TWO-DIMENSIONAL l-ADIC REPRESENTATIONS OF THE FUNDAMENTAL GROUP OF A CURVE OVER A FINITE FIELD AND AUTOMORPHIC FORMS ON GL(2)

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Introduction. Let X be a smooth projective absolutely irreducible curve over $\mathbf{F}_q$, k the field of rational functions on X, $\mathfrak{A}$ its adèle ring. We denote the completion of k at v by k_v, the ring of integers of k_v by O_v and the order of its residue field by q_v. Put $O = \Pi_v O_v$. Fix a prime number l which does not divide q.

Definition. An unramified cusp form on $\mathrm{GL}(2, \mathfrak{A})$ is a function $f: \mathrm{GL}(2, \mathfrak{A}) \to \bar{\mathbf{Q}}_l$ such that

1) $f(x\gamma) = f(x)$ for all $x \in \mathrm{GL}(2, \mathfrak{A})$, $\gamma \in \mathrm{GL}(2, k)$;
2) $f(ux) = f(x)$ for all $x \in \mathrm{GL}(2, \mathfrak{A})$, $u \in \mathrm{GL}(2, O)$;
3) $\int_{z \in \mathfrak{A}/k} f\left(x\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)dz = 0$ for all $x \in \mathrm{GL}(2, \mathfrak{A})$.

For every closed point $v \in X$ there are two Hecke operators T_v and U_v acting on the space of unramified cusp forms on $\mathrm{GL}(2, \mathfrak{A})$. Recall that

$$(1)\qquad \begin{aligned}(T_v f)(x) &= \int_{g \in M_v} f(g^{-1}x)dg\\ (U_v f)(x) &= f\left(\begin{pmatrix}\pi_v^{-1} & 0\\ 0 & \pi_v^{-1}\end{pmatrix}x\right)\end{aligned}$$

Here π_v denotes a prime element of O_v, $M_v \stackrel{\text{def}}{=} \{h \in \mathrm{Mat}(2, O_v) \mid \det h \in \pi_v O_v^*\}$, dg is the Haar measure on $\mathrm{GL}(2, k_v)$ such that $\operatorname{mes} \mathrm{GL}(2, O_v) = 1$. The geometric Frobenius element of $\pi_1(X)$ corresponding to a closed point $v \in X$ is denoted by Fr_v.

MAIN THEOREM. *Let $E \subseteq \bar{\mathbf{Q}}_l$ be a finite extension of $\mathbf{Q}_l$, ρ an absolutely irreducible two-dimensional representation of $\pi_1(X)$ over E continuous in the l-adic topology. For every closed point $v \in X$ put $t_v = \operatorname{Tr} \rho(\mathrm{Fr}_v)$, $u_v = q_v^{-1} \det \rho(\mathrm{Fr}_v)$. Then there exists a nonzero unramified cusp form f on*

$\mathrm{GL}(2, \mathfrak{A})$ *such that for every closed point* $v \in X$ *the equalities* $T_v f = t_v f$, $U_v f = u_v f$ *hold.*

Remarks. 1) It follows from Jacquet-Langlands' theory [2] that the form f mentioned in the theorem is unique up to a constant factor.

2) For representations ρ contained in a compatible system of l-adic representations the theorem was proved by P. Deligne in [3] (not only for representations of $\pi_1(X)$ but also for those of $\mathrm{Gal}(\bar{k}/k)$). This proof does not work for a single l-adic representation because it has not been proved that the constant appearing in Grothendieck's functional equation for the L-functions of l-adic representations of $\mathrm{Gal}(\bar{k}/k)$ [3] is "correct" (in [3] the "correctness" of the constant was proved only for compatible systems of l-adic representations). P. Deligne managed to prove the "correctness" of the constant in Grothendieck's functional equation for the L-functions of unramified representations of $\mathrm{Gal}(\bar{k}/k)$ (probably the proof has not been published yet). But in order to prove the theorem formulated above it is necessary to show that the constant is "correct" not only in the unramified case but also for representations of $\mathrm{Gal}(\bar{k}/k)$ of the form $\rho \otimes \omega$ where ρ is unramified and ω is an arbitrary one-dimensional representation.

3) It is shown in [1] that if a nonzero unramified cusp form f on $\mathrm{GL}(2, \mathfrak{A})$ exists such that $T_v f = t_v f$, $U_v f = u_v f$ and $u_v \in \bar{\mathbf{Z}}_l^*$, then there exists a finite extension E of $\mathbf{Q}_l$ and an absolutely irreducible continuous two-dimensional representation ρ of $\pi_1(X)$ over E such that $\mathrm{Tr}\, \rho(\mathrm{Fr}_v) = t_v$, $\det \rho(\mathrm{Fr}_v) = q_v u_v$.

4) It is easy to deduce from the main theorem that for every irreducible two-dimensional representation ρ of $\pi_1(X)$ over a finite extension of $\mathbf{Q}_l$, there is a representation ω of dimension 1 such that $\rho \otimes \omega$ is contained in a compatible system of l-adic representations.

5) The method developed in this paper works also in the case when ρ is a two-dimensional l-adic representation of $\mathrm{Gal}(\bar{k}/k)$ such that the inertia group of each point of X acts unipotently. Using this fact and the Saito-Shintani-Langlands base change theorem [7] it is possible to prove that every irreducible two-dimensional l-adic representation of $\mathrm{Gal}(\bar{k}/k)$ (ramified at finitely many places only) corresponds to a cuspidal automorphic representation of $\mathrm{GL}(2, \mathfrak{A})$. The complete proof of this theorem will be given elsewhere.

The works and ideas of Deligne contributed much to this paper. In particular, the idea of using the fact that $\mathbf{P}^n$ is simply connected, which plays a fundamental role in the proof of the main theorem (see Section 2), was taken from [5], pp. 491–519. The technique for studying symmetric

powers of an l-adic sheaf on a curve used in the present paper is also due to Deligne.

The following feature of this work seems to be essentially new: we have constructed an automorphic form without using L-functions.

Now some remarks as to the words and symbols which will be used in a nonstandard manner. We shall write "scheme" instead of "Noetherian scheme over $\mathbf{F}_q$". Accordingly, if X and Y are schemes then $X \times Y$ denotes the product of X and Y over $\mathbf{F}_q$. For every field K the symbol $\bar{K}$ means the separable closure of K. We denote by Pic X the Picard *group* of X, while $\underline{\mathrm{Pic}}\, X$ is used for the Picard *scheme* of X. The fiber of a coherent sheaf $\mathfrak{F}$ on a scheme S at a point $x \in S$ is defined as $\mathfrak{F}_x/M_x\mathfrak{F}_x$, where $\mathfrak{F}_x$ is the stalk of $\mathfrak{F}$ at x and M_x is the maximal ideal of the local ring of x. If $\mathfrak{F}$ is an l-adic sheaf on a scheme S, the trace of the geometric Frobenius automorphism on $H^j(S \otimes \bar{\mathbf{F}}_q, \mathfrak{F})$ is denoted by $\mathrm{Tr}(\mathrm{Fr}, H^j(S \otimes \bar{\mathbf{F}}_q, \mathfrak{F}))$. We denote by $\mathrm{Tr}(\mathrm{Fr}_u, \mathfrak{F})$ the trace of the geometric Frobenius automorphism acting on the stalk of $\mathfrak{F}$ at a closed point $u \in S$. The constant étale sheaf on S, whose stalks are equal to Λ, is denoted by $\underline{\Lambda}$.

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1. Unramified cusp forms on GL(2, $\mathfrak{A}$). Let B denote the group of upper triangular matrices over k. Denote by V the space of locally constant functions $f: \mathrm{GL}(2, \mathfrak{A}) \to \bar{\mathbf{Q}}_l$ such that

1) $f(x\gamma) = f(x)$ for $x \in GL(2, \mathfrak{A})$, $\gamma \in B$;

2) $\int_{z \in \mathfrak{A}/k} f\left(x\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)dz = 0$ for $x \in \mathrm{GL}(2, \mathfrak{A})$;

3) $f(ux) = f(x)$ for $u \in \mathrm{GL}(2, O)$, $x \in \mathrm{GL}(2, \mathfrak{A})$.

Fix a nontrivial locally constant homomorphism $\Psi: \mathfrak{A}/k \to \bar{\mathbf{Q}}_l^*$ and denote by W the space of locally constant functions $\varphi: \mathrm{GL}(2, \mathfrak{A}) \to \bar{\mathbf{Q}}_l$ such that

1) $\varphi\left(x\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right) = \Psi(z)\varphi(x)$ for $x \in \mathrm{GL}(2, \mathfrak{A})$, $z \in \mathfrak{A}$;

2) $\varphi\left(x\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}\right) = \varphi(x)$ for $a \in k^*$, $x \in \mathrm{GL}(2, \mathfrak{A})$;

3) $\varphi(ux) = \varphi(x)$ for $u \in \mathrm{GL}(2, O)$, $x \in \mathrm{GL}(2, \mathfrak{A})$.

If $f \in V$ the function $\varphi: \mathrm{GL}(2, \mathfrak{A}) \to \bar{\mathbf{Q}}_l$ defined by

$$\varphi(x) = \int_{z \in \mathfrak{A}/k} f\left(x\begin{pmatrix}1 & z\\ 0 & 1\end{pmatrix}\right)\Psi(-z)dz$$

belongs to W. It is shown in [2] that this mapping $V \to W$ is an isomorphism and the inverse mapping $W \to V$ is given by

$$f(x) = \sum_{a \in k^*} \varphi\left(x\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{2}$$

The operators in V and W defined by formula (1) are denoted by T_v and U_v as before. It is clear that the isomorphism $W \rightrightarrows V$ commutes with T_v and U_v.

Suppose that a homomorphism $\eta: \operatorname{Pic} X \to \bar{\mathbf{Q}}_l^*$ and numbers $t_v \in \bar{\mathbf{Q}}_l$, $v \in X$ are given. Put $u_v = \eta(\bar{v})$ where $\bar{v}$ is the divisor class of v. It is proved in [6] that the space

$$U \overset{\text{def}}{=} \{\varphi \in W \mid \forall v T_v \varphi = t_v \varphi,\ U_v \varphi = u_v \varphi\}$$

has dimension 1. Besides that a nonzero function $\varphi \in U$ is found. To write the formula for this function we shall introduce some notations. For every $a \in \mathfrak{A}^*$ let us denote the corresponding divisor by $\operatorname{div} a$ and the adèle norm of a by $|a|$. Suppose that the largest O-submodule of $\mathfrak{A}$ on which Ψ is trivial equals to Ou; then put $\delta \overset{\text{def}}{=} -\operatorname{div} u$. For every closed point $v \in X$ we denote the coefficient of z^n in the formal series $(1 - t_v z + q_v u_v z^2)^{-1}$ by $c_v(n)$ (in particular, $c_v(n) = 0$ if $n < 0$). For every divisor D on X put $r(D) = \Pi_v c_v(n_v)$, where n_v is the multiplicity of D at v. Note that $r(D) = 0$ if D is not effective. In [6] (Chapter VI, Proposition 6) U is shown to be generated by the function φ whose restriction to the group of upper triangular matrices is given by

$$\varphi\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \Psi(z)\left|\frac{b}{a}\right| \eta(\operatorname{div} b)^{-1} r(\operatorname{div} b + \delta - \operatorname{div} a) \tag{3}$$

Note that φ is uniquely defined by formula (3) and the condition $\varphi(ux) = \varphi(x)$ for $u \in \mathrm{GL}(2, O)$.

As $V \simeq W$, the space $\{f \in V \mid \forall v T_v f = t_v f,\ U_v f = u_v f\}$ has dimension 1 and is generated by the function f whose restriction to the set of upper triangular matrices is given by (2) and (3). We have

$$f\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right) = \left|\frac{b}{a}\right| \eta(\operatorname{div} b)^{-1} \cdot \sum_{h \in k^*} r(\operatorname{div} h + \operatorname{div} b + \delta - \operatorname{div} a)\Psi(hz) \tag{4}$$

Note that Ψ may be represented as $\Psi(z) = \Psi_0(\langle z, \omega_0\rangle)$ where $\Psi_0 : \mathbf{F}_q \to \bar{\mathbf{Q}}_l^*$ is a non-trivial character, ω_0 is a rational differential form on X (the symbol $\langle\ ,\ \rangle$ denotes the canonical pairing between $\mathfrak{A}/k$ and the space of rational differentials Ω_k). δ is nothing else but the divisor of ω_0. Therefore

$$f\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)$$
$$= \left|\frac{b}{a}\right| \eta(\operatorname{div} b)^{-1} \sum_{\omega\in\Omega_k^*} r(\operatorname{div}\omega + \operatorname{div} b - \operatorname{div} a)\Psi_0(\langle z, \omega\rangle)$$

where $\Omega_k^* = \Omega_k \backslash \{0\}$. The number $r(\operatorname{div}\omega + \operatorname{div} b - \operatorname{div} a)$ does not change if ω is multiplied by an element of $\mathbf{F}_q^*$. Hence

$$(5)\quad f\left(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\right)$$
$$= \left|\frac{b}{a}\right| \eta(\operatorname{div} b)^{-1}\Big(q \sum_{\omega\in\Omega_k^*/\mathbf{F}_k^*, \langle z,\omega\rangle=0} r(\operatorname{div}\omega + \operatorname{div} b - \operatorname{div} a)$$
$$- \sum_{\omega\in\Omega_k^*/\mathbf{F}_q^*} r(\operatorname{div}\omega + \operatorname{div} b - \operatorname{div} a)\Big)$$

Now we are going to give a geometric interpretation of formula (5).

Definition. Let $\mathcal{L}$ be a sheaf of $\mathcal{O}_X$-modules. An invertible subsheaf $\mathfrak{A} \subset \mathcal{L}$ is said to be maximal if it is not contained in any larger invertible subsheaf.

Denote by Bun_2 the set of isomorphism classes of two-dimensional locally free sheaves of $\mathcal{O}_X$-modules. We shall denote by Flag_2 the set of isomorphism classes of pairs $(\mathcal{L}, \mathfrak{A})$ consisting of a two-dimensional locally free sheaf $\mathcal{L}$ and a maximal invertible subsheaf $\mathfrak{A} \subset \mathcal{L}$. There are canonical bijections

$$\mathrm{GL}(2, O)\backslash \mathrm{GL}(2, \mathfrak{A})/\mathrm{GL}(2, k) \rightrightarrows \mathrm{Bun}_2, \quad \mathrm{GL}(2, O)\backslash \mathrm{GL}(2, \mathfrak{A})/B \rightrightarrows \mathrm{Flag}_2.$$

Therefore unramified cusp forms on $\mathrm{GL}(2, \mathfrak{A})$ may be considered as functions on Bun_2 and elements of V as functions of Flag_2. Note that the bijection $\mathrm{GL}(2, O)\backslash \mathrm{GL}(2, \mathfrak{A})/B \rightrightarrows \mathrm{Flag}_2$ associates to

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

the pair $\mathcal{L}$, $\mathcal{A}$ where $\mathcal{A}$ is the invertible sheaf corresponding to div a, $\mathcal{L}$ is the two-dimensional locally free sheaf of $\mathcal{O}_X$-modules such that 1) $\mathcal{L} \supseteq \mathcal{A}$, 2) $\mathcal{L}/\mathcal{A} = \mathcal{B}$ where $\mathcal{B}$ is the invertible sheaf corresponding to div b, 3) the element of $\operatorname{Ext}(\mathcal{B}, \mathcal{A})$ corresponding to the exact sequence $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$ is the image of z under the homomorphism $\mathfrak{A} \to \mathfrak{A}/(k + a^{-1}bO) = H^1(X, \mathcal{B}^{-1} \otimes \mathcal{A}) = \operatorname{Ext}(\mathcal{B}, \mathcal{A})$. Also note that $r(\operatorname{div}\omega + \operatorname{div} b - \operatorname{div} a) \neq 0$ only if $\operatorname{div}\omega + \operatorname{div} b - \operatorname{div} a \geq 0$, i.e. if $\omega \in H^0(X, \mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega)$, where Ω is the cotangent sheaf of X. Therefore, if the function f given by (5) is considered as a function on Flag_2, the formula for f may be rewritten as

$$(6) \quad f(\mathcal{L}, \mathcal{A}) = q^{\deg \mathcal{A} - \deg \mathcal{B}} \eta(\mathcal{B})^{-1}\Big(q \sum_{D \in H(\mathcal{L},\mathcal{A})} r(D) - \sum_{D \in P(\mathcal{L},\mathcal{A})} r(D)\Big)$$

where $\mathcal{B} = \mathcal{L}/\mathcal{A}$, $P(\mathcal{L}, \mathcal{A})$ is the set of one-dimensional vector subspaces of $H^0(X, \mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega)$ (i.e. the set of effective divisors in the class $\mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega$), $H(\mathcal{L}, \mathcal{A}) \subset P(\mathcal{L},\mathcal{A})$ is the zero set of the functional $\lambda: H^0(X, \mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega) \to \mathbf{F}_q$ which corresponds by Serre's duality to the element of $\operatorname{Ext}(\mathcal{B}, \mathcal{A}) = H^1(X, \mathcal{B}^{-1} \otimes \mathcal{A})$ defined by the exact sequence $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$.

Now we are going to formulate a sufficient condition for a function $f: \mathrm{GL}(2, O)\backslash \mathrm{GL}(2, \mathfrak{A})/B \to \bar{\mathbf{Q}}_l$ to be right invariant under $\mathrm{GL}(2, k)$ provided it is an eigenfunction of Hecke operators. This condition will be formulated geometrically. Recall that we identify $\mathrm{GL}(2, O)\backslash \mathrm{GL}(2, \mathfrak{A})/B$ with Flag_2. To describe geometrically the action of Hecke operators in the space of functions $\mathrm{Flag}_2 \to \bar{\mathbf{Q}}_l$ let us introduce the following definition.

