MHM PROJECT

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PREFACE

PART 0

INTRODUCTORY PART

CONVENTIONS AND NOTATION

0.1. Numbers.

- $\mathbb{N}$: non-negative integers.
- $\mathbb{Z}$: integers.
- $\mathbb{Q}$: rational numbers.
- $\mathbb{R}$: real numbers.
- $\mathbb{C} = \mathbb{R} + \mathsf{i}\,\mathbb{R}$: complex numbers.
- $\mathbb{S}^1 \subset \mathbb{C}$: complex numbers with absolute value equal to 1.

0.2. Signs. For every $k \in \mathbb{Z}$, we define the orientation number by the equality $\varepsilon(k) = (-1)^{k(k-1)/2}$. We have

$$\begin{aligned} \varepsilon(k+1) &= -\,\varepsilon(-k) = (-1)^k\,\varepsilon(k), & \varepsilon(k+2) &= -\,\varepsilon(k),\\ \varepsilon(k+\ell) &= (-1)^{k\ell}\,\varepsilon(k)\,\varepsilon(\ell), & \varepsilon(n-k) &= (-1)^k\,\varepsilon(n+k). \end{aligned}$$

If we take complex coordinates $z_1, \dots, z_n$ on $\mathbb{C}^n$ and set $z_j = x_j + \mathsf{i}\,y_j$, then

$$(-1)^n \frac{\varepsilon(n)}{(2\pi\mathsf{i})^n}\,(\mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_n) \wedge (\mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}\overline{z}_n) = (1/\pi)^n \mathrm{d}x_1 \wedge \mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}x_n \wedge \mathrm{d}y_n.$$

The following coefficients will often occur later and deserve a specific notation:

$$\begin{aligned} \operatorname{Sgn}(n,k) &= (-1)^n \frac{\varepsilon(n+k)}{(2\pi\mathsf{i})^n} \quad (n,k \in \mathbb{Z}),\\ \operatorname{Sgn}(n) &= \operatorname{Sgn}(n,0) = (-1)^n \frac{\varepsilon(n)}{(2\pi\mathsf{i})^n} = \frac{\varepsilon(n+1)}{(2\pi\mathsf{i})^n}. \end{aligned} \tag{0.2 $*$}$$

Let us notice the following relations:

$$\begin{aligned} \operatorname{Sgn}(n) &= \operatorname{Sgn}(n-1) \cdot \frac{(-1)^{n-1}}{2\pi\mathsf{i}},\\ \operatorname{Sgn}(n,k) &= (-1)^{m(p+k)} \operatorname{Sgn}(m) \operatorname{Sgn}(p,k), \quad n = m+p,\ m,p \geqslant 0,\\ \operatorname{Sgn}(n,-k) &= (-1)^k \operatorname{Sgn}(n,k). \end{aligned} \tag{0.2 $**$}$$

0.3. Categories. Given a category A, we say that a subcategory A' is *full* if, for every pair of objects A, B of A', we have $\mathrm{Hom}_{\mathsf{A}'}(A,B) = \mathrm{Hom}_{\mathsf{A}}(A,B)$. A stronger notion is that of a *strictly full* subcategory, which has the supplementary condition that every object in A which is isomorphic (in A) to an object in A' is already in A'. All along this text, we will use the latter notion, but simply call it a full subcategory, in order to avoid too many different meanings for the word "strict". This should not cause any trouble, since all full subcategories we define consist of all objects satisfying some properties which are obviously stable by isomorphism.

For a sheaf of rings $\mathcal{A}_X$ on a topological space X, we denote by $\mathsf{Mod}(\mathcal{A}_X)$, or simply $\mathsf{Mod}(\mathcal{A}_X)$ the category of $\mathcal{A}_X$-modules. It is an abelian category. The category of complexes on $\mathsf{Mod}(\mathcal{A}_X)$ is denoted by $\mathsf{C}^\star(\mathcal{A}_X)$, with

- $\star$ empty: no condition,
- $\star = +$: complexes bounded from below,
- $\star = -$: complexes bounded from above,
- $\star = \mathrm{b}$: bounded complexes.

By considering the morphisms up to homotopy we obtain the category $\mathsf{K}^\star(\mathcal{A}_X)$ of complexes up to homotopy. Finally, $\mathsf{D}^\star(\mathcal{A}_X)$ denotes the corresponding derived category. We refer for example to [**KS90**, Chap. 1] for the fundamental properties of the derived category of a triangulated category.

0.4. Filtrations.

- Filtrations denoted by F are indexed by $\mathbb{Z}$.
- Increasing filtrations are indicated by a lower index, and decreasing filtrations by an upper index. The usual rule for passing from one kind to the other one is to set $F^\bullet = F_{-\bullet}$.
- However, this is not the rule used for V-filtrations, which are indexed by $\mathbb{R}$, and where we set $V^\bullet = V_{-\bullet-1}$.
- The shift $[k]$ of a filtration by an integer k is defined by

$$F[k]_\bullet := F_{\bullet-k}, \quad \text{equivalently,} \quad F[k]^\bullet := F^{\bullet+k}.$$

- Given a filtered sheaf of rings $(\mathcal{A}_X, F_\bullet\mathcal{A}_X)$ on a topological space, the Rees sheaf of rings $R_F\mathcal{A}_X := \bigoplus_p F_p\mathcal{A}_X \cdot z^p$ is denoted by a calligraphic letter $\mathcal{A}_X$.
- The Rees ring attached to the field $\mathbb{C}$ of complex numbers equipped with the filtration $F_0\mathbb{C} = \mathbb{C}$ and $F_{-1}\mathbb{C} = 0$ is $\widetilde{\mathbb{C}} = \mathbb{C}[z]$.
- In general, for a filtered object $(\mathcal{M}, F_\bullet\mathcal{M})$, the associated Rees object is denoted by the calligraphic letter $\mathcal{M}$. The Rees construction for a filtered morphism φ is indicated by the decoration $\widetilde{\varphi}$.

0.5. Vector spaces and sesquilinear pairings

- The *conjugate* $\overline{\mathcal{H}}$ of a $\mathbb{C}$-vector space $\mathcal{H}$ is $\mathcal{H}_\mathbb{R}$, i.e., $\mathcal{H}$ considered as an $\mathbb{R}$-vector space, together with the $\mathbb{C}$ action defined by $\lambda \cdot x = \overline{\lambda}x$ (for all $\lambda \in \mathbb{C}$ and $x \in \mathcal{H}_\mathbb{R}$) (see Exercise 2.1). In order to make clear the structure, we denote by $\overline{x}$ the element x of $\mathcal{H}_\mathbb{R}$ when considered as being in $\overline{\mathcal{H}}$. Conjugation is a covariant functor on the category of $\mathbb{C}$-vector spaces. For a morphism $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$, we denote by $\overline{\varphi} : \overline{\mathcal{H}}_1 \to \overline{\mathcal{H}}_2$ the corresponding morphism.

• We denote by $\langle \bullet, \bullet \rangle : \mathcal{H} \otimes_{\mathbb{C}} \mathcal{H}^\vee \to \mathbb{C}$ the tautological duality pairing. We set $\mathcal{H}^* = \overline{\mathcal{H}}^\vee = (\overline{\mathcal{H}})^\vee$, that we call the *Hermitian dual vector space*. Duality and Hermitian duality are contravariant functors on the category of $\mathbb{C}$-vector spaces. For a morphism $\varphi : \mathcal{H}_1 \to \mathcal{H}_2$, we denote by $\varphi^* : \mathcal{H}_2^* \to \mathcal{H}_1^*$ the corresponding morphism.

• A *sesquilinear pairing* between $\mathbb{C}$-vector spaces $\mathcal{H}', \mathcal{H}''$ is a $\mathbb{C}$-linear morphism

$$\mathfrak{s} : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}''} \longrightarrow \mathbb{C}.$$

We identify such a sesquilinear pairing $\mathfrak{s}$ with a linear morphism

$$\mathfrak{s} : \mathcal{H}'' \longrightarrow \mathcal{H}'^*$$

by setting, for $x \in \mathcal{H}'$, $y \in \mathcal{H}''$,

$$\mathfrak{s}(x, \overline{y}) := \langle x, \overline{\mathfrak{s}(y)} \rangle_{\mathcal{H}'} \quad (\overline{\mathfrak{s}(y)} \in \mathcal{H}'^\vee).$$

If $\varphi'' : \mathcal{H}_1'' \to \mathcal{H}''$ is a linear morphism, then $\mathfrak{s} \circ \varphi'' : \mathcal{H}_1'' \to \mathcal{H}'^*$ corresponds to the pairing $\mathcal{H}' \otimes \overline{\mathcal{H}_1''} \to \mathbb{C}$ given by $\mathfrak{s}(x, \overline{\varphi''(y_1)})$, and if $\varphi' : \mathcal{H}_1' \to \mathcal{H}'$ is a linear morphism, then $\varphi'^* \circ \mathfrak{s} : \mathcal{H}'' \to \mathcal{H}_1'^*$ corresponds to the pairing $\mathfrak{s}(\varphi'(x_1), \overline{y})$.

The Hermitian adjoint $\mathfrak{s}^* : \mathcal{H}' \to \mathcal{H}''^*$ of a linear morphism $\mathfrak{s} : \mathcal{H}'' \to \mathcal{H}'^*$ corresponds to the sesquilinear pairing

$$\mathfrak{s}^*(y, \overline{x}) = \overline{\mathfrak{s}(x, \overline{y})}.$$

Assume now that $\mathcal{H}' = \mathcal{H}'' = \mathcal{H}$. We say that $\mathfrak{s}$ is $\pm$-Hermitian if $\mathfrak{s}^* = \pm\mathfrak{s}$. For a linear morphism $\varphi : \mathcal{H} \to \mathcal{H}$, we say that φ is self- resp. skew-adjoint (with respect to $\mathfrak{s}$) if

$$\mathfrak{s}(\varphi(x), \overline{y}) = \mathfrak{s}(x, \overline{\varphi(y)}) \quad \text{resp.} \; -\mathfrak{s}(x, \overline{\varphi(y)}).$$

Considering $\mathfrak{s}$ as a linear morphism as above and denoting by φ^* the Hermitian adjoint of φ, this is translated as

$$\varphi^* \circ \mathfrak{s} = \pm \mathfrak{s} \circ \varphi.$$

Then φ is self- resp. skew-adjoint (with respect to $\mathfrak{s}$) if and only if it is so with respect to $\mathfrak{s}^*$.

If $\mathfrak{s}$ is *non-degenerate*, that is, if $\mathfrak{s} : \mathcal{H} \to \mathcal{H}^*$ is an isomorphism, then one can define the $\mathfrak{s}$-adjoint $\varphi^\star$ (not to be confused with φ^*) of any linear morphism $\varphi : \mathcal{H} \to \mathcal{H}$ by the formula

$$\varphi^\star = \mathfrak{s}^{(-1)} \circ \varphi^* \circ \mathfrak{s} : \mathcal{H} \longrightarrow \mathcal{H}.$$

In such a case, φ is self- (resp. skew-) adjoint with respect to $\mathfrak{s}$ if and only if $\varphi^\star = \pm\varphi$.

0.6. Complex manifolds and their basic sheaves of rings

• We consider complex manifolds, usually denoted by X, Y of complex dimension d_X, d_Y, and holomorphic maps $f : X \to Y$ between them, of relative dimension $d_{X/Y} = d_X - d_Y$. We will often use the following shortcuts:

$$\boxed{n = d_X = \dim X, \quad m = d_Y = \dim Y, \quad n - m = d_X - d_Y = d_{X/Y}.}$$

• A smooth hypersurface of X (i.e., a closed complex submanifold everywhere of codimension 1 in X) will usually be denoted by H.

• A divisor D is a reduced complex analytic subspace of X everywhere of codimension 1. A local defining equation for D or H is usually denoted by g.

• At many places, the divisor D is assume to have only normal crossings as singularities. It is however in general not necessary to assume that its irreducible components are smooth.

• The structure sheaf of holomorphic functions on X is denoted by $\mathcal{O}_X$. The sheaf of holomorphic differential forms is Ω^1_X and the sheaf of holomorphic vector fields is Θ_X. Their k-th wedge product is Ω^k_X resp. $\Theta_{X,k} = \Theta_{X,k}$. The dualizing sheaf $\wedge^n\Omega^1_X$ is denoted by ω_X. The sheaf of holomorphic differential operators is $\mathcal{D}_X$.

• The sheaf of C^∞ function on the underlying C^∞ manifold is denoted by $\mathcal{C}^\infty_X$. Correspondingly, the sheaf of C^∞ differential forms of degree k is $\mathcal{E}^k_X = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} \Omega^k_X$ and is decomposed with respect to the holomorphic and anti-holomorphic degrees with components $\mathcal{E}^{(p,q)}_X$. The sheaf of C^∞ poly-vector fields of degree k is denoted by $\mathcal{T}_{X,k} = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} \Theta_{X,k}$ and is similarly decomposed with components $\mathcal{T}_{X,(p,q)}$.

• The Rees construction for $\mathcal{O}_X$, equipped with the filtration $F_0\mathcal{O}_X = \mathcal{O}_X$ and $F_{-1}\mathcal{O}_X = 0$ leads to $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z]$.

• Correspondingly, we set $\widetilde{\Omega}^1_X = z^{-1}\Omega^1_X[z]$ and $\widetilde{\Theta}_X = z\Theta_X[z]$. We have $\widetilde{\Omega}^k_X = \wedge^k\widetilde{\Omega}^1_X$ and $\widetilde{\Theta}_{X,k} = \wedge^k\widetilde{\Theta}_X$.

• The Rees construction for $\omega_X = \Omega^n_X$ equipped with the filtration $F_{-n}\omega_X = \omega_X$ and $F_{-n-1}\omega_X = 0$ gives rise to $\widetilde{\omega}_X = \widetilde{\Omega}^n_X$.

• The Rees construction for $\mathcal{D}_X$ with its filtration by the order of differential operators $F_\bullet\mathcal{D}_X$ gives rise to $\widetilde{\mathcal{D}}_X$.

• The C^∞ analogues are $\widetilde{\mathcal{C}}^\infty_X = \mathcal{C}^\infty_X[z]$, $\widetilde{\mathcal{E}}^k_X = \widetilde{\mathcal{C}}^\infty_X \otimes_{\mathcal{O}_X} \widetilde{\Omega}^k_X$, and $\mathcal{T}^k_X = \widetilde{\mathcal{C}}^\infty_X \otimes_{\mathcal{O}_X} \widetilde{\Theta}_X$.

0.7. Sheaves of rings and modules

• $\mathbb{C}$-vector spaces are denoted by $\mathcal{H}, \mathcal{H}', \mathcal{H}''$.

• C^∞ vector bundles on X are denoted by $\mathcal{H}$, and a C^∞ connection is denoted by $D : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$. It is decomposed into its $(1,0)$ and $(0,1)$ components $D' : \mathcal{H} \to \mathcal{E}^{1,0}_X \otimes \mathcal{H}$ and $D'' : \mathcal{H} \to \mathcal{E}^{0,1}_X \otimes \mathcal{H}$.

• A holomorphic vector bundle is denoted by $\mathcal{H}', \mathcal{H}''$ or $\mathcal{V}$. For a C^∞ vector bundle $\mathcal{H}$ with integrable connection D, we regard $\mathcal{H}' = \operatorname{Ker} D''$ as a holomorphic vector bundle with a integrable holomorphic connection $\nabla : \mathcal{H}' \to \Omega^1_X \otimes \mathcal{H}'$.

• Modules over a sheaf of rings $\mathcal{A}_X = \mathcal{O}_X, \mathcal{D}_X$ are denoted by $\mathcal{M}, \mathcal{N}$.

• The Rees objects associated to filtered holomorphic vector bundles, $\mathcal{O}_X$-modules or $\mathcal{D}_X$-modules are denoted by the corresponding calligraphic letter $\mathcal{H}', \mathcal{H}'', \mathcal{M}, \mathcal{N}$.

0.8. Basic operators of Hodge theory. Some operators will have an invariable notation, whatever the category they belong to.

• Sesquilinear pairings used for categories of triples are denoted by $\mathfrak{s}$.

• Nilpotent endomorphisms are denoted by N, and their monodromy filtration is denoted by $\mathrm{M}_\bullet(\mathrm{N})$ or simply $\mathrm{M}_\bullet$.

• Polarizations are denoted by S, but their components may be denoted by $\mathcal{S}$ or $\widetilde{\mathcal{S}}$, depending on the context.

Cohomology functors.

• Whatever derived category it acts on, the k-th cohomology functor is denoted by H^k.

• Pushforward or pullback functors are mostly defined as functors on derived categories, but not defined as right or left derived functors. In order to simplify the notation, the cohomology functors like $\mathcal{H}^k \boldsymbol{R} f_*$ or ${}_{\mathrm{D}}f_*$ are denoted by $f_*^{(k)}$ or ${}_{\mathrm{D}}f_*^{(k)}$.

CHAPTER 1

OVERVIEW

CHAPTER 2

HODGE THEORY: REVIEW OF CLASSICAL RESULTS

Summary. This chapter reviews classical results of Hodge theory. It introduces the general notion of Hodge structure and various extensions of this notion: polarized Hodge structure and mixed Hodge structure. These notions are the model (on finite dimensional vector spaces) of the corresponding notions on complex manifolds, called Hodge module, polarized Hodge module and mixed Hodge module. Although Hodge structures are usually defined over $\mathbb{Q}$ (and even over $\mathbb{Z}$), we emphasize the notion of a $\mathbb{C}$-Hodge structure.

2.1. Introduction

The notion of (polarized) Hodge structure has emerged from the properties of the cohomology of smooth complex projective varieties. In this chapter, as a prelude to the theory of complex Hodge modules, we focus on the notion of (polarized) complex Hodge structure. In doing so, we forget the integral structure in the cohomology of a smooth complex projective variety, and even the rational structure and the real structure.

We are then left with a very simple structure: a complex Hodge structure is nothing but a finite-dimensional graded vector space, and a morphism between Hodge structures is a graded morphism of degree zero between these vector spaces. Hodge structures obviously form an abelian category.

A polarization is nothing but a positive definite Hermitian form on the underlying vector space, which is compatible with the grading, that is, such that the decomposition given by the grading is orthogonal with respect to the Hermitian form.

It is then clear that any Hodge substructure of a polarized Hodge structure is itself polarized by the induced Hermitian form and, as such, is a direct summand of the original polarized Hodge structure.

Why should the reader continue reading this chapter, since the main definitions and properties have been given above?

The reason is that this description does not have a good behaviour when considering holomorphic families of such object. Such families arise, for example, when considering the cohomology of the smooth varieties occurring in a flat family of smooth complex projective varieties. It is known that the grading does not deform holomorphically. Both the grading and the Hermitian form vary real-analytically, and this causes troubles when applying arguments of complex algebraic geometry.

Instead of the grading, it is then suitable to consider the two natural filtrations giving rise to this grading. One then varies holomorphically and the other one anti-holomorphically. From this richer point of view, one can introduce the notion of weight, which is fundamental in the theory, as it leads to the notion of mixed Hodge structure.

Similarly, instead of the positive definite Hermitian form, one should consider the Hermitian form which is $\pm$-definite on each graded term in order to have an object which varies in a locally constant way, as does the cohomology of the varieties. The sign will be made precise. Explanations on our sign conventions are given later, after that enough material has been developed, in an appendix (see Page 543).

This chapter moves around the notion of (polarized) complex Hodge structure by shedding light on its different aspects. In Chapter 5, we will emphasize the point of view of "triples", which will be the one chosen here for the theory of polarizable Hodge modules.

2.2. Hodge-Tate structure and highest dimensional cohomology

Let X be a connected compact complex manifold of dimension n. The highest dimensional cohomology $H^{2n}(X,\mathbb{Z})$ is a free $\mathbb{Z}$-module of rank 1, and the cap product with the fundamental homology class $[X]$ induces an isomorphism

$$\int_{[X]} : H^{2n}(X,\mathbb{Z}) \xrightarrow{\sim} \mathbb{Z}.$$

The complex cohomology $H^{2n}(X,\mathbb{C})$ can be realized with C^∞ differential forms $\mathcal{E}_X^\bullet$ as the cohomology $H^{2n}_{\mathrm{d}}(X) = \Gamma(X,\mathcal{E}_X^{2n})/\mathrm{d}\Gamma(X,\mathcal{E}_X^{2n-1})$ and as the Dolbeault cohomology $H^{n,n}_{\mathrm{d}''}(X)$. We say that $H^{2n}(X,\mathbb{C})$ *is pure of weight* $2n$. Integration of C^∞ forms of maximal degree on X induces a $\mathbb{C}$-linear isomorphism

$$\int_X : H^{2n}_{\mathrm{d}}(X) \xrightarrow{\sim} \mathbb{C}.$$

Differential forms are equipped with a conjugation operator:

$$\eta_{I,J}(z)\,\mathrm{d}z_I\wedge\mathrm{d}\overline{z}_J \longmapsto \overline{\eta_{I,J}(z)}\,\mathrm{d}\overline{z}_I\wedge\mathrm{d}z_J = (-1)^{\#I\#J}\overline{\eta_{I,J}(z)}\,\mathrm{d}z_J\wedge\mathrm{d}\overline{z}_I,$$

and integration on X commutes with conjugation. Moreover, the natural diagram commutes:

$$\begin{tikzcd} H^{2n}(X,\mathbb{Z}) \arrow[r, "\int_{[X]}"] \arrow[d, hook] & \mathbb{Z} \arrow[d, hook] \\ H^{2n}(X,\mathbb{C}) \arrow[r, "\int_X"] & \mathbb{C} \end{tikzcd}$$

Changing the choice of a square root of -1, i.e., i to $-\mathsf{i}$, has the effect to changing the orientation of $\mathbb{C}$ to its opposite, hence to multiplying that of $\mathbb{C}^n$ by $(-1)^n$. Since X

is a complex manifold, it also has the effect to to multiplying its orientation by $(-1)^n$, in other words to change the fundamental class $[X]$ to $(-1)^n[X]$. This change has the effect of replacing X with the complex conjugate manifold (i.e., the same underlying C^∞ manifold equipped with the sheaf of anti-holomorphic functions as structural sheaf). Therefore, it has the effect of multiplying $\int_X$ by $(-1)^n$.

In order to make $\int_{[X]}$ and $\int_X$ independent of the choice of a square root of -1, one replaces them with

$$\operatorname{tr}_{[X]} := (2\pi \mathrm{i})^{-n} \int_{[X]} \quad \text{and} \quad \operatorname{tr}_X := (2\pi \mathrm{i})^{-n} \int_X .$$

The *Hodge-Tate structure of weight* $2n$, also denoted by $\mathbb{Z}^{\mathrm{H}}(-n)$ or simply $\mathbb{Z}(-n)$, consists of the following set of data:

- the $\mathbb{C}$-vector space $\mathbb{C}$, equipped with
- the (trivial) bigrading of bidegree (n, n): $\mathbb{C} = \mathbb{C}^{n,n}$.
- and its $\mathbb{Z}$-lattice $(2\pi \mathrm{i})^{-n}\,\mathbb{Z}$

The very first result in Hodge theory can thus be stated as follows.

2.2.1. Proposition. *The normalized integration morphism*

$$\operatorname{tr}_X : \big(H^{2n}(X, \mathbb{C}), H^{n,n}_{\mathrm{d}''}(X), H^{2n}(X, \mathbb{Z})\big) \longrightarrow \mathbb{Z}^{\mathrm{H}}(-n)$$

is an isomorphism.

***2.2.2. Remark* (Forgetting the $\mathbb{Z}$-structure).** One can define $\mathbb{Q}^{\mathrm{H}}(-n)$ and $\mathbb{R}^{\mathrm{H}}(-n)$. If we completely forget the $\mathbb{R}$-structure, we are left with $\mathbb{C}^{\mathrm{H}}(-n)$ which consists only of the first two pieces of data. In the next two sections, we will avoid possible $\mathbb{Z}$-torsion in abelian groups by working over one of the previous fields, say $\mathbb{Q}$.

2.3. Complex Hodge theory on compact Riemann surfaces

Let X be a compact Riemann surface of genus $g \geqslant 0$. Let us assume for simplicity that it is connected. Then $H^0(X, \mathbb{Z})$ and $H^2(X, \mathbb{Z})$ are both isomorphic to $\mathbb{Z}$ (as X is orientable). The only interesting cohomology group is $H^1(X, \mathbb{Z})$, isomorphic to $\mathbb{Z}^{2g}$.

The Poincaré duality isomorphism induces a skew-symmetric non-degenerate bilinear form

$$\langle \bullet, \bullet \rangle : H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X, \mathbb{Z}) \xrightarrow{\;\bullet \cup \bullet\;} H^2(X, \mathbb{Z}) \xrightarrow{\;\int_{[X]}\;} \mathbb{Z}.$$

One of the main analytic results of the theory asserts that the space $H^1(X, \mathcal{O}_X)$ is finite dimensional and has dimension equal to the genus g (see e.g. [**Rey89**, Chap. IX] for a direct approach). Then, *Serre duality* $H^1(X, \mathcal{O}_X) \xrightarrow{\sim} H^0(X, \Omega^1_X)^\vee$ also gives $\dim H^0(X, \Omega^1_X) = g$. A dimension count implies then the *Hodge decomposition*

$$H^1(X, \mathbb{C}) \simeq H^{1,0}(X) \oplus H^{0,1}(X), \quad H^{0,1}(X) = H^1(X, \mathcal{O}_X), \quad H^{1,0}(X) = H^0(X, \Omega^1_X).$$

We also interpret the space $H^{1,0}(X)$ resp. $H^{0,1}(X)$ as the Dolbeault cohomology space $H^{0,1}_{\mathrm{d}''}(X)$ resp. $H^{0,1}_{\mathrm{d}''}(X)$. If we regard Serre duality as the non-degenerate pairing

$$H^{1,0} \otimes_{\mathbb{C}} H^{0,1} \xrightarrow{\;\bullet\wedge\bullet\;} H^{1,1} \xrightarrow{\;\int_X\;} \mathbb{C},$$

then Serre duality is equivalent to the complexified Poincaré duality pairing

$$\langle\bullet,\bullet\rangle_{\mathbb{C}} : H^1(X,\mathbb{C}) \otimes_{\mathbb{C}} H^1(X,\mathbb{C}) \longrightarrow \mathbb{C},$$

since $\langle H^{1,0}, H^{1,0}\rangle_{\mathbb{C}} = 0$ and $\langle H^{0,1}, H^{0,1}\rangle_{\mathbb{C}} = 0$.

With respect to the real structure $H^1(X,\mathbb{C}) = \mathbb{C} \otimes_{\mathbb{R}} H^1(X,\mathbb{R})$, $H^{1,0}$ is conjugate to $H^{0,1}$, and using Serre duality (or Poincaré duality) we get a *skew-Hermitian* sesquilinear pairing (see Exercise 2.1 for the notion of conjugate $\mathbb{C}$-vector space and sesquilinear pairing)

$$\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}} : H^1(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^1(X,\mathbb{C})} \longrightarrow \mathbb{C},$$

whose restriction to $H^1(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^1(X,\mathbb{Z})$ is $\langle\bullet,\bullet\rangle$, and whose restriction to $H^{1,0}_{\mathrm{d}''}$ is

$$\begin{aligned} H^{1,0} \otimes_{\mathbb{C}} \overline{H^{1,0}} &\longrightarrow \mathbb{C} \\ \eta' \otimes \overline{\eta''} &\longmapsto \int_X \eta' \wedge \overline{\eta''}. \end{aligned}$$

Then, the Riemann bilinear relations assert that the Hermitian pairing

$$\mathrm{h}(\bullet,\overline{\bullet}) := \frac{\mathsf{i}}{2\pi}\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}} = -\frac{1}{2\pi\mathsf{i}}\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}} : \quad \eta' \otimes \overline{\eta''} \longmapsto -\frac{1}{2\pi\mathsf{i}}\int_X \eta' \wedge \overline{\eta''} = -\operatorname{tr}_X(\eta' \wedge \overline{\eta''})$$

is *positive definite* on $H^{1,0}$. In a similar way one finds that $\frac{1}{2\pi\mathsf{i}}\langle\bullet,\overline{\bullet}\rangle_{\mathbb{C}}$ is positive definite on $H^{0,1}$.

2.4. Complex Hodge theory of smooth projective varieties

Let X be a smooth complex projective variety of pure complex dimension n (i.e., each of its connected components has dimension n). It will be equipped with the usual topology, which makes it a complex analytic manifold. Classical Hodge theory asserts that each cohomology space $H^k(X,\mathbb{C})$ decomposes as the direct sum

$$H^k(X,\mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X), \tag{2.4.1}$$

where $H^{p,q}(X)$ stands for $H^q(X,\Omega^p_X)$ or, equivalently, for the Dolbeault cohomology space $H^{p,q}_{\mathrm{d}''}(X)$. Although this result is classically proved by methods of analysis (Hodge theory for the Laplace operator), it can be expressed in a purely algebraic way, by means of the de Rham complex.

The holomorphic de Rham complex is the complex of sheaves $(\Omega^\bullet_X, \mathrm{d})$, where d is the differential, sending a k-form to a $(k+1)$-form. Recall (holomorphic Poincaré lemma) that $(\Omega^\bullet_X, \mathrm{d})$ is a resolution of the constant sheaf. Therefore, the cohomology $H^k(X,\mathbb{C})$ is canonically identified with the hypercohomology $\boldsymbol{H}^k\big(X, (\Omega^\bullet_X, \mathrm{d})\big)$ of the holomorphic de Rham complex.

The de Rham complex can be filtered in a natural way by sub-complexes ("filtration bête" in [**Del71b**]).

2.4.2. Remark. In general, we denote by an upper index a *decreasing filtration* and by a lower index an *increasing filtration*. Filtrations are indexed by $\mathbb{Z}$ unless otherwise specified.

We define the "stupid" (increasing) filtration on $\mathcal{O}_X$ by setting

$$F_p\mathcal{O}_X = \begin{cases} \mathcal{O}_X & \text{if } p \geqslant 0, \\ 0 & \text{if } p \leqslant -1. \end{cases}$$

Observe that, trivially, $\mathrm{d}(F_p\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X^k) \subset F_{p+1}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X^{k+1}$. Therefore, the de Rham complex can be (decreasingly) filtered by

$$F^p(\Omega_X^\bullet, \mathrm{d}) = \{0 \longrightarrow F_{-p}\mathcal{O}_X \xrightarrow{\ \mathrm{d}\ } F_{-p+1}\mathcal{O}_X \otimes_{\mathcal{O}_X} \Omega_X^1 \xrightarrow{\ \mathrm{d}\ } \cdots\}. \tag{2.4.3}$$

If $p \leqslant 0$, $F^p(\Omega_X^\bullet, \mathrm{d}) = (\Omega_X^\bullet, \mathrm{d})$, although if $p \geqslant 1$,

$$F^p(\Omega_X^\bullet, \mathrm{d}) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow \cdots \longrightarrow \Omega_X^{\dim X} \longrightarrow 0\}.$$

As a consequence, the p-th graded complex is 0 if $p \leqslant -1$ and, if $p \geqslant 0$, it is given by

$$\mathrm{gr}_F^p(\Omega_X^\bullet, \mathrm{d}) = \{0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{p}{\Omega_X^p} \longrightarrow 0 \longrightarrow \cdots \longrightarrow 0\}.$$

In other words, the graded complex $\mathrm{gr}_F(\Omega_X^\bullet, \mathrm{d}) = \bigoplus_p \mathrm{gr}_F^p(\Omega_X^\bullet, \mathrm{d})$ is the complex $(\Omega_X^\bullet, 0)$ (i.e., the same terms as for the de Rham complex, but with differential equal to 0).

From general results on filtered complexes, the filtration of the de Rham complex induces a (decreasing) filtration on the hypercohomology spaces (that is, on the de Rham cohomology of X) and there is a spectral sequence starting from $\boldsymbol{H}^\bullet\big(X, \mathrm{gr}_F(\Omega_X^\bullet, \mathrm{d})\big)$ and abutting to $\mathrm{gr}_F H^\bullet(X, \mathbb{C})$. Let us note that $\boldsymbol{H}^\bullet\big(X, \mathrm{gr}_F(\Omega_X^\bullet, \mathrm{d})\big)$ is nothing but $\bigoplus_{p,q} H^q(X, \Omega_X^p)$.

2.4.4. Theorem. *The spectral sequence of the filtered de Rham complex on a smooth projective variety degenerates at E_1, that is,*

$$H^\bullet(X, \mathbb{C}) \simeq H^\bullet_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p,q} H^q(X, \Omega_X^p).$$

2.4.5. Remark. Although the classical proof uses Hodge theory for the Laplace operator which is valid in the general case of compact Kähler manifolds, there is a purely algebraic/arithmetic proof in the projective case, due to Deligne and Illusie [**DI87**].

For each $j \in \mathbb{N}$, let us consider the following set of data $H^j(X, \mathbb{C})^{\mathrm{H}}$ (also called a *pure $\mathbb{C}$-Hodge structure of weight j*) consisting of:

- the complex vector space $H^j(X, \mathbb{C})$, equipped with
- the bigrading $H^j(X, \mathbb{C}) = \bigoplus_{p+q=j} H^{p,q}$,

For every $k \in \mathbb{Z}$, Poincaré duality is the non-degenerate bilinear pairing

$$\langle\bullet,\bullet\rangle_{(n+k,n-k)} : H^{n+k}(X,\mathbb{Z}) \otimes_{\mathbb{Z}} H^{n-k}(X,\mathbb{Z}) \xrightarrow{\bullet\cup\bullet} H^{2n}(X,\mathbb{Z}) \xrightarrow{\int_{[X]}} \mathbb{Z},$$

whose complexification reads in C^∞ de Rham cohomology

$$\langle\bullet,\bullet\rangle_{\mathbb{C},(n+k,n-k)} : H^{n+k}_{\mathrm{d}}(X) \otimes_{\mathbb{C}} H^{n-k}_{\mathrm{d}}(X) \xrightarrow{\bullet\wedge\bullet} H^{2n}_{\mathrm{d}}(X) \xrightarrow{\int_X} \mathbb{C}.$$

It is $(-1)^{n\pm k}$-symmetric.

In analogy with the setting of Riemann surfaces, let us consider the case $k=0$. Then $\langle\bullet,\bullet\rangle_n$ is a non-degenerate $(-1)^n$-symmetric bilinear form on $H^n(X,\mathbb{Z})$ and its complexified bilinear form satisfies

$$\langle H^{p',n-p'}, H^{p,n-p}\rangle_{\mathbb{C},n} = 0 \quad \text{if } p+p' \neq n. \tag{2.4.6}$$

Let us define the sesquilinear pairing

$$\mathrm{S}_0 : H^n(X,\mathbb{C}) \otimes \overline{H^n(X,\mathbb{C})} \longrightarrow \mathbb{C}$$

by (recall $\varepsilon(n) = (1)^{n(n-1)/2}$)

$$\mathrm{S}_0(\eta',\overline{\eta''}) = (-1)^n \frac{\varepsilon(n)}{(2\pi\mathsf{i})^n} \int_X \eta' \wedge \overline{\eta''}. \tag{2.4.7}$$

It is Hermitian and the Hodge decomposition is S_0-orthogonal. More generally, for any $k \in \mathbb{Z}$, we define

$$\mathrm{S}_k : H^{n+k}(X,\mathbb{C}) \otimes \overline{H^{n-k}(X,\mathbb{C})} \longrightarrow \mathbb{C}$$

by (see Notation (0.2 $*$))

$$\mathrm{S}_k(\eta',\overline{\eta''}) = (-1)^n \frac{\varepsilon(n+k)}{(2\pi\mathsf{i})^n} \int_X \eta' \wedge \overline{\eta''} = \mathrm{Sgn}(n,k) \int_X \eta' \wedge \overline{\eta''}. \tag{2.4.8}$$

We refer to Section A.3 in the appendix for explanations on how we derive such a formula.

Classical Hodge theory identifies $H^j(X,\mathbb{C})$ with the finite-dimensional space of harmonic j-forms on X. This space is equipped with the metric induced by that used on the space of C^∞-forms by means of the Hodge star operator. However, this is not the metric to be considered later in Hodge theory. Instead of the Hodge operator, one uses the Lefschetz operator induced by the class of the Kähler form or the first Chern class of an ample line bundle on X. This leads to Hodge-Lefschetz theory. The corresponding Hermitian form on $H^j(X,\mathbb{C})$ is defined in a subtler way, and its positivity is then a theorem, whose direct consequence is the Hard Lefschetz theorem.

The Lefschetz operator. Fix an ample line bundle $\mathcal{L}$ on X (for instance, any embedding of X in a projective space defines a very ample bundle, by restricting the canonical line bundle $\mathcal{O}(1)$ of the projective space to X). The first Chern class $c_1(\mathcal{L}) \in H^2(X,\mathbb{Z})$ defines a Lefschetz operator

$$\mathrm{L}_{\mathcal{L}} := c_1(\mathcal{L}) \cup \bullet : H^j(X,\mathbb{Z}) \longrightarrow H^{j+2}(X,\mathbb{Z}). \tag{2.4.9}$$

(Note that wedging on the left or on the right amounts to the same, as c_1 has degree 2.) In such a case, one can choose as a Kähler form ω on X a real $(1,1)$-form whose cohomology class in $H^2(X,\mathbb{R})$ is $c_1(\mathcal{L})$, and the Lefschetz operator $\mathrm{L}_{\mathcal{L}}$ can be lifted as the operator on differential forms obtained by wedging with ω. The Lefschetz operator has thus type $(1,1)$ with respect to the Hodge decomposition, hence sends $H^{p,q}$ to $H^{p+1,q+1}$. Denoting the latter Hodge structure by the Tate twist notation $H(1)$, we regard $\mathrm{L}_{\mathcal{L}}$ as a morphism of Hodge structures $H \to H(1)$.

Polarization in the middle dimension. It is mostly obvious that the category of pure Hodge structures of a given weight is abelian, that is, we can consider kernels and cokernels in this category in a natural way. In particular, the pure Hodge structure of weight n

$$\mathrm{P}_0(X,\mathbb{Q}) = \mathrm{Ker}[\mathrm{L}_{\mathcal{L}} : H^n(X,\mathbb{C}) \to H^{n+2}(X,\mathbb{C})(1)]$$

whose underlying $\mathbb{C}$-vector space consists of *primitive classes* in $H^n(X,\mathbb{C})$, can thus be decomposed correspondingly as $\bigoplus_{p+q=n} \mathrm{P}_0^{p,q}(X)$. Moreover, $\mathrm{P}_0(X,\mathbb{Q})$ is a direct summand of $H^n(X,\mathbb{Q})$. The orthogonality relations (2.4.6) imply that the restriction of S_0 (defined by (2.4.7)) to $\mathrm{P}_0(X,\mathbb{C})$ induces a morphism of pure $\mathbb{C}$-Hodge structures of weight n:

$$\mathrm{S}_0 : \mathrm{P}_0(X,\mathbb{C}) \otimes \overline{\mathrm{P}_0(X,\mathbb{C})} \longrightarrow \mathbb{C}(-n).$$

Classical Hodge theory states that the Hermitian form h_0 on $\mathrm{P}^n(X,\mathbb{C})$, defined by

$$\mathrm{h}_0 = (-1)^q \mathrm{S}_0 \quad \text{on } \mathrm{P}_0^{p,q}(X,\mathbb{C}) \tag{2.4.10}$$

and for which the Hodge decomposition is orthogonal, is positive definite.

Polarization in any dimension. Set now $H = \bigoplus_{k\in\mathbb{Z}} H^{n+k}(X,\mathbb{C})$ and let

$$\mathrm{S} : H \otimes \overline{H} \longrightarrow \mathbb{C}(-n) \tag{2.4.11}$$

be the sesquilinear pairing defined in such a way that $\mathrm{S}(H^{n+k}, \overline{H^{n-\ell}}) = 0$ if $k \neq \ell$ and, for every k, its restriction to $H^{n+k}(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^{n-k}(X,\mathbb{C})}$ is equal to S_k as defined by (2.4.8). Then S is Hermitian.

In order to obtain positivity results, it is necessary to choose an isomorphism between the pure Hodge structures $H^{n-k}(X,\mathbb{C})$ and $H^{n+k}(X,\mathbb{C})(k)$ for any $k \geqslant 0$ (we know that the underlying vector spaces have the same dimension, as Poincaré duality is non-degenerate). A class of good morphisms is given by the *Lefschetz operators*

$$\mathrm{X}_{\mathcal{L}} = (2\pi\mathsf{i})\mathrm{L}_{\mathcal{L}} \tag{2.4.12}$$

with $\mathrm{L}_{\mathcal{L}}$ defined above. Since $\mathrm{L}_{\mathcal{L}}$ is real, we have $\langle u, \overline{\mathrm{L}_{\mathcal{L}} v}\rangle = \langle \mathrm{L}_{\mathcal{L}} u, \overline{v}\rangle$ and from the properties of ε one deduces that

$$\mathrm{S}(u, \overline{\mathrm{X}_{\mathcal{L}} v}) = \mathrm{S}(\mathrm{X}_{\mathcal{L}} u, \overline{v}). \tag{2.4.13}$$

The *Hard Lefschetz theorem*, usually proved together with the previous results of Hodge theory, asserts that, *for any smooth complex projective variety X, any ample* *line bundle $\mathcal{L}$, and any $\ell \geqslant 1$, the ℓ-th power*

$$\mathrm{X}_{\mathcal{L}}^{\ell} : H^{n-\ell}(X, \mathbb{C}) \longrightarrow H^{n+\ell}(X, \mathbb{C})(\ell)$$

is an isomorphism. In order to express the corresponding positivity property, we consider the *primitive sub-Hodge* structure (of weight $n - \ell$)

$$\mathrm{P}_{-\ell}(X, \mathbb{C}) := \mathrm{Ker}\big[\mathrm{X}_{\mathcal{L}}^{\ell+1} : H^{n-\ell}(X, \mathbb{C}) \to H^{n+\ell+2}(X, \mathbb{C})(\ell+1)\big].$$

For $\ell \geqslant 0$, we consider the sesquilinear form

$$\mathrm{S}(\mathrm{X}_{\mathcal{L}}^{\ell}\bullet, \overline{\bullet}) = \mathrm{S}(\bullet, \overline{\mathrm{X}_{\mathcal{L}}^{\ell}\bullet}) : H^{n-\ell}(X, \mathbb{C}) \otimes \overline{H^{n-\ell}(X, \mathbb{C})} \longrightarrow \mathbb{C}, \tag{2.4.14}$$

which is in fact Hermitian. Classical Hodge theory then asserts that, for each $\ell \geqslant 0$, its restriction $\mathrm{P}_{-\ell}\mathrm{S}$ to $\mathrm{P}_{-\ell}(X, \mathbb{C}) \otimes \overline{\mathrm{P}_{-\ell}(X, \mathbb{C})}$ is a polarization, in the sense that, for each $q \geqslant 0$,

$$(-1)^q \mathrm{P}_{-\ell}\mathrm{S}(\eta, \overline{\mathrm{X}_{\mathcal{L}}^{\ell}\eta}) > 0 \quad \text{for } \eta \in \mathrm{P}_{-\ell}^{p,q}(X, \mathbb{C}) \smallsetminus \{0\}. \tag{2.4.15}$$

Anticipating the definitions in Chapter 3, we regard the graded vector space $H^{n+\bullet}(X, \mathbb{C})$ as an $\mathfrak{sl}_2$-Hodge structure, and considering the modified Weil operator $\mathrm{C}_{\mathrm{D}} = (-1)^q$ on $H^{p,q}$ and the Weil element w, the positivity property can be concisely rephrased by saying that the Hermitian form $\mathrm{S}(\bullet, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}\bullet})$ on the total cohomology space $H = \bigoplus_k H^{n+k}(X, \mathbb{C})$ is positive definite (see Section 3.2 for an interpretation in terms of $\mathfrak{sl}_2$-representations).

2.5. Polarizable Hodge structures

The previous properties of the cohomology of a projective variety can be put in an axiomatic form. This will happen to be useful as a first step to Hodge modules. We will first emphasize the notion of a $\mathbb{C}$-Hodge structure and we will indicate the additional properties brought by a $\mathbb{Q}$-structure.

2.5.a. Category of $\mathbb{C}$-Hodge structures. This is, in some sense, a category looking like that of finite dimensional complex vector spaces. In particular, it is *abelian*, that is, the kernel and cokernel of a morphism exist in this category. This category is very useful as an intermediate category for building that of mixed Hodge structures, but the main results in Hodge theory use a supplementary property, namely the existence of a polarization (see Section 2.5.b). Let us start with the oppositeness property.

***2.5.1. Definition* (Opposite filtrations).** Let us fix $w \in \mathbb{Z}$. Given two decreasing filtrations $F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}$ of a vector space $\mathcal{H}$ by vector subspaces, we say that the filtrations $F'^{\bullet}\mathcal{H}$ and $F''^{\bullet}\mathcal{H}$ are *w-opposite* if

$$\begin{cases} F'^{p}\mathcal{H} \cap F''^{w-p+1}\mathcal{H} = 0 \\ F'^{p}\mathcal{H} + F''^{w-p+1}\mathcal{H} = \mathcal{H} \end{cases} \qquad \text{for every } p \in \mathbb{Z},$$

i.e., $F'^p\mathcal{H} \oplus F''^{w-p+1}\mathcal{H} \xrightarrow{\sim} \mathcal{H}$ for every $p \in \mathbb{Z}$. Equivalently setting

$$\mathcal{H}^{p,w-p} = F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H},$$

then $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ (see Exercise 2.5(1b)).

***2.5.2. Definition* (ℂ-Hodge structure).** A $\mathbb{C}$-*Hodge structure of weight* $w \in \mathbb{Z}$

$$H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$$

consists of a *finite dimensional* complex vector space $\mathcal{H}$ equipped with two decreasing filtrations $F'^{\bullet}\mathcal{H}$ and $F''^{\bullet}\mathcal{H}$ which are w-opposite. A morphism between $\mathbb{C}$-Hodge structures is a linear morphism between the underlying vector spaces compatible with both filtrations. We denote by $\mathsf{HS}(\mathbb{C})$ the category of $\mathbb{C}$-Hodge structures of some weight w and by $\mathsf{HS}(\mathbb{C}, w)$ the full category whose objects have weight w.

The category $\mathsf{HS}(\mathbb{C})$ has the following functors lifting those existing on $\mathbb{C}$-vector spaces (see Exercise 2.7):

- tensor product $H_1 \otimes H_2$, of weight $w_1 + w_2$,
- Homomorphisms $\mathrm{Hom}(H_1, H_2)$ of weight $w_2 - w_1$,
- dual $H^\vee$ of weight $-w$,
- conjugate $\overline{H}$ of weight w,
- Hermitian dual $H^* = \overline{H}^\vee = \overline{H^\vee}$ of weight $-w$.

Let us emphasize the following statement (see Exercise 2.5).

2.5.3. Proposition. *The category* $\mathsf{HS}(\mathbb{C}, w)$ *of complex Hodge structures of weight* w *is abelian, and any morphism is strictly compatible with both filtrations and with the decomposition.* □

2.5.4. Caveat. On the other hand, the category $\mathsf{HS}(\mathbb{C})$ is *not abelian* (see an example in Exercise 2.6).

***2.5.5. Proposition* (Morphisms in $\mathsf{HS}(\mathbb{C})$).**

(1) *Let* $\varphi : H_1 \to H_2$ *be a morphism between objects of* $\mathsf{HS}(\mathbb{C}, w)$ *such that the induced morphism* $\mathcal{H}_1 \to \mathcal{H}_2$ *is injective resp. surjective. Then* $F^\bullet\mathcal{H}_1 = \varphi^{-1}F^\bullet\mathcal{H}_2$ *resp.* $F^\bullet\mathcal{H}_2 = \varphi(F^\bullet\mathcal{H}_1)$, *and* φ *is a monomorphism resp. an epimorphism in* $\mathsf{HS}(\mathbb{C}, w)$. *If moreover the induced morphism* $\mathcal{H}_1 \to \mathcal{H}_2$ *is an isomorphism, then* φ *is an isomorphism in* $\mathsf{HS}(\mathbb{C}, w)$.

(2) *There is no non-zero morphism* $\varphi : H_1 \to H_2$ *in* $\mathsf{HS}(\mathbb{C})$ *if* $w_1 > w_2$.

Proof.

(1) The first point is nothing but the reformulation that φ is strict.

(2) The image of $\mathcal{H}_1^{p,w_1-p}$ is contained in $F'^p\mathcal{H}_2 \cap F''^{w_1-p}\mathcal{H}_2$, hence in $F'^p\mathcal{H}_2 \cap F''^{w_2+1-p}\mathcal{H}_2$ since $w_1 > w_2$, and the latter space is zero by Definition 2.5.1. □

2.5.6. Twists. Given a $\mathbb{C}$-Hodge structure H of weight w and integers k, ℓ, we set $H(k,\ell) := (\mathcal{H}, F[k]'^{\bullet}\mathcal{H}, F[\ell]''^{\bullet}\mathcal{H})$ (see Convention 0.4). Then $H(k,\ell)$ is a $\mathbb{C}$-Hodge structure of weight $w-k-\ell$. If $\varphi : H_1 \to H_2$ is a morphism of $\mathbb{C}$-Hodge structures of weight w, then it is also a morphism $H_1(k,\ell) \to H_2(k,\ell)$. The twist (k,ℓ) is then an equivalence between the category $\mathsf{HS}(\mathbb{C}, w)$ with $\mathsf{HS}(\mathbb{C}, w-k-\ell)$ (morphisms are unchanged). Let us note in particular that $H^*(k,\ell) = H(-k,-\ell)^*$.

2.5.7. Definition **(Tate twist).** The symmetric twists (k,k) are called *Tate twists*. We also regard them as the tensor product with $\mathbb{C}^{\mathrm{H}}(k)$ as defined in Remark 2.2.2. We will use the notation (k,k) when we only want to consider bi-filtered objects, and (k) when we want to keep in mind the relation with classical Hodge theory. Given a morphism $\varphi : H_1 \to H_2$, we still denote by φ the morphism $\varphi \otimes \mathrm{Id} : H_1 \otimes \mathbb{C}^{\mathrm{H}}(k) \to H_2 \otimes \mathbb{C}^{\mathrm{H}}(k)$.

2.5.8. Complex Hodge structures and representations of $\mathbb{S}^1$**.** A $\mathbb{C}$-Hodge structure of weight 0 on a complex vector space $\mathcal{H}$ is nothing but a grading of this space indexed by $\mathbb{Z}$, and a morphism between such Hodge structures is nothing but a graded morphism of degree zero. Indeed, in weight 0, the summand $\mathcal{H}^{p,-p}$ can simply be written $\mathcal{H}^p$. This grading defines a continuous representation $\rho : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})$ by setting $\rho(\lambda)_{|\mathcal{H}^p} = \lambda^p \,\mathrm{Id}_{\mathcal{H}^p}$.

Conversely, any continuous representation $\rho : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})$ is of this form. This can be seen as follows. Since $\mathbb{S}^1$ is compact, one can construct a Hermitian metric on $\mathcal{H}$ which is invariant by any $\rho(\lambda)$. It follows that each $\rho(\lambda)$ is semi-simple and there is a common eigen-decomposition of $\mathcal{H}$. The eigenvalues are continuous characters on $\mathbb{S}^1$. Any such character χ takes the form $\chi(\lambda) = \lambda^p$ (note first that $|\chi| = 1$ since $|\chi(\mathbb{S}^1)|$ is compact in $\mathbb{R}_+^*$ and, if $|\chi(\lambda_o)| \neq 1$, then $|\chi(\lambda_o^k)| = |\chi(\lambda_o)|^k$ tends to 0 or ∞ if $k \to \infty$; therefore, χ is a continuous group homomorphism $\mathbb{S}^1 \to \mathbb{S}^1$, and the assertion is standard).

Recall (Schur's lemma) that the center of $\mathrm{Aut}(\mathcal{H})$ is $\mathbb{C}^*\,\mathrm{Id}$. We claim that a continuous representation $\widetilde{\rho} : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})/\mathbb{C}^*\,\mathrm{Id}$ determines a $\mathbb{C}$-Hodge structure of weight 0, up to a shift by an integer of the indices. In other words, one can lift $\widetilde{\rho}$ to a representation ρ. We first note that the morphism

$$\begin{aligned}\mathbb{R}_+^* \times \mathrm{Ker}\,|\det| &\longrightarrow \mathrm{Aut}(\mathcal{H})\\ (c,T) &\longmapsto c^{1/d}\,T \quad (d = \dim \mathcal{H})\end{aligned}$$

is an isomorphism. It follows that $\mathrm{Ker}\,|\det| \to \mathrm{Aut}(\mathcal{H})/\mathbb{R}_+^*\,\mathrm{Id}$ is an isomorphism. Similarly, $\mathrm{Ker}\,|\det|/\mathbb{S}^1\,\mathrm{Id} \simeq \mathrm{Aut}(\mathcal{H})/\mathbb{C}^*\,\mathrm{Id}$. It follows that any continuous representation $\widetilde{\rho}$ lifts as a continuous representation $\widehat{\rho} : \mathbb{S}^1 \to \mathrm{Aut}(\mathcal{H})/\mathbb{S}^1\,\mathrm{Id}$. Given a Hermitian metric h and $[T] \in \mathrm{Aut}(\mathcal{H})/\mathbb{S}^1\,\mathrm{Id}$, then $h(Tu, \overline{Tv})$ does not depend on the lift T of $[T]$ in $\mathrm{Aut}(\mathcal{H})$, and one can thus construct a $\widehat{\rho}$-invariant metric on $\mathcal{H}$. The eigenspace decomposition is well-defined, although the eigenvalues of $\widehat{\rho}(\lambda)$ are defined up to a multiplicative constant. One can fix the constant to 1 on some eigenspace, and argue as above for the other eigenspaces. The lift is not unique, and the indeterminacy produces a shift in the filtration.

2.5.9. Example.

(1) Let X be a smooth complex projective variety. Then $H^k(X,\mathbb{C})$ defines a $\mathbb{C}$-Hodge structure of weight k by setting $F'^pH^k(X,\mathbb{C}) = F^pH^k(X,\mathbb{C})$ and $F''^qH^k(X,\mathbb{C}) = \overline{F^qH^k(X,\mathbb{C})}$ and by using the isomorphism $\overline{H^k(X,\mathbb{C})} \simeq H^k(X,\mathbb{C})$ coming from the real structure $H^k(X,\mathbb{C}) \simeq \mathbb{C} \otimes_{\mathbb{R}} H^k(X,\mathbb{R})$.

(2) Let $f : X \to Y$ be a morphism between smooth projective varieties. Then the induced morphism $f^* : H^k(Y,\mathbb{C}) \to H^k(X,\mathbb{C})$ is a morphism of Hodge structures of weight k.

2.5.b. Polarized/polarizable $\mathbb{C}$-Hodge structures. In the same way Hodge structures look like complex vector spaces, *polarized $\mathbb{C}$-Hodge structures* look like vector spaces equipped with a positive definite Hermitian form. Any such object can be decomposed into an orthogonal direct sum of irreducible objects, which have dimension 1 (this follows from the classification of positive definite Hermitian forms). We will see that this remains true for polarized $\mathbb{C}$-Hodge structures (for polarizable Hodge modules in higher dimensions, the decomposition remains true, but the irreducible objects may have rank bigger than 1, fortunately). From a categorical point of view, i.e., when considering morphisms between objects, it will be convenient not to restrict to morphisms compatible with polarizations (see Section 2.5.18).

***2.5.10. Definition* (Polarization of a $\mathbb{C}$-Hodge structure, first definition)**
Given a Hodge structure H of weight w, regarded as a grading $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ of the finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$, a *polarization* is a positive definite Hermitian form h on $\mathcal{H}$ such that the grading is h-orthogonal (so h induces a positive definite Hermitian form on each $\mathcal{H}^{p,w-p}$).

Although this definition is natural and quite simple, it does not extend “flatly” in higher dimension, and this leads to emphasize the polarization S below, which is also the right object to consider when working with $\mathbb{Q}$-Hodge structures.

Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a $\mathbb{C}$-Hodge structure of weight w. By a *pre-polarization* of the $\mathbb{C}$-Hodge structure H, we mean a morphism $\mathrm{S} : H \otimes \overline{H} \to \mathbb{C}^{\mathrm{H}}(-w)$ of $\mathbb{C}$-Hodge structures of weight $2w$ (see Exercise 2.7(1) for the tensor product) such that the morphism $\mathrm{S} : H \to H^*(-w)$ that it defines is an *isomorphism.* This is nothing but a *non-degenerate* sesquilinear pairing $\mathcal{S} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$ satisfying

$$\mathcal{S}(F'^p\mathcal{H}, \overline{F''^q\mathcal{H}}) = 0 \quad \text{and} \quad \mathcal{S}(F''^p\mathcal{H}, \overline{F'^q\mathcal{H}}) = 0 \quad \text{for } p+q > w, \tag{2.5.11}$$

or, equivalently, such that the decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ is $\mathcal{S}$-orthogonal. *In the following, for the sake of simplicity, we will not distinguish between the $\mathbb{C}$-linear morphism $\mathcal{S}$ and the morphism of Hodge structures* S *that it underlies.*

The Hermitian adjoint S^* of S is the sesquilinear pairing defined by

$$\mathcal{S}^*(y, \overline{x}) := \overline{\mathcal{S}(x, \overline{y})}.$$

It also defines a morphism $\mathrm{S}^* : H\otimes\overline{H}\to\mathbb{C}^{\mathrm{H}}(-w)$ of $\mathbb{C}$-Hodge structures of weight $2w$, hence is also a pre-polarization of H. It is also an isomorphism $H(w)\to H^*$, that we identify with the twisted isomorphism $\mathrm{S}^* : H\to H^*(-w)$.

2.5.12. Definition **(The Weil operator).**

(1) The Weil operator C is the automorphism of H equal to i^{p-q} on $\mathcal{H}^{p,q}$.

(2) The Deligne-Weil operator C_{D} is the automorphism of H equal to $(-1)^q$ on $\mathcal{H}^{p,q}$.

2.5.13. Remark **(Weil operator and Tate twist).** Interpreting $H(k)$ as $H\otimes\mathbb{C}^{\mathrm{H}}(k)$, we denote by $\mathrm{C}(k)$ resp. $\mathrm{C}_{\mathrm{D}}(k)$ the tensor product of the Weil operators, and *not* the morphism C resp. C_{D} induced by the Weil operator after Tate twist (i.e., by tensoring C resp. C_{D} on H with Id on $\mathbb{C}^{\mathrm{H}}(k)$, see Definition 2.5.7). In such a way, we have $\mathrm{C}(k)=\mathrm{C}$, and $\mathrm{C}_{\mathrm{D}}(k)=(-1)^k\mathrm{C}_{\mathrm{D}}$.

Let S be a pre-polarization of H. By the S-orthogonality of the Hodge decomposition, the only nonzero pairings $\mathcal{S}(x,\overline{y})$ occur when both x,y are in the same $\mathcal{H}^{p,q}$. We conclude that

$$\mathcal{S}(\mathrm{C}_{\mathrm{D}}x,\overline{y})=\mathcal{S}(x,\overline{\mathrm{C}_{\mathrm{D}}y}).$$

This is translated as $\mathrm{C}_{\mathrm{D}}^*\circ\mathrm{S}=\mathrm{S}\circ\mathrm{C}_{\mathrm{D}}$, and also follows from the property that the Hermitian adjoint $\mathrm{C}_{\mathrm{D}}^* : \mathcal{H}^*\to\mathcal{H}^*$ of the Deligne-Weil operator C_{D} on H is the Deligne-Weil operator of $H^*(-w)$ (see Exercise 2.7(6)).

2.5.14. Definition **(Polarization of a $\mathbb{C}$-Hodge structure, second definition)**

Let $H=(\mathcal{H},F'^{\bullet}\mathcal{H},F''^{\bullet}\mathcal{H})$ be a $\mathbb{C}$-Hodge structure of weight w. A *polarization* of H is a pre-polarization $\mathrm{S} : H\otimes\overline{H}\to\mathbb{C}^{\mathrm{H}}(-w)$ satisfying

(1) S is Hermitian, i.e., $\mathcal{S}^*(y,\overline{x})=\mathcal{S}(y,\overline{x})$ for all $x,y\in\mathcal{H}$,

(2) the pairing $\mathrm{h}(x,\overline{y}):=\mathcal{S}(\mathrm{C}_{\mathrm{D}}x,\overline{y})=\mathcal{S}(x,\overline{\mathrm{C}_{\mathrm{D}}y})$ on $\mathcal{H}$ is (Hermitian) positive definite.

2.5.15. Remark **(Deligne's convention).** We adopt here a sign convention which differs by multiplication by $(-1)^w$ to the usual one, where one would instead consider the operator $\bigoplus_p(-1)^p\,\mathrm{Id}_{|\mathcal{H}^{p,q}}$.

2.5.16. Remarks **(Polarized $\mathbb{C}$-Hodge structures).** Let H be a $\mathbb{C}$-Hodge structure of weight w with polarization S.

(1) Let H^* denote the Hermitian dual complex Hodge structure (Exercise 2.7(6)). We can regard S as a morphism $H\to H^*(-w)$. Its Hermitian adjoint morphism S^* is a morphism $H(w)\to H^*$, that we can also regard as a morphism $H\to H^*(-w)$. Condition 2.5.14(1) can then be expressed by saying that S is Hermitian as such, that is, $\mathrm{S}^*=\mathrm{S}$.

(2) Regarding S as a morphism $H\to H^*(-w)$, Condition 2.5.14(2) simply says that the Hermitian form underlying $\mathrm{S}\circ\mathrm{C}_{\mathrm{D}}=(\mathrm{C}_{\mathrm{D}})^*\circ\mathrm{S}$ is positive definite.

(3) Similarly, defining the form $\overline{\mathrm{S}} : \overline{H} \otimes H \to \mathbb{C}^{\mathrm{H}}(-w)$ by $\overline{\mathrm{S}}(\overline{x}, y) = \overline{\mathrm{S}(y, \overline{x})}$, one checks that $(-1)^w \overline{\mathrm{S}}$ a polarization of $\overline{H}$, as defined by Exercise 2.7(5). One can also regard $\overline{\mathrm{S}}$ as the conjugate Hermitian morphism $\overline{H} \to \overline{H}^*(-w)$ obtained from S as given by (1).

(4) It follows from 2.5.14(2) that the decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$ is also h-orthogonal, so a polarization in the sense of the second definition 2.5.14 gives rise to a polarization in the sense of the first one 2.5.10. Notice also that

$$\mathrm{h}(x, \overline{y}) = (-1)^{w-p} \mathcal{S}(x, \overline{y}) \quad \text{on } \mathcal{H}^{p,w-p}.$$

Conversely, from h as in the first definition 2.5.10 one defines S by $\mathcal{S}(x, \overline{y}) = \mathrm{h}((\mathrm{C_D})^{-1} x, \overline{y})$ and, the decomposition being S-orthogonal, one recovers a polarization in the sense of the second definition 2.5.14.

(5) If S is a polarization of H, then $(-1)^k \mathrm{S}$ is a polarization of $H(k)$ for any $k \in \mathbb{Z}$ (this follows from the behaviour of $\mathrm{C_D}$ with respect to Tate twist).

2.5.17. Polarized Hodge structure as a filtered Hermitian pair. The definition of a polarized Hodge structure as a pair (H, S) contains some redundancy. However, it has the advantage of exhibiting the underlying Hodge structure. We give a simplified presentation, which only needs *one* filtration, together with the sesquilinear form S.

By a *filtered Hermitian pair of weight* w we mean the data $(\mathcal{H}, F^\bullet\mathcal{H}, \mathcal{S}, w)$, where w is an integer, $(\mathcal{H}, F^\bullet\mathcal{H})$ is a filtered vector space, and $\mathcal{S} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathbb{C}$ is a Hermitian sesquilinear pairing, i.e., a morphism $\mathcal{S} : \mathcal{H} \to \mathcal{H}^*$ satisfying $\mathcal{S}^* = \mathcal{S}$.

A polarized Hodge structure of weight w can be described as the data of a filtered Hermitian pair $(\mathcal{H}, F^\bullet\mathcal{H}, \mathcal{S}, w)$ subject to the following conditions:

(1) $\mathcal{S}$ is non-degenerate, i.e., induces an isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{H}^*$,

(2) if $F^\bullet\mathcal{H}^*$ is the filtration on the Hermitian dual space $\mathcal{H}^*$ naturally defined by $F^\bullet\mathcal{H}$, then $F^\bullet\mathcal{H}$ is 0-opposite to the filtration $\mathcal{S}^{-1}(F^\bullet\mathcal{H}^*)$ (which corresponds thus to $F''[w]^\bullet\mathcal{H}$),

(3) the positivity condition 2.5.14(2) holds.

A filtered Hermitian pair $(\mathcal{H}, F^\bullet\mathcal{H}, \mathcal{S}, w)$ satisfying these conditions will also be called a *polarized Hodge structure of weight* w. We then define the filtration $F''^\bullet\mathcal{H}$ by

$$F''^{w-p+1}\mathcal{H} = \overline{F^p\mathcal{H}^{\perp_{\mathcal{S}}}},$$

and (2) means that $F''^\bullet\mathcal{H}$ is w-opposite to $F^p\mathcal{H}$, then denoted by $F'^\bullet\mathcal{H}$, and the corresponding decomposition is $\mathcal{S}$-orthogonal. In this setting, the weight w can be chosen freely.

2.5.18. Category of polarizable $\mathbb{C}$-Hodge structures. A $\mathbb{C}$-Hodge structure may be polarized by many polarizations. At many places, we do not want to make a choice of a polarization, and it is enough to know that there exists one. Nevertheless, any $\mathbb{C}$-Hodge structure admits at least one polarization, as is obvious from Definition 2.5.10. Notice that this property will not remain true when considering $\mathbb{Q}$-Hodge structures (see Section 2.5.c below) or variations of $\mathbb{C}$-Hodge structure on a complex manifold, and this will lead us to distinguish the full subcategory of polarizable (instead of polarized) objects (see Definition 4.1.9). This is not needed here.

Recall that the category $\mathsf{HS}(\mathbb{C})$ is equipped with tensor product, Hom, duality and conjugation. If we are moreover given a polarization of the source terms of these operations, we naturally obtain a polarization on the resulting $\mathbb{C}$-Hodge structure (see Exercise 2.11). For example, if $H = H_1 \otimes H_2$, then

$$\mathcal{H}^{p,w-p} = \bigoplus_{p_1+p_2=p} \mathcal{H}_1^{p_1,w_1-p_1} \otimes \mathcal{H}_2^{p_2,w_2-p_2}$$

and the positive definite Hermitian forms $\mathrm{h}_1, \mathrm{h}_2$ induce such a form h on each $\mathcal{H}_1^{p_1,w-p_1} \otimes \mathcal{H}_2^{p_2,w-p_2}$, and thus on $\mathcal{H}^{p,w-p}$ by imposing that the above decomposition is h-orthogonal.

2.5.c. Real and rational (polarized) Hodge structures. A real structure on a $\mathbb{C}$-Hodge structure H is an isomorphism $\kappa : H \xrightarrow{\sim} \overline{H}$ (see Exercise 2.7(5)) such that $\overline{\kappa} \circ \kappa = \mathrm{Id}$ and $\kappa \circ \overline{\kappa} = \mathrm{Id}$. In other words, a real Hodge structure of weight w consists of the data $(\mathcal{H}_\mathbb{R}, F^\bullet\mathcal{H})$, where

(i) $\mathcal{H}_\mathbb{R}$ is a finite-dimensional $\mathbb{R}$-vector space,

(ii) $\mathcal{H} = \mathbb{C} \otimes_\mathbb{R} \mathcal{H}_\mathbb{R}$,

(iii) the filtration $F^\bullet\mathcal{H}$ is w-opposite to the conjugate filtration; equivalently, the Hodge decomposition satisfies $\mathcal{H}^{q,p} = \overline{\mathcal{H}^{p,q}}$, where the conjugation is taken with respect to the real structure $\mathcal{H}_\mathbb{R}$.

A $\mathbb{Q}$-Hodge structure $H_\mathbb{Q}$ consists of the data $(\mathcal{H}_\mathbb{Q}, H_\mathbb{R}, \mathrm{iso})$, where $\mathcal{H}_\mathbb{Q}$ is a finite-dimensional $\mathbb{Q}$-vector space and $H_\mathbb{R}$ is a real Hodge structure and iso is an isomorphism $\mathbb{R} \otimes_\mathbb{Q} \mathcal{H}_\mathbb{Q} \xrightarrow{\sim} \mathcal{H}_\mathbb{R}$. Morphisms should be compatible with the data, so that we can assume that $\mathbb{R} \otimes_\mathbb{Q} \mathcal{H}_\mathbb{Q} = \mathcal{H}_\mathbb{R}$ and iso $= \mathrm{Id}$.

Real and rational Hodge structures are preserved by the operations tensor product, Hom and duality considered in Exercise 2.7. By definition, conjugation is the identity on such Hodge structures, and therefore Hermitian duality reduces to duality. We obtain in a natural way an abelian category $\mathsf{HS}(\mathbb{Q}, w)$ (morphisms should preserve the $\mathbb{Q}$-structure on $\mathcal{H}_\mathbb{Q}$) for each integer w and a forgetful functor $\mathsf{HS}(\mathbb{Q}, w) \to \mathsf{HS}(\mathbb{C}, w)$.

A polarization of a $\mathbb{Q}$-Hodge structure of weight w is a morphism $\mathrm{S}_\mathbb{Q} : H_\mathbb{Q} \otimes H_\mathbb{Q} \to \mathbb{Q}^\mathrm{H}(-w)$ inducing a polarization of the associated $\mathbb{C}$-Hodge structure. A typical example is given by the geometric setting (2.4.11). Although $\mathbb{C}$-Hodge structures can always be polarized, imposing that S is defined over $\mathbb{Q}$ is a constraint that cannot be always satisfied for a $\mathbb{Q}$-Hodge structure. This makes stronger the notion of polarizability for a $\mathbb{Q}$-Hodge structure, and leads us to denote the category of polarizable $\mathbb{Q}$-Hodge structures of weight w by $\mathsf{pHS}(\mathbb{Q}, w)$. Exercises 2.11 and 2.12 can be adapted to the rational setting:

2.5.19. Proposition. *The full subcategory $\mathsf{pHS}(\mathbb{Q}, w)$ of $\mathsf{HS}(\mathbb{Q}, w)$ is abelian and stable by direct summand in $\mathsf{HS}(\mathbb{Q}, w)$. The tensor product, Hom and duality functors on $\mathsf{HS}(\mathbb{Q})$ preserve $\mathsf{pHS}(\mathbb{Q})$.* $\square$

2.6. Mixed Hodge structures

Our aim is to construct an abelian category which contains all the categories $\mathsf{HS}(\mathbb{C}, j)$ as full subcategories. The category $\mathsf{HS}(\mathbb{C})$ of Hodge structures of arbitrary weight is not suitable, since it is not abelian (see Exercise 2.6). Instead, we will use the category T of triples defined in Remark 2.6.a below, and we will regard an object of $\mathsf{HS}(\mathbb{C}, j)$ as an object of T of weight j.

2.6.a. An ambient abelian category. In order to regard all categories $\mathsf{HS}(\mathbb{C}, w)$ ($w \in \mathbb{Z}$) as full subcategories of a single *abelian* category, one has to modify a little the presentation of $\mathsf{HS}(\mathbb{C}, w)$. We anticipate here the constructions in Chapter 5, which we refer to for details (see also Convention 0.4). The starting point is that the category of filtered vector spaces and filtered morphisms is not abelian, and one can use the Rees trick (see Section 5.1.3) to replace it with an abelian category.

A finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ with an exhaustive filtration $F^\bullet\mathcal{H}$ defines a free graded $\mathbb{C}[z]$-module $\widetilde{\mathcal{H}}$ of finite rank by the formula $\widetilde{\mathcal{H}} = \bigoplus_p F^p\mathcal{H}z^{-p}$ (the term $F^p\mathcal{H}z^{-p}$ is in degree p). On the other hand, the category $\mathsf{Mod}_{\mathrm{gr\,ft}}(\mathbb{C}[z])$ of graded $\mathbb{C}[z]$-modules of finite type (whose morphisms are graded of degree zero) is abelian, but not all its objects are free. The free modules in this category are also called *strict objects*. Strict objects are in one-two-one correspondence with filtered vector spaces: from a strict object $\widetilde{\mathcal{H}}$ one recovers the vector space $\mathcal{H} := \widetilde{\mathcal{H}}/(z-1)\widetilde{\mathcal{H}}$, and the grading $\widetilde{\mathcal{H}} = \bigoplus \widetilde{\mathcal{H}}^p$ induces a filtration $F^p\mathcal{H} := \widetilde{\mathcal{H}}^p/\widetilde{\mathcal{H}}^p \cap (z-1)\widetilde{\mathcal{H}}$.

Similarly, we say that a morphism in this category is *strict* if its kernel and cokernel are strict. A morphism between strict objects corresponds to a filtered morphism between the corresponding filtered vector spaces. A morphism between strict objects is strict if and only if its cokernel is strict.

To a bi-filtered vector space $(\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$ we associate the following pair of filtered vector spaces:

- $(\mathcal{H}', F^\bullet\mathcal{H}') := (\mathcal{H}, F'^\bullet\mathcal{H})$,
- $(\mathcal{H}'', F^\bullet\mathcal{H}'') := (\overline{\mathcal{H}}, \overline{F''^\bullet\mathcal{H}})$.

We thus have an isomorphism $\gamma : \mathcal{H}' \xrightarrow{\sim} \overline{\mathcal{H}''}$ (the identity). We associate to $(\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$ the object $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \gamma)$ (where we regard γ as an homogeneous isomorphism of degree zero). In such a way, we embed the (non abelian) category of bi-filtered vector spaces (and morphisms compatible with both filtrations) as a full subcategory of the category T of triples $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \gamma)$ consisting of two graded $\mathbb{C}[z]$-modules $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$ and an isomorphism $\gamma : \mathcal{H}' \xrightarrow{\sim} \overline{\mathcal{H}''}$. Morphisms are pairs of graded morphisms (φ', φ'') of degree zero whose restriction to $z = 1$ are compatible with γ. One recovers a bi-filtered vector space if $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$ are *strict* (i.e., $\mathbb{C}[z]$-flat, see Exercise 5.2) by setting $\mathcal{H} = \mathcal{H}'$, by getting the filtrations $F^\bullet\mathcal{H}', \overline{F^\bullet\mathcal{H}''}$ from $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$, and by transporting them to $\mathcal{H}$ by the isomorphisms Id and γ^{-1}.

2.6.1. Lemma. *For every $j \in \mathbb{Z}$, the category $\mathsf{HS}(\mathbb{C}, j)$ is a full subcategory of T which satisfies the following properties.*

(1) $\mathsf{HS}(\mathbb{C}, j)$ *is stable by* Ker *and* Coker *in* T.
(2) *For every* $j > k$, $\mathrm{Hom}_{\mathsf{T}}(\mathsf{HS}(\mathbb{C}, j), \mathsf{HS}(\mathbb{C}, k)) = 0$.

Proof. The first point follows from the abelianity of the full subcategory $\mathsf{HS}(\mathbb{C}, j)$ of T, and the second one is Proposition 2.5.5(2). $\square$

2.6.b. Abelian categories and W-filtrations. Let A be an abelian category. The category WA consisting of objects of A equipped with a finite exhaustive[(1)] increasing filtration indexed by $\mathbb{Z}$, and morphisms compatible with filtrations, is an additive category which has kernels and cokernels, but which is not abelian in general. For a filtered object $(H, W_\bullet H)$ and for every $k \leqslant \ell$, the object $(W_\ell H, W_\bullet H)_{\bullet \leqslant \ell}$ is a subobject of $(H, W_\bullet H)$ (i.e., the kernel of $(W_\ell H, W_\bullet H)_{\bullet \leqslant \ell} \to (H, W_\bullet H)$ is zero) and the object $(W_\ell H/W_k H, W_\bullet H/W_k H)_{k \leqslant \bullet \leqslant \ell}$ is a quotient object of $(W_\ell H, W_\bullet H)_{\bullet \leqslant \ell}$ (i.e., the cokernel of $(W_\ell H, W_\bullet H)_{\bullet \leqslant \ell} \to (W_\ell H/W_k H, W_\bullet H/W_k H)_{k \leqslant \bullet \leqslant \ell}$ is zero).

2.6.2. Definition. Let A_j ($j \in \mathbb{Z}$) be full abelian subcategories of A which are stable by Ker and Coker in A an such that, for every $j > k$, $\mathrm{Hom}_{\mathsf{A}}(\mathsf{A}_j, \mathsf{A}_k) = 0$. We will denote by $\mathsf{A}_\bullet$ the data $(\mathsf{A}, (\mathsf{A}_j)_{j \in \mathbb{Z}})$ and by $\mathsf{WA}_\bullet$ the full subcategory of WA consisting of objects such that for every j, $\mathrm{gr}_j^W \in \mathsf{A}_j$.

2.6.3. Proposition. *The category* $\mathsf{WA}_\bullet$ *is abelian, and morphisms are strictly compatible with* $W_\bullet$.

Proof. It suffices to show the second assertion. Let $\varphi : (H, W_\bullet H) \to (H', W_\bullet H')$ be a morphism. It is proved by induction on the length of $W_\bullet$. Consider the diagram of exact sequences in A:

$$
\begin{tikzcd}
0 \arrow[r] & W_{j-1}H \arrow[r] \arrow[d, "\varphi_{j-1}"'] & W_j H \arrow[r] \arrow[d, "\varphi_j"] & \mathrm{gr}_j^W H \arrow[r] \arrow[d, "\mathrm{gr}_j^W \varphi"] & 0 \\
0 \arrow[r] & W_{j-1}H' \arrow[r] & W_j H' \arrow[r] & \mathrm{gr}_j^W H' \arrow[r] & 0
\end{tikzcd}
\tag{2.6.4}
$$

Due to the inductive assumption, the assertion reduces to proving in A:

$$\mathrm{Im}\,\varphi_{j-1} = \mathrm{Im}\,\varphi_j \cap W_{j-1}H',$$

equivalently, $\mathrm{Coker}\,\varphi_{j-1} \to \mathrm{Coker}\,\varphi_j$ is a monomorphism. This follows from the assumption on the categories A_j and the snake lemma, which imply that the short sequences of Ker's and that of Coker's are exact. $\square$

---

(1) Exhaustivity means that, for a given object H in A, we have $W_\ell H = 0$ for $\ell \ll 0$ and $W_\ell H = H$ for $\ell \gg 0$.

2.6.c. Mixed Hodge structures. Following Definition 2.6.2, we will denote by $\mathsf{HS}_\bullet(\mathbb{C})$ the data $\big(\mathsf{T}, \mathsf{HS}(\mathbb{C}, j)_{j\in\mathbb{Z}}\big)$.

***2.6.5. Definition* (Mixed Hodge structures).** The category $\mathsf{MHS}(\mathbb{C})$ is the category $\mathsf{WHS}_\bullet(\mathbb{C})$.

Proposition 2.6.3 and Lemma 2.6.1 immediately imply the following corollary.

2.6.6. Corollary. *The category $\mathsf{MHS}(\mathbb{C})$ is abelian, and morphisms are strictly compatible with $W_\bullet$.* ☐

2.6.7. Remark. Let us make explicit the notion of mixed Hodge structure.

(1) A *mixed $\mathbb{C}$-Hodge structure* consists of

(a) a finite dimensional $\mathbb{C}$-vector space $\mathcal{H}$ equipped with an exhaustive increasing filtration $W_\bullet\mathcal{H}$ indexed by $\mathbb{Z}$,

(b) decreasing filtrations $F^\bullet\mathcal{H}$ ($F = F'$ or F''),

such that each quotient space $\mathrm{gr}_\ell^W\mathcal{H} := W_\ell\mathcal{H}/W_{\ell-1}\mathcal{H}$, when equipped with the *induced filtrations*

$$F^p\mathrm{gr}_\ell^W\mathcal{H} := \frac{F^p\mathcal{H}\cap W_\ell\mathcal{H}}{F^p\mathcal{H}\cap W_{\ell-1}\mathcal{H}}$$

is a $\mathbb{C}$-Hodge structure of weight ℓ. From the point of view of $\mathbb{C}$-Hodge triples (the category A), a mixed $\mathbb{C}$-Hodge triple consists of a W-filtered triple $(H, W_\bullet H)$ such that H is strict and each $\mathrm{gr}_\ell H$ is a $\mathbb{C}$-Hodge triple of weight ℓ. In particular it is strict, hence Remark 5.2.2(5) applies.

(2) A morphism of mixed $\mathbb{C}$-Hodge structures

$$(H_1, W_\bullet H_1) \longrightarrow (H_2, W_\bullet H_2)$$

is a morphism $\mathcal{H}_1 \to \mathcal{H}_2$ which is compatible with the filtrations $W_\bullet$ and with the filtrations $F'^\bullet, F''^\bullet$. Equivalently, it consists of a pair of bi-filtered morphisms

$$\begin{cases}(\mathcal{H}'_1, F^\bullet\mathcal{H}'_1, W_\bullet\mathcal{H}'_1) \to (\mathcal{H}'_2, F^\bullet\mathcal{H}'_2, W_\bullet\mathcal{H}'_2),\\ (\mathcal{H}''_2, F^\bullet\mathcal{H}''_2, W_\bullet\mathcal{H}''_2) \to (\mathcal{H}''_1, F^\bullet\mathcal{H}''_1, W_\bullet\mathcal{H}''_1)\end{cases}$$

compatible with γ_1, γ_2.

(3) The category $\mathsf{MHS}(\mathbb{C})$ of mixed Hodge structures defined by 2.6.5, i.e., as in (1) and (2), is equipped with endofunctors, the twists (k, ℓ) ($k, \ell \in \mathbb{Z}$) defined by

$$(H, W_\bullet H)(k, \ell) := \big((H(k, \ell), W[-(k+\ell)]_\bullet H(k, \ell))\big).$$

(4) We say that a mixed Hodge structure H is

- *pure* (of weight w) if $\mathrm{gr}_\ell^W H = 0$ for $\ell \neq w$,
- *graded-polarizable* if $\mathrm{gr}_\ell^W H$ is polarizable for every $\ell \in \mathbb{Z}$.

2.6.8. Proposition. *Any morphism in the abelian category $\mathsf{MHS}(\mathbb{C})$ is strictly compatible with both filtrations $F^\bullet$ and $W_\bullet$.*

Proof. Note that for every morphism φ, the graded morphism $\mathrm{gr}_\ell^W\varphi$ is F-strict, according to Exercise 2.5(2). The proof is then by induction on the length of $W_\bullet$, by considering the diagram (2.6.4). Since the sequence of cokernels is exact, the cokernel of φ_j is strict, and we can apply the criterion of Exercise 5.1(3). $\square$

Since any $\mathbb{C}$-Hodge structure is polarizable, any mixed Hodge structure is *graded-polarizable.*

2.6.d. Mixed $\mathbb{Q}$-Hodge structures. A real mixed Hodge structure is a complex mixed Hodge structure together with an isomorphism $\kappa : (H, W_\bullet H) \xrightarrow{\sim} (\overline{H}, W_\bullet\overline{H})$ satisfying $\kappa\circ\overline{\kappa} = \mathrm{Id}$ and $\overline{\kappa}\circ\kappa = \mathrm{Id}$. We have a description similar to that of Section 2.5.c.

The category $\mathsf{MHS}(\mathbb{Q})$ of (graded-polarizable) mixed $\mathbb{Q}$-Hodge structure consists of objects $\big(\mathcal{H}_\mathbb{Q}, W_\bullet\mathcal{H}_\mathbb{Q}), (H_\mathbb{R}, W_\bullet H_\mathbb{R}), \mathrm{iso}\big)$, where

- $(H_\mathbb{R}, W_\bullet H_\mathbb{R})$ is a real mixed Hodge structure
- $W_\bullet\mathcal{H}_\mathbb{Q}$ is an exhaustive filtration of the finite-dimensional $\mathbb{Q}$-vector space $\mathcal{H}_\mathbb{Q}$,
- iso is a filtered isomorphism $\mathbb{R}\otimes(\mathcal{H}_\mathbb{Q}, W_\bullet\mathcal{H}_\mathbb{Q}) \xrightarrow{\sim} (\mathcal{H}_\mathbb{R}, W_\bullet\mathcal{H}_\mathbb{R})$
- for each $\ell\in\mathbb{Z}$, $(\mathrm{gr}_\ell^W\mathcal{H}_\mathbb{Q}, \mathrm{gr}_\ell^W H_\mathbb{R})$ is a polarizable $\mathbb{Q}$-Hodge structure, i.e., an object of $\mathsf{pHS}(\mathbb{Q},\ell)$.

The morphisms are $\mathbb{Q}$-linear morphisms between the $\mathbb{Q}$-vector spaces which preserve the filtrations. In a way analogous to Corollary 2.6.6 and Proposition2.6.8, we obtain the fundamental result:

2.6.9. Proposition. *The category $\mathsf{MHS}(\mathbb{Q})$ is abelian. Any morphism is strictly compatible with both filtrations $F^\bullet$ and $W_\bullet$ (on the $\mathbb{C}$- and $\mathbb{Q}$-vector spaces respectively).* $\square$

The main result in the theory of mixed Hodge structures is due to Deligne [**Del71b, Del74**].

***2.6.10. Theorem* (Hodge-Deligne Theorem, mixed case).** *Let X be a complex quasi-projective variety. Then the cohomology $H^k(X,\mathbb{Q})$ and the cohomology with compact supports $H^k_\mathrm{c}(X,\mathbb{Q})$ admit a canonical (graded-polarizable) mixed Hodge structure for each k. The weights of $H^k(X,\mathbb{Q})$ are $\geqslant k$ and those of $H^k_\mathrm{c}(X,\mathbb{Q})$ are $\leqslant k$.* $\square$

2.7. Exercises

***Exercise 2.1* (Conjugate vector space).** Let $\mathcal{H}$ be a complex vector space. If we only remember the $\mathbb{R}$-structure it is an $\mathbb{R}$-vector space. Show that the action of $\mathbb{C}$ defined by

$$\lambda\cdot x := \overline{\lambda}x, \quad \lambda\in\mathbb{C},\ x\in\mathcal{H}_\mathbb{R},$$

defines a new complex vector space, *the conjugate $\overline{\mathcal{H}}$ of* $\mathcal{H}$, which has the same underlying $\mathbb{R}$-vector space as $\mathcal{H}$. Given an element $x\in\mathcal{H}$, we denote by $\overline{x}$ the same element regarded as belonging to $\overline{\mathcal{H}}$. Show the following (tautological) formula

$$\lambda\overline{x} = \overline{\overline{\lambda}x}.$$

Show that $\mathbb{C}$-linear morphisms are transformed by the rule

$$\overline{\varphi}(\overline{x}) = \overline{\varphi(x)}.$$

Show that $\overline{\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)} = \mathrm{Hom}_{\mathbb{C}}(\overline{\mathcal{H}}_1, \overline{\mathcal{H}}_2)$ by the correspondence given above. Similarly, show that

$$\overline{\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2} = \overline{\mathcal{H}}_1 \otimes_{\mathbb{C}} \overline{\mathcal{H}}_2.$$

***Exercise 2.2* (Finite dimensional Hilbert spaces).** Consider the category of finite-dimensional $\mathbb{C}$-vector spaces equipped with a positive definite Hermitian form h. For two objects $(\mathcal{H}_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, \mathrm{h}_2)$ in this category, equip $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ and $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ of natural positive definite Hermitian forms. Show that the Hermitian forms on $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ and $\mathcal{H}_1^\vee \otimes_{\mathbb{C}} \mathcal{H}_2$ coincide (where $\mathcal{H}_1^\vee := \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathbb{C})$). [*Hint*: Fix a h_i-orthonormal basis $\boldsymbol{\varepsilon}_i$ of $\mathcal{H}_i$ $(i = 1, 2)$ and define h on $\mathcal{H}_1 \otimes \mathcal{H}_2$ so that $\boldsymbol{\varepsilon}_1 \otimes \boldsymbol{\varepsilon}_2$ is an orthonormal basis, etc.]

***Exercise 2.3* (Algebraic de Rham complex).** Using the *Zariski topology* on X, we get an algebraic variety denoted by X^{alg}. In the algebraic category, it is also possible to define a de Rham complex, called the *algebraic de Rham complex*.

(1) Is the algebraic de Rham complex a resolution of the constant sheaf $\mathbb{C}_{X^{\mathrm{alg}}}$?
(2) Do we have $H^\bullet(X^{\mathrm{alg}}, \mathbb{C}) = \boldsymbol{H}^\bullet\big(X^{\mathrm{alg}}, (\Omega^\bullet_{X^{\mathrm{alg}}}, \mathrm{d})\big)$?

Exercise 2.4. Check that the sesquilinear form of (2.4.14) is Hermitian.

***Exercise 2.5* (The category $\mathsf{HS}(\mathbb{C}, w)$ is abelian).**

(1) Given two decreasing filtrations $F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}$ of a vector space $\mathcal{H}$ by vector subspaces, show that the following properties are equivalent:

(a) the filtrations $F'^\bullet\mathcal{H}$ and $F''^\bullet\mathcal{H}$ are *w-opposite*;
(b) setting $\mathcal{H}^{p,w-p} = F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H}$, then $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$.

(2) (*Strictness of morphisms*) Show that a morphism $\varphi : H_1 \to H_2$ between objects of $\mathsf{HS}(\mathbb{C}, w)$ preserves the decomposition (1b) as well. Conclude that it is *strictly* compatible with both filtrations, that is, $\varphi(F^\bullet\mathcal{H}_1) = \varphi(\mathcal{H}_1) \cap F^\bullet\mathcal{H}_2$ (with $F = F'$ or $F = F''$). Deduce that, if H' is a sub-object of H in $\mathsf{HS}(\mathbb{C}, w)$, i.e., there is a morphism $H' \to H$ in $\mathsf{HS}(\mathbb{C}, w)$ whose induced morphism $\mathcal{H}' \to \mathcal{H}$ is injective, then $F^\bullet\mathcal{H}' = \mathcal{H}' \cap F^\bullet\mathcal{H}$ for $F = F'$ and $F = F''$, and $\mathcal{H}'^{p,q} = \mathcal{H}' \cap \mathcal{H}^{p,q}$.

(3) (*Abelianity*) Conclude that the category $\mathsf{HS}(\mathbb{C}, w)$ is *abelian*.

***Exercise 2.6* (Non-abelianity).** Consider a linear morphism $\varphi : \mathcal{H}_1^{1,0} \oplus \mathcal{H}_1^{0,1} \to \mathcal{H}_2^{2,0} \oplus \mathcal{H}_2^{1,1} \oplus \mathcal{H}_2^{0,2}$ sending $\mathcal{H}_1^{1,0}$ into $\mathcal{H}_2^{2,0} \oplus \mathcal{H}_2^{1,1}$ and $\mathcal{H}_1^{0,1}$ into $\mathcal{H}_2^{1,1} \oplus \mathcal{H}_2^{0,2}$, and check when it is strict. [*Hint*: Write $\varphi = \varphi^1 \oplus \varphi^0$ with $\varphi^1 = \varphi^1_{2,0} \oplus \varphi^1_{1,1}$ and $\varphi^0 = \varphi^0_{1,1} \oplus \varphi^0_{0,2}$, so that $\mathrm{gr}^1\varphi = \varphi^1_{1,1}$ and $\mathrm{gr}^0\varphi = \varphi^0_{0,2}$.] Conclude that the category $\mathsf{HS}(\mathbb{C})$ is not abelian.

***Exercise 2.7* (Operations on filtrations and oppositeness).** Let H_1, H_2, H be $\mathbb{C}$-Hodge structures of respective weights w_1, w_2, w.

(1) (*Tensor product*) One defines $H_1 \otimes H_2$ so that the underlying vector space is $\mathcal{H}_1 \otimes_{\mathbb{C}} \mathcal{H}_2$ and the filtration on the tensor product is

$$F^p(\mathcal{H}_1 \otimes \mathcal{H}_2) = \sum_{p_1+p_2=p} F^{p_1}\mathcal{H}_1 \otimes F^{p_2}\mathcal{H}_2.$$

Show that $(F'^{\bullet}(\mathcal{H}_1 \otimes \mathcal{H}_2), F''^{\bullet}(\mathcal{H}_1 \otimes \mathcal{H}_2))$ are $(w_1 + w_2)$-opposite.

(2) (*Hom*) One defines $\mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2)$ so that the underlying vector space is $\mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2)$ and the filtration on the space of linear morphisms is

$$F^p \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2) = \{f \in \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_1, \mathcal{H}_2) \mid \forall\, k \in \mathbb{Z},\ f(F^k\mathcal{H}_1) \subset F^{p+k}\mathcal{H}_2\}.$$

Show that $(F'^{\bullet} \mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2), F''^{\bullet} \mathrm{Hom}(\mathcal{H}_1, \mathcal{H}_2))$ are $(w_2 - w_1)$-opposite.

(3) (*Dual*) One sets $H^\vee = \mathrm{Hom}(H, \mathbb{C}^{\mathrm{H}})$ with $\mathbb{C}^{\mathrm{H}} := \mathbb{C}^{\mathrm{H}}(0)$ (see Section 2.2), which is a pure Hodge structure of weight $-w$ according to (2). Show that the filtrations on the dual space $\mathcal{H}^\vee$ are given by

$$F'^p\mathcal{H}^\vee = (F'^{-p+1}\mathcal{H})^\perp, \qquad F''^p\mathcal{H}^\vee = (F''^{-p+1}\mathcal{H})^\perp,$$

and that we have

$$\mathrm{gr}^p_{F'}\mathcal{H}^\vee \simeq \big(\mathrm{gr}^{-p}_{F'}\mathcal{H}\big)^\vee, \qquad \mathrm{gr}^p_{F''}\mathcal{H}^\vee \simeq \big(\mathrm{gr}^{-p}_{F''}\mathcal{H}\big)^\vee, \quad (\mathcal{H}^\vee)^{p,q} = (\mathcal{H}^{-p,-q})^\vee.$$

(4) Identify $\mathrm{Hom}(H_1, H_2)$ with $H_1^\vee \otimes H_2$.

(5) (*Conjugation*) Let $\overline{\mathcal{H}}$ be the complex conjugate of $\mathcal{H}$ (see Exercise 2.1). Consider the bi-filtered vector space $\overline{H} := (\overline{\mathcal{H}}, \overline{F''^{\bullet}\mathcal{H}}, \overline{F'^{\bullet}\mathcal{H}})$. Show that $\overline{H} \in \mathsf{HS}(\mathbb{C}, w)$ and $\overline{\mathcal{H}^{p,q}} = \overline{\mathcal{H}}^{q,p}$.

(6) (*Hermitian duality*) Define the Hermitian dual Hodge structure H^* as the conjugate dual Hodge structure $\overline{H}^\vee$. Deduce that it is an object of $\mathsf{HS}(\mathbb{C}, -w)$ and that

$$(\mathcal{H}^*)^{p,q} = (\mathcal{H}^{-q,-p})^*.$$

***Exercise 2.8* (Behaviour with respect to Tate twist).** Show the following behaviour of the functors of Exercise 2.7 with respect to Tate twist:

- $H_1(k) \otimes H_2 = H_1 \otimes H_2(k) = (H_1 \otimes H_2)(k)$,
- $\mathrm{Hom}(H_1(k), H_2) = \mathrm{Hom}(H_1, H_2(-k)) = \mathrm{Hom}(H_1, H_2)(k)$,
- $H^\vee(k) = H(-k)^\vee$,
- $\overline{H}(k) = \overline{H(k)}$,
- $H^*(k) = H(-k)^*$.

***Exercise 2.9* (The Hodge polynomial).** Let H be a Hodge structure of weight w with Hodge decomposition $\mathcal{H} = \bigoplus_{p+q=w} \mathcal{H}^{p,q}$. The Hodge polynomial $P_h(H) \in \mathbb{Z}[u, v, u^{-1}, v^{-1}]$ is the two-variable Laurent polynomial defined as $\sum_{p,q\in\mathbb{Z}} h^{p,q}u^pv^q$ with $h^{p,q} = \dim \mathcal{H}^{p,q}$. This is a homogeneous Laurent polynomial of degree w. Show the following formulas:

$$\begin{aligned}
P_h(H_1\otimes H_2)(u,v) &= P_h(H_1)(u,v)\cdot P_h(H_2)(u,v),\\
P_h(\mathrm{Hom}(H_1,H_2))(u,v) &= P_h(H_1)(u^{-1},v^{-1})\cdot P_h(H_2)(u,v),\\
P_h(H^\vee)(u,v) &= P_h(H)(u^{-1},v^{-1}),\\
P_h(H(k))(u,v) &= P_h(H)(u,v)\cdot(uv)^{-k}.
\end{aligned}$$

Exercise 2.10 **(Polarization and twist).** Show that, if (H,S) is a polarized Hodge structure of weight w, then $(H(k,\ell),(-1)^\ell\mathrm{S})$ is a polarized Hodge structure of weight $w-k-\ell$. In particular, considering the Tate twist, $(H(k),(-1)^k\mathrm{S})$ is a polarized Hodge structure of weight $w-2k$.

Exercise 2.11 **(Operations on polarized Hodge structures).** Show the following for polarized Hodge structures $(H_1,\mathrm{S}_1),(H_2,\mathrm{S}_2),(H,\mathrm{S})$:

(1) (*Tensor product*) $\mathrm{S}_1\otimes\mathrm{S}_2 : (H_1\otimes H_2)\otimes\overline{H_1\otimes H_2}) = H_1\otimes\overline{H}_1\otimes H_2\otimes\overline{H}_2\to\mathbb{C}^{\mathrm{H}}(-(w_1+w_2))$ is a polarization of $H_1\otimes H_2$.

(2) (*Dual*) Using the interpretation (Remark 2.5.16(1)) of S as a Hermitian morphism $H\to H^*(-w)$, and the definition of $\overline{\mathrm{S}}$ in Remark 2.5.16(3)), show that $\mathrm{S}^\vee := (-1)^w\overline{\mathrm{S}^*}$ is a polarization of $H^\vee$.

Exercise 2.12 **(Polarization on $\mathbb{C}$-Hodge sub or quotient structures)**

Let S be a polarization (Definition 2.5.14) of a $\mathbb{C}$-Hodge structure H of weight w. Let H_1 be a $\mathbb{C}$-Hodge sub-structure of weight w of H (see Proposition 2.5.5(1)).

(1) Show that the restriction S_1 of S to H_1 is a polarization of H_1. [*Hint*: Use that the restriction of a positive definite Hermitian form to a subspace remains positive definite.]

(2) Deduce that (H_1,S_1) is a direct summand of (H,S) in the category of polarized $\mathbb{C}$-Hodge structures of weight w. [*Hint*: Define $\mathcal{H}_2$ to be $\mathcal{H}_1^\perp$, where the orthogonal is taken with respect to $\mathcal{S}$; use (1) to show that $(\mathcal{H},\mathcal{S})=(\mathcal{H}_1,\mathcal{S}_1)\oplus(\mathcal{H}_2,\mathcal{S}_2)$; show similarly that $\mathcal{H}_2^{p,w-p}:=\mathcal{H}_2\cap\mathcal{H}^{p,w-p}=\mathcal{H}_1^{p,w-p,\perp}$ for every p and conclude that H_2 is a $\mathbb{C}$-Hodge structure of weight w, which is polarized by S_2.]

(3) Argue similarly with a quotient $\mathbb{C}$-Hodge structure.

Exercise 2.13 **(Semi-simple $\mathbb{C}$-Hodge structures).** Show the following:

(1) A $\mathbb{C}$-Hodge structure H of weight w is simple (i.e., does not admit any nontrivial $\mathbb{C}$-Hodge sub-structure) if and only if $\dim_{\mathbb{C}}\mathcal{H}=1$.

(2) Any $\mathbb{C}$-Hodge structure is semi-simple as such.

2.8. Comments

Sections 2.3 and 2.4 give a very brief abstract of classical Hodge theory, for which various references exist: Hodge's book [**Hod41**] is of course the first one; more recently, Griffiths and Harris' book [**GH78**], Demailly's introductory article [**Dem96**] and Voisin's book [**Voi02**] are modern references. The point of view of an abstract Hodge structure, as emphasized by Deligne in [**Del71a, Del71b**], is taken up in Peters and Steenbrink's book [**PS08**], which we have tried to follow with respect to notation at least.

In Hodge theory, the $\mathbb{Q}$-structure (or, better, the $\mathbb{Z}$-structure) is usually emphasized, as both Hodge and $\mathbb{Q}$-structures give information on the transcendental aspects of algebraic varieties, by means of the periods for example. It may then look strange to focus, as we did in this chapter, and as is also done in [**Kas86b, KK87**] and [**SV11**], on one aspect of the theory, namely that of complex Hodge structures, where the $\mathbb{Q}$-structure is absent, and so is any real structure. The main reason is that this is a preparation to the theory in higher dimensions, where the analytic and the rational structures diverge with respect to the tools needed for expressing them. On the one hand, the analytic part of the theory needs the introduction of holonomic $\mathcal{D}$-modules (replacing $\mathbb{C}$-vector spaces), while on the other hand the rational structure makes use of the theory of $\mathbb{Q}$-perverse sheaves (replacing $\mathbb{Q}$-vector spaces). The relation between both theories is provided by the Riemann-Hilbert correspondence, in the general framework developed by Kashiwara [**Kas84**] and Mebkhout [**Meb84a, Meb84b**] (see also [**Meb89**] and [**Meb04**]). The theory of Hodge modules developed by Saito [**Sai88, Sai90**] combines both structures, as desirable, but this leads to developing fine comparison results between the analytic and the rational theory by means of the Riemann-Hilbert correspondence. This is done in [**Sai88**] and also in [**Sai89a**]. In order to simplify the text and focus on the very Hodge aspects of the theory, we emphasize on $\mathbb{C}$-Hodge structures, and consider the $\mathbb{Q}$-structure as an additional property, whose relations with the $\mathbb{C}$-Hodge structure are governed by the Riemann-Hilbert correspondence.

Developing the theory from the complex point of view also has the advantage of emphasizing the relation with the theory of twistor $\mathcal{D}$-modules, as developed in [**Sab05, Moc02, Moc07, Moc15**]. In fact, the idea of introducing a sesquilinear pairing $\mathfrak{s}$ is inspired by the latter theory, where one does not expect any $\mathbb{Q}$- or $\mathbb{R}$-structure in general, and where one is forced to develop the theory with a complex approach only. The category of triples that will be introduced in Section 5.2 mimics the notion of twistor structure, introduced by Simpson in [**Sim97**], and adapted for a higher dimensional use in [**Sab05**]. The somewhat strange idea to replace an isomorphism by a sesquilinear pairing is motivated by the higher dimensional case, already for a variation of Hodge structure, where among the two filtrations considered in Definition 2.5.1, one varies in a holomorphic way and the other one in an anti-holomorphic way. Also, the idea of emphasizing the Rees module of a filtration, as in Remark 2.6.a, is much inspired by the theory of twistor $\mathcal{D}$-modules.

Also, in complex Hodge theory, the (Tate) twist is more flexible since we can reduce to weight zero any complex Hodge structure of weight $w \in \mathbb{Z}$. However, we will not use this possibility in order to keep the relation with standard Hodge theory as close as possible.

Mixed Hodge structures are quickly introduced in Section 2.6. This fundamental notion, envisioned by Grothendieck as part of the realization properties of a theory of motives, and realized by Deligne in [**Del71a, Del71b, Del74**], is explained carefully in [**PS08**, Chap. 3]. In the theory of pure Hodge modules, it only appears through the disguise of a Hodge-Lefschetz structure considered in Chapter 3.

CHAPTER 3

HODGE-LEFSCHETZ STRUCTURES

Summary. We develop the notion of a Hodge-Lefschetz structure as the first example of a mixed Hodge structure. The total cohomology of a smooth complex projective variety, together with the Chern class of an ample line bundle, gives rise to the notion of $\mathfrak{sl}_2$-Hodge structure. On the other hand, degenerations of 1-parameter families of smooth complex projective varieties are the main provider of Hodge-Lefschetz structures. Vanishing cycles of holomorphic functions with isolated critical points also produce such structures. The S-decomposition theorem 3.4.22 is the main result in this chapter.

3.1. $\mathfrak{sl}_2$-representations and quivers

3.1.a. $\mathfrak{sl}_2$-representations. The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ is generated by the three elements usually denoted by $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ which satisfy the relations

$$[\mathrm{X}, \mathrm{Y}] = \mathrm{H}, \quad [\mathrm{H}, \mathrm{X}] = 2\mathrm{X}, \quad [\mathrm{H}, \mathrm{Y}] = -2\mathrm{Y}.$$

(See Exercise 3.1 for a few properties of $\mathrm{X}, \mathrm{Y}, \mathrm{H}$.) With respect to the standard basis of $\mathbb{C}^2$, the matrices of $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ are respectively

$$\mathsf{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathsf{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \mathsf{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Let H be a finite-dimensional $\mathbb{C}$-vector space equipped with a representation $\rho : \mathfrak{sl}_2 \to \mathrm{End}(H)$ (i.e., a Lie algebra morphism $\mathfrak{sl}_2 \to \mathrm{End}(H)$). We still denote by $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ the endomorphisms $\rho(\mathrm{X}), \rho(\mathrm{Y}), \rho(\mathrm{H})$. The following lemma is classical.

3.1.1. Lemma.

(1) *The endomorphism* H *is semi-simple and its eigenvalues are integers. The eigenspace corresponding to the eigenvalue k is denoted H_k.*

(2) *For each $k \in \mathbb{Z}$,* X *(resp.* Y*) sends H_k to H_{k+2} (resp. H_{k-2}).*

(3) *For each $\ell \geqslant 0$,* X^ℓ*, resp.* Y^ℓ*, induces an isomorphism*

$$\mathrm{X}^\ell : H_{-\ell} \xrightarrow{\sim} H_\ell, \quad \textit{resp. } \mathrm{Y}^\ell : H_\ell \xrightarrow{\sim} H_{-\ell}. \qquad \square$$

Let H^* denote the Hermitian dual vector space of H. Then the Hermitian adjoint endomorphisms $(\mathrm{X}^*, \mathrm{Y}^*, -\mathrm{H}^*)$ define and $\mathfrak{sl}_2$-representation on H^*.

It is useful to enlarge the previous setting to $\mathfrak{sl}_2$-representations on objects of an abelian category. Let us introduce the corresponding notation. Let $\boldsymbol{k}$ be a field of characteristic zero (we will mainly use $\boldsymbol{k} = \mathbb{C}$ in the subsequent sections). We fix a $\boldsymbol{k}$-linear abelian category A (i.e., the Hom's are $\boldsymbol{k}$-vector spaces). We have in mind the category of Hodge structures $\mathsf{HS}(\mathbb{C}, w)$, the category of mixed Hodge structures $\mathsf{MHS}(\mathbb{C})$, or the category of holonomic $\mathcal{D}$-modules for example.

Let H be an object of A. By an $\mathfrak{sl}_2$-representation $\rho : \mathfrak{sl}_2 \to \operatorname{End}_{\mathsf{A}}(H)$ we mean a morphism of Lie algebras satisfying the following properties (by analogy to the case of finite-dimensional vector spaces):

• The endomorphism $\rho(\mathrm{H})$ is semi-simple and its eigenvalues are integers. The eigenspace corresponding to the eigenvalue k is denoted H_k. (Hence the object H decomposes as the direct sum $\bigoplus_k \operatorname{Ker}(\rho(\mathrm{H}) - k \operatorname{Id}) = \bigoplus_k H_k$ and $\rho(\mathrm{X})$, resp. $\rho(\mathrm{Y})$, send H_k to H_{k+2}, resp. to H_{k-2}.)

• The endomorphisms $\rho(\mathrm{X}), \rho(\mathrm{Y})$ are nilpotent.

• For each $\ell \geqslant 1$, $\rho(\mathrm{X})^\ell : H_{-\ell} \to H_\ell$ and $\rho(\mathrm{Y})^\ell : H_\ell \to H_{-\ell}$ are isomorphisms (hence the decomposition $H = \bigoplus_k H_k$ is finite).

In the following, we will omit ρ in the notation of an $\mathfrak{sl}_2$-representation, and we denote by $\mathrm{X}, \mathrm{Y}, \mathrm{H}$ the endomorphisms that ρ induces. A morphism between $\mathfrak{sl}_2$-representations in A is a morphism in A which commutes with the $\mathfrak{sl}_2$-action. It is then graded, and its kernel, image and cokernel in A are $\mathfrak{sl}_2$-representations in A, so that the category of $\mathfrak{sl}_2$-representations in A is abelian.

3.1.2. σ-$\mathfrak{sl}_2$-representations. We will have to apply the previous notions in a slightly more general setting. We assume that the abelian category A is equipped with an automorphism $\sigma : \mathsf{A} \mapsto \mathsf{A}$. By a *$\sigma$-endomorphism* of an object H of A we mean a morphism $H \to \sigma^{-1}H$. It defines for every k a morphism $\sigma^{-k}H \to \sigma^{-k-1}H$. We say that a *$\sigma$-endomorphism* N is *nilpotent* if there exists $k \geqslant 0$ such that $\sigma^{-k}\mathrm{N} \circ \cdots \circ \sigma^{-1}\mathrm{N} \circ \mathrm{N} = 0$. By a σ-$\mathfrak{sl}_2$-representation ρ we mean the data of nilpotent $\rho(\mathrm{X}) \in \operatorname{Hom}(H, \sigma H)$ and $\rho(\mathrm{Y}) \in \operatorname{Hom}(H, \sigma^{-1}H)$, and semi-simple $\rho(\mathrm{H}) \in \operatorname{End}(H)$ satisfying the $\mathfrak{sl}_2$-relations. We will mainly use the case where σ is the Tate twist (1) in the category of Hodge structures. We will omit the reference to σ when there is no possible confusion.

***3.1.3. Definition* (Primitive subobjects).** For each $\ell \geqslant 0$, the primitive subobject $\mathrm{P}_{-\ell} \subset H_{-\ell}$ of an $\mathfrak{sl}_2$-representation is $\operatorname{Ker} \mathrm{Y} : H_{-\ell} \to H_{-\ell-2}$. Similarly, the primitive subobject P_ℓ is $\operatorname{Ker} \mathrm{X} : H_\ell \to H_{\ell+2}$.

Note that P_0 is equal to both $\operatorname{Ker} \mathrm{X}$ and $\operatorname{Ker} \mathrm{Y}$ acting on H_0. One also checks the following.

***3.1.4. Lemma* (Lefschetz decomposition).**

• *For each $\ell \geqslant 0$, X^ℓ induces an isomorphism $\mathrm{P}_{-\ell} \xrightarrow{\sim} \mathrm{P}_\ell = \mathrm{X}^\ell(\mathrm{P}_{-\ell})$. Similarly, Y^ℓ induces an isomorphism $\mathrm{P}_\ell \xrightarrow{\sim} \mathrm{P}_{-\ell} = \mathrm{Y}^\ell(\mathrm{P}_\ell)$.*

• *For each $\ell \geqslant 0$, we have*

$$(3.1.4\,*)\qquad \begin{aligned} \mathrm{P}_{-\ell} &= \operatorname{Ker} \mathrm{X}^{\ell+1} : H_{-\ell} \longrightarrow H_{\ell+2},\\ \mathrm{P}_{\ell} &= \operatorname{Ker} \mathrm{Y}^{\ell+1} : H_{\ell} \longrightarrow H_{-\ell-2}. \end{aligned}$$

• *For every $k \geqslant 0$ we have*

$$(3.1.4\,**)\qquad \begin{aligned} H_{-k} &= \bigoplus_{j\geqslant 0} \mathrm{X}^j \mathrm{P}_{-k+2j} \quad \text{and} \quad H_k = \bigoplus_{j\geqslant 0} \mathrm{X}^{k+j} \mathrm{P}_{-k+2j},\\ H_k &= \bigoplus_{j\geqslant 0} \mathrm{Y}^j \mathrm{P}_{k+2j} \quad \text{and} \quad H_{-k} = \bigoplus_{j\geqslant 0} \mathrm{Y}^{k+j} \mathrm{P}_{k+2j}. \end{aligned}$$

• *The morphism $\mathrm{Y} : H_k \to H_{k-2}$ is a monomorphism if $k \geqslant 1$ and an epimorphism if $k \leqslant -1$, and the morphism $\mathrm{X} : H_k \to H_{k+2}$ is a monomorphism if $k \leqslant -1$ and an epimorphism if $k \geqslant 0$.* $\square$

This structure is pictured in Figure 3.1. By exponentiating the action of $\mathrm{X}, \mathrm{Y}, \mathrm{H}$, an $\mathfrak{sl}_2$-representation leads to an action of the group SL_2. There is a distinguished element in this group, called *the Weil element* and denoted by w, which induces an automorphism (also denoted by) w of H. It is defined by the formula

$$\mathrm{w} = \mathrm{e}^X \mathrm{e}^{-Y} \mathrm{e}^X.$$

In the standard basis of $\mathbb{C}^2$, its matrix is

$$\mathsf{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Some of its properties are considered in Exercise 3.1.

3.1.5. Lemma. *The Weil element* w *induces isomorphisms* $\mathrm{w} : H_k \xrightarrow{\sim} H_{-k}$ *and* $\mathrm{P}_k \xrightarrow{\sim} \mathrm{P}_{-k}$ *for any $k \in \mathbb{Z}$.*

Proof. We use the relations of Exercise 3.1(3). The first assertion follows from the relation $\mathrm{wHw}^{-1} = -\mathrm{H}$. If $k \geqslant 0$ and $x \in \mathrm{P}_k$ for example, then $\mathrm{X}x = 0$, hence $\mathrm{Y}(\mathrm{w}x) = -\mathrm{w}(\mathrm{X}x) = 0$, so $\mathrm{w}x \in \mathrm{P}_{-k}$. $\square$

3.1.6. Proposition. *Let $(H_\bullet, \mathrm{N})$ be a finitely graded object in A endowed with a nilpotent endomorphism* N *sending H_k to H_{k-2} for each k and such that $\mathrm{N}^\ell : H_\ell \to H_{-\ell}$ is an isomorphism for each $\ell \geqslant 0$. Then there exists a unique A-representation of $\mathfrak{sl}_2$ on H mapping* Y *to* N *and such that $\mathrm{H}_{|H_\ell} = \ell\, \mathrm{Id}_{H_\ell}$ for every $\ell \in \mathbb{Z}$. Last, any endomorphism $Z \in \operatorname{End}(H)$ which commutes with* Y *and* H *also commutes with* X.

Proof. Indeed, if X exists, the relation $[\mathrm{H}, \mathrm{X}] = 2\mathrm{X}$ implies that X sends H_ℓ to $H_{\ell+2}$ for every $\ell \in \mathbb{Z}$. Then, for $\ell \geqslant 0$ and $0 \leqslant j \leqslant \ell - 1$, let us denote by $\mathrm{N}_{\ell,j} : \mathrm{N}^j \mathrm{P}_\ell \xrightarrow{\sim} \mathrm{N}^{j+1} \mathrm{P}_\ell$ the isomorphism induced by N. We define the morphism $\mathrm{X}_{\ell,j+1} : \mathrm{N}^{j+1} \mathrm{P}_\ell \xrightarrow{\sim} \mathrm{N}^j \mathrm{P}_\ell$ as $c_{\ell,j} \mathrm{Y}_{\ell,j}^{-1}$, where $c_{\ell,j}$ are positive integers uniquely determined by the relations $c_{\ell,j+1} = c_{\ell,j} + \ell - j$. This determines X, according to the Lefschetz decomposition for N.

For the uniqueness it suffices to check that if $[Z, \mathrm{Y}] = 0$ and $[\mathrm{H}, Z] = 2Z$, then $Z = 0$. For $\ell \geqslant 0$, the composition $\mathrm{Y}^{\ell+2} Z : \mathrm{P}_\ell H \to H_{-\ell-2}$, being equal to $Z\mathrm{Y}^{\ell+2}$, is

$$
\begin{array}{rcl}
H_\ell & = & \mathrm{P}_\ell \\
H_{\ell-1} & = & \mathrm{P}_{\ell-1} \oplus 0 \\
H_{\ell-2} & = & \mathrm{P}_{\ell-2} \oplus 0 \oplus \mathrm{Y}\mathrm{P}_\ell \\
\vdots & & \vdots \\
\vdots & & \vdots \\
H_2 & = & \mathrm{P}_2 \oplus 0 \\
H_1 & = & \mathrm{P}_1 \oplus 0 \oplus \mathrm{Y}\mathrm{P}_3 \oplus \dots \\
H_0 & = & \mathrm{P}_0 \oplus 0 \oplus \mathrm{Y}\mathrm{P}_2 \oplus 0 \oplus \mathrm{Y}^2\mathrm{P}_4 \oplus \dots \\
H_{-1} & = & \mathrm{Y}\mathrm{P}_1 \oplus 0 \oplus \mathrm{Y}^2\mathrm{P}_3 \oplus \dots \\
H_{-2} & = & \mathrm{Y}^2\mathrm{P}_2 \oplus 0 \\
\vdots & & \vdots \\
\vdots & & \vdots \\
H_{-\ell+2} & = & 0 \oplus \mathrm{Y}^{-\ell+1}\mathrm{P}_{-\ell} \\
H_{-\ell+1} & = & \mathrm{Y}^{\ell-1}\mathrm{P}_{\ell-1} \oplus 0 \\
H_{-\ell} & = & \mathrm{Y}^\ell\mathrm{P}_\ell
\end{array}
$$

FIGURE 3.1. A graphical way of representing the Lefschetz decomposition (with $\ell \geqslant 0$): the arrows represent the isomorphisms induced by Y; each H_k is the direct sum of the terms of its line, where empty places are replaced with 0. The Lefschetz decomposition relative to X is obtained by reversing the vertical arrows.

zero, so Z is zero on $\mathrm{P}_\ell H$. It is then easy to conclude that Z is zero on each $\mathrm{Y}^j\mathrm{P}_\ell H$ $(j \geqslant 0)$.

Let now $Z \in \mathrm{End}(H)$ be such that Z commutes with Y and H. Then for $c \in \boldsymbol{k}$ nonzero, the Jacobi identity shows that $(\mathrm{X} + c[Z,\mathrm{X}], \mathrm{Y}, \mathrm{H})$ also defines an $\mathfrak{sl}_2$-representation on H, hence $[Z,\mathrm{X}] = 0$ by uniqueness. $\square$

3.1.7. Remark. One can obviously exchange the roles of X and Y in the previous proposition.

3.1.b. $\mathfrak{sl}_2$-quivers

By an $\mathfrak{sl}_2$-*quiver* we mean a data $(H, G, \mathrm{c}, \mathrm{v})$ consisting of a pair (H, G) of $\mathfrak{sl}_2$-representations and A-morphisms $\mathrm{c} : H \to G$, $\mathrm{v} : G \to H$, with

$$\mathrm{c} : H_k \longrightarrow G_{k-1} \quad \text{and} \quad \mathrm{v} : G_k \longrightarrow H_{k-1}, \quad \text{for each } k \in \mathbb{Z},$$

such that $\mathrm{c} \circ \mathrm{v} = \mathrm{Y}_G$ and $\mathrm{v} \circ \mathrm{c} = \mathrm{Y}_H$. The $\mathfrak{sl}_2$-quivers form in an obvious way an abelian category (morphisms of $\mathfrak{sl}_2$-quivers consist of pairs of morphisms $H \to H'$, $G \to G'$, of $\mathfrak{sl}_2$-representations which commute both with c and v). We denote such an object (omitting the shift in the notation) by

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c}"] & G. \arrow[l, bend left, "\mathrm{v}"] \end{tikzcd} \tag{3.1.8}$$

Note that c, v commute with Y, but are not morphisms of $\mathfrak{sl}_2$-representations in A since they do not commute with H (hence neither with X). The properties of Y in Lemma 3.1.4 imply that

- $\mathrm{c} : H_k \to G_{k-1}$ *and* $\mathrm{v} : G_k \to H_{k-1}$ *are monomorphisms for* $k \geqslant 1$ *and epimorphisms for* $k \leqslant -1$.

3.1.9. Remark **(X-$\mathfrak{sl}_2$-quiver).** One can also develop the notion of $\mathfrak{sl}_2$-quiver by replacing Y with X, in which case we speak of an X-$\mathfrak{sl}_2$-quiver to distinguish the notion. In such a case, c sends H_k to G_{k+1} and v sends G_k to H_{k+1}, and satisfy $\mathrm{c} \circ \mathrm{v} = \mathrm{X}_G$, $\mathrm{v} \circ \mathrm{c} = \mathrm{X}_\mathrm{H}$. Then $\mathrm{c} : H_k \to G_{k+1}$ and $\mathrm{v} : G_k \to H_{k+1}$ are monomorphisms for $k \leqslant -1$ and epimorphisms for $k \geqslant 1$.

3.1.10. Definition **(Middle extension, punctual support, S-decomposability)**
Let $(H, G, \mathrm{c}, \mathrm{v})$ be an $\mathfrak{sl}_2$-quiver.

- We say that it is *a middle extension* if c is an epimorphism and v is a monomorphism in A.
- We say that it has a *punctual support* if $H = 0$, hence $G = G_0$ is endowed with the zero $\mathfrak{sl}_2$-representation.
- We say that $(H, G, \mathrm{c}, \mathrm{v})$ is *Support-decomposable*, or simply *S-decomposable*, if it can be decomposed as the direct sum of a middle extension quiver and a quiver with punctual support.

Let H be an $\mathfrak{sl}_2$-representation. Set $G_k = \mathrm{Im}[\mathrm{Y} : H_{k+1} \to H_{k-1}]$. Then $G = \bigoplus_k G_k$ is left invariant by H and Y (but not by X) and $(G, (\mathrm{H} + \mathrm{Id})_{|G}, \mathrm{Y}_{|G})$ can be completed as an $\mathfrak{sl}_2$-representation, according to Proposition 3.1.6. The $\mathfrak{sl}_2$-quiver

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c} = \mathrm{Y}"] & G \arrow[l, bend left, "\mathrm{v} = \mathrm{incl}"] \end{tikzcd} \tag{3.1.10 $*$}$$

is called the *middle extension quiver* attached to H (see Remark 3.3.12 for an explanation of the terminology).

The following proposition is easily checked by using the Lefschetz decomposition for Y.

3.1.11. Proposition. *For a middle extension quiver* $(H, G, \mathrm{c}, \mathrm{v})$, *we have the following properties. For each* $k \in \mathbb{Z}$,

(a) $\mathrm{c} : H_k \to G_{k-1}$ *is an epimorphism and, if* $k \geqslant 1$, *an isomorphism,*
$\mathrm{v} : G_k \to H_{k-1}$ *is a monomorphism and, if* $k \leqslant -1$, *an isomorphism*

(b) $\mathrm{v}(G_k) = \mathrm{Im}\big[\mathrm{Y} : H_{k+1} \to H_{k-1}\big] \simeq \begin{cases} H_{k+1} & \text{if } k \geqslant 0, \\ H_{k-1} & \text{if } k \leqslant 0, \end{cases}$

(c) $\mathrm{P}_k(G) = \mathrm{c}(\mathrm{P}_{k+1}(H))$ *if* $k \geqslant 0$. □

3.1.12. Remark **(A criterion for S-decomposability).** An $\mathfrak{sl}_2$-quiver $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable if and only if the $\mathfrak{sl}_2$-representation G decomposes as $\mathrm{Im}\,\mathrm{c} \oplus \mathrm{Ker}\,\mathrm{v}$, in which case

$$(H, G, \mathrm{c}, \mathrm{v}) = (H, \mathrm{Im}\,\mathrm{c}, \mathrm{c}, \mathrm{v}_{|\,\mathrm{Im}\,\mathrm{c}}) \oplus (0, \mathrm{Ker}\,\mathrm{v}, 0, 0).$$

The following weaker property is modeled on the classical weak Lefschetz theorem for a smooth projective variety.

3.1.13. Definition **(Weak Lefschetz property).** We say that an $\mathfrak{sl}_2$-quiver $(H, G, \mathrm{c}, \mathrm{v})$ satisfies the *weak Lefschetz property* if v is an isomorphism for $k \leqslant -1$ (and an epimorphism for $k = 0$). For an X-$\mathfrak{sl}_2$-quiver, the condition is that v is an isomorphism for $k \geqslant 1$ (and a epimorphism for $k = 0$).

3.1.14. Remarks.

(1) Clearly, if $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable, it satisfies the weak Lefschetz property.

(2) If $(H, G, \mathrm{c}, \mathrm{v})$ satisfies the weak Lefschetz property, then $\mathrm{v} : G_{-1} \to H_{-2}$ is an isomorphism, and therefore $\mathrm{P}_0(H) = \mathrm{Ker}[\mathrm{Y} \colon H_0 \to H_{-2}]$ is equal to $\mathrm{Ker}[\mathrm{c} \colon H_0 \to G_{-1}]$. For an X-$\mathfrak{sl}_2$-Hodge quiver, $\mathrm{P}_0(H) = \mathrm{Ker}[\mathrm{c} : H_0 \to G_1]$.

3.2. Polarized $\mathfrak{sl}_2$-Hodge structures

3.2.a. $\mathfrak{sl}_2$-Hodge structures and quivers. We say that an $\mathfrak{sl}_2$-representation H is an $\mathfrak{sl}_2$-Hodge structure with central weight $w \in \mathbb{Z}$ if for each $k \in \mathbb{Z}$, H_k is (equipped with) a pure Hodge structure of weight $w + k$, and if $\mathfrak{sl}_2$ acts by morphisms of Hodge structure as follows, for $k \in \mathbb{Z}$,

$$\mathrm{X} : H_k \longrightarrow H_{k+2}(1), \quad \mathrm{Y} : H_k \longrightarrow H_{k-2}(-1).$$

(Note that H acts by $k\,\mathrm{Id}$ on H_k, hence is trivially a morphism of Hodge structure). It follows from (3.1.4 $*$) that $\mathrm{P}_k H$ is a pure Hodge structure of weight $w + k$ for each $k \in \mathbb{Z}$ and that the Lefschetz decompositions (3.1.4 $**$) are decompositions in the category of Hodge structures of weight $w+k$. The notion of Tate twist is meaningful in this context, and the twist by (k) shifts the central weight by $-2k$. Last, the Hermitian dual $\mathfrak{sl}_2$-representation H^* is an $\mathfrak{sl}_2$-Hodge structure with central weight $(-w)$.

3.2.1. Remark **($\mathfrak{sl}_2$-Hodge structures are mixed Hodge structures)**

The $\mathfrak{sl}_2$-Hodge structures are examples of *mixed Hodge structures*, with (increasing) weight filtration $W_\bullet$ defined by

$$W_k H = \bigoplus_{k' \leqslant k} H_{k'-w}.$$

The symmetry of Lemma 3.1.1(3) reads, for $\ell \geqslant 0$,

$$\mathrm{X}^\ell : \mathrm{gr}^W_{w-\ell} H \xrightarrow{\sim} \mathrm{gr}^W_{w+\ell} H(\ell) \quad \text{and} \quad \mathrm{Y}^\ell : \mathrm{gr}^W_{w+\ell} H \xrightarrow{\sim} \mathrm{gr}^W_{w-\ell} H(-\ell),$$

justifying the expression "with central weight w".

An $\mathfrak{sl}_2$-quiver $(H, G, \mathrm{c}, \mathrm{v})$ is an $\mathfrak{sl}_2$-*Hodge quiver with central weight* w if H resp. G is an $\mathfrak{sl}_2$-Hodge structure with central weight $w-1$ resp. w and c, v are graded morphisms of degree -1 of mixed Hodge structures:

$$\mathrm{c} : H \longrightarrow G, \quad \mathrm{v} : G \longrightarrow H(-1).$$

More precisely, for each k, c, resp. v, is a morphism of pure Hodge structure of weight $w+k$:

$$\mathrm{c}_k : H_{k+1} \longrightarrow G_k, \quad \text{resp. } \mathrm{v}_k : G_k \longrightarrow H_{k-1}(-1). \tag{3.2.2}$$

We will use the notation

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c}"] & G \arrow[l, bend left, "\mathrm{v}", "(-1)"'] \end{tikzcd}. \tag{3.2.3}$$

We say that $(H, G, \mathrm{c}, \mathrm{v})$ is a *middle extension* $\mathfrak{sl}_2$-*Hodge quiver* if the morphisms (3.2.2) are respectively epimorphisms and monomorphisms in the category of pure Hodge structures of weight $w+k$ for each $k \in \mathbb{Z}$ (equivalently, c, v, are graded epi (resp. mono) morphisms of degree -1 of mixed Hodge structures). We also have similar definitions for punctual support and S-decomposability. Last, the notion of X-$\mathfrak{sl}_2$-*Hodge quiver* is defined similarly (see Remark 3.1.9), with the Tate twist shift by v being equal to (1).

3.2.4. Remark. The criterion of S-decomposability given in Remark 3.1.12 holds for $\mathfrak{sl}_2$-Hodge quivers, by replacing $\mathfrak{sl}_2$-quiver, resp. $\mathfrak{sl}_2$-representation, with $\mathfrak{sl}_2$-Hodge quiver, resp. $\mathfrak{sl}_2$-Hodge structure.

3.2.5. Example. If H is an $\mathfrak{sl}_2$-Hodge structure with central weight $w-1$, then the middle extension quiver (3.1.10 $*$) is an $\mathfrak{sl}_2$-Hodge quiver with central weight w. Indeed, since $\mathrm{Y} : H_{k+1} \to H_{k-1}(-1)$ is a morphism of pure Hodge structures of weight $w+k$, its image G_k is of the same kind, and is a Hodge sub-structure of $H_{k-1}(-1)$, since $\mathsf{HS}(w+k)$ is an abelian category.

3.2.6. Example **(see [Voi02, §13.2.2]).** Let $X \subset \mathbb{P}^N$ be a smooth projective variety of dimension n and let Y be a smooth hyperplane section of X. The cohomology $H = \bigoplus_k H_k = \bigoplus_k H^{n+k}(X, \mathbb{C})$, endowed with the action of the cup product with $(2\pi\mathrm{i})[Y] = \mathrm{X}$ is an $\mathfrak{sl}_2$-Hodge structure centered at n. The cohomology $G = \bigoplus_k G_k =$

$\bigoplus_k H^{n-1+k}(Y,\mathbb{C})$ of Y is also endowed with a natural action of X. If we denote by $\mathrm{c}: H^{n+k}(X,\mathbb{C}) \to H^{n-1+(k+1)}(Y,\mathbb{C})$ the restriction morphism ι_Y^* and by $\mathrm{v}: H^{n-1+k}(Y,\mathbb{C}) \to H^{n+(k+1)}(X,\mathbb{C})(1)$ the Gysin morphism $(2\pi\mathrm{i})\iota_{Y*}$, then $(H,G,\mathrm{c},\mathrm{v})$ is an X-$\mathfrak{sl}_2$-Hodge quiver.

3.2.b. Polarization of $\mathfrak{sl}_2$-Hodge structures and quivers

3.2.7. Definition. Let H be an $\mathfrak{sl}_2$-Hodge structure with central weight w.

(1) A *pre-polarization* of H is an isomorphism $\mathrm{S}: H \xrightarrow{\sim} H^*(-w)$ of $\mathfrak{sl}_2$-Hodge structures with central weight w. Equivalently, S is a morphism of mixed Hodge structures

$$\mathrm{S}: H \otimes \overline{H} \longrightarrow \mathbb{C}^{\mathrm{H}}(-w)$$

which is non-degenerate on the underlying vector spaces and satisfies the identities, for $x, y \in H$,

$$\mathrm{S}(\mathrm{H}x, \overline{y}) = -\mathrm{S}(x, \overline{\mathrm{H}y}), \quad \mathrm{S}(\mathrm{X}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{X}y}), \quad \mathrm{S}(\mathrm{Y}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{Y}y}),$$

hence also $\mathrm{S}(\mathrm{w}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}y})$.

(2) We say that a pre-polarization S of H is a *polarization* if the form $\mathrm{S}(\mathrm{w}\bullet, \overline{\bullet})$ induces a polarization

$$\mathrm{S}_k : H_k \otimes \overline{H}_k \longrightarrow \mathbb{C}(-w-k)$$

of each Hodge structure H_k of weight $w+k$ ($k \in \mathbb{Z}$), i.e., the Hermitian form

$$\mathrm{h}_k(x, \overline{y}) = \mathrm{S}_k(x, \overline{\mathrm{C_D}y}) = \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C_D}y}) \quad (x, y \in H_k)$$

is positive definite on H_k.

3.2.8. Remarks.

(1) If S is a pre-polarization of H, we have $\mathrm{S}(H_k \otimes \overline{H}_\ell) = 0$ if $k+\ell \neq 0$. It follows that the direct sum decomposition $H = \bigoplus_k H_k$ is orthogonal for $\mathrm{S}(\mathrm{w}\bullet, \overline{\bullet})$. If moreover S is a polarization, it is Hermitian by (2). From the equalities

$$\overline{\mathrm{S}(\mathrm{w}y, \overline{x})} = \mathrm{S}(\mathrm{w}x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}y}),$$

we deduce that S is Hermitian on H. Furthermore, $\mathrm{h}(\bullet, \overline{\bullet}) = \mathrm{S}(\mathrm{w}\bullet, \overline{\mathrm{C_D}\bullet})$ is positive definite on H.

(2) With respect to h, X, Y, H satisfy the following relations:

$$\mathrm{h}(\mathrm{X}\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{Y}\bullet}), \quad \mathrm{h}(\mathrm{Y}\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{X}\bullet}), \quad \mathrm{h}(\mathrm{H}\bullet, \overline{\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{H}\bullet}).$$

Let us check the first one for example: we have

$$\begin{aligned}\mathrm{h}(\mathrm{X}x, \overline{y}) &= \mathrm{S}(\mathrm{wX}x, \overline{\mathrm{C_D}y}) = -\mathrm{S}(\mathrm{Yw}x, \overline{\mathrm{C_D}y})\\ &= -\mathrm{S}(\mathrm{w}x, \overline{\mathrm{YC_D}y}) = \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C_D}\mathrm{Y}y}) = \mathrm{h}(x, \overline{\mathrm{Y}y}).\end{aligned}$$

3.2.9. Equivalent definitions of a polarized $\mathfrak{sl}_2$-Hodge structure (1)

We can describe a polarized $\mathfrak{sl}_2$-Hodge structure by means of the metric h in a way similar to Definition 2.5.10.

Let H be an $\mathfrak{sl}_2$-Hodge structure and let h be a positive definite Hermitian form on H such that

(1) the direct sum $H = \bigoplus_k H_k$ is orthogonal for h,
(2) for each k, the Hodge decomposition $H_k = \bigoplus H_k^{p,q}$ is h-orthogonal,
(3) X, Y are adjoint with respect to h and H is h-self-adjoint.

If we define S such that $\mathrm{h}(\bullet, \overline{\bullet}) = \mathrm{S}(\mathrm{w}\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet})$, then S is a polarization of H.

3.2.10. Equivalent definitions of a polarized $\mathfrak{sl}_2$-Hodge structure (2)

From the last identities in 3.2.7(1) and those of Exercise 3.1(3), one deduces that, for each $k \in \mathbb{Z}$, the Lefschetz decomposition of H_k is S_k-orthogonal. The relation $\mathrm{w}_{|\mathrm{P}_{-\ell}} = \mathrm{X}^{\ell}_{|\mathrm{P}_{-\ell}}$ for $\ell \geqslant 0$ (Exercise 3.1(5)) implies that the restriction to $\mathrm{P}_{-\ell}$ of the form

$$\mathrm{P}_{-\ell}\mathrm{S}(x, \overline{y}) = \mathrm{S}(\mathrm{X}^{\ell}x, \overline{y})$$

is a polarization of $\mathrm{P}_{-\ell}$ *if* $\ell \geqslant 0$. Indeed, for $x \neq 0 \in \mathrm{P}_{-\ell}$, we have $\mathrm{C}_{\mathrm{D}}x \in \mathrm{P}_{-\ell}$ and $\mathrm{w}x = \mathrm{X}^{\ell}x/\ell!$, hence

$$0 < \mathrm{h}(x, \overline{x}) = \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}x}) = \mathrm{S}(\mathrm{X}^{\ell}x, \overline{\mathrm{C}_{\mathrm{D}}x})/\ell! = \mathrm{P}_{-\ell}\mathrm{S}(x, \overline{\mathrm{C}_{\mathrm{D}}y})/\ell!.$$

Conversely, *if* S *as in Definition* 3.2.7 *satisfies* 3.2.7(1) *and*

(2′) $\mathrm{P}_{-\ell}\mathrm{S}$ *is a polarization of* $\mathrm{P}_{-\ell}$ *for each* $\ell \geqslant 0$,

then S *is a polarization of* H *in the sense of Definition* 3.2.7*, that is, it also satisfies* 3.2.7(2). Indeed, let us fix $k, \ell \geqslant 0$ and, for $i, j \geqslant 0$, let us first compute $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y})$ for $x = \mathrm{X}^i x_{-k}$ and $y = \mathrm{X}^j y_{-\ell}$ with $x_{-k} \in \mathrm{P}_{-k}$ and $y_{-\ell} \in \mathrm{P}_{-\ell}$. Since X is of type $(1, 1)$, it anti-commutes with C_{D}, so that

$$\mathrm{C}_{\mathrm{D}}\mathrm{X}^j y_{-\ell} = (-1)^j \mathrm{X}^j \mathrm{C}_{\mathrm{D}} y_{-\ell}.$$

Therefore,

$$\begin{aligned}\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) &= \mathrm{S}(\mathrm{w}\mathrm{X}^i x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}\mathrm{X}^j y_{-\ell}}) \\ &= (-1)^j \mathrm{S}(\mathrm{w}\mathrm{X}^i x_{-k}, \overline{\mathrm{X}^j \mathrm{C}_{\mathrm{D}} y_{-\ell}}) \\ &= \mathrm{S}(\mathrm{w}\mathrm{Y}^j \mathrm{X}^i x_{-k}, \overline{\mathrm{C}_{\mathrm{D}} y_{-\ell}}) \quad \text{since } \mathrm{wX} = -\mathrm{Yw} \text{ (Exercise 3.1(3))}.\end{aligned}$$

According to the computation of Exercise 3.1(2), this term vanishes if we do not have $0 \leqslant j \leqslant i \leqslant k$, and is equal to $\star\mathrm{S}(\mathrm{w}\mathrm{X}^{i-j}x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{-\ell}}) = \star\mathrm{S}(\mathrm{X}^{i-j}x_{-k}, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}y_{-\ell}})$ if this condition holds, where $\star$ is a positive constant. Furthermore, this term vanishes if $k - \ell \neq 2(i - j)$. Since $\mathrm{C}_{\mathrm{D}}y_{-\ell} \in \mathrm{P}_{-\ell}$, we have $\mathrm{w}\mathrm{C}_{\mathrm{D}}y_{-\ell} = \mathrm{X}^{\ell}\mathrm{C}_{\mathrm{D}}y_{-\ell}/\ell!$, so finally $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y})$ may be nonzero only if $0 \leqslant j \leqslant i \leqslant k$ and $k \geqslant 2(i - j)$, in which case

$$\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) = \star\mathrm{S}(\mathrm{X}^{\ell+i-j}x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{-\ell}}) = \star\mathrm{S}(\mathrm{X}^{k-(i-j)}x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{2(i-j)-k}}), \quad \star > 0.$$

Last, if $k-(i-j) > k-2(i-j)$, we have $\mathrm{X}^{k-(i-j)}y_{2(i-j)-k} = 0$, so the only remaining possibility for $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y})$ to be nonzero is the case where $i=j$. Then

$$\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}y}) = \star\mathrm{S}(\mathrm{X}^k x_{-k}, \overline{\mathrm{C}_{\mathrm{D}}y_{-k}}) = \star\mathrm{P}_{-k}\mathrm{S}(x, \overline{\mathrm{C}_{\mathrm{D}}y}).$$

By using the Lefschetz decomposition with respect to X, we finally find that, with the assumption that all $\mathrm{P}_{-\ell}\mathrm{S}$ are polarizations, $\mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}x}) > 0$ for any nonzero $x \in H$. ☐

3.2.11. *Equivalent definitions of a polarized $\mathfrak{sl}_2$-Hodge structure (3)*

For $\ell \geqslant 0$, let us define similarly $\mathrm{P}_\ell\mathrm{S}$ on P_ℓ as the restriction to P_ℓ of $\mathrm{S}\circ(\mathrm{Y}^\ell \otimes \overline{\mathrm{Id}})$. *If* S *as in Definition* 3.2.7 *satisfies* 3.2.7(1) *and*

(2″) $(-1)^\ell\mathrm{P}_\ell\mathrm{S}$ *is a polarization of* P_ℓ *for each* $\ell \geqslant 0$,

then S *is a polarization of* H *in the sense of Definition* 3.2.7, *that is, it also satisfies* 3.2.7(2). Indeed, for $x' \in \mathrm{P}_\ell \smallsetminus \{0\}$, we have $x = \mathrm{Y}^\ell x' \in \mathrm{P}_{-\ell}$ and thus, by 3.2.10,

$$\begin{aligned}0 < \mathrm{S}(\mathrm{X}^\ell x, \overline{\mathrm{C}_{\mathrm{D}}x}) &= \mathrm{S}(\mathrm{X}^\ell\mathrm{Y}^\ell x', \overline{\mathrm{C}_{\mathrm{D}}\mathrm{Y}^\ell x'}) = \star\mathrm{S}(x', \overline{\mathrm{C}_{\mathrm{D}}\mathrm{Y}^\ell x'})\\ &= (-1)^\ell \star \mathrm{S}(x', \overline{\mathrm{Y}^\ell\mathrm{C}_{\mathrm{D}}x'}) \quad (\mathrm{Y} \text{ of type } (-1,-1))\\ &= (-1)^\ell \star \mathrm{S}(\mathrm{Y}^\ell x', \overline{\mathrm{C}_{\mathrm{D}}x'}).\end{aligned}$$ ☐

3.2.12. Definition. Let $(H, G, \mathrm{c}, \mathrm{v})$ be an $\mathfrak{sl}_2$-Hodge quiver with central weight w. A (pre-)polarization of $(H, G, \mathrm{c}, \mathrm{v})$ is a pair $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ of (pre-)polarizations of the $\mathfrak{sl}_2$-Hodge structures H, G of respective central weights $w-1$ and w, which satisfy the following relations:

$$\mathrm{S}_G(\mathrm{c}x, \overline{y}) = -\mathrm{S}_H(x, \overline{\mathrm{v}y}) \quad \text{and} \quad \mathrm{S}_G(y, \overline{\mathrm{c}x}) = -\mathrm{S}_H(\mathrm{v}y, \overline{x}), \quad \forall\, x \in H,\ y \in G.$$

3.2.13. Remark. It can be convenient to interpret the pairings as morphisms and the above relations in terms of commutativity of a diagram. Let H^*, G^* be the Hermitian duals of H, G respectively (Exercises 2.7 and 2.8) endowed with $\rho^*(\mathrm{X}) = \mathrm{X}^*$, $\rho^*(\mathrm{Y}) = \mathrm{Y}^*$, $\rho^*(\mathrm{H}) = -\mathrm{H}^*$, and let $\mathrm{c}^* : G^* \to H^*$ and $\mathrm{v}^* : H(-1)^* = H^*(1) \to G^*$ denote the Hermitian adjoint morphisms. Then, defining the Hermitian dual $(H, G, \mathrm{c}, \mathrm{v})^*$ as

$$(H, G, \mathrm{c}, \mathrm{v})^* := (H^*(1), G^*, -\mathrm{v}^*, -\mathrm{c}^*),$$

we conclude that the Hermitian dual of an $\mathfrak{sl}_2$-Hodge quiver centered at w is an $\mathfrak{sl}_2$-Hodge quiver centered at $-w$. The signs $-\mathrm{v}^*, -\mathrm{c}^*$ are justified as follows.

We interpret the pre-polarizations S_H of H and S_G of G as $\mathfrak{sl}_2$- isomorphisms

$$\mathrm{S}_H : H \xrightarrow{\sim} H^*(-w+1), \quad \mathrm{S}_G : G \xrightarrow{\sim} G^*(-w).$$

Then the relations in Definition 3.2.7(1) are equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc}H & \xrightarrow{\mathrm{S}_H} & H^*(-w+1) = H^*(1)(-w)\\ {\scriptstyle\mathrm{c}}\downarrow & & \downarrow{\scriptstyle -\mathrm{v}^*}\\ G & \xrightarrow{\mathrm{S}_G} & G^*(-w)\end{array} \quad \text{and} \quad \begin{array}{ccc}G & \xrightarrow{\mathrm{S}_G} & G^*(-w)\\ {\scriptstyle\mathrm{v}}\downarrow & & \downarrow{\scriptstyle -\mathrm{c}^*}\\ H(-1) & \xrightarrow{\mathrm{S}_H} & H^*(-w)\end{array} \tag{3.2.14}$$

In other words, we can regard the pair $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ as an isomorphism

$$\mathrm{S} : (H, G, \mathrm{c}, \mathrm{v}) \xrightarrow{\sim} (H, G, \mathrm{c}, \mathrm{v})^*(-w).$$

3.2.15. Proposition. *If $(H, G, \mathrm{c}, \mathrm{v})$ is a middle extension $\mathfrak{sl}_2$-Hodge quiver with central weight w, and if H is a polarizable $\mathfrak{sl}_2$-Hodge structure, then $(H, G, \mathrm{c}, \mathrm{v})$ is polarizable.*

Proof. Let S_H be a polarization of H. It defines a morphism of mixed Hodge structures

$$-\mathrm{S}_H(\bullet, \overline{\bullet}) : H \otimes \overline{H}(-1) \longrightarrow \mathbb{C}(-w),$$

that induces morphism $-\mathrm{S}_H(\bullet, \overline{\mathrm{v}\bullet}) : H \otimes \overline{G} \to \mathbb{C}(-w)$. Since $\mathrm{c} : H \to G$ is an epimorphism, this morphism induces a well-defined morphism $\mathrm{S}_G : G \otimes \overline{G} \to \mathbb{C}(-w)$ if and only if $\mathrm{S}_H(x, \overline{\mathrm{v}y}) = 0$ whenever $x \in \operatorname{Ker} \mathrm{c} = \operatorname{Ker} \mathrm{Y}_H$ and $y \in G$. We can write $\mathrm{v}y = \mathrm{Y}_H y'$ for some $y' \in H$, and then

$$\mathrm{S}_H(x, \overline{\mathrm{v}y}) = \mathrm{S}_H(x, \overline{\mathrm{Y}_H y'}) = \mathrm{S}_H(\mathrm{Y}_H x, \overline{y'}) = 0.$$

We thus obtain the existence of $\mathrm{S}_G : G \otimes \overline{G} \to \mathbb{C}(-w)$. Let us check polarizability. We will use the criterion of Section 3.2.11. Let us fix $\ell \geqslant 0$. We have $\mathrm{P}_\ell(G) = \mathrm{c}(\mathrm{P}_{\ell+1}(H))$. For $x', y' \in \mathrm{P}_\ell(G)$, we set $x' = \mathrm{c}x$ and $y' = \mathrm{c}y$ with $x, y \in \mathrm{P}_{\ell+1}(H)$, so that

$$\begin{aligned}
\mathrm{P}_\ell \mathrm{S}_G(x', \overline{\mathrm{C}_{\mathrm{D}} y'}) &= \mathrm{S}_G(\mathrm{Y}_G^\ell x', \overline{\mathrm{C}_{\mathrm{D}} y'}) = \mathrm{S}_G(\mathrm{c}\mathrm{Y}_H^\ell x, \overline{\mathrm{C}_{\mathrm{D}} \mathrm{c} y}) \quad (\mathrm{Y}_G \mathrm{c} = \mathrm{c}\mathrm{Y}_H)\\
&= -\mathrm{S}_H(\mathrm{Y}_H^\ell x, \overline{\mathrm{v}\mathrm{C}_{\mathrm{D}} \mathrm{c} y})\\
&= \mathrm{S}_H(\mathrm{Y}_H^\ell x, \overline{\mathrm{C}_{\mathrm{D}} \mathrm{v}\mathrm{c} y}) \quad (\mathrm{v} \text{ of type } (-1,-1))\\
&= \mathrm{S}_H(\mathrm{Y}_H^\ell x, \overline{\mathrm{C}_{\mathrm{D}} \mathrm{Y}_H y})\\
&= -\mathrm{S}_H(\mathrm{Y}_H^\ell x, \overline{\mathrm{Y}_H \mathrm{C}_{\mathrm{D}} y}) \quad (\mathrm{Y}_H \text{ of type } (-1,-1))\\
&= -\mathrm{S}_H(\mathrm{Y}_H^{\ell+1} x, \overline{\mathrm{C}_{\mathrm{D}} y}) = -\mathrm{P}_{\ell+1}\mathrm{S}_H(x, \overline{\mathrm{C}_{\mathrm{D}}, y}).
\end{aligned} \tag{3.2.16}$$

Since $(-1)^{\ell+1}\mathrm{P}_{\ell+1}\mathrm{S}_H$ is positive definite on $\mathrm{P}_{\ell+1}(H)$, we conclude that $(-1)^\ell \mathrm{P}_\ell \mathrm{S}_G$ is positive definite on $\mathrm{P}_\ell(G)$, as desired. ∎

3.2.c. The S-decomposition theorem for polarizable $\mathfrak{sl}_2$-Hodge quivers

The following result is at the source of the decomposition theorem for the pushforward of pure Hodge modules (see Definition 3.1.10).

***3.2.17. Theorem* (S-decomposition theorem for polarizable $\mathfrak{sl}_2$-Hodge quivers)**
Let $(H, G, \mathrm{c}, \mathrm{v})$ be a polarizable $\mathfrak{sl}_2$-Hodge quiver with central weight w. Then the $\mathfrak{sl}_2$-Hodge structure G decomposes as $G = \operatorname{Im} \mathrm{c} \oplus \operatorname{Ker} \mathrm{v}$ in the category of $\mathfrak{sl}_2$-Hodge structures and $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable.

***Proof of Theorem* 3.2.17.** Recall that $\mathrm{Y}_H : H_k \to H_{k-2}(-1)$ and $\mathrm{v} : G_k \to H_{k-1}(-1)$ anti-commute with the Weil operator C_{D}, and $\mathrm{c} : H_k \to G_{k-1}$ commutes with it. On the other hand, $\mathrm{c}\mathrm{Y}_H = \mathrm{Y}_G \mathrm{c}$ and $\mathrm{v}\mathrm{Y}_G = \mathrm{Y}_H \mathrm{v}$. We first notice the following inclusions for $\ell \geqslant 0$:

$$\mathrm{c}(\mathrm{P}_\ell H) \subset \begin{cases} \mathrm{Y}_G(\mathrm{P}_1 G(1)) & \text{if } \ell = 0, \\ \mathrm{P}_{\ell-1}G \oplus \mathrm{Y}_G(\mathrm{P}_{\ell+1}G(1)) & \text{if } \ell \geqslant 1, \end{cases} \tag{3.2.18}$$

$$\mathrm{v}(\mathrm{P}_\ell G) \subset \begin{cases} \mathrm{Y}_H(\mathrm{P}_1 H) & \text{if } \ell = 0, \\ \mathrm{P}_{\ell-1}H(-1) \oplus \mathrm{Y}_H(\mathrm{P}_{\ell+1}H) & \text{if } \ell \geqslant 1, \end{cases} \tag{3.2.19}$$

Let us check the inclusions (3.2.18) for example. According to Exercise 3.2 if $\ell \geqslant 1$ and obviously if $\ell = 0$, it is enough to prove that $\mathrm{Y}_G^{\ell+1}\mathrm{c}(\mathrm{P}_\ell(H)) = 0$. Since $\mathrm{Y}_G\mathrm{c} = \mathrm{c}\mathrm{Y}_H$, the result follows from the definition of $\mathrm{P}_\ell(H)$.

We will prove by induction the following properties for all $\ell \geqslant 0$ (below we use the convention that $\mathrm{P}_{-1}H = 0$ and $\mathrm{P}_{-1}G = 0$).

(a_ℓ) $\mathrm{c}(\mathrm{P}_{\ell+2}H) = \mathrm{P}_{\ell+1}G$,

(b_ℓ) $\mathrm{c}(\mathrm{P}_\ell H) \subset \mathrm{P}_{\ell-1}G$.

Let us fix a polarization $(\mathrm{S}_H, \mathrm{S}_G)$ of $(H, G, \mathrm{c}, \mathrm{v})$.

Step* 1*: For each* $\ell \geqslant 0$*,* $\mathrm{v}(\mathrm{P}_{\ell+1}G) \cap \mathrm{P}_\ell H = 0$*. We have to prove, if $\ell \geqslant 0$,

$$y_{\ell+1} \in \mathrm{P}_{\ell+1}G \text{ and } \mathrm{v}y_{\ell+1} \in \mathrm{P}_\ell H \Longrightarrow y_{\ell+1} = 0.$$

Assume $y_{\ell+1} \neq 0$. We have, by 3.2.11

$$(-1)^{\ell+1}\mathrm{S}_G(\mathrm{Y}_G^{\ell+1}y_{\ell+1}, \overline{\mathrm{C}_{\mathrm{D}}y_{\ell+1}}) > 0 \quad \text{and} \quad (-1)^{\ell}\mathrm{S}_H(\mathrm{Y}_H^{\ell}(\mathrm{v}y_{\ell+1}), \overline{\mathrm{C}_{\mathrm{D}}(\mathrm{v}y_{\ell+1})}) \geqslant 0.$$

Then, since v anticommutes with C_{D},

$$\begin{aligned} 0 \leqslant (-1)^{\ell}\mathrm{S}_H(\mathrm{Y}_H^{\ell}(\mathrm{v}y_{\ell+1}), \overline{\mathrm{C}_{\mathrm{D}}(\mathrm{v}y_{\ell+1})}) &= (-1)^{\ell+1}\mathrm{S}_H(\mathrm{v}\mathrm{Y}_G^{\ell}y_{\ell+1}, \overline{\mathrm{v}\mathrm{C}_{\mathrm{D}}(y_{\ell+1})}) \\ &= (-1)^{\ell}\mathrm{S}_G(\mathrm{Y}_G^{\ell+1}y_{\ell+1}, \overline{\mathrm{C}_{\mathrm{D}}y_{\ell+1}}) \quad \text{(by definition)} \\ &< 0, \quad \text{a contradiction.} \end{aligned}$$

Step* 2*: Proof that* (a_ℓ) *holds for* $\ell \gg 0$*. For $\ell \gg 0$ we have $\mathrm{P}_\ell H = 0$ and $\mathrm{P}_{\ell+2}H = 0$, so (a_ℓ) amounts to $\mathrm{P}_{\ell+1}G = 0$. By (3.2.19), $\mathrm{v}(\mathrm{P}_{\ell+1}G) = 0$. Since $\ell \geqslant 0$, this implies that $\mathrm{P}_{\ell+1}G = 0$ because v is injective on $G_{\ell+1}$.

Step* 3*: Proof of* (a_ℓ) $\Rightarrow$ (b_ℓ) *if* $\ell \geqslant 0$*. By (a_ℓ) we have $\mathrm{P}_{\ell+1}G = \mathrm{c}(\mathrm{P}_{\ell+2}H)$, so

$$\mathrm{c}(\mathrm{P}_\ell H) \subset \mathrm{P}_{\ell-1}G \oplus \mathrm{c}\mathrm{Y}_H(\mathrm{P}_{\ell+2}H).$$

Since $\mathrm{c}(\mathrm{P}_\ell H) \subset \operatorname{Ker} \mathrm{Y}_H^{\ell}\mathrm{v}$ and, by (3.2.19), $\mathrm{P}_{\ell-1}G \subset \operatorname{Ker} \mathrm{Y}_H^{\ell}\mathrm{v}$, it is enough to prove $\operatorname{Ker} \mathrm{Y}_H^{\ell}\mathrm{v} \cap \mathrm{c}\mathrm{Y}_H(\mathrm{P}_{\ell+2}H) = 0$, that is, $\operatorname{Ker} \mathrm{Y}_H^{\ell+2} \cap \mathrm{P}_{\ell+2}H = 0$, which holds by definition.

Step* 4*: Proof of* (b_ℓ) $\Rightarrow$ ($\mathrm{a}_{\ell-2}$) *for* $\ell \geqslant 2$*. Let us assume that $\ell \geqslant 2$. Let $y_{\ell-1} \in \mathrm{P}_{\ell-1}G$. We have $\mathrm{v}y_{\ell-1} \in \mathrm{P}_{\ell-2}H \oplus \mathrm{Y}_H\mathrm{P}_\ell H$ by (3.2.19), that is, $\mathrm{v}y_{\ell-1} = x_{\ell-2} + \mathrm{v}\mathrm{c}x_\ell$. By ($\mathrm{b}_\ell$), $\mathrm{c}x_\ell \in \mathrm{P}_{\ell-1}G$. Therefore, since $\mathrm{v}(y_{\ell-1} - \mathrm{c}x_\ell) = x_{\ell-2} \in \mathrm{P}_{\ell-2}H$ and since $\ell \geqslant 2$, Step 1 implies $x_{\ell-2} = 0$. By the injectivity of v on $G_{\ell-1}$, this implies $y_{\ell-1} = \mathrm{c}x_\ell$.

We can now conclude the proof of the theorem. We notice that (b_ℓ) for all $\ell \geqslant 0$ implies that the morphism c decomposes with respect to the Lefschetz decomposition. Similarly, Step 1 together with (3.2.19) implies that $\mathrm{v}(\mathrm{P}_\ell G) \subset \mathrm{Y}_H\mathrm{P}_{\ell+1}H$, so v is also compatible with the Lefschetz decomposition. Proving the decomposition $G = \operatorname{Im} \mathrm{c} \oplus \operatorname{Ker} \mathrm{v}$ amounts thus to proving the decomposition on each primitive subspace

$\mathrm{P}_\ell G$ ($\ell \geqslant 0$). We have $\mathrm{P}_{\ell+1}G = \mathrm{c}(\mathrm{P}_{\ell+2}H)$ by (a_ℓ), and $\operatorname{Ker}\mathrm{v}_{|\mathrm{P}_{\ell+1}G} = 0$ so the decomposition is trivial. We are left with proving

$$\mathrm{P}_0 G = \mathrm{c}(\mathrm{P}_1 H) \oplus \operatorname{Ker}\mathrm{v}_{|\mathrm{P}_0 G}.$$

This follows from Exercise 3.5 applied to the category of Hodge structures of weight w. $\square$

One can replace the polarizability property of $(H, G, \mathrm{c}, \mathrm{v})$ in Theorem 3.2.17 by a weaker condition, involving the weak Lefschetz property (Definition 3.1.13).

3.2.20. Theorem. *Let $(H, G, \mathrm{c}, \mathrm{v})$ be an $\mathfrak{sl}_2$-Hodge quiver with central weight w such that*

(a) *$(H, G, \mathrm{c}, \mathrm{v})$ satisfies the weak Lefschetz property,*

(b) *there exists a pre-polarization $(\mathrm{S}_H, \mathrm{S}_G)$ of $(H, G, \mathrm{c}, \mathrm{v})$ such that S_G is a polarization of G and $\mathrm{P}_0\mathrm{S}_H$ is a polarization of $\mathrm{P}_0 H$.*

Then S_H is a polarization of H and $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable.

Proof. In view of Theorem 3.2.17, it is enough to prove that S_H is a polarization of H and it is enough to check that $(-1)^\ell \mathrm{P}_\ell \mathrm{S}_H$ is a polarization of $\mathrm{P}_\ell H$ if $\ell \geqslant 1$ since this property is assumed if $\ell = 0$.

We first claim that, for $\ell \geqslant 1$, we have the inclusion $\mathrm{c}(\mathrm{P}_\ell H) \subset \mathrm{P}_{\ell-1}G$. Indeed, let $x_\ell \in \mathrm{P}_\ell H$, so that $\mathrm{Y}_H^{\ell+1} x_\ell = 0$, hence $\mathrm{v}\mathrm{Y}_G^\ell \mathrm{c}(x_\ell) = 0$. We have $\mathrm{Y}_G^\ell \mathrm{c}(x_\ell) \in G_{-\ell-1}$ and $-\ell - 1 \leqslant -2$, so the weak Lefschetz property implies that $\mathrm{Y}_G^\ell \mathrm{c}(x_\ell) = 0$, that is, $\mathrm{c}(x_\ell) \in \mathrm{P}_{\ell-1}G$.

Assume that $x_\ell \neq 0$ with $\ell \geqslant 1$. Since c is a monomorphism for $\ell \geqslant 1$, we have $\mathrm{c}x_\ell \neq 0$. Assumption (b) then implies

$$\begin{aligned}(-1)^\ell \mathrm{P}_\ell \mathrm{S}_H(x_\ell, \overline{\mathrm{C}_{\mathrm{D}} x_\ell}) &= (-1)^\ell \mathrm{S}_H(\mathrm{Y}_H^\ell x_\ell, \overline{\mathrm{C}_{\mathrm{D}} x_\ell}) = (-1)^\ell \mathrm{S}_H(\mathrm{v}\mathrm{Y}_G^{\ell-1}\mathrm{c}x_\ell, \overline{\mathrm{C}_{\mathrm{D}} x_\ell})\\ &= (-1)^{\ell-1}\mathrm{S}_G(\mathrm{Y}_G^{\ell-1}\mathrm{c}x_\ell, \overline{\mathrm{c}\mathrm{C}_{\mathrm{D}} x_\ell}) = (-1)^{\ell-1}\mathrm{S}_G(\mathrm{Y}_G^{\ell-1}\mathrm{c}x_\ell, \overline{\mathrm{C}_{\mathrm{D}}\mathrm{c} x_\ell}) > 0. \ \square\end{aligned}$$

3.2.d. Differential polarized (bi-) $\mathfrak{sl}_2$-Hodge structures

3.2.21. Definition **(Differential polarized $\mathfrak{sl}_2$-Hodge structure)**

Let (H, S) be a polarized $\mathfrak{sl}_2$-Hodge structure with central weight w. A *differential* on (H, S) is a morphism $d : H \to H(-1)$ of mixed Hodge structures which satisfies the following properties:

- $d \circ d = 0$,
- d is self-adjoint with respect to S,
- $[\mathrm{H}, d] = -d$ and $[\mathrm{Y}, d] = 0$.

We say that (H, S, d) is a differential polarized $\mathfrak{sl}_2$-Hodge structure with central weight w.

The breaking of symmetry between X and Y is clarified with the next lemma. Note that, since h (defined by $\mathrm{h}(\bullet, \overline{\bullet}) = \mathrm{S}(\mathrm{w}\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet})$) is non-degenerate, X can be defined as the h-adjoint of Y.

3.2.22. Lemma. *Let $d^\star$ be the* h*-adjoint of d. Then $d^\star$ is a morphism of mixed Hodge structures $H \to H(1)$ which satisfies the following properties:*

- $d^\star \circ d^\star = 0$,
- *$d^\star$ is self-adjoint with respect to* S,
- $[\mathrm{H}, d^\star] = d^\star$ *and* $[\mathrm{X}, d^\star] = 0$.

Proof. Since $\mathrm{h}(x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}} y})$, we have the relation

$$d^\star \mathrm{w}\, \mathrm{C}_{\mathrm{D}} = \mathrm{w}\, \mathrm{C}_{\mathrm{D}}\, d,$$

and as d anti-commutes with C_{D}, we obtain

$$d^\star = -\mathrm{w}\, d\, \mathrm{w}^{-1}.$$

Since w and d are self-adjoint with respect to S, so is $d^\star$. The other properties are obtained by means of the relations of Exercise 3.1(3). □

It is instructive to interpret d and $d^\star$ as elements of the $\mathfrak{sl}_2$-representation $\mathrm{End}(H)$ (see Exercise 3.3). Here, we omit the Hodge structure in order not to deal with the Tate twist.

3.2.23. Lemma. *Let d and $d^\star$ be as above. Then d belongs to $\mathrm{P}_{-1}\,\mathrm{End}(H)$, $d^\star$ belongs to $\mathrm{P}_1\,\mathrm{End}(H)$, and we have*

$$d^\star = -\mathrm{X}(d) \quad \textit{and} \quad d = -\mathrm{Y}(d^\star).$$

Furthermore, the subspace $\mathbb{C}d \oplus \mathbb{C}d^\star$ of $\mathrm{End}(H)$ is an $\mathfrak{sl}_2$-sub-representation.

Proof. Due to the commutation relations with H, we have $d \in \mathrm{End}(H)_{-1}$ and $d^\star \in \mathrm{End}(H)_1$. The commutation relations with X and Y show the primitivity of d and $d^\star$. Since $\mathrm{w}_{|\mathrm{P}_{-1}} = \mathrm{X}_{|\mathrm{P}_{-1}}$ and $\mathrm{w}^{-1}_{|\mathrm{P}_1} = \mathrm{Y}_{|\mathrm{P}_1}$ according to the formulas of Exercise 3.1(5) and (6), we deduce

$$d^\star = -\mathrm{w}(d) = -\mathrm{X}(d) \quad \text{and} \quad d = -\mathrm{w}^{-1}(d^\star) = -\mathrm{Y}(d^\star).$$

The last assertion is then clear, and with respect to the $\mathfrak{sl}_2$-representation, we can write $\mathbb{C}d \oplus \mathbb{C}d^\star = \mathrm{P}_{-1}(\mathbb{C}d \oplus \mathbb{C}d^\star) \oplus \mathrm{P}_1(\mathbb{C}d \oplus \mathbb{C}d^\star)$. □

Let (H, S, d) be a differential polarized $\mathfrak{sl}_2$-Hodge structure with central weight w. The grading of H defined by the action of H induces a grading on the cohomology $\mathrm{Ker}\, d/\mathrm{Im}\, d$, and Y induces a nilpotent endomorphism on it, which is a graded morphism of degree -2, since Y commutes with d. Moreover, since d is S-self-adjoint, S induces a sesquilinear pairing on $\mathrm{Ker}\, d/\mathrm{Im}\, d$.

3.2.24. Proposition. *If (H, S, d) is a differential polarized $\mathfrak{sl}_2$-Hodge structure with central weight w, then its cohomology $\mathrm{Ker}\, d/\mathrm{Im}\, d$, equipped with the previous grading, nilpotent endomorphism and sesquilinear pairing, is a polarized $\mathfrak{sl}_2$-Hodge structure with central weight w.*

Proof. The first point is to prove that, for $\ell \geqslant 1$, $\mathrm{Y}^\ell : (\operatorname{Ker} d/\operatorname{Im} d)_\ell \to (\operatorname{Ker} d/\operatorname{Im} d)_{-\ell}$ is an isomorphism. Let $d^\star$ be the h-adjoint of d and consider the "Laplacian" $\Delta := dd^\star + d^\star d$. It is graded of degree zero. Due to the positivity of h, we have, in a way compatible with the grading,

$$\operatorname{Ker} d/\operatorname{Im} d = \operatorname{Ker} d \cap \operatorname{Ker} d^\star = \operatorname{Ker}\Delta, \quad H = \operatorname{Ker}\Delta \overset{\perp}{\oplus} \operatorname{Im}\Delta$$

where the sum is orthogonal with respect to h. We first notice that H commutes with $dd^\star$ and $d^\star d$, hence with Δ, so that H preserves the decomposition. We will prove that Δ commutes with Y. Since Δ is h-self-adjoint, it also commutes with X, hence with w.

Furthermore, Δ is a morphism of mixed Hodge structures $H \to H$, hence induces for each $k \in \mathbb{Z}$ a morphism of pure Hodge structures $H_k \to H_k$, and therefore commutes with C_{D}. In particular, $\operatorname{Ker}\Delta$ is an $\mathfrak{sl}_2$-Hodge structure.

On the other hand, if we denote by an index Δ the restriction of the objects to $\operatorname{Ker}\Delta$, the sesquilinear form $\mathrm{h}_\Delta(\mathrm{w}_\Delta^{-1}\bullet, \overline{\mathrm{C}_{\mathrm{D}\Delta}\bullet})$ on $\operatorname{Ker}\Delta$ is a polarization of $\operatorname{Ker}\Delta$, since h_Δ is Hermitian positive definite. But by the previous commutation relations, this form is equal to the restriction S_Δ of S to $\operatorname{Ker}\Delta$. In such a way, we have obtained all the desired properties.

Let us thus prove the commutation of Δ with Y. Let us consider the graded subspace $D = \mathbb{C}d^\star \oplus \mathbb{C}d$ of the $\mathfrak{sl}_2$-representation $\operatorname{End} H$ (see Lemma 3.2.23; note that we now forget the Hodge structure) and the morphism induced by the composition

$$\mathrm{Comp} : D \otimes D \longrightarrow \operatorname{End} H,$$

which is a morphism of $\mathfrak{sl}_2$-representations (see Exercise 3.3(2)). The image of $d^\star \otimes d + d \otimes d^\star$ is equal to Δ. We wish to prove that $\Delta \in \mathrm{P}_0 \operatorname{End} H$ (see Exercise 3.4). Since Comp sends $\mathrm{P}_0(D \otimes D)$ to $\mathrm{P}_0 \operatorname{End} H$, the assertion will follow from the property

$$d^\star \otimes d + d \otimes d^\star \in \mathrm{P}_0(D \otimes D) + \operatorname{Ker}\mathrm{Comp}. \tag{3.2.25}$$

The Lefschetz decomposition of the four-dimensional vector space $D \otimes D$ is easy to describe (a particular case of the Clebsch-Gordan formula):

- $(D \otimes D)_2 = \mathbb{C}(d^\star \otimes d^\star)$,
- $(D \otimes D)_{-2} = \mathbb{C}(d \otimes d)$,
- $(D \otimes D)_0 = \mathrm{Y}\mathbb{C}(d^\star \otimes d^\star) \oplus \mathrm{P}_0(D \otimes D)$.

The assumption $d \circ d = 0$ implies that $\mathrm{Comp}(D \otimes D)_{-2} = 0$, hence $\mathrm{Comp}(D \otimes D)_2 = 0$, $\mathrm{Comp}\mathrm{Y}(D \otimes D)_2 = 0$. In other words, $D \otimes D = \mathrm{P}_0(D \otimes D) + \operatorname{Ker}\mathrm{Comp}$, so (3.2.25) is clear. $\square$

We will meet the following bi-graded situation when dealing with spectral sequences. A bi-$\mathfrak{sl}_2$-Hodge structure with central weight w on a mixed Hodge structure H consists of the data of two commuting $\mathfrak{sl}_2$-representation ρ_1, ρ_2 on H making it an $\mathfrak{sl}_2$-Hodge structures with central weight w in two ways. The basic operators of one structure commute with those of the other structure. We denote them $\mathrm{X}_1, \mathrm{X}_2$, etc. The space H is equipped with a bi-grading, induced by the commuting actions of H_1

and H_2, and a Lefschetz bi-decomposition involving the bi-primitive subspaces, which are pure Hodge structures of suitable weight.

We note that $\mathrm{X} := \mathrm{X}_1 + \mathrm{X}_2$, $\mathrm{Y} := \mathrm{Y}_1 + \mathrm{Y}_2$ and $\mathrm{H} := \mathrm{H}_1 + \mathrm{H}_2$ form an $\mathfrak{sl}_2$-triple, and define an $\mathfrak{sl}_2$-Hodge structure with central weight w, with $H_\ell = \bigoplus_{\ell_1+\ell_2=\ell} H_{\ell_1,\ell_2}$. The corresponding w is $\mathrm{w}_1\mathrm{w}_2$, due to the commutation properties.

3.2.26. Proposition. *Let $(H, \rho_1, \rho_2, \mathrm{S})$ be a polarized bi-$\mathfrak{sl}_2$-Hodge structure. Then the associated $\mathfrak{sl}_2$-Hodge structure $(H, \rho_1 + \rho_2)$, equipped with the same sesquilinear pairing S, is a polarized $\mathfrak{sl}_2$-Hodge structure.*

Sketch of proof. By analyzing the action on each term of the Lefschetz bi-decomposition in terms of bi-primitive subspaces, in a way similar to that in the proof of Section 3.2.11, one checks that the sesquilinear form $\mathrm{S}(x, \overline{\mathrm{w}_1\mathrm{w}_2\mathrm{C}_{\mathrm{D}}y})$ is Hermitian positive definite on H. The statement follows from the identity $\mathrm{w} = \mathrm{w}_1\mathrm{w}_2$. Let us emphasize that this proof enables us not to give an explicit expression for $\mathrm{P}_\ell(H, \rho_1 + \rho_2)$. $\square$

This leads to the bi-graded analogue of Proposition 3.2.24. Let $(H, \rho_1, \rho_2, \mathrm{S})$ be a polarized bi-$\mathfrak{sl}_2$-Hodge structure with central weight w. A differential d on it is a morphism $d : H \to H(-1)$ of mixed Hodge structures such that $(H, \rho_i, \mathrm{S}, d)$ $(i = 1, 2)$ are both differential polarized $\mathfrak{sl}_2$-Hodge structures with central weight w.

3.2.27. Proposition. *If $(H, \rho_1, \rho_2, \mathrm{S}, d)$ is a differential polarized bi-$\mathfrak{sl}_2$-Hodge structure with central weight w, then its cohomology $\operatorname{Ker} d/\operatorname{Im} d$, equipped with the natural bi-grading, nilpotent endomorphisms and sesquilinear pairing, is a polarized bi-$\mathfrak{sl}_2$-Hodge structure with central weight w.*

Proof. We consider the positive definite Hermitian form $\mathrm{h}(x, \overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}y})$ with $\mathrm{w} := \mathrm{w}_1\mathrm{w}_2$ and the Laplacian $\Delta = dd^\star + d^\star d$ corresponding to h, with $d^\star = -\mathrm{w}d\mathrm{w}^{-1}$. Then Δ is bi-graded of bi-degree zero. As in Proposition 3.2.24, we consider the bi-graded space $D = \mathbb{C}d^\star \oplus \mathbb{C}\mathrm{Y}_1(d^\star) \oplus \mathrm{Y}_2(d^\star) \oplus \mathbb{C}d$, with $d = \mathrm{Y}_1\mathrm{Y}_2(d^\star)$. Arguing similarly, we only need to prove that

$$(d \otimes d^\star + d^\star \otimes d) \in \mathrm{P}_{0,0}(D \otimes D) + \operatorname{Ker}\operatorname{Comp}, \tag{3.2.28}$$

where $\mathrm{P}_{0,0}(D \otimes D) = \operatorname{Ker}\mathrm{Y}_1 \cap \operatorname{Ker}\mathrm{Y}_2 \cap (D \otimes D)_{(0,0)}$. We have

$$\operatorname{Ker}\operatorname{Comp} \ni \mathrm{Y}_1\mathrm{Y}_2(d^\star \otimes d^\star) = (d \otimes d^\star + d^\star \otimes d) + \big[(\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_2(d^\star)) + (\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\big].$$

On the other hand,

$$\begin{aligned}\mathrm{Y}_1\big[(\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_2(d^\star)) &+ (\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\big]\\ &= (\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_1\mathrm{Y}_2(d^\star)) + (\mathrm{Y}_1\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\\ &= \mathrm{Y}_1\big[(d^\star \otimes \mathrm{Y}_1\mathrm{Y}_2(d^\star)) + (\mathrm{Y}_1\mathrm{Y}_2(d^\star) \otimes d^\star)\big]\\ &= \mathrm{Y}_1(d \otimes d^\star + d^\star \otimes d),\end{aligned}$$

and similarly with Y_2, so we obtain

$$(d \otimes d^\star + d^\star \otimes d) - \big[(\mathrm{Y}_1(d^\star) \otimes \mathrm{Y}_2(d^\star)) + (\mathrm{Y}_2(d^\star) \otimes \mathrm{Y}_1(d^\star))\big] \in \mathrm{P}_{0,0}(D \otimes D),$$

since this element is annihilated by Y_1, Y_2 and has bi-degree $(0,0)$. We conclude that (3.2.28) holds. □

3.3. A-Lefschetz structures

We use the notation of Section 3.1.a.

3.3.a. The monodromy filtration.

Let H be an object of A equipped with a nilpotent endomorphism N (i.e., $\mathrm{N}^{k+1}=0$ for k large).

***3.3.1. Lemma* (Jakobson-Morosov).** *There exists a unique increasing exhaustive filtration of H indexed by $\mathbb{Z}$, called the* monodromy filtration *relative to* N *and denoted by* $\mathrm{M}_\bullet(\mathrm{N})H$ *or simply* $\mathrm{M}_\bullet H$, *satisfying the following properties:*

(a) *For every* $\ell\in\mathbb{Z}$, $\mathrm{N}(\mathrm{M}_\ell H)\subset \mathrm{M}_{\ell-2}H$,

(b) *For every* $\ell\geqslant 1$, N^ℓ *induces an isomorphism* $\mathrm{gr}^{\mathrm{M}}_\ell H \xrightarrow{\sim} \mathrm{gr}^{\mathrm{M}}_{-\ell}H$.

The proof of Lemma 3.3.1 is left as an exercise. In case of finite-dimensional vector spaces, one can prove the existence by using the decomposition into Jordan blocks and Example 3.3.2. In general, one proves it by induction on the index of nilpotence. The uniqueness is interesting to prove. In fact, there is an explicit formula for this filtration in terms of the kernel filtration of N and of its image filtration (see **[SZ85]**).

3.3.2. Example. If H is a finite dimensional vector space and if N consists only of one lower Jordan block of size $k+1$, one can write the basis as $e_k, e_{k-2},\dots,e_{-k}$, with $\mathrm{N}e_j=e_{j-2}$. Then M_ℓ is the space generated by the e_j's with $j\leqslant\ell$.

***3.3.3. Definition* ((Graded) Lefschetz structure).**

(1) We call such a pair (H,N) an A-*Lefschetz structure*. A morphism between two such pairs is a morphism in A which commutes with the nilpotent endomorphisms.

(2) Assume moreover that H is a graded object in A. We then say that (H,N) a *graded* A-*Lefschetz structure* if $H_\ell=\mathrm{gr}^{\mathrm{M}}_\ell H$ for every ℓ.

For a pair (H,N), we will denote by grN the induced morphism $\mathrm{gr}^{\mathrm{M}}_\ell H\to\mathrm{gr}^{\mathrm{M}}_{\ell-2}H$. Therefore, an A-Lefschetz structure (H,N) gives rise to a graded A-Lefschetz structure, namely, the graded pair $(\mathrm{gr}^{\mathrm{M}}_\bullet H,\mathrm{grN})$. Any morphism $\varphi:(H_1,\mathrm{N}_1)\to(H_2,\mathrm{N}_2)$ is compatible with the monodromy filtrations and induces a graded morphism of degree zero $\mathrm{gr}\varphi:(\mathrm{gr}^{\mathrm{M}}_\bullet H_1,\mathrm{grN}_1)\to(\mathrm{gr}^{\mathrm{M}}_\bullet H_2,\mathrm{grN}_2)$.

3.3.4. Remarks.

(1) According to Proposition 3.1.6, a graded A-Lefschetz structure is nothing but an $\mathfrak{sl}_2$-representation in the category A. The results of Section 3.1.a apply thus to graded A-Lefschetz structures. We will emphasize some of these properties in the setting of A-Lefschetz structures.

(2) The case of a category A with an automorphism σ can (and will) be considered in the realm of A-Lefschetz structures. The arguments of 3.1.2 readily apply to this case.

3.3.5. Lefschetz decomposition. For vector spaces, the choice of a splitting of the filtration (which always exists for a filtration on a finite dimensional vector space) corresponds to the choice of a Jordan decomposition of N. The decomposition (hence the splitting) is not unique, although the filtration is. In general, there exists a decomposition of the *graded object*, called the *Lefschetz decomposition* (see Figure 3.1). For every $\ell \geqslant 0$, we define the ℓ-th N-primitive subspace as

$$\mathrm{P}_\ell(H) := \mathrm{Ker}(\mathrm{grN})^{\ell+1} : \mathrm{gr}^{\mathrm{M}}_\ell(H) \longrightarrow \mathrm{gr}^{\mathrm{M}}_{-\ell-2}(H). \tag{3.3.5 $*$}$$

Then for every $k \geqslant 0$, we have

$$\mathrm{gr}^{\mathrm{M}}_k(H) = \bigoplus_{j\geqslant 0} \mathrm{N}^j \mathrm{P}_{k+2j}(H) \quad \text{and} \quad \mathrm{gr}^{\mathrm{M}}_{-k}(H) = \bigoplus_{j\geqslant 0} \mathrm{N}^{k+j} \mathrm{P}_{k+2j}(H). \tag{3.3.5 $**$}$$

3.3.6. Lemma. *Let H_1, H_2 be two objects of the abelian category A, equipped with nilpotent endomorphisms $\mathrm{N}_1, \mathrm{N}_2$. Let $\varphi : (H_1, \mathrm{N}_1) \to (H_2, \mathrm{N}_2)$ be a morphism which is strictly compatible with the corresponding monodromy filtrations $\mathrm{M}(\mathrm{N}_1), \mathrm{M}(\mathrm{N}_2)$. Then*

$$\mathrm{Im}\,\mathrm{N}_1 \cap \mathrm{Ker}\,\varphi = \mathrm{N}_1(\mathrm{Ker}\,\varphi) \quad \textit{and} \quad \mathrm{Im}\,\mathrm{N}_2 \cap \mathrm{Im}\,\varphi = \mathrm{N}_2(\mathrm{Im}\,\varphi).$$

Proof. Let us first consider the graded morphisms $\mathrm{gr}^{\mathrm{M}}_\ell \varphi : \mathrm{gr}^{\mathrm{M}}_\ell H_1 \to \mathrm{gr}^{\mathrm{M}}_\ell H_2$. One easily checks that it decomposes with respect to the Lefschetz decomposition (see Exercise 3.9). It follows that the property of the lemma is true at the graded level.

Let us now denote by $\mathrm{M}(\mathrm{N}_1)_\bullet \mathrm{Ker}\,\varphi$ (resp. $\mathrm{M}(\mathrm{N}_2)_\bullet \mathrm{Coker}\,\varphi$) the induced filtration on $\mathrm{Ker}\,\varphi$ (resp. $\mathrm{Coker}\,\varphi$). Since φ is strictly compatible with $\mathrm{M}(\mathrm{N}_1), \mathrm{M}(\mathrm{N}_2)$, we have for every ℓ an exact sequence

$$0 \longrightarrow \mathrm{gr}^{\mathrm{M}(\mathrm{N}_1)}_\ell \mathrm{Ker}\,\varphi \longrightarrow \mathrm{gr}^{\mathrm{M}(\mathrm{N}_1)}_\ell H_1 \xrightarrow{\mathrm{gr}^{\mathrm{M}}_\ell \varphi} \mathrm{gr}^{\mathrm{M}(\mathrm{N}_2)}_\ell H_2 \longrightarrow \mathrm{gr}^{\mathrm{M}(\mathrm{N}_2)}_\ell \mathrm{Coker}\,\varphi \longrightarrow 0,$$

from which we conclude that $\mathrm{M}(\mathrm{N}_1)\,\mathrm{Ker}\,\varphi$ (resp. $\mathrm{M}(\mathrm{N}_2)\,\mathrm{Coker}\,\varphi$) satisfies the characteristic properties of the monodromy filtration of $\mathrm{N}_{1|\mathrm{Ker}\,\varphi}$ (resp. $\mathrm{N}_{2|\mathrm{Coker}\,\varphi}$). As a consequence, $\mathrm{Ker}\,\varphi \cap \mathrm{M}(\mathrm{N}_1)_\ell = \mathrm{M}(\mathrm{N}_{1|\mathrm{Ker}\,\varphi})_\ell$ and $\mathrm{Im}\,\varphi \cap \mathrm{M}(\mathrm{N}_2)_\ell = \mathrm{M}(\mathrm{N}_{2|\mathrm{Im}\,\varphi})_\ell$ for every ℓ.

Let us show the first equality, the second one being similar. By the result at the graded level we have

$$\mathrm{Im}\,\mathrm{N}_1 \cap \mathrm{Ker}\,\varphi \cap \mathrm{M}(\mathrm{N}_1)_\ell = \mathrm{N}_1\big(\mathrm{Ker}\,\varphi \cap \mathrm{M}(\mathrm{N}_1)_{\ell+2}\big) + \mathrm{Im}\,\mathrm{N}_1 \cap \mathrm{Ker}\,\varphi \cap \mathrm{M}(\mathrm{N}_1)_{\ell-1},$$

and we can argue by induction on ℓ to conclude. □

***3.3.7. Lemma* (Strictness of $\mathrm{N} : (H, \mathrm{M}_\bullet H) \to (H, \mathrm{M}[2]_\bullet H)$).** *The morphism N, regarded as a filtered morphism $(H, \mathrm{M}_\bullet H) \to (H, \mathrm{M}[2]_\bullet H)$ is strictly compatible with the filtrations, i.e., for every ℓ, $\mathrm{N}(\mathrm{M}_\ell) = \mathrm{Im}\,\mathrm{N} \cap \mathrm{M}_{\ell-2}$. Moreover, considering the induced filtrations $\mathrm{M}_\ell\,\mathrm{Ker}\,\mathrm{N} := \mathrm{M}_\ell H \cap \mathrm{Ker}\,\mathrm{N}$ and $\mathrm{M}_\ell\,\mathrm{Coker}\,\mathrm{N} = \mathrm{M}_\ell H/(\mathrm{M}_\ell H \cap \mathrm{Im}\,\mathrm{N})$, we have*

$$\mathrm{gr}^{\mathrm{M}}\,\mathrm{Coker}\,\mathrm{N} \simeq \mathrm{Coker}\,\mathrm{grN} = \bigoplus_{\ell\geqslant 0} \mathrm{P}_\ell, \quad \mathrm{gr}^{\mathrm{M}}\,\mathrm{Ker}\,\mathrm{N} \simeq \mathrm{Ker}\,\mathrm{grN} = \bigoplus_{\ell\geqslant 0} \mathrm{N}^\ell \mathrm{P}_\ell.$$

In particular, $\mathrm{Ker}\,\mathrm{N} \subset M_0 H$ *and* $M_{-1}\,\mathrm{Coker}\,\mathrm{N} = 0$.

Proof. The first assertion is equivalent to the following two properties:

(1) if $\ell \leqslant 1$, $\mathrm{N} : \mathrm{M}_\ell H \to \mathrm{M}_{\ell-2} H$ is onto,
(2) if $\ell \geqslant -2$, $\mathrm{N} : H/M_{\ell+2}H \to H/\mathrm{M}_\ell H$ is injective.

Let us prove the first one for example. By looking at Figure 3.1, one checks that $\mathrm{M}_{\ell-2}H \subset \mathrm{N}(\mathrm{M}_\ell H) + \mathrm{M}_{\ell-1}H$ for $\ell \leqslant 1$. Iterating this inclusion for $\ell-1, \ell-2, \dots$ gives (1).

Once we know that N is M-strict, we deduce that $\mathrm{gr}^{\mathrm{M}} \operatorname{Ker} \mathrm{N} \simeq \operatorname{Ker} \mathrm{grN}$ and $\mathrm{gr}^{\mathrm{M}} \operatorname{Coker} \mathrm{N} \simeq \operatorname{Coker} \mathrm{grN}$, so that the second part follows from the Lefschetz decomposition (3.3.5 $**$). $\square$

The following criterion, whose proof will not be reproduced here, is at the heart of the decomposition theorem 14.3.2. The notation $\mathsf{D}^{\mathrm{b}}(\mathsf{A})$ is for the bounded derived category of the abelian category A.

***3.3.8. Theorem* (Deligne's criterion).** *Let $C^\bullet$ be an object of $\mathsf{D}^{\mathrm{b}}(\mathsf{A})$ equipped with an endomorphism $\mathrm{N} : C^\bullet \to C^{\bullet+2}$. Assume that $(\bigoplus_k H^k(C^\bullet), \mathrm{N})$ is a graded A-Lefschetz structure (see Definition 3.3.3 and set $H_{-k}(C^\bullet) = H^k(C^\bullet)$). Then $C^\bullet$ is isomorphic to $\bigoplus_k H^k(C^\bullet)[-k]$ in $\mathsf{D}^{\mathrm{b}}(\mathsf{A})$.* $\square$

3.3.b. Lefschetz quivers

By a *Lefschetz quiver* on an abelian category A we mean a data $(H, G, \mathrm{c}, \mathrm{v})$ consisting of a pair (H, G) of objects of A and a pair of morphisms

$$\begin{tikzcd} H \arrow[r, bend left, "\mathrm{c}"] & G \arrow[l, bend left, "\mathrm{v}"] \end{tikzcd} \tag{3.3.9}$$

such that $\mathrm{c} \circ \mathrm{v}$ is nilpotent (on G) and $\mathrm{v} \circ \mathrm{c}$ is nilpotent (on H). We denote by $\mathrm{N}_H, \mathrm{N}_G$ the corresponding nilpotent endomorphisms, so that c, c are morphisms between (H, N_H) and (G, N_G). Lefschetz quivers form in an obvious way an abelian category.

***3.3.10. Definition* (Middle extension, punctual support, S-decomposability)**

We say that a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is *a middle extension* if c is an epimorphism and v is a monomorphism. We say that it has a *punctual support* if $H = 0$. We say that a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is *Support-decomposable*, or simply *S-decomposable*, if it can be decomposed as the direct sum of a middle extension quiver and a quiver with punctual support.

Let (H, N) be an A-Lefschetz structure. Set $G = \operatorname{Im} \mathrm{N}$ and $\mathrm{N}_G = \mathrm{N}_{|G}$. The Lefschetz quiver

$$\begin{tikzcd} (H, \mathrm{N}) \arrow[r, bend left, "\mathrm{c} = \mathrm{N}"] & (G, \mathrm{N}_G) \arrow[l, bend left, "\mathrm{v} = \mathrm{incl}"] \end{tikzcd} \tag{3.3.11}$$

is called the *middle extension quiver* attached to (H, N_H).

***3.3.12. Remark* (on the terminology).** Given an A-Lefschetz structure, one can associate with it in a canonical way, i.e., without any other choice, three natural Lefschetz quivers, that we call "extensions of (H,N)":

- $(H,\mathrm{N})_!$ is the quiver $(H,H,\mathrm{c}=\mathrm{Id},\mathrm{v}=\mathrm{N})$,
- $(H,\mathrm{N})_*$ is the quiver $(H,H,\mathrm{c}=\mathrm{N},\mathrm{v}=\mathrm{Id})$,
- $(H,\mathrm{N})_{!*}$ is the middle extension quiver $(H,\mathrm{Im}\,\mathrm{N},\mathrm{c}=\mathrm{N},\mathrm{v}=\mathrm{incl})$.

There are canonical epi and mono morphisms in the abelian category of Lefschetz quivers:

$$(H,\mathrm{N})_! \twoheadrightarrow (H,\mathrm{N})_{!*} \hookrightarrow (H,\mathrm{N})_*,$$

justifying the name "middle extension" for $(H,\mathrm{N})_{!*}$. These morphisms are obtained through the following diagram:

$$\begin{tikzcd}
H \arrow[r, "\mathrm{Id}"] \arrow[d, bend left, "\mathrm{Id}"] & H \arrow[r, "\mathrm{Id}"] \arrow[d, bend left, "\mathrm{N}"] & H \arrow[d, bend left, "\mathrm{N}"] \\
H \arrow[r, "\mathrm{N}"] \arrow[u, bend left, "\mathrm{N}"] & \mathrm{Im}\,\mathrm{N} \arrow[r, "\mathrm{incl}"] \arrow[u, bend left, "\mathrm{incl}"] & H \arrow[u, bend left, "\mathrm{Id}"]
\end{tikzcd}$$

***3.3.13. Lemma* (The middle extension quiver).** *For the middle extension quiver* (3.3.11)*, we have the following properties.*

(a) $\mathrm{M}_\bullet(\mathrm{N}_G)=G\cap\mathrm{M}_{\bullet-1}(\mathrm{N})=\mathrm{N}(\mathrm{M}_{\bullet+1}(\mathrm{N}))$.
(b) $\mathrm{c}(\mathrm{M}_\bullet H)\subset\mathrm{M}_{\bullet-1}G$, $\mathrm{v}(\mathrm{M}_\bullet G)\subset\mathrm{M}_{\bullet-1}H$,
(c) *the filtered morphisms*

$$\mathrm{c}:(H,\mathrm{M}_\bullet(\mathrm{N}))\longrightarrow(G,\mathrm{M}_{\bullet-1}(\mathrm{N}_G))\quad\textit{and}\quad\mathrm{v}:(G,\mathrm{M}_\bullet(\mathrm{N}_G))\longrightarrow(H,\mathrm{M}_{\bullet-1}(\mathrm{N}))$$

are strictly filtered and the associated graded morphisms are the corresponding canonical morphisms at the graded level. They satisfy the properties of Proposition 3.1.11*.*

Proof. Assume that $\ell\geqslant 0$. We first check that the morphism $\mathrm{N}^\ell:\mathrm{Im}\,\mathrm{N}\cap\mathrm{M}_{\ell-1}(\mathrm{N})\to\mathrm{Im}\,\mathrm{N}\cap\mathrm{M}_{-\ell-1}(\mathrm{N})$ is an isomorphism. By Lemma 3.3.7, this amounts to showing that $\mathrm{N}^\ell:\mathrm{N}(\mathrm{M}_{\ell+1})\to\mathrm{N}(\mathrm{M}_{-\ell+1})$ is an isomorphism. This is a consequence of the following properties: $\mathrm{N}:\mathrm{M}_{\ell+1}\to\mathrm{N}(\mathrm{M}_{\ell+1})$ is an isomorphism, $\mathrm{N}:\mathrm{M}_{-\ell+1}\to\mathrm{M}_{-\ell-1}$ is onto, and $\mathrm{N}^{\ell+1}:\mathrm{M}_{\ell+1}\to\mathrm{M}_{-\ell-1}$ is an isomorphism. Now, (b) and (c) follow from the strictness of $\mathrm{N}:(H,\mathrm{M}_\bullet H)\to(H,\mathrm{M}[2]_\bullet H)$. The remaining part of the lemma is straightforward. $\square$

3.4. Polarizable Hodge-Lefschetz structures

3.4.a. Hodge-Lefschetz structures. We adapt the general framework of Section 3.3 on the Lefschetz decomposition to the case of Hodge structures. Let $H=(\mathcal{H},F'^\bullet\mathcal{H},F''^\bullet\mathcal{H})$ be a bi-filtered vector space and let $\mathrm{N}:\mathcal{H}\to\mathcal{H}$ be a nilpotent endomorphism. In the case of Hodge structures, as we expect that the nilpotent operator $\mathrm{N}:\mathcal{H}\to\mathcal{H}$ sends F^k into F^{k-1} (this is an infinitesimal version of Griffiths transversality property, see Section 4.1), we regard N as a morphism $H\to H(-1)$ (see Definition 2.5.7 for the Tate twist).

Let $\mathrm{M}_\bullet\mathcal{H}$ be the monodromy filtration of $(\mathcal{H},\mathrm{N})$. For each $\ell\in\mathbb{Z}$, we define the bifiltered object $(\mathrm{M}_\ell\mathcal{H},F'^\bullet\mathrm{M}_\ell\mathcal{H},F''^\bullet\mathrm{M}_\ell\mathcal{H})$ as the sub-object for which, for $F=F',F''$, $F^p\mathrm{M}_\ell\mathcal{H}=F^p\mathcal{H}\cap\mathrm{M}_\ell\mathcal{H}$. The quotient space $\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H}=\mathrm{M}_\ell\mathcal{H}/\mathrm{M}_{\ell-1}\mathcal{H}$ is thus bifiltered by setting, for $F=F',F''$,

$$F^p\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H}:=\frac{F^p\mathcal{H}\cap\mathrm{M}_\ell\mathcal{H}}{F^p\mathcal{H}\cap\mathrm{M}_{\ell-1}\mathcal{H}}. \tag{3.4.1}$$

By assumption on N, we obtain for each ℓ a bi-filtered morphism (with $F=F',F''$)

$$\mathrm{grN}:(\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H},F^\bullet\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H})\longrightarrow(\mathrm{gr}_{\ell-2}^{\mathrm{M}}\mathcal{H},F^{\bullet-1}\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{H}). \tag{3.4.2}$$

By definition, Condition (a) in Lemma 3.3.1 holds in the setting of bi-filtered vector spaces. Without any other condition on H, there is no reason that, for $\ell\geqslant 0$, Condition (b) holds when considering grN^ℓ as a bi-filtered morphism. The main reason is that grN in (3.4.2) may not be *strictly* bi-filtered. If we add the condition that each bi-filtered vector space is a Hodge structure of suitable weight, then suddenly everything gets better.

***3.4.3. Definition* (Hodge-Lefschetz structure).** Let $H=(\mathcal{H},F'^\bullet\mathcal{H},F''^\bullet\mathcal{H})$ be a bi-filtered vector space and let $\mathrm{N}:H\to H(-1)$ be a nilpotent endomorphism. We say that (H,N) is a *Hodge-Lefschetz structure with central weight* w if for every ℓ, the object $\mathrm{gr}_\ell^{\mathrm{M}}H$ belongs to $\mathsf{HS}(\mathbb{C},w+\ell)$.

3.4.4. Remarks.

(1) We can consider a bi-filtered vector space H as an object of the abelian category T (see Remark 2.6.a). The general setting of Section 3.3.a applies: the ambient abelian category A is the category T of triples considered in Remark 2.6.a and we choose for σ the Tate twist (1) by the Hodge-Tate structure $\mathbb{C}^{\mathrm{H}}(1)$ of weight -2 (see Section 2.2). (In Section 5.2 we also consider the abelian category of triples as in Definition 5.2.1, and we use the Tate twist as in Notation 5.2.3.) The monodromy filtration $\mathrm{M}_\bullet H$ in T is then well-defined. What goes wrong in general is that the quotient objects $\mathrm{gr}_\ell^{\mathrm{M}}H$ in T may not be bi-strict, hence do not necessarily correspond to bi-filtered vector spaces. If we assume they are *bi-strict*, then the corresponding bi-filtered vector spaces are given by the formula (3.4.1). Therefore, we could have defined a Hodge-Lefschetz structure by simply imposing that $\mathrm{gr}_\ell^{\mathrm{M}}H$ in T belong to $\mathsf{HS}(\mathbb{C},w+\ell)$.

(2) Notice also that the Hodge property implies that, for each ℓ, the bi-filtered morphism (3.4.2) is bi-strict.

(3) One can equivalently define the notion of Hodge-Lefschetz structure by asking that the graded object $\mathrm{gr}^{\mathrm{M}}H=\bigoplus_\ell\mathrm{gr}_\ell^{\mathrm{M}}H$, equipped with the nilpotent endomorphism grN, is part of a (unique) $\mathfrak{sl}_2$-Hodge structure with central weight w. That this second definition is equivalent to the first one follows from the variant of Proposition 3.1.6 in the Hodge setting.

(4) It is important to notice, as in Remark 3.2.1, that Hodge-Lefschetz structures are mixed Hodge structures. Furthermore, $\mathrm{M}_\bullet H$ is a filtration in MHS, and each

object $\mathrm{gr}_\ell^{\mathrm{M}} H$ is a pure object of MHS (of weight $w+\ell$). In other words, the weight filtration $W_\bullet H$ is equal to the shifted filtration $\mathrm{M}_{\bullet-w}H$. Then, for each $\ell\in\mathbb{Z}$,

$$\mathrm{grN}:\mathrm{gr}_\ell^{\mathrm{M}} H \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}} H(-1)$$

is a morphism in $\mathsf{HS}(\mathbb{C}, w+\ell)$.

***3.4.5. Definition* (Category of Hodge-Lefschetz structures).** The category HLS of Hodge-Lefschetz structures is the category whose objects consist of Hodge-Lefschetz structures with central weight some $w\in\mathbb{Z}$, and whose morphisms are morphisms of mixed Hodge structures compatible with N. The category $\mathsf{HLS}(w)$ is the full sub-category consisting of objects with central weight w. It is an *abelian category* (see Exercise 3.14).

3.4.6. Proposition. *Let (H,N) be an object in $\mathsf{HLS}(w)$. Then*

(1) $(H,\mathrm{N})(k) := (H(k),\mathrm{N})$ *is an object in* $\mathsf{HLS}(w+k)$ *for every* $k\in\mathbb{Z}$,

(2) $(G := \operatorname{Im}\mathrm{N}, \mathrm{N}_G)$ *is an object of* $\mathsf{HLS}(w+1)$. *Furthermore, it satisfies* $\mathrm{gr}^{\mathrm{M}}(\operatorname{Im}\mathrm{N}) = \operatorname{Im}(\mathrm{grN})$.

Proof. The first point is clear. Let us check (2). The image of N is regarded in the abelian category T considered at the beginning of this section: it consists of the triple $(\mathrm{N}(\mathcal{H}), \mathrm{N}(F'^\bullet\mathcal{H}), \mathrm{N}(F''^\bullet\mathcal{H}))$. Since $\mathrm{N}: H\to H(-1)$ is a morphism of mixed Hodge structures, it is F-strict and we can also write

$$\begin{aligned}\operatorname{Im}\mathrm{N} &= (\mathrm{N}(\mathcal{H}), \mathrm{N}(F'^\bullet\mathcal{H}), \mathrm{N}(F''^\bullet\mathcal{H}))\\ &= (\mathrm{N}(\mathcal{H}), F'^{\bullet-1}\mathcal{H}\cap\mathrm{N}(\mathcal{H}), F''^{\bullet-1}\mathcal{H}\cap\mathrm{N}(\mathcal{H})).\end{aligned}\tag{3.4.6 $*$}$$

We can thus consider $G=\operatorname{Im}\mathrm{N}$ as an object of the abelian category MHS. It is equipped with a weight filtration which satisfies $W_\bullet G := \mathrm{N}(W_\bullet H)$, by W-strictness of N. Then (2) amounts to identifying the weight filtration $W_\bullet G$ with $\mathrm{M}_{\bullet-(w+1)}G$. This follows from Lemma 3.3.13, provided that we work in the abelian category $\mathsf{MHS}^\oplus$ and extend our objects to objects in this category (see Exercise 3.14(4)). Lastly, the property $\mathrm{gr}^{\mathrm{M}}(\operatorname{Im}\mathrm{N}) = \operatorname{Im}(\mathrm{grN})$ is a consequence of W-strictness of N as a morphism in MHS, that is, $\mathrm{gr}^W(\operatorname{Im}\mathrm{N}) = \operatorname{Im}(\mathrm{gr}^W\mathrm{N})$. □

3.4.b. Hodge-Lefschetz quivers. The definition of a *Hodge-Lefschetz quiver* will be a little different from the general definition (3.3.9) of a Lefschetz quiver, since we will impose that the nilpotent morphisms $\mathrm{N}_H, \mathrm{N}_G$ are those of the corresponding Hodge-Lefschetz structures, hence are (1)-morphisms (we use the terminology of 3.1.2, see Remark 3.3.4).

***3.4.7. Definition* (Hodge-Lefschetz quiver).** A *Hodge-Lefschetz quiver with central weight* w consists of data

$$(H,\mathrm{N}),\ (G,\mathrm{N}),\ \mathrm{c},\mathrm{v},$$

such that

- (H,N) is a Hodge-Lefschetz structure with central weight $w-1$,
- (G,N) is a Hodge-Lefschetz structure with central weight w,

- c, v are morphisms in HLS, hence in MHS:

$$\mathrm{c} : (H, \mathrm{N}) \longrightarrow (G, \mathrm{N}), \qquad \mathrm{v} : (G, \mathrm{N}) \longrightarrow (H, \mathrm{N})(-1),$$

- $\mathrm{c} \circ \mathrm{v} = \mathrm{N}_G$ and $\mathrm{v} \circ \mathrm{c} = \mathrm{N}_H$.

We will use the notation reminiscent to that of (3.2.3):

$$\begin{tikzcd} (H, \mathrm{N}) \arrow[rr, bend left, "\mathrm{c}"] & & (G, \mathrm{N}). \arrow[ll, bend left, "\mathrm{v}", "(-1)"'] \end{tikzcd} \tag{3.4.8}$$

3.4.9. Proposition. *Let* $((H, \mathrm{N}), (G, \mathrm{N}), \mathrm{c}, \mathrm{v})$ *be a Hodge-Lefschetz quiver with central weight* w. *Then*

(1) $(\mathrm{Im}\, \mathrm{c}, \mathrm{N})$ *and* $(\mathrm{Ker}\, \mathrm{v}, \mathrm{N})$ *are objects of* $\mathsf{HLS}(w)$,

(2) *grading all data with respect to the monodromy filtrations* M *(in the sense of Lemma* 3.3.13*) produces an* $\mathfrak{sl}_2$*-Hodge quiver.*

Proof. We will use in an essential way that H and G are in MHS and that c, v are strict with respect to the weight filtrations $W_\bullet$. Let us prove the statement (1) for $\mathrm{Im}\, \mathrm{c}$. W-strictness of c shows that $\mathrm{c}(M_{\ell-1}(\mathrm{N}_H)) = \mathrm{c}(H) \cap \mathrm{M}_\ell(\mathrm{N}_G)$ for every ℓ, by interpreting $\mathrm{M}_\bullet$ in terms of the weight filtrations. We will prove that this term is equal to $\mathrm{M}_\ell(\mathrm{N}_{\mathrm{c}(H)})$ in MHS. The point is to check that, for $\ell \geqslant 0$, N^ℓ induces an isomorphism

$$\frac{\mathrm{c}(H) \cap \mathrm{M}_\ell(\mathrm{N}_G)}{\mathrm{c}(H) \cap \mathrm{M}_{\ell-1}(\mathrm{N}_G)} \xrightarrow{\ \sim\ } \frac{\mathrm{c}(H) \cap \mathrm{M}_{-\ell}(\mathrm{N}_G)}{\mathrm{c}(H) \cap \mathrm{M}_{-\ell-1}(\mathrm{N}_G)}(-\ell), \tag{$*$}$$

also expressed equivalently by means of $\mathrm{c}(\mathrm{M}_\bullet(\mathrm{N}_H))$. Since

$$\mathrm{gr}\mathrm{N}_G^\ell : \mathrm{gr}_\ell^{\mathrm{M}}(G) \longrightarrow \mathrm{gr}_\ell^{\mathrm{M}}(G)(-\ell)$$

is a monomorphism and the left-hand term of $(*)$ is contained in $\mathrm{gr}_\ell^{\mathrm{M}}(G)$, we conclude that $(*)$ is a monomorphism. On the other hand, $\mathrm{gr}\mathrm{N}_H^\ell : \mathrm{gr}_{\ell-1}^{\mathrm{M}} H \to \mathrm{gr}_{-\ell-1}^{\mathrm{M}} H(-\ell)$ is an epimorphism. Since c is strict with respect to the weight filtrations, we also have $\mathrm{c}(\mathrm{gr}_{\ell-1}^{M}(\mathrm{N}_H)) = \mathrm{c}(M_{\ell-1}(\mathrm{N}_H))/\mathrm{c}(M_{\ell-2}(\mathrm{N}_H))$, and thus

$$\mathrm{gr}\mathrm{N}^\ell : \mathrm{c}(M_{\ell-1}(\mathrm{N}_H))/\mathrm{c}(M_{\ell-2}(\mathrm{N}_H)) \longrightarrow \mathrm{c}(M_{-\ell-1}(\mathrm{N}_H))/\mathrm{c}(M_{-\ell-2}(\mathrm{N}_H))(-\ell)$$

is also an epimorphism, concluding the proof that $(*)$ is an isomorphism. It is then straightforward to check that $(\mathrm{Im}\, \mathrm{c}, \mathrm{N})$ is a subobject of (G, N) in $\mathsf{HLS}(w)$.

The proof of (2) is obtained similarly by using strictness of all involved morphisms with respect to $W_\bullet$, hence to $\mathrm{M}_\bullet$ up to a suitable shift. $\square$

3.4.10. Example. We say that a Hodge-Lefschetz quiver is a middle extension if c is an epimorphism and v is a monomorphism (when considered as morphisms in the abelian category MHS). According to Proposition 3.4.6, the set of data

$$\big((H, \mathrm{N}), (\mathrm{Im}\, \mathrm{N}, \mathrm{N}_{|\mathrm{Im}\, \mathrm{N}}), \mathrm{c} = \mathrm{N}, \mathrm{v} = \mathrm{incl}\big)$$

forms a middle extension quiver. Here, we consider c as the morphism $\mathrm{N} : (H, \mathrm{N}) \to (\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}})$ and v as the inclusion $(\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}}) \hookrightarrow (H, \mathrm{N})(-1)$ (see (3.4.6 $*$)). Similarly, we have the notion of *S-decomposable* quiver (see Definition 3.3.10).

3.4.11. Lemma. *A Hodge-Lefschetz quiver is a middle extension, resp. with punctual support, resp. S-decomposable if and only if its associated* M*-graded quiver is so.*

Proof. Similar to that of Proposition 3.4.9. □

3.4.12. Remark. The criterion of S-decomposability of Remark 3.2.4 holds for Hodge-Lefschetz quivers, by replacing there $\mathfrak{sl}_2$-Hodge quiver, resp. $\mathfrak{sl}_2$-Hodge structure, with Hodge-Lefschetz quiver, resp. Hodge-Lefschetz structure.

The proof of the following proposition is straightforward, once we know that $\mathsf{HLS}(w)$ is abelian (Exercise 3.14) and according to Proposition 3.4.9. We emphasize that the criterion in item (4) or (5) below will be essential in the construction of Hodge modules.

***3.4.13. Proposition* (The category $\mathsf{HLQ}(w)$ of Hodge-Lefschetz quivers with central weight w)**

(1) *The Hodge-Lefschetz quivers with central weight w form an abelian category $\mathsf{HLQ}(w)$ in an obvious way.*

(2) *There is no nonzero morphism from a middle extension to an object with punctual support.*

(3) *There is no nonzero morphism from an object with punctual support to a middle extension.*

(4) *A Hodge-Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable if and only if $G = \operatorname{Im}\mathrm{c} \oplus \operatorname{Ker}\mathrm{v}$ in $\mathsf{HLS}(w)$. Then, the decomposition is unique.*

(5) *The latter condition is also equivalent to the conjunction of the following two conditions:*

- *the natural morphism $\operatorname{Im}(\mathrm{v} \circ \mathrm{c}) \to \operatorname{Im}\mathrm{v}$ is an isomorphism,*
- *the natural morphism $\operatorname{Ker}\mathrm{c} \to \operatorname{Ker}(\mathrm{v} \circ \mathrm{c})$ is an isomorphism.* □

3.4.c. Polarization. Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$ be a bi-filtered vector space and let $\mathrm{N} : \mathcal{H} \to \mathcal{H}$ be a nilpotent endomorphism. Let w be an integer and let

$$\mathrm{S} : H \otimes \overline{H} \longrightarrow \mathbb{C}^{\mathrm{H}}(-w)$$

be a bi-filtered morphism. Assume that N is self-adjoint with respect to S, that is, $\mathrm{S}(\bullet, \overline{\mathrm{N}\bullet}) = \mathrm{S}(\mathrm{N}\bullet, \overline{\bullet}) = 0$. Then S induces a sesquilinear pairing

$$\mathrm{gr}^{\mathrm{M}}\mathrm{S} : \mathrm{gr}^{\mathrm{M}} H \otimes \overline{\mathrm{gr}^{\mathrm{M}} H} \longrightarrow \mathbb{C}^{\mathrm{H}}(-w)$$

with respect to which grN is self-adjoint.

***3.4.14. Definition* (Polarization of a Hodge-Lefschetz structure)**

Let (H, N) be a Hodge-Lefschetz structure with central weight w. We say that a sesquilinear pairing $\mathrm{S} : H \otimes \overline{H} \to \mathbb{C}^{\mathrm{H}}(-w)$ is a *polarization* of (H, N) if

(1) N is self-adjoint with respect to S,

(2) $\mathrm{gr}^{\mathrm{M}}\mathrm{S}$ is a polarization of the $\mathfrak{sl}_2$-Hodge structure $(\mathrm{gr}^{\mathrm{M}}H, \mathrm{grN})$ centered at w (see Definition 3.2.7).

3.4.15. Remark. If S is a polarization of (H, N), then

(1) $(-1)^k\mathrm{S}$ is a polarization of $(H, \mathrm{N})(k)$ for every $k \in \mathbb{Z}$ (see Remark 2.5.16(5)),

(2) S is non-degenerate and Hermitian. Indeed, we can regard S as a morphism of mixed Hodge structures $H \to H^*(-w)$, where H^* is the Hermitian dual of H. By definition and Remark 3.2.8(1), $\mathrm{gr}^W\mathrm{S}$ is an isomorphism (non-degenerate) and equal to its Hermitian dual (Hermitian). One deduces that S satisfies the same properties.

3.4.16. Hodge-Lefschetz Hermitian pairs. We can simplify the data of a polarized Hodge-Lefschetz structure with central weight w by giving a *Hodge-Lefschetz Hermitian pair* $((\mathcal{H}, F^\bullet\mathcal{H}), \mathrm{N}, \mathcal{S}, w)$, where N is a filtered morphism

$$(\mathcal{H}, F^\bullet\mathcal{H}) \longrightarrow (\mathcal{H}, F^\bullet\mathcal{H})(-1)$$

and S is a Hermitian isomorphism $\mathcal{S} : (\mathcal{H}, \mathrm{N}) \to (\mathcal{H}, \mathrm{N})^*$ in such a way that, defining $F''^\bullet\mathcal{H}$ as in Section 2.5.17, we obtain data $(H, \mathrm{N}, \mathrm{S})$ as in Definition 3.4.14.

3.4.17. Mixed Hodge structure polarized by **N.** The terminology *mixed Hodge structure polarized by* N is also used in the literature for a polarized Hodge-Lefschetz structure.

Let us summarize a few properties of the categories $\mathsf{HLS}(w)$ and $\mathsf{pHLS}(w)$ (polarizable Hodge-Lefschetz structures of weight w).

3.4.18. Proposition.

(1) *The category* $\mathsf{HLS}(w)$ *is abelian, and a morphism in* $\mathsf{HLS}(w)$ *is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if it is injective (resp. ...) on the underlying vector spaces.*

(2) *Let* $\big((H, \mathrm{N}), \mathrm{S}\big)$ *be a polarized Hodge-Lefschetz structure with central weight* w*, and let* (H_1, N) *be a sub-object in* $\mathsf{HLS}(w)$*. Then* S *induces a polarization* S_1 *on* (H_1, N) *and* $\big((H_1, \mathrm{N}), \mathrm{S}_1\big)$ *is a direct summand of* $\big((H, \mathrm{N}), \mathrm{S}\big)$.

(3) *The category* $\mathsf{pHLS}(w)$ *of polarizable Hodge-Lefschetz structures with central weight* w *is abelian and semi-simple.*

Proof. Assertion (1) is treated in Exercise 3.14. For (2), we know by Exercise 3.14(6) that the inclusion $(H_1, \mathrm{N}) \hookrightarrow (H, \mathrm{N})$ is strict for $\mathrm{M}_\bullet(\mathrm{N})$. Therefore, $\mathrm{gr}^{\mathrm{M}}_\ell H_1$ is a sub Hodge structure of $\mathrm{gr}^{\mathrm{M}}_\ell H$ for each ℓ. Let S_1 be the sesquilinear pairing induced by S on H_1. Then $\mathrm{gr}^{\mathrm{M}}\mathrm{S}_1$ is the sesquilinear pairing induced by $\mathrm{gr}^{\mathrm{M}}\mathrm{S}$ on $\mathrm{gr}^{\mathrm{M}}H_1 \otimes \overline{\mathrm{gr}^{\mathrm{M}}H_1}$, and $\mathrm{gr}^{\mathrm{M}}\mathrm{S}_1(\bullet, \overline{\mathrm{wC_D}\bullet})$ that induced by $\mathrm{gr}^{\mathrm{M}}\mathrm{S}(\bullet, \overline{\mathrm{wC_D}\bullet})$. Since the latter is Hermitian positive definite by assumption, so is the former, meaning that S_1 is a a polarization of (H_1, N). That $\big((H_1, \mathrm{N}), \mathrm{S}_1\big)$ is a direct summand of $\big((H, \mathrm{N}), \mathrm{S}\big)$ is proved in a way similar to Exercise 2.12(2).

Finally, (3) directly follows from (2). □

3.4.d. Polarization of Hodge-Lefschetz quivers and the S-decomposition theorem. In analogy with Definition 3.2.12, we introduce the notion of polarization of a Hodge-Lefschetz quiver.

3.4.19. Definition. Let $(H, G, \mathrm{c}, \mathrm{v})$ be a Hodge-Lefschetz quiver with central weight w. A polarization of $(H, G, \mathrm{c}, \mathrm{v})$ is a pair $\mathrm{S} = (\mathrm{S}_H, \mathrm{S}_G)$ of polarizations of the Hodge-Lefschetz structures H, G of respective central weights $w-1$ and w, which satisfy the following relations:

$$\mathrm{S}_G(\mathrm{c}x, \overline{y}) = -\mathrm{S}_H(x, \overline{\mathrm{v}y}) \quad \text{and} \quad \mathrm{S}_G(y, \overline{\mathrm{c}x}) = -\mathrm{S}_H(\mathrm{v}y, \overline{x}), \quad \forall\, x \in H,\ y \in G.$$

Remark 3.2.13 applies as well for polarizations of Hodge-Lefschetz quivers.

3.4.20. Proposition. *If (H, N) is a Hodge-Lefschetz structure with central weight $w-1$, then the middle extension quiver of Example 3.2.5 is polarizable. More precisely, let S_H be a polarization of (H, N) and let $(G, \mathrm{N}_G) = (\operatorname{Im}\mathrm{N}, \mathrm{N}_{|\operatorname{Im}\mathrm{N}})$ be the image of N regarded as an object of $\mathsf{HLS}(w)$ (Proposition 3.4.6). Using the quiver notation of Example 3.4.10, the formula*

$$\mathrm{S}_G(\mathrm{c}x, \overline{\mathrm{c}y}) := -\mathrm{S}_H(\mathrm{N}x, \overline{y}) = -\mathrm{S}_H(x, \overline{\mathrm{N}y})$$

well-defines a sesquilinear pairing on G, which is a polarization of (G, N_G).

Proof. We argue as in the proof of Proposition 3.2.15 to show that S_G is well-defined as a morphism of mixed Hodge structures $G \otimes \overline{G} \to \mathbb{C}^{\mathrm{H}}(-w)$. Furthermore, grading with respect to M gives back the formula of Proposition 3.2.15, whose conclusion yields the conclusion of the present proposition. □

3.4.21. Examples.

(1) By Proposition 3.4.20, if (H, N) is a polarizable Hodge-Lefschetz structure, then the associated middle extension quiver is polarizable.

(2) If (G, N_G) is a polarizable Hodge-Lefschetz structure, then the quiver with punctual support $(0, (G, \mathrm{N}_G), 0, 0)$ is polarizable.

The following theorem is one of the main results in this chapter.

***3.4.22. Theorem* (S-decomposition theorem for polarizable Hodge-Lefschetz quivers)** *Let $(H, G, \mathrm{c}, \mathrm{v})$ be a polarizable Hodge-Lefschetz quiver with central weight w. Then the polarizable Hodge-Lefschetz structure (G, N_G) decomposes as $(G, \mathrm{N}_G) = \operatorname{Im}\mathrm{c} \oplus \operatorname{Ker}\mathrm{v}$ in $\mathsf{pHLS}(w)$ and $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable.*

Proof. S-decomposability follows from the decomposition of (G, N_G) and Remark 3.4.12.

Recall (Proposition 3.4.9) that $(\operatorname{Im}\mathrm{c}, \mathrm{N})$ and $(\operatorname{Ker}\mathrm{v}, \mathrm{N})$ are subobjects of (G, N_G) in $\mathsf{HLS}(w)$. By Proposition 3.4.18, a polarization on (G, N_G) induces a polarization on each of them, hence they also belong to $\mathsf{pHLS}(w)$. There is a natural morphism in $\mathsf{HLS}(w)$:

$$(\operatorname{Im}\mathrm{c}, \mathrm{N}) \oplus (\operatorname{Ker}\mathrm{v}, \mathrm{N}) \longrightarrow (G, \mathrm{N}_G).$$

It is enough to prove that it is an isomorphism. Since it is strict with respect to $\mathrm{M}_\bullet$ (because it is so with respect to $W_\bullet$), it is enough to prove that gr^{M} of this morphism is an isomorphism. This is provided by the S-decomposition theorem for $\mathfrak{sl}_2$-Hodge quivers (Theorem 3.2.17). $\square$

Proposition 3.4.18 and Theorem 3.4.22 have the following consequence for Hodge-Lefschetz quivers.

3.4.23. Proposition.

(1) *The category* $\mathsf{HLQ}(w)$ *is abelian, and a morphism in* $\mathsf{HLQ}(w)$ *is a monomorphism (resp. epi, resp. iso) if and only if it is injective (resp. onto, resp. iso) on the underlying vector spaces.*

(2) *Let* $\big((H,G,\mathrm{c},\mathrm{v}),\mathrm{S}\big)$ *be a polarized Hodge-Lefschetz quiver with central weight* w*, and let* $(H_1,G_1,\mathrm{c},\mathrm{v})$ *be a sub-object in* $\mathsf{HLQ}(w)$*. Then* S *induces a polarization* S_1 *on* $(H_1,G_1,\mathrm{c},\mathrm{v})$ *and* $\big((H_1,G_1,\mathrm{c},\mathrm{v}),\mathrm{S}_1\big)$ *is a direct summand of* $\big((H,G,\mathrm{c},\mathrm{v}),\mathrm{S}\big)$*.*

(3) *The category* $\mathsf{pHLQ}(w)$ *of polarizable Hodge-Lefschetz quivers of with central weight* w *is abelian and semi-simple.* $\square$

3.5. Exercises

Exercise 3.1.

(1) Show the following identities in $\mathrm{End}(H)$:

$$\mathrm{e}^{\mathrm{Y}}\mathrm{H}\mathrm{e}^{-\mathrm{Y}} = \mathrm{H} + 2\mathrm{Y}, \qquad \mathrm{e}^{-\mathrm{X}}\mathrm{Y}\mathrm{e}^{\mathrm{X}} = \mathrm{Y} - \mathrm{H} - \mathrm{X},$$
$$\mathrm{e}^{-\mathrm{X}}\mathrm{H}\mathrm{e}^{\mathrm{X}} = \mathrm{H} + 2\mathrm{X}, \qquad \mathrm{e}^{\mathrm{Y}}\mathrm{X}\mathrm{e}^{-\mathrm{Y}} = \mathrm{X} - \mathrm{H} - \mathrm{Y}.$$

[*Hint*: Denote by $\mathrm{ad}\,\mathrm{Y} : \mathrm{End}(H) \to \mathrm{End}(H)$ the Lie algebra morphism $A \mapsto [\mathrm{Y}, A]$; show that $\mathrm{e}^{\mathrm{Y}}\mathrm{H}\mathrm{e}^{-\mathrm{Y}} = \mathrm{e}^{\mathrm{ad}\,\mathrm{Y}}(\mathrm{H}) = \mathrm{H} + [\mathrm{Y},\mathrm{H}] + \frac{1}{2}[\mathrm{Y},[\mathrm{Y},\mathrm{H}]] + \cdots$ and conclude for the first equality; argue similarly for the other ones.]

(2) Show that, for $j,k,\ell \geqslant 0$,

$$\mathrm{Y}^j\mathrm{X}^k_{|\mathrm{P}_{-\ell}H} = \begin{cases} a^{(\ell)}_{j,k}\mathrm{X}^{k-j}_{|\mathrm{P}_{-\ell}H} & \text{if } 0 \leqslant j \leqslant k \leqslant \ell \text{ and with } a^{(\ell)}_{j,k} = \dfrac{k!(\ell-k+j)!}{(k-j)!(\ell-k)!}, \\ 0 & \text{otherwise.} \end{cases}$$

[*Hint*: Compute first $a^{(\ell)}_{1,k}$ by noticing that $\mathrm{Y}_{|\mathrm{P}_{-\ell}H} = 0$ and $\mathrm{H}\mathrm{X}^m_{|\mathrm{P}_{-\ell}H} = (2m-\ell)\mathrm{X}^m$ if $0 \leqslant m \leqslant \ell$ and is zero otherwise.]

Show similarly that $\mathrm{X}^j\mathrm{Y}^k_{|\mathrm{P}_{\ell}H} = a^{(\ell)}_{j,k}\mathrm{Y}^{k-j}_{|\mathrm{P}_{-\ell}H}$ or zero in the same range. Conclude that the isomorphism inverse to $\mathrm{X}^{\ell}_{|\mathrm{P}_{-\ell}H}$ is $\mathrm{Y}^{\ell}_{|\mathrm{P}_{\ell}H}/(\ell!)^2$.

(3) Let $\mathrm{w} := \mathrm{e}^{\mathrm{X}}\mathrm{e}^{-\mathrm{Y}}\mathrm{e}^{\mathrm{X}} \in \mathrm{Aut}(H)$ denote the Weil element. Show that

$$\mathrm{w}\mathrm{H} = -\mathrm{H}\mathrm{w}, \quad \mathrm{w}\mathrm{X} = -\mathrm{Y}\mathrm{w}, \quad \mathrm{w}\mathrm{Y} = -\mathrm{X}\mathrm{w}.$$

Conclude that w sends H_ℓ to $H_{-\ell}$ for every ℓ.

(4) Deduce that $\mathrm{w}\mathrm{e}^{-\mathrm{X}} = \mathrm{e}^{\mathrm{Y}}\mathrm{w}$ and

$$\mathrm{w} = \mathrm{e}^{-\mathrm{Y}}\mathrm{e}^{\mathrm{X}}\mathrm{e}^{-\mathrm{Y}}.$$

Conclude also that, if h is a Hermitian metric on H such that the h-adjoints $\mathrm{X}^*, \mathrm{Y}^*$ satisfy $\mathrm{X}^* = \mathrm{Y}$ and $\mathrm{Y}^* = \mathrm{X}$ (hence $\mathrm{H}^* = \mathrm{H}$), then $\mathrm{w}^* = \mathrm{w}^{-1}$.

(5) For $\ell \geqslant 0$, show that

$$\mathrm{w}_{|\mathrm{P}_\ell H} = \frac{(-1)^\ell}{\ell!}\mathrm{Y}^\ell_{|\mathrm{P}_\ell H} \quad \text{and} \quad \mathrm{w}_{|\mathrm{P}_{-\ell} H} = \frac{1}{\ell!}\mathrm{X}^\ell_{|\mathrm{P}_{-\ell} H}.$$

[*Hint*: Use (3) to avoid any computation.]

(6) Deduce that, for $\ell \geqslant 0$ and $0 \leqslant j \leqslant \ell$,

$$\mathrm{w}\mathrm{Y}^j_{|\mathrm{P}_\ell H} = \frac{(-1)^{\ell-j}}{\ell!}\,\mathrm{X}^j\mathrm{Y}^\ell_{|\mathrm{P}_\ell H} = \frac{(-1)^{\ell-j}j!}{(\ell-j)!}\,\mathrm{Y}^{\ell-j}_{|\mathrm{P}_\ell H} \quad \text{and} \quad \mathrm{w}^{-1}\mathrm{Y}^j_{|\mathrm{P}_\ell H} = \frac{(-1)^j j!}{(\ell-j)!}\,\mathrm{Y}^{\ell-j}_{|\mathrm{P}_\ell H}.$$

Exercise 3.2. Let H be an $\mathfrak{sl}_2$-representation in A. Assume that $\ell \geqslant 0$. Show that

$$\mathrm{P}_\ell H \oplus \mathrm{Y}\mathrm{P}_{\ell+2} H = \mathrm{Ker}[\mathrm{Y}^{\ell+2} : H_\ell \to H_{-\ell-4}].$$

[*Hint*: Consider the rough Lefschetz decomposition

$$H_\ell = \mathrm{P}_\ell H \oplus \mathrm{Y}\mathrm{P}_{\ell+2} H \oplus \mathrm{Y}^2 H_{\ell+4},$$

and show that the first two terms are contained in $\mathrm{Ker}\,\mathrm{Y}^{\ell+2}$, while $\mathrm{Y}^{\ell+2}$ is injective on the third term.]

***Exercise 3.3* (The $\mathfrak{sl}_2$ representation on** $\mathrm{End}(H)$**).**

(1) Let H be an $\mathfrak{sl}_2$-representation. Consider the grading $\mathrm{End}_\bullet(H)$ defined by $\mathrm{End}_\ell(H) := \bigoplus_k \mathrm{Hom}(H_k, H_{k+\ell})$, and the nilpotent endomorphism $\mathrm{ad}\,\mathrm{Y} = [\mathrm{Y}, \bullet]$. Show that this defines the $\mathfrak{sl}_2$ representation for which H acts by $\mathrm{ad}\,\mathrm{H}$, X by $\mathrm{ad}\,\mathrm{X}$, and w by $\mathrm{Ad}\,\mathrm{w}(\bullet) := \mathrm{w} \bullet \mathrm{w}^{-1}$.

(2) Show that the composition morphism $\mathrm{Comp} : \mathrm{End}(H) \to \mathrm{End}(H)$ is a morphism of $\mathfrak{sl}_2$-representations:

(a) Since any $\varphi \in \mathrm{End}(H)$ decomposes with respect to the grading, prove commutation with H by showing that if φ is of degree k and φ' of degree ℓ, then $\varphi \circ \varphi'$ is of degree $k + \ell$.

(b) Show the commutation with X, Y by means of the formula $[\mathrm{X}, \varphi\varphi'] = [\mathrm{X}, \varphi]\varphi' + \varphi[\mathrm{X}, \varphi']$, and similarly for Y.

(3) Show that if $d \in \mathrm{End}_{-\ell}(H)$ ($\ell \geqslant 0$) commutes with Y, then $\mathrm{w}^{-1}d\mathrm{w}$ and $\mathrm{w}d\mathrm{w}^{-1} \in \mathrm{End}_\ell(H)$ belong to $\mathrm{P}_\ell\,\mathrm{End}(H)$, i.e., commute with X.

Exercise 3.4. This exercise complements Proposition 3.1.6. Let $\varphi : (H_{1,\bullet}, \mathrm{N}_1) \to (H_{2,\bullet}, \mathrm{N}_2)$ be a morphism between graded Lefschetz structures. Show that φ commutes with the action of X. [*Hint*: Equip $\mathrm{Hom}(H_1, H_2)$ with an $\mathfrak{sl}_2$-action as in 3.3(1) above; with respect to this action, show that $\mathrm{H}(\varphi) = 0$, i.e., $\varphi \in \mathrm{Hom}_0(H_1, H_2)$, and $\mathrm{Y}(\varphi) = 0$, i.e., $\mathrm{N}_2 \circ \varphi - \varphi \circ \mathrm{N}_1 = 0$, and deduce that $\varphi \in \mathrm{P}_0\,\mathrm{Hom}(H_1, H_2)$; conclude that $\mathrm{X}(\varphi) = 0$.]

Exercise 3.5. Let P_1, P'_0, P_{-1} be objects of an abelian category A. Let $c : P_1 \to P'_0$ and $v : P'_0 \to P_{-1}$ be two morphisms such that $v \circ c : P_1 \to P_{-1}$ is an isomorphism. Show that $P'_0 = \operatorname{Im} c \oplus \operatorname{Ker} v$. [*Hint*: Check that it amounts to proving that the composed morphism $\varphi : \operatorname{Im} c \to P'_0/\operatorname{Ker} v$ is an isomorphism; with the commutative diagram

$$\begin{tikzcd} P_1 \arrow[r, "v \circ c", "\sim"'] \arrow[d, two heads, "c"'] & P_{-1} \\ \operatorname{Im} c \arrow[r, "\varphi"] & P'_0/\operatorname{Ker} v \arrow[u, hook, "v"'] \end{tikzcd}$$

show that $\operatorname{Ker} \varphi = \operatorname{Ker} v \circ \varphi = c(\operatorname{Ker} v \circ c) = 0$, and similarly, $\operatorname{Im} v \circ \varphi = \operatorname{Im} v \circ \varphi \circ c = \operatorname{Im} v \circ c = P_{-1}$, hence conclude that $v \circ \varphi$ is an epimorphism, then that v is both an epimorphism and a monomorphism, thus an isomorphism, and φ is an isomorphism.]

Exercise 3.6. Show that an $\mathfrak{sl}_2$-Hodge structure is completely determined by the Hodge structures $\mathrm{P}_\ell H$ ($\ell \geqslant 0$).

Exercise 3.7. Let H be a finite dimensional vector space and let N, N' be nilpotent endomorphisms with monodromy filtrations $\mathrm{M}_\bullet(\mathrm{N}), \mathrm{M}_\bullet(\mathrm{N}')$.

(1) Show that if $\mathrm{N}' - \mathrm{N}$ sends $\mathrm{M}_\ell(\mathrm{N})$ to $\mathrm{M}_{\ell-3}(\mathrm{N})$, then $\mathrm{M}_\bullet(\mathrm{N}') = \mathrm{M}_\bullet(\mathrm{N})$. [*Hint*: Show that $\mathrm{M}_\bullet(\mathrm{N})$ satisfies the characteristic properties of $\mathrm{M}_\bullet(\mathrm{N}')$.]

(2) Deduce that, in such a case, N' is then conjugate to N. [*Hint*: Show that N' and N have the same Jordan normal form.]

***Exercise 3.8* (Morphisms and monodromy filtration).** Let $\varphi : H_1 \to H_2$ be a morphism such that $\mathrm{N}_2 \circ \varphi = \varphi \circ \mathrm{N}_1$, in other words, φ is a morphism of pairs $(H_1, \mathrm{N}_1) \to (H_2, \mathrm{N}_2)$.

(1) Show that φ is compatible with the monodromy filtrations.

(2) Let $\mathrm{gr}^{\mathrm{M}}\varphi$ be the associated graded morphism $\mathrm{gr}^{\mathrm{M}} H_1 \to \mathrm{gr}^{\mathrm{M}} H_2$. Show that φ is an isomorphism if and only if $\mathrm{gr}^{\mathrm{M}}\varphi$ is an isomorphism. [*Hint*: If φ is an isomorphism, identify $\varphi(\mathrm{M}_\ell H_1)$ with $\mathrm{M}_\ell H_2$ by uniqueness of the monodromy filtration.]

***Exercise 3.9* (Morphisms and Lefschetz decomposition).** Let $\varphi : H_1 \to H_2$ be a morphism between A-Lefschetz structures, and assume that they are graded. Show that φ is graded with respect to the Lefschetz decomposition. [*Hint*: Show that, for $\ell \geqslant 0$, φ maps $\mathrm{P}_\ell H_1$ to $\mathrm{P}_\ell H_2$.]

***Exercise 3.10* (Inductive construction of the monodromy filtration)**

Assume $\mathrm{N}^{\ell+1} = 0$ on H. Show the following properties:

(1) $\mathrm{M}_\ell H = H$, $\mathrm{M}_{\ell-1} H = \operatorname{Ker} \mathrm{N}^\ell$, $\mathrm{M}_{-\ell} H = \operatorname{Im} \mathrm{N}^\ell$, $\mathrm{M}_{-\ell-1} H = 0$.

(2) Set $H' = \operatorname{Ker} \mathrm{N}^\ell/\operatorname{Im} \mathrm{N}^\ell$ and $\mathrm{N}' : H' \to H'$ is induced by N. Then $\mathrm{N}'^\ell = 0$ and for $j \in [-\ell+1, \ell-1]$, $\mathrm{M}_j H$ is the pullback of $\mathrm{M}_j H'$ by the projection $H \to H'$.

(3) Conclude that any morphism of A-Lefschetz structures is compatible with the monodromy filtrations.

Exercise 3.11.

(1) Show that the Lefschetz quivers on A form an abelian category in an obvious way.

(2) Show that there is no nonzero morphism from a middle extension to an object with punctual support.

(3) Show that there is no nonzero morphism from an object with punctual support to a middle extension.

(4) Show that a Lefschetz quiver $(H, G, \mathrm{c}, \mathrm{v})$ is S-decomposable if and only if $G = \operatorname{Im}\mathrm{c} \oplus \operatorname{Ker}\mathrm{v}$. Show then that the decomposition is unique.

(5) Show that the latter condition is also equivalent to the conjunction of the following two conditions:

- The natural morphism $\operatorname{Im}(\mathrm{v}\circ\mathrm{c}) \to \operatorname{Im}\mathrm{v}$ is an isomorphism.
- The natural morphism $\operatorname{Ker}\mathrm{c} \to \operatorname{Ker}(\mathrm{v}\circ\mathrm{c})$ is an isomorphism.

Exercise 3.12. The goal of this exercise is to show that any Hodge-Lefschetz structure is isomorphic (non-canonically) to its associated $\mathfrak{sl}_2$-Hodge structure obtained by grading with the monodromy filtration. In (1)–(4) below, the filtration F is either F' or F''.

(1) For every $\ell \geqslant 0$ and p, choose a section $s_{j,p} : \operatorname{gr}_F^p \mathrm{P}_\ell H \to F^p\mathrm{M}_\ell H$ of the projection $F^p\mathrm{M}_\ell H \to \operatorname{gr}_F^p\operatorname{gr}_\ell^{\mathrm{M}} H$ and show that $\operatorname{Im}\mathrm{N}^{\ell+1}s_{\ell,p} \subset F^{p-\ell-1}\mathrm{M}_{-\ell-3}H$. The next questions aim at modifying this section in such a way that its image is contained in $\operatorname{Ker}\mathrm{N}^{\ell+1}$.

(2) Show that, for every $j \geqslant 0$, and any p, $\ell \geq 0$

$$F^{p-\ell-1}\mathrm{M}_{-\ell-3-j}H \subset \mathrm{N}^{\ell+j+3}F^{p+j+2}\mathrm{M}_{\ell+j+3}H + F^{p-\ell-1}\mathrm{M}_{-\ell-3-(j+1)}H.$$

(3) Conclude that, for every $j \geqslant 0$,

$$F^{p-\ell-1}\mathrm{M}_{-\ell-3-j}H \subset \mathrm{N}^{\ell+1}F^{p}\mathrm{M}_{\ell-1}H + F^{p-\ell-1}\mathrm{M}_{-\ell-3-(j+1)}H.$$

(4) Show that if for some $j \geqslant 0$ we have constructed a section $s_{\ell,p}^{(j)}$ such that $\operatorname{Im}\mathrm{N}^{\ell+1}s_{\ell,p}^{(j)} \subset F^{p-\ell-1}\mathrm{M}_{-\ell-3-j}H$, then one can find a section $s_{\ell,p}^{(j+1)}$ such that $\operatorname{Im}\mathrm{N}^{\ell+1}s_{\ell,p}^{(j+1)} \subset F^{p-\ell-1}\mathrm{M}_{-\ell-3-(j+1)}H$. Use then $s_{\ell,p} = s_{\ell,p}^{(0)}$ to obtain a section $s_{\ell,p}^{(\infty)}$ such that $\mathrm{N}^{\ell+1}s_{\ell,p}^{(\infty)} = 0$.

(5) Use the Lefschetz decomposition to obtained the desired isomorphism.

***Exercise 3.13* (Twist of Hodge-Lefschetz structures).** Define the twist (k,ℓ) of an Hodge-Lefschetz structure (H,N) with central weight w as $(H(k,\ell),\mathrm{N})$ and leaving N unchanged. Show that $(H,\mathrm{N})(k,\ell)$ is a Hodge-Lefschetz structure with central weight $w-(k+\ell)$. In particular, the Tate twisted object $(H,\mathrm{N})(k)$ is a Hodge-Lefschetz structure with central weight $w-2k$.

***Exercise 3.14* (The category $\mathsf{HLS}(w)$ is abelian).** Show the following properties.

(1) In the category HLS, any morphism is strict with respect to the filtrations $F^\bullet$ and the filtration $W_\bullet$. [*Hint*: Use Proposition 2.6.8.]

(2) $\mathrm{N}:(H,\mathrm{N})\to(H,\mathrm{N})(-1)$ is a morphism in this category. In particular, $\mathrm{N}(F^p\mathcal{H})=F^{p-1}\mathcal{H}\cap\operatorname{Im}\mathrm{N}$ for $F=F'$ or F''.

(3) The filtration $\mathrm{M}_\bullet(\mathrm{N})H$ is a filtration in the category of mixed Hodge structures.

(4) Consider the category $\mathsf{MHS}^\oplus$ whose objects are $H^\oplus:=\bigoplus_{k,\ell\in\mathbb{Z}}H(k,\ell)$, where H is a mixed Hodge structure, and morphisms $\varphi^\oplus:H_1^\oplus\to H_2^\oplus$ are the direct sums of the *same* morphism of mixed Hodge structures $\varphi:H_1\to H_2(k_o,\ell_o)$ for some (k_o,ℓ_o), twisted by any $(k,\ell)\in\mathbb{Z}$. Show that

(a) the category $\mathsf{MHS}^\oplus$ is abelian,

(b) for (H,N) in $\mathsf{HLS}(w)$, N defines a *nilpotent endomorphism* $\mathrm{N}^\oplus$ in the category $\mathsf{MHS}^\oplus$ on $H^\oplus$,

(c) $\bigoplus_{k,\ell}\mathrm{M}_\bullet(\mathrm{N})H(k,\ell)$ is the monodromy filtration of $\mathrm{N}^\oplus$ in the abelian category $\mathsf{MHS}^\oplus$.

(5) Let $\varphi:(H_1,\mathrm{N}_1)\to(H_2,\mathrm{N}_2)$ be a morphism in HLS. Then $\varphi=0$ if $w_1>w_2$. [*Hint*: Use that φ is compatible with both $\mathrm{M}_\bullet$ and $W_\bullet$.]

(6) Let φ be a morphism in $\mathsf{HLS}(w)$. Show that φ is strictly compatible with $\mathrm{M}_\bullet$. Conclude that $\mathsf{HLS}(w)$ is abelian.

(7) Let φ be a morphism in $\mathsf{HLS}(w)$. Show that φ is a monomorphism (resp. an epimorphism, resp. an isomorphism) if and only if it is injective (resp. ...) on the underlying vector spaces. [*Hint*: Use that the forgetful functor $(H,\mathrm{N})\mapsto\mathcal{H}$ from $\mathsf{HLS}(w)$ to the category of vector spaces is faithful.]

(8) Show that, for such a φ, the conclusion of Lemma 3.3.6 holds in the category of mixed Hodge structures. [*Hint*: Use the auxiliary category $\mathsf{MHS}^\oplus$ and the nilpotent endomorphisms $\mathrm{N}_1^\oplus,\mathrm{N}_2^\oplus$; this trick is useful since N is not an endomorphism of (H,N) in $\mathsf{HLS}(w)$, due to the twist by (-1).]

(9) Similar results hold for $\mathfrak{sl}_2$-Hodge structures.

3.6. Comments

The Hard Lefschetz theorem for complex projective varieties equipped with an ample line bundle, named so after the fundamental memoir of Lefschetz [**Lef24**], and for which there does not exist up to now a purely topological proof (see [**Lam81**] for an overview of the topology of complex algebraic varieties), is intrinsically present in classical Hodge theory (see e.g. [**GH78, Dem96, Voi02**]). That a relative version of this theorem is instrumental in proving the decomposition theorem (one of the main goals of the theory of pure Hodge modules) had been emphasized and proved by Deligne in [**Del68**], by introducing the criterion 3.3.8. On the other hand, the theory of degeneration of polarized variations of Hodge structure [**Sch73, GS75**] also gives rise to such Hodge-Lefschetz structures, not necessarily graded however. Note also that such structures have been discovered by Steenbrink [**Ste77**] and Varchenko [**Var82**] on the space of vanishing cycles attached to an isolated critical point of a holomorphic function. This property was at the source of the definition of pure Hodge modules by Saito in [**Sai88**].

Since the very definition of a pure Hodge module by Saito [**Sai88**] is modeled on the theory of degenerations, we devote a complete chapter to the notion of a Hodge-Lefschetz structure. Together with the criterion 3.3.8, a few results are used in an essential way in the decomposition theorem for pure polarized Hodge modules as proved by Saito [**Sai88**], namely the S-decomposition theorem 3.4.22 and those of Section 3.2.d. They are originally proved in [**Sai88**, §4]. We follow here the proof given by Guillén and Navarro Aznar in [**GNA90**], according to the idea, due to Deligne, of using harmonic theory in finite dimensions and the full strength of the action of SL_2 by means of the Weil element denoted by w. The polarization property is often reduced to saying that the primitive part of the Hodge-Lefschetz structure is a polarized Hodge structure, and is is rarely emphasized that each graded part of a polarized $\mathfrak{sl}_2$-Hodge structure (like any cohomology space of a smooth complex projective variety) is also a polarized pure Hodge structure. The latter approach makes it explicit.

Basic results on the monodromy filtration, which gives rise to the Hodge-theoretic weight filtration, are explained in [**Sch73, CK82, SZ85**]. The notion of a polarized Hodge-Lefschetz structure is also known under the name of *polarized mixed Hodge structure* [**CK82**], and it is also said that the nilpotent operator *polarizes the mixed Hodge structure*. This is justified by the fact that the choice of an ample line bundle on a smooth complex projective variety is regarded as a polarization, and it determines a polarization form on the cohomology. Such data also give rise to a *nilpotent orbit* (see [**Sch73, CK82**] and also [**Kas85**, Def. 2.3.1]). We do not use this terminology here, since we also want to use a Hodge-Lefschetz structure without any polarization, as we did for Hodge structures.

For the purpose of pure Hodge modules, the notion of *middle extension Lefschetz quiver* is a basic tool, corresponding to the notion of middle extension for perverse sheaves or holonomic $\mathcal{D}$-modules. It consists of two objects, called respectively *nearby cycles* and *vanishing cycles* related by two morphisms usually called can and var. The middle extension property is that can is an epimorphism and var is a monomorphism, so that the vanishing cycles are identified with the image of $\mathrm{N} := \mathrm{var}\circ\mathrm{can}$ in the nearby cycles. Hodge theory for vanishing cycles can then be deduced from Hodge theory for nearby cycles, as already remarked by Kashiwara and Kawai [**KK87**]. In particular, Lemma 3.3.13 is much inspired from [**KK87**, Prop. 2.1.1], and also of [**Sai88**, Lem. 5.1.12].

The basic decomposition result of Exercise 3.5 is at the heart of the notion of *Support-decomposability*, which is a fundamental property of Saito's pure Hodge modules [**Sai88**]. Exercise 3.12 is taken from [**Sai89b**, Prop. 3.7].

CHAPTER 4

VARIATIONS OF HODGE STRUCTURE ON A COMPLEX MANIFOLD

Summary. The notion of a variation of Hodge structure on a complex manifold is the first possible generalization of a Hodge structure. It naturally occurs when considering holomorphic families of smooth projective varieties. Later, we will identify this notion with the notion of a *smooth Hodge module.* We consider global properties of polarizable variations of Hodge structure on a smooth projective variety. On the one hand, the Hodge theorem asserts that the de Rham cohomology of a polarizable variation of Hodge structure on a smooth projective variety is itself a polarizable $\mathfrak{sl}_2$-Hodge structure. On the other hand, we show that the local system underlying a polarizable variation of Hodge structure on a smooth projective variety is semi-simple, and we classify all such variations with a given underlying semi-simple local system.

4.1. Variations of Hodge structure

4.1.a. Variations of $\mathbb{C}$-Hodge structure. The definition of a variation of $\mathbb{C}$-Hodge structure is modeled on the behaviour of the cohomology of a family of smooth projective varieties parametrized by a smooth algebraic variety, that is, a smooth projective morphism $f: Y \to X$, that we call below the "geometric setting".

Let us first motivate the definition. Let X be a connected (possibly non compact) complex manifold. In such a setting, the generalization of a vector space $\mathcal{H}^o$ is a *locally constant sheaf of vector spaces* $\underline{\mathcal{H}}$ *on* X. Let us choose a universal covering $\widetilde{X} \to X$ of X and let us denote by Π its group of deck-transformations, which is isomorphic to $\pi_1(X, \star)$ for any choice of a base-point $\star \in X$. Let us denote by $\widetilde{\mathcal{H}}$ the space of global sections of the pullback $\widetilde{\underline{\mathcal{H}}}$ of $\underline{\mathcal{H}}$ to $\widetilde{X}$. Then, giving $\underline{\mathcal{H}}$ is equivalent to giving the *monodromy representation* $\Pi \to \mathrm{GL}(\widetilde{\mathcal{H}})$. However, it is known that, in the geometric setting, the Hodge decomposition in each fiber of the family does not give rise to locally constant sheaves, but to C^∞-bundles.

