat{\Gamma}_x^\circ} \circ g \circ \beta\big|_{\hat{\Gamma}_x^\circ}\colon \mathcal{E}\big|_{\hat{\Gamma}_x^\circ} \to \mathcal{E}\big|_{\hat{\Gamma}_x^\circ}$, and $g\beta\colon g\mathcal{E}\big|_{X_R - \Gamma_x} \xrightarrow{\sim} \mathcal{E}\big|_{X_R - \Gamma_x} \xrightarrow{\beta} \mathcal{E}_0\big|_{X_R - \Gamma_x}$. We refer to [Ric14, §3.1] for more details on these group actions.

Definition 4.9. For any nonempty finite set I, we define the *Hecke prestack* by $\mathrm{Hck}_{G,I} := L_I^+G \backslash \mathrm{Gr}_{G,I}$. We will denote the canonical quotient by map $u : \mathrm{Gr}_{G,I} \to \mathrm{Hck}_{G,I}$.

The following result should be well-known, but we include it as we were unable to find a reference.

Proposition 4.10. *Let K, L, M be smooth affine S-group schemes, and $K \to M$ and $L \to M$ group homomorphisms, with $L \to M$ surjective. Suppose $K \times_M L$ is represented by a smooth affine S-group scheme. Then the natural morphisms $\mathrm{Gr}_{K \times_M L} \to \mathrm{Gr}_K \times_{\mathrm{Gr}_M} \mathrm{Gr}_L$, and $\mathrm{Gr}_{K \times_M L, I} \to \mathrm{Gr}_{K,I} \times_{\mathrm{Gr}_{M,I}} \mathrm{Gr}_{L,I}$ are isomorphisms for any finite set I.*

Proof. We will only show the first assertion; the case of Beilinson–Drinfeld Grassmannians can be handled analogously. To construct the inverse, let R be a scheme over S. Let $x \in X(R)$ be the point $\mathrm{Spec}(R) \to X$ corresponding to the origin. An element of $(\mathrm{Gr}_K \times_{\mathrm{Gr}_M} \mathrm{Gr}_L)(R)$ can be represented by a pair $(\mathcal{E}_K, \beta_K)$, with $\mathcal{E}_K$ a K-torsor on $\hat{\Gamma}_x$ and $\beta_K : \mathcal{E}_K\big|_{\hat{\Gamma}_x^\circ} \cong \mathcal{E}_{K,0}\big|_{\hat{\Gamma}_x^\circ}$, a similar pair $(\mathcal{E}_L, \beta_L)$ for L, and an isomorphism $\alpha : \mathcal{E}_K \times^K M \cong \mathcal{E}_L \times^L M$, commuting with β_K and β_L under the natural identifications $\mathcal{E}_{0,K} \times^K M \cong \mathcal{E}_{0,M} \cong \mathcal{E}_{0,L} \times^L M$. Let us denote $\mathcal{E}_K \times^K M \cong \mathcal{E}_L \times^L M$ by $\mathcal{E}_M$. Using the natural morphisms $\mathcal{E}_K \cong \mathcal{E}_K \times^K K \to \mathcal{E}_K \times^K M \cong \mathcal{E}_M$ and $\mathcal{E}_L \to \mathcal{E}_M$, we can consider the fiber product $\mathcal{E}_K \times_{\mathcal{E}_M} \mathcal{E}_L$. Moreover, the isomorphisms β_K and β_L induce an isomorphism $\beta_{K\times_M L}\colon \mathcal{E}_K \times_{\mathcal{E}_M} \mathcal{E}_L\big|_{\hat{\Gamma}_x^\circ} \cong \mathcal{E}_{0,K\times_M L}\big|_{\hat{\Gamma}_x^\circ}$. We leave it to the reader to verify that the inverse is

$$(\mathcal{E}_K, \beta_K, \mathcal{E}_L, \beta_L, \alpha) \mapsto (\mathcal{E}_K \times_{\mathcal{E}_M} \mathcal{E}_L, \beta_{K\times_M L}).$$

□

4.2. **Convolution Grassmannians.**

4.2.1. *Local case.* Recall that $\mathcal{P} \subset LG$ denotes a parahoric subgroup.

Definition and Lemma 4.11. The *convolution product* is defined to be the functor

$$\star : \mathrm{DM}_{(\mathrm{r})}(\mathcal{P}\backslash LG/\mathcal{P}) \times \mathrm{DM}_{(\mathrm{r})}(\mathcal{P}\backslash LG/\mathcal{P}) \to \mathrm{DM}_{(\mathrm{r})}(\mathcal{P}\backslash LG/\mathcal{P})$$
$$\mathcal{F}_1 \star \mathcal{F}_2 := m_! p^! (\mathcal{F}_1 \boxtimes \mathcal{F}_2), \tag{4.5}$$

where the maps are the natural quotient and multiplication maps (which are maps of prestacks):

$$\mathcal{P}\backslash LG/\mathcal{P} \times \mathcal{P}\backslash LG/\mathcal{P} \xleftarrow{\ p\ } \mathcal{P}\backslash LG \times^{\mathcal{P}} LG/\mathcal{P} \xrightarrow{\ m\ } \mathcal{P}\backslash LG/\mathcal{P}. \tag{4.6}$$

The functor $\star$ preserves anti-effective (resp. all) stratified Tate motives (Definition 2.22). It endows (at least) the homotopy category $\mathrm{Ho}(\mathrm{DTM}_{(\mathrm{r})}(\mathcal{P}\backslash LG/\mathcal{P})^{(\mathrm{anti})})$ with the structure of a monoidal category.

Remark 4.12. Recall the existence of the functors in the right hand side of (4.5). For any map of prestacks $f : Y \to Z$, we have a !-pullback functor by construction, cf. (2.2). If f, or its Zariski- or Nisnevich sheafification, is an ind-schematic map (as is the case for m), then $f^!$ admits a left adjoint $f_!$, cf. [RS20, Lemma 2.2.9, Proposition 2.3.3], joint with Remark 2.20. Finally, the exterior product functor for motives on placid prestacks has been constructed in [RS21, Corollary A.15]. The prestacks $\mathcal{P}\backslash LG/\mathcal{P}$ and also the Hecke prestacks $\mathrm{Hck}_{G,I}$ are placid, the point being that the quotient is formed with respect to a pro-smooth group $\mathcal{P}$, resp. L_I^+G. For later use, we note that the !-pullback along a pro-smooth quotient map (such as $\mathrm{Gr}_G \to \mathcal{P}\backslash \mathrm{Gr}_P$) is compatible with $\boxtimes$.

Proof. The proofs of [RS21, Lemma 3.7, Theorem 3.17] carry over to show that Tate motives are preserved under convolution. What is more, these proofs also show more generally that the convolution product defined in [RS20, Definition 3.1] (for a triple of parahorics) preserves anti-effective Tate motives. More precisely, the argument for $\mathcal{P} = \mathcal{I}$ in [RS20, Proposition 3.19] applies verbatim. The argument for the next case in [RS20, Proposition 3.26] needs to be modified by using the functors a^* and b^* instead of $a^!$ and $b^!$ in the diagram in loc. cit. so as to avoid introducing positive twists. This proves the theorem for arbitrary $\mathcal{P}$ and motives which are anti-effective with respect to the Iwahori stratification. The general case of a coarser stratification follows as in the the third case of Proposition 3.7. □

4.2.2. *Global case: Type I.* By replacing LG by L_IG and $\mathcal{P}$ by L_I^+G in (4.6), and using any number of factors of $\mathrm{Hck}_{G,I}$, we obtain a convolution product

$$\star : \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \times \cdots \times \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}).$$
$$\mathcal{F}_1 \star \ldots \star \mathcal{F}_n := m_! p^! (\mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_n)[-(n-1)|I|]. \tag{4.7}$$

The box product is formed with respect to X^I. As above, this functor turns the homotopy category of $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ into a monoidal category. It will be clear from context if we mean the local or global version of $\star$. The shift by $-(n-1)|I|$ ensures the box product will be right exact for a t-structure introduced later, cf. Theorem 5.32 and Proposition 5.37.

4.2.3. *Beilinson–Drinfeld convolution Grassmannians.* If I is an ordered finite set we identify I with $\{1, \ldots, |I|\}$. For the rest of this subsection, fix an order-preserving surjection of nonempty ordered finite sets $\phi\colon I \twoheadrightarrow J$. If $(x_i) \in X^I$, let $x_{I_j} \in X^{I_j}$ be the corresponding component.

Definition 4.13. The *convolution Beilinson–Drinfeld Grassmannian* over X^I is the functor

$$\widetilde{\mathrm{Gr}}_{G,\phi}(R) = \left\{ (x, \mathcal{E}_j, \beta_j)_{j=1,\ldots,|J|} \ : \ x \in X^I(R),\ \mathcal{E}_j \text{ is a } G\text{-torsor on } X_R,\ \beta_j\colon \mathcal{E}_j\big|_{X_R - \Gamma_{x_{I_j}}} \cong \mathcal{E}_{j-1}\big|_{X_R - \Gamma_{x_{I_j}}} \right\}, \tag{4.8}$$

where $\mathcal{E}_0$ is the trivial G-torsor.

There is a factorization property for $\widetilde{\mathrm{Gr}}_{G,\phi}$ similar to (4.4). There is also a convolution morphism

$$m_\phi\colon \widetilde{\mathrm{Gr}}_{G,\phi} \to \mathrm{Gr}_{G,I}, \quad (x, \mathcal{E}_i, \beta_i) \mapsto (x, \mathcal{E}_{|J|}, \beta_1 \circ \cdots \circ \beta_{|J|}) \tag{4.9}$$

which restricts to an isomorphism over the locus of X^I with pairwise distinct coordinates. More generally, for $j \in J$ there is a morphism $\widetilde{\mathrm{Gr}}_{G,\phi} \to \mathrm{Gr}_{G,X^{I_1 \sqcup \cdots \sqcup I_j}}$ which records $\mathcal{E}_j$ and the trivialization $\beta_1 \circ \cdots \circ \beta_j$. The product of these $|J|$ morphisms is a closed embedding, so $\widetilde{\mathrm{Gr}}_{G,\phi}$ is represented by an ind-projective scheme over X^I. The group

L_I^+G acts on $\widetilde{\mathrm{Gr}}_{G,\phi}$ on the left by $(x,\mathcal{E}_j,\beta_j)\mapsto(x,g\mathcal{E}_j,g\beta_jg^{-1})$, cf. [Ric14, Corollary 3.10 ff.], and m_ϕ is equivariant for this action. If $\phi=\mathrm{id}$ we let $\widetilde{\mathrm{Gr}}_{G,I}:=\widetilde{\mathrm{Gr}}_{G,\phi}$ and $m_I:=m_\phi$.

4.2.4. *Global case: Type II.* As in previous approaches to geometric Satake, we will relate the convolution product $\star$ on $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ to a fusion product. In order to prove facts about the fusion product, we will use another type of global convolution product described below.

Let $I=I_1\sqcup I_2$ be a partition into two nonempty finite sets associated to a surjection $\phi\colon I\to\{1,2\}$. We define a functor on AffSch_S by

$$\begin{aligned}L_{I_1I_2}G(R)=\{((x_i,\mathcal{E}_i,\beta_i)_{i=1,2},\sigma)\colon x_i\in X^{I_i}(R),\mathcal{E}_i\text{ is a }G\text{-torsor on }X_R,\\ \beta_i\colon \mathcal{E}_i\big|_{X_R-\Gamma_{x_i}}\cong\mathcal{E}_0\big|_{X_R-\Gamma_{x_i}},\sigma\colon\mathcal{E}_0\big|_{\hat\Gamma_{x_2}}\cong\mathcal{E}_1\big|_{\hat\Gamma_{x_2}}\}.\end{aligned}$$

There is a commutative diagram, which we explain below.

$$\begin{tikzcd}
\mathrm{Gr}_{G,I_1}\times_S\mathrm{Gr}_{G,I_2} \arrow[d, "u"] & L_{I_1I_2}G \arrow[l, "p"'] \arrow[d, "q"] & \\
\mathrm{Gr}_{G,I_1}\times_S\mathrm{Hck}_{G,I_2} \arrow[d, "v"] & \widetilde{\mathrm{Gr}}_{G,\phi} \arrow[l, "\tilde p"'] \arrow[r, "m_\phi"] \arrow[d, "w"] & \mathrm{Gr}_{G,I} \arrow[d, "w"] \\
\mathrm{Hck}_{G,I_1}\times_S\mathrm{Hck}_{G,I_2} & L_I^+G\setminus\widetilde{\mathrm{Gr}}_{G,\phi} \arrow[l, "\overline{p}"'] \arrow[r, "\overline{m}_\phi"] & \mathrm{Hck}_{G,I}
\end{tikzcd} \tag{4.10}$$

Lemma 4.14. *The projection $p\colon L_{I_1I_2}G\to\mathrm{Gr}_{G,I_1}\times\mathrm{Gr}_{G,I_2}$ which forgets σ is a Zariski-locally trivial $L_{I_2}^+G\times_{X^{I_2}}X^I$-torsor. Consequently, $L_{I_1I_2}G$ is represented by an ind-scheme over X^I.*

Proof. The group $L_{I_2}^+G\times_{X^{I_2}}X^I$ acts on $L_{I_1I_2}G$ by changing σ, and p is a torsor for this group. For every point in $\mathrm{Gr}_{G,I_1}(R)$, $\mathcal{E}_1$ is trivializable on $X_R-\Gamma_{x_1\sqcup x_2}$, so by Lemma 4.8, $\mathcal{E}_1$ is trivializable on $\hat\Gamma_{x_1\sqcup x_2}$ Zariski-locally with respect to R. By pulling back along the map $\hat\Gamma_{x_1}\to\hat\Gamma_{x_1\sqcup x_2}$ this shows that a trivialization σ as in the definition of $L_{I_1I_2}G(R)$ exists Zariski-locally on R. ☐

For a point in $LG_{I_1I_2}(R)$, we can construct a point $(x,\mathcal{E}'_j,\beta'_j)_{j=1,2}\in\widetilde{\mathrm{Gr}}_{G,\phi}(R)$ as follows. Let $x=x_1\times x_2$, $\mathcal{E}'_1=\mathcal{E}_1$ and $\beta'_1=\beta_1$. Let $\mathcal{E}'_2$ be the bundle obtained using [BL95] (see also [BD99, §2.12]) to glue $\mathcal{E}_1\big|_{X_R-\Gamma_{x_2}}$ to $\mathcal{E}_2\big|_{\hat\Gamma_{x_2}}$ along $\sigma\circ\beta_2\big|_{\hat\Gamma^\circ_{x_2}}$. By construction there is an isomorphism $\beta'_2\colon\mathcal{E}'_2\big|_{X_R-\Gamma_{x_1}}\cong\mathcal{E}_1\big|_{X_R-\Gamma_{x_1}}$. This defines the map q, which is a torsor for the action of $L_{I_2}^+G\times_{X^{I_2}}X^I$, that fixes $(\mathcal{E}_1,\beta_1)$, and sends $(\mathcal{E}_2,\beta_2,\sigma)$ to $(g\mathcal{E}_2,g\beta_2,\sigma g^{-1})$, cf. [BR18, §1.7.4].

Lemma 4.15. *The map q is a Zariski-locally trivial $L_{I_2}^+G\times_{X^{I_2}}X^I$-torsor.*

Proof. Fix a point $(x_i,\mathcal{E}_i,\beta_i)_{i=1,2}\in\widetilde{\mathrm{Gr}}_{G,\phi}(R)$. By Lemma 4.8, after an affine Zariski cover $\mathrm{Spec}(R')\to\mathrm{Spec}(R)$, both bundles $\mathcal{E}_i$ become trivial on $\hat\Gamma_{x_1\sqcup x_2}$. Then the isomorphism $\beta_2\colon\mathcal{E}_2\big|_{X_{R'}-\Gamma_{x_2}}\cong\mathcal{E}_1\big|_{X_{R'}-\Gamma_{x_2}}$ shows that $\mathcal{E}_2$ is obtained by gluing $\mathcal{E}_1\big|_{X_{R'}-\Gamma_{x_2}}$ to the trivial bundle $\mathcal{E}_0\big|_{\hat\Gamma_{x_2}}$. We can then construct a section over R' by taking $\mathcal{E}'_2=\mathcal{E}_0$ to be the trivial bundle on $X_{R'}$. ☐

The maps labelled with u, v, and w are the natural maps to the prestack quotients by the left $L_{I_2}^+G\times_{X^{I_2}}X^I$, $L_{I_1}^+G\times_{X^{I_1}}X^I$, and L_I^+G-actions. The map $\tilde p$ exists because p is equivariant for the left action of $L_{I_2}^+G$ on Gr_{G,I_2}. Technically, $\tilde p$ only exists after taking the Zariski-sheafification of the target. However, for any prestack Z the !-pullback along the sheafification map $Z\to Z_{\mathrm{Zar}}$ induces an equivalence $\mathrm{DM}_{(\mathrm{r})}(Z_{\mathrm{Zar}})\xrightarrow{\cong}\mathrm{DM}_{(\mathrm{r})}(Z)$ (and likewise with the Nisnevich topology), so we can safely pretend $\tilde p$ exists as stated. Likewise, the map $\overline{p}$ exists because we only take pre-stack quotients on the bottom row.

For $\mathcal{F}_i\in\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_i})$, we can form the Type II convolution product

$$\overline{m}_{\phi!}\overline{p}^!(\mathcal{F}_1\boxtimes\mathcal{F}_2)\in\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}).$$

Lemma 4.16. *The motive on $\widetilde{\mathrm{Gr}}_{G,\phi}$ underlying $\overline{p}^!(\mathcal{F}_1\boxtimes\mathcal{F}_2)$ agrees with what is in the literature often denoted by $\mathcal{F}_1\widetilde{\boxtimes}\mathcal{F}_2$, cf.* [Zhu17b, A.1.2]. *Similarly, the motive on $\mathrm{Gr}_{G,I}$ underlying $\overline{m}_{\phi!}\overline{p}^!(\mathcal{F}_1\boxtimes\mathcal{F}_2)$ agrees with $m_{\phi!}(\mathcal{F}_1\widetilde{\boxtimes}\mathcal{F}_2)$. This specializes to the construction in* [MV07, Eqn. (5.6)] *(without the perverse truncation of the box product) under Betti realization ρ_{B} for compact motives over $S=\mathrm{Spec}\,\mathbf{Q}$.*

Proof. The left most functor $v^!$ is, by definition, compatible with $\boxtimes$. The map m_ϕ is ind-schematic and proper, so that the natural map $m_{\phi !}w^! \to w^! \overline{m}_{\phi !}$ is an isomorphism. Thus, the underlying motive of $\overline{m}_{\phi !}\overline{p}^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ in $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ is given by $m_{\phi !}\tilde{p}^!(v^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2))$. By the setup of equivariant motives, $\overline{p}^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$ is the unique object in $\mathrm{DM}_{(\mathrm{r})}(L_I^+ G \backslash \widetilde{\mathrm{Gr}}_{G,\phi})$ whose image under $q^! w^!$ in $\mathrm{DM}_{(\mathrm{r})}(L_{I_1 I_2} G)$ is $p^!(u \circ v)^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2)$. Again by definition, $u^!$ and $v^!$ are compatible with $\boxtimes$. In other words, the motive $\tilde{p}^!(v^!(\mathcal{F}_1 \boxtimes \mathcal{F}_2))$ is a twisted external product in the sense of [Zhu17b, A.1.2]. The compatibility with [MV07, Eqn. (5.6)] then follows from the fact that Betti realization commutes with the six functors. $\square$

Remark 4.17. There is an analogue of $L_{I_1 I_2} G$ for n factors $I_1, \dots, I_n$, cf. [Ric14, Definition 3.11]. Lemma 4.16 and Proposition 4.27 below generalize to n-fold convolution products.

4.2.5. *One further convolution Grassmannian.* We will need one further object similar to $L_{I_1 I_2} G$ in order to show admissibility of a certain stratification of $\mathrm{Gr}_{G,I}$. Specifically, for $n > 1$ there is an étale $L^+_{\{n\}} G$-torsor

$$\mathbf{E} \to \widetilde{\mathrm{Gr}}_{G,\{1,\dots,n-1\}} \times X$$

whose functor of points records $(x, \mathcal{E}_i, \beta_i)_{i=1,\dots,n-1} \in \widetilde{\mathrm{Gr}}_{G,\{1,\dots,n-1\}}(R)$, $x_n \in X(R)$, and a trivialization of $\mathcal{E}_{n-1}$ on $\hat{\Gamma}_{x_n}$. There is an $L^+_{\{n\}} G$-action on $\mathbf{E}$ given by changing the trivialization over $\hat{\Gamma}_{x_n}$. Then $\widetilde{\mathrm{Gr}}_{G,I} \cong \mathbf{E} \times_X^{L^+_{\{n\}} G} \mathrm{Gr}_{G,\{n\}}$, where $L^+_{\{n\}} G$ acts diagonally. Given an R-point of $\mathbf{E}$, the last bundle $\mathcal{E}_{n-1}$ is trivial on $X_R - \Gamma_{x_1 \sqcup \dots \sqcup x_{n-1}}$, and therefore also on $X_R - \Gamma_{x_1 \sqcup \dots \sqcup x_{n-1} \sqcup x_n}$. Hence $\mathcal{E}_{n-1}$ becomes trivial on $\hat{\Gamma}_{x_1 \sqcup \dots \sqcup x_n}$ after passing to some Zariski cover of R, so $\mathbf{E} \to \widetilde{\mathrm{Gr}}_{G,\{1,\dots,n-1\}} \times X$ is Zariski-locally trivial and $\mathbf{E}$ is represented by an ind-scheme over X^I. Iterating this procedure shows that $\widetilde{\mathrm{Gr}}_{G,I}$ can be written as a twisted product

$$\widetilde{\mathrm{Gr}}_{G,I} \cong \mathrm{Gr}_{G,\{1\}} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_{G,\{n\}} . \tag{4.11}$$

See [Zhu17b, Eq. (3.1.22)] for more details.

4.3. **Stratifications of Beilinson–Drinfeld Grassmannians.** In this section we show that a certain stratification of Beilinson–Drinfeld affine Grassmannians is Whitney–Tate.

Definition 4.18. For a surjection of nonempty finite sets $\phi \colon I \twoheadrightarrow J$ and $\mu = (\mu_j) \in (X_*(T)^+)^J$, the corresponding stratum of $\mathrm{Gr}_{G,I}$ is

$$\mathrm{Gr}_{G,I}^{\phi,\mu} := \prod_{j \in J} \mathrm{Gr}_G^{\mu_j} \times X^\phi \subset \mathrm{Gr}_{G,I},$$

where the inclusion is induced by the factorization isomorphism (4.4). Let

$$\iota \colon \mathrm{Gr}_{G,I}^\dagger \to \mathrm{Gr}_{G,I} \tag{4.12}$$

be the inclusion of the disjoint union of the strata.

Example 4.19. As a preparation for the proof below, we consider the case of the trivial group G, where $\mathrm{Gr}_{G,I} \cong X^I = \mathbf{A}^I$. For $\phi \colon I \twoheadrightarrow J$, let

$$j_\phi \colon X^\phi \to X^I \tag{4.13}$$

be the inclusion of the corresponding stratum. For example, if $I = \{1,2\}$, we have $X^\circ = X^2 \setminus \Delta_X$, the complement of the diagonal. The closure of X^ϕ is the diagonal

$$\overline{X_\phi} := \{(x_i) \in X^I \ : \ x_i = x_i' \text{ if } \phi(i) = \phi(i')\}.$$

In particular, this is smooth and it is a union of strata, so that we have a Whitney–Tate stratification by Remark 2.5.

Lemma 4.20. *The closure of* $\mathrm{Gr}_{G,I}^{\phi,\mu}$ *is a union of strata.*

Proof. By restricting to $\overline{X_\phi}$ and using (4.4), we reduce to the case when ϕ is a bijection. Thus we can assume our initial stratum is of the form $\mathrm{Gr}_{G,I}^{\circ,\mu}$. In this case, using (4.11) we can form the closed subscheme

$$Y := (\mathrm{Gr}_G^{\leq \mu_1} \times X) \tilde{\times} \cdots \tilde{\times} (\mathrm{Gr}_G^{\leq \mu_n} \times X) \subset \widetilde{\mathrm{Gr}}_{G,I}.$$

Note that m_I (cf. (4.9)) is an isomorphism over X°, and that by computing smooth-locally it follows that Y is the closure of $\mathrm{Gr}_{G,I}^{\phi,\mu}$ in $\widetilde{\mathrm{Gr}}_{G,I}$. Hence $m_I(Y) \subseteq \overline{\mathrm{Gr}_{G,I}^{\circ,\mu}} \subset \mathrm{Gr}_{G,I}$. As $m_I(Y)$ is closed and it contains $\mathrm{Gr}_{G,I}^{\circ,\mu}$, this containment is an equality.

It remains to see that $m_I(Y)$ is a union of strata. For this, let $\phi' \colon I \twoheadrightarrow K$ be an arbitrary surjection and put $\lambda_k = \sum_{i \in (\phi')^{-1}(k)} \mu_i$ for $k \in K$. Using that the image of a local convolution morphism $\mathrm{Gr}_G^{\leq \mu_1} \tilde{\times} \cdots \tilde{\times} \mathrm{Gr}_G^{\leq \mu_n} \xrightarrow{m} \mathrm{Gr}_G$ is $\mathrm{Gr}_G^{\leq \mu_1 + \cdots + \mu_n}$, it follows that

$$m_I(Y)\big|_{X^{\phi'}} \cong \mathrm{Gr}_G^{\leq \lambda_1} \times \cdots \times \mathrm{Gr}_G^{\leq \lambda_{|K|}} \times X^{\phi'}.$$

$\square$

Theorem 4.21. *The stratification of* $\mathrm{Gr}_{G,I}$ *in (4.12) is universally admissible.*

The key idea for the proof is to interpret restrictions along partial diagonals in $\mathrm{Gr}_{G,I}$ as convolution. Then we can apply Definition and Lemma 4.11.

Proof. Let $\iota^{\phi,\mu}\colon \mathrm{Gr}_{G,I}^{\phi;\mu} \to \mathrm{Gr}_{G,I}$ be the inclusion of a stratum as in Definition 4.18, where $\phi\colon I \twoheadrightarrow J$. Then we have to prove $\iota^* \circ \iota_*^{\phi,\mu}(\mathbf{Z}) \in \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I}^{\dagger})$.

Fix a bijection $I \xrightarrow{\sim} \{1,\dots,n\}$. We will induct on n. If $n=1$, the proposition follows from smooth base change applied to the projection $\mathrm{Gr}_G \times X \to \mathrm{Gr}_G$ and the universal Whitney–Tate property of Gr_G (Proposition 3.7). In fact, the Iwahori-orbit stratification on Gr_G by affine spaces is Whitney–Tate, so we get a cellular Whitney–Tate stratification in the case $n=1$, also using that $X = \mathbf{A}^1$ is cellular itself.

Now assume $n>1$. By Lemma 2.9, the schemes X^ϕ and therefore also the strata $\mathrm{Gr}_{G,I}^{\phi;\mu}$ are admissible S-schemes in the sense of Definition 2.7. Thus, it remains to show the stratification is universally Whitney–Tate. If ϕ is not injective, there exist i, $i' \in I$ such that $\overline{X^\phi}$ is contained in the diagonal

$$X^{x_i = x_{i'}} = \{(x_i) \in X^I \,:\, x_i = x_{i'}\}.$$

This diagonal is a union of strata, so under the obvious identification $X^{x_i=x_{i'}} \cong X^{I-\{i'\}}$ and the factorization property (4.4) this case is covered by induction.

Now assume ϕ is injective. We can assume $I=J$ and $\phi=\mathrm{id}$. Write $X^\circ := X^{\mathrm{id}}$. To ease notation we let $\phi\colon I \to K$ be a new surjection, and we will compute the fiber of $\iota_*^{\circ,\mu}\mathbf{Z}$ over X^ϕ. We consider two further cases.

If there exists $i \in I$ such that $\phi^{-1}\phi(i)$ is a singleton, let $X_i^\circ \subset X^{I-\{i\}}$ be the locus with pairwise distinct coordinates. Consider the open subset $X^{(\phi)} \subseteq X^I$ from (4.3), for which X°, $X^\phi \subset X^{(\phi)}$. Then

$$\mathrm{Gr}_{G,I}\big|_{X^\circ} \cong \left(\mathrm{Gr}_{G,I-\{i\}}\big|_{X_i^\circ} \times \mathrm{Gr}_{G,\{i\}}\right) \times_{X^I} X^{(\phi)}$$

(note that the right hand side already lives over X°) and

$$\mathrm{Gr}_{G,I}\big|_{X^{(\phi)}} \cong \left(\mathrm{Gr}_{G,I-\{i\}} \times \mathrm{Gr}_{G,\{i\}}\right) \times_{X^I} X^{(\phi)}.$$

Hence this case follows by induction applied to the factor $\mathrm{Gr}_{G,I-\{i\}}$ and universality of the Whitney–Tate stratification of $\mathrm{Gr}_{G,\{i\}}$.

For the final case, suppose the fibers of ϕ have at least two elements each. Write $j^\circ\colon X^\circ \to X$ for the embedding. Let $\mu = (\mu_i) \in (X_*(T)^+)^I$, and let $j^{\mu_i}\colon \mathrm{Gr}_G^{\mu_i} \to \mathrm{Gr}_G$ be the inclusion of the corresponding L^+G-orbit. Let us require that ϕ is order-preserving so that there is an induced order on K.

The proof is based on the following diagram:

$$\begin{tikzcd}
 & \prod_{i\in I}\mathrm{Gr}_G \times X^\circ \times L^+G \arrow[r, "j^2"] \arrow[d, "q^\circ"] & \mathbf{E} \times_X \mathrm{Gr}_{G,\{n\}} \arrow[d, "q"] & Y_{<m} \times Y_m \arrow[l, "i_2"'] \arrow[d, "q^\phi"] \\
 & \prod_{i\in I}\mathrm{Gr}_G \times X^\circ \arrow[r, "j^1"] \arrow[d, "\sim"] & \widetilde{\mathrm{Gr}}_{G,I} \arrow[d, "m_I"] & \prod_{k\in K}\mathrm{Gr}_G^{I_k,\mathrm{conv}} \times X^\phi \arrow[l, "i_1"'] \arrow[d, "m_I^\phi"] \\
\prod_{i\in I}\mathrm{Gr}_G^{\mu_i} \times X^\circ \arrow[r, "j^{\mu_i}"] & \prod_{i\in I}\mathrm{Gr}_G \times X^\circ \arrow[r, "j^0"] \arrow[d] & \mathrm{Gr}_{G,I} \arrow[d] & \prod_{k\in K}\mathrm{Gr}_G \times X^\phi \arrow[l, "i_0"'] \arrow[d] \\
 & X^\circ \arrow[r, "j^\circ"] & X & X^\phi. \arrow[l, "j_\phi"']
\end{tikzcd}$$

The bottom squares are cartesian by the factorization property of Gr_G. The remaining squares, all of which are also cartesian, will be discussed below.

It suffices to show

$$i_0^* \circ j_*^0 \left(\boxtimes_{i\in I} j_*^{\mu_i}\mathbf{Z} \boxtimes \mathbf{Z}\big|_{X^\circ} \right) \cong \boxtimes_{k\in K} \left(\star_{i\in I_k} j_*^{\mu_i}\mathbf{Z} \right) \boxtimes \left(j_\phi^* j_*^\circ(\mathbf{Z}) \right). \tag{4.14}$$

Indeed, the convolution product $\star_{i\in I_k} j_*^{\mu_i}\mathbf{Z}$ is an object of $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_G)$ by Definition and Lemma 4.11. This holds independently of the base scheme since the stratification on Gr_G is *universally* Whitney–Tate by Corollary 3.10. Also, $j_\phi^* j_*^\circ(\mathbf{Z}) \in \mathrm{DTM}_{(\mathrm{r})}(X^\phi)$ by Example 4.19. Again, this is independent of the base scheme because the computation behind Remark 2.5 only uses relative purity.

We will prove this formula by induction on the number of cocharacters such that $\mu_i = 0$, starting with the case where all $\mu_i = 0$. In this case the above formula holds because the relevant geometry is supported on the image of the trivial section $X^I \to \mathrm{Gr}_{G,I}$.

Now suppose $\mu_i \neq 0$ for some i. After possibly relabelling we can assume $\mu_n \neq 0$. The middle part of the above diagram is cartesian, where $I_k = \phi^{-1}(k)$ and we write, for a nonempty finite ordered set J,

$$\mathrm{Gr}_G^{J,\mathrm{conv}} := \underbrace{LG \times^{L^+G} \cdots \times^{L^+G} LG \times^{L^+G} \mathrm{Gr}}_{|J| \text{ factors}}.$$

The morphism m_I^ϕ is a product of $|K|$ local convolution morphisms, times the identity morphism on X^ϕ. The map q at the top is the L^+G-torsor introduced in (4.11). Let $m \in K$ be the largest element. The fiber of $\mathbf{E} \times_X \mathrm{Gr}_{G,\{n\}}$ over X^ϕ is the product of

$$Y_{<m} := \prod_{k<m} \mathrm{Gr}_G^{I_k,\mathrm{conv}} \times X^\phi$$

and

$$Y_m := \underbrace{LG \times^{L^+G} \cdots \times^{L^+G} LG \times \mathrm{Gr}}_{|I_m| \text{ factors}}.$$

Here we set $Y_{<m} = S$ if $|K| = 1$. The map q° is a trivial L^+G-torsor, cf. [BR18, (1.7.5) ff.]. The map q^ϕ is the quotient by the diagonal action of L^+G on the last two factors $LG \times \mathrm{Gr}$ in Y_m, which is possible by our assumption that $|I_m| > 1$. The point is that in the top row we have split off the factor Gr_G which supports $j_*^{\mu_n}\mathbf{Z}$.

