Categorical traces

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Before we go into derived categorical algebra, it might be useful to review usual algebra, then derived algebra. Let A be an associative algebra, and M an A-bimodule. We define the center and trace of M:

$$Z(A, M) = \{m \in M \mid am = ma \ \forall a \in A\}, \qquad \mathrm{Tr}(A, M) = M/\{am - ma \mid m \in M, a \in A\}.$$

Of special interest is the case $M = A$, where we will write the above by $Z(A)$ and $\mathrm{Tr}(A)$. The center of $Z(A)$ is a commutative algebra, and both $Z(A, M)$ and $\mathrm{Tr}(A, M)$ are modules for it.

We write $A^e = A \otimes A^{\mathrm{op}}$ as shorthand. The above formulas may be rewritten:

$$Z(A, M) = \mathrm{Hom}_{A^e}(A, M), \qquad \mathrm{Tr}(A, M) = A \otimes_{A^e} M.$$

Let M be a right A-module and N a left A-module. We will use the following obvious formula a lot:

$$M \otimes_A N = A \otimes_{A^e} (M \boxtimes N).$$

Let A, B be associative algebras. Given an (A, B)-bimodule K, we define the left and right dual (B, A)-bimodules:

$${}^\vee K = \mathrm{Hom}_A(K, A), \qquad K^\vee = \mathrm{Hom}_B(K, B)$$

which only really deserve to be called duals if the coevaluation and evaluation maps

$$\eta : B \to {}^\vee K \otimes_A K, \qquad \epsilon : K \otimes_B {}^\vee K \to A$$

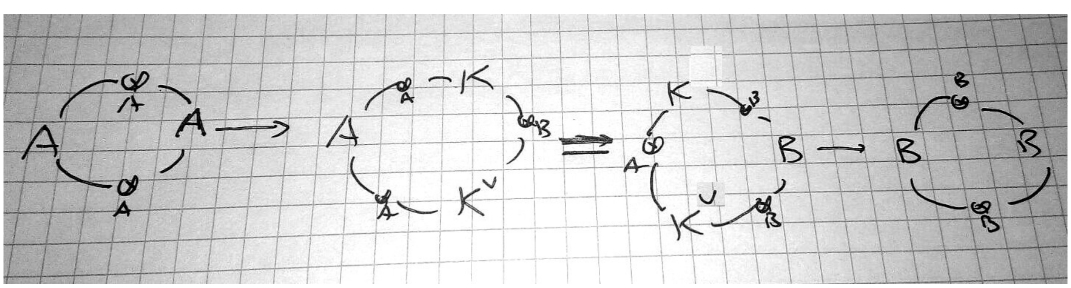
$$\eta : A \to K \otimes_B K^\vee, \qquad \epsilon : K^\vee \otimes_A K \to B$$

satisfy "Zorro's identities", i.e. the compositions

$$K \to K \otimes_B K^\vee \otimes_A K \to K, \qquad K^\vee \to K^\vee \otimes_A K \otimes_B K \to K$$

are the identity. A dual pair $(K, K^\vee)$ gives rise to a functoriality of traces:[1]

$$\mathrm{Tr}(A) = A \otimes_{A^e} A \to A \otimes_{A^e} (K \otimes_B K^\vee) \simeq B \otimes_{B^e} (K^\vee \otimes_A K) \to B \otimes_{B^e} B = \mathrm{Tr}(B).$$



[1] By convention, we will consider right modules, thus an (A, B)-bimodule K defines a functor $\mathrm{Mod}(A) \to \mathrm{Mod}(B)$, thus we have covariant functoriality for right dualizable bimodules, and contravariant functoriality for left dualizable bimodules.

Explicitly, right dualizability of K is equivalent to K being projective as a B-module. For simplicity, let us assume that K is free as a B-module; then this functoriality assigns to an element $a \in A$ the trace of its action on K, viewed as a B-valued matrix (assuming B is commutative). One may verify that it factors through the quotient by commutators in A.

There is also a coefficients version, i.e. given an A-bimodule M and a B-bimodule N, one can work out what extra data is needed to write down a map $\mathrm{Tr}(A, M) \to \mathrm{Tr}(B, N)$: we need a way to commute M with K, i.e. a map $M \otimes_A K \to K \otimes_B N$ (exercise: justify using the above picture).

Example 0.0.1. Let $A = M = k$, and $N = B_F$ where $\phi : B \to B$ is an endomorphism of the algebra, and B_ϕ is the B-bimodule where $b_1 \cdot b \cdot b_2 = b_1 b \phi(b_2)$. Let K be a right B-module; a commuting structure is given by a ϕ-semilinear map $f : K \to K$. Now, the trace of the map defined by K, which we call its *character* $[K, f]$, is the trace of the endomorphism f viewed as a B-valued matrix. In particular, if $\phi = f = \mathrm{id}$, then $[K, f]$ is the rank of K.

As far as I know there is not a nice functoriality for centers.

Deriving algebra is not hard. The main tool is the *bar resolution*, i.e. the following resolution of A as an A^e-module:

$$B(A) := \Big(\cdots \;\; A \otimes A \otimes A \otimes A \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; A \otimes A \otimes A \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; A \otimes A \;\Big) \simeq A$$

where the maps are given by adjacent multiplication. More generally, if A is a R-algebra one can take a relative bar resolution

$$B(A/R) := \Big(\cdots \;\; A \otimes_R A \otimes_R A \otimes_R A \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; A \otimes_R A \otimes_R A \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; A \otimes_R A \;\Big) \simeq A.$$

This complex will not be a resolution unless A is projective over R. However, if we have a good notion of derived tensor product, we may derive the tensor products in the terms of the relative bar resolution to obtain a resolution. This is less useful in algebra, but will be useful in categorical algebra.

We can now define *derived centers* (also known as *Hochschild cohomology*) and *derived traces* (also known as *Hochschild homology*)

$$HH^\bullet(A, M) = \mathrm{Ext}^\bullet_{A^e}(A, M) \simeq \mathrm{Hom}^\bullet_{A^e}(B(A), M), \qquad HH_\bullet(A, M) = \mathrm{Tor}^{A^e}_\bullet(A, M) \simeq B(A) \otimes_{A^e} M.$$

The utility of the bar construction is not limited to computing Hochschild (co)homology. It can be used to produce a resolution of any relative tensor product via the formula

$$M \otimes_A N \simeq (M \boxtimes N) \otimes_{A^e} A \simeq \Big(\cdots \;\; M \otimes A \otimes A \otimes N \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; M \otimes A \otimes N \;\substack{\longrightarrow\\[-2pt]\longrightarrow}\; M \otimes N \;\Big).$$

This gives a universal way to compute derived tensor products. This will be useful when categorifying, because we can't do things like choose generators and relations.

1 Generalities on traces

We will now categorify the above discussion. Confusingly, there are two things we might mean by categorifying the algebra A, and one can discuss centers and traces for both, leading to two different notions. Furthermore, these two notions are related, and we will require discussion of both.

- We can view the algebra A itself as a k-linear category with one object (or pass to its category of modules $\mathrm{Mod}(A)$), i.e. view A as living in k-linear categories. Note that $\mathrm{Mod}(A)$ is only monoidal if A is commutative, so the monoidal structure is not necessary to define this trace. Here, a module is a functor $\mathbf{A}^{\mathrm{op}} \to \mathbf{Vect}_k$. The corresponding trace is sometimes called the *1-categorical* or *vertical* or *ordinary* trace. The corresponding Morita theory is the dg Morita theory of Toën.[To07]

- We may categorify the notion of algebras, leading to associative monoidal k-linear categories $\mathbf{A}$. Note that $\mathbf{A}$ may be considered in the above set-up as well. Here, a module is a category $\mathbf{M}$ with an action functor $\mathbf{A} \times \mathbf{M} \to \mathbf{M}$ satisfying identities. The corresponding trace is sometimes called the *2-categorical* or *horizontal* or *monoidal* trace.

- If $\mathbf{A}$ is a monoidal category satisfying some compactness conditions, its vertical trace becomes an algebra. The horiontal trace need not be monoidal. This hints that the two traces are being taken in two directions. We will discuss this later.

Example 1.0.1. Let us now introduce the examples of monoidal categories we want to keep in mind.

1. Let X, Y be perfect stacks over k (e.g. quasiprojective schemes modulo affine groups). Then, the category $\mathrm{QC}(X \times_Y X)$ is monoidal under convolution.
   (a) If we take $X = Y$, we obtain $\mathrm{QC}(X)$ under tensor product.
   (b) If we take $Y = BG$ and $X = \mathrm{pt}$, we obtain $\mathrm{QC}(G)$ under group multiplication.
2. In the set-up above, assume that X, Y are smooth and QCA (e.g. smooth quasiprojective schemes modulo affine groups), and furthermore that $f : X \to Y$ is *proper*. Then, $\mathrm{QC}^!(X \times_Y X)$ is monoidal under convolution. This is a renormalization of QC, and may be ignored at a first pass.
3. Let Y be a stack with smooth diagonal (e.g. a classifying stack) and assume $f : X \to Y$ is proper. Then, $\mathcal{D}(X \times_Y X)$ is monoidal under convolution.

Not all the conditions above are necessary to define monoidal structures, but they will be for the monoidal categories to have some basic good properties we desire, e.g. semi-rigidity and control over compact objects.

1.1 Trace in a symmetric monoidal category

We will be vague in this section about what kind of objects (1-category, ∞-categorical, et cetera) we are dealing with. Let $(\mathbf{U}, \otimes, \mathbb{1})$ denote some kind of symmetric monoidal category, with operation $\otimes$ and unit $\mathbb{1}$. We are mostly interested in the following examples of $(\mathbf{U}, \otimes, \mathbb{1})$ for k a field:

(0) $\mathbf{U} = \mathbf{Vect}_k$,[2] the usual tensor product of chain complexes, and $\mathbb{1} = k$.

(1) $\mathbf{U} = \mathbf{1Mor}_k$ is the dg[3] 1-Morita category, whose objects are the dg categories $\mathrm{Mod}(A)$ for dg algebras A, and $\mathrm{Hom}_{\mathbf{U}}(A, B)$ is the set of (A, B)-bimodules. We define $\mathrm{Mod}(A) \otimes \mathrm{Mod}(B) := \mathrm{Mod}(A \otimes B)$, and $\mathbb{1} := \mathrm{Mod}(k)$.

(1') $\mathbf{U} = \mathbf{dgCat}_k$ is the category of dg categories, with colimit-presrving functors, over k under the Lurie tensor product $\otimes$, and $\mathbb{1} := \mathrm{Mod}(k)$. Not every dg category is equivalent to $\mathrm{Mod}(A)$, for example, $\mathrm{Rep}(G)$ for G a reductive group (there is not a single compact generator but an infinite collection of them).