Definition. A lower modification of a two-dimensional locally free $\mathcal{O}_X$-module sheaf $\mathcal{L}$ at a point $v \in X$ is a subsheaf $\mathcal{L}' \subset \mathcal{L}$ such that $\mathcal{L}' \neq \mathcal{L}$, $\mathcal{L}' \supset \mathcal{L}(-v)$, $\mathcal{L}' \neq \mathcal{L}(-v)$. $\mathcal{L}'$ is said to be an upper modification of $\mathcal{L}$ at v if $\mathcal{L}$ is a lower modification of $\mathcal{L}'$ at v.

It is easy to see that Hecke operators act on functions $f: \mathrm{Flag}_2 \to \bar{\mathbf{Q}}_l$ in the following way:

$$(U_v f)(\mathcal{L}, \mathcal{A}) = f(\mathcal{L}(-v), \mathcal{A}(-v))$$

$$(T_v f)(\mathcal{L}, \mathcal{A}) = \sum_{\mathcal{L}' \in S} f(\mathcal{L}', \mathcal{A} \cap \mathcal{L}')$$

where S is the set of lower modifications of $\mathcal{L}$ at v.

For every $\mathcal{L} \in \mathrm{Bun}_2$ we denote by $h(\mathcal{L})$ the least degree of invertible quotient sheaves of $\mathcal{L} \otimes \bar{\mathbf{F}}_q$, where $\mathcal{L} \otimes \bar{\mathbf{F}}_q$ is the inverse image of $\mathcal{L}$ on $X \otimes \bar{\mathbf{F}}_q$ ($h(\mathcal{L}) \neq -\infty$ since $\dim H^1(X, \mathcal{L}) < \infty$).

PROPOSITION 1.1. *Let* $f: \mathrm{Flag}_2 \to \bar{\mathbf{Q}}_l$ *be an eigenfunction of Hecke operators. Suppose that for some* $N \in \mathbf{Z}$ *the following condition is satisfied*: $f(\mathcal{L}, \mathcal{A}) = f(\mathcal{L}, \mathcal{A}')$ *for all* $\mathcal{L}$, $\mathcal{A}$, $\mathcal{A}'$ *such that the degrees of* $\mathcal{A}$ *and* $\mathcal{A}'$ *are less than* $h(\mathcal{L}) - N$. *Then there exists a function* $g: \mathrm{Bun}_2 \to \bar{\mathbf{Q}}_l$ *such that* $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$.

Proof. Put $g(\mathcal{L}) = f(\mathcal{L}, \mathcal{A})$ where $\mathcal{A}$ is a maximal invertible subsheaf of $\mathcal{L}$ such that $\deg \mathcal{A} < h(\mathcal{L}) - N$. In order to show that g is well defined we have to prove the following lemma.

LEMMA 1. *Let* $\mathcal{L}$ *be a two-dimensional locally free sheaf of* $\mathcal{O}_X$*-modules. Then there exist maximal invertible subsheaves of* $\mathcal{L}$ *whose degree is arbitrarily small.*

Proof. The set of degrees of invertible subsheaves of $\mathcal{L}$ is bounded from above because $\dim H^0(X, \mathcal{L}) < \infty$. The set of invertible subsheaves of $\mathcal{L}$ having a fixed degree is finite since $\mathrm{Pic}^0 X$ is finite and for every $\mathcal{A} \in \mathrm{Pic}\, X$ the set $\mathrm{Hom}(\mathcal{A}, \mathcal{L})$ is finite. On the other hand the set of all maximal invertible subsheaves of $\mathcal{L}$ is infinite because such subsheaves correspond bijectively to one-dimensional vector subspaces of the generic fiber of $\mathcal{L}$. □

Let v be a closed point of X, $t_v, u_v \in \bar{\mathbf{Q}}_l$, $T_v f = t_v f$, $U_v f = u_v f$. Then $T_v g = t_v g$, $U_v g = u_v g$.

LEMMA 2. *Let* $(\mathcal{L}, \mathcal{A}) \in \mathrm{Flag}_2$ *and let* $\mathcal{L}'$ *be an upper modification of* $\mathcal{L}$ *at* v *such that* $\mathcal{A}$ *is maximal as a subsheaf of* $\mathcal{L}'$. *Suppose that for every two-dimensional locally free* $\mathcal{O}_X$*-module sheaf* $\bar{\mathcal{L}} \supset \mathcal{L}'$ *such that* $\mathcal{A}$ *is maximal as a subsheaf of* $\bar{\mathcal{L}}$, *the equality* $f(\bar{\mathcal{L}}, \mathcal{A}) = g(\bar{\mathcal{L}})$ *holds. Then* $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$.

Proof. Denote by S the set of lower modifications of $\mathcal{L}'$ at v different from $\mathcal{L}$. We have

$$f(\mathcal{L}, \mathcal{A}) + \sum_{\mathcal{L}'' \in S} f(\mathcal{L}'', \mathcal{A} \cap \mathcal{L}'') = t_v f(\mathcal{L}', \mathcal{A}) = t_v g(\mathcal{L}')$$

$$g(\mathcal{L}) + \sum_{\mathcal{L}'' \in S} g(\mathcal{L}'') = t_v g(\mathcal{L}')$$

On the other hand, if $\mathcal{L}'' \in S$ then $\mathcal{A} \cap \mathcal{L}'' = \mathcal{A}(-v)$. Hence we may put $\bar{\mathcal{L}} = \mathcal{L}''(v)$. Therefore $f(\mathcal{L}'', \mathcal{A} \cap \mathcal{L}'') = u_v f(\mathcal{L}''(v), \mathcal{A}) = u_v g(\mathcal{L}''(v)) = g(\mathcal{L}'')$ for every $\mathcal{L}'' \in S$, whence it follows that $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$. □

LEMMA 3. *Let* $(\mathcal{L}, \mathcal{A}) \in \mathrm{Flag}_2$, $\deg \mathcal{L} > 2h(\mathcal{L})$. *Then there exists an upper modification* $\mathcal{L}'$ *of* $\mathcal{L}$ *at* v *such that* $h(\mathcal{L}') > h(\mathcal{L})$ *and* $\mathcal{A}$ *is maximal as a subsheaf of* $\mathcal{L}'$.

Proof. Let $\mathcal{M} \subset \mathcal{L} \otimes \overline{\mathbf{F}}_q$ be an invertible subsheaf of the highest degree. Then $\deg \mathcal{M} = \deg \mathcal{L} - h(\mathcal{L}) > h(\mathcal{L})$. There exists an upper modification $\mathcal{L}' \supset \mathcal{L}$ such that $\mathcal{A}$ and $\mathcal{M}$ are maximal as subsheaves of $\mathcal{L}'$ and $\mathcal{L}' \otimes \overline{\mathbf{F}}_q$, respectively. We are going to show that $h(\mathcal{L}') > h(\mathcal{L})$. Assume that there is an invertible quotient sheaf $\mathcal{N}$ of $\mathcal{L}' \otimes \overline{\mathbf{F}}_q$ such that $\deg \mathcal{N} \leq h(\mathcal{L})$. The superposition $\mathcal{M} \to \mathcal{L}' \otimes \overline{\mathbf{F}}_q \to \mathcal{N}$ must be equal to zero. Hence $\mathcal{N} = (\mathcal{L}' \otimes \overline{\mathbf{F}}_q)/\mathcal{M}$ and $\deg \mathcal{L}' = \deg \mathcal{M} + \deg \mathcal{N} \leq \deg \mathcal{L}$, which is impossible. □

Fix $\mathcal{A} \in \operatorname{Pic} X$ and denote by $P(m, n)$ the following statement: "For every two-dimensional locally free sheaf $\mathcal{L}$ on X containing $\mathcal{A}$ as a maximal invertible subsheaf and such that $h(\mathcal{L}) \geq m$, $\deg \mathcal{L} \geq n$ the equality $f(\mathcal{L}, \mathcal{A}) = g(\mathcal{L})$ holds." By the condition of the proposition, $P(m, n)$ is true for $m > \deg \mathcal{A} + N$. It follows from Lemma 2 that $\forall m, n(P(m, n+1) \Rightarrow P(m, n))$. Lemmas 2 and 3 show that $\forall m \forall n > 2m(P(m+1, n) \Rightarrow P(m, n))$. Thus $P(m, n)$ is true for all $m, n \in \mathbf{Z}$. □

2. Derivation of the main theorem from vanishing cycle theorems. Let E, ρ, t_v, u_v denote the same objects as in the main theorem. If the restriction of ρ to $\pi_1(X \otimes \overline{\mathbf{F}}_q)$ is not absolutely irreducible then after replacing E by its finite extension, ρ becomes induced by a one-dimensional representation of $\pi_1(X \otimes \mathbf{F}_{q^2})$. The existence of a cusp form corresponding to such a representation ρ is well-known ([2], Theorem 12.2). So we shall suppose that the restriction of ρ to $\pi_1(X \otimes \overline{\mathbf{F}}_q)$ is absolutely irreducible.

According to class field theory the homomorphism $\det \rho : \pi_1(X \otimes \overline{\mathbf{F}}_q) \to \overline{\mathbf{Q}}_l^*$ corresponds to a homomorphism $\mu : \operatorname{Pic} X \to \overline{\mathbf{Q}}_l^*$. Define $\eta : \operatorname{Pic} X \to \overline{\mathbf{Q}}_l^*$ by $\eta(\mathcal{B}) = \mu(\mathcal{B}) \mathrm{q}^{-\deg \mathcal{B}}$. Then $u_v = \eta(v)$. Therefore the results of Section 1 are applicable. Thus, in order to prove the main theorem we have to show that the function $f : \mathrm{Flag}_2 \to \overline{\mathbf{Q}}_l$ defined by (6) is the pullback of a function $\mathrm{Bun}_2 \to \overline{\mathbf{Q}}_l$.

Let us give a cohomological interpretation of formula (6). Denote by $\mathcal{E}$ the locally constant sheaf on X corresponding to ρ. If $\mathcal{F}_i$ is an l-adic sheaf on a scheme Y_i then $\mathcal{F}_1 \boxtimes \mathcal{F}_2$ will denote the following sheaf on $Y_1 \times Y_2$: $\mathcal{F}_1 \boxtimes \mathcal{F}_2 \overset{\text{def}}{=} \mathrm{pr}_1{}^* \mathcal{F}_1 \otimes \mathrm{pr}_2{}^* \mathcal{F}_2$ where pr_i is the projection $Y_1 \times Y_2 \to Y_i$. Put

$$\boxtimes^n \mathcal{E} = \underbrace{\mathcal{E} \boxtimes \mathcal{E} \boxtimes \cdots \boxtimes \mathcal{E}}_{n \text{ times}}$$

Put $\operatorname{Sym}^n X = X^n/S_n$, $\mathcal{E}^{(n)} = (\varphi_* \boxtimes^n \mathcal{E})^{S_n}$ where $\varphi: X^n \to \operatorname{Sym}^n X$ is the natural mapping. Let $(\mathcal{L}, \mathcal{A}) \in \mathrm{Flag}_2$. In formula (6) $H(\mathcal{L}, \mathcal{A})$ denotes a projective space over $\mathbf{F}_q$ considered as a set. Denote by $\mathcal{H}(\mathcal{L}, \mathcal{A})$ "the same" space considered as a scheme over $\mathbf{F}_q$. We have $\mathcal{H}(\mathcal{L}, \mathcal{A}) \subset \operatorname{Sym}^m X$, where $m = \deg \mathcal{L} - 2 \deg \mathcal{A} + 2g - 2$. The restriction of $\mathcal{E}^{(m)}$ to $\mathcal{H}(\mathcal{L}, \mathcal{A})$ will be denoted by $\mathcal{F}(\mathcal{L}, \mathcal{A})$.

PROPOSITION 2.1. *Let* $(\mathcal{L}, \mathcal{A}) \in \mathrm{Flag}_2$, $\deg \mathcal{L} - 2 \deg \mathcal{A} > 2g - 2$. *Put* $\mathcal{H} = \mathcal{H}(\mathcal{L}, \mathcal{A})$, $\mathcal{F} = \mathcal{F}(\mathcal{L}, \mathcal{A})$. *Then* $f(\mathcal{L}, \mathcal{A}) = q^{1+\deg \mathcal{A}} \mu(\det \mathcal{L})^{-1} \mu(\mathcal{A}) \times \Sigma_j (-1)^j \operatorname{Tr}(\mathrm{Fr}, H^j(\mathcal{H} \otimes \bar{\mathbf{F}}_q, \mathcal{F}))$.

Proof. According to the Lefschetz-Grothendieck formula

$$\sum_j (-1)^j \operatorname{Tr}(\mathrm{Fr}, H^j(\mathcal{H} \otimes \bar{\mathbf{F}}_q, \mathcal{F}))$$

$$= \sum_{D \in H(\mathcal{L},\mathcal{A})} \operatorname{Tr}(\mathrm{Fr}_D, \mathcal{F}) = \sum_{D \in H(\mathcal{L},\mathcal{A})} \operatorname{Tr}(\mathrm{Fr}_D, \mathcal{E}^{(m)})$$

It is easy to show that $\operatorname{Tr}(\mathrm{Fr}_D, \mathcal{E}^{(m)}) = r(D)$. It remains to prove the equality

$$\sum_{D \in P(\mathcal{L},\mathcal{A})} r(D) = 0 \tag{7}$$

Let ω be a homomorphism $\operatorname{Pic} X \to \bar{\mathbf{Q}}_l^*$. We shall also denote by ω the corresponding homomorphism $\pi_1(X) \to \bar{\mathbf{Q}}_l^*$. Denote by $\deg v$ the degree over $\mathbf{F}_q$ of the residue field of a closed point $v \in X$.

According to Grothendieck's theorem [3] the formal series

$$L(\rho \otimes \omega, t) \overset{\text{def}}{=} \prod_v \det(1 - t^{\deg v} \omega(\bar{v}) \rho(\mathrm{Fr}_v))^{-1}$$

is a polynomial in t of degree $4g - 4$. Since $m = \deg \mathcal{L} - 2 \deg \mathcal{A} + 2g - 2 > 4g - 4$ the coefficient of t^m in $L(\rho \otimes \omega, t)$ is equal to zero. On the other hand, it is equal to $\Sigma_{D \in S}\, r(D)\omega(D)$, where S is the set of effective divisors of degree m. Thus $\Sigma_{D \in S}\, r(D)\omega(D) = 0$ for all ω. Therefore $\Sigma_{D \in T}\, r(D) = 0$ for every complete linear system T of degree m. In particular, the equality (7) holds. □

Fix a locally free $\mathcal{O}_X$-module sheaf $\mathcal{L}$ of dimension 2. Let us try to understand how $H^j(\mathcal{H}(\mathcal{L}, \mathcal{A}) \otimes \bar{\mathbf{F}}_q, \mathcal{F}(\mathcal{L}, \mathcal{A}))$ depends on $\mathcal{A}$. To do this we need some notations. Denote by $\bar{V}_n$ the moduli scheme of invertible subsheaves of $\mathcal{L}$ having degree n. The scheme $\bar{V}_n$ is projective. There are canonical morphisms $\varphi_{rn}: \bar{V}_{r+n} \times \operatorname{Sym}^r X \to \bar{V}_n$ (they act on $\mathbf{F}_q$-points as follows: if $\mathcal{A}$ is an invertible subsheaf of $\mathcal{L}$, D is an effective divisor on X, deg $\mathcal{A} = r + n$, $\deg D = r$, then $\varphi_{rn}(\mathcal{A}, D) = \mathcal{A}(-D)$). Put $V_n = \bar{V}_n \backslash \varphi_{1n}(\bar{V}_{n+1} \times X)$, $\tilde{V}_n = \bar{V}_n \backslash \varphi_{2n}(\bar{V}_{n+2} \times \mathrm{Sym}^2 X)$. V_n and $\tilde{V}_n$ are open subsets of $\bar{V}_n$. Note that $V_n \subset \tilde{V}_n$. V_n is the moduli scheme of maximal invertible subsheaves of $\mathcal{L}$ having degree n. The morphism φ_{1n} maps $V_{n+1} \times X$ isomorphically onto a closed subscheme $\Delta_n \subset \tilde{V}_n$ such that Supp $\Delta_n = \tilde{V}_n \backslash V_n$. There is a canonical morphism $\bar{V}_n \to \underline{\mathrm{Pic}}^n\, X$ (associating to an invertible subsheaf $\mathcal{A} \subset \mathcal{L}$ its isomorphism class). Denote by g the genus of X.

PROPOSITION 2.2. *If $n < h(\mathcal{L}) - 2g + 2$ the morphism $\bar{V}_n \to \underline{\mathrm{Pic}}^n X$ is a locally trivial bundle whose fibers are isomorphic to $\mathbf{P}^N$, $N = \deg \mathcal{L} - 2n + 1 - 2g$.*

Proof. From now on we denote by $\bar{\mathcal{L}}$ and $\bar{\Omega}$ the inverse images of $\mathcal{L}$ and Ω on $X \otimes \bar{\mathbf{F}}_q$. It is enough to prove that for every $\mathcal{A} \in \mathrm{Pic}(X \otimes \bar{\mathbf{F}}_q)$ the equalities $\dim H^0(X \otimes \bar{\mathbf{F}}_q, \mathcal{A}^{-1} \otimes \bar{\mathcal{L}}) = N + 1$, $H^1(X \otimes \bar{\mathbf{F}}_q, \mathcal{A}^{-1} \otimes \bar{\mathcal{L}}) = 0$ hold. Indeed, $\dim H^1(X \otimes \bar{\mathbf{F}}_q, \mathcal{A}^{-1} \otimes \bar{\mathcal{L}}) = \dim H^0(X \otimes \bar{\mathbf{F}}_q, \bar{\mathcal{L}}^* \otimes \mathcal{A} \otimes \bar{\Omega}) = \dim \mathrm{Hom}(\bar{\mathcal{L}}, \mathcal{A} \otimes \bar{\Omega}) = 0$ because $\deg \mathcal{A} \otimes \bar{\Omega} = n + 2g - 2 < h(\mathcal{L})$. It remains to use the Riemann-Roch theorem. □

COROLLARY. *If $n < h(\mathcal{L}) - 2g$, then* $\dim(\bar{V}_n \backslash \tilde{V}_n) = \dim \bar{V}_n - 2$, $\dim \Delta_n = \dim \bar{V}_n - 1$, *$\Delta_n$ is an irreducible smooth variety.*

From now on n will denote an integer less than $h(\mathcal{L}) - 2g$.