By proper base change,

$$i_0^* \circ j_*^0 \left(\boxtimes_{i\in I} j_*^{\mu_i}\mathbf{Z} \boxtimes \mathbf{Z}\big|_{X^\circ}\right) \cong m^\phi_{I*} \circ i_1^* \circ j_*^1 \left(\boxtimes_{i\in I} j_*^{\mu_i}\mathbf{Z} \boxtimes \mathbf{Z}\big|_{X^\circ}\right).$$

Let

$$\mathcal{E} = \boxtimes_{1\le i\le n-1} j_*^{\mu_i}\mathbf{Z} \boxtimes \mathbf{Z}\big|_{\mathrm{Gr}_G^0} \boxtimes \mathbf{Z}\big|_{X^\circ},$$

i.e., the object we would like to push-pull but where we set the last cocharacter $\mu_n = 0$. By smooth base change and Lemma 3.8 we have

$$q^{\phi *} \circ i_1^* \circ j_*^1(\boxtimes_{i\in I} j_*^{\mu_i}\mathbf{Z} \boxtimes \mathbf{Z}) \cong q^{\phi *} \circ i_1^* \circ j_*^1(\mathcal{E}) \boxtimes j_*^{\mu_n}\mathbf{Z},$$

cf. [Zhu14, Prop. 7.4(ii)]. This isomorphism is equivariant for the diagonal action of L^+G, so by descent we have

$$i_1^* \circ j_*^1(\boxtimes_{i\in I} j_*^{\mu_i}\mathbf{Z} \boxtimes \mathbf{Z}) \cong i_1^* \circ j_*^1(\mathcal{E}) \widetilde{\boxtimes} j_*^{\mu_n}\mathbf{Z}.$$

Here the twisted product is formed on

$$\prod_{k\in K} \mathrm{Gr}_G^{I_k,\mathrm{conv}} \times X^\phi \cong \prod_{k<m} \mathrm{Gr}_G^{I_k,\mathrm{conv}} \times \mathrm{Gr}_G^{I_m-\{n\},\mathrm{conv}} \times X^\phi \mathbin{\tilde{\times}} \mathrm{Gr}_G,$$

which comes from the identification (obtained by restricting (4.11) to the diagonal)

$$\mathrm{Gr}_G^{I_m,\mathrm{conv}} \cong \mathrm{Gr}_G^{I_m-\{n\},\mathrm{conv}} \mathbin{\tilde{\times}} \mathrm{Gr}_G.$$

The construction of the twisted product of motivic sheaves is analogous to Lemma 4.16 and is purely local, cf. [RS21, Lemma 3.11 ff.].

Now we factor m_I^ϕ by first convolving the left $n-1$ factors, and then convolving with the final factor supporting $j_*^{\mu_n}\mathbf{Z}$, as below.

$$\begin{tikzcd}
\prod_{k\in K} \mathrm{Gr}_G^{I_k,\mathrm{conv}} \times X^\phi \arrow[r, "m_1"] \arrow[dr, "m_I^\phi"'] & \prod_{k<m} \mathrm{Gr}_G \times (\mathrm{Gr} \mathbin{\tilde{\times}} \mathrm{Gr}) \times X^\phi \arrow[d, "m_2"] \\
& \prod_{k\in K} \mathrm{Gr}_G \times X^\phi.
\end{tikzcd}$$

By proper base change and Lemma 3.8,

$$m_{1*}(i_1^* \circ j_*^1(\mathcal{E}) \widetilde{\boxtimes} j_*^{\mu_n}\mathbf{Z}) \cong (m^\phi_{I*} \circ i_1^* \circ j_*^1(\mathcal{E})) \widetilde{\boxtimes} j_*^{\mu_n}\mathbf{Z}.$$

By induction the above is isomorphic to

$$\boxtimes_{k<m} \left(\mathop{\star}_{i\in I_k} j_*^{\mu_i}\mathbf{Z} \right) \boxtimes \left(\left(\mathop{\star}_{i\in I_m-\{n\}} j_*^{\mu_i}\mathbf{Z} \right) \widetilde{\boxtimes} j_*^{\mu_n}\mathbf{Z} \right) \boxtimes j_\phi^* j_*^\circ(\mathbf{Z}).$$

Applying the projection formula to $m_{2*} = m_{2!}$, we deduce (4.14), completing the proof. ☐

Remark 4.22. The proof of Theorem 4.21 implies that the convolution product $\mathop{\star}_{i\in I} j_*^{\mu_i}\mathbf{Z}$ is independent of the ordering on I, modulo the factor $j_\phi^* j_*^\circ(\mathbf{Z})$. This is similar to Gaitsgory's construction of the commutativity constraint following [Gai01, Gai04], where this factor disappears when one uses nearby cycles; see also [Zhu15].

Theorem 4.21 entitles us to the following definitions, cf. Definition 2.4 and Definition 2.22.

Definition and Lemma 4.23. We denote by $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ the category of *stratified Tate motives* on the Beilinson–Drinfeld Grassmannian. The category $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ of *Tate motives on the Hecke prestack* is defined to be the full subcategory of $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ of objects whose underlying motive in $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ lies in $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$.

Both categories carry a natural t-structure such that the forgetful functor $u^! : \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ is t-exact. As usual, the heart of these t-structures is denoted by $\mathrm{MTM}_{(\mathrm{r})}$. We refer to these motives as *mixed (stratified) Tate motives.*

Proof. The t-structure on $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ is an instance of Remark 2.13. For the t-structure on the Hecke prestack we first note that this t-structure is glued by the t-structures on $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I} \times_{X^I} X^{\phi})$, for all the surjections $\phi : I \twoheadrightarrow J$ (cf. Definition 4.18). By factorization, we have $\mathrm{Gr}_{G,I} \times_{X^I} X^{\phi} = \mathrm{Gr}_{G^J} \times X^{\phi}$, compatibly with the stratification and the $L_I G\big|_{X^{\phi}}$-action. Therefore the existence of the t-structure on equivariant objects follows from [RS20, Proposition 3.2.15]. □

Corollary 4.24. *The natural map $\pi_G\colon \mathrm{Gr}_{G,I} \to X^I$ is Whitney–Tate, i.e., π_{G*} preserves Tate motives, with respect to the stratification of X^I in Example 4.19.*

Proof. It suffices to show that $\pi_{G!}$ maps the generators of $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$, namely $\iota_!^{\phi,\mu}\mathbf{Z}$, to an object in $\mathrm{DTM}_{(\mathrm{r})}(X^I)$, for $\phi : I \twoheadrightarrow J$. Using the factorization property (4.4), we have

$$\pi_{G!}\iota_!^{\phi,\mu}\mathbf{Z} \cong j_!^{\phi}\left(\boxtimes_{j\in J} \pi_!^{(j)}\mathbf{Z}\Big|_{X^{\phi}} \right),$$

where $j^{\phi}\colon X^{\phi} \to X^I$ is as in (4.13), and $\pi^{(j)} : \prod_{i\in\phi^{-1}(j)}(\mathrm{Gr}_G^{\mu_i} \times_S X) \to X^{\phi^{-1}(j)}$ is the projection onto the diagonal. The stratification of $\mathrm{Gr}_G^{\mu_i}$ by Iwahori-orbits is a stratification by affine spaces, so that pushforward along the structural map $\mathrm{Gr}_G^{\mu_i} \to S$ preserves Tate motives [RS20, Lemma 3.1.19]. Thus, the above expression is an object of $\mathrm{DTM}_{(\mathrm{r})}(X^I)$. □

4.4. Tate motives on Beilinson–Drinfeld Grassmannians.

Notation 4.25. When working on $\mathrm{Gr}_{G,I}$, for some finite index set I, we will often need to shift by $|I|$. We will denote $[|I|]$ and $[-|I|]$ by $[I]$ and $[-I]$, and similarly for twists.

4.4.1. *Convolution.* In this subsection, we prove that the various convolution functors preserve Tate motives, and we relate Type I and II convolution products of Tate motives.

In the context of the Type II convolution diagram, suppose $I_1 = I_2$. The fiber of part of (4.10) over the diagonal embedding $X^{I_1} \to X^I$, where $I = I_1 \sqcup I_1$, is the global version of the classical convolution diagram [MV07, (Eqn. (4.1)],

$$\mathrm{Gr}_{G,I_1} \times_{X^{I_1}} \mathrm{Gr}_{G,I_1} \longleftarrow L_{I_1}G \times_{X^{I_1}} \mathrm{Gr}_{G,I_1} \longrightarrow (L_{I_1}G \times_{X^{I_1}}^{L^+_{I_1}G} \mathrm{Gr}_{G,I_1})_{\mathrm{Zar}} \xrightarrow{m} \mathrm{Gr}_{G,I_1}. \tag{4.15}$$

The left map is the quotient on the left factor and the right map is the quotient by the diagonal action of $L^+_{I_1}G$. Denoting the base of a box product with a subscript, for $\mathcal{F}_i \in \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_i})$ we can form the twisted product $\mathcal{F}_1 \widetilde{\boxtimes}_{X^{I_1}} \mathcal{F}_2 \in \mathrm{DM}_{(\mathrm{r})}(L_{I_1}G \times_{X^{I_1}}^{L^+_{I_1}G} \mathrm{Gr}_{G,I_1})$ by applying the same arguments as in Lemma 4.16. Let $f_{I_1}\colon \mathrm{Gr}_{G,I_1} \to \mathrm{Hck}_{G,I_1}$ be the quotient map. Base change along the quotient of m by the left action of $L^+_{I_1}G$ shows that

$$m_!(\mathcal{F}_1 \widetilde{\boxtimes}_{X^{I_1}} \mathcal{F}_2) \cong f_{I_1}^!(\mathcal{F}_1 \star \mathcal{F}_2)[I_1]. \tag{4.16}$$

This result generalizes to an n-fold convolution product.

Proposition 4.26. *The convolution product $\star$ in (4.7) preserves Tate motives, i.e., it restricts to a functor*

$$\star : \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \times \ldots \times \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}).$$

Likewise, for a surjection $\phi\colon I \twoheadrightarrow J$ and $I_j = \phi^{-1}(j)$, the convolution product $m_{\phi!}(-\widetilde{\boxtimes}\ldots\widetilde{\boxtimes}-)$ from Lemma 4.16 (for any number of factors) restricts to a functor

$$\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_1}) \times \ldots \times \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_{|J|}}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I}).$$

Proof. By continuity we may restrict to compact objects. Then we may replace the torsors used to construct the twisted products in both types of convolution by finite-type quotients. In this case, the twisted products can be formed using either !- or $*$-pullback. The fibers of (4.15) and (4.10) over the strata of X^I are products of local convolution diagrams. Hence, by base change and the compatibility between box products and the two operations of $*$-pullback and !-pushforward [JY21, Theorem 2.4.6], the claim follows from the local case (Definition and Lemma 4.11). □

Proposition 4.27. *Suppose $I_1 = I_2$. Let*

$$i_1\colon \mathrm{Gr}_{G,I_1} \to \mathrm{Gr}_{G,I}, \quad i\colon \mathrm{Hck}_{G,I_1} \times_{X^{I_1}} \mathrm{Hck}_{G,I_1} \to \mathrm{Hck}_{G,I_1} \times_S \mathrm{Hck}_{G,I_1}$$

be the diagonal embeddings. For $\mathcal{F}_k \in \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_k})$, there is a canonical isomorphism

$$i_1^! m_{\phi!}(\mathcal{F}_1 \widetilde{\boxtimes} \mathcal{F}_2) \cong f_{I_1}^!(\mathcal{F}_1 \star \mathcal{F}_2)(-I_1)[-I_1]. \tag{4.17}$$

Proof. Note that the definition of $\star$ includes a shift by $-|I_1|$. By base change and (4.16), it suffices to identify $\mathcal{F}_1 \widetilde{\boxtimes}_{X^{I_1}} \mathcal{F}_2(-I_1)[-2|I_1|]$ with the corestriction of $\mathcal{F}_1 \widetilde{\boxtimes}_S \mathcal{F}_2$ to the diagonal. This amounts to identifying $\mathcal{F}_1 \boxtimes_{X^{I_1}} \mathcal{F}_2(-I_1)[-2|I_1|]$ with $i^!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2)$. To get a map

$$(\mathcal{F}_1 \boxtimes_{X^{I_1}} \mathcal{F}_2)(-I_1)[-2|I_1|] \to i^!(\mathcal{F}_1 \boxtimes_S \mathcal{F}_2),$$

rewrite i as $i' \times \mathrm{id}\colon \mathrm{Hck}_{G,I_1} \times_{X^{I_1}} \mathrm{Hck}_{G,I_1} \to (\mathrm{Hck}_{G,I_1} \times_S X^{I_1}) \times_{X^{I_1}} \mathrm{Hck}_{G,I_1}$. Here i' is the product of the identity map of Hck_{G,I_1} and the structure map to X^{I_1}. The map we seek is the canonical map in [JY21, Theorem 2.4.6],

$$i'^!(\mathcal{F}_1 \boxtimes_S \mathbf{Z}\big|_{X^{I_1}}) \boxtimes_{X^{I_1}} \mathcal{F}_2 \to (i' \times \mathrm{id})^!(\mathcal{F}_1 \boxtimes_S \mathbf{Z}\big|_{X^{I_1}} \boxtimes_{X^{I_1}} \mathcal{F}_2). \tag{4.18}$$

Indeed, by using relative purity to rewrite $\mathcal{F}_1 \boxtimes_S \mathbf{Z}\big|_{X^{I_1}}$ as a !-pullback from Hck_{G,I_1}, we have $i'^!(\mathcal{F}_1 \boxtimes_S \mathbf{Z}\big|_{X^{I_1}}) \cong \mathcal{F}_1(-I_1)[-2|I_1|]$.

To show that the map (4.18) is an isomorphism, we may replace Hck_{G,I_i} by Gr_{G,I_i} throughout (by definition of $\boxtimes$ on the Hecke prestacks). By the Whitney–Tateness of the stratification on Gr_{G,I_1}, we may assume $\mathcal{F}_2$ is *-pushforward of a Tate motive on $\mathrm{Gr}_{G,I_2}|_{X^{(\phi)}} = \prod_{j \in J} \mathrm{Gr}_{G,\{*\}}|_{X^{(\phi)}}$ for some $I_2 \overset{\phi}{\twoheadrightarrow} J$. By base change we may thus replace Gr_{G,I_2} by $\prod_{j \in J} \mathrm{Gr}_{G,\{*\}}$. We are then done by Lemma 3.8. □

4.4.2. *Independence of the base.* The following statement, which is false for non-reduced motives, allows to connect Satake categories over various base schemes.

Lemma 4.28. *Let $\pi : S' \to S$ be a scheme over S. For $G/\operatorname{Spec}\mathbf{Z}$, we consider $G' = G \times_S S'$ etc. and the associated Beilinson–Drinfeld affine Grassmannian $\mathrm{Gr}_{G',I}$ over S'. The functor*

$$\pi^* : \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G',I})$$

is a monoidal functor with respect to the convolution product $\star$ and also with respect to $m_\phi(-\widetilde{\boxtimes}\ldots\widetilde{\boxtimes}-)$. For reduced motives (but not for non-reduced ones), these functors are equivalences.

Proof. The functor π^* exists since $\mathrm{Gr}_{G',I} \to \mathrm{Gr}_{G,I}$ is schematic. Up to taking reduced subschemes, the stratification on $\mathrm{Gr}_{G',I}$ is just the preimage stratification of the one on $\mathrm{Gr}_{G,I}$. This follows from [RS20, Proposition 4.4.3]. Thus π^* preserves Tate motives. The functor π^* is compatible with these convolution functors since it commutes with $\boxtimes$, the !-pullback functors along pro-smooth maps (the maps p in (4.6), resp. in (4.10)) and !-pushforward (along the maps m, resp. m_ϕ). The functor π^* is an equivalence for reduced motives by Lemma 2.10, which is applicable by Theorem 4.21. □

4.4.3. *Forgetting the equivariance.* The following local result will be used in the proof of Corollary 6.14. The proceeding global result (Proposition 4.31) will be used implicitly in several places, cf. Remark 5.16.

Proposition 4.29. *Let $\mu \in X_*(T)^+$ and let $e\colon S \to L^+G\backslash \mathrm{Gr}_G^\mu$ be the identity section. Then we have an equivalence*

$$e^![\dim \mathrm{Gr}_G^\mu]\colon \mathrm{MTM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G^\mu) \cong \mathrm{MTM}_{(\mathrm{r})}(S).$$

Proof. By [RS20, Proposition 3.1.27] we may replace L^+G by L^nG for $n \gg 0$. Then using the isomorphism $\mathrm{Gr}_G^\mu \cong (L^nG/\mathcal{P}_\mu^n)_{\mathrm{Zar}}$ from Proposition 3.2, the proposition follows from the arguments in [RS20, Proposition 3.2.23]. □

Lemma 4.30. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be a fully faithful and exact functor between abelian categories. Then the Ind-completion $\mathrm{Ind}(F)\colon \mathrm{Ind}(\mathcal{A}) \to \mathrm{Ind}(\mathcal{B})$ is fully faithful and exact. Furthermore, if $\mathcal{B}$ is Noetherian (sequences of subobjects stabilize eventually) and the image of F is stable under subquotients, so is the image of $\mathrm{Ind}(F)$.*

Proof. The full faithfulness of $\mathrm{Ind}(F)$ is clear by definition, and the exactness is [KS90, Corollary 8.6.8]. For stability under subquotients, let $Y \to X$ be a subobject in $\mathrm{Ind}(\mathcal{B})$, where $X = \operatorname{colim} X_i$ with $X_i \in \mathcal{A}$ and $Y = \operatorname{colim} Y_j$ with $Y_j \in \mathcal{B}$. Replacing each X_i with its image in X, and likewise for each Y_j, we can assume these are filtered colimits of injective maps [Hub93, Lemma 1.3]. Then each Y_j is a subobject of some X_i, and hence belongs to the image of F. This shows stability under subobjects, and then stability under quotients follows using additionally exactness and full faithfulness. □

Proposition 4.31. *The pullback functor $u^!\colon \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ is fully faithful, and the image is stable under subquotients.*

Proof. By Lemma 4.30, it suffices to prove this for the respective subcategories of compact objects. These are supported on a finite-type L_I^+G-stable closed subscheme $Y \subset \mathrm{Gr}_{G,I}$, and admit a finite filtration by IC motives of compact mixed Tate motives on the strata. The action of L_I^+G on Y factors through a smooth quotient $L_I^+G \twoheadrightarrow H$ such that the kernel of this quotient map is split pro-unipotent by [RS20, Lemma A.3.5, Proposition A.4.9]. By [RS20, Proposition 3.1.27], we reduce to considering H-equivariant motives on Y.

Let $a, p\colon H \times_{X^I} Y \to Y$ be the action and projection maps. Since H is smooth and the fiber of H over each stratum $X^\phi \subset X^I$ is cellular, the preimage stratification on $H \times_{X^I} Y$ is admissible. By smooth base change, we therefore get an exact functor $p^![-d] = p^*(d)[d]\colon \mathrm{MTM}_{(\mathrm{r})}(Y) \to \mathrm{MTM}_{(\mathrm{r})}(H \times_{X^I} Y)$, where H has relative dimension d. The restriction of $p^![-d]$ to the fiber over each stratum $X^\phi \subset X^I$ is fully faithful by [RS20, Proposition 3.2.12]. The functor $p^![-d]$ also preserves IC motives. Since homomorphisms between IC motives associated to the same stratum are determined by their restriction to the stratum, and there are no nonzero homomorphisms between IC motives associated to different strata, this implies $p^![-d]$ is fully faithful when restricted to IC motives. Full faithfulness in general then follows by induction on the lengths of filtrations by IC motives as in [Cas22, Lemma 3.4]. By the proof of [RS20, Proposition 3.1.20] this implies that $u^!$ is fully faithful, with image consisting of mixed Tate motives such that there exists an isomorphism $a^!\mathcal{F} \cong p^!\mathcal{F}$.

To show stability under subquotients, by a standard argument as in [Let05, Proposition 4.2.13] it suffices to show that the image of the exact functor $p^![-d]\colon \mathrm{MTM}_{(\mathrm{r})}(Y)^{\mathrm{c}} \to \mathrm{MTM}_{(\mathrm{r})}(H \times_{X^I} Y)^{\mathrm{c}}$ is stable under subquotients. Here we give $H \times_{X^I} Y$ the preimage stratification. The functor $p^![-d]$ has a left adjoint ${}^{\mathrm{p}}p_![d]$ and a right adjoint ${}^{\mathrm{p}}p_*(-d)[-d]$. Then by [BBD82, §4.2.6], it suffices to show that the natural map ${}^{\mathrm{p}}p_!p^!\mathcal{F} \to \mathcal{F}$ is an injection for all $\mathcal{F} \in \mathrm{MTM}_{(\mathrm{r})}(Y)^{\mathrm{c}}$. To prove we have an injection, let K be the homotopy fiber of $p_!p^!\mathcal{F} \to \mathcal{F}$. It suffices to show that if $\mathcal{F} \in \mathrm{DTM}_{(\mathrm{r})}(Y)^{\leq 0}$, then $K \in \mathrm{DTM}_{(\mathrm{r})}(Y)^{\leq -1}$. By base change, the formation of the map $p_!p^!\mathcal{F} \to \mathcal{F}$ commutes with $*$-pullback to the strata $X^\phi \subset X^I$. Hence, to prove the claim we can assume H is cellular relative to X^I. Then because H is connected, by dévissage we have an exact triangle $K' \to p_!p^!\mathbf{Z} \to \mathbf{Z}$ where $K' \in \mathrm{DTM}_{(\mathrm{r})}(Y)$ lies in negative cohomological degrees for the classical t-structure (defined like the t-structure in Lemma 2.11 etc., but omitting the shift by $\dim X_w/S$ in (2.5); see also [RS20, Remark 3.2.7]). Hence $K' \otimes \mathcal{F} \in \mathrm{DTM}_{(\mathrm{r})}(Y)^{\leq -1}$, and by the projection formula $K = K' \otimes \mathcal{F}$. $\square$

5. The global Satake category

In this section, we construct and study the Satake category. We do this in a global situation, i.e., as certain motives on the Beilinson–Drinfeld affine Grassmannians. Recall the category of (mixed) Tate motives on the Hecke prestack, cf. Definitions 4.9 and 4.23.

5.1. **Constant terms.** Given a cocharacter $\chi \in X_*(T)$, consider the induced conjugation action $\mathbf{G}_{\mathrm{m}} \times G \to G\colon (t, g) \mapsto \chi(t) \cdot g \cdot \chi(t)^{-1}$. The attractor and repeller for this action are opposite parabolics P^+ and P^- of G, and the fixed points are given by the Levi subgroup $M = P^+ \cap P^-$. We will often abbreviate $P := P^+$. If χ is dominant regular, then $P = B$ is the Borel, $P^- = B^-$ the opposite Borel, and $M = T$ is the maximal torus.

Now, $\mathbf{G}_{\mathrm{m}}$ also acts on $\mathrm{Gr}_{G,I}$ via $\mathbf{G}_{\mathrm{m}} \to L_I^+\mathbf{G}_{\mathrm{m}} \xrightarrow{L_I^+\chi} L_I^+T \to L_I^+G$. For this action, the fixed points, attractor and repeller are given by $\mathrm{Gr}_{M,I}$, $\mathrm{Gr}_{P^+,I}$ and $\mathrm{Gr}_{P^-,I}$ respectively, compatibly with the natural morphisms between them, cf. [HR18b, Theorem 3.17]. In particular, the natural projections and inclusions corresponding to these affine Grassmannians only depend on the parabolic P^+, not on χ. We obtain the corresponding hyperbolic localization diagram as follows. Since the top horizontal maps are L_I^+M-equivariant, we get the corresponding diagram of prestacks underneath:

$$\begin{tikzcd}
\mathrm{Gr}_{M,I} \arrow[d] & \mathrm{Gr}_{P^\pm,I} \arrow[l, "q_P^\pm"'] \arrow[r, "p_P^\pm"] \arrow[d] & \mathrm{Gr}_{G,I} \arrow[d] \\
L_I^+M\backslash \mathrm{Gr}_{M,I} & L_I^+M\backslash \mathrm{Gr}_{P^\pm,I} \arrow[l, "\overline{q}_P^\pm"'] \arrow[r, "\overline{p}_P^\pm"] & L_I^+M\backslash \mathrm{Gr}_{G,I}\,.
\end{tikzcd} \tag{5.1}$$

By a proof analogous to that of Lemma 3.3, the morphisms $p_P^\pm$ are locally closed immersions on connected components. Recall that these are indexed by $\pi_1(M)$, and are exactly the preimages of the connected components of $\mathrm{Gr}_{M,I}$.

If χ is dominant regular, so that $P = B$ is a Borel, the connected components of $\mathrm{Gr}_{B,I}$ are denoted $S_{\nu,I}$, and called the semi-infinite orbits as for the usual affine Grassmannian.

Proposition 5.1. *The semi-infinite orbits determine a stratification of* $\mathrm{Gr}_{G,I}$.

Proof. We claim that $\overline{S_{\nu,I}} = \bigcup_{\nu' \leq \nu} S_{\nu',I}$. To prove this we can assume S is the spectrum of an algebraically closed field. In this case, consider the usual affine Grassmannians $\mathrm{Gr}_T \leftarrow \mathrm{Gr}_B \to \mathrm{Gr}_G$, and let $S_\nu \subseteq \mathrm{Gr}_B$ be the preimage of the connected component $[\nu] \in \pi_0(\mathrm{Gr}_T)$. Taking the closure inside Gr_G, we get $\overline{S_\nu} = \bigcup_{\nu' \leq \nu} S_{\nu'}$ by [Zhu17b, Proposition 5.3.6]. From this, we can immediately conclude our lemma in the case $I = \{*\}$. The case of general I is a straightforward generalization of arguments in the proof of [BR18, Proposition 1.8.3] for $I = \{1, 2\}$, which involve the factorization property (4.4) and the identification of the $S_{\nu,I}$ with the attractors for a $\mathbf{G}_{\mathrm{m}}$-action. ☐

The previous proposition also shows that the semi-infinite orbits $T_{\nu,I} \subseteq \mathrm{Gr}_{B^-,I}$ for the opposite Borel determine a stratification.

We consider again a general parabolic. If $(\mathrm{Gr}_{P^\pm,I})_\nu$ and $(\mathrm{Gr}_{M,I})_\nu$ are the connected components corresponding to ν, we denote the restriction of $p_P^\pm$ and $q_P^\pm$ by

$$(\mathrm{Gr}_{M,I})_\nu \xleftarrow{q_\nu^\pm} (\mathrm{Gr}_{P^\pm,I})_\nu \xrightarrow{p_\nu^\pm} \mathrm{Gr}_{G,I}\,. \tag{5.2}$$

The map $q_P^\pm$ is map between ind-schemes, so that also the functor q_{P*} exists [RS20, Theorem 2.4.2]. The geometry of hyperbolic localization, see e.g. [Ric19, Construction 2.2], induces a map $q^-_{P*}p_P^{-!} \to q^+_{P!}p_P^{+*}$. Since L_I^+M is pro-smooth, we obtain functors $\overline{q}^-_{P*}$ etc. which are compatible with forgetting the L_I^+M-action (cf. Remark 2.20), and therefore a natural transformation

$$(\overline{q}^-_P)_*(\overline{p}^-_P)^! \to (\overline{q}^+_P)_!(\overline{p}^+_P)^* \tag{5.3}$$

of functors $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{M,I})$. By Proposition 2.3, this is an equivalence after forgetting the L_I^+M-equivariance, for $\mathbf{G}_{\mathrm{m}}$-monodromic objects. However, as forgetting the equivariance is conservative, and the $\mathbf{G}_{\mathrm{m}}$-action factors through L_I^+M, we see that (5.3) is already a natural equivalence. Hence, the following definition makes sense.

Definition 5.2. Using the maximal torus quotient M/M_{der} of M, we define the degree map as the locally constant function

$$\deg_P : \mathrm{Gr}_{M,I} \to \mathrm{Gr}_{M/M_{\mathrm{der}},I} \to X_*(M/M_{\mathrm{der}}) \xrightarrow{\langle 2\rho_G - 2\rho_M, -\rangle} \mathbf{Z},$$

where the middle map is given by summing the relative positions, and ρ_- indicates in which group we take the half-sum of the positive roots. If $P = B$, we will usually write $\deg := \deg_B$.

The *constant term functor* associated to P is

$$\mathrm{CT}_P^I := (\overline{q}^+_P)_!(\overline{p}^+_P)^*[\deg_P] \cong (\overline{q}^-_P)_*(\overline{p}^-_P)^![\deg_P] \colon \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{M,I}). \tag{5.4}$$

Implicit in this definition is the functor $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(L_I^+M \backslash \mathrm{Gr}_{G,I})$ forgetting part of the equivariance.

Remark 5.3. By the usual properties of morphisms of quotient stacks, the constant term functors satisfy

$$u^!\mathrm{CT}_P^I \cong (q_P^+)_!(p_P^+)^*u^! \cong (q_P^-)_*(p_P^-)^!u^!,$$

where u denotes both quotient maps $\mathrm{Gr}_{G,I} \to \mathrm{Hck}_{G,I}$ and $\mathrm{Gr}_{M,I} \to \mathrm{Hck}_{M,I}$.

Remark 5.4. The same discussion as above also works in the setting of usual affine Grassmannians, so that we can define constant term functors

$$\mathrm{CT}_P := (\overline{q}^+_P)_!(\overline{p}^+_P)^*[\deg_P] \cong (\overline{q}^-_P)_*(\overline{p}^-_P)^![\deg_P] \colon \mathrm{DM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G) \to \mathrm{DM}_{(\mathrm{r})}(L^+M\backslash \mathrm{Gr}_M).$$

Although we will not mention this explicitly, all properties we prove for CT_P^I also hold for CT_P.

The following lemma can be compared to [FS21, Proposition VI.7.13] and the proof of [BD99, 5.3.29].

Lemma 5.5. *Let $P' \subseteq P \subseteq G$ be parabolic subgroups with Levi quotients $M' \subseteq M$, and let $Q := \mathrm{im}(P' \to M)$ be the parabolic of M with Levi quotient M'. Then there is a natural equivalence $\mathrm{CT}_{P'}^I \cong \mathrm{CT}_Q^I \circ \mathrm{CT}_P^I$ of functors $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{M',I})$.*

Proof. This follows from base change and Proposition 4.10. ☐

The following result is crucial in order to prove the t-exactness of the fiber functor. Although we expect it to hold in general, we only prove it in case M is of semisimple rank at most one. In particular, we will always need to assume this.

Proposition 5.6. *If $P = B$ or if M is of semisimple rank one, then the constant term functor CT_P^I preserves Tate motives.*

Proof. Since CT_P^I commutes with restriction along the maps j_ϕ (4.13), we may replace X^I by X^ϕ. By using factorization properties, we may assume ϕ is injective, and then have to consider $\boxtimes_{i\in I}\mathrm{CT}_P^{\{i\}}$. By the Künneth formula, i.e., the compatibility of !-pushforwards and *-pullbacks with exterior products, and preservation of Tate motives by $\boxtimes$, it suffices to consider the individual $\mathrm{CT}_P^{\{i\}}$'s. For these individual constant term functors, our claim holds by Theorem 3.44 (or the weaker Theorem 3.32 if $P = B$). ☐

The following result will allow us to reduce many proofs to the case of tori, which is easier to handle by e.g. Lemma 5.8. Recall that for an ind-scheme Y, a motive $\mathcal{F} \in \mathrm{DM}_{(\mathrm{r})}(Y)$ is said to have *bounded support*, if there exists a closed subscheme $i : Z \subseteq Y$, such that $\mathcal{F}$ lies in the essential image of $i_* \colon \mathrm{DM}_{(\mathrm{r})}(Z) \to \mathrm{DM}_{(\mathrm{r})}(Y)$. We say $\mathcal{F} \in \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ has bounded support if $u^!\mathcal{F} \in \mathrm{DM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ has bounded support.

Lemma 5.7. *The restriction of the constant term functor* $\mathrm{CT}_P^I \colon \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{M,I})$ *to the subcategory of motives with bounded support is conservative.*

Proof. As the property of having bounded support is preserved by the constant terms, we can assume $P = B$ is the Borel by Lemma 5.5. Given some $\mathcal{F} \in \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ with bounded support that satisfies $\mathrm{CT}_B^I(\mathcal{F}) = 0$, we will prove $\mathcal{F} = 0$. Recall that $\mathcal{F}$ being trivial can be checked on the strata of X^I, as hyperbolic localization commutes with the restriction functors. So assume $\mathcal{F} \neq 0$, and let $X^\phi \subseteq X^I$ be a stratum, on which the restriction of $\mathcal{F}$ does not vanish. By the factorization property [Zhu17b, Theorem 3.2.1], we can assume ϕ is bijective, so that $X^\phi = X^\circ$. Now, let $(\mu_i)_i \in (X_*(T)^+)^I$ correspond to a maximal stratum $L_I^+G\backslash \mathrm{Gr}_{G,I}^{\circ,(\mu_i)_i}$ on which $\mathcal{F}$ is supported. This stratum is isomorphic to the prestack quotient $\left(\prod_{i\in I}(L_i^+G)_{\mu_i} \times_X X^\circ\right)\backslash X^\circ$, where $(L_i^+G)_{\mu_i}$ is the stabilizer of μ_i in L_i^+G. Consider $T_{(\mu_i)_i} := \left(\prod_{i\in I} T_{\mu_i,\{i\}} \times X\right)\big|_{X^\circ}$, a connected component of the restriction of the semi-infinite orbit $T_{\sum_i \mu_i, I}$ for the opposite Borel. As $(\mathrm{Gr}_{G,I}^{\circ,(\mu_i)_i} \cap T_{(\mu_i)_i,I}) \times_{X^I} X^\circ \cong X^\circ$, we see that after forgetting the L_I^+T-equivariance, the restriction of $\mathrm{CT}_B^I(\mathcal{F})$ to $\prod_{i\in I} X_{\mu_i}\big|_{X^\circ} \subseteq [\sum_i \mu_i] \in \pi_0(\mathrm{Gr}_{T,I})$, where $X_{\mu_i} \cong X$ denotes the connected component of $\mathrm{Gr}_{T,\{i\}}$ corresponding to μ_i, is given by a shift of the !-pullback of $\mathcal{F}$ along $X^\circ \to \left(\prod_{i\in I}(L_i^+G)_{\mu_i} \times_X X^\circ\right)\backslash X^\circ$, which does not vanish. ☐

Lemma 5.8. *The pushforward* $\pi_{T!} : \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{T,I}) \to \mathrm{DTM}_{(\mathrm{r})}(X^I)$ *is t-exact and conservative. (The stratifications are those of Definition 4.18 and Example 4.19.)*

Proof. It suffices to check this after replacing X^I by X^ϕ. Over X^ϕ, the reduced subschemes of the connected components of $\mathrm{Gr}_{T,I}$ are just X^ϕ, so the claim is immediate from the definitions. ☐

For the following proposition, we assume M is of semisimple rank at most one, so that CT_P preserves Tate motives.

