Chapter 1: Semigroups and affine toric varieties

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The variety associated to a semigroup. We will work over a fixed algebraically closed field k of arbitrary characteristic. Note that we may, and will, restrict our attention to the closed points of the varieties under consideration.

By definition, a **semigroup** is a set S endowed with an operation $+$ (we will use in general the additive notation) which is commutative, associative and has a unit element 0. For example, $(k, \cdot)$ is a semigroup. Note that k has also a semigroup structure (even a group structure) given by addition. A map $\phi\colon S \to S'$ between two semigroups is a **semigroup morphism** if $\phi(u_1+u_2) = \phi(u_1)+\phi(u_2)$ for all u_1 and u_2, and if $\phi(0) = 0$.

To a semigroup S we associate the semigroup algebra $k[S]$. This has a basis over k indexed by the elements of S. We denote the elements of this basis by χ^u, for $u \in S$. The multiplication is defined by $\chi^{u_1} \cdot \chi^{u_2} = \chi^{u_1+u_2}$. If we put $1 = \chi^0$, we see that $k[S]$ becomes a k-algebra.

If $\phi\colon S \to S'$ is a morphism of semigroups, we have an induced k-algebra homomorphism $f\colon k[S] \to k[S']$ such that $f(\chi^u) = \chi^{\phi(u)}$.

Let us consider a few examples. If we take the nonnegative integers $\mathbb{N}$, with the operation given by addition, then $k[\mathbb{N}]$ is canonically isomorphic to the polynomial ring $k[t]$ such that χ^1 corresponds to t. More generally, we have an identification $k[\mathbb{N}^n] = k[t_1, \ldots, t_n]$ such that if e_i is the ith vector of the standard basis of $\mathbb{Z}^n$, χ^{e_i} corresponds to t_i. If we consider now $S = \mathbb{Z}^n$, then the previous isomorphism extends to give an isomorphism between $k[\mathbb{Z}^n]$ and the ring of Laurent polynomials $k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, the localization of $k[t_1, \ldots, t_n]$ at the element $t_1 \cdot \ldots \cdot t_n$. Under this identification, we will also use the notation t^u for the monomial corresponding to χ^u.

Note that if S_1 are two semigroups, and if $S = S_1 \times S_2$, with the componentwise operation, then $k[S]$ is canonically isomorphic to $k[S_1] \otimes_k k[S_2]$.

We will consider only semigroups S which are **finitely generated**: this means that there are $u_1, \ldots, u_n \in S$ such that every $u \in S$ can be written as $u = a_1u_1 + \ldots + a_nu_n$ for some $a_i \in \mathbb{N}$. Equivalently, these elements define a surjective semigroup morphism

$$\phi\colon \mathbb{N}^n \to S$$

such that $\phi(e_i) = u_i$. The corresponding k-algebra homomorphism $f\colon k[\mathbb{N}^n] \to k[S]$ is again surjective, so $k[S]$ is a finitely generated algebra. Note that the converse is also true: if $k[S]$ is finitely generated as a k-algebra, then S is a finitely generated semigroup.

We will also assume that S is **integral**: this means that it can be embedded as a sub-semigroup in a finitely generated, free abelian group. The embedding of S in some $\mathbb{Z}^m$ induces an injective homomorphism $k[S] \hookrightarrow k[t_1^{\pm 1}, \ldots, t_m^{\pm 1}]$, so $k[S]$ is a domain.

However, we will occasionally refer to semigroups which do not satisfy the above requirements. Such an example is $(k, \cdot)$.

If S is a finitely generated, integral semigroup, then $k[S]$ is a finitely generated k-algebra with no zero-divisors, so it defines a variety over k, $X = \mathrm{Spec}(k[S])$. A (closed) point of X is given by a k-algebra homomorphism $f\colon k[S] \to k$. This corresponds to a semigroup morphism $\phi\colon S \to (k, \cdot)$ such that $f(\chi^u) = \phi(u)$ for $u \in S$.