In the geometric setting, to the locally constant sheaf $R^k f_* \mathbb{C}_X$ ($k \in \mathbb{N}$) is associated the *Gauss-Manin connection*, which is a holomorphic vector bundle on Y equipped with a holomorphic flat connection. In such a case, the Hodge filtration can be naturally defined and it is known to produce holomorphic bundles. Therefore, in the general setting of a variation of $\mathbb{C}$-Hodge structure that we intend to define, a better analogue of the complex vector space $\mathcal{H}^o$ is a holomorphic vector bundle $\mathcal{H}'$ equipped with a flat holomorphic connection $\nabla : \mathcal{H}' \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{H}'$, so that the locally constant sheaf $\underline{\mathcal{H}}' = \operatorname{Ker} \nabla$, that we also denote by $\mathcal{H}'^{\nabla}$, is the desired local system. Note that we can recover $(\mathcal{H}', \nabla)$ from $\underline{\mathcal{H}}'$ since the natural morphism of flat bundles

$$(\mathcal{O}_X \otimes_{\mathbb{C}} \underline{\mathcal{H}}', \mathrm{d} \otimes \mathrm{Id}) \longrightarrow (\mathcal{H}', \nabla)$$

is an isomorphism. A filtration is then a finite (exhaustive) decreasing filtration by sub-bundles $F^\bullet\mathcal{H}'$ (recall that a sub-bundle $F^p\mathcal{H}'$ of $\mathcal{H}'$ is a locally free $\mathcal{O}_X$-submodule of $\mathcal{H}'$ such that $\mathcal{H}'/F^p\mathcal{H}'$ is also a locally free $\mathcal{O}_X$-module; $F^\bullet\mathcal{H}'$ is a filtration by sub-bundles if each $F^p\mathcal{H}'/F^{p+1}\mathcal{H}'$ is a locally free $\mathcal{O}_X$-module). The main property, known as *Griffiths transversality property* is that the filtration should satisfy

$$\nabla(F^p\mathcal{H}') \subset \Omega^1_X \otimes_{\mathcal{O}_X} F^{p-1}\mathcal{H}' \quad \forall p \in \mathbb{Z}. \tag{4.1.1}$$

However, the analogue of a bi-filtered vector space is not a bi-filtered holomorphic flat bundle, since one knows in the geometric setting that one of the filtrations should behave holomorphically, while the other one should behave anti-holomorphically. This leads to a presentation by C^∞-bundles.

Let $\mathcal{H} = \mathcal{C}^\infty_X \otimes_{\mathcal{O}_X} \mathcal{H}'$ be the associated C^∞ bundle and let D be the connection on $\mathcal{H}$ defined, for any C^∞ function φ and any local holomorphic section v of $\mathcal{H}'$, by $D(\varphi \otimes v) = \mathrm{d}\varphi \otimes v + \varphi \otimes \nabla v$ (this is a flat connection which decomposes with respect to types as $D = D' + D''$ and $D'' = \mathrm{d}'' \otimes \mathrm{Id}$). Then D'' is a holomorphic structure on $\mathcal{H}$, i.e., $\operatorname{Ker} D''$ is a holomorphic bundle with connection ∇ induced by D': this is $(\mathcal{H}', \nabla)$ by construction. Each bundle $F^p\mathcal{H}'$ gives rise similarly to a C^∞-bundle $F'^p\mathcal{H}$ which is holomorphic in the sense that $D''F'^p\mathcal{H} \subset \mathcal{E}^{0,1}_X \otimes F'^p\mathcal{H}$ (and thus $(D'')^2 = 0$ on $F'^p\mathcal{H}$).

On the other hand, D' defines an anti-holomorphic structure on $\mathcal{H}$ (see below), and $\operatorname{Ker} D'$ is an anti-holomorphic bundle with a flat anti-holomorphic connection $\overline{\nabla}$ induced by D''. If we wish to work with holomorphic bundle, we can thus consider the conjugate bundle[1] $\mathcal{H}'' = \overline{\operatorname{Ker} D'}$, that we equip with the holomorphic flat connection $\nabla = \overline{D''_{|\operatorname{Ker} D'}}$. A filtration of $\mathcal{H}$ by anti-holomorphic sub-bundles is by definition a filtration $F''^\bullet\mathcal{H}$ by C^∞-sub-bundles on which $D' = 0$. It corresponds to a filtration of $\mathcal{H}''$ by holomorphic sub-bundles $F^\bullet\mathcal{H}''$.

Conversely, given a flat C^∞ bundle $(\mathcal{H}, D)$, we decompose the flat connection into its $(1,0)$ part D' and its $(0,1)$ part D''. By considering types, one checks that flatness is equivalent to the three properties

$$(D')^2 = 0, \quad (D'')^2 = 0, \quad D'D'' + D''D' = 0.$$

Since, by flatness, $(D'')^2 = 0$, the Koszul-Malgrange theorem **[KM58]** implies that $\operatorname{Ker} D''$ is a holomorphic bundle $\mathcal{H}'$, that we can equip with the restriction ∇ to

[1] The precise definition is as follows. Let $\overline{\mathcal{O}}_X$ denote the sheaf of anti-holomorphic functions on X and regard $\mathcal{O}_X$ as an $\overline{\mathcal{O}}_X$-module: the action of an anti-holomorphic function $\overline{g}$ on a holomorphic function f is by definition $\overline{g} \cdot f := gf$. Then any $\overline{\mathcal{O}}_X$-module E'' determines an $\mathcal{O}_X$-module $\overline{E''}$ by setting $\overline{E''} := \mathcal{O}_X \otimes_{\overline{\mathcal{O}}_X} E''$.

$\operatorname{Ker} D''$ of the connection D'. Flatness of D also implies that ∇ is a flat holomorphic connection on $\mathcal{H}'$.

The conjugate C^∞ bundle $\overline{\mathcal{H}}$ is equipped with the conjugate connection $\overline{D}$, which is also flat. Conjugation exchanges of course the $(1,0)$-part and the $(0,1)$-part, that is, $\overline{D}' = \overline{D''}$ and $\overline{D}'' = \overline{D'}$. The corresponding holomorphic sub-bundle is $\mathcal{H}'' := (\overline{\mathcal{H}})' = \operatorname{Ker} \overline{D}''$. We can also express it as $\mathcal{H}'' = \overline{\operatorname{Ker} D'}$, and it is equipped with the flat holomorphic connection induced by $\overline{D}' = \overline{D''}$.

Similarly, we set $F'^p\overline{\mathcal{H}} = \overline{F''^p\mathcal{H}}$, etc.

***4.1.2. Definition* (Flat sesquilinear pairings).**

(1) A *sesquilinear pairing* $\mathfrak{s}$ on a C^∞ bundle $\mathcal{H}$ is a pairing on $\mathcal{H}$ with values in the sheaf $\mathcal{C}^\infty_X$, which satisfies, for local sections u, v of $\mathcal{H}$ and a C^∞ function g the relation $\mathfrak{s}(gu, \overline{v}) = \mathfrak{s}(u, \overline{gv}) = g\mathfrak{s}(u, \overline{v})$. We regard it as a $\mathcal{C}^\infty_X$-linear morphism

$$\mathfrak{s} : \mathcal{H} \otimes_{\mathcal{C}^\infty_X} \overline{\mathcal{H}} \longrightarrow \mathcal{C}^\infty_X.$$

(2) A *flat sesquilinear pairing* $\mathfrak{s}$ on a flat C^∞ bundle $(\mathcal{H}, D)$ is a sesquilinear pairing which satisfies

$$\mathrm{d}\mathfrak{s}(u, \overline{v}) = \mathfrak{s}(Du, \overline{v}) + \mathfrak{s}(u, \overline{Dv});$$

equivalently, decomposing into types,

$$\begin{cases} \mathrm{d}'\mathfrak{s}(u, \overline{v}) = \mathfrak{s}(D'u, \overline{v}) + \mathfrak{s}(u, \overline{D''v}), \\ \mathrm{d}''\mathfrak{s}(u, \overline{v}) = \mathfrak{s}(D''u, \overline{v}) + \mathfrak{s}(u, \overline{D'v}). \end{cases}$$

4.1.3. Lemma. *Giving a flat sesquilinear pairing $\mathfrak{s}$ on $(\mathcal{H}, D)$ is equivalent to giving an $\mathcal{O}_X \otimes_{\mathbb{C}} \overline{\mathcal{O}}_X$-linear morphism $\mathfrak{s} : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}'} \to \mathcal{C}^\infty_X$, that is, which satisfies*

$$\begin{cases} \mathfrak{s}(gu, \overline{v}) = g\mathfrak{s}(u, \overline{v}), \\ \mathfrak{s}(u, \overline{gv}) = \overline{g}\mathfrak{s}(u, \overline{v}), \end{cases} \qquad g \in \mathcal{O}_X,\ u, v \in \mathcal{H}',$$

and

$$\begin{cases} \mathrm{d}'\mathfrak{s}(u, \overline{v}) = \mathfrak{s}(\nabla u, \overline{v}), \\ \mathrm{d}''\mathfrak{s}(u, \overline{v}) = \mathfrak{s}(u, \overline{\nabla v}). \end{cases}$$

***Proof*.** Immediate from the definitions. □

***4.1.4. Definition* (Variation of $\mathbb{C}$-Hodge structure, first definition)**

A variation of $\mathbb{C}$-Hodge structure H of weight w consists of the data of a flat C^∞ bundle $(\mathcal{H}, D)$, equipped with a filtration $F'^\bullet\mathcal{H}$ by holomorphic sub-bundles satisfying Griffiths transversality (4.1.1), and with a filtration $F''^\bullet\mathcal{H}$ by anti-holomorphic sub-bundles satisfying anti-Griffiths transversality, such that the restriction of these data at each point $x \in X$ is a $\mathbb{C}$-Hodge structure of weight w (Definition 2.5.2).

A *morphism* $\varphi : H_1 \to H_2$ is a flat morphism of C^∞-bundles compatible with both the holomorphic and the anti-holomorphic filtrations.

A *polarization* S is a morphism $H \otimes \overline{H} \to \mathcal{C}^\infty_X(-w)$ of flat filtered bundles, where $\mathcal{C}^\infty_X(-w)$, is equipped with the natural connection d and w-shifted trivial filtrations, whose restriction to each $x \in X$ is a polarization of the Hodge structure H_x

(see Definition 2.5.10). We usually denote by $\mathcal{S} : \mathcal{H} \otimes \overline{\mathcal{H}} \to \mathcal{C}^\infty$ the underlying flat morphism and by $\underline{\mathcal{S}} : \underline{\mathcal{H}} \otimes \overline{\underline{\mathcal{H}}} \to \mathbb{C}$ its restriction to the local system $\underline{\mathcal{H}}$.

***4.1.5. Definition* (Variation of $\mathbb{C}$-Hodge structure, second definition)**
A variation of $\mathbb{C}$-Hodge structure H of weight w consists of the data of a flat C^∞ bundle $(\mathcal{H}, D)$, equipped with a *Hodge decomposition* by C^∞-sub-bundles

$$\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$$

satisfying *Griffiths transversality*:

$$\begin{aligned} D'\mathcal{H}^{p,q} &\subset \Omega^1_X \otimes (\mathcal{H}^{p,q} \oplus \mathcal{H}^{p-1,q+1}), \\ D''\mathcal{H}^{p,q} &\subset \overline{\Omega^1_X} \otimes (\mathcal{H}^{p,q} \oplus \mathcal{H}^{p+1,q-1}). \end{aligned} \tag{4.1.5 $*$}$$

A *morphism* $H_1 \to H_2$ is a D-flat morphism $(\mathcal{H}_1, D) \to (\mathcal{H}_2, D)$ which is compatible with the Hodge decomposition.

A *polarization* is a C^∞ Hermitian metric h on the C^∞-bundle $\mathcal{H}$ such that

- the Hodge decomposition is orthogonal with respect to h,
- The polarization form $\mathcal{S}$, defined by the property that (see Definition 2.5.14)
    - the decomposition is S-orthogonal and
    - $\mathrm{h}_{|\mathcal{H}^{p,w-p}} := (-1)^q \mathrm{S}_{|\mathcal{H}^{p,w-p}}$,

induces a D-flat $\mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}$-linear pairing $\mathcal{S} : \mathcal{H} \otimes_{\mathbb{C}} \overline{\mathcal{H}} \to \mathcal{C}^\infty_X$.

4.1.6. Lemma. *The two definitions are equivalent.*

Proof. Given a C^∞ flat bundle $(\mathcal{H}, D)$ equipped with a C^∞ Hodge decomposition satisfying (4.1.5 $*$), we define $F'^p\mathcal{H} = \bigoplus_{p' \geqslant p} \mathcal{H}^{p',w-p'}$, and (4.1.5 $*$) implies $D''(F'^p\mathcal{H}) \subset \mathcal{E}^{1,0}_X \otimes F'^{p-1}\mathcal{H}$, so that D'' induces a holomorphic structure on $F'^p\mathcal{H}$ and $\mathrm{gr}^p_F\mathcal{H} \simeq \mathcal{H}^{p,q-p}$ is a C^∞ bundle. We argue similarly to obtain the properties of $F''^\bullet\mathcal{H}$. By construction, the restriction of the filtrations to any point of X is give rise to the Hodge decomposition $\bigoplus \mathcal{H}^{p,w-p}_x$.

Conversely, assume that we are given $(\mathcal{H}, D, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H})$ as in Definition 4.1.4. The dimension of each fiber of $\mathcal{H}^{p,w-p} := F'^p\mathcal{H} \cap F''^{w-p}\mathcal{H}$ is constant, since it is equal to that of the bundle $\mathrm{gr}^p_F\mathcal{H}$. We now use that, if all fibers of the intersection of two sub-bundles of a vector bundle have the same dimension, then this intersection is also a sub-bundle. This implies that $\mathcal{H}^{p,w-p}$ is a sub-bundle of $\mathcal{H}$, and the decomposition follows from the Hodge property in each fiber. In order to obtain (4.1.5 $*$), we notice that $F''^q\mathcal{H}$, being anti-holomorphic, is preserved by D', so

$$D'(F'^p\mathcal{H} \cap F''^q\mathcal{H}) \subset \Omega^1_X \otimes (F'^{p-1}\mathcal{H} \cap F''^q\mathcal{H}).$$

We argue similarly with D''. □

4.1.7. Remark. While it is easy, by using a partition of unity, to construct a Hermitian metric compatible with the Hodge decomposition, the condition of flatness of $\mathcal{S}$ is a true constraint if $\dim X \geqslant 1$. For example, any flat C^∞-bundle $(\mathcal{H}, D)$ can be regarded as a variation of $\mathbb{C}$-Hodge structure of type $(0,0)$, and it admits many Hermitian metrics, but the polarization condition imposes that the Hermitian metric is flat, which only occurs when the monodromy representation of the flat bundle is (conjugate to) a unitary representation.

4.1.8. Remark **(Polarized variation of Hodge structure as a flat filtered Hermitian pair)**
In analogy with Section 2.5.17, we can describe a polarized variation of Hodge structure by using only one filtration. By a *flat filtered Hermitian pair* we mean the data $((\mathcal{H}', \nabla), F^\bullet\mathcal{H}', \mathcal{S})$, where

(i) $(\mathcal{H}', \nabla)$ is a flat holomorphic vector bundle and $F^\bullet\mathcal{H}'$ is a filtration by holomorphic sub-bundles satisfying Griffiths transversality,

(ii) $\mathcal{S} : \mathcal{H}' \otimes_{\mathbb{C}} \overline{\mathcal{H}'} \to \mathcal{C}^\infty_X$ is a ∇-flat Hermitian pairing as defined in Lemma 4.1.3.

This can also be described as C^∞ data $((\mathcal{H}, D), F^\bullet\mathcal{H}, \mathcal{S})$ as in Definition 4.1.2, the equivalence being given by Lemma 4.1.3.

A flat filtered Hermitian pair $((\mathcal{H}', \nabla), F^\bullet\mathcal{H}', \mathcal{S})$ is a *polarized variation of Hodge structure of weight* w if its restriction to every $x \in X$ is a polarized Hodge structure in the sense of 2.5.17(1)–(3).

4.1.9. Definition **(The categories $\mathsf{VHS}(X, \mathbb{C}, w)$ and $\mathsf{pVHS}(X, \mathbb{C}, w)$)**
Definitions 4.1.4 and 4.1.5 produce the category $\mathsf{VHS}(X, \mathbb{C}, w)$ of variations of $\mathbb{C}$-Hodge structures of weight w on X. The category $\mathsf{pVHS}(X, \mathbb{C}, w)$ of polarizable variations of $\mathbb{C}$-Hodge structures of weight w is the full subcategory of $\mathsf{VHS}(X, \mathbb{C}, w)$ whose objects admit a polarization.

We refer to Exercises 4.1 and 4.2 for the following result.

4.1.10. Proposition.

(1) *The category* $\mathsf{VHS}(X, \mathbb{C}, w)$ *is abelian and each morphism is strictly compatible with the Hodge filtration. It is equipped with the operations tensor product, Hom, dual, conjugation and Hermitian dual.*

(2) *The full subcategory* $\mathsf{pVHS}(X, \mathbb{C}, w)$ *is abelian and stable by the previous operations. It is stable by direct summand in* $\mathsf{VHS}(X, \mathbb{C}, w)$ *and is semi-simple.* □

4.1.b. Variations of $\mathbb{Q}$-Hodge structure. We can now mimic the definition of Section 2.5.c. An object of $\mathsf{VHS}(X, \mathbb{Q}, w)$ is a tuple $(\underline{\mathcal{H}}_{\mathbb{Q}}, H, \mathrm{iso})$, where

- $\mathcal{H}_{\mathbb{Q}}$ is a $\mathbb{Q}$-local system on X,
- H is an object of $\mathsf{VHS}(X, \mathbb{C}, w)$,
- iso is an isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} \underline{\mathcal{H}}_{\mathbb{Q}} \xrightarrow{\sim} \underline{\mathcal{H}}$,

with the condition that at each $x \in X$, these data restrict to a $\mathbb{Q}$-Hodge structure. Morphisms are the obvious ones which are compatible with the data. The definition of $\mathsf{pVHS}(X, \mathbb{Q}, w)$ is similar, by imposing that the polarization form $\underline{\mathcal{S}}$ comes, after tensoring with $\mathbb{C}$, from a bilinear form

$$\underline{\mathcal{S}}_{\mathbb{Q}} : \underline{\mathcal{H}}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \underline{\mathcal{H}}_{\mathbb{Q}} \longrightarrow (2\pi \mathsf{i})^{-w}\mathbb{Q}.$$

4.1.11. Proposition.

(1) *The category* $\mathsf{VHS}(X,\mathbb{Q},w)$ *is abelian and each morphism is strictly compatible with the Hodge filtration. It is equipped with the operations tensor product, Hom, and dual.*

(2) *The full subcategory* $\mathsf{pVHS}(X,\mathbb{Q},w)$ *is abelian and stable by the previous operations. It is stable by direct summand in* $\mathsf{VHS}(X,\mathbb{Q},w)$ *and is semi-simple.* □

4.1.12. Remark. One can define the category $\mathsf{VHS}(X,\mathbb{R},w)$ of variations of $\mathbb{R}$-Hodge structure without referring to the Riemann-Hilbert correspondence, i.e., without using the local system $\underline{\mathcal{H}}$, by using instead a complex involution $\kappa:(\mathcal{H},D)\xrightarrow{\sim}(\overline{\mathcal{H}},\overline{D})$.

4.2. The Hodge theorem

4.2.a. The Hodge theorem for unitary representations. We will extend the Hodge theorem (Theorem 2.4.4 and the results indicated after its statement concerning the polarization) to the case of the cohomology with coefficients in a unitary representation.

Let us start with a holomorphic vector bundle $\mathcal{H}'$ of rank d on a complex projective manifold X equipped with a flat holomorphic connection ∇. The local system $\underline{\mathcal{H}}=\mathcal{H}'^{\nabla}$ corresponds to a representation $\pi_1(X,\star)\to\mathrm{GL}_d(\mathbb{C})$, up to conjugation. The unitary assumption means that we can conjugate the given representation in such a way that it takes values in the unitary group.

In other words, there exists a Hermitian metric h on the associated C^∞-bundle $\mathcal{H}=\mathcal{C}^\infty\otimes_{\mathcal{O}_X}\mathcal{H}'$ such that, if we denote as above by D the connection on $\mathcal{H}$ defined by $D(\varphi\otimes v)=\mathrm{d}\varphi\otimes v+\varphi\otimes\nabla v$, the connection D is compatible with the metric h (i.e., is the Chern connection of the metric h).

That D is a connection *compatible with the metric* implies that its formal adjoint (with respect to the metric) is obtained with a suitably defined Hodge $\star$ operator by the formula $D^\star=-\star D\star$. This leads to the decomposition of the space of C^∞ k-forms on X with coefficients in $\mathcal{H}$ (resp. (p,q)-forms) as the orthogonal sum of the kernel of the Laplace operator with respect to D (resp. D' or D''), that is, the space of harmonic sections, and its image.

As the connection D is *flat*, there is a C^∞ de Rham *complex* $(\mathcal{E}_X^\bullet\otimes\mathcal{H},D)$, and standard arguments give

$$H^k(X,\underline{\mathcal{H}})=\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{H}',\nabla))=H^k\big(\Gamma(X,(\mathcal{E}_X^\bullet\otimes\mathcal{H},D))\big).$$

One can also define the Dolbeault cohomology groups by decomposing $\mathcal{E}^\bullet$ into $\mathcal{E}^{p,q}$'s and by decomposing D as $D'+D''$. Then $H^{p,q}_{D''}(X,\mathcal{H})=H^q(X,\Omega^p_X\otimes\mathcal{H}')$.

As the projective manifold X is *Kähler*, we obtain the Kähler identities for the various Laplace operators: $\Delta_D=2\Delta_{D'}=2\Delta_{D''}$.

Then, exactly as in Theorem 2.4.4, we get:

4.2.1. Theorem. *Under these conditions, one has a canonical decomposition*

$$\boldsymbol{H}^k(X, \mathrm{DR}(\mathcal{H}', \nabla)) = \bigoplus_{p+q=k} H^{p,q}_{D''}(X, \mathcal{H})$$

and $H^{q,p}_{D''}(X, \mathcal{H})$ is identified with $\overline{H^{p,q}_{D''}(X, \mathcal{H}^\vee)}$, where $\mathcal{H}^\vee$ is the dual bundle.[(2)]

The Hard Lefschetz theorem also holds in this context.

4.2.b. Harmonic bundles. If we do not assume anymore that $\underline{\mathcal{H}}$ is unitary, but only assume that it underlies a polarized variation of Hodge structure of some weight w (so that the unitary case is the particular case of a variation of pure type $(0,0)$), we have a flat connection D on the C^∞-bundle $\mathcal{H}$ associated to $\mathcal{H}'$, with $D = D' + \mathrm{d}''$, and we also have a Hermitian metric h on $\mathcal{H}$ associated with S, but D is possibly *not* compatible with the metric. The argument using the Hodge $\star$ operator is not valid anymore. We first consider a general situation.

Let X be a complex manifold, let $(\mathcal{H}, D)$ be a flat C^∞ bundle on X, and let h be a Hermitian metric on $\mathcal{H}$. We decompose D into its $(1,0)$ and $(0,1)$ parts: $D = D' + D''$.

4.2.2. Lemma. *Given $(\mathcal{H}, D, \mathrm{h})$, there exists a unique connection $D_{\mathrm{h}} = D'_{\mathrm{h}} + D''_{\mathrm{h}}$ on $\mathcal{H}$ and a unique C^∞-linear morphism $\theta = \theta' + \theta'' : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$ satisfying the following properties:*

(1) *D_{h} is compatible with* h, *i.e.,* $\mathrm{dh}(u, \overline{v}) = \mathrm{h}(D_{\mathrm{h}} u, \overline{v}) + \mathrm{h}(u, \overline{D_{\mathrm{h}} v})$, *or equivalently, decomposing into types,*

$$\mathrm{d}'\mathrm{h}(u, \overline{v}) = \mathrm{h}(D'_{\mathrm{h}} u, \overline{v}) + \mathrm{h}(u, \overline{D''_{\mathrm{h}} v}), \quad \mathrm{d}''\mathrm{h}(u, \overline{v}) = \mathrm{h}(D''_{\mathrm{h}} u, \overline{v}) + \mathrm{h}(u, \overline{D'_{\mathrm{h}} v}).$$

(2) *θ is self-adjoint with respect to* h, *i.e.,* $\mathrm{h}(\theta u, \overline{v}) = \mathrm{h}(u, \overline{\theta v})$, *or equivalently, decomposing into types,*

$$\mathrm{h}(\theta' u, \overline{v}) = \mathrm{h}(u, \overline{\theta'' v}), \quad \mathrm{h}(\theta'' u, \overline{v}) = \mathrm{h}(u, \overline{\theta' v}).$$

(3) *$D = D_{\mathrm{h}} + \theta$, or equivalently, decomposing into types,*

$$D' = D'_{\mathrm{h}} + \theta', \quad D'' = D''_{\mathrm{h}} + \theta''.$$

4.2.3. Remark. In 4.2.2(1), we have extended the metric h in a natural way as a sesquilinear operator $(\mathcal{E}^1_X \otimes \mathcal{H}) \otimes \overline{\mathcal{H}} \to \mathcal{E}^1_X$ resp. $\mathcal{H} \otimes (\overline{\mathcal{E}^1_X \otimes \mathcal{H}}) \to \mathcal{E}^1_X$.

Proof. Let D_{h} be a connection on $\mathcal{H}$ which is compatible with h. Let A be a $\mathcal{C}^\infty_X$-linear morphism $A : \mathcal{H} \to \mathcal{E}^\infty_X \otimes \mathcal{H}$ which is skew-adjoint with respect to h, that is, such that $\mathrm{h}(Au, \overline{v}) = -\mathrm{h}(u, \overline{Av}) = 0$ for every local sections u, v of $\mathcal{H}$. Then the connection $D_{\mathrm{h}} + A$ is also compatible with the metric. So let us choose any h-compatible connection $\widetilde{D}_{\mathrm{h}}$ (for example the Chern connection, also compatible with the holomorphic structure D'') and let us set $A = D - \widetilde{D}_{\mathrm{h}}$. Let us decompose A as $A^+ + A^-$, with A^+ self-adjoint and A^- skew-adjoint. We can thus set $D_{\mathrm{h}} = \widetilde{D}_{\mathrm{h}} + A^-$ and $\theta = A^+$. Uniqueness is seen similarly. $\square$

(2) When we work with a polarized variation of Hodge structure, the polarization S identifies $(\mathcal{H}, D)$ and $(\mathcal{H}^\vee, D^\vee)$ and we recover the usual conjugation relation between $H^{q,p}$ and $H^{p,q}$.

4.2.4. Remarks.

(1) Iterating 4.2.2(2), we find that $\theta'' \wedge \theta''$ is h-adjoint to $-\theta' \wedge \theta'$ and $\theta' \wedge \theta'' + \theta'' \wedge \theta'$ is skew-adjoint. By applying d' or d'' to 4.2.2(1) and (2), we see that D''^2_{h} is adjoint to $-D'^2_{\mathrm{h}}$, $D''_{\mathrm{h}}(\theta')$ is adjoint to $-D'_{\mathrm{h}}(\theta'')$, $D''_{\mathrm{h}}(\theta'')$ is adjoint to $-D'_{\mathrm{h}}(\theta')$, and $D'_{\mathrm{h}}D''_{\mathrm{h}} + D''_{\mathrm{h}}D'_{\mathrm{h}}$ is skew-adjoint with respect to h.

(2) Let us set $\widehat{D}' = D'_{\mathrm{h}} - \theta'$. Then the Chern connection for the metric h on the holomorphic bundle $(\mathcal{H}, D'')$ is equal to $\widehat{D}' + D''$ (see Exercise 4.4). Similarly, setting $\widehat{D}'' = D''_{\mathrm{h}} - \theta''$, the Chern connection for the anti-holomorphic bundle $(\mathcal{H}, D')$ is $D' + \widehat{D}''$. We will set

$$D^{\mathrm{c}} = \widehat{D}'' - \widehat{D}'.$$

We refer to Exercise 4.4 for the properties of these operators.

***4.2.5. Definition* (Harmonic bundle).** Let $(\mathcal{H}, D, \mathrm{h})$ be a flat C^∞-bundle equipped with a Hermitian metric h. We say that $(\mathcal{H}, D, \mathrm{h})$ is a *harmonic bundle* if the operator $D''_{\mathrm{h}} + \theta' = \frac{1}{2}(D + D^{\mathrm{c}})$ has square 0. We also set

$$\mathcal{D} = D'_{\mathrm{h}} + \theta'', \quad \overline{\mathcal{D}} = D''_{\mathrm{h}} + \theta', \tag{4.2.5 $*$}$$

so that $D = D' + D'' = \mathcal{D} + \overline{\mathcal{D}}$ and $D^{\mathrm{c}} = \overline{\mathcal{D}} - \mathcal{D}$.

By looking at types, the harmonicity condition is equivalent to

$$D''^2_{\mathrm{h}} = 0, \quad D''_{\mathrm{h}}(\theta') = 0, \quad \theta' \wedge \theta' = 0. \tag{4.2.6}$$

By adjunction, this implies

$$D'^2_{\mathrm{h}} = 0, \quad D'_{\mathrm{h}}(\theta'') = 0, \quad \theta'' \wedge \theta'' = 0.$$

Moreover, the flatness of D implies then

$$D'_{\mathrm{h}}(\theta') = 0, \quad D''_{\mathrm{h}}(\theta'') = 0, \quad D'_{\mathrm{h}}D''_{\mathrm{h}} + D''_{\mathrm{h}}D'_{\mathrm{h}} = -(\theta' \wedge \theta'' + \theta'' \wedge \theta').$$

4.2.7. Lemma. *Let $(\mathcal{H}, D)$ be a flat bundle and let* h *be a Hermitian metric on $\mathcal{H}$. Then*

$$(\overline{\mathcal{D}})^2 = 0 \Longrightarrow \begin{cases} DD^{\mathrm{c}} + D^{\mathrm{c}}D = 0, \\ \mathcal{D}\overline{\mathcal{D}} + \overline{\mathcal{D}}\mathcal{D} = 0, \quad \mathcal{D}^2 = 0. \end{cases}$$

Proof. Since $D^2 = 0$, it is a matter of proving $(D^{\mathrm{c}})^2 = 0$. From the vanishing above, we find $(\widehat{D}')^2 = 0$, $(\widehat{D}'')^2 = 0$. We also get $\widehat{D}''\widehat{D}' + \widehat{D}'\widehat{D}'' = 0$. The properties for $\mathcal{D}, \overline{\mathcal{D}}$ are obtained similarly. $\square$

***4.2.8. Definition* (Higgs bundle).** Set $\mathcal{E} = \operatorname{Ker} D''_{\mathrm{h}} : H \to H$. If $(\mathcal{H}, D, \mathrm{h})$ is harmonic, then $\mathcal{E}$ is a holomorphic vector bundle equipped with a holomorphic $\mathrm{End}(\mathcal{E})$-valued 1-form θ induced by θ', which satisfies $\theta \wedge \theta = 0$. It is called the *Higgs bundle* associated to the harmonic bundle, and θ is its *associated holomorphic Higgs field.* Let us also notice that, by definition, D_{h} is the Chern connection for the Hermitian holomorphic bundle $(\mathcal{E}, \mathrm{h})$.

4.2.9. Kähler identities for harmonic bundles. We assume that X is a compact Kähler complex manifold. We can apply the Kähler identities to the holomorphic bundle $(\mathcal{H}, D'')$ with Hermitian metric h and Chern connection $\widehat{D}' + D''$, as well as to the holomorphic bundle $(\mathcal{H}, D''_{\mathrm{h}})$ with Hermitian metric h and Chern connection $D'_{\mathrm{h}}+D''_{\mathrm{h}}$. Denoting by $P^\star$ the formal L^2-adjoint of an operator P, the classical Kähler identities take the form

$$\begin{aligned}\widehat{D}'^\star &= \mathsf{i}[\Lambda, D''], \quad & D''^\star &= -\,\mathsf{i}[\Lambda, \widehat{D}'], \quad & \Delta_{D''} &= -\,\mathsf{i}[D'', [\Lambda, \widehat{D}']],\\ D'^\star_{\mathrm{h}} &= \mathsf{i}[\Lambda, D''_{\mathrm{h}}], \quad & D''^\star_{\mathrm{h}} &= -\,\mathsf{i}[\Lambda, D'_{\mathrm{h}}], \quad & \Delta_{D''_{\mathrm{h}}} &= -\,\mathsf{i}[D''_{\mathrm{h}}, [\Lambda, D'_{\mathrm{h}}]].\end{aligned}$$

As a consequence, we find

$$D^\star_{\mathrm{h}} := D'^\star_{\mathrm{h}} + D''^\star_{\mathrm{h}} = \mathsf{i}[\Lambda, D^{\mathrm{c}}_{\mathrm{h}}].$$

Note also the following identity (see Exercise 4.9):

$$\Delta_D = 2\Delta_{\mathcal{D}} = 2\Delta_{\overline{\mathcal{D}}}. \tag{4.2.10}$$

4.2.c. Polarized variations of Hodge structure on a compact Kähler manifold: the Hodge-Deligne theorem. Let us come back to the setting of Section 4.2.a.

4.2.11. Proposition. *Let $(\mathcal{H}, D, \mathrm{h})$ be a flat bundle with metric underlying a polarized variation of $\mathbb{C}$-Hodge structure on X. Then $(\mathcal{H}, D, \mathrm{h})$ is a harmonic bundle.*

Proof. This is the content of Exercise 4.3. □

Let us emphasize that the h-compatible connections D'_{h} resp. D''_{h} of Lemma 4.2.2 are given by the first projection in Griffiths' transversality relations (4.1.5 $*$), and θ' resp. θ'' as the second projections. Recall that $(\mathcal{H}, D)$ is the flat C^∞ bundle associated with the flat holomorphic bundle $(\mathcal{H}', \nabla)$. Then D''_{h} defines a holomorphic structure on $\mathcal{H}^{p,w-p}$, with associated holomorphic bundle $\operatorname{Ker} D''_{\mathrm{h}}$ isomorphic to the holomorphic bundle $\mathrm{gr}^p_F\mathcal{H}'$. Moreover, $\theta' : \mathcal{H}^{p,w-p} \to \mathcal{H}^{p-1,w-p+1}$ is the C^∞ morphism associated with the $\mathcal{O}_X$-linear morphism induced by ∇:

$$\theta := \mathrm{gr}^{-1}_F\nabla : \mathrm{gr}^p_F\mathcal{H}' \longrightarrow \Omega^1_X \otimes \mathrm{gr}^{p-1}_F\mathcal{H}'. \tag{4.2.12}$$

The decomposition $D = D' + D''$ is thus replaced with the decomposition $D = \mathcal{D} + \overline{\mathcal{D}}$. The disadvantage is that we loose the decomposition into types $(1,0)$ and $(0,1)$, but we keep the flatness property. On the other hand, we also keep the Kähler identities (4.2.10).

We did not really loose the decomposition into types: the operator $\overline{\mathcal{D}}$ sends a section of $\mathcal{H}^{p,q}$ to a section of $(\Omega^1_X \otimes \mathcal{H}^{p-1,q+1}) + (\overline{\Omega^1_X} \otimes \mathcal{H}^{p,q})$. Counting the total type, we find $(p, q+1)$ for both terms. In other word, taking into account the Hodge type of a section, the operator $\overline{\mathcal{D}}$ is indeed of type $(0,1)$. A similar argument applies to $\mathcal{D}$. By Definition 4.2.5, $(\mathcal{E}^\bullet_X \otimes \mathcal{H}, \overline{\mathcal{D}})$ is a complex. Let us set

$$F^p(\mathcal{E}^m_X \otimes \mathcal{H}) = \bigoplus_{\substack{i+j=m\\ i+k\geqslant p}} \mathcal{E}^{i,j}_X \otimes \mathcal{H}^{k,w-k}.$$

We note that
$$\overline{\mathcal{D}}F^p(\mathcal{E}_X^m\otimes\mathcal{H})\subset F^p(\mathcal{E}_X^{m+1}\otimes\mathcal{H}),$$
since D''_{h} sends $\mathcal{E}_X^{i,j}\otimes\mathcal{H}^{k,w-k}$ to $\mathcal{E}_X^{i,j+1}\otimes\mathcal{H}^{k,w-k}$ and θ' sends $\mathcal{E}_X^{i,j}\otimes\mathcal{H}^{k,w-k}$ to $\mathcal{E}_X^{i+1,j}\otimes\mathcal{H}^{k-1,w-k+1}$. We thus get a filtered complex by setting
$$F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})=\Big\{F^p(\mathcal{E}_X^0\otimes\mathcal{H})\xrightarrow{\overline{\mathcal{D}}}F^p(\mathcal{E}_X^1\otimes\mathcal{H})\longrightarrow\cdots\Big\},$$
and the associated graded complex has the following degree-m term:
$$\bigoplus_{i=0}^m(\mathcal{E}_X^{i,m-i}\otimes\mathcal{H}^{p-i,w-p+i}).$$

On the other hand, we filter $\mathrm{gr}_F\mathcal{H}'$ by setting $F^p\mathrm{gr}_F\mathcal{H}'=\bigoplus_{p'\geqslant p}\mathrm{gr}_F^{p'}\mathcal{H}'$, so that, according to (4.2.12), we obtain the *holomorphic Dolbeault complex*

$$\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta):=\{0\to\mathrm{gr}_F\mathcal{H}'\xrightarrow{\theta}\Omega_X^1\otimes\mathrm{gr}_F\mathcal{H}'\xrightarrow{\theta}\cdots\}.\tag{4.2.13}$$

which is filtered by setting
$$F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)=\Big\{F^p\mathrm{gr}_F\mathcal{H}'\xrightarrow{\theta}\Omega_X^1\otimes F^{p-1}\mathrm{gr}_F\mathcal{H}'\longrightarrow\cdots\Big\},$$
so that
$$\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)=\Big\{\mathrm{gr}_F^p\mathcal{H}'\xrightarrow{\theta}\Omega_X^1\otimes\mathrm{gr}_F^{p-1}\mathcal{H}'\longrightarrow\cdots\Big\}.$$

***4.2.14. Proposition* (Dolbeault resolution).** *For each p, the complex $F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})$ is a resolution of $F^p(\Omega_X^\bullet\otimes\mathrm{gr}_F\mathcal{H}',\theta)$ and $\mathrm{gr}_F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})$ is a resolution of $\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)$.*

***Proof*.** Since the filtration $F^\bullet$ is finite, it is enough to prove the second statement. Due to the relations (4.2.6), we can regard (up to signs) $\mathrm{gr}_F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})$ as the simple complex associated with the double complex

$$\begin{tikzcd}
\mathcal{E}^{i,j}\otimes\mathcal{H}^{p-i,w-p+i} \arrow[r, "\theta'"] \arrow[d, "D''_{\mathrm{h}}"'] & \mathcal{E}^{i+1,j}\otimes\mathcal{H}^{p-i-1,w-p+i+1} \arrow[d, "D''_{\mathrm{h}}"] \\
\mathcal{E}^{i,j+1}\otimes\mathcal{H}^{p-i,w-p+i} \arrow[r, "\theta'"] & \mathcal{E}^{i+1,j+1}\otimes\mathcal{H}^{p-i-1,w-p+i+1}
\end{tikzcd}$$

The i-th vertical complex a resolution of $\Omega_X^i\otimes\mathrm{gr}_F^{p-i}\mathcal{H}'$. □

***4.2.15. Corollary* (Dolbeault Lemma).** *We have for each $p,q\in\mathbb{Z}$:*
$$\boldsymbol{H}^q\big(X,\mathrm{gr}_F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)\big)\simeq H^q\big(\Gamma(X,\mathrm{gr}_F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})\big),$$
$$\boldsymbol{H}^q\big(X,F^p\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta)\big)\simeq H^q\big(\Gamma(X,F^p(\mathcal{E}_X^\bullet\otimes\mathcal{H},\overline{\mathcal{D}})\big).$$
□

This being understood, the arguments of Hodge theory apply to this situation as in the case considered in Section 4.2.a, to get the Hodge-Deligne theorem.

4.2.16. Theorem **(Hodge-Deligne theorem).** *Let (H, S) be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on a smooth complex projective variety X of pure dimension n and let $\mathcal{L}$ be an ample line bundle on X. Then $\big(H^\bullet(X, \underline{\mathcal{H}}), \mathrm{X}_{\mathcal{L}}\big)$ is naturally equipped with a polarizable $\mathfrak{sl}_2$-Hodge structure with central weight $w+n$ (see Definition 3.2.7). In particular, each $H^k(X, \underline{\mathcal{H}})$ comes equipped with a polarized $\mathbb{C}$-Hodge structure of weight $w+k$. If (H, S) is a polarized variation of $\mathbb{Q}$-Hodge structure of weight w, then each $H^k(X, \underline{\mathcal{H}_{\mathbb{Q}}})$ is equipped with a polarized $\mathbb{Q}$-Hodge structure of weight $w+k$.*

Sketch of proof. We refer to [**Zuc79**, §2] for the detailed adaptation to this setting of the Kähler identities and their consequences. One realizes each cohomology class in $H^\bullet(X, \underline{\mathcal{H}})$ by a unique Δ_D-harmonic section, by the arguments of Hodge theory, which extend if one takes into account the total type, as above.

The polarization is obtained from S on H and Poincaré duality as we did for S in Section 2.4, and from it we cook up the form S on the cohomology. More precisely, the flat pairing

$$\mathcal{S} : \mathcal{H} \otimes \overline{\mathcal{H}} \longrightarrow \mathcal{C}_X^\infty$$

induces a pairing of locally constant sheaves

$$\underline{\mathcal{S}} : \underline{\mathcal{H}} \otimes \overline{\underline{\mathcal{H}}} \longrightarrow \mathbb{C}_X,$$

Then $\mathrm{S} : H^\bullet(X, \underline{\mathcal{H}})^{\mathrm{H}} \otimes \overline{H^\bullet(X, \underline{\mathcal{H}})^{\mathrm{H}}} \to \mathbb{C}^{\mathrm{H}}(-(w+n))$ satisfies $\mathrm{S}(H^{n+k}, \overline{H^{n-\ell}}) = 0$ if $k \neq \ell$ and otherwise induces for every $k \in \mathbb{Z}$ with $|k| \leqslant n$ a pairing S_k of $\mathbb{C}$-Hodge structures

$$H^{n+k}(X, \underline{\mathcal{H}})^{\mathrm{H}} \otimes \overline{H^{n-k}(X, \underline{\mathcal{H}})^{\mathrm{H}}} \longrightarrow H^{2n}(X, \mathbb{C})^{\mathrm{H}}(-(w+n)) = \mathbb{C}^{\mathrm{H}}(-(w+n))$$

by the formula (see Notation (0.2 ∗))

$$\mathrm{S}_k(\bullet, \overline{\bullet}) := \mathrm{Sgn}(n,k) \int_{[X]} \underline{\mathcal{S}}(\bullet, \overline{\bullet}). \tag{4.2.17}$$

Since the Lefschetz operator $\mathrm{X}_{\mathcal{L}}$ only acts on the forms and not on the sections of $\mathcal{H}$, it is self-adjoint with respect to S in the sense that $\mathrm{S}_k(\mathrm{X}_{\mathcal{L}}\eta', \overline{\eta''}) = \mathrm{S}_{k-2}(\eta', \overline{\mathrm{X}_{\mathcal{L}}\eta''})$, since it is so for the modified Poincaré duality pairing $\mathrm{Sgn}(n,k)\langle\bullet, \overline{\bullet}\rangle_{\mathbb{C}}$ (see (2.4.13)). Then S is a sesquilinear pairing on the $\mathfrak{sl}_2$-Hodge structure $H^\bullet(X, \underline{\mathcal{H}})$ (see Section 3.4.c).

Due to the Kähler identities and the commutation of $\mathrm{L}_{\mathcal{L}}$ with $\Delta_{\mathcal{D}}$, a harmonic section of $\mathcal{E}_X^{n-\ell} \otimes \mathcal{H}$ ($\ell \geqslant 0$) is primitive if and only if each of its components with respect to the total bigrading is so, and since $\mathrm{L}_{\mathcal{L}}$ only acts on the differential form part of such a component, this occurs if an only if each of its component on $\mathcal{E}_X^{p,q} \otimes \mathcal{H}^{a,b}$ is primitive, with $p+q = n-\ell$ and $a+b = w$. Fixing an h-orthonormal basis $(v_i)_i$ of $\mathcal{H}^{a,b}$, such a component can be written in a unique way as $\sum_i \eta_i^{p,q} \otimes v_i$ with $\eta_i^{p,q}$ primitive. Then the positivity property of $\mathrm{P}_{-\ell}\mathrm{S}$ defined in 3.2.10 on $\eta_i^{p,q} \otimes v_i$ amounts to the positivity of

$$\mathrm{Sgn}(n, -\ell) \int_X (-1)^{q+b} \mathcal{S}(v_i, \overline{v_i}) \cdot \eta_i^{p,q} \wedge \overline{\mathrm{X}_{\mathcal{L}}^{\ell} \eta_i^{p,q}}.$$

By the positivity for $\mathcal{S}$, there exists a C^∞ function g_i such that $(-1)^b\mathcal{S}(v_i,\overline{v_i})=|g_i|^2$. Therefore, (2.4.15) applied to $g_i\eta_i^{p,q}$ gives the desired positivity.

Last, in the presence of a $\mathbb{Q}$-structure, we define the bilinear form $\mathrm{S}_{\mathbb{Q}}$ by replacing $\underline{\mathcal{S}}$ with $\underline{\mathcal{S}}_{\mathbb{Q}}$ in (4.2.17) in order to obtain a pairing of $\mathbb{Q}$-Hodge structures

$$H^{n+k}(X,\underline{\mathcal{H}_{\mathbb{Q}}})^{\mathrm{H}}\otimes\overline{H^{n-k}(X,\underline{\mathcal{H}_{\mathbb{Q}}})^{\mathrm{H}}}\longrightarrow H^{2n}(X,\mathbb{Q})(-(w+n))^{\mathrm{H}}=\mathbb{Q}^{\mathrm{H}}(-(w+n)).\quad\square$$

4.2.18. Remarks. Let H be a polarizable variation of $\mathbb{C}$-Hodge structure of weight w on a smooth complex projective variety X.

(1) (*The Hodge filtration*) Consider the de Rham complex $\mathrm{DR}(\mathcal{H}',\nabla)$. According to the Griffiths transversality property, it comes equipped with a filtration, by setting (see Definition 8.4.1):

$$F^p\,\mathrm{DR}(\mathcal{H}',\nabla)=\{0\to F^p\mathcal{H}'\xrightarrow{\nabla}\Omega^1_X\otimes F^{p-1}\mathcal{H}'\xrightarrow{\nabla}\cdots\xrightarrow{\nabla}\Omega^n_X\otimes F^{p-n}\mathcal{H}'\to 0\}.$$

The natural inclusion of complexes $F^p\,\mathrm{DR}(\mathcal{H}',\nabla)\hookrightarrow\mathrm{DR}(\mathcal{H}',\nabla)$ induces a morphism

$$\boldsymbol{H}^k(X,F^p\,\mathrm{DR}(\mathcal{H}',\nabla))\longrightarrow\boldsymbol{H}^k(X,\mathrm{DR}(\mathcal{H}',\nabla))=H^k(X,\underline{\mathcal{H}}),\tag{4.2.18 $*$}$$

whose image is the filtration $F'^pH^k(X,\underline{\mathcal{H}})$. Working anti-holomorphically with the filtration $F''^{\bullet}\mathcal{H}$ by anti-holomorphic sub-bundles and the anti-holomorphic connection induced by D'' on $\operatorname{Ker}D'$, one obtains the filtration $F''^{\bullet}H^k(X,\underline{\mathcal{H}})$. The Hodge-Deligne theorem implies that these filtrations are $(w+k)$-opposed.

(2) (*Degeneration at E_1 of the Hodge-to-de Rham spectral sequence*)

Moreover, we claim that, for every p,k, *the morphism* (4.2.18 $*$) *is injective.* In other words, the filtered complex $\boldsymbol{R}\Gamma\big(X,F^{\bullet}\,\mathrm{DR}(\mathcal{H}',\nabla)\big)$ is *strict* (see Section 5.1.b). The graded complex $\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla)$ is the complex

$$\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla)=\{0\to\mathrm{gr}^p_F\mathcal{H}'\xrightarrow{\theta}\Omega^1_X\otimes\mathrm{gr}^{p-1}_F\mathcal{H}'\xrightarrow{\theta}\cdots\xrightarrow{\theta}\Omega^n_X\otimes\mathrm{gr}^{p-n}_F\mathcal{H}'\to 0\},$$

that is,

$$\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla)=\mathrm{gr}^p_F\,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H}',\theta).$$

Since each term of this complex is $\mathcal{O}_X$-locally free of finite rank and since θ is $\mathcal{O}_X$-linear, the hypercohomology spaces $\boldsymbol{H}^k(X,\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla))$ are finite-dimensional. The strictness property is then equivalent to

$$\forall\,p,k,\quad\mathrm{gr}^p_FH^k(X,\underline{\mathcal{H}})=\boldsymbol{H}^k(X,\mathrm{gr}^p_F\,\mathrm{DR}(\mathcal{H}',\nabla)),$$

where $\mathrm{gr}^p_FH^k(X,\underline{\mathcal{H}})\simeq H^{p,w+k-p}(X,\underline{\mathcal{H}})$. This property is also equivalent to

$$\forall\,k,\quad\dim H^k(X,\underline{\mathcal{H}})=\dim\boldsymbol{H}^k(X,\mathrm{Dol}(\mathrm{gr}_F\mathcal{H},\theta)).$$

This statement is obtained by standard arguments of Hodge theory applied to the operators $D,\mathcal{D},\overline{\mathcal{D}}$ and their Laplacians. $\square$

(3) For any smooth projective variety X, the space $H^0(X,\underline{\mathcal{H}})$ is primitive (for any $\mathcal{L}$) and, given a polarization $\mathcal{S}$ of H, a polarization of the pure $\mathbb{C}$-Hodge structure of weight w is obtained by taking integral of the polarization function against a volume form (defined from $\mathcal{L}$, in order to match with the Hodge-Deligne theorem).

If X is a compact Riemann surface, then $H^1(X,\underline{\mathcal{H}})$ is also primitive, and there is no need to choose a polarization bundle $\mathcal{L}$ in order to obtain the polarized pure $\mathbb{C}$-Hodge structure on $H^1(X,\underline{\mathcal{H}})$. The polarization (4.2.17) on $H^1(X,\underline{\mathcal{H}})$ is simply written as

$$\mathrm{S}^{(1)} = -\frac{1}{2\pi\mathrm{i}}\int_{[X]}\underline{\mathcal{S}}(\bullet,\bar{\bullet}).$$

(4) (*The fixed-part theorem*) The maximal constant subsheaf of $\underline{\mathcal{H}}$ is the constant subsheaf with stalk $H^0(X,\underline{\mathcal{H}})$ at each point, by means of a natural injective morphism $H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathbb{C}_X\to\underline{\mathcal{H}}$. By the Hodge-Deligne theorem 4.2.16, $H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathbb{C}_X$ is equipped with a constant variation of Hodge structure of weight w. We claim that *the previous morphism is compatible with the Hodge filtrations, i.e., is a morphism in* $\mathsf{VHS}(X,\mathbb{C},w)$, that is, the injective morphism

$$\varphi : H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow \underline{\mathcal{H}}\otimes_{\mathbb{C}}\mathcal{O}_X = \mathcal{H}'$$

is compatible with the Hodge filtration $F^\bullet$ on both terms.

Since X is compact and $F^p\mathcal{H}'$ is $\mathcal{O}_X$-coherent (being $\mathcal{O}_X$-locally free of finite rank), the space $H^0(X,F^p\mathcal{H}')$ is finite dimensional, and we have a natural injective morphism

$$\varphi_p : H^0(X,F^p\mathcal{H}')\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow F^p\mathcal{H}'$$

by sending a global section of $F^p\mathcal{H}'$ to its germ at every point of X. On the other hand, regarding $F^p\mathcal{H}'$ as a complex with only one term in degree zero, we have an obvious morphism of complexes

$$F^p\,\mathrm{DR}\,\mathcal{H}' \longrightarrow F^p\mathcal{H}',$$

which induces a morphism $\boldsymbol{H}^0(X,F^p\,\mathrm{DR}\,\mathcal{H}')\to H^0(X,F^p\mathcal{H}')$, from which, together with φ_p, we obtain a morphism

$$\boldsymbol{H}^0(X,F^p\,\mathrm{DR}\,\mathcal{H}')\otimes_{\mathbb{C}}\mathcal{O}_X \longrightarrow F^p\mathcal{H}'.$$

For p small enough so that $F^p\mathcal{H}'=\mathcal{H}'$, we recover the morphism φ above. By the degeneration property (2), $\boldsymbol{H}^0(X,F^p\,\mathrm{DR}\,\mathcal{H}')$ is identified with $F^pH^0(X,\underline{\mathcal{H}})$, hence the assertion.

As a consequence, *if a global horizontal section of* $(\mathcal{H}',\nabla)$*, i.e., a global section of* $\underline{\mathcal{H}}$*, regarded as a global section of* $\mathcal{H}'$*, is in* $F^p\mathcal{H}'$ *at one point, it is a global section of* $F^p\mathcal{H}'$.

Arguing similarly with the anti-holomorphic Hodge filtration, and then with the Hodge decomposition of the C^∞-bundle $\mathcal{H}$, we find that the natural injective morphism

$$H^0(X,\underline{\mathcal{H}})\otimes_{\mathbb{C}}\mathcal{C}^\infty_X \longrightarrow \underline{\mathcal{H}}\otimes_{\mathbb{C}}\mathcal{C}^\infty_X = \mathcal{H}$$

is compatible with the Hodge decomposition of each term. As a consequence, *for any global horizontal section of* $(\mathcal{H},D)$*, i.e., any global section of* $\underline{\mathcal{H}}$*, regarded as a global section of* $\mathcal{H}$*, the Hodge* (p,q)*-components are also* D*-horizontal.* In particular, if the global section is of type (p,q) at one point, it is of type (p,q) at every point of X.

Concerning the polarization, let us notice that the restriction of the polarization $\underline{\mathcal{S}}$ to the constant sub local system $H^0(X,\underline{\mathcal{H}})\otimes\mathbb{C}_X$ is constant. The polarization on

$H^0(X, \underline{\mathcal{H}})$ is thus equal, up to a positive multiplicative constant, to the restriction of $\underline{\mathcal{S}}$ to $H^0(X, \underline{\mathcal{H}})$ regarded as the stalk of $H^0(X, \underline{\mathcal{H}}) \otimes \mathbb{C}_X$ at any chosen point of X.

4.2.d. The L^2 de Rham and Dolbeault complexes. The compactness assumption in Hodge theory is not mandatory. One can relax it, provided that the metric on the manifold remains *complete* (see e.g. [**Dem96**, §12]). We will indicate the new phenomena that occur in the setting of Section 4.2.a.

One has to work with C^∞ sections v of $\mathcal{E}_X^\bullet \otimes \mathcal{H}$ which are globally L^2 with respect to the metric h and to the complete metric on X, and whose differential Dv is L^2. The analysis of the Laplace operator is now similar to that of the compact case. One uses a L^2 de Rham complex and a L^2 Dolbeault complex (i.e., one puts a L^2 condition on sections and their derivatives), that we analyze in this section.

We assume that the complex manifold X is equipped with a metric, hence a volume form vol, that enables to define the space $L^2(U, \mathrm{vol})$ for each open set $U \subset X$, and we will omit vol in the notation from now on. The locally free sheaves of differential forms are then equipped with a metric.

We can regard the sheaf of C^∞ functions on X at a subsheaf of the sheaf of locally integrable function, simply denoted by $\mathcal{L}_{1,\mathrm{loc}}$ (with respect to any metric on X). For any relatively compact open set $U \subset X$, $\Gamma(U, \mathcal{L}_{1,\mathrm{loc}}) := L^1_{\mathrm{loc}}(U, \mathrm{vol})$. One checks easily that the assignment $U \mapsto L^1_{\mathrm{loc}}(U, \mathrm{vol})$ is a sheaf, which does not depend on the choice of the metric.

Let $(\mathcal{H}, \mathrm{h})$ be a C^∞ vector bundle on X with a Hermitian metric h and let $\mathcal{L}_{1,\mathrm{loc}} \otimes_{\mathcal{C}_X^\infty} \mathcal{H}$ be the associated sheaf of L^1_{loc} sections of $\mathcal{H}$. The h-norm of any local section of the latter is a locally integrable function.

4.2.19. Definition. The space $L^2(X, \mathcal{H}, \mathrm{h})$ is the subspace of $\Gamma(X, \mathcal{L}_{1,\mathrm{loc}} \otimes \mathcal{H})$ consisting of sections whose h-norm belongs to $L^2(X)$.

The following lemmas are standard.

4.2.20. Lemma. *A section $u \in \Gamma(X, \mathcal{L}_{1,\mathrm{loc}} \otimes_{\mathcal{C}_X^\infty} \mathcal{H})$ belongs to $L^2(X, \mathcal{H}, \mathrm{h})$ if and only if there exists a sequence $u_n \in \Gamma(X, \mathcal{H}) \cap L^2(X, \mathcal{H}, \mathrm{h})$ such that $\|u - u_n\|_{2,\mathrm{h}} \to 0$ in $L^2(X, \mathcal{H}, \mathrm{h})$. In such a case, $u_n \to u$ weakly, that is, for any $\chi \in \Gamma_{\mathrm{c}}(X, \mathcal{H})$,*

$$\int_X (\mathrm{h}(u, \chi) - \mathrm{h}(u_n, \chi)) \,\mathrm{vol} \longrightarrow 0. \qquad \square$$

Let $\boldsymbol{\varepsilon}$ be an h-orthonormal frame of $\mathcal{H}$ on X. Then a section $u = \sum_i f_i \varepsilon_i$, with $f_i \in \mathcal{L}^1_{\mathrm{loc}}(X)$, belongs to $L^2(X, \mathcal{H}, \mathrm{h}))$ if and only if each f_i belongs to $L^2(X)$. Orthonormal frames may not be easy to find and in order to check the L^2 property, other frames may be more convenient.

4.2.21. Definition **(L^2-adaptedness).** A frame $\boldsymbol{v}$ of $\mathcal{H}$ on X is said to be *L^2-adapted* if there exists a positive constant $C_{\boldsymbol{v}}$ such that , for any section $u = \sum_i f_i v_i$ in

$\Gamma(X, \mathcal{L}^1_{\mathrm{loc}} \otimes \mathcal{H})$, the following inequality holds

$$\sum_i \|f_i v_i\|_2 \leqslant C_{\boldsymbol{v}} \|u\|_2 \Big(\leqslant C_{\boldsymbol{v}} \sum_i \|f_i v_i\|_2\Big).$$

In other words, an L^2-adapted frame $\boldsymbol{v}$ gives rise to a decomposition $L^2(X, \mathcal{H}, \mathrm{h}) = L^2(X, \mathcal{H}_i, \mathrm{h})$, where $\mathcal{H}_i$ is the C^∞ subbundle generated by v_i with induced metric. Let us state some properties.

4.2.22. Lemma.

(1) *An orthonormal frame $\boldsymbol{\varepsilon}$ is L^2-adapted.*

(2) *If* h *and* h' *are comparable metrics on $\mathcal{H}$, then a frame of $\mathcal{H}$ is L^2-adapted for* h *if and only if it is so for* h'.

(3) *If a frame $\boldsymbol{v} = (v_1, \dots, v_r)$ is L^2-adapted, any rescaled frame $(h_1 v_1, \dots, h_r v_r)$, with $h_i \in L^1_{\mathrm{loc}}(X)$ nowhere vanishing, is also L^2-adapted.*

(4) *A sufficient condition for a frame $\boldsymbol{v}$ to be L^2-adapted is that the functions $\|v_i\|_{\mathrm{h}}$ are locally bounded and each entry of the matrix $\mathsf{h}_{\boldsymbol{v}}^{-1} := (\mathrm{h}(v_i, v_j)_{i,j})^{-1}$ is locally bounded.*

Proof. The first three points are clear. For the last one, let $C > 0$ denote a bounding constant, let $u = \sum_i f_i v_i$ be a section of $\mathcal{L}^1_{\mathrm{loc}} \otimes \mathcal{H}$ and set $g_i := \mathrm{h}(u, v_i)$. Then $|g_i| \leqslant \|u\|_{\mathrm{h}} \|v_i\|_{\mathrm{h}} \leqslant C\|u\|_{\mathrm{h}}$, since $\|v_i\|_{\mathrm{h}} \leqslant C$. The column vectors G and F (of the g_i's and the f_i's respectively) are related by $G = \mathsf{h}_{\boldsymbol{v}} \cdot F$, so that $F = \mathsf{h}_{\boldsymbol{v}}^{-1} \cdot G$. It follows that, for each i, $|f_i| \leqslant rC^2 \|u\|_{\mathrm{h}}$ and thus $\|f_i v_i\|_{\mathrm{h}} \leqslant rC^3 \|u\|_{\mathrm{h}}$. Then $\boldsymbol{v}$ is L^2-adapted with constant $C_{\boldsymbol{v}} = rC^3$. $\square$

Assume now that $\mathcal{H}$ is equipped with a flat connection $D : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$.

4.2.23. Definition. The space $L^2(X, \mathcal{H}, \mathrm{h}, D)$ is the subspace of $L^2(X, \mathcal{H}, \mathrm{h})$ consisting of sections u such that

- Du, considered in the weak sense, (i.e., distributional sense) is a section of $\mathcal{L}_{1,\mathrm{loc}} \otimes (\mathcal{E}^1_X \otimes \mathcal{H})$,
- as such, its h-norm belongs to $L^2(X)$.

C^∞-approximation also holds in this case.

4.2.24. Lemma. *A section $u \in \Gamma(X, \mathcal{L}_{1,\mathrm{loc}} \otimes_{\mathcal{C}^\infty_X} \mathcal{H})$ belongs to $L^2(X, \mathcal{H}, \mathrm{h}, D)$ if and only if there exists a sequence $u_n \in \Gamma(X, \mathcal{H}) \cap L^2(X, \mathcal{H}, \mathrm{h})$ such that*

- $\|u - u_n\|_{2,\mathrm{h}} \to 0$ *in* $L^2(X, \mathcal{H}, \mathrm{h})$,
- Du_n *belongs to* $L^2(X, \mathcal{E}^1_X \otimes \mathcal{H}, \mathrm{h})$ *and is a Cauchy sequence in this space.* $\square$

The L^2 de Rham complex is then well-defined as the complex

$$0 \longrightarrow L^2(X, \mathcal{H}, \mathrm{h}, D) \xrightarrow{\ D\ } L^2(X, \mathcal{E}^1_X \otimes \mathcal{H}, \mathrm{h}, D) \xrightarrow{\ D\ } \cdots \xrightarrow{\ D\ } L^2(X, \mathcal{E}^{2n}_X \otimes \mathcal{H}, \mathrm{h}, D) \longrightarrow 0, \tag{4.2.25}$$

whose cohomology is denoted by $H^k_{D,L^2}(X, \mathcal{H})$.

4.2.26. Definition. The assignment $U \mapsto L^2(U, \mathcal{H}, \mathrm{h})$ defines a presheaf, which is a sheaf on X, denoted by $\mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h})$. If U is relatively compact in X, we have

$$\Gamma(U, \mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h})) = L^2(U, \mathcal{H}, \mathrm{h}).$$

We define similarly the sheaf $\mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h}, D)$ and we set

$$\mathcal{L}^i_{(2)}(\mathcal{H}, \mathrm{h}, D) := \mathcal{L}_{(2)}(\mathcal{E}^i_X \otimes \mathcal{H}, \mathrm{h}, D).$$

We obtain therefore a complex of sheaves

$$(4.2.27)\qquad 0 \longrightarrow \mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h}) \xrightarrow{\;D\;} \mathcal{L}^1_{(2)}(\mathcal{H}, \mathrm{h}, D) \xrightarrow{\;D\;} \cdots \xrightarrow{\;D\;} \mathcal{L}^{2n}_{(2)}(\mathcal{H}, \mathrm{h}, D) \longrightarrow 0.$$

4.2.28. Lemma **(L^2 Poincaré lemma).** *The complex* (4.2.27) *is a resolution of the locally constant sheaf* $\underline{\mathcal{H}} = \operatorname{Ker} D$.

Proof. Near a given point of X, we can find a local isomorphism $(\mathcal{H}, D) \simeq (\mathcal{C}^\infty_X)^{\operatorname{rk}\mathcal{H}}, \mathrm{d})$ and the metric h is equivalent to the standard metric on $(\mathcal{C}^\infty_X)^{\operatorname{rk}\mathcal{H}}$ in which the canonical basis is orthonormal. Then both assertions of the theorem need only to be proved for $(\mathcal{C}^\infty_X, \mathrm{d}, \|\cdot\|)$ where $\|1\| = 1$. On the other hand, the Poincaré metric is locally equivalent to the standard Euclidean metric near this point. The proof is then obtained by a standard regularization procedure (see Corollary 12.2.5 for a statement in any dimension). $\square$

4.2.29. Definition. By considering the decomposition (4.2.5 ∗) one defines similarly $L^2(X, \mathcal{H}, \mathrm{h}, \overline{\mathcal{D}})$ and the L^2 Dolbeault complexes ($p = 0, \dots, d_X$)

$$(4.2.30)\qquad 0 \to L^2(X, \operatorname{gr}^p_F(\mathcal{E}^0_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \xrightarrow{\;\overline{\mathcal{D}}\;} L^2(X, \operatorname{gr}^p_F(\mathcal{E}^1_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}})$$
$$\xrightarrow{\;\overline{\mathcal{D}}\;} \cdots \xrightarrow{\;\overline{\mathcal{D}}\;} L^2(X, \operatorname{gr}^p_F(\mathcal{E}^n_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \to 0,$$

whose m-th cohomology is denoted by $H^{p,m+w-p}_{\overline{\mathcal{D}},L^2}(X, \mathcal{H})$.

The analogue of the C^∞-approximation lemma 4.2.24 also holds in this case, and we define the sheaves $\mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^i_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}})$ in a way similar to Definition 4.2.26. Then the Dolbeault complex (4.2.30) sheafifies as

$$(4.2.31)\qquad 0 \to \mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^0_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \xrightarrow{\;\overline{\mathcal{D}}\;} \mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^1_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}) \xrightarrow{\;\overline{\mathcal{D}}\;} \cdots$$

4.2.32. Lemma **(L^2 Dolbeault lemma).** *The inclusion of complexes*

$$\operatorname{gr}^p_F \operatorname{Dol}(\operatorname{gr}_F \mathcal{H}', \theta) \hookrightarrow \big(\mathcal{L}_{(2)}(\operatorname{gr}^p_F(\mathcal{E}^\bullet_X \otimes \mathcal{H}), \mathrm{h}, \overline{\mathcal{D}}), \overline{\mathcal{D}}\big)$$

is a quasi-isomorphism.

Proof. We write $\overline{\mathcal{D}} = \mathrm{d}''_{\mathrm{h}} - \theta'$. Since θ' is C^∞, it is locally bounded, so the local L^2-condition on the $\overline{\mathcal{D}}$-derivative only concerns d'', and we can consider (up to signs) the complex (4.2.31) as the simple complex associated with the double complex with differentials d'' and θ'. For the complex with differential d'' we can apply the standard L^2-Dolbeault lemma (recalled as Theorem 12.2.6 in Chapter 12). Then the statement is clear. $\square$

4.2.e. Polarized variations of Hodge structure on a complex manifold with a complete metric

One missing point in the general context of noncompact complex manifolds is the finite dimensionality of the L^2-cohomologies involved. In the compact case, it is ensured, for instance, by the finiteness of the Betti cohomology $H^k(X,\underline{\mathcal{H}})$. So the Hodge theorem is stated as

4.2.33. Theorem. *Let (X,ω) be a complete Kähler manifold and let $(H,\mathcal{S})$ be a polarized variation of Hodge structure on X. Let $(\mathcal{H},D)$ be the associated flat C^∞ bundle. Then, with the assumption that all the terms involved are finite dimensional, one has a canonical isomorphisms*

$$H^k_{D,L^2}(X,\mathcal{H})\simeq\bigoplus_{p+q=k+w}H^{p,q}_{\overline{\mathcal{D}},L^2}(X,\mathcal{H}),\quad H^{q,p}_{\overline{\mathcal{D}},L^2}(X,\mathcal{H})\simeq\overline{H^{p,q}_{\overline{\mathcal{D}},L^2}(X,\mathcal{H}^\vee)}.\qquad\square$$

We refer to [**Zuc79**, §7] for a proof of this result. The finiteness assumption can be obtained by relating the L^2 de Rham cohomology with topology. If we are lucky, then this will not only provide a relation with Betti cohomology, but the Betti cohomology will be finite-dimensional and this will also provide finiteness of the L^2 de Rham cohomology.

There is also a need for finiteness of the L^2 Dolbeault cohomology. In the case that will occupy us later, where X is a punctured compact Riemann surface, this will be done by relating L^2 Dolbeault cohomology with the cohomology of a coherent sheaf on the compact Riemann surface.

We will indicate in Sections 6.12 and 6.14 the way to solve these two problems in dimension 1, by means of the L^2 Poincaré lemma and the L^2 Dolbeault lemma.

What about the Lefschetz aspect of Hodge theory in this context? The the complete Kähler metric acts as a Lefschetz operator on the L^2 cohomology, and gives rise to a polarization of the Hodge structure. On the other hand, the theory of Hodge modules takes place on smooth complex projective varieties (or for some aspects compact Kähler manifolds). The non-compact manifold that occurs is the complement of divisor with normal crossings. Such a manifold can be equipped with a complete Kähler metric having a controlled behaviour at infinity, that is, in the neighbourhood of the normal crossing divisor of the compactification: the metric has a Poincaré-like behaviour locally at infinity. However, in the theory of Hodge modules, we only consider the Lefschetz operator coming from an ample line bundle on the projective variety.[3]

4.3. Semi-simplicity

4.3.a. A review on completely reducible representations. We review here some classical results concerning the theory of finite-dimensional linear representations. Let Π be a group and let ρ be a linear representation of Π on a finite-dimensional

[3] See [**KK87**, §6.4] for the comparison between both Lefschetz operators.

$\boldsymbol{k}$-vector space V. In other words, ρ is a group homomorphism $\Pi \to \mathrm{GL}(V)$. We will say that V is a Π-module (it would be more correct to introduce the associative algebra $\boldsymbol{k}[\Pi]$ of the group Π, consisting of $\boldsymbol{k}$-linear combinations of the elements of Π, and to speak of a left $\boldsymbol{k}[\Pi]$-module). The subspaces of V stable by $\rho(\Pi)$ correspond thus to the sub-Π-modules of V.

We say that a Π-module V is *irreducible* if it does not admit any nontrivial sub-Π-module. Then, any homomorphism between two irreducible Π-modules is either zero, or an isomorphism (Schur's lemma). If k is algebraically closed, any automorphism of an irreducible Π-module is a nonzero multiple of the identity (consider an eigenspace of the automorphism).

4.3.1. Proposition. *Given a Π-module V, the following properties are equivalent:*

(1) *The Π-module V is semi-simple, i.e., every sub-Π-module has a supplementary sub-Π-module.*

(2) *The Π-module V is completely reducible, i.e., V has a decomposition (in general non unique) into the direct sum of irreducible sub-Π-modules.*

(3) *The Π-module V is generated by its irreducible sub-Π-modules.*

Proof. The only non-obvious point is (3) $\Rightarrow$ (1). Let then W be a sub-Π-module of V. We will show the result by induction on $\operatorname{codim} W$, this being clear for $\operatorname{codim} W = 0$. If $\operatorname{codim} W \geqslant 1$, there exists by assumption a nontrivial irreducible sub-Π-module $V_1 \subset V$ not contained in W. Since V_1 is irreducible, we have $W \cap V_1 = \{0\}$, so $W_1 := W \oplus V_1$ is a sub-Π-module of V to which one can apply the inductive assumption. If W_1' is a supplementary Π-module of W_1, then $W' = W_1' \oplus V_1$ is a supplementary Π-module of W. $\square$

It follows then from Schur's lemma that a completely reducible Π-module has a *unique* decomposition as the direct sum

$$V = \bigoplus_i V_i = \bigoplus_i (V_i^o \otimes E_i),$$

in which the *isotypic components* V_i are sub-Π-modules of the form $V_i^o \otimes E_i$, where V_i^o is an irreducible Π-module, V_i^o is not isomorphic to V_j^o for $i \neq j$, and E_i is a trivial Π-module, i.e., on which Π acts by the identity.

One also notes that if W is a sub-Π-module of a completely reducible Π-module V, then W is completely reducible and its isotypical decomposition is

$$W = \bigoplus_i (W \cap V_i),$$

in which $W \cap V_i = V_i^o \otimes F_i$ for some subspace F_i of E_i. A Π-module supplementary to W can be obtained by choosing for every i a $\boldsymbol{k}$-vector space supplementary to F_i in E_i.

4.3.2. Remarks. The previous properties have easy consequences.

(1) A $\boldsymbol{k}$-vector space V is a semi-simple Π-module *if and only if* the associated complex space $V_{\mathbb{C}} = \mathbb{C} \otimes V$ is a semi-simple Π-module (for the complexified representation).

Indeed, let us first recall that the group $\mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$ acts on $V_{\mathbb{C}}$: if $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_n)$ is some $\boldsymbol{k}$-basis of V then, for $a_1, \dots, a_n \in \mathbb{C}$ and $\sigma \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$, one sets $\sigma\big(\sum_i a_i\varepsilon_i\big) = \sum_i \sigma(a_i)\varepsilon_i$. A subspace $W_{\mathbb{C}}$ of $V_{\mathbb{C}}$ is "defined over $\boldsymbol{k}$", i.e., of the form $\mathbb{C} \otimes W$ for some sub-space W of V, if and only if it is stable by any automorphism $\sigma \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$: indeed, if $d = \dim_{\mathbb{C}} W_{\mathbb{C}}$, one can find, up to renumbering the basis $\boldsymbol{\varepsilon}$, a basis $e_1, \dots, e_d$ of $W_{\mathbb{C}}$ such that

$$
\begin{aligned}
e_1 &= \varepsilon_1 + a_{1,2}\varepsilon_2 + \cdots + a_{1,d}\varepsilon_d + \cdots + a_{1,n}\varepsilon_n \\
e_2 &= \qquad\qquad \varepsilon_2 + \cdots + a_{1,d}\varepsilon_d + \cdots + a_{2,n}\varepsilon_n \\
&\ \vdots \\
e_d &= \qquad\qquad\qquad\qquad\quad \varepsilon_d + \cdots + a_{d,n}\varepsilon_n,
\end{aligned}
$$

with $a_{i,j} \in \mathbb{C}$; one then shows by descending induction on $i \in \{d, \dots, 1\}$ that, if $W_{\mathbb{C}}$ is stable by $\mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$, then $a_{i,j}$ are invariant by any automorphism of $\mathbb{C}$ over $\boldsymbol{k}$, i.e., belong to $\boldsymbol{k}$ since $\mathbb{C}$ is separable over $\boldsymbol{k}$.

Let us now prove the assertion. Let us first assume that V is irreducible and let us consider the subspace $W_{\mathbb{C}}$ of $V_{\mathbb{C}}$ generated by the sub-Π-modules of minimal dimension (hence irreducible). Since the representation of Π is defined over $\boldsymbol{k}$, if $E_{\mathbb{C}}$ is a Π-module, so is $\sigma(E_{\mathbb{C}})$ for every $\sigma \in \mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$; therefore the space $W_{\mathbb{C}}$ is invariant by $\mathrm{Aut}_{\boldsymbol{k}}(\mathbb{C})$, in other words takes the form $\mathbb{C} \otimes_{\boldsymbol{k}} W$ for some subspace W of V. It is clear that W is a sub-Π-module of V, hence $W = V$. According to 4.3.1(3), $V_{\mathbb{C}}$ is semi-simple.

Conversely, let us assume that $V_{\mathbb{C}}$ is semi-simple. Let us choose a $\boldsymbol{k}$-linear form $\ell : \mathbb{C} \to \boldsymbol{k}$ such that $\ell(1) = 1$. It defines a $\boldsymbol{k}$-linear map $L : V_{\mathbb{C}} \to V$ which is Π-invariant and which induces the identity on V. Let W be a sub-Π-module of V. We have a Π-invariant projection $V_{\mathbb{C}} \to W_{\mathbb{C}}$, hence a composed projection p which is Π-invariant:

$$
\begin{tikzcd}
V_{\mathbb{C}} \arrow[r] & W_{\mathbb{C}} \arrow[r, "L"] & W \\
V \arrow[u, hook] \arrow[urr, "p"'] & &
\end{tikzcd}
$$

from which one obtains a Π-module supplementary to W in V.

(2) If $\Pi'' \to \Pi$ is a surjective group-homomorphism and ρ'' is the composed representation, then V is a semi-simple Π-module if and only if it is a semi-simple Π''-module. Indeed, the Π-module structure only depends on the image $\rho(\Pi) \subset \mathrm{GL}(V)$.

(3) Let $\Pi' \lhd \Pi$ be a *normal* subgroup, and let V be a Π-module. Then, if V is semi-simple as a Π-module, it so as a Π'-module. Indeed, if V' is an irreducible sub-Π'-module of V, then $\rho(\pi)V'$ remains so for every $\pi \in \Pi$. If V is Π-irreducible and if V' is a nonzero irreducible sub-Π'-module, the sub-Π'-module generated by the $\rho(\pi)V'$ is a Π-module, hence coincides with V. As a consequence, V is generated by its irreducible sub-Π'-modules, hence is Π'-semi-simple, according to 4.3.1(3).

(4) A real representation $\Pi \to \mathrm{Aut}(V_{\mathbb{R}})$ is simple if and only if the associated complexified representation $\Pi \to \mathrm{Aut}(V_{\mathbb{C}})$ has at most two simple components. [*Hint*: Any simple component of the complexified representation can be summed with its conjugate to produce a sub-representation of the real representation.]

4.3.b. The semi-simplicity theorem. Let X be a smooth projective variety. Let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}, D, \mathrm{S})$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on X (see Definition 4.1.4), and let $\underline{\mathcal{H}} = \operatorname{Ker} D$ be the associated complex local system.

4.3.3. Theorem. *Under these assumptions, the complex local system $\underline{\mathcal{H}}$ is semi-simple.*

Let us already note that the result is easy for unitary local systems (underlying thus polarized variations of type $(0,0)$, as in Section 4.2.a). The general case will use the objects introduced in Section 4.2.b, and will not be specific to polarized variations of Hodge structure. The proof of the semi-simplicity theorem will apply to these more general objects called harmonic bundle (see Section 4.2.b). Moreover, we can relax the property that the smooth variety is projective, and only assume that it is a compact Kähler manifold, since we will only use the Kähler identities.

4.3.4. Remark. If $\underline{\mathcal{H}}$ is obtained from a local system $\underline{\mathcal{H}}_{\mathbb{Q}}$ defined over $\mathbb{Q}$, then $\underline{\mathcal{H}}_{\mathbb{Q}}$ is also semi-simple as such, according to Remark 4.3.2(1).

Let $\mathcal{H}$ be a C^∞-bundle with metric h. The group of C^∞ automorphisms g of $\mathcal{H}$ acts on a given connection D by the formula ${}_gD := g \circ D \circ g^{-1} = D - D(g) \circ g^{-1}$, where we have extended the action of D in a natural way on the bundle $\mathcal{E}nd(\mathcal{H})$. If D is flat, then so is ${}_gD$. We then set ${}_gD = {}_gD_{\mathrm{h}} + {}_g\theta$. Let us also set $\widehat{D} = D_{\mathrm{h}} - \theta$ (see Lemma 4.2.2).

4.3.5. Lemma. *We have ${}_g\theta = \theta - \frac{1}{2}\big(D(g)g^{-1} + g^{*-1}\widehat{D}(g^*)\big)$, where g^* is the* h*-adjoint of g.*

Proof. We have

$${}_gD = D_{\mathrm{h}} + \theta - \big(D_{\mathrm{h}}(g)g^{-1} + [\theta, g]g^{-1}\big) = D_{\mathrm{h}} - D_{\mathrm{h}}(g)g^{-1} + g^{-1}\theta g.$$

It follows that ${}_g\theta$ is the self-adjoint part of $-D_{\mathrm{h}}(g)g^{-1} + g^{-1}\theta g$, that is, taking into account that the adjoint of $D_{\mathrm{h}}(g)$ is $D_{\mathrm{h}}(g^*)$ (by working in a local h-orthonormal basis),

$$(4.3.6) \qquad {}_g\theta = \frac{1}{2}\big(-(D_{\mathrm{h}}(g)g^{-1} + g^{*-1}D_{\mathrm{h}}(g^*)) + g^{-1}\theta g + g^*\theta g^{*-1}\big).$$

The lemma follows from a straightforward computation. □

If we fix a metric on X, we deduce with h a metric on $\mathcal{E}^1_X \otimes \mathcal{H}$ and then a metric $\|\cdot\|$ on $\mathcal{H}om(\mathcal{H}, \mathcal{E}^1_X \otimes \mathcal{H})$ with associated scalar product $(\cdot\,,\cdot)$. We then denote by $\langle\cdot\,,\cdot\rangle$ the integrated product using the volume form on X:

$$\langle\cdot\,,\cdot\rangle = \int_X (\cdot\,,\cdot)\,\mathrm{vol}\,.$$

4.3.7. Definition. The *energy* of $g \in \mathrm{Aut}(\mathcal{H})$ with respect to $(\mathcal{H}, D, \mathrm{h})$ is defined as

$$\mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g) := \|{}_g\theta\|^2 = \langle {}_g\theta, {}_g\theta\rangle.$$

Let $\xi \in \mathrm{End}(\mathcal{H})$ and let $\xi = \xi^+ + \xi^-$ be its decomposition into its self-adjoint part $\xi^+ = \frac{1}{2}(\xi + \xi^*)$ and its skew-adjoint part $\xi^- = \frac{1}{2}(\xi - \xi^*)$.

4.3.8. Proposition. *For t varying in $\mathbb{R}$, we have*

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(e^{t\xi})\big|_{t=0} = 2\langle D_{\mathrm{h}}\xi^+, \theta\rangle.$$

Proof. We have $D(e^{t\xi})e^{-t\xi} = tD\xi \bmod t^2$ and $e^{-t\xi^*}\widehat{D}(e^{t\xi^*}) = t\widehat{D}\xi^* \bmod t^2$. From Lemma 4.3.5 we deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(e^{t\xi})\big|_{t=0} &= -\langle D\xi + \widehat{D}\xi^*, \theta\rangle - \langle \theta, D\xi + \widehat{D}\xi^*\rangle \\ &= -\langle D_{\mathrm{h}}\xi^+ + [\theta, \xi^-], \theta\rangle - \langle \theta, D_{\mathrm{h}}\xi^+ + [\theta, \xi^-]\rangle \\ &= -2\,\mathrm{Re}\langle D_{\mathrm{h}}\xi^+, \theta\rangle = -2\langle D_{\mathrm{h}}\xi^+, \theta\rangle,\end{aligned}$$

since $\langle \theta\xi^-, \theta\rangle = -\langle \theta, \theta\xi^-\rangle$, $\langle \xi^-\theta, \theta\rangle = -\langle \theta, \xi^-\theta\rangle$, and both θ and $D_{\mathrm{h}}\xi^+$ are self-adjoint. □

The property of being semi-simple or not for $(\mathcal{H}, D)$ is seen on the energy functional.

4.3.9. Proposition. *Let $(\mathcal{H}, D)$ be a flat bundle. Assume that there exists a metric* h *such that the energy functional $g \mapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g)$ has a critical point at $g = \mathrm{Id}$. Then $(\mathcal{H}, D)$ is semi-simple.*

Proof. Let us argue by contraposition and let us assume that $(\mathcal{H}, D)$ is not semi-simple. Let h be any metric on $\mathcal{H}$. We will prove that Id is not a critical point for $g \mapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g)$. It is enough to prove that there exists $\xi \in \mathrm{End}(\mathcal{H})$ such that the function

$$f : \mathbb{R} \longrightarrow \mathbb{R}, \quad t \longmapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(e^{t\xi})$$

has no critical point at $t = 0$. By assumption, there exists a sub-bundle $\mathcal{H}_1$ of $\mathcal{H}$ stable by D such that its orthogonal $\mathcal{H}_2$ is not stable by D. Set $n_i = \mathrm{rk}\,\mathcal{H}_i$ $(i = 1, 2)$. With respect to this decomposition we have

$$D = \begin{pmatrix} D_1 & 2\eta \\ 0 & D_2 \end{pmatrix},$$

with $\eta : \mathcal{H}_2 \to \mathcal{E}^1_X \otimes \mathcal{H}_1$ nonzero. Set $\xi = n_2\,\mathrm{Id}_{\mathcal{H}_1} - n_1\,\mathrm{Id}_{\mathcal{H}_2}$ and $g = e^{t\xi}$ $(t \in \mathbb{R})$. We have

$$\begin{aligned} D(e^{t\xi})e^{-t\xi} &= \left[\begin{pmatrix} 0 & 2\eta \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} e^{n_2 t} & 0 \\ 0 & e^{-n_1 t} \end{pmatrix}\right] \cdot \begin{pmatrix} e^{-n_2 t} & 0 \\ 0 & e^{n_1 t} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2\eta \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} e^{-n_2 t} & 0 \\ 0 & e^{n_1 t} \end{pmatrix}\begin{pmatrix} 0 & 2\eta \\ 0 & 0 \end{pmatrix}\begin{pmatrix} e^{n_2 t} & 0 \\ 0 & e^{-n_1 t} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 2(1 - e^{-(n_1+n_2)t})\eta \\ 0 & 0 \end{pmatrix},\end{aligned}$$

so

$$ {}_gD = D - D(e^{t\xi})e^{-t\xi} = \begin{pmatrix} D_1 & 2e^{-(n_1+n_2)t}\eta \\ 0 & D_2 \end{pmatrix}, $$

and

$$ {}_g\theta = \begin{pmatrix} \theta_1 & e^{-(n_1+n_2)t}\eta \\ e^{-(n_1+n_2)t}\eta^* & \theta_2 \end{pmatrix}. $$

It follows that

$$ f(t) = c_0 + c_1 e^{-(n_1+n_2)t}, \quad c_0 \geqslant 0,\, c_1 > 0, $$

and it is clear that $f'(0) \neq 0$. □

Proof of the semi-simplicity theorem. In view of Propositions 4.2.11 and 4.3.9, the semi-simplicity theorem 4.3.3 is a consequence of the following.

4.3.10. Proposition. *Assume that X is compact Kähler and that $(\mathcal{H}, D, \mathrm{h})$ is a harmonic bundle. Then $g \mapsto \mathrm{E}_{(\mathcal{H},D,\mathrm{h})}(g)$ has a critical point at $g = \mathrm{Id}$.*

Proof. According to Proposition 4.3.8, it is enough to show that $D_{\mathrm{h}}^\star(\theta) = 0$, where $D_{\mathrm{h}}^\star$ denotes the formal adjoint of D_{h}. Setting $D_{\mathrm{h}}^{\mathrm{c}} := D_{\mathrm{h}}'' - D_{\mathrm{h}}'$, the Kähler identities for a Hermitian vector bundle (see Section 4.2.9) imply that $D_{\mathrm{h}}^\star$ is a multiple of $[\Lambda, D_{\mathrm{h}}^{\mathrm{c}}]$. Since θ is a matrix of 1-forms and Λ is an operator of type $(-1,-1)$, we have $\Lambda(\theta) = 0$. On the other hand, by the properties after Definition 4.2.5, we have $D_{\mathrm{h}}^{\mathrm{c}}(\theta) = 0$. □

4.3.c. Structure of polarized variations of $\mathbb{C}$-Hodge structure

Let X be a complex manifold. We will say that two polarized variations of $\mathbb{C}$-Hodge structures are *equivalent* if one is obtained from the other one by a twist (k, ℓ) (see Exercise 2.10) and by multiplying the polarization form by a positive constant.

4.3.11. Lemma. *There exists at most one equivalence class of polarized variations of $\mathbb{C}$-Hodge structure on a simple (i.e., irreducible) $\mathbb{C}$-local system $\underline{\mathcal{H}}$ on a compact complex manifold X.*

4.3.12. Remark. A criterion for the *existence* of a polarized variation of $\mathbb{C}$-Hodge structure on a simple $\mathbb{C}$-local system $\underline{\mathcal{H}}$ is given in [**Sim92**, §4] in terms of rigidity.

Proof. If we are given two polarizable variations of $\mathbb{C}$-Hodge structure on an irreducible local system $\underline{\mathcal{H}}$, we deduce such a polarizable variation on $\mathcal{E}nd(\underline{\mathcal{H}})$ (Remark 5.4.5), and the dimension 1 vector space $\mathrm{End}(\underline{\mathcal{H}}) := H^0(X, \mathcal{E}nd(\underline{\mathcal{H}}))$ is equipped with a $\mathbb{C}$-Hodge structure of some type (k, ℓ) by the Hodge-Deligne theorem 4.2.16. The identity morphism $\mathrm{Id}_{\mathcal{H}} \in \mathrm{End}(\underline{\mathcal{H}})$ defines thus a morphism of type (k, ℓ) between the two variations. Therefore, the first one is obtained from the second one by a twist (k, ℓ). It remains to check that, on a given polarizable variation of $\mathbb{C}$-Hodge structure on an irreducible local system $\underline{\mathcal{H}}$, there exists exactly one polarization up to a positive multiplicative constant. Note that such a polarization is an isomorphism $\underline{\mathcal{H}} \xrightarrow{\sim} \underline{\mathcal{H}}^*$, so one polarization is obtained from another one by multiplying by a nonzero constant. This constant must be positive, by the positivity property of the associated Hermitian form. □

Let X be a compact complex manifold and let $H = (\mathcal{H}, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H}, D, \mathcal{S})$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on X. If the associated local system $\underline{\mathcal{H}}$ is semi-simple, which is the case when X is Kähler according to Theorem 4.3.3, it decomposes as $\underline{\mathcal{H}} = \bigoplus_{\alpha\in A} \underline{\mathcal{H}}_\alpha^{n_\alpha}$, where $\underline{\mathcal{H}}_\alpha$ are irreducible and pairwise non isomorphic, and $\underline{\mathcal{H}}_\alpha^{n_\alpha}$ means the direct sum of n_α copies of $\underline{\mathcal{H}}_\alpha$. Similarly, $(\mathcal{H}, D) = \bigoplus_{\alpha\in A}(\mathcal{H}_\alpha, D)^{n_\alpha}$, and the polarization $\mathcal{S}$, being D-horizontal, decomposes with respect to $\alpha \in A$ as $\mathcal{S} = \bigoplus \mathcal{S}_{\alpha,n_\alpha}$. Let us set $\mathcal{H}_\alpha^o := \mathbb{C}^{n_\alpha}$ and let us write $\underline{\mathcal{H}} = \bigoplus_{\alpha\in A} \mathcal{H}_\alpha^o \otimes \underline{\mathcal{H}}_\alpha$. If we are given a basis $\mathcal{S}_\alpha$ of the dimension 1 vector space $\mathrm{Hom}(\underline{\mathcal{H}}_\alpha, \underline{\mathcal{H}}_\alpha^*)$, there exists a unique morphism $\mathrm{S}_\alpha^o \in \mathrm{Hom}(\mathcal{H}_\alpha^o, \mathcal{H}_\alpha^{o*})$ such that $\mathcal{S}_{\alpha,n_\alpha} = \mathrm{S}_\alpha^o \otimes \mathcal{S}_\alpha$.

4.3.13. Theorem. *Under these conditions, the following holds:*

(1) *For every $\alpha \in A$, there exists a unique equivalence class of polarized variation of $\mathbb{C}$-Hodge structure of weight w on $\underline{\mathcal{H}}_\alpha$.*

(2) *For every $\alpha \in A$, let us fix a representative $H_\alpha = (\mathcal{H}_\alpha, F'^{\bullet}\mathcal{H}_\alpha, F''^{\bullet}\mathcal{H}_\alpha, D, \mathcal{S}_\alpha)$ of such an equivalence class. There exists then a polarized $\mathbb{C}$-Hodge structure*

$$H_\alpha^o = (\mathcal{H}_\alpha^o, F'^{\bullet}\mathcal{H}_\alpha^o, F''^{\bullet}\mathcal{H}_\alpha^o, \mathrm{S}_\alpha^o)$$

of weight 0 *with* $\dim \mathcal{H}_\alpha^o = n_\alpha$ *such that*

$$H = \bigoplus_{\alpha\in A} (H_\alpha^o \otimes_{\mathbb{C}} H_\alpha). \tag{4.3.13 $*$}$$

4.3.14. Remark. Let us emphasize the following statement: given a polarized variation of Hodge structure on a compact complex manifold X such that the underlying local system is semi-simple (which is the case if X is Kähler), then each irreducible component of this local system underlies a polarized variation of Hodge structure (which is essentially unique). The proposition also explains how to reconstruct the original variation from its irreducible components.

***Proof of Theorem* 4.3.13.**

(1) The uniqueness statement is given by Lemma 4.3.11. In order to prove the existence in 4.3.13(1), it is enough to exhibit for every $\alpha \in A$ a sub-variation of Hodge structure of H of weight w with underlying local system $\underline{\mathcal{H}}_\alpha$. The polarization $\mathcal{S}$ will then induce a polarization $\mathcal{S}_\alpha$, according to Exercise 4.2(1). For that purpose, it is enough to exhibit $\underline{\mathcal{H}}_\alpha$ as the image of an endomorphism $\underline{\mathcal{H}} \to \underline{\mathcal{H}}$ which is compatible with the Hodge structures: by abelianity (Proposition 4.1.10), this image is an object of $\mathsf{VHS}(X, \mathbb{C}, w)$. Let us therefore analyze $\mathrm{End}(\underline{\mathcal{H}}) = H^0(X, \mathcal{E}nd(\underline{\mathcal{H}}))$.

If we set $\mathcal{H}_\alpha^o = \mathbb{C}^{n_\alpha}$, so that $\underline{\mathcal{H}} = \bigoplus_\alpha (\mathcal{H}_\alpha^o \otimes_{\mathbb{C}} \underline{\mathcal{H}}_\alpha)$, we have an algebra isomorphism $\mathrm{End}(\underline{\mathcal{H}}) \simeq \prod_\alpha \mathrm{End}(\mathcal{H}_\alpha^o)$ (where $x_\alpha x_\beta = 0$ if $x_\alpha \in \mathcal{H}_\alpha^o$, $x_\beta \in \mathcal{H}_\beta^o$ and $\alpha \neq \beta$). We know that the local system $\mathcal{E}nd(\underline{\mathcal{H}})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure of weight 0. Therefore, $\mathrm{End}(\underline{\mathcal{H}}) = \Gamma(X, \mathcal{E}nd(\underline{\mathcal{H}}))$ underlies a $\mathbb{C}$-Hodge structure of weight 0 by the Hodge-Deligne theorem 4.2.16. It is then enough to show that *each $\mathcal{H}_\alpha^o$ underlies a $\mathbb{C}$-Hodge structure H_α^o of weight* 0 *such that the equality* $\mathrm{End}(\underline{\mathcal{H}}) = \prod_\alpha \mathrm{End}(\mathcal{H}_\alpha^o)$ *is compatible with the Hodge structures on both terms.* Indeed, choose then any rank 1 endomorphism p_α of some nonzero vector space $\mathcal{H}_\alpha^{o,(k,-k)}$. Extend it as a rank 1 endomorphism of $\mathcal{H}_\alpha^o$ of type $(0,0)$ by mapping every other summand $\mathcal{H}_\alpha^{o,(\ell,-\ell)}$ to zero, and extend it similarly as a rank 1 endomorphism of $\bigoplus_\beta \mathcal{H}_\beta^o$ of type $(0,0)$. One obtains thus a rank 1 endomorphism in $\mathrm{End}(\underline{\mathcal{H}})^{0,0}$. With respect to this identification, its image is $(\mathrm{Im}\, p_\alpha)\otimes_{\mathbb{C}} \underline{\mathcal{H}}_\alpha \simeq \underline{\mathcal{H}}_\alpha$, as wanted.

Let us prove the assertion, which reduces to proving the existence of a grading of each $\mathcal{H}_\alpha^o$ giving rise to the Hodge grading of $\mathrm{End}(\underline{\mathcal{H}})$. By the product formula above, the $\mathbb{C}$-algebra $\mathrm{End}(\underline{\mathcal{H}})$ is semi-simple, with center $Z=\prod_\alpha \mathbb{C}\cdot \mathrm{Id}_{\mathcal{H}_\alpha^o}$. An algebra automorphism φ of $\mathrm{End}(\underline{\mathcal{H}})$ induces an automorphism of the ring Z, whose matrix in the basis above only consists of zeros and ones. By the Skolem-Noether theorem (see e.g. [**Bou12**, §14, N° 5, Th. 4]), algebra automorphisms for which the corresponding matrix is the identity are interior automorphisms, that is, products of interior automorphisms of each $\mathrm{End}(\mathcal{H}_\alpha^o)$. Any algebra automorphism can be composed with an automorphism with matrix having block entries Id or 0 in order that the matrix on Z is the identity. As a consequence, the identity component of the group of algebra automorphisms $\mathrm{Aut}^{\mathrm{alg}}(\mathrm{End}(\underline{\mathcal{H}}))$ is identified with $\prod_{\alpha\in A}(\mathrm{Aut}(\mathcal{H}_\alpha^o)/\mathbb{C}^*\,\mathrm{Id}_\alpha)$.

As in Section 2.5.8, the $\mathbb{C}$-Hodge structure of weight 0 on $\mathrm{End}(\underline{\mathcal{H}})$ defines a continuous representation $\rho:\mathbb{S}^1\to \mathrm{Aut}(\mathrm{End}(\underline{\mathcal{H}}))$, such that $\rho(\lambda)=\lambda^p$ on $\mathrm{End}(\underline{\mathcal{H}})^{p,-p}$. Since the grading is compatible with the algebra structure, the continuous representation ρ takes values in the group of algebra automorphisms $\mathrm{Aut}^{\mathrm{alg}}(\mathrm{End}(\underline{\mathcal{H}}))$. Since $\rho(1)=\mathrm{Id}$, it takes values in the identity component of $\mathrm{Aut}^{\mathrm{alg}}(\mathrm{End}(\underline{\mathcal{H}}))$, i.e., in $\prod_{\alpha\in A}(\mathrm{Aut}(\mathcal{H}_\alpha^o)/\mathbb{C}^*\,\mathrm{Id}_\alpha)$. By the argument given in Section 2.5.8, it defines a grading, up to a shift, on each $\mathcal{H}_\alpha^o$, as wanted.

(2) Let us now equip $\mathcal{H}_\alpha^o$ with a polarized $\mathbb{C}$-Hodge structure of weight 0 so that (4.3.13 $*$) holds. We already have obtained a grading, i.e., a $\mathbb{C}$-Hodge structure of weight 0. In order to obtain a polarization of this $\mathbb{C}$-Hodge structure satisfying (4.3.13 $*$), we note the equality $\mathcal{H}_\alpha^o=\mathrm{Hom}(\underline{\mathcal{H}}_\alpha,\underline{\mathcal{H}})$, and since $\mathcal{H}om(\underline{\mathcal{H}}_\alpha,\underline{\mathcal{H}})$ underlies a polarized variation of Hodge structure of weight 0 according to 4.3.13(1), $\mathcal{H}_\alpha^o$ comes equipped with a polarized Hodge structure of weight 0 by the Hodge-Deligne theorem 4.2.16. By definition, the natural morphism $\mathcal{H}_\alpha^o\otimes\underline{\mathcal{H}}_\alpha\to\underline{\mathcal{H}}$ underlies a morphism of polarized variations of Hodge structure. □

4.4. Exercises

Exercise 4.1.

(1) Show that the category $\mathsf{VHS}(X,\mathbb{C},w)$ is abelian. [*Hint*: Use that, according to Definition 4.1.5, any morphism is bigraded with respect to the Hodge decomposition, hence so are its kernel, image and cokernel.]

(2) Define the tensor product

$$\mathsf{VHS}(X,\mathbb{C},w_1)\otimes\mathsf{VHS}(X,\mathbb{C},w_2)\longrightarrow\mathsf{VHS}(X,\mathbb{C},w_1+w_2)$$

and the external Hom

$$\mathsf{VHS}(X,\mathbb{C},w_1)\otimes\mathsf{VHS}(X,\mathbb{C},w_2)\longrightarrow\mathsf{VHS}(X,\mathbb{C},w_2-w_1).$$

(3) Show that these operations preserve the subcategories of polarizable objects.

***Exercise 4.2* (Abelianity and semi-simplicity).** Let $\big(H, \mathrm{S}\big)$ be a polarized variation of Hodge structure of weight w on X.

(1) Show that any subobject of H in $\mathsf{VHS}(X, \mathbb{C}, w)$ is a direct summand of the given variation, and that the polarization S induces a polarization. [*Hint*: Use Exercise 2.12.]

(2) Conclude that the full subcategory $\mathsf{pVHS}(X, \mathbb{C}, w)$ of polarizable variations of Hodge structure is abelian and semi-simple (i.e., any object decomposes as the direct sum of its irreducible components). [*Hint*: Use the C^∞ interpretation of Definition 4.1.5.]

Exercise 4.3. Let $(H, \mathcal{S})$ be a polarized variation of Hodge structure of weight w on X (see Definition 4.1.4). Let h be the Hermitian metric deduced from $\mathcal{S}$ and let $D = D' + D''$ be the flat C^∞ connection. Let $\mathcal{H} = \bigoplus_{p+q=w} \mathcal{H}^{p,q}$ be the Hodge decomposition (which is h-orthogonal by construction). Show the following properties.

(1) In the Griffiths transversality relations (4.1.5 $*$), the composition of D' (resp. D'') with the projection on the first summand defines a $(1,0)$ (resp. $(0,1)$)-connection D'_{h} (resp. D''_{h}), and that the projection to the second summand defines a C^∞-linear morphism θ' (resp. θ'').

(2) The connection $D_{\mathrm{h}} := D'_{\mathrm{h}} + D''_{\mathrm{h}}$ is compatible with the metric h, but is possibly not flat.

(3) The morphism θ'' is the h-adjoint of θ'.

(4) The connection $\overline{\mathcal{D}} := D''_{\mathrm{h}} + \theta'$ has square zero, as well as the connection $\mathcal{D} := D'_{\mathrm{h}} + \theta''$.

For each $p \in \mathbb{Z}$, set $\theta'_p : \mathcal{H}^{p,w-p} \to \mathcal{E}_X^{(1,0)} \otimes \mathcal{H}^{p-1,w-p+1}$ be the component of θ' on $\mathcal{H}^{p,w-p}$ and set θ''_p similarly.

(5) Show that θ''_p is the h-adjoint of θ'_{p-1}.

(6) Show that the Hermitian holomorphic bundle $(F^p\mathcal{H}, D'')$ has Chern connection equal to $(D'_{\mathrm{h}} + \sum_{p' \geqslant p+1} \theta'_{p'}) + (D''_{\mathrm{h}} + \sum_{p' \geqslant p} \theta''_{p'})$. [*Hint*: Recall that each $\mathcal{H}^{p,w-p}$ is stable by D_{h} and write the holomorphic structure D'' on $F^p\mathcal{H}$ as $D_{\mathrm{h}} + \sum_{p' \geqslant p} \theta''_{p'}$.]

Exercise 4.4. Let $(\mathcal{H}, D)$ be a flat bundle and let h be a Hermitian metric on $\mathcal{H}$.

(1) Show that there exist a unique $(1,0)$-connection $\widehat{D}'$ and a unique $(0,1)$-connection $\widehat{D}''$ such that $D' + \widehat{D}''$ and $\widehat{D}' + D''$ preserve the metric h.

(2) Show that $\widehat{D}' = D'_{\mathrm{h}} - \theta'$ and $\widehat{D}'' = D''_{\mathrm{h}} - \theta''$.

(3) Conclude that $\widehat{D}' + D''$ is the Chern connection of the Hermitian holomorphic bundle $(\mathcal{H}, D'', \mathrm{h})$.

(4) We set $D^{\mathrm{c}} := \widehat{D}'' - \widehat{D}'$. Show that $D^{\mathrm{c}} = \overline{\mathcal{D}} - \mathcal{D}$ and $\frac{1}{2}(D + D^{\mathrm{c}}) = D''_{\mathrm{h}} + \theta' = \overline{\mathcal{D}}$.

(5) Let $R_{\mathrm{h}} = (\widehat{D}' + D'')^2$ be the curvature of the Hermitian holomorphic bundle $(\mathcal{H}, \mathrm{h}, D'')$. Show that

$$R_{\mathrm{h}} = -2(\overline{\mathcal{D}}^2 + \theta' \wedge \theta'' + \theta'' \wedge \theta').$$

(6) Show that, *if* $(\mathcal{H}, D, \mathrm{h})$ *is harmonic*, then $\operatorname{tr} R_{\mathrm{h}} = 0$. [*Hint*: Use that, in any case, $\operatorname{tr}(\theta' \wedge \theta'' + \theta'' \wedge \theta') = 0$.]

(7) In the setting of Exercise 4.3, show that, for each $p \in \mathbb{Z}$, the curvature R^p_{h} of the Hermitian holomorphic bundle $(F^p\mathcal{H}, D'')$ satisfies $\|R^p_{\mathrm{h}}\|_{\mathrm{h}} \leqslant C_p\|\theta'\|^2_{\mathrm{h}}$ for a suitable positive constant C_p. Here, the curvature R^p_{h} is regarded as a section of $\mathcal{E}nd(F^p\mathcal{H}) \otimes \mathcal{E}^2_X$ and its norm is computed with respect to the metric on $\mathcal{E}nd(F^p\mathcal{H})$ induced by h and any fixed norm on differential forms. A similar definition holds for $\|\theta'\|_{\mathrm{h}}$.

Exercise 4.5 **(Norm of horizontal sections).** Let $(\mathcal{H}, D, \mathrm{h})$ be a harmonic flat bundle with Higgs fields θ', θ''. Let v be a horizontal section of $(\mathcal{H}, D)$ (equivalently, a horizontal section of the associated holomorphic flat bundle (V, ∇)). Show that the h-norm of v satisfies

$$\mathrm{d}'\|v\|^2_{\mathrm{h}} = -2\mathrm{h}(\theta' v, \overline{v}), \quad \mathrm{d}''\|v\|^2_{\mathrm{h}} = -2\mathrm{h}(\theta'' v, \overline{v}).$$

[*Hint*: Use that $D'_{\mathrm{h}}v = -\theta' v$ and $D''_{\mathrm{h}}v = -\theta'' v$ by horizontality.]

Exercise 4.6 **(Rescaling the Higgs field).**

(1) Let $(\mathcal{H}, D, \mathrm{h})$ be a harmonic flat bundle with Higgs fields θ', θ''. Show that, for any nonzero complex number t, there exists a harmonic flat bundle $(\mathcal{H}, D_t, \mathrm{h})$ whose Higgs fields are $(t\theta', \overline{t}\theta'')$. [*Hint*: Set $D'_t = D'_{\mathrm{h}} + t\theta'$, $D''_t = D''_{\mathrm{h}} + \overline{t}\theta''$ and show that $D_t = D'_t + D''_t$ is flat.]

(2) In case $(\mathcal{H}, D, \mathrm{h})$ is attached to a polarized variation of Hodge structure of weight w as in Exercise 4.3, show that $(\mathcal{H}, D_t, \mathrm{h}) \simeq (\mathcal{H}, D, \mathrm{h})$ if $|t| = 1$. [*Hint*: Compare with Section 2.5.8.]

Exercise 4.7.

(1) Given C^∞ bundles with flat connection and Hermitian metric $(\mathcal{H}_1, D_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, D_2, \mathrm{h}_2)$, equip the C^∞ bundles $\mathcal{H}_1 \otimes \mathcal{H}_2$ and $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ with a natural flat connection D and a natural Hermitian metric h, and identify connection and metric on $\mathcal{H}_1^* \otimes \mathcal{H}_2$ and $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$. [*Hint*: Use Exercise 2.2.]

(2) Show that, for a section φ of $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$, we have $D(\varphi) = D_2 \circ \varphi - \varphi \circ D_1$.

(3) Prove that if $(\mathcal{H}_1, D_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, D_2, \mathrm{h}_2)$ are harmonic, then $(\mathcal{H}_1 \otimes \mathcal{H}_2, D, \mathrm{h})$ and $(\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2), D, \mathrm{h})$ are also harmonic.

(4) Show that θ on $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ is given by the following formula. For an open set U and a local section $\varphi : \mathcal{H}_{1|U} \to \mathcal{H}_{2|U}$ of $\mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ on U,

$$\forall\, V \subset U,\ \forall\, u_1 \in \Gamma(V, \mathcal{H}_1), \quad [\theta(\varphi)](u_1) = \theta_2(\varphi(u_1)) - \varphi(\theta_1(u_1)) \in \Gamma(V, \mathcal{E}_X \otimes \mathcal{H}_2).$$

Exercise 4.8 **(Formal adjoint).** Let $(\mathcal{H}, \mathrm{h})$ be a Hermitian vector bundle on a complex manifold X equipped with a Hermitian metric on its tangent bundle, which induces a Hermitian metric on the sheaves $\mathcal{E}^k_X$, simply denoted by $\langle\,,\rangle$. Then the sheaves $\mathcal{E}^k_X \otimes \mathcal{H}$ are equipped with a natural Hermitian metric denoted by $\langle\,,\rangle_{\mathrm{h}}$ (see Exercise 2.2). For the differential operator of order 1

$$D : \mathcal{E}^k_X \otimes \mathcal{H} \longrightarrow \mathcal{E}^{k+1}_X \otimes \mathcal{H}$$

induced by a connection $D : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$, the formal adjoint $D^\star : \mathcal{E}^{k+1}_X \otimes \mathcal{H} \to \mathcal{E}^k_X \otimes \mathcal{H}$ is the operator defined by

$$\int_X \langle Du, \overline{v} \rangle_{\mathrm{h}} \ \mathrm{vol}_X = \int_X \langle u, \overline{D^\star v} \rangle_{\mathrm{h}} \ \mathrm{vol}_X$$

for any pair of local sections with compact support of $\mathcal{E}^k_X \otimes \mathcal{H}$ and $\mathcal{E}^{k+1}_X \otimes \mathcal{H}$.

(1) Show that the formal adjoint $\varphi^\star$ of a $\mathcal{C}^\infty_X$-linear morphism $\varphi \in \mathcal{H}om(\mathcal{H}_1, \mathcal{H}_2)$ is nothing but its h-adjoint φ^*.

(2) Show that, if φ is self-adjoint with respect to h, then regarding $D(\varphi) = [D, \varphi]$ as a $\mathcal{C}^\infty_X$-linear morphism $\mathcal{E}^k_X \otimes \mathcal{H} \to \mathcal{E}^{k+1}_X \otimes \mathcal{H}$, we have $D(\varphi)^* = -[D^\star, \varphi]$.

Exercise 4.9. The goal of this exercise is to prove the identity (4.2.10). Recall $\mathcal{D} = D'_{\mathrm{h}} + \theta''$ and $\overline{\mathcal{D}} = D''_{\mathrm{h}} + \theta'$.

(1) Prove that $\theta'^\star = -\,\mathsf{i}[\Lambda, \theta'']$ and $\theta''^\star = \mathsf{i}[\Lambda, \theta']$.

(2) Deduce from the Kähler identities of Section 4.2.9 that

$$\widehat{D}''^\star = -\,\mathsf{i}[\Lambda, D'], \quad \mathcal{D}^\star = \mathsf{i}[\Lambda, \overline{\mathcal{D}}], \quad \overline{\mathcal{D}}^\star = -\,\mathsf{i}[\Lambda, \mathcal{D}].$$

(3) Conclude by using Lemma 4.2.7 and the computation of standard Kähler identities.

(4) Show also that $D^\star = \mathsf{i}[\Lambda, D^{\mathrm{c}}]$.

***Exercise 4.10* (Formulas for a holomorphic subbundle).** We keep the setting of Exercise 4.4. Let $(\mathcal{H}_1, D_1)$ be a flat holomorphic subbundle of $(\mathcal{H}, D)$, i.e., $\mathcal{H}_1$ is stable by D' and D''. Let $\pi : \mathcal{H} \to \mathcal{H}_1$ be the orthogonal projection (so that $\pi \circ \pi = \pi$). We still denote by D the connection on $\mathcal{E}nd(\mathcal{H})$, so that $D\pi - \pi D = D(\pi)$.

(1) Show the following relations for D, D_1 and π:

(a) $\pi D(\pi) = D(\pi)$ and $D(\pi)\pi = 0$. [*Hint*: for the first one, use that $\pi \circ D \circ \pi = D \circ \pi$; for the second one, use that, for a section v of $\mathcal{H}_1$, $D(v)$ is a section of $\mathcal{E}^1_X \otimes \mathcal{H}_1$, so that $\pi(D(v)) = D(v)$.]

(b) $D_1 = \pi \circ D \circ \pi = D \circ \pi = \pi \circ D + D(\pi)$.

(2) Show that $(\widehat{D}'_1 + D''_1) = \pi \circ (\widehat{D}' + D'') \circ \pi$. [*Hint*: recall that $(\widehat{D}' + D'')$ is the Chern connection for $(\mathcal{H}, \mathrm{h}, D'')$ and use [**GH78**, Lem. p. 73]).]

(3) Show a similar relation for $(\widehat{D}'' + D')$ and deduce a similar relation for D^{c}.

(4) Conclude that $D_1 D^{\mathrm{c}}_1 + D^{\mathrm{c}}_1 D_1 = \pi(DD^{\mathrm{c}} + D^{\mathrm{c}} D)\pi + D(\pi) D^{\mathrm{c}}(\pi)$.

4.5. Comments

Although one can trace back the notion of variation of Hodge structure to the study of the Legendre family of elliptic curves in the nineteenth century, the modern approach using the Gauss-Manin connection goes back to the fundamental work of Griffiths [**Gri68, Gri70a, Gri70b**] motivated by the properties of the period domain (see also [**Del71c**], [**CMSP03**]), a subject that is not considered in the present text. In the work of Griffiths, the transversality property (4.1.1) has been emphasized. From the point of view of $\mathcal{D}$-modules, this property is now encoded in the notion of a coherent filtration, and is at the heart of the notion of filtered $\mathcal{D}$-module, which is part of a Hodge module as defined by Saito.

The notion of a polarized variation of Hodge structure can be regarded as equivalent to the notion of a smooth polarized Hodge module. However, this equivalence is not obvious since the definition of a polarized Hodge module imposes properties on nearby cycles along any germ of holomorphic function, while the notion of variation only requires to consider coordinate functions.

The C^∞ approach as in Definition 4.1.5 proves useful for extending the Hodge theorem on smooth complex projective varieties and constant coefficients to the case when the coefficient system is a unitary local system (see [**Dem96**]) and the more general case when it underlies a polarized variation of Hodge structure (Hodge-Deligne theorem 4.2.16 explained in the introduction of [**Zuc79**]). It is also well-adapted to the extension of this theorem to harmonic bundles, as explained by Simpson in [**Sim92**]. In this smooth context, the flat sesquilinear pairing $\mathfrak{s}$ gives rise in a natural way to the (non-flat in general) Hermitian Hodge metric. The fixed-part theorem, proved in Remark 4.2.18(4), is originally due to Griffiths [**Gri70a**] in a geometric setting, and has been proved in a more general context by Deligne [**Del71b**, Cor. 4.1.2], and also by Schmid [**Sch73**, Th. 7.22].

We have also mentioned the case of complete Kähler manifolds, going back to Andreotti and Vesentini [**AV65**] and Hörmander [**Hör65, Hör66**]. Theorem 4.2.33 is taken from [**Dem96**, §12B]. They are useful for understanding the L^2 approach as in Zucker's theorem 6.11.1 of [**Zuc79**].

It is remarkable that the local system underlying a polarized variation of Hodge structure on a smooth complex projective variety (or a compact Kähler manifold) is semi-simple. This property, proved by Deligne in the presence of a $\mathbb{Z}$-structure (see [**Del71b**, Th. 4.2.6]), can be regarded as a special case of a result of Corlette [**Cor88**] and [**Sim92**], since the Hodge metric is a pluri-harmonic metric on the corresponding flat holomorphic bundle. These articles are at the source of Sections 4.2.b and 4.3.b. Exercises 4.4 and 4.10 are extracted from [**Sim90**] and [**Sim92**].

Last, the structure theorem for polarized variations of Hodge structure (Theorem 4.3.13) is nothing but [**Del87**, Prop. 1.13].

CHAPTER 5

THE REES CONSTRUCTION FOR HODGE STRUCTURES

Summary. In this chapter, we revisit the notion of Hodge structure in order to adapt it to $\mathcal{D}$-modules. There are two major changes of point of view. On the one hand, we replace a vector space equipped with two filtrations with two free modules over the ring $\mathbb{C}[z]$ and we express oppositeness in this language as a gluing property. On the other hand, in order to handle singularities in the gluing properties for filtered $\mathcal{D}$-modules, we express the gluing as a nondegenerate pairing, in order to relax the nondegeneracy condition when necessary. The notion of sesquilinear pairing, which was mainly used for expressing the polarization in the previous chapters, is now used for expressing the oppositeness property. This leads to the general notion of *triples*, which form an abelian category, equipped with Hermitian duality. The polarization is now expressed as an isomorphism between a triple and its Hermitian dual, satisfying a suitable positivity condition. We will make clear the way to pass from one approach to the other one.

5.1. Filtered objects and the Rees construction

5.1.1. Convention. We denote with a *lower* index the *increasing* filtrations and with an *upper* index the *decreasing* ones. A standard rule is to pass from one type to the other one by changing the sign of the index. However, this rule is slightly modified for V-filtrations (see Chapter 9).

5.1.a. Filtered rings and modules

5.1.2. Definition. Let $(\mathcal{A}, F_\bullet)$ be a filtered $\mathbb{C}$-algebra. A *filtered $\mathcal{A}$-module* $(\mathcal{M}, F_\bullet\mathcal{M})$ is an $\mathcal{A}$-module $\mathcal{M}$ together with an increasing filtration indexed by $\mathbb{Z}$ satisfying (for left modules for instance)

$$F_k\mathcal{A} \cdot F_\ell\mathcal{M} \subset F_{k+\ell}\mathcal{M} \quad \forall\, k, \ell \in \mathbb{Z}.$$

We always assume that the filtration is *exhaustive*, i.e., $\bigcup_\ell F_\ell\mathcal{M} = \mathcal{M}$. We also say that $F_\bullet\mathcal{M}$ is an $F_\bullet\mathcal{A}$-filtration, or simply an *F-filtration*.

A filtered morphism between filtered $\mathcal{A}$-modules is a morphism of $\mathcal{A}$-modules which is compatible with the filtrations.

A simple way to treat a filtered module as a module is to consider the Rees object associated to any filtered object. Let us introduce a new variable z. We will replace the base field $\mathbb{C}$ with the polynomial ring $\mathbb{C}[z]$.

5.1.3. Rees ring and Rees module. If $(\mathcal{A}, F_\bullet)$ is a filtered $\mathbb{C}$-algebra, we denote by $\widetilde{\mathcal{A}}$ (or $R_F\mathcal{A}$ if we want to insist on the dependence with respect to the filtration) the *graded* subring $\bigoplus_p F_p\mathcal{A}\cdot z^p$ of $\mathcal{A}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$ (the term $F_p\mathcal{A}\cdot z^p$ is in degree p). For example, if $F_p\mathcal{A}=0$ for $p\leqslant -1$ and $F_p\mathcal{A}=\mathcal{A}$ for $p\geqslant 0$, we have $\widetilde{\mathcal{A}}=\mathcal{A}\otimes_{\mathbb{C}}\mathbb{C}[z]$.

It will be convenient to set $\widetilde{\mathbb{C}}=\mathbb{C}[z]$ (i.e., we apply the Rees construction to $\mathbb{C}$ equipped with its trivial filtration $F_0\mathbb{C}=\mathbb{C}$ and $F_{-1}\mathbb{C}=0$).

Any filtered module $(\mathcal{M}, F_\bullet)$ on the filtered ring $(\mathcal{A}, F_\bullet)$ gives rise similarly to a graded $\widetilde{\mathcal{A}}$-module $R_F\mathcal{M}=\bigoplus_p F_p\mathcal{M}\cdot z^p\subset\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[z,z^{-1}]$, and a filtered morphism gives rise to a graded morphism (of degree zero) between the associated Rees modules.

Conversely, any graded $\widetilde{\mathcal{A}}$-module $\widetilde{\mathcal{M}}$ can be written as $\bigoplus_p\mathcal{M}_p z^p$ ($\widetilde{\mathcal{M}}_p=\mathcal{M}_p z^p$ is in degree p), where each $\mathcal{M}_p$ is an $\mathcal{A}$-module, and the $\mathbb{C}[z]$-structure is given by $\mathcal{A}$-linear morphisms $\mathcal{M}_p\to\mathcal{M}_{p+1}$. The $\mathcal{A}$-module $\mathcal{M}=\varinjlim_p\mathcal{M}_p$ is called the $\mathcal{A}$-module *associated with the graded $\widetilde{\mathcal{A}}$-module* $\widetilde{\mathcal{M}}$. The natural morphism

$$\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}[z]}\widetilde{\mathcal{M}}\longrightarrow\mathbb{C}[z,z^{-1}]\otimes_{\mathbb{C}}\mathcal{M}=:\mathcal{M}[z,z^{-1}]$$

is an isomorphism of $\mathcal{A}[z,z^{-1}]$-modules.

The category $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ is the category whose objects are graded $\widetilde{\mathcal{A}}$-modules and whose morphisms are graded morphisms of degree zero. It is an abelian category. It comes equipped with an automorphism σ: given an object of $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ written as $\widetilde{\mathcal{M}}=\bigoplus_p\mathcal{M}_p z^p$, we set

$$\sigma(\widetilde{\mathcal{M}})=\widetilde{\mathcal{M}}(1)=z\widetilde{\mathcal{M}}\quad\text{so that}\quad\widetilde{\mathcal{M}}(1)_p=\mathcal{M}_{p-1}z^p. \tag{5.1.4}$$

In other words, we regard multiplication by z as an isomorphism $\widetilde{\mathcal{M}}\xrightarrow{\sim}\widetilde{\mathcal{M}}(1)$.

5.1.5. Remark **(Shift of the filtration and twist of the Rees module)**

(1) The shift $F[k]$ of an increasing filtration is defined by

$$F[k]_\bullet\mathcal{M}=F_{\bullet-k}\mathcal{M}\qquad(\text{hence }\mathrm{gr}_p^{F[k]}\mathcal{M}=\mathrm{gr}_{p-k}^F\mathcal{M},\ \forall\,p). \tag{5.1.5$\,*$}$$

For example, if $F_\bullet\mathcal{M}$ only jumps at p_o, then $F[k]_\bullet\mathcal{M}$ only jumps at p_o+k.

(2) If $\widetilde{\mathcal{M}}=R_F\mathcal{M}$, then $F_p\mathcal{M}=\mathcal{M}_p$ and $\widetilde{\mathcal{M}}(k)_p=F_{p-k}\mathcal{M}z^p$. In other words,

$$\widetilde{\mathcal{M}}(k)\xrightarrow{\sim}R_{F[k]}\mathcal{M}. \tag{5.1.5$\,**$}$$

5.1.b. Strictness. Strictness is a property which enables one to faithfully pass properties from a filtered object to the associated graded object.

5.1.6. Definition **(Strictness in $\mathsf{Mod}(\widetilde{\mathcal{A}})$ and $\mathsf{Modgr}(\widetilde{\mathcal{A}})$).**

(1) An object of $\mathsf{Mod}(\widetilde{\mathcal{A}})$ is said to be *strict* if it has no $\widetilde{\mathbb{C}}[z]$-torsion.

(2) A morphism in $\mathsf{Mod}(\widetilde{\mathcal{A}})$ is said to be *strict* if its kernel and cokernel are strict (note that the composition of two strict morphisms need not be strict).

(3) A complex $\widetilde{\mathcal{M}}^\bullet$ of $\mathsf{Mod}(\widetilde{\mathcal{A}})$ is said to be *strict* if each of its cohomology modules is a strict object of $\mathsf{Mod}(\widetilde{\mathcal{A}})$.

An object, resp. morphism, resp. complex in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ is strict if it is so when considered in $\mathsf{Mod}(\widetilde{\mathcal{A}})$.

5.1.7. Caveat. The composition of strict morphisms between strict objects need not be strict. One cannot form a category (which would be abelian) by only considering these morphisms. On the other hand, the full subcategory $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$ of $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ whose objects are strict (morphisms need not be strict) is in general not abelian.

***5.1.8. Proposition* (Strict objects).**

(1) *An object of* $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ *is strict if and only if it comes from a filtered $\mathcal{A}$-module by the Rees construction.*

(2) *The Rees construction induces an equivalence between the category of filtered $\mathcal{A}$-modules (and morphisms preserving filtrations) ant the category* $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$.

(3) *The restriction functor* $\widetilde{\mathcal{M}} \mapsto \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ *from* $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$ *to* $\mathsf{Mod}(\mathcal{A})$ *is faithful.*

Proof.

(1) One checks that $\widetilde{\mathcal{M}}$ is strict if and only if the $\mathcal{A}$-linear morphisms $\mathcal{M}_p \to \mathcal{M}_{p+1}$ are all injective. In such a case, $\mathcal{M} = \varinjlim_p \mathcal{M}_p = \bigcup_p \mathcal{M}_p$ and the $\mathcal{M}_p$ form an increasing filtration $F_\bullet\mathcal{M}$, so that, by definition, $\widetilde{\mathcal{M}} = R_F\mathcal{M}$.

(2) We are left with considering morphisms. Let $\widetilde{\varphi} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a graded morphism of degree zero. Its restriction $\widetilde{\varphi}_p$ to $\widetilde{\mathcal{M}}_p$ satisfies $\widetilde{\varphi}_{p+1} z m_p = z\varphi_p m$ by $\widetilde{\mathbb{C}}[z]$-linearity. Therefore, $\widetilde{\varphi}_p$ is the restriction of $\widetilde{\varphi}_{p+1}$ by the inclusion $z\widetilde{\mathcal{M}}_p \hookrightarrow \widetilde{\mathcal{M}}_{p+1}$, hence the family $(\varphi_p)_p$ defines a morphism $\varphi : \mathcal{M} \to \mathcal{N}$, and we obviously have $\widetilde{\varphi} = R_F\varphi$.

(3) See Exercise 5.2. □

We now consider the category WA (see Section 2.6.b) of W-filtered objects of the category $\mathsf{A} := \mathsf{Modgr}(\widetilde{\mathcal{A}})$ with $\widetilde{\mathcal{A}} = R_F\mathcal{A}$, and the notion of strictness is as in Definition 5.1.6.

5.1.9. Lemma. *We set* $\mathsf{Modgr}(\widetilde{\mathcal{A}}) = \mathsf{A}$.

(1) *Let $\widetilde{\mathcal{M}}$ be a an object of* WA. *If each* $\mathrm{gr}_k^W \widetilde{\mathcal{M}}$ *is strict, then* $\widetilde{\mathcal{M}}$ *is strict.*

(2) *Let* $\widetilde{\varphi} : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ *be a morphism in* WA. *If* $\mathrm{gr}_k^W \widetilde{\mathcal{M}}_1, \mathrm{gr}_k^W \widetilde{\mathcal{M}}_2$ *are strict for all k, and if $\widetilde{\varphi}$ is strictly compatible with W, i.e., satisfies* $\widetilde{\varphi}(W_k\widetilde{\mathcal{M}}) = W_k\widetilde{\mathcal{N}} \cap \varphi(\widetilde{\mathcal{M}})$ *for all k, then $\widetilde{\varphi}$ is strict.*

Proof. The first point is treated in Exercise 5.1(2). Let us prove (2). Let $W_\bullet \operatorname{Ker}\widetilde{\varphi}$ and $W_\bullet \operatorname{Coker}\widetilde{\varphi}$ be the induced filtrations. By strict compatibility, the sequence

$$0 \longrightarrow \mathrm{gr}_k^W \operatorname{Ker}\widetilde{\varphi} \longrightarrow \mathrm{gr}_k^W \widetilde{\mathcal{M}} \xrightarrow{\mathrm{gr}_k^W \widetilde{\varphi}} \mathrm{gr}_k^W \widetilde{\mathcal{N}} \longrightarrow \mathrm{gr}_k^W \operatorname{Coker}\widetilde{\varphi} \longrightarrow 0$$

is exact. By strictness of $\mathrm{gr}_k^W \widetilde{\varphi}$, and applying (1) to $\operatorname{Ker}\widetilde{\varphi}$ and $\operatorname{Coker}\widetilde{\varphi}$, one gets that $\operatorname{Ker}\widetilde{\varphi}$ and $\operatorname{Coker}\widetilde{\varphi}$ are strict, i.e., $\widetilde{\varphi}$ is strict. □

Strictness with respect to the monodromy filtration. We consider the setup of Chapter 3 on Lefschetz structures and take up the notation of Remark 3.3.4. We equip the category $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ of graded $\widetilde{\mathcal{A}}$-modules with the automorphism σ shifting the grading by 1, so that $\sigma(\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}}(1)$ (see (5.1.4)). Let $\widetilde{\mathcal{M}}$ be an object of $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ and let $\mathrm{N} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(-1) = \sigma^{-1}\widetilde{\mathcal{M}}$ be a nilpotent endomorphism.

5.1.10. Proposition. *Let* $\mathrm{M}_\bullet(\mathrm{N})\widetilde{\mathcal{M}}$ *be the monodromy filtration of* $(\widetilde{\mathcal{M}}, \mathrm{N})$ *in the abelian category* $\mathsf{Modgr}(\widetilde{\mathcal{A}})$ *(see Lemma* 3.3.1*). Assume that* $\widetilde{\mathcal{M}}$ *is strict. Then the following properties are equivalent:*

(1) *For every* $\ell \geqslant 1$, $\mathrm{N}^\ell : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(-\ell)$ *is a strict morphism.*
(2) *For every* $\ell \in \mathbb{Z}$, $\mathrm{gr}_\ell^{\mathrm{M}}\widetilde{\mathcal{M}}$ *is strict.*
(3) *For every* $\ell \geqslant 0$, $\mathrm{P}_\ell\widetilde{\mathcal{M}}$ *is strict.*

Proof. The equivalence between (2) and (3) comes from the Lefschetz decomposition in the category $\mathsf{Modgr}(\widetilde{\mathcal{A}})$.

(2) $\Rightarrow$ (1) Assume $\ell \geqslant 1$. The Lefschetz decomposition implies that each morphism $\mathrm{gr}_{-2\ell}\mathrm{N}^\ell$ on $\mathrm{gr}_\bullet^{\mathrm{M}}\widetilde{\mathcal{M}}$ is strict. Since N^ℓ, regarded as a filtered morphism

$$(\widetilde{\mathcal{M}}, \mathrm{M}_\bullet\widetilde{\mathcal{M}}) \longrightarrow (\widetilde{\mathcal{M}}(-\ell), \mathrm{M}[2\ell]_\bullet\widetilde{\mathcal{M}})$$

is strictly compatible with the filtrations M (Lemma 3.3.7), the result follows from Lemma 5.1.9(2).

(1) $\Rightarrow$ (2) We will use the inductive construction of the monodromy filtration given in Exercise 3.10. We argue by induction on the order of nilpotence of N. Assume that $\mathrm{N}^{\ell+1} = 0$. The strictness of $\widetilde{\mathcal{M}}$ implies that $\mathrm{M}_\ell\widetilde{\mathcal{M}}$, $\mathrm{M}_{\ell-1}\widetilde{\mathcal{M}}$, $\mathrm{M}_{-\ell}\widetilde{\mathcal{M}} = \mathrm{gr}_{-\ell}^{\mathrm{M}}\widetilde{\mathcal{M}}$ and $\mathrm{P}_\ell\widetilde{\mathcal{M}} = \mathrm{gr}_\ell^{\mathrm{M}}\widetilde{\mathcal{M}} \simeq \mathrm{gr}_{-\ell}^{\mathrm{M}}\widetilde{\mathcal{M}}$ are strict. The strictness of $\widetilde{\mathcal{M}}' := \widetilde{\mathcal{M}}/\mathrm{M}_{-\ell}\widetilde{\mathcal{M}} = \operatorname{Coker}\mathrm{N}^\ell$ follows from the strictness of N^ℓ. Moreover, $(\widetilde{\mathcal{M}}', \mathrm{N}')$ satisfies (1) with $\mathrm{N}'^\ell = 0$, hence by induction each $\mathrm{gr}_j^{\mathrm{M}}\widetilde{\mathcal{M}}'$ is strict. Now, the relation between $\mathrm{gr}_\bullet^{\mathrm{M}}\widetilde{\mathcal{M}}'$ and $\mathrm{gr}_\bullet^{\mathrm{M}}\widetilde{\mathcal{M}}$ is easily seen from the Lefschetz decomposition (see Figure 3.1), and (2) for $\mathrm{gr}_\bullet^{\mathrm{M}}\widetilde{\mathcal{M}}$ follows. $\square$

5.1.c. Filtered holomorphic flat bundles. Let $(\mathcal{H}', \nabla)$ be an $\mathcal{O}_X$-module with connection on a complex manifold X, equipped with a decreasing filtration $F^\bullet\mathcal{H}'$ by $\mathcal{O}_X$-submodules (here, we do not make any coherence or local freeness assumption). The filtration on the sheaf of rings $\mathcal{O}_X$ is simply defined by $F^0\mathcal{O}_X = \mathcal{O}_X$ and $F^1\mathcal{O}_X = 0$, so that $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z]$ as a sheaf of graded rings. The Rees module attached to $F^\bullet\mathcal{H}'$ is the graded coherent $\mathcal{O}_X[z]$-module $\widetilde{\mathcal{H}}' := \bigoplus_p F^p\mathcal{H}'z^{-p}$.

By a *holomorphic z-connection* on $\widetilde{\mathcal{H}}'$, we mean a morphism

$$\widetilde{\nabla} : \widetilde{\mathcal{H}}' \longrightarrow \widetilde{\Omega}_X^1 \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{H}}'(-1)$$

in the category $\mathsf{Modgr}(\widetilde{\mathbb{C}})$ (in particular it is $\widetilde{\mathbb{C}}$-linear) which satisfies the z-Leibniz rule

$$\widetilde{\nabla}(fv) = \widetilde{\mathrm{d}}f \otimes v + f\widetilde{\nabla}v, \quad \widetilde{\mathrm{d}} := z\mathrm{d},\ f \in \widetilde{\mathcal{O}}_X,\ v \in \widetilde{\mathcal{H}}'.$$

We say that $\widetilde{\nabla}$ is *flat* if its curvature $\widetilde{\nabla} \circ \widetilde{\nabla}$ (taken in the usual sense) is zero.

5.1.11. Lemma. *The connection ∇ on $\mathcal{H}'$ extends to a z-connection $\widetilde{\nabla}$ on $\widetilde{\mathcal{H}}'$ if and only if the filtration $F^{\bullet}\mathcal{H}'$ satisfies the Griffiths transversality property. In such a case, ∇ is flat if and only if $\widetilde{\nabla}$ is flat.*

Proof. For the "only if" part, define first $\widetilde{\nabla} : \mathcal{H}'[z, z^{-1}] \to \widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{H}'[z, z^{-1}](-1)$ as $z\nabla$. This is a z-connection. Griffiths transversality implies that it sends $\widetilde{\mathcal{H}}'$ to $\widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{H}}'(-1)$, defining thus $\widetilde{\nabla}$. If ∇ is flat, then so is $z\nabla$, hence $\widetilde{\nabla}$.

Conversely, starting from $\widetilde{\nabla}$, one extends it by z-linearity to $\mathcal{H}'[z, z^{-1}]$ and, dividing by z and then restricting to $z = 1$, one obtains the desired ∇. □

We can now restate Proposition 5.1.8 in the present setting.

5.1.12. Proposition. *The Rees construction induces an equivalence between*

- *the category of filtered $\mathcal{O}_X$-modules with flat connection and with a filtration satisfying the Griffiths transversality property,*
- *and the full subcategory of strict objects in the category of graded $\widetilde{\mathcal{O}}_X$-modules with flat z-connection.* □

5.2. The category of $\widetilde{\mathbb{C}}$-triples

5.2.a. A geometric interpretation of a bi-filtered vector space

Let $(\mathcal{H}, F'^{\bullet})$ be a filtered vector space. We introduce a new variable z and we consider, in the free $\mathbb{C}[z, z^{-1}]$-module $\widetilde{\mathcal{H}} := \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathcal{H}$, the $\mathbb{C}$-vector space $\widetilde{\mathcal{H}}' := \bigoplus_p F'^p\mathcal{H}z^{-p}$. Then $\widetilde{\mathcal{H}}'$ is a $\widetilde{\mathbb{C}}$-submodule of $\widetilde{\mathcal{H}}$ which generates $\widetilde{\mathcal{H}}$, that is, $\widetilde{\mathcal{H}} = \mathbb{C}[z, z^{-1}] \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{H}}'$. It is a free $\widetilde{\mathbb{C}}$-module. Indeed, Let us choose for each p a family $\boldsymbol{v}^p$ in $F'^p\mathcal{H}$ inducing a basis of $\mathrm{gr}^p_{F'}\mathcal{H}$; then $\widetilde{\mathcal{H}}' = \bigoplus_p \widetilde{\mathbb{C}} z^{-p}\boldsymbol{v}^p$.

Similarly, denote by $\widetilde{\mathcal{H}}''$ the object $\bigoplus_q F''^q\mathcal{H}z^q$. Then $\widetilde{\mathcal{H}}''$ is a free $\mathbb{C}[z^{-1}]$-submodule of $\widetilde{\mathcal{H}}$ which generates $\widetilde{\mathcal{H}}$, that is, $\widetilde{\mathcal{H}} = \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}[z^{-1}]} \widetilde{\mathcal{H}}''$. Using the gluing

$$\begin{tikzcd} \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z]} \widetilde{\mathcal{H}}' \arrow[r, "\sim"] \arrow[rr, bend left, "\sim"] & \widetilde{\mathcal{H}} & \mathbb{C}[z,z^{-1}] \otimes_{\mathbb{C}[z^{-1}]} \widetilde{\mathcal{H}}'' \arrow[l, "\sim"'] \end{tikzcd}$$

the pair $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'')$ defines an algebraic vector bundle $\mathcal{F}$ on $\mathbb{P}^1$ of rank $\dim \mathcal{H}$. The properties 2.5(1a) (oppositeness) and (1b) (Hodge decomposition) are also equivalent to (see Exercise 5.5)

(c) The vector bundle $\mathcal{F}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(w)^{\dim \mathcal{H}}$.

The gluing isomorphism $\mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathcal{H} \xrightarrow{\sim} \mathbb{C}[z, z^{-1}] \otimes_{\mathbb{C}} \mathcal{H}$ described above has the property of being homogeneous of degree zero with respect to the z-grading, since it is induced by a constant isomorphism $\mathcal{H} \xrightarrow{\sim} \mathcal{H}$ (the identity).

5.2.b. $\widetilde{\mathbb{C}}$-Triples. We will introduce another language for dealing with polarizable complex Hodge structures. This is similar to the presentation given in Section 2.6.a, but compared with it, we replace $\mathcal{H}''$ with its dual $\mathcal{H}''^\vee$. This approach will be useful in higher dimensions.

***5.2.1. Definition* ($\widetilde{\mathbb{C}}$-Triples).** The category $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$ is the category whose objects

$$T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$$

consist of a pair of $\widetilde{\mathbb{C}}$-modules of finite type $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$ and a sesquilinear pairing $\mathfrak{s} : \mathcal{H}' \otimes \overline{\mathcal{H}''} \to \mathbb{C}$ between the associated vector spaces (see Section 5.1.3) that we also regard as a morphism $\mathfrak{s} : \mathcal{H}'' \to \mathcal{H}'^*$ (see Section 0.5), and whose morphisms $\varphi : T_1 \to T_2$ are pairs $(\widetilde{\varphi}', \widetilde{\varphi}'')$ of morphisms (graded of degree zero)

$$\widetilde{\varphi}' : \widetilde{\mathcal{H}}'_1 \longrightarrow \widetilde{\mathcal{H}}'_2, \qquad \widetilde{\varphi}'' : \widetilde{\mathcal{H}}''_2 \longrightarrow \widetilde{\mathcal{H}}''_1 \tag{5.2.1 $*$}$$

such that, for every $v'_1 \in \mathcal{H}'_1$ and $v''_2 \in \mathcal{H}''_2$, denoting by φ', φ'' the morphisms induced by $\widetilde{\varphi}', \widetilde{\varphi}''$ on $\mathcal{H}', \mathcal{H}''$, we have

$$\mathfrak{s}_1(v'_1, \overline{\varphi''(v''_2)}) = \mathfrak{s}_2(\varphi'(v'_1), \overline{v''_2}), \tag{5.2.1 $**$}$$

or equivalently

$$\varphi'^* \circ \mathfrak{s}_2 = \mathfrak{s}_1 \circ \varphi'' : \mathcal{H}''_2 \longrightarrow \mathcal{H}'^*_1.$$

***5.2.2. Operations on the category* $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$**

(1) The category $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$ is abelian, the "prime" part is covariant, while the "double-prime" part is contravariant. In other words, the "prime" part is an object of the category of $\widetilde{\mathbb{C}}$-vector spaces, while the 'double-prime" part is an object of the opposite category.

For example, the triple $\operatorname{Ker}\varphi$ is the triple $(\operatorname{Ker}\widetilde{\varphi}', \operatorname{Coker}\widetilde{\varphi}'', \mathfrak{s}_1^\varphi)$, where $\mathfrak{s}_1^\varphi$ is the pairing between $\operatorname{Ker}\varphi'$ and $\operatorname{Coker}\varphi''$ induced by $\mathfrak{s}_1$, which is well-defined because of (5.2.1 $**$). Similarly, we have

$$\operatorname{Coker}\varphi = (\operatorname{Coker}\widetilde{\varphi}', \operatorname{Ker}\widetilde{\varphi}'', \mathfrak{s}_2^\varphi), \quad \operatorname{Im}\varphi = (\operatorname{Im}\widetilde{\varphi}', \widetilde{\mathcal{H}}''_2/\operatorname{Ker}\widetilde{\varphi}'', \mathfrak{s}_2^\varphi).$$

(2) An increasing filtration $W_\bullet T$ of a triple T consists of increasing filtrations $W_\bullet\widetilde{\mathcal{H}}', W_\bullet\widetilde{\mathcal{H}}''$ such that $\mathfrak{s}(W_\ell\mathcal{H}', \overline{W_{-\ell-1}\mathcal{H}''}) = 0$ for every ℓ. Then $\mathfrak{s}$ induces a pairing

$$\mathfrak{s}_\ell : W_\ell\mathcal{H}' \otimes \overline{\mathcal{H}''/W_{-\ell-1}\mathcal{H}''} \longrightarrow \mathbb{C}.$$

We set $W_\ell T = (W_\ell\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''/W_{-\ell-1}\widetilde{\mathcal{H}}'', \mathfrak{s}_\ell)$. We have

$$\operatorname{gr}_\ell^W T = (\operatorname{gr}_\ell^W\widetilde{\mathcal{H}}', \operatorname{gr}_{-\ell}^W\widetilde{\mathcal{H}}'', \mathfrak{s}_\ell).$$

(3) We say that a triple is *strict* if $\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}''$ are strict. Strict triples are in one-two-one correspondence with *filtered triples* $((\mathcal{H}', F^\bullet\mathcal{H}'), (\mathcal{H}'', F^\bullet\mathcal{H}''), \mathfrak{s})$. *We will not distinguish between $\widetilde{\mathbb{C}}$-triples and filtered $\mathbb{C}$-triples.*

We say that a morphism $\varphi : T_1 \to T_2$ is strict if its components $\widetilde{\varphi}', \widetilde{\varphi}''$ are strict. Strict morphisms between strict triples are in one-to-one correspondence with strict morphisms between filtered triples.

(4) The difference with the construction made in Section 2.6.a is that the "double prime" part is now contravariant, and the isomorphism γ is replaced with a pairing. This gives more flexibility since the pairing is not assumed to be non-degenerate a priori. We say that a $\widetilde{\mathbb{C}}$-triple T is *non-degenerate* if $\mathfrak{s}$ is so. If $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is strict and non-degenerate, one can associate a triple like in Section 2.6.a by replacing $(\mathcal{H}'', F^\bullet\mathcal{H}'')$ defined from $\widetilde{\mathcal{H}}''$ with $(\mathcal{H}''^*, F^\bullet\mathcal{H}''^*)$ and by defining γ as the isomorphism $\mathcal{H}' \to \mathcal{H}''^*$ obtained by Hermitian adjunction from $\mathfrak{s} : \mathcal{H}'' \to \mathcal{H}'^*$.

(5) Let $(T, W_\bullet T)$ be a W-filtered $\widetilde{\mathbb{C}}$-triple as in (2). Assume that T and all $\mathrm{gr}_\ell^W T$ are strict, i.e., all inclusions $W_\ell T \hookrightarrow W_{\ell+1}T$ are strict morphisms. Then $\mathrm{gr}_\ell^W \widetilde{\mathcal{H}}'$ is the Rees object attached with the filtered vector space

$$F^p \mathrm{gr}_\ell^W \mathcal{H}' := \frac{F^p\mathcal{H}' \cap W_\ell\mathcal{H}'}{F^p\mathcal{H}' \cap W_{\ell-1}\mathcal{H}'},$$

and a similar equality for $\mathrm{gr}_{-\ell}^W \widetilde{\mathcal{H}}''$.

(6) The *Hermitian dual* of a $\widetilde{\mathbb{C}}$-triple $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is the $\widetilde{\mathbb{C}}$-triple $T^* := (\widetilde{\mathcal{H}}'', \widetilde{\mathcal{H}}', \mathfrak{s}^*)$, where $\mathfrak{s}^*$ is defined by

$$\mathfrak{s}^*(v'', \overline{v'}) := \overline{\mathfrak{s}(v', \overline{v''})}.$$

We have $T^{**} = T$. If $\varphi = (\widetilde{\varphi}', \widetilde{\varphi}'') : T_1 \to T_2$ is a morphism, its Hermitian adjoint $\varphi^* : T_2^* \to T_1^*$ is the morphism $(\widetilde{\varphi}'', \widetilde{\varphi}')$.

The Hermitian dual of a strict $\widetilde{\mathbb{C}}$-triple T is also strict, and T^* corresponds to the filtered triple $T^* := ((\mathcal{H}'', F^\bullet\mathcal{H}''), (\mathcal{H}', F^\bullet\mathcal{H}'), \mathfrak{s}^*)$.

(7) Given a pair of integers (k, ℓ), the twist $T(k, \ell)$ is defined by

$$T(k, \ell) := (z^k\widetilde{\mathcal{H}}', z^{-\ell}\widetilde{\mathcal{H}}'', \mathfrak{s}).$$

We have $(T(k, \ell))^* = T^*(-\ell, -k)$. If $\varphi : T_1 \to T_2$ is a morphism, then it is also a morphism $T_1(k, \ell) \to T_2(k, \ell)$.

If T is strict with associated filtered triple T, the twisted object $T(k, \ell)$ is also strict and its associated filtered triple is

$$(F[k]^\bullet\mathcal{H}', F[-\ell]^\bullet\mathcal{H}'', \mathfrak{s}).$$

This is compatible with the twist as defined in Section 2.5.6, by means of the equivalence of Lemma 5.2.7 below. (Recall that $F[k]^p := F^{p+k}$.)

5.2.3. Notation. As in Definition 2.5.7, we simply use the notation (w) for the (symmetric) Tate twist (w, w): $T(w) = (z^w\widetilde{\mathcal{H}}', z^{-w}\widetilde{\mathcal{H}}'', \mathfrak{s})$.

***5.2.4. Definition* (w-oppositeness condition).** Let T be a filtered triple and let $w \in \mathbb{Z}$. The filtration $F^\bullet\mathcal{H}''$ naturally induces a filtration $F^\bullet\mathcal{H}''^*$ on the Hermitian dual space $\mathcal{H}''^* = \overline{\mathcal{H}''}^\vee$. We say that T satisfies the w-oppositeness condition if $\mathfrak{s}$ is non-degenerate and if the filtration $F^\bullet\mathcal{H}'$ is w-opposite to the filtration obtained from $F^\bullet\mathcal{H}''^*$ by means of the isomorphism $\mathcal{H}' \xrightarrow{\sim} \mathcal{H}''^*$ induced by $\mathfrak{s}^*$.

***5.2.5. Definition* ($\mathbb{C}$-Hodge triples).** The category of $\mathbb{C}$-Hodge triples of weight $w \in \mathbb{Z}$ is the full subcategory of $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$ (i.e., morphisms are described by (5.2.1 $*$) and satisfying (5.2.1 $**$)) whose objects are strict and satisfy the w-oppositeness condition. In particular, $\mathfrak{s}$ is assumed to be non-degenerate. A $\mathbb{C}$-Hodge triple will be denoted by $H = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ or $((\mathcal{H}', F^\bullet\mathcal{H}'), (\mathcal{H}'', F^\bullet\mathcal{H}''), \mathfrak{s})$.

***5.2.6. Remark* (Hermitian duality and twist).** The category of $\mathbb{C}$-Hodge triples of weight w is changed to that of $\mathbb{C}$-Hodge triples of weight $-w$ by the Hermitian duality functor 5.2.2(6) and, for a $\mathbb{C}$-Hodge triple H of weight w, the twisted $\widetilde{\mathbb{C}}$-triple $H(k,\ell)$ is a $\mathbb{C}$-Hodge triple H of weight $w-(k+\ell)$. In particular, the Tate twisted $\widetilde{\mathbb{C}}$-triple $H(k)$ is a $\mathbb{C}$-Hodge triple H of weight $w-2k$.

5.2.7. Lemma. *The correspondence*

$$H = (\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}) \longmapsto H = \big(R_F\mathcal{H}', R_F\mathcal{H}'', \mathfrak{s}\big),$$

obtained by setting

$$(\mathcal{H}', F^\bullet\mathcal{H}') := (\mathcal{H}, F'^\bullet\mathcal{H}), \quad (\mathcal{H}'', F^\bullet\mathcal{H}'') := (\mathcal{H}^*, F''^\bullet\mathcal{H}^*), \quad \mathfrak{s} := \langle\bullet,\bullet\rangle : \mathcal{H}\otimes\mathcal{H}^\vee \to \mathbb{C}$$

(recall that $F''^\bullet\mathcal{H}^*$ is obtained by duality from $\overline{F''^\bullet\mathcal{H}}$) is an equivalence between $\mathsf{HS}(\mathbb{C}, w)$ *and the category of $\mathbb{C}$-Hodge triples of weight w.* ☐

From now on, we will not distinguish between $\mathbb{C}$-Hodge structures of weight w and $\mathbb{C}$-Hodge triples of weight w.

5.2.8. Lemma. *Assume we have a decomposition $H = H_1 \oplus H_2$ of $\widetilde{\mathbb{C}}$-triples. If H is $\mathbb{C}$-Hodge of weight w, so are H_1 and H_2.*

Proof. First, H_1 and H_2 must be strict, hence correspond to filtered triples. The non-degeneracy of $\mathfrak{s}_1$ and $\mathfrak{s}_2$ is also clear. Last, we use the interpretation 5.2.a(c) of w-oppositeness and the standard property that, if a vector bundle on $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(w)^d$, then any direct summand is isomorphic to a power of $\mathcal{O}_{\mathbb{P}^1}(w)$. ☐

***5.2.9. Definition* (Pre-polarization of weight w of a $\widetilde{\mathbb{C}}$-triple)**

A *pre-polarization* of weight w of a $\widetilde{\mathbb{C}}$-triple T is an isomorphism

$$\mathrm{S} = (\widetilde{\mathrm{S}}', \widetilde{\mathrm{S}}'') : T \xrightarrow{\sim} T^*(-w)$$

which is Hermitian, in the sense that its Hermitian adjoint

$$\mathrm{S}^* = (\widetilde{\mathrm{S}}'', \widetilde{\mathrm{S}}') : (T^*(-w))^* = T(w) \xrightarrow{\sim} T^*,$$

which defines an isomorphism denoted in the same way $\mathrm{S}^* : T \xrightarrow{\sim} T^*(-w)$, satisfies

$$\mathrm{S}^* = \mathrm{S}.$$

The Tate twist acts on a pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S) of weight w by the formula

$$(T, \mathrm{S})(\ell) = (T(\ell), (-1)^\ell \mathrm{S})$$

(see Notation 5.2.3 for the notation $H(\ell)$).

Let us make explicit this definition. Since the Hermitian dual T^* of a $\widetilde{\mathbb{C}}$-triple $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ is nothing but the triple $(\widetilde{\mathcal{H}}'', \widetilde{\mathcal{H}}', \mathfrak{s}^*)$, we have

$$T^*(-w) = (z^{-w}\widetilde{\mathcal{H}}'', z^w\widetilde{\mathcal{H}}', \mathfrak{s}^*),$$

and a morphism $\mathrm{S} : T \to T^*(-w)$ is nothing but a pair $(\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'')$, with $\widetilde{\mathcal{S}}' : \widetilde{\mathcal{H}}' \to z^{-w}\widetilde{\mathcal{H}}''$ and $\widetilde{\mathcal{S}}'' : z^w\widetilde{\mathcal{H}}' \to \widetilde{\mathcal{H}}''$ satisfying the compatibility property with $\mathfrak{s}$ and $\mathfrak{s}^*$, that is, for every $v_1', v_2' \in \mathcal{H}'$,

$$\mathfrak{s}(v_1', \overline{\mathcal{S}''v_2'}) = \mathfrak{s}^*(\mathcal{S}'v_1', \overline{v_2'}) =: \overline{\mathfrak{s}(v_2', \overline{\mathcal{S}'v_1'})}.$$

That S is Hermitian, i.e., $\mathrm{S}^* = \mathrm{S}$, means $\widetilde{\mathcal{S}}'' = z^w\widetilde{\mathcal{S}}'$. In other words, considering morphisms of filtered vector spaces, we have

$$\mathcal{S}' = \mathcal{S}'' : (\mathcal{H}', F^\bullet\mathcal{H}') \xrightarrow{\sim} (\mathcal{H}'', F^\bullet\mathcal{H}'')(-w). \tag{5.2.10}$$

As a consequence of $\mathcal{S}' = \mathcal{S}''$, the compatibility property reads

$$\mathfrak{s}(v_1', \overline{\mathcal{S}'v_2'}) = \overline{\mathfrak{s}(v_2', \overline{\mathcal{S}'v_1'})}. \tag{5.2.11}$$

This is equivalent to the property that the pairing $\mathcal{S}$ of $\mathbb{C}$-vector spaces defined by

$$\mathcal{S}(\bullet, \overline{\bullet}) := \mathfrak{s}(\bullet, \overline{\mathcal{S}'\bullet}) : \mathcal{H}' \otimes \overline{\mathcal{H}'} \longrightarrow \mathbb{C} \tag{5.2.12}$$

is Hermitian in the usual sense. We call $(\widetilde{\mathcal{H}}', \mathcal{S})$ the *Hermitian pair* attached to the pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S) of weight w. Note that the weight w does not appear in the definition of a Hermitian pair. In fact, a Hermitian pair can give rise to a pre-polarized $\widetilde{\mathbb{C}}$-triple of any weight, as a consequence of the lemma below.

5.2.13. Lemma. *A pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S) of weight w is isomorphic to the pre-polarized $\widetilde{\mathbb{C}}$-triple $((\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(w), \mathcal{S}), (\mathrm{Id}, \mathrm{Id}))$ of weight w. Two pre-polarized $\widetilde{\mathbb{C}}$-triples of the same weight w are isomorphic if and only if their associated Hermitian pairs are isomorphic.*

Proof. The second part follows from the first one. Let $\mathrm{S} = (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'' = \widetilde{\mathcal{S}}')$ be a pre-polarization of T of weight w. Then $(\mathrm{Id}, \widetilde{\mathcal{S}}')$ is an isomorphism $T \xrightarrow{\sim} (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(w), \mathfrak{s}')$ with $\mathfrak{s}'(\bullet, \overline{\bullet}) = \mathfrak{s}(\bullet, \overline{\mathcal{S}'\bullet}) = \mathcal{S}$. ☐

We can now give the definition of a polarized $\mathbb{C}$-Hodge triple.

***5.2.14. Definition* (Polarization of a $\mathbb{C}$-Hodge triple).** Let H be a $\mathbb{C}$-Hodge triple of weight w. A *polarization of H* is a pre-polarization $\mathrm{S} = (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'')$ of weight w of the underlying $\widetilde{\mathbb{C}}$-triple such that the associated filtered Hermitian pair $((\mathcal{H}, F^\bullet\mathcal{H}), \mathcal{S})$, with $(\mathcal{H}, F^\bullet\mathcal{H}) = (\mathcal{H}', F^\bullet\mathcal{H}')$ and $\mathcal{S}$ defined by (5.2.12), is a polarized Hodge structure of weight w in the sense of Section 2.5.17.

5.2.15. Tate twist of a Hermitian pair. The isomorphisms of Lemma 5.2.13 behave well with respect to Tate twist, that is, for a pre-polarized $\widetilde{\mathbb{C}}$-triple (T, S), we have

$$(T,\mathrm{S})(\ell) \simeq ((\widetilde{\mathcal{H}}'(\ell), \widetilde{\mathcal{H}}'(w-\ell), (-1)^\ell \mathcal{S}), (\mathrm{Id}, \mathrm{Id})),$$

and Tate twist reads as follows on the associated Hermitian pair $(\widetilde{\mathcal{H}}', \mathcal{S})$:

$$(\widetilde{\mathcal{H}}', \mathcal{S})(\ell) = (\widetilde{\mathcal{H}}'(\ell), (-1)^\ell \mathcal{S}).$$

If $(H, \mathrm{S}) = ((\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s}), \mathrm{S})$ is a polarized $\mathbb{C}$-Hodge triple of weight w, the pair $(H,\mathrm{S})(\ell) := (H(\ell), (-1)^\ell \mathrm{S})$ is a polarized $\mathbb{C}$-Hodge triple of weight $w - 2\ell$.

The relation with polarized $\mathbb{C}$-Hodge structures in the form of Hodge-Hermitian pairs (see Section 2.5.17) can now be expressed in a simpler way.

5.2.16. Proposition. *Let $T = (\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'', \mathfrak{s})$ be an object of $\widetilde{\mathbb{C}}$-Triples. It is a polarizable $\mathbb{C}$-Hodge triple of weight w if and only if it is isomorphic (in $\widetilde{\mathbb{C}}$-Triples) to the object $(\widetilde{\mathcal{H}}', \widetilde{\mathcal{H}}'(w), \mathfrak{s}')$ for some suitable $\mathfrak{s}'$, such that $\widetilde{\mathcal{H}}'$ is strict and the corresponding filtered Hermitian pair $((\mathcal{H}, F^\bullet\mathcal{H}), \mathcal{S}) := ((\mathcal{H}', F^\bullet\mathcal{H}'), \mathfrak{s}')$ is a polarized Hodge structure of weight w (in particular, $\mathfrak{s}'$ is Hermitian).*

Proof. The "only if" part directly follows from Lemma 5.2.13 and the definition. Conversely, given a polarized Hodge structure $((\mathcal{H}, F^\bullet\mathcal{H}), \mathcal{S})$ of weight w, one checks that $((\mathcal{H}, F^\bullet\mathcal{H}), (\mathcal{H}, F[w]^\bullet\mathcal{H}), \mathfrak{s}' := \mathcal{S})$ is a $\mathbb{C}$-Hodge triple of weight w and that $(\mathrm{Id}, \mathrm{Id})$ is a polarization of it. If

$$\varphi = (\widetilde{\varphi}', \widetilde{\varphi}'') : T \xrightarrow{\sim} ((\mathcal{H}, F^\bullet\mathcal{H}), (\mathcal{H}, F[w]^\bullet\mathcal{H}), \mathfrak{s}')$$

is an isomorphism in $\widetilde{\mathbb{C}}$-Triples, then T is a $\mathbb{C}$-Hodge triple and, setting $\widetilde{\mathrm{S}}' := \widetilde{\varphi}''^{-1}\widetilde{\varphi}'$, $\mathrm{S} := (\widetilde{\mathrm{S}}', \widetilde{\mathrm{S}}'' = \widetilde{\mathrm{S}}')$ is a polarization of T. ∎

5.2.17. Remark **(Two points of view on (pre-)polarized triples)**

The sesquilinear pairing $\mathfrak{s}$ is constitutive of the notion of a triple and is only used to reflect the oppositeness of filtrations (with no positivity involved). On the other hand, a (pre-)polarization can be regarded as a "sesquilinear pairing on triples". We thus have two distinct roles for a sesquilinear pairing, that we also distinguish with the notation.

Lemma 5.2.13 helps us to simplify the setting, by reducing the polarization S to identity, and transferring the positivity property to the sesquilinear pairing of the triple. There remains only one sesquilinear pairing involved.

While we can simplify in that way the presentation of *polarized* triples, we still have to keep the ordinary sesquilinear pairing $\mathfrak{s}$ for *polarizable* triples.

5.3. Hodge-Lefschetz triples

We now make explicit the notion of Hodge-Lefschetz structures, and $\mathfrak{sl}_2$-Hodge structure (Definitions 3.3.3, 3.4.3 and 3.2.7) in the language of $\widetilde{\mathbb{C}}$-triples of Section 5.2.

5.3.a. Lefschetz triples. The abelian category A is that of $\widetilde{\mathbb{C}}$-triples with its automorphism σ (see Section 3.3.4) given by the Tate twist (1). Let $H=(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}'',\mathfrak{s})$ be a $\widetilde{\mathbb{C}}$-triple. Recall that $(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}'',\mathfrak{s})(-1)=(\widetilde{\mathcal{H}}'(-1),\widetilde{\mathcal{H}}''(1),\mathfrak{s})$. Note also that giving a morphism $\widetilde{\mathcal{H}}''(1)\to\widetilde{\mathcal{H}}''$ is equivalent to giving a morphism $\widetilde{\mathcal{H}}''\to\widetilde{\mathcal{H}}''(-1)$.

Assume that $H=(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}'',\mathfrak{s})$ is equipped with a nilpotent endomorphism $\mathrm{N}=(\mathrm{N}',\mathrm{N}''):H\to H(-1)$, that is,

$$\mathrm{N}':\widetilde{\mathcal{H}}'\longrightarrow\widetilde{\mathcal{H}}'(-1)\qquad\text{and}\qquad \mathrm{N}'':\widetilde{\mathcal{H}}''(1)\longrightarrow\widetilde{\mathcal{H}}''$$

which also reads, when H is *strict*,

$$\mathrm{N}':(\mathcal{H}',F^\bullet\mathcal{H}')\longrightarrow(\mathcal{H}',F[-1]^\bullet\mathcal{H}')\quad\text{and}\quad \mathrm{N}'':(\mathcal{H}'',F[1]^\bullet\mathcal{H}'')\longrightarrow(\mathcal{H}'',F^\bullet\mathcal{H}''),$$

are two nilpotent morphisms which satisfy, when forgetting the filtration,

$$\mathfrak{s}(v',\overline{\mathrm{N}''v''})=\mathfrak{s}(\mathrm{N}'v',\overline{v''}),\quad v'\in\mathcal{H}',\ v''\in\mathcal{H}''. \tag{5.3.1}$$

***5.3.2. Definition* (Hermitian dual of (H,N)).** The Hermitian dual $(H,\mathrm{N})^*$ of (H,N) is (H^*,N^*), where H^* is the Hermitian dual of H and N^* is the Hermitian adjoint of the morphism N, regarded as a morphism $H^*\to H^*(-1)$.

In other words, $H^*=(\widetilde{\mathcal{H}}'',\widetilde{\mathcal{H}}',\mathfrak{s}^*)$ and $\mathrm{N}^*=(\mathrm{N}'',\mathrm{N}')$. The monodromy filtration is defined in the abelian category $\widetilde{\mathbb{C}}$-$\mathsf{Triples}$. Let us make it explicit. The monodromy filtration $\mathrm{M}(\mathrm{N}')_\bullet\widetilde{\mathcal{H}}'$ exists in the abelian category of graded $\widetilde{\mathbb{C}}$-modules, as well as $\mathrm{M}(\mathrm{N}'')_\bullet\widetilde{\mathcal{H}}''$. By restricting to $z=1$, it induces the monodromy filtration of $(\mathcal{H}',\mathrm{N}')$ resp. $(\mathcal{H}'',\mathrm{N}'')$. We note that, according to (5.3.1), $\mathfrak{s}$ induces zero on $\mathrm{M}_\ell\mathcal{H}'\otimes\overline{\mathrm{M}_{-\ell-1}\mathcal{H}''}$ for every ℓ, hence induces a sesquilinear pairing

$$\mathfrak{s}:\mathrm{M}_\ell\mathcal{H}'\otimes\overline{\mathcal{H}''/\mathrm{M}_{-\ell-1}\mathcal{H}''}\longrightarrow\mathbb{C}.$$

We then have

$$\mathrm{M}_\ell T=(\mathrm{M}_\ell\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''/\mathrm{M}_{-\ell-1}\widetilde{\mathcal{H}}'',\mathfrak{s}).$$

Let us also consider the induced pairing

$$\mathfrak{s}_{\ell,-\ell}:\mathrm{gr}^{\mathrm{M}}_\ell\mathcal{H}'\otimes_{\widetilde{\mathbb{C}}}\overline{\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathcal{H}''}\longrightarrow\mathbb{C}.$$

Then

$$\mathrm{gr}^{\mathrm{M}}_\ell H=(\mathrm{gr}^{\mathrm{M}}_\ell\widetilde{\mathcal{H}}',\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{H}}'',\mathfrak{s}_{\ell,-\ell}).$$

On the other hand, $\mathrm{grN}:=(\mathrm{grN}',\mathrm{grN}'')$ induces a morphism

$$\begin{aligned}\mathrm{gr}^{\mathrm{M}}_\ell H&=(\mathrm{gr}^{\mathrm{M}}_\ell\widetilde{\mathcal{H}}',\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{H}}'',\mathfrak{s}_{\ell,-\ell})\\&\longrightarrow(\mathrm{gr}^{\mathrm{M}}_{\ell-2}\widetilde{\mathcal{H}}'(-1),\mathrm{gr}^{\mathrm{M}}_{-\ell+2}\widetilde{\mathcal{H}}''(1),\mathfrak{s}_{\ell-2,-\ell+2})=\mathrm{gr}^{\mathrm{M}}_{\ell-2}H(-1).\end{aligned}$$

5.3.1. ***Pre-polarization of weight*** w. A pre-polarization S of weight w of a Lefschetz triple (H,N) is a Hermitian isomorphism $(H,\mathrm{N}) \xrightarrow{\sim} (H,\mathrm{N})^*(-w)$, i.e., an isomorphism $H \xrightarrow{\sim} H^*(-w)$ such that

$$\mathrm{S}\circ\mathrm{N} = \mathrm{N}^*\circ\mathrm{S} : H \longrightarrow H^*(-w-1).$$

More explicitly, setting $\mathrm{S} = (\widetilde{\mathcal{S}}', \widetilde{\mathcal{S}}'')$, we have $\widetilde{\mathcal{S}}'' = z^w\widetilde{\mathcal{S}}'$ and

$$\widetilde{\mathcal{S}}'\circ\mathrm{N}' = \mathrm{N}''\circ\widetilde{\mathcal{S}}' : \widetilde{\mathcal{H}}' \longrightarrow \widetilde{\mathcal{H}}''(-w-1).$$

In particular, the associated sesquilinear pairing $\mathcal{S}(\bullet,\overline{\bullet}) = \mathfrak{s}(\bullet,\overline{\mathcal{S}'\bullet})$ on $\mathcal{H}'$ (see (5.2.12)) satisfies

$$\mathcal{S}(\mathrm{N}'\bullet,\overline{\bullet}) = \mathcal{S}(\bullet,\overline{\mathrm{N}'\bullet})$$

because

$$\mathfrak{s}(\bullet,\overline{\mathcal{S}'\mathrm{N}'\bullet}) = \mathfrak{s}(\bullet,\overline{\mathrm{N}''\mathcal{S}'\bullet}) = \mathfrak{s}(\mathrm{N}'\bullet,\overline{\mathcal{S}'\bullet}).$$

Since S is a morphism, it is compatible with the monodromy filtrations and S induces a pre-polarization

$$\mathrm{gr}^{\mathrm{M}}\mathrm{S} : \mathrm{gr}^{\mathrm{M}}_{\bullet} H \longrightarrow \mathrm{gr}^{\mathrm{M}}_{\bullet}(H^*)(-w) = (\mathrm{gr}^{\mathrm{M}}_{-\bullet}H)^*(-w)$$

of the graded Lefschetz triple $(\mathrm{gr}^{\mathrm{M}}_{\bullet}H, \mathrm{grN})$.

5.3.b. Hodge-Lefschetz triples. Let (H,N) be a Lefschetz triple. We say that (H,N) is a Hodge-Lefschetz triple with central weight w if $\mathrm{gr}^{\mathrm{M}}_{\ell}H$ is a Hodge triple of weight $w+\ell$ for every ℓ. In such a case, for every $j,k\in\mathbb{Z}$,

$$(H,\mathrm{N})(j,k) := ((\widetilde{\mathcal{H}}'(j),\mathrm{N}'),(\widetilde{\mathcal{H}}''(-k),\mathrm{N}''),\mathfrak{s})$$

is a Hodge-Lefschetz triple with central weight $w-(k+\ell)$ and $(H,\mathrm{N})^*$ is a Hodge-Lefschetz triple with central weight $-w$, with monodromy filtration satisfying $\mathrm{gr}^{\mathrm{M}}_{\ell}(H^*) = (\mathrm{gr}^{\mathrm{M}}_{-\ell}H)^*$.

Therefore, the data $(\mathrm{gr}^{\mathrm{M}}_{\bullet}H, \mathrm{grN})$ defined as

$$\bigoplus_{\ell}(\mathrm{gr}^{\mathrm{M}}_{\ell}\widetilde{\mathcal{H}}', \mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathcal{H}}'', \mathfrak{s}_{\ell,-\ell}), \qquad \mathrm{grN} := (\mathrm{grN}', \mathrm{grN}'')$$

form an $\mathfrak{sl}_2$-Hodge triple. In particular, each $\mathfrak{s}_{\ell,-\ell}$ is non-degenerate, which implies that $\mathfrak{s}$ itself is non-degenerate. Its Hermitian dual $(\mathrm{gr}^{\mathrm{M}}_{\bullet}H, \mathrm{grN})^*$ is also an $\mathfrak{sl}_2$-Hodge triple.

5.3.3. Remark **(Stability by extension).** We consider the abelian category of graded $\widetilde{\mathbb{C}}$-triples $H = \bigoplus_{\ell} H_{\ell}$ equipped with a nilpotent endomorphism $\mathrm{N} : H_{\ell} \to H_{\ell-2}(-1)$. Let

$$0 \longrightarrow (H_1,\mathrm{N}_1) \longrightarrow (H,\mathrm{N}) \longrightarrow (H_2,\mathrm{N}_2) \longrightarrow 0$$

be an exact sequence in this category. Assume that $(H_1,\mathrm{N}_1),(H_2,\mathrm{N}_2)$ are $\mathfrak{sl}_2$-Hodge triples of the same weight w. Then (H,N) is of the same kind. Indeed, by Exercise 5.7, each summand H_{ℓ} is a $\mathbb{C}$-Hodge triple of weight $w+\ell$. It is then clear that N^{ℓ} is an isomorphism $H_{\ell} \xrightarrow{\sim} H_{-\ell}(-\ell)$ if this holds on H_1,H_2.

5.3.4. Polarization. Let (H, N) be a Hodge-Lefschetz triple of weight w. By a *polarization* S of (H, N) we mean a pre-polarization of weight w of the Lefschetz triple (Section 5.3.1) which satisfies the properties as in Definition 3.4.14.

Last, we remark as in Proposition 5.2.16 that any polarized Hodge-Lefschetz triple with central weight w is isomorphic to $((\widetilde{\mathcal{H}}, \widetilde{\mathcal{H}}(w), \mathcal{S}), \mathrm{N})$ for a suitable polarized Hodge-Lefschetz structure $((\mathcal{H}, F^\bullet\mathcal{H}), \mathrm{N}, \mathcal{S})$ with central weight w, as in Remark 3.4.16 (in particular, $\mathcal{S} : \mathcal{H} \to \mathcal{H}^*$ is Hermitian).

5.3.5. The polarized Hodge-Lefschetz triple **Im N.** Let $(H, \mathrm{N}, \mathrm{S})$ be a polarized Hodge-Lefschetz triple with central weight $w-1$. We have by definition a commutative diagram

$$\begin{CD} H @>{\mathrm{S}}>> H^*(-w+1) \\ @V{\mathrm{N}}VV @VV{\mathrm{N}^*}V \\ H(-1) @>{\mathrm{S}}>> H^*(-w) \end{CD}$$

Then $G = \operatorname{Im} \mathrm{N}$ equipped with the nilpotent endomorphism $\mathrm{N}_G = \mathrm{N}_{|\operatorname{Im}\mathrm{N}}$ is a Hodge-Lefschetz triple with central weight w polarized by S_G, which is the sesquilinear pairing

$$\mathrm{S}_G : \operatorname{Im}\mathrm{N} \longrightarrow (\operatorname{Im}\mathrm{N})^* = \operatorname{Coker}(\mathrm{N}^*)$$

induced by $-\mathrm{S}$. The image $\operatorname{Im}\mathrm{N} = G$ is expressed as follows:

$$\begin{aligned} (\mathcal{G}', F^\bullet\mathcal{G}') &= \big(\mathrm{N}'(\mathcal{H}'), \mathrm{N}'(F[-1]^\bullet\mathcal{H}') = F[-1]^\bullet\mathcal{H}' \cap \mathrm{N}'(\mathcal{H}')\big) \\ (\mathcal{G}'', F^\bullet\mathcal{G}'') &= \big((\mathcal{H}''/\operatorname{Ker}\mathrm{N}''), (F[1]^\bullet\mathcal{H}'')/(F[1]^\bullet\mathcal{H}'' \cap \operatorname{Ker}\mathrm{N}'')\big) \\ \mathfrak{s}_G &= \mathfrak{s}_{|\mathrm{N}'(\mathcal{H}')\otimes\overline{(\mathcal{H}''/\operatorname{Ker}\mathrm{N}'')}}, \end{aligned}$$

and $\mathrm{N}'_{\mathcal{G}}, \mathrm{N}''_{\mathcal{G}}$ are the naturally induced nilpotent endomorphisms. It can also be presented as a filtered Hermitian pair

$$\big((\mathcal{G}, F^\bullet\mathcal{G}), \mathrm{N}_{\mathcal{G}}, \mathcal{S}_{\mathcal{G}}\big)$$

obtained from the filtered Hermitian pair $\big((\mathcal{H}, F^\bullet\mathcal{H}), \mathrm{N}, \mathcal{S}\big)$ by setting

$$(\mathcal{G}, F^\bullet\mathcal{G}) = (\mathrm{N}(\mathcal{H}), F[-1]^\bullet\mathcal{H} \cap \mathrm{N}(\mathcal{H}))$$

with the induced action $\mathrm{N}_{\mathcal{G}}$ of N and by defining $\mathrm{S}_{\mathcal{G}}$ by (see Definition 3.2.12)

$$\mathcal{S}_{\mathcal{G}}(\mathrm{N}\bullet, \overline{\mathrm{N}\bullet}) = -\mathcal{S}(\mathrm{N}\bullet, \bar{\bullet}) = -\mathcal{S}(\bullet, \overline{\mathrm{N}\bullet}).$$

5.3.6. Polarized Hodge-Lefschetz quivers. The definition of a polarized Hodge-Lefschetz quiver in the setting of triples can be mimicked from that of a polarized Hodge-Lefschetz quiver of Section 3.4.d.

Given two Lefschetz quivers in the category of $\widetilde{\mathbb{C}}$-modules

$$\begin{tikzcd} \widetilde{\mathcal{H}}' \arrow[r, bend left, "\mathrm{c}'"] & \widetilde{\mathcal{G}}' \arrow[l, bend left, "\mathrm{v}'", "(-1)"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \widetilde{\mathcal{H}}'' \arrow[r, bend left, "\mathrm{c}''"] & \widetilde{\mathcal{G}}'' \arrow[l, bend left, "\mathrm{v}''", "(-1)"'] \end{tikzcd}$$

(notation of Remark 5.1.5) with $\mathrm{N}'=\mathrm{v}'\mathrm{c}'$, etc., and sesquilinear pairings $\mathfrak{s}_H,\mathfrak{s}_G$ giving rise to $\widetilde{\mathbb{C}}$-triples

$$H=(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''(-1),\mathfrak{s}_H)\quad\text{and}\quad G=(\widetilde{\mathcal{G}}',\widetilde{\mathcal{G}}'',\mathfrak{s}_G)$$

we can build up a Lefschetz quiver $(H,G,\mathrm{c},\mathrm{v})$ in the category $\widetilde{\mathbb{C}}$-Triples by setting

$$\begin{aligned}\mathrm{c}&=(\mathrm{c}',-\mathrm{v}''):(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''(-1),\mathfrak{s}_H)\longrightarrow(\widetilde{\mathcal{G}}',\widetilde{\mathcal{G}}'',\mathfrak{s}_G)\\ \mathrm{v}&=(\mathrm{v}',-\mathrm{c}''):(\widetilde{\mathcal{G}}',\widetilde{\mathcal{G}}'',\mathfrak{s}_G)\longrightarrow(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''(-1),\mathfrak{s}_H)(-1)=(\widetilde{\mathcal{H}}'(-1),\widetilde{\mathcal{H}}'',\mathfrak{s}_H),\end{aligned}\tag{5.3.7}$$

provided the the compatibility relations of $\mathrm{c}',\dots,\mathrm{v}''$ with the sesquilinear pairings $\mathfrak{s}_H,\mathfrak{s}_G$ hold, that is,

$$\begin{aligned}\mathfrak{s}_G(\mathrm{c}'x',\overline{y''})&=-\mathfrak{s}_H(x',\overline{\mathrm{v}''y''})\quad x'\in\mathcal{H}',\ y''\in\mathcal{G}'',\\ \mathfrak{s}_G(\mathrm{v}'y',\overline{x''})&=-\mathfrak{s}_H(y',\overline{\mathrm{c}''x''})\quad x''\in\mathcal{H}'',\ y'\in\mathcal{G}'.\end{aligned}\tag{5.3.8}$$

The choice of signs is made to ensure later compatibility with the signs occurring in Definition 3.4.19. The signs cancel out when defining $\mathrm{N}_H=\mathrm{vc}$ and $\mathrm{N}_G=\mathrm{cv}$. Furthermore, they are compatible with the definition of the Hermitian dual of a Lefschetz quiver given in Remark 3.2.13. Indeed, working now in the category of $\mathbb{C}$-vector spaces, and recalling that the Hermitian dual of $(\mathcal{H}'',\mathcal{G}'',\mathrm{c}'',\mathrm{v}'')$ is $(\mathcal{H}'',\mathcal{G}'',\mathrm{c}'',\mathrm{v}'')^*=(\mathcal{H}''^*,\mathcal{G}''^*,-\mathrm{v}'',-\mathrm{c}'')$, the relations (5.3.8) amount to the property that the pair $(\mathfrak{s}_H,\mathfrak{s}_G)$ is a morphism of Lefschetz quivers

$$(\mathfrak{s}_H,\mathfrak{s}_G):(\mathcal{H}',\mathcal{G}',\mathrm{c}',\mathrm{v}')\longrightarrow(\mathcal{H}'',\mathcal{G}'',\mathrm{c}'',\mathrm{v}'')^*.$$

One can check that, conversely, any Lefschetz quiver $(H,G,\mathrm{c},\mathrm{v})$ in the category $\widetilde{\mathbb{C}}$-Triples is obtained by the previous construction.

A Hodge-Lefschetz quiver with central weight w is defined as a Lefschetz quiver in $\widetilde{\mathbb{C}}$-Triples such that $(H,\mathrm{N}_H),(G,\mathrm{N}_G)$ are Hodge-Lefschetz triples with respective weights $w-1$ and w (Section 5.3.b), and $\mathrm{c}:(H,\mathrm{N}_H)\to(G,\mathrm{N}_G)$ and $\mathrm{v}:(G,\mathrm{N}_G)\to(H(-1),\mathrm{N}_H)$ are morphisms in the category of Hodge-Lefschetz triples, so that they are morphisms of mixed Hodge structures.

Defining the Hermitian dual $(H,G,\mathrm{c},\mathrm{v})^*=(H^*(1),G^*,-\mathrm{v}^*,-\mathrm{c}^*)$ as in Remark 3.2.13, a polarization of a Hodge-Lefschetz quiver $(H,G,\mathrm{c},\mathrm{v})$ is a pair of polarizations $\mathrm{S}=(\mathrm{S}_H,\mathrm{S}_G)$ of H and G respectively, defining an isomorphism

$$\mathrm{S}:(H,G,\mathrm{c},\mathrm{v})\xrightarrow{\sim}(H,G,\mathrm{c},\mathrm{v})^*(-w).\tag{5.3.9}$$

Assume now that H and G are presented as Hermitian pairs

$$H=(\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}'(-w+1),\mathcal{S}_H)\quad\text{and}\quad G=(\widetilde{\mathcal{G}}',\widetilde{\mathcal{G}}'(-w),\mathcal{S}_G),$$

so that $\mathrm{S}_H=(\mathrm{Id},\mathrm{Id})$ and $\mathrm{S}_G=(\mathrm{Id},\mathrm{Id})$, and $(\widetilde{\mathcal{H}}',\mathcal{S}_H),(\widetilde{\mathcal{G}}',\mathcal{S}_G)$ are polarized Hodge structures of respective weights $w-1,w$ in the sense of Section 2.5.17. In this setting, (5.3.9) implies that $\mathrm{c}=-\mathrm{v}^*$, equivalently $\mathrm{v}=-\mathrm{c}^*$, so that, by their definition (5.3.7), we obtain $\mathrm{c}'=\mathrm{c}''$ and $\mathrm{v}'=\mathrm{v}''$. Therefore, (5.3.8) reads

$$\mathcal{S}_G(\mathrm{c}'x',\overline{y'})=-\mathcal{S}_H(x',\overline{\mathrm{v}'y'})\quad\text{and}\quad\mathcal{S}_G(\mathrm{v}'y',\overline{x'})=-\mathcal{S}_H(y',\overline{\mathrm{c}'x'}).$$

5.4. Variations of Hodge triple

5.4.a. Variations of Hodge structure as triples. We now revisit the notion of variation of Hodge structure, by the using the language of $\widetilde{\mathbb{C}}$-triples of Section 5.2. It enables us to keep holomorphy for both filtrations, by putting the non-holomorphic behaviour in the sesquilinear pairing $\mathfrak{s}$. This approach will be convenient in presence of singularities.

When working with a pairing $\mathfrak{s}$, we start by introducing a larger category, which can be enlarged to an abelian category.

We denote by $\widetilde{\mathcal{O}}_X$ the sheaf of graded rings $\mathcal{O}_X[z]$ and by an $\widetilde{\mathcal{O}}_X$-module we mean a *graded* $\mathcal{O}_X[z]$-module. By a locally free $\widetilde{\mathcal{O}}_X$-module of rank $r<\infty$ we mean an $\widetilde{\mathcal{O}}_X$-module locally isomorphic to the direct sum (see Exercise 5.4)

$$\bigoplus_{i=1}^{r}\widetilde{\mathcal{O}}_X(k_i)\quad (k_i\in\mathbb{Z}).$$

We replace the filtered flat bundles $(\mathcal{H}',\nabla,F^\bullet\mathcal{H}')$ and $(\mathcal{H}'',\nabla,F^\bullet\mathcal{H}'')$ by graded $\widetilde{\mathcal{O}}_X$-modules with a flat z-connection (see Section 5.1.c). (This point of view will be expanded in Section 8.1.)

5.4.1. Definition **(Flat $\widetilde{\mathcal{O}}$-triples with pairing $\mathfrak{s}$).** A *flat $\widetilde{\mathcal{O}}$-triple* on X consists of the data of

- a pair of $\widetilde{\mathcal{O}}_X$-modules $\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''$ equipped with a flat z-connection $\widetilde{\nabla}$,
- a flat $\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}$-linear morphism $\mathfrak{s}:\mathcal{H}'\otimes_{\mathbb{C}}\overline{\mathcal{H}''}\to\mathcal{C}^\infty_X$, i.e., for local holomorphic sections m',m'' of $\mathcal{H}',\mathcal{H}''$, we have

$$\begin{aligned}\partial\mathfrak{s}(m',\overline{m''})&=\mathfrak{s}(\nabla m',\overline{m''}),\\ \overline{\partial}\mathfrak{s}(m',\overline{m''})&=\mathfrak{s}(m',\overline{\nabla m''}).\end{aligned}$$

5.4.2. Remark **(Flatness of $\mathfrak{s}$).** The restriction $\underline{\mathfrak{s}}$ of $\mathfrak{s}$ to the local system $\underline{\mathcal{H}}'\otimes_{\mathbb{C}}\overline{\underline{\mathcal{H}}''}$ takes values in the constant sheaf $\mathbb{C}_X$ since for local sections m' of $\underline{\mathcal{H}}'$ and m'' of $\underline{\mathcal{H}}''$, we have, by the previous formulas, $\partial\mathfrak{s}(m',\overline{m''})=\overline{\partial}\mathfrak{s}(m',\overline{m''})=0$. Moreover, we can recover $\mathfrak{s}$ from its restriction $\underline{\mathfrak{s}}$ by $\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}$-linearity. As a consequence, we see that if X is connected, $\mathfrak{s}$ is non-degenerate if and only if its restriction at some point $x\in X$ is a non-degenerate pairing $\mathcal{H}'_x\otimes_{\mathbb{C}}\overline{\mathcal{H}''_x}\to\mathbb{C}$, since this obviously holds for $\underline{\mathfrak{s}}$.

5.4.3. Definition **(Variation of $\mathbb{C}$-Hodge structure, third definition)**

A variation of $\mathbb{C}$-Hodge structure of weight w is a flat $\widetilde{\mathcal{O}}_X$-triple

$$H=((\widetilde{\mathcal{H}}',\widetilde{\nabla}),(\widetilde{\mathcal{H}}'',\widetilde{\nabla}),\mathfrak{s})$$

such that $\widetilde{\mathcal{H}}',\widetilde{\mathcal{H}}''$ are $\widetilde{\mathcal{O}}_X$-locally free of finite rank and whose restriction $H_x=(\widetilde{\mathcal{H}}'_x,\widetilde{\mathcal{H}}''_x,\mathfrak{s}_x)$ at each $x\in X$ is a $\mathbb{C}$-Hodge triple of weight w. In particular, $\mathfrak{s}$ is non-degenerate.

A *polarization* is a *flat* morphism $\mathrm{S}:H\to H^*(-w)$ inducing a polarization at each $x\in X$. Equivalently (see Section 2.5.17), a polarized variation of $\mathbb{C}$-Hodge structure of weight w consists of the data $((\widetilde{\mathcal{H}},\widetilde{\nabla}),\mathcal{S})$, where $\mathcal{S}$ is a flat sesquilinear pairing on $(\mathcal{H},\nabla)$, inducing a polarized $\mathbb{C}$-Hodge structure at every $x\in X$.

5.4.4. Example. The triple

$$ {}_{\mathrm{T}}\mathcal{O}_X := ((\widetilde{\mathcal{O}}_X, \widetilde{\mathrm{d}}), (\widetilde{\mathcal{O}}_X, \widetilde{\mathrm{d}}), \mathfrak{s}_n), \qquad \mathfrak{s}_n(1,1) := 1, $$

is a variation of $\mathbb{C}$-Hodge triple of weight 0. It is polarized by $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$. The associated Hodge-Hermitian pair is $(\widetilde{\mathcal{O}}_X, \mathfrak{s}_n)$.

5.4.5. Remarks.

(1) One can also define the category $\mathsf{VHS}(X, \mathbb{C}, w)$ as the full subcategory of that of filtered flat triples whose objects are variations of $\mathbb{C}$-Hodge structures of weight w on X. The category $\mathsf{pVHS}(X, \mathbb{C}, w)$ of polarizable objects is defined correspondingly.

The category VHS can be naturally equipped with the operations Hom, tensor product, duality, and conjugation. The full subcategory pVHS is stable by these operations, since the polarization can be constructed in a natural way in each of these operations (see Section 2.5.18).

(2) Let $f : X \to Y$ be a holomorphic map between smooth complex manifolds. The pullback ${}_{\mathrm{T}}f^*H$ of a triple H is defined as $(f^*(\widetilde{\mathcal{H}}', \widetilde{\nabla}), f^*(\widetilde{\mathcal{H}}'', \widetilde{\nabla}), f^*\mathfrak{s})$, where $f^*\mathfrak{s} : f^*\mathcal{H}' \otimes \overline{f^*\mathcal{H}''} \to \mathcal{C}_X^\infty$ is defined by $f^*\mathfrak{s}(1 \otimes m', \overline{1 \otimes m''}) := \mathfrak{s}(m', \overline{m''}) \circ f$. If H is a variation of $\mathbb{C}$-Hodge triple of weight w, then so is ${}_{\mathrm{T}}f^*H$.

5.4.b. Smooth $\mathbb{C}$-Hodge triples. Let us now introduce a different normalization of the objects, in order to fit with the notion of polarizable Hodge module developed in Chapters 7 and 14. Recall that we set $n = \dim X$.

***5.4.6. Definition* (The polarized $\mathbb{C}$-Hodge triple ${}_{\mathrm{H}}\mathcal{O}_X$).** We denote by ${}_{\mathrm{H}}\mathcal{O}_X$ the triple ${}_{\mathrm{T}}\mathcal{O}_X(0, n)$ (note the half-twist), that is,

$$ {}_{\mathrm{H}}\mathcal{O}_X = ((\widetilde{\mathcal{O}}_X, \widetilde{\mathrm{d}}), (\widetilde{\mathcal{O}}_X(n), \widetilde{\mathrm{d}}), \mathfrak{s}_n), \qquad \mathfrak{s}_n(1,1) := 1. $$

It is a $\mathbb{C}$-Hodge triple of weight $n = \dim X$ with polarization

$$ {}_{\mathrm{H}}\mathrm{S} = (\mathrm{Id}, \mathrm{Id}) : {}_{\mathrm{H}}\mathcal{O}_X \longrightarrow {}_{\mathrm{H}}\mathcal{O}_X(-n). $$

The associated Hermitian pair is $((\mathcal{O}_X, F^\bullet\mathcal{O}_X), \mathrm{d}), \mathfrak{s}_n)$ where $F^0\mathcal{O}_X = \mathcal{O}_X$ and $F^1\mathcal{O}_X = 0$.

A smooth $\mathbb{C}$-Hodge triple of weight w is defined to be a variation of Hodge triple of weight $w - \dim X$ (Definition 5.4.3) twisted by ${}_{\mathrm{H}}\mathcal{O}_X$.

***5.4.7. Definition* (Smooth $\mathbb{C}$-Hodge triples).**

(1) A smooth $\mathbb{C}$-Hodge triple of weight w is a triple

$$ {}_{\mathrm{H}}H := ((\widetilde{\mathcal{H}}', \widetilde{\nabla}), (\widetilde{\mathcal{H}}'', \widetilde{\nabla}), \mathfrak{s}), $$

such that the triple

$$ H = {}_{\mathrm{H}}H(0, -n) $$

is a variation of $\mathbb{C}$-Hodge structure of weight $w - n$ on X.

(2) A polarization ${}_{\mathrm{H}}\mathrm{S}$ of ${}_{\mathrm{H}}H$ is a Hermitian morphism ${}_{\mathrm{H}}\mathrm{S} : {}_{\mathrm{H}}H \to {}_{\mathrm{H}}H^*(-w)$ such that, when regarded as a morphism $H \to H^*(-(w-n))$, $\mathrm{S} := {}_{\mathrm{H}}\mathrm{S}$ is a polarization of H.

(3) A smooth polarized $\mathbb{C}$-Hodge triple of weight w on X consists of the data $((\widetilde{\mathcal{H}},\widetilde{\nabla}),{}_{\mathrm{H}}\mathcal{S})$, where ${}_{\mathrm{H}}\mathcal{S}$ is a non-degenerate Hermitian pairing on $\mathcal{H}$ and ${}_{\mathrm{H}}H := ((\widetilde{\mathcal{H}},\widetilde{\nabla}),(\widetilde{\mathcal{H}},\widetilde{\nabla})(w),{}_{\mathrm{H}}\mathcal{S})$ is a smooth $\mathbb{C}$-Hodge triple of weight w polarized by ${}_{\mathrm{H}}\mathrm{S} = (\mathrm{Id},\mathrm{Id})$. Tate twist reads $((\widetilde{\mathcal{H}},\widetilde{\nabla}),{}_{\mathrm{H}}\mathcal{S})(\ell) = ((\widetilde{\mathcal{H}}(\ell),\widetilde{\nabla}),(-1)^{\ell}{}_{\mathrm{H}}\mathcal{S})$.

***5.4.8. Definition* (Pullback of a smooth $\mathbb{C}$-Hodge triple).** Let $f : X \to Y$ be a holomorphic map between smooth manifolds of relative dimension $p = n - m$, and let ${}_{\mathrm{H}}H = ((\widetilde{\mathcal{H}}',\widetilde{\nabla}),(\widetilde{\mathcal{H}}'',\widetilde{\nabla}),\mathfrak{s})$ be a smooth $\mathbb{C}$-Hodge triple of weight w on Y. The pullback ${}_{\mathrm{H}}f^*{}_{\mathrm{H}}H$ is the triple defined as

$$ {}_{\mathrm{H}}f^*{}_{\mathrm{H}}H := (f^*(\widetilde{\mathcal{H}}',\widetilde{\nabla}), f^*(\widetilde{\mathcal{H}}'',\widetilde{\nabla})(p), f^*\mathfrak{s}). $$

Since ${}_{\mathrm{H}}f^*{}_{\mathrm{H}}\mathcal{O}_Y = {}_{\mathrm{H}}\mathcal{O}_X$, we see that ${}_{\mathrm{H}}f^*{}_{\mathrm{H}}H = {}_{\mathrm{H}}({}_{\mathrm{T}}f^*H)$ is a smooth $\mathbb{C}$-Hodge triple of weight $w+p$. Moreover, the pullback ${}_{\mathrm{H}}f^*{}_{\mathrm{H}}\mathrm{S} := f^*{}_{\mathrm{H}}\mathrm{S}$ of a polarization is a polarization. Correspondingly, the pullback of a smooth polarized $\mathbb{C}$-Hodge triple $((\widetilde{\mathcal{H}},\widetilde{\nabla}),{}_{\mathrm{H}}\mathcal{S})$ of weight w is the smooth polarized $\mathbb{C}$-Hodge triple $(f^*(\widetilde{\mathcal{H}},\widetilde{\nabla}),f^*\mathcal{S})$ of weight $w+p$.

***5.4.9. Remark* (on the symmetry breaking).** Definition 5.4.7 clearly breaks the symmetry between the "prime" (or holomorphic) part and the "double prime" (or anti-holomorphic) part of a triple, in order to obtain a formalism of weights similar to that of the theory of mixed Hodge modules of M. Saito. Similarly, the definition of the pullback functor is not symmetric, and the same will occur for other functors in Chapter 12. However, *pre-polarized* triples can be reduced to Hermitian pairs, for which the problem disappears since the behaviour of weights by functors is reflected by simply changing the sign of the pre-polarization.

5.5. Exercises

Exercise 5.1. Show the following properties in $\mathsf{Mod}(\widetilde{\mathcal{A}})$ or in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$.

(1) A subobject of a strict object is strict.

(2) An extension in of two strict objects is strict.

(3) A morphism between two strict objects is strict if and only if its cokernel is strict.

(4) A complex which consists of strict objects and which is bounded from above is a strict complex if and only if each differential is a strict morphism.

Exercise 5.2.

(1) If $\widetilde{\mathcal{M}}$ is a graded $\widetilde{\mathcal{A}}$-module, show that its $\widetilde{\mathbb{C}}$-torsion is also graded and each torsion element is annihilated by some power of z.

(2) Conclude that $(z-a) : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is injective for every $a \in \mathbb{C} \smallsetminus \{0\}$, equivalently that $\widetilde{\mathcal{M}}[z^{-1}] := \mathbb{C}[z,z^{-1}] \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}$ is $\mathbb{C}[z,z^{-1}]$-flat, and that a graded $\widetilde{\mathcal{A}}$-module is $\widetilde{\mathbb{C}}$-flat if and only if it has no z-torsion.

(3) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$. Assume that φ is injective. Show that the induced morphism $\varphi_a : \widetilde{\mathcal{M}}/(z-a)\widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}/(z-a)\widetilde{\mathcal{N}}$ is injective

- if $a \neq 0$,

- and also if $a=0$ provided that $\widetilde{\mathcal{M}}$ is strict.

[*Hint*: Use (2) for Coker φ.]

(4) Let $\widetilde{\mathcal{M}}^\bullet$ be a complex in $\mathsf{Modgr}(\widetilde{\mathcal{A}})$. Show that, for every i, there is a natural isomorphism

$$H^i(\widetilde{\mathcal{M}}^\bullet/(z-a)\widetilde{\mathcal{M}}^\bullet)\simeq H^i\widetilde{\mathcal{M}}^\bullet/(z-a)H^i\widetilde{\mathcal{M}}^\bullet,$$

- if $a\neq 0$,
- and also if $a=0$ provided that $\widetilde{\mathcal{M}}^\bullet$ is strict (see Definition 5.1.6(3)).

[*Hint*: Consider the long exact sequence

$$\cdots H^i\widetilde{\mathcal{M}}^\bullet\xrightarrow{\ z-a\ }H^i\widetilde{\mathcal{M}}^\bullet\longrightarrow H^i(\widetilde{\mathcal{M}}^\bullet/(z-a)\widetilde{\mathcal{M}}^\bullet)\longrightarrow\cdots$$

attached to the exact sequence of complexes (according to (3))

$$0\longrightarrow\widetilde{\mathcal{M}}^\bullet\xrightarrow{\ z-a\ }\widetilde{\mathcal{M}}^\bullet\longrightarrow\widetilde{\mathcal{M}}^\bullet/(z-a)\widetilde{\mathcal{M}}^\bullet\longrightarrow 0$$

and apply (3).]

(5) Recover the associated $\mathcal{A}$-module $\mathcal{M}$ of $\widetilde{\mathcal{M}}$ as $\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ and, if $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ is strict, $\mathrm{gr}^F\mathcal{M}$ as $R_F\mathcal{M}/zR_F\mathcal{M}$ (as a graded $\mathrm{gr}^F\mathcal{A}$-module).

(6) Let $\varphi:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ be a morphism in $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$ (i.e., between strict objects). Assume that $\varphi_{|z=1}$ is zero. Show that $\varphi=0$. Deduce the faithfulness of the restriction functor $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}\mapsto\mathsf{Mod}(\mathcal{A})$ given by $\widetilde{\mathcal{M}}\mapsto\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$.

(7) Let $\varphi:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ be a morphism in $\mathsf{Modgr}(\widetilde{\mathcal{A}})_{\mathrm{st}}$. Show that φ is strict if and only if the associated morphism $\varphi:(\mathcal{M},F_\bullet\mathcal{M})\to(\mathcal{N},F_\bullet\mathcal{N})$ is strict, i.e., satisfies $\varphi(F_k\mathcal{M})=\varphi(\mathcal{M})\cap F_k\mathcal{N}$ for every index k. [*Hint*: Use Exercise 5.1(3).]

Exercise 5.3. If $(\mathcal{M},F_\bullet\mathcal{M})$ is a filtered object of $\mathsf{Mod}(\mathcal{A})$, then a subobject $\mathcal{M}'$ of $\mathcal{M}$ carries the induced filtration $(F_p\mathcal{M}\cap\mathcal{M}')_{p\in\mathbb{Z}}$, while a quotient object $\mathcal{M}/\mathcal{M}''$ carries the induced filtration $((F_p\mathcal{M}+\mathcal{M}'')/\mathcal{M}'')_{p\in\mathbb{Z}}$. Show the following properties.

(1) $R_F\mathcal{M}'=R_F\mathcal{M}\cap\mathcal{M}'[z,z^{-1}]$ and $R_F(\mathcal{M}/\mathcal{M}'')=R_F\mathcal{M}\cap\mathcal{M}''[z,z^{-1}]/\mathcal{M}''[z,z^{-1}]$.

(2) The two possible induced filtrations on a subquotient $\mathcal{M}'\cap\mathcal{M}''/\mathcal{M}''$ of $\mathcal{M}$ agree.

(3) For every filtered complex $(\mathcal{M}^\bullet,F)$, the i-th cohomology of the complex is a subquotient of $\mathcal{M}^i$, hence it carries an induced filtration. Then there is a canonical morphism $H^i(F_p\mathcal{M}^\bullet)\to H^i(\mathcal{M}^\bullet)$, whose image is denoted by $F_pH^i(\mathcal{M}^\bullet)$.

***Exercise 5.4* (Locally free $\widetilde{\mathcal{O}}_X$-modules and filtrations by sub-bundles)**

Let $\widetilde{\mathcal{H}}$ be a locally free $\widetilde{\mathcal{O}}_X$-module of rank r. Show that the corresponding filtration $F^\bullet\mathcal{H}$ is a filtration by sub-bundles, i.e., $F^p\mathcal{H}/F^{p+1}\mathcal{H}$ is $\widetilde{\mathcal{O}}_X$-locally free for each $p\in\mathbb{Z}$. [*Hint*: reduce the statement to the case where $r=1$.]

Conversely, let $(\mathcal{H},F^\bullet\mathcal{H})$ be a filtered holomorphic bundle. Show that $\widetilde{\mathcal{H}}:=R_F\mathcal{H}$ is $\widetilde{\mathcal{O}}_X$-locally free. [*Hint*: use local bases of $F^p\mathcal{H}/F^{p+1}\mathcal{H}$ for each p.]

Exercise 5.5. We take the notation of Section 5.2.a.

(1) Let $F'^\bullet\mathcal{H}$ and $F''^\bullet\mathcal{H}$ be 0-opposite filtrations of $\mathcal{H}$ in the sense of Definition 2.5.1. Show that $\widetilde{\mathcal{H}}'\simeq\bigoplus_p\mathcal{H}^{p,-p}z^{-p}\mathbb{C}[z]$ (where the sum is finite), and similarly $\widetilde{\mathcal{H}}''\simeq\bigoplus_p\mathcal{H}^{p,-p}z^p\mathbb{C}[z^{-1}]$. Using that the gluing of $z^{-p}\mathbb{C}[z]$ with $z^p\mathbb{C}[z^{-1}]$ in $\mathbb{C}[z,z^{-1}]$

gives rise to the trivial bundle $\mathcal{O}_{\mathbb{P}^1}$, conclude that the bundle $\mathcal{F}$ of (c) in Section 5.2.a is isomorphic to the trivial bundle on $\mathbb{P}^1$.

(2) Argue similarly in weight w.

(3) In order to prove that Condition (c) on $\mathcal{F}$ in Section 5.2.a implies oppositeness, reduce first to the case where $w = 0$ by identifying the effect of tensoring with $\mathcal{O}_{\mathbb{P}^1}(-w)$ with a shift of one filtration.

(4) Assume that $\mathcal{F}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}^{\dim \mathcal{H}}$. Show that the two filtrations giving rise to $\mathcal{F}$ are opposite.

Exercise 5.6.

(1) (Another proof of 2.5.5(2)) Show that a morphism in $\mathsf{HS}(\mathbb{C})$ induces a morphism between the associated vector bundles on $\mathbb{P}^1$ (see Section 5.2.a). Conclude that there is no non-zero morphism if $w_1 > w_2$. [*Hint*: Use standard properties of vector bundles on $\mathbb{P}^1$.]

(2) Let H_1 and H_2 be objects of $\mathsf{HS}(\mathbb{C}, w)$, let $\mathcal{F}_1, \mathcal{F}_2$ be the associated $\mathcal{O}_{\mathbb{P}^1}$-modules (see Section 5.2.a) and let H be a bi-filtered vector space whose associated $\mathcal{O}_{\mathbb{P}^1}$-module $\mathcal{F}$ is an extension of $\mathcal{F}_1, \mathcal{F}_2$ in the category of $\mathcal{O}_{\mathbb{P}^1}$-modules. Show that H is an object of $\mathsf{HS}(\mathbb{C}, w)$. [*Hint*: Use standard properties of vector bundles on $\mathbb{P}^1$.]

***Exercise 5.7* (Stability by extension).** Let $0 \to T_1 \to T \to T_2 \to 0$ be a short exact sequence of $\widetilde{\mathbb{C}}$-triples. Show that, if T_1, T_2 are $\mathbb{C}$-Hodge triples of weight w, then so is T. [*Hint*: By using the interpretation (c) of w-oppositeness in Section5.2.a, reduce the question to showing that, if a locally free $\mathcal{O}_{\mathbb{P}^1}$-module is an extension of two trivial bundles $\mathcal{O}_{\mathbb{P}^1}^{d_1}$ and $\mathcal{O}_{\mathbb{P}^1}^{d_2}$, then it is itself a trivial bundle.]

Exercise 5.8. Show that the category $\mathsf{VHS}(X, \mathbb{C}, w)$ as defined by 5.4.3 is equivalent to $\mathsf{VHS}(X, \mathbb{C}, w)$ as defined by 4.1.4, and hence to $\mathsf{VHS}(X, \mathbb{C}, w)$ as defined by 4.1.5. Show a similar result for $\mathsf{pVHS}(X, \mathbb{C}, w)$.

5.6. Comments

The Rees construction for filtered objects, embedding the non-abelian category of filtered objects into the abelian category of modules over a ring, is a well-known trick to treat filtered objects. The main application has been the proof of the Artin-Rees lemma, that we will reproduce in the context of filtered $\mathcal{D}$-modules in Chapters 7, 8 and 9.

The notion of triple has been instrumental in defining generalizations of the categories of Hodge modules, called twistor $\mathcal{D}$-modules (see [**Sab05, Moc07, Moc11a**]. When working with the Hodge metric (or more generally a harmonic metric) as a primary object on flat vector bundles, one is lead to the problem of extending the notion at the singularities of the vector bundle. The notion of metric is difficult to extend, because it contains in it the property of being non-degenerate. Similarly, the notion of C^∞ vector bundle does not extend across singularities, because the sheaf $\mathcal{C}_X^\infty$ is not coherent. The notion of sesquilinear pairing with values in distributions is a good replacement of the C^∞ isomorphism between a holomorphic vector bundle and its conjugate, as it allows "degenerate" gluings.

This new point of view, which will be present all along this book, is explained at all levels of Hodge theory, starting from classical Hodge theory.

PART I

POLARIZABLE HODGE MODULES ON CURVES

CHAPTER 6

VARIATIONS OF HODGE STRUCTURE ON CURVES
PART 1: METRIC PROPERTIES NEAR PUNCTURES

Summary. We consider polarizable variations of $\mathbb{C}$-Hodge structure on a punctured smooth projective curve. This is the first occurrence of polarizable variations of $\mathbb{C}$-Hodge structure with singularities. It is essential to understand their local behaviour in the neighbourhood of a singular point. In this part of Chapter 6, we state the main results and, as an application, we prove the semi-simplicity theorem analogue to that proved in Chapter 4.

6.1. Introduction

A Hodge structure, as explained in Section 2.5, can be considered as a Hodge structure on a vector bundle supported by a point, that is, a vector space. The case where the underlying space is a complex manifold is called a *variation of Hodge structure*. It has been explained in Section 4.1 from a local point of view. The global properties have been considered in Section 4.2.

The question we address in this chapter is the definition and properties of Hodge structures on a vector bundle on a punctured complex projective curve (punctured compact Riemann surface) in the neighbourhood of the punctures (also called the *singularities* of the variation). The notion of a polarized variation of Hodge structure on a non-compact curve is analytic in nature, and a control near the punctures is needed in order to obtain interesting global results. Let us emphasize that, nevertheless, the approach is local, and we will mainly restrict the study to a local setting, where the base manifold is a disc Δ centered at the origin in $\mathbb{C}$ of radius 1 for convenience (or simply the germ of Δ at the origin), and we will denote by t its coordinate.

This chapter is divided in three parts, due to the length of the arguments. In this part, we state the fundamental properties of the variation near a puncture. We first focus on metric properties without paying much attention to the Hodge filtration itself. Our interest lies in the relations between two possible extensions of the holomorphic bundle with connection and Hermitian metric underlying a variation of $\mathbb{C}$-Hodge structure from the punctured Riemann surface X^* to the compact one X. We then explain how to extend the Hodge filtration at the punctures and provide the main statement for the limiting Hodge-Lefschetz structure. As an application of the metric properties, we prove the semi-simplicity theorem analogue of Theorem 4.3.3.

6.2. Variations of Hodge structure on a punctured disc

We consider the behaviour of a variation of $\mathbb{C}$-Hodge structure near a singular point. From now on, we will work on a disc Δ of radius 1 with coordinate t, as indicated in the introduction of this chapter and we will denote by Δ^* the punctured disc $\Delta \smallsetminus \{0\}$. Assume that H is a variation of Hodge structure on Δ^* (Definitions 4.1.4, 4.1.5 and 5.4.3). Our goal is to define a suitable restriction of these data to the origin. As for the case of a point in Δ^*, the underlying vector space of the restricted object should have a dimension equal to the rank of the bundle on Δ^*.

6.2.a. Reminder on holomorphic vector bundles with connection. We recall in this section the equivalence between the category of holomorphic vector bundle with connection $(\mathcal{V}, \nabla)$ on Δ^* and the category of finite dimensional vector spaces equipped with an automorphism. We shall first construct a functor from the first one to the second one.

If we are given a holomorphic vector bundle with connection $(\mathcal{V}, \nabla)$ on Δ^*, there exists a canonical meromorphic extension, called the *Deligne meromorphic extension*, of the bundle $\mathcal{V}$ to a meromorphic bundle $\mathcal{V}_*$ (that is, a free sheaf of $\mathcal{O}_\Delta[1/t]$-modules) equipped with a connection ∇. It consists of all local sections of $j_*\mathcal{V}$ (where $j : \Delta^* \hookrightarrow \Delta$ is the inclusion) whose coefficients in some (or any) basis of multivalued ∇-horizontal sections have moderate growth in any sector with bounded arguments. Equivalently, it is characterized by the property that the coefficients of any multivalued horizontal section expressed in some basis of $\mathcal{V}_*$ are multivalued functions on Δ^* with moderate growth in any sector with bounded arguments.

Similarly, there exists a canonical free $\mathcal{O}_\Delta$-submodule $\mathcal{V}_*^0$ of $\mathcal{V}_*$, called the *Deligne canonical lattice*, consisting of *all* local sections of $j_*\mathcal{V}$ whose coefficients in any basis of horizontal sections on any bounded sector are holomorphic functions on this sector *with at most logarithmic growth*. On this bundle $\mathcal{V}_*^0$, the connection ∇ has a pole of order 1. The residue $\mathcal{R}$ of the connection on $\mathcal{V}_*^0$ is an endomorphism of the vector space $\mathcal{V}_*^0/t\mathcal{V}_*^0$. The real parts of its eigenvalues belong to $[0, 1)$. The latter two properties also characterize $\mathcal{V}_*^0$ among all lattices of $\mathcal{V}_*$ (i.e., free $\mathcal{O}_\Delta$-submodules of $\mathcal{V}_*$ which generate $\mathcal{V}_*$ as a $\mathcal{O}_\Delta[t^{-1}]$-module).

The existence of a free $\mathcal{O}_\Delta$-submodule $\mathcal{V}_*^0$ of $\mathcal{V}_*$ such that $\mathcal{O}_\Delta[t^{-1}] \otimes \mathcal{V}_*^0 = \mathcal{V}_*$ and on which ∇ has a pole of order 1 is by definition the condition ensuring that $(\mathcal{V}_*, \nabla)$ has a *regular singularity* at the origin of Δ.

A classical result (see e.g. [**Mal91**, (2.6) p. 24]) asserts that $\mathcal{V}_*^0$ has an $\mathcal{O}_\Delta$-basis with respect to which the matrix of ∇ is constant. More precisely, any $\mathbb{C}$-basis of $\mathcal{V}_*^0/t\mathcal{V}_*^0$ can be lifted to an $\mathcal{O}_\Delta$-basis of $\mathcal{V}_*^0$, and the matrix of ∇ is then equal to the matrix of the residue $\mathcal{R}$ in the given basis of $\mathcal{V}_*^0/t\mathcal{V}_*^0$. These results can be reformulated as follows.

6.2.1. Theorem. *The construction $(\mathcal{V}, \nabla) \mapsto (\mathcal{V}_*, \nabla)$ induces an equivalence between the category of vector bundles with connection on the punctured disc Δ^* and that of free $\mathcal{O}_\Delta[1/t]$-modules with a connection ∇ having a regular singularity at the origin.* $\square$

Of course, an inverse functor is the restriction of $(\mathcal{V}_*, \nabla)$ to Δ^*. Notice also that this result implies that any morphism $\varphi : (\mathcal{V}_1, \nabla) \to (\mathcal{V}_2, \nabla)$ can be extended in a unique way as a morphism $(\mathcal{V}_{1*}, \nabla) \to (\mathcal{V}_{2*}, \nabla)$. The proof is obtained by interpreting φ as a horizontal section of $\mathcal{H}om_{\mathcal{O}_{\Delta^*}}(\mathcal{V}_1, \mathcal{V}_2)$ and by using the property that, for a connection with regular singularity (as ∇ on $\mathcal{H}om_{\mathcal{O}_\Delta[1/t]}(\mathcal{V}_{1*}, \mathcal{V}_{2*})$), any horizontal section on Δ^* extends in a unique way as a ∇-horizontal section on Δ (see Exercise 6.1(4)).

We can then more generally consider a whole family of Deligne canonical lattices: for every $\beta \in \mathbb{R}$, we denote by $\mathcal{V}_*^\beta$ the lattice defined by the property that the eigenvalues of the residue of the connection have their real part in $[\beta, \beta+1)$. If we set $\mathcal{V}_*^{>\beta} = \bigcup_{\beta'>\beta} \mathcal{V}_*^{\beta'}$, then $\mathcal{V}_*^{>\beta}$ is the Deligne canonical lattice for which the eigenvalues of the residue of the connection have real part in $(\beta, \beta+1]$. We use the notation

$$\operatorname{gr}^\beta \mathcal{V}_* := \mathcal{V}_*^\beta / \mathcal{V}_*^{>\beta}. \tag{6.2.2}$$

See Exercise 6.2 for the properties of the canonical lattices.

If we denote by $\mathcal{V}_*^{>-1}$ the lattice on which $\operatorname{Res}\nabla$ has eigenvalues with real part in $(-1, 0]$, and if $\beta \in (-1, 0]$, then $\operatorname{gr}^\beta \mathcal{V}_*$ is identified with the generalized eigenspace of $\operatorname{Res}\nabla$ on $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ corresponding to the eigenvalues $\beta + \mathsf{i}\,\beta''$ ($\beta'' \in \mathbb{R}$) with real part β. We set $\mathrm{N} = -(\operatorname{Res}\nabla)^{\mathrm{nilp}}$ (nilpotent part). This is the endomorphism induced by $\bigoplus_{\beta''}[-(t\partial_t - \beta - \mathsf{i}\,\beta'')]$ on $\operatorname{gr}^\beta \mathcal{V}_*$. [This choice is suggested by the property that the unipotent part of the monodromy operator on the locally constant sheaf $\mathcal{V}^\nabla := \operatorname{Ker}\nabla$ can be identified with $\exp 2\pi\mathsf{i}\,\mathrm{N}$.]

***6.2.3. Remark* (Behaviour with respect to operations).** Let $(\mathcal{V}, \nabla)$ be a holomorphic bundle with connection on Δ^*.

(1) Let $\mathcal{V}_1$ be a holomorphic subbundle of $\mathcal{V}$ which is preserved by the connection. Then, by construction, the Deligne canonical lattice $\mathcal{V}_{1,*}^0$ of $(\mathcal{V}_1, \nabla)$ is nothing but $j_*\mathcal{V}_1 \cap \mathcal{V}_*^0$, and similarly, for any β, $\mathcal{V}_{1,*}^\beta = j_*\mathcal{V}_1 \cap \mathcal{V}_*^\beta$.

(2) Let $(\mathcal{V}^\vee, \nabla)$ be the dual bundle with the dual connection. Using that the residue of the connection on $(\mathcal{V}_*^\beta)^\vee$ is minus the transposed of that on $\mathcal{V}_*^\beta$, one deduces that

$$(\mathcal{V}_*^\vee)^\beta \simeq (\mathcal{V}_*^{>-\beta-1})^\vee.$$

As a consequence, the natural pairing

$$(\mathcal{V}_*^\vee)^\beta \otimes \mathcal{V}_*^{-\beta-1} \longrightarrow \mathcal{O}_\Delta[1/t]$$

induces, by composing with the residue at $t = 0$, a perfect pairing

$$\operatorname{gr}^\beta \mathcal{V}_*^\vee \otimes \operatorname{gr}^{-\beta-1} \mathcal{V}_* \longrightarrow \mathbb{C}.$$

Equivalently, after multiplication by t, the natural pairing

$$\langle\bullet\,,\bullet\rangle : (\mathcal{V}_*^\vee)^\beta \otimes \mathcal{V}_*^{-\beta} \longrightarrow \mathcal{O}_\Delta$$

induces, by composing with restriction at $t = 0$, a perfect pairing

$$\operatorname{gr}^\beta \mathcal{V}_*^\vee \otimes \operatorname{gr}^{-\beta} \mathcal{V}_* \longrightarrow \mathbb{C}.$$

In particular, for any section v of $\mathcal{V}_*^{-\beta}$ whose class in $\operatorname{gr}^{-\beta}\mathcal{V}_*$ is nonzero, there exists a section $v^\vee$ of $(\mathcal{V}_*^\vee)^\beta$ (whose class in $\operatorname{gr}^\beta \mathcal{V}_*^\vee$ is nonzero) such that $\langle v^\vee, v\rangle = 1$.

(3) Let $\det(\mathcal{V},\nabla)$ be the determinant bundle (maximal exterior power) with connection. Given a frame $\boldsymbol{e}$ of $\mathcal{V}$, the matrix of the connection on $\det\mathcal{V}$ in the frame $e_1\wedge\cdots\wedge e_r$ is the trace of that of ∇ on $\mathcal{V}$ in the frame $\boldsymbol{e}$. Let $\gamma\geqslant 0$ be the sum of the real parts of the eigenvalues of the residue at the origin of ∇ on $\mathcal{V}^0_*$. We thus find

$$(\det\mathcal{V})^\gamma_* = \det(\mathcal{V}^0_*)\quad\text{and}\quad \dim\operatorname{gr}^\gamma(\det\mathcal{V})_* = 1.$$

6.2.4. Theorem. *The correspondence*

$$(\mathcal{V}_*,\nabla)\longmapsto(\mathcal{H}^o,\mathrm{T}) = \bigoplus_{\beta\in(-1,0]}(\operatorname{gr}^\beta\mathcal{V}_*, \mathrm{e}^{-2\pi\mathrm{i}\beta}\,\mathrm{T}_\beta\cdot\mathrm{e}^{2\pi\mathrm{i}\,\mathrm{N}}),$$

with T_β semi-simple with positive eigenvalues, is an equivalence between the category of free $\mathcal{O}_\Delta[1/t]$-modules with a connection ∇ having a regular singularity at the origin and the category of finite dimensional vector spaces with an automorphism.

Here is a quasi-inverse functor. Given $(\mathcal{H}^o,\mathrm{T})$, we group the generalized eigenspaces corresponding to the eigenvalues μ of T which share the same value $\lambda=\mu/|\mu|$, and denote this space $\mathcal{H}^o_\lambda$. On such a subspace, the action of T reads $\lambda\mathrm{T}_\lambda\mathrm{e}^{2\pi\mathrm{i}\,\mathrm{N}}$ with N nilpotent and T_λ semi-simple with positive eigenvalue. We thus obtain a decomposition $(\mathcal{H}^o,\mathrm{T}) = \bigoplus_{|\lambda|=1}(\mathcal{H}^o_\lambda, \lambda\mathrm{T}_\lambda\mathrm{e}^{2\pi\mathrm{i}\,\mathrm{N}})$. Furthermore, we write each λ as $\exp-2\pi\mathrm{i}\,\beta$ with $\beta\in(-1,0]$. We then associate to $(\mathcal{H}^o_\lambda,\lambda\mathrm{T}_\lambda\mathrm{e}^{2\pi\mathrm{i}\,\mathrm{N}})$ the free $\mathcal{O}_\Delta[1/t]$-module $\mathcal{H}^o_\lambda\otimes_{\mathbb{C}}\mathcal{O}_\Delta[1/t]$ with connection $\nabla=\mathrm{Id}\otimes\mathrm{d}+(\beta\,\mathrm{Id}+\frac{1}{2\pi}\log\mathrm{T}_\lambda-\mathrm{N})\mathrm{d}t/t$.

The canonical decomposition of the right-hand side of the correspondence of Theorem 6.2.4 corresponds to a canonical decomposition of the left-hand side:

6.2.5. Corollary. *There exists a canonical decomposition*

$$(\mathcal{V}_*,\nabla)\simeq\bigoplus_{\beta\in(-1,0]}(\mathcal{V}_{*\beta},\nabla)\tag{6.2.5 $*$}$$

for which $\mathcal{V}_{*\beta}$ has a frame $\boldsymbol{v}_\beta$ in which ∇ has matrix $(\beta\,\mathrm{Id}+\frac{1}{2\pi}\,\mathsf{D}_\beta-\mathsf{N})\mathrm{d}t/t$ with N nilpotent and D_β diagonal with positive eigenvalues.

Proof. We denote by $\mathcal{V}_\beta$ the subsheaf of $\mathcal{V}$ consisting of $\mathcal{O}_{\Delta^*}$-linear combinations of local sections of $\mathcal{V}$ annihilated by some power of $t\partial_t-(\beta+\mathrm{i}\,b'')$ for all possible $b''\in\mathbb{R}$. We then have a canonical decomposition

$$(\mathcal{V},\nabla)\simeq\bigoplus_{\beta\in(-1,0]}(\mathcal{V}_\beta,\nabla).\tag{6.2.5 $**$}$$

The correspondence of Theorem 6.2.1 induces a canonical decomposition $(\mathcal{V}_*,\nabla)\simeq\bigoplus_{\beta\in(-1,0]}(\mathcal{V}_{\beta*},\nabla)$ and we set $\mathcal{V}_{*\beta}=\mathcal{V}_{\beta*}$. $\square$

It follows from this decomposition that the space of multi-valued horizontal sections of $(\mathcal{V},\nabla)$ on Δ^* decomposes correspondingly with respect to the eigenvalues λ of the monodromy, which take the form $\lambda=\exp(-2\pi\mathrm{i}(\beta+ib''))$ for any $(\beta+ib'')$ occurring in (6.2.5 $*$). In particular, the absolute value of the eigenvalues of the monodromy are all equal to one if and only if $\mathsf{D}_\beta=0$ for any β.

6.2.b. Reminder on Hermitian bundles on the punctured disc. Let $\mathcal{V}$ be a holomorphic vector bundle on Δ^* and let h be a Hermitian metric on the associated C^∞-bundle $\mathcal{H} := \mathcal{C}^\infty_{\Delta^*} \otimes_{\mathcal{O}_{\Delta^*}} \mathcal{V}$. We denote by $\mathcal{V}_{\mathrm{mod}}$ the subsheaf of $j_*\mathcal{V}$ consisting of local sections whose h-norms have *moderate growth* in the neighbourhood of the origin, i.e., bounded by some (negative) power of $|t|$. This is an $\mathcal{O}_\Delta[1/t]$-module, which coincides with $\mathcal{V}$ when restricted to Δ^*.

The *parabolic filtration* $\mathcal{V}^\bullet_{\mathrm{mod}}$ is the decreasing filtration, indexed by $\mathbb{R}$, defined as follows. For any $\beta \in \mathbb{R}$, we define $\mathcal{V}^\beta_{\mathrm{mod}}$ as consisting of local sections v of $j_*\mathcal{V}$ such that, for any $\varepsilon > 0$, there exists $C_\varepsilon(v) > 0$ such that $\|v\|_{\mathrm{h}} \leqslant C_\varepsilon(v)|t|^{\beta-\varepsilon}$. For $\beta' > \beta$, we have $\mathcal{V}^{\beta'}_{\mathrm{mod}} \subset \mathcal{V}^\beta_{\mathrm{mod}}$ and we set $\mathcal{V}^{>\beta}_{\mathrm{mod}} = \bigcup_{\beta'>\beta} \mathcal{V}^{\beta'}_{\mathrm{mod}}$.

Clearly, each $\mathcal{V}^\beta_{\mathrm{mod}}$ is an $\mathcal{O}_\Delta$-submodule of $\mathcal{V}_{\mathrm{mod}}$, which coincides with $\mathcal{V}$ when restricted to Δ^*, and we have

$$\mathcal{V}_{\mathrm{mod}} = \bigcup_\beta \mathcal{V}^\beta_{\mathrm{mod}}, \qquad \text{and} \qquad \forall\, k \in \mathbb{Z}, \quad t^k\mathcal{V}^\bullet_{\mathrm{mod}} = \mathcal{V}^{\bullet+k}_{\mathrm{mod}}.$$

A *jump* (or, more correctly, jumping index) of the parabolic filtration is a real number β such that the quotient $\mathrm{gr}^\beta(\mathcal{V}_{\mathrm{mod}}) := \mathcal{V}^\beta_{\mathrm{mod}}/\mathcal{V}^{>\beta}_{\mathrm{mod}}$ is nonzero. Clearly, if β is a jump, then $\beta + k$ is a jump for every $k \in \mathbb{Z}$. We denote by $J(\beta)$ the set of jumping indices which belong to $[\beta, \beta + 1)$. We have $J(\beta + k) = J(\beta) + k$ for every $k \in \mathbb{Z}$.

6.2.6. Definition. We say that the metric is *moderate* if each $\mathcal{V}^\beta_{\mathrm{mod}}$ ($\beta \in (-1, 0]$) is $\mathcal{O}_\Delta$-locally free.

If the metric is moderate, $\mathcal{V}^\beta_{\mathrm{mod}}$ is $\mathcal{O}_\Delta$-locally free for any $\beta \in \mathbb{R}$ and $\mathcal{V}_{\mathrm{mod}} = \mathcal{O}_\Delta[1/t]\otimes_{\mathcal{O}_\Delta} \mathcal{V}^\beta_{\mathrm{mod}}$ (any β) is $\mathcal{O}_\Delta[1/t]$-locally free. Furthermore, the induced decreasing filtration $\mathcal{V}^\bullet_{\mathrm{mod}}(\mathcal{V}^\beta_{\mathrm{mod}}/\mathcal{V}^{\beta+1}_{\mathrm{mod}})$ is finite, so that $J(\beta)$ is finite. It follows that $\mathcal{V}^{>\beta}_{\mathrm{mod}} = \mathcal{V}^{\beta'}_{\mathrm{mod}}$ for some $\beta' > \beta$. We also have

$$\mathcal{V}^\beta_{\mathrm{mod}}/t\mathcal{V}^\beta_{\mathrm{mod}} = \bigoplus_{\beta' \in J(\beta)} \mathrm{gr}^{\beta'}\mathcal{V}_{\mathrm{mod}}.$$

***6.2.7. Remark* (Behaviour with respect to operations).** Let $(\mathcal{V}, \mathrm{h})$ be a holomorphic bundle with a *moderate* Hermitian metric.

(1) Let $\mathcal{V}_1$ be a holomorphic subbundle of $\mathcal{V}$ and let h_1 be the Hermitian metric induced by h on $\mathcal{V}_1$. Then, by construction, $\mathcal{V}_{1,\mathrm{mod}} = j_*\mathcal{V}_1 \cap \mathcal{V}_{\mathrm{mod}}$ and, for any β, $\mathcal{V}^\beta_{1,\mathrm{mod}} = j_*\mathcal{V}_1 \cap \mathcal{V}^\beta_{\mathrm{mod}}$. However, *we cannot claim that* $(\mathcal{V}_1, \mathrm{h}_1)$ *is moderate,* i.e., that $\mathcal{V}^\beta_{1,\mathrm{mod}}$ is locally free for any β (see Exercise 6.3).

(2) Let $\boldsymbol{v}$ be a frame of $\mathcal{V}^0_{\mathrm{mod}}$ lifting a basis of $\mathcal{V}^0_{\mathrm{mod}}/t\mathcal{V}^0_{\mathrm{mod}}$ adapted to the filtration induced by $\mathcal{V}^\bullet_{\mathrm{mod}}$. The diagonal entries of the matrix A of h in this frame have thus a controlled behaviour. The determinant bundle $\det\mathcal{V}$ is naturally equipped with a metric, and using this frame, one finds that it is moderate. Furthermore, setting $\gamma = \sum_{\beta\in J(0)} \beta$, one has $(\det\mathcal{V})^\gamma_{\mathrm{mod}} = \det\mathcal{V}^0_{\mathrm{mod}}$.

(3) We do not have much information on the other entries of the matrix A. Similarly, we do not have much information on the matrix ${}^t\mathsf{A}^{-1}$ of the metric on the dual bundle $\mathcal{V}^\vee$ in the dual frame $\boldsymbol{v}^\vee$.

We will make use in Part 2 of a criterion of moderateness in terms of the curvature, which goes back to [**CG75**] and [**Sim88**], and that we will not prove here (see [**Sim88**, §10] and [**Sim90**, Prop. 3.1]). For a Hermitian holomorphic bundle $(\mathcal{V}, \mathrm{h})$, the curvature operator R_{h} of the Chern connection of the metric is a linear morphism $\mathcal{H} \to \mathcal{E}^2_{\Delta^*} \otimes \mathcal{H}$, where $\mathcal{H}$ is the C^∞ bundle associated with $\mathcal{V}$. By fixing a constant norm on the trivial bundle $\mathcal{E}^2_{\Delta^*}$ (e.g. $\mathrm{d}t \wedge \mathrm{d}\bar{t}$ has norm one), we can consider the norm of R_{h} considered as a section of $\mathcal{E}nd(\mathcal{V}) \otimes \mathcal{E}^2_{\Delta^*}$, that we denote by $\|R_{\mathrm{h}}\|_{\mathrm{h}}$.

***6.2.8. Notation* (for $\mathrm{L}(t)$).** We consider on Δ^* the function

$$\mathrm{L}(t) = -\log |t|^2 = -\log t\bar{t}.$$

The main properties we use are given as an exercise (see Exercise 6.5).

***6.2.9. Theorem* (Criterion of moderateness).** *Assume that the curvature R_{h} satisfies $\|R_{\mathrm{h}}\|_{\mathrm{h}} \leqslant C/|t|^2\mathrm{L}(t)^2$ for some constant $C > 0$. Then the Hermitian holomorphic bundle $(\mathcal{V}, \mathrm{h})$ is moderate.* □

6.3. Metric properties near a puncture

6.3.a. The Deligne and parabolic filtrations for a polarized variation of Hodge structure. Let us consider a polarized variation of $\mathbb{C}$-Hodge structure $(H, \mathcal{S})$ of weight w on the punctured disc Δ^* (see Definitions 4.1.4 and 4.1.5). We set $H = (\mathcal{H}, D, F'^{\bullet}\mathcal{H}, F''^{\bullet}\mathcal{H})$. We thus have a positive definite Hermitian metric h on $\mathcal{H}$. On the other hand, we set $\mathcal{V} = \operatorname{Ker} D''$, on which the filtration $F'^{\bullet}\mathcal{H}$ induces a filtration $F^{\bullet}\mathcal{V}$ by holomorphic sub-bundles. We aim at comparing the canonical filtration $\mathcal{V}^{\bullet}_*$ and the filtration $\mathcal{V}^{\bullet}_{\mathrm{mod}}$ relative to the Hodge metric h, and more precisely at showing that they coincide. In particular, this implies that the Hodge metric is moderate.

***6.3.1. Example* (The unitary case).** In the simple case where the connection is compatible with the Hermitian metric h, we claim that the metric is moderate.

The assumption corresponds to a variation of Hodge structure of pure type $(0, 0)$. Then the norm of any horizontal section of $\mathcal{V}$ is constant, hence bounded. The monodromy matrix being unitary, its eigenvalues have absolute value equal to 1, and the matrices T_β considered in Corollary 6.2.5 are the identity matrices, so that $\log \mathsf{T}_\beta = 0$.

The decomposition (6.2.5 $**$) is compatible with the metric, and we are reduced to proving the claim on each term. We can then assume for simplicity that $\beta = 0$ by multiplying by $|t|^{2\beta}$. It is then enough to identify $\mathcal{V}^0_*$ and $\mathcal{V}^0_{\mathrm{mod}}$.

Given any section v of $\mathcal{V}$, we express it on a unitary frame of multivalued horizontal sections, and v is a section of $\mathcal{V}^0_{\mathrm{mod}}$ if and only if its multivalued coefficients are bounded by $|t|^{-\varepsilon}$ in any bounded angular sector of sufficiently small radius. Similarly, by definition, a section of $\mathcal{V}$ is a section of $\mathcal{V}^0_*$ if and only if its mutlivalued coefficients have logarithmic growth, and equivalently satisfy the same growth condition as for $\mathcal{V}^0_{\mathrm{mod}}$, hence the claim.

The properties of the previous example hold true for any polarized variation of $\mathbb{C}$-Hodge structure: this is the main results in this part of Chapter 6.

6.3.2. Theorem. *Let $(\mathcal{V},\nabla,\mathrm{h})$ be a Hermitian holomorphic bundle with connection underlying a polarized variation of $\mathbb{C}$-Hodge structure on Δ^*. Then,*

(1) *the metric h on $\mathcal{H}$ is moderate and the parabolic filtration $\mathcal{V}^\bullet_{\mathrm{mod}}$ on $\mathcal{V}_*$ induced by the metric h is equal to the filtration $\mathcal{V}^\bullet_*$;*

(2) *furthermore, the eigenvalues of the monodromy have absolute value equal to* 1.

6.3.3. Remark. This result justifies the need of refining the filtration $\mathcal{V}^\bullet_*$ indexed by $\mathbb{Z}$ and its graded spaces with a filtration indexed by $\mathbb{R}$ and the corresponding graded spaces (6.2.2).

Theorem 6.3.2 characterizes sections of $\mathcal{V}^\beta_*$ in terms of growth of their norm with respect to real powers of $|t|$. In order to analyze the L^2 behaviour of the norm, we will need to refine this result by using a logarithmic scale.

***6.3.4. Definition* (Lift of the monodromy filtration).** For each $\beta\in\mathbb{R}$, we denote by $\mathrm{M}_\bullet\mathrm{gr}^\beta\mathcal{V}_*$ the monodromy filtration relative to the nilpotent endomorphism N of $\mathrm{gr}^\beta\mathcal{V}_*$ (see Theorem 6.2.4). The *lift* $\mathrm{M}_\bullet\mathcal{V}^\beta_*$ of $\mathrm{M}_\bullet\mathrm{gr}^\beta\mathcal{V}_*$ is the pullback by the projection $\mathcal{V}^\beta_*\to\mathrm{gr}^\beta\mathcal{V}_*$ of $\mathrm{M}_\bullet\mathrm{gr}^\beta\mathcal{V}_*$. This is a locally free extension of $\mathcal{V}$ to Δ.

***6.3.5. Theorem* (Finer norm estimates).** *A section of $\mathcal{V}$ on Δ^* extends as a section of $\mathrm{M}_\ell\mathcal{V}^\beta_*$ and not as a section of $\mathrm{M}_{\ell-1}\mathcal{V}^\beta_*$ (i.e., with non-zero image in $\mathrm{gr}^{\mathrm{M}}_\ell\mathrm{gr}^\beta\mathcal{V}_*$) if and only if its h-norm has the same order of growth as $|t|^\beta\mathrm{L}(t)^{\ell/2}$.*

Theorems 6.3.2 and 6.3.5, while depending on the Hodge structure in their assumptions, do not involve Hodge properties in their conclusions. As a matter of fact, the statements hold for harmonic flat bundles (Definition 4.2.5) on the punctured disc whose Higgs field is nilpotent. We will prove them in that setting. We thus forget the Hodge filtration for a while and consider a vector bundle $(\mathcal{V},\nabla)$ equipped with a *harmonic metric* h. We now assume that $(\mathcal{H},\mathrm{h},D)$ is a harmonic flat bundle on Δ^* and we consider the associated metric connection $D_{\mathrm{h}}=D'_{\mathrm{h}}+D''_{\mathrm{h}}$ and Higgs field $\theta=\theta'+\theta''$. We recall that the Hermitian holomorphic Higgs bundle $(\mathcal{E},\mathrm{h},\theta)$ is defined by $\mathcal{E}=\operatorname{Ker}D''_{\mathrm{h}}$ and θ is induced by θ' (see Definition 4.2.8). In other words, for a polarized variation of Hodge structure, we also pay attention to the graded bundle $\mathcal{E}=\mathrm{gr}_F\mathcal{V}$ equipped with its Higgs field induced by $\theta:=\mathrm{gr}_F^{-1}\nabla$, as in (4.2.12). However, we forget the grading of this bundle and only remember that θ is nilpotent.

***6.3.6. Definition* (Nilpotent harmonic bundle).** We say that the harmonic bundle is *nilpotent* if the coefficient of $\mathrm{d}t$ in θ' is a nilpotent endomorphism of $\mathcal{H}$.

6.3.7. Remarks.

(1) By Hermitian adjunction, the coefficient of $\mathrm{d}t$ in θ' is nilpotent if and only if the coefficient of $\mathrm{d}\bar t$ in θ'' is nilpotent.

(2) The harmonic bundle associated with a polarized variation of Hodge structure on Δ^* is nilpotent. Indeed, θ' has bidegree $(-1,1)$ with respect to the Hodge decomposition.

In this part, we give a proof of these theorems and we give some important consequences, in particular concerning semi-simplicity.

6.3.8. Remarks.

(1) In Section 6.2.a, when extending the vector bundle $\mathcal{V}$ with holomorphic connection ∇ from Δ^* to Δ, we have chosen Deligne's meromorphic extension, that is, we have chosen the (unique) meromorphic extension on which the extended connection is meromorphic and has *regular singularities*. Such a choice, while being canonical and, in some sense, as simple as possible, was not the only possible one. We could have chosen other kinds of meromorphic extensions, on which the extended meromorphic connection has irregular singularities. A posteriori, when considering variations of *polarized* Hodge structures, Theorem 6.3.2 strongly justify the previous choice.

(2) One may wonder why we have considered the filtration $\mathcal{V}_*^\bullet$ decreasing and the filtration $\mathrm{M}_\bullet\mathrm{gr}^\beta\mathcal{V}_*$ increasing. The answer is that this reflects the scale of growth of the family of functions $|t|^\beta\mathrm{L}(t)^{\ell/2}$ ($\beta\in(-1,0]$ and $\ell\in\mathbb{Z}$): the function $|t|^\beta\mathrm{L}(t)^{\ell/2}$ grows faster (or decreases slower) than $|t|^{\beta'}\mathrm{L}(t)^{\ell'/2}$ when $t\to 0$ if and only if either $\beta<\beta'$ or $\beta=\beta'$ and $\ell>\ell'$.

6.3.b. Sketch of the proof of Theorems 6.3.2 and 6.3.5 for nilpotent harmonic bundles.

Let $(\mathcal{V},\nabla,\mathrm{h})$ be a nilpotent harmonic flat bundle.

Step 1. The first objective is to show that the eigenvalues of the monodromy have absolute value one (Theorem 6.3.2(2)). This point relies on the estimate of the h-norm of a multi-valued horizontal section of $\mathcal{V}$ which is an eigenvector for the monodromy operator. Due to Exercise 4.5, the h-norm of any multi-valued horizontal section v satisfies

$$\partial_t\|v\|_{\mathrm{h}}^2=-2\mathrm{h}(\theta_0'v,\overline{v}),\quad \partial_{\overline{t}}\|v\|_{\mathrm{h}}^2=-2\mathrm{h}(\theta_0''v,\overline{v}),$$

where we have set $\theta'=\theta_0'\mathrm{d}t$ and $\theta''=\theta_0''\mathrm{d}\overline{t}$. Making use of the norm of the Higgs field computed with the metric on the bundle of endomorphisms of $\mathcal{E}$, we deduce

$$\big|\partial_t\|v\|_{\mathrm{h}}\big|\leqslant 2\|v\|_{\mathrm{h}}\|\theta_0'\|_{\mathrm{h}}^{1/2},\quad \big|\partial_{\overline{t}}\|v\|_{\mathrm{h}}\big|\leqslant 2\|v\|_{\mathrm{h}}\|\theta_0''\|_{\mathrm{h}}^{1/2}=2\|v\|_{\mathrm{h}}\|\theta_0'\|_{\mathrm{h}}^{1/2},$$

where the latter equality follows from the fact that θ_0'' is the h-adjoint of θ_0'. The main tool for the proof is then an estimate for the norm of the Higgs field, proved in Section 6.3.d.

***6.3.9. Theorem* (Simpson's estimate).** *If $(\mathcal{H},D,\mathrm{h})$ is a nilpotent harmonic bundle, the Higgs field $\theta'=\theta_0'\mathrm{d}t$ satisfies $\|\theta_0'\|_{\mathrm{h}}\leqslant C/|t|\mathrm{L}(t)$ on Δ^*, for some $C>0$.*

6.3.10. Remark. By choosing a volume form vol on Δ^*, giving rise to a norm on differential forms, one can consider the norm $\|\theta'\|_{\mathrm{h},\mathrm{vol}}$. In the Poincaré metric that we will consider in Section 6.12.c, the norm of $\mathrm{d}t/t$ and $\mathrm{d}\overline{t}/\overline{t}$ is $\mathrm{L}(t)$. The theorem thus asserts that the norm $\|\theta'\|_{\mathrm{h},\mathrm{vol}}$ (and that of $\|\theta''\|_{\mathrm{h},\mathrm{vol}}$ since θ'' is the h-adjoint of θ') is bounded.

This estimate leads to the following:

$$\big|t\partial_t \log \|v\|_{\mathrm{h}}\big| \leqslant C'/\mathrm{L}(t), \quad \big|\bar{t}\partial_{\bar{t}} \log \|v\|_{\mathrm{h}}\big| \leqslant C'/\mathrm{L}(t).$$

If v is an eigenvector of the monodromy operator T corresponding to the eigenvalue λ, then $\log \|\mathrm{T}v\|_{\mathrm{h}} - \log \|v\|_{\mathrm{h}} = \log |\lambda|$. Expressing this difference by an integral formula in the universal covering of Δ^* involving the above partial derivatives of $\log \|v\|_{\mathrm{h}}$ one finds

$$\big|\log |\lambda|\big| \leqslant C''/\mathrm{L}(t)$$

for a suitable constant $C'' > 0$. Since the right-hand side tends to zero when t tends to 0, this implies $\log |\lambda| = 0$, that is, $|\lambda| = 1$.

Step 2. The next step (Section 6.3.c) is, starting with the only data of $(\mathcal{V}, \nabla)$ without any other assumption, to construct a model harmonic metric, that we call the Deligne harmonic model, and to show that, if we moreover assume that the eigenvalues of the monodromy have absolute value equal to one, this model satisfies the conclusions of Theorems 6.3.2(1) and 6.3.5 (the conclusion of Theorem 6.3.2(2) being part of the assumption).

Step 3. If $(\mathcal{V}, \nabla)$ satisfying 6.3.2(2) is equipped with two comparable harmonic metrics, and if it satisfies the conclusions of Theorems 6.3.2(1) and 6.3.5 for one of both, it does so for the other one. These theorems are thus a consequence of the following.

6.3.11. Theorem. *Let $(\mathcal{V}, \nabla, \mathrm{h})$ be a nilpotent harmonic flat bundle. Then the metrics* h *and* $\mathrm{h}^{\mathrm{Del}}$ *are mutually bounded, that is, there exist constants $C_1, C_2 > 0$ such that, on Δ^*,*

$$C_1 |\mathrm{h}^{\mathrm{Del}}(\bullet, \bar{\bullet})| \leqslant |\mathrm{h}(\bullet, \bar{\bullet})| \leqslant C_2 |\mathrm{h}^{\mathrm{Del}}(\bullet, \bar{\bullet})|.$$

Proof. The filtration $\mathcal{V}_*^\bullet$ is the parabolic filtration both for h and $\mathrm{h}^{\mathrm{Del}}$. The identity morphism $(\mathcal{H}, D, \mathrm{h}) \to (\mathcal{H}, D, \mathrm{h}^{\mathrm{Del}})$ or vice versa, regarded as a flat section of $\mathcal{H}om(\mathcal{H}, \mathcal{H})$ satisfies thus the metric assumption of Lemma 6.3.12 below. Recall that $\mathcal{H}om(\mathcal{H}, \mathcal{H})$, equipped with its natural metric and flat connection, is harmonic (Exercise 4.7). By Lemma 6.3.12 below, the identity morphism, in both directions, is bounded, which is equivalent to the mutual boundedness of h and $\mathrm{h}^{\mathrm{Del}}$. □

6.3.12. Lemma. *Let $(\mathcal{H}, D, \mathrm{h})$ be a flat bundle with metric on Δ^*. Assume that $(\mathcal{H}, D, \mathrm{h})$ is harmonic. Let $u \in \Gamma(\Delta^*, \mathcal{H})$ be a D-flat section of $\mathcal{H}$ such that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying $\|u(t)\|_{\mathrm{h}} \leqslant C_\varepsilon |t|^{-\varepsilon}$ on Δ^*. Then $\|u(t)\|_{\mathrm{h}}$ is bounded near the origin.*

A proof of this lemma is given in Section 6.3.d. This concludes the proof of Theorems 6.3.2 and 6.3.5 for nilpotent harmonic bundles. □

6.3.c. The Deligne harmonic model. We will now construct a model of such a vector bundle, starting from an $\mathfrak{sl}_2$-representation with a positive definite Hermitian form, and we will check whether Theorem 6.3.2 holds on such a model. We will link Property 6.3.2(2) with the nilpotency of the Higgs field of the model. This model only relies on the datum of the flat vector bundle $(\mathcal{V},\nabla)$ on Δ^* and is built so that the Deligne canonical filtration $\mathcal{V}_*^\bullet$ is equal to the parabolic filtration of the harmonic metric. This is why we call it the Deligne harmonic model.

Let $\mathcal{H}^o$ be a $\mathbb{C}$-vector space of dimension d equipped with an $\mathfrak{sl}_2$-representation, hence of endomorphisms $\mathrm{X},\mathrm{Y},\mathrm{H}$ (see Section 3.1.a). Since we do not deal with Hodge filtrations for the moment, we do not introduce a polarization S and only consider the resulting positive definite Hermitian form h^o. In order to prepare compatibility with the notion of polarization, we impose that

$$\mathrm{h}^o(\mathrm{X}\bullet,\overline{\bullet}) = \mathrm{h}^o(\bullet,\overline{\mathrm{Y}\bullet}),\quad \mathrm{h}^o(\mathrm{Y}\bullet,\overline{\bullet}) = \mathrm{h}^o(\bullet,\overline{\mathrm{X}\bullet}),\quad \mathrm{h}^o(\mathrm{H}\bullet,\overline{\bullet}) = \mathrm{h}^o(\bullet,\overline{\mathrm{H}\bullet}), \tag{6.3.13}$$

as suggested by Remark 3.2.8(2). Let us fix an h^o-orthonormal basis $\boldsymbol{v}^o=(v_1^o,\dots,v_d^o)$ consisting of eigenvectors for H. If we denote by A the matrix of the endomorphism A in a given basis, this identification leads to the notation

$$(\mathrm{X}v_1^o,\dots,\mathrm{X}v_d^o) = (v_1^o,\dots,v_d^o)\cdot\mathsf{X}\,.$$

Similarly, for a Hermitian form $\mathfrak{s}^o$ on $\mathcal{H}^o$, we also denote by s^o the matrix (s^o_{ij}) defined by $\mathsf{s}^o_{ij}=\mathfrak{s}^o(v_i^o,\overline{v_j^o})$.

