1. Deformations of $\mathrm{Rep}(G_{\mathbb{Z}})$ and integral Whittaker categories

There is an interest in considering *topological twistings*[1] of Langlands, as [7] and [8] in geometric and arithmetic setting respectively. Fundamental to this is an understanding of the moduli of $\mathbb{E}_2$ deformations of the representation category.

$$\mathcal{D}\mathrm{ef}^{(2)}(\mathrm{Rep}_{\mathbb{C}}G) : \mathrm{Art}^{(4)}_{\mathbb{C}} \to \mathcal{S}$$

$$R \mapsto \mathbb{E}_2(\mathrm{LCat}_R) \times_{\mathbb{E}_2(\mathrm{LCat}_{\mathbb{C}})} \{\mathrm{Rep}_{\mathbb{C}}G\}$$

where $\mathrm{Art}^{(4)}_{\mathbb{C}}$ is the category of $\mathbb{E}_4$ Artinian $\mathbb{C}$-algebras. [2] This is classically the moduli problem of *braided tensor deformations*, studied by Yetter and Crane [16], [3]. Its R-points consists of the groupoid of pairs

$$(\mathcal{C}, \alpha) : \mathcal{C} \otimes_R \mathbb{C} \simeq \mathrm{Rep}_{\mathbb{C}}G, \mathcal{C} \in \mathrm{LCat}_R$$

We will now describe this moduli problem from the perspective of geometric representation theory. The *Whittaker construction/category* yields an equivalence of *stable (triangulated) categories*, [4],

$$\mathrm{Rep}_{\mathbb{C}}(G) \simeq \mathrm{Whit}(\mathrm{Dmod}_{\mathbb{C}}(\mathrm{Gr}_{\check{G}})) \tag{1}$$

The Grothendieck group of the right hand side is the module of Whittaker functions, in the sense of B. Casselman and J. Shalika, [2].

Consider the topological parameter space of *multiplicative* $\mathbb{G}_m$-gerbes:

$$\mathrm{Ge}_{\mathbb{G}_m}(\mathrm{Gr}_{\check{G}}) : R \mapsto \mathrm{Map}_{\mathbb{E}_2(\mathcal{S})}(\mathrm{Gr}_{\check{G}}, B^2\mathbb{G}_m(R))$$

where $\mathbb{G}_m(R) := (\Omega^\infty R)^\times$, the groupoid of invertible elements.

Theorem 1.1. [9, 10] The map on R-points given by *twisting* [7, 2]

$$\eta \mapsto \mathrm{Whit}^\eta(\mathrm{Dmod}_{\mathbb{C}}(\mathrm{Gr}_{\check{G}}))$$

yields an equivalence

$$\widehat{\mathrm{Ge}_{\mathbb{G}_m}(\mathrm{Gr}_{\check{G}})} \simeq \mathcal{D}\mathrm{ef}^{(2)}\mathrm{Rep}_{\mathbb{C}}G$$

where the left hand side is the formal completion at the trivial gerbe.

[1] there are *twists* of different flavours:

	algebro-geometric	differential geometric	Topological
Parameter Type	K-theoretic, [15]	Quantum, [5]	Metaplectic, [7].

The metaplectic version gives rise to *gerbes*, which exists in most sheaf theoretic context, see [17].

[2] The $\mathbb{E}_4$ condition is technical and it can be ignored: it is so that LCat_R is an $\mathbb{E}_2$ monoidal category.

2. Question: is there an integral version of whittaker category?

We describe some main obstructions in this setting in 3. Can one give

- an *integral* Whittaker category/ integral version of (1)?
- a topological description of the $\mathbb{E}_2$ deformations of $\mathrm{Rep}_{\mathbb{Z}} G_{\mathbb{Z}}$?
- a $l \neq p$, mixed characteristic version (for the geometry) of the above construction? This is my current focus with Konrad Zou and Ashwin Iyengar.