Exercise. Show that if P is a point on X corresponding to the semigroup morphism $\phi\colon S \to k$ and if u is in S, then the value of $\chi^u \in \mathcal{O}(X)$ at P is $\phi(u)$.

Let us reconsider the previous examples. If $S = \mathbb{N}^n$, then $X = \mathbb{A}^n$ and if $S = \mathbb{Z}^n$, then $X = (k^*)^n$. Note that in general we have a canonical isomorphism between $\mathrm{Spec}(k[S_1 \times S_2])$ and $\mathrm{Spec}(k[S_1]) \times \mathrm{Spec}(k[S_2])$.

For a slightly less trivial example, let $S = \{m \in \mathbb{N} \mid m \neq 1\}$. Note that S is generated by 2 and 3, which gives a surjective morphism $f\colon k[u, v] \to k[S]$ such that $f(u) = \chi^2$ and $f(v) = \chi^3$.

Exercise. Show that the kernel of f is generated by $u^3 - v^2$.

In general, an integral finitely generated semigroup can be given as the image of a semigroup morphism $\phi\colon \mathbb{N}^n \to \mathbb{Z}^m$.

Proposition 1. *The kernel of the surjective k-algebra homomorphism $f\colon k[t_1, \ldots, t_n] \to k[S]$ associated to ϕ is the ideal*

$$I := (t^a - t^b \mid a, b \in \mathbb{N}^n, \phi(a) = \phi(b)).$$

Proof. We just need to prove that $\ker(f) \subseteq I$. The statement follows from homogeneity considerations: the embedding of S in $\mathbb{Z}^m$ gives a $\mathbb{Z}^m$-grading of $k[S]$. Moreover, we have a $\mathbb{Z}^m$-grading on $k[\mathbb{N}^n] = k[t_1, \ldots, t_n]$ such that f is homogeneous: this is given by $\deg(t^w) = \phi(w)$ for $w \in \mathbb{N}^n$.

Since f is homogeneous, so is its kernel. An element of $\ker(f)$ of degree $u \in \mathbb{Z}^m$ can be written as $\sum_{\phi(w)=u} \alpha_w t^w$ with $\sum_w \alpha_w = 0$, and such an element is in I.

Note that with the notation in this Proposition, $\ker(f)$ is a prime ideal, and $t^w \notin \ker(f)$ for any w. Therefore, it is enough to consider only those $t^a - t^b$ as above such that $a_i b_i = 0$ for all i. In any case, the generators we obtain are infinite. Finding finitely many such generators is nontrivial. For some algorithms to do this, based on Gröbner bases, we refer to [St].

The toric structure. The semigroup S induces some extra structure on the variety $X = \mathrm{Spec}(k[S])$ which we now describe. We define first an operation $X \times X \to X$. At the level of k-algebras this is given by the morphism of k-algebras $\Phi\colon k[S] \to k[S] \otimes k[S]$ such that

$$\Phi(\chi^u) = \chi^u \otimes \chi^u, \text{ for } u \in S.$$

At the level of points, it associates to a pair (ϕ, ψ) of semigroup morphisms $S \to (k, \cdot)$ the morphism $\phi \cdot \psi$ defined by $(\phi \cdot \psi)(u) = \phi(u) \cdot \psi(u)$ for $u \in S$.

It is clear that this operation is associative, commutative, and has as unit element the pointcorresponding to the constant morphism $S \to k$ with value one. This defines on X a structure of "semigroup variety". Note that in general this is not a group structure. For an example, if $S = \mathbb{N}^n$ so $X = \mathbb{A}^n$, then the operation on X is given by $(x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = (x_1 y_1, \ldots, x_n y_n)$. In particular, the set of invertible elements is given by the torus $(k^*)^n \subset \mathbb{A}^n$.