PROPOSITION 2.3. $\deg \mathcal{L} - 2h(\mathcal{L}) \geq -2g$.

Proof. Let $\mathcal{M}$ be an invertible sheaf on $X \otimes \bar{\mathbf{F}}_q$ of degree $h(\mathcal{L}) - 1$. Then $\mathrm{Hom}(\bar{\mathcal{L}}, \mathcal{M}) = H^0(X \otimes \bar{\mathbf{F}}_q, \bar{\mathcal{L}}^* \otimes \mathcal{M}) = 0$. Hence $\chi(\bar{\mathcal{L}}^* \otimes \mathcal{M}) \leq 0$, i.e. $2(h(\mathcal{L}) - 1) - \deg \mathcal{L} + 2 - 2g \leq 0$. □

Put $m = \deg \mathcal{L} - 2n + 2g - 2$. By Proposition 2.3, $m \geq 2(h(\mathcal{L}) - n - 1) \geq 4g$ (recall that $n \leq h(\mathcal{L}) - 2g - 1$). Denote by ν_n the morphism $\bar{V}_n \to \underline{\mathrm{Pic}}^m\, X$ mapping $x \in \bar{V}_n(\bar{\mathbf{F}}_q)$ to $\mathcal{A}^{-2} \otimes \bar{\Omega} \otimes \det \bar{L} \in \underline{\mathrm{Pic}}^m(X \otimes \bar{\mathbf{F}}_q)$, where $\mathcal{A}$ is the isomorphism class of the invertible subsheaf of $\bar{\mathcal{L}}$ corresponding to x. Denote by jac_m the canonical morphism $\mathrm{Sym}^m X \to \mathrm{Pic}^m X$. Consider the Cartesian square

$$\begin{array}{ccc} \bar{\mathcal{P}}_n & \xrightarrow{\bar{\pi}_n} & \bar{V}_n \\ {\scriptstyle \bar{\sigma}_n}\downarrow & & \downarrow{\scriptstyle \nu_n} \\ \mathrm{Sym}^m X & \xrightarrow{\mathrm{jac}_m} & \underline{\mathrm{Pic}}^m X \end{array}$$

Since $m \geq 4g > 2g - 2$, jac_m and, therefore, $\bar{\pi}_n$ are locally trivial bundles whose fibers are isomorphic to $\mathbf{P}^{m-g}$. Put $\bar{\mathcal{G}}_n = \bar{\sigma}_n{}^*\mathcal{E}^{(m)}$, $\mathcal{P}_n = \bar{\pi}_n{}^{-1}(V_n)$, $\tilde{\mathcal{P}}_n = \bar{\pi}_n{}^{-1}(\tilde{V}_n)$, $\mathcal{P}_n{}^{\Delta} = \bar{\pi}_n{}^{-1}(\Delta_n)$. The restrictions of $\bar{\mathcal{G}}_n$ to $\mathcal{P}_n$, $\tilde{\mathcal{P}}_n$ and $\mathcal{P}_n{}^{\Delta}$ will be denoted by $\mathcal{G}_n$, $\tilde{\mathcal{G}}_n$ and $\mathcal{G}_n{}^{\Delta}$. $\bar{\pi}_n$ induces morphisms $\pi_n : \mathcal{P}_n \to V_n$, $\tilde{\pi}_n : \tilde{\mathcal{P}}_n \to \tilde{V}_n$, $\pi_n{}^{\Delta} : \mathcal{P}_n{}^{\Delta} \to \Delta_n$.

Let $u \in V_n(\bar{\mathbf{F}}_q)$ and $\mathcal{A}$ be the corresponding invertible subsheaf of $\bar{\mathcal{L}}$. Put $\mathcal{B} = \bar{\mathcal{L}}/\mathcal{A}$. As $\mathcal{A}$ is maximal we have $\mathcal{B} = \mathcal{A}^{-1} \otimes \det \bar{\mathcal{L}}$. So $\pi_n{}^{-1}(u)$ is the variety of one-dimensional vector subspaces of $\mathrm{Hom}(\mathcal{A}, \mathcal{B} \otimes \bar{\Omega})$. Note that $\mathrm{Hom}(\bar{\mathcal{L}}, \mathcal{A}) = 0$ because $\deg \mathcal{A} = n < h(\mathcal{L})$. Therefore the exact sequence $0 \to \mathcal{A} \to \mathcal{L} \to \mathcal{B} \to 0$ does not split. Thus we obtain a nonzero element $\lambda \in \mathrm{Ext}(\mathcal{B}, \mathcal{A}) = \mathrm{Hom}(\mathcal{A}, \mathcal{B} \otimes \bar{\Omega})^*$. We shall denote by $\mathcal{H}_u$ the hyperplane in the projective space $\pi_n{}^{-1}(u)$ defined by the functional $\lambda : \mathrm{Hom}(\mathcal{A}, \mathcal{B} \otimes \bar{\Omega}) \to \bar{\mathbf{F}}_q$. It is clear that there exists a subscheme $\mathcal{H}_n \subset \mathcal{P}_n$ such that the morphism $\mathcal{H}_n \to V_n$ is a locally trivial bundle and its fiber over every point $u \in V_n(\bar{\mathbf{F}}_q)$ is equal to $\mathcal{H}_u$. Put $\mathcal{F}_n = t_* t^* \mathcal{G}_n$ where t is the embedding $\mathcal{H}_n \to \mathcal{P}_n$.

PROPOSITION 2.4. *Let $\mathcal{A} \subset \mathcal{L}$ be a maximal invertible subsheaf of degree n, denote by u the corresponding F_q-point of V_n. Then*

$$f(\mathcal{L}, \mathcal{A}) = q^{1+\deg \mathcal{A}} \mu(\det \mathcal{L})^{-1} \mu(\mathcal{A}) \sum_j (-1)^j \mathrm{Tr}(\mathrm{Fr}_u, R^j(\pi_n)_* \mathcal{F}_n)$$

Proof. Using Proposition 2.3 and the inequality $n < h(L) - 2g$ we get $\deg \mathcal{L} - 2 \deg \mathcal{A} > 2g$. Therefore Proposition 2.1 is applicable. It is clear that $\mathcal{H}(\mathcal{L}, \mathcal{A})$ is the fiber of $\mathcal{H}_n$ over u and the restriction of $\mathcal{F}_n$ to $\pi_n{}^{-1}(u)$ is equal to the direct image of $\mathcal{F}(\mathcal{L}, \mathcal{A})$. It remains to use the base change theorem. □

Let us explain the general scheme of the proof of the main theorem. We are going to formulate two "vanishing cycle theorems" (their proof will be given in Sections 3, 4). It follows from these theorems that the sheaves $R^j(\pi_n)_*\mathcal{F}_n$ are locally constant. Moreover, $R^j(\pi_n)_*\mathcal{F}_n$ is the restriction of a locally constant sheaf on $\bar{V}_n$. Since the fibers of the morphism $\bar{V}_n \to \underline{\mathrm{Pic}}^n X$ are connected and simply connected (they are projective spaces), $R^j(\pi_n)_*\mathcal{F}_n$ is the inverse image of a locally constant sheaf $\mathcal{M}_n{}^j$ on $\underline{\mathrm{Pic}}^n X$. Taking into account Proposition 2.4 we see that $f(\mathcal{L}, \mathcal{A}) = f(\mathcal{L}, \mathcal{A}')$ if $\mathcal{A} \simeq \mathcal{A}'$ and $\deg \mathcal{A} < h(\mathcal{L}) - 2g$. In order to show that $f(\mathcal{L}, \mathcal{A})$ does not depend on the isomorphism class of $\mathcal{A}$ we shall prove that the sheaves $\mathcal{M}_n{}^j$ are "almost constant" (the difference from the constant sheaf compensates the factor $q^{\deg \mathcal{A}} \mu(\mathcal{A})$ in the formula for $f(\mathcal{L}, \mathcal{A})$).

VANISHING CYCLE THEOREM I. *The sheaves* $R^j(\pi_n)_*\mathfrak{F}_n$ *are locally constant.*

Let O_n denote the henselization of the local ring of the generic point of Δ_n considered as a point of $\tilde{V}_n$. The strict henselization of the same ring is denoted by $\bar{O}_n$. Denote by K_n the field of fractions of O_n and by κ_n the field of rational functions on Δ_n. The residue fields of O_n and $\bar{O}_n$ are equal to κ_n and $\bar{\kappa}_n$. Denote by $\mathcal{P}_n^{\bar{O}}$ the fiber product of $\tilde{\mathcal{P}}_n$ and Spec $\bar{O}_n$ over $\tilde{V}_n$. Let $\mathcal{P}_n^{\bar{K}}$ and $\mathcal{P}_n^{\bar{\kappa}}$ denote the fibers of $\mathcal{P}_n^{\bar{O}}$ over Spec $\bar{K}_n$ and Spec $\bar{\kappa}_n$. Denote by $\mathfrak{F}_n^{\bar{K}}$ the inverse image of $\mathfrak{F}_n$ on $\mathcal{P}_n^{\bar{K}}$. Let $s:\mathcal{P}_n^{\bar{K}} \to \mathcal{P}_n^{\bar{O}}$, $i:\mathcal{P}_n^{\bar{\kappa}} \to \mathcal{P}_n^{\bar{O}}$ be natural morphisms.

The second vanishing cycle theorem describes the sheaves $i^*R^js_*\mathfrak{F}_n^{\bar{K}}$. In order to formulate it we shall introduce some notations. Denote by τ the superposition $\mathcal{P}_n^{\Delta} \to \Delta_n \overset{\varphi_{1n}}{\rightleftarrows} V_{n+1} \times X \to X$, by $D \subset \mathrm{Sym}^m X \times X$ the incidence correspondence, by σ_n^{Δ} the restriction of $\bar{\sigma}_n$ to $\mathcal{P}_n^{\Delta}$. The inverse image of D under the morphism $(\sigma_n^{\Delta}, \tau):\mathcal{P}_n^{\Delta} \to \mathrm{Sym}^m X \times X$ is denoted by $\mathcal{K}_n^{\Delta}$. Denote by $\mathfrak{F}_n^{\Delta}$ the restriction of $\mathcal{G}_n^{\Delta}$ to $\mathcal{K}_n^{\Delta}$ (this restriction is considered as a sheaf on $\mathcal{P}_n^{\Delta}$). Denote by $\mathcal{G}_n'$ the kernel of the homomorphism $\mathcal{G}_n \to \mathfrak{F}_n$. The isomorphism $V_{n+1} \times X \to \Delta_n$ induced by $\varphi_{1n}:\bar{V}_{n+1} \times X \hookrightarrow \bar{V}_n$ can be lifted canonically to an embedding $\Phi:\mathcal{P}_{n+1} \times X \to \mathcal{P}_n^{\Delta}$ (if $u \in V_{n+1}(\bar{\mathbf{F}}_q)$, $z \in X(\bar{\mathbf{F}}_q)$, $u' = \varphi_{1n}(u, z)$, $\mathfrak{A}$ is the invertible subsheaf of $\bar{\mathcal{L}}$ corresponding to u, then $(\pi_{n+1})^{-1}(u)$ is the set of one-dimensional vector subspaces of $H^0(X \otimes \bar{\mathbf{F}}_q, \mathfrak{A}^{-2} \otimes \bar{\Omega} \otimes \det \mathcal{L})$ and $\bar{\pi}_n^{-1}(u')$ is the set of one-dimensional vector subspaces of $H^0(X \otimes \bar{\mathbf{F}}_q, \mathfrak{A}^{-2} \otimes \bar{\Omega}(2z) \otimes \det \bar{\mathcal{L}})$ so that $(\pi_{n+1})^{-1}(u) \subset \bar{\pi}_n^{-1}(u'))$. Put $\mathfrak{R}_n = \Phi_*(\mathcal{G}'_{n+1}(-1) \boxtimes \det \mathcal{E})$. The inverse images of $\mathfrak{F}_n^{\Delta}$ and $\mathfrak{R}_n$ under the morphism $\mathcal{P}_n^{\bar{\kappa}} \to \mathcal{P}_n^{\Delta}$ will be denoted by $\mathfrak{F}_n^{\bar{\kappa}}$ and $\mathfrak{R}_n^{\bar{\kappa}}$.

The action of $\mathrm{Gal}(\bar{\kappa}_n/\kappa_n)$ on $\mathcal{P}_n^{\bar{\kappa}}$ can be extended canonically to an action of this group on $\mathfrak{F}_n^{\bar{\kappa}}$ and $\mathfrak{R}_n^{\bar{\kappa}}$. By means of the homomorphism $\mathrm{Gal}(\bar{K}_n/K_n) \to \mathrm{Gal}(\bar{\kappa}_n/k_n)$ we obtain an action of $\mathrm{Gal}(\bar{K}_n/K_n)$ on $\mathcal{P}_n^{\bar{\kappa}}$, $\mathfrak{F}_n^{\bar{\kappa}}$ and $\mathfrak{R}_n^{\bar{\kappa}}$. The action of $\mathrm{Gal}(\bar{K}_n/K_n)$ on $\mathcal{P}_n^{\bar{\kappa}}$ can be extended canonically to its action on the sheaves $i^*R^js_*\mathfrak{F}_n^{\bar{K}}$.

VANISHING CYCLE THEOREM II.

1) $i^*R^js_*\mathfrak{F}_n^{\bar{K}} = 0$ *if* $j > 1$.

2) *There exist isomorphisms* $i^*s_*\mathfrak{F}_n^{\bar{K}} \cong \mathfrak{F}_n^{\bar{\kappa}}$, $i^*R^1s_*\mathfrak{F}_n^{\bar{K}} \cong \mathfrak{R}_n^{\bar{\kappa}}$ *compatible with the action of* $\mathrm{Gal}(\bar{K}_n/K_n)$.

Remark. In particular, the theorem asserts that the inertia subgroup of $\mathrm{Gal}(\bar{K}_n/K_n)$ acts trivially on the sheaves $i^*R^js_*\mathfrak{F}_n^{\bar{K}}$.

We shall need the following theorem.

DELIGNE'S THEOREM. *Suppose that* $r > 4g - 4$. *Denote by* jac_r *the canonical morphism* $\mathrm{Sym}^r X \to \underline{\mathrm{Pic}}^r X$. *Then* $R^j(\mathrm{jac}_r)_* \mathcal{E}^{(r)} = 0$ *for all* j.

The proof of this theorem will be given in the appendix of the paper.

Remark. The assumption of the absolute irreducibility of the representation of $\pi_1(X \otimes \bar{\mathbf{F}}_q)$ corresponding to $\mathcal{E}$ is essential to Deligne's theorem but not to the vanishing cycle theorems.

PROPOSITION 2.5. *There exist* $\mathrm{Gal}(\bar{K}_n/K_n)$*-equivariant isomorphisms* $H^j(\mathcal{P}_n^{\bar{K}}, \mathfrak{F}_n^{\bar{K}}) \rightrightarrows H^{j-1}(\mathcal{P}_n^{\bar{\kappa}}, \mathcal{R}_n^{\bar{\kappa}})$.

Proof. Consider the Leray spectral sequence for the morphism $s: \mathcal{P}_n^{\bar{K}} \to \mathcal{P}_n^{\bar{O}}$ and the sheaf $\mathfrak{F}_n^{\bar{K}}$. By the base change theorem, $H^*(\mathcal{P}_n^{\bar{O}}, R^j s_* \mathfrak{F}_n^{\bar{K}}) = H^*(\mathcal{P}_n^{\bar{\kappa}}, i^* R^j s_* \mathfrak{F}_n^{\bar{K}})$. According to the second vanishing cycle theorem, $i^* R^j s_* \mathfrak{F}_n^{\bar{K}} = 0$ for $j > 1$, $i^* R^1 s_* \mathfrak{F}_n^{\bar{K}} = \mathcal{R}_n^{\bar{\kappa}}$, $i^* s_* \mathfrak{F}_n^{\bar{K}} = \mathfrak{F}_n^{\bar{\kappa}}$. It remains for us to prove that $H^*(\mathcal{P}_n^{\bar{\kappa}}, \mathfrak{F}_n^{\bar{\kappa}}) = 0$.