Proposition 5.9. *The constant term functor* $\mathrm{CT}_P^I \colon \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{M,I})$ *is t-exact. In particular, if* $\mathcal{F} \in \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ *has bounded support, then* $\mathcal{F}$ *lies in positive (resp. negative) degrees if and only if this is true for* $\mathrm{CT}_P^I(\mathcal{F})$.

Proof. It is enough to show CT_P^I is t-exact, the second statement follows from Lemma 5.7. By Lemma 5.5, we can then also assume $P = B$, as t-exactness can be checked on compact objects, which always have bounded support.

We will show that $(q_B^+)_!(p_B^+)^*[\deg]$ is right t-exact, while $(q_B^-)_*(p_B^-)^![\deg]$ is left t-exact. For the right t-exactness, note that $\mathrm{DTM}_{(\mathrm{r})}^{\leq 0}(\mathrm{Gr}_{G,I})$ is generated by $\iota_!\mathrm{DTM}_{(\mathrm{r})}^{\leq 0}(\coprod_{\phi,\mu} \mathrm{Gr}_{G,I}^{\phi,\mu})$. So consider some $\phi \colon I \twoheadrightarrow J$ and $\mu = (\mu_j)_j \in (X_*(T)^+)^J$, and let us denote $\langle 2\rho, \mu\rangle := \sum_{j\in J}\langle 2\rho, \mu_j\rangle$. As $\mathrm{DTM}_{(\mathrm{r})}^{\leq 0}(\mathrm{Gr}_{G,I}^{\phi,\mu})$ is generated under twists, extensions, and coproducts by the $\mathbf{Z}[k]$ for $k \geq \langle 2\rho, \mu\rangle + |J| (= \dim \mathrm{Gr}_{G,I}^{\phi,\mu})$, it is enough to show that $(q_B^+)_!(p_B^+)^*(\iota_{G!}^{\phi,\mu}\mathbf{Z}[\langle 2\rho,\mu\rangle + |J|])[\deg] \in \mathrm{DTM}_{(\mathrm{r})}^{\leq 0}(\mathrm{Gr}_{T,I})$. Now, let $\nu = (\nu_j)_j \in X_*(T)^J$, denote by $\mathrm{Gr}_{B,I}^{\phi,\mu,\nu}$ the intersection of the preimages of $\mathrm{Gr}_{G,I}^{\phi,\mu}$ and $\mathrm{Gr}_{T,I}^{\phi,\nu}$ in $\mathrm{Gr}_{B,I}$, and consider the diagram

$$\begin{tikzcd}
\mathrm{Gr}_{T,I}^{\phi,\nu} \arrow[d, "\iota_T^{\phi,\nu}"'] & \mathrm{Gr}_{B,I}^{\phi,\mu,\nu} \arrow[l, "q^+_{\phi,\mu,\nu}"'] \arrow[r, "p^+_{\phi,\mu,\nu}"] \arrow[d] & \mathrm{Gr}_{G,I}^{\phi,\mu} \arrow[d, "\iota_G^{\phi,\mu}"] \\
\mathrm{Gr}_{T,I} & \mathrm{Gr}_{B,I} \arrow[l, "q_B^+"'] \arrow[r, "p_B^+"] & \mathrm{Gr}_{G,I},
\end{tikzcd}$$

We can assume $\mathrm{Gr}_{B,I}^{\phi,\mu,\nu} \neq \varnothing$. By base change, and as $\mathrm{Gr}_{B,I}^{\phi,\mu,\nu}$ is a union of connected components of $\mathrm{Gr}_{B,I} \times_{\mathrm{Gr}_{G,I}} \mathrm{Gr}_{G,I}^{\phi,\mu}$, it is enough to show that $(q_{\phi,\mu,\nu}^+)_!\mathbf{Z}[\langle 2\rho,\mu\rangle + |J|] \in \mathrm{DTM}^{\leq -\langle 2\rho,\nu\rangle}(\mathrm{Gr}_{T,I}^{\phi,\nu})$. Using the product description of Beilinson–Drinfeld Grassmannians over X^ϕ, we see via Theorem 3.44 that $\mathrm{Gr}_{B,I}^{\phi,\mu,\nu}$ admits a filtrable cellular decomposition, of relative dimension $\langle \rho, \mu+\nu\rangle$ over $\mathrm{Gr}_{T,I}^{\phi,\nu}$. We conclude right t-exactness by the usual computation of $f_!\mathbf{Z}$ for f the product of vector bundles and punctured vector bundles, which is an extension of twists of $\mathbf{Z}[-2(\dim f)]$, and the observation that

$$\langle 2\rho, \mu\rangle + |J| - 2\langle \rho, \mu+\nu\rangle + \langle 2\rho,\nu\rangle = |J|$$

is exactly the dimension of $\mathrm{Gr}_{T,I}^{\phi,\nu}$.

To show that $(q_P^-)_*(p_P^+)^!$ is left t-exact, we cannot apply duality, even when restricted to equivariant motives, as we are working with integral coefficients. Instead, arguing as above, and using the description of the heart of the t-structure in Lemma 2.11, we reduce to the computation of $(q_B^-)_*(p_B^-)^!(\iota_*^{\phi,\mu}(\mathbf{Z}/m[\langle 2\rho,\mu\rangle+|J|]))$, for $m\in\mathbf{Z}$ (including $m=0$). Note that for a locally closed immersion i of codimension c between two smooth S-schemes, we have $i^!\mathbf{Z}/m\cong i^*\mathbf{Z}/m(-c)[-2c]$. For f as above, one computes that $f_*\mathbf{Z}/m$ is an extension of twists of $\mathbf{Z}/m[k]$'s, with $-\dim f\leq k\leq 0$. We can then proceed as above, replacing $\mathrm{Gr}_{B,I}^{\phi,\mu,\nu}$ by a similar subset of $\mathrm{Gr}_{B^-,I}$, which is stratified cellular of dimension $\langle\rho,\mu-\nu\rangle$ over $\mathrm{Gr}_{T,I}^{\phi,\nu}$. □

We denote the natural projection $\mathrm{Gr}_{G,I}\to X^I$ by π_G, and similarly for other groups.

Proposition 5.10. *If $\mathcal{F}\in\mathrm{DTM}(\mathrm{Hck}_{G,I})$ satisfies $\pi_{T!}u^!\mathrm{CT}_B^I(\mathcal{F})\in\mathrm{DTM}_{(\mathrm{r})}(X^I)$, then also $\pi_{G!}u^!(\mathcal{F})\in\mathrm{DTM}_{(\mathrm{r})}(X^I)$, where Tate motives on X^I are defined with respect to the trivial stratification.*

Proof. We use the notation of (5.2). By assumption,

$$\pi_{T!}u^!\mathrm{CT}_B^I(\mathcal{F})\cong\bigoplus_{\nu\in X_*(T)}\pi_{T!}(q_\nu^+)_!(p_\nu^+)^*(u^!\mathcal{F})[\langle 2\rho,\nu\rangle]$$

lies in $\mathrm{DTM}_{(\mathrm{r})}(X^I)$, which is idempotent-closed, so that each of the summands is also contained in $\mathrm{DTM}_{(\mathrm{r})}(X^I)$. On the other hand, the stratification of $\mathrm{Gr}_{G,I}$ into the semi-infinite orbits $S_{\nu,I}$ (Proposition 5.1) gives a filtration on $\pi_{G!}(u^!\mathcal{F})$ with graded pieces

$$\pi_{G!}(p_\nu^+)_!(p_\nu^+)^*(u^!\mathcal{F})\cong\pi_{T!}(q_\nu^+)_!(p_\nu^+)^*(u^!\mathcal{F}).$$

So $\pi_{G!}(u^!\mathcal{F})$ is a colimit of extensions of Tate motives on X^I, and hence Tate itself. □

The following proposition gives two ways to describe the fiber functor we will use later on, similar to [BR18, Theorem 1.5.9].

Proposition 5.11. *There is a natural equivalence*

$$\bigoplus_{n\in\mathbf{Z}}{}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}u^!\cong\pi_{T!}u^!\mathrm{CT}_B^I$$

of functors $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})\to\mathrm{MTM}_{(\mathrm{r})}(X^I)$.

Proof. More precisely, we will construct a natural equivalence

$${}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}u^!\cong\bigoplus_{\nu\in X_*(T)\colon\langle 2\rho,\nu\rangle=n}\pi_{T!}(q_\nu^+)_!(p_\nu^+)^*u^![n]$$

of functors $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})\to\mathrm{MTM}_{(\mathrm{r})}(X^I)$, for each $n\in\mathbf{Z}$, where p_ν^+ and q_ν^+ are the restrictions of the hyperbolic localization functors to the semi-infinite orbit $S_{\nu,I}\subseteq\mathrm{Gr}_{B,I}$, as in the proof of Proposition 5.10.

For any $n\in\mathbf{Z}$, let $S_n=\bigsqcup_{\nu\in X_*(T)\colon\langle 2\rho,\nu\rangle=n}S_{\nu,I}\subseteq\mathrm{Gr}_{G,I}$. These $i_n\colon S_n\to\mathrm{Gr}_{G,I}$ determine a decomposition of $\mathrm{Gr}_{G,I}$. Then we have a natural equivalence ${}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}i_{n!}i_n^*u^!\cong\bigoplus_{\langle 2\rho,\nu\rangle=n}\pi_{T!}(q_\nu^+)_!(p_\nu^+)^*u^![n]$ of functors $\mathrm{MTM}(\mathrm{Hck}_{G,I})\to\mathrm{MTM}_{(\mathrm{r})}(X^I)$, while ${}^{\mathrm{p}}\mathrm{H}^k\pi_{G!}i_{n!}i_n^*u^!$ is trivial if $k\neq n$; this follows from Proposition 5.9 and Lemma 5.8.

As the Bruhat ordering can only compare tuples of cocharacters μ_i for which $\sum_i\langle 2\rho,\mu_i\rangle$ have the same parity, we can decompose $\mathrm{Gr}_{G,I}=\mathrm{Gr}_{G,I}^{\mathrm{even}}\coprod\mathrm{Gr}_{G,I}^{\mathrm{odd}}$ into two clopen sub-ind-schemes, each containing the Schubert cells corresponding to $(\mu_i)_i\in(X_*(T)^+)^I$ for which $\sum_i\langle 2\rho,\mu_i\rangle$ is even, respectively odd. Additionally using that $\overline{S_{\nu,I}}=\bigsqcup_{\nu'\leq\nu}S_{\nu',I}$ in $\mathrm{Gr}_{G,I}$, cf. the proof of Proposition 5.1, we get the closure relations $\overline{S_n}=S_n\sqcup S_{n-2}\sqcup S_{n-4}\sqcup\ldots=S_n\sqcup\overline{S_{n-2}}$. Denote the corresponding inclusion by $\overline{i}_n\colon\overline{S_n}\to\mathrm{Gr}_{G,I}$. We claim that the two natural morphisms

$${}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}i_{n!}i_n^*u^!\to{}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}\overline{i}_{n!}\overline{i}_n^*u^!\leftarrow{}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}u^!$$

of functors are equivalences, which will finish the proof. This can be checked on compact objects, so let $\mathcal{F}\in\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ be the image of a compact object in $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$; in particular it has bounded support. Moreover, using the decomposition of $\mathrm{Gr}_{G,I}$ into clopen sub-ind-schemes as above, we can assume the support of $\mathcal{F}$ is contained in $\mathrm{Gr}_{G,I}^{\mathrm{even}}$; the case where its support is contained in $\mathrm{Gr}_{G,I}^{\mathrm{odd}}$ can be handled analogously.

Consider the closed immersion with complementary open immersion $\overline{S_{n-2}}\xrightarrow{i}\overline{S_n}\xleftarrow{j}S_n$. Applying $\overline{i}_{n!}$ to the localization sequence $j_!j^*\overline{i}_n^*\mathcal{F}\to\overline{i}_n^*\mathcal{F}\to i_!i^*\overline{i}_n^*\mathcal{F}$ gives the exact triangle $i_{n!}i_n^*\mathcal{F}\to\overline{i}_{n!}\overline{i}_n^*\mathcal{F}\to\overline{i}_{n-2,!}\overline{i}_{n-2}^*\mathcal{F}$, which in turns gives a long exact sequence

$$\ldots\to{}^{\mathrm{p}}\mathrm{H}^k(\pi_{G!}i_{n!}i_n^*\mathcal{F})\to{}^{\mathrm{p}}\mathrm{H}^k(\pi_{G!}\overline{i}_{n!}\overline{i}_n^*\mathcal{F})\to{}^{\mathrm{p}}\mathrm{H}^k(\pi_{G!}\overline{i}_{n-2,!}\overline{i}_{n-2}^*\mathcal{F})\to{}^{\mathrm{p}}\mathrm{H}^{k+1}(\pi_{G!}i_{n!}i_n^*\mathcal{F})\to\ldots$$

We claim that

$${}^{\mathrm{p}}\mathrm{H}^k(\pi_{G!}\overline{i}_{n!}\overline{i}_n^*\mathcal{F})=0\text{ if }k>n\text{ or }k\text{ is odd.}$$

This can be proved by induction on n, starting with the observation that $\pi_{G!}\bar{i}_{n!}\bar{i}_n^*\mathcal{F} = 0$ for $n \ll 0$ as the support of $\mathcal{F}$ is bounded. The claim also holds for $k \ll 0$ by compactness. Now the claim follows in general by induction on k, using the long exact sequence and the fact that ${}^{\mathrm{p}}\mathrm{H}^k(\pi_{G!}i_{n!}i_n^*\mathcal{F}) = 0$ if $k \neq n$.

We further claim that the natural localization morphisms give isomorphisms

$$ {}^{\mathrm{p}}\mathrm{H}^n(\pi_{G!}i_{n!}i_n^*\mathcal{F}) \xrightarrow{\cong} {}^{\mathrm{p}}\mathrm{H}^n(\pi_{G!}\bar{i}_{n!}\bar{i}_n^*\mathcal{F}) \xleftarrow{\cong} {}^{\mathrm{p}}\mathrm{H}^n(\pi_{G!}\bar{i}_{m!}\bar{i}_m^*\mathcal{F}) $$

for all $m \geq n$ such that $m \equiv n \mod 2$. The first isomorphism is immediate from the previous claim; the second follows from induction on m. As the support of $\mathcal{F}$ is bounded, we conclude by noting that $\bar{i}_{m!}\bar{i}_m^*\mathcal{F} \cong \mathcal{F}$ for $m \gg 0$. $\square$

Corollary 5.12. *The functor $\pi_{T!}u^!\mathrm{CT}_B^I \cong \bigoplus_{n\in\mathbf{Z}} {}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}u^! \colon \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{MTM}(X^I)$ is exact, conservative, and faithful.*

Proof. The exactness combines Proposition 5.11 and Proposition 5.9. The functor is conservative on objects with bounded support by Lemma 5.7 and Lemma 5.8, and therefore on compact objects. The conservativity of an exact functor is equivalent to its faithfulness. We now conclude using that for an exact faithful functor $C \to D$ between abelian categories, the induced functor $\mathrm{Ind}\, C \to \mathrm{Ind}\, D$ is also faithful. $\square$

5.2. **The global Satake category.** For a surjection $\phi\colon I \twoheadrightarrow J$ of nonempty finite sets, recall that we have defined the locally closed subscheme $X^\phi \subset X^I$ (4.2) and the open subscheme (4.3) $X^{(\phi)} \subset X^I$. We denote the corresponding open immersion and complementary closed immersion into $\mathrm{Gr}_{G,I}$ by

$$ j^{(\phi)}\colon \mathrm{Gr}_{G,I}\big|_{X^{(\phi)}} \to \mathrm{Gr}_{G,I}, \quad i_{(\phi)}\colon \mathrm{Gr}_{G,I}\big|_{X-X^{(\phi)}} \to \mathrm{Gr}_{G,I}\,. $$

If $\phi = \mathrm{id}$, then $X^{\mathrm{id}} \subset X^I$ is the locus with distinct coordinates. We denote the base change of an X^I-scheme to X^{id} with the symbol $\circ$. For example, $X^\circ = X^{\mathrm{id}}$ and $\mathrm{Gr}_{G,I}^\circ = \mathrm{Gr}_{G,I} \times_{X^I} X^\circ$.

Definition 5.13. Consider some $\mu \in (X_*(T)^+)^I$ and let $j^{\circ,\mu}\colon\ \mathrm{Gr}_{G,I}^{\circ;\mu} \to \mathrm{Gr}_{G,I}$ be the inclusion of the corresponding open stratum, as defined in Definition 4.18 when $\phi = \mathrm{id}$. By Lemma 2.23, we may for $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$ define

$$ \mathrm{IC}_{\mu,L} \in \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) $$

to be the (reduced) intersection motive of this stratum.

Definition 5.14. The *global Satake category* $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$ is defined as the Ind-completion of the full subcategory of $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ consisting of objects $\mathcal{F}$ that admit a finite filtration

$$ 0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \mathcal{F} \tag{5.5} $$

for some integer k such that $\mathcal{F}_i \in \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ and

$$ \mathcal{F}_i/\mathcal{F}_{i-1} \cong \mathrm{IC}_{\mu_i,L_i} $$

for some $L_i \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$ and $\mu_i \in (X_*(T)^+)^I$ for all $1 \leq i \leq k$.

Remark 5.15. Since the IC_{μ_i,L_i} above are compact objects in $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$, $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$ is again a full subcategory thereof. It can also be described as the smallest full subcategory containing the $\mathrm{IC}_{\mu,L}$, and being closed under extensions and filtered colimits.

The compact objects in $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$ are precisely the direct summands of objects admitting a finite filtration as above.

Remark 5.16. By Proposition 4.31, every filtration of a compact object in $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$ by objects in $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ comes from a filtration by objects in $\mathrm{MTM}_{(\mathrm{r})}(L_I^+G\backslash \mathrm{Gr}_{G,I})$.

Proposition 5.17. *Let $p\colon \mathrm{Gr}_{G,\{*\}} \to \mathrm{Gr}_G$ be the projection coming from the identification (4.1). Then $p^![-1]$ induces a t-exact equivalence*

$$ p^![-1]\colon \mathrm{DTM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{DTM}_{(\mathrm{r})}(L^+_{\{*\}}G\backslash \mathrm{Gr}_{G,\{*\}}) $$

with quasi-inverse $p_![1]$.

Corollary 5.18. *The functor $p^!(-1)[-1] \cong p^*[1]$ induces an equivalence*

$$ \mathrm{MTM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{Sat}_{(\mathrm{r})}^{G,\{*\}}, $$

which identifies the IC motives in these two categories.

Proof. First note that $\mathrm{Sat}^{G,\{*\}}_{(\mathrm{r})} = \mathrm{MTM}_{(\mathrm{r})}(L^+_{\{*\}}G\backslash \mathrm{Gr}_{G,\{*\}})$ by Lemma 2.18, since $X^\circ = X$ if $I = \{*\}$. By smooth base change and the isomorphism $L^+_{\{*\}}G \cong L^+G \times X$, it follows that $p^!$ preserves equivariance. Hence $p^![-1]$ induces a functor as in the proposition. Since p is smooth of relative dimension one and $\mathrm{Gr}_{G,\{*\}}$ has the preimage stratification, it also follows that $p^![-1]$ is t-exact (recall the convention on the normalization of the t-structure from (2.5)). As $p^!$ commutes with both types of pushforwards and pullbacks between strata, the argument in [ES22, Proposition 4.25] reduces us to the case of a single stratum. Here the result follows from homotopy invariance. □

Proposition 5.19. *For any surjection $\phi\colon I \twoheadrightarrow J$ and $\mathcal{F} \in \mathrm{Sat}^{G,I}_{(\mathrm{r})}$, there is a canonical isomorphism*

$$j^{(\phi)}_{!*}(j^{(\phi),*}(\mathcal{F})) \cong \mathcal{F}.$$

Proof. Both objects are perverse and canonically identified over $X^{(\phi)}$, so it suffices to show ${}^{\mathrm{p}}i^*_{(\phi)}\mathcal{F} = 0$ and ${}^{\mathrm{p}}i^!_{(\phi)}\mathcal{F} = 0$. Since $i^*_{(\phi)}$ and $i^!_{(\phi)}$ preserve colimits and both halves of the t-structure on $\mathrm{DTM}_{(\mathrm{r})}$ are stable under filtered colimits (cf. Section 2.3), we may assume $\mathcal{F}$ has a filtration as in (5.5). By induction on the length of that filtration, it suffices to consider $\mathcal{F} = \mathrm{IC}_{\mu,L}$ for some $\mu \in (X_*(T)^+)^I$ and $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$. This case is immediate because $\mathrm{IC}_{\mu,L}$ is an intermediate extension from the open subset $X^\circ \subset X^{(\phi)}$. □

Example 5.20. For the trivial group $G = 1$, we have $\mathrm{Gr}_{1,I} = X^I$. Since the projection $p : X^I \to S$ and also the structural map of all strata are smooth, the functor $p^*[I]$ is t-exact, so that the intersection motives (in (2.6), with respect to the trivial stratification of X^I) are just given by $p^*L[I]$.

It is a generality about recollement of t-structures that $j^{(\phi)}_{!*}$ is fully faithful, cf. [BBD82, Remarque 1.4.14.1], so we arrive at the following corollary.

Corollary 5.21. *The restriction functor*

$$j^{(\phi),*}\colon \mathrm{Sat}^I_{(\mathrm{r})} \to \mathrm{MTM}_{(\mathrm{r})}(L^+_I G\backslash \mathrm{Gr}_{G,I}\big|_{X^{(\phi)}})$$

is fully faithful.

We now determine the structure of $\mathrm{Sat}^{T,I}_{(\mathrm{r})}$. First, note that

$$(\mathrm{Gr}^\circ_{T,I})_{\mathrm{red}} = \coprod_{\mu \in X_*(T)^I} X^\circ.$$

It follows that

$$\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}^\circ_{T,I}) \cong \mathrm{Fun}(X_*(T)^I, \mathrm{MTM}_{(\mathrm{r})}(X^\circ)). \tag{5.6}$$

Proposition 5.22. *Let $j_I\colon X^\circ \to X^I$ and $j^\circ\colon \mathrm{Gr}^\circ_{T,I} \to \mathrm{Gr}_{T,I}$ be the inclusions. The following composition of functors, denoted $\mathcal{S}_T$,*

$$\begin{tikzcd}
\mathrm{Fun}(X_*(T)^I, \mathrm{MTM}_{(\mathrm{r})}(X^I)) \arrow[r, "\prod_{X_*(T)^I} j_I^*"] \arrow[ddr, "\mathcal{S}_T"] & \mathrm{Fun}(X_*(T)^I, \mathrm{MTM}_{(\mathrm{r})}(X^\circ)) \arrow[d, "\cong"', "(5.6)"] \\
& \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}^\circ_{T,I}) \arrow[d, "j^\circ_{!*}"] \\
& \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{T,I})
\end{tikzcd}$$

is fully faithful, with essential image being $\mathrm{Sat}^{T,I}_{(\mathrm{r})}$. Here mixed Tate motives on X^I are with respect to the trivial stratification *(so that $\mathrm{MTM}_{(\mathrm{r})}(X^I) \cong \mathrm{MTM}_{(\mathrm{r})}(S)$).*

Proof. Let $L \in \mathrm{MTM}_{(\mathrm{r})}(X^I)^{\mathrm{c}}$. Then $\mathcal{S}_T$ sends the object of $\mathrm{Fun}(X_*(T)^I, \mathrm{MTM}_{(\mathrm{r})}(X^I))$ supported at μ with value L to $\mathrm{IC}_{\mu,L} \in \mathrm{Sat}^{T,I}_{(\mathrm{r})}$. Since all functors appearing are continuous, $\mathcal{S}_T$ takes values in $\mathrm{Sat}^{T,I}_{(\mathrm{r})}$.

The connected components of $\mathrm{Gr}_{T,I}$ are in bijection with $X_*(T)$. The closure of the connected component in $\mathrm{Gr}^\circ_{T,I}$ associated to $(\mu_i) \in X_*(T)^I$ is isomorphic to X^I, and it is an irreducible component of the connected component of $\mathrm{Gr}_{T,I}$ associated to $\sum_i \mu_i$. Note that by Example 5.20, $\mathrm{IC}_{\mu,L}$ is the pullback of $L[I]$ along the structure morphism $X^I \to S$ of the corresponding irreducible component of $\mathrm{Gr}_{T,I}$. Hence $\mathcal{S}_T$ is exact. This description of $\mathrm{IC}_{\mu,L}$, along with the fact that $j^\circ_{!*}$ is fully faithful, implies that $\mathcal{S}_T$ is also fully faithful. Since $\mathrm{Sat}^{T,I}_{(\mathrm{r})}$ is generated under filtered colimits by objects that admit a finite filtration with subquotients isomorphic to the $\mathrm{IC}_{\mu,L}$, to see that $\mathcal{S}_T$ is an equivalence, it suffices to show that if $\mu \neq \lambda$, then $\mathrm{Ext}^1_{\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{T,I})}(\mathrm{IC}_{\mu,L_1}, \mathrm{IC}_{\lambda,L_2}) = 0$ for any L_1, $L_2 \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$.

To prove this, let Y be the (reduced) union of the supports of IC_{μ,L_1} and $\mathrm{IC}_{\lambda,L_2}$. We can assume Y is connected. Let $i\colon Z \to Y$ be the inclusion of the intersection of the supports, and let $j\colon U \to Z$ be the inclusion of the complement. Then $Z \cong \mathbf{A}^c$ where $c > 0$ is the number of $i \in I$ such that $\mu_i = \lambda_i$. By localization, it suffices to prove

$$\mathrm{Hom}_{\mathrm{DTM}_{(\mathrm{r})}(Y)}(\mathrm{IC}_{\mu,L_1}, i_*i^!\mathrm{IC}_{\lambda,L_2}[1]) = 0, \quad \mathrm{Hom}_{\mathrm{DTM}_{(\mathrm{r})}(Y)}(\mathrm{IC}_{\mu,L_1}, j_*j^*\mathrm{IC}_{\lambda,L_2}[1]) = 0.$$

Using adjunction, the right group is a Hom group between sheaves on U with disjoint support, so it vanishes. Now apply adjunction to identify the left group with a Hom group on Z. Then by relative purity and the homotopy invariance $\mathrm{DTM}_{(\mathrm{r})}(Z) \cong \mathrm{DTM}_{(\mathrm{r})}(S)$, the group on the left is

$$\mathrm{Hom}_{\mathrm{DTM}_{(\mathrm{r})}(S)}(L_1, L_2(c - |I|)[1 - 2(|I| - c)].$$

Since $\mu \neq \lambda$ we have $1 - 2(|I| - c) < 0$, so this group vanishes. $\square$

Theorem 5.23. *The constant term functor restricts to a functor*

$$\mathrm{CT}_B^I \colon \mathrm{Sat}_{(\mathrm{r})}^{G,I} \to \mathrm{Sat}_{(\mathrm{r})}^{T,I}.$$

Proof. By Proposition 4.31 (cf. Remark 5.16) and the continuity of CT-functors, it suffices to show $\mathrm{CT}_B^I(\mathrm{IC}_{\mu,L}) \in \mathrm{Sat}_{(\mathrm{r})}^{T,I}$ for $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$. To prove this, note that Proposition 5.9 implies $\mathrm{CT}_B^I(\mathrm{IC}_{\mu,L}) \cong j^\circ_{!*}j^{\circ*}(\mathrm{CT}_B^I(\mathrm{IC}_{\mu,L}))$. Hence by Proposition 5.22, it suffices to identify $j^{\circ*}(\mathrm{CT}_B^I(\mathrm{IC}_{\mu,L}))$ with an object in the image of $\mathrm{Fun}(X_*(T)^I, \mathrm{MTM}_{(\mathrm{r})}(X^I)) \to \mathrm{Fun}(X_*(T)^I, \mathrm{MTM}_{(\mathrm{r})}(X^\circ))$. Let $\mathrm{IC}_{\mu,L}^I$ be the IC-motive associated to L and the open embedding $\prod_{i\in I}(\mathrm{Gr}_G^{\mu_i} \times X) \to \prod_{i\in I}(\mathrm{Gr}_G^{\leq\mu_i} \times X)$. By (4.4), $j^{\circ*}(\mathrm{CT}_B^I(\mathrm{IC}_{\mu,L}))$ is isomorphic to restriction of $(\prod_i q_B^+)_!(\prod_i p_B^+)^*\mathrm{IC}_{\mu,L}^I$ to X°. Analogous reasoning as in the proof of Proposition 5.17 yields a homotopy equivalence $\mathrm{DTM}(\mathrm{Gr}_{G^I} \times X^I) \cong \mathrm{DTM}(\mathrm{Gr}_{G^I,\{*\}})$ which extends the equivalence $\mathrm{DTM}(X^I) \cong \mathrm{DTM}(X)$. Using the identification $\mathrm{Gr}_{G^I} \cong (\mathrm{Gr}_G)^I$ of Proposition 4.10, $(\prod_i q_B^+)_!(\prod_i p_B^+)^*$ corresponds under these homotopy equivalences to a constant term functor $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G^I,\{*\}}) \to \mathrm{DTM}(X)$. It follows that $(\prod_i q_B^+)_!(\prod_i p_B^+)^*\mathrm{IC}_{\mu,L}^I \in \mathrm{DTM}(X^I)$ is unstratified Tate. $\square$

Proposition 5.24. *If $\mathcal{F} \in (\mathrm{Sat}_{(\mathrm{r})}^{G,I})^{\mathrm{c}}$, then $\mathcal{F}$ has no subquotients in $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ supported over $X^I \setminus X^\circ$.*

Proof. Suppose $\mathcal{F}$ has a subquotient supported over $X^I \setminus X^\circ$. Then $\mathrm{CT}_B^I(\mathcal{F})$ has a subquotient supported over $X^I \setminus X^\circ$. Since CT_B^I preserves the Satake category, we may assume $G = T$ and we must show $\mathcal{F}$ does not belong to $\mathrm{Sat}_{(\mathrm{r})}^{T,I}$. We will argue by contradiction. Since $\mathcal{F}$ has a subquotient supported over $X^I \setminus X^\circ$ there exists a finite filtration $0 = \mathcal{F}_0 \subset \cdots \subset \mathcal{F}_k = \mathcal{F}$ for some integer k such that $\mathcal{F}_i \in \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{T,I})$, each $\mathcal{F}_i/\mathcal{F}_{i-1}$ is an intersection motive as in Definition 2.16, and at least one of these intersection motives is supported over $X^I \setminus X^\circ$. Because $\mathcal{F}$ is an intermediate extension over X° (Proposition 5.19), it follows that $\mathcal{F}_1$ has support over X°. Thus $\mathcal{F}_1 \in \mathrm{Sat}_{(\mathrm{r})}^{T,I}$, and since $\mathrm{Sat}_{(\mathrm{r})}^{T,I}$ is an abelian category by Proposition 5.22 it follows that $\mathcal{F}/\mathcal{F}_1 \in \mathrm{Sat}_{(\mathrm{r})}^{T,I}$. Replacing $\mathcal{F}$ with $\mathcal{F}/\mathcal{F}_1$ and proceeding in this manner, we eventually reach an object of $\mathrm{Sat}_{(\mathrm{r})}^{T,I}$ with a subobject supported over $X^I \setminus X^\circ$, a contradiction. $\square$

Corollary 5.25. *The following conditions on $\mathcal{F} \in \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{c}}$ are equivalent.*

(1) $\mathcal{F}$ belongs to $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$.
(2) $\mathcal{F}$ has no subquotients supported over $X^I \setminus X^\circ$.
(3) $\mathrm{CT}_B^I(\mathcal{F})$ belongs to $\mathrm{Sat}_{(\mathrm{r})}^{T,I}$.

Therefore, the category $(\mathrm{Sat}_{(\mathrm{r})}^{G,I})^{\mathrm{c}} \subset \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{c}}$ is stable under subquotients and extensions. In particular, it is abelian, and therefore so is $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$.

Proof. If $\mathcal{F} \in \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{c}}$ has no subquotients supported over $X^I \setminus X^\circ$, then every finite filtration of $\mathcal{F}$ with subquotients given by intersection motives as in Definition 2.16 satisfies the conditions of Definition 5.14. Thus $\mathcal{F} \in \mathrm{Sat}_{(\mathrm{r})}^{G,I}$. Hence by Proposition 5.24, (1) and (2) are equivalent. Condition (1) implies (3) by Theorem 5.23. Because CT_B^I is t-exact (Proposition 5.9) and conservative (on objects with bounded support, Lemma 5.7), if $\mathrm{CT}_B^I(\mathcal{F})$ has no subquotients supported over $X^I \setminus X^\circ$, the same is true of $\mathcal{F}$. Thus (3) implies (2).

The final statement follows from the characterization in (2). $\square$

Now that we know the property of lying in the Satake category can be checked after applying CT_B^I, the following corollary is immediate.

Corollary 5.26. *For any parabolic P with Levi M of semisimple rank at most one, the constant term functor restricts to*

$$\mathrm{CT}_P^I \colon \mathrm{Sat}_{(\mathrm{r})}^{G,I} \to \mathrm{Sat}_{(\mathrm{r})}^{M,I}.$$

Note that $\overline{X^\phi} \cong X^J$. Let $i_{\overline{\phi}}\colon \mathrm{Gr}_J \to \mathrm{Gr}_I$ be the corresponding closed immersion induced by the factorization isomorphisms [Zhu17b, 3.2.1]. Let $d_\phi = |I| - |J|$. By Remark 2.20, there are functors

$$i_{\overline{\phi}}^*,\ i_{\overline{\phi}}^!\colon \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,J})$$

which restrict the usual pullback functors on the non-equivariant derived categories.