This is true more generally. Let us consider first the case when $S = M$ is a group (which by our assumptions has to be free and finitely generated). With the above operation $T = \mathrm{Spec}(k[M])$ becomes an algebraic group: taking the inverse corresponds to the k-algebra homomorphism $k[M] \to k[M]$, $\chi^u \to \chi^{-u}$. The set of closed points of T as a group is identified via a canonical isomorphism with

$$\mathrm{Hom}_{\mathbb{Z}}(M, k^*) = N \otimes_{\mathbb{Z}} k^*$$

where $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. If we choose an isomorphism $N \simeq \mathbb{Z}^n$, we see that $T \simeq (k^*)^n$, so it is a **torus**.

We return now to the case of an arbitrary semigroup S. Because we assume that all our semigroups are integral, we may embed S in a finitely generated, free abelian group L. Take S^{gp} to be the subgroup of L spanned by S:

$$S^{\mathrm{gp}} := \{u_1 - u_2 \in L \mid u_1, u_2 \in S\}.$$

It follows from the next exercise that S^{gp} is canonically associated to S, so it does not depend on the embedding we have chosen.

Exercise. The inclusion morphism $i : S \hookrightarrow S^{\mathrm{gp}}$ satisfies the following universal property: for every semigroup morphism $\phi\colon S \to A$, where A is an abelian group, there is a unique group morphism $\widetilde{\phi}\colon S^{\mathrm{gp}} \to A$ such that $\widetilde{\phi} \circ i = \phi$.

The inclusion $i : S \hookrightarrow M := S^{\mathrm{gp}}$ induces a morphism $j\colon T := \mathrm{Spec}(k[M]) \to X = \mathrm{Spec}(k[S])$.

Proposition 2. *j embeds T as a principal affine open subset of X.*

Proof. Let us choose finitely many generators $u_1, \ldots, u_n$ for the semigroup S. Then S^{gp} is generated as a group by S and $-(u_1 + \ldots + u_n)$. This implies that $k[M]$ is the localization of $k[S]$ at the element $\chi^{u_1+\ldots+u_n}$.

Corollary 1. *The dimension of X is equal to the rank of M.*

Via the above embedding $T \subseteq X$, the operation on X induces $T \times X \to X$, which is clearly a group action. At the level of k-algebras, it is induced by $k[S] \to k[M] \otimes k[S]$, $\chi^u \to \chi^u \otimes \chi^u$. Note that a morphism $\phi\colon S \to (k, \cdot)$, considered as a point of X, lies in T if and only if $\phi(S) \subseteq k^*$.

By definition, the **affine toric variety** associated to S is the variety X together with the open embedding of the torus T in X, and with an action of T on X, extending the natural action of T on itself. As a matter of terminology, toric varieties are usually assumed to be normal, and we follow this convention. Therefore whenever we drop this hypothesis, we call the corresponding object a not necessarily normal (nnn for short) toric variety.

Remark 1. It turns out that the right structure to consider on X is the action of T on X, and not the multiplication $X \times X \to X$. One reason is that as we will see, the torus action is strong enough to recover the semigroup. More importantly, however, is that this action is the one which globalizes when we want to consider not necessarily affine toric varieties. In fact, Boyarchenko shows in [Boy] that every toric variety X on which the action of the torus on itself extends to a morphism $X \times X \longrightarrow X$ has to be affine.

Let us consider some examples. If $S = \{m \in \mathbb{N} \mid m \neq 1\}$, then $S^{\mathrm{gp}} = \mathbb{Z}$. If we embed X in $\mathbb{A}^2$ as the curve with equation $u^3 - v^2 = 0$, then the embedding $T \simeq k^* \hookrightarrow X$ is given by $\lambda \to (\lambda^2, \lambda^3)$. The action of T on X is described by $\lambda \cdot (u, v) = (\lambda^2 u, \lambda^3 v)$.