LEMMA. *Let* $\psi: \mathrm{Sym}^{m-1} X \times X \to \mathrm{Sym}^m X$, $\alpha: \mathrm{Sym}^{m-2} X \times X \hookrightarrow \mathrm{Sym}^{m-1} X \times X$ *be given by* $\psi(D, z) = D + z$, $\alpha(D', z) = (D' + z, z)$, *where* $z \in X(\bar{\mathbf{F}}_q)$, $D \in \mathrm{Sym}^{m-1} X(\bar{\mathbf{F}}_q)$ *and* $D' \in \mathrm{Sym}^{m-2} X(\bar{\mathbf{F}}_q)$ *are considered as divisors on* $X \otimes \bar{\mathbf{F}}_q$. *Then there is an exact sequence*

$$0 \to \psi^* \mathcal{E}^{(m)} \xrightarrow{\beta} \mathcal{E}^{(m-1)} \boxtimes \mathcal{E} \xrightarrow{\gamma} \alpha_*(\mathcal{E}^{(m-2)} \boxtimes \det \mathcal{E}) \to 0 \tag{8}$$

Proof. For a positive integer $i \leq m$ denote by Y_i the following submanifold of $X^m \times X$: $Y_i(\bar{\mathbf{F}}_q) \stackrel{\mathrm{def}}{=} \{(x_1, \ldots, x_m, z) \in X^{m+1}(\bar{\mathbf{F}}_q) \mid x_i = z\}$. Put $Y = \cup^m{}_{i=1} Y_i$. Denote by π the projection $Y \to X^m$. Define $h: Y \to \mathrm{Sym}^{m-1} X \times X$ by $h(x_1, \ldots, x_m, z) = (\Sigma^m{}_{i=1} x_i - z, z)$. Then the diagram

$$\begin{tikzcd} Y \arrow[r, "\pi"] \arrow[d, "h"'] & X^m \arrow[d] \\ \mathrm{Sym}^{m-1} X \times X \arrow[r, "\psi"] & \mathrm{Sym}^m X \end{tikzcd}$$

is a Cartesian square. Therefore $\psi^* \mathcal{E}^{(m)} = (h_* \pi^* \boxtimes^m \mathcal{E})^{S_m}$. For any l-adic sheaf $\mathfrak{N}$ on Y there is a canonical resolution $0 \to \mathfrak{N} \to \bigoplus_{1 \leq i \leq m} \mathfrak{N}_i \to \bigoplus_{1 \leq i_1 < i_2 < m} \mathfrak{N}_{i_1 i_2} \to \cdots$ where $\mathfrak{N}_{i_1 \cdots i_k}$ is the restriction of $\mathfrak{N}$ to $Y_{i_1} \cap \cdots \cap Y_{i_k}$ considered as a sheaf on Y. If $\mathfrak{N} = \pi^* \boxtimes^m \mathcal{E}$ this resolution yields (8). □

Denote by t the natural morphism $\mathrm{Sym}^{m-1} X \times X \to \underline{\mathrm{Pic}}^{m-1} X \times X$. It is easy to see that $H^j(\mathcal{P}_n^{\bar{\kappa}}, \mathfrak{F}_n^{\bar{\kappa}})$ is the stalk of $R^j t_* \psi^* \mathcal{E}^{(m)}$ over a $\bar{\kappa}_n$-point of

$\underline{\mathrm{Pic}}^{m-1} X \times X$. Therefore it suffices to prove that $R^j t_* \psi^* \mathcal{E}^{(m)} = 0$. In view of the lemma it is enough to show that $R^j t_*(\mathcal{E}^{(m-2)} \boxtimes \mathcal{E}) = 0$ and $R^j t_* \alpha_* (\mathcal{E}^{(m-1)} \boxtimes \det \mathcal{E}) = 0$ for all j. This follows from Deligne's theorem since $m - 2 \geq 4g - 2 > 4g - 4$. $\square$

It follows from Proposition 2.5 that the inertia subgroup of $\mathrm{Gal}(\bar{K}_n / K_n)$ acts trivially on $H^j(\mathcal{P}_n^{\bar{K}}, \mathfrak{F}_n^{\bar{K}})$. Hence $R^j(\pi_n)_* \mathfrak{F}_n$, which is a locally constant sheaf on V_n by the first vanishing cycle theorem, can be extended to a locally constant sheaf on $\bar{V}_n$ (we have used the following theorem due to M. Nagata and O. Zariski: if M is a smooth variety, Z is a closed subset of M such that $\operatorname{codim} Z \geq 2$, then any étale covering of $M \backslash Z$ can be extended to an étale covering of M). As the geometric fibers of the morphism $\bar{V}_n \to \underline{\mathrm{Pic}}^n X$ are connected and simply connected (being projective spaces), $R^j(\pi_n)_* \mathfrak{F}_n$ is the inverse image of a unique locally constant sheaf $\mathfrak{M}_n^j$ on $\underline{\mathrm{Pic}}^n X$. It follows from Proposition 2.4 that if $\mathcal{A} \subset \mathcal{L}$ is a maximal invertible subsheaf of degree n and u is the corresponding F_q-point of $\underline{\mathrm{Pic}}^n X$, then

$$(9) \qquad f(\mathcal{L}, \mathcal{A}) = q^{1+\deg \mathcal{A}} \mu(\det \mathcal{L})^{-1} \mu(\mathcal{A}) \sum_j (-1)^j \mathrm{Tr}(\mathrm{Fr}_u, \mathfrak{M}_n^j)$$

Proposition 2.6. *Suppose that* $n < h(\mathcal{L}) - 2g - 1$ *and denote by* α *the subtraction morphism* $\underline{\mathrm{Pic}}^{n+1} X \times X \to \underline{\mathrm{Pic}}^n X$. *Then* $\alpha^* \mathfrak{M}_n^j \simeq \mathfrak{M}_{n+1}^{j-2}(-1) \boxtimes \det \mathcal{E}$.

Proof. Consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec} K_n \arrow[r, "\xi"] & \operatorname{Spec} O_n \arrow[r] \arrow[rrr, bend left, "\gamma"] & \tilde{V}_n \arrow[rr] & & \underline{\mathrm{Pic}}^n X \\
 & \operatorname{Spec} \kappa_n \arrow[u, "\tau"] \arrow[r, "\beta"] & \Delta_n \arrow[r, "\delta", "\sim"'] & V_{n+1} \times X \arrow[r, "\epsilon"] & \underline{\mathrm{Pic}}^{n+1} X \times X \arrow[u, "\alpha"']
\end{tikzcd}$$

The geometric fibers of ϵ are connected. Therefore it is sufficient to prove that $\beta^* \delta^* \epsilon^* \alpha^* \mathfrak{M}_n^j \simeq \beta^* \delta^* \epsilon^* (\mathfrak{M}_{n+1}^{j-2}(-1) \boxtimes \det \mathcal{E})$. The sheaf $\xi^* \gamma^* \mathfrak{M}_n^j$ corresponds to the representation of $\mathrm{Gal}(\bar{K}_n / K_n)$ on $H^j(\mathcal{P}_n^{\bar{K}}, \mathfrak{F}_n^{\bar{K}})$, which is isomorphic to $H^{j-1}(\mathcal{P}_n^{\bar{\kappa}}, \mathcal{R}_n^{\bar{\kappa}})$ (see Proposition 2.5). The representation of $\mathrm{Gal}(\bar{K}_n / K_n)$ on $H^{j-1}(\mathcal{P}_n^{\bar{\kappa}}, \mathcal{R}_n^{\bar{\kappa}})$ is in fact a representation of $\mathrm{Gal}(\bar{\kappa}_n / \kappa_n)$. This representation of $\mathrm{Gal}(\bar{\kappa}_n / \kappa_n)$ corresponds to the sheaf $\tau^* \gamma^* \mathfrak{M}_n^j = \beta^* \delta^* \epsilon^* \alpha^* \mathfrak{M}_n^j$. Thus $\beta^* \delta^* \epsilon^* \alpha^* \mathfrak{M}_n^j \simeq \beta^* R^{j-1}(\pi_n^{\Delta})_* \mathcal{R}_n$.

It remains for us to prove that $R^{j-1}(\pi_n{}^{\Delta})_*\mathcal{R}_n = \delta^*\epsilon^*(\mathfrak{M}_{n+1}{}^{j-2}(-1) \boxtimes \det \mathcal{E})$. By the definition of $\mathcal{R}_n$ we have $R^{j-1}(\pi_n{}^{\Delta})_*\mathcal{R}_n \simeq \delta^*R^{j-1}h_*(\mathcal{G}'_{n+1}(-1) \boxtimes \det \mathcal{E})$, where $h:\mathcal{P}_{n+1} \times X \to V_{n+1} \times X$ is obtained by base change from $\pi_{n+1}:\mathcal{P}_{n+1} \to V_{n+1}$. It is clear that $R^{j-1}h_*(\mathcal{G}'_{n+1}(-1) \boxtimes \det \mathcal{E}) = (R^{j-2}(\pi_{n+1})_*\mathcal{G}'_{n+1}(-1)) \boxtimes \det \mathcal{E}$. On the other hand, $\epsilon^*(\mathfrak{M}_{n+1}{}^{j-2} \boxtimes \det \mathcal{E}) = (R^{j-2}(\pi_{n+1})_*\mathcal{F}_{n+1}) \boxtimes \det \mathcal{E}$. It remains for us to prove that $R^{j-1}(\pi_{n+1})_*\mathcal{G}'_{n+1} \simeq R^{j-2}(\pi_{n+1})_*\mathcal{F}_{n+1}$.

Recall that $\mathcal{G}'_{n+1}$ is the kernel of the epimorphism $\mathcal{G}_{n+1} \to \mathcal{F}_{n+1}$. Therefore it is sufficient to show that $R^j(\pi_{n+1})_*\mathcal{G}_{n+1} = 0$ for all j. This follows from Deligne's theorem since the morphism $\pi_{n+1}:\mathcal{P}_{n+1} \to V_{n+1}$ is obtained by base change from $\mathrm{jac}_{m-2}:\mathrm{Sym}^{m-2}X \to \underline{\mathrm{Pic}}^{m-2}X$ and $\mathcal{G}_{n+1}$ is obtained by the same base change from $\mathcal{E}^{(m-2)}$ (note that $m-2 \geq 4g-2 > 4g-4$, so Deligne's theorem is applicable). □

Put $\mathcal{C} = \det \mathcal{E}$. Denote by α_{nr} the subtraction morphism $\underline{\mathrm{Pic}}^n X \times \mathrm{Sym}^r X \to \underline{\mathrm{Pic}}^{n-r}X$.

Proposition 2.7. *Suppose that* $n < h(\mathcal{L}) - 2g$. *Then* $(\alpha_{nr})^*\mathfrak{M}^j{}_{n-r} \simeq \mathfrak{M}_n{}^{j-2r}(-r) \boxtimes \mathcal{C}^{(r)}$.

Proof. Denote by τ the natural morphism $\underline{\mathrm{Pic}}^n X \times X^r \to \underline{\mathrm{Pic}}^n X \times \mathrm{Sym}^r X$. After applying Proposition 2.6 r times we obtain an isomorphism $\tau^*(\alpha_{nr})^*\mathfrak{M}^j_{n-r} \rightrightarrows \tau^*(\mathfrak{M}_n{}^{j-2r}(-r) \boxtimes \mathcal{C}^{(r)})$. It is easy to show that this isomorphism is compatible with the action of S_r (consider the restriction of our sheaves to $\underline{\mathrm{Pic}}^n X \times T$, where $T \subset X^r$ is the diagonal). Therefore our isomorphism induces an isomorphism $(\alpha_{nr})^*\mathfrak{M}^j{}_{n-r} \rightrightarrows \mathfrak{M}_n{}^{j-2r}(-r) \boxtimes \mathcal{C}^{(r)}$. □

Proposition 2.8. *Let* $\mathcal{A}$ *and* $\mathcal{A}'$ *be maximal invertible subsheaves of* $\mathcal{L}$ *such that* $\deg \mathcal{A} < h(\mathcal{L}) - 2g$, $\deg \mathcal{A}' < h(\mathcal{L}) - 2g$. *Then* $f(\mathcal{L}, \mathcal{A}) = f(\mathcal{L}, \mathcal{A}')$.

Proof. In view of formula (9) it is sufficient to prove that if $m, n < h(\mathcal{L}) - 2g$, $u \in \mathrm{Pic}^m X$, $v \in \mathrm{Pic}^n X$, then

$$q^m \mu(u)\mathrm{Tr}(\mathrm{Fr}_u, \mathfrak{M}_m{}^{j-2m}) = q^n \mu(v)\mathrm{Tr}(\mathrm{Fr}_v, \mathfrak{M}_n{}^{j-2n}) \tag{10}$$

It is enough to consider the case when $v - u$ is the class of an effective divisor D. Put $r = n - m$. Then, according to Proposition 2.7, $(\alpha_{nr})^* \mathfrak{M}_m{}^{j-2m} \simeq \mathfrak{M}_n{}^{j-2n}(-r) \boxtimes \mathcal{C}^{(r)}$. Put $w = (v, D) \in \mathrm{Pic}^n X \times \mathrm{Sym}^r X)(\mathbf{F}_q)$. Then $\mathrm{Tr}(\mathrm{Fr}_u, \mathfrak{M}_m{}^{j-2m}) = \mathrm{Tr}(\mathrm{Fr}_w, (\alpha_{nr})^*\mathfrak{M}_m{}^{j-2m}) = \mathrm{Tr}(\mathrm{Fr}_w, \mathfrak{M}_n{}^{j-2n}(-r) \boxtimes \mathcal{C}^{(r)}) = q^r \mu(D)\mathrm{Tr}(\mathrm{Fr}_v, \mathfrak{M}_n{}^{j-2n})$, which is equivalent to (10). □

The main theorem follows from Proposition 1.1 and 2.8.

3. Proof of the first vanishing cycle theorem. The theorem is formulated in Section 2. Here we shall slightly change our notations. In this section the number n is fixed ($n < h(\mathcal{L}) - 2g$), therefore we shall write π instead of π_n, $\mathcal{F}$ instead of $\mathcal{F}_n$ and so on. Because of the geometric nature of the theorem we may replace the varieties $\mathcal{P}$ and V occurring in it by $\mathcal{P} \otimes \overline{\mathbf{F}}_q$ and $V \otimes \overline{\mathbf{F}}_q$ ($\mathcal{F}$ is replaced by its inverse image on $\mathcal{P} \otimes \overline{\mathbf{F}}_q$). While proving the theorem we deal only with schemes over $\overline{\mathbf{F}}_q$. Therefore we shall write X, $\mathcal{P}$, V, etc. instead of $X \otimes \overline{\mathbf{F}}_q$, $\mathcal{P}_n \otimes \overline{\mathbf{F}}_q$, $V_n \otimes \overline{\mathbf{F}}_q$ etc. It is clear that, while proving the theorem, we may suppose that the sheaf $\mathcal{E}$, which is implicit in its formulation, is not an l-adic sheaf but a locally constant Λ-module sheaf whose stalks are equal to Λ^2, where $\Lambda = O_E/I$, O_E being the ring of integers of E, I a nonzero ideal of O_E.

We shall prove more than the local constancy of $R^j \pi_* \mathcal{F}$. Namely, it will be proved that π is locally acyclic with respect to $\mathcal{F}$ (for the definition of local acyclicity see [4], p. 242). While proving the local acyclicity we may suppose $\mathcal{E}$ to be a constant sheaf. In this case $\mathcal{E}^{(m)} = \bigoplus^m{}_{\alpha=0} (t^\alpha)_* \underline{\Lambda}$, where t^α is the natural morphism $\mathrm{Sym}^\alpha X \times \mathrm{Sym}^{m-\alpha} X \to \mathrm{Sym}^m X$. For an integer $\alpha \in [0, m]$ denote by $\mathcal{H}^\alpha$ the fiber product of $\mathcal{H}$ and $\mathrm{Sym}^\alpha X \times \mathrm{Sym}^{m-\alpha} X$ over $\mathrm{Sym}^m X$, and by τ^α the natural morphism $\mathcal{H}^\alpha \to \mathcal{P}$. Put $\pi^\alpha = \pi \circ \tau^\alpha$. As $\mathcal{E}^{(m)} = \bigoplus^m{}_{\alpha=0} (t^\alpha)_* \underline{\Lambda}$ we have $\mathcal{F} = \bigoplus^m{}_{\alpha=0} (\tau^\alpha)_* \underline{\Lambda}$. Hence, to prove the local acyclicity of π with respect to $\mathcal{F}$ it will be enough to show that for every $\alpha \in [0, m]$ the morphism $\pi^\alpha : \mathcal{H}^\alpha \to V$ is locally acyclic (with respect to the constant sheaf). Because of the symmetry between α and $m - \alpha$ we may suppose that $\alpha \geq \frac{1}{2} \cdot m \geq 2g$.

It is easy to see that the fibers of the natural morphism $\gamma : \mathcal{H}^\alpha \to V \times \mathrm{Sym}^{m-\alpha} X$ are projective spaces. To be precise: let u be a closed point of V, D an effective divisor on X of degree $m - \alpha$; $\mathcal{A}$ the invertible subsheaf of $\mathcal{L}$ corresponding to u, $\mathcal{B} \overset{\text{def}}{=} \mathcal{L}/\mathcal{A}$, $W(u, D) \overset{\text{def}}{=} H^0(X, \mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega(-D))^* = \mathrm{Ext}(\mathcal{B}(-D), \mathcal{A})$, $s(u, D) \in W(u, D)$ the image of the canonical element of $\mathrm{Ext}(\mathcal{B}, \mathcal{A})$, $C_{u,D}$ the quotient of $W(u, D)$ by the subspace generated by $s(u, D)$; then $\gamma^{-1}(u, D) = \mathbf{P}(C_{u,D})$. As $\deg(\mathcal{A}^{-1} \otimes \mathcal{B} \otimes \Omega(-D)) = \alpha \geq 2g$, $W(u, D)$ is the fiber at (u, D) of a naturally defined locally free sheaf $\mathcal{W}$ on $V \times \mathrm{Sym}^{m-\alpha} X$ of dimension $\alpha - g + 1$. It is clear that $s(u, D)$ comes from a section s of this sheaf. Denote by $\mathcal{C}$ the quotient of $\mathcal{W}$ by the subsheaf generated by s. Then $\mathcal{H}^\alpha$ and $\mathbf{P}(\mathcal{C})$ are isomorphic as schemes over $V \times \mathrm{Sym}^{m-\alpha} X$.

Denote by Y the subscheme of $V \times \mathrm{Sym}^{m-\alpha} X$ defined by the equation

$s = 0$. If $Y = \emptyset$, the morphism π^α would have been smooth and, hence, locally acyclic. In fact $Y \neq \emptyset$ (for some α), but the following proposition shows that Y is a trivial bundle over V. This saves the situation.

PROPOSITION 3.1. *Y is V-isomorphic to $V \times \bar{V}_j$, where $j = n + \alpha - 2g + 2$.*

Proof. We shall only construct a bijection $Y(\bar{\mathbf{F}}_q) \rightrightarrows V(\bar{\mathbf{F}}_q) \times \bar{V}_j(\bar{\mathbf{F}}_q)$. The construction of the bijection $Y(A) \rightrightarrows V(A) \times \bar{V}_j(A)$ for any $\bar{\mathbf{F}}_q$-algebra A is quite similar.