Proposition 5.27. *For $\mathcal{F} \in \mathrm{Sat}_{(\mathrm{r})}^{G,I}$ we have*

$$i_{\overline{\phi}}^*\mathcal{F}[-d_\phi] \in \mathrm{Sat}_{(\mathrm{r})}^{G,J}, \quad i_{\overline{\phi}}^!\mathcal{F}[d_\phi] \in \mathrm{Sat}_{(\mathrm{r})}^{G,J}.$$

Proof. The functors $i_{\overline{\phi}}^*$ and $i_{\overline{\phi}}^!$ commute with CT_B^I, so by Corollary 5.25 we can assume $G = T$. By induction on the length of a filtration of $\mathcal{F}$ we can assume $\mathcal{F} = \mathrm{IC}_{\mu,L}$ for some $\mu \in X_*(T)^I$ and $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$. Then $\mathrm{IC}_{\mu,L}$ is a shifted constant sheaf supported on X^I, so the result follows from relative purity. $\square$

Proposition 5.28. *Pushforward along π_G induces a functor*

$$\pi_{G!}u^!\colon \mathrm{Sat}_{(\mathrm{r})}^{G,I} \to \mathrm{DTM}_{(\mathrm{r})}(X^I),$$

where Tate motives on X^I are defined with respect to the trivial stratification.

Proof. We can assume $G = T$ by Theorem 5.23 and Proposition 5.10. It suffices to handle the case of intersection motives, so let $\mu \in X_*(T)^I$ and $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$. Recall the intersection motives $\mathrm{IC}_{\mu,L}$ associated to the stratum $\mathrm{Gr}_{G,I}^{\circ,\mu}$ (Definition 5.13). If we denote by $p\colon X^I \to S$ the structure morphism, then by Example 5.20 we can identify $\mathrm{IC}_{\mu,L}$ with $p^*L[I]$ supported on the irreducible component X^I of $\mathrm{Gr}_{T,I}$ associated to μ. Then $\pi_{T!}(\mathrm{IC}_{L,\mu}) \cong p^*L[I]$ is Tate because π_T restricts to the identity morphism on this irreducible component. $\square$

Definition 5.29. Recall that $X^I \xleftarrow{\pi_T} \mathrm{Gr}_{T,I} \xrightarrow{u} \mathrm{Hck}_{T,I}$ are the natural maps. Using the constant term functor from Definition 5.2, we define a functor F^I as the composite

$$F^I := \pi_{T!}u^!\mathrm{CT}_B^I\colon \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(X^I).$$

We denote its restriction to $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$ the same way, in which case it takes values in unstratified mixed Tate motives by Proposition 5.11 and Proposition 5.28:

$$F^I := \pi_{T!}u^!\mathrm{CT}_B^I\colon \mathrm{Sat}_{(\mathrm{r})}^{G,I} \to \mathrm{MTM}_{(\mathrm{r})}(X^I).$$

We call this restriction the *fiber functor*. (Since $X = \mathbf{A}_S^1$, we have $\mathrm{MTM}_{(\mathrm{r})}(X^I) \cong \mathrm{MTM}_{(\mathrm{r})}(S)$, but we prefer to write X^I to emphasize the rôle of I.) Using the natural isomorphism $\pi_0(\mathrm{Gr}_{T,I}) \cong X_*(T)$, we see that F^I decomposes as a direct sum, which we denote by $F^I = \bigoplus_{\nu \in X_*(T)} F_\nu^I$.

We will mostly be interested in the restriction of F^I to $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$, but the general functor will be useful when constructing adjoints in Subsection 6.1. See also Remark 5.3 for an equivalent way of defining F^I.

Remark 5.30. By Corollary 5.12, the fiber functor F^I is exact, conservative, and faithful, and hence deserves its name.

Recall from Proposition 5.11 that the fiber functor F^I is isomorphic to $\bigoplus_{n\in\mathbf{Z}} {}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}u^!$, hence independent of the choice of $T \subseteq B \subseteq G$. Note also that ${}^{\mathrm{p}}\mathrm{H}^n\pi_{G!}u^! \cong \bigoplus_{\langle 2\rho,\nu\rangle = n} F_\nu^I$.

In the context of motives with rational coefficients in the case $I = \{*\}$, the fiber functor appearing in [RS21, Definition 5.11] is the composite of F^I and taking the associated graded of the weight filtration. The weight filtration is less useful in the context of integral coefficients, e.g., $\mathbf{Z}/n$ is not pure of weight 0. Moreover, by not taking the associated graded we are able to construct a Hopf algebra in $\mathrm{MTM}_{(\mathrm{r})}(S)$, which is helpful for showing it is reduced, and thus independent of S, in Theorem 6.24.

Remark 5.31. As in Remark 5.4, we can define a functor

$$F := \pi_{T!}u^!\mathrm{CT}_B\colon \mathrm{DM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G) \to \mathrm{DM}_{(\mathrm{r})}(S),$$

which satisfies a similar decomposition $F = \bigoplus_{\nu\in X_*(T)} F_\nu$.

5.3. **Fusion.** As usual, $\phi\colon I \twoheadrightarrow J$ denotes a surjection of nonempty finite index sets, and we write $I_j := \phi^{-1}(j)$ for $j \in J$. Recall the setup and notation from Lemma 4.16 (and its generalization to $|J|$-fold convolution products): for $\mathcal{F}_j \in \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_j})$, we defined the twisted product $\widetilde{\boxtimes}_{j\in J}\mathcal{F}_j \in \mathrm{DM}_{(\mathrm{r})}(\widetilde{\mathrm{Gr}}_{G,\phi})$. Recall also the convolution morphism $m_\phi\colon \widetilde{\mathrm{Gr}}_{G,\phi} \to \mathrm{Gr}_{G,I}$. By Proposition 4.26, the functor $m_{\phi!}\widetilde{\boxtimes}_{j\in J}(-)$ preserves Tate motives. We use the same notation to denote the restriction of this functor to the product of Satake categories.

Theorem 5.32. *The functor $m_{\phi!}\widetilde{\boxtimes}_{j\in J}(-)\colon \prod_{j\in J}\mathrm{Sat}_{(\mathrm{r})}^{G,I_j} \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ is right t-exact, and ${}^{\mathrm{p}}\mathrm{H}^n(m_{\phi!}\widetilde{\boxtimes}_{j\in J}(-))$ restricts to a functor $\prod_{j\in J}\mathrm{Sat}_{(\mathrm{r})}^{G,I_j} \to \mathrm{Sat}_{(\mathrm{r})}^{G,I}$ for all $n \le 0$.*

In the proof below we make crucial use of a Betti realization functor. It would be interesting to find a proof which avoids this.

Proof. We proceed by several reductions.

Reduction to compact objects and $|J|=2$. As all relevant functors preserve colimits, we may restrict to compact objects. There is a natural associativity constraint on $m_{\phi!}\widetilde{\boxtimes}_{j\in J}(-)$ coming from the associativity of $\boxtimes$ and proper base change. This can be constructed using the general version of $L_{I_1I_2}G$ for any number of factors in analogy to the associativity constraint on the local Satake category, cf. [RS21, Lemma 3.7]. Associativity and the closure of the Satake category under subquotients and extensions in $\mathrm{MTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})^{\mathrm{c}}$ (Corollary 5.25) allows us to reduce to the case $J = \{1,2\}$.

Standard objects. We take a brief detour and consider certain standard objects in $\mathrm{Sat}^{G,I}$; see Subsection 6.1.4 for more details when $I = \{*\}$. For $\mu \in (X_*(T)^+)^I$, let $j\colon \mathrm{Gr}_{G,I}^{\circ,\mu} \to \overline{\mathrm{Gr}_{G,I}^{\circ,\mu}}\Big|_{X^\circ}$ be the inclusion (take $\phi = \mathrm{id}$ in Definition 4.18). Here j is a product of the identity map on X° with the embeddings $\mathrm{Gr}_G^{\mu_i} \to \mathrm{Gr}_G^{\le\mu_i}$, and $j^\circ\colon \overline{\mathrm{Gr}_{G,I}^{\circ,\mu}}\Big|_{X^\circ} \to \overline{\mathrm{Gr}_{G,I}^{\circ,\mu}}$. Fix $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$, and let $\mathcal{J}_{\mu!}^I(L) = j^\circ_{!*}{}^p j_!(L[\dim \mathrm{Gr}_{G,I}^{\circ,\mu}])$. By a proof similar to Theorem 5.23, $\mathcal{J}_{\mu!}^I(L) \in \mathrm{Sat}^{G,I}$. In the particular case $L = \mathbf{Z}/N$ for $N \ge 0$, we claim

$$\mathcal{J}_{\mu!}^I(\mathbf{Z}/N)\big|_{X^\circ} \cong \boxtimes_{i\in I}\mathcal{J}_{\mu_i!}^{\{i\}}(\mathbf{Z}) \otimes \mathbf{Z}/N\Big|_{X^\circ}. \tag{5.7}$$

Indeed, since $\mathrm{Gr}_G^I \cong \mathrm{Gr}_{G^I}$, then $\mathcal{J}_{\mu!}^I(\mathbf{Z}/N)\Big|_{X^\circ} \cong \mathcal{J}_{\mu!}^I(\mathbf{Z})\Big|_{X^\circ} \otimes \mathbf{Z}/N$ by Proposition 6.11. (The reader can check that this computation does not use anything from the present section. See also [BR18, Proposition 1.11.3] for the corresponding assertion in the context of Betti sheaves.) This reduces us to $N = 0$, where the claim follows from the Künneth formula for !-pushforward, right t-exactness of $\boxtimes$, and flatness of the $\mathrm{CT}_B^{\{i\}}\mathcal{J}_{\mu_i!}^{\{i\}}(\mathbf{Z})$ (see Lemma 6.12 or [BR18, Proposition 1.11.1]). By applying CT_B^I in order to check the left side below lies in $\mathrm{Sat}^{G,I}$, it follows from (5.7) and Proposition 5.19 that

$$\mathcal{J}_{\mu!}^I(\mathbf{Z}) \otimes \mathbf{Z}/N \cong \mathcal{J}_{\mu!}^I(\mathbf{Z}/N). \tag{5.8}$$

Reduction to standard objects. Suppose we have proved the theorem for the standard objects as above; we now show how to deduce the general case. Let $I = I_1 \sqcup I_2$, and let $\phi\colon I \to \{1,2\}$ map I_1 to $\{1\}$ and I_2 to $\{2\}$. Fix a standard object $\mathcal{F}_2$. We will prove the theorem holds for all $\mathcal{F}_1$ by induction on the support of $\mathcal{F}_1\big|_{X^\circ}$. If the theorem holds for a given $\mathcal{F}_1$, then it hold for any retract of $\mathcal{F}_1$ as well. Thus, we may assume $\mathcal{F}_1$ has a filtration with subquotients given by compact IC-motives. Again, using the closure of the Satake category under subquotients and extensions, it suffices to consider the case $\mathcal{F}_1 = \mathrm{IC}_{\mu,L}$ for arbitrary $\mu = (\mu_i) \in (X_*(T)^+)^{I_1}$ and $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$. In the base case $\mathrm{Gr}_G^{\mu_i} = \mathrm{Gr}_G^{\le\mu_i}$ for all $i \in I_1$, so we are done because $\mathcal{F}_1$ is a standard object. In general, there is an exact sequence $0 \to K \to \mathcal{J}_{\mu!}^{I_1}(L) \to \mathrm{IC}_{\mu,L} \to 0$ in Sat^{G,I_1}, so this case follows by applying induction to K and the case of standard objects. To finish the proof we may consider a similar induction on the support of $\mathcal{F}_2\big|_{X^\circ}$.

Reduction to constructible sheaves. It remains to prove the theorem for standard objects. In particular, the rest of the proof amounts to checking the vanishing of certain perverse cohomology sheaves of the ! and $*$-restrictions of $m_{\phi!}(\mathcal{J}_{\mu^1!}^{I_1}(-)\widetilde{\boxtimes}\mathcal{J}_{\mu^2!}^{I_2}(-))$ and its perverse truncations over the strata of $X \setminus X^\circ$, for $\mu^i \in (X_*(T)^+)^{I_i}$. We will shortly prove the theorem for constructible sheaves in the complex analytic topology over $\mathbf{C}$. Given this, the theorem follows in general by the following steps:

(1) *Regular motives motives over $S = \mathrm{Spec}\,\mathbf{Q}$.* Betti realization ρ_{B} commutes with all relevant functors and is t-exact and conservative by Lemma 2.15, so this case follows from the case of constructible sheaves over $\mathbf{C}$.
(2) *Reduced motives over any base S.* By Lemma 4.28, it suffices to consider $S = \mathrm{Spec}\,\mathbf{Q}$. Then this case follows from the first step since ρ_{r} is t-exact and commutes with all relevant functors.

(3) *Regular motives over any base S satisfying (2.4).* This follows from the previous case by using the reduction functor, this time additionally using that it is conservative by Lemma 2.15.

The case of constructible sheaves. We now work with constructible sheaves over $\mathbf{C}$, but we keep our notation for motives.

For constructible sheaves, the theorem holds for arbitrary J and $\mathcal{F}_j$ in the Satake category if ϕ is a bijection. Indeed, this follows from the stratified smallness of m_ϕ as observed in [MV07, Eqn. (5.10) ff.]. This amounts to the stratified semi-smallness of local convolution morphisms, first observed in [Lus83], and requires one to check that $\widetilde{\boxtimes}_{j\in J}\mathcal{F}_j$ is constructible with respect to the stratification of $\widetilde{\mathrm{Gr}}_{G,\phi}$ by $|I|$-fold twisted products $(\mathrm{Gr}_G^{\lambda_1}\times X)\,\widetilde{\times}\dots\widetilde{\times}\,(\mathrm{Gr}_G^{\lambda_{|I|}}\times X)$ for $\lambda_i\in X_*(T)^+$ as in (4.11). Constructibility with respect to this stratification follows from the Whitney–Tateness of Gr_G, or one can use $(L^+_{\{*\}}G)^I$-equivariance of $\boxtimes_{j\in J}\mathcal{F}_j$ on $(\mathrm{Gr}_{G,\{*\}})^I$. Here we use that $\mathrm{Sat}^{G,\{*\}}\cong\mathrm{MTM}(L^+G\backslash\mathrm{Gr}_G)$. If the box product is perverse, so is the convolution product. See also [Ach21, Lemma 9.5.8] for more details.

Returning to the task at hand, we consider standard objects in the case $I=I_1\sqcup I_2\to\{1,2\}=J$. Let $(\mu^i_j)_{j\in I_i}\in(X_*(T)^+)^{I_i}$ for $j=1,2$. Using the case where ϕ is a bijection and (5.7), we have

$$\mathcal{J}^{I_i}_{\mu^i!}(\mathbf{Z})\cong m_{\phi_i!}(\mathcal{J}^{\{1\}}_{\mu^i_1}(\mathbf{Z})\widetilde{\boxtimes}\dots\widetilde{\boxtimes}\mathcal{J}^{\{|I_i|\}}_{\mu^i_{|I_i|}}(\mathbf{Z})).$$

Thus, by associativity of $m_{\phi!}\widetilde{\boxtimes}_{j\in J}(-)$ and using (5.7) again, we have $\mathcal{J}^{I_1\sqcup I_2}_{\mu^1\sqcup\mu^2}(\mathbf{Z})\cong m_{\phi!}(\mathcal{J}^{I_1}_{\mu^1!}(\mathbf{Z})\widetilde{\boxtimes}\mathcal{J}^{I_2}_{\mu^2!}(\mathbf{Z}))$. This proves the theorem for standard objects associated to free, finitely abelian groups. To get torsion coefficients, we note that since all functors appearing are $\mathbf{Z}$-linear, by (5.8) it follows that for all $N_1,N_2\geq 0$,

$$\mathcal{J}^{I_1\sqcup I_2}_{\mu^1\sqcup\mu^2}(\mathbf{Z})\otimes(\mathbf{Z}/N_1\otimes\mathbf{Z}/N_2)\cong m_{\phi!}(\mathcal{J}^{I_1}_{\mu^1!}(\mathbf{Z}/N_1)\widetilde{\boxtimes}\mathcal{J}^{I_2}_{\mu^2!}(\mathbf{Z}/N_2)).$$

□

Remark 5.33. The reason for using standard objects in the proof of Theorem 5.32 instead of IC-motives is that (5.7) and (5.8) are both false in general due to torsion. For coefficients in a field, we could work directly with IC-motives.

Remark 5.34. Recall the fully faithful functor $j^{(\phi),*}\colon\mathrm{Sat}^I_{(\mathrm{r})}\to\mathrm{MTM}_{(\mathrm{r})}(L^+_IG\backslash\mathrm{Gr}_{G,I}\big|_{X^{(\phi)}})$ from Corollary 5.21. By Proposition 5.19 and Theorem 5.32, there is a natural isomorphism

$${}^{\mathrm{p}}m_{\phi!}\widetilde{\boxtimes}_{j\in J}(-)\cong j^{(\phi)}_{!*}({}^{\mathrm{p}}\mathrm{H}^0(\boxtimes_{j\in J}(-)\big|_{X^{(\phi)}}))\colon\prod_{j\in J}\mathrm{Sat}^{I_j}_{(\mathrm{r})}\to\mathrm{Sat}^I_{(\mathrm{r})}.$$

This functor satisfies natural commutativity and associativity constraints, induced from those of the exterior products over $X^{(\phi)}$. However, this naive commutativity constraint is not compatible with that of $\prod_{j\in J}\mathrm{MTM}(X^{I_j})\xrightarrow{\boxtimes}\mathrm{MTM}(X^I)$ under the fiber functors F^{I_j}, as the two will differ by some signs. To correct this we modify the commutativity constraint by hand as in [FS21, VI.9.4 ff.]. Namely, let us decompose $\mathrm{Gr}_{G,I}=\mathrm{Gr}^{\mathrm{even}}_{G,I}\coprod\mathrm{Gr}^{\mathrm{odd}}_{G,I}$ into open and closed subsets, where $\mathrm{Gr}^{\mathrm{even}}_{G,I}$ is the union of the Schubert cells corresponding to $(\mu_i)_i\in(X_*(T)^+)^I$ for which $\sum_{i\in I}\langle 2\rho,\mu_i\rangle$ is even, and likewise for $\mathrm{Gr}^{\mathrm{odd}}_{G,I}$. This induces a similar decomposition of $\mathrm{Hck}_{G,I}$. Then, we change the commutativity constraint by adding a minus sign when commuting the exterior product of motives concentrated on $\mathrm{Gr}^{\mathrm{odd}}_{G,I}$. If we denote the resulting functor equipped with this commutativity constraint by $*$, we have changed the signs such that the diagram

$$\begin{tikzcd}
\prod_{j\in J}\mathrm{Sat}^{I_j}_{(\mathrm{r})} \arrow[r, "*"] \arrow[d, "\prod_{j\in J}F^{I_j}"'] & \mathrm{Sat}^I_{(\mathrm{r})} \arrow[d, "F^I"] \\
\prod_{j\in J}\mathrm{MTM}(X^{I_j}) \arrow[r, "\boxtimes"] & \mathrm{MTM}(X^I)
\end{tikzcd}$$

is functorial in the I_j and under permutations of $I_1,\dots,I_{|J|}$; this follows from the implicit shifts appearing in the fiber functors, via Proposition 5.11. The diagram is commutative because this can be checked over $X^{(\phi)}$ by full faithfulness of $j^{(\phi),*}$, where it is immediate because the twisted product in the definition of $*$ becomes a box product. Here we also use the $(-)_!(-)^*$-description of CT^I_B and the Künneth formula, as well as Lemma 5.8.

Definition 5.35. The functor $*\colon\prod_{j\in J}\mathrm{Sat}^{I_j}_{(\mathrm{r})}\to\mathrm{Sat}^I_{(\mathrm{r})}$ in Remark 5.34 equipped with the modified commutativity constraint is called the *fusion product*.

Definition 5.36. We endow the category $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ with the tensor product defined as ${}^{\mathrm{p}}\mathrm{H}^0((\text{-})\otimes(\text{-})[-I])$. The normalization ensures that the monoidal unit is $\mathbf{Z}[I]$. We refer to this as the *underived* tensor product.

Proposition 5.37. *Let*
$$-^{\mathrm{p}}\star - := {}^{\mathrm{p}}\mathrm{H}^0(-\star-)\colon \mathrm{MTM}(\mathrm{Hck}_{G,I}) \times \mathrm{MTM}(\mathrm{Hck}_{G,I}) \to \mathrm{MTM}(\mathrm{Hck}_{G,I})$$
be the perverse truncation of the functor constructed in Proposition 4.26 (recall it includes a shift by $[-I]$, cf. (4.7)). Then $(\mathrm{Sat}^{G,I}_{(\mathrm{r})}, {}^{\mathrm{p}}\star)$ has the structure of a symmetric monoidal category coming from the fusion product (constructed in the proof), and CT^I_P (where $P \subseteq G$ is a parabolic whose Levi has semisimple rank at most one) is a symmetric monoidal functor. Moreover, $F^I : \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{MTM}_{(\mathrm{r})}(X^I)$ is a symmetric monoidal functor, where the tensor structure on the target is as in Definition 5.36.

Proof. The construction is analogous to [FS21, VI.9.4. ff.]; see also [Ric14, Theorem 3.24] for more details. Briefly, for an integer $n > 0$, consider the natural surjection $\phi : \sqcup_{i=1}^n I \to I$ and the diagonal embedding $i_{\overline{\phi}}\colon \mathrm{Gr}_{G,I} \to \mathrm{Gr}_{G,I^n}$. Let $d_\phi = |I|^{n-1}$. The fusion product and Proposition 5.27 determine a functor
$$\mathrm{Sat}^I_{(\mathrm{r})} \times \ldots \times \mathrm{Sat}^I_{(\mathrm{r})} \xrightarrow{*} \mathrm{Sat}^{I\sqcup\ldots\sqcup I}_{(\mathrm{r})} \xrightarrow{i^!_{\overline{\phi}}(-d_\phi)[d_\phi]} \mathrm{Sat}^I_{(\mathrm{r})}, \tag{5.9}$$
which makes $\mathrm{Sat}^I_{(\mathrm{r})}$ into a symmetric monoidal category. By Proposition 4.27, this convolution product agrees with ${}^{\mathrm{p}}\star$, so $(\mathrm{Sat}^{G,I}_{(\mathrm{r})}, {}^{\mathrm{p}}\star)$ is a symmetric monoidal category.

Over $X^{(\phi)} \subseteq X^I$, the functor CT^I_P decomposes as $\prod_{j\in J} \mathrm{CT}^{I_j}_P$. As in Remark 5.34, there is a diagram expressing the compatibility between fusion and the constant term functors, which is also functorial in the I_j and under permutations of $I_1, \ldots, I_{|J|}$. Base change for !-pullback along the diagonal embedding $X^I \to X^{I\sqcup\ldots\sqcup I}$ then shows that CT^I_P and F^I are symmetric monoidal. ☐

5.3.1. *Dualizability.* The following theorem, which is similar to [MV07, §11] and [FS21, VI.7.2], will be used in Theorem 6.20 to establish inverses for the dual group.

Let sw be the involution of $\mathrm{DM}(\mathrm{Hck}_{G,I})$ induced by !-pullback along the inversion map $L_IG \to L_IG, g \mapsto g^{-1}$. Recall that for a prestack $Z \xrightarrow{\pi} Y$ over a scheme Y, the *(relative) Verdier duality functor* is defined as $\mathrm{D}_{Z/Y}(\mathcal{F}) := \underline{\mathrm{Hom}}(\mathcal{F}, \omega_{Z/Y})$, for $\mathcal{F} \in \mathrm{DM}(Z)$ and $\omega_{Z/Y} := \pi^!\mathbf{Z}_Y$. At least if Z is an ind-scheme and $Y \to S$ a smooth scheme, $D_{Z/Y}$ is an involution on compact objects by relative purity. We write $\mathrm{D}_Z := \mathrm{D}_{Z/S}$ and we also sometimes omit the subscript in D_Z if the choice of Z is clear.

Definition 5.38. Following [AG15, §12.2.3], the subcategory of *locally compact motives* $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{lc}} \subset \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ is the full subcategory of motives whose image in $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ is compact.

Lemma 5.39. *Verdier duality (relative to X^I) is an anti-equivalence on $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})^{\mathrm{c}}$ and on $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{lc}}$.*

Proof. We show that the natural map $M \to \mathrm{D}(\mathrm{D}(M))$ is an isomorphism for these motives. For compact Tate motives on Gr^W_I this holds since Tate motives are generated by twists of $\iota^{\mu,\phi}_*\mathbf{Z}$. For motives on the Hecke prestack, we additionally use the isomorphism $\mathrm{D}\,\mathrm{coav} = \mathrm{av}\,\mathrm{D}$ (Lemma 2.21(3)) which implies $u^!\mathrm{D}_{\mathrm{Hck}^W_{G,I}} = \mathrm{D}_{\mathrm{Gr}^W_{G,I}}u^!$ by passing to adjoints. For reduced motives we additionally use Lemma 2.6. ☐

Lemma 5.40. *The dualizing functor with respect to the (derived) convolution product $\star$ on $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{lc}}$, i.e., the internal Hom functor $\underline{\mathrm{Hom}}_\star(-,1)$, is given by*
$$\mathrm{D}^- := \mathrm{D}_{\mathrm{Hck}_{G,I}/X^I} \circ \mathrm{sw}. \tag{5.10}$$
An object $M \in \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{lc}}$ is dualizable iff the resulting natural map
$$M \star N \to \mathrm{D}^-(\mathrm{D}^-M \star \mathrm{D}^-N) \tag{5.11}$$
is an isomorphism with $N = \mathrm{D}^-(M)$.

Proof. We have a cartesian diagram as follows.
$$\begin{tikzcd}
L_I^+G \backslash L_IG/L_I^+G \arrow[r, "\mathrm{inv}"] \arrow[d, "f"] & L_I^+G \backslash L_IG \times^{L_I^+G} L_IG/L_I^+G \arrow[r, "p"] \arrow[d, "m"] & (L_I^+G \backslash L_IG/L_I^+G)^2 \\
L_I^+G \backslash L_I^+G/L_I^+G \arrow[r, "i"] & L_I^+G \backslash L_IG/L_I^+G &
\end{tikzcd}$$
Here f is the quotient by L_I^+G of the structural map $L_IG/L_I^+G \to L_I^+G/L_I^+G \cong S$, the map i is induced by the inclusion $L_I^+G \to L_IG$, and inv is induced by the identity map on the first factor and inversion on the second factor. Let p be the natural quotient map (obtained by modding out two copies of L_I^+G instead of one acting diagonally). Then the composite $\mathrm{inv}^*p^! = \Delta^*(\mathrm{id} \times \mathrm{sw})$, where $\mathrm{id} \times \mathrm{sw}$ is the involution of $\mathrm{DM}(\mathrm{Hck}^2_{G,I})$ induced by inversion on the second factor.

The first claim is a formal manipulation similar to, say, [BD14, Lemma A.10]:

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{DM}(\mathrm{Hck}_{G,I})}(M, \mathrm{D}^-(N)) &= \mathrm{Hom}_{\mathrm{DM}(\mathrm{Hck}_{G,I})}(M \otimes \mathrm{sw}(\mathrm{N}), \omega_{\mathrm{Hck_{G,I}}}) \\
&= \mathrm{Hom}_{\mathrm{DM}(\mathrm{Hck}_{G,I})}(\mathrm{inv}^* p^!(M \boxtimes N), f^! \mathbf{Z}) \\
&= \mathrm{Hom}_{\mathrm{DM}(X^I/L_I^+G)}(f_! \mathrm{inv}^* p^!(M \boxtimes N), \mathbf{Z}) \\
&= \mathrm{Hom}_{\mathrm{DM}(X^I/L_I^+G)}(i^* m_! p^!(M \boxtimes N), \mathbf{Z}) \\
&= \mathrm{Hom}_{\mathrm{DM}(X^I/L_I^+G)}(i^*(M \star N)[-I], \mathbf{Z}) \\
&= \mathrm{Hom}_{\mathrm{DM}(\mathrm{Hck}_{G,I})}(M \star N, i_* \mathbf{Z}[I]) \\
&= \mathrm{Hom}_{\mathrm{DM}(\mathrm{Hck}_{G,I})}(M \star N, 1).
\end{aligned}$$

Given this and Lemma 5.39, the category $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{lc}}$ is thus an r-category in the sense of [BD13, Definition 1.5], which gives a natural morphism as in (5.11). The final claim is then just [BD13, Corollary 4.5]. □

Theorem 5.41. *Let $\mathcal{F} \in \mathrm{Sat}_{(\mathrm{r})}^{G,I}$ be compact. Then $\mathcal{F}$ is dualizable with respect to ${}^{\mathrm{p}}\star$ if and only if $F^I(\mathcal{F})$ is dualizable in $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ with respect to the underived tensor product (Definition 5.36). In the case of reduced motives, this means that the graded components of $F^I(\mathcal{F})$ are flat abelian groups. If $\mathcal{F}$ is dualizable, its dual agrees with (the derived dual) $\mathrm{D}^-(\mathcal{F})$ (cf. (5.10)).*

Proof. Every compact object in $\mathrm{DTM}_{(\mathrm{r})}(X^I)$ is dualizable with respect to $\otimes$. (For compatibility with $\mathrm{MTM}_{(\mathrm{r})}$ we add a shift by $-|I|$ so that $\mathbf{Z}[I]$ is the unit.) A compact object in $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ is dualizable with respect to the underived tensor product if and only if its derived dual lies in $\mathrm{MTM}_{(\mathrm{r})}(X^I)$. This follows because the reduction functor is conservative and preserves inner Hom, so it reduces to the statement that a finitely generated abelian group is dualizable if and only if it is free.

Now let $\mathcal{C} \subset (\mathrm{Sat}_{(\mathrm{r})}^{G,I})^{\mathrm{c}}$ be the subcategory of objects such that $F^I(\mathcal{F})$ is dualizable in $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ with respect to the underived tensor product. We first show that $\mathcal{C}$ is stable under D and sw, so the last statement makes sense. As in [FS21, Proposition IV.6.13], D commutes with hyperbolic localization up to taking the inverse of the $\mathbf{G}_{\mathrm{m}}$-action. Also, CT_B^I commutes with sw because the diagram (5.1) is induced by homomorphisms of loop groups. Then by Corollary 5.25 we are reduced to proving that D and sw preserve $\mathcal{C}$ in the case $G = T$. This in turn follows from the explicit description of $\mathrm{Sat}_{(\mathrm{r})}^{T,I}$ in Proposition 5.22. Moreover, $\pi_{T!}$ is invariant under sw, and it commutes with D because π_T is proper. As F^I is independent of the choice of B when restricted to $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$, the functor $F^I\big|_{\mathcal{C}}$ preserves dualizing functors (and the derived duals agree with the underived duals).

Next, we claim that the natural map $\mathcal{F} \star \mathcal{F}' \to \mathcal{F} \, {}^{\mathrm{p}}\star \, \mathcal{F}'$ is an isomorphism for any $\mathcal{F} \in \mathcal{C}$ and $\mathcal{F}' \in \mathrm{Sat}_{(\mathrm{r})}^{G,I}$. To see this it suffices to apply the conservative functor F^I, which is monoidal (both with respect to $\star$ and to ${}^{\mathrm{p}}\star$). To check that the resulting map $F^I(\mathcal{F}) \otimes F^I(\mathcal{F}')[-I] \to {}^{\mathrm{p}}\mathrm{H}^0(F^I(\mathcal{F}) \otimes F^I(\mathcal{F}')[-I])$ is an isomorphism, we may pass to reduced motives, which again reduces to a statement about finite abelian groups.

Putting everything together, it follows that $F^I\big|_{\mathcal{C}}$ commutes with the formation of the map (5.11). Thus, every object $\mathcal{F} \in \mathcal{C}$ is dualizable with dual $\mathrm{D}(\mathrm{sw}(\mathcal{F}))$. For the converse, if $\mathcal{F} \in \mathrm{Sat}_{(\mathrm{r})}^{G,I}$ is dualizable with dual $\mathcal{F}^\vee$, by monoidality of F^I we have $F(\mathcal{F}^\vee) = F(\mathcal{F})^\vee$. □

6. Tannakian reconstruction

6.1. **The Hopf algebra object.** The goal of this section is to construct a Hopf algebra object $H_{(\mathrm{r})}^{G,I} \in \mathrm{MTM}_{(\mathrm{r})}(X^I)$ such that the Satake category is equivalent to comodules over that Hopf algebra (Theorem 6.20). Based on the results of the previous sections, the Satake category appears for formal reasons in a comonadic adjunction (Proposition 6.7). Several steps, including an analysis of (co)standard motives, are needed to show the relevant comonad is given by tensoring with a Hopf algebra.

Throughout, I denotes a nonempty finite set, and $W \subset (X_*(T)^+)^I$ a finite subset closed under the Bruhat order. Let $\mathrm{Gr}_{G,I}^W$ be the closure of the union of the strata $\mathrm{Gr}_{G,I}^{\circ,\mu}$ for $\mu \in W$, and let $i_W : \mathrm{Gr}_{G,I}^W \to \mathrm{Gr}_{G,I}$ be the closed embedding. In addition to the Satake category $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$, we consider its full subcategory $\mathrm{Sat}_{(\mathrm{r}),W}^{G,I}$ consisting of motives supported on $\mathrm{Gr}_{G,I}^W$.

6.1.1. *Adjunctions between motives on the Hecke prestack and on the curve.* We establish two adjoints for $F^I := \pi_{T!} u^! \mathrm{CT}_B^I$ (Definition 5.29). Recall that the restriction of this functor to $\mathrm{Sat}_{(\mathrm{r})}^{G,I}$ is the fiber functor.

Lemma 6.1. *The restriction of F^I to $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I})$ admits a left and a right adjoint given by*

$$\begin{aligned} L^I_W &= \mathrm{coav}\, p^-_{W!} q^{-*}_W \pi^*_{T,W}[-\deg], \\ R^I_W &= \mathrm{av}\, p^+_{W*} q^{+!}_W \pi^!_{T,W}[-\deg]. \end{aligned} \tag{6.1}$$

Here the subscript W denotes the restriction of (5.1) to $\mathrm{Gr}^W_{G,I} \subset \mathrm{Gr}_{G,I}$, and $\mathrm{Gr}^W_{T,I} = \mathrm{Gr}_{T,I} \cap \mathrm{Gr}^W_{G,I}$.