Exercise. Let S be the sub-semigroup of $\mathbb{Z}^3$ generated by e_1, e_2, e_3 and $e_1+e_2-e_3$. These generators induce a surjective morphism $f\colon k[\mathbb{N}^4] = k[t_1,\dots,t_4] \to k[S]$. Show that the kernel of f is generated by $t_1t_2-t_3t_4$. We have $S^{\mathrm{gp}} = \mathbb{Z}^3$, the embedding of $T = (k^*)^3 \hookrightarrow X$ is given by $(\lambda_1,\lambda_2,\lambda_3) \to (\lambda_1,\lambda_2,\lambda_3,\lambda_1\lambda_2/\lambda_3)$, and the action of T on X is induced via this embedding by coordinate-wise multiplication.

Remark 2. The left action of an algebraic group G on an affine variety X induces a left action of G on the coordinate ring $\mathcal{O}(X)$ by $(g\cdot F)(x) = F(g^{-1}x)$. We will follow this usual convention, despite the fact that in our case the groups are commutative.

If $\phi\colon M \to k^*$ is a closed point of T, then by this definition we have $\phi\cdot\chi^u = \phi(-u)\chi^u$.

Suppose that $\phi\colon S \to S'$ is a morphism of semigroups. By the universal property of S^{gp}, we have an induced group morphism $\phi^{\mathrm{gp}}\colon S^{\mathrm{gp}} \to S'^{\mathrm{gp}}$. If we denote by X' and T' the varieties associated to S' and S'^{gp}, respectively, then we have a commutative diagram

$$\begin{tikzcd} T' \arrow[r, "g"] \arrow[d] & T \arrow[d] \\ X' \arrow[r, "f"] & X. \end{tikzcd}$$

such that g is a morphism of algebraic groups and f preserves the actions: $f(\lambda\cdot q) = g(\lambda)\cdot f(q)$ for all points $\lambda \in T'$ and $q \in X'$. We will call a morphism f like this a **toric morphism**.

Semigroups versus toric varieties. We want to show now that the category of nnn toric varieties is equivalent with the dual of the category of (finitely generated, integral) semigroups.

Proposition 3. *Let X be an affine variety, $T \subseteq X$ an open subset which is a torus such that the action of T on itself extends to an action on X. Then there is a finitely generated, integral semigroup S and and isomorphism $X \simeq \mathrm{Spec}(k[S])$ which induces an isomorphism of algebraic groups $T \simeq \mathrm{Spec}(k[S^{\mathrm{gp}}])$, and which is compatible with the action.*

Proof. There is a finitely generated, free abelian group M such that $T \simeq \mathrm{Spec}(k[M])$. The open immersion $T \subseteq X$ induces $\mathcal{O}(X) \subseteq k[M]$. Since the action of T on itself extends to X, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}(X) & \longrightarrow & k[M] \otimes \mathcal{O}(X) \\ \Big\downarrow & & \Big\downarrow \\ k[M] & \xrightarrow{\Phi} & k[M] \otimes k[M] \end{array}$$

where the vertical maps are inclusions, and $\Phi(\chi^u) = \chi^u \otimes \chi^u$ for $u \in M$. It follows that if $\sum_{u \in M} \alpha_u \chi^u$ is in $\mathcal{O}(X)$, then $\sum_u \alpha_u \chi^u \otimes \chi^u$ is in $k[M] \otimes \mathcal{O}(X)$, so $\alpha_u \chi^u \in \mathcal{O}(X)$ for every $u \in M$. This shows that there is a subset S of M such that $\mathcal{O}(X) = \bigoplus_{u \in S} k\chi^u$. Since $\mathcal{O}(X)$ is a subring of $k[M]$, it follows that S is a sub-semigroup of M. It is obviously integral, and it is finitely generated since $\mathcal{O}(X)$ is a finitely generated k-algebra.