Let $u \in V(\bar{\mathbf{F}}_q)$, $\mathcal{A}$ be the corresponding invertible subsheaf of $\mathcal{L}$, $\mathcal{B} \overset{\text{def}}{=} \mathcal{L}/\mathcal{A}$, c the canonical element of $\operatorname{Ext}(\mathcal{B}, \mathcal{A})$. The fiber of $Y(\bar{\mathbf{F}}_q)$ over u is the set of effective divisors D on X of degree $m - \alpha$ such that the image of c in $\operatorname{Ext}(\mathcal{B}(-D), \mathcal{A})$ is equal to 0, i.e. the embedding $\mathcal{B}(-D) \to \mathcal{B}$ lifts to a morphism $\mathcal{B}(-D) \to \mathcal{L}$. The lifting is unique since $\deg \mathcal{B}(-D) = \deg \mathcal{L} - n - m + \alpha = n + \alpha - 2g + 2 > n = \deg \mathcal{A}$. The image of $\mathcal{B}(-D)$ in $\mathcal{L}$ is an invertible subsheaf of degree $n + \alpha - 2g + 2 = j$. Thus we have constructed a mapping $Y(\bar{\mathbf{F}}_q) \to V(\bar{\mathbf{F}}_q) \times \bar{V}_j(\bar{\mathbf{F}}_q)$. Its bijectiveness is clear. □

Remark. For every $u \in V(\bar{\mathbf{F}}_q)$ the embedding $V \times \bar{V}_j \rightrightarrows Y \hookrightarrow V \times \operatorname{Sym}^{m-\alpha} X$ induces an embedding $i_u : \bar{V}_j \to Sym^{m-\alpha} X$. Note that i_u does depend on u.

Fix $\bar{\mathbf{F}}_q$-points $u \in V$, $z \in \operatorname{Sym}^{m-\alpha} X$. Put $x = (u, z) \in V \times \operatorname{Sym}^{m-\alpha} X$. Denote by pr_u the composition $V \times \operatorname{Sym}^{m-\alpha} X \to \operatorname{Sym}^{m-\alpha} X \rightrightarrows u \times \operatorname{Sym}^{m-\alpha} X \hookrightarrow V \times \operatorname{Sym}^{m-\alpha} X$. Put $\mathcal{C}' = (\mathrm{pr}_u)^* \mathcal{C}$. Denote by O the henselization of the local ring of x. The inverse images of $\mathcal{C}$ and $\mathcal{C}'$ on Spec O will be denoted by $\mathcal{C}_O$ and $\mathcal{C}_O'$.

PROPOSITION 3.2. *There exists an automorphism σ of* Spec O *over V such that $\sigma^* \mathcal{C}_O \simeq \mathcal{C}_O'$.*

Proof. If $x \in Y$ then $\mathcal{C}$ and $\mathcal{C}'$ are locally free at x, so we may put $\sigma = \mathrm{id}$.

Now suppose that $x \in Y$. Put $Y' = (\mathrm{pr}_u)^{-1}(Y)$. Denote by Y_O and Y_O' the inverse images of Y and Y' under the morphism Spec $O \to V \times \operatorname{Sym}^{m-\alpha} X$.

LEMMA. *There exists an automorphism σ of* Spec O *over V such that $\sigma^{-1}(Y_O) = Y_O'$.*

Proof. Denote by S the fiber of Spec O over u. Then $S \cap Y_O = S \cap Y_O'$. It follows from Proposition 3.1 that the identity automorphism of $S \cap$

Y_O' can be extended to a V-isomorphism $\psi: Y_O' \xrightarrow{\sim} Y_O$. Since O is the henselization of a local ring of a scheme which is smooth over V, ψ may be extended to a V-morphism $\sigma: \operatorname{Spec} O \to \operatorname{Spec} O$ such that the restriction of σ to S is equal to the identity automorphism. σ is an isomorphism because $\sigma|_S = \mathrm{id}$. □

Let σ be the automorphism of Spec O constructed in the lemma. Put $r = \alpha - g + 1$, $M = H^0(\operatorname{Spec} O, \sigma^*\mathcal{C}_O)$, $M' = H^0(\operatorname{Spec} O, \mathcal{C}_O')$. Then $M \simeq O^r/Oh$, $M' \simeq O^r/Oh'$, where $h = (h_1, \ldots, h_r) \in O^r$, $h' = (h_1', \ldots, h_r') \in O^r$, the equations $h_1 = 0, \ldots, h_r = 0$ define the subscheme $\sigma^{-1}(Y_O) \subset \operatorname{Spec} O$ and the equations $h_1' = 0, \ldots, h_r' = 0$ define Y_O'. Since $\sigma^{-1}(Y_O) = Y_O'$ the elements $h_1, \ldots, h_r$ generate the same ideal of O as $h_1', \ldots, h_r'$. It is easy to deduce from this fact that there exists an automorphism γ of O^r such that $h' = \gamma(h)$. Therefore $M \simeq M'$, whence it follows that $\sigma^*\mathcal{C}_O \simeq \mathcal{C}_O'$. □

Since $\mathbf{P}(\mathcal{C}')$ is V-isomorphic to $V \times U$ for some scheme U over $\overline{\mathbf{F}}_q$, the morphism $\mathbf{P}(\mathcal{C}_O') \to V$ is locally acyclic ([4], p. 243, Corollary 2.16). It follows from this fact and Proposition 3.2 that the morphism $\mathbf{P}(\mathcal{C}) \to V$ is locally acyclic at every point of $\mathbf{P}(\mathcal{C})$ lying over u. As u may be arbitrary the morphism $\pi^\alpha: \mathcal{K}^\alpha = \mathbf{P}(\mathcal{C}) \to V$ is locally acyclic. We have shown at the beginning of the paragraph that the first vanishing cycle theorem follows from the local acyclicity of π^α.

4. Proof of the second vanishing cycle theorem.

The theorem is formulated in Section 2. The notations used in this paragraph are the same as in Section 2 (and differ from those of Section 3).

PROPOSITION 4.1. *There is a closed subscheme* $\tilde{\mathcal{K}}_n \subset \tilde{\mathcal{P}}_n$ *such that*

1) *the morphism* $\tilde{\mathcal{K}}_n \to \tilde{V}_n$ *is a locally trivial bundle with* $\mathbf{P}^{m-g-1}$ *as a fiber*;

2) $\tilde{\mathcal{K}}_n \cap \mathcal{P}_n = \mathcal{K}_n$, $\tilde{\mathcal{K}}_n \cap \mathcal{P}_n^\Delta = \mathcal{K}_n^\Delta$

Proof. Denote by $\tilde{\mathcal{L}}$ the inverse image of $\mathcal{L}$ on $X \times \tilde{V}_n$. Let $\mathcal{A} \subset \tilde{\mathcal{L}}$ be the universal invertible subsheaf. Put $\mathcal{B} = \mathcal{A}^{-1} \otimes \det \tilde{\mathcal{L}}$, $\mathcal{C} = \tilde{\mathcal{L}}/\mathcal{A}$. The superposition $\mathcal{A} \otimes \tilde{\mathcal{L}} \hookrightarrow \tilde{\mathcal{L}} \otimes \tilde{\mathcal{L}} \to \det \tilde{\mathcal{L}}$ is trivial on $\mathcal{A} \otimes \mathcal{A}$ and, therefore, induces a morphism $\mathcal{A} \otimes \mathcal{C} \to \det \tilde{\mathcal{L}}$, i.e. a morphism $\alpha: \mathcal{C} \to \mathcal{B}$. Note that α is an isomorphism over the open subset $X \times V_n \subset X \times \tilde{V}_n$. If $x \in \Delta_n(\overline{\mathbf{F}}_q)$, $\mathcal{B}_x$ and $\mathcal{C}_x$ are the restrictions of $\mathcal{B}$ and $\mathcal{C}$ to $X \times x$, $\alpha_x: \mathcal{C}_x \to \mathcal{B}_x$ is induced by α, then Ker α_x and Coker α_x are sky-scraper sheaves concentrated in one point, the stalks of Ker α_x and Coker α_x at this point being one-dimensional.

Denote by pr the projection $X \times \tilde{V}_n \to \tilde{V}_n$ and by $\tilde{\Omega}$ the inverse image of Ω on $X \times \tilde{V}_n$. The exact sequence $0 \to \mathcal{A} \to \tilde{\mathcal{L}} \to \mathcal{C} \to 0$ induces a morphism $f: \mathrm{pr}_*(\mathcal{A}^{-1} \otimes \mathcal{C} \otimes \tilde{\Omega}) \to R^1\mathrm{pr}_*\tilde{\Omega} = \mathcal{O}_{\tilde{V}_n}$. Put $\mathcal{K} = \mathrm{Ker} f$. If $x \in \tilde{V}_n$, $\mathcal{A}_x$, $\mathcal{C}_x$ and $\tilde{\Omega}_x$ are the restrictions of $\mathcal{A}$, $\mathcal{C}$ and $\tilde{\Omega}$ to $X \times x$, then $H^1(X \times x, \mathcal{A}_x{}^{-1} \otimes \mathcal{C}_x \otimes \tilde{\Omega}_x) = 0$ (because the quotient of $\mathcal{A}_x{}^{-1} \otimes \mathcal{C}_x \otimes \tilde{\Omega}_x$ by its torsion subsheaf has a degree not less than $\deg \mathcal{L} - 2n - 1 + 2g - 2 = m - 1 \geq 4g - 1 > 2g - 2$. Therefore $\mathrm{pr}_*(\mathcal{A}^{-1} \otimes \mathcal{C} \otimes \tilde{\Omega})$ is a locally free sheaf whose fiber at a point x is equal to $H^0(X \times x, \mathcal{A}_x{}^{-1} \otimes \mathcal{C}_x \otimes \tilde{\Omega}_x)$. f is surjective because in the opposite case the exact sequence $0 \to \mathcal{A} \to \tilde{\mathcal{L}} \to \mathcal{C} \to 0$ would have split over some point $x \in \tilde{V}_n$, which is impossible since $\deg \mathcal{A}_x = n < h(\mathcal{L})$. As f is surjective and $\mathrm{pr}_*(\mathcal{A}^{-1} \otimes \mathcal{C} \otimes \tilde{\Omega})$ is locally free, $\mathcal{K}$ is also locally free. Put $\mathcal{M} = \mathrm{pr}_*(\mathcal{A}^{-1} \otimes \mathcal{B} \otimes \tilde{\Omega})$. $\mathcal{M}$ is locally free and the fiber of $\mathcal{M}$ at a point $x \in \tilde{V}_n$ is equal to $H^0(X \times x, \mathcal{A}_x{}^{-1} \otimes \mathcal{B}_x \otimes \tilde{\Omega}_x)$. It is clear that $\tilde{\mathcal{P}}_n = \mathbf{P}(\mathcal{M}^*)$. The morphism $\alpha: \mathcal{C} \to \mathcal{B}$ induces a morphism $h: \mathcal{K} \to \mathcal{M}$. It is easy to see that h induces injective mappings of the fibers. Therefore h induces an isomorphism between $\mathbf{P}(\mathcal{K}^*)$ and a closed subscheme $\tilde{\mathcal{K}}_n \subset \tilde{\mathcal{P}}_n$. The subscheme $\tilde{\mathcal{K}}_n$ has the desired properties. □

Let us slightly change our notations. Recall that the support of $\mathcal{F}_n$ is equal to $\mathcal{K}_n$. Henceforth we shall consider $\mathcal{F}_n$ as a sheaf on $\mathcal{K}_n$ (not on $\mathcal{P}_n$). Accordingly, $\mathcal{F}_n{}^{\bar{\kappa}}$ and $\mathcal{R}_n{}^{\bar{\kappa}}$ will be considered as sheaves on $\mathcal{K}_n{}^{\bar{\kappa}}$, and $\mathcal{F}_n{}^{\bar{K}}$ as that on $\mathcal{K}_n{}^{\bar{K}}$. Denote by $\mathcal{K}_n{}^{\bar{O}}$ the fiber product of $\tilde{\mathcal{K}}_n$ and Spec $\bar{O}_n$ over $\tilde{V}_n$. In this paragraph s will denote the morphism $\mathcal{K}_n{}^{\bar{K}} \to \mathcal{K}_n{}^{\bar{O}}$ (not $\mathcal{P}_n{}^{\bar{K}} \to \mathcal{P}_n{}^{\bar{O}}$) and i will denote the morphism $\mathcal{K}_n{}^{\bar{\kappa}} \to \mathcal{K}_n{}^{\bar{O}}$ (not $\mathcal{P}_n{}^{\bar{\kappa}} \to \mathcal{P}_n{}^{\bar{O}}$).

We shall formulate and then prove a more precise statement than the second vanishing cycle theorem. To do this we need some notations. Denote by $\tilde{K}_n$ the field of fractions of $\bar{O}_n$. Denote by $\mathcal{K}_n{}^{\tilde{K}}$ the fiber of $\mathcal{K}_n{}^{\bar{O}}$ over Spec $\tilde{K}_n$ and by $\underline{s}$ the embedding $\mathcal{K}_n{}^{\tilde{K}} \to \mathcal{K}_n{}^{\bar{O}}$. Denote by $\mathcal{F}_n{}^{\tilde{K}}$ and $\mathcal{F}_n{}^{\bar{O}}$ the inverse images of $\tilde{\mathcal{G}}_n$ under the morphisms $\mathcal{K}_n{}^{\tilde{K}} \to \tilde{\mathcal{P}}_n$, $\mathcal{K}_n{}^{\bar{O}} \to \tilde{\mathcal{P}}_n$.

Put $Z_n = \Phi(\mathcal{K}_{n+1} \times X)$, where $\Phi: \mathcal{P}_{n+1} \times X \hookrightarrow \mathcal{P}_n{}^{\Delta}$ is the embedding defined in Section 2 (before the formulation of the second vanishing cycle theorem). It is clear that $Z_n \subset \mathcal{K}_n{}^{\Delta}$. Denote by $Z_n{}^{\bar{\kappa}}$ the fiber of Z_n over Spec $\bar{\kappa}_n$ and by t the natural embedding $\mathcal{K}_n{}^{\bar{\kappa}} \backslash Z_n{}^{\bar{\kappa}} \to \mathcal{K}_n{}^{\bar{\kappa}}$. Denote by h the embedding of the generic point of $\mathcal{K}_n{}^{\bar{\kappa}}$ into the whole scheme $\mathcal{K}_n{}^{\bar{\kappa}}$.

Recall that in Section 2 $\mathcal{K}_n{}^{\Delta}$ was defined as the inverse image of the incidence correspondence $D \subset \mathrm{Sym}^m X \times X$ under a morphism

$\mathcal{P}_n{}^{\Delta} \to \mathrm{Sym}^m X \times X$. In fact $D = \mathrm{Sym}^{m-1} X \times X$, so we obtain a morphism $\mathcal{H}_n{}^{\Delta} \to \mathrm{Sym}^{m-1} X \times X$. Denote by f the corresponding morphism $\mathcal{H}_n{}^{\bar{K}} \to \mathrm{Sym}^{m-1} X \times X$.

Let ψ, α, β denote the same objects as in the exact sequence (8) (recall that ψ is a morphism $\mathrm{Sym}^{m-1} X \times X \to \mathrm{Sym}^m X$, α is a morphism $\mathrm{Sym}^{m-2} X \times X \hookrightarrow \mathrm{Sym}^{m-1} X \times X$, β is a morphism $\psi^*\mathcal{E}^{(m)} \to \mathcal{E}^{(m-1)} \boxtimes \mathcal{E}$). It is clear that $\mathfrak{F}_n{}^{\bar{K}} = f^*\psi^*\mathcal{E}^{(m)}$, $\mathcal{R}_n{}^{\bar{K}} = t_! t^* f^* \alpha_*(\mathcal{E}^{(m-2)}(-1) \boxtimes \det \mathcal{E})$.

Note that if $\varphi: \mathcal{K} \to \mathfrak{M}$ is a morphism of sheaves on $\mathcal{H}_n{}^{\bar{K}}$ inducing an isomorphism of their generic stalks, then there is a natural morphism $\mathfrak{M} \to h_* h^* \mathcal{K}$. In particular, $\beta: \psi^*\mathcal{E}^{(m)} \to \mathcal{E}^{(m-1)} \boxtimes \mathcal{E}$ induces an isomorphism $h^* f^* \psi^* \mathcal{E}^{(m)} \to h^* f^*(\mathcal{E}^{(m-1)} \boxtimes \mathcal{E})$ and, therefore, a morphism $\bar{\beta}: f^*(\mathcal{E}^{(m-1)} \boxtimes \mathcal{E}) \to h_* h^* f^* \psi^* \mathcal{E}^{(m)} = h_* h^* \mathfrak{F}_n{}^{\bar{K}}$.

Besides, note that if $\mathfrak{N}$ is an l-adic sheaf on $\mathcal{H}_n{}^{K}$, $\tilde{\mathfrak{N}}$ and $\bar{\mathfrak{N}}$ are its inverse images on $\mathcal{H}_n{}^{\tilde{K}}$ and $\mathcal{H}_n{}^{\bar{K}}$, then there is a canonical exact sequence

$$0 \to H^1(\mathrm{Gal}(\bar{\mathbf{K}}_n/\tilde{K}_n), i^*s_*\bar{\mathfrak{N}})$$

$$\to i^*R^1\underline{s}_*\tilde{\mathfrak{N}} \to H^0(\mathrm{Gal}(\bar{K}_n/\tilde{K}_n), i^*R^1 s_*\bar{\mathfrak{N}}) \to 0$$

In particular, if the action of $\mathrm{Gal}(\bar{K}_n/\tilde{K}_n)$ on $i^*R^j s_*\bar{\mathfrak{N}}$ is trivial for $j = 0, 1$, then we obtain a canonical exact sequence

$$0 \to i^*s_*\bar{\mathfrak{N}}(-1) \to i^*R^1\underline{s}_*\tilde{\mathfrak{N}} \to i^*R^1 s_*\bar{\mathfrak{N}} \to 0$$

Finally, note that the isomorphism $s^*\mathfrak{F}_n{}^{\bar{O}} \xrightarrow{\sim} \mathfrak{F}_n{}^{\bar{K}}$ induces a morphism $\mathfrak{F}_n{}^{\bar{O}} \to s_*\mathfrak{F}_n{}^{\bar{K}}$. Thus we obtain a canonical morphism $\mathfrak{F}_n{}^{\bar{\kappa}} \to i^*s_*\mathfrak{F}_n{}^{\bar{K}}$.