(2) $\mathbf{U} = \mathbf{2Mod}_k$ is the 2-Morita category,[4] whose objects are $\mathrm{Mod}(\mathbf{A})$ for monoidal k-linear ∞-categories $\mathbf{A}$, and morphisms $\mathrm{Hom}_{\mathbf{U}}(\mathbf{A}, \mathbf{B})$ are $(\mathbf{A}, \mathbf{B})$-bimodule categories. We define the tensor product analogously, and $\mathbb{1} := \mathrm{Mod}(\mathbf{dgCat}_k)$.

Definition 1.1.1. We say that $X \in \mathbf{U}$ has a *dual* $X^\vee$ if there are maps

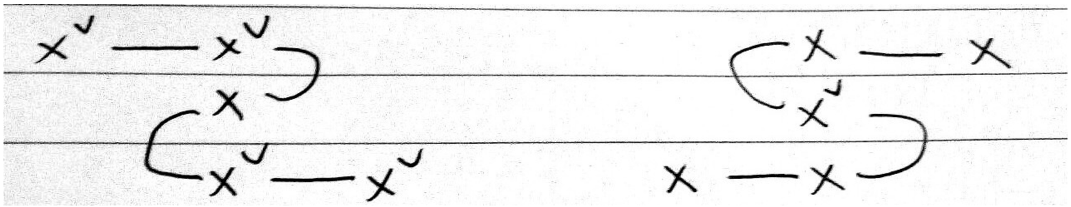
$$\eta_X : \mathbb{1} \to X^\vee \otimes X, \qquad \epsilon_X : X \otimes X^\vee \to \mathbb{1}$$

satisfying "Zorro's axioms":

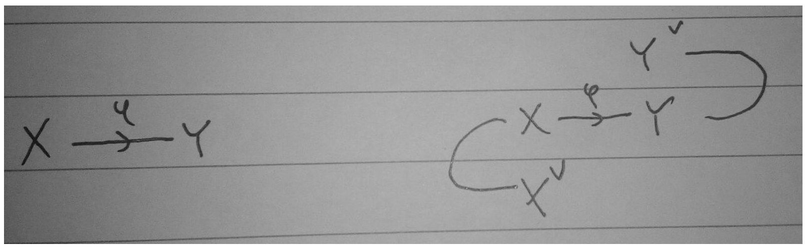
[2] Note we use this notation to mean the category of chain complexes, i.e. derived vector spaces.

[3] The difference between this and usual Morita theory is that we "derive everything." The functors are no longer required to be exact.

[4] We now move from dg categories to stable ∞-categories. This is necessary because of the abstract nature of various constructions here which are difficult to formulate in the explicit dg setting.



Exercise: use Zorro's axioms to prove that duals are unique if they exist. Hint: if there are two duals $X^\vee$ and ${}'X^\vee$, write down the diagram corresponding to the first Zorro axiom above using both. Note that for any map $\phi : X \to Y$ between dualizable objects, there is a dual map $\phi^\vee : Y^\vee \to X^\vee$:



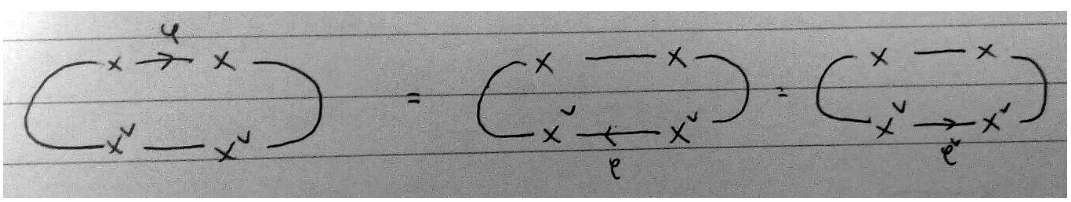
For the next definition, we need the following observation. For $X \in \mathbf{U}$, the functor $\mathrm{act}_X : \mathbf{U} \to \mathbf{U}$ has a right adjoint $\underline{\mathrm{Hom}}(X, -) : \mathbf{U} \to \mathbf{U}$. Taking $X = \mathbb{1}$ and applying the right adjoint to $\mathbb{1}$, we find that $\mathrm{End}_{\mathbf{U}}(\mathbb{1})$ is enriched in $\mathbf{U}$, and since the adjoint functors are equivalences in this case, we have an isomorphism in $\mathbf{U}$:

$$\underline{\mathrm{End}}_{\mathbf{U}}(\mathbb{1}) \simeq \mathbb{1}.$$

Note that this implies that $\mathrm{End}_{\mathbf{U}}(\mathbb{1})$, which is a priori only associative, is in fact commutative.

Definition 1.1.2. Assume X is dualizable and let $\phi : X \to X$ be an endomorphism of X. We define the trace

$$\mathrm{tr}(X, \phi) = (\epsilon_X \circ (\phi \otimes \mathrm{id}_{X^\vee}) \circ \eta_X) \in \underline{\mathrm{End}}_{\mathbf{U}}(\mathbb{1}) \simeq \mathbb{1}.$$



We remark, without clarifying terms, that the trace of the identity $\mathrm{tr}(X, \mathrm{id}_X)$ has a canonical S^1-action, while in general traces have a "paracyclic action."

In our examples:

(0) $\mathbf{U} = \mathbf{Vect}_k$: a chain complex is dualizable if and only if it has bounded and finite-dimensional cohomology, and we have $\mathrm{tr}(V^\bullet, \phi) = \sum_{n \in \mathbb{Z}} (-1)^n \,\mathrm{tr}(H^n(V), H^n(\phi)) \in k$ is the alternating sum of the usual trace.

(1) $\mathbf{U} = \mathbf{1Mor}_k$: every category $\mathrm{Mod}(A)$ is dualizable, with $\mathrm{Mod}(A)^\vee = \mathrm{Mod}(A^{op})$, and the trace of a bimodule is its Hochschild homology $\mathrm{tr}(A, M) = HH_\bullet(A, M) \in \mathbf{Vect}_k$.

(1') $\mathbf{U} = \mathbf{dgCat}_k$: it is not easy to say precisely which categories are dualizable. If $\mathbf{C} = \mathrm{Ind}(\mathbf{C}_0)$ is compactly generated, then its dual is $\mathbf{C}^\vee \simeq \mathrm{Ind}(\mathbf{C}_0^{op})$. The trace is also the Hochschild homology $HH_\bullet(\mathbf{C})$; there is a notion of cyclic bar complex of a compactly generated category. Note that this notion generalizes that one, i.e. we only require the category to be dualizable, which is weaker than being compactly generated. One particular example of interest is the case where $F : \mathbf{C} \to \mathbf{C}$ is a colimit-preserving endofunctor, and we define $\mathbf{C}_F$ to be the F-twisted diagonal bimodule, i.e. sending $(X, Y) \mapsto \mathrm{Hom}_{\mathbf{C}}(X, F(Y))$.

For example, if X is perfect then $\mathbf{C} = \mathrm{QC}(X)$, then $\mathbf{C}^\vee := \mathrm{QC}(X)$, the coevaluation is $\Delta_* p^*$ and the evaluation is $p_*\Delta^*$, where $p : X \to \operatorname{Spec} k$ and $\Delta : X \hookrightarrow X \times X$. Thus the trace of $\phi : X \to X$ is the global sections on the derived intersection (we will define $\mathcal{L}_q$ later):

$$\mathrm{tr}(\mathrm{QC}(X), \phi_*) \simeq \mathcal{O}(\Gamma_\phi \cap^R_{X\times X} \Delta_X) \simeq \mathcal{O}(\mathcal{L}_q(X)).$$

Definition 1.1.3. The *derived loop space* is defined

$$\mathcal{L}X = \mathrm{Map}(S^1, X) = \mathrm{Map}(\Sigma S^0, X) = \mathrm{Map}(* \coprod_{* *} *, X) = X \times_{X \times X} X.$$

For $\phi : X \to X$ a self-map, the *ϕ-twisted derived loop space* is defined

$$\mathcal{L}_\phi X = \Gamma_\phi \cap_{X\times X} \Delta_X.$$

(2) $\mathbf{U} = \mathbf{2Mod}_k$: every category $\mathrm{Mod}(\mathbf{A})$ is dualizable with $\mathrm{Mod}(\mathbf{A}) = \mathrm{Mod}(\mathbf{A}^{rv})$. The Hochschild homology is computed by the cyclic bar complex and lives in $\mathbf{dgCat}_k$. We similarly define for any *monoidal* endofunctor $F : \mathbf{A} \to \mathbf{A}$, the corresponding F-twisted diagonal bimodule $\mathbf{A}_F$.

1.2 Functoriality of traces and characters

Furthermore, we have functoriality of traces. This notion only makes sense if $\mathbf{U}$ is a 2-category, i.e. in the 1 and 2-categorical contexts.

Definition 1.2.1. Let X, Y be dualizable objects with endomorphisms ϕ_X, ϕ_Y. Let $f : X \to Y$ be a right-dualizable morphism in $\mathbf{U}$ with a commuting structure $\eta : f \circ \phi_X \to \phi_Y \circ f$. Then we have a map on traces

$$\mathrm{tr}(f, \eta) : \mathrm{tr}(X, \phi_X) \to \mathrm{tr}(Y, \phi_Y)$$

defined in the usual way (see the picture, and examples).

Example 1.2.2. Again, we go to the examples.

(0) There is no functoriality for 0-categorical traces, since there are no 2-morphisms.

(1) First, let's recall that the 0-categorical trace is literally just taking the trace of a matrix. Now, recall how we defined functoriality of 1-categorical traces earlier. Let A, B be algebras, and let K be a right-dualizable (A, B)-bimodule (i.e. perfect as a B-module). Then we have a map on traces:

$$HH_\bullet(A) = A/[A, A] \longrightarrow HH_\bullet(B) = B/[B, B].$$

This map is given by the *0-categorical trace* for the A-action on K.

(1') A right dualizable functor $F : \mathbf{C} \to \mathbf{D}$ is one that admits a colimit-preserving right adjoint F^R. If $\mathbf{C}$ is compactly generated, this is equivalent to F preserving compact objects. In this case, we have a map $\mathrm{Tr}(F) : HH_\bullet(\mathbf{C}) \to HH_\bullet(\mathbf{D})$. More generally, given $\phi_{\mathbf{C}}$ and $\phi_{\mathbf{D}}$ endofunctors, we need to provide a "ϕ-equivariant" structure $F \circ \phi_{\mathbf{C}} \to \phi_{\mathbf{D}} \circ F$.

As an explicit example, let $f : X \to Y$ be a map of schemes, equivariant for a $\mathbb{Z}$-action generated by ϕ. Then, f^* acquires a ϕ_*-equivariant structure, and we have

$$HH(f^*, \phi_*) : HH(\mathrm{QC}(Y), \phi_*) = \mathcal{L}_\phi f^* \simeq \mathcal{O}(\mathcal{L}_\phi(Y)) \to HH_\bullet(\mathrm{QC}(X), \phi_*) \simeq \mathcal{O}(\mathcal{L}_\phi(X)).$$

(2) We will discuss this later, but things are more complicated. As far as I know there isn't a general easy criterion for determining if a module category is right dualizable. By [GKRV22], if $\mathbf{A}, \mathbf{B}$ are rigid monoidal categories and $F : \mathbf{A} \to \mathbf{B}$ a monoidal endofunctor, then $\mathbf{B}$ is a $(\mathbf{A}, \mathbf{B})$-bimodule and its right dual is itself.