In order to finish, it is enough to show that $S^{\mathrm{gp}} = M$. Note that we have $\mathrm{rank}(S^{\mathrm{gp}}) = \mathrm{rank}(M) = \dim(X)$. On the other hand, if $S^{\mathrm{gp}} \neq M$, then $\mathrm{Spec}(k[M]) \to \mathrm{Spec}(k[S^{\mathrm{gp}}])$ is finite of degree $\#(M/S^{\mathrm{gp}})$. To see this, it is enough to choose a basis $e_1, \ldots, e_r$ of M such that $a_1 e_1, \ldots, a_r e_r$ is a basis of S^{gp} for some positive integers $a_1, \ldots a_r$.

This shows that if $M \neq S^{\mathrm{gp}}$, then the composition

$$T \to \mathrm{Spec}(k[S^{\mathrm{gp}}]) \to X$$

is not an open embedding, and this finishes the proof.

We deal now with toric morphisms.

Proposition 4. *Let S and S' be semigroups as above, and consider a morphism $f\colon \mathrm{Spec}(k[S]) \to \mathrm{Spec}(k[S'])$ which induces a morphism of algebraic groups $g\colon \mathrm{Spec}(k[S^{\mathrm{gp}}]) \to \mathrm{Spec}(k[S'^{\mathrm{gp}}])$ such that f is compatible with the corresponding actions. Then there is a unique semigroup morphism $\phi\colon S' \to S$ such that f is induced by ϕ.*

Proof. Let $M = S^{\mathrm{gp}}$ and $M' = S'^{\mathrm{gp}}$. Since g is a morphism of algebraic groups, we have a commutative diagram

$$\begin{array}{ccc} k[M'] & \xrightarrow{\Phi'} & k[M'] \otimes k[M'] \\ \Big\downarrow{\scriptstyle g^*} & & \Big\downarrow{\scriptstyle g^* \otimes g^*} \\ k[M] & \xrightarrow{\Phi} & k[M] \otimes k[M]. \end{array}$$

Given $u' \in M'$, let us write $g^*(\chi^{u'}) = \sum_{u \in M} \alpha_u \chi^u$. It follows from the above diagram that

$$\sum_u \alpha_u \chi^u \otimes \chi^u = \sum_{u_1, u_2} \alpha_{u_1} \alpha_{u_2} \chi^{u_1} \otimes \chi^{u_2}.$$

This shows that there is at most one u with $\alpha_u \neq 0$ and in this case $\alpha_u = 1$. Note also that $\chi^{u'}$ is invertible, so $g^*(\chi^{u'}) \neq 0$. These two facts imply that there is a unique $\phi\colon M' \to M$ such that $g^*(\chi^{u'}) = \chi^{\phi(u')}$ for every $u' \in M$. Since g^* is a ring homomorphism, it follows that ϕ is a morphism of groups.

The fact that g can be extended to f is equivalent with $\phi(S') \subseteq S$, which completes the proof.

This proposition shows that in particular, if we have two tori $T = \mathrm{Spec}(k[M])$ and $T' = \mathrm{Spec}(k[M'])$, then we can identify via a canonical isomorphism

$$\mathrm{Hom}_{\mathrm{alg-gp}}(T, T') = \mathrm{Hom}_{\mathbb{Z}}(M', M).$$

In particular, the group of **one-parameter subgroups** of a torus $T = \mathrm{Spec}(k[M])$ (these are morphisms of algebraic groups $k^* \to T$) is canonically isomorphic to $\mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$. Dually, we can recover M as the group of **characters** of T (these are morphisms of algebraic groups $T \to k^*$). Note that if $c_u\colon T \to k^*$ is the character associated to $u \in M$, and if $\phi\colon M \to k^*$ is a closed point of T, then $c_u(\phi) = \phi(u)$.