6.3.14. Simple example. We suggest the reader to follow the next computations on the simple example where Y consists of a single Jordan block of size $\ell+1$, so that $\dim\mathrm{P}_\ell=1$ and $\mathrm{P}_{\ell'}=0$ if $\ell'\neq 0$. Then each $\boldsymbol{v}^o_{\ell,j}$ consists of a single element $v_{\ell,j}$.

It will be convenient to assume that the basis $\boldsymbol{v}^o$ is obtained as follows: for each $\ell\geqslant 0$, let us fix an h^o-orthonormal basis $\boldsymbol{v}^o_{\ell,0}$ of the ℓ-th primitive part $\mathrm{P}_\ell\mathcal{H}^o\subset\operatorname{Ker}\mathrm{X}$ made with eigenvectors of H (with eigenvalue ℓ); for any $j\geqslant 0$, consider the basis of $\mathrm{Y}^j\mathrm{P}_\ell\mathcal{H}^o$

$$\boldsymbol{v}^o_{\ell,j}=\star_{\ell,j}\boldsymbol{v}^o_{\ell,0}\,\mathsf{Y}^j, \tag{6.3.15}$$

where $\star_{\ell,j}$ is some constant. In such a way, $\boldsymbol{v}^o_{\ell,j}$ is a basis of the Lefschetz component $\mathrm{Y}^j\mathrm{P}_\ell$, the basis $\boldsymbol{v}^o:=(\boldsymbol{v}^o_{\ell,j})_{\ell,j}$ is h^o-orthogonal, and one can (and will) choose $\star_{\ell,j}$ (with $\star_{\ell,0}=1$) such that this basis is h^o-orthonormal. The formula of Exercise 3.1(2) shows that these constants are positive. Then the matrix H of H in this basis is diagonal with integral entries, while X (resp. Y) is block upper (resp. lower) triangular whose entries are positive or zero, X being the transpose of Y.

6.3.16. Definition **(The model bundle with connection).** Let $\mathcal{H}=\mathcal{C}^\infty_{\Delta^*}\otimes_{\mathbb{C}}\mathcal{H}^o$ be the trivial C^∞-bundle on Δ^* and let $\boldsymbol{v}$ be the basis $\boldsymbol{v}=1\otimes\boldsymbol{v}^o$. Let b be a complex number, that we write $b=b'+\mathrm{i}\,b''$ $(b',b''\in\mathbb{R})$ and a real number $\beta\in(-1,0]$. We endow $\mathcal{H}$ with the connection D such that

$$D''\boldsymbol{v}=0,\quad D'\boldsymbol{v}=\boldsymbol{v}\cdot(b\operatorname{Id}-\mathsf{Y})\frac{\mathrm{d}t}{t}, \tag{6.3.16 $*$}$$

so that $\mathcal{V} := \operatorname{Ker} D''$ is the holomorphic trivial bundle $\mathcal{O}_{\Delta^*} \cdot \boldsymbol{v}$ and the connection ∇ on $\mathcal{V}$ induced by D' has matrix $(b\operatorname{Id} - \mathsf{Y})\mathrm{d}t/t$.

Let $\boldsymbol{\varepsilon}$ be the basis obtained from $\boldsymbol{v}$ by the change of basis of having inverse matrix

$$\mathsf{P}_\beta(t) = \mathrm{e}^{\mathsf{X}}|t|^\beta \mathrm{L}(t)^{\mathsf{H}/2} = |t|^\beta \mathrm{L}(t)^{\mathsf{H}/2}\mathrm{e}^{\mathsf{X}/\mathrm{L}(t)} \quad \text{(see Exercise 6.5)}, \tag{6.3.17}$$

that is,

$$\boldsymbol{v} = \boldsymbol{\varepsilon} \cdot \mathsf{P}_\beta(t). \tag{6.3.18}$$

The bases $\boldsymbol{\varepsilon}$ and $\boldsymbol{v}$ are decomposed as $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_{\ell,j})_{\ell,j}$ and $\boldsymbol{v} = (\boldsymbol{v}_{\ell,j})_{\ell,j}$, so that (6.3.18) reads

$$\boldsymbol{v}_{\ell,j} = |t|^\beta \sum_{k\geqslant 0} c_{\ell,j,k}\boldsymbol{\varepsilon}_{\ell,j+k}\mathrm{L}(t)^{\mathsf{H}/2} = |t|^\beta \mathrm{L}(t)^{\ell/2-j}\sum_{k\geqslant 0} c_{\ell,j,k}\mathrm{L}(t)^{-k}\boldsymbol{\varepsilon}_{\ell,j+k}, \tag{6.3.19}$$

for some nonnegative numbers $c_{\ell,j,k}$ with $c_{\ell,j,0} = 1$.

***6.3.20. Definition* (The model metric on the model bundle with connection)**
We equip $\mathcal{H}$ with the Hermitian metric h such that $\boldsymbol{\varepsilon}$ is an orthonormal basis.

We now group the terms $\boldsymbol{v}_{\ell,j}, \boldsymbol{\varepsilon}_{\ell,j}$ corresponding to the same $w = \ell - 2j$ and we set

$$\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_w)_{w\in\mathbb{Z}}, \quad \boldsymbol{v} = (\boldsymbol{v}_w)_{w\in\mathbb{Z}} \quad \text{with } v \in \boldsymbol{v}_w \iff \|v\|_{\mathrm{h}} \underset{t\to 0}{\sim} |t|^\beta \mathrm{L}(t)^{w/2}. \tag{6.3.21}$$

Moreover, the basis $\boldsymbol{v}$ is *asymptotically* h-*orthogonal*, with logarithmic decay.

The metric h and the connection D on $\mathcal{H}$ enable us to define operators D'_{h}, D''_{h}, θ' and θ'' (see Lemma 4.2.2).

6.3.22. Proposition. *With the previous assumptions, the metric* h *on* $(\mathcal{H}, D)$ *is harmonic.*

Proof. Let us write

$$D'\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\cdot \mathsf{M}'\,\frac{\mathrm{d}t}{t}, \quad D''\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\cdot \mathsf{M}''\,\frac{\mathrm{d}\bar t}{\bar t}.$$

Applying the base change formula for connections, we find

$$\mathsf{M}' = b\operatorname{Id} - \mathsf{P}_\beta\,\mathsf{Y}(\mathsf{P}_\beta)^{-1} + \mathsf{P}_\beta\, t\partial_t(\mathsf{P}_\beta)^{-1} \quad \text{and} \quad \mathsf{M}'' = \mathsf{P}_\beta\,\bar t\partial_{\bar t}(\mathsf{P}_\beta)^{-1}.$$

According to the identities of Exercise 6.5 we obtain

$$\begin{aligned}\mathsf{M}' &= \Big(b - \frac{\beta}{2}\Big)\operatorname{Id} - \frac{\mathsf{Y} + \mathsf{H}/2}{\mathrm{L}(t)}\\ \mathsf{M}'' &= -\frac{\beta}{2}\operatorname{Id} + \frac{\mathsf{H}/2 - \mathsf{X}}{\mathrm{L}(t)}\end{aligned} \tag{6.3.23}$$

$$\begin{aligned}\theta' &= \frac{1}{2}(\mathsf{M}' + \mathsf{M}''^*)\frac{\mathrm{d}t}{t} = \Big(\frac{(b-\beta)}{2}\operatorname{Id} - \frac{\mathsf{Y}}{\mathrm{L}(t)}\Big)\frac{\mathrm{d}t}{t}\\ \theta'' &= \frac{1}{2}(\mathsf{M}'^* + \mathsf{M}'')\frac{\mathrm{d}\bar t}{\bar t} = \Big(\frac{(\bar b-\beta)}{2}\operatorname{Id} - \frac{\mathsf{X}}{\mathrm{L}(t)}\Big)\frac{\mathrm{d}\bar t}{\bar t}\end{aligned} \tag{6.3.24}$$

and

$$D''_{\mathrm{h}}\boldsymbol{\varepsilon} = (D'' - \theta'')\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}\cdot\Big(-\frac{\bar b}{2}\operatorname{Id} + \frac{\mathsf{H}/2}{\mathrm{L}(t)}\Big)\frac{\mathrm{d}\bar t}{\bar t}.$$

We need to prove that the matrix of θ' is holomorphic when expressed in a D''_{h}-holomorphic basis of $\mathcal{H}$. We note that, for any complex number c, the diagonal matrix

$$\mathsf{A}_c(t) = |t|^c \,\mathrm{L}(t)^{\mathsf{H}/2} \tag{6.3.25}$$

satisfies, according to Exercise 6.5,

$$t\partial_t \mathsf{A}_c(t) = \bar t\partial_{\bar t} \mathsf{A}_c(t) = \Big(\frac{c}{2}\,\mathrm{Id} - \frac{\mathsf{H}/2}{\mathrm{L}(t)}\Big).$$

Therefore, after the base change with matrix

$$\mathsf{A}_{\bar b}(t) := |t|^{\bar b}\,\mathrm{L}(t)^{\mathsf{H}/2}, \tag{6.3.26}$$

the basis $\boldsymbol{e} = \boldsymbol{\varepsilon}\cdot\mathsf{A}_{\bar b}(t)$ is D''_{h}-holomorphic: indeed, the coefficient of $\mathrm{d}\bar t/\bar t$ in the matrix of D''_{h} with respect to the basis $\boldsymbol{e}$ is

$$(\mathsf{A}_{\bar b})^{-1}\frac{\mathsf{H}/2 - \bar b\,\mathrm{Id}}{\mathrm{L}(t)}\,\mathsf{A}_{\bar b} + (\mathsf{A}_{\bar b})^{-1}\bar t\partial_{\bar t}\,\mathsf{A}_{\bar b} = 0,$$

hence the assertion. Let us notice that $\boldsymbol{e}$ decomposes as $(\boldsymbol{e}_w)_{w\in\mathbb{Z}}$ according to the decomposition $\boldsymbol{\varepsilon} = (\boldsymbol{\varepsilon}_w)_{w\in\mathbb{Z}}$ analogous to (6.3.21), and each element of $\boldsymbol{e}_w$ has norm $|t|^{b'}\mathrm{L}(t)^{w/2}$. Moreover, $\boldsymbol{e}$ is h-orthogonal.

The coefficient of $\mathrm{d}t/t$ in the matrix of θ' in the basis $\boldsymbol{e}$ is therefore

$$(\mathsf{A}_{\bar b})^{-1}\Big(\frac{(b-\beta)}{2}\,\mathrm{Id} - \frac{\mathsf{Y}}{\mathrm{L}(t)}\Big)\,\mathsf{A}_{\bar b} = \frac{(b-\beta)}{2}\,\mathrm{Id} - \mathsf{Y}, \tag{6.3.27}$$

according to Exercise 6.5. □

Proof of Theorem* 6.3.2 *for the model. The norm of a holomorphic section of $\mathcal{V}$ is easily computed with its coefficients in the orthonormal basis $\boldsymbol{\varepsilon}$. Since the entries of the matrices P_β and $(\mathsf{P}_\beta)^{-1}$ defined by (6.3.17) have moderate growth, this norm is moderate if and only if the coefficients of the section in the holomorphic basis $\boldsymbol{v}$ have moderate growth on Δ^*, i.e., if and only if they are meromorphic functions. Therefore, $\mathcal{V}_*$ is determined by the moderate growth condition on the norm of holomorphic sections.

In order to determine the parabolic filtration, we need to compute the norm of the elements v of $\boldsymbol{v}$. This norm is computed by (6.3.21). The parabolic filtration $\mathcal{V}^{\bullet}_{\mathrm{mod}}$ is thus given by $\mathcal{V}^{\beta+k}_{\mathrm{mod}} = t^k\mathcal{O}_\Delta\cdot\boldsymbol{v}$ (the jumps occur only at $\beta+\mathbb{Z}$).

On the other hand, the filtration by Deligne canonical lattices $\mathcal{V}^{\bullet}_*$ is given by $\mathcal{V}^{b'+k}_* = t^k\mathcal{O}_\Delta\cdot\boldsymbol{v}$ (the jumps occur only at $b'+\mathbb{Z}$).

If $b'=\beta$, then both filtrations coincide, and (6.3.21) also shows that $\mathrm{M}_w\mathcal{V}^{\beta}_*$ is determined by the norm condition of Theorem 6.3.5.

We notice that the model is a *nilpotent* harmonic bundle if and only if $b=\beta$. The nilpotency condition thus implies both properties in Theorem 6.3.2 as well as the conclusion of Theorem 6.3.5. □

Let $(\mathcal{V},\nabla)$ be a holomorphic bundle with connection. Tensoring the decomposition of (6.2.5 $**$) with $\mathcal{C}^\infty_{\Delta^*}$ leads to a decomposition

$$(\mathcal{H}, D) \simeq \bigoplus_{\beta\in(-1,0]} (\mathcal{H}_\beta, D), \tag{6.3.28}$$

and $\boldsymbol{v}_\beta$ is a holomorphic frame of $(\mathcal{H}_\beta, D)$ on Δ^*. Besides, the C^∞ bundle $\mathcal{H}_\beta$ is in fact defined all over Δ, since the frame $\boldsymbol{v}_\beta$ is so, and its fiber $\mathcal{H}^\circ_\beta$ at the origin is isomorphic to $\mathrm{gr}^\beta\mathcal{V}_*$ via an identification of bases. For every $b'' \in \mathbb{R}$ such that $\exp(2\pi b'')$ is an eigenvalue of T_β in Theorem 6.2.4, we associate a model metric $\mathrm{h}^{\mathrm{Del}}_b$ as in Section 6.3.c with $b = \beta + \mathrm{i}\, b''$ and the nilpotent endomorphism N given by that theorem. Summing over all such possible $b'' \in \mathbb{R}$, we obtain a model metric $\mathrm{h}^{\mathrm{Del}}_\beta$ on each $(\mathcal{H}_\beta, D)$, and summing over all $\beta \in (-1, 0]$, we obtain a model metric $\mathrm{h}^{\mathrm{Del}}$ in such a way that the decomposition (6.3.28) is $\mathrm{h}^{\mathrm{Del}}$-orthogonal and that the restriction of $\mathrm{h}^{\mathrm{Del}}$ to each $\mathcal{H}_\beta$ is the model metric $\mathrm{h}^{\mathrm{Del}}_\beta$. Since $b' = \beta$, the parabolic filtration of $(\mathcal{V}, \mathrm{h}^{\mathrm{Del}})$ is the canonical filtration $\mathcal{V}_*$. If we consider the Higgs bundle $(\mathcal{E}^{\mathrm{Del}}, \mathrm{h}^{\mathrm{Del}}, \theta^{\mathrm{Del}})$, the parabolic filtration is such that the frame $(\boldsymbol{e}_\beta)_{\beta\in(-1,0]}$ forms an adapted basis of $\mathcal{E}^{\mathrm{Del}}$.

6.3.29. Definition. We call $(\mathcal{V}, \mathrm{h}^{\mathrm{Del}}, \nabla)$ the *Deligne harmonic model* for $(\mathcal{V}, \nabla)$.

6.3.30. Remark. For the Deligne harmonic model, the statement of Theorem 6.3.2(2) is equivalent to the property that it is nilpotent.

6.3.d. Proof of Theorem 6.3.9 and Lemma 6.3.12. We continue assuming that $(\mathcal{V}, \nabla, \mathrm{h})$ is a harmonic flat bundle on Δ^*. Let us start with a corollary of Theorem 6.3.9 that will be used when proving semi-simplicity in Section 6.4.

***6.3.31. Corollary* (Curvature properties).** *The curvature $R_{\mathcal{V}}$ of $(\mathcal{V}, \mathrm{h})$ and the curvature $R_{\mathcal{E}}$ of $(\mathcal{E}, \mathrm{h})$ satisfy an inequality*

$$\|R\|_{\mathrm{h}} \leqslant C/|t|^2\mathrm{L}(t)^2 \quad \textit{for some } C > 0, \tag{6.3.31 $*$}$$

in particular they are L^1_{loc} on Δ.

6.3.32. Remark. This corollary follows from Simpson's estimate (Theorem 6.3.9) and can be combined with the criterion of moderateness provided by Theorem 6.2.9 to yield moderateness of the metric h. However, we do not make use of this criterion here, as moderateness follows from the identification of the parabolic filtration $\mathcal{V}^\bullet_{\mathrm{mod}}$ with the Deligne canonical filtration $\mathcal{V}^\bullet_*$, as follows from Theorem 6.3.11 and the properties of the Deligne harmonic model.

Proof. Let us emphasize that $R_{\mathcal{V}}$ is the curvature of the Chern connection of h on $\mathcal{H}$ with the holomorphic structure D'', while $R_{\mathcal{E}}$ is that of h on $\mathcal{H}$ with the holomorphic structure D''_{h}. The formula of Exercise 4.4(5) and the identities following (4.2.6) give

$$R_{\mathcal{V}} = -2(\theta' \wedge \theta'' + \theta'' \wedge \theta') = 2R_{\mathcal{E}}.$$

Since θ'' has the same h-norm as θ', it follows that both $R_{\mathcal{V}}$ and $R_{\mathcal{E}}$ satisfy (6.3.31 $*$), hence are L^1_{loc} on Δ, according to Exercise 6.6. $\square$

***Proof of Theorem* 6.3.9.** We start with a variant of Ahlfors Lemma, whose proof is given as an exercise (Exercise 6.8). We denote by Δ_t the Euclidean Laplacian on the disc, that is, $\Delta_t = 4\partial_t\partial_{\bar t}$.

6.3.33. Lemma. *Let f be a C^2 function with nonnegative real values on the unit punctured disc Δ^*. Let us assume that the following inequality holds:*

$$\Delta_t \log f(t) \geqslant 4f(t). \tag{6.3.33 $*$}$$

Then

$$f(t) \leqslant \frac{1}{|t|^2 \mathrm{L}(t)^2} \quad \text{on } \Delta^*. \tag{6.3.33 $**$}$$

We thus aim at proving that $f = c\|\theta'\|_{\mathrm{h}}^2$ (for some $c > 0$) satisfies the assumption of the lemma. Let us set $\theta' = \theta'_0 \mathrm{d}t$ and $\theta'' = \theta''_0 \mathrm{d}\bar{t}$. Regarding θ'_0 as generating a line subbundle of $\mathcal{E}nd(\mathcal{E})$ with induced metric h, so that $\|\theta'\|_{\mathrm{h}}^2 = 2\|\theta'_0\|_{\mathrm{h}}$, the inequality for the curvature of a subbundle (see [**GH78**, p. 79]) implies

$$\|\theta'_0\|_{\mathrm{h}}^2 \cdot \mathrm{d}''\mathrm{d}' \log \|\theta'_0\|_{\mathrm{h}}^2 \leqslant \mathrm{h}\big(\mathrm{ad}(R_{\mathcal{E}})(\theta'_0), \overline{\theta'_0}\big),$$

in the sense that the coefficients of $\mathrm{d}t \wedge \mathrm{d}\bar{t}$ satisfy the corresponding inequality. The above expression of $R_{\mathcal{E}}$ amounts to $R_{\mathcal{E}} = -[\theta'_0, \theta''_0]\mathrm{d}t \wedge \mathrm{d}\bar{t}$ and the previous inequality reads

$$-\partial_t \partial_{\bar{t}} \log \|\theta'_0\|_{\mathrm{h}}^2 \leqslant -\frac{\mathrm{h}([[\theta'_0, \theta''_0], \theta'_0], \overline{\theta'_0})}{\|\theta'_0\|_{\mathrm{h}}^2}.$$

We write

$$\begin{aligned} -\mathrm{h}([[\theta'_0, \theta''_0], \theta'_0], \theta'_0) &= \mathrm{h}(\mathrm{ad}(\theta'_0)([\theta'_0, \theta''_0]), \overline{\theta'_0}) \\ &= \mathrm{h}([\theta'_0, \theta''_0], \overline{\mathrm{ad}(\theta''_0)(\theta'_0)}) = -\big\|[\theta'_0, \theta''_0]\big\|_{\mathrm{h}}^2, \end{aligned}$$

and the previous inequality reads

$$\Delta_t \log \|\theta'_0\|_{\mathrm{h}}^2 \geqslant 4\, \frac{\big\|[\theta'_0, \theta''_0]\big\|_{\mathrm{h}}^2}{\|\theta'_0\|_{\mathrm{h}}^2}.$$

Here comes the assumption that θ'_0 is nilpotent. We claim that there exists a constant $c > 0$ only depending on the rank of $\mathcal{E}$ such that $\big\|[\theta'_0, \theta''_0]\big\|_{\mathrm{h}} \geqslant c\|\theta'_0\|_{\mathrm{h}}^2$. Indeed, because we look for a universal constant c, it is enough to solve the question independently on each fiber, and we are reduced to a question on vector spaces, which is treated in see Exercise 6.12. As a consequence,

$$\Delta_t \log \|\theta'_0\|_{\mathrm{h}}^2 \geqslant 4c^2 \|\theta'_0\|_{\mathrm{h}}^2.$$

We conclude the proof of Theorem 6.3.9 by applying Lemma 6.3.33 to $f(t) = c^2\|\theta'_0\|_{\mathrm{h}}^2$. □

***Proof of Lemma* 6.3.12.** This lemma is a direct consequence of the following lemma, together with Exercise 6.7.

6.3.34. Lemma. *For u as in Lemma* 6.3.12*, the function $\log \|u\|_{\mathrm{h}}^2$ is subharmonic in Δ^*, that is, we have the inequality*

$$\Delta_t \log \|u\|_{\mathrm{h}}^2 \geqslant 0.$$

Proof. Let us start by computing $\Delta_t\|u\|^2_{\mathrm{h}}$. On the one hand, we have

$$(\Delta_t\|u\|^2_{\mathrm{h}})\,\mathrm{d}t\wedge\mathrm{d}\bar t = 4\mathrm{d}'\mathrm{d}''\|u\|^2_{\mathrm{h}}.$$

On the other hand, u satisfies $D'u = 0$ and $D''u = 0$, that is, $D'_{\mathrm{h}}u = -\theta'u$ and $D''_{\mathrm{h}}u = -\theta''u$ (recall the notation of Section 4.2.b). Moreover, since $D''_{\mathrm{h}}(\theta') = 0$ (see (4.2.6)), we find

$$D''_{\mathrm{h}}\theta'u = -\theta'D''_{\mathrm{h}}u = \theta'\theta''u,$$

and similarly $D'_{\mathrm{h}}\theta''u = \theta''\theta'u$. We thus obtain, since θ'' is the h-adjoint of θ' (we use the convention of Remark 4.2.3),

$$\begin{aligned}\mathrm{d}'\mathrm{d}''\|u\|^2_{\mathrm{h}} &= -\mathrm{d}'\big[\mathrm{h}(\theta''u,\overline{u}) + \mathrm{h}(u,\overline{\theta'u})\big] = -2\mathrm{d}'\mathrm{h}(u,\overline{\theta'u})\\ &= 2\big[\mathrm{h}(\theta'u,\overline{\theta'u}) - \mathrm{h}(u,\overline{\theta'\theta''u})\big]\\ &= 2\big[\mathrm{h}(\theta'u,\overline{\theta'u}) - \mathrm{h}(\theta''u,\overline{\theta''u})\big].\end{aligned}$$

Since $\|\mathrm{d}t\| = \|\mathrm{d}\bar t\| = 2$ with the metric induced by the Euclidean volume form, we find

$$\mathrm{h}(\theta'u,\overline{\theta'u}) = \frac14\,\|\theta'u\|^2_{\mathrm{h}}\,\mathrm{d}t\wedge\mathrm{d}\bar t \quad\text{and}\quad \mathrm{h}(\theta''u,\overline{\theta''u}) = -\frac14\,\|\theta''u\|^2_{\mathrm{h}}\,\mathrm{d}t\wedge\mathrm{d}\bar t,$$

we finally obtain

$$\Delta_t\|u\|^2_{\mathrm{h}} = 2\big(\|\theta'u\|^2_{\mathrm{h}} + \|\theta''u\|^2_{\mathrm{h}}\big). \tag{6.3.35}$$

Now,

$$\Delta_t\log\|u\|^2_{\mathrm{h}} = \frac{\Delta_t\|u\|^2_{\mathrm{h}}}{\|u\|^2_{\mathrm{h}}} - 4\,\frac{\partial_t\|u\|^2_{\mathrm{h}}}{\|u\|^2_{\mathrm{h}}}\,\frac{\partial_{\bar t}\|u\|^2_{\mathrm{h}}}{\|u\|^2_{\mathrm{h}}},$$

and $\partial_t\|u\|^2_{\mathrm{h}}\cdot\partial_{\bar t}\|u\|^2_{\mathrm{h}}$ is the coefficient of $\mathrm{d}t\wedge\mathrm{d}\bar t$ in $\mathrm{d}'\mathrm{h}(u,\overline{u})\wedge\mathrm{d}''\mathrm{h}(u,\overline{u})$. The previous arguments give

$$\begin{aligned}\mathrm{d}'\mathrm{h}(u,\overline{u})\wedge\mathrm{d}''\mathrm{h}(u,\overline{u}) &= 4\mathrm{h}(\theta'u,\overline{u})\wedge\mathrm{h}(u,\overline{\theta'u})\\ &= 4\mathrm{h}(\theta'u,\overline{u})\wedge\overline{\mathrm{h}(\theta'u,\overline{u})} = \|\mathrm{h}(\theta'u,\overline{u})\|^2\,\mathrm{d}t\wedge\mathrm{d}\bar t.\end{aligned}$$

Therefore, noticing that $\|\mathrm{h}(\theta'u,\overline{u})\| = \|\mathrm{h}(\theta''u,\overline{u})\|$,

$$\begin{aligned}\partial_t\|u\|^2_{\mathrm{h}}\cdot\partial_{\bar t}\|u\|^2_{\mathrm{h}} = \|\mathrm{h}(\theta'u,\overline{u})\|^2 &= \frac12\,(\|\mathrm{h}(\theta'u,\overline{u})\|^2 + \|\mathrm{h}(\theta''u,\overline{u})\|^2)\\ &\leqslant \frac12\,\|u\|^2_{\mathrm{h}}\cdot(\|\theta'u\|^2_{\mathrm{h}} + \|\theta''u\|^2_{\mathrm{h}}),\end{aligned}$$

and the desired inequality follows. □

6.3.36. Remarks.

(a) Together with the conclusion of Lemma 6.3.12, (6.3.35) also implies that $\Delta_t\|u\|^2_{\mathrm{h}}$ and $\|\theta'u\|^2_{\mathrm{h}} + \|\theta''u\|^2_{\mathrm{h}}$ are L^1_{loc} at the origin (see Exercise 6.9).

(b) On a Riemann surface X equipped with a Kähler metric, the Laplacian Δ satisfies $\Delta = 2\Delta'' = -2\,\mathrm{i}\,\Lambda\mathrm{d}'\mathrm{d}''$. In the setting of Lemma 6.3.12 with a punctured X^* instead of Δ^*, then an argument similar to that leading to (6.3.35) gives

$$\Delta\|u\|^2_{\mathrm{h}} = -4\big(\|\theta'u\|^2_{\mathrm{h}} + \|\theta''u\|^2_{\mathrm{h}}\big).$$

Moreover, (a) implies that the right-hand side—hence the left-hand side also—is L^1_{loc} on X.

6.4. Semi-simplicity

As an application of the metric properties of Section 6.3, we extend in this section the results of Section 4.3 to the case of a punctured projective curve. Let X be a smooth projective curve and let X^* be a Zariski open subset of X (i.e., the complement of a finite set of points).

6.4.1. Theorem. *Let* $(\mathcal{H},\mathrm{h},D)$ *be a* nilpotent harmonic bundle *on* X^* *and let* $\underline{\mathcal{H}}$ *be the associated local system* $\operatorname{Ker} D$. *Assume that* $(\mathcal{V},\mathrm{h},\nabla)$ *and its dual* $(\mathcal{V}^\vee,\mathrm{h},\nabla)$ *satisfy* 6.3.2(1). *Then the complex local system* $\underline{\mathcal{H}}$ *is semi-simple.*

***6.4.2. Corollary* (of Theorems 6.3.2(1) and 6.4.1).** *Let* $H=(\mathcal{H},F'^{\bullet}\mathcal{H},F''^{\bullet}\mathcal{H},D,\mathrm{S})$ *be a polarized variation of* $\mathbb{C}$*-Hodge structure of weight* w *on* X^* *(see Definition* 4.1.4*), and let* $\underline{\mathcal{H}}=\operatorname{Ker}\nabla$ *be the associated complex local system. Then* $\underline{\mathcal{H}}$ *is semi-simple.* □

6.4.3. Remark. After having proved the Hodge-Zucker theorem 6.11.1, we will complement this corollary in a way analogous to that of Theorem 4.3.13, by showing (Theorem 6.14.17) that each irreducible component of $\underline{\mathcal{H}}$ underlies an essentially unique polarized variation of Hodge structure and we will show how to recover the polarized variation of Hodge structure H from its irreducible components. In other words, the underlying local system of a simple object in the category of polarized variations of Hodge structure on X^* is irreducible.

Before starting the proof of the semi-simplicity theorem, we notice useful consequences of Property (1) of Theorem 6.3.2.

6.4.4. Proposition. *Assume that* $(\mathcal{V},\mathrm{h},\nabla)$ *satisfies* 6.3.2(1). *Then*

(1) *any flat holomorphic subbundle with induced metric and connection* $(\mathcal{V}_1,\mathrm{h},\nabla)$ *also satisfies* 6.3.2(1)*;*

(2) *the determinant* $\det(\mathcal{V},\mathrm{h},\nabla)$ *also satisfies* 6.3.2(1).

Proof.

(1) In view of Remark 6.2.7(1), the question reduces to the $\mathcal{O}_\Delta$-coherence of $j_*\mathcal{V}_1\cap\mathcal{V}^\beta_{\mathrm{mod}}$. But the latter is equal to $j_*\mathcal{V}_1\cap\mathcal{V}^\beta_*$, which $\mathcal{O}_\Delta$-locally free, being equal to $\mathcal{V}^\beta_{1,*}$ (see Remark 6.2.3(1)).

(2) This point follows from Remarks 6.2.3(3) and 6.2.7(2). □

6.4.5. Corollary. *Assume that* $(\mathcal{V},\mathrm{h},\nabla)$ *and its dual* $(\mathcal{V}^\vee,\mathrm{h},\nabla)$ *satisfy* 6.3.2(1). *Then so does any direct summand* $(\mathcal{V}_1,\mathrm{h}_1,\nabla_1)$ *of* $(\mathcal{V},\mathrm{h},\nabla)$. □

Let $(\mathcal{V},\nabla,\mathrm{h})$ be a Hermitian holomorphic bundle with connection on a punctured compact Riemann surface X^*. Let R_{h} denote the curvature of $(\mathcal{V},\mathrm{h})$: it is a section of $\mathcal{E}^{1,1}_{X^*}\otimes\mathcal{E}nd(\mathcal{V})$. The determinant bundle $\det(\mathcal{V},\mathrm{h},\nabla)$ has curvature $\operatorname{tr}(R_{\mathrm{h}})$. Assume that the curvature $\mathrm{tr}(R_{\mathrm{h}})$ of $\det(\mathcal{V},\mathrm{h})$ is L^1_{loc} on X, i.e., in local coordinates near a puncture, it can be written as $k\,\mathrm{d}t\wedge\mathrm{d}\bar{t}$ with k being L^1_{loc}. We then set

$$\deg^{\mathrm{an}}(\mathcal{V},\mathrm{h})=\frac{\mathrm{i}}{2\pi}\int_X \mathrm{tr}(R_{\mathrm{h}}).$$

***6.4.6. Proposition* (Vanishing of the analytic degree).** *Assume that $(\mathcal{V},\mathrm{h},\nabla)$ is a Hermitian holomorphic bundle with connection on a punctured compact Riemann surface X^* that satisfies* 6.3.2(1) *as well as its dual $(\mathcal{V}^\vee,\mathrm{h},\nabla)$. Assume that the curvature* $\mathrm{tr}(R_{\mathrm{h}})$ *of* $\det(\mathcal{V},\mathrm{h})$ *is L^1_{loc} on X. Then for any flat holomorphic subbundle $(\mathcal{V}_1,\nabla)$ of $(\mathcal{V},\nabla)$ equipped with the induced Hermitian metric, the curvature of* $\det(\mathcal{V}_1,\mathrm{h})$ *is L^1_{loc} on X and we have*

$$\deg^{\mathrm{an}}(\mathcal{V}_1,\mathrm{h})=0.$$

6.4.7. Lemma. *If $(\mathcal{V},\mathrm{h},\nabla)$ and its dual $(\mathcal{V}^\vee,\mathrm{h},\nabla)$ satisfy* 6.3.2(1) *on Δ^*, then the* h*-norm of any local section v of $\mathcal{V}^\beta_*$ whose image in $\mathrm{gr}^\beta\mathcal{V}_*$ is nonzero satisfies the inequalities*

$$\forall\,\varepsilon>0,\quad |t|^{\beta+\varepsilon}\leqslant\|v\|_{\mathrm{h}}\leqslant|t|^{\beta-\varepsilon}\quad(|t|<R_\varepsilon).\tag{6.4.7$\,*$}$$

Proof. The right inequality is by assumption. Let $v^\vee$ be a local section of $(\mathcal{V}^\vee_*)^{-\beta}$ such that $\langle v^\vee,v\rangle=1$ (see Remark 6.2.3(1)). Then $\|v^\vee\|_{\mathrm{h}}\leqslant|t|^{-\beta-\varepsilon}$ for all $\varepsilon>0$ and $|t|$ correspondingly small enough, by assumption. By computing in an orthonormal frame, Schwartz inequality implies $\|v^\vee\|_{\mathrm{h}}\|v\|_{\mathrm{h}}\geqslant|\langle v^\vee,v\rangle|=1$. Therefore, $\|v\|_{\mathrm{h}}\geqslant|t|^{\beta+\varepsilon}$, hence the assertion. ☐

***Proof of Proposition* 6.4.6.** Let $x\in X$ be a puncture and let $\gamma_x\in[0,1)$ be the unique jumping index of the filtration of $\mathcal{L}_*:=\det\mathcal{V}_{1,*}$ (this is justified by Proposition 6.4.4(2)). For a local frame v at x of the Deligne canonical extension $\mathcal{L}^0_*$ obtained from a frame adapted to the filtration of $\mathcal{V}_{1,*}$, we deduce that $\|v\|_{\mathrm{h}}$ satisfies the inequalities of the lemma with γ_x instead of β.

Let us prove the statement on the curvature of $(\mathcal{V}_1,\mathrm{h})$. This is a local statement near each puncture, so that we assume that $X^*=\Delta^*$. Let $\boldsymbol{v}$ be a frame of $\mathcal{V}$ inducing a frame of $\mathcal{V}^0_*/t\mathcal{V}^0_*$ adapted to the filtration induced by $\mathcal{V}^\bullet_*$ and to $\mathcal{V}_{1,*}$, so that part of $\boldsymbol{v}$, denoted $\boldsymbol{v}_1$, is a frame of $\mathcal{V}^0_{1,*}/t\mathcal{V}^0_{1,*}$ adapted to the filtration induced by $\mathcal{V}^\bullet_{1,*}$. Then the curvature matrix of $(\mathcal{V}_1,\mathrm{h})$ in the frame $\boldsymbol{v}_1$ is smaller than that of $(\mathcal{V},\mathrm{h})$ in the frame $\boldsymbol{v}$, in the sense of [**GH78**, p. 79]. Taking traces, the same property holds for $\det\mathcal{V}_1$ with its frame $v_1=\det\boldsymbol{v}_1$ and $\det\mathcal{V}$ with its frame $v=\det\boldsymbol{v}$. This reads

$$-\partial_t\partial_{\bar t}\log\|v_1\|^2_{\mathrm{h}}\leqslant-\partial_t\partial_{\bar t}\log\|v\|^2_{\mathrm{h}},$$

that is, with the Laplacian $\Delta_t=4\partial_t\partial_{\bar t}$,

$$\Delta_t\log\|v_1\|^2_{\mathrm{h}}\geqslant\Delta_t\log\|v\|^2_{\mathrm{h}}.$$

We have $\Delta_t\log\|v_1\|^2_{\mathrm{h}}=\Delta_t f_1$ on Δ^* with $f_1=\log\|v_1\|^2_{\mathrm{h}}-\log|t|^{2\gamma_1}$, where γ_1 is the exponent of (6.4.7$\,*$) for v_1, and similarly f and γ for v. By assumption, f is L^1_{loc} on Δ. On the other hand, (6.4.7$\,*$) for v_1 implies that for all $\varepsilon>0$, $|f_1(t)|\leqslant\varepsilon\mathrm{L}(t)$ on $\Delta^*_{R_\varepsilon}(x)$ for $R_\varepsilon>0$ small enough. Therefore, $\lim_{t\to0}|f_1(t)|/\mathrm{L}(t)=0$. The assumptions in

Exercise 6.10 are thus fulfilled and we can conclude that $\Delta_t f_1 = \Delta_t \log \|v_1\|^2_{\mathrm{h}}$ is L^1_{loc} on Δ, as wanted.

By the residue theorem we have $\deg(\mathcal{L}^0_*)^\vee = -\deg \mathcal{L}^0_* = \sum_x \gamma_x$. Let us fix an arbitrary C^∞ metric on $(\mathcal{L}^0_*)^\vee$. We thus obtain a metric, that we still denote by h, on the trivial bundle $\mathcal{O} = (\mathcal{L}^0_*)^\vee \otimes \mathcal{L}^0_*$, such that the norm of the unit section 1 satisfies (6.4.7 $*$) (up to constants). We aim at proving that $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h})$ is well-defined and is equal to $\sum_x \gamma_x$.

Let us consider a model metric h^o on $\mathcal{O}$, such that $\|1\|_{\mathrm{h}^o}$ is C^∞ on X^*, equal to h on the complement of discs centered at the punctures, and *equal to* $|t|^{\gamma_x}$ for some local coordinate t at each puncture x. The curvature of h^o is $\mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o}$, and is meaningful as a $(1,1)$-current on X. In the neighbourhood of a puncture, we have

$$\frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o} = \frac{\gamma_x \mathrm{i}}{\pi}\,\mathrm{d}''\mathrm{d}' \log |t| = -\gamma_x \delta_x,$$

so that, on X, we have $\frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o} = \eta - \sum_x \gamma_x \delta_x$, where $\eta \in \mathcal{E}^{1,1}_{\mathrm{c}}(X^*)$. Furthermore, $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) = \int_X \eta$. We then find

$$0 = \deg \mathcal{O} = \frac{\mathrm{i}}{2\pi}\langle 1, \mathrm{d}''\mathrm{d}' \log \|1\|^2_{\mathrm{h}^o}\rangle = \deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) - \sum_x \gamma_x,$$

and thus $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) = \sum_x \gamma_x$.

Let us set $f = \log \|1\|_{\mathrm{h}} - \log \|1\|_{\mathrm{h}^o}$. It is supported on the union of discs $\Delta^*_R(x)$, where x is a puncture. Then, as above, (6.4.7 $*$) implies that for each puncture x, $\lim_{t\to 0} |f(t)|/\mathrm{L}(t) = 0$. On the other hand, we have seen that $\mathrm{d}''\mathrm{d}' f_{|X^*}$ is L^1_{loc} on X, so that $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h})$ is well-defined. The assumptions of Exercise 6.9 are thus fulfilled and we conclude that the current $\mathrm{d}''\mathrm{d}' f$ is L^1_{loc} on X. Furthermore,

$$\frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' \log \|1\|_{\mathrm{h}} = \eta + \frac{\mathrm{i}}{2\pi}\,\mathrm{d}''\mathrm{d}' f - \sum_x \gamma_x \delta_x$$

as currents on X, where the first two terms of the right-hand side are L^1_{loc} on X. Since

$$\int_{X^*} \mathrm{d}''\mathrm{d}' f = \int_X \mathrm{d}''\mathrm{d}' f = \langle 1, \mathrm{d}''\mathrm{d}' f\rangle = 0,$$

we find $\deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}) = \deg^{\mathrm{an}}(\mathcal{O}, \mathrm{h}^o) = \sum_x \gamma_x$, as wanted. □

***Proof of the semi-simplicity theorem* 6.4.1.** We argue by induction on the rank of $\mathcal{H}$, the case of rank 1 being clear. Let $(\mathcal{H}_1, D)$ be a flat subbundle of $(\mathcal{H}, D)$, that we equip with the Hermitian metric h_1 induced by h. Proving that $(\mathcal{H}_1, \mathrm{h}_1, D)$ is a direct summand amounts to proving that the h-orthogonal projection $\pi : \mathcal{H} \to \mathcal{H}_1$ is compatible with D. However, in order to apply Theorem 6.4.1 by induction on the rank, we also need to prove that $(\mathcal{H}_1, \mathrm{h}_1, D)$ is a nilpotent harmonic bundle and that it satisfies 6.3.2(1) together with its dual. Corollary 6.4.5 provides the latter property. Considering π as a section of the nilpotent harmonic bundle $(\mathcal{E}nd\,\mathcal{H}, \mathrm{h}, D)$ (see Exercise 6.11), we are thus left with proving

(1) $D(\pi) = 0$,
(2) $\theta(\pi) = 0$.

We first claim that the second property is a consequence of the first one. By (1), π is a flat section of $\mathcal{E}nd\,\mathcal{V}$ which preserves the metric, hence the filtration $\mathcal{V}^{\bullet}_{\mathrm{mod}}$, and satisfies thus the hypotheses in Lemma 6.3.12. It follows that $\|\pi\|^2_{\mathrm{h}}$ is bounded. Furthermore, according to Remark 6.3.36(b), the function $\|\theta'(\pi)\|^2_{\mathrm{h}}+\|\theta''(\pi)\|^2_{\mathrm{h}}$ is L^1_{loc} on X with integral equal to zero, since X is compact and $\langle 1,\Delta\|\pi\|^2_{\mathrm{h}}\rangle=0$. Therefore, $\|\theta'(\pi)\|^2_{\mathrm{h}}+\|\theta''(\pi)\|^2_{\mathrm{h}}=0$, as claimed.

Let us prove (1), that is, $D(\pi)=0$. We set $(\mathcal{V}_1,\nabla)=\operatorname{Ker}D''$. Let us denote by h_1 the metric on $\mathcal{V}_1$ to avoid confusion, and let R_{h_1} denote the corresponding curvature.

6.4.8. Lemma. *With the previous notation we have, denoting by $\|\bullet\|^2_{\mathrm{HS}}$ the Hilbert-Schmidt norm,*

$$\operatorname{tr}R_{\mathrm{h}_1}=\frac{\mathrm{i}}{2}\,\|D(\pi)\|^2_{\mathrm{HS}}\,\mathrm{vol}\,.$$

By Corollary 6.3.31 and Proposition 6.4.6, we have $\deg^{\mathrm{an}}(\mathcal{V}_1,\mathrm{h}_1)=0$. On the other hand, the above lemma yields

$$\deg^{\mathrm{an}}(\mathcal{V}_1,\mathrm{h}_1)=-\frac{1}{4\pi}\int_X\|D(\pi)\|^2_{\mathrm{HS}}\,\mathrm{vol},$$

hence $D(\pi)=0$, and this concludes the proof of the theorem. □

***Proof of Lemma* 6.4.8.** We will use the formulas in Exercises 4.4–4.10 to compute the curvature of $\det(\mathcal{V}_1,\mathrm{h}_1)$. For any Hermitian holomorphic bundle with flat connection $(\mathcal{H},\mathrm{h},D)$, since $\dim X^*=1$, we have

- $D''_{\mathrm{h}}(\theta')+D'_{\mathrm{h}}(\theta'')=-(\theta'\wedge\theta''+\theta''\wedge\theta')$,
- $(D^{\mathrm{c}})^2=D''_{\mathrm{h}}(\theta')+D'_{\mathrm{h}}(\theta'')-(\theta'\wedge\theta''+\theta''\wedge\theta')=-2(\theta'\wedge\theta''+\theta''\wedge\theta')$,
- $4\overline{\mathcal{D}}^2=DD^{\mathrm{c}}+D^{\mathrm{c}}D-2(\theta'\wedge\theta''+\theta''\wedge\theta')$,

and the formula of Exercise 4.4(5) becomes

$$R_{\mathrm{h}}=-\tfrac{1}{2}(DD^{\mathrm{c}}+D^{\mathrm{c}}D)-(\theta'\wedge\theta''+\theta''\wedge\theta').$$

Taking trace, we obtain, since the trace of $(\theta'\wedge\theta''+\theta''\wedge\theta')$ is zero,

$$\operatorname{tr}R_{\mathrm{h}}=-\tfrac{1}{2}\operatorname{tr}(DD^{\mathrm{c}}+D^{\mathrm{c}}D).$$

Then Exercise 4.10(4) implies

$$\operatorname{tr}R_{\mathrm{h}_1}=-\tfrac{1}{2}\operatorname{tr}(D(\pi)D^{\mathrm{c}}(\pi))=\tfrac{1}{2}\operatorname{tr}(D^{\mathrm{c}}(\pi)D(\pi)),$$

and this yields

$$\operatorname{tr}R_{\mathrm{h}_1}=(\Lambda\operatorname{tr}R_{\mathrm{h}_1})\,\mathrm{vol}=\tfrac{1}{2}\,\Lambda\operatorname{tr}(D^{\mathrm{c}}(\pi)D(\pi))\,\mathrm{vol}\,.$$

Since Λ commutes with π and acts by 0 except on $(1,1)$-forms with values in $\mathcal{H}$, we can write, according to Exercise 4.9(4),

$$\Lambda D^{\mathrm{c}}(\pi)D(\pi)=[[\Lambda,D^{\mathrm{c}}],\pi]D(\pi)=-\,\mathrm{i}\,D^{\star}(\pi)D(\pi).$$

But π being obviously self-adjoint with respect to h, and recalling that $f^{\star}=f^{*}$ for a $\mathcal{C}^{\infty}_{X^*}$-linear morphism between Hermitian bundles, we deduce

$$D^{\star}(\pi)=[D^{\star},\pi]=-[D,\pi]^{\star}=-[D,\pi]^{*}=-D(\pi)^{*}.$$

If $\|D(\pi)\|^2_{\mathrm{HS}}$ denotes the square of the Hilbert-Schmidt norm of the $\mathcal{C}^\infty_{X^*}$-linear morphism $D(\pi) : \mathcal{H} \to \mathcal{E}^1_X \otimes \mathcal{H}$, i.e., $\|D(\pi)\|^2_{\mathrm{HS}} = \operatorname{tr}(D(\pi)^* D(\pi))$, we finally obtain the desired formula. $\square$

6.5. Exercises

***Exercise 6.1* (The structure of $(\mathcal{V}_*, \nabla)$).** Assume that $(\mathcal{V}_*, \nabla)$ has a regular singularity at the origin of Δ and no other singularity.

(1) Show that $(\mathcal{V}_*, \nabla)$ is a successive extension of rank 1 meromorphic connections. [*Hint*: Use a Jordan basis for $\mathcal{R}$ of $\mathcal{V}^0_*/t\mathcal{V}^0_*$.]

(2) Assume that $\mathcal{V}$ has rank 1. Let v_γ be an $\mathcal{O}_\Delta$-basis of $\mathcal{V}^0_*$ in which the matrix of $t\nabla_{\partial_t}$ is constant. Show that $t\nabla_{\partial_t} v_\gamma = \gamma v_\gamma$ with $\operatorname{Re}\gamma \in [0, 1)$. Identify $\mathcal{V}^\nabla$ with the subsheaf of $\rho_* \mathcal{O}_{\widetilde{\Delta}^*}$ consisting of multiples of some (or any) branch of the multivalued function $t^{-\gamma}$, by sending $ct^{-\gamma}$ to $ct^{-\gamma} v_\gamma$.

(3) For $\operatorname{Re}\gamma \in [0, 1)$ and $p \geqslant 0$, set $\mathcal{J}_{\gamma,p} = (\mathcal{O}_\Delta[1/t]^{p+1}, \nabla)$, where the matrix of ∇_{∂_t} in the canonical basis $\boldsymbol{v}_{\gamma,p} = (v_{\gamma,0}, \dots, v_{\gamma,p})$ is given by $t\nabla_{\partial_t} v_{\gamma,k} = \gamma v_{\gamma,k} - v_{\gamma,k-1}$ (so that $v_{\gamma,p}$ is a generating section with respect to $t\nabla_{\partial_t}$). Show that $(\mathcal{V}_*, \nabla)$ has a decomposition

$$(\mathcal{V}_*, \nabla) \simeq \bigoplus_{\gamma \in [0,1)} \Big[\bigoplus_p (\mathcal{J}_{\gamma,p}, \nabla) \Big]. \tag{6.5.1}$$

[*Hint*: Use a Jordan decomposition for $\mathcal{R}$.]

(4) Compute $\operatorname{Ker}\nabla$ on $\mathcal{V}_*$ in terms of this decomposition.

(5) Show that there is no nonzero morphism $\mathcal{J}_{\gamma_1,p} \to \mathcal{J}_{\gamma_2,q}$ if $\gamma_1 \neq \gamma_2 \in [0, 1)$, and conclude that the decomposition indexed by γ above is unique.

Exercise 6.2. Show the following properties.

(1) $\mathcal{V}^{\beta+k}_* = t^k \mathcal{V}^\beta_*$ for every $k \in \mathbb{Z}$.

(2) $\operatorname{gr}^\beta \mathcal{V}_*$ can be identified with the generalized β-eigenspace of the residue of ∇ on $\mathcal{V}^{[\beta]}_*/t\mathcal{V}^{[\beta]}_*$.

(3) The map induced by ∇_{∂_t} sends $\operatorname{gr}^\beta \mathcal{V}_*$ to $\operatorname{gr}^{\beta-1} \mathcal{V}_*$ and, if $\beta \neq 0$, it is an isomorphism. [*Hint*: Use that the composition $t\nabla_{\partial_t} : \operatorname{gr}^\beta \mathcal{V}_* \to \operatorname{gr}^\beta \mathcal{V}_*$ is identified with the restriction of the residue of ∇ on $\mathcal{V}^{[\beta]}_*/t\mathcal{V}^{[\beta]}_*$ to its generalized β-eigenspace.]

(4) The map $\nabla_{\partial_t} : \mathcal{V}^\beta_* \to \mathcal{V}^{\beta-1}_*$ is onto (equivalently, $t\nabla_{\partial_t} : \mathcal{V}^\beta_* \to \mathcal{V}^\beta_*$ is onto) provided that $\beta > 0$. [*Hint*: Reduce to the case where $\mathcal{V}_*$ has rank 1 by using Exercise 6.1 and has a basis v_γ which satisfies $t\nabla_{\partial_t} v_\gamma = \gamma v_\gamma$ for some $\gamma \in [0, 1)$, and show that $\mathcal{V}^{\gamma+k}_* = t^k \mathcal{O}_\Delta v_\gamma$ for $k \in \mathbb{Z}$.]

(5) With respect to a decomposition of $(\mathcal{V}_*, \nabla)$ as in Exercise 6.1(3), show that, for $\gamma \in [0, 1)$, we have, for $k \in \mathbb{Z}$,

$$\mathcal{V}^{\gamma+k}_* = \bigoplus_{i,\, \gamma_i \geqslant \gamma} t^k \mathcal{O}_\Delta \cdot \boldsymbol{v}_{\gamma_i,p_i} \oplus \bigoplus_{i,\, \gamma_i < \gamma} t^{k+1} \mathcal{O}_\Delta \cdot \boldsymbol{v}_{\gamma_i,p_i}.$$

(6) The subsheaf $\sum_{j \geqslant 0} (\nabla_{\partial_t})^j \mathcal{V}^\beta_*$ of $\mathcal{V}_*$ is an $\mathcal{O}_\Delta$-module equipped with a connection ∇, and

- does not depend on $\beta > -1$, or on $\beta \leqslant -1$,
- in the latter case, it is equal to $\mathcal{V}_*$,
- in the former case, we call it the *middle extension* of $(\mathcal{V}, \nabla)$ and denote it by $\mathcal{V}_{\text{mid}}$; then $\nabla_{\partial_t} : \mathcal{V}_{\text{mid}} \to \mathcal{V}_{\text{mid}}$ is *onto* and has kernel equal to the sheaf $j_*(\mathcal{V}^\nabla)$.

***Exercise 6.3* (Local freeness and subbundles).** Let F be a rank two free bundle on Δ, with basis f_1, f_2. Let $E \subset j^*F$ be the subbundle on Δ^* with basis $e = \exp(1/t)f_1 + f_2$. Show that $j_*E \cap F$ is not locally free. [*Hint*: Show that the germ $(j_*E \cap F)_0$ consists of sections $a(t)e$, with $a(t)$ holomorphic on some punctured neighbourhood of 0 in Δ, such that both $a(t)$ and $\exp(1/t)a(t)$ belong to $\mathbb{C}\{t\}$; conclude that $(j_*E \cap F)_0 = 0$.]

Exercise 6.4. Prove the result of Theorem 6.3.2 in the unitary case of Example 6.3.1.

Exercise 6.5. Show the following identities on Δ^* for the function $\mathrm{L}(t) = -\log|t|^2 = -\log t\bar{t}$:

$$(6.5*)\qquad \begin{aligned} &\mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{Y}\,\mathrm{L}(t)^{\mp\mathrm{H}/2} = \mathrm{L}(t)^{\mp1}\,\mathrm{Y}, \quad \mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{X}\,\mathrm{L}(t)^{\mp\mathrm{H}/2} = \mathrm{L}(t)^{\pm1}\,\mathrm{X}\\ &\mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{e}^{\mathrm{Y}}\,\mathrm{L}(t)^{\mp\mathrm{H}/2} = \mathrm{e}^{\mathrm{L}(t)^{\mp1}\,\mathrm{Y}}, \quad \mathrm{L}(t)^{\pm\mathrm{H}/2}\,\mathrm{e}^{\mathrm{X}}\,\mathrm{L}(t)^{\mp\mathrm{H}/2} = \mathrm{e}^{\mathrm{L}(t)^{\pm1}\,\mathrm{X}}\end{aligned}$$

[*Hint*: Use Exercise 3.1(1)] and

$$(6.5**)\qquad \begin{aligned} &-t\partial_t\mathrm{L}(t)^k/k! = -\bar{t}\partial_{\bar{t}}\mathrm{L}(t)^k/k! = \mathrm{L}(t)^{k-1}/(k-1)! \quad (k \geqslant 0),\\ &\mathrm{L}(t)^{\mathrm{H}/2}\,t\,\frac{\partial}{\partial t}\big(\mathrm{L}(t)^{-\mathrm{H}/2}\big) = \mathrm{L}(t)^{\mathrm{H}/2}\,\bar{t}\,\frac{\partial}{\partial \bar{t}}\big(\mathrm{L}(t)^{-\mathrm{H}/2}\big) = \frac{\mathrm{H}/2}{\mathrm{L}(t)}.\end{aligned}$$

Exercise 6.6. Let $R \in (0,1)$, let $\beta \in \mathbb{R}$ and $\ell \in \mathbb{Z}$. Show that the integral

$$\int_0^R r^{2\beta}\mathrm{L}(r)^\ell\,\frac{\mathrm{d}r}{r}$$

is finite iff $\beta > 0$ or $\beta = 0$ and $\ell \leqslant -2$ (recall that $\mathrm{L}(r) := 2|\log r| = -2\log r$). Conclude that the function $t \mapsto |t|^{-2}\mathrm{L}(t)^{-2}$ is L^1_{loc} near the origin. [*Hint*: Recall that the volume form in polar coordinates is a multiple of $r\mathrm{d}r\mathrm{d}\theta$.]

***Exercise 6.7* (Subharmonic functions).** Let $R \in (0,1)$ and let Δ^*_R be the punctured open disc of radius R. Let f be a continuous subharmonic function on Δ^*_R.

(1) Assume that $\limsup_{t\to0} f(t)/\mathrm{L}(t) \leqslant 0$. Show that $f \leqslant \sup_{\partial\Delta_{R'}} f(t)$ on $\Delta^*_{R'}$. [*Hint*: Reduce first to the case where $\sup_{\partial\Delta_{R'}} f(t) = 0$ by considering $f - \sup_{\partial\Delta_{R'}} f(t)$. Then, prove that, for any $\varepsilon > 0$, $f(t) - \varepsilon\mathrm{L}(t) \leqslant 0$ on $\Delta^*_{R'}$ by showing first that $\limsup_{t\to0}(f(t) - \varepsilon\mathrm{L}(t)) \leqslant 0$ and by applying the maximum principle on $\Delta^*_{R'}$ for subharmonic functions, i.e., if g is subharmonic on $\Delta^*_{R'}$ and if for any $t_o \in \{0\} \cup \partial\Delta_{R'}$ it satisfies $\limsup_{t\to t_o} g(t) \leqslant 0$, then $g \leqslant 0$ on $\Delta^*_{R'}$.]

(2) Assume that, for any $\varepsilon > 0$, there exists $C_\varepsilon > 0$ satisfying $f(t) \leqslant \log C_\varepsilon + \varepsilon\mathrm{L}(t)$ on $\Delta^*_{R'}$. Prove that $f \leqslant \sup_{\partial\Delta_{R'}} f(t)$ on $\Delta^*_{R'}$. [*Hint*: Show that f satisfies the assumptions in (1).]

Exercise 6.8 **(Proof of Lemma 6.3.33).** In this exercise, Δ, resp. Δ^*, denotes the open disc, resp. the punctured open disc, of radius 1 with coordinate τ, resp. t, and $\rho:\Delta\to\Delta^*$ is the model of a universal covering defined by $\rho(\tau)=\exp(\mathrm{i}(1+\tau)/(1-\tau))$. The Poincaré metric on Δ, resp. Δ^*, has volume form $\mathrm{vol}_\Delta=(1-|\tau|^2)^{-2}|\mathrm{d}\tau\,\mathrm{d}\overline{\tau}|$, resp. $\mathrm{vol}_{\Delta^*}=(|t|\mathrm{L}(t))^{-2}|\mathrm{d}t\,\mathrm{d}\overline{t}|$. Furthermore, $\rho^*\mathrm{vol}_{\Delta^*}=\mathrm{vol}_\Delta$. Let $\Delta_\tau=4\partial_\tau\partial_{\overline{\tau}}$, resp. $\Delta_t=4\partial_t\partial_{\overline{t}}$, be the corresponding Laplacians. Let $f:\Delta^*\to\mathbb{R}_+$ be a C^2 function satisfying the assumptions of Lemma 6.3.33. We first transfer the assumption on Δ^* to an assumption on Δ.

(1) For $R\in(0,1]$, set $\mathrm{v}_R(\tau)=R^2/(R-|\tau|^2)^2$ on the open disc Δ_R. Show that $\Delta_\tau\log\mathrm{v}_R=4\mathrm{v}_R$ and $\mathrm{v}_{1|\Delta_R}\leqslant\mathrm{v}_R$.

(2) Express the equality $\rho^*\mathrm{vol}_{\Delta^*}=\mathrm{vol}_\Delta$ as $(|\rho(\tau)||\log\rho(\tau)|)^{-2}=\mathrm{v}_1\cdot|\rho'(\tau)|^{-2}$.

(3) Set $g(\tau)=f\circ\rho(\tau)\cdot|\rho'(\tau)|^2$. Prove that the nonnegative real function g satisfies $\Delta_\tau\log g(\tau)\geqslant 4g(\tau)$. [*Hint*: Show that for a C^2 function $h(t)$, $\Delta_\tau(h\circ\rho)=(\Delta_t h)\circ\rho\cdot|\rho'(\tau)|^2$.]

(4) Let $U(R)\subset\Delta_R$ be the open set where $g(\tau)>\mathrm{v}_R(\tau)$. Show that $\log(g/\mathrm{v}_R)$ is subharmonic on $U(R)$. [*Hint*: Use that $U(R)\subset U(1)$.]

(5) Show that $\partial U(R)\cap\partial\Delta_R=\varnothing$. Deduce that $U(R)=\varnothing$. [*Hint*: Use that $\log(g/\mathrm{v}_R)=0$ on $\partial U(R)$ and the maximum principle.]

(6) Conclude that $g\leqslant\mathrm{v}_R$ on Δ_R and the proof of Lemma 6.3.33. [*Hint*: Pass to the limit $R\to 1$.]

Exercise 6.9. Let $R\in(0,1)$ and Δ_R^* be as in Exercise 6.7. Let f be a C^2 function on Δ_R^* such that $\lim_{t\to 0}|f(t)|/\mathrm{L}(t)=0$ (in particular, f is L^1_{loc} on Δ_R). We consider the Laplace operator $\Delta_t=4\partial_t\partial_{\overline{t}}$. Then $\Delta_t f$ is a distribution on Δ_R and $\Delta_t(f_{|\Delta_R^*})$ is a continuous function on Δ_R^*. The aim of this exercise is to prove that *if $\eta:=\Delta_t(f_{|\Delta_R^*})$ is L^1_{loc} on Δ_R, then $\Delta_t f=\eta$ as distributions on Δ_R, i.e., $\Delta_t f$ does not have components supported at the origin.*

(1) Let $\psi:\mathbb{R}_+\to[0,1]$ be decreasing a C^∞ function such that $\psi(r)=1$ for $r\in[0,1/2]$ and $\psi\equiv 0$ for $r\geqslant 1$. For any $N>0$, set

$$\psi_N(r)=N\psi(r\mathrm{e}^N)+\mathrm{L}(r)(1-\psi(r\mathrm{e}^N)).$$

Show the following properties of the C^∞ function ψ_N on $(0,1)$:

(a) $0\leqslant\psi_N(r)\leqslant\min(N+\log 2,\mathrm{L}(r))$ and $\psi_N(r)\equiv N$ if $\mathrm{L}(r)\geqslant N+\log 2$,

(b) $\psi_N(t)\to\mathrm{L}(r)$ and $\psi_N(r)/N\to 0$ pointwise when $N\to\infty$,

(c) setting $\Delta_t\psi_N(r)=\partial_r^2\psi_N(r)$ and $\partial_t\psi_N(r)=\partial_{\overline{t}}\psi_N(r)=\frac{1}{2}\partial_r\psi_N(r)$, show that the functions $\Delta_t\psi_N,\partial_t\psi_N,\partial_{\overline{t}}\psi_N$ are supported on the set

$$\{r\mid\mathrm{L}(r)\leqslant N+\log 2\}\subset\{r\mid\mathrm{L}(r)\leqslant 2N\},$$

and

$$\int_{\Delta_R}|\partial_t\psi_N(r)|\,\mathrm{vol},\quad\int_{\Delta_R}|\partial_{\overline{t}}\psi_N(r)|\,\mathrm{vol},\quad\int_{\Delta_R}|\psi_N(r)|\,\mathrm{vol}$$

are bounded by a constant independent of N.

(2) Let $\chi \in C^\infty_{\mathrm{c}}(\Delta_R)$ be a test function. Show that

$$\int_{\Delta_R^*} f\Delta_t[(1-\psi_N/N)\chi]\ \mathrm{vol} \xrightarrow[N\to\infty]{} \int_{\Delta_R^*} f\Delta_t\chi\ \mathrm{vol}$$

by showing first

$$\int_{\Delta_R^*} f(1-\psi_N/N)\Delta_t\chi\ \mathrm{vol} \xrightarrow[N\to\infty]{} \int_{\Delta_R^*} f\Delta_t\chi\ \mathrm{vol}\,.$$

[*Hint*: Use that $|f|\,|\Delta_t\psi_N|/N \leqslant 2(|f|/\mathrm{L}(r))|\Delta_t\psi_N|$ and similarly with $\partial_r\psi_N$.]

(3) Using that $(1-\psi_N/N)\chi$ is a test function on Δ_R^*, show that

$$\int_{\Delta_R^*} f\Delta_t[(1-\psi_N/N)\chi]\ \mathrm{vol} = \int_{\Delta_R^*} \eta(t)[(1-\psi_N/N)\chi]\ \mathrm{vol} \xrightarrow[N\to\infty]{} \int_{\Delta_R^*} \eta(t)\chi(t)\ \mathrm{vol},$$

and conclude.

Exercise 6.10. Same setting as in Exercise 6.9. Prove that *if there exists $\eta \in L^1_{\mathrm{loc}}(\Delta_R)$ such that $\Delta_t(f_{|\Delta_R^*}) \geqslant \eta_{|\Delta_R^*}$, then the distribution $\Delta_t f$ on Δ_R is in fact L^1_{loc}, i.e., $\Delta_t(f_{|\Delta_R^*})$ is L^1_{loc} on Δ_R and coincide with $\Delta_t f$ as distributions.*

(1) Prove that $\Delta_t f \geqslant \eta$ as distributions on Δ_R, i.e., for any nonnegative test function χ on Δ_R,

$$\langle \Delta_t f, \chi\rangle \geqslant \int_{\Delta_R^*} \eta\cdot\chi\ \mathrm{vol}\,.$$

[*Hint*: Keep 6.9(1) and (2) as they are with a nonnegative χ, and in (3) replace the equality with an inequality.]

(2) Deduce that the distribution $\Delta_t f - \eta$, hence also $\Delta_t f$, is the sum of a L^1_{loc} function on Δ_R and a multiple of the Dirac mass at the origin. [*Hint*: Use [**Hör03**, Th. 2.1.7] and the theorem of Radon-Nikodym.]

(3) Apply Exercise 6.9 to conclude.

Exercise 6.11. Let $(\mathcal{H}_1, D_1, \mathrm{h}_1)$ and $(\mathcal{H}_2, D_2, \mathrm{h}_2)$ be nilpotent harmonic bundles. Show that $(\mathcal{H}_1\otimes\mathcal{H}_2, D, \mathrm{h})$ and $\mathcal{H}om(\mathcal{H}_1,\mathcal{H}_2), D, \mathrm{h})$ are also nilpotent. [*Hint*: Use Exercise 4.7.]

Exercise 6.12. Let E be a finite dimensional $\mathbb{C}$-vector space with a Hermitian metric h and a nilpotent endomorphism θ'_0.

(1) Show that there exists a h-orthonormal basis $\boldsymbol{\varepsilon}$ of E in which the matrix A of θ'_0 is strictly upper triangular.

(2) Let θ''_0 be the h-adjoint of θ'_0, with matrix ${}^t\overline{A}$. Show that there exists a positive constant c depending only on $\dim E$ such that $\|[A, {}^t\overline{A}]\| \geqslant c\|A\|^2$. [*Hint*: By homogeneity and compactness of the sphere $\|A\| = 1$, it is enough to show that the function $A \mapsto \|[A, {}^t\overline{A}]\|$ (A strictly upper triangular) does not vanish on the sphere; use then that the only normal and nilpotent matrix is the zero matrix.]

6.6. Comments

The fundamental work of Griffiths on the period mapping attached to a polarized variation of Hodge structure (see [**Gri70b, Del71c**] and the references therein) leads to the analysis of degenerations of such variations, which was achieved in the fundamental article of Schmid [**Sch73**] (see also [**GS75**] and the references therein). Theorems 6.3.2(1) and 6.3.5, together with Theorems 6.7.3 and 6.8.7 are due to Schmid in loc. cit., and Theorem 6.3.2(2) is due to Borel (see [**Sch73**, Lem. 4.5]).

While Schmid's theory focuses on variations having unipotent local monodromies, it is well-known that the results can be extended to the case of quasi-unipotent local monodromies. The more general case treated here of local monodromies whose eigenvalues have absolute value equal to 1 is known to be a consequence of the methods of Schmid (see [**Del87**, §1.11]).

The idea of focusing on the harmonic aspect of the theory is due to Simpson [**Sim88, Sim90**]. A similar approach is considered in [**S-Sch22**], with a more precise estimate on constants involved, that can prove useful in higher dimensions. The proof of the semi-simplicity theorem 6.4.1 given here, in the framework of nilpotent harmonic bundles, is due to Simpson. The idea of considering the analytic degree $\deg^{\mathrm{an}}$ is instrumental in his proof of stability of general harmonic flat bundles.

CHAPTER 6

VARIATIONS OF HODGE STRUCTURE ON CURVES
PART 2: LIMITING HODGE PROPERTIES

Summary. We keep the local setting of Part 1. We state the fundamental theorems of Schmid concerning the limiting behavior of the Hodge filtration and give an idea of the proof, together with the example of the Deligne harmonic model.

6.7. The holomorphic Hodge filtration

We keep the setting of Section 6.3 and we assume (as justified by Theorem 6.3.2(2)) that the eigenvalues of the monodromy have absolute value equal to 1. We wish to extend the filtration $F^\bullet\mathcal{V}$ as a filtration $F^\bullet\mathcal{V}_*$ by sub-bundles satisfying the Griffiths transversality property with respect to the meromorphic connection ∇. A first natural choice would be to set

$$F^p\mathcal{V}_* := j_*F^p\mathcal{H}\cap\mathcal{V}_*,$$

where $j:\Delta^*\hookrightarrow\Delta$ denotes the inclusion. This choice can lead to a non-coherent $\mathcal{O}_\Delta$-module: for example, if $p\ll 0$, we have $F^p\mathcal{V}=\mathcal{V}$ and we would get $F^p\mathcal{V}_*=\mathcal{V}_*$, which is not $\mathcal{O}_\Delta$-coherent. Since we have at our disposal the locally free $\mathcal{O}_\Delta$-modules $\mathcal{V}_*^\beta$ for any $\beta\in\mathbb{R}$, it may be more clever to consider, for any such β,

$$F^p\mathcal{V}_*^\beta := j_*F^p\mathcal{H}\cap\mathcal{V}_*^\beta, \tag{6.7.1}$$

where the intersection is taken in $j_*\mathcal{V}$. The main question to address is whether these sheaves are $\mathcal{O}_\Delta$-coherent. If so, being torsion free, they would be $\mathcal{O}_\Delta$-locally free. Furthermore, we may wonder whether the filtration $F^\bullet\mathcal{V}_*^\beta$ of $\mathcal{V}_*^\beta$ which clearly satisfies $F^p\mathcal{V}_*^\beta=0$ for $p\gg 0$ and $F^p\mathcal{V}_*^\beta=\mathcal{V}_*^\beta$ for $p\ll 0$) is a filtration by *sub-bundles*, i.e., whether the quotients $F^p\mathcal{V}_*^\beta/F^{p+1}\mathcal{V}_*^\beta$ are locally free for any $p\in\mathbb{Z}$.

According to Theorem 6.3.2, we can interpret sections of $F^p\mathcal{V}_*^\beta$ on Δ as being the sections of $F^p\mathcal{V}$ on Δ^* whose h-norm on any punctured closed sub-disc $(\overline{\Delta'})^*$ ($\overline{\Delta'}\subset\Delta$) is bounded by $C_\varepsilon|t|^{\beta-\varepsilon}$ for any $\varepsilon>0$ and some $C_\varepsilon>0$. Let us already notice:

6.7.2. Lemma.

(1) *For $k \geqslant 0$ and any $\beta \in \mathbb{R}$, we have*
$$F^p\mathcal{V}_*^{\beta+k} = t^k F^p\mathcal{V}_*^\beta.$$

(2) *The following properties are equivalent:*
   (a) *there exists $\beta \in \mathbb{R}$ such that, for any $p \in \mathbb{Z}$, $F^p\mathcal{V}_*^\beta$ is $\mathcal{O}_X$-coherent,*
   (b) *for any $\beta \in \mathbb{R}$, the filtration $F^\bullet\mathcal{V}_*^\beta$ of $\mathcal{V}_*^\beta$ is a filtration by sub-bundles.*

Proof. The first point is clear since $\mathcal{V}_*^{\beta+k} = t^k\mathcal{V}_*^\beta$, as well as the implication (2b) $\Rightarrow$ (2a). Let us show (2a) $\Rightarrow$ (2b). Let β be such that $F^p\mathcal{V}_*^\beta$ is $\mathcal{O}_X$-coherent for any p and let γ in $\mathbb{R}$. By the first point, any $F^p\mathcal{V}_*^{\beta+k}$ is $\mathcal{O}_X$-coherent, so we can assume that $\gamma \leqslant \beta$. Then $F^p\mathcal{V}_*^\gamma = (F^p\mathcal{V}_*^\beta) \cap \mathcal{V}_*^\gamma$ and, since both terms in the right-hand side are coherent, so is their intersection.[1]

Moreover, by the coherence property and the first point, $\dim(\mathrm{gr}_F^p\mathcal{V}_*^\gamma/\mathrm{gr}_F^p\mathcal{V}_*^{\gamma+1}) \geqslant \mathrm{rk}\,\mathrm{gr}_F^p\mathcal{V}$ for each p. Since the sum over p of both sides are equal (as $\mathcal{V}_*^\gamma$ is locally free), they are equal for each p, hence $\mathrm{gr}_F^p\mathcal{V}_*^\gamma$ is locally free. $\square$

6.7.3. Theorem. *For any $\beta \in \mathbb{R}$, the filtration $F^p\mathcal{V}_*^\beta$ is a filtration of $\mathcal{V}_*^\beta$ by sub-bundles.*

Proof. According to Lemma 6.7.2, it is enough to prove that, for any p and any β, the $\mathcal{O}_X$-module $F^p\mathcal{V}_*^\beta$ is coherent. Let us fix p. Since we already know that $\mathcal{V}_*^\beta = \mathcal{V}_{\mathrm{mod}}^\beta$ (Theorem 6.3.2), it is enough to show that the Hermitian holomorphic bundle $(F^p\mathcal{V}, \mathrm{h})$, where h is the metric induced by h on $\mathcal{V}$, is *moderate*. As noticed in Remark 6.2.7(1), some care has to be taken. Exercise 4.4(7) together with Simpson's estimate show that $(F^p\mathcal{V}, \mathrm{h})$ satisfies the criterion of Theorem 6.2.9 for each $p \in \mathbb{Z}$. Therefore, $(F^p\mathcal{V}, \mathrm{h})$ is moderate. $\square$

6.8. The limiting Hodge-Lefschetz structure

We will now describe the limiting Hodge-Lefschetz structure attached to a polarized variation of $\mathbb{C}$-Hodge structure (H, S) of weight w on Δ^*.

6.8.1. Convention. We use the simplified setting as in Proposition 5.2.16 and we now write (H, S) as $((\mathcal{V}, \nabla, F^\bullet\mathcal{V}), \mathcal{S})$ (see Definition 5.4.3, and 5.4.1 for $\mathcal{S}$).

For every $\beta \in (-1, 0]$, we define the object $\mathrm{gr}^\beta H$ as follows. We set
$$\mathrm{gr}^\beta(\mathcal{V}, \nabla, F^\bullet\mathcal{V}) = (\mathrm{gr}^\beta\mathcal{V}_*, F^\bullet\mathrm{gr}^\beta\mathcal{V}_*),$$
which is equipped with the nilpotent endomorphism N induced by the action of $-(t\partial_t - \beta)$:
$$(\mathrm{gr}^\beta\mathcal{V}_*, F^\bullet\mathrm{gr}^\beta\mathcal{V}_*) \xrightarrow{\mathrm{N}} (\mathrm{gr}^\beta\mathcal{V}_*, F[-1]^\bullet\mathrm{gr}^\beta\mathcal{V}_*).$$
It remains to define the sesquilinear pairing $\mathrm{gr}^\beta\mathcal{S}$

[1] Indeed, the sum $(F^p\mathcal{V}_*^\beta) + \mathcal{V}_*^\gamma$ is clearly locally of finite type in $\mathcal{V}_*$, hence coherent. Then one deduces the desired coherence from the isomorphism $\big[(F^p\mathcal{V}_*^\beta) + \mathcal{V}_*^\gamma\big]/\mathcal{V}_*^\gamma \simeq (F^p\mathcal{V}_*^\beta)/(F^p\mathcal{V}_*^\beta) \cap \mathcal{V}_*^\gamma$.

6.8.a. Behaviour of sesquilinear pairings. We will make explicit the behaviour of sesquilinear pairings (see Definition 4.1.2) with respect to the functor $(\mathcal{V},\nabla)\mapsto(\mathcal{H}^o,\mathrm{T})$ of Theorem 6.2.4. We assume in this section that the eigenvalues of the residue of ∇ are real, that is, each matrix D_β occurring in Corollary 6.2.5 is equal to zero. This is justified by Theorem 6.3.2(2).

We keep the notation of Exercise 6.1(3), but we choose the indices in $(-1,0]$ instead of $[0,1)$. Let $\beta',\beta''\in(-1,0]$ and let $\mathfrak{s}:\mathcal{J}_{\beta',p|\Delta^*}\otimes\overline{\mathcal{J}_{\beta'',q|\Delta^*}}\to\mathcal{C}^\infty_{\Delta^*}$ be a sesquilinear pairing as in Definition 5.4.1. We denote by $\boldsymbol{v}'_{\beta',p}$ (resp. $\boldsymbol{v}''_{\beta'',q}$) the basis considered in Exercise 6.1. Recall Notation 6.2.8. The compatibility of $\mathfrak{s}$ with the connection enables us to simplify its expression.

6.8.2. Lemma. *For $i=0,\dots,p$ and $j=0,\dots,q$, there exist complex numbers $c_k(i,j)$ such that*

$$\mathfrak{s}(v'_{\beta',i},\overline{v''_{\beta'',j}})=\begin{cases}0 & \text{if } \beta'\neq\beta'',\\ |t|^{2\beta}\sum_{k=0}^{\min(i,j)}c_k(i,j)\,\mathrm{L}(t)^k/k! & \text{if } \beta'=\beta''=:\beta.\end{cases}\tag{6.8.2 $*$}$$

Proof. Let us first assume that $i=j=0$. If we restrict on an open sector centered at the origin on which $t^{\beta'}$ and $t^{\beta''}$ are univalued holomorphic functions, then $\mathfrak{s}(t^{-\beta'}v'_{\beta',0},\overline{t^{-\beta''}v''_{\beta'',0}})$ is constant since it is annihilated by ∂_t and $\partial_{\bar t}$. Therefore, $\mathfrak{s}(v'_{\beta',0},\overline{v''_{\beta'',0}})=c\bar t^{\beta''}t^{\beta'}$ on such a sector. But $\mathfrak{s}(v'_{\beta',0},\overline{v''_{\beta'',0}})$ is a C^∞ function on the whole Δ^*, hence $\beta'-\beta''\in\mathbb{Z}$ unless $\mathfrak{s}(v'_{\beta',0},\overline{v''_{\beta'',0}})=0$. Since we assume $\beta',\beta''\in(-1,0]$, we obtain the assertion in this case.

In general, we argue similarly by using that, if $\eta\in C^\infty(\Delta^*)$ satisfies $(t\partial_t)^{i+1}\eta=(\bar t\partial_{\bar t})^{j+1}\eta=0$, then $\eta=\sum_{k=0}^{\min(i,j)}c_k\mathrm{L}(t)^k/k!$. $\square$

We conclude that any sesquilinear pairing $\mathfrak{s}:\mathcal{J}_{\beta',p|\Delta^*}\otimes\overline{\mathcal{J}_{\beta'',q|\Delta^*}}\to\mathcal{C}^\infty_{\Delta^*}$ is zero if $\beta'\neq\beta''$, and we are reduced to considering sesquilinear pairings

$$\mathfrak{s}:\mathcal{J}_{\beta,p|\Delta^*}\otimes\overline{\mathcal{J}_{\beta,q|\Delta^*}}\longrightarrow\mathcal{C}^\infty_{\Delta^*}.$$

Let us notice that, due to the explicit expression of $\mathfrak{s}$, we have

$$\mathfrak{s}(v',\overline{t\partial_t v''})=\mathfrak{s}(t\partial_t v',\overline{v''}).$$

We still denote by $\boldsymbol{v}'_{\beta,p}$ (resp. $\boldsymbol{v}''_{\beta,q}$) the basis induced on $\mathrm{gr}^\beta\mathcal{J}'_{\beta,p}=\mathcal{O}_\Delta\boldsymbol{v}'_{\beta,p}/t\mathcal{O}_\Delta\boldsymbol{v}'_{\beta,p}$ (resp. $\mathrm{gr}^\beta\mathcal{J}''_{\beta,q}$). We define $\mathrm{gr}^\beta\mathfrak{s}$ by the formula

$$(\mathrm{gr}^\beta\mathfrak{s})(v'_{\beta,i},\overline{v''_{\beta,j}})=c_0(i,j).\tag{6.8.3}$$

We conclude from the previous remark that $(\mathrm{gr}^\beta\mathfrak{s})(v',\overline{\mathrm{N}v''})=(\mathrm{gr}^\beta\mathfrak{s})(\mathrm{N}v',\overline{v''})$ (with N induced by $-(t\partial_t-\beta)$), that is, N is self-adjoint with respect to $\mathrm{gr}^\beta\mathfrak{s}$.

We can now define the pairing $\mathrm{gr}^\beta\mathfrak{s}:\mathrm{gr}^\beta\mathcal{V}_*\otimes_{\mathbb{C}}\overline{\mathrm{gr}^\beta\mathcal{V}_*}\to\mathbb{C}$ by using the decomposition (6.5.1) for $(\mathcal{V}_*,\nabla)$ and by applying (6.8.3) to each pair of terms corresponding to the same $\beta\in(-1,0]$. This can also be obtained by a residue formula, without explicitly referring to such a decomposition and showing also the independence with respect to it (see Exercise 6.13). We can regard $\mathrm{gr}^\beta\mathfrak{s}$ as a morphism of Lefschetz pairs

$$\mathrm{gr}^\beta\mathfrak{s}:(\mathrm{gr}^\beta\mathcal{V}_*,2\pi\mathsf{i}\,\mathrm{N})\longrightarrow(\mathrm{gr}^\beta\mathcal{V}_*,2\pi\mathsf{i}\,\mathrm{N})^*,\tag{6.8.4}$$

as $2\pi\mathsf{i}\,\mathrm{N}$ is skew-adjoint with respect to $\mathfrak{s}$.

We note that the coefficients $c_0(i,j)$ (for i,j varying) determine all the coefficients $c_k(i,j)$ $(0\leqslant k\leqslant \min(i,j))$. Indeed, if $i\geqslant 1$ we find, by compatibility of $\mathfrak{s}$ with ∇,

$$\begin{aligned}|t|^{2\beta}\sum_{k=1}^{\min(i,j)} c_k(i,j)\,\frac{\mathrm{L}(t)^{k-1}}{(k-1)!} &= -(t\partial_t-\beta)\mathfrak{s}(v'_{\beta,i},\overline{v''_{\beta,j}})\\ &= \mathfrak{s}(v'_{\beta,i-1},\overline{v''_{\beta,j}}) = |t|^{2\beta}\sum_{k=0}^{\min(i-1,j)} c_k(i-1,j)\,\frac{\mathrm{L}(t)^k}{k!},\end{aligned}$$

hence $c_k(i,j)=c_{k-1}(i-1,j)$ for $k\geqslant 1$. In such a way one reconstructs $\mathfrak{s}$ from the sesquilinear pairings $\mathrm{gr}^\beta\mathfrak{s}$ by means of (6.8.2 $*$).

6.8.5. Lemma. *The pairing* $\mathrm{gr}^\beta\mathfrak{s}$ *induces a pairing* $\mathrm{gr}^{\mathrm{M}}_{\bullet}\mathrm{gr}^\beta\mathcal{V}_*\otimes_{\mathbb{C}}\overline{\mathrm{gr}^{\mathrm{M}}_{-\bullet}\mathrm{gr}^\beta\mathcal{V}_*}\to\mathbb{C}$, *which is non-degenerate if and only if* $\mathfrak{s}$ *is non-degenerate.*

Proof. Being a morphism of Lefschetz pairs, $\mathrm{gr}^\beta\mathfrak{s}$ is therefore compatible with the monodromy filtrations (see Section 3.3.a). For the second assertion, we can assume that only terms $\mathcal{J}_{\beta,p}$ (with the same $\beta\in(-1,0]$) occur in the decomposition (6.5.1). Note that $\mathrm{gr}^{\mathrm{M}}\mathrm{gr}^\beta\mathfrak{s}$ is an isomorphism if and only if $\mathrm{gr}^\beta\mathfrak{s}$ is so (Exercise 3.8). In order to conclude, we can now interpret Lemma 6.8.2 as giving an asymptotic expansion of $\mathfrak{s}$ when $|t|\to 0$, and (6.8.3) as taking its dominant part. We then clearly obtain that $\mathfrak{s}$ is non-degenerate near the origin if and only if $\mathrm{gr}^\beta\mathfrak{s}$ is non-degenerate. The equivalence with non-degeneracy on the whole disk follows then from Remark 5.4.2. $\square$

***6.8.6. Example* (A symbolic identity).** Let $\eta\in C^\infty_{\mathrm{c}}(\Delta)$ be any test function. Exercise 6.13(1) shows that the function

$$F(s)=\int_\Delta |t|^{2s-2}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t$$

is holomorphic on the half-space $\mathrm{Re}\,s>0$ and extends as a meromorphic function on the s-plane with a simple pole at $s=0$. An integration by parts gives

$$F(s)=\frac{1}{s^2}\int_\Delta |t|^{2s}\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t, \tag{6.8.6 $*$}$$

[apply Stokes formula first to $\mathrm{d}(|t|^{2s}\eta(t)\mathrm{d}\bar t/\bar t)$ and then to $\mathrm{d}(|t|^{2s}\partial_t\eta(t)\mathrm{d}t)$] and expanding with respect to s (taking into account that $|t|^{2s}=\mathrm{e}^{-s\mathrm{L}(t)}$) gives the residue:

$$\mathrm{Res}_{s=0}\,F(s)=-\int_\Delta \mathrm{L}(t)\,\partial_t\partial_{\bar t}\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t.$$

Note that, by Exercise 6.13(1) and the residue interpretation, if χ is a cut-off function, we have

$$\int_\Delta \mathrm{L}(t)\,\partial_t\partial_{\bar t}\chi(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t=2\pi\mathrm{i}\,.$$

We are interested in rewriting the symbolic expression, where N is a nilpotent element of some $\mathbb{C}$-algebra,

$$\int_\Delta |t|^{2s-2-2\mathrm{N}}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t=\sum_{n=0}^\infty\Big(\int_\Delta\frac{\mathrm{L}(t)^n}{n!}\,|t|^{2s-2}\,\eta(t)\,\mathrm{d}t\wedge\mathrm{d}\bar t\Big)\mathrm{N}^n$$

in a way that lets us analyze how it behaves near $s = 0$. As long as $\operatorname{Re} s > 0$, iterating differentiation under the integral sign gives the n-th derivative of $F(s)$:

$$(-1)^n \frac{F^{(n)}(s)}{n!} = \int_\Delta \frac{\mathrm{L}(t)^n}{n!} |t|^{2s-2} \eta(t) \,\mathrm{d}t \wedge \mathrm{d}\bar{t},$$

and since $F(s)$ has a simple pole at $s = 0$, we have $\operatorname{Res}_{s=0} F^{(n)}(s) = 0$ for $n \geqslant 1$. Consequently,

$$\int_\Delta |t|^{2s-2-2\mathrm{N}} \eta(t) \,\mathrm{d}t \wedge \mathrm{d}\bar{t} = \sum_{n=0}^{\infty} (-1)^n \mathrm{N}^n \frac{F^{(n)}(s)}{n!},$$

and this function of s has residue at $s = 0$ equal to that of $F(s)$.

On the other hand, we can expand the expression (6.8.6 $*$) for $F(s)$ into a power series near $s = 0$; the result is that

$$F(s) = \frac{1}{s^2} \sum_{p=0}^{\infty} (-1)^p s^p \int_\Delta \frac{\mathrm{L}(t)^p}{p!} \partial_t \partial_{\bar{t}} \eta(t) \,\mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

After we insert this into the previous expression and simplify the result, we eventually arrive at the symbolic identity

$$\int_\Delta |t|^{2s-2-2\mathrm{N}} \eta(t) \,\mathrm{d}t \wedge \mathrm{d}\bar{t} = \int_\Delta \frac{|t|^{2s-2\mathrm{N}} - 1}{(\mathrm{N} - s)^2} \partial_t \partial_{\bar{t}} \eta(t) \,\mathrm{d}t \wedge \mathrm{d}\bar{t}. \tag{6.8.6 $**$}$$

It should be understood as an identity between two families of holomorphic functions – namely the coefficients at N^p on both sides – on the half-space $\operatorname{Re} s > 0$.

6.8.b. The limiting Hodge-Lefschetz structure. We continue with Convention 6.8.1. In order to obtain a Hodge-Lefschetz structure, we use the sesquilinear pairing $\mathrm{gr}^\beta \mathcal{S} : \mathrm{gr}^\beta \mathcal{V}_* \otimes_{\mathbb{C}} \overline{\mathrm{gr}^\beta \mathcal{V}_*} \to \mathbb{C}$ defined by (6.8.3) (see also Exercise 6.13).

6.8.7. Theorem. *Let (H, S) be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on Δ^*. Then for every $\beta \in (-1, 0]$, the data*

$$(\mathrm{gr}^\beta H, \mathrm{N}, \mathrm{gr}^\beta \mathrm{S})$$

form a polarized Hodge-Lefschetz structure with central weight w (Definitions 3.4.3 and 3.4.14).

We will not give a proof of this theorem and refer to **[S-Sch22]** for a proof of it, by means of the analysis of the period mapping. We will content ourselves with illustrating it on the model of Section 6.3.c (from which we keep the notation), that we enrich with a Hodge filtration. So we start from a polarized $\mathfrak{sl}_2$-Hodge structure $(H^o, \mathrm{N}, \mathrm{S}^o)$ with central weight $w \in \mathbb{Z}$, so that S^o and h^o are related by (see Definition 3.2.7(2))

$$\mathrm{h}^o(u^o, \overline{v^o}) = \mathrm{S}^o(\mathrm{w} u^o, \overline{\mathrm{C}^o_{\mathrm{D}} v^o}).$$

Recall that, since X, Y are of type $(-1,-1)$, they anti-commute with $\mathrm{C}^o_{\mathrm{D}}$, while H commutes with $\mathrm{C}^o_{\mathrm{D}}$. Let us now examine the commutation of w with $\mathrm{C}^o_{\mathrm{D}}$. Let us consider the modified Weil operator $\mathrm{C}_{\mathrm{D}}^{\mathrm{abs}}$ on H^o obtained by removing in C_{D} the dependence in ℓ but keeping the dependence in p, that is, by setting

$$\mathrm{C}_{\mathrm{D}}^{\mathrm{abs}} = (-1)^{w-p} = (-1)^{\ell}\mathrm{C}_{\mathrm{D}}^{o} \quad \text{on } (H_\ell^o)^{p,w+\ell-p} \tag{6.8.8}$$

for any $\ell \in \mathbb{Z}$. Since w sends $(H_\ell^o)^{p-\ell,w-p}$ to $(H_{-\ell}^o)^{p,w+\ell-p}$, we have

$$\mathrm{C}_{\mathrm{D}}^{o}\,\mathrm{w} = \mathrm{w}\,\mathrm{C}_{\mathrm{D}}^{\mathrm{abs}}.$$

We can then express the metric $\mathrm{h}(x,\overline{y}) := \mathrm{S}(\mathrm{w}x, \overline{\mathrm{C}_{\mathrm{D}}^{o}y})$ as (see Exercise 3.1(6))

$$\mathrm{h}(x,\overline{y}) = \mathrm{S}(x, \overline{\mathrm{w}\mathrm{C}_{\mathrm{D}}^{o}y}) = \mathrm{S}(x, \overline{\mathrm{C}_{\mathrm{D}}^{\mathrm{abs}}\mathrm{w}y}).$$

Let $\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}$ and w be the matrices of $\mathrm{C}_{\mathrm{D}}^{\mathrm{abs}}$ and w in the orthonormal basis $\boldsymbol{v}^o$. Then the matrix of S^o in this basis is

$$\mathsf{S}^o := \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}} \cdot \mathsf{w},$$

since $\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}$ is real (its entries are ± 1 or 0), as well as the matrix w.

We consider the C^∞ bundle $\mathcal{H}$ on Δ^* with flat connection D as in Definition 6.3.16, that we equip with the metric h and orthonormal frame $\boldsymbol{\varepsilon}$ as in Definition 6.3.20. It has a holomorphic frame $\boldsymbol{v} = 1 \otimes \boldsymbol{v}^o$, which is now further decomposed as $(\boldsymbol{v}^p)_{p\in\mathbb{Z}}$, as well as the basis $\boldsymbol{\varepsilon}$ defined by (6.3.18).

Since, for each $p \in \mathbb{Z}$, Y sends F^pH^o to $F^{p-1}H^o$ and X sends it to $F^{p+1}H^o$, while H preserves F^pH^o, (6.3.19) now reads

$$\boldsymbol{v}_{\ell,j}^p = |t|^\beta \mathrm{L}(t)^{\ell/2-j}\Big[\boldsymbol{\varepsilon}_{\ell,j}^p + \textstyle\sum_{k\geqslant 1} c_{\ell,j,k}\mathrm{L}(t)^{-k}\boldsymbol{\varepsilon}_{\ell,j+k}^{p+k}\Big] \tag{6.8.9}$$

We denote by $\mathcal{H}^{p,w-p}$ the C^∞ bundle with basis $\boldsymbol{\varepsilon}^p$ on Δ^*, giving rise to a decomposition $\mathcal{H} = \bigoplus_p \mathcal{H}^{p,w-p}$, and by $F^p\mathcal{H}$ the C^∞ sub-bundle of $\mathcal{H}$ generated by the sub-basis $(\boldsymbol{\varepsilon}^{p'})_{p'\geqslant p}$, equivalently (due to (6.8.9)), the sub-basis $(\boldsymbol{v}^{p'})_{p'\geqslant p}$. This is clearly a holomorphic sub-bundle, either because $D''\boldsymbol{v}^p = 0$, or because the matrix M'' does not decrease p. Then $F^p\mathcal{V} := \operatorname{Ker} D''_{\mathcal{V}|F^p\mathcal{H}}$ is the $\mathcal{O}_{\Delta^*}$-submodule of $\mathcal{V} = \mathcal{O}_{\Delta^*}\cdot\boldsymbol{v}$ generated by the elements of $\boldsymbol{v}_{\ell,j}^{p'}$ for $p' \geqslant p$ and ℓ, j arbitrary. Since $\mathcal{V}_*^\beta = \mathcal{O}_\Delta \cdot \boldsymbol{v}$, we have $F^p\mathcal{V}_*^\beta = \mathcal{O}_\Delta \cdot \boldsymbol{v}^{\geqslant p}$, and the $\mathcal{O}_\Delta$-coherence is clear. Moreover, by construction, Griffiths transversality holds for $F^\bullet\mathcal{V}_*^\beta$ and the filtration induced on $\mathrm{gr}^\beta\mathcal{V}_* = \mathcal{H}^o$ is equal to $F^p\mathcal{H}^o$.

Let us now analyze the polarization. We define the sesquilinear form $\mathcal{S}$ on $\mathcal{H}$ by the expected rule

$$\mathcal{S}(\bullet,\overline{\bullet}) = \mathrm{h}(\bullet, \overline{(\mathrm{C}_{\mathrm{D}})^{-1}\bullet}) = \mathrm{h}(\bullet, \overline{\mathrm{C}_{\mathrm{D}}\bullet}),$$

where h is the metric for which $\boldsymbol{\varepsilon}$ is an orthonormal basis and C_{D} is relative to the decomposition $\mathcal{H} = \bigoplus \mathcal{H}^{p,w-p}$. By definition, when restricted to any point x of Δ^*, the sesquilinear form $\mathcal{S}$ is a polarization of the Hodge structure H_x. Furthermore, the matrix of C_{D} in the h-orthonormal frame $\boldsymbol{\varepsilon}$ is equal to $\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}$, so the matrix of $\mathcal{S}$ in the frame $\boldsymbol{\varepsilon}$ is $\mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}$.

6.8.10. Lemma. *The sesquilinear form $\mathcal{S}$ is a polarization of the variation of Hodge structure H on Δ^* which satisfies $\mathrm{gr}^\beta\mathcal{S} = \mathrm{S}^o$.*

We thus find that $(\mathrm{gr}^\beta H, \mathrm{N}, \mathrm{gr}^\beta \mathcal{S})$, being identified with $(H^o, \mathrm{N}, \mathrm{S}^o)$, is a polarized Hodge-Lefschetz structure with central weight w.

Proof. In order to prove that $\mathcal{S}$ is D-horizontal, let us first compute the matrix S of $\mathcal{S}$ in the holomorphic basis $\boldsymbol{v}$. According to (6.3.18) and Exercise 6.5, we find

$$\begin{aligned}\mathsf{S} = {}^t\mathsf{P}_\beta\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \overline{\mathsf{P}}_\beta &= |t|^{2\beta}\mathrm{L}(t)^{\mathsf{H}/2}\, \mathrm{e}^{\mathsf{Y}}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathrm{e}^{\mathsf{X}}\, \mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, |t|^{2\beta}\mathrm{L}(t)^{\mathsf{H}/2}\, \mathrm{e}^{-\mathsf{Y}}\mathrm{e}^{\mathsf{X}}\, \mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, |t|^{2\beta}\mathrm{L}(t)^{\mathsf{H}/2}\, \mathrm{e}^{-\mathsf{X}}\, \mathsf{w}\, \mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, |t|^{2\beta}\mathrm{e}^{-\mathrm{L}(t)\,\mathsf{X}}\, \mathrm{L}(t)^{\mathsf{H}/2}\, \mathsf{w}\, \mathrm{L}(t)^{\mathsf{H}/2}\\ &= \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, |t|^{2\beta}\mathrm{e}^{-\mathrm{L}(t)\,\mathsf{X}}\, \mathsf{w}\\ &= |t|^{2\beta}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathsf{w} = \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathsf{w}\; |t|^{2\beta}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}.\end{aligned}$$

Recall (see (6.3.16 $*$)) that $\boldsymbol{v}\cdot t^{-\beta\,\mathrm{Id}+\mathsf{Y}}$ is a horizontal basis of the connection, and the matrix of $\mathcal{S}$ in this basis is, since the transpose of Y is X and both are real,

$$t^{-\beta\,\mathrm{Id}+\mathsf{X}}|t|^{2\beta}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathsf{w}\, \overline{t}^{-\beta\,\mathrm{Id}+\mathsf{Y}} = t^{\mathsf{X}}\, \mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \overline{t}^{\mathsf{Y}} = t^{\mathsf{X}}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}\, \overline{t}^{\mathsf{Y}}.$$

Horizontality of $\mathcal{S}$ follows thus from the identities:

$$t\partial_t(t^{\mathsf{X}}\mathrm{e}^{\mathrm{L}(t)\,\mathsf{X}}) = 0 \quad\text{and}\quad \overline{t}\partial_{\overline{t}}(\mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}\,\overline{t}^{\mathsf{Y}}) = 0.$$

In order to prove the second part of the lemma, let us show that the matrix of $\mathrm{gr}^\beta\mathcal{S}$ is equal to S^o. By Exercise 6.13 (items (2) and (1)) the matrix of $\mathrm{gr}^\beta\mathcal{S}$ in the basis $\boldsymbol{v}^o$ is given by

$$\begin{aligned}\mathrm{Res}_{s=-\beta-1}\int_{\mathbb{C}}|t|^{2s}\, \mathsf{S}\, \chi(t)\, \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}\qquad&\\ = \mathrm{Res}_{s+\beta=-1}&\int_{\mathbb{C}}|t|^{2(s+\beta)}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathsf{w}\, \mathrm{e}^{\mathrm{L}(t)\,\mathsf{Y}}\chi(t)\, \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}\\ = \mathrm{Res}_{s+\beta=-1}&\int_{\mathbb{C}}|t|^{2(s+\beta)}\, \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathsf{w}\, \chi(t)\, \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}\\ = \mathsf{C}_{\mathrm{H}}^{\mathrm{abs}}\, \mathsf{w} &= \mathsf{S}^o.\end{aligned}$$

∎

6.9. Exercises

***Exercise 6.13* (A residue formula for $\mathrm{gr}^\beta\mathfrak{s}$).** Let $\chi(t)$ be a C^∞ function with compact support on Δ which is $\equiv 1$ near $t=0$ (that we simply call a *cut-off function near* $t=0$). Assume that $\chi(t)$ only depends on $|t|$ (e.g. $\chi(t)=\widetilde{\chi}(|t|^2)$ where $\widetilde{\chi}$ is C^∞).

(1) Show that the function

$$s\longmapsto (s+1)\int_{\mathbb{C}}|t|^{2s}\chi(t)\, \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}$$

is holomorphic for $\operatorname{Re} s>-1$ and extends as an entire function. Show that

$$\mathrm{Res}_{s=-1}\int_{\mathbb{C}}|t|^{2s}\chi(t)\, \tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t} = 1.$$

[*Hint*: By expressing the integrand with respect to the real variables x, y with $t = x + \mathrm{i}\, y$, check the sign of the left-hand side; then compute with polar coordinates up to sign.]

(2) By differentiating k times for $\operatorname{Re} s > -1$, show that

$$\int_{\mathbb{C}} |t|^{2s} \, \frac{\mathrm{L}(t)^k}{k!} \, \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t} = \frac{(-1)^k}{(s+1)^{k+1}} + F_k(s),$$

where $F_k(s)$ is holomorphic for $\operatorname{Re} s > -1$ and extends as an entire function. Conclude that, for $k \geqslant 1$,

$$\operatorname{Res}_{s=-1} \int_{\mathbb{C}} |t|^{2s} \, \frac{\mathrm{L}(t)^k}{k!} \, \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t} = 0.$$

(3) Let $\mathfrak{s} : \mathcal{V}' \otimes \overline{\mathcal{V}}'' \to \mathcal{C}^\infty_{\Delta^*}$ be a sesquilinear pairing. For $\beta \in (-1, 0]$ and sections v' of $\mathcal{V}'^{\beta}_*$ and v'' of $\mathcal{V}''^{\beta}_*$, with respective classes $[v']$ and $[v'']$ in $\mathrm{gr}^\beta \mathcal{V}'_*$ and $\mathrm{gr}^\beta \mathcal{V}''_*$, show the formula

$$(\mathrm{gr}^\beta \mathfrak{s})([v'], \overline{[v'']}) = \operatorname{Res}_{s=-\beta-1} \int_{\mathbb{C}} |t|^{2s} \mathfrak{s}(v', \overline{v''}) \, \chi(t) \, \tfrac{\mathrm{i}}{2\pi} \, \mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

[*Hint*: Argue (in a simpler way) as in Proposition 12.5.4.]

6.10. Comments

The idea of defining the limiting Hodge filtration by a formula like (6.7.1) goes back to [**Sai84**], where M. Saito was inspired by the work of Steenbrink and Varchenko. This idea was further developed in his subsequent works, abutting to [**Sai88**]. The approach followed for the proof of Theorem 6.7.3 is that of Simpson [**Sim88, Sim90**], which was then extended in higher dimension by T. Mochizuki [**Moc11a**, Chap. 21] and revisited more recently by Deng [**Den22**]. In [**S-Sch22**], the results are obtained by means of the analysis of the period mapping and its convergence properties, more in the spirit of the fundamental work of Schmid [**Sch73**].

CHAPTER 6

VARIATIONS OF HODGE STRUCTURE ON CURVES
PART 3: THE HODGE-ZUCKER THEOREM

Summary. This part provides a proof of the Hodge-Zucker theorem 6.11.1. The notion of middle extension of a local system appears as the topological analogue of the L^2 extension of a Hermitian bundle with flat connection, and the main results consist in the algebraic computation of the L^2 de Rham and Dolbeault complexes.

6.11. Introduction

Our aim in this part is to present the proof of the *Hodge-Zucker theorem* 6.11.1 on a punctured compact Riemann surface, which is a Hodge theorem "with singularities". We mix the setting of Sections 4.2.c and 4.2.e, that is, we consider a polarized variation of Hodge structure (H, S) of weight w on a punctured compact Riemann surface $X^* \xhookrightarrow{j} X$.

***6.11.1. Theorem* (Hodge-Zucker).** *In such a case, the cohomology $H^k(X, j_*\underline{\mathcal{H}})$ carries a natural polarized Hodge structure of weight $w+k$ ($k = 0, 1, 2$).*

The way of using L^2 cohomology is exactly the same as in Section 4.2.e, provided that we replace D' and D'' with $\mathcal{D}'$ and $\mathcal{D}''$. Then we are left with the corresponding L^2 Poincaré and Dolbeault lemmas.

In any case, it is important to extend in some way the variation to the projective curve in order to apply algebraic techniques. What kind of an object should we expect on the projective curve? On the one hand, the theorems of Schmid enable us to extend each step of the Hodge filtration as an algebraic bundle over the curve. On the other hand, Zucker selects the interesting extension among all possible extensions in order to obtain the Hodge-Zucker theorem. This is the *middle extension* $(\mathcal{V}_{\mathrm{mid}}, \nabla)$ of the polarized variation of Hodge structure. This selection is suggested by the L^2 approach to the Hodge theorem. As in the previous parts of this chapter, we mainly work in a neighbourhood Δ of a puncture.

6.12. The holomorphic de Rham complexes

6.12.a. The meromorphic de Rham complex. Let $(\mathcal{V},\nabla)$ be any holomorphic bundle with connection on Δ^*. Recall that the holomorphic de Rham complex $\mathrm{DR}(\mathcal{V},\nabla)$ is the complex

$$0 \longrightarrow \mathcal{V} \xrightarrow{\nabla} \Omega^1_{\Delta^*} \otimes \mathcal{V} \longrightarrow 0,$$

whose cohomology is nonzero only in degree zero, with $H^0\,\mathrm{DR}(\mathcal{V},\nabla) = \mathcal{H}^\nabla := \operatorname{Ker}\nabla$.

Assume now that $(\mathcal{V}_*,\nabla)$ is a meromorphic bundle with connection on Δ, having a regular singularity at the origin and set $\mathcal{V} = \mathcal{V}_{*|\Delta^*}$. Let us consider the meromorphic de Rham complex $\mathrm{DR}(\mathcal{V}_*,\nabla)$, defined as the complex

$$0 \longrightarrow \mathcal{V}_* \xrightarrow{\nabla} \Omega^1_{\Delta} \otimes \mathcal{V}_* \longrightarrow 0.$$

Its restriction to Δ^* coincides with $\mathrm{DR}(\mathcal{V},\nabla)$, hence has nonzero cohomology in degree zero only. In other words, $H^1\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a skyscraper sheaf supported at the origin, and $H^0\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is some sheaf extension (across the origin) of the locally constant sheaf $\mathcal{V}^\nabla := \operatorname{Ker}\nabla$. We will determine these sheaves.

One can filter the de Rham complex, so that each term of the filtration is a complex whose terms are free $\mathcal{O}_\Delta$-modules of finite rank: for every β, we set

$$V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla) = \{0 \longrightarrow \mathcal{V}_*^\beta \xrightarrow{\nabla} \Omega^1_\Delta \otimes \mathcal{V}_*^{\beta-1} \longrightarrow 0\}. \tag{6.12.1}$$

Since the action of t is invertible on $\mathcal{V}_*$, the latter complex is quasi-isomorphic to the complex

$$V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla) = \{0 \longrightarrow \mathcal{V}_*^\beta \xrightarrow{t\nabla} \Omega^1_\Delta \otimes \mathcal{V}_*^{\beta} \longrightarrow 0\}.$$

***6.12.2. Lemma* (The de Rham complex of the canonical meromorphic extension)**

The inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla) \hookrightarrow \mathrm{DR}(\mathcal{V}_*,\nabla)$ is a quasi-isomorphism provided $\beta \leqslant 0$. Moreover, the germs at the origin of these complexes can be computed as the complex of finite dimensional vector spaces

$$0 \longrightarrow \mathrm{gr}^0\mathcal{V}_* \xrightarrow{t\partial_t} \mathrm{gr}^0\mathcal{V}_* \longrightarrow 0.$$

As a consequence, the natural morphism (in the derived category)

$$\mathrm{DR}(\mathcal{V}_*,\nabla) \longrightarrow \boldsymbol{R}j_*j^{-1}\,\mathrm{DR}(\mathcal{V}_*,\nabla) = \boldsymbol{R}j_*\,\mathrm{DR}(\mathcal{V},\nabla) \xleftarrow{\sim} \boldsymbol{R}j_*\mathcal{V}^\nabla$$

is an isomorphism.

Proof. For the first statement, we notice that it is enough to check that for every $\beta \leqslant 0$ and any $\gamma < \beta$, the inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_*,\nabla) \hookrightarrow V^\gamma\,\mathrm{DR}(\mathcal{V}_*,\nabla)$ is a quasi-isomorphism. This amounts to showing that the quotient complex

$$0 \longrightarrow \mathcal{V}_*^\gamma/\mathcal{V}_*^\beta \xrightarrow{\partial_t} \mathcal{V}_*^{\gamma-1}/\mathcal{V}_*^{\beta-1} \longrightarrow 0$$

is quasi-isomorphic to zero for such pairs (β,γ), and an easy inductive argument reduces to proving that, for every $\gamma < 0$, the complex

$$0 \longrightarrow \mathrm{gr}^\gamma\mathcal{V}_* \xrightarrow{t\partial_t} \mathrm{gr}^\gamma\mathcal{V}_* \longrightarrow 0$$

is quasi-isomorphic to zero. The result is now easy since $t\partial_t - \gamma$ is nilpotent on $\mathrm{gr}^\gamma \mathcal{V}_*$.

For the second statement, we are reduced to proving that the germ at the origin of the complex

$$0 \longrightarrow \mathcal{V}_*^{>0} \xrightarrow{\ t\partial_t\ } \mathcal{V}_*^{>0} \longrightarrow 0$$

is quasi-isomorphic to zero.[(2)]

Arguing as in Exercise 6.1, one can assume that $\mathcal{V}_*$ has rank 1, and has a basis v_γ ($\gamma \in [0,1)$) such that $t\nabla_{\partial_t} v_\gamma = \gamma \cdot v_\gamma$.

(1) If $\gamma \neq 0$, then $\mathcal{V}_*^{>0} = \mathcal{V}_*^0 = \mathcal{O}_\Delta v_\gamma$ and, setting $\mathcal{O} = \mathcal{O}_{\Delta,0}$, the result follows from the property that $(t\partial_t + \gamma) : \mathcal{O} \to \mathcal{O}$ is an isomorphism (easily checked on series expansions).

(2) If $\gamma = 0$, then $\mathcal{V}_*^{>0} = t\mathcal{V}_*^0 = t\mathcal{O}_\Delta v_0$, and the result follows from the property that $(t\partial_t + 1) : \mathcal{O} \to \mathcal{O}$ is an isomorphism, proved as above.

For the last statement, we first note that the morphism is functorial in $(\mathcal{V}_*, \nabla)$. We can therefore reduce to the case of rank 1 by the argument of Exercise 6.1. If $\gamma \neq 0$, the isomorphism is obvious since both complexes are quasi-isomorphic to zero. If $\gamma = 0$, the isomorphism property is checked in a straightforward way. □

6.12.b. The de Rham complex of the middle extension. This de Rham complex will be the main object for the Hodge-Zucker theorem 6.11.1. We first introduce the middle extension $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. We know that $\mathcal{V}_*$ is generated by $\mathcal{V}_*^{>-1}$ as an $\mathcal{O}_\Delta(*0)$-module (with connection). On the other hand, we define $\mathcal{V}_{\mathrm{mid}}$ as the $\mathcal{O}_\Delta$-submodule of $\mathcal{V}_*$ generated by $\mathcal{V}_*^{>-1}$ through the iterated action of ∇_{∂_t} (and not t^{-1}). In other words,

$$\mathcal{V}_{\mathrm{mid}} := \sum_{j\geqslant 0} (\nabla_{\partial_t})^j\, \mathcal{V}_*^{>-1} \subset \mathcal{V}_*. \tag{6.12.3}$$

(See Exercise 6.2(6).) The main properties of $\mathcal{V}_{\mathrm{mid}}$ are developed in Exercise 6.14.

We now compute the de Rham complex of the middle extension $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. For $\beta \in \mathbb{R}$, let us denote by $\lceil \beta \rceil = -[-\beta]$ the smallest integer bigger than or equal to β. We have $\gamma := \beta - \lceil \beta \rceil \in (-1, 0]$. We set, for any $\beta \in \mathbb{R}$ (inductively if $\beta \leqslant -1$),

$$\mathcal{V}_{\mathrm{mid}}^\beta = \begin{cases} \mathcal{V}_*^\beta & \text{if } \beta > -1, \\ (\nabla_{\partial_t})^k\, \mathcal{V}_*^\gamma + \mathcal{V}_*^{>\beta} & \text{if } \beta \leqslant -1, \\ & \text{with } k = -\lceil \beta \rceil = [-\beta],\ \gamma = \beta - \lceil \beta \rceil, \end{cases} \tag{6.12.4}$$

where $>\beta$ is the next β' such that $\mathrm{gr}^{\beta'} \mathcal{V}_* \neq 0$. For $\beta \leqslant -1$, the formula also reads

$$\mathcal{V}_*^\beta = (\nabla_{\partial_t})^k\, \mathcal{V}_*^\gamma + \sum_{j=0}^{k-1} (\nabla_{\partial_t})^j\, \mathcal{V}_*^{>-1}. \tag{6.12.5}$$

(2) This is obviously *not true* away from the origin.

For example, $\mathcal{V}_{\mathrm{mid}}^{-1} = \partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>-1}$. We also set $\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}} := \mathcal{V}_{\mathrm{mid}}^\beta/\mathcal{V}_{\mathrm{mid}}^{>\beta}$. We note that, by Exercise 6.14(4), $\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}}$ is naturally included in $\mathrm{gr}^\beta\mathcal{V}_*$ for each β and is preserved by the nilpotent endomorphism N.

***6.12.6. Definition* (The morphisms can and var).** We define $\mathrm{can}: \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}$ as the homomorphism induced by $-\partial_t$ and $\mathrm{var}: \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}$ as that induced by t, so that

$$\mathrm{var}\circ\mathrm{can} = \mathrm{N} : \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \longrightarrow \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \quad\text{and}\quad \mathrm{can}\circ\mathrm{var} = \mathrm{N} : \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}.$$

By the definition of $\mathcal{V}_{\mathrm{mid}}$, can is onto and var is injective. In other words, the corresponding quiver $(\mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}, \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}, \mathrm{can}, \mathrm{var})$ is a middle extension quiver, in the sense of Definition 3.3.10.

In a way similar to (6.12.1), the complex $\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ is filtered by the subcomplexes $V^\beta\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ whose terms are thus $\mathcal{O}_\Delta$-free of finite rank.

***6.12.7. Lemma* (The de Rham complex of the middle extension)**

The inclusion of complexes $V^\beta\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \hookrightarrow \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ is a quasi-isomorphism provided $\beta \leqslant 0$. Moreover, the germs at the origin of these complexes can be computed as the complex of finite dimensional vector spaces

$$0 \longrightarrow \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \xrightarrow{\partial_t} \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0.$$

As a consequence, $H^1\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) = 0$ and the natural morphism

$$H^0\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \longrightarrow j_*\mathcal{V}^\nabla$$

is an isomorphism.

***Proof*.** For the first statement, we argue as in Lemma 6.12.2, together with Exercise 6.14(5). The second statement is obtained similarly by using Exercise 6.14(6). The last statement follows then from that of Lemma 6.12.2. □

In particular, since $t : \mathcal{V}_*^{-1} \to \mathcal{V}_*^0$ is injective, it induces an isomorphism

$$(\partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>-1}) \xrightarrow{\sim} (t\partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>0})$$

and we have

$$\{0 \to \mathcal{V}_*^0 \xrightarrow{t\partial_t} (t\partial_t\mathcal{V}_*^0 + \mathcal{V}_*^{>0}) \to 0\} \simeq V^0\,\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla) \xrightarrow{\sim} \mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla). \tag{6.12.8}$$

We can refine the presentation (6.12.8) by using the lifted monodromy filtration $\mathrm{M}_\bullet\mathcal{V}_*^0$. Indeed, the finite dimensional vector space $\mathrm{gr}^0\mathcal{V}_*$ is equipped with the nilpotent endomorphism induced by $\mathrm{N} = -t\partial_t$, hence is equipped with the corresponding monodromy filtration $\mathrm{M}_\bullet\mathrm{gr}^0\mathcal{V}_*$ (see Lemma 3.3.1). We can then consider the lifted monodromy filtration $\mathrm{M}_\ell\mathcal{V}_*^0$ (see Definition 6.3.4).

6.12.9. Lemma. *The complex $\mathrm{DR}(\mathcal{V}_{\mathrm{mid}}, \nabla)$ is quasi-isomorphic to*

$$\{0 \longrightarrow \mathrm{M}_0\mathcal{V}_*^0 \xrightarrow{t\partial_t} \mathrm{M}_{-2}\mathcal{V}_*^0 \longrightarrow 0\}.$$

Proof. Clearly, the complex in the lemma is a subcomplex of (6.12.8). Let us consider the quotient complex. This is

$$0 \longrightarrow (\mathrm{gr}^0\mathcal{V}_*/\mathrm{M}_0\mathrm{gr}^0\mathcal{V}_*) \xrightarrow{t\partial_t} (\mathrm{image}\, t\partial_t/\mathrm{M}_{-2}\mathrm{gr}^0\mathcal{V}_*) \longrightarrow 0. \tag{6.12.10}$$

Applying Lemma 3.3.7, we find that this complex is quasi-isomorphic to 0 (i.e., the middle morphism is an isomorphism). □

6.12.c. The holomorphic L^2 de Rham complex. The Hodge-Zucker theorem 6.11.1 relies on the L^2 computation of the hypercohomology of a de Rham complex, since this L^2 approach naturally furnishes a Hermitian form on the hypercohomology spaces (see Section 4.2.e). In order to analyze the global L^2 condition on a Riemann surface, it is convenient to introduce it in a local way, in the form of an L^2 de Rham complex. We will find in Theorem 6.12.15 the justification for focusing on the de Rham complex of the middle extension.

Hermitian bundle and volume form. Assume that the holomorphic vector bundle $\mathcal{V}$ on Δ^* is equipped with a metric h (equivalently, the C^∞ bundle $\mathcal{H} = \mathcal{C}^\infty_{\Delta^*} \otimes_{\mathcal{O}_{\Delta^*}} \mathcal{V}$ is equipped with such a metric). If we fix a metric on the punctured disc, with volume element vol, we can define the L^2-norm of a section v of $\mathcal{V}$ on an open set $U \subset \Delta^*$ by the formula

$$\|v\|_2^2 = \int_U \mathrm{h}(v, \overline{v})\ \mathrm{vol}\,.$$

In order to be able to apply the techniques of Section 4.2.e, we choose a metric on Δ^* which is complete in the neighbourhood of the puncture. We will assume that, near the puncture, the volume form is given by

$$\mathrm{vol} = \frac{dx^2 + \mathrm{d}y^2}{|t|^2\mathrm{L}(t)^2}, \quad \text{with } x = \mathrm{Re}\, t,\ y = \mathrm{Im}\, t,\ \mathrm{L}(t) := \bigl|\log|t|^2\bigr| = -\log t\bar{t}. \tag{6.12.11}$$

Let us be more explicit concerning the *Poincaré metric*. Working in polar coordinates $t = r\mathrm{e}^{i\theta}$ and volume element $\mathrm{d}\theta\,\mathrm{d}r/r$, vol can also be written as

$$\mathrm{vol} = \mathrm{L}(r)^{-2} \cdot \mathrm{d}\theta\,\mathrm{d}r/r$$

and the metric on $\mathcal{E}^1_{\Delta^*}$ is given by

$$\|\mathrm{d}r/r\| = \|\mathrm{d}\theta\| = \mathrm{L}(r).$$

We thus get a characterization of the L^2 behaviour of forms near the puncture:

$$f \in L^2(\mathrm{vol}) \Longleftrightarrow |\log r|^{-1} f \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r); \tag{$6.12.12_0$}$$

$$\omega = f\,\mathrm{d}r/r + g\,\mathrm{d}\theta \in L^2(\mathrm{vol}) \Longleftrightarrow f \text{ and } g \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r); \tag{$6.12.12_1$}$$

$$\eta = h\,\mathrm{d}\theta\,\mathrm{d}r/r \in L^2(\mathrm{vol}) \Longleftrightarrow |\log r|\, h \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r). \tag{$6.12.12_2$}$$

For example, given a section $\omega \otimes v$ of $\Omega^1_{\Delta^*} \otimes \mathcal{V}$ on an open subset of Δ^*, where ω is written in polar coordinates as $f\,\mathrm{d}r/r + g\,\mathrm{d}\theta$, its L^2-norm with respect to the metric h and the volume vol is

$$\|\omega \otimes v\|_2^2 = \|fv\|_2^2 + \|gv\|_2^2\,. \tag{6.12.13}$$

On the other hand, by Exercise 6.6, we have

$$r^{\beta}\,|\log r|^{\ell/2} \in L^2(\mathrm{d}\theta\,\mathrm{d}r/r) \iff \beta > 0 \text{ or } (\beta = 0 \text{ and } \ell \leqslant -2). \tag{6.12.14}$$

The holomorphic L^2 de Rham complex. We will consider the *holomorphic L^2 de Rham complex*

$$\mathrm{DR}(\mathcal{V}_*, \nabla)_{(2)} = \{0 \to \mathcal{V}_{*(2)} \xrightarrow{\ \nabla\ } (\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)} \to 0\},$$

which is the subcomplex of the meromorphic de Rham complex $\mathrm{DR}(\mathcal{V}_*, \nabla)$ defined in the following way:

• $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)}$ is the subsheaf of $\Omega^1_\Delta \otimes \mathcal{V}_*$ consisting of sections whose restriction to Δ^* is L^2 (with respect to the metric h on $\mathcal{V}$ and the volume vol on Δ^*),

• $\mathcal{V}_{*(2)}$ is the subsheaf of $\mathcal{V}_*$ consisting of sections v whose restriction to Δ^* is L^2 *and* such that ∇v belongs to $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)}$ defined above.

Let us note that, by the very definition, we get a complex. The following theorem is the first step toward an L^2 computation of $j_*\mathcal{V}^\nabla$.

6.12.15. Theorem. *If $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, we have $(\mathrm{DR}\,\mathcal{V}_{*(2)}) \simeq \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} = j_*\mathcal{V}^\nabla$.*

Proof. We start by identifying the terms in degree one, since the L^2 condition is simpler for them.

6.12.16. Lemma. *We have $(\Omega^1_\Delta \otimes \mathcal{V}_*)_{(2)} = (\mathrm{d}t/t) \otimes \mathrm{M}_{-2}\mathcal{V}^0_*$ and $\mathcal{V}_{*(2)} = \mathrm{M}_0\mathcal{V}^0_*$.*

Proof. Let v be a section of $\mathcal{V}_*$ such that $(\mathrm{d}t/t)\otimes v$ is L^2. Equivalently, both $(\mathrm{d}r/r)\otimes v$ and $\mathrm{d}\theta \otimes v$ are L^2, that is, v is L^2, according to $(6.12.12)_1$. If v is a section of $\mathrm{M}_\ell\mathcal{V}^\beta_*$, its norm behaves like $r^\beta \mathrm{L}(r)^{\ell/2}$ near the origin, and (6.12.14) implies that the L^2 condition is achieved iff $\beta > 0$ or $\beta = 0$ and $\ell \leqslant -2$.

Similarly, one checks that the holomorphic sections of $\mathcal{V}$ which are L^2 near the origin are the sections of $\mathrm{M}_0\mathcal{V}^0_*$, since one is led to test whether $\mathrm{L}(r)^{-1}\,\|v\|_{\mathrm{h}}$ is L^2 or not. In order to conclude that $\mathcal{V}_{*(2)} = \mathrm{M}_0\mathcal{V}^0_*$, it is enough to check that $t\partial_t(\mathrm{M}_0\mathcal{V}^0_*) \subset \mathrm{M}_{-2}\mathcal{V}^0_*$. This immediately follows from the definition of the monodromy filtration $\mathrm{M}_\bullet$. ◻

This conclude the proof of Theorem 6.12.15, since $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is expressed by the formula of Lemma 6.12.9. ◻

6.13. The L^2 de Rham complex and the L^2 Poincaré lemma

We take up the definitions of Section 4.2.d. The role of the complex manifold X is played by Δ^* with its Poincaré metric, which induces a metric on the sheaves of C^∞ differential forms on Δ^*, and the value of the L^2-norm of forms up to a positive constant is given by the formulas (6.12.12).

Let $\mathcal{H}$ be a C^∞ bundle $\mathcal{H}$ on Δ^*, equipped with a Hermitian metric h. Correspondingly, the sheaf $\mathcal{E}^i_{\Delta^*} \otimes \mathcal{H}$ is equipped with a metric, and the L^2 norm of a section of this sheaf is given by a formula like (6.12.13). The various L^2 sheaves are thus defined on Δ^*, and we can use the notion of L^2-adapted basis (see Definition 4.2.21).

6.13.1. Examples **(of L^2-adapted frames).**

(1) The frame $(\mathrm{d}r/r, \mathrm{d}\theta)$ is an L^2-adapted frame of $\mathcal{E}^1_{\Delta^*}$. If $\boldsymbol{v}$ is an L^2-adapted frame of $\mathcal{H}$, then $(\mathrm{d}r/r \otimes \boldsymbol{v}, \mathrm{d}\theta \otimes \boldsymbol{v})$ is an L^2-adapted frame of $\mathcal{E}^1_{\Delta^*} \otimes \mathcal{H}$.

(2) In the setting of the model of Section 6.3.c, the frame $\boldsymbol{v}$ is L^2-adapted. Indeed, the frame $\boldsymbol{\varepsilon} \cdot \mathrm{e}^{\mathsf{X}}$ is L^2-adapted by 4.2.22(4), and $\boldsymbol{v}$ is obtained by a rescaling of the latter, so 4.2.22(3) gives the assertion.

Let us group (with respect to $\beta \in (-1, 0]$) the model frames of Section 6.3.c to get a frame $\boldsymbol{v} = (\boldsymbol{v}_\beta)_\beta$ of $\mathcal{V}_*^{>-1}$ and let $\boldsymbol{v}^o$ denote its restriction to $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$. It corresponds, via the canonical decomposition $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1} = \bigoplus_{\beta>-1} \mathrm{gr}^\beta \mathcal{V}_*$, to grouping of the bases $\boldsymbol{v}^o$ of Section 6.3.c.

Assume that $(\mathcal{V}_*, \nabla)$ underlies a polarized variation of Hodge structure (H, S) on Δ^*. By Theorem 6.8.7, $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ underlies a polarized Hodge-Lefschetz structure, and we can define on it the model basis $\boldsymbol{v}^o$ as above.

6.13.2. Proposition **(A criterion for L^2-adaptedness).** *With these assumptions, let $\boldsymbol{v}'$ be any holomorphic frame of $\mathcal{V}_*^{>-1}$ such that its restriction to $\mathcal{V}_*^{>-1}/t\mathcal{V}_*^{>-1}$ is equal to $\boldsymbol{v}^o$. Then $\boldsymbol{v}'$ is L^2-adapted with respect to the Hodge metric.*

Proof. According to Theorem 6.3.11 and Lemma 4.2.22(2), we can replace the Hodge metric by the model metric, that we still denote by h. Then the model frame $\boldsymbol{v}$ is expressed as

$$\boldsymbol{v} = \boldsymbol{\varepsilon} \cdot \mathrm{e}^{\mathsf{X}} \, \mathsf{P}_{\mathrm{diag}}(t),$$

where now X denotes the diagonal bloc matrix with diagonal β-bloc corresponding to that of Section 6.3.c, and similarly $\mathsf{P}_{\mathrm{diag}}$ has diagonal blocs P_β. On the other hand, we can write $\boldsymbol{v}' = \boldsymbol{v} \cdot (\mathrm{Id} + t\,\mathsf{A}(t))$ for some holomorphic matrix $\mathsf{A}(t)$. Then

$$\boldsymbol{v}' = \boldsymbol{\varepsilon} \cdot \mathrm{e}^{\mathsf{X}} \, \mathsf{P}_{\mathrm{diag}}(t)(\mathrm{Id} + t\,\mathsf{A}(t)) = \boldsymbol{\varepsilon} \cdot \mathrm{e}^{\mathsf{X}}(\mathrm{Id} + t\,\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1})\,\mathsf{P}_{\mathrm{diag}}(t).$$

An entry of $\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1}$ is obtained from the corresponding one of A by multiplying it by a term of the form $|t|^{\beta'-\beta}\mathrm{L}(t)^{k/2}$ for some suitable $\beta, \beta' \in (-1, 0]$ and $k \in \mathbb{Z}$. Since $|\beta' - \beta| < 1$, it follows that $\mathrm{Id} + t\,\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1}$ is bounded as well as its inverse matrix, so that $\boldsymbol{\varepsilon} \cdot \mathrm{e}^{\mathsf{X}}(\mathrm{Id} + t\,\mathsf{P}_{\mathrm{diag}}\,\mathsf{A}\,\mathsf{P}_{\mathrm{diag}}^{-1})$ is L^2-adapted, according to Lemma 4.2.22(4). Since $\boldsymbol{v}'$ is obtained from the latter by applying a rescaling, it is also L^2-adapted (Lemma 4.2.22(3)). $\square$

The L^2 sheaves $\mathcal{L}_{(2)}(\mathcal{E}^i_{\Delta^*} \otimes \mathcal{H}, \mathrm{h})$ can be extended as sheaves on Δ by the assignment $U \mapsto L^2(U \cap \Delta^*, \mathcal{E}^i_{\Delta^*} \otimes \mathcal{H}, \mathrm{h})$. We simply denote them by $\mathcal{L}^i_{(2)}(\mathcal{H}, \mathrm{h})$.

Assume moreover that $\mathcal{H}$ is equipped with a flat connection

$$D = D' + D'' : \mathcal{H} \longrightarrow \mathcal{E}^1_{\Delta^*} \otimes \mathcal{H}.$$

By flatness, the bundle $\mathcal{V} = \operatorname{Ker} D''$ equipped with the connection ∇ induced by D' is a holomorphic bundle with holomorphic connection on Δ^*. Moreover, $\underline{\mathcal{H}} := \operatorname{Ker} D = \mathcal{V}^\nabla := \operatorname{Ker} \nabla$ is a locally constant sheaf on Δ^*. The sheaf $\mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h}, D)$ on Δ^* (see Definition 4.2.26) can similarly be extended as a sheaf on Δ. If $U \subset \Delta$ is an open subset containing the origin, a section $u \in L^2(U, \mathcal{H}, \mathrm{h})$ belongs to $\Gamma(U, \mathcal{L}_{(2)}(\mathcal{H}, \mathrm{h}, D))$

if its restriction to $U\cap\Delta^*$ belongs to $\Gamma(U\cap\Delta^*,\mathcal{L}_{(2)}(\mathcal{H},\mathrm{h},D))$ and if $Du\in L^2(U,\mathcal{H},\mathrm{h})$. One can use the approximation lemma 4.2.24.

The L^2 de Rham complex (4.2.27) reads

$$0\longrightarrow \mathcal{L}^0_{(2)}(\mathcal{H},\mathrm{h},D)\xrightarrow{\ D\ }\mathcal{L}^1_{(2)}(\mathcal{H},\mathrm{h},D)\xrightarrow{\ D\ }\mathcal{L}^2_{(2)}(\mathcal{H},\mathrm{h},D)\longrightarrow 0, \tag{6.13.3}$$

where the upper index refers to the degree of forms, as a complex of sheaves on Δ. Clearly, $\mathcal{L}^2_{(2)}(\mathcal{H},\mathrm{h},D)=\mathcal{L}^2_{(2)}(\mathcal{H},\mathrm{h})$ since the latter condition is tautologically satisfied. When restricted to Δ^* the L^2 Poincaré lemma 4.2.28 shows that the complex (6.13.3) is a resolution of the locally constant sheaf $\underline{\mathcal{H}}$.

Without further conditions on $(\mathcal{H},\mathrm{h},D)$, one cannot give much information on (6.13.3) near the origin. The polarized Hodge property provides the formula we expect.

6.13.4. Theorem **(L^2 Poincaré lemma).** *If $(\mathcal{V},\nabla,\mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure, the natural inclusion of complexes $(\mathrm{DR}\,\mathcal{V}_{*(2)})\hookrightarrow\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$ is a quasi-isomorphism. Equivalently (see Theorem 6.12.15),*

(1) *the L^2 complex $\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$ has nonzero cohomology in degree zero at most,*
(2) *the inclusion $j_*\underline{\mathcal{H}}\simeq H^0(\mathrm{DR}\,\mathcal{V}_{*(2)})\hookrightarrow H^0\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$ is an isomorphism.*

By Lemma 4.2.28, its suffices to prove the theorem for the germ of the L^2 de Rham complex at the origin. The assertions amount then to

(1) $H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)_0)=(j_*\underline{\mathcal{H}})_0$,
(2) $H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)_0)$ and $H^2(\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)_0)$ are zero.

Applying the hypercohomology functor to Theorems 6.13.4 and 6.12.15, we obtain:

6.13.5. Theorem. *Let $j:X^*\hookrightarrow X$ be the inclusion of the complement of a finite set in a compact Riemann surface X. If $(\mathcal{V},\nabla,\mathrm{h})$ underlies a polarized variation of $\mathbb{C}$-Hodge structure on X^*, the cohomology $H^\bullet(X,j_*\mathcal{V}^\nabla)$ is equal to the L^2 cohomology of the C^∞-bundle with flat connection $(\mathcal{H},D)$ associated with the holomorphic bundle $(\mathcal{V},\nabla)$, the L^2 condition being taken with respect to the Hodge metric* h *on $\mathcal{H}$ and a complete metric on X^*, locally equivalent near each puncture to the Poincaré metric.* □

The L^2 Poincaré pairing. For $i,j\geqslant 0$ with $i+j=2$, we have a natural pairing of sheaves

$$\mathcal{L}^i_{(2)}(\mathcal{H})\otimes\mathcal{L}^j_{(2)}(\mathcal{H})\longrightarrow\mathcal{L}^2_{(1)}(\mathcal{H}), \tag{6.13.6}$$

where $\mathcal{L}^2_{(1)}(\mathcal{H})$ denotes the sheaf of L^1_{loc} 2-forms (i.e., $(1,1)$-forms) on X, which can thus be integrated. This pairing is compatible with the differential, and induces therefore a pairing of graded complexes, which in turn produces, by taking global sections, a pairing on cohomology.

Proof of Theorem* 6.13.4*: first reduction. We consider the decomposition (6.2.5 $**$) and we work with the corresponding decomposition (6.3.28) of $(\mathcal{H}, D)$. According to Theorem 6.3.11, we can replace the metric h with the model metric $\mathrm{h}^{\mathrm{Del}}$ without changing the L^2 de Rham complex. We now simply denote by h the model metric on each $\mathcal{H}_\beta$. We thus have

$$\mathcal{L}^\bullet_{(2)}(\mathcal{H}, \mathrm{h}, D) \simeq \bigoplus_{\beta\in(-1,0]} \mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D). \tag{6.13.7}$$

The assertions (1) and (2) above can thus be shown for each $\mathcal{H}_\beta$ separately. We notice that, if $\beta \neq 0$, (1) is also a vanishing assertion.

Let us fix $\beta \in (-1, 0]$ and let us work with the model frame $(\boldsymbol{v}_{\beta,\ell})_\ell$ of Section 6.3.c (we now do not distinguish the components in the Lefschetz decomposition and set $\boldsymbol{v}_{\beta,\ell} = (\boldsymbol{v}_{\beta,\ell',j})_{\ell'-2j=\ell}$ so that $\boldsymbol{v}^o_{\beta,\ell}$ is a basis of $\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^\beta \mathcal{V}_*$). We have seen in Example 6.13.1(2) that this frame is L^2-adapted. Denoting by $\mathcal{H}_{\beta,\ell}$ the subbundle framed by $(\boldsymbol{v}_{\beta,\ell})$, and setting $\mathrm{M}_\ell\mathcal{H}_\beta = \bigoplus_{\ell'\leqslant\ell} \mathcal{H}_{\beta,\ell'}$ that we equip with the induced metric, L^2-adaptedness implies an exact sequence for each i

$$0 \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_{\ell-1}\mathcal{H}_\beta, \mathrm{h}) \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}) \longrightarrow \mathcal{L}^i_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}) \longrightarrow 0.$$

On the other hand, since $\mathrm{M}_\ell\mathcal{H}_\beta$ is preserved by the connection, we can equip $\mathcal{H}_{\beta,\ell}$ with the quotient connection by means of the identification with $\mathrm{gr}^{\mathrm{M}}_\ell\mathcal{H}_\beta$, that is, $D\boldsymbol{v}_{\beta,\ell} = \beta(\mathrm{d}t/t) \otimes \boldsymbol{v}_{\beta,\ell}$. We thus have an exact sequence

$$0 \longrightarrow (\mathrm{M}_{\ell-1}\mathcal{H}_\beta, D) \longrightarrow (\mathrm{M}_\ell\mathcal{H}_\beta, D) \longrightarrow (\mathcal{H}_{\beta,\ell}, D) \longrightarrow 0.$$

Then one checks that the sequence

$$0 \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_{\ell-1}\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^i_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^i_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}, D) \longrightarrow 0$$

is exact, leading to an exact sequence of complexes

$$0 \longrightarrow \mathcal{L}^\bullet_{(2)}(\mathrm{M}_{\ell-1}\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^\bullet_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}, D) \longrightarrow \mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}, D) \longrightarrow 0.$$

We can thus regard $\mathcal{L}^\bullet_{(2)}(\mathrm{M}_\ell\mathcal{H}_\beta, \mathrm{h}, D)$ as defining an increasing filtration of the complex $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D)$ with associated graded complexes $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, \mathrm{h}, D)$. Since this filtration is finite, $H^k(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D)_0)$ is the abutment of a spectral sequence with E_1 term defined as (taking into account that M_ℓ is increasing, that we make decreasing by setting $\mathrm{M}^p = \mathrm{M}_{-p}$)

$$E_1^{p,q} = H^{p+q}(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,-p}, \mathrm{h}, D)_0) \implies H^{p+q}(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta, \mathrm{h}, D)_0). \tag{6.13.8}$$

We first aim at computing $E_1^{p,q}$. The main tool will be Hardy's inequalities.

Proof of Theorem* 6.13.4*: Hardy's inequalities and an application. We will make use of the following type of inequalities, called Hardy's inequalities.

6.13.9. Theorem **(L^2 Hardy inequalities, see e.g. [OK90,** Th. 1.14**])**

Let R be a real number in $(0,1)$ and let v, w be two functions (weights) on $I_R = (0,R)$, which are measurable and almost everywhere positive and finite. Let f be a C^1

function[3] *on* I. *Then the following inequality holds between* L^2 *norms with respect to the measure* $\mathrm{d}r$*:*

$$\big\|f\cdot w\big\|_2 \leqslant C\big\|f'\cdot v\big\|_2,$$

with

$$C=\begin{cases}\displaystyle\sup_{r\in I}\int_r^R w(\rho)^2\,\mathrm{d}\rho\cdot\int_0^r v(\rho)^{-2}\,\mathrm{d}\rho & \text{if } \displaystyle\lim_{r\to 0_+} f(r)=0,\\ \displaystyle\sup_{r\in I}\int_0^r w(\rho)^2\,\mathrm{d}\rho\cdot\int_r^R v(\rho)^{-2}\,\mathrm{d}\rho & \text{if } \displaystyle\lim_{r\to R_-} f(r)=0.\end{cases}$$

□

6.13.10. Corollary. *Let* $(b,k)\in\mathbb{R}\times\mathbb{Z}$ *with* $(b,k)\neq(0,1)$*. Given* $g(r)$ *continuous and integrable on* I_R*, let us set*

$$f(r)=\begin{cases}\displaystyle\int_0^r g(\rho)\,\mathrm{d}\rho & \text{if } b<0 \text{ or if } (k\geqslant 2 \text{ and } b=0),\\ \displaystyle\int_{\min(R,\mathrm{e}^{-k/2b})}^r g(\rho)\,\mathrm{d}\rho & \text{if } b>0 \text{ or if } (k\leqslant 0 \text{ and } b=0).\end{cases}$$

(In the second case, we replace $\mathrm{e}^{-k/2b}$ *with its limit* $+\infty$ *when* $b\to 0_+$*.) Then there exists a constant* $C=C(k,b)>0$ *such that the following inequality holds (we consider* $L^2(I_R;\mathrm{d}r/r)$ *norms)*

$$\begin{aligned}\big\|f(r)\cdot r^b\mathrm{L}(r)^{k/2-1}\big\|_{2,\mathrm{d}r/r} &\leqslant C\big\|g(r)\cdot r^b\mathrm{L}(r)^{k/2-1}\cdot r\mathrm{L}(r)\big\|_{2,\mathrm{d}r/r}\\ &= C\big\|g(r)\cdot r^{b+1}\mathrm{L}(r)^{k/2}\big\|_{2,\mathrm{d}r/r}.\end{aligned}$$

Moreover, for k *fixed, there exists* $b_o=b_o(R)>0$ *such that, for* $|b|\geqslant b_o$*, the constant* $C(k,b)$ *can be chosen equal to* 1*.*

The case where $b=0$ and $k=1$ is missing. This leads to the following definition, where we are only interested in germs at the origin, so that $R\in(0,1)$ can be arbitrary small.

6.13.11. Definition. The "Hardy bad space" $\mathfrak{H}$ is the quotient of the space of measurable functions g on I_R for some $R\in(0,1)$ such that $\|g(r)\cdot r\mathrm{L}(r)^{1/2}\|_{2,\mathrm{d}r/r}<\infty$, modulo the space of such g's which can be realized (maybe with a smaller R) as the weak derivative f' of functions f which are L^1_{loc} on I_R and satisfy $\|f(r)\cdot\mathrm{L}(r)^{-1/2}\|_{2,\mathrm{d}r/r}<\infty$.

Proof. We will choose the following weight functions with respect to the measure dr: $w(r)=r^{b-1/2}\mathrm{L}(r)^{k/2-1}$ and $v(r)=r^{b+1/2}\mathrm{L}(r)^{k/2}$.

[3] A weaker property (absolute continuity on every closed subinterval) is sufficient.

The case $b>0$***, and the case*** $b=0$ ***with*** $k\leqslant 0$***.***

(1) If $b>0$ and $R\leqslant \mathrm{e}^{-k/2b}$, i.e., $k/2b\leqslant \mathrm{L}(R)$, we set $b_o=|k|/2\mathrm{L}(R)$ and we have

$$f(r)=-\int_r^R g(\rho)\,\mathrm{d}\rho$$

and $\lim_{r\to R_-}f(r)=0$. We will show the finiteness of

$$\sup_{r\in[0,R]}\Bigl(\int_0^r \rho^{2b-1}\mathrm{L}(\rho)^{k-2}\,\mathrm{d}\rho\cdot\int_r^R \rho^{-2b-1}\mathrm{L}(\rho)^{-k}\,\mathrm{d}\rho\Bigr).$$

After the change of variable $y=\mathrm{L}(\rho)$ and setting $x=\mathrm{L}(r)$, we have to estimate

$$\sup_{x\in(\mathrm{L}(R),+\infty)}\Bigl(\int_x^{+\infty}\mathrm{e}^{-2by}y^k\,\frac{\mathrm{d}y}{y^2}\cdot\int_{\mathrm{L}(R)}^x \mathrm{e}^{2by}y^{-k}\,\mathrm{d}y\Bigr).$$

The function $y\mapsto \mathrm{e}^{-2by}y^k$ is decreasing on $(\mathrm{L}(R),+\infty)$, hence the first integral is bounded by $\mathrm{e}^{-2bx}x^{k-1}$, and the second one by $\mathrm{e}^{2bx}x^{-k}(x-\mathrm{L}(R))$, so the sup is bounded by one. Hardy's inequality holds with $C=1$.

If $k\leqslant 0$ and $b=0$, the same argument applies and gives the same constant $C=1$.

(2) Assume now $b>0$ and $k/2b>\mathrm{L}(R)>0$, so that $k\geqslant 1$. We have

$$f(r)=\int_{\mathrm{e}^{-k/2b}}^r g(\rho)\,\mathrm{d}\rho\quad\text{and}\quad\lim_{r\to 0+}f(r)=0.$$

We will show the finiteness of

$$\sup_{r\in[0,R]}\Bigl(\int_r^R \rho^{2b-1}\mathrm{L}(\rho)^{k-2}\,\mathrm{d}\rho\cdot\int_0^r \rho^{-2b-1}\mathrm{L}(\rho)^{-k}\,\mathrm{d}\rho\Bigr).$$

We decompose the argument following whether $r\in(0,\mathrm{e}^{-k/2b})$ or $r\in(\mathrm{e}^{-k/2b},R)$.

(a) If $r\in(0,\mathrm{e}^{-k/2b})$, we can apply the same argument as in (1) after replacing $\mathrm{L}(R)$ with $k/2b$, and we can therefore choose $C=1$.

(b) If $r\in(\mathrm{e}^{-k/2b},R)$, we want show the finiteness of

$$\int_{\mathrm{L}(R)}^x \mathrm{e}^{-2by}y^k\,\frac{\mathrm{d}y}{y^2}\cdot\int_x^{k/2b}\mathrm{e}^{2by}y^{-k}\,\mathrm{d}y,$$

with $x\in(k/2b,+\infty)$. The function $\mathrm{e}^{-2by}y^k$ is increasing, and the second integral is bounded by $\mathrm{e}^k(k/2b)^{-k}(k/2b-x)$, hence by $\mathrm{e}^k(k/2b)^{-k+1}$. Similarly, the first one is bounded by $\mathrm{e}^{-2bx}x^k(1/\mathrm{L}(R)-1/x)=\mathrm{e}^{-2bx}x^{k-1}(x/\mathrm{L}(R)-1)$ which has limit zero when $x\to\infty$.

The case $b<0$ ***and the case*** $b=0$ ***with*** $k\geqslant 2$***.*** If $b<0$, we have

$$f(r)=\int_0^r g(\rho)\mathrm{d}\rho\quad\text{and}\quad\lim_{r\to 0_+}f(r)=0.$$

(1) We assume that $\mathrm{e}^{(k-2)/2|b|}\geqslant R$, i.e., $k\geqslant 2(1-|b|\,\mathrm{L}(R))$, which is satisfied in particular whenever $k\geqslant 2$. We also set $b_o=|2-k|/\mathrm{L}(R)$. Then the function $\mathrm{e}^{-2by}y^{k-2}$ is increasing on $(\mathrm{L}(R),+\infty)$. An upper bound of

$$\int_{\mathrm{L}(R)}^x \mathrm{e}^{-2by}y^{k-2}\,\mathrm{d}y\cdot\int_x^{+\infty}\mathrm{e}^{2by}y^{2-k}\,\frac{\mathrm{d}y}{y^2}$$

is given by

$$(x - \mathrm{L}(R))\mathrm{e}^{-2bx}x^{k-2} \cdot \mathrm{e}^{2bx}x^{-k+2}x^{-1} = (1 - \mathrm{L}(R)/x) \leqslant 1.$$

The case when $b = 0$ and $k \geqslant 2$ can be treated in a similar way.

(2) If $\mathrm{e}^{(k-2)/2|b|} < R$, i.e., $k < 2(1 - |b|\,\mathrm{L}(R))$, the function $\mathrm{e}^{-2by}y^{k-2}$ is decreasing on $(\mathrm{L}(R), +\infty)$. The first integral is bounded by $\mathrm{e}^{2|b|\mathrm{L}(R)}\mathrm{L}(R)^{k-2}(x - \mathrm{L}(r))$. For the second one, we can choose $\varepsilon > 0$ small enough such that $\mathrm{e}^{2by}y^{-k}$ is bounded by $C_\varepsilon \mathrm{e}^{-2(|b|-\varepsilon)y}$ on $[\mathrm{L}(R), +\infty)$ and we bound the second integral by $C_\varepsilon \mathrm{e}^{-2(|b|-\varepsilon)x}/2(|b|-\varepsilon)$. Hence, the product of both integrals tends to 0 when $x \to +\infty$. $\square$

Proof of Theorem* 6.13.4*: computation of of the* E_1*-term of the spectral sequence (6.13.8)

We will prove the following precise result as a consequence of Hardy's inequalities.

6.13.12. Lemma.

(1) *If $\beta \neq 0$, the cohomology spaces of $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, D)_0$ all vanish.*

(2) *If $\beta = 0$, the cohomology spaces of $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell}, D)_0$ are given by the following formulas.*

$$(6.13.12)_0 \qquad H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell}, D)_0) = \begin{cases} \mathcal{H}^o_{0,\ell} & \text{if } \ell \leqslant 0, \\ 0 & \text{if } \ell \geqslant 1, \end{cases}$$

$$(6.13.12)_1 \qquad H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell}, D)_0) = \begin{cases} \mathcal{H}^o_{0,\ell} \otimes \mathrm{d}\theta & \text{if } \ell \leqslant -2, \\ \mathfrak{H} \otimes_{\mathbb{C}} \mathcal{H}^o_{0,2} \otimes (\mathrm{d}r/r) & \text{if } \ell = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$(6.13.12)_2 \qquad H^2(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell}, D)_0) = \begin{cases} 0 & \text{if } \ell \neq -1, \\ \mathfrak{H} \otimes_{\mathbb{C}} \mathcal{H}^o_{0,0} \otimes ((\mathrm{d}r/r) \wedge \mathrm{d}\theta) & \text{if } \ell = -1. \end{cases}$$

Proof. Recall that $\mathfrak{H}$ is introduced in Definition 6.13.11. Since D is diagonal with respect to the frame $\boldsymbol{v}_{\beta,\ell}$ of $\mathcal{H}_{\beta,\ell}$, and by L^2-adaptedness, we may, and will, assume during the proof that $\mathcal{H}_{\beta,\ell}$ has rank 1 with frame $v_{\beta,\ell}$. We will use the following lemma.

6.13.13. Lemma.

(1) *Let $f(r) \in L^2(I_R, r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$. Then there exists a sequence $f_m \in C^0(I_R)$ such that $f_m \to f$ in $L^2(I_R, r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$.*

(2) *Let $f(r) \in L^2(I_R, r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$ be such that $f'(r) \in L^2(I_R, r^{2\beta}\mathrm{L}(r)^{k+2}\mathrm{d}r/r)$. Then there exists a sequence $f_m \in C^1(I_R)$ such that $f_m \to f$ in $L^2(I_R, r^{2\beta}\mathrm{L}(r)^k\mathrm{d}r/r)$ and $f'm \to f'$ in $L^2(I_R, r^{2\beta}\mathrm{L}(r)^{k+2}\mathrm{d}r/r)$.*

Computation of $H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell}, D)_0)$**.** If $\beta \in (-1, 0)$, there is no nonzero germ of horizontal section of $(j_*\mathcal{L}^1_{\mathrm{loc}} \otimes \mathcal{H}_{\beta,\ell})_0$, a fortiori no nonzero L^2 section. Let us thus assume $\beta = 0$, so that the connection is simply d. Then $H^0(j_*\mathcal{L}^1_{\mathrm{loc}} \otimes \mathcal{H}_{\beta,\ell})_0 = \mathbb{C}v_{0,\ell}$ and the question reduces to whether $\mathrm{L}(t)^{-1}\cdot \mathrm{L}(t)^{\ell/2}\in L^2(\mathrm{d}\theta\,\mathrm{d}r/r)$, according to $(6.12.12)_0$. In conclusion, due to (6.12.14),

$$H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0)=\begin{cases}\mathbb{C}v_{0,\ell} & \text{if } \beta=0 \text{ and } \ell\leqslant 0,\\ 0 & \text{if } \beta\neq 0 \text{ or if } (\beta=0 \text{ and } \ell\geqslant 1).\end{cases}$$

Computation of $H^2(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0)$**.** As a prelude to the Dolbeault case, we wish to solve $D(f(r,\theta)(\mathrm{d}t/t)\otimes v_{\beta,\ell})=\eta$ weakly, that is, $\bar t\partial_{\bar t}f=-h$ weakly, or equivalently $\frac12(r\partial_r+\mathrm{i}\,\partial_\theta)f=-h$ weakly on Δ^*_R with $R<1$ small enough, with $f\cdot(\mathrm{d}t/t)\otimes v_{\beta,\ell}$ in $\Gamma(\Delta^*_R,\mathcal{L}^1_{(2)}(\mathcal{H}_{\beta,\ell},D))$.

Let us develop a section $\eta=h(r,\theta)(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)$ of $j_*\mathcal{E}^2_{\Delta^*_R}$ in Fourier series, with $h(r,\theta)=\sum_{n\in\mathbb{Z}}h_n(r)\mathrm{e}^{\mathrm{i}n\theta}$. The L^2 condition $(6.12.12)_2$ twisted by $v_{\beta,\ell}$ reads

$$\sum_n \|h_n(r)\cdot r^\beta\mathrm{L}(r)^{1+\ell/2}\|^2_{2,\mathrm{d}r/r}<+\infty.$$

Solving termwise the above differential equation amounts to solving in the weak sense

$$(r^{-n}f_n(r))'=-2r^{-n-1}h_n(r),\tag{6.13.14}$$

with $f_n(r)\in L^1_{\mathrm{loc}}(I_R)$ and

$$\|f_n(r)r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}\leqslant C\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{1+\ell/2}\|_{2,\mathrm{d}r/r}\tag{6.13.15}$$

for each n and a constant C independent of n. According to Lemma 6.13.13(1), for n fixed, we can choose a sequence $h_{n,m}\in C^0(I_R)$ such that

$$h_{n,m}\longrightarrow h_n\quad\text{in } L^2(I_R,r^{2\beta}\mathrm{L}(r)^{\ell+2}\mathrm{d}r/r)\text{ when } m\longrightarrow\infty.$$

In particular, for m large, $h_{n,m}\in L^2(I_R,r^{2\beta}\mathrm{L}(r)^{\ell+2}\mathrm{d}r/r)$. Assume that we have solved (6.13.14) for $h_{n,m}$ with $f_{n,m}$ being C^1 on I_R and satisfying (6.13.15) for a constant C independent of n,m. Then, by arguing with Cauchy sequences, $f_n=\lim_{m\to\infty}f_{n,m}$ exists in $L^2(I_R,r^{2\beta}\mathrm{L}(r)^\ell\mathrm{d}r/r)$ and solves (6.13.14) for h_n in the weak sense.

According to Lemma 6.13.13, we can thus assume that h_n is continuous on I_R. Let us set $b=\beta+n$. Due to Corollary 6.13.10 we can solve (6.13.14) with

$$\|f_n(r)r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}\leqslant C\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{1+\ell/2}\|_{2,\mathrm{d}r/r}$$

- for any ℓ, if $\beta\in(-1,0)$, or if $\beta=0$ and $n\neq 0$,
- for $\ell\neq -1$, if $\beta=0$ and $n=0$.

Notice that the constant C can be chosen independent of n since, for $|n|$ large, i.e., $|b|$ large, it can be chosen equal to 1. Therefore, we obtain the first line of $(6.13.12)_2$, as well as the second line by definition of $\mathfrak{H}$.

Computation of $H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0)$**.** As in the previous case, we start with $\omega\otimes v_{\beta,\ell}=[f\mathrm{d}r/r+g\mathrm{d}\theta]\otimes v_{\beta,\ell}$ with f and g expanded as Fourier series with coefficients f_n,g_n satisfying

$$\sum_n\|f_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}+\sum_n\|g_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}<+\infty.$$

The closedness of $D(\omega\otimes v_{\beta,\ell})$ reads $(r\partial_r+\beta)g=(\partial_\theta+i\beta)f$ weakly, and we wish to solve $D(h\otimes v_{\beta,\ell})=\omega\otimes v_{\beta,\ell}$ weakly, that is, $(r\partial_r+\beta)h=f$ and $(\partial_\theta+i\beta)h=g$ weakly, with appropriate L^2 conditions. Written on the Fourier coefficients, the closedness condition reads

$$rg_n'(r)+\beta g_n(r)=i(n+\beta)f_n(r)\quad\text{weakly.}$$

We look for h_n such that

$$\sum_n\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}<+\infty$$

and $$(rh_n'+\beta h_n)=f_n,\quad i(n+\beta)h_n=g_n\quad\text{weakly.}$$

If $n+\beta\neq0$, then h_n is given by $g_n/i(n+\beta)$ and satisfies the left equation above, by the integrability property. The only point is to bound $\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}$. We have

$$\begin{aligned}\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}&=|n+\beta|^{-1}\|g_n(r)\mathrm{L}(r)^{-1}\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}\\&\leqslant|n+\beta|^{-1}\mathrm{L}(R)^{-1}\|g_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r},\end{aligned}$$

so there exists $C>0$ such that

$$\sum_{n|n+\beta\neq0}\|h_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2-1}\|_{2,\mathrm{d}r/r}\leqslant C\sum_{n|n+\beta\neq0}\|g_n(r)\cdot r^\beta\mathrm{L}(r)^{\ell/2}\|_{2,\mathrm{d}r/r}.$$

If $\beta\neq0$, there is no restriction on n and thus

$$H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{\beta,\ell},D)_0)=0\quad\text{if }\beta\neq0.$$

If $\beta=0$, any class in $H^1(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0)$ has a representative $f_0(r)(\mathrm{d}r/r)+g_0(r)\mathrm{d}\theta$, with $g_0(r)$ constant. This constant may be nonzero only if $\mathrm{L}(r)^{\ell/2}\in L^2(I_R,\mathrm{d}r/r)$, that is, $\ell\leqslant-2$ (Exercise 6.6). On the other hand, we look for h_0 such that $h_0'=r^{-1}f_0$. By the reasoning done for H^2, this equation has a solution if $\ell\neq1$. This concludes the proof of $(6.13.12)_1$. □

End of the proof of Theorem* 6.13.4*: analysis of the spectral sequence. In the decomposition (6.13.7), we immediately conclude by induction on ℓ from Lemma 6.13.12 that $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_\beta,D)_0$ is quasi-isomorphic to 0 if $\beta\neq0$. We are thus left with computing the cohomology of $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_0,D)_0$, a complex which is filtered by $\mathcal{L}^\bullet_{(2)}(\mathcal{H}_{0,\ell},D)_0$. We will analyze the spectral sequence (6.13.8) when $\beta=0$, whose nonzero terms $E_1^{p,q}$ are given by Lemma 6.13.12:

$$\begin{aligned}E_1^{p,-p}&=\mathcal{H}^o_{0,-p}\quad\text{for any }p\geqslant0,\\E_1^{p,1-p}&=\mathcal{H}^o_{0,-p}\otimes\mathrm{d}\theta\quad\text{for any }p\geqslant2,\\E_1^{-1,2}&=\mathfrak{H}\otimes\mathcal{H}^o_{0,1}\otimes(\mathrm{d}r/r),\\E_1^{1,1}&=\mathfrak{H}\otimes\mathcal{H}^o_{0,-1}\otimes((\mathrm{d}r/r)\wedge\mathrm{d}\theta).\end{aligned}$$

The only possible nonzero d_1's are $d_1:E_1^{p,-p}\to E_1^{p+1,-p}$ for $p\geqslant0$, induced by D. The only term in D which does not preserve the filtration is $-\mathrm{N}\mathrm{d}t/t$, and it shifts the filtration by -2, so $d_1=0$ and the previous equalities also hold for the corresponding E_2's.

Now, for $p \geqslant 0$, $d_2 : E_2^{p,-p} \to E_2^{p+2,-p-1}$ is induced by $-\mathrm{N} : \mathcal{H}^o_{0,-p} \to \mathcal{H}^o_{0,-p-2}$, which is surjective (see (1) in the proof of Lemma 3.3.7). On the other hand, $d_2 : E_2^{-1,2} \to E_2^{1,1}$ is equivalent to $\mathrm{N} : \mathcal{H}^o_{0,1} \to \mathcal{H}^o_{0,-1}$. Since $\mathrm{N} : \mathrm{gr}_1^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_* \to \mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_*$ is an isomorphism, we conclude that $E_3^{p,q} = 0$ except possibly $E_3^{p,-p}$ with $p \geqslant 0$, and we have

$$E_3^{-\ell,\ell} \simeq \mathrm{Ker}\big[\mathrm{N} : \mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_*\big] \quad \text{for } \ell \leqslant 0.$$

The spectral sequence (6.13.8) degenerates thus at E_3, and $H^i(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_0, D)_0) = 0$ if $i = 1, 2$. Moreover, the inclusion $(j_*\underline{\mathcal{H}})_0 \hookrightarrow H^0(\mathcal{L}^\bullet_{(2)}(\mathcal{H}_0, D)_0)$ is an isomorphism, since both spaces have the same dimension

$$\begin{aligned}\dim \mathrm{Ker}\big[\mathrm{N} : \mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{gr}^0\mathcal{V}_*\big] &= \dim \mathrm{Ker}\big[\mathrm{N} : \mathrm{M}_0\mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{M}_{-2}\mathrm{gr}^0\mathcal{V}_*\big] \\ &= \sum_{\ell\leqslant 0} \dim \mathrm{Ker}\big[\mathrm{N} : \mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_* \longrightarrow \mathrm{gr}_{\ell-2}^{\mathrm{M}}\mathrm{gr}^0\mathcal{V}_*\big].\end{aligned}$$

This concludes the proof of Theorem 6.13.4. $\square$

6.14. The Hodge filtration

In this section, we assume that $(\mathcal{V}, \nabla, \mathrm{h})$ underlies a polarized variation of Hodge structure. Our aim is to define a Hodge filtration on the cohomology $H^\bullet(X, j_*\mathcal{V}^\nabla)$, and to prove that it endows this cohomology with a polarizable Hodge structure. We will also make precise the polarization. The method will be of a local nature, in a way similar to the computation of the L^2 cohomology.

6.14.a. The Hodge filtration on $\mathcal{V}_{\mathrm{mid}}$. We first define the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ from that on $\mathcal{V}^{>-1}$ by the formula

$$F^p\mathcal{V}_{\mathrm{mid}} = \sum_{j\geqslant 0} (\nabla_{\partial_t})^j F^{p+j}\mathcal{V}_*^{>-1}, \tag{6.14.1}$$

in order to obtain Griffiths transversality (recall that $\mathcal{V}_*^{>-1} = \mathcal{V}_{\mathrm{mid}}^{>-1}$, see (6.12.4)). One first checks that this formula defines an $\mathcal{O}_\Delta$-module by using he standard commutation rule. For example, for a local section m of $F^{p+1}\mathcal{V}_*^{>-1}$,

$$\begin{aligned}g(t)\nabla_{\partial_t} m = \nabla_{\partial_t} g(t)m - g'(t)m \in \nabla_{\partial_t} F^{p+1}\mathcal{V}_*^{>-1} + F^{p+1}\mathcal{V}_*^{>-1} \\ \subset \nabla_{\partial_t} F^{p+1}\mathcal{V}_*^{>-1} + F^{p}\mathcal{V}_*^{>-1}.\end{aligned}$$

With this definition, the relation $\nabla_{\partial_t} F^p\mathcal{V}_{\mathrm{mid}} \subset F^{p-1}\mathcal{V}_{\mathrm{mid}}$ is clearly satisfied. We now give more properties of the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$. For $p \in \mathbb{Z}$ and $\beta \in \mathbb{R}$, we set $F^p\mathcal{V}^\beta_{\mathrm{mid}} := F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}^\beta_{\mathrm{mid}}$ and $F^p\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}} := F^p\mathcal{V}^\beta_{\mathrm{mid}}/F^p\mathcal{V}^{>\beta}_{\mathrm{mid}}$.

6.14.2. Proposition **(Properties of the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$).**

(1) *The filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$ is exhaustive, that is,* $\bigcup_p F^p\mathcal{V}_{\mathrm{mid}} = \mathcal{V}_{\mathrm{mid}}$.

(2) *For every $\beta > -1$, we have*

$$F^p\mathcal{V}^\beta_{\mathrm{mid}} = j_*F^p\mathcal{V} \cap \mathcal{V}^\beta_{\mathrm{mid}} = j_*F^p\mathcal{V} \cap \mathcal{V}^\beta_*.$$

(3) *Moreover,*
   (a) *for every* $\beta > -1$, $t(F^p\mathcal{V}^\beta_{\mathrm{mid}}) = F^p\mathcal{V}^{\beta+1}_{\mathrm{mid}}$*;*
   (b) *for every* $\beta < 0$, $\partial_t F^p\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}} = F^{p-1}\mathrm{gr}^{\beta-1}\mathcal{V}_{\mathrm{mid}}$*;*
   (c) *The latter property also holds for* $\beta = 0$.

(4) *Conversely, a filtration* $F^\bullet\mathcal{V}_{\mathrm{mid}}$ *by* $\mathcal{O}_\Delta$*-submodules which satisfies* (6.7.1), (3b) *and* (3c) *also satisfies* (6.14.1).

The inclusions $\subset$ in (3a) and (3b) are easy; the remarkable property is the existence of inclusions $\supset$; we will call the conjunction of (3a) and (3b) the property of *strict* $\mathbb{R}$*-specializability.* Property (3c) involves a Hodge-theoretical argument. we will call the conjunction of (3a)–(3c) the property of filtered middle extension (see Section 9.3.c).

Proof. The statement (1) is clear by (6.12.3).

For (2), it is enough to prove the assertion with $\beta = > -1$ and we start by showing that for any $k \geqslant 0$,

$$F^p\mathcal{V}^{>-k-1}_{\mathrm{mid}} = \sum_{j=0}^k \partial_t^j F^{p+j}\mathcal{V}^{>-1}_*, \tag{6.14.3}$$

which will give the conclusion in case $k = 0$. It is enough to prove that, for any $\ell \geqslant k+1$,

$$\Big(\sum_{j=k+1}^{\ell} \partial_t^j F^{p+j}\mathcal{V}^{>-1}_*\Big) \cap \mathcal{V}^{>-\ell}_{\mathrm{mid}} \subset \Big(\sum_{j=k+1}^{\ell-1} \partial_t^j F^{p+j}\mathcal{V}^{>-1}_*\Big),$$

and this reduces to

$$(\partial_t^\ell F^{p+\ell}\mathcal{V}^{>-1}_*) \cap \mathcal{V}^{>-\ell}_{\mathrm{mid}} \subset \partial_t^{\ell-1}F^{p+\ell-1}\mathcal{V}^{>-1}_* \quad \text{for } \ell \geqslant 1.$$

Let $m \in \mathcal{V}^{>-1}_*$ be such that $\partial_t^\ell m \in \mathcal{V}^{>-\ell}_{\mathrm{mid}}$. Let β be such that $\partial_t m \in \mathcal{V}^\beta_{\mathrm{mid}}$ with $[\partial_t m] \neq 0$ in $\mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}}$. If $\beta \geqslant -1$, $\partial_t^{\ell-1} : \mathrm{gr}^\beta\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\beta-\ell+1}\mathcal{V}_{\mathrm{mid}}$ is an isomorphism and $\partial_t^{\ell-1}(\partial_t m) \notin \mathcal{V}^{>\beta-\ell+1}_{\mathrm{mid}} \supset \mathcal{V}^{>-\ell}_{\mathrm{mid}}$, a contradiction. We must then have $\beta > -1$. Therefore, $\partial_t m \in F^{p+\ell-1}\mathcal{V}^{>-1}_*$, as wanted.

(3a) follows from (2) since t acts in an invertible way on $j_*F^p\mathcal{V}$. Let us check (3c), which amounts to

$$F^{p-1}\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \subset \partial_t F^p\mathrm{gr}^0\mathcal{V}_{\mathrm{mid}}.$$

Since $t : \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} = \mathrm{gr}^0\mathcal{V}_*$ is injective, this is implied by

$$tF^{p-1}\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \subset t\partial_t F^p\mathrm{gr}^0\mathcal{V}_*.$$

The left-hand side is included in $F^{p-1}\mathrm{gr}^0\mathcal{V}_* \cap \mathrm{Im}(t\partial_t)$. By Theorem 6.8.7, $\mathrm{N} = -t\partial_t : (\mathrm{gr}^0\mathcal{V}_*, F^\bullet) \to (\mathrm{gr}^0\mathcal{V}_*, F^\bullet)(-1)$ is a *morphism of Hodge structure, hence is* F*-strict*, which amounts to $F^{p-1}\mathrm{gr}^0\mathcal{V}_* \cap \mathrm{Im}(t\partial_t) \subset t\partial_t F^p\mathrm{gr}^0\mathcal{V}_*$, as wanted.

Let us now check (3b), which amounts to

$$F^{p-1}\mathcal{V}^{\beta-1}_{\mathrm{mid}} \subset \partial_t(F^p\mathcal{V}^\beta_{\mathrm{mid}}) + \mathcal{V}^{>\beta-1}_{\mathrm{mid}} \quad \text{if } \beta < 0.$$

For example, let us assume $\beta \in (-1,0)$. Then

$$\begin{aligned} F^{p-1}\mathcal{V}_{\mathrm{mid}}^{\beta-1} &= F^{p-1}\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{>-2} \cap \mathcal{V}_{\mathrm{mid}}^{\beta-1}\\ &= (F^{p-1}\mathcal{V}_{*}^{>-1} + \partial_t F^p\mathcal{V}_{*}^{>-1}) \cap \mathcal{V}_{\mathrm{mid}}^{\beta-1} \quad \text{(after (6.14.3))}\\ &\subset \mathcal{V}_{\mathrm{mid}}^{>\beta-1} + (\partial_t F^p\mathcal{V}_{*}^{>-1}) \cap \mathcal{V}_{\mathrm{mid}}^{\beta-1} \quad (\beta > 0). \end{aligned}$$

Since $\partial_t : \mathrm{gr}^\gamma\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\gamma-1}\mathcal{V}_{\mathrm{mid}}$ is an isomorphism for $\gamma < 0$ (see Exercise 6.14(5)), we have

$$(\partial_t F^p\mathcal{V}_{*}^{>-1}) \cap \mathcal{V}_{\mathrm{mid}}^{\beta-1} = (\partial_t F^p\mathcal{V}_{*}^{\beta}) + \mathcal{V}_{\mathrm{mid}}^{>\beta-1}.$$

The general case of $\beta > 0$ is treated similarly.

Let us end with (4). One first easily checks that (6.7.1) implies (2) and (3a). Then, by a simple induction on k, (3b) and (3c) imply (6.14.3), hence (6.14.1) by passing to the limit on k. □

***6.14.4. Corollary* (of Theorem 6.7.3).** *The $\mathcal{O}_\Delta$-modules*

$$F^p\mathcal{V}_{\mathrm{mid}}, \quad F^p\mathcal{V}_{\mathrm{mid}}^{\beta} := F^p\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_{\mathrm{mid}}^{\beta}, \quad F^p\mathrm{M}_\ell\mathcal{V}_{\mathrm{mid}}^{\beta} := F^p\mathcal{V}_{\mathrm{mid}} \cap \mathrm{M}_\ell\mathcal{V}_{\mathrm{mid}}^{\beta}$$

are $\mathcal{O}_\Delta$-locally free, hence free, of finite rank.

Proof. Since these sheaves are contained in $\mathcal{V}_{\mathrm{mid}}$, it is enough to prove that they are locally finitely generated. For $\beta > -1$, we simply use Schmid's theorem 6.7.3 and that $F^p\mathcal{V}_{\mathrm{mid}}^{\beta} = F^p\mathcal{V}_{*}^{\beta}$. For $\beta = -1$, we have $F^p\mathcal{V}_{\mathrm{mid}}^{-1} = \partial_t F^{p+1}\mathcal{V}_{\mathrm{mid}}^{0} + F^p\mathcal{V}^{>-1}$ according to 6.14.2(3c), which implies the desired finiteness. The argument for $\beta < -1$ is similar, by using 6.14.2(3b) instead. Last, the finiteness for $F^p\mathcal{V}_{\mathrm{mid}} \cap \mathrm{M}_\ell\mathcal{V}_{\mathrm{mid}}^{\beta}$ is obtained by induction on ℓ, due to the fact that $\mathrm{gr}_\ell^{\mathrm{M}}\mathcal{V}_{\mathrm{mid}}^{\beta}$ is a finite-dimensional vector space. □

6.14.b. The filtered de Rham complex. The de Rham complex $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ has various presentations (Lemmas 6.12.7 and 6.12.9), the latter being linked with the holomorphic L^2 de Rham complex (Theorem 6.12.15). Each of these complexes can naturally be filtered by the usual procedure as in (2.4.3). For $\mathcal{V}_{\mathrm{mid}}$, starting from the filtration $F^\bullet\mathcal{V}_{\mathrm{mid}}$, we define

$$\begin{aligned} F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &:= \{0 \longrightarrow F^p\mathcal{V}_{\mathrm{mid}} \xrightarrow{\nabla} \Omega^1_\Delta \otimes F^{p-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0\}\\ &\simeq \{0 \longrightarrow F^p\mathcal{V}_{\mathrm{mid}} \xrightarrow{\partial_t} F^{p-1}\mathcal{V}_{\mathrm{mid}} \longrightarrow 0\}. \end{aligned} \tag{6.14.5}$$

We also define

$$\begin{aligned} F^pV^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &:= \{0 \longrightarrow F^p\mathcal{V}_{*}^{0} \xrightarrow{\nabla} \Omega^1_\Delta \otimes F^{p-1}\mathcal{V}_{\mathrm{mid}}{-1} \longrightarrow 0\}\\ &\simeq \{0 \longrightarrow F^p\mathcal{V}_{\mathrm{mid}}^{0} \xrightarrow{\partial_t} F^{p-1}\mathcal{V}_{\mathrm{mid}}^{-1} \longrightarrow 0\}. \end{aligned} \tag{6.14.6}$$

Last, taking advantage of Theorem 6.12.15, we define

$$F^p\,\mathrm{DR}\,\mathcal{V}_{*(2)} := \{0 \longrightarrow F^p\mathrm{M}_0\mathcal{V}_{*}^{0} \xrightarrow{t\partial_t} F^{p-1}\mathrm{M}_{-2}\mathcal{V}_{*}^{0} \longrightarrow 0\}. \tag{6.14.7}$$

6.14.8. Proposition. *The inclusions of filtered complexes*

$$F^\bullet \operatorname{DR} \mathcal{V}_{*(2)} \hookrightarrow F^\bullet V^0 \operatorname{DR} \mathcal{V}_{\mathrm{mid}} \hookrightarrow F^\bullet \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$$

are filtered quasi-isomorphisms.

Proof. For the second inclusion, we are reduced to proving that, when $\beta < 0$, the complex

$$0 \longrightarrow F^p \mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \xrightarrow{\ \partial_t\ } F^{p-1} \mathrm{gr}^{\beta-1} \mathcal{V}_{\mathrm{mid}} \longrightarrow 0$$

is quasi-isomorphic to zero. This is precisely 6.14.2(3b), since we know that $\partial_t : \mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\beta-1} \mathcal{V}_{\mathrm{mid}}$ is an isomorphism for such β's.

For the first inclusion, we first argue as for (6.12.8) (by using 6.14.2(3a)) to identify $F^p V^0 \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$ with the complex

$$0 \longrightarrow F^p \mathcal{V}^0_* \xrightarrow{\ t\partial_t\ } (t\partial_t F^p \mathcal{V}^0_* + F^{p-1} \mathcal{V}^{>0}) \longrightarrow 0.$$

The cokernel complex of the first inclusion is then isomorphic to the complex

$$0 \longrightarrow F^p(\mathrm{gr}^0 \mathcal{V}_* / \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^*) \xrightarrow{\ -\mathrm{N}\ } \big(\mathrm{N} F^p \mathrm{gr}^0 \mathcal{V}_* / F^{p-1} \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*\big) \longrightarrow 0,$$

and we wish to prove that the middle arrow is an isomorphism. Surjectivity is clear, and injectivity amounts to the equality

$$\mathrm{N} F^p \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^* = F^{p-1} \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*.$$

We know that $\mathrm{N} : \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^* \to \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*$ is surjective, but we need a supplementary argument for the compatibility with the Hodge filtration. This argument is furnished by the Hodge-Lefschetz property provided by Theorem 6.8.7. Indeed, we know that

$$\mathrm{N} : (\mathrm{gr}^0 \mathcal{V}^*, F^\bullet, \mathrm{M}_{w+\bullet}) \longrightarrow (\mathrm{gr}^0 \mathcal{V}^*, F^\bullet, \mathrm{M}_{w+\bullet})(-1)$$

is a morphism of mixed Hodge structures (see Remark 3.2.1), hence it is strictly compatible with both $F^\bullet$ and $\mathrm{M}_{w+\bullet}$ (see Proposition 2.6.8), hence $\mathrm{N} : F^p \mathrm{M}_0 \mathrm{gr}^0 \mathcal{V}^* \to F^{p-1} \mathrm{M}_{-2} \mathrm{gr}^0 \mathcal{V}^*$ is surjective too. □

6.14.9. Remarks.

(1) We *do not* claim that the filtered complex $F^\bullet \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$ is *strict*, that is, that $H^1 F^p \operatorname{DR} \mathcal{V}_{\mathrm{mid}} = 0$ for any p.

(2) The graded complex $\mathrm{gr}^p_F \operatorname{DR} \mathcal{V}_{*(2)} \simeq \mathrm{gr}^p_F \operatorname{DR} \mathcal{V}_{\mathrm{mid}}$ is a complex in the category of $\mathcal{O}_\Delta$-modules whose terms are $\mathcal{O}_\Delta$-coherent. Reading this property on a compact Riemann surface X, this implies that the hypercohomology spaces $\boldsymbol{H}^q(X, \mathrm{gr}^p_F \operatorname{DR} \mathcal{V}_{*(2)})$ are finite-dimensional vector spaces.

6.14.c. The L^2 Dolbeault lemma. One of the important points in order to prove E_1-degeneration of the Hodge-to-de Rham spectral sequence in the context of the Hodge-Zucker theorem 6.11.1 is the Dolbeault lemma, making the bridge between the holomorphic world and the L^2 world of harmonic sections. It will ensure finite dimensionality needed in the proof of the Hodge-Deligne theorem 4.2.33 in the case of a complex manifold with a complete metric, here a Riemann surface.

Let us now come back to the Dolbeault lemma in the context of the Hodge-Zucker theorem 6.11.1, where X is a compact Riemann surface and X^* is the same surface with isolated punctures. Given a polarized variation of Hodge structure (H, S) of weight w on X^*, we consider the associated flat bundle with metric $(\mathcal{H}, \mathrm{h}, D)$ and the associated flat holomorphic bundle $(\mathcal{H}', \nabla)$, also denoted $(\mathcal{V}, \nabla)$.

The L^2 Dolbeault complex (4.2.30) reads

$$0 \to L^2(X^*, \mathrm{gr}_F^p \mathcal{H}, \mathrm{h}, \mathcal{D}'') \xrightarrow{\mathcal{D}''} L^2(X^*, \mathrm{gr}_F^p(\mathcal{E}^1_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'') \xrightarrow{\mathcal{D}''} L^2(X^*, \mathrm{gr}_F^p(\mathcal{E}^2_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'') \to 0.$$

It will be useful to regard $L^2(X^*, \mathrm{gr}_F^p \mathcal{H}, \mathrm{h}, \mathcal{D}'')$, as well as its relatives, as the space of global sections of a flabby sheaf $\mathcal{L}_{(2)}(\mathrm{gr}_F^p \mathcal{H}, \mathrm{h}, \mathcal{D}'')$ on X, defined by the assignment

$$X \supset U \longmapsto L^2(U \cap X^*, \mathrm{gr}_F^p \mathcal{H}, \mathrm{h}, \mathcal{D}'').$$

This gives rise to a complex of sheaves $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^\bullet_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')$ on X with differential $\mathcal{D}''$.

On the holomorphic side, we regard the holomorphic Dolbeault complex (4.2.13) not only on X^* but its extension to X with the L^2 condition. Namely, $\mathrm{gr}_F^p \mathrm{DR}(\mathcal{V}, \nabla) = \mathrm{gr}_F^p \mathrm{Dol}(\mathrm{gr}_F \mathcal{V}, \theta)$ on X^* is extended to X as $\mathrm{gr}_F^p \mathrm{DR}\, \mathcal{V}_{\mathrm{mid}}$, that we now can write as $\mathrm{gr}_F^p(\mathrm{DR}\, \mathcal{V}_{*(2)})$, a form which will help us to compare with the L^2 side.

6.14.10. Theorem **(L^2 Dolbeault lemma).** *With the assumptions of Theorem* 6.13.5, *there is a natural inclusion of complexes*

$$\mathrm{gr}_F^p(\mathrm{DR}\, \mathcal{V}_{*(2)}) \hookrightarrow \mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^\bullet_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')$$

which is a quasi-isomorphism.

Away from the punctures, Lemma 4.2.32 shows that the inclusion is a quasi-isomorphism. We are thus reduced to analyzing the germ $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}^\bullet_{X^*} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0$ of the sheaf L^2 complex at the origin of the disc Δ.

Proof of Theorem* 6.14.10*: choice of an L^2-adapted basis. As in the proof of the L^2 Poincaré lemma, we can replace the Hodge metric h by an equivalent one, and we can work with an L^2-adapted frame with respect to this metric. However, we cannot use anymore the decomposition (6.2.5 $**$), which much simplified the expression of the connection when analyzing the L^2 de Rham complex, since it is a priori not compatible with the Hodge filtration. We will use Proposition 6.13.2 instead, in a way compatible with the Hodge filtration.

For that purpose, we specify that the basis $(\boldsymbol{v}^o_{\beta,\ell})_{\beta,\ell}$ of $\mathcal{V}_*^{-1}/t\mathcal{V}_*^{-1} \simeq \bigoplus_{\beta \in (-1,0]} \mathrm{gr}^\beta \mathcal{V}_*$ is compatible with the filtration induced on each $\mathrm{gr}^\beta \mathcal{V}_*$ by the Hodge filtration, which is the Hodge filtration of the polarized Hodge-Lefschetz structure $\bigoplus_\beta(\mathrm{gr}^\beta H, \mathrm{N}, \mathrm{gr}^\beta \mathrm{S})$ (Theorem 6.8.7). We thus decompose each $\boldsymbol{v}^o_{\beta,\ell}$ as $\boldsymbol{v}^{o;p}_{\beta,\ell}$ (recall that we now set $\boldsymbol{v}^o_{\beta,\ell} = (\boldsymbol{v}^o_{\beta,\ell',j})_{\ell'-2j=\ell}$ in order to obtain a basis of $\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^\beta \mathcal{V}_*$), so that $\boldsymbol{v}^{o;p}_{\beta,\ell}$ is a basis of $\mathrm{gr}_F^p \mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^\beta \mathcal{V}_*$.

Let us fix p. Since $\mathrm{gr}_F^p\mathcal{V}_*^{>-1}$ is locally free (Theorem 6.7.3), we can lift $\boldsymbol{v}_{\beta,\ell}^{o,p}$ as a family $\boldsymbol{v}_{\beta,\ell}^p$ in $\mathrm{gr}_F^p\mathrm{M}_\ell\mathcal{V}_*^\beta$ so that $(\boldsymbol{v}_{\beta,\ell}^p)_{\beta,\ell}$ is a frame of $\mathrm{gr}_F^p\mathcal{V}_*^{>-1}$. By Proposition 6.13.2, (the restriction to Δ^* of) $(\boldsymbol{v}_{\beta,\ell}^p)_{\beta,\ell}$ is L^2-adapted with respect to the metric induced by the Hodge metric h on $\mathcal{H}^{p,w-p}\simeq\mathrm{gr}_F^p\mathcal{V}$.

Proof of Theorem* 6.14.10*: simplification of the L^2 complex. We present the L^2 Dolbeault complex as the simple complex associated with a double complex, by decoupling d'' and θ'. This relies on the following lemma.

6.14.11. Lemma. *For $q=0,1$, the morphism $\theta':\mathcal{E}_\Delta^{0,q}\otimes\mathrm{gr}_F^p\mathcal{V}\to\mathcal{E}_\Delta^{1,q}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}$ has bounded L^2-norm.*

Proof. The morphism θ' is the C^∞ morphism associated with the holomorphic morphism $\theta:\mathrm{gr}_F^p\mathcal{V}\to\Omega_{\Delta^*}^1\otimes\mathrm{gr}_F^{p-1}\mathcal{V}$, which is itself induced by

$$\theta:\mathrm{gr}_F^p\mathcal{V}_*^{>-1}\longrightarrow\Omega_\Delta^1\otimes\mathrm{gr}_F^{p-1}\mathcal{V}_*^{>-1}.$$

The restriction of θ at $t=0$ being that of $\mathrm{gr}_F^{-1}\nabla$, it has matrix $-\mathrm{gr}_F^p\mathsf{N}$ in the bases $\boldsymbol{v}^{o,p},\boldsymbol{v}^{o,p-1}$. The image by θ of a section $u=\sum_{\beta,\ell,k}u_{\beta,\ell,k}\,v_{\beta,\ell,k}^p$ reads thus

$$\sum_{\beta,\ell,k}u_{\beta,\ell+2,k}\,v_{\beta,\ell,k}^p\,\frac{\mathrm{d}t}{t}+t\sum_{\beta,\ell,k}\widetilde{u}_{\beta,\ell,k}\,v_{\beta,\ell,k}^p\,\frac{\mathrm{d}t}{t},$$

where $\widetilde{u}_{\beta,\ell,k}$ belongs to $\sum_{\beta',\ell',k'}\mathcal{O}_\Delta\cdot u_{\beta',\ell',k'}$. Therefore,

$$\|\theta u\|_2\leqslant\sum_{\beta,\ell,k}\|(u_{\beta,\ell+2,k}+t\widetilde{u}_{\beta,\ell,k})\mathrm{L}(t)v_{\beta,\ell,k}^p\|_2\sim\sum_{\beta,\ell,k}\|(u_{\beta,\ell+2,k}+t\widetilde{u}_{\beta,\ell,k})|t|^\beta\mathrm{L}(t)^{1+\ell/2}\|_2,$$

according to Theorem 6.3.5. On the other hand, by the argument already used in the proof of Proposition 6.13.2, we have

$$\|(u_{\beta,\ell+2,k}+t\widetilde{u}_{\beta,\ell,k})|t|^\beta\mathrm{L}(t)^{1+\ell/2}\|_{\mathrm{h}}\sim\|u_{\beta,\ell+2,k}|t|^\beta\mathrm{L}(t)^{1+\ell/2}\|_{\mathrm{h}}.$$

Since $\boldsymbol{v}^p$ is L^2-adapted we have (see Definition 4.2.21), still using Theorem 6.3.5,

$$\|u_{\beta,\ell+2,k}|t|^\beta\mathrm{L}(t)^{1+\ell/2}\|_2\sim\|u_{\beta,\ell+2,k}v_{\beta,\ell+2,k}^p\|_2\leqslant C_{\boldsymbol{v}}\|a\|_2.$$

We conclude that there exists $C>0$ such that $\|\theta u\|_2\leqslant C\|u\|_2$. ☐

This lemma implies that

$$\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^k\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0=\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^k\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\quad k=0,1,2.\tag{6.14.12}$$

Moreover, we claim that

$$\theta'\big[\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,q}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h},\mathrm{d}'')_0\big]\subset\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,q}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h},\mathrm{d}'')_0.$$

Indeed, this also follows from the lemma if $q=1$ since, in that case,

$$\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,1}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h},\mathrm{d}'')_0=\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,1}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h})_0.$$

On the other hand, we need to prove that, given $u\in\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,0}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h},\mathrm{d}'')_0$, we have $\theta'u\in\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,0}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h},\mathrm{d}'')_0$, that is, $\mathrm{d}''(\theta'u)\in\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,1}\otimes\mathrm{gr}_F^{p-1}\mathcal{V}),\mathrm{h},\mathrm{d}'')_0$. But we have, in the weak sense, $\mathrm{d}''(\theta'u)=-\theta'(\mathrm{d}''u)$, and since

$$\mathrm{d}''u\in\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,1}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h},\mathrm{d}'')_0=\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,1}\otimes\mathrm{gr}_F^p\mathcal{V}),\mathrm{h})_0$$

by assumption, the lemma allows us to conclude.

We can now regard (up to sign) the complex $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^\bullet \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')$ as the simple complex associated with the double complex

$$
\begin{tikzcd}
\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,0} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0 \arrow[r, "\theta'"] \arrow[d, "\mathrm{d}''"'] & \mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,0} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0 \arrow[d, "\mathrm{d}''"] \\
\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,1} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h})_0 \arrow[r, "\theta'"] & \mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,1} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h})_0
\end{tikzcd}
\tag{6.14.13}
$$

and the inclusion $\mathrm{gr}_F^p(\mathrm{DR}\, \mathcal{V}_{*(2)})_0 \hookrightarrow \mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^\bullet \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0$ is obtained by means of the inclusions

$$
\begin{aligned}
(\mathrm{gr}_F^p \mathcal{V}_{*(2)})_0 &\subset \mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,0} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h}, \mathrm{d}'')_0 \\
(\mathrm{gr}_F^{p-1}(\Omega_\Delta^1 \otimes \mathcal{V}_{*(2)}))_0 &\subset \mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{1,0} \otimes \mathrm{gr}_F^{p-1} \mathcal{V}), \mathrm{h})_0.
\end{aligned}
$$

***Proof of Theorem* 6.14.10*: analysis of the vertical morphisms* d'' *in* (6.14.13).** Since these morphisms are diagonal with respect to the L^2-adapted basis $\boldsymbol{v}^p$, the question of the surjectivity of these morphisms will reduce to checking Hardy's inequalities. Let us fix p, β, ℓ. In polar coordinates, we wish to check the surjectivity (or not) of $\bar{t}\partial_{\bar{t}} = \frac{1}{2}(r\partial_r + \mathrm{i}\,\partial_\theta)$:

$$
\bar{t}\partial_{\bar{t}} : \begin{cases} \mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell-2}\mathrm{d}\theta\,\mathrm{d}r/r; (r\partial_r + \mathrm{i}\,\partial_\theta)\big)_0 \to \mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell}\mathrm{d}\theta\,\mathrm{d}r/r\big)_0, \\ \mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell}\mathrm{d}\theta\,\mathrm{d}r/r; (r\partial_r + \mathrm{i}\,\partial_\theta)\big)_0 \to \mathcal{L}_{(2)}\big(r^{2\beta}\mathrm{L}(r)^{\ell+2}\mathrm{d}\theta\,\mathrm{d}r/r\big)_0. \end{cases}
$$

The result has already been obtained in the proof of $(6.13.12)_2$: the first (resp. the second) morphism is onto if $(\beta,\ell) \neq (0,1)$ (resp. $(\beta,\ell) \neq (0,-1)$). Moreover, if $(\beta,\ell) = (0,1)$ (resp. $(\beta,\ell) = (0,-1)$), the subspace $\mathcal{L}_{(2)}\big(\mathrm{L}(r)\mathrm{d}r/r\big)_0$, i.e., consisting of functions only depending on r, surjects to the cokernel.

***Proof of Theorem* 6.14.10*: vanishing of* $H^2\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^\bullet \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0$.** The previous analysis shows that only combinations of terms $u(r)v_{0,-1}^p(\mathrm{d}t/t)\wedge(\mathrm{d}\bar{t}/\bar{t})$, where $v_{0,-1}^p$ is any element of the subfamily $\boldsymbol{v}_{0,-1}^p$ (i.e., $\beta = 0$ and $\ell = -1$) may not belong to $\operatorname{Im}\mathrm{d}''$. However, one then checks that $u(r)v_{0,1}^p(\mathrm{d}\bar{t}/\bar{t})$ belongs to $\mathcal{L}_{(2)}((\mathcal{E}_{X^*}^{0,1} \otimes \mathrm{gr}_F^p \mathcal{V}), \mathrm{h})_0$ and, by the previous analysis,

$$
\theta'\big(u(r)v_{0,1}^p(\mathrm{d}\bar{t}/\bar{t})\big) \equiv u(r)v_{0,-1}^p(\mathrm{d}t/t) \wedge (\mathrm{d}\bar{t}/\bar{t}) \mod \operatorname{Im}\mathrm{d}''.
$$

This implies the vanishing of $H^2\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^\bullet \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0$. □

***End of the proof of Theorem* 6.14.10.** The previous step identifies, up to a quasi-isomorphism, the complex $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^\bullet \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0$ with its subcomplex

$$
0 \longrightarrow \mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{H}), \mathrm{h}, \mathcal{D}'')_0 \xrightarrow{\ \mathcal{D}''\ } \operatorname{Ker}\mathcal{D}'' \longrightarrow 0,
$$

where

$$
\operatorname{Ker}\mathcal{D}'' = \operatorname{Ker}\Big[\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^1 \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0 \xrightarrow{\ \mathcal{D}''\ } \mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^2 \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0\Big].
$$

6.14.14. Lemma. *Any local section* $u' \cdot (\mathrm{d}t/t) + u'' \cdot (\mathrm{d}\bar{t}/\bar{t})$ *of* $\operatorname{Ker}\mathcal{D}''$ *is equivalent, modulo* $\operatorname{Im}\mathcal{D}''$*, to a local section of* $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0} \otimes \mathcal{H}), \mathrm{h}, \mathcal{D}'')_0 \cap \operatorname{Ker}\mathcal{D}''$.

Proof. Since $u'\cdot(\mathrm{d}t/t)+u''\cdot(\mathrm{d}\bar t/\bar t)$ is assumed to belong to $\operatorname{Ker}\mathcal{D}''$, it is enough to show that it belongs to $\operatorname{Im}\mathcal{D}''+\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$, and it is also enough to show that such is the case for $u''\cdot(\mathrm{d}\bar t/\bar t)$.

First, we write $u''=u''_{\neq(0,1)}+u''_{(0,1)}$, where $u''_{\neq(0,1)}$ resp. $u''_{(0,1)}$ is a combination of basis sections $v^p_{\beta,\ell}$ with $(\beta,\ell)\neq(0,1)$ resp. $(\beta,\ell)=(0,1)$. Since $u''_{\neq(0,1)}\in\operatorname{Im}\mathrm{d}''$ by the previous analysis, it belongs to $\operatorname{Im}\mathcal{D}''+\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$. We can thus write our original section (up to changing notation for u') as $u'\cdot(\mathrm{d}t/t)+u''_{(0,1)}\cdot(\mathrm{d}\bar t/\bar t)$, and as such it still belongs to $\operatorname{Ker}\mathcal{D}''$.

Let us denote by $u''_{(0,-1)}$ the combination of basis sections $v^p_{0,-1}$ where the coefficient of $v^p_{0,-1}$ is that of $u''_{(0,1)}$ on $v^p_{0,1}$. Arguing as in the proof of the vanishing of H^2, we find

$$\theta'(u''_{(0,1)}(\mathrm{d}\bar t/\bar t))\equiv u''_{(0,-1)}(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)\mod\operatorname{Im}\mathrm{d}''.$$

On the other hand, by assumption, $\theta'(u''_{(0,1)}(\mathrm{d}\bar t/\bar t))=\mathrm{d}''\big(u'\cdot(\mathrm{d}t/t)\big)$, so that $u''_{(0,-1)}(\mathrm{d}t/t)\wedge(\mathrm{d}\bar t/\bar t)\in\operatorname{Im}\mathrm{d}''$. But the preliminary analysis of $\operatorname{Im}\mathrm{d}''$ done above shows that this is equivalent to $u''_{(0,1)}(\mathrm{d}\bar t/\bar t)\in\operatorname{Im}\mathrm{d}''$. As a consequence, $u''_{(0,1)}(\mathrm{d}\bar t/\bar t)$ belongs to $\operatorname{Im}\mathcal{D}''+\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$, as wanted. $\square$

We note that, because of (6.14.12) and by considering types,

$$\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0\cap\operatorname{Ker}\mathcal{D}''=\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\cap\operatorname{Ker}\mathrm{d}''.$$

Then the L^2 Dolbeault complex $\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{\bullet}\otimes\mathcal{H}),\mathrm{h},\mathcal{D}'')_0$ is now seen to be quasi-isomorphic to its subcomplex

$$0\to\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\xrightarrow{\ \mathcal{D}''\ }\big[\operatorname{Im}\mathcal{D}''+\big(\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\cap\operatorname{Ker}\mathrm{d}''\big)\big]\to0.$$

Besides, by considering types, the latter is isomorphic to its subcomplex (up to sign)

$$0\longrightarrow\operatorname{Ker}\mathrm{d}''\xrightarrow{\ \theta'\ }\mathcal{L}_{(2)}(\mathrm{gr}_F^p(\mathcal{E}_{X^*}^{1,0}\otimes\mathcal{H}),\mathrm{h},\mathrm{d}'')_0\cap\operatorname{Ker}\mathrm{d}''.\tag{6.14.15}$$

Extending the germs to a small disc Δ, the restriction of the above complex to Δ^* is isomorphic to the holomorphic Dolbeault complex

$$0\longrightarrow\mathrm{gr}^p\mathcal{V}\xrightarrow{\ \theta\ }\Omega^1_{\Delta^*}\otimes\mathrm{gr}^p\mathcal{V},$$

as already mentioned. Then, by definition of the L^2 condition, (6.14.15) is nothing but $\mathrm{gr}_F^p\operatorname{DR}\mathcal{V}_{*(2)}$, and this ends the proof of Theorem 6.14.10. $\square$

6.14.d. Conclusion: proof of the Hodge-Zucker theorem. We are now in position to apply Hodge theory on complete non-compact complex manifolds as in Section 4.2.e. Starting from a polarized variation of Hodge structure (H,S) on the punctured Riemann surface X^* equipped with a complete metric locally like the Poincaré metric near each puncture, we consider the corresponding L^2 de Rham complex $\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D)$. By Theorem 6.13.5, the cohomology of the complex $\Gamma(X,\mathcal{L}^\bullet_{(2)}(\mathcal{H},\mathrm{h},D))$ is isomorphic to $H^*(X,j_*\underline{\mathcal{H}})$, hence is finite dimensional. On the other hand, by the L^2 Dolbeault lemma 6.14.10, each cohomology space $H^k\big(\Gamma(X,\mathcal{L}^\bullet_{(2)}(\mathrm{gr}^F_p\mathcal{H},\mathrm{h},\mathcal{D}''))\big)$ is finite-dimensional, being isomorphic to the cohomology on X of a complex whose terms are $\mathcal{O}_X$-coherent (see Remark 6.14.9(2)). The finiteness conditions in Theorem 4.2.33 are thus fulfilled, and we obtain the desired Hodge decomposition. It is important to remark that, according to Theorems 6.13.5 and 6.14.10 read in the reverse direction, we can express the Hodge structure on $H^*(X, j_*\underline{\mathcal{H}})$ only in terms of the algebraic object $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}}, \nabla)$.

Let us now consider the polarization. The cohomology $H^1(X, j_*\underline{\mathcal{H}})$ is primitive, so the polarization on it can be expressed without referring to an ample line bundle. The positivity property of the polarization on H^0 and H^1 is proved exactly as in Theorem 4.2.16 in the case of compact Riemann surfaces, by replacing sections of the C^∞ de Rham complex on X with sections of the L^2 complex, with respect to the complete metric fixed on X^*, and using the pairing (6.13.6). There is no need here to argue on primitivity of L^2 sections.

6.14.16. Remarks.

(1) As in Remark 4.2.18(4), a consequence of the Hodge-Zucker theorem 6.11.1 is that the maximal constant subsheaf of $\underline{\mathcal{H}}$ has stalk $H^0(X^*, \underline{\mathcal{H}}) = H^0(X, j_*\underline{\mathcal{H}})$, and thus underlies a constant polarizable variation of Hodge structure of weight w whose restriction at any point of X is a direct summand in H on which the polarization of H induces a polarization (see Exercise 2.12). Poincaré duality enables us to transport this polarized Hodge structure to $H^2(X, j_*\underline{\mathcal{H}})$.

(2) (Degeneration at E_1 of the Hodge-to-de Rham spectral sequence) One checks that the filtered complex $\boldsymbol{R}\Gamma\big(X, F^\bullet(\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})_{(2)}\big)$ is *strict*, exactly as in Remark 4.2.18(2). This reads here as the injectivity of the natural horizontal morphisms

$$\begin{array}{ccc}\boldsymbol{H}^k(X, F^pV^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) & \hookrightarrow & \boldsymbol{H}^k(X, V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\\ \wr\big\downarrow & & \big\downarrow\wr\\ \boldsymbol{H}^k(X, F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) & \hookrightarrow & \boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\end{array}$$

6.14.e. Structure of polarized variations of $\mathbb{C}$-Hodge structure

Let X be a compact Riemann surface, let X^* be the complement of a finite set of point, and let $(\mathcal{H}, F'^\bullet\mathcal{H}, F''^\bullet\mathcal{H}, D, \mathrm{S})$ be a polarized variation of $\mathbb{C}$-Hodge structure of weight w on X^*. By Corollary 6.4.2, the local system $\underline{\mathcal{H}}$ is semi-simple, that we write as $\underline{\mathcal{H}} = \bigoplus_{\alpha\in A}\mathcal{H}_\alpha^o\otimes\underline{\mathcal{H}}_\alpha$, with the same notation as in Section 4.3.c. Moreover, the polarization decomposes as well.

6.14.17. Theorem. *The statements of Lemma* 4.3.11 *and Theorem* 4.3.13 *hold in this setting.*

Proof. Indeed, the reference to Theorem 4.3.3 is replaced with a reference to Corollary 6.4.2, so the new argument needed both for Lemma 4.3.11 and for Theorem 4.3.13 only concerns the existence of a pure Hodge structure on

$$\mathrm{End}(\underline{\mathcal{H}}) = H^0(X^*, \mathcal{E}nd(\underline{\mathcal{H}})) = H^0(X, j_*\,\mathcal{E}nd(\underline{\mathcal{H}})),$$

and similarly on $\mathrm{Hom}(\underline{\mathcal{H}}_\alpha, \underline{\mathcal{H}})$, which is provided by the Hodge-Zucker theorem 6.11.1, according to Remark 6.14.16(1). $\square$

6.15. Exercises

Exercise 6.14. Show the following properties (see (6.12.4) for $\mathcal{V}^\beta_{\mathrm{mid}}$, $\beta \in \mathbb{R}$).

(1) $\mathcal{V}^\beta_{\mathrm{mid}}$ is an $\mathcal{O}_\Delta$-coherent module, which is free of rank equal to $\operatorname{rk}\mathcal{V}$, since, being included in $\mathcal{V}_*$, it is torsion-free.

(2) For $\gamma \in (-1,0]$ and $k \geqslant 0$, $\partial_t^k : \mathrm{gr}^\gamma \mathcal{V}_{\mathrm{mid}} = \mathrm{gr}^\gamma \mathcal{V}_* \to \mathrm{gr}^{\gamma-k}\mathcal{V}_{\mathrm{mid}}$ is onto.

(3) $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \subset \mathrm{gr}^\beta \mathcal{V}_*$. [*Hint*: Clear if $\beta > -1$; for $\gamma \in (-1,0)$ and $\beta = \gamma - k < -1$, use $\partial_t^k : \mathrm{gr}^\gamma \mathcal{V}_* \xrightarrow{\sim} \mathrm{gr}^{\gamma-k}\mathcal{V}_*$; for $\beta = -1$, show the inclusion directly; for $\beta = -1-k \leqslant -2$, use the inclusion for $\beta = -1$ and the bijectivity of $\partial_t^k : \mathrm{gr}^{-1}\mathcal{V}_* \to \mathrm{gr}^{-1-k}\mathcal{V}_*$.]

(4) $\mathcal{V}^\beta_{\mathrm{mid}} = \mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}^\beta_*$. [*Hint*: Inclusion $\subset$ is clear; for $\supset$, let $m \in \mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}^\beta_*$ with $[m] \neq 0$ in $\mathrm{gr}^\beta \mathcal{V}_*$; there exists $\beta' \leqslant \beta$ such that $m \in \mathcal{V}^{\beta'}_{\mathrm{mid}} \smallsetminus \mathcal{V}^{>\beta'}_{\mathrm{mid}}$; then (3) implies $\beta' = \beta$.]

(5) For $\beta \neq 0$, $\partial_t : \mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{\beta-1}\mathcal{V}_{\mathrm{mid}}$ is bijective. Deduce that $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} = \mathrm{gr}^\beta \mathcal{V}_*$ for $\beta \neq -1,-2,\dots$ [*Hint*: For the injectivity, use (4) to show that $\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}} \subset \mathrm{gr}^\beta \mathcal{V}_*$.]

(6) $\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}} \subset \mathrm{gr}^{-1}\mathcal{V}_*$ is identified with the image of $\partial_t : \mathrm{gr}^0 \mathcal{V}_* \to \mathrm{gr}^{-1}\mathcal{V}_*$. Conclude that $\partial_t : \mathrm{gr}^0\mathcal{V}_{\mathrm{mid}} \to \mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}$ is onto. Using the isomorphism $t : \mathrm{gr}^{-1}\mathcal{V}_* \xrightarrow{\sim} \mathrm{gr}^0\mathcal{V}_*$ identify also $\mathrm{gr}^{-1}\mathcal{V}_{\mathrm{mid}}$ with the image of $t\partial_t : \mathrm{gr}^0\mathcal{V}_* \to \mathrm{gr}^0\mathcal{V}_*$.

Exercise 6.15. The goal of this exercise is to illustrate the degeneration property of Remark 6.14.16(2) in a case where Hodge theory is not needed. The punctured Riemann surface is the Riemann sphere $X = \mathbb{P}^1$ with $r \geqslant 3$ punctures $x_1, \dots, x_r$ and $\mathcal{V}$ is a rank 1 bundle with connection on X^*. For each $i = 1, \dots, r$, the residue α_i of the connection on $\mathcal{V}^0_{\mathrm{mid}}$ at x_i, is assumed to have its real part in $(0,1)$.

(1) Show that $d := \sum_i \alpha_i \in (0,r)$ is an integer (hence $1 \leqslant d \leqslant r-1$) and that $\mathcal{V}^0_{\mathrm{mid}} = \mathcal{O}_{\mathbb{P}^1}(-d)$. Conclude that $H^0(\mathbb{P}^1, \mathcal{V}^0_{\mathrm{mid}}) = 0$. [*Hint*: Use the residue theorem for connections.]

(2) Show that $\mathcal{V}^{-1}_{\mathrm{mid}} = \mathcal{V}^{>-1}_{\mathrm{mid}} \simeq \mathcal{O}_{\mathbb{P}^1}(r-d)$. Conclude that $H^1(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes \mathcal{V}^{-1}_{\mathrm{mid}}) = 0$. [*Hint*: Compute the residue of the connection on $\mathcal{V}^{-1}_{\mathrm{mid}}$; use that $\Omega^1_{\mathbb{P}^1} \simeq \mathcal{O}_{\mathbb{P}^1}(-2)$.]

(3) Show that the long exact sequence

$$\cdots \longrightarrow \boldsymbol{H}^k(\mathbb{P}^1, V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) \longrightarrow H^k(\mathbb{P}^1, \mathcal{V}^0_{\mathrm{mid}}) \longrightarrow H^k(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes \mathcal{V}^{-1}_{\mathrm{mid}}) \longrightarrow \cdots$$

reduces to the short exact sequence

$$0 \longrightarrow H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1} \otimes \mathcal{V}^{-1}_{\mathrm{mid}}) \longrightarrow \boldsymbol{H}^1(\mathbb{P}^1, V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}) \longrightarrow H^1(\mathbb{P}^1, \mathcal{V}^0_{\mathrm{mid}}) \longrightarrow 0.$$

(4) Interpret this result as the degeneration at E_1 of the spectral sequence associated with the filtration of $V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ defined by

$$\begin{aligned} F^1 V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &= 0, \\ F^0 V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &= \Omega^1_{\mathbb{P}^1} \otimes \mathcal{V}^{>-1}_{\mathrm{mid}}[-1], \\ F^{-1} V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}} &= V^0 \,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}. \end{aligned}$$

(5) If all α_i's are real, relate this result with Remark 6.14.16(2). [*Hint*: The local system $\mathcal{V}^\nabla$ is then unitary.]

Exercise 6.16. Let X be a compact Riemann surface and let $(\mathcal{V},\nabla)$ be any non constant irreducible bundle with connection on X^*. Consider the filtration of $V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ defined by

$$\begin{aligned}F^1V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}&=0,\\ F^0V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}&=\Omega^1_{\mathbb{P}^1}\otimes\mathcal{V}_{\mathrm{mid}}^{>-1}[-1],\\ F^{-1}V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}&=V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}.\end{aligned}$$

(1) Show that degeneration at E_1 of the associated spectral sequence on hypercohomology both for $\mathcal{V}$ and $\mathcal{V}^\vee$ is equivalent to the property that both $\mathcal{V}^0_*$ and $(\mathcal{V}^\vee)^0_*$ do not have nonzero global sections on X. [*Hint*:

- $\Rightarrow$ Show first that $H^2(X,j_*\mathcal{V})$ and $H^2(X,j_*\mathcal{V}^\vee)$ are zero; show then that degeneration at E_1 is equivalent to the properties $H^1(X,\Omega^1_X\otimes\mathcal{V}^{>-1})=0$ and $H^0(X,\Omega^1_X\otimes\mathcal{V}^{>-1})\hookrightarrow\boldsymbol{H}^1(X,V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})$ and their dual analogues; use Serre duality and Remark 6.2.3(2) to get the vanishing of $H^0(X,\mathcal{V}^0_*)$ and $H^0(X,(\mathcal{V}^\vee)^0_*)$.
- $\Leftarrow$ The vanishing of $H^1(X,\Omega^1_X\otimes\mathcal{V}^{>-1})$ and its dual analogue is obtained by Serre duality as above; in order to obtain the inclusion property for H^0, use the exact sequence

$$\cdots\longrightarrow H^0(X,\mathcal{V}^0_*)\longrightarrow H^0(X,\mathcal{V}^{-1}_{\mathrm{mid}})\longrightarrow\boldsymbol{H}^1(X,V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\simeq H^1(X,j_*\mathcal{V}^\nabla)\longrightarrow\cdots$$

together with the inclusion $H^0(X,\mathcal{V}^{>-1}_{\mathrm{mid}})\subset H^0(X,\mathcal{V}^{-1}_{\mathrm{mid}})$, and the analogous results for $\mathcal{V}^\vee$.]

(2) Show that for a unitary local system $\mathcal{V}^\nabla$ with no nonzero constant global section on X^*, the vector bundle $\mathcal{V}^0_*$ has no nonzero global section. [*Hint*: Prove that the local system is semi-simple with no constant simple component, and that its dual local system satisfies the same property; show that the Hodge filtration of $V^0\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is that considered in (1); use the degeneration property of Remark 6.14.16(2) to conclude.]

6.16. Comments

The Hodge-Zucker theorem [**Zuc79**] makes use of the fundamental results of Schmid (Parts 1 and 2 of this chapter), and is the first occurrence of the purity theorem of the intermediate (or minimal) extension of a polarizable variation of Hodge structure. The proof given here is taken from loc. cit., with a small difference in the proof of the L^2 Dolbeault lemma (Theorem 6.14.10), for which we give a local result, while that of Zucker is global (on the cohomology).

In the approach of M. Saito [**Sai88**] to polarizable Hodge modules, the Hodge-Zucker theorem is the only analytic result that needs to be used. Nevertheless, for the extension of the theory to the mixed case, Zucker's theorem in higher dimensions ([**CK82, CKS86, CKS87, KK87**] and the more recent [**Moc22**]) are needed.

CHAPTER 7

POLARIZABLE HODGE MODULES ON CURVES

Summary. The aim of this chapter is to introduce the general notion of polarized pure Hodge module on a Riemann surface, as the right notion of a singular analogue of a polarized variation of Hodge structure. We will define it by *local* properties, as we do for polarized variations of Hodge structure. For that purpose, we first recall basics on $\mathcal{D}$-modules, which are much more developed in Chapters 8–12. While the notion of a variation of $\mathbb{C}$-Hodge structure on a punctured compact Riemann surface is purely analytic, that of a pure Hodge module on the corresponding smooth projective curve is partly algebraic.

7.1. Introduction

Let $j : X^* \hookrightarrow X$ be the inclusion of the complement of a finite set of points D in a compact Riemann surface[(1)] X, and let (H, S) be a polarized variation of Hodge structure on X^*, with associated local system $\underline{\mathcal{H}}$ and filtered holomorphic bundle $(\mathcal{V}, \nabla, F^\bullet\mathcal{V})$, as considered in Chapter 6. The Hodge-Zucker theorem gives importance to the differential object $(\mathcal{V}_{\mathrm{mid}}, \nabla)$ (see Exercise 6.2(6)). However it is, in general, *not* a coherent $\mathcal{O}_X$-module with connection. It is neither a meromorphic bundle with connection in general, i.e., it is not an $\mathcal{O}_X(*D)$-module (where $\mathcal{O}_X(*D)$ denotes the sheaf of meromorphic functions on X with poles on D at most). We have to consider it as a coherent $\mathcal{D}_X$-module, where $\mathcal{D}_X$ denotes the sheaf of holomorphic differential operators. In order to do so, we recall in Section 7.2 the basic notions on $\mathcal{D}$-modules in one complex variable, the general case being treated in Chapter 8.

The punctured Riemann surface will then be a punctured disc Δ^* in the remaining part of this introduction. The object analogue to $(\mathcal{V}, \nabla, F^\bullet\mathcal{V})$ on Δ is a holonomic $\mathcal{D}_\Delta$-module $\mathcal{M}$ equipped with an F-filtration $F^\bullet\mathcal{M}$ (this encodes the Griffiths transversality property). Here, the language of triples introduced in Section 5.2 becomes useful in order to avoid using "C^∞ bundles with singularities". On the other hand, we can increase the domain (C^∞ functions) where sesquilinear pairing takes

(1) In order to simplify some statements, we will always assume in this chapter that X is *connected*.

values: as in our discussion of Schmid's theorem, we should add to C^∞ functions on Δ functions like $|t|^{2\beta}\mathrm{L}(t)^k/k!$. More generally, we should also accept Dirac "delta functions", so that the sheaf of distributions on Δ is a possible candidate as the target sheaf of sesquilinear pairings, as it is acted on by holomorphic and anti-holomorphic differential operators.

The idea of M. Saito for defining the Hodge property of a filtered $\mathcal{D}$-module (more precisely, triples) in an axiomatic way is to impose the Hodge property on the restriction of the data—a filtered triple in the sense of Section 5.2—at each point of Δ, in order to apply the corresponding definitions. While this does not cause any trouble at points of $\Delta^* := \Delta \smallsetminus \{0\}$, this leads to problems at the origin for the following reason: the restriction of $\mathcal{M}$ in the sense of $\mathcal{D}$-modules is a complex, which has two cohomology vector spaces in general. The right way to consider the restriction consists in introducing *nearby cycles*. Therefore, the compatibility of the data with the nearby and vanishing cycle functors will be the main tool in the theory of Hodge modules.

However, not all $\mathcal{D}_\Delta$-modules underlie a Hodge module. On the one hand, we have to restrict the category by only considering holonomic $\mathcal{D}_\Delta$-modules having a regular singularity at the origin. This is "forced" by the theorem of Griffiths-Schmid (see Remark 6.3.8(1)) stating the regularity of the connection on the extended Hodge bundles. Moreover, the Hodge-Zucker theorem leads us to focus on regular holonomic $\mathcal{D}_\Delta$-modules which are middle extensions of their restriction to Δ^*. Now, a new phenomenon appears when dealing with $\mathcal{D}_\Delta$-modules, when compared to the case of vector bundles with connection, namely, there do exist $\mathcal{D}_\Delta$-modules supported at the origin, like those generated by Dirac distributions. But their Hodge variants are easy to define.

There are thus two kinds of $\mathcal{D}_\Delta$-modules that should underlie a pure Hodge module. Which extensions between these two kinds can we allow? Since our goal is to define the category of polarizable Hodge modules as an analogue over Δ of the category of polarizable Hodge structures, we expect to obtain a semi-simple category. The polarizability condition we impose solves this question for us: only direct sums of objects of each kind may appear as a polarizable Hodge module. This is called *Support-decomposability* (S-decomposability), and is obtained as a consequence of the S-decomposability theorem for polarizable Hodge-Lefschetz structures 3.4.22.

In this chapter, we will consider *left* $\mathcal{D}$-modules in order to keep the analogy with vector bundles with connections and variations of Hodge structure considered in Chapter 6.

The Hodge theorem takes the following form in the framework of $\mathbb{C}$-Hodge modules on a compact Riemann surface X. We consider the constant map $a: X \to \mathrm{pt}$. For a given $\mathbb{C}$-Hodge module M polarized by S, we define for $k=-1,0,1$ the k-th de Rham cohomology ${}_{\mathrm{T}}a_{X*}^{(k)}M$ in the category $\mathbb{C}$-$\mathsf{Triples}$ (see Definition 5.2.1).

***7.1.1. Theorem* (Hodge-Saito).** *If M is a polarizable Hodge module of weight w on a compact Riemann surface X, the triple ${}_{\mathrm{T}}a_{X*}^{(k)}M$ is a polarizable Hodge structure of weight $w+k$.*

7.2. Basics on holonomic $\mathcal{D}$-modules in one variable

We refer to Chapter 8 for a more general setting. We denote by t a coordinate on the disc Δ, by $\mathbb{C}\{t\}$ the ring of convergent power series in the variable t. Let us denote by $\mathcal{D} = \mathbb{C}\{t\}\langle\partial_t\rangle$ the ring of germs at $t = 0$ of holomorphic differential operators: this is the quotient of the free algebra generated by $\mathbb{C}\{t\}$ and the ring $\mathbb{C}[\partial]$ of polynomials in one variable ∂ by the two-sided ideal generated by the elements $\partial g - g\partial - g'$ for any $g \in \mathbb{C}\{t\}$ (where g' denotes the derivative). We denote by ∂_t the class of ∂. This is a noncommutative algebra, which operates in a natural way on $\mathbb{C}\{t\}$: the subalgebra $\mathbb{C}\{t\}$ acts by multiplication and ∂_t acts as the usual derivation. There is a natural increasing filtration $F_\bullet\mathcal{D}$ indexed by $\mathbb{Z}$ defined by

$$F_k\mathcal{D} = \begin{cases} 0 & \text{if } k \leqslant -1, \\ \sum_{j=0}^{k} \mathbb{C}\{t\} \cdot \partial_t^j & \text{if } k \geqslant 0. \end{cases}$$

This filtration is compatible with the ring structure (i.e., $F_k \cdot F_\ell \subset F_{k+\ell}$ for every $k, \ell \in \mathbb{Z}$). The graded ring $\mathrm{gr}^F\mathcal{D} := \bigoplus_k \mathrm{gr}_k^F\mathcal{D} = \bigoplus_k F_k/F_{k-1}$ is isomorphic to the polynomial ring $\mathbb{C}\{t\}[\tau]$ (graded with respect to the degree in τ).

We also denote by $\mathcal{D}_\Delta$ the *sheaf* of differential operators with holomorphic coefficients on Δ. This is a coherent sheaf, similarly equipped with an increasing filtration $F_\bullet\mathcal{D}_\Delta$ by free $\mathcal{O}_\Delta$-modules of finite rank. The graded sheaf $\mathrm{gr}^F\mathcal{D}_\Delta$ is identified with the sheaf on Δ of functions on the cotangent bundle $T^*\Delta$ which are polynomial in the fibers of the fibration $T^*\Delta \to \Delta$.

7.2.a. Coherent F-filtrations, holonomic modules. Let M be a finitely generated $\mathcal{D}$-module (we basically use left $\mathcal{D}$-modules, but similar properties can be applied to right ones). By an F-filtration of M we mean increasing filtration $F_\bullet M$ by $\mathcal{O} = \mathbb{C}\{t\}$-submodules, indexed by $\mathbb{Z}$, such that, for every $k, \ell \in \mathbb{Z}$, $F_k\mathcal{D} \cdot F_\ell M \subset F_{k+\ell}M$. Such a filtration is said to be *coherent* if it satisfies the following properties:

(1) $F_kM = 0$ for $k \ll 0$,

(2) each F_kM is finitely generated over $\mathcal{O}$,

(3) there exists $\ell_0 \in \mathbb{Z}$ such that, for every $k \geqslant 0$ and any $\ell \geqslant \ell_0$, $F_k\mathcal{D} \cdot F_\ell M = F_{k+\ell}M$.

7.2.1. Remark **(Increasing or decreasing?)** In Hodge theory, one usually uses decreasing filtrations. The trick (see Notation 0.4) to pass from increasing (lower index) to decreasing (upper index) filtrations is to set, for every $p \in \mathbb{Z}$,

$$F^pM := F_{-p}M.$$

The notion of shift is compatible with this convention:

$$F[k]^pM = F^{p+k}M, \quad F[k]_pM = F_{p-k}M.$$

7.2.2. Definition. We say that M is *holonomic* if it is finitely generated and any element of M is annihilated by some nonzero $P \in \mathcal{D}$.

One can prove that any holonomic $\mathcal{D}$-module can be generated by one element (i.e., it is cyclic), hence of the form $\mathcal{D}/I$ where I is a left ideal in $\mathcal{D}$, and that this ideal can be generated by two elements (see [**BM84**]).

7.2.b. The V-filtration. In order to analyze the behaviour of a holonomic module near the origin, we will use another kind of filtration, called the Kashiwara-Malgrange filtration. It is an extension to holonomic modules of the notion of Deligne lattice for meromorphic bundle with connection.

We first define the increasing filtration $V_\bullet\mathcal{D}$ indexed by $\mathbb{Z}$, by giving to any monomial $t^{a_1}\partial_t^{b_1}\cdots t^{a_n}\partial_t^{b_n}$ the V-degree $\sum_i b_i - \sum_i a_i$, and by defining the V-order of an operator $P \in \mathcal{D}$ as the biggest V-degree of its monomials. (See Exercise 7.2.)

7.2.3. Definition. Let M be a left $\mathcal{D}$-module. By a *V-filtration* we mean an decreasing filtration $U^\bullet M$ of M, indexed by $\mathbb{Z}$, which satisfies $V_k\mathcal{D}\cdot U^\ell M \subset U^{\ell-k}M$ for every $k,\ell \in \mathbb{Z}$. We say that $U^\bullet M$ is *coherent* if there exists $\ell_0 \in \mathbb{N}$ such that the previous inclusion is an equality for every $k \geqslant 0$ and $\ell \leqslant -\ell_0$, and for every $k \leqslant 0$ and $\ell \geqslant \ell_0$.

Some properties of V-filtrations are given in Exercise 7.3. In particular, for any V-filtration $U^\bullet M$ of a holonomic $\mathcal{D}$-module M, the graded spaces $\mathrm{gr}_U^k M$ are finite-dimensional and we denote by E the action of $t\partial_t$ on each $\mathrm{gr}_U^k M$, which has thus a minimal polynomial on each such space.

***7.2.4. Proposition* (The Kashiwara-Malgrange filtration).** *Let M be a holonomic $\mathcal{D}$-module. Then there exists a unique coherent V-filtration denoted by $V^\bullet M$ and called* the Kashiwara-Malgrange filtration of M*, such that the eigenvalues of* E *acting on the finite dimensional vector spaces $\mathrm{gr}_V^k M$ have their real part in $[k, k+1)$.*

Proof. Adapt Exercise 9.16 to the present setting. □

See Exercises 7.4–7.7 for more properties of the Kashiwara-Malgrange filtration.

7.2.5. Caveat. It may happen that the V-filtration is constant, so that all V-graded modules are zero. The regularity condition explained below prevents such a behaviour.

7.2.c. Nearby and vanishing cycles. For simplicity, in the following we always assume that M is holonomic. We will also assume that the eigenvalues of E (Exercise 7.3) acting on $\mathrm{gr}_V^k M$ are *real*, i.e., belong to $[k, k+1)$. This will be the only case of interest in Hodge theory, according to Theorem 6.3.2(6.3.2). Let $B \subset [0,1)$ be the finite set of eigenvalues of E acting on $\mathrm{gr}_V^0 M$, to which we add 0 if 0 is not an eigenvalue. By Exercise 7.5, the set B_k of eigenvalues of E acting on $\mathrm{gr}_V^k M$ satisfies $k + (B \smallsetminus \{0\}) \subset B_k \subset k + B$.

For every $\beta \in \mathbb{R}$, we denote by $V^\beta M \subset V^{[\beta]}M$ the pullback by $V^{[\beta]}M \to \mathrm{gr}_V^{[\beta]}M$ of the sum of the generalized eigenspaces of $\mathrm{gr}_V^{[\beta]}M$ corresponding to eigenvalues of E which are $\geqslant \beta$, i.e., the subspace $\bigoplus_{\gamma\in[\beta,[\beta]+1)}\mathrm{Ker}(\mathrm{E}-\gamma\,\mathrm{Id})^N$, $N \gg 0$.

In such a way, we obtain a decreasing filtration $V^\bullet M$ indexed by $B + \mathbb{Z} \subset \mathbb{R}$, and we now denote by $\mathrm{gr}_V^\beta M$ the quotient space $V^\beta M/V^{>\beta}M$. It is identified with the generalized eigenspace of E with eigenvalue β in $V^{[\beta]}M/V^{[\beta]+1}M$, and we still denote by E the induced action of $t\partial_t$ on it. As a consequence, $\mathrm{E}-\beta\,\mathrm{Id}$ is *nilpotent* on $\mathrm{gr}_V^\beta M$. We can also consider $V^\bullet M$ as a filtration indexed by $\mathbb{R}$ which jumps at most at $B+\mathbb{Z}$ (see Exercise 7.8).

Exercise 7.5 implies:

(1) for every $\beta > -1$, the morphism $V^\beta M \to V^{\beta+1}M$ induced by t is an isomorphism, and so is the morphism $t : \mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta+1} M$; in particular, $V^\beta M$ is $\mathcal{O}$-free if $\beta > -1$;

(2) for every $\beta < 0$, the morphism $\mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta-1} M$ induced by ∂_t is an isomorphism;

(3) for every $\beta \in [-1, 0)$ and $k \geqslant 1$,

$$V^{\beta-k}M = \partial_t^k V^\beta M + \sum_{j=0}^{k-1} \partial_t^j V^{-1}M.$$

In particular, the knowledge of $\mathrm{gr}_V^\beta M$ for $\beta \in [-1, 0]$ implies that for all β. The following notation will be used.

7.2.6. Notation.

- $\psi_{t,\lambda}M := \mathrm{gr}_V^\beta M$, if $\lambda = \exp(-2\pi\mathrm{i}\,\beta)$ with $\beta \in (-1, 0]$,
- $\phi_{t,1}M := \mathrm{gr}^{-1} M$.

***7.2.7. Definition* (The morphisms $\mathrm{N}, \mathrm{can}, \mathrm{var}$).** Let M be a holonomic $\mathcal{D}$-module.

(a) We denote by N the nilpotent part of the endomorphism induced by $-\mathrm{E}$ on $\mathrm{gr}_V^\beta M$ for every β (we will only consider $\beta \in [-1, 0]$, according to (1) and (2) above). So we have $\mathrm{N} = -(\mathrm{E}-\beta\,\mathrm{Id})$ on $\mathrm{gr}_V^\beta M$ for $\beta \in [-1, 0]$.

(b) We define $\mathrm{can} : \psi_{t,1}M \to \phi_{t,1}M$ as the homomorphism induced by $-\partial_t$ and $\mathrm{var} : \phi_{t,1}M \to \psi_{t,1}M$ as that induced by t, so that $\mathrm{var} \circ \mathrm{can} = \mathrm{N} : \psi_{t,1}M \to \psi_{t,1}M$ and $\mathrm{can} \circ \mathrm{var} = \mathrm{N} : \phi_{t,1}M \to \phi_{t,1}M$.

(c) We also denote by $\mathrm{M}_\bullet \mathrm{gr}_V^\beta M$ the monodromy filtration defined by the nilpotent endomorphism N (see Section 3.4.a).

(See Exercise 7.9 for various properties.)

7.2.8. Examples.

(1) If 0 is not a singular point of M, then M is $\mathcal{O}$-free of finite rank and $\mathrm{gr}_V^\beta M = 0$ unless $\beta \in \mathbb{N}$ (i.e., $\psi_{t,\lambda}M = 0$ if $\lambda \neq 1$ and $\phi_{t,1}M = 0$). Then $\mathrm{can} = 0$, $\mathrm{var} = 0$ and $\mathrm{N} = 0$.

(2) If M is supported at the origin, i.e., if any element of M is annihilated by some power of t, then $\psi_{t,\lambda}M = 0$ for any λ, so that $\mathrm{can}, \mathrm{var}, \mathrm{N}$ are zero, and M is identified with $(\phi_{t,1}M)[\partial_t]$.

(3) If M is *purely irregular*, e.g. $M = (\mathcal{O}[t^{-1}], \nabla)$ with $\nabla = \mathrm{d} + \mathrm{d}t/t^2$, then $\mathrm{gr}_V^\beta M = 0$ for every β. In such a case, the gr_V^β-functors do not bring any interesting information on M.

7.2.9. Definition **(Regular singularity).** We say that M has a regular singularity (or is *regular*) at the origin if V^0M (equivalently, any $V^\beta M$) has finite type over $\mathcal{O}$.

(See Exercises 7.10 and 7.11.)

Structure of regular holonomic $\mathcal{D}$-modules. Let M be regular holonomic. For $\beta \in \mathbb{R}$, set

$$M^\beta := \bigcup_k \operatorname{Ker}\big[(t\partial_t - \beta)^k : M \to M\big].$$

Then $M^\beta \cap M_\gamma = 0$ if $\beta \neq \gamma$. Moreover, $M^\beta \cap V^{>\beta}M = 0$: indeed, if $(t\partial_t - \beta)^k m = 0$ and $b(t\partial_t)m = tP(t, t\partial_t)m$ with b having roots $> \beta$, we conclude a relation $m = tQ(t, t\partial_t)m$ by Bézout, so the $\mathcal{D}$-module $\mathcal{D}\cdot m$ satisfies $\mathcal{D}\cdot m = V^1(\mathcal{D}\cdot m)$, and its V-filtration is constant; iterating, we find $V^1(\mathcal{D}\cdot m) = tV^1(\mathcal{D}\cdot m)$; however, the $\mathcal{O}$-finiteness of $V^1(\mathcal{D}\cdot m)$ implies $V^1(\mathcal{D}\cdot m) = 0$ (Nakayama), hence $\mathcal{D}\cdot m = 0$, and therefore $m = 0$. As a consequence, M^β injects in $\operatorname{gr}_V^\beta M$ and thus has finite dimension. Obviously, multiplication by t sends M^β to $M^{\beta+1}$ and ∂_t goes in the reverse direction. Moreover, $t : M^\beta \to M^{\beta+1}$ is an isomorphism if $\beta > -1$ and $\partial_t : M^{\beta+1} \to M^\beta$ is an isomorphism if $\beta < 0$.

The set consisting of β's such that $M^\beta \neq 0$ is therefore contained in $B + \mathbb{Z}$ (B is defined at the beginning of Section 7.2.c), and $M^{\mathrm{alg}} := \bigoplus_\beta M^\beta$ is a regular holonomic $\mathbb{C}[t]\langle\partial_t\rangle$-module.

7.2.10. Proposition. *If M is regular holonomic, Then the natural morphism*

$$\mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}} \longrightarrow M$$

is an isomorphism of $\mathcal{D}$-modules, and induces an $\mathbb{R}$-graded isomorphism

$$M^{\mathrm{alg}} \xrightarrow{\sim} \operatorname{gr}_V M^{\mathrm{alg}} \xrightarrow{\sim} \operatorname{gr}_V M.$$

Sketch of proof. If M is supported at the origin, the result is easy. One can then assume that M has no section supported at the origin. Let us first set $V^{>-1}M^{\mathrm{alg}} := \bigoplus_{\beta>-1} M^\beta$ and prove $\mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} V^{>-1}M^{\mathrm{alg}} \xrightarrow{\sim} V^{>-1}M$. Note that $V^{>-1}M$ is $\mathcal{O}$-free and the matrix $A(t)$ of the action of $t\partial_t$ on $V^{>-1}M$ is holomorphic and the eigenvalues of $A(0)$ belong to $(-1, 0]$. It is standard that there exists an $\mathcal{O}$-basis $(m_1, \dots, m_r)$ of $V^{>-1}M$ for which the matrix of $t\partial_t$ is equal to $A(0)$. This gives the desired isomorphism.

Let us extend this isomorphism to $V^{-1}M^{\mathrm{alg}}$ and $V^{-1}M$ for example. If $m \in V^{-1}M$, then $tm = \sum_{i=1}^r a_i(t)m_i$ with a_i holomorphic. Let us set $a_i(t) = a_i(0) + tb_i(t)$. Then $t(t\partial_t + 1)^k(m - \sum_i b_i(t)m_i) = 0$ for some $k \geqslant 1$ and, by our assumption, $m - \sum_i b_i(t)m_i \in M^{-1}$. Continuing this way, we get the result. □

7.2.11. Definition **(Middle extension).** We say that a regular holonomic M is the *middle (or minimal) extension* of $M[t^{-1}] := \mathcal{O}[t^{-1}] \otimes_{\mathcal{O}} M$ if can *is onto and* var *is injective*, that is, if M has neither a quotient nor a submodule supported at the origin (see Exercise 7.9).

Clearly, there is non non-zero morphism between a middle extension and a $\mathcal{D}$-module supported at the origin.

7.2.12. Definition **(S-decomposability).** We say that a regular holonomic $\mathcal{D}$-module M is S(upport)-decomposable if it can be decomposed as $M_1 \oplus M_2$, where M_1 is a middle extension and M_2 is supported at the origin.

See Exercise 7.9 for details. In particular, such a decomposition is unique if it exists, and there is a criterion for S-decomposability, obtained by considering M^{alg} first:

7.2.13. Proposition. *A holonomic M is S-decomposable if and only if*

$$\phi_{t,1}M = \operatorname{Im}\operatorname{can} \oplus \operatorname{Ker}\operatorname{var}. \qquad \square$$

The following proposition makes the link between the $\mathcal{D}$-module approach and the approach of Section 6.2.a.

7.2.14. Proposition. *Assume that M has a regular singularity at the origin. Then $M[t^{-1}]$ is equal to the germ at 0 of $(\mathcal{V}_*, \nabla)$ (Deligne's canonical meromorphic extension), where $(\mathcal{V}, \nabla)$ is the restriction of M to a punctured small neighbourhood of the origin. Moreover, if M is a middle extension, then M is equal to the germ at 0 of $(\mathcal{V}_{\mathrm{mid}}, \nabla)$. Last, the filtration $\mathcal{V}_*^\bullet$ (resp. $\mathcal{V}_{\mathrm{mid}}^\bullet$) is equal to the Kashiwara-Malgrange filtration.*

Proof. Let $\mathcal{M}$ be a coherent $\mathcal{D}_\Delta$-module that represents the germ M on a small disc Δ, having a singularity at 0 only. Set $(\mathcal{V}, \nabla) = \mathcal{M}_{|\Delta^*}$. By the uniqueness of the Deligne lattices with given range of eigenvalues of the residue, we have $\mathcal{V}_*^{>-1} = V^{>-1}\mathcal{M}$. We then have $\mathcal{M}[t^{-1}] = V^{>-1}\mathcal{M}[t^{-1}] = \mathcal{V}_*$, according to Exercise 7.10(1). If M is a middle extension, the assertion follows from 7.10(2). The last assertion is proved similarly. $\square$

F-filtration on nearby and vanishing cycles. Let M be holonomic and equipped with a coherent F-filtration $F_\bullet M$. In order to keep notations analogous to that of Chapter 6, we rather use the associated decreasing filtration $F^\bullet M$ (see Remark 7.2.1). There is a natural way to induce a filtration on each vector space $\mathrm{gr}_V^\beta M$ by setting

$$F^p\mathrm{gr}_V^\beta M := \frac{F^pM \cap V^\beta M}{F^pM \cap V^{>\beta}M}. \tag{7.2.15}$$

Notation 7.2.6 is convenient for the following convention.

$$\begin{aligned} F^p\psi_{t,\lambda}M &:= F^p\mathrm{gr}_V^\beta M = \frac{F^pM \cap V^\beta M}{F^pM \cap V^{>\beta}M} \\ F^p\phi_{t,1}M &:= F^{p-1}\mathrm{gr}_V^{-1} M = F[-1]^p\mathrm{gr}_V^{-1} M = \frac{F^{p-1}M \cap V^{-1}M}{F^{p-1}M \cap V^{>-1}M}. \end{aligned} \tag{7.2.16}$$

We also write (see (5.1.5 $**$))

$$\psi_{t,\lambda}(M, F^\bullet) = (\mathrm{gr}_V^\beta M, F^\bullet), \qquad \phi_{t,1}(M, F^\bullet) = (\mathrm{gr}_V^{-1} M, F^\bullet)(-1). \tag{7.2.17}$$

We then have a Lefschetz quiver (see Exercise 7.17)

$$\begin{tikzcd}(\psi_{t,1}M, F^{\bullet}) \arrow[rr, bend left, "\mathrm{can} = -\partial_t\cdot"] && (\phi_{t,1}M, F^{\bullet}) \arrow[ll, bend left, "\mathrm{var} = t\cdot", "(-1)"']\end{tikzcd}. \tag{7.2.18}$$

The notion of strict $\mathbb{R}$-specializability models a good behaviour of the filtration $F^{\bullet}M$ with respect to the V-filtration. In the following, we will set

$$F^pV^\beta M := F^pM \cap V^\beta M.$$

***7.2.19. Definition* (Strict $\mathbb{R}$-specializability).** An F-filtered $\mathcal{D}$-module $(M, F^{\bullet}M)$ is said to be *strictly* $\mathbb{R}$*-specializable* if the properties 6.14.2(3a) and (3b) are satisfied, that is,

(a) for every $\beta > -1$ and p, $t(F^pV^\beta M) = F^pV^{\beta+1}M$,
(b) for every $\beta < 0$ and p, $\partial_t(F^p\mathrm{gr}_V^\beta M) = F^{p-1}\mathrm{gr}_V^{\beta-1}M$.

(See also Definition 9.3.14 together with Proposition 10.8.3.) We note that strict $\mathbb{R}$-specializability implies regularity:

7.2.20. Proposition. *Let $(M, F^{\bullet}M)$ be a coherently F-filtered $\mathcal{D}$-module with M holonomic. If $(M, F^{\bullet}M)$ is strictly $\mathbb{R}$-specializable, then M is regular holonomic.*

See Exercise 7.15 for the proof.

7.2.21. Lemma. *For a coherently F-filtered $\mathcal{D}$-module $(M, F^{\bullet}M)$,* 7.2.19(a) *and* (b) *are respectively equivalent to*

(a) *for every $\beta > -1$ and p, $t : F^p\mathrm{gr}_V^\beta M \to F^p\mathrm{gr}_V^{\beta+1}M$ is an isomorphism,*
(b) *for every $\beta < 0$ and p, $\partial_t : F^p\mathrm{gr}_V^{\beta-1}M \to F^{p-1}\mathrm{gr}_V^{\beta-1}M$ is an isomorphism.*

Proof. 7.2.19(a) $\Leftrightarrow$ 7.2.21(a):

• For $\Rightarrow$, we note that since $t : \mathrm{gr}_V^\beta M \to \mathrm{gr}^{\beta+1}M$ is injective ($\beta > -1$), it remains so when restricted to $F^p\mathrm{gr}_V^\beta M$. Surjectivity in 7.2.21(a) is then clear.

• For $\Leftarrow$, we know by regularity that $V^\beta M$ has finite type over $\mathbb{C}\{t\}$. Recall that Artin-Rees implies that $tF^pV^\beta M \supset F^p \cap t^qV^\beta M$ for $q \gg 0$. On the other hand, 7.2.21(a) means that $F^pV^{\beta+1}M = tF^pV^\beta M + F^pV^{>\beta+1}M$ and, by an easy induction, $F^pV^{\beta+1}M = tF^pV^\beta M + F^pV^{\beta+q}M$ for any $q \geqslant 1$. We can thus conclude by Artin-Rees.

7.2.19(b) $\Leftrightarrow$ 7.2.21(b): 7.2.19(b) means surjectivity in 7.2.21(b). Injectivity is automatic since it holds when forgetting filtrations. □

7.2.22. Caveat. Even if $(M, F^{\bullet}M)$ is strictly $\mathbb{R}$-specializable, Proposition 7.2.10 may not hold with filtration.

The full subcategory of that of coherently F-filtered $\mathcal{D}$-modules which are strictly $\mathbb{R}$-specializable is not abelian. Nevertheless, strictly $\mathbb{R}$-specializable morphisms have kernels and cokernels in this category.

7.2.23. Proposition. *Let $\varphi : (M_1, F^\bullet M_1) \to (M_2, F^\bullet M_2)$ be a morphism between strictly $\mathbb{R}$-specializable coherently F-filtered $\mathcal{D}$-modules. If φ is strictly $\mathbb{R}$-specializable, that is, if $\mathrm{gr}_V^\beta \varphi$ is strict for any $\beta \in [0,1]$, then φ is strict and $\operatorname{Ker}\varphi, \operatorname{Im}\varphi, \operatorname{Coker}\varphi$ are strictly $\mathbb{R}$-specializable.*

Proof.

Step* 1*: strictness of* φ*. It is enough to prove that, for any β and p, we have

$$\varphi(V^\beta M_1) \cap F^p V^\beta M_2 = \varphi(F^p V^\beta M_1), \tag{7.2.24}$$

where we have set $F^p V^\beta M := F^p M \cap V^\beta M$. We know that all objects involved have finite type over $\mathbb{C}\{t\}$, and the inclusion $\supset$ is clear. By assumption, $\mathrm{gr}_V^\beta \varphi$ is strict for any $\beta \in [-1,0]$. Now, strict $\mathbb{R}$-specializability of M_1, M_2 implies that it is so for any $\beta \in \mathbb{R}$. This is translated as

$$\begin{aligned} \varphi(V^\beta M_1) \cap F^p V^\beta M_2 &= \varphi(F^p V^\beta M_1) + V^{>\beta} M_2 \\ &= \varphi(F^p V^\beta M_1) + (\varphi(V^\beta M_1) \cap F^p V^{>\beta} M_2) \end{aligned} \tag{7.2.25}$$

for any β and p. By an easy induction, one can replace in the right-hand side the term $F^p V^{>\beta} M_2$ with $F^p V^{\beta+k} M_2$ for any $k \geqslant 1$. If $\beta > -1$, we have $F^p V^{\beta+1} M_2 = t F^p V^\beta M_2$ and, by V-strictness of φ,

$$\varphi(V^\beta M_1) \cap V^{\beta+1} M_2 = \varphi(V^{\beta+1} M_1) = t\varphi(V^\beta M_1),$$

hence

$$\varphi(V^\beta M_1) \cap F^p V^{\beta+1} M_2 = t(\varphi(V^\beta M_1) \cap F^p V^\beta M_2),$$

so (7.2.24) holds by Nakayama's lemma. Assuming now that (7.2.24) holds for $\beta' > \beta$, (7.2.25) reads

$$\varphi(V^\beta M_1) \cap F^p V^\beta M_2 = \varphi(F^p V^\beta M_1) + \varphi(F^p V^{>\beta} M_1) = \varphi(F^p V^\beta M_1),$$

as wanted.

Step 2. We prove that $\operatorname{Ker} \mathrm{gr}_V^\beta \varphi$ (with filtration induced by that of $\mathrm{gr}_V^\beta M$) is equal to $\mathrm{gr}_V^\beta \operatorname{Ker}\varphi$ (with filtration coming from that on $\operatorname{Ker}\varphi$ induced by that of M), and similarly for Coker.

The case of $\operatorname{Coker} \mathrm{gr}_V^\beta \varphi$ is clear, since both induced filtrations are equal to the image of $F^p V^\beta M_2$.

Let us consider the case of $\operatorname{Ker}\varphi$. The assertion amounts to the following property (for all β, p):

$$\{m \in F^p V^\beta M_1 \mid \varphi(m) \in V^{>\beta} M_2\} \subset \{m \in F^p V^\beta M_1 \mid \varphi(m) = 0\} + V^{>\beta} M_1.$$

By the V-strictness of φ, the equality holds if we forget F^p. Let us fix m in the left-hand side, and let us write it as $m = m_1 - m_1'$, with $m_1 \in V^\beta \operatorname{Ker}\varphi$ and $m_1' \in V^{>\beta} M_1$. We aim at proving that $m_1 \in F^p V^\beta M_1$. We thus write $m_1 = m + m_1'$, $m \in F^p V^\beta M_1$ and $m_1' \in V^{>\beta} M_1$.

Assume that $m_1' \in V^\gamma M_1$ with $\gamma > \beta$, and let $[m_1']$ its class in $\mathrm{gr}_V^\gamma M_1$. Its image by $\mathrm{gr}_V^\gamma \varphi$, being the class of $\varphi(m)$, belongs to $F^p\mathrm{gr}_V^\gamma M_2$ and, by F-strictness of $\mathrm{gr}_V^\gamma \varphi$, is also the image of $[\widetilde{m}] \in F^p\mathrm{gr}_V^\gamma M_1$. It follows that

$$m_1 = m + \widetilde{m} + m_1'', \quad \widetilde{m} \in F^pV^\gamma M_1,\ m_1'' \in V^{>\gamma}M_1.$$

Continuing this way, we can write for each $k \geqslant 1$

$$m_1 = m^{(k)} + m_1^{(k)}, \quad m^{(k)} \in F^pV^\beta M_1,\ m_1^{(k)} \in t^kV^\beta M_1.$$

In other words, let us denote by $[m_1]$ the image of m_1 in $V := V^\beta M_1/F^pV^\beta M_1$. Then $[m_1]$ becomes zero in V/t^kV for any k, hence in $\widehat{V} = \varprojlim_k V/t^kV$. Since V has finite type over $\mathbb{C}\{t\}$, we have $\widehat{V} = \mathbb{C}[\![t]\!] \otimes_{\mathbb{C}\{t\}} V$ and the natural morphism $V \to \widehat{V}$ is injective. Therefore, $[m_1] = 0$, as wanted.

Step 3. We prove that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$, as F-filtered $\mathcal{D}_\Delta$-modules, are strictly $\mathbb{R}$-specializable at the origin. Properties 7.2.21(a) and (b) hold for $\mathrm{gr}_V^\beta M_i$ ($i = 1, 2$, any $\beta \in \mathbb{R}$), hence they old for $\operatorname{Ker}\mathrm{gr}_V^\beta\varphi$ and $\operatorname{Coker}\mathrm{gr}_V^\beta\varphi$. But by Step 2, these are $\mathrm{gr}_V^\beta \operatorname{Ker}\varphi$ and $\mathrm{gr}_V^\beta \operatorname{Coker}\varphi$, so the assertion holds, according to Lemma 7.2.21. □

The definition of middle extension for a coherently F-filtered $\mathcal{D}$-module similar to that of Definition 7.2.12 is not sufficient for our purposes (see Proposition 9.7.2). If we restrict to those coherently F-filtered $\mathcal{D}$-modules which are strictly $\mathbb{R}$-specializable, the definition in terms of injectivity of var and surjectivity of can is stronger and more convenient. Let us make precise that, for a morphism of filtered vector spaces, surjectivity means means subjectivity of F^p to F^p for each p.

***7.2.26. Definition* (Filtered middle extension).** Let $(M, F^\bullet M)$ be a coherently F-filtered holonomic $\mathcal{D}$-module which is strictly $\mathbb{R}$-specializable. We say that $(M, F^\bullet M)$ is a middle extension if M is a middle extension, i.e.,

(a) $t : \mathrm{gr}_V^{-1}M \to \mathrm{gr}_V^0 M$ is injective,
(b) $\partial_t : \mathrm{gr}_V^0 M \to \mathrm{gr}_V^{-1}M$ is onto,

and moreover

(c) $F^p\mathrm{gr}_V^{-1}M = \partial_t F^{p+1}\mathrm{gr}_V^0 M$ for all p.

Then the notion of S-decomposability for a coherently F-filtered $\mathcal{D}$-module with M strictly $\mathbb{R}$-specializable is similar to that of Definition 7.2.12. The criterion of Proposition 7.2.13 extends to the filtered case:

7.2.27. Proposition. *If $(M, F^\bullet M)$ is coherent, holonomic and strictly $\mathbb{R}$-specializable, then it is S-decomposable if and only if*

$$\phi_{t,1}(M, F^\bullet M) = \operatorname{Im}\operatorname{can} \oplus \operatorname{Ker}\operatorname{var}. \qquad \square$$

One should be careful with the notion of image and kernel, since the category of filtered $\mathcal{D}$-modules is not abelian. Here, we take the image filtration $\operatorname{can}(F^\bullet\psi_{t,1}M)$ and the induced filtration $\operatorname{Ker}\operatorname{var} \cap F^\bullet\phi_{t,1}M$. The proof is left as an exercise. A similar statement in higher dimension is given in Proposition 9.7.5.