Functoriality leads to the notion of characters.

Definition 1.2.3. Suppose $X \in \mathbf{U}$ is dualizable, and $\phi : X \to X$ an endomorphism. Let $(F_x, \eta) : (\mathbb{1}, \mathrm{id}_{\mathbb{1}}) \to (X, \phi)$ be a right dualizable morphism in $\mathbf{U}$ equipped with a compatible commuting structure $\eta : x \to \phi(x)$, which we can view as defining a "finite-dimensional element" of X via the action of $\mathbb{1}$ on $x \in X$ (plus a ϕ-semilinear endomorphism). By functoriality of traces, we have

$$\mathrm{tr}(F_x) : \mathrm{tr}(\mathbb{1}) \to \mathrm{tr}(X, \phi).$$

Assume that $\mathrm{tr}(\mathbb{1})$ is an algebra object with unit 1.[5] We define $[x, \eta] := \mathrm{tr}(F_x)(1)$.

Just how functoriality "looks like" a lower trace, the same is true for characters, but we avoid using this term so as to not overload the word trace.

Example 1.2.4. We compute some examples.

(1a) Let M be a perfect A-module, which we view as a (k, A)-bimodule. The right dual is $M^\vee = \mathrm{Hom}_A(M, A)$. This defines a map

$$k \to \mathrm{End}_A(M) \simeq (M^\vee \otimes_k M) \otimes_{A \otimes A^{op}} A \to A \otimes_{A \otimes A^{op}} A \simeq HH_\bullet(A)$$

and we denote the image of 1 by $[M] \in HH_\bullet(A)$. Note, in particular, that it always defines an element of $HH_0(A) = A/[A, A]$, i.e. the derived structure is not relevant. If A is a connective dg algebra, then this is still true, i.e. $HH_0(A) = \pi_0(A)/[\pi_0(A), \pi_0(A)]$. If A is coconnective, this can fail.

Example: let $A = \mathrm{End}(V)$ for a finite vector space V. Show that $[V] = 1$, thus $[A] = \dim(V)$. Hint: the coevaluation map $k \to V^* \otimes_A V$ sends $1 \mapsto f \otimes e$ for any choices such that $f(e) = 1$.

(1b) The map $A \to A/[A, A]$ is realized as follows. Take $M = A$ to be the right regular A-module, and let $\eta_a : A \to A$ be the right action by $a \in A$ (which is an endomorphism as a right A-module). Then, the character $[a] := [A, \eta_a]$ is given by the composition

$$k \to A \simeq \mathrm{End}_A(A) \simeq (A^\vee \otimes_k A) \otimes_{A \otimes A^{op}} A \to A \otimes_{A \otimes A^{op}} A \simeq A/[A, A]$$

where now the first map sends $1 \mapsto a$. One can check that this is the obvious map.

(1') Let $\mathbf{C}$ be a category, and $X \in \mathbf{C}$ a compact object. Then, the functor $F_X : \mathbf{Vect}_k \to \mathbf{C}$ where $F_X(V) = V \otimes X$ is right dualizable, so we can define $[X] \in HH_\bullet(\mathbf{C})$ to be image of k.

Example: take $\mathbf{C} = \mathrm{QC}(X)$ where X is a smooth scheme. For $\mathcal{E}$ a vector bundle on X, we have $[\mathcal{E}] \in \bigoplus H^p(X, \Omega_X^p)$ is the Chern character of $\mathcal{E}$. Explicitly, it is the composition

$$k \to R\,\mathrm{End}_X(\mathcal{E}) \simeq R\Gamma(X, \Delta^*(\mathcal{E}^\vee \boxtimes \mathcal{E})) \to R\Gamma(X, L\Delta^* \Delta_* \mathcal{O}_X)$$

where the latter map is $R\Gamma \circ L\Delta^*$ applied to the map $\mathcal{E}^\vee \boxtimes \mathcal{E} \to \Delta_* \mathcal{O}_X$, adjoint to the usual evaluation map $\Delta^*(\mathcal{E}^\vee \boxtimes \mathcal{E}) \simeq \mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_X$ on X. With some work, one can calculate (see [Ca05]) that this is in fact the Chern character for $X = \mathbb{P}^1$. Note that here, because we mixed a right and left adjoint, it is important to take derived global sections: $R^0\Gamma(X, \Delta^* \Delta_* \mathcal{O}_X) \simeq H^0(X, \mathcal{O}_X)$, but $H^0(R\Gamma(X, \Delta_* \Delta^* \mathcal{O}_X) \simeq \bigoplus_p H^p(X, \Omega_X^p)$.

Example:

(2) Let $\mathbf{M}$ be an $\mathbf{A}$-module category which is right-dualizable. We are interested in computing $[\mathbf{M}]$. Alternatively, for any $A \in \mathbf{A}$, we can compute $[A] := [\mathbf{A}, A \otimes -]$, i.e. consider the trace of the left action of A on the right regular $\mathbf{A}$-module, and produce a functor $\mathbf{A} \to \mathrm{tr}(\mathbf{A})$. Some of these will be discussed later.

Finally, we discuss a compatibility of 1 and 2-categorical traces, which realizes the 0-categorical and 1-categorical traces in the 2-categorical setting. There are twisted versions, et cetera, as well. This is the main structural theorem from which we conclude our results.

[5] I am not sure if this is automatic or not. Probably?

Theorem 1.2.5 (Gaitsgory, Kazhdan, Rozenblyum, Varshavsky; Campbell, Ponto)**.** *Let* $\mathbf{A}$ *be a rigid monoidal category and* $\mathbf{M}$ *a* $\mathbf{A}$*-module category. There is an equivalence*

$$\mathrm{Hom}_{\mathbf{Tr}(\mathbf{A})}([\mathbf{A}],[\mathbf{M}]) \simeq \mathrm{Hom}_{HH_\bullet(\mathbf{A})}(HH_\bullet(\mathbf{A}), HH_\bullet(\mathbf{M})) = HH_\bullet(\mathbf{M})$$

as $\mathrm{End}_{\mathbf{Tr}(\mathbf{A})}([\mathbf{A}]) \simeq HH_\bullet(\mathbf{A})$*-modules.*

We won't prove or justify it. A few remarks:

1. The rigidity assumption is used to make various modules automatically dualizable.
2. The theorem is about a "secdonary functoriality" or lower-level functoriality. Namely, characters are given by functoriality for
$$\mathbf{dgCat}_k \overset{F_{\mathbf{A}}}{\underset{F_{\mathbf{M}}}{\rightrightarrows}} \mathbf{Mod}(\mathbf{A})$$
so a map $[\mathbf{A}] \to [\mathbf{M}]$ should roughly have something to do with a natural
3. The reference [CP22], though in a different language, gives some intriguing TFT-style pictures explaining this compatibility.

1.3 Digression: 1-categorical traces

This section is lazily copy-pasted from some notes from another talk, so it doesn't fit in so well. The calculation of the 1-categorical trace is essentially by Toën's dg Morita theory [To07]. The basic example is the diagonal $\mathbf{C}^{\mathrm{op}} \otimes \mathbf{C}$-bimodule $\mathbf{C}$ is the Yoneda Hom-functor, i.e.

$$\mathbf{C}(x,y) := \mathrm{Hom}_{\mathbf{C}}(x,y).$$

Let M be a $\mathbf{C}$-module and N a $\mathbf{D}^{\mathrm{op}}$-module. This defines a $\mathbf{C}\otimes\mathbf{D}^{\mathrm{op}}$-module (taking the "pointwise" tensor product of dg categories)

$$(M \boxtimes N)(x \otimes y) = M(x) \otimes N(y) \qquad x \in \mathbf{C}, y \in \mathbf{D}.$$

We can combine these two to compute relative tensor products. Non-derived version: say M is a right $\mathbf{C}$-module and N a left $\mathbf{C}$-module, then we should have

$$M \otimes_{\mathbf{C}} N = \mathbf{C} \otimes_{\mathbf{C}\otimes\mathbf{C}^{\mathrm{op}}} (M \boxtimes N) = \mathrm{coeq}\left(\bigoplus_{f: x_0 \to x_1} M(x_0) \otimes N(x_1) \overset{f_* \otimes \mathrm{id}}{\underset{\mathrm{id}\otimes f^*}{\rightrightarrows}} \bigoplus_{x \in \mathbf{C}} M(x) \otimes N(x) \right)$$

The derived version just continues the bar complex:

$$M \otimes_{\mathbf{C}} N = \left(\cdots \substack{\longrightarrow\\\longrightarrow\\\longrightarrow\\\longrightarrow} \bigoplus_{\substack{f: x_0 \to x_1\\ g: x_1 \to x_2}} M(x_0) \otimes N(x_2) \substack{\longrightarrow\\\longrightarrow\\\longrightarrow} \bigoplus_{f: x_0 \to x_1} M(x_0) \otimes N(x_1) \overset{f_* \otimes \mathrm{id}}{\underset{\mathrm{id}\otimes f^*}{\rightrightarrows}} \bigoplus_{x \in \mathbf{C}} M(x) \otimes N(x) \right)$$

Note that since M is a right module, we act by pushforwards, and since N is a left module, we act by pullback. Also note it might look more natural to write the terms in the complex (where $\mathbf{C}(-,-) = \mathrm{Hom}_{\mathbf{C}}(-,-)$):

$$\bigoplus_{x_0,x_1,x_2 \in \mathbf{C}} M(x_0) \otimes \mathbf{C}(x_0,x_1) \otimes \mathbf{C}(x_1,x_2) \otimes N(x_2) \substack{\rightarrow\\\rightarrow\\\rightarrow} \bigoplus_{x_0,x_1 \in \mathbf{C}} M(x_0) \otimes \mathbf{C}(x_0,x_1) \otimes N(x_1) \rightrightarrows \bigoplus_{x \in \mathbf{C}} M(x) \otimes N(x).$$

Exercise 1.3.1. Some exercises to get a feeling for this notion.

1. Let A be an algebra with central orthogonal idempotents $e_1, \ldots, e_r$. Then, one may realize A as a category with one object as usual, or as a category with r objects $x_1, \ldots, x_r$ where $\mathrm{End}(x_r) = e_r A = A e_r$. Check the compatibility of the usual tensor product in this setting with the notion above.

2. Check that $M \otimes_{\mathbf{C}} \mathbf{C} \simeq M$.

3. Let A be an idempotented (possibly non-unital) algebra, e.g. compactly supported functions on a p-adic group under convolution. We cannot realize A as a category with one object since there is no unit, but we can realize it as a category with many objects. For each idempotent e, we define an object x_e with endomorphisms $\mathrm{End}(x_e) = eAe$, and for two idempotents e_1, e_2 we define $\mathrm{Hom}(x_1, x_2) = e_1 A e_2$. Check that the category of modules in the above sense is the category of *non-degenerate modules* (i.e. those such that $AM = M$).