Proof. We will show that the forgetful functor $u^! : \mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \to \mathrm{DM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I})$ admits a right adjoint denoted $\mathrm{av} := \mathrm{av}_{L^+_I G}$ and that the restriction of $u^!$ to objects with bounded support, i.e., $u^!|_{\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I})}$, admits a left adjoint coav_W. Once this is shown, the formulae in (6.1) do define the advertised adjoints by Remark 5.3. Note that the restriction of (5.1) to $\mathrm{Gr}^W_{G,I} \subset \mathrm{Gr}_{G,I}$ consists of maps of schemes (as opposed to ind-schemes), so that the functors appearing in the definitions of L^I_W (notably q^{-*}_W) are well defined on the categories $\mathrm{DM}_{(\mathrm{r})}$.

By definition of motives on prestacks, see around (2.2), each !-pullback functor admits a right adjoint, so av exists. For each W, the $L^+_I G$-action on $\mathrm{Gr}^W_{G,I}$ factors over a smooth algebraic quotient group $L^+_I G \twoheadrightarrow H_W$, such that the kernel of this quotient map is split pro-unipotent by [RS20, Lemma A.3.5, Proposition A.4.9]. By the computation of equivariant motives in [RS20, Proposition 3.1.27], $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I}) = \mathrm{DM}_{(\mathrm{r})}(H_W \setminus \mathrm{Gr}^W_{G,I})$. Thus, the coaveraging functor coav_{H_W} from Lemma 2.21 is left adjoint to $u^!$, as claimed. □

Proposition 6.2. *The adjunction $L^I_W \dashv F^I \dashv R^I_W$ restricts to an adjunction on the categories $\mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I})$ and $\mathrm{DTM}_{(\mathrm{r})}(X^I)$.*

Proof. We need to show L^I_W and R^I_W preserves stratified Tate motives. The composites $p^+_{W*} q^{+!}_W \pi^!_{T,W}$ and $p^-_{W!} q^{-*}_W \pi^*_{T,W}$ map $\mathrm{DTM}_{(\mathrm{r})}(X^I, (X^I)^\dagger)$ to the category of motives on $\mathrm{Gr}_{G,I}$ that are Tate motives with respect to the stratification by the $S_{\nu,I} \cap \mathrm{Gr}^{\phi,\mu}_{G,I}$. It remains to be shown that av and coav map these motives to Tate motives with respect to the (coarser!) stratification by the $\mathrm{Gr}^{\phi,\mu}_{G,I}$.

The formation of av commutes with !-pullback over the strata $X^\phi \subset X^I$, and likewise for coav and $*$-pullback. Using the factorization property (4.4) and the fact that $\mathrm{Gr}^J_G \cong \mathrm{Gr}_{G^J}$, we only need to consider Gr_G. We treat the functor av; the argument for coav is dual. If $i\colon S_\nu \cap \mathrm{Gr}^\mu_G \to \mathrm{Gr}^\mu_G$ is the inclusion, it suffices to show that $u^! \,\mathrm{av}\, i_* \mathbf{Z}$ is Tate. Since this motive is $L^n G$-equivariant for $n \gg 0$, it suffices to show its !-restriction along the base point $t^\mu\colon S \to \mathrm{Gr}^\mu_G$ is a Tate motive [RS20, Lemma 2.2.21, Proposition 3.1.23]. Using Lemma 2.21 to compute this !-restriction, it suffices to show that $f_*(\mathbf{Z}) \in \mathrm{DM}(S)$ is Tate, where $f\colon a^{-1}(t^\mu) \to S$ and $a\colon L^n G \times (S_\nu \cap \mathrm{Gr}^\mu_G) \to \mathrm{Gr}^\mu_G$ is the action map. By Proposition 3.5, $a^{-1}(t^\mu) \cong \mathcal{P}^\mu_n \times (S_\nu \cap \mathrm{Gr}^\mu_G)$, where $\mathcal{P}^n_\mu$ is an extension of a split reductive $\mathbf{Z}$-group by a split unipotent $\mathbf{Z}$-group (Proposition 3.2). For the structural map $\pi : \mathcal{P}^n_\mu \to S$, we have $\pi_* \pi^* \mathbf{Z} \in \mathrm{DTM}(S)$ by virtue of the cellular Bruhat stratification. Since $S_\nu \cap \mathrm{Gr}^\mu_G$ is also stratified cellular, we conclude that $f_*(\mathbf{Z})$ is Tate. □

Corollary 6.3. *(1) Let $W' \subseteq W$ be a second finite subset, also closed under the Bruhat order. For $i : \mathrm{Gr}^{W'}_{G,I} \subset \mathrm{Gr}^W_{G,I}$, the adjoints are related by*

$$i^* L^I_W = L^I_{W'}, \ \ i^! R^I_W = R^I_{W'}. \tag{6.2}$$

Thus, the right adjoints assemble to a globally defined adjunction

$$F^I : \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) \rightleftarrows \mathrm{DTM}_{(\mathrm{r})}(X^I, (X^I)^\dagger) : R^I$$

which is given by

$$R^I = \operatorname*{colim}_W R^I_W. \tag{6.3}$$

(2) The preceding adjunctions restrict to adjunctions

$$\begin{aligned} F^I : \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I}) &\rightleftarrows \mathrm{MTM}_{(\mathrm{r})}(X^I) : {}^{\mathrm{p}}R^I = {}^{\mathrm{p}}\mathrm{H}^0\, \mathrm{av}\, p^+_* q^{+!} \pi^!_T[-\deg]. \\ {}^{\mathrm{p}}L^I_W = {}^{\mathrm{p}}\mathrm{H}^0\, \mathrm{coav}\, p^-_{W!} q^{-*}_W \pi^*_{T,W}[-\deg] : \mathrm{MTM}_{(\mathrm{r})}(X^I) &\rightleftarrows \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I}) : F^I. \end{aligned} \tag{6.4}$$

(3) The functors F^I, R^I, R^I_W and L^I_W are compatible with the reduction functor ρ_{r}.

Proof. (1): This holds by passing to adjoints starting from $F^I i_* = F^I$. For the existence of R^I recall that objects in $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$ are compatible systems–under !-pullback–of objects in $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I})$, for all W. The equation (6.3) is a generality about filtered colimits of presentable ∞-categories: for a filtered colimit $C = \mathrm{colim}_W C_W$ in $\mathrm{Pr}^{\mathrm{St}}$, there is an equivalence $\mathrm{colim}_W \iota_W \iota^R_W \xrightarrow{\cong} \mathrm{id}_C$, where $\iota_W : C_W \to C$ is the natural functor and ι^R_W its right adjoint.

(2): This holds since F^I is t-exact, so that R^I is left t-exact and L^I_W is right t-exact.

(3): The functor ρ_{r} commutes with F^I since the latter is a composite of the standard six functors (cf. Section 2.1.3). By construction, to check that ρ_{r} also commutes with R^I, we again only have to consider the finite-dimensional situation. There, the claim holds by Lemma 2.21(4). ☐

6.1.2. *Compatibility with Verdier duality and exterior products.* Recall the notation for Verdier duality from Section 5.3.1.

Lemma 6.4. *Up to inverting the $\mathbf{G}_{\mathrm{m}}$-action, the right and left adjoints are exchanged under Verdier duality. More precisely, let $L^I_{W^-}$ be the left adjoint of $\pi_{T!}u^!\mathrm{CT}^I_{B^-}$, where B^- is the opposite Borel. Then there is an isomorphism of functors $\mathrm{DTM}_{(\mathrm{r})}(X^I) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})^{\mathrm{op}}$:*

$$\mathrm{D}_{\mathrm{Hck}^W_{G,I}/S} L^I_{W^-} = R^I_W \mathrm{D}_{X^I/S}.$$

Proof. By Lemma 5.39, D is adjoint to itself on $\mathrm{DTM}(\mathrm{Gr}^W_{G,I})^{\mathrm{c}}$ and thus also on $\mathrm{DTM}(\mathrm{Hck}^W_{G,I})^{\mathrm{lc}}$. By Proposition 6.2, $L^I_{W^-}$ and R^I_W preserve Tate motives. Thus our claim is equivalent, by passing to right adjoints, to having a natural isomorphism (of functors $\mathrm{DTM}(\mathrm{Hck}^W_{G,I})^{\mathrm{lc}} \to (\mathrm{DTM}(X^I)^{\mathrm{c}})^{\mathrm{op}}$)

$$\pi_{T,W!}u^!\mathrm{CT}^I_{B^-}\mathrm{D}_{\mathrm{Hck}^W_{G,I}} = \mathrm{D}_{X^I}\pi_{T,W!}u^!\mathrm{CT}^I_B. \tag{6.5}$$

Hyperbolic localization commutes with Verdier duality, provided we invert the $\mathbf{G}_{\mathrm{m}}$-action [FS21, Proposition IV.6.13]. As $\deg_{B^-} = -\deg_B$, this gives an isomorphism $\mathrm{CT}^I_{B^-} \circ \mathrm{D} = \mathrm{D} \circ \mathrm{CT}^I_B$. We conclude using the above compatibility of D with $u^!$, as well as $\mathrm{D}\pi_{T,W!} = \pi_{T,W*}\mathrm{D} = \pi_{T,W!}\mathrm{D}$, since $\pi_{T,W}$ is proper. ☐

Proposition 6.5. *The adjunctions $L^I_W \dashv F^I \dashv R^I_W$ for Tate motives are compatible with the exterior product in the following sense. Consider the diagrams for $k = 1, 2$*

$$X^{I_k} \xleftarrow{\pi_k} \mathrm{Gr}^{W_k}_{T,I_k} \xleftarrow{q_k^+} \mathrm{Gr}^{W_k}_{B,I_k} \xrightarrow{p_k^+} \mathrm{Gr}^{W_k}_{G,I_k} \xrightarrow{u_k} \mathrm{Hck}_{G,I_k}$$

as in (5.1), where $I_k = \{*\}$ is a singleton. Let $R_k := R^{W_k}_{I_k}$ etc. and write

$$\begin{aligned}
R_{12} &:= \mathrm{av}_{12}(p_1^+ \times p_2^+)_*(q_1^+ \times q_2^+)^!(\pi_1 \times \pi_2)^!,\\
F_{12} &:= (u_1 \times u_2)^!(p_1^+ \times p_2^+)^*(q_1^+ \times q_2^+)_!(\pi_1 \times \pi_2)_!,\\
L_{12} &:= \mathrm{coav}_{12}(p_1^- \times p_2^-)_!(q_1^- \times q_2^-)^*(\pi_1 \times \pi_2)^*,
\end{aligned}$$

where $\mathrm{av}_{12} : \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Gr}_{G,I_1} \times \mathrm{Gr}_{G,I_2}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_1} \times \mathrm{Hck}_{G,I_2})$ is the right adjoint to $(u_1 \times u_2)^!$ and coav_{12} is its left adjoint. Then there are isomorphisms (of functors $\mathrm{DTM}_{(\mathrm{r})}(X^{I_1}) \times \mathrm{DTM}_{(\mathrm{r})}(X^{I_2}) \to \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_1} \times_S \mathrm{Hck}_{G,I_2})$)

$$L_{12}(- \boxtimes -) \to L_1(-) \boxtimes L_2(-). \tag{6.6}$$

$$R_1(-) \boxtimes R_2(-) \to R_{12}(- \boxtimes -). \tag{6.7}$$

as well as

$$F_1(-) \boxtimes F_2(-) \to F_{12}(- \boxtimes -). \tag{6.8}$$

Proof. First of all, the existence of the av_{12}, coav_{12} as stated is proven exactly the same way as for the single (co)averaging functors. The isomorphism (6.8) exists since *-pullbacks and !-pushforwards are compatible with $\boxtimes$. By adjunction, this gives the maps (6.6) and (6.7). The former map is an isomorphism even when evaluated on $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I_k})$ again because $\boxtimes$ commutes with *-pullbacks and !-pushforwards.

It remains to prove (6.7) is an isomorphism. Since Verdier duality is an involution on $\mathrm{DTM}(X^{I_k})^{\mathrm{c}}$, it suffices to show that the natural map

$$(R_1(\mathrm{D}\mathcal{F}_1)) \boxtimes (R_2(\mathrm{D}\mathcal{F}_2)) \to R_{12}(\mathrm{D}\mathcal{F}_1 \boxtimes \mathrm{D}\mathcal{F}_2)$$

is an isomorphism for $\mathcal{F}_k \in \mathrm{DTM}(X^{I_k})^{\mathrm{c}}$. We compute this map as the composite of the following isomorphisms

$$\begin{aligned}
R_1\mathrm{D}\mathcal{F}_1 \boxtimes R_2\mathrm{D}\mathcal{F}_2 &= \mathrm{D}L_1\mathcal{F}_1 \boxtimes \mathrm{D}L_2\mathcal{F}_2 \text{ (by Lemma 6.4)}\\
&= \mathrm{D}(L_1\mathcal{F}_1 \boxtimes L_2\mathcal{F}_2) \text{ (*)}\\
&= \mathrm{D}L_{12}(\mathcal{F}_1 \boxtimes \mathcal{F}_2) \text{ (by (6.6))}\\
&= R_{12}(\mathrm{D}(\mathcal{F}_1 \boxtimes \mathcal{F}_2)) \text{ (by Lemma 6.4)}\\
&= R_{12}(\mathrm{D}\mathcal{F}_1 \boxtimes \mathrm{D}\mathcal{F}_2) \text{ (*)}.
\end{aligned}$$

The isomorphisms marked (*) hold by Lemma 6.6 below, which is applicable since L_k preserves Tate motives. ☐

The following lemma is the technical replacement of the notion of ULA-ness.

Lemma 6.6. *Consider again two singletons $I_k = \{*\}$ and write $\mathrm{Hck}_k := \mathrm{Hck}_{G,I_k}$ for clarity. For $\mathcal{F}_k \in \mathrm{DTM}_{(\mathrm{r})}(\mathrm{Hck}_k)$, there are natural isomorphisms*

$$\mathrm{D}(\mathcal{F}_1) \boxtimes \mathrm{D}(\mathcal{F}_2) \overset{\cong}{\to} \underline{\mathrm{Hom}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \omega_{\mathrm{Hck}_1} \boxtimes \omega_{\mathrm{Hck}_2}) \overset{\cong}{\to} \underline{\mathrm{Hom}}(\mathcal{F}_1 \boxtimes \mathcal{F}_2, \omega_{\mathrm{Hck}_1 \times \mathrm{Hck}_2}) = \mathrm{D}(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

Proof. By definition of $\boxtimes$ on the Hecke prestacks, we may replace Hck_k by $\mathrm{Gr}_k := \mathrm{Gr}_{G,I_k}$. Let $f_k : \mathrm{Gr}_k \to S$ be the structural map. Following the method of [Cis21, Theorem 3.1.10], we first observe that the natural map $f_1^!\mathbf{Z} \boxtimes f_2^!\mathbf{Z} \to (f_1 \times f_2)^!\mathbf{Z}$ is an isomorphism. This follows from Lemma 3.8 and the observation that $\omega_{\mathrm{Gr}_k} = f_k^!\mathbf{Z}$ is a (non-compact) Tate motive. Since $\mathrm{DTM}(\mathrm{Gr}_k)$ is generated by $\iota_{k!}\mathbf{Z}$ (ι_k is the stratification map of Gr_k), we conclude by using $\mathrm{D}\iota_! = \iota_*\mathrm{D}$, and using $\iota_{1*}\mathbf{Z} \boxtimes \iota_{2*}\mathbf{Z} = (\iota_1 \times \iota_2)_*\mathbf{Z}$, which again follows from Lemma 3.8. ☐

6.1.3. *Adjunction for the Satake category.* We now construct an adjunction involving the fiber functor $F^I : \mathrm{Sat}^{G,I}_{(\mathrm{r})} \to \mathrm{MTM}_{(\mathrm{r})}(X^I)$. The right adjoint R^I_{Sat} will computed explicitly in Proposition 6.17.

Proposition 6.7. *There is a comonadic adjunction*

$$F^I : \mathrm{Sat}^{G,I}_{(\mathrm{r})} \rightleftarrows \mathrm{MTM}_{(\mathrm{r})}(X^I) : R^I_{\mathrm{Sat}}. \tag{6.9}$$

In other words, there is an equivalence

$$\mathrm{Sat}^{G,I}_{(\mathrm{r})} = \mathrm{coMod}_{T^I}(\mathrm{MTM}_{(\mathrm{r})}(X^I)),$$

where $T^I := F^I \circ R^I_{\mathrm{Sat}}$ is the comonad induced by the adjunction and coMod *denotes the category of comodules over that comonad. Here $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ denotes the category of mixed unstratified Tate motives on X^I.*

Similarly, the restriction of F^I to $\mathrm{Sat}^{G,I}_{(\mathrm{r}),W}$ yields a comonadic adjunction with right adjoint denoted $R^I_{\mathrm{Sat},W}$, so that with $T^{G,I}_W := F^I \circ R^I_{\mathrm{Sat},W}$:

$$\mathrm{Sat}^{G,I}_{(\mathrm{r}),W} = \mathrm{coMod}_{T^I_W}(\mathrm{MTM}_{(\mathrm{r})}(X^I)).$$

Proof. The restriction of F^I to $\mathrm{Sat}^{G,I}_{(\mathrm{r})}$ takes values in unstratified Tate motives by Proposition 5.28. On the level of DTM, the functor $\pi_{T!}$ is left adjoint to $\pi_T^!$, hence it preserves coproducts. The constant term functor also preserves coproducts. The functor F^I is t-exact by Corollary 5.12, and therefore preserves all colimits. Both categories are presentable (in fact compactly generated) so the adjoint functor theorem guarantees the existence of a right adjoint R^I_{Sat}.

In addition, F^I is conservative again by Corollary 5.12. Being t-exact, it also preserves finite limits, so that the adjunction is comonadic by the Barr–Beck comonadicity theorem. ☐

Our eventual goal is to show that T^I preserves colimits. Part of this is immediate:

Lemma 6.8. *The functors R^I_{Sat} and $R^I_{\mathrm{Sat},W}$, and thus the comonads T^I and T^I_W, preserve filtered colimits.*

Proof. Both $\mathrm{Sat}^{G,I}_{(\mathrm{r})}$ and $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ are compactly generated. Thus R^I_{Sat} preserves filtered colimits as soon as F^I preserves compact objects. Generally, for a category C, compact objects in the ind-completion $\mathrm{Ind}(C)$ are the retracts of objects in C. Thus, compact objects in $\mathrm{Sat}^{G,I}_{(\mathrm{r})}$ are precisely the retracts of sheaves $\mathcal{F}$ as in (5.5). Thus, it suffices to see that $\mathrm{IC}_{\mu,L}$ (Definition 2.16) with $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$ is mapped to a compact object under $\pi_{T!}\mathrm{CT}^I_B$. Indeed, $\pi_!$, for any map π of ind-schemes, preserves compact objects (even for DM). In addition, p^{+*} preserves compact objects since the map is schematic. ☐

6.1.4. *Standard and costandard motives.* Lemma 6.8 does not cover the preservation of finite colimits, i.e., cokernels. To handle these, we adapt the method of (co)standard sheaves [BR18, §1.11] to a motivic context. In this subsection, we only consider the local case, i.e., Gr_G instead of $\mathrm{Gr}_{G,I}$. Using the isomorphism $\mathrm{Gr}_{G,\{*\}} = \mathrm{Gr}_G \times X$ and homotopy invariance, the results in this subsection immediately yield similar results for Beilinson–Drinfeld Grassmannians in the case $I = \{*\}$. In particular, if $I = \{*\}$ in Proposition 6.2, we may suppress this extra factor of X and consider the triple $L_W \dashv F \dashv R_W$ as an adjunction on the categories $\mathrm{DTM}_{(\mathrm{r})}(L^+G \setminus \mathrm{Gr}^W_G)$ and $\mathrm{DTM}_{(\mathrm{r})}(S)$.

Definition 6.9. The *standard* and *costandard functors* are the functors $\mathrm{MTM}_{(\mathrm{r})}(S) \to \mathrm{MTM}_{(\mathrm{r})}(L^+G \backslash \mathrm{Gr}_G)$ defined as

$$\mathcal{J}^\mu_! := {}^{\mathrm{p}}\iota^\mu_!(p^*_\mu(-)[\dim \mathrm{Gr}^\mu_G]), \quad \mathcal{J}^\mu_* := {}^{\mathrm{p}}\iota^\mu_*(p^*_\mu(-)[\dim \mathrm{Gr}^\mu_G]),$$

where $p_\mu : L^+G \setminus \mathrm{Gr}^\mu_G \to X$ and $\iota^\mu : L^+G \setminus \mathrm{Gr}^\mu_G \to L^+G \setminus \mathrm{Gr}_G$.

Here $p^*_\mu : \mathrm{DM}_{(\mathrm{r})}(S) \to \mathrm{DM}_{(\mathrm{r})}(L^+G \setminus \mathrm{Gr}^\mu_G)$ denotes the functor whose composition with the forgetful functor to $\mathrm{DM}_{(\mathrm{r})}(\mathrm{Gr}^\mu_G)$ is the usual functor p^*_μ (and the components in further terms of the Čech nerve of the L^+G-action, as in (2.7), are given by !-pullbacks along the action, resp. projection maps).

Remark 6.10. Using the equivalence $\mathrm{MTM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G) \cong \mathrm{Sat}_{(\mathrm{r})}^{G,\{*\}}$, we get corresponding (co)standard functors $\mathrm{MTM}_{(\mathrm{r})}(S) \to \mathrm{Sat}_{(\mathrm{r})}^{G,\{*\}}$ which we denote the same way. When using this notation, the target of the functors will clear from the context.

Proposition 6.11. *For $\mu \in X_*(T)^+$ and $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$ there are functorial isomorphisms*

$$\mathcal{J}_!^\mu(\mathbf{Z}) \otimes L \xrightarrow{\cong} \mathcal{J}_!^\mu(L), \quad \mathcal{J}_*^\mu(L) \xrightarrow{\cong} \mathcal{J}_*^\mu(\mathbf{Z}) \otimes L.$$

(The tensor product is formed in $\mathrm{DTM}_{(\mathrm{r})}(L^+G \setminus \mathrm{Gr}_G)$, i.e., it is a derived tensor product.)

Proof. The first map is the composite ${}^{\mathrm{p}}\mathrm{H}^0\iota_!^\mu(\mathbf{Z}[\dim \mathrm{Gr}_G^\mu])\otimes L \to \iota_!^\mu(\mathbf{Z}[\dim \mathrm{Gr}_G^\mu])\otimes L = \iota_!^\mu(L[\dim \mathrm{Gr}_G^\mu]) = \iota_!^\mu\iota^{\mu!}\mathcal{J}_!^\mu(L) \to \mathcal{J}_!^\mu(L)$ and dually for the second one.

Being supported on $\mathrm{Gr}_G^{\leq\mu}$, both motives have bounded support, so our claim follows from Corollary 5.12 and the computation in Lemma 6.12. □

Lemma 6.12. *The composite $F_\nu\mathcal{J}_!^\mu$ is isomorphic to the endofunctor on $\mathrm{MTM}_{(\mathrm{r})}(S)$ given by*

$$L \mapsto \bigoplus_{\mathrm{Irr}(S_\nu\cap\mathrm{Gr}^\mu)} L(-\dim S_\nu \cap \mathrm{Gr}^\mu),$$

where the direct sum runs over the irreducible components of $S_\nu \cap \mathrm{Gr}^\mu$. A similar statement holds for $F_\nu\mathcal{J}_*^\mu$, where the direct sum is indexed by $\mathrm{Irr}(T_\nu \cap \mathrm{Gr}^\mu)$.

Proof. By t-exactness of F_ν we have

$$F_\nu\mathcal{J}_!^\mu \cong {}^{\mathrm{p}}\mathrm{H}^{\langle 2\rho,\nu\rangle}((q_\nu^+)_!(p_\nu^+)^*\iota_!^\mu[\langle 2\rho,\mu\rangle]) = {}^{\mathrm{p}}\mathrm{H}^{\langle 2\rho,\mu+\nu\rangle}(q_\nu^+)_!\iota_!'^\mu.$$

By Theorem 3.32, $S_\nu \cap \mathrm{Gr}_G^\mu$ is cellular of equidimension $\langle \rho, \mu+\nu\rangle$ relative to S, so this expression computes the top-dimensional cohomology group with compact support.

For a cellular stratified scheme $Z \xrightarrow{\pi} X$ that is of equidimension d, let $Z^\circ = \bigsqcup_i \mathbf{A}^{n_i} \times_X \mathbf{G}_{\mathrm{m}}^{m_i}$ be the disjoint union of the top-dimensional cells (i.e., $n_i + m_i = d$) and $j : Z^\circ \to Z$. Then

$${}^{\mathrm{p}}\mathrm{H}^{2d}\pi_!j_!j^*\pi^* \to {}^{\mathrm{p}}\mathrm{H}^{2d}\pi_!\pi^*$$

is an isomorphism (of endofunctors on $\mathrm{MTM}_{(\mathrm{r})}(S)$) by localization. The left hand functor is tensoring with $\bigoplus_i \mathbf{Z}(-d)$.

The argument for $\mathcal{J}_*$ is dual to this, using instead the affine Grassmannian Gr_{B^-} for the opposite Borel. In this case it is cohomology in degree 0 of the top-dimensional cells which contributes. □

Example 6.13. Following up on Example 3.33 we consider $G = \mathrm{PGL}_2$. In this case $F_\nu\mathcal{J}_!^\mu L = L(-\frac{\mu+\nu}{2})$ if $|\nu| \leq \mu$ and $\nu \equiv \mu \mod 2$. In all other cases, $F_\nu\mathcal{J}_!^\mu = 0$.

Corollary 6.14. *Let $W \subset X_*(T)^+$ be a finite subset closed under the Bruhat order. For a maximal element $\mu \in W$, let $j_\mu\colon \mathrm{Gr}_G^\mu \to \mathrm{Gr}_G^W$ be the open immersion. Then for $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$ we have*

$${}^{\mathrm{p}}j_\mu^*R_W(L) \cong \bigoplus_{\nu\in X_*(T)} p_\mu^*L(\dim S_\nu \cap \mathrm{Gr}^\mu)[\dim \mathrm{Gr}_G^\mu]^{\bigoplus|\mathrm{Irr}(S_\nu\cap\mathrm{Gr}^\mu)|}.$$

A similar statement holds for the functor ${}^{\mathrm{p}}j_\mu^*L_{W^-}$.

Proof. By Proposition 4.31 we can take quotients by L^+G. Then we have adjunctions, with left adjoints on top of their right adjoints:

$$\mathrm{MTM}_{(\mathrm{r})}(S) \underset{\cong}{\overset{p_\mu^*[\dim \mathrm{Gr}_G^\mu]}{\rightleftarrows}} \mathrm{MTM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G^\mu) \underset{{}^{\mathrm{p}}j_\mu^*}{\overset{{}^{\mathrm{p}}j_{\mu,!}}{\rightleftarrows}} \mathrm{MTM}_{(\mathrm{r})}(L^+G\backslash \mathrm{Gr}_G^W) \underset{{}^{\mathrm{p}}R_W}{\overset{F}{\rightleftarrows}} \mathrm{MTM}_{(\mathrm{r})}(S).$$

The equivalence on the left follows from Proposition 4.29. By Lemma 6.12, the functor going right is given by tensoring with $\bigoplus_\nu \mathbf{Z}(-\dim S_\nu \cap \mathrm{Gr}_G^\mu)^{\bigoplus|\mathrm{Irr}(S_\nu\cap\mathrm{Gr}_G^\mu)|}$, hence its right adjoint is given by $\underline{\mathrm{Hom}}(\bigoplus_\nu \mathbf{Z}(-\dim S_\nu \cap \mathrm{Gr}^\mu)^{\bigoplus|\mathrm{Irr}(S_\nu\cap\mathrm{Gr}_G^\mu)|}, -)$. This implies that ${}^{\mathrm{p}}j_\mu^*R_W(L)$ has the stated form. The argument for ${}^{\mathrm{p}}j_\mu^*L_{W^-}$ is dual to this, using the computation of $F_\nu\mathcal{J}_*^\mu$ instead. □

Proposition 6.15. *The kernel and cokernel of the natural morphism $\alpha_L\colon \mathcal{J}_!^\mu(L) \to \mathcal{J}_*^\mu(L)$ are killed by an integer $N > 0$ independent of $L \in \mathrm{MTM}_{(\mathrm{r})}(S)$. Furthermore, for $\mu \in X_*(T)^+$ the canonical surjection $\mathcal{J}_!^\mu(\mathbf{Z}) \to \mathrm{IC}_{\mu,\mathbf{Z}}$ is an isomorphism.*

Proof. For the first statement, let $\mathrm{fib}(\alpha_L)$ be the homotopy fiber of $\mathcal{J}_!^\mu(L) \to \mathcal{J}_*^\mu(L)$. By Lemma 6.12, $\mathrm{fib}(\alpha_L) \cong \mathrm{fib}(\alpha_{\mathbf{Z}}) \otimes L$. Thus, it suffices to show that $\mathrm{fib}(\alpha_{\mathbf{Z}})$ is N-torsion for some $N > 0$. Since $\alpha_{\mathbf{Z}}$ is a map between objects with bounded support, it suffices to prove the statement after applying F_ν. Then we may restrict to reduced motives by Lemma 2.15. By Lemma 6.12, $F_\nu(\alpha_{\mathbf{Z}})$ is a map between finite free $\mathbf{Z}$-modules, so it suffices to check it becomes an isomorphism after applying $-\otimes_{\mathbf{Z}} \mathbf{Q}$. The category $\mathrm{MTM_r}(\mathrm{Gr}_G, \mathbf{Q})^{\mathrm{c}}$ is semisimple: by Lemma 2.10 (applicable by Theorem 4.21), we may assume $S = \operatorname{Spec} \mathbf{F}_p$ for this, then apply Lemma 2.24 and [RS21, Corollary 6.4]. In addition $\mathrm{IC}_{\mu,\mathbf{Q}} \in \mathrm{MTM_r}(\mathrm{Gr}_G, \mathbf{Q})$ is a simple object, so the natural morphisms $\mathcal{J}_!^\mu(\mathbf{Q}) \to \mathrm{IC}_{\mu,\mathbf{Q}}$ and $\mathrm{IC}_{\mu,\mathbf{Q}} \to \mathcal{J}_*^\mu(\mathbf{Q})$ are isomorphisms. Since $F_\nu(\alpha_{\mathbf{Z}}) \otimes_{\mathbf{Z}} \mathbf{Q}$ is the composite of these morphisms, it is an isomorphism. The fact that $\mathcal{J}_!^\mu(\mathbf{Z}) \to \mathrm{IC}_{\mu,\mathbf{Z}}$ is an isomorphism may also be checked on reduced motives, so this completes the proof. ☐

Lemma 6.16. *In the notation of Corollary 6.14, let $\overline{W} = W \setminus \{\mu\}$.*

(1) The mixed Tate motive ${}^{\mathrm{p}}\mathrm{H}^n F(R_W(\mathbf{Z}))$ can be identified with a free graded $\mathbf{Z}$-module for $n = 0, 1$ (regarded as an object in $\mathrm{MTM}(S)$ via the natural functor, cf. (6.15)). Furthermore, $F({}^{\mathrm{p}}R(\mathbf{Z}))$ is flat.

(2) For $L \in \mathrm{MTM_{(r)}}(S)$, localization yields an exact sequence (in $\mathrm{MTM_{(r)}}(\mathrm{Hck}_G)$)

$$0 \to {}^{\mathrm{p}}R_{\overline{W}}(L) \to {}^{\mathrm{p}}R_W(L) \to {}^{\mathrm{p}}j_{\mu*}j_\mu^{*\mathrm{p}}R_W(L) \to 0.$$

(3) For $L \in \mathrm{MTM_{(r)}}(S)$, there is a canonical isomorphism

$${}^{\mathrm{p}}R(\mathbf{Z}) \otimes L \cong {}^{\mathrm{p}}R(L).$$

Proof. (1): The proof is similar to [FS21, VI.10.1]. First, flatness of $F({}^{\mathrm{p}}R(\mathbf{Z})) = {}^{\mathrm{p}}\mathrm{H}^0 F(R(\mathbf{Z}))$ will follow from freeness of ${}^{\mathrm{p}}\mathrm{H}^0 F(R_W(\mathbf{Z}))$. By Lemma 6.4, $R_W(\mathbf{Z})$ is the Verdier dual of $L_{W^-}(\mathbf{Z})$. We again use the isomorphism $\mathrm{D}F = F_-\mathrm{D}$, cf. (6.5), where $F_- = \pi_{T!}u^!\mathrm{CT}_{B^-}$. Thus, $F(R_W(\mathbf{Z}))$ is Verdier dual to $F_-(L_{W^-}(\mathbf{Z}))$. Since $F_-(L_{W^-}(\mathbf{Z})) \in \mathrm{DTM_{(r)}}(S)^{\leq 0}$, then ${}^{\mathrm{p}}\mathrm{H}^n F(R_W(\mathbf{Z}))$ is free for $n = 0, 1$ if ${}^{\mathrm{p}}\mathrm{H}^0 F_-(L_{W^-}(\mathbf{Z}))$ is a free, finitely graded $\mathbf{Z}$-module. By Corollary 6.14, $j_\mu^{*\mathrm{p}}L_{W^-}(\mathbf{Z})$ can be identified with a free, finitely generated graded $\mathbf{Z}$-module. Then by localization there is an exact sequence

$$0 \to {}^{\mathrm{p}}j_{\mu!}j_\mu^{*\mathrm{p}}L_{W^-}(\mathbf{Z}) \to {}^{\mathrm{p}}L_{W^-}(\mathbf{Z}) \to {}^{\mathrm{p}}L_{\overline{W}^-}(\mathbf{Z}) \to 0.$$

Injectivity of the left map follows from Proposition 6.15, because the kernel is contained in the kernel of ${}^{\mathrm{p}}j_{\mu!}j_\mu^{*\mathrm{p}}L_{W^-}(\mathbf{Z}) \to {}^{\mathrm{p}}j_{\mu*}j_\mu^{*\mathrm{p}}L_{W^-}(\mathbf{Z})$. By Lemma 6.12, $F_-({}^{\mathrm{p}}j_{\mu!}j_\mu^{*\mathrm{p}}L_W(\mathbf{Z}))$ is a free graded $\mathbf{Z}$-module, and by induction $F_-({}^{\mathrm{p}}L_{\overline{W}}(\mathbf{Z}))$ is free. Thus ${}^{\mathrm{p}}\mathrm{H}^0 F_-(L_W(\mathbf{Z}))$ is also free.