The germic version of the de Rham complex. Let us first consider the de Rham complex of M. The holomorphic de Rham complex $\mathrm{DR}\,M$ is defined as the complex

$$\mathrm{DR}\,M = \{0 \to M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} M \to 0\},$$

with the standard grading, i.e., M is in degree 0 and $\Omega^1 \otimes_{\mathcal{O}} M$ in degree 1. The de Rham complex can be V-filtered, by setting

$$V^\beta \,\mathrm{DR}\,M = \{0 \to V^\beta M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} V^{\beta-1}M \to 0\},$$

for every $\beta \in \mathbb{R}$. As the morphism $\mathrm{gr}_V^\beta M \to \mathrm{gr}_V^{\beta-1} M$ induced by ∂_t is an isomorphism for every $\beta < 0$, it follows that the inclusion of complexes

$$V^0 \,\mathrm{DR}\,M \hookrightarrow \mathrm{DR}\,M \tag{7.2.28}$$

is a quasi-isomorphism. If M has a regular singularity, the terms of the left-hand complex have finite type as $\mathcal{O}$-modules.

If M comes equipped with a coherent filtration $F^\bullet M$, we set, in accordance with the future definition 8.4.1 (see also Remark 8.4.9),

$$F^p \,\mathrm{DR}\,M = \{0 \to F^p M \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} F^{p-1}M \to 0\}.$$

7.2.d. F-Filtered holonomic $\mathcal{D}_\Delta$-modules. We now sheafify the previous constructions and consider a $\mathcal{D}_\Delta$-module $\mathcal{M}$. We assume it is holonomic, that is, its germ at any point of the open disc $\Delta \subset \mathbb{C}$ centered at 0 is holonomic in the previous sense. Then the $\mathcal{D}_\Delta$-module $\mathcal{M}$ is an $\mathcal{O}_\Delta$-module and is equipped with a connection. Moreover, we always assume that the origin of Δ is the only singularity of $\mathcal{M}$ on Δ, that is, away from the origin $\mathcal{M}$ is locally $\mathcal{O}_{\Delta^*}$-free of finite rank.

All the notions of the previous subsection extend in a straightforward way to the present setting. In particular, for a holonomic $\mathcal{D}_\Delta$-module $\mathcal{M}$ having a regular singularity at the origin, Proposition 7.2.10 reads

$$\mathcal{M} \simeq \mathcal{O}_\Delta \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}.$$

There are filtered analogues of these notions. We only work with coherently F-filtered $\mathcal{D}_\Delta$-modules, that is, we assume that each $F^p\mathcal{M}$ is $\mathcal{O}_X$-coherent and that there exists p_o such that $F^{p_o-p}\mathcal{M} = F_p\mathcal{D}_X \cdot F^{p_o}\mathcal{M}$.

7.2.29. Definition **(Pure support).**

(1) We say that $\mathcal{M}$ as above has *pure support* the disc Δ if its germ M at the origin is a middle extension, as defined in 7.2.11.

(2) We say that $(\mathcal{M}, F^\bullet\mathcal{M})$ as above has *pure support* the disc Δ if its germ $(M, F^\bullet)$ at the origin is a filtered middle extension, as defined in 7.2.26.

Clearly, if $(\mathcal{M}, F^\bullet\mathcal{M})$ has pure support Δ, then so does the underlying $\mathcal{M}$, but the latter condition is not sufficient to ensure the former.

7.2.30. Remark. For the sheaf version, the conditions 7.2.19(a) and (b) are respectively equivalent to

(a) for $\beta > -1$ and any p, $F^pV^\beta\mathcal{M} = (j_*j^{-1}F^p\mathcal{M}) \cap V^\beta\mathcal{M}$,
(b) for $\beta \in [-1, 0)$, $k \geqslant 1$ and any p,

$$F^pV^{\beta-k}\mathcal{M} = \partial_t^k F^{p+k}V^\beta\mathcal{M} + \sum_{j=0}^{k-1} \partial_t^j F^{p+j}V^{-1}\mathcal{M}.$$

In particular, $F^p\mathcal{M} = \sum_{j\geqslant 0} \partial_t^j F^{p+j}V^{-1}\mathcal{M}$.

Moreover, if $(\mathcal{M}, F^\bullet\mathcal{M})$ is a filtered middle extension (Definition 7.2.26), 7.2.19(b) together with 7.2.26(c) are equivalent to

(c) for $\beta \in (-1, 0]$, $k \geqslant 1$ and any p,

$$F^pV^{\beta-k}\mathcal{M} = \partial_t^k F^{p+k}V^\beta\mathcal{M} + \sum_{j=0}^{k-1} \partial_t^j F^{p+j}V^{>-1}\mathcal{M}.$$

In particular, $F^p\mathcal{M} = \sum_{j\geqslant 0} \partial_t^j F^{p+j}V^{>-1}\mathcal{M}$.

As a consequence, *if* $(\mathcal{M}, F^\bullet\mathcal{M})$ *is a filtered middle extension,* $F^\bullet\mathcal{M}$ *is uniquely determined from* $j^{-1}F^\bullet\mathcal{M}$.

7.2.19(a) $\Leftrightarrow$ 7.2.30(a): The direction $\Leftarrow$ is clear. Let us prove $\Rightarrow$. Let m be a local section of $(j_*j^{-1}F^p\mathcal{M} \cap V^\beta\mathcal{M})$. Then m is a local section of $(F^qV^\beta\mathcal{M})$ for some $q > p$, and m induces a section of $(F^qV^\beta\mathcal{M})/(F^pV^\beta\mathcal{M})$ supported at the origin. Since the latter quotient is $\mathcal{O}_\Delta$-coherent, it follows that t^Nm is a local section of $F^pV^\beta\mathcal{M}$ for some N, hence also a local section of $(F^pV^\beta\mathcal{M}) \cap V^{\beta+N}\mathcal{M} = F^pV^{\beta+N}\mathcal{M} = t^NF^pV^\beta\mathcal{M}$, according to Property 7.2.19(a). Since t^N is injective on $V^\beta\mathcal{M}$, this implies that m is a local section of $F^pV^\beta\mathcal{M}$, hence the desired assertion.

7.2.19(b) $\Leftrightarrow$ 7.2.30(b): This is obvious by an easy induction on β.

By definition of $F^p\mathcal{V}_{\mathrm{mid}}$ (see (6.14.1)), we deduce from this remark and Proposition 6.14.2:

7.2.31. Corollary.

(1) *Assume that* $(\mathcal{M}, F^\bullet\mathcal{M})$ *is a filtered middle extension. With the identification* $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ *of Proposition* 7.2.14*, we have* $F^p\mathcal{M} = F^p\mathcal{V}_{\mathrm{mid}}$.

(2) *If* $(\mathcal{V}, F^\bullet\mathcal{V})$ *underlies a polarizable variation of Hodge structure, then the pair* $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ *is a filtered middle extension.* □

On the other hand, we say that $\mathcal{M}$ (resp. $(\mathcal{M}, F^\bullet\mathcal{M})$) has support the origin if any local section m of $M = \mathcal{M}_0$ (resp. F^pM for any p) is annihilated by some power of t. Here, the condition on $(\mathcal{M}, F^\bullet\mathcal{M})$ is equivalent to that on $\mathcal{M}$. Let us denote by $\iota : \{0\} \hookrightarrow \Delta$ the inclusion.

7.2.32. Proposition. *Let* $(\mathcal{M}, F^\bullet\mathcal{M})$ *be a coherently* F*-filtered* $\mathcal{D}_\Delta$*-module which is strictly* $\mathbb{R}$*-specializable. Then it has support the origin if and only if it takes the form* ${}_{\mathrm{D}}\iota_*(\mathcal{H}, F^\bullet\mathcal{H})$ *for some filtered finite dimensional* $\mathbb{C}$*-vector space* $(\mathcal{H}, F^\bullet\mathcal{H})$*. We then have* $(\mathcal{H}, F^\bullet\mathcal{H}) = \phi_{t,1}(\mathcal{M}, F^\bullet\mathcal{M})$.

Proof. Since $\mathcal{M}$ is supported at $\{0\}$, there exists a finite-dimensional vector space $\mathcal{H}$ (equal to $\mathrm{gr}_V^{-1}\mathcal{M}$) such that $\mathcal{M} = \iota_*\mathcal{H}[\partial_t]$ (Exercise 7.7). Considering the finite-dimensional $\mathbb{C}$-vector space $\mathcal{H}$ as a holonomic $\mathcal{D}$-module on a point, we regard $\mathcal{M}$ as the $\mathcal{D}$-module pushforward of $\mathcal{H}$ by the inclusion ι, a relation that we denote

$$\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H} := \iota_*\mathcal{H}[\partial_t].$$

For $k \geqslant 0$ we have $V^k\mathcal{M} = 0$ and $V^{-k-1}\mathcal{M} = \sum_{j\leqslant k} \iota_*\mathcal{H}\partial_t^j$, so that one recovers $\mathcal{H}$ from $\mathcal{M}$ as

$$\mathcal{H} = \phi_{t,1}\mathcal{M}.$$

Let now $(\mathcal{H}, F^\bullet\mathcal{H})$ be a filtered vector space. The F-filtration on $\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H}$ is defined by (see also Example 8.7.7(2))

$$F^p\mathcal{M} = F^p{}_{\mathrm{D}}\iota_*\mathcal{H} = \bigoplus_{j\geqslant 0} \iota_*(F[1]^{p+j}\mathcal{H}) \cdot \partial_t^j = \bigoplus_{k\geqslant 0} \iota_*(F^{p+j+1}\mathcal{H}) \cdot \partial_t^j. \tag{7.2.32 $*$}$$

This defines the pushforward ${}_{\mathrm{D}}\iota_*(\mathcal{H}, F^\bullet\mathcal{H})$ as a filtered holonomic $\mathcal{D}_X$-module supported at the origin. Note that it is strictly $\mathbb{R}$-specializable at the origin. We recover $F^\bullet\mathcal{H}$ from $F^\bullet\mathcal{M}$ by the formula

$$F^p\mathcal{H} = F^p\phi_{t,1}\mathcal{M},$$

due to the shift in the definition of $F^\bullet\mathcal{M}$ and the opposite shift in that of $F^\bullet\phi_{t,1}\mathcal{M}$ (see (7.2.16)).

The converse is left as an exercise (see Exercise 7.14). □

7.2.e. Pushforward of regular holonomic left $\mathcal{D}_X$-modules. The holomorphic de Rham complex $\mathrm{DR}\,\mathcal{M}$ is defined as the complex (degrees as above)

$$\mathrm{DR}\,\mathcal{M} = \{0 \to \mathcal{M} \xrightarrow{\nabla} \Omega^1_\Delta \otimes_{\mathcal{O}_\Delta} \mathcal{M} \to 0\},$$

and its filtered version is

$$F^p\,\mathrm{DR}\,\mathcal{M} = \{0 \to F^p\mathcal{M} \xrightarrow{\nabla} \Omega^1 \otimes_{\mathcal{O}} F^{p-1}\mathcal{M} \to 0\}.$$

Away from the origin, the de Rham complex has cohomology in degree 0 only, and $H^0\,\mathrm{DR}\,\mathcal{M}_{|\Delta^*} = \mathcal{V}^\nabla$ is a local system of finite dimensional $\mathbb{C}$-vector spaces on Δ^*. In general, $\mathrm{DR}\,\mathcal{M}$ is a constructible complex on Δ, that is, it is such a locally constant sheaf on Δ^* and its cohomology spaces at the origin are finite dimensional $\mathbb{C}$-vector spaces. The subcomplex $V^0\,\mathrm{DR}\,\mathcal{M}$ is quasi-isomorphic to $\mathrm{DR}\,\mathcal{M}$ and, if $\mathcal{M}$ has a regular singularity at the origin, $V^0\,\mathrm{DR}\,\mathcal{M}$ is a complex whose terms are $\mathcal{O}_\Delta$-coherent (in fact $V^0\mathcal{M}$ is $\mathcal{O}_\Delta$ free).

If $\mathcal{M}$ has pure support the disc Δ, the de Rham complex $\mathrm{DR}\,\mathcal{M}$ has cohomology in degree 0 only, and $H^0\,\mathrm{DR}\,\mathcal{M} = j_*\mathcal{V}^\nabla$, with $j : \Delta^* \hookrightarrow \Delta$. In such a case, both terms of $V^0\,\mathrm{DR}\,\mathcal{M}$ are $\mathcal{O}_\Delta$-free. On the other hand, if $\mathcal{M}$ is supported at the origin, then $\mathrm{DR}\,\mathcal{M} \simeq V^0\,\mathrm{DR}\,\mathcal{M}$ reduces to the complex with the single term $V^{-1}\mathcal{M} = \mathrm{gr}_V^{-1}\mathcal{M}$ in degree 1.

We now consider the global setting of a compact Riemann surface and a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with singularities at a finite set $D \subset X$. The pushforward (in the sense of left $\mathcal{D}_X$-modules) of $\mathcal{M}$ by the constant map $a_X : X \to \mathrm{pt}$ is the complex

$$\boldsymbol{R}\Gamma(X, \mathrm{DR}\,\mathcal{M}),$$

that we regard as a complex of $\mathcal{D}$-modules on a point, that is, a complex of $\mathbb{C}$-vector spaces. It follows that $\boldsymbol{R}\Gamma(X, \mathrm{DR}\,\mathcal{M})$ has cohomology in degrees $0, 1, 2$.

For a regular holonomic $\mathcal{D}_X$-module $\mathcal{M}$, it is immediate to check that the hypercohomology space $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})$ is finite dimensional for every k. Indeed, denote by $V^\beta\mathcal{M}$ the subsheaf of $\mathcal{M}$ which coincides with $V^\beta(\mathcal{M}_{|\Delta})$ on each disc Δ near a singularity and is equal to $\mathcal{M}$ away from the singularities. Then $V^\beta\mathcal{M}$ is $\mathcal{O}_X$-coherent and (7.2.28) gives $V^0(\mathrm{DR}\,\mathcal{M}) \simeq \mathrm{DR}\,\mathcal{M}$, so $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = \boldsymbol{H}^k(X, V^0\,\mathrm{DR}\,\mathcal{M})$ is finite dimensional since each term of the complex $V^0\,\mathrm{DR}\,\mathcal{M}$ is $\mathcal{O}_X$-coherent and X is compact.

If $(\mathcal{M}, F^\bullet\mathcal{M})$ is a coherently F-filtered $\mathcal{D}$-module, then $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})$ is filtered by the formula

$$F^p\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) := \mathrm{image}[\boldsymbol{H}^k(X, F^p\,\mathrm{DR}\,\mathcal{M}) \longrightarrow \boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M})].$$

7.2.33. Examples.

(1) Assume that $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ and set $\underline{\mathcal{H}} = \mathcal{V}^\nabla$. Then $\mathrm{DR}\,\mathcal{M} = j_*\underline{\mathcal{H}}$ and $\boldsymbol{H}^k(X, \mathrm{DR}\,\mathcal{M}) = H^k(X, j_*\underline{\mathcal{H}})$. As explained in Remark 6.14.16, the only interesting cohomology is $\boldsymbol{H}^1(X, \mathrm{DR}\,\mathcal{M}) = H^1(X, j_*\underline{\mathcal{H}})$.

(2) Assume $\mathcal{M}$ is supported at one point in X, and let Δ be a small disc centered at that point, with coordinate t. We can then assume that $X = \Delta$. We denote by $\iota : \{0\} \hookrightarrow \Delta$ the inclusion. Then $V^0(\mathrm{DR}\,\mathcal{M})$ is the complex having the skyscraper sheaf with stalk $\mathcal{H}$ at the origin as its term in degree 1, and all other terms of the complex are zero. We can thus write

$$\mathrm{DR}\,\mathcal{M} = \iota_*\mathcal{H}[-1],$$

and we find

$$H^k(X, \mathrm{DR}\,\mathcal{M}) = \begin{cases} \mathcal{H} & \text{if } k = 1, \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand, for the same reason of shift in the definition, we obtain

$$F^p\,\mathrm{DR}\,\mathcal{M} = \iota_*F^p\mathcal{H}[-1],$$

so that, if we recover $\mathcal{H}$ from $\mathcal{M}$ as $H^1(X, \mathrm{DR}\,\mathcal{M})$, we also recover $F^\bullet\mathcal{H}$ by the formula

$$F^p\mathcal{H} = F^pH^1(X, \mathrm{DR}\,\mathcal{M}).$$

7.2.34. Caveat. In order to treat on the same footing $\mathcal{D}$-modules with pure support in dimension zero and one, we replace the de Rham functor DR by its shifted version ${}^{\mathrm{p}}\mathrm{DR} = \mathrm{DR}[1]$. This shift does not affect the filtrations, in the sense that, for a filtered $\mathcal{D}$-module $(\mathcal{M}, F^\bullet\mathcal{M})$, we set

$$F^p\,{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}) = (F^p\,\mathrm{DR}\,\mathcal{M})[1].$$

As a consequence, the notion of weight has to be shifted for variations of Hodge structure on Δ^*.

7.3. Sesquilinear pairings between $\mathcal{D}$-modules on a Riemann surface

We have seen in Section 4.1 that the notion of a sesquilinear pairing is instrumental in order to define the polarization of a variation of $\mathbb{C}$-Hodge structure and even, taking the approach of triples (Section 5.2), in defining the notion of variation of $\mathbb{C}$-Hodge structure. It takes values in the space of C^∞ functions. In order to extend this notion to that of pairing on $\mathcal{D}$-modules, we need to extend the target space, as suggested by the formula in Lemma 6.8.2. When working with left $\mathcal{D}$-modules, the target space for sesquilinear pairings will be the spaces of distributions on the Riemann surface X. A general presentation of sesquilinear pairing will be given in Chapter 12. We also refer to Section 8.3.4 for general properties of distributions and currents.

7.3.a. Basic distributions. Let us start by noticing that the C^∞ functions on Δ^* (punctured unit disc) considered in Lemma 6.8.2, and that we denote by

$$u_{\beta,p} := |t|^{2\beta}\,\frac{\mathrm{L}(t)^p}{p!},\quad \beta > -1,\, p\in\mathbb{N},\quad (\mathrm{L}(t) = -\log|t|^2),$$

define distributions on Δ by the formula

$$\langle \eta, u_{\beta,p}\rangle = \int_\Delta u_{\beta,p}\eta,$$

for any C^∞ $(1,1)$-form η with compact support on Δ. In fact, a direct computation in polar coordinates shows that $u_{\beta,p}$ is a locally integrable function on Δ. These distributions are related by the formula

$$-(t\partial_t-\beta)u_{\beta,p} = -(\bar t\partial_{\bar t}-\beta)u_{\beta,p} = u_{\beta,p-1}, \tag{7.3.1}$$

as can be seen by using integration by parts ($u_{\beta,-1}:=0$).

7.3.2. Proposition. *Suppose that a distribution $u\in\mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t-\beta')^k u = (\bar t\partial_{\bar t}-\beta'')^k u = 0$$

for real numbers $\beta',\beta''>-1$ and an integer $k\geqslant 0$. Then

(a) *$u=0$ unless $\beta'-\beta''\in\mathbb{Z}$,*

(b) *if $\beta'=\beta''=\beta$, u is a linear combination of the distributions $u_{\beta,p}$ with $p\in[0,k-1]$.*

Proof. Let us first show that if $\operatorname{Supp}u\subseteq\{0\}$, then $u=0$. By continuity, u is annihilated by some large power of t; let $m\in\mathbb{N}$ be the least integer such that $t^m u=0$. If $m\geqslant 1$, we have

$$\begin{aligned}0 = t^{m-1}(t\partial_t-\beta')^k u &= (t\partial_t-\beta'-(m-1))^k t^{m-1}u\\ &= (\partial_t t-\beta'-m)^k t^{m-1}u = (-1)^k(\beta'+m)^k t^{m-1}u,\end{aligned}$$

hence $t^{m-1}u=0$, due to the fact that $\beta'>-1$. The conclusion is that $m=0$, and hence that $u=0$.

Now let us prove the general case. We recall (see Section 12.2.c for details) that the restriction $\mathfrak{Db}(\Delta)\to\mathfrak{Db}(\Delta^*)$ has kernel consisting of distributions supported at the origin. The preliminary result implies that it is enough to prove the proposition for distributions on Δ^*. The pullback by the exponential mapping

$$\mathbb{H} := \{\operatorname{Re}\tau < 0\} \xrightarrow{\exp} \Delta^*, \quad \tau \longmapsto \mathrm{e}^{\tau}$$

of such a distribution is then well-defined: for a test $(1,1)$-form η on $\mathbb{H}$, that we write $a(\tau)\mathrm{d}\tau \wedge \mathrm{d}\overline{\tau}$ with $a \in C^\infty_{\mathrm{c}}(\mathbb{H})$, the *trace* $\operatorname{tr}\eta$ is the $(1,1)$-form on Δ^* defined as

$$\operatorname{tr}\eta = (\operatorname{tr}a)(t)\,\frac{\mathrm{d}t}{t} \wedge \frac{\mathrm{d}\overline{t}}{\overline{t}}, \qquad \text{with } (\operatorname{tr}a)(t) := \sum_{\tau \mapsto t} a(\tau).$$

Since the exponential mapping is a covering and a has compact support, the sum above is finite and $\operatorname{tr}a \in C^\infty_{\mathrm{c}}(\Delta^*)$ satisfies $t\partial_t \operatorname{tr}a = \operatorname{tr}(\partial_\tau a)$ and a conjugate analogue. We can thus define a distribution $\widetilde{u} := \exp^* u$ on $\mathbb{H}$ by

$$\langle \eta, \widetilde{u}\rangle = \langle \operatorname{tr}\eta, u\rangle,$$

with the property that

$$(\partial_\tau - \beta')^k \widetilde{u} = (\partial_{\overline{\tau}} - \beta'')^k \widetilde{u} = 0.$$

The equations imply that the product

$$v = e^{-\beta'\tau} e^{-\beta''\overline{\tau}} \cdot \widetilde{u}$$

is annihilated by the k-th power of ∂_τ and $\partial_{\overline{\tau}}$, and in particular by the k-th power $(\partial_\tau\partial_{\overline{\tau}})^k$ of the Laplacian. By the regularity of the Laplacian, v is C^∞, and the above equations imply that v is a polynomial $P(\tau, \overline{\tau})$ of degree $\leqslant k$. Consequently,

$$\widetilde{u} = P(\tau, \overline{\tau}) \cdot e^{\beta'\tau} e^{\beta''\overline{\tau}}.$$

By construction, $\widetilde{u}$ is invariant under the translation $\tau \mapsto \tau + 2\pi\mathrm{i}$; if $\widetilde{u} \neq 0$, this forces $P(\tau, \overline{\tau})$ to be a polynomial in $\tau + \overline{\tau}$ and $\beta' - \beta'' \in \mathbb{Z}$.

Now there are two cases. If $\beta' - \beta'' \notin \mathbb{Z}$, then $\widetilde{u} = 0$, hence $u = 0$ in $\mathfrak{Db}(\Delta^*)$, as wanted. If $\beta' = \beta'' = \beta$, then u is a linear combination of the C^∞ functions $u_{\beta,p|\Delta^*}$ with $0 \leqslant p \leqslant k-1$. $\square$

To include the case $\beta' = \beta'' = -1$ into the picture, we need the following simple facts about distributions. Since we do not consider currents in this chapter, we consider the Dirac distribution δ_0 as defined by

$$\langle \eta(t)\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t}), \delta_0\rangle = \eta(0),$$

which thus depends on the choice of the coordinate t through the identification $\mathcal{E}^{1,1}_\Delta = \mathcal{C}^\infty_\Delta \cdot \mathrm{d}t \wedge \mathrm{d}\overline{t}$. Since the form $\frac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t})$ is real, the distribution δ_0 is *real*, in the sense that, defining its conjugate $\overline{\delta}_0$ by

$$\langle \eta\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t}), \overline{\delta}_0\rangle := \overline{\langle \overline{\eta\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t})}, \delta_0\rangle},$$

we have $\overline{\delta}_0 = \delta_0$.

Cauchy's formula reads (see Exercise 7.19)

$$\partial_t \partial_{\overline{t}} \mathrm{L}(t) = -\delta_0.$$

For the sake of simplicity, we will set for $p \geqslant 0$

$$u_{-1,p} := \partial_t \partial_{\bar t} u_{0,p+1} = \partial_t \partial_{\bar t} (\mathrm{L}(t)^{p+1})/(p+1)!.$$

In particular, $u_{-1,0} = -\delta_o$. Note that the basic relations (7.3.1) also hold for $u_{-1,p}$, that is,

$$-(t\partial_t + 1)u_{-1,p} = -(\bar t \partial_{\bar t} + 1)u_{-1,p} = u_{-1,p-1} \qquad (u_{-1,-1} := 0).$$

7.3.3. Proposition. *Suppose that a distribution $u \in \mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t + 1)^k u = (\bar t \partial_{\bar t} + 1)^k u = 0$$

for some $k \geqslant 1$. Then u is a linear combination of $u_{-1,p}$ with $0 \leqslant p \leqslant k-1$.

Proof. Using the relation $t(t\partial_t + 1) = t\partial_t t$, we find $(t\partial_t)^k |t|^2 u = (\bar t \partial_{\bar t})^k |t|^2 u = 0$, and by Proposition 7.3.2 we deduce

$$|t|^2 u = \sum_{p=0}^{k-1} c_{p+2} u_{0,p} = |t|^2 \partial_t \partial_{\bar t} \sum_{q=2}^{k+1} c_q u_{0,q},$$

according to the basic relations (7.3.1). On the other hand, distributions solutions of $|t|^2 v = 0$ are $\mathbb{C}$-linear combinations of δ_0, $\partial_t^j \delta_0$, $\partial_{\bar t}^j \delta_0$ $(j \geqslant 1)$. As a consequence, and using Cauchy's formula above, we find an expression

$$u = \partial_t \partial_{\bar t} \sum_{q=1}^{k+1} c_q u_{0,q} + \sum_{j \geqslant 1} (a_j \partial_t^j \delta_0 + b_j \partial_{\bar t}^j \delta_0),$$

and we are left with showing $c_{k+1} = a_j = b_j = 0$ for all $j \geqslant 1$. For that purpose, we note that, for $p = 1, \dots, k+1$,

$$(\partial_t t)^k \partial_t \partial_{\bar t} u_{0,p} = \partial_t \partial_{\bar t} (t\partial_t)^k u_{0,p} = (-1)^k \partial_t \partial_{\bar t} u_{0,p-k} = \begin{cases} 0 & \text{if } p \leqslant k, \\ (-1)^{k+1}\delta_0 & \text{if } p = k+1. \end{cases}$$

On the other hand, since $k \geqslant 1$, we have $(\partial_t t)^k \partial_{\bar t}^j \delta_0 = \partial_{\bar t}^j (\partial_t t)^k \delta_0 = 0$ and thus

$$\begin{aligned} (\partial_t t)^k \sum_{j \geqslant 1} (a_j \partial_t^j \delta_0 + b_j \partial_{\bar t}^j \delta_0) &= \sum_{j \geqslant 1} a_j \delta_0 (\partial_t t)^k \partial_t^j \delta_0 \\ &= \sum_{j \geqslant 1} a_j \partial_t^j (\partial_t t - j)^k \delta_0 = \sum_{j \geqslant 1} (-j)^k a_j \partial_t^j \delta_0, \end{aligned}$$

and similarly

$$(\partial_{\bar t} \bar t)^k \sum_{j \geqslant 1} (a_j \partial_t^j \delta_0 + b_j \partial_{\bar t}^j \delta_0) = \sum_{j \geqslant 1} (-j)^k b_j \partial_{\bar t}^j \delta_0,$$

so the equations satisfied by u imply

$$-c_{k+1}\delta_0 + \sum_{j \geqslant 1} j^k a_j \partial_t^j \delta_0 = 0 \quad \text{and} \quad -c_{k+1}\delta_0 + \sum_{j \geqslant 1} j^k b_j \partial_{\bar t}^j \delta_0 = 0,$$

hence $c_{k+1} = a_j = b_j = 0$, as was to be proved. $\square$

In the same vein, we solve the mixed case:

7.3.4. Proposition. *Suppose that a distribution $u \in \mathfrak{Db}(\Delta)$ solves the equations*

$$(t\partial_t)^k u = (\bar{t}\partial_{\bar{t}} + 1)^k u = 0$$

for some $k \geqslant 1$. Then u is a linear combination of $\partial_{\bar{t}} u_{0,p}$ with $1 \leqslant p \leqslant k$.

Proof. We notice that $\partial_t u$ solves the equations in Proposition 7.3.3, so we can write

$$\partial_t u = \sum_{p=0}^{k-1} c_{p+1} u_{-1,p} = \partial_t \sum_{q=1}^{k} c_q \partial_{\bar{t}} \mathrm{L}(t)^q/q!,$$

and thus $u = h(\bar{t}) + \sum_{q=1}^{k} c_q \partial_{\bar{t}} \mathrm{L}(t)^q/q!$ for some anti-holomorphic function $h(\bar{t})$. One checks that

$$(\bar{t}\partial_{\bar{t}} + 1)^k \partial_{\bar{t}} \mathrm{L}(t)^q/q! = \partial_{\bar{t}} (\bar{t}\partial_{\bar{t}})^k \mathrm{L}(t)^q/q! = 0 \quad \text{if } q \leqslant k,$$

so $h(\bar{t})$ must satisfy $(\bar{t}\partial_{\bar{t}} + 1)^k h(\bar{t}) = 0$, which implies $h = 0$. □

7.3.b. Sesquilinear pairings. Let $\mathcal{M}', \mathcal{M}''$ be regular holonomic $\mathcal{D}_\Delta$-modules, each of which written as $\mathcal{M} \simeq \mathcal{O}_\Delta \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}$ (see Section 7.2.d). We will consider the conjugate module $\overline{\mathcal{M}''}$: this is $\mathcal{M}''$ as a sheaf of $\mathbb{R}$-vector spaces, equipped with the structure of a module over the sheaf $\overline{\mathcal{D}}_\Delta$ of anti-holomorphic differential operators as follows. Any anti-holomorphic function $b_j(\bar{t})$ can be written as the conjugate $\overline{a(t)}$ of a holomorphic function $a(t)$, and any anti-holomorphic differential operator $\sum_j b_j(\bar{t})\partial_{\bar{t}}^j$, where b_j are anti-holomorphic functions, can be written as the conjugate $\overline{P(t,\partial_t)}$ of a holomorphic differential operator $P(t,\partial_t) = \sum_j a_j(t)\partial_t$. When regarded as a section of $\overline{\mathcal{M}''}$, we write a section m'' of the sheaf $\mathcal{M}''$ as $\overline{m''}$, and the action of $\overline{\mathcal{D}}_\Delta$ is defined by

$$\overline{P(t,\partial_t)} \cdot \overline{m''} := \overline{P(t,\partial_t)m''}.$$

A *sesquilinear pairing* $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_\Delta$ is, by definition (see also Definition 5.4.1), a $\mathbb{C}$-linear pairing which satisfies, for any local sections m', m'' of $\mathcal{M}', \mathcal{M}''$,

$$\begin{aligned} P(t,\partial_t)\mathfrak{s}(m',\overline{m''}) &= \mathfrak{s}(P(t,\partial_t)m',\overline{m''}),\\ \overline{P(t,\partial_t)}\mathfrak{s}(m',\overline{m''}) &= \mathfrak{s}(m',\overline{P(t,\partial_t)m''}). \end{aligned} \tag{7.3.5}$$

Propositions 7.3.2 and 7.3.3 immediately imply:

7.3.6. Proposition. *Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$.*

(1) *The induced pairing $\mathfrak{s} : M'^{\beta'} \otimes \overline{M''^{\beta''}} \to \mathfrak{Db}(\Delta)$ vanishes if $\beta' - \beta'' \notin \mathbb{Z}$.*

(2) *For $\beta \geqslant -1$, $m' \in M'^{\beta}$ and $m'' \in M''^{\beta}$, the induced pairing $\mathfrak{s}^{(\beta)}(m',\overline{m''})$ is a $\mathbb{C}$-linear combination of the basic distributions $u_{\beta,p}$ ($p \geqslant 0$).* □

As a consequence, the pairing $\mathfrak{s}^{(\beta)}$, which is a sesquilinear pairing between the finite-dimensional $\mathbb{C}$-vector spaces M'^{β} and M''^{β} with values in $\mathfrak{Db}(\Delta)$, has a unique expansion $\sum_{p\geqslant 0} \mathfrak{s}_p^{(\beta)} u_{\beta,p}$, where $\mathfrak{s}_p^{(\beta)}$ ($\beta \geqslant -1$) is a sesquilinear pairing $M'^{\beta} \otimes \overline{M''^{\beta}} \to \mathbb{C}$.

Using the relations in (7.3.1) and (7.3.5), we get (recall that $\mathrm{E} = t\partial_t$)

$$\begin{aligned}\sum_{p\geqslant 0}\mathfrak{s}_p^{(\beta)}(-(\mathrm{E}-\beta)m',\overline{m''})u_{\beta,p} = \mathfrak{s}(-(\mathrm{E}-\beta)m',\overline{m''}) &= -(t\partial_t-\beta)\mathfrak{s}(m',\overline{m''})\\ &= \sum_{p\geqslant 0}\mathfrak{s}_{p+1}^{(\beta)}(m',\overline{m''})u_{\beta,p},\end{aligned}$$

and therefore $\mathfrak{s}_{p+1}^{(\beta)}(m',\overline{m''}) = \mathfrak{s}_p^{(\beta)}(-(\mathrm{E}-\beta)m',\overline{m''})$. So, if we denote by N' or N'' the nilpotent operator $-(\mathrm{E}-\beta)$, we have

$$\mathfrak{s}^{(\beta)}(m',\overline{m''}) = \sum_{p\geqslant 0}\mathfrak{s}_0^{(\beta)}(\mathrm{N}'^p m',\overline{m''})u_{\beta,p} = \sum_{p\geqslant 0}\mathfrak{s}_0^{(\beta)}(m',\overline{\mathrm{N}''^p m''})u_{\beta,p}$$

(the latter equality is a consequence of (7.3.1)).

7.3.7. Corollary.

(1) *For $\beta\geqslant -1$, the pairing $\mathfrak{s}_0^{(\beta)} : M'^\beta\otimes\overline{M''^\beta}\to\mathbb{C}$ satisfies the equality*

$$(\mathfrak{s}^*)_0^{(\beta)} = (\mathfrak{s}_0^{(\beta)})^*.$$

(2) *For $\beta\geqslant -1$, the pairing $\mathfrak{s}_0^{(\beta)} : M'^\beta\otimes\overline{M''^\beta}\to\mathbb{C}$ satisfies the relation*

$$\mathfrak{s}_0^{(\beta)}\circ(\mathrm{N}'\otimes\overline{\mathrm{Id}}) = \mathfrak{s}_0^{(\beta)}\circ(\mathrm{Id}\otimes\overline{\mathrm{N}''}). \tag{7.3.7 $*$}$$

(3) *The pairings $\mathfrak{s}_0^{(0)},\mathfrak{s}_0^{(-1)}$ satisfy the relations*

$$\mathfrak{s}_0^{(-1)}\circ(\mathrm{can}\otimes\overline{\mathrm{Id}}) = \mathfrak{s}_0^{(0)}\circ(\mathrm{Id}\otimes\overline{\mathrm{var}}),\quad \mathfrak{s}^{(-1)}\circ(\mathrm{Id}\otimes\overline{\mathrm{can}}) = \mathfrak{s}_0^{(0)}\circ(\mathrm{var}\otimes\overline{\mathrm{Id}}). \tag{7.3.7 $**$}$$

Proof. The first point is a consequence from the fact that the basic distributions are real. The second point has already been noticed. Let us prove for example the first equality in (7.3.7 $**$). Assume $m'\in M'^0$ and $m''\in M''^{-1}$. Then $\mathfrak{s}(m',\overline{m''})$ satisfies the assumption of Proposition 7.3.4, hence $\mathfrak{s}(m',\overline{m''}) = \sum_{p=0}^{k-1}c_p\partial_{\bar t}u_{0,p+1}$. Therefore,

$$\mathfrak{s}(\mathrm{can}\,m',\overline{m''}) = -\partial_t\sum_{p=0}^{k-1}c_p\partial_{\bar t}u_{0,p+1} = -\sum_{p=0}^{k-1}c_p u_{-1,p}.$$

On the other hand,

$$\mathfrak{s}(m',\overline{\mathrm{var}\,m''}) = \bar t\sum_{p=0}^{k-1}c_p\partial_{\bar t}u_{0,p+1} = -\sum_{p=0}^{k-1}c_p u_{0,p}.$$

Therefore, $-c_0 = \mathfrak{s}_0^{(-1)}(\mathrm{can}\,m',\overline{m''}) = \mathfrak{s}_0^{(0)}(m',\overline{\mathrm{var}\,m''})$. $\square$

Using the power series expansion of the exponential function, we may write the above formula for $\mathfrak{s}^{(\beta)}$ in a purely symbolic way as ($m'\in M'^\beta$, $m''\in M''^\beta$)

$$\mathfrak{s}^{(\beta)}(m',\overline{m''}) = \begin{cases}\mathfrak{s}_0^{(\beta)}(|t|^{2(\beta\,\mathrm{Id}-\mathrm{N})}m',\overline{m''}) & \text{if } \beta>-1,\\ \partial_t\partial_{\bar t}\mathfrak{s}_0^{(-1)}\Big(\dfrac{|t|^{-2\mathrm{N}}-1}{\mathrm{N}}m',\overline{m''}\Big) & \text{if } \beta=-1.\end{cases} \tag{7.3.8}$$

7.3.9. Example. We make more explicit the possible sesquilinear pairings when M' and M'' are either middle extensions or supported at the origin.

(1) The "mixed case", where for example $\mathcal{M}'$ is a middle extension and $\mathcal{M}''$ is supported at the origin, is easily treated: in such a case, we have $\mathfrak{s} = 0$ (see Lemma 12.3.10 for a similar statement in higher dimension). The assumption implies that $M''^{\beta} = 0$ for $\beta \neq -1, -2, \dots$, and on the other hand, $t : M'^{k-1} \to M'^{k}$ is bijective except if $k = 0$, in which case it is only injective, and $\partial_t : M'^{k} \to M'^{k-1}$ is bijective except if $k = 0$, where it is only onto. If $k \neq 0$, we have

$$\mathfrak{s}(M'^{k}, \overline{M''^{-1}}) = \mathfrak{s}(tM'^{k-1}, \overline{M''^{-1}}) = \mathfrak{s}(M'^{k-1}, \overline{tM''^{-1}}) = 0.$$

Therefore, we also have

$$\mathfrak{s}(M'^{0}, \overline{M''^{-1}}) = \mathfrak{s}(\partial_t M'^{1}, \overline{M''^{-1}}) = \partial_t \mathfrak{s}(M'^{1}, \overline{M''^{-1}}) = 0.$$

Last, for $\ell \geqslant 0$,

$$\mathfrak{s}(M'^{k}, \overline{M''^{-1-\ell}}) = \mathfrak{s}(M'^{k}, \overline{\partial_t^{\ell} M''^{-1}}) = \partial_{\bar t}^{\ell} \mathfrak{s}(M'^{k}, \overline{M''^{-1}}) = 0.$$

(2) If $\mathcal{M}', \mathcal{M}''$ are supported at the origin, then $\mathfrak{s}$ is determined by $\mathfrak{s}^{(-1)}$ and, for $m' \in M'^{-1}, m'' \in M''^{-1}$

$$\mathfrak{s}^{(-1)}(m', \overline{m''}) = \mathfrak{s}_0^{(-1)}(m', \overline{m''}) u_{-1,0} = -\mathfrak{s}_0^{(-1)}(m', \overline{m''}) \delta_o,$$

where $\mathfrak{s}^{(-1)}$ can be any complex-valued sesquilinear pairing between M'^{-1} and M''^{-1}.

(3) If $\mathcal{M}', \mathcal{M}''$ are middle extensions, then $\mathfrak{s}$ is uniquely determined by its restriction $\mathfrak{s}^{(\beta)}$ to $M'^{\beta} \otimes_{\mathbb{C}} \overline{M''^{\beta}}$ for $\beta \in (-1, 0]$, hence by the $\mathbb{C}$-valued sesquilinear pairings $\mathfrak{s}_0^{(\beta)}$ for $\beta \in (-1, 0]$, according to (7.3.8).

Indeed, let us first assume that $\beta \in (-1, 0)$. If $k \geqslant 0$ we have $M'^{\beta+k} = t^k M'^{\beta}$ and $M'^{\beta-k} = \partial_t^k M'^{\beta}$ and similar equalities for M''. By $\mathcal{D} \otimes \overline{\mathcal{D}}$-linearity, the restriction of $\mathfrak{s}$ to $M'^{\beta+k} \otimes \overline{M''^{\beta+\ell}}$ $(k, \ell \in \mathbb{Z})$ is then uniquely determined by $\mathfrak{s}^{(\beta)}$.

If $\beta = 0$, we can argue similarly for the restriction of $\mathfrak{s}$ to $M'^{k} \otimes \overline{M''^{\ell}}$, according to the middle extension property.

7.3.c. Sesquilinear pairing on nearby cycles.

We have seen in Exercise 6.13(3) a way to define the sesquilinear pairing $\mathrm{gr}_V^{\beta} \mathfrak{s}$ by means of a residue formula, if $\beta > -1$. Notice that, for such a β, the distribution $\mathfrak{s}^{(\beta)}$ is L^1_{loc}, and it follows that the restriction of $\mathfrak{s}$ to $V^{\beta}\mathcal{M}' \otimes \overline{V^{\beta}\mathcal{M}''}$ takes values in $L^1_{\mathrm{loc}}(\Delta)$. We can conclude:

7.3.10. Lemma. *For every $\beta > -1$, the sesquilinear pairing on $V^{\beta}\mathcal{M}' \otimes \overline{V^{\beta}\mathcal{M}''}$ defined by the formula*

$$(m', \overline{m''}) \longmapsto \mathrm{Res}_{s=-\beta-1} \int_{\Delta} |t|^{2s} \mathfrak{s}(m', \overline{m''})\, \chi(t)\, \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}t \wedge \mathrm{d}\bar t$$

(for some, or any, cut-off function $\chi \in C_c^{\infty}(\Delta)$) induces a well-defined sesquilinear pairing

$$\mathrm{gr}_V^{\beta} \mathfrak{s} : \mathrm{gr}_V^{\beta} \mathcal{M}' \otimes \overline{\mathrm{gr}_V^{\beta} \mathcal{M}''} \longrightarrow \mathbb{C}$$

which coincides with $\mathfrak{s}_0^{(\beta)}$ via the identification $M^{\beta} \simeq \mathrm{gr}_V^{\beta} \mathcal{M}$ ($\mathcal{M} = \mathcal{M}', \mathcal{M}''$) of Proposition 7.2.10 and satisfies (see (7.3.7 $*$))

$$\mathrm{gr}_V^{\beta} \mathfrak{s}(\mathrm{N}'\bullet, \bar\bullet) = \mathrm{gr}_V^{\beta} \mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet}). \qquad \square$$

7.3.11. Remark. For $m' \in M'^{\beta}$ and $m'' \in M''^{\beta}$, we recover the equality $\mathrm{gr}_V^{\beta}\mathfrak{s}(m', \overline{m''}) = \mathfrak{s}_0^{(\beta)}(m', \overline{m''})$ (by using the identification $M^{\beta} = \mathrm{gr}_V^{\beta}\mathcal{M}$) as already checked in Exercise 6.13(3), by means of the formula above for $\mathfrak{s}^{(\beta)}$. Indeed,

$$\begin{aligned}\mathrm{Res}_{s=-\beta-1}\int_{\Delta}|t|^{2s}\mathfrak{s}^{(\beta)}(m',\overline{m''})\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t \\ &= \mathrm{Res}_{\sigma=0}\int_{\Delta}\mathfrak{s}^{(\beta)}(|t|^{2(\sigma-1-\mathrm{N})}m',\overline{m''})\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t\\ &= \mathfrak{s}_0^{(\beta)}\Big(\mathrm{Res}_{\sigma=0}\Big(\int_{\Delta}|t|^{2(\sigma-1-\mathrm{N})}\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t\Big)\,m',\overline{m''}\Big),\end{aligned}$$

and from Example 6.8.6 and Exercise 6.13(1) we have

$$\mathrm{Res}_{\sigma=0}\int_{\Delta}|t|^{2(\sigma-1-\mathrm{N})}\,\chi(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t = 1.$$

7.3.12. Definition **(Sesquilinear pairing on nearby cycles).** Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$. For $\lambda = \exp -2\pi\mathrm{i}\,\beta$ with $\beta \in (-1, 0]$, we set

$$\psi_{t,\lambda}\mathfrak{s} = \mathrm{gr}_V^{\beta}\mathfrak{s} : \psi_{t,\lambda}\mathcal{M}' \otimes \overline{\psi_{t,\lambda}\mathcal{M}''} \longrightarrow \mathbb{C},$$

which satisfies $\psi_{t,\lambda}(\mathfrak{s}^*) = (\psi_{t,\lambda}\mathfrak{s})^*$ and

$$\psi_{t,\lambda}\mathfrak{s}(\mathrm{N}'\bullet, \bar\bullet) = \psi_{t,\lambda}\mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet}).$$

7.3.d. Sesquilinear pairing on vanishing cycles. We note that, if $\beta = -1$, the residue formula of Lemma 7.3.10 is identically zero, since $|t|^{2s}\mathfrak{s}(m', m'') = 0$ for $\mathrm{Re}(s) \gg 0$, and this lemma cannot be used for defining $\phi_{t,1}\mathfrak{s}$. On the other hand, if a distribution u is a $\mathbb{C}$-linear combination of distributions $u_{\beta,p}$ ($\beta \geqslant -1$, $p \geqslant 0$), one can recover the coefficient of $u_{-1,0}$ by a residue formula applied to the Fourier transform of u. This justifies the considerations below.

Let $\widehat{\chi}(\theta)$ be a C^{∞} function of the complex variable $\theta \in \mathbb{C}$ such that $\widehat{\chi}$ is a cut-off function near $\theta = 0$. For s such that $\mathrm{Re}\, s > 0$, we consider the function

$$I_{\widehat{\chi}}(t, s) := \int_{\mathbb{C}} e^{\bar t/\bar\theta - t/\theta}\,|\theta|^{2(s-1)}\,\widehat{\chi}(\theta)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\theta\wedge\mathrm{d}\bar\theta,$$

and we define $I_{\widehat{\chi},k,\ell}$ by replacing $|\theta|^{2(s-1)}$ with $\theta^k\bar\theta^{\ell}\,|\theta|^{2(s-1)}$ in the integral defining $I_{\widehat{\chi}}$; in particular, we have $I_{\widehat{\chi}} = I_{\widehat{\chi},0,0}$ and $I_{\widehat{\chi},k,k}(t, s) = I_{\widehat{\chi}}(t, s+k)$ for any $k \in \mathbb{Z}$. We refer to Exercise 7.21 for the properties of these functions that we will use.

7.3.13. Remark. We can also use the coordinate $\tau = 1/\theta$ to write $I_{\widehat{\chi}}(t, s)$ as

$$I_{\widehat{\chi}}(t, s) = \int e^{\overline{t\tau} - t\tau}\,|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\bar\tau$$

where now $\widehat{\chi}$ is a cut-off function near $\tau = \infty$. $I_{\widehat{\chi}}(t, s)$ is the Fourier transform of $|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)$ (see Exercise 7.20): put $\tau = (\xi + i\eta)/\sqrt{2}$ and $t = (x + iy)/\sqrt{2}$; then

$$I_{\widehat{\chi}}(t, s) = \frac{1}{2\pi}\int e^{-i(\xi y+\eta x)}\,|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)\,\mathrm{d}\xi\wedge\mathrm{d}\eta.$$

By applying the properties of the functions $I_{\widehat{\chi},k,k}$ obtained in Exercise 7.21 and by arguing as in Exercise 6.13(3), we obtain that, for any test function χ on Δ (we will use a cut-off function near 0), the function

$$s \longmapsto \big\langle I_{\widehat{\chi}}(t,s)\chi(t)\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t}), \mathfrak{s}(m', \overline{m''})\big\rangle$$

extends as a meromorphic function on the plane $\mathbb{C}$ with possible poles contained in $\mathbb{R}_{\leqslant 0}$ (we do not use here the symbol $\int$ since $\mathfrak{s}(m', \overline{m''})$ is a distribution which is possibly not a function, like δ_0).

7.3.14. Lemma. *The sesquilinear pairing on $V^{-1}\mathcal{M}' \otimes \overline{V^{-1}\mathcal{M}''}$ defined by the formula*

$$(m', \overline{m''}) \longmapsto \operatorname{Res}_{s=0}\big\langle I_{\widehat{\chi}}(t,s)\chi(t)\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t}), \mathfrak{s}(m', \overline{m''})\big\rangle$$

(for some, or any, cut-off function $\chi \in C_c^\infty(\Delta)$) induces a well-defined sesquilinear pairing

$$\mathrm{gr}_V^{-1}\mathfrak{s} : \mathrm{gr}_V^{-1}\mathcal{M}' \otimes \overline{\mathrm{gr}_V^{-1}\mathcal{M}''} \longrightarrow \mathbb{C}$$

which coincides with $-\mathfrak{s}_0^{(-1)}$ via the identification $M^{-1} \simeq \mathrm{gr}_V^{-1}\mathcal{M}$ ($\mathcal{M} = \mathcal{M}', \mathcal{M}''$) of Proposition 7.2.10.

Sketch of proof. We note that the basic distributions $u_{\beta,p}$ (with $\beta \geqslant -1$ and $p \geqslant 0$) are temperate distributions on $\mathbb{C}$. Hence so are their Fourier transforms $\widehat{u}_{\beta,p} := \mathcal{F}(u_{\beta,p})$. Assume first that $\beta > -1$. Then $\widehat{u}_{\beta,p}$ solves the equations

$$(\tau\partial_\tau + \beta + 1)^{p+1}\widehat{u}_{\beta,p} = (\overline{\tau}\partial_{\overline{\tau}} + \beta + 1)^{p+1}\widehat{u}_{\beta,p} = 0,$$

and thus the restriction of $\widehat{u}_{\beta,p}$ to $\tau \neq 0$ is a $\mathbb{C}$-linear combination of the functions $|\tau|^{-2(\beta+1)}\mathrm{L}(\tau)^k/k!$ for $k \leqslant p$. It follows from Exercise 6.13(3), applied with the variable $\theta = 1/\tau$, that

$$s \longmapsto \int_{\mathbb{C}} |\tau|^{-2(s+1)}\widehat{\chi}(\tau)\widehat{u}_{\beta,p}\, \tfrac{\mathrm{i}}{2\pi}\, \mathrm{d}\tau \wedge \mathrm{d}\overline{\tau}$$

extends as a meromorphic function with no pole at $s = 0$. One can refine this reasoning in order to get the first statement.

For the second statement, we are reduced to showing

$$\operatorname{Res}_{s=0}\big\langle I_{\widehat{\chi}}(t,s)\chi(t)\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t}), u_{-1,p})\big\rangle = \begin{cases} -1 & \text{if } p = 0, \\ 0 & \text{if } p \geqslant 1. \end{cases}$$

The first case follows from the identity $\operatorname{Res}_{s=0} I_{\widehat{\chi}}(0,s) = 1$ (see Exercise 7.21(2)), since $u_{-1,0} = -\delta_0$. For $p \geqslant 1$, one uses Exercise 7.21(1) and (4) to show that $((t\partial_t)^p I_{\widehat{\chi}})(0,s)$ has no pole at $s = 0$. □

7.3.15. Definition. The sesquilinear pairing

$$\phi_{t,1}\mathfrak{s} : \mathrm{gr}_V^{-1}\mathcal{M}' \otimes \overline{\mathrm{gr}_V^{-1}\mathcal{M}''} \longrightarrow \mathbb{C}$$

is well-defined by the formula

$$([m'], [m'']) \longmapsto \operatorname{Res}_{s=0}\big\langle I_{\widehat{\chi}}(g,s)\chi(t)\, \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \mathrm{d}\overline{t}), \mathfrak{s}(m', \overline{m''})\big\rangle, \tag{7.3.15 $*$}$$

where m', m'' are local liftings of $[m'], [m'']$ and $\chi(t)$ is any cut-off function. It satisfies (see Corollary (7.3.7)) $\phi_{t,1}(\mathfrak{s}^*) = (\phi_{t,1}\mathfrak{s})^*$ and

$$\begin{aligned}&\phi_{t,1}\mathfrak{s}(\mathrm{N}'\bullet, \overline{\bullet}) = \phi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet}),\\ &\phi_{t,1}\mathfrak{s}(\mathrm{can}'\bullet, \overline{\bullet}) = -\psi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{var}''\bullet}), \quad \phi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{can}''\bullet}) = -\psi_{t,1}\mathfrak{s}(\mathrm{var}'\bullet, \overline{\bullet}).\end{aligned} \tag{7.3.15 $**$}$$

7.3.16. Examples **(Sesquilinear pairing on vanishing cycles).** Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$. We denote by $\mathcal{M}$ either $\mathcal{M}'$ or $\mathcal{M}''$.

(1) If $\mathcal{M}', \mathcal{M}''$ are supported at the origin, we have $\mathcal{M} = M^{-1}[\partial_t]$ and we recover (see Example 7.3.9(2)) that $\phi_{t,1}\mathfrak{s}$ on $M'^{-1} \otimes \overline{M''^{-1}}$ is the coefficient of δ_0 in $\mathfrak{s}^{(-1)}$. This explains the minus sign occurring in the second line of (7.3.15 $**$), while there is no minus sign in (7.3.7 $**$).

(2) If $\mathcal{M}$ is a middle extension, we have $\phi_{t,1}\mathcal{M} = \operatorname{Im}\mathrm{N} : \psi_{t,1}\mathcal{M} \to \psi_{t,1}\mathcal{M}$, with $\mathrm{can} = \mathrm{N}$ and $\mathrm{var} = \mathrm{incl}$. Formulas (7.3.15 $**$) give

$$\phi_{t,1}\mathfrak{s}(\mathrm{N}'\bullet, \overline{\mathrm{N}''\bullet}) := -\psi_{t,1}\mathfrak{s}(\mathrm{N}'\bullet, \overline{\bullet}) = -\psi_{t,1}\mathfrak{s}(\bullet, \overline{\mathrm{N}''\bullet}).$$

Note that this is compatible with Proposition 3.4.20.

7.3.e. Pushforward of a sesquilinear pairing.

We will consider the case of the closed inclusion $\iota : \{0\} \hookrightarrow \Delta$ and, in the global setting, the case of the constant map $X \to \mathrm{pt}$ on a Riemann surface X.

7.3.17. Pushforward of a sesquilinear pairing by a closed inclusion. Let $\iota : \{0\} \hookrightarrow \Delta$ denote the inclusion and let $\mathfrak{s} : \mathcal{H}' \otimes \overline{\mathcal{H}''}$ be a sesquilinear pairing between $\mathbb{C}$-vector spaces. We set the following, for $\mathcal{H} = \mathcal{H}', \mathcal{H}''$:

- $\iota_*\mathcal{H}$ is the skyscraper sheaf with stalk $\mathcal{H}$ at the origin.
- $\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H}$ is the sheaf supported at the origin

$$\iota_*\mathcal{H}[\partial_t] := \iota_*\mathcal{H} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] = \bigoplus_{k\geqslant 0} \iota_*\mathcal{H} \cdot \partial_t^k,$$

where we regard ∂_t as a new variable, and that we equip with the left $\mathcal{D}_\Delta$-module structure for which the action of t defined by $t \cdot v\partial_t^k = -kv\partial_t^{k-1}$ ($v \in \mathcal{H}$), and the action of ∂_t is the obvious one $\partial_t \cdot v\partial_t^k = v\partial_t^{k+1}$.

- The pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_\Delta$ is defined by $\mathcal{D}_\Delta \otimes_{\mathbb{C}} \overline{\mathcal{D}_\Delta}$-linearity from its restriction to $\iota_*\mathcal{H}' \otimes_{\mathbb{C}} \iota_*\overline{\mathcal{H}''}$ as follows:

$$({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s})(v', \overline{v''}) = \mathfrak{s}(v', \overline{v''})\delta_0.$$

Note that, since δ_0 is real, we have ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*(\mathfrak{s}^*) = ({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s})^*$.

Pushforward of a sesquilinear pairing by a constant map. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing. We wish to "integrate" it on X, that is, to define for each k, by integration, a sesquilinear pairing

$$\textstyle\int_X^{(k,-k)}\mathfrak{s} : \boldsymbol{H}^{1+k}(X, \mathrm{DR}\,\mathcal{M}') \otimes \overline{\boldsymbol{H}^{1-k}(X, \mathrm{DR}\,\mathcal{M}'')} \longrightarrow \mathbb{C}. \tag{7.3.18}$$

It is convenient to realize elements of the de Rham cohomology $\boldsymbol{H}^j(X, \mathrm{DR}\,\mathcal{M})$ as differential forms with coefficients in $\mathcal{M}$. For that purpose, we replace the complex $\mathrm{DR}\,\mathcal{M}$ with its C^∞ resolution $(\mathcal{E}_X^\bullet \otimes \mathcal{M}, \mathrm{d} + \nabla)$. An element of $\boldsymbol{H}^j(X, \mathrm{DR}\,\mathcal{M})$ can then be represented by a global section of $\mathcal{E}_X^j \otimes \mathcal{M}$ which is closed under $\mathrm{d} + \nabla$ (a shortcut for $\mathrm{d} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla$), modulo exact global sections. By using a partition of unity, each global section can be written as a sum of terms $\eta \otimes m$, where m is a section of $\mathcal{M}$ on some open set of X and η is a C^∞ j-form with compact support contained in this open subset. For η' of degree $1+k$ and η'' of degree $1-k$, we set

$$(\textstyle\int_X^{(k,-k)} \mathfrak{s})(\eta' \otimes m', \overline{\eta'' \otimes m''}) := \langle \eta' \wedge \overline{\eta''}, \mathfrak{s}(m', \overline{m''}) \rangle, \tag{7.3.19}$$

where $\mathfrak{s}(m', \overline{m''})$ is regarded as a distribution on the intersection of the domains of m' and m'', which contains the support of the C^∞ 2-form $\eta' \wedge \overline{\eta''}$.

7.3.20. Proposition. *Formula* (7.3.19) *(extended by linearity on both sides) well defines a sesquilinear pairing* (7.3.18).

Proof. If we denote by D the differential of the C^∞ de Rham complex, the assertion would follow from the property

$$(\textstyle\int_X \mathfrak{s})(D(\eta' \otimes m'), \overline{\eta'' \otimes m''}) = \pm (\textstyle\int_X \mathfrak{s})(\eta' \otimes m', \overline{D(\eta'' \otimes m'')}), \tag{7.3.21}$$

where $\pm$ depends on k. Assume for example that η' is a C^∞ function and η'' a 1-form. Stokes formula implies

$$\langle \eta' \overline{\eta''}, \mathrm{d}' \mathfrak{s}(m', \overline{m''}) \rangle = -\langle \mathrm{d}'(\eta' \overline{\eta''}), \mathfrak{s}(m', \overline{m''}) \rangle$$

and similarly with d''. Since $D(\eta' \otimes m') = \mathrm{d}\eta' \otimes m' + \eta' \wedge \nabla m'$ and since $\mathfrak{s}(\nabla m', \overline{m''}) = \mathrm{d}' \mathfrak{s}(m', \overline{m''})$, the left-hand side of (7.3.21) is equal to

$$\langle (\mathrm{d}\eta') \wedge \overline{\eta''}, \mathfrak{s}(m', \overline{m''}) \rangle - \langle \mathrm{d}'(\eta' \overline{\eta''}), \mathfrak{s}(m', \overline{m''}) \rangle$$

while the right-hand side of (7.3.21) is similarly

$$\langle \eta' (\overline{\mathrm{d}\eta''}), \mathfrak{s}(m', \overline{m''}) \rangle - \langle \mathrm{d}''(\eta' \overline{\eta''}), \mathfrak{s}(m', \overline{m''}) \rangle,$$

and the sum of the two sides is equal to zero. $\square$

7.3.22. Definition. The pushforward

$$_{\mathrm{D},\overline{\mathrm{D}}}a_*^{(k,-k)} \mathfrak{s} : \boldsymbol{H}^{1+k}(X, \mathrm{DR}\, \mathcal{M}') \otimes \overline{\boldsymbol{H}^{1-k}(X, \mathrm{DR}\, \mathcal{M}'')} \longrightarrow \mathbb{C}$$

is defined as

$$_{\mathrm{D},\overline{\mathrm{D}}}a_*^{(k,-k)} \mathfrak{s} := \mathrm{Sgn}(1,k) \textstyle\int_X^{(k,-k)} \mathfrak{s}.$$

7.4. Hodge $\mathcal{D}$-modules on a Riemann surface and the Hodge-Saito theorem

What kind of an algebraic object do we get by considering $\mathcal{V}_{\mathrm{mid}}$ together with its connection and its filtration? How to describe it axiomatically, as we did for variations of Hodge structure? Is there a wider class of filtered $\mathcal{D}$-modules which would give rise to a Hodge theorem? We give an answer to these questions in this section.

7.4.a. The category of triples of filtered $\mathcal{D}_X$-modules and its functors

The category of triples, as considered in Section 5.4, will prove much convenient as an ambient abelian category for Hodge modules. We develop here the language of triples for filtered $\mathcal{D}_X$-modules.

A filtered $\mathcal{D}_X$-triple

$$\mathcal{T} = ((\mathcal{M}', F^\bullet\mathcal{M}'), (\mathcal{M}'', F^\bullet\mathcal{M}''), \mathfrak{s})$$

consists of filtered $\mathcal{D}_X$-modules together with a sesquilinear pairing between the underlying $\mathcal{D}_X$-modules. We say that a triple is coherent, holonomic, regular, strictly $\mathbb{R}$-specializable, S-decomposable, middle extension, with punctual support, if both its filtered $\mathcal{D}_X$-module components are so. We note that, by Example 7.3.9(1), if $\mathcal{T}$ is holonomic, strictly $\mathbb{R}$-specializable at any point, hence also regular (Proposition 7.2.20), and S-decomposable, then $\mathcal{T}$ decomposes in a unique way as $\mathcal{T}_1 \oplus \mathcal{T}_2$, where $\mathcal{T}_1$ has pure support X and $\mathcal{T}_2$ has punctual support.

7.4.1. Morphisms, Hermitian duality, twist

(1) The notion of morphism is the obvious one, as in the category of triples. A morphism $\varphi : \mathcal{T}_1 \to \mathcal{T}_2$ is a pair (φ', φ''), where φ' is a filtered morphism $(\mathcal{M}'_1, F^\bullet\mathcal{M}'_1) \to (\mathcal{M}'_2, F^\bullet\mathcal{M}'_2)$ and φ'' a filtered morphism $(\mathcal{M}''_2, F^\bullet\mathcal{M}''_2) \to (\mathcal{M}''_1, F^\bullet\mathcal{M}''_1)$, both satisfying the compatibility relation (5.2.1 $**$) in $\mathfrak{Db}_X$.

(2) It is convenient to embed the category of triples of filtered $\mathcal{D}_X$-modules as a full subcategory of that of triples of $R_F\mathcal{D}$-modules, which is abelian. In order to do so, we start by applying the Rees construction of Section 5.1.3, and we denote by $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$ the Rees ring obtained from the filtered ring $(\mathcal{D}_X, F_\bullet\mathcal{D}_X)$. We the consider the triples consisting of pairs $(\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'')$ of graded $R_F\mathcal{D}_X$-modules and a sesquilinear pairing between the associated $\mathcal{D}_X$-modules $\mathcal{M} = \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ with values in $\mathfrak{Db}_X$, and we associate with a triple $\mathcal{T}$ as above the triple consisting of the Rees modules $R_F\mathcal{M}', R_F\mathcal{M}''$ (in particular they are *strict* as graded $R_F\mathcal{D}_X$-modules) and the sesquilinear pairing $\mathfrak{s}$ between $\mathcal{M}'$ and $\mathcal{M}''$. This category of triples of is abelian, since one does not insist on the torsion freeness with respect to z.

(3) Hermitian duality is defined as in Section 5.2.2(6):

$$\mathcal{T}^* = ((\mathcal{M}'', F^\bullet\mathcal{M}''), (\mathcal{M}', F^\bullet\mathcal{M}'), \mathfrak{s}^*).$$

(4) Tate twist is defined as in Section 5.2.2(7), so

$$\mathcal{T}(k) = ((\mathcal{M}', F[k]^\bullet\mathcal{M}'), (\mathcal{M}'', F[-k]^\bullet\mathcal{M}''), \mathfrak{s}).$$

(5) A pre-polarization of $\mathcal{T}$ of weight w is an isomorphism $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ which is Hermitian.

(6) The data of a pre-polarized filtered triple $(\mathcal{T}, \mathrm{S})$ of weight w is equivalent to the data of a filtered Hermitian pair $((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S})$ together with the weight w.

The normalization of Section 5.4.b leads us to de-symmetrize the nearby cycle functors, in a way similar to that of the pullback functor.

7.4.2. Nearby and vanishing cycles. We assume that $X = \Delta$. Let

$$\mathcal{T} = ((\mathcal{M}', F^\bullet\mathcal{M}'), (\mathcal{M}'', F^\bullet\mathcal{M}''), \mathfrak{s})$$

be coherent, holonomic and strictly $\mathbb{R}$-specializable at the origin. We set (see (7.2.16))

$$\begin{aligned}
\psi_{t,\lambda}\mathcal{T} &:= ((\psi_{t,\lambda}\mathcal{M}', F^\bullet\psi_{t,\lambda}\mathcal{M}'), (\psi_{t,\lambda}\mathcal{M}'', F^\bullet\psi_{t,\lambda}\mathcal{M}'')(-1), \psi_{t,\lambda}\mathfrak{s})\\
\phi_{t,1}\mathcal{T} &:= ((\phi_{t,1}\mathcal{M}', F^\bullet\phi_{t,1}\mathcal{M}'), (\phi_{t,1}\mathcal{M}'', F^\bullet\phi_{t,1}\mathcal{M}''), \phi_{t,1}\mathfrak{s}),\\
\mathrm{N} &= (\mathrm{N}', \mathrm{N}''), \quad \mathrm{can} = (\mathrm{can}', -\mathrm{var}''), \quad \mathrm{var} = (\mathrm{var}', -\mathrm{can}'').
\end{aligned}$$

The signs are reminiscent of (5.3.7). We have

$$(\psi_{t,\lambda}\mathcal{T})^* = \psi_{t,\lambda}(\mathcal{T}^*)(-1), \quad (\phi_{t,1}\mathcal{T})^* = \phi_{t,1}(\mathcal{T}^*).$$

Since can' is a morphism $(\psi_{t,1}\mathcal{M}', F^\bullet\psi_{t,1}\mathcal{M}') \to (\phi_{t,1}\mathcal{M}', F^\bullet\phi_{t,1}\mathcal{M}')$ and var'' is a morphism $(\phi_{t,1}\mathcal{M}'', F^\bullet\phi_{t,1}\mathcal{M}'') \to (\psi_{t,1}\mathcal{M}'', F^\bullet\psi_{t,1}\mathcal{M}'')(-1)$, and similarly when exchanging the prime and double prime parts, we deduce from (7.3.15 $**$) a nearby/vanishing cycle Lefschetz quiver

$$\begin{tikzcd}
\psi_{t,1}\mathcal{T} \arrow[rr, bend left, "\mathrm{can}"] && \phi_{t,1}\mathcal{T}. \arrow[ll, bend left, "\mathrm{var}", "(-1)"']
\end{tikzcd}$$

If $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ is a pre-polarization, it induces pre-polarizations

$$\begin{aligned}
\psi_{t,\lambda}\mathrm{S} &: (\psi_{t,\lambda}\mathcal{T}, \mathrm{N}) \longrightarrow (\psi_{t,\lambda}\mathcal{T}, \mathrm{N})^*(-(w-1)),\\
\phi_{t,1}\mathrm{S} &: (\phi_{t,1}\mathcal{T}, \mathrm{N}) \longrightarrow (\phi_{t,1}\mathcal{T}, \mathrm{N})^*(-w).
\end{aligned}$$

where we have set $(\psi_{t,\lambda}\mathcal{T}, \mathrm{N})^* = (\psi_{t,\lambda}\mathcal{T}^*, \mathrm{N}^*)$ and similarly for $\phi_{t,1}$. We then set

$$\begin{aligned}
\psi_{t,\lambda}(\mathcal{T}, \mathrm{S}) &:= (\psi_{t,\lambda}\mathcal{T}, \psi_{t,\lambda}\mathrm{S}),\\
\phi_{t,1}(\mathcal{T}, \mathrm{S}) &:= (\phi_{t,1}\mathcal{T}, \phi_{t,1}\mathrm{S}).
\end{aligned}$$

For the corresponding filtered Hermitian pair $((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S}, w)$, this reads as

$$\begin{aligned}
\psi_{t,\lambda}((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S}, w) &:= (\psi_{t,\lambda}(\mathcal{M}', F^\bullet\mathcal{M}'), \psi_{t,\lambda}\mathcal{S}, w-1),\\
\phi_{t,1}((\mathcal{M}', F^\bullet\mathcal{M}'), \mathcal{S}, w) &:= (\phi_{t,1}(\mathcal{M}', F^\bullet\mathcal{M}'), \phi_{t,1}\mathcal{S}, w).
\end{aligned}$$

7.4.3. S-decomposability. In the local setting above, we say that $\mathcal{T}$ is S-decomposable if its filtered $\mathcal{D}$-module components are so. It follows from Example 7.3.9(1) that the sesquilinear pairing decomposes correspondingly, and thus $\mathcal{T} = \mathcal{T}_1 \oplus \mathcal{T}_2$ with $\mathcal{T}_2$ supported at the origin and where $\mathcal{T}_1$ is a middle extension. The criterion of Proposition 7.2.27 extends as well: $\mathcal{T}$ is S-decomposable if and only if $\phi_{t,1}\mathcal{T} = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$.

7.4.4. Pushforward by a closed inclusion. For a filtered $\mathbb{C}$-triple

$$\mathcal{T} = ((\mathcal{H}', F^\bullet\mathcal{H}'), (\mathcal{H}'', F^\bullet\mathcal{H}''), \mathfrak{s}),$$

we use the notation of (7.2.32 $*$) and of Section 7.3.17, and we set

$${}_{\mathrm{T}}\iota_*\mathcal{T} := \big({}_{\mathrm{D}}\iota_*(\mathcal{H}', F^\bullet\mathcal{H}'), {}_{\mathrm{D}}\iota_*(\mathcal{H}'', F^\bullet\mathcal{H}''), {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}\big).$$

We recover $\mathcal{T}$ as $\phi_{t,1}({}_{\mathrm{T}}\iota_*\mathcal{T})$ (see Proposition 7.2.32 and Section 7.3.17). A pre-polarization S of weight w is pushforwarded to a pre-polarization ${}_{\mathrm{T}}\iota_*\mathrm{S} : {}_{\mathrm{T}}\iota_*\mathcal{T} \to {}_{\mathrm{T}}\iota_*(\mathcal{T}^*(-w)) = ({}_{\mathrm{T}}\iota_*\mathcal{T})^*(-w)$ of weight w.

7.4.5. Pushforward by the constant map. Let $a_X : X \to \mathrm{pt}$ be the constant map. Recall (see Section 7.2.e and Caveat 7.2.34) that, for a coherently F-filtered holonomic $(\mathcal{M}, F^\bullet\mathcal{M})$, we have set ${}_{\mathrm{D}}a_X^{(k)}\mathcal{M} = \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$ and correspondingly we set

$$F^p\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = \mathrm{image}\big[\boldsymbol{H}^k(X, F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})\big],$$

defining thus ${}_{\mathrm{D}}a_X^{(k)}(\mathcal{M}, F^\bullet\mathcal{M})$. We define

$${}_{\mathrm{T}}a_*^{(k)}\mathcal{T} = \big({}_{\mathrm{D}}a_X^{(k)}(\mathcal{M}', F^\bullet\mathcal{M}'), {}_{\mathrm{D}}a_X^{(-k)}(\mathcal{M}'', F^\bullet\mathcal{M}''), {}_{\mathrm{D},\overline{\mathrm{D}}}a_*^{(k,-k)}\mathfrak{s}\big).$$

With this definition we have

$${}_{\mathrm{T}}a_*^{(k)}(\mathcal{T}^*) = ({}_{\mathrm{T}}a_*^{(-k)}\mathcal{T})^*,$$

and if S is a pre-polarization $\mathcal{T} \to \mathcal{T}^*(-w)$, it defines a pre-polarization

$${}_{\mathrm{T}}a_*^{(k)}\mathrm{S} : {}_{\mathrm{T}}a_*^{(k)}\mathcal{T} \longrightarrow {}_{\mathrm{T}}a_*^{(k)}(\mathcal{T}^*)(-w) = ({}_{\mathrm{T}}a_*^{(-k)}\mathcal{T})^*(-w).$$

7.4.6. Example. If $\mathcal{T}$ is a polarizable smooth $\mathbb{C}$-Hodge triple of weight w (see Definition 5.4.7) and if X is compact, then ${}_{\mathrm{T}}a_*^{(k)}\mathcal{T}$ is a $\mathbb{C}$-Hodge triple of weight $w+k$, according to the Hodge-Deligne theorem 4.2.16.

7.4.b. Polarizable $\mathbb{C}$-Hodge modules. Let us introduce the main objects of this section.

***7.4.7. Definition* (of a polarized $\mathbb{C}$-Hodge module of weight w)**

Let $\mathcal{T}$ be a holonomic coherently F-filtered $\mathcal{D}_X$-triple with singular set $\Sigma \subset X$, and let $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ be a morphism ($w \in \mathbb{Z}$). We say that $(\mathcal{T}, \mathrm{S})$ is a *polarized Hodge module of weight* w on X if the following properties hold:

(1) $(\mathcal{T}, \mathrm{S})_{|X\smallsetminus\Sigma}$ is a polarized smooth $\mathbb{C}$-Hodge triple of weight w (Definition 5.4.7),

(2) For each $x_o \in \Sigma$ and some local coordinate t vanishing at x_o, $\mathcal{T}$ is strictly $\mathbb{R}$-specializable at x_o and

(a) for any $\lambda \in \mathbb{S}^1$, $\psi_{t,\lambda}(\mathcal{T}, \mathrm{S}) := (\psi_{t,\lambda}\mathcal{T}, \psi_{t,\lambda}\mathrm{S})$ is a polarized Hodge-Lefschetz triple with central weight $w-1$,

(b) $\phi_{t,1}(\mathcal{T}, \mathrm{S}) := (\phi_{t,1}\mathcal{T}, \phi_{t,1}\mathrm{S})$ is a polarized Hodge-Lefschetz triple with central weight w.

See Section 5.3 for the notion of polarized Hodge-Lefschetz triple. We note that a morphism S satisfying (1) and (2) is a Hermitian isomorphism, i.e., it is a filtered isomorphism which satisfies $\mathrm{S}^* = \mathrm{S}$. In other words, it is a pre-polarization of weight w. Indeed, this property holds on $X \smallsetminus \Sigma$, and at each point x_o of Σ we have $\psi_{t,\lambda}\mathrm{S}^* = \psi_{t,\lambda}\mathrm{S}$ ($\forall\,\lambda \in \mathbb{S}^1$) and $\phi_{t,1}\mathrm{S}^* = \phi_{t,1}\mathrm{S}$, by definition of a (pre-)polarization of a Hodge-Lefschetz triple (see Section 5.3.4). That S is Hermitian follows from Exercise 7.13(2). The filtered-isomorphism property follows from Exercise 7.16. We then call S a *polarization* of the Hodge module $\mathcal{T}$ of weight w (a positivity property has been added to the notion of pre-polarization).

7.4.8. Definition **(of a polarizable $\mathbb{C}$-Hodge module of weight w)**

Let $\mathcal{T}$ be a holonomic coherently F-filtered $\mathcal{D}_X$-triple. We say that $\mathcal{T}$ is a *polarizable Hodge module of weight* w on X if there exists a pre-polarization $\mathrm{S} : \mathcal{T} \to \mathcal{T}^*(-w)$ of weight w such that $(\mathcal{T}, \mathrm{S})$ is a polarized Hodge module of weight w on X in the sense of Definition 7.4.7.

We will denote by M a triple which is a polarizable Hodge module and by $\mathsf{pHM}(X, w)$ the full subcategory of the category of holonomic coherently F-filtered $\mathcal{D}_X$-triples whose objects are polarizable $\mathbb{C}$-Hodge modules of weight w.

7.4.9. Proposition **(Simplified form for an object of $\mathsf{pHM}(X, w)$)**

Any object M of $\mathsf{pHM}(X, w)$ is isomorphic to an object of the form

$$\big((\mathcal{M}, F^\bullet\mathcal{M}), (\mathcal{M}, F^\bullet\mathcal{M})(w), \mathcal{S}\big)$$

such that $\mathcal{S}^* = \mathcal{S}$ and with polarization $(\mathrm{Id}, \mathrm{Id}) : M \to M^*(-w)$.

We also call the data $\big((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{S}, w\big)$ a *polarized $\mathbb{C}$-Hodge module of weight w* if the corresponding triple $\big((\mathcal{M}, F^\bullet\mathcal{M}), (\mathcal{M}, F^\bullet\mathcal{M})(w), \mathcal{S}\big)$ with polarization $(\mathrm{Id}, \mathrm{Id})$ is polarized Hodge module of weight w.

Proof. Let $\mathrm{S} = (\mathcal{S}', \mathcal{S}'') : M \to M^*(-w)$ be a polarization. It enables us to identify $(\mathcal{M}'', F^\bullet\mathcal{M}'')$ with $(\mathcal{M}', F^\bullet\mathcal{M}')(w)$ by $\mathcal{S}' = \mathcal{S}''$. We then argue as in Proposition 5.2.16. □

7.4.10. Theorem **(The S-decomposition theorem for polarizable Hodge modules)**

Let M be a polarizable Hodge module of weight w on X. Then M decomposes in a unique way in $\mathsf{pHM}(X, w)$ as the direct sum $M = M_1 \oplus M_2$, where M_1 has pure support X and M_2 has punctual support.

Proof. Assume that M has weight w and let $\mathrm{S} : M \to M^*(-w)$ be a polarization. Due to uniqueness, the question is local at each singular point of M. We can moreover replace (M, S) with the corresponding Hodge-Hermitian pair $\big((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{S}, w\big)$. In order to apply the S-decomposition theorem for Hodge-Lefschetz structures 3.4.22 to the Hodge-Lefschetz quiver of this Hodge-Hermitian pair, we need to check that it is polarizable as such. This amounts to checking the equality

$$\phi_{t,1}\mathcal{S}(m, \overline{\operatorname{can} n}) = -\psi_{t,1}\mathcal{S}(\operatorname{var} m, \overline{n}),$$

which holds, as seen in (7.3.15 $**$).

The S-decomposability criterion of Section 7.4.3 implies that M decomposes as wanted in the category of filtered $\mathcal{D}$-triples. It remains to check that both M_1 and M_2 are objects of $\mathsf{pHM}(X, w)$. But by construction, the corresponding decomposition of $\psi_{t,\lambda}M$ and $\phi_{t,1}M$ is that given in Theorem 3.4.22, hence is by polarized Hodge-Lefschetz structures, as wanted. □

Clearly, there is no non-zero morphism between M_1 and M_2 in the S-decomposition of M, as this already holds for the underlying $\mathcal{D}$-modules. Therefore, any morphism in $\mathsf{pHM}(X, w)$ S-decomposes correspondingly. We denote by $\mathsf{pHM}_X(X, w)$

resp. $\mathsf{pHM}_\Sigma(X,w)$ the full subcategory of $\mathsf{pHM}(X,w)$ consisting of objects with pure support X resp. with punctual support Σ. Any object M and morphism φ of $\mathsf{pHM}_\Sigma(X,w)$ decomposes therefore as the direct sum of objects M_1 and M_2 and morphisms φ_1 and φ_2, one in each subcategory, for a suitable discrete set $\Sigma\subset X$.

Most reasonings concerning polarizable Hodge modules are therefore divided in two cases, that of middle extensions and that of objects with punctual support. The latter case is usually reduced to that of polarizable Hodge structures by the previous remark, and the former is reduced to that of polarizable Hodge-Lefschetz structures by means of $\psi_{t,\lambda}$, while the case of $\phi_{t,1}$ is deduced from that of $\psi_{t,1}$ by Proposition 3.4.20.

Let us analyze the local structure (on Δ, with $\iota:\Sigma=\{0\}\hookrightarrow\Delta$) of M_1 and M_2.

7.4.11. Proposition. *The functor* ${}_{\mathrm{T}}\iota_*$ *of Section* 7.4.4 *induces an equivalence between* $\mathsf{pHS}(w)$ *and* $\mathsf{pHM}_{\{0\}}(\Delta,w)$*, a quasi-inverse functor being* $\phi_{t,1}$. □

7.4.12. Proposition. *If* $(\mathcal{M},F^\bullet\mathcal{M})$ *underlies a Hodge module with pure support* Δ *of weight* w*, then* $(\mathcal{M},F^\bullet\mathcal{M})\simeq(\mathcal{V}_{\mathrm{mid}},F^\bullet\mathcal{V}_{\mathrm{mid}})$ *as defined by* (6.14.1)*, with* $\mathcal{V}=\mathcal{M}_{|\Delta^*}$*. Furthermore,* $(\mathcal{V},F^\bullet\mathcal{V})$ *underlies a variation of Hodge structure of weight* $w-1$.

Proof. That $\mathcal{M}\simeq\mathcal{V}_{\mathrm{mid}}$ follows from Definition 7.2.29. It remains to check that the filtrations coincide. By Proposition 6.14.2, it is enough to check that $F^\bullet\mathcal{M}\cap V^{>-1}\mathcal{M}=F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ and that $F^\bullet\mathcal{M}$ satisfies 6.14.2(3c), since we assume that $(\mathcal{M},F^\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable (Definition 7.2.19).

Let us first show that

$$F^\bullet\mathcal{M}\cap V^{>-1}\mathcal{M}=(j_*j^{-1}F^\bullet\mathcal{M})\cap V^{>-1}\mathcal{M},$$

the latter term being equal to $F^p\mathcal{V}_{\mathrm{mid}}^{>-1}$ by (6.7.1). Let m be a local section of $(j_*j^{-1}F^p\mathcal{M}\cap V^{>-1}\mathcal{M})\cap(F^q\mathcal{M}\cap V^{>-1}\mathcal{M})$ for $q>p$. Then m defines a section of $(F^q\mathcal{M}\cap V^{>-1}\mathcal{M})/(F^p\mathcal{M}\cap V^{>-1}\mathcal{M})$ supported at the origin. Since the latter quotient is $\mathcal{O}_\Delta$-coherent, it follows that t^Nm is a local section of $F^p\mathcal{M}\cap V^{>-1}\mathcal{M}$ for some N, hence a local section of $F^p\mathcal{M}\cap V^{>-1+N}\mathcal{M}$. Now, Property 6.14.2(3a) implies that m is a local section of $F^p\mathcal{M}\cap V^{>-1}\mathcal{M}$, hence the desired assertion.

It remains to check 6.14.2(3c). This amounts to proving that can is an epimorphism in the category of filtered vector spaces. This follows from the property that $\mathrm{Im}[\mathrm{N}:\psi_{t,0}\mathcal{M}\to\psi_{t,0}\mathcal{M}]$ is a Hodge-Lefschetz structure with central weight w (see Exercise 3.14(2), Proposition 3.4.6 and its translation in the language of triples in Section 5.3). □

7.4.13. Remark. Strict $\mathbb{R}$-specializability, as defined by 7.2.19 and assumed in Definition 7.4.7, would not have been enough to prove Proposition 7.4.12. Hodge theory is used in an essential way here, by means of Exercise 3.14, to ensure Property 6.14.2(3c).

It M has pure support the disc, it follows from Theorem 6.8.7 that 7.4.7(1) implies 7.4.7(2a), and Proposition 3.4.20 implies that 7.4.7(2b) also holds. The definition of a polarized Hodge module consists therefore in taking Theorem 6.8.7 as a defining property. This leads to the definition of the category $\mathsf{pHM}_X(X,w)$ of polarizable $\mathbb{C}$-Hodge modules of weight w with pure support X.

Let j denote the inclusion $\Delta^* \hookrightarrow \Delta$.

***7.4.14. Corollary* (of the results of Chapter 6).** *The restriction functor j^*, from the category of polarizable $\mathbb{C}$-Hodge modules with pure support Δ, weight w and singularity at 0 at most to the category of polarizable variations of $\mathbb{C}$-Hodge structure on Δ^* of weight $w-1$ is an equivalence of categories.*

***Proof*.** Let us prove essential surjectivity. Given a polarized variation of Hodge structure (H,S) on Δ^* (i.e., we choose a polarization on H), we know by Formula (6.14.1), Corollary 6.14.4 and Proposition 6.14.2 that $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ is a holonomic filtered $\mathcal{D}_\Delta$-module which is strictly $\mathbb{R}$-specializable at the origin. The sesquilinear pairing $\mathcal{S}$ extends as a sesquilinear pairing on $\mathcal{V}_*^\beta$ ($\beta \in (-1,0]$) as explained in Section 6.8.a, and this uniquely defines an extension of $\mathcal{S}$ to $\mathcal{V}_{\mathrm{mid}}$, as noticed in Example 7.3.9(3). Then Theorem 6.8.7 implies that Property 7.4.7(2a) holds, and this is enough, as noticed in Remark 7.4.13.

For the full faithfulness, it is enough to prove that a morphism $\varphi : (\mathcal{V}_1, F^\bullet\mathcal{V}_1) \to (\mathcal{V}_2, F^\bullet\mathcal{V}_2)$ extends in a unique way as a morphism

$$(\mathcal{V}_{1\mathrm{mid}}, F^\bullet\mathcal{V}_{1\mathrm{mid}}) \longrightarrow (\mathcal{V}_{2\mathrm{mid}}, F^\bullet\mathcal{V}_{2\mathrm{mid}}).$$

First, φ extends in a unique way as a morphism $\mathcal{V}_{1*} \to \mathcal{V}_{2*}$, by the equivalence of Theorem 6.2.1, and this morphism sends $\mathcal{V}_1^{>-1}$ to $\mathcal{V}_2^{>-1}$, hence $\mathcal{V}_{1\mathrm{mid}}$ to $\mathcal{V}_{2\mathrm{mid}}$. The compatibility with filtrations follows from (6.7.1) and (6.14.1). $\square$

***7.4.15. Proposition*.** *There is no nonzero morphism $M_1 \to M_2$ between polarizable $\mathbb{C}$-Hodge modules of weight w_1, w_2 if $w_1 > w_2$.*

***Proof*.** We can treat separately the case of pure support and the case with punctual support. The latter case follows from Proposition 2.5.5(2).

Let us consider the case of a middle extension. The $\mathcal{D}$-module part of $\operatorname{Im}\varphi$ has support $\{0\}$, by applying Proposition 2.5.5(2) at points of Δ^*, but is included in a $\mathcal{D}$-module with pure support of dimension 1, hence is zero. $\square$

***7.4.16. Proposition* (Abelianity).** *The category $\mathsf{pHM}(\Delta,w)$ of polarizable Hodge modules is* abelian *and any morphism is strict with respect to the F-filtrations.*

***Proof*.** The case of punctual support follows from Proposition 2.5.3, so we only consider the subcategory $\mathsf{pHM}_\Delta(\Delta,w)$. Let $\varphi : M_1 \to M_2$ be a morphism.

The morphisms $\psi_{t,\lambda}\varphi, \phi_{t,1}\varphi$ are morphisms in MHS, hence are strict on the filtered $\mathcal{D}$-module components. In other words, φ is strictly $\mathbb{R}$-specializable in the sense of Proposition 7.2.23 and, by loc. cit., Ker and Coker commute with $\psi_{t,\lambda}, \phi_{t,1}$ for φ on the filtered $\mathcal{D}$-module components. The same property holds with the sesquilinear pairing, so $\operatorname{Ker}\psi_{t,\lambda}\varphi$ resp. $\operatorname{Ker}\phi_{t,1}\varphi$ (and similarly for Coker) are kernel of morphisms in $\mathsf{pHLS}(w-1)$ resp. in $\mathsf{pHLS}(w)$. Since the latter categories are abelian (Proposition 3.4.18), these kernels and cokernels belong to the corresponding categories, and so do the objects obtained by commuting Ker, Coker with $\psi_{t,\lambda}, \phi_{t,1}$. We note that Proposition 3.4.18 makes precise that the sesquilinear form induced on these objects by polarizations $\mathrm{S}_1, \mathrm{S}_2$ of M_1, M_2 are polarizations. This means that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$, equipped with the induced $\mathrm{S}_1, \mathrm{S}_2$, are polarized Hodge modules of weight w. $\square$

7.4.17. Corollary. *Let φ be a morphism in $\mathsf{pHM}(\Delta, w)$. Assume that it is injective on the $\mathcal{D}_X$-module components (i.e., φ' is injective and φ'' is onto). Then it is a monomorphism, i.e., the Hodge filtration on the source of φ is the filtration induced by that on its target.* $\square$

According to Exercises 4.2(2) and 2.12, and due to the S-decomposition theorem, we obtain:

7.4.18. Corollary. *Let X be a Riemann surface. The category $\mathsf{pHM}(X, w)$ is semi-simple.* $\square$

The Hodge-Saito theorem. Let (M, S) be a polarized Hodge module of weight w on a compact Riemann surface X (see Definition 7.4.7), that we can represent as a Hodge-Hermitian pair $((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{S})$ of weight w. Away from a finite set $\Sigma \overset{\iota}{\hookrightarrow} X$, it corresponds to a polarized variation of Hodge structure of weight $w-1$. The de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is naturally filtered (see Caveat (7.2.34)), so that we get in a natural way a filtration on its hypercohomology.

7.4.19. Theorem. *Let $((\mathcal{M}, F^\bullet\mathcal{M}), \mathcal{S})$ be a polarized Hodge module of weight w on X. Then*

(1) *the filtered complex $\boldsymbol{R}\Gamma(X, F^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$ is strict, i.e., for every k, p, the natural morphism $\boldsymbol{H}^k(X, F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$ is injective,*

(2) *the filtered Hermitian pair*

$$\big(\boldsymbol{H}^0(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), F^\bullet\boldsymbol{H}^0({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), \mathrm{S} = {}_{\mathrm{D},\overline{\mathrm{D}}}a_*\mathcal{S}\big)$$

is a polarized Hodge structure of weight w,

(3) *for $k = 1, -1$, the triple*

$$(\big(\boldsymbol{H}^k(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), F^\bullet), (\boldsymbol{H}^{-k}(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}), F^\bullet), \mathrm{S} = {}_{\mathrm{D},\overline{\mathrm{D}}}a_*^{(k,-k)}\mathcal{S}\big)$$

is a polarizable Hodge triple of weight $w + k$.

Proof. We treat the case of pure support X and punctual support separately. Assume first that M has pure support X. Then $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ and $F^\bullet\mathcal{M} = F^\bullet\mathcal{V}_{\mathrm{mid}}$ (see Corollary 7.2.31). We recall (see §7.2.d) that $\mathrm{DR}\,\mathcal{M}$ is a resolution of $j_*\underline{\mathcal{H}}$. The Hodge-Zucker theorem 6.11.1 (as made precise in Section 6.14.d) implies the theorem in a straightforward way (recall Definition 7.3.22 for ${}_{\mathrm{D},\overline{\mathrm{D}}}a_*\mathcal{S}$ and that, for $n = 1$ and $k = 0$, $\operatorname{Sgn}(n, k) = \operatorname{Sgn}(1, 0) = \mathsf{i}/2\pi$, see (0.2 $*$)). We also recall that $\boldsymbol{H}^k(X, F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) := \boldsymbol{H}^{k+1}(X, F^p\,\mathrm{DR}\,\mathcal{M})$ for all p.

Assume now that M has support equal to the origin in Δ. Then $(M, \mathrm{S}) = {}_{\mathrm{T}}\iota_*(H, \mathrm{S})$ for some polarized Hodge structure H of weight w (see Proposition 7.4.11). According to Example 7.2.33(2), we thus have $\boldsymbol{H}^k(\Delta, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) = 0$ for $k \neq 0$, and the morphism of complexes $F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \to {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is nothing but $\iota_* F^p\mathcal{H} \to \iota_*\mathcal{H}$. Therefore, the map

$$\boldsymbol{H}^0(\Delta, F^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \to \boldsymbol{H}^0(\Delta, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})$$

is nothing but the map $F^p\mathcal{H} \to \mathcal{H}$, hence is *injective*. This proves the first point. It remains to identify the polarization. For that purpose, it is useful to replace the holomorphic de Rham complex with its C^∞ resolution $(\mathcal{E}^\bullet_\Delta \otimes_{\mathcal{O}_\Delta} \mathcal{M}, D)$, with $D = \mathrm{d}\otimes\mathrm{Id} + \mathrm{Id}\otimes\nabla$, in order to deal with global sections on Δ. Let us fix a basis $\boldsymbol{v} = (v_i)_i$ of $\mathcal{H}$ and let us denote with the same letter the corresponding section of $\iota_*\mathcal{H}$. Any section $m \in \Gamma(\Delta, \mathcal{E}^1_\Delta \otimes \mathcal{M})$ can be written as a finite sum $\sum_i \eta_{i,k} \otimes v_i\partial_t^k$.

7.4.20. Lemma. *Any D-closed section of $\Gamma(\Delta, \mathcal{E}^1_\Delta \otimes \mathcal{M})$ is equivalent, modulo $\operatorname{Im} D$, to a section of the form $\sum_i f_i\mathrm{d}t \otimes v_i$, with $f_i \in \mathcal{C}^\infty(\Delta)$. Moreover, the cohomology class of a closed section of the form $\sum_i f_i\mathrm{d}t\otimes v_i$ is equal to $\sum_i f_i(0)v_i \in \mathcal{H}$.*

Proof. For the first point, we argue by induction on the degree k_o of the section with respect to ∂_t. Let us consider the highest degree term $\sum_i \eta_{i,k_o} \otimes v_i\partial_t^{k_o}$. The highest degree term of the differential of the section is $\sum_i(\eta_{i,k_o} \wedge \mathrm{d}t) \otimes v_i\partial_t^{k_o+1}$, hence it is equal to zero since the section is D-closed. It follows that $\eta_{i,k_o} = f_{i,k_o}\mathrm{d}_t$ for some C^∞ function f_{i,k_o}. If $k_o \geqslant 1$, we subtract $D(\sum_i f_{i,k_o} \otimes v_i\partial_t^{k_o-1})$ to the section, and we get a closed section of degree $< k_o$.

For the second point, it is enough to check that, if $\sum_i f_i\mathrm{d}t \otimes v_i$ is D-closed, it is equal to $\sum_i f_i(0)\mathrm{d}t \otimes v_i$, since we know that the cohomology admits $\boldsymbol{v}$ as a basis by the first part of the proof of the theorem. First, the closedness property implies that each f_i is holomorphic on Δ. Then, according to the $\mathcal{O}_\Delta$-module structure of $\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{H}$, we have $tv_i = 0$ for each i, hence the result. $\square$

We can now compute, for m', m'' as above, with cohomology classes $[m'], [m'']$,

$$\begin{aligned}\mathrm{Sgn}(1,0)\int_X \mathcal{S}(m', \overline{m''}) &= \frac{\mathsf{i}}{2\pi}\sum_{i,j}\langle f'_i\overline{f''_j}\mathrm{d}t\wedge\mathrm{d}\bar{t}, \mathrm{S}(v_i, \overline{v_j})\delta_o\rangle\\ &= \mathrm{S}\big(\textstyle\sum_i f'_i(0)v_i, \overline{\sum_j f''_j(0)v_j}\big) = \mathrm{S}([m'], \overline{[m'']}).\end{aligned} \qquad \square$$

7.5. Semi-simplicity

Let X be a Riemann surface. Corollary 7.4.18 tells us that the category $\mathsf{pHM}(X, w)$ is semi-simple. If X is compact, we will determine the simple objects and show more precisely that their underlying $\mathcal{D}$-modules are themselves simple as such. The main argument will of course be Theorem 4.3.13 and Corollary 6.4.2.

***7.5.1. Theorem* (Semi-simplicity).** *Let X be a compact Riemann surface and let (M, S) be a polarized Hodge module of weight w. Then the underlying $\mathcal{D}_X$-module $\mathcal{M}$ is semi-simple. Furthermore, any simple component $\mathcal{M}_\alpha$ of $\mathcal{M}$ underlies a unique (up to equivalence) polarized Hodge module $(M_\alpha, \mathrm{S}_\alpha)$ of the same weight w and there exists* *a polarized Hodge structure* $(H^o_\alpha, \mathrm{S}^o_\alpha)$ *of weight* 0 *such that* $(M, \mathrm{S}) \simeq \bigoplus_\alpha((H^o_\alpha, \mathrm{S}^o_\alpha) \otimes (M_\alpha, \mathrm{S}_\alpha))$.

(See Section 4.3.c for the notion of equivalence.) Let us start by describing the simple objects in the category of regular holonomic $\mathcal{D}_X$-modules on a Riemann surface (not necessarily compact).

7.5.2. Proposition. *Let X be a Riemann surface. A regular holonomic $\mathcal{D}_X$-module is simple if*

- *either $\mathcal{M}$ is supported on a point $x \in X$ and in a local coordinate t vanishing at x, $\mathcal{M} \simeq \mathbb{C}[\partial_t]$,*
- *or there exists a discrete subset $\Sigma \subset X$ and an irreducible bundle with connection $(\mathcal{V}, \nabla)$ on $X \smallsetminus \Sigma$ (i.e., such that the local system $\mathcal{V}^\nabla$ on $X \smallsetminus \Sigma$ is irreducible) such that $\mathcal{M} \simeq \mathcal{V}_{\mathrm{mid}}$.*

Proof. If $\mathcal{M}$ is supported on a point, then it is isomorphic to $(\mathrm{gr}_V^{-1}\mathcal{M})[\partial_t]$ (Exercise 7.7(2)), and simplicity implies $\dim \mathrm{gr}_V^{-1}\mathcal{M} = 1$. Otherwise, $\mathcal{M}$ has no submodule and no quotient module supported on a point. If Σ denotes the singular set of $\mathcal{M}$, then $\mathcal{M}_{|X\smallsetminus\Sigma} = \mathcal{V}$ is a holomorphic bundle with connection ∇ and $\mathcal{M} = \mathcal{V}_{\mathrm{mid}}$ (Definition 7.2.11). Simplicity of $\mathcal{M}$ implies simplicity of $\mathcal{V}$ (if $0 \neq \mathcal{V}_1 \subsetneq \mathcal{V}$, then $0 \neq \mathcal{V}_{1\mathrm{mid}} \subsetneq \mathcal{V}_{\mathrm{mid}}$), that is, irreducibility of $\mathcal{V}$. □

***Proof of Theorem* 7.5.1.** The S-decomposition theorem 7.4.10 already solves part of the problem: we can assume that M either is supported on a point or has pure support X. The first case is solved by Proposition 7.4.11. For the second case we use Corollary 7.4.14 to reduce to Corollary 6.4.2 and Theorem 6.14.17 in case X is compact. □

7.6. Numerical invariants of variations of $\mathbb{C}$-Hodge structure

Let X be a compact Riemann surface of genus g and let $(\mathcal{V}, F^\bullet, \mathrm{S})$ be a polarized variation of Hodge structure of weight w on $X \smallsetminus \Sigma$, for some finite subset $\Sigma \subset X$ (hence corresponding to a smooth Hodge triple of weight $w+1$). It can be extended as a polarized $\mathbb{C}$-Hodge module of weight $w+1$ with pure support X, whose underlying $\mathcal{D}_X$-module is $\mathcal{V}_{\mathrm{mid}}$. In this section, we relate the local and global numerical invariants attached to such data. The local numerical invariants are

- the Hodge numbers of the variation,
- and the Hodge numbers of the nearby and vanishing cycles at each point of Σ.

The global numerical invariants are

- the degrees of the Hodge bundles $F^p\mathcal{V}_{\mathrm{mid}}$,
- the Hodge numbers of the de Rham cohomology.

We also analyze the behaviour of some of these invariants under the tensor product operation.

7.6.a. Local Hodge numerical invariants. We consider the local setting $(\Delta, 0)$ of Section 6.2. We define a bunch of numbers attached to a polarizable variation $\mathbb{C}$-Hodge structure on Δ^* (Definitions 7.6.1 and 7.6.9).

7.6.1. Definition **(The local invariant h^p).** For a filtered holomorphic bundle $(\mathcal{V}, F^\bullet\mathcal{V})$ on Δ^*, we will set $h^p(\mathcal{V}) = h^p(\mathcal{V}, F^\bullet\mathcal{V}) = \operatorname{rk}\operatorname{gr}_F^p\mathcal{V}$.

For a variation of $\mathbb{C}$-Hodge structure, we thus have

$$h^p(\mathcal{V}) = \operatorname{rk}\mathcal{H}^{p,w-p}.$$

From the freeness of $F^p\mathcal{V}_*^\beta$ for every β we obtain:

$$h^p(\mathcal{V}) = \sum_{\lambda\in\mathbb{S}^1} h^p\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.2}$$

Nearby cycles. In the following, we will set $\nu_\lambda^p(\mathcal{V}) = h^p\psi_{t,\lambda}(\mathcal{V}_*) = h^p\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$ for $\lambda \in \mathbb{S}^1$. Note that the associated graded Hodge-Lefschetz structure has the same numbers $h^p(\operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_*)) = h^p(\psi_{t,\lambda}(\mathcal{V}_*))$. The Hodge filtration on $\operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}}) = \operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_*)$ splits with respect to the Lefschetz decomposition associated with N. The primitive part $\mathrm{P}_\ell\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$, equipped with the filtration induced by that on $\operatorname{gr}_\ell^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$ and a suitable polarization, is a polarizable $\mathbb{C}$-Hodge structure (Theorem 6.8.7). We can thus define the numbers

$$\nu_{\lambda,\ell}^p(\mathcal{V}_{\mathrm{mid}}) = \nu_{\lambda,\ell}^p(\mathcal{V}_*) := h^p\big(\mathrm{P}_\ell\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})\big) = \dim\operatorname{gr}_F^p\mathrm{P}_\ell\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}}).$$

According to the F-strictness of N and the Lefschetz decomposition of $\operatorname{gr}^{\mathrm{M}}\psi_{t,\lambda}(\mathcal{V}_{\mathrm{mid}})$, we have

$$\nu_\lambda^p(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0}\sum_{k=0}^{\ell} \nu_{\lambda,\ell}^{p+k}(\mathcal{V}_{\mathrm{mid}}), \tag{7.6.3}$$

and we set

$$\nu_{\lambda,\mathrm{prim}}^p(\mathcal{V}_{\mathrm{mid}}) := \sum_{\ell\geqslant 0}\nu_{\lambda,\ell}^p(\mathcal{V}_{\mathrm{mid}}), \quad \nu_{\lambda,\mathrm{coprim}}^p(\mathcal{V}_{\mathrm{mid}}) := \sum_{\ell\geqslant 0}\nu_{\lambda,\ell}^{p+\ell}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.4}$$

We have

$$\nu_\lambda^p(\mathcal{V}_{\mathrm{mid}}) - \nu_\lambda^{p-1}(\mathcal{V}_{\mathrm{mid}}) = \nu_{\lambda,\mathrm{coprim}}^p(\mathcal{V}_{\mathrm{mid}}) - \nu_{\lambda,\mathrm{prim}}^{p-1}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.5}$$

Vanishing cycles. For $\lambda \neq 1$, we set

$$\mu_{\lambda,\ell}^p(\mathcal{V}_{\mathrm{mid}}) = \nu_{\lambda,\ell}^p(\mathcal{V}_{\mathrm{mid}}) \quad \forall\, p.$$

Let us now focus on $\lambda = 1$. We have by definition

$$\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) = \operatorname{gr}^{-1}(\mathcal{V}_{\mathrm{mid}}).$$

On the other hand, the filtration $F^\bullet\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$ is defined so that we have natural morphisms

$$(\psi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet) \xrightarrow{\ \mathrm{can}\ } (\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet) \xhookrightarrow{\ \mathrm{var}\ } (\psi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet)(-1).$$

Since can is strictly onto and var is injective, $(\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N}, F^\bullet)$ is identified with $\operatorname{Im}\mathrm{N}$ together with the filtration $F^p\operatorname{Im}\mathrm{N} = \mathrm{N}(F^p)$. We also have, by definition of the Hodge filtration on $\mathcal{V}_{\mathrm{mid}}$,

$$F^p\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) = \frac{F^{p-1}\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_*^{-1}\mathcal{V}_{\mathrm{mid}}}{F^{p-1}\mathcal{V}_{\mathrm{mid}} \cap \mathcal{V}_*^{>-1}\mathcal{V}_{\mathrm{mid}}}.$$

For $\ell \geqslant 0$, we thus have

$$F^p\mathrm{P}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) = \mathrm{N}(F^p\mathrm{P}_{\ell+1}\psi_{t,1}(\mathcal{V}_{\mathrm{mid}})),$$

and therefore

$$\mu^p_{1,\ell}(\mathcal{V}_{\mathrm{mid}}) = \nu^p_{1,\ell+1}(\mathcal{V}_{\mathrm{mid}}).$$

From (7.6.4) and (7.6.5) we obtain:

$$\mu^p_1(\mathcal{V}_{\mathrm{mid}}) = \nu^p_1(\mathcal{V}_{\mathrm{mid}}) - \nu^p_{1,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}}) = \nu^{p-1}_1(\mathcal{V}_{\mathrm{mid}}) - \nu^{p-1}_{1,\mathrm{prim}}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.6}$$

Note that, using the Lefschetz decomposition for the graded pieces of the monodromy filtration of $(\phi_{t,1}(\mathcal{V}_{\mathrm{mid}}), \mathrm{N})$, we also have

$$\mu^p_1(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0}\sum_{k=0}^{\ell}\mu^{p+k}_{1,\ell}. \tag{7.6.7}$$

We will set, similarly to (7.6.4):

$$\mu^p_{\lambda,\mathrm{prim}}(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0}\mu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}), \quad \mu^p_{\lambda,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}}) = \sum_{\ell\geqslant 0}\mu^{p+\ell}_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}). \tag{7.6.8}$$

The various numbers that we already introduced are recovered from the following Hodge numbers. We use the notation $(\mathcal{V}_*, F^\bullet\mathcal{V}_*)$ and $(\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}})$ as above.

***7.6.9. Definition* (Local Hodge numerical invariants for $\mathcal{V}_{\mathrm{mid}}$)**

- $h^p(\mathcal{V})$,
- $\mu^p_{1,\ell}(\mathcal{V}_{\mathrm{mid}}) = \dim\mathrm{gr}^p_F\mathrm{P}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$, where $\mathrm{P}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$ denotes the primitive part of $\mathrm{gr}^{\mathrm{M}}_\ell\phi_{t,1}(\mathcal{V}_{\mathrm{mid}})$, and $\mu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}}) = \nu^p_{\lambda,\ell}(\mathcal{V}_{\mathrm{mid}})$ if $\lambda \neq 1$,
- $\mu^p_\lambda(\mathcal{V}_{\mathrm{mid}})$ given by (7.6.7) and $\mu^p(\mathcal{V}_{\mathrm{mid}}) = \sum_\lambda \mu^p_\lambda(\mathcal{V}_{\mathrm{mid}})$.

7.6.10. Remark. The data ν^p_1 are recovered from the data $\mu^p_\bullet$ together with $h^p(\mathcal{V})$:

$$\nu^p_{1,\ell}(\mathcal{V}_{\mathrm{mid}}) = \begin{cases} \mu^p_{1,\ell-1}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \ell \geqslant 1, \\ h^p(\mathcal{V}) - \mu^p(\mathcal{V}_{\mathrm{mid}}) - \mu^{p+1}_{1,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \ell = 0. \end{cases}$$

7.6.b. Example: twist with a unitary rank 1 local system. Let $\underline{\mathcal{L}}$ be a nontrivial unitary rank 1 local system on Δ^*, determined by its monodromy $\lambda_o \in \mathbb{S}^1 \smallsetminus \{1\}$, and let $(\mathcal{L}, \nabla)$ be the associated bundle with connection. We simply denote by $\mathcal{L}^\bullet$ the various Deligne extensions of $(\mathcal{L}, \nabla)$, and $\mathcal{L}_*$ is the meromorphic Deligne extension. It will be easier to work with $\mathcal{L}^0$ (i.e., $\beta = 0$). We set $\lambda_o = \exp(-2\pi\mathsf{i}\,\beta_o)$ with $\beta_o \in (0,1)$. Then, $\mathcal{L}^0 = \mathcal{L}^{\beta_o}$ and, for every $\beta \in \mathbb{R}$,

$$\mathcal{V}^\beta_* \otimes \mathcal{L}^0 = (\mathcal{V}_* \otimes \mathcal{L}_*)^{\beta+\beta_o} \subset (\mathcal{V}_* \otimes \mathcal{L}_*)^\beta.$$

On the other hand, the Hodge bundles on $\mathcal{V}\otimes\mathcal{L}$ are $F^p\mathcal{V}\otimes\mathcal{L}$ so that, by Schmid's procedure, for every β,

$$F^p(\mathcal{V}_*\otimes\mathcal{L}_*)^\beta := j_*(F^p\mathcal{V}\otimes\mathcal{L})\cap(\mathcal{V}_*\otimes\mathcal{L}_*)^\beta$$

(intersection taken in $\mathcal{V}_*\otimes\mathcal{L}_*$) is a bundle, and we have a mixed Hodge structure by inducing $F^p(\mathcal{V}_*\otimes\mathcal{L}_*)^\beta$ on $\mathrm{gr}^\beta_{\mathcal{V}}(\mathcal{V}_*\otimes\mathcal{L}_*)$. We claim that

$$F^p\mathcal{V}^\beta_*\otimes\mathcal{L}^0 = F^p(\mathcal{V}_*\otimes\mathcal{L}_*)^{\beta+\beta_o}. \tag{7.6.11}$$

This amounts to showing

$$(j_*F^p\cap\mathcal{V}^\beta_*)\otimes\mathcal{L}^0 = j_*(F^p\mathcal{H}\otimes L)\cap(\mathcal{V}_*\otimes\mathcal{L}_*)^{\beta+\beta_o},$$

intersection taken in $\mathcal{V}_*\otimes\mathcal{L}_*$. The inclusion $\subset$ is clear, and the equality is shown by working with a local basis of $\mathcal{L}^0$, which can also serve as a basis for L and $\mathcal{L}_*$.

We deduce:

$$h^p(\mathcal{V}\otimes\mathcal{L}) = h^p(\mathcal{V}), \tag{7.6.12}$$

$$\nu^p_{\lambda,\ell}(\mathcal{V}_*\otimes\mathcal{L}_*) = \nu^p_{\lambda/\lambda_o,\ell}(\mathcal{V}_*) \tag{7.6.13}$$

$$\mu^p_{\lambda,\ell}((\mathcal{V}_*\otimes\mathcal{L}_*)_{\mathrm{mid}}) = \begin{cases} \mu^p_{\lambda/\lambda_o,\ell}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \lambda\neq 1,\lambda_o,\\ \mu^p_{1/\lambda_o,\ell+1}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \lambda=1,\\ \mu^p_{1,\ell-1}(\mathcal{V}_{\mathrm{mid}}) & \text{if } \lambda=\lambda_o \text{ and } \ell\geqslant 1,\\ \left.\begin{matrix} h^p(\mathcal{V})-\mu^p(\mathcal{V}_{\mathrm{mid}})\\ -\mu^{p+1}_{1,\mathrm{coprim}}(\mathcal{V}_{\mathrm{mid}})\end{matrix}\right\} & \text{if } \lambda=\lambda_o \text{ and } \ell=0.\end{cases} \tag{7.6.14}$$

7.6.c. Hodge numerical invariants for a variation on X^*.

Assume now that X is a compact Riemann surface and let Σ be the finite set of points in X complementary to X^*. Let $(\mathcal{V},F'^\bullet\mathcal{V},F''^\bullet\mathcal{V},\nabla)$ be a polarizable variation of $\mathbb{C}$-Hodge structure on $X^*=X\smallsetminus\Sigma$. Together with the local Hodge numerical invariants at each $x\in\Sigma$ we consider the following global Hodge numbers. We consider for every p the Hodge bundle $\mathrm{gr}^p_F\mathcal{V}^0_*=\mathrm{gr}^p_F\mathcal{V}^0_{\mathrm{mid}}$, whose rank is $h^p(\mathcal{V})$.

***7.6.15. Caveat* (Apparent singular points).** A polarizable variation $(\mathcal{V},F'^\bullet\mathcal{V},F''^\bullet\mathcal{V},\nabla)$ can extend smoothly at some $x\in\Sigma$. In such a case, all vanishing cycle numbers μ at x vanish, as well as all nearby cycle numbers $\nu^p_{\lambda,\ell}$ for $\lambda\neq 1$ or $\lambda=1$ and $\ell\leqslant 0$. There only remains $\nu^p_{1,0}$ at x, which is nothing but h^p.

***7.6.16. Definition* (Degree of the Hodge bundles).** For every p, we set

$$\delta^p(\mathcal{V}) = \deg\mathrm{gr}^p_F\mathcal{V}^0_*.$$

7.6.d. Example: degree of the Hodge bundles for a tensor product

Let $(\mathcal{L},\nabla)$ be the holomorphic *line bundle* with connection associated to a unitary rank 1 local system on X^*. (Up to adding apparent singular points as introduced in 7.6.15, we can assume that $\mathcal{L}$ and $\mathcal{V}$ are defined on the same open set X^*.) We denote by $\alpha_x\in[0,1)$ the residue of the connection $(\mathcal{L}^0,\nabla)$ at x, so that $\deg\mathcal{L}^0=-\sum_{x\in\Sigma}\alpha_x$, and $\alpha_x=0$ if and only if x is an apparent singular point for $\mathcal{L}$. We now denote by

$\nu^p_{x,\lambda}(\mathcal{V})$ etc. the local Hodge numbers of $\mathcal{V}$ at x whenever $\lambda\neq 1$, and for $\boldsymbol{\beta}=(\beta_x)_{x\in\Sigma}$ we denote by $\mathcal{V}^{\boldsymbol{\beta}}_*$ the extension of $\mathcal{V}$ equal to $\mathcal{V}^{\beta_x}_*$ near x.

7.6.17. Proposition. *With the notation as above, we have*

$$\delta^p(\mathcal{V}\otimes\mathcal{L})=\delta^p(\mathcal{V})+h^p(\mathcal{V})\deg\mathcal{L}^0+\sum_{x\in\Sigma}\sum_{\substack{\beta\in[-\alpha_x,0)\\ \lambda=\exp(-2\pi\mathrm{i}\,\beta)}}\nu^p_{x,\lambda}(\mathcal{V}).$$

(See Exercise 7.26 for the general formula (7.26 $*$) when $\operatorname{rk}\mathcal{L}>1$.)

Proof. We deduce from (7.6.11) (at each $x\in\Sigma$) that

$$\begin{aligned}\delta^p(\mathcal{V}\otimes\mathcal{L})&=\deg\operatorname{gr}^p_F(\mathcal{V}\otimes\mathcal{L})^0=\deg\big[(\operatorname{gr}^p_F\mathcal{V}^{-\boldsymbol{\alpha}}_*)\otimes\mathcal{L}^0\big]\quad\text{after (7.6.11)}\\&=\deg\operatorname{gr}^p_F\mathcal{V}^{-\boldsymbol{\alpha}}_*+h^p(\mathcal{V})\deg\mathcal{L}^0\\&=\delta^p(\mathcal{V})+h^p(\mathcal{V})\deg\mathcal{L}^0+\sum_{x\in\Sigma}\sum_{\substack{\beta\in[-\alpha_x,0)\\ \lambda=\exp(-2\pi\mathrm{i}\,\beta)}}\nu^p_{x,\lambda}(\mathcal{V}).\end{aligned}\qquad\square$$

7.6.e. Hodge numbers of the de Rham cohomology. Let $\mathcal{V}_{\mathrm{mid}}$ denote the middle extension of $\mathcal{V}_*$ at each of the singular points $x\in\Sigma$ and let $F^\bullet\mathcal{V}_{\mathrm{mid}}$ be the extended Hodge filtration as in (6.7.1) and (6.14.1). The de Rham complex $\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}$ is filtered by

$$F^p\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}=\{0\longrightarrow F^p\mathcal{V}_{\mathrm{mid}}\longrightarrow\Omega^1_{\mathbb{P}^1}\otimes F^{p-1}\mathcal{V}_{\mathrm{mid}}\longrightarrow 0\},$$

and this induces a filtration on the hypercohomology $\boldsymbol{H}^\bullet(X,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})=H^\bullet(X,j_*\underline{\mathcal{H}})$, where $j:X^*\hookrightarrow X$ denotes the open inclusion. By the Hodge-Zucker Theorem 6.11.1, $F^\bullet H^k(X,j_*\underline{\mathcal{H}})$ underlies a polarizable $\mathbb{C}$-Hodge structure. Note that, if $\underline{\mathcal{H}}$ is irreducible and non constant, then $H^k(X,j_*\underline{\mathcal{H}})=0$ for $k\neq 1$. Let $g=g(X)$ denote the genus of X.

7.6.18. Proposition. *Assume that $\underline{\mathcal{H}}$ is irreducible and non constant. Then*

$$\begin{aligned}(7.6.18\,*)\quad h^p(H^1(X,j_*\underline{\mathcal{H}}))&=\delta^{p-1}(\mathcal{V})-\delta^p(\mathcal{V})+(h^{p-1}(\mathcal{V})+h^p(\mathcal{V}))(g-1)\\&\qquad+\sum_{x\in\Sigma}^{r}\big(\nu^{p-1}_{x,\neq1}(\mathcal{V})+\mu^p_{x,1}(\mathcal{V}_{\mathrm{mid}})\big).\end{aligned}$$

Proof. It follows from Proposition 6.14.8 that the inclusion of the filtered subcomplex

$$F^\bullet\mathcal{V}^0_*\,\mathrm{DR}\,\mathcal{V}_{\mathrm{mid}}:=\{0\longrightarrow F^\bullet\mathcal{V}^0_{\mathrm{mid}}\longrightarrow\Omega^1_X\otimes F^{\bullet-1}\mathcal{V}^{-1}_{\mathrm{mid}}\longrightarrow 0\}$$

into the filtered de Rham complex is a filtered quasi-isomorphism. By the degeneration at E_1 of the Hodge-to-de Rham spectral sequence (see Remark 6.14.16(2)), we conclude that

$$\begin{aligned}
-h^p(H^1(X, j_*\underline{\mathcal{H}})) &= \chi\big(\mathrm{gr}_F^p \boldsymbol{H}^\bullet(X, \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\big) \text{ (irreducibility and non-constancy of } \underline{\mathcal{H}})\\
&= \chi\big(\boldsymbol{H}^\bullet(X, \mathrm{gr}_F^p \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\big) \text{ (degeneration at } E_1)\\
&= \chi\big(\boldsymbol{H}^\bullet(X, \mathrm{gr}_F^p \mathcal{V}_*^0 \mathrm{DR}\,\mathcal{V}_{\mathrm{mid}})\big) \quad \text{(after Proposition 6.14.8)}\\
&= \chi\big(H^\bullet(X, \mathrm{gr}_F^p \mathcal{V}_{\mathrm{mid}}^0)\big) - \chi\big(H^\bullet(X, \Omega_X^1 \otimes \mathrm{gr}_F^{p-1}\mathcal{V}_{\mathrm{mid}}^{-1})\big)\\
&\qquad\qquad\qquad (\mathcal{O}\text{-linearity of the differential})\\
&= \delta^p(\mathcal{V}) - \deg(\Omega_X^1 \otimes \mathrm{gr}_F^{p-1}\mathcal{V}_{\mathrm{mid}}^{-1}) + (h^p(\mathcal{V}) - h^{p-1}(\mathcal{V}))(1-g)\\
&\qquad\qquad\qquad \text{(Riemann-Roch)}\\
&= \delta^p(\mathcal{V}) - \deg(\mathrm{gr}_F^{p-1}\mathcal{V}_{\mathrm{mid}}^{-1}) + (h^p(\mathcal{V}) + h^{p-1}(\mathcal{V}))(1-g).
\end{aligned}$$

We now have

$$\begin{aligned}
\deg(\mathrm{gr}_F^{p-1}\mathcal{V}_{\mathrm{mid}}^{-1}) &= \delta^{p-1}(\mathcal{V}) + \dim \mathrm{gr}_F^{p-1}(\mathcal{V}_{\mathrm{mid}}^{-1}/\mathcal{H}_{\mathrm{mid}}^0)\\
&= \delta^{p-1}(\mathcal{V}) + \textstyle\sum_{\beta\in[-1,0)} \dim \mathrm{gr}_F^{p-1}(\mathrm{gr}^\beta \mathcal{V}_{\mathrm{mid}})\\
&= \delta^{p-1}(\mathcal{V}) + \textstyle\sum_{x\in\Sigma}^r \big(\nu_{x,\neq 1}^{p-1}(\mathcal{V}) + \mu_{x,1}^p(\mathcal{V}_{\mathrm{mid}})\big). \qquad \square
\end{aligned}$$

7.7. Exercises

***Exercise 7.1* (The Rees module).** The properties of a coherent filtration (Section 7.2.a) can be expressed in a simpler way by adding a dummy variable. Let M be a left $\mathcal{D}$-module and let $F_\bullet M$ be an F-filtration of M. Let z be such a variable and let us set $R_F\mathcal{D} = \bigoplus_{k\in\mathbb{Z}} F_k\mathcal{D}\cdot z^k$ and $R_F M = \bigoplus_{k\in\mathbb{Z}} F_k M\cdot z^k$.

(1) Prove that $R_F\mathcal{D}$ is a Noetherian ring.

(2) Prove that $R_F M$ has no $\mathbb{C}[z]$-torsion.

(3) Prove that the F-filtration condition is equivalent to: *$R_F M$ is a left graded $R_F\mathcal{D}$-module.*

(4) Prove that $R_F M/zR_F M = \mathrm{gr}^F M$ and $R_F M/(z-1)R_F M = M$.

(5) Prove that the coherence of $F_\bullet M$ is equivalent to: *$R_F M$ is a finitely generated left $R_F\mathcal{D}$-module.*

(6) Prove that M has a coherent F-filtration if and only if it is finitely generated.

Exercise 7.2.

(1) Check that the V-order of an operator $P\in\mathcal{D}$ does not depend on the way we write its monomials (due to the non-commutativity of $\mathcal{D}$).

(2) Check that each $V_k\mathcal{D}$ is a free $\mathcal{O}$-module, and that, for $k\leqslant 0$, $V_k\mathcal{D} = t^{-k}V_0\mathcal{D}$.

(3) Check that the filtration by the V-order is compatible with the product, and more precisely that

$$V_k\mathcal{D}\cdot V_\ell\mathcal{D} \begin{cases} \subset V_{k+\ell}\mathcal{D} & \text{for every } k,\ell\in\mathbb{Z},\\ = V_{k+\ell}\mathcal{D} & \text{if } k,\ell\leqslant 0 \text{ or if } k,\ell\geqslant 0.\end{cases}$$

Conclude that $V_0\mathcal{D}$ is a ring and that each $V_k\mathcal{D}$ is a left and right $V_0\mathcal{D}$-module.

(4) Check that the Rees object $R_V\mathcal{D} := \bigoplus_{k\in\mathbb{Z}} V_k\mathcal{D}\cdot v^k$ is a Noetherian ring.

(5) Show that $\mathrm{gr}_0^V\mathcal{D}$ can be identified with the polynomial ring $\mathbb{C}[\mathrm{E}]$, where E is the class of $t\partial_t$ in $\mathrm{gr}_0^V\mathcal{D}$.

(6) Show that E does not depend on the choice of the coordinate t on the disc.

Exercise 7.3.

(1) Show that a filtration $U^\bullet M$ is a V-filtration if and only if the Rees object $R_U M := \bigoplus_{k\in\mathbb{Z}} U^k M v^{-k}$ is naturally a left graded $R_V\mathcal{D}$-module.

(2) Show that, for every V-filtration $U^\bullet M$ on M, $R_U M/vR_U M = \mathrm{gr}^U M$ and $R_U M/(v-1)R_U M = M$.

(3) Show that any finitely generated $\mathcal{D}$-module has a coherent V-filtration.

(4) Show that a V-filtration is coherent if and only if the Rees module $R_U M$ is finitely generated over $R_V\mathcal{D}$. Conclude that if M' is a submodule of M, then a coherent V-filtration $U^\bullet M$ induces a coherent V-filtration $U^\bullet M' := M' \cap U^\bullet M$. [*Hint*: Use Artin-Rees lemma.]

(5) Show that, if M is holonomic, then for any coherent V-filtration the graded spaces $\mathrm{gr}_U^k M$ are finite dimensional $\mathbb{C}$-vector spaces equipped with a linear action of E. [*Hint*: Prove the result for holonomic $\mathcal{D}$-modules of the form $\mathcal{D}/(P)$, where (P) is the left ideal generated by $P \in \mathcal{D} \smallsetminus \{0\}$; conclude by using the property that any holonomic $\mathcal{D}$-module is a successive extension of such modules together with (4).]

(6) Show that, if $U^\bullet M$ is a V-filtration of M, then the left multiplication by t induces for every $k \in \mathbb{Z}$ a $\mathbb{C}$-linear homomorphism $\mathrm{gr}_U^k M \to \mathrm{gr}_U^{k+1} M$ and that the action of ∂_t induces $\mathrm{gr}_U^k M \to \mathrm{gr}_U^{k-1} M$. How does E commute with these morphisms?

(7) Show that if a V-filtration is coherent, then $t : U^k M \to U^{k+1} M$ is an isomorphism for every $k \gg 0$ and $\partial_t : \mathrm{gr}_U^k M \to \mathrm{gr}_U^{k-1} M$ is so for every $k \ll 0$.

***Exercise 7.4* (V-strictness of morphisms).** Show that any morphism $\varphi : M \to M'$ between holonomic $\mathcal{D}$-modules is V-strict, i.e., satisfies $\varphi(V^k M) = \varphi(M) \cap V^k M'$ for every $k \in \mathbb{Z}$. [*Hint*: Show that the right-hand side defines a coherent filtration of $\varphi(M)$ and use the uniqueness of the Kashiwara-Malgrange filtration.]

Exercise 7.5. Show that the Kashiwara-Malgrange filtration satisfies the following properties:

(1) for every $k \geqslant 0$, the morphism $V^k M \to V^{k+1} M$ induced by t is an isomorphism;

(2) for every $k \geqslant 0$, the morphism $\mathrm{gr}_V^{-1-k} M \to \mathrm{gr}_V^{-2-k} M$ induced by ∂_t is an isomorphism.

Exercise 7.6. Show that, for any holonomic module M, the module $M[t^{-1}] := \mathcal{O}[t^{-1}] \otimes_{\mathcal{O}} M$ is still holonomic and is a finite dimensional vector space over the field of Laurent series $\mathcal{O}[t^{-1}]$, equipped with a connection. Show that its Kashiwara-Malgrange filtration satisfies $V^k M[t^{-1}] = t^k V^0 M[t^{-1}]$ for every $k \in \mathbb{Z}$ (while this only holds for $k \geqslant 0$ for a general holonomic $\mathcal{D}$-module. Conversely, prove that any finite dimensional vector space $(\mathcal{V}_*, \nabla)$ over the field of Laurent series $\mathcal{O}[t^{-1}]$ equipped with a connection is a holonomic $\mathcal{D}$-module.

Exercise 7.7 **($\mathcal{D}$-modules with support the origin).** Let M be a finitely generated left $\mathcal{D}$-module with support the origin, i.e., each element is annihilated by some power of t (hence M is holonomic). Show that

(1) $V^\beta M = 0$ for $\beta > -1$ and $\mathrm{gr}_V^\beta M = 0$ for $\beta \notin -\mathbb{N}^*$,

(2) $M \simeq (\mathrm{gr}_V^{-1} M)[\partial_t]$, where the action of $\mathcal{D}$ on the right-hand side is given by

$$\begin{aligned}\partial_t \cdot m\partial_t^k &= m\partial_t^{k+1},\\ t \cdot m\partial_t^k &= -km\partial_t^{k-1},\end{aligned}$$

(3) M has also the structure of a *right* $\mathcal{D}$-module (denoted by M^{right} in Section 8.2) by setting

$$\begin{aligned}m\partial_t^k \cdot \partial_t &= m\partial_t^{k+1},\\ m\partial_t^k \cdot t &= km\partial_t^{k-1}.\end{aligned}$$

Exercise 7.8 **(V-strictness of morphisms).** Show the V-strictness of morphisms for the V-filtration indexed by $\mathbb{R}$ (see Exercise 7.4).

Exercise 7.9. Let $M \neq 0$ be a holonomic $\mathcal{D}$-module. One can assume for simplicity that M is *regular* and use Proposition 7.2.10. Prove that

(1) the construction of $\mathrm{gr}_V^\beta M$, $\mathrm{gr}_V^{-1} M$, can, var, N, is functorial with respect to M, and gr_V^β is an exact functor (i.e., compatible with short exact sequences);

(2) $M \neq 0$ is supported at the origin if and only if $\mathrm{gr}_V^\beta M = 0$ for every $\beta > -1$ and $\mathrm{gr}_V^{-1} M \neq 0$;

(3) can is onto iff M has no nonzero quotient supported at the origin (i.e., there is no surjective morphism $M \to N \neq 0$ where each element of N is annihilated by some power of t);

(4) var is injective if and only if M has no nonzero submodule supported at the origin (i.e., whose elements are annihilated by some power of t);

(5) $\mathrm{gr}_V^{-1} M = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$ if and only if $M = M_1 \oplus M_2$, where M_2 is supported at the origin and M_1 has neither a nonzero quotient nor a nonzero submodule supported at the origin (in such a case, we say that M is *S(upport)-decomposable*).

Exercise 7.10.

(1) Show that the Kashiwara-Malgrange filtration of $M[t^{-1}]$ satisfies

- (a) $V^{>-1} M[t^{-1}] = V^{>-1} M$,
- (b) $V^{\beta+k} M[t^{-1}] = t^k V^\beta M[t^{-1}]$ for all $k \in \mathbb{Z}$.

(2) Show that M is a middle extension $\mathcal{D}$-module if and only if M is equal to the $\mathcal{D}$-submodule of $M[t^{-1}]$ generated by $V^{>-1} M$.

(3) Show that the Kashiwara-Malgrange filtration of a middle extension $\mathcal{D}$-module M satisfies, for $\beta \in (-1, 0]$ and $k \geqslant 1$,

$$V^{\beta-k} M = \partial_t^k V^\beta M + \sum_{j=0}^{k-1} \partial_t^j V^{>-1} M.$$

[*Hint*: Check this first with $\beta = 0$ and $k = 1$.]

Exercise 7.11. Prove that, if M has finite type over $\mathcal{D}$ and is supported at the origin, then M has a regular singularity at the origin.

Exercise 7.12. Let M_1, M_2 be holonomic $\mathcal{D}$-modules. Let $\varphi : M_1 \to M_2$ be a $\mathcal{D}$-linear morphism. Show that, if M_1 is a middle extension, then φ is zero as soon as the induced morphism $M_1[t^{-1}] \to M_2[t^{-1}]$. [*Hint*: If $\varphi = 0$ on $M_1[t^{-1}]$, show first that φ is zero on $V^{>-1}M_1$ because $V^{>-1}M_2$ is $\mathcal{O}$-free, and then use Exercise 7.10(2).]

Exercise 7.13. Let $\varphi : M_1 \to M_2$ be a morphism between *regular* holonomic $\mathcal{D}$-modules. Show that

(1) φ is an isomorphism if and only if $\mathrm{gr}_V^\beta \varphi$ is an isomorphism for any β having real part in $[-1, 0]$. [*Hint*: use the isomorphism $M \simeq \mathbb{C}\{t\} \otimes_{\mathbb{C}[t]} M^{\mathrm{alg}}$.]

(2) $\varphi = 0$ if and only if $\mathrm{gr}_V^\beta \varphi = 0$ for any such β.

Exercise 7.14. Show that any filtered holonomic $\mathcal{D}_X$-module supported at the origin, and which is strictly $\mathbb{R}$-specializable there, is of the form ${}_{\mathrm{D}}\iota_*(\mathcal{H}, F^\bullet\mathcal{H})$. [*Hint*: Use the relation 7.2.19(b).]

Exercise 7.15. The purpose of this exercise is to show that, if $(M, F_\bullet M)$ is a filtered holonomic $\mathcal{D}$-module which is strictly $\mathbb{R}$-specializable at the origin, then it is regular (Proposition 7.2.20), i.e., any $V^\beta M$ has finite type over $\mathcal{O}$.

(1) Show that $F^pV^\beta M := F^pM \cap V^\beta M$ has finite type over $\mathcal{O}$ for any p, β.

(2) Show that it is enough to prove the property for some β, that we now fix > -1.

(3) Show that, for p small enough, the filtration $F^p(V^\beta M/V^{\beta-1}M)$ is stationary.

(4) Deduce from strict $\mathbb{R}$-specializability that

$$F^pV^\beta M/tF^pV^\beta M = F^{p-1}V^\beta M/tF^{p-1}V^\beta M$$

for p small enough.

(5) Use Nakayama's lemma to conclude that, $\mathrm{gr}_F^p V^\beta M = 0$ for $p \leqslant p_o$ small enough, and thus $V^\beta M = F^{p_o}V^\beta M$.

Exercise 7.16. Let $(M_i, F^\bullet M_i)$ $(i = 1, 2)$ be as in Exercise 7.15. Let

$$\varphi : (M_1, F^\bullet M_1) \longrightarrow (M_2, F^\bullet M_2)$$

be a morphism. Show that φ is a *strict* isomorphism if and only if $\mathrm{gr}_V^\beta \varphi$ is a strict isomorphism for any β having real part in $[-1, 0]$. For the direction $\Leftarrow$:

(a) Show that $\varphi : M_1 \to M_2$ is an isomorphism. [*Hint*: Use Exercise 7.13.]

(b) Show that $\mathrm{gr}_V^\beta \varphi$ is a strict isomorphism for any β.

(c) If $\beta > -1$, show that $F^pV^\beta M_2 = \varphi(F^pV^\beta M_1) + tF^pV^\beta M_2$ and conclude that $F^pV^\beta M_2 = \varphi(F^pV^\beta M_1)$ by Nakayama's lemma. [*Hint*: Use 7.2.19(a).]

(d) Deduce that $F^pV^\beta M_2 = \varphi(F^pV^\beta M_1)$ for any β. [*Hint*: Use 7.2.19(b).]

Exercise 7.17. With respect to (7.2.17), show that $\mathrm{N}\cdot F^p\mathrm{gr}_V^\beta M \subset F^{p-1}\mathrm{gr}_V^\beta M$ for every $\beta \in \mathbb{R}$ and that

$$\mathrm{can}(F^p\mathrm{gr}_V^0 M) \subset F^{p-1}\mathrm{gr}_V^{-1} M, \quad \mathrm{var}(F^p\mathrm{gr}_V^{-1} M) \subset F^p\mathrm{gr}_V^0 M.$$

***Exercise 7.18* (Invariance by Tate twist).** Show that (see (7.2.17))
$$\big(\psi_{t,\lambda}(\mathcal{M},F^\bullet)\big)(k)=\psi_{t,\lambda}\big((\mathcal{M},F^\bullet)(k)\big),\quad \big(\phi_{t,1}(\mathcal{M},F^\bullet)\big)(k)=\phi_{t,1}\big((\mathcal{M},F^\bullet)(k)\big).$$

Exercise 7.19. Let δ_0 be the Dirac distribution, defined by
$$\langle\eta(t)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t,\delta_0\rangle=\eta(0).$$
Using that $1/t$ and $1/\bar t$ are in $L^1_{\mathrm{loc}}(\Delta)$, and Cauchy's formula, show the formulas:
$$\partial_t\mathrm{L}(t)=-1/t,\quad \partial_{\bar t}\mathrm{L}(t)=-1/\bar t,\quad \partial_t\partial_{\bar t}\mathrm{L}(t)=-\delta_0$$
as distributions on Δ.

***Exercise 7.20* (Fourier transform with a complex variable).** Set $\tau=(\xi+i\eta)/\sqrt2$ and $t=(x+iy)/\sqrt2$. We denote by $\mathcal{F}$ the Fourier transform with kernel
$$e^{\overline{t\tau}-t\tau}\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau}=\frac{1}{2\pi}\,e^{-\mathrm{i}(\xi y+\eta x)}\,\mathrm{d}\xi\wedge\mathrm{d}\eta.$$
Show that the inverse Fourier transform $\mathcal{F}^{-1}$ has kernel
$$e^{-\overline{t\tau}+t\tau}\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\bar t=\frac{1}{2\pi}\,e^{\mathrm{i}(x\eta+y\xi)}\,\mathrm{d}x\wedge\mathrm{d}y.$$
[*Hint*: Show that the assertion holds up to sign (i.e., orientation) by using the standard result on $\mathbb{R}^2$; to fix the sign, show that $\mathcal{F}(e^{-|\tau|^2})=e^{-|t|^2}$ and $\overline{\mathcal{F}}(e^{-|t|^2})=e^{-|\tau|^2}$.]

***Exercise 7.21* (The function $I_{\widehat\chi}$).** The functions $I_{\widehat\chi,k,\ell}$ are defined by the formula
$$I_{\widehat\chi,k,\ell}(t,s)=\int e^{\overline{t\tau}-t\tau}\tau^{-k}\overline{\tau}^{-\ell}\,|\tau|^{-2(s+1)}\,\widehat\chi(\tau)\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau},$$
and we set $I_{\widehat\chi}(s,t):=I_{\widehat\chi,0,0}(t,s)$.

(1) Show that if $\operatorname{Re}s>0$, $I_{\widehat\chi}(t,s)$ is continuous with respect to t and holomorphic with respect to s. [*Hint*: Notice that the exponent $\bar t/\overline\theta-t/\theta$ is purely imaginary; use polar coordinates $\theta=\varrho e^{i\vartheta}$ and write $|\theta|^{2(s-1)}\,\frac{\mathrm{i}}{2\pi}\,\mathrm{d}\theta\wedge\mathrm{d}\overline\theta$ as $\varrho^{2s}\frac1\pi\,(\mathrm{d}\varrho/\varrho)\wedge\mathrm{d}\vartheta$ and conclude.]

(2) Deduce that $I_{\widehat\chi}(0,s)$ extends meromorphically on $\operatorname{Re}s>-\varepsilon$ with a simple pole at $s=0$, and $\operatorname{Res}_{s=0}I_{\widehat\chi}(0,s)=1$. [*Hint*: Write the integral in terms of the variable θ and use Exercise 6.13(1).]

(3) For any $p\in\mathbb{N}$, show that $I_{\widehat\chi}$, when restricted to the domain $2\operatorname{Re}s>p$, is C^p in t and holomorphic with respect to s.

(4) By using Stokes formula, show that, for $\operatorname{Re}s$ large enough, the following identities hold:
$$\begin{aligned}tI_{\widehat\chi,k-1,\ell}(t,s)&=-(s+k)I_{\widehat\chi,k,\ell}(t,s)-I_{\partial\widehat\chi/\partial\theta,k+1,\ell}(t,s)\\ \bar tI_{\widehat\chi,k,\ell-1}(t,s)&=-(s+\ell)I_{\widehat\chi,k,\ell}(t,s)-I_{\partial\widehat\chi/\partial\overline\theta,k,\ell+1}(t,s),\end{aligned}$$
with $I_{\partial\widehat\chi/\partial\theta,k+1,\ell},I_{\partial\widehat\chi/\partial\overline\theta,k,\ell+1}\in C^\infty(\mathbb{C}\times\mathbb{C})$, holomorphic with respect to $s\in\mathbb{C}$.

(5) In particular, deduce that
$$|t|^2I_{\widehat\chi}(t,s-1)=-s^2I_{\widehat\chi}(t,s)+\cdots,$$
where "$\cdots$" is C^∞ in (t,s) and holomorphic with respect to $s\in\mathbb{C}$. This equality holds on $\operatorname{Re}s>1$.

(6) Conclude that $I_{\widehat{\chi}}$ can be extended as a C^∞ function on $\{t \neq 0\} \times \mathbb{C}$, holomorphic with respect to s.

(7) For $\operatorname{Re} s > 1$, show that

$$\partial_t I_{\widehat{\chi}}(t,s) = -I_{\widehat{\chi},-1,0}(t,s) \quad \text{and} \quad \partial_{\bar{t}} I_{\widehat{\chi}}(t,s) = -I_{\widehat{\chi},0,-1}(t,s),$$

and deduce

$$t\partial_t I_{\widehat{\chi}} = sI_{\widehat{\chi}} + I_{\partial\widehat{\chi}/\partial\theta,1,0} \quad \text{and} \quad \bar{t}\partial_{\bar{t}} I_{\widehat{\chi}} = sI_{\widehat{\chi}} + I_{\partial\widehat{\chi}/\partial\bar{\theta},0,1}.$$

(8) Show that, by analytic extension, these equalities hold on $\{t \neq 0\} \times \mathbb{C}$.

***Exercise 7.22* (The functions $\widehat{I}_{\chi,k,\ell}$).** Let $\chi(t)$ be a cut-off function near $t = 0$. For $k, \ell \in \mathbb{Z}$, we consider the functions

$$\widehat{I}_{\chi,k,\ell}(\tau,s) := \mathcal{F}^{-1}(|t|^{2s} t^k \bar{t}^\ell \chi(t)).$$

(1) Show that, for any $s \in \mathbb{C}$ with $\operatorname{Re}(s + 1 + (k+\ell)/2) > 0$, the function $(\tau, s) \mapsto \widehat{I}_{\chi,k,\ell}(\tau,s)$ is C^∞, depends holomorphically on s, and satisfies

$$\lim_{\tau\to\infty} \widehat{I}_{\chi,k,\ell}(\tau,s) = 0$$

locally uniformly with respect to s [*Hint*: apply the classical Riemann-Lebesgue lemma saying that the Fourier transform of a function in L^1 is continuous and tends to 0 at infinity.]

(2) Show that

$$(7.7.0\,*)\qquad \begin{aligned} \tau\widehat{I}_{\chi,k,\ell} &= -(s+k)\widehat{I}_{\chi,k-1,\ell} - \widehat{I}_{\partial\chi/\partial t,k,\ell} \qquad & \partial_\tau \widehat{I}_{\chi,k,\ell} &= \widehat{I}_{\chi,k+1,\ell} \\ \bar{\tau}\widehat{I}_{\chi,k,\ell} &= -(s+\ell)\widehat{I}_{\chi,k,\ell-1} - \widehat{I}_{\partial\chi/\partial\bar{t},k,\ell} \qquad & \bar{\partial}_\tau \widehat{I}_{\chi,k,\ell} &= \widehat{I}_{\chi,k,\ell+1}, \end{aligned}$$

where the equalities hold on the common domain of definition (with respect to s) of the functions involved.

(3) Deduce that, for $\operatorname{Re}(s+1) + (k+\ell)/2 > 0$,

$$(7.7.0\,**)\qquad \begin{aligned} \tau\partial_\tau \widehat{I}_{\chi,k,\ell} &= -(s+k+1)\widehat{I}_{\chi,k,\ell} - \widehat{I}_{\partial\chi/\partial t,k+1,\ell}, \\ \bar{\tau}\partial_{\bar{\tau}} \widehat{I}_{\chi,k,\ell} &= -(s+\ell+1)\widehat{I}_{\chi,k,\ell} - \widehat{I}_{\partial\chi/\partial\bar{t},k,\ell+1}. \end{aligned}$$

(4) Show that the functions $\widehat{I}_{\partial\chi/\partial t,k,\ell}$ and $\widehat{I}_{\partial\chi/\partial\bar{t},k,\ell}$ are C^∞ on $\mathbb{P}^1 \times \mathbb{C}$, depend holomorphically on s, and are infinitely flat at $\tau = \infty$. [*Hint*: use that $t^k \bar{t}^\ell |t|^{2s} \partial_{t,\bar{t}}\chi$ is C^∞ in t with compact support, and holomorphic with respect to s, so that its Fourier transform is in the Schwartz class, holomorphically with respect to s.]

(5) Consider the variable $\theta = \tau^{-1}$ with corresponding derivation $\partial_\theta = -\tau^2\partial_\tau$, and write $\widehat{I}_{\chi,k,\ell}(\theta,s)$ the function $\widehat{I}_{\chi,k,\ell}$ in this variable. Show that, for any $p \geqslant 0$, any $s \in \mathbb{C}$ with $\operatorname{Re}(s+1+(k+\ell)/2) > p$, all derivatives up to order p of $\widehat{I}_{\chi,k,\ell}(\theta,s)$ with respect to θ tend to 0 when $\theta \to 0$, locally uniformly with respect to s. [*Hint*: Use (7.7.0 $**$) and (7.7.0 $*$).]

(6) Deduce that the function $\widehat{I}_{\chi,k,\ell}(\tau,s)$ extends as a function of class C^p on the set $\mathbb{P}^1 \times \{\operatorname{Re}(s+1+(k+\ell)/2) > p\}$, holomorphic with respect to s.

(7) Conclude that the function $\widehat{I}_{\chi,1,0}(\tau,s)$ is C^∞ in τ and holomorphic in s on $\mathbb{C}_\tau \times \{s \mid \operatorname{Re} s > -3/2\}$.

Exercise 7.23. Let M be a Hodge module of weight w with pure support the disc Δ and let $(\psi_{t,1}M, \mathrm{N})$ be the associated Hodge-Lefschetz structure with central weight w. Consider the associated Hodge-Lefschetz middle extension quiver (see Definition 3.4.7). Show that $\operatorname{Im}\mathrm{N}$ has underlying vector spaces $\phi_{t,1}\mathcal{M}', \phi_{t,1}\mathcal{M}''$, equipped with the filtration induced on $\phi_{t,1}\mathcal{M}$ as in (7.2.16).

Exercise 7.24. Same as Exercise 7.23 with polarization.

Exercise 7.25. Show that the sequence

$$0 \longrightarrow (\mathcal{V}_{\mathrm{mid}}, F^\bullet\mathcal{V}_{\mathrm{mid}}) \longrightarrow (\mathcal{V}_*, F^\bullet\mathcal{V}_*) \longrightarrow (C, F^\bullet C) \longrightarrow 0$$

is exact and strict, and that $(C, F^\bullet C)$ can be identified with the cokernel of the morphism $\operatorname{var} : \phi_{t,1}(\mathcal{V}_{\mathrm{mid}}) \to \psi_{t,1}(\mathcal{V}_{\mathrm{mid}})(-1)$ of mixed Hodge structures. Conclude in particular that

$$h^p(C) = 0, \quad \mu_1^p(C) = \dim \operatorname{gr}_F^p(C) = \nu_{1,\mathrm{prim}}^{p-1}(\mathcal{V}_{\mathrm{mid}}).$$

***Exercise 7.26* (Degree for a tensor product).** Assume that $\mathcal{V}_1, \mathcal{V}_2$ underlie polarizable variations of Hodge structure with Hodge filtration $F^\bullet\mathcal{V}_i$ $(i = 1, 2)$ on $X^* = X \smallsetminus \Sigma$, where X is a compact Riemann surface. Then $\mathcal{V} = \mathcal{V}_1 \otimes \mathcal{V}_2$ also underlies such a variation, with Hodge filtration $F^p\mathcal{V} = \sum_{p_1+p_2=p} F^{p_1}\mathcal{V}_1 \otimes F^{p_2}\mathcal{V}_2$. At each $x \in \Sigma$, set

$$\nu_x^p(\mathcal{V}_1, \mathcal{V}_2) := \sum_{p_1+p_2=p} \sum_{\substack{\lambda_j=\exp(-2\pi\mathrm{i}\,\beta_j)\\ \beta_i\in[0,1)\ (i=1,2)\\ \beta_1+\beta_2\geqslant 1}} \nu_{x,\lambda_1}^{p_1}(\mathcal{V}_1) \cdot \nu_{x,\lambda_2}^{p_2}(\mathcal{V}_2).$$

The aim of this exercise is to prove the formula

$$(7.26\,*)\quad \delta^p(\mathcal{V}_1 \otimes \mathcal{V}_2) = \sum_{p_1+p_2=p} \big(\delta^{p_1}(\mathcal{V}_1)h^{p_2}(\mathcal{V}_2) + h^{p_1}(\mathcal{V}_1)\delta^{p_2}(\mathcal{V}_2)\big) + \sum_{x\in\Sigma} \nu_x^p(\mathcal{V}_1, \mathcal{V}_2).$$

The question consists mainly in comparing $\mathcal{V}^0 = (\mathcal{V}_1 \otimes \mathcal{V}_2)^0$ equipped with the filtration $F^\bullet\mathcal{V}^0 = j_*F^\bullet\mathcal{V} \cap \mathcal{V}^0$, with $\mathcal{V}_1^0 \otimes \mathcal{V}_2^0$ equipped with the filtration

$$F^p(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) := \sum_{p_1+p_2=p} F^{p_1}\mathcal{V}_1^0 \otimes F^{p_2}\mathcal{V}_2^0,$$

and the first part of the exercise is local on Δ with coordinate t.

(1) Show that there are natural inclusions compatible with the F-filtrations

$$\big(\mathcal{V}^1, F^\bullet\mathcal{V}^1\big) \subset \big((\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), F^\bullet(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)\big) \subset \big(\mathcal{V}^0, F^\bullet\mathcal{V}^0\big).$$

(2) The aim of this question is to show that the inclusion $\big((\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), F^\bullet(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)\big) \subset \big(\mathcal{V}^0, F^\bullet\mathcal{V}^0\big)$ is *strict*, that is,

$$F^p(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) = F^p\mathcal{V}^0 \cap (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), \quad \forall\, p.$$

(a) By using Proposition 7.2.10 for $\mathcal{V}_{1*}, \mathcal{V}_{2*}$ and $\mathcal{V}_*$, show that

$$\mathcal{V}^0/(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) \simeq t^{-1} \cdot \bigoplus_{\substack{\beta_1,\beta_2\in[0,1)\\ \beta_1+\beta_2\geqslant 1}} \mathrm{gr}^{\beta_1}\mathcal{V}_1 \otimes \mathrm{gr}^{\beta_2}\mathcal{V}_2$$

and
$$(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t\mathcal{V}^0 \simeq \bigoplus_{\substack{\beta_1,\beta_2\in[0,1)\\ \beta_1+\beta_2<1}} \mathrm{gr}^{\beta_1}\mathcal{V}_1 \otimes \mathrm{gr}^{\beta_2}\mathcal{V}_2.$$

(b) Show that the natural composed morphism

$$(\mathcal{V}_1^0/t\mathcal{V}_1^0) \otimes (\mathcal{V}_2^0/t\mathcal{V}_2^0) = (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0) \twoheadrightarrow (\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)/t\mathcal{V}^0 \hookrightarrow \mathcal{V}^0/t\mathcal{V}^0$$

is compatible with the F-filtrations naturally induced on each quotient space.

(c) Filter its source and target with respect to the filtrations induced respectively by $\mathcal{V}_1^\bullet, \mathcal{V}_2^\bullet, \mathcal{V}^\bullet$, and induce on the graded spaces the F-filtrations in order to obtain an F-filtered morphism

$$\bigoplus_{\beta_1,\beta_2\in[0,1)} (\mathrm{gr}^{\beta_1}\mathcal{V}_1, F^\bullet) \otimes (\mathrm{gr}^{\beta_2}\mathcal{V}_2, F^\bullet) \longrightarrow \bigoplus_{\beta\in[0,1)} (\mathrm{gr}^{\beta}\mathcal{V}, F^\bullet).$$

(d) Show that the latter morphism is F-strict. [*Hint*: Use that it underlies a morphism of mixed Hodge structures.]

(e) Conclude that the morphism of (2b) is also F-strict as well as the natural morphism

$$(\mathcal{V}_1^0/t^k\mathcal{V}_1^0) \otimes (\mathcal{V}_2^0/t^k\mathcal{V}_2^0) \longrightarrow \mathcal{V}^0/t^k\mathcal{V}^0, \quad \forall\, k \geqslant 0.$$

(f) Set $\widehat{\mathcal{V}}^0 := \varprojlim_k \mathcal{V}^0/t^k\mathcal{V}^0$ and $F^p\widehat{\mathcal{V}}^0 := \varprojlim_k F^p(\mathcal{V}^0/t^k\mathcal{V}^0)$. Conclude from (2e) that the inclusion

$$\big((\widehat{\mathcal{V}}_1^0 \otimes \widehat{\mathcal{V}}_2^0), F^\bullet(\widehat{\mathcal{V}}_1^0 \otimes \widehat{\mathcal{V}}_2^0)\big) \hookrightarrow \big(\widehat{\mathcal{V}}^0, F^\bullet\widehat{\mathcal{V}}^0\big)$$

is strict.

(g) Show that $F^p\widehat{\mathcal{V}}^0 = \widehat{\mathcal{O}} \otimes F^p\mathcal{V}^0$. [*Hint*: Argue as in the end of the proof of Proposition 7.4.16.] By using that $\widehat{\mathcal{O}}$ is faithfully flat over $\mathcal{O}$, conclude that the inclusion

$$\big((\mathcal{V}_1^0 \otimes \mathcal{V}_2^0), F^\bullet(\mathcal{V}_1^0 \otimes \mathcal{V}_2^0)\big) \hookrightarrow \big(\mathcal{V}^0, F^\bullet\mathcal{V}^0\big)$$

is strict.

(3) Deduce from (2) that there exists for each p an injective morphism

$$\bigoplus_{p_1+p_2=p} \mathrm{gr}_F^{p_1}\mathcal{V}_1^0 \otimes \mathrm{gr}_F^{p_2}\mathcal{V}_2^0 \hookrightarrow \mathrm{gr}_F^p\mathcal{V}^0$$

whose cokernel is supported at the origin of Δ and has dimension

$$\dim \mathrm{gr}_F^p\Big(\frac{\mathcal{V}^0}{\mathcal{V}_1^0 \otimes \mathcal{V}_2^0}\Big).$$

(4) The aim of this question is to prove the equality

$$\dim \mathrm{gr}_F^p\Big(\frac{\mathcal{V}^0}{\mathcal{V}_1^0 \otimes \mathcal{V}_2^0}\Big) = \nu^p(\mathcal{V}_1, \mathcal{V}_2). \tag{7.26 $**$}$$

(a) Consider the F-filtered composed morphism

$$\mathcal{V}^1/\mathcal{V}^2 \longrightarrow\!\!\!\!\rightarrow \mathcal{V}^1/t(\mathcal{V}_1^0\otimes\mathcal{V}_2^0) \hookrightarrow (\mathcal{V}_1^0\otimes\mathcal{V}_2^0)/t(\mathcal{V}_1^0\otimes\mathcal{V}_2^0) = (\mathcal{V}_1^0/t\mathcal{V}_1^0)\otimes(\mathcal{V}_2^0/t\mathcal{V}_2^0).$$

After grading as in (2c), show that it is F-strict and has image

$$\bigoplus_{\substack{\beta_i\in[0,1)\ i=1,2\\ \beta_1+\beta_2\geqslant 1}} \mathrm{gr}^{\beta_1}\mathcal{V}_1\otimes\mathrm{gr}^{\beta_2}\mathcal{V}_2.$$

(b) By using the F-strictness of $t:\mathcal{V}^0\to\mathcal{V}^1$ and similarly for $\mathcal{V}_1,\mathcal{V}_2$, show that $t:\mathcal{V}^0/(\mathcal{V}_1^0\otimes\mathcal{V}_2^0)\to\mathcal{V}^1/t(\mathcal{V}_1^0\otimes\mathcal{V}_2^0)$ is an F-strict isomorphism. Conclude that (7.26 $**$) holds true.

(5) Conclude the proof of (7.26 $*$) by using (3) globally on X together with (7.26 $**$) at each point $x\in\Sigma$. [*Hint*: Use the standard formula for computing the degree of a tensor product of two vector bundles on a compact Riemann surface.]

7.8. Comments

This chapter aims at explaining the point of view of Hodge modules of M. Saito [**Sai88**] in the simplest case of curves. Many technical points of the general theory are thus avoided, and this case sheds light on the importance of the Kashiwara-Malgrange filtration and the notion of nearby and vanishing cycles, which will be instrumental in the general case. It also emphasizes the notion of pure support and S-decomposable modules. The emphasis on sesquilinear pairings is inspired by the work of Barlet and Maire (in dimension 1, see [**BM87, BM89**]), and by the notion of complex conjugation for holonomic $\mathcal{D}$-modules as developed by Kashiwara [**Kas86a**] (see also [**Bjö93**]).

The definitions and computations of Hodge invariants introduced in Section7.6 are taken from [**DS13**]. Exercise 7.26 is taken from [**DR20**].

PART II

TOOLS FROM THE THEORY OF $\mathcal{D}$-MODULES

CHAPTER 8

TRAINING ON $\mathcal{D}$-MODULES

Summary. In this chapter, we introduce the fundamental functors on $\mathcal{D}$-modules that we use in order to define supplementary structures, and we also introduce various operations: pullback and pushforward by a holomorphic map between complex manifolds or a morphism between smooth algebraic varieties. Most results are presented as exercises. They mainly rely on Leibniz rule. The main references for this chapter are [**Bjö93**], [**Kas03**] and [**GM93**].

8.1. The sheaf of holomorphic differential operators

Let $(X, \mathcal{O}_X)$ be a complex manifold equipped with its sheaf of holomorphic functions. We also denote by $\mathcal{C}_X^\infty$ the sheaf of complex-valued C^∞ functions on the underlying C^∞ manifold $X_\mathbb{R}$. This sheaf is a c-soft sheaf.

8.1.a. Vector fields, derivations, differential forms, contractions. We will denote by Θ_X the sheaf of holomorphic vector fields on X. This is the $\mathcal{O}_X$-locally free sheaf generated in local coordinates by $\partial_{x_1}, \dots, \partial_{x_n}$. It is a sheaf of $\mathcal{O}_X$-Lie algebras, and vector fields act (on the left) on functions by derivation, in a way compatible with the Lie algebra structure: given local vector fields ξ, η acting on functions as a derivations and given a local holomorphic function f,

- $f\xi$ is the vector field acting as $(f\xi)(g) = f \cdot \xi(g)$,
- the bracket $[\xi, \eta]$ defined as the operator $[\xi, \eta](g) := \xi(\eta(g)) - \eta(\xi(g))$ is still a derivation, hence defines a vector field.

We will denote by $\Theta_{X,k}$ the exterior product $\wedge^k \Theta_X$, which is also a locally free $\mathcal{O}_X$-module.

Dually, we denote by Ω_X^1 the sheaf of holomorphic 1-forms on X. We will set $\Omega_X^k = \wedge^k \Omega_X^1$ and $\omega_X = \Omega_X^n$. We denote by $\mathrm{d} : \Omega_X^k \to \Omega_X^{k+1}$ the differential.

The natural nondegenerate pairing $\langle \bullet, \bullet \rangle : \Omega_X^1 \otimes \Theta_X \to \mathcal{O}_X$ extends in a natural way as a nondegenerate pairing $\Omega_X^k \otimes \Theta_{X,k} \to \mathcal{O}_X$. In local coordinates $(x_1, \dots, x_n)$, a basis of Ω_X^k is given by the family $(\mathrm{d}x_I)_I$, where I runs among the subsets of cardinal k of $\{1, \dots, n\}$ and $\mathrm{d}x_I$ is defined as $\mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}$, with $I = \{i_1, \dots, i_k\}$

and $i_1 < \cdots < i_k$. Dually, the partial derivatives ∂_{x_i} lead to the basis $(\partial_{x_I})_I$ of $\Theta_{X,k}$, with a similar meaning. Due to anti-commutativity of the wedge product, $(\partial_{x_I})_I$ is the basis dual to $(\mathrm{d}x_I)_I$ up to sign, that is, denoting by δ the Kronecker index,

$$\langle \mathrm{d}x_I, \partial_{x'_I}\rangle = \varepsilon(k)\delta_{I,I'} \quad (\varepsilon(k) := (-1)^{k(k-1)/2}).$$

We can thus regard sections of Ω^k_X as $\mathcal{O}_X$-linear forms on $\Theta_{X,k}$. For a local section η of Ω^k_X, we may denote $\langle \eta, \bullet\rangle$ as $\eta(\bullet)$.

The *contraction* by a vector field ξ is the $\mathcal{O}_X$-linear morphism $\xi \lrcorner\colon \Omega^k_X \to \Omega^{k-1}_X$ defined by $\eta \mapsto \eta(\xi \wedge \bullet)$, where $\bullet$ is local section of $\Theta_{X,k-1}$. More generally, for a local section ξ of $\Theta_{X,j}$, the contraction $\eta \mapsto \eta(\xi \wedge \bullet)$ sends Ω^k_X to Ω^{k-j}_X.

For example, if $k = n = \dim X$, set

$$\mathrm{d}\boldsymbol{x} := \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n \quad \text{and} \quad \mathrm{d}\boldsymbol{x}_{\widehat{\imath}} := \mathrm{d}x_1 \wedge \cdots \wedge \widehat{\mathrm{d}x_i} \wedge \cdots \wedge \mathrm{d}x_n.$$

Then we have

$$\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x} = (-1)^{n-i}\mathrm{d}\boldsymbol{x}_{\widehat{\imath}},$$

since

$$\begin{aligned}(\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x})(\partial_{\boldsymbol{x}_{\widehat{\imath}}}) &= \mathrm{d}\boldsymbol{x}(\partial_{x_i} \wedge \partial_{\boldsymbol{x}_{\widehat{\imath}}}) = (-1)^{i-1}\mathrm{d}\boldsymbol{x}(\partial_{\boldsymbol{x}})\\ &= (-1)^{i-1}\,\varepsilon(n) = (-1)^{n-i}\,\varepsilon(n-1) = (-1)^{n-i}\mathrm{d}\boldsymbol{x}_{\widehat{\imath}}(\partial_{\boldsymbol{x}_{\widehat{\imath}}}).\end{aligned}$$

As a consequence, for $f \in \mathcal{O}_X$, we have $\mathrm{d}(f\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x}) = (-1)^{n-1}\partial f/\partial_{x_i} \cdot \mathrm{d}\boldsymbol{x}$.

The *Lie derivative* of $\mathrm{d}\boldsymbol{x}$ along ξ is defined as $\mathcal{L}_\xi(\mathrm{d}\boldsymbol{x}) := \mathrm{d}(\xi \lrcorner\, \mathrm{d}\boldsymbol{x})$. Similarly, we rename the action of ξ as a derivation on f as $\mathcal{L}_\xi(f) = \partial f/\partial_{x_i}$. Note that $\mathcal{L}_{\partial_{x_i}}(\mathrm{d}\boldsymbol{x}) = 0$. We conclude from these formulas that there is a natural *right* action (in a compatible way with the Lie algebra structure) of Θ_X on ω_X, defined by

$$\omega \cdot \xi = (-1)^n \mathcal{L}_\xi\omega := (-1)^n\mathrm{d}(\xi \lrcorner\, \omega). \tag{8.1.1}$$

Indeed, the relation $\xi(f)\omega = \omega \cdot [\xi, f] = (\omega\cdot\xi)f - (\omega f)\cdot\xi$ holds, as for example, taking $\xi = \partial_{x_i}$, we find $(f\mathrm{d}\boldsymbol{x}) \cdot \partial_{x_i} = -(\partial f/\partial x_i)\mathrm{d}\boldsymbol{x}$ and

$$(\partial f/\partial x_i)\mathrm{d}\boldsymbol{x} = (-1)^{n-1}\mathrm{d}(f\partial_{x_i} \lrcorner\, \mathrm{d}\boldsymbol{x}) = -(f\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i} \quad \text{and} \quad (\mathrm{d}\boldsymbol{x}\cdot\partial_{x_i})f = 0.$$

Similarly, let us check $\omega \cdot [\xi, \xi'] = (\omega \cdot \xi)\cdot \xi' - (\omega \cdot \xi')\cdot\xi$ with $\omega = \mathrm{d}\boldsymbol{x}$, $\xi = f\partial_{x_i}$, $\xi' = \partial_{x_j}$. We have $[\xi, \xi'] = -(\partial f/\partial_{x_j})\partial_{x_i}$ and $\omega\cdot\xi' = 0$, so we only have to check

$$-(\partial f/\partial x_j\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i} = ((f\mathrm{d}\boldsymbol{x})\cdot\partial_{x_i})\cdot\partial_{x_j},$$

which follows from the commutativity of the partial derivatives of f.

***8.1.2. Definition* (The sheaf of holomorphic differential operators)**

For any open set U of X, the ring $\mathcal{D}_X(U)$ of *holomorphic differential operators on* U is the subring of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{O}_U, \mathcal{O}_U)$ generated by

- multiplication by holomorphic functions on U,
- derivation by holomorphic vector fields on U.

The sheaf $\mathcal{D}_X$ is defined by $\Gamma(U, \mathcal{D}_X) = \mathcal{D}_X(U)$ for every open set U of X.

By construction, the sheaf $\mathcal{D}_X$ acts on the left on $\mathcal{O}_X$, i.e., $\mathcal{O}_X$ is a left $\mathcal{D}_X$-module.

8.1.3. Definition **(The filtration of $\mathcal{D}_X$ by the order).** The increasing family of subsheaves $F_k\mathcal{D}_X \subset \mathcal{D}_X$ is defined inductively:

- $F_k\mathcal{D}_X = 0$ if $k \leqslant -1$,
- $F_0\mathcal{D}_X = \mathcal{O}_X$ (*via* the canonical injection $\mathcal{O}_X \hookrightarrow \mathcal{H}om_{\mathbb{C}}(\mathcal{O}_X, \mathcal{O}_X)$),
- the local sections P of $F_{k+1}\mathcal{D}_X$ are characterized by the fact that $[P, f]$ is a local section of $F_k\mathcal{D}_X$ for any holomorphic function g.

8.1.4. Proposition. *Giving a left $\mathcal{D}_X$-module $\mathcal{M}$ is equivalent to giving an $\mathcal{O}_X$-module $\mathcal{M}$ together with an integrable connection ∇.*

Proof. Exercises 8.1, 8.5 and 8.6. ☐

8.1.b. Vector fields and differential forms in presence of a filtration

We now apply the constructions of Section 5.1 to the filtered ring $(\mathcal{D}_X, F_\bullet\mathcal{D}_X)$ and its (left or right) modules. We obtain the following properties:

- $\widetilde{\mathcal{O}}_X := R_F\mathcal{O}_X = \mathcal{O}_X[z]$.
- in local coordinates, we have

$$\widetilde{\mathcal{D}}_X := R_F\mathcal{D}_X = \mathcal{O}_X[z]\langle\widetilde{\partial}_{x_1}, \dots, \widetilde{\partial}_{x_n}\rangle, \tag{8.1.5}$$

i.e., any germ of section of $\widetilde{\mathcal{D}}_X$ may be written in a unique way as

$$\sum_\alpha a_\alpha(x,z)\widetilde{\partial}_x^\alpha = \sum_\alpha \widetilde{\partial}_x^\beta b_\alpha(x,z),$$

where $a_\alpha, b_\alpha \in \widetilde{\mathcal{O}}_X$, and where we set

$$\widetilde{\partial}_{x_i} := z\partial_{x_i}. \tag{8.1.6}$$

- The ring $\widetilde{\mathcal{D}}_X$ is equipped with a natural filtration $F_\bullet\widetilde{\mathcal{D}}_X$ by the order in $\widetilde{\partial}_x$. If we write $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, then this filtration is defined by the formula

$$F_k\widetilde{\mathcal{D}}_X = \bigoplus_{j=0}^{k-1} F_j\mathcal{D}_X z^j \oplus F_k\mathcal{D}_X z^k\mathbb{C}[z]. \tag{8.1.7}$$

- The sheaf $\widetilde{\Theta}_X$ is the locally free $\widetilde{\mathcal{O}}_X$-module locally generated by $\widetilde{\partial}_{x_1}, \dots, \widetilde{\partial}_{x_n}$ (having degree 1, due to our convention in Section 5.1.3) and we have $[\widetilde{\partial}_{x_i}, f] = z\partial f/\partial x_i$ for any local section g of $\widetilde{\mathcal{O}}_X$; we also set $\widetilde{\Theta}_{X,k} = \wedge^k\widetilde{\Theta}_X$;
- $\widetilde{\Omega}^1_X$ is the locally free graded $\widetilde{\mathcal{O}}_X$-module $z^{-1}\mathbb{C}[z] \otimes_{\mathbb{C}} \Omega^1_X$, and $\widetilde{\Omega}^k_X = \wedge^k\widetilde{\Omega}^1_X$; the differential $\widetilde{\mathrm{d}}$ is induced by $1 \otimes \mathrm{d}$ on $\widetilde{\Omega}^k_X = z^{-k}\mathbb{C}[z] \otimes_{\mathbb{C}} \Omega^k_X$; we set $\widetilde{\omega}_X = \widetilde{\Omega}^n_X$; we regard the differential as a graded morphism of degree zero

$$\widetilde{\mathrm{d}} : \widetilde{\Omega}^k_X \longrightarrow \widetilde{\Omega}^{k+1}_X;$$

the local basis $(\widetilde{\mathrm{d}}x_i = z^{-1}\mathrm{d}x_i)_i$ (having degree -1) is dual to the basis $(\widetilde{\partial}_{x_i})_i$ of $\widetilde{\Theta}_X$.
- We also set $\widetilde{\mathcal{C}}^\infty_X := \mathcal{C}^\infty_X[z]$. This is a c-soft sheaf on the underlying C^∞ manifold $X_{\mathbb{R}}$.
- Contraction of a z-differential form of degree k by a z-vector field is defined as in Section 8.1.a.

• We have natural Lie algebra actions of $\widetilde{\Theta}_X$ on $\widetilde{\mathcal{O}}_X$ (action on the left) and on $\widetilde{\omega}_X$ (action on the right).

***8.1.8. Example* (Filtered flat bundles).** Let $(\mathcal{L}, \nabla)$ be a flat holomorphic bundle on X and let $F^\bullet\mathcal{L}$ be a decreasing filtration of $\mathcal{L}$ by $\mathcal{O}_X$-locally free sheaves. Then the flat connection ∇ endows $\mathcal{L}$ with the structure of a left $\mathcal{D}_X$-module (Proposition 8.1.4). The *Griffiths transversality property*

$$\nabla F^p\mathcal{L} \subset \Omega^1_X \otimes F^{p-1}\mathcal{L}, \quad \forall\, p \in \mathbb{Z} \tag{8.1.8 $*$}$$

is equivalent to the property that the corresponding increasing filtration $F_\bullet\mathcal{L}$ is an $F\mathcal{D}_X$-filtration of the $\mathcal{D}_X$-module $\mathcal{L}$.

***8.1.9. Definition* (Connection).** Let $\widetilde{\mathcal{M}}$ be a graded $\widetilde{\mathcal{O}}_X$-module. A *connection* on $\widetilde{\mathcal{M}}$ is a graded $\widetilde{\mathbb{C}}$-linear morphism $\widetilde{\nabla} : \widetilde{\mathcal{M}} \to \widetilde{\Omega}^1_X \otimes \widetilde{\mathcal{M}}$ (of degree zero) which satisfies the Leibniz rule

$$\forall\, f \in \widetilde{\mathcal{O}}_X, \quad \widetilde{\nabla}(fm) = f\,\widetilde{\nabla}m + \widetilde{\mathrm{d}}\,f \otimes m.$$

Proposition 8.1.4 holds true in this filtered setting (Exercise 8.7).

8.1.10. Example. The fundamental examples of filtered left and right $\mathcal{D}_X$-modules are:

• $(\mathcal{O}_X, F_\bullet\mathcal{O}_X)$ with $\mathrm{gr}^F_p\mathcal{O}_X = 0$ for $p \neq 0$, so $R_F\mathcal{O}_X = \mathcal{O}_X[z]$,

• $(\omega_X, F_\bullet\omega_X)$ with $\mathrm{gr}^F_p\omega_X = 0$ for $p \neq -n$, so $R_F\omega_X = \widetilde{\omega}_X = \widetilde{\Omega}^n_X = z^{-n}\omega_X[z]$.

8.1.11. Convention. We will use the following convention.

(i) $\widetilde{\mathcal{O}}_X$ (resp. $\widetilde{\mathcal{C}}^\infty_X$) denotes either the sheaf rings $\mathcal{O}_X$ (resp. $\mathcal{C}^\infty_X$) or the sheaf of graded rings $\mathcal{O}_X[z] = R_F\mathcal{O}_X$ (resp. $\mathcal{C}^\infty_X[z]$), and $\mathsf{Mod}(\widetilde{\mathcal{O}}_X)$ denotes the category of $\mathcal{O}_X$-modules or that of *graded* $\mathcal{O}_X[z]$-modules.

(ii) The notation $\widetilde{\Theta}_X$, $\widetilde{\Omega}^k_X$, $\wedge^k\widetilde{\Theta}_X$ has a similar double meaning.

(iii) Similarly, $\widetilde{\mathcal{D}}_X$ denotes either the sheaf rings $\mathcal{D}_X$ or the sheaf of graded rings $R_F\mathcal{D}_X$, and $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ denotes the category of $\mathcal{D}_X$-modules or that of *graded* $R_F\mathcal{D}_X$-modules.

(iv) It will also be convenient to denote by $\widetilde{\mathbb{C}}$ either the field $\mathbb{C}$ or the *graded* ring $\mathbb{C}[z]$.

(v) In each of the second cases above, we will usually omit the word "graded", although it is always understood.

(vi) One recovers standard results for $\mathcal{D}_X$-modules by setting $z = 1$ and $\widetilde{\partial} = \partial$.

(vii) The strictness condition that we may consider only refers to the second cases above, it is empty in the first cases.

8.2. Left and right

Considering left or right $\widetilde{\mathcal{D}}_X$-modules is not completely symmetric. The main reason is that the *left* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{O}}_X$ is a sheaf of rings, while its right analogue $\widetilde{\omega}_X := \widetilde{\Omega}^n_X$, is not a sheaf of rings. So for example the behaviour with respect to tensor products over $\widetilde{\mathcal{O}}_X$ is not the same for left and right $\widetilde{\mathcal{D}}$-modules. Also, the side changing functor defined below sends $\widetilde{\mathcal{D}}_X^{\mathrm{left}}$ to $\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, and not to $\widetilde{\mathcal{D}}_X$ regarded as a right $\widetilde{\mathcal{D}}_X$-module over itself.

The categories of left (resp. right) $\widetilde{\mathcal{D}}_X$-modules are denoted by $\mathsf{Mod}^{\mathrm{left}}(\widetilde{\mathcal{D}}_X)$ (resp. $\mathsf{Mod}^{\mathrm{right}}(\widetilde{\mathcal{D}}_X)$ (recall that we consider graded modules and morphisms of degree zero in the case of $\widetilde{\mathcal{D}} = R_F\mathcal{D}$). We analyze the relations between both categories in this section.

The main rule to be followed is that the side-changing functor changes a functor in the category of left objects to the functor denoted in the same way in the category of right objects, and conversely.

Exercises 8.8 and 8.9 give the basic tools for generating left or right $\widetilde{\mathcal{D}}$-modules.

***8.2.1. Example* (Example 8.1.10 continued).**

(1) $\widetilde{\mathcal{D}}_X$ is a left and a right $\widetilde{\mathcal{D}}_X$-module.

(2) $\widetilde{\mathcal{O}}_X$ is a left $\widetilde{\mathcal{D}}_X$-module (Exercise 8.10), with grading

$$(\widetilde{\mathcal{O}}_X)_p = \begin{cases} \mathcal{O}_X & \text{if } p \geqslant 0, \\ 0 & \text{if } p < 0. \end{cases}$$

(3) $\widetilde{\omega}_X := \widetilde{\Omega}_X^{\dim X}$ is a right $\widetilde{\mathcal{D}}_X$-module (Exercise 8.11), with grading

$$(\widetilde{\omega}_X)_p = \begin{cases} \omega_X & \text{if } p \geqslant -n, \\ 0 & \text{if } p < -n. \end{cases}$$

***8.2.2. Definition* (Side-changing of $\widetilde{\mathcal{D}}_X$-modules).** Any left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{left}}$ gives rise to a right one $\widetilde{\mathcal{M}}^{\mathrm{right}}$ by setting $\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ and, for any vector field ξ and any function g,

$$(\omega \otimes m) \cdot f = f\omega \otimes m = \omega \otimes fm, \quad (\omega \otimes m) \cdot \xi = \omega\xi \otimes m - \omega \otimes \xi m.$$

Conversely, set $\widetilde{\mathcal{M}}^{\mathrm{left}} = \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{M}}^{\mathrm{right}})$, which also has in a natural way the structure of a left $\widetilde{\mathcal{D}}_X$-module (see Exercise 8.13(2)). The grading behaves as follows (see Example 8.1.10 and (5.1.4)):

$$\begin{aligned} \widetilde{\mathcal{M}}^{\mathrm{right}} &= z^{-n}\omega_X \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} = \omega_X \otimes_{\mathcal{O}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}(-n), \\ \mathcal{M}_p^{\mathrm{right}} &= \omega_X \otimes_{\mathcal{O}_X} \mathcal{M}_{p+n}^{\mathrm{left}}. \end{aligned} \tag{8.2.2 $*$}$$

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ is the Rees module of a filtration, then the side-changing functor is written as

$$F_p\mathcal{M}^{\mathrm{right}} = F_p(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}^{\mathrm{left}}) = \omega_X \otimes_{\mathcal{O}_X} F_{p+n}\mathcal{M}^{\mathrm{left}}. \tag{8.2.2 $**$}$$

8.2.3. Caveat. Let $\widetilde{\omega}_X^\vee = \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{O}}_X)$ as an $\widetilde{\mathcal{O}}_X$-module. One often finds in the literature the formula $\widetilde{\mathcal{M}}^{\mathrm{left}} = \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\omega}_X^\vee$, which give the $\widetilde{\mathcal{O}}_X$-module structure of $\widetilde{\mathcal{M}}^{\mathrm{left}}$. However, the left $\widetilde{\mathcal{D}}_X$-module structure is not obtained with a "tensor product formula" as in Exercise 8.12, but uses the interpretation as $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{M}}^{\mathrm{right}})$.

On the other hand, let U be a chart of X with coordinates $x_1, \dots, x_n$. Set $\widetilde{\mathrm{d}}\boldsymbol{x} = \widetilde{\mathrm{d}}x_1 \wedge \dots \wedge \widetilde{\mathrm{d}}x_n$. This is an $\widetilde{\mathcal{O}}_U$-basis of $\widetilde{\omega}_X$. Let $\widetilde{\mathrm{d}}\boldsymbol{x}^\vee$ denote the dual basis of $\widetilde{\omega}_X^\vee$. It is often convenient, for a right $\widetilde{\mathcal{D}}_U$-module $\widetilde{\mathcal{M}}^{\mathrm{right}}$, to write $\widetilde{\mathcal{M}}^{\mathrm{left}} = \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes \widetilde{\mathrm{d}}\boldsymbol{x}^\vee$ with the convention that a local section $\widetilde{\mathrm{d}}\boldsymbol{x}^\vee \otimes m$ is regarded as the morphism sending $\widetilde{\mathrm{d}}\boldsymbol{x}$ to m. In view of the duality between $\widetilde{\Omega}^1_X$ and $\widetilde{\Theta}_X$, one can identify $\widetilde{\omega}_X^\vee$ with $\wedge^n\widetilde{\Theta}_X$ and choose the local basis $\widetilde{\partial}_{\boldsymbol{x}}^{\wedge n} := \widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_n}$ of $\wedge^n\widetilde{\Theta}_U$. Both bases are related by $\widetilde{\mathrm{d}}\boldsymbol{x}^\vee = \varepsilon(n)\widetilde{\partial}_{\boldsymbol{x}}^{\wedge n}$. See also Exercise 8.17.

The following is obvious from Exercises 8.14 and 8.15.

8.2.4. Proposition. *The side-changing functors left-to-right and right-to-left are isomorphisms of between the categories of left and right graded $\widetilde{\mathcal{D}}_X$-modules, which are inverse one another. The left-to-right functor induces a twist $(-n)$, while the right-to-left functor induces a twist (n) ($n = \dim X$).* □

8.2.5. Remark. The ring $\widetilde{\mathcal{D}}_X$ considered as a right $\widetilde{\mathcal{D}}_X$-module over itself *is not equal* to the right $\widetilde{\mathcal{D}}_X$-module associated with $\widetilde{\mathcal{D}}_X$ regarded as a left $\widetilde{\mathcal{D}}_X$-module over itself by the side-changing functor.

***8.2.6. Caveat* (Side-changing and grading).** For a filtered left $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, side-changing and grading are related by the formula (according to example 8.2.1(3))

$$\mathrm{gr}^F(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \omega_X \otimes_{\mathcal{O}_X} \mathrm{gr}^F\mathcal{M}(-n),$$

as $\mathcal{O}_X$-modules. The action of $\mathrm{gr}^F\mathcal{D}_X$ is not exactly preserved by this isomorphism. Indeed, recall that, for a vector field ξ, we have $(\omega \otimes m)\xi = \omega\xi \otimes m - \omega \otimes \xi m$ and, taking classes in the suitable graded piece, we find $[\omega \otimes m][\xi] = -\omega \otimes [\xi m]$. We can thus write, as $\mathrm{gr}^F\mathcal{D}_X$-modules,

$$\mathrm{gr}^F(\omega_X \otimes_{\mathcal{O}_X} \mathcal{M}) = \omega_X \otimes_{\mathcal{O}_X} \mathrm{inv}^*\mathrm{gr}^F\mathcal{M}(-n),$$

where $\mathrm{inv}^*\mathrm{gr}^F\mathcal{M}$ denotes the $\mathcal{O}_X$-module $\mathrm{gr}^F\mathcal{M}$ on which the action of $\mathrm{gr}^F\mathcal{D}_X$ is modified in such a way that $\mathrm{gr}^F_k\mathcal{D}_X$ acts by multiplying by $(-1)^k$ the usual action.

8.3. Examples of $\widetilde{\mathcal{D}}$-modules

We list here some classical examples of $\widetilde{\mathcal{D}}$-modules. One can get many other examples by applying various operations on $\widetilde{\mathcal{D}}$-modules.

8.3.1. Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. There is a very simple way to get a right $\widetilde{\mathcal{D}}_X$-module from $\widetilde{\mathcal{L}}$: consider $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ equipped with the natural right action of $\widetilde{\mathcal{D}}_X$. This is called an *induced* $\widetilde{\mathcal{D}}_X$-module. Although this construction is very simple, it is also very useful to get cohomological properties of $\widetilde{\mathcal{D}}_X$-modules. One can also consider the left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}$ (however, this is not the left $\widetilde{\mathcal{D}}_X$-module attached to the right one $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ by the side-changing functor of Definition 8.2.2).

8.3.2. One of the main geometrical examples of $\mathcal{D}_X$-modules are the vector bundles on X equipped with an *integrable* connection. Recall (Proposition 8.1.4) that left $\mathcal{D}_X$-modules are $\mathcal{O}_X$-modules with an integrable connection. Among them, the coherent $\mathcal{D}_X$-modules are of particular interest. One can show that such modules are $\mathcal{O}_X$-locally free, i.e., correspond to holomorphic vector bundles of finite rank on X.

It may happen that, for some X, such a category does not give any interesting geometric object. Indeed, if for instance X has a trivial fundamental group (e.g. $X = \mathbb{P}^1(\mathbb{C})$), then any vector bundle with integrable connection is isomorphic to the trivial bundle $\mathcal{O}_X$ with the connection d. However, on non simply connected Zariski open sets of X, there exist interesting vector bundles with connections. This leads to the notion of meromorphic vector bundle with connection.

Let D be a divisor in X and denote by $\mathcal{O}_X(*D)$ the sheaf of meromorphic functions on X with poles along D at most. This is a sheaf of left $\mathcal{D}_X$-modules, being an $\mathcal{O}_X$-module equipped with the natural connection $\mathrm{d} : \mathcal{O}_X(*D) \to \Omega^1_X(*D)$.

By definition, a *meromorphic bundle* is a locally free $\mathcal{O}_X(*D)$ module of finite rank. When it is equipped with an integrable connection, it becomes a left $\mathcal{D}_X$-module.

8.3.3. One can *twist* the previous examples. Assume that ω is a *closed* holomorphic form on X. Define $\nabla : \mathcal{O}_X \to \Omega^1_X$ by the formula $\nabla = \mathrm{d} + \omega$. As ω is closed, ∇ is an integrable connection on the trivial bundle $\mathcal{O}_X$.

Usually, the nonzero closed form on X are meromorphic, with poles on some divisor D. Then ∇ is an integrable connection on $\mathcal{O}_X(*D)$.

If ω is exact, $\omega = \mathrm{d}f$ for some meromorphic function g on X, then ∇ can be written as $e^{-f} \circ \mathrm{d} \circ e^f$.

More generally, if $\mathcal{M}$ is any meromorphic bundle with an integrable connection ∇, then, for any such ω, $\nabla + \omega\,\mathrm{Id}$ defines a new $\mathcal{D}_X$-module structure on $\mathcal{M}$.

8.3.4. Denote by $\mathfrak{Db}_X$ the sheaf of distributions on the complex manifold X of dimension n: given any open set U of X, $\mathfrak{Db}_X(U)$ is the space of distributions on U, which is by definition the weak dual of the space of C^∞ forms with compact support on U, of type (n,n). By Exercise 8.11, there is a right action of $\mathcal{D}_X$ on such forms. The left action of $\mathcal{D}_X$ on distributions is defined by adjunction: denote by $\langle \eta, u\rangle$ the natural pairing between a compactly supported C^∞-form η and a distribution u on U; let P be a holomorphic differential operator on U; define then $P \cdot u$ in such a way that, for every η, on has

$$\langle \eta, P\cdot u\rangle = \langle \eta \cdot P, u\rangle.$$

Given any distribution u on X, the subsheaf $\mathcal{D}_X \cdot u \subset \mathfrak{Db}_X$ is the $\mathcal{D}_X$-module generated by this distribution. Saying that a distribution is a solution of a family $P_1, \dots, P_k$ of differential equation is equivalent to saying that the morphism $\mathcal{D}_X \to \mathcal{D}_X \cdot u$ sending 1 to u induces a surjective morphism $\mathcal{D}_X/(P_1, \dots, P_k) \to \mathcal{D}_X \cdot u$.

Similarly, the sheaf $\mathfrak{C}_X$ of currents of degree 0 on X is defined in such a way that, for any open set $U \subset X$, $\mathfrak{C}_X(U)$ is dual to $C^\infty_{\mathrm{c}}(U)$ with a suitable topology. It is a right $\mathcal{D}_X$-module.

In local coordinates $x_1, \dots, x_n$, a current of degree 0 is nothing but a distribution times the volume form $dx_1 \wedge \cdots \wedge dx_n \wedge d\overline{x}_1 \wedge \cdots \wedge d\overline{x}_n$.

As we are now working with C^∞ forms or with currents, it is natural not to forget the anti-holomorphic part of these objects. Denote by $\mathcal{O}_{\overline{X}}$ the sheaf of anti-holomorphic functions on X and by $\mathcal{D}_{\overline{X}}$ the sheaf of anti-holomorphic differential operators. Then $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) are similarly left (resp. right) $\mathcal{D}_{\overline{X}}$-modules. Of course, the $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$ actions do commute, and they coincide when considering multiplication by constants.

The *conjugation* exchanges both structures. For example, if u is a distribution on U, its conjugate $\overline{u}$ is defined by the formula

$$\langle \eta, \overline{u} \rangle := \overline{\langle \overline{\eta}, u \rangle} \qquad (\eta \in \mathcal{E}_c^{n,n}(U)). \tag{8.3.4 $*$}$$

This is of course compatible with the usual conjugation of L^1_{loc} functions.

It is therefore natural to introduce the following sheaves of rings:

$$\mathcal{O}_{X,\overline{X}} := \mathcal{O}_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}, \qquad \mathcal{D}_{X,\overline{X}} := \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}, \tag{8.3.4 $**$}$$

and consider $\mathfrak{Db}_X$ (resp. $\mathfrak{C}_X$) as left (resp. right) $\mathcal{D}_{X,\overline{X}}$-modules.

8.3.5. One can construct new examples from old ones by using various operations.

- Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Then $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$ has a natural structure of right $\widetilde{\mathcal{D}}_X$-module. Using a resolution $\widetilde{\mathcal{N}}^\bullet$ of $\widetilde{\mathcal{M}}$ by *left* $\widetilde{\mathcal{D}}_X$-modules which are acyclic for $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\bullet, \widetilde{\mathcal{D}}_X)$, one gets a right $\widetilde{\mathcal{D}}_X$-module structure on $\mathcal{E}xt^k_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$ for $k \geqslant 0$.
- Given two left (resp. a left and a right) $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$, the same argument enables one to put on the various $\mathcal{T}or_{i,\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{N}}, \widetilde{\mathcal{M}})$ a left (resp. a right) $\widetilde{\mathcal{D}}_X$-module structure.

8.4. The de Rham functor

***8.4.1. Definition* (de Rham).** For a *left* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the de Rham complex $\mathrm{DR}\,\widetilde{\mathcal{M}}$ is the bounded complex (with $\bullet$ in degree zero and all nonzero terms in non-negative degrees)

$$\mathrm{DR}\,\widetilde{\mathcal{M}} := \{0 \to \underset{\bullet}{\widetilde{\mathcal{M}}} \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^1_X \otimes \widetilde{\mathcal{M}} \xrightarrow{\widetilde{\nabla}} \cdots \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^n_X \otimes \widetilde{\mathcal{M}} \to 0\}.$$

The terms are the $\widetilde{\mathcal{O}}_X$-modules $\widetilde{\Omega}^\bullet_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ and the differentials the $\widetilde{\mathbb{C}}$-linear morphisms $\widetilde{\nabla}$ defined in Exercise 8.6 or 8.7.

The *shifted* de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is defined as

$${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}} := \{0 \to \widetilde{\mathcal{M}} \xrightarrow{(-1)^n\widetilde{\nabla}} \widetilde{\Omega}^1_X \otimes \widetilde{\mathcal{M}} \xrightarrow{(-1)^n\widetilde{\nabla}} \cdots \xrightarrow{(-1)^n\widetilde{\nabla}} \widetilde{\Omega}^n_X \underset{\bullet}{\otimes} \widetilde{\mathcal{M}} \to 0\}.$$

The previous definition produces a *complex* since $\widetilde{\nabla} \circ \widetilde{\nabla} = 0$, according to the integrability condition on $\widetilde{\nabla}$, as remarked in Exercise 8.6 or 8.7. The notation ${}^{\mathrm{p}}\mathrm{DR}$ is motivated by the property that, for a holonomic $\mathcal{D}_X$-module $\mathcal{M}$, the complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is a perverse sheaf (a theorem of Kashiwara).

8.4.2. Remark **(Shift of a complex).** Given a complex $(C^\bullet, \widetilde{\delta})$, the shifted complex $(C^\bullet, \widetilde{\delta})[n]$ is the complex $(C^{n+\bullet}, (-1)^n\widetilde{\delta})$. Thus the complex ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is equal to $\mathrm{DR}\,\widetilde{\mathcal{M}}[n]$. The shifted de Rham complex is implicitly considered in Formula (8.1.1).

8.4.3. Definition **(Spencer).** The *Spencer complex* $\mathrm{Sp}(\widetilde{\mathcal{M}})$ of a *right* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is the bounded complex (with $\bullet$ in degree zero and all nonzero terms in non-positive degrees; recall also the notation $\widetilde{\Theta}_{X,k} = \wedge^k\widetilde{\Theta}_X$)

$$\mathrm{Sp}(\widetilde{\mathcal{M}}) := \{0 \to \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,n} \xrightarrow{\widetilde{\delta}_{\widetilde{\mathcal{M}}}} \cdots \xrightarrow{\widetilde{\delta}_{\widetilde{\mathcal{M}}}} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,1} \xrightarrow{\widetilde{\delta}_{\widetilde{\mathcal{M}}}} \underset{\bullet}{\widetilde{\mathcal{M}}} \to 0\},$$

where the differential $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ is the $\widetilde{\mathbb{C}}$-linear map given by

$$\begin{aligned} m \otimes (\xi_1 \wedge \cdots \wedge \xi_k) \xmapsto{\widetilde{\delta}_{\widetilde{\mathcal{M}}}} &\sum_{i=1}^{k} (-1)^{i-1}(m\xi_i) \otimes (\xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \xi_k) \\ &+ \sum_{i<j} (-1)^{i+j} m \otimes \big([\xi_i, \xi_j] \wedge \xi_1 \wedge \cdots \wedge \widehat{\xi_i} \wedge \cdots \wedge \widehat{\xi_j} \wedge \cdots \wedge \xi_k\big), \end{aligned}$$

where $\widehat{\xi_i}$ means that we omit ξ_i in the wedge product.

Of special interest will be, of course, the de Rham or Spencer complex of the ring $\widetilde{\mathcal{D}}_X$, considered as a left or right $\widetilde{\mathcal{D}}_X$-module. Notice that in $\mathrm{DR}(\widetilde{\mathcal{D}}_X)$ the differentials are *right* $\widetilde{\mathcal{D}}_X$-linear, and in $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ they are *left* $\widetilde{\mathcal{D}}_X$-linear. See Exercises 8.21–8.24 for some of their properties.

8.4.4. Remark.

(1) For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the complex $\mathrm{Sp}(\widetilde{\mathcal{M}})$ is isomorphic to $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ (Exercise 8.24). It is then possible to prove some statements on $\mathrm{Sp}(\widetilde{\mathcal{M}})$ by only considering the case where $\widetilde{\mathcal{M}} = \widetilde{\mathcal{D}}_X$.

(2) For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, it is usual to find in the literature the definition of ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ as $\boldsymbol{R}\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X\widetilde{\mathcal{M}})$ (in a suitable derived category). Since $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ by locally free $\widetilde{\mathcal{D}}_X$-modules, this isomorphism amounts to the isomorphism ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}} \simeq \mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X), \widetilde{\mathcal{M}})$. This is shown in Exercise 8.25.

Side-changing. Given any $k \geqslant 0$, the *contraction* is the morphism (see Section 8.1.a)

$$\begin{aligned} \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k} &\xrightarrow{\ \lrcorner\ } \widetilde{\Omega}_X^{n-k} \\ \omega \otimes \xi &\longmapsto (\xi \lrcorner \omega)(\bullet) = \omega(\xi \wedge \bullet). \end{aligned} \tag{8.4.5}$$

8.4.6. Example. In local coordinates $(x_1, \ldots, x_n)$, let us set $\widetilde{\mathrm{d}}\boldsymbol{x} = \widetilde{\mathrm{d}}x_1 \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n$. For $i = 1, \ldots, k \leqslant n$, let us set $\widetilde{\partial}_{\widehat{x_i}} := \widetilde{\partial}_{x_1} \wedge \cdots \wedge \widehat{\widetilde{\partial}_{x_i}} \wedge \cdots \wedge \widetilde{\partial}_{x_k}$ (i.e., omitting $\widetilde{\partial}_{x_i}$ in the wedge product) for simplicity. Then the following formulas hold, for $k \leqslant n$:

$$(\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_n}) \lrcorner \widetilde{\mathrm{d}}\boldsymbol{x} = \varepsilon(n),$$

$$(\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}) \lrcorner \widetilde{\mathrm{d}}\boldsymbol{x} = \varepsilon(n)\,\varepsilon(n-k)\widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n, \tag{8.4.6$*$}$$

$$\widetilde{\partial}_{\widehat{x_i}} \lrcorner \widetilde{\mathrm{d}}\boldsymbol{x} = (-1)^{k-i}\,\varepsilon(n)\,\varepsilon(n-k+1)\widetilde{\mathrm{d}}x_i \wedge \widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n. \tag{8.4.6$**$}$$

8.4.7. Lemma. *There exists a natural isomorphism of complexes of right $\widetilde{\mathcal{D}}_X$-modules (i.e., is compatible with the differentials of these complexes)*

$$\iota : \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)$$

which induces the identity

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}^0(\widetilde{\mathcal{D}}_X) = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X = \mathrm{DR}^n \widetilde{\mathcal{D}}_X.$$

It is induced by the isomorphisms of right $\widetilde{\mathcal{D}}_X$-modules

$$\begin{aligned}\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}\big) &\xrightarrow[\sim]{\iota} \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\\ \big[\omega \otimes (1 \otimes \xi)\big] \cdot P &\longmapsto (\xi \lrcorner\, \omega) \otimes P\end{aligned}$$

(where the right structure of the right-hand term is the natural one and that of the left-hand term is nothing but that induced by the natural left structure of $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}$ by side-changing).

Proof. It is enough to prove that the diagram

$$\begin{tikzcd}
\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}\big) \arrow[r, "\iota"] \arrow[d, "\widetilde{\delta}"'] & \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \arrow[d, "(-1)^n\widetilde{\nabla}"] \\
\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k-1}\big) \arrow[r, "\iota"] & \widetilde{\Omega}_X^{n-k+1} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X
\end{tikzcd}$$

commutes. We will make use of the relations satisfied by the function ε (see Notation 0.2). It is also enough to check this locally, and, in local coordinates $(x_1, \dots, x_n)$, we are reduced by right $\widetilde{\mathcal{D}}_X$-linearity to checking this on sections of the form $\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes (\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}))$. We use the notation of Example 8.4.6.

On the one hand, we have $\widetilde{\mathrm{d}}\boldsymbol{x} \cdot \widetilde{\partial}_{x_i} = 0$ and, according to (8.4.6∗∗) we find

$$\begin{aligned}\widetilde{\delta}\big[\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes (\widetilde{\partial}_{x_1} \wedge \cdots \wedge \widetilde{\partial}_{x_k}))\big] &= \sum_{i=1}^k (-1)^{i-1} \widetilde{\mathrm{d}}\boldsymbol{x} \otimes (\widetilde{\partial}_{x_i} \otimes \widetilde{\partial}_{\widehat{x_i}})\\ &= \sum_{i=1}^k (-1)^i \big[\widetilde{\mathrm{d}}\boldsymbol{x} \otimes (1 \otimes \widetilde{\partial}_{\widehat{x_i}})\big] \cdot \widetilde{\partial}_{x_i}\\ &\xrightarrow{\iota} \sum_{i=1}^k (-1)^i (\widetilde{\partial}_{\widehat{x_i}} \lrcorner\, \widetilde{\mathrm{d}}\boldsymbol{x}) \otimes \widetilde{\partial}_{x_i}\\ &= (-1)^k \varepsilon(n)\, \varepsilon(n-k+1) \sum_{i=1}^k (\widetilde{\mathrm{d}}x_i \wedge \widetilde{\mathrm{d}}x_{k+1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_n) \otimes \widetilde{\partial}_{x_i}.\end{aligned}$$

On the other hand, we have, according to (8.4.6∗) (see Exercises 8.5 and 8.7),

$$\begin{aligned}(-1)^n\widetilde{\nabla}\iota\big[\widetilde{\mathrm{d}}\boldsymbol{x}\otimes&(1\otimes(\widetilde{\partial}_{x_1}\wedge\cdots\wedge\widetilde{\partial}_{x_k}))\big]\\ &=(-1)^n\widetilde{\nabla}\big[(\widetilde{\partial}_{x_1}\wedge\cdots\wedge\widetilde{\partial}_{x_k})\lrcorner\,\widetilde{\mathrm{d}}\boldsymbol{x})\otimes 1\big]\\ &=(-1)^n\,\varepsilon(n)\,\varepsilon(n-k)\widetilde{\nabla}\big[(\widetilde{\mathrm{d}}x_{k+1}\wedge\cdots\wedge\widetilde{\mathrm{d}}x_n)\otimes 1\big]\\ &=(-1)^n\,\varepsilon(n)\,\varepsilon(n-k)\sum_{i=1}^k(-1)^{n-k}(\widetilde{\mathrm{d}}x_{k+1}\wedge\cdots\wedge\widetilde{\mathrm{d}}x_n\wedge\widetilde{\mathrm{d}}x_i)\otimes\widetilde{\partial}_{x_i}\\ &=(-1)^n\,\varepsilon(n)\,\varepsilon(n-k)\sum_{i=1}^k(\widetilde{\mathrm{d}}x_i\wedge\widetilde{\mathrm{d}}x_{k+1}\wedge\cdots\wedge\widetilde{\mathrm{d}}x_n)\otimes\widetilde{\partial}_{x_i}.\end{aligned}$$

and the desired equality follows from the relation $\varepsilon(n-k+1)=(-1)^{n-k}\,\varepsilon(n-k)$. □

Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ the associated right module. We will now compare ${}^{\mathrm{p}}\mathrm{DR}_X(\widetilde{\mathcal{M}})$ and $\mathrm{Sp}(\widetilde{\mathcal{M}}^{\mathrm{right}})$. We will denote by ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})$ the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{M}}^{\mathrm{right}})$ and we keep the notation ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})$ for the de Rham complex of a left $\widetilde{\mathcal{D}}_X$-module. Exercise 8.26 gives an isomorphism

$$
{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})\xrightarrow{\ \sim\ }{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}). \tag{8.4.8}
$$

8.4.9. The grading of ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$. In the left and right case, ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is a bounded complex of sheaves of graded $\widetilde{\mathbb{C}}$-modules and the isomorphism (8.4.8) is an isomorphism as such (i.e., preserves the grading). Indeed, we note that, for $k\geqslant 0$, $\widetilde{\Omega}^k_X$ (resp. $\widetilde{\Theta}_{X,k}$) is homogeneous of degree $-k$ (resp. k); therefore, the degree p component of ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ is the complex of $\mathbb{C}$-vector spaces

$$\begin{aligned}({}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{left}})_p&:=\{0\to\mathcal{M}^{\mathrm{left}}_p\longrightarrow\Omega^1_X\otimes\mathcal{M}^{\mathrm{left}}_{p+1}\longrightarrow\cdots\longrightarrow\Omega^n_X\otimes\underset{\bullet}{\mathcal{M}^{\mathrm{left}}_{p+n}}\to 0\}\cdot z^p,\\ ({}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{right}})_p&:=\{0\to\mathcal{M}^{\mathrm{right}}_{p-n}\otimes\Theta_{X,n}\to\cdots\to\mathcal{M}^{\mathrm{right}}_{p-1}\otimes\Theta_{X,1}\to\underset{\bullet}{\mathcal{M}^{\mathrm{right}}_p}\to 0\}\cdot z^p,\end{aligned}$$

and the side-changing functors preserve the grading (see (8.2.2 ∗)). If $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ is the Rees module of an F-filtered $\mathcal{D}_X$-module $\mathcal{M}$, we regard ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}$ as the Rees complex of the filtered complex

$$\begin{aligned}F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}}&:=\{0\to F_p\mathcal{M}^{\mathrm{left}}\to\Omega^1_X\otimes F_{p+1}\mathcal{M}^{\mathrm{left}}\to\cdots\to\Omega^n_X\otimes\underset{\bullet}{F_{p+n}}\mathcal{M}^{\mathrm{left}}\to 0\},\\ F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{right}}&\\ &:=\{0\to F_{p-n}\mathcal{M}^{\mathrm{right}}\otimes\Theta_{X,n}\to\cdots\to F_{p-1}\mathcal{M}^{\mathrm{right}}\otimes\Theta_{X,1}\to\underset{\bullet}{F_p}\mathcal{M}^{\mathrm{right}}\to 0\}.\end{aligned}$$

Recall that the side-changing functor for filtered $\mathcal{D}_X$-modules (8.2.2 ∗∗) amounts to

$$F_p\mathcal{M}^{\mathrm{right}}=\omega_X\otimes F_{p+n}\mathcal{M}^{\mathrm{left}}.$$

Exercise 8.24 clearly shows that ${}^{\mathrm{p}}\mathrm{DR}$ is a functor from the category of left (resp. right) $\widetilde{\mathcal{D}}_X$-modules to the category of bounded complex of sheaves of $\widetilde{\mathbb{C}}$-modules. It can be extended to a functor between the corresponding bounded derived categories.

***8.4.10. Definition* (Contraction by a one-form).** The contraction morphism

$$\widetilde{\Theta}_{X,k} \otimes \widetilde{\Omega}^1_X \xrightarrow{\;\lrcorner\;} \widetilde{\Theta}_{X,k-1}$$

is the unique morphism such that the following diagram commutes:

$$\begin{array}{ccc}
\widetilde{\omega}_X \otimes \widetilde{\Theta}_{X,k} \otimes \widetilde{\Omega}^1_X & \xrightarrow{\mathrm{Id}\otimes \lrcorner} & \widetilde{\omega}_X \otimes \widetilde{\Theta}_{X,k-1} \\
\big\downarrow & & \big\downarrow \\
\widetilde{\Omega}^{n-k}_X \otimes \widetilde{\Omega}^1_X & \xrightarrow{\quad\wedge\quad} & \widetilde{\Omega}^{n-k+1}_X
\end{array}$$

where the vertical morphisms are induced by (8.4.5), i.e., $\widetilde{\omega}(\xi \lrcorner \eta^1) = \widetilde{\omega}(\xi) \wedge \eta^1$.

8.4.11. Action of a closed one-form on the de Rham complex. Let η be a *closed* holomorphic one-form on X. Then the exterior product by η induces a morphism

$$\eta \wedge \bullet : {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})[1].$$

Indeed, for a local section m of $\widetilde{\mathcal{M}}$ and a k-form ω, we have

$$\widetilde{\nabla}((\eta\wedge\omega)\otimes m) = (\widetilde{\mathrm{d}}\eta\wedge\omega)\otimes m - \eta\wedge\widetilde{\nabla}(\omega\otimes m) = -\eta\wedge\widetilde{\nabla}(\omega\otimes m),$$

so that the morphism $\eta\wedge$ commutes with the differentials (see Remark 8.4.2).

According to Lemma 8.4.7, we can define the contraction

$$\bullet \lrcorner\, \eta : \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \longrightarrow \mathrm{Sp}(\widetilde{\mathcal{D}}_X)[1]$$

as the unique morphism which corresponds to $\eta\wedge\bullet$ on ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)$ via ι. According to Remark 8.4.4(1), we can define in a similar way a morphism of complexes

$$\bullet \lrcorner\, \eta : {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})[1]. \tag{8.4.12}$$

Note that, if $\eta = \widetilde{\mathrm{d}}f$ is *exact*, then the induced morphism

$$\eta\wedge\; : H^i\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}) \longrightarrow H^{i+1}\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}})$$

is zero. Indeed, if a local section μ of $\widetilde{\Omega}^k_X\otimes\widetilde{\mathcal{M}}^{\mathrm{left}}$ satisfies $\widetilde{\nabla}\mu = 0$, then $\widetilde{\mathrm{d}}f\wedge\mu = \widetilde{\nabla}(f\mu)$. In other words, the morphism $\eta\wedge$ on the cohomology only depends on the class of η in $H^1\Gamma(X,(\widetilde{\Omega}^\bullet_X,\widetilde{\mathrm{d}}))$. The same result holds when we make η acting on the complex $\Gamma(X,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}))$, and a similar result holds for the action $\bullet\lrcorner\,\eta$ on ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}})$.

8.4.13. C^∞ de Rham and Spencer complexes. Let us denote by $(\widetilde{\mathcal{E}}^{(\bullet,0)}_X,\widetilde{\mathrm{d}}')$ the complex $\widetilde{\mathcal{C}}^\infty_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Omega}^\bullet_X$ with the differential induced by $\widetilde{\mathrm{d}}$ (here, we assume $\bullet\geqslant 0$). More generally, let us set

$$\widetilde{\mathcal{E}}^{(p,q)}_X = \widetilde{\Omega}^p_X\wedge\widetilde{\mathcal{E}}^{(0,q)}_X = \widetilde{\mathcal{E}}^{(p,0)}_X\wedge\widetilde{\mathcal{E}}^{(0,q)}_X$$

and let d'' be the (usual) anti-holomorphic differential. For every p, the complex $(\widetilde{\mathcal{E}}^{(p,\bullet)}_X,\mathrm{d}'')$ is a resolution of $\widetilde{\Omega}^p_X$ (note that, here, d'' is not affected by z, hence is homogeneous of degree zero with respect to the grading). We therefore have a complex $(\widetilde{\mathcal{E}}^\bullet_X,\widetilde{\mathrm{d}})$, which is the single complex associated to the double complex $(\widetilde{\mathcal{E}}^{(\bullet,\bullet)}_X,\widetilde{\mathrm{d}}',\mathrm{d}'')$. In particular, since $\widetilde{\mathcal{D}}_X$ is $\widetilde{\mathcal{O}}_X$-locally free, we have a natural quasi-isomorphism of complexes of right $\widetilde{\mathcal{D}}_X$-modules:

$$(\widetilde{\Omega}^\bullet_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,\widetilde{\nabla}) \xrightarrow{\;\sim\;} (\widetilde{\mathcal{E}}^\bullet_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X,\widetilde{D}) =: \mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X),\quad \widetilde{D} := \mathrm{Id}\otimes\widetilde{\nabla} + \mathrm{d}''\otimes\mathrm{Id},$$

by sending holomorphic k-forms to $(k,0)$-forms. Given a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{left}}$, we can define similarly the C^∞ de Rham complex

$$ {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}}) := (\widetilde{\mathcal{E}}_X^{n+\bullet} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}, \widetilde{D}), \quad \widetilde{D} := (-1)^n(\mathrm{Id} \otimes \widetilde{\nabla} + \mathrm{d}'' \otimes \mathrm{Id}). $$

As in Exercise 8.24(2), by using that $(\widetilde{\mathcal{E}}_X^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{D})$ is a complex of right $\widetilde{\mathcal{D}}_X$-modules, we obtain a quasi-isomorphism:

$$ {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}}). $$

From the commutative diagram

$$ \begin{array}{ccc} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} & \xrightarrow{\sim} & {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}) \\ \wr\big\downarrow & & \big\downarrow \\ {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} & \xrightarrow{\sim} & {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}}) \end{array} $$

we conclude that the right vertical morphism is a quasi-isomorphism.

We can argue similarly for defining the C^∞ Spencer complex of a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{right}}$. We resolve

$$ \widetilde{\Theta}_{X,k} \xrightarrow{\sim} (\widetilde{\Theta}_{X,k} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}_X^{(0,\bullet)}, \mathrm{Id} \otimes \mathrm{d}''). $$

Let us set, for each $\ell \in \mathbb{Z}$,

$$ \widetilde{\mathrm{Sp}}_X^{\infty,\ell} = \bigoplus_{j-i=\ell} (\widetilde{\Theta}_{X,i} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}_X^{(0,j)}). $$

For any right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{right}}$, we define

$$ {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}) := (\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}_X^{\infty,\bullet}, \widetilde{\delta}_{\widetilde{\mathcal{M}}}^{\infty}), $$

where the differential $\widetilde{\delta}_{\widetilde{\mathcal{M}}}^{\infty}$ is defined in Exercise 8.28. We will use the notation $\mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X)$ for the C^∞ Spencer complex of $\widetilde{\mathcal{D}}_X$ with its right structure. Then, arguing as in Exercise 8.24(1), we obtain a quasi-isomorphism

$$ \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}), $$

from which we deduce as above a quasi-isomorphism

$$ {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}). $$

Recall that $\widetilde{\mathcal{C}}_X^\infty$ is flat over $\widetilde{\mathcal{O}}_X$, hence so are $\widetilde{\mathcal{E}}_X^k$ and $\widetilde{\mathrm{Sp}}_X^{\infty,\ell}$. The terms of ${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{D}}_X)$ and $\mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X)$ are flat over $\widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{D}}_X$, and are c-soft sheaves, so that any short exact sequence $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ gives rise to an exact sequence of the corresponding C^∞ de Rham complexes, which consist of c-soft sheaves.

Moreover, by Exercise 8.28, if $\widetilde{\mathcal{M}}^{\mathrm{right}}$ corresponds to $\widetilde{\mathcal{M}}^{\mathrm{left}}$ by side-changing, then ${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}})$.

8.5. Induced $\widetilde{\mathcal{D}}$-modules

A subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ proves very useful in many places, namely that of *induced right* $\widetilde{\mathcal{D}}_X$*-modules.* Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. It induces a *right* $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, called an *induced right* $\widetilde{\mathcal{D}}_X$*-module.*

8.5.1. Remark. We note that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has two structures of $\widetilde{\mathcal{O}}_X$-module, one coming from the action on $\widetilde{\mathcal{L}}$ and the other one from the right $\widetilde{\mathcal{D}}_X$-module structure, and they do not coincide. We will mainly use the right one. The "left" $\widetilde{\mathcal{O}}_X$-module structure on $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ will only be used when noticing that some naturally defined sheaves of $\widetilde{\mathbb{C}}$-vector spaces are in fact sheaves of $\widetilde{\mathcal{O}}_X$-modules. On the other hand, $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has a canonical structure of right $\widetilde{\mathcal{D}}_X$-module.

The category $\mathsf{Mod}_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ of right induced differential modules is the full subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ consisting of induced $\widetilde{\mathcal{D}}_X$-modules (i.e., we consider as morphisms all $\widetilde{\mathcal{D}}_X$-linear morphisms). It is an additive category (but not an abelian category).

***8.5.2. Proposition* (The canonical resolution by induced $\widetilde{\mathcal{D}}_X$-modules)**
Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Then the complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is isomorphic to a complex of right induced $\widetilde{\mathcal{D}}_X$-modules which is a resolution of $\widetilde{\mathcal{M}}$ as such.

One should not confuse $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ with $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}(\widetilde{\mathcal{M}})$ as in Exercise 8.24(1), where a tensor product over $\widetilde{\mathcal{D}}_X$ is considered. A good preliminary for the following proof is Exercise 8.29.

***Proof*.** (See Exercise 8.30 for a detailed proof.) That the terms of the complex are induced $\widetilde{\mathcal{D}}_X$-modules follows from Exercise 8.19(4) applied to $\widetilde{\mathcal{L}} = \widetilde{\Theta}_{X,k}$. Since $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a $\widetilde{\mathcal{D}}_X$-module, hence as an $\widetilde{\mathcal{O}}_X$-module, and since the terms of $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ are $\widetilde{\mathcal{O}}_X$-locally free, we conclude that $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{M}}$. $\square$

Let $\mathsf{C}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ the category of $\star$-bounded complexes of the additive category $\mathsf{Mod}_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ and let $\mathsf{K}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ be the corresponding homotopy category. Since $\mathrm{Sp}\,\widetilde{\mathcal{D}}_X$ is a complex of locally free $\widetilde{\mathcal{O}}_X$-modules, the functor $\widetilde{\mathcal{M}}^\bullet \to \widetilde{\mathcal{M}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\,\widetilde{\mathcal{D}}_X$ is a functor of triangulated categories, and sends acyclic complexes to acyclic complexes according to the previous proposition. It induces therefore a functor $\mathsf{D}^\star(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$.

***8.5.3. Corollary* (Equivalence of $\mathsf{D}^\star(\widetilde{\mathcal{D}}_X)$ with $\mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$).** *The natural functor $\mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^\star(\widetilde{\mathcal{D}}_X)$ is an equivalence of categories, and the functor $\mathsf{D}^\star(\widetilde{\mathcal{D}}_X) \to \mathsf{D}^\star_{\mathrm{i}}(\widetilde{\mathcal{D}}_X)$ induced by $\widetilde{\mathcal{M}}^\bullet \mapsto \widetilde{\mathcal{M}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\,\widetilde{\mathcal{D}}_X$ is a quasi-inverse functor.* $\square$

8.6. Pullback and external product of $\widetilde{\mathcal{D}}$-modules

8.6.a. Pullback of left $\widetilde{\mathcal{D}}$-modules. Let us begin with some relative complements to Section 8.2. Let $f : X \to Y$ be a holomorphic map between analytic manifolds.

For any local section ξ of the sheaf $\widetilde{\Theta}_X$ of z-vector fields on X, $Tf(\xi)$ is a local section of $\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\Theta}_Y$. We hence have an $\widetilde{\mathcal{O}}_X$-linear map

$$Tf : \widetilde{\Theta}_X \longrightarrow \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\Theta}_Y,$$

and dually

$$T^*f : \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} \widetilde{\Omega}^1_Y \longrightarrow \widetilde{\Omega}^1_X.$$

Therefore, if $\widetilde{\mathcal{N}}$ is any left $\widetilde{\mathcal{D}}_Y$-module, the connection $\widetilde{\nabla}^Y$ on $\widetilde{\mathcal{N}}$ can be lifted as a connection

$$\widetilde{\nabla}^X : f^*\widetilde{\mathcal{N}} := \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}} \longrightarrow \widetilde{\Omega}^1_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}} = \widetilde{\Omega}^1_X \otimes_{\widetilde{\mathcal{O}}_X} f^*\widetilde{\mathcal{N}}$$

by setting

$$\widetilde{\nabla}^X = \widetilde{\mathrm{d}} \otimes \mathrm{Id} + (T^*f \otimes \mathrm{Id}_{\widetilde{\mathcal{N}}}) \circ (1 \otimes \widetilde{\nabla}^Y). \tag{8.6.1}$$

8.6.2. Lemma. *The connection $\widetilde{\nabla}^X$ on $f^*\widetilde{\mathcal{N}}$ is integrable and defines the structure of a left $\widetilde{\mathcal{D}}_X$-module on $\widetilde{\mathcal{N}}$.*

Proof. Exercise 8.31(1). □

This leads to the first definition of the pullback functor for $\widetilde{\mathcal{D}}_Y$-modules.

8.6.3. Definition. The left $\widetilde{\mathcal{D}}_X$-module corresponding to $(f^*\widetilde{\mathcal{N}}, \widetilde{\nabla}_X)$ is the pullback of $\widetilde{\mathcal{N}}$ in the sense of $\widetilde{\mathcal{D}}$-modules, and is denoted ${}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$.

However, this definition is not suited for considering derived inverse images, since the sheaves $\mathcal{T}or_j^{f^{-1}\widetilde{\mathcal{O}}_Y}(\widetilde{\mathcal{O}}_X, f^{-1}\widetilde{\mathcal{N}})$ are not obviously equipped with an integrable connection. In order to overcome this difficulty, we introduce the transfer modules.

***8.6.4. Definition* (Transfer modules).**

(1) The sheaf $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = {}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_Y$ is a left-right $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodule when using the natural right $f^{-1}\widetilde{\mathcal{D}}_Y$-module structure and the left $\widetilde{\mathcal{D}}_X$-module introduced above (see Exercise 8.31(2)). It has a canonical section $\mathbf{1}$.

Correspondingly, we have $F_p\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}F_p\mathcal{D}_Y$ and the previous definition reads $R_F\mathcal{D}_{X\to Y} = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}R_F\mathcal{D}_Y$ (with $\widetilde{\mathcal{O}}_X = \mathcal{O}_X[z]$ and $\widetilde{\mathcal{O}}_Y = \mathcal{O}_Y[z]$).

(2) The sheaf $\widetilde{\mathcal{D}}_{Y\leftarrow X}$ is obtained from $\widetilde{\mathcal{D}}_{X\to Y}$ by using the usual side-changing functor on both sides:

$$\widetilde{\mathcal{D}}_{Y\leftarrow X} = \mathcal{H}om_{f^{-1}\widetilde{\mathcal{O}}_Y}\big(\widetilde{\omega}_Y, \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}\big).$$

In the filtered/graded setting, this definition reads

$$F_p\mathcal{D}_{Y\leftarrow X} = \mathcal{H}om_{f^{-1}\mathcal{O}_Y}\big(\omega_Y, \omega_X \otimes_{\mathcal{O}_X} F_{p+n-m}\mathcal{D}_{X\to Y}\big).$$

8.6.5. Example.

(1) One recovers $\widetilde{\mathcal{D}}_X$ as $\widetilde{\mathcal{D}}_{X\to X}$ for the identity map $\mathrm{Id} : X \to X$, so that $\widetilde{\mathcal{D}}_{X\leftarrow X}$ is identified with $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}\big(\widetilde{\omega}_X, \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\big)$.

(2) On the other hand, if Y is reduced to a point, so that f^{-1} is the constant map, we have $\widetilde{\mathcal{D}}_{X\to \mathrm{pt}} = \widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{D}}_{X\leftarrow \mathrm{pt}} = \widetilde{\omega}_X$.

We can now give a better definition of the pullback of a left $\widetilde{\mathcal{D}}_Y$-module $\widetilde{\mathcal{N}}$, better in the sense that it is defined inside of the category of $\widetilde{\mathcal{D}}$-modules. It also enables one to give a definition of a derived inverse image. The coincidence between both definitions can be obtained by Exercise 8.38.

***8.6.6. Definition* (of the pullback of a left $\widetilde{\mathcal{D}}_Y$-module).** Let $\widetilde{\mathcal{N}}$ be a left $\widetilde{\mathcal{D}}_Y$-module. The pullback ${}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$ is the left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y}\widetilde{\mathcal{N}}$.

The *derived pullback* $\boldsymbol{L}\,{}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$ is now defined by the usual method, i.e., by taking a flat resolution of $\widetilde{\mathcal{N}}$ as a left $\widetilde{\mathcal{D}}_Y$-module, or by taking a right $f^{-1}\widetilde{\mathcal{D}}_Y$-flat resolution of $\widetilde{\mathcal{D}}_{X\to Y}$ by $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodules. The cohomology modules $\boldsymbol{L}^j\,{}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} := \mathcal{T}or_j^{f^{-1}\widetilde{\mathcal{D}}_Y}(\widetilde{\mathcal{D}}_{X\to Y}, f^{-1}\widetilde{\mathcal{N}})$ are left $\widetilde{\mathcal{D}}_X$-modules.

8.6.7. Remark. If $f:X\to Y$ is a *smooth* morphism, that is, locally expressed as the projection of a product, then for any left $\widetilde{\mathcal{D}}_Y$-module $\widetilde{\mathcal{N}}$, we have $\boldsymbol{L}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} = {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$, i.e., $\boldsymbol{L}^j\,{}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} = 0$ for $j\neq 0$. Indeed, in such a case, $\widetilde{\mathcal{D}}_{X\to Y}$ is $f^{-1}\widetilde{\mathcal{D}}_Y$-flat (Exercise 8.35).

8.6.8. Side-changing and pullback. The pullback for a right $\widetilde{\mathcal{D}}_Y$-module $\widetilde{\mathcal{N}}^{\mathrm{right}}$ is obtained by applying the side-changing functor at the source and the target. Let $\widetilde{\mathcal{N}}^{\mathrm{left}}$ be the left $\widetilde{\mathcal{D}}_Y$-module associated with $\widetilde{\mathcal{N}}^{\mathrm{right}}$, so that $\widetilde{\mathcal{N}}^{\mathrm{right}} = \widetilde{\omega}_Y\otimes\widetilde{\mathcal{N}}^{\mathrm{left}}$. Then we set

$$
{}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}^{\mathrm{right}} := \widetilde{\omega}_X\otimes{}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}^{\mathrm{left}},
$$

and similarly with $\boldsymbol{L}\,{}_{\mathrm{D}}f^*$. Notice the change of grading by $\dim Y-\dim X$, due to the grading of $\widetilde{\omega}_X\otimes f^{-1}\widetilde{\omega}_Y^\vee$, i.e., we have

$$
({}_{\mathrm{D}}f^*\mathcal{N}^{\mathrm{right}})_p := \omega_X\otimes({}_{\mathrm{D}}f^*\mathcal{N}^{\mathrm{left}})(m-n)_p = \omega_X\otimes f^*\mathcal{N}^{\mathrm{left}}_{p+n-m}.
$$

***8.6.9. Example* (Pull-back of a filtered module).** Assume that $\widetilde{\mathcal{N}}$ is the Rees module $R_F\mathcal{N}$ of a filtered left $\mathcal{D}_Y$-module $(\mathcal{N}, F_\bullet\mathcal{N})$. Then $f^*\mathcal{N} = \mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N}$ is equipped with the filtration

$$
F_pf^*\mathcal{N} = \mathrm{image}\big[\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}F_p\mathcal{N}\to\mathcal{O}_X\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{N}\big],
$$

and the corresponding Rees module $R_Ff^*\mathcal{N}$ is equal to $f^*\widetilde{\mathcal{N}}/z$-torsion. If for example f is a smooth morphism, so that $\mathcal{O}_X$ is $f^{-1}\mathcal{O}_Y$-flat, then $\widetilde{\mathcal{O}}_X$ is also $f^{-1}\widetilde{\mathcal{O}}_Y$-flat and $f^*\widetilde{\mathcal{N}} = R_Ff^*\mathcal{N}$.

We also have $F_pf^*\mathcal{N}^{\mathrm{right}} = \omega_X\otimes F_{p+n-m}\mathcal{N}^{\mathrm{left}}$, after (8.2.2 $**$).

8.6.b. External product. We start with the case of $\mathcal{D}_X$-modules. Let X,Y be two complex manifolds and let p_X,p_Y be the projections from $X\times Y$ to X and Y respectively. For any pair of sheaves $\mathcal{F}_X,\mathcal{F}_Y$ of $\mathbb{C}$-vector spaces on X and Y respectively, let us set $\mathcal{F}_X\boxtimes_{\mathbb{C}}\mathcal{F}_Y := p_X^{-1}\mathcal{F}_X\otimes_{\mathbb{C}} p_Y^{-1}\mathcal{F}_Y$.

By using an analogue of Theorem 8.8.6(2), one obtains that $\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y$ and $\mathcal{D}_X\boxtimes_{\mathbb{C}}\mathcal{D}_Y$ are coherent sheaves of rings on $X\times Y$. Moreover, $\mathcal{O}_{X\times Y}$ is flat over $\mathcal{O}_X \boxtimes_{\mathbb{C}} \mathcal{O}_Y$ (as can be seen by applying [**Ser56**, Prop. 28] to each germ $\mathcal{O}_{X\times Y,(x,y)}$ and the localization of $\mathcal{O}_{X,x} \boxtimes_{\mathbb{C}} \mathcal{O}_{Y,y}$), and we also have

$$\mathcal{D}_{X\times Y} = \mathcal{O}_{X\times Y} \otimes_{(\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y)} (\mathcal{D}_X \boxtimes_{\mathbb{C}} \mathcal{D}_Y) = (\mathcal{D}_X \boxtimes_{\mathbb{C}} \mathcal{D}_Y) \otimes_{(\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y)} \mathcal{O}_{X\times Y}.$$

For an $\mathcal{O}_X$-module $\mathcal{L}_X$ (resp. a $\mathcal{D}_X$-module $\mathcal{M}_X$) and an $\mathcal{O}_Y$-module $\mathcal{L}_Y$ (resp. a $\mathcal{D}_Y$-module $\mathcal{M}_Y$), set

$$\begin{aligned} \mathcal{L}_X \boxtimes_{\mathcal{O}} \mathcal{L}_Y &= (\mathcal{L}_X \boxtimes_{\mathbb{C}} \mathcal{L}_Y) \otimes_{\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y} \mathcal{O}_{X\times Y} \\ \text{resp.}\qquad \mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y &= (\mathcal{M}_X \boxtimes_{\mathbb{C}} \mathcal{M}_Y) \otimes_{\mathcal{O}_X\boxtimes_{\mathbb{C}}\mathcal{O}_Y} \mathcal{O}_{X\times Y} \\ &= (\mathcal{M}_X \boxtimes_{\mathbb{C}} \mathcal{M}_Y) \otimes_{\mathcal{D}_X\boxtimes_{\mathbb{C}}\mathcal{D}_Y} \mathcal{D}_{X\times Y}. \end{aligned}$$

Clearly, if $\mathcal{L}_X, \mathcal{L}_Y$ are $\mathcal{O}$-coherent, then $\mathcal{L}_X \boxtimes_{\mathbb{C}} \mathcal{L}_Y$ is $\mathcal{O}_X \boxtimes_{\mathbb{C}} \mathcal{O}_Y$-coherent. It follows that $\mathcal{L}_X \boxtimes_{\mathcal{O}} \mathcal{L}_Y$ is $\mathcal{O}_{X\times Y}$-coherent. Similarly, if $\mathcal{M}_X, \mathcal{M}_Y$ are $\mathcal{D}$-coherent, $\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y$ is $\mathcal{D}_{X\times Y}$-coherent.

We now consider the case of $\widetilde{\mathcal{D}}_X$-modules. For any pair of sheaves $\mathcal{F}_X, \mathcal{F}_Y$ of $\widetilde{\mathbb{C}}$-modules on X and Y respectively, we set $\mathcal{F}_X \boxtimes_{\widetilde{\mathbb{C}}} \mathcal{F}_Y := p_X^{-1}\mathcal{F}_X \otimes_{\widetilde{\mathbb{C}}} p_Y^{-1}\mathcal{F}_Y$. If we identify $\widetilde{\mathbb{C}} \boxtimes_{\mathbb{C}} \widetilde{\mathbb{C}}$ with $\mathbb{C}[z_1, z_2]$, then $\mathcal{F}_X \boxtimes_{\mathbb{C}} \mathcal{F}_Y$ is a $\mathbb{C}[z_1, z_2]$-module and

$$\mathcal{F}_X \boxtimes_{\widetilde{\mathbb{C}}} \mathcal{F}_Y = \operatorname{Coker}\bigl[\mathcal{F}_X \boxtimes_{\mathbb{C}} \mathcal{F}_Y \xrightarrow{z_1 - z_2} \mathcal{F}_X \boxtimes_{\mathbb{C}} \mathcal{F}_Y\bigr].$$

As a consequence, we will obtain a behaviour of $\boxtimes_{\widetilde{\mathbb{C}}}$ similar to that of $\boxtimes_{\mathbb{C}}$ only with a supplementary $\mathbb{C}[z]$-flatness (i.e., strictness) condition for $\mathcal{F}_X, \mathcal{F}_Y$.

We have $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y = (\mathcal{O}_X \boxtimes_{\mathbb{C}} \mathcal{O}_Y) \otimes_{\mathbb{C}} \mathbb{C}[z]$, therefore $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y$ is a coherent sheaf of rings, and one also checks that $\widetilde{\mathcal{D}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{D}}_Y$ is coherent. Moreover, from the above flatness result, we find that $\widetilde{\mathcal{O}}_{X\times Y}$ is flat over $\widetilde{\mathcal{O}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{O}}_Y$.

For *strict* $\widetilde{\mathcal{O}}$-modules $\widetilde{\mathcal{L}}_X, \widetilde{\mathcal{L}}_Y$ (resp. $\widetilde{\mathcal{D}}$-modules $\widetilde{\mathcal{M}}_X, \widetilde{\mathcal{M}}_Y$), one defines the external product $\widetilde{\mathcal{L}}_X \boxtimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{L}}_Y$ (resp. $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$) as for $\mathcal{O}$-modules (resp. $\mathcal{D}$-modules). In such a case, we have $\widetilde{\mathcal{M}}_X = R_F\mathcal{M}_X$ for some $F_\bullet\mathcal{D}_X$-filtration $F_\bullet\mathcal{M}$, and similarly for Y, according to Proposition 5.1.8(1).

***8.6.10. Lemma* (See [Kas03, §4.3]).** *If $F_\bullet\mathcal{M}_X, F_\bullet\mathcal{M}_Y$ are $F_\bullet\mathcal{D}$-filtrations, then*

$$F_j(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y) := \sum_{k+\ell=j} F_k\mathcal{M}_X \boxtimes_{\mathcal{O}} F_\ell\mathcal{M}_Y$$

is an $F_\bullet\mathcal{D}$-filtration of $\mathcal{M}_X \boxtimes_{\widetilde{\mathcal{D}}} \mathcal{M}_Y$ for which

$$\operatorname{gr}^F(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y) = \operatorname{gr}^F\mathcal{M}_X \boxtimes_{\operatorname{gr}^F\mathcal{D}} \operatorname{gr}^F\mathcal{M}_Y.$$

Proof. We set $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. Let us start by considering $\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y$ as a $\mathbb{C}[z_1, z_2]$-module. One checks that multiplication by $z_1 - z_2$ is injective on $\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y$. Its cokernel is identified with $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y$, where the action of z is induced either by that of $z_1 \boxtimes 1$ or that of $1 \boxtimes z_2$. But $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y$ is also $\widetilde{\mathbb{C}}$-torsion free, and defining $F_\bullet(\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y)$ by a formula similar to that of the lemma amounts to setting (due to torsion-freeness)

$$R_F(\mathcal{M}_X \boxtimes_{\mathbb{C}} \mathcal{M}_Y) = \widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y.$$

We have a commutative diagram of short exact sequences

$$\begin{tikzcd}
\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y \arrow[r, hook, "z_1 - z_2"] \arrow[d, hook, "z_1"'] & \widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y \arrow[r, two heads] \arrow[d, hook, "z_1"'] & \widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y \arrow[d, hook, "z"'] \\
\widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y \arrow[r, hook, "z_1 - z_2"] \arrow[d, two heads] & \widetilde{\mathcal{M}}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y \arrow[r, two heads] \arrow[d, two heads] & \widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{M}}_Y \arrow[d, two heads] \\
\mathrm{gr}^F \mathcal{M}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y \arrow[r, hook, "-z_2"] & \mathrm{gr}^F \mathcal{M}_X \boxtimes_{\mathbb{C}} \widetilde{\mathcal{M}}_Y \arrow[r, two heads] & C
\end{tikzcd}$$

and the term C is identified with $\mathrm{gr}^F(\mathcal{M}_X \boxtimes_{\mathbb{C}} \mathcal{M}_Y)$ when considered as the cokernel of the vertical arrow, while it is identified with $\mathrm{gr}^F \mathcal{M}_X \boxtimes_{\mathbb{C}} \mathrm{gr}^F \mathcal{M}_Y$ when considered as the cokernel of the horizontal one.

Once this identification is obtained, the formula of the lemma is simply deduced by tensoring with $\mathcal{O}_{X\times Y}$ over $\mathcal{O}_X \boxtimes_{\mathbb{C}} \mathcal{O}_Y$. $\square$

8.6.11. Remark. We will interpret this property in terms of flatness in Exercise 10.12.

8.7. Pushforward of $\widetilde{\mathcal{D}}$-modules

Let $f : X \to Y$ be a holomorphic map between complex manifolds. The pullback of a C^∞ function on Y is easy to define and, by adjunction, the pushforward of a current of degree 0 is easily defined provided that f is proper. On the other hand, the pullback of a form of maximal degree on Y is usually not of maximal degree on X, so the pushforward of a distribution is not defined in an easy way. This example is an instance of the fact that the pushforward of $\widetilde{\mathcal{D}}_X$-modules by a proper holomorphic map should be defined in a simple way for right $\widetilde{\mathcal{D}}_X$-modules, while for left $\widetilde{\mathcal{D}}_X$-modules one should use the side-changing functors.

8.7.1. Remark.

(1) We will distinguish the usual direct image and the direct image with proper supports for the sake of completeness. However, in the main part of this text, we always assume properness of the map on the support of the object to which it is applied. Therefore, this distinction will not be useful.

(2) The pushforward functor by a map $f : X \to Y$ applied to a $\widetilde{\mathcal{D}}_X$-module takes values in the derived category $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)$.

8.7.a. Definition and examples. We aim at defining the derived pushforward of a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ by a formula using the transfer module (see Definition 8.6.4(1)) like

$$\boldsymbol{R}f_*\big(\widetilde{\mathcal{M}} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_{X\to Y}\big).$$

However, the derived tensor product $\widetilde{\mathcal{M}} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is a priori an object of $\mathsf{D}^-(\widetilde{\mathcal{D}}_X)^{\mathrm{right}}$ and we need to argue that f has finite cohomological dimension in order to apply $\boldsymbol{R}f_\star$ to it. In order to avoid such an argument, we will simply make explicit a finite resolution of $\widetilde{\mathcal{D}}_{X\to Y}$ as a $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodule whose terms are $\widetilde{\mathcal{D}}_X$-locally free: this is the *relative Spencer complex* $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ (see Exercise 8.39). Recall also that the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$, which was defined in 8.4.3, is a complex of locally free left $\widetilde{\mathcal{D}}_X$-modules (hence locally free $\widetilde{\mathcal{O}}_X$-modules) and is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module. There is an isomorphism of complexes of bi-modules (see Exercise 8.39)

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}. \tag{8.7.2}$$

On the right-hand term, the left $\widetilde{\mathcal{O}}_X$-structure on each factor is used for the tensor product, and it is a complex of $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodules: the right $f^{-1}\widetilde{\mathcal{D}}_Y$ structure is the trivial one; the left $\widetilde{\mathcal{D}}_X$-structure is that defined by Exercise 8.12(1). It is a resolution of

$$\widetilde{\mathcal{O}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{D}}_{X\to Y}$$

as a left $\widetilde{\mathcal{D}}_X$-module, in a way compatible with the right $f^{-1}\widetilde{\mathcal{D}}_Y$-module structure (see Exercise 8.40).

For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we will use the identification

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$$

(see Exercise 8.39).

***8.7.3. Definition* (Pushforward of a $\widetilde{\mathcal{D}}$-module).** Setting $\star = *$ or $\star = !$, the *direct image* ${}_{\mathrm{D}}f_\star$ is the functor from $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)^{\mathrm{right}}$ to $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)^{\mathrm{right}}$ defined[(1)] by

$$ {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} := \boldsymbol{R}f_*\big(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big) \simeq \boldsymbol{R}f_*\,\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}). \tag{8.7.3 $*$}$$

For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we set

$$ {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} := \big({}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}^{\mathrm{right}}\big)^{\mathrm{left}}. \tag{8.7.3 $**$}$$

The cohomology modules are objects of $\mathsf{Mod}(\widetilde{\mathcal{D}}_Y)$ (right or left, respectively) and are denoted by

$$ {}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}} := H^j\,{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}.$$

One can give a formula for the pushforward of left $\widetilde{\mathcal{D}}_X$-modules which looks like that for the right $\widetilde{\mathcal{D}}_X$-modules.

8.7.4. Lemma. *For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we have*

$$ {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} \simeq \boldsymbol{R}f_\star\big(\widetilde{\mathcal{D}}_{Y\leftarrow X} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}\big).$$

Proof. See Definition 8.6.4(2) for the transfer module. The meaning of $\widetilde{\mathcal{D}}_{Y\leftarrow X} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}$ is $\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$, with an obvious notation. For the proof, see Exercise 8.42. ◻

[(1)]Recall that, if $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, then $\mathsf{Mod}(\widetilde{\mathcal{D}}_X) := \mathsf{Modgr}(R_F\mathcal{D}_X)$.

8.7.5. Remarks.

(1) If f is proper, or proper on the support of $\widetilde{\mathcal{M}}$, we have an isomorphism in the category $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)$:

$$ {}_{\mathrm{D}}f_!\widetilde{\mathcal{M}} \xrightarrow{\sim} {}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}. $$

(2) If $\mathcal{F}$ is any sheaf on X, we have $R^jf_*\mathcal{F}=0$ and $R^jf_!\mathcal{F}=0$ for $j\notin[0,2\dim X]$. Therefore, taking into account the length $\dim X$ of the relative Spencer complex, we find that ${}_{\mathrm{D}}f_*^{(j)}\widetilde{\mathcal{M}}$ and ${}_{\mathrm{D}}f_!^{(j)}\widetilde{\mathcal{M}}$ are zero for $j\notin[-\dim X,2\dim X]$: we say that ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}},{}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}$ have *bounded amplitude* (see Remark 8.7.13 for a more precise estimate of the amplitude).

(3) See Exercise 8.51 for a simple expression of the pushforward in terms of differential forms.

Let us give natural examples of pushforward of $\widetilde{\mathcal{D}}_X$-modules.

8.7.6. Example **(Pushforward of a $\widetilde{\mathcal{D}}$-module by a closed embedding)**

If ι is a closed embedding, it is proper, so the ordinary pushforward and the pushforward with proper support will be the same. Since $\widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\mathcal{D}}_X$-locally free in this case (Exercise 8.34), we have, for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$,

$$ {}_{\mathrm{D}}\iota_*^{(0)}\widetilde{\mathcal{M}}=\iota_*(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_{X\to Y}),\qquad {}_{\mathrm{D}}\iota_*^{(k)}\widetilde{\mathcal{M}}=0 \text{ if } k\neq 0, $$

so that we will simply denote ${}_{\mathrm{D}}\iota_*^{(0)}$ by ${}_{\mathrm{D}}\iota_*$, and it is a functor $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)\mapsto\mathsf{Mod}(\widetilde{\mathcal{D}}_Y)$. Similarly, for a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ we can write

$$ {}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}=\iota_*(\widetilde{\mathcal{D}}_{Y\leftarrow X}\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}). $$

8.7.7. Example **(Pushforward by a graph inclusion (see also Exercise 8.45))**

Let $g:X\to\mathbb{C}$ be a holomorphic function and let $\iota_g:X\hookrightarrow X\times\mathbb{C}$ denote the graph embedding of g, with coordinate t on the factor $\mathbb{C}$. A special case is when $g\equiv 0$, so that the formulas below can be simplified by replacing every occurrence of g by zero. We denote $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ by $\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$.

(1) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Recall (Exercise 8.34) that $\widetilde{\mathcal{D}}_{X\to X\times\mathbb{C}}\simeq\iota_{g*}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t]$. Then ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}\simeq\iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ with the right $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-action defined locally be the following formulas (recall that for a holomorphic function $h(\boldsymbol{x},t,z)$, the bracket $[\widetilde{\partial}_t^k,h]$ can be written as $\sum_{j<k}a_{h,j}(\boldsymbol{x},t,z)\widetilde{\partial}_t^j=\sum_{j<k}\widetilde{\partial}_t^jb_{h,j}(\boldsymbol{x},t,z)$):

$$ \begin{aligned} (m\otimes\widetilde{\partial}_t^k)\cdot\widetilde{\partial}_{x_i}&=(m\widetilde{\partial}_{x_i})\otimes\widetilde{\partial}_t^k-\Big(m\,\frac{\partial g}{\partial x_i}\Big)\otimes\widetilde{\partial}_t^{k+1},\\ (m\otimes\widetilde{\partial}_t^k)\cdot\widetilde{\partial}_t&=m\otimes\widetilde{\partial}_t^{k+1},\\ (m\otimes\widetilde{\partial}_t^k)\cdot h(\boldsymbol{x},t,z)&=\sum_{j<k}ma_{h,j}(\boldsymbol{x},g(\boldsymbol{x}),z)\otimes\widetilde{\partial}_t^j+mh(\boldsymbol{x},g(\boldsymbol{x}),z)\otimes\widetilde{\partial}_t^k. \end{aligned}\tag{8.7.7 $*$} $$

If $\widetilde{\mathcal{M}}=R_F\mathcal{M}$, then the filtration of ${}_{\mathrm{D}}\iota_{g*}\mathcal{M}\simeq\mathcal{M}[\partial_t]$ is simply given by

$$ F_p(\mathcal{M}^{\mathrm{right}}[\partial_t])=\sum_{q+r=p}F_q\mathcal{M}^{\mathrm{right}}\,\partial_t^r. $$

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Since the coordinate t on $\mathbb{C}$ is fixed, a generator $\widetilde{\mathrm{d}t}$ of $\widetilde{\omega}_{\mathbb{C}}$ is also fixed, and we identify (see Caveat 8.2.3 for the notation $\widetilde{\mathrm{d}t}^\vee$)

$$ {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}} \simeq \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t] \otimes \widetilde{\mathrm{d}t}^\vee, $$

i.e., the remaining right action of $\widetilde{\partial}_t$ is changed to a left action. Note that the term $\widetilde{\mathrm{d}t}^\vee$ also shifts the grading of the right-hand side. In other words, the left-hand side is obtained from ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{right}}$ by applying the left-to-right functor on $X \times \widetilde{\mathbb{C}}$, which introduces a twist $(\dim X + 1)$, while the right-hand side is obtained from $\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{right}}[\widetilde{\partial}_t]$ by applying the right-to-left functor on X, which introduces a twist $(\dim X)$ (see Proposition 8.2.4). We will usually omit the term $\widetilde{\mathrm{d}t}^\vee$ in the notation. For example, if $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, the right-hand term corresponds to the $\mathcal{D}_X$-module $\iota_{g*}\mathcal{M}[\partial_t]$ equipped with the filtration

$$ F_p(\iota_{g*}\mathcal{M}^{\mathrm{left}}[\partial_t]) = \sum_{q+r=p} \iota_{g*}F[1]_q\mathcal{M}^{\mathrm{left}}\,\partial_t^r = \sum_{q+r=p-1} \iota_{g*}F_q\mathcal{M}^{\mathrm{left}}\,\partial_t^r. $$

The left $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-action is defined locally be the following formulas (by using Exercise 8.17; note the sign at the second line due to (the omitted) $\widetilde{\mathrm{d}t}^\vee$):

$$ \begin{aligned} \widetilde{\partial}_{x_i}(m\otimes\widetilde{\partial}_t^k) &= (\widetilde{\partial}_{x_i}m)\otimes\widetilde{\partial}_t^k - \Big(\frac{\partial g}{\partial x_i}\,m\Big)\otimes\widetilde{\partial}_t^{k+1},\\ \widetilde{\partial}_t(m\otimes\widetilde{\partial}_t^k) &= -m\otimes\widetilde{\partial}_t^{k+1},\\ h(\boldsymbol{x},t,z)(m\otimes\widetilde{\partial}_t^k) &= \sum_{j<k}(-1)^{k-1-j}b_{h,j}(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^j + h(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^k. \end{aligned} \tag{8.7.7 $**$} $$

(3) In both left and right cases, we can also consider $\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ as a module over the ring $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$, i.e., algebraically with respect to the variable t, with the action of t given by

$$ (m\otimes\widetilde{\partial}_t^k)\cdot t = mg\otimes\widetilde{\partial}_t^k + kzm\otimes\widetilde{\partial}_t^{k-1}, \quad \text{resp. } t\cdot(m\otimes\widetilde{\partial}_t^k) = gm\otimes\widetilde{\partial}_t^k - kzm\otimes\widetilde{\partial}_t^{k-1}. $$

This corresponds to the third lines in (8.7.7 $*$) and (8.7.7 $**$), according to the equality $[\widetilde{\partial}_t^k, t] = kz\widetilde{\partial}_t^{k-1}$.

8.7.8. Remark. If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, one can also consider the left $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-module structure on $\iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t] := \iota_{g*}\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ defined by setting (without a sign on the second line)

$$ \begin{aligned} \widetilde{\partial}_{x_i}(m\otimes\widetilde{\partial}_t^k) &= (\widetilde{\partial}_{x_i}m)\otimes\widetilde{\partial}_t^k - \Big(\frac{\partial g}{\partial x_i}\,m\Big)\otimes\widetilde{\partial}_t^{k+1},\\ \widetilde{\partial}_t(m\otimes\widetilde{\partial}_t^k) &= m\otimes\widetilde{\partial}_t^{k+1},\\ h(\boldsymbol{x},t,z)(m\otimes\widetilde{\partial}_t^k) &= -\sum_{j<k}b_{h,j}(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^j + h(\boldsymbol{x},g,z)m\otimes\widetilde{\partial}_t^k. \end{aligned} $$

However, there exists a natural $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-linear isomorphism

$$ \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t] \xrightarrow{\ \sim\ } \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t]\otimes\widetilde{\mathrm{d}t}^\vee(-1), \quad m\otimes\widetilde{\partial}_t^k \longmapsto m\otimes(-\widetilde{\partial}_t^k)\otimes\widetilde{\mathrm{d}t}^\vee. $$

8.7.9. Example **(Pushforward by a constant map).** If $Y = \mathrm{pt}$ we denote by a_X the constant map on X. For a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we have (recall that, as a graded complex, ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{right}} \simeq {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}^{\mathrm{left}}$)

$$
{}_{\mathrm{D}}a_{X,*}\widetilde{\mathcal{M}} = \boldsymbol{R}\Gamma(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}), \quad {}_{\mathrm{D}}a_{X,!}\widetilde{\mathcal{M}} = \boldsymbol{R}\Gamma_c(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}).
$$

These are bounded complexes of $\widetilde{\mathbb{C}}$-modules. If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, then for every $j \in \mathbb{Z}$, ${}_{\mathrm{D}}a_*^{(j)}\mathcal{M}$ is equipped with the filtration

$$
F_p({}_{\mathrm{D}}a_*^{(j)}\mathcal{M}) : \mathrm{image}[\boldsymbol{H}^j(X, F_p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \longrightarrow \boldsymbol{H}^j(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})],
$$

where the filtration $F_\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is defined in Remark 8.4.9, and

$$
R_F({}_{\mathrm{D}}a_*^{(j)}\mathcal{M}) \simeq ({}_{\mathrm{D}}a_*^{(j)}R_F\mathcal{M})/z\text{-torsion}.
$$

8.7.10. Example **(Pushforward by a projection, right case).** If $X = Y \times T$ and f is the projection $Y \times T \to Y$, denote by $\widetilde{\Theta}_{X/Y}$ the sheaf of *relative* tangent vector fields, i.e., which do not contain $\widetilde{\partial}_{y_j}$ in their local expression in coordinates adapted to the product $Y \times T$. It leads to the subsheaf of *relative* differential operators $\widetilde{\mathcal{D}}_{X/Y} \subset \widetilde{\mathcal{D}}_X$. On the other hand, ${}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_Y = \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = \widetilde{\mathcal{D}}_{X\to Y}$ can also be regarded as a subsheaf of $\widetilde{\mathcal{D}}_X$ (differential operators only containing $\widetilde{\partial}_{y_j}$ in their expression).