Vanishing cycle theorem II (refined version). 1. *The canonical morphism* $\mathfrak{F}_n{}^{\bar{\kappa}} \to i^*s_*\mathfrak{F}_n{}^{\bar{K}}$ *is an isomorphism.*

2) $i^*R^j s_*\mathfrak{F}_n{}^{\bar{K}} = 0$ *for* $j > 1$.

3) *The restriction of* $R^1 s_*\mathfrak{F}_n{}^{\bar{K}}$ *to* $Z_n{}^{\bar{\kappa}}$ *is equal to zero.*

4) *The action of* $\mathrm{Gal}(\bar{K}_n/\tilde{K}_n)$ *on* $R^1 s_*\mathfrak{F}_n{}^{\bar{K}}$ *is trivial.*

5) *The canonical morphism* $i^*s_*\mathfrak{F}_n{}^{\bar{K}}(-1) \to i^*R^1\underline{s}_*\mathfrak{F}_n{}^{\tilde{K}}$ *induces an isomorphism* $h^*i^*s_*\mathfrak{F}_n{}^{\bar{K}}(-1) \xrightarrow{\sim} h^*i^*R^1\underline{s}_*\mathfrak{F}_n{}^{\tilde{K}}$. *The resulting morphism* $i^*R^1\underline{s}_*\mathfrak{F}_n{}^{\tilde{K}} \to h_*h^*i^*s_*\mathfrak{F}_n{}^{\bar{K}}(-1)$ *induces an isomorphism* $t^*i^*R^1\underline{s}_*\mathfrak{F}_n{}^{\tilde{K}} \xrightarrow{\sim} t^*h_*h^*i^*s_*\mathfrak{F}_n{}^{\bar{K}}(-1)$.

6) *The morphism* $\bar{\beta}: f^*(\mathcal{E}^{(m-1)} \boxtimes \mathcal{E}) \to h_*h^*\mathfrak{F}_n{}^{\bar{\kappa}}$ *is an isomorphism.*

The second vanishing cycle theorem in its former formulation results from the theorem just formulated. Indeed, from the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & t^*i^*s_*\mathfrak{F}_n^{\bar K}(-1) \arrow[r] & t^*i^*R^1\underline{s}_*\mathfrak{F}_n^{\tilde K} \arrow[r] \arrow[d] & t^*i^*R^1 s_*\mathfrak{F}_n^{\bar K} \arrow[r] & 0 \\
 & & t^*h_*h^*i^*s_*\mathfrak{F}_n^{\bar\kappa}(-1) & & \\
 & & t^*h_*h^*\mathfrak{F}_n^{\bar\kappa}(-1) \arrow[u] & & \\
0 \arrow[r] & t^*\mathfrak{F}_n^{\bar\kappa}(-1) \arrow[r] \arrow[uuu] & t^*f^*(\mathcal{E}^{(m-1)}(-1) \boxtimes \mathcal{E}) \arrow[r] \arrow[u] & t^*f^*(\mathcal{E}^{(m-2)}(-1) \boxtimes \det \mathcal{E}) \arrow[r] & 0
\end{tikzcd}$$

having exact rows (the lower row being the inverse image of the exact sequence (8)) it follows that $t^*i^*R^1s_*\mathfrak{F}_n^{\bar K} \simeq t^*f^*(\mathcal{E}^{(m-2)}(-1) \boxtimes \det \mathcal{E})$. By statement 3) this isomorphism induces an isomorphism $i^*R^1s_*\mathfrak{F}_n^{\bar K} \simeq t_!t^*f^*(\mathcal{E}^{(m-2)}(-1) \boxtimes \det \mathcal{E}) = \mathcal{R}_n^{\bar\kappa}$.

When proving the theorem we may suppose $\mathcal{E}$ to be not a locally constant l-adic sheaf but the constant sheaf $\underline{\Lambda}^2$, where Λ denotes the quotient of the ring of integers of E by a nonzero ideal. For any integer $r \in [0, m]$ we denote by $\tilde{\mathcal{H}}_{n,r}$ the fiber product of $\tilde{\mathcal{H}}_n$ and $\mathrm{Sym}^r X \times \mathrm{Sym}^{m-r}X$ over $\mathrm{Sym}^m X$ and by $\mathcal{H}^{\bar O}_{n,r}$ the fiber product of $\tilde{\mathcal{H}}_{n,r}$ and $\mathrm{Spec}\, O_n$ over $\tilde V_n$. The fibers of $\mathcal{H}^{\bar O}_{n,r}$ over $\mathrm{Spec}\, \bar K_n$, $\mathrm{Spec}\, \tilde K_n$ and $\mathrm{Spec}\, \bar\kappa_n$ will be denoted by $\mathcal{H}^{\bar K}_{n,r}$, $\mathcal{H}^{\tilde K}_{n,r}$, $\mathcal{H}^{\bar\kappa}_{n,r}$. Let $i_r : \mathcal{H}^{\bar\kappa}_{n,r} \hookrightarrow \mathcal{H}^{\bar O}_{n,r}$, $\underline{s}^r : \mathcal{H}^{\tilde K}_{n,r} \to \mathcal{H}^{\bar O}_{n,r}$, $s^r : \mathcal{H}^{\bar K}_{n,r} \to \mathcal{H}^{\bar O}_{n,r}$, $\tau^r : \mathcal{H}^{\bar\kappa}_{n,r} \to \mathcal{H}_n^{\bar\kappa}$ be the natural morphisms. Denote by h^r the embedding into $\mathcal{H}^{\bar\kappa}_{n,r}$ of the fiber of $\mathcal{H}^{\bar\kappa}_{n,r}$ over the generic point of $\mathcal{H}_n^{\bar\kappa}$.

Denote by $Z^{\bar\kappa}_{n,r}$ the inverse image of $Z_n^{\bar\kappa}$ in $\mathcal{H}^{\bar\kappa}_{n,r}$. Recall that $\mathcal{P}_n^{\bar\kappa}(\bar\kappa_n)$ is the set of effective divisors on $X \otimes \bar\kappa_n$ belonging to a certain divisor class of degree m, $\mathcal{H}_n^{\bar\kappa}(\bar\kappa_n)$ is the set of divisors from $\mathcal{P}_n^{\bar\kappa}(\bar\kappa_n)$ containing a fixed point $\zeta \in X(\bar\kappa_n)$. Therefore $\mathcal{H}^{\bar\kappa}_{n,r}(\bar\kappa_n)$ is the set of pairs (D, D'), where D and D' are effective divisors on $X \otimes \bar\kappa_n$, $\deg D = r$, $\deg D' = m - r$, $\zeta \in D + D'$, $D + D' \in \mathcal{P}_n^{\bar\kappa}(\bar\kappa_n)$. It is clear that $\mathcal{H}^{\bar\kappa}_{n,r}$ is the union of its closed subsets B_r and P_r, where $B_r(\bar\kappa_n) = \{(D, D') \in \mathcal{H}^{\bar\kappa}_{n,r}(\bar\kappa_n) \mid \zeta \in D\}$, $P_r(\bar\kappa_n) = \{(D, D') \in \mathcal{H}^{\bar\kappa}_{n,r}(\bar\kappa_n) \mid \zeta \in D'\}$. Let $\gamma^r : P_r \to \mathcal{H}^{\bar\kappa}_{n,r}$, $\delta^r : B_r \to \mathcal{H}^{\bar\kappa}_{n,r}$, $t_r : \mathcal{H}^{\bar\kappa}_{n,r} \setminus Z^{\bar\kappa}_{n,r} \to \mathcal{H}^{\bar\kappa}_{n,r}$ be the natural embeddings.

We have canonical isomorphisms

$$i^*R^j s_*\mathfrak{F}_n^{\bar K} \simeq \oplus_{r=0}^m (\tau^r)_*(i_r)^*R^j(s^r)_*\underline{\Lambda},$$

$$i^*R^j \underline{s}_*\mathfrak{F}_n^{\tilde K} \simeq \oplus_{r=0}^m (\tau^r)_*(i_r)^*R^j(\underline{s}^r)_*\underline{\Lambda},$$

$$\mathfrak{F}_n^{\bar\kappa} \simeq \oplus_{r=0}^m (\tau^r)_*\underline{\Lambda},$$

$f^*(\mathcal{E}^{(m-1)} \boxtimes \mathcal{E}) = \oplus_{r=0}^{m}(\tau^r)_*((\gamma^r)_*\underline{\Lambda} \oplus (\delta^r)_*\underline{\Lambda})$. Thus we have reduced the second vanishing cycle theorem to the following one.

THEOREM A. *Let $r \in \mathbf{Z}$, $0 \leq r \leq m$.*

1) *The canonical isomorphism $\underline{\Lambda} \to (i_r)^*(s^r)_*\underline{\Lambda}$ is an isomorphism.*

2) $(i_r)^*R^j(s^r)_*\underline{\Lambda} = 0$ *for* $j > 1$.

3) *The restriction of $R^1(s^r)_*\underline{\Lambda}$ to $Z_{n,r}^{\bar{\kappa}}$ is equal to zero.*

4) *The action of* $\mathrm{Gal}(\bar{K}_n/\tilde{K}_n)$ *on $R^1(s^r)_*\underline{\Lambda}$ is trivial.*

5) *There is an isomorphism $(t_r)^*(i_r)^*R^1(\underline{s}^r)_*\underline{\Lambda} \xrightarrow{\sim} (t_r)^*(\gamma^r)_*\underline{\Lambda}(-1) \oplus (t_r)^*(\delta^r)_*\underline{\Lambda}(-1)$ such that the following diagram is commutative*:

$$\begin{array}{ccc} (t_r)^*(i_r)^*R^1(\underline{s}^r)_*\underline{\Lambda} & \xrightarrow{\sim} & (t_r)^*(\gamma^r)_*\underline{\Lambda}(-1) \oplus (t_r)^*(\delta^r)_*\underline{\Lambda}(-1) \\ \uparrow & & \uparrow \\ (t_r)^*(i_r)^*(s^r)_*\underline{\Lambda}(-1) & \xleftarrow{\sim} & (t_r)^*\underline{\Lambda}(-1) \end{array}$$

6) *The natural morphism $(\gamma^r)_*\underline{\Lambda} \oplus (\delta^r)_*\underline{\Lambda} \to (h^r)_*\underline{\Lambda}$ is a isomorphism.*

Proof of statement 6). It suffices to show that a) $\tau^r(B_r \cap P_r) \neq \mathcal{H}_n^{\bar{\kappa}}$, b) B_r and P_r are smooth varieties, whose irreducible components have the same dimension as $\mathcal{H}_n^{\bar{\kappa}}$. Since $\tau^r(B_r \cap P_r)$ is the set of divisors from $P_n^{\bar{\kappa}}$ containing ζ with multiplicity greater than one, we have $\tau^r(B_r \cap P_r) \neq \mathcal{H}_n^{\bar{\kappa}}$. It is clear that B_r is the fiber of the natural morphism $\varphi: \mathrm{Sym}^{r-1}X \times \mathrm{Sym}^{m-r}X \to \underline{\mathrm{Pic}}^{m-1}X$ over a $\bar{\kappa}_n$-point of $\underline{\mathrm{Pic}}^{m-1}X$. As $m \geq 4g$ we have $r-1 > 2g-2$ or $m-r > 2g-2$. Therefore either $\mathrm{jac}_{r-1}: \mathrm{Sym}^{r-1}X \to \underline{\mathrm{Pic}}^{r-1}X$, or $\mathrm{jac}_{m-r}: \mathrm{Sym}^{m-r}X \to \underline{\mathrm{Pic}}^{m-r}X$ or both morphisms are smooth. Hence φ is smooth. So B_r is a smooth variety whose irreducible components have dimension $m-1-g = \dim \mathcal{H}_n^{\bar{\kappa}}$. P_r has the same properties because $P_r \simeq B_{m-r}$.

Statements 1)–5) of Theorem A will be proved in the remaining part of the section. Due to the symmetry between r and $m-r$ we may suppose that $r \leq \frac{1}{2} \cdot m$.

To calculate $(i_r)^*R^j(s^r)_*\underline{\Lambda}$ and $(t_r)^*(i_r)^*R^j(\underline{s}^r)_*\underline{\Lambda}$ we shall construct a kind of singularity resolution of $\tilde{\mathcal{H}}_{n,r}$. Let $\mathcal{C}$ denote the same sheaf as in the proof of Proposition 4.1. Denote by $\mathrm{Quot}_r(C)$ the scheme over $\tilde{V}_n$ representing the following functor: if S is a scheme over $\tilde{V}_n$, $\mathrm{pr}: X \times S \to S$ the projection, $\mathcal{C}_S$ the inverse image of $\mathcal{C}$ on $X \times S$, then $\mathrm{Mor}_{\tilde{V}_n}(S, \mathrm{Quot}_r(\mathcal{C}))$ is defined to be the set of quotient sheaves $\mathcal{G}$ of $\mathcal{C}_S$ such that the support of $\mathcal{G}$ is finite over S and $\mathrm{pr}_*\mathcal{G}$ is a locally free sheaf of rank r.

We shall define a canonical morphism $\mathrm{Quot}_r(\mathcal{C}) \to \mathrm{Sym}^r X$. To do this we need the following construction. Let $\mathcal{G}$ be a coherent sheaf on $X \times S$ such that the support of $\mathcal{G}$ is finite over S and $\mathrm{pr}_*\mathcal{G}$ is a locally free $\mathcal{O}_S$-module sheaf of rank r. Every point of $X \times S$ has a neighborhood U such that the restriction of $\mathcal{G}$ to U may be represented as the cokernel of a monomorphism $f: \mathcal{O}_U{}^k \to \mathcal{O}_U{}^k$. Denote by D_U the closed subscheme of U defined by $\det f = 0$. It is easy to show that 1) D_U does not depend on the choice of f, 2) there is a closed subscheme $D \subset X \times S$ such that $D_U = D \cap U$ for every U, 3) the morphism $D \to S$ is flat and finite of degree r. D will be called the determinant of G.

Now we proceed to the construction of $\mathrm{Quot}_r(\mathcal{C}) \to \mathrm{Sym}^r X$. Let S be a scheme, φ a morphism $S \to \mathrm{Quot}_r(\mathcal{C})$, $\mathcal{G}$ the corresponding quotient sheaf of $\mathcal{C}_S$. Then the determinant of $\mathcal{G}$ defines a morphism $S \to \mathrm{Sym}^r X$. Thus we have constructed a functor morphism $\mathrm{Mor}(?, \mathrm{Quot}_r(\mathcal{C})) \to \mathrm{Mor}(?, \mathrm{Sym}^r X)$ and therefore a morphism $\mathrm{Quot}_r(\mathcal{C}) \to \mathrm{Sym}^r X$.

Denote by $\bar{\mathcal{A}}$, $\bar{\mathcal{C}}$, $\bar{\Omega}$, $\bar{\mathcal{L}}$ the inverse images of $\mathcal{A}$, $\mathcal{C}$, Ω, $\mathcal{L}$ on $X \times \mathrm{Quot}_r(\mathcal{C})$ and by pr the projection $X \times \mathrm{Quot}_r(\mathcal{C}) \to \mathrm{Quot}_r(\mathcal{C})$. Let $\mathcal{N}$ be the subsheaf of $\bar{\mathcal{C}}$ such that $\bar{\mathcal{C}}/\mathcal{N}$ is the universal quotient sheaf of $\bar{\mathcal{C}}$. Put $\mathfrak{J} = \mathrm{pr}_*(\bar{\mathcal{A}}^{-1} \otimes \mathcal{N} \otimes \bar{\Omega})$. If $x \in \mathrm{Quot}_r(\mathcal{C})$, $\bar{\mathcal{A}}_x$, $\mathcal{N}_x$, $\bar{\Omega}_x$ are the restrictions of $\bar{\mathcal{A}}$, $\mathcal{N}$, $\bar{\Omega}$ to $X \times x$, then $H^1(X \times x, \bar{\mathcal{A}}_x{}^{-1} \otimes \mathcal{N}_x \otimes \bar{\Omega}_x) = 0$ (since the quotient of $\bar{\mathcal{A}}_x{}^{-1} \otimes \mathcal{N}_x \otimes \bar{\Omega}_x$ by its torsion subsheaf has a degree not less than $\deg \mathcal{L} - 2n - 1 - r + 2g - 2 = m - r - 1 \geq (m/2) - 1 \geq 2g - 1$). Therefore $\mathfrak{J}$ is locally free and its fiber over x is equal to $H^0(X \times x, \bar{\mathcal{A}}_x{}^{-1} \otimes \mathcal{N}_x \otimes \bar{\Omega}_x)$. Consider the superposition $\mathfrak{J} = \mathrm{pr}_*(\bar{\mathcal{A}}^{-1} \otimes \mathcal{N} \otimes \bar{\Omega}) \to \mathrm{pr}_*(\bar{\mathcal{A}}^{-1} \otimes \bar{\mathcal{C}} \otimes \bar{\Omega}) \to R^1 \mathrm{pr}_* \bar{\Omega}$, where the second morphism is defined via the exact sequence $0 \to \bar{\mathcal{A}} \to \bar{\mathcal{L}} \to \bar{\mathcal{C}} \to 0$. As $R^1 \mathrm{pr}_* \bar{\Omega}$ is equal to the structure sheaf, the morphism $\mathfrak{J} \to R^1 \mathrm{pr}_* \bar{\Omega}$ defines a section $\sigma \in H^0(\mathrm{Quot}_r(\mathcal{C}), \mathfrak{J}^*)$. Denote by $\mathcal{R}$ the quotient of $\mathfrak{J}^*$ by the subsheaf generated by σ and put $\hat{\mathcal{K}}_{n,r} = \mathbf{P}(\mathcal{R})$.