4. Note that there is no "forgetful functor" to **Vect**, i.e. any functor $\mathbf{Vect} \to \mathbf{C}$ necessitates choosing a favored object of $c \in \mathbf{C}$. Given such an object, we can define $\mathbf{C}_c(x) = \mathrm{Hom}(x, c)$ to be a left $\mathbf{C}$-module, and check that $M \otimes_{\mathbf{C}} \mathbf{C}_c = M(c)$.

5. Check that the Hochschild homology $HH(\mathbf{C}) = \mathbf{C} \otimes_{\mathbf{C}^{\mathrm{op}} \otimes \mathbf{C}} \mathbf{C}$ is computed by the cyclic bar complex:

$$\cdots \overset{\longrightarrow}{\underset{\longrightarrow}{\overset{\longrightarrow}{\longrightarrow}}} \bigoplus_{x_0, x_1, x_2 \in \mathbf{C}} \mathbf{C}(x_0, x_1) \otimes \mathbf{C}(x_1, x_2) \otimes \mathbf{C}(x_2, x_0) \overset{\longrightarrow}{\underset{\longrightarrow}{\longrightarrow}} \bigoplus_{x_0, x_1 \in \mathbf{C}} \mathbf{C}(x_0, x_1) \otimes \mathbf{C}(x_1, x_0) \rightrightarrows \bigoplus_{x_0 \in \mathbf{C}} \mathbf{C}(x_0, x_0)$$

6. Not every category has a single compact generator, e.g. $\mathrm{Rep}(G)$ where G is a reductive algebraic group. Check that
$$\mathrm{Tr}(\mathrm{Rep}(G)) = \bigoplus_{\mathrm{Irr}(G)} k = \mathcal{O}(G)^G$$
and that the character of a G-representation $[V] \in \mathcal{O}(G)^G$ is its character in the usual sense.

In particular, this allows for an explicit algebraic description of the 1-categorical trace. We summarize some of its main properties.

1. One can check directly using the above definitions that the 1-categorical trace takes semi-orthogonal decompositions (i.e. upper triangular categories) to direct sums.

2. The 1-categorical trace is Morita invariant, i.e. we have $HH(\mathrm{Perf}(A)) = HH(A)$.

3. There is a comparison "Chern character" map from the connective K-theory spectrum $K_\bullet(\mathbf{C}) \to HH(\mathbf{C})$. This is via the Blumberg-Gepner-Tabuada characterization of (connective/non-connective) K-theory as the universal additive/localizing invariant.

4. The horizontal trace plays well with tensor products of categories. Namely, if $\mathbf{A}$ is a monoidal category acting on a module category $\mathbf{M}$ in a sufficiently nice way (i.e. everything can be formulated in terms of compact objects), then $HH(\mathbf{A})$ is an algebra, with module $HH(\mathbf{M})$. This more or less follows from functoriality, and the Eilenberg-Zilberg identification of $HH(\mathbf{A} \otimes \mathbf{B}) \simeq HH(\mathbf{A}) \otimes HH(\mathbf{B})$.

1.4 Digression: topological field theories

There is a TFT-style way to picture traces and centers. We will avoid making this picture too precise.

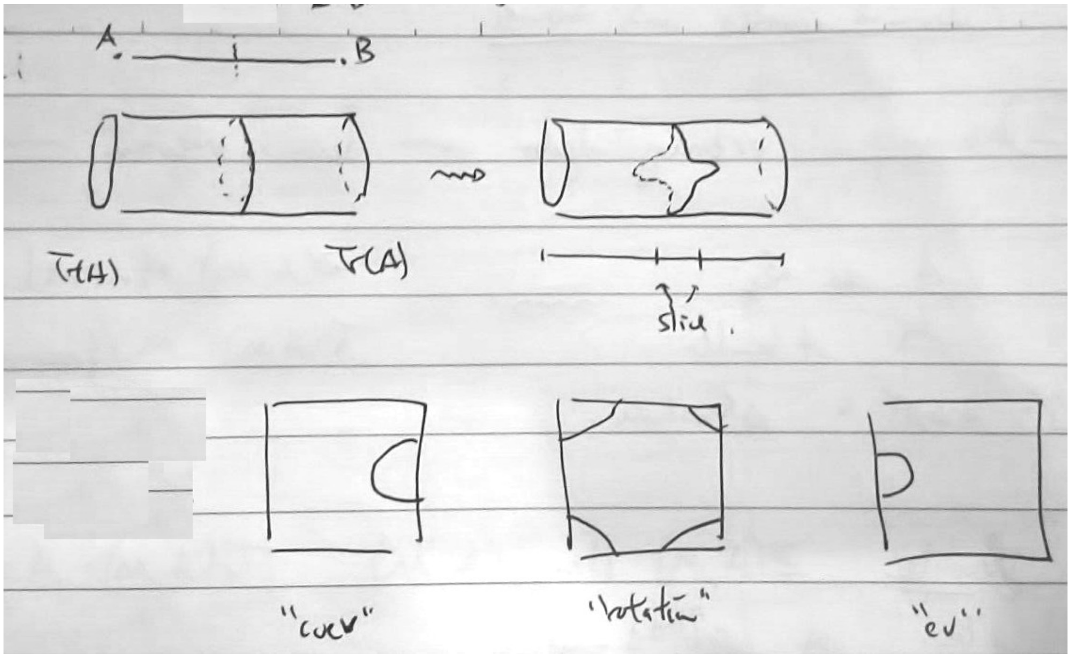
Definition 1.4.1. An *n-framing* of an d-manifold M is a choice of basis of sections of $TM \times \mathbb{R}^{n-d}$. Let $\mathbf{Bord}_n$ denote the (∞, n)-category whose objects are 0-manifolds with an n-framing, 1-morphisms are 1-manifolds with an n-framing, et cetera, up to the n-morphisms being n-manifolds with a framing. Let $\mathbf{C}$ be a monoidal (∞, n)-category; a *fully extended framed topological field theory* is a (∞, n)-functor $\mathcal{Z} : \mathbf{Bord}_n \to \mathbf{C}$.

Theorem 1.4.2 (Cobordism hypothesis (Baez, Dolan, Lurie, Grady, Pavlov))**.** *The category of fully extended framed n-dimensional TFTs in $\mathbf{C}$ is equivalent to fully n-dualizable objects in $\mathbf{C}$.*

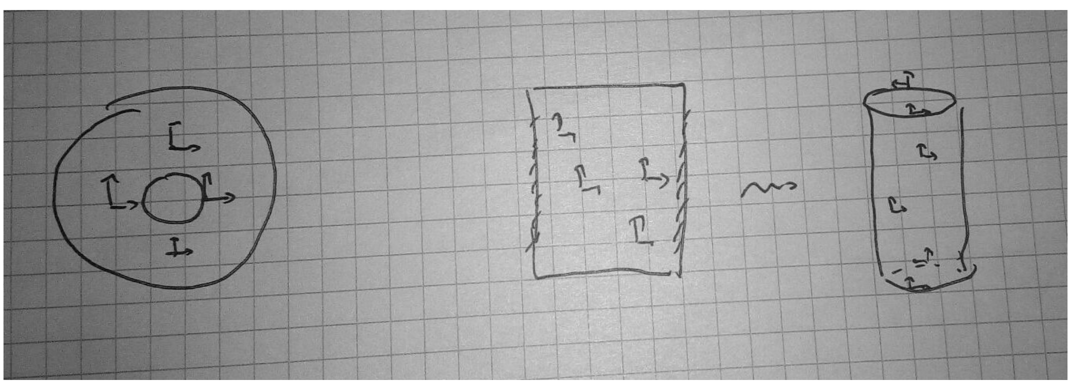
We stress this is mostly a guideline that we are not quite able to make precise all the time. The first exercise to understand the following.

Example 1.4.3. Let $(\mathbf{U}, \otimes, \mathbb{1})$ be a monoidal category. A dualizable object X is given by a 1-dimensional TFT $\mathcal{Z}$, where $\mathcal{Z}(*) = X$, and $\mathcal{Z}(S^1) = \mathrm{tr}(X, \mathrm{id}_X)$. Traces can be understood by inserting an "event" in S^1 corresponding to ϕ.

Functoriality can be depicted in the following way.



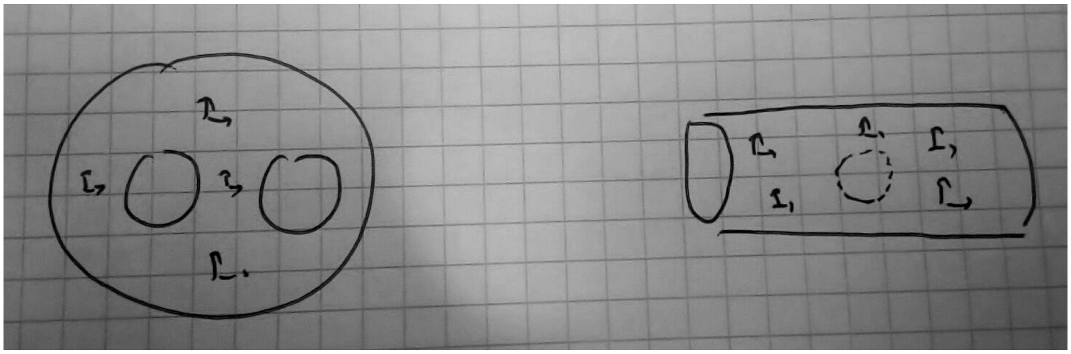
Can centers be realized in this TFT picture? I'm not able to say anything very precise, but can try to draw some pictures. When $n = 2$, a convenient way to try to view framings on 1-manifolds is to write down the identity map as a cobordism and give a framing of identity.



The left is the annulus framing of the circle S^1_{an}, and the right is the cylinder S^1_{cy}. If $\mathcal{Z}$ is the TFT with $\mathcal{Z}(*) = \mathbf{C}$, then we might hope for something like

$$\mathcal{Z}(S^1_{an}) \simeq HH^\bullet(\mathbf{C}), \qquad \mathcal{Z}(S^1_{cy}) \simeq HH_\bullet(\mathbf{C}).$$

Heuristically, we can see the multiplication and actions via pictures:



On the left, evidently the restriction of the framing to the boundary gives rise to three annular framings, one with opposite orientation (i.e. the target). On the right, we take the identity map with the cylindrical framing and cut out a hole in the side, which restricts to an anular framing.