Hints of this *rationally* are in the work of T. Richarz and J. Scholbach [13]. To a prestack X, one constructs presentable stable categories $\mathrm{DM}(X, \mathbb{Q})$ of motives, extending the theory of Ayoub and Cisinki-Déglise.

When $X = \mathrm{LG}^+_{\mathbb{Z}} \backslash LG_{\mathbb{Z}} / \mathrm{LG}^+_{\mathbb{Z}}$ is the automoprhic Hecke stack, X is a stratified space. One has a full subcategory of *stratified Tate motives* $\mathrm{DTM}(X; \mathbb{Q})$. This has a t structure whose heart is the *mixed (stratified) Tate motives*:

Theorem 2.1. [14, Thm. C] For each finite field $\mathbb{F}_q$, there is a symmetric monoidal equivalence,

$$\mathrm{MTM}(\mathrm{LG}^+_{\mathbb{F}_q} \backslash \mathrm{LG}_{\mathbb{F}_q} / \mathrm{LG}^+_{\mathbb{F}_q}) \simeq \mathrm{Rep}_{\mathbb{Q}}(\widehat{G}_1)^{\heartsuit}$$

where $\widehat{G}_1$ is the *modified Langlands dual*, [1, 5].

Our goal is to give a Whittaker construction/category in this context, obtaining a similar equivalence as (1) at the level of *stable (triangulated) categories.*

3. What are the obstructions?

Recall for GL_2 the universal principal series

$$\mathrm{Fun}(G/U) := \mathrm{Ind}_U^G k := \{f : G \to \mathbb{C} \,:\, f(ug) = f(g) \,:\, u \in U, g \in G\}$$

the induction from the trivial representation of the unipotent, fails to contain the most interesting representations of GL_2 and being multiplicity free. Classically, he *Whittaker module* for GL_2 can address this phenomena via a twist

$$\mathrm{Fun}^{\chi}(G/U) := \mathrm{Ind}_U^G k_{\chi} \, \{f : G \to k \,:\, f(ug) = \chi(u) f(g), \quad u \in U, v \in G\}$$

where χ is a *nondegenerate character*. The geomerization of this condition is easy in *equal* characteristic set up. The condition of a function being χ equivariant can be captured by a compatibility condition.

$$\mathrm{act}^* \mathcal{F} \simeq \mathcal{F} \boxtimes \mathcal{L}_{\psi}$$

where $\mathcal{L}_{\psi}$ is often referred as the *Artin-Schrier sheaf* induced from the *Lang isogeny*

$$\mathbb{A} \to \mathbb{A} \quad x^q - x, \quad |k| = q$$

. To make sense of the above equation see the works of Ngô and Polo, in particular [11, Lem 11.1].

In defining the Whittaker category when the characteristic is *zero*. One can give a definition of Whittaker category, [6], using the *Kirillov model*. The basic object of study is sheaves on $\mathbb{G}_a \rtimes \mathbb{G}_m$, and the appropriate fourier transform.

3.1. Further questions. We may ask:

(1) To what generalirty can we allow our geometric objects to be? In de Rham sheaf theory: $\mathbb{G}_a$ canont be viewed as a $\mathbb{E}_\infty$-group scheme over $\mathbb{S}$, [10, 1.6.20].
(2) What the appropriate [3] sheaf theory? [4] In the Betti sheaf theory: we need a notion of *constructible sheaves* with *singular support* in Sp. This is unclear but seems more plausible. It may be possible to work out integral case using recent development in *motivic sheaves*.

[3] A litmus test to a correct sheaf theory, denoted Shv, is the property

$$\mathrm{Whit}(\mathrm{Shv}(G/B; \mathrm{Sp})) \simeq \mathrm{Shv}(T, \mathrm{Sp})$$

for triplet $(B, N, T) = (\text{Borel}, \text{maximal unipotent subgroup}, \text{ maximal torus subgroup})$.

[4] using terminology of [12].

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