The relative Spencer complex $\widetilde{\mathcal{D}}_{X/Y} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}$ (with $\widetilde{\Theta}_{X/Y,k} := \wedge^{-k}\widetilde{\Theta}_{X/Y}$)is defined in the same way as its absolute analogue, and is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_{X/Y}$-module. As a consequence, $\widetilde{\mathcal{D}}_{X/Y} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$ is also a resolution of $\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y = \widetilde{\mathcal{D}}_{X\to Y}$ as a bimodule by locally free left $\widetilde{\mathcal{D}}_X$-modules. By identifying $\widetilde{\mathcal{D}}_X$ with $\widetilde{\mathcal{D}}_{X/Y} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$, we can also write this resolution as $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}$. There is moreover a canonical quasi-isomorphism as bimodules

$$
\begin{aligned}
\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) &= \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\big(\widetilde{\Theta}_{Y,\bullet} \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y\big) \\
&= \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big) \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\big(\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y\big) \\
&\xrightarrow{\sim} \big(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big) \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Y} \\
&= \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}.
\end{aligned}
$$

Definition 8.7.3 now reads

$$
{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} = \boldsymbol{R}f_\star\big(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big), \tag{8.7.10 $*$}
$$

where the right $\widetilde{\mathcal{D}}_Y$ structure is naturally induced from that of ${}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_Y \subset \widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{M}}$.

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, the p-th term of the filtration $F_\bullet(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet})$ of the complex $\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet}$ has $F_{p+k}\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,k}$ in degree $-k$ and for every $j \in \mathbb{Z}$,

$$
{}_{\mathrm{D}}f_\star^{(j)}\widetilde{\mathcal{M}}/z\text{-torsion} \simeq R_F({}_{\mathrm{D}}f_\star^{(j)}\mathcal{M})
$$

with

$$
F_p({}_{\mathrm{D}}f_\star^{(j)}\mathcal{M}) = \mathrm{image}[f_\star^{(j)}F_p(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet}) \to f_\star^{(j)}(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet})].
$$

8.7.11. Example **(Pushforward by a projection, left case).** We take up the setting of Example 8.7.10 and we make explicit the formula in the case of left $\widetilde{\mathcal{D}}_X$-modules (See Exercise 8.43). Let us denote by $\widetilde{\Omega}^1_{X/Y}$ the sheaf of *relative* differential forms, i.e., which do not contain $\widetilde{\mathrm{d}}y_j$ in their local expression in coordinates adapted to the product $Y \times T$. If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, we can form the relative de Rham complex ${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\widetilde{\mathcal{M}}$ by mimicking Definition 8.4.1 and by using the relative connection $\widetilde{\nabla}_{X/Y}$. On the other hand, there remains an action of $\widetilde{\nabla}_Y$ on $\widetilde{\mathcal{M}}$. Due to the integrability property of $\widetilde{\nabla}$ on $\widetilde{\mathcal{M}}$, both connections $\widetilde{\nabla}_{X/Y}$ and $\widetilde{\nabla}_Y$ commute, so that the relative de Rham complex ${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\widetilde{\mathcal{M}}$ (the shift is by $d_{X/Y} := \dim X - \dim Y$) is naturally equipped with an $f^{-1}\widetilde{\mathcal{O}}_Y$-connection $\widetilde{\nabla}_Y$. Then we have (Exercise 8.43), for $\star = *$ or $\star = !$,

$$ {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} = (\boldsymbol{R}f_\star\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\widetilde{\mathcal{M}}, \widetilde{\nabla}_Y). $$

If $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, the p-th term of the filtration $F_\bullet({}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M})$ of the complex ${}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} = \Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}$ has $\Omega^k_{X/Y} \otimes_{\mathcal{O}_X} F_{p+k}\mathcal{M}$ in cohomological degree $k - \dim X$ and for every $j \in \mathbb{Z}$,

$$ {}_{\mathrm{D}}f^{(j)}_\star\widetilde{\mathcal{M}}/z\text{-torsion} \simeq R_F\,{}_{\mathrm{D}}f^{(j)}_\star\mathcal{M} $$

with

$$ F_p\,{}_{\mathrm{D}}f^{(j)}_\star\mathcal{M} = \mathrm{image}[f^{(j)}_\star F_p({}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}) \to f^{(j)}_\star({}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M})]. $$

8.7.12. Remark. Since any morphism can be decomposed as a closed embedding followed be a projection, through the graph embedding, we could simply say that the pushforward by a closed embedding (resp. a projection) of a right $\widetilde{\mathcal{D}}_X$-module is obtained by the definition of Example 8.7.6 (resp. Example 8.7.10), and define the pushforward by any holomorphic map f by composing the pushforward functors in these simple cases. Nevertheless, in order to check various other properties, it is useful to have the intrinsic definition 8.7.3 for any holomorphic mapping f.

8.7.13. Remark **(Amplitude of the pushforward).** Formula (8.7.17) below shows that ${}_{\mathrm{D}}f^{(j)}_\star\widetilde{\mathcal{M}} = 0$ for $j \notin [-n, n]$. On the other hand, if f is a closed inclusion, the amplitude is equal to zero, and if f is a projection, the C^∞ resolutions for Examples 8.7.10 or 8.7.11 show that ${}_{\mathrm{D}}f^{(j)}_\star\widetilde{\mathcal{M}} = 0$ for $j \notin [-(n-m), (n-m)]$.

8.7.b. Explicit constructions with the pushforward functor

There are two natural ways (at least) to make explicit the functor $\boldsymbol{R}f_\star$ entering the definition of ${}_{\mathrm{D}}f_\star$: one can use the canonical Godement resolution by flabby sheaves, which is a very general procedure but with few geometric content, or one can replace the relative Spencer or de Rham complexes by their C^∞ counterparts as in Remark 8.4.13. We will mainly use the latter, but it can be useful to have the former at hand.

Godement resolution. Recall that the flabby sheaves are injective with respect to the functor f_* (direct image) in the category of sheaves (of modules over a ring) and, being c-soft, are injective with respect to the functor $f_!$ (direct image with proper support). The Godement canonical resolution is an explicit functorial flabby resolution for any sheaf (see Exercise 8.48 for details).

***8.7.14. Definition* (Godement resolution).**

(1) The *Godement functor* $\mathcal{C}^0$ (see [**God64**, p. 167]) associates to any sheaf $\widetilde{\mathcal{L}}$ the *flabby* sheaf $\mathcal{C}^0(\widetilde{\mathcal{L}})$ of its discontinuous sections and to any morphism the corresponding family of germs of morphisms. Then there is a canonical injection $\widetilde{\mathcal{L}} \hookrightarrow \mathcal{C}^0(\widetilde{\mathcal{L}})$.

(2) Set inductively (see [**God64**, p. 168]) $\mathcal{Z}^0(\widetilde{\mathcal{L}}) = \widetilde{\mathcal{L}}$, $\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}) = \mathcal{C}^k(\widetilde{\mathcal{L}})/\mathcal{Z}^k(\widetilde{\mathcal{L}})$, $\mathcal{C}^{k+1}(\widetilde{\mathcal{L}}) = \mathcal{C}^0(\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}))$ and define $\delta : \mathcal{C}^k(\widetilde{\mathcal{L}}) \to \mathcal{C}^{k+1}(\widetilde{\mathcal{L}})$ as the composition $\mathcal{C}^k(\widetilde{\mathcal{L}}) \to \mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}) \to \mathcal{C}^0(\mathcal{Z}^{k+1}(\widetilde{\mathcal{L}}))$. This defines a complex $(\mathcal{C}^\bullet(\widetilde{\mathcal{L}}), \delta)$, that we will denote as $(\mathrm{God}^\bullet \widetilde{\mathcal{L}}, \delta)$.

(3) Given any sheaf $\widetilde{\mathcal{L}}$, $(\mathrm{God}^\bullet \widetilde{\mathcal{L}}, \delta)$ is a resolution of $\widetilde{\mathcal{L}}$ by flabby sheaves. For a complex $(\widetilde{\mathcal{L}}^\bullet, d)$, we regard $\mathrm{God}^\bullet \widetilde{\mathcal{L}}^\bullet$ as a double complex ordered as written, i.e., with differential $(\delta_i, (-1)^i d_j)$ on $\mathrm{God}^i \widetilde{\mathcal{L}}^j$, and therefore also as the associated simple complex.

8.7.15. Corollary. *We have, by taking the single complex associated to the double complex, and for $\star = *$ or $\star = !$,*

$$ {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} = f_\star \,\mathrm{God}^\bullet \,\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}). \qquad \square$$

C^∞ ***resolution.*** Recall (see Remark 8.4.13) that $\mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X)$ is a resolution of $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ in the category of left $\widetilde{\mathcal{D}}_X$-modules by flat $\widetilde{\mathcal{O}}_X$-modules. Therefore,

$$\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X) \simeq \mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$$

is a resolution of $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ in the category of $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$ bi-modules, so that, for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, (8.7.3 $*$) becomes

$$ {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} \simeq f_\star(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X)) \simeq f_\star \,\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}}). \tag{8.7.16}$$

On the other hand, for a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we can use Exercise 8.51(5) to obtain

$$ {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} \simeq f_\star \big[\widetilde{\mathcal{E}}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big]^{\mathrm{left}}. \tag{8.7.17}$$

8.7.18. The Lefschetz morphism. As a consequence of Exercise 8.51(5), given a $(1,1)$-form $\widetilde{\eta} \in \Gamma(X, \widetilde{\mathcal{E}}_X^{(1,1)})$ which $\widetilde{\mathrm{d}}$-closed (equivalently, $\widetilde{\mathrm{d}}'$ and d''-closed), there is a well-defined morphism for a left $\widetilde{\mathcal{D}}_X$-module ($\star = *$ or $\star = !$)

$$\widetilde{\eta} \wedge\, : {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} \longrightarrow {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}}[2](1),$$

induced by $\widetilde{\eta} \wedge\, : \widetilde{\mathcal{E}}_X^\bullet \to \widetilde{\mathcal{E}}_X^\bullet[2](1)$. (Here, [2] means the shift by 2 of the complex, which occurs since $\widetilde{\eta}$ has total degree 2, while (1) is the Tate twist shift, which occurs since $\widetilde{\eta}$ has a degree-one holomorphic part.) It is clearly functorial with respect to $\widetilde{\mathcal{M}}$, that is, given any morphism $\varphi:\widetilde{\mathcal{M}}_1\to\widetilde{\mathcal{M}}_2$, the following diagram commutes (where $\star$ is either for $*$ or for $!$):

$$\begin{tikzcd}
{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}_1 \arrow[r, "\widetilde{\eta}\wedge"] \arrow[d, "{}_{\mathrm{D}}f_\star\varphi"'] & {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}_1[2](1) \arrow[d, "{}_{\mathrm{D}}f_\star\varphi"] \\
{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}_2 \arrow[r, "\widetilde{\eta}\wedge"] & {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}_2[2](1)
\end{tikzcd}$$

8.7.19. Definition **(The Lefschetz morphism attached to a closed $(1,1)$-form)**
For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the Lefschetz morphism associated to a (usual) closed $(1,1)$-form η on X is the morphism

$$\mathrm{L}_\eta := \frac{1}{z}\,\eta\wedge\ :{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}\longrightarrow{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}[2](1).$$

It is functorial with respect to $\widetilde{\mathcal{M}}$.

8.7.20. The Lefschetz morphism attached to a line bundle. Let $f:X\to Y$ be any morphism between complex manifolds and let $\mathcal{L}$ be a line bundle on X, with Chern class $c_1(\mathcal{L})\in H^2(X,\mathbb{Z})$. We will define a Lefschetz morphism

$$\mathrm{L}_{\mathcal{L}}:{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}\longrightarrow{}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}[2](1).$$

We can choose a closed $(1,1)$-form η on X whose class in $H^2(X,\mathbb{C})$ is equal to the complexified class $c_1(\mathcal{L})_{\mathbb{C}}$. We regard η as a closed relative $(1,1)$-form with respect to the projection. As noticed in Remark 8.4.11, namely by using a similar argument, the action of L_η given in Definition 8.7.19 only depends on the class of η in $H^2(X,\mathbb{C})$. Notice also that, since η has degree two, wedging (or contracting) with η on the left or on the right gives the same result.

We thus *define* $\mathrm{L}_{\mathcal{L}}$ as L_η. This operator only depends on $c_1(\mathcal{L})_{\mathbb{C}}$. It is functorial with respect to $\widetilde{\mathcal{M}}$.

8.7.21. Remark **(Restriction to $z=1$ of the Lefschetz morphism)**
It is obvious that the restriction to $z=1$ of the morphism $\mathrm{L}_{\mathcal{L}}$ is the morphism

$$\mathrm{L}_{\mathcal{L}}:{}_{\mathrm{D}}f_\star\mathcal{M}\longrightarrow{}_{\mathrm{D}}f_\star\mathcal{M}[2].$$

8.7.c. Composition of direct images and the Leray spectral sequence

We compare the result of the pushforward functor by the composition of two maps with the pushforward by the second map of the pushforward by the first map. We find an isomorphism at the level of derived categories, that we will translate as a spectral sequence, which is the $\widetilde{\mathcal{D}}$-module analogue of the Leray spectral sequence (see Section 8.9.c).

8.7.22. Theorem **(Composition of direct images).** *Let*

$$f:X\longrightarrow Y\quad\textit{and}\quad f':Y\longrightarrow Z$$

be two holomorphic maps. There is a functorial canonical isomorphism of functors

$${}_{\mathrm{D}}(f'\circ f)_!(\bullet)={}_{\mathrm{D}}f'_!({}_{\mathrm{D}}f_!(\bullet)).$$

If f is proper, we also have

$$_{\mathrm{D}}(f'\circ f)_*(\bullet) = {}_{\mathrm{D}}f'_*({}_{\mathrm{D}}f_*(\bullet)).$$

Proof. We start from the canonical isomorphism of $(\widetilde{\mathcal{D}}_X, (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules (Exercise 8.36):

$$\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\xrightarrow{\sim}\widetilde{\mathcal{D}}_{X\to Z}. \tag{8.7.23}$$

We deduce an isomorphism of complexes of $(\widetilde{\mathcal{D}}_X, (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules

$$\Big[\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}\Big]\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\xrightarrow{\sim}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Z}$$

lifting (8.7.23), that is, a natural isomorphism

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\xrightarrow{\sim}\mathrm{Sp}_{X\to Z}(\widetilde{\mathcal{D}}_X).$$

On the other hand, there exists a natural morphism of complexes

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\longrightarrow\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z},$$

obtained by tensoring the augmentation morphism $\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\to\widetilde{\mathcal{D}}_{Y\to Z}$ with $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$, and the left-hand term is a resolution of $\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}$ (in the category of $(\widetilde{\mathcal{D}}_X, (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules) by locally free $\widetilde{\mathcal{D}}_X$-modules. Indeed, remark that, as $\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)$ is $\widetilde{\mathcal{D}}_Y$ locally free, one has

$$\begin{aligned}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)&\xrightarrow{\sim}\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\\&=\widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\\&=\widetilde{\mathcal{O}}_X\otimes^{\boldsymbol{L}}_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\\&=\widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}\quad(\widetilde{\mathcal{D}}_{Y\to Z}\text{ is }\widetilde{\mathcal{O}}_Y\text{ locally free})\\&=\widetilde{\mathcal{D}}_{X\to Y}\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z}.\end{aligned}$$

Altogether, we have found a morphism, lifting (8.7.23),

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\longrightarrow\mathrm{Sp}_{X\to Z}(\widetilde{\mathcal{D}}_X),$$

between two resolutions (in the category of $(\widetilde{\mathcal{D}}_X, (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)$-bimodules). This morphism is therefore a quasi-isomorphism. We now have, for an object $\widetilde{\mathcal{M}}$ of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ or of $\mathsf{D}^+(\widetilde{\mathcal{D}}_X)$

$$\begin{aligned}{}_{\mathrm{D}}(f'\circ f)_!(\widetilde{\mathcal{M}})&=\boldsymbol{R}(f'\circ f)_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Z}(\widetilde{\mathcal{D}}_X)\big)\\&\simeq\boldsymbol{R}(f'\circ f)_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\big)\\&\simeq\boldsymbol{R}f'_!\boldsymbol{R}f_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\big)\\&\simeq\boldsymbol{R}f'_!\Big[\boldsymbol{R}f_!\big(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{\widetilde{\mathcal{D}}_Y}\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big]\\&={}_{\mathrm{D}}f'_!({}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}).\end{aligned}$$

The above arguments also apply if we replace Sp with Sp^∞ as defined in Remark 8.4.13, according to the $\widetilde{\mathcal{D}}_X$-flatness of $\mathrm{Sp}^\infty_{X\to Y}$ and the $\widetilde{\mathcal{D}}_Y$-flatness of $\mathrm{Sp}^\infty_{Y\to Z}$

(see Exercise 8.50(1)). All terms of the corresponding complexes are c-soft and we have

$$
\begin{aligned}
{}_{\mathrm{D}}(f'\circ f)_!(\widetilde{\mathcal{M}}) &\simeq (f'\circ f)_!\,\mathrm{Sp}^\infty_{X\to Z}(\widetilde{\mathcal{M}})\\
&\simeq f'_!\Big[f_!\big(\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})\big)\otimes_{\widetilde{\mathcal{D}}_Y}\mathrm{Sp}^\infty_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big].
\end{aligned}
\tag{8.7.24}
$$

The same result holds with ${}_{\mathrm{D}}f_*$ if we only assume that f is proper on the support of $\widetilde{\mathcal{M}}$. On the other hand, if f is proper or proper on the support of $\widetilde{\mathcal{M}}$, but f' is possibly not proper, then the same results are valid for $*$ instead of !: indeed, $f_! = f_*$ and $f_!\,\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ is flabby, so the last isomorphism in (8.7.24) still holds with f'_*, and the same reasoning gives ${}_{\mathrm{D}}(f'\circ f)_* = {}_{\mathrm{D}}f'_*\,{}_{\mathrm{D}}f_*$. □

8.7.25. Remark. If f is not proper, we cannot assert in general that ${}_{\mathrm{D}}(f'\circ f)_*(\bullet) = {}_{\mathrm{D}}f'_*({}_{\mathrm{D}}f_*(\bullet))$. However, such an identity still holds when applied to suitable subcategories of $\mathsf{D}^+(\widetilde{\mathcal{D}}_X)$, the main examples being:

- the restriction of f to the support of $\widetilde{\mathcal{M}}$ is proper, as already seen,
- $\widetilde{\mathcal{M}}$ has $\widetilde{\mathcal{D}}_X$-coherent cohomology.

In such cases, the natural morphism coming in the projection formula for f_* is a quasi-isomorphism (see [**MN93**, §II.5.4] for the coherent case).

This theorem reduces the computation of the direct image by any morphism $f: X\to Y$ by decomposing it as $f = p\circ\iota_f$, where $\iota_f: X\hookrightarrow X\times Y$ denotes the graph inclusion $x\mapsto(x,f(x))$. As ι_f is an embedding, it is proper, so we have ${}_{\mathrm{D}}f_* = {}_{\mathrm{D}}p_*{}_{\mathrm{D}}\iota_{f*}$. The following corollary is a direct consequence of Example 8.7.6.

8.7.26. Corollary **(Composition with a closed embedding).**

(1) *Assume that f is a closed embedding. Then, for each $k\in\mathbb{Z}$, we have a functorial isomorphism ${}_{\mathrm{D}}(f'\circ f)_!^{(k)}\simeq{}_{\mathrm{D}}f'^{(k)}_!\circ{}_{\mathrm{D}}f_!$.*

(2) *Assume that f' is a closed embedding. Then, for each $k\in\mathbb{Z}$, we have a functorial isomorphism ${}_{\mathrm{D}}(f'\circ f)_!^{(k)}\simeq{}_{\mathrm{D}}f'_!\circ{}_{\mathrm{D}}f_!^{(k)}$.* □

The Leray spectral sequence exists in this setting.

8.7.27. Corollary **(Leray spectral sequence for the composition of maps)**

There exists a bounded spectral sequence with $E_2^{p,q} = {}_{\mathrm{D}}f'^{(p)}_!({}_{\mathrm{D}}f_!^{(q)}\widetilde{\mathcal{M}})$ which converges to ${}_{\mathrm{D}}(f'\circ f)_!^{p+q}\widetilde{\mathcal{M}}$. There are corresponding spectral sequences with ${}_{\mathrm{D}}f_*$ and ${}_{\mathrm{D}}f'_*$ under the properness assumptions above.

Proof. Let us consider the expression (8.7.24). First, $f_!\,\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ is a bounded complex having cohomology ${}_{\mathrm{D}}f_!^{(q)}\widetilde{\mathcal{M}}$. The second line of (8.7.24) is a double complex $(K^{\bullet,\bullet},\delta_1,\delta_2)$. The single complex attached to $(K^{\bullet,\bullet},\delta_1,\delta_2)$ has cohomology ${}_{\mathrm{D}}(f'\circ f)_!^{(k)}(\widetilde{\mathcal{M}})$, according to our previous computation. The spectral sequence attached to this double complex has E_2 term

$$E_2^{p,q} = H^p_{\delta_2}(H^q_{\delta_1}(K^{\bullet,\bullet})) = {}_{\mathrm{D}}f'^{(p)}_!({}_{\mathrm{D}}f_!^{(q)}\widetilde{\mathcal{M}}).$$

The spectral sequence degenerates at a finite step. We have a similar result for ${}_{\mathrm{D}}(f'\circ f)_*^{(k)}(\widetilde{\mathcal{M}})$ if f is proper. $\square$

We call this spectral sequence the *Leray spectral sequence* for the composition $f'\circ f$. In such a way, the abutment ${}_{\mathrm{D}}(f'\circ f)_!^{(k)}(\widetilde{\mathcal{M}})$ comes equipped with a natural filtration, that we call the *Leray filtration*, such that

$$E_\infty^{p,q} = \mathrm{gr}^p_{\mathrm{Ler}}\big[{}_{\mathrm{D}}(f'\circ f)_!^{(p+q)}(\widetilde{\mathcal{M}})\big].$$

It is clear that the restriction to $z=1$ of the Leray spectral sequence is the Leray spectral sequence for $\mathcal{D}_X$-modules.

8.7.28. Behaviour of the Spencer complex by pushforward. In the proof of Theorem 8.7.22, let us set $Z=\mathrm{pt}$, so that $\mathrm{Sp}_{Y\to Z}(\widetilde{\mathcal{D}}_Y)=\mathrm{Sp}(\widetilde{\mathcal{D}}_Y)$. By the same argument, but not applying the functor $\boldsymbol{R}f'_!$, we obtain

$$\mathrm{Sp}({}_{\mathrm{D}}f_!\widetilde{\mathcal{M}})\simeq \boldsymbol{R}f_!\,\mathrm{Sp}(\widetilde{\mathcal{M}}).$$

We already have an identification on X as follows: considering the right $\widetilde{\mathcal{D}}_Y$-structure on $\widetilde{\mathcal{D}}_{X\to Y}$, the Spencer complex $\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})$ is well defined, and is nothing but

$$\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})=\widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\,\mathrm{Sp}(\widetilde{\mathcal{D}}_Y)\simeq \widetilde{\mathcal{O}}_X\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{O}}_Y=\widetilde{\mathcal{O}}_X$$

as a left $\widetilde{\mathcal{D}}_X$-module. Similarly, regarding $\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ as a complex of right $f^{-1}(\widetilde{\mathcal{D}}_Y)$-modules, we obtain

$$\begin{aligned}\mathrm{Sp}_Y(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)) &= \mathrm{Sp}_Y((\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X))\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y})\\ &\simeq (\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X))\otimes_{\widetilde{\mathcal{O}}_X}\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})\\ &\simeq \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\simeq\mathrm{Sp}(\widetilde{\mathcal{M}}).\end{aligned}$$

We also conclude that, for a left or right $\widetilde{\mathcal{D}}_X$-module, we have

$${}^{\mathrm{p}}\mathrm{DR}({}_{\mathrm{D}}f_!\widetilde{\mathcal{M}})\simeq \boldsymbol{R}f_!\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}).$$

8.7.d. A morphism of adjunction.

There are various adjunction morphisms for $\widetilde{\mathcal{D}}$-modules in the literature (see [**Kas03, HTT08**]). We will give here a simple one, in the case where the source and target of the *proper holomorphic map* $f:X\to Y$ have *the same dimension*. In such a case, the cotangent map T^*f induces a morphism

$$f^{-1}\widetilde{\Omega}^k_Y\longrightarrow\widetilde{\Omega}^k_X$$

for every k, which is compatible with the differential $\widetilde{\mathrm{d}}$, and similarly for C^∞ forms.

8.7.29. Proposition. *Under this assumption, if $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_Y$-module, there is a functorial morphism*

$$\mathrm{adj}_f:\widetilde{\mathcal{M}}\longrightarrow {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}).$$

Proof. Set $n = \dim X = \dim Y$. By Exercise 8.51 we have

$$\begin{aligned}({}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}))^{\mathrm{right}} &\simeq f_*\Big(\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}\widetilde{\mathcal{D}}_Y)\Big)\\ &\simeq f_*\Big(\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y}f^{-1}(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y)\Big)\\ &\simeq f_*\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes_{\widetilde{\mathcal{O}}_Y}(\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y),\end{aligned}$$

where the last isomorphism is the sheaf-theoretic projection formula for a proper map. By using the isomorphism of Exercise 8.18(3), we finally obtain

$$({}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}))^{\mathrm{right}} \simeq (f_*\widetilde{\mathcal{E}}_X^{n+\bullet}\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{D}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}},$$

where the left $\widetilde{\mathcal{D}}_Y$-module structure of $\widetilde{\mathcal{D}}_Y$ is used for the C^∞-complex ($\bullet$), and the right $\widetilde{\mathcal{D}}_Y$-module structure of $\widetilde{\mathcal{D}}_Y$ is used in the tensor product with $\widetilde{\mathcal{M}}$ in order to obtain the final right $\widetilde{\mathcal{D}}_Y$-module structure (see Exercise 8.18(2)). The cotangent map $f^{-1}\widetilde{\mathcal{E}}_Y^k \to \widetilde{\mathcal{E}}_X^k$ induces, by using the sheaf-theoretic adjunction $\mathrm{Id} \to f_*f^{-1}$, a morphism $\widetilde{\mathcal{E}}_Y^k \to f_*\widetilde{\mathcal{E}}_X^k$ compatible with differentials, hence a morphism

$$(\widetilde{\mathcal{E}}_Y^{n+\bullet}\otimes\widetilde{\mathcal{D}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}} \longrightarrow ({}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}))^{\mathrm{right}}.$$

Last, by using Exercise 8.30(2), we find

$$(\widetilde{\mathcal{E}}_Y^{n+\bullet}\otimes\widetilde{\mathcal{D}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}} \xleftarrow{\sim} (\widetilde{\Omega}_Y^{n+\bullet}\otimes\widetilde{\mathcal{D}}_Y)\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\omega}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}^{\mathrm{right}}. \qquad \square$$

8.7.30. Example. Let us consider the simple case of a finite morphism, with $X = \mathbb{C}^n$ (coordinates $x_1, \dots, x_n$), $Y = \mathbb{C}^n$ (coordinates $y_1, \dots, y_n$) and $f = (f_1, \dots, f_n) : X \to Y$ is the finite morphism defined by $f_i(x_1, \dots, x_n) = x_i^{r_i}$ with $r_i \in \mathbb{N}^*$. Then (see Exercise 8.57) there exists a trace morphism

$$\mathrm{Tr}_f : {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}) \longrightarrow \widetilde{\mathcal{M}}$$

such that the composition $\mathrm{Tr}_f \circ \mathrm{adj}_f : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is the identity. As a consequence, $\widetilde{\mathcal{M}}$ is a direct summand of ${}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}})$.

8.7.e. Pushforward of $\mathcal{D}_{X,\overline{X}}$-modules. As we will apply the pushforward functor by a holomorphic map $f : X \to Y$ to the sheaf of distributions on X or to the sheaf of currents of maximal degree (see Example 8.3.4), we will make precise the adaptation of the previous properties to the category of $\mathcal{D}_{X,\overline{X}}$-modules, where we recall that $\mathcal{D}_{X,\overline{X}} := \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$ (see (8.3.4 $**$)). We will denote the corresponding pushforward functor by ${}_{\mathrm{D},\overline{\mathrm{D}}}f_!$ or ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*$. This notation was already use, with that meaning, in Section 7.3.17, for the pushforward by a closed inclusion.

We define $\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}$ as $\mathcal{D}_{X\to Y} \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}\to\overline{Y}}$. This sheaf can also be described as $\mathcal{O}_{X,\overline{X}} \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}\mathcal{D}_{Y,\overline{Y}}$. The Spencer complex $\mathrm{Sp}(\mathcal{D}_{X,\overline{X}})$ is the simple complex associated with the double complex $\mathrm{Sp}(\mathcal{D}_X) \otimes_{\mathbb{C}} \mathrm{Sp}(\mathcal{D}_{\overline{X}})$. Defining $\Theta^k_{X,\overline{X}} = \bigoplus_{i+j=k}(\Theta^i_X \otimes_{\mathbb{C}} \Theta^j_{\overline{X}})$, the $(-k)$-th term of the Spencer complex $\mathrm{Sp}(\mathcal{D}_{X,\overline{X}})$ is equal to $\mathcal{D}_{X,\overline{X}} \otimes \Theta^k_{X,\overline{X}}$, which is $\mathcal{D}_{X,\overline{X}}$-locally free of finite rank, and the differentials are expressed in a way similar to that in Definition 8.4.3. It is a $\mathcal{D}_{X,\overline{X}}$-resolution of $\mathcal{O}_{X,\overline{X}}$ by locally free $\mathcal{D}_{X,\overline{X}}$-modules.

The relative Spencer complex is defined similarly to (8.7.2), by

$$\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}}) = \mathrm{Sp}(\mathcal{D}_{X,\overline{X}}) \otimes_{\mathcal{O}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}\to Y,\overline{Y}},$$

and is a resolution of $\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}$ as a $(\mathcal{D}_{X,\overline{X}}, f^{-1}\mathcal{D}_{Y,\overline{Y}})$-bimodule by locally free $\mathcal{D}_{X,\overline{X}}$-modules.

The pushforward functor ${}_{\mathrm{D},\overline{\mathrm{D}}}f_\star$ ($\star = !, *$) is defined, for a right $\mathcal{D}_{X,\overline{X}}$-module $\mathcal{N}$, or a bounded complex of such, by

$${}_{\mathrm{D},\overline{\mathrm{D}}}f_\star(\mathcal{N}) = \boldsymbol{R}f_\star(\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{N})) \simeq \boldsymbol{R}f_\star\big(\mathcal{N} \otimes_{\mathcal{D}_{X,\overline{X}}} \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big).$$

In a way similar to what is done in Theorem 8.7.22 and Corollary 8.7.27, we obtain the following result. In the present setting, it is enough to use the Godement flabby resolution $\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{N})$ when a flabby resolution is needed.

8.7.31. Proposition. *Let*

$$f : X \longrightarrow Y \quad \textit{and} \quad f' : Y \longrightarrow Z$$

be two holomorphic maps. There is a functorial canonical isomorphism of functors

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f' \circ f)_!(\bullet) = {}_{\mathrm{D},\overline{\mathrm{D}}}f'_!({}_{\mathrm{D},\overline{\mathrm{D}}}f_!(\bullet)).$$

If f is proper, we also have

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f' \circ f)_*(\bullet) = {}_{\mathrm{D},\overline{\mathrm{D}}}f'_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*(\bullet)).$$

Furthermore, there exists a bounded spectral sequence with $E_2^{p,q} = {}_{\mathrm{D},\overline{\mathrm{D}}}f'^{(p)}_!({}_{\mathrm{D},\overline{\mathrm{D}}}f^{(q)}_!\mathcal{M})$ which converges to ${}_{\mathrm{D},\overline{\mathrm{D}}}(f' \circ f)^{p+q}_!\mathcal{M}$. There are corresponding spectral sequences with ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*$ and ${}_{\mathrm{D},\overline{\mathrm{D}}}f'_*$ under the properness assumptions above. $\square$

8.8. Coherent $\widetilde{\mathcal{D}}_X$-modules and coherent filtrations

Although it would be natural to develop the theory of coherent $\widetilde{\mathcal{D}}_X$-modules in a way similar to that of $\widetilde{\mathcal{O}}_X$-modules, some points of the theory are not known to extend to $\widetilde{\mathcal{D}}_X$-modules (the lemma on holomorphic matrices). The approach which is therefore classically used consists in using the $\widetilde{\mathcal{O}}_X$-theory, and the main tools for that purpose are the coherent filtrations.

8.8.a. Coherence of $\widetilde{\mathcal{D}}_X$. Let us begin by recalling the definition of coherence. Let $\widetilde{\mathcal{A}}$ be a sheaf of rings on a space X.

8.8.1. Definition.

(1) A sheaf of $\widetilde{\mathcal{A}}$-modules $\mathcal{F}$ is said to be $\widetilde{\mathcal{A}}$-*coherent* if it is locally of finite type:

$$\forall\, x \in X,\ \exists\, U_x,\ \exists\, q, \quad \exists\, \widetilde{\mathcal{A}}^q_{|U_x} \twoheadrightarrow \mathcal{F}_{|U_x},$$

and if, for any open set U of X and any $\widetilde{\mathcal{A}}$-linear morphism $\varphi : \widetilde{\mathcal{A}}^r_{|U} \to \mathcal{F}_{|U}$, the kernel of φ is locally of finite type.

(2) The sheaf $\widetilde{\mathcal{A}}$ is a coherent sheaf of rings if it is coherent as a (left and right) module over itself.

8.8.2. Lemma. *Assume $\widetilde{\mathcal{A}}$ coherent. Let $\mathcal{F}$ be a sheaf of $\widetilde{\mathcal{A}}$-module. Then $\mathcal{F}$ is $\widetilde{\mathcal{A}}$-coherent if and only if $\mathcal{F}$ is* locally of finite presentation*: $\forall\, x \in X$, $\exists\, U_x$, $\exists\, p, q$ and an exact sequence*

$$\widetilde{\mathcal{A}}^p_{|U_x} \longrightarrow \widetilde{\mathcal{A}}^q_{|U_x} \longrightarrow \mathcal{F}_{|U_x} \longrightarrow 0.$$

Classical theorems of Cartan and Oka claim the *coherence of* $\widetilde{\mathcal{O}}_X$, and a theorem of Frisch asserts that, if K is a compact polycylinder, $\widetilde{\mathcal{O}}_X(K)$ is a Noetherian ring. It follows that $\mathrm{gr}^F\widetilde{\mathcal{D}}_X(K)$ is a Noetherian ring, and one deduces that $\widetilde{\mathcal{D}}_X(K)$ is left and right Noetherian. From this one concludes that the sheaf of rings $\widetilde{\mathcal{D}}_X$ is coherent (see [**GM93, Kas03**] for details).

8.8.b. Coherent $\widetilde{\mathcal{D}}$-modules and filtrations

Let $\widetilde{\mathcal{M}}$ be a $\widetilde{\mathcal{D}}_X$-module. From the preliminary reminder on coherence, we know that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$*-coherent* if it is locally finitely presented, i.e., if for any $x \in X$ there exists an open neighbourhood U_x of x an an exact sequence $\widetilde{\mathcal{D}}^q_{X|U_x} \to \widetilde{\mathcal{D}}^p_{X|U_x} \to \widetilde{\mathcal{M}}_{|U_x}$.

***8.8.3. Definition* (Coherent filtrations).** Let $F_\bullet\widetilde{\mathcal{M}}$ be a filtration of $\widetilde{\mathcal{M}}$ (see Section 5.1). We say that the filtration is *coherent* if the Rees module $R_F\widetilde{\mathcal{M}}$ is coherent over the coherent sheaf $R_F\widetilde{\mathcal{D}}_X$ (i.e., locally finitely presented).

It is useful to have various criteria for a filtration to be coherent.

***8.8.4. Proposition* (Existence of coherent filtrations).**

(1) *If $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, then it admits* locally on X *a coherent filtration.*

(2) *If $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$ and if $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strict, it admits* globally on X *a coherent filtration.*

Proof. For (1), see Exercise 8.62. Let us prove (2). By Proposition 5.1.8(1), we have $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ for some filtered $\mathcal{D}_X$-module $\mathcal{M}$ and since $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, $F_\bullet\mathcal{M}$ is a coherent $F_\bullet\mathcal{D}$-filtration. Then one can apply Exercise 8.65. $\square$

The notion of a coherent filtration is the main tool to obtain results on coherent $\widetilde{\mathcal{D}}_X$-modules from theorems on coherent $\widetilde{\mathcal{O}}_X$-modules, and the main results concerning coherent $\widetilde{\mathcal{D}}_X$-modules are obtained from the theorems of Cartan and Oka for $\widetilde{\mathcal{O}}_X$-modules.

***8.8.5. Theorem* (Theorems of Cartan-Oka for $\widetilde{\mathcal{D}}_X$-modules)**

Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let K be a compact polycylinder contained in an open subset U of X, such that $\widetilde{\mathcal{M}}$ has a coherent filtration on U. Then,

(1) *$\Gamma(K, \widetilde{\mathcal{M}})$ generates $\widetilde{\mathcal{M}}_{|K}$ as an $\widetilde{\mathcal{O}}_K$-module,*

(2) *For every $i \geqslant 1$, $H^i(K, \widetilde{\mathcal{M}}) = 0$.*

Proof. This is easily obtained from the theorems A and B for $\widetilde{\mathcal{O}}_X$-modules, by using inductive limits (for A it is obvious and, for B, see [**God64**, Th. 4.12.1]). $\square$

***8.8.6. Theorem* (Characterization of coherence for $\widetilde{\mathcal{D}}_X$-modules, see [GM93])**

(1) *Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Then, for any small enough compact polycylinder K, we have the following properties:*

(a) *$\widetilde{\mathcal{M}}(K)$ is a finite type $\widetilde{\mathcal{D}}(K)$-module,*

(b) *For every $x \in K$, $\widetilde{\mathcal{O}}_x \otimes_{\widetilde{\mathcal{O}}(K)} \widetilde{\mathcal{M}}(K) \to \widetilde{\mathcal{M}}_x$ is an isomorphism.*

(2) *Conversely, if there exists a covering $\{K_\alpha\}$ by polycylinders K_α such that X is the union of the interiors of the K_α and that on any K_α the properties* (1a) *and* (1b) *are fulfilled, then $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent.* □

A first application of Theorem 8.8.6 is a variant of the classical Artin-Rees lemma:

8.8.7. Corollary. *Let $\widetilde{\mathcal{M}}$ be a $\widetilde{\mathcal{D}}_X$-module with a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$ and let $\widetilde{\mathcal{N}}$ be a coherent $\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\mathcal{M}}$. Then the filtration $F_\bullet\widetilde{\mathcal{N}} := \widetilde{\mathcal{N}} \cap F_\bullet\widetilde{\mathcal{M}}$ is coherent.*

Proof. Let K be a small compact polycylinder for $R_F\widetilde{\mathcal{M}}$. Then $\Gamma(K, R_F\widetilde{\mathcal{M}})$ is finitely generated, hence so is $\Gamma(K, R_F\widetilde{\mathcal{N}})$, as $\Gamma(K, R_F\widetilde{\mathcal{D}}_X)$ is Noetherian. It remains to be proved that, for any $x \in K$ and any k, the natural morphism

$$\widetilde{\mathcal{O}}_x \otimes_{\widetilde{\mathcal{O}}(K)} (F_k\widetilde{\mathcal{M}}(K) \cap \widetilde{\mathcal{N}}(K)) \longrightarrow F_k\widetilde{\mathcal{M}}_x \cap \widetilde{\mathcal{N}}_x$$

is an isomorphism. This follows from the flatness of $\widetilde{\mathcal{O}}_x$ over $\widetilde{\mathcal{O}}(K)$ (see [**Fri67**]). □

8.8.8. Structure of coherent $\widetilde{\mathcal{D}}_X$-modules. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Its z-torsion submodule is the submodule $\widetilde{\mathcal{M}}' := \bigcup_{k\geqslant 1} \operatorname{Ker}[z^k : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}]$. Since each submodule $\operatorname{Ker}[z^k : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}]$ is coherent (see Exercise 8.59) the union is locally finite and $\widetilde{\mathcal{M}}'$ has a locally finite filtration such that each successive quotient is a coherent $\widetilde{\mathcal{D}}_X$-module annihilated by z. Such a module is thus a coherent (graded) $(\widetilde{\mathcal{D}}_X/z\widetilde{\mathcal{D}}_X) = \operatorname{gr}^F\widetilde{\mathcal{D}}_X$-module, on which the z-action is zero.

On the other hand, the quotient module $\widetilde{\mathcal{M}}'' := \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}'$ is strict by definition.

8.8.c. Support and characteristic variety. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Being a sheaf on X, $\widetilde{\mathcal{M}}$ has a support $\operatorname{Supp}\widetilde{\mathcal{M}}$, which is the closed subset complement to the set of $x \in X$ in the neighbourhood of which $\widetilde{\mathcal{M}}$ is zero.

8.8.9. Lemma. *The support of a coherent $\widetilde{\mathcal{O}}_X$-module is a closed analytic subset of X.*

Proof. This is standard if $\widetilde{\mathcal{O}}_X = \mathcal{O}_X$. On the other hand, if $\widetilde{\mathcal{O}}_X = R_F\mathcal{O}_X$, let $\widetilde{\mathcal{I}}$ be a *graded* ideal of $\widetilde{\mathcal{O}}_X$, locally generated by functions f_jz^j with $f_j \in \mathcal{O}_X$. Then the support of $\widetilde{\mathcal{O}}_X/\widetilde{\mathcal{I}}$ is that of $\mathcal{O}_X/(f_j)_j$. □

Such a property extends to coherent $\widetilde{\mathcal{D}}_X$-modules:

8.8.10. Proposition. *The support* $\operatorname{Supp}\widetilde{\mathcal{M}}$ *of a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is a closed analytic subset of X.*

Proof. The property of being an analytic subset being local, we may assume that $\widetilde{\mathcal{M}}$ is generated over $\widetilde{\mathcal{D}}_X$ by a coherent $\widetilde{\mathcal{O}}_X$-submodule $\mathcal{F}$ (see Exercise 8.62(4)). Then the support of $\widetilde{\mathcal{M}}$ is equal to the support of $\mathcal{F}$. □

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module and let Z be a closed analytic subset of X. It follows from Exercise 8.66 that the subsheaf $\Gamma_Z\widetilde{\mathcal{M}}$ consisting of local sections of $\widetilde{\mathcal{M}}$ annihilated by some power of the ideal $\mathcal{I}_Z$ is $\widetilde{\mathcal{D}}_X$-coherent. In particular, let us denote by $\bigcup_j S_j$ the decomposition of $\operatorname{Supp}\widetilde{\mathcal{M}}$ into its irreducible components. Then $\Gamma_{S_j}\widetilde{\mathcal{M}}$ is a coherent sub $\widetilde{\mathcal{D}}_X$-module of $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}/\Gamma_{S_j}\widetilde{\mathcal{M}}$ is supported on $\bigcup_{k\neq j} S_k$. The following lemma is then obvious.

8.8.11. Lemma. *The kernel and cokernel of the natural morphism*

$$\bigoplus_j \Gamma_{S_j}\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$$

have support everywhere of codimension $\geqslant 1$ *in* $\operatorname{Supp}\widetilde{\mathcal{M}}$. □

The support is usually not the right geometric object attached to a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, as it does not provide enough information on $\widetilde{\mathcal{M}}$. A finer object is the *characteristic variety*. Using the convention 8.1.11, we set $\widetilde{T}^*X = T^*X$ or $\widetilde{T}^*X = T^*X \times \mathbb{C}_z$.

***8.8.12. Definition* (Characteristic variety).** Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. The *characteristic variety* $\operatorname{Char}\widetilde{\mathcal{M}}$ is the subset of the cotangent space $\widetilde{T}^*X$ defined locally as the support of $\operatorname{gr}^F\widetilde{\mathcal{M}}$ for some (or any) local coherent filtration of $\widetilde{\mathcal{M}}$.

8.8.13. Structure of the characteristic variety. The characteristic variety is additive (see Exercise 8.67), so by using the notation of Remark 8.8.8 and after Exercise 8.65, we have a decomposition

$$\operatorname{Char}\widetilde{\mathcal{M}} = \operatorname{Char}\widetilde{\mathcal{M}}' \cup (\operatorname{Char}\mathcal{M}'' \times \mathbb{C}_z),$$

where $\operatorname{Char}\widetilde{\mathcal{M}}'$ is contained in $T^*X = T^*X \times \{0\} \subset \widetilde{T}^*X$.

It is known that $\operatorname{Char}\mathcal{M}''$ is involutive in T^*X: the first proof has been given by Sato, Kawai, Kashiwara [**SKK73**]. Next, Malgrange gave a very simple proof in a seminar Bourbaki talk ([**Mal78**], see also [**GM93**, p. 165]). And finally, Gabber gave the proof of a general algebraic version of this theorem (see [**Gab81**], see also [**Bjö93**, p. 473]). A consequence is that any irreducible component of $\operatorname{Char}\mathcal{M}''$ has a dimension $\geqslant \dim X$.

On the other hand, there is no restriction on $\operatorname{Char}\mathcal{M}'$, which is nothing but the support of $\operatorname{Char}\mathcal{M}'$ when $\mathcal{M}'$ is considered as a $\operatorname{gr}^F\mathcal{D}_X$-module.

8.8.d. Holonomic $\widetilde{\mathcal{D}}_X$-modules and duality

***8.8.14. Definition* (Smooth $\widetilde{\mathcal{D}}_X$-modules).** A coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is said to be *smooth* if it is $\widetilde{\mathcal{O}}_X$-locally free.

In particular, a smooth $\widetilde{\mathcal{D}}_X$-module is strict, and its characteristic variety is equal to $(T^*_X X) \times \widetilde{\mathbb{C}}_z$. (See Exercise 8.68 for the converse.)

***8.8.15. Definition* (Holonomic $\widetilde{\mathcal{D}}_X$-modules).** A coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is said to be holonomic if $\operatorname{Char}\widetilde{\mathcal{M}} \subset \Lambda \times \mathbb{C}_z$, where Λ is a Lagrangian closed subvariety of T^*X.

8.8.16. Remark. By Remarks 8.8.8 and 8.8.13, this is equivalent to asking that $\mathcal{M}''$ is holonomic and that the support of $\mathcal{M}'$ is Lagrangian in T^*X. In particular, if $\widetilde{\mathcal{M}}$ is strict, holonomicity of $\widetilde{\mathcal{M}}$ is equivalent to that of the underlying $\mathcal{D}_X$-module $\mathcal{M}$.

Such a Lagrangian subvariety is the union of its irreducible components, each of which is usually written as $T^*_Z X$, where Z is a closed irreducible subvariety of X and $T^*_Z X$ means the closure, in the cotangent space T^*X of the conormal bundle $T^*_{Z^o}X$ of the smooth part Z^o of Z. It is also known that, due to the existence of stratifications satisfying Whitney condition (a), there exist a locally finite family $(Z_i^o)_{i\in I}$ of locally closed sub-manifolds Z_i^o of Z, with analytic closure and one of them being Z^o, such that $T^*_Z X \subset \bigsqcup_i T^*_{Z_i^o}X$.

For example, a smooth $\widetilde{\mathcal{D}}_X$-module, or a coherent $\widetilde{\mathcal{D}}_X$-module as in Exercise 8.68 or 8.69, is holonomic. The case of $\mathcal{D}_X$-modules is the most useful. We will recall some fundamental results.

8.8.17. Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We have*

$$\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0 \quad \textit{for } i \geqslant n+1. \qquad \square$$

8.8.18. Theorem. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module and $x \in \operatorname{Supp}\mathcal{M}$. Then*

$$2n - \dim_x \operatorname{Char}\mathcal{M} = \inf\{i \in \mathbb{N} \mid \mathcal{E}xt^i_{\mathcal{D}_{X,x}}(\mathcal{M}_x, \mathcal{D}_{X,x}) = 0\}. \qquad \square$$

8.8.19. Corollary. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. Then $\mathcal{M}$ is holonomic if and only if $\mathcal{E}xt^i_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) = 0$ for $i \neq \dim X$. The $\mathcal{D}_X$-module $\mathcal{E}xt^{\dim X}_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$ (see Section 8.3.5), after having applied the suitable side changing functor to it, is called the dual of $\mathcal{M}$, and denoted by $\boldsymbol{D}\mathcal{M}$.*

***8.8.20. Theorem* (Bi-duality, see [Kas76]).** *Let $\mathcal{M}$ be a holonomic $\mathcal{D}_X$-module. Then its dual module $\boldsymbol{D}\mathcal{M}$ is holonomic and the natural functorial morphism from $\mathcal{M}$ to its bi-dual module $\boldsymbol{D}\boldsymbol{D}\mathcal{M}$ is an isomorphism.* $\square$

Let us now consider holonomicity and duality for strict coherent $\widetilde{\mathcal{D}}_X$-modules.

8.8.21. Definition. Let $\widetilde{\mathcal{M}}$ be a strict coherent $\widetilde{\mathcal{D}}_X$-modules. We say that $\widetilde{\mathcal{M}}$ is *strict holonomic* if $\widetilde{\mathcal{M}}$ is holonomic and $\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$ is a strict $\widetilde{\mathcal{D}}_X$-module for every i.

As a consequence, we obtain the following results from Exercise 5.2.

***8.8.22. Corollary* (Cohen-Macaulay property of the graded module)**

Assume that $\widetilde{\mathcal{M}}$ is strict holonomic. Then the following properties hold.

(1) *$\mathcal{E}xt^i_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X) = 0$ for $i \neq n = \dim X$, and we denote by $\boldsymbol{D}\widetilde{\mathcal{M}}$ the $\widetilde{\mathcal{D}}_X$-module obtained after side-changing from $\mathcal{E}xt^n_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$;*

(2) *$\mathcal{E}xt^n_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$ takes the form $R_F\mathcal{M}^\vee$ for some holonomic $\mathcal{D}_X$-module $\mathcal{M}^\vee$;*

(3) *$\mathcal{M}^\vee$ is nothing but $\mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X)$;*

(4) *if $\widetilde{\mathcal{M}}$ is written as $R_F\mathcal{M}$, then*

$$\operatorname{gr}^F \mathcal{E}xt^n_{\mathcal{D}_X}(\mathcal{M}, \mathcal{D}_X) \simeq \mathcal{E}xt^n_{\operatorname{gr}^F\mathcal{D}_X}(\operatorname{gr}^F\mathcal{M}, \operatorname{gr}^F\mathcal{D}_X) =: \boldsymbol{D}(\operatorname{gr}^F\mathcal{M}),$$

(5) *and (Cohen-Macaulay property)* $\mathcal{E}xt^i_{\mathrm{gr}^F\mathcal{D}_X}(\mathrm{gr}^F\mathcal{M}, \mathrm{gr}^F\mathcal{D}_X) = 0$ *for* $i \neq n$*;*

(6) *if for example* $\widetilde{\mathcal{M}}$ *is a right* $\widetilde{\mathcal{D}}_X$*-module, then after side-changing we obtain the following isomorphism of right* $\mathrm{gr}^F\mathcal{D}_X$*-module (see Caveat* 8.2.6*)*

$$\mathrm{gr}^F(\boldsymbol{D}\widetilde{\mathcal{M}}) \simeq \widetilde{\omega}_X \otimes \mathrm{inv}^* \boldsymbol{D}\mathrm{gr}^F\mathcal{M}(-n). \qquad \square$$

8.8.23. Proposition. *Assume that* $\widetilde{\mathcal{M}}$ *is strict holonomic with* $\mathrm{Char}\,\mathcal{M} = \Lambda$*. Then* $\mathcal{E}xt^n_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{D}}_X)$ *is also strict holonomic, with characteristic variety equal to that of* $\widetilde{\mathcal{M}}$*, and biduality holds.* $\square$

8.8.e. Coherence of the pushforward

***8.8.24. Theorem* (Coherence of the pushforward).** *Let* $f : X \to X'$ *be a holomorphic map between complex manifolds and let* $\widetilde{\mathcal{M}}$ *be a coherent* $\widetilde{\mathcal{D}}_X$*-module. Assume that* $\widetilde{\mathcal{M}}$ *admits a coherent filtration* $F_\bullet\widetilde{\mathcal{M}}$*. Then, if* f *is proper, the pushforward complex* ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ *has* $\widetilde{\mathcal{D}}_{X'}$*-coherent cohomology.*

Proof. Assume first that $\widetilde{\mathcal{M}}$ is an induced right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ where $\widetilde{\mathcal{L}}$ is a coherent $\widetilde{\mathcal{O}}_X$-module. Due to the formula of Exercise 8.52(3), the result follows from Grauert's direct image theorem. As a consequence, the same result holds for any bounded complex of such induced right $\widetilde{\mathcal{D}}_X$-modules.

For $\widetilde{\mathcal{M}}$ arbitrary, it is enough by Remark 8.7.5(2) to prove the coherence of ${}_{\mathrm{D}}f^{(j)}_*\widetilde{\mathcal{M}}$ for $j \in [-\dim X, 2\dim X]$. Since the $\widetilde{\mathcal{D}}_{X'}$-coherence is a local property on X', it is enough to prove the coherence property in the neighbourhood of any $x' \in X'$, and therefore it is enough to show the existence, in the neighbourhood of the compact set $f^{-1}(x')$, of a resolution of $\widetilde{\mathcal{M}}_{-N-1} \to \cdots \to \widetilde{\mathcal{M}}_0 \to \widetilde{\mathcal{M}} \to 0$ of sufficiently large length $N+2$, such that $\widetilde{\mathcal{M}}_j$ is a coherent induced $\widetilde{\mathcal{D}}_X$-module for $j = -N, \dots, 0$.

Since $f^{-1}(x')$ is compact, there exists p such that $F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ is onto in some neighbourhood of $f^{-1}(x')$ (i.e., the coherent $\widetilde{\mathcal{O}}_X$-module $F_p\widetilde{\mathcal{M}}$ generates $\widetilde{\mathcal{M}}$ as a $\widetilde{\mathcal{D}}_X$-module). Set $F_q(F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X) = F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} F_{q-p}\widetilde{\mathcal{D}}_X$. This is a coherent filtration of $F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, which therefore induces a coherent filtration on $\mathrm{Ker}[F_p\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}]$. Continuing this way $N+2$ times, we obtained a resolution of length $N+2$ of $\widetilde{\mathcal{M}}$ by coherent induced right $\widetilde{\mathcal{D}}_X$-modules on some neighbourhood of $f^{-1}(x')$. $\square$

8.8.25. Pushforward of a holonomic $\widetilde{\mathcal{D}}_X$***-module.*** Assume that the coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ has a coherent filtration. For example, assume that $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$ and $\widetilde{\mathcal{M}}$ is strict (Proposition 8.8.4(2)). Then, the pushforward of $\widetilde{\mathcal{M}}$ by a proper holomorphic map $f : X \to X'$ has coherent cohomology. Moreover, a theorem of Kashiwara **[Kas76]** complements Theorem 8.8.24 with an estimate of the characteristic variety of the pushforward cohomology $\widetilde{\mathcal{D}}_{X'}$-modules in terms of the characteristic variety of the source $\widetilde{\mathcal{D}}_X$-module. This estimate shows that holonomicity is preserved by proper pushforward. (The theorem of Kashiwara is proved for holonomic $\mathcal{D}_X$-modules, but it extends in a straightforward way to holonomic $\widetilde{\mathcal{D}}_X$-modules.) Therefore, the pushforward by a proper holomorphic map of a strict coherent $\widetilde{\mathcal{D}}_X$-module which is holonomic has holonomic cohomologies when $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$.

Notice also that a holonomic $\mathcal{D}_X$-module $\mathcal{M}$ has a coherent filtration, by a theorem of Malgrange [**Mal04**].

8.8.26. Laumon's formula. Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module. The behaviour of grading with respect to pushforward is treated in Exercises 8.54 and 8.55. For example, for right $\mathcal{D}_X$-modules, Laumon's formula is that, if $f : X \to Y$ is a holomorphic map and *if* ${}_{\mathrm{D}}f_*\mathcal{M}$ *is a strict complex*, then for every i,

$$\mathrm{gr}^F{}_{\mathrm{D}}f_\star^{(i)}\mathcal{M} \simeq H^i\boldsymbol{R}f_\star\big(\mathrm{gr}^F\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y\big). \tag{8.8.26 $*$}$$

8.8.f. Künneth formula.

Assume that X, Y are compact complex manifolds. Let $\widetilde{\mathcal{M}}_X, \widetilde{\mathcal{M}}_Y$ be strict coherent $\widetilde{\mathcal{D}}$-modules. The Künneth formula compares the de Rham cohomology of the external product $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$ with that of the factors.

***8.8.27. Theorem* (Künneth formula).** *Let $\widetilde{\mathcal{M}}_X, \widetilde{\mathcal{M}}_Y$ be coherent $\widetilde{\mathcal{D}}$-modules having a coherent filtration. Assume that $\boldsymbol{R}\Gamma(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y)$ is strict, i.e., has strict cohomologies. Then for each k we have*

$$\boldsymbol{H}^k\big(X\times Y, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y)\big) \simeq \bigoplus_{i+j=k} \boldsymbol{H}^i(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_X) \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{H}^j(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y). \tag{8.8.27 $*$}$$

Note that, if $\widetilde{\mathcal{D}} = R_F\mathcal{D}$, the existence of a coherent filtration for $\widetilde{\mathcal{M}}$ is ensured by Proposition 8.8.4(2). Note also that the roles of $\widetilde{\mathcal{M}}_X$ and $\widetilde{\mathcal{M}}_Y$ can be exchanged.

Proof. We denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections. Let us assume that $\widetilde{\mathcal{M}}_X = \widetilde{\mathcal{L}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ and $\widetilde{\mathcal{M}}_Y = \widetilde{\mathcal{L}}_Y \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y$ are induced $\widetilde{\mathcal{D}}$-modules such that $\widetilde{\mathcal{L}}_X$ is an inductive limit of coherent $\widetilde{\mathcal{O}}_X$-modules and $\widetilde{\mathcal{L}}_Y$ is strict. One computes that

$$\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y \simeq (\widetilde{\mathcal{L}}_X \boxtimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{L}}_Y) \otimes_{\widetilde{\mathcal{O}}_{X\times Y}} \widetilde{\mathcal{D}}_{X\times Y},$$

and thus

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y) \simeq \widetilde{\mathcal{L}}_X \boxtimes_{\widetilde{\mathcal{O}}} \widetilde{\mathcal{L}}_Y = p^*\widetilde{\mathcal{L}}_X \otimes_{q^{-1}\widetilde{\mathcal{O}}_Y} q^{-1}\widetilde{\mathcal{L}}_Y.$$

By the projection formula (see e.g. [**KS90**, Prop. 2.6.6]) and using the strictness of $\widetilde{\mathcal{L}}_Y$, we obtain

$$\boldsymbol{R}q_*\,{}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y) \simeq (\boldsymbol{R}q_*p^*\widetilde{\mathcal{L}}_X) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{L}}_Y,$$

and by Exercise 8.72 the latter term is isomorphic to

$$(\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X)) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{L}}_Y \simeq \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{L}}_Y. \tag{8.8.28}$$

Applying once more the projection formula we finally obtain in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathbb{C}})$:

$$\begin{aligned}\boldsymbol{R}\Gamma\big({}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y)\big) &\simeq \boldsymbol{R}\Gamma\big(Y, \boldsymbol{R}q_* \, {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y)\big)\\ &\simeq \boldsymbol{R}\Gamma\big(Y, \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathcal{L}}_Y\big)\\ &\simeq \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \overset{\boldsymbol{L}}{\otimes}_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(Y, \widetilde{\mathcal{L}}_Y)\\ &\simeq \boldsymbol{R}\Gamma(X, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_X) \overset{\boldsymbol{L}}{\otimes}_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y).\end{aligned} \tag{8.8.29}$$

Let now $\widetilde{\mathcal{M}}_X$ and $\widetilde{\mathcal{M}}_Y$ be as in the theorem. Each term of their canonical resolution (Proposition 8.5.2) satisfies the corresponding assumptions on $\widetilde{\mathcal{L}}_X, \widetilde{\mathcal{L}}_Y$ and thus (8.8.29) holds for each term of the corresponding resolution of $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$. As a consequence, (8.8.29) holds for $\widetilde{\mathcal{M}}_X \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_Y$. Strictness of $\boldsymbol{R}\Gamma(Y, {}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}}_Y)$ then implies Künneth formula (8.8.27 $*$) (see e.g. [**God64**, Th. 5.5.2]). □

8.9. Appendix: Differential complexes and the Gauss-Manin connection

In this section we switch to the case of $\mathcal{D}_X$-modules as in Section 8.1 (see Remark 8.9.9). Let $\mathcal{M}$ be a *left* $\mathcal{D}_X$-module and let $f : X \to Y$ be a holomorphic mapping. On the one hand, we have defined the direct images ${}_{\mathrm{D}}f_*\mathcal{M}$ or ${}_{\mathrm{D}}f_!\mathcal{M}$ of $\mathcal{M}$ viewed as $\mathcal{D}_X$-modules. These are objects in $\mathsf{D}^+(\mathcal{D}_Y)^{\mathrm{left}}$. On the other hand, *when f is a smooth holomorphic mapping*, a flat connection called the *Gauss-Manin connection* is defined on the relative de Rham cohomology of $\mathcal{M}$. We will compare both constructions, when f is smooth. Such a comparison has essentially already been done *when f is the projection of a product $X = Y \times T \to Y$* (see Examples 8.7.10 and 8.7.11).

In this section we also introduce the derived category of differential complexes on a complex manifold X, that is, complexes of $\mathcal{O}_X$-modules with differential morphisms as differential. This derived category is shown to be equivalent to that of $\mathcal{D}_X$-modules (Theorem 8.9.15). It is sometimes useful to work in this category (see e.g. the proof of Theorem 8.9.21).

8.9.a. Differential complexes.

Given an $\mathcal{O}_X$-module $\mathcal{L}$, there is a natural $\mathcal{O}_X$-linear morphism (with the right structure on the right-hand term)

$$\mathcal{L} \longrightarrow \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \quad \ell \longmapsto \ell \otimes 1.$$

There is also a (only) $\mathbb{C}$-linear morphism

$$\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{L} \tag{8.9.1}$$

defined at the level of local sections by $\ell \otimes P \mapsto P(1)\ell$, where $P(1)$ is the result of the action of the differential operator P on 1, which is equal to the degree 0 coefficient of P if P is locally written as $\sum_\alpha a_\alpha(x)\partial_x^\alpha$. In an intrinsic way, consider the natural augmentation morphism $\mathcal{D}_X \to \mathcal{O}_X$, which is left $\mathcal{D}_X$-linear, hence left $\mathcal{O}_X$-linear; then apply $\mathcal{L} \otimes_{\mathcal{O}_X} \bullet$ to it. Notice however that (8.9.1) is an $\mathcal{O}_X$-linear morphism by using the left $\mathcal{O}_X$-module structure on $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$.

Let $\mathcal{L}, \mathcal{L}'$ be two $\mathcal{O}_X$-modules. A (right) $\mathcal{D}_X$-linear morphism

$$v : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \tag{8.9.2}$$

is uniquely determined by the $\mathcal{O}_X$-linear morphism

$$w : \mathcal{L} \longrightarrow \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \tag{8.9.3}$$

that it induces (where the right $\mathcal{O}_X$-module structure is chosen on $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$). In other words, the natural morphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \longrightarrow \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$$

is an isomorphism. We also have, at the sheaf level,

$$\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \xrightarrow{\sim} \mathcal{H}om_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X). \tag{8.9.4}$$

Notice that $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is naturally equipped with an $\mathcal{O}_X$-module structure by using the left $\mathcal{O}_X$-module structure on $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X$ (see Remark 8.5.1), and similarly $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)$ is a $\Gamma(X, \mathcal{O}_X)$-module.

Now, w induces a $\mathbb{C}$-linear morphism

$$u : \mathcal{L} \longrightarrow \mathcal{L}', \tag{8.9.5}$$

by composition with (8.9.1): $\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X \to \mathcal{L}'$. By Exercise 8.73, u is nothing but the morphism

$$H^0\big({}^{\mathrm{p}}\mathrm{DR}(v)\big) : H^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X)\big) \longrightarrow H^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X)\big).$$

8.9.6. Definition **(Differential operators between two $\mathcal{O}_X$-modules)**
The $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$ of *differential operators from $\mathcal{L}$ to $\mathcal{L}'$* is the image of the morphism $\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) \to \mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}')$.

Similarly we define the sheaf of $\mathbb{C}$-vector spaces $\mathcal{H}om_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$.

8.9.7. Definition **(The category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$).** We denote by $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ the category whose objects are $\mathcal{O}_X$-modules and morphisms are differential operators between $\mathcal{O}_X$-modules (this is justified by Exercise 8.74(4)).

In particular, $\mathsf{Mod}(\mathcal{O}_X)$ is a subcategory of $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$, since any $\mathcal{O}_X$-linear morphism is a differential operator (of degree zero).

We will now show that the correspondence $\mathcal{L} \mapsto \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ induces a functor $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X)$. In order to do so, one first needs to show that to any differential morphism u corresponds at most one v.

8.9.8. Lemma. *The morphism*

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X) &\longrightarrow \mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}') \\ v &\longmapsto u\end{aligned}$$

is injective.

Proof. Recall that, for any multi-index β, we have $\partial_x^\alpha(x^\beta)=0$ if $\beta_i<\alpha_i$ for some i, and $\partial_x^\alpha(x^\alpha)=\alpha!$. Assume that $u=0$. Let ℓ be a local section of $\mathcal{L}$ and, using local coordinates $(x_1,\dots,x_n)$, write in a unique way $w(\ell)=\sum_\alpha w(\ell)_\alpha\otimes\partial_x^\alpha$, where the sum is taken on multi-indices α and w is as in (8.9.3). If $w(\ell)\neq 0$, let β be minimal (with respect to the usual partial ordering on $\mathbb{N}^n$) among the multi-indices α such that $w(\ell)_\alpha\neq 0$. Then,

$$0=u(x^\beta\ell)=\sum_\alpha \partial_x^\alpha(x^\beta)w(\ell)_\alpha=\beta!w(\ell)_\beta,$$

a contradiction. $\square$

8.9.9. Remark. A similar lemma would not hold in the category of induced graded $R_F\mathcal{D}_X$-modules because of possible z-torsion: one would only get that $z^k u(x^\beta\ell)=0$ for some k.

According to Lemma 8.9.8, the following definition is meaningful.

***8.9.10. Definition* (The inverse de Rham functor).** The functor

$$^{\mathrm{diff}}\mathrm{DR}^{-1}:\mathsf{Mod}(\mathcal{O}_X,\mathrm{Diff}_X)\longrightarrow\mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X)$$

is defined by $^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L})=\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X$ and $^{\mathrm{diff}}\mathrm{DR}^{-1}(u)=v$.

8.9.11. Remarks.

(1) The notation is justified by the fact that $^{\mathrm{p}}\mathrm{DR}(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\simeq\mathcal{L}$ (see Exercise 8.29(5)).

(2) By the isomorphism of Exercise 8.77, $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L},\mathcal{L}')$ is equipped with the structure of a $\Gamma(X,\mathcal{O}_X)$-module. Similarly,

$$\mathcal{H}om_{\mathcal{D}_X}(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X,\mathcal{L}'\otimes_{\mathcal{O}_X}\mathcal{D}_X)\longrightarrow\mathcal{H}om_{\mathbb{C}}(\mathcal{L},\mathcal{L}')$$

is injective, and and this equips the image sheaf $\mathcal{H}om_{\mathrm{Diff}}(\mathcal{L},\mathcal{L}')$ with the structure of an $\mathcal{O}_X$-module.

(3) When considered as taking values in $\mathsf{Mod}(\mathcal{D}_X)$, the functor $^{\mathrm{diff}}\mathrm{DR}^{-1}$ is not, however, an equivalence of categories, i.e., is not essentially surjective. The reason is that, first, not all $\mathcal{D}_X$-modules are isomorphic to some $\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X$ and, next, its natural quasi-inverse would be the de Rham functor $^{\mathrm{p}}\mathrm{DR}$ which takes values in a category of complexes. Nevertheless, if one extends suitably these functors to categories of complexes, they become equivalences (see below Theorem 8.9.15).

8.9.b. The de Rham complex as a differential complex. Given an induced $\mathcal{D}$-module, its de Rham complex gives enough information to recover it, according to Remark 8.9.11(1). On the other hand, given a bounded complex of induced $\mathcal{D}$-modules, its de Rham complex does not give enough information to recover its differentials. We will refine the functor $^{\mathrm{p}}\mathrm{DR}$ to a functor $^{\mathrm{diff}}\mathrm{DR}$, which takes values in differential complexes, and has quasi-inverse induced by $^{\mathrm{diff}}\mathrm{DR}^{-1}$.

According to Exercise 8.76, one may consider the category $\mathsf{C}^\star(\mathcal{O}_X,\mathrm{Diff}_X)$ of $\star$-bounded complexes of objects of $\mathsf{Mod}(\mathcal{O}_X,\mathrm{Diff}_X)$ (with $\star=\varnothing,+,-,\mathrm{b}$), and the category $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ of $\star$-bounded complexes up to homotopy (see [**KS90**, Def. 1.3.4]). These are called $\star$-bounded *differential complexes*.

There is a natural forgetful functor Forget from $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ to $\mathsf{Mod}(\mathbb{C}_X)$, and by extension a functor Forget at the level of $\mathsf{C}^\star$ and $\mathsf{K}^\star$. Exercise 8.78 shows that we can decompose the ${}^{\mathrm{p}}\mathrm{DR}$ functor as

$$\begin{tikzcd}
\mathsf{Mod}(\mathcal{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{C}^{\mathrm{b}}(\mathcal{O}_X, \mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{C}^{\mathrm{b}}(\mathbb{C}_X)
\end{tikzcd}$$

and

$$\begin{tikzcd}
\mathsf{K}^\star(\mathcal{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{K}^\star(\mathbb{C}_X)
\end{tikzcd}$$

In order to define the "derived category" of the additive category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$, one needs to define the notion of null system in $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ and localize the category with respect to the associated multiplicative system. A possible choice would be to say that an object belongs to the null system if it belongs to the null system of $\mathsf{C}^\star(\mathbb{C}_X)$ when forgetting the Diff structure, i.e., which has zero cohomology when considered as a complex of sheaves of $\mathbb{C}$-vector spaces. This is not the choice made below. One says that a differential morphism $u : \mathcal{L} \to \mathcal{L}'$ as in (8.9.5) is a Diff-quasi-isomorphism if the corresponding v as in (8.9.2) is a quasi-isomorphism of right $\mathcal{D}_X$-modules.

The functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ of Definition 8.9.10 extends as a functor $\mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{C}^\star_{\mathrm{i}}(\mathcal{D}_X)$ and $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{K}^\star_{\mathrm{i}}(\mathcal{D}_X)$ in a natural way, and is a functor of triangulated categories on K. Moreover, according to the last part of Exercise 8.77, it is an equivalence of triangulated categories.

We wish now to define *acyclic objects* in the triangulated category $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, and show that they form a *null system* in the sense of [**KS90**, Def. 1.6.6].

8.9.12. Definition. We say that a object $\mathcal{L}^\bullet$ of $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ is Diff-*acyclic* if ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^\bullet)$ is acyclic in $\mathsf{K}^\star_{\mathrm{i}}(\mathcal{D}_X)$ (equivalently, in $\mathsf{K}^\star(\mathcal{D}_X)$).

Define, as in [**KS90**, (1.6.4)], the family $S(\mathsf{N})$ as the family of morphisms which can be embedded in a distinguished triangle of $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, with the third term being an object of N. We call such morphisms Diff-*quasi-isomorphisms*. Clearly, they correspond exactly *via* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ to quasi-isomorphisms in $\mathsf{K}^\star(\mathcal{D}_X)$.

We now may localize the category $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ with respect to the null system N and get a category denoted by $\mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$. By construction, we still get a functor

$$(8.9.13)\qquad {}^{\mathrm{diff}}\mathrm{DR}^{-1} : \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \longrightarrow \mathsf{D}^\star_{\mathrm{i}}(\mathcal{D}_X) \longrightarrow \mathsf{D}^\star(\mathcal{D}_X).$$

We note that the first component is an equivalence by definition of the null system (since we have an equivalence at the level of the categories $\mathsf{K}^\star$). The second component is also an equivalence, according to Corollary 8.5.3. We will show below (Theorem 8.9.15) that ${}^{\mathrm{diff}}\mathrm{DR}$ is a quasi-inverse functor.

8.9.14. Remark. The category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ is also naturally a subcategory of the category $\mathsf{Mod}(\mathbb{C}_X)$ of sheaves of $\mathbb{C}$-vector spaces because $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}, \mathcal{L}')$ is a subset of $\mathrm{Hom}_{\mathbb{C}}(\mathcal{L}, \mathcal{L}')$. We therefore have a natural functor $\mathsf{Forget} : \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \to \mathsf{K}^\star(\mathbb{C}_X)$, forgetting that the differentials of a complex are differential operators, and forgetting also that the homotopies should be differential operators too. As a consequence of Theorem 8.9.15, we will see in Exercise 8.83 that any object in the null system N defined above is sent to an object in the usual null system of $\mathsf{K}^\star(\mathbb{C}_X)$, i.e., objects with zero cohomology. In other words, a Diff-quasi-isomorphism is sent into a usual quasi-isomorphism. But there may exist morphisms in $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ which are quasi-isomorphisms when viewed in $\mathsf{K}^\star(\mathbb{C}_X)$, but are not Diff-quasi-isomorphisms.

8.9.15. Theorem. *The functors* ${}^{\mathrm{diff}}\mathrm{DR}$ *and* ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ *induce quasi-inverse and induce equivalences of categories*

$$\begin{tikzcd} \mathsf{D}^\star(\mathcal{D}_X) \arrow[rr, bend left, "{}^{\mathrm{diff}}\mathrm{DR}"] && \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X). \arrow[ll, bend left, "{}^{\mathrm{diff}}\mathrm{DR}^{-1}"] \end{tikzcd}$$

8.9.16. Lemma. *There is an isomorphism of functors* ${}^{\mathrm{diff}}\mathrm{DR}^{-1} \circ {}^{\mathrm{diff}}\mathrm{DR} \xrightarrow{\sim} \mathrm{Id}$ *from* $\mathsf{D}^\star(\mathcal{D}_X)$ *(right* $\mathcal{D}_X$*-modules) to itself.*

This lemma enables one to attach to each object of $\mathsf{D}^\star(\mathcal{D}_X)$ a canonical resolution by induced $\mathcal{D}_X$-modules since ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ takes values in $\mathsf{D}^\star_{\mathrm{i}}(\mathcal{D}_X)$.

Proof. Let us recall that there exists an explicit side-changing isomorphism of complexes ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}} \simeq {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}^{\mathrm{right}}$ which is given by termwise $\mathcal{O}_X$-linear morphisms. If we regard these complexes as objects of $\mathsf{C}^{\mathrm{b}}(\mathcal{O}_X, \mathrm{Diff})$, we deduce that the side-changing isomorphism is an isomorphism in this category. In other words, we have ${}^{\mathrm{diff}}\mathrm{DR}(\mathcal{M}^{\mathrm{left}}) \simeq {}^{\mathrm{diff}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}})$.

For the proof of the lemma, start with a left $\mathcal{D}_X$-module $\mathcal{M}^{\mathrm{left}}$. By definition, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}} = (\Omega_X^{n+\bullet} \otimes \mathcal{M}^{\mathrm{left}}) \otimes \mathcal{D}_X$ with differential ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\nabla)$. This is nothing but the complex $\Omega_X^{n+\bullet} \otimes (\mathcal{M}^{\mathrm{left}} \otimes \mathcal{D}_X)$ where the differential is the connection on the left $\mathcal{D}_X$-module $(\mathcal{M}^{\mathrm{left}} \otimes \mathcal{D}_X)_{\mathrm{tens}}$. Furthermore, this identification is right $\mathcal{D}_X$-linear with respect to the $(\mathrm{right})_{\mathrm{triv}}$ structure on both terms.

We note that $\big[(\mathcal{M}^{\mathrm{left}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)^{\mathrm{right}}\big]_{\mathrm{tens}} \simeq (\mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\mathrm{tens}}$, i.e., both with the tensor structure, respectively left and right, and this isomorphism is compatible with the right $\mathcal{D}_X$-structure $(\mathrm{right})_{\mathrm{triv}}$ on both terms. By side-changing we find

$$\big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{left}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}} \simeq \big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}},$$

and by using the involution of Exercise 8.19,

$$\big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\mathrm{tens}}\big]_{\mathrm{triv}} \simeq \big[{}^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\mathrm{triv}}\big]_{\mathrm{tens}}.$$

Lastly, we have

$$^{\mathrm{p}}\mathrm{DR}(\mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathcal{D}_X)_{\mathrm{triv}} = \mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathrm{Sp}^\bullet(\mathcal{D}_X) \simeq \mathcal{M}^{\mathrm{right}} \otimes_{\mathcal{O}_X} \mathcal{O}_X = \mathcal{M}^{\mathrm{right}},$$

and the remaining right $\mathcal{D}_X$-structure is deduced from the tens one, which is the natural right structure on $\mathcal{M}^{\mathrm{right}}$. We conclude that, functorially, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\, {}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}} \simeq \mathcal{M}^{\mathrm{right}}$. Since ${}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{left}} \simeq {}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}^{\mathrm{right}}$, the lemma follows. □

***Proof of Theorem* 8.9.15.** From the previous lemma, it is now enough to show that, if $\mathcal{L}^\bullet$ is a complex in $\mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, there exists a a Diff-quasi-isomorphism ${}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^\bullet \to \mathcal{L}^\bullet$, and, by definition, this is equivalent to showing the existence of a quasi-isomorphism ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^\bullet \to {}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^\bullet$, that we know from the previous result applied to $\mathcal{M} = {}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^\bullet$. □

8.9.17. Remark. The functor ${}^{\mathrm{diff}}\mathrm{DR}^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}$, regarded as a functor $\mathsf{D}^\star(\mathcal{D}_X) \to \mathsf{D}^\star_{\mathrm{i}}(\mathcal{D}_X)$, is nothing but that of Corollary 8.5.3.

***8.9.18. Remark* (The Godement resolution of a differential complex)**

Let $\mathcal{L}^\bullet$ be an object of $\mathsf{C}^+(\mathcal{O}_X, \mathrm{Diff}_X)$. Then $\mathrm{God}^\bullet\,\mathcal{L}^\bullet$ is maybe not a differential complex (see Exercise 8.48(2)). However, $\mathrm{God}^\bullet\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^\bullet$ is a differential complex, being equal to ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^\bullet\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}\,\mathcal{L}^\bullet$. Therefore, the composite functor $\mathrm{God}^\bullet\,{}^{\mathrm{diff}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}$ plays the role of Godement resolutions in the category of differential complexes.

8.9.c. The Gauss-Manin connexion. We assume in this section that $f : X \to Y$ is a *smooth* holomorphic map. The cotangent map $T^*f : f^*\Omega^1_Y \to \Omega^1_X$ is then injective, and we will identify $f^*\Omega^1_Y$ with its image. We set $n = \dim X$, $m = \dim Y$ and $d = n - m$ (we assume that X and Y are pure dimensional, otherwise one works on each connected component of X and Y).

Consider the *Leray filtration* $\mathrm{Ler}^\bullet$ on the complex $(\Omega^\bullet_X, \mathrm{d})$, defined by

$$\mathrm{Ler}^p\Omega^i_X = \mathrm{Im}(f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{i-p}_X \longrightarrow \Omega^i_X).$$

[With this notation, $\mathrm{Ler}^p\Omega^i_X$ can be nonzero only when $i \in [0, n]$ and $p \in [0, \min(i, m)]$.]

Then, as f is smooth, we have (by computing with local coordinates adapted to f),

$$\mathrm{gr}^p_{\mathrm{Ler}}\Omega^i_X = f^*\Omega^p_Y \otimes_{\mathcal{O}_X} \Omega^{i-p}_{X/Y},$$

where $\Omega^k_{X/Y}$ is the sheaf of relative differential forms: $\Omega^k_{X/Y} = \wedge^k\Omega^1_{X/Y}$ and $\Omega^1_{X/Y} = \Omega^1_X \big/ f^*\Omega^1_Y$. Notice that $\Omega^k_{X/Y}$ is $\mathcal{O}_X$-locally free.

Let $\mathcal{M}$ be a left $\mathcal{D}_X$-module or an object of $\mathsf{D}^+(\mathcal{D}_X)^{\mathrm{left}}$. As f is smooth, the sheaf $\mathcal{D}_{X/Y}$ of relative differential operators is well-defined and by composing the flat connection $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$ with the projection $\Omega^1_X \to \Omega^1_{X/Y}$ we get a relative flat connection $\nabla_{X/Y}$ on $\mathcal{M}$, and thus the structure of a left $\mathcal{D}_{X/Y}$-module on $\mathcal{M}$. In particular, the relative de Rham complex is defined as

$$^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} = (\Omega^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}, \nabla_{X/Y}).$$

We have ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = (\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}, \nabla)$ (see Definition 8.4.1) and the Leray filtration $\mathrm{Ler}^p\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$ is preserved by the differential ∇. We can therefore induce the filtration $\mathrm{Ler}^\bullet$ on the complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$. We then have an equality of complexes

$$\mathrm{gr}^p_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} = f^*\Omega_Y^p \otimes_{\mathcal{O}_X} {}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}[-p].$$

Notice that the differential of these complexes are $f^{-1}\mathcal{O}_Y$-linear.

The complex $f_*\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ (resp. the complex $f_!\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$) is filtered by subcomplexes $f_*\,\mathrm{God}^\bullet\,\mathrm{Ler}^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ (resp. $f_!\,\mathrm{God}^\bullet\,\mathrm{Ler}^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$). We therefore get a spectral sequence (the Leray spectral sequence in the category of sheaves of $\mathbb{C}$-vector spaces, see, e.g. [**God64**]). Using the projection formula for $f_!$ and the fact that Ω_Y^p is $\mathcal{O}_Y$-locally free, one obtains that the E_1 term for the complex $f_!\,\mathrm{God}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is given by

$$E_{1,!}^{p,q} = \Omega_Y^p \otimes_{\mathcal{O}_Y} R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}, \tag{8.9.19}$$

and the spectral sequence converges to (a suitable graded object associated with) $R^{p+q} f_!\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$. If f is proper on $\mathrm{Supp}\,\mathcal{M}$ or if $\mathcal{M}$ has $\mathcal{D}_X$-coherent cohomology, one can also apply the projection formula to f_* (see [**MN93**, §II.5.4]).

By definition of the spectral sequence, the differential $d_1 : E_1^{p,q} \to E_1^{p+1,q}$ is the connecting morphism (see Exercise 8.86 below) in the long exact sequence associated to the short exact sequence of complexes

$$0 \longrightarrow \mathrm{gr}^{p+1}_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \longrightarrow \mathrm{Ler}^p\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}/\mathrm{Ler}^{p+2}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \longrightarrow \mathrm{gr}^p_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M} \longrightarrow 0$$

after applying $f_!\,\mathrm{God}^\bullet$ (or $f_*\,\mathrm{God}^\bullet$ if one of the previous properties is satisfied).

***8.9.20. Lemma* (The Gauss-Manin connection).** *The morphism*

$$\nabla^{\mathrm{GM}} := d_1 : R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M} = E_1^{0,q} \longrightarrow E_1^{1,q} = \Omega_Y^1 \otimes_{\mathcal{O}_Y} R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}$$

is a flat connection on $R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}$*, called the* Gauss-Manin connection *and the complex* $(E_1^{\bullet,q}, d_1)$ *is equal to the de Rham complex* ${}^{\mathrm{diff}}\mathrm{DR}_Y(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}, \nabla^{\mathrm{GM}})$.

***Sketch of proof of Lemma* 8.9.20.** Instead of using the Godement resolution, one can use the C^∞ de Rham complex $\mathcal{E}_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}$, with the differential D defined by

$$D(\eta \otimes m) = \mathrm{d}\eta \otimes m + (-1)^k \eta \wedge \nabla m,$$

if η is C^∞ differential k-form, that is, a local section of $\mathcal{E}_X^k$ ($k \leqslant 0$). By a standard argument (Dolbeault resolution) analogous to that of Exercise 8.51(5), this C^∞ de Rham complex is quasi-isomorphic to the holomorphic one, and is equipped with the Leray filtration. The quasi-isomorphism is strict with respect to $\mathrm{Ler}^\bullet$. One can therefore compute with the C^∞ de Rham complex. Moreover, the assertion is local with respect to Y.

Assume first that, in the neighbourhood of $f^{-1}(y)$, X is diffeomorphic to a product $X \simeq Z \times Y$. This occurs for example if f is proper (Ehresmann's theorem). Then we identify $\mathcal{E}_X^{p+q}$ with $\mathcal{E}_Y^p \otimes \mathcal{E}_{X/Y}^q$ and the differential D decomposes accordingly as $D_Y + D_{X/Y}$. The flatness of D implies the flatness of $D_{X/Y}$ and D_Y. Given a section μ of $f_!\big(\mathcal{E}_Y^p \otimes (\mathcal{E}_{X/Y}^q \otimes \mathcal{M})\big)$ which is closed with respect to $D_{X/Y}$, we can identify it with its lift $\widetilde{\mu}$ (see Exercise 8.86), and $d_1\mu$ is thus the class of $D_Y\mu$, so the C^∞ Gauss-Manin connection D^{GM} in degree zero induces d_1 in any degree.

In general, choose a partition of unity (χ_α) such that for every α, when restricted to some open neighbourhood of $\operatorname{Supp}\chi_\alpha$, f is locally the projection from a product to one of its factors. We set $D=\sum_\alpha \chi_\alpha D=\sum_\alpha D^{(\alpha)}$ and we apply the previous argument to each $D^{(\alpha)}$. □

8.9.21. Theorem. *Let $f: X\to Y$ be a smooth holomorphic map and let $\mathcal{M}$ be left $\mathcal{D}_X$-module—or more generally an object of $\mathsf{D}^+(\mathcal{D}_X)^{\mathrm{left}}$. Then there is a functorial isomorphism of left $\mathcal{D}_Y$-modules*

$$R^kf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\longrightarrow {}_{\mathrm{D}}f_!^{(k)}\mathcal{M}$$

when one endows the left-hand term with the Gauss-Manin connection ∇^{GM}. The same result holds for ${}_{\mathrm{D}}f_*$ instead of ${}_{\mathrm{D}}f_!$ if f is proper on $\operatorname{Supp}\mathcal{M}$ *or $\mathcal{M}$ is $\mathcal{D}_X$-coherent (or has coherent cohomology).*

Proof. Recall (Exercise 8.26) that, for a left $\mathcal{D}_X$-module $\mathcal{M}$, we have

$$\mathcal{M}^{\mathrm{right}}\otimes_{\mathcal{D}_X}\mathrm{Sp}^\bullet_{X\to Y}(\mathcal{D}_X)\simeq\Omega^\bullet_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)[n],$$

so that the direct image of $\mathcal{M}$, regarded as a right $\mathcal{D}_Y$-module, is

$$({}_{\mathrm{D}}f_!\mathcal{M})^{\mathrm{right}}=\boldsymbol{R}f_!\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)[m],\tag{8.9.22}$$

by Exercise 8.51(3). We conclude that

$${}^{\mathrm{diff}}\mathrm{DR}_Y\,{}_{\mathrm{D}}f_!\mathcal{M}\simeq{}^{\mathrm{diff}}\mathrm{DR}_Y\big(\boldsymbol{R}f_!\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)\big).$$

There is a Leray filtration $\mathrm{Ler}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)$. Notice that the graded complex $\mathrm{gr}^p_{\mathrm{Ler}}\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)$ is equal to the complex

$$f^{-1}\Omega^p_Y\otimes_{f^{-1}\mathcal{O}_Y}{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y[-p],$$

with differential induced by $\nabla_{X/Y}$ on $\mathcal{M}$ (remark that the part of the differential involving T^*f is killed by taking $\mathrm{gr}^p_{\mathrm{Ler}}$). The differential is now $f^{-1}\mathcal{O}_Y$-linear.

The filtered complex $\boldsymbol{R}f_!\mathrm{Ler}^\bullet\,{}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y)$ gives rise to a spectral sequence in the category of right $\mathcal{D}_Y$-modules. By the previous computation, the $E_1^{p,q}$ term of this spectral sequence is the right $\mathcal{D}_Y$-module

$$\begin{aligned}R^{p+q}f_!\big(f^{-1}\Omega^p_Y\otimes_{f^{-1}\mathcal{O}_Y}{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\otimes_{f^{-1}\mathcal{O}_Y}f^{-1}\mathcal{D}_Y[-p]\big)&\\ \simeq\Omega^p_Y\otimes_{\mathcal{O}_Y}R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M}\otimes_{\mathcal{O}_Y}\mathcal{D}_Y&,\end{aligned}$$

which is an induced $\mathcal{D}_Y$-module, whose ${}^{\mathrm{diff}}\mathrm{DR}_Y$ is equal to the corresponding Gauss-Manin term (8.9.19). We claim, as will show below, that the differential d_1 becomes the Gauss-Manin d_1 after applying ${}^{\mathrm{diff}}\mathrm{DR}_Y$. This will prove that the Gauss-Manin E_1 complex is equal to ${}^{\mathrm{diff}}\mathrm{DR}_Y$ of the E_1 complex of right $\mathcal{D}_Y$-modules.

Notice now that Lemma 8.9.20 shows in particular that the E_1 complex considered there is a complex in $\mathsf{C}^+(\mathcal{O}_Y,\mathrm{Diff}_Y)$, and

$${}^{\mathrm{diff}}\mathrm{DR}_Y^{-1}(E_1^{\bullet,q},d_1)\simeq(R^qf_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\,\mathcal{M},\nabla^{\mathrm{GM}})^{\mathrm{right}}[-m],$$

since, for a left $\mathcal{D}_Y$-module $\mathcal{N}$, we have, according to Theorem 8.9.15,

$$ {}^{\mathrm{diff}}\mathrm{DR}_Y^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}_Y(\mathcal{N}) = {}^{\mathrm{diff}}\mathrm{DR}_Y^{-1}\,{}^{\mathrm{diff}}\mathrm{DR}_Y(\mathcal{N}^{\mathrm{right}})[-m] \simeq \mathcal{N}^{\mathrm{right}}[-m]. $$

The claim above, together with Lemma 8.9.16, implies that the E_1 complex of the $\mathcal{D}_Y$-Leray spectral sequence has cohomology in degree m only, hence this spectral sequence degenerates at E_2, this cohomology being isomorphic to $(R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathcal{M}, \nabla^{\mathrm{GM}})^{\mathrm{right}}[-m]$. But the spectral sequences converges (the Leray filtration is finite) and its limit is $\bigoplus_p \mathrm{gr}^p({}_{\mathrm{D}}f_!^{(q-m)}\mathcal{M})^{\mathrm{right}}$ for the induced filtration on $({}_{\mathrm{D}}f_!\mathcal{M}^{(q-m)})^{\mathrm{right}}$, according to (8.9.22). We conclude that this implicit filtration is trivial and that $({}_{\mathrm{D}}f_!^{(q)}\mathcal{M})^{\mathrm{right}} = (R^q f_!\,{}^{\mathrm{p}}\mathrm{DR}_{X/Y}\mathcal{M}, \nabla^{\mathrm{GM}})^{\mathrm{right}}$, as wanted, after side changing.

Let us now compare the d_1 of both spectral sequences. As the construction is clearly functorial with respect to $\mathcal{M}$, we can replace $\mathcal{M}$ by the flabby sheaf $\mathrm{God}^\ell\mathcal{M}$ for every ℓ. We then have

$$\begin{aligned}
&\boldsymbol{R}f_!\big(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathrm{God}^\ell\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\big)\\
&\qquad = \boldsymbol{R}f_!\big(\mathrm{God}^\ell(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M}) \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\big) \quad \text{(Exercise 8.48)}\\
&\qquad = \boldsymbol{R}f_!\big(\mathrm{God}^\ell(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M})\big) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \quad \text{(projection formula)}\\
&\qquad = f_!\big(\mathrm{God}^\ell(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathcal{M})\big) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \quad \text{(flabbiness of } \mathrm{God}^\ell)\\
&\qquad = f_!\big(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathrm{God}^\ell\mathcal{M}\big) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \quad \text{(Exercise 8.48)}\\
&\qquad = f_!\big(\Omega_X^\bullet \otimes_{\mathcal{O}_X} \mathrm{God}^\ell\mathcal{M} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y\big) \quad \text{(projection formula).}
\end{aligned}$$

It is also enough to make the computation locally on Y, so that we can write $f = (f_1, \dots, f_m)$, using local coordinates $(y_1, \dots, y_m)$. If μ is a section of $\Omega_X^k \otimes \mathcal{M}$ and $\mathbf{1}_Y$ is the unit of $\mathcal{D}_Y$, then (8.6.1) can be written as

$$ \nabla^X(\mu \otimes \mathbf{1}_Y) = (\nabla\mu) \otimes \mathbf{1}_Y + \sum_{j=1}^m \mu \wedge df_j \otimes \partial_{y_j}. $$

Using the definition of d_1 given by Exercise 8.86 and an argument similar to that of Exercise 8.84, one gets the desired assertion. □

8.10. Exercises

8.10.a. Exercises for Section 8.1

Exercise 8.1. Let $\mathcal{E}$ be a locally free $\mathcal{O}_X$-module of rank d and let $\mathcal{E}^\vee$ be its dual. Show that, given any local basis $\boldsymbol{e} = (e_1, \dots, e_d)$ of $\mathcal{E}$ with dual basis $\boldsymbol{e}^\vee$, the section $\sum_{i=1}^d e_i \otimes e_i^\vee$ of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$ does not depend on the choice of the local basis $\boldsymbol{e}$ and extends as a global section of $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$. Show that it defines, up to a constant, an $\mathcal{O}_X$-linear section $\mathcal{O}_X \to \mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee$ of the natural duality pairing $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E}^\vee \to \mathcal{O}_X$. Conclude that we have a natural global section of $\Omega_X^1 \otimes_{\mathcal{O}_X} \Theta_X$ given, in local coordinates, by $\sum_i \mathrm{d}x_i \otimes \partial_{x_i}$.

Exercise 8.2. Show that a differential operator P *of order* $\leqslant 1$ satisfying $P(1) = 0$ is a derivation of $\mathcal{O}_X$, i.e., a section of Θ_X.

Exercise 8.3 **(Local computations).** Let U be an open set of $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$. Denote by $\partial_{x_1}, \dots, \partial_{x_n}$ the corresponding vector fields.

(1) Show that the following relations are satisfied in $\mathcal{D}(U)$:

$$[\partial_{x_i}, f] = \frac{\partial f}{\partial x_i}, \quad \forall f \in \mathcal{O}(U),\ \forall i \in \{1, \dots, n\},$$
$$[\partial_{x_i}, \partial_{x_j}] = 0 \quad \forall i, j \in \{1, \dots, n\}.$$

with standard notation concerning multi-indices α, β.

(2) Show that any element $P \in \mathcal{D}(U)$ can be written in a unique way as $\sum_\alpha a_\alpha \partial_x^\alpha$ or $\sum_\alpha \partial_x^\alpha b_\alpha$with $a_\alpha, b_\alpha \in \mathcal{O}(U)$. Conclude that $\mathcal{D}_X$ is a locally free module over $\mathcal{O}_X$ with respect to the action on the left and that on the right.

(3) Show that $\max\{|\alpha| \ ; \ a_\alpha \neq 0\} = \max\{|\alpha| \ ; \ b_\alpha \neq 0\}$. It is denoted by $\mathrm{ord}_x P$.

(4) Show that $\mathrm{ord}_x P$ does not depend on the coordinate system chosen on U.

(5) Show that $PQ = 0$ in $\mathcal{D}(U) \Rightarrow P = 0$ or $Q = 0$.

(6) Identify $F_k\mathcal{D}_X$ with the subsheaf of local sections of $\mathcal{D}_X$ having order $\leqslant k$ (in some or any local coordinate system). Show that it is a locally free $\mathcal{O}_X$-module of finite rank.

(7) Show that the filtration $F_\bullet\mathcal{D}_X$ is exhaustive (i.e., $\mathcal{D}_X = \bigcup_k F_k\mathcal{D}_X$) and that it satisfies

$$F_k\mathcal{D}_X \cdot F_\ell\mathcal{D}_X = F_{k+\ell}\mathcal{D}_X.$$

(The left-hand term consists by definition of all sums of products of a section of $F_k\mathcal{D}_X$ and a section of $F_\ell\mathcal{D}_X$.)

(8) Show that the bracket $[P, Q] := PQ - QP$ induces for every k, ℓ a $\mathbb{C}$-bilinear morphism $F_k\mathcal{D}_X \otimes_{\mathbb{C}} F_\ell\mathcal{D}_X \to F_{k+\ell-1}\mathcal{D}_X$.

(9) Conclude that the graded ring $\mathrm{gr}^F\mathcal{D}_X$ is commutative.

Exercise 8.4 **(The graded sheaf $\mathrm{gr}^F\mathcal{D}_X$).** The goal of this exercise is to show that the sheaf of graded rings $\mathrm{gr}^F\mathcal{D}_X$ may be canonically identified with the sheaf of graded rings $\mathrm{Sym}\,\Theta_X$. If one identifies Θ_X with the sheaf of functions on the cotangent space T^*X which are linear in the fibers, then $\mathrm{Sym}\,\Theta_X$ is the sheaf of functions on T^*X which are polynomial in the fibers. In particular, $\mathrm{gr}^F\mathcal{D}_X$ is a sheaf of commutative rings.

(1) Identify the $\mathcal{O}_X$-module $\mathrm{Sym}^k\,\Theta_X$ with the sheaf of symmetric $\mathbb{C}$-linear forms $\boldsymbol{\xi} : \mathcal{O}_X \otimes_{\mathbb{C}} \cdots \otimes_{\mathbb{C}} \mathcal{O}_X \to \mathcal{O}_X$ on the k-fold tensor product, which behave like a derivation with respect to each factor.

(2) Show that $\mathrm{Sym}\,\Theta_X := \bigoplus_k \mathrm{Sym}^k\,\Theta_X$ is a sheaf of graded $\mathcal{O}_X$-algebras on X and identify it with the sheaf of functions on T^*X which are polynomial in the fibers.

(3) Show that the map $F_k\mathcal{D}_X \to \mathcal{H}om_{\mathbb{C}}\big(\otimes^k_{\mathbb{C}}\mathcal{O}_X, \mathcal{O}_X\big)$ which sends any section P of $F_k\mathcal{D}_X$ to

$$f_1 \otimes \cdots \otimes f_k \longmapsto [\cdots[[P, f_1]f_2]\cdots f_k]$$

induces an isomorphism of $\mathcal{O}_X$-modules $\mathrm{gr}^F_k \mathcal{D}_X \to \mathrm{Sym}^k \Theta_X$.

(4) Show that the induced morphism

$$\mathrm{gr}^F \mathcal{D}_X := \bigoplus_k \mathrm{gr}^F_k \mathcal{D}_X \longrightarrow \mathrm{Sym}\,\Theta_X$$

is an isomorphism of sheaves of graded $\mathcal{O}_X$-algebras.

***Exercise 8.5* (The universal connection).**

(1) Show that the natural left multiplication of Θ_X on $\mathcal{D}_X$ can be written as a *connection*

$$\nabla : \mathcal{D}_X \longrightarrow \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X,$$

i.e., as a $\mathbb{C}$-linear morphism satisfying the *Leibniz rule* $\nabla(fP) = \mathrm{d}f \otimes P + f\nabla P$, where g is any local section of $\mathcal{O}_X$ and P any local section of $\mathcal{D}_X$. [*Hint*: $\nabla(1)$ is the global section of $\Omega^1_X \otimes_{\mathcal{O}_X} \Theta_X$ considered in Exercise 8.1.]

(2) Extend this connection for every $k \geqslant 1$ as a $\mathbb{C}$-linear morphism

$${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$$

satisfying the Leibniz rule written as

$${}^{(k)}\nabla(\omega \otimes P) = d\omega \otimes P + (-1)^k \omega \wedge \nabla P.$$

(3) Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$ for every $k \geqslant 0$ (i.e., ∇ is *integrable* or *flat*).

(4) Show that the morphisms ${}^{(k)}\nabla$ are *right* $\mathcal{D}_X$-linear (but not left $\mathcal{O}_X$-linear).

Exercise 8.6. More generally, show that a left $\mathcal{D}_X$-module $\mathcal{M}$ is nothing but an $\mathcal{O}_X$-module with an *integrable* connection $\nabla : \mathcal{M} \to \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{M}$. [*Hint*: To get the connection, tensor the left $\mathcal{D}_X$-action $\mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \mathcal{M}$ by Ω^1_X on the left and compose with the universal connection to get $\mathcal{D}_X \otimes \mathcal{M} \to \Omega^1_X \otimes \mathcal{M}$; compose it on the left with $\mathcal{M} \to \mathcal{D}_X \otimes \mathcal{M}$ given by $m \mapsto 1 \otimes m$.] Define similarly the iterated connections ${}^{(k)}\nabla : \Omega^k_X \otimes_{\mathcal{O}_X} \mathcal{M} \to \Omega^{k+1}_X \otimes_{\mathcal{O}_X} \mathcal{M}$. Show that ${}^{(k+1)}\nabla \circ {}^{(k)}\nabla = 0$.

Exercise 8.7.

(1) Show that $\widetilde{\mathcal{D}}_X$ has a universal connection $\widetilde{\nabla}$ for which $\widetilde{\nabla}(1) = \sum_i \widetilde{\mathrm{d}}x_i \otimes \widetilde{\partial}_{x_i}$.

(2) Show the equivalence between graded left $\widetilde{\mathcal{D}}_X$-modules and graded $\widetilde{\mathcal{O}}_X$-modules equipped with an integrable connection.

(3) Extend the properties shown in Exercises 8.5 and 8.6 to the present case.

8.10.b. Exercises for Section 8.2

***Exercise 8.8* (Generating left $\widetilde{\mathcal{D}}_X$-modules).** Let $\widetilde{\mathcal{M}}$ be an $\widetilde{\mathcal{O}}_X$-module and let $\varphi^{\mathrm{left}} : \widetilde{\Theta}_X \otimes_{\widetilde{\mathbb{C}}_X} \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ be a $\widetilde{\mathbb{C}}$-linear morphism such that, for any local sections g of $\widetilde{\mathcal{O}}_X$, ξ, ξ' of $\widetilde{\Theta}_X$ and m of $\widetilde{\mathcal{M}}$, one has

(1) $\varphi^{\mathrm{left}}(g\xi \otimes m) = g\varphi^{\mathrm{left}}(\xi \otimes m)$,

(2) $\varphi^{\mathrm{left}}(\xi \otimes gm) = g\varphi^{\mathrm{left}}(\xi \otimes m) + \xi(g)m$,

(3) $\varphi^{\mathrm{left}}([\xi, \xi'] \otimes m) = \varphi^{\mathrm{left}}(\xi \otimes \varphi^{\mathrm{left}}(\xi' \otimes m)) - \varphi^{\mathrm{left}}(\xi' \otimes \varphi^{\mathrm{left}}(\xi \otimes m))$.

Show that there exists a unique structure $\widetilde{\mathcal{M}}^{\mathrm{left}}$ of left $\widetilde{\mathcal{D}}_X$-module on $\widetilde{\mathcal{M}}$ such that $\xi m = \varphi^{\mathrm{left}}(\xi \otimes m)$ for every ξ, m.

Exercise 8.9 **(Generating right $\widetilde{\mathcal{D}}_X$-modules).** Let $\widetilde{\mathcal{M}}$ be an $\widetilde{\mathcal{O}}_X$-module and let $\varphi^{\mathrm{right}}: \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathbb{C}}_X} \widetilde{\Theta}_X \to \widetilde{\mathcal{M}}$ be a $\widetilde{\mathbb{C}}$-linear morphism such that, for any local sections g of $\widetilde{\mathcal{O}}_X$, ξ, ξ' of $\widetilde{\Theta}_X$ and m of $\widetilde{\mathcal{M}}$, one has

(1) $\varphi^{\mathrm{right}}(mg \otimes \xi) = \varphi^{\mathrm{right}}(m \otimes g\xi)$ (φ^{right} is in fact defined on $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_X$),

(2) $\varphi^{\mathrm{right}}(m \otimes g\xi) = \varphi^{\mathrm{right}}(m \otimes \xi)g - m\xi(g)$,

(3) $\varphi^{\mathrm{right}}(m \otimes [\xi, \xi']) = \varphi^{\mathrm{right}}(\varphi^{\mathrm{right}}(m \otimes \xi) \otimes \xi') - \varphi^{\mathrm{right}}(\varphi^{\mathrm{right}}(m \otimes \xi') \otimes \xi)$.

Show that there exists a unique structure $\widetilde{\mathcal{M}}^{\mathrm{right}}$ of right $\widetilde{\mathcal{D}}_X$-module on $\widetilde{\mathcal{M}}$ such that $m\xi = \varphi^{\mathrm{right}}(m \otimes \xi)$ for every ξ, m.

Exercise 8.10 **($\mathcal{O}_X$ is a *simple* left $\mathcal{D}_X$-module).** We consider here the setting of Section 8.1.

(1) Use the left action of Θ_X on $\mathcal{O}_X$ to define on $\mathcal{O}_X$ the structure of a left $\mathcal{D}_X$-module.

(2) Let g be a nonzero holomorphic function on $\mathbb{C}^n$. Show that there exists a multi-index $\alpha \in \mathbb{N}^n$ such that $(\partial^\alpha gm)(0) \neq 0$.

(3) Conclude that $\mathcal{O}_X$ is a simple left $\mathcal{D}_X$-module, i.e., does not contain any proper non trivial $\mathcal{D}_X$-submodule. Is it simple as a left $\mathcal{O}_X$-module?

(4) Show that $R_F\mathcal{O}_X$ is not a simple graded $R_F\mathcal{D}_X$-module. [*Hint*: Consider $zR_F\mathcal{O}_X \subset R_F\mathcal{O}_X$.]

Exercise 8.11 **(ω_X is a *simple* right $\mathcal{D}_X$-module).** Same setting as in Exercise 8.10.

(1) Use the right action of Θ_X on ω_X to define on ω_X the structure of a right $\mathcal{D}_X$-module.

(2) Show that it is simple as a right $\mathcal{D}_X$-module.

(3) Show that $R_F\omega_X$ is not a simple graded right $R_F\mathcal{D}_X$-module.

Exercise 8.12 **(Tensor products over $\widetilde{\mathcal{O}}_X$).**

(1) Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{left}}$ be two left $\widetilde{\mathcal{D}}_X$-modules.

(a) Show that the $\widetilde{\mathcal{O}}_X$-module $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}$ has the structure of a *left* $\widetilde{\mathcal{D}}_X$-module when setting, by analogy with the Leibniz rule,

$$\xi \cdot (m \otimes n) = \xi m \otimes n + m \otimes \xi n.$$

(b) If $\widetilde{\mathcal{M}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{left}}$ are regarded as $\widetilde{\mathcal{O}}_X$-modules with connection (Proposition 8.1.4 and Exercise 8.7), show that the connection on $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}$ coming from the left $\widetilde{\mathcal{D}}_X$-module structure above is equal to $\widetilde{\nabla} \otimes \mathrm{Id}_{\widetilde{\mathcal{N}}} + \mathrm{Id}_{\widetilde{\mathcal{M}}} \otimes \widetilde{\nabla}$.

(c) Notice that, in general, $m \otimes n \mapsto (\xi m) \otimes n$ (or $m \otimes n \mapsto m \otimes (\xi n)$) does not define a left $\widetilde{\mathcal{D}}_X$-action on the $\widetilde{\mathcal{O}}_X$-module $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}$.

(d) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ and $\psi : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}'$ be $\widetilde{\mathcal{D}}_X$-linear morphisms. Show that $\varphi \otimes \psi$ is $\widetilde{\mathcal{D}}_X$-linear.

(e) Show the associativity

$$(\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\mathrm{left}} = \widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{N}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\mathrm{left}}).$$

(2) Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ be a left $\widetilde{\mathcal{D}}_X$-module and $\widetilde{\mathcal{N}}^{\mathrm{right}}$ be a *right* $\widetilde{\mathcal{D}}_X$-module.

(a) Show that $\widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ has the structure of a right $\widetilde{\mathcal{D}}_X$-module by setting

$$(n \otimes m) \cdot \xi = n\xi \otimes m - n \otimes \xi m,$$

and prove the analogue of (1d).

Remark: one can define a right $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{right}}$ by using the natural involution $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{right}} \xrightarrow{\sim} \widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$, so this brings no new structure.

(b) Show the associativity

$$(\widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\mathrm{left}} = \widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{P}^{\mathrm{left}}).$$

(3) Assume that $\widetilde{\mathcal{M}}^{\mathrm{right}}$ and $\widetilde{\mathcal{N}}^{\mathrm{right}}$ are right $\widetilde{\mathcal{D}}_X$-modules. Does there exist a (left or right) $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{right}}$ defined with analogous formulas?

Exercise 8.13 **($\mathcal{H}om$ over $\widetilde{\mathcal{O}}_X$).**

(1) Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ be left $\widetilde{\mathcal{D}}_X$-modules. Show that $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ has a natural structure of left $\widetilde{\mathcal{D}}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \xi \cdot (\varphi(m)) + \varphi(\xi \cdot m),$$

for any local sections ξ of $\widetilde{\Theta}_X$, m of $\widetilde{\mathcal{M}}$ and φ of $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$.

(2) Similarly, if $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ are right $\widetilde{\mathcal{D}}_X$-modules, then $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ has a natural structure of left $\widetilde{\mathcal{D}}_X$-module defined by

$$(\xi \cdot \varphi)(m) = \varphi(m \cdot \xi) - \varphi(m) \cdot \xi.$$

Exercise 8.14 **(Compatibility of side-changing functors).** Show that the natural morphisms

$$\widetilde{\mathcal{M}}^{\mathrm{left}} \longrightarrow \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}), \quad \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\omega}_X, \widetilde{\mathcal{M}}^{\mathrm{right}}) \longrightarrow \widetilde{\mathcal{M}}^{\mathrm{right}}$$

are isomorphisms of graded $\widetilde{\mathcal{D}}_X$-modules.

Exercise 8.15 **(Side-changing on morphisms).** To any left $\widetilde{\mathcal{D}}_X$-linear morphism $\varphi^{\mathrm{left}} : \widetilde{\mathcal{M}}_1^{\mathrm{left}} \to \widetilde{\mathcal{M}}_2^{\mathrm{left}}$ is associated the $\widetilde{\mathcal{O}}_X$-linear morphism $\varphi^{\mathrm{right}} = \mathrm{Id}_{\widetilde{\omega}_X} \otimes \varphi^{\mathrm{left}} : \widetilde{\mathcal{M}}_1^{\mathrm{right}} \to \widetilde{\mathcal{M}}_2^{\mathrm{right}}$.

(1) Show that φ^{right} is right $\widetilde{\mathcal{D}}_X$-linear.

(2) Define the reverse correspondence $\varphi^{\mathrm{right}} \mapsto \varphi^{\mathrm{left}}$.

(3) Conclude that the left-right correspondence $\mathsf{Mod}^{\mathrm{left}}(\widetilde{\mathcal{D}}_X) \mapsto \mathsf{Mod}^{\mathrm{right}}(\widetilde{\mathcal{D}}_X)$ is a functor, as well as the right-left correspondence $\mathsf{Mod}^{\mathrm{right}}(\widetilde{\mathcal{D}}_X) \mapsto \mathsf{Mod}^{\mathrm{left}}(\widetilde{\mathcal{D}}_X)$.

Exercise 8.16 **(Compatibility of side-changing functors with tensor product)**

Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{left}}$ be two left $\widetilde{\mathcal{D}}_X$-modules and denote by $\widetilde{\mathcal{M}}^{\mathrm{right}}, \widetilde{\mathcal{N}}^{\mathrm{right}}$ the corresponding right $\widetilde{\mathcal{D}}_X$-modules (see Definition 8.2.2). Show that there is a natural isomorphism of graded right $\widetilde{\mathcal{D}}_X$-modules (by using the right structure given in

Exercise 8.12(2)):

$$\begin{aligned}\widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} &\xrightarrow{\sim} \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}} \\ (\omega\otimes n)\otimes m &\longmapsto (\omega\otimes m)\otimes n\end{aligned}$$

and that this isomorphism is functorial in $\widetilde{\mathcal{M}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{left}}$.

Exercise 8.17 **(Local expression of the side-changing functors)**

Let U be an open set of $\mathbb{C}^n$.

(1) Show that there exists a unique $\widetilde{\mathbb{C}}$-linear involution $P \mapsto {}^tP$ from $\widetilde{\mathcal{D}}(U)$ to itself such that

- $\forall\, g \in \widetilde{\mathcal{O}}(U)$, ${}^tg = g$,
- $\forall\, i \in \{1,\dots,n\}$, ${}^t\widetilde{\partial}_{x_i} = -\widetilde{\partial}_{x_i}$,
- $\forall\, P,Q \in \widetilde{\mathcal{D}}(U)$, ${}^t(PQ) = {}^tQ\cdot{}^tP$.

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let ${}^t\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{M}}$ equipped with the right $\widetilde{\mathcal{D}}_X$-module structure

$$m\cdot P := {}^tPm.$$

Show that $z^{-n}\,{}^t\widetilde{\mathcal{M}} \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\mathrm{right}}$, that is, ${}^t\widetilde{\mathcal{M}}(n) \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\mathrm{right}}$. [*Hint*: Use that $F_p{}^t\mathcal{O}_X = F_{p-n}\omega_X$, hence $R_F{}^t\mathcal{O}_X = R_{F[n]}\omega_X$, so ${}^t\widetilde{\mathcal{O}}_X = \widetilde{\omega}_X(-n)$, according to Remark 5.1.5(2).] Argue similarly starting with a right $\widetilde{\mathcal{D}}_X$-module.

Exercise 8.18 **(Tensor product of a left $\widetilde{\mathcal{D}}_X$-module with $\widetilde{\mathcal{D}}_X$)**

Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Notice that $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has two commuting structures of $\widetilde{\mathcal{O}}_X$-module. Similarly $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ has two such structures. The goal of this exercise is to extend them as $\widetilde{\mathcal{D}}_X$-structures and examine their relations.

(1) Show that $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has the structure of a left and of a right $\widetilde{\mathcal{D}}_X$-module *which commute*, given by the formulas:

$$\text{(left)}\qquad (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\mathrm{tens}}: \quad \begin{cases} f\cdot(m\otimes P) = (fm)\otimes P = m\otimes(fP),\\ \xi\cdot(m\otimes P) = (\xi m)\otimes P + m\otimes \xi P,\end{cases}$$

$$\text{(right)}\qquad (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\mathrm{triv}}: \quad \begin{cases} (m\otimes P)\cdot f = m\otimes(Pf),\\ (m\otimes P)\cdot\xi = m\otimes(P\xi),\end{cases}$$

for any local vector field ξ and any local holomorphic function g. Show that a left $\widetilde{\mathcal{D}}_X$-linear morphism $\varphi: \widetilde{\mathcal{M}}_1^{\mathrm{left}} \to \widetilde{\mathcal{M}}_2^{\mathrm{left}}$ extends as a bi-$\widetilde{\mathcal{D}}_X$-linear morphism $\varphi\otimes 1: \widetilde{\mathcal{M}}_1^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}_2^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$.

(2) Similarly, show that $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ also has such structures *which commute* and are functorial, given by formulas:

$$\text{(left)}\qquad (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}})_{\mathrm{triv}}: \quad \begin{cases} f\cdot(P\otimes m) = (fP)\otimes m,\\ \xi\cdot(P\otimes m) = (\xi P)\otimes m,\end{cases}$$

$$\text{(right)}\qquad (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}})_{\mathrm{tens}}: \quad \begin{cases} (P\otimes m)\cdot f = P\otimes(fm) = (Pf)\otimes m,\\ (P\otimes m)\cdot\xi = P\xi\otimes m - P\otimes\xi m.\end{cases}$$

(3) Show that both morphisms

$$\begin{array}{cc} \widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} & \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} \longrightarrow \widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \\ m \otimes P \longmapsto (1 \otimes m) \cdot P & P \otimes m \longmapsto P \cdot (m \otimes 1) \end{array}$$

are left and right $\widetilde{\mathcal{D}}_X$-linear, induce the identity $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes 1 = 1 \otimes \widetilde{\mathcal{M}}^{\mathrm{left}}$, and their composition is the identity of $\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ or $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$, hence both are reciprocal isomorphisms. Show that this correspondence is functorial (i.e., compatible with morphisms).

(4) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. Justify the following isomorphisms of left $\widetilde{\mathcal{D}}_X$-modules and $\widetilde{\mathcal{O}}_X$-modules for the action on the right:

$$\begin{aligned} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) &\simeq (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} \\ &\simeq (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} \simeq \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}). \end{aligned}$$

Assume moreover that $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{L}}$ are $\widetilde{\mathcal{O}}_X$-locally free. Show that $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})$ is $\widetilde{\mathcal{D}}_X$-locally free.

***Exercise 8.19* (Tensor product of a right $\widetilde{\mathcal{D}}_X$-module with $\widetilde{\mathcal{D}}_X$)**

Let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ be a right $\widetilde{\mathcal{D}}_X$-module.

(1) Show that $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ has two structures of right $\widetilde{\mathcal{D}}_X$-module denoted triv and tens (tensor; the latter defined by using the left structure on $\widetilde{\mathcal{D}}_X$ and Exercise 8.12(2)), given by:

$$\text{(right)} \qquad (\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\mathrm{triv}} : \quad \begin{cases} (m \otimes P) \cdot_{\mathrm{triv}} f = m \otimes (Pf), \\ (m \otimes P) \cdot_{\mathrm{triv}} \xi = m \otimes (P\xi), \end{cases}$$

$$\text{(right)} \qquad (\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)_{\mathrm{tens}} : \quad \begin{cases} (m \otimes P) \cdot_{\mathrm{tens}} f = mf \otimes P = m \otimes fP, \\ (m \otimes P) \cdot_{\mathrm{tens}} \xi = m\xi \otimes P - m \otimes (\xi P). \end{cases}$$

(2) Show that there is a unique involution $\iota : \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \xrightarrow{\sim} \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ which exchanges both structures and is the identity on $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes 1$, given by $(m \otimes P)_{\mathrm{triv}} \mapsto (m \otimes 1) \cdot_{\mathrm{tens}} P$. [*Hint*: Show first the properties of ι by using local coordinates, and glue the local constructions by uniqueness of ι.]

(3) For every $p \geqslant 0$, consider the p-th term $F_p\widetilde{\mathcal{D}}_X$ of the filtration of $\widetilde{\mathcal{D}}_X$ by the order (see Exercise 8.1.3) with both structures of $\widetilde{\mathcal{O}}_X$-module (one on the left, one on the right) and equip similarly $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} F_p\widetilde{\mathcal{D}}_X$ with two structures of $\widetilde{\mathcal{O}}_X$-modules. Show that, for every p, ι preserves $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} F_p\widetilde{\mathcal{D}}_X$ and exchanges the two structures of $\widetilde{\mathcal{O}}_X$-modules.

(4) Let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ be a right $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module. By considering the natural $\widetilde{\mathcal{O}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}$, we define an induced right $\widetilde{\mathcal{D}}_X$-module $\big[(\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\big]_{\mathrm{triv}}$. Here, the $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\mathrm{right}}$ is not used.

On the other hand, considering the canonical left $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}$ and using Exercise 8.12(2), we obtain a right $\widetilde{\mathcal{D}}_X$-module structure

$\big[\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})\big]_{\text{tens}}$. Here, the $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}^{\text{right}}$ is used in an essential way.

Prove that the canonical $\widetilde{\mathcal{O}}_X$-linear morphism

$$\begin{aligned}\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} &\longrightarrow \widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})\\ m\otimes \ell &\longmapsto m\otimes(1\otimes \ell)\end{aligned}$$

extends in a unique way as a $\widetilde{\mathcal{D}}_X$-linear morphism

$$\Big[(\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\Big]_{\text{triv}} \longrightarrow \Big[\widetilde{\mathcal{M}}^{\text{right}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})\Big]_{\text{tens}}$$

which is an isomorphism. [*Hint*: Argue as in (2).]

8.10.c. Exercises for Section 8.4

Exercise 8.20. Check that $\mathrm{Sp}(\widetilde{\mathcal{M}})$ is indeed a complex, i.e., that $\widetilde{\delta}\circ\widetilde{\delta}=0$.

Exercise 8.21 **($\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module)**

The natural surjective morphism $\widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{O}}_X$ of left $\widetilde{\mathcal{D}}_X$-modules has kernel the image of $\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_X \to \widetilde{\mathcal{D}}_X$. In other words, we have a morphism of complexes of left $\widetilde{\mathcal{D}}_X$-modules

$$\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \longrightarrow \widetilde{\mathcal{O}}_X$$

(where $\widetilde{\mathcal{O}}_X$ is regarded as a complex with a nonzero term in degree zero only), which induces an isomorphism

$$H^0\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} \widetilde{\mathcal{O}}_X.$$

In this exercise, one proves that $H^k(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))=0$ for $k\neq 0$, so that the morphism above is a quasi-isomorphism.

Let $F_\bullet\widetilde{\mathcal{D}}_X$ be the filtration of $\widetilde{\mathcal{D}}_X$ by the order of differential operators. Filter the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ by the subcomplexes $F_p(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))$ defined as

$$\cdots \xrightarrow{\widetilde{\delta}} F_{p-k}\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k} \xrightarrow{\widetilde{\delta}} F_{p-k+1}\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k-1} \xrightarrow{\widetilde{\delta}} \cdots$$

(1) Show that, locally on X, using coordinates $x_1,\dots,x_n$, the graded complex $\mathrm{gr}^F\mathrm{Sp}(\widetilde{\mathcal{D}}_X) := \bigoplus_p \mathrm{gr}^F_p\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is equal to the Koszul complex of the ring $\widetilde{\mathcal{O}}_X[\xi_1,\dots,\xi_n]$ with respect to the regular sequence $\xi_1,\dots,\xi_n$.

(2) Conclude that $\mathrm{gr}^F\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$.

(3) Check that $F_p\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X)=0$ for $p<0$, $F_0\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X)=\mathrm{gr}^F_0\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is isomorphic to $\widetilde{\mathcal{O}}_X$ and deduce that the complex

$$\mathrm{gr}^F_p\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X) := \{\cdots \xrightarrow{\widetilde{\delta}} \mathrm{gr}^F_{p-k}\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k} \xrightarrow{\widetilde{\delta}} \mathrm{gr}^F_{p-k+1}\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k-1} \xrightarrow{\widetilde{\delta}} \cdots\}$$

is acyclic (i.e., quasi-isomorphic to 0) for $p>0$.

(4) Show that the inclusion $F_0\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \hookrightarrow F_p\,\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a quasi-isomorphism for every $p\geqslant 0$ and deduce, by passing to the inductive limit, that the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module by locally free left $\widetilde{\mathcal{D}}_X$-modules.

Exercise 8.22 **($^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\omega}_X$ as a right $\widetilde{\mathcal{D}}_X$-module)**
Show similarly that the natural morphism of right $\widetilde{\mathcal{D}}_X$-modules

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\omega}_X$$

defined as the right action of $\widetilde{\mathcal{D}}_X$ on $\widetilde{\omega}_X$ extends as a morphism of complexes of right $\widetilde{\mathcal{D}}_X$-modules

$$^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \longrightarrow \widetilde{\omega}_X.$$

Show that $H^k(\mathrm{DR}\,\widetilde{\mathcal{D}}_X) = 0$ for $k \neq n$, so that the shifted complex $\mathrm{DR}(\widetilde{\mathcal{D}}_X)[n]$ is a resolution of $\widetilde{\omega}_X$ as a right $\widetilde{\mathcal{D}}_X$-module by locally free right $\widetilde{\mathcal{D}}_X$-modules.

Exercise 8.23 **(Tensor product over $\widetilde{\mathcal{D}}_X$).** Let $\widetilde{\mathcal{M}}^{\mathrm{left}}, \widetilde{\mathcal{N}}^{\mathrm{left}}$ be two left $\widetilde{\mathcal{D}}_X$-modules. One can consider the tensor products $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}$ and $\widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$. Both are bi-functors with values in the category of sheaves of $\widetilde{\mathbb{C}}$-vector spaces (a priori they do not have any other structure). Show that there is a natural $\widetilde{\mathbb{C}}$-linear isomorphism $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}} \xrightarrow{\sim} \widetilde{\mathcal{N}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$ induced by

$$(\widetilde{\omega} \otimes_{\widetilde{\mathcal{O}}} m) \otimes_{\widetilde{\mathcal{D}}} n \longmapsto (\widetilde{\omega} \otimes_{\widetilde{\mathcal{O}}} n) \otimes_{\widetilde{\mathcal{D}}} m.$$

[*Hint*: Show that, for any holomorphic vector field ξ, one has the equality $(\widetilde{\omega} \otimes m) \otimes \xi n = (\widetilde{\omega} \otimes n) \otimes \xi m$.]

Exercise 8.24 **(The Spencer complex: tensoring over $\widetilde{\mathcal{D}}_X$ with $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$)**

(1) Let $\widetilde{\mathcal{M}}^{\mathrm{right}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that the natural morphism

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \longrightarrow \widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}$$

defined by $m \otimes (P \otimes \xi) \mapsto mP \otimes \xi$ induces an isomorphism of complexes

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}).$$

[*Hint*: The point is to check that the differential $\mathrm{Id} \otimes \widetilde{\delta}_{\widetilde{\mathcal{D}}}$ on the left corresponds to the differential $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ on the right.]

(2) Let $\widetilde{\mathcal{M}}^{\mathrm{left}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Similar question for

$$^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} \longrightarrow {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{left}}).$$

Exercise 8.25 **(The de Rham complex: $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X), \widetilde{\mathcal{M}})$)**

For left $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$, the sheaf $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ is a priori only a $\widetilde{\mathbb{C}}$-module. If $\widetilde{\mathcal{N}}$ is also a right $\widetilde{\mathcal{D}}_X$-module, like $\widetilde{\mathcal{D}}_X$, $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}})$ comes equipped with the structure of right $\widetilde{\mathcal{D}}_X$-module inherited from that of $\widetilde{\mathcal{N}}$. In particular, for each k, $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}, \widetilde{\mathcal{D}}_X)$ is a right $\widetilde{\mathcal{D}}_X$-module and $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X), \widetilde{\mathcal{D}}_X)$ is a complex of right $\widetilde{\mathcal{D}}_X$-modules.

(1) Identify the complex of right $\widetilde{\mathcal{D}}_X$-modules $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X), \widetilde{\mathcal{D}}_X)$ (where the right structure comes from the second term $\widetilde{\mathcal{D}}_X$) with the complex ${}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{D}}_X$ up to changing the sign of the differential in the latter complex. [*Hint*:

(a) Identify first the right $\widetilde{\mathcal{D}}_X$-module $\mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}, \widetilde{\mathcal{D}}_X)$ with $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\Theta}_{X,k}, \widetilde{\mathcal{D}}_X)$, then to $\mathcal{H}om_{\widetilde{\mathcal{O}}_X}(\widetilde{\Theta}_{X,k}, \widetilde{\mathcal{O}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$, hence to $\widetilde{\Omega}^k_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$;

(b) In local coordinates, for $I, I' \subset \{1,\dots,n\}$ such that $\#I, \#I' = k$, set $\widetilde{\mathrm{d}}x_I = \widetilde{\mathrm{d}}x_{i_1} \wedge \cdots \wedge \widetilde{\mathrm{d}}x_{i_k}$ with $i_1 < \cdots < i_k$, and similarly for $\widetilde{\partial}_{x_{I'}}$; consider the pairing $\langle \widetilde{\mathrm{d}}x_I, \widetilde{\partial}_{x_{I'}}\rangle = (-1)^{k(k-1)/2}$ if $I = I'$, and $= 0$ otherwise (see Section 8.1.a); recall that $\widetilde{\mathrm{d}}(\widetilde{\mathrm{d}}x_I \otimes 1) = \sum_{j\notin I} \widetilde{\mathrm{d}}x_I \wedge \widetilde{\mathrm{d}}x_j \otimes \widetilde{\partial}_{x_j}$ in $\widetilde{\Omega}_X^{k+1} \otimes \widetilde{\mathcal{D}}_X$, and if $J = \{i_1,\dots,i_{k+1}\}$, $\widetilde{\delta}(1 \otimes \widetilde{\partial}_{x_J}) = \sum_{j=1}^{k+1} (-1)^j \widetilde{\partial}_{x_{i_j}} \otimes \widetilde{\partial}_{J\smallsetminus i_j}$ in $\widetilde{\mathcal{D}}_X \otimes \widetilde{\Theta}_{X,k}$; then show that for any such I, J, one has $\langle \widetilde{\mathrm{d}}x_I \otimes 1, \widetilde{\delta}(1\otimes\widetilde{\partial}_{x_J})\rangle = -\langle \widetilde{\mathrm{d}}(\widetilde{\mathrm{d}}x_I\otimes 1), 1\otimes\widetilde{\partial}_{x_J}\rangle$ and conclude.]

(2) Conclude that, for a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, one has

$$\begin{aligned}{}^{\mathrm{p}}\mathrm{DR}\,\widetilde{\mathcal{M}} \simeq {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}} &\simeq \mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X), \widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}\\ &\simeq \mathcal{H}om_{\widetilde{\mathcal{D}}_X}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X), \widetilde{\mathcal{M}}).\end{aligned}$$

***Exercise 8.26* (Side-changing for the de Rham functors).**

(1) If $\widetilde{\mathcal{M}}$ is any left $\widetilde{\mathcal{D}}_X$-module and $\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$ is the associated right $\widetilde{\mathcal{D}}_X$-module, show that ι defined in Lemma 8.4.7 induces an isomorphism

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \xrightarrow{\ \sim\ } {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}$$

which is termwise $\widetilde{\mathcal{O}}_X$-linear. [*Hint*: Use Exercise 8.23 to identify $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ with $(\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}$.]

(2) Interpret the isomorphism ι of Lemma 8.4.7 as the composition of the inverse of the isomorphism

$$\big[(\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X\big]_{\mathrm{triv}} \xrightarrow{\ \sim\ } \big[\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k})\big]_{\mathrm{tens}}$$

of Exercise 8.19(4), with $(-1)^{kn}\,\varepsilon(k+1) \lrcorner$.

(3) Argue as in Lemma 8.4.7 (with the interpretation above) to show that the $\widetilde{\mathcal{O}}_X$-linear isomorphism

$$\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k} \xrightarrow{\ \sim\ } \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}} \xrightarrow{\ \sim\ } \widetilde{\Omega}_X^{n-k} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$$

given on $\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}$ by

$$\omega \otimes m \otimes \xi \longmapsto (-1)^{kn}\,\varepsilon(k+1)\omega(\xi \wedge \bullet) \otimes m$$

induces a functorial isomorphism ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\ \sim\ } {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}})$ for any left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, which is termwise $\widetilde{\mathcal{O}}_X$-linear.

***Exercise 8.27* (Interior product with a 1-form).** Let $x_1,\dots,x_n$ be local coordinates. Fix $k \geqslant 1$ and set $\widetilde{\partial}_x = \widetilde{\partial}_{x_1} \wedge\cdots\wedge \widetilde{\partial}_{x_k}$ and, for $i \in \{1,\dots,k\}$, $\widetilde{\partial}_{x_{\widehat{\imath}}} = \widetilde{\partial}_{x_1} \wedge\cdots\wedge \widehat{\widetilde{\partial}_{x_i}} \wedge\cdots\wedge \widetilde{\partial}_{x_k}$. Show the following equalities for $i \neq j \in \{1,\dots,k\}$:

$$\widetilde{\partial}_x \lrcorner \widetilde{\mathrm{d}}x_i = (-1)^{k-i}\widetilde{\partial}_{x_{\widehat{\imath}}}, \qquad \widetilde{\partial}_{x_{\widehat{\jmath}}} \lrcorner \widetilde{\mathrm{d}}x_i = \begin{cases} (-1)^{k-i+1}\widetilde{\partial}_{x_{\widehat{\imath\jmath}}} & \text{if } i<j,\\ (-1)^{k-i}\widetilde{\partial}_{x_{\widehat{\imath\jmath}}} & \text{if } i>j.\end{cases}$$

[*Hint*: Use (8.4.6∗∗) and (8.4.6∗).]

Exercise 8.28 **(The C^∞ Spencer complex).** Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module and let us denote by $\widetilde{\delta}'_{\widetilde{\mathcal{M}}}$ the differential of the Spencer complex ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}})$.

(1) Show that, for each j, the formula (for $i, j \geqslant 0$)

$$\begin{aligned}\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,i} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}^{(0,j)} &\xrightarrow{\widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}}} \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,i-1} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}^{(0,j)}\\ m \otimes \boldsymbol{\xi}_i \otimes \varphi &\longmapsto \widetilde{\delta}'_{\widetilde{\mathcal{M}}}(m \otimes \boldsymbol{\xi}_i) \otimes \varphi + m \otimes \boldsymbol{\xi}_i \lrcorner \widetilde{\mathrm{d}}'\varphi\end{aligned}$$

defines the differential of a complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{E}}^{(0,j)}$. Show that

$$\widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}}\mathrm{d}'' + \mathrm{d}''\widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}} = 0,$$

and deduce a complex ${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}) := (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}_X^{\infty,\bullet}, \widetilde{\delta}'^{\infty}_{\widetilde{\mathcal{M}}} + \mathrm{d}'')$ (notation of Section 8.4.13).

(2) Show that the natural morphism

$${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}})$$

is a quasi-isomorphism.

(3) Argue as in Exercise 8.24(1) to define an isomorphism of complexes

$$\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}^\infty(\widetilde{\mathcal{D}}_X) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}).$$

(4) Argue as in Exercise 8.26 to define the side-changing isomorphism

$${}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{right}}) \xrightarrow{\sim} {}^{\mathrm{p}}\mathrm{DR}^\infty(\widetilde{\mathcal{M}}^{\mathrm{left}}).$$

8.10.d. Exercises for Section 8.5

Exercise 8.29. Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module.

(1) Show that, for every k, we have a (termwise) exact sequence of complexes

$$0 \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} F_{k-1}(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} F_k(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}_k^F(\mathrm{Sp}(\widetilde{\mathcal{D}}_X)) \to 0.$$

[*Hint*: Use that the terms of the complexes $F_j(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))$ and $\mathrm{gr}_k^F(\mathrm{Sp}(\widetilde{\mathcal{D}}_X))$ are $\widetilde{\mathcal{O}}_X$-locally free.]

(2) Show that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{L}}$ as an $\widetilde{\mathcal{O}}_X$-module.

(3) Show that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{L}}$ as an $\widetilde{\mathcal{O}}_X$-module.

(4) Identify the Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)$ with $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ as complexes of "left" $\widetilde{\mathcal{O}}_X$-modules.

(5) Conclude that ${}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X) \simeq \widetilde{\mathcal{L}}$.

Exercise 8.30 **(Canonical resolution of $\widetilde{\mathcal{M}}$: tensoring over $\widetilde{\mathcal{O}}_X$ with $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$)**

(1) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Regarding $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ as a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module, the complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is regarded as a complex of right $\widetilde{\mathcal{D}}_X$-module, by using the tensor right $\widetilde{\mathcal{D}}_X$-module structure on each term.

(a) Show that $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{M}}$. [*Hint*: use the functoriality of the tensor right $\widetilde{\mathcal{D}}_X$-module structure and the local $\widetilde{\mathcal{O}}_X$-freeness of each term of $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)$.]

(b) Show that the differential of this complex is expressed as follows, for local sections m of $\widetilde{\mathcal{M}}$, ξ_i of $\widetilde{\Theta}_X$ and P of $\widetilde{\mathcal{D}}_X$, and setting

$$\widehat{\boldsymbol{\xi}}_i = \xi_1 \wedge \cdots \wedge \xi_{i-1} \wedge \xi_{i+1} \wedge \cdots \wedge \xi_k,$$

and a similar meaning for $\widehat{\boldsymbol{\xi}}_{i,j}$:

$$\begin{aligned}(\mathrm{Id} \otimes \widetilde{\delta})\big[(m \otimes (1 \otimes \boldsymbol{\xi})) \cdot_{\mathrm{tens}} P\big] &= \big[(\mathrm{Id} \otimes \widetilde{\delta})(m \otimes (1 \otimes \boldsymbol{\xi}))\big] \cdot_{\mathrm{tens}} P \\ &= \Big[m \otimes \Big[\sum_{i=1}^{k} (-1)^{i-1} \xi_i \otimes \widehat{\boldsymbol{\xi}}_i + \sum_{i<j} (-1)^{i+j} 1 \otimes ([\xi_i, \xi_j] \wedge \widehat{\boldsymbol{\xi}}_{i,j})\Big]\Big] \cdot_{\mathrm{tens}} P.\end{aligned}$$

(c) Consider the involution

$$\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \simeq (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$$

exchanging the tens structure on the left-hand side with the triv structure on the right hand side. Show that the differential becomes $\widetilde{\delta}_{\mathrm{triv}}$, with

$$\widetilde{\delta}_{\mathrm{triv}}\big[(m \otimes \boldsymbol{\xi}) \otimes P\big] = \widetilde{\delta}_{\mathrm{triv}}\big[(m \otimes \boldsymbol{\xi}) \otimes 1\big] \cdot_{\mathrm{triv}} P$$

and

$$\begin{aligned}\widetilde{\delta}_{\mathrm{triv}}\big[(m \otimes \boldsymbol{\xi}) \otimes 1\big] &= \sum_{i=1}^{k} (-1)^{i-1} (m\xi_i \otimes \widehat{\boldsymbol{\xi}}_i) \otimes 1 \\ &\quad - \sum_{i=1}^{k} (-1)^{i-1} (m \otimes \widehat{\boldsymbol{\xi}}_i) \otimes \xi_i + \sum_{i<j} (-1)^{i+j} (m \otimes ([\xi_i, \xi_j] \wedge \widehat{\boldsymbol{\xi}}_{i,j})) \otimes 1 \\ &= \big[\widetilde{\delta}_{\widetilde{\mathcal{M}}}(m \otimes \boldsymbol{\xi})\big] \otimes 1 - \sum_{i=1}^{k} (-1)^{i-1} (m \otimes \widehat{\boldsymbol{\xi}}_i) \otimes \xi_i,\end{aligned}$$

where $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ is the differential occurring in the complex $\mathrm{Sp}\,\widetilde{\mathcal{M}}$. [*Hint*: write

$$m \otimes (\xi_i \otimes \widehat{\boldsymbol{\xi}}_i) = m\xi_i \otimes (1 \otimes \widehat{\boldsymbol{\xi}}_i) - [m \otimes (1 \otimes \widehat{\boldsymbol{\xi}}_i)] \cdot \xi_i.]$$

(d) Conclude that the complex of induced $\widetilde{\mathcal{D}}_X$-modules

$$\big((\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}}\big)$$

is a resolution of $\widetilde{\mathcal{M}}$.

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Show that the complex

$$ {}^{\mathrm{p}}\mathrm{DR}(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}})$$

is a resolution of $\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$ by right $\widetilde{\mathcal{D}}_X$-modules, where the left and right structures of $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$ are those of Exercise 8.18(2), and the left one is used to compute the de Rham complex.

8.10.e. Exercises for Section 8.6

Exercise 8.31 **(Definition of the pullback of a left $\widetilde{\mathcal{D}}_X$-module)**

(1) Show that the connection $\widetilde{\nabla}^X$ on $f^*\widetilde{\mathcal{N}} := \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}}$ is integrable.

(2) Show that, if $\widetilde{\mathcal{N}}$ also has a right $\widetilde{\mathcal{D}}_Y$-module structure commuting with the left one, then $\widetilde{\nabla}^X$ is right $f^{-1}\widetilde{\mathcal{D}}_Y$-linear, and ${}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$ is a right $f^{-1}\widetilde{\mathcal{D}}_Y$-module.

Exercise 8.32.

(1) Express the previous connection in local coordinates on X and Y.

(2) Show that, if $\widetilde{\mathcal{M}}$ is any left $\widetilde{\mathcal{D}}_X$-module and $\widetilde{\mathcal{N}}$ any left $f^{-1}\widetilde{\mathcal{D}}_Y$-module, then $\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}}$ may be equipped with a left $\widetilde{\mathcal{D}}_X$-module structure: if ξ is a local z-vector field on X, i.e., a local section of $\widetilde{\Theta}_X$, set

$$\xi \cdot (m \otimes n) = (\xi m) \otimes n + Tf(\xi)(m \otimes n).$$

[*Hint*: Identify $\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{N}}$ with $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}$ and use Exercise 8.31.]

Exercise 8.33 **(Local computation of $\widetilde{\mathcal{D}}_{X\to Y}$).**

(1) Show that ${}_{\mathrm{D}}f^*\widetilde{\mathcal{D}}_Y$ is a locally free $\widetilde{\mathcal{O}}_X$-module. [*Hint*: Use that $\widetilde{\mathcal{D}}_Y$ is a locally free $\widetilde{\mathcal{O}}_Y$-module.]

(2) Choose local coordinates $x_1, \dots, x_n$ on X and $y_1, \dots, y_m$ on Y. Show that $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{O}}_X[\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_m}]$ and, with this identification, the left $\widetilde{\mathcal{D}}_X$-structure is given by

$$\widetilde{\partial}_{x_i} \cdot \sum_\alpha a_\alpha(x)\widetilde{\partial}_y^\alpha = \sum_\alpha \Big(z\frac{\partial a_\alpha}{\partial x_i} + \sum_{j=1}^m a_\alpha(x)\frac{\partial f_j}{\partial x_i}\widetilde{\partial}_{y_j}\Big)\widetilde{\partial}_y^\alpha.$$

Exercise 8.34 **($\widetilde{\mathcal{D}}_{X\to Y}$ for a closed embedding).** Assume that $\iota : X \hookrightarrow Y$ is the closed immersion of a complex submanifold of Y of codimension d.

(1) Show that the canonical section $\mathbf{1}$ of $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{O}}_X \otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y} \iota^{-1}\widetilde{\mathcal{D}}_Y$ is a generator of $\widetilde{\mathcal{D}}_{X\to Y}$ as a right $\iota^{-1}\widetilde{\mathcal{D}}_Y$-module.

(2) Assume that X is defined by $g_1 = \cdots = g_d = 0$, where the g_i are holomorphic functions on Y. Show that

$$\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{D}}_Y \big/ \textstyle\sum_{i=1}^d g_i\widetilde{\mathcal{D}}_Y$$

with its natural right $\widetilde{\mathcal{D}}_Y$ structure. In local coordinates $(x_1, \dots, x_n, y_1, \dots, y_d)$ such that $g_i = y_i$, show that $\widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_d}]$.

Conclude that, if f is an embedding, the sheaves $\widetilde{\mathcal{D}}_{X\to Y}$ and $\widetilde{\mathcal{D}}_{Y\leftarrow X}$ are locally free over $\widetilde{\mathcal{D}}_X$.

Exercise 8.35 **($\widetilde{\mathcal{D}}_{X\to Y}$ for a smooth morphism).** Let $0 \to \widetilde{\mathcal{N}}' \to \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}'' \to 0$ be an exact sequence of left $\widetilde{\mathcal{D}}_Y$-modules.

(1) Show that the sequence ${}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}' \to {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} \to {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}'' \to 0$ is exact.

(2) Assume that $f : X \to Y$ is smooth, i.e., locally isomorphic to the projection of a product. Show that the sequence $0 \to {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}' \to {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}} \to {}_{\mathrm{D}}f^*\widetilde{\mathcal{N}}'' \to 0$ is exact. [*Hint*: Use that $\widetilde{\mathcal{O}}_X$ is $f^{-1}\widetilde{\mathcal{O}}_Y$-flat.]

(3) Conclude that $\widetilde{\mathcal{D}}_{X\to Y}$ is $f^{-1}\widetilde{\mathcal{D}}_Y$-flat.

Exercise 8.36 **(The chain rule).** Consider holomorphic maps $f : X \to Y$ and $f' : Y \to Z$.

(1) Construct a canonical isomorphism $\widetilde{\mathcal{D}}_{X\to Y} \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} f^{-1}\widetilde{\mathcal{D}}_{Y\to Z} \xrightarrow{\sim} \widetilde{\mathcal{D}}_{X\to Z}$ as right $(f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z$-modules. [*Hint*: Show that the contraction morphisms

$$\begin{aligned}(\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \otimes_{f^{-1}\widetilde{\mathcal{D}}_Y} (f^{-1}\widetilde{\mathcal{O}}_Y \otimes_{(f'\circ f)^{-1}\widetilde{\mathcal{O}}_Z} (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)&\\ \longrightarrow \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} (f^{-1}\widetilde{\mathcal{O}}_Y \otimes_{(f'\circ f)^{-1}\widetilde{\mathcal{O}}_Z} (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z)&\\ \longrightarrow \widetilde{\mathcal{O}}_X \otimes_{(f'\circ f)^{-1}\widetilde{\mathcal{O}}_Z} (f'\circ f)^{-1}\widetilde{\mathcal{D}}_Z&\end{aligned}$$

yield such an isomorphism, whose inverse is the morphism $\varphi\otimes Q \mapsto (\varphi\otimes 1)\otimes(1\otimes Q)$.]

(2) Use the chain rule to show that this isomorphism is left $\widetilde{\mathcal{D}}_X$-linear.

Exercise 8.37 **(Restriction to $z = 1$).** Show that

$$({}_{\mathrm{D}}f^*\widetilde{\mathcal{N}})/(z-1)_{\mathrm{D}}f^*\widetilde{\mathcal{N}} = {}_{\mathrm{D}}f^*(\widetilde{\mathcal{N}}/(z-1)\widetilde{\mathcal{N}}).$$

Exercise 8.38.

(1) Show that Definition 8.6.6 coincides with that of Exercise 8.31(1).

(2) Let $f : X \to Y$, $f' : Y \to Z$ be holomorphic maps and let $\widetilde{\mathcal{N}}$ be a left $\widetilde{\mathcal{D}}_Z$-module. Show that ${}_{\mathrm{D}}(f'\circ f)^*\widetilde{\mathcal{N}} \simeq {}_{\mathrm{D}}f^*({}_{\mathrm{D}}f'^*\widetilde{\mathcal{N}})$.

8.10.f. Exercises for Section 8.7

Exercise 8.39 **(The relative Spencer complex $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})$).** Let $f : X \to Y$ be a holomorphic map and let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. The goal of the exercise is to identify the complex $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ entering in the definition of the pushforward with the complex

$$\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) := \big((\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y, \widetilde{\delta}_{\widetilde{\mathcal{M}},Y}\big),$$

where $\widetilde{\delta}_{\widetilde{\mathcal{M}},Y}$ is given by the formula

$$\widetilde{\delta}_{\widetilde{\mathcal{M}},Y}((m\otimes\boldsymbol{\xi})\otimes Q) = \widetilde{\delta}_{\widetilde{\mathcal{M}}}(m\otimes\boldsymbol{\xi})\otimes Q + \sum_{i=1}^k (-1)^i (m\otimes\boldsymbol{\xi}_{\widehat{\imath}})\otimes Tf(\xi_i)Q.$$

Here, $\widetilde{\delta}_{\widetilde{\mathcal{M}}}$ is given by the formula of Definition 8.4.3 and we use the notation of Exercise 8.30. The first part concerns the complex $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_Y)$.

(1) Let $\widetilde{\mathcal{L}}$ be a locally free $\widetilde{\mathcal{O}}_X$-module. Consider on $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ the following $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$ bi-module structures:

(a) $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$ also called the tens structure, where the right $f^{-1}\widetilde{\mathcal{D}}_Y$ is the trivial one and the left $\widetilde{\mathcal{D}}_X$-module structure is the left tensor one on $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$ (see Exercise 8.12(1)). In particular, $f^{-1}\widetilde{\mathcal{O}}_Y$ acts on the left on $(\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}})$.

(b) $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$ also called the triv structure, where we use the trivial $f^{-1}\widetilde{\mathcal{D}}_Y$-module structure on the right and the trivial $\widetilde{\mathcal{D}}_X$-module structure on the left (on the other hand, the right $\widetilde{\mathcal{O}}_X$-module structure is used on $\widetilde{\mathcal{D}}_X$ for the tensor product).

Show that there exists a unique isomorphism of $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$ bi-modules

$$\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \xrightarrow{\sim} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$$

inducing the identity on $\widetilde{\mathcal{L}} = \widetilde{\mathcal{O}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{O}}_Y$. [*Hint*: Show that the morphism $P \otimes \ell \otimes Q \mapsto P \cdot_{\mathrm{tens}} (1 \otimes \ell \otimes Q)$ is well-defined by using that $\widetilde{\mathcal{D}}_X$ is locally free over $\widetilde{\mathcal{O}}_X$, and is an isomorphism by considering the the top degree part of P.]

(2) Recall that the differential on the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\delta} \otimes \mathrm{Id}$, with $\widetilde{\delta} = \widetilde{\delta}_{\widetilde{\mathcal{D}}}$ (see Exercise 8.12(1d)). Show that $\widetilde{\delta}_{\widetilde{\mathcal{D}},Y}$ is linear with respect to the triv $(\widetilde{\mathcal{D}}_X, f^{-1}\widetilde{\mathcal{D}}_Y)$-bimodule structure, and that the following diagram commutes, by checking first on $\mathbf{1} \otimes \widetilde{\Theta}_{X,\bullet} \otimes f^{-1}\widetilde{\mathcal{D}}_Y$:

$$\begin{tikzcd}
\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\Theta}_{X,k} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \arrow[r, "\sim"] \arrow[d, "\widetilde{\delta}_{\widetilde{\mathcal{D}},Y}"'] & (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y \arrow[d, "\widetilde{\delta} \otimes \mathrm{Id}"] \\
\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\Theta}_{X,k-1} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \arrow[r, "\sim"] & (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k-1}) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y
\end{tikzcd}$$

and conclude that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) \to \mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is an isomorphism.

(3) Deduce that the terms of the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ are locally free left $\widetilde{\mathcal{D}}_X$-modules. [*Hint*: Check this for the complex $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.]

(4) Conclude that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) \to \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} (\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y})$ is an isomorphism. [*Hint*: Check that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.]

Exercise 8.40 **(The relative Spencer complex of $\widetilde{\mathcal{D}}_X$).**

(1) Let $\widetilde{\mathcal{L}}^\bullet$ be a bounded resolution by left $\widetilde{\mathcal{D}}_X$-modules of $\widetilde{\mathcal{O}}_X$ (as a left $\widetilde{\mathcal{D}}_X$-module). Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Show that, if the terms L^k are $\widetilde{\mathcal{O}}_X$-locally free, $L^{\otimes}_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$ (with the tensor product structure of left $\widetilde{\mathcal{D}}_X$-module) is a resolution of $\widetilde{\mathcal{M}}$ as a $\widetilde{\mathcal{D}}_X$-module.

(2) Deduce that $\mathrm{Sp}(\widetilde{\mathcal{D}}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is a resolution of $\widetilde{\mathcal{D}}_{X\to Y}$ as a bimodule.

(3) Let $\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})$ be the Spencer complex of $\widetilde{\mathcal{D}}_{X\to Y}$ considered as a right $\widetilde{\mathcal{D}}_Y$-module. Show that $\mathrm{Sp}_Y(\widetilde{\mathcal{D}}_{X\to Y})$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $\widetilde{\mathcal{D}}_X$-module.

(4) Show that $\mathrm{gr}^F\mathcal{D}_{X\to Y} = R_F\mathcal{D}_{X\to Y}/zR_F\mathcal{D}_{X\to Y}$ is identified with $\pi^*\operatorname{Sym}\Theta_Y$ as a graded $(\operatorname{Sym}\Theta_X)$-module (see Exercise 8.4). For example, if $Y = \mathrm{pt}$, so that

$\mathcal{D}_{X\to Y} = \mathcal{O}_X$, $\mathrm{gr}^F\mathcal{O}_X = \mathcal{O}_X$ is regarded as a $(\mathrm{Sym}\,\Theta_X)$-module: in local coordinates, we have $\mathrm{Sym}\,\Theta_X = \mathbb{C}\{x_1,\dots,x_n\}[\xi_1,\dots,\xi_n]$ and

$$\mathbb{C}\{x_1,\dots,x_n\} = \mathbb{C}\{x_1,\dots,x_n\}[\xi_1,\dots,\xi_n]/(\xi_1,\dots,\xi_n).$$

(5) For $f = \mathrm{Id} : X \to X$, the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to X} = \mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_X$ is a resolution of $\widetilde{\mathcal{D}}_{X\to X} = \widetilde{\mathcal{D}}_X$ as a left and right $\widetilde{\mathcal{D}}_X$-module (notice that the left structure of $\widetilde{\mathcal{D}}_X$ is used for the tensor product).

(6) For $f : X \to \mathrm{pt}$, the complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to \mathrm{pt}} = \mathrm{Sp}(\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{D}}_{X\to\mathrm{pt}} = \widetilde{\mathcal{O}}_X$.

Exercise 8.41. Extend ${}_{\mathrm{D}}f_*$ and ${}_{\mathrm{D}}f_!$ as functors from $\mathsf{D}^+(\widetilde{\mathcal{D}}_X)$ (or $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_X)$) to $\mathsf{D}^+(\widetilde{\mathcal{D}}_Y)$. [*Hint*: Replace first $\widetilde{\mathcal{M}}^\bullet\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$ with the associated single complex.]

As in Remark 8.7.5(2), show that if $\widetilde{\mathcal{M}}^\bullet$ has bounded amplitude, then so has ${}_{\mathrm{D}}f_!\widetilde{\mathcal{M}}^\bullet$.

Exercise 8.42. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module.

(1) Show that

$$\big[\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{D}}_{X\to Y}\big]\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}} \simeq (\widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{M}})\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_{X\to Y}$$

as right $f^{-1}\widetilde{\mathcal{D}}_Y$-modules. [*Hint*: Use Exercise 8.16 and show that the corresponding isomorphism is compatible with the right $f^{-1}\widetilde{\mathcal{D}}_Y$-action.]

(2) Same question by replacing $\widetilde{\mathcal{D}}_{X\to Y}$ with $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.

(3) Conclude that

$$\begin{aligned}\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}} &\simeq \mathcal{H}om_{f^{-1}\widetilde{\mathcal{O}}_Y}\big(f^{-1}\widetilde{\omega}_Y, \widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\\ &\simeq \big(\widetilde{\mathcal{M}}^{\mathrm{right}}\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\omega}_Y.\end{aligned}$$

(4) Deduce from the first line, by using that f^{-1} is left adjoint to $\boldsymbol{R}f_*$, that

$$\boldsymbol{R}f_*(\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}) \simeq \big[{}_{\mathrm{D}}f_*(\widetilde{\mathcal{M}}^{\mathrm{right}})\big]^{\mathrm{left}},$$

and deduce from the second line (and justify the identification of the $\widetilde{\mathcal{D}}_Y$-module structures), by the projection formula for $f_!$, that

$$\boldsymbol{R}f_!(\mathrm{Sp}_{Y\leftarrow X}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}) \simeq \big[{}_{\mathrm{D}}f_!(\widetilde{\mathcal{M}}^{\mathrm{right}})\big]^{\mathrm{left}}.$$

Exercise 8.43. Show that the formula for the pushforward in Example 8.7.11 is obtained by side-changing from that of Example 8.7.10. [*Hint*: Adapt Exercise 8.26 in the relative case of a projection.]

***Exercise 8.44* (Pushforward by a closed inclusion).** Assume that $\iota : X \hookrightarrow Y$ is a closed inclusion. For a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, show that ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$ is generated by $\widetilde{\mathcal{M}}\otimes\mathbf{1}$ over $\widetilde{\mathcal{D}}_Y$. [*Hint*: Use Exercise 8.34.]

***Exercise 8.45* (Pushforward by a graph inclusion (see Example 8.7.7))**

Let $f : X \to Y$ a holomorphic map and let $\iota_f : X \hookrightarrow X\times Y$ be the the graph inclusion. In local coordinates $y_1,\dots,y_m$ on Y, set $f_j = y_j\circ f$.

(1) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that ${}_{\mathrm{D}}\iota_{f*}\widetilde{\mathcal{M}} \simeq \iota_{f*}\widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_m}]$ with right $\widetilde{\mathcal{D}}_{X\times Y}$ structure given locally by

$$\begin{aligned}\mu\widetilde{\partial}_y^\alpha\cdot\widetilde{\partial}_{y_j} &= \mu\widetilde{\partial}_y^{\alpha+\mathbf{1}_j},\\ \mu\widetilde{\partial}_y^\alpha\cdot\widetilde{\partial}_{x_i} &= (\mu\widetilde{\partial}_{x_i})\widetilde{\partial}_y^\alpha-\sum_{j=1}^m\mu\,\frac{\partial f_j}{\partial x_i}\,\widetilde{\partial}_y^{\alpha+\mathbf{1}_j}.\end{aligned}$$

(2) Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Show that ${}_{\mathrm{D}}\iota_{f*}\widetilde{\mathcal{M}} \simeq \iota_{f*}\widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_m}](m)$ with left $\widetilde{\mathcal{D}}_{X\times Y}$ structure given locally by (omitting $\widetilde{\mathrm{d}}\boldsymbol{y}^\vee$ in the notation)

$$\begin{aligned}\widetilde{\partial}_{y_j}\cdot\mu\widetilde{\partial}_y^\alpha &= -\mu\widetilde{\partial}_y^{\alpha+\mathbf{1}_j},\\ \widetilde{\partial}_{x_i}\cdot\mu\widetilde{\partial}_y^\alpha &= (\widetilde{\partial}_{x_i}\mu)\widetilde{\partial}_y^\alpha-\sum_{j=1}^m\frac{\partial f_j}{\partial x_i}\,\mu\,\widetilde{\partial}_y^{\alpha+\mathbf{1}_j}.\end{aligned}$$

[*Hint*: For the shift (m) of the grading, use Remark (8.2.2 $*$).]

***Exercise 8.46* (Compatibility of Spencer with ${}_{\mathrm{D}}\iota_*$ (right case))**

Let $\iota : X \hookrightarrow Y$ be a closed embedding. The goal of this exercise is to make explicit the isomorphism $\operatorname{Sp}_Y({}_{\mathrm{D}}\iota_!\widetilde{\mathcal{M}}) \simeq \iota_*\operatorname{Sp}_X\widetilde{\mathcal{M}}$ (equivalently, $\iota^{-1}\operatorname{Sp}_Y({}_{\mathrm{D}}\iota_!\widetilde{\mathcal{M}}) \simeq \operatorname{Sp}_X\widetilde{\mathcal{M}}$) for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$.

(1) By using that $\widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\mathcal{D}}_X$-locally free (Exercise 8.34), show that

$$\begin{aligned}\iota^{-1}\operatorname{Sp}_Y({}_{\mathrm{D}}\iota_!\widetilde{\mathcal{M}}) &\simeq (\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_{X\to Y})\otimes_{\iota^{-1}\widetilde{\mathcal{D}}_Y}\iota^{-1}\operatorname{Sp}_Y(\widetilde{\mathcal{D}}_Y)\\ &\simeq \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}\widetilde{\mathcal{D}}_Y)\otimes_{\iota^{-1}\widetilde{\mathcal{D}}_Y}\iota^{-1}\operatorname{Sp}_Y(\widetilde{\mathcal{D}}_Y)\\ &\simeq \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{D}}_X}(\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}\operatorname{Sp}_Y(\widetilde{\mathcal{D}}_Y)).\end{aligned}$$

(2) By using the natural $\widetilde{\mathcal{O}}_X$-linear injective morphism $\widetilde{\Theta}_X\to\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}\widetilde{\Theta}_Y$, deduce a natural $\widetilde{\mathcal{O}}_X$-linear injective morphism for each $k\geqslant 0$:

$$\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}\longrightarrow\widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}(\widetilde{\mathcal{D}}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\Theta}_{Y,k}).$$

(3) In local coordinates $(x_1,\dots,x_n,y_1,\dots,y_p)$ where X is defined by $y_1=\cdots=y_p=0$, and for multi-indices $\alpha\in\mathbb{N}^n$ and $\beta\in\mathbb{N}^p$, we use the notation $\widetilde{\partial}_x^{\wedge\alpha}$ for $\widetilde{\partial}_{x_1}^{\alpha_1}\wedge\cdots\wedge\widetilde{\partial}_{x_n}^{\alpha_n}$, and similarly for $\widetilde{\partial}_y^{\wedge\beta}$. Then express the above morphism as the composition of the two natural inclusions

$$\bigoplus_{|\alpha|=k}\widetilde{\mathcal{D}}_X\otimes\widetilde{\partial}_x^{\wedge\alpha}\longleftrightarrow\bigoplus_{|\alpha|=k}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes\widetilde{\partial}_x^{\wedge\alpha}\subset\bigoplus_{|\alpha|+|\beta|=k}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes(\widetilde{\partial}_x^{\wedge\alpha}\wedge\widetilde{\partial}_y^{\wedge\beta})$$

(4) Show that the left action of $\widetilde{\mathcal{D}}_X$ on the right-hand side of the morphism in (2) comes from the standard left action on $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{y_1},\dots,\widetilde{\partial}_{y_p}]\otimes(\widetilde{\partial}_x^{\wedge\alpha}\wedge\widetilde{\partial}_y^{\wedge\beta})$.

(5) Show that the following diagram commutes:

$$\begin{tikzcd}\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k}\arrow[r]\arrow[d,"\widetilde{\delta}_X"'] & \widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}(\widetilde{\mathcal{D}}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\Theta}_{Y,k})\arrow[d,"\mathrm{Id}\otimes\widetilde{\delta}_Y"]\\ \widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\Theta}_{X,k-1}\arrow[r] & \widetilde{\mathcal{O}}_X\otimes_{\iota^{-1}\widetilde{\mathcal{O}}_Y}\iota^{-1}(\widetilde{\mathcal{D}}_Y\otimes_{\widetilde{\mathcal{O}}_Y}\widetilde{\Theta}_{Y,k-1})\end{tikzcd}$$

[*Hint*: Use the local expression of (3) for the horizontal morphisms.]

(6) Show similarly that for a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, the natural quasi-isomorphism of complexes $\mathrm{Sp}_X(\widetilde{\mathcal{M}}) \to \iota^{-1}\mathrm{Sp}_Y({}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}})$ is locally termwise described as

$$\bigoplus_{|\alpha|=k} \widetilde{\mathcal{M}} \otimes \widetilde{\partial}_x^{\wedge\alpha} \hookrightarrow \bigoplus_{|\alpha|=k} \widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_p}] \otimes \widetilde{\partial}_x^{\wedge\alpha} \subset \bigoplus_{|\alpha|+|\beta|=k} \widetilde{\mathcal{M}}[\widetilde{\partial}_{y_1}, \dots, \widetilde{\partial}_{y_p}] \otimes (\widetilde{\partial}_x^{\wedge\alpha} \wedge \widetilde{\partial}_y^{\wedge\beta}).$$

***Exercise 8.47* (Compatibility of Spencer with ${}_{\mathrm{D}}\iota_*$ (left case))**

The setting is as in Exercise 8.46. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module.

(1) Show that ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}} \simeq \iota_*\widetilde{\mathcal{M}}[\widetilde{\partial}_y] \otimes \widetilde{\mathrm{d}}\boldsymbol{y}^\vee$. [*Hint*: let $\widetilde{\mathcal{N}}$ be the RHS; prove that $(\widetilde{\mathrm{d}}\boldsymbol{x} \wedge \widetilde{\mathrm{d}}\boldsymbol{y}) \otimes \widetilde{\mathcal{N}} \simeq (\widetilde{\mathrm{d}}\boldsymbol{x} \otimes \iota_*\widetilde{\mathcal{M}})[\widetilde{\partial}_y]$.]

(2) Show that the isomorphism ${}^{\mathrm{p}}\mathrm{DR}_X(\widetilde{\mathcal{M}}) \simeq \iota^{-1}\,{}^{\mathrm{p}}\mathrm{DR}_Y({}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}})$ is given termwise, for any local section η_x of $\widetilde{\Omega}_X^{n+k}$ by

$$\widetilde{\Omega}_X^{n+k} \otimes \widetilde{\mathcal{M}} \ni \eta_x \otimes m \longmapsto (\eta_x \wedge \widetilde{\mathrm{d}}\boldsymbol{y}) \otimes (m \otimes \widetilde{\mathrm{d}}\boldsymbol{y}^\vee) \in \widetilde{\Omega}_X^{n+p+k} \otimes {}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}.$$

[*Hint*: Apply Exercise 8.46(6) to $\widetilde{\mathcal{N}}$ considered in (1), and then the side-changing formula of Lemma 8.4.7.]

***Exercise 8.48* (Compatibility with the Godement functor).**

(1) Show by induction on k that, for every $k \geqslant 0$, the functor God^k is exact (see [**God64**, p. 168]). Given an exact sequence $0 \to \widetilde{\mathcal{L}}' \to \widetilde{\mathcal{L}} \to \widetilde{\mathcal{L}}'' \to 0$ of sheaves, show that we have an exact sequence of complexes

$$0 \longrightarrow \mathrm{God}^\bullet \widetilde{\mathcal{L}}' \longrightarrow \mathrm{God}^\bullet \widetilde{\mathcal{L}} \longrightarrow \mathrm{God}^\bullet \widetilde{\mathcal{L}}'' \longrightarrow 0.$$

Similarly, show that the functors $f_\star \mathrm{God}^k$ are exact (with $\star = *$ or $\star = !$) and deduce an exact sequence of complexes

$$0 \longrightarrow f_\star \mathrm{God}^\bullet \widetilde{\mathcal{L}}' \longrightarrow f_\star \mathrm{God}^\bullet \widetilde{\mathcal{L}} \longrightarrow f_\star \mathrm{God}^\bullet \widetilde{\mathcal{L}}'' \longrightarrow 0.$$

Deduce also that, for every $k \geqslant 0$ and a complex $\widetilde{\mathcal{L}}^\bullet$, we have

$$H^i(f_\star \mathrm{God}^k \widetilde{\mathcal{L}}^\bullet) \simeq f_\star \mathrm{God}^k H^i \widetilde{\mathcal{L}}^\bullet.$$

(2) Show that, if $\widetilde{\mathcal{L}}$ and $\mathcal{F}$ are $\widetilde{\mathcal{O}}_X$-modules and if $\mathcal{F}$ is locally free, then we have a natural inclusion $\mathcal{C}^0(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F} \hookrightarrow \mathcal{C}^0(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F})$, which is an equality if $\mathcal{F}$ has finite rank. More generally, show by induction that we have a natural morphism $\mathcal{C}^k(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F} \to \mathcal{C}^k(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F})$, which is an equality if $\mathcal{F}$ has finite rank.

(3) With the same assumptions, show that both complexes $\mathrm{God}^\bullet(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F}$ and $\mathrm{God}^\bullet(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F})$ are resolutions of $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F}$. Conclude that the natural morphism of complexes $\mathrm{God}^\bullet(\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F} \to \mathrm{God}^\bullet(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathcal{F})$ is a quasi-isomorphism, and an equality if $\mathcal{F}$ has finite rank.

(4) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that the natural morphism of complex

$$(\mathrm{God}^\bullet \widetilde{\mathcal{M}}) \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\,\widetilde{\mathcal{D}}_X \longrightarrow \mathrm{God}^\bullet(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\,\widetilde{\mathcal{D}}_X)$$

is a quasi-isomorphism.

(5) Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. Show that

$$\mathrm{Sp}(\mathrm{God}^\bullet \widetilde{\mathcal{M}}) = \mathrm{God}^\bullet \mathrm{Sp}\,\widetilde{\mathcal{M}}.$$

(6) If $f : X = Y \times T \to Y$ is the projection, show that, for $\star = *, !$,

$$ {}_{\mathrm{D}}f_\star \widetilde{\mathcal{M}} = f_\star \operatorname{God}^\bullet\big(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X/Y,\bullet}\big). $$

[*Hint*: Use Example 8.7.10.]

Exercise 8.49 **(Restriction to $z = 1$).**

(1) Show that the Godement functor applied to sheaves of $\widetilde{\mathbb{C}}$-modules restricts, for $z = 1$, to the Godement functor applied to sheaves of $\mathbb{C}$-vector spaces.

(2) Show that $\mathrm{Sp}_{X\to Y}(\mathcal{D}_X) = \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)/(z-1)\,\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$.

(3) Conclude that ${}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}}/(z-1){}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} = {}_{\mathrm{D}}f_\star(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ and, for every i, ${}_{\mathrm{D}}f_\star^{(i)}\widetilde{\mathcal{M}}/(z-1){}_{\mathrm{D}}f_\star^{(i)}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}f_\star^{(i)}(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ $(\star = *, !)$.

Exercise 8.50 **(Computation of the pushforward with the C^∞ Spencer complex)**

We take up the notation of Exercise 8.28. Let $f : X \to Y$ be a holomorphic map. For a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we define on

$$ \mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}}) := \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}^{\infty,\bullet}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y} \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}^{\infty,\bullet}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y $$

the right $f^{-1}\widetilde{\mathcal{D}}_Y$-linear differential

$$ \widetilde{\delta}^\infty_{\widetilde{\mathcal{M}},Y}\big((m\otimes\xi)\otimes\varphi\otimes Q\big) := \widetilde{\delta}'_{\widetilde{\mathcal{M}},Y}\big((m\otimes\xi)\otimes\varphi\otimes Q\big) + (m\otimes(\xi \lrcorner \widetilde{\mathrm{d}}'\varphi))\otimes Q + (m\otimes\xi)\otimes\widetilde{\mathrm{d}}''\varphi\otimes Q, $$

where the first term is naturally defined from the formula in Exercise 8.39, and the second and third terms are as in Exercise 8.28.

(1) Show that each term of the complex $\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X) = \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathrm{Sp}}^{\infty,\bullet}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is $\widetilde{\mathcal{D}}_X$-flat.

(2) Show that $\widetilde{\delta}^\infty_{\widetilde{\mathcal{M}},Y}$ is indeed a differential and that $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}}) \to \mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ is a quasi-isomorphism.

(3) Show that $\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}}) \to \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X)$ is an isomorphism. [*Hint*: Argue as in Exercises 8.24 and 8.28.]

Exercise 8.51 **(Computation of the pushforward with differential forms)**

Let $f : X \to Y$ be a holomorphic map. The formula for the pushforward has a simpler expression when we regard it as producing, from a *left* $\widetilde{\mathcal{D}}_X$-module, a complex of *right* $\widetilde{\mathcal{D}}_Y$-modules. This exercise gives such a formula.

Let $\widetilde{\mathcal{M}}$ be a *left* $\widetilde{\mathcal{D}}_X$-module. As $\widetilde{\mathcal{D}}_{X\to Y}$ is a left $\widetilde{\mathcal{D}}_X$-module,

$$ \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y $$

has a natural structure of left $\widetilde{\mathcal{D}}_X$-module (by setting $\xi(\mu\otimes\mathbf{1}_Y) = \xi\mu\otimes\mathbf{1}_Y + \mu\otimes Tf(\xi)$, see Exercise 8.12(2)) and of course a compatible structure of right $f^{-1}\widetilde{\mathcal{D}}_Y$-module.

(1) Show that the de Rham complex

$$ \widetilde{\Omega}^{n+\bullet}_X \otimes (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}) = \widetilde{\Omega}^{n+\bullet}_X \otimes (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} f^*\widetilde{\mathcal{D}}_Y) = \widetilde{\Omega}^{n+\bullet}_X \otimes (\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) $$

is isomorphic to $\widetilde{\mathcal{M}}^{\mathrm{right}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X)$, as a complex of right $f^{-1}\widetilde{\mathcal{D}}_Y$-modules, by using the isomorphism (see Lemma 8.4.7)

$$ \omega\otimes\mu\otimes\xi\otimes\mathbf{1}_Y \longmapsto (-1)^{kn}\,\varepsilon(k+1)\omega(\xi\wedge\bullet)\otimes\mu\otimes\mathbf{1}_Y \quad (\xi \in \wedge^k\widetilde{\Theta}_X). $$

[*Hint*: see Exercise 8.26.]

(2) Check that the connection induced on $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} f^*\widetilde{\mathcal{D}}_Y$ by the left $\widetilde{\mathcal{D}}_X$-module structure is $\widetilde{\nabla} \otimes \mathrm{Id} + \mathrm{Id}_{\widetilde{\mathcal{M}}} \otimes \widetilde{\nabla}^X$, where $\widetilde{\nabla}^X$ is obtained from the universal connection $\widetilde{\nabla}^Y$ on $\widetilde{\mathcal{D}}_Y$ by the formula (8.6.1).

(3) Conclude that, for $\star = *, !$,

$$ {}_{\mathrm{D}}f_\star(\widetilde{\mathcal{M}}^{\mathrm{right}}) = \boldsymbol{R}f_\star\big[\widetilde{\Omega}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big], \tag{8.51 $*$}$$

$$ {}_{\mathrm{D}}f_\star(\widetilde{\mathcal{M}}^{\mathrm{left}}) = \boldsymbol{R}f_\star\big[\widetilde{\Omega}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big]^{\mathrm{left}}, \tag{8.51 $**$}$$

where (8.51 $*$) is the complex of right $\widetilde{\mathcal{D}}_Y$-modules associated to the double complex

$$f_\star \operatorname{God}^\bullet\big[\widetilde{\Omega}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big].$$

Show that this complex is quasi-isomorphic to the complex

$$f_\star\big[\widetilde{\Omega}_X^{n+\bullet} \otimes (\operatorname{God}^\bullet \widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big].$$

[*Hint*: Use Exercise 8.48.]

(4) Show that the latter complex is the single complex associated with the double complex having terms $f_\star(\widetilde{\Omega}_X^{n+i} \otimes \operatorname{God}^j \widetilde{\mathcal{M}}) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y$ and first differential $f_\star(\widetilde{\nabla} \otimes \mathrm{Id} + \mathrm{Id}_{\widetilde{\mathcal{M}}} \otimes \widetilde{\nabla}^X)$ (the second differential is induced by the Godement differential).

(5) It is often more convenient to replace the Godement resolution by a Dolbeault resolution. Prove that

$$\begin{aligned}({}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}})^{\mathrm{right}} &\simeq f_\star\big[\widetilde{\mathcal{E}}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big],\\ {}_{\mathrm{D}}f_\star\widetilde{\mathcal{M}} &\simeq f_\star\big[\widetilde{\mathcal{E}}_X^{n+\bullet} \otimes (\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)\big]^{\mathrm{left}},\end{aligned}$$

where the differential in the latter complexes is obtained in the usual way from the holomorphic differential of 8.51(1) and the anti-holomorphic differential d''.

Other properties of the pushforward functor

***Exercise 8.52* (Pushforward of induced $\widetilde{\mathcal{D}}$-modules).** Let $\widetilde{\mathcal{L}}$ be an $\widetilde{\mathcal{O}}_X$-module and let $\widetilde{\mathcal{M}} = \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ be the associated induced right $\widetilde{\mathcal{D}}_X$-module. Let $f : X \to Y$ be a holomorphic map.

(1) Show that $\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) \to \widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ is a quasi-isomorphism. [*Hint*: Use that $\widetilde{\mathcal{D}}_X$ is $\widetilde{\mathcal{O}}_X$-locally free.]

(2) Deduce that

$$\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X) = \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_{X\to Y} = \widetilde{\mathcal{L}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y.$$

(3) Show that ${}_{\mathrm{D}}f_!(\widetilde{\mathcal{L}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X)$ is quasi-isomorphic to $(\boldsymbol{R}f_!\widetilde{\mathcal{L}}) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y$. [*Hint*: Use the projection formula.]

***Exercise 8.53* (Pushforward of $\widetilde{\mathcal{D}}$-modules and pushforward of $\widetilde{\mathcal{O}}$-modules)**

Let $f : X \to Y$ be a holomorphic map and let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. It is also an $\widetilde{\mathcal{O}}_X$-module. The goal of this exercise is to exhibit natural $\widetilde{\mathcal{O}}_Y$-linear morphisms $(\star = *, !)$

$$R^i f_\star \widetilde{\mathcal{M}} \longrightarrow {}_{\mathrm{D}}f_\star^{(i)} \widetilde{\mathcal{M}}.$$

(1) Show that $\widetilde{\mathcal{D}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$ has a natural global section $\mathbf{1}$.

(2) Show that there is a natural $f^{-1}\widetilde{\mathcal{O}}_Y$-linear morphism of complexes

$$\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} \mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{D}}_X), \quad m \longmapsto m \otimes \mathbf{1},$$

where $\widetilde{\mathcal{M}}$ is considered as a complex with $\widetilde{\mathcal{M}}$ in degree 0 and all other terms equal to 0, so the differential are all equal to 0. [*Hint*: Use Exercise 8.18(3) to identify $\mathrm{Sp}^0_{X\to Y}(\widetilde{\mathcal{D}}_X) = \widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$ with its twisted left $\widetilde{\mathcal{D}}_X$-structure (denoted by $\widetilde{\mathcal{D}}_{X\to Y} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ in loc. cit.) with $\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y}$, where the tensor product uses the right $\widetilde{\mathcal{O}}_X$-structure on $\widetilde{\mathcal{D}}_X$ and the left $\widetilde{\mathcal{D}}_X$ structure is the trivial one, and then with $\widetilde{\mathcal{D}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{O}}_Y$ with trivial left $\widetilde{\mathcal{D}}_X$-structure and tensor product using the right $\widetilde{\mathcal{O}}_X$-structure of $\widetilde{\mathcal{D}}_X$. Identify then $\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{D}}_X} (\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_{X\to Y})$ with $\widetilde{\mathcal{M}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y$.]

(3) Conclude with the existence of the desired morphisms.

***Exercise 8.54* (Grading and pushforward, right case).** Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a filtered right $\mathcal{D}_X$-module. Set $\mathcal{M} = R_F\mathcal{M}$, so that $\mathrm{gr}^F\mathcal{M} = \mathcal{M}/z\mathcal{M}$.

(1) Show that

$$(\mathcal{M} \otimes_{R_F\mathcal{D}_X} \mathrm{Sp}\, R_F\mathcal{D}_{X\to Y}) \otimes_{\mathbb{C}[z]} \mathbb{C}[z]/z\mathbb{C}[z] \simeq \mathrm{gr}^F\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y.$$

[*Hint*: Use the associativity of $\otimes$ and Exercise 8.40(4).]

(2) Assume that ${}_{\mathrm{D}}f_\star\mathcal{M}$ is *strict* (i.e., the complex of Corollary 8.7.15 is strict in the sense of Definition 5.1.6 or 10.2.2). Show that, for every i, we have, as graded modules

$$\mathrm{gr}^F{}_{\mathrm{D}}f_\star^{(i)}\mathcal{M} \simeq H^i \boldsymbol{R}f_\star\big(\mathrm{gr}^F\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y\big).$$

***Exercise 8.55* (Grading and pushforward, left case).** With the assumptions as in Exercise 8.54(2), but assuming that $\mathcal{M}$ is a left $\mathcal{D}_X$-module, show that

$$\mathrm{gr}^F{}_{\mathrm{D}}f_\star^{(i)}\mathcal{M} \simeq H^i \boldsymbol{R}f_\star\big(\omega_{X/Y} \otimes_{\mathcal{O}_X} \mathrm{gr}^F_{\bullet+n-m}\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} f^*\,\mathrm{Sym}\,\Theta_Y\big),$$

where $\omega_{X/Y} := \omega_X \otimes_{\mathcal{O}_X} f^*\omega_Y^\vee$, and we have set $n = \dim X$, $m = \dim Y$. (Notice the shift of the filtration, which comes from $\widetilde{\omega}_{X/Y} = z^{n-m}\omega_{X/Y}$.) For example, if $Y = \mathrm{pt}$, deduce that

$$\mathrm{gr}^F \boldsymbol{H}^i(X, {}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}) \simeq \boldsymbol{H}^i\big(X, \omega_X \otimes (\mathrm{gr}^F_{\bullet+n}\mathcal{M} \otimes^{\boldsymbol{L}}_{\mathrm{Sym}\,\Theta_X} \mathcal{O}_X)\big).$$

***Exercise 8.56* (Trace for a finite map, preliminaries).** We take up the notation of Example 8.7.30, so that $f : X = \mathbb{C}^n \to Y = \mathbb{C}^n$ is defined by $f_i(x_1, \dots, x_n) = x_i^{r_i}$, with $r_i \in \mathbb{N}^*$ and $r_i \geqslant 2$ if and only if $i = 1, \dots, \ell$. We set $D = \{\prod_{i=1}^\ell x_i = 0\}$ and we have $f(D) = \{\prod_{i=1}^\ell y_i = 0\}$.

(1) Define $\mathrm{Tr}_f : f_*\widetilde{\mathcal{O}}_X \to \widetilde{\mathcal{O}}_Y$ as an $\widetilde{\mathcal{O}}_Y$-linear morphism such that, composed with $\mathrm{adj}_f : \widetilde{\mathcal{O}}_Y \to f_*\widetilde{\mathcal{O}}_X$, it yields the identity $\widetilde{\mathcal{O}}_Y \to \widetilde{\mathcal{O}}_Y$. [*Hint*: Set $\mathrm{Tr}_f(g)(y) = (1/\#g^{-1}(y))\sum_{x\in f^{-1}(y)} g(x)$.]

(2) Show that for any holomorphic function g on X, there exists a holomorphic function g' on Y such that $\widetilde{\mathrm{d}}g/g = f^*(\widetilde{\mathrm{d}}g'/g')$ (where f^* means T^*f).

(3) Show that there exists an $\widetilde{\mathcal{O}}_Y$-linear morphism

$$\mathrm{Tr}_f : f_*\widetilde{\Omega}^1_X(\log D) \longrightarrow \widetilde{\Omega}^1_Y(\log f(D))$$

satisfying the following properties:

(a) $\mathrm{Tr}_f(\widetilde{\mathrm{d}}x_i/x_i) = (1/r_i)\widetilde{\mathrm{d}}y_i/y_i$ for $i = 1, \dots, \ell$ and $\mathrm{Tr}_f(\widetilde{\mathrm{d}}x_j) = \widetilde{\mathrm{d}}x_j$ for $j \geqslant \ell + 1$,

(b) $\mathrm{Tr}_f(\widetilde{\mathrm{d}}g/g) = \widetilde{\mathrm{d}}g'/g'$, with g' as above,

(c) $\mathrm{Tr}_f(h \cdot \widetilde{\mathrm{d}}g/g) = \mathrm{Tr}_f(h) \cdot \widetilde{\mathrm{d}}g'/g'$.

(4) Deduce that there exists an $\widetilde{\mathcal{O}}_Y$-linear morphism $\mathrm{Tr}_f : f_*\widetilde{\Omega}^1_X \to \widetilde{\Omega}^1_Y$ such that the composition

$$\widetilde{\Omega}^1_Y \xrightarrow{f_*(T^*f)} f_*\widetilde{\Omega}^1_X \xrightarrow{\mathrm{Tr}_f} \widetilde{\Omega}^1_Y$$

is the identity, and satisfies $\widetilde{\mathrm{d}}\,\mathrm{Tr}_f(g) = \mathrm{Tr}_f(\widetilde{\mathrm{d}}g)$ for any holomorphic function g on X.

(5) Extend Tr_f as a morphism of complexes $(f_*(\widetilde{\Omega}^\bullet_X), f_*(\widetilde{\mathrm{d}})) \to (\widetilde{\Omega}^\bullet_Y, \widetilde{\mathrm{d}})$ such that the composition

$$(\widetilde{\Omega}^\bullet_Y, \widetilde{\mathrm{d}}) \xrightarrow{f_*(T^*f)} (f_*(\widetilde{\Omega}^\bullet_X), f_*(\widetilde{\mathrm{d}})) \xrightarrow{\mathrm{Tr}_f} (\widetilde{\Omega}^\bullet_Y, \widetilde{\mathrm{d}})$$

is the identity.

***Exercise 8.57* (Trace for a finite map).** Let $f : X \to Y$ be as in Exercise 8.56 and let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_Y$-module. Show that

$$({}_{\mathrm{D}}f_*({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}))^{\mathrm{right}} \simeq (f_*\widetilde{\Omega}^{n+\bullet}_X \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{M}}.$$

[*Hint*: Use that $R^i f_*(\bullet) = 0$ for $i \geqslant 0$ and argue as in the proof of Proposition 8.7.29.]

Deduce that there exists morphisms whose composition is the identity:

$$\widetilde{\mathcal{M}}^{\mathrm{right}} \xrightarrow{\mathrm{adj}_f} ({}_{\mathrm{D}}f^{(0)}_*({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}))^{\mathrm{right}} \xrightarrow{\mathrm{Tr}_f} \widetilde{\mathcal{M}}^{\mathrm{right}},$$

and conclude that $\widetilde{\mathcal{M}}^{\mathrm{right}}$ is a direct summand in $({}_{\mathrm{D}}f^{(0)}_*({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}))^{\mathrm{right}}$.

8.10.g. Exercises for Section 8.8

Exercise 8.58.

(1) Prove the coherence of the sheaf of rings $\mathrm{gr}^F\widetilde{\mathcal{D}}_X$ in a way similar to that of $\widetilde{\mathcal{D}}_X$.

(2) Let $D \subset X$ be a hypersurface and let $\widetilde{\mathcal{O}}_X(*D)$ be the sheaf of meromorphic functions on X with poles on D at most (with arbitrary order). Prove similarly that $\widetilde{\mathcal{O}}_X(*D)$ is a coherent sheaf of rings.

(3) Prove that $\widetilde{\mathcal{D}}_X(*D) := \widetilde{\mathcal{O}}_X(*D) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ is a coherent sheaf of rings.

(4) Let $\iota : Y \hookrightarrow X$ denote the inclusion of a smooth submanifold. Show that $i^*\widetilde{\mathcal{D}}_X := \widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X$ is a coherent sheaf of rings on Y.

(5) Let $Y \subset X$ be a smooth hypersurface of X. Show that $V_0\widetilde{\mathcal{D}}_X$ (see Section 9.2) is a coherent sheaf of rings.

Exercise 8.59.

(1) Let $\widetilde{\mathcal{M}}\subset\widetilde{\mathcal{N}}$ be a $\widetilde{\mathcal{D}}_X$-submodule of a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$. Show that, if $\widetilde{\mathcal{M}}$ is locally finitely generated, then it is coherent.

(2) Let $\phi:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ be a morphism between coherent $\widetilde{\mathcal{D}}_X$-modules. Show that $\operatorname{Ker}\phi$ and $\operatorname{Coker}\phi$ are coherent.

Exercise 8.60 **(Non-validity of Cartan Theorem B for $\widetilde{\mathcal{D}}$-modules)**

(1) Let X be an open disc with coordinate x, of radius r (possibly ∞) in $\widetilde{\mathbb{C}}$, and let $(x_i)_{i\in\mathbb{N}}$ be a sequence of points in X such that $\lim_i(r-|x_i|)=0$. Let $\varphi:\widetilde{\mathcal{O}}_X^{\mathbb{N}}\to\widetilde{\mathcal{O}}_X^{\mathbb{N}}$ be the diagonal morphism equal to $(x-x_i)$ on the i-th component. Let $\widetilde{\mathbb{C}}_{x_i}$ denote the skyscraper sheaf supported on x_i. Show that

(a) $\operatorname{Coker}\varphi=\bigoplus_i\widetilde{\mathbb{C}}_{x_i}$ and $H^0(X,\operatorname{Coker}\varphi)=\prod_i H^0(X,\widetilde{\mathbb{C}}_{x_i})$;

(b) $\operatorname{Coker}[H^0\varphi:H^0(X,\widetilde{\mathcal{O}}_X^{\mathbb{N}})\to H^0(X,\widetilde{\mathcal{O}}_X^{\mathbb{N}})]=\bigoplus_i H^0(X,\widetilde{\mathbb{C}}_{x_i})$.

(2) Deduce that $H^1(X,\widetilde{\mathcal{O}}_X^{\mathbb{N}})\neq 0$.

(3) Let $\widetilde{\mathcal{O}}_X(*0)$ be the sheaf of meromorphic functions on X with poles at $x=0$ at most. Show that $H^1(X,\widetilde{\mathcal{O}}_X(*0))\neq 0$. [*Hint*: Use Cartan Theorem B for $\widetilde{\mathcal{O}}_X$ and apply the previous result to $\widetilde{\mathcal{O}}_X(*0)/\widetilde{\mathcal{O}}_X$.]

Exercise 8.61 **(Characterization of coherent filtrations).**

(1) Show that the following properties are equivalent:

(a) $F_\bullet\widetilde{\mathcal{M}}$ is a coherent filtration;

(b) for every $k\in\mathbb{Z}$, $F_k\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{O}}_X$-coherent, and, for every $x\in X$, there exists a neighbourhood U of x and $k_0\in\mathbb{Z}$ such that, for every $k\geqslant 0$, $F_k\widetilde{\mathcal{D}}_{X|U}\cdot F_{k_0}\widetilde{\mathcal{M}}_{|U}=F_{k+k_0}\widetilde{\mathcal{M}}_{|U}$;

(c) the graded module $\operatorname{gr}^F\widetilde{\mathcal{M}}$ is $\operatorname{gr}^F\widetilde{\mathcal{D}}_X$-coherent.

(2) Conclude that, if $F_\bullet\widetilde{\mathcal{M}},G_\bullet\widetilde{\mathcal{M}}$ are two coherent filtrations of $\widetilde{\mathcal{M}}$, then, locally on X, there exists k_0 such that, for every k, we have

$$F_{k-k_0}\widetilde{\mathcal{M}}\subset G_k\widetilde{\mathcal{M}}\subset F_{k+k_0}\widetilde{\mathcal{M}}.$$

Exercise 8.62 **(Local existence of coherent filtrations).** Let $F_\bullet\widetilde{\mathcal{M}}$ be a filtration of $\widetilde{\mathcal{M}}$.

(1) Write $R_F\widetilde{\mathcal{M}}=\bigoplus_k F_k\widetilde{\mathcal{M}}\zeta^k$, where ζ is a new variable, and show that, if $\widetilde{\mathcal{M}}$ has a coherent filtration, then it is $\widetilde{\mathcal{D}}_X$-coherent. [*Hint*: Use that the tensor product $\mathbb{C}[\zeta]/(\zeta-1)\otimes_{\mathbb{C}[\zeta]}\bullet$ is right exact.]

(2) Conversely, show that any coherent $\widetilde{\mathcal{D}}_X$-module admits locally a coherent filtration. [*Hint*: Choose a local presentation $\widetilde{\mathcal{D}}^q_{X|U}\xrightarrow{\varphi}\widetilde{\mathcal{D}}^p_{X|U}\to\widetilde{\mathcal{M}}_{|U}\to 0$, and show that the filtration induced on $\widetilde{\mathcal{M}}_{|U}$ by $F_\bullet\widetilde{\mathcal{D}}^p_{X|U}$ is coherent by using Exercise 8.61: Set $\mathcal{K}=\operatorname{Im}\varphi$ and reduce the assertion to showing that $F_j\widetilde{\mathcal{D}}_X\cap\mathcal{K}$ is $\widetilde{\mathcal{O}}_X$-coherent; prove that, up to shrinking U, there exists $k_o\in\mathbb{N}$ such that $\varphi(F_k\widetilde{\mathcal{D}}^q_{X|U})\subset F_{k+k_o}\widetilde{\mathcal{D}}^p_{X|U}$ for every k; deduce that $\varphi(F_k\widetilde{\mathcal{D}}^q_{X|U})$, being locally of finite type and contained in a coherent $\widetilde{\mathcal{O}}_X$-module, is $\widetilde{\mathcal{O}}_X$-coherent for every k; conclude by using the fact that

an increasing sequence of coherent $\widetilde{\mathcal{O}}_X$-modules in a coherent $\widetilde{\mathcal{O}}_X$-module is locally stationary.]

(3) Show that a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$ satisfies $F_p\widetilde{\mathcal{M}} = 0$ for $p \ll 0$ locally [*Hint*: Use that this holds for the filtration constructed in (2) and apply Exercise 8.61(2).]

(4) Show that, locally, any coherent $\widetilde{\mathcal{D}}_X$-module is generated over $\widetilde{\mathcal{D}}_X$ by a coherent $\widetilde{\mathcal{O}}_X$-submodule.

(5) Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module and let $\mathcal{F}$ be an $\widetilde{\mathcal{O}}_X$-submodule which is locally finitely generated. Show that $\mathcal{F}$ is $\widetilde{\mathcal{O}}_X$-coherent. [*Hint*: Choose a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$ and show that, locally, $\mathcal{F} \subset F_k\widetilde{\mathcal{M}}$ for some k; apply then the analogue of Exercise 8.59(1) for $\widetilde{\mathcal{O}}_X$-modules.]

Exercise 8.63.

(1) Show statements similar to those of Theorem 8.8.6 for $R_F\widetilde{\mathcal{D}}_X$-modules, $\mathrm{gr}^F\widetilde{\mathcal{D}}_X$-modules, $\widetilde{\mathcal{O}}_X(*D)$-modules, $\widetilde{\mathcal{D}}_X(*D)$-modules and $i^*\widetilde{\mathcal{D}}_X$-modules (see Exercise 8.58).

(2) Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Show that $\widetilde{\mathcal{D}}_X(*D) \otimes_{\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X(*D)$-coherent and that $i^*\widetilde{\mathcal{M}}$ is $i^*\widetilde{\mathcal{D}}_X$-coherent.

Exercise 8.64. Similarly to Corollary 8.8.7, prove that if $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ is a surjective morphism of coherent $\widetilde{\mathcal{D}}_X$-modules and if $F_\bullet\widetilde{\mathcal{M}}$ is coherent, then $F_\bullet\widetilde{\mathcal{N}} := \varphi(F_\bullet\widetilde{\mathcal{M}})$ is coherent as well.

Exercise 8.65.

(1) Show that $R_F\mathcal{D}_X$ is naturally filtered by locally free graded $\mathcal{O}_X[z]$-modules of finite rank by setting (locally)

$$F_k(R_F\mathcal{D}_X) = \sum_{|\alpha|\leqslant k} \mathcal{O}_X[z]\partial_x^\alpha.$$

(2) Show that $\mathrm{gr}^F(R_F\mathcal{D}_X) = \mathbb{C}[z] \otimes_{\mathbb{C}} \mathrm{gr}^F\mathcal{D}_X$ with the tensor product grading.

(3) For a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, show that, if one defines the filtration

$$F_k(R_F\mathcal{M}) = \sum_{j\leqslant k} F_j\mathcal{M} \otimes_{\mathbb{C}} z^j\mathbb{C}[z],$$

then $F_\bullet(R_F\mathcal{M})$ is an $F_\bullet(R_F\mathcal{D}_X)$-filtration and $\mathrm{gr}^F(R_F\mathcal{M})$ can be identified with $\mathbb{C}[z] \otimes_{\mathbb{C}} \mathrm{gr}^F\mathcal{M}$, equipped with the tensor product grading.

Exercise 8.66. Recall (see e.g. [**ST71**, Prop. 1.9]) that, for a coherent sheaf $\mathcal{F}$ of $\widetilde{\mathcal{O}}_X$-modules and a closed analytic subset $Z \subset X$, the sheaf $\Gamma_Z\mathcal{F}$ consisting of local sections which vanish away from Z is also the sheaf of local sections annihilated by some power of $\mathcal{I}_Z$, and is a coherent sheaf of $\widetilde{\mathcal{O}}_X$-modules. Deduce a similar property for coherent $\widetilde{\mathcal{D}}_X$-modules. [*Hint*: Prove that the assertion is local and apply the result for $\widetilde{\mathcal{O}}_X$-modules for a large step of a coherent filtration of $\widetilde{\mathcal{M}}$.]

Exercise 8.67. Let $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ be an exact sequence of $\widetilde{\mathcal{D}}_X$-modules. Show that $\mathrm{Char}\,\widetilde{\mathcal{M}} = \mathrm{Char}\,\widetilde{\mathcal{M}}' \cup \mathrm{Char}\,\widetilde{\mathcal{M}}''$. [*Hint*: Take a coherent filtration on $\widetilde{\mathcal{M}}$ and induce it on $\widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{M}}''$.]

Exercise 8.68 **(Coherent $\widetilde{\mathcal{D}}_X$-modules with characteristic variety $T^*_X X$)**
Assume that $\widetilde{\mathcal{M}}$ is coherent with characteristic variety contained in $T^*_X X \times \mathbb{C}_z$.

(1) Show that, for any local coherent filtration $F_\bullet\widetilde{\mathcal{M}}$, the graded module $\mathrm{gr}^F\widetilde{\mathcal{M}}$ is locally of finite type, hence coherent (see Exercise 8.62(5)) over $\widetilde{\mathcal{O}}_X$.

(2) Deduce that, locally on X, there exists p_o such that $\mathrm{gr}^F_p\widetilde{\mathcal{M}} = 0$ for $p \geqslant p_o$.

(3) For a $\mathcal{D}_X$-module $\mathcal{M}$, deduce that $\mathcal{M}$ is locally free of finite rank.

Exercise 8.69 **(Coherent $\mathcal{D}_X$-modules with characteristic variety contained in $T^*_Y X$)**
In this exercise, we switch to the case of $\mathcal{D}_X$-modules. Let $\iota : Y \hookrightarrow X$ be the inclusion of a smooth codimension p closed submanifold. Define the p-th algebraic local cohomology with support in Y by $R^p\Gamma_{[Y]}\mathcal{O}_X = \varinjlim_k \mathcal{E}xt^p(\mathcal{O}_X/\mathcal{I}_Y^k, \mathcal{O}_X)$, where $\mathcal{I}_Y$ is the ideal defining Y. $R^p\Gamma_{[Y]}\mathcal{O}_X$ has a natural structure of $\mathcal{D}_X$-module. In local coordinates $(x_1, \dots, x_n)$ where Y is defined by $x_1 = \cdots = x_p = 0$, we have

$$R^p\Gamma_{[Y]}\mathcal{O}_X \simeq \frac{\mathcal{O}_{\mathbb{C}^n}[1/x_1\cdots x_n]}{\sum_{i=1}^p \mathcal{O}_{\mathbb{C}^n}(x_i/x_1\cdots x_n)}.$$

Denote this $\mathcal{D}_X$-module by $\mathcal{B}_Y X$.

(1) Show that $\mathcal{B}_Y X$ has support contained in Y and characteristic variety equal to $T^*_Y X$.

(2) Identify $\mathcal{B}_Y X$ with ${}_{\mathrm{D}}\iota_*\mathcal{O}_Y$.

(3) Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module with characteristic variety equal to $T^*_Y X$. Show that $\mathcal{M}$ is locally isomorphic to $(\mathcal{B}_Y X)^d$ for some d.

Exercise 8.70. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module equipped with a coherent filtration $F_\bullet\mathcal{M}$. Set $\mathcal{M} = R_F\mathcal{M}$.

(1) Show that $\mathrm{Char}(R_F\mathcal{M}) = (\mathrm{Char}\,\mathcal{M}) \times \mathbb{C}_z$, so that $\mathcal{M}$ is holonomic (in the sense of Definition 8.8.15) if and only if $\mathcal{M}$ is holonomic. (In other words, for a strict coherent $\widetilde{\mathcal{D}}_X$-module $\mathcal{M}$, $\mathcal{M}/(z-1)\mathcal{M}$ is holonomic if and only if $\mathcal{M}$ itself is holonomic.)

(2) In such a case, show that $\mathcal{E}xt^i_{R_F\mathcal{D}_X}(R_F\mathcal{M}, R_F\mathcal{D}_X)$ consists of z-torsion if $i \neq \dim X$.

Exercise 8.71 **(Characteristic variety of the external product, see [Kas03, §4.3])**
Consider the setting of Lemma 8.6.10. Assume moreover that the filtrations $F_\bullet\mathcal{M}_X, F_\bullet\mathcal{M}_Y$ are coherent. Show that $F_\bullet(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y)$ is coherent. Conclude that $\mathrm{Char}(\mathcal{M}_X \boxtimes_{\mathcal{D}} \mathcal{M}_Y) = \mathrm{Char}\,\mathcal{M}_X \times \mathrm{Char}\,\mathcal{M}_Y$.

Exercise 8.72 **(Projection formula for $\widetilde{\mathcal{O}}$-modules).** Let X, Y be complex manifolds, X being compact, let $\widetilde{\mathcal{L}}_X$ be an $\widetilde{\mathcal{O}}_X$-modules and let us denote by $p : X \times Y \to X$ and $q : X \times Y \to Y$ the projections.

(1) Show that there exists a natural morphism $\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \to \boldsymbol{R}q_*p^*\widetilde{\mathcal{L}}_X$. [*Hint*: Justify the following composition of morphisms

$$\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}\Gamma(X, \widetilde{\mathcal{L}}_X) \xrightarrow{\sim} \widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} \boldsymbol{R}q_*p^{-1}\widetilde{\mathcal{L}}_X \simeq \boldsymbol{R}q_*(q^{-1}\widetilde{\mathcal{O}}_Y \otimes_{\widetilde{\mathbb{C}}} p^{-1}\widetilde{\mathcal{L}}_X) \longrightarrow \boldsymbol{R}q_*p^*\widetilde{\mathcal{L}}_X$$

and conclude.]

The goal of the remaining part is to prove (8.8.28), that is, if $\widetilde{\mathcal{L}}_X$ is the inductive limit of its coherent $\widetilde{\mathcal{O}}_X$-submodules this morphism is an isomorphism.

(2) Reduce the statement to the case where $\widetilde{\mathcal{L}}_X$ is $\widetilde{\mathcal{O}}_X$-coherent. [*Hint*: Proper pushforward commutes with inductive limits.]

(3) Consider first the case of $\mathcal{O}_X$-modules. Use Grauert's theorem (see e.g. [**BS76**, Th. 4.12]) to prove the result.

(4) For $\widetilde{\mathcal{O}}_X$-modules, apply the previous result to each graded piece and conclude.

8.10.h. Exercises for Section 8.9

Exercise 8.73. Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Recall (Exercise 8.29) that $\mathrm{Sp}(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X)$ is a resolution of $\mathcal{L}$ as an $\mathcal{O}_X$-module. Show that the morphism (8.9.1) is the augmentation morphism $\mathrm{Sp}^0(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\to\mathcal{L}$.

Exercise 8.74.

(1) Show that any $\mathcal{O}_X$-linear morphism $u:\mathcal{L}\to\mathcal{L}'$ is a differential operator from $\mathcal{L}$ to $\mathcal{L}'$ and that a corresponding v is $u\otimes 1$.

(2) Assume that $\mathcal{L},\mathcal{L}'$ are right $\mathcal{D}_X$-modules. Let $u:\mathcal{L}\to\mathcal{L}'$ be $\mathcal{D}_X$-linear. Show that the corresponding v is $\mathcal{D}_X$-linear for both structures $(\mathrm{right})_{\mathrm{triv}}$ and $(\mathrm{right})_{\mathrm{tens}}$ (see Exercise 8.19) on $\mathcal{L}^{(\prime)}\otimes_{\mathcal{O}_X}\mathcal{D}_X$.

(3) Show that $\mathcal{H}om_{\mathrm{Diff}}(\mathcal{O}_X,\mathcal{O}_X)=\mathcal{D}_X$.

(4) Show that the morphism in Definition 8.9.6 is compatible with composition. Conclude that the composition of differential operators is a differential operator and that it is associative.

Exercise 8.75 **(Integrable connections are differential operators)**

Let $\mathcal{M}$ be an $\mathcal{O}_X$-module and let $\nabla:\mathcal{M}\to\Omega^1_X\otimes_{\mathcal{O}_X}\mathcal{M}$ be an integrable connection on $\mathcal{M}$.

(1) Show that ∇ is a differential morphism, by considering the right $\mathcal{D}_X$-linear morphism

$$v(m\otimes P):=\nabla(m)\otimes P+m\otimes\nabla(P),$$

for any local section m of $\mathcal{M}$ and P of $\mathcal{D}_X$, and where ∇P is defined in Exercise 8.5. Extend this result to connections ${}^{(k)}\nabla$.

(2) Let $\mathcal{M}',\mathcal{M}''$ be $\mathcal{O}_X$-submodules of $\mathcal{M}$ such that ${}^{(k)}\nabla$ induces a $\mathbb{C}$-linear morphism ${}^{(k)}\nabla':\Omega^k_X\otimes_{\mathcal{O}_X}\mathcal{M}'\to\Omega^{k+1}_X\otimes_{\mathcal{O}_X}\mathcal{M}''$. Show that ${}^{(k)}\nabla'$ is a differential morphism.

Exercise 8.76. Show that $\mathsf{Mod}(\mathcal{O}_X,\mathrm{Diff}_X)$ is an additive category, i.e.,

- $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L},\mathcal{L}')$ is a $\mathbb{C}$-vector space and the composition is $\mathbb{C}$-bilinear,
- the 0 $\mathcal{O}_X$-module satisfies $\mathrm{Hom}_{\mathrm{Diff}}(0,0)=0$,
- $\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}_1\oplus\mathcal{L}_2,\mathcal{L}')=\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}_1,\mathcal{L}')\oplus\mathrm{Hom}_{\mathrm{Diff}}(\mathcal{L}_2,\mathcal{L}')$ and similarly with $\mathcal{L}'_1,\mathcal{L}'_2$.

Exercise 8.77 **(De Rham and inverse de Rham on induced $\mathcal{D}$-modules)**

(1) Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. Show that $H^k\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\big)=0$ for $k\neq 0$ and $H^0\big({}^{\mathrm{p}}\mathrm{DR}(\mathcal{L}\otimes_{\mathcal{O}_X}\mathcal{D}_X)\big)=\mathcal{L}$. [*Hint*: Use Exercise 8.29.]

(2) Show that $H^0({}^{\mathrm{p}}\mathrm{DR})$ defines a functor $\mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X) \mapsto \mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$, which will be denoted by ${}^{\mathrm{diff}}\mathrm{DR}$.

(3) Show that ${}^{\mathrm{diff}}\mathrm{DR}^{-1} : \mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X)$ is an equivalence of categories, a quasi-inverse functor being ${}^{\mathrm{diff}}\mathrm{DR} : \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \mapsto \mathcal{L}$, ${}^{\mathrm{diff}}\mathrm{DR}(v) = u$.

(4) Show that the composed functor $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{Mod}_{\mathrm{i}}(\mathcal{D}_X) \mapsto \mathsf{Mod}(\mathcal{D}_X)$, still denoted by ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$, is *fully faithful*, i.e., it induces a bijective morphism

$$\mathrm{Hom}_{\mathsf{Diff}}(\mathcal{L}, \mathcal{L}') \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}_X}(\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X, \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X).$$

(One may think that we "embed" the additive (non abelian) category $\mathsf{Mod}(\mathcal{O}_X, \mathrm{Diff}_X)$ in the abelian category $\mathsf{Mod}(\mathcal{D}_X)$; we will use this "embedding" to define below acyclic objects).

***Exercise 8.78* (The de Rham functor ${}^{\mathrm{diff}}\mathrm{DR}$).**

(1) Show that the de Rham complex of a left $\mathcal{D}_X$-module $\mathcal{M}$ is a complex in $\mathsf{C}^{\mathrm{b}}(\mathcal{O}_X, \mathrm{Diff}_X)$. [*Hint*: Use Exercise 8.74(1).]

(2) By using Exercise 8.26(1), show that the de Rham complex of a right $\mathcal{D}_X$-module $\mathcal{M}$ is a complex in $\mathsf{C}^{\mathrm{b}}(\mathcal{O}_X, \mathrm{Diff}_X)$

(3) Show that the de Rham complex of a $\star$-bounded complex of right $\mathcal{D}_X$-modules has its associated single complex in $\mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$. [*Hint*: Use Exercise 8.24.]

(4) Conclude that ${}^{\mathrm{p}}\mathrm{DR}$ induces a functor ${}^{\mathrm{diff}}\mathrm{DR} : \mathsf{C}^\star(\mathcal{D}_X) \mapsto \mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$.

(5) Extend this functor as a functor of triangulated categories $\mathsf{K}^\star(\mathcal{D}_X) \to \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$.

Exercise 8.79. Let $\mathcal{M}$ be a $\mathcal{D}_X$-module. Show that $\mathrm{God}^\bullet\, {}^{\mathrm{diff}}\mathrm{DR}\,\mathcal{M}$ is a differential complex. [*Hint*: Identify this complex with ${}^{\mathrm{diff}}\mathrm{DR}\,\mathrm{God}^\bullet\,\mathcal{M}$.]

Exercise 8.80. Show that the family N of Diff-acyclic objects forms a *null system* in $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, i.e.,

- the object 0 belongs to N,
- an object $\mathcal{L}^\bullet$ belongs to N iff $\mathcal{L}^\bullet[1]$ does so,
- if $\mathcal{L}^\bullet \to \mathcal{L}'^\bullet \to \mathcal{L}''^\bullet \to \mathcal{L}^\bullet[1]$ is a distinguished triangle of $\mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$, and if $\mathcal{L}^\bullet, \mathcal{L}'^\bullet$ are objects in N, then so is $\mathcal{L}''^\bullet$.

[*Hint*: Use that the extension of ${}^{\mathrm{diff}}\mathrm{DR}^{-1}$ to the categories $\mathsf{K}^\star$ is a functor of triangulated categories.]

***Exercise 8.81* (The functor $\mathsf{D}^\star(\mathcal{O}_X) \mapsto \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$).** Using Exercise 8.74(1), define a functor $\mathsf{C}^\star(\mathcal{O}_X) \mapsto \mathsf{C}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$ and $\mathsf{K}^\star(\mathcal{O}_X) \mapsto \mathsf{K}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$. By using that $\mathcal{D}_X$ is $\mathcal{O}_X$-flat, show that if $\mathcal{L}^\bullet$ is acyclic in $\mathsf{K}^\star(\mathcal{O}_X)$, then $\mathcal{L}^\bullet \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is acyclic in $\mathsf{K}^\star(\mathcal{D}_X)$. Conclude that the previous functor extends as a functor $\mathsf{D}^\star(\mathcal{O}_X) \mapsto \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X)$.

Exercise 8.82. Show that the following diagram commutes:

$$\begin{tikzcd} \mathsf{D}^\star(\mathcal{D}_X) \arrow[r, "{}^{\mathrm{diff}}\mathrm{DR}"'] \arrow[rr, bend left, "{}^{\mathrm{p}}\mathrm{DR}"] & \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \arrow[r, "\mathsf{Forget}"'] & \mathsf{D}^\star(\mathbb{C}_X) \end{tikzcd}$$

Exercise 8.83. Assume that $\mathcal{L}^\bullet$ is Diff-acyclic. Show that $\mathsf{Forget}\,\mathcal{L}^\bullet$ is acyclic. [*Hint*: By definition, ${}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^\bullet)$ is acyclic; then ${}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1}(\mathcal{L}^\bullet)$ is also acyclic and quasi-isomorphic to $\mathsf{Forget}\,\mathcal{L}^\bullet$.]

Conclude that Forget induces a functor $\mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \mapsto \mathsf{D}^\star(\mathbb{C}_X)$, and that we have an isomorphism of functors

$$ {}^{\mathrm{p}}\mathrm{DR}\,{}^{\mathrm{diff}}\mathrm{DR}^{-1} \xrightarrow{\sim} \mathsf{Forget} : \mathsf{D}^\star(\mathcal{O}_X, \mathrm{Diff}_X) \longmapsto \mathsf{D}^\star(\mathbb{C}_X). $$

Compare with Exercise 8.29.

Exercise 8.84. Let $\mathcal{L}, \mathcal{L}'$ be two $\mathcal{O}_X$-modules and

$$ v : \mathcal{M} = \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X \longrightarrow \mathcal{M}' = \mathcal{L}' \otimes_{\mathcal{O}_X} \mathcal{D}_X $$

a $\mathcal{D}_X$-linear morphism. It defines a $f^{-1}\mathcal{D}_Y$-linear morphism

$$ v \otimes \mathbf{1} : \mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y} \longrightarrow \mathcal{M}' \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}, $$

where $\mathbf{1}$ is the section introduced in Exercise 8.53(1). This is therefore a morphism

$$ \widetilde{v} : \mathcal{L} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \longrightarrow \mathcal{L}' \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y. $$

Show that ${}^{\mathrm{diff}}\mathrm{DR}_Y(\widetilde{v}) = {}^{\mathrm{diff}}\mathrm{DR}_X(v)$.

[*Hint*: Since the problem is local, argue with coordinates on X and Y and write $f = (f_1, \dots, f_m)$. Let ℓ be a local section of $\mathcal{L}$, and let $\mathbf{1}_X$ be the unit of $\mathcal{D}_X$. Set $v(\ell \otimes \mathbf{1}_X) = w(\ell) = \sum_\alpha w(\ell)_\alpha \otimes \partial_x^\alpha$ and $\widetilde{v}(\ell \otimes \mathbf{1}_X) = v(\ell \otimes \mathbf{1}_X) \otimes \mathbf{1}_{X\to Y}$. Show that, if $\alpha_i \neq 0$,

$$ \partial_{x_i}^{\alpha_i} \otimes \mathbf{1}_{X\to Y} = \partial_{x_i}^{\alpha_i - 1} \sum_j \frac{\partial f_j}{\partial x_i} \otimes \partial_{y_j}. $$

Deduce that the image of $\widetilde{v}(\ell \otimes \mathbf{1}_X)$ by the map $\mathcal{L} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y \to \mathcal{L}$ is equal to the image of $w(\ell)_0$, which is nothing but $u(\ell)$ by definition of $u := H^0\,\mathrm{DR}_X(v)$.]

Exercise 8.85.

(1) Show that the Leray filtration is a decreasing finite filtration and that it is compatible with the differential.

(2) Show that, locally, being in Ler^p means having at least p factors dy_j in any summand.

***Exercise 8.86* (The connecting morphism).** Let $0 \to C_1^\bullet \to C_2^\bullet \to C_3^\bullet \to 0$ be an exact sequence of complexes. Let $[\mu] \in H^k C_3^\bullet$ and choose a representative in C_3^k with $d\mu = 0$. Lift μ as $\widetilde{\mu} \in C_2^k$.

(1) Show that $d\widetilde{\mu} \in C_1^{k+1}$ and that its differential is zero, so that the class $[d\widetilde{\mu}] \in H^{k+1}C_1^\bullet$ is well-defined.

(2) Show that $\delta : [\mu] \mapsto [d\widetilde{\mu}]$ is a well-defined morphism $H^k C_3^\bullet \to H^{k+1}C_1^\bullet$.

(3) Deduce the existence of the cohomology long exact sequence, having δ as its connecting morphism.

8.11. Comments

Most of the results in this chapter are now classical and explained in various reference books (e.g. [**MS93a, MS93b**], [**Bjö93**], [**MN04**], [**Kas03**], [**HTT08**]). We have emphasized their adaptation to the case of filtered $\mathcal{D}$-modules, or more precisely to the case of $\widetilde{\mathcal{D}}$-modules, in a way similar to what is done in [**Sab05**] with the analytification $\mathcal{R}$ with respect to the variable z of the sheaf $\widetilde{\mathcal{D}}$, and [**Moc07, Moc11a, Moc15**].

The notion of induced $\mathcal{D}$-module and the idea of inverting the de Rham functor is due to M. Saito [**Sai89a**]. The comparison of the notion of pushforward of a $\mathcal{D}$-module with the Katz-Oda construction of the Gauss-Manin connection is taken from [**DMSS00**].

The pushforward of a holonomic $\mathcal{D}$-module $\mathcal{M}$ by a finite morphism (or finite on the support of $\mathcal{M}$) is worth considering in detail. This is done in [**Käl18**] in the algebraic setting. In particular, the decomposition theorem holds without any Hodge assumption for such morphisms.

CHAPTER 9

NEARBY AND VANISHING CYCLES OF $\widetilde{\mathcal{D}}$-MODULES

Summary. We introduce the Kashiwara-Malgrange filtration for a $\widetilde{\mathcal{D}}_X$-module, and the notion of strict $\mathbb{R}$-specializability. This leads to the construction of the nearby and vanishing cycle functors. One of the main results is a criterion for the compatibility of this functor with the proper pushforward functor of $\widetilde{\mathcal{D}}$-modules.

Throughout this chapter we use the following notation.

9.0.1. Notation.

- X denotes a complex manifold.
- H denotes a smooth hypersurface in X.
- Locally on H, we choose a decomposition $X = H \times \Delta_t$, where Δ_t is a small disc in $\mathbb{C}$ with coordinate t. We have the corresponding z-vector field $\widetilde{\partial}_t$.
- D denotes an effective divisor on X with support denoted by $|D|$. Locally on D, we choose a holomorphic function $g : X \to \mathbb{C}$ such that $D = (g)$. We then have $|D| = g^{-1}(0)$.
- Recall that $\widetilde{\mathcal{D}}_X$ means $\mathcal{D}_X$ or $R_F\mathcal{D}_X$ and, in the latter case, $\widetilde{\mathcal{D}}_X$-modules mean *graded* $\widetilde{\mathcal{D}}_X$-modules (see Chapter 8). We then use (k) for the shift by k of the grading (see Section 5.1.a). When the information on the grading is not essential, we just omit to indicate the corresponding shift. We use the convention that, whenever $\widetilde{\mathcal{D}}_X$ means $\mathcal{D}_X$, all conditions and statements relying on gradedness or strictness are understood to be empty or tautological.

9.0.2. Remark **(Left and right $\widetilde{\mathcal{D}}$-modules).** For various purposes, it is more convenient to work with right $\widetilde{\mathcal{D}}$-modules. However, left $\widetilde{\mathcal{D}}$-modules are more commonly used in applications. We will therefore mainly treat right $\widetilde{\mathcal{D}}$-modules and give the corresponding formulas for left $\widetilde{\mathcal{D}}$-modules in various remarks.

9.0.3. Remark **(Restriction to $z = 1$).** Throughout this chapter we keep the Convention 8.1.11. All the constructions can be done either for $\mathcal{D}_X$-modules or for graded $R_F\mathcal{D}_X$-modules, in which case a strictness assumption (strict $\mathbb{R}$-specializability) is most often needed. By "good behaviour with respect to the restriction $z = 1$", we mean that the restriction functor $\widetilde{\mathcal{M}} \mapsto \mathcal{M} := \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is compatible with the constructions. We will see that many, *but not all*, of the constructions in this chapter have good behaviour with respect to setting $z=1$. We will make this precise for each such construction.

9.1. Introduction

This chapter has one main purpose: Given a coherent $\widetilde{\mathcal{D}}_X$-module, to give a sufficient condition such that the restriction functor to a divisor D, producing a complex of $\widetilde{\mathcal{D}}_X$-modules supported on the divisor D which corresponds to the functor ${}_{\mathrm{D}}\iota_{H*}{}_{\mathrm{D}}\iota_H^*$ when $\iota_H : H \hookrightarrow X$ is the inclusion of a smooth hypersurface, gives rise to a complex of $\widetilde{\mathcal{D}}_X$-modules with coherent cohomology.

The property of being *specializable* along D will answer this first requirement. However, in the case where $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, strictness comes into play in a fundamental way in order to ensure a good behaviour. This leads to the notion of *strict specializability* along D. When forgetting the F-filtration, i.e., when considering $\mathcal{D}_X$-modules, the strictness condition is empty.

Given any holomorphic function g on X with associated divisor D and for every strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ along D, we introduce the nearby cycle $\widetilde{\mathcal{D}}_X$-modules $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ ($\lambda \in \mathbb{C}^*$ with $|\lambda| = 1$) and the vanishing cycle module $\phi_{g,1}\widetilde{\mathcal{M}}$. They are the "generalized restriction functors", which the usual restriction functors can be deduced from.

The construction is possible when the Kashiwara-Malgrange V-filtration exists on a given $\widetilde{\mathcal{D}}_X$-module. More precisely, the notion of V-filtration is well-defined in the case when D is a smooth divisor. We reduce to this case by considering, when more generally $D=(g)$, the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}$. The V-filtration can exist on the pushforward ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$. We then say that $\widetilde{\mathcal{M}}$ is strictly specializable along D.

Kashiwara's equivalence is an equivalence (via the pushforward functor $\iota_Y : Y \hookrightarrow X$) between the category of coherent $\mathcal{D}_Y$-modules and that of coherent $\mathcal{D}_X$-modules supported on the submanifold Y. When Y has codimension 1 in X, this equivalence can be extended as an equivalence between strict coherent $\widetilde{\mathcal{D}}_Y$-modules and coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along Y.

Complex Hodge modules will satisfy a property of semi-simplicity with respect to their support that we introduce in this chapter under the name of *strict S-decomposability* ("S" is for "support"). The support of a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is a closed analytic subspace in X. It may have various irreducible components. We introduce a condition that ensures the following to properties.

- The $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ decomposes as the direct sum of $\widetilde{\mathcal{D}}_X$-modules, each of which supported by a single component.
- Moreover, each such summand decomposes itself as the direct sum of $\widetilde{\mathcal{D}}_X$-modules, each of which supported on an irreducible closed analytic subset of the support of the given summand, in order to satisfy a "geometric simplicity property", namely each such new summand has no coherent sub- nor quotient module supported on a

strictly smaller closed analytic subset. We then say that such a summand has *pure support*.

In Section 9.8, we give a criterion in order that the functors $\psi_{g,\lambda}$ and $\phi_{g,1}$ commute with proper pushforward. This will be an essential step in the theory of complex Hodge modules (see Chapter 14), where we need to prove that the property of strict S-decomposability (i.e., geometric semi-simplicity) is preserved by projective pushforward.

9.2. The filtration $V_\bullet\widetilde{\mathcal{D}}_X$ relative to a smooth hypersurface

Let $H \subset X$ be a smooth hypersurface[(1)] of X with defining ideal $\mathcal{I}_H \subset \mathcal{O}_X$. Let us set $\widetilde{\mathcal{I}}_H^\ell = \widetilde{\mathcal{O}}_X$ for $\ell < 0$ and $\widetilde{\mathcal{I}}_H^\ell = \mathcal{I}_H^\ell\widetilde{\mathcal{O}}_X$ for $\ell \geqslant 0$. The sheaf of *logarithmic vector fields along* H, denoted by $\widetilde{\Theta}_X(\log H)$ is the subsheaf of the sheaf $\widetilde{\Theta}_X$ of holomorphic vector fields on X which preserve the ideal $\widetilde{\mathcal{I}}_H$. This is a sheaf of Lie sub-algebras of $\widetilde{\Theta}_X$. The subsheaf of algebras of $\widetilde{\mathcal{D}}_X$ generated by $\widetilde{\mathcal{O}}_X$ and $\widetilde{\Theta}_X(\log H)$ is called the *sheaf of logarithmic differential operators along* H.[(2)] We will denote it by $V_0\widetilde{\mathcal{D}}_X$. This sheaf shares many properties with $\widetilde{\mathcal{D}}_X$ (see Exercise 9.1).

In local coordinates $(t, x_2, \dots, x_n)$ where H has equation $t = 0$, a local section of $\widetilde{\Theta}_X(\log H)$ can be written as $a_1 t\widetilde{\partial}_t + a_2\widetilde{\partial}_{x_2} + \cdots + a_n\widetilde{\partial}_{x_n}$, where a_i are local sections of $\widetilde{\mathcal{O}}_X$. Local sections of $V_0\widetilde{\mathcal{D}}_X$ consist of local sections of $\widetilde{\mathcal{D}}_X$ expressed only with $t\widetilde{\partial}_t, \widetilde{\partial}_{x_2}, \widetilde{\partial}_{x_n}$. One can characterize sections of $V_0\widetilde{\mathcal{D}}_X$ on an open subset U of X as follows:

$$V_0\widetilde{\mathcal{D}}_X(U) = \{P \in \widetilde{\mathcal{D}}_X(U) \mid P \cdot \widetilde{\mathcal{I}}_H^j(U) \subset \widetilde{\mathcal{I}}_H^j(U),\ \forall\, j \in \mathbb{Z}\}.$$

This leads us to define a canonical increasing filtration of $\widetilde{\mathcal{D}}_X$ indexed by $\mathbb{Z}$. For every $k \in \mathbb{Z}$, the subsheaf $V_k\widetilde{\mathcal{D}}_X \subset \widetilde{\mathcal{D}}_X$ ($k \in \mathbb{Z}$) consists of operators P such that $P\widetilde{\mathcal{I}}_H^j \subset \widetilde{\mathcal{I}}_H^{j-k}$ for every $j \in \mathbb{Z}$. For every open subset U of X we thus have

$$V_k\widetilde{\mathcal{D}}_X(U) = \{P \in \widetilde{\mathcal{D}}_X(U) \mid P \cdot \widetilde{\mathcal{I}}_H^j(U) \subset \widetilde{\mathcal{I}}_H^{j-k}(U),\ \forall\, j \in \mathbb{Z}\}. \tag{9.2.1}$$

This defines an increasing filtration $V_\bullet\widetilde{\mathcal{D}}_X$ of $\widetilde{\mathcal{D}}_X$ indexed by $\mathbb{Z}$. Note that one can also define $V_k\widetilde{\mathcal{D}}_X(U)$ with the right action, that is, as the set of $Q \in \widetilde{\mathcal{D}}_X(U)$ such that $\widetilde{\mathcal{I}}_H^j(U) \cdot Q \subset \widetilde{\mathcal{I}}_H^{j-k}(U)$, $\forall\, j \in \mathbb{Z}$. See Exercise 9.3 for basic properties of $V_\bullet\widetilde{\mathcal{D}}_X$.

The *Euler vector field* E is the class E of $t\widetilde{\partial}_t$ in $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$ in some local product decomposition as in Exercise 9.3. See Exercise 9.4 for its basic properties. Let us insist on the fact that E only depends on H, not on the generator chosen in the ideal $\widetilde{\mathcal{I}}_H$.

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(1) Other settings can be considered, for example a smooth subvariety, or a finite family of smooth subvarieties, but they will not be needed for our purpose.

(2) The notation $\widetilde{\Theta}_X(-\log H)$ is also used, since such vector fields vanish along H.

9.2.2. Structure of $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$ ***and*** $\mathrm{gr}_V\widetilde{\mathcal{D}}_X$**.** What is the geometric meaning of the sheaf of rings $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$? A natural question is to relate the sheaf $\widetilde{\mathcal{D}}_H$ of differential operators on H with it. While $\widetilde{\mathcal{D}}_H$ can be identified to the quotient $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X/\mathrm{E}\,\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X = \mathrm{gr}_0^V\widetilde{\mathcal{D}}_X/\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X\,\mathrm{E}$, one cannot in general consider it as a subsheaf of $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$. This is related to the possible non-triviality of the normal bundle of H in X.

When H is globally defined by a holomorphic function g, Exercise 9.4(3) shows an identification $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X \simeq \widetilde{\mathcal{D}}_H[\mathrm{E}]$. More generally, for any effective divisor D defined by a holomorphic function $g : X \to \mathbb{C}$, we will often use the trick of the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}$ and we will then consider the filtration $V_\bullet\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$ with respect to $X \times \{0\}$, so that we will be able to identify $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$ with the ring $\widetilde{\mathcal{D}}_X[\mathrm{E}]$.

What about the sheaf $\mathrm{gr}_V\widetilde{\mathcal{D}}_X$? Let $\nu : N_HX \to H$ denote the normal bundle of H in X. Let us define the sheaf $\widetilde{\mathcal{D}}_{[N_HX]}$ of differential operators which are algebraic in the fibers of ν. We first consider the sheaf $\widetilde{\mathcal{O}}_{[N_HX]}$ on X of holomorphic functions which are algebraic in the fibers of ν. It is locally defined by using a local trivialization of ν as a product $X \times \mathbb{C}$, where $\mathbb{C}$ has coordinate t. Then $\widetilde{\mathcal{O}}_{[N_HX]} = \widetilde{\mathcal{O}}_X[t]$. For an intrinsic definitions, one extends in a canonical way ν as a projective fibration $\widetilde{\nu} : \mathbb{P}(N_HX \oplus \mathcal{O})$ with fibers $\mathbb{P}^1$ and we denote by X_∞ the section ∞ of this bundle. Then $\widetilde{\mathcal{O}}_{[N_HX]} := \widetilde{\nu}_*\widetilde{\mathcal{O}}_{\mathbb{P}(N_HX\oplus\mathcal{O})}(*X_\infty)$. Now, $\widetilde{\mathcal{D}}_{[N_HX]}$ is by definition the sheaf of differential operators with coefficients in $\widetilde{\mathcal{O}}_{[N_HX]}$. It is similarly equipped with its V-filtration $V_\bullet\widetilde{\mathcal{D}}_{[N_HX]}$. Then there is a canonical isomorphism (as graded objects) $\mathrm{gr}^V\widetilde{\mathcal{D}}_X \simeq \mathrm{gr}^V\widetilde{\mathcal{D}}_{[N_HX]}$, and the latter sheaf is isomorphic (forgetting the grading) to $\widetilde{\mathcal{D}}_{[N_HX]}$.

9.2.3. V***-filtration indexed by*** $A+\mathbb{Z}$**.** The following construction of extending the set of indices will prove useful. Let $A \subset (-1, 0]$ be a finite subset containing 0. Let us fix the numbering of $A+\mathbb{Z} = \{\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots\}$ which respect the order and such that $\alpha_0 = 0$. We thus have $1 = \alpha_{\#A}$. We denote by ${}^A V_\bullet\widetilde{\mathcal{D}}_X$ the filtration indexed by $A+\mathbb{Z}$ defined by ${}^A V_\alpha\widetilde{\mathcal{D}}_X := V_{[\alpha]}\widetilde{\mathcal{D}}_X$. We consider it as a filtration indexed by $\mathbb{Z}$ by using the previous order-preserving bijection $A+\mathbb{Z} \simeq \mathbb{Z}$. Since $[\alpha]+[\beta] \leqslant [\alpha+\beta]$, we have ${}^A V_\alpha\widetilde{\mathcal{D}}_X \cdot {}^A V_\beta\widetilde{\mathcal{D}}_X \subset {}^A V_{\alpha+\beta}\widetilde{\mathcal{D}}_X$, and on the other hand, ${}^A V_{\alpha_o}\widetilde{\mathcal{D}}_X = V_0\widetilde{\mathcal{D}}_X$. The Rees ring is $R_{{}^A V}\widetilde{\mathcal{D}}_X := \bigoplus_{k\in\mathbb{Z}} {}^A V_{\alpha_k}\widetilde{\mathcal{D}}_X v^k$. Note also that $\mathrm{gr}_\alpha^{{}^A V}\widetilde{\mathcal{D}}_X = 0$ if $\alpha \notin \mathbb{Z}$ and

$$\mathrm{gr}^{{}^A V}\widetilde{\mathcal{D}}_X = \bigoplus_{k\in\mathbb{Z}} \mathrm{gr}_{\alpha_k}^{{}^A V}\widetilde{\mathcal{D}}_X = \bigoplus_{k\in\#A\cdot\mathbb{Z}} \mathrm{gr}_{(k/\#A)}^V\widetilde{\mathcal{D}}_X.$$

It will sometimes be convenient to write, for short, $R_{{}^A V}\widetilde{\mathcal{D}}_X := \bigoplus_{\alpha\in A+\mathbb{Z}} {}^A V_\alpha\widetilde{\mathcal{D}}_X v^\alpha$.

9.2.4. Remark **(Restriction to** $z=1$**).** The V-filtration restricts well when setting $z = 1$, that is, $V_k\mathcal{D}_X = V_k\widetilde{\mathcal{D}}_X/(z-1)V_k\widetilde{\mathcal{D}}_X = V_k\widetilde{\mathcal{D}}_X/(z-1)\widetilde{\mathcal{D}}_X \cap V_k\widetilde{\mathcal{D}}_X$.

9.3. Specialization of coherent $\widetilde{\mathcal{D}}_X$-modules

In this section, we denote by H a smooth hypersurface of a complex manifold X and by t a local generator of $\mathcal{I}_H$. We use the definitions and notation of Section 9.2.

9.3.1. Caveat. In Subsections 9.3.a–9.3.c, when $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$, we will forget about the grading of the $\widetilde{\mathcal{D}}_X$-modules and morphisms involved, in order to keep the notation similar to the case of $\mathcal{D}_X$-modules. From Section 9.4, we will remember the shift of grading for various morphisms, in the case of $R_F\mathcal{D}_X$-modules (this shift has no effect in the case of $\mathcal{D}_X$-modules).

9.3.a. Coherent V-filtrations

The coherence of the Rees sheaf of rings $R_V\widetilde{\mathcal{D}}_X$ is proved in Exercise 9.9.

***9.3.2. Definition* (Coherent V-filtrations).** Let $\widetilde{\mathcal{M}}$ be a coherent right $\widetilde{\mathcal{D}}_X$-module. A V-filtration indexed by $\mathbb{Z}$ is a decreasing filtration $U_\bullet\widetilde{\mathcal{M}}$ which satisfies

$$V_k\widetilde{\mathcal{D}}_X \cdot U_\ell\widetilde{\mathcal{M}} \subset U_{\ell+k}\widetilde{\mathcal{M}} \qquad \forall\, k, \ell \in \mathbb{Z}.$$

In particular, each $U_\ell\widetilde{\mathcal{M}}$ is a right $V_0\widetilde{\mathcal{D}}_X$-module. We say that it is a *coherent V-filtration* if each $U_\ell\widetilde{\mathcal{M}}$ is $V_0\widetilde{\mathcal{D}}_X$-coherent, locally on X, there exists $\ell_o \geqslant 0$ such that, for all $k \geqslant 0$,

$$U_{-k-\ell_0}\widetilde{\mathcal{M}} = U_{-\ell_o}\widetilde{\mathcal{M}}t^k \quad \text{and} \quad U_{k+\ell_0}\widetilde{\mathcal{M}} = \sum_{j=0}^{k} U_{\ell_o}\widetilde{\mathcal{M}}\widetilde{\partial}_t^j.$$

The definition is similar for left $\widetilde{\mathcal{D}}_X$-modules.

***9.3.3. Remark* (Left and right).** In the following, it will be more natural to consider decreasing V-filtrations on left $\widetilde{\mathcal{D}}_X$-modules, mimicking the t-adic filtration on $\widetilde{\mathcal{O}}_X$, while the V-filtrations on right $\widetilde{\mathcal{D}}_X$-modules will remain increasing. In such a way, the formulas for the Bernstein polynomial below remain very similar. As usual, decreasing filtrations are denoted with an upper index. We will mainly work in the context of right $\widetilde{\mathcal{D}}$-modules, and we will give the main formulas in both cases. Let us insist, however, that both cases are interchanged naturally by the side changing functor (Exercise 9.28) and that the final formulas in terms of the functors ψ, ϕ are identical.

9.3.b. Specializable coherent $\widetilde{\mathcal{D}}_X$-modules. Let $H \subset X$ be a smooth hypersurface. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module and let m be a germ of section of $\widetilde{\mathcal{M}}$. In the following, we abuse notation by denoting $\mathrm{E} \in V_0\widetilde{\mathcal{D}}_X$ any local lifting of the Euler operator $\mathrm{E} \in \mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$, being understood that the corresponding formula does not depend on the choice of such a lifting.

9.3.4. Definition.

(1) A *weak Bernstein equation* for m is a relation (right resp. left case)

$$m \cdot (z^\ell b(\mathrm{E}) - P) = 0 \quad \text{resp. } (z^\ell b(\mathrm{E}) - P)m = 0, \tag{9.3.4 $*$}$$

where

- ℓ is some non-negative integer,
- $b(s)$ is a nonzero polynomial in a variable s with coefficients in $\mathbb{C}$, which takes the form $\prod_{\alpha\in A}(s - \alpha z)^{\nu_\alpha}$ for some finite subset $A \in \mathbb{C}$ (depending on m),
- P is a germ in $V_{-1}\widetilde{\mathcal{D}}_X$, i.e., $P = tQ = Q't$ with Q, Q' germs in $V_0\widetilde{\mathcal{D}}_X$.

(2) We say that $\widetilde{\mathcal{M}}$ is *specializable along* H if any germ of section of $\widetilde{\mathcal{M}}$ is the solution of some weak Bernstein equation (9.3.4 $*$).

9.3.5. Remark. The full subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X)$ consisting of $\widetilde{\mathcal{D}}_X$-modules which are specializable along H is abelian (see Exercises 9.18 and 9.19).

Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and specializable along H. According to Bézout, for every local section m of $\widetilde{\mathcal{M}}$, there exists a minimal polynomial

$$b_m(s) = \prod_{\alpha\in R(m)} (s-\alpha z)^{\nu_\alpha}, \quad R(m)\subset\widetilde{\mathbb{C}} \text{ finite},$$

giving rise to a weak Bernstein equation (9.3.4 $*$). We say that $\widetilde{\mathcal{M}}$ is $\mathbb{R}$*-specializable along* H if for every local section m, we have $R(m)\subset\mathbb{R}$. We then set:

$$\operatorname{ord}_H(m) = \max R(m), \quad \text{resp. } \operatorname{ord}_H(m) = \min R(m). \tag{9.3.6}$$

***9.3.7. Definition* (Filtration by the order along H).** Assume that $\widetilde{\mathcal{M}}$ is a right $\widetilde{\mathcal{D}}_X$-module. The *filtration by the order along* H is the increasing filtration $V_\bullet\widetilde{\mathcal{M}}_{x_o}$ indexed by $\mathbb{R}$ defined by

$$V_\alpha\widetilde{\mathcal{M}}_{x_o} = \{m\in\widetilde{\mathcal{M}}_{x_o} \mid \operatorname{ord}_{H,x_o}(m)\leqslant\alpha\}, \tag{9.3.8}$$

$$V_{<\alpha}\widetilde{\mathcal{M}}_{x_o} = \{m\in\widetilde{\mathcal{M}}_{x_o} \mid \operatorname{ord}_{H,x_o}(m)<\alpha\}. \tag{9.3.9}$$

We do not claim that it is a coherent V-filtration. The order filtration satisfies (see Exercise 9.17):

$$\forall\, k\in\mathbb{Z},\ \forall\,\alpha,\beta\in\mathbb{R},\quad V_\alpha\widetilde{\mathcal{M}}_{x_o}\cdot V_k\widetilde{\mathcal{D}}_{X,x_o}\subset V_{\alpha+k}\widetilde{\mathcal{M}}_{x_o}.$$

It is a filtration of $\widetilde{\mathcal{M}}$ by subsheaves $V_\alpha\widetilde{\mathcal{M}}$ of $V_0\widetilde{\mathcal{D}}_X$-modules. We set

$$\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}} := V_\alpha\widetilde{\mathcal{M}}/V_{<\alpha}\widetilde{\mathcal{M}}. \tag{9.3.10}$$

These are $\operatorname{gr}^V_0\widetilde{\mathcal{D}}_X$-modules. In particular, they are equipped with an action of the Euler field E. We already notice, as a preparation to strict $\mathbb{R}$-specializability, that they satisfy part of the strictness condition.

9.3.11. Lemma. *The $\operatorname{gr}^V_0\widetilde{\mathcal{D}}_X$-module $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}$ has no z-torsion.*

Proof. It is a matter of proving that, for a section m of $V_\alpha\widetilde{\mathcal{M}}$, if mz^j is a section of $V_{<\alpha}\widetilde{\mathcal{M}}$ for some $j\geqslant 0$, then so does m. But one checks in a straightforward way that, if P in Exercise 9.17 is equal to z^j, then the inequality there is an equality (with $k=0$). □

9.3.12. The case of left $\widetilde{\mathcal{D}}_X$-modules. Recall that the order of a local section m is defined as $\operatorname{ord}_H(m)=\min R(m)$. In Exercise 9.17 we have $\operatorname{ord}_{H,x_o}(Pm)\geqslant\operatorname{ord}_{H,x_o}(m)-k$. The filtration by the order along H is the decreasing filtration $V^\bullet\widetilde{\mathcal{M}}_{x_o}$ indexed by $\mathbb{R}$ defined by

$$V^\beta\widetilde{\mathcal{M}}_{x_o} = \{m\in\widetilde{\mathcal{M}}_{x_o} \mid \operatorname{ord}_{H,x_o}(m)\geqslant\beta\},$$
$$V^{>\beta}\widetilde{\mathcal{M}}_{x_o} = \{m\in\widetilde{\mathcal{M}}_{x_o} \mid \operatorname{ord}_{H,x_o}(m)>\beta\}.$$

The order filtration satisfies

$$\forall\, k \in \mathbb{Z},\ \forall\, \alpha, \beta \in \mathbb{R}, \quad V_k\widetilde{\mathcal{D}}_{X,x_o} \cdot V^\beta\widetilde{\mathcal{M}}_{x_o} \subset V^{\beta-k}\widetilde{\mathcal{M}}_{x_o}.$$

We set $\mathrm{gr}_V^\beta\widetilde{\mathcal{M}} := V^\beta\widetilde{\mathcal{M}}/V^{>\beta}\widetilde{\mathcal{M}}$. Lemma 9.3.11 also applies. See Exercise 9.28 for the side-changing properties.

9.3.c. Strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathcal{D}}_X$-modules. A drawback of the setting of Section 9.3.b is that we cannot ensure that the order filtration is *a coherent V-filtration.*

9.3.13. Lemma **(Kashiwara-Malgrange V-filtration).** *Let $\widetilde{\mathcal{M}}$ be an $\mathbb{R}$-specializable coherent $\widetilde{\mathcal{D}}_X$-module. Assume that, in the neighbourhood of any $x_o \in X$ there exists a coherent V-filtration $U_\bullet\widetilde{\mathcal{M}}$ with the following two properties:*

(1) *its minimal weak Bernstein polynomial $b_U(s) = \prod_{\alpha\in A(U)}(s-\alpha z)^{\nu_\alpha}$ satisfies $A(U) \subset (-1,0]$,*

(2) *for every k, $U_k\widetilde{\mathcal{M}}/U_{k-1}\widetilde{\mathcal{M}}$ has no z-torsion.*

Then such a filtration is unique and equal to the order filtration when considered indexed by integers, which is therefore a coherent V-filtration as such. It is called the Kashiwara-Malgrange filtration of $\widetilde{\mathcal{M}}$.

Proof. Assume $U_\bullet\mathcal{M}$ satisfies (1) and (2). Let m be a local section of $U_k\widetilde{\mathcal{M}}$ and let $U_\bullet(m\cdot\widetilde{\mathcal{D}}_X)$ be the V-filtration induced by $U_\bullet\widetilde{\mathcal{M}}$ on $m\cdot\widetilde{\mathcal{D}}_X$. By Exercise 9.12(1), it is a coherent V-filtration. There exists thus $k_o \geqslant 1$ such that $U_{k-k_o}(m\cdot\widetilde{\mathcal{D}}_X) \subset m\cdot V_{-1}\widetilde{\mathcal{D}}_X$. It follows that $R(m) \subset (A(U)+k)\cup\cdots\cup(A(U)+k-k_o+1)$ and thus $\mathrm{ord}_H m = \max R(m) \leqslant k$, so $m \subset V_k\widetilde{\mathcal{M}}$.

Conversely, assume m is a local section of $V_k\widetilde{\mathcal{M}}$. It is also a local section of $U_{k+k_o}\widetilde{\mathcal{M}}$ for some $k_o \geqslant 0$. Its class in $\mathrm{gr}^U_{k+k_o}\widetilde{\mathcal{M}}$ is annihilated both by $z^\ell b_m(\mathrm{E})$ and by $z^{\ell'}b_U(\mathrm{E}-(k+k_o)z)$ (for some $\ell,\ell' \geqslant 0$), so if $k_o > 0$, both polynomials have no common z-root, and this class is annihilated by some non-negative power of z, according to Bézout. By Assumption (2), it is zero, and m is a local section of $U_{k+k_o-1}\widetilde{\mathcal{M}}$, from which we conclude by induction that m is a local section of $U_k\widetilde{\mathcal{M}}$, as wanted. $\square$

9.3.14. Definition **(Strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules).** Assume that $\widetilde{\mathcal{M}}$ is $\mathbb{R}$-specializable along H. We say that it is *strictly $\mathbb{R}$-specializable* along H if

(1) there exists a finite set $A \subset (-1,0]$ such that the filtration by the order along H is a coherent V-filtration indexed by $A+\mathbb{Z}$,

and for some (or every) local decomposition $X \simeq H\times\Delta_t$,

(2) for every $\alpha < 0$, $t : \mathrm{gr}_\alpha^V\widetilde{\mathcal{M}} \to \mathrm{gr}_{\alpha-1}^V\widetilde{\mathcal{M}}$ is onto,

(3) for every $\alpha > -1$, $\widetilde{\partial}_t : \mathrm{gr}_\alpha^V\widetilde{\mathcal{M}} \to \mathrm{gr}_{\alpha+1}^V\widetilde{\mathcal{M}}(-1)$ is onto.

See Exercise 9.29 for the relation between Definition 9.3.14 and Lemma 9.3.13, and Exercise 9.20 for the equivalence between "some" and "every" in the definition above.

9.3.15. Remark **(The need of a shift).** We will now remember explicitly the grading in the case of $R_F\mathcal{D}_X$-modules. Recall (see (5.1.4) and (5.1.5 $**$)) that, given a graded object $M = \bigoplus_p M_p$ (with M_p in degree $-p$), we set $M(k) = \bigoplus_p M(k)_p$ with $M(k)_p = M_{p-k}$.

If we regard the actions of t and $\widetilde{\partial}_t$ as morphisms in $\mathsf{Mod}(\widetilde{\mathcal{D}}_H)$-modules, that is, graded morphisms of degree zero, we have to introduce a shift by -1 (see Remark 5.1.5) for the action of $\widetilde{\partial}_t$, which sends $F_p z^p$ to $F_{p+1} z^{p+1}$. The same shift has to be introduced for the action of E, as well as for that of $\mathrm{N} = (\mathrm{E} - \alpha z)$.

We have seen that, for strictly $\mathbb{R}$-specializable $R_F\mathcal{D}$-modules, the module $\mathrm{gr}_\alpha^V \widetilde{\mathcal{M}}$ are graded $R_F\mathcal{D}$-modules in a natural way. Let us emphasize that, in Definition 9.3.14(2) and (3),

- the morphism t is graded of degree zero,
- the morphism $\widetilde{\partial}_t$ is graded of degree one; this explains why we write 9.3.14(3) as
$$\widetilde{\partial}_t : \mathrm{gr}_\alpha^V \widetilde{\mathcal{M}} \xrightarrow{\sim} \mathrm{gr}_\alpha^V \widetilde{\mathcal{M}}(-1) \quad \text{for } \alpha > -1.$$

9.3.16. Proposition. *Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H. Then, every $\mathrm{gr}_\alpha^V \widetilde{\mathcal{M}}$ is a graded $\mathrm{gr}_0^V \widetilde{\mathcal{D}}_X$-module, and is strict as such (see Definition* 5.1.6*).*

Proof. Recall that, for a graded module, strictness is equivalent to absence of z-torsion (see Exercise 5.2(1)). Therefore, the second point follows from the first one and from Lemma 9.3.11.

Let us consider the first point. We first claim that a local section m of $\widetilde{\mathcal{M}}$ is a local section of $V_\alpha \widetilde{\mathcal{M}}$ if and only if it satisfies a relation
$$m \cdot b(\mathrm{E}) \in V_\alpha \widetilde{\mathcal{M}}$$
for some b with z-roots $\leqslant \alpha$. Indeed, if m is a local section of $V_\beta \widetilde{\mathcal{M}}$ with $\beta > \alpha$ and satisfying such a relation, the Bézout argument already used and the absence of z-torsion on each $\mathrm{gr}_\gamma^V \widetilde{\mathcal{M}}$ (Lemma 9.3.11) implies that m is a local section of $V_{<\beta} \widetilde{\mathcal{M}}$. Property 9.3.14(1) implies that there is only a finite set of jumps of the V-filtration between α and β, so by induction we conclude that $m \in V_\alpha \widetilde{\mathcal{M}}$. The converse is clear.