2 2-categorical traces and coherent sheaves

2.1 Colimits of categories

To compute the trace we need to compute a colimit in categories. First, let's do a very basic review in 1-categories: it is generally easy to take a pullback of categories but not a pushout. Consider a pullback diagram

$$\begin{tikzcd} \mathbf{C}' \arrow[r, "\mathrm{ev}_1"] \arrow[d, "\mathrm{ev}_2"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbf{C}_1 \arrow[d, "F_1"] \\ \mathbf{C}_2 \arrow[r, "F_2"'] & \mathbf{C}_0. \end{tikzcd}$$

One can either formulate the universal property of categories in a 1-categorical way ("strict") or a 2-categorical way ("weak"). For weak pullbacks, the objects of $\mathbf{C}'$ consist of pairs $(X_1, X_2) \in \mathrm{Ob}(\mathbf{C}_1) \times \mathrm{Ob}(\mathbf{C}_2)$ along with an isomorphism $\alpha : F_1(X_1) \simeq F_2(X_2)$ (in the strict case this is required to be an equality, and therefore not an extra datum). We leave it to the reader to formulate what the morphisms are and check that, in particular, the pullback of abelian categories is again abelian, and the inclusion $\mathbf{Ab} \hookrightarrow \mathbf{1Cat}$ preserves limits. For example, given an exact functor between abelian categories, it is easy to say what the kernel of the functor is. In general, we can (and often do) easily talk about fibers of functors.

Why weak pullbacks instead of strong pullbacks? Consider the following example.

Example 2.1.1. Let G be an affine algebraic group and $p : \mathbf{Vect}_k \simeq \mathrm{QC}(\mathrm{Spec}\, k) \to \mathrm{QC}(G)$ the pullback, i.e. sending k to $\mathcal{O}_G$ (here QC means the abelian 1-category). The strict pullback $\mathbf{Vect}_k \times_{\mathrm{QC}(G)} \mathbf{Vect}_k \simeq \mathbf{Vect}_k$ is not very interesting, and evidently can never be. An object in the weak pullback is a pair of vector spaces (V, W) and an $\mathcal{O}(G)$-module isomorphism $\alpha : V \otimes \mathcal{O}(G) \simeq W \otimes \mathcal{O}(G)$. This is close to descent datum for a sheaf on BG, i.e. a rational G-representation, but we are missing the cocycle condition since we haven't included the next "level" in the simplicial diagram.

Pushouts, on the other hand, are more difficult. They can be understood in a few examples.

1. Let $\mathbf{C}$ be a category, and let $\mathbf{R}$ be an equivalence relation on arrows realized as a category with $\mathrm{Ob}(\mathbf{R}) = \mathrm{Ob}(\mathbf{C})$ and $\mathrm{Hom}_{\mathbf{R}}(x, y) \subset \mathrm{Hom}_{\mathbf{C}}(x, y) \times \mathrm{Hom}_{\mathbf{C}}(x, y)$ satisfying the usual conditions. The (strict or weak) coequalizer

$$\mathbf{R} \rightrightarrows \mathbf{C}$$

can be realized as a quotient in the usual way. One can see that this may not appear so much in practice.

2. The (strict and weak) coequalizer

$$\{\bullet\} \rightrightarrows \{\bullet \to \bullet\}$$

is (equivalent to) the category corresponding to the monoid $\mathbb{N}$. From this example, we see it is necessary to "freely generate" morphisms when identifying objects. One can imagine this process getting complicated.

3. Let $\mathbf{C}_0 \subset \mathbf{C}_1$ be a full subcategory, and consider the coequalizer

$$\begin{tikzcd} \mathbf{C}_0 \arrow[r, hook, shift left] \arrow[r, shift right, "0"'] & \mathbf{C}_1 \end{tikzcd}$$

If $\mathbf{C}_0$ is a *Serre subcategory*, i.e. closed under isomorphisms, subquotients and extensions, then there is a construction of the *Serre quotient* very roughly by "freely adjoining isomorphisms $0 \simeq x$ for $x \in \mathbf{C}_0$ in "an abelian sense", which means very roughly "freely' adjoining isomorphisms $x \simeq x/x'$ for subobjects $x' \in \mathbf{C}_0$ and $x \simeq x'$ for subobjects x' such that $x/x' \in \mathbf{C}_0$. The Serre quotient is the coequalizer in $\mathbf{Ab}$, but the coequalizer in $\mathbf{Add}$ or $\mathbf{1Cat}$ may look different.

For example, take $\mathbf{C}_1 \subset \mathbf{Vect}_k$ the full subcategory generated by k^n for $n = 0, 1, \ldots$, and $\mathbf{C}_0$ the full subcategory generated by $0, k$. The quotient in $\mathbf{Ab}$ is the zero category, but the quotient in $\mathbf{Add}_k$ is equivalent to the full subcategory of $\mathbf{Vect}_k$ generated by $k, k, k^2, \ldots$ (i.e. we now have two copies of k).

2.2 Colimits of presentable ∞-categories

So, as we've seen:

1. Taking limits and colimits of 1-categories "correctly" involves thinking $(2,1)$-categorically. Since we are interested in derived categories, we will work in the setting of $(\infty, 1)$-categories.

2. Colimits tend to depend on what kind of categories we restrict ourselves to, so we need to be fix a category of ∞-categories where we take colimits.

3. Taking general colimits is difficult, and we need a strategy to do so. The strategy will be to turn them into limits.

We will ignore (1); roughly, we will work with ∞-categories as though they were ordinary categories, except that we are not able to directly manipulate objects, morphisms, et cetera. For (2), we introduce the following.

Definition 2.2.1. The category $\mathbf{Pr}_k^L$ is the *category of k-linear presentable ∞-categories.* Morally, they are categories with all small colimits plus a technical condition ("accessibility") that we won't discuss. The morphisms are given by colimit-preserving functors (equivalently, by the adjoint functor theorem, left adjoint functors).

One indication that this is the correct setting is that (1) for any dg algebra A, the (dg nerve of the) dg derived category $\mathrm{Mod}(A)$ is presentable, (2) the Lurie tensor product is well-defined for presentable ∞-categories, and (3) $\mathrm{Mod}(A) \otimes \mathrm{Mod}(B) \simeq \mathrm{Mod}(A \otimes B)$.

For (3), we will commonly use the following strategy available to us in the context of presentable ∞-categories. Here are some facts:

1. The category $\mathbf{Pr}_k^L$ has all limits and colimits, and the inclusion $\mathbf{Pr}_k^L \hookrightarrow \infty\mathbf{Cat}$ commutes with limits.

2. By passing to right adjoints, we have $\mathbf{Pr}_k^L \simeq (\mathbf{Pr}_k^R)^{op}$.

3. The inclusion $\mathbf{Pr}_k^R \hookrightarrow \mathbf{Cat}_\infty$ commutes with limits.

So, if there is a colimit we want to compute in $\mathbf{Pr}_k^L$ then we can *pass to right adjoints* to obtain a limit in $\mathbf{Pr}_k^R$, then regard it as a limit in $\mathbf{Cat}_\infty$, and compute the limit there (or basically anywhere, since it's irrelevant for limits).

Example 2.2.2. There are many well-established examples of categories of sheaves being presented as limits, e.g. by descent. More concretely, say X is a scheme and $p : U \to X$ is a fpqc morphism. Descent tends to be formulated in one of several ways.

1. By descent data, i.e. the most classical formulation: a sheaf $\mathcal{F}$ on U, with an isomorphism identifying its two pullbacks to $U \times_X U$ satisfying a cocycle condition.

2. By a monad, i.e. the adjoint pair (p^*, p_*) defines a monad $p^* p_*$ on $\mathrm{QC}(U)$, and take modules for the monad.

3. By a limit, i.e. the totalization $\mathrm{Tot}(\mathrm{QC}(U \times_X \cdots \times_X U)$ of the simplicial diagram of categories given by the Cech complex.

Formulations (1) and (2) are (in the 1-categorical setting) related by the Bénabou-Roubaud theorem. Formulations (2) and (3) are related by Beck-Chevalley theorems.

2.3 Example: quasicoherent sheaves

We now compute some examples in the setting of quasi-coherent sheaves. The main reference here is [BFN10].

Proposition 2.3.1 (Ben-Zvi, Francis Nadler)**.** *Let X_i, Y be perfect stacks. Then,*

$$\mathrm{QC}(X_1) \otimes_{\mathrm{QC}(Y)} \mathrm{QC}(X_2) \simeq \mathrm{QC}(X_1 \times_Y X_2).$$

Proof. First, we note that $\mathrm{QC}(X_1) \otimes \mathrm{QC}(X_2) \simeq \mathrm{QC}(X_1 \times X_2)$. This is established in Proposition 4.6 of [BFN10]. We sketch the proof of the relative statement, which is Theorem 4.7 of [BFN10]. We first try to compute the tensor product using the passing to right adjoints trick as above. That is, we have

$$\mathrm{QC}(X_1) \otimes_{\mathrm{QC}(Y)} \mathrm{QC}(X_2) \simeq \mathrm{colim}\left(\mathrm{QC}(X_1 \times X_2) \leftleftarrows \mathrm{QC}(X_1 \times Y \times X_2) \Lleftarrow \cdots \right)$$

$$\simeq \lim \left(\mathrm{QC}(X_1 \times X_2) \overset{p_{1*}}{\underset{p_{2*}}{\rightrightarrows}} \mathrm{QC}(X_1 \times Y \times X_2) \Rrightarrow \cdots \right).$$

By general nonsense, the limit $\mathbf{C} \simeq \mathrm{Mod}_{\mathrm{QC}(X_1 \times X_2)}(\mathrm{ev} \circ \mathrm{ev}^L)$, i.e. modules for the monad given by the evaluation functor. We identify this monad using a "Beck-Chevalley" argument, i.e. letting $\mathbf{C}$ denote the above limit we have adjoint maps of augmented simplicial diagrams

$$\begin{tikzcd}
\mathbf{C} \arrow[r, "\mathrm{ev}"] \arrow[d, dotted, shift left, "\mathrm{ev}"] & \mathrm{QC}(X_1 \times X_2) \arrow[r, shift left, "p_{1*}"] \arrow[r, shift right, "p_{2*}"'] \arrow[d, shift left, "p_{1*}"] & \mathrm{QC}(X_1 \times Y \times X_2) \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[d, shift left, "p_{1*}"] & \cdots \\
\mathrm{QC}(X_1 \times X_2) \arrow[r, "p_{2*}"'] \arrow[u, dotted, shift left, "\mathrm{ev}^L"] & \mathrm{QC}(X_1 \times Y \times X_2) \arrow[r, shift left] \arrow[r, shift right] \arrow[u, shift left, "p_1^*"] & \mathrm{QC}(X_1 \times Y^2 \times X_2) \arrow[r, shift left] \arrow[r] \arrow[r, shift right] \arrow[u, shift left, "p_1^*"] & \cdots
\end{tikzcd}.$$

The bottom diagram is a limit since the diagram admits a splitting via the extra degeneracy. Since the squares satisfy base change, by Beck-Chevalley we have that $\mathbf{C} \simeq \mathrm{Mod}_{\mathrm{QC}(X_1 \times X_2)}(p_1^* p_{2*})$.