Recall that the action of a torus $T = \mathrm{Spec}(k[M])$ on itself induces an action of T on $k[M]$. Using the above notation, we see that the action is described by $\phi \cdot \chi^u = c_{-u}(\phi)\chi^u$, i.e. $k\chi^u$ is the **eigenspace corresponding to the character** c_{-u}. Therefore the M-grading of $k[M]$ gives precisely the eigenspace decomposition.

Exercise. Use this to show that if $V \subseteq k[M]$ is a vector subspace, then V is invariant under the action of T if and only if it is an M-graded subspace.

Invariant subvarieties. We have the general principle that in this affine setting, T-invariant geometric objects correspond to M-graded algebraic ones. We will see this when we describe the invariant subvarieties of a toric variety.

More generally, suppose that Y is a closed subscheme of an nnn toric variety $X = \mathrm{Spec}(k[S])$. If Y is defined by the ideal I, then by definition Y is invariant under the torus action if and only if the action of T on X induces an action on Y, i.e. we have a commutative diagram

$$\begin{array}{ccc} k[S] & \xrightarrow{\Phi} & k[M] \otimes k[S] \\ \downarrow & & \downarrow \\ \mathcal{O}(Y) & \longrightarrow & k[M] \otimes \mathcal{O}(Y), \end{array}$$

where the vertical maps are the canonical projections.

This is the case if and only if $\Phi(I) \subseteq k[M] \otimes I$. As in the proof of Proposition 3, this is equivalent with the fact that I is an M-graded ideal of $k[S]$. If we write $I = \bigoplus_{u \in S'} k\chi^u$, then the condition for I to be an ideal is equivalent with the fact that for every $u_1 \in S$ and $u_2 \in S'$, we have $u_1 + u_2 \in S'$. Moreover, I is prime if and only if, in addition, $S \smallsetminus S'$ is a sub-semigroup of S.

We define a **face** of a semigroup S to be a sub-semigroup F such that whenever $u_1, u_2 \in S$ and $u_1 + u_2 \in F$, then we have u_1 and u_2 in F. In particular, $S \smallsetminus F$ is a sub-semigroup of S. Note that if F is a face of S, and if S is generated by $u_1, \ldots, u_m$, then F is generated as a semigroup by a subset of these generators. In particular, every face is an integral, finitely generated semigroup. Moreover, it follows that S has only finitely many faces.

Therefore we have proved the following

Proposition 5. *The (irreducible) invariant subvarieties of the nnn toric variety $X = \mathrm{Spec}(k[S])$ are in an inclusion-preserving bijection with the faces of S, such that the ideal of the variety corresponding to the face F is defined by $\bigoplus_{u \in S \smallsetminus F} k\chi^u$.*

It follows from this proposition that every invariant subvariety Y of X is again an nnn toric variety. Indeed, we have $Y = \mathrm{Spec}(k[F])$ and the corresponding torus is $T_Y := \mathrm{Spec}(k[F^{\mathrm{gp}}])$. Note that the inclusion $Y \subseteq X$ is not a toric morphism unless $F = S$.

On the other hand, the inclusion $F \subseteq S$ induces a morphism of toric varieties $f \colon X \to Y$ which is a retract. The corresponding morphism of tori $g \colon T \to T_Y$ is also surjective (for example, this follows from the fact that k^* is injective as an abelian group). Since g is surjective and since $f(tx) = g(t)f(x)$ for every t in T and x in X, we see that T_Y is an orbit for the action of T on X. It consists of those $\phi \colon S \to k$ such that $S \smallsetminus F = \phi^{-1}(0)$.

Proposition 6. *The above correspondence gives a bijection between the faces of S and the orbits of T in X.*

Proof. Given a closed point ϕ on X, let $F := \phi^{-1}(k^*)$. It is clear that F is a face of S. The orbit of ϕ in X is precisely T_Y, where Y is the closed subvariety $\mathrm{Spec}(k[F])$. Moreover, it is clear that two different faces give two distinct orbits, which proves our statement.