The grading on $\widetilde{\mathcal{M}}$ induces a natural left action of $z\partial_z$ on $\widetilde{\mathcal{M}}$: for a local section $m = \bigoplus_p m_p$ of $\widetilde{\mathcal{M}} = \bigoplus_p \widetilde{\mathcal{M}}^p$, we set $z\partial_z m := \bigoplus_p p m_p$. We define a right action of $-\partial_z z$ by the trick of Exercise 8.17: we set $m(-\partial_z z) := z\partial_z m$. This action is natural in the sense that it satisfies the usual commutation relations with the right action of $\widetilde{\mathcal{D}}_X$. We claim that, for every $\alpha \in \mathbb{R}$, we have $V_\alpha \widetilde{\mathcal{M}}(-\partial_z z) \subset V_\alpha \widetilde{\mathcal{M}}$. Let m be a local section of $V_\alpha \widetilde{\mathcal{M}}$, which satisfies a relation $m b_m(\mathrm{E}) = m \cdot P$ with $P \in V_{-1} \widetilde{\mathcal{D}}_X$. Then one checks that
$$\begin{aligned} m(-\partial_z z) b_m(\mathrm{E}) &= m b_m(\mathrm{E})(-\partial_z z) + mQ, \quad Q \in V_0 \widetilde{\mathcal{D}}_X \\ &= mP(-\partial_z z) + mQ, \quad P \in V_{-1} \widetilde{\mathcal{D}}_X \\ &= m(-\partial_z z)P + mR, \quad R \in V_0 \widetilde{\mathcal{D}}_X. \end{aligned}$$
We conclude that $m(-\partial_z z) \in V_\alpha \widetilde{\mathcal{M}}$ by applying the first claim above.

Since the eigenvalues of $(-\partial_z z)$ on $\widetilde{\mathcal{M}}$ are integers and are simple, the same property holds for $V_\alpha \widetilde{\mathcal{M}}$, showing that $V_\alpha \widetilde{\mathcal{M}}$ decomposes as the direct sum of its $(-\partial_z z)$-eigenspaces, which are its graded components of various degrees. $\square$

9.3.17. Caveat. For a morphism φ between $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along H, the kernel and cokernel of φ, while being $\mathbb{R}$-specializable along H, need not be strictly $\mathbb{R}$-specializable. See Exercises 9.22, 9.26, 9.32, as well as Definition 9.3.28 and Caveat 9.3.29 for further properties.

9.3.18. Remark **(The case of left $\widetilde{\mathcal{D}}_X$-modules).** For left $\widetilde{\mathcal{D}}_X$-modules, we take $\beta > -1$ in 9.3.14(2) and $\beta < 0$ in 9.3.14(3) for $\mathrm{gr}_V^\beta \widetilde{\mathcal{M}}$. The nilpotent endomorphism N of $\mathrm{gr}_V^\beta \widetilde{\mathcal{M}}$ is induced by the action of $-(\mathrm{E} - \beta z)$.

9.3.19. Side-changing. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_X \otimes \widetilde{\mathcal{M}}$ denote the associated right $\widetilde{\mathcal{D}}_X$-module. Let us assume that H is defined by one equation $g = 0$, so that $\mathrm{gr}_V^\beta \widetilde{\mathcal{M}}$ and $\mathrm{gr}^V_\alpha \widetilde{\mathcal{M}}^{\mathrm{right}}$ are respectively left and right $\widetilde{\mathcal{D}}_H$-modules equipped with an action of E (see Exercise 9.4(3)).

Assume first that $\widetilde{\mathcal{M}} = \widetilde{\mathcal{O}}_X$ and $\widetilde{\mathcal{M}}^{\mathrm{right}} = \widetilde{\omega}_X$. We have

$$V^k \widetilde{\mathcal{O}}_X = \begin{cases} \widetilde{\mathcal{O}}_X & \text{if } k \leqslant 0, \\ g^k \widetilde{\mathcal{O}}_X & \text{if } k \geqslant 0, \end{cases} \qquad \text{and} \qquad V_k \widetilde{\omega}_X = \begin{cases} \widetilde{\omega}_X & \text{if } k \geqslant -1, \\ g^{-(k+1)} \widetilde{\omega}_X & \text{if } k \leqslant -1. \end{cases}$$

We have $\mathrm{gr}^V_{-1} \widetilde{\omega}_X = \widetilde{\omega}_H \otimes \mathrm{d}g/z$, so that $\mathrm{d}g/z$ induces an isomorphism (see Remark 5.1.5)

$$\widetilde{\omega}_H(-1) \xrightarrow{\sim} \mathrm{gr}^V_{-1} \widetilde{\omega}_X, \quad \text{that is,} \quad \mathrm{gr}^V_{-1}(\widetilde{\mathcal{O}}_X^{\mathrm{right}}) \simeq (\mathrm{gr}_V^0 \widetilde{\mathcal{O}}_X)^{\mathrm{right}}(-1).$$

Arguing similarly for $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}^{\mathrm{right}}$ gives an identification

$$\mathrm{gr}^V_\alpha(\widetilde{\mathcal{M}}^{\mathrm{right}}) \simeq (\mathrm{gr}_V^\beta \widetilde{\mathcal{M}})^{\mathrm{right}}(-1), \quad \beta = -\alpha - 1.$$

With this identification, the actions of E (resp. N) on both sides coincide.

9.3.20. Proposition. *Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H. Then, in any local decomposition $X \simeq H \times \Delta_t$ we have*

(a) $\quad \forall\, \alpha < 0,\ t : V_\alpha \widetilde{\mathcal{M}} \longrightarrow V_{\alpha-1} \widetilde{\mathcal{M}}$ *is an isomorphism;*

(b) $\quad \forall\, \alpha \geqslant 0,\ V_\alpha \widetilde{\mathcal{M}} = V_{<\alpha} \widetilde{\mathcal{M}} + (V_{\alpha-1} \widetilde{\mathcal{M}}) \widetilde{\partial}_t;$

(c) $\quad t : \mathrm{gr}^V_\alpha \widetilde{\mathcal{M}} \longrightarrow \mathrm{gr}^V_{\alpha-1} \widetilde{\mathcal{M}}$ *is* $\begin{cases} \textit{an isomorphism} & \textit{if } \alpha < 0, \\ \textit{injective} & \textit{if } \alpha > 0; \end{cases}$

(d) $\quad \widetilde{\partial}_t : \mathrm{gr}^V_\alpha \widetilde{\mathcal{M}} \longrightarrow \mathrm{gr}^V_{\alpha+1} \widetilde{\mathcal{M}}(-1)$ *is* $\begin{cases} \textit{an isomorphism} & \textit{if } \alpha > -1, \\ \textit{injective} & \textit{if } \alpha < -1; \end{cases}$

In particular (from (b)*), $\widetilde{\mathcal{M}}$ is generated as a $\widetilde{\mathcal{D}}_X$-module by $V_0 \widetilde{\mathcal{M}}$.*

Proof. Because $V_{\alpha+\bullet} \widetilde{\mathcal{M}}$ is a coherent V-filtration, (a) holds for $\alpha \ll 0$ locally and (b) for $\alpha \gg 0$ locally. Therefore, (a) follows from (c) and (b) follows from (d). By 9.3.14(2) (resp. (3)), the map in (c) (resp. (d)) is onto. The composition $t \widetilde{\partial}_t = (\mathrm{E} - \alpha z) + \alpha z$ is injective on $\mathrm{gr}^V_\alpha \widetilde{\mathcal{M}}$ for $\alpha \neq 0$ since $(\mathrm{E} - \alpha z)$ is nilpotent and $\mathrm{gr}^V_\alpha \widetilde{\mathcal{M}}$ is strict, hence (c) holds. The argument for (d) is similar. $\square$

In Exercises 9.32 and 9.33, we explain which set of data is needed to recover coherent $V_0\widetilde{\mathcal{D}}_X$-modules and morphisms between them. This will be used from a more general point of view in Chapter 11.

Assume that $X = H \times \Delta_t$. Consider the category whose objects consist of the data $(\widetilde{\mathcal{M}}_{\leqslant -1}, \widetilde{\mathcal{M}}_0, \mathrm{c}, \mathrm{v})$, where

- $\widetilde{\mathcal{M}}_{\leqslant -1}$ is a coherent $V_0\widetilde{\mathcal{D}}_X$-module on which t is torsion-free and such that $\widetilde{\mathcal{M}}_{-1} := \widetilde{\mathcal{M}}_{\leqslant -1}/\widetilde{\mathcal{M}}_{\leqslant -1}t$ is strict and the induced action of $\widetilde{\partial}_t t$ on it is nilpotent with index of nilpotence locally bounded on H,
- $\widetilde{\mathcal{M}}_0$ is a strict coherent $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-module on which the action of $t\widetilde{\partial}_t$ is nilpotent with index of nilpotence locally bounded on H,
- the data c, v are $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-linear morphisms

$$\begin{tikzcd} \widetilde{\mathcal{M}}_{-1} \arrow[r, bend left, "\mathrm{c}\quad(-1)"] & \widetilde{\mathcal{M}}_0 \arrow[l, bend left, "\mathrm{v}"] \end{tikzcd}$$

such that $\mathrm{c}\circ\mathrm{v} = \widetilde{\partial}_t t$ on $\widetilde{\mathcal{M}}_{-1}$ and $\mathrm{v}\circ\mathrm{c} = t\widetilde{\partial}_t$ on $\widetilde{\mathcal{M}}_0$.

Morphisms in this category consist of pairs $(\varphi_{\leqslant -1}, \varphi_0)$, where $\varphi_{\leqslant -1} : \widetilde{\mathcal{M}}_{\leqslant -1} \to \widetilde{\mathcal{N}}_{\leqslant -1}$ is $V_0\widetilde{\mathcal{D}}_X$-linear, $\varphi_0 : \widetilde{\mathcal{M}}_0 \to \widetilde{\mathcal{N}}_0$ is $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-linear, and the restriction φ_{-1} of $\varphi_{\leqslant -1}$ to $\widetilde{\mathcal{M}}_{-1}$ satisfies

$$\mathrm{c}\circ\varphi_{-1} = \varphi_0\circ\mathrm{c}, \quad \varphi_{-1}\circ\mathrm{v} = \mathrm{v}\circ\varphi_0.$$

We have a functor from the category of coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along H to the above category:

$$\widetilde{\mathcal{M}} \longmapsto (V_{-1}\widetilde{\mathcal{M}}, \mathrm{gr}_0^V\widetilde{\mathcal{M}}, \widetilde{\partial}_t, t).$$

***9.3.21. Corollary* (Recovering morphisms from their restriction to V_{-1} and gr_0^V)**

This functor is fully faithful, i.e., any morphism $(\varphi_{\leqslant -1}, \varphi_0)$ can be lifted in a unique way as a morphism φ.

***Proof*.** Consider the category whose objects are coherent $V_0\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}_{\leqslant 0}$ such that

- $\widetilde{\mathcal{M}}_{\leqslant 0}/\widetilde{\mathcal{M}}_{\leqslant 0}t$ is strict,
- $t\widetilde{\partial}_t$ acting on $\widetilde{\mathcal{M}}_{\leqslant 0}/\widetilde{\mathcal{M}}_{\leqslant 0}t$ has a minimal polynomial with roots αz satisfying $\alpha \in (-1, 0]$,
- defining $V_\alpha\widetilde{\mathcal{M}}$ for $\alpha < 0$ as in Exercise 9.29, every $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_{\leqslant 0}$ is strict and 9.3.20(a) holds,

and whose morphisms are $V_0\widetilde{\mathcal{D}}_X$-linear morphisms such that the following diagram commutes:

(9.3.22)
$$\begin{tikzcd} V_{-1}\widetilde{\mathcal{M}}_{1,\leqslant 0} \arrow[r, "\varphi_{\leqslant 0}"] \arrow[d, "\widetilde{\partial}_t"'] & V_{-1}\widetilde{\mathcal{M}}_{2,\leqslant 0} \arrow[d, "\widetilde{\partial}_t"] \\ \widetilde{\mathcal{M}}_{1,\leqslant 0} \arrow[r, "\varphi_{\leqslant 0}"] & \widetilde{\mathcal{M}}_{2,\leqslant 0} \end{tikzcd}$$

According to Exercise 9.32, the functor $\widetilde{\mathcal{M}} \mapsto \widetilde{\mathcal{M}}_{\leqslant 0} := V_0\widetilde{\mathcal{M}}$ is fully faithful. Now, the functor $\widetilde{\mathcal{M}}_{\leqslant 0} \mapsto (V_{-1}\widetilde{\mathcal{M}}_{\leqslant 0}, \mathrm{gr}_0^V\widetilde{\mathcal{M}}_{\leqslant 0}, \widetilde{\partial}_t, t)$ is an *equivalence of categories*. Indeed, Exercise 9.33 shows that it is essentially surjective and, since the reconstruction is functorial in an obvious way, it enables one to lift in a unique way a pair $(\varphi_{\leqslant -1}, \varphi_0)$ as a $V_0\widetilde{\mathcal{D}}_X$-linear morphism $\varphi_{\leqslant 0}$ such that (9.3.22) commutes, showing the full faithfulness. $\square$

***9.3.23. Remark* (Restriction to $z = 1$).** Let us keep the notation of Exercise 9.27. For a coherent $\mathcal{D}_X$-module $\mathcal{M}$ which is $\mathbb{R}$-specializable, 9.3.14(2) and (3) are automatically satisfied. Moreover, the morphisms in 9.3.20(c) and (d) are isomorphisms for the given values of α. In other words, for coherent $\mathcal{D}_X$-modules, being $\mathbb{R}$-specializable is equivalent to being strictly $\mathbb{R}$-specializable. In particular, Exercise 9.27 applies to coherent $R_F\mathcal{D}_X$-modules which are strictly $\mathbb{R}$-specializable along H.

We note that, if $z = 1$, the morphisms in 9.3.20(c) and (d) are isomorphisms in all cases. We introduce now a stronger notion, but see Remark 9.3.26 below.

***9.3.24. Definition* (Strong strict $\mathbb{R}$-specializability).** We will say that $\widetilde{\mathcal{M}}$ is *strongly strictly $\mathbb{R}$-specializable along H* if it is strictly $\mathbb{R}$-specializable and for any $\alpha \in \mathbb{R}$, the morphisms $t : \mathrm{gr}_\alpha^V\widetilde{\mathcal{M}} \to \mathrm{gr}_{\alpha-1}^V\widetilde{\mathcal{M}}$ and the morphism $\widetilde{\partial}_t : gr_\alpha^V\widetilde{\mathcal{M}} \to \mathrm{gr}_{\alpha+1}^V\widetilde{\mathcal{M}}(-1)$ are strict.

9.3.25. Lemma. *If $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along H, the morphisms in* 9.3.20(c) *and* (d) *are isomorphisms in all cases.*

***Proof*.** In such a case, taking Coker commutes with restricting to $z = 1$. We are thus reduced to proving that, for a strict $\widetilde{\mathcal{D}}_H$-module $\widetilde{\mathcal{N}}$, we have $\mathcal{N} := \widetilde{\mathcal{N}}/(z-1)\widetilde{\mathcal{N}} = 0$ if and only if $\widetilde{\mathcal{N}} = 0$. This follows from the property that $\widetilde{\mathcal{N}} = R_F\mathcal{N}$ for some filtration $F_\bullet\mathcal{N}$ (Exercise 5.2(5)). $\square$

9.3.26. Remark. See Exercise 9.23 for a characterization of strong strict $\mathbb{R}$-specializability. A priori, strong strict $\mathbb{R}$-specializability is stronger than strict $\mathbb{R}$-specializability. In the application to pure or mixed Hodge modules, we will see that the morphisms $t : \mathrm{gr}_\alpha^V\widetilde{\mathcal{M}} \to \mathrm{gr}_{\alpha-1}^V\widetilde{\mathcal{M}}$ and $\widetilde{\partial}_t : gr_\alpha^V\widetilde{\mathcal{M}} \to \mathrm{gr}_{\alpha+1}^V\widetilde{\mathcal{M}}(-1)$ underlie morphisms of mixed Hodge modules, and their strictness will follow from this property. As a consequence, in the category of Hodge modules, strong strict $\mathbb{R}$-specializability will be fulfilled although only strict $\mathbb{R}$-specializability will be assumed.

***9.3.27. Example* (Morphisms inducing an isomorphism on $V_{<0}$)**

Assume that $X = H \times \Delta_t$. Let $\widetilde{\mathcal{M}}, \widetilde{\mathcal{N}}$ be strictly $\mathbb{R}$-specializable along H and let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a $\widetilde{\mathcal{D}}_X$-linear morphism. Since φ is also $V_0\widetilde{\mathcal{D}}_X$-linear, it induces a morphism $\widetilde{\mathcal{M}}/V_{\alpha_o}\widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}/V_{<\alpha_o}\widetilde{\mathcal{N}}$ for each α_o, which decomposes with respect to the decomposition 9.34(2). Each summand is then identified with $\mathrm{gr}_\alpha^V\varphi$. We will consider more specifically the case where φ induces an isomorphism on $V_{<0}$.

We first claim that this condition implies that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are supported on H, that is, every local section of $\operatorname{Ker}\varphi, \operatorname{Coker}\varphi$ is annihilated by some power of t (due to the $\widetilde{\mathcal{D}}_H$-coherence of these modules). For $\operatorname{Ker}\varphi$, this follows from $\operatorname{Ker}\varphi\cap V_{<0}\widetilde{\mathcal{M}}=0$ together with the property that t is nilpotent on $\widetilde{\mathcal{M}}/V_{<0}\widetilde{\mathcal{M}}$. For $\operatorname{Coker}\varphi$, we note that any local section n of $\widetilde{\mathcal{N}}$ satisfies $nt^k\in V_{<0}\widetilde{\mathcal{N}}=\varphi(V_{<0}\widetilde{\mathcal{M}})$ for some k, hence t^k is nilpotent on $\operatorname{Coker}\varphi$.

The decomposition 9.34(2) induces decompositions $\operatorname{Ker}\varphi=\bigoplus_{k\geqslant0}\operatorname{Ker}\operatorname{gr}_k^V\varphi$ and $\operatorname{Coker}\varphi=\bigoplus_{k\geqslant0}\operatorname{Coker}\operatorname{gr}_k^V\varphi$ as $V_0\widetilde{\mathcal{D}}_X$-modules. Moreover, since E acts as 0 on $\operatorname{Ker}\operatorname{gr}_0^V\varphi,\operatorname{Coker}\operatorname{gr}_0^V\varphi$, the obstruction in 9.34(6) (adapted to the present setting) to extending the $V_0\widetilde{\mathcal{D}}_X$-structure to a $\widetilde{\mathcal{D}}_X$-structure vanishes, and we conclude that the $\widetilde{\mathcal{D}}_X$-module $\operatorname{Ker}\varphi$, resp. $\operatorname{Coker}\varphi$, is identified with the $\widetilde{\mathcal{D}}_X$-module ${}_{\mathrm{D}}i_{H*}\operatorname{Ker}\operatorname{gr}_0^V\varphi$, resp. ${}_{\mathrm{D}}i_{H*}\operatorname{Coker}\operatorname{gr}_0^V\varphi$.

***9.3.28. Definition* (Strictly $\mathbb{R}$-specializable morphisms).** A morphism φ between strictly $\mathbb{R}$-specializable coherent left $\widetilde{\mathcal{D}}_X$-modules is said to be *strictly $\mathbb{R}$-specializable* if for every $\alpha\in[-1,0]$, the induced morphism $\operatorname{gr}_\alpha^V\varphi$ is *strict* (i.e., its cokernel is strict), and a similar property for right modules.

9.3.29. Caveat. The composition of strictly $\mathbb{R}$-specializable morphisms need not be strictly $\mathbb{R}$-specializable (see Caveat 5.1.7).

9.3.30. Proposition. *If φ is strictly $\mathbb{R}$-specializable, then $\operatorname{gr}_\alpha^V\varphi$ is strict for every $\alpha\in\mathbb{R}$, and $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$ and $\operatorname{Coker}\varphi$ are strictly $\mathbb{R}$-specializable along H and their V-filtrations are given by*

$$V_\alpha\operatorname{Ker}\varphi=V_\alpha\widetilde{\mathcal{M}}\cap\operatorname{Ker}\varphi,\quad V_\alpha\operatorname{Coker}\varphi=\operatorname{Coker}(\varphi_{|V_\alpha\widetilde{\mathcal{M}}}),$$
$$V_\alpha\operatorname{Im}\varphi=\operatorname{Im}(\varphi_{|V_\alpha\widetilde{\mathcal{M}}})=V_\alpha\widetilde{\mathcal{N}}\cap\operatorname{Im}\varphi. \tag{9.3.30 $*$}$$

Proof. Let us equip $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ with the filtration $U_\bullet$ naturally induced by $V_\bullet\widetilde{\mathcal{M}},V_\bullet\widetilde{\mathcal{N}}$. By using 9.3.20(c) and (d) for $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$, we find that $\operatorname{gr}_\alpha^U\operatorname{Ker}\varphi$ and $\operatorname{gr}_\alpha^U\operatorname{Coker}\varphi$ are strict for every $\alpha\in\mathbb{R}$. By the uniqueness of the V-filtration, the first line in (9.3.30 $*$) holds, and therefore all properties of Definition 9.3.14 hold for $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$. Now, $\operatorname{Im}\varphi$ has two possible coherent V-filtrations, one induced by $V_\bullet\widetilde{\mathcal{N}}$ and the other one being the image of $V_\bullet\widetilde{\mathcal{M}}$. For the first one, strictness of $\operatorname{gr}_\alpha\operatorname{Im}\varphi$ holds, hence $\operatorname{Im}\varphi$ is strictly $\mathbb{R}$-specializable and $V_\alpha\operatorname{Im}\varphi=\operatorname{Im}\varphi\cap V_\alpha\widetilde{\mathcal{N}}$. For the second one $U_\alpha\operatorname{Im}\varphi$, $\operatorname{gr}_\alpha^U\operatorname{Im}\varphi$ is identified with the image of $\operatorname{gr}_\alpha^V\varphi$, hence is also strict, so $U_\bullet\operatorname{Im}\varphi$ is also equal to $V_\bullet\operatorname{Im}\varphi$. ∎

9.3.31. Corollary. *Let $\widetilde{\mathcal{M}}^\bullet=\{\cdots\xrightarrow{d_i}\widetilde{\mathcal{M}}^i\xrightarrow{d_{i+1}}\cdots\}$ be a complex bounded above whose terms are $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Assume that, for every $\alpha\in[-1,0]$, the graded complex $\operatorname{gr}_\alpha^V\widetilde{\mathcal{M}}^\bullet$ is strict, i.e., its cohomology is strict. Then each differential d_i and each $H^i\widetilde{\mathcal{M}}^\bullet$ is strictly $\mathbb{R}$-specializable along H and $\operatorname{gr}_\alpha^V$ commutes with taking cohomology.*

Proof. By using 9.3.20(c) and (d) for each term of the complex $\operatorname{gr}_\alpha^V\widetilde{\mathcal{M}}^\bullet$, we find that strictness of the cohomology holds for every $\alpha\in\mathbb{R}$. We argue by decreasing induction. Assume $\widetilde{\mathcal{M}}^{k+1}=0$. Then the assumption implies that $d_k:\widetilde{\mathcal{M}}^{k-1}\to\widetilde{\mathcal{M}}^k$ is strictly $\mathbb{R}$-specializable, so we can apply Proposition 9.3.30 to it. We then replace the complex by $\cdots \widetilde{\mathcal{M}}^{k-2} \xrightarrow{d_{k-1}} \operatorname{Ker} d_k \to 0$ and apply the inductive assumption. Moreover, the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}^k/\operatorname{Ker} d_k \simeq \operatorname{Im} d_{k+1}$ implies that of d_{k-1}. $\square$

***9.3.32. Definition* (Strictly $\mathbb{R}$-specializable W-filtered $\widetilde{\mathcal{D}}_X$-module)**
Let $(\widetilde{\mathcal{M}}, W_\bullet\widetilde{\mathcal{M}})$ be a coherent $\widetilde{\mathcal{D}}_X$-module equipped with a locally finite filtration by coherent $\widetilde{\mathcal{D}}_X$-submodules. We say that $(\widetilde{\mathcal{M}}, W_\bullet\widetilde{\mathcal{M}})$ is a *strictly $\mathbb{R}$-specializable filtered $\widetilde{\mathcal{D}}_X$-module (along H)* if each $W_\ell\widetilde{\mathcal{M}}$ and each $\mathrm{gr}_\ell^W\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable.

9.3.33. Lemma. *Let $(\widetilde{\mathcal{M}}, W_\bullet\widetilde{\mathcal{M}})$ be a strictly $\mathbb{R}$-specializable filtered $\widetilde{\mathcal{D}}_X$-module. Then each $W_\ell\widetilde{\mathcal{M}}/W_k\widetilde{\mathcal{M}}$ $(k < \ell)$ is strictly $\mathbb{R}$-specializable along H.*

Proof. By induction on $\ell - k \geqslant 1$, the case $\ell - k = 1$ holding true by assumption. Let $U_\bullet(W_\ell\widetilde{\mathcal{M}}/W_k\widetilde{\mathcal{M}})$ be the V-filtration naturally induced by $V_\bullet W_\ell\widetilde{\mathcal{M}}$. It is a coherent filtration. By induction we have $U_\bullet(W_{\ell-1}\widetilde{\mathcal{M}}/W_k\widetilde{\mathcal{M}}) = V_\bullet(W_{\ell-1}\widetilde{\mathcal{M}}/W_k\widetilde{\mathcal{M}})$ and $U_\bullet\mathrm{gr}_\ell^W\widetilde{\mathcal{M}} = V_\bullet\mathrm{gr}_\ell^W\widetilde{\mathcal{M}}$. Similarly, $V_\bullet W_\ell\widetilde{\mathcal{M}} \cap W_{\ell-1}\widetilde{\mathcal{M}} = V_\bullet W_{\ell-1}\widetilde{\mathcal{M}}$. We conclude that the sequence

$$0 \longrightarrow \mathrm{gr}_\bullet^V(W_{\ell-1}\widetilde{\mathcal{M}}/W_k\widetilde{\mathcal{M}}) \longrightarrow \mathrm{gr}_\bullet^U(W_\ell\widetilde{\mathcal{M}}/W_k\widetilde{\mathcal{M}}) \longrightarrow \mathrm{gr}_\bullet^V\mathrm{gr}_\ell^W\widetilde{\mathcal{M}} \longrightarrow 0$$

is exact, hence the strictness of the middle term. $\square$

9.4. Nearby and vanishing cycle functors

***9.4.1. Definition* (Strict $\mathbb{R}$-specializability along D).** Let D be an effective divisor in X and let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. We say that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along D if for some (or any) local equation g defining D, denoting by $X \xhookrightarrow{\iota_g} X \times \mathbb{C}$ the graph inclusion of g, ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $X \times \{0\}$.

In order to justify this definition, one has to check that the condition does not depend on the choice of g. If $u(x)$ is a local invertible function, consider the isomorphism $\varphi_u : (x,t) \mapsto (x, u(x)t)$. Then $\iota_{ug} = \varphi_u \circ \iota_g$, and one deduces the assertion from the property that ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $(u(x)t)$ (see Exercise 9.20).

***9.4.2. Remark* (strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$).** While $\mathcal{O}_X$ and ω_X are $\mathbb{R}$-specializable along any divisor D, as provided by the theory of the Bernstein-Sato polynomial, the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$ does not follow from that theory. It relies on Hodge properties and will only be obtained in Section 14.6, as a particular case of Theorem 14.6.1.

***9.4.3. Definition* (Nearby and vanishing cycle functors).** Assume that $\widetilde{\mathcal{M}}$ is coherent and strictly $\mathbb{R}$-specializable along (g). We then set

- (Left case)

$$(9.4.3\,*)\qquad \begin{cases} \psi_{g,\lambda}\widetilde{\mathcal{M}}^{\mathrm{left}} := \mathrm{gr}_V^\beta({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{left}}), & \lambda = \exp(-2\pi\mathsf{i}\,\beta),\ \beta \in (-1, 0], \\ \phi_{g,1}\widetilde{\mathcal{M}}^{\mathrm{left}} := \mathrm{gr}_V^{-1}({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{left}})(-1). \end{cases}$$

- (Right case)

$$\begin{cases}\psi_{g,\lambda}\widetilde{\mathcal{M}} := \mathrm{gr}_\alpha^V({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})(1), & \lambda = \exp(2\pi\mathrm{i}\,\alpha),\ \alpha \in [-1,0),\\ \phi_{g,1}\widetilde{\mathcal{M}} := \mathrm{gr}_0^V({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}).\end{cases} \tag{9.4.3 $**$}$$

Then $\psi_{g,\lambda}\widetilde{\mathcal{M}}, \phi_{g,1}\widetilde{\mathcal{M}}$ are $\widetilde{\mathcal{D}}_X$-modules supported on $g^{-1}(0)$, equipped with an endomorphism E induced by $t\widetilde{\partial}_t$. We set

$$\mathrm{N} = \begin{cases}-(\mathrm{E}-\beta z) & \text{in the left case,}\\ (\mathrm{E}-\alpha z) & \text{in the right case.}\end{cases}$$

***9.4.4. Remark* (Choice of the shift).** The choice of a shift (-1) for $\phi_{g,1}$ in the left case has already been justified in dimension 1 (see (7.2.16)). In the right case, the choice of a shift (1) for $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and no shift for $\phi_{g,1}\widetilde{\mathcal{M}}$ is justified by the following examples.

(1) If $\widetilde{\mathcal{M}} = \widetilde{\omega}_{X\times\mathbb{C}}$ we have $\mathrm{gr}_{-1}^V\widetilde{\omega}_{X\times\mathbb{C}}(1) \simeq \widetilde{\omega}_X$ by identifying $\widetilde{\omega}_{X\times\mathbb{C}}$ with $\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_{X\times\mathbb{C}}\,\mathrm{d}t/z$ (see Remark 9.3.19).

(2) If $\widetilde{\mathcal{M}}$ is a right $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-module of the form ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{N}}$ where $\widetilde{\mathcal{N}}$ is a right $\widetilde{\mathcal{D}}_{X\times\{0\}}$-module and $\iota: X\times\{0\} \hookrightarrow X\times\mathbb{C}$ is the inclusion, then $\mathrm{gr}_0^V\widetilde{\mathcal{M}} = \widetilde{\mathcal{N}}$.

***9.4.5. Lemma* (Side-changing for the nearby/vanishing cycle functors)**

The side-changing functor commutes with the nearby/vanishing cycle functors, namely

$$\psi_{g,\lambda}(\widetilde{\mathcal{M}}^{\mathrm{right}}) = (\psi_{g,\lambda}\widetilde{\mathcal{M}}^{\mathrm{left}})^{\mathrm{right}}, \quad \phi_{g,1}(\widetilde{\mathcal{M}}^{\mathrm{right}}) = (\phi_{g,1}\widetilde{\mathcal{M}}^{\mathrm{left}})^{\mathrm{right}}.$$

It is moreover compatible with the actions of N.

Proof. If $\widetilde{\mathcal{N}}$ is a left $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-module which is strictly $\mathbb{R}$-specializable along $X\times\{0\}$, we have (see Remark 9.3.19)

$$\mathrm{gr}_\alpha^V(\widetilde{\omega}_{X\times\mathbb{C}}\otimes\widetilde{\mathcal{N}})(1) \simeq \widetilde{\omega}_X\otimes\mathrm{gr}_V^\beta(\widetilde{\mathcal{N}}) \quad \forall\,\alpha\in\mathbb{R},\ \beta=-\alpha-1.$$

We apply this to $\widetilde{\mathcal{N}} = {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{left}}$, so that $\widetilde{\mathcal{N}}^{\mathrm{right}} = {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}^{\mathrm{right}}$. The right action of $t\widetilde{\partial}_t$ corresponds to the left action of $-\widetilde{\partial}_t t = -(t\widetilde{\partial}_t+z)$, so the right action of $\mathrm{N} = (t\widetilde{\partial}_t-\alpha z)$ corresponds to that of $-(t\widetilde{\partial}_t + z + \alpha z) = -(\mathrm{E}-\beta z) = \mathrm{N}$. $\square$

9.4.6. Proposition. *Let $g: X\to\mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $g=0$. Then $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}$ are $\widetilde{\mathcal{D}}_X$-coherent.*

Proof. By assumption, $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}$ are $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_{X\times\mathbb{C}} = \widetilde{\mathcal{D}}_X[\mathrm{E}]$-coherent. Since N is nilpotent on $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}$, the $\widetilde{\mathcal{D}}_X$-coherence follows. $\square$

***9.4.7. Definition* (Morphisms N, can and var, nearby/vanishing Lefschetz quiver)**

Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $g=0$. The nilpotent operator N is a morphism

$$\psi_{g,\lambda}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{N}} \psi_{g,\lambda}\widetilde{\mathcal{M}}(-1), \qquad \phi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{N}} \phi_{g,1}\widetilde{\mathcal{M}}(-1).$$

When $\lambda = 1$, the nilpotent operator N on $\psi_{g,1}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}$ is the operator obtained as the composition $\mathrm{var}\circ\mathrm{can}$ and $\mathrm{can}\circ\mathrm{var}$ in the *nearby/vanishing Lefschetz quiver*:

$$\begin{tikzcd}\psi_{g,1}\widetilde{\mathcal{M}} \arrow[r, bend left, "\mathrm{can}\,=\,-\widetilde{\eth}_t\,\cdot"] & \phi_{g,1}\widetilde{\mathcal{M}} \arrow[l, bend left, "\mathrm{var}\,=\,t\,\cdot", "(-1)"']\end{tikzcd} \qquad \text{(left case)} \tag{9.4.7 $*$}$$

$$\begin{tikzcd}\psi_{g,1}\widetilde{\mathcal{M}} \arrow[r, bend left, "\mathrm{can}\,=\,\cdot\,\widetilde{\eth}_t"] & \phi_{g,1}\widetilde{\mathcal{M}} \arrow[l, bend left, "\mathrm{var}\,=\,\cdot\,t", "(-1)"']\end{tikzcd} \qquad \text{(right case)} \tag{9.4.7 $**$}$$

with the same convention as in (3.4.8).

***9.4.8. Definition* (Monodromy operator).** We work with right $\mathcal{D}_X$-modules. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along (g). The monodromy operator T on $\psi_{g,\lambda}\mathcal{M}$ is the operator induced by $\exp(2\pi\mathsf{i}\, t\partial_t)$ (for left $\mathcal{D}_X$-modules $\mathrm{T} = \exp(-2\pi\mathsf{i}\, t\partial_t)$). On $\psi_{g,\lambda}\mathcal{M}$, we have $\mathrm{T} = \lambda\exp 2\pi\mathsf{i}\,\mathrm{N}$, and the nilpotent operator N is given by $\frac{1}{2\pi\mathsf{i}}\log(\lambda^{-1}\mathrm{T})$. On $\phi_{g,1}\mathcal{M}$ we have $\mathrm{T} = \exp 2\pi\mathsf{i}\,\mathrm{N}$ and $\mathrm{N} = \frac{1}{2\pi\mathsf{i}}\log\mathrm{T}$.

***9.4.9. Remark* (Monodromy filtration on nearby and vanishing cycles)**

The monodromy filtration relative to N on $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}$ (see Exercise 3.3.1 and Remark 3.3.4) is well-defined in the abelian category of graded $\widetilde{\mathcal{D}}_X$-modules with the automorphism σ induced by the shift (1) of the grading (or in the abelian category of $\mathcal{D}_X$-modules). The Lefschetz decomposition holds in this category, with respect to the corresponding primitive submodules $\mathrm{P}_\ell\psi_{g,\lambda}\widetilde{\mathcal{M}}$, $\mathrm{P}_\ell\phi_{g,1}\widetilde{\mathcal{M}}$ for $\ell \geqslant 0$.

Nevertheless, strict $\mathbb{R}$-specializability is not sufficient to ensure that each such primitive submodule (hence each graded piece of the monodromy filtration) is *strict*. The following proposition gives a criterion for the strictness of the primitive parts.

9.4.10. Proposition. *Assume $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (g) and fix $\lambda \in \mathbb{S}^1$. The following properties are equivalent.*

(1) *For every $\ell \geqslant 1$, $\mathrm{N}^\ell : \psi_{g,\lambda}\widetilde{\mathcal{M}} \to \psi_{g,\lambda}\widetilde{\mathcal{M}}(-\ell)$ is a strict morphism.*
(2) *For every $\ell \in \mathbb{Z}$, $\mathrm{gr}^{\mathrm{M}}_\ell\psi_{g,\lambda}\widetilde{\mathcal{M}}$ is strict.*
(3) *For every $\ell \geqslant 0$, $\mathrm{P}_\ell\psi_{g,\lambda}\widetilde{\mathcal{M}}$ is strict.*

We have similar assertions for $\phi_{g,1}\mathcal{M}$.

Proof. This is Proposition 5.1.10. □

***9.4.11. Remark* (Restriction to $z = 1$ of the monodromy filtration)**

If $\mathcal{M}$ is a coherent $R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable along D and setting $\mathcal{M} = \mathcal{M}/(z-1)\mathcal{M}$, we have $\psi_{g,\lambda}\mathcal{M} = \psi_{g,\lambda}\mathcal{M}/(z-1)\psi_{g,\lambda}\mathcal{M}$ and $\phi_{g,1}\mathcal{M} = \phi_{g,1}\mathcal{M}/(z-1)\phi_{g,1}\mathcal{M}$, according to Exercise 9.27, and the morphisms can and var for $\mathcal{M}$ obviously restrict to the morphisms can and var for $\mathcal{M}$, as well as the nilpotent endomorphism N.

Similarly, the monodromy filtration $\mathrm{M}_\bullet(\mathrm{N})$ on $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$ restricts to the monodromy filtration $\mathrm{M}_\bullet(\mathrm{N})$ on $\psi_{g,\lambda}\mathcal{M}, \phi_{g,1}\mathcal{M}$, since everything behaves $\mathbb{C}[z,z^{-1}]$-flatly after tensoring with $\mathbb{C}[z,z^{-1}]$.

9.5. Strict restrictions

9.5.a. Strict restriction along a principal divisor. Let us start with a smooth divisor H of X defined as the zero locus of a function $t : X \to \mathbb{C}$. Let $\iota_H : H \hookrightarrow X$ denote the inclusion. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Its pullback $\boldsymbol{L}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}}$ is defined in Section 8.6.a. We can simply represent it by the complex

$$\widetilde{\mathcal{M}} \xrightarrow{\ t\ } \underset{\bullet}{\widetilde{\mathcal{M}}},$$

where the $\bullet$ indicates the term in degree zero.

9.5.1. Proposition. *Assume that $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along H (see Definition 9.3.24). Then the complex $\boldsymbol{L}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}}$ is quasi-isomorphic to the complex*

$$\mathrm{gr}_0^V\widetilde{\mathcal{M}} \xrightarrow{\ t\ } \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}.$$

Proof. Let us consider the morphisms of complexes

$$\begin{array}{ccccc}
\mathrm{gr}_0^V\widetilde{\mathcal{M}} & \longleftarrow & V_0\widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{M}}\\
\big\downarrow & & \big\downarrow & & \big\downarrow\\
\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}} & \longleftarrow & V_{-1}\widetilde{\mathcal{M}} & \longrightarrow & \widetilde{\mathcal{M}}
\end{array}$$

where the vertical morphisms are induced by multiplication by t. Then 9.3.20(a) implies that the left morphism is a quasi-isomorphism, and 9.3.20(c) together with strong strict $\mathbb{R}$-specializability implies that $t : \widetilde{\mathcal{M}}/V_0\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}/V_{-1}\widetilde{\mathcal{M}}$ is also a quasi-isomorphism. $\square$

Let now $D = (g)$ be a principal divisor and let us set $H = X \times \{0\} \subset X \times \mathbb{C}_t$. We set

$$({}_{\mathrm{D}}\iota_{g*}\boldsymbol{L}_{\mathrm{D}}\iota_g^*)\widetilde{\mathcal{M}} := \boldsymbol{L}_{\mathrm{D}}\iota_H^*({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}). \tag{9.5.2}$$

9.5.3. Corollary. *Assume that $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along (g), i.e., ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ is so along H. Then $({}_{\mathrm{D}}\iota_{g*}\boldsymbol{L}_{\mathrm{D}}\iota_g^*)\widetilde{\mathcal{M}}$ is quasi-isomorphic to the complex*

$$\phi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\ \mathrm{var}\ } \psi_{g,1}\widetilde{\mathcal{M}}(-1). \qquad \square$$

9.5.b. Strict non-characteristic restrictions. Let $\iota_Y : Y \hookrightarrow X$ denote the inclusion of a closed submanifold with ideal $\mathcal{I}_Y$ (in local coordinates $(x_1,\dots,x_n)$, $\mathcal{I}_Y$ is generated by $x_1,\dots,x_p$, where $p = \operatorname{codim} Y$). The pullback functor ${}_{\mathrm{D}}\iota_Y^*$ is defined in Section 8.6.a. The case of left $\widetilde{\mathcal{D}}_X$-modules is easier to treat, so we will consider *left $\widetilde{\mathcal{D}}_X$-modules* and the corresponding setting for the V-filtration in this section.

Let us make the construction explicit in the case of a closed inclusion. A local section ξ of $\iota_Y^{-1}\widetilde{\Theta}_X$ (vector field on X, considered at points of Y only; we denote by ι_Y^{-1} the sheaf-theoretic pullback) is said to be tangent to Y if, for every local section g of $\widetilde{\mathcal{I}}_Y$, $\xi(g) \in \widetilde{\mathcal{I}}_Y$. This defines a subsheaf $\widetilde{\Theta}_{X|Y}$ of $\iota_Y^{-1}\widetilde{\Theta}_X$. Then $\widetilde{\Theta}_Y = \widetilde{\mathcal{O}}_Y \otimes_{\iota_Y^{-1}\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X|Y} = \iota_Y^* \widetilde{\Theta}_{X|Y}$ is a subsheaf of $\iota_Y^*\widetilde{\Theta}_X$.

Given a *left* $\widetilde{\mathcal{D}}_X$-module, the action of $\iota_Y^{-1}\widetilde{\Theta}_X$ on $\iota_Y^{-1}\widetilde{\mathcal{M}}$ restricts to an action of $\widetilde{\Theta}_Y$ on $\iota_Y^*\widetilde{\mathcal{M}} = \widetilde{\mathcal{O}}_Y \otimes_{\iota_Y^{-1}\widetilde{\mathcal{O}}_X} \iota_Y^{-1}\widetilde{\mathcal{M}}$. The criterion of Exercise 8.8 is fulfilled since it is fulfilled for $\widetilde{\Theta}_X$ and $\widetilde{\mathcal{M}}$, defining therefore a left $\widetilde{\mathcal{D}}_Y$-module structure on $\iota_Y^*\widetilde{\mathcal{M}}$: this is ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$.

Without any other assumption, coherence is not preserved by ${}_{\mathrm{D}}\iota_Y^*$. For example, ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{D}}_X$ is not $\widetilde{\mathcal{D}}_Y$-coherent if $\operatorname{codim} Y \geqslant 1$. A criterion for coherence of the pullback is given below in terms of the characteristic variety.

The cotangent map to the inclusion defines a natural bundle morphism

$$\varpi : T^*X_{|Y} \times \mathbb{C}_z \longrightarrow T^*Y \times \mathbb{C}_z,$$

the kernel of which is by definition the conormal bundle $T^*_Y X \times \mathbb{C}_z$ of $Y \times \mathbb{C}_z$ in $X \times \mathbb{C}_z$.

***9.5.4. Definition* (Non-characteristic property).** Let $\widetilde{\mathcal{M}}$ be a holonomic $\widetilde{\mathcal{D}}_X$-module with characteristic variety $\operatorname{Char}\widetilde{\mathcal{M}}$ contained in $\Lambda \times \mathbb{C}_z$, where $\Lambda \subset T^*X$ is Lagrangean (see Section 8.8.d). Let $Y \subset X$ be a submanifold of X. We say that Y is *non-characteristic* with respect to the holonomic $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, or that $\widetilde{\mathcal{M}}$ *is non-characteristic along* Y, if one of the following equivalent conditions is satisfied:

- $(T^*_Y X \times \mathbb{C}_z) \cap \operatorname{Char}\widetilde{\mathcal{M}} \subset T^*_X X \times \mathbb{C}_z$,
- $\varpi : \operatorname{Char}\widetilde{\mathcal{M}}_{|Y\times\mathbb{C}_z} \to T^*Y \times \mathbb{C}_z$ is finite, i.e., proper with finite fibers.

9.5.5. Example. Assume that $Y = H$ is a hypersurface defined by a coordinate function $t : X \to \mathbb{C}$ and that $\mathcal{M}$ is a holonomic (more generally, coherent) $\mathcal{D}_X$-module with characteristic variety $\operatorname{Char}\mathcal{M} \subset T^*X$. Then, if H is non-characteristic with respect to $\mathcal{M}$, $\mathcal{M}$ is $\mathcal{D}_{X/\mathbb{C}}$-coherent in the neighbourhood of H and $t : \mathcal{M} \to \mathcal{M}$ is injective (see e.g. [**MT04**, Prop. II.3.4 & Prop. III.3.3] and the references therein). It follows that the filtration $U^k\mathcal{M} = t^k\mathcal{M}$ for $k \geqslant 0$ and $U^k\mathcal{M} = \mathcal{M}$ for $k \leqslant 0$ is a good V-filtration, which is equal to the Kashiwara-Malgrange filtration, so that $\mathcal{M} = V^0\mathcal{M}$.

***9.5.6. Theorem* (Coherence of non-characteristic restrictions)**

Assume that $\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_X$*-coherent and that* Y *is non-characteristic with respect to* $\widetilde{\mathcal{M}}$. *Then* ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_Y$*-coherent and* $\operatorname{Char}{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} \subset \varpi(\operatorname{Char}\widetilde{\mathcal{M}}_{|Y})$.

Sketch of proof. The question is local near a point $x \in Y$. We may therefore assume that $\widetilde{\mathcal{M}}$ has a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$.

(1) Set $F_k{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} = \operatorname{image}[\iota_Y^*F_k\widetilde{\mathcal{M}} \to \iota_Y^*\widetilde{\mathcal{M}}]$. Then, using Exercise 8.63(2), one shows that $F_\bullet{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$ is a coherent filtration with respect to $F_\bullet{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{D}}_X$.

(2) The module $\operatorname{gr}^F{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$ is a quotient of $\iota_Y^*\operatorname{gr}^F\widetilde{\mathcal{M}}$, hence its support is contained in $\operatorname{Char}\widetilde{\mathcal{M}}_{|Y}$. By Remmert's Theorem, it is a coherent $\operatorname{gr}^F\widetilde{\mathcal{D}}_Y$-module.

(3) The filtration $F_\bullet{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$ is thus a coherent filtration of the $\widetilde{\mathcal{D}}_Y$-module ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$. By Exercise 8.62(1), ${}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_Y$-coherent. Using the coherent filtration above, it is clear that $\operatorname{Char}{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} \subset \varpi(\operatorname{Char}\widetilde{\mathcal{M}}_{|Y})$. $\square$

***9.5.7. Definition* (Strict non-characteristic property).** In the setting of Definition 9.5.4, we say that $\widetilde{\mathcal{M}}$ is *strictly non-characteristic along* Y if $\widetilde{\mathcal{M}}$ is non-characteristic along Y and, moreover, $\boldsymbol{L}\,{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} = \widetilde{\mathcal{O}}_Y \otimes^{\boldsymbol{L}}_{\iota_Y^{-1}\widetilde{\mathcal{O}}_X} \iota_Y^{-1}\widetilde{\mathcal{M}}$ is *strict*.

9.5.8. Proposition. *If $\widetilde{\mathcal{M}}$ is strictly non-characteristic along Y, then $\boldsymbol{L}\,{}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_Y^*\widetilde{\mathcal{M}}$.*

Proof. The result holds for $\mathcal{D}_X$-modules, and therefore it holds after tensoring with $\mathbb{C}[z, z^{-1}]$. As a consequence, ${}_{\mathrm{D}}\iota^{*(j)}\widetilde{\mathcal{M}}$ is a z-torsion module if $j \neq 0$. It is strict if and only if it is zero. $\square$

9.5.9. Proposition. *Assume that* $\operatorname{codim} Y = 1$ *and denote it by H. Then*

(1) *if $\widetilde{\mathcal{M}}$ is strictly non-characteristic along H, it is also strictly $\mathbb{R}$-specializable along H,*

(2) *if $\widetilde{\mathcal{M}}$ is non-characteristic and strictly $\mathbb{R}$-specializable along H, it is strictly non-characteristic along H.*

In such a case, $\mathrm{gr}_V^\beta\widetilde{\mathcal{M}} = 0$ unless $\beta \in \mathbb{N}$, the nilpotent endomorphism $t\widetilde{\partial}_t$ on $\mathrm{gr}_V^0\widetilde{\mathcal{M}}$ is equal to zero, and $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along H. Last, ${}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}}$ is naturally identified with $\mathrm{gr}_V^0\widetilde{\mathcal{M}}$.

Proof. Since the question is local, we may assume that $X \simeq H \times \Delta_t$.

(1) The previous proposition says that $t : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is injective and the definition amounts to the strictness of $\widetilde{\mathcal{M}}/t\widetilde{\mathcal{M}}$.

Since $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_{X/\mathbb{C}}$-coherent (Exercise 9.40), the filtration defined by $U^k\widetilde{\mathcal{M}} = t^k\widetilde{\mathcal{M}}$ ($k \in \mathbb{N}$) is a coherent V-filtration and $\mathrm{E} : \mathrm{gr}_U^0\widetilde{\mathcal{M}} \to \mathrm{gr}_U^0\widetilde{\mathcal{M}}$ acts by 0 since $\widetilde{\partial}_t U^0\widetilde{\mathcal{M}} \subset U^0\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}$. It follows that $\widetilde{\mathcal{M}}$ is specializable along H and that the Bernstein polynomial of the filtration $U^\bullet\widetilde{\mathcal{M}}$ has only integral roots. Moreover, $t : \mathrm{gr}_U^k\widetilde{\mathcal{M}} \to \mathrm{gr}_U^{k+1}\widetilde{\mathcal{M}}$ is onto for $k \geqslant 0$. We will show by induction on k that each $\mathrm{gr}_U^k\widetilde{\mathcal{M}}$ is strict. The assumption is that $\mathrm{gr}_U^0\widetilde{\mathcal{M}}$ is strict. We note that $\mathrm{E} - kz$ acts by zero on $\mathrm{gr}_U^k\widetilde{\mathcal{M}}$. If $\mathrm{gr}_U^k\widetilde{\mathcal{M}}$ is strict, then the composition $\widetilde{\partial}_t t$, that acts by $(k+1)z$ on $\mathrm{gr}_U^k\widetilde{\mathcal{M}}$, is injective, so $t : \mathrm{gr}_U^k\widetilde{\mathcal{M}} \to \mathrm{gr}_U^{k+1}\widetilde{\mathcal{M}}$ is bijective, and $\mathrm{gr}_U^{k+1}\widetilde{\mathcal{M}}$ is thus strict. It follows that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, and the t-adic filtration $U^\bullet\widetilde{\mathcal{M}}$ is equal to the V-filtration.

(2) It follows from the assumption that $\mathcal{M}$ is non-characteristic along H, hence $\mathcal{M} = V^0\mathcal{M}$ by Example 9.5.5, and $\mathrm{gr}_V^\beta\mathcal{M} = 0$ for any $\beta < 0$. By strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}$, we deduce that $\mathrm{gr}_V^\beta\widetilde{\mathcal{M}} = 0$ for any $\beta < 0$, hence $\widetilde{\mathcal{M}} = V^0\widetilde{\mathcal{M}}$, that $t : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ is injective, and that $\widetilde{\mathcal{M}}/t\widetilde{\mathcal{M}} = \mathrm{gr}_V^0\widetilde{\mathcal{M}}$ is strict.

If $\widetilde{\mathcal{M}}$ satisfies (1), equivalently (2), we have seen in the proof of (1) that $\mathrm{gr}_V^\beta\widetilde{\mathcal{M}} = 0$ for $\beta \notin \mathbb{N}$. Since $\mathrm{gr}_V^{-1}\widetilde{\mathcal{M}} = 0$, we deduce that $t\widetilde{\partial}_t$ acts by zero on $\mathrm{gr}_V^\beta\widetilde{\mathcal{M}}$. The criterion for strong strict $\mathbb{R}$-specializability given in Exercise 9.23 implies immediately that $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable.

We note that $\mathrm{gr}_V^0\widetilde{\mathcal{M}}$ is naturally a $\widetilde{\mathcal{D}}_H$-module since E acts by 0, and $\widetilde{\mathcal{D}}_H = \mathrm{gr}_0^V\widetilde{\mathcal{D}}_X / \mathrm{E}\,\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$, and one checks that the identification ${}_{\mathrm{D}}\iota_H^*\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}/\mathcal{I}_H\widetilde{\mathcal{M}} = \mathrm{gr}_V^0\widetilde{\mathcal{M}}$ is compatible with the action of $\widetilde{\mathcal{D}}_H$. $\square$

***9.5.10. Remark* (The case of right $\widetilde{\mathcal{D}}_X$-modules).** Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{M}}^{\mathrm{right}} := \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$ be the associated right $\widetilde{\mathcal{D}}_X$-module (with grading). If $\widetilde{\mathcal{M}}$ is strictly non-characteristic along H, then so is $\widetilde{\mathcal{M}}^{\mathrm{right}}$. We have

$$_{\mathrm{D}}\iota_H^* \widetilde{\mathcal{M}}^{\mathrm{right}} := \widetilde{\omega}_H \otimes_{\widetilde{\mathcal{O}}_H} {_{\mathrm{D}}\iota_H^*}\widetilde{\mathcal{M}} = \widetilde{\omega}_H \otimes_{\widetilde{\mathcal{O}}_H} \mathrm{gr}_V^0 \widetilde{\mathcal{M}} = \mathrm{gr}_{-1}^V \widetilde{\mathcal{M}}^{\mathrm{right}}(1),$$

according to Remark 9.3.19.

Assume that H is globally defined by the smooth function g. Then

$$_{\mathrm{D}}\iota_{H*}({_{\mathrm{D}}\iota_H^*}\widetilde{\mathcal{M}}^{\mathrm{right}}) = {_{\mathrm{D}}\iota_{H*}}(\mathrm{gr}_V^0 \widetilde{\mathcal{M}}) = {_{\mathrm{D}}\iota_{H*}}(\mathrm{gr}_{-1}^V \widetilde{\mathcal{M}}^{\mathrm{right}})(1) = \psi_{g,1}\widetilde{\mathcal{M}}^{\mathrm{right}},$$

according to Exercise 9.36.

9.6. Strict Kashiwara's equivalence

We now return to the case of right $\widetilde{\mathcal{D}}_X$-module when considering the pushforward functor.

Let $\iota_Z : Z \subset X$ be the inclusion of a complex submanifold. The following is known as "Kashiwara's equivalence".

***9.6.1. Proposition* (see [Kas03, §4.8]).** *The pushforward functor $_{\mathrm{D}}\iota_{Z*}$ induces a natural equivalence between coherent $\mathcal{D}_Z$-modules and coherent $\mathcal{D}_X$-modules supported on Z, whose quasi-inverse is the restriction functor $_{\mathrm{D}}\iota_Z^*$.* □

Be aware however that this result does not hold for graded coherent $R_F\mathcal{D}_X$-modules. For example, if $X = \mathbb{C}$ with coordinate s and $\iota_Z : Z = \{0\} \hookrightarrow X$ denotes the inclusion, $_{\mathrm{D}}\iota_{Z*}\mathbb{C}[z] = \delta_s \cdot \mathbb{C}[z, \widetilde{\partial}_s]$ with $\delta_s s = 0$. On the other hand, consider the $\mathbb{C}[z,s]\langle\widetilde{\partial}_s\rangle$-submodule of $\mathbb{C}[z] \otimes_{\mathbb{C}} {_{\mathrm{D}}\iota_{Z*}}\mathbb{C} = \delta_s\mathbb{C}[z, \partial_s]$ generated by $\delta_s\partial_s$ (note: ∂_s and not $\widetilde{\partial}_s$). This submodule is written $\delta_s\mathbb{C}[z] \oplus \bigoplus_{k\geqslant 0} \delta_s\widetilde{\partial}_s^k\partial_s$. It has finite type over $\mathbb{C}[z,s]\langle\widetilde{\partial}_s\rangle$ by construction, each element is annihilated by some power of s, and $_{\mathrm{D}}\iota_Z^{*(-1)}(\delta_s\partial_s \cdot \mathbb{C}[z,s]\langle\widetilde{\partial}_s\rangle) = \delta_s\mathbb{C}[z]$, but it is not equal to $_{\mathrm{D}}\iota_{Z*}\mathbb{C}[z]$.

Note also that this proposition implies in particular that $_{\mathrm{D}}\iota_Z^{*(k)}\,{_{\mathrm{D}}\iota_{Z*}}\mathcal{M} = 0$ for $k \neq -1$, if $\mathcal{M}$ is $\mathcal{D}_X$-coherent. In the example above, we have $_{\mathrm{D}}\iota_{Z*}\mathbb{C} = \mathbb{C}[\partial_s]$ and the complex $_{\mathrm{D}}\iota_Z^*\,{_{\mathrm{D}}\iota_{Z*}}\mathbb{C}$ is the complex $\mathbb{C}[\partial_s] \xrightarrow{\cdot s} \mathbb{C}[\partial_s]$ with terms in degrees -1 and 0. It has cohomology in degree -1 only.

However, this is not true for graded coherent $R_F\mathcal{D}_X$-modules. With the similar example, the complex $_{\mathrm{D}}\iota_Z^*\,{_{\mathrm{D}}\iota_{Z*}}\mathbb{C}[z]$ is the complex $\mathbb{C}[z,\widetilde{\partial}_s] \xrightarrow{\cdot s} \mathbb{C}[z,\widetilde{\partial}_s]$. We have $\widetilde{\partial}_s^k \cdot s = kz\widetilde{\partial}_s^{k-1}$, so the cokernel of s is not equal to zero.

***9.6.2. Proposition* (Strict Kashiwara's equivalence).** *Let Z be a smooth closed submanifold of X, and let $\iota_Z : Z \hookrightarrow X$ denote the inclusion. Then the functor $_{\mathrm{D}}\iota_{Z*}$: $\mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_Z) \mapsto \mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ is fully faithful. If moreover $Z = H$ is smooth of codimension 1 in X, it induces an equivalence between the full subcategory of $\mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_H)$ whose objects are strict, and the full subcategory of $\mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_X)$ whose objects are strictly $\mathbb{R}$-specializable along H and supported on H. An inverse functor is $\widetilde{\mathcal{M}} \mapsto \mathrm{gr}_0^V\widetilde{\mathcal{M}}$.*

Proof the full faithfulness. It is enough to prove that each morphism $\varphi : {}_{\mathrm{D}}\iota_{Z*}\widetilde{\mathcal{N}}_1 \to {}_{\mathrm{D}}\iota_{Z*}\widetilde{\mathcal{N}}_2$ takes the form ${}_{\mathrm{D}}\iota_{Z*}\psi$ for a unique $\psi : \widetilde{\mathcal{N}}_1 \to \widetilde{\mathcal{N}}_2$. Because of uniqueness, the assertion is local with respect to Z, hence we can assume that there exist local coordinates $(x_1, \dots, x_r)$ defining Z. Assume $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{Z*}\widetilde{\mathcal{N}}$ for some coherent $\widetilde{\mathcal{D}}_Z$-module $\widetilde{\mathcal{N}}$. Then one can recover $\widetilde{\mathcal{N}}$ from $\widetilde{\mathcal{M}}$ as the $\widetilde{\mathcal{D}}_Z$-module $\widetilde{\mathcal{M}}/\sum_i \widetilde{\mathcal{M}} \cdot \widetilde{\partial}_{x_i}$. As a consequence, ψ must be the morphism induced by φ on $\widetilde{\mathcal{M}}/\sum_i \widetilde{\mathcal{M}} \cdot \widetilde{\partial}_{x_i}$, hence its uniqueness. On the other hand, since $\widetilde{\mathcal{M}}_1$ is generated by $\widetilde{\mathcal{N}}_1 \otimes \mathbf{1}$ over $\widetilde{\mathcal{D}}_X$ (see Exercise 8.44), φ is determined by its restriction to $\widetilde{\mathcal{N}}_1 \otimes \mathbf{1}$, that is by ψ, and the formula is $\varphi = {}_{\mathrm{D}}\iota_{Z*}\psi$. $\square$

9.6.3. Lemma. *Assume $X \simeq H \times \mathbb{C}$ with coordinate s on the second factor. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module supported on $H \times \{0\}$.*

(1) *Assume that there exists a strict $\widetilde{\mathcal{D}}_H$-module $\widetilde{\mathcal{N}}$ such that $\widetilde{\mathcal{M}} \simeq {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$. Then*
   - (a) $\widetilde{\mathcal{N}} = \mathrm{Ker}[s : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}]$,
   - (b) $\widetilde{\mathcal{N}}$ *is $\widetilde{\mathcal{D}}_H$-coherent,*
   - (c) $\widetilde{\mathcal{M}}$ *is strict and strictly $\mathbb{R}$-specializable along H,*
   - (d) $\widetilde{\mathcal{N}} = \mathrm{gr}_0^V \widetilde{\mathcal{M}}$.

(2) *Conversely, if $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, then such an $\widetilde{\mathcal{N}}$ exists. In particular, $\widetilde{\mathcal{M}}$ is also strict.*

***9.6.4. Remark* (Strictness and strict $\mathbb{R}$-specializability).** Let $\widetilde{\mathcal{M}}$ be as in Lemma 9.6.3, that is, $\widetilde{\mathcal{D}}_X$-coherent and supported on $H \times \{0\}$. Then the filtration $U_0\widetilde{\mathcal{M}} = \mathrm{Ker}\, s \subset U_1\widetilde{\mathcal{M}} = \mathrm{Ker}\, s^2 \subset \cdots$ is a filtration by $V_0\widetilde{\mathcal{D}}_X$-submodules and obviously admits a weak Bernstein polynomial. Assume moreover that $\widetilde{\mathcal{M}}$ is *strict*. Then every $\mathrm{gr}_k^U \widetilde{\mathcal{M}}$ is also strict: if $m \in U_k\widetilde{\mathcal{M}}$ and $z^\ell m \in U_{k-1}\widetilde{\mathcal{M}}$, that is, if $s^{k+1}m = 0$ and $s^k z^\ell m = 0$, then $s^k m = 0$ by strictness of $\widetilde{\mathcal{M}}$ and thus $m = 0$ in $\mathrm{gr}_k^U \widetilde{\mathcal{M}}$. Therefore, $U_\bullet\widetilde{\mathcal{M}}$ is the Kashiwara-Malgrange filtration $V_\bullet\widetilde{\mathcal{M}}$ in the sense of Lemma 9.3.13, and Properties 9.3.14(1) and (2) are satisfied.

However, the condition that $\widetilde{\mathcal{M}}$ is strict is not enough to obtain the conclusion of 9.6.3(1), as shown by the following example. The point is that 9.3.14(3) may not hold. Assume that H is reduced to a point and let $\widetilde{\mathcal{M}}$ be the $\widetilde{\mathcal{D}}_X$-submodule of the $\mathcal{D}_X[z]$-module $\widetilde{\mathbb{C}}\langle\partial_s\rangle$ generated by 1 and ∂_s (recall that $\widetilde{\mathbb{C}} := \mathbb{C}[z]$), that we denote by [1] and $[\partial_s]$ for the sake of clarity. By definition, we have $[1]s = 0$ and $[\partial_s]s^2 = 0$. For the Kashiwara-Malgrange filtration $V_\bullet\widetilde{\mathcal{M}}$ defined above, $\widetilde{\partial}_s : \mathrm{gr}_0^V \widetilde{\mathcal{M}} = \widetilde{\mathbb{C}} \to \mathrm{gr}_1^V \widetilde{\mathcal{M}} = [\partial_s]\widetilde{\mathbb{C}}$ is not onto, for its cokernel is $[\partial_s]\mathbb{C}$. In other words, $\widetilde{\mathcal{M}}$ is not strictly $\mathbb{R}$-specializable at $s = 0$ and not of the form ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$.

***Proof of Lemma* 9.6.3.**

(1) Assume $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$ for some strict $\widetilde{\mathcal{D}}_H$-module $\widetilde{\mathcal{N}}$. We have ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}} = \bigoplus_{k\geqslant 0} \iota_{H*}\widetilde{\mathcal{N}} \otimes \delta_s \widetilde{\partial}_s^k$ with $\delta_s s = 0$ (see Exercise 8.45(1)). The action of s on ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$ is the z-shift $n \otimes \delta_s \widetilde{\partial}_s^k \mapsto zkn \otimes \delta_s \widetilde{\partial}_s^{k-1}$, hence $\widetilde{\mathcal{N}} = \mathrm{Ker}\, s$ because $\widetilde{\mathcal{N}}$ is strict. Given a finite family of local $\widetilde{\mathcal{D}}_X$-generators of $\widetilde{\mathcal{M}}$, we produce another such family made of homogeneous elements, by taking the components on the previous decomposition. Therefore, there exists a finite family of local sections n_i of $\widetilde{\mathcal{N}}$ such that $n_i \otimes \delta_s$ generate $\widetilde{\mathcal{M}}$. Let

$\widetilde{\mathcal{N}}' \subset \widetilde{\mathcal{N}}$ be the $\widetilde{\mathcal{D}}_H$-submodule they generate. Then ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}' \to {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}}$ is onto. On the other hand, since $\widetilde{\mathcal{N}}'$ is also strict, this map is injective: If $\sum_{k=1}^N n'_k \otimes \delta_s \widetilde{\partial}_s^k \mapsto 0$, then $n'_N \otimes \delta_s \widetilde{\partial}_s^N \mapsto 0$, and $s^N n'_N \otimes \delta_s \widetilde{\partial}_s^N = \star z^N n'_N \otimes \delta_s \widetilde{\partial}_s^N \mapsto 0$, where $\star$ is a nonzero constant; so $z^N n'_N = 0$ in $\widetilde{\mathcal{N}}$, hence $n'_N = 0$. We conclude $\widetilde{\mathcal{N}}' = \widetilde{\mathcal{N}}$ since both are equal to $\operatorname{Ker} s$ in ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$. Therefore, $\widetilde{\mathcal{N}}$ is locally finitely $\widetilde{\mathcal{D}}_H$-generated in $\widetilde{\mathcal{M}}$, and then is $\widetilde{\mathcal{D}}_H$-coherent. One then checks that the filtration $U_j\widetilde{\mathcal{M}} := \bigoplus_{k\geqslant 0}^j \iota_{H*}\widetilde{\mathcal{N}} \otimes \delta_s \widetilde{\partial}_s^k$ is a coherent V-filtration of $\widetilde{\mathcal{M}}$, and $\widetilde{\mathcal{N}} = \mathrm{gr}_0^U \widetilde{\mathcal{M}}$. We deduce that each $\mathrm{gr}_k^U \widetilde{\mathcal{M}}$ is strict, and $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable. Last, $n \otimes \delta_s$ satisfies $(n \otimes \delta_s)s\widetilde{\partial}_s = 0$, so $V_\bullet\widetilde{\mathcal{M}} = U_\bullet\widetilde{\mathcal{M}}$ jumps at non-negative integers only.

(2) Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H. Then $V_{<0}\widetilde{\mathcal{M}} = 0$, according to 9.3.20(a). Similarly, $\mathrm{gr}_\alpha^V \widetilde{\mathcal{M}} = 0$ for $\alpha \notin \mathbb{Z}$. As $s : \mathrm{gr}_k^V \widetilde{\mathcal{M}} \to \mathrm{gr}_{k-1}^V \widetilde{\mathcal{M}}$ is injective for $k \neq 0$ (see 9.3.20(c)), we conclude that

$$\mathrm{gr}_0^V \widetilde{\mathcal{M}} \simeq V_0\widetilde{\mathcal{M}} = \operatorname{Ker}[s : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}].$$

Since $\widetilde{\partial}_s : \mathrm{gr}_k^V \widetilde{\mathcal{M}} \to \mathrm{gr}_{k-1}^V \widetilde{\mathcal{M}}$ is an isomorphism for $k \leqslant 0$, we obtain

$$\widetilde{\mathcal{M}} = \bigoplus_{\ell\geqslant 0} \mathrm{gr}_0^V \widetilde{\mathcal{M}} \widetilde{\partial}_s^\ell = {}_{\mathrm{D}}\iota_* \mathrm{gr}_0^V \widetilde{\mathcal{M}}.$$

Last, E acts by zero on $\mathrm{gr}_0^V \widetilde{\mathcal{M}}$, which is therefore a coherent $\widetilde{\mathcal{D}}_H$-module by means of the isomorphism $\mathrm{gr}_0^V \widetilde{\mathcal{D}}_X / \mathrm{E}\, \mathrm{gr}_0^V \widetilde{\mathcal{D}}_X \simeq \widetilde{\mathcal{D}}_H$. It is strict since $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable. $\square$

***End of the proof of Proposition* 9.6.2.** It remains to prove essential surjectivity. Let $V_\bullet\widetilde{\mathcal{M}}$ be the V-filtration of $\widetilde{\mathcal{M}}$ along H. By the argument in the second part of the proof of Lemma 9.6.3, we have local isomorphisms $\widetilde{\mathcal{M}} \xrightarrow{\sim} {}_{\mathrm{D}}\iota_* \mathrm{gr}_0^V \widetilde{\mathcal{M}}$ which induce the identity on $V_0\widetilde{\mathcal{M}} = \mathrm{gr}_0^V \widetilde{\mathcal{M}}$. By full faithfulness they glue in a unique way as a global isomorphism $\widetilde{\mathcal{M}} \simeq {}_{\mathrm{D}}\iota_* \mathrm{gr}_0^V \widetilde{\mathcal{M}}$. $\square$

9.6.5. Corollary. *Assume* $\operatorname{codim} H = 1$. *Let* $\widetilde{\mathcal{N}}$ *be* $\widetilde{\mathcal{D}}_H$*-coherent and set* $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$. *If* $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 \oplus \widetilde{\mathcal{M}}_2$ *with* $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2$ *being* $\widetilde{\mathcal{D}}_X$*-coherent, then there exist coherent* $\widetilde{\mathcal{D}}_H$*-submodules* $\widetilde{\mathcal{N}}_1, \widetilde{\mathcal{N}}_2$ *of* $\widetilde{\mathcal{N}}$ *such that* $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}_1 \oplus \widetilde{\mathcal{N}}_2$ *and* $\widetilde{\mathcal{M}}_j = {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}_j$ *for* $j = 1, 2$.

Proof. Each $\widetilde{\mathcal{M}}_i$ is coherent and supported on H. We set $\widetilde{\mathcal{N}}_i = \widetilde{\mathcal{M}}_i \cap \widetilde{\mathcal{N}}$. Locally, choose a coordinate s defining H and set $\widetilde{\mathcal{N}}'_i = \widetilde{\mathcal{M}}_i / \widetilde{\mathcal{M}}_i \cdot \widetilde{\partial}_s$. Since $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}} / \widetilde{\mathcal{M}} \cdot \widetilde{\partial}_s$, we deduce that $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}'_1 \oplus \widetilde{\mathcal{N}}'_2$, and we have a (local) isomorphism $\widetilde{\mathcal{M}}_i \simeq {}_{\mathrm{D}}\iota_* \widetilde{\mathcal{N}}'_i$. Then one checks that $\widetilde{\mathcal{N}}' i = \widetilde{\mathcal{N}}_i$, so it is globally defined. $\square$

We now consider the behaviour of strict $\mathbb{R}$-specializability along a function $g : X \to \mathbb{C}$ with respect to strict Kashiwara's equivalence along H. We can regard 9.6.6(1) as the particular case of Theorem 9.8.8 below where f is a closed embedding ι.

9.6.6. Proposition. *Let* $\widetilde{\mathcal{N}}$ *be a coherent* $\widetilde{\mathcal{D}}_H$*-module and set* $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$.

(1) *Assume that* $\widetilde{\mathcal{N}}$ *is strictly* $\mathbb{R}$*-specializable along* $(g_{|H})$. *Then* $\widetilde{\mathcal{M}}$ *is strictly* $\mathbb{R}$*-specializable along* (g).

(2) *Assume that* $\widetilde{\mathcal{M}}$ *is strictly* $\mathbb{R}$*-specializable along* (g)*. Then* $\widetilde{\mathcal{N}}$ *is strictly* $\mathbb{R}$*-specializable along* $(g_{|H})$*.*

In such a case, we have $\psi_{g,\lambda}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\psi_{g_{|H},\lambda}\widetilde{\mathcal{N}}$ *and* $\phi_{g,1}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\phi_{g_{|H},1}\widetilde{\mathcal{N}}$*. Moreover,* $\mathrm{can}_{\widetilde{\mathcal{M}}} = {}_{\mathrm{D}}\iota_{H*}\,\mathrm{can}_{\widetilde{\mathcal{N}}}$ *and* $\mathrm{var}_{\widetilde{\mathcal{M}}} = {}_{\mathrm{D}}\iota_{H*}\,\mathrm{var}_{\widetilde{\mathcal{N}}}$*.*

Proof. The first statement is easy to check. Let us consider the second one. We first consider the case where $X = H_g\times\mathbb{C}_t\times\mathbb{C}_s$, with $H = H_g\times\mathbb{C}_t$ and g is the projection to $\mathbb{C}_t$. We denote by V the V-filtration along t. We have $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}} = \bigoplus_k \iota_{H*}\widetilde{\mathcal{N}}\otimes\delta_s\widetilde{\partial}_s^k$.

Let n be a local section of $\widetilde{\mathcal{N}}$. If $b(t\widetilde{\partial}_t) - tP(x,t,s,\widetilde{\partial}_x,t\widetilde{\partial}_t,\widetilde{\partial}_s)$ is a Bernstein equation for $n\otimes\delta_s$ in ${}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$, and if $P = P_0 + sQ$, where P_0 does not depend on s, then $b(t\widetilde{\partial}_t) - tP_0(x,t,\widetilde{\partial}_x,t\widetilde{\partial}_t,\widetilde{\partial}_s)$ is also a Bernstein equation for $n\otimes\delta_s$. The degree-zero part with respect to $\widetilde{\partial}_s$ of this equation still gives a Bernstein equation for $n\otimes\delta_s$, and thus a Bernstein equation for n in $\widetilde{\mathcal{N}}$. We conclude that $\widetilde{\mathcal{N}}$ is $\mathbb{R}$-specializable along H_g and that $\mathrm{ord}_V(n) \geqslant \mathrm{ord}_V(n\otimes\delta_s)$.

Let us now prove that the V-filtration of $\widetilde{\mathcal{M}}$ is compatible with the decomposition. Let $\sum_{i=0}^N n_i\otimes\delta_s\widetilde{\partial}_s^i$ be a section in $V_\alpha\widetilde{\mathcal{M}}$. We will prove by induction on N that $\mathrm{ord}_V(n_i) \leqslant \alpha$ for every i. It is enough to prove it for $i = N$. We have $\sum_{i=0}^N n_i\otimes\delta_s\widetilde{\partial}_s^i\cdot s^N = \star z^N n_N\otimes\delta_s \in V_\alpha\widetilde{\mathcal{M}}$ for some nonzero constant $\star$. If $n_N\otimes\delta_s \in V_\gamma\widetilde{\mathcal{M}}$ for $\gamma > \alpha$, then the class of $n_N\otimes\delta_s$ in $\mathrm{gr}_\gamma^V\widetilde{\mathcal{M}}$ is annihilated by z^N, hence is zero since $\mathrm{gr}_\gamma^V\widetilde{\mathcal{M}}$ is strict. Therefore, $n_N\otimes\delta_s \in V_\alpha\widetilde{\mathcal{M}}$, and by the preliminary remark, $\mathrm{ord}_V(n_N) \leqslant \alpha$. If we denote by $U_\bullet\widetilde{\mathcal{N}}$ the (possibly not coherent) V-filtration by the V-order, then one has $V_\alpha\widetilde{\mathcal{M}} = \bigoplus_i \iota_{H*}U_\alpha\widetilde{\mathcal{N}}\otimes\delta_s\widetilde{\partial}_s^i$ and $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}} = \bigoplus_i \iota_{H*}\mathrm{gr}_\alpha^V\widetilde{\mathcal{N}}\otimes\delta_s\widetilde{\partial}_s^i$. It follows that $U_\bullet\widetilde{\mathcal{N}}$ is a coherent V-filtration of $\widetilde{\mathcal{N}}$ and that each $\mathrm{gr}_\alpha^V\widetilde{\mathcal{N}}$ is strict. By uniqueness of the V-filtration, we have $U_\bullet\widetilde{\mathcal{N}} = V_\bullet\widetilde{\mathcal{N}}$, and Properties 9.3.14(2) and (3) are clearly satisfied, as they hold for $\widetilde{\mathcal{M}}$.

For the general case, the question is local and we can assume that H is defined by a smooth function h. By assumption, ${}_{\mathrm{D}}\iota_{g*}({}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}})$ is strictly $\mathbb{R}$-specializable along t, and thus so is ${}_{\mathrm{D}}\iota_{(h,g)*}({}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}) = {}_{\mathrm{D}}\iota_{s=0*}\iota_{g_{|H}}\widetilde{\mathcal{N}}$, after (1). The previous argument implies that $\iota_{g_{|H}}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along t, that is, $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(g_{|H})$.

The last statement is then clear by the computation of the V-filtrations above. $\square$

9.7. Support-decomposable $\widetilde{\mathcal{D}}$-modules

Let $g : X \to \mathbb{C}$ be a holomorphic function. We set $D := (g)$ and $|D| = g^{-1}(0)$. Let $\iota_g : X \hookrightarrow X\times\mathbb{C}$ denote the graph embedding associated with g. We set $H = X\times\{0\} \subset X\times\mathbb{C}$.

Let us make precise the behaviour of the support of nearby and vanishing cycles.

9.7.1. Proposition. *Assume that* $\widetilde{\mathcal{M}}$ *is* $\widetilde{\mathcal{D}}_X$*-coherent and strictly* $\mathbb{R}$*-specializable along* D*.*

(1) *For every* $\lambda\in\mathbb{S}^1$*,* $\dim\mathrm{Supp}\,\psi_{g,\lambda}\widetilde{\mathcal{M}} < \dim\mathrm{Supp}\,\widetilde{\mathcal{M}}$*.*

(2) *If* $\mathrm{Supp}\,\widetilde{\mathcal{M}} \subset |D|$*, then* $\psi_{g,\lambda}\widetilde{\mathcal{M}} = 0$ *for any* $\lambda\in\mathbb{S}^1$*, and* $\widetilde{\mathcal{M}} \simeq \phi_{g,1}\widetilde{\mathcal{M}}$*.*

Proof.

(1) Clearly, the support is contained in $g^{-1}(0)\cap\operatorname{Supp}\widetilde{\mathcal{M}}$. The question is local. Let $x_o\in g^{-1}(0)\cap\operatorname{Supp}\widetilde{\mathcal{M}}$. Assume that a local component S_{x_o} of $\operatorname{Supp}\widetilde{\mathcal{M}}$ at x_o is contained in $f^{-1}(0)$. It is enough to prove the vanishing of $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ in the neighbourhood of a point $x'_o\in S_{x_o}\cap(\operatorname{Supp}\widetilde{\mathcal{M}})^{\mathrm{smooth}}$. We can choose local coordinates at x'_o such that $g=t^r$ for some $r\geqslant 1$. By the example of Section 9.9.a below, we are reduced to proving that, near x'_o, we have $\psi_{t,\lambda}\widetilde{\mathcal{M}}=0$ for every $\lambda\in\mathbb{S}^1$. This follows from Lemma 9.6.3(2).

(2) The first statement follows from the first point. By Proposition 9.6.2 we have ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}={}_{\mathrm{D}}\iota_{t*}\mathrm{gr}_0^V{}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}=:{}_{\mathrm{D}}\iota_{t*}\phi_{g,1}\widetilde{\mathcal{M}}$. On the other hand, we recover $\widetilde{\mathcal{M}}$ from ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ as $\widetilde{\mathcal{M}}={}_{\mathrm{D}}p_{*}{}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$, where $p:X\times\mathbb{C}\to\mathbb{C}$ is the projection. We then use that $p\circ\iota_t=\mathrm{Id}_X$. $\square$

9.7.2. Proposition. *Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g).*

(1) *The following properties are equivalent:*

(a) $\mathrm{var}:\phi_{g,1}\widetilde{\mathcal{M}}\to\psi_{g,1}\widetilde{\mathcal{M}}(-1)$ *is injective,*

(b) ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ *has no proper subobject in* $\mathsf{Mod}_{\mathrm{coh}}(\widetilde{\mathcal{D}}_{X\times\mathbb{C}})$ *supported on H,*

(c) *There is no strictly $\mathbb{R}$-specializable inclusion $\widetilde{\mathcal{N}}\hookrightarrow{}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ with $\widetilde{\mathcal{N}}$ strictly $\mathbb{R}$-specializable supported on H.*

(2) *If* $\mathrm{can}:\psi_{g,1}\widetilde{\mathcal{M}}\to\phi_{g,1}\widetilde{\mathcal{M}}$ *is onto, then ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ has no proper quotient satisfying* 9.3.14(1) *and supported on H.*

***9.7.3. Definition* (Middle extension along (g)).** Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). We say that $\widetilde{\mathcal{M}}$ is a *middle extension along* (g) if $\mathrm{var}:\phi_{g,1}\widetilde{\mathcal{M}}\to\psi_{g,1}\widetilde{\mathcal{M}}(-1)$ is injective and $\mathrm{can}:\psi_{g,1}\widetilde{\mathcal{M}}\to\phi_{g,1}\widetilde{\mathcal{M}}$ is onto. (See Remark 3.3.12 for the terminology.)

The nearby/vanishing Lefschetz quiver of a middle extension is isomorphic to the Lefschetz quiver

$$\begin{tikzcd} \psi_{g,1}\widetilde{\mathcal{M}} \arrow[rr, bend left, "\mathrm{can}=\mathrm{N}"] && \mathrm{Im}\,\mathrm{N}. \arrow[ll, bend left, "\mathrm{var}=\mathrm{incl}", "(-1)"'] \end{tikzcd} \tag{9.7.4}$$

(Proof as Exercise 9.42.)

9.7.5. Proposition. *Let $\widetilde{\mathcal{M}}$ be as in Proposition* 9.7.2. *The following properties are equivalent:*

(1) $\phi_{g,1}\widetilde{\mathcal{M}}=\mathrm{Im}\,\mathrm{can}\oplus\mathrm{Ker}\,\mathrm{var}$,

(2) $\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}'\oplus\widetilde{\mathcal{M}}''$ *with $\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}''$ strictly $\mathbb{R}$-specializable along (g), $\widetilde{\mathcal{M}}'$ being a middle extension along (g) and $\widetilde{\mathcal{M}}''$ supported on $g^{-1}(0)$.*

Moreover, such a decomposition is unique, and if $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}$ satisfy these properties, any morphism $\varphi:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ decomposes correspondingly.

***Proof of Propositions* 9.7.2 *and* 9.7.5.** All along this proof, we set $\widetilde{\mathcal{N}} = {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ for short.

9.7.2(1) (1a) $\Leftrightarrow$ (1b): It is enough to show that the morphisms

$$\begin{tikzcd} & \operatorname{Ker}[t : V_0\widetilde{\mathcal{N}} \to V_{-1}\widetilde{\mathcal{N}}] \arrow[dl, hook'] \arrow[dr] & \\ \operatorname{Ker}[t : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}] & & \operatorname{Ker}[t : \operatorname{gr}_0^V\widetilde{\mathcal{N}} \to \operatorname{gr}_{-1}^V\widetilde{\mathcal{N}}] \end{tikzcd}$$

are isomorphisms. It is clear for the right one, since $t : V^{<0}\widetilde{\mathcal{N}} \to V^{<-1}\widetilde{\mathcal{N}}$ is an isomorphism, according to 9.3.20(a). For the left one this follows from the fact that t is injective on $\operatorname{gr}_\alpha^V\widetilde{\mathcal{N}}$ for $\alpha \neq 0$ according to 9.3.20(c).

(1b) $\Leftrightarrow$ (1c): let us check $\Leftarrow$ (the other implication is clear). Let $\widetilde{\mathcal{T}}$ denote the t-torsion submodule of $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{T}}'$ the $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-submodule generated by

$$\widetilde{\mathcal{T}}_0 := \operatorname{Ker}[t : \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{N}}].$$

9.7.6. Assertion. *$\widetilde{\mathcal{T}}'$ is strictly $\mathbb{R}$-specializable and the inclusion $\widetilde{\mathcal{T}}' \hookrightarrow \widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable.*

This assertion gives the implication $\Leftarrow$ because Assumption (1c) implies $\widetilde{\mathcal{T}}' = 0$, hence $t : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ is injective, so $\widetilde{\mathcal{T}} = 0$.

***Proof of the assertion*.** Let us show first that $\widetilde{\mathcal{T}}'$ is $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-coherent. As we remarked above, we have $\widetilde{\mathcal{T}}_0 = \operatorname{Ker}[t : \operatorname{gr}_0^V\widetilde{\mathcal{N}} \to \operatorname{gr}_{-1}^V\widetilde{\mathcal{N}}]$. Now, $\widetilde{\mathcal{T}}_0$ is the kernel of a linear morphism between $\widetilde{\mathcal{D}}_H$-coherent modules ($H = X \times \{0\}$), hence is also $\widetilde{\mathcal{D}}_H$-coherent. It follows that $\widetilde{\mathcal{T}}'$ is $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-coherent.

Let us now show that $\widetilde{\mathcal{T}}'$ is strictly $\mathbb{R}$-specializable. We note that $\widetilde{\mathcal{T}}_0$ is strict because it is isomorphic to a submodule of $\operatorname{gr}_0^V\widetilde{\mathcal{N}}$. Let $U_\bullet\widetilde{\mathcal{T}}'$ be the filtration induced by $V_\bullet\widetilde{\mathcal{N}}$ on $\widetilde{\mathcal{T}}'$. Then $U_{<0}\widetilde{\mathcal{T}}' = 0$, according to 9.3.20(a), and $\operatorname{gr}_\alpha^U\widetilde{\mathcal{T}}' = 0$ for $\alpha \notin \mathbb{N}$. Let us show by induction on k that

$$U_k\widetilde{\mathcal{T}}' = \widetilde{\mathcal{T}}_0 + \widetilde{\mathcal{T}}_0\widetilde{\partial}_t + \cdots + \widetilde{\mathcal{T}}_0\widetilde{\partial}_t^k.$$

Let us denote by $U'_k\widetilde{\mathcal{T}}'$ the right-hand term. The inclusion $\supset$ is clear. Let $x_o \in H$, $m \in U_k\widetilde{\mathcal{T}}'_{x_o}$ and let $\ell \geqslant k$ such that $m \in U'_\ell\widetilde{\mathcal{T}}'_{x_o}$. If $\ell > k$ one has $m \in \widetilde{\mathcal{T}}'_{x_o} \cap V_{\ell-1}\widetilde{\mathcal{N}}_{x_o}$ hence $mt^\ell \in \widetilde{\mathcal{T}}'_{x_o} \cap V_{-1}\widetilde{\mathcal{N}}_{x_o} = 0$. Set

$$m = m_0 + m_1\widetilde{\partial}_t + \cdots + m_\ell\widetilde{\partial}_t^\ell,$$

with $m_jt = 0$ ($j = 0, \dots, \ell$). One then has $m_\ell\widetilde{\partial}_t^\ell t^\ell = 0$, and since

$$m_\ell\widetilde{\partial}_t^\ell t^\ell = m_\ell \cdot \prod_{j=1}^{\ell}(t\widetilde{\partial}_t + jz) = \ell!\, m_\ell z^\ell$$

and $\widetilde{\mathcal{T}}_0$ is strict, one concludes that $m_\ell = 0$, hence $m \in U'_{\ell-1}\widetilde{\mathcal{T}}'_{x_o}$. By induction, this implies the other inclusion.

As $\mathrm{gr}^U_\alpha \widetilde{\mathfrak{T}}'$ is contained in $\mathrm{gr}^V_\alpha \widetilde{\mathcal{N}}$, one deduces from 9.3.20(d) that $\widetilde{\partial}_t : \mathrm{gr}^U_k \widetilde{\mathfrak{T}}' \to \mathrm{gr}^U_{k+1} \widetilde{\mathfrak{T}}'$ is injective for $k \geqslant 0$. The previous computation shows that it is onto, hence $\widetilde{\mathfrak{T}}'$ is strictly $\mathbb{R}$-specializable and $U_\bullet \widetilde{\mathfrak{T}}'$ is its Malgrange-Kashiwara filtration.

It is now enough to prove that the injective morphism $\mathrm{gr}^U_0 \widetilde{\mathfrak{T}}' \to \mathrm{gr}^V_0 \widetilde{\mathcal{N}}$ is strict. But its cokernel is identified with the submodule $\mathrm{Im}[t : \mathrm{gr}^V_0 \widetilde{\mathcal{N}} \to \mathrm{gr}^V_{-1} \widetilde{\mathcal{N}}]$ of $\mathrm{gr}^V_{-1} \widetilde{\mathcal{N}}$, which is strict. $\square$

9.7.2(2) If can is onto, then $\widetilde{\mathcal{N}} = \widetilde{\mathcal{D}}_{X\times\mathbb{C}} \cdot V_{<0} \widetilde{\mathcal{N}}$. If $\widetilde{\mathcal{N}}$ has a t-torsion quotient $\widetilde{\mathfrak{T}}$ satisfying 9.3.14(1), then $V_{<0} \widetilde{\mathfrak{T}} = 0$, so $V_{<0} \widetilde{\mathcal{N}}$ is contained in $\mathrm{Ker}[\widetilde{\mathcal{N}} \to \widetilde{\mathfrak{T}}]$ and thus $\widetilde{\mathcal{N}} = \widetilde{\mathcal{D}}_{X\times\mathbb{C}} \cdot V_{<0} \widetilde{\mathcal{N}}$ is also contained in this kernel, that is, $\widetilde{\mathfrak{T}} = 0$.

9.7.5(1) $\Rightarrow$ 9.7.5(2) Set

$$U_0 \widetilde{\mathcal{N}}' = V_{<0} \widetilde{\mathcal{N}} + \widetilde{\partial}_t V_{-1} \widetilde{\mathcal{N}} \quad \text{and} \quad \widetilde{\mathfrak{T}}_0 = \mathrm{Ker}[t : \widetilde{\mathcal{N}} \longrightarrow \widetilde{\mathcal{N}}].$$

The assumption (1) is equivalent to $V_0 \widetilde{\mathcal{N}} = U_0 \widetilde{\mathcal{N}}' \oplus \widetilde{\mathfrak{T}}_0$. Define

$$U_k \widetilde{\mathcal{N}}' = V_k \widetilde{\mathcal{D}}_X \cdot U_0 \widetilde{\mathcal{N}}' \quad \text{and} \quad U_k \widetilde{\mathcal{N}}'' = V_k \widetilde{\mathcal{D}}_X \cdot \widetilde{\mathfrak{T}}_0$$

for $k \geqslant 0$. As $V_k \widetilde{\mathcal{N}} = V_{k-1} \widetilde{\mathcal{N}} + \widetilde{\partial}_t V_{k-1} \widetilde{\mathcal{N}}$ for $k \geqslant 1$, one has $V_k \widetilde{\mathcal{N}} = U_k \widetilde{\mathcal{N}}' + U_k \widetilde{\mathcal{N}}''$ for $k \geqslant 0$. Let us show by induction on $k \geqslant 0$ that this sum is direct. Fix $k \geqslant 1$ and let $m \in U_k \widetilde{\mathcal{N}}' \cap U_k \widetilde{\mathcal{N}}''$. Write

$$m = m'_{k-1} + n'_{k-1} \widetilde{\partial}_t = m''_{k-1} + n''_{k-1} \widetilde{\partial}_t, \quad \begin{cases} m'_{k-1}, n'_{k-1} \in U_{k-1} \widetilde{\mathcal{N}}', \\ m''_{k-1}, n''_{k-1} \in U_{k-1} \widetilde{\mathcal{N}}''. \end{cases}$$

One has $[n'_{k-1}] \widetilde{\partial}_t = [n''_{k-1}] \widetilde{\partial}_t$ in $V_k \widetilde{\mathcal{N}} / V_{k-1} \widetilde{\mathcal{N}}$, hence, as

$$\widetilde{\partial}_t : V_{k-1} \widetilde{\mathcal{N}} / V_{k-2} \widetilde{\mathcal{N}} \longrightarrow V_k \widetilde{\mathcal{N}} / V_{k-1} \widetilde{\mathcal{N}}$$

is bijective for $k \geqslant 1$, one gets $[n'_{k-1}] = [n''_{k-1}]$ in $V_{k-1} \widetilde{\mathcal{N}} / V_{k-2} \widetilde{\mathcal{N}}$ and by induction one deduces that both terms are zero. One concludes that $m \in U_{k-1} \widetilde{\mathcal{N}}' \cap U_{k-1} \widetilde{\mathcal{N}}'' = 0$ by induction.

This implies that $\widetilde{\mathcal{N}} = \widetilde{\mathcal{N}}' \oplus \widetilde{\mathcal{N}}''$ with $\widetilde{\mathcal{N}}' := \bigcup_k U_k \widetilde{\mathcal{N}}'$ and $\widetilde{\mathcal{N}}''$ defined similarly. It follows from Exercise 9.22(1) that both $\widetilde{\mathcal{N}}'$ and $\widetilde{\mathcal{N}}''$ are strictly $\mathbb{R}$-specializable along H and the filtrations $U_\bullet$ above are their Malgrange-Kashiwara filtrations. In particular, $\widetilde{\mathcal{N}}'$ satisfies (1) and (2). By Corollary 9.6.5 we also know that $\widetilde{\mathcal{N}}' = {}_{\mathrm{D}}\iota_{g*} \widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{N}}'' = {}_{\mathrm{D}}\iota_{g*} \widetilde{\mathcal{M}}''$ for some coherent $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$.

9.7.5(2) $\Rightarrow$ 9.7.5(1): One has $V_{<0} \widetilde{\mathcal{N}}'' = 0$. Let us show that $\mathrm{Im}\,\mathrm{can} = \mathrm{gr}^V_0 \widetilde{\mathcal{N}}'$ and $\mathrm{Ker}\,\mathrm{var} = \mathrm{gr}^V_0 \widetilde{\mathcal{N}}''$. The inclusions $\mathrm{Im}\,\mathrm{can} \subset \mathrm{gr}^V_0 \widetilde{\mathcal{N}}'$ and $\mathrm{Ker}\,\mathrm{var} \supset \mathrm{gr}^V_0 \widetilde{\mathcal{N}}''$ are clear. Moreover $\mathrm{gr}^V_0 \widetilde{\mathcal{N}}' \cap \mathrm{Ker}\,\mathrm{var} = 0$ as $\widetilde{\mathcal{N}}'$ satisfies (1). Last, $\mathrm{can} : \mathrm{gr}^V_{-1} \widetilde{\mathcal{N}}' \to \mathrm{gr}^V_0 \widetilde{\mathcal{N}}'$ is onto, as $\widetilde{\mathcal{N}}'$ satisfies (2). Hence $\mathrm{gr}^V_0 \widetilde{\mathcal{N}} = \mathrm{Im}\,\mathrm{can} \oplus \mathrm{Ker}\,\mathrm{var}$.

Let us now prove the last assertion. We first note that the uniqueness statement follows from the statement on morphisms: if we have two decomposition $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}'_1 \oplus \widetilde{\mathcal{M}}''_1 = \widetilde{\mathcal{M}}'_2 \oplus \widetilde{\mathcal{M}}''_2$, then the identity morphism decomposes correspondingly.

Let us consider a morphism $\varphi : \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}'' \to \widetilde{\mathcal{N}}' \oplus \widetilde{\mathcal{N}}''$. First, by (1b) in Proposition 9.7.2, the component $\widetilde{\mathcal{M}}'' \to \widetilde{\mathcal{N}}'$ is zero. For the component $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{N}}''$, let us denote by $\widetilde{\mathcal{M}}'_1$ its image. The V-filtration on ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}'_1$ induced by $V_\bullet {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{N}}''$ is coherent (Exercise 9.12(1)) and satisfies 9.3.14(1), hence by Proposition 9.7.2(2) we have ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}'_1 = 0$. □

9.7.7. Definition **(S(upport)-decomposable $\widetilde{\mathcal{D}}_X$-modules).** We say that a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is

- *S-decomposable along* (g) if it is strictly $\mathbb{R}$-specializable along (g) and satisfies the equivalent conditions 9.7.5;
- *S-decomposable at* $x_o \in X$ if for any analytic germ $g : (X, x_o) \to (\mathbb{C}, 0)$ such that $g^{-1}(0) \cap \operatorname{Supp} \widetilde{\mathcal{M}}$ has everywhere codimension 1 in $\operatorname{Supp} \widetilde{\mathcal{M}}$, $\widetilde{\mathcal{M}}$ is S-decomposable along (g) in some neighbourhood of x_o;
- *S-decomposable* if it is S-decomposable at all points $x_o \in X$.

9.7.8. Lemma.

(1) *If* $\widetilde{\mathcal{M}}$ *is S-decomposable along* (g)*, then it is S-decomposable along* (g^r) *for every* $r \geqslant 1$.

(2) *If* $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 \oplus \widetilde{\mathcal{M}}_2$*, then* $\widetilde{\mathcal{M}}$ *is S-decomposable of some kind if and only if* $\widetilde{\mathcal{M}}_1$ *and* $\widetilde{\mathcal{M}}_2$ *are so.*

(3) *We assume that* $\widetilde{\mathcal{M}}$ *is S-decomposable and its support* Z *is a pure dimensional closed analytic subset of* X*. Then the following conditions are equivalent:*

(a) *for any analytic germ* $g : (X, x_o) \to (\mathbb{C}, 0)$ *such that* $g^{-1}(0) \cap Z$ *has everywhere codimension* 1 *in* Z, ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ *is a middle extension along* (g)*;*

(b) *near any* x_o*, there is no* $\widetilde{\mathcal{D}}_X$*-coherent submodule of* $\widetilde{\mathcal{M}}$ *with support having codimension* $\geqslant 1$ *in* Z*;*

(c) *near any* x_o*, there is no nonzero morphism* $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$*, with* $\widetilde{\mathcal{N}}$ *S-decomposable at* x_o*, such that* $\operatorname{Im} \varphi$ *is supported in codimension* $\geqslant 1$ *in* Z.

Proof. Property (1) is an immediate consequence of the example of Section 9.9.a, and Property (2) follows from the fact that for any germ g, the corresponding can and var decompose with respect to the given decomposition of $\widetilde{\mathcal{M}}$. Let us now prove (3). Both conditions (3a) and (3b) reduce to the property that, for any analytic germ $g : (X, x_o) \to (\mathbb{C}, 0)$ which does not vanish identically on any local irreducible component of Z at x_o, the corresponding decomposition $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}''$ of 9.7.5(2) reduces to $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}'$, i.e., $\widetilde{\mathcal{M}}'' = 0$. For the equivalence with (3c), we note that, according to the last assertion in Proposition 9.7.5, and with respect to the decomposition $\varphi = \varphi' \oplus \varphi''$ along a germ g, we have $\operatorname{Im} \varphi \neq 0$ and supported in $g^{-1}(0)$ if and only if $\operatorname{Im} \varphi'' \neq 0$, and thus $\widetilde{\mathcal{M}}'' \neq 0$. Conversely, if $\widetilde{\mathcal{M}}'' \neq 0$, the projection $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}''$ gives a morphism contradicting (3c). □

9.7.9. Definition **(Pure support).** Let $\widetilde{\mathcal{M}}$ be S-decomposable and having support a pure dimensional closed analytic subset Z of X. We say that $\widetilde{\mathcal{M}}$ has *pure support* Z if the equivalent conditions of 9.7.8(3) are satisfied.

9.7.10. Proposition **(Generic structure of a S-decomposable module)**

Assume that $\widetilde{\mathcal{M}}$ is holonomic and S-decomposable with pure support Z, where Z is smooth. Then there exists a unique holonomic and S-decomposable $\widetilde{\mathcal{D}}_Z$-module $\widetilde{\mathcal{N}}$ such that $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{Z*}\widetilde{\mathcal{N}}$. Moreover, there exists a Zariski dense open subset $Z^o \subset Z$ such that $\widetilde{\mathcal{N}}_{|Z^o}$ is $\widetilde{\mathcal{O}}_{Z^o}$-coherent and strict.

Proof. Let us consider the first statement. By uniqueness, the question is local on Z. We argue by induction on $\dim X$. Let H be a smooth hypersurface containing Z such that $H = \{t = 0\}$ of some local coordinate system $(t, x_2, \dots, x_d)$. Since $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along t, the strict Kashiwara's equivalence implies that $\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{H*}\widetilde{\mathcal{N}}$ for a unique coherent $\widetilde{\mathcal{D}}_H$-module $\widetilde{\mathcal{N}}$. Moreover, $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along any function g on H, according to Proposition 9.6.6. If $g = g_{|H}$, then one checks that a decomposition 9.7.5(2) for $\widetilde{\mathcal{M}}$ along (g) comes from a decomposition 9.7.5(2) for $\widetilde{\mathcal{N}}$ along g. We conclude that $\widetilde{\mathcal{N}}$ is also S-decomposable, and has pure support $Z \subset H$. Continuing this way, we reach a coherent $\widetilde{\mathcal{D}}_Z$-module which is S-decomposable. It is easy to check that $\widetilde{\mathcal{N}}$ is holonomic since, if $\operatorname{Char}\widetilde{\mathcal{M}}$ denotes the characteristic variety of $\widetilde{\mathcal{M}}$, it is obtained by a straightforward formula from $\operatorname{Char}\widetilde{\mathcal{N}}$.

Coming back to the global setting, we consider the characteristic variety $\operatorname{Char}\widetilde{\mathcal{N}}$ of $\widetilde{\mathcal{N}}$, which is contained, by definition, in a set of the form $\big(\bigcup_i T^*_{Z_i}Z\big) \times \mathbb{C}_z$, where Z_i is an irreducible closed analytic subset of Z, one of them being Z. We set $Z^o = Z \smallsetminus \bigcup_{i|Z_i\neq Z} Z_i$. In such a way, we obtain a Zariski-dense open subset Z^o of Z such that $\operatorname{Char}\widetilde{\mathcal{N}}_{|Z^o} \subset T^*_{Z^o}Z^o \times \mathbb{C}_z$. We conclude from Exercise 8.68 that $\widetilde{\mathcal{N}}_{|Z^o}$ is $\widetilde{\mathcal{O}}_{Z^o}$-coherent.

Let us now restrict to Z^o and prove that $\widetilde{\mathcal{N}}$ is strict there. If t is a local coordinate, notice that each term of the V-filtration $V_\bullet\widetilde{\mathcal{N}}$ is also $\widetilde{\mathcal{O}}_{Z^o}$-coherent (recall that we know that $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along t). It follows that the V-filtration is locally stationary, hence $\widetilde{\mathcal{N}} = V_0\widetilde{\mathcal{N}}$, since $\operatorname{gr}^V_\alpha\widetilde{\mathcal{N}} = 0$ for $\alpha \gg 0$ (Proposition 9.3.20(d)), hence for all $\alpha > 0$. Let m be a local section of $\widetilde{\mathcal{N}}$ killed by z. Then m is zero in $\widetilde{\mathcal{N}}/\widetilde{\mathcal{N}}t$ by strict $\mathbb{R}$-specializability. As $\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{O}}_{Z^o}$-coherent, Nakayama's lemma (applied to $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_{Z^o}} \mathcal{O}_{Z^o\times\mathbb{C}_z}$) implies that $m = 0$. $\square$

9.7.11. Corollary. *Let $\widetilde{\mathcal{M}}$ be holonomic and S-decomposable. Then $\widetilde{\mathcal{M}}$ is strict.*

Proof. The question is local, and we can assume that $\widetilde{\mathcal{M}}$ has pure support Z with Z closed irreducible analytic near x_o. Proposition 9.7.10 applied to the smooth part of Z produces a dense open subset Z^o of Z such that $\widetilde{\mathcal{M}}_{|Z^o}$ is strict. Let m be a local section of $\widetilde{\mathcal{M}}$ near $x_o \in Z$ killed by z. Then $m \cdot \widetilde{\mathcal{D}}_X$ is supported by a proper analytic subset of Z near x_o by the previous argument. As $\widetilde{\mathcal{M}}$ has pure support Z, we conclude that $m = 0$. $\square$

9.7.12. Corollary. *Let $\widetilde{\mathcal{M}}$ be holonomic and S-decomposable. Then there exist irreducible closed analytic subsets Z_i of X such that* $\operatorname{Char}\widetilde{\mathcal{M}} = \big(\bigcup_i T^*_{Z_i}X\big) \times \mathbb{C}_z$.

Proof. Since $\widetilde{\mathcal{M}}$ is strict, there exists a coherently F-filtered $\mathcal{D}_X$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ such that $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. We can thus apply Exercise 8.70(1). $\square$

9.7.13. Corollary. *Let $Z \subset X$ be a closed analytic subset of X and let $\widetilde{\mathcal{M}}$ be holonomic and S-decomposable with pure support Z. Then there exists a dense open subset Z^o of Z, a neighbourhood $\mathrm{nb}(Z^o)$ in X, and a $\widetilde{\mathcal{D}}_{Z^o}$-holonomic module $\widetilde{\mathcal{N}}$ which is $\widetilde{\mathcal{O}}_{Z^o}$-locally free of finite rank, such that $\widetilde{\mathcal{M}}_{|\mathrm{nb}(Z^o)} = {}_{\mathrm{D}}\iota_{Z^o *}\widetilde{\mathcal{N}}$.*

Proof. By restricting first to a neighbourhood of the smooth locus of Z, we can assume that Z is smooth, so that the setting is that of Proposition 9.7.10, and we can also assume that $X = Z$. Recall that, by strictness, $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. According to the same proposition, we can also assume that $\mathcal{M}$ is $\mathcal{O}_X$-coherent, hence $\mathcal{O}_X$-locally free (see Exercise 8.68(3)).

The filtration $F_\bullet\mathcal{M}$ has then only a finite number of jumps, and $\mathrm{gr}^F\mathcal{M}$ is also $\mathcal{O}_X$-coherent. Up to restricting to a dense open subset, we can assume that $\mathrm{gr}^F\mathcal{M}$ is $\mathcal{O}_X$-locally free. For each p, let $\boldsymbol{v}_p$ be a local family of elements of $F_p\mathcal{M}$ whose classes in $\mathrm{gr}^F_p\mathcal{M}$ form a local frame. Then $(\boldsymbol{v}_p)_p$ is a local frame of $\mathcal{M}$. We have a natural surjective morphism $\bigoplus_p z^p\widetilde{\mathcal{O}}_X\boldsymbol{v}_p \to R_F\mathcal{M}$, which induces an isomorphism after tensoring with $\widetilde{\mathcal{O}}_X[z^{-1}]$ over $\widetilde{\mathcal{O}}_X$, since both terms have $(\boldsymbol{v}_p)_p$ as an $\widetilde{\mathcal{O}}_X[z^{-1}]$-basis. Each local section of the kernel is thus annihilated by some power of z, hence is zero since the left-hand term is obviously strict. Therefore, $R_F\mathcal{M}$ is $\widetilde{\mathcal{O}}_X$-locally free. □

We will now show that a S-decomposable holonomic $\widetilde{\mathcal{D}}_X$-module (see Definition 8.8.15) can indeed be decomposed as the direct sum of holonomic $\widetilde{\mathcal{D}}_X$-modules having as pure support closed irreducible analytic subsets. These subsets are then called the *pure components of (the support of)* $\widetilde{\mathcal{M}}$ (note that a pure component could be included in another one). We first consider the local decomposition and, by uniqueness, we get the global one. It is important for that to be able to define *a priori* the pure components. They are obtained from the characteristic variety of $\widetilde{\mathcal{M}}$, equivalently of $\mathcal{M}$, according to Corollary 9.7.12.

9.7.14. Proposition. *Let $\widetilde{\mathcal{M}}$ be holonomic and S-decomposable at x_o, and let $(Z_i, x_o)_{i\in I}$ be the family of closed irreducible analytic germs (Z_i, x_o) such that $\mathrm{Char}\,\widetilde{\mathcal{M}} = \bigcup_i T^*_{Z_i}X \times \mathbb{C}_z$ near x_o. There exists a unique decomposition $\widetilde{\mathcal{M}}_{x_o} = \oplus_{i\in I}\widetilde{\mathcal{M}}_{Z_i,x_o}$ of germs at x_o such that $\widetilde{\mathcal{M}}_{Z_i,x_o} = 0$ or has pure support (Z_i, x_o).*

Proof. For the existence of the decomposition, we will argue by induction on $\dim\mathrm{Supp}\,\widetilde{\mathcal{M}}$. The case where $\dim\mathrm{Supp}\,\widetilde{\mathcal{M}}$ is clear. First, we reduce to the case when the support S of $\widetilde{\mathcal{M}}$ (see Proposition 8.8.10) is irreducible at x_o. For this purpose, let us decompose the germ of S at $x_o$into its irreducible components $\bigcup_j S_j$. Let g be a germ of holomorphic function at x_o such that $g^{-1}(0) \cap S$ has everywhere codimension 1 in S and contains the support of the kernel and cokernel of

$$\bigoplus_j \Gamma_{S_j}\widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}$$

(see Lemma 8.8.11). Let us consider the decomposition $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}''$ with $\widetilde{\mathcal{M}}'$ being a middle extension along (g) and $\widetilde{\mathcal{M}}''$ supported on $g^{-1}(0)$. Since the kernel and cokernel of the above morphism have support contained in $g^{-1}(0)$, we conclude that it induces an isomorphism $\bigoplus_j \Gamma_{S_j}\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$. Moreover, since S-decomposability is stable by direct summand (Lemma 9.7.8(2)), each $\Gamma_{S_j}\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}''$ are S-decomposable. We can apply the induction hypothesis to $\widetilde{\mathcal{M}}''$, and we are reduced to treat each $\Gamma_{S_j}\widetilde{\mathcal{M}}$, so we can assume that S is irreducible and has dimension $\geqslant 1$.

Let us now choose a germ $g : (X, x_o) \to (\mathbb{C}, 0)$ which is non-constant on S and such that $g^{-1}(0)$ contains all the components Z_i defined by $\operatorname{Char}\widetilde{\mathcal{M}}$, except S. We have, as above, a unique decomposition $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}''$ of germs at x_o, where $\widetilde{\mathcal{M}}'$ is a middle extension along (g), and $\widetilde{\mathcal{M}}''$ is supported on $g^{-1}(0)$, by the assumption of S-decomposability along (g) at x_o. Moreover, $\widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{M}}''$ are also S-decomposable at x_o. We can apply the inductive assumption to $\widetilde{\mathcal{M}}''$.

Let us show that $\widetilde{\mathcal{M}}'$ has pure support S near x_o: if $\widetilde{\mathcal{M}}'_1$ is a coherent submodule of $\widetilde{\mathcal{M}}'$ supported on a strict analytic subset $Z \subset S$, then Z is contained in the union of the components Z_i, hence $\widetilde{\mathcal{M}}'_1$ is supported in $g^{-1}(0)$, so is zero. We conclude by 9.7.8(3b).

For the uniqueness of the decomposition, we note that, given two local decompositions with components $\widetilde{\mathcal{M}}_{Z_i,x_o}, \widetilde{\mathcal{M}}'_{Z_i,x_o}$, the components φ_{ij} of any morphism $\varphi : \widetilde{\mathcal{M}}_{x_o} \to \widetilde{\mathcal{M}}_{x_o}$ vanishes as soon as $i \neq j$. Indeed, we have either $\operatorname{codim}_{Z_i}(Z_i \cap Z_j) \geqslant 1$, or $\operatorname{codim}_{Z_j}(Z_i \cap Z_j) \geqslant 1$. In the first case we apply Lemma 9.7.8(3c) to $\widetilde{\mathcal{M}}_{Z_i,x_o}$. In the second case, we apply Lemma 9.7.8(3b) to $\widetilde{\mathcal{M}}'_{Z_j,x_o}$. We apply this same result to $\varphi = \mathrm{Id} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}$ to obtain uniqueness. $\square$

By uniqueness of the local decomposition, we get:

9.7.15. Corollary. *Let $\widetilde{\mathcal{M}}$ be holonomic and S-decomposable on X and let $(Z_i)_{i\in I}$ be the (locally finite) family of closed irreducible analytic subsets Z_i such that* $\operatorname{Char}\widetilde{\mathcal{M}} = \bigcup_i T^*_{Z_i}X \times \mathbb{C}_z$. *There exists a unique decomposition $\widetilde{\mathcal{M}} = \oplus_i \widetilde{\mathcal{M}}_{Z_i}$ such that each $\widetilde{\mathcal{M}}_{Z_i} = 0$ or has pure support Z_i.*

As indicated above, a closed analytic irreducible subset Z of X such that $\widetilde{\mathcal{M}}_Z \neq 0$ is called a *pure component* of $\widetilde{\mathcal{M}}$.

***Proof of Corollary* 9.7.15.** Given the family $(Z_i)_{i\in I}$ and $x_o \in X$, the germs (Z_i, x_o) are equidimensional, and Proposition 9.7.14 gives a unique decomposition $\widetilde{\mathcal{M}}_{x_o} = \oplus_{i\in I}\widetilde{\mathcal{M}}_{Z_i,x_o}$ by gathering the local irreducible components of (Z_i, x_o). The uniqueness enables us to glue all along Z_i the various germs $\widetilde{\mathcal{M}}_{Z_i,x}$. $\square$

9.7.16. Corollary. *Let $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ be two holonomic $\widetilde{\mathcal{D}}_X$-module which are S-decomposable and let $(Z_i)_{i\in I}$ be the family of their pure components. Then any morphism $\varphi : \widetilde{\mathcal{M}}'_{Z_i} \to \widetilde{\mathcal{M}}''_{Z_j}$ vanishes identically if $Z_i \neq Z_j$.*

Proof. The image of φ is supported on $Z_i \cap Z_j$, which has everywhere codimension $\geqslant 1$ in Z_i or Z_j if $Z_i \neq Z_j$. We then apply Lemma 9.7.8. $\square$

***9.7.17. Remark* (Restriction to $z = 1$).** Let us keep the notation of Exercise 9.27 and let us assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable. It is obvious that, if can is onto for $\widetilde{\mathcal{M}}$, it is also onto for $\mathcal{M} := \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}(z-1)$. On the other hand, it is also true that, if var in injective for $\widetilde{\mathcal{M}}$, it is also injective for $\mathcal{M}$ (see Exercise 5.2(3)). As a consequence, if $\widetilde{\mathcal{M}}$ is a middle extension along (g), so is $\mathcal{M}$. Moreover, if $\widetilde{\mathcal{M}}$ is S-decomposable along (g) at x_o, so is $\mathcal{M}$, and the strict decomposition $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}' \oplus \widetilde{\mathcal{M}}''$ restricts to the decomposition $\mathcal{M} = \mathcal{M}' \oplus \mathcal{M}''$ given by 9.7.5(2).

We conclude that, if $\widetilde{\mathcal{M}}$ is S-decomposable, then $\mathcal{M}$ is S-decomposable, and the pure components are in one-to-one correspondence.

9.8. Direct image of strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathcal{D}}_X$-modules

Let us consider the setting of Theorem 8.8.24. So $f : X \to X'$ is a proper holomorphic map, and $\widetilde{\mathcal{M}}$ is a coherent *right* $\widetilde{\mathcal{D}}_X$-module. Let $H' \subset X'$ be a smooth hypersurface. We will assume that $H := f^*(H')$ is also a smooth hypersurface of X.

If $\widetilde{\mathcal{M}}$ has a coherent V-filtration $U_\bullet\widetilde{\mathcal{M}}$ along H, the $R_V\widetilde{\mathcal{D}}_X$-module $R_U\widetilde{\mathcal{M}}$ is therefore coherent. With the assumptions above it is possible to define a sheaf $R_V\widetilde{\mathcal{D}}_{X\to X'}$ and therefore the pushforward ${}_{\mathrm{D}}f_*R_U\widetilde{\mathcal{M}}$ as an $R_V\widetilde{\mathcal{D}}_{X'}$-module (where $V_\bullet\widetilde{\mathcal{D}}_{X'}$ is the V-filtration relative to H').

We will show the $R_V\widetilde{\mathcal{D}}_{X'}$-coherence of the cohomology sheaves ${}_{\mathrm{D}}f_*^{(k)}R_U\widetilde{\mathcal{M}}$ of the pushforward ${}_{\mathrm{D}}f_*R_U\widetilde{\mathcal{M}}$ if $\widetilde{\mathcal{M}}$ is equipped with a coherent filtration. By the argument of Exercise 9.11, by quotienting by the v-torsion, we obtain a coherent V-filtration on the cohomology sheaves ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ of the pushforward ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$.

The v-torsion part contains much information however, and this supplementary operation killing the v-torsion looses it. The main result of this section is that, if $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, then so are the cohomology sheaves ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ along H', and moreover, when considering the filtration by the order, the corresponding Rees modules ${}_{\mathrm{D}}f_*^{(k)}R_V\widetilde{\mathcal{M}}$ have no v-torsion, and can thus be written as $R_U{}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ for some coherent V-filtration $U_\bullet{}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$. This coherent V-filtration is nothing but the Kashiwara-Malgrange filtration of ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$. We say that the Kashiwara-Malgrange filtration behaves *strictly* with respect to the pushforward functor ${}_{\mathrm{D}}f_*$.

Another way to present this property is to consider the individual terms $V_\alpha\widetilde{\mathcal{M}}$ of the Kashiwara-Malgrange filtration as $V_0\widetilde{\mathcal{D}}_X$-modules. By introducing the sheaf $V_0\widetilde{\mathcal{D}}_{X\to X'}$, one can define the pushforward complex ${}_{\mathrm{D}}f_*V_\alpha\widetilde{\mathcal{M}}$ for every α, and the strictness property amounts to saying that for every k and α, the morphisms ${}_{\mathrm{D}}f_*^{(k)}V_\alpha\widetilde{\mathcal{M}} \to {}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ are *injective.* In the following, we work with right $\widetilde{\mathcal{D}}_X$-modules and increasing V-filtrations.

9.8.a. Definition of the pushforward functor and the coherence theorem

We first note that our assumption on H, H', f is equivalent to the property that, locally at $x_o \in H$, setting $x'_o = f(x_o)$, there exist local decompositions $(X, x_o) \simeq (H, x_o) \times (\mathbb{C}, 0)$ and $(X', x'_o) \simeq (H', x'_o) \times (\mathbb{C}, 0)$ such that $f(y,t) = (f_{|H}(y), t)$.

Let $U_\bullet\widetilde{\mathcal{M}}$ be a V-filtration of $\widetilde{\mathcal{M}}$ and let $R_U\widetilde{\mathcal{M}}$ be the associated graded $R_V\widetilde{\mathcal{D}}_X$-module. Our first objective is to apply the same reasoning as in Theorem 8.8.24 by replacing the category of $\widetilde{\mathcal{D}}$-modules with that of graded $R_V\widetilde{\mathcal{D}}_X$-modules.

The sheaf $\widetilde{\mathcal{D}}_{X\to X'}$ has a V-filtration: we set $V_k\widetilde{\mathcal{D}}_{X\to X'} := \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}V_k\widetilde{\mathcal{D}}_{X'}$. One checks in local decompositions as above that, with respect to the left $\widetilde{\mathcal{D}}_X$-structure one has $V_\ell\widetilde{\mathcal{D}}_X \cdot V_k\widetilde{\mathcal{D}}_{X\to X'} \subset V_{k+\ell}\widetilde{\mathcal{D}}_{X\to X'}$. We can write

$$R_V\widetilde{\mathcal{D}}_{X\to X'} := \widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}R_V\widetilde{\mathcal{D}}_{X'} = R_V\widetilde{\mathcal{O}}_X \otimes_{f^{-1}R_V\widetilde{\mathcal{O}}_{X'}} f^{-1}R_V\widetilde{\mathcal{D}}_{X'}. \tag{9.8.1}$$

Indeed, this amounts to checking that

$$\widetilde{\mathcal{O}}_X \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}R_V\widetilde{\mathcal{O}}_{X'} = R_V\widetilde{\mathcal{O}}_X,$$

which is clear. According to Exercise 9.7, $R_V\widetilde{\mathcal{D}}_{X'}$ is $R_V\widetilde{\mathcal{O}}_{X'}$-locally free, so $R_V\widetilde{\mathcal{D}}_{X\to X'}$ is $R_V\widetilde{\mathcal{O}}_X$-locally free.

We define

$$_{\mathrm{D}}f_*R_U\widetilde{\mathcal{M}} := \boldsymbol{R}f_*\big(R_U\widetilde{\mathcal{M}} \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{D}}_X} R_V\widetilde{\mathcal{D}}_{X\to X'}\big) \tag{9.8.2}$$

as an object of $\mathsf{D}^{\mathrm{b}}(R_V\widetilde{\mathcal{D}}_{X'})$.

9.8.3. Theorem. *Let $\widetilde{\mathcal{M}}$ be a $\widetilde{\mathcal{D}}_X$-module equipped with a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$. Let $U_\bullet\widetilde{\mathcal{M}}$ be a coherent V-filtration of $\widetilde{\mathcal{M}}$. Then the cohomology modules of $_{\mathrm{D}}f_*R_U\widetilde{\mathcal{M}}$ have coherent $R_V\widetilde{\mathcal{D}}_{X'}$-cohomology.*

9.8.4. Lemma. *Let $\widetilde{\mathcal{L}}$ be an $R_V\widetilde{\mathcal{O}}_X$-module. Then*

$$(\widetilde{\mathcal{L}} \otimes_{R_V\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X) \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{D}}_X} R_V\widetilde{\mathcal{D}}_{X\to X'} = \widetilde{\mathcal{L}} \otimes_{f^{-1}R_V\widetilde{\mathcal{O}}_{X'}} f^{-1}R_V\widetilde{\mathcal{D}}_{X'}.$$

Proof. It is a matter of proving that the left-hand side has cohomology in degree 0 only, since this cohomology is easily seen to be equal to the right-hand side. This can be checked on germs at $x \in X$. Let $\widetilde{\mathcal{L}}^\bullet_x$ be a resolution of $\widetilde{\mathcal{L}}_x$ by free $R_V\widetilde{\mathcal{O}}_{X,x}$-modules. We have

$$\begin{aligned}
(\widetilde{\mathcal{L}}_x&\otimes_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X,x}) \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{D}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x}\\
&= (\widetilde{\mathcal{L}}_x \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X,x}) \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{D}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x} \quad \text{(Ex. 9.7)}\\
&= (\widetilde{\mathcal{L}}^\bullet_x \otimes_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X,x}) \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{D}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x}\\
&= (\widetilde{\mathcal{L}}^\bullet_x \otimes_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X,x}) \otimes_{R_V\widetilde{\mathcal{D}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x}\\
&= \widetilde{\mathcal{L}}^\bullet_x \otimes_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x} = \widetilde{\mathcal{L}}_x \otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x}\\
&= \widetilde{\mathcal{L}}_x \otimes_{R_V\widetilde{\mathcal{O}}_{X,x}} R_V\widetilde{\mathcal{D}}_{X\to X',x} \quad (R_V\widetilde{\mathcal{D}}_{X\to X',x} \text{ is } R_V\widetilde{\mathcal{O}}_{X,x}\text{-free})\\
&= \widetilde{\mathcal{L}}_x \otimes_{f^{-1}R_V\widetilde{\mathcal{O}}_{X',x'}} f^{-1}R_V\widetilde{\mathcal{D}}_{X',x'}.
\end{aligned}$$
$\square$

As a consequence of this lemma, we have

$$_{\mathrm{D}}f_*(\widetilde{\mathcal{L}} \otimes_{R_V\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X) = (\boldsymbol{R}f_*\widetilde{\mathcal{L}}) \otimes_{R_V\widetilde{\mathcal{O}}_{X'}} R_V\widetilde{\mathcal{D}}_{X'} \tag{9.8.5}$$

and the cohomology of this complex is $R_V\widetilde{\mathcal{D}}_{X'}$-coherent.

9.8.6. Remark. Assume that $\widetilde{\mathcal{L}} = \mathcal{K} \otimes_{\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{O}}_X$ for some $\widetilde{\mathcal{O}}_X$-module $\mathcal{K}$. Note that, by flatness (see Exercise 9.7),

$$\mathcal{K} \otimes^{\boldsymbol{L}}_{\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X = \mathcal{K} \otimes_{\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X = \widetilde{\mathcal{L}} \otimes_{R_V\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X.$$

Hence, by Lemma 9.8.4 and (9.8.1),

$$(\mathcal{K}\otimes_{\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X)\otimes^{\boldsymbol{L}}_{R_V\widetilde{\mathcal{D}}_X} R_V\widetilde{\mathcal{D}}_{X\to X'} = \mathcal{K}\otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}R_V\widetilde{\mathcal{D}}_{X'} = \mathcal{K}\otimes^{\boldsymbol{L}}_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}R_V\widetilde{\mathcal{D}}_{X'},$$

and thus (9.8.5) becomes

$$_{\mathrm{D}}f_*(\mathcal{K}\otimes_{\widetilde{\mathcal{O}}_X} R_V\widetilde{\mathcal{D}}_X) = \boldsymbol{R}f_*\mathcal{K}\otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} R_V\widetilde{\mathcal{D}}_{X'}.$$

9.8.7. Lemma. *Assume that $\widetilde{\mathcal{M}}$ is a $\widetilde{\mathcal{D}}_X$-module having a coherent filtration $F_\bullet\widetilde{\mathcal{M}}$ and let $U_\bullet\widetilde{\mathcal{M}}$ be a coherent V-filtration of $\widetilde{\mathcal{M}}$. Then in the neighbourhood of any compact set of X, $R_U\widetilde{\mathcal{M}}$ has a coherent $F_\bullet R_V\widetilde{\mathcal{D}}_X$-filtration.*

Proof. Fix a compact set $K\subset X$. We can thus assume that $\widetilde{\mathcal{M}}$ is generated by a coherent $\widetilde{\mathcal{O}}_X$-module $\mathcal{F}$ in some neighbourhood of K, i.e., $\widetilde{\mathcal{M}}=\widetilde{\mathcal{D}}_X\cdot\mathcal{F}$. Consider the V-filtration $U'_\bullet\widetilde{\mathcal{M}}$ generated by $\mathcal{F}$, i.e., $U'_\bullet\widetilde{\mathcal{M}}=V_\bullet\widetilde{\mathcal{D}}_X\cdot\mathcal{F}$. Then, clearly, $R_V\widetilde{\mathcal{O}}_X\cdot\mathcal{F}=\bigoplus_k V_k\widetilde{\mathcal{O}}_X\cdot\mathcal{F}v^k$ is a coherent graded $R_V\widetilde{\mathcal{O}}_X$-module which generates $R_{U'}\widetilde{\mathcal{M}}$ as an $R_V\widetilde{\mathcal{D}}_X$-module.

If the filtration $U''_\bullet\widetilde{\mathcal{M}}$ is obtained from $U'_\bullet\widetilde{\mathcal{M}}$ by a shift by $-\ell\in\mathbb{Z}$, i.e., if $R_{U''}\widetilde{\mathcal{M}}=v^\ell R_{U'}\widetilde{\mathcal{M}}\subset\widetilde{\mathcal{M}}[v,v^{-1}]$, then $R_{U''}\widetilde{\mathcal{M}}$ is generated by the $R_V\widetilde{\mathcal{O}}_X$-coherent submodule $v^\ell R_V\widetilde{\mathcal{O}}_X\cdot\mathcal{F}$.

On the other hand, let $U''_\bullet\widetilde{\mathcal{M}}$ be a coherent V-filtration such that $R_{U''}\widetilde{\mathcal{M}}$ has a coherent $F_\bullet R_V\widetilde{\mathcal{D}}_X$-filtration. Then any coherent V-filtration $U_\bullet\widetilde{\mathcal{M}}$ such that $U_k\widetilde{\mathcal{M}}\subset U''_k\widetilde{\mathcal{M}}$ for every k satisfies the same property, because $R_U\widetilde{\mathcal{M}}$ is thus a coherent graded $R_V\widetilde{\mathcal{D}}_X$-submodule of $R_{U''}\widetilde{\mathcal{M}}$, so a coherent filtration on the latter induces a coherent filtration on the former.

As any coherent V-filtration $U_\bullet\widetilde{\mathcal{M}}$ is contained, in some neighbourhood of K, in the coherent V-filtration $U'_\bullet\widetilde{\mathcal{M}}$ suitably shifted, we get the lemma. □

***Proof of Theorem* 9.8.3.** The proof now ends exactly as that for Theorem 8.8.24. □

9.8.b. Strictness of the Kashiwara-Malgrange filtration by pushforward

9.8.8. Theorem **(Pushforward of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}$-modules)**

Let $f:X\to X'$ be a proper morphism of complex manifolds, let H' be a smooth hypersurface of X' and assume that $\mathcal{I}_H:=\mathcal{I}_{H'}\mathcal{O}_X$ defines a smooth hypersurface H of X. Let $\widetilde{\mathcal{M}}$ be a coherent right $\widetilde{\mathcal{D}}_X$-module equipped with a coherent filtration. Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H with Kashiwara-Malgrange filtration $V_\bullet\widetilde{\mathcal{M}}$ indexed by $A+\mathbb{Z}$ with A finite contained in $(-1,0]$, and that each cohomology module $_{\mathrm{D}}f^{(i)}_{|H*}\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}$ is strict *($\alpha\in[-1,0]$).*

Then each cohomology module $_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$, which is $\widetilde{\mathcal{D}}_{X'}$-coherent according to Theorem 8.8.24*, is strictly $\mathbb{R}$-specializable along H' and moreover,*

1. *for every α,i, the natural morphism $_{\mathrm{D}}f^{(i)}_*(V_\alpha\widetilde{\mathcal{M}})\to{}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ is injective,*
2. *its image is the Kashiwara-Malgrange filtration of $_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ along H',*
3. *for every α,i, $\mathrm{gr}^V_\alpha({}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}})={}_{\mathrm{D}}f^{(i)}_{|H*}(\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}})$.*

As an important corollary we obtain in a straightforward way:

9.8.9. Corollary. *Let $f : X \to X'$ be a proper morphism of complex manifolds. Let $g' : X' \to \mathbb{C}$ be any holomorphic function on X' and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along (g) with $g = g' \circ f$. Assume that for all i and λ, ${}_{\mathrm{D}}f_*^{(i)}(\psi_{g,\lambda}\widetilde{\mathcal{M}})$ and ${}_{\mathrm{D}}f_*^{(i)}(\phi_{g,1}\widetilde{\mathcal{M}})$ are* strict.

Then ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_{X'}$-coherent and strictly $\mathbb{R}$-specializable along (g'), we have for all i and λ,

$$(\psi_{g,\lambda}({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}), \mathrm{N}) = {}_{\mathrm{D}}f_*^{(i)}(\psi_{g,\lambda}\widetilde{\mathcal{M}}, \mathrm{N}),$$
$$(\phi_{g,1}({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}), \mathrm{N}) = {}_{\mathrm{D}}f_*^{(i)}(\phi_{g,1}\widetilde{\mathcal{M}}, \mathrm{N}),$$

and the morphisms can, var *for ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ are the morphisms ${}_{\mathrm{D}}f_*^{(i)}$ can, ${}_{\mathrm{D}}f_*^{(i)}$ var.* □

We first explain the mechanism which leads to the strictness property stated in Theorem 9.8.8(1).

9.8.10. Proposition. *Let $H' \subset X'$ be a smooth hypersurface. Let $(\widetilde{\mathcal{N}}^\bullet, U_\bullet\widetilde{\mathcal{N}}^\bullet)$ be a V-filtered complex of $\widetilde{\mathcal{D}}_{X'}$-modules, where $U_\bullet$ is indexed by $A+\mathbb{Z}$, $A \subset (-1,0]$ finite. Let $N \geqslant 0$ and assume that*

(1) *$H^i(\mathrm{gr}_\alpha^U\widetilde{\mathcal{N}}^\bullet)$ is strict for all $\alpha \in A+\mathbb{Z}$ and all $i \geqslant -N-1$;*

(2) *for every $\alpha \in A+\mathbb{Z}$, there exists $\nu_\alpha \geqslant 0$ such that $(\mathrm{E}-\alpha z)^{\nu_\alpha}$ acts by zero on $H^i(\mathrm{gr}_\alpha^U\widetilde{\mathcal{N}}^\bullet)$ for every $i \geqslant -N-1$;*

(3) *there exists α_o such that for all $\alpha \leqslant \alpha_o$ and all $i \geqslant -N-1$, the right multiplication by some (or any) local reduced equation t of H' induces an isomorphism $t : U_\alpha\widetilde{\mathcal{N}}^i \xrightarrow{\sim} U_{\alpha-1}\widetilde{\mathcal{N}}^i$;*

(4) *there exists $i_o \in \mathbb{Z}$ such that, for all $i \geqslant i_o$ and any α, one has $H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet) = 0$;*

(5) *$H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)$ is $V_0\widetilde{\mathcal{D}}_{X'}$-coherent for all $\alpha \in A+\mathbb{Z}$ and all $i \geqslant -N-1$.*

Then for every α and $i \geqslant -N$ the morphism $H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet) \to H^i(\widetilde{\mathcal{N}}^\bullet)$ is injective. Moreover, the filtration $U_\bullet H^i(\widetilde{\mathcal{N}}^\bullet)$ defined by

$$U_\alpha H^i(\widetilde{\mathcal{N}}^\bullet) = \mathrm{image}\big[H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet) \longrightarrow H^i(\widetilde{\mathcal{N}}^\bullet)\big]$$

satisfies $\mathrm{gr}_\alpha^U H^i(\widetilde{\mathcal{N}}^\bullet) = H^i(\mathrm{gr}_\alpha^U\widetilde{\mathcal{N}}^\bullet)$ for all $\alpha \in A+\mathbb{Z}$.

Proof. It will have three steps. During the proof, the indices α, β, γ will run in $A+\mathbb{Z}$.

First step. This step proves a formal analogue of the conclusion of the proposition. Put

$$\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet} = \varprojlim_\gamma U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet \quad \text{and} \quad \widehat{\widetilde{\mathcal{N}}}^\bullet = \varinjlim_\alpha \widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}.$$

Under the assumption of Proposition 9.8.10, we will prove the following:

(a) For all $\beta \leqslant \alpha$, $\widehat{U_\beta\widetilde{\mathcal{N}}^\bullet} \to \widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}$ is injective (hence, for all α, $\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet} \to \widehat{\widetilde{\mathcal{N}}}^\bullet$ is injective) and $\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}/\widehat{U_{<\alpha}\widetilde{\mathcal{N}}^\bullet} = U_\alpha\widetilde{\mathcal{N}}^\bullet/U_{<\alpha}\widetilde{\mathcal{N}}^\bullet$.

(b) For every $\beta \leqslant \alpha$ and any i, $H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\beta\widetilde{\mathcal{N}}^\bullet)$ is strict.

(c) $H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}) = \varprojlim_\gamma H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet)$ $(i \geqslant -N)$.

(d) $H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}) \to H^i(\widehat{\widetilde{\mathcal{N}}}^\bullet)$ is injective $(i \geqslant -N)$.

(e) $H^i(\widehat{\widetilde{\mathcal{N}}}^\bullet) = \varinjlim_\alpha H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet})$ $(i \geqslant -N)$.

We note that the statements (b)–(d) imply that $H^i(\widehat{\widetilde{\mathcal{N}}}^\bullet)$ is strict for $i \geqslant -N$, although $H^i(\widetilde{\mathcal{N}}^\bullet)$ need not be strict.

Define $U_\alpha H^i(\widehat{\widetilde{\mathcal{N}}}^\bullet) = \operatorname{image}\big[H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}) \to H^i(\widehat{\widetilde{\mathcal{N}}}^\bullet)\big]$. Then the statements (a) and (d) imply that

$$\operatorname{gr}^U_\alpha H^i(\widehat{\widetilde{\mathcal{N}}}^\bullet) = H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}/\widehat{U_{<\alpha}\widetilde{\mathcal{N}}^\bullet}) = H^i(\operatorname{gr}^U_\alpha\widetilde{\mathcal{N}}^\bullet) \quad (i \geqslant -N).$$

For $\gamma < \beta < \alpha$ consider the exact sequence of complexes

$$0 \longrightarrow U_\beta\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet \longrightarrow U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet \longrightarrow U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\beta\widetilde{\mathcal{N}}^\bullet \longrightarrow 0.$$

As the projective system $(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet)_\gamma$ trivially satisfies the Mittag-Leffler condition (ML) (see e.g. [**KS90**, Prop. 1.12.4]), the sequence remains exact after passing to the projective limit, so we get an exact sequence of complexes

$$0 \longrightarrow \widehat{U_\beta\widetilde{\mathcal{N}}^\bullet} \longrightarrow \widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet} \longrightarrow U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\beta\widetilde{\mathcal{N}}^\bullet \longrightarrow 0,$$

hence (a).

Let us show by induction on $\rho = \alpha - \gamma \in A + \mathbb{N}$ that, for all $\gamma < \alpha$ and $i \geqslant -N$,

(i) $\prod_{\gamma<\beta\leqslant\alpha}(\mathrm{E} - \beta z)^{\nu_\beta}$ annihilates $H^i(U_\alpha/U_\gamma)$,

(ii) for all β such that $\gamma < \beta < \alpha$, we have an exact sequence,

$$0 \to H^i(U_\beta\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet) \to H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet) \to H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\beta\widetilde{\mathcal{N}}^\bullet) \to 0. \tag{9.8.11}$$

(iii) $H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet)$ is strict.

If γ is the predecessor of α in $A + \mathbb{Z}$, (i) and (iii) are true by assumption and (ii) is empty. Moreover, $(\text{ii})_\rho$ and $(\text{iii})_{<\rho}$ imply $(\text{iii})_\rho$. For $\gamma < \beta < \alpha$ and $\alpha - \gamma = \rho$, consider the exact sequence

$$\cdots \xrightarrow{\ \psi^i\ } H^i(U_\beta/U_\gamma) \longrightarrow H^i(U_\alpha/U_\gamma) \longrightarrow H^i(U_\alpha/U_\beta) \xrightarrow{\ \psi^{i+1}\ } H^{i+1}(U_\beta/U_\gamma) \longrightarrow \cdots$$

For any $i \geqslant -N$, any local section of $\operatorname{Im}\psi^{i+1}$ is then killed by $\prod_{\beta<\delta\leqslant\alpha}(\mathrm{E} - \delta z)$ and by $\prod_{\gamma<\delta\leqslant\beta}(\mathrm{E} - \delta z)$ according to $(\text{i})_{<\rho}$, hence is zero by $(\text{iii})_{<\rho}$, and the same property holds for $\operatorname{Im}\psi^i$, so the previous sequence of H^i is exact. Arguing similarly, we get the exactness of (9.8.11) for $\alpha - \gamma = \rho$, hence $(\text{ii})_\rho$, from which $(\text{i})_\rho$ follows.

Consequently, the projective system $(H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet))_\gamma$ satisfies (ML), so we get (c). Moreover, taking the limit on γ in (9.8.11) gives, according to (ML), an exact sequence

$$0 \longrightarrow H^i(\widehat{U_\beta\widetilde{\mathcal{N}}^\bullet}) \longrightarrow H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet}) \longrightarrow H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\beta\widetilde{\mathcal{N}}^\bullet) \longrightarrow 0,$$

hence (d). Now, (e) is clear.

Second step. For every i,α, denote by $\widetilde{\mathfrak{T}}^i_\alpha \subset H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)$ the $\mathcal{I}_{H'}$-torsion subsheaf of $H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)$. We set locally $\mathcal{I}_{H'}=t\mathcal{O}_{X'}$. We will now prove that it is enough to show

$$\exists\,\alpha_o,\quad \alpha\leqslant\alpha_o \Longrightarrow \widetilde{\mathfrak{T}}^i_\alpha=0\ \ \forall\, i\geqslant -N. \tag{9.8.12}$$

We assume that (9.8.12) is proved (step 3). Let $\gamma\leqslant\alpha_o$ and $i\geqslant -N$, so that $\widetilde{\mathfrak{T}}^i_\gamma=0$, and let $\alpha\geqslant\gamma$. Then, by definition of a V-filtration, $t^{\lceil\alpha-\gamma\rceil}$ acts by 0 on $U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet$, so that the image of $H^{i-1}(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet)$ in $H^i(U_\gamma\widetilde{\mathcal{N}}^\bullet)$ is contained in $\widetilde{\mathfrak{T}}^i_\gamma$, and thus is zero. We therefore have an exact sequence for every $i\geqslant -N$:

$$0\longrightarrow H^i(U_\gamma\widetilde{\mathcal{N}}^\bullet)\longrightarrow H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)\longrightarrow H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\gamma\widetilde{\mathcal{N}}^\bullet)\longrightarrow 0.$$

Using (9.8.11), we get for every $\beta<\alpha$ the exact sequence

$$0\longrightarrow H^i(U_\beta\widetilde{\mathcal{N}}^\bullet)\longrightarrow H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)\longrightarrow H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_\beta\widetilde{\mathcal{N}}^\bullet)\longrightarrow 0.$$

This implies that $H^i(U_\beta\widetilde{\mathcal{N}}^\bullet)\to H^i(\widetilde{\mathcal{N}}^\bullet)=\varinjlim_\alpha H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)$ is injective. For every α, let us set

$$U_\alpha H^i(\widetilde{\mathcal{N}}^\bullet):=\operatorname{image}\bigl[H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)\hookrightarrow H^i(\widetilde{\mathcal{N}}^\bullet)\bigr].$$

We thus have, for every $\alpha\in A+\mathbb{Z}$ and $i\geqslant -N$,

$$\operatorname{gr}^U_\alpha H^i(\widetilde{\mathcal{N}}^\bullet)=H^i(\operatorname{gr}^U_\alpha\widetilde{\mathcal{N}}^\bullet).$$

Third step: proof of (9.8.12)**.** Let us choose α_o as in 9.8.10(3). We notice that the multiplication by t induces an isomorphism $t:\widehat{U_\alpha\widetilde{\mathcal{N}}^i}\xrightarrow{\sim}\widehat{U_{\alpha-1}\widetilde{\mathcal{N}}^i}$ for $\alpha\leqslant\alpha_o$ and $i\geqslant -N-1$, hence an isomorphism $t:H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet})\xrightarrow{\sim}H^i(\widehat{U_{\alpha-1}\widetilde{\mathcal{N}}^\bullet})$, and that (d) in Step 1 implies that, for all $i\geqslant -N$ and all $\alpha\leqslant\alpha_o$, the multiplication by t on $H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet})$ is injective.

The proof of (9.8.12) is done by decreasing induction on i. It clearly hods for $i\geqslant i_o$ (given by 9.8.10(4)). We assume that, for every $\alpha\leqslant\alpha_o$, we have $\widetilde{\mathfrak{T}}^{i+1}_\alpha=0$. We have (after 9.8.10(3)) an exact sequence of complexes, for every $k\in\mathbb{N}$ and $\bullet\geqslant -N-1$,

$$0\longrightarrow U_\alpha\widetilde{\mathcal{N}}^\bullet\xrightarrow{\ t^k\ }U_\alpha\widetilde{\mathcal{N}}^\bullet\longrightarrow U_\alpha\widetilde{\mathcal{N}}^\bullet/U_{\alpha-k}\widetilde{\mathcal{N}}^\bullet\longrightarrow 0.$$

As $\widetilde{\mathfrak{T}}^{i+1}_\alpha=0$, we have, for every $k\geqslant 1$ an exact sequence

$$H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)\xrightarrow{\ t^k\ }H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)\longrightarrow H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_{\alpha-k}\widetilde{\mathcal{N}}^\bullet)\longrightarrow 0,$$

hence, according to Step 1,

$$H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet})/H^i(\widehat{U_{\alpha-k}\widetilde{\mathcal{N}}^\bullet})=H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet/U_{\alpha-k}\widetilde{\mathcal{N}}^\bullet)=H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)/t^kH^i(U_\alpha\widetilde{\mathcal{N}}^\bullet).$$

According to Assumption 9.8.10(5) and Exercise 9.13, for k big enough (locally on X'), the map $\widetilde{\mathfrak{T}}^i_\alpha\to H^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)/t^kH^i(U_\alpha\widetilde{\mathcal{N}}^\bullet)$ is injective. It follows that $\widetilde{\mathfrak{T}}^i_\alpha\to H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet})$ is injective too. But we know that t is injective on $H^i(\widehat{U_\alpha\widetilde{\mathcal{N}}^\bullet})$ for $\alpha\leqslant\alpha_o$, hence $\widetilde{\mathfrak{T}}^i_\alpha=0$, thus concluding Step 3. $\square$

***Proof of Theorem* 9.8.8**

9.8.13. Lemma. *Let $U_\bullet\widetilde{\mathcal{M}}$ be a V-filtration indexed by $A+\mathbb{Z}$ of a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ which satisfies the following properties:*

(a) *$t : U_\alpha\widetilde{\mathcal{M}} \to U_{\alpha-1}\widetilde{\mathcal{M}}$ is bijective for every $\alpha < 0$,*

(b) *$\widetilde{\partial}_t : \mathrm{gr}^U_\alpha\widetilde{\mathcal{M}} \to \mathrm{gr}^U_{\alpha+1}\widetilde{\mathcal{M}}$ is bijective for every $\alpha > -1$.*

We define $R_U\widetilde{\mathcal{M}}$ as in Remark 9.2.3*, which is thus an $R_{AV}\widetilde{\mathcal{D}}_X$-module. Then $R_U\widetilde{\mathcal{M}}$ has a resolution $\widetilde{\mathcal{L}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} R_{AV}\widetilde{\mathcal{D}}_X$, where each $\widetilde{\mathcal{L}}^i$ is an $\widetilde{\mathcal{O}}_X$-module.*

Proof. By assumption, the morphism $\varphi : \bigoplus_{\gamma\in[-1,0]} U_\gamma\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}$ is surjective and induces surjective morphisms $\bigoplus_{\gamma\in[-1,0]} U_\gamma\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} {}^A V_{\alpha-\gamma}\widetilde{\mathcal{D}}_X \to U_\alpha\widetilde{\mathcal{M}}$ for every $\alpha \in A+\mathbb{Z}$, hence a surjective morphism $\bigoplus_{\gamma\in[-1,0]} U_\gamma\widetilde{\mathcal{M}}v^\gamma \otimes_{\widetilde{\mathcal{O}}_X} R_{AV}\widetilde{\mathcal{D}}_X \to R_U\widetilde{\mathcal{M}}$, with the convention of Remark 9.2.3. We note that the V-filtered induced $\widetilde{\mathcal{D}}_X$-module that we have introduced also satisfies (a) and (b). Set $\mathcal{K} = \operatorname{Ker}\varphi$, that we equip with the induced filtration $U_\bullet\mathcal{K}$. We thus have an exact sequence for every α:

$$0 \longrightarrow U_\alpha\mathcal{K} \longrightarrow \bigoplus_{\gamma\in[-1,0]} U_\gamma\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} {}^A V_{\alpha-\gamma}\widetilde{\mathcal{D}}_X \longrightarrow U_\alpha\widetilde{\mathcal{M}} \longrightarrow 0,$$

from which we deduce that $U_\bullet\mathcal{K}$ satisfies (a) and (b), enabling us to continue the process. ☐

The assertion of the theorem is local on X', and we will work in the neighbourhood of a point $x'_o \in H'$. The Kashiwara-Malgrange filtration $V_\bullet\widetilde{\mathcal{M}}$ satisfies the properties 9.8.13(a) and (b), according to Proposition 9.3.20. We can then use a resolution as in Lemma 9.8.13, that we stop at a finite step chosen large enough (due to the cohomological finiteness of f) such that, for the corresponding bounded complex $\widetilde{\mathcal{L}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} R_{AV}\widetilde{\mathcal{D}}_X$, one has

$${}_{\mathrm{D}}f_*^{(i)}(R_V\widetilde{\mathcal{M}}) \neq 0 \Longrightarrow {}_{\mathrm{D}}f_*^{(i)}(R_V\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{L}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} R_{AV}\widetilde{\mathcal{D}}_X)$$

and similarly for every α,

$${}_{\mathrm{D}}f_{|H*}^{(i)}(\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}) \neq 0 \Longrightarrow {}_{\mathrm{D}}f_{|H*}^{(i)}(\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}f_{|H*}^{(i)}(\widetilde{\mathcal{L}}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^{AV}_\alpha\widetilde{\mathcal{D}}_X).$$

In such a case, ${}_{\mathrm{D}}f_*^{(i)}(R_V\widetilde{\mathcal{M}}) = H^i\big(f_*\operatorname{God}^\bullet(\widetilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}R_{AV}\widetilde{\mathcal{D}}_{X'})\big)$, according to Remark 9.8.6. We thus set

$$(\widetilde{\mathcal{N}}^\bullet, U_\bullet\widetilde{\mathcal{N}}^\bullet) = \big(f_*\operatorname{God}^\bullet(\widetilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}\widetilde{\mathcal{D}}_{X'}), f_*\operatorname{God}^\bullet(\widetilde{\mathcal{L}}^\bullet \otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1A}V_\bullet\widetilde{\mathcal{D}}_{X'})\big).$$

Since the sequences

$$0 \longrightarrow {}^A V_\alpha\widetilde{\mathcal{D}}_{X'} \longrightarrow \widetilde{\mathcal{D}}_{X'} \longrightarrow \widetilde{\mathcal{D}}_{X'}/{}^A V_\alpha\widetilde{\mathcal{D}}_{X'} \longrightarrow 0$$

and

$$0 \longrightarrow {}^A V_{<\alpha}\widetilde{\mathcal{D}}_{X'} \longrightarrow {}^A V_\alpha\widetilde{\mathcal{D}}_{X'} \longrightarrow \mathrm{gr}^{AV}_\alpha\widetilde{\mathcal{D}}_{X'} \longrightarrow 0$$

are exact sequences of locally free $\widetilde{\mathcal{O}}_{X'}$-modules, they remain exact after applying $\widetilde{\mathcal{L}}^\bullet\otimes_{\widetilde{\mathcal{O}}_{X'}}$, then also after applying the Godement functor (see Exercise 8.48(1)), and then after applying f_* since the latter complexes consist of flabby sheaves.

This implies that $U_\alpha\widetilde{\mathcal{N}}^\bullet$ is indeed a subcomplex of $\widetilde{\mathcal{N}}^\bullet$ and

$$\mathrm{gr}_\alpha^U\widetilde{\mathcal{N}}^\bullet = f_*\,\mathrm{God}^\bullet(\widetilde{\mathcal{L}}^\bullet\otimes_{f^{-1}\widetilde{\mathcal{O}}_{X'}} f^{-1}\mathrm{gr}_\alpha^{{}^AV}\widetilde{\mathcal{D}}_{X'}).$$

Property 9.8.10(5) is satisfied, according to Theorem 9.8.3, and Properties 9.8.10(3) and (4) are clear.

We have $\widetilde{\mathcal{H}}^i(\mathrm{gr}_\alpha^U\widetilde{\mathcal{N}}^\bullet) = {}_{\mathrm{D}}f^{(i)}_{|H*}\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}$ for $i\geqslant -N$ for some N such that ${}_{\mathrm{D}}f^{(i)}_{|H*}\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}=0$ if $i<-N$, so that 9.8.10(1) holds by assumption and 9.8.10(2) is satisfied by taking the maximum of the local values ν_α along the compact fiber $f^{-1}(x'_o)$.

From Proposition 9.8.10 we conclude that 9.8.8(1) holds for $\alpha\in A+\mathbb{Z}$ and any i. Denoting by $U_\bullet{}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ the image filtration in 9.8.8(1), we thus have $R_U{}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}f^{(i)}_*R_V\widetilde{\mathcal{M}}$ and therefore

$$\mathrm{gr}_\alpha^U({}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}f^{(i)}_{|H*}\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}.$$

In particular, the left-hand term is strict by assumption on the right-hand term.

By the coherence theorem 9.8.3, we conclude that $U_\bullet{}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ is a coherent ${}^AV$-filtration of ${}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$. Therefore, $U_\bullet{}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ satisfies the assumptions of Lemma 9.3.13 (extended to filtrations indexed by $A+\mathbb{Z}$). Moreover, the properties 9.3.14(2) and (3) are also satisfied since they hold for $\widetilde{\mathcal{M}}$. We conclude that ${}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H' and that $U_\bullet{}_{\mathrm{D}}f^{(i)}_*\widetilde{\mathcal{M}}$ is its Kashiwara-Malgrange filtration. Now, Properties (1)–(3) in Theorem 9.8.8 are clear. □

9.9. Examples of computations of nearby and vanishing cycles

In this section, we make explicit some examples of computation of nearby and vanishing cycles simple situations, anticipating more complicated computations in Chapter 13.

9.9.a. Strict $\mathbb{R}$-specializability along (g^r).

Let g be a holomorphic function on X and let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). The purpose of this example is to show that $\widetilde{\mathcal{M}}$ is then also strictly $\mathbb{R}$-specializable along (g^r) for every $r\geqslant 2$, and to compare nearby and vanishing cycles of $\widetilde{\mathcal{M}}$ with respect to g and to $h:=g^r$.

9.9.1. Proposition. *Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). Then $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (h) and*

(a) $(\psi_{h,\lambda}\widetilde{\mathcal{M}},\mathrm{N}) = (\psi_{g,\lambda^r}\widetilde{\mathcal{M}},\mathrm{N}/r)$ *for every λ,*
(b) $(\phi_{h,1}\widetilde{\mathcal{M}},\mathrm{N}) = (\phi_{g,1}\widetilde{\mathcal{M}},\mathrm{N}/r)$,

(c) *denoting by* $\iota_g : X \hookrightarrow X \times \mathbb{C}$ *the graph inclusion and setting* $\widetilde{\mathcal{N}} = {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$, *there is an isomorphism*

$$\left\{\begin{tikzcd}\psi_{h,1}\widetilde{\mathcal{M}} \arrow[r, bend left, "\mathrm{can}_h"] & \phi_{h,1}\widetilde{\mathcal{M}} \arrow[l, bend left, "(-1)\ \ \mathrm{var}_h"]\end{tikzcd}\right\} \simeq \left\{\begin{tikzcd}\mathrm{gr}^V_{-r}\widetilde{\mathcal{N}} \arrow[rr, bend left=40, "\mathrm{can}_h := \mathrm{can}_g \circ (rg^{r-1})^{-1}"] & \psi_{g,1}\widetilde{\mathcal{M}} \arrow[l, dashed, "g^{r-1}"', "\sim"] \arrow[r, dashed, bend left, "\mathrm{can}_g"] & \phi_{g,1}\widetilde{\mathcal{M}} \arrow[l, dashed, bend left, "(-1)\ \mathrm{var}_g"] \arrow[ll, bend left=40, "(-1)\quad \mathrm{var}_h := g^{r-1} \circ \mathrm{var}_g"]\end{tikzcd}\right\}$$

Proof. It is equivalent to prove the assertion with $\widetilde{\mathcal{M}} = \widetilde{\mathcal{N}}$, $g = t$ and $h = t^r$, so we will only consider this setting. We can then write ${}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}} = \bigoplus_{k\in\mathbb{N}} \widetilde{\mathcal{M}} \otimes \delta\widetilde{\partial}_u^k$ as a $\widetilde{\mathcal{D}}_X[u]\langle\widetilde{\partial}_u\rangle$-module, with

$$\begin{aligned}(m\otimes\delta)\widetilde{\partial}_u^k &= m\otimes\delta\widetilde{\partial}_u^k \quad \forall\, k\geqslant 0,\\ (m\otimes\delta)\widetilde{\partial}_t &= (m\widetilde{\partial}_t)\otimes\delta - (rg^{r-1}m)\otimes\delta\widetilde{\partial}_u,\\ (m\otimes\delta)u &= (mt^r)\otimes\delta,\\ (m\otimes\delta)\widetilde{\mathcal{O}}_X &= (m\widetilde{\mathcal{O}}_X)\otimes\delta,\end{aligned}$$

and with the usual commutation rules. We then have the relation

$$r(m\otimes\delta)u\widetilde{\partial}_u = [mt\widetilde{\partial}_t]\otimes\delta - (mt\otimes\delta)\widetilde{\partial}_t.$$

We will denote by V^t the V-filtration with respect to the variable t and by V^u that with respect to the variable u.

For $\alpha \leqslant 0$, we set

$$U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) := \bigl(V^t_{r\alpha}\widetilde{\mathcal{M}}\otimes\delta\bigr)\cdot V^u_0(\widetilde{\mathcal{D}}_X[u]\langle\widetilde{\partial}_u\rangle),$$

and for $\alpha > 0$ we define inductively

$$U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) := U_{<\alpha}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) + U_{\alpha-1}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})\widetilde{\partial}_u.$$

We will prove that the filtration $U_\bullet({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ is the V-filtration $V^u({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$.

- Let us assume that $\alpha \leqslant 0$. Using the above relation we obtain that, if

$$V^t_{r\alpha}\widetilde{\mathcal{M}}(t\widetilde{\partial}_t - r\alpha z)^{\nu_{r\alpha}} \subset V^t_{<r\alpha}\widetilde{\mathcal{M}},$$

then

$$U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})(u\widetilde{\partial}_u - \alpha z)^{\nu_{r\alpha}} \subset U_{<\alpha}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}),$$

from which we conclude that $(u\widetilde{\partial}_u - \alpha z)$ is nilpotent on $\mathrm{gr}^U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ for $\alpha \leqslant 0$.

- By using the relation

$$(mt\widetilde{\partial}_t)\otimes\delta = (m\otimes\delta)(t\widetilde{\partial}_t - ru\widetilde{\partial}_u),$$

we see that, if $m_1,\dots,m_\ell$ generate $V^t_{r\alpha}\widetilde{\mathcal{M}}$ over $V^t_0\widetilde{\mathcal{D}}_X$ ($\alpha\leqslant 0$), then $m_1\otimes\delta,\dots,m_\ell\otimes\delta$ generate $U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ over $V^u_0(\widetilde{\mathcal{D}}_X[u]\langle\widetilde{\partial}_u\rangle)$, from which we conclude that $U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ is $V^u_0(\widetilde{\mathcal{D}}_X[u]\langle\widetilde{\partial}_u\rangle)$-coherent for every $\alpha\leqslant 0$, hence for every α.

By using the analogous property for $\widetilde{\mathcal{M}}$ we obtain that, for every α,

$$U_{\alpha-1}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) \subset U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})u,$$

resp.
$$U_{\alpha+1}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) \subset U_{<\alpha+1}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) + U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})\widetilde{\partial}_u,$$

with equality if $\alpha < 0$ (resp. if $\alpha \geqslant -1$), from which we deduce that $U_\bullet({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ is a coherent V-filtration.

- For $\alpha \leqslant 0$, we check that

$$U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) = U_{<\alpha}({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}}) + \sum_{k\geqslant 0}(V^t_{r\alpha}\widetilde{\mathcal{M}} \otimes \delta)\widetilde{\partial}_t^k.$$

We deduce, by considering the degree in $\widetilde{\partial}_t$, that the natural morphism

$$\begin{aligned}\bigoplus_k(\mathrm{gr}^{V^t}_{r\alpha}\widetilde{\mathcal{M}} \otimes \widetilde{\partial}_t^k) &\longrightarrow \mathrm{gr}^U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})\\ \bigoplus_k[m_k] \otimes \widetilde{\partial}_t^k &\longmapsto \Big[\sum_k(m_k \otimes \delta)\widetilde{\partial}_t^k\Big]\end{aligned}$$

is an isomorphism of $\widetilde{\mathcal{D}}_X$-modules. It follows that $\mathrm{gr}^U_\alpha({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ is strict for any $\alpha \leqslant 0$. Since Properties (2) and (3) of Definition 9.3.14 clearly hold for $U_\bullet({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$, we conclude from Exercise 9.31 that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (h) with Kashiwara-Malgrange filtration $V^u_\bullet({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$ equal to $U_\bullet({}_{\mathrm{D}}\iota_{h*}\widetilde{\mathcal{M}})$. The assertions (a), (b) and (c) follow in a straightforward way. ∎

9.9.b. Specialization along a strictly non-characteristic divisor

Let $D = D_1 \cup D_2$ be a divisor with normal crossings in X and smooth irreducible components D_1, D_2. We set $D_{1,2} = D_1 \cap D_2$, which is a smooth manifold of codimension two in X. Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module which is *strictly non-characteristic* along D_1, D_2 and $D_{1,2}$. Let us summarize some consequences of the assumption on nearby cycles. In local coordinates we will set $D_i = \{x_i = 0\}$ $(i = 1, 2)$ and we denote by $\iota_i : D_i \hookrightarrow X$ the inclusion, and similarly $\iota_{1,2}$.

(a) $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along D_1 and D_2. We denote by $V^{(i)}_\bullet\widetilde{\mathcal{M}}$ the V-filtration of $\widetilde{\mathcal{M}}$ along D_i $(i = 1, 2)$.

(b) $\mathrm{gr}^{V^{(i)}}_\alpha\widetilde{\mathcal{M}} = 0$ if $\beta \notin \mathbb{N}$.

(c) $\mathrm{gr}^{V^{(i)}}_{-1}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_i^*\widetilde{\mathcal{M}} = \iota_i^*\widetilde{\mathcal{M}}$. In local coordinates, $\mathrm{gr}^{V^{(i)}}_{-1}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}x_i$.

9.9.2. Lemma. *For $i = 1, 2$, the $\widetilde{\mathcal{D}}_{D_i}$-module ${}_{\mathrm{D}}\iota_i^*\widetilde{\mathcal{M}}$ is strictly non-characteristic, hence strictly $\mathbb{R}$-specializable, along $D_{1,2}$ and $V^{(j)}_\bullet\mathrm{gr}^{V^{(i)}}_{-1}\widetilde{\mathcal{M}}$ is the filtration induced by $V^{(j)}_\bullet\widetilde{\mathcal{M}}$ ($\{i, j\} = \{1, 2\}$), so that*

$$\mathrm{gr}^{V^{(2)}}_{-1}\mathrm{gr}^{V^{(1)}}_{-1}\widetilde{\mathcal{M}} = \mathrm{gr}^{V^{(1)}}_{-1}\mathrm{gr}^{V^{(2)}}_{-1}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_{1,2}^*\widetilde{\mathcal{M}} = \iota_{1,2}^*\widetilde{\mathcal{M}}.$$

Proof. The first point is mostly obvious, giving rise to the last formula, according to (c). For the second point, we have to check in local coordinates that $(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}x_1)x_2^k = \widetilde{\mathcal{M}}x_2^k/\widetilde{\mathcal{M}}x_1x_2^k$ for every $k \geqslant 1$, that is, the morphism

$$\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}x_1 \xrightarrow{x_2^k} \widetilde{\mathcal{M}}x_2^k/\widetilde{\mathcal{M}}x_1x_2^k$$

is an isomorphism. Recall (see Exercise 9.40) that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_{X/\mathbb{C}^2}$-coherent, so by taking a local resolution by free $\widetilde{\mathcal{D}}_{X/\mathbb{C}^2}$-modules, we are reduced to proving the assertion for $\widetilde{\mathcal{M}} = \widetilde{\mathcal{D}}^{\ell}_{X/\mathbb{C}^2}$, for which it is obvious. □

Our aim is to compute, in the local setting, the nearby cycles of $\widetilde{\mathcal{M}}$ along $g = x_1x_2$ (after having proved that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (g), of course). We consider then the graph inclusion $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$. The following proposition also holds in the left case after side-changing.

9.9.3. Proposition. *Under the previous assumptions, the $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is a middle extension along (g), we have $\psi_{g,\lambda}\widetilde{\mathcal{M}} = 0$ for $\lambda \neq 1$ and there are functorial isomorphisms*

$$\mathrm{P}_\ell\psi_{g,1}\widetilde{\mathcal{M}} \simeq \begin{cases} \psi_{x_1,1}\widetilde{\mathcal{M}} \oplus \psi_{x_2,1}\widetilde{\mathcal{M}} & \text{if } \ell = 0,\\ \psi_{x_1,1}\psi_{x_2,1}\widetilde{\mathcal{M}}(-1) = \psi_{x_2,1}\psi_{x_1,1}\widetilde{\mathcal{M}}(-1) & \text{if } \ell = 1,\\ 0 & \text{otherwise.}\end{cases} \tag{9.9.3 $*$}$$

Proof. We set $\widetilde{\mathcal{N}} = {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$. We have $\widetilde{\mathcal{N}} = \iota_{g*}\widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ with the usual structure of a right $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$-module (see Example 8.7.7). We identify $\iota_{g*}\widetilde{\mathcal{M}}$ as the component of $\widetilde{\partial}_t$-degree zero in $\widetilde{\mathcal{N}}$. Let $U_\bullet\widetilde{\mathcal{N}}$ denote the filtration defined by

$$U_{-1}(\widetilde{\mathcal{N}}) = \iota_{g*}\widetilde{\mathcal{M}}\cdot\widetilde{\mathcal{D}}_X \subset \widetilde{\mathcal{N}}, \qquad U_{-k-1}(\widetilde{\mathcal{N}}) = \begin{cases} U_{-1}(\widetilde{\mathcal{N}})\cdot t^k & \text{if } k \geqslant 0,\\ \sum_{\ell\leqslant -k} U_{-1}(\widetilde{\mathcal{N}})\cdot\widetilde{\partial}_t^{\ell} & \text{if } k \leqslant 0.\end{cases}$$

We wish to prove that $U_\bullet\widetilde{\mathcal{N}}$ satisfies all the properties of the V-filtration of $\widetilde{\mathcal{N}}$.

Let m be a local section of $\widetilde{\mathcal{M}}$. From the relation

$$(m\otimes 1)\widetilde{\partial}_{x_1} = (m\widetilde{\partial}_{x_1})\otimes 1 - mx_2\otimes\widetilde{\partial}_t \tag{9.9.4}$$

we deduce

$$\begin{aligned}(m\otimes 1)\widetilde{\partial}_t t &= (m\widetilde{\partial}_{x_1}x_1)\otimes 1 - (m\otimes 1)x_1\widetilde{\partial}_{x_1}\\ &= (m\widetilde{\partial}_{x_2}x_2)\otimes 1 - (m\otimes 1)x_2\widetilde{\partial}_{x_2},\end{aligned} \tag{9.9.5}$$

showing that $U_{-1}(\widetilde{\mathcal{N}})$ is a $V_0\widetilde{\mathcal{D}}_{X\times\mathbb{C}_t}$-module. If $(m_i)_{i\in I}$ is a finite set of local $\widetilde{\mathcal{D}}_{X/\mathbb{C}^2}$-generators of $\widetilde{\mathcal{M}}$ (see Exercise 9.40), we deduce that it is a set of $\widetilde{\mathcal{D}}_X$-generators, hence of $V_0\widetilde{\mathcal{D}}_{X\times\mathbb{C}_t}$-generators, of $U_{-1}(\widetilde{\mathcal{N}})$. It follows that $U^\bullet(\widetilde{\mathcal{N}})$ is a good V-filtration of $\widetilde{\mathcal{N}}$. Moreover, the formulas above imply

$$(m\otimes 1)(\widetilde{\partial}_t t)^2 = \big((m\widetilde{\partial}_{x_1}\widetilde{\partial}_{x_2}\otimes 1) + (m\otimes 1)\widetilde{\partial}_{x_1}\widetilde{\partial}_{x_2} - (m\widetilde{\partial}_{x_2}\otimes 1)\widetilde{\partial}_{x_1} - (m\widetilde{\partial}_{x_1}\otimes 1)\widetilde{\partial}_{x_2}\big)\cdot t,$$

giving a Bernstein relation. Since $(\widetilde{\partial}_t t)^2$ vanishes on $\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}})$, the monodromy filtration is given by

$$\mathrm{M}_{-2}\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) = 0, \qquad \mathrm{M}_{-1}\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) = \mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}})\cdot\widetilde{\partial}_t t,$$
$$\mathrm{M}_0\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) = \mathrm{Ker}[\widetilde{\partial}_t t : \mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) \to \mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}})], \qquad \mathrm{M}_1\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) = \mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}).$$

As a consequence,

$$\begin{aligned}
\mathrm{P}_0\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) &= \mathrm{gr}^{\mathrm{M}}_0\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) = \operatorname{Ker}\widetilde{\partial}_t t/\operatorname{Im}\widetilde{\partial}_t t,\\
\mathrm{P}_1\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) &= \mathrm{gr}^{\mathrm{M}}_1\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}}) = \mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}})/\operatorname{Ker}\widetilde{\partial}_t t \xrightarrow{\sim} \mathrm{M}_{-1}\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}})(-1).
\end{aligned}$$

We will identify these $\widetilde{\mathcal{D}}_X$-modules with those given in the statement. This will also prove that $\mathrm{gr}^U_{-1}(\widetilde{\mathcal{N}})$ is strict, because $\psi_{x_1,1}\widetilde{\mathcal{M}}, \psi_{x_2,1}\widetilde{\mathcal{M}}, \psi_{x_1,1}\psi_{x_2,1}\widetilde{\mathcal{M}}$ are strict.

Let $G_\bullet\widetilde{\mathcal{N}}$ denote the filtration by the order with respect to $\widetilde{\partial}_t$. It will be useful to get control on the various objects occurring in the computations, mainly because when working on $\mathrm{gr}^G\widetilde{\mathcal{N}}$, the action of $\widetilde{\partial}_{x_1}$ amounts to that of $-x_2\otimes\widetilde{\partial}_t$ and similarly for $\widetilde{\partial}_{x_2}$, and the action of x_1, x_2 on $\widetilde{\mathcal{M}}$ is well understood, due to Exercise 9.43.

9.9.6. Lemma. *We have* $U_{-1}(\widetilde{\mathcal{N}})\cap G_p(\widetilde{\mathcal{N}}) = \sum_{k_1+k_2\leqslant p}(\widetilde{\mathcal{M}}\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}$.

Proof. Any local section ν of $U_{-1}(\widetilde{\mathcal{N}})$ can be written as $\sum_{k_1,k_2\geqslant 0}(m_{k_1,k_2}\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}$ for some local sections m_{k_1,k_2} of $\widetilde{\mathcal{M}}$ and, if $q=\max\{k_1+k_2 \mid m_{k_1,k_2}\neq 0\}$, the degree of ν with respect to $\widetilde{\partial}_t$ is $\leqslant q$ and the coefficient of $\widetilde{\partial}_t^q$ is

$$(-1)^q\sum_{k_1+k_2=q} m_{k_1,k_2}x_2^{k_1}x_1^{k_2}.$$

If this coefficient vanishes, Exercise 9.43 implies that

$$\nu = \sum_{k_1+k_2\leqslant q}((\mu_{k_1-1,k_2}x_1-\mu_{k_1,k_2-1}x_2)\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}.$$

The operator against $\mu_{i,j}\otimes 1$ is $(x_1\widetilde{\partial}_{x_1}-x_2\widetilde{\partial}_{x_2})\widetilde{\partial}_{x_1}^{i}\widetilde{\partial}_{x_2}^{j}$, and (9.9.5) implies

$$(\mu_{i,j}\otimes 1)(x_1\widetilde{\partial}_{x_1}-x_2\widetilde{\partial}_{x_2}) = (\mu_{i,j}(x_1\widetilde{\partial}_{x_1}-x_2\widetilde{\partial}_{x_2}))\otimes 1,$$

so that $\nu\in\sum_{k_1+k_2\leqslant q-1}(\widetilde{\mathcal{M}}\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}$. $\square$

As a consequence, let us prove the equality

$$\widetilde{\partial}_t^{-1}(U_{-1}(\widetilde{\mathcal{N}}))\cap U_{-1}\widetilde{\mathcal{N}} = \sum_{k_1,k_2}(\widetilde{\mathcal{M}}\cdot(x_1,x_2)\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}, \tag{9.9.7}$$

where $\widetilde{\partial}_t^{-1}(U_{-1}(\widetilde{\mathcal{N}})) := \{\nu\in U_{-1}(\widetilde{\mathcal{N}}) \mid \nu\widetilde{\partial}_t\in U_{-1}(\widetilde{\mathcal{N}})\}$, and that t *acts injectively on* $U_{-1}\widetilde{\mathcal{N}}$.

Let $\nu=\sum_{q\leqslant p}\nu_q\otimes\widetilde{\partial}_t^q$ be a nonzero local section of $U_{-1}(\widetilde{\mathcal{N}})$ of G-order p, so that $\nu_p\neq 0$. We will argue by induction on p. By the lemma we have

$$\nu_p = \sum_{k_1+k_2=p}(m_{k_1,k_2}\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}\quad\text{with}\quad \sum_{k_1+k_2=p}m_{k_1,k_2}x_2^{k_1}x_1^{k_2}\neq 0 \text{ in } \widetilde{\mathcal{M}}.$$

Assume $\nu\widetilde{\partial}_t$ is a local section of $U_{-1}(\widetilde{\mathcal{N}})$. Then $\sum_{k_1+k_2=p}m_{k_1,k_2}x_2^{k_1}x_1^{k_2}$ is a local section of $\widetilde{\mathcal{M}}\cdot(x_1,x_2)^{p+1}$, that is, is equal to

$$\sum_{k_1+k_2=p}\mu_{k_1,k_2}x_2^{k_1}x_1^{k_2}\quad\text{with}\quad \mu_{k_1,k_2}\in\widetilde{\mathcal{M}}\cdot(x_1,x_2),$$

so $\nu-\sum_{k_1+k_2=p}(\mu_{k_1,k_2}\otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}$ a local section of $U_{-1}(\widetilde{\mathcal{N}})\widetilde{\partial}_t\cap U_{-1}\widetilde{\mathcal{N}}$ and has G-order $\leqslant p-1$. We can conclude by induction.

Assume now that $\nu t = 0$. We have

$$0 = (\nu t)_p = \big[(\nu_p \otimes \widetilde{\partial}_t^p)t\big]_p = \nu_p \otimes t\widetilde{\partial}_t^p = \nu_p x_1 x_2 \otimes \widetilde{\partial}_t^p,$$

so $\nu_p x_1 x_2 = 0$ in $\widetilde{\mathcal{M}}$, and thus $\nu_p = 0$, a contradiction. $\square$

Recall that $\widetilde{\mathcal{M}} = V_{-1}^{(1)}\widetilde{\mathcal{M}}$ (V-filtration relative to x_1), so that $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}x_1 = \mathrm{gr}_{-1}^{V^{(1)}}\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}_1 := (\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}x_1)[\widetilde{\partial}_{x_1}] \simeq \psi_{x_1,1}\widetilde{\mathcal{M}}(-1)$, according to Exercise 9.36. Similarly, $\widetilde{\mathcal{N}}_{12} \simeq \psi_{x_1,1}\psi_{x_2,1}\widetilde{\mathcal{M}}(-2)$. The map

$$m_{k_1,k_2} \otimes \widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2} \longmapsto (m_{k_1,k_2} \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2} \cdot \widetilde{\partial}_t t \tag{9.9.8}$$

sends $\widetilde{\mathcal{M}}{\cdot}(x_1,x_2)[\widetilde{\partial}_{x_1},\widetilde{\partial}_{x_1}]$ to $U_{-2}\widetilde{\mathcal{N}}(-1)$, according to (9.9.4) and defines thus a surjective morphism

$$\psi_{x_1,1}\psi_{x_2,1}\widetilde{\mathcal{M}}(-2) = \widetilde{\mathcal{N}}_{12} \longrightarrow \mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}}(-1).$$

Let us prove that it is also injective. Let us denote by $[m_{k_1,k_2}]$ the class of m_{k_1,k_2} in $\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}{\cdot}(x_1,x_2)$. Let $\sum[m_{k_1,k_2}] \otimes \widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}$ be nonzero and of degree equal to p and set

$$\nu = \sum_{k_1+k_2\leqslant p} (m_{k_1,k_2} \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}.$$

Assume that $\nu\widetilde{\partial}_t t \in U_{-2}\widetilde{\mathcal{N}}$, hence, by the injectivity of t, $\nu\widetilde{\partial}_t \in U_{-1}\widetilde{\mathcal{N}}$. The proof of (9.9.7) above shows that, for $k_1 + k_2 = p$, there exists $\mu_{k_1,k_2} \in \widetilde{\mathcal{M}}{\cdot}(x_1,x_2)$ such that $\sum_{k_1+k_2=p}(m_{k_1,k_2} - \mu_{k_1,k_2})x_2^{k_1}x_1^{k_2} = 0$, and by Exercise 9.43 we conclude that $m_{k_1,k_2} \in \widetilde{\mathcal{M}}{\cdot}(x_1,x_2)$, so $[m_{k_1,k_2}] = 0$, a contradiction.

As a consequence, if $\nu\widetilde{\partial}_t t = \sum(m_{k_1,k_2} \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}\widetilde{\partial}_t t$ belongs to $U_{-2}\widetilde{\mathcal{N}} = U_{-1}\widetilde{\mathcal{N}} \cdot t$, (9.9.7) implies $\nu \in \sum(\widetilde{\mathcal{M}}{\cdot}(x_1,x_2) \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}$. We obtain therefore

$$\mathrm{gr}_1^{\mathrm{M}}\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}} \xrightarrow[\sim]{\mathrm{N}} \mathrm{gr}_{-1}^{\mathrm{M}}\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}}(-1) \simeq \psi_{x_1,1}\psi_{x_2,1}\widetilde{\mathcal{M}}(-2), \tag{9.9.9}$$

and these modules are strict. Note that the isomorphism $\widetilde{\mathcal{N}}_{12} \xrightarrow{\sim} \mathrm{gr}_1^{\mathrm{M}}\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}} = U_{-1}\widetilde{\mathcal{N}}/(\widetilde{\partial}_t t)^{-1}U_{-1}\widetilde{\mathcal{M}})$ is induced by

$$m_{k_1,k_2} \otimes \widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2} \longmapsto (m_{k_1,k_2} \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2}. \tag{9.9.10}$$

Let us now consider M_0. Note that (9.9.7) and the injectivity of t imply

$$\mathrm{M}_0\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}} = \sum_{k_1,k_2} (\widetilde{\mathcal{M}}{\cdot}(x_1,x_2) \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2} \mod U_{-2}\widetilde{\mathcal{N}},$$

and clearly $\sum_{k_1,k_2}(\widetilde{\mathcal{M}}x_1x_2 \otimes 1)\widetilde{\partial}_{x_1}^{k_1}\widetilde{\partial}_{x_2}^{k_2} \subset U_{-2}\widetilde{\mathcal{N}}$. Note also that $(mx_1 \otimes 1)\widetilde{\partial}_{x_1}^{k_1} \equiv (m\widetilde{\partial}_{x_1}^{k_1}x_1) \otimes 1 \bmod \mathrm{Im}\,\widetilde{\partial}_t t$, according to (9.9.5). As a consequence,

$$\mathrm{M}_0\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}} = \sum_{k_1}(\widetilde{\mathcal{M}}x_2 \otimes 1)\widetilde{\partial}_{x_1}^{k_1} + \sum_{k_2}(\widetilde{\mathcal{M}}x_1 \otimes 1)\widetilde{\partial}_{x_2}^{k_2} \mod (U_{-1}\widetilde{\mathcal{N}}\widetilde{\partial}_t t + U_{-2}\widetilde{\mathcal{N}}),$$

and we have a surjective morphism

$$\psi_{x_1,1}\widetilde{\mathcal{M}}(-1) \oplus \psi_{x_2,1}\widetilde{\mathcal{M}}(-1) = \widetilde{\mathcal{N}}_1 \oplus \widetilde{\mathcal{N}}_2 \longrightarrow \mathrm{gr}_0^{\mathrm{M}}\mathrm{gr}_{-1}^{U}\widetilde{\mathcal{N}}, \tag{9.9.11}$$

sending $m_{k_1,0} \otimes \widetilde{\partial}_{x_1}^{k_1}$ to $(m_{k_1,0}x_2 \otimes 1)\widetilde{\partial}_{x_1}^{k_1}$ and $m_{0,k_2} \otimes \widetilde{\partial}_{x_2}^{k_2}$ to $(m_{0,k_2}x_1 \otimes 1)\widetilde{\partial}_{x_2}^{k_2}$. In order to show injectivity, we first check that it is strict with respect to the filtration $G_\bullet\widetilde{\mathcal{N}}$ and the filtration by the degree in $\widetilde{\partial}_{x_1}, \widetilde{\partial}_{x_2}$ on $\widetilde{\mathcal{N}}_1, \widetilde{\mathcal{N}}_2$.

Assume that $(m_{k_1,0}x_2\otimes 1)\widetilde{\partial}_{x_1}^{k_1}+(m_{0,k_2}x_1\otimes 1)\widetilde{\partial}_{x_2}^{k_2}\in G_{p-1}\widetilde{\mathcal{N}}$ for $k_1,k_2\leqslant p$. Then we find that $m_{p,0}\in\widetilde{\mathcal{M}}x_1$ and $m_{0,p}\widetilde{\mathcal{M}}x_2$, as wanted. By the same argument we deduce the injectivity.

Due to the strictness of $\widetilde{\mathcal{N}}_1,\widetilde{\mathcal{N}}_2,\widetilde{\mathcal{N}}_{12}$, we conclude at this point that $\mathrm{gr}^U_{-1}\widetilde{\mathcal{M}}$ is strict. If we show that $\mathrm{gr}^U_k\widetilde{\mathcal{N}}$ is also strict for any k, then $U_\bullet\widetilde{\mathcal{N}}$ satisfies all properties characterizing the V-filtration. As a consequence, $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (g), $\mathrm{gr}^U_{-1}\widetilde{\mathcal{N}}=\psi_{g,1}\widetilde{\mathcal{M}}(-1)$, and (9.9.3 $*$) holds.

Clearly, $\widetilde{\partial}_t:\mathrm{gr}^U_{-1}\widetilde{\mathcal{N}}\to\mathrm{gr}^U_0\widetilde{\mathcal{N}}$ is onto. So we are left with proving the following assertions:

(i) $t^k:\mathrm{gr}^U_{-1}\widetilde{\mathcal{N}}\to\mathrm{gr}^U_{-1-k}\widetilde{\mathcal{N}}$ is an isomorphism (equivalently, injective) for $k\geqslant 1$,

(ii) $t:\mathrm{gr}^U_0\widetilde{\mathcal{N}}\to\mathrm{gr}^U_{-1}\widetilde{\mathcal{N}}$ is injective (so $\mathrm{gr}^U_0\widetilde{\mathcal{N}}$ is strict),

(iii) $\widetilde{\partial}_t^k:\mathrm{gr}^U_0\widetilde{\mathcal{N}}\to\mathrm{gr}^U_k\widetilde{\mathcal{N}}$ is an isomorphism (equivalently, injective) for $k\geqslant 1$.

Proof of the assertions.

(i) If $\nu\in U_{-1}\widetilde{\mathcal{N}}$ satisfies $\nu t^k=\mu t^{k+1}$ for some $\mu\in U_{-1}\widetilde{\mathcal{N}}$ then, by injectivity of t on $U_{-1}\widetilde{\mathcal{N}}$, $\nu=\mu t$, so $\nu\in U_{-2}\widetilde{\mathcal{N}}$.

(ii) If $\nu\in U_{-1}\widetilde{\mathcal{N}}$ is such that $\nu\widetilde{\partial}_t\cdot t\in U_{-2}\widetilde{\mathcal{N}}$, then there exists $\mu\in U_{-1}\widetilde{\mathcal{N}}$ such that $(\nu\widetilde{\partial}_t-\mu)t=0$ hence, by t-injectivity, $\nu\widetilde{\partial}_t\in U_{-1}\widetilde{\mathcal{N}}$.

(iii) We prove the injectivity by induction on $k\geqslant 1$. Let $\nu\in U_{-1}\widetilde{\mathcal{M}}$ and consider $\nu\widetilde{\partial}_t \bmod U_{-1}\widetilde{\mathcal{N}}$ as an element of $\mathrm{gr}^U_0\widetilde{\mathcal{N}}$. If $(\nu\widetilde{\partial}_t)\widetilde{\partial}_t^k\in U_{k-1}\widetilde{\mathcal{N}}$, then $(\nu\widetilde{\partial}_t^k)\widetilde{\partial}_t t=0$ in $\mathrm{gr}^U_{k-1}\widetilde{\mathcal{N}}$. Since $\widetilde{\partial}_t t-kz$ is nilpotent on $\mathrm{gr}^U_{k-1}\widetilde{\mathcal{N}}$ and since $\mathrm{gr}^U_{k-1}\widetilde{\mathcal{N}}$ is strict (by (ii) and the inductive assumption), $\widetilde{\partial}_t t$ is injective on $\mathrm{gr}^U_{k-1}\widetilde{\mathcal{N}}$, so $(\nu\widetilde{\partial}_t)\widetilde{\partial}_t^{k-1}=0$ in $\mathrm{gr}^U_{k-1}\widetilde{\mathcal{N}}$, and by induction $\nu\widetilde{\partial}_t=0$ in $\mathrm{gr}^U_0\widetilde{\mathcal{N}}$. $\square$

This concludes the proof of Proposition 9.9.3. $\square$

9.9.c. Nearby cycles along a monomial function of a smooth $\widetilde{\mathcal{D}}$-module

We consider a situation similar to that of the previous example, where we increase the number of active variables but we simplify the $\widetilde{\mathcal{D}}_X$-module. We will work in the left setting, which is more natural in this context.

Let $\widetilde{\mathcal{M}}$ be a smooth $\widetilde{\mathcal{D}}_X$-module (see Definition 8.8.14). The purpose of this section is to compute the nearby cycles of $\widetilde{\mathcal{M}}$ with respect to a function g which takes the form $g(x_1,\dots,x_n)=x_1\cdots x_p$ for some local coordinates $x_1,\dots,x_n$ on X and for some $p\geqslant 1$. The goal is to show that, first, $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $(g)=D$, and to compute the primitive parts in terms of the restriction of $\widetilde{\mathcal{M}}$ to various coordinate planes.

The computation is *local* on X. Thus X denotes a neighbourhood of the origin in $\mathbb{C}^n$ with coordinates $(x_1,\dots,x_p,y)$, $y=(x_{p+1},\dots,x_n)$, and D is the divisor (g) in this neighbourhood.

We set $\widetilde{\mathcal{O}}=\widetilde{\mathcal{O}}_{X,0}$. For a (possibly empty) subset $I\subset\{1,\dots,p\}$, we denote by $J=I^{\mathrm{c}}$ its complementary subset, by $\widetilde{\mathcal{O}}_I$ the ring $\mathbb{C}\{(x_i)_{i\in I},y\}[z]$ and by ι_I the inclusion $\{x_j=0,\forall\, j\in J\}\hookrightarrow X$. In particular, the ring $\widetilde{\mathcal{O}}_\varnothing$ contains no variables

$x_1,\dots,x_p$. For $\ell\leqslant p$, let us denote by $\mathcal{J}_{\ell+1}$ the set of subsets $J\subset\{1,\dots,p\}$ having cardinal equal to $\ell+1$.

9.9.12. Proposition. *Under these assumptions*

(1) $\widetilde{\mathcal{M}}$ *is strictly* $\mathbb{R}$*-specializable and a middle extension along* (g)*;*

(2) *The morphisms* $\mathrm{N},\mathrm{can},\mathrm{var}$ *are strict;*

(3) *for* $\lambda\in\mathbb{S}^1$*, we have* $\psi_{g,\lambda}\widetilde{\mathcal{M}}=0$ *unless* $\lambda=1$ *and, for any* $\ell\geqslant 0$*, there is a functorial isomorphism*

$$\mathrm{P}_\ell\psi_{g,1}(\widetilde{\mathcal{M}})\xrightarrow{\ \sim\ }\bigoplus_{J\in\mathcal{J}_{\ell+1}}{}_{\mathrm{D}}\iota_{I*}({}_{\mathrm{D}}\iota_I^*\widetilde{\mathcal{M}})(-\ell)\qquad(I=J^{\mathrm{c}}),\tag{9.9.12 $*$}$$

where $\mathrm{P}_\ell\psi_{g,1}\widetilde{\mathcal{M}}$ *denotes the primitive part of* $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,1}\widetilde{\mathcal{M}}$.

9.9.13. Remarks.

(1) According to Proposition 9.4.10, (3) implies that N^ℓ is strict for any $\ell\geqslant 1$.

(2) Since $\phi_{g,1}\widetilde{\mathcal{M}}=\mathrm{Im}\,\mathrm{N}$ after (1), we have a similar formula for $\mathrm{P}_\ell\phi_{g,1}(\widetilde{\mathcal{M}})$, according to Lemma 3.3.13.

Proof when $\widetilde{\mathcal{M}}=\widetilde{\mathcal{O}}$***.*** Let us set (see Example 8.7.7(2))

$$\widetilde{\mathcal{N}}={}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{O}}(-1)=\iota_*\widetilde{\mathcal{O}}[\widetilde{\partial}_t],$$

where $\iota_g:X\hookrightarrow X\times\mathbb{C}_t$ is the graph inclusion of g. Once we know that $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along (t), we have $\psi_{g,1}\widetilde{\mathcal{O}}=\mathrm{gr}_V^0\widetilde{\mathcal{N}}(1)$.

We set $y_j=x_{p+j}$ for $j=1,\dots,n-p$. If $\widetilde{\delta}$ denotes the $\widetilde{\mathcal{D}}$-generator 1 of $\widetilde{\mathcal{N}}$, we have the following relations:

$$t\widetilde{\delta}=g(x)\widetilde{\delta},\quad x_i\widetilde{\partial}_{x_i}\widetilde{\delta}=-(t\widetilde{\partial}_t+z)\widetilde{\delta},\quad \widetilde{\partial}_{y_j}\widetilde{\delta}=0,\quad t\Bigl(\prod_{i=1}^p\widetilde{\partial}_{x_i}\Bigr)\widetilde{\delta}=(-t\widetilde{\partial}_t)^p\widetilde{\delta}.\tag{9.9.14}$$

If we set $U^0\widetilde{\mathcal{N}}=(V_0\widetilde{\mathcal{D}})\cdot\widetilde{\delta}$ and, for $k\geqslant 1$,

$$U^k\widetilde{\mathcal{N}}=t^kU^0\widetilde{\mathcal{N}}=(V_0\widetilde{\mathcal{D}})\cdot t^k\widetilde{\delta},\qquad U^{-k}\widetilde{\mathcal{N}}=(V_k\widetilde{\mathcal{D}})\cdot U^0\widetilde{\mathcal{N}}=(V_k\widetilde{\mathcal{D}})\cdot\widetilde{\delta},$$

this shows that the coherent V-filtration $U^\bullet\widetilde{\mathcal{N}}$ satisfies the $\mathbb{R}$-specializability property:

$$b(-t\widetilde{\partial}_t+kz)U^k\widetilde{\mathcal{N}}\subset U^{k+1}\widetilde{\mathcal{N}}\quad\text{with } b(s)=s^p.$$

Each $\mathrm{gr}_U^k\widetilde{\mathcal{N}}$ is thus equipped with a nilpotent operator N satisfying $\mathrm{N}^p=0$, and with monodromy filtration $\mathrm{M}_\bullet\mathrm{gr}_U^k\widetilde{\mathcal{N}}$.

Claim. $\mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}_U^k\widetilde{\mathcal{N}}$ *is strict for any* k,ℓ.

As a consequence of this claim, we obtain that $\mathrm{gr}_U^k\widetilde{\mathcal{N}}$ is strict for any k, hence $U^\bullet\widetilde{\mathcal{N}}$ is the order filtration $V^\bullet\widetilde{\mathcal{N}}$, which is indexed by $\mathbb{Z}$, according to Lemma 9.3.13. Moreover, Properties 9.3.14(2) and (3) are obviously satisfied, due the definition of $U^\bullet\widetilde{\mathcal{N}}$, so $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along (t). Also by construction, the morphism can is onto.

It will be convenient to work within the localized module $\widetilde{\mathcal{O}}(*D):=\widetilde{\mathcal{O}}[1/g]$ and its direct image $\widetilde{\mathcal{N}}(*D)={}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{O}}(*D)=\widetilde{\mathcal{N}}[1/g]$, so that we can invert the variables x_i for

$i = 1, \dots, p$. In such a way, we highlight and make simple the action of $-t\widetilde{\partial}_t$, while the action of other operators are less obvious. We consider $\widetilde{\mathcal{N}}$ as a sub $\widetilde{\mathcal{D}}$-module of $\widetilde{\mathcal{N}}(*D)$.

9.9.15. Lemma. *$\widetilde{\mathcal{N}}(*D)$ is a free rank 1 module over $\widetilde{\mathcal{O}}(*D)[t\widetilde{\partial}_t]$ with generator $\widetilde{\delta}$.*

Proof. We have $\widetilde{\partial}_t^j\widetilde{\delta} = x^{-j\mathbf{1}}(\widetilde{\partial}_t^j t^j)\widetilde{\delta}$, showing that $\widetilde{\mathcal{N}}(*D) = \bigoplus_j \widetilde{\mathcal{O}}(*D)(\widetilde{\partial}_t^j t^j)\widetilde{\delta}$, hence also $\widetilde{\mathcal{N}}(*D) = \bigoplus_j \widetilde{\mathcal{O}}(*D)(t\widetilde{\partial}_t)^j\widetilde{\delta}$. $\square$

In order to prove the claim, it is necessary to have a canonical expression of local sections of $U^k\widetilde{\mathcal{N}}$ modulo $U^{k+1}\widetilde{\mathcal{N}}$. For that purpose, we introduce a family of polynomials of one variable s indexed by an integer k and a multi-index $\boldsymbol{a} \in \mathbb{Z}^p$. For $k \in \mathbb{Z}$ we set

$$q_{\boldsymbol{a},k}(s) = \prod_{i=1}^{p} \prod_{\ell\in(-k,a_i]} (s - \ell z),$$

where the index ℓ a priori runs in $\mathbb{Z}$ and we take the convention that the product indexed by the empty set is 1. For $\boldsymbol{a} \in \mathbb{Z}^p$ and $k \in \mathbb{Z}$, we set

$$J_k(\boldsymbol{a}) = \{i \in \{1, \dots, p\} \mid a_i \geqslant -k\}, \quad x_{J_k(\boldsymbol{a})} = (x_i)_{i\in J_k(\boldsymbol{a})}.$$

The following relations are easily checked:

$$\begin{aligned} q_{\boldsymbol{a},k+1}(s) &= (s + kz)^{\#J_k(\boldsymbol{a})} q_{\boldsymbol{a},k}(s)\\ q_{\boldsymbol{a}-\mathbf{1},k+1}(s - z) &= q_{\boldsymbol{a},k}(s)\\ q_{\boldsymbol{a},k}(s) &= q_{\boldsymbol{a}-\mathbf{1}_i,k}(s) \cdot \begin{cases} 1 & \text{if } i \notin J_k(\boldsymbol{a} - \mathbf{1}_i),\\ (s - a_i z) & \text{if } i \in J_k(\boldsymbol{a} - \mathbf{1}_i). \end{cases} \end{aligned}$$

We also set

$$Q_{\boldsymbol{a},k}(s) = \begin{cases} q_{\boldsymbol{a},k}(s) & \text{if } k \geqslant 0,\\ q_{\boldsymbol{a},k}(s) \cdot \prod_{j\in[k,0)} (s + jz)^{\min(1,\#J_j(\boldsymbol{a}))} & \text{if } k \leqslant -1, \end{cases}$$

that is, for $k \leqslant -1$, $Q_{\boldsymbol{a},k}(s)$ is the gcd of the polynomials $q_{\boldsymbol{a},k}(s) \cdot \prod_{j\in[\ell,0)}(s + jz)$ for ℓ varying in $[k, 0)$, and

$$\nu_k(\boldsymbol{a}) = \begin{cases} \#J_k(\boldsymbol{a}) & \text{if } k \geqslant 0,\\ \#J_k(\boldsymbol{a}) - \min(1, \#J_k(\boldsymbol{a})) = \begin{cases} \#J_k(\boldsymbol{a}) - 1 & \text{if } \#J_k(\boldsymbol{a}) \geqslant 1,\\ 0 & \text{if } \#J_k(\boldsymbol{a}) = 0, \end{cases} & \text{if } k \leqslant -1. \end{cases}$$

(so that $\nu_k(\boldsymbol{a}) \leqslant p \leqslant n$). We have the relation

$$Q_{\boldsymbol{a},k+1}(s) = \star(s + kz)^{\nu_k(\boldsymbol{a})} Q_{\boldsymbol{a},k}(s). \tag{9.9.16}$$

Let us also notice that

$$Q_{\boldsymbol{a},k}(s) \text{ is a multiple of } Q_{\boldsymbol{a}-\mathbf{1}_i,k}(s) \quad \forall\, i \in \{1, \dots, p\},\ \forall\, k \in \mathbb{Z}. \tag{9.9.17}$$

Indeed, this is clear for $q_{\boldsymbol{a},k}$, hence if $k \geqslant 0$. On the other hand, we have $J_k(\boldsymbol{a} - \mathbf{1}_i) \subset J_k(\boldsymbol{a})$, so $\min(1, \#J_j(\boldsymbol{a} - \mathbf{1}_i)) \leqslant \min(1, \#J_j(\boldsymbol{a}))$ and the assertion also holds for $k \geqslant -1$.

9.9.18. Lemma. *For $k\in\mathbb{Z}$, the filtration $U^\bullet\widetilde{\mathcal{N}}$ has the following expression:*

$$U^k\widetilde{\mathcal{N}}=\sum_{\boldsymbol{a}\in\mathbb{Z}^p}\widetilde{\mathcal{O}}[t\widetilde{\partial}_t]x^{-\boldsymbol{a}}Q_{\boldsymbol{a},k}(-t\widetilde{\partial}_t)\widetilde{\delta}.$$

Proof. Let us start with $U^0\widetilde{\mathcal{N}}$. Let us rewrite a section $P(x,\widetilde{\partial}_x,t,t\widetilde{\partial}_t)\cdot\widetilde{\delta}$ of $U^0\widetilde{\mathcal{N}}$. The differential operator $P\in V_0(\widetilde{\mathcal{D}})$ can be written as a sum of monomials of the form $(t\widetilde{\partial}_t)^q\widetilde{\partial}_x^{\boldsymbol{a}}h(x,t)$ with h holomorphic in its variables. Since $h(x,t)\widetilde{\delta}=h(x,g(x))\widetilde{\delta}$, we can simply consider (by using commutation relations) monomials of the form $(t\widetilde{\partial}_t)^qh(x)\widetilde{\partial}_x^{\boldsymbol{a}}$. Moreover, since $\widetilde{\partial}_{y_j}\widetilde{\delta}=0$, we can assume that $\boldsymbol{a}\in\mathbb{N}^p$. Using now the relation $x_i\widetilde{\partial}_{x_i}\widetilde{\delta}=-(t\widetilde{\partial}_t+z)\widetilde{\delta}$, we write

$$\widetilde{\partial}_x^{\boldsymbol{a}}\widetilde{\delta}=x^{-\boldsymbol{a}}\prod_{i=1}^p\prod_{\ell=1}^{a_i}(-t\widetilde{\partial}_t-\ell z)\cdot\widetilde{\delta}=x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t)\cdot\widetilde{\delta}.$$

At this point, we have obtained

$$U^0\widetilde{\mathcal{N}}=\sum_{\boldsymbol{a}\in\mathbb{N}^p}\widetilde{\mathcal{O}}\langle t\widetilde{\partial}_t\rangle x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t)\widetilde{\delta}.$$

We note that, if $a_i\leqslant 0$ for some $i\in\{1,\dots,p\}$, then $Q_{\boldsymbol{a}-\boldsymbol{1}_i,0}(s)=Q_{\boldsymbol{a},0}(s)$, and thus

$$x^{-(\boldsymbol{a}-\boldsymbol{1}_i)}Q_{\boldsymbol{a}-\boldsymbol{1}_i,0}(-t\widetilde{\partial}_t)=x_ix^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t)\in\widetilde{\mathcal{O}}x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t).$$

Therefore, the above expression of $U^0\widetilde{\mathcal{N}}$ is equal to that in the statement. For $k\geqslant 0$, we write

$$\begin{aligned}U^k\widetilde{\mathcal{N}}=t^k\sum_{\boldsymbol{a}\in\mathbb{Z}^p}\widetilde{\mathcal{O}}\langle t\widetilde{\partial}_t\rangle x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t)\widetilde{\delta}&=\sum_{\boldsymbol{a}\in\mathbb{Z}^p}\widetilde{\mathcal{O}}\langle t\widetilde{\partial}_t\rangle x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t+kz)t^k\widetilde{\delta}\\&=\sum_{\boldsymbol{a}\in\mathbb{Z}^p}\widetilde{\mathcal{O}}\langle t\widetilde{\partial}_t\rangle x^{-\boldsymbol{a}+k\boldsymbol{1}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t+kz)\widetilde{\delta}=\sum_{\boldsymbol{a}\in\mathbb{Z}^p}\widetilde{\mathcal{O}}\langle t\widetilde{\partial}_t\rangle x^{-\boldsymbol{a}}Q_{\boldsymbol{a},k}(-t\widetilde{\partial}_t)\widetilde{\delta}.\end{aligned}$$

Let us now consider $U^{-k}\widetilde{\mathcal{N}}$ for $k\geqslant 1$. We write

$$\begin{aligned}\widetilde{\partial}_t^kx^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t)\widetilde{\delta}&=x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t-kz)\widetilde{\partial}_t^k\widetilde{\delta}\\&=(-1)^kx^{-(\boldsymbol{a}+k\boldsymbol{1})}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t-kz)\prod_{j=1}^k(-t\widetilde{\partial}_t-jz)\widetilde{\delta},\end{aligned}$$

and we note that $Q_{\boldsymbol{a},0}(s-kz)=q_{\boldsymbol{a},0}(s-kz)=q_{\boldsymbol{a}+k\boldsymbol{1},-k}(s)$. One obtains the desired assertion by induction on k. $\square$

The algebraic case. We consider a similar setting as above with the simplification that the variables $x_1,\dots,x_p$ are polynomial variables. Namely, we now set $\widetilde{\mathcal{O}}=\widetilde{\mathcal{O}}_\varnothing[x_1,\dots,x_p]$ and we keep the notation for the corresponding objects $\widetilde{\mathcal{D}},\widetilde{\mathcal{N}},U^\bullet\widetilde{\mathcal{N}}$. We will prove Proposition 9.9.12 in this setting. The above results can be expressed in a more precise way.

9.9.19. Lemma. *For every $k\in\mathbb{Z}$, $U^k\widetilde{\mathcal{N}}$ can be decomposed as*

$$U^k\widetilde{\mathcal{N}}=\bigoplus_{\boldsymbol{a}\in\mathbb{Z}^p}\widetilde{\mathcal{O}}_\varnothing[t\widetilde{\partial}_t]\cdot x^{-\boldsymbol{a}}Q_{\boldsymbol{a},k}(-t\widetilde{\partial}_t)\widetilde{\delta}.\tag{9.9.20}$$

Proof. Recall that $\widetilde{\mathcal{O}}[t\widetilde{\partial}_t] = \bigoplus_{\boldsymbol{a}\in\mathbb{Z}^p} \widetilde{\mathcal{O}}_\varnothing[t\widetilde{\partial}_t]\cdot x^{-\boldsymbol{a}}$. The lemma characterizes an element of $U^k\widetilde{\mathcal{N}}$ through the possible coefficients of $x^{-\boldsymbol{a}}\widetilde{\delta}$ with respect to such a decomposition.

We start from the expression of Lemma 9.9.18 and we argue by induction on $\boldsymbol{a}$. It suffices to consider a term $x_i x^{-\boldsymbol{a}} Q_{\boldsymbol{a},k}(-t\widetilde{\partial}_t)\widetilde{\delta}$, $i \in \{1,\dots,p\}$. Since $Q_{\boldsymbol{a},k}$ is a multiple of $Q_{\boldsymbol{a}-\boldsymbol{1}_i,k}$ in $\widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ (see (9.9.17)), $x_i x^{-\boldsymbol{a}} Q_{\boldsymbol{a},k}$ is a multiple of $x^{-(\boldsymbol{a}-\boldsymbol{1}_i)} Q_{\boldsymbol{a}-\boldsymbol{1}_i,k}$. $\square$

From (9.9.16) we deduce that, as an $\widetilde{\mathcal{O}}_\varnothing[t\widetilde{\partial}_t]$-module, $U^k\widetilde{\mathcal{N}}/U^{k+1}\widetilde{\mathcal{N}}$ admits a similar direct sum decomposition, for which the coefficient of $x^{-\boldsymbol{a}}Q_{\boldsymbol{a},k}\widetilde{\delta}$ can vary in the quotient module $\widetilde{\mathcal{O}}_\varnothing[t\widetilde{\partial}_t]/(-t\widetilde{\partial}_t + kz)^{\nu_k(\boldsymbol{a})}$. In particular, it is strict, and N, which is induced by the action of $(-t\widetilde{\partial}_t + kz)$, is a strict morphism. The elements $x^{-\boldsymbol{a}}Q_{\boldsymbol{a},k}\cdot(-t\widetilde{\partial}_t + kz)^{\ell}\widetilde{\delta}$ $(0 \leqslant \ell \leqslant \nu_k(\boldsymbol{a}) - 1)$ lift a $\widetilde{\mathcal{O}}_\varnothing$-basis of this component and N has only one Jordan block of size $\nu_k(\boldsymbol{a})$ on this component.

We can now denote the filtration $U^\bullet\widetilde{\mathcal{N}}$ by $V^\bullet\widetilde{\mathcal{N}}$. Since N has only one Jordan block of size $\ell \geqslant 0$ on each term such that $\nu_k(\boldsymbol{a}) = \ell + 1$, we deduce that, as an $\widetilde{\mathcal{O}}_\varnothing$-module,

$$\mathrm{P}_\ell \mathrm{gr}_V^k \widetilde{\mathcal{N}} \simeq \bigoplus_{\substack{\boldsymbol{a}\in\mathbb{Z}^p\\ \nu_k(\boldsymbol{a})=\ell+1}} \widetilde{\mathcal{O}}_\varnothing \cdot x^{-\boldsymbol{a}}.$$

Let us now focus on $\mathrm{gr}_V^0\widetilde{\mathcal{N}} = \psi_{g,1}\widetilde{\mathcal{O}}$ and $\mathrm{gr}_V^{-1}\widetilde{\mathcal{N}} = \phi_{g,1}\widetilde{\mathcal{O}}(1)$. We have already seen that $\widetilde{\partial}_t : \mathrm{gr}_V^0\widetilde{\mathcal{N}} \to \mathrm{gr}_V^{-1}\widetilde{\mathcal{N}}(1)$ is onto. Let us check that $t : \mathrm{gr}_V^{-1}\widetilde{\mathcal{N}} \to \mathrm{gr}_V^0$ is injective and strict. Let us fix $\boldsymbol{a} \in \mathbb{Z}^p$. The corresponding component of $\mathrm{gr}_V^{-1}\widetilde{\mathcal{N}}$ is nonzero only if $\nu_{-1}(\boldsymbol{a}) \geqslant 1$, that is, $\#J_{-1}(\boldsymbol{a}) \geqslant 2$. We note that $J_{-1}(\boldsymbol{a}) = J_0(\boldsymbol{a} - \boldsymbol{1})$. A lift $x^{-\boldsymbol{a}}Q_{\boldsymbol{a},-1}(-\widetilde{\partial}_t t)(-\widetilde{\partial}_t t)^\ell\widetilde{\delta}$ $(0 \leqslant \ell \leqslant \nu_{-1}(\boldsymbol{a}) - 1)$ of a basis element of $\mathrm{gr}_V^{-1}\widetilde{\mathcal{N}}$ is sent by t to

$$x^{-(\boldsymbol{a}-\boldsymbol{1})}Q_{\boldsymbol{a},-1}(-t\widetilde{\partial}_t)(-t\widetilde{\partial}_t)^\ell\widetilde{\delta} = x^{-(\boldsymbol{a}-\boldsymbol{1})}Q_{\boldsymbol{a}-\boldsymbol{1},0}(-t\widetilde{\partial}_t)(-t\widetilde{\partial}_t)^{\ell+1}\widetilde{\delta}$$

since $Q_{\boldsymbol{a},-1}(s) = Q_{\boldsymbol{a}-\boldsymbol{1},0}(s)\cdot s^{\min(1,\#J_0(\boldsymbol{a}-\boldsymbol{1}))} = sQ_{\boldsymbol{a}-\boldsymbol{1},0}(s)$. We now note that $\nu_0(\boldsymbol{a}-\boldsymbol{1}) = \nu_{-1}(\boldsymbol{a}) + 1$, so $\ell+1 \leqslant \nu_0(\boldsymbol{a}-\boldsymbol{1})$ and the image in $\mathrm{gr}_V^0\widetilde{\mathcal{N}}$ is a basis element. The cokernel of t on this component is identified with $\widetilde{\mathcal{O}}_\varnothing x^{-(\boldsymbol{a}-\boldsymbol{1})}Q_{\boldsymbol{a}-\boldsymbol{1},0}(-t\widetilde{\partial}_t)\widetilde{\delta}$, hence is strict. One similarly checks that N is strict on $\mathrm{gr}_V^0\widetilde{\mathcal{N}}$ and $\mathrm{gr}_V^{-1}\widetilde{\mathcal{N}}$, and $\widetilde{\partial}_t : \mathrm{gr}_V^0\widetilde{\mathcal{N}} \to \mathrm{gr}_V^{-1}\widetilde{\mathcal{N}}$ is obviously strict, being onto. At this point, we have proved all the statements of Proposition 9.9.12 except the second part of (3) that we now consider.

We wish to identify $\mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}}$ as a $\widetilde{\mathcal{D}}$-module. We have the decomposition as an $\widetilde{\mathcal{O}}_\varnothing$-module:

$$\mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}} = \bigoplus_{\substack{J\subset\{1,\dots,p\}\\ \#J=\ell+1}} \bigoplus_{\substack{\boldsymbol{a}\\ J_0(\boldsymbol{a})=J}} (\mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}})_{\boldsymbol{a}},$$

and if $\#J_0(\boldsymbol{a}) = \ell+1$, the image of $\widetilde{\mathcal{O}}_\varnothing x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t)$ by the projection $V^0\widetilde{\mathcal{N}} \to \mathrm{gr}_V^0\widetilde{\mathcal{N}}$ is contained in $\mathrm{M}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}}$ and the morphism $\widetilde{\mathcal{O}}_\varnothing x^{-\boldsymbol{a}}Q_{\boldsymbol{a},0}(-t\widetilde{\partial}_t) \to \mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}_V^0\widetilde{\mathcal{N}}$ induces an isomorphism onto $(\mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}})_{\boldsymbol{a}}$. It is now convenient to go back to the original expression of the elements of $\widetilde{\mathcal{N}}$.

Recall that, for $J = J_0(\boldsymbol{a})$, $x_J^{-\boldsymbol{a}_J}Q_{\boldsymbol{a}_J,0}(-t\widetilde{\partial}_t)\widetilde{\delta}$ is nothing but $\widetilde{\partial}_{x_J}^{\boldsymbol{a}_J}\widetilde{\delta}$. For $J \subset \{1,\dots,p\}$, we denote by $I = J^{\mathrm{c}}$ its complement. We conclude that

- $\bigoplus_{J\mid \#J=\ell+1} x_I^{\mathbf{1}_I}\widetilde{\mathcal{O}}_I[\widetilde{\partial}_{x_J}]\widetilde{\delta}$ is contained in $\mathrm{M}_\ell V^0\widetilde{\mathcal{N}}$,
- and maps $\widetilde{\mathcal{O}}_\varnothing$-linearly isomorphically onto $\mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}}$.

Let us denote by ι_I the inclusion $\{x_j=0\mid j\in J\}\hookrightarrow X$.

9.9.21. Lemma. *The $\widetilde{\mathcal{O}}_\varnothing$-linear isomorphism defined as the composition*

$$\bigoplus_{\#J=\ell+1} {}_{\mathrm{D}}\iota_{I*}\widetilde{\mathcal{O}}_I(-(\ell+1)) = \bigoplus_{\#J=\ell+1}\widetilde{\mathcal{O}}_I[\widetilde{\partial}_{x_J}]\widetilde{\delta}_J \xrightarrow{\sim} \bigoplus_{\#J=\ell+1} x_I^{\mathbf{1}_I}\widetilde{\mathcal{O}}_I[\widetilde{\partial}_{x_J}]\widetilde{\delta} \xrightarrow{\sim} \mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}}$$

sending $\widetilde{\delta}_J$ to the class of $x_I^{\mathbf{1}_I}\widetilde{\delta}$ is a $\widetilde{\mathcal{D}}$-linear isomorphism.

The shift $-(\ell+1)$ comes from the definition of the pushforward of left $\widetilde{\mathcal{D}}$-modules by a closed embedding (see Exercise 8.45(2)). Since $\mathrm{P}_\ell\psi_{g,1}\widetilde{\mathcal{O}} = \mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}}(1)$, this ends the proof of Proposition 9.9.12 for $\widetilde{\mathcal{O}}$ in the algebraic case.

Proof of the lemma. We are left with proving $\widetilde{\mathcal{D}}$-linearity. This amounts to proving that $x_jx_I^{\mathbf{1}_I}\widetilde{\delta}$ and $\widetilde{\partial}_{x_i}x^{\mathbf{1}_I}\widetilde{\delta}$ have image zero in $\mathrm{gr}_\ell^{\mathrm{M}}\mathrm{gr}_V^0\widetilde{\mathcal{N}}$. This follows from the previous computations with the polynomials $Q_{\boldsymbol{a},0}$. For example, the set J' associated with $x_jx_I^{\mathbf{1}_i}\widetilde{\delta}$ satisfies $\#J'=\ell$, so this element is mapped to $\mathrm{M}_{\ell-1}\mathrm{gr}_V^0\widetilde{\mathcal{N}}$. ☐

Analytic case for $\widetilde{\mathcal{O}}$. Le us denote by $\widetilde{\mathcal{O}}^{\mathrm{alg}}$ the ring denoted by $\widetilde{\mathcal{O}}$ above, and $\widetilde{\mathcal{O}}^{\mathrm{an}}$ the analytic version considered in the proposition. We have similarly $\widetilde{\mathcal{N}}^{\mathrm{an}} = \widetilde{\mathcal{O}}^{\mathrm{an}}\otimes_{\widetilde{\mathcal{O}}^{\mathrm{alg}}}\widetilde{\mathcal{N}}^{\mathrm{alg}}$. By flatness of $\widetilde{\mathcal{O}}^{\mathrm{an}}$ over $\widetilde{\mathcal{O}}^{\mathrm{alg}}$, the filtration defined by $\widetilde{\mathcal{O}}^{\mathrm{an}}\otimes_{\widetilde{\mathcal{O}}^{\mathrm{alg}}} V^\bullet\widetilde{\mathcal{N}}^{\mathrm{alg}}$ satisfies all the properties necessary for $\widetilde{\mathcal{N}}^{\mathrm{an}}$ to be strictly $\mathbb{R}$-specializable along (t). Moreover, $\mathrm{gr}_V^k\widetilde{\mathcal{N}}^{\mathrm{an}}$ is obtained in the same way from $\mathrm{gr}_V^k\widetilde{\mathcal{N}}^{\mathrm{alg}}$, and similarly for $\mathrm{P}_\ell\mathrm{gr}_V^0\widetilde{\mathcal{N}}^{\mathrm{an}}$. Also, Lemma 9.9.21 holds in this analytic setting. We conclude Proposition 9.9.12 holds for $\widetilde{\mathcal{N}}^{\mathrm{an}}$ if it holds for $\widetilde{\mathcal{N}}^{\mathrm{alg}}$. ☐

Proof for any smooth $\widetilde{\mathcal{D}}_X$-module . If now $\widetilde{\mathcal{M}}$ is any smooth $\widetilde{\mathcal{D}}_X$-module, we note that ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{N}}$ with its usual twisted structure of $\widetilde{\mathcal{D}}_X$-module, and that the action of t resp. $\widetilde{\partial}_t$ comes from that on $\widetilde{\mathcal{N}}$. As $\widetilde{\mathcal{M}}$ is assumed to be $\widetilde{\mathcal{O}}_X$-locally free, the filtration of ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ defined by $V_\alpha({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}) = \widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}V_\alpha(\widetilde{\mathcal{N}})$ satisfies all properties of the Malgrange-Kashiwara filtration. Notice also that Lemma 9.9.21 holds if we replace ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{O}}_I$ with ${}_{\mathrm{D}}\iota_*({}_{\mathrm{D}}\iota^*\widetilde{\mathcal{M}})$. It is then easy to deduce all assertions of the proposition for $\widetilde{\mathcal{M}}$ from the corresponding statement for $\widetilde{\mathcal{N}}$. ☐

9.10. Exercises

Exercise 9.1 **($V_0\widetilde{\mathcal{D}}_X$-modules).** Let H be a smooth hypersurface of X.

(1) Denote by $\widetilde{\Omega}_X^1(\log H)$ (sheaf of logarithmic 1-forms along H) the $\widetilde{\mathcal{O}}_X$-dual of $\widetilde{\Theta}_X(\log H)$. Express a local section of $\widetilde{\Omega}_X^1(\log H)$ in local coordinates.

(2) Show that $\wedge^n(\widetilde{\Omega}_X^1(\log H)) = \widetilde{\omega}_X(H) := \widetilde{\omega}_X\otimes_{\widetilde{\mathcal{O}}_X}\widetilde{\mathcal{O}}_X(H)$.

(3) Show that $\widetilde{\omega}_X(H)$ is a right $V_0\widetilde{\mathcal{D}}_X$-module.

(4) Define the side-changing functors for $V_0\widetilde{\mathcal{D}}_X$-modules by means of $\widetilde{\omega}_X(H)$.

(5) Define the logarithmic de Rham complex and the logarithmic Spencer complex for a left resp. right $V_0\widetilde{\mathcal{D}}_X$-module in a way similar to that of Section 8.4 by means of logarithmic forms and vector fields.

(6) Show that $\mathrm{Sp}(V_0\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ as a left $V_0\widetilde{\mathcal{D}}_X$-module and ${}^{\mathrm{p}}\mathrm{DR}(V_0\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\omega}_X(H)$ as a right $V_0\widetilde{\mathcal{D}}_X$-module. [*Hint*: Argue as in Exercises 8.21 and 8.22.]

(7) Show the analogues of Exercises 8.30, 8.24 and 8.26.

Exercise 9.2 **(The Spencer complex of $\widetilde{\mathcal{D}}_X$ regarded as a right $V_0\widetilde{\mathcal{D}}_X$-module)**

Let H be a smooth hypersurface of X. We regard $\widetilde{\mathcal{D}}_X$ as a right $V_0\widetilde{\mathcal{D}}_X$-module and consider the corresponding Spencer complex $\mathrm{Sp}(\widetilde{\mathcal{D}}_X; V_0\widetilde{\mathcal{D}}_X) := \widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \mathrm{Sp}(V_0\widetilde{\mathcal{D}}_X)$.

(1) Choose local coordinates $(t, x_2, \dots, x_n)$ such that $H = \{t = 0\}$ and let $\tau, \xi_2, \dots, \xi_n$ be the corresponding logarithmic vector fields. Show that $(\xi_2, \dots, \xi_n, t\tau)$ is a regular sequence on the ring $\widetilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]$ and deduce that the corresponding Koszul complex is a resolution of $\widetilde{\mathcal{O}}_X[\tau]/(t\tau)$.

(2) Arguing as in Exercise 8.21, show that $\mathrm{Sp}(\widetilde{\mathcal{D}}_X; V_0\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{D}}_X/\widetilde{\mathcal{D}}_X \cdot \widetilde{\Theta}_X(\log H)$ by locally free left $\widetilde{\mathcal{D}}_X$-modules.

(3) Identify locally $\widetilde{\mathcal{D}}_X/\widetilde{\mathcal{D}}_X \cdot \widetilde{\Theta}_X(\log H)$ with $\widetilde{\mathcal{O}}_X\langle\widetilde{\partial}_t\rangle/(\widetilde{\mathcal{O}}_X\langle\widetilde{\partial}_t\rangle \cdot t\widetilde{\partial}_t)$.

(4) Let $\widetilde{\mathcal{N}}$ be a right $V_0\widetilde{\mathcal{D}}_X$-module. Show that, if $t : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ is *injective*, then $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X; V_0\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X/\widetilde{\mathcal{D}}_X \cdot \widetilde{\Theta}_X(\log H))$ as a right $V_0\widetilde{\mathcal{D}}_X$-module, by using the tens right $V_0\widetilde{\mathcal{D}}_X$-module structures. [*Hint*: Use that the terms of $\mathrm{Sp}(\widetilde{\mathcal{D}}_X; V_0\widetilde{\mathcal{D}}_X)$ are left $\widetilde{\mathcal{D}}_X$-locally free, hence $\widetilde{\mathcal{O}}_X$-locally free to conclude that $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X; V_0\widetilde{\mathcal{D}}_X) \simeq \widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X/\widetilde{\mathcal{D}}_X \cdot \widetilde{\Theta}_X(\log H))$; express locally $\widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X/\widetilde{\mathcal{D}}_X \cdot \widetilde{\Theta}_X(\log H))$ as the complex

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X\langle\widetilde{\partial}_t\rangle \xrightarrow{\;\cdot\, t\widetilde{\partial}_t\;} \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X\langle\widetilde{\partial}_t\rangle$$

and check that the differential is injective.]

(5) Conclude that, under the previous assumption on $\widetilde{\mathcal{N}}$, we have

$$H^i(\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}(\widetilde{\mathcal{D}}_X; V_0\widetilde{\mathcal{D}}_X)) = 0 \quad \text{for } i \neq 0.$$

Exercise 9.3 **(The V-filtration of $\widetilde{\mathcal{D}}_X$).** Show the following properties.

(1) Let us fix a local decomposition $X \simeq H \times \Delta_t$ (where $\Delta_t \subset \mathbb{C}$ is a disc with coordinate t). With respect to this decomposition we have

$$V_0\widetilde{\mathcal{D}}_X = \widetilde{\mathcal{O}}_X\langle\widetilde{\partial}_x, t\widetilde{\partial}_t\rangle, \quad V_{-j}\widetilde{\mathcal{D}}_X = \begin{cases} t^j \cdot V_0\widetilde{\mathcal{D}}_X, \\ V_0\widetilde{\mathcal{D}}_X \cdot t^j, \end{cases} \quad V_j\widetilde{\mathcal{D}}_X = \begin{cases} \displaystyle\sum_{k=0}^{j} \widetilde{\partial}_t^k \cdot V_0\widetilde{\mathcal{D}}_X, \\ \displaystyle\sum_{k=0}^{j} V_0\widetilde{\mathcal{D}}_X \cdot \widetilde{\partial}_t^k, \end{cases} \quad (j \geqslant 0)$$

(2) For every k, $V_k\widetilde{\mathcal{D}}_X$ is a locally free $V_0\widetilde{\mathcal{D}}_X$-module.

(3) $\widetilde{\mathcal{D}}_X = \bigcup_k V_k\widetilde{\mathcal{D}}_X$ (the filtration is exhaustive).

(4) $V_k\widetilde{\mathcal{D}}_X \cdot V_\ell\widetilde{\mathcal{D}}_X \subset V_{k+\ell}\widetilde{\mathcal{D}}_X$ with equality for $k, \ell \leqslant 0$ or $k, \ell \geqslant 0$.

(5) $V_0\widetilde{\mathcal{D}}_X$ is a sheaf of subalgebras of $\widetilde{\mathcal{D}}_X$.

(6) $V_k\widetilde{\mathcal{D}}_{X|X\smallsetminus H} = \widetilde{\mathcal{D}}_{X|X\smallsetminus H}$ for all $k \in \mathbb{Z}$.

(7) $\mathrm{gr}_k^V\widetilde{\mathcal{D}}_X$ is supported on H for all $k \in \mathbb{Z}$,

(8) The induced filtration $V_k\widetilde{\mathcal{D}}_X \cap \widetilde{\mathcal{O}}_X = \widetilde{\mathcal{I}}_H^{-k}\widetilde{\mathcal{O}}_X$ is the $\widetilde{\mathcal{I}}_H$-adic filtration of $\widetilde{\mathcal{O}}_X$ made increasing.

(9) $\big(\bigcap_k V_k\widetilde{\mathcal{D}}_X\big)_{|H} = \{0\}$.

Exercise 9.4 **(Euler vector field).**

(1) Show that the class E of $t\widetilde{\partial}_t$ in $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$ in some local product decomposition as above does not depend on the choice of such a local product decomposition. [*Hint*: see [**MM04**, Lem. 4.1-12].]

(2) Show that $V_0\widetilde{\mathcal{D}}_X$ acts on $\widetilde{\mathcal{O}}_H = \widetilde{\mathcal{O}}_X/\widetilde{\mathcal{I}}_H$ and with respect to this action that $V_{<0}\widetilde{\mathcal{D}}_X$ acts by 0, so that $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$ acts on $\widetilde{\mathcal{O}}_H$, and that E acts by 0. Conclude that there exists a morphism $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X/\,\mathrm{E}\,\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{D}}_H$ and check by a local computation that it is an isomorphism.

(3) Show that if H has a global equation g, then $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X \simeq \widetilde{\mathcal{D}}_H[\mathrm{E}]$.

(4) Conclude that $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$ is a sheaf of rings and that E belongs to its center.

Exercise 9.5 **(Euler vector field, continued).**

(1) Show the identification (which forgets the grading) between $\mathrm{gr}^V\widetilde{\mathcal{D}}_X$ and $\widetilde{\mathcal{D}}_{[N_HX]}$. [*Hint*: see [**MM04**, Lem. 4.1-12].]

(2) Let $\mathcal{M}$ be a *monodromic* $\mathcal{D}_{[N_HX]}$-module, i.e., a $\mathcal{D}_{[N_HX]}$-module for which the action of E has a minimal polynomial with coefficients in $\mathbb{C}$. Show that $\mathcal{M}$ has a finite filtration by $\mathcal{D}_H$-submodules. [*Hint*: Reduce first to the case where the minimal polynomial has only one root α; in this case, filter $\mathcal{M}$ so that $\mathrm{E}-\alpha\,\mathrm{Id}$ vanishes on each graded piece; identify then $\mathrm{gr}_0^V\mathcal{D}_X/(\mathrm{E}-\alpha)\mathrm{gr}_0^V\mathcal{D}_X$ with $\mathcal{D}_H$.]

Exercise 9.6. Show the equivalence between the category of $\widetilde{\mathcal{O}}_X$-modules with integrable logarithmic connection $\widetilde{\nabla} : \widetilde{\mathcal{M}} \to \widetilde{\Omega}_X^1(\log H) \otimes \widetilde{\mathcal{M}}$ and the category of left $V_0\widetilde{\mathcal{D}}_X$-modules. Show that the residue $\mathrm{Res}\,\widetilde{\nabla}$ corresponds to the induced action of E on $\widetilde{\mathcal{M}}/\widetilde{\mathcal{I}}_H\widetilde{\mathcal{M}}$.

Exercise 9.7 **(The Rees sheaf of rings $R_V\widetilde{\mathcal{D}}_X$).** Introduce the Rees sheaf of rings $R_V\widetilde{\mathcal{D}}_X := \bigoplus_k V_k\widetilde{\mathcal{D}}_X \cdot v^k \subset \widetilde{\mathcal{D}}_X[v,v^{-1}]$ associated to the filtered sheaf $(\widetilde{\mathcal{D}}_X, V_\bullet\widetilde{\mathcal{D}}_X)$ (see Section 5.1.3), and similarly $R_V\widetilde{\mathcal{O}}_X = \bigoplus_k V_k\widetilde{\mathcal{O}}_X \cdot v^k \subset \widetilde{\mathcal{O}}_X[v,v^{-1}]$, which is the Rees ring associated to the $\widetilde{\mathcal{I}}_H$-adic filtration of $\widetilde{\mathcal{O}}_X$.

(1) Show that $R_V\widetilde{\mathcal{O}}_X = \widetilde{\mathcal{O}}_X[v,tv^{-1}]$, where $t=0$ is a local equation of H. Identify this sheaf of rings with $\widetilde{\mathcal{O}}_X[v,w]/(t-vw)$ and show that, as an $\widetilde{\mathcal{O}}_X$-module, it is isomorphic to $\widetilde{\mathcal{O}}_X[v] \oplus w\widetilde{\mathcal{O}}_X[w]$. Conclude that $R_V\widetilde{\mathcal{O}}_X$ is $\widetilde{\mathcal{O}}_X$-flat.

(2) Show that $R_V\widetilde{\mathcal{D}}_X = \widetilde{\mathcal{O}}_X[v,tv^{-1}]\langle v\widetilde{\partial}_t, \widetilde{\partial}_{x_2}, \dots, \widetilde{\partial}_{x_n}\rangle$.

(3) Conclude that $R_V\widetilde{\mathcal{D}}_X$ is locally free over $R_V\widetilde{\mathcal{O}}_X$ and is $\widetilde{\mathcal{O}}_X$-flat.

Exercise 9.8. Define ${}^A V_\alpha\widetilde{\mathcal{O}}_X$ in a way similar to that of ${}^A V_\alpha\widetilde{\mathcal{D}}_X$ (Remark 9.2.3) and show that $R_{{}^AV}\widetilde{\mathcal{D}}_X$ is locally free over $R_{{}^AV}\widetilde{\mathcal{O}}_X$.

***Exercise 9.9* (Coherence of $R_V\widetilde{\mathcal{D}}_X$).** We consider the Rees sheaf of rings $R_V\widetilde{\mathcal{D}}_X := \bigoplus_k V_k\widetilde{\mathcal{D}}_X \cdot v^k$ as in Exercise 9.7. The aim of this exercise is to show the coherence of the sheaf of rings $R_V\widetilde{\mathcal{D}}_X$. Since the problem is local, we can assume that there are coordinates $(t, x_2, \dots, x_n)$ such that $H = \{t = 0\}$.

(1) Let K be a compact polycylinder in X. Show that $R_V\widetilde{\mathcal{O}}_X(K) = R_V(\widetilde{\mathcal{O}}_X(K))$ is Noetherian, being the Rees ring of the $\widetilde{\mathcal{I}}_H$-adic filtration on the ring $\widetilde{\mathcal{O}}_X(K)$ (which is Noetherian, by a theorem of Frisch). Similarly, as $\widetilde{\mathcal{O}}_{X,x}$ is flat on $\widetilde{\mathcal{O}}_X(K)$ for every $x \in K$, show that the ring $(R_V\widetilde{\mathcal{O}}_X)_x = R_V\widetilde{\mathcal{O}}_X(K) \otimes_{\widetilde{\mathcal{O}}_X(K)} \widetilde{\mathcal{O}}_{X,x}$ is flat on $R_V\widetilde{\mathcal{O}}_X(K)$.

(2) Show that $R_V\widetilde{\mathcal{O}}_X$ is coherent on X by following the strategy developed in **[GM93]**. [*Hint*: Let $\widetilde{\Omega}$ be any open set in X and let $\varphi : (R_V\widetilde{\mathcal{O}}_X)^q_{|\widetilde{\Omega}} \to (R_V\widetilde{\mathcal{O}}_X)^p_{|\widetilde{\Omega}}$ be any morphism. Let K be a polycylinder contained in $\widetilde{\Omega}$. Show that $\operatorname{Ker}\varphi(K)$ is finitely generated over $R_V\widetilde{\mathcal{O}}_X(K)$ and, if K° is the interior of K, show that $\operatorname{Ker}\varphi_{|K^\circ} = \operatorname{Ker}\varphi(K) \otimes_{R_V\widetilde{\mathcal{O}}_X(K)} (R_V\widetilde{\mathcal{O}}_X)_{|K^\circ}$. Conclude that $\operatorname{Ker}\varphi_{|K^\circ}$ is finitely generated, whence the coherence of $R_V\widetilde{\mathcal{O}}_X$.]

(3) Consider the sheaf $\widetilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]$ equipped with the V-filtration for which τ has degree 1, the variables $\xi_2, \dots, \xi_n$ have degree 0, and inducing the V-filtration (i.e., t-adic in the reverse order) on $\widetilde{\mathcal{O}}_X$. First, forgetting τ, Show that $R_V(\widetilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n]) = (R_V\widetilde{\mathcal{O}}_X)[\xi_2, \dots, \xi_n]$. Secondly, using $V_k(\widetilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]) = \sum_{j\geqslant 0} V_{k-j}(\widetilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n])\tau^j$ for every $k \in \mathbb{Z}$, show that we have a surjective morphism

$$\begin{aligned} R_V\widetilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n] \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\tau'] &\longrightarrow R_V(\widetilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n]) \\ V_\ell\widetilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n]q^\ell\tau'^j &\longmapsto V_\ell\widetilde{\mathcal{O}}_X[\xi_2, \dots, \xi_n]\tau^j q^{\ell+j}. \end{aligned}$$

If $K \subset X$ is any polycylinder show that $R_V(\widetilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n])(K)$ is Noetherian, by using that $\big(R_V\widetilde{\mathcal{O}}_X(K)\big)[\tau', \xi_2, \dots, \xi_n]$ is Noetherian.

(4) As $R_V\widetilde{\mathcal{D}}_X$ can be filtered (by the degree of the operators) in such a way that, locally on X, $\operatorname{gr}R_V\widetilde{\mathcal{D}}_X$ is isomorphic to $R_V(\widetilde{\mathcal{O}}_X[\tau, \xi_2, \dots, \xi_n])$, conclude that, if K is any sufficiently small polycylinder, then $R_V\widetilde{\mathcal{D}}_X(K)$ is Noetherian.

(5) Use now arguments similar to that of **[GM93]** to concludes that $R_V\widetilde{\mathcal{D}}_X$ is coherent.

***Exercise 9.10* (Characterization of coherent V-filtrations).** Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Show that the following properties are equivalent for a V-filtration $U_\bullet\widetilde{\mathcal{M}}$.

(1) $U_\bullet\widetilde{\mathcal{M}}$ is a coherent filtration.

(2) The Rees module $R_U\widetilde{\mathcal{M}} := \bigoplus_\ell U_\ell\widetilde{\mathcal{M}}v^\ell$ is $R_V\widetilde{\mathcal{D}}_X$-coherent.

(3) For every $x \in X$, replacing X with a small neighbourhood of x, there exist integers $\lambda_{j=1,\dots,q}, \mu_{i=1,\dots,p}, k_{i=1,\dots,p}$ and a presentation (recall that $[\bullet]$ means a shift of the grading)

$$\bigoplus_{j=1}^q \widetilde{\mathcal{D}}_X[\lambda_j] \longrightarrow \bigoplus_{i=1}^p \widetilde{\mathcal{D}}_X[\mu_i] \longrightarrow \widetilde{\mathcal{M}} \longrightarrow 0$$

such that $U_\ell\widetilde{\mathcal{M}} = \operatorname{image}(\bigoplus_{i=1}^p V_{k_i+\ell}\widetilde{\mathcal{D}}_X[\mu_i])$.

Note that, as for $\widetilde{\mathcal{I}}_H$-adic filtrations on coherent $\widetilde{\mathcal{O}}_X$-modules, it is not enough to check the coherence of $\mathrm{gr}_U\widetilde{\mathcal{M}}$ as a $\mathrm{gr}^V\widetilde{\mathcal{D}}_X$-module in order to deduce that $U_\bullet\widetilde{\mathcal{M}}$ is a coherent V-filtration.

Exercise 9.11 **(From coherent $R_V\widetilde{\mathcal{D}}_X$-modules to $\widetilde{\mathcal{D}}_X$-modules with a coherent V-filtration)**

(1) Show that a graded $R_V\widetilde{\mathcal{D}}_X$-module $\mathcal{M}$ can be written as $R_U\widetilde{\mathcal{M}}$ for some V-filtration on some $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ if and only if it has no v-torsion.

(2) Show that, if $\mathcal{M}$ is a graded coherent $R_V\widetilde{\mathcal{D}}_X$-module, then its v-torsion is a graded coherent $R_V\widetilde{\mathcal{D}}_X$-module.

(3) Conclude that, for any graded coherent $R_V\widetilde{\mathcal{D}}_X$-module $\mathcal{M}$, there exists a unique coherent $\widetilde{\mathcal{D}}_X$-module and a unique coherent V-filtration $U^\bullet\widetilde{\mathcal{M}}$ such that $\mathcal{M}/v\text{-torsion} = R_U\widetilde{\mathcal{M}}$.

Exercise 9.12 **(Some basic properties of coherent V-filtrations)**

(1) Show that the filtration naturally induced by a coherent V-filtration on a coherent $\widetilde{\mathcal{D}}_X$-module on a coherent sub or quotient $\widetilde{\mathcal{D}}_X$-modules is a coherent V-filtration. [*Hint*: For the case of a submodule, use the characterization of Exercise 9.10(2) and the classical Artin-Rees lemma, as in Corollary 8.8.7. This proof shows the interest of considering $R_U\widetilde{\mathcal{M}}$.]

(2) Deduce that, locally on X, there exist integers $\lambda_{j=1,\dots,q}$, $\ell_{j=1,\dots,q}$, $\mu_{i=1,\dots,p}$, $k_{i=1,\dots,p}$ and a presentation $\bigoplus_{j=1}^q \widetilde{\mathcal{D}}_X[\lambda_j] \to \bigoplus_{i=1}^p \widetilde{\mathcal{D}}_X[\mu_i] \to \widetilde{\mathcal{M}} \to 0$ inducing for every ℓ a presentation

$$\bigoplus_{j=1}^q V_{\ell_j+\ell}\widetilde{\mathcal{D}}_X[\lambda_j] \longrightarrow \bigoplus_{i=1}^p V_{k_i+\ell}\widetilde{\mathcal{D}}_X[\mu_i] \longrightarrow U_\ell\widetilde{\mathcal{M}} \longrightarrow 0.$$

(3) Show that two coherent V-filtrations $U_\bullet\widetilde{\mathcal{M}}$ and $U'{}_\bullet\widetilde{\mathcal{M}}$ are *locally comparable*, that is, locally on X there exists $\ell_o \geqslant 0$ such that, for every $\ell \in \mathbb{Z}$,

$$U_{\ell-\ell_o}\widetilde{\mathcal{M}} \subset U'_\ell\widetilde{\mathcal{M}} \subset U_{\ell+\ell_o}\widetilde{\mathcal{M}}.$$

(4) If $U_\bullet\widetilde{\mathcal{M}}$ is a coherent V-filtration, then for every $\ell_o \in \mathbb{Z}$, the filtration $U_{\bullet+\ell_o}\widetilde{\mathcal{M}}$ is also coherent.

(5) If $U_\bullet\widetilde{\mathcal{M}}$ and $U'_\bullet\widetilde{\mathcal{M}}$ are two coherent V-filtrations, then the filtration $U''_\ell\widetilde{\mathcal{M}} := U_\ell\widetilde{\mathcal{M}} + U'_\ell\widetilde{\mathcal{M}}$ is also coherent.

(6) Assume that H is defined by an equation $t = 0$. Prove that, locally on X, there exists k_0 such that, for every $k \leqslant k_0$, $t : U_k \to U_{k-1}$ is bijective. [*Hint*: Use (2) above.]

Exercise 9.13. Let $\mathcal{U}$ be a coherent left $V_0\widetilde{\mathcal{D}}_X$-module and let $\widetilde{\mathcal{T}}$ be its t-torsion subsheaf, i.e., the subsheaf of local sections locally killed by some power of t. Show that, locally on X, there exists ℓ such that $\widetilde{\mathcal{T}} \cap t^\ell\mathcal{U} = 0$. Adapt to the right case. [*Hint*: Consider the t-adic filtration on $V_0\widetilde{\mathcal{D}}_X$, i.e., the filtration $V_{-j}\widetilde{\mathcal{D}}_X$ with $j \geqslant 0$. Show (e.g. in the left case) that the filtration $t^j\mathcal{U}$ is coherent with respect to it, and locally there is a surjective morphism $(V_0\widetilde{\mathcal{D}}_X)^n \to \mathcal{U}$ which is strict with respect to the V-filtration. Deduce that its kernel $\mathcal{K}$ is coherent and comes equipped with the induced V-filtration, which is coherent. Conclude that, locally on X, there exists $j_0 \geqslant 0$ such that $V_{j_0-j}\mathcal{K} = t^j V^{j_0}\mathcal{K}$ for every $j \geqslant 0$. Show that, for every $j \geqslant 0$ there is locally an exact sequence (up to shifting the grading on each $V_\bullet\widetilde{\mathcal{D}}_X$ summand)

$$(V_{-j}\widetilde{\mathcal{D}}_X)^m \longrightarrow (V_{-(j+j_0)}\widetilde{\mathcal{D}}_X)^n \longrightarrow t^{(j+j_0)}\mathcal{U} \longrightarrow 0.$$

As $t : V_k\widetilde{\mathcal{D}}_X \to V_{k-1}\widetilde{\mathcal{D}}_X$ is bijective for $k \leqslant 0$, conclude that $t : t^{j_0}\mathcal{U} \to t^{j_0+1}\mathcal{U}$ is so, hence $\widetilde{\mathcal{T}} \cap t^{j_0}\mathcal{U} = 0$.]

***Exercise 9.14* (Coherent V-filtration indexed by $A+\mathbb{Z}$).** Extend the properties of coherent V-filtrations indexed by $\mathbb{Z}$ to coherent V-filtrations indexed by $A+\mathbb{Z}$, where $A \subset (-1,0]$ is some finite set (see Remark 9.2.3).

Exercise 9.15. Show that a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is specializable along H if and only if one of the following properties holds:

(1) locally on X, *some* coherent V-filtration $U_\bullet\widetilde{\mathcal{M}}$ (resp. $U^\bullet\widetilde{\mathcal{M}}$, left case) has a weak Bernstein polynomial, i.e., there exists a nonzero $b(s)$ and a non-negative integer ℓ such that

$$\forall\, k \in \mathbb{Z},\quad \mathrm{gr}_k^U\widetilde{\mathcal{M}} \cdot z^\ell b(\mathrm{E} - kz) = 0, \quad \text{resp. } z^\ell b(\mathrm{E} - kz)\mathrm{gr}_U^k\widetilde{\mathcal{M}} = 0; \tag{9.10.0 $*$}$$

(2) locally on X, *any* coherent V-filtration $U^\bullet\widetilde{\mathcal{M}}$ (resp. $U_\bullet\widetilde{\mathcal{M}}$) has a weak Bernstein polynomial.

[*Hint*: In one direction, take the V-filtration generated by a finite number of local generators of $\widetilde{\mathcal{M}}$; in the other direction, use that two coherent filtrations are locally comparable.]

Exercise 9.16. Assume that $\widetilde{\mathcal{M}}$ is (right) $\widetilde{\mathcal{D}}_X$-coherent and specializable along H.

(1) Fix $\ell_o \in \mathbb{Z}$ and set $U'_\ell\widetilde{\mathcal{M}} = U_{\ell+\ell_o}\widetilde{\mathcal{M}}$. Show that $b_{U'}(s)$ can be chosen as $b_U(s - \ell_o z)$.

(2) Set $b_U = b_1 b_2$ where b_1 and b_2 have no common root. Show that the filtration $U'_k\widetilde{\mathcal{M}} := U_{k-1}\widetilde{\mathcal{M}} + b_2(\mathrm{E} - kz)U_k\widetilde{\mathcal{M}}$ is a coherent filtration and compute a polynomial $b_{U'}$ in terms of b_1, b_2.

(3) Conclude that there exists locally a coherent filtration $U_\bullet\widetilde{\mathcal{M}}$ for which $b_U(s) = \prod_{\alpha\in A}(s - \alpha z)^{\nu_\alpha}$ and $\mathrm{Re}(A) \subset (-1,0]$.

(4) Adapt the result to the left case.

Exercise 9.17. Assume that $\widetilde{\mathcal{M}}$ is an $\mathbb{R}$-specializable coherent right $\widetilde{\mathcal{D}}_X$-module. Show that, for $m \in \widetilde{\mathcal{M}}_{x_o}$ and $P \in V_k\widetilde{\mathcal{D}}_{X,x_o}$, we have

$$\mathrm{ord}_{H,x_o}(mP) \leqslant \mathrm{ord}_{H,x_o}(m) + k.$$

[*Hint*: Use that $[\mathrm{E}, V_{-1}\widetilde{\mathcal{D}}_X] \subset V_0\widetilde{\mathcal{D}}_X$ and that the coherent V-filtrations $(mP \cdot \widetilde{\mathcal{D}}_X) \cap m \cdot V_\bullet\widetilde{\mathcal{D}}_X$ and $mP \cdot V_\bullet\widetilde{\mathcal{D}}_X$ of $mP \cdot \widetilde{\mathcal{D}}_X$ are locally comparable.]

In the left case, show that

$$\mathrm{ord}_{H,x_o}(Pm) \geqslant \mathrm{ord}_{H,x_o}(m) - k.$$

Exercise 9.18 **($\mathbb{R}$-specializability).**

(1) In a short exact sequence $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ of coherent $\widetilde{\mathcal{D}}_X$-modules, show that $\widetilde{\mathcal{M}}$ is $\mathbb{R}$-specializable along H if and only if $\widetilde{\mathcal{M}}'$ and $\widetilde{\mathcal{M}}''$ are so.

(2) Let $\varphi : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$ be a morphism between $\mathbb{R}$-specializable modules along H. Show that φ is compatible with the order filtrations along H. Conclude that, on the full subcategory consisting of $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules of the category of $\widetilde{\mathcal{D}}_X$-modules (and morphisms consist of all morphisms of $\widetilde{\mathcal{D}}_X$-modules), gr^V_α is a functor to the category of $\mathrm{gr}^V_0\widetilde{\mathcal{D}}_X$-modules.

Exercise 9.19 **($\mathbb{R}$-specializability for $\mathcal{D}_X$-modules).**

(1) Show that, for an $\mathbb{R}$-specializable $\mathcal{D}_X$-module $\mathcal{M}$, the assumption of Lemma 9.3.13 is satisfied. [*Hint*: Choose a finite set of local sections generating $\mathcal{M}$ and consider the V-filtration they generate.] Conclude that Properties (1)–(3) of Definition 9.3.14 are also satisfied.

(2) Show that any morphism between coherent $\mathbb{R}$-specializable $\mathcal{D}_X$-modules is strictly compatible with the V-filtrations and its kernel and cokernel are coherent $\mathbb{R}$-specializable $\mathcal{D}_X$-modules.

Exercise 9.20. Show that the notion of strict $\mathbb{R}$-specializability does not depend on the choice of a local decomposition $X \simeq H \times \Delta_t$. [*Hint*: Use the formulas in [**MM04**, Lem. 4.1-12].]

Exercise 9.21 **(Strict $\mathbb{R}$-specializability and Bernstein polynomials)**

Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and let m be a local section of $\widetilde{\mathcal{M}}$, with Bernstein polynomial b_m. We have seen in the proof of Proposition 9.3.16 that m is a local section of $V_\alpha\widetilde{\mathcal{M}}$ if and only if the z-roots of b_m are $\leqslant \alpha$. Prove that any z-root γ of b_m is such that $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}} \neq 0$. [*Hint*: Since $\widetilde{\mathcal{D}}_X \cdot m \cap V_\bullet\widetilde{\mathcal{M}}$ is a good V-filtration of $\widetilde{\mathcal{D}}_X \cdot m$ (see Exercise 9.12(1)), there exists $N \geqslant 0$ such that $\widetilde{\mathcal{D}}_X \cdot m \cap V_{\alpha-N}\widetilde{\mathcal{M}} \subset V_{-1}\widetilde{\mathcal{D}}_X \cdot m$; let $\nu(\gamma)$ be the order of nilpotence of $\mathrm{E} - \gamma$ on $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}}$; show that the product $\prod_{\gamma\in(\alpha-N,\alpha]}(\mathrm{E}-\gamma)^{\nu(\gamma)}$ sends m to $V_{-1}\widetilde{\mathcal{D}}_X \cdot m$ and conclude.]

Exercise 9.22 **(Strict $\mathbb{R}$-specializability and exact sequences)**

We consider an exact sequence $0 \to \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_2 \to 0$ of coherent $\widetilde{\mathcal{D}}_X$-modules.