On the right hand side, we can identify $\mathrm{QC}(X_1 \times_Y X_2)$ via Barr-Beck for the affine (since Y has affine diagonal) map[6] $i : X_1 \times_Y X_2 \to X_1 \times X_2$. One verifies that the adjoint pair (i^*, i_*) is monadic, thus $\mathrm{QC}(X_1 \times_Y X_2) \simeq \mathrm{Mod}_{\mathrm{QC}(X_1 \times X_2)}(i_* i^*)$. Finally, by base change:

$$\begin{tikzcd}
X_1 \times_Y X_2 \arrow[r, "i"] \arrow[d, "i"] & X_1 \times X_2 \arrow[d, "p_2"] \\
X_1 \times X_2 \arrow[r, "p_1"] & X_1 \times Y \times X_2
\end{tikzcd}$$

we may identify $i_* i^* \simeq p_1^* p_{2*}$. ∎

Example 2.3.2. It may be worth doing out the the example where $X_1 = X_2 = \mathrm{Spec}\, k$ and $Y = BG$ to get a feeling for the above.

[6] A decent toy model for this is: given an affine map of schemes, we can understand sheaves upstairs as modules for some sheaf of algebras downstairs.

Corollary 2.3.3. *Let X be a perfect stack and $\phi : X \to X$ be a self-map. Then,*

$$\mathrm{tr}(\mathrm{QC}(X), \phi_*) \simeq \mathrm{QC}(X \times_{X \times X} X) = \mathrm{QC}(\mathcal{L}_\phi X).$$

Example 2.3.4. We introduced the derived loop space in Definition 1.1.3. Some examples to get a sense of this derived stack:

1. The classical points of $\mathcal{L}X$ are just the inertia stack, i.e. $\pi_0(\mathcal{L}X) = IX$. Recall that inetia stack has points

$$IX(S) = \{x \in X(S), \alpha \in \mathrm{Aut}(x) = S_x \times_X S_x\}.$$

2. In particular, $\pi_0(\mathcal{L}X) = X$ when X is a scheme, i.e. for a scheme the derived loop space is entirely in cohomological dimensions over X. More explicitly, if X is smooth,

$$\mathcal{L}X = \mathrm{Spec}_X \mathrm{Sym}_X \Omega^1_X[1].$$

3. If $X = BG$, then $\mathcal{L}X = IX = G/G$, i.e. there are no derived directions.

4. If $X = Y/G$, then $\mathcal{L}X = \dfrac{(G \times Y) \times_{Y \times Y} Y)}{G}$.

5. Take $X = \mathfrak{g}/G$ and $\phi = q_*$, where q is the scaling by q map. Then we have

$$\mathcal{L}_q X = \{(x,g) \in \mathfrak{g} \times G \mid gxg^{-1} = qx\}/G \simeq \{(x,g) \in \mathcal{N} \times G \mid gxg^{-1} = qx\}/G.$$

   We may take the nilpotent cone instead by applying $\mathcal{L}_q$ (which commutes with limits) to the fiber square:

$$\begin{tikzcd} \mathcal{N}/G \arrow[r] \arrow[d] & \mathfrak{g}/G \arrow[d] \\ \{0\} \arrow[r] & \mathfrak{h}//W \end{tikzcd} \qquad \begin{tikzcd} \mathcal{L}_q(\mathcal{N}/G) \arrow[r] \arrow[d] & \mathcal{L}_q(\mathfrak{g}/G) \arrow[d] \\ \mathcal{L}_q(\{0\}) = \{0\} \arrow[r] & \mathcal{L}_q(\mathfrak{h}//W) \simeq \{0\} \end{tikzcd}$$

   where $\mathcal{L}_q(\mathfrak{h}//W) \simeq \{0\}$ as long as q is not a root of unity for any of the degrees of the polynomial generators of $k[\mathfrak{h}]^W$. Furthermore, this derived stack has no derived structure, which one can see via a dimension count: G acts on $\mathcal{N}$ by finitely many orbits, so $\pi_0(\mathcal{L}_q(\mathcal{N}/G))$ is the union of irreducible components of dimension $\dim(G)$ indexed by the orbits, and since $\pi_0(\mathcal{L}_q(\mathcal{N}/G)) = \pi_0(\mathcal{L}_q(\mathfrak{g}/G))$, the claim follows since $\mathfrak{g}/G$ is smooth.

2.4 Example: convolution categories

I will use the term convolution category to vague refer to the following kind of category of quasi/ind-coherent sheaves. A very similar story works for D-modules with more or less all the geometric diagrams the same, just with different functors; see [BN15, BN19].

Let $f_i : X_i \to Y$ be maps of stacks. Convolution is the push-pull:

$$Z_{12} \times Z_{23} := X_1 \times_Y X_2 \times X_2 \times_Y X_3 \longleftarrow Z_{123} := X_1 \times_Y X_2 \times_Y X_3 \longrightarrow Z_{13} := X_1 \times_Y X_3.$$

This defines a functor on quasi-coherent sheaves

$$\mathrm{QC}(Z_{12}) \otimes \mathrm{QC}(Z_{23}) \longrightarrow \mathrm{QC}(Z_{13}).$$

One can make similar definitions for other sheaves, as long as they satisfy a Kunneth formula, though we often impose other conditions. We have the following theorem.[BFN10]

Theorem 2.4.1 (Ben-Zvi, Francis, Nadler)**.** *Suppose that X, Y are perfect stacks. Then we have a monoidal equivalence*

$$\mathrm{Fun}^L_{\mathrm{QC}(Y)}(\mathrm{QC}(X_1), \mathrm{QC}(X_2)) \simeq \mathrm{QC}(X_1 \times_Y X_2)$$

exchanging composition and convolution.

2.4.2. Ind-coherent sheaves We need to introduce the category of ind-coherent sheaves. This is a technical issue that ends up being somewhat important. The point is that for X a perfect stack, we have $\mathrm{QC}(X) = \mathrm{Ind}(\mathrm{Perf}(X))$. One can alternatively take as a small category, $\mathrm{Coh}(X)$, which when X is classical is an enlargement of $\mathrm{Perf}(X)$. We then define $\mathrm{QC}^!(X) = \mathrm{Ind}(\mathrm{Coh}(X))$.[7] There are also singular support conditions Λ that give intermediate categories $\mathrm{QC}(X) = \mathrm{QC}^!{}_{\{0\}_X}(X) \subset \mathrm{QC}^!{}_\Lambda(X) \subset \mathrm{QC}^!(X)$. We give three justifications for the use of ind-coherent sheaves (or singular support conditions) when discussing convolution categories.

1. In ind-coherent sheaves, under some mild conditions the monoidal product preserves compact objects, and the monoidal unit is compact. This means it is well-behaved with respect to traces.
2. Singular support conditions give natural characterizations of the relative tensor product of two convolution categories.
3. Categorical Morita theory says that quasi-coherent sheaves don't give us anything new; ind-coherent sheaves do.

First, we note that if $X \to Y$ is proper and X, Y are smooth, then pullback along the diagonal preserves coherent objects, and pushforward also preserves coherent objects. Thus, the convolution functor restricts:

$$\mathrm{QC}^!(Z_{12}) \otimes \mathrm{QC}^!(Z_{23}) \to \mathrm{QC}^!(Z_{13})$$

We also have the following theorem.[BNP17b]

Theorem 2.4.3 (Ben-Zvi, Francis, Nadler, Preygel)**.** *Suppose that X, Y are smooth perfect stacks and that X_i are proper over Y. Then we have on small categories*

$$\mathrm{Fun}^{ex}_{\mathrm{Perf}(Y)}(\mathrm{Perf}(X_1), \mathrm{Perf}(X_2)) \simeq \mathrm{Coh}(X_1 \times_Y X_2).$$

In particular, the identity functor correpsonds to the sheaf $\Delta_* \mathcal{O}_X$. This is generally not compact in $\mathrm{QC}(X \times_Y X)$, but it is often compact in $\mathrm{QC}^!(X \times_Y X)$. So:

1. The 1-categorical trace of $\mathrm{QC}(X \times_Y X)$ does not have a unit, and is not an algebra. But $\mathrm{QC}^!(X \times_Y X)$ is a unital algebra.
2. Theorem 1.2.5 requires rigidity. The category $\mathrm{QC}(X \times_Y X)$ is only semi-rigid, while $\mathrm{QC}^!(X \times_Y X)$ is rigid.

Why else should we consider ind-coherent sheaves, or more generally, singular support conditions? It turns out they give very natural characterizations of convolution categories.

Proposition 2.4.4. *The convolution functor descends to a relative tensor product*

$$\star : \mathrm{QC}^!(Z_{12}) \otimes_{\mathrm{QC}^!(Z_{22})} \mathrm{QC}^!(Z_{23}) \to \mathrm{QC}^!(X_1 \times_Y X_3).$$

Furthermore, this functor is fully faithful. Its essential image may be characterized by a singular support condition.

Example 2.4.5. Sanity check: if $T = B_0 A_0 = BA$ (with A' and A the same dimensions, and likewise for B) then then there is a sequence of matrices (A_i, B_i) (for $i = 1, \dots, r$) such that $A_i = PA_{i-1}$ and $B_i P = B_{i-1}$ or a matrix Q such that $QA_i = A_{i-1}$ and $B_i = BQ_{i-1}$, such that $A_r = A$ and $B_r = B$.

Finally, there is a "dictionary" between the usual algebraic Morita theory and a geometric Morita theory.

[7] This is technically only the right definition when X is QCA, which is a theorem, but let's ignore this.

algebraic, 1-categorical	geometric, 2-categorical
R commutative algebra	$\mathcal{R} = (\mathrm{QC}(Y), \otimes_{\mathcal{O}_Y})$ symmetric monoidal
P an R-module	$\mathcal{P} = \mathrm{QC}(X)$ module under pullback
$A = \mathrm{End}_R(P)$ an R-algebra	$\mathcal{A} = (\mathrm{QC}(X \times_Y X), \star)$
$M = \mathrm{Hom}_R(P, Q)$ or $Q \otimes_R P$ a right A-module	$\mathcal{M} = \mathrm{QC}(W \times_Y X)$
A the regular A-representation	$\mathrm{QC}(X \times_Y X)$
P the standard/vector A-representation	$\mathrm{QC}(X) = \mathrm{QC}^!(X)$
if P finite rank projective over R, Morita equivalence $\mathrm{Mod}(R) \simeq \mathrm{Mod}(\mathrm{End}_R(P))$	if X proper surjective over Y, $\mathrm{Mod}(\mathrm{QC}(Y)) \simeq \mathrm{Mod}(\mathrm{QC}(X \times_Y X))$

In particular,

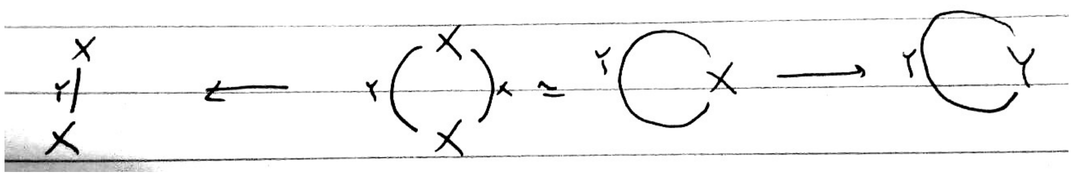
$$\mathrm{Mod}(\mathrm{QC}(Y)) \simeq \mathrm{Mod}(\mathrm{QC}(X \times_Y X)) \not\simeq \mathrm{Mod}(\mathrm{QC}^!(X \times_Y X)).$$

The point is that module categories for $\mathrm{QC}(X \times_Y X)$ is the same module categories for $\mathrm{QC}(Y)$, which is sometimes not interesting enough. For example, in local categorical Langlands, we ware interested in $X = \widetilde{\mathcal{N}}/G$ and $Y = G/G$, and $\mathrm{QC}(G/G)$ is the spectral bi-Whittaker category, which doesn't capture all representations of the p-adic group.