(2): By localization, the sequence is exact except possibly at the right. Thus we have to show that $C(L) := \mathrm{coker}({}^{\mathrm{p}}R_W(L) \to {}^{\mathrm{p}}j_{\mu*}j_\mu^{*\mathrm{p}}R_W(L)) \in \mathrm{MTM_{(r)}}(L^+G\backslash \mathrm{Gr}_G)$ vanishes. The composite $j_\mu^{*\mathrm{p}}R_W$ preserves surjections by Corollary 6.14, and hence so does ${}^{\mathrm{p}}j_{\mu*}j_\mu^{*\mathrm{p}}R_W$ by Proposition 6.11. Thus C also preserves surjections. Since it also preserves filtered colimits (Lemma 6.8) and is compatible with Tate twists, it suffices to show $C(\mathbf{Z}) = 0$.

The map ${}^{\mathrm{p}}j_{\mu!}j_\mu^{*\mathrm{p}}R_W(\mathbf{Z}) \to {}^{\mathrm{p}}j_{\mu*}j_\mu^{*\mathrm{p}}R_W(\mathbf{Z})$ factors through ${}^{\mathrm{p}}R_W(\mathbf{Z})$, so by Proposition 6.15, $C(\mathbf{Z})$ is killed by an integer $N > 0$. Thus, the claim will follow from freeness of ${}^{\mathrm{p}}\mathrm{H}^1 F(R_{\overline{W}}(\mathbf{Z}))$, once we show that $F(C(\mathbf{Z})) \subset {}^{\mathrm{p}}\mathrm{H}^1 F(R_{\overline{W}}(\mathbf{Z}))$ is a torsion submodule of a free module. But for this, if i is the complementary embedding to j_μ, we have a sequence of injections $C(\mathbf{Z}) \to {}^{\mathrm{p}}\mathrm{H}^1(i^{!\mathrm{p}}R_W(\mathbf{Z})) \to {}^{\mathrm{p}}\mathrm{H}^1(i^! R_W(\mathbf{Z}))$, where $i^! R_W(\mathbf{Z}) = R_{\overline{W}}(\mathbf{Z})$. Here the first map is an injection by the localization sequence for ${}^{\mathrm{p}}R_W(\mathbf{Z})$, and the second map is an injection because $i^!$ is left t-exact.

(3): We will construct isomorphisms ${}^{\mathrm{p}}R_W(\mathbf{Z}) \otimes L \cong {}^{\mathrm{p}}R_W(L)$ functorial in W and L. The functors defining R_W and F are $\mathrm{DTM_{(r)}}(S)$-linear, so there is a natural isomorphism $R_W(\mathbf{Z}) \otimes L \cong R_W(L)$. Thus, it suffices to show that the natural map ${}^{\mathrm{p}}R_W(\mathbf{Z}) \otimes L \to R_W(\mathbf{Z}) \otimes L$ identifies the source with ${}^{\mathrm{p}}\mathrm{H}^0(R_W(\mathbf{Z}) \otimes L)$. We may assume L is a compact object. Then, using Lemma 2.15, it suffices to consider reduced motives. Thus, $L \in \mathrm{MTM_r}(S)^{\mathrm{c}}$ is a graded abelian group. We may then check the claim after applying F, where it follows from the fact that $F(R_W(\mathbf{Z})) \in \mathrm{DTM_r}(S)^{\geq 0}$ is free in degrees 0 and 1, and because $\mathbf{Z}$ has global dimension one. ☐

6.1.5. *Compatibility with the fusion product.*

Proposition 6.17. *(1) The right adjoint in (6.9) is given by*

$$R^I_{\mathrm{Sat}} = j_{!*}j^{*\mathrm{p}}R^I$$

where $j : \mathrm{Hck}^\circ_{G,I} \to \mathrm{Hck}_{G,I}$ is the inclusion over the open locus of pairwise distinct points.

(2) For $W = \prod_i W_i \subset (X_*(T)^+)^I$ and $L \in \mathrm{MTM_{(r)}}(X^I)$ there is a functorial isomorphism

$$R^I_{W,\mathrm{Sat}}(L) \cong *_{i\in I}{}^{\mathrm{p}}R^{\{i\}}_{W_i}(\mathbf{Z}[1]) \otimes L[-I].$$

Proof. (1) We first show that ${}^{\mathrm{p}}R^I(L)$, equivalently each ${}^{\mathrm{p}}R^I_W(L)$, has no subobjects supported over $X^I \setminus X^\circ$. We can assume W is of the form $W = \prod W_i$. Let $j\colon \mathrm{Hck}^{\circ,W}_{G,I} \to \mathrm{Hck}^W_{G,I}$ be the inclusion over X° and let $i\colon \mathrm{Hck}^W_{G,I} \setminus \mathrm{Hck}^{\circ,W}_{G,I} \to \mathrm{Hck}^W_{G,I}$ be its complement. We claim that if $\mathcal{F} \in \mathrm{DTM}^{\geq 0}_{(\mathrm{r})}(X^I)$ is an unstratified Tate motive, then $i^! R^I_W(\mathcal{F}) \in \mathrm{DTM}^{\geq 1}_{(\mathrm{r})}(\mathrm{Hck}^W_{G,I} \setminus \mathrm{Hck}^{\circ,W}_{G,I})$. By base change applied to $\mathrm{av}\, p^+_*$ (Lemma 2.21(5)), the formation of $i^! R^I_W$ commutes with !-restriction to any of the $\binom{|I|}{2}$ hyperplanes which comprise $X^I \setminus X^\circ$. Hence the claim follows from relative purity and the left t-exactness of $R^{I\setminus\{*\}}_W$. Because $i^!$ is also left t-exact, ${}^{\mathrm{p}}R^I_W(L)$ indeed has no subobjects supported over $X \setminus X^\circ$.

Next we will show that $j_{!*}j^* {}^{\mathrm{p}}R^I$ takes values in the subcategory $\mathrm{Sat}^{G,I}_{(\mathrm{r})} \subset \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,I})$. Using this claim, we conclude using Lemma 2.19: for $\mathcal{F} \in \mathrm{Sat}^{G,I}_{(\mathrm{r})}$ we have

$$\mathrm{Hom}(\mathcal{F}, {}^{\mathrm{p}}R^I(L)) \cong \mathrm{Hom}(\mathcal{F}, j_{!*}j^* {}^{\mathrm{p}}R^I(L)).$$

Over X°, we have a Künneth formula for the right adjoints by Proposition 6.5. Hence $j^* R^I_W(L) \cong \boxtimes_{i\in I} R^{\{i\}}_{W_i}(\mathbf{Z}) \otimes L\Big|_{X^\circ}$. We claim that the natural maps ${}^{\mathrm{p}}R^{\{i\}}_{W_i}(\mathbf{Z}[1]) \to R^{\{i\}}_{W_i}(\mathbf{Z}[1])$ induce an isomorphism $\boxtimes_{i\in I} {}^{\mathrm{p}}R^{\{i\}}_{W_i}(\mathbf{Z}[1]) \otimes L[-I] \cong {}^{\mathrm{p}}\mathrm{H}^0(\boxtimes_{i\in I} R^{\{i\}}_{W_i}(\mathbf{Z}[1]) \otimes L[-I])$. Arguing as in the proof of Lemma 6.16(3), this follows from Lemma 6.16(1). Thus,

$$\boxtimes_{i\in I} {}^{\mathrm{p}}R^{\{i\}}_{W_i}(\mathbf{Z}[1]) \otimes L[-I]\Big|_{X^\circ} \cong j^*({}^{\mathrm{p}}R^I_W(L)). \tag{6.10}$$

Moreover, if $|I| = 1$, we have $\mathrm{Sat}_{G,\{*\}} = \mathrm{MTM}(\mathrm{Hck}_{G,\{*\}})$. Now we conclude by using that fusion preserves the Satake categories by Theorem 5.32.

(2): By $\mathrm{DTM}_{(\mathrm{r})}(X^I)$-linearity, there is an isomorphism $R^I_W(\mathbf{Z}[I]) \otimes L[-I] \cong R^I_W(L)$ functorial in W and L. Using (6.10), applying $j_{!*}{}^{\mathrm{p}}j^*$ produces the desired isomorphism. $\square$

Corollary 6.18. *The object $T^I(\mathbf{Z}[I])$ is a flat object with respect to the underived tensor product in $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ (Definition 5.36).*

Proof. In order to show that $\mathrm{MTM}_{(\mathrm{r})}(X^I) \ni L \mapsto F^I R^I_{\mathrm{Sat}}(\mathbf{Z}[I]) \otimes L = F^I(R^I_{\mathrm{Sat}}(\mathbf{Z}[I]) \otimes L)$ (underived tensor products, cf. Definition 5.36) preserves finite limits, we may replace R^I_{Sat} by $R^I_{W,\mathrm{Sat}}$ as in Proposition 6.17(2) above, according to which we may assume $I = \{*\}$. We have $R^{\{*\}}_{\mathrm{Sat}} = R^{\{*\}}$, so we are done by Lemma 6.16(1). $\square$

Corollary 6.19. *The comonads $T^{G,I}$ and $T^{G,I}_W$ preserve colimits.*

Proof. By the continuity of F^I and (6.3), we only need to show this for $R^I_{\mathrm{Sat},W}$. By Lemma 6.8, it suffices to prove $R^I_{\mathrm{Sat},W}$ preserves cokernels. This follows from Proposition 6.17(2). $\square$

6.1.6. *The Hopf algebra.*

Theorem 6.20. *The object*

$$H^{G,I}_{(\mathrm{r})} := F^I R^I_{\mathrm{Sat}} \mathbf{Z}[I]$$

is a commutative Hopf algebra in $\mathrm{MTM}_{(\mathrm{r})}(X^I)$. If $\widetilde{G}_{I,(\mathrm{r})} \in \mathrm{MTM}_{(\mathrm{r})}(X^I)^{\mathrm{op}}$ denotes the corresponding monoid, there is an equivalence of symmetric monoidal categories, for the underived tensor product on $\mathrm{MTM}_{(\mathrm{r})}(X^I)$ (Definition 5.36):

$$(\mathrm{Sat}^{G,I}_{(\mathrm{r})}, {}^{\mathrm{p}}\star) \cong (\mathrm{Rep}_{\widetilde{G}_{I,(\mathrm{r})}}(\mathrm{MTM}_{(\mathrm{r})}(X^I)), \otimes). \tag{6.11}$$

Proof. The functor F^I is symmetric monoidal (Proposition 5.37), so that R^I_{Sat} is symmetric lax monoidal. Thus the composite $T^I = F^I \circ R^I_{\mathrm{Sat}}$ is also lax monoidal, so that in particular $H^{G,I}_{(\mathrm{r})} = T^I(\mathbf{Z}[I])$ is a commutative algebra object in $\mathrm{MTM}_{(\mathrm{r})}(S)$. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{MTM}(X^I)$, the monoidality of F^I and adjunctions yield a natural map

$$T^I(\mathcal{F}_1) \otimes T^I(\mathcal{F}_2) \to T^I(\mathcal{F}_1 \otimes \mathcal{F}_2) \tag{6.12}$$

and

$$T^I(\mathbf{Z}[I]) \otimes \mathcal{F}_1 \to T^I(\mathcal{F}_1). \tag{6.13}$$

We claim this latter map is an isomorphism.

To see this, we observe that both functors are exact: the right hand side by Corollary 6.19 and Corollary 5.12, and the left hand side by Corollary 6.18. In addition, both functors preserve filtered colimits. Thus, to see that the natural transformation (6.13) is an isomorphism, it suffices to check it for $\mathcal{F}_1 = \mathbf{Z}(k)[I]$, which is clear.

In particular $H^{G,I}_{(\mathrm{r})}$ is a coalgebra, and (6.11) holds on the level of abelian categories. A routine, if tedious argument (e.g., [Moe02, Theorem 7.1] dualized; our map (6.12) is the one in Definition 1.1 there) shows that the monoidality of F^I (i.e., the forgetful functor $\mathrm{coMod}_H(\mathrm{MTM}_{(\mathrm{r})}(X^I)) \to \mathrm{MTM}_{(\mathrm{r})}(X^I))$) implies that $H^{G,I}_{(\mathrm{r})}$ is then a bialgebra, and hence (6.11) holds on the level of symmetric monoidal categories.

It remains to show the existence of an antipode. For each $W \subseteq (X_*(T)^*)^I$ as usual, $T_W^{G,I}(\mathbf{Z}[I])$ is flat by Lemma 6.16 (1) and compact, and hence dualizable. Thus, $R^I_{\mathrm{Sat},W}(\mathbf{Z}[I])$ is dualizable by Theorem 5.41. Using $^\vee$ to denote duals, it follows formally that

$$\begin{aligned} T_W^{G,I}(\mathbf{Z}[I]) &\cong \mathrm{Hom}((F^I \circ R^I_{\mathrm{Sat},W}(\mathbf{Z}[I]))^\vee, \mathbf{Z}) \\ &\cong \mathrm{Hom}((R^I_{\mathrm{Sat},W}(\mathbf{Z}[I]))^\vee, R^I_{\mathrm{Sat},W}(\mathbf{Z}[I])) \\ &\cong \mathrm{Hom}((R^I_{\mathrm{Sat},W}(\mathbf{Z}[I]))^\vee \otimes (R^I_{\mathrm{Sat},W}(\mathbf{Z}[I]))^\vee, \mathbf{Z}[I]). \end{aligned}$$

Switching the two copies of $(R^I_{\mathrm{Sat},W}(\mathbf{Z}[I]))^\vee$ as in [FS21, Proposition VI.10.2] gives an involution of $T_W^{G,I}(\mathbf{Z}[I])$, functorial in W. In particular, we get an involution of $T^{G,I}(\mathbf{Z}[I]) = \mathrm{colim}_W T_W^{G,I}(\mathbf{Z}[I])$. It is routine to check that this defines an antipode of $H^{G,I}_{(\mathrm{r})} = T^{G,I}(\mathbf{Z}[I])$. ☐

Corollary 6.21. *There is a natural isomorphism of Hopf algebras*

$$\otimes_{i\in I} H^{G,\{i\}}_{(\mathrm{r})} \cong H^{G,I}_{(\mathrm{r})}.$$

Proof. This follows from Proposition 6.17(2) and the fact that F^I is symmetric monoidal. ☐

6.1.7. *Rational and modular coefficients.* In the identification of the dual group below, we also need to work with $\mathbf{Q}$- and $\mathbf{F}_p$-coefficients. Let $\Lambda = \mathbf{Q}$ or $\mathbf{F}_p$. We define the category

$$\mathrm{Sat}^{G,\Lambda}_{(\mathrm{r})} := \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,\{*\}}, \Lambda) \subset \mathrm{MTM}_{(\mathrm{r})}(\mathrm{Hck}_{G,\{*\}}) = \mathrm{Sat}^{G,\{*\}}_{(\mathrm{r})}$$

of mixed Tate motives with rational, resp. modular coefficients as in Subsection 2.5.

Lemma 6.22. *The full subcategory $\mathrm{Sat}^{G,\Lambda}_{(\mathrm{r})} \subset \mathrm{Sat}^{G,\{*\}}_{(\mathrm{r})}$ is stable under the fusion product (Definition 5.35). In addition, the underived tensor product functor*

$$^{\mathrm{p}}\mathrm{H}^0(- \otimes \Lambda) : \mathrm{Sat}^{G,\{*\}}_{(\mathrm{r})} \to \mathrm{Sat}^{G,\Lambda}_{(\mathrm{r})}$$

is symmetric monoidal.

Proof. This is clear for $\Lambda = \mathbf{Q}$, since $\mathcal{F} \otimes \mathbf{Q} = \mathrm{colim}(\dots \mathcal{F} \xrightarrow{n} \mathcal{F} \dots)$ and all our functors are additive and preserve filtered colimits.

For $\Lambda = \mathbf{F}_p$, we write $\mathcal{F}/p := {}^{\mathrm{p}}\mathrm{H}^0(\mathcal{F} \otimes \mathbf{F}_p) = \mathrm{coker}(\mathcal{F} \xrightarrow{p} \mathcal{F})$ for an object $\mathcal{F}$ in an abelian category. Then $\mathrm{Sat}^{G,\mathbf{F}_p}_{(\mathrm{r})} = \{\mathcal{F} \in \mathrm{Sat}^{G,\{*\}}_{(\mathrm{r})}, \mathcal{F} = \mathcal{F}/p\}$. Both functors in (5.9) are right exact (combine Theorem 5.32 and the t-exactness of $i^!_{\bar\phi}[d_\phi]$, which follows from Proposition 5.27) so that in particular $(\mathcal{F}_1 {}^{\mathrm{p}}\star \mathcal{F}_2)/p = \mathcal{F}_1 {}^{\mathrm{p}}\star (\mathcal{F}_2/p)$. This shows our claims for $\mathbf{F}_p$-coefficients. ☐

The adjunction in (6.4) restricts to an adjunction

$$\mathrm{Sat}^{G,\Lambda}_{(\mathrm{r})} \underset{R}{\overset{F}{\rightleftarrows}} \mathrm{MTM}_{(\mathrm{r})}(X, \Lambda).$$

and $FR(\Lambda[1]) = FR(\mathbf{Z}[1]) \otimes \Lambda = {}^{\mathrm{p}}\mathrm{H}^0(FR(\mathbf{Z}[1]) \otimes \Lambda)$, according to Lemma 6.16(3). The image of $H^{G,\{*\}}_{(\mathrm{r})}$ under ${}^{\mathrm{p}}\mathrm{H}^0(-\otimes\Lambda)$ is again a commutative Hopf algebra object. If we denote the corresponding group by $\widetilde{G}_{(\mathrm{r}),\Lambda} \in \mathrm{Sat}^{G,\Lambda,\mathrm{op}}_{(\mathrm{r})}$, we get a version of (6.11) for Λ-coefficients.

Corollary 6.23. *There is an equivalence of symmetric monoidal categories*

$$\mathrm{Sat}^{G,\Lambda}_{(\mathrm{r})} = \mathrm{Rep}_{\widetilde{G}_{(\mathrm{r}),\Lambda}}(\mathrm{MTM}_{(\mathrm{r})}(X, \Lambda)).$$

6.2. **The dual group.** In this section we identify the group associated to the Hopf algebra object $\mathrm{H}^{G,I}_{(\mathrm{r})} \in \mathrm{MTM}_{(\mathrm{r})}(X^I)$. We first show in Theorem 6.24 that the unreduced Hopf algebra $H^{G,I}$ is in fact reduced. We then move on to compute the reduced Hopf algebra $H^{G,I}_{\mathrm{r}}$.

6.2.1. *Independence of the base.*

Theorem 6.24. *There is a natural isomorphism of Hopf algebras*

$$H^{G,I} = iH^{G,I}_{\mathrm{r}}, \tag{6.14}$$

where

$$i : \mathrm{MTM}_{\mathrm{r}}(X^I) = \mathrm{grAb} \to \mathrm{MTM}(X^I), \mathbf{Z}(k) \mapsto \mathbf{Z}(k) \tag{6.15}$$

is the natural symmetric monoidal functor.

As an immediate consequence we obtain that the Betti realization functor (for $S = \operatorname{Spec}\mathbf{Q}$) factors over the reduction functor:

$$\begin{tikzcd} \mathrm{MTM}(L^+G \setminus \mathrm{Gr}_G) \arrow[r, "\rho_{\mathrm{B}}"] \arrow[d, "\rho_{\mathrm{r}}"] & \mathrm{Perv}_{L^+G}(\mathrm{Gr}_G) \\ \mathrm{MTM}_{\mathrm{r}}(L^+G \setminus \mathrm{Gr}_G) \arrow[ur, dotted] & \end{tikzcd}$$

This answers affirmatively a question posed in [ES22, §1.6.1].

The following result provides the key geometric input in the proof of Theorem 6.24. To begin, fix $\nu \in X_*(T)$ such that $T_\nu \cap \mathrm{Gr}_G^\mu \neq \emptyset$. For $n \gg 0$, we have the action map $a\colon L^nG \times (T_\nu \cap \mathrm{Gr}_G^\mu) \to \mathrm{Gr}_G^\mu$. Let $Y(\mu,\lambda,\nu) = a^{-1}(S_\lambda \cap \mathrm{Gr}_G^\mu)$ for one such n.

Proposition 6.25. *We have* $\dim Y(\mu,\lambda,\nu) = \dim L^nG - \langle\rho,\nu\rangle + \langle\rho,\lambda\rangle$ *and this scheme has a cellular stratification.*

Proof. Let F_0 be the fiber of $Y(\mu,\lambda,\nu)$ over t^μ, and let $s\colon S_\lambda \cap \mathrm{Gr}_G^\mu \to L^nG$ be a section as per Lemma 3.4. Using the left action of L^nG on $L^nG \times (T_\nu \cap \mathrm{Gr}_G^{\leq\mu})$, there is an isomorphism $(S_\lambda \cap \mathrm{Gr}_G^\mu) \times F_0 \to Y(\mu,\lambda,\nu)$, $(x,f) \mapsto s(x)\cdot f$. Since $S_\lambda \cap \mathrm{Gr}_G^\mu$ has a filtrable cellular decomposition (Theorem 3.32), it suffices to decompose F_0. By Proposition 3.5, $F_0 \cong \mathcal{P}_\mu^n \times (T_\nu \cap \mathrm{Gr}_G^\mu)$. Now we are done because $T_\nu \cap \mathrm{Gr}_G^\mu$ is cellular (Theorem 3.32), and $\mathcal{P}_\mu^n$ is cellular by the Bruhat decomposition. For the dimension, note that $\dim S_\lambda \cap \mathrm{Gr}_G^\mu = \langle\rho,\mu+\lambda\rangle$, $\dim T_\nu \cap \mathrm{Gr}_G^\mu = \langle\rho,\mu-\nu\rangle$, and $\dim \mathcal{P}_\mu^n = \dim L^nG - 2\langle\rho,\mu\rangle$. ☐

Proof of Theorem 6.24. The isomorphism $\rho_{\mathrm{r}}(H^{G,I}) = H_{\mathrm{r}}^{G,I}$ (Corollary 6.3(3)) is an isomorphism of Hopf algebras by construction. It therefore suffices to prove the isomorphism (6.14) after forgetting the (co)multiplication. By Corollary 6.21, it suffices to consider the case $I = \{*\}$. Note that i is a colimit-preserving fully faithful functor. By (6.3), it suffices to identify $F^{\{*\}}({}^{\mathrm{p}}R_W(\mathbf{Z}[1])) \in \mathrm{MTM}(X)$ with an object in the image of i for all finite $W \subset X_*(T)^+$ closed under the Bruhat order. As in the proof of Lemma 6.16(1), $F^{\{*\}}({}^{\mathrm{p}}R_W(\mathbf{Z}[1]))$ is Verdier dual to $F^{\{*\}}({}^{\mathrm{p}}L_{W^-}(\mathbf{Z}[1]))$. By restricting to connected components of $\mathrm{Gr}_{T,\{*\}}$, we get a direct sum decomposition $L_{W^-} = \oplus_{\nu\in X_*(T)} L_{W^-}^\nu$. Let $\widetilde{a}\colon L^nG \times (T_\nu \cap \mathrm{Gr}_G^{\leq\mu}) \to \mathrm{Gr}_G^{\leq\mu}$ be the action map and let $\widetilde{Y}(\mu,\lambda,\nu) = \widetilde{a}^{-1}(S_\lambda \cap \mathrm{Gr}_G^{\leq\mu})$. Then

$$F_\lambda^{\{*\}}(L_{W^-}^\nu(\mathbf{Z}[1])) = f_!\mathbf{Z}(2\dim L^nG)[2\dim L^nG - \langle 2\rho,\nu\rangle + \langle 2\rho,\lambda\rangle + 1],$$

where $f\colon \widetilde{Y}(\mu,\lambda,\nu) \times X \to X$ is the projection, cf. [BR18, Proposition 1.12.1]. Here we are using the isomorphism $\mathrm{Gr}_{G,\{*\}} = \mathrm{Gr}_G \times X$. The scheme $\widetilde{Y}(\mu,\lambda,\nu)$ is stratified by the $Y(\mu',\lambda,\nu)$ for $\mu' \leq \mu$, so by Proposition 6.25, $\widetilde{Y}(\mu,\lambda,\nu)$ is cellular of dimension $\dim L^nG - \langle\rho,\nu\rangle + \langle\rho,\lambda\rangle$. Thus, $F_\lambda^{\{*\}}({}^{\mathrm{p}}L_{W^-}^\nu(\mathbf{Z}[1]))$ computes the top cohomology of a cellular scheme, so as in the proof of Lemma 6.12 it is identified with a free graded $\mathbf{Z}$-module. ☐

6.2.2. *Identification of the Hopf algebra for reduced motives.* We now identify the group $\widetilde{G}_{I,(\mathrm{r})} \in \mathrm{MTM}_{(\mathrm{r})}(X^I)^{\mathrm{op}}$. By Corollary 6.21, it suffices to do this for $I = \{*\}$. By Theorem 6.24, it is enough to consider reduced motives, in which case we have $\mathrm{MTM}_{\mathrm{r}}(X) \cong \mathrm{grAb}$. In particular, we can describe $H_{\mathrm{r}}^G := H_{\mathrm{r}}^{G,\{*\}}$ by the affine $\mathbf{Z}$-group scheme $\widetilde{G}$ which underlies $\widetilde{G}_{\{*\},\mathrm{r}}$, together with a grading of its global sections. This is similar to [FS21, §VI.11], where we have a $\mathbf{G}_{\mathrm{m}}$-action instead of a W_E-action. We will therefore follow the methods of loc. cit. in the proof of the theorem below.

Consider the Langlands dual group $\hat{G}$, which is the pinned Chevalley group scheme with root datum dual to the root datum of G. In particular, it comes with a fixed choice $\hat{T} \subseteq \hat{B} \subseteq \hat{G}$ of maximal Torus and Borel. As in [FS21], we need to modify it to get a canonical identification of $\widetilde{G}$. Namely, for any simple root a of $\hat{G}$, instead of requiring an isomorphism $\mathrm{Lie}(\hat{U}_a) \cong \mathbf{Z}$ in the data of our pinning, we require an isomorphism $\mathrm{Lie}(\hat{U}_a) \cong \mathbf{Z}(1)$ (note that $\mathbf{Z}(1)$ does not have a preferred basis). In particular, this induces a $\mathbf{G}_{\mathrm{m}}$-action on all the root groups of $\hat{G}$. Letting $\mathbf{G}_{\mathrm{m}}$ act trivially on the root datum $(X_*(T), \Phi^\vee, X^*(T), \Phi)$, we get a $\mathbf{G}_{\mathrm{m}}$-action on (the modified) $\hat{G}$.

Now, let $\hat{T}_{\mathrm{adj}} \subseteq \hat{G}_{\mathrm{adj}}$ be the adjoint torus of $\hat{G}$, and let $2\rho_{\mathrm{adj}}\colon \mathbf{G}_{\mathrm{m}} \to \hat{T}_{\mathrm{adj}}$ be the composition of 2ρ with the projection $\hat{T} \twoheadrightarrow \hat{T}_{\mathrm{adj}}$. Then we get a $\mathbf{G}_{\mathrm{m}}$-action on $\hat{G}$ by

$$\mathbf{G}_{\mathrm{m}} \xrightarrow{\rho_{\mathrm{adj}}} \hat{T}_{\mathrm{adj}} \to \mathrm{Aut}(\hat{G}), \tag{6.16}$$

where $\rho_{\mathrm{adj}}\colon \mathbf{G}_{\mathrm{m}} \to \hat{T}_{\mathrm{adj}}$ is the unique square root of $2\rho_{\mathrm{adj}}$ and the action is by conjugation.

Remark 6.26. To ensure that the above $\mathbf{G}_{\mathrm{m}}$-action gives the correct grading in the following theorem, we consider the Tate twist $\mathbf{Z}(1)$ to be in degree -1 under the equivalence $\mathrm{MTM}_{\mathrm{r}}(S) \cong \mathrm{grAb}$. We keep this convention for the rest of the paper. This agrees with the usual convention, and also with [Zhu20] and [ES22].

Theorem 6.27. *There is a canonical* $\mathbf{G}_{\mathrm{m}}$*-equivariant isomorphism* $\widetilde{G} \cong \hat{G}$*. Moreover, this* $\mathbf{G}_{\mathrm{m}}$*-action agrees with the one given by (6.16).*

The existence of a (non-equivariant) isomorphism of $\mathbf{Z}$-group schemes $\widetilde{G} \cong \hat{G}$ can be deduced from Theorem 6.24 ff. and [MV07]. Following [FS21, §VI.11], we will prove Theorem 6.27 from scratch in a way which also gives the $\mathbf{G}_{\mathrm{m}}$-action, or equivalently the $\mathbf{Z}$-grading, and makes the isomorphism canonical. We start by fixing a pinning of G, which extends the choice of (T, B). We will use this pinning to construct the isomorphism $\widetilde{G} \cong \hat{G}$, and afterwards show this isomorphism does not depend on the choice of pinning.

By Proposition 5.22, we have $\mathrm{Sat}_{\mathrm{r}}^{T,\{*\}} = \mathrm{Fun}(X_*(T), \mathrm{MTM}_{\mathrm{r}}(X))$. This implies that H_{r}^T has degree 0, and that $\widetilde{T}$ is the torus with character group $X_*(T)$, i.e., $\widetilde{T} \cong \hat{T}$ canonically.

As the constant term functor $\mathrm{CT}_B^{\{*\}} \colon \mathrm{Sat}_{\mathrm{r}}^{G,\{*\}} \to \mathrm{Sat}_{\mathrm{r}}^{T,\{*\}}$ is symmetric monoidal and commutes with the fiber functors, we get an induced morphism $H_{\mathrm{r}}^G \to H_{\mathrm{r}}^T$ of Hopf algebras in $\mathrm{MTM}_{\mathrm{r}}(X)$, and hence a homomorphism $\widetilde{T} \to \widetilde{G}$. To show this is a closed immersion, it is enough to show that each $\mathrm{IC}_{\mu,\mathbf{Z}} \in \mathrm{Sat}_{\mathrm{r}}^{T,\{*\}}$, for $\mu \in X_*(T)^+$, is a quotient of an object lying in the image of $\mathrm{CT}_B^{\{*\}}$. This holds since $F_\nu^{\{*\}}(\mathcal{J}_!^\mu(\mathbf{Z}))$ for $\mu \in X_*(T)^+$ is free and nonzero for each ν in the Weyl-orbit of μ by Lemma 6.12.

Proposition 6.28. *The generic fiber $\widetilde{G}_{\mathbf{Q}}$ is a split connected reductive group and $\widetilde{T}_{\mathbf{Q}} \subset \widetilde{G}_{\mathbf{Q}}$ is a maximal torus.*

Proof. The proof follows [MV07, §7]. Restricting the equivalence of Corollary 6.23 to compact objects with $\Lambda = \mathbf{Q}$, we see that $\mathrm{Rep}_{\widetilde{G}_{\mathbf{Q}}}(\mathrm{Vect}_{\mathbf{Q}}^{\mathrm{fd}})$ is semisimple, cf. the proof of Proposition 6.15, and the irreducible objects are parametrized by $X_*(T)^+$. As $X_*(T)^+$ is a finitely generated monoid and $\mathrm{IC}_{\mu_1+\mu_2,\mathbf{Q}}$ is a subquotient of $\mathrm{IC}_{\mu_1,\mathbf{Q}} \star \mathrm{IC}_{\mu_2,\mathbf{Q}}$ for all $\mu_1, \mu_2 \in X_*(T)^+$, we see that $\widetilde{G}_{\mathbf{Q}}$ is algebraic by [DM82, Proposition 2.20]. As for any $0 \neq \mu \in X_*(T)^+$, the intersection motive $\mathrm{IC}_{2\mu,\mathbf{Q}}$ is a subquotient of $\mathrm{IC}_{\mu,\mathbf{Q}} \star \mathrm{IC}_{\mu,\mathbf{Q}}$, [DM82, Corollary 2.22] tells us that $\widetilde{G}_{\mathbf{Q}}$ is connected. Using this, we deduce by [DM82, Proposition 2.23] that $\widetilde{G}_{\mathbf{Q}}$ is reductive. The rank of a reductive group over $\mathbf{Q}$ agrees with the rank of its representation ring by [Tit71], so $\widetilde{T}_{\mathbf{Q}}$ is a maximal torus of $\widetilde{G}_{\mathbf{Q}}$, and hence $\widetilde{G}_{\mathbf{Q}}$ is split. ☐

We have the following generalization of [FS21, Lemma VI.11.3].

Lemma 6.29. *Let $f \colon M \to N$ be a morphism of flat abelian groups. If $M/p \to N/p$ is injective for all primes p and $M \otimes_{\mathbf{Z}} \mathbf{Q} \to N \otimes_{\mathbf{Z}} \mathbf{Q}$ is an isomorphism, then f is an isomorphism.*

Proof. Injectivity of f follows from flatness of M. For surjectivity, consider some $x \in N$. There exists some $n \geq 1$ such that $nx = f(y)$ lies in the image of f; let $n \geq 1$ be minimal with this property. Then y is a nontrivial element in the kernel of $M/p \to N/p$ for any prime p dividing n. Our assumption on the maps $M/p \to N/p$ then tells us that $n = 1$, so that f is surjective. ☐

Consider the quotient $H_{\mathrm{r}}^G \twoheadrightarrow K$, stabilizing the filtration $\bigoplus_{i \leq n} {}^{\mathrm{p}}\mathrm{H}^i \pi_{G!} u^!$ of the fiber functor. This corresponds to a subgroup $\widetilde{B} \subseteq \widetilde{G}$, such that $\widetilde{B}_{\mathbf{Q}}$ is a Borel containing $\widetilde{T}_{\mathbf{Q}}$.