(1) Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and that the exact sequence splits, i.e., $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}_1 \oplus \widetilde{\mathcal{M}}_2$. Show that $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2$ are strictly $\mathbb{R}$-specializable along H. [*Hint*: Show that the order filtration of $\widetilde{\mathcal{M}}$ splits, and deduce the V-coherence of the summands.]

(2) If $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, but the exact sequence does not split, set

$$U_\alpha\widetilde{\mathcal{M}}_1 = V_\alpha\widetilde{\mathcal{M}} \cap \widetilde{\mathcal{M}}_1, \quad U_\alpha\widetilde{\mathcal{M}}_2 = \mathrm{image}(V_\alpha\widetilde{\mathcal{M}}).$$

- Show that these V-filtrations are coherent (see Exercise 9.12(1)) and that, for every α, the sequence

$$0 \longrightarrow \mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}_1 \longrightarrow \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}} \longrightarrow \mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}_2 \longrightarrow 0$$

is exact.

• Conclude that $U_\bullet\widetilde{\mathcal{M}}_1$ satisfies the Bernstein property 9.3.13(1) and the strictness property 9.3.13(2) (with index set $\mathbb{R}$), and thus injectivity in 9.3.20(a) and (d), but possibly not 9.3.14(2) and (3). Deduce that $U_\alpha\widetilde{\mathcal{M}}_1 = V_\alpha\widetilde{\mathcal{M}}_1$. [*Hint*: Use the uniqueness property of Lemma 9.3.13.]

• If each $\mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}_2$ is also strict, show that $U_\alpha\widetilde{\mathcal{M}}_2 = V_\alpha\widetilde{\mathcal{M}}_2$.

• If moreover one of both $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2$ is strictly $\mathbb{R}$-specializable, show that so is the other one.

(3) Conclude that if $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{M}}_2$ are strictly $\mathbb{R}$-specializable, then so is $\widetilde{\mathcal{M}}_1$ and for every α, the sequence

$$0 \longrightarrow \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}_1 \longrightarrow \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}} \longrightarrow \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}_2 \longrightarrow 0$$

is exact.

Exercise 9.23 **(A characterization of strong strict $\mathbb{R}$-specializability)**

Let $\widetilde{\mathcal{M}}$ be a coherent right $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H. Show that the following properties are equivalent:

(1) $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along H (Definition 9.3.24),

(2) $\widetilde{\mathcal{M}}$ satisfies

(a) the morphisms $t : \mathrm{gr}^V_0\widetilde{\mathcal{M}} \to \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}$ and $\widetilde{\partial}_t : \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}} \to \mathrm{gr}^V_0\widetilde{\mathcal{M}}(-1)$ are strict,

(b) for each $\alpha \in (-1, 0]$ and for every $k \geqslant 1$, the morphism $\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}} \to \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}(-k)$ induced by $\widetilde{\partial}_t^k t^k = \prod_{i=0}^{k-1}(\widetilde{\partial}_t t + iz)$ is strict,

(c) for each $\alpha \in [-1, 0)$ and for every $k \geqslant 1$, the morphism $\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}} \to \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}(-k)$ induced by $t^k\widetilde{\partial}_t^k = \prod_{i=0}^{k-1}(t\widetilde{\partial}_t - iz)$ is strict.

Exercise 9.24 **(Strictness of submodules supported on the divisor H)**

Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and let $\widetilde{\mathcal{M}}_1$ be a coherent $\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\mathcal{M}}$ supported on H. Show that $\widetilde{\mathcal{M}}_1$ is strict. [*Hint*: Use Exercise 9.22(9.22) and show that $V_{<0}\widetilde{\mathcal{M}}_1 = 0$; from strictness of each $\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}_1$, deduce that each $V_\alpha\widetilde{\mathcal{M}}_1$ is strict and conclude.]

Exercise 9.25 **(Compatibility with Kashiwara's equivalence)**

Let $\iota : X \hookrightarrow X_1$ be a closed inclusion of complex manifolds, and let $H_1 \subset X_1$ be a smooth hypersurface such that $H := X \cap H_1$ is a smooth hypersurface of X. Show that a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H if and only if $\widetilde{\mathcal{M}}_1 := {}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$ is so along H_1, and we have, for every α,

$$(\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}_1, \mathrm{N}) = ({}_{\mathrm{D}}\iota_*\mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}, \mathrm{N}).$$

[*Hint*: Assume that $X_1 = H \times \Delta_t \times \Delta_x$ and $X = H \times \Delta_t \times \{0\}$, so that $\widetilde{\mathcal{M}}_1 = \iota_*\widetilde{\mathcal{M}}[\widetilde{\partial}_x]$; show that the filtration $U_\alpha\widetilde{\mathcal{M}}_1 := \iota_*V_\alpha\widetilde{\mathcal{M}}[\widetilde{\partial}_x]$ satisfies all the characteristic properties of the V-filtration of $\widetilde{\mathcal{M}}_1$ along H_1.]

Exercise 9.26 **(Strict $\mathbb{R}$-specializability and morphisms).**

(1) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be an isomorphism between strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules. Show that it is strictly compatible with the V-filtrations and for any α, $\mathrm{gr}^V_\alpha \varphi$ is an isomorphism. [*Hint*: Use the uniqueness in Lemma 9.3.13.]

(2) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be any morphism between coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along H. Show that the order filtration on $\mathrm{Im}\,\varphi$ is a coherent V-filtration, and that $\mathrm{Im}\,\varphi$ is strictly $\mathbb{R}$-specializable if and only if so is $\mathrm{Ker}\,\varphi$. [*Hint*: Apply Exercise 9.22(2).]

(3) Let $\varphi : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{N}}$ be a morphism between strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules. It induces a morphism $\mathrm{gr}^V_\alpha \varphi : \mathrm{gr}^V_\alpha \widetilde{\mathcal{M}} \to \mathrm{gr}^V_\alpha \widetilde{\mathcal{N}}$. Show that if $\mathrm{gr}^V_\alpha \varphi$ is a strict morphism for every α, then $\mathrm{Coker}\,\varphi$ is also strictly $\mathbb{R}$-specializable and φ is strictly compatible with V, so that the sequence

$$0 \longrightarrow \mathrm{gr}^V_\alpha \mathrm{Ker}\,\varphi \longrightarrow \mathrm{gr}^V_\alpha \widetilde{\mathcal{M}} \longrightarrow \mathrm{gr}^V_\alpha \widetilde{\mathcal{N}} \longrightarrow \mathrm{gr}^V_\alpha \mathrm{Coker}\,\varphi \longrightarrow 0$$

is exact for every α.

Exercise 9.27 **(Restriction to $z = 1$).** Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. Assume that $\widetilde{\mathcal{M}}$ is $\mathbb{R}$-specializable along H.

(1) Show that for every α,

$$(z-1)\widetilde{\mathcal{M}} \cap V_\alpha \widetilde{\mathcal{M}} = (z-1) V_\alpha \widetilde{\mathcal{M}}.$$

[*Hint*: Let $m = (z-1)n$ be a local section of $(z-1)\widetilde{\mathcal{M}} \cap V_\alpha \widetilde{\mathcal{M}}$; then n is a local section of $V_\gamma \widetilde{\mathcal{M}}$ for some γ; if $\gamma > \alpha$, show that the class of n in $\mathrm{gr}^V_\gamma \widetilde{\mathcal{M}}$ is a annihilated by $z-1$; conclude with Exercise 5.2(1).]

(2) Conclude that $\mathcal{M} := \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is $\mathbb{R}$-specializable along H and that, for every α,

$$\begin{aligned} V_\alpha \mathcal{M} &= V_\alpha \widetilde{\mathcal{M}}/(z-1)V_\alpha \widetilde{\mathcal{M}} = V_\alpha \widetilde{\mathcal{M}}/\big((z-1)\widetilde{\mathcal{M}} \cap V_\alpha \widetilde{\mathcal{M}}\big), \\ \mathrm{gr}^V_\alpha \mathcal{M} &= \mathrm{gr}^V_\alpha \widetilde{\mathcal{M}}/(z-1)\mathrm{gr}^V_\alpha \widetilde{\mathcal{M}}. \end{aligned}$$

(3) Show that $(V_\alpha \widetilde{\mathcal{M}}) \otimes_{\widetilde{\mathbb{C}}[z]} \widetilde{\mathbb{C}}[z, z^{-1}] = V_\alpha \mathcal{M}[z, z^{-1}]$.

Exercise 9.28 **(Side changing).** Define the side changing functor for $V_0\widetilde{\mathcal{D}}_X$-modules by replacing $\widetilde{\mathcal{D}}_X$ with $V_0\widetilde{\mathcal{D}}_X$ in Definition 8.2.2. Show that $\widetilde{\mathcal{M}}^{\mathrm{left}}$ is $\mathbb{R}$-specializable along H if and only if $\widetilde{\mathcal{M}}^{\mathrm{right}}$ is so and, for every $\beta \in \mathbb{R}$, $V^\beta(\widetilde{\mathcal{M}}^{\mathrm{left}}) = \big[V_{-\beta-1}(\widetilde{\mathcal{M}}^{\mathrm{right}})\big]^{\mathrm{left}}$. [*Hint*: Use the local computation of Exercise 8.17.]

Exercise 9.29 **(Indexing with $\mathbb{Z}$ or with $\mathbb{R}$).** The order filtration is naturally indexed by $\mathbb{R}$, while the notion of V-filtration considers filtrations indexed by $\mathbb{Z}$. The purpose of this exercise is to show how both notions match when the properties of Lemma 9.3.13 are satisfied. Let $U_\bullet \widetilde{\mathcal{M}}$ be a filtration for which the properties of Lemma 9.3.13 are satisfied. Then we have seen that $U_\bullet \widetilde{\mathcal{M}}$ coincides with the "integral part" of the order filtration $V_\bullet \widetilde{\mathcal{M}}$. Show the following properties.

(1) The weak Bernstein equations (9.3.4 $*$) and (9.10.0 $*$) hold without any power of z, i.e., for every k the operator $\mathrm{E} - kz$ has a minimal polynomial on $U_k\widetilde{\mathcal{M}}/U_{k-1}\widetilde{\mathcal{M}} = V_k\widetilde{\mathcal{M}}/V_{k-1}\widetilde{\mathcal{M}}$ which does not depend on k.

(2) The eigen module of $\mathrm{E} - kz$ on this quotient module corresponding to the eigenvalue αz isomorphic to $\mathrm{gr}^V_{\alpha+k}\widetilde{\mathcal{M}}$ and the corresponding nilpotent endomorphism is

$$\mathrm{N} := (\mathrm{E} - (k+\alpha)z). \tag{9.10.0 $*$}$$

In particular, each $\mathrm{gr}^V_{\alpha+k}\widetilde{\mathcal{M}}$ is strict and we have a canonical identification

$$V_k\widetilde{\mathcal{M}}/V_{k-1}\widetilde{\mathcal{M}} = \bigoplus_{-1<\alpha\leqslant 0} \mathrm{gr}^V_{\alpha+k}\widetilde{\mathcal{M}}.$$

(3) For every $\alpha \in (-1, 0]$, identify $V_{\alpha+k}\widetilde{\mathcal{M}}$ with the pullback of

$$\bigoplus_{-1<\alpha'\leqslant\alpha} \mathrm{gr}^V_{\alpha'+k}\widetilde{\mathcal{M}}$$

by the projection $V_k\widetilde{\mathcal{M}} \to V_k\widetilde{\mathcal{M}}/V_{k-1}\widetilde{\mathcal{M}}$, and show that the shifted order filtration indexed by integers $V_{\alpha+\bullet}\widetilde{\mathcal{M}}$ is a coherent V-filtration.

(4) Conclude that there exists a finite set $A \subset (-1, 0]$ such that the order filtration is indexed by $A + \mathbb{Z}$, and is coherent as such (see Exercise 9.14).

Exercise 9.30. Check that if 9.3.14(2) and 9.3.14(3) hold for some local decomposition $X \simeq H \times \Delta_t$ at $x_o \in H$, then they hold for any such decomposition.

Exercise 9.31 **(A criterion to recognize the V-filtration).** Assume that $\widetilde{\mathcal{M}}$ is coherent and $\mathbb{R}$-specializable along H and let $U_\bullet\widetilde{\mathcal{M}}$ be a good V-filtration indexed by $A+\mathbb{Z}$ for some finite set $A \subset [0, 1)$. Assume that $U_\bullet\widetilde{\mathcal{M}}$ satisfies the following properties:

(1) $\mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}$ is strict for any $\alpha \leqslant 0$,
(2) same as 9.3.14(2),
(3) same as 9.3.14(3).

Argue as in the proof of Proposition 9.3.20 to deduce that $t : \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}} \to \mathrm{gr}^V_{\alpha-1}\widetilde{\mathcal{M}}$ is an isomorphism for any $\alpha < 0$ and, inductively, that $\widetilde{\eth}_t : \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}} \to \mathrm{gr}^V_{\alpha+1}\widetilde{\mathcal{M}}(-1)$ is an isomorphism for any $\alpha > -1$. Conclude that $\mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}$ is strict for any α, that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, and that $U_\bullet\widetilde{\mathcal{M}}$ is the V-filtration of $\widetilde{\mathcal{M}}$.

Exercise 9.32 **(Recovering morphisms from their restriction to V_0)**

Assume that $X = H \times \Delta_t$ and that $\widetilde{\mathcal{M}}_1, \widetilde{\mathcal{M}}_2$ are strictly $\mathbb{R}$-specializable along H. Let $\varphi_{\leqslant 0} : V_0\widetilde{\mathcal{M}}_1 \to V_0\widetilde{\mathcal{M}}_2$ be a morphism in $\mathsf{Mod}(V_0\widetilde{\mathcal{D}}_X)$ such that the diagram (9.3.22) commutes. Show that $\varphi_{\leqslant 0}$ extends in a unique way as a morphism $\varphi : \widetilde{\mathcal{M}}_1 \to \widetilde{\mathcal{M}}_2$. [*Hint*: For the uniqueness, use 9.3.20(b); show inductively the existence of $\varphi_{\leqslant k} : V_k\widetilde{\mathcal{M}}_1 \to V_k\widetilde{\mathcal{M}}_2$ ($k \geqslant 1$) such that the diagram

$$\begin{tikzcd} V_{k-1}\widetilde{\mathcal{M}}_1 \arrow[r, "\varphi_{\leqslant k}"] \arrow[d, "\widetilde{\eth}_t"'] & V_{k-1}\widetilde{\mathcal{M}}_2 \arrow[d, "\widetilde{\eth}_t"] \\ V_k\widetilde{\mathcal{M}}_1 \arrow[r, "\varphi_{\leqslant k}"] & V_k\widetilde{\mathcal{M}}_2 \end{tikzcd} \tag{D_k}$$

commutes; for example, if $k=1$, use 9.3.20(d) to show that, for $m,m',n,n'\in V_0\widetilde{\mathcal{M}}_1$, if $m-m'=(n'-n)\widetilde{\partial}_t$, then $n'-n\in V_{-1}\widetilde{\mathcal{M}}_2$ and deduce that setting $\varphi_{\leqslant 1}(m+n\widetilde{\partial}_t)=\varphi_{\leqslant 0}(m)+\varphi_{\leqslant 0}(n)\widetilde{\partial}_t$ well defines a $V_0\widetilde{\mathcal{D}}_X$-linear morphism $\varphi_{\leqslant 1}:V_1\widetilde{\mathcal{M}}_1\to V_1\widetilde{\mathcal{M}}_2$ for which (D_1) commutes.]

Exercise 9.33 **(Recovering $V_0\widetilde{\mathcal{M}}$).** Assume that $X=H\times\Delta_t$ and that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H. We have a natural exact sequence of $V_0\widetilde{\mathcal{D}}_X$-modules

$$0\longrightarrow V_{<0}\widetilde{\mathcal{M}}\longrightarrow V_0\widetilde{\mathcal{M}}\longrightarrow \mathrm{gr}_0^V\widetilde{\mathcal{M}}\longrightarrow 0.$$

We wish to recover explicitly the middle term in terms of the extreme ones and of the morphisms (c) and (d) in Proposition 9.3.20, for the most interesting value $\alpha=0$.

(1) Consider the morphisms

$$\begin{array}{ccccc}
\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1) & \xrightarrow{A} & V_{-1}\widetilde{\mathcal{M}}\oplus\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)\oplus\mathrm{gr}_0^V\widetilde{\mathcal{M}} & \xrightarrow{\quad B\quad} & \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}\\
e & \longmapsto & (0,e,e\widetilde{\partial}_t) & & \\
 & & (m,e,\varepsilon) & \longmapsto & [m]+e\cdot\widetilde{\partial}_t t-\varepsilon\cdot t
\end{array}$$

where, for $m\in V_{-1}\widetilde{\mathcal{M}}$, $[m]$ denotes its class in $\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}$. Show that the composition is zero, hence they define a complex $C^\bullet$ of $V_0\widetilde{\mathcal{D}}_X$-modules (by regarding each $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}$ as a $V_0\widetilde{\mathcal{D}}_X$-module). Show that $H^j(C^\bullet)=0$ for $j\neq 1$.

(2) Consider the morphism from $V_0\widetilde{\mathcal{M}}$ to the middle term given by $\mu\mapsto(\mu\cdot t,0,[\mu])$, where $[\mu]$ denotes the class of μ in $\mathrm{gr}_0^V\widetilde{\mathcal{M}}$. Show that it injects into $\operatorname{Ker}B$ and that its intersection with $\operatorname{Im}A$ is zero. [*Hint*: Use 9.3.20(a).]

(3) Show that the induced morphism $V_0\widetilde{\mathcal{M}}\to H^1(C^\bullet)$ is an isomorphism. [*Hint*: Injectivity follows from (2) above; modulo $\operatorname{Im}A$, any element of $\operatorname{Ker}B$ can be represented in a unique way as $(m,0,\delta)$ with $[m]=\delta\cdot t$; choose any lifting $\widetilde{\delta}\in V_0\widetilde{\mathcal{M}}$ of δ and show that there exists $\eta\in V_{<0}\widetilde{\mathcal{M}}$ such that $m-\widetilde{\delta}\cdot t=\eta\cdot t$ by using 9.3.20(a); conclude by setting $\mu=\widetilde{\delta}+\eta$.]

(4) Show that, for any $V_0\widetilde{\mathcal{D}}_X$-linear morphism $\varphi_{\leqslant -1}:V_{-1}\widetilde{\mathcal{M}}_1\to V_{-1}\widetilde{\mathcal{M}}_2$, the diagram (D_{-1}) commutes, and conclude that giving a morphism $\varphi_{\leqslant 0}V_0\widetilde{\mathcal{M}}_1\to V_0\widetilde{\mathcal{M}}_2$ such that (D_0) commutes is equivalent to giving a pair $(\varphi_{\leqslant -1},\varphi_0)$ such that, with respect to the morphisms

$$\begin{tikzcd}
\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}} \arrow[r, bend left, "\cdot\widetilde{\partial}_t", "(1)"'] & \mathrm{gr}_0^V\widetilde{\mathcal{M}} \arrow[l, bend left, "\cdot t"]
\end{tikzcd}$$

and setting $\varphi_{-1}=\mathrm{gr}_{-1}^V\varphi_{\leqslant -1}$, we have

$$\widetilde{\partial}_t\circ\varphi_{-1}=\varphi_0\circ\widetilde{\partial}_t,\quad \varphi_{-1}\circ t=t\circ\varphi_0.$$

Exercise 9.34 **(Structure of $\widetilde{\mathcal{M}}/V_{<\alpha_o}\widetilde{\mathcal{M}}$).** Let $\widetilde{\mathcal{M}}$ be a coherent right $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H. Let us fix $\alpha_o\in\mathbb{R}$. Then $\widetilde{\mathcal{M}}/V_{<\alpha_o}\widetilde{\mathcal{M}}$ is a $V_0\widetilde{\mathcal{D}}_X$-module.

(1) Show that $\widetilde{\mathcal{M}}/V_{<\alpha_o}\widetilde{\mathcal{M}}$ is strict.

(2) Show that $\widetilde{\mathcal{M}}/V_{<\alpha_o}\widetilde{\mathcal{M}}$ decomposes as $\bigoplus_{\alpha\geqslant\alpha_o}\operatorname{Ker}(\mathrm{E}-\alpha z)^N$ with $N\gg 0$.

(3) Show that the α-summand can be identified with $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}$.

(4) Show that $\widetilde{\mathcal{M}}/V_{<\alpha_o}\widetilde{\mathcal{M}}$ can be identified with $\bigoplus_{\alpha\geqslant\alpha_o}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}$ as a $V_0\widetilde{\mathcal{D}}_X$-module. Does the $V_0\widetilde{\mathcal{D}}_X$-module structure of $\widetilde{\mathcal{M}}/V_{<\alpha_o}\widetilde{\mathcal{M}}$ extend to a $\widetilde{\mathcal{D}}_X$-module structure? [*Hint*: In local coordinates, what about the relation $[\widetilde{\partial}_t,t]=z$ applied to a nonzero section of $\operatorname{gr}^V_{\alpha_o}\widetilde{\mathcal{M}}$?]

(5) Assume now that $X\simeq H\times\Delta_t$. Let s be a new variable and let us equip $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s]:=\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}\otimes_{\mathbb{C}}\mathbb{C}[s]$ with the following right $V_0\widetilde{\mathcal{D}}_X$-structure defined by

$$m_\alpha^{(j)}s^j\cdot t=\begin{cases}0 & \text{if } j=0,\\ \big(m_\alpha^{(j)}(\mathrm{E}+jz)\big)s^{j-1} & \text{if } j\geqslant 1,\end{cases}$$
$$(m_\alpha^{(j)}s^j)t\widetilde{\partial}_t=\big(m_\alpha^{(j)}(\mathrm{E}+jz)\big)s^j.$$

Check that this is indeed a $V_0\widetilde{\mathcal{D}}_X$-module structure (i.e., $[t\widetilde{\partial}_t,t]$ acts as zt). Show that $\widetilde{\mathcal{M}}/V_{-1}\widetilde{\mathcal{M}}$ can be identified with $\bigoplus_{\alpha\in(-1,0]}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s]$. With this structure, show that $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}s^j=\operatorname{Ker}(t\widetilde{\partial}_t-(\alpha+j)z)^N$ (with $N\gg 0$ locally).
[*Hint*: Use that $\widetilde{\partial}_t:\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}\to\operatorname{gr}^V_{\alpha+1}\widetilde{\mathcal{M}}$ is an isomorphism for $\alpha>-1$ to identify $\bigoplus_{\alpha>-1}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}$ with $\bigoplus_{\alpha\in(-1,0]}\bigoplus_{j\geqslant 0}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}\widetilde{\partial}_t^j$.]

(6) Equip $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s]$ with the action of $\widetilde{\partial}_t$ defined by $(m_\alpha^{(j)}s^j)\widetilde{\partial}_t=m_\alpha^{(j)}s^{j+1}$. Show that the relation $[\widetilde{\partial}_t,t]=z$ holds on $s\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s]$, but that $[\widetilde{\partial}_t,t]=z+(\mathrm{E}+z)$ on $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}$. Conclude that this action does not define a $\widetilde{\mathcal{D}}_X$-module structure on $\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s]$.

(7) Show that $\widetilde{\mathcal{M}}/V_{-2}\widetilde{\mathcal{M}}$ can be identified, as a $V_0\widetilde{\mathcal{D}}_X$-module, to

$$\bigoplus_{\beta\in(-2,-1]}\operatorname{gr}^V_\beta\widetilde{\mathcal{M}}\oplus\bigoplus_{\alpha\in(-1,0]}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s],$$

where the $V_0\widetilde{\mathcal{D}}_X$-module structure on the latter term is a little modified with respect to that of $\bigoplus_{\alpha\in(-1,0]}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}[s]$, namely:

- t acts by zero on $\bigoplus_{\beta\in(-2,-1]}\operatorname{gr}^V_\beta\widetilde{\mathcal{M}}$,
- for $\alpha\in(-1,0]$ and $j=0$, $m_0^\alpha\cdot t=m_0^\alpha t\in\operatorname{gr}^V_{\alpha-1}\widetilde{\mathcal{M}}$ (instead of 0),
- all the remaining actions are the same as in (5).

Exercise 9.35. Justify that $\psi_{g,\lambda}$ and $\phi_{g,1}$ are functors from the category of $\mathbb{R}$-specializable right $\widetilde{\mathcal{D}}_X$-modules to the category of right $\widetilde{\mathcal{D}}_X$-modules supported on $g^{-1}(0)$. [*Hint*: Use Exercise 9.18(2).]

Exercise 9.36. Let $\widetilde{\mathcal{M}}$ be a right $\widetilde{\mathcal{D}}_X$-module. When g is smooth and $g^{-1}(0)=H$, show that we have $\psi_{g,\lambda}\widetilde{\mathcal{M}}\simeq{}_{\mathrm{D}}\iota_{H*}\operatorname{gr}^V_\alpha\widetilde{\mathcal{M}}(1)$ and $\phi_{g,1}\widetilde{\mathcal{M}}={}_{\mathrm{D}}\iota_{H*}\operatorname{gr}^V_0\widetilde{\mathcal{M}}$, where $i_H:H\hookrightarrow X$ denotes the inclusion.

Exercise 9.37. Similarly to Exercise 9.36, show that, if $X=H\times\Delta_t$ and g is the projection to Δ_t, so that ι_g is induced by the diagonal embedding $\Delta_t\hookrightarrow\Delta_{t_1}\times\Delta_{t_2}$, then $\mathrm{can}=\widetilde{\partial}_{t_2}$ and $\mathrm{var}=t_2$ for ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ are ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\partial}_{t_1})$ and ${}_{\mathrm{D}}\iota_{g*}(t_1)$, with $\widetilde{\partial}_{t_1}=\widetilde{\partial}_t:\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}\to\operatorname{gr}^V_0\widetilde{\mathcal{M}}(-1)$ and $t_1=t:\operatorname{gr}^V_0\widetilde{\mathcal{M}}\to\operatorname{gr}^V_{-1}\widetilde{\mathcal{M}}$.

Exercise 9.38 **(Strict $\mathbb{R}$-specializability and ramification).** We take up the notation of Section 9.9.a. Let $q \geqslant 1$ be an integer and let $\rho_q : \mathbb{C} \to \mathbb{C}$ be the ramification $u \mapsto t = u^q$. We set $X_q = X_0 \times \mathbb{C}_u$ and we still denote by ρ_q the induced map $X_q \to X$. Since we will deal with pullbacks of $\widetilde{\mathcal{D}}_X$-modules, we will work in the *left* setting. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module.

(1) Show that the pullback ${}_{\mathrm{D}}\rho^*\widetilde{\mathcal{M}}$ (Definitions 8.6.3 and 8.6.6) can also be defined as follows:

- as an $\widetilde{\mathcal{O}}_{X_q}$-module, we set ${}_{\mathrm{D}}\rho_q^*\widetilde{\mathcal{M}} = \rho_q^*\widetilde{\mathcal{M}} = \widetilde{\mathcal{O}}_{X_q} \otimes_{\rho_q^{-1}\widetilde{\mathcal{O}}_X} \rho_q^{-1}\widetilde{\mathcal{M}}$;
- for coordinates x_i on X_0, the action of $\widetilde{\partial}_{x_i}$ is the natural one, i.e., $\widetilde{\partial}_{x_i}(1 \otimes m) = 1 \otimes \widetilde{\partial}_{x_i} m$;
- the action of $\widetilde{\partial}_u$ is defined, by a natural extension using Leibniz rule, from

$$\widetilde{\partial}_u(1 \otimes m) = qu^{q-1} \otimes \widetilde{\partial}_t m.$$

(2) Identify ${}_{\mathrm{D}}\rho_q^*\widetilde{\mathcal{M}}$ with $\bigoplus_{k=0}^{q-1} u^k \otimes \widetilde{\mathcal{M}}$ and make precise the $\widetilde{\mathcal{D}}_{X_q}$-module structure on the right-hand term.

(3) Assume that $\widetilde{\mathcal{M}}$ is $\mathbb{R}$-specializable along (t). Show that any local section of ${}_{\mathrm{D}}\rho_q^*\widetilde{\mathcal{M}}$ satisfies a weak Bernstein functional equation, by using that

$$u^k \otimes t\widetilde{\partial}_t m = \frac{1}{q}(u\widetilde{\partial}_u - kz)(u^k \otimes m).$$

(4) Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (t). Show that the filtration defined by the formula

$$V^\beta {}_{\mathrm{D}}\rho_q^*\widetilde{\mathcal{M}} = \bigoplus_{k=0}^{q-1} (u^k \otimes V^{(\beta-k)/q}\widetilde{\mathcal{M}}),$$

satisfies all properties required for the Kashiwara-Malgrange filtration.

(5) Show that, for any $\mu \in \mathbb{S}^1$,

$$\psi_{u,\mu}({}_{\mathrm{D}}\rho_q^*\widetilde{\mathcal{M}}) \simeq \bigoplus_{\lambda^q=\mu} \psi_{t,\lambda}\widetilde{\mathcal{M}},$$

and, under this identification, the nilpotent endomorphism N_u corresponds to the direct sum of the nilpotent endomorphisms $q\mathrm{N}_t$. Conclude that we have a similar relation for the graded modules with respect to the monodromy filtration and the corresponding primitive submodules.

Exercise 9.39. Show that both conditions in Definition 9.5.4 are indeed equivalent. [*Hint*: Use the homogeneity property of $\operatorname{Char}\widetilde{\mathcal{M}}$.]

Exercise 9.40. With the assumptions of Theorem 9.5.6, show similarly that, if Y is defined by $x_1 = \cdots = x_p = 0$ then, considering the map $\boldsymbol{x} : X \to \mathbb{C}^p$ induced by $\boldsymbol{x} := (x_1, \dots, x_p)$, then $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_{X/\mathbb{C}^p}$-coherent.

Exercise 9.41 **(Middle extension property for holonomic $\mathcal{D}_X$-modules)**

(1) Show that an $\mathbb{R}$-specializable $\mathcal{D}_X$-module $\mathcal{M}$, the property of being a middle extension along (g) (i.e., can is onto and var is injective) is equivalent to the property that $\mathcal{M}$ has no submodule not quotient module supported in $\{g = 0\}$. [*Hint*: Notice that Property 9.3.14(1) is empty in Proposition 9.7.2(2).]

(2) Show that, if $\mathcal{M}$ is holonomic, this property is equivalent to the property that both $\mathcal{M}$ and its dual $\mathcal{D}_X$-module have no submodules supported in $\{g = 0\}$.

(3) Show that if $\mathcal{M}$ is smooth (i.e., is a vector bundle with flat connection), then it is a middle extension along any divisor (g). [*Hint*: Use that the dual module is also smooth.]

Exercise 9.42 **(Nearby/vanishing Lefschetz quiver for a middle extension)**

Show that the nearby/vanishing Lefschetz quivers (9.4.7 $**$) and (9.4.7 $*$) are isomorphic to the quiver

$$\begin{tikzcd} \psi_{g,1}\widetilde{\mathcal{M}} \arrow[rr, bend left, "\mathrm{can} = \mathrm{N}"] & & \mathrm{Im}\,\mathrm{N}. \arrow[ll, bend left, "\mathrm{var} = \mathrm{incl}", "(-1)"'] \end{tikzcd}$$

Exercise 9.43. In the setting of Lemma 9.9.2, prove that (x_1, x_2) is a regular sequence on $\widetilde{\mathcal{M}}$, i.e., $x_1\widetilde{\mathcal{M}} \cap x_2\widetilde{\mathcal{M}} = x_1x_2\widetilde{\mathcal{M}}$. Show that, for every $k \geqslant 1$, if we have a relation $\sum_{k_1+k_2=k} x_2^{k_1}x_1^{k_2}m_{k_1,k_2} = 0$ in $\widetilde{\mathcal{M}}$, then there exist $\mu_{i,j} \in \widetilde{\mathcal{M}}$ for $i, j \geqslant 0$ (and the convention that $\mu_{i,j} = 0$ if i or $j \leqslant -1$) such that $m_{k_1,k_2} = x_1\mu_{k_1-1,k_2} - x_2\mu_{k_1,k_2-1}$ for every k_1, k_2.

9.11. Comments

The idea of computing the monodromy of a differential equation with regular singularities only in terms of the coefficients of the differential equation itself, that is, in an algebraic way with respect to the differential equation, goes back to the work of Fuchs. In higher dimension, this has been extended in terms of vector bundles and connections by Deligne [**Del70**]. On the other hand, the algebraic computation of the monodromy by Brieskorn [**Bri70**] opened the way to the differential treatment of the monodromy as done by Malgrange in [**Mal74**], and generalized in [**Mal83**]. The general definition of the V-filtration has been obtained by Kashiwara [**Kas83**]. It has been developed for the purpose of the theory of Hodge modules by M. Saito [**Sai88**], and an account has been given in [**Sab87**]. The theory of the V-filtration is intimately related to that of the Bernstein-Sato polynomial [**Ber68, BG69, Ber72**] and [**Kas76, Kas78**].

For the purpose of the theory of Hodge modules, M. Saito has developed the notion of V-filtration for filtered $\mathcal{D}_X$-modules. His approach will be explained in Chapter 10. In the present chapter, we have followed the adaptation of M. Saito's approach for $\widetilde{\mathcal{D}}_X$-modules, inspired by [**Sab05**]. For example, the proof of the pushforward theorem 9.8.8 is a direct adaptation of loc. cit., which in turn is an adaptation of a similar result of M. Saito in [**Sai88**], namely, Theorem 10.6.4. The computation of Section 9.9.b followed the same path.

CHAPTER 10

SPECIALIZATION OF FILTERED $\mathcal{D}$-MODULES

Summary. In this chapter, we take up the notion of specialization and the compatibility property with proper pushforward for filtered $\mathcal{D}_X$-modules. Compared with the approach of Sections 9.3–9.8, we insist in keeping the strictness property, that is, we only work with filtered $\mathcal{D}_X$-modules, not graded modules over the Rees ring $R_F\mathcal{D}_X$. We will compare the two approaches in Section 10.8.

10.1. Introduction

One can introduce the notion of filtered $\mathcal{D}$-module by keeping the data of the $\mathcal{D}$-module and its filtration. The advantage is to keep a hand on the filtration at each step. The main goal of this chapter (Theorem 10.6.4) is to give a proof of the criterion given in Theorem 9.8.8 from this point of view. One should be careful since the category of filtered $\mathcal{D}$-modules is not abelian anymore. As a consequence, dealing with derived categories, as needed when considering pushforward, needs some care, as well as strictness for bi-filtered complexes.

On the way, we will introduce the notion of compatible filtrations, which will be important in Chapter 13. The comparison between the present approach and that of Chapter 9 will be done in Section 10.8. Of particular interest is the property that, for a strict graded $R_F\mathcal{D}_X$-module, strict $\mathbb{R}$-specializability along a smooth divisor H implies a regularity property, which has not been emphasized up to now, but which is essential for the approach in this chapter. In particular, the approach of Section 9.8 does not give as a result the strictness of the pushforward, only its strict specializability. We will show in Section 10.8 how to recover strictness properties from this point of view. On the other hand, the advantage of the approach of Section 9.8 is to allow generalization to cases where the regularity property is not fulfilled (twistor $\mathcal{D}$-modules), since strictness is not used for proving Theorem 9.8.8, only strict specializability is used. Last, localization and maximalization also have a natural formalism in the framework of graded $R_F\mathcal{D}_X$-module. We will not take up the corresponding formalism in the setting of filtered $\mathcal{D}_X$-modules.

10.2. Strict and bi-strict complexes

In this section we review the definition and basic properties of strictness for filtered and bi-filtered complexes. We will consider the case of several filtrations in Section 10.3. In particular, when dealing with at least three filtrations, an important role is played by the compatibility condition on filtrations. However, this condition does not arise when dealing with one or two filtrations and the strictness condition on complexes is also very easy to treat directly.

10.2.1. Convention. We work in an abelian category A in which all filtered direct limits exist and are exact. Given an object A in this category, we only consider increasing filtrations $F_\bullet A$ that are indexed by $\mathbb{Z}$ and satisfy $\varinjlim_k F_k A = A$. We write a filtered object in A as (A, F), where $F = (F_k A)_{k\in\mathbb{Z}}$.

Note that if (A, F) is a filtered object, then a subobject B of A carries the induced filtration $(F_k A \cap B)_{k\in\mathbb{Z}}$, while a quotient object A/A' carries the induced filtration $((F_k A + A')/A')_{k\in\mathbb{Z}}$. It is easy to see that the two possible induced filtrations on a subquotient B/A' of A agree.

***10.2.2. Definition* (Strictness of filtered complexes).** Consider a complex $(C^\bullet, F)$ of filtered objects in A. This is a *strict complex* if all morphisms $d\colon C^i \to C^{i+1}$ are strict, in the sense that the isomorphism $\mathrm{Coim}(d) \to \mathrm{Im}(d)$ is an isomorphism of filtered objects, that is, we have

$$d(F_k C^i) = F_k C^{i+1} \cap d(C^i) \quad \text{for all } k, i \in \mathbb{Z}.$$

We will be interested in complexes of bi-filtered objects in A. These are objects of A carrying two filtrations (A, F', F''). We write

$$F'_k F''_\ell A := F'_k A \cap F''_\ell A. \tag{10.2.3}$$

The morphisms in this case are required to be compatible with each of the two filtrations.

10.2.4. Definition. Let $(C^\bullet, F', F'')$ be a complex of bi-filtered objects. We say that the complex is *strict* (or *bi-strict*, if we want to emphasize the fact that we consider two filtrations) if for every i, p, and q, the natural maps in the commutative square

$$\begin{tikzcd} H^i(F'_k F''_\ell C^\bullet) \arrow[r] \arrow[d] & H^i(F'_k C^\bullet) \arrow[d] \\ H^i(F''_\ell C^\bullet) \arrow[r] & H^i(C^\bullet) \end{tikzcd}$$

are injective, and furthermore, the square is Cartesian, that is, $H^i(F'_k F''_\ell C^\bullet) = H^i(F'_k C^\bullet) \cap H^i(F''_\ell C^\bullet)$.

10.2.5. Remark. It follows from Remark 10.2 that $(C^\bullet, F', F'')$ is strict if and only if all canonical morphisms

$$H^i(F'_kC^\bullet) \longrightarrow F'_kH^i(C^\bullet), \quad H^i(F''_\ell C^\bullet) \longrightarrow F''_\ell H^i(C_\bullet),$$

and

$$H^i(F'_kF''_\ell C^\bullet) \longrightarrow F'_kF''_\ell H^i(C^\bullet)$$

are isomorphisms. (See Exercise 10.6 for the case of a bi-strict morphism.)

10.2.6. Lemma. *If $(C^\bullet, F', F'')$ is a strict complex of bi-filtered objects, then the complexes $(C^\bullet, F')$ and $(F'_kC^\bullet, F'')$ are strict for every $k \in \mathbb{Z}$. In particular, we have a short exact sequence*

$$0 \longrightarrow H^i(F'_kF''_\ell C^\bullet) \longrightarrow H^i(F'_kF''_mC^\bullet) \longrightarrow H^i\left(F'_k(F''_mC^\bullet/F''_\ell C^\bullet)\right) \longrightarrow 0$$

for every $\ell < m$ and every i. Furthermore, for every k, every $\ell < m < n$, and every i, we have short exact sequences

$$0 \to H^i\Big(F'_k(F''_mC^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(C^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(C^\bullet/F''_mC^\bullet)\Big) \to 0$$

and

$$0 \to H^i\Big(F'_k(F''_mC^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(F''_nC^\bullet/F''_\ell C^\bullet)\Big) \longrightarrow H^i\Big(F'_k(F''_nC^\bullet/F''_mC^\bullet)\Big) \to 0.$$

Proof. The first assertion is an immediate consequence of the definition, while the exact sequences follow from the strictness of $(F'_kC^\bullet, F'')$, using Remarks 10.3 and 10.4. □

10.2.7. Lemma. *If $(C^\bullet, F', F'')$ is a strict complex of bi-filtered objects, then for every $k < q$, the complex $(F''_kC^\bullet/F''_\ell C^\bullet, F')$ is strict. In particular, each complex $(\mathrm{gr}_k^{F''}(C^\bullet), F')$ is strict.*

Proof. It follows from Lemma 10.2.6 that for every s and i, in the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & H^i(F'_mF''_kC^\bullet) & \longrightarrow & H^i(F'_mF''_\ell C^\bullet) & \longrightarrow & H^i\Big(F'_m(F''_\ell C^\bullet/F''_kC^\bullet)\Big) & \longrightarrow & 0 \\
 & & \big\downarrow{\scriptstyle u} & & \big\downarrow{\scriptstyle v} & & \big\downarrow{\scriptstyle w} & & \\
0 & \longrightarrow & H^i(F''_kC^\bullet) & \longrightarrow & H^i(F''_\ell C^\bullet) & \longrightarrow & H^i\left(F''_\ell C^\bullet/F''_kC^\bullet\right) & \longrightarrow & 0
\end{array}$$

the rows are exact. Furthermore, since $(C^\bullet, F', F'')$ is a strict complex, it follows that u and v are injective and the left square is Cartesian (this follows by describing all the objects that appear in that square as subobjects of $H^i(C^\bullet)$). This implies that w is injective, hence $(F''_\ell C^\bullet/F''_kC^\bullet, F')$ is a strict complex. □

10.3. Compatible filtrations and strictness

We keep Convention 10.2.1.

10.3.a. Compatible filtrations. Suppose that A is an object of our category A, and $A_1, \dots, A_n \subseteq A$ are finitely many subobjects. When $n = 1$, we have a short exact sequence of the form

$$A_1 \longrightarrow A \longrightarrow *$$

where $*$ is of course just an abbreviation for the quotient A/A_1. When $n = 2$, we similarly have a commutative diagram of the form

$$\begin{tikzcd}
* \arrow[r] & * \arrow[r] & * \\
A_2 \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & * \arrow[u] \\
* \arrow[r] \arrow[u] & A_1 \arrow[r] \arrow[u] & * \arrow[u]
\end{tikzcd}$$

in which all rows and all columns are short exact sequences. (For example, the entry in the upper-right corner is $A/(A_1 + A_2)$, the entry in the lower-left corner $A_1 \cap A_2$.) Once $n \geqslant 3$, such a diagram no longer exists in general; if it does exist, one says that $A_1, \dots, A_n$ are *compatible subobjects* of A. More precisely, the condition is the following: there should exist an n-dimensional commutative diagram $C(A_1, \dots, A_n; A)$, consisting of 3^n objects placed at the points $\{-1, 0, 1\}^n$ and $2n \cdot 3^{n-1}$ morphisms corresponding to the line segments connecting those points, such that A sits at the point $(0, \dots, 0)$, each A_i sits at the point $(0, \dots, -1, \dots, 0)$ on the i-th coordinate axis, and all lines parallel to the coordinate axes form short exact sequences in the abelian category. It is easy to see that the objects at points in $\{-1, 0\}^n$ are just intersections: if the i-th coordinate of such a point is -1 for $i \in I \subset \{1, \dots, n\}$ and 0 for $i \notin I$, then the exactness of the diagram forces the corresponding object to be

$$\bigcap_{i \in I} A_i,$$

with the convention that the intersection equals A when I is empty. In particular, the object $A_1 \cap \cdots \cap A_n$ always sits at the point with coordinates $(-1, \dots, -1)$.

On the other hand, given a subset $I \subset \{1, \dots, n\}$, fixing the coordinate $\varepsilon_i^o \in \{-1, 0, 1\}$ for every $i \in I$ produces a sub-diagram of size $n - \#I$, hence $n - \#I$ compatible sub-objects of the term placed at $(\varepsilon^o_{i\in I}, 0_{i\notin I})$, that we denote by $A(\varepsilon^o_{i\in I}, 0_{i\notin I})$. For example, fixing $\varepsilon_i^o = 0$ shows that the sub-family $(A_i)_{i\notin I}$ is a compatible family.

As another example, fix $\varepsilon_n^o = -1$. Then the induced family $(A_i \cap A_n)_{i\in\{0,\dots,n-1\}}$ of sub-objects of A_n is also compatible.

As still another example, let us fix $\varepsilon_n^o = 1$. We have an exact sequence

$$A_n = A(0, \dots, 0, -1) \longrightarrow A = A(0, \dots, 0) \longrightarrow A/A_n = A(0, \dots, 0, 1).$$

Our new diagram has central term A/A_n and the term placed at $(0, \dots, (-1)_i, \dots, 0, 1)$ is $A_i/A_i \cap A_n$. This means that the induced family $(A_i/A_i \cap A_n)_{i\in\{0,\dots,n-1\}}$ is also compatible.

In the definition of compatibility, the object A does not play a relevant role and one can replace it by a sub-object provided that all A_i are contained in it. Similarly one can replace it by a sup-object. This is shown in Exercise 10.8.

10.3.1. Lemma. *Let $A_1, \dots, A_n \subset A$ be a family of sub-objects of A. Assume the following properties:*

(1) *$A_1 \subset A_2$.*
(2) *Both sub-families $A_1, A_3, \dots, A_n$ and $A_2, A_3, \dots, A_n$ are compatible.*

Then the family $A_1, \dots, A_n$ is compatible. Moreover, the family $(A_i \cap A_2)/(A_i \cap A_1)$ $(i = 3, \dots, n)$ of sub-objects of A_2/A_1 is also compatible.

Proof. We wish to define a diagram with vertices $A(\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ ($\varepsilon_i \in \{-1, 0, 1\}$) satisfying the properties above. The second assumption means that we have the diagrams with vertices $A(\varepsilon_1, 0, \varepsilon_3, \dots, \varepsilon_n)$ and $A(0, \varepsilon_2, \dots, \varepsilon_n)$. On the other hand, if the diagram we search for exists, the inclusion $A_1 \cap A_2 = A_1 \subset A_2$ is satisfied for all terms of the diagram, namely

$$A(-1,-1,\varepsilon_{\geqslant 3}) = A(-1,0,\varepsilon_{\geqslant 3}) \subset A(0,-1,\varepsilon_{\geqslant 3}). \tag{10.3.2}$$

We are thus forced to set

$$\begin{aligned} A(1,-1,\varepsilon_{\geqslant 3}) &:= A(0,-1,\varepsilon_{\geqslant 3})/A(-1,-1,\varepsilon_{\geqslant 3}) \\ A(1,1,\varepsilon_{\geqslant 3}) &:= A(0,1,\varepsilon_{\geqslant 3}). \end{aligned} \tag{10.3.3}$$

In such a way, we obtain a commutative diagram where the columns are exact sequences (by assumption for the middle one, by our setting for the left and right ones), as well as the middle horizontal line

$$\begin{tikzcd} A(1,-1,\varepsilon_{\geqslant 3}) \arrow[r] & A(1,0,\varepsilon_{\geqslant 3}) \arrow[r] & A(1,1,\varepsilon_{\geqslant 3}) \\ A(0,-1,\varepsilon_{\geqslant 3}) \arrow[u, two heads] \arrow[r, hook] & A(0,0,\varepsilon_{\geqslant 3}) \arrow[u, two heads] \arrow[r, two heads] & A(0,1,\varepsilon_{\geqslant 3}) \arrow[u, equal] \\ A(-1,-1,\varepsilon_{\geqslant 3}) \arrow[u, hook] \arrow[r, equal] & A(-1,0,\varepsilon_{\geqslant 3}) \arrow[u, hook] \arrow[r] & A(-1,1,\varepsilon_{\geqslant 3}) = 0 \arrow[u] \end{tikzcd} \tag{10.3.4}$$

It is then easy to check that the upper horizontal line is exact. This shows that, in the diagram of size n, the lines where ε_1 varies in $\{-1, 0, 1\}$ and all other ε_i fixed, as well as the lines where ε_2 varies and all other ε_i are fixed, are exact. Let us now vary ε_3, say, by fixing all other ε_i and let us omit ε_i for $i \geqslant 4$ in the notation. From the diagram above, we see that the only possibly non-obvious exact sequence has terms $A(1,-1,\varepsilon_3)_{\varepsilon_3=-1,0,1}$. We now consider the commutative diagram where the columns are exact and only the upper horizontal line is possibly non-exact:

$$\begin{tikzcd} A(1,-1,-1) \arrow[r] & A(1,-1,0) \arrow[r] & A(1,-1,1) \\ A(0,-1,-1) \arrow[u, two heads] \arrow[r, hook] & A(0,-1,0) \arrow[u, two heads] \arrow[r, two heads] & A(0,-1,1) \arrow[u, two heads] \\ A(-1,-1,-1) \arrow[u, hook] \arrow[r, hook] & A(-1,-1,0) \arrow[u, hook] \arrow[r, two heads] & A(-1,-1,1) \arrow[u, hook] \end{tikzcd} \tag{10.3.5}$$

But the snake lemma shows its exactness. We conclude that the family $A_1, A_2, \dots, A_n$ is compatible. We now remark that

$$A_2/A_1 = A_2/(A_1 \cap A_2) = A(1, -1, 0, \dots, 0).$$

The compatibility of the family $(A_i \cap A_2/A_i \cap A_1)_{i=3,\dots,n}$ will be proved if we prove $(A_3 \cap A_2)/(A_3 \cap A_1) = A(1, -1, -1, 0, \dots, 0)$, and similarly for $i \geqslant 4$. Let us consider the previous diagram when fixing $\varepsilon_i = 0$ for $i \geqslant 4$. The left vertical inclusion reads $A_1 \cap A_2 \cap A_3 \hookrightarrow A_2 \cap A_3$, hence the desired equality. $\square$

The previous lemma can be taken the other way round, which can be used for an inductive construction of compatible families.

10.3.6. Lemma. *Let $A_1, \dots, A_n \subset A$ be a family of sub-objects of A. Assume the following properties:*

(1) *$A_1 \subset A_2$.*

(2) *Both families $A_1, A_3, \dots, A_n$ in A and $A_2/A_1, A_3/(A_3 \cap A_1), \dots, A_n/(A_n \cap A_1)$ in A/A_1 are compatible.*

Then the family $A_1, A_2, \dots, A_n$ is compatible in A.

Proof. We argue as in Lemma 10.3.1, from which we keep the notation. We have the diagrams of exact sequences with $A(\varepsilon_1, 0, \varepsilon_{\geqslant 3})$ and $A(1, \varepsilon_2, \varepsilon_{\geqslant 3})$. We must also have (10.3.2). It remains to determine $A(0, \varepsilon_2, \varepsilon_{\geqslant 3})$ and check that the sequences, when ε_i varies in $\{-1, 0, 1\}$ and all other ε_j fixed, are exact. We know $A(0, 0, \varepsilon_{\geqslant 3})$, and as for (10.3.3), we must set

$$A(0, 1, \varepsilon_{\geqslant 3}) := A(1, 1, \varepsilon_{\geqslant 3}),$$

the latter term being known. We then search for a diagram similar to (10.3.4):

$$\begin{tikzcd}
A(1,-1,\varepsilon_{\geqslant 3}) \arrow[r, hook] & A(1,0,\varepsilon_{\geqslant 3}) \arrow[r, two heads] & A(1,1,\varepsilon_{\geqslant 3}) \\
A(0,-1,\varepsilon_{\geqslant 3})? \arrow[u] \arrow[r, "a"] & A(0,0,\varepsilon_{\geqslant 3}) \arrow[u, two heads] \arrow[r, "b"] & A(0,1,\varepsilon_{\geqslant 3}) \arrow[u, equal] \\
A(-1,-1,\varepsilon_{\geqslant 3}) \arrow[u] \arrow[r, equal] & A(-1,0,\varepsilon_{\geqslant 3}) \arrow[u, hook] \arrow[r] & A(-1,1,\varepsilon_{\geqslant 3}) = 0 \arrow[u]
\end{tikzcd} \tag{10.3.7}$$

where $A(0, -1, \varepsilon_{\geqslant 3})$ has to be chosen so that the left column is exact (the other ones being so), and we must then show that the middle line is exact (the other ones being so). Clearly b is onto, so we are forced to set $A(0, -1, \varepsilon_{\geqslant 3}) = \operatorname{Ker} b$. The exactness of the left column is then left as an exercise. We now vary ε_3, say, and consider the corresponding sequences. The only possibly non-exact ones have $\varepsilon_1 = 0$

and we are left with examining the diagram

$$\begin{tikzcd} A(0,1,-1) \arrow[r, hook] & A(0,1,0) \arrow[r, two heads] & A(0,1,1) \\ A(0,0,-1) \arrow[r, hook] \arrow[u, two heads] & A(0,0,0) \arrow[r, two heads] \arrow[u, two heads] & A(0,0,1) \arrow[u, two heads] \\ A(0,-1,-1) \arrow[r] \arrow[u, hook] & A(0,-1,0) \arrow[r] \arrow[u, hook] & A(0,-1,1) \arrow[u, hook] \end{tikzcd} \tag{10.3.8}$$

By the exactness of the middle line in (10.3.7), all columns are exact, and by assumption the middle line is exact. On the other hand, the upper line is identified with the similar line with $\varepsilon_1 = 1$, so is exact. Therefore, the lower line is also exact. □

***10.3.9. Definition* (Compatible filtrations).** Given finitely many increasing filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ of an object A in the abelian category, we call them *compatible* if

$$F^{(1)}_{k_1} A, \dots, F^{(n)}_{k_n} A \subseteq A$$

are compatible sub-objects for every choice of $k_1, \dots, k_n \in \mathbb{Z}$.

10.3.10. Remark.

(1) As a consequence of our previous remarks, any sub-family of filtrations of a compatible family remains compatible. Moreover, by Lemma 10.3.1, any finite family of sub-objects consisting of terms of the filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ is compatible, and the last assertion of this lemma implies that the induced filtrations $F^{(1)}_\bullet, \dots, F^{n-1}_\bullet$ on each $\mathrm{gr}^{F^{(n)}}_\ell A$ are compatible.

(2) Let $B = F^{(1)}_{j_1} \cap \cdots \cap F^{(n)}_{j_n}$ for some $j_1, \dots, j_n$. Then the family of filtrations $F^{(1)}_\bullet B, \dots, F^{(n)}_\bullet B$ naturally induced on B is compatible, as follows from the compatibility of the family of $2n$ sub-objects $F^{(1)}_{k_1} A, \dots, F^{(n)}_{k_n} A, F^{(1)}_{j_1} A, \dots, F^{(n)}_{j_n} A$ and that of the induced family on B.

10.3.b. Reformulation in terms of flatness.

Let A be an object with n filtrations $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$. As usual, we can pass from filtered to graded objects by the Rees construction. Let $R = \mathbb{C}[z_1, \dots, z_n]$ denote the polynomial ring in n variables, with the $\mathbb{Z}^n$-grading that gives z_i the weight $e_i = (0, \dots, 1, \dots, 0)$. For $k \in \mathbb{Z}^n$, we define

$$M_k = M_{k_1,\dots,k_n} = F^{(1)}_{k_1} A \cap \cdots \cap F^{(n)}_{k_n} A \subseteq A.$$

We then obtain a $\mathbb{Z}^n$-graded module M over the ring R by taking the direct sum

$$R_{F^{(1)},\dots,F^{(n)}} A := M = \bigoplus_{k \in \mathbb{Z}^n} M_k,$$

with the obvious $\mathbb{Z}^{(n)}$-grading: for $m \in M_k$, the product $z_i m$ is simply the image of m under the inclusion $M_k \subseteq M_{k+e_i}$. From now on, we use the term "graded" to mean "$\mathbb{Z}^n$-graded".

10.3.11. Theorem. *A graded R-module comes from an object with n compatible filtrations if and only if it is flat over R.*

Before giving the proof, we recall a few general facts about flatness. For any commutative ring R, flatness of an R-module M is equivalent to the condition that

$$\operatorname{Tor}_1^R(M, R/I) = 0$$

for every finitely generated ideal $I \subseteq R$; when R is Noetherian, it is enough to check this for all prime ideals $P \subseteq R$. In our setting, the ring R is graded, and by a similar argument as in the ungraded case, flatness is equivalent to

$$\operatorname{Tor}_1^R(M, R/P) = 0$$

for every *graded* prime ideal $P \subseteq R$. Of course, there are only finitely many graded prime ideals in $R = \mathbb{C}[z_1, \ldots, z_n]$, namely those that are generated by the 2^n possible subsets of the set $\{z_1, \ldots, z_n\}$. Moreover, the quotient R/P always has a canonical free resolution given by the Koszul complex.

10.3.12. Example. For $n = 1$, a graded R-module M is flat if and only if $z_1 \colon M \to M$ is injective. For $n = 2$, a graded R-module M is flat if and only if $z_1 \colon M \to M$ and $z_2 \colon M \to M$ are both injective and the Koszul complex

$$M \xrightarrow{(-z_2, z_1)} M \oplus M \xrightarrow{z_1\bullet + z_2\bullet} M$$

is exact in the middle. (Here we are ignoring the grading in the notation.) The Koszul complex is just the simple complex associated to the double complex

$$\begin{tikzcd} M \arrow[r, "z_1"] \arrow[d, "z_2"'] & M \arrow[d, "z_2"] \\ M \arrow[r, "z_1"] & M \end{tikzcd}$$

with Deligne's sign conventions. The exactness of the Koszul complex in the middle can be read on each graded term as $M_{k_1-1,k_2} \cap M_{k_1,k_2-1} = M_{k_1-1,k_2-1}$. In this way, it is clear that two filtrations give rise to a flat R-module, illustrating thereby Theorem 10.3.11.

Exactness of the Koszul complex is closely related to the concept of regular sequences. Recall that $z_1, \ldots, z_n$ form a *regular sequence* on M if multiplication by z_1 is injective on M, multiplication by z_2 is injective on M/z_1M, multiplication by z_3 is injective on $M/(z_1, z_2)M$, and so on.

10.3.13. Lemma. *A graded R-module M is flat over R if and only if any permutation of $z_1, \ldots, z_n$ is a regular sequence on M.*

Proof. This is one of the basic properties of the Koszul complex. The point is that multiplication by z_1 is injective on M if and only if the Koszul complex

$$M \xrightarrow{z_1} M$$

is a resolution of M/z_1M. If this is the case, multiplication by z_2 is injective on M/z_1M if and only if the Koszul complex

$$M \xrightarrow{(-z_2, z_1)} M \oplus M \xrightarrow{z_1\bullet + z_2\bullet} M$$

is a resolution of $M/(z_1, z_2)M$, etc. □

***Proof of Theorem* 10.3.11.** Let us first show that if $F^{(1)}_\bullet A, \dots, F^{(n)}_\bullet A$ are compatible filtrations, then the associated Rees module M is flat over R. Because of the inherent symmetry, it is enough to prove that $z_n, \dots, z_1$ form a regular sequence on M. Because M comes from a filtered object, multiplication by z_n is injective and

$$M/z_nM = \bigoplus_{k\in\mathbb{Z}^n} M_{k_1,\dots,k_n}/M_{k_1,\dots,k_{n-1},k_n-1}.$$

This is now a $\mathbb{Z}^n$-graded module over the polynomial ring $\mathbb{C}[z_1, \dots, z_{n-1}]$. We remarked, after Definition 10.3.9, that for every $\ell \in \mathbb{Z}$, the $n-1$ induced filtrations on

$$A_\ell = \mathrm{gr}_\ell^{F^{(n)}} A = F^{(n)}_\ell A / F^{(n)}_{\ell-1} A$$

are still compatible, and that

$$F^{(1)}_{k_1} A_\ell \cap \cdots \cap F^{n-1}_{k_{n-1}} A_\ell \simeq M_{k_1,\cdots,k_{n-1},\ell}/M_{k_1,\dots,k_{n-1},\ell-1}.$$

By induction, this implies that $z_{n-1}, \dots, z_1$ form a regular sequence on M/z_nM, which is what we wanted to show.

For the converse, suppose that M is now an arbitrary graded R-module that is flat over R. We need to construct from M an object with n compatible filtrations. We consider the graded components M_k as a directed system, indexed by $k \in \mathbb{Z}^n$, with morphisms given by multiplication by $z_1, \dots, z_n$; since M is flat, all these morphisms are injective. Since we are working in an abelian category in which all filtered direct limits exist and are exact, we can define

$$A = \varinjlim_{k\in\mathbb{Z}^n} M_k.$$

If we hold the i-th index fixed, the resulting direct limit determines a subobject $F^{(i)}_{k_i} A$, and in fact an increasing filtration $F^{(i)}_\bullet A$. We can use the flatness of M to prove that these n filtrations are compatible, and that

$$M_{k_1,\dots,k_n} = F^{(1)}_{k_1} A \cap \cdots \cap F^{(n)}_{k_n} A, \tag{10.3.14}$$

as subobjects of A.

Fix $k, \ell \in \mathbb{Z}^n$. Observe that because R is graded, the graded submodules $z_1^{\ell_1} R, \dots, z_n^{\ell_n} R$ are trivially compatible; in fact, the required n-dimensional commutative diagram exists in the category of graded R-modules. If we tensor this diagram by M, it remains exact everywhere, due to the fact that M is flat. Take the graded piece of degree $k+\ell$ everywhere; for $n=2$, for example, the result looks like this:

$$\begin{tikzcd}
* \arrow[r] & * \arrow[r] & * \\
M_{k_1+\ell_1,k_2} \arrow[r] \arrow[u] & M_{k_1+\ell_1,k_2+\ell_2} \arrow[r] \arrow[u] & * \arrow[u] \\
M_{k_1,k_2} \arrow[r] \arrow[u] & M_{k_1,k_2+\ell_2} \arrow[r] \arrow[u] & * \arrow[u]
\end{tikzcd}$$

Apply the direct limit over $\ell \in \mathbb{Z}^n$; this operation preserves exactness. For $n=2$, for example, the resulting n-dimensional commutative diagram looks like this:

$$\begin{tikzcd}
* \arrow[r] & * \arrow[r] & * \\
F^2_{k_2}A \arrow[r] \arrow[u] & A \arrow[r] \arrow[u] & * \arrow[u] \\
M_{k_1,k_2} \arrow[r] \arrow[u] & F^{(1)}_{k_1}A \arrow[r] \arrow[u] & * \arrow[u]
\end{tikzcd}$$

The existence of such a diagram proves that $F^{(1)}_{k_1}A,\dots,F^{(n)}_{k_n}A$ are compatible subobjects of A, and also that (10.3.14) holds. $\square$

***10.3.15. Remark* (Interpretation of flatness in terms of multi-grading)**

Lemma 10.3.13 has the following practical consequence: for compatible filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$, the n-graded object obtained by inducing iteratively the filtrations on the j-graded object $\mathrm{gr}^{F^{(i_j)}}_{k_{i_j}}\cdots\mathrm{gr}^{F^{(i_1)}}_{k_{i_1}}A$ ($j=1,\dots,n$) does not depend on the order $\{i_1,\dots,i_n\}=\{1,\dots,n\}$, and is equal to

$$\frac{F^{(1)}_{k_1}A\cap\cdots\cap F^{(n)}_{k_n}A}{\sum_j F^{(1)}_{k_1}A\cap\cdots\cap F^{(j)}_{k_j-1}A\cap\cdots\cap F^{(n)}_{k_n}A}.$$

***10.3.16. Remark* (Multi-filtered morphisms).** Given two multi-filtered objects

$$(A,(F^{(i)}_\bullet A)_{i=1,\dots,n}) \quad\text{and}\quad (B,(F^{(i)}_\bullet B)_{i=1,\dots,n})$$

in A, let $\varphi: A\to B$ be a morphism compatible with the filtrations. It induces various morphisms $\mathrm{gr}^{F^{(i_j)}}_{k_{i_j}}\cdots\mathrm{gr}^{F^{(i_1)}}_{k_{i_1}}\varphi$. Assume that the filtrations in A and in B are compatible. Then the source and the target of these morphisms are independent of the order of multi-grading, as remarked above. We claim that *the morphisms $\mathrm{gr}^{F^{(i_j)}}_{k_{i_j}}\cdots\mathrm{gr}^{F^{(i_1)}}_{k_{i_1}}\varphi$ are also independent of the order of multi-grading.* Indeed, φ induces a graded morphism $R_F\varphi: M\to N$ between the associated Rees objects, and due to the compatibility assumption, we are led to checking that the restriction of $R_F\varphi$ to $M/(z_{k_{i_1}},\dots,z_{k_{i_j}})M$ is independent of the order, which is clear.

10.3.c. Flatness criterion. Under certain conditions on the graded R-module M, one can deduce flatness from the vanishing of the single R-module

$$\operatorname{Tor}_1^R\big(M, R/(z_1, \dots, z_n)R\big).$$

In the case of local rings, this kind of result is usually called the "local criterion for flatness". The simplest example is when M is finitely generated as an R-module, which is to say that all the filtrations are bounded from below.

10.3.17. Proposition. *If M is a finitely generated graded R-module, then the vanishing of $\operatorname{Tor}_1^R\big(M, R/(z_1, \dots, z_n)R\big)$ implies that M is flat.*

Proof. This is a general result in commutative algebra. To show what is going on, let us give a direct proof in the case $n = 2$. By assumption, the Koszul complex

$$M \xrightarrow{\;(-z_2, z_1)\;} M \oplus M \xrightarrow{\;z_1 \bullet + z_2 \bullet\;} M$$

is exact in the middle. It follows quite easily that multiplication by z_1 is injective. Indeed, if there is an element $m \in M_{i,j}$ with $z_1 m = 0$, then the pair $(m, 0)$ is in the kernel of the differential (z_1, z_2), and therefore $m = -z_2 m'$ and $0 = z_1 m'$ for some $m' \in M_{i,j-1}$. Continuing in this way, we eventually arrive at the conclusion that $m = 0$, because $M_{i,j} = 0$ for $j \ll 0$. For the same reason, multiplication by z_2 is injective; but now we have checked the condition in the definition of flatness for all graded prime ideals in R. $\square$

10.3.d. Strictness of morphisms. Let A and B be two objects in our abelian category A, each with n compatible filtrations

$$F_\bullet^{(1)} A, \dots, F_\bullet^{(n)} A, \quad \text{respectively} \quad F_\bullet^{(1)} B, \dots, F_\bullet^{(n)} B.$$

Denote by M and N the graded R-modules that are obtained by the Rees construction; both are flat by Theorem 10.3.11. Now consider a filtered morphism $\varphi\colon A \to B$. It induces an R-linear morphism $R_F\varphi\colon M \to N$ between the two Rees modules.

10.3.18. Definition. We say that $\varphi\colon A \to B$ is *strict* if $\operatorname{Coker} R_F\varphi$ is again a flat R-module.

Flatness of $\operatorname{Coker} R_F\varphi$ implies also that $\operatorname{Ker} R_F\varphi$ and $\operatorname{Im} R_F\varphi$ are flat: the reason is that we have two short exact sequences

$$0 \longrightarrow \operatorname{Ker} R_F\varphi \longrightarrow M \longrightarrow \operatorname{Im} R_F\varphi \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{Im} R_F\varphi \longrightarrow N \longrightarrow \operatorname{Coker} R_F\varphi \longrightarrow 0,$$

and because M and N are both flat, flatness of $\operatorname{Coker} R_F\varphi$ implies that of $\operatorname{Im} R_F\varphi$, which implies that of $\operatorname{Ker} R_F\varphi$. Note that $\operatorname{Ker}\varphi$ and $\operatorname{Coker}\varphi$ are equipped with filtrations $F_\bullet^{(1)} \operatorname{Ker}\varphi, \dots, F_\bullet^{(n)} \operatorname{Ker}\varphi$ respectively $F_\bullet^{(1)} \operatorname{Coker}\varphi, \dots, F_\bullet^{(n)} \operatorname{Coker}\varphi$ naturally induced from those on A and B. If φ is strict, we have

$$\operatorname{Ker} R_F\varphi = R_F \operatorname{Ker}\varphi, \quad \operatorname{Im} R_F\varphi = R_F \operatorname{Im}\varphi, \quad \operatorname{Coker} R_F\varphi = R_F \operatorname{Coker}\varphi.$$

Indeed, we know by Theorem 10.3.11 that the graded modules $\operatorname{Ker} R_F\varphi$, $\operatorname{Im} R_F\varphi$ and $\operatorname{Coker} R_F\varphi$ are attached to compatible filtrations, and (for $\operatorname{Coker}\varphi$ for example) the term in degree $k\in\mathbb{Z}^n$ is $(F^{(1)}_{k_1}B\cap\cdots\cap F^{(n)}_{k_n}B)+\operatorname{Im}\varphi/\operatorname{Im}\varphi$, so that the compatible filtrations on $\operatorname{Coker}\varphi$ given by the theorem are nothing but the filtrations induced by $F^i_\bullet B$.

For example, in the case of two filtrations F',F'' as considered in Definition 10.2.4, the last equality in bi-degree k,ℓ gives

$$F'_kF''_\ell B/\varphi(F'_kF''_\ell A)=(F'_kB+\operatorname{Im}\varphi)\cap(F''_\ell B+\operatorname{Im}\varphi)/\operatorname{Im}\varphi,$$

which corresponds to the condition of Definition 10.2.4.

10.3.19. Caveat. The strictness of φ implies that the induced filtrations (on $\operatorname{Ker}\varphi$, $\operatorname{Im}\varphi$ and $\operatorname{Coker}\varphi$) are compatible. However, the latter condition is not enough for strictness. For example, two filtrations are always compatible, while a morphism between bi-filtered objects need not be strict.

***10.3.20. Example* (Strict inclusions).** The composition of strict morphisms need not be strict in general. However, the composition of strict monomorphisms i_1,i_2 between objects with compatible filtrations remains a strict monomorphism since $\operatorname{Coker}R_F(i_2\circ i_1)=\operatorname{Coker}(R_Fi_2\circ R_Fi_1)$ is an extension of $\operatorname{Coker}R_Fi_2$ by $\operatorname{Coker}R_Fi_1$, and flatness is preserved by extensions.

Given n compatible filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$, they induce compatible filtrations on $M_k:=F^{(1)}_{k_1}A\cap\cdots\cap F^{(n)}_{k_n}A$ for every $k=(k_1,\dots,k_n)\in\mathbb{Z}^n$ (see Remark 10.3.10). Moreover, for $k\leqslant\ell\in\mathbb{Z}^n$ (i.e., $k_i\leqslant\ell_i$ for all $i=1,\dots,n$), the inclusion $M_k\hookrightarrow M_\ell$ is strict. Indeed, by the preliminary remark, it is enough to show that the inclusion $M_{k-1_i}\hookrightarrow M_k$ is strict for all i. This has been explained in the first part of the proof of Theorem 10.3.11.

***10.3.21. Lemma* (A criterion for strictness of inclusions).**

Let $(A,F^{(0)}_\bullet A,F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A)$ and $(B,F^{(0)}_\bullet B,F^{(1)}_\bullet B,\dots,F^{(n)}_\bullet B)$ be multi-filtered objects of A and let φ be a multi-filtered monomorphism between them. Assume the following properties:

(a) *$F^{(0)}_pB=0$ for $p\ll 0$,*

(b) *φ is $F^{(i)}$-strict for $i=0,1,\dots,n$ (i.e., $F^{(i)}_pA=F^{(i)}_pB\cap A$),*

(c) *for each p, the induced filtrations $F^{(1)},\dots,F^{(n)}$ on $\operatorname{gr}^{F^{(0)}}_pA,\operatorname{gr}^{F^{(0)}}_pB$ are compatible and $\operatorname{gr}^{F^{(0)}}_p\varphi:\operatorname{gr}^{F^{(0)}}_pA\to\operatorname{gr}^{F^{(0)}}_pB$ is an n-strict monomorphism.*

Then the filtrations $F^{(0)},F^{(1)},\dots,F^{(n)}$ on A,B are compatible and φ is an $(n+1)$-strict monomorphism.

Proof. We first prove that the filtrations $F^{(1)},\dots,F^{(n)}$ on A,B are compatible and φ is an n-strict monomorphism. We denote by F' the n-multi-filtration forgetting $F^{(0)}$ and by C the cokernel of $\varphi:A\to B$, equipped with the induced filtrations. By the second assumption, the sequence

$$0\longrightarrow R_{F'}A\longrightarrow R_{F'}B\longrightarrow R_{F'}C$$

is exact, and we wish to complete it into a short exact sequence. By the same assumption, $\operatorname{Coker}\operatorname{gr}_p^{F^{(0)}}\varphi = \operatorname{gr}_p^{F^{(0)}}C$ for every p. Then the third assumption says that $R_{F'}\operatorname{gr}_p^{F^{(0)}}B \to R_{F'}\operatorname{gr}_p^{F^{(0)}}C$ is an epimorphism for every p, and $\operatorname{gr}_p^{F^{(0)}}R_{F'}C$ is $\mathbb{C}[z_1,\dots,z_n]$-flat. Let us also note that $R_{F'}\operatorname{gr}_p^{F^{(0)}} = \operatorname{gr}_p^{F^{(0)}}R_{F'}$.

By induction on p and because of the first assumption, we have a diagram where all terms except possibly those of the middle line are $\mathbb{C}[z_1,\dots,z_n]$-flat and all sequences are exact:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F^{(0)}_{p-1}R_{F'}A \arrow[r] \arrow[d] & F^{(0)}_{p-1}R_{F'}B \arrow[r] \arrow[d] & F^{(0)}_{p-1}R_{F'}C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F^{(0)}_{p}R_{F'}A \arrow[r] \arrow[d] & F^{(0)}_{p}R_{F'}B \arrow[r] \arrow[d] & F^{(0)}_{p}R_{F'}C \arrow[d] & \\
0 \arrow[r] & \operatorname{gr}_p^{F^{(0)}}R_{F'}A \arrow[r] \arrow[d] & \operatorname{gr}_p^{F^{(0)}}R_{F'}B \arrow[r] \arrow[d] & \operatorname{gr}_p^{F^{(0)}}R_{F'}C \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

It follows that all terms of the middle line are $\mathbb{C}[z_1,\dots,z_n]$-flat, since flatness is preserved by extensions, and the middle line can be completed as an exact sequence. Taking the limit for $p\to\infty$ implies that the sequence

$$0 \longrightarrow R_{F'}A \longrightarrow R_{F'}B \longrightarrow R_{F'}C \longrightarrow 0$$

is exact and all its terms are $\mathbb{C}[z_1,\dots,z_n]$-flat.

Let us now prove that the filtrations $F^{(0)},F^{(1)},\dots,F^{(n)}$ on A,B are compatible and φ is an $(n+1)$-strict monomorphism. We consider the family of filtrations $F^{(0)},F^{(1)},\dots,F^{(n)},F^{(n+1)}$ with $F^{(n+1)} = F^{(0)}$. Then (b) holds trivially, and (c) holds since the filtration induced by $F^{(n+1)}$ on $\operatorname{gr}_p^{F^{(0)}}$ jumps at p only (see Exercise 10.11(1)). The first part of the proof for $F^{(1)},\dots,F^{(n)},F^{(n+1)}$ gives the desired statement. $\square$

If we have a complex of objects with n compatible filtrations and differentials that preserve the filtrations, we consider the associated complex of flat graded R-modules; if all of its cohomology modules are again flat over R, we say that the original filtered complex is *strict*. At least if the complex is bounded from above, a similar argument with short exact sequences proves that strictness of a complex is equivalent to strictness of the individual differentials.

10.4. Bi-filtered $\mathcal{D}_X$-modules

In the remaining part of this chapter, we will work with right $\mathcal{D}_X$-modules, since we are mainly interested in the pushforward theorem. Accordingly, we will consider increasing V-filtrations.

We consider the setting of Section 9.2 with a smooth hypersurface $H \subset X$ locally defined by a coordinate t. The ring $\mathcal{D}_X$ is then equipped with the F-filtration, and the V-filtration corresponding to H. We consider correspondingly $\mathcal{D}_X$-modules $\mathcal{M}$ equipped with an F-filtration and a V-filtration. In view of future use, we consider V-filtrations indexed by $A+\mathbb{Z}$ for some finite subset $A \subset (-1,0]$. We extend in a trivial way the filtration $V_\bullet\mathcal{D}_X$ indexed by $\mathbb{Z}$ to a filtration indexed by $A+\mathbb{Z}$, that is, we set for any $\alpha \in A+\mathbb{Z}$,

$$V_\alpha\mathcal{D}_X = V_{[\alpha]}\mathcal{D}_X.$$

In this section, we mainly consider the interaction of both filtrations, without consideration of coherence or $\mathbb{R}$-specializability.

For a $\mathcal{D}_X$-module $\mathcal{M}$ equipped with filtrations $F_\bullet\mathcal{M}$ ($p \in \mathbb{Z}$) and $V_\bullet\mathcal{M}$ ($\alpha \in A+\mathbb{Z}$), we set $F_pV_\alpha\mathcal{M} := F_p\mathcal{M} \cap V_\alpha\mathcal{M}$ as in (10.2.3), and $F_p\mathrm{gr}_\alpha^V\mathcal{M} := F_pV_\alpha\mathcal{M}/F_pV_{<\alpha}\mathcal{M}$. We notice that $(\mathcal{D}_X, F_\bullet, V_\bullet)$ satisfies the following properties:

(a) Multiplication by t induces an isomorphism $F_pV_\alpha\mathcal{D}_X \simeq F_pV_{\alpha-1}\mathcal{D}_X$ whenever $\alpha \leqslant 0$.

(b) Multiplication by ∂_t induces an isomorphism $F_p\mathrm{gr}_\alpha^V\mathcal{D}_X \simeq F_{p+1}\mathrm{gr}_{\alpha+1}^V\mathcal{D}_X$ whenever $\alpha > -1$.

These will be the basic relations we impose to bi-filtered $\mathcal{D}_X$-modules. We consider then the category $\mathsf{FV}(\mathcal{D}_X)$ consisting of triples $(\mathcal{M}, F, V)$, where $\mathcal{M}$ is a right $\mathcal{D}_X$-module, F is a (usual) filtration and V is a V-filtration indexed by $A+\mathbb{Z}$ on $\mathcal{M}$, for some finite set $A \in (-1,0]$, such that the following conditions are satisfied (no coherence assumption is made here):

(i) the F and V-filtrations $F_\bullet\mathcal{M}$ and $V_\bullet\mathcal{M}$ are exhaustive,

(ii) $F_p\mathcal{M} = 0$ for $p \ll 0$,

(iii) multiplication by t induces an isomorphism $F_pV_\alpha\mathcal{M} \simeq F_pV_{\alpha-1}\mathcal{M}$ whenever $\alpha < 0$,

(iv) multiplication by ∂_t induces an isomorphism $F_p\mathrm{gr}_\alpha^V\mathcal{M} \simeq F_{p+1}\mathrm{gr}_{\alpha+1}^V\mathcal{M}$ whenever $\alpha > -1$.

The morphisms in $\mathsf{FV}(\mathcal{D}_X)$ are morphisms of right $\mathcal{D}_X$-modules that are compatible with both filtrations. We usually refer to the objects of $\mathsf{FV}(\mathcal{D}_X)$ simply as *bi-filtered $\mathcal{D}_X$-modules*. We note that Condition (iii) implies in particular that, for all $\alpha < 0$, $V_\alpha\mathcal{M}$ has no t-torsion and $V_\alpha\mathcal{M}\cdot t = V_{\alpha-1}\mathcal{M}$ (note that we do not include $\alpha = 0$ for arbitrary bi-filtered $\mathcal{D}_X$-modules). However, we do not assume that $(V_\alpha\mathcal{M})_{\alpha\in\mathbb{R}}$ is a coherent V-filtration with respect to H (more precisely, we do not require any coherence condition or the fact that $t\partial_t - \alpha$ is nilpotent on $\mathrm{gr}_\alpha^V\mathcal{M}$).

10.4.1. Remark. It is not true that a morphism φ in $\mathsf{FV}(\mathcal{D}_X)$ has kernels and cokernels (it is not necessarily true that the induced filtrations on the $\mathcal{D}_X$-modules kernels or cokernels satisfy conditions (iii) and (iv) above). However, this is the case if φ is bi-strict: indeed, since taking F_pV_α and $F_p\mathrm{gr}_\alpha^V$ both commute with taking Ker and Coker, (see Exercise 10.6(3)), the isomorphism condition on t and ∂_t is preserved. If φ is bi-strict, we have an isomorphism $\mathrm{Coim}(\varphi) \simeq \mathrm{Im}(\varphi)$ in $\mathsf{FV}(\mathcal{D}_X)$.

Let $\mathcal{L}$ be an $\mathcal{O}_X$-module. It defines an induced $\mathcal{D}_X$-module $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{D}_X$ (see Section 8.5), equipped with filtrations $\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F_\bullet)$ and $\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, , V_\bullet)$. By shifting the filtrations on $\mathcal{D}_X$, we obtain for each $p \in \mathbb{Z}$ and $\alpha \in A + \mathbb{Z}$ an induced $\mathcal{D}_X$-module with filtrations $\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p]_\bullet)$ and $\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, V[\alpha]_\bullet)$:

$$F_q(\mathcal{L} \otimes \mathcal{D}_X) = \operatorname{Im}(\mathcal{L} \otimes F_{q-p}\mathcal{D}_X \hookrightarrow \mathcal{L} \otimes \mathcal{D}_X)$$
$$V_\beta(\mathcal{L} \otimes \mathcal{D}_X) = \operatorname{Im}(\mathcal{L} \otimes V_{\beta-\alpha}\mathcal{D}_X \hookrightarrow \mathcal{L} \otimes \mathcal{D}_X).$$

In order to also obtain "induced properties" for $F_q V_\beta$, we are led to the following definition.

***10.4.2. Definition* (Induced bi-filtered $\mathcal{D}_X$-modules).**

(1) An *elementary induced bi-filtered $\mathcal{D}_X$-module* is a bi-filtered $\mathcal{D}_X$-module of the form

$$\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p]_\bullet, V[\alpha]_\bullet), \quad p \in \mathbb{Z},\ \alpha \in [-1, 0],$$

such that

(a) if $\alpha \in [-1, 0)$, $\mathcal{L}$ *has no t-torsion*,

(b) if $\alpha = 0$, $\mathcal{L}$ has *t-torsion of order at most one*, that is,

$$\{u \in \mathcal{L} \mid ut^j = 0 \text{ for some } j \geqslant 1\} = \{u \in \mathcal{L} \mid ut = 0\}.$$

(2) An *induced bi-filtered $\mathcal{D}_X$-module* is an object of $\mathsf{FV}(\mathcal{D}_X)$ that is isomorphic to a direct sum of elementary induced bi-filtered $\mathcal{D}_X$-modules

$$\bigoplus_i \big(\mathcal{L}_i \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p_i], V[\alpha_i])\big), \quad p_i \in \mathbb{Z},\ \alpha_i \in [-1, 0].$$

The full subcategory of $\mathsf{FV}(\mathcal{D}_X)$ consisting of induced objects is denoted $\mathsf{FV}_i(\mathcal{D}_X)$.

We nevertheless need to justify that elementary induced bi-filtered $\mathcal{D}_X$-modules as defined above belong to $\mathsf{FV}(\mathcal{D}_X)$, that is, satisfy Properties (i)–(iv) above. In order to do so, it is convenient to treat separately the case when $\mathcal{L}$ has no t-torsion and when $\mathcal{L}t = 0$, the general case following using the existence of an exact sequence

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}'' \longrightarrow 0,$$

with $\mathcal{L}'t = 0$ and $\mathcal{L}''$ without t-torsion.

It is useful to note that since $\mathcal{L}'t = 0$, we have locally

$$\begin{aligned} \mathcal{L}' \otimes \mathcal{D}_X &= \mathcal{L}' \otimes_{\mathcal{O}_H} \mathcal{D}_H[\partial_t], \\ F_q(\mathcal{L}' \otimes \mathcal{D}_X) &= \oplus_{j \geqslant 0}(\mathcal{L}' \otimes_{\mathcal{O}_H} (F_{q-p-j}\mathcal{D}_H)\partial_t^j) \\ V_\beta(\mathcal{L}' \otimes \mathcal{D}_X) &= \oplus_{j=0}^{\lfloor \beta \rfloor}(\mathcal{L}' \otimes_{\mathcal{O}_H} \mathcal{D}_H \partial_t^j). \end{aligned} \tag{10.4.3}$$

10.4.4. Lemma. *With the above notation, for every q and β, we have*

(i) $F_q V_\beta(\mathcal{L} \otimes \mathcal{D}_X) = \operatorname{Im}(\mathcal{L} \otimes F_{q-p} V_{\beta-\alpha}\mathcal{D}_X \to \mathcal{L} \otimes \mathcal{D}_X)$.

(ii) *There is an exact sequence*

$$0 \longrightarrow F_q V_\beta(\mathcal{L}' \otimes \mathcal{D}_X) \longrightarrow F_q V_\beta(\mathcal{L} \otimes \mathcal{D}_X) \longrightarrow F_q V_\beta(\mathcal{L}'' \otimes \mathcal{D}_X) \longrightarrow 0.$$

Furthermore, we have $\mathcal{L} \otimes (\mathcal{D}_X, F[p], V[\alpha]) \in \mathsf{FV}(\mathcal{D}_X)$.

Proof. The assertion in (i) follows easily when $\mathcal{L}$ has no t-torsion, using the fact that the following maps are injective:

$$\mathcal{L}\otimes V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L}\otimes\mathcal{D}_X,\quad \mathcal{L}\otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X \longrightarrow \mathcal{L}\otimes\mathcal{D}_X,$$

and
$$\mathcal{L}\otimes(V_{\beta-\alpha}\mathcal{D}_X/F_{q-p}V_{\beta-\alpha}\mathcal{D}_X) \longrightarrow \mathcal{L}\otimes\mathcal{D}_X/F_{q-p}\mathcal{D}_X.$$

When $\mathcal{L}t=0$, we deduce (i) from the explicit description in (10.4.3).

We now note that we have exact sequences

$$0\longrightarrow F_q(\mathcal{L}'\otimes\mathcal{D}_X)\longrightarrow F_q(\mathcal{L}\otimes\mathcal{D}_X)\longrightarrow F_q(\mathcal{L}''\otimes\mathcal{D}_X)\longrightarrow 0$$

and
$$0\longrightarrow V_\beta(\mathcal{L}'\otimes\mathcal{D}_X)\longrightarrow V_\beta(\mathcal{L}\otimes\mathcal{D}_X)\longrightarrow V_\beta(\mathcal{L}''\otimes\mathcal{D}_X)\longrightarrow 0$$

(exactness follows from definition and the fact that the maps

$$\mathcal{L}''\otimes F_{q-p}\mathcal{D}_X\longrightarrow\mathcal{L}''\otimes\mathcal{D}_X\quad\text{and}\quad\mathcal{L}''\otimes V_{\beta-\alpha}\mathcal{D}_X\longrightarrow\mathcal{L}''\otimes\mathcal{D}_X$$

are injective. Let

$$M=\operatorname{Im}(\mathcal{L}\otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X\longrightarrow\mathcal{L}\otimes\mathcal{D}_X)$$

and we similarly define M' and M''. We deduce that we have a commutative diagram with exact rows and injective vertical maps

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "j'"] & M \arrow[r] \arrow[d, "j"] & M'' \arrow[r] \arrow[d, "j''"] & 0\\
0 \arrow[r] & F_qV_\beta(\mathcal{L}'\otimes\mathcal{D}_X) \arrow[r] & F_qV_\beta(\mathcal{L}\otimes\mathcal{D}_X) \arrow[r] & F_qV_\beta(\mathcal{L}''\otimes\mathcal{D}_X)
\end{tikzcd}$$

(for the exactness of the top row we use the fact that the map

$$\mathcal{L}''\otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X\longrightarrow\mathcal{L}''\otimes\mathcal{D}_X$$

is injective; the exactness of the bottom row follows from the above two exact sequences). Since we know that j' and j'' are surjective, it follows that j is also surjective. This completes the proof of both (i) and (ii). The last assertion in the lemma is easy to check for $\mathcal{L}'$ and $\mathcal{L}''$, and we deduce it also for $\mathcal{L}$ using (ii). $\square$

10.4.5. Remark. Given $(\mathcal{M},F,V)\in\mathsf{FV}(\mathcal{D}_X)$, note that for every $\alpha\in[-1,0]$ and every $p\in\mathbb{Z}$, we obtain an elementary induced bi-filtered $\mathcal{D}_X$-module as

$$F_pV_\alpha\mathcal{M}\otimes(\mathcal{D}_X,F[p],V[\alpha]).$$

Indeed, we know by Condition (iii) that $F_pV_\alpha\mathcal{M}$ has no t-torsion when $\alpha<0$. Furthermore, if $u\in F_pV_0\mathcal{M}$ is such that $t^ju=0$ for some $j\geqslant 2$, then $tu\in F_pV_{-1}\mathcal{M}$ and $t(tu)=0$, hence $tu=0$. We have a strict surjective morphism

$$\bigoplus_{\substack{p\in\mathbb{Z}\\ \alpha\in[0,1]}} F_pV_\alpha\mathcal{M}\otimes(\mathcal{D}_X,F[p],V[\alpha])\longrightarrow(\mathcal{M},F,V)$$

(in this case strictness simply means that the filtrations on the target are induced by the ones on the source, see Exercise 10.6(1)). Indeed, the surjectivity is a consequence of Conditions (iii) and (iv) in the definition of the category $\mathsf{FV}(\mathcal{D}_X)$. By applying the same argument to the kernel, with the induced filtrations (note that this lies in $\mathsf{FV}(\mathcal{D}_X)$), we obtain a (possibly infinite) resolution of $(\mathcal{M},F,V)$ by induced objects.

We consider the category of complexes $\mathsf{C}^*(\mathsf{FV}(\mathcal{D}_X))$, where $*$ stands for $+$, $-$, b, or for the empty set. We assume that all complexes $C^\bullet$ in this category satisfy the following assumptions:

(i) For $p \ll 0$, we have $F_pC^\bullet = 0$.

(ii) There exists a finite set $A \subset [-1, 0)$ suitable for each term of $C^\bullet$.

We have a corresponding homotopic category $\mathsf{K}^*(\mathsf{FV}(\mathcal{D}_X))$. A morphism $C_1^\bullet \to C_2^\bullet$ in $\mathsf{K}(\mathsf{FV}(\mathcal{D}_X))$ is a *filtered quasi-isomorphism* if $\mathcal{H}^i(F_pV_\alpha C_1^\bullet) \to \mathcal{H}^i(F_pV_\alpha C_2^\bullet)$ is an isomorphism for all $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. Note that since we work with exhaustive filtrations, every filtered quasi-isomorphism is, in particular, a quasi-isomorphism.

We obtain the filtered derived categories $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X))$ by localizing $\mathsf{K}^*(\mathsf{FV}(\mathcal{D}_X))$ at the class of filtered quasi-isomorphisms. As in the case of the derived category of an abelian category, one shows that each $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X))$ is a triangulated category. It follows from the universal property of the localization that we get exact functors

$$H^iF_pV_\alpha(-)\colon \mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X)) \longrightarrow \mathsf{D}^*(\mathcal{O}_X),$$

where $\mathsf{D}^*(\mathcal{O}_X)$ is the derived category of $\mathcal{O}_X$-modules, with the suitable boundedness condition.

10.4.6. Remark. Let us assume that X is a product $X \simeq H \times \Delta_t$. Note that for every $\alpha \in \mathbb{R}$, taking $(C^\bullet, F, V)$ to $(\mathrm{gr}_\alpha^V(C^\bullet), F)$ defines an exact functor $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{D}^*(\mathsf{F}(\mathcal{D}_H))$, where $\mathsf{D}^*(\mathsf{F}(\mathcal{D}_H))$ is the filtered derived category of filtered $\mathcal{D}_H$-modules (with suitable boundedness conditions).

Let $\mathsf{K}^*(\mathsf{FV}_i(\mathcal{D}_X))$ be the homotopic category of complexes of induced objects in $\mathsf{FV}(\mathcal{D}_X)$, with suitable boundedness conditions. By localizing this with respect to filtered quasi-isomorphisms, we get $\mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_X))$.

10.4.7. Lemma. *The exact functor*

$$\mathsf{D}^-(\mathsf{FV}_i(\mathcal{D}_X)) \longrightarrow \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X))$$

induced by inclusion is an equivalence of categories.

Proof. For every $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_X)$, we construct the resolution $\mathcal{I}^\bullet(\mathcal{M}, F, V)$ by induced bi-filtered $\mathcal{D}_X$-modules as in Remark 10.4.5. It is clear that this is functorial and we extend the construction to a functor $\mathsf{C}^-(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{C}^-(\mathsf{FV}_i(\mathcal{D}_X))$, by mapping a complex $(\mathcal{M}^\bullet, F, V)$ to the total complex of the double complex $\mathcal{I}^\bullet(\mathcal{M}^\bullet, F, V)$. It is standard to check that this induces a functor between the corresponding filtered derived categories and that this gives an inverse for the functor induced by the inclusion. $\square$

10.4.8. Remark. If $(\mathcal{M}, F, V)$ is a bi-filtered $\mathcal{D}_X$-module, we can choose a finite subset $A \subset [-1, 0]$ such that $\mathrm{gr}_\alpha^V(\mathcal{M}) = 0$ for all $\alpha \in [-1, 0] \smallsetminus A$. As in Remark 10.4.5, we obtain a strict surjective morphism

$$\bigoplus_{\substack{p\in\mathbb{Z}\\ \alpha\in A}} F_pV_\alpha\mathcal{M} \otimes (\mathcal{D}_X, F[p], V[\alpha]) \longrightarrow (\mathcal{M}, F, V),$$

and by iterating this construction, we obtain a resolution $(\mathcal{I}^\bullet, F, V)$ of $(\mathcal{M}, F, V)$ by induced objects such that each $(\mathcal{I}^i, F, V)$ is the direct sum of elementary bi-filtered modules $\mathcal{L} \otimes \mathcal{O}_X(\mathcal{D}_X, F[p_i], V[\alpha_i])$, with the α_i varying over a finite set. In particular, since for every q and β we have

$$F_q V_\beta(\mathcal{L} \otimes \mathcal{O}_X(\mathcal{D}_X, F[p_i], V[\alpha_i])) = 0 \text{ unless } p_i \leqslant q,$$

we conclude that if $F_p V_\alpha \mathcal{M}$ is a coherent $\mathcal{O}_X$-module for every p and α, then $F_p V_\alpha \mathcal{I}^j$ is a coherent $\mathcal{O}_X$-module for every p, α, and j.

10.4.9. Lemma. *Consider two elementary induced bi-filtered $\mathcal{D}_X$-modules*

$$(\mathcal{M}_i, F, V) = \mathcal{L}_i \otimes (\mathcal{D}_X, F[p], V[\alpha]) \quad i = 1, 2,$$

and consider the exact sequences

$$0 \longrightarrow \mathcal{L}'_i \longrightarrow \mathcal{L}_i \longrightarrow \mathcal{L}''_i \longrightarrow 0,$$

where $\mathcal{L}'_i t = 0$ and $\mathcal{L}''_i$ has no t-torsion. If $u\colon \mathcal{L}_1 \to \mathcal{L}_2$ is an injective morphism such that the induced morphism $u''\colon \mathcal{L}''_1 \to \mathcal{L}''_2$ has the property that $\mathrm{Coker}(u'')$ *has no t-torsion, then the induced morphism $\overline{u}\colon (\mathcal{M}_1, F, V) \to (\mathcal{M}_2, F, V)$ is strict and* $\mathrm{Coker}(\overline{u}) \simeq \mathrm{Coker}(u) \otimes (\mathcal{D}_X, F[p], V[\alpha])$.

Proof. We need to show that if we consider on $\mathrm{Coker}(\overline{u}) \simeq \mathrm{Coker}(u) \otimes \mathcal{D}_X$ the induced filtrations, then for every q and β, the sequence

$$0 \longrightarrow F_q V_\beta(\mathcal{L}_1 \otimes \mathcal{D}_X) \longrightarrow F_q V_\beta(\mathcal{L}_2 \otimes \mathcal{D}_X) \longrightarrow F_q V_\beta(\mathrm{Coker}(u) \otimes \mathcal{D}_X) \longrightarrow 0$$

is exact. This is easy to check when both $\mathcal{L}_i$ have no t-torsion and it follows from the explicit description in (10.4.3) when $\mathcal{L}_i t = 0$ for $i = 1, 2$.

We now consider the general case. Let $u'\colon \mathcal{L}'_1 \to \mathcal{L}'_2$ be the morphism induced by u. Note first that the Snake lemma gives an exact sequence

$$0 \longrightarrow \mathrm{Coker}(u') \longrightarrow \mathrm{Coker}(u) \longrightarrow \mathrm{Coker}(u'') \longrightarrow 0.$$

(since $\mathrm{Ker}(u'')$ has no t-torsion, it has to be zero). This exact sequence is the canonical one associated to $\mathrm{Coker}(u)$ such that the first term is annihilated by t and the third one has no t-torsion.

Consider the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & F_qV_\beta(\mathcal{L}'_1 \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}_1 \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}''_1 \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_qV_\beta(\mathcal{L}'_2 \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}_2 \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathcal{L}''_2 \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & F_qV_\beta(\mathrm{Coker}(u') \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathrm{Coker}(u) \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & F_qV_\beta(\mathrm{Coker}(u'') \otimes \mathcal{D}_X) \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0. &
\end{tikzcd}$$

The first and the third columns are exact by what we have already discussed. Moreover, the rows are all exact by Lemma 10.4.4. Therefore the middle column is also exact, which is what we had to prove. $\square$

In order to define functors between filtered derived categories, it will be convenient to use the Godement resolution (see Definition 8.7.14), that we now extend to our bi-filtered setting.

For $(\mathcal{M}, F, V) \in \mathrm{FV}(\mathcal{D}_X)$, we define $\mathcal{C}^0(\mathcal{M}, F, V)$ to be the bi-filtered $\mathcal{D}_X$-module $\mathcal{N} = \bigcup_{p,\alpha} \mathcal{C}^0(F_pV_\alpha\mathcal{M}) \subseteq \mathcal{C}^0(\mathcal{M})$, with the filtrations given by $F_p\mathcal{N} = \bigcup_\alpha \mathcal{C}^0(F_pV_\alpha\mathcal{M})$ and $V_\alpha\mathcal{N} = \bigcup_p \mathcal{C}^0(F_pV_\alpha\mathcal{M})$ for $p \in \mathbb{Z}$, $\alpha \in \mathbb{R}$. One checks that

$$\mathcal{C}^0(F_pV_\alpha\mathcal{M}) \cap \mathcal{C}^0(F_qV_\beta\mathcal{M}) = \mathcal{C}^0(F_{\min(p,q)}V_{\min(\alpha,\beta)}\mathcal{M}).$$

It follows that $F_pV_\alpha\mathcal{N} = \mathcal{C}^0(F_pV_\alpha\mathcal{M})$, hence each $F_pV_\alpha\mathcal{N}$ is flabby. We have a natural strict monomorphism $(\mathcal{M}, F, V) \hookrightarrow \mathcal{C}^0(\mathcal{M}, F, V)$, whose cokernel is also a bi-filtered $\mathcal{D}_X$-module, and we can proceed inductively as in Definition 8.7.14 to define the complex $\mathrm{God}^\bullet(\mathcal{M}, F, V)$ in $\mathsf{C}^+(\mathsf{FV}(\mathcal{D}_X))$ that is filtered quasi-isomorphic to $(\mathcal{M}, F, V)$.

10.4.10. Lemma. *Given an elementary induced bi-filtered $\mathcal{D}_X$-module*

$$(\mathcal{M}, F, V) \simeq \mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p], V[\alpha]),$$

we have

$$\mathcal{C}^0(\mathcal{M}, F, V) \simeq \mathcal{C}^0(\mathcal{L}) \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p], V[\alpha]).$$

Proof. If we consider the exact sequence

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}'' \longrightarrow 0,$$

where $\mathcal{L}'t = 0$ and $\mathcal{L}''$ has no t-torsion, then we have an induced exact sequence

$$0 \longrightarrow \mathcal{C}^0(\mathcal{L}') \longrightarrow \mathcal{C}^0(\mathcal{L}) \longrightarrow \mathcal{C}^0(\mathcal{L}'') \longrightarrow 0$$

and $\mathcal{C}^0(\mathcal{L}')t = 0$, while $\mathcal{C}^0(\mathcal{L}'')$ has no t-torsion. In particular, we see that every t-torsion element in $\mathcal{C}^0(\mathcal{L})$ is annihilated by t. We also deduce from this that it is enough to prove the lemma when either $\mathcal{L}$ has no t-torsion or when $\mathcal{L}t = 0$.

Suppose first that $\mathcal{L}$ has no t-torsion. In this case we have

$$F_qV_\beta(\mathcal{C}^0(\mathcal{L}) \otimes \mathcal{D}_X) = \mathcal{C}^0(\mathcal{L}) \otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X \simeq \mathcal{C}^0(\mathcal{L} \otimes F_{q-p}V_{\beta-\alpha}\mathcal{D}_X),$$

since $F_{q-p}V_{\beta-\alpha}\mathcal{D}_X$ is a locally free $\mathcal{O}_X$-module, of finite type (see Exercise 8.48(2)). This implies the isomorphism in the lemma. The case when $\mathcal{L}t = 0$ follows similarly, using the explicit description in (10.4.3). $\square$

10.4.11. Corollary. *If $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_X)$ is induced, then its filtered resolution*

$$0 \longrightarrow (\mathcal{M}, F, V) \longrightarrow \mathcal{C}^0(\mathcal{M}, F, V) \longrightarrow \mathcal{C}^1(\mathcal{M}, F, V) \longrightarrow \cdots$$

consists of induced objects and the morphisms are strict and they correspond to morphisms of $\mathcal{O}_X$-modules.

Proof. This follows by combining Lemmas 10.4.9 and 10.4.10. The only thing to note is that if we have a short exact sequence of $\mathcal{O}_X$-modules

$$0 \longrightarrow \mathcal{L}' \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}'' \longrightarrow 0,$$

with $\mathcal{L}'t = 0$ and $\mathcal{L}''$ without t-torsion, then $\operatorname{Coker}(\mathcal{L}'' \to \mathcal{C}^0(\mathcal{L}''))$ has no t-torsion. $\square$

10.5. The direct image of bi-filtered $\mathcal{D}_X$-modules

Let $f\colon X \to X'$ be a morphism between complex manifolds. We assume that $X' = H' \times \Delta_t$ and $X = H \times \Delta_t$ such that $f = f_{|H} \times \mathrm{Id}_t$. We set $X_0 = H \times \{0\}$ and $X'_0 = H' \times 0$. Our first goal is to define a functor ${}_{\mathrm{D}}f_*\colon \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_{X'}))$.

In addition to the sheaf $\mathcal{D}_X$, we also have on X the sheaf $f^{-1}(\mathcal{D}_{X'})$. This carries the F-filtration and the V-filtration induced from $\mathcal{D}_{X'}$ (the V-filtration being the one with respect to X'_0). In particular, we may consider the categories $\mathsf{FV}(f^{-1}(\mathcal{D}_{X'}))$ and $\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$. For example, an object in $\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$ is one that is isomorphic to a direct sum of objects of the form $\mathcal{L} \otimes_{f^{-1}(\mathcal{O}_{X'})} (f^{-1}(\mathcal{D}_{X'}), F[p], V[\alpha])$, where $\mathcal{L}$ is an $f^{-1}(\mathcal{O}_{X'})$-module that has no t-torsion, unless $\alpha = 0$, in which case the all local sections of $\mathcal{L}$ that are annihilated by some power of t are actually annihilated by t. The same construction from before (see Lemma 10.4.7) shows that the inclusion functor determines an equivalence of categories

$$\mathsf{D}^-(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))) \longrightarrow \mathsf{D}^-(\mathsf{FV}(f^{-1}(\mathcal{D}_{X'}))).$$

As in the case of the direct image of non-filtered $\mathcal{D}_X$-modules, the key player in the definition of the direct image for bi-filtered $\mathcal{D}_X$-modules is

$$\mathcal{D}_{X\to X'} := \mathcal{O}_X \otimes_{f^{-1}(\mathcal{O}_{X'})} f^{-1}(\mathcal{D}_{X'}).$$

This has a structure of left $\mathcal{D}_X$-module and right $f^{-1}(\mathcal{D}_{X'})$-bimodule and carries an F-filtration and a V-filtration induced from $f^{-1}(\mathcal{D}_{X'})$. These are compatible not only with the F and V-filtrations on $f^{-1}(\mathcal{D}_{X'})$, via right multiplication, but also with the F and V-filtrations on $\mathcal{D}_X$, via left multiplication.

10.5.1. Example. The two main examples are when f is smooth and when f is a closed immersion. The typical case for f being smooth is when $f\colon X = X' \times W \to X'$ is the projection onto the first factor. In this case we have a surjection $\mathcal{D}_{X'\times W} \to \mathcal{D}_{X'\times W\to X'}$ such that in local coordinates $w_1, \dots, w_r$ on W, we get an isomorphism

$$\mathcal{D}_{X'\times W\to X'} \simeq \mathcal{D}_{X'\times W}/\mathcal{D}_{X'\times W} \cdot (\partial_{w_1}, \dots, \partial_{w_r}).$$

On the other hand, the typical case when f is a closed immersion is when $f\colon X \hookrightarrow X' = X \times Z$ is given by $f(x) = (x, z_0)$. If we have coordinates $z_1, \dots, z_r$ on Z, then

$$\mathcal{D}_{X\to X\times Z} \simeq \mathcal{D}_X \otimes \mathbb{C}[\partial_{z_1}, \dots, \partial_{z_r}].$$

We first claim that

$$({10.5.2}) \qquad {}^{\mathrm{p}}\mathrm{DR}_{X/X'}(\mathcal{M}, F, V) = (\mathcal{M}, F, V) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X\to X'}, F, V),$$

with the tensor product of the filtrations from the two factors, defines a functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}\colon \mathsf{FV}_i(\mathcal{D}_X) \to \mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$. Indeed, we have

$$\mathcal{L} \otimes_{\mathcal{O}_X} (\mathcal{D}_X, F[p], V[\alpha]) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X\to X'}, F, V) \simeq \mathcal{L} \otimes_{f^{-1}(\mathcal{O}_{X'})} (f^{-1}(\mathcal{D}_{X'}), F[p], V[\alpha]).$$

It is clear on such a formula that this functor is compatible with grading with respect to $V_\bullet$, as defined in Remark 10.4.6.

10.5.3. Lemma. *The functor* ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$ *maps a filtered quasi-isomorphism in the category* $\mathsf{K}(\mathsf{FV}_i(\mathcal{D}_X))$ *to a filtered quasi-isomorphism.*

Proof. We need to prove that if $(C^\bullet, F, V)$ is a complex of bi-filtered $\mathcal{D}_X$-modules such that all complexes $F_pV_\alpha C^\bullet$ are exact, then $F_pV_\alpha\, {}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet)$ is exact for all $p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$. By factoring f as $X \xrightarrow{j} X \times X' \xrightarrow{p} X'$, where p is the second projection and j is the graph of f, we reduce the proof for f to proving the assertion separately for j and p (note that $\mathcal{D}_{X\to X'} \simeq \mathcal{D}_{X\to X\times X'} \otimes_{j^{-1}(\mathcal{D}_{X\times X'})} j^{-1}(\mathcal{D}_{X\times X'\to X'})$).

The assertion for j is trivial since we may assume that we have coordinates $y_1, \dots, y_r$ on X', so that ${}^{\mathrm{p}}\mathrm{DR}_{X/X\times X'}$ can be identified with $\mathbb{C}[\partial_{y_1}, \dots, \partial_{y_r}] \otimes_{\mathbb{C}} (-)$. Let us prove now the assertion for the projection $p\colon X \times X' \to X'$. For every object $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_{X\times X'})$, consider the complex ${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}, F, V)$ consisting of $\mathcal{M} \otimes_{\mathcal{D}_{X\times X'}} \wedge^{-\bullet}\Theta_X$ (given local coordinates $x_1, \dots, x_n$ on X, this complex can be identified to the Koszul-type complex corresponding to $\partial_{x_1}, \dots, \partial_{x_r}$). The filtrations are defined by

$$F_p(\mathcal{M} \otimes \wedge^{-i}\Theta_X) = F_{p+i}\mathcal{M} \otimes \wedge^{-i}\Theta_X, \quad V_\alpha(\mathcal{M} \otimes \wedge^{-i}\Theta_X) = V_\alpha\mathcal{M} \otimes \wedge^{-i}\Theta_X.$$

Note that the morphism $\mathcal{D}_{X\times X'} \otimes_{\mathcal{O}_X} \wedge^{-\bullet}\Theta_X \to \mathcal{D}_{X\to X'}$ induces a morphism

$${}^{\mathrm{p}}\mathrm{DR}_X(\mathcal{M}, F, V) \longrightarrow (\mathcal{M}, F, V) \otimes_{\mathcal{D}_X} (\mathcal{D}_{X\to X'}, F, V)$$

for every bi-filtered $\mathcal{D}_X$-module $(\mathcal{M}, F, V)$. This is a filtered quasi-isomorphism if $(\mathcal{M}, F, V) \simeq \mathcal{L} \otimes (\mathcal{D}_{X\times X'}, F[p], V[\alpha])$, hence for all induced bi-filtered $\mathcal{D}_{X\times X'}$-modules. Indeed, it is enough to check the assertion when either $\mathcal{L}$ has no t-torsion, or when $\mathcal{L}t = 0$; in each case, the verification is straightforward.

On the other hand, it is clear that if all $F_pV_\alpha C^\bullet$ are exact, then also all complexes $F_pV_\alpha(C^\bullet \otimes \wedge^{-i}\Theta_W)$ are exact, hence each $F_pV_\alpha\, {}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet)$ is exact by the above discussion. This completes the proof of the lemma. $\square$

As a consequence of the lemma, the functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$ we have defined in (10.5.2) induces an exact functor ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}\colon \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_X)) \to \mathsf{D}^*(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'})))$, where $*$ stands for $+$, $-$, b, or the empty set, and this functor is compatible with V-grading.

We now introduce the topological direct image. We first define it at the level of bi-filtered D-modules. Suppose that $f\colon X \to X'$ is as above. If $(\mathcal{M}, F, V)$ is a bi-filtered $f^{-1}(\mathcal{D}_{X'})$-module, we define $f_*(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_{X'})$ to be given by $(\mathcal{N}, F, V)$, where $\mathcal{N} = \bigcup_{p,\alpha} f_*(F_pV_\alpha\mathcal{M})$, with $F_p\mathcal{N} = \bigcup_\alpha f_*(F_pV_\alpha\mathcal{M})$ and $V_\alpha\mathcal{N} = \bigcup_p f_*(F_pV_\alpha\mathcal{M})$. We obtain in this way a functor $f_*\colon \mathsf{FV}(f^{-1}(\mathcal{D}_{X'})) \to \mathsf{FV}(\mathcal{D}_{X'})$. Note that if $\mathcal{L}$ is an $f^{-1}(\mathcal{O}_{X'})$-module, then $f_*(\mathcal{L} \otimes (f^{-1}(\mathcal{D}_{X'}), F[p], V[\alpha])) \simeq f_*(\mathcal{L}) \otimes (\mathcal{D}_{X'}, F[p], V[\alpha])$

by the projection formula (we use the fact that $\mathcal{D}_{X'}$ is a locally free $\mathcal{O}_{X'}$-module). Therefore we also have a functor $f_*\colon \mathsf{FV}_i(f^{-1}(\mathcal{D}_X)) \to \mathsf{FV}_i(\mathcal{D}_{X'})$.

We next define a version of the topological direct image functor at the level of filtered derived categories

$$f_*\colon \mathsf{D}^*(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))) \longrightarrow \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_{X'})),$$

as follows. By a variant of Corollary 10.4.11, we associate functorially to every $(\mathcal{M}, F, V) \in \mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$ a strict complex $\mathcal{C}^\bullet(\mathcal{M}, F, V)$

$$0 \longrightarrow \mathcal{C}^0(\mathcal{M}, F, V) \longrightarrow \mathcal{C}^1(\mathcal{M}, F, V) \longrightarrow \cdots$$

that gives a filtered resolution of $(\mathcal{M}, F, V)$ by induced bi-filtered modules. It is convenient to replace this by a bounded complex, hence if $\dim_{\mathbb{R}}(X) = 2n$, we consider the complex

$$\widetilde{\mathcal{C}}^\bullet(\mathcal{M}, F, V) : \{0 \to \widetilde{\mathcal{C}}^0(\mathcal{M}, F, V) \to \widetilde{\mathcal{C}}^1(\mathcal{M}, F, V) \to \cdots \to \widetilde{\mathcal{C}}^{2n}(\mathcal{M}, F, V) \to 0\},$$

where

$$\widetilde{\mathcal{C}}^j(\mathcal{M}, F, V) = \begin{cases} \mathcal{C}^j(\mathcal{M}, F, V), & 0 \leq j \leqslant 2n-1; \\ \operatorname{Coker}(\mathcal{C}^{2n-2}(\mathcal{M}, F, V) \to \mathcal{C}^{2n-1}(\mathcal{M}, F, V)), & j = 2n; \\ 0, & j \geqslant 2n+1. \end{cases}$$

It follows from the construction that $\widetilde{\mathcal{C}}^\bullet(\mathcal{M}, F, V)$ is a strict complex, giving a filtered resolution of $(\mathcal{M}, F, V)$ by induced bi-filtered $f^{-1}(\mathcal{D}_{X'})$-modules. Moreover, since we truncated at the dimension of X, we have $R^m f_*(F_pV_\alpha\widetilde{\mathcal{C}}^j(\mathcal{M}, F, V)) = 0$ for every $m \geqslant 1$ and every j, p, and α. Given a complex $(\mathcal{M}^\bullet, F, V)$ in $\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))$, we consider the total complex of the double complex $\widetilde{\mathcal{C}}^\bullet(\mathcal{M}^\bullet, F, V)$. It is now standard to see that this induces exact functors $f_*\colon \mathsf{D}^*(\mathsf{FV}_i(f^{-1}(\mathcal{D}_{X'}))) \to \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_{X'}))$, where $*$ stands for $+$, $-$, b, or for the empty set. According to Lemma 10.4.10, the above construction $\widetilde{\mathcal{C}}^\bullet$ is compatible with the V-grading functor of Remark 10.4.6 and, by the exactness of $f*$ on the terms of the complexes $\widetilde{\mathcal{C}}^\bullet$, V-grading commutes with f_* as defined by the previous construction.

By composing f_* and ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$, we obtain an exact functor

$${}_{\mathrm{D}}f_*\colon \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_X)) \longrightarrow \mathsf{D}^*(\mathsf{FV}_i(\mathcal{D}_{X'})),$$

which in light of Lemma 10.4.7 also gives a functor $\mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X)) \to \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_{X'}))$.

It is also clear, by applying the arguments separately to f_* and ${}^{\mathrm{p}}\mathrm{DR}_{X/X'}$, that taking the direct image commutes with taking the graded pieces of the V-filtration. More precisely, if $f_0\colon X_0 \to X_0'$ is the restriction of f, then given any $(\mathcal{M}^\bullet, F, V) \in \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_X))$ and any $\alpha \in \mathbb{R}$, we have an isomorphism in $\mathsf{D}^-(\mathsf{F}(\mathcal{D}_{X_0'}))$:

$$(\mathrm{gr}^V_\alpha\, {}_{\mathrm{D}}f_*(\mathcal{M}^\bullet, F, V), F) \simeq {}_{\mathrm{D}}f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}^\bullet, F, V), F). \tag{10.5.4}$$

We consider two conditions on an object $C^\bullet$ of $\mathsf{C}(\mathsf{FV}(\mathcal{D}_X))$:

(a) The action of $t\partial_t - \alpha$ on $H^i(\mathrm{gr}^V_\alpha C^\bullet)$ is nilpotent for all $i \in \mathbb{Z}$, $\alpha \in \mathbb{R}$,

(b) Each $H^i(F_pV_\alpha C^\bullet)$ is a coherent $\mathcal{O}_X$-module.

Let $\mathsf{C}^*_m(\mathsf{FV}(\mathcal{D}_X))$ and $\mathsf{C}^*_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_X))$ be the full subcategories of $\mathsf{C}^*(\mathsf{FV}(\mathcal{D}_X))$ consisting of those objects that satisfy condition (a), respectively (b), and we similarly define $\mathsf{D}^*_m(\mathsf{FV}(\mathcal{D}_X))$ and $\mathsf{D}^*_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_X))$ as full subcategories of $\mathsf{D}^*(\mathsf{FV}(\mathcal{D}_X))$

10.5.5. Lemma. *With the above notation, suppose also that f is proper and* $(\mathcal{M}, F, V) \in \mathsf{FV}(\mathcal{D}_X)$.

(i) *If* $(\mathcal{M}, F, V) \in \mathsf{C}_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_X))$, *then* $f_*(\mathcal{M}, F, V) \in \mathsf{D}^-_{\mathrm{c}}(\mathsf{FV}_{\mathrm{c}}(\mathcal{D}_{X'}))$.

(ii) *If* $(\mathcal{M}, F, V) \in \mathsf{C}(\mathsf{FV}(\mathcal{D}_X))$, *then* $f_*(\mathcal{M}, F, V) \in \mathsf{D}^-_m(\mathsf{FV}_m(\mathcal{D}_{X'}))$.

Proof. Let $(C^\bullet, F, V) \to (\mathcal{M}, F, V)$ be a filtered resolution by induced bi-filtered $\mathcal{D}_X$-modules constructed as in Remark 10.4.8. If $F_pV_\alpha\mathcal{M}$ is a coherent $\mathcal{O}_X$-module for every p, α, then $F_pV_\alpha C^k$ is a coherent $\mathcal{O}_X$-module for every p, α, and k. One can then deduce that all $H^k(F_pV_\alpha\,{}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet))$ are coherent $f^{-1}(\mathcal{O}_{X'})$-modules, and then that all $H^k(F_pV_\alpha f_*({}^{\mathrm{p}}\mathrm{DR}_{X/X'}(C^\bullet)))$ are coherent $\mathcal{O}_{X'}$-modules.

If the action of $(t\partial_t - \alpha)^m$ on $\mathrm{gr}^V_\alpha(\mathcal{M})$ is zero, then also its action on

$$f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}), F) \simeq \mathrm{gr}^V_\alpha f_*(\mathcal{M}, F, V)$$

is zero, hence the same holds for the action on $H^k(\mathrm{gr}^V_\alpha f_*(\mathcal{M}, F, V))$. $\square$

10.6. Specializability of filtered $\mathcal{D}_X$-modules

We assume that $X = H \times \Delta_t$, where Δ_t is a disc with coordinate t and we set $X_0 = H \times \{0\} \subset X$. We use the notion of a coherent V-filtration for a coherent $\mathcal{D}_X$-module $\mathcal{M}$ as defined in Section 9.3, as well as the notion of $\mathbb{R}$-specializability. Since we are dealing with $\mathcal{D}_X$-modules, the strictness property is not involved.

We now turn to strict $\mathbb{R}$-specializability for filtered $\mathcal{D}_X$-modules. Suppose that $(\mathcal{M}, F)$ is a coherently F-filtered $\mathcal{D}_X$-module (recall that the *coherence* condition means that the $\mathrm{gr}^F(\mathcal{D}_X)$-module $\mathrm{gr}^F\mathcal{M} := \oplus_m F_m\mathcal{M}/F_{m-1}\mathcal{M}$ is coherent, see Exercise 8.61).

10.6.1. Definition. One says that $(\mathcal{M}, F)$ is *strictly $\mathbb{R}$-specializable along H* if $\mathcal{M}$ is strictly $\mathbb{R}$-specializable along H with V-filtration denoted by $V_\bullet\mathcal{M}$ and if $(\mathcal{M}, F_\bullet, V_\bullet)$ belongs to $\mathsf{FV}(\mathcal{D}_X)$.

In other words, arguing as in Proposition 9.3.20, we have

(a) $(F_pV_\alpha\mathcal{M}) \cdot t = F_pV_{\alpha-1}\mathcal{M}$ for all $p \in \mathbb{Z}$ and $\alpha < 0$.

(b) $(F_p\mathrm{gr}^V_\alpha\mathcal{M}) \cdot \partial_t = F_{p+1}\mathrm{gr}^V_{\alpha+1}\mathcal{M}$ for all $p \in \mathbb{Z}$ and $\alpha > -1$.

Furthermore, we say that $(\mathcal{M}, F)$ is *a filtered middle extension along H* $\mathcal{M}$ is a middle extension along H, i.e., the non filtered morphism var resp. can is injective resp. is onto, and moreover if (b) holds also for $\alpha = -1$, that is, the filtered can is onto.

Of course, the inclusions "$\subseteq$" in (a) and (b) always hold for every $\alpha \in \mathbb{R}$. We also note that each $(\mathrm{gr}^V_\alpha\mathcal{M}, F)$ is a filtered $\mathcal{D}_H$-module. The first condition can be called a *regularity condition.* Indeed, for a nonzero holonomic $\mathcal{D}_X$-module $\mathcal{M}$ with irregular singularities, we can have $V_\alpha\mathcal{M} = \mathcal{M}$ for every α (e.g. when $\dim X = 1$,

$\mathcal{M} = \mathcal{D}_X/\mathcal{D}_X(t^2\partial_t + 1))$, and the condition $tF_p\mathcal{M} = F_p\mathcal{M}$ cannot be satisfied by a nonzero coherent $\mathcal{O}_X$-module $F_p\mathcal{M}$.

10.6.2. Remark. As in Remark 7.2.30, the conditions (a) and (b) are respectively equivalent to

(a) for $\alpha < 0$ and any p, $F_pV_\alpha\mathcal{M} = (j_*j^{-1}F_p\mathcal{M}) \cap V_\alpha\mathcal{M}$,

(b) for $\alpha \in (-1, 0]$, $k \geqslant 1$ and any p,

$$F_pV_{\alpha+k}\mathcal{M} = \partial_t^k F_pV_\alpha\mathcal{M} + \sum_{j=0}^{k-1} \partial_t^j F_{p-j}V_0\mathcal{M}.$$

Furthermore, if $(\mathcal{M}, F_\bullet)$ is a filtered middle extension, (b) is replaced with

(c) for $\alpha \in [-1, 0)$, $k \geqslant 1$ and any p,

$$F_pV_{\alpha+k}\mathcal{M} = \partial_t^k F_pV_\alpha\mathcal{M} + \sum_{j=0}^{k-1} \partial_t^j F_{p-j}V_{<0}\mathcal{M}.$$

In particular, $F_p\mathcal{M} = \sum_{j\geqslant 0} \partial_t^j F_{p-j}V_{<0}\mathcal{M}$ and $F_\bullet\mathcal{M}$ is uniquely determined from $j^{-1}F_\bullet\mathcal{M}$.

As above, in the presence of a nonzero $g \in \mathcal{O}(X)$, we consider the graph embedding $\iota_g\colon X \to X \times \mathbb{A}^1_{\mathbb{C}}$. Given a filtered $\mathcal{D}_X$-module $(\mathcal{M}, F)$ on X, we say that $(\mathcal{M}, F)$ is strictly $\mathbb{R}$-specializable along (g) if $\iota_{g*}(\mathcal{M}, F)$ is so along $H \subset X \times \mathbb{A}^1_{\mathbb{C}}$. One can show that if $(g = 0)$ is smooth, then this condition holds if and only if $(\mathcal{M}, F)$ is strictly $\mathbb{R}$-specializable along (g) (see Exercise 9.25).

10.6.3. Lemma. *Let $(\mathcal{M}, F)$ be a coherently F-filtered $\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable along H. Then for each $\alpha \in A + \mathbb{Z}$, $F_\bullet V_\alpha\mathcal{M}$ is a coherent $FV_0\mathcal{D}_X$-filtration of $V_\alpha\mathcal{M}$ and $F_\bullet\mathrm{gr}^V_\alpha\mathcal{M}$ is a coherent F-filtration of $\mathrm{gr}^V_\alpha\mathcal{M}$.*

Proof. We first prove that $F_pV_\alpha\mathcal{M}$ is $\mathcal{O}_X$-coherent. This is a local question, and we can then assume that $V_\alpha\mathcal{M}$ is the union of $\mathcal{O}_X$-coherent submodules. The intersection of each such with $F_p\mathcal{M}$ is coherent, according to Corollary 8.8.7, and their union in $F_p\mathcal{M}$ is also coherent, as wanted. Applying a similar reasoning to $R_FV_\alpha\mathcal{M}$ in $V_\alpha\mathcal{M}[z, z^{-1}]$ gives the coherence of $(V_\alpha\mathcal{M}, F_\bullet)$. Since this holds for each α, it follows that $(\mathrm{gr}^V_\alpha\mathcal{M}, F_\bullet)$ is coherent as a filtered $\mathrm{gr}^V_0\mathcal{D}_X$-module. Using now strict $\mathbb{R}$-specializability implies the coherence as a filtered $\mathcal{D}_X$-module. $\square$

We now come to the main result of this chapter.

10.6.4. Theorem. *Let $f\colon X \to X'$ be a morphism as in Section 10.5 and let $f_0\colon X_0 \to X'_0$ be the restriction of f. Suppose that f is proper and that $(\mathcal{M}, F)$ is a coherently F-filtered $\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable, with V-filtration $V_\bullet\mathcal{M}$. If ${}_{\mathrm{D}}f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}), F)$ is strict for every $\alpha \in \mathbb{R}$, then ${}_{\mathrm{D}}f_*(\mathcal{M}, F, V)$ is strict in a neighborhood of X'_0.*

The strictness assumption means that the natural morphism

$$R^k f_{0*}(F_p\,{}^{\mathrm{p}}\mathrm{DR}_{X_0/X_0'}\,\mathrm{gr}_\alpha^V\mathcal{M}) \longrightarrow R^k f_{0*}\,{}^{\mathrm{p}}\mathrm{DR}_{X_0/X_0'}\,\mathrm{gr}_\alpha^V\mathcal{M} = {}_{\mathrm{D}}f_{0*}^{(k)}\mathrm{gr}_\alpha^V\mathcal{M}$$

is injective for every k, p, α.

The proof of the theorem will be given at the end of Section 10.7. Let us emphasize some consequences of the theorem.

10.6.5. Consequences. Under all assumptions of Theorem 10.6.4, the following holds.

(1) The complex ${}_{\mathrm{D}}f_*(\mathcal{M}, F)$ is strict, i.e., for all k and all $p \in \mathbb{Z}$, the natural morphism $R^k f_*(F_p\,{}^{\mathrm{p}}\mathrm{DR}_{X/X'}\,\mathcal{M}) \to R^k f_*\,{}^{\mathrm{p}}\mathrm{DR}_{X/X'}\,\mathcal{M} = {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ is injective.

(2) For each k, the $\mathcal{D}_{X'}$-module ${}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ is $\mathbb{R}$-specializable along H' and for each $\alpha \in \mathbb{R}$, $\mathrm{gr}_\alpha^V({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}) \simeq {}_{\mathrm{D}}f_{0*}^{(k)}\mathrm{gr}_\alpha^V(\mathcal{M})$.

(3) Moreover, for each $p \in \mathbb{Z}$, $F_p\mathrm{gr}_\alpha^V\,{}_{\mathrm{D}}f_*^{(k)}\mathcal{M} \simeq F_p\,{}_{\mathrm{D}}f_{0*}^{(k)}\mathrm{gr}_\alpha^V\mathcal{M}$.

10.7. A strictness criterion for complexes of filtered $\mathcal{D}$-modules

10.7.a. Setup. Assume that $X = H \times \Delta_t$ and set $X_0 = X \times \{0\}$. We consider a *bounded* complex

$$\cdots \longrightarrow \mathcal{M}^{i-1} \xrightarrow{\ d\ } \mathcal{M}^i \xrightarrow{\ d\ } \mathcal{M}^{i+1} \longrightarrow \cdots$$

of $\mathcal{D}_X$-modules. We set $X = H \times \Delta_t$. We make the following assumptions:

(a) Each $\mathcal{M}^i$ has an increasing filtration $F_\bullet\mathcal{M}^i$ by $\mathcal{O}_X$-submodules, exhaustive, locally bounded below, and compatible with the order filtration on $\mathcal{D}_X$.

(b) Each $\mathcal{M}^i$ has an increasing filtration $V_\bullet\mathcal{M}^i$ by $\mathcal{O}_X$-submodules, discretely indexed by $\mathbb{R}$, on which t and ∂_t act in the usual way.

(c) The differentials $d\colon \mathcal{M}^i \to \mathcal{M}^{i+1}$ respect both filtrations $F_\bullet\mathcal{M}^i$ and $V_\bullet\mathcal{M}^i$.

(d) The $\mathcal{O}_X$-modules $H^i(F_pV_\alpha\mathcal{M}^\bullet)$ are coherent for every $i, p \in \mathbb{Z}$ and $\alpha \in \mathbb{R}$.

(e) The morphism $t\colon F_pV_\alpha\mathcal{M}^i \to F_pV_{\alpha-1}\mathcal{M}^i$ is an isomorphism for $i, p \in \mathbb{Z}$ and $\alpha < 0$.

(f) The morphism $\partial_t\colon F_p\mathrm{gr}_\alpha^V\mathcal{M}^i \to F_{p+1}\mathrm{gr}_{\alpha+1}^V\mathcal{M}^i$ is an isomorphism for $i, p \in \mathbb{Z}$ and $\alpha > -1$.

(g) For every $\alpha \in \mathbb{R}$, the operator $t\partial_t - \alpha$ acts nilpotently on $H^i(\mathrm{gr}_\alpha^V\mathcal{M}^\bullet)$.

(h) For every $\alpha \in [-1, 0]$, the complex $\mathrm{gr}_\alpha^V\mathcal{M}^\bullet$, with the induced differential and the filtration induced by $F_\bullet\mathcal{M}^\bullet$, is strict.

(i) For every $i \in \mathbb{R}$, the Rees module $\bigoplus_{p\in\mathbb{Z}} H^i(F_p\mathcal{M}^\bullet)z^p$ is coherent over $R_F\mathcal{D}_X$.

Let us denote by $(M^\bullet, d)$ the resulting complex of graded modules over the ring $R = \mathbb{C}[z, v]$; here the z-variable goes with the filtration $F_\bullet\mathcal{M}^i$, and the v-variable with the filtration $V_\bullet\mathcal{M}^i$. Since the latter is indexed by $\mathbb{R}$, this needs a little bit of care. Because we are dealing with a bounded complex, we can choose an increasing sequence of real numbers $\alpha_k \in \mathbb{R}$, indexed by $k \in \mathbb{Z}$, such that all the jumps in the filtrations $V_\bullet\mathcal{M}^b$ happen at some α_k; we then define

$$M_{j,k}^i = F_jV_{\alpha_k}\mathcal{M}^i$$

for $i, j, k \in \mathbb{Z}$. This makes each

$$M^i = \bigoplus_{j,k\in\mathbb{Z}} M^i_{j,k}$$

into a $\mathbb{Z}^2$-graded module over the ring R; since the differentials in the original complex are compatible with both filtrations, they induce morphisms of graded R-modules $d\colon M^i \to M^{i+1}$.

10.7.1. Theorem. *The complex $(\mathcal{M}^\bullet, d)$, equipped with the two filtrations $F_\bullet\mathcal{M}^\bullet$ and $V_\bullet\mathcal{M}^\bullet$, is strict on an open neighborhood of X_0.*

In contrast with the analogous proposition 9.8.10, the proof we give here does not use completions.

10.7.b. Proof of Theorem 10.7.1. Note first that each M^i is a flat R-module. Using the above definition of the complex $(M^\bullet, d)$, we clearly have

$$(M^\bullet/vM^\bullet)_{j,k} = \frac{F_j V_{\alpha_k}\mathcal{M}^\bullet}{F_j V_{\alpha_{k-1}}\mathcal{M}^\bullet} = F_j \mathrm{gr}^V_{\alpha_k}\mathcal{M}^\bullet.$$

The condition in (h) has the following interpretation.

10.7.2. Lemma. *All cohomology modules of the complex $(M^\bullet/vM^\bullet, d)$ are flat over the ring $R/vR = \mathbb{C}[z]$.*

Proof. Together with (e) and (f), the condition in (h) says that the complex $\mathrm{gr}^V_\alpha\mathcal{M}^\bullet$ is strict for every $\alpha \in \mathbb{R}$. In terms of graded modules, this means that multiplication by z is injective on the cohomology of the complex $M^\bullet/vM^\bullet$, which is equivalent to flatness over the ring $\mathbb{C}[z]$. $\square$

The next step in the proof involves a local argument, and so we fix a point $x \in X_0$ and localize everything at x. Although we keep the same notation as above, in the remainder of this section, each $\mathcal{M}^i$ is a $\mathcal{D}_{X,x}$-module, the condition in (d) reads $H^i(F_p V_\alpha \mathcal{M}^\bullet)$ is a finitely generated $\mathcal{O}_{X,x}$-module, etc. With this convention in place, consider the short exact sequence of complexes

$$0 \longrightarrow M^\bullet \longrightarrow M^\bullet \longrightarrow M^\bullet/vM^\bullet \longrightarrow 0,$$

in which the morphism from $M^\bullet$ to $M^\bullet$ is multiplication by v. (To keep the notation simple, we are leaving out the change in the grading.) The resulting long exact sequence in cohomology looks like this:

$$\cdots \longrightarrow H^i(M^\bullet) \longrightarrow H^i(M^\bullet) \longrightarrow H^i(M^\bullet/vM^\bullet) \longrightarrow H^{i+1}(M^\bullet) \longrightarrow H^{i+1}M^\bullet \longrightarrow \cdots$$

The following result constitutes the heart of the proof.

10.7.3. Proposition. *The connecting homomorphisms $\delta\colon H^i(M^\bullet/vM^\bullet) \to H^{i+1}(M^\bullet)$ in the long exact sequence are trivial.*

Once we have proved the proposition, we will know that the multiplication morphisms $v\colon H^i(M^\bullet) \to H^i(M^\bullet)$ are injective and that

$$\frac{H^i(M^\bullet)}{vH^i(M^\bullet)} \simeq H^i(M^\bullet/vM^\bullet).$$

Together with Lemma 10.7.2, this will tell us that v, z is a regular sequence on $H^i(M^\bullet)$, which is two thirds of what we need to prove that $H^i(M^\bullet)$ is a flat R-module.

In preparation for the proof, let us consider the graded pieces in a fixed bidegree (j,k) in the long exact sequence; to simplify the notation, set $p = j$ and $\alpha = \alpha_k$. We then have the following commutative diagram with exact rows and columns:

$$\begin{tikzcd}
 & H^{i+1}(F_p V_\beta \mathcal{M}^\bullet) \arrow[d] & \\
H^i(F_p \mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \arrow[r, "\delta"] \arrow[dr, "\varepsilon"'] & H^{i+1}(F_p V_{<\alpha} \mathcal{M}^\bullet) \arrow[r] \arrow[d] & H^{i+1}(F_p V_\alpha \mathcal{M}^\bullet) \\
 & H^{i+1}(F_p V_{(\beta,\alpha)} \mathcal{M}^\bullet) &
\end{tikzcd}$$

Here $\beta < \alpha$, and the notation $V_{(\beta,\alpha)}\mathcal{M}^\bullet$ is an abbreviation for $V_{<\alpha}\mathcal{M}^\bullet/V_\beta\mathcal{M}^\bullet$. We observe that the morphism ε is trivial because the source and the target have different "weights" with respect to the action of the operator $t\partial_t$.

10.7.4. Lemma. *With notation as above, the morphism*

$$\varepsilon\colon H^i(F_p \mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \longrightarrow H^{i+1}(F_p V_{(\beta,\alpha)} \mathcal{M}^\bullet)$$

is trivial.

Proof. We have a commutative diagram

$$\begin{tikzcd}
H^i(F_p \mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \arrow[r, "\varepsilon"] \arrow[d, hook] & H^{i+1}(F_p V_{(\beta,\alpha)} \mathcal{M}^\bullet) \arrow[d, hook] \\
H^i(\mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \arrow[r] & H^{i+1}(V_{(\beta,\alpha)} \mathcal{M}^\bullet)
\end{tikzcd}$$

in which the two vertical morphisms are injective because of (h). Now the operator $t\partial_t$ acts on the $\mathcal{O}_X$-module in the lower left corner with α as its only eigenvalue, and on the $\mathcal{O}_X$-module in the lower right corner with eigenvalues contained in the interval (β, α); this is a consequence of (g). Since the bottom arrow is compatible with the action of $t\partial_t$, it must be zero; but then ε is also zero. $\square$

We conclude from the lemma that the image of

$$\delta\colon H^i(F_p \mathrm{gr}_\alpha^V \mathcal{M}^\bullet) \to H^{i+1}(F_p V_{<\alpha} \mathcal{M}^\bullet)$$

is contained in the intersection

$$\bigcap_{\beta<\alpha} \mathrm{Im}\Big(H^{i+1}(F_p V_\beta \mathcal{M}^\bullet) \to H^{i+1}(F_p V_{<\alpha} \mathcal{M}^\bullet)\Big)$$

We can now use (e) and Krull's intersection theorem to prove that this intersection is trivial (in the local ring $\mathcal{O}_{X,x}$).

10.7.5. Lemma. *We have*

$$\bigcap_{\beta<\alpha} \operatorname{Im}\Big(H^i(F_pV_\beta\mathcal{M}^\bullet) \to H^i(F_pV_\alpha\mathcal{M}^\bullet)\Big) = \{0\}.$$

Proof. Consider the following commutative diagram:

$$\begin{tikzcd}
F_pV_\beta\mathcal{M}^{i-1} \arrow[r, "d"] \arrow[d, hook] & F_pV_\beta\mathcal{M}^{i} \arrow[r, "d"] \arrow[d, hook] & F_pV_\beta\mathcal{M}^{i+1} \arrow[d, hook] \\
F_pV_\alpha\mathcal{M}^{i-1} \arrow[r, "d"] & F_pV_\alpha\mathcal{M}^{i} \arrow[r, "d"] & F_pV_\alpha\mathcal{M}^{i+1}
\end{tikzcd}$$

Suppose that we have an element $m \in F_pV_\alpha\mathcal{M}^i$ with $dm = 0$ that belongs to the image of $H^i(F_pV_\beta\mathcal{M}^\bullet)$. Then

$$m = dm_0 + m_1$$

for some $m_0 \in F_pV_\alpha\mathcal{M}^{i-1}$ and some $m_1 \in F_pV_\beta\mathcal{M}^i$. Now if $\beta < -1$, then by (e), we have $m_1 = m_2t$ for a unique $m_2 \in F_pV_{\beta+1}\mathcal{M}^i$. Since multiplication by t is injective on $F_pV_{\beta+1}\mathcal{M}^{i+1}$, the fact that $dm_1 = 0$ implies that $dm_2 = 0$. As long as $\beta + 1 \leqslant \alpha$, we also have

$$m_2t \in (F_pV_{\beta+1}\mathcal{M}^i) \cdot t \subseteq (F_pV_\alpha\mathcal{M}^i) \cdot t,$$

and therefore $m \in d(F_pV_\alpha\mathcal{M}^{i-1}) + (F_pV_\alpha\mathcal{M}^i) \cdot t$. By this type of argument, one shows more generally that

$$\bigcap_{\beta<\alpha} \operatorname{Im}\Big(H^i(F_pV_\beta\mathcal{M}^\bullet) \to H^i(F_pV_\alpha\mathcal{M}^\bullet)\Big) \subseteq \bigcap_{m\in\mathbb{N}} H^i(F_pV_\alpha\mathcal{M}^\bullet) \cdot t^m.$$

Since $H^i(F_pV_\alpha\mathcal{M}^\bullet)$ is finitely generated as an $\mathcal{O}_{X,x}$-module by (d), Krull's intersection theorem implies that the right-hand side is equal to zero. □

The conclusion is that $\delta = 0$, and hence that v, z form a regular sequence on $H^i(M^\bullet)$. Together with the following result, this proves that $H^i(M^\bullet)$ is flat as an R-module.

10.7.6. Lemma. *The morphism $z\colon H^i(M^\bullet) \to H^i(M^\bullet)$ is injective.*

Proof. Since v, z form a regular sequence on $H^i(M^\bullet)$, the corresponding Koszul complex is exact. By the same argument as in the proof of Proposition 10.3.17, every element in the kernel of $z\colon H^i(M^\bullet) \to H^i(M^\bullet)$ can be written as v times another element in the kernel; consequently,

$$\operatorname{Ker}\Big(z\colon H^i(M^\bullet) \to H^i(M^\bullet)\Big) \subseteq \bigcap_{m\geqslant 1} v^m H^i(M^\bullet).$$

Looking at a fixed bidegree (j,k) and setting $p = j$ and $\alpha = \alpha_k$ as above, the right-hand side equals

$$\bigcap_{\beta<\alpha} \operatorname{Im}\Big(H^i(F_pV_\beta\mathcal{M}^\bullet) \to H^i(F_pV_\alpha\mathcal{M}^\bullet)\Big),$$

which is equal to zero by Lemma 10.7.5. □

In summary, we have shown that for every point $x \in X_0$, the localization of the complex $(\mathcal{M}^\bullet, d)$ is strict (as a complex of $\mathcal{D}_{X,x}$-modules with two filtrations). Now it remains to prove that the complex $(\mathcal{M}^\bullet, d)$ is strict on an open neighborhood of X_0, using the coherence condition in (i). This will end the proof of Theorem 10.7.1.

10.7.7. Lemma. *If $(\mathcal{M}^\bullet, F, V)$ is a complex of bi-filtered $\mathcal{D}_X$-modules whose restriction to X_0 is strict and which satisfies the following two conditions:*

(1) *for every j, the $R_F\mathcal{D}_X$-module $\oplus_{p\in\mathbb{Z}} H^j(F_p\mathcal{M}^\bullet)z^p$ is coherent;*
(2) *we have $H^j(F_p\mathcal{M}^\bullet) = 0$ for $|j| \gg 0$ and all p.*

Then $(\mathcal{M}^\bullet, F, V)$ is strict in a neighborhood of X_0.

Proof. Note that over $X \smallsetminus X_0$ we have $V_\alpha\mathcal{D}_X = \mathcal{D}_X$ for every α. Since $\bigcup_\alpha V_\alpha\mathcal{M} = \mathcal{M}$, it is easy to deduce that over this open subset, $V_\alpha\mathcal{M} = \mathcal{M}$ for every α. Therefore $(\mathcal{M}^\bullet, F, V)$ is strict over an open subset $U \subseteq X \smallsetminus X_0$ if and only if $(\mathcal{M}^\bullet, F)$ is strict over U.

By assumption, $(\mathcal{M}^\bullet, F, V)$ is strict at the points $x \in X_0$, hence in order to complete the proof of the lemma, it is enough to show that if $(\mathcal{M}^\bullet, F)$ is strict at a point $x \in X$, then it is strict in an open neighborhood of x. Since the F-filtration on $\mathcal{M}^\bullet$ is exhaustive, it follows from Exercise 10.5 that $(\mathcal{M}^\bullet, F)$ is strict at $x \in X$ if and only if the natural map $H^j(F_p\mathcal{M}^\bullet)_x \to H^j(F_{p+1}\mathcal{M}^\bullet)_x$ is injective for all p and j. For every j, consider the coherent $\mathcal{B}_X$-module $\mathcal{M}_j := \oplus_{p\in\mathbb{Z}} H^j(F_p\mathcal{M}^\bullet)$. We see that $(\mathcal{M}^\bullet, F)$ is strict at $x \in X$ if and only if the map $u_j\colon \mathcal{M}_j \to \mathcal{M}_j$ given by multiplication with z is injective for all j. Furthermore, by (2) we only need to consider finitely many j. Since $\mathcal{M}_j$ is a coherent $\mathcal{B}_X$-module, it follows that $\operatorname{Ker}(u_j)$ is a coherent $R_F\mathcal{D}_X$-module. In the neighborhood of a given point $x \in X$, we have a finite set of generators $s_1, \dots, s_r$ of $\operatorname{Ker}(u_j)$ over $R_F\mathcal{D}_X$. If all the s_i vanish at x, then they also vanish in an open neighborhood of x and u_j is injective in this neighborhood. Since we can argue in this way simultaneously for finitely many j, this concludes the proof of the lemma. □

10.7.c. Proof of Theorem 10.6.4. We will apply Theorem 10.7.1 to the bounded complex ${}_{\mathrm{D}}f_*(\mathcal{M}, F, V)$. We first check that the conditions (a)–(i) of Section 10.7.a are fulfilled.

Since $(\mathcal{M}, F, V)$ is an object of $\mathsf{FV}(\mathcal{D}_X)$, an application of Lemma 10.5.5 gives that ${}_{\mathrm{D}}f_*(\mathcal{M}, F, V) \in \mathsf{D}^-(\mathsf{FV}(\mathcal{D}_{X'}))$. Moreover, by hypothesis we have that

$$(\mathrm{gr}^V_\alpha\, {}_{\mathrm{D}}f_*(\mathcal{M}^\bullet, F, V), F) \simeq {}_{\mathrm{D}}f_{0*}(\mathrm{gr}^V_\alpha(\mathcal{M}^\bullet, F, V), F)$$

is strict (the isomorphism is given by (10.5.4)). On the other hand, since $(\mathcal{M}, F)$ is coherent, $(\mathcal{M}, F, V) \in \mathsf{FV}_{\mathrm{c}}(\mathcal{D}_X)$. Therefore another application of Lemma 10.5.5 implies that ${}_{\mathrm{D}}f_*(\mathcal{M}, F, V) \in \mathsf{D}^-_{\mathrm{c}}(\mathsf{FV}(\mathcal{D}_{X'}))$. As a consequence, the conditions (a)–(h) are thus fulfilled by ${}_{\mathrm{D}}f_*(\mathcal{M}, F, V)$. Last, the coherence condition (i) follows from the coherence theorem 8.8.24. Therefore Theorem 10.7.1 implies that ${}_{\mathrm{D}}f_*(\mathcal{M}, F, V)$ is strict in a neighborhood of X'_0. □

10.8. Strictness of strictly $\mathbb{R}$-specializable $R_F\mathcal{D}_X$-modules

In this section we compare the notion of specializability for filtered $\mathcal{D}_X$-modules, as developed in this chapter, and that for a strict $R_F\mathcal{D}_X$-module, as considered in Chapter 9 (see Definition 9.3.14).

Let $\widetilde{\mathcal{M}}$ be a (right) coherent graded $R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable along H and let $V_\bullet\widetilde{\mathcal{M}}$ denotes its Kashiwara-Malgrange filtration. Then $\widetilde{\mathcal{M}}$ is strict if and only if $V_\alpha\widetilde{\mathcal{M}}$ is strict for some α, since all $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}}$ are assumed to be strict. The former property is equivalent to the existence of a coherent $\mathcal{D}_X$-module $\mathcal{M}$ equipped with a coherent F-filtration $F_\bullet\mathcal{M}$ such that $\widetilde{\mathcal{M}} = R_F\mathcal{M}$, while the latter is equivalent to the existence of a coherent $V_0\mathcal{D}_X$-module $V_\alpha\mathcal{M}$ equipped with a coherent F-filtration $F_\bullet V_\alpha\mathcal{M}$ such that $V_\alpha\widetilde{\mathcal{M}} = R_F V_\alpha\mathcal{M}$.

10.8.1. Lemma. *Let $\widetilde{\mathcal{M}}$ be as above. If $\widetilde{\mathcal{M}}$ is strict, then the Kashiwara-Malgrange filtration of $\widetilde{\mathcal{M}}$ satisfies*

$$V_\alpha\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}} \cap (V_\alpha\widetilde{\mathcal{M}}[z^{-1}]), \tag{10.8.1 $*$}$$

where the intersection takes place in $\widetilde{\mathcal{M}}[z^{-1}]$.

Proof. For $\gamma > \alpha$, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & V_\alpha\widetilde{\mathcal{M}} \arrow[r, hook] \arrow[d] & V_\gamma\widetilde{\mathcal{M}} \arrow[r] \arrow[d] & V_\gamma\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & V_\alpha\widetilde{\mathcal{M}}[z^{-1}] \arrow[r, hook] & V_\gamma\widetilde{\mathcal{M}}[z^{-1}] \arrow[r] & (V_\gamma\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}})[z^{-1}] \arrow[r] & 0
\end{tikzcd}$$

The upper horizontal line is clearly exact, and the lower one is so because $\mathbb{C}[z, z^{-1}]$ is flat over $\mathbb{C}[z]$. The first two vertical maps are injective since $\widetilde{\mathcal{M}}$ is strict. The third vertical map is injective since $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable. It follows that $V_\alpha\widetilde{\mathcal{M}} = V_\gamma\widetilde{\mathcal{M}} \cap V_\alpha\widetilde{\mathcal{M}}[z^{-1}]$ in $\widetilde{\mathcal{M}}[z^{-1}]$. Taking the limit for $\gamma \to \infty$ gives the assertion. $\square$

Assume moreover that $\widetilde{\mathcal{M}}$ is strict and let $(\mathcal{M}, F_\bullet\mathcal{M})$ be the coherently F-filtered $\mathcal{D}_X$-module such that $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. Then $\mathcal{M}$ is $\mathbb{R}$-specializable and we have (see Exercise 9.27):

$$V_\alpha\mathcal{M} = V_\alpha\widetilde{\mathcal{M}}/(z-1)V_\alpha\widetilde{\mathcal{M}} \quad \text{and} \quad V_\alpha\widetilde{\mathcal{M}}[z^{-1}] = V_\alpha\mathcal{M}[z, z^{-1}].$$

Consider on $\mathcal{M}$ the bi-filtration $F_pV_\alpha\mathcal{M} := F_p\mathcal{M} \cap V_\alpha\mathcal{M}$. Then (10.8.1 $*$) means that the filtration $U_\bullet\widetilde{\mathcal{M}}$ defined by $U_\alpha\widetilde{\mathcal{M}} := \bigoplus_p (F_pV_\alpha\mathcal{M})z^p$ satisfies the properties of the Kashiwara-Malgrange filtration of a strictly $\mathbb{R}$-specializable $R_F\mathcal{D}_X$-module. In particular we get, according to 9.3.20(a) and (d):

(a) $\forall\, p$ and $\forall\, \alpha < 0$, $t : F_pV_\alpha\mathcal{M} \to F_pV_{\alpha-1}\mathcal{M}$ is an isomorphism,

(b) $\forall\, p$ and $\forall\, \alpha > -1$, $\partial_t : F_p\mathrm{gr}^V_\alpha\mathcal{M} \to F_{p+1}\mathrm{gr}^V_{\alpha+1}\mathcal{M}$ is an isomorphism.

In other words, $(\mathcal{M}, F_\bullet, V_\bullet)$ is an object of $\mathsf{FV}(\mathcal{D}_X)$ (see Definition 10.6.1).

10.8.2. Remark. Due to the coherence of each $F_p\mathcal{M}$, the property (a) is equivalent to

(a′) $\forall\, p$ and $\forall\, \alpha < 0$, $F_pV_\alpha\mathcal{M} = (j_*j^{-1}F_p\mathcal{M}) \cap V_\alpha\mathcal{M}$, where $j : X \smallsetminus H \hookrightarrow X$ denotes the open inclusion.

Indeed, let us check the nontrivial implication (a) $\Rightarrow$ (a$'$). The inclusion $\subset$ is clear and it is enough to check the inclusion $\supset$. For a local section m of $\mathcal{M}$, there exists q such that m is a local section of $F_q\mathcal{M}$. If the restriction of m to $X \smallsetminus H$ is a local section of $F_p\mathcal{M}$ for some $p < q$, there exists $k \geqslant 1$ such that $m \cdot t^k$ is a local section of $F_p\mathcal{M}$. Therefore, if m is a local section of $(j_*j^{-1}F_p\mathcal{M}) \cap V_\alpha\mathcal{M}$, $m \cdot t^k$ is a local section of $F_p\mathcal{M} \cap V_{\alpha-k}\mathcal{M} = (F_p\mathcal{M} \cap V_\alpha\mathcal{M}) \cdot t^k$. Since $t^k : V_\alpha\mathcal{M} \to V_{\alpha-k}\mathcal{M}$ is bijective for $\alpha < 0$, the conclusion follows.

10.8.3. Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is $\mathbb{R}$-specializable along H, equipped with a coherent F-filtration. The following properties are equivalent:*

(1) *$R_F\mathcal{M}$ is strictly $\mathbb{R}$-specializable along H,*

(2) *$(\mathcal{M}, F_\bullet, V_\bullet)$ is an object of $\mathsf{FV}(\mathcal{D}_X)$.*

Moreover, when these conditions are fulfilled, the filtration $F_\bullet\mathcal{M}$ induces in some neighbourhood of H on each $V_\alpha\mathcal{M}$ a coherent $F_\bullet\mathcal{D}_{X/\mathbb{C}}$-filtration with respect to any local reduced equation $t : X \to \mathbb{C}$ of H, i.e., each $V_\alpha R_F\mathcal{M} = R_F V_\alpha\mathcal{M}$ is $R_F\mathcal{D}_{X/\mathbb{C}}$-coherent in some neighbourhood of H.

Proof. We have already seen that (1) implies (2). Conversely, let us assume (2) and let us set

$$U_\alpha R_F\mathcal{M} = \bigoplus_p (F_p V_\alpha\mathcal{M}) z^p.$$

For a local section mz^p of $(F_pV_\alpha\mathcal{M})z^p$, we have $m(t\partial_t - \alpha z)^{\nu_m} z^p \in (F_{p+\nu_m} V_{<\alpha}\mathcal{M}) z^{p+\nu_m}$, showing the $\mathbb{R}$-specializability of $R_F\mathcal{M}$ and the fact that $U_\alpha R_F\mathcal{M} \subset V_\alpha R_F\mathcal{M}$. It is enough to show that $U_\bullet\mathcal{M}$ is a coherent filtration indexed by $A + \mathbb{Z}$, since we obviously have $\mathrm{gr}^U_\alpha R_F\mathcal{M} = R_F \mathrm{gr}^V_\alpha\mathcal{M}$, hence the strictness. According to 10.6.1(a) and (b), it is enough to show the $V_0R_F\mathcal{D}_X$-coherence of $U_\alpha R_F\mathcal{M}$ for $\alpha \in [-1, 0)$. For a local reduced equation $t : X \to \mathbb{C}$ of H, we will more precisely show the $R_F\mathcal{D}_{X/\mathbb{C}}$-coherence of $U_\alpha R_F\mathcal{M}$ in some neighbourhood of H, showing both the reverse implication (2) $\Rightarrow$ (1) and the last part of the proposition.

We have already seen (see Lemma 10.6.3) that each $F_pV_\alpha\mathcal{M}$ is $\mathcal{O}_X$-coherent. It is thus enough to show that, locally on X, there exists p_o such that $(F_{p_o}V_\alpha\mathcal{M}) \cdot F_p\mathcal{D}_{X/\mathbb{C}} = F_{p+p_o}V_\alpha\mathcal{M}$ for all $p \geqslant 0$. Since $\mathrm{E} - \alpha$ is nilpotent on $\mathrm{gr}^V_\alpha\mathcal{M}$, the filtration $F_\bullet \mathrm{gr}^V_\alpha\mathcal{M}$, being $F_\bullet\mathrm{gr}^V_0\mathcal{D}_X$-coherent for every α, is also $F_\bullet\mathcal{D}_H$-coherent. The same argument applies to the induced filtration $(F_\bullet V_\alpha\mathcal{M})/(F_\bullet V_{\alpha-1}\mathcal{M})$ and therefore there exists locally p_o such that

$$\big[(F_{p_o}V_\alpha\mathcal{M})/(F_{p_o}V_{\alpha-1}\mathcal{M})\big] \cdot F_p\mathcal{D}_H = (F_{p+p_o}V_\alpha\mathcal{M})/(F_{p+p_o}V_{\alpha-1}\mathcal{M}).$$

Let us set $U_{\alpha,p} = (F_{p_o}V_\alpha\mathcal{M}) \cdot F_p\mathcal{D}_{X/\mathbb{C}}$. By 10.6.1(a) and since α has been chosen in $[-1, 0)$, the left-hand term above can be written as $U_{\alpha,p}/U_{\alpha,p}t$, while the right-hand term is

$$(F_{p+p_o}V_\alpha\mathcal{M})/(F_{p+p_o}V_\alpha\mathcal{M})t,$$

so Nakayama's lemma implies finally $(F_{p_o}V_\alpha\mathcal{M}) \cdot F_p\mathcal{D}_{X/\mathbb{C}} = F_{p+p_o}V_\alpha\mathcal{M}$ in some neighbourhood of H, as wanted. $\square$

10.8.4. Corollary. *Let $\widetilde{\mathcal{M}}$ be a coherent graded $R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable along H. Then $\widetilde{\mathcal{M}}$ strict in some neighbourhood of H if and only if, for some $\alpha < 0$ and all p, the p-th graded component $(V_\alpha\widetilde{\mathcal{M}})_p$ is $\mathcal{O}_X$-coherent. In such a case, the properties of Proposition* 10.8.3 *hold true and in particular, $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ and $(V_\alpha\widetilde{\mathcal{M}})_p = F_p\mathcal{M}\cap V_\alpha\mathcal{M}$ for every α, p, where $\mathcal{M} := \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is a coherent $\mathcal{D}_X$-module which is $\mathbb{R}$-specializable along H and $F_\bullet\mathcal{M}$ is a coherent F-filtration of $\mathcal{M}$.*

Proof. If $\widetilde{\mathcal{M}}$ is strict, we can write $\widetilde{\mathcal{M}} = R_F\mathcal{M}$ for some coherent F-filtration on $\mathcal{M} := \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$, and we have, according to Proposition 10.8.3, $(V_\alpha\widetilde{\mathcal{M}})_p = F_p\mathcal{M}\cap V_\alpha\mathcal{M}$, which is $\mathcal{O}_X$-coherent as we have seen in the proof of Proposition 10.8.3.

Conversely, since $\widetilde{\mathcal{M}}$ is assumed to be strictly $\mathbb{R}$-specializable, each $\mathrm{gr}^V_\gamma\widetilde{\mathcal{M}}$ is strict, and it is enough to prove that $V_\alpha\widetilde{\mathcal{M}}$ is strict for some $\alpha < 0$. For such an α, $V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j$ is also strict for every $j \geqslant 1$. By left exactness of $\varprojlim_j$, we deduce that $\varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)$ is also strict. It is thus enough to show that the natural morphism $V_\alpha\widetilde{\mathcal{M}} \to \varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)$ is injective.

We choose $\alpha < 0$ as given by the assumption of the proposition, and we have $(V_\alpha\widetilde{\mathcal{M}})_pt^j = (V_{\alpha-j}\widetilde{\mathcal{M}})_p$ for $j \geqslant 0$ and any p, due to 9.3.20(a). Then

$$\big(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j\big)_p = (V_\alpha\widetilde{\mathcal{M}})_p/(V_\alpha\widetilde{\mathcal{M}})_pt^j$$

for every $j \geqslant 0$ and p, and therefore

$$\big(\varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)\big)_p = \varprojlim_j\big((V_\alpha\widetilde{\mathcal{M}})_p/(V_\alpha\widetilde{\mathcal{M}})_pt^j\big).$$

Since $(V_\alpha\widetilde{\mathcal{M}})_p$ is $\mathcal{O}_X$-coherent, $\varprojlim_j(V_\alpha\widetilde{\mathcal{M}})_p/(V_\alpha\widetilde{\mathcal{M}})_pt^j = \mathcal{O}_{\widehat{X|H}}\otimes_{\mathcal{O}_{X|H}}(V_\alpha\widetilde{\mathcal{M}})_p$ and the natural morphism $(V_\alpha\widetilde{\mathcal{M}})_{p|H} \to \varprojlim_j(V_\alpha\widetilde{\mathcal{M}})_p/(V_\alpha\widetilde{\mathcal{M}})_pt^j$ is injective. It follows that

$$(V_\alpha\widetilde{\mathcal{M}})_{p|H} \longrightarrow \big(\varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)\big)_p$$

is injective for every p, and thus so is $(V_\alpha\widetilde{\mathcal{M}})_{|H} \to \big(\varprojlim_j(V_\alpha\widetilde{\mathcal{M}}/V_\alpha\widetilde{\mathcal{M}}t^j)\big)$, as wanted. $\square$

***10.8.5. Corollary* (Regularity).** *Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be strict and strictly $\mathbb{R}$-specializable along g. Then* $\operatorname{Supp}\widetilde{\mathcal{M}} \subset g^{-1}(0)$ *if and only if $\psi_{g,\lambda}\widetilde{\mathcal{M}} = 0$ for all $\lambda \in \mathbb{S}^1$. If moreover $\phi_{g,1}\widetilde{\mathcal{M}} = 0$, then $\widetilde{\mathcal{M}} = 0$.*

Proof. Let $\iota_g : X \hookrightarrow X\times\mathbb{C}_t$ be the inclusion of the graph of g. The properties hold for $\widetilde{\mathcal{M}}$ and g if and only if they hold for ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$ and t. We can thus assume that we are in the setting of Corollary 10.8.4. The direction $\Rightarrow$ is clear, since the assumption implies that each local section of $\widetilde{\mathcal{M}}$ is annihilated by some power of t, and strict $\mathbb{R}$-specializability implies then that $V_{<0}\widetilde{\mathcal{M}} = 0$. For the direction $\Leftarrow$, we note that the assumption implies that the filtration $V_\alpha\widetilde{\mathcal{M}}$ is constant for $\alpha < 0$, hence so is the filtration $F_p\mathcal{M}\cap V_\alpha\mathcal{M}$ (p being fixed). Since this is a coherent $\mathcal{O}_X$-module and t induces an isomorphism on it by strict $\mathbb{R}$-specializability, we conclude by Nakayama that it is zero.

The remaining assertion on $\phi_{g,1}\widetilde{\mathcal{M}}$ is then clear. $\square$

10.8.6. Corollary **(Complement to Corollary 9.3.30).** *Assume that* $\widetilde{\mathcal{M}},\widetilde{\mathcal{N}}$ *are strictly* $\mathbb{R}$*-specializable along* (g) *and strict. If* $\varphi:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{N}}$ *is strictly* $\mathbb{R}$*-specializable along* (g)*, then* φ *is strict in some neighbourhood of* $g^{-1}(0)$*.*

Proof. It is a matter of proving strictness of $\operatorname{Coker}\varphi$. As in the corollary above, we can assume that we are in the setting of Corollary 10.8.4. By Corollary 9.3.30, we have $(V_\alpha(\operatorname{Coker}\varphi))_p=\operatorname{Coker}\varphi_{|(V_\alpha\widetilde{\mathcal{M}})_p}$, hence is $\mathcal{O}_X$-coherent. ☐

We can now add the strictness property in Theorem 9.8.8, obtaining thus a complete analogue of Theorem 10.6.4.

10.8.7. Corollary. *With the notation and assumptions of Theorem* 9.8.8*,*

(4) *if* $\widetilde{\mathcal{M}}$ *is strict in the neighbourhood of* H*, then* ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ *is strict in the neighbourhood of* H'*.*

Proof. We replace X' by a suitable neighbourhood of H' and X by the pullback of this neighbourhood, so that $\widetilde{\mathcal{M}}$ is strict on X. By Corollary 10.8.4 it is enough to show the $\mathcal{O}_X$-coherence of $U_\alpha({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})_p=({}_{\mathrm{D}}f_*^{(i)}V_\alpha\widetilde{\mathcal{M}})_p$ for some $\alpha<0$ and each p,i, where the equality holds according to 9.8.8(1).

If $f:X=X'\times Z\to X'$ is a projection, we have, in a way similar to Exercise 8.48(6), ${}_{\mathrm{D}}f_*V_\alpha\widetilde{\mathcal{M}}=\boldsymbol{R}f_*(V_\alpha\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathcal{O}}_X}\wedge^{-\bullet}\widetilde{\Theta}_{X/X'})$, and $({}_{\mathrm{D}}f_*^{(i)}V_\alpha\widetilde{\mathcal{M}})_p$ is the i-th cohomology of the relative Spencer complex ($m=\dim X/X'$)

$$\boldsymbol{R}f_*\Big(0\to(V_\alpha\widetilde{\mathcal{M}})_{p-m}\otimes\wedge^m\Theta_{X/X'}\to\cdots\to(V_\alpha\widetilde{\mathcal{M}})_{p-1}\otimes\Theta_{X/X'}\to(V_\alpha\widetilde{\mathcal{M}})_p\to0\Big)$$

whose differentials are $\mathcal{O}_{X'}$-linear. Since each term of the complex is $\mathcal{O}_X$-coherent by our assumption of strictness of $\widetilde{\mathcal{M}}$ and since f is proper, Grauert's coherence theorem together with a standard spectral sequence argument in the category of $\mathcal{O}_{X'}$-complexes show that $({}_{\mathrm{D}}f_*^{(i)}V_\alpha\widetilde{\mathcal{M}})_p$ is $\mathcal{O}_{X'}$-coherent.

If $f:X\hookrightarrow X'$ is a closed immersion, it is locally of the form $(t,x_2,\dots,x_n)\mapsto(t,x_2,\dots,x_n,0,\dots,0)$. Then

$${}_{\mathrm{D}}f_*V_\alpha\widetilde{\mathcal{M}}={}_{\mathrm{D}}f_*^{(0)}V_\alpha\widetilde{\mathcal{M}}=f_*V_\alpha\widetilde{\mathcal{M}}[\widetilde{\partial}_{x'_1},\dots,\widetilde{\partial}_{x'_m}]$$

and

$$({}_{\mathrm{D}}f_*^{(0)}V_\alpha\widetilde{\mathcal{M}})_p=\sum_{|a|\leqslant p}f_*(V_\alpha\widetilde{\mathcal{M}})_{p-|a|}\widetilde{\partial}_{x'}^a,$$

which is $\widetilde{\mathcal{O}}_{X'}$-coherent since $(V_\alpha\widetilde{\mathcal{M}})_q=0$ for $q\ll0$ locally (use that $(V_\alpha\widetilde{\mathcal{M}})_q=F_q(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})\cap V_\alpha(\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})$ according to Corollary 10.8.4, and apply Exercise 8.62(3)). ☐

10.8.8. Corollary. *With the notation and assumptions of Corollary* 9.8.9*, if* $\widetilde{\mathcal{M}}$ *is strict in the neighbourhood of* $g^{-1}(0)$*, so is* ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ *in the neighbourhood of* $g'^{-1}(0)$*.* ☐

10.8.a. Holonomicity and specialization. Let us end this chapter with the behaviour of holonomicity by specialization. If $\mathcal{M}$ is a holonomic $\mathcal{D}_X$-module, it is known ([**Kas78**]) that $\mathcal{M}$ is specializable (but possibly not $\mathbb{R}$-specializable) along each hypersurface and that nearby and vanishing cycles of $\mathcal{M}$ with respect to any holomorphic function g are holonomic.

Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module and set $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. Recall that $\widetilde{\mathcal{M}}$ is holonomic iff $\mathcal{M}$ is holonomic (see Remark 8.8.16). The following result is thus a straightforward consequence of the above properties.

10.8.9. Corollary. *Assume that $\widetilde{\mathcal{M}}$ is holonomic, strict, and strict $\mathbb{R}$-specializability along (g). Then, $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ ($\lambda \in \mathbb{S}^1$) and $\phi_{g,1}\widetilde{\mathcal{M}}$ are holonomic and strict.* $\square$

10.9. Exercises

Exercise 10.1. Show that a complex $(C^\bullet, F)$ which is *bounded from above* is strict if and only if the associated Rees complex $R_FC^\bullet$ is strict in the sense of Definition 5.1.6.

Exercise 10.2. Show that $(C^\bullet, F)$ is strict if and only if the canonical morphism $H^i(F_kC^\bullet) \to H^i(C^\bullet)$ is a monomorphism for all $k, i \in \mathbb{Z}$.

Exercise 10.3. By considering the long exact sequence in cohomology for the exact sequence
$$0 \longrightarrow F_kC^\bullet \longrightarrow C^\bullet \longrightarrow C^\bullet/F_kC^\bullet \longrightarrow 0,$$
show that if $(C^\bullet, F)$ is strict, then for every i and k we have a short exact sequence
$$0 \longrightarrow H^i(F_kC^\bullet) \longrightarrow H^i(C^\bullet) \longrightarrow H^i(C^\bullet/F_kC^\bullet) \longrightarrow 0.$$
Furthermore, show also that the map $H^i(F_kC^\bullet) \to H^i(F_\ell C^\bullet)$ is a monomorphism for every $k<\ell$, by considering the long exact sequence in cohomology corresponding to
$$0 \longrightarrow F_kC^\bullet \longrightarrow F_\ell C^\bullet \longrightarrow F_\ell C^\bullet/F_kC^\bullet \longrightarrow 0,$$
and obtain a short exact sequence
$$0 \longrightarrow H^i(F_kC^\bullet) \longrightarrow H^i(F_\ell C^\bullet) \longrightarrow H^i(F_\ell C^\bullet/F_kC^\bullet) \longrightarrow 0$$
for every $i \in \mathbb{Z}$.

Exercise 10.4. Show that if $(C^\bullet, F)$ is a strict complex, then for every $k \in \mathbb{Z}$, the complexes $(F_kC^\bullet, F)$ and $(C^\bullet/F_kC^\bullet, F)$, with the induced filtrations, are strict. In particular, using the second complex and Exercise 10.3, deduce that for every $k < \ell < m$ and every i, we have short exact sequences
$$0 \longrightarrow H^i\left(F_\ell C^\bullet/F_kC^\bullet\right) \longrightarrow H^i\left(C^\bullet/F_kC^\bullet\right) \longrightarrow H^i\left(C^\bullet/F_\ell C^\bullet\right) \longrightarrow 0$$
and
$$0 \longrightarrow H^i\left(F_\ell C^\bullet/F_kC^\bullet\right) \longrightarrow H^i\left(F_mC^\bullet/F_kC^\bullet\right) \longrightarrow H^i\left(F_mC^\bullet/F_\ell C^\bullet\right) \longrightarrow 0.$$

Exercise 10.5. Show that the complex $(C^\bullet, F)$ is strict if and only if all canonical morphisms $H^i(F_k C^\bullet) \to H^i(F_{k+1}C^\bullet)$ are monomorphisms. [*Hint*: It is clear that this condition is necessary; prove sufficiency by showing that the condition implies that $H^i(F_k C^\bullet) \to H^i(F_\ell C^\bullet)$ is a monomorphism for every $k < \ell$; use the exhaustivity of the filtration and the exactness of filtering direct limits to prove that $H^i(C^\bullet) \simeq \varinjlim_\ell H^i(F_\ell C^\bullet)$.]

Exercise 10.6. Let $\varphi : (C^0, F'_\bullet, F''_\bullet) \to (C^1, F'_\bullet, F''_\bullet)$ be a bi-filtered morphism, that we consider as defining a complex with two terms.

(1) Assume that φ is onto. Show that φ is bi-strict if and only if $F'_k F''_\ell C^1 = \varphi(F'_k F''_\ell C^0)$ for all $k, \ell \in \mathbb{Z}$.

(2) In general, show that φ is bi-strict if and only if it is strict with respect to each filtration and moreover

$$\begin{aligned}(F'_k C^1 + \operatorname{Im}\varphi) \cap (F''_\ell C^1 + \operatorname{Im}\varphi) &= F'_k F''_\ell C^1 + \operatorname{Im}\varphi,\\ F'_k F''_\ell C^1 \cap \operatorname{Im}\varphi &= \varphi(F'_k F''_\ell C^0).\end{aligned}$$

(3) Show that, if φ is bi-strict, then taking Ker and Coker commutes with grading with respect to $F'_\bullet$, $F''_\bullet$, or both in any order.

Exercise 10.7. Show as in Section 10.3.a that the object $A(1_{i\in I}, 0_{i\notin I})$ is equal to $A/\sum_{i\in I} A_i$.

Exercise 10.8.

(1) Let $A_1, \dots, A_n \subset A$ be a compatible family of sub-objects of A and let $B \supset A$. Show that $A_1, \dots, A_n, A$ is a compatible family in B (in particular, $A_1, \dots, A_n$ is a compatible family in B). [*Hint*: Note first that, for $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i \geqslant 0$ for all i, A surjects to $A(\varepsilon)$ and set $A(\varepsilon) = A/I(\varepsilon)$, with $I(0) = 0$; define then $B(\varepsilon, \varepsilon_{n+1})$ by

$$\begin{aligned}B(\varepsilon, -1) &= A(\varepsilon) \quad \forall\, \varepsilon,\\ B(\varepsilon, 0) &= \begin{cases} A(\varepsilon) & \text{if } \exists\, i,\ \varepsilon_i = -1,\\ B/I(\varepsilon) & \text{if } \forall\, i,\ \varepsilon_i \geqslant 0,\end{cases}\\ B(\varepsilon, 1) &= \begin{cases} 0 & \text{if } \exists\, i,\ \varepsilon_i = -1,\\ B/A & \text{if } \forall\, i,\ \varepsilon_i \geqslant 0;\end{cases}\end{aligned}$$

check the exactness of sequences like $B(-1, \varepsilon', 0) \to B(0, \varepsilon', 0) \to B(1, \varepsilon', 0)$.]

(2) Let $A_1, \dots, A_n \subset A$ be a family of sub-objects of A which is compatible in B, for some $B \supset A$. Then this family is compatible in A. [*Hint*: Set $A(\varepsilon) = B(\varepsilon)$ if $\varepsilon_i = -1$ for some i, and if $\varepsilon_i \geqslant 0$ for all i, set $A(\varepsilon) = A/I(\varepsilon)$, where $B(\varepsilon) = B/I(\varepsilon)$ and show first that $I(\varepsilon) \subset \sum_i A_i$ by using Exercise 10.7.]

(3) Let $A_0, A_1, \dots, A_n \subset A$ be a family of sub-objects of A. Assume that $A_1, \dots, A_{n-1} \subset A_n$. Show that the family $A_0, A_1, \dots, A_n$ is compatible if and only

if the family $A_0 \cap A_n, A_1, \dots, A_n$ of sub-objects of A_n is compatible. [*Hint*: If the diagram $C(A_0, \dots, A_n; A)$ exists, there should be an exact sequence

$$0 \to C(A_0 \cap A_n, \dots, A_n; A_n) \to C(A_0, \dots, A_n; A) \to C\Big(\frac{A_0}{A_0 \cap A_n}, 0, \dots, 0; \frac{A}{A_n}\Big) \to 0,$$

corresponding to exact sequences

$$0 \longrightarrow A(\varepsilon_0, \varepsilon', -1) \longrightarrow A(\varepsilon_0, \varepsilon', 0) \longrightarrow A(\varepsilon_0, \varepsilon', 1);$$

show that $A(\varepsilon_0, \varepsilon', 1) = 0$ if $\varepsilon'_i = -1$ for some $i = 1, \dots, n-1$; set thus $A(\varepsilon_0, \varepsilon', 0) := A(\varepsilon_0, \varepsilon', -1)$ for such an ε'; to determine $A(\varepsilon_0, \varepsilon', 0)$ for $\varepsilon'_i \geqslant 0$ for all i, use Exercise 10.7 if $\varepsilon_0 \geqslant 0$ and deduce the case $\varepsilon_0 = -1$; end by checking that all possibly exact sequences are indeed exact.]

Exercise 10.9 **(Basics on Rees modules).** We take up the notation of Section 10.3.b. Set $\mathbb{C}[z] = \mathbb{C}[z_1, \dots, z_n]$. Let $M = \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} M_{\boldsymbol{k}}$ be a $\mathbb{Z}^n$-graded $\mathbb{C}[z]$-module.

(1) Show that the subset $T_{\mathrm{m}}M \subset M$ consisting of elements $m \in M$ annihilated by a monomial in $z_1, \dots, z_n$ is a graded $\mathbb{C}[z]$-submodule of M. Conclude that M/T_{m} is a graded $\mathbb{C}[z]$-module.

(2) Let $T \subset M$ be the $\mathbb{C}[z]$-torsion submodule of M. Show that $T = T_{\mathrm{m}}$. [*Hint*: Assume that $T_{\mathrm{m}} = 0$ by working in M/T_{m}; if $pm = 0$ with $p = \sum p_{\boldsymbol{j}} z^{\boldsymbol{j}} \in \mathbb{C}[z]$ and $m = \sum_{\boldsymbol{k}} m_{\boldsymbol{k}} \in M$, choose a linear form L with non-negative coefficients such that $\max\{L(\boldsymbol{j}) \mid p_{\boldsymbol{j}} \neq 0\}$ is achieved for a unique index $\boldsymbol{j} = \boldsymbol{j}_o$ and similarly for $\boldsymbol{k}$ and $\boldsymbol{k}_o$; show that $z^{\boldsymbol{j}_o} m_{\boldsymbol{k}_o} = 0$ and conclude that $m = 0$.]

(3) Show that M is $\mathbb{C}[z]$-torsion free if and only if the natural morphism $M \to M[z^{-1}] := M \otimes_{\mathbb{C}[z]} \mathbb{C}[z^{-1}]$ is injective.

(4) Set $A = M/\sum_i (z_i - 1)M$. Show that M is $\mathbb{C}[z]$-torsion free if and only if there exists an exhaustive $\mathbb{Z}^n$-filtration $F_\bullet A$ such that $M = R_F A := \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} (F_{\boldsymbol{k}} A) z^{\boldsymbol{k}}$. [*Hint*: Show first that $A = M[z^{-1}]/\sum_i (z_i - 1)M[z^{-1}]$ and $M[z^{-1}] = A \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$; consider then the graded inclusion $M \hookrightarrow A \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$.]

(5) *Forgetting indices.* Let $(A, F_\bullet A)$ be a multi-filtered vector space, let $I \subset \{1, \dots, n\}$ be a subset and denote by I^{c} its complement. Let $F^{(I)}_\bullet A$ be the $\mathbb{Z}^I$-filtration defined by $F^{(I)}_{\boldsymbol{k}_I} A := \bigcup_{\boldsymbol{k}_{I^{\mathrm{c}}}\in\mathbb{Z}^{I^{\mathrm{c}}}} F^{(I)}_{(\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})} A$. Show that

$$R_{F^{(I)}} A = \big(R_F A / \textstyle\sum_{i\in I^{\mathrm{c}}} (z_i - 1) R_F A\big) \,/\, \mathbb{C}[z_I]\text{-torsion}.$$

Conclude that if $R_F A$ is $\mathbb{C}[z]$-flat, then $R_{F^{(I)}} A$ is $\mathbb{C}[z_I]$-flat. [*Hint*: Use that flatness is preserved by base change.]

(6) *Grading.* Set

$$F^{(I)}_{<\boldsymbol{k}_I} A = \sum_{i\in I} F^{(I)}_{\boldsymbol{k}_I - \mathbf{1}_i} A, \quad \mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I} A = F^{(I)}_{\boldsymbol{k}_I} A / F^{(I)}_{<\boldsymbol{k}_I} A, \quad F_{(<\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})} A = \sum_{i\in I} F_{(\boldsymbol{k}_I - \mathbf{1}_i, \boldsymbol{k}_{I^{\mathrm{c}}})} A,$$
$$F^{(I^{\mathrm{c}})}_{\boldsymbol{k}_{I^{\mathrm{c}}}} \mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I} A = F_{(\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})} A \,/\, \big[F_{(\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})} A \cap F^{(I)}_{<\boldsymbol{k}_I} A\big].$$

Show that, as $\mathbb{Z}^n$-graded modules,

$$R_F A / \textstyle\sum_{i\in I} z_i R_F A \simeq \displaystyle\bigoplus_{(\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})\in\mathbb{Z}^n} F_{(\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})} A / F_{(<\boldsymbol{k}_I, \boldsymbol{k}_{I^{\mathrm{c}}})} A \cdot z^{\boldsymbol{k}_{I^{\mathrm{c}}}}$$

and

$$\bigoplus_{\boldsymbol{k}_I\in\mathbb{Z}^I} R_{F^{(I^{\mathrm{c}})}} \mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I} A \simeq \big(R_F A / \textstyle\sum_{i\in I} z_i R_F A\big) \,/\, \mathbb{C}[z_{I^{\mathrm{c}}}]\text{-torsion}.$$

Identify $R_{F^{(I^c)}}\mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A$ with the term of I-degree $\boldsymbol{k}_I$ in the right-hand side. Conclude that if R_FA is $\mathbb{C}[z]$-flat, then $R_{F^{(I^c)}}\mathrm{gr}^{F^{(I)}}_{\boldsymbol{k}_I}A$ is $\mathbb{C}[z_{I^c}]$-flat. [*Hint*: Use that flatness is preserved by base change.] Conclude that, if R_FA is $\mathbb{C}[z]$-flat, the inclusion $F_{(<\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A \subset F_{(\boldsymbol{k}_I,\boldsymbol{k}_{I^c})}A \cap F^{(I)}_{<\boldsymbol{k}_I}A$ is an equality.

(7) Show that if R_FA is $\mathbb{C}[z]$-flat, then $F_{\boldsymbol{k}}A = \bigcap_{i=1}^n F^{(i)}_{k_i}A$. [*Hint*: Argue by induction on n and prove $F_{\boldsymbol{k}}A = F^{(1)}_{k_1}A \cap F^{\{1\}^c}_{\boldsymbol{k}_{\{1\}^c}}A$ by using the last result of (6).]

Exercise 10.10. We keep the notation as in Lemma 10.3.13.

(1) Show that the sequence $z_1,\dots,z_n$ is a regular sequence on M if and only if for every $k=1,\dots,n$, the Koszul complex $K(z_1,\dots,z_k;M)$ is a graded resolution of $M/(z_1,\dots,z_k)M$.

(2) Deduce that the following properties are equivalent:

(a) any permutation of $z_1,\dots,z_n$ is a regular sequence on M,

(b) any subsequence of $z_1,\dots,z_n$ is a regular sequence on M,

(c) for every subset $J\subset\{1,\dots,n\}$ the Koszul complex $K((z_j)_{j\in J};M)$ is a graded resolution of $M/(z_j)_{j\in J}M$.

Exercise 10.11 **(Applications of the flatness criterion).**

(1) Let A be an object with n compatible filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$ and let $F^{(n+1)}A$ be a filtration which jumps at one index at most, for example $F^{(n+1)}_{-1}A=0$ and $F^{(n+1)}_0A=A$. Show that the family $F^{(1)}_\bullet A,\dots,F^{(n+1)}_\bullet A$ is still compatible. [*Hint*: Show that the new Rees module is obtained from the old one by tensoring over $\mathbb{C}$ with $\mathbb{C}[z_{n+1}]$.]

(2) Let A be an object with n compatible filtrations $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$. Show that any family of filtrations $G^{(1)}_\bullet A,\dots,G^{(m)}_\bullet A$ where each $G^{(i)}_\bullet A$ is obtained by *convolution* of some of the filtrations $F^j_\bullet A$, i.e.,

$$G^{(i)}_pA = \sum_{q_1+\cdots+q_k=p} F^{(j_1)}_{q_1}A+\cdots+F^{(j_k)}_{q_k}A,$$

(also denoted by $G^i_\bullet A = F^{(j_1)}_\bullet A\star\cdots\star F^{(j_k)}_\bullet A$) is also a compatible family. [*Hint*: Express the Rees module R^i_GA as obtained by base change from $R_{F^{(j_1)},\dots,F^{(j_k)}}A$ and, more generally express $R_{G^{(1)},\dots,G^{(m)}}A$ as obtained by base change from $R_{F^{(1)},\dots,F^{(n)}}A$; conclude by using that flatness is preserved by base change.]

(3) Let $F^{(1)}_\bullet A,\dots,F^{(n)}_\bullet A$ be filtrations on A. Let B be a sub-object of A and let $F^{(i)}_\bullet B$ and $F^{(i)}_\bullet(A/B)$ be the induced filtrations. Assume that

(a) the families $(F^{(i)}_\bullet B)_i$ and $(F^{(i)}_\bullet(A/B))_i$ are compatible,

(b) for all $k_1,\dots,k_n$, the following sequence is exact:

$$0\longrightarrow \bigcap_{i=1}^n F^{(i)}_{k_i}B \longrightarrow \bigcap_{i=1}^n F^{(i)}_{k_i}A \longrightarrow \bigcap_{i=1}^n F^{(i)}_{k_i}(A/B)\longrightarrow 0.$$

Then the family $(F^{(i)}_\bullet A)_i$ is compatible. [*Hint*: Show that there is an exact sequence of the associated Rees modules, and use that flatness of the extreme terms implies flatness of the middle term.]

Exercise 10.12 **(External products and flatness).**

(1) Let $R=\mathbb{C}[z_1,\dots,z_n]$ and $R'=\mathbb{C}[z'_1,\dots,z'_m]$ be polynomial rings set $R''=R\otimes_{\mathbb{C}}R'=\mathbb{C}[z_1,\dots,z'_m]$. Let M resp. M' be a graded flat R- resp. R'- module. Show that $M'':=M\otimes_{\mathbb{C}}M'$ is R''-flat as a graded R''-module. [*Hint*: Use the criterion of Exercise 10.10.]

(2) Assume now that R and R' are polynomial rings (with variables as above) over a polynomial ring $\mathbb{C}[z''_1,\dots,z''_p]$. Let M,M' be as above. Show that $M'':=M\otimes_{\mathbb{C}[z'']}M'$ is R''-flat as a graded R''-module. [*Hint*: Define M'' in terms of $M\otimes_{\mathbb{C}}M'$.]

(3) Reprove Lemma 8.6.10 by using the argument of (2) and that flatness commutes with base change (in a way similar to that of Remark 10.3.15). [*Hint*: Set $\widetilde{\mathcal{M}}=R_F\mathcal{M}$ and consider $\widetilde{\mathcal{M}}_X\boxtimes_{\mathbb{C}}\widetilde{\mathcal{M}}_Y$; show that this is a flat bi-graded $\mathbb{C}[z_1,z_2]$-module; deduce that restricting first to $z_1=z_2$ and then to $z=0$, or restricting to $z_1=0$ and then to $z_2=0$ give the same result.]

(4) Let $\widetilde{\mathcal{M}}_X,\widetilde{\mathcal{M}}_Y$ be *strict* $\widetilde{\mathcal{D}}$-modules equipped with coherent $F_\bullet\widetilde{\mathcal{D}}$-filtrations $F_\bullet\widetilde{\mathcal{M}}_X,F_\bullet\widetilde{\mathcal{M}}_Y$. Assume that $\mathrm{gr}^F\widetilde{\mathcal{M}}_X,\mathrm{gr}^F\widetilde{\mathcal{M}}_Y$ are strict. Show that

$$\mathrm{gr}^F(\widetilde{\mathcal{M}}_X\boxtimes_{\widetilde{\mathcal{D}}}\widetilde{\mathcal{M}}_Y)\simeq \mathrm{gr}^F\widetilde{\mathcal{M}}_X\boxtimes_{\mathrm{gr}^F\widetilde{\mathcal{D}}}\mathrm{gr}^F\widetilde{\mathcal{M}}_Y.$$

[*Hint*: Show with Exercise 10.10 that $R_F\widetilde{\mathcal{M}}:=\bigoplus_k F_k\widetilde{\mathcal{M}}z_1^k$ is $\mathbb{C}[z,z_1]$-flat and use (2).]

10.10. Comments

The aim of this chapter, which covers part of the content of [**Sai88**, §1 & 3] and whose first sketch has been written by Mircea Mustaţă, is to give a proof of Theorem 10.6.4 which closely follows the original proof of Saito [**Sai88**, Prop. 3.3.17], from which is extracted the formalism of bi-filtered derived categories (see also Section 8.9 which is inspired form [**Sai89a**]). However, the original argument using formal completions, which has been reproduced in the proof of Proposition 9.8.10, has been replaced here (Section 10.7.b) by an argument, due to Christian Schnell, using his interpretation of compatibility of a finite family of filtrations in terms of flatness, which somewhat clarifies [**Sai88**, §1.1]. This interpretation is explained with details in Section 10.3, ending with Exercise 10.10 due to Matthieu Kochersperger. The conclusion of Proposition 10.8.3 is an adaptation of [**Sai88**, Cor. 3.4.7], and is inspired from [**ESY17**, Prop. 2.2.4].

CHAPTER 11

LOCALIZATION, DUAL LOCALIZATION AND MAXIMAL EXTENSION

Summary. We introduce the localization functor along a divisor $D \subset X$. Although it only consists in tensoring with $\mathcal{O}_X(*D)$ in the case of $\mathcal{D}_X$-modules, the definition for modules over $R_F\mathcal{D}_X$ is more subtle. It strongly uses the Kashiwara-Malgrange filtration. This construction can also be made for the dual localization functor, and this leads to the notion of middle extension along D. On the other hand, the maximal extension functor enables one to describe a $\widetilde{\mathcal{D}}_X$-module in terms of the localized object along D and of a $\widetilde{\mathcal{D}}_X$-module supported on D.

In this chapter, we keep the notation and setting as in Chapter 9. In particular, we keep Notation 9.0.1, and Remarks 9.0.2 and 9.0.3 continue to be applied. We continue to treat the case of right $\widetilde{\mathcal{D}}_X$-modules.

***11.0.1. Remark* (The case of left $\widetilde{\mathcal{D}}_X$-modules).** The case of left $\widetilde{\mathcal{D}}_X$-modules is very similar, and the only changes to be made are the following:

- to consider $V^{>-1}$ instead of $V_{<0}$,
- to modify the definition of $\psi_{t,\lambda}$ with a shift,
- to change the definition of can (with a sign).

11.1. Introduction

We consider the following question in this chapter: given a coherent $\widetilde{\mathcal{D}}_X$-module, to classify all coherent $\widetilde{\mathcal{D}}_X$-modules which coincide with it on the complement of a divisor D. This has to be understood in the algebraic sense, i.e., the $\widetilde{\mathcal{D}}_X$-modules coincide after tensoring with the sheaf $\mathcal{O}_X(*D)$ of meromorphic functions with poles along D.

For each $\mathcal{D}_X$-module $\mathcal{M}$ which is $\mathbb{R}$-specializable along D, e.g. holonomic $\mathcal{D}_X$-modules (with the restriction that the roots of the Bernstein-Sato polynomials are real) it is known that the localized $\mathcal{D}_X$-module $\mathcal{M}(*D) := \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} \mathcal{M}$ is $\mathcal{D}_X$-coherent and $\mathbb{R}$-specializable along D. There is a dual notion, giving rise to $\mathcal{M}(!D)$, and we obtain natural morphisms

$$\mathcal{M}(!D) \longrightarrow \mathcal{M} \longrightarrow \mathcal{M}(*D).$$

The notion of localization is subtler when taking into account the coherent F-filtration. Indeed, for a coherent graded $R_F\mathcal{D}_X$-module $\widetilde{\mathcal{M}}$, we cannot just consider $\widetilde{\mathcal{M}}(*D)$, since this would correspond to tensoring each term of the underlying coherent filtration by $\mathcal{O}_X(*D)$, that would produce a non-coherent $\mathcal{O}_X$-module.

It is nevertheless useful to first consider this "naive" localization of a coherent graded $R_F\mathcal{D}_X$-module $\widetilde{\mathcal{M}}$. Let D be an effective divisor in X. The sheaf $\mathcal{O}_X(*D)$ of meromorphic functions on X with arbitrary poles along the support of D at most is a coherent sheaf of ring. So are the sheaves $\mathcal{D}_X(*D) := \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} \mathcal{D}_X = \mathcal{D}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X(*D)$, and $\widetilde{\mathcal{O}}_X(*D), \widetilde{\mathcal{D}}(*D)$ defined similarly. Given a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, its "naive" localization $\widetilde{\mathcal{M}}(*D) := \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(*D)$ is a coherent $\widetilde{\mathcal{D}}_X(*D)$-module.

Assume that D is smooth. We then denote it by H and its ideal by $\mathcal{I}_H$, and we keep the notation of Section 9.2. The $\mathcal{I}_H$-adic filtration of $\widetilde{\mathcal{O}}_X(*H)$ is now indexed by $\mathbb{Z}$, and the corresponding V-filtration (9.2.1) of $\widetilde{\mathcal{D}}_X(*H)$ is nothing but the corresponding $\mathcal{I}_H$-adic filtration. We can then define the notion of a coherent V-filtration for a coherent $\widetilde{\mathcal{D}}_X(*H)$-module, and the notion of strict $\mathbb{R}$-specializability of Definition 9.3.14 can be adapted in the following way: we replace both conditions 9.3.14(2) and (3) by the only condition 9.3.14(2) which should hold for every for every $\alpha \in \mathbb{R}$. By using a local graph embedding, one defines similarly, for every effective divisor D, the notion of strict $\mathbb{R}$-specializability along D. The following lemma is then mostly obvious.

11.1.1. Lemma. *Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module, strictly $\mathbb{R}$-specializable along D. Then the coherent $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}(*D)$ is strictly $\mathbb{R}$-specializable along D.* $\square$

If $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along a *smooth* hypersurface H, one can construct a substitute to the "naive" localized module $\widetilde{\mathcal{M}}(*H)$, that we call the *localized $\widetilde{\mathcal{D}}_X$-module*, denoted by $\widetilde{\mathcal{M}}[*H]$, and a dual version $\widetilde{\mathcal{M}}[!H]$. Both are $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H, and we have natural morphisms

$$\widetilde{\mathcal{M}}[!H] \longrightarrow \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}[*H].$$

Due to the possible failure of Kashiwara's equivalence for $R_F\mathcal{D}_X$-modules, the trick of considering the graph inclusion ι_g when $D = (g)$ is not enough to ensure localizability for arbitrary D, so we are forced to considering the possibly smaller category of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules along D which are *localizable* along D, in order to have well-defined functors $[!D]$ and $[*D]$, and a sequence

$$\widetilde{\mathcal{M}}[!D] \longrightarrow \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{M}}[*D].$$

The purpose of this chapter is to introduce a method for recovering any strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ from a pair of $\widetilde{\mathcal{D}}_X$-modules, one of them being supported on D and the other one being localizable along D, and of morphisms between them. This leads to the construction of the *maximal extension* $\Xi\widetilde{\mathcal{M}}$ of $\widetilde{\mathcal{M}}$ along D. It can be done when $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along D, at least when $D = H$ is a smooth hypersurface (with multiplicity one). For a general divisor D, we encounter the same problem as for localization, and the existence of the maximal extension is not guaranteed by the strict specializability condition only. We say that $\widetilde{\mathcal{M}}$ is *maximalizable* along D when this maximal extension exists.

Assume that $D=(g)$. Given a strictly $\mathbb{R}$-specializable, localizable and maximalizable (along D) $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}_*$, we will construct a functor $\mathrm{G}_{\widetilde{\mathcal{M}}_*}$ from the category consisting of triples $(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})$, where $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along D and supported on D, and c,v are morphisms

$$\begin{tikzcd} \psi_{g,1}\widetilde{\mathcal{M}}_* \arrow[r, bend left, "\mathrm{c}"] & \widetilde{\mathcal{N}} \arrow[l, bend left, "(-1)\ \ \mathrm{v}"] \end{tikzcd}$$

to that of strictly $\mathbb{R}$-specializable and localizable $\widetilde{\mathcal{D}}_X$-modules, so that

(a) $\mathrm{G}_{\widetilde{\mathcal{M}}_*}(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v})(*D)=\widetilde{\mathcal{M}}_*$,

(b) the above diagram is isomorphic to the specialization diagram

$$\begin{tikzcd} \psi_{g,1}\mathrm{G}_{\widetilde{\mathcal{M}}_*}(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v}) \arrow[rr, bend left, "\mathrm{can}"] && \phi_{g,1}\mathrm{G}_{\widetilde{\mathcal{M}}_*}(\widetilde{\mathcal{N}},\mathrm{c},\mathrm{v}) \arrow[ll, bend left, "(-1)\ \ \mathrm{var}"] \end{tikzcd}.$$

This classifies all such $\widetilde{\mathcal{D}}_X$-modules $\widetilde{\mathcal{M}}'$ such that $\widetilde{\mathcal{M}}'(*D)=\widetilde{\mathcal{M}}_*$. A first approximation of this construction was obtained in Exercise 9.33.

11.2. Localization and dual localization in the strictly non-characteristic case

In section, we fix a smooth hypersurface H of X and we simply write strictly $\mathbb{R}$-specializable instead of strictly $\mathbb{R}$-specializable along H. We also denote by ι (instead of ι_H) the inclusion $H\hookrightarrow X$. The coherent $\mathcal{D}_X$-module $\mathcal{O}_X(*H)$ is generated as such by the $\mathcal{O}_X$-submodule $\mathcal{O}_X(H)$ consisting of meromorphic functions having a pole of order at most one along H. If we interpret $\mathcal{O}_X(H)$ as $V^{-1}\mathcal{O}_X(*H)$, we then have the equality $\mathcal{O}_X(*H)=\mathcal{D}_X\cdot\mathcal{O}_X(H)=\mathcal{D}_X\cdot V^{-1}\mathcal{O}_X(*H)$.

11.2.a. Localization of $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$. Working with $\widetilde{\mathcal{D}}_X$-modules, we note that $\widetilde{\mathcal{O}}_X(*H)$ is not locally of finite type over $\widetilde{\mathcal{D}}_X$: for example, if t is a local equation for H, the $\widetilde{\mathcal{D}}_X$-submodule generated by $1/t$ does not contain $1/t^2$ (but contains z/t^2), that generated by $1/t,1/t^2$ does not contain $1/t^3$, etc.

We then define the coherent $\widetilde{\mathcal{D}}_X$-submodule $\widetilde{\mathcal{O}}_X[*H]$ of $\widetilde{\mathcal{O}}_X(*H)$ as the $\widetilde{\mathcal{D}}_X$-submodule generated by $\widetilde{\mathcal{O}}_X(H)$, that is, $\widetilde{\mathcal{D}}_X\cdot V^{-1}\widetilde{\mathcal{O}}_X(*H)$. It is a proper coherent submodule of $\widetilde{\mathcal{O}}_X(*H)$, as shown above. If we equip $\mathcal{O}_X(*H)$ with the increasing filtration by the order of the pole, i.e., such that $F_k\mathcal{O}_X(*)=\mathcal{O}_X((k+1)H)$ for $k\geqslant 0$ and $F_k\mathcal{O}_X(*H)=0$ for $k<0$, then $\widetilde{\mathcal{O}}_X[*H]=R_F\mathcal{O}_X(*H)$. We define $\widetilde{\omega}_X[*H]$ similarly, as the $\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\omega}_X(*H)$ generated by $V_0\widetilde{\omega}_X$.

11.2.1. Lemma. *The right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\omega}_X[*H]$ is is isomorphic to that obtained by side-changing from $\widetilde{\mathcal{O}}_X[*H]$.*

Proof. It is a matter of proving that $\widetilde{\omega}_X[*H] \simeq \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]$. This is obtained as follows:

$$\begin{aligned}V_0(\widetilde{\omega}_X(*H)) \cdot \widetilde{\mathcal{D}}_X &= (\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(H)) \cdot \widetilde{\mathcal{D}}_X \\ &\simeq \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \cdot \widetilde{\mathcal{O}}_X(H)) = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]. \qquad \square\end{aligned}$$

11.2.2. Proposition.

(1) *The natural surjective morphism $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H) \to \widetilde{\mathcal{O}}_X[*H]$ (resp. the surjective morphism $\widetilde{\omega}_X(H) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\omega}_X[*H]$) is an isomorphism.*

(2) *The coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{O}}_X[*H]$ (resp. $\widetilde{\omega}_X[*H]$) is strictly $\mathbb{R}$-specializable and for any k, the V-filtration is given by the formula $V^{-k-1}\widetilde{\mathcal{O}}_X[*H] = V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H)$ (resp. $V_k\widetilde{\omega}_X[*H] = \widetilde{\omega}_X(H) \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$).*

(3) *The cokernel of the morphism $\mathrm{loc} : \widetilde{\mathcal{O}}_X \hookrightarrow \widetilde{\mathcal{O}}_X[*H]$ (resp. $\mathrm{loc} : \widetilde{\omega}_X \hookrightarrow \widetilde{\omega}_X[*H]$) is strictly $\mathbb{R}$-specializable and isomorphic to ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H)(-1)$ (resp. ${}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H)(-1)$).*

Proof.

(1) Since this is a local question, we can assume that $X = H \times \Delta$ and use adapted local coordinates. Then $\widetilde{\mathcal{O}}_X(H) = (1/t)\widetilde{\mathcal{O}}_X = V_0\widetilde{\mathcal{D}}_X / \big(\sum_i V_0\widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + V_0\widetilde{\mathcal{D}}_X(\widetilde{\partial}_t t)\big)$, so $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H) \simeq \widetilde{\mathcal{D}}_X / \big(\sum_i \widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + \widetilde{\mathcal{D}}_X(\widetilde{\partial}_t t)\big)$, and the natural morphism is $P \mapsto P \cdot (1/t) \in \widetilde{\mathcal{O}}_X(*H)$. For the injectivity of the morphism, we are led to showing that $P \cdot (1/t) = 0$ implies $P \in \big(\sum_i \widetilde{\mathcal{D}}_X\widetilde{\partial}_{x_i} + \widetilde{\mathcal{D}}_X(\widetilde{\partial}_t t)\big)$, which can be checked in a straightforward way.

(2) A direct computation shows that the following formula define a V-filtration of $\widetilde{\mathcal{O}}_X[*H]$:

$$\begin{aligned}V^0(\widetilde{\mathcal{O}}_X[*H]) &= V^0(\widetilde{\mathcal{O}}_X(*H)) = \widetilde{\mathcal{O}}_X, \\ V^{-1}(\widetilde{\mathcal{O}}_X[*H]) &= V^{-1}(\widetilde{\mathcal{O}}_X(*H)) = \widetilde{\mathcal{O}}_X(H), \\ V^{-k-1}(\widetilde{\mathcal{O}}_X[*H]) &= \sum_{j=0}^{k} z^j\widetilde{\mathcal{O}}_X((j+1)H) \quad (k \geqslant 1).\end{aligned}$$

The graded objects read

$$\mathrm{gr}_V^{-k-1}(\widetilde{\mathcal{O}}_X[*H]) = \begin{cases}\mathrm{gr}_V^{-k-1}\widetilde{\mathcal{O}}_X & \text{if } k < 0, \\ \widetilde{\mathcal{O}}_X(H)/\widetilde{\mathcal{O}}_X & \text{if } k = 0, \\ z^k\widetilde{\mathcal{O}}_X((k+1)H)/\widetilde{\mathcal{O}}_X(kH) & \text{if } k > 0,\end{cases}$$

hence are strict, showing strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{O}}[*H]$. Note that the Euler vector field E acts by zero on each graded piece, hence the $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-module structure descends to a $\widetilde{\mathcal{D}}_H$-module structure (see Exercise 9.4).

(3) The filtration induced by $V^\bullet(\widetilde{\mathcal{O}}[*H])$ on $\widetilde{\mathcal{O}}[*H]/\widetilde{\mathcal{O}}_X$ satisfies

$$\operatorname{gr}_V^k(\widetilde{\mathcal{O}}_X[*H]/\widetilde{\mathcal{O}}_X) \simeq \begin{cases} 0 & \text{if } k \geqslant 0, \\ \operatorname{gr}_V^k(\widetilde{\mathcal{O}}_X[*H]) & \text{if } k \leqslant -1. \end{cases}$$

Therefore, $\widetilde{\mathcal{O}}[*H]/\widetilde{\mathcal{O}}_X$ is strictly $\mathbb{R}$-specializable and $\operatorname{gr}_V^{-1}(\widetilde{\mathcal{O}}[*H]/\widetilde{\mathcal{O}}_X) \simeq \widetilde{\mathcal{O}}_X(H)/\widetilde{\mathcal{O}}_X$. Similar results hold for $\widetilde{\omega}_X$ by side-changing (Lemma 11.2.1). We can regard $\operatorname{gr}_0^V \widetilde{\omega}_X[*H] = \widetilde{\omega}_X(H)/\widetilde{\omega}_X$ as a right $\widetilde{\mathcal{D}}_H$-module and strict Kashiwara's equivalence of Proposition 9.6.2 implies

$$\widetilde{\omega}_X[*H]/\widetilde{\omega}_X \simeq {}_{\mathrm{D}}\iota_*\big(\widetilde{\omega}_X(H)/\widetilde{\omega}_X\big).$$

11.2.3. Lemma. *The residue morphism induces an isomorphism of right $\widetilde{\mathcal{D}}_X$-modules*

$$\operatorname{Res}_H : \widetilde{\omega}_X(H)/\widetilde{\omega}_X \xrightarrow{\ \sim\ } \widetilde{\omega}_H(-1).$$

Proof. This is easily checked in local coordinates. The twist (-1) is due to the "division by $\widetilde{\mathrm{d}}t$", which induces a multiplication by z. $\square$

As a consequence, we obtain the exact sequence via the residue:

$$0 \longrightarrow \widetilde{\omega}_X \xrightarrow{\ \mathrm{loc}\ } \widetilde{\omega}_X[*H] \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H)(-1) \longrightarrow 0.$$

Since ${}_{\mathrm{D}}\iota_*$ commutes with side-changing, we deduce an exact sequence

$$0 \longrightarrow \widetilde{\mathcal{O}}_X \xrightarrow{\ \mathrm{loc}\ } \widetilde{\mathcal{O}}_X[*H] \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H)(-1) \longrightarrow 0. \qquad \square$$

11.2.4. Example. Assume that $X = H \times \Delta$. Then any local section of ${}_{\mathrm{D}}\iota_*\widetilde{\mathcal{O}}_H$ can be written as (see (8.7.7 $**$) with $g = 0$)

$$\bigoplus_{k\geqslant 0} (-1)^k \eta_{ok} \otimes \widetilde{\partial}_t^k \otimes \widetilde{\mathrm{d}}t^\vee = \bigoplus_{k\geqslant 0} \widetilde{\partial}_t^k(\eta_{ok} \otimes 1 \otimes \widetilde{\mathrm{d}}t^\vee)$$

with $\eta_{ok} \in \widetilde{\mathcal{O}}_H$. One can obtain a lift of such a local section in $\widetilde{\mathcal{O}}_X[*H]$ by the formula

$$\sum_{k\geqslant 0} \widetilde{\partial}_t^k(\eta_k/t)$$

where η_k is a local holomorphic function on X such that $\eta_{k|H} = \eta_{ok}$.

11.2.b. Dual localization of $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$. We now consider a dual setting, although strictly speaking the duality functor is not involved in the next construction.

We set $\widetilde{\mathcal{O}}_X[!H] := \widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$ (resp. $\widetilde{\omega}_X[!H] := \widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$), where the right (resp. left) $V_0\widetilde{\mathcal{D}}_X$-module structure of $\widetilde{\mathcal{D}}_X$ is used for the tensor product. The trivial left (resp. right) action of $\widetilde{\mathcal{D}}_X$ makes $\widetilde{\mathcal{O}}_X[!H]$ (resp. $\widetilde{\omega}_X[!H]$) a coherent left (resp. right) $\widetilde{\mathcal{D}}_X$-module equipped with a surjective morphism $\mathrm{dloc} : \widetilde{\mathcal{O}}_X[!H] \to \widetilde{\mathcal{O}}_X$ (resp. $\mathrm{dloc} : \widetilde{\omega}_X[!H] \to \widetilde{\omega}_X$) whose kernel is supported on H. We will analyze its kernel. Let us first check:

11.2.5. Lemma. *The right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\omega}_X[!H]$ is is isomorphic to that obtained by side-changing from $\widetilde{\mathcal{O}}_X[!H]$.*

Proof. Using notation similar to that of Exercise 8.19, it is a matter of showing that $[\widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X]_{\mathrm{triv}} \simeq [\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X)]_{\mathrm{tens}}$. The proof is completely similar to that of loc. cit. □

11.2.6. Proposition.

(1) *The coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{O}}_X[!H]$ (resp. $\widetilde{\omega}_X[!H]$) is strictly $\mathbb{R}$-specializable and for any k, the V-filtration is given by the formula $V^{-k}\widetilde{\mathcal{O}}_X[!H] = V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$ (resp. $V_{k-1}\widetilde{\omega}_X[!H] = \widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$).*

(2) *The kernel of* $\mathrm{dloc} : \widetilde{\mathcal{O}}_X[!H] \to \widetilde{\mathcal{O}}_X$ *(resp.* $\mathrm{dloc} : \widetilde{\omega}_X[!H] \to \widetilde{\omega}_X$*) is also strictly $\mathbb{R}$-specializable and isomorphic to ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H)$ (resp. ${}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H)$).*

Proof. It is a priori not clear that the formula for the V-filtration defines a filtration, i.e., that $V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$ injects into $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X$. We will check this by a local computation. Let us consider the local setting with $X = H \times \Delta$, where Δ has coordinate t. Then $\widetilde{\mathcal{O}}_H$, which is a quotient sheaf of $\widetilde{\mathcal{O}}_X$ on which t acts by zero, is also regarded as a subsheaf of $\widetilde{\mathcal{O}}_X$ (functions which do not depend on t). Then

$$V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X = V_k\widetilde{\mathcal{D}}_X \big/ \big[\textstyle\sum_i V_k\widetilde{\mathcal{D}}_X \widetilde{\partial}_{x_i} + V_k\widetilde{\mathcal{D}}_X(t\widetilde{\partial}_t)\big]$$

admits the local decomposition

$$V_k\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X \simeq \widetilde{\mathcal{O}}_X \oplus \bigoplus_{i=0}^{k-1} \widetilde{\mathcal{O}}_H \cdot \widetilde{\partial}_t^{i+1},$$

which makes clear the injectivity property, as well as the strict $\mathbb{R}$-specializability of the kernel of $\mathrm{Ker}[\mathrm{dloc} : \widetilde{\mathcal{O}}_X[!H] \to \widetilde{\mathcal{O}}_X]$, whose V^{-k} reads $\bigoplus_{i=0}^{k-1} \widetilde{\mathcal{O}}_H \cdot \widetilde{\partial}_t^{i+1}$. After side-changing, we obtain similar results for $\widetilde{\omega}_X[!H]$. In the right setting, we find the local identification

$$\widetilde{\omega}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X \simeq \widetilde{\omega}_X \oplus \bigoplus_{i=0}^{k-1} (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H) \cdot \widetilde{\partial}_t^{i+1}.$$

We note that $\mathrm{Ker}\,\mathrm{dloc} : \widetilde{\omega}_X[!H] \to \widetilde{\omega}_X$ is supported on H, so, by strict Kashiwara's equivalence (Proposition 9.6.2), $\mathrm{Ker}\,\mathrm{dloc} \simeq \iota_*(\mathrm{gr}_0^V \mathrm{Ker}\,\mathrm{dloc})$. The first point of the proposition yields an isomorphism of $\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$-modules

$$\mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{dloc}) = \mathrm{gr}_0^V(\widetilde{\omega}_X[!H]) \simeq \widetilde{\omega}_X \otimes_{V^0\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X \simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H) \otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X,$$

since $\mathcal{I}_H$ acts by zero on $\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X$. The Euler vector field E acts by zero on both sides: this is clear by definition for the left-hand side, and for the right-hand side, in local coordinates, the right action of $\widetilde{\partial}_t t$ sends $\widetilde{\omega}_X$ to $\widetilde{\omega}_X\mathcal{I}_H$. Therefore, both sides are $\widetilde{\mathcal{D}}_H$-modules by means of the identification $\widetilde{\mathcal{D}}_H = \mathrm{gr}_0^V\widetilde{\mathcal{D}}_X / \mathrm{E}\,\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X$, and the isomorphism is as such.

***11.2.7. Lemma* (Dual residue lemma).** *In the right setting, we have a natural isomorphism of right $\widetilde{\mathcal{D}}_H$-modules:*

$$\mathrm{dRes} : \mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{dloc}) \simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\widetilde{\mathcal{I}}_H) \otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X \xrightarrow{\sim} \widetilde{\omega}_H$$

given in a local decomposition $X = H \times \Delta$ *by*

$$\big(a(x,t,z)\widetilde{\mathrm{d}}x_1 \wedge \cdots \wedge \widetilde{\mathrm{d}}x_{n-1} \wedge \widetilde{\mathrm{d}}t\big) \otimes \widetilde{\partial}_t \longmapsto a(x,0,z)\widetilde{\mathrm{d}}x_1 \wedge \cdots \wedge \widetilde{\mathrm{d}}x_{n-1}.$$

Proof. Note that dRes does not involve a Tate twist since $\widetilde{\mathrm{d}}t \otimes \widetilde{\partial}_t$ has z-degree equal to zero. Is is a matter of showing that the formula in the lemma is independent of the choice of the decomposition $X \simeq H \times \Delta$ and of local the coordinates on H and Δ. This is easily checked by considering the coordinate changes on X which preserve H, that is, of the form $x'_i = p_i(x,t,z)$ and $t' = t\mu(x,t,z)$ (argue as in Exercise 9.4). $\square$

The lemma ends the proof of the proposition, which thus provides two exact sequences obtained one from the other by side-changing:

$$\begin{gathered} 0 \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{O}}_H) \longrightarrow \widetilde{\mathcal{O}}_X[!H] \longrightarrow \widetilde{\mathcal{O}}_X \longrightarrow 0, \\ 0 \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\omega}_H) \longrightarrow \widetilde{\omega}_X[!H] \longrightarrow \widetilde{\omega}_X \longrightarrow 0. \end{gathered} \qquad \square$$

11.2.c. Generalization for strictly non-characteristic $\widetilde{\mathcal{D}}_X$-modules

The properties of localization and dual localization for $\widetilde{\mathcal{O}}_X$ and $\widetilde{\omega}_X$ extend to arbitrary coherent $\widetilde{\mathcal{D}}_X$-modules provided that they are strictly non-characteristic along H (the general case of coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along H will be treated in Sections 11.3.a and 11.4.b)). In this section 11.2.c, we consider a coherent right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$ for simplicity and we assume that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$. Then $\widetilde{\mathcal{M}} = V_{-1}\widetilde{\mathcal{M}}$ is $V_0\widetilde{\mathcal{D}}_X$-coherent.

Localization. The naive localization $\widetilde{\mathcal{M}}(*H)$ is strictly $\mathbb{R}$-specializable as a $\widetilde{\mathcal{D}}_X(*H)$-module and $V_0\widetilde{\mathcal{M}}(*H) = \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(H)$, as seen by computing in a local chart. We then denote by $\widetilde{\mathcal{M}}[*H]$ the $\widetilde{\mathcal{D}}_X$-submodule $V_0\widetilde{\mathcal{M}}(*H)\cdot\widetilde{\mathcal{D}}_X \subset \widetilde{\mathcal{M}}(*H)$. The natural morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(*H)$ is injective since the action of t is injective on $\widetilde{\mathcal{M}} = V_{-1}\widetilde{\mathcal{M}}$. Hence the natural morphism $\mathrm{loc}: \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ is also injective.

Let us check that $\widetilde{\mathcal{M}}[*H] \simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]$ (where the right-hand side is equipped with its tensor structure of right $\widetilde{\mathcal{D}}_X$-module). We have

$$\begin{aligned} V_0(\widetilde{\mathcal{M}}(*H))\cdot\widetilde{\mathcal{D}}_X &= (\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X(H))\cdot\widetilde{\mathcal{D}}_X \\ &\simeq \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{D}}_X \cdot \widetilde{\mathcal{O}}_X(H)) = \widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{O}}_X[*H]. \end{aligned} \tag{11.2.8}$$

11.2.9. Proposition. *The natural morphism* $V_0\widetilde{\mathcal{M}}(*H) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}[*H]$ *is an isomorphism. Furthermore,* $\widetilde{\mathcal{M}}[*H]$ *is strictly* $\mathbb{R}$*-specializable, as well as* $\widetilde{\mathcal{M}}[*H]/\widetilde{\mathcal{M}}$*, the latter being supported on* H*, hence isomorphic to* ${}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H]/\widetilde{\mathcal{M}})\big) = {}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H])\big)$*. Last, there exists a natural isomorphism* $\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H]/\widetilde{\mathcal{M}}) \simeq \widetilde{\mathcal{M}}_H(-1)$*, where* $\widetilde{\mathcal{M}}_H = {}_{\mathrm{D}}\iota_H^*(\widetilde{\mathcal{M}})$ *is the restriction of* $\widetilde{\mathcal{M}}$ *to* H*, giving rise to an exact sequence*

$$0 \longrightarrow \widetilde{\mathcal{M}} \xrightarrow{\ \mathrm{loc}\ } \widetilde{\mathcal{M}}[*H] \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1) \longrightarrow 0.$$

Proof. For the first assertion, we replace $\cdot\widetilde{\mathcal{D}}_X$ with $\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{D}}_X$ in the sequence of isomorphisms (11.2.8) and we use the isomorphism $\widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{O}}_X(H) \simeq \widetilde{\mathcal{O}}_X[*H]$. The formulas for the V-filtration given in the proof of Proposition 11.2.2, when considered in the right setting, extend in a straightforward way by replacing there $\widetilde{\omega}_X$ with $\widetilde{\mathcal{M}}$.

Let us give details on the identification of Coker loc with ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)$. We consider $\widetilde{\mathcal{M}}$ as obtained by side-changing: $\widetilde{\mathcal{M}} = \widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}$. Then we have a residue morphism

$$(\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}})(H)/(\widetilde{\omega}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}) = (\widetilde{\omega}_X(H)/\widetilde{\omega}_X) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}} \xrightarrow{\mathrm{Res}\otimes\mathrm{Id}} \widetilde{\omega}_H(-1) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}^{\mathrm{left}}.$$

Furthermore, since $\widetilde{\omega}_H \cdot \mathcal{I}_H = 0$, the latter term is isomorphic to

$$\widetilde{\omega}_H(-1) \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{M}}^{\mathrm{left}}/\mathcal{I}_H\widetilde{\mathcal{M}}^{\mathrm{left}}) = \widetilde{\omega}_H(-1) \otimes_{\widetilde{\mathcal{O}}_H} \widetilde{\mathcal{M}}^{\mathrm{left}}_H = \widetilde{\mathcal{M}}_H(-1), \tag{11.2.10}$$

which gives the desired identification, according to Kashiwara's equivalence.

On the other hand, one can also argue as follows: we have $\widetilde{\mathcal{M}} = V_{-1}\widetilde{\mathcal{M}}$, so that $\mathrm{gr}_0^V(\mathrm{Coker\,loc}) \simeq \mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H])$ and, as

$$\mathrm{var} : \mathrm{gr}_0^V(\widetilde{\mathcal{M}}[*H]) \longrightarrow \mathrm{gr}_{-1}^V(\widetilde{\mathcal{M}}[*H])(-1) \simeq \mathrm{gr}_{-1}^V(\widetilde{\mathcal{M}})(-1) = \widetilde{\mathcal{M}}_H(-1)$$

is an isomorphism, this provides the desired identification $\mathrm{gr}_0^V(\mathrm{Coker\,loc}) \simeq \widetilde{\mathcal{M}}_H(-1)$. □

11.2.11. Example. Let us consider the setting of Example 11.2.4. A lift of a local section $\bigoplus_{k\geqslant 0} \widetilde{\partial}_t^k(m_{ok} \otimes 1 \otimes \widetilde{\mathrm{d}}t^\vee)$ of ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)$ in $\widetilde{\mathcal{M}}^{\mathrm{left}}[*H]$ is given by the formula $\sum_{k\geqslant 0} \widetilde{\partial}_t^k(m_k/t)$, where m_k is a lift of $m_{ok} \in \widetilde{\mathcal{M}}^{\mathrm{left}}_H$ in $\widetilde{\mathcal{M}}^{\mathrm{left}}$.

Dual localization. We define $\widetilde{\mathcal{M}}[!H] = V_{-1}\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X = \widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$.

11.2.12. Proposition. *The coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!H]$ is strictly $\mathbb{R}$-specializable with V-filtration given by $V_{k-1}(\widetilde{\mathcal{M}}[!H]) = \widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$, as well as the kernel of the surjective morphism* dloc : $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$, *the latter being supported on H, hence isomorphic to ${}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\mathrm{Ker\,dloc})\big) = {}_{\mathrm{D}}\iota_*\big(\mathrm{gr}_0^V(\widetilde{\mathcal{M}}[!H])\big)$. Last, there exists a natural isomorphism $\mathrm{gr}_0^V(\mathrm{Ker\,dloc}) \simeq \widetilde{\mathcal{M}}_H$ giving rise to an exact sequence*

$$0 \longrightarrow {}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H) \longrightarrow \widetilde{\mathcal{M}}[!H] \xrightarrow{\mathrm{dloc}} \widetilde{\mathcal{M}} \longrightarrow 0.$$

Proof. With the same argument as in the proof of Proposition 11.2.6, we find a local decomposition

$$\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X \simeq \widetilde{\mathcal{M}} \oplus \bigoplus_{i=0}^{k-1} (\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}\mathcal{I}_H) \cdot \widetilde{\partial}_t^{i+1},$$

showing the first properties and the fact that $\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} V_k\widetilde{\mathcal{D}}_X$ is the V-filtration $V_{k-1}(\widetilde{\mathcal{M}}[!H])$. It remains to prove the identification $\mathrm{gr}_0^V(\mathrm{Ker\,dloc}) \simeq \widetilde{\mathcal{M}}_H$. In a first step, we find

$$\mathrm{gr}_0^V(\mathrm{Ker\,dloc}) = \mathrm{gr}_0^V(\widetilde{\mathcal{M}}[!H]) \simeq \widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X = (\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}\mathcal{I}_H) \otimes_{\widetilde{\mathcal{D}}_H} \mathrm{gr}_1^V\widetilde{\mathcal{D}}_X.$$

Arguing as in (11.2.10), but with the dual residue map of Lemma 11.2.7, we find (see Exercise 8.19)

$$\begin{aligned}(\widetilde{\mathcal{M}}/\widetilde{\mathcal{M}}\mathcal{I}_H)\otimes_{\widetilde{\mathcal{D}}_H}\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X &\simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H)\otimes_{\widetilde{\mathcal{O}}_H}(\widetilde{\mathcal{M}}^{\mathrm{left}}/\mathcal{I}_H\widetilde{\mathcal{M}}^{\mathrm{left}})\otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X}\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X\\ &\simeq (\widetilde{\omega}_X/\widetilde{\omega}_X\mathcal{I}_H)\otimes_{\mathrm{gr}_0^V\widetilde{\mathcal{D}}_X}\mathrm{gr}_1^V\widetilde{\mathcal{D}}_X\otimes_{\widetilde{\mathcal{O}}_H}(\widetilde{\mathcal{M}}^{\mathrm{left}}/\mathcal{I}_H\widetilde{\mathcal{M}}^{\mathrm{left}})\\ &\xrightarrow{\mathrm{dRes}\otimes\mathrm{Id}}\widetilde{\omega}_H\otimes_{\widetilde{\mathcal{O}}_H}\widetilde{\mathcal{M}}_H^{\mathrm{left}}=\widetilde{\mathcal{M}}_H.\end{aligned}$$

On the other hand, as in the proof for Coker loc, we can also argue by noticing that can : $\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}[!H]\to\mathrm{gr}_0^V\widetilde{\mathcal{M}}[!H]$ is an isomorphism. Then

$$\mathrm{gr}_0^V(\mathrm{Ker}\,\mathrm{dloc})\simeq\mathrm{gr}_0^V\widetilde{\mathcal{M}}[!H]\xleftarrow{\sim}\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}[!H]\simeq\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}_H.\qquad\square$$

Side-changing. For a left $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}=\widetilde{\mathcal{M}}^{\mathrm{left}}$, the dual localized module $\widetilde{\mathcal{M}}[!H]$ is defined as $\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$. Arguing as in Lemma 11.2.5, it is obtained by side-changing from $\widetilde{\mathcal{M}}^{\mathrm{right}}[!H]$, and the V-filtration is given by $V^{-k}\widetilde{\mathcal{M}}[!H]=V_k\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}$. It admits local decompositions

$$V_k\widetilde{\mathcal{D}}_X\otimes_{V_0\widetilde{\mathcal{D}}_X}\widetilde{\mathcal{M}}\simeq\widetilde{\mathcal{M}}\oplus\bigoplus_{i=0}^{k-1}\widetilde{\mathcal{M}}_H\cdot\widetilde{\partial}_t^{i+1}.\tag{11.2.13}$$

Let us make explicit the left action of $\widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{M}}[!H]$ in the local decomposition (11.2.13). The action of $\widetilde{\mathcal{D}}_H$ is the natural one on each coefficient, while

$$\widetilde{\partial}_t\cdot\Bigl(m_0+\textstyle\sum_{i=0}^{k-1}m_{i+1}\widetilde{\partial}_t^{i+1}\Bigr)=\widetilde{\partial}_t(m_0)+m_{0|H}\widetilde{\partial}_t+\textstyle\sum_{i=0}^{k-1}m_{i+1}\widetilde{\partial}_t^{i+2},\tag{11.2.14}$$

and

$$t\cdot\Bigl(m_0+\textstyle\sum_{i=0}^{k-1}m_{i+1}\widetilde{\partial}_t^{i+1}\Bigr)=tm_0-\textstyle\sum_{i=0}^{k-2}(i+2)m_{i+2}\widetilde{\partial}_t^{i+1}.$$

In particular, the decomposition (11.2.13) is stable under the action of $V_0\widetilde{\mathcal{D}}_X$. Furthermore, the natural morphism $\widetilde{\mathcal{M}}[!H]\to\widetilde{\mathcal{M}}$ is induced by the projection to $\widetilde{\mathcal{M}}$, and the morphism

$$\bigoplus_{i\geqslant0}\widetilde{\mathcal{M}}_H\cdot\widetilde{\partial}_t^{i+1}\longrightarrow\bigoplus_{i\geqslant0}\widetilde{\mathcal{M}}_H\cdot\widetilde{\partial}_t^{i}\otimes\widetilde{\mathrm{d}}t^\vee,\quad m_{i+1}\cdot\widetilde{\partial}_t^{i+1}\longmapsto m_{i+1}\cdot(-\widetilde{\partial}_t)^i\otimes\widetilde{\mathrm{d}}t^\vee\tag{11.2.15}$$

identifies (see Example 8.7.7(2)) its kernel with the pushforward ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)$ (note that the supplementary term $\widetilde{\partial}_t$ on the left-hand side adjusts the gradings).

Conclusion. It follows that, in the sequence

$$\widetilde{\mathcal{M}}[!H]\xrightarrow{\mathrm{dloc}}\widetilde{\mathcal{M}}\xrightarrow{\mathrm{loc}}\widetilde{\mathcal{M}}[*H],$$

the natural morphisms $\mathrm{Ker}\,\mathrm{dloc}\to{}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)$ and ${}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)\to\mathrm{Coker}\,\mathrm{loc}$ are isomorphisms. Furthermore, we have two exact sequences whose terms are strictly $\mathbb{R}$-specializable along H:

$$\begin{gathered}0\longrightarrow{}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)\longrightarrow\widetilde{\mathcal{M}}[!H]\xrightarrow{\mathrm{dloc}}\widetilde{\mathcal{M}}\longrightarrow0,\\ 0\longrightarrow\widetilde{\mathcal{M}}\xrightarrow{\mathrm{loc}}\widetilde{\mathcal{M}}[*H]\longrightarrow{}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)(-1)\longrightarrow0.\end{gathered}\tag{11.2.16}$$

11.2.d. The restriction and Gysin morphisms associated to a strictly non-characteristic hypersurface. Let us assume that X is compact and let $a_X : X \to \mathrm{pt}$ denote the constant map. The constant map $a_H : H \to \mathrm{pt}$ is equal to $a_X \circ \iota$, and we denote both a_X and a_H by a for simplicity.

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module. We assume that H *is strictly non-characteristic with respect to* $\widetilde{\mathcal{M}}$. Let $\widetilde{\mathcal{M}}_H$ denotes the pullback ${}_{\mathrm{D}}\iota^*\widetilde{\mathcal{M}}$.

The exact sequences (11.2.16) give rise to two connecting morphisms

$$\begin{aligned}{}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}} &\xrightarrow{\ \mathrm{restr}_H\ } {}_{\mathrm{D}}a_*^{(k+1)}\widetilde{\mathcal{M}}_H \simeq {}_{\mathrm{D}}a_*^{(k+1)}({}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)),\\ {}_{\mathrm{D}}a_*^{(k)}({}_{\mathrm{D}}\iota_*(\widetilde{\mathcal{M}}_H)) \simeq {}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}}_H &\xrightarrow{\ \mathrm{Gys}_H\ } {}_{\mathrm{D}}a_*^{(k+1)}\widetilde{\mathcal{M}}(1),\end{aligned} \tag{11.2.17}$$

where the isomorphisms are given by Corollary 8.7.26.

Let us set $\mathcal{L} = \mathcal{O}_X(H)$, and let $\mathrm{X}_{\mathcal{L}}$ denote the Lefschetz operator on ${}_{\mathrm{D}}a_*^{(\bullet)}\widetilde{\mathcal{M}}$ associated with the Tate-twisted Chern class $(2\pi\mathrm{i})c_1(\mathcal{L}) \in H^2(X, \mathbb{Q}(1))$. Recall that, if η is a closed de Rham representative of $(2\pi\mathrm{i})c_1(\mathcal{L})$ in $\Gamma(X, \mathcal{E}_X^2)$, then $\mathrm{X}_{\mathcal{L}}$ is induced by the wedge product with $\widetilde{\eta} = \eta/z$.

11.2.18. Proposition. *Under the previous assumptions, the following diagram commutes:*

$$\begin{tikzcd} & {}_{\mathrm{D}}a_*^{(k)}\widetilde{\mathcal{M}} \arrow[rr, "\mathrm{X}_{\mathcal{L}}"] \arrow[dr, "\mathrm{restr}_H"] & & {}_{\mathrm{D}}a_*^{(k+2)}\widetilde{\mathcal{M}}(1) \\ {}_{\mathrm{D}}a_*^{(k-1)}\widetilde{\mathcal{M}}_H(-1) \arrow[ur, "\mathrm{Gys}_H"] \arrow[rr, "\mathrm{X}_{\mathcal{L}}"] & & {}_{\mathrm{D}}a_*^{(k+1)}\widetilde{\mathcal{M}}_H \arrow[ur, "\mathrm{Gys}_H"'] & \end{tikzcd}$$

Proof. Each term in the diagram is the hypercohomology of a de Rham complex ${}^{\mathrm{p}}\mathrm{DR} = \mathrm{DR}[n]$. The shift has the effect of multiplying the differentials of the complexes by $(-1)^n$, and it follows that the connecting morphisms restr_H and Gys_H are also multiplied by $(-1)^n$. For the sake of simplicity, we will then argue with the non shifted de Rham complexes and the result for the shifted complexes will follow.

On the other hand, it will be convenient to make use of a different realization (11.2.20) below of the complexes involved in the exact sequences (11.2.17). This is why we first introduce the *logarithmic de Rham complexes.*

Logarithmic de Rham complexes. The sheaf of logarithmic 1-forms $\widetilde{\Omega}_X^1(\log H)$ is the locally free $\widetilde{\mathcal{O}}_X$-module locally generated by $\widetilde{\Omega}_X^1$ and $\widetilde{\mathrm{d}}g/g$ for any local equation g of H. We set $\widetilde{\Omega}_X^k(\log H) = \wedge^k\widetilde{\Omega}_X^1(\log H)$ and we consider the logarithmic de Rham complex $(\widetilde{\Omega}_X^\bullet(\log H), \widetilde{\mathrm{d}})$, which contains $(\widetilde{\Omega}_X^\bullet, \widetilde{\mathrm{d}})$ as a sub-complex. We also consider the corresponding complex where we tensor each term with $\widetilde{\mathcal{O}}_X(-H)$, and with induced differential, that we denote by $\widetilde{\Omega}_X^\bullet(\log H)(-H)$. For each $k \geqslant 0$, the sheaf $\widetilde{\Omega}_X^k(\log H)(-H)$ maps injectively to $\widetilde{\Omega}_X^k$ and the cokernel is $\iota_*\widetilde{\Omega}_H^k$. The morphism $T^*\iota : \widetilde{\Omega}_X^k \to \iota_*\widetilde{\Omega}_H^k$ is the pullback of forms. We then have a natural exact sequence of complexes

$$0 \longrightarrow (\widetilde{\Omega}_X^\bullet(\log H)(-H), \widetilde{\mathrm{d}}) \longrightarrow (\widetilde{\Omega}_X^\bullet, \widetilde{\mathrm{d}}) \longrightarrow \iota_*(\widetilde{\Omega}_H^\bullet, \widetilde{\mathrm{d}}) \longrightarrow 0.$$

(In local coordinates $(t, x_2, \dots, x_n)$, $\widetilde{\Omega}_X^k(\log H)(-H)$ is generated by $\widetilde{\mathrm{d}}t, t\widetilde{\mathrm{d}}x_2, \dots, t\widetilde{\mathrm{d}}x_n$.)

On the other hand, we have an exact sequence

$$0 \longrightarrow (\widetilde{\Omega}^\bullet_X, \widetilde{\mathrm{d}}) \longrightarrow (\widetilde{\Omega}^\bullet_X(\log H), \widetilde{\mathrm{d}}) \xrightarrow{\mathrm{Res}} \iota_*(\widetilde{\Omega}^{\bullet-1}_H, -\widetilde{\mathrm{d}})(-1) \longrightarrow 0,$$

where Res is defined in local coordinates by (setting $I = i_1, \dots, i_k$)

$$\mathrm{Res}\Big(\varphi(t,x)\frac{\widetilde{\mathrm{d}}t}{t} \wedge \widetilde{\mathrm{d}}x_I\Big) = \varphi(0,x)\widetilde{\mathrm{d}}x_I.$$

The Tate twist (-1) is due to the division by $\widetilde{\mathrm{d}}t$, as already noticed in Lemma 11.2.3.

Logarithmic de Rham complexes of $V_0\widetilde{\mathcal{D}}_X$-modules. Let $\widetilde{\mathcal{M}}$ be a left $\widetilde{\mathcal{D}}_X$-module. Let us assume that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$, and let $V^\bullet\widetilde{\mathcal{M}}$ denote the V-filtration of $\widetilde{\mathcal{M}}$ along H. Then $\widetilde{\mathcal{M}} = V^0\widetilde{\mathcal{M}}$ and $V^1\widetilde{\mathcal{M}} = (V^0\widetilde{\mathcal{M}})(-H) := \widetilde{\mathcal{O}}_X(-H) \otimes_{\widetilde{\mathcal{O}}_X} V^0\widetilde{\mathcal{M}}$ (see Section 9.5.b). For a left $V_0\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$, the (unshifted) logarithmic de Rham complex $\mathrm{DR}_{\log H}\widetilde{\mathcal{N}}$ is defined by means of logarithmic forms:

$$\mathrm{DR}_{\log H}\widetilde{\mathcal{N}} = \{0 \to \widetilde{\mathcal{N}} \xrightarrow{\widetilde{\nabla}} \widetilde{\Omega}^1_X(\log H) \otimes \widetilde{\mathcal{N}} \longrightarrow \cdots \longrightarrow \widetilde{\Omega}^n_X(\log H) \otimes \widetilde{\mathcal{N}} \to 0\}.$$

11.2.19. Lemma. *The natural morphism*

$$\mathrm{DR}(\widetilde{\mathcal{M}}[!H]) \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}})$$

is isomorphic to the natural morphism

$$\mathrm{DR}_{\log H}(V^1\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}(V^0\widetilde{\mathcal{M}}) = \mathrm{DR}(\widetilde{\mathcal{M}})$$

and the natural morphism

$$\mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}}[*H])$$

is isomorphic to the natural morphism

$$\mathrm{DR}(\widetilde{\mathcal{M}}) = \mathrm{DR}(V^0\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}_{\log H}(V^0\widetilde{\mathcal{M}}).$$

Proof. Let us treat the case of $\widetilde{\mathcal{M}}[!H]$ for example. The question is local in the neighbourhood of H, and we can assume that $X = \Delta \times H$, where Δ is a disc with coordinate t. Then $\mathrm{DR}(\widetilde{\mathcal{M}})$ is realized as the total complex of the double complex $\mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{M}}) \xrightarrow{\widetilde{\partial}_t} \mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{M}})$ and $\mathrm{DR}_{\log H}(\widetilde{\mathcal{N}})$ as that of $\mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{N}}) \xrightarrow{t\widetilde{\partial}_t} \mathrm{DR}_{X/\Delta}(\widetilde{\mathcal{N}})$.

Since $\widetilde{\mathcal{M}}$ is strictly non-characteristic along H, we have $\widetilde{\mathcal{M}} = V^0\widetilde{\mathcal{M}}$, so that the complex $\{\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\partial}_t} \widetilde{\mathcal{M}}\}$ is equal to $\{V^0\widetilde{\mathcal{M}} \xrightarrow{\widetilde{\partial}_t} V^0\widetilde{\mathcal{M}}\}$. On the other hand, since $\widetilde{\partial}_t : \mathrm{gr}^k_V(\widetilde{\mathcal{M}}[!H]) \to \mathrm{gr}^{k+1}_V(\widetilde{\mathcal{M}}[!H])$ is an isomorphism for any $k \leqslant 0$, the inclusion of complexes

$$\{V^1(\widetilde{\mathcal{M}}[!H]) \xrightarrow{\widetilde{\partial}_t} V^0(\widetilde{\mathcal{M}}[!H])\} \hookrightarrow \{\widetilde{\mathcal{M}}[!H] \xrightarrow{\widetilde{\partial}_t} \widetilde{\mathcal{M}}[!H]\}$$

is a quasi-isomorphism, and since $t : V^0(\widetilde{\mathcal{M}}[!H]) \to V^1(\widetilde{\mathcal{M}}[!H])$ is an isomorphism, we find a quasi-isomorphism

$$\{V^1(\widetilde{\mathcal{M}}[!H]) \xrightarrow{t\widetilde{\partial}_t} V^1(\widetilde{\mathcal{M}}[!H])\} \simeq \{\widetilde{\mathcal{M}}[!H] \xrightarrow{\widetilde{\partial}_t} \widetilde{\mathcal{M}}[!H]\}.$$

Finally, note that $V^k(\widetilde{\mathcal{M}}[!H]) = V^k\widetilde{\mathcal{M}}$ for $k \geqslant 0$. Applying the functor $\mathrm{DR}_{X/\Delta}$ concludes the proof. $\square$

After applying the de Rham functor DR to the exact sequences (11.2.16), we obtain therefore two exact sequences of complexes that are quasi-isomorphic to the exact sequences

$$
\begin{aligned}
0 \longrightarrow \mathrm{DR}_{\log H}(V^1\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H) \longrightarrow 0,\\
0 \longrightarrow \mathrm{DR}(\widetilde{\mathcal{M}}) \longrightarrow \mathrm{DR}_{\log H}(\widetilde{\mathcal{M}}) \xrightarrow{\mathrm{Res}} \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H)[-1](-1) \longrightarrow 0,
\end{aligned}
\tag{11.2.20}
$$

and we can replace restr_H and Gys_H by the connecting morphisms of the hypercohomology sequences attached to these exact sequences, for which we use the same notation.

C^∞ logarithmic de Rham complexes. We will also consider C^∞ variants of these complexes. On the one hand, the pullback of forms extends to the Dolbeault resolutions of $\widetilde{\Omega}^\bullet_X$ and $\widetilde{\Omega}^\bullet_H$. Arguing as in Section 8.4.13, we can realize the right terms of the first line of (11.2.20) by the corresponding C^∞ complexes, so that we have a commutative diagram of exact sequences of complexes in which the last two vertical morphisms are quasi-isomorphisms, hence so is the first one:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{DR}_{\log H}(V^1\widetilde{\mathcal{M}}) \arrow[r] \arrow[d] & \mathrm{DR}(\widetilde{\mathcal{M}}) \arrow[r] \arrow[d, "\wr"'] & \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H) \arrow[r] \arrow[d, "\wr"'] & 0\\
0 \arrow[r] & \widetilde{\mathcal{K}}^\bullet_1 \arrow[r] & \mathrm{DR}^\infty(\widetilde{\mathcal{M}}) \arrow[r] & \iota_* \mathrm{DR}^\infty(\widetilde{\mathcal{M}}_H) \arrow[r] & 0
\end{tikzcd}
\tag{11.2.21}
$$

where $\widetilde{\mathcal{K}}^\bullet_1$ is defined as the kernel of the horizontal morphism of complexes.

On the other hand, we introduce the sheaf $\widetilde{\mathcal{E}}^1_X(\log H)$ of C^∞ logarithmic 1-forms, having as local basis near a point of H the forms $\widetilde{\mathrm{d}}t/t, \mathrm{d}\overline{t}, \widetilde{\mathrm{d}}x_i, \mathrm{d}\overline{x}_i$. The C^∞ logarithmic de Rham complex $(\widetilde{\mathcal{E}}^\bullet_X(\log H), \widetilde{\mathrm{d}})$ contains $(\widetilde{\mathcal{E}}^\bullet_X, \widetilde{\mathrm{d}})$ as a subcomplex, and the corresponding inclusion is quasi-isomorphic to the inclusion $(\widetilde{\Omega}^\bullet_X, \widetilde{\mathrm{d}}) \hookrightarrow (\widetilde{\Omega}^\bullet_X(\log H), \widetilde{\mathrm{d}})$. There is also a residue morphism

$$
\mathrm{Res} : (\widetilde{\mathcal{E}}^\bullet_X(\log H), \widetilde{\mathrm{d}}) \longrightarrow (\iota_*\widetilde{\mathcal{E}}^{\bullet-1}_H, -\widetilde{\mathrm{d}})(-1) = (\iota_*\widetilde{\mathcal{E}}^\bullet_H, \widetilde{\mathrm{d}})[-1](-1)
$$

that is compatible with the holomorphic one. We choose the latter for constructing the C^∞ exact sequence below, leading to a commutative diagram:

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{DR}(\widetilde{\mathcal{M}}) \arrow[r] \arrow[d] & \mathrm{DR}_{\log H}(\widetilde{\mathcal{M}}) \arrow[r, "\mathrm{Res}"] \arrow[d, "\wr"'] & \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H)[-1](-1) \arrow[r] \arrow[d, "\wr"'] & 0\\
0 \arrow[r] & \widetilde{\mathcal{K}}^\bullet_2 \arrow[r] & \mathrm{DR}^\infty_{\log H}(\widetilde{\mathcal{M}}) \arrow[r, "\mathrm{Res}"] & \iota_* \mathrm{DR}^\infty(\widetilde{\mathcal{M}}_H)[-1](-1) \arrow[r] & 0
\end{tikzcd}
\tag{11.2.22}
$$

where $\widetilde{\mathcal{K}}^\bullet_2$ is defined as the kernel of the horizontal morphism of complexes and the left vertical morphism is thus a quasi-isomorphism.

We can instead define the complex $\widetilde{\mathcal{C}}^\bullet$ by replacing the commutative diagram (11.2.21) with

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{DR}_{\log H}(V^1\widetilde{\mathcal{M}}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{DR}(\widetilde{\mathcal{M}}) \arrow[r] \arrow[d, "\wr"'] & \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathrm{DR}^\infty_{\log H}(V^1\widetilde{\mathcal{M}}) \arrow[r] & \mathrm{DR}^\infty(\widetilde{\mathcal{M}}) \arrow[r] & \widetilde{\mathcal{C}}^\bullet \arrow[r] & 0 \end{tikzcd} \tag{11.2.23}$$

and deduce that the right vertical arrow is a quasi-isomorphism. Explicitly, we have $\widetilde{\mathcal{C}}^k = [\widetilde{\mathcal{E}}^k_X \otimes \widetilde{\mathcal{M}}/\widetilde{\mathcal{E}}^k_X(\log H)(-H)] \otimes \widetilde{\mathcal{M}}$. The restriction $T^*\iota$ factorizes through $\widetilde{\mathcal{C}}^k$. We denote by ι^* the natural morphism $\widetilde{\mathcal{E}}^k_X \otimes \widetilde{\mathcal{M}} \to \widetilde{\mathcal{C}}^k$ and by $T'^*\iota$ the morphism $\widetilde{\mathcal{C}}^k \to \iota_*\widetilde{\mathcal{E}}^k_H$, so that $T^*\iota = T'^*\iota \circ \iota^*$. Then $T'^*\iota : \widetilde{\mathcal{C}}^\bullet \to \iota_* \mathrm{DR}^\infty(\widetilde{\mathcal{M}}_H)$ is a quasi-isomorphism.

For the corresponding diagram (11.2.22), we define the morphism

$$\mathrm{Res} : \widetilde{\mathcal{E}}^k_X(\log H) \otimes \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{C}}^k$$

by sending a local section $((\widetilde{\mathrm{d}t/t}) \wedge \psi + \mu) \otimes m$ to $\iota^*(\psi \otimes m)$. We then obtain the commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{DR}(\widetilde{\mathcal{M}}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{DR}_{\log H}(\widetilde{\mathcal{M}}) \arrow[r, "\mathrm{Res}"] \arrow[d, "\wr"'] & \iota_* \mathrm{DR}(\widetilde{\mathcal{M}}_H)[-1](-1) \arrow[r] \arrow[d, "\wr"'] & 0 \\ 0 \arrow[r] & \mathrm{DR}^\infty(\widetilde{\mathcal{M}}) \arrow[r] & \mathrm{DR}^\infty_{\log H}(\widetilde{\mathcal{M}}) \arrow[r, "\mathrm{Res}"] & \widetilde{\mathcal{C}}^\bullet[-1](-1) \arrow[r] & 0 \end{tikzcd} \tag{11.2.24}$$

A representative of the Chern class. Let $\theta \in \Gamma(X, \mathcal{E}^1_X(\log H))$ be any C^∞ logarithmic 1-form on X that can be locally written as $\mathrm{d}t/t + \varphi$ for some C^∞ 1-form φ, where $t = 0$ is a local equation for H. Then $\eta := \mathrm{d}\theta \in \Gamma(X, \mathcal{E}^2_X(\log H))$ belongs to the subspace $\Gamma(X, \mathcal{E}^2_X)$ and is closed.

Let $\mathcal{U} = (U_\alpha)$ be an open covering of X by charts in which $H \cap U_\alpha$ is defined by the equation $t_\alpha = 0$, where t_α is part of a local coordinate system in U_α. If (χ_α) is a partition of unity adapted to this covering, then $\theta = \sum \chi_\alpha \mathrm{d}t_\alpha/t_\alpha$ satisfies the above hypothesis.

11.2.25. Lemma. *For θ as above, the cohomology class of $\eta = \mathrm{d}\theta$ in $H^2(X, \mathcal{E}^\bullet_X) \simeq H^2(X, \mathbb{C})$ is equal to the complexified Chern class $(2\pi\mathrm{i})c_1(\mathcal{O}_X(H))$.*

Proof. We can realize $H^2(X, \mathbb{C})$ as the cohomology of total complex of the Čech double complex $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{E}^\bullet_X)$, with Čech differential δ and de Rham differential d. We consider the cochain $(\theta_\alpha) \in \Gamma(\mathcal{U}, \mathcal{E}^1_X(\log H)) = \mathcal{C}^0(\mathcal{U}, \mathcal{E}^1_X(\log H))$ defined as $\theta_\alpha = \mathrm{d}t_\alpha/t_\alpha$. We have $\mathrm{d}\theta_\alpha = 0$ and the class of $\delta(\theta_\alpha)$ in $H^2(X, \mathbb{C})$ is $(2\pi\mathrm{i})c_1(\mathcal{O}_X(H))$. So the class of $(\delta + \mathrm{d})(\theta_\alpha)$ is equal to $(2\pi\mathrm{i})c_1(\mathcal{O}_X(H))$.

On the other hand, let us consider the cochain $(\theta_{|U_\alpha}) \in \Gamma(\mathcal{U}, \mathcal{E}^1_X(\log H))$. Its δ-differential is zero, and the class of $(\delta + \mathrm{d})(\theta_{|U_\alpha})$ in $H^2(X, \mathbb{C})$ is equal to that of $\mathrm{d}\theta$.

We end the proof by noting that the difference $(\delta+\mathrm{d})\big((\theta_{|U_\alpha})-(\theta_\alpha)\big)$ is a coboundary in the total complex, since $\theta_{|U_\alpha} - \theta_\alpha \in \Gamma(U_\alpha, \mathcal{E}^1_X)$. $\square$

***End of the proof of Proposition* 11.2.18.** Let us start with the right triangle. We will make use of the complexes in (11.2.21) and (11.2.22). Let $m \in \Gamma(X, \widetilde{\mathcal{E}}_X^k \otimes \widetilde{\mathcal{M}})$ be a closed global section of $\widetilde{\mathcal{E}}_X^k \otimes \widetilde{\mathcal{M}}$ and let $[m]$ denote its cohomology class. Then $\mathrm{restr}_H([m])$ is the cohomology class of the image $\mathrm{restr}_H(m)$ of m by the restriction morphism induced by the lower line of (11.2.21). In order to compute $\mathrm{Gys}_H(\mathrm{restr}_H([m]))$, one has to make explicit the connecting morphism coming from the lower line of (11.2.22). One has to choose a lift μ of $\mathrm{restr}_H(m)$ in the space $\Gamma(X, \widetilde{\mathcal{E}}_X^{k+1}(\log H) \otimes \widetilde{\mathcal{M}})$, differentiate is as $D\mu$, where D is the differential of the complex $\mathrm{DR}^\infty_{\log H} \widetilde{\mathcal{M}}$; then $D\mu$ belongs to $\Gamma(X, \mathcal{K}_2^{k+2})$ and is closed there, defining thus a class in $\boldsymbol{H}^{k+2}(X, \mathcal{K}_2^\bullet) \simeq \boldsymbol{H}^{k+2}(X, \mathrm{DR}\, \widetilde{\mathcal{M}})$.