Saturation. Suppose that S is a sub-semigroup of a finitely generated, free abelian group L. We say that S is **saturated in** L if for every $u \in L$, such that there is a positive integer m with $mu \in S$, we have $u \in S$. If S is saturated in S^{gp}, we will simply say that S is **saturated**.

Proposition 7. *The variety X associated to S is normal if and only if S is saturated.*

Proof. The rings $k[S] \subseteq k[S^{\mathrm{gp}}]$ have the same fraction field, and $k[S^{\mathrm{gp}}] \simeq k[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ for some n, so $k[S^{\mathrm{gp}}]$ is normal. Therefore $k[S]$ is normal if and only if it is integrally closed in $k[S^{\mathrm{gp}}]$.

Suppose first that $k[S]$ is normal. If $u \in S^{\mathrm{gp}}$ and if $mu \in S$, then $(\chi^u)^m \in k[S]$ and $\chi^u \in k[S^{\mathrm{gp}}]$. As $k[S]$ is integrally closed in $k[S^{\mathrm{gp}}]$, it follows that $\chi^u \in k[S]$, so $u \in S$.

Conversely, let us assume that S is saturated, and let R be the integral closure of $k[S]$ in $k[S^{\mathrm{gp}}]$. It is clear that R is invariant under the torus action, so it is S^{gp}-graded. Let $\chi^u \in R$ be a homogeneous element. By taking the suitable homogeneous component of an equation over $k[S]$ satisfied by χ^u, we get an equation of the form

$$(\chi^u)^m + \alpha_1 \chi^{v_1} (\chi^u)^{m-1} + \ldots + \alpha_m \chi^{v_m} = 0,$$

where $\alpha_i \in k$ and $v_i \in S$. It follows that $v_i + (m-i)u = mu$ if $\alpha_i \neq 0$, i.e. $iu = v_i$ if $\alpha_i \neq 0$. Therefore iu is in S for some $i \geq 1$, and because S is saturated we deduce $u \in S$, so $R = k[S]$.

For an arbitrary semigroup S (assumed, as always integral and finitely generated), let

$$S^{\mathrm{sat}} := \{u \in S^{\mathrm{gp}} \mid mu \in S, \text{ for some } m \geq 1\}.$$

It is clear that S^{sat} is a saturated semi-group, and in fact it is the smallest saturated sub-semigroup of S^{gp} containing S.

Proposition 8. *The morphism* $\mathrm{Spec}(k[S^{\mathrm{sat}}]) \to \mathrm{Spec}(k[S])$ *induced by* $S \hookrightarrow S^{\mathrm{sat}}$ *is the normalization of* $\mathrm{Spec}(k[S])$.

Proof. We have $k[S] \subseteq k[S^{\mathrm{sat}}] \subseteq k[S^{\mathrm{gp}}]$, and as in the proof of Proposition 7, $k[S^{\mathrm{sat}}]$ is contained in the integral closure R of $k[S]$ in $k[S^{\mathrm{gp}}]$. Moreover, from the proof of Proposition 7 we see that $k[S^{\mathrm{sat}}]$ is normal, so it must equal R and we are done.

Remark 3. We know that the normalization of a domain of finite type over k is finitely generated (see for example [Eis], Cor. 13.13). It follows from Proposition 8 that S^{sat} is a finitely generated semigroup.

In general, we will call $\mathrm{Spec}(k[S])$ the **toric variety associated to** S if S is saturated. Starting from Chapter 3, we will be mostly concerned with such semigroups. As we will see in the next section, in this case S is determined by the cone it spans in $M \otimes_{\mathbb{Z}} \mathbb{R}$, so the geometry of the corresponding variety depends only on combinatorics.