Now we shall define a morphism $\nu_r: \hat{\mathcal{K}}_{n,r} \to \tilde{\mathcal{K}}_{n,r}$. Put $\mathcal{M} = \mathrm{pr}_*(\bar{\mathcal{A}}^{-1} \otimes \bar{\mathcal{C}} \otimes \bar{\Omega})$. The embedding $\mathfrak{J} \to \mathcal{M}$ induces an epimorphism $\varphi: \mathcal{M}^* \to \mathfrak{J}^*$. φ in its turn induces an epimorphism $\bar{\mathcal{K}}^* \to \mathcal{R}$, where $\bar{\mathcal{K}}$ is the inverse image under the morphism $\mathrm{Quot}_r(\mathcal{C}) \to \tilde{V}_n$ of the sheaf $\mathcal{K}$ from the proof of Proposition 4.1. Denote by ψ the superposition $\hat{\mathcal{K}}_{n,r} = \mathbf{P}(\mathcal{R}) \to \mathbf{P}(\bar{\mathcal{K}}^*) \hookrightarrow \mathbf{P}(\mathcal{K}^*) \rightrightarrows \tilde{\mathcal{K}}_n$ and by f the superposition $\hat{\mathcal{K}}_{n,r} \to \mathrm{Quot}_r(\mathcal{C}) \to \mathrm{Sym}^r X$. $\tilde{\mathcal{K}}_{n,r}$ may be considered as a closed subscheme of $\tilde{\mathcal{K}}_n \times \mathrm{Sym}^r X$. It is easy to show that the image of $(\psi, f): \hat{\mathcal{K}}_{n,r} \to \tilde{\mathcal{K}}_n \times \mathrm{Sym}^r X$ is contained in $\tilde{\mathcal{K}}_{n,r}$. Thus we obtain a morphism $\hat{\mathcal{K}}_{n,r} \to \tilde{\mathcal{K}}_{n,r}$, which will be denoted by ν_r.

The morphism $\nu_r : \hat{\mathfrak{K}}_{n,r} \to \tilde{\mathfrak{K}}_{n,r}$ is the desired "resolution of singularities" of $\tilde{\mathfrak{K}}_{n,r}$. In fact, $\hat{\mathfrak{K}}_{n,r}$ is not smooth in general. However the singularities of the morphism $\hat{\mathfrak{K}}_{n,r} \to \tilde{V}_n$ are rather simple (see Proposition 4.3).

Before studying the morphism $\hat{\mathfrak{K}}_{n,r} \to \tilde{V}_n$ we shall investigate the canonical morphism $\mathrm{Quot}_r(\mathfrak{C}) \to \tilde{V}_n \times \mathrm{Sym}^r X$. Denote by z the superposition $\Delta_n \rightrightarrows V_{n+1} \times X \to X$. Let $M \subset \Delta_n \times \mathrm{Sym}^r X$ be the inverse image of the incidence correspondence under the morphism $(z, \mathrm{id}) : \Delta_n \times \mathrm{Sym}^r X \to X \times \mathrm{Sym}^r X$. It is clear that M is a smooth subvariety of $\tilde{V}_n \times \mathrm{Sym}^r X$ having codimension 2.

PROPOSITION 4.2. $\mathrm{Quot}_r(\mathfrak{C})$ *is the blow-up of* $\tilde{V}_n \times \mathrm{Sym}^r X$ *along* M.

Proof. Denote by Γ the graph of z and by $\mathfrak{J}_\Gamma$ the sheaf of ideals of $\Gamma \subset X \times \tilde{V}_n$.

LEMMA 1. $\mathfrak{C} \simeq \mathfrak{J}_\Gamma \otimes \mathfrak{B}$.

Proof. The canonical morphism $\alpha : \mathfrak{C} \to \mathfrak{B}$ is injective since $\mathfrak{C}$ and $\mathfrak{B}$ are flat over $\tilde{V}_n$ and α is an isomorphism over $X \times V_n$. The cokernel of α is the restriction of $\mathfrak{B}$ to a closed subscheme $\Gamma' \subset X \times \tilde{V}_n$. It is clear that Γ' and Γ are equal as sets. We must prove that Γ' and Γ are equal as schemes.

Since the non-empty geometric fibers of the morphism $\Gamma' \to \tilde{V}_n$ consist of a single point (without nilpotents), the projection $X \times \tilde{V}_n \to \tilde{V}_n$ maps Γ' isomorphically onto a closed subscheme $\Delta_n' \subset \tilde{V}_n$. Denote by $\mathfrak{A}'$ and $\mathfrak{L}'$ the restrictions of $\mathfrak{A}$ and $\tilde{\mathfrak{L}}$ to $X \times \Delta_n'$. It is clear that $\mathfrak{A}'$ is contained in an invertible subsheaf of $\mathfrak{L}'$ of degree $n + 1$. Therefore Δ_n' lies in the image of the morphism $V_{n+1} \times X \to \tilde{V}_n$, i.e. $\Delta_n' \subset \Delta_n$. It is clear that $\Delta_n \subset \Delta_n'$. Thus $\Delta_n' = \Delta_n$, and hence $\Gamma' = \Gamma$. □

It follows from Lemma 1 that $\mathrm{Quot}_r(\mathfrak{C}) = \mathrm{Quot}_r(\mathfrak{J}_\Gamma)$.

Consider the following functor F from the category of schemes over $\tilde{V}_n \times \mathrm{Sym}^r X$ to the category of sets: let S be a scheme over $\tilde{V}_n \times \mathrm{Sym}^r X$, $\mathfrak{J}^S$ the inverse image of $\mathfrak{J}_\Gamma$ under the morphism $X \times S \to X \times \tilde{V}_n$, $D \subset X \times S$ the relative divisor defined by the morphism $S \to \mathrm{Sym}^r X$, $\mathfrak{J}^D$ the restriction of $\mathfrak{J}^S$ to D, $\mathrm{pr} : X \times S \to S$ the projection; then $F(S)$ is defined to be the set of those quotient sheaves of $\mathrm{pr}_* \mathfrak{J}^D$ which are locally free and r-dimensional. We have an obvious embedding $\mathrm{Mor}(S, \mathrm{Quot}_r(\mathfrak{J}_\Gamma)) \hookrightarrow F(S)$ (here "Mor" denotes the set of morphisms over $\tilde{V}_n \times \mathrm{Sym}^r X$). Put $Y = X \times \tilde{V}_n$, $Z = X \times \mathrm{Sym}^{r-1} X \times \tilde{V}_n$. Let $\alpha : Z \to Y$, $\beta : Z \to \mathrm{Sym}^r X \times \tilde{V}_n$ be the natural morphisms.

LEMMA 2. *F is represented by* $\mathbf{P}(\wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma)$.

Proof. Let S be a scheme over $\mathrm{Sym}^r X \times \tilde{V}_n$, $\mathcal{G}$ the inverse image of $\beta_*\alpha^*\mathfrak{J}_\Gamma$ under the morphism $S \to \mathrm{Sym}^r X \times \tilde{V}_n$. Then $F(S)$ is the set of r-dimensional locally free quotient sheaves of $\mathcal{G}$. Therefore F is represented by a closed subscheme $T \subset \mathbf{P}(\wedge^r \beta_*\alpha^*\mathfrak{J}_\Gamma)$. To prove that $T = \mathbf{P}(\wedge^r \beta_*\alpha^*\mathfrak{J}_\Gamma)$ it suffices to show that the dimensions of the geometric fibers of $\beta_*\alpha^*\mathfrak{J}_\Gamma$ do not exceed $r + 1$.

Apply the functor $\beta_*\alpha^*$ to the exact sequence

$$0 \to \mathfrak{J}_\Gamma \to \mathcal{O}_Y \to \mathcal{O}_\Gamma \to 0 \tag{11}$$

where $\mathcal{O}_\Gamma$ is considered as a sheaf on Y. Note that β_* and α^* are exact, $\alpha^*\mathcal{O}_Y = \mathcal{O}_Z$, $\beta_*\alpha^*\mathcal{O}_\Gamma = \mathcal{O}_M$ (here $\mathcal{O}_M$ is considered as a sheaf on $\mathrm{Sym}^r X \times \tilde{V}_n$). Thus we obtain an exact sequence

$$0 \to \beta_*\alpha^*\mathfrak{J}_\Gamma \to \beta_*\mathcal{O}_Z \to \mathcal{O}_M \to 0 \tag{12}$$

It follows from (12) that the dimensions of the geometric fibers of $\beta_*\alpha^*\mathfrak{J}_\Gamma$ do not exceed $r + 1$ (indeed, the restriction of $\beta_*\alpha^*\mathfrak{J}_\Gamma$ to the complement of M is a locally free sheaf of rank r, and the restriction of $\beta_*\alpha^*\mathfrak{J}_\Gamma$ to M is a locally free sheaf of rank $r + 1$). □

So $\mathrm{Quot}_r(\mathfrak{J}_\Gamma)$ is a closed subscheme of $\mathbf{P}(\wedge^r\beta_*\alpha^*\mathfrak{J}_\Gamma)$.

LEMMA 3. *Let $\mathcal{G}$ be a coherent sheaf on $X \times S$ flat over S and such that the support of $\mathcal{G}$ is finite over S. Let $D \subset X \times S$ be the determinant of $\mathcal{G}$. Denote by* pr *the projection $X \times S \to S$. Then every function $a \in H^0(D, \mathcal{O}_D)$ acts on the sheaf* $\det \mathrm{pr}_*\mathcal{G}$ *as multiplication by $N(a)$, where $N: H^0(D, \mathcal{O}_D) \to H^0(S, \mathcal{O}_S)$ is the norm mapping.*

Proof. a) Suppose S is the spectrum of a field. If the statement to be proved holds for a subsheaf $\mathcal{G}' \subset \mathcal{G}$ and for $\mathcal{G}/\mathcal{G}'$, then it holds for $\mathcal{G}$. Therefore, it suffices to consider the case when there are no non-trivial subsheaves in $\mathcal{G}$. In this case the statement is obvious.

b) Now let us consider the general case. We may assume S to be affine. In this case $\mathcal{G}$ can be represented as a quotient of $\mathcal{O}^k_{X \times S}$ (for some k). Therefore it is enough to consider the case $S = \mathrm{Quot}_r(\mathcal{O}_X{}^k)$. A standard argument shows that $\mathrm{Quot}_r(\mathcal{O}_X{}^k)$ is smooth. Therefore a function on $\mathrm{Quot}_r(\mathcal{O}_X{}^k)$ is uniquely defined by its values at the points of $\mathrm{Quot}_r(\mathcal{O}_X{}^k)$. Thus, we have reduced the general case to the already studied case when S is the spectrum of a field. □

Denote by H the group scheme over $\mathrm{Sym}^r X \times \tilde{V}_n$ representing the following functor: for a scheme S over $\mathrm{Sym}^r X \times \tilde{V}_n$, $\mathrm{Mor}(S, H) \overset{\mathrm{def}}{=} H^0(\bar{S}, \mathcal{O}_{\bar{S}}^*)$, where $\bar{S}$ is obtained from S by the base change $\beta: Z \to \mathrm{Sym}^r X \times \tilde{V}_n$. Let $\nu: H \to \mathbf{G}_m$ be the norm homomorphism. For every coherent sheaf $\mathfrak{M}$ on Z, H acts naturally on $\beta_* \mathfrak{M}$. In particular, H acts on $\beta_* \alpha^* \mathfrak{J}_\Gamma$ and hence on $\wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma$. Denote by $\mathfrak{N}$ the maximal quotient sheaf of $\wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma$ on which H acts via ν. It follows from Lemma 3 that $\mathrm{Quot}_r(\mathfrak{J}_\Gamma) \subset \mathbf{P}(\mathfrak{N})$ and that the embedding $\mathrm{Quot}_r(\mathfrak{J}_\Gamma) \to \mathbf{P}(\mathfrak{N})$ induces a bijection of the sets of $\bar{\mathbf{F}}_q$-points. It is easy to show that 1) the fiber of $\mathrm{Quot}_r(\mathfrak{J}_\Gamma)$ over every point of M is isomorphic to $\mathbf{P}^1$ (at least up to nilpotents), 2) the fiber of $\mathrm{Quot}_r(\mathfrak{J}_\Gamma)$ over every point of $(\mathrm{Sym}^r X \times \tilde{V}_n) \backslash M$ consists of a single point. Therefore the dimension of a geometric fiber of $\mathfrak{N}$ is equal to 1 or to 2.

LEMMA 4. *$\mathfrak{N} = \mathfrak{J}_M \otimes \det \beta_* \mathcal{O}_Z$, where $\mathfrak{J}_M$ is the sheaf of ideals of the subscheme $M \subset \mathrm{Sym}^r X \times \tilde{V}_n$.*

Proof. The embedding $\beta_* \alpha^* \mathfrak{J}_\Gamma \to \beta_* \mathcal{O}_Z$ (see the exact sequence (12)) induces a morphism $\wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma \to \det \beta_* \mathcal{O}_Z$. We shall denote its kernel and image by $\mathcal{K}'$ and $\mathfrak{N}'$, respectively. Local calculations by means of (12) show that 1) $\mathcal{K}'$ is the direct image of a locally free sheaf on M of rank $r-1$, 2) $\mathfrak{N}' = \mathfrak{J}_M \otimes \det \beta_* \mathcal{O}_Z$, 3) the exact sequence

$$0 \to \mathcal{K}' \to \wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma \to \mathfrak{N}' \to 0 \tag{13}$$

splits locally.

Denote by $\mathcal{K}$ the subsheaf of $\wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma$ such that $\mathfrak{N} = (\wedge^r \beta_* \alpha^* \mathfrak{J}_\Gamma)/\mathcal{K}$. Let us show that $\mathcal{K} = \mathcal{K}'$. Indeed, if $\mathcal{K} \not\subset \mathcal{K}'$ the fibers of $\mathfrak{N}$ over some points of $(\mathrm{Sym}^r X \times \tilde{V}_n) \backslash M$ would have been equal to zero, and if $\mathcal{K} \subset \mathcal{K}'$, $\mathcal{K} \neq \mathcal{K}'$, then the dimension of the fiber of $\mathfrak{N}$ at some point of M would have been greater than 2. Therefore $\mathcal{K} = \mathcal{K}'$ and hence $\mathfrak{N} = \mathfrak{N}' = \mathfrak{J}_M \otimes \det \beta_* \mathcal{O}_Z$.

It follows from Lemma 4 that $\mathbf{P}(\mathfrak{N}) = \mathbf{P}(\mathfrak{J}_M)$, i.e. $\mathbf{P}(\mathfrak{N})$ is the blow-up of $\mathrm{Sym}^r X \times \tilde{V}_n$ along M. $\mathrm{Quot}_r(\mathfrak{J}_\Gamma)$ is a subscheme of $\mathbf{P}(\mathfrak{N})$ which is equal to $\mathbf{P}(\mathfrak{N})$ as a set. Since $\mathbf{P}(\mathfrak{N})$ is reduced we have $\mathrm{Quot}_r(\mathfrak{J}_\Gamma) = \mathbf{P}(\mathfrak{N})$. □

Denote by Blow_r the inverse image of M in $\mathrm{Quot}_r(\mathcal{C})$ and by Prop_r the proper transform of $\Delta_n \times \mathrm{Sym}^r X$, i.e. the irreducible component of the inverse image of $\Delta_n \times \mathrm{Sym}^r X$ in $\mathrm{Quot}_r(\mathcal{C})$ which is not contained in Blow_r. Denote by $\hat{\mathcal{H}}_{n,r}^{\Delta}$ the inverse image of Δ_n under the morphism $\hat{\mathcal{H}}_{n,r} \to \tilde{V}_n$. Denote by $\widehat{\mathrm{Prop}}_r$, $\widehat{\mathrm{Blow}}_r$ the inverse images of Prop_r, Blow_r under the morphism $\hat{\mathcal{H}}_{n,r} \to \mathrm{Quot}_r(\mathcal{C})$. Put $\widehat{\mathrm{Sing}}_r \overset{\mathrm{def}}{=} \widehat{\mathrm{Prop}}_r \cap \widehat{\mathrm{Blow}}_r$.

PROPOSITION 4.3. 1) *$\hat{\mathfrak{K}}^{\Delta}_{n,r}$ is equal to* $\widehat{\mathrm{Prop}}_r \cup \widehat{\mathrm{Blow}}_r$ *as a scheme.*

2) *The morphism* $\hat{\mathfrak{K}}_{n,r} \setminus \widehat{\mathrm{Sing}}_r \to \tilde{V}_n$ *is locally acyclic.*

3) *There is an open smooth subvariety* $U \subset \hat{\mathfrak{K}}_{n,r}$ *such that* a) $U \supset \widehat{\mathrm{Sing}}_r$, b) $U \cap \widehat{\mathrm{Prop}}_r$ *and* $U \cap \widehat{\mathrm{Blow}}_r$ *are smooth subvarieties of U having codimension* 1 *and intersecting transversally.*

4) *The morphism* $\mathrm{Sing}_r \to \Delta_n$ *is smooth.*

Proof. Recall that $\hat{\mathfrak{K}}_{n,r} = \mathbf{P}(\mathfrak{R})$, where $\mathfrak{R}$ is the quotient of a locally free sheaf $\mathfrak{J}^*$ on $\mathrm{Quot}_r(\mathfrak{C})$ by a subsheaf generated by a certain section $\sigma \in H^0(\mathrm{Quot}_r(\mathfrak{C}), \mathfrak{J}^*)$. Denote by W the open subset of $\mathrm{Quot}_r(\mathfrak{C})$ defined by the inequality $\sigma \neq 0$.