Proposition 6.30. *For $G = \mathrm{PGL}_2$, the standard pinning of G induces a graded isomorphism $\widetilde{G} \cong \hat{G}$. Moreover, the induced $\mathbf{G}_{\mathrm{m}}$-action on $\hat{G}$ agrees with (6.16).*

Proof. The Langlands dual group of PGL_2 is SL_2. Consider the minuscule dominant cocharacter μ, for which $\mathrm{Gr}_{\mathrm{PGL}_2,\{*\}}^\mu \cong \mathbf{P}_S^1 \times_S X$. Moreover, we have $\mathrm{Gr}_{\mathrm{PGL}_2,\{*\}}^\mu \cap S_{\mu,\{*\}} \cong \mathbf{A}_S^1 \times_S X$, and $\mathrm{Gr}_{\mathrm{PGL}_2,\{*\}}^\mu \cap S_{-\mu,\{*\}} \cong X$, while the intersection of $\mathrm{Gr}_{\mathrm{PGL}_2,\{*\}}^\mu$ with the other semi-infinite orbits is empty. In particular, $F^{\{*\}}(\mathrm{IC}_{\mu,\mathbf{Z}}) \cong \mathbf{Z} \oplus \mathbf{Z}(-1)$ is an $H_{\mathrm{r}}^{\mathrm{PGL}_2}$-comodule, where we omit the shifts for simplicity. This induces a homomorphism $\widetilde{G} \to \mathrm{GL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$. We claim this is a closed immersion, with image $\mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1)) \subseteq \mathrm{GL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$. Then $\widetilde{G}$ is reduced by [PY06, Theorem 1.2], so that $\widetilde{G} \cong \mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$.

Indeed, $\hat{T}$ acts on $\mathbf{Z} \oplus \mathbf{Z}(-1)$ by weights ± 1, as $\mathrm{CT}_B^{\{*\}}(\mathrm{IC}_{\mu,\mathbf{Z}})$ is concentrated on the connected components corresponding to ± 1 under the isomorphism $\pi_0(\mathrm{Gr}_{T,\{*\}}) \cong \mathbf{Z}$. Thus, the image of $\hat{T}$ lands in $\mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$. In particular, the claim over $\operatorname{Spec} \mathbf{Q}$ follows as we already know $\widetilde{G}_{\mathbf{Q}}$ is reductive with maximal torus $\hat{T}_{\mathbf{Q}} \cong \mathbf{G}_{m,\mathbf{Q}}$, and $\widetilde{G}_{\mathbf{Q}}$ is not a torus by considering its representation ring. By flatness, the $\mathbf{Z}$-morphism $\widetilde{G} \to \mathrm{GL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$ factors through $\mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$, and we get a map $\widetilde{G}_{\mathbf{F}_p} \to \mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1))_{\mathbf{F}_p}$ for any prime p. Let K_p denote the image of this map, so that we have a surjection $\widetilde{G}_{\mathbf{F}_p} \to K_p$. The irreducible (ungraded) representations of $\widetilde{G}_{\mathbf{F}_p}$ are parametrized by $X_*(T)^+$. In particular, the irreducible representations of K_p can be indexed by a subset of $X_*(T)^+$, so that $K_p = \mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1))_{\mathbf{F}_p}$ by [FS21, Lemma VI.11.2]. Then Lemma 6.29, used on the level of (ungraded) Hopf algebras, tells us that $\widetilde{G} \to \mathrm{SL}(\mathbf{Z} \oplus \mathbf{Z}(-1))$ is an isomorphism.

As $\widetilde{B}$ stabilizes $\mathbf{Z} \subseteq \mathbf{Z} \oplus \mathbf{Z}(-1)$, the Lie algebra of its unipotent radical $\widetilde{U}$ can be identified with $\underline{\mathrm{Hom}}(\mathbf{Z}(-1), \mathbf{Z}) \cong \mathbf{Z}(1)$. This gives the $\mathbf{G}_{\mathrm{m}}$-equivariant isomorphism $\widetilde{G} \cong \hat{G}$ when $G = \mathrm{PGL}_2$.

To identify the $\mathbf{Z}$-grading of $\widetilde{G}$, we again consider $\widetilde{U} \subset \widetilde{B} \subset \widetilde{G} \cong \mathrm{SL}_2$. It is clear that $\hat{T} \cong \widetilde{T} \to \widetilde{G}$ is the maximal torus appearing in the pinning of SL_2. Restricting the action of $\widetilde{U}$ on $\mathbf{Z} \oplus \mathbf{Z}(-1)$ to $\mathbf{Z}$ gives a (graded) morphism

$$\mathbf{Z}(-1) \to \mathbf{Z}[t] \otimes (\mathbf{Z} \oplus \mathbf{Z}(-1)) = (\mathbf{Z}[t] \otimes \mathbf{Z}) \oplus (\mathbf{Z}[t] \otimes \mathbf{Z}(-1)) : n \mapsto (t \otimes n, 1 \otimes n),$$

where $\mathbf{Z}[t]$ is the Hopf algebra of $\widetilde{U}$, equipped with a grading as a quotient of H_{r}^G. In particular, t lies in degree 1, and similarly we see that the coordinate of the unipotent radical of the opposite Borel of $\widetilde{G}$ lies in degree -1. This shows that the grading on $\widetilde{G}$ corresponds to the $\mathbf{G}_{\mathrm{m}}$-action $\mathbf{G}_{\mathrm{m}} \to \mathrm{Aut}(\mathrm{SL}_2)$, where an invertible element x acts by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & xb \\ x^{-1}c & d \end{pmatrix} = \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x^{-1} & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.17}$$

This is the grading (6.16), so this finishes the case $G = \mathrm{PGL}_2$. □

Returning to arbitrary G, the adjoint quotient $G \twoheadrightarrow G_{\mathrm{adj}}$ induces a map $\mathrm{Gr}_{G,\{*\}} \to \mathrm{Gr}_{G_{\mathrm{adj}},\{*\}}$ which restricts to a universal homeomorphism on each connected component. As DM does not satisfy étale descent we cannot conclude that this map induces equivalences of motives on these connected components. However, we can show that we get equivalences for Tate motives, even for unreduced motives.

Lemma 6.31. *Let X be a connected component of $\mathrm{Gr}_{G,\{*\}}$ and Y the connected component of $\mathrm{Gr}_{G_{\mathrm{adj}},\{*\}}$ to which it maps under the natural morphism. If we denote the induced morphism by $\alpha\colon X \to Y$, then $\alpha^*\colon \mathrm{DTM}_{(\mathrm{r})}(Y) \to \mathrm{DTM}_{(\mathrm{r})}(X)$ is an equivalence.*

Proof. It suffices to prove the lemma for the stratification by Iwahori-orbits. In this case, we will show that the unit $\mathrm{id} \to \alpha_*\alpha^*$ and counit $\alpha^*\alpha_* \to \mathrm{id}$ are equivalences. The map α is an LG-equivariant universal homeomorphism by Lemma 3.31, so it induces an isomorphism on Iwahori-orbits, which are isomorphic to affine spaces. Hence, the lemma is immediate if we have a single cell. Because α is ind-proper, the unit and counit maps commute with $*$-pullback to any union of cells. We thus conclude by localization and induction on the number of cells. □

Proposition 6.32. *If G has semisimple rank 1, then a pinning of G induces a graded isomorphism $\widetilde{G} \cong \hat{G}$, and the grading agrees with (6.16).*

Proof. The adjoint quotient G_{adj} can be identified with PGL_2 via the pinning of G. Since $\pi_0(\mathrm{Gr}_{G,\{*\}}) \cong \pi_1(G)$ canonically, it follows from Lemma 6.31 that $\mathrm{DTM}_{\mathrm{r}}(\mathrm{Gr}_{G,\{*\}}) \cong \mathrm{DTM}_{\mathrm{r}}(\pi_1(G) \times_{\pi_1(\mathrm{PGL}_2)} \mathrm{Gr}_{\mathrm{PGL}_2,\{*\}})$. As $\pi_1(\mathrm{PGL}_2) \cong \mathbf{Z}/2$, every object in $\mathrm{DTM}_{\mathrm{r}}(\mathrm{PGL}_2)$, and hence every object in the Satake category, is equipped with a $\mathbf{Z}/2$-grading. Then $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\}}$ is equivalent to the category of objects of $\mathrm{Sat}_{\mathrm{r}}^{\mathrm{PGL}_2,\{*\}}$, equipped with a $\pi_1(G)$-grading that refines the $\mathbf{Z}/2$-grading. In particular, we get $\widetilde{G} \cong \widetilde{\mathrm{PGL}_2} \overset{\mu_2}{\times} \widetilde{Z}$, where $\widetilde{Z}$ is the multiplicative group scheme with character group $\pi_1(G)$; note that $\widetilde{Z}$ is a torus exactly when $\pi_1(G)$ is torsion-free. By [SGA70a, XII, Proposition 4.11], the center $\hat{Z}$ of $\hat{G}$ is multiplicative, with character group canonically isomorphic to $\pi_1(G)$ by [Bor98, Proposition 1.10]. Thus, the isomorphism $\hat{G} \cong \hat{G}_{\mathrm{sc}} \overset{\mu_2}{\times} \hat{Z}$ implies that $\hat{G} \cong \widetilde{G}$ canonically, inducing $\hat{T} \cong \widetilde{T}$ and $\hat{B} \cong \widetilde{B}$. Moreover, these isomorphisms are compatible with the gradings. □

Finally, we consider a general group G, still equipped with a fixed pinning. To any simple coroot a of G, we can associate a parabolic with Levi quotient containing the maximal torus: $T \subseteq M_a \subseteq P_a \subseteq G$. As M_a is of semisimple rank 1, we have the symmetric monoidal constant term functor $\mathrm{CT}_{P_a}\colon \mathrm{Sat}_{\mathrm{r}}^{G,\{*\}} \to \mathrm{Sat}_{\mathrm{r}}^{M_a,\{*\}}$. This commutes with the fiber functors, so it induces a morphism $H_{\mathrm{r}}^G \to H_{\mathrm{r}}^{M_a}$ of Hopf algebras, and hence a homomorphism $\hat{M}_a \cong \widetilde{M}_a \to \widetilde{G}$. By Lemma 5.5, the morphism $\widetilde{M}_a \to \widetilde{G}$ commutes with the closed immersions $\hat{T} \to \hat{M}_a \cong \widetilde{M}_a$ and $\hat{T} \to \widetilde{G}$. To show this is a closed immersion on the generic fiber, we apply [DM82, Proposition 2.21]: consider objects of the form $\mathrm{IC}_{\lambda,\mathbf{Q}} \in \mathrm{Sat}_{\mathrm{r}}^{M_a,\{*\}}$, with λ dominant for M_a. By Theorem 3.44, $\mathrm{IC}_{\lambda,\mathbf{Q}}$ is a quotient of a twist of $\mathrm{IC}_{\mu,\mathbf{Q}}$, where μ is the unique dominant (for G) representative of λ. Thus $\hat{M}_{a,\mathbf{Q}} \to \widetilde{G}_{\mathbf{Q}}$ is a closed immersion.

Proposition 6.33. *The closed immersions $\hat{T}_{\mathbf{Q}} \to \widetilde{G}_{\mathbf{Q}}$ and $\hat{M}_{a,\mathbf{Q}} \to \widetilde{G}_{\mathbf{Q}}$, which involve a choice of pinning of G, extend uniquely to a graded isomorphism $\hat{G}_{\mathbf{Q}} \cong \widetilde{G}_{\mathbf{Q}}$, and the grading agrees with (6.16).*

Proof. By Proposition 6.28, $\widetilde{G}_{\mathbf{Q}}$ is a split reductive group with maximal torus $\tilde{T}$. As $\widetilde{M}_{a,\mathbf{Q}} \to \widetilde{G}_{\mathbf{Q}}$ is a closed immersion, it induces an embedding on Lie algebras, so that $a \in X_*(T) = X^*(\hat{T})$ determines a root of $\widetilde{G}_{\mathbf{Q}}$, while the root $a^\vee$ associated to a determines a coroot of $\widetilde{G}_{\mathbf{Q}}$. Note that, as $\widetilde{T} \cong \hat{T}$, passing to dual groups preserves the pairing between roots and coroots, up to reversing their roles. Hence, the simple reflections are also contained in the Weyl group of $\widetilde{G}_{\mathbf{Q}}$, giving an inclusion $W = W(G,T) \subseteq \widetilde{W} := W(\widetilde{G}_{\mathbf{Q}}, \widetilde{T}_{\mathbf{Q}})$, as subgroups of $\mathrm{Aut}(X_*(T))$. Let us denote as before the roots of G by $\Phi := \Phi(G,T)$, and the coroots by $\Phi^\vee := \Phi^\vee(G,T)$. Then, as all (co)roots are a W-translate of a simple (co)root, this implies that $\Phi^\vee \subseteq \Phi(\widetilde{G}_{\mathbf{Q}}, \widetilde{T}_{\mathbf{Q}})$ and $\Phi \subseteq \Phi^\vee(\widetilde{G}_{\mathbf{Q}}, \widetilde{T}_{\mathbf{Q}})$, as subsets of $X_*(T) \cong X^*(\hat{T})$ and $X^*(T) \cong X_*(\hat{T})$ respectively.

To show that these inclusions are equalities, note that for $\lambda \in X^*(\widetilde{T})^+$, the weights of the simple $\widetilde{G}_{\mathbf{Q}}$-representation of highest weight λ are those weights λ' in the convex hull of the $\widetilde{W}$-orbit of λ such that $\lambda - \lambda'$ is in the root lattice of $\widetilde{G}_{\mathbf{Q}}$. Let $a \in \Phi(\widetilde{G}_{\mathbf{Q}}, \widetilde{T}_{\mathbf{Q}})$, and choose λ regular, so that the corresponding simple $\widetilde{G}_{\mathbf{Q}}$-representation has a weight λ' such that a appears in $\lambda - \lambda'$ with non-zero coefficient. Then the restriction of $\mathrm{CT}_B^{\{*\}}(\mathrm{IC}_{\lambda,\mathbf{Q}})$ to $\lambda' \in \pi_0(\mathrm{Gr}_{T,\{*\}})$

does not vanish. But this implies that $\lambda - \lambda'$ is in the coroot lattice of G, so that a must be a coroot. Thus, the root data of $\widetilde{G}_{\mathbf{Q}}$ and $\hat{G}_{\mathbf{Q}}$ agree, so we get the desired isomorphism $\hat{G}_{\mathbf{Q}} \cong \widetilde{G}_{\mathbf{Q}}$. Finally, compatibility with the gradings follows from the case where G is a torus or has semisimple rank 1, as the closed immersions induced by constant term functors are compatible with the gradings. ☐

Proposition 6.34. *The isomorphism in Proposition 6.33 extends uniquely to an integral isomorphism $\hat{G} \cong \widetilde{G}$.*

Proof. Consider a prime p, and the ring of integers $\breve{\mathbf{Z}}_p \subseteq \breve{\mathbf{Q}}_p$ of the completion of the maximal unramified extension of $\mathbf{Q}_p$. Viewing $\hat{G}(\breve{\mathbf{Z}}_p)$ and $\widetilde{G}(\breve{\mathbf{Z}}_p)$ as subsets of $\hat{G}(\breve{\mathbf{Q}}_p) \cong \widetilde{G}(\breve{\mathbf{Q}}_p)$, we know that $\hat{G}(\breve{\mathbf{Z}}_p) \subseteq \widetilde{G}(\breve{\mathbf{Z}}_p)$, as the former is generated by the $\hat{M}_a(\breve{\mathbf{Z}}_p) \cong \widetilde{M}_a(\breve{\mathbf{Z}}_p)$. Let $\widetilde{G} \to \mathrm{GL}_n$ be a representation corresponding to some object in $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\}}$ which is a closed immersion on the generic fiber. Then $\hat{G}_{\mathbf{Q}_p} \cong \widetilde{G}_{\mathbf{Q}_p} \to \mathrm{GL}_{n,\mathbf{Q}_p}$ extends to a morphism $\hat{G}_{\mathbf{Z}_p} \to \mathrm{GL}_{n,\mathbf{Z}_p}$ by [BT84, Proposition 1.7.6]. Using a reduction to the adjoint group as before, in which case $\hat{G}$ is simply connected, [PY06, Corollary 1.3] tells us that $\hat{G}_{\mathbf{Z}_p} \to \mathrm{GL}_{n,\mathbf{Z}_p}$ is also a closed immersion. By flatness of $\widetilde{G}$, this map then factors as $\widetilde{G}_{\mathbf{Z}_p} \to \hat{G}_{\mathbf{Z}_p}$, which is an isomorphism on the generic fiber. But it is also surjective on the special fiber, as any point in $\hat{G}(\overline{\mathbf{F}}_p)$ can be lifted to a point in $\hat{G}(\breve{\mathbf{Z}}_p)$ by smoothness of $\hat{G}$ and completeness of $\breve{\mathbf{Z}}_p$, and we already know that $\hat{G}(\breve{\mathbf{Z}}_p) \subseteq \widetilde{G}(\breve{\mathbf{Z}}_p)$. Lemma 6.29 then tells us that $\widetilde{G}_{\mathbf{Z}_p} \to \hat{G}_{\mathbf{Z}_p}$ is an isomorphism.

As the previous paragraph is valid for all primes p, we see that all fibers of $\widetilde{G} \to \operatorname{Spec} \mathbf{Z}$ are reductive, so that $\widetilde{G}$ is reductive. Since the closed immersion $\hat{T} \to \widetilde{G}$ determines a maximal torus on the generic fiber, $\hat{T}$ is already a maximal torus of $\widetilde{G}$, so that $\widetilde{G}$ is split. The corresponding root datum can be determined on the generic fiber, so that the previously obtained identification of $\widetilde{G}$ on the generic fiber gives us $\hat{G} \cong \widetilde{G}$, although a priori only non-canonically. But then [SGA70b, XXIII, Théorème 4.1] gives us a unique isomorphism $\hat{G} \cong \widetilde{G}$ extending the isomorphisms $\hat{G}_{\mathbf{Z}_p} \cong \widetilde{G}_{\mathbf{Z}_p}$ for each prime p. ☐

This finishes the proof of Theorem 6.27 when G is equipped with a pinning; it remains to show the isomorphism $\hat{G} \cong \widetilde{G}$ is independent of the choices.

Proposition 6.35. *The isomorphism in Proposition 6.34 is independent of the pinning of G.*

Proof. As $\widetilde{T}$ is the stabilizer of the cohomological grading of the fiber functor, and $\widetilde{B}$ is the stabilizer of the corresponding filtration, they are independent of (T, B) by Corollary 5.12.

For the independence of the rest of the pinning, let us consider the flag variety $\mathcal{F}\ell$, non-canonically isomorphic to G/B, parametrizing the Borels of G. The quotient of the universal Borel $B_{\mathcal{F}\ell} \subseteq G_{\mathcal{F}\ell} := G \times_S \mathcal{F}\ell$ by its unipotent radical is a torus, the universal Cartan $T_{\mathcal{F}\ell}$. It is defined over $\mathbf{Z}$, as $\mathcal{F}\ell$ is simply connected, and split. Consider a simple coroot a. Then there is the universal parabolic $P_{a,\mathcal{F}\ell}$ with Levi quotient $M_{a,\mathcal{F}\ell}$. Let $\widetilde{\mathcal{F}\ell}_a$ be the $\mathbf{G}_{\mathrm{m}}$-torsor over $\mathcal{F}\ell$ parametrizing the pinnings of $M_{a,\mathcal{F}\ell}$. The group $M_{a,\widetilde{\mathcal{F}\ell}_a}$ is constant by [SGA70b, XXIII, Théorème 4.1]. Now, we can repeat the whole story, replacing our base S by $\widetilde{\mathcal{F}\ell}_a$. In particular, we have the symmetric monoidal constant term functor

$$\mathrm{CT}^{\{*\}}_{P_{a,\widetilde{\mathcal{F}\ell}_a}} : \mathrm{DTM}_{\mathrm{r}}(\mathrm{Hck}_{G_{\mathcal{F}\ell},\{*\},\widetilde{\mathcal{F}\ell}_a}) \to \mathrm{DTM}_{\mathrm{r}}(\mathrm{Hck}_{M_{a,\mathcal{F}\ell},\{*\},\widetilde{\mathcal{F}\ell}_a}),$$

where the reduced motives are defined using the base $\widetilde{\mathcal{F}\ell}_a$. The fact that $\mathrm{CT}^{\{*\}}_{P_{a,\widetilde{\mathcal{F}\ell}_a}}$ preserves Tate motives is not immediate because $P_{a,\widetilde{\mathcal{F}\ell}_a}$ is not split, but we can argue as follows. Since $G \to G/B$ is Zariski-locally trivial, $P_{a,\widetilde{\mathcal{F}\ell}_a}$ is Zariski-locally split. Thus, $\mathrm{CT}^{\{*\}}_{P_{a,\widetilde{\mathcal{F}\ell}_a}}$ preserves Tateness Zariski-locally on $\widetilde{\mathcal{F}\ell}_a$ by Theorem 3.44. By $L^+_{\{*\}}M_{a,\widetilde{\mathcal{F}\ell}_a}$-equivariance of the image, $\mathrm{CT}^{\{*\}}_{P_{a,\widetilde{\mathcal{F}\ell}_a}}$ preserves Tateness [RS20, Lemma 2.2.21, Proposition 3.1.23]. Now by independence of the base, Lemma 4.28, *-restricting to a fiber of $\widetilde{\mathcal{F}\ell}_a$ and shifting by $\dim(\widetilde{\mathcal{F}\ell}_a)$ induces an equivalence of Satake categories (defined using different bases), compatible with the constant term functors. This shows that the constant term functor $\mathrm{CT}^{\{*\}}_{P_a}$ is independent of the choice of Borel. We can hence replace G by a simple Levi, i.e., assume that G is of semisimple rank 1. There, we only used the pinning to identify G_{adj}, reducing us to $G \cong \mathrm{PGL}_2$. In that case, we had a canonical graded isomorphism $\mathrm{Lie}(\widetilde{U}) \cong \underline{\mathrm{Hom}}(\mathbf{Z}(-1), \mathbf{Z}) \cong \mathbf{Z}(1)$, cf. the discussion preceding (6.17). ☐

Remark 6.36. Let the *C-group* of G be the semidirect product ${}^{C}G := \hat{G} \rtimes \mathbf{G}_{\mathrm{m}}$, with $\mathbf{G}_{\mathrm{m}}$ acting on $\hat{G}$ via (6.16), cf. [Zhu20, §1.1]. When k is a field and $S = \operatorname{Spec} k$, this group is related to the C-group defined in [BG14], which is a semidirect product of $\hat{G} \rtimes \mathbf{G}_{\mathrm{m}}$ with the absolute Galois group of the field k.

Modulo the identification of the root groups of $\hat{G}$ with Tate twists, ${}^{C}G$ agrees with Deligne's modification of the dual group from [FG09], defined as $\hat{G}_1 := (\hat{G} \times \mathbf{G}_{\mathrm{m}})/(2\rho \times \mathrm{id})(\mu_2)$, cf. also [Del07]. As the group we obtain below arises naturally as a semidirect product, we prefer to keep the notation ${}^{C}G := \hat{G} \rtimes \mathbf{G}_{\mathrm{m}}$.

Since graded abelian groups are equivalent to abelian groups equipped with a $\mathbf{G}_{\mathrm{m}}$-action, we can also describe the Satake category as a representation category of (ungraded) abelian groups as follows.

Corollary 6.37. *There is a canonical equivalence* $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\}} \cong \mathrm{Rep}_{{}^{C}G}(\mathrm{Ab})$.

This gives further evidence for Bernstein's suggestion that the C-group might be more appropriate in the Langlands program than the L-group [Zhu20, Remark 9 (2)].

Remark 6.38. In order to get canonical equivalences in Theorem 6.27 and Corollary 6.37, we have to identify the simple root groups of $\hat{G}$ with a Tate twist, even after forgetting the grading. However, any isomorphism $\mathbf{Z} \cong \mathbf{Z}(1)$ of abelian groups induces a $\mathbf{G}_{\mathrm{m}}$-equivariant isomorphism of $\widetilde{G}$ with the usual Langlands dual group, equipped with the $\mathbf{G}_{\mathrm{m}}$-action from (6.16).

6.3. **The Vinberg monoid.** In this section we consider a subcategory of anti-effective stratified Tate motives for the purpose of geometrizing Hecke algebras over $\mathbf{Z}[\mathbf{q}]$, where $\mathbf{q}$ is an indeterminate. Recall that for ℓ-adic sheaves over $S = \mathbf{F}_q$, the trace of the geometric Frobenius element on $\mathbf{Q}_\ell(-1)$ is q. Thus, it is natural to single out anti-effective motives, so that ℓ-adic realization geometrizes the specialization map on Hecke algebras $\mathbf{q} \mapsto q$.

By Proposition 3.7 and Definition and Lemma 4.11, we have the symmetric monoidal category

$$\mathrm{Sat}_{(\mathrm{r})}^{G,\mathrm{anti}} := \mathrm{Sat}_{(\mathrm{r})}^{G,\{*\},\mathrm{anti}} := \mathrm{MTM}(\mathrm{Hck}_{G,\{*\}})^{\mathrm{anti}}.$$

For $G = 1$, $\mathrm{Sat}_{\mathrm{r}}^{G,\mathrm{anti}} = \mathrm{MTM}_{\mathrm{r}}(X)^{\mathrm{anti}} \subset \mathrm{Sat}_{\mathrm{r}}^{G,\{*\}} = \mathrm{MTM}_{\mathrm{r}}(X)$ identifies with the full subcategory consisting of those graded abelian groups concentrated in nonnegative degrees (under the monoidal isomorphism $f^*[1] \colon \mathrm{MTM}_{\mathrm{r}}(S) \to \mathrm{MTM}_{\mathrm{r}}(X)$ for $f \colon X \to S$, where we still consider the Tate twist $\mathbf{Z}(1)$ to be negatively graded). For general G, $\mathrm{Sat}_{\mathrm{r}}^{G,\mathrm{anti}}$ is generated by the $\mathrm{IC}_{\mu,L}$ for $\mu \in X_*(T)^+$ and $L \in \mathrm{MTM}_{\mathrm{r}}(S)^{\mathrm{anti}} \subseteq \mathrm{MTM}_{\mathrm{r}}(S)$.

Theorem 6.39. *Let $M \in \mathrm{Sat}_{(\mathrm{r})}^{G,\{*\}}$. Then $M \in \mathrm{Sat}_{(\mathrm{r})}^{G,\mathrm{anti}}$ if and only if $F^{\{*\}}(M) \in \mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$.*

Proof. If $M \in \mathrm{Sat}_{(\mathrm{r})}^{G,\mathrm{anti}}$, by excision and the filtrable decomposition in Theorem 3.32 we have $F^{\{*\}}(M) \in \mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$. For the converse, suppose $F^{\{*\}}(M) \in \mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$. If M is compact, it admits a finite filtration with subquotients given by IC-motives of compact objects on the strata. Since we are in the equivariant situation, by Proposition 4.29 these subquotients are of the form $\mathrm{IC}_{\mu,L}$ for $L \in \mathrm{MTM}_{(\mathrm{r})}(S)^{\mathrm{c}}$ and $\mu \in X_*(T)^+$. Each $F^{\{*\}}(\mathrm{IC}_{\mu,L})$ is a subquotient of $F^{\{*\}}(M)$, and consequently each $f^*L[1]$ is also a subquotient of $F^{\{*\}}(M)$, as $\mathrm{Gr}^{\mu}_{G,\{*\}} \cap S_{w_0(\mu),\{*\}} = X$ where w_0 is the longest element in the Weyl group. Because $\mathrm{MTM}_{(\mathrm{r})}(X)^{\mathrm{anti}}$ is closed under subquotients (Lemma 2.12), this implies each L is anti-effective, and hence so is $\mathrm{IC}_{\mu,L}$ by Remark 2.13. This implies that M is also anti-effective. If M is not necessarily compact, we can present it as a filtered colimit $M = \operatorname{colim} M_i$ of compact subobjects $M_i \subset M$ in $\mathrm{MTM}(\mathrm{Hck}_{G,\{*\}})$. Being a subobject of $F^{\{*\}}(M)$, $F^{\{*\}}(M_i)$ is anti-effective (Lemma 2.12), hence so is M_i and therefore also $M = \operatorname{colim} M_i$. ☐

In Zhu's integral Satake isomorphism [Zhu20, Proposition 5], the Vinberg monoid appears instead of the usual dual group. We now explain how this monoid naturally appears from our motivic Satake equivalence. Afterwards, we construct a generic Satake isomorphism between the generic spherical Hecke algebra and the representation ring of the Vinberg monoid, which are naturally $\mathbf{Z}[\mathbf{q}]$-algebras for some indeterminate $\mathbf{q}$. In an attempt to not further lengthen the paper, we will only recall the necessary definitions for the Vinberg monoid, and we refer to [Vin95, XZ19, Zhu20] and the references there for more details.

Denote by $X^*(\hat{T}_{\mathrm{adj}})_{\mathrm{pos}} \subseteq X^*(\hat{T})^+_{\mathrm{pos}} \subseteq X^*(\hat{T})$ the submonoids of characters generated by the simple roots, respectively the dominant characters and the simple roots. Viewing $\mathbf{Z}[\hat{G}]$ as a $\hat{G} \times \hat{G}$-module via left and right multiplication, the global sections $\mathbf{Z}[\hat{G}]$ admit an $X^*(\hat{T})^+_{\mathrm{pos}}$-multi-filtration $\mathbf{Z}[\hat{G}] = \sum_{\mu \in X^*(\hat{T})^+_{\mathrm{pos}}} \mathrm{fil}_\mu \mathbf{Z}[\hat{G}]$, where $\mathrm{fil}_\mu \mathbf{Z}[\hat{G}]$ is the maximal $\hat{G} \times \hat{G}$-submodule such that all its weights $(\lambda, \lambda') \in X^*(\hat{T}) \times X^*(\hat{T})$ satisfy $\lambda \leq -w_0(\mu)$ and $\lambda' \leq \mu$. Here w_0 is the longest element of the Weyl group of G. We then define Vinberg's universal monoid $V_{\hat{G}} = \operatorname{Spec} \bigoplus_{\mu \in X^*(\hat{T})^+_{\mathrm{pos}}} \mathrm{fil}_\mu \mathbf{Z}[\hat{G}]$, with the natural (co)multiplication map and monoid morphism $d_{\rho_{\mathrm{adj}}} \colon V_{\hat{G}} \to \operatorname{Spec} \mathbf{Z}[X^*(\hat{T}_{\mathrm{adj}})_{\mathrm{pos}}] =: \hat{T}^+_{\mathrm{adj}}$. The dominant cocharacter ρ_{adj} extends to a monoid morphism $\rho_{\mathrm{adj}} \colon \mathbf{A}^1 \to \hat{T}^+_{\mathrm{adj}}$, and the monoid $V_{\hat{G},\rho_{\mathrm{adj}}}$ is defined as in the commutative diagram below, in which all squares are cartesian.

$$\begin{tikzcd}
\hat{G} \times \hat{T} \arrow[r, two heads] & \hat{G} \overset{Z_{\hat{G}}}{\times} \hat{T} \arrow[r] \arrow[d] & V_{\hat{G}} \arrow[d] & V_{\hat{G},\rho_{\mathrm{adj}}} \arrow[l] \arrow[d, "d_{\rho_{\mathrm{adj}}}"] & (\hat{G} \times \mathbf{G}_{\mathrm{m}})/(2\rho \times \mathrm{id})(\mu_2) \arrow[l] \arrow[d] \arrow[lll, bend right, "\mathrm{id} \times 2\rho"'] & \hat{G} \times \mathbf{G}_{\mathrm{m}} \arrow[l, two heads] \\
 & \hat{T}_{\mathrm{adj}} \arrow[r] & \hat{T}^+_{\mathrm{adj}} & \mathbf{A}^1 \arrow[l, "\rho_{\mathrm{adj}}"'] & \mathbf{G}_{\mathrm{m}} \arrow[l] &
\end{tikzcd}$$

There is an isomorphism $(\hat{G}\times\mathbf{G}_{\mathrm{m}})/(2\rho\times\mathrm{id})(\mu_2)\cong\hat{G}\rtimes\mathbf{G}_{\mathrm{m}}={}^{C}G$, $(g,t)\mapsto(g2\rho(t)^{-1},t^2)$. In particular, $\hat{G}\overset{Z_{\hat{G}}}{\times}\hat{T}\subseteq V_{\hat{G}}$ and ${}^{C}G\subseteq V_{\hat{G},\rho_{\mathrm{adj}}}$ are the respective groups of units.

The following theorem was already explained by T. Richarz during a talk in the Harvard Number Theory Seminar in April 2021 for rational coefficients. In the same talk, he asked whether it was possible to do this integrally, and mentioned Lemma 6.44. The theorem below gives an affirmative answer to this question.

Theorem 6.40. *The equivalence* $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\}}\cong\mathrm{Rep}_{{}^{C}G}(\mathrm{Ab})$ *restricts to an equivalence* $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\},\mathrm{anti}}\cong\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab})$.

Remark 6.41. As observed by T. Richarz, the reason we have to consider anti-effective, instead of effective, motives is that the fiber functor and convolution use $(-)_!$ functors, which are cohomological in nature. As the Tate twist is of a homological nature, this leads to a change of signs.

Proof. The proof is a generalization of the arguments in [Zhu20, Lemma 21] to integral coefficients. First, as ${}^{C}G\subseteq V_{\hat{G},\rho_{\mathrm{adj}}}$ is open and dense, the restriction functor $\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab})\to\mathrm{Rep}_{{}^{C}G}(\mathrm{Ab})$ is fully faithful. In particular, it suffices to identify the two full subcategories under the equivalence $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\}}\cong\mathrm{Rep}_{{}^{C}G}(\mathrm{Ab})$.

The composite $\hat{G}\times\hat{T}\to V_{\hat{G}}$ in the above diagram corresponds to the natural inclusion of coalgebras

$$\bigoplus_{\mu\in X^*(\hat{T})^+_{\mathrm{pos}}}\mathrm{fil}_\mu\mathbf{Z}[\hat{G}]\to\mathbf{Z}[\hat{G}]\otimes_{\mathbf{Z}}\mathbf{Z}[X^*(\hat{T})].$$

This map is given by sending each $\mathrm{fil}_\mu\mathbf{Z}[\hat{G}]$ into $\mathbf{Z}[\hat{G}]\otimes\mathbf{Z}e^\mu$, where $e^\mu\in\mathbf{Z}[X^*(\hat{T})]$ generates the rank 1 subgroup corresponding to μ; this follows from [XZ19, (3.2.3) ff.].