Exercise. If F is a face of S, show that

$$\widetilde{F} := \{u \in S^{\mathrm{sat}} \mid mu \in F \,\text{for some}\, m \geq 1\}$$

is a face of S^{sat}. Moreover, if G is a face of S^{sat}, then $G \cap S$ is a face of S, and these two maps give mutually inverse bijections between the faces of S and those of S^{sat}. Hence the normalization map $f\colon \widetilde{X} \to X$ of an nnn toric variety induces a bijection between the invariant subvarieties of $\widetilde{X}$ and those of X, by taking V to $f(V)$.

Exercise. Show that every face of a saturated semigroup is saturated. Hence every invariant subvariety of a toric variety is again a toric variety.

Smooth affine toric varieties. We start with the following

Lemma 1. *If S is a saturated semigroup, then S is isomorphic to $S' \times \mathbb{Z}^r$ for some r and some semigroup S' in which no nonzero element is invertible.*

Proof. Let $L := \{u \in S \mid -u \in S\}$, so L is the largest sub-semigroup of S which is a group. We may form the quotient $S' = S/L$ which is a finitely generated semigroup.

If $M = S^{\mathrm{gp}}$, then M/L is free: if $mu \in L$ for some $m \geq 1$, then u and $-u$ are in S, as S is saturated, so $u \in L$. Therefore $M/L = S'^{\mathrm{gp}}$.

As M/L is free, there is a section $i\colon M/L \to M$ of the canonical projection. We have $i(S') \subseteq S$, so the morphism $L \times S' \to S$, $(u, t) \to u + i(t)$ is an isomorphism.

This lemma may not hold if S is not saturated:

Exercise. Let S be the sub-semigroup of $\mathbb{Z}^2$ generated by e_1, e_2, $me_1 - me_2$ and $me_2 - me_1$ for some $m \geq 2$. Show that, with the above notation, S/L is not an integral semigroup.

From now on, let us assume that S is a (not necessarily saturated) semigroup in which no nonzero element is invertible. Note that this is equivalent with saying that $\{0\}$ is a face of S. It is clear that it is the unique minimal face, corresponding to the unique torus-invariant point ϕ_0 of $X = \mathrm{Spec}(k[S])$. This point is given by $\phi_0(u) = 1$, if $u = 0$ and $\phi_0(u) = 0$ if $u \in S \smallsetminus \{0\}$.

Let us fix a minimal system of generators $u_1, \ldots, u_m$ for S as a semigroup. These induce a surjection $f\colon k[t_1, \ldots, t_m] \to k[S]$, whose ideal I is generated by those $t^a - t^b$ such that $\sum_i a_i u_i = \sum_i b_i u_i$ and $a_i b_i = 0$ for all i.

Note that since $\{0\}$ is a face of S, we have $I \subseteq (t_1, \ldots, t_m)$. In addition, since we have assumed that our system of generators is minimal we deduce that $I \subseteq (t_1, \ldots, t_m)^2$. This proves the following

Proposition 9. *If S is a semigroup with no nonzero invertible elements, and if $\phi_0 \in X = \mathrm{Spec}(k[S])$ is the unique torus-fixed point, then the dimension of the tangent space to X at ϕ_0 is equal to the number of elements in any minimal system of generators of S.*

Corollary 2. *Let S be an arbitrary semigroup (integral and finitely generated). The variety $\mathrm{Spec}(k[S])$ is smooth if and only if S is isomorphic to $\mathbb{Z}^r \times \mathbb{N}^m$ for some r and m.*

Proof. If $\mathrm{Spec}(k[S])$ is smooth, then in particular it is normal, so S is saturated. By Lemma 1 we may write $S \simeq \mathbb{Z}^r \times S'$, where S' has no nonzero invertible elements. If m is the rank of S'^{gp}, then the dimension of $\mathrm{Spec}(k[S'])$ is equal to m. As $\mathrm{Spec}(k[S'])$ is smooth, the proposition shows that S' can be generated by m elements. These elements therefore induce an isomorphism $S' \simeq \mathbb{N}^m$.

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