LEMMA. $W \supset \mathrm{Prop}_r$.

Proof. Let $\bar{\mathfrak{L}}$, $\bar{\mathfrak{C}}$, $\mathfrak{N}$ denote the same objects as in the definition of $\mathfrak{J}$. For a point $x \in \mathrm{Quot}_r(\mathfrak{C})$ denote by $\bar{\mathfrak{L}}_x$, $\bar{\mathfrak{C}}_x$, $\mathfrak{N}_x$ the restrictions of $\bar{\mathfrak{L}}$, $\bar{\mathfrak{C}}$, $\mathfrak{N}$ to $X \times x$. If $x \in \mathrm{Quot}_r(\mathfrak{C}) \setminus W$, the embedding $\mathfrak{N}_x \to \bar{\mathfrak{C}}_x$ lifts to an embedding $\mathfrak{N}_x \hookrightarrow \bar{\mathfrak{L}}_x$ and therefore $\mathfrak{N}_x$ is invertible. On the other hand, if $x \in \mathrm{Prop}_r$ the torsion of $\mathfrak{N}_x$ is nonzero. □

Since the morphism $\hat{\mathfrak{K}}^{\Delta}_{n,r} \to \mathrm{Quot}_r(\mathfrak{C})$ is smooth over W, statements 1), 3) and 4) result from the lemma. Statement 2) is proved in the same way as the analogous statement in Section 3. □

Denote by $Z_{n,r}$ the inverse image of Z_n in $\tilde{\mathfrak{K}}^{\Delta}_{n,r}$.

PROPOSITION 4.4. 1) *The morphism* $\nu_r : \hat{\mathfrak{K}}_{n,r} \to \tilde{\mathfrak{K}}_{n,r}$ *induces an isomorphism* $\nu_r^{-1}(\tilde{\mathfrak{K}}_{n,r} \setminus Z_{n,r}) \to \tilde{\mathfrak{K}}_{n,r} \setminus Z_{n,r}$.

2) *The geometric fiber of ν_r over a point of $Z_{n,r}$ either consists of a single point, which does not belong to* $\widehat{\mathrm{Sing}}_r$*, or is isomorphic to* $\mathbf{P}^1$*. In the second case the fiber lies in* $\widehat{\mathrm{Blow}}_r$ *and intersects* $\widehat{\mathrm{Sing}}_r$ *in exactly one point, the intersection being transversal.*

Proof. Let $x \in \tilde{\mathfrak{K}}_{n,r}(\bar{\mathbf{F}}_q)$, D be the image of x in $\mathrm{Sym}^r X(\bar{\mathbf{F}}_q)$, y be the image of x in $\tilde{\mathfrak{K}}_n(\bar{\mathbf{F}}_q)$. Denote by $\underline{\mathfrak{L}}$ and $\underline{\Omega}$ the inverse images of $\mathfrak{L}$ and Ω on $X \otimes \bar{\mathbf{F}}_q$. The point y corresponds to an invertible subsheaf $\underline{\mathfrak{A}} \subset \underline{\mathfrak{L}}$ and an embedding $\alpha : \underline{\mathfrak{A}} \otimes \underline{\Omega}^{-1} \to \underline{\mathfrak{C}}$, where $\underline{\mathfrak{C}} = \underline{\mathfrak{L}}/\underline{\mathfrak{A}}$ (α is defined up to an element of $\bar{\mathbf{F}}_q^*$). Put $\mathfrak{N} = \mathrm{Coker}\,\alpha$. Let $D' \in \mathrm{Sym}^m X(\bar{\mathbf{F}}_q)$ be the determinant of $\mathfrak{N}$. It is easy to see that $D' \geq D$. Denote by F_x the fiber of ν_r over x. F_x is the variety of quotient sheaves of $\mathfrak{N}$ whose determinant is equal to D.

If $x \notin Z_{n,r}(\overline{\mathbf{F}}_q)$, then $\mathfrak{N} \simeq \mathcal{O}_{D'}$, (where $\mathcal{O}_{D'}$ is the direct image of the structure sheaf of the subscheme $D' \subset X \otimes \overline{\mathbf{F}}_q$). Therefore F_x consists of a single point. A similar argument also works if $x \in (\tilde{\mathcal{H}}_{n,r} \backslash Z_{n,r})(A)$, where A is an arbitrary algebra over $\mathbf{F}_q$. This proves statement 1).

Now let $x \in Z_{n,r}(\overline{\mathbf{F}}_q)$. Then $\mathfrak{N} \simeq \mathcal{O}_{D'-\xi} \oplus \mathcal{O}_\xi$, where ξ is an $\overline{\mathbf{F}}_q$-point of X is contained in D'. Denote by k and k' the multiplicities of ξ in D and D'. It is clear that $0 \leq k \leq k'$. It is easy to show that if $k = 0$ then F_x consists of a single point lying in $\widehat{\mathrm{Prop}}_r \backslash \widehat{\mathrm{Blow}}_r$, and if $k = k'$ then F_x consists of a single point lying in $\widehat{\mathrm{Blow}}_r \backslash \widehat{\mathrm{Prop}}_r$. Now let $0 < k < k'$, $\bar{x}$ be the image of x in $\tilde{V}_n(\overline{\mathbf{F}}_q) \times \mathrm{Sym}^r X(\overline{\mathbf{F}}_q)$, $\overline{F}$ the fiber of the morphism $\mathrm{Quot}_r(\mathcal{C}) \to \tilde{V}_n \times \mathrm{Sym}^r X$ over $\bar{x}$. It is easy to see that $\bar{x} \in M(\overline{\mathbf{F}}_q)$ and the natural morphism $F_x \to \overline{F}$ is an isomorphism. Thus statement 2) is proved. □

Now let us prove statements 1)–5) of Theorem A. Denote by $\hat{\mathcal{H}}^{\bar{\kappa}}_{n,r}$, $\widehat{\mathrm{Blow}}_r^{\bar{\kappa}}$, $\widehat{\mathrm{Prop}}_r^{\bar{\kappa}}$ and $\widehat{\mathrm{Sing}}_r^{\bar{\kappa}}$ the fibers of $\hat{\mathcal{H}}_{n,r}$, $\widehat{\mathrm{Blow}}_r$, $\widehat{\mathrm{Prop}}_r$ and $\widehat{\mathrm{Sing}}_r$ over $\mathrm{Spec}\, \bar{\kappa}_n$. Denote by $\bar{\nu}_r$ the morphism $\hat{\mathcal{H}}^{\bar{\kappa}}_{n,r} \to \mathcal{H}^{\bar{\kappa}}_{n,r}$ induced by ν_r. By the first part of Proposition 4.4, $\bar{\nu}_r$ maps $\bar{\nu}_r^{-1}(\mathcal{H}^{\bar{\kappa}}_{n,r} \backslash Z^{\bar{\kappa}}_{n,r})$ isomorphically onto $\mathcal{H}^{\bar{\kappa}}_{n,r} \backslash Z^{\bar{\kappa}}_{n,r}$. It is easy to show that $\widehat{\mathrm{Blow}}_r^{\bar{\kappa}} \cap \bar{\nu}_r^{-1}(\mathcal{H}^{\bar{\kappa}}_{n,r} \backslash Z^{\bar{\kappa}}_{n,r})$ is mapped onto $B_r \backslash Z^{\bar{\kappa}}_{n,r}$ and $\widehat{\mathrm{Prop}}_r^{\bar{\kappa}} \cap \bar{\nu}_r^{-1}(\mathcal{H}^{\bar{\kappa}}_{n,r} \backslash Z^{\bar{\kappa}}_{n,r})$ is mapped onto $P_r \backslash Z^{\bar{\kappa}}_{n,r}$. Therefore, using Proposition 4.3 we obtain statement 5) of Theorem A and the following statements, which are weaker than statements 1), 2), 4) of the theorem: 1′) the canonical morphism $(t_r)^*\underline{\Lambda} \to (t_r)^*(i_r)^*(s^r)_*\underline{\Lambda}$ is an isomorphism; 2′) $(t_r)^*(i_r)^*R^j(s^r)_*\underline{\Lambda} = 0$ for $j > 1$; 4′) the action of $\mathrm{Gal}(\overline{K}_n/\tilde{K}_n)$ on $(t_r)^*R^1(s^r)_*\underline{\Lambda}$ is trivial. It remains for us to prove that if $x \in Z^{\bar{\kappa}}_{n,r}(\bar{\kappa}_n)$ and W_x^j is the fiber of $R^j(s^r)_*\underline{\Lambda}$ over x, then $W_x^j = 0$ for $j > 0$ and the natural homomorphism $\Lambda \to W_x^0$ is an isomorphism.

Denote by $\hat{\mathcal{H}}^{\bar{O}}_{n,r}$ the fiber product of $\hat{\mathcal{H}}_{n,r}$ and $\mathrm{Spec}\, \bar{O}_n$ over $\tilde{V}_n$, by $\hat{s}^r$ the natural morphism $\mathcal{H}^{\overline{K}}_{n,r} \to \hat{\mathcal{H}}^{\bar{O}}_{n,r}$. Denote by F_x the inverse image of x in $\hat{\mathcal{H}}^{\bar{\kappa}}_{n,r}$ and by K_x the restriction of $R(\hat{s}^r)_*\underline{\Lambda}$ to F_x ($R(\hat{s}^r)_*\underline{\Lambda}$ is the direct image of $\underline{\Lambda}$ in the derived category). Then $W_x^j = \mathbf{H}^j(F_x, K_x)$. It follows from Proposition 4.3 that 1) the restriction of $(\hat{s}^r)_*\underline{\Lambda}$ to $\hat{\mathcal{H}}^{\bar{\kappa}}_{n,r}$ is equal to $\underline{\Lambda}$, 2) $R^1(\hat{s}^r)_*\underline{\Lambda} = \alpha_*\underline{\Lambda}(-1)$ where α is the natural embedding $\widehat{\mathrm{Sing}}_r^{\bar{\kappa}} \to \hat{\mathcal{H}}^{\bar{O}}_{n,r}$, 3) $R^j(\hat{s}^r)_*\underline{\Lambda} = 0$ for $j > 0$, 4) the restriction of $R(\hat{s}^r)_*\underline{\Lambda}$ to $\widehat{\mathrm{Blow}}_r^{\bar{\kappa}}$ is equal to $R(\mu^r)_*\underline{\Lambda}$, μ^r being the natural embedding $\widehat{\mathrm{Blow}}_r^{\bar{\kappa}} \backslash \widehat{\mathrm{Sing}}_r^{\bar{\kappa}} \to \widehat{\mathrm{Blow}}_r^{\bar{\kappa}}$. It follows from these facts and the second part of Proposition 4.4 that either F_x consists of a single point and $K_x = \underline{\Lambda}$, or F_x is a projective line, $F_x \cap \widehat{\mathrm{Sing}}_r^{\bar{\kappa}}$ consists of a single point a and $K_x = R\gamma_*\underline{\Lambda}$, where γ is the natural embedding $F_x \backslash \{a\} \to F_x$. In both cases $\mathbf{H}^j(F_x, K_x) = 0$ for $j > 0$, $\mathbf{H}^0(F_x, K_x) = \Lambda$. Thus Theorem A is proved.

Appendix (proof of Deligne's theorem).

DELIGNE'S THEOREM. *Let X be a smooth connected projective curve of genus g over an algebraically closed field F,* $\mathrm{jac}_r\colon \mathrm{Sym}^r X \to \underline{\mathrm{Pic}}^r X$ *the canonical morphism, ρ a continuous absolutely irreducible representation of $\pi_1(X)$ in a vector space of dimension d over a finite extension E of $\mathbf{Q}_l$, l being a prime number different from the characteristic of F. Denote by $\mathcal{E}$ the locally constant sheaf on X corresponding to ρ and by $\mathcal{E}^{(r)}$ the corresponding sheaf on* $\mathrm{Sym}^r X$ *(see the beginning of Section* 2*). Assume that $d > 1$, $r > d(2g-2)$. Then $R^j(\mathrm{jac}_r)_*\mathcal{E}^{(r)} = 0$ for all j.*

The proof given below is due to Deligne and is published by his consent. The author of this paper bears the responsibility for possible shortcomings of the account.

Proof. First of all let us prove the following statement which is weaker than Deligne's theorem.

LEMMA. *The sheaves $R^j(\mathrm{jac}_r)_*\mathcal{E}^{(r)}$ are locally constant.*

Proof. It suffices to prove the analogous statement in the case when $\mathcal{E}$ is a locally constant sheaf of Λ-modules whose stalks are isomorphic to Λ^d, where Λ is a quotient of the ring of integers of E by a nonzero ideal. To this end we shall prove that jac_r is locally acyclic with respect to $\mathcal{E}^{(r)}$. When proving the local acyclicity we may assume that $\mathcal{E} = \underline{\Lambda}^d$. In this case $\mathcal{E}^{(r)} = \bigoplus_{r_1+\cdots+r_d=r}\mathcal{F}(r_1, \ldots, r_d)$, where $\mathcal{F}(r_1, \ldots, r_d)$ is the direct image of the constant sheaf on $\mathrm{Sym}^{r_1}X \times \cdots \times \mathrm{Sym}^{r_d}X$. Therefore, it is enough to prove that the natural morphism $\mathrm{Sym}^{r_1}X \times \cdots \times \mathrm{Sym}^{r_d}X \to \underline{\mathrm{Pic}}^r X$ is smooth provided $r_1 + \cdots + r_d = r > d(2g-2)$. To do this it suffices to show that the morphism $\mathrm{Sym}^{r_i}X \to \underline{\mathrm{Pic}}^{r_i}X$ is smooth for some i. Indeed, $r_i > 2g-2$ for some i, because $r_1 + \cdots + r_d > d(2g-2)$. $\square$

Now let us suppose that the theorem is wrong. Let j be the least number such that $R^j(\mathrm{jac}_r)_*\mathcal{E}^{(r)} \neq 0$. According to the lemma $R^j(\mathrm{jac}_r)_*\mathcal{E}^{(r)}$ is locally constant and, therefore, corresponds to a representation ω of $\pi_1(\underline{\mathrm{Pic}}^r X)$. Since $\pi_1(\underline{\mathrm{Pic}}^r X)$ is abelian, ω contains a subrepresentation of dimension 1 (maybe after replacing E by its finite extension). Therefore, having replaced E by its finite extension we obtain a one-dimensional locally constant sheaf $\mathfrak{M}$ on $\underline{\mathrm{Pic}}^r X$ such that $H^0(\underline{\mathrm{Pic}}^r X, \mathfrak{M} \otimes R^j(\mathrm{jac}_r)_*\mathcal{E}^{(r)}) \neq 0$. Then according to the Leray spectral sequence $H^j(\mathrm{Sym}^r X, \mathcal{E}^{(r)} \otimes (\mathrm{jac}_r)^*\mathfrak{M}) \neq 0$. Put $\mathfrak{A} = \varphi^*\mathfrak{M}$, where φ is the natural embedding $X \to \underline{\mathrm{Pic}}^r X$ ($\mathfrak{A}$ does not depend on the choice of φ). It is well known that $(\mathrm{jac}_r)^*\mathfrak{M}$

$= \mathcal{A}^{(r)}$. Therefore $\mathcal{E}^{(r)} \otimes (\mathrm{jac}_r)^* \mathfrak{M} = \underline{\mathcal{E}}^{(r)}$, where $\underline{\mathcal{E}} = \mathcal{E} \otimes \mathcal{A}$. Hence $H^j(\mathrm{Sym}^r X, \underline{\mathcal{E}}^{(r)}) \neq 0$.

On the other hand, let us calculate $H^*(\mathrm{Sym}^r X, \underline{\mathcal{E}}^{(r)})$ using the Künneth formula. As ρ is irreducible we have $H^0(X, \underline{\mathcal{E}}) = H^2(X, \underline{\mathcal{E}}) = 0$. Therefore $H^i(\mathrm{Sym}^r X, \underline{\mathcal{E}}^{(r)}) = 0$ for $i \neq r$, $H^r(\mathrm{Sym}^r X, \underline{\mathcal{E}}^{(r)}) = \wedge^r H^1(X, \underline{\mathcal{E}})$. But $\dim H^1(X, \underline{\mathcal{E}}) = d(2g-2) < r$, so $\wedge^r H^1(X, \underline{\mathcal{E}}) = 0$. Thus $H^*(\mathrm{Sym}^r X, \underline{\mathcal{E}}^{(r)}) = 0$.

So we have obtained a contradiction, which proves the theorem.

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REFERENCES

[1] V. G. Drinfeld, Langlands' conjecture for GL(2) over functional fields, *Proceedings of the International Congress of Mathematicians* (Helsinki, 1978), vol. 2, pp. 565–574.

[2] H. Jacquet and R. P. Langlands, Automorphic forms on GL(2), *Lecture Notes in Mathematics* N114, Springer-Verlag, Berlin (1970).

[3] P. Deligne, Les constantes des équations fonctionnelles des fonctions L, in *Modular Functions of One Variable* II, pp. 501–597 *Lecture Notes in Mathematics* N349, Springer-Verlag, Berlin (1973).

[4] Cohomologie Étale (SGA 4½), *Lecture Notes in Mathematics* N569, Springer-Verlag, Berlin (1977).

[5] Théorie des Topos et Cohomologie Étale des Schémas (SGA 4), tome 3, *Lecture Notes in Mathematics* N305, Springer-Verlag, Berlin (1973).

[6] A. Weil, Dirichlet Series and Automorphic Forms, *Lecture Notes in Mathematics* N189, Springer-Verlag, Berlin (1971).

[7] R. P. Langlands, *Base Change for* GL(2), Princeton University Press and University of Tokyo Press (1980).