2.4.6. Traces of convolution categories Anyway, let's compute the trace. We'll use the trace correspondence:

$$Z = X \times_Y X \xleftarrow{\delta} Z \times_{X \times X} X \simeq \mathcal{L}Y \times_Y X \xrightarrow{\pi} \mathcal{L}Y.$$

One can think about δ as a base change of the diagonal map $X \to X \times X$ and π as a base change of the map $f : X \to Y$. This can be depicted:



Informally, one can think of the left term as two points in X and a path in Y connecting their images, the middle term as a single point in X and a loop in Y with base point its image, and the right term as a loop in Y.

We didn't talk about singular supports, but it is possible to pullback and pushforward singular support conditions along these maps. Let $\Lambda_{X/Y}$ denote the singular support condition on $\mathcal{L}Y$ obtained by applying this correspondence to the maximal singular support on Z. See [BNP17a] for the following theorem.

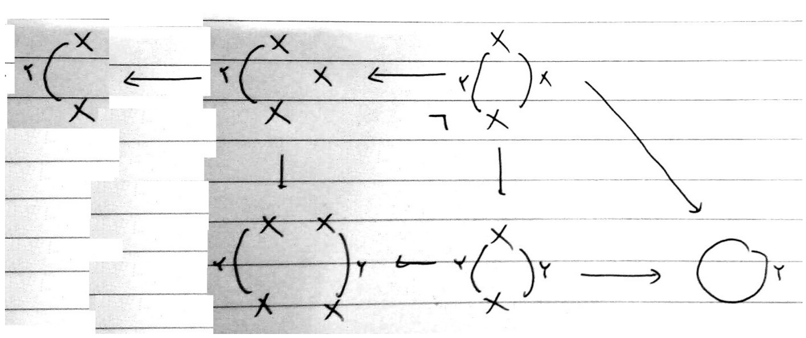
Theorem 2.4.7 (Ben-Zvi, Nadler, Preygel). *Let ϕ_X, ϕ_Y be compatible proper self-maps of X, Y, inducing ϕ_Z on Z. We have*

$$\mathrm{Tr}(\mathrm{QC}^!(Z), \phi_*) \simeq \mathrm{QC}^!{}_{\Lambda_{X/Y}}(\mathcal{L}_\phi Y).$$

The character is given by

$$[-] = \pi_* \delta^! : \mathrm{QC}^!(Z) \to \mathrm{Tr}(\mathrm{QC}^!(Z)) = \mathrm{QC}^!{}_{\Lambda_{X/Y}}(\mathcal{L}Y).$$

Proof. Here is a proof by picture, using Proposition 2.4.4:



That is, the bottom row depicts what happens when we "compose" $\mathrm{QC}^!(Z) \otimes_{\mathrm{QC}^!(Z)\otimes \mathrm{QC}^!(Z)} \mathrm{QC}^!(Z)$ "in a circle" and the top base change is a simplification of it. The singular support condition arises via the same principle as the one in *loc. cit.* $\square$

Example 2.4.8. Let $f : X \to Y$ be surjective; then $\mathrm{QC}(X)$ is naturally a $\mathrm{QC}(Y)$-module category, and we have

$$[\mathrm{QC}(X)]_{\mathrm{QC}(Y)} = \mathcal{L}f_*\mathcal{O}_{\mathcal{L}X} \in \mathrm{Tr}(\mathrm{QC}(Y)) = \mathrm{QC}(\mathcal{L}Y).$$

On the other hand, $\mathrm{QC}(X)$ is naturally a $\mathrm{QC}(Z) = \mathrm{QC}(X \times_Y X)$-module category. It is induced from $\mathrm{QC}(Y)$, and by Morita invariance (here is where we needed surjectivity) we have

$$[\mathrm{QC}(X)]_{\mathrm{QC}(Z)} = [\mathrm{QC}(Y)]_{\mathrm{QC}(Y)} = \mathcal{O}_{\mathcal{L}Y} \in \mathrm{QC}(\mathcal{L}Y).$$

Inducing the $\mathrm{QC}(Y)$-module category $\mathrm{QC}(X)$ gives the regular $\mathrm{QC}(Z)$-representation $\mathrm{QC}(Z)$, and we have

$$[\mathrm{QC}(Z)]_{\mathrm{QC}(Z)} = \mathcal{L}f_*\mathcal{O}_{\mathcal{L}X}.$$

This can be seen directly via a base-change, i.e. it is given via the trace correspondence on the diagonal $\Delta_*\mathcal{O}_X$:

$$\begin{tikzcd} \mathcal{L}X = X \times_{X\times X} X \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ Z \times_{X\times X} X \arrow[r] \arrow[d] & Z \\ \mathcal{L}Y & \end{tikzcd}$$

The regular representation is always the trace of the identity (i.e. the diagonal), but the vector representation is not always (obviously) realizable as the trace of an object of $\mathrm{QC}^!(Z)$. The following is from [BCHN22], which roughly says that for the regular and vector representation, the calculation is exactly the same as the one above.

Proposition 2.4.9 (Ben-Zvi, C, Helm, Nadler)**.** *We have calculations of the trace of the regular and vector representations*

$$[\mathrm{QC}^!(Z)] = [\Delta_*\mathcal{O}_X] = \mathcal{L}f_*\mathcal{O}_{\mathcal{L}X}, \qquad [\mathrm{QC}^!(X)] = R\Gamma_{f(X)}(\mathcal{O}_{\mathcal{L}Y}).$$

Note that since $\mathcal{L}Y$ is Calabi-Yau, we have $\mathcal{O}_{\mathcal{L}Y} = \omega_{\mathcal{L}Y}$. It is sometimes more natural to write the latter, i.e.

$$[\mathrm{QC}^!(X)] = R\Gamma_{f(X)}\omega_{\mathcal{L}Y} = \omega_{\widehat{\mathcal{L}Y}_{f(X)}}.$$

Proof. Let us sketch the idea and ignore singular support conditions; keeping track of them carefully is an essential part of the argument but it takes us too far afield. First, we show that $\mathrm{QC}(X)$ is self-dual as a $\mathrm{QC}(Y)$-module by checking the Zorro axioms. Then, we compute:

$$\mathrm{QC}(X) \otimes_{\mathrm{QC}(Z)} \mathrm{QC}(X) \simeq \mathrm{QC}(Y), \qquad \mathrm{QC}(X) \otimes_{\mathrm{QC}(Y)} \mathrm{QC}(X) \simeq \mathrm{QC}(Z)$$

using Proposition 2.4.4. Note there is no room for singular support conditions. Now, when we "glue the ends" in the trace, we obtain $\mathrm{QC}(\mathcal{L}Y)$, with various singular supports, for which we either include or take "local cohomology" with respect to. $\square$

Example 2.4.10. Take $X = BB$ and $Y = BG$. Then, the above corespondence is just the horocycle correspondence:

$$B\backslash G/B \xleftarrow{\delta} \frac{G}{B} \xrightarrow{\pi} \frac{G}{G}.$$

Furthermore, the bar complex is

$$\cdots \qquad \frac{G \times^B G \times^B G}{B} \;\substack{\longrightarrow\\[-2pt]\longrightarrow\\[-2pt]\longrightarrow}\; \frac{G \times^B G}{B} \rightrightarrows \frac{G}{B} \longrightarrow \frac{G}{G}.$$

The trace of the identity (i.e. regular representation) is the "Springer sheaf", i.e. letting $\mu : B/B \to G/G$ we have

$$[\mathrm{QC}(B\backslash G/B)] = [\mathcal{O}_{B\backslash B/B}] = \pi_* \mathcal{O}_{B/B}.$$

By the above, the trace of the standard representation is

$$[\mathrm{QC}(BB)] = \mathcal{O}_{G/G}.$$

The same example works in the setting of sheaves or D-modules as well.

3 Application to spectral geometric/geometrized local Langlands

3.1 Spectral side

Let's briefly discuss the following example. First, letting F be a non-Archimedian local field with residue $\mathbb{F}_q$, we let

$$\mathcal{H}_q = \mathcal{C}_c(I\backslash G_F/I), \qquad \mathcal{W}_q = \mathcal{C}_c(I\backslash G_F/I^0, \psi)$$

denote the affine Hecke algebra specialized to q and the Iwahori-Whittaker module. Next, we let $F = \mathbb{F}_q((t))$, and $F^u = \overline{\mathbb{F}}_q((t))$, and let LG be the group ind-scheme whose such that $LG(\overline{\mathbb{F}}_q) = G_{F^u}$, and define the *affine Hecke category*, with the equivalence by Arhipov and Bezrukavnikov:[AB09, Be16]

$$\mathbf{H} := \mathrm{Sh}(I\backslash LG/I) \simeq \mathrm{Coh}(\widetilde{\mathcal{N}}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \widetilde{\mathcal{N}}/\check{G})$$

$$\mathbf{W} := \mathrm{Sh}(I\backslash LG/I^0, \psi) \simeq \mathrm{Coh}(\widetilde{\mathcal{N}}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \check{\mathfrak{g}}/\check{G}) \simeq \mathrm{Coh}(\widetilde{\mathcal{N}}/\check{G}).$$

The equivalences intertwine pullback along geometric Frobenius and pushforward along scaling by q.

We have the following theorem. We define the *coherent Springer sheaf* $\mathcal{S}_q := \mathcal{L}_q \mu_* \mathcal{O}_{\mathcal{L}_q(\widetilde{\mathcal{N}}/G)}$, where $\mu : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the Springer resolution.