We first determine when a $\hat{G}\overset{Z_{\hat{G}}}{\times}\hat{T}$-representation M extends (necessarily uniquely) to a $V_{\hat{G}}$-representation. Consider the decomposition of M into its $\hat{T}\times\hat{T}$-weight spaces. We may assume that $1\times\hat{T}$ acts on M by a fixed weight $\mu\in X^*(\hat{T})$. Then M extends to $V_{\hat{G}}$-representation if only if $\mu\in X^*(\hat{T})^+_{\mathrm{pos}}$, and as a $\hat{G}$-representation, the coaction map sends M into $\mathrm{fil}_\mu\mathbf{Z}[\hat{G}]\otimes_{\mathbf{Z}}M$. The second condition means that $\lambda\leq -w_0(\mu)$ for each weight λ of $T\times 1$ appearing in M. Since these weights are symmetric under the Weyl group, it follows that M extends to a $V_{\hat{G}}$-representation if and only if each weight (λ,μ) of $\hat{T}\times\hat{T}$ satisfies $\mu+\lambda_-\in X^*(\hat{T}_{\mathrm{adj}})_{\mathrm{pos}}$, where λ_- is the unique anti-dominant weight in the Weyl-orbit of λ.

We now specialize the previous argument to $V_{\hat{G},\rho_{\mathrm{adj}}}$. The composite $\hat{G}\times\mathbf{G}_{\mathrm{m}}\to V_{\hat{G}}$ sends $\mathrm{fil}_\mu\mathbf{Z}[\hat{G}]$ into $\mathrm{fil}_\mu\mathbf{Z}[\hat{G}]\otimes t^{\langle 2\rho,\mu\rangle}$, where t is the coordinate of $\mathbf{G}_{\mathrm{m}}$. Thus, a representation of $\hat{G}\times\mathbf{G}_{\mathrm{m}}$ extends to $V_{\hat{G},\rho_{\mathrm{adj}}}$ if and only if for each weight (λ,n) appearing, there exists some $\mu\in X^*(\hat{T})^+_{\mathrm{pos}}$ such that $\langle 2\rho,\mu\rangle=n$ and $\mu+\lambda_-\in X^*(\hat{T}_{\mathrm{adj}})_{\mathrm{pos}}$. We claim this condition is equivalent to the following two conditions on the weights (λ,n):

$$(-1)^{\langle 2\rho,\lambda\rangle}=(-1)^n\quad\text{and}\quad\langle 2\rho,\lambda_-\rangle\geq -n.\tag{6.18}$$

The necessity of these conditions is straightforward; to see sufficiency take $\mu=-\lambda_-+\nu$, where $\nu\in X^*(\hat{T})_{\mathrm{pos}}$ is any element such that $\langle 2\rho,\nu\rangle=n+\langle 2\rho,\lambda_-\rangle$.

Under the isomorphism $(\hat{G}\times\mathbf{G}_{\mathrm{m}})/(2\rho\times\mathrm{id})(\mu_2)\cong{}^{C}G$, $(g,t)\mapsto(g2\rho(t)^{-1},t^2)$, a weight (μ,n) of $\hat{T}\rtimes\mathbf{G}_{\mathrm{m}}\subset{}^{C}G$ pulls back to the weight $(\mu,-\langle 2\rho,\mu\rangle+2n)$. Thus, using (6.18), the previous isomorphism identifies $\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab})$ with the subcategory of $\mathrm{Rep}_{{}^{C}G}(\mathrm{Ab})$ of representations with nonnegative $\mathbf{G}_{\mathrm{m}}$-weights. Now we conclude using Theorem 6.39. $\square$

Remark 6.42. The criterion we obtained for extending a representation to $V_{\hat{G}}$ or $V_{\hat{G},\rho_{\mathrm{adj}}}$ is equivalent to the condition that the representation extends to the closure of a maximal torus. For normal reductive monoids over an algebraically closed field, this condition is always sufficient, cf. [Ren05, Remark 5.3].

We conclude with a generalization of [Zhu20, Proposition 5] for generic Hecke algebras. Recall that for $S=\operatorname{Spec}\mathbf{F}_q$ the spectrum of a finite field, the *spherical Hecke algebra* of G is the ring $\mathcal{H}_G^{\mathrm{sph}}:=C_{\mathrm{c}}(G(\mathbf{F}_q[\![t]\!])\backslash G(\mathbf{F}_q(\!(t)\!))/G(\mathbf{F}_q[\![t]\!]),\mathbf{Z})$ of locally constant, compactly supported, bi-$G(\mathbf{F}_q[\![t]\!])$-invariant, $\mathbf{Z}$-valued functions on $G(\mathbf{F}_q(\!(t)\!))$, equipped with the convolution product $\star$. This is a free $\mathbf{Z}$-module, with a basis given by the characteristic functions $\mathbf{1}_\mu$ of $G([\![t]\!])\mu(t)G([\![t]\!])$, for $\mu\in X_*(T)^+$. The convolution is given by

$$\mathbf{1}_\mu\star\mathbf{1}_\lambda=\sum_{\nu\in X_*(T)^+}N_{\mu,\lambda,\nu}(q)\cdot\mathbf{1}_\nu,$$

for uniquely determined polynomials $N_{\mu,\lambda,\nu}$ with integral coefficients. This follows by considering a second basis of $\mathcal{H}_G^{\mathrm{sph}}\otimes_{\mathbf{Z}}\mathbf{Z}[q^{\frac{1}{2}}]$ consisting of the $\phi_\mu:=q^{\langle\rho,\mu\rangle}\chi_\mu$, where $\chi_\mu\in\mathbf{Z}[X^*(\hat{T})]$ is the character of the irreducible complex representation of $\hat{G}$ of highest weight μ. Indeed, the change of basis between $\{\phi_\mu\}$ and $\{\mathbf{1}_\mu\}$ is given by integral polynomials in q [Gro98, Proposition 4.4], while the multiplication for the basis $\{\phi_\mu\}$ is determined by integers independent of q [Lus83, Corollary 8.7]. This suggests the following definition, generalizing [PS20, Definition 6.2.3] for $G = \mathrm{GL}_2$.

Definition 6.43. Let $\mathbf{q}$ be an indeterminate. The *generic spherical Hecke algebra* $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q})$ of G is the free $\mathbf{Z}[\mathbf{q}]$-module with basis $\{T_\mu \mid \mu \in X_*(T)^+\}$, and multiplication

$$T_\mu \cdot T_\lambda = \sum_{\nu \in X_*(T)^+} N_{\mu,\lambda,\nu}(\mathbf{q}) T_\nu.$$

To show that this generic spherical Hecke algebra agrees with the representation ring of the Vinberg monoid, we first show that the representation ring is not affected by rationalizing. Recall that the representation ring of an (algebraic) group or monoid M is defined as the Grothendieck group of the category of finitely generated representations of M, and we denote it by $R(M)$.

Lemma 6.44. *The rationalization functor* $\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab})^{\mathrm{c}} \to \mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}} \otimes \mathbf{Q}}(\mathrm{Vect}_{\mathbf{Q}})^{\mathrm{c}}$ *induces an isomorphism on Grothendieck rings.*

Proof. As the rationalization functor is symmetric monoidal, it induces a ring morphism on Grothendieck rings. This morphism is surjective, as $\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}} \otimes \mathbf{Q}}(\mathrm{Vect}_{\mathbf{Q}})^{\mathrm{c}} \cong \mathrm{Sat}_{\mathrm{r}}^{G,\mathbf{Q},\mathrm{anti,c}}$ is semisimple with simple objects $\mathcal{J}_!^\mu(\mathbf{Q})(n)$ for $\mu \in X_*(T)^+$ and $n \leq 0$, and $\mathcal{J}_!^\mu(\mathbf{Z})(n) \otimes \mathbf{Q} \cong \mathcal{J}_!(\mathbf{Q})(n)$ by Proposition 6.11.

For injectivity, we first claim that any torsion $V_{\hat{G},\rho_{\mathrm{adj}}}$-representation vanishes in the Grothendieck ring. Indeed, we can reduce to the case of $\mathcal{J}_!^\mu(L)$ and $\mathcal{J}_*^\mu(L)$ for $\mu \in X_*(T)^+$ and $L \in \mathrm{MTM}_{\mathrm{r}}(S)^{\mathrm{anti,c}}$ torsion. But then we can find some degreewise free L' and a presentation $0 \to L' \to L' \to L \to 0$. This gives a short exact sequence $0 \to \mathcal{J}_!^\mu(L') \to \mathcal{J}_!^\mu(L') \to \mathcal{J}_!^\mu(L) \to 0$, so that $[\mathcal{J}_!^\mu(L)] = 0 \in K_0(\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab})^{\mathrm{c}})$, and similarly for $\mathcal{J}_*^\mu$.

Finally, we note that if $M_1, M_2 \in \mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab})^{\mathrm{c}}$ become isomorphic after rationalizing, where we may assume the M_i are torsion-free, we can scale such a rational isomorphism such that it preserves the integral subrepresentations. Then the kernel and cokernel of this integral morphism are torsion representations, which are trivial in the Grothendieck ring. ☐

We can now construct a *generic Satake isomorphism*, again generalizing [PS20, Theorem 6.2.4] for $G = \mathrm{GL}_2$. (The authors informed us they also knew how to generalize their proof to general split reductive groups.) Note that the Grothendieck ring of $\mathrm{Sat}_{\mathrm{r}}^{G,\{*\},\mathrm{anti}}$ is naturally a $\mathbf{Z}[\mathbf{q}]$-algebra, where multiplication by $\mathbf{q}$ corresponds to twisting by (-1).

Corollary 6.45. *There is a unique natural isomorphism* Ψ *between* $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q})$ *and the representation ring of* $V_{\hat{G},\rho_{\mathrm{adj}}}$ *such that for any prime power* q*, the diagram*

$$\begin{tikzcd}
\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q}) \arrow[rr, "\Psi"] \arrow[d, "\mathbf{q}=q"] & & R(V_{\hat{G},\rho_{\mathrm{adj}}}) \arrow[d, "{[d_{\rho_{\mathrm{adj}}}]=q}"] \\
\mathcal{H}^{\mathrm{sph}}_G \otimes_{\mathbf{Z}} \mathbf{Z}[q^{\pm\frac{1}{2}}] \arrow[r, "\Psi_{\mathrm{cl}}"] & \mathbf{Z}[q^{\pm\frac{1}{2}}][X_*(T)]^{W_0} \arrow[r, phantom, "\cong"] & R(\hat{G}) \otimes_{\mathbf{Z}} \mathbf{Z}[q^{\pm\frac{1}{2}}]
\end{tikzcd} \tag{6.19}$$

commutes. Here, Ψ_{cl} *denotes the classical Satake isomorphism, cf.* [Gro98, Proposition 3.6]*, and the rightmost map is obtained by taking the character* $R(V_{\hat{G},\rho_{\mathrm{adj}}}) \to R({}^{C}G) \to \mathbf{Z}[X^*(\hat{T} \rtimes \mathbf{G}_{\mathrm{m}})]$*, and then setting the character of the projection* $\hat{T} \rtimes \mathbf{G}_{\mathrm{m}} \to \mathbf{G}_{\mathrm{m}}$ *equal to* q*. In particular,* $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q})$ *is commutative and unital.*

Proof. By Lemma 6.44, it suffices to show that $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q})$ is isomorphic to the Grothendieck ring of $\mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}},\mathbf{Q}}(\mathrm{Vect}_{\mathbf{Q}})^{\mathrm{c}} \cong \mathrm{Sat}^{G,\mathbf{Q},\mathrm{anti,c}}$, which we denote by R^{anti}. Recall that $\mathbf{Z}[q^{\pm\frac{1}{2}}][X_*(T)]^{W_0}$ admits a natural $\mathbf{Z}[q^{\pm\frac{1}{2}}]$-basis given by the characters χ_μ of the simple complex algebraic representations of $\hat{G}$, for $\mu \in X_*(T)^+$. Define $f_\mu := \Psi_{\mathrm{cl}}^{-1}(q^{\langle\rho,\mu\rangle}\chi_\mu)$. By [Gro98, (3.12) and Proposition 4.4], we have $f_\mu = \mathbf{1}_\mu + \sum_{\lambda<\mu} d_{\mu,\lambda}(q)\mathbf{1}_\lambda$, for uniquely determined polynomials $d_{\mu,\lambda}(\mathbf{q}) \in \mathbf{Z}[\mathbf{q}]$. Setting $f_\mu(\mathbf{q}) := T_\mu + \sum_{\lambda<\mu} d_{\mu,\lambda}(\mathbf{q})T_\lambda$, we get a second $\mathbf{Z}[\mathbf{q}]$-basis $\{f_\mu(\mathbf{q}) \mid \mu \in X_*(T)^+\}$ of $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q})$.

Now, consider the group homomorphism $\Psi \colon \mathcal{H}^{\mathrm{sph}}_G(\mathbf{q}) \to R^{\mathrm{anti}}$ sending $\mathbf{q}^n \cdot f_\mu(\mathbf{q})$ to $[\mathrm{IC}_{\mu,\mathbf{Q}}(-n)]$, where $[-]$ denotes the class of an object in R^{anti}. As the simple objects in $\mathrm{Sat}_{\mathrm{r}}^{G,\mathbf{Q},\mathrm{anti,c}}$ are exactly the $\mathrm{IC}_{\mu,\mathbf{Q}}(-n)$ for $\mu \in X_*(T)^+$ and $n \in \mathbf{Z}_{\geq 0}$, and because the $f_\mu(\mathbf{q})$ form a $\mathbf{Z}[\mathbf{q}]$-basis of $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q})$, it follows that Ψ is both injective and surjective. For Ψ to be an isomorphism of rings, we need to show certain equalities of polynomials. But since the classical Satake isomorphism is a ring morphism, the polynomials in question agree for all prime powers q, so that they must be equal. Hence Ψ gives the desired isomorphism $\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q}) \cong R^{\mathrm{anti}}$ of rings. Since we defined Ψ by Ψ_{cl} and scaling by some power of q, the $\mathbf{G}_{\mathrm{m}}$-action on $\hat{T}$ appearing in $\hat{T} \rtimes \mathbf{G}_{\mathrm{m}}$ ensures that (6.19) commutes. ☐

Remark 6.46. In [Zhu20, (1.12)], Zhu modifies the Satake isomorphism so that it is defined over $\mathbf{Z}[q]$. Using the commutative diagram in [Zhu20, Lemma 25 ff.], one can also construct a commutative diagram as in Corollary 6.45 involving Ψ, [Zhu20, (1.12)], and a subring of $R({}^{C}G) \otimes_{\mathbf{Z}} \mathbf{Z}[q]$. Setting $\mathbf{q} = q$ and base changing along $\mathbf{Z} \to \mathbf{F}_p$ recovers the mod p Satake isomorphism as in [Her11, HV15], cf. also [Zhu20, Corollary 7].

References

[Ach21] Pramod N. Achar. *Perverse sheaves and applications to representation theory*, volume 258 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, [2021] ©2021. doi:10.1090/surv/258. 8, 41

[AG15] D. Arinkin and D. Gaitsgory. Singular support of coherent sheaves and the geometric Langlands conjecture. *Selecta Math. (N.S.)*, 21(1):1–199, 2015. URL: http://dx.doi.org/10.1007/s00029-014-0167-5, doi:10.1007/s00029-014-0167-5. 42

[Alp14] Jarod Alper. Adequate moduli spaces and geometrically reductive group schemes. *Algebr. Geom.*, 1(4):489–531, 2014. doi:10.14231/AG-2014-022. 23

[AR18] Pramod N. Achar and Simon Riche. Reductive groups, the loop Grassmannian, and the Springer resolution. *Invent. Math.*, 214(1):289–436, 2018. doi:10.1007/s00222-018-0805-1. 1

[Ayo10] Joseph Ayoub. Note sur les opérations de Grothendieck et la réalisation de Betti. *J. Inst. Math. Jussieu*, 9(2):225–263, 2010. URL: http://dx.doi.org/10.1017/S1474748009000127, doi:10.1017/S1474748009000127. 4

[BB73] A. Białynicki-Birula. Some theorems on actions of algebraic groups. *Ann. of Math. (2)*, 98:480–497, 1973. doi:10.2307/1970915. 13, 17

[BB76] A. Białynicki-Birula. Some properties of the decompositions of algebraic varieties determined by actions of a torus. *Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.*, 24(9):667–674, 1976. 13, 17

[BBD82] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982. 8, 32, 37

[BD99] Alexander Beilinson and Vladimir Drinfeld. Quantization of Hitchin's integrable system and Hecke eigensheaves. 1999. URL: http://www.math.utexas.edu/users/benzvi/Langlands.html. 1, 26, 33

[BD13] Mitya Boyarchenko and Vladimir Drinfeld. A duality formalism in the spirit of Grothendieck and Verdier. *Quantum Topol.*, 4(4):447–489, 2013. doi:10.4171/QT/45. 43

[BD14] Mitya Boyarchenko and Vladimir Drinfeld. Character sheaves on unipotent groups in positive characteristic: foundations. *Sel. Math., New Ser.*, 20(1):125–235, 2014. doi:10.1007/s00029-013-0133-7. 43

[BG14] Kevin Buzzard and Toby Gee. The conjectural connections between automorphic representations and Galois representations. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 135–187. Cambridge Univ. Press, Cambridge, 2014. doi:10.1017/CBO9781107446335.006. 54

[BL95] Arnaud Beauville and Yves Laszlo. Un lemme de descente. *C. R. Acad. Sci. Paris Sér. I Math.*, 320(3):335–340, 1995. 23, 26

[Bor91] Armand Borel. *Linear algebraic groups*, volume 126 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. doi:10.1007/978-1-4612-0941-6. 20

[Bor98] Mikhail Borovoi. Abelian Galois cohomology of reductive groups. *Mem. Amer. Math. Soc.*, 132(626):viii+50, 1998. doi:10.1090/memo/0626. 53

[BR18] Pierre Baumann and Simon Riche. Notes on the geometric Satake equivalance. In *Relative aspects in representation theory, Langlands functoriality and automorphic forms*, volume 2221 of *Lecture Notes in Math.*, pages 1–134. Springer, Cham, 2018. 26, 29, 33, 35, 40, 46, 51

[BT72] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. *Inst. Hautes Études Sci. Publ. Math.*, (41):5–251, 1972. URL: http://www.numdam.org/item?id=PMIHES_1972__41__5_0. 22

[BT84] F. Bruhat and J. Tits. Groupes réductifs sur un corps local. II. Schémas en groupes. Existence d'une donnée radicielle valuée. *Inst. Hautes Études Sci. Publ. Math.*, (60):197–376, 1984. URL: http://www.numdam.org/item?id=PMIHES_1984__60__5_0. 54

[Cas22] Robert Cass. Perverse $\mathbb{F}_p$-sheaves on the affine Grassmannian. *J. Reine Angew. Math.*, 785:219–272, 2022. doi:10.1515/crelle-2021-0089. 2, 32

[CD19] Denis-Charles Cisinski and Frédéric Déglise. *Triangulated categories of mixed motives*. Springer Monographs in Mathematics. Springer, Cham, [2019] ©2019. URL: http://arxiv.org/abs/0912.2110, doi:10.1007/978-3-030-33242-6. 4, 9, 10

[Cho21] Chirantan Chowdhury. Motivic Homotopy Theory of Algebraic Stacks, 2021. arXiv:arXiv:2112.15097. 4

[Cis21] Denis-Charles Cisinski. Cohomological methods in intersection theory, 2021. doi:10.1007/978-3-030-78977-0_3. 46

[dCHL18] Mark Andrea de Cataldo, Thomas J. Haines, and Li Li. Frobenius semisimplicity for convolution morphisms. *Math. Z.*, 289(1-2):119–169, 2018. URL: https://doi.org/10.1007/s00209-017-1946-4. 10

[Del07] Pierre Deligne. Letter to Serre. 2007. URL: https://publications.ias.edu/sites/default/files/2007%20Serre_0.pdf. 54

[Deo85] Vinay V. Deodhar. On some geometric aspects of Bruhat orderings. I. A finer decomposition of Bruhat cells. *Invent. Math.*, 79(3):499–511, 1985. doi:10.1007/BF01388520. 3, 19, 21

[DG14] V. Drinfeld and D. Gaitsgory. On a theorem of Braden. *Transform. Groups*, 19(2):313–358, 2014. doi:10.1007/s00031-014-9267-8. 5

[DM82] Pierre Deligne and James S Milne. Tannakian categories. In *Hodge cycles, motives, and Shimura varieties*, pages 101–228. Springer, 1982. 52, 53

[Dri18] Vladimir Drinfeld. On algebraic spaces with an action of $\mathbb{G}_m$. 2018. URL: https://arxiv.org/abs/1308.2604. 17

[Dud08] Olivier Dudas. Note on the Deodhar decomposition of a double Schubert cell. 2008. arXiv:0807.2198. 21

[ES22] Jens Niklas Eberhardt and Jakob Scholbach. Integral Motivic Sheaves and Geometric Representation Theory, 2022. arXiv:2202.11477. 1, 2, 4, 6, 7, 10, 37, 51

[Fal03] Gerd Faltings. Algebraic loop groups and moduli spaces of bundles. *J. Eur. Math. Soc. (JEMS)*, 5(1):41–68, 2003. URL: https://doi.org/10.1007/s10097-002-0045-x. 10, 24

[FG09] Edward Frenkel and Benedict Gross. A rigid irregular connection on the projective line. *Ann. of Math. (2)*, 170(3):1469–1512, 2009. doi:10.4007/annals.2009.170.1469. 54

[FM99] Michael Finkelberg and Ivan Mirković. Semi-infinite flags. I. Case of global curve $\mathbf{P}^1$. In *Differential topology, infinite-dimensional Lie algebras, and applications*, volume 194 of *Amer. Math. Soc. Transl. Ser. 2*, pages 81–112. Amer. Math. Soc., Providence, RI, 1999. doi:10.1090/trans2/194/05. 1

[FS21] Laurent Fargues and Peter Scholze. Geometrization of the local Langlands correspondence, 2021. arXiv:2102.13459. 1, 2, 3, 33, 41, 42, 43, 45, 48, 50, 51, 52

[Gai01] D. Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001. doi:10.1007/s002220100122. 29

[Gai04] Dennis Gaitsgory. Appendix: braiding compatibilities. In *Representation theory of algebraic groups and quantum groups*, volume 40 of *Adv. Stud. Pure Math.*, pages 91–100. Math. Soc. Japan, Tokyo, 2004. doi:10.2969/aspm/04010091. 29

[Gai07] D. Gaitsgory. On de Jong's conjecture. *Israel J. Math.*, 157:155–191, 2007. doi:10.1007/s11856-006-0006-2. 1

[Gin00] Victor Ginzburg. Perverse sheaves on a Loop group and Langlands' duality. 2000. URL: https://arxiv.org/abs/alg-geom/9511007. 1

[GL05] S. Gaussent and P. Littelmann. LS galleries, the path model, and MV cycles. *Duke Math. J.*, 127(1):35–88, 2005. doi:10.1215/S0012-7094-04-12712-5. 3, 14, 15, 16, 17, 18, 19

[Gro98] Benedict H. Gross. On the Satake isomorphism. In *Galois representations in arithmetic algebraic geometry (Durham, 1996)*, volume 254 of *London Math. Soc. Lecture Note Ser.*, pages 223–237. Cambridge Univ. Press, Cambridge, 1998. doi:10.1017/CBO9780511662010.006. 56, 57

[Her11] Florian Herzig. A Satake isomorphism in characteristic p. *Compos. Math.*, 147(1):263–283, 2011. doi:10.1112/S0010437X10004951. 58

[Hoy17] Marc Hoyois. The six operations in equivariant motivic homotopy theory. *Adv. Math.*, 305:197–279, 2017. URL: https://doi.org/10.1016/j.aim.2016.09.031. 3

[HR18a] T. J. Haines and T. Richarz. On the normality of Schubert varieties: remaining cases in positive characteristic, 2018. arXiv:1806.11001. 10

[HR18b] T. J. Haines and T. Richarz. The Test Function Conjecture for Local Models of Weil-restricted groups, 2018. arXiv:1805.07081. 10, 11, 23, 24, 32

[HR22] Thomas J Haines and Timo Richarz. Normality and cohen–macaulayness of parahoric local models. *Journal of the European Mathematical Society*, 2022. 18

[Hub93] Annette Huber. Calculation of derived functors via Ind-categories. *J. Pure Appl. Algebra*, 90(1):39–48, 1993. doi:10.1016/0022-4049(93)90134-F. 31

[HV15] Guy Henniart and Marie-France Vignéras. A Satake isomorphism for representations modulo p of reductive groups over local fields. *J. Reine Angew. Math.*, 701:33–75, 2015. doi:10.1515/crelle-2013-0021. 58

[Jan87] Jens Carsten Jantzen. *Representations of algebraic groups*, volume 131 of *Pure and Applied Mathematics*. Academic Press, Inc., Boston, MA, 1987. 11

[JMW14] Daniel Juteau, Carl Mautner, and Geordie Williamson. Parity Sheaves. *Journal of the American Mathematical Society*, 27(4):1169–1212, May 2014. arXiv:0906.2994, doi:10.1090/S0894-0347-2014-00804-3. 13

[JY21] Fangzhou Jin and Enlin Yang. Künneth formulas for motives and additivity of traces. *Adv. Math.*, 376:Paper No. 107446, 83, 2021. doi:10.1016/j.aim.2020.107446. 9, 12, 30, 31

[KR21] Adeel A. Khan and Charanya Ravi. Generalized cohomology theories for algebraic stacks, 2021. arXiv:2106.15001. 4

[KS90] Masaki Kashiwara and Pierre Schapira. *Sheaves on manifolds*, volume 292 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1990. With a chapter in French by Christian Houzel. 31

[Laf18] Vincent Lafforgue. Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale. *J. Amer. Math. Soc.*, 31(3):719–891, 2018. 1

[Let05] Emmanuel Letellier. *Fourier transforms of invariant functions on finite reductive Lie algebras*, volume 1859 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2005. doi:10.1007/b104209. 32

[Lev93] Marc Levine. Tate motives and the vanishing conjectures for algebraic K-theory. In *Algebraic K-theory and algebraic topology (Lake Louise, AB, 1991)*, volume 407 of *NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci.*, pages 167–188. Kluwer Acad. Publ., Dordrecht, 1993. 6

[Lev14] Marc Levine. A comparison of motivic and classical stable homotopy theories. *J. Topol.*, 7(2):327–362, 2014. doi:10.1112/jtopol/jtt031. 4

[Lus83] George Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. In *Analysis and topology on singular spaces, II, III (Luminy, 1981)*, volume 101 of *Astérisque*, pages 208–229. Soc. Math. France, Paris, 1983. 41, 57

[Moe02] I. Moerdijk. Monads on tensor categories. *J. Pure Appl. Algebra*, 168(2-3):189–208, 2002. doi:10.1016/S0022-4049(01)00096-2. 49

[MR18] Carl Mautner and Simon Riche. Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković–Vilonen conjecture. *Journal of the European Mathematical Society*, 20(9):2259–2332, July 2018. doi:10.4171/JEMS/812. 1

[Mum99] David Mumford. The red book of varieties and schemes, second, expanded ed. *Lecture notes in mathematics*, 1358, 1999. 17

[MV07] I. Mirković and K. Vilonen. Geometric Langlands duality and representations of algebraic groups over commutative rings. *Ann. Math. (2)*, 166(1):95–143, 2007. 1, 2, 26, 27, 30, 41, 42, 52

[NP01] B. C. Ngô and P. Polo. Résolutions de Demazure affines et formule de Casselman-Shalika géométrique. *J. Algebraic Geom.*, 10(3):515–547, 2001. 11

[PS20] Cédric Pépin and Tobias Schmidt. Generic and mod p Kazhdan-Lusztig theory for GL_2. 2020. arXiv:2007.01364. 57

[PY06] Gopal Prasad and Jiu-Kang Yu. On quasi-reductive group schemes. *Journal of algebraic geometry*, 15(3):507–549, 2006. 52, 54

[PZ13] G. Pappas and X. Zhu. Local models of Shimura varieties and a conjecture of Kottwitz. *Invent. Math.*, 194(1):147–254, 2013. URL: https://doi.org/10.1007/s00222-012-0442-z. 10

[Qui76] Daniel Quillen. Projective modules over polynomial rings. *Invent. Math.*, 36:167–171, 1976. doi:10.1007/BF01390008. 11

[Ren05] Lex E. Renner. *Linear algebraic monoids*, volume 134 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 2005. Invariant Theory and Algebraic Transformation Groups, V. 56

[Ric13] Timo Richarz. Schubert varieties in twisted affine flag varieties and local models. *J. Algebra*, 375:121–147, 2013. URL: https://doi.org/10.1016/j.jalgebra.2012.11.013. 20

[Ric14] Timo Richarz. A new approach to the geometric Satake equivalence. *Doc. Math.*, 19:209–246, 2014. URL: http://arxiv.org/abs/math/arXiv:1207.5314v2. 1, 24, 26, 27, 42

[Ric19] Timo Richarz. Spaces with $\mathbb{G}_m$-action, hyperbolic localization and nearby cycles. *J. Algebraic Geom.*, 28(2):251–289, 2019. doi:10.1090/jag/710. 5, 11, 17, 33

[RS20] Timo Richarz and Jakob Scholbach. The intersection motive of the moduli stack of shtukas. *Forum of Mathematics (Sigma)*, 8(e8), 2020. URL: https://arxiv.org/abs/1901.04919, doi:10.1017/fms.2019.32. 1, 4, 5, 8, 9, 10, 11, 12, 13, 21, 25, 30, 31, 32, 33, 44, 54

[RS21] Timo Richarz and Jakob Scholbach. The motivic Satake equivalence. *Math. Ann.*, 380(3-4):1595–1653, 2021. doi:10.1007/s00208-021-02176-9. 1, 2, 9, 10, 25, 29, 39, 40, 48

[SGA70a] *Schémas en groupes. II: Groupes de type multiplicatif, et structure des schémas en groupes généraux*. Lecture Notes in Mathematics, Vol. 152. Springer-Verlag, Berlin-New York, 1970. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Dirigé par M. Demazure et A. Grothendieck. 53

[SGA70b] *Schémas en groupes. III: Structure des schémas en groupes réductifs*. Lecture Notes in Mathematics, Vol. 153. Springer-Verlag, Berlin-New York, 1970. Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), Dirigé par M. Demazure et A. Grothendieck. 54

[Spi16] Markus Spitzweck. Notes for a mini-course on "Mixed Tate Motives and Fundamental Groups" given in Bonn, 2016. URL: https://www.him.uni-bonn.de/fileadmin/him/Spitzweck-minicourse-HIM.pdf. 6

[Spi18] Markus Spitzweck. A commutative $\mathbb{P}^1$-spectrum representing motivic cohomology over Dedekind domains. *Mém. Soc. Math. Fr. (N.S.)*, (157):110, 2018. doi:10.24033/msmf.465. 2, 3

[Sta22] Stacks Project Authors. Stacks Project. http://stacks.math.columbia.edu, 2022. 17, 21

[Ste75] Robert Steinberg. Torsion in reductive groups. *Advances in Math.*, 15:63–92, 1975. doi:10.1016/0001-8708(75)90125-5. 22

[Sus76] A. A. Suslin. Projective modules over polynomial rings are free. *Dokl. Akad. Nauk SSSR*, 229(5):1063–1066, 1976. 11

[SW18] Wolfgang Soergel and Matthias Wendt. Perverse motives and graded derived category $\mathcal{O}$. *J. Inst. Math. Jussieu*, 17(2):347–395, 2018. URL: https://arxiv.org/abs/1404.6333v3, doi:10.1017/S1474748016000013. 5

[Tit71] J. Tits. Représentations linéaires irréductibles d'un groupe réductif sur un corps quelconque. *J. Reine Angew. Math.*, 247:196–220, 1971. doi:10.1515/crll.1971.247.196. 52

[Vin95] E. B. Vinberg. On reductive algebraic semigroups. In *Lie groups and Lie algebras: E. B. Dynkin's Seminar*, volume 169 of *Amer. Math. Soc. Transl. Ser. 2*, pages 145–182. Amer. Math. Soc., Providence, RI, 1995. doi:10.1090/trans2/169/10. 55

[Wil17] Jörg Wildeshaus. Intermediate extension of Chow motives of Abelian type. *Adv. Math.*, 305:515–600, 2017. doi:10.1016/j.aim.2016.09.032. 5

[XZ19] Liang Xiao and Xinwen Zhu. On vector-valued twisted conjugation invariant functions on a group. In *Representations of reductive groups*, volume 101 of *Proc. Sympos. Pure Math.*, pages 361–425. Amer. Math. Soc., Providence, RI, 2019. With an appendix by Stephen Donkin. 55, 56

[XZ22] Daxin Xu and Xinwen Zhu. Bessel F-isocrystals for reductive groups. *Invent. Math.*, 227(3):997–1092, 2022. doi:10.1007/s00222-021-01079-5. 1

[Zhu14] Xinwen Zhu. On the coherence conjecture of Pappas and Rapoport. *Ann. of Math. (2)*, 180(1):1–85, 2014. doi:10.4007/annals.2014.180.1.1. 29

[Zhu15] Xinwen Zhu. The geometric Satake correspondence for ramified groups. *Ann. Sci. Éc. Norm. Supér. (4)*, 48(2):409–451, 2015. doi:10.24033/asens.2248. 1, 29

[Zhu17a] Xinwen Zhu. Affine Grassmannians and the geometric Satake in mixed characteristic. *Ann. of Math. (2)*, 185(2):403–492, 2017. doi:10.4007/annals.2017.185.2.2. 1

[Zhu17b] Xinwen Zhu. An introduction to affine Grassmannians and the geometric Satake equivalence. In *Geometry of moduli spaces and representation theory*, volume 24 of *IAS/Park City Math. Ser.*, pages 59–154. Amer. Math. Soc., Providence, RI, 2017. 1, 11, 13, 23, 24, 26, 27, 33, 34, 39

[Zhu18] Xinwen Zhu. Geometric Satake, categorical traces, and arithmetic of Shimura varieties. In *Current developments in mathematics 2016*, pages 145–206. Int. Press, Somerville, MA, 2018. 1

[Zhu20] Xinwen Zhu. A note on integral Satake isomorphisms, 2020. arXiv:2005.13056. 1, 3, 51, 54, 55, 56, 58