Theorem 3.1.1 (Ben-Zvi, C, Helm, Nadler)**.** *Consider the two cases.*

1. *Assume q is not a root of unity. We have equivalences of complexes*

$$HH_\bullet(\mathbf{H}, q_*) \simeq \mathcal{H}_q, \qquad HH_\bullet(\mathbf{W}, q_*) \simeq \mathcal{W}.$$

Furthermore, we have an equivalence

$$\mathrm{Tr}(\mathbf{H}, q_*) \simeq \mathrm{Coh}(\mathcal{L}_q(\mathcal{N}/G)).$$

Furthermore, $\mathcal{O}_{\mathcal{L}_q(\mathcal{N}/G)}$ is a summand of $\mathcal{S}_q$. From this, we may conclude

$$\mathcal{S}_q = [\mathbf{H}, q_*], \qquad \mathcal{O}_{\mathcal{L}_q(\mathcal{N}/G)} = [\mathbf{W}, q_*].$$

$$\mathrm{End}(\mathcal{S}_q) \simeq \mathcal{H}_q, \qquad \mathrm{Hom}(\mathcal{S}_q, \mathcal{O}) \simeq \mathcal{W}, \qquad \mathrm{End}(\mathcal{O}) \simeq \mathcal{O}(\mathcal{L}_q(\mathcal{N}/G)).$$

Furthermore, there is an embedding $\mathrm{Mod}(\mathcal{H}_q) \hookrightarrow \mathrm{QC}^!(\mathcal{L}_q(\mathcal{N}/G))$ whose right adjoint is $\mathrm{Hom}(\mathcal{S}_q, -)$ which takes

$$\mathcal{H}_q \mapsto \mathcal{S}_q, \qquad \mathcal{W}_q \mapsto \mathcal{O}_{\mathcal{L}_q(\mathcal{N}/G)}.$$

2. *When $q = 1$ we have equivalences of complexes (where $\mathfrak{h}$ is the Lie algebra of the universal Cartan)*

$$HH_\bullet(\mathbf{H}) \simeq k\widetilde{W}^a \otimes \mathrm{Sym}(\mathfrak{h}^*[1] \oplus \mathfrak{h}^*[2]), \qquad HH_\bullet(\mathbf{W}) \simeq \mathrm{Ind}_{W^f}^{\widetilde{W}^a} k.$$

Furthermore, we have an equivalence

$$\mathrm{Tr}(\mathbf{H}, q_*) \simeq \mathrm{Coh}_{\mathcal{N}}(\mathcal{L}(\mathfrak{g}/G)).$$

From this, we may conclude

$$\mathcal{S} = [\mathbf{H}], \qquad \omega_{\mathcal{L}(\widehat{\mathcal{N}}/G)} = [\mathbf{W}].$$

$$\mathrm{End}(\mathcal{S}) \simeq k\widetilde{W}^a \otimes \mathrm{Sym}(\mathfrak{h}^*[1] \oplus \mathfrak{h}^*[2]), \qquad \mathrm{Hom}(\mathcal{S}, \omega) \simeq \mathrm{Ind}_{W^f}^{\widetilde{W}^a} k.$$

Furthermore, there is an embedding $\mathrm{Mod}(k\widetilde{W}^a \otimes \mathrm{Sym}(\mathfrak{h}^*[1] \oplus \mathfrak{h}^*[2])) \hookrightarrow \mathrm{QC}^!_{\mathcal{N}}(\mathcal{L}(\mathfrak{g}/G))$ *whose right adjoint is* $\mathrm{Hom}(\mathcal{S}, -)$ *which takes*

$$k\widetilde{W}^a \otimes \mathrm{Sym}(\mathfrak{h}^*[1] \oplus \mathfrak{h}^*[2]) \mapsto \mathcal{S}, \qquad W \mapsto \mathrm{pr}_{\mathcal{S}}\omega_{\mathcal{L}(\widehat{\mathcal{N}}/G)}$$

where $\mathrm{pr}_{\mathcal{S}}$ *is the projection to the Springer subcategory.*

3.2 Automorphic side

Let G_F be a reductive group over a non-archimedean local field F, and let LG denote the loop group defined over $\overline{\mathbb{F}_q}$ where $\mathbb{F}_q$ is the residue field of F, and L^+G the arc group. First, let us recall Lang's theorem; I won't attempt to prove it, there are many standard references.

Theorem 3.2.1 (Lang)**.** *Let G be a connected linear algebraic group defined over $\mathbb{F}_q$ and based changed to $\overline{\mathbb{F}}_q$, and let* Fr *denote the geometric Frobenius automorphism. The map*

$$G \to G, \qquad g \mapsto g^{-1}\mathrm{Fr}(g)$$

is surjective.

Corollary 3.2.2. *The same is true for L^+G, or for any compact open subgroup $K \subset LG$. In particular, $K/_{\mathrm{Fr}} K \simeq BK_F$.*

Proof. The quotient maps $L^+G \to L^{(n)}G$ are compatible with Frobenius, thus surjectivity for each n implies surjectivity. It remains to verify that $L^{(n)}G$ are connected, but this follows since the reductive quoteint is just G. A similar argument works for K. □

We now apply the traces formalism to the example: $X = B(L^+G)$ and $Y = B(LG)$ (pretend these make sense as stacks, and see Example 2.4.10). I want to keep this discussion informal, and refer the reader to [Zhu21, HZ] for details. Let's define the following moduli stacks of G-bundles (we will suppress G from the notation) on a curve X, for a fixed point $x_0 \in X$:

1. Bun_X is the stack whose points are vector bundles on X. Its local model at x_0 is given by the quotient stack $LG\backslash LG/L^+G = B(L^+G)$.

2. Gr_X is the *Beilinson-Drinfeld Grassmannian*, whose points are $(x, \mathcal{E}, \beta)$ where $x \in X$, $\mathcal{E} \in \mathrm{Bun}_X$, and $\beta : \mathcal{E}|_{X-x} \simeq \mathcal{O}_G|_{X-x}$. Its local model at x_0 is given by the *affine Grassmannian* $\mathrm{Gr}_{X,x_0} = LG/L^+G$. It has a forgetful map $\mathrm{Gr}_X \to \mathrm{Bun}_X$.

3. Hk_X is the *Hecke stack*, whose points are $(x, \mathcal{E}_0, \mathcal{E}_1, \beta)$ where $x \in X$, $\mathcal{E}_i \in \mathrm{Bun}_X$, and $\beta : \mathcal{E}_0|_{X-x} \simeq \mathcal{E}_1|_{X-x}$. Its local model at x_0 is $LG \times^{L^+G} LG$. It has a forgetful map $\mathrm{Hk}_X \to \mathrm{Bun}_X \times \mathrm{Bun}_X$.

4. Sht_X is the *moduli stack of Shtukas*, whose points are $(x, \mathcal{E}, \beta)$ where $x \in X$, $\mathcal{E} \in \mathrm{Bun}_X$, and $\beta : \mathcal{E}|_{X-x} \simeq {}^\tau\mathcal{E}|_{X-x}$, where $^\tau$ denotes a "twisting" by an automorphism of Bun_X (e.g. twist by Frobenius). Its local model is $\dfrac{LG}{L^+G}$, where L^+G acts by the twist on one side. It has a forgetful map $\mathrm{Sht}_X \to \mathrm{Bun}_X$.

5. $\mathrm{Hk}_X^{(n)}$ and $\mathrm{Sht}_X^{(n)}$ are the iterated Hecke stack and moduli of Shtukas, i.e. where we have a sequence of isomorophisms $\mathcal{E}_0|_{X-x} \simeq \mathcal{E}_1|_{X-x} \simeq \cdots \simeq \mathcal{E}_n|_{X-x}$ (and for Shtukas, take $\mathcal{E}_n = {}^\tau\mathcal{E}_0$). They have local models $LG \times^{L^+G} \cdots \times^{L^+G} LG$ and $\dfrac{LG \times^{L^+G} \cdots \times^{L^+G} LG}{L^+G}$ ($n+1$ and n factors respectively).

6. There are versions for level structures, but let me ignore this to keep things simple. Of course for the affine Hecke category the relevant thing is Iwahori level structure.

We may wish to consider two cases: where τ is the trivial automorphism, and where τ is the geometric Frobenius.

1. When $\tau = 1$, then $\mathrm{Tr}(\mathrm{Sh}(LG)) = \mathrm{Sh}(LG/LG)$ is the category of *affine character sheaves*. It has an intricate geometry that I do not understand.

2. When $\tau = \mathrm{Fr}$, then $\mathrm{Tr}(\mathrm{Sh}(LG)) = \mathrm{Sh}(LG/_{\mathrm{Fr}}\, LG)$ is the *stack of G-isocrystals* or *Kottwitz stack*. It is, in some sense, "smaller" than LG/LG.[8] Its points are parameterized by the set of G-isocrystals $B(G)$. Each isocrystal determines a form H of G_F, and sheaves supported on that isocrystal are equivalent to $\mathrm{Rep}(H)$. In particular, for the unique closed orbit, we have $\mathrm{Rep}(G_F)$.

By Example 2.4.10, we see that

$$[\mathbf{H}] = \pi_* \mathcal{C}_{B/B}.$$

is the affine Springer sheaf. For the Frobenius-twisted version, we have a map for any compact open which factors:

$$BK \to BG_F \hookrightarrow LG/_{\mathrm{Fr}}\, LG.$$

We denote the pushforward of the constant sheaf by $\delta_K \in \mathrm{Sh}(LG/_{\mathrm{Fr}}\, LG)$, and

$$[\mathbf{H}, \mathrm{Fr}^*] = \delta_I.$$

Now, for the Iwahori-level setting we have Bezrukavnikov's equivalence, whence we have the following commuting diagram

$$\begin{tikzcd} \mathrm{Sh}(I\backslash LG/I) \arrow[r, "\simeq"] \arrow[d, hook, "\mathrm{Tr}(-,\mathrm{Fr}^*)"'] & \mathrm{QC}^!(\widetilde{\mathcal{N}}_{\check{G}}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \widetilde{\mathcal{N}}_{\check{G}}/\check{G}) \arrow[d, "\mathrm{Tr}(-,q_*)"] \\ \mathrm{Sh}(LG/_{\mathrm{Fr}}\, LG) \arrow[r] & \mathrm{QC}^!(\mathcal{L}_q(\mathcal{N}_{\check{G}}/\check{G})) \end{tikzcd}$$

which, furthermore, identifies the trace of the identity and the trace of the Whittaker module, i.e.

$$\delta_I \longleftrightarrow \mathcal{S}_q$$

In [HZ] a precise description of the essential image of the left map will be given.

Taking trace of the identity instead, we obtain the following relationship between affine Springer theory and coherent Springer theory

$$\begin{tikzcd} \mathrm{Sh}(I\backslash LG/I) \arrow[r, "\simeq"] \arrow[d, hook, "\mathrm{Tr}"'] & \mathrm{QC}^!(\widetilde{\mathcal{N}}_{\check{G}}/\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \widetilde{\mathcal{N}}_{\check{G}}/\check{G}) \arrow[d, "\mathrm{Tr}"] \\ \mathrm{Sh}(LG/LG) \arrow[r] & \mathrm{QC}^!(\mathcal{L}_{\mathcal{N}}(\check{\mathfrak{g}}_{\check{G}}/\check{G})) \end{tikzcd}$$

where $\mathcal{L}_{\mathcal{N}}$ here means both nilpotent classical (but not singular) support. We note this gives an automorphic explanation for why the ring $\mathrm{Sym}(\mathfrak{h}^*[1] \oplus \mathfrak{h}^*[2])$ appears for $q = 1$ but not otherwise; this ring is the cohomology ring $H^\bullet(I/I; k)$, which appears in Springer theory but not in the Frobenius-twisted version.

[8] In the finite analogue, by Lang's theorem, $G/_{\mathrm{Fr}}\, G = G(\mathbb{F}_q)$.

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