Let us make explicit this process. We set $\widetilde{\theta} = \theta/z$ with θ as in Lemma 11.2.25. One can take $\mu = \widetilde{\theta} \wedge m$ as a lift of $\mathrm{restr}_H(m)$. Since m is closed, we have $D\mu = \widetilde{\mathrm{d}}\widetilde{\theta} \wedge m = \widetilde{\eta} \wedge m$, whose cohomology class is $\mathrm{X}_{\mathcal{L}}([m])$, according to Lemma 11.2.25, as desired.

Let us consider now the left triangle, for which we will make use of the complexes in (11.2.23) and (11.2.24). Let $[m]$ be a cohomology class in $H^{k-1}(X, \widetilde{\mathcal{C}}^\bullet)$. We also denote by $[m]$ a representative in $\Gamma(X, \widetilde{\mathcal{C}}^{k-1})$. Let $m \in \Gamma(X, \widetilde{\mathcal{E}}_X^{k-1} \otimes \widetilde{\mathcal{M}})$ be a lift of $[m]$, that is, such that $\iota^* m = [m]$. A lift of $[m]$ by Res can be represented as $\widetilde{\theta} \wedge m$ and the composition $\mathrm{restr}_H \circ \mathrm{Gys}_H([m])$ is the class of $\iota^*(D(\widetilde{\theta} \wedge m))$. Since the class of $\iota^*(\widetilde{\mathrm{d}}\widetilde{\theta} \wedge m)$ is the desired class $\mathrm{X}_{\mathcal{L}}([m])$, it remains to show that the class of $\iota^*(\widetilde{\theta} \wedge Dm)$ is zero. Since $[m]$ is closed, Dm is a section of $\widetilde{\mathcal{E}}_X^k(\log H)(-H) \otimes \widetilde{\mathcal{M}}$. It follows that $\widetilde{\theta} \wedge Dm$ is a section of $\widetilde{\mathcal{E}}_X^{k+1} \otimes \widetilde{\mathcal{M}}$ which is locally a multiple of $\widetilde{\mathrm{d}}t$. As a consequence, the class of $T^*\iota((\widetilde{\theta} \wedge Dm)) = T'^*\iota \circ \iota^*((\widetilde{\theta} \wedge Dm))$ is zero and since $T'^*\iota$ is a quasi-isomorphism, the class of $\iota^*(\widetilde{\theta} \wedge Dm)$ is zero. □

11.2.e. The weak Lefschetz property. Although we cannot assert in such generality that the diagram of Proposition 11.2.18 defines an X-$\mathfrak{sl}_2$-quiver with $H_k = \bigoplus_k {}_{\mathrm{D}}a_*^{(k)} \widetilde{\mathcal{M}}$, $G_k = {}_{\mathrm{D}}a_*^{(k)} \widetilde{\mathcal{M}}_H$, $\mathrm{c} = \mathrm{restr}_H$, $\mathrm{v} = \mathrm{Gys}_H$ (see Remark 3.1.9), we give a criterion for the weak Lefschetz property of this quiver to hold (see Definition 3.1.13). It will be used in the proof of the Hodge-Saito theorem 14.3.1.

***11.2.26. Proposition* (A criterion for the weak Lefschetz property)**

Let $f: X \to Y$ be a morphism between smooth projective varieties, and let H be a smooth hypersurface of X.

(1) *Assume that H is a divisor of the line bundle $\mathcal{O}_X(1)$.*

(2) *Assume that $\widetilde{\mathcal{M}}$ is coherent, strict, and that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$.*

Let $\mathrm{restr}_H : {}_{\mathrm{D}}f_*^{(k)} \widetilde{\mathcal{M}} \to {}_{\mathrm{D}}f_*^{(k+1)} \widetilde{\mathcal{M}}_H$ *(resp.* $\mathrm{Gys}_H : {}_{\mathrm{D}}f_*^{(k)} \widetilde{\mathcal{M}}_H \to {}_{\mathrm{D}}f_*^{(k+1)} \widetilde{\mathcal{M}}(1)$*) be the connecting morphisms obtained by applying ${}_{\mathrm{D}}f_*$ to the exact sequences* (11.2.16).

(3) *Last, assume that, for all $k \in \mathbb{Z}$,* restr_H *(resp.* Gys_H*) is a strict morphism.*

Then $\mathrm{restr}_H : {}_{\mathrm{D}}f_*^{(k)} \widetilde{\mathcal{M}} \to {}_{\mathrm{D}}f_*^{(k+1)} \widetilde{\mathcal{M}}_H$ *(resp.* $\mathrm{Gys}_H : {}_{\mathrm{D}}f_*^{(k)} \widetilde{\mathcal{M}}_H \to {}_{\mathrm{D}}f_*^{(k+1)} \widetilde{\mathcal{M}}(1)$*) is an isomorphism if $k \geqslant 1$ and is onto if $k = 0$.*

Proof. According to the long exact sequence deduced from the first (resp. second) line (11.2.16), it is a matter of proving that ${}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}[*H]) = 0$ for $k \geqslant 1$. The strictness assumption (3) implies that ${}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}[*H])$ is strict for any k. It is then enough to prove that the $\mathcal{D}_Y$-module underlying ${}_{\mathrm{D}}f_*^{(k)}(\widetilde{\mathcal{M}}[*H])$ is zero for $k \geqslant 1$. This module is nothing but the pushforward of the $\mathcal{D}_X$-module underlying $\widetilde{\mathcal{M}}[*H]$, that is, $\mathcal{M}(*H)$.

For the final part of the argument, it will be convenient to express the pushforward complex ${}_{\mathrm{D}}f_*\mathcal{M}(*H)$ as a complex in nonnegative degrees. We will thus make use of Formula (8.51 $*$) with no shift. Furthermore, we recall that, since $X \smallsetminus H$ is affine, for any coherent $\mathcal{O}_X$-module $\mathcal{F}$, the pushforward $R^{n+k}f_*\mathcal{F}(*H)$ vanishes for $k \geqslant 1$.[(1)]

Since $\widetilde{\mathcal{M}}$ is strict, $\mathcal{M}$ admits a coherent filtration $F_\bullet\mathcal{M}$ (Proposition 8.8.4(2)). Together with the filtration of $\mathcal{D}_Y$ by the order of differential operators, we obtain a filtration $\widetilde{\Omega}_X^k \otimes (\widetilde{\mathcal{M}}^{\mathrm{left}} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$ by $\mathcal{O}_X$-coherent modules for any k, by means of which we derive a filtration $F_pC^\bullet$ of the complex

$$C^\bullet := \widetilde{\Omega}_X^\bullet \otimes (\widetilde{\mathcal{M}}^{\mathrm{left}}(*H) \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y)$$

whose terms take the form $\mathcal{F}(*H)$ with $\mathcal{F}$ being $\mathcal{O}_X$-coherent (see 8.4.9). This complex is in nonnegative degrees (we did not shift it as in (8.51 $*$)) and $R^{n+k}f_*$ of each of its terms vanishes for $k \geqslant 1$. Therefore, $R^{n+k}f_*(F_pC^\bullet) = 0$ for each p and each $k \geqslant 1$. Passing to the inductive limit (f is proper), we conclude that $R^{n+k}f_*(C^\bullet) = 0$ for $k \geqslant 1$, which is the desired assertion. $\square$

11.3. Localization of $\widetilde{\mathcal{D}}_X$-modules

Our aim in this section is to define, for any effective divisor D in X, a localization functor with values in the category of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules along D. In the case of $\mathcal{D}_X$-modules, the localization coincides with the naive localization, but we will present the localization in a uniform way for $\mathcal{D}_X$-modules and graded $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$-modules with our usual convention for the meaning of $\widetilde{\mathcal{D}}_X$ and of strictness.

11.3.a. Localization along a smooth hypersurface for $\widetilde{\mathcal{D}}_X$-modules

If $\widetilde{\mathcal{M}}$ is a coherent graded $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$-module which is strictly $\mathbb{R}$-specializable, we cannot assert that $\widetilde{\mathcal{M}}(*H)$ is coherent. However, the natural morphism $V_{<0}\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(*H)$ is injective since $V_{<0}\widetilde{\mathcal{M}}$ has no $\mathcal{I}_H$-torsion. For $\alpha \in [-1, 0)$ and $k \geqslant 1$, let us set

$$V_{\alpha+k}\widetilde{\mathcal{M}}(*H) = V_\alpha\widetilde{\mathcal{M}}t^{-k} \subset \widetilde{\mathcal{M}}(*H),$$

[(1)] Let us recall the proof: by considering the order of the pole along H, $\mathcal{F}(*H)$ is the inductive limit of $\mathcal{O}_X$-coherent submodules $\mathcal{F}(*H)_\ell$ and, since f is proper, $R^{n+k}f_*\mathcal{F}(*H) = \varinjlim_\ell R^{n+k}f_*\mathcal{F}(*H)_\ell$; by GAGA, each $\mathcal{F}(*H)_\ell$ is the analytification of a coherent $\mathcal{O}_{X^{\mathrm{alg}}}$-module, and the pushforward, as well as its inductive limit, can be computed with the Zariski topology; the latter is then equal to $R^{n+k}f_*^{\mathrm{alg}}\mathcal{F}^{\mathrm{alg}}_{|X\smallsetminus H}$, whose germ in $y \in Y$ is the inductive limit, taken on the affine open neighborhoods V of y, of the cohomologies $H^{n+k}(f^{-1}_{|X\smallsetminus H}(V), \mathcal{F}^{\mathrm{alg}}_{|X\smallsetminus H})$; since $f^{-1}_{|X\smallsetminus H}(V)$ is affine, each such cohomology vanishes if $k \geqslant 1$.

where t is any local reduced equation of H. Each $V_\gamma\widetilde{\mathcal{M}}(*H)$ is a coherent $V_0\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\mathcal{M}}(*H)$, which satisfies $V_\gamma\widetilde{\mathcal{M}}(*H)t = V_{\gamma-1}\widetilde{\mathcal{M}}(*H)$ and $V_\gamma\widetilde{\mathcal{M}}(*H)\widetilde{\partial}_t \subset V_{\gamma+1}\widetilde{\mathcal{M}}(*H)$ (multiply both terms by t). Last, each $\mathrm{gr}_\gamma^V\widetilde{\mathcal{M}}(*H)$ is strict, being isomorphic to $\mathrm{gr}^V_{\gamma-[\gamma]-1}\widetilde{\mathcal{M}}$ if $\gamma \geqslant 0$.

11.3.1. Definition **(Localization of strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X$-modules)**

For a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H, the localized module is (see 9.3.20(b))

$$\widetilde{\mathcal{M}}[*H] = V_0(\widetilde{\mathcal{M}}(*H)) \cdot \widetilde{\mathcal{D}}_X \subset \widetilde{\mathcal{M}}(*H).$$

11.3.2. Remark. The construction of $\widetilde{\mathcal{M}}[*H]$ only depends on the $\widetilde{\mathcal{D}}_X(*H)$-module $\widetilde{\mathcal{M}}(*H)$, provided it is strictly $\mathbb{R}$-specializable in the sense given in the introduction of this chapter. In Proposition 11.3.3 below, we could have started from such a module.

11.3.3. Proposition **(Properties of the localization along H).** *Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Then we have the following properties.*

(1) *$\widetilde{\mathcal{M}}[*H]$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H.*

(2) *The natural morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}(*H)$ factorizes through $\widetilde{\mathcal{M}}[*H]$, so defines a morphism* $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ *and induces an isomorphism*

$$V_{<0}\widetilde{\mathcal{M}} \longrightarrow V_{<0}\widetilde{\mathcal{M}}[*H],$$

and in particular

$$\mathrm{gr}_\gamma^V \mathrm{loc} : \mathrm{gr}_\gamma^V\widetilde{\mathcal{M}} \xrightarrow{\sim} \mathrm{gr}_\gamma^V\widetilde{\mathcal{M}}[*H] \quad \textit{for any } \gamma \in [-1, 0).$$

Moreover, if $X \simeq H \times \Delta_t$, $\mathrm{Ker}\,\mathrm{loc}$ *(resp.* $\mathrm{Coker}\,\mathrm{loc}$*) is isomorphic to the kernel (resp. cokernel) of ${}_{\mathrm{D}}\iota_{H*}t : {}_{\mathrm{D}}\iota_{H*}(\mathrm{gr}_0^V\widetilde{\mathcal{M}}) \to {}_{\mathrm{D}}\iota_{H*}(\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}})$.*

(3) *For every γ, we have $V_\gamma\widetilde{\mathcal{M}}[*H] = V_\gamma\widetilde{\mathcal{M}}(*H) \cap \widetilde{\mathcal{M}}[*H]$ and, for $\gamma \leqslant 0$, we have $V_\gamma\widetilde{\mathcal{M}}[*H] = V_\gamma\widetilde{\mathcal{M}}(*H)$.*

(4) *We have, with respect to a local product decomposition $X \simeq H \times \Delta_t$,*

$$V_\gamma\widetilde{\mathcal{M}}[*H] = \begin{cases} V_\gamma\widetilde{\mathcal{M}} & \textit{if } \gamma < 0, \\ V_0\widetilde{\mathcal{M}}(*H) = V_{-1}\widetilde{\mathcal{M}} \cdot t^{-1} & \textit{if } \gamma = 0, \\ V_{\gamma-[\gamma]-1}\widetilde{\mathcal{M}}\widetilde{\partial}_t^{[\gamma]+1} + \displaystyle\sum_{j=0}^{[\gamma]} V_0\widetilde{\mathcal{M}}(*H)\widetilde{\partial}_t^j & \textit{in } \widetilde{\mathcal{M}}(*H),\ \textit{if } \gamma > 0. \end{cases}$$

(5) $(\widetilde{\mathcal{M}}[*H]/(z-1)\widetilde{\mathcal{M}}[*H]) = (\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}})(*H)$, *and* $\widetilde{\mathcal{M}}[*H][z^{-1}] = \widetilde{\mathcal{M}}(*H)[z^{-1}]$.

(6) *If t is a local generator of $\mathcal{I}_H$, the multiplication by t induces an isomorphism $\mathrm{gr}_0^V\widetilde{\mathcal{M}}[*H] \xrightarrow{\sim} \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}[*H]$.*

(7) $\widetilde{\mathcal{M}}[*H] = V_0(\widetilde{\mathcal{M}}(*H)) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$.

(8) *Assume $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'$ is a morphism between strictly $\mathbb{R}$-specializable coherent $\widetilde{\mathcal{D}}_X$-modules which induces an isomorphism $\widetilde{\mathcal{M}}(*H) \to \widetilde{\mathcal{M}}'(*H)$ (i.e., whose restriction to $V_{<0}$ is an isomorphism). Assume moreover that $\widetilde{\mathcal{M}}'$ satisfies* (6)*, i.e., the multiplication by t induces an isomorphism $\mathrm{gr}_0^V\widetilde{\mathcal{M}}' \xrightarrow{\sim} \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}'$. Then $\widetilde{\mathcal{M}}' \simeq \widetilde{\mathcal{M}}[*H]$.*

More precisely, the induced morphism $\widetilde{\mathcal{M}}[*H] \to \widetilde{\mathcal{M}}'[*H]$ *is an isomorphism, as well as* $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}'[*H]$.

(9) *Let* $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'$ *be as in* (8). *Then any morphism* $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ *factorizes through* $\widetilde{\mathcal{M}}'[*H]$. *In particular, if* $\widetilde{\mathcal{M}}'$ *is supported on* H, *such a morphism is zero.*

(10) *If* $\widetilde{\mathcal{M}}$ *is strict, then so is* $\widetilde{\mathcal{M}}[*H]$.

(11) *Let* $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ *be an exact sequence of coherent strictly* $\mathbb{R}$*-specializable* $\widetilde{\mathcal{D}}_X$*-modules. Then the sequence*

$$0 \longrightarrow \widetilde{\mathcal{M}}'[*H] \longrightarrow \widetilde{\mathcal{M}}[*H] \longrightarrow \widetilde{\mathcal{M}}''[*H] \longrightarrow 0$$

is exact.

Proof. The $\widetilde{\mathcal{D}}_X$-coherence of $\widetilde{\mathcal{M}}[*H]$ is clear, by definition. Let us set $U_\alpha\widetilde{\mathcal{M}}[*H] = V_\alpha(\widetilde{\mathcal{M}}(*H)) \cap \widetilde{\mathcal{M}}[*H]$ as in (3). Our first goal is to show both that $\widetilde{\mathcal{M}}[*H]$ is strictly $\mathbb{R}$-specializable and that $U_\bullet\widetilde{\mathcal{M}}[*H]$ is its Kashiwara-Malgrange filtration.

Note that $U_\alpha\widetilde{\mathcal{M}}[*H]$ is a coherent $V_0\widetilde{\mathcal{D}}_X$-submodule of $\widetilde{\mathcal{M}}[*H]$ (locally, $\widetilde{\mathcal{M}}[*H]$ has a coherent V-filtration, which induces on $V_\alpha(\widetilde{\mathcal{M}}(*H))$ a filtration by coherent $V_0\widetilde{\mathcal{D}}_X$-submodules, which is thus locally stationary since $V_\alpha(\widetilde{\mathcal{M}}(*H))$ is $V_0\widetilde{\mathcal{D}}_X$-coherent). It satisfies in an obvious way the following local properties:

- $U_\alpha\widetilde{\mathcal{M}}[*H]t \subset U_{\alpha-1}\widetilde{\mathcal{M}}[*H]$,
- $U_\alpha\widetilde{\mathcal{M}}[*H]\widetilde{\partial}_t \subset U_{\alpha+1}\widetilde{\mathcal{M}}[*H]$,
- $\mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}[*H] \subset \mathrm{gr}^V_\alpha\widetilde{\mathcal{M}}(*H)$ is strict.

Since by definition $V_0\widetilde{\mathcal{M}}(*H) \subset \widetilde{\mathcal{M}}[*H]$, it is clear that $U_\alpha\widetilde{\mathcal{M}}[*H] = V_\alpha\widetilde{\mathcal{M}}(*H)$ for $\alpha \leqslant 0$, and thus $U_\alpha\widetilde{\mathcal{M}}[*H]t = U_{\alpha-1}\widetilde{\mathcal{M}}[*H]$ for such an α. To prove our assertion, we will check that $U_\alpha\widetilde{\mathcal{M}}[*H] = U_{<\alpha}\widetilde{\mathcal{M}}[*H] + U_{\alpha-1}\widetilde{\mathcal{M}}[*H]\widetilde{\partial}_t$ for $\alpha > 0$, i.e., $\widetilde{\partial}_t : \mathrm{gr}^U_{\alpha-1}\widetilde{\mathcal{M}}[*H] \to \mathrm{gr}^U_\alpha\widetilde{\mathcal{M}}[*H]$ is onto. We will prove the following assertion, which is enough for our purpose:

11.3.4. Assertion. *For every* $\alpha \in [-1, 0)$ *and* $k \geqslant 1$, *if* $m := \sum_{j=0}^N m_j\widetilde{\partial}_t^j \in V_{\alpha+k}\widetilde{\mathcal{M}}(*H)$ *with* $m_j \in V_0\widetilde{\mathcal{M}}(*H)$ $(j = 0, \dots, N)$, *then one can re-write* m *as a similar sum with* $N \leqslant k$ *and* $m_k \in V_\alpha\widetilde{\mathcal{M}}(*H)$.

Let us first reduce to $N \leqslant k$. If $N > k$, we have $m_N\widetilde{\partial}_t^N \in V_{N-1}\widetilde{\mathcal{M}}(*H)$, which is equivalent to $m_N\widetilde{\partial}_t^N t^N \in V_{-1}\widetilde{\mathcal{M}}(*H)$ by definition. We note that, by strictness, $\widetilde{\partial}_t^N t^N$ is injective on $\mathrm{gr}^V_\delta\widetilde{\mathcal{M}}(*H)$ for $\delta > -1$. We conclude that $m_N \in V_{-1}\widetilde{\mathcal{M}}(*H)$. We can set $m'_{N-1} = m_{N-1} + m_N\widetilde{\partial}_t \in V_0\widetilde{\mathcal{M}}(*H)$ and decrease N by one. We can thus assume that $N = k$.

If $m_k \in V_\gamma\widetilde{\mathcal{M}}(*H)$ with $\gamma > \alpha$, we argue as above that $m_k t^k\widetilde{\partial}_t^k \in V_\alpha\widetilde{\mathcal{M}}(*H)$, hence $m_k \in V_{<\gamma}\widetilde{\mathcal{M}}(*H)$ by the same argument as above, and we finally find $m_k \in V_\alpha\widetilde{\mathcal{M}}(*H)$. Now, (1) and (3) are proved, and (2) is then clear (according to Example 9.3.27 for the last statement), as well as (4). Then (5) means that, for $\mathcal{D}_X$-modules, there is no difference between $\widetilde{\mathcal{M}}[*H]$ and $\widetilde{\mathcal{M}}(*H)$, which is true since $\widetilde{\mathcal{M}}(*H)$ is $\mathbb{R}$-specializable, so $\mathcal{D}_X$-generated by $V_0\widetilde{\mathcal{M}}(*H)$.

For (6), we note that, by (3), $\mathrm{gr}^V_0\widetilde{\mathcal{M}}[*H] = \mathrm{gr}^V_0\widetilde{\mathcal{M}}(*H)$ and $\mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[*H] = \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}(*H)$, and by definition $t : \mathrm{gr}^V_0\widetilde{\mathcal{M}}(*H) \xrightarrow{\sim} \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}(*H)$ is an isomorphism.

Let us now prove (7). Set $\widetilde{\mathcal{M}}' = V_0(\widetilde{\mathcal{M}}(*H)) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$. By definition, we have a natural surjective morphism $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ and the composition $V_0(\widetilde{\mathcal{M}}(*H)) \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ is injective, where the first morphism is defined by $m \mapsto m \otimes 1$. We thus have $V_0(\widetilde{\mathcal{M}}(*H)) \subset \widetilde{\mathcal{M}}'$ and we set $V_k\widetilde{\mathcal{M}}' = \sum_{j=0}^k V_0\widetilde{\mathcal{M}}(*H)\widetilde{\partial}_t^j$ for $k \geqslant 0$. Let us check that, for $k \geqslant 1$, $\widetilde{\partial}_t^k : \mathrm{gr}_0^V\widetilde{\mathcal{M}}' \to \mathrm{gr}_k^V\widetilde{\mathcal{M}}'$ is injective. We have a commutative diagram (here gr_k^V means V_k/V_{k-1})

$$\begin{tikzcd}
\mathrm{gr}_0^V\widetilde{\mathcal{M}}' \arrow[r, "\widetilde{\partial}_t^k"] \arrow[d, "\wr"'] & \mathrm{gr}_k^V\widetilde{\mathcal{M}}' \arrow[d] \\
\mathrm{gr}_0^V\widetilde{\mathcal{M}}[*H] \arrow[r, "\widetilde{\partial}_t^k", "\sim"'] & \mathrm{gr}_k^V\widetilde{\mathcal{M}}[*H]
\end{tikzcd}$$

where the lower horizontal isomorphism follows from strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}[*H]$ and Proposition 9.3.20(d). Therefore, the upper horizontal arrow is injective. Note that it is onto by definition. As a consequence, all arrows are isomorphisms, and it follows, by taking the inductive limit on k, that $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ is an isomorphism.

For (8) we notice that, since $V_0\widetilde{\mathcal{M}}(*H) \xrightarrow{\sim} V_0\widetilde{\mathcal{M}}'(*H)$ and according to (7), we have $\widetilde{\mathcal{M}}[*H] \xrightarrow{\sim} \widetilde{\mathcal{M}}'[*H]$. Since $\widetilde{\mathcal{M}}'$ is strictly $\mathbb{R}$-specializable and satisfies (6), we have $\widetilde{\mathcal{M}}' \subset \widetilde{\mathcal{M}}'(*H)$ and $V_0\widetilde{\mathcal{M}}' = V_0\widetilde{\mathcal{M}}'(*H)$. Still due to the strict $\mathbb{R}$-specializability, $\widetilde{\mathcal{M}}'$ is generated by $V_0\widetilde{\mathcal{M}}'$, hence we conclude by Definition 11.3.1.

For (9), we remark that a morphism $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}[*H]$ induces a morphism $\widetilde{\mathcal{M}}'(*H) \to \widetilde{\mathcal{M}}[*H](*H) = \widetilde{\mathcal{M}}(*H)$ and thus $V_0\widetilde{\mathcal{M}}'(*H) \to V_0\widetilde{\mathcal{M}}(*H)$, hence the first assertion follows (7). The second assertion is then clear, since $\widetilde{\mathcal{M}}'[*H] \subset \widetilde{\mathcal{M}}'(*H)$.

(10) holds since, if $\widetilde{\mathcal{M}}$ is strict, then $\widetilde{\mathcal{M}}(*H)$ is also strict, and thus so is $\widetilde{\mathcal{M}}[*H]$.

It remains to prove (11). By flatness of $\widetilde{\mathcal{O}}_X(*H)$ over $\widetilde{\mathcal{O}}_X$, the sequence

$$0 \longrightarrow \widetilde{\mathcal{M}}'(*H) \longrightarrow \widetilde{\mathcal{M}}(*H) \longrightarrow \widetilde{\mathcal{M}}''(*H) \longrightarrow 0$$

is exact, and by Exercise 9.22(2), the sub-sequence

$$0 \longrightarrow V_{-1}\widetilde{\mathcal{M}}' \longrightarrow V_{-1}\widetilde{\mathcal{M}} \longrightarrow V_{-1}\widetilde{\mathcal{M}}'' \longrightarrow 0$$

is also exact. It follows that the sequence

$$0 \longrightarrow V_0\widetilde{\mathcal{M}}'(*H) \longrightarrow V_0\widetilde{\mathcal{M}}(*H) \longrightarrow V_0\widetilde{\mathcal{M}}''(*H) \longrightarrow 0$$

is exact. By (7) we conclude that the sequence

$$\widetilde{\mathcal{M}}'[*H] \longrightarrow \widetilde{\mathcal{M}}[*H] \longrightarrow \widetilde{\mathcal{M}}''[*H] \longrightarrow 0$$

is exact. Since $\widetilde{\mathcal{M}}[*H] \subset \widetilde{\mathcal{M}}(*H)$, the injectivity of $\widetilde{\mathcal{M}}'[*H] \to \widetilde{\mathcal{M}}[*H]$ is clear. $\square$

11.3.5. Remark. The kernel of $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ is strictly $\mathbb{R}$-specializable along H and supported on H. Indeed, by Proposition 11.3.3(2), in any local setting $X = H \times \Delta_t$, it is equal to the pushforward by ι_H of the kernel of $\mathrm{var} : \phi_{t,1}\widetilde{\mathcal{M}} \to \psi_{t,1}\widetilde{\mathcal{M}}(-1)$, which is strict. In particular, $\mathrm{loc} : \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ is injective if and only if, in any local setting, var_t is injective.

On the other hand, $\mathrm{Coker}\,\mathrm{loc}$, which is also supported on H, may not be strictly $\mathbb{R}$-specializable along H without any further hypothesis. It is so if and only if, in any local setting, Coker var_t is strict, i.e., the morphism var_t is strictly $\mathbb{R}$-specializable. For example, it is so if $\widetilde{\mathcal{M}}$ is strongly strictly $\mathbb{R}$-specializable along H (Definition 9.3.24).

11.3.6. Remark **(Side-changing and localization).** If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, we define $\widetilde{\mathcal{M}}[*H]$ as the submodule of $\widetilde{\mathcal{M}}(*H)$ generated by $V^{-1}\widetilde{\mathcal{M}}$. We will check that $\widetilde{\mathcal{M}}^{\mathrm{right}}[*H] \simeq (\widetilde{\mathcal{M}}[*H])^{\mathrm{right}}$. This relation clearly holds for the naive localization, i.e., if we replace $[*H]$ with $(*H)$. Then the morphism $\widetilde{\mathcal{M}}^{\mathrm{right}} \to (\widetilde{\mathcal{M}}[*H])^{\mathrm{right}} = \widetilde{\mathcal{M}}'$ obtained by side-changing from the natural morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*H]$ satisfies the assumptions of Proposition 11.3.3(8), proving the desired isomorphism. If H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$, we then have $\widetilde{\mathcal{M}}[*H] \simeq \widetilde{\mathcal{O}}_X[*H] \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{M}}$.

11.3.b. Localization along an effective divisor

Let $g : X \to \mathbb{C}$ be a holomorphic function. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). We say that $\widetilde{\mathcal{M}}$ is *localizable along* (g) if there exists a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$ such that $({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[*H] = {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{N}}$. Recall indeed that Kashiwara's equivalence is not strong enough in the filtered case in order to ensure the existence of $\widetilde{\mathcal{N}}$. Nevertheless, by full faithfulness, if $\widetilde{\mathcal{N}}$ exists, it is unique, and we denote it by $\widetilde{\mathcal{M}}[*g]$. At this point, some checks are in order.

- Assume that g is smooth. Then one can check (Exercise 11.1) that $\widetilde{\mathcal{M}}[*g]$ as defined by 11.3.1 satisfies the defining property above, so there is no discrepancy between Definition 11.3.1 and the definition above.
- By uniqueness, the local existence of $\widetilde{\mathcal{M}}[*g]$ implies its global existence.
- Let u be an invertible holomorphic function on X. We denote by $\varphi_u : X \times \mathbb{C} \to X \times \mathbb{C}$ the isomorphism defined by $(x,t) \mapsto (x, u(x)t)$, so that $\iota_{ug} = \varphi_u \circ \iota_g$. We continue to set $H = X \times \{0\}$, so that φ_u induces the identity on H.

Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along (g). If $\widetilde{\mathcal{M}}$ is localizable along (g), then it is so along (ug) and we have $\widetilde{\mathcal{M}}[*g] = \widetilde{\mathcal{M}}[*ug]$. Indeed, one checks that

$$ {}_{\mathrm{D}}\varphi_{u*}\big(({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[*H]\big) = ({}_{\mathrm{D}}\iota_{ug*}\widetilde{\mathcal{M}})[*H], $$

and this implies $({}_{\mathrm{D}}\iota_{ug*}\widetilde{\mathcal{M}})[*H] = {}_{\mathrm{D}}\iota_{ug*}(\widetilde{\mathcal{M}}[*g])$, hence the assertion by uniqueness.

11.3.7. Definition **(Localization along an effective divisor).** Let D be an effective divisor on X. We then say that $\widetilde{\mathcal{M}}$ is *localizable along* D if $\widetilde{\mathcal{M}}$ is a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along D (see Definition 9.4.1) and such that $\widetilde{\mathcal{M}}[*g]$ exists locally for some (or any) local equation g defining the divisor D. The localized module, obtained by gluing the various local $\widetilde{\mathcal{M}}[*g]$, is denoted by $\widetilde{\mathcal{M}}[*D]$, and the complex $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ is denoted by $R\Gamma_{[D]}\widetilde{\mathcal{M}}$.

11.3.8. Corollary **(Properties of the localization along (g)).** *Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along (g). Set $H = X \times \{0\} \subset X \times \mathbb{C}$. Assume moreover that $\widetilde{\mathcal{M}}$ is localizable along (g).*

(1) *The $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[*g]$ is strictly $\mathbb{R}$-specializable along (g) and*

$$ \mathrm{var} : \phi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big) \longrightarrow \psi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big)(-1) $$

is an isomorphism.

(2) *There exists a natural morphism* $\mathrm{loc}:\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}[*g]$. *This morphism induces an isomorphism*

$$\widetilde{\mathcal{M}}(*g)\xrightarrow{\sim}\big(\widetilde{\mathcal{M}}[*g]\big)(*g),$$

and isomorphisms

$$\psi_{g,\lambda}\widetilde{\mathcal{M}}\xrightarrow{\sim}\psi_{g,\lambda}\big(\widetilde{\mathcal{M}}[*g]\big)\quad \textit{for every } \lambda.$$

Moreover, we have a commutative diagram

$$\begin{tikzcd}
\phi_{g,1}\widetilde{\mathcal{M}} \arrow[r, "\phi_{g,1}\mathrm{loc}"] \arrow[d, "\mathrm{var}_{\widetilde{\mathcal{M}}}"'] & \phi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big) \arrow[d, "\mathrm{var}_{\widetilde{\mathcal{M}}[*g]}", "\wr"'] \\
\psi_{g,1}\widetilde{\mathcal{M}}(-1) \arrow[r, "\psi_{g,1}\mathrm{loc}", "\sim"'] & \psi_{g,1}\big(\widetilde{\mathcal{M}}[*g]\big)(-1)
\end{tikzcd}$$

and $\mathrm{Ker}\,\mathrm{loc}$ *(resp.* $\mathrm{Coker}\,\mathrm{loc}$*) is identified with* $\mathrm{Ker}\,\mathrm{var}_{\widetilde{\mathcal{M}}}$ *(resp.* $\mathrm{Coker}\,\mathrm{var}_{\widetilde{\mathcal{M}}}$*).*

(3) *Given a short exact sequence of coherent* $\widetilde{\mathcal{D}}_X$*-modules which are strictly* $\mathbb{R}$*-specializable and localizable along* (g)*, the* $[*g]$ *sequence is exact.*

Proof. This follows from Proposition 11.3.3 by using full faithfulness of ${}_{\mathrm{D}}\iota_{g*}$ (Proposition 9.6.2) and Proposition 9.6.6. □

11.3.9. Remark. The proof gives in particular that ${}_{\mathrm{D}}\iota_{g*}\mathrm{loc}_g=\mathrm{loc}_t$.

***11.3.10. Remark* (Remark 11.3.2 continued).** One easily checks that ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}(*g))=({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})(*H)$, so that, in Corollary 11.3.8, we could start from a coherent $\widetilde{\mathcal{D}}_X(*g)$-module $\widetilde{\mathcal{M}}_*$ which is strictly $\mathbb{R}$-specializable. One deduces that the construction $\widetilde{\mathcal{M}}[*g]$ only depends on the naively localized module $\widetilde{\mathcal{M}}(*D)$. Similarly, for an effective divisor D, $\widetilde{\mathcal{M}}[*D]$ (when it exists) only depends on $\widetilde{\mathcal{M}}(*D)$.

***11.3.11. Remark* (Restriction to $z=1$).** Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and is strictly $\mathbb{R}$-specializable and localizable along (g). Then, setting $\mathcal{M}=\widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$,

$$({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})(*H)/(z-1)({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})(*H)=({}_{\mathrm{D}}\iota_{g*}\mathcal{M})(*H),$$

the same holds for V_0, and thus $({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[*H]/(z-1)({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[*H]=({}_{\mathrm{D}}\iota_{g*}\mathcal{M})(*H)$. As a consequence,

$$\widetilde{\mathcal{M}}[*g]/(z-1)\widetilde{\mathcal{M}}[*g]=\mathcal{M}(*g).$$

***11.3.12. Example* (The case of holonomic $\mathcal{D}_X$-modules).** The main example of specializable coherent $\mathcal{D}_X$-modules are the holonomic $\mathcal{D}_X$-modules. This is the origin of the theory of the Bernstein-Sato polynomial. The roots of the Bernstein polynomials are not necessarily real, but a similar theory applies. For such a $\mathcal{D}_X$-module, the localized module $\mathcal{M}(*D)$ is $\mathcal{D}_X$-holonomic (see e.g. [**Bjö93**, Prop. 3.2.14]). As a consequence, if $\mathcal{M}$ is smooth on $X\smallsetminus D$, $\mathcal{M}(*D)$ is coherent over $\mathcal{O}_X(*D)$. Indeed, the assertion is local, so we can assume that $\mathcal{M}$ has a coherent filtration $F_\bullet\mathcal{M}$ such that $F_0\mathcal{M}_{|X\smallsetminus D}=\mathcal{M}_{|X\smallsetminus D}$. For $k\geqslant 0$, the inclusion $F_0\mathcal{M}\hookrightarrow F_k\mathcal{M}$ has an $\mathcal{O}_X$-coherent cokernel supported on D, hence it induces an isomorphism $F_0\mathcal{M}(*D) \xrightarrow{\sim} F_k\mathcal{M}(*D)$. Passing to the limit $k \to \infty$, we find $F_0\mathcal{M} \xrightarrow{\sim} \mathcal{M}(*D)$.

11.4. Dual localization

In this section, we treat simultaneously the case of $\mathcal{D}_X$-modules and that of graded $R_F\mathcal{D}_X$-modules, so $\widetilde{\mathcal{D}}_X$ means either of these sheaves. The Kashiwara-Malgrange filtration enables one to give a comprehensive definition of the dual localization functor, which should be thought of as the adjoint of the localization functor by the $\widetilde{\mathcal{D}}_X$-module duality functor. We will give a direct definition and we will not need the duality functor.

11.4.a. Tensoring with respect to $V_0\widetilde{\mathcal{D}}_X$. Let $H \subset X$ be a smooth hypersurface in X with defining ideal $\mathcal{I}_H$ in $\mathcal{O}_X$. Locally, there exists a smooth function $t \in \mathcal{O}_X$ such that $\mathcal{I}_H = t\mathcal{O}_X$. In this section, we let $V_\bullet$ denote the corresponding V-filtration and we analyze, for a right $V_0\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$, the tensor product $\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$, where the tensor product uses the left action of $V_0\widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{D}}_X$, and the right $\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$ is the trivial one. Such a tensor product already occurred in Proposition 11.3.3(7), and a similar tensor product will occur in Definition 11.4.6 below. Let us already notice that, if $\widetilde{\mathcal{N}}$ is a $V_0\widetilde{\mathcal{D}}_X$-submodule of a right $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, then by using the right action of $\widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{M}}$ we find a natural morphism of right $\widetilde{\mathcal{D}}_X$-modules

$$\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\mathcal{M}}.$$

In particular, in Proposition 11.3.3, we exhibited a morphism

$$V_0(\widetilde{\mathcal{M}}(*H)) \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\mathcal{M}}(*H).$$

11.4.1. Proposition. *Let $\widetilde{\mathcal{N}}$ be a right $V_0\widetilde{\mathcal{D}}_X$-module such that*

$$\widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{O}}_X/\widetilde{\mathcal{I}}_H) = \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (\widetilde{\mathcal{O}}_X/\widetilde{\mathcal{I}}_H)$$

(i.e., for some or any local equation t of H, $t : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ is injective). Then

$$H^i(\widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X) = 0 \quad \textit{for } i \neq 0,$$

that is,

$$\widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \simeq \widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X.$$

Furthermore,

$$\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \simeq \operatorname{Coker}\begin{bmatrix} \big(\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_X(\log H)\big) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \\ \hfill (n \otimes \theta) \otimes P \longmapsto (n\theta \otimes P - n \otimes \theta P) \end{bmatrix}.$$

Proof. We first revisit Exercise 9.2. Recall (see Exercise 9.1) that $\operatorname{Sp} V_0\widetilde{\mathcal{D}}_X$ is the complex having $V_0\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}(\log H)$ as its term in degree $-k$, and differential the left $V_0\widetilde{\mathcal{D}}_X$-linear morphism

$$V_0\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}(\log H) \xrightarrow{\widetilde{\delta}} V_0\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k-1}(\log H)$$

given, for $\boldsymbol{\theta} = \theta_1 \wedge \cdots \wedge \theta_k$

$$\widetilde{\delta}(P \otimes \boldsymbol{\theta}) = \sum_{i=1}^{k} (-1)^{i-1} (P\theta_i) \otimes \widehat{\boldsymbol{\theta}}_i + \sum_{i<j} (-1)^{i+j} P \otimes ([\theta_i, \theta_j] \wedge \widehat{\boldsymbol{\theta}}_{i,j}),$$

with $\widehat{\boldsymbol{\theta}}_i = \theta_1 \wedge \cdots \wedge \theta_{i-1} \wedge \theta_{i+1} \wedge \cdots \wedge \theta_k$, and a similar meaning for $\widehat{\boldsymbol{\theta}}_{i,j}$ (see Exercise 9.1). Since $\mathrm{Sp}(V_0\widetilde{\mathcal{D}}_X)$ is a resolution of $\widetilde{\mathcal{O}}_X$ by locally free left $V_0\widetilde{\mathcal{D}}_X$-modules which are $\widetilde{\mathcal{O}}_X$-locally free, we have

$$\widetilde{\mathcal{N}} \simeq \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\, V_0\widetilde{\mathcal{D}}_X,$$

with their right $V_0\widetilde{\mathcal{D}}_X$-module structure, by using the tensor right structure on the right-hand side. The complex $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}\, V_0\widetilde{\mathcal{D}}_X$ has $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (V_0\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}(\log H))$ as its term in degree $-k$, and differential $\mathrm{Id} \otimes \widetilde{\delta}$, which is right $V_0\widetilde{\mathcal{D}}_X$-linear for the tensor right structure (see Exercise 8.12(2a)). Let us make explicit the differential. For $P \in V_0\widetilde{\mathcal{D}}_X$, the element $[n \otimes (1 \otimes \boldsymbol{\theta})] \cdot P$ is complicated to express, but we must have, by right $V_0\widetilde{\mathcal{D}}_X$-linearity of $\mathrm{Id} \otimes \widetilde{\delta}$,

$$\begin{aligned}(\mathrm{Id} \otimes \widetilde{\delta})\big[(n \otimes (1 \otimes \boldsymbol{\theta})) \cdot P\big] &= \big[(\mathrm{Id} \otimes \widetilde{\delta})(n \otimes (1 \otimes \boldsymbol{\theta}))\big] \cdot P \\ &= \Big[n \otimes \Big[\sum_{i=1}^{k} (-1)^{i-1} \theta_i \otimes \widehat{\boldsymbol{\theta}}_i + \sum_{i<j} (-1)^{i+j} 1 \otimes ([\theta_i, \theta_j] \wedge \widehat{\boldsymbol{\theta}}_{i,j})\Big]\Big] \cdot P.\end{aligned}$$

We now write

$$n \otimes (\theta_i \otimes \widehat{\boldsymbol{\theta}}_i) = n\theta_i \otimes (1 \otimes \widehat{\boldsymbol{\theta}}_i) - [n \otimes (1 \otimes \widehat{\boldsymbol{\theta}}_i)] \cdot \theta_i,$$

so the previous formula reads, after the involution

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} (V_0\widetilde{\mathcal{D}}_X \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}(\log H)) \simeq (\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,k}(\log H)) \otimes_{\widetilde{\mathcal{O}}_X} V_0\widetilde{\mathcal{D}}_X$$

transforming the tens structure to the triv one, by denoting $\widetilde{\delta}_{\mathrm{triv}}$ the corresponding differential:

$$\begin{aligned}\text{(11.4.2)}\quad \widetilde{\delta}_{\mathrm{triv}}\big[(n \otimes \boldsymbol{\theta}) \otimes P\big] &= \sum_{i=1}^{k} (-1)^{i-1} (n\theta_i \otimes \widehat{\boldsymbol{\theta}}_i) \otimes P \\ &\quad - \sum_{i=1}^{k} (-1)^{i-1} (n \otimes \widehat{\boldsymbol{\theta}}_i) \otimes (\theta_i P) + \sum_{i<j} (-1)^{i+j} (n \otimes ([\theta_i, \theta_j] \wedge \widehat{\boldsymbol{\theta}}_{i,j})) \otimes P \\ &= \big[\widetilde{\delta}_{\widetilde{\mathcal{N}}}(n \otimes \boldsymbol{\theta})\big] \otimes P - \sum_{i=1}^{k} (-1)^{i-1} (n \otimes \widehat{\boldsymbol{\theta}}_i) \otimes (\theta_i P),\end{aligned}$$

where $\widetilde{\delta}_{\widetilde{\mathcal{N}}}$ is the differential of the Spencer complex $\mathrm{Sp}_{\log} \widetilde{\mathcal{N}}$ of $\widetilde{\mathcal{N}}$ as a right $V_0\widetilde{\mathcal{D}}_X$-module.

We obtain, due to the local $\widetilde{\mathcal{O}}_X$-freeness of $V_0\widetilde{\mathcal{D}}_X$ and $\widetilde{\mathcal{D}}_X$,

$$\begin{aligned}\widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X &\simeq (\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \operatorname{Sp} V_0\widetilde{\mathcal{D}}_X) \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \\ &\simeq \big((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet}(\log H)) \otimes_{\widetilde{\mathcal{O}}_X} V_0\widetilde{\mathcal{D}}_X,\ \widetilde{\delta}_{\mathrm{triv}}\big) \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \\ &\simeq \big((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet}(\log H)) \otimes^{\mathrm{L}}_{\widetilde{\mathcal{O}}_X} V_0\widetilde{\mathcal{D}}_X,\ \widetilde{\delta}_{\mathrm{triv}}\big) \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \\ &\simeq \big((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet}(\log H)) \otimes^{\mathrm{L}}_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X,\ \widetilde{\delta}_{\mathrm{triv}}\big) \\ &\simeq \big((\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_{X,\bullet}(\log H)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X,\ \widetilde{\delta}_{\mathrm{triv}}\big).\end{aligned}$$

In the last two lines, $\widetilde{\delta}_{\mathrm{triv}}$ is given by (11.4.2), where P is now a local section of $\widetilde{\mathcal{D}}_X$.

We have thus realized $\widetilde{\mathcal{N}} \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$ as a complex $(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$, where each term $\mathcal{F}^k$ is an $\widetilde{\mathcal{O}}_X$-module (here, we forget the right $V_0\widetilde{\mathcal{D}}_X$-module structure of $\widetilde{\mathcal{N}}$).

With respect to the filtration $\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} F_k\widetilde{\mathcal{D}}_X$, $\widetilde{\delta}_{\mathrm{triv}}$ has degree one, and the differential $\mathrm{gr}^F_1\widetilde{\delta}_{\mathrm{triv}}$ of the graded complex $\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F\widetilde{\mathcal{D}}_X$ is expressed as

$$\widetilde{\delta}_{\mathrm{triv}}\big[(n \otimes \boldsymbol{\theta}) \otimes Q\big] = \sum_{i=1}^{k} (-1)^i (n \otimes \widehat{\boldsymbol{\theta}}_i) \otimes (\theta_i \cdot Q)$$

for a local section Q of $\mathrm{gr}^F\widetilde{\mathcal{D}}_X$. The filtration $F_p(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$ whose term in degree $-k$ is $\mathcal{F}^{-k} \otimes_{\widetilde{\mathcal{O}}_X} F_{p-k}\widetilde{\mathcal{D}}_X$ satisfies $F_p(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}}) = 0$ for $p < 0$ and we have

$$\mathrm{gr}^F(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}}) = (\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F\widetilde{\mathcal{D}}_X, \mathrm{gr}^F_1\widetilde{\delta}_{\mathrm{triv}}), \tag{11.4.3}$$

compatible with the grading.

11.4.4. Assertion. *The graded complex $(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F\widetilde{\mathcal{D}}_X, \mathrm{gr}^F_1\widetilde{\delta}_{\mathrm{triv}})$ has zero cohomology in any degree $i \neq 0$.*

Proof. In local coordinates $t, x_2, \dots, x_n$ such that $H = \{t=0\}$, let us choose a basis $\widetilde{\partial}_t, \widetilde{\partial}_{x_2}, \dots, \widetilde{\partial}_{x_n}$ as a basis of local vector fields, and let us replace $\widetilde{\partial}_t$ with $t\widetilde{\partial}_t$ to obtain a basis of logarithmic vector fields. Let $\tau, \xi_2, \dots, \xi_n$ resp. $t\tau, \xi_2, \xi_2, \dots, \xi_n$ be the corresponding basis of $\mathrm{gr}^F_1\widetilde{\mathcal{D}}_X$ resp. $\mathrm{gr}^F_1 V_0\widetilde{\mathcal{D}}_X$. Then $\mathrm{gr}^F(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$ is identified with a Koszul complex. More precisely, it isomorphic to the simple complex associated to the n-cube with vertices $\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F\widetilde{\mathcal{D}}_X = \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\tau, \xi_2, \dots, \xi_n]$ and arrows in the i-th direction all equal to multiplication by ξ_i if $i \neq 1$ and by $t \otimes \tau$ if $i = 1$.

In such a way we obtain that $(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{gr}^F\widetilde{\mathcal{D}}_X, \mathrm{gr}^F_1\widetilde{\delta}_{\mathrm{triv}})$ is quasi-isomorphic to the complex

$$\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\xi_1] \xrightarrow{\ t \otimes \tau\ } \underset{\bullet}{\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\xi_1]},$$

where $\bullet$ indicates the term in degree zero. Injectivity of the differential immediately follows from the injectivity assumption on $t : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$. $\square$

By (11.4.3), the assertion applies to the graded complex $\mathrm{gr}^F(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$ and therefore each $\mathrm{gr}^F_p(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$ has cohomology in degree zero at most.

It follows that each $F_p(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$ satisfies the same property, and passing to the inductive limit, so does the complex $(\mathcal{F}^\bullet \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X, \widetilde{\delta}_{\mathrm{triv}})$.

Last, $\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X$ is isomorphic to the cokernel of

$$\widetilde{\delta}_{\mathrm{triv}} : (\widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\Theta}_X(\log H)) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\mathcal{N}} \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{D}}_X,$$

and the last formula of the proposition follows from the expression (11.4.2) of $\widetilde{\delta}_{\mathrm{triv}}$. $\square$

Let $\widetilde{\mathcal{N}}$ be a left $\widetilde{\mathcal{D}}_X$-module. We consider similarly the tensor product $\widetilde{\mathcal{D}}_X \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}$ with the trivial left $\widetilde{\mathcal{D}}_X$-action, and where the right $V_0\widetilde{\mathcal{D}}_X$-action on $\widetilde{\mathcal{D}}_X$ is used for the tensor product.

11.4.5. Corollary. *Let $\widetilde{\mathcal{N}}$ be a coherent left $V_0\widetilde{\mathcal{D}}_X$-module such that for any local equation t of H, $t : \widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}$ is injective. Then $H^i(\widetilde{\mathcal{D}}_X \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}) = 0$ for $i \neq 0$.*

Proof. Here, the right action of $V_0\widetilde{\mathcal{D}}_X$ on $\widetilde{\mathcal{D}}_X$ is used. The question is local, and we can interpret the side-changing functor for $V_0\widetilde{\mathcal{D}}_X$-modules (given by $\widetilde{\mathcal{N}}^{\mathrm{left}} \mapsto \widetilde{\mathcal{N}}^{\mathrm{right}} = \widetilde{\omega}_X(H) \otimes_{\widetilde{\mathcal{O}}_X} \widetilde{\mathcal{N}}^{\mathrm{left}}$) as coming from an involution of $V_0\widetilde{\mathcal{D}}_X$ induced by an involution of $\widetilde{\mathcal{D}}_X$ (see Exercise 8.17). If $(V_0\widetilde{\mathcal{D}}_X)^\bullet$ is a finite resolution of $\widetilde{\mathcal{N}}^{\mathrm{left}}$ by free $V_0\widetilde{\mathcal{D}}_X$-module, it gives rise to a $\widetilde{\mathcal{D}}_X$-free resolution $(\widetilde{\mathcal{D}}_X)^\bullet$ of $\widetilde{\mathcal{D}}_X \otimes^{\mathrm{L}}_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{N}}$. Regarding these modules as right $\widetilde{\mathcal{D}}_X$-modules via the involution above, Proposition 11.4.1 implies that the cohomology of this complex vanishes in any nonzero degree. $\square$

11.4.b. Dual localization along a smooth hypersurface

We switch back to the case of right $\widetilde{\mathcal{D}}_X$-modules.

***11.4.6. Definition* (Dual localization along a smooth hypersurface)**

Let $H \subset X$ be a smooth hypersurface and let $\widetilde{\mathcal{M}}$ be a coherent right $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along H. The dual localization of $\widetilde{\mathcal{M}}$ along H is defined as

$$\widetilde{\mathcal{M}}[!H] := V_{<0}\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X.$$

***11.4.7. Proposition* (Properties of the dual localization along H)**

Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H. Then the following properties hold.

(1) *$\widetilde{\mathcal{M}}[!H]$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along H.*

(2) *The natural morphism* $\mathrm{dloc} : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ *induces an isomorphism*

$$V_{<0}\widetilde{\mathcal{M}}[!H] \xrightarrow{\sim} V_{<0}\widetilde{\mathcal{M}},$$

and in particular

$$\mathrm{gr}^V_{-1}\mathrm{dloc} : \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[!H] \xrightarrow{\sim} \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}.$$

(3) *With respect to a local decomposition $X \simeq H \times \Delta_t$,*

$$\widetilde{\eth}_t : \mathrm{gr}^V_{-1}\widetilde{\mathcal{M}}[!H] \longrightarrow \mathrm{gr}^V_0\widetilde{\mathcal{M}}[!H](-1)$$

is an isomorphism, and $\operatorname{Ker}\operatorname{gr}_0^V\mathrm{dloc}$ *(resp.* $\operatorname{Coker}\operatorname{gr}_0^V\mathrm{dloc}$*) is isomorphic to the kernel (resp. cokernel) of* $\widetilde{\partial}_t : \operatorname{gr}_{-1}^V\widetilde{\mathcal{M}}(1) \to \operatorname{gr}_0^V\widetilde{\mathcal{M}}$*, that is, the kernel (resp. cokernel) of* $\mathrm{can} : \psi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}$.

(4) *Assume* $\widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}}$ *is a morphism between strictly* $\mathbb{R}$*-specializable coherent* $\widetilde{\mathcal{D}}_X$*-modules which induces an isomorphism* $\widetilde{\mathcal{M}}'(*H) \to \widetilde{\mathcal{M}}(*H)$ *(i.e., whose restriction to* $V_{<0}$ *is an isomorphism). Assume moreover that* $\widetilde{\mathcal{M}}'$ *satisfies* (3)*, i.e., the action of* $\widetilde{\partial}_t$ *induces an isomorphism* $\operatorname{gr}_{-1}^V\widetilde{\mathcal{M}}' \xrightarrow{\sim} \operatorname{gr}_0^V\widetilde{\mathcal{M}}'(-1)$*. Then* $\widetilde{\mathcal{M}}' \simeq \widetilde{\mathcal{M}}[!H]$*. More precisely, the induced morphism* $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}[!H]$ *is an isomorphism, as well as* $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}'$*.*

(5) *Let* $\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}'$ *be as in* (4)*. Then any morphism* $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}'$ *factorizes through* $\widetilde{\mathcal{M}}'[!H]$*. In particular, if* $\widetilde{\mathcal{M}}'$ *is supported on* H*, such a morphism is zero.*

(6) *If* $\widetilde{\mathcal{M}}$ *is strict, then so is* $\widetilde{\mathcal{M}}[!H]$*.*

(7) *Let* $0 \to \widetilde{\mathcal{M}}' \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}'' \to 0$ *be an exact sequence of coherent strictly* $\mathbb{R}$*-specializable* $\widetilde{\mathcal{D}}_X$*-modules. Then the sequence*

$$0 \longrightarrow \widetilde{\mathcal{M}}'[!H] \longrightarrow \widetilde{\mathcal{M}}[!H] \longrightarrow \widetilde{\mathcal{M}}''[!H] \longrightarrow 0$$

is exact.

Proof. We first locally construct a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}_!$ which satisfies all properties described in Proposition 11.4.7, and we then identify it with the globally defined $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!H]$. The question is therefore local on X and we can assume that $X \simeq H \times \Delta_t$. We will use the notation and results of Exercise 9.34.

Step 1. We search for $\widetilde{\mathcal{M}}_!$ with a morphism $\widetilde{\mathcal{M}}_! \to \widetilde{\mathcal{M}}$ inducing an isomorphism $V_{<0}\widetilde{\mathcal{M}}_! \to V_{<0}\widetilde{\mathcal{M}}$, hence $\psi_{t,\lambda}\widetilde{\mathcal{M}}_! \xrightarrow{\sim} \psi_{t,\lambda}\widetilde{\mathcal{M}}$ for every $\lambda \in \mathbb{S}^1$, and such that $\phi_{t,1}\widetilde{\mathcal{M}}_!$ is naturally identified to the graph of $\mathrm{can}_{\widetilde{\mathcal{M}}} : \psi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}$, hence to $\psi_{t,1}\widetilde{\mathcal{M}}$, so that $\psi_{t,1}\widetilde{\mathcal{M}}_! \to \psi_{t,1}\widetilde{\mathcal{M}}$ is the identity, while $\phi_{t,1}\widetilde{\mathcal{M}}_! \to \psi_{t,1}\widetilde{\mathcal{M}}$ is induced by the second projection $\psi_{t,1}\widetilde{\mathcal{M}} \oplus \phi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}$, hence can be identified with $\mathrm{can}_{\widetilde{\mathcal{M}}}$.

We use the identification of Exercise 9.34(5) of $\widetilde{\mathcal{M}}/V_{-1}\widetilde{\mathcal{M}}$ with $\bigoplus_{\alpha\in(-1,0]}\operatorname{gr}_\alpha^V\widetilde{\mathcal{M}}[s]$. On the other hand, we introduce a similar $V_0\widetilde{\mathcal{D}}_X$-module structure on $\operatorname{gr}_{-1}^V\widetilde{\mathcal{M}}(1)[s]$ by setting

$$\begin{aligned}\mu_{-1}^{(j)}s^j \cdot t &= \begin{cases}0 & \text{if } j = 0,\\ \big(\mu_{-1}^{(j)}(\mathrm{E}+(j-1)z)\big)s^{j-1} & \text{if } j \geqslant 1,\end{cases}\\ (\mu_{-1}^{(j)}s^j)t\widetilde{\partial}_t &= \big(\mu_{-1}^{(j)}(\mathrm{E}+(j-1)z)\big)s^j.\end{aligned}$$

One checks similarly that this is indeed a $V_0\widetilde{\mathcal{D}}_X$-module structure (i.e., $[t\widetilde{\partial}_t, t]$ acts as zt), but the action of $\widetilde{\partial}_t$, defined as the multiplication by s, does not extend this structure as a $\widetilde{\mathcal{D}}_X$-module structure (see Exercise 9.34(6)). We then notice that the morphism

$$\begin{aligned}\rho : \operatorname{gr}_{-1}^V\widetilde{\mathcal{M}}(1)[s] &\longrightarrow \operatorname{gr}_0^V\widetilde{\mathcal{M}}[s] \subset \widetilde{\mathcal{M}}/V_{-1}\widetilde{\mathcal{M}}\\ \mu_{-1}^{(j)}s^j &\longmapsto (\mu_{-1}^{(j)}\widetilde{\partial}_t)s^j\end{aligned}$$

is $V_0\widetilde{\mathcal{D}}_X$-linear.

Given a local section m of $\widetilde{\mathcal{M}}$, we denote by $[m]$ its class in $\widetilde{\mathcal{M}}/V_{-1}\widetilde{\mathcal{M}} = \bigoplus_{\alpha\in(-1,0]}\mathrm{gr}_\alpha\widetilde{\mathcal{M}}[s]$, and by $[m]_0 = \sum_{j\geqslant 0}[m]_0^{(j)}s^j$ the component of this class in $\mathrm{gr}_0^V\widetilde{\mathcal{M}}[s]$. Let us consider the $V_0\widetilde{\mathcal{D}}_X$-submodule $\widetilde{\mathcal{M}}_! \subset \widetilde{\mathcal{M}}\oplus\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)[s]$ consisting of pairs (m,μ_{-1}) of local sections such that $[m]_0 = \rho(\mu_{-1})$ (since the maps ρ and $m\mapsto [m]_0$ are $V_0\widetilde{\mathcal{D}}_X$-linear, $\widetilde{\mathcal{M}}_!$ is indeed a $V_0\widetilde{\mathcal{D}}_X$-submodule). We will extend the $V_0\widetilde{\mathcal{D}}_X$-module structure on $\widetilde{\mathcal{M}}_!$ to a $\widetilde{\mathcal{D}}_X$-module structure so that the natural morphism $\widetilde{\mathcal{M}}_!\to\widetilde{\mathcal{M}}$ induced by the first projection is $\widetilde{\mathcal{D}}_X$-linear.

We have a decomposition $\widetilde{\mathcal{M}}/V_{<-1}\widetilde{\mathcal{M}} \simeq \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}\oplus\bigoplus_{\alpha\in(-1,0]}\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}[s]$ and, for a local section m of $\widetilde{\mathcal{M}}$, we can write

$$[m\widetilde{\partial}_t]_0 = \mathrm{can}_{\widetilde{\mathcal{M}}}[m]_{-1}^{(0)} + \sum_{j\geqslant 1}[m]_0^{(j-1)}s^j = \mathrm{can}_{\widetilde{\mathcal{M}}}[m]_{-1}^{(0)} + [m]_0 s,$$

where $[m]_{-1}^{(0)}$ obviously denotes the component of $m \bmod V_{<-1}\widetilde{\mathcal{M}}$ in $\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}$. For any local section (m,μ_{-1}) of $\widetilde{\mathcal{M}}_!$ we define

$$(m,\mu_{-1})\widetilde{\partial}_t := (m\widetilde{\partial}_t, [m]_{-1}^{(0)} + \mu_{-1}s).$$

The right-hand term is easily checked to belong to $\widetilde{\mathcal{M}}_!$. We now check that $(m,\mu_{-1})[\widetilde{\partial}_t,t] = z(m,\mu_{-1})$. On the one hand, we have

$$\begin{aligned}(m,\mu_{-1})\widetilde{\partial}_t t = \big(m\widetilde{\partial}_t t, ([m]_{-1}^{(0)} + \mu_{-1}s)t\big) &= \Big(m\widetilde{\partial}_t t, \textstyle\sum_{j\geqslant 0}(\mathrm{N}+jz)\mu_{-1}^{(j)}s^j\Big)\\ &= (m\widetilde{\partial}_t t, \mu_{-1}\widetilde{\partial}_t t),\end{aligned}$$

and, on the other hand,

$$\begin{aligned}(m,\mu_{-1})t\widetilde{\partial}_t &= \Big(mt, \textstyle\sum_{j\geqslant 1}(\mathrm{N}+(j-1)z)\mu_{-1}^{(j)}s^{j-1}\Big)\widetilde{\partial}_t\\ &= \Big(mt\widetilde{\partial}_t, [mt]_{-1}^{(0)} + \textstyle\sum_{j\geqslant 1}(\mathrm{N}+(j-1)z)\mu_{-1}^{(j)}s^j\Big).\end{aligned}$$

Moreover, we have $[mt]_{-1}^{(0)} = \mathrm{var}_{\widetilde{\mathcal{M}}}[m]_0^{(0)} = \mathrm{var}_{\widetilde{\mathcal{M}}}(\mathrm{can}_{\widetilde{\mathcal{M}}}\,\mu_{-1}^{(0)}) = \mathrm{N}\mu_{-1}^{(0)}$. As a consequence,

$$(m,\mu_{-1})[\widetilde{\partial}_t,t] = (zm, z\mu_{-1} + \mathrm{var}_{\widetilde{\mathcal{M}}}[m]_0^{(0)} - \mathrm{N}\mu_{-1}^{(0)}) = z(m,\mu_{-1}).$$

Since $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent and $\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_H$-coherent, one concludes easily that $\widetilde{\mathcal{M}}_!$ is $\widetilde{\mathcal{D}}_X$-coherent.

We set

$$V_\alpha\big(\widetilde{\mathcal{M}}\oplus\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)[s]\big) := V_\alpha\widetilde{\mathcal{M}}\oplus\bigoplus_{j=0}^{[\alpha]}\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)s^j.$$

The induced filtration $V_\alpha\widetilde{\mathcal{M}}_! := \widetilde{\mathcal{M}}_!\cap V_\alpha\big(\widetilde{\mathcal{M}}\oplus\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)[s]\big)$ satisfies $V_\alpha\widetilde{\mathcal{M}}_! \xrightarrow{\sim} V_\alpha\widetilde{\mathcal{M}}$ for $\alpha<0$ and

$$\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}_! = \begin{cases}\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}} & \text{if } \alpha\notin\mathbb{N},\\ \big\{([m]_0^{(j)},\mu_{-1}^{(j)})\in\mathrm{gr}_0^V\widetilde{\mathcal{M}}\oplus\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)\mid [m]_0^{(j)} = \mathrm{can}_{\widetilde{\mathcal{M}}}\,\mu_{-1}^{(j)}\big\}\cdot s^j & \text{if } \alpha = j\\ \qquad\simeq \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)s^j.\end{cases}$$

It is clear that this is a coherent V-filtration and that $\widetilde{\mathcal{M}}_!$ satisfies 11.4.7(1)–(3).

Identification between $\widetilde{\mathcal{M}}[!H]$ ***and*** $\widetilde{\mathcal{M}}_!$**.** Since $V_{<0}\widetilde{\mathcal{M}}_! \xrightarrow{\sim} V_{<0}\widetilde{\mathcal{M}}$, the natural morphism $\widetilde{\mathcal{M}}_![!H] \to \widetilde{\mathcal{M}}[!H]$ is an isomorphism, and we will prove that the natural morphism

$$\widetilde{\mathcal{M}}_![!H] = V_{<0}\widetilde{\mathcal{M}}_! \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \longrightarrow \widetilde{\mathcal{M}}_! \tag{11.4.8}$$

is an isomorphism. For any coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{N}}$ which is strictly $\mathbb{R}$-specializable along H, the natural morphism $V_0\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{N}}$ is onto, and if $\mathrm{can}_{\widetilde{\mathcal{N}}}$ is onto, then $V_{<0}\widetilde{\mathcal{N}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{N}}$ is also onto. Since $\mathrm{can}_{\widetilde{\mathcal{M}}_!}$ is an isomorphism, (11.4.8) is onto.

The composition $V_{<0}\widetilde{\mathcal{M}}_! \simeq V_{<0}\widetilde{\mathcal{M}}_![!H] \to \widetilde{\mathcal{M}}_![!H] \to \widetilde{\mathcal{M}}_!$, so (11.4.8) is injective when restricted to the $V_{<0}$ part. We V-filter $\widetilde{\mathcal{M}}_![!H]$ by setting $U_{<k}\widetilde{\mathcal{M}}_![!H] = \sum_{j\leqslant k} V_{<0}\widetilde{\mathcal{M}}_!\widetilde{\eth}_t^j$, so that $U_{<0}\widetilde{\mathcal{M}}_![!H] = V_{<0}\widetilde{\mathcal{M}}_!$. For $k \geqslant 1$ we have a commutative diagram

$$\begin{tikzcd}
(U_{<0}/U_{<-1})\widetilde{\mathcal{M}}_![!H] \arrow[r, "\sim"] \arrow[d, "\widetilde{\eth}_t^k"'] & (V_{<0}/V_{<-1})\widetilde{\mathcal{M}}_! \arrow[d, "\widetilde{\eth}_t^k", "\wr"'] \\
(U_{<k}/U_{<k-1})\widetilde{\mathcal{M}}_{!}[{!}H](-k) \arrow[r] & (V_{<k}/V_{<k-1})\widetilde{\mathcal{M}}_!(-k)
\end{tikzcd}$$

The left down arrow is onto by definition, and since the right down arrow is an isomorphism by the properties of $\widetilde{\mathcal{M}}_!$, the left down arrow is also an isomorphism, as well as the lower horizontal arrow, showing by induction on k that $\widetilde{\mathcal{M}}_![!H] \to \widetilde{\mathcal{M}}_!$ is an isomorphism, so $\widetilde{\mathcal{M}}_![!H] = \widetilde{\mathcal{M}}[!H]$ satisfies 11.4.7(1)–(3).

We now prove (4). Since $V_{<0}\widetilde{\mathcal{M}}' \xrightarrow{\sim} V_{<0}\widetilde{\mathcal{M}}$, Definition 11.4.6 implies $\widetilde{\mathcal{M}}'[!H] \xrightarrow{\sim} \widetilde{\mathcal{M}}[!H]$. It remains to check that $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}'$ is an isomorphism. Since the question is local, it is enough to check that the morphism $\widetilde{\mathcal{M}}'_! \to \widetilde{\mathcal{M}}'$ is an isomorphism, which is straightforward from the construction of $\widetilde{\mathcal{M}}'_!$, with the assumption that $\mathrm{can}_{\widetilde{\mathcal{M}}'}$ is an isomorphism.

For (5), we remark that the morphism $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}'$ restricts to a morphism $V_{<0}\widetilde{\mathcal{M}}[!H] = V_{<0}\widetilde{\mathcal{M}} \to V_{<0}\widetilde{\mathcal{M}}'$, so the first assertion follows from Definition 11.4.6. The second one is then obvious since $V_{<0}\widetilde{\mathcal{M}}' = 0$ if $\widetilde{\mathcal{M}}'$ is supported on H.

Let us now check (6), that is, the strictness of $\widetilde{\mathcal{M}}[!H]$. One check it locally for $\widetilde{\mathcal{M}}_!$, for which it is clear since $\widetilde{\mathcal{M}}_! \subset \widetilde{\mathcal{M}} \oplus \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}(1)[s]$.

It remains to prove (7). The argument is the same as for 11.3.3(11) except for the injectivity of $\widetilde{\mathcal{M}}'[!H] \to \widetilde{\mathcal{M}}[!H]$. In order to prove this property, we notice that $V_{<0}\widetilde{\mathcal{M}}'[!H] \to V_{<0}\widetilde{\mathcal{M}}[!H]$ is injective, according to (2). It is then enough to check the injectivity of $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}'[!H] \to \mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}[!H]$ for every $\alpha \geqslant 0$. Due to the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}'[!H], \widetilde{\mathcal{M}}[!H]$, injectivity holds for every $\alpha \notin \mathbb{Z}$ since $\mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}' \to \mathrm{gr}_\alpha^V\widetilde{\mathcal{M}}$ is injective. Similarly, if α is a non-negative integer, the injectivity at α holds if and only if it holds at $\alpha = 0$. Now, (3) reduces this check to the case $\alpha = -1$, where the injectivity holds since $\mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}' \to \mathrm{gr}_{-1}^V\widetilde{\mathcal{M}}$ is injective. $\square$

11.4.9. Remark **(Remark 11.3.2 continued).** Clearly, $\widetilde{\mathcal{M}}[!H]$ only depends on $\widetilde{\mathcal{M}}(*H)$, so that, in Proposition 11.4.7, we could start from a coherent $\widetilde{\mathcal{D}}_X(*H)$-module $\widetilde{\mathcal{M}}$ which is strictly $\mathbb{R}$-specializable.

***11.4.10. Remark* (Uniqueness of the morphism dloc).** Let $\mathrm{dloc}' : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ be a morphism whose naive localization $\mathrm{dloc}'_{(*H)} : \widetilde{\mathcal{M}}[!H](*H) \to \widetilde{\mathcal{M}}(*H)$ coincides with the naive localization $\mathrm{dloc}_{(*H)}$ of dloc. Then $\mathrm{dloc}' = \mathrm{dloc}$. Indeed, the assumption implies that dloc' coincides with dloc when restricted to $V_{<0}\widetilde{\mathcal{M}}[!H] = V_{<0}\widetilde{\mathcal{M}}$. Both induce then the same morphism $\widetilde{\mathcal{M}}[!H] = V_{<0}\widetilde{\mathcal{M}} \otimes_{V_0\widetilde{\mathcal{D}}_X} \widetilde{\mathcal{D}}_X \to \widetilde{\mathcal{M}}$.

11.4.11. Remark. The kernel of the morphism $\mathrm{dloc} : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and supported on H. Indeed, in any local setting $X = H \times \Delta_t$, it is identified with the pushforward by ι_H of $\mathrm{Ker}[\mathrm{can} : \psi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}]$, which is strict.

On the other hand, $\mathrm{Coker}[\mathrm{dloc} : \widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}]$, which is supported on H, need not be strictly $\mathbb{R}$-specializable without any further assumption. It is so if and only if, in any local setting, Coker can is strict, i.e., $\mathrm{can} : \psi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H.

***11.4.12. Remark* (Side-changing and dual localization).** If $\widetilde{\mathcal{M}}$ is a left $\widetilde{\mathcal{D}}_X$-module, we define $\widetilde{\mathcal{M}}[!H] = \widetilde{\mathcal{D}}_X \otimes_{V_0\widetilde{\mathcal{D}}_X} V^{>-1}\widetilde{\mathcal{M}}$. Let us check that $\widetilde{\mathcal{M}}^{\mathrm{right}}[!H] \simeq (\widetilde{\mathcal{M}}[!H])^{\mathrm{right}}$. This relation clearly holds for the naive localization, i.e., if we replace $[!H]$ with $(*H)$. Then the morphism $\widetilde{\mathcal{M}}' = (\widetilde{\mathcal{M}}[!H])^{\mathrm{right}} \to \widetilde{\mathcal{M}}^{\mathrm{right}}$ obtained by side-changing from the natural morphism $\widetilde{\mathcal{M}}[!H] \to \widetilde{\mathcal{M}}$ satisfies the assumptions of Proposition 11.4.7(4), proving the desired isomorphism.

11.4.c. Dual localization along an arbitrary effective divisor

We keep the same notation as in Section 11.3.b. Let D be an effective divisor on X and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along D. We say that $\widetilde{\mathcal{M}}$ is *dual-localizable along D* if for some (or any) local equation g defining D, there exists a coherent $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!g]$ such that ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!g]) = ({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[!H]$. The various checks done in Section 11.3.b can be done similarly here in order to fully justify this definition.

***11.4.13. Corollary* (Properties of the dual localization along (g))**

Let $g : X \to \mathbb{C}$ be a holomorphic function and let $\widetilde{\mathcal{M}}$ be $\widetilde{\mathcal{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and dual-localizable along (g). Set $H = X \times \{0\} \subset X \times \mathbb{C}$.

(1) *The $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}[!g]$ is strictly $\mathbb{R}$-specializable along (g) and*

$$\mathrm{can} : \psi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big) \longrightarrow \phi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big)$$

is an isomorphism.

(2) *There is a natural morphism* $\mathrm{dloc} : \widetilde{\mathcal{M}}[!g] \to \widetilde{\mathcal{M}}$. *This morphism induces an isomorphism*

$$\big(\widetilde{\mathcal{M}}[!g]\big)(*g) \xrightarrow{\ \sim\ } \widetilde{\mathcal{M}}(*g),$$

and therefore isomorphisms

$$\psi_{g,\lambda}\big(\widetilde{\mathcal{M}}[!g]\big) \xrightarrow{\ \sim\ } \psi_{g,\lambda}\widetilde{\mathcal{M}} \quad \textit{for every } \lambda.$$

Moreover, we have a commutative diagram

$$\begin{array}{ccc}\psi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big) & \xrightarrow[\sim]{\psi_{g,1}\mathrm{dloc}} & \psi_{g,1}\widetilde{\mathcal{M}} \\ {\scriptstyle \mathrm{can}_{\widetilde{\mathcal{M}}[!g]}}\big\downarrow\wr & & \big\downarrow{\scriptstyle \mathrm{can}_{\widetilde{\mathcal{M}}}} \\ \phi_{g,1}\big(\widetilde{\mathcal{M}}[!g]\big) & \xrightarrow{\phi_{g,1}\mathrm{dloc}} & \phi_{g,1}\widetilde{\mathcal{M}}\end{array}$$

and $\operatorname{Ker}\mathrm{dloc}$ *(resp.* $\operatorname{Coker}\mathrm{dloc}$*) is identified with* $\operatorname{Ker}\mathrm{can}_{\widetilde{\mathcal{M}}}$ *(resp.* $\operatorname{Coker}\mathrm{can}_{\widetilde{\mathcal{M}}}$*).*

(3) *Given a short exact sequence of coherent $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and dual-localizable along (g), the $[!g]$ sequence is exact.*

Proof. Similar to that of Corollary 11.3.8. □

11.4.14. Remark. Denoting by dloc^g resp. dloc^t the morphism given by 11.4.13(2) resp. the same for t, the proof gives in particular that ${}_{\mathrm{D}}\iota_{g*}(\mathrm{dloc}^g) = \mathrm{dloc}^t$.

***11.4.15. Remark* (Remark 11.3.2 continued).** In Corollary 11.4.13, we could start from a coherent $\widetilde{\mathcal{D}}_X(*g)$-module $\widetilde{\mathcal{M}}$ which is strictly $\mathbb{R}$-specializable and, globally, we could start from a coherent $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}$ which is strictly $\mathbb{R}$-specializable.

***11.4.16. Remark* (Restriction to $z = 1$).** One proves as in Remark 11.3.11 that dual localization behaves well with respect to the restriction $z = 1$.

11.5. D-localizable $\widetilde{\mathcal{D}}_X$-modules and middle extension

Let D be an arbitrary effective divisor.

***11.5.1. Definition* (D-localizable $\widetilde{\mathcal{D}}_X$-modules).** Assume that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along D. We say that it is *D-localizable* if it is localizable and dual-localizable along D. The localized (resp. dual localized) module $\widetilde{\mathcal{M}}[\star D]$ ($\star = *$, resp. $\star = !$) is then well-defined and is strictly $\mathbb{R}$-specializable along D.

Recall that, if $D = H$ is smooth, any $\widetilde{\mathcal{M}}$ which is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along D is D-localizable. On the other hand, for $\mathcal{D}_X$-modules, $\mathbb{R}$-specializability implies D-localizability, whatever D is.

***11.5.2. Definition* (Middle extension).** Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and localizable along an effective divisor D. The image of the composed morphism $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ is called the *middle extension of $\widetilde{\mathcal{M}}$ along D* and denoted by $\widetilde{\mathcal{M}}[!*D]$.

Note however that we do not assert that $\widetilde{\mathcal{M}}[!*D]$ is *strictly* $\mathbb{R}$-specializable along D. Nevertheless, if $D = (g)$, ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!*D])$ is the image of ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!D]) \to {}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[*D])$, that is, the image of $({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[!H] \to ({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[*H]$, and it is $\mathbb{R}$-specializable along H with strict V-graded objects, according to Exercise 9.26(2). We will still use the notation $\psi_{g,\lambda}\widetilde{\mathcal{M}}[!*D]$ and $\phi_{g,1}\widetilde{\mathcal{M}}[!*D]$ for $\mathrm{gr}^V_\alpha\big({}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!*D])\big)(1)$ for $\alpha \in [-1, 0)$ and $\mathrm{gr}^V_0\big({}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[!*D])\big)$ respectively.

11.5.3. Example. Assume that $D = (g)$ and that $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable and localizable along D (if $D = H$ is smooth, the latter condition holds if the former holds). Assume moreover that can is onto and var is injective, that is, $\widetilde{\mathcal{M}}$ is a middle extension along (g). Then, according to Remarks 11.3.5 and 11.4.11, $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ is onto and $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ is injective, so $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}[!{*}D]$, and in particular $\widetilde{\mathcal{M}}[!{*}D]$ is strictly $\mathbb{R}$-specializable along D. (See also Remark 3.3.12.) This property holds for example if $\widetilde{\mathcal{M}}$ is strictly non-characteristic along H.

11.5.4. Proposition **(A criterion for the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{M}}[!{*}g]$)**
Assume that $\widetilde{\mathcal{M}}$ is $\widetilde{\mathcal{D}}_X$-coherent, strictly $\mathbb{R}$-specializable and localizable along (g). If $\mathrm{N} = \mathrm{var} \circ \mathrm{can} : \psi_{g,1}\widetilde{\mathcal{M}} \to \psi_{g,1}\widetilde{\mathcal{M}}(-1)$ is a strict morphism, then $\widetilde{\mathcal{M}}[!{*}g]$ is strictly $\mathbb{R}$-specializable along (g).

Proof. This follows from Exercise 11.4. $\square$

11.5.5. Remark **(Restriction to $z = 1$).** If $\widetilde{\mathcal{M}}$ satisfies the assumptions in Definition 11.5.2, then the restriction to $z = 1$ of the middle extension $\widetilde{\mathcal{M}}[!{*}D]$ is equal to the $\mathcal{D}_X$-module middle extension $\mathcal{M}(!{*}D)$. Indeed, by tensoring over $\widetilde{\mathbb{C}}$ with $\widetilde{\mathbb{C}}[z^{-1}]$ we obtain that $\widetilde{\mathcal{M}}[!{*}D][z^{-1}]$ is the image of $\widetilde{\mathcal{M}}[!D][z^{-1}]$ in $\widetilde{\mathcal{M}}[*D][z^{-1}]$. According to Remarks 11.4.16 and 11.3.11 we have $\widetilde{\mathcal{M}}[!D][z^{-1}] = \mathcal{M}(!D) \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$ and $\widetilde{\mathcal{M}}[*D][z^{-1}] = \mathcal{M}(*D) \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$, and there exists a $\mathcal{D}_X$-module $\mathcal{M}'$ such that $\widetilde{\mathcal{M}}[!{*}D][z^{-1}] = \mathcal{M}' \otimes_{\mathbb{C}} \mathbb{C}[z, z^{-1}]$. Restricting to $z = 1$ shows that $\mathcal{M}'$ is the image of $\mathcal{M}(!D)$ in $\mathcal{M}(*D)$, that is, $\mathcal{M}(!{*}D)$.

11.6. Beilinson's maximal extension and applications

In this section, we continue working with right $\widetilde{\mathcal{D}}_X$-modules.

11.6.1. Remark **(The case of left $\widetilde{\mathcal{D}}_X$-modules).** The same changes as in Remark 11.0.1 have to be made for left $\widetilde{\mathcal{D}}_X$-modules.

11.6.a. Properties of Beilinson's maximal extension.

Let $g : X \to \mathbb{C}$ be a holomorphic function. Let $\widetilde{\mathcal{M}}$ be a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along $D := (g)$. When D is not smooth, we also assume that $\widetilde{\mathcal{M}}$ is D-localizable, and *maximalizable* (see Definition 11.6.11 below). We aim at constructing a *coherent* $\widetilde{\mathcal{D}}_X$-module $\Xi_g\widetilde{\mathcal{M}}$, called *Beilinson's maximal extension of $\widetilde{\mathcal{M}}$ along* D, which is also strictly $\mathbb{R}$-specializable along D. It comes with two exact sequences

$$0 \longrightarrow \widetilde{\mathcal{M}}[!g] \xrightarrow{\ a\ } \Xi_g\widetilde{\mathcal{M}} \xrightarrow{\ b\ } \psi_{g,1}\widetilde{\mathcal{M}}(-1) \longrightarrow 0, \tag{11.6.2 !}$$

$$0 \longrightarrow \psi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\ b^\vee\ } \Xi_g\widetilde{\mathcal{M}} \xrightarrow{\ a^\vee\ } \widetilde{\mathcal{M}}[*g] \longrightarrow 0, \tag{11.6.2 $*$}$$

such that $b \circ b^\vee = -\mathrm{N}$ and $a^\vee \circ a = \mathrm{loc} \circ \mathrm{dloc}$, where dloc, loc are the natural morphisms (see Corollaries 11.3.8(2) and 11.4.13(2))

$$\widetilde{\mathcal{M}}[!g] \xrightarrow{\ \mathrm{dloc}\ } \widetilde{\mathcal{M}} \quad \text{and} \quad \widetilde{\mathcal{M}} \xrightarrow{\ \mathrm{loc}\ } \widetilde{\mathcal{M}}[*g].$$

The construction and the exact sequences only depend on the naively localized module $\widetilde{\mathcal{M}}(*D)$ (recall also that $\widetilde{\mathcal{M}}[!g]$ and $\widetilde{\mathcal{M}}[*g]$ only depend on $\widetilde{\mathcal{M}}(*D)$). It can be done for any coherent $\widetilde{\mathcal{D}}_X(*D)$-module $\widetilde{\mathcal{M}}_*$ which is strictly $\mathbb{R}$-specializable along D and gives rise nevertheless to a coherent $\widetilde{\mathcal{D}}_X$-module which is strictly $\mathbb{R}$-specializable along D. The usefulness of Beilinson's maximal extension comes from Corollary 11.6.5 below, which enables one to treat some questions on $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable along D by reducing to the case of $\widetilde{\mathcal{D}}_X(*D)$-modules strictly $\mathbb{R}$-specializable along D on the one hand, and to the case of $\widetilde{\mathcal{D}}_X$-modules supported on D and strictly $\mathbb{R}$-specializable along D on the other hand, the latter case being subject to an induction argument.

***11.6.3. Theorem* (Gluing construction).** *Let $\widetilde{\mathcal{M}}_*$ be a coherent $\widetilde{\mathcal{D}}_X(*D)$-module which is strictly $\mathbb{R}$-specializable, D-localizable and maximalizable along $D = (g)$. Let $(\widetilde{\mathcal{N}}, \mathrm{c}, \mathrm{v})$ be a triple consisting of a coherent $\widetilde{\mathcal{D}}_X$-module supported on D and strictly $\mathbb{R}$-specializable along D, and a pair morphisms $\mathrm{c} : \psi_{g,1}\widetilde{\mathcal{M}}_* \to \widetilde{\mathcal{N}}$ and $\mathrm{v} : \widetilde{\mathcal{N}} \to \psi_{g,1}\widetilde{\mathcal{M}}_*(-1)$ such that $\mathrm{v} \circ \mathrm{c} = \mathrm{N}$. Then the complex*

$$\psi_{g,1}\widetilde{\mathcal{M}}_* \xrightarrow{\;b^\vee \oplus \mathrm{c}\;} \Xi_g\widetilde{\mathcal{M}}_* \oplus \widetilde{\mathcal{N}} \xrightarrow{\;b + \mathrm{v}\;} \psi_{g,1}\widetilde{\mathcal{M}}_*(-1) \tag{11.6.3 $*$}$$

has nonzero cohomology in degree one at most, its H^1 is a coherent $\widetilde{\mathcal{D}}_X$-module $\mathrm{G}(\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{N}}, \mathrm{c}, \mathrm{v})$ which is strictly $\mathbb{R}$-specializable along D and we have an isomorphism of diagrams

$$\left[\begin{tikzcd} \psi_{g,1}\mathrm{G}(\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{N}}, \mathrm{c}, \mathrm{v}) \arrow[r, bend left, "\mathrm{can}"] & \phi_{g,1}\mathrm{G}(\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{N}}, \mathrm{c}, \mathrm{v}) \arrow[l, bend left, "\mathrm{var}", "(-1)"'] \end{tikzcd}\right] \simeq \left[\begin{tikzcd} \psi_{g,1}\widetilde{\mathcal{M}}_* \arrow[r, bend left, "\mathrm{c}"] & \widetilde{\mathcal{N}} \arrow[l, bend left, "\mathrm{v}", "(-1)"'] \end{tikzcd}\right].$$

11.6.4. Remarks.

(1) We obviously have $\mathrm{G}(\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{N}}, \mathrm{c}, \mathrm{v})(*D) = (\Xi_g\widetilde{\mathcal{M}}_*)(*D) = \widetilde{\mathcal{M}}_*$.

(2) If $D = H$ is smooth and g is a projection, the conditions "D-localizable" and "maximalizable" along D follow from the condition "strictly $\mathbb{R}$-specializable along D".

Set $D = (g)$ and consider the category $\mathrm{Glue}(X, D)$ whose objects consist of data $(\widetilde{\mathcal{M}}_*, \widetilde{\mathcal{N}}, \mathrm{c}, \mathrm{v})$ satisfying the properties as in the theorem, and whose morphisms are pairs of morphisms $\widetilde{\mathcal{M}}_* \to \widetilde{\mathcal{M}}'_*$ and $\widetilde{\mathcal{N}} \to \widetilde{\mathcal{N}}'$ which are naturally compatible with c, v and c', v'.

We have a natural functor

$$\widetilde{\mathcal{M}} \longmapsto \mathrm{G}(\widetilde{\mathcal{M}}(*D), \phi_{g,1}\widetilde{\mathcal{M}}, \mathrm{can}, \mathrm{var}).$$

from the category of $\widetilde{\mathcal{D}}_X$-coherent modules which are strictly $\mathbb{R}$-specializable, localizable and maximalizable along D, to the category $\mathrm{Glue}(X, D)$.

11.6.5. Corollary. *This functor is an equivalence of categories.*

The proof will occupy Sections 11.6.b–11.6.c, where we consider the case of a projection $t : X \simeq H \times \Delta_t \to \Delta_t$, and 11.6.d for the case of a principal divisor. Before starting, we give some examples.

11.6.6. Example **(Local identification of $R\Gamma_{[D]}\widetilde{\mathcal{M}}$).** Assume that $D = (g)$ and $\widetilde{\mathcal{M}}$ corresponds to the object $\mathrm{G}(\widetilde{\mathcal{M}}(*D), \phi_{g,1}\widetilde{\mathcal{M}}, \mathrm{can}, \mathrm{var})$, then we have

$$\begin{aligned}\widetilde{\mathcal{M}}[*D] &\longmapsto \mathrm{G}(\widetilde{\mathcal{M}}(*D), \psi_{g,1}\widetilde{\mathcal{M}}(-1), \mathrm{N}, \mathrm{Id}),\\ \widetilde{\mathcal{M}}[!D] &\longmapsto \mathrm{G}(\widetilde{\mathcal{M}}(*D), \psi_{g,1}\widetilde{\mathcal{M}}, \mathrm{Id}, \mathrm{N}).\end{aligned}$$

The morphisms $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ resp. $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ corresponds to the pairs $(\mathrm{Id}, \mathrm{can})$ resp. $(\mathrm{Id}, \mathrm{var})$. The inclusion of horizontal complexes

$$\begin{tikzcd} \mathrm{G}(0, \phi_{g,1}\widetilde{\mathcal{M}}, 0, 0) \arrow[r, "{(0,\mathrm{var})}"] \arrow[d, "{(0,\mathrm{Id})}"'] & \mathrm{G}(0, \psi_{g,1}\widetilde{\mathcal{M}}(-1), 0, 0) \arrow[d, "{(0,\mathrm{Id})}"'] \\ \mathrm{G}(\widetilde{\mathcal{M}}(*D), \phi_{g,1}\widetilde{\mathcal{M}}, \mathrm{can}, \mathrm{var}) \arrow[r, "{(\mathrm{Id},\mathrm{var})}"] & \mathrm{G}(\widetilde{\mathcal{M}}(*D), \psi_{g,1}\widetilde{\mathcal{M}}(-1), \mathrm{N}, \mathrm{Id}) \end{tikzcd}$$

is a quasi-isomorphism. It follows that we have a quasi-isomorphism

$$R\Gamma_{[D]}\widetilde{\mathcal{M}} \simeq \{\phi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{var}} \psi_{g,1}\widetilde{\mathcal{M}}(-1)\}.$$

If we add to the assumptions on $\widetilde{\mathcal{M}}$ made in Theorem 11.6.3 the assumption of strong strict $\mathbb{R}$-specializability (see Definition 9.3.24), Corollary 9.5.3 implies that we have a quasi-isomorphism

$$R\Gamma_{[D]}\widetilde{\mathcal{M}} \simeq ({}_{\mathrm{D}}\iota_{g*}\boldsymbol{L}_{\mathrm{D}}\iota_g^*)\widetilde{\mathcal{M}}. \tag{11.6.6 $*$}$$

Similarly, the complex $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ is quasi-isomorphic to $\psi_{g,1}\widetilde{\mathcal{M}} \xrightarrow{\mathrm{can}} \phi_{g,1}\widetilde{\mathcal{M}}$.

11.6.b. A construction of $\psi_{t,1}$ starting from localization. We will give another construction of $\psi_{t,1}\widetilde{\mathcal{M}}_*$ for a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X(*H)$-module $\widetilde{\mathcal{M}}_*$ (see the introduction of this chapter for this notion).

Let k be a non-negative integer, set $\varepsilon = 0, 1$, and let $\mathcal{J}^{(\varepsilon,k)}$ denote the upper Jordan block of size $k+1-\varepsilon$, that is, the filtered vector space $\mathbb{C}e_\varepsilon \oplus \cdots \oplus \mathbb{C}e_k$, where $e_i \in F^{i-\varepsilon}$ $(i \geqslant \varepsilon)$, so $\mathcal{J}^{(\varepsilon,k)}$ is in fact graded, with the nilpotent endomorphism

$$\begin{aligned}\mathcal{J}^{(\varepsilon,k)} &\xrightarrow{\ \mathrm{J}^{(\varepsilon,k)}\ } \mathcal{J}^{(\varepsilon,k)}(-1)\\ e_i &\longmapsto e_{i-1} \qquad \text{(convention: } e_{\varepsilon-1} = 0).\end{aligned}$$

Similarly, we denote by $\mathcal{J}_{(\varepsilon,k)}$ the lower Jordan block $\mathbb{C}e_\varepsilon \oplus \cdots \oplus \mathbb{C}e_k$ increasingly filtered (in fact graded) so that $e_i \in F_{i-\varepsilon}$, with the nilpotent endomorphism

$$\begin{aligned}\mathcal{J}_{(\varepsilon,k)} &\xrightarrow{\ \mathrm{J}_{(\varepsilon,k)}\ } \mathcal{J}_{(\varepsilon,k)}(-1)\\ e_i &\longmapsto e_{i+1} \qquad \text{(convention: } e_{k+1} = 0).\end{aligned}$$

We have natural morphisms (graded of degree zero and compatible with the nilpotent endomorphisms):

$$
\begin{array}{ccccc}
\mathcal{J}^{(1,k)}(-1) & \twoheadleftarrow & \mathcal{J}^{(0,k)} & \hookrightarrow & \mathcal{J}^{(0,k+1)} \\
e_i & \longleftarrow\!\!\mid & e_{i=1,\dots,k} & \longmapsto & e_i \\
0 & \longleftarrow\!\!\mid & e_0 & \longmapsto & e_0
\end{array}
\qquad
\begin{array}{ccccc}
\mathcal{J}_{(1,k)}(1) & \hookrightarrow & \mathcal{J}_{(0,k)} & \twoheadleftarrow & \mathcal{J}_{(0,k+1)} \\
e_{i=1,\dots,k} & \longmapsto & e_i & \longleftarrow\!\!\mid & e_{i=0,\dots,k} \\
 & & 0 & \longleftarrow\!\!\mid & e_{k+1}
\end{array}
$$

Let $\widetilde{\mathcal{M}}_*$ be a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_X(*H)$-module. We set $\widetilde{\mathcal{M}}_{*(\varepsilon,k)} = \widetilde{\mathcal{M}}_* \otimes_{\mathbb{C}} \mathcal{J}_{(\varepsilon,k)}$ with the action of $t\widetilde{\partial}_t$ given by

$$(m \otimes e_i)t\widetilde{\partial}_t := (mt\widetilde{\partial}_t) \otimes e_i + m \otimes \mathrm{J}_{(\varepsilon,k)}e_i,$$

and we define $\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}$ similarly. Both are strictly $\mathbb{R}$-specializable, according to Exercise 11.8.

11.6.7. Proposition. *Assume that $\widetilde{\mathcal{M}}_*$ is strictly $\mathbb{R}$-specializable along H.*

(1) *The morphisms*

$$(\mathrm{loc} \circ \mathrm{dloc})^{(\varepsilon,k)} : \widetilde{\mathcal{M}}_*^{(\varepsilon,k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(\varepsilon,k)}[*H]$$

and

$$(\mathrm{loc} \circ \mathrm{dloc})_{(\varepsilon,k)} : \widetilde{\mathcal{M}}_{*(\varepsilon,k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_{*(\varepsilon,k)}[*H]$$

are strictly $\mathbb{R}$-specializable for k large enough, locally on H.

(2) *We have functorial isomorphisms*

$$\varinjlim_k \mathrm{Ker}(\mathrm{loc} \circ \mathrm{dloc})^{(\varepsilon,k)} \simeq \psi_{t,1}\widetilde{\mathcal{M}}_* \simeq \varprojlim_k \mathrm{Coker}(\mathrm{loc} \circ \mathrm{dloc})_{(\varepsilon,k)},$$

and the limits are achieved for k large enough, locally on H.

(3) *The composed natural morphisms*

$$\widetilde{\mathcal{M}}_*^{(0,k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(0,k)}[*H] \longrightarrow \widetilde{\mathcal{M}}_*^{(1,k)}[*H](-1)$$

and

$$\widetilde{\mathcal{M}}_{*(1,k)}[!H](1) \longrightarrow \widetilde{\mathcal{M}}_{*(0,k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_{*(0,k)}[*H]$$

are strictly $\mathbb{R}$-specializable for k large enough, locally on H.

Proof.

(1) Since the morphisms considered induce isomorphisms on $V_{<0}$, it is enough to check that their $\phi_{t,1}$ are strict for k large enough (Example 9.3.27). By Exercise 11.4(3), this amounts to the strictness of $\mathrm{N}^{(\varepsilon,k)} : \psi_{t,1}\widetilde{\mathcal{M}}_*^{(\varepsilon,k)} \to \psi_{t,1}\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}(-1)$ and, by Exercise 11.8, to the strictness of $\mathrm{N}^{(\varepsilon,k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(\varepsilon,k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(\varepsilon,k)}(-1)$, and similarly for $\mathrm{N}_{(\varepsilon,k)}$. For k large enough locally on H, the cokernel of $\mathrm{N}^{(\varepsilon,k)}$ is identified with $\psi_{t,1}\widetilde{\mathcal{M}}_*(\varepsilon-(k+1))$, and similarly for $\mathrm{N}_{(\varepsilon,k)}$, according to Exercise 11.6, hence the strictness.

(2) By Exercises 11.4(1) and 11.8, we have

$$\mathrm{Ker}(\mathrm{loc} \circ \mathrm{dloc})^{(\varepsilon,k)} \simeq \mathrm{Ker}\big[\mathrm{N}^{(\varepsilon,k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(\varepsilon,k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(\varepsilon,k)}(-1)\big],$$

which is identified with $\psi_{t,1}\widetilde{\mathcal{M}}_*$ according to Exercise 11.6. We argue similarly for the lower case.

(3) Arguing as above, we are reduced to checking the strictness of $\phi_{t,1}$ of the composed morphisms. The upper one reads

$$(\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(0,k)} \xrightarrow{\mathrm{N}^{(0,k)}} (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(0,k)}(-1) \longrightarrow (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(1,k)}(-2)$$

and, according to Exercise 11.7(1), coincides with the composed morphism

$$(\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(0,k)} \longrightarrow (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(1,k)}(-1) \xrightarrow{\mathrm{N}^{(1,k)}} (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(1,k)}(-2)$$

whose cokernel, which is the cokernel of $\mathrm{N}^{(1,k)}$ since the first morphism is onto, is identified with $\psi_{t,1}\widetilde{\mathcal{M}}_*(-k-1)$ for k large, hence the strictness. The argument for the lower one is similar. □

11.6.c. The maximal extension along $H \times \{0\}$

11.6.8. Definition **(Maximal extension along** H**).** Let $\widetilde{\mathcal{M}}_*$ be a coherent $\widetilde{\mathcal{D}}_X(*H)$-module which is strictly $\mathbb{R}$-specializable along H. We set

$$\Xi_t\widetilde{\mathcal{M}}_* := \varinjlim_k \mathrm{Ker}\big(\widetilde{\mathcal{M}}_*^{(0,k)}[!H] \to \widetilde{\mathcal{M}}_*^{(1,k)}[*H](-1)\big).$$

11.6.9. Proposition **(The basic exact sequences).** *The limit in the definition of* $\Xi_t\widetilde{\mathcal{M}}_*$ *is achieved for* k *large enough, locally on* H*, and* $\Xi_t\widetilde{\mathcal{M}}_*$ *is a coherent* $\widetilde{\mathcal{D}}_X$*-module which is strictly* $\mathbb{R}$*-specializable along* H*. We have two functorial exact sequences*

$$0 \longrightarrow \widetilde{\mathcal{M}}_*[!H] \xrightarrow{a} \Xi_t\widetilde{\mathcal{M}}_* \xrightarrow{b} \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \longrightarrow 0, \tag{11.6.9 !}$$

$$0 \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}_* \xrightarrow{b^\vee} \Xi_t\widetilde{\mathcal{M}}_* \xrightarrow{a^\vee} \widetilde{\mathcal{M}}_*[*H] \longrightarrow 0, \tag{11.6.9 ∗}$$

with $b \circ b^\vee = -\mathrm{N}$ *and* $a^\vee \circ a = \mathrm{loc} \circ \mathrm{dloc}$ *(see Corollaries* 11.3.8(2) *and* 11.4.13(2)*). Moreover, we also have*

$$\Xi_t\widetilde{\mathcal{M}}_* := \varprojlim_k \mathrm{Coker}\big(\widetilde{\mathcal{M}}_{*(1,k)}[!H](1) \to \widetilde{\mathcal{M}}_{*(0,k)}[*H]\big).$$

Proof. Arguing as in Proposition 9.3.30, one checks that the kernel of the morphism $\widetilde{\mathcal{M}}_*^{(0,k)}[!H] \to \widetilde{\mathcal{M}}_*^{(1,k)}[*H](-1)$ is strictly $\mathbb{R}$-specializable along H. We decompose this morphism either as

$$\widetilde{\mathcal{M}}_*^{(0,k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(1,k)}[!H](-1) \longrightarrow \widetilde{\mathcal{M}}_*^{(1,k)}[*H](-1)$$

or as

$$\widetilde{\mathcal{M}}_*^{(0,k)}[!H] \longrightarrow \widetilde{\mathcal{M}}_*^{(0,k)}[*H] \longrightarrow \widetilde{\mathcal{M}}_*^{(1,k)}[*H](-1).$$

In the first case, its kernel is the middle term of a short exact sequence having the kernel of the right-hand morphism as right-hand term, that is, $\psi_{t,1}\widetilde{\mathcal{M}}_*(-1)$ for k large enough locally, according to Proposition 11.6.7, and the kernel of the left-hand morphism as left-hand term, that is, $\widetilde{\mathcal{M}}_*[!H]$, according to Proposition 11.4.7(7). The kernel is thus independent of k if k is large enough locally, and we have thus obtained (11.6.9 !).

In the second case, its kernel is the middle term of a short exact sequence having the kernel of the right-hand morphism as right-hand term, that is, $\widetilde{\mathcal{M}}_*[*H]$, according to Proposition 11.3.3(11), and the kernel of the left-hand morphism as left-hand term, that is, $\psi_{t,1}\widetilde{\mathcal{M}}_*$ for k large enough locally, according to Proposition 11.6.7. We have thus obtained (11.6.9 $*$).

The composed morphism $a^\vee \circ a$ is the composition

$$\widetilde{\mathcal{M}}_*[!H] \simeq \widetilde{\mathcal{M}}_*[!H] \otimes e_0 \hookrightarrow \widetilde{\mathcal{M}}_*^{(0,k)}[!H] \xrightarrow{\mathrm{dloc}^{\vee(0,k)} \circ \mathrm{dloc}^{(0,k)}} \widetilde{\mathcal{M}}_*^{(0,k)}[*H] \\ \longrightarrow \widetilde{\mathcal{M}}_*[*H] \otimes e_0 \simeq \widetilde{\mathcal{M}}_*[*H],$$

which is equal to $\mathrm{loc} \circ \mathrm{dloc}$. On the other hand, the morphism $b \circ b^\vee : \psi_{t,1}\widetilde{\mathcal{M}}_* \to \psi_{t,1}\widetilde{\mathcal{M}}_*(-1)$ is identified with the natural morphism

$$\mathrm{Ker}(\mathrm{dloc}^{\vee(0,k)} \circ \mathrm{dloc}^{(0,k)}) \longrightarrow \mathrm{Ker}(\mathrm{dloc}^{\vee(1,k)} \circ \mathrm{dloc}^{(1,k)})$$

for k large enough locally. It is identified with the natural morphism

$$\mathrm{Ker}\big[\mathrm{N}^{(0,k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(0,k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(0,k)}(-1)\big] \\ \longrightarrow \mathrm{Ker}\big[\mathrm{N}^{(1,k)} : (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(1,k)} \to (\psi_{t,1}\widetilde{\mathcal{M}}_*)^{(1,k)}(-1)\big],$$

which is identified, as in Exercise 11.6, to the morphism (k large enough locally)

$$-\mathrm{N} : \mathrm{Ker}\,\mathrm{N}^{k+1} \simeq \psi_{t,1}\widetilde{\mathcal{M}}_* \longrightarrow \mathrm{Ker}\,\mathrm{N}^{k}(-1) \simeq \psi_{t,1}\widetilde{\mathcal{M}}_*(-1). \qquad \square$$

11.6.10. Proposition **(Nearby and vanishing cycles of the maximal extension)**

(1) *The morphisms* $a : \widetilde{\mathcal{M}}_*[!H] \to \Xi_t\widetilde{\mathcal{M}}_*$ *and* $a^\vee : \Xi_t\widetilde{\mathcal{M}}_* \to \widetilde{\mathcal{M}}_*[*H]$ *induce isomorphisms when restricted to* $V_{<0}$, *and thus isomorphisms of the* $\psi_{t,\lambda}$ *objects.*

(2) *The exact sequence* $\phi_{t,1}(11.6.9\,!)$ *is isomorphic to the naturally split exact sequence* $0 \to \psi_{t,1}\widetilde{\mathcal{M}}_* \xrightarrow{i_1} \psi_{t,1}\widetilde{\mathcal{M}}_* \oplus \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \xrightarrow{p_2} \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \to 0$. *With respect to this isomorphism, the exact sequence* $\phi_{t,1}(11.6.9\,*)$ *reads*

$$0 \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}_* \xrightarrow{(\mathrm{Id}, -\mathrm{N})} \psi_{t,1}\widetilde{\mathcal{M}}_* \oplus \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \xrightarrow{\mathrm{N} + \mathrm{Id}} \psi_{t,1}\widetilde{\mathcal{M}}_*(-1) \longrightarrow 0.$$

Proof.

(1) We notice that, since all modules in (11.6.9 !) and (11.6.9 $*$) are strictly $\mathbb{R}$-specializable, the morphisms a and $a^\vee$ are strictly $\mathbb{R}$-specializable, in the sense of Definition 9.3.28. The result follows from Proposition 9.3.30, since $\psi_{t,1}\widetilde{\mathcal{M}}_*$ is supported on H.

(2) This follows from Exercise 11.7. $\square$

***Proof of Theorem* 11.6.3 *for the function* t.** The complex $C^\bullet$ considered in the theorem has nonzero cohomology in degree one only, since $b^\vee$ is injective and b is onto. We show that $\psi_{t,\lambda}C^\bullet$ and $\phi_{t,1}C^\bullet$ are strict. We have $\psi_{t,\lambda}C^\bullet = \{0 \to \psi_{t,\lambda}\Xi_t\widetilde{\mathcal{M}} \to 0\}$, so the strictness follows from Proposition 11.6.9. On the other hand, according to Proposition 11.6.10, $\phi_{t,1}C^\bullet$ is identified with the complex

$$\begin{array}{ccccc} \psi_{t,1}\widetilde{\mathcal{M}} & \longrightarrow & \psi_{t,1}\widetilde{\mathcal{M}} \oplus \psi_{t,1}\widetilde{\mathcal{M}}(-1) \oplus \widetilde{\mathcal{N}} & \longrightarrow & \psi_{t,1}\widetilde{\mathcal{M}}(-1) \\ e & \longmapsto & (e, -\mathrm{N}e, ce) & & \\ & & (e, m, \varepsilon) & \longmapsto & m + \mathrm{v}\varepsilon. \end{array}$$

Its cohomology in degree one is then identified with $\widetilde{\mathcal{N}}$. Since $\widetilde{\mathcal{N}}$ is assumed to be strict, $H^1\phi_{t,1}C^\bullet$ is strict, and we clearly have $H^j\phi_{t,1}C^\bullet = 0$ for $j \neq 1$. We deduce from Corollary 9.3.31 that $H^1C^\bullet$ is strictly $\mathbb{R}$-specializable along H and $\psi_{t,\lambda}H^1C^\bullet = H^1\psi_{t,\lambda}C^\bullet$, and $\phi_{t,1}H^1C^\bullet = H^1\phi_{t,1}C^\bullet$. □

***Proof of Corollary* 11.6.5 *for the function* t.** The construction G of Theorem 11.6.3 gives a right inverse of the functor considered in Corollary 11.6.5, implying that the latter is essentially surjective. That it is fully faithful now follows from Corollary 9.3.21. □

11.6.d. The maximal extension along an arbitrary effective divisor

11.6.11. Definition. Let D be an arbitrary effective divisor in X and let $\widetilde{\mathcal{M}}_*$ be $\widetilde{\mathcal{D}}_X(*D)$-coherent and strictly $\mathbb{R}$-specializable along D.

(1) If $D = (g)$, where $g : X \to \mathbb{C}$ is a holomorphic function, set $H = X \times \{0\} \subset X \times \mathbb{C}$. We say that $\widetilde{\mathcal{M}}_*$ is *maximalizable* along (g) if $\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}$ is (g)-localizable for every k and $\varepsilon \in \{0,1\}$ (see Definition 11.5.1).

(2) In general, we say that $\widetilde{\mathcal{M}}_*$ is *maximalizable* along D if for each point $x_o \in D$ and some (or any) local equation g of D near x_o, $\widetilde{\mathcal{M}}_*$ is *maximalizable* along (g).

11.6.12. Proposition. *Assume that $\widetilde{\mathcal{M}}_*$ is maximalizable along $D = (g)$. Set*

$$\Xi_g\widetilde{\mathcal{M}}_* := \varinjlim_k \operatorname{Ker}\big(\widetilde{\mathcal{M}}_*^{(0,k)}[!D] \to \widetilde{\mathcal{M}}_*^{(1,k)}[*D](-1)\big).$$

Then the analogues of Propositions 11.6.9 *and* 11.6.10 *hold for $\Xi_g\widetilde{\mathcal{M}}_*$.*

Sketch of proof. One first checks that the analogue of Proposition 11.6.7 holds, by checking that it holds after applying ${}_{\mathrm{D}}\iota_{g*}$. This follows from the fact that the morphisms dloc and loc behave well under ${}_{\mathrm{D}}\iota_{g*}$ (see Remarks 11.4.14 and 11.3.9). The remaining part of the proof is done with similar arguments. □

11.6.13. Remark. If we denote by $a_g, a_g^\vee, b_g, b_g^\vee$ and $a_t, a_t^\vee, b_t, b_t^\vee$ the morphisms $a, a^\vee, b, b^\vee$ given by (11.6.2 !), (11.6.2 ∗) and Proposition 11.6.9 respectively, we have $a_t = {}_{\mathrm{D}}\iota_{g*}a_g$, etc.

***Proof of Theorem* 11.6.3 *and Corollary* 11.6.5.** Let us apply the exact functor ${}_{\mathrm{D}}\iota_{g*}$ to $(11.6.3\,*)_g$. Since $\widetilde{\mathcal{M}}_*$ is maximalizable along D, this produces $(11.6.3\,*)_t$, to which we apply the theorem. Since ${}_{\mathrm{D}}\iota_{g*}^{(j)}(11.6.3\,*)_g \simeq {}_{\mathrm{D}}\iota_{g*}H^j(11.6.3\,*)_t$, we deduce the theorem for $(11.6.3\,*)_g$, and thus the functor of Corollary 11.6.5 is essentially surjective. It is fully faithful because it is so when $g = t$ and ${}_{\mathrm{D}}\iota_{g*}$ is fully faithful by Proposition 9.6.2. □

***11.6.14. Proposition* (Recovering $\phi_{g,1}$ from localization and maximalization)**

Let $\widetilde{\mathcal{M}}$ be as above and set $\widetilde{\mathcal{M}}_* = \widetilde{\mathcal{M}}(*D)$. Then the complex

$$\Phi_g^\bullet\widetilde{\mathcal{M}} := \Big\{\widetilde{\mathcal{M}}_*[!g] \xrightarrow{a\,\oplus\,\mathrm{dloc}} \Xi_g\widetilde{\mathcal{M}}_* \oplus \widetilde{\mathcal{M}} \xrightarrow{a^\vee - \mathrm{loc}} \widetilde{\mathcal{M}}_*[*g]\Big\} \tag{11.6.14 ∗}$$

satisfies $H^k\Phi_g^\bullet\widetilde{\mathcal{M}} = 0$ for $k \neq 1$ and $H^1\Phi_g^\bullet\widetilde{\mathcal{M}} \simeq \phi_{g,1}\widetilde{\mathcal{M}}$.

Proof. We first consider the case of $X = H \times \mathbb{C}$ and $g = t$. Injectivity of $a \oplus \mathrm{dloc}$ follows from that of a, and surjectivity of $a^\vee - \mathrm{loc}$ follows from that of $a^\vee$. Since, for every $\lambda \in \mathbb{S}^1$, $\psi_{t,\lambda} a$ and $\psi_{t,\lambda} a^\vee$ are isomorphisms inverse one to the other, and the same property holds for $\psi_{t,\lambda}\mathrm{dloc}$ and $\psi_{t,\lambda}\mathrm{loc}$, it follows that $\psi_{t,\lambda}\Phi_t^\bullet\widetilde{\mathcal{M}} \simeq 0$. On the other hand, the complex $\phi_{t,1}\Phi_t^\bullet\widetilde{\mathcal{M}}$ is isomorphic to the complex

$$\begin{array}{ccccc} 0 \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}} \oplus \psi_{t,1}\widetilde{\mathcal{M}}(-1) \oplus \phi_{t,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{t,1}\widetilde{\mathcal{M}}(-1) \longrightarrow 0 \\ e \longmapsto (e, 0, \operatorname{can} e) \\ (e, n, \varepsilon) \longmapsto \mathrm{N}e + n - \operatorname{var}\varepsilon \end{array}$$

so $H^1\phi_{t,1}\Phi_t^\bullet\widetilde{\mathcal{M}} \simeq (\psi_{t,1}\widetilde{\mathcal{M}} \oplus \phi_{t,1}\widetilde{\mathcal{M}})/\operatorname{Im}(\mathrm{Id} \oplus \operatorname{can})$, and therefore the projection $\psi_{t,1}\widetilde{\mathcal{M}} \oplus \phi_{t,1}\widetilde{\mathcal{M}} \to \phi_{t,1}\widetilde{\mathcal{M}}$ induces an isomorphism $H^1\phi_{t,1}\Phi_t^\bullet\widetilde{\mathcal{M}} \xrightarrow{\sim} \phi_{t,1}\widetilde{\mathcal{M}}$. As a consequence of Corollary 9.3.31, the cohomology of complex $\Phi_t^\bullet\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H and in particular $\phi_{t,1}H^1\Phi_t^\bullet\widetilde{\mathcal{M}} \simeq H^1\phi_{t,1}\Phi_t^\bullet\widetilde{\mathcal{M}}$. The first part of the proof also shows that $H^1\Phi_t^\bullet\widetilde{\mathcal{M}} \simeq \phi_{t,1}H^1\Phi_t^\bullet\widetilde{\mathcal{M}}$, so $H^1\Phi_t^\bullet\widetilde{\mathcal{M}} \simeq \phi_{t,1}\widetilde{\mathcal{M}}$.

The general case is obtained by using the exactness of ${}_{\mathrm{D}}\iota_{g*}$. □

11.7. Localizability, maximalizability and pushforward

Let us keep the notation and assumptions of Corollary 9.8.9.

11.7.1. Corollary.

(1) *Assume moreover that $\widetilde{\mathcal{M}}$ is localizable along (g). Then ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ are so along (g') for all i, we have $({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})[\star g'] \simeq {}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[\star g])$ $(\star = *, !)$ and the morphisms* dloc, loc *behave well under ${}_{\mathrm{D}}f_*^{(i)}$.*

(2) *Assume moreover that $\widetilde{\mathcal{M}}$ is maximalizable along (g). Then ${}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}$ are so along (g') for all i, we have $\Xi_{g'}({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}) \simeq {}_{\mathrm{D}}f_*^{(i)}(\Xi_g\widetilde{\mathcal{M}})$, and the exact sequences* (11.6.2 !) *and* (11.6.2 $*$) *behave well under ${}_{\mathrm{D}}f_*^{(i)}$.*

Proof.

(1) Assume first that f takes the form $f_H \times \mathrm{Id} : H \times \Delta_t \to H' \times \Delta_t$. Then from Theorem 9.8.8 one deduces that ${}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[\star H])$ satisfies the characteristic properties 11.3.3(8) or 11.4.7(4) for $({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})[\star H']$, so the statement holds in this case.

For the general case, we note that we have a Cartesian diagram

$$\begin{tikzcd} X \arrow[r, hook, "\iota_g"] \arrow[d, "f"'] & X \times \Delta_t \arrow[d, "f \times \mathrm{Id}"] \\ X' \arrow[r, hook, "\iota_{g'}"] & X' \times \Delta_t \end{tikzcd}$$

and we set $H = X \times \{0\}$, $H' = X' \times \{0\}$. Then

$$\begin{aligned} ({}_{\mathrm{D}}(f \times \mathrm{Id})_*^{(i)}\widetilde{\mathcal{M}})[\star H'] &\simeq {}_{\mathrm{D}}(f \times \mathrm{Id})_*^{(i)}\big(({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[\star H]\big) \\ &\simeq {}_{\mathrm{D}}(f \times \mathrm{Id})_*^{(i)}\big({}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[\star g])\big) \simeq {}_{\mathrm{D}}\iota_{g'*}({}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[\star g])), \end{aligned}$$

and the assertion holds according to the first case.

(2) Let us indicate the proof in the case where $f = f_H \times \mathrm{Id}$, as above. We first notice that ${}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}^{(\varepsilon,k)}) \simeq ({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})^{(\varepsilon,k)}$, and since f is proper, we can locally on X' choose k big enough so that the limits involved are already obtained for k. Let us denote by φ_k the morphism $\widetilde{\mathcal{M}}^{(0,k)}[!H] \to \widetilde{\mathcal{M}}^{(1,k)}[*H]$. We have a natural morphism ${}_{\mathrm{D}}f_*^{(i)} \operatorname{Ker}\varphi_k \to \operatorname{Ker}{}_{\mathrm{D}}f_*^{(i)}\varphi_k$ and, according to (1), it induces a morphism between sequences

$$\begin{aligned} {}_{\mathrm{D}}f_*^{(i)}\big((11.6.9\,!)(\widetilde{\mathcal{M}})\big) &\longrightarrow (11.6.9\,!)({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}),\\ {}_{\mathrm{D}}f_*^{(i)}\big((11.6.9\,*)(\widetilde{\mathcal{M}})\big) &\longrightarrow (11.6.9\,*)({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}}). \end{aligned}$$

The right-hand sequences are short exact, while the left-hand ones are a priori only exact in the middle. Moreover, the extreme morphisms between these sequences are isomorphisms, by the previous results. Let us show that the left-hand sequences are indeed short exact and that the morphisms (in the middle) are isomorphisms. We will treat (11.6.9 !) for example. The composed (diagonal) morphism

$$\begin{tikzcd} {}_{\mathrm{D}}f_*^{(i)}(\widetilde{\mathcal{M}}[!H]) \arrow[r, "{}_{\mathrm{D}}f_*^{(i)}a"] \arrow[d, "\wr"'] \arrow[dr] & {}_{\mathrm{D}}f_*^{(i)}\Xi_g(\widetilde{\mathcal{M}}) \arrow[d] \\ ({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})[!H'] \arrow[r, hook, "a"] & \Xi_{g'}\,{}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}} \end{tikzcd}$$

is injective by assumption, hence so is ${}_{\mathrm{D}}f_*^{(i)}a$, and by applying this with $i+1$, we find that ${}_{\mathrm{D}}f_*^{(i)}\Xi_g(\widetilde{\mathcal{M}}) \to {}_{\mathrm{D}}f_*^{(i)}(\psi_{t,1}\widetilde{\mathcal{M}})$ is onto, so that the sequence ${}_{\mathrm{D}}f_*^{(i)}\big((11.6.9\,!)(\widetilde{\mathcal{M}})\big)$ is short exact. Now, it is clear that it is isomorphic to $(11.6.9\,!)({}_{\mathrm{D}}f_*^{(i)}\widetilde{\mathcal{M}})$. $\square$

11.8. The Thom-Sebastiani formula for the vanishing cycles

The Thom-Sebastiani formula for the vanishing cycle functor is analogous to the Künneth formula for the pushforward functor of Section 8.8.f. The setting is as follows. We are given, for $i = 1, 2$, a holomorphic function $g_i : X_i \to \mathbb{C}$ and a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}_{X_i}$-module $\widetilde{\mathcal{M}}_i$ along g_i. We consider the *Thom-Sebastiani sum* $g : X := X_1 \times X_2 \to \mathbb{C}$ defined by $g(x_1, x_2) = g_1(x_1) + g_2(x_2)$. In other words, g is the composition of the map $(g_1, g_2) : X_1 \times X_2 \to \mathbb{C} \times \mathbb{C}$ with the sum map $\mathbb{C} \times \mathbb{C} \to \mathbb{C}$ defined by $(t_1, t_2) \mapsto t_1 + t_2$. In order to state this formula in a uniform way, we will set $\phi_{g,\lambda} = \psi_{g,\lambda}$ as defined by (9.4.3 $**$) if $\lambda \neq 1$ (and we keep the notation $\phi_{g,1}$ as it is). Moreover, given λ with $|\lambda| = 1$, we set $\lambda = \exp 2\pi\mathrm{i}\,\alpha$ with $\alpha \in (-1, 0]$.

***11.8.1. Theorem* (Thom-Sebastiani formula).** *Assume that $\widetilde{\mathcal{M}}_i$ $(i = 1, 2)$ are strict and strictly $\mathbb{R}$-specializable along g_i. Then, if $\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{M}}_2$ is strictly $\mathbb{R}$-specializable* *along g, we have*

$$\phi_{g,\lambda}\widetilde{\mathcal{M}} \simeq \bigoplus_{\alpha_1\in(-1,\alpha]\cup\{0\}} (\phi_{g_1,\lambda_1}\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \phi_{g_2,\lambda/\lambda_1}\widetilde{\mathcal{M}}_2) \\ \oplus \bigoplus_{\alpha_1\in(\alpha,0)} (\phi_{g_1,\lambda_1}\widetilde{\mathcal{M}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \phi_{g_2,\lambda/\lambda_1}\widetilde{\mathcal{M}}_2)(-1).$$

We will denote by $\widetilde{\mathcal{N}}_i$ the pushforward ${}_{\mathrm{D}}\iota_{g_i*}\widetilde{\mathcal{M}}_i$ and set similarly $\widetilde{\mathcal{N}} := {}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$. Recall that $\widetilde{\mathcal{N}}_i = \widetilde{\mathcal{M}}_i \otimes_{\widetilde{\mathbb{C}}} \widetilde{\mathbb{C}}[\widetilde{\partial}_{t_i}]$ with a suitable right action of $\widetilde{\mathcal{D}}_{X_i\times\widetilde{\mathbb{C}}}$ (see Example 8.7.7). We will regard it as a $\widetilde{\mathcal{D}}_{X_i}[t_i]\langle\widetilde{\partial}_{t_i}\rangle$-module.

11.8.2. Lemma. *The following sequence of $\widetilde{\mathcal{D}}_X$-modules is exact:*

$$0 \longrightarrow \iota^{-1}_{(g_1,g_2)}(\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_2)(1) \xrightarrow{\widetilde{\partial}_{t_1}\boxtimes 1 - 1\boxtimes\widetilde{\partial}_{t_2}} \iota^{-1}_{(g_1,g_2)}(\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_2) \longrightarrow \iota_g^{-1}\widetilde{\mathcal{N}} \longrightarrow 0,$$

and the right action of $\widetilde{\partial}_t$ on $\iota_g^{-1}\widetilde{\mathcal{N}}$ is the action naturally induced by that of $\widetilde{\partial}_{t_1}\boxtimes 1$ and that of $1\boxtimes\widetilde{\partial}_{t_2}$.

Proof. Let us first make precise that $\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_2$ is supported on the image of $\iota_{(g_1,g_2)}$, and similarly for $\widetilde{\mathcal{N}}$, so that the functors dloc^{-1} only serve to identify the supports of all the terms to $X = X_1 \times X_2$. In the following, we will neglect to write them down.

Considering only the $\widetilde{\mathbb{C}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}]$-module structure, the sequence is written

$$\widetilde{\mathcal{M}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}](1) \xrightarrow{\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2}} \widetilde{\mathcal{M}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}] \longrightarrow \widetilde{\mathcal{M}}[\widetilde{\partial}_t],$$

where the second map is obtained by sending $\widetilde{\partial}_{t_i}$ to $\widetilde{\partial}_t$ ($i = 1, 2$). This obviously forms a short exact sequence. One then checks, by using Exercise 8.45, that the sequence is compatible with the $\widetilde{\mathcal{D}}_X$-actions. ∎

11.8.a. Naive algebraic microlocalization. In order to understand the behaviour of the V-filtrations, we will need to invert $\widetilde{\partial}_t$. We first make clear the corresponding framework. We will work in the setting of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$-modules. The ring $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$ is obtained by inverting $\widetilde{\partial}_t$ (so that the degree of $\widetilde{\partial}_x^{\boldsymbol{\ell}}\widetilde{\partial}_t^k$ is $|\boldsymbol{\ell}| + k$ for every $\boldsymbol{\ell} \in \mathbb{N}^{\dim X}$ and $k \in \mathbb{Z}$, and the grading of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$ is indexed by $\mathbb{Z}$). The only possible way to define it as a ring containing $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$ as a subring is to impose that $\widetilde{\partial}_t^{-1}$ commutes with $\widetilde{\mathcal{D}}_X$, to set

$$[\widetilde{\partial}_t^k, t] = kz\widetilde{\partial}_t^{k-1}, \quad k \in \mathbb{Z},$$

(extending thus the formula for $k \in \mathbb{N}$) and to define $[\widetilde{\partial}_t^{-k}, t^\ell]$ by similar (more complicated) formulas. For example, we have

$$t\widetilde{\partial}_t^k = (t\widetilde{\partial}_t)\widetilde{\partial}_t^{k-1} = \widetilde{\partial}_t^{k-1}((t\widetilde{\partial}_t) - (k-1)z) = \widetilde{\partial}_t^k t - k\widetilde{\partial}_t^{k-1}, \quad k \in \mathbb{Z}.$$

Note that working with $\widetilde{\mathcal{D}}_{X\times\mathbb{C}}$ instead of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$ would have led us to introduce (non convergent) series in $\widetilde{\partial}_t^{-1}$, and this justifies our choice of keeping the variable t algebraic. Note also that, if instead of inverting the action of $\widetilde{\partial}_t$ we invert that of t, we recover the notion of naive localization of the introduction of this chapter.

We will write $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}\rangle=\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$, so that we consider $\widetilde{\partial}_t^{-1}$ as the "variable" in the t-direction. In such a way, we set

$$\begin{aligned}&V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle=\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle,\\ &V_k\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle=\widetilde{\partial}_t^kV_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle=V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle\widetilde{\partial}_t^k.\end{aligned}$$

We clearly have $tV_k\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle\subset V_{k-1}\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$. For a $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$-module ${}^{\mu}\widetilde{\mathcal{N}}$, a coherent V-filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ indexed by $A+\mathbb{Z}$ for $A\subset(-1,0]$ finite, is an exhaustive filtration such that $U_{\alpha+k}{}^{\mu}\widetilde{\mathcal{N}}=U_\alpha{}^{\mu}\widetilde{\mathcal{N}}(k)\widetilde{\partial}_t^k$ ($k\in\mathbb{Z}$, $\alpha\in A$) and each $U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ is $V_0\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]\langle t\widetilde{\partial}_t\rangle$-coherent. We say that ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$ if there exists a coherent V-filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ indexed by $A+\mathbb{Z}$ such that $t\widetilde{\partial}_t-\alpha z$ is nilpotent on $\mathrm{gr}_\alpha^U{}^{\mu}\widetilde{\mathcal{N}}$ ($\alpha\in A$) and each $\mathrm{gr}_\alpha^U{}^{\mu}\widetilde{\mathcal{N}}$ is strict.

11.8.3. Remark. Let us denote by θ the 'variable' $\widetilde{\partial}_t^{-1}$. The commutation relations above show that t behaves like $\theta^2\widetilde{\partial}_\theta$. Then $\theta\widetilde{\partial}_\theta$ is identified with $\widetilde{\partial}_tt=t\widetilde{\partial}_t+z$. The setting is then completely similar to that of Section 9.2.

11.8.4. Lemma. *If ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$, such a filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ is unique.*

This filtration is then denoted by $V_\bullet{}^{\mu}\widetilde{\mathcal{N}}$.

Proof. Assume we are given two such filtrations U,U', that we can assume to be indexed by the same index set $A+\mathbb{Z}$, by taking the union of both index sets. Fix $\alpha\in A$. We will prove that $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}}\subset U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ and the reverse inclusion is proved similarly. There exists $\beta\geqslant\alpha$ (locally on X) such that

$$U'_\alpha{}^{\mu}\widetilde{\mathcal{N}}\subset U_\beta{}^{\mu}\widetilde{\mathcal{N}},\quad U'_{<\alpha}{}^{\mu}\widetilde{\mathcal{N}}\subset U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}.$$

Let m be a local section of $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}}$. It satisfies $m(t\widetilde{\partial}_t-\beta z)^N\in U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}$ on the one hand, and $m(t\widetilde{\partial}_t-\alpha z)^M\in U'_{<\alpha}{}^{\mu}\widetilde{\mathcal{N}}\subset U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}$ on the other hand. Therefore, the class $[m]$ of m in $\mathrm{gr}_\beta^U{}^{\mu}\widetilde{\mathcal{N}}$ is annihilated by a power of $z(\alpha-\beta)$. If $\beta>\alpha$, it is thus zero by strictness of $\mathrm{gr}_\beta^U{}^{\mu}\widetilde{\mathcal{N}}$. We conclude that $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}}\subset U_{<\beta}{}^{\mu}\widetilde{\mathcal{N}}$ and, by induction, we obtain $U'_\alpha{}^{\mu}\widetilde{\mathcal{N}}\subset U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$. □

***11.8.5. Definition* (Naive algebraic microlocalization).** Let $\widetilde{\mathcal{N}}$ be a $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle$-module. The associated microlocalized module is ${}^{\mu}\widetilde{\mathcal{N}}:=\widetilde{\mathcal{N}}\otimes_{\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t\rangle}\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}\rangle$.

11.8.6. Lemma. *Assume that $\widetilde{\mathcal{N}}={}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}\simeq\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathbb{C}}[\widetilde{\partial}_t]$ and that $\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along (t). Then ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$ and we have $\mathrm{gr}_\alpha^V{}^{\mu}\widetilde{\mathcal{N}}\simeq\mathrm{gr}_\alpha^V\widetilde{\mathcal{N}}$ for $\alpha>-1$.*

Proof. With the first assumption, we have a natural inclusion $\widetilde{\mathcal{N}}\hookrightarrow{}^{\mu}\widetilde{\mathcal{N}}$ since ${}^{\mu}\widetilde{\mathcal{N}}\simeq\widetilde{\mathcal{M}}\otimes_{\widetilde{\mathbb{C}}}\widetilde{\mathbb{C}}[\widetilde{\partial}_t,\widetilde{\partial}_t^{-1}]$ as a $\widetilde{\mathcal{D}}_X$-module. We define a V-filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ by

$$U_\alpha{}^{\mu}\widetilde{\mathcal{N}}=\sum_{i\geqslant0}V_{\alpha+i}\widetilde{\mathcal{N}}\cdot\widetilde{\partial}_t^{-i},\quad\forall\,\alpha\in\mathbb{R}.\tag{11.8.7}$$

(See Exercise 11.9.) It is straightforward to check that it is a coherent V-filtration of ${}^{\mu}\widetilde{\mathcal{N}}$ as defined before Lemma 11.8.4.

(1) We claim that the morphism $\widetilde{\mathcal{N}} \hookrightarrow {}^{\mu}\widetilde{\mathcal{N}}$ is strictly compatible with the filtrations $V_{>-1}\widetilde{\mathcal{N}}$ and $U_{>-1}{}^{\mu}\widetilde{\mathcal{N}}$, i.e., we have $V_\alpha{}^{\mu}\widetilde{\mathcal{N}} \cap \widetilde{\mathcal{N}} = V_\alpha\widetilde{\mathcal{N}}$ for every $\alpha > -1$. Let $\sum_{i=0}^k n_i\widetilde{\partial}_t^{-i}$ be a local section of $U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ for some fixed $\alpha > -1$. In particular, n_k is a section of $V_{\alpha+k}\widetilde{\mathcal{N}}$. Assume that it belongs to $\widetilde{\mathcal{N}}\widetilde{\partial}^{-k+1}$. We claim that $n_k = n'_{k-1}\widetilde{\partial}_t$, where n'_{k-1} is a local section of $V_{\alpha+k-1}\widetilde{\mathcal{N}}$. This claim implies that the sum above can be rewritten with i running from 0 to $k-1$. Arguing inductively, we find that the sum can be rewritten with $i = 0$ only, i.e., belongs to $V_\alpha\widetilde{\mathcal{N}}$.

In order to prove the claim, we note that $n_k\widetilde{\partial}_t^{-k}$ also belongs to $\widetilde{\mathcal{N}}\widetilde{\partial}^{-k+1}$ and n_k belongs to $V_{\alpha+k}\widetilde{\mathcal{N}} \cap \widetilde{\mathcal{N}}\cdot\widetilde{\partial}_t$. Let us write $n_k = n'_\beta\widetilde{\partial}_t$ with $n'_\beta \in V_\beta\widetilde{\mathcal{N}}$. If $\beta > \alpha+k-1$, we deduce that $n_\beta\widetilde{\partial}_t = 0$ in $\mathrm{gr}_{\beta+1}\widetilde{\mathcal{N}}$. By the strict $\mathbb{R}$-specializability of $\widetilde{\mathcal{N}}$ and Proposition 9.3.20(d) we conclude that $n_\beta \in V_{<\beta}\widetilde{\mathcal{N}}$, and by induction this implies that $n_\beta \in V_{\alpha+k-1}\widetilde{\mathcal{N}}$, as wanted.

(2) We claim that the filtration $U_\bullet({}^{\mu}\widetilde{\mathcal{N}}/\widetilde{\mathcal{N}})$ naturally induced by $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ satisfies $\mathrm{gr}^U_\alpha({}^{\mu}\widetilde{\mathcal{N}}/\widetilde{\mathcal{N}}) = 0$ for $\alpha > -1$. Indeed, this amounts to proving that $U_\alpha{}^{\mu}\widetilde{\mathcal{N}} = U_{<\alpha}{}^{\mu}\widetilde{\mathcal{N}} + \widetilde{\mathcal{N}}$ for $\alpha > -1$. This immediately follows from the property that, for $\alpha > -1$ and $k \geqslant 1$, $V_{\alpha+k}\widetilde{\mathcal{N}} = V_{\alpha+k-1}\widetilde{\mathcal{N}}\cdot\widetilde{\partial}_t + V_{<\alpha+k}\widetilde{\mathcal{N}}$ (Proposition 9.3.20(b)).

We conclude from (1) and (2) that, for every $\alpha > -1$, $\mathrm{gr}^V_\alpha\widetilde{\mathcal{N}} \to \mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ is an isomorphism. As a consequence, $\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ is strict, and ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$ with $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ as V-filtration. This concludes the proof. ☐

11.8.b. External product of $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}\rangle$-modules. We consider in this section coherent $\widetilde{\mathcal{D}}_{X_i}[\widetilde{\partial}_{t_i}, \widetilde{\partial}_{t_i}^{-1}]$-modules ${}^{\mu}\widetilde{\mathcal{N}}_i$ $(i = 1, 2)$ equipped with a compatible action of $t_i\widetilde{\partial}_{t_i}$, that we denote E_i for short. This covers the case considered in Lemma 11.8.6. Then ${}^{\mu}\widetilde{\mathcal{N}}_i$ are also $\widetilde{\mathcal{D}}_{X_i}[t_i]\langle\widetilde{\partial}_{t_i}, \widetilde{\partial}_{t_i}^{-1}\rangle$-coherent. We denote by ${}^{\mu}\widetilde{\mathcal{N}}$ the cokernel of

$$
{}^{\mu}\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} {}^{\mu}\widetilde{\mathcal{N}}_2 \xrightarrow{\widetilde{\partial}_{t_1} \boxtimes 1 - 1 \boxtimes \widetilde{\partial}_{t_2}} {}^{\mu}\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} {}^{\mu}\widetilde{\mathcal{N}}_2. \tag{11.8.8}
$$

It is a $\widetilde{\mathcal{D}}_X$-module, that we equip with the structure of a $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]$-module by defining the action of $\widetilde{\partial}_t$ as induced by that of $\widetilde{\partial}_{t_1} \boxtimes 1$ or, equivalently, that of $1 \boxtimes \widetilde{\partial}_{t_2}$. Then ${}^{\mu}\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]$-coherent (as this is already true if ${}^{\mu}\widetilde{\mathcal{N}}_i = \widetilde{\mathcal{D}}_{X_i}[\widetilde{\partial}_{t_i}, \widetilde{\partial}_{t_i}^{-1}]^{k_i}$).

We now neglect to write the $\boxtimes$ symbol. From the relations $\mathrm{E}_i\,\widetilde{\partial}_{t_i} = \widetilde{\partial}_{t_i}(\mathrm{E}_i - z)$ and $\mathrm{E}_i\,\widetilde{\partial}_{t_j} = \widetilde{\partial}_{t_j}\,\mathrm{E}_i$ if $i \neq j$, we deduce that $(\mathrm{E}_1 + \mathrm{E}_2)(\widetilde{\partial}_{t_1} - \widetilde{\partial}_{t_2}) = (\widetilde{\partial}_{t_1} - \widetilde{\partial}_{t_2})(\mathrm{E}_1 + \mathrm{E}_2 - z)$, and $\mathrm{E}_1 + \mathrm{E}_2$ induces a well-defined action on ${}^{\mu}\widetilde{\mathcal{N}}$ in a way compatible with the $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t, \widetilde{\partial}_t^{-1}]$-action.

11.8.9. Lemma. *Assume that, for $i = 1, 2$, ${}^{\mu}\widetilde{\mathcal{N}}_i$ are strict and strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_{t_i}^{-1})$. Let us set*

$$
U_\alpha({}^{\mu}\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} {}^{\mu}\widetilde{\mathcal{N}}_2) = \sum_{\alpha_1+\alpha_2=\alpha} (V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1 \boxtimes_{\widetilde{\mathcal{D}}} V_{\alpha_2}{}^{\mu}\widetilde{\mathcal{N}}_2).
$$

Then we have for every $\alpha\in\mathbb{R}$

$$\mathrm{gr}^U_\alpha({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)=\bigoplus_{\alpha_1+\alpha_2=\alpha}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}\mathrm{gr}^V_{\alpha_2}{}^{\mu}\widetilde{\mathcal{N}}_2).$$

Proof. Same proof as in Exercise 10.12(4), by replacing the F-filtration there with the V-filtration here. □

11.8.10. Lemma. *The morphism* (11.8.8) *is strictly filtered of degree one with respect to the filtration* $U_\bullet({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)$.

Proof. For $\alpha\in(-1,0]$ and $\ell\in\mathbb{Z}$, we have, due to Lemma 11.8.9,

$$\mathrm{gr}^U_{\alpha+\ell}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\simeq\bigoplus_{k\in\mathbb{Z}}\bigoplus_{\alpha_1\in(-1,0]}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\widetilde{\partial}_{t_1}^{-k}\boxtimes\mathrm{gr}^V_{\alpha-\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_2\widetilde{\partial}_{t_2}^{k+\ell}).\tag{11.8.11}$$

The graded morphism at the level $\alpha+\ell$

$$\mathrm{gr}^U_{\alpha+\ell-1}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\xrightarrow{\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2}}\mathrm{gr}^U_{\alpha+\ell}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)$$

is then clearly injective. Let m be a local section of $\mathrm{Im}\,(11.8.8)\cap U_{\alpha+\ell}$. Let us write $m=n(\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})$ for $n\in U_{\beta-1}$ for β big enough. Assume that $\beta>\alpha+\ell$ and $[n]\neq0$ in $\mathrm{gr}^U_{\beta-1}$. Then the class $[n]$ of n in $\mathrm{gr}^U_{\beta-1}$ satisfies $[n](\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})=0$ in gr^U_β, hence is zero, a contradiction. This shows that $\mathrm{Im}\,(11.8.8)\cap U_{\alpha+\ell}=U_{\alpha+\ell-1}(\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2})$, as wanted (see Exercise 5.2(7)). □

11.8.12. Remark. Let us keep the assumptions of Lemma 11.8.9 and let us equip ${}^{\mu}\widetilde{\mathcal{N}}$ with the filtration $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ naturally induced by $U_\bullet({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)$. We then have

$$\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}=\mathrm{Coker}\Big[\mathrm{gr}^U_{\alpha-1}({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\xrightarrow{\widetilde{\partial}_{t_1}-\widetilde{\partial}_{t_2}}\mathrm{gr}^U_\alpha({}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes_{\widetilde{\mathcal{D}}}{}^{\mu}\widetilde{\mathcal{N}}_2)\Big].$$

Formula (11.8.11) leads to

$$\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}\simeq\bigoplus_{\alpha_1\in(-1,0]}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes\mathrm{gr}^V_{\alpha-\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_2).\tag{11.8.12 $*$}$$

In particular, each $\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ is strict, and $U_\bullet{}^{\mu}\widetilde{\mathcal{N}}$ satisfies all properties of the V-filtration except possibly the $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t^{-1}\rangle$-coherence, so we cannot infer that ${}^{\mu}\widetilde{\mathcal{N}}$ is strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_t^{-1})$.

Nevertheless, if $\alpha_1\in(-1,\alpha)$, we have $\alpha-\alpha_1\in(0,\alpha+1)$ and we replace isomorphically $\mathrm{gr}^V_{\alpha-\alpha_1}$ with $\mathrm{gr}^V_{\alpha-\alpha_1-1}(1)$ so that all indices belong to $(-1,0]$ and (11.8.12 $*$) reads

$$\mathrm{gr}^U_\alpha{}^{\mu}\widetilde{\mathcal{N}}\simeq\bigoplus_{\alpha_1\in(-1,\alpha)}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes\mathrm{gr}^V_{\alpha-\alpha_1-1}{}^{\mu}\widetilde{\mathcal{N}}_2(1))\oplus\bigoplus_{\alpha_1\in[\alpha,0]}(\mathrm{gr}^V_{\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes\mathrm{gr}^V_{\alpha-\alpha_1}{}^{\mu}\widetilde{\mathcal{N}}_2).$$

If we write ϕ_λ for $\mathrm{gr}^U_\alpha(1)$ if $\lambda=\exp2\pi\mathrm{i}\,\alpha\neq1$ and $\phi_1=\mathrm{gr}^U_0$, we find

$$\phi_\lambda{}^{\mu}\widetilde{\mathcal{N}}\simeq\bigoplus_{\alpha_1\in(-1,\alpha]\cup\{0\}}(\phi_{\lambda_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes\phi_{\lambda/\lambda_1}{}^{\mu}\widetilde{\mathcal{N}}_2)\oplus\bigoplus_{\alpha_1\in(\alpha,0)}(\phi_{\lambda_1}{}^{\mu}\widetilde{\mathcal{N}}_1\boxtimes\phi_{\lambda/\lambda_1}{}^{\mu}\widetilde{\mathcal{N}}_2)(-1).$$

11.8.c. Proof of Theorem 11.8.1. We take up the notation of Lemma 11.8.2 and we will apply the results of Section 11.8.b. From the exact sequence of Lemma 11.8.2 we obtain, by tensoring over $\widetilde{\mathbb{C}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}]$ with $\widetilde{\mathbb{C}}[\widetilde{\partial}_{t_1}, \widetilde{\partial}_{t_2}, \widetilde{\partial}_{t_1}^{-1}, \widetilde{\partial}_{t_2}^{-1}]$ (and since the latter is flat over the former), the exact sequence

$$0 \longrightarrow \widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu} \xrightarrow{\widetilde{\partial}_{t_1} - \widetilde{\partial}_{t_2}} \widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu} \longrightarrow {}^{\mu}\widetilde{\mathcal{N}} \longrightarrow 0.$$

By the assumptions in the Theorem and Lemma 11.8.6, $\widetilde{\mathcal{N}}_{1\mu}$, $\widetilde{\mathcal{N}}_{2\mu}$ and ${}^{\mu}\widetilde{\mathcal{N}}$ are strict and strictly $\mathbb{R}$-specializable along $(\widetilde{\partial}_{t_1}^{-1})$, $(\widetilde{\partial}_{t_2}^{-1})$ and $(\widetilde{\partial}_{t}^{-1})$ respectively. In view of Lemmas 11.8.6 and 11.8.10 and of Remark 11.8.12, we are reduced to proving that $U_{\bullet}(\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu})$ induces $V_{\bullet}{}^{\mu}\widetilde{\mathcal{N}}$, that is, the image of each $U_{\alpha}(\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu})$ is $\widetilde{\mathcal{D}}_X[t]\langle\widetilde{\partial}_t^{-1}\rangle$-coherent.

The finiteness on $\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu}$ a priori involves the independent actions of t_1 and t_2, while we can only use the action of $t_1 + t_2$ on ${}^{\mu}\widetilde{\mathcal{N}}$. We will thus prove that finiteness for $V_{\alpha}{}^{\mu}\widetilde{\mathcal{N}}_i$ already holds without taking into account the action of E_i, that is, $V_{\alpha_i}{}^{\mu}\widetilde{\mathcal{N}}_i$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{t_i}^{-1}]$-coherent. This will imply that each $U_{\alpha}(\widetilde{\mathcal{N}}_{1\mu} \boxtimes_{\widetilde{\mathcal{D}}} \widetilde{\mathcal{N}}_{2\mu})$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_{t_1}^{-1}, \widetilde{\partial}_{t_2}^{-1}]$-coherent and thus the module $U_{\alpha}{}^{\mu}\widetilde{\mathcal{N}}$ induced on ${}^{\mu}\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]$-coherent. As noticed in Remark 11.8.12, Formula (11.8.12 $*$) gives then the Thom-Sebastiani formula in the theorem.

Let us therefore show the $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]$-coherence of $V_{\alpha}{}^{\mu}\widetilde{\mathcal{N}}$, if $\widetilde{\mathcal{N}} = \widetilde{\mathcal{M}}[\widetilde{\partial}_t]$ is strict and strictly $\mathbb{R}$-specializable along (t). By Proposition 10.8.3, these two properties imply that each $V_{\alpha}\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X[t]$-coherent. But $\widetilde{\mathcal{N}}$ is supported on the graph of g, hence $t - g$ acts in a nilpotent way on any section of $\widetilde{\mathcal{N}}$. This implies that $V_{\alpha}\widetilde{\mathcal{N}}$ is $\widetilde{\mathcal{D}}_X$-coherent. The formula of Exercise 11.9 gives the $\widetilde{\mathcal{D}}_X[\widetilde{\partial}_t^{-1}]$-coherence of $V_{\alpha}{}^{\mu}\widetilde{\mathcal{N}}$ if $\alpha > -1$, hence that of $V_{\alpha}{}^{\mu}\widetilde{\mathcal{N}}$ for any α. ☐

11.9. Exercises

Exercise 11.1. In the setting of Section 11.3.b, assume that g is smooth and set $D = g^{-1}(0) = (g)$. Let $\iota_g : X \hookrightarrow X \times \mathbb{C}$ be the graph inclusion and set $H = X \times \{0\}$. Show that $\widetilde{\mathcal{M}}[*D]$ as defined by 11.3.1 satisfies

$${}_{\mathrm{D}}\iota_{*}\widetilde{\mathcal{M}}[*|D|] = ({}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}})[*H].$$

Conclude that $\widetilde{\mathcal{M}}[*g]$ exists and is equal to $\widetilde{\mathcal{M}}[*|D|]$.

Exercise 11.2. In the proof of Proposition 11.4.7, show however that the action of $\widetilde{\partial}_t$ induces a $\widetilde{\mathcal{D}}_X$-module structure on $\operatorname{Ker}\rho$ and on $\operatorname{Coker}\rho$, and identify these $\widetilde{\mathcal{D}}_X$-modules with $\operatorname{Ker}\mathrm{can}_{\widetilde{\mathcal{M}}}$ and $\operatorname{Coker}\mathrm{can}_{\widetilde{\mathcal{M}}}$ respectively. [*Hint*: Argue as in Example 9.3.27.]

Exercise 11.3. We work within the full subcategory of $\widetilde{\mathcal{D}}_X$-modules which are strictly $\mathbb{R}$-specializable and localizable along D.

(1) Show that $\widetilde{\mathcal{M}}[*D]$ and $\widetilde{\mathcal{M}}[!D]$ are localizable along D and

(a) the morphisms $(\widetilde{\mathcal{M}}[!D])[*D] \to \widetilde{\mathcal{M}}[*D]$ and $(\widetilde{\mathcal{M}}[!D])[!D] \to \widetilde{\mathcal{M}}[!D]$ induced by $\widetilde{\mathcal{M}}[!D] \to \widetilde{\mathcal{M}}$ are isomorphisms,

(b) the morphisms $\widetilde{\mathcal{M}}[!D] \to (\widetilde{\mathcal{M}}[*D])[!D]$ and $\widetilde{\mathcal{M}}[*D] \to (\widetilde{\mathcal{M}}[*D])[*D]$ induced by $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*D]$ are isomorphisms.

(2) Let g be a local equation of D. Show that there are isomorphisms of diagrams (see Definition 9.7.3)

$$\begin{tikzcd}
\psi_{g,1}\widetilde{\mathcal{M}}[*g] \arrow[r, bend left, "\mathrm{can}"] & \phi_{g,1}\widetilde{\mathcal{M}}[*g] \arrow[l, bend left, "\mathrm{var}", "\sim"', "(-1)" near end] & \simeq & \psi_{g,1}\widetilde{\mathcal{M}} \arrow[r, bend left, "\mathrm{N}"] & \psi_{g,1}\widetilde{\mathcal{M}}(-1) \arrow[l, bend left, "\mathrm{Id}", "(-1)" near end]
\end{tikzcd}$$

and

$$\begin{tikzcd}
\psi_{g,1}\widetilde{\mathcal{M}}[!g] \arrow[r, bend left, "\mathrm{can}", "\sim"'] & \phi_{g,1}\widetilde{\mathcal{M}}[!g] \arrow[l, bend left, "\mathrm{var}", "(-1)" near end] & \simeq & \psi_{g,1}\widetilde{\mathcal{M}} \arrow[r, bend left, "\mathrm{Id}"] & \psi_{g,1}\widetilde{\mathcal{M}}. \arrow[l, bend left, "\mathrm{N}", "(-1)" near end]
\end{tikzcd}$$

Exercise 11.4. We keep the assumptions as in Definition 11.5.2 and we also assume also that $D = (g)$. Recall that loc (resp. dloc) have been defined in 11.3.3(2) (resp. 11.4.7(2)).

(1) Show that the kernel and cokernel of the natural morphism

$$\mathrm{loc} \circ \mathrm{dloc} : \widetilde{\mathcal{M}}[!g] \longrightarrow \widetilde{\mathcal{M}}[*g]$$

are equal respectively to the kernel and cokernel of

$$\phi_{g,1}(\mathrm{loc} \circ \mathrm{dloc}) : \phi_{g,1}\widetilde{\mathcal{M}}[!g] \longrightarrow \phi_{g,1}\widetilde{\mathcal{M}}[*g],$$

and also to the kernel and cokernel of

$$\mathrm{N} : \psi_{g,1}\widetilde{\mathcal{M}} \longrightarrow \psi_{g,1}\widetilde{\mathcal{M}}(-1).$$

[*Hint*: Show that $\mathrm{loc} \circ \mathrm{dloc}$ induces an isomorphism on $V_{<0}$ and argue as in Example 9.3.27 for ${}_{\mathrm{D}}\iota_{g*}(\widetilde{\mathcal{M}}[*g])$.]

(2) Identify $\psi_{g,\lambda}\widetilde{\mathcal{M}}[!{*}g]$ with $\psi_{g,\lambda}\widetilde{\mathcal{M}}$ and $\phi_{g,1}\widetilde{\mathcal{M}}[!{*}g]$ with image(N).

(3) Show that if $\mathrm{N} : \psi_{g,1}\widetilde{\mathcal{M}} \to \psi_{g,1}\widetilde{\mathcal{M}}(-1)$ is strict, then $\mathrm{loc} \circ \mathrm{dloc} : \widetilde{\mathcal{M}}[!g] \to \widetilde{\mathcal{M}}[*g]$ is strictly $\mathbb{R}$-specializable.

Exercise 11.5. With the assumptions of Proposition 11.5.4, show similarly that the morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}[*g]$ (resp. $\widetilde{\mathcal{M}}[!g] \to \widetilde{\mathcal{M}}$) is strictly $\mathbb{R}$-specializable along (g) if and only if the morphism $\mathrm{var} : \phi_{g,1}\widetilde{\mathcal{M}} \to \psi_{g,1}\widetilde{\mathcal{M}}(-1)$ (resp. $\mathrm{can} : \psi_{g,1}\widetilde{\mathcal{M}} \to \phi_{g,1}\widetilde{\mathcal{M}}$) is strict.

***Exercise 11.6* (Linear algebra 1).** Let (M, N) be a graded $\mathbb{C}$-vector space with a nilpotent endomorphism $\mathrm{N} : M \to M(-1)$. For $\varepsilon = 0, 1$, set $M^{(\varepsilon,k)} = M \otimes_{\mathbb{C}} \mathcal{J}^{(\varepsilon,k)}$ ($\mathcal{J}^{(\varepsilon,k)}$ as in Section 11.6.b) with nilpotent endomorphism

$$\mathrm{N}^{(\varepsilon,k)} := \mathrm{N} \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{J}^{(\varepsilon,k)} : M^{(\varepsilon,k)} \longrightarrow M^{(\varepsilon,k)}(-1)$$

and similarly for $\mathrm{N}_{(\varepsilon,k)}$. Show the following properties.

(1) The morphism

$$\begin{aligned} M &\longrightarrow M^{(\varepsilon,k)} \\ m &\longmapsto \textstyle\sum_{i=\varepsilon}^k (-\mathrm{N})^{i-\varepsilon} m \otimes e_i \end{aligned}$$

induces an isomorphism $\operatorname{Ker}\mathrm{N}^{k+1-\varepsilon} \xrightarrow{\sim} \operatorname{Ker}\mathrm{N}^{(\varepsilon,k)}$ with respect to which the natural morphism $\operatorname{Ker}\mathrm{N}^{(\varepsilon,k)} \to \operatorname{Ker}\mathrm{N}^{(\varepsilon,k+1)}$ correspond to the natural morphism $\operatorname{Ker}\mathrm{N}^{k+1-\varepsilon} \hookrightarrow \operatorname{Ker}\mathrm{N}^{k+2-\varepsilon}$ and the natural morphism $\operatorname{Ker}\mathrm{N}^{(0,k)} \to \operatorname{Ker}\mathrm{N}^{(1,k)}(-1)$ correspond to the natural morphism $\operatorname{Ker}\mathrm{N}^{k+1} \xrightarrow{-\mathrm{N}} \operatorname{Ker}\mathrm{N}^{k}(-1)$. In particular, if N has finite order on M, show that have natural commutative diagrams

$$\begin{tikzcd}
\varinjlim_k \operatorname{Ker}\mathrm{N}^{(0,k)} \arrow[d] & \varinjlim_k \operatorname{Ker}\mathrm{N}^{k+1} \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "-\mathrm{N}"] & M \arrow[d, "-\mathrm{N}"] \\
\varinjlim_k \operatorname{Ker}\mathrm{N}^{(1,k)}(-1) & \varinjlim_k \operatorname{Ker}\mathrm{N}^{k}(-1) \arrow[l, "\sim"'] \arrow[r, "\sim"] & M(-1)
\end{tikzcd}$$

and the limits are achieved for $k > \operatorname{ord}(\mathrm{N})$.

(2) Show that the morphism

$$\begin{aligned} M^{(\varepsilon,k)} &\longrightarrow M(\varepsilon-k) \\ \sum_{i=\varepsilon}^k m_i \otimes e_i &\longmapsto \sum_{i=\varepsilon}^k (-\mathrm{N})^{k-i} m_i \end{aligned}$$

induces an isomorphism

$$\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} := M^{(\varepsilon,k)}(-1)/\operatorname{Im}\mathrm{N}^{(\varepsilon,k)} \xrightarrow{\sim} M(\varepsilon-(k+1))/\operatorname{Im}\mathrm{N}^{k+1-\varepsilon},$$

and thus, if $k > \operatorname{ord}(\mathrm{N})$,

$$\operatorname{Coker}\mathrm{N}^{(\varepsilon,k)} \simeq M(\varepsilon-(k+1)).$$

(3) Show similar properties for the lower Jordan block. Note that the previous diagram becomes

$$\begin{tikzcd}
M \arrow[r, "\sim"] \arrow[d, "-\mathrm{N}"'] & \varprojlim_k \operatorname{Coker}\mathrm{N}^{k} \arrow[d, "-\mathrm{N}"'] & \varprojlim_k \operatorname{Coker}\mathrm{N}_{(1,k)} \arrow[l, "\sim"'] \arrow[d] \\
M(-1) \arrow[r, "\sim"] & \varprojlim_k \operatorname{Coker}\mathrm{N}^{k+1}(-1) & \varprojlim_k \operatorname{Coker}\mathrm{N}_{(0,k)}(-1) \arrow[l, "\sim"']
\end{tikzcd}$$

***Exercise 11.7* (Linear algebra 2).** We keep the notation as in Exercise 11.6.

(1) Show that the two composed natural maps

$$M^{(0,k)} \longrightarrow M^{(1,k)}(-1) \xrightarrow{\mathrm{N}^{(1,k)}} M^{(1,k)}(-2)$$

and

$$M^{(0,k)} \xrightarrow{N^{(0,k)}} M^{(0,k)}(-1) \longrightarrow M^{(1,k)}(-2)$$

coincide. Let $\Xi^k M$ denote their kernel. In particular, $N^{(0,k)}$ induces a map

$$\mathrm{N}^{(0,k)}_{|\Xi^k M} : \Xi^k M \longrightarrow \operatorname{Ker}\big[M^{(0,k)}(-1) \to M^{(1,k)}(-2)\big] \simeq (M \otimes e_0)(-1) \simeq M(-1).$$

(2) Show that the map

$$\begin{aligned} M \oplus \operatorname{Ker} \mathrm{N}^k(-1) &\longrightarrow M^{(0,k)} \\ (n,m) &\longmapsto n \otimes e_0 + \textstyle\sum_{i=1}^k (-\mathrm{N})^{i-1} m \otimes e_i \end{aligned}$$

induces an isomorphism onto $\Xi^k M$.

(3) Show that, under this isomorphism, $\mathrm{N}^{(0,k)}_{|\Xi^k M} : \Xi^k M \to M(-1)$ is identified with $(n,m) \mapsto \mathrm{N}n + m$.

(4) Conclude that, if $\operatorname{ord}(\mathrm{N})$ is finite and $k > \operatorname{ord}(\mathrm{N})$, then the exact sequence

$$0 \longrightarrow \operatorname{Ker}\big[M^{(0,k)} \to M^{(1,k)}(-1)\big] \longrightarrow \Xi^k M \longrightarrow \operatorname{Ker} \mathrm{N}^{(1,k)} \longrightarrow 0$$

is isomorphic to the naturally split sequence $0 \to M \to M \oplus M(-1) \to M(-1) \to 0$ with respect to which the exact sequence

$$0 \longrightarrow \operatorname{Ker} \mathrm{N}^{(0,k)} \longrightarrow \Xi^k M \longrightarrow \operatorname{Ker}\big[M^{(0,k)}(-1) \to M^{(1,k)}(-2)\big] \longrightarrow 0$$

corresponds to

$$0 \longrightarrow \operatorname{Ker}(\mathrm{N} + \mathrm{Id}) \longrightarrow M \oplus M(-1) \xrightarrow{\ \mathrm{N} + \mathrm{Id}\ } M(-1) \longrightarrow 0.$$

(5) Show similar properties for the lower Jordan block.

Exercise 11.8. Show that, if $\widetilde{\mathcal{M}}_*$ is strictly $\mathbb{R}$-specializable along H, then so are $\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}$ and $\widetilde{\mathcal{M}}_{*(\varepsilon,k)}$, we have $V_\bullet \widetilde{\mathcal{M}}_*^{(\varepsilon,k)} = (V_\bullet \widetilde{\mathcal{M}}_*)^{(\varepsilon,k)}$ and the lower similar equalities, and for every λ, $\psi_{t,\lambda}(\widetilde{\mathcal{M}}_*^{(\varepsilon,k)}) \simeq (\psi_{t,\lambda} \widetilde{\mathcal{M}}_*)^{(\varepsilon,k)}$, and other similar equalities with $\phi_{t,1}$, together with the lower similar equalities.

Exercise 11.9. Show that, for $\alpha > -1$, $U_\alpha{}^{\mu}\widetilde{\mathcal{N}}$ defined by (11.8.7) is equal to $V_\alpha \widetilde{\mathcal{N}} + \sum_{i \geqslant 1} V_0 \widetilde{\mathcal{N}} \cdot \widetilde{\partial}_t^{-i}$. [*Hint*: Use that, for such an α and for $k \geqslant 1$, $V_{\alpha+k} \widetilde{\mathcal{N}} = V_{<\alpha+k} \widetilde{\mathcal{N}} + V_{\alpha+k-1} \widetilde{\mathcal{N}} \cdot \widetilde{\partial}_t$, see Proposition 9.3.20(b).]

11.10. Comments

The property that the localization along a hypersurface of holonomic $\mathcal{D}_X$-module remains coherent and, better, holonomic, is one of the main applications of the theory of the Bernstein polynomial (see [**Ber72, Kas76, Kas78**], see also [**Bjö79**] and [**Ehl87**]).

The notion of localizable filtered $\mathcal{D}_X$-module has been introduced (with a different terminology) by M. Saito in [**Sai90**] as an essential step for the theory of *mixed* Hodge modules. The approach given here follows that of T. Mochizuki in [**Moc15**]. In particular, the parallel way to present localization and dual localization is due to him. The proof of Proposition 11.2.18 owes much to that of [**Voi02**, Prop. 8.34].

The gluing construction for perverse sheaves goes back to the work of Verdier [**Ver85**] and Beilinson [**Bei87**]. It plays an important role in M. Saito's theory of mixed Hodge modules [**Sai90**], where the construction of the maximal extension is given in a geometric way. The approach given here is closer to that of Beilinson, and has been much inspired by the treatment made by T. Mochizuki in [**Moc15**], where this gluing construction is also fundamental for the theory of mixed twistor D-modules. Section 11.2.d is also much inspired from the treatment in [**Moc11a**, §17.3].

The result in Section 11.4.a is due to [**Wei20**] and the proof is taken from [**ES19**]. The Thom-Sebastiani formula proved in Section 11.8 is due to M. Saito. An initial proof had been given in an unpublished preprint dated 1990 [**Sai11**]. A simpler proof has been given in [**MSS20**]. The proof of Theorem 11.8.1 is much inspired by the latter.

CHAPTER 12

THE CATEGORY OF TRIPLES OF $\widetilde{\mathcal{D}}_X$-MODULES

Summary. The category of triple of filtered $\widetilde{\mathcal{D}}_X$-modules has been considered in dimension one as a suitable abelian category containing the category of polarizable Hodge modules as a full subcategory (see Section 7.4.a). Our aim in this chapter is to extend the notion of triples in any dimension. For that purpose, we are led to extend the notion of $(\nabla, \overline{\nabla})$-flat sesquilinear pairing, used for the definition of a polarized variation of $\mathbb{C}$-Hodge structure (see Definition 4.1.4), to the case where the flat bundle is replaced with a $\mathcal{D}$-module. Such a sesquilinear pairing takes values in the sheaf of distributions or of currents of maximal degree. We also make precise its behaviour with respect to functors like pushforward, smooth pullback, nearby and vanishing cycles and localization.

In this chapter, we keep Notation 9.0.1. In the first sections, we will only consider $\mathcal{O}_X$-modules and $\mathcal{D}_X$-modules, as coherent filtrations will not play any role here. We will use the constructions and results of Chapter 9 in this framework. We come back to the filtered case in Section 12.7.

12.1. Introduction

One of the ingredients of a polarized variation of Hodge structure is a flat Hermitian pairing (that we have denoted by S), which is $(-1)^{w-p}$-definite on $\mathcal{H}^{p,w-p}$. In this chapter, we introduce the notion of sesquilinear pairing between holonomic $\mathcal{D}_X$-modules. It takes values in the sheaf of distributions (in fact a smaller sheaf, but we are not interested in characterizing the image). This notion will not be used directly as in classical Hodge theory to furnish the notion of polarization. Instead, we will take up the definition of a $\mathbb{C}$-Hodge structure as a triple (see Section 5.2) and mimic this definition in higher dimension. Our aim is therefore to define a category of $\mathcal{D}$-triples (an object consists of a pair of $\mathcal{D}_X$-modules and a sesquilinear pairing between the underlying $\mathcal{D}_X$-modules) and to extend to this abelian category the various functors considered in Chapter 9.

12.2. Distributions and currents on a complex manifold

12.2.a. Distributions and currents. Let $\overline{X}$ denote the complex manifold conjugate to X, i.e., with structure sheaf $\mathcal{O}_{\overline{X}}$ defined as the sheaf of anti-holomorphic functions $\overline{\mathcal{O}_X}$. Correspondingly is defined the sheaf of anti-holomorphic differential operators $\mathcal{D}_{\overline{X}}$. The sheaf of C^∞ functions on X is acted on by $\mathcal{D}_X$ and $\mathcal{D}_{\overline{X}}$ on the left and both actions commute, i.e., $\mathcal{C}^\infty_X$ is a left $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-module. Similarly, the sheaf of distributions $\mathfrak{Db}_X$ is a left $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-module: by definition, on any open set $U \subset X$, $\mathfrak{Db}_X(U)$ is dual to the space $\mathcal{E}^{2n}_{\mathrm{c}}(U)$ of C^∞ $2n$-forms with compact support, equipped with a suitable topology, and the presheaf defined in this way is a sheaf. On the other hand, the space of $\mathfrak{C}_X(U)$ of currents of degree 0 on X is dual to $C^\infty_{\mathrm{c}}(U)$ with suitable topology. Then $\mathfrak{C}_X$ is the right $\mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$-module obtained from $\mathfrak{Db}_X$ by the left-to-right transformation for such objects, i.e.,

$$\mathfrak{C}_X = (\omega_X \otimes_{\mathbb{C}} \omega_{\overline{X}}) \otimes_{(\mathcal{O}_X \otimes \mathcal{O}_{\overline{X}})} \mathfrak{Db}_X .$$

12.2.1. Notation. From now on, the notation $\mathcal{A}_{X,\overline{X}}$ will mean $\mathcal{A}_X \otimes_{\mathbb{C}} \mathcal{A}_{\overline{X}}$ ($\mathcal{A} = \mathcal{O}$ or $\mathcal{D}$).

De Rham complex. One can easily adapt Exercise 8.22 to prove that the C^∞-de Rham complex $\mathcal{E}^{2n+\bullet}_X \otimes_{\mathcal{C}^\infty_X} \mathcal{D}_{X,\overline{X}} = \mathcal{E}^\bullet_X[2n] \otimes_{\mathcal{O}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}}$, where the differential is obtained from the standard differential on C^∞ k-forms and the universal connection $\nabla_X + \overline{\nabla}_X$ on $\mathcal{D}_{X,\overline{X}}$, is a resolution of $\mathcal{E}^{n,n}_X = \mathcal{E}^{2n}_X$ as a right $\mathcal{D}_{X,\overline{X}}$-module.

We denote by $\mathfrak{Db}^{n-p,n-q}_X = \mathcal{E}^{n-p,n-q}_X \otimes_{\mathcal{C}^\infty_X} \mathfrak{Db}_X$ or $\mathfrak{Db}_{X,p,q}$ the sheaf of currents of degree (p,q) (we also say of type $(n-p,n-q)$), that is, continuous linear forms on C^∞_{c} differential forms of degree p,q.

The distributional de Rham complex yields then a resolution of $\mathfrak{C}_X$ as a right $\mathcal{D}_{X,\overline{X}}$-module:

$$\mathfrak{Db}^\bullet_X[2n] \otimes_{\mathcal{O}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}} \xrightarrow{\sim} \mathfrak{C}_X . \tag{12.2.2}$$

Let us make precise that the morphism is induced by

$$\mathfrak{Db}^{n,n}_X \otimes \mathcal{D}_{X,\overline{X}} = \mathfrak{C}_X \otimes \mathcal{D}_{X,\overline{X}} \longrightarrow \mathfrak{C}_X, \quad u \otimes P \longmapsto u \cdot P.$$

The Poincaré and Dolbeault lemmas for distributions and for L^2_{loc} forms. Here, the complex structure of X is not needed. It could be any differentiable manifold. The Poincaré lemma for distributions, due to de Rham **[dR73, dR84]**, asserts that the complex $(\mathfrak{Db}^\bullet_X, \mathrm{d})$ is quasi-isomorphic to its subcomplex $(\mathcal{E}^\bullet_X, \mathrm{d})$, which is itself, by the standard Poincaré lemma, quasi-isomorphic to the constant sheaf $\mathbb{C}_X$. The notion of current is important in the proof since it involves integration of currents (see Definition 12.2.9 below). The basic regularization procedure is given by the following lemma.

***12.2.3. Lemma* (Regularization lemma, [dR73, dR84]).** *Let U be an open subset of $\mathbb{R}^m$. For each $\varepsilon \in (0,1)$ and each integer p, there exist $\mathbb{C}$-linear morphisms $R_\varepsilon : \mathfrak{Db}^p(U) \to \mathfrak{Db}^p(U)$ and $S_\varepsilon : \mathfrak{Db}^{p-1}(U) \to \mathfrak{Db}^p(U)$ such that*

1. *R_ε takes values in $\mathcal{E}^p(U)$,*
2. *$R_\varepsilon(u) - u = \mathrm{d}S_\varepsilon(u) - S_\varepsilon(\mathrm{d}u)$ for any $u \in \mathfrak{Db}^p(U)$,*
3. *$R_\varepsilon(\mathrm{d}u) = \mathrm{d}R_\varepsilon(u)$ and $\lim_{\varepsilon\to 0} R_\varepsilon(u) = u$ weakly.*

On the other hand, the L^2 de Rham complex $(\mathcal{L}^{\bullet}_{(2),X}, \mathrm{d})$ on X is the subcomplex of $(\mathfrak{Db}^{\bullet}_X, \mathrm{d})$ such that $\mathcal{L}^p_{(2),X}$ consists of p-forms with coefficients in L^2_{loc} in some (or any) basis $(\mathrm{d}x_I)_{\#I=p}$ *and* whose differentials are also L^2_{loc}. The construction of R_ε and S_ε also gives:

12.2.4. Lemma. *If u has L^2_{loc} coefficients, then so does $S_\varepsilon(u)$.*

12.2.5. Corollary **(Poincaré lemmas).** *The inclusions of complexes*

$$(\mathbb{C}_X, 0) \hookrightarrow (\mathcal{E}^{\bullet}_X, \mathrm{d}) \hookrightarrow (\mathcal{L}^{\bullet}_{(2),X}, \mathrm{d}) \hookrightarrow (\mathfrak{Db}^{\bullet}_X, \mathrm{d})$$

are quasi-isomorphisms.

Proof. That the first inclusion is a quasi-isomorphism is the standard Poincaré lemma. Let us check the second inclusion for example. Let u be a local section of $\mathcal{L}^p_{(2),X}$ which is d closed. Then (for any $\varepsilon \in (0,1)$) $S_\varepsilon(u)$ has L^2_{loc} coefficients, and so does $\mathrm{d}S_\varepsilon(u) = u - R_\varepsilon(u)$, so that $S_\varepsilon(u)$ is a local section of $\mathcal{L}^{p-1}_{(2),X}$. It follows that u is cohomologous to $R_\varepsilon(u)$ in $\mathcal{L}^p_{(2),X}$, showing surjectivity $\mathcal{H}^p(\mathcal{E}^{\bullet}_X, \mathrm{d}) \to \mathcal{H}^p(\mathcal{L}^{\bullet}_{(2),X}, \mathrm{d})$, hence the latter is zero for $p > 0$, by the standard Poincaré lemma. For $p = 0$, we use that a distribution all of whose derivatives are zero is a locally constant function. $\square$

One defines in a similar way the Dolbeault complexes $(\mathcal{E}^{p,\bullet}_X, \mathrm{d}'')$, $(\mathcal{L}^{p,\bullet}_{(2),X}, \mathrm{d}'')$ and $(\mathfrak{Db}^{p,\bullet}_X, \mathrm{d}'')$.

12.2.6. Theorem **(Dolbeault lemmas).** *The inclusion of complexes*

$$(\Omega^p_X, 0) \hookrightarrow (\mathcal{E}^{p,\bullet}_X, \mathrm{d}'') \hookrightarrow (\mathcal{L}^{p,\bullet}_{(2),X}, \mathrm{d}'') \hookrightarrow (\mathfrak{Db}^{p,\bullet}_X, \mathrm{d}'')$$

are quasi-isomorphisms.

We refer e.g. to [**GH78**, p. 382–385] for the Dolbeault-Grothendieck theorem, i.e., $(\mathfrak{Db}^{p,\bullet}_X, \mathrm{d}'')$ is a resolution of Ω^p_X, and to [**Hör66**, Th. 4.2.2] for the L^2-Dolbeault lemma (the result proved in loc. cit. is stronger).

Distributions and currents depending continuously on a parameter. We wish to define the notion of distribution depending continuously on the parameter $s \in S$. We will define such a sheaf on $X \times S$ by $\mathfrak{Db}_{X\times S/S}$.

Let S be a C^∞ real manifold (we will mainly use $S = \mathbb{C}_s$). Let $\mathcal{E}^{(n,n)}_{X\times S/S,\mathrm{c}}$ the sheaf of C^∞ relative (with respect to the projection $X \times S \to S$) (n,n) forms with compact support. The sheaf on $X \times S$ of distributions which are continuous with respect to S is defined as follows. Given any open set W of $X \times S$, an element of $\mathfrak{Db}_{X\times S/S}(W)$ is a $C^\infty(S)$-linear map $\mathcal{E}^{(n,n)}_{X\times S/S,\mathrm{c}}(W) \to C^0_{\mathrm{c}}(S)$ which is continuous with respect to the usual sup norm on $C^0_{\mathrm{c}}(S)$ and the family of semi-norms on $\mathcal{E}^{(n,n)}_{X\times S/S,\mathrm{c}}(W)$ obtained by taking the sup on some compact set of W of the module of partial derivatives up to some order with respect to X. Given a compact set in W, the smallest order in ∂_x which is needed is called the *order* of u.

Currents of maximal degree which are continuous with respect to S are defined similarly. A section on W of $\mathfrak{C}_{X\times S/S}$ is a continuous $C^\infty(S)$-linear map $\mathcal{C}^\infty_{X\times S,\mathrm{c}}(W)\to C^0_{\mathrm{c}}(S)$.

12.2.7. Lemma. *Let $u\in\mathfrak{Db}_{X\times S/S}(U\times V)$. Then, for every $s_o\in V$, its restriction to $s=s_o$ is well-defined as a distribution on U, and similarly for currents.*

Proof. Let $\eta_o\in\mathcal{E}^{(n,n)}_{\mathrm{c}}(U)$ and let χ be a C^∞ function with compact support on V such that $\chi(s_o)=1$. Then $u(\eta_o\cdot\chi)$ is a continuous function on S that we can evaluate at $s=s_o$. The correspondence $\eta_o\mapsto u(\eta_o\cdot\chi)_{|s=s_o}$ obviously defines a distribution on U, because $|u(\eta_o\cdot\chi)_{|s=s_o}|\leqslant\sup_S|u(\eta_o\cdot\chi)|$. If $\sigma(s)$ is another such function on S we have, by $C^\infty(S)$-linearity,

$$u(\eta_o\cdot\sigma\chi)=\sigma u(\eta_o\cdot\chi)=\chi u(\eta_o\cdot\sigma),$$

hence both take the same value at s_o. $\square$

12.2.b. Pushforward of currents. Let η be a C^∞ form of maximal degree on X. If $f:X\to Y$ is a proper holomorphic map which is *smooth*, then the integral of η in the fibers of f is a C^∞ form of maximal degree on Y, that one denotes by $\int_f\eta$.

If f is not smooth, then $\int_f\eta$ is only defined as a current of degree 0 on Y, and the definition extends to the case where η is itself a current of degree 0 on X (see Section 8.3.4 for the notion of current).

12.2.8. Remark. The definitions and properties below extend to the case when f is only assumed to be proper on the support of the currents involved, see Exercise 12.4.

12.2.9. Definition **(Integration of currents of degree (p,q)).** Let $f:X\to Y$ be a proper holomorphic map and let u be a current of degree (p,q) on X. The current $\int_f u$ of degree (p,q) on Y is defined by

$$\Big\langle\int_f u,\eta\Big\rangle=\langle u,\eta\circ f\rangle,\quad\forall\,\eta\in\mathcal{E}^{p,q}_{\mathrm{c}}(Y).\tag{12.2.9 $*$}$$

This definition extends in a straightforward way if f is only assumed to be proper on the support of u.

We continue to assume that f is proper. We will now show how the integration of currents is used to defined a natural $\mathcal{D}_{Y,\overline{Y}}$-linear morphism ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_X\to\mathfrak{C}_Y$. The simpler case of a closed embedding is treated in Exercise 12.2.

The integration of currents is a morphism

$$\int_f\;:\;f_*\,\mathfrak{Db}_{X,p,q}\longrightarrow\mathfrak{Db}_{Y,p,q},$$

which is compatible with the d' and d'' differentials of currents on X and Y. In other words, taking the associated simple complex, it is a morphism of complexes

$$\int_f\;:\;f_*\,\mathfrak{Db}^\bullet_X[2n]\longrightarrow\mathfrak{Db}^\bullet_Y[2m].$$

Let us notice that the integration of currents is compatible with conjugation. Namely, given a current $u_{p,q} \in \Gamma(X, \mathfrak{Db}_X^{n-p,n-q})$, its conjugate $\overline{u_{p,q}} \in \Gamma(X, \mathfrak{Db}_X^{n-q,n-p})$ is defined by the relation

$$\langle \overline{u_{p,q}}, \eta^{q,p} \rangle := \overline{\langle u_{p,q}, \overline{\eta^{q,p}} \rangle}$$

for any test form $\eta^{q,p}$. Then we clearly have

$$\int_f \overline{u_{p,q}} = \overline{\int_f u_{p,q}}. \tag{12.2.10}$$

The notion of pushforward of a $\mathcal{D}_{X,\overline{X}}$-module is modeled on that of a $\mathcal{D}_X$-module in an obvious way (see Section 8.7.e). Since $\mathfrak{C}_X = (\mathfrak{Db}_X)^{\mathrm{right,right}}$ as a right $\mathcal{D}_{X,\overline{X}}$-module, we can apply the $\mathcal{D}_{X,\overline{X}}$-variant of Exercise 8.51(5) (more precisely, (8.51 $*$)) to get, since f is proper,

$$_{\mathrm{D},\overline{\mathrm{D}}}f_* \,\mathfrak{C}_X \simeq f_*(\mathfrak{Db}_X^\bullet[2n] \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}\mathcal{D}_{Y,\overline{Y}}) = f_* \,\mathfrak{Db}_X^\bullet[2n] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}. \tag{12.2.11}$$

The integration of currents $\int_f$ induces then a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism of complexes

$$_{\mathrm{D},\overline{\mathrm{D}}}f_* \,\mathfrak{C}_X \xrightarrow{\int_f^\bullet} \mathfrak{Db}_Y^\bullet[2m] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}} \simeq \mathfrak{C}_Y, \tag{12.2.12}$$

where we recall that the differential on the complex $\mathfrak{Db}_Y^\bullet[2m] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$ uses the universal connection $\nabla^Y + \overline{\nabla^Y}$ on $\mathcal{D}_{Y,\overline{Y}}$, and the isomorphism with $\mathfrak{C}_Y$ is given by (12.2.2).

On the left-hand side of (12.2.12), the term of degree zero reads $f_* \,\mathfrak{C}_X \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$ and, for a current u of degree zero on X and a differential operator P on Y, the morphism (12.2.12) sends $u \otimes P$ to the current $(\int_f u) \cdot P$ of degree zero on Y. It descends to a morphism $\mathcal{H}^0(_{\mathrm{D},\overline{\mathrm{D}}}f_* \,\mathfrak{C}_X) \to \mathfrak{C}_Y$.

If we start from $\mathfrak{Db}_X$, considered as a left $\mathcal{D}_{X,\overline{X}}$-module, we have similarly by the $\mathcal{D}_{X,\overline{X}}$-variant of Exercise 8.51(5) (more precisely, (8.51 $**$)):

$$_{\mathrm{D},\overline{\mathrm{D}}}f_* \,\mathfrak{Db}_X \simeq \mathcal{H}om_{\mathcal{O}_{Y,\overline{Y}}}(\omega_{Y,\overline{Y}}, f_* \,\mathfrak{Db}_X^\bullet[2n] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}})$$

and the integration of currents induces a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism

$$_{\mathrm{D},\overline{\mathrm{D}}}f_* \,\mathfrak{Db}_X \xrightarrow{\int_f} \mathcal{H}om_{\mathcal{O}_{Y,\overline{Y}}}(\omega_{Y,\overline{Y}}, \mathfrak{C}_Y) = \mathfrak{Db}_Y\,. \tag{12.2.13}$$

12.2.14. Remark **(Proper support).** One can relax the assumption that f is proper on X. If f is only proper on a closed analytic subset $S \subset X$, one replaces $\mathfrak{C}_X$ resp. $\mathfrak{Db}_X$ in the previous arguments with the sheaves $\mathfrak{C}_{X,S}$ resp. $\mathfrak{Db}_{X,S}$ of currents resp. distributions supported on S, i.e., vanishing when applied to any test function resp. form with compact support in $X \smallsetminus S$ (see Exercise 12.4).

More generally, considering the functor $_{\mathrm{D},\overline{\mathrm{D}}}f_!$ instead of $_{\mathrm{D},\overline{\mathrm{D}}}f_*$, by replacing f_* with $f_!$, enables one to only take into account currents with f-proper support, on which $\int_f$ is defined, so that (12.2.12) and (12.2.13) are well defined on $_{\mathrm{D},\overline{\mathrm{D}}}f_! \,\mathfrak{C}_X$ and $_{\mathrm{D},\overline{\mathrm{D}}}f_! \,\mathfrak{Db}_X$.

12.2.c. Moderate distributions. We refer to [**Mal66**, Chap. VII] for the results in this subsection.

Let D be a reduced divisor in X, let $\mathcal{O}_X(*D)$ be the sheaf of meromorphic functions on X with poles along D, and let $\mathfrak{Db}_{X,D}$ be the subsheaf of $\mathfrak{Db}_X$ consisting of distributions supported on D.

On the other hand, let $j : X \smallsetminus D \hookrightarrow X$ denote the open inclusion. By definition, there is an exact sequence of left $\mathcal{D}_{X,\overline{X}}$-modules

$$0 \longrightarrow \mathfrak{Db}_{X,D} \longrightarrow \mathfrak{Db}_X \longrightarrow j_* \mathfrak{Db}_{X\smallsetminus D}\,.$$

The image of the latter morphism is the sheaf on X of distributions on $X \smallsetminus D$ which are extendable as distributions on X. It can be characterized as the subsheaf of $j_* \mathfrak{Db}_{X\smallsetminus D}$ consisting of distributions which can be tested along C^∞ forms of maximal degree on $X \smallsetminus D$ having rapid decay along D. It is denoted by $\mathfrak{Db}_X^{\mathrm{mod}\,D}$ (sheaf on X of distributions having moderate growth along D). It can be characterized more algebraically. Indeed, we have

$$\mathfrak{Db}_X^{\mathrm{mod}\,D} = \mathcal{O}_X(*D) \otimes_{\mathcal{O}_X} \mathfrak{Db}_X = \mathcal{O}_{\overline{X}}(*\overline{D}) \otimes_{\mathcal{O}_{\overline{X}}} \mathfrak{Db}_X\,.$$

In other words, $\mathfrak{Db}_{X,D}$ is equal to the subsheaf of $\mathfrak{Db}_X$ consisting of local sections annihilated some power of g (or $\overline{f}$), and we have a short exact sequence

$$0 \longrightarrow \mathfrak{Db}_{X,D} \longrightarrow \mathfrak{Db}_X \longrightarrow \mathfrak{Db}_X^{\mathrm{mod}\,D} \longrightarrow 0.$$

The previous results apply to currents of degree 0 as well, and we keep similar notation.

12.2.15. Example **(The case where D is smooth).** If D is smooth, so that we denote it by H, the sheaf $\mathfrak{Db}_{X,H}$ is identified with the push-forward, in the sense of $\mathcal{D}_{X,\overline{X}}$-modules, of $\mathfrak{Db}_H$. If for example $X = H \times \mathbb{C}$, then, according to Exercise 12.2, we find exact sequences

$$\begin{gathered}0 \longrightarrow \iota_* \mathfrak{Db}_H[\partial_t, \partial_{\overline{t}}] \longrightarrow \mathfrak{Db}_X \longrightarrow \mathfrak{Db}_X[1/t] \longrightarrow 0,\\ 0 \longrightarrow \iota_* \mathfrak{C}_H[\partial_t, \partial_{\overline{t}}] \longrightarrow \mathfrak{C}_X \longrightarrow \mathfrak{C}_X[1/t] \longrightarrow 0.\end{gathered}$$

12.3. Sesquilinear pairings between $\mathcal{D}_X$-modules

The naive conjugation functor $\mathcal{M} \mapsto \overline{\mathcal{M}}$ transforms $\mathcal{O}_X$-modules (resp. $\mathcal{D}_X$-modules) into $\mathcal{O}_{\overline{X}}$-modules (resp. $\mathcal{D}_{\overline{X}}$-modules): let us regard $\mathcal{O}_{\overline{X}}$ as an $\mathcal{O}_X$-module by setting $f\cdot\overline{g} := \overline{fg}$, and similarly let us regard $\mathcal{D}_{\overline{X}}$ as a $\mathcal{D}_X$-module; for an $\mathcal{O}_X$-module (resp. a $\mathcal{D}_X$-module) $\mathcal{M}$ we then define $\overline{\mathcal{M}}$ as $\mathcal{O}_{\overline{X}}\otimes_{\mathcal{O}_X} \mathcal{M}$ (resp. $\mathcal{D}_{\overline{X}}\otimes_{\mathcal{D}_X} \mathcal{M}$). In other words, for a local section m of $\mathcal{M}$, we denote by $\overline{m}$ the same local section, that we act on by $\overline{f} \in \mathcal{O}_{\overline{X}}$ (resp. $\mathcal{D}_{\overline{X}}$) with the formula $\overline{f}\cdot\overline{m} := \overline{fm}$.

12.3.1. Definition **(Left sesquilinear pairing).**

(1) A *sesquilinear pairing* $\mathfrak{s}$ between left $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ is a $\mathcal{D}_{X,\overline{X}}$-linear morphism $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$. When $\mathcal{M}' = \mathcal{M}'' = \mathcal{M}$, we speak of a sesquilinear pairing on $\mathcal{M}$.

(2) The *Hermitian adjoint* of a *left* sesquilinear pairing $\mathfrak{s}$ is $\mathfrak{s}^* : \mathcal{M}'' \otimes_{\mathbb{C}} \overline{\mathcal{M}'} \to \mathfrak{Db}_X$ defined by $\mathfrak{s}^*(m'', \overline{m'}) = \overline{\mathfrak{s}(m', \overline{m''})}$, that is,

$$\langle \eta, \mathfrak{s}^*(m'', \overline{m'})\rangle := \overline{\langle \overline{\eta}, \mathfrak{s}(m', \overline{m''})\rangle}$$

for any test form of maximal degree η (see (8.3.4 $*$)).

12.3.2. Definition **(Right sesquilinear pairing).**

(1) A *sesquilinear pairing* $\mathfrak{s}$ between right $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ is a $\mathcal{D}_{X,\overline{X}}$-linear morphism $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$. When $\mathcal{M}' = \mathcal{M}'' = \mathcal{M}$, we speak of a sesquilinear pairing on $\mathcal{M}$.

(2) The *Hermitian adjoint* of a *right* sesquilinear pairing $\mathfrak{s}$ is $\mathfrak{s}^* : \mathcal{M}'' \otimes_{\mathbb{C}} \overline{\mathcal{M}'} \to \mathfrak{C}_X$ defined by $\mathfrak{s}^*(m'', \overline{m'}) = \overline{\mathfrak{s}(m', \overline{m''})}$, that is,

$$\langle \mathfrak{s}^*(m'', \overline{m'}), \eta\rangle := \overline{\langle \mathfrak{s}(m', \overline{m''}), \overline{\eta}\rangle}$$

for any test function η (see (8.3.4 $*$)).

12.3.3. Side-changing. If $\mathfrak{s} = \mathfrak{s}^{\mathrm{left}} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ is a sesquilinear pairing between left $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$, then it determines in a canonical way a sesquilinear pairing (recall $\mathrm{Sgn}(n) := \varepsilon(n+1)/(2\pi\mathsf{i})^n$, see Notation (0.2 $*$))

$$\begin{aligned}(\omega_X \otimes \mathcal{M}') \otimes_{\mathbb{C}} (\overline{\omega_X \otimes \mathcal{M}''}) &\xrightarrow{\mathfrak{s}^{\mathrm{right}}} \omega_X \otimes \overline{\omega_X} \otimes \mathfrak{Db}_X = \mathfrak{C}_X \\ (\omega' \otimes m', \overline{\omega'' \otimes m''}) &\longmapsto \mathrm{Sgn}(n)(\omega' \wedge \overline{\omega''}) \otimes \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}).\end{aligned} \tag{12.3.3 $*$}$$

Conversely, from a sesquilinear pairing between right $\mathcal{D}_X$-modules one recovers one for left $\mathcal{D}_X$-modules.

The compatibility with Hermitian adjunction is given by the following relation:

$$(\mathfrak{s}^{\mathrm{right}})^* = (\mathfrak{s}^{\mathrm{left}*})^{\mathrm{right}}, \tag{12.3.3 $**$}$$

since $\overline{\mathrm{Sgn}(n)(\omega'' \wedge \overline{\omega'})} = \mathrm{Sgn}(n)(\omega' \wedge \overline{\omega''})$. In both left and right cases we have $\mathfrak{s}^{**} = \mathfrak{s}$.

12.3.4. Extension to C^∞ coefficients. Let us consider the right case for example. Let us define a right action of $\mathcal{D}_{X,\overline{X}}$ on $\mathcal{M} \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty$ by setting

$$(m \otimes \eta) \cdot \partial_{x_i} = m\partial_{x_i} \otimes \eta - m \otimes \partial\eta/\partial x_i \quad \text{and} \quad (m \otimes \eta) \cdot \partial_{\overline{x}_i} = -m \otimes \partial\eta/\partial\overline{x}_i.$$

Then $\mathfrak{s}$ extends in a unique way as a $\mathcal{C}_X^\infty$-linear morphism

$$(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty) \otimes_{\mathcal{C}_X^\infty} \overline{(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathcal{C}_X^\infty)} \longrightarrow \mathfrak{C}_X$$

which satisfies, for any local section ξ of Θ_X or $\overline{\Theta_X}$,

$$\mathfrak{s}(\mu', \overline{\mu''})\xi = \mathfrak{s}(\mu'\xi, \overline{\mu''}) + \mathfrak{s}(\mu', \overline{\mu''}\xi),$$

by setting

$$\mathfrak{s}(m' \otimes \eta', \overline{m'' \otimes \eta''}) := \mathfrak{s}(m', \overline{m''})\eta'\overline{\eta''}.$$

Conversely, given such a pairing, one recovers the original $\mathfrak{s}$ by restricting to $\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''}$.

12.3.5. Example.

(1) Assume $\mathcal{M}' = \mathcal{M}'' = \mathcal{O}_X$ with $\dim X = n$. We have a standard sesquilinear pairing $\mathfrak{s}_n = \mathfrak{s}_n^{\text{left}}$ defined by

$$\boxed{\mathfrak{s}_n^{\text{left}}(1,1) = 1.} \tag{12.3.5 $*$}$$

(2) If $\mathcal{M}' = \mathcal{M}'' = \omega_X$, then $\mathfrak{s}_n = \mathfrak{s}_n^{\text{right}}$ is defined by

$$\boxed{\mathfrak{s}_n^{\text{right}}(\omega', \overline{\omega''}) = \operatorname{Sgn}(n)(\omega' \wedge \overline{\omega''}),} \tag{12.3.5 $**$}$$

in such a way that $\mathfrak{s}_n^{\text{right}} = (\mathfrak{s}_n^{\text{left}})^{\text{right}}$.

Let us notice the following.

12.3.6. Lemma. *If $\mathcal{M}'$ and $\mathcal{M}''$ are $\mathcal{O}_X$-coherent (hence $\mathcal{O}_X$-locally free of finite rank), the pairing $\mathfrak{s}$ takes values in C^∞ functions (resp. forms of maximal degree).*

Proof. We know (see Example 8.3.2) that $\mathcal{M}', \mathcal{M}''$ are $\mathcal{O}_X$-generated by their flat local sections. For such local sections m', m'', the distribution (resp. current) $\mathfrak{s}(m', \overline{m''})$ is annihilated by d' and d'', hence is locally a constant. It follows that, for any local sections m', m'', $\mathfrak{s}(m', \overline{m''})$ is real-analytic, so in particular C^∞. More precisely, for local horizontal sections μ', μ'' and holomorphic functions h', h'', we can write

$$\mathfrak{s}(\mu' \otimes h', \overline{\mu'' \otimes h''}) = h'\overline{h''} \cdot \mathfrak{s}^\nabla(\mu', \overline{\mu''}), \tag{12.3.7}$$

where $\mathfrak{s}^\nabla : \mathcal{M}'^\nabla \otimes \overline{\mathcal{M}''^\nabla} \to \mathbb{C}$ is the sesquilinear pairing induced by $\mathfrak{s}$ on the underlying local systems. In other words, with respect to the above identification, we have $\mathfrak{s} = \mathfrak{s}^\nabla \cdot \mathfrak{s}_n$. ◻

12.3.8. Proposition **(Uniqueness across a non-characteristic divisor)** *Let $\mathcal{M}', \mathcal{M}''$ be coherent $\mathcal{D}_X$-modules and let H be a hypersurface which is non-characteristic for them. If two sesquilinear pairings $\mathfrak{s}_1, \mathfrak{s}_2 : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$ (or $\mathfrak{Db}_X$) coincide when restricted to the open set $X \smallsetminus H$, then they coincide.*

Proof. We will treat the case of right $\mathcal{D}_X$-modules. The question is local, so we can assume that $X = H \times \Delta_t$ and we can shrink Δ_t if needed. Set $\mathfrak{s} = \mathfrak{s}_1 - \mathfrak{s}_2$ and let m', m'' be local sections of $\mathcal{M}', \mathcal{M}''$ defined on some neighbourhood $\mathrm{nb}(x_o) = \mathrm{nb}_H \times \Delta_t$ of $x_o \in H \times \{0\}$. Let $\eta \in C_{\mathrm{c}}^\infty(\mathrm{nb}(x_o))$, and let p be the order of $\mathfrak{s}(m', \overline{m''})$ on the compact set $\operatorname{Supp} \eta$. We aim at proving that $\langle \mathfrak{s}(m', \overline{m''}), \eta \rangle = 0$.

We consider the current $\mathfrak{s}(m', \overline{m''})_\eta$ on Δ_t defined by

$$\chi \longmapsto \langle \mathfrak{s}(m', \overline{m''})_\eta, \chi \rangle := \langle \mathfrak{s}(m', \overline{m''}), \chi \cdot \eta \rangle \quad \text{for } \chi \in C_{\mathrm{c}}^\infty(\Delta_t).$$

It is enough to prove that $\mathfrak{s}(m', \overline{m''})_\eta = 0$ (by choosing $\chi \equiv 1$ on the projection to Δ_t of $\operatorname{Supp} \eta$). This current has order $\leqslant p$ and is supported at the origin, hence can be written in a unique way, by using the Dirac current δ_0 at the origin, as

$$\mathfrak{s}(m', \overline{m''})_\eta = \sum_{0 \leqslant a+b \leqslant p} c_{a,b}(\eta) \delta_0 \partial_t^a \partial_{\bar{t}}^b, \quad c_{a,b}(\eta) \in \mathbb{C}.$$

We will prove that all the coefficients $c_{a,b}(\eta)$ vanish. This is obvious if $\eta = t^q\bar{t}^r\eta_{q,r}$ with $q+r>p$ for some C^∞ function $\eta_{q,r}$. We can thus assume that $\eta = \sum_{p+q\leqslant p} t^q\bar{t}^r\eta_{q,r}$, where $\eta_{q,r}$ is a test function H, and we are finally reduced to treating the case where η is equal to such an $\eta_{q,r}$, i.e., does not depend on $t,\bar{t}$.

We claim that there exists N large enough such that m' satisfies an equation of the form

$$m'\cdot b(t\partial_t) := m'\cdot\prod_{k=1}^N (t\partial_t + k) = m'\cdot t^{p+1}\sum_j P_j(t,x,\partial_x)(t\partial_t)^j,$$

where x is a local coordinate system on H. Indeed, H is also non-characteristic for the coherent sub-module $m'\cdot\mathcal{D}_X$, and the filtration $m'\cdot V_k\mathcal{D}_X$ is comparable with the V-filtration $V_\bullet(m'\cdot\mathcal{D}_X)$, so there exists N such that $V_{-N-1}(m'\cdot\mathcal{D}_X)\subset m'\cdot V_{-(p+1)}\mathcal{D}_X$. Since $m'\partial_t^N\in(m'\cdot\mathcal{D}_X)=V_{-1}(m'\cdot\mathcal{D}_X)$, we have $m'\partial_t^N t^N\in V_{-N-1}(m'\cdot\mathcal{D}_X)$, hence the assertion.

It follows that $\mathfrak{s}(m',\overline{m''})_\eta\cdot b(t\partial_t)=0$. Since $\delta_0\partial_t^a\partial_{\bar{t}}^b\cdot(t\partial_t+k)=(a+k)\delta_0\partial_t^a\partial_{\bar{t}}^b$, we conclude that for every a,b, we have $c_{a,b}(\eta)\cdot\prod_{k=1}^N(a+k)=0$, so $c_{a,b}(\eta)=0$. $\square$

We also have an analogue of Corollary 9.7.16 for sesquilinear pairings.

12.3.9. Proposition. *Let $\mathcal{M}',\mathcal{M}''$ be two holonomic $\mathcal{D}_X$-modules which are S-decomposable and let $(S_i)_{i\in I}$ be the family of their pure components. Then any sesquilinear pairing $\mathfrak{s}:\mathcal{M}'_{S_i}\otimes_{\mathbb{C}}\overline{\mathcal{M}''_{S_j}}\to\mathfrak{C}_X$ resp. $\mathfrak{Db}_X$ vanishes identically if $S_i\neq S_j$.*

This is reminiscent of Example 7.3.9(1). We will first prove a similar result related to the S-decomposition along a function.

12.3.10. Lemma. *Let $g:X\to\mathbb{C}$ be a holomorphic function and let $\mathcal{M}',\mathcal{M}''$ be two coherent $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g). Assume that one of them, say $\mathcal{M}'$, is a middle extension along (g), and the other one, say $\mathcal{M}''$, is supported on $g^{-1}(0)$. Then any sesquilinear pairing $\mathfrak{s}:\mathcal{M}'\otimes\overline{\mathcal{M}''}\to\mathfrak{C}_X$ vanishes identically.*

Proof. By Kashiwara's equivalence (Proposition 12.4.7 below and Exercise 12.8), we can assume that g is the projection $X_0\times\mathbb{C}\to\mathbb{C}$, and we choose a coordinate t on $\mathbb{C}$. We work locally near $x_o\in X_0$. Consider $\mathfrak{s}$ as a morphism $\mathcal{M}'\to\mathcal{H}om_{\mathcal{D}_{\overline{X}}}(\overline{\mathcal{M}''},\mathfrak{C}_X)$. Fix local $\mathcal{D}_X$-generators $m''_1,\dots,m''_\ell$ of $\mathcal{M}''_{x_o}$. By Kashiwara's equivalence 9.6.1, there exists $q\geqslant 0$ such that $m''_k t^q=0$ for all $k=1,\dots,\ell$. Let $m'\in\mathcal{M}'_{x_o}$ and let p be the maximum of the orders of the currents $\mathfrak{s}(m')(\overline{m''_k})$ on some neighbourhood of x_o. As $t^{p+1+q}/\bar{t}^q$ is C^p, we have, for every $k=1,\dots,\ell$,

$$\mathfrak{s}(m')(\overline{m''_k})t^{p+1+q}=\mathfrak{s}(m')(\overline{m''_k})\bar{t}^q\cdot\frac{t^{p+1+q}}{\bar{t}^q}=\mathfrak{s}(m')(\overline{m''_k t^q})\cdot\frac{t^{p+1+q}}{\bar{t}^q}=0,$$

hence $\mathfrak{s}(m')t^{p+1+q}\equiv 0$. Applying this to generators of $\mathcal{M}'_{x_o}$ shows that all local sections of $\mathfrak{s}(\mathcal{M}'_{x_o})$ are killed by some power of t.

As $\mathcal{M}'$ is a middle extension along (t), we know from Proposition 9.7.2(2) that $V_{<0}\mathcal{M}'_{x_o}$ generates $\mathcal{M}'_{x_o}$ over $\mathcal{D}_X$. It is therefore enough to show that $\mathfrak{s}(V_{<0}\mathcal{M}'_{x_o})=0$. Let us fix a finite set of $V_0\mathcal{D}_X$-generators m'_i of $V_{<0}\mathcal{M}'_{x_o}$, and let N be such that

$\mathfrak{s}(m'_i)t^N = 0$ for all i. Since $t\partial_t \cdot t^N = t^N(t\partial_t + N)$, we conclude that t^N annihilates $\mathfrak{s}(V_<0\mathcal{M}'_{x_o})$. It follows that $\mathfrak{s}(V_{<-N}\mathcal{M}'_{x_o}) = 0$, since $V_{<-N}\mathcal{M}'_{x_o} = V_{<0}\mathcal{M}'_{x_o}t^N$.

Let now $\alpha < 0$ be such that $\mathfrak{s}(V_{<\alpha}\mathcal{M}'_{x_o}) = 0$, and let m' be a section of $V_\alpha\mathcal{M}'_{x_o}$; there exists $\nu_\alpha \geqslant 0$ such that, setting $b(s) = (s-\alpha)^{\nu_\alpha}$, we have $m'b(t\partial_t) \in V_{<\alpha}\mathcal{M}'_{x_o}$, hence $\mathfrak{s}(m')b(t\partial_t) = 0$; on the other hand, we have seen that there exists N such that $\mathfrak{s}(m')t^N = 0$, hence, putting $B(s) = \prod_{\ell=0}^{N-1}(s-\ell)$, it also satisfies $\mathfrak{s}(m')B(t\partial_t) = 0$; notice now that $b(s)$ and $B(s)$ have no common root, so $\mathfrak{s}(m') = 0$. $\square$

***Proof of Proposition* 12.3.9.** The assertion is local on X, so we fix $x_o \in X$ and we work with germs at x_o. Assume for example that S_i is not contained in S_j and consider a germ g of analytic function, such that $g \equiv 0$ on S_j and $g \not\equiv 0$ on S_i. Then we can apply Lemma 12.3.10 to $\mathcal{M}'_{S_i}$ and $\mathcal{M}''_{S_j}$. $\square$

12.4. Pushforward of sesquilinear pairings

12.4.a. General definition. Let $\mathcal{M}', \mathcal{M}''$ be coherent $\mathcal{D}_X$-modules and let $f\colon X \to Y$ be a holomorphic map which is *proper* when restricted to $S := \operatorname{Supp}\mathcal{M}' \cap \operatorname{Supp}\mathcal{M}''$. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$ or $\mathfrak{Db}_X$ be a sesquilinear pairing. Note that it takes values in currents or distributions supported on S. Our aim is to define, for every $k \in \mathbb{Z}$, sesquilinear pairings:

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s} : ({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}') \otimes_{\mathbb{C}} \overline{({}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}'')} \longrightarrow \mathfrak{C}_Y \quad \text{resp. } \mathfrak{Db}_Y\,. $$

Of course, the sesquilinear pairing for left $\mathcal{D}_X$-modules is expected to be obtained from the one for right $\mathcal{D}_X$-modules by side-changing at the source and the target and conversely. We call ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}$ the *k-th pushforward* of the sesquilinear pairing $\mathfrak{s}$.

It is easier to start with right $\mathcal{D}_X$-modules. So, let $\mathcal{M}', \mathcal{M}''$ be coherent right $\mathcal{D}_X$-modules and let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_{X,S}$ be a sesquilinear pairing between them. We set (see Section 8.7.e)

$$ \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}} := \operatorname{Sp}_{X\to Y}(\mathcal{D}_X) \otimes_{\mathbb{C}} \overline{\operatorname{Sp}_{X\to Y}(\mathcal{D}_X)}, \tag{12.4.1} $$

which is a complex of left $\mathcal{D}_{X,\overline{X}}$-modules. Therefore, $\mathfrak{s}$ yields a morphism of complexes

$$ \begin{aligned} \big(\mathcal{M}' \otimes_{\mathcal{D}_X} \operatorname{Sp}_{X\to Y}(\mathcal{D}_X)\big) \otimes_{\mathbb{C}} \big(\overline{\mathcal{M}'' \otimes_{\mathcal{D}_X} \operatorname{Sp}_{X\to Y}(\mathcal{D}_X)}\big) \simeq (\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''}) \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}} \\ \longrightarrow \mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}\,. \end{aligned} $$

By applying $\boldsymbol{R}f_*$, we thus obtained a morphism in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_{Y,\overline{Y}})$:

$$ \begin{aligned} \boldsymbol{R}f_*\big(\mathcal{M}' \otimes_{\mathcal{D}_X} \operatorname{Sp}_{X\to Y}(\mathcal{D}_X)\big) \otimes_{\mathbb{C}} \boldsymbol{R}f_*\big(\overline{\mathcal{M}'' \otimes_{\mathcal{D}_X} \operatorname{Sp}_{X\to Y}(\mathcal{D}_X)}\big) \\ \longrightarrow \boldsymbol{R}f_*\Big((\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''}) \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}\Big) \\ \longrightarrow \boldsymbol{R}f_*(\mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}) = {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S} \xrightarrow{\int_f} \mathfrak{C}_Y \end{aligned} $$

(see Exercise 12.1 and (12.2.12)), and thus, for each $k \in \mathbb{Z}$, a morphism

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s} : ({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}') \otimes_{\mathbb{C}} \overline{{}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}''} \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\,\mathfrak{C}_{X,S} \xrightarrow{\int_f} \mathfrak{C}_Y\,. \tag{12.4.2} $$

12.4.3. Definition.

(1) The sesquilinear pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}$ is the k-th pushforward of $\mathfrak{s}$. The k-th pushforward of $\mathfrak{s}$ in the left setting is obtained by side-changing at the source and target of f.

(2) The *signed right k-th pushforward* of $\mathfrak{s}$ is defined as (see Notation 0.2)

$$ {}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s} := \varepsilon(k)\,{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}, $$

and the signed left k-th pushforward of $\mathfrak{s}$ is obtained from the latter by side-changing at the source and target of f.

12.4.4. Pushforward and Hermitian adjunction. For sections ξ'_k, ξ''_ℓ of $\Theta_{X,k}$ and $\Theta_{X,\ell}$ respectively, we have the relation $\overline{\xi''_\ell \wedge \overline{\xi'_k}} = (-1)^{k\ell}\xi'_k \wedge \overline{\xi''_\ell}$. It follows that

$$ ({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(-k,k)}\mathfrak{s})^* = (-1)^k\,{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}(\mathfrak{s}^*). $$

The relation $\varepsilon(k) = (-1)^k\,\varepsilon(-k)$ enables us to absorb the sign, so it yields

$$ ({}_{\mathrm{T}}f_*^{(-k,k)}\mathfrak{s})^* = {}_{\mathrm{T}}f_*^{(k,-k)}(\mathfrak{s}^*). \tag{12.4.4 $*$} $$

12.4.5. C^∞ Computation of the pushforward of a sesquilinear pairing. Let us use the notation of Exercise 8.50, in particular $\mathrm{Sp}_X^{\infty,k} = \bigoplus_\ell \Theta_{X,\ell} \otimes \mathcal{E}_X^{(0,k+\ell)}$. We set

$$ K'^{\bullet} = f_!(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathrm{Sp}_X^{\infty,\bullet} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \simeq f_!(\mathcal{M}' \otimes_{\mathcal{O}_X} \mathrm{Sp}_X^{\infty,\bullet}) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y $$

and similarly for $K''^{\bullet}$, both equipped with the differentials $(d', d'') = (f_!\delta^\infty_{\mathcal{M}',Y}, f_!\delta^\infty_{\mathcal{M}'',Y})$ obtained by applying $f_!$ to $\delta^\infty_{\mathcal{M},Y} = \delta'^\infty_{\mathcal{M},Y} + \mathrm{d}''$ defined in Exercise 8.28. Furthermore, we can replace $f_!$ with f_* if f is assumed to be proper on $S = \operatorname{Supp}\mathcal{M}' \cup \operatorname{Supp}\mathcal{M}''$.

The sesquilinear pairing $\mathfrak{s}$ enables us to define termwise a pairing

$$ \mathrm{Sp}^\infty(\mathfrak{s}) : (\mathcal{M}' \otimes_{\mathcal{O}_X} \mathrm{Sp}_X^{\infty,k}) \otimes_{\mathbb{C}} \overline{(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathrm{Sp}_X^{\infty,-k})} \longrightarrow \mathfrak{C}_{X,S} $$

defined on local sections as

$$ (m' \otimes \xi'_\ell \otimes \eta'_{k+\ell}) \otimes \overline{(m'' \otimes \xi''_{k+j} \otimes \eta''_j)} \\ \longmapsto \begin{cases} 0 & \text{if } \ell \neq j, \\ (-1)^\ell \mathfrak{s}(m', \overline{m''}) \cdot \overline{\eta''_\ell}(\xi'_\ell) \cdot \eta'_{k+\ell}(\overline{\xi''_{k+\ell}}), & \text{if } \ell = j. \end{cases} $$

We extend in a natural way this pairing as an $f^{-1}\mathcal{D}_{Y,\overline{Y}}$-sesquilinear pairing

$$ \mathrm{Sp}_Y^\infty(\mathfrak{s}) : (\mathcal{M}' \otimes_{\mathcal{O}_X} \mathrm{Sp}_X^{\infty,k} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y) \otimes_{\mathbb{C}} \overline{(\mathcal{M}'' \otimes_{\mathcal{O}_X} \mathrm{Sp}_X^{\infty,-k} \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)} \\ \longrightarrow \mathfrak{C}_{X,S} \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}\mathcal{D}_{Y,\overline{Y}}. $$

One checks that this defines a morphism of complexes. Applying $f_!$ (or f_*), yields a $\mathcal{D}_{Y,\overline{Y}}$-sesquilinear pairing

$$ f_*\,\mathrm{Sp}_Y^\infty(\mathfrak{s}) : K'^{\bullet} \otimes_{\mathbb{C}} \overline{K''^{\bullet}} \longrightarrow f_*\,\mathfrak{C}_{X,S} \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}. $$

Integrating along f we finally obtain

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\mathfrak{s} = \textstyle\int f_*\,\mathrm{Sp}_Y^\infty(\mathfrak{s}) : K'^{\bullet} \otimes_{\mathbb{C}} \overline{K''^{\bullet}} \longrightarrow \mathfrak{C}_Y \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}} \longrightarrow \mathfrak{C}_Y, \tag{12.4.5 $*$} $$

where the second morphism is the right action of $\mathcal{D}_{Y,\overline{Y}}$ on $\mathfrak{C}_Y$.

12.4.b. Pushforward of a sesquilinear pairing by a closed embedding

Assume that $\iota : X \hookrightarrow Y$ is a closed immersion and let $\mathcal{M} = \mathcal{M}', \mathcal{M}''$ be *right* $\mathcal{D}_X$-modules. We then have ${}_{\mathrm{D}}\iota_*\mathcal{M} = {}_{\mathrm{D}}\iota_*\mathcal{M} = \iota_*(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y})$. Let $\mathbf{1}$ denote the canonical section of $\mathcal{D}_{X\to Y} = \mathcal{O}_X \otimes_{\iota^{-1}\mathcal{O}_Y} \iota^{-1}\mathcal{D}_Y$. It is a generator of $\mathcal{D}_{X\to Y}$ as a right $\iota^{-1}\mathcal{D}_Y$-module. Any sesquilinear pairing $\mathfrak{s}_Y : {}_{\mathrm{D}}\iota_*\mathcal{M}' \otimes \overline{{}_{\mathrm{D}}\iota_*\mathcal{M}''} \to \mathfrak{C}_Y$ takes values in $\mathfrak{C}_{Y,X}$ (i.e., has support in X) and, by $\mathcal{D}_{Y,\overline{Y}}$-linearity, is determined by its restriction to $\iota_*(\mathcal{M}' \otimes \mathbf{1}) \otimes \overline{\iota_*(\mathcal{M}'' \otimes \mathbf{1})}$. Hence, for local sections m', m'' of $\mathcal{M}', \mathcal{M}''$, $\mathfrak{s}_Y(m' \otimes \mathbf{1}, \overline{m'' \otimes \mathbf{1}})$ must be the pushforward of some current on X. Conversely, given $\mathfrak{s}_X$, we *define* the sesquilinear pairing ${}_{\mathrm{T}}\iota_*\mathfrak{s}_X = {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_X^{(0,0)}$ in such a way that[1]

$$
{}_{\mathrm{T}}\iota_*\mathfrak{s}_X(m' \otimes \mathbf{1}, \overline{m'' \otimes \mathbf{1}}) = \int_\iota \mathfrak{s}_X(m', \overline{m''}), \tag{12.4.6}
$$

that is, for any test function η on Y,

$$
\big\langle ({}_{\mathrm{T}}\iota_*\mathfrak{s}_X)(m' \otimes \mathbf{1}, \overline{m'' \otimes \mathbf{1}}), \eta \big\rangle = \big\langle \mathfrak{s}_X(m', \overline{m''}), \eta_{|X} \big\rangle,
$$

and we extend it by $\mathcal{D}_{Y,\overline{Y}}$-linearity.

12.4.7. Proposition **(Kashiwara's equivalence for sesquilinear pairings)**

Let $Z \xhookrightarrow{\iota} X$ be the inclusion of a closed submanifold and let $\mathcal{M}', \mathcal{M}''$ be coherent $\mathcal{D}_Z$-modules. There is a one-to-one correspondence between sesquilinear pairings $\mathfrak{s}_Z : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_Z$ and sesquilinear pairings $\mathfrak{s} : {}_{\mathrm{D}}\iota_*\mathcal{M}' \otimes \overline{{}_{\mathrm{D}}\iota_*\mathcal{M}''} \to \mathfrak{C}_X$. In one direction, $\mathfrak{s} = {}_{\mathrm{T}}\iota_*\mathfrak{s}_Z$. In the other direction, $\mathfrak{s}_Z$ is the pairing defined from $\mathfrak{s}$ by the formula

$$
\langle \mathfrak{s}_Z(m', \overline{m''}), \eta_Z \rangle = \langle \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})}), \eta \rangle
$$

for any test function η on X such that $\eta_{|Z} = \eta_Z$.

Proof. See Exercise 12.8. □

According to (12.2.10), the behaviour by Hermitian adjunction is expressed by

$$
({}_{\mathrm{T}}\iota_*\mathfrak{s}_X)^* = {}_{\mathrm{T}}\iota_*(\mathfrak{s}_X^*).
$$

The pushforward for a left sesquilinear pairing is defined by side-changing:

$$
{}_{\mathrm{T}}\iota_*(\mathfrak{s}_X^{\mathrm{left}}) := ({}_{\mathrm{T}}\iota_*\mathfrak{s}_X^{\mathrm{right}})^{\mathrm{left}}.
$$

12.4.8. Example **(Pushforward of a left sesquilinear pairing by a closed embedding)**

Let us denote by $\iota_g : X \hookrightarrow X \times \mathbb{C}$ the graph embedding attached to a holomorphic function $g : X \to \mathbb{C}$. Let $\mathcal{M}', \mathcal{M}''$ be left $\mathcal{D}_X$-modules and let $\mathfrak{s}$ be a sesquilinear pairing between them. Let us identify ${}_{\mathrm{D}}\iota_{g*}\mathcal{M}$ with $\iota_{g*}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t] \otimes \mathrm{d}t^\vee$ (see Example 8.7.7(2)). Let $m' \in \mathcal{M}'_{x_o}$ and $m'' \in \mathcal{M}''_{x_o}$ be local sections and let η be a test form of maximal degree on a neighbourhood of $(x_o, 0)$ in $X \times \mathbb{C}$ where m', m'' are defined. Set

$$
\eta = \eta_1 \wedge \tfrac{\mathrm{i}}{2\pi}(\mathrm{d}(t-g) \wedge \mathrm{d}\overline{(t-g)}),
$$

[1] Since ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_X^{(k,-k)} = 0$ for $k \neq 0$ and $\varepsilon(0) = 1$, we do not distinguish between ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_X^{(0,0)}$ and ${}_{\mathrm{T}}\iota_*\mathfrak{s}_X^{(0,0)}$.

where η_1 is a relative form of degree (n,n), and set $\eta_o = \iota_g^*\eta_1$. Then we have

$$\big\langle \eta, ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(m'\otimes 1, \overline{m''\otimes 1})\big\rangle = \big\langle \eta_o, \mathfrak{s}(m', \overline{m''})\big\rangle.$$

Indeed, let us write $\eta_1 = \varphi \mathrm{d}x\wedge \mathrm{d}\overline{x}$ in local coordinates, so that $\eta_o = \varphi_o \mathrm{d}x\wedge\mathrm{d}\overline{x}$, with $\varphi_o = \varphi_{|X\times\{0\}}$. Then, identifying m' with $m'\otimes 1$ in $\iota_{g*}\mathcal{M}'[\partial_t]$ and similarly with m'', it yields

$$\begin{aligned}
&\langle \eta, ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(m'\otimes \mathrm{d}t^\vee, \overline{m''\otimes \mathrm{d}t^\vee})\rangle\\
&\qquad = (-1)^n \tfrac{\mathrm{i}}{2\pi}\langle \varphi(\omega'\otimes \mathrm{d}t)\wedge(\overline{\omega''\otimes \mathrm{d}t}), ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(m'\otimes \mathrm{d}t^\vee, \overline{m''\otimes \mathrm{d}t^\vee})\rangle\\
&\qquad = (-1)^n \tfrac{\mathrm{i}}{2\pi}\operatorname{Sgn}(n+1)^{-1}\\
&\qquad\qquad \cdot\langle ({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}^{\mathrm{right}})\big((\omega'\otimes\mathrm{d}t)\otimes(m'\otimes \mathrm{d}t^\vee), \overline{(\omega''\otimes \mathrm{d}t)\otimes(m''\otimes\mathrm{d}t^\vee)}\big), \varphi)\rangle\\
&\qquad = (-1)^n \tfrac{\mathrm{i}}{2\pi}\operatorname{Sgn}(n+1)^{-1}\langle \mathfrak{s}^{\mathrm{right}}(\omega'\otimes m', \overline{\omega''\otimes m''}), \varphi_o\rangle\\
&\qquad = (-1)^n \tfrac{\mathrm{i}}{2\pi}\operatorname{Sgn}(n+1)^{-1}\operatorname{Sgn}(n)\langle \eta_o, \mathfrak{s}(m', \overline{m''})\rangle.
\end{aligned}$$

The conclusion follows from the identity $(-1)^n \frac{\mathrm{i}}{2\pi}\operatorname{Sgn}(n+1)^{-1}\operatorname{Sgn}(n) = 1$ (see Notation 0.2).

12.4.c. Pushforward of a sesquilinear pairing with differential forms

Let us now return to the general case of a map $f : X \to Y$ which is proper on $S = \operatorname{Supp}\mathcal{M}' \cup \operatorname{Supp}\mathcal{M}''$. We will use the formulas of Exercise 8.51 (i.e., (8.51 $*$) and (8.51 $**$)) for computing the direct image, as they happen to be more convenient at some place. Note that we already used them when expressing the integration morphism (12.2.12).

Starting from a *left* sesquilinear pairing, we aim at giving the formula for its direct images as *right* sesquilinear pairings, that is defined by side-changing at the source from ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{right}}$, and that we denote by $({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$. Similarly, $({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$ is defined by side-changing at the source from ${}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{right}}$, and in Proposition 12.4.12 we make precise the sign in the formula for $({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$.

Starting from a left $\mathcal{D}_X$-module $\mathcal{M}^{\mathrm{left}}$, let us consider the complex of right $\mathcal{D}_Y$-modules (we only indicate the shift of the complex, the sign change in the differential is understood)

$$K^\bullet := \boldsymbol{R}f_*\Omega_X^{n+\bullet}(\mathcal{M}^{\mathrm{left}}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y).$$

If $\mathcal{M}^{\mathrm{right}}$ is the right $\mathcal{D}_X$-module associated with $\mathcal{M}^{\mathrm{left}}$, we thus have

$${}_{\mathrm{D}}f_*(\mathcal{M}^{\mathrm{right}}) \simeq K^\bullet \quad\text{and}\quad {}_{\mathrm{D}}f_*(\mathcal{M}^{\mathrm{left}}) \simeq (K^\bullet)^{\mathrm{left}},$$

where the isomorphisms are induced termwise by the morphisms in Lemma 8.4.7. Moreover, it will be convenient to compute the direct image $\boldsymbol{R}f_*$ by using flabby sheaves more adapted to the computation than the Godement sheaves, so we will use the formula

$$\boldsymbol{R}f_*\Omega_X^{n+\bullet}(\mathcal{M}^{\mathrm{left}}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y)\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty \xrightarrow{\ \sim\ } f_*\mathcal{E}_X^{n+\bullet}(\mathcal{M}^{\mathrm{left}}\otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{D}_Y),$$

obtained from the Dolbeault resolution $\Omega_X^i \xrightarrow{\sim} (\mathcal{E}^{(i,\bullet)}, \mathrm{d}'')$ and by taking the associated simple complex. Last, we identify each term of this complex with

$$K_\infty^{n+\bullet} := f_*(\mathcal{E}_X^{n+\bullet} \otimes_{\mathcal{O}_X} \mathcal{M}^{\mathrm{left}}) \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \tag{12.4.9}$$

and, with this identification, the differential is given by the formula

$$(-1)^n \cdot \Big(\big[(\mathrm{d} \otimes \mathrm{Id}_{\mathcal{M}^{\mathrm{left}}}) \otimes \mathrm{Id}\big] + \big[(\mathrm{Id} \otimes \nabla) \otimes \mathrm{Id}\big] + \big[(\mathrm{Id} \otimes \mathrm{Id}) \otimes f_* f^* \nabla^Y\big]\Big),$$

where ∇^Y is the universal connection on $\mathcal{D}_Y$.

Let $\mathfrak{s} = \mathfrak{s}^{\mathrm{left}} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing between left $\mathcal{D}_X$-modules, and let $\mathfrak{s}_\infty$ denote its C^∞ extension (that suffices for our purpose, according to Remark 12.3.4). We first aim at defining a graded sesquilinear pairing whose degree k term ($k \in \mathbb{Z}$) is a pairing

$$\mathcal{H}^k(K'^{n+\bullet}) \otimes \overline{\mathcal{H}^{-k}(K''^{n+\bullet})} \longrightarrow \mathfrak{C}_Y\,. \tag{12.4.10}$$

The C^∞ extension $\mathfrak{s}_\infty$ of $\mathfrak{s}$ induces a morphism

$$\mathfrak{s}_\infty^{k,\ell} : (\mathcal{E}_X^{n+k} \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes_{\mathcal{C}_X^\infty} \overline{(\mathcal{E}_X^{n+\ell} \otimes_{\mathcal{O}_X} \mathcal{M}'')} \longrightarrow \mathfrak{Db}_{X,S}^{2n+k+\ell} \tag{12.4.11}$$

by the formula

$$(\eta'^{n+k} \otimes m') \otimes \overline{\eta''^{n+\ell} \otimes m''} \longmapsto \eta'^{n+k} \wedge \overline{\eta''^{n+\ell}}\, \mathfrak{s}(m', \overline{m''}),$$

and by applying f_*,

$$(f_*\mathfrak{s}_\infty)^{k,\ell} : f_*(\mathcal{E}_X^{n+k} \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes_{f_*\mathcal{C}_X^\infty} f_*(\overline{\mathcal{E}_X^{n+\ell} \otimes_{\mathcal{O}_X} \mathcal{M}''}) \longrightarrow f_*\, \mathfrak{Db}_{X,S}^{2n+k+\ell},$$

so, by right $\mathcal{D}_{Y,\overline{Y}}$-linearity, a morphism

$$\begin{aligned}(f_*\mathfrak{s}_\infty)^{k,\ell} : \big[f_*(\mathcal{E}_X^{n+k} \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes_{\mathcal{O}_Y} \mathcal{D}_Y\big] \otimes_{\mathcal{C}_Y^\infty} \overline{f_*(\mathcal{E}_X^{n+\ell} \otimes_{\mathcal{O}_X} \mathcal{M}'') \otimes_{\mathcal{O}_Y} \mathcal{D}_Y}& \\ \longrightarrow f_*(\mathfrak{Db}_{X,S}^{2n+k+\ell}) \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}&.\end{aligned}$$

The compatibility of $\mathfrak{s}$ with the connections on $\mathcal{M}', \mathcal{M}''$ implies that this morphism is compatible with the differentials, so that, with respect to the identifications above and according to (12.2.11), we get a morphism of complexes of right $\mathcal{D}_{Y,\overline{Y}}$-modules

$$(f_*\mathfrak{s}_\infty)^{\bullet,\bullet} : (K'^{n+\bullet}_\infty \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty) \otimes_{\mathcal{C}_Y^\infty} \overline{(K''^{n+\bullet}_\infty \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty)} \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\, \mathfrak{C}_{X,S}\,.$$

Composing with the integration of currents (see Exercise 12.4)

$$\int_f : {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\, \mathfrak{C}_{X,S} \longrightarrow \mathfrak{C}_Y$$

we finally get a morphism of complexes of right $\mathcal{D}_{Y,\overline{Y}}$-modules (where $\mathfrak{C}_Y$ is regarded as a complex having a single term in degree zero) that we denote by the same symbol:

$$(\textstyle\int f_*\mathfrak{s}_\infty)^{\bullet,\bullet} : (K'^{n+\bullet}_\infty \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty) \otimes_{\mathcal{C}_Y^\infty} \overline{(K''^{n+\bullet}_\infty \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty)} \longrightarrow \mathfrak{C}_Y\,.$$

At the cohomology level, we regard $\mathcal{H}^0 \int f_*\mathfrak{s}_\infty$ as a graded pairing, the degree k term being the induced pairing

$$(\mathcal{H}^0 \textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)} : \big(\mathcal{H}^k(K'^{n+\bullet}_\infty \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty\big) \otimes \overline{\big(\mathcal{H}^{-k}(K''^{n+\bullet}_\infty \otimes_{\mathcal{O}_Y} \mathcal{C}_Y^\infty)\big)} \longrightarrow \mathfrak{C}_Y\,.$$

The natural morphism $\mathcal{H}^j(K^{n+\bullet}) \to \mathcal{H}^j(K_\infty^{n+\bullet})$ is an isomorphism and $\mathcal{H}^j(K^{n+\bullet})$ is thus an $\mathcal{O}_Y$-submodule of $\mathcal{H}^j(K_\infty^{n+\bullet})\otimes_{\mathcal{O}_Y}\mathcal{C}_Y^\infty$. We can hence restrict $(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)}$ to obtain a graded sesquilinear pairing whose degree k term is

$$(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)} : \mathcal{H}^k(K'^{n+\bullet}) \otimes \overline{\big(\mathcal{H}^{-k}(K''^{n+\bullet})\big)} \longrightarrow \mathfrak{C}_Y\,.$$

We finally adjust the sign in order to ensure compatibility with (12.4.2) (Recall $\mathrm{Sgn}(n,k)=(-1)^n\,\varepsilon(n+k)/(2\pi\mathsf{i})^n$, see Notation (0.2 $*$)).

12.4.12. Proposition. *The following equality holds between right sesquilinear pairings:*

$$({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}} = \mathrm{Sgn}(n,k)(\mathcal{H}^0\textstyle\int f_*\mathfrak{s}_\infty)^{(k,-k)}.$$

Proof. We prove sign-commutativity, up to a precise constant in each bi-degree, of the following diagram:

$$\begin{array}{ccc}
\left.\begin{array}{c}(\mathcal{M}'^{\mathrm{right}}\otimes \mathrm{Sp}_{X\to Y})\\ \otimes\\ \overline{(\mathcal{M}''^{\mathrm{right}}\otimes \mathrm{Sp}_{X\to Y})}\end{array}\right\} & \xrightarrow{\ \sim\ } & \left\{\begin{array}{c}(\Omega_X^{n+\bullet}\otimes\mathcal{M}'^{\mathrm{left}}\otimes f^{-1}\mathcal{D}_Y)\\ \otimes\\ \overline{(\Omega_X^{n+\bullet}\otimes\mathcal{M}''^{\mathrm{left}}\otimes f^{-1}\mathcal{D}_Y)}\end{array}\right.\\
\big\downarrow & & \big\downarrow\\
(\mathcal{M}'^{\mathrm{right}}\otimes\mathcal{M}''^{\mathrm{right}})\otimes\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}} & & (\Omega_{X,\overline{X}}^{n+\bullet,n+\bullet}\otimes(\mathcal{M}'^{\mathrm{left}}\otimes\mathcal{M}''^{\mathrm{left}})\otimes f^{-1}\mathcal{D}_{Y,\overline{Y}}\\
\Big\downarrow{\scriptstyle\mathfrak{s}^{\mathrm{right}}} & & \Big\downarrow{\scriptstyle\mathfrak{s}^{\mathrm{left}}}\\
\mathfrak{C}_X\otimes\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}} & \xrightarrow{\ \sim\ } & \Omega_{X,\overline{X}}^{n+\bullet,n+\bullet}\otimes\mathfrak{Db}_X\otimes f^{-1}\mathcal{D}_{Y,\overline{Y}}
\end{array}$$

Let us consider local sections $\omega'\otimes m'$ of $\mathcal{M}'^{\mathrm{right}}$ and $\omega''\otimes m''$ of $\mathcal{M}''^{\mathrm{right}}$, where m',m'' are local sections of $\mathcal{M}'^{\mathrm{left}},\mathcal{M}''^{\mathrm{left}}$, and let ξ'_k,ξ''_ℓ be poly-vector fields of respective degree $k,\ell\geqslant 1$. Following the arrows downward for the local sections $\omega'\otimes m'\otimes\xi'_k\otimes\mathbf{1}_Y$ and $\overline{\omega''\otimes m''\otimes\xi''_\ell\otimes\mathbf{1}_Y}$ of the upper left terms, we obtain

$$\begin{aligned}
&(\omega'\otimes m'\otimes\xi'_k\otimes\mathbf{1}_Y)\otimes(\overline{\omega''\otimes m''\otimes\xi''_\ell\otimes\mathbf{1}_Y})\\
&\qquad\longmapsto(\omega'\otimes\overline{\omega''})\otimes(m'\otimes\overline{m''})\otimes(\xi'_k\otimes\overline{\xi''_\ell})\otimes\mathbf{1}_{Y,\overline{Y}}\\
&\qquad\qquad\longmapsto\mathfrak{s}^{\mathrm{right}}\big((\omega'\otimes\overline{\omega''})\otimes(m'\otimes\overline{m''})\big)\otimes(\xi'_k\otimes\overline{\xi''_\ell})\otimes\mathbf{1}_{Y,\overline{Y}}
\end{aligned}$$

and the image by the lower horizontal isomorphism of the last term above is, by mimicking Lemma 8.4.7,

$$\begin{aligned}
&\mathrm{Sgn}(n)(\omega'\otimes\overline{\omega''})\otimes\mathfrak{s}^{\mathrm{left}}(m'\otimes\overline{m''})\otimes(\xi'_k\otimes\overline{\xi''_\ell})\otimes\mathbf{1}_{Y,\overline{Y}}\\
&\qquad=\mathrm{Sgn}(n)(-1)^{nk}((\xi'_k\lrcorner\,\omega')\otimes(\overline{\xi''_\ell\lrcorner\,\omega''}))\otimes\mathbf{1}_{Y,\overline{Y}},
\end{aligned}$$

where the sign $(-1)^{nk}$ comes from the commutation of ξ'_k with $\overline{\omega''}$.

On the other hand, the image of the first term above by the first horizontal isomorphism is, according to Lemma 8.4.7,

$$\big((\xi'_k \lrcorner\, \omega') \otimes m' \otimes \mathbf{1}_Y\big) \otimes \big(\overline{(\xi''_\ell \lrcorner\, \omega'') \otimes m'' \otimes \mathbf{1}_Y}\big)\\ = \big((\xi'_k \lrcorner\, \omega') \otimes m' \otimes \mathbf{1}_Y\big) \otimes \big(\overline{(\xi''_\ell \lrcorner\, \omega'') \otimes m'' \otimes \mathbf{1}_Y}\big),$$

and the image of the latter term by the right vertical morphisms is

$$((\xi'_k \lrcorner\, \omega') \wedge (\overline{\xi''_\ell \lrcorner\, \omega''})) \otimes \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}) \otimes \mathbf{1}_{Y,\overline{Y}}.$$

Therefore, in bi-degree k, ℓ, the diagram commutes up to $(-1)^{nk}\operatorname{Sgn}(n)$.

The computation of $(\mathcal{H}^0 \textstyle\int f_* \mathfrak{s}_\infty)^{(k,-k)}$ makes use of the morphism of complexes deduced from the right vertical arrows, from the upper right term to the lower right one, while the computation of $({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}$ uses the morphism obtained by composing the arrows between the same terms in the other path. It follows that

$$({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}} = (-1)^{nk}\operatorname{Sgn}(n)(\mathcal{H}^0 \textstyle\int f_* \mathfrak{s}_\infty)^{(k,-k)},$$

and the desired equality follows from the relation $(-1)^{nk}\,\varepsilon(k)\,\varepsilon(n) = \varepsilon(n+k)$. □

12.4.13. The Lefschetz morphism. The left-to-right pushforward of a sesquilinear pairing is best suited to analyze the action of the Lefschetz morphism, i.e., the action of the external product by a closed $(1,1)$-form.

In the previous setting, let η be a closed $(1,1)$-form on X which is *real*, i.e., such that $\overline{\eta} = \eta$. This condition is satisfied if the cohomology class of η is equal to c_1 of some line bundle on X. The corresponding Lefschetz morphism $\mathrm{L}_\eta : {}_{\mathrm{D}}f_*\mathcal{M} \to {}_{\mathrm{D}}f_*\mathcal{M}[2]$ with $\mathcal{M} = \mathcal{M}', \mathcal{M}''$ (see Definition 8.7.19) satisfies

$$\int_f f_*\big(\mathfrak{s}_\infty^{k,-k}(\mathrm{L}_\eta m_\infty'^{k-2}, \overline{m_\infty''^{-k}})\big) = \int_f f_*\big(\mathfrak{s}_\infty^{k-2,-k+2}(m_\infty'^{k-2}, \overline{\mathrm{L}_\eta m_\infty''^{-k}})\big),$$

according to the definition of $\mathfrak{s}_\infty^{k,\ell}$ in (12.4.11), and therefore, by Proposition 12.4.12, since $\operatorname{Sgn}(n, k-2) = -\operatorname{Sgn}(n,k)$,

$${}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}(\mathrm{L}_\eta m', \overline{m''}) = -{}_{\mathrm{T}}f_*^{(k-2,-k+2)}\mathfrak{s}(m', \overline{\mathrm{L}_\eta m''}),$$

if m' (resp. m'') is a local section of ${}_{\mathrm{D}}f_*^{k-2}\mathcal{M}'$ (resp. of ${}_{\mathrm{D}}f_*^{-k}\mathcal{M}''$). In order to eliminate the sign, we work as in (2.4.12) with

$$\mathrm{X}_\eta := (2\pi\mathsf{i})\mathrm{L}_\eta,$$

so that

$$\text{(12.4.13}\,*\text{)}\qquad {}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}(\mathrm{X}_\eta m', \overline{m''}) = {}_{\mathrm{T}}f_*^{(k-2,-k+2)}\mathfrak{s}(m', \overline{\mathrm{X}_\eta m''}).$$

12.4.14. Composition with a closed embedding. Let us consider a composition

$$X \xrightarrow{\ f\ } Y \xrightarrow{\ f'\ } Z,$$

where X, Y, Z are complex manifolds. Let $\mathcal{M}', \mathcal{M}''$ be right $\mathcal{D}_X$-modules such that $f' \circ f$ is proper on $S = \operatorname{Supp}\mathcal{M}' \cup \operatorname{Supp}\mathcal{M}''$ (or $S = \operatorname{Supp}\mathcal{M}' \cap \operatorname{Supp}\mathcal{M}''$ if $\mathcal{M}', \mathcal{M}''$ are $\mathcal{D}_X$-coherent, see Remark 8.7.25), and let $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_{X,S}$ be a sesquilinear pairing. Let us assume that f or f' is a closed embedding. We will prove that there is a natural identification for any $k\in\mathbb{Z}$:

$$(12.4.14\,*)\qquad {}_{\mathrm{T}}(f'\circ f)_*^{(k,-k)}\mathfrak{s}=\begin{cases}{}_{\mathrm{T}}f_*'^{(k,-k)}\big({}_{\mathrm{T}}f_*\mathfrak{s}\big) & \text{if } f \text{ is a closed embedding,}\\ {}_{\mathrm{T}}f'_*\big({}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s}\big) & \text{if } f' \text{ is a closed embedding.}\end{cases}$$

Using Notation (12.4.1) and arguing as in Theorem 8.7.22 with $\mathcal{D}_{X,\overline{X}}$-modules, we find, for any right $\mathcal{D}_{X,\overline{X}}$-module $\mathcal{N}$, a natural isomorphism

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_!(\mathcal{N})\simeq {}_{\mathrm{D},\overline{\mathrm{D}}}f'_!({}_{\mathrm{D},\overline{\mathrm{D}}}f_!\mathcal{N}).$$

We consider the case where and $f'\circ f$ is proper on $\operatorname{Supp}\mathcal{N}$. For example, let us assume that f is a closed embedding, the other case being treated in a similar way (a more general situation will be treated in Section 12.7.d). Then the isomorphism reads as follows:

$$\begin{aligned}{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*(\mathcal{N})&\simeq \boldsymbol{R}(f'\circ f)_*(\mathcal{N}\otimes_{\mathcal{D}_{X,\overline{X}}}\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}\otimes_{f^{-1}\mathcal{D}_{Y,\overline{Y}}}f^{-1}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}})\\ &\simeq \boldsymbol{R}f'_*(f_*((\mathcal{N}\otimes_{\mathcal{D}_{X,\overline{X}}}\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}})\otimes_{\mathcal{D}_{Y,\overline{Y}}}\operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}))\\ &\simeq {}_{\mathrm{D},\overline{\mathrm{D}}}f'_*({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\mathcal{N}).\end{aligned}$$

Taking cohomology in degree zero yields a functorial isomorphism

$${}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathcal{N})\simeq {}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\mathcal{N}).$$

We apply this to $\mathcal{N}=\mathcal{M}'\otimes\overline{\mathcal{M}''}$ and to $\mathcal{N}=\mathfrak{C}_{X,S}$. By functoriality, we get a commutative diagram

$$\begin{tikzcd}{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathcal{M}'\otimes\overline{\mathcal{M}''}) \arrow[r, "\sim"] \arrow[d, "{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathfrak{s})"'] & {}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*(\mathcal{M}'\otimes\overline{\mathcal{M}''})) \arrow[d, "{}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*(\mathfrak{s}))"] \\ {}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathfrak{C}_{X,S}) \arrow[r, "\sim"] & {}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S})\end{tikzcd}$$

On the one hand, we can complete this commutative diagram from above by adding the line

$${}_{\mathrm{D}}(f'\circ f)_*^{(k)}(\mathcal{M}')\otimes\overline{{}_{\mathrm{D}}(f'\circ f)_*^{(-k)}\mathcal{M}''}\xrightarrow{\sim}{}_{\mathrm{D}}f'^{(k)}({}_{\mathrm{D}}f_*(\mathcal{M}'))\otimes\overline{{}_{\mathrm{D}}f'^{(-k)}({}_{\mathrm{D}}f_*(\mathcal{M}''))}$$

with the natural morphisms to the upper line of the diagram. On the other hand, we claim that the following diagram is commutative:

$$\begin{tikzcd}{}_{\mathrm{D},\overline{\mathrm{D}}}(f'\circ f)_*^{(0)}(\mathfrak{C}_{X,S}) \arrow[r, "\sim"] \arrow[dd, "\int_{f'\circ f}"'] & {}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S}) \arrow[d, "{}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}(\int_f)"] \\ & {}_{\mathrm{D},\overline{\mathrm{D}}}f_*'^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{Y,f(S)}) \arrow[d, "\int_{f'}"] \\ \mathfrak{C}_Z \arrow[r, equal] & \mathfrak{C}_Z\end{tikzcd}$$

This follows from the property that $\int_{f'}(\int_f u_{p,q}) = \int_{f'\circ f} u_{p,q}$ for a current $u_{p,q}$ such that $f'\circ f$ is proper on $\operatorname{Supp} u_{p,q}$. All together, we obtain the commutativity corresponding to the first line of (12.4.14 $*$). The second line of (12.4.14 $*$) is obtained in a similar way. $\square$

12.4.d. An adjunction formula. We will verify the compatibility of the morphism of adjunction adj_f of Section 8.7.d with sesquilinear pairings. We consider a proper holomorphic map $f: X\to Y$ between complex manifolds of the same dimension $m=n$. In order to avoid any delicate question concerning the pullback of a sesquilinear pairing, we will assume that $\mathfrak{s}$ is a sesquilinear pairing between the left $\mathcal{D}_Y$-modules $\mathcal{M}',\mathcal{M}''$ that takes values in $\mathcal{C}^\infty_Y$. The main example is the case where $\mathcal{M}',\mathcal{M}''$ are holomorphic bundles with flat connection (Lemma 12.3.6). In such a case, since ${}_{\mathrm{D}}f^*\mathcal{M}$ is the $\mathcal{O}_X$-pullback module equipped with the pullback connection, we have a well-defined pullback sesquilinear pairing ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}$ which satisfies

$$
{}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}(1\otimes m',\overline{1\otimes m''}) = f^*\mathfrak{s}(m',\overline{m''}) := \mathfrak{s}(m',\overline{m''})\circ f,
$$

which is a C^∞ function on X. We can then consider the pushforward

$$
{}_{\mathrm{T}}f_*^{(0,0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}) : {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}')\otimes \overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \longrightarrow \mathfrak{Db}_Y\,.
$$

12.4.15. Proposition. *The following diagram is commutative:*

$$
\begin{tikzcd}
\mathcal{M}'\otimes\overline{\mathcal{M}''} \arrow[rr, "\mathfrak{s}"] \arrow[d, "\mathrm{adj}_f"', "\overline{\mathrm{adj}_f}"] && \mathfrak{Db}_Y \arrow[d, equal] \\
{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}')\otimes\overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \arrow[rr, "{}_{\mathrm{T}}f_*^{(0,0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})"] && \mathfrak{Db}_Y
\end{tikzcd}
$$

Proof. It will be more convenient to work in the right setting. Extending the proof of Proposition 8.7.29 to the case of right $\mathcal{D}_{Y,\overline{Y}}$-modules, we obtain a functorial morphism

$$
\mathrm{adj}:\mathcal{M}'\otimes\overline{\mathcal{M}''}\longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{M}'\otimes\overline{\mathcal{M}''})),
$$

which factorizes as

$$
\mathcal{M}'\otimes\overline{\mathcal{M}''}\xrightarrow{\mathrm{adj}_f\otimes\overline{\mathrm{adj}_f}} {}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}')\otimes\overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')}\longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{M}'\otimes\overline{\mathcal{M}''})\big).
$$

We therefore obtain, by functoriality of adj, a commutative diagram

$$
\begin{tikzcd}
\mathcal{M}'\otimes\overline{\mathcal{M}''} \arrow[rr, "\mathfrak{s}"] \arrow[d, "\mathrm{adj}"'] && \mathcal{E}_Y^{n,n} \arrow[d, "\mathrm{adj}"] \\
{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{M}'\otimes\overline{\mathcal{M}''})\big) \arrow[rr, "{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})"] && {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\big)
\end{tikzcd}
$$

We have a natural morphism

$$
{}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{C}^\infty_Y) = \mathcal{O}_{X,\overline{X}}\otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}(\mathcal{C}^\infty_Y)\longrightarrow\mathcal{C}^\infty_X,
$$

so that, by side-changing, a natural morphism

$$
{}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\longrightarrow\mathcal{E}_X^{n,n},
$$

and thus a composed morphism

$$
\tag{12.4.16} {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\big) \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}(\mathcal{E}_X^{n,n}) \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}(\mathfrak{C}_X) \xrightarrow{\int_f} \mathfrak{C}_Y\,.
$$

In order to prove the proposition, it is enough to check that the composition on the left of (12.4.16) with adj is the natural inclusion $\mathcal{E}_Y^{n,n} \hookrightarrow \mathfrak{C}_Y$. Let us describe this morphism. Starting from a local section η of $\mathcal{E}_Y^{n,n}$, we lift it as the section $\eta \otimes 1$ of $\mathcal{E}_Y^{n,n} \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$, then we consider its image in $(f_*\mathcal{E}_X^{n,n}) \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}}$, that we integrate along f. We are thus left with checking that, for such an η, the integral of $u = f^*\eta$ as a current is equal to η. This follows from the property that, for any test function χ on Y and any (n,n)-form u on X, we have

$$
\big\langle (\textstyle\int_f u), \chi \big\rangle = \langle u, \chi \circ f\rangle = \displaystyle\int_X (\chi \circ f) u,
$$

so that, if $u = f^*\eta$,

$$
\big\langle (\textstyle\int_f u), \chi \big\rangle = \displaystyle\int_X f^*(\chi\eta) = \int_Y \chi\eta = \langle \eta, \chi\rangle. \qquad \square
$$

12.4.17. Example. In the setting of Example 8.7.30, the diagram of Proposition 12.4.15 can be completed as a commutative diagram

$$
\begin{tikzcd}
\mathcal{M}' \otimes \overline{\mathcal{M}''} \arrow[rr, "\mathfrak{s}"] \arrow[d, "\mathrm{adj}_f"', shift right=6] \arrow[d, "\overline{\mathrm{adj}_f}", shift left=6] \arrow[dd, bend left, "\mathrm{Id}"'] \arrow[dd, bend right] & & \mathfrak{Db}_Y \arrow[d, equal] \\
{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}') \otimes \overline{{}_{\mathrm{D}}f_*^{(0)}({}_{\mathrm{D}}f^*\mathcal{M}'')} \arrow[rr, "{}_{\mathrm{T}}f_*^{(0,0)}({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})"] \arrow[d, "\mathrm{Tr}_f"', shift right=6] \arrow[d, "\overline{\mathrm{Tr}_f}", shift left=6] & & \mathfrak{Db}_Y \arrow[d, equal] \\
\mathcal{M}' \otimes \overline{\mathcal{M}''} \arrow[rr, "\mathfrak{s}"] & & \mathfrak{Db}_Y
\end{tikzcd}
$$

The proof of commutativity of the lower diagram is very similar to that for the upper diagram. We define the trace Tr_f for $\mathcal{D}_{Y,\overline{Y}}$-modules and commutativity follows from identifying $\mathrm{Tr}_f : {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(0)}\big({}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathcal{E}_Y^{n,n})\big) \to \mathcal{E}_Y^{n,n}$ with the morphism (12.4.16). This identification follows from that of Exercise 12.7.

12.5. Pullback, specialization and localization of sesquilinear pairings

12.5.a. Pullback by a smooth morphism. The case of left $\mathcal{D}_X$-modules is easier to treat first. Let $f : X \to Y$ be a *smooth* holomorphic map (i.e., everywhere of maximal rank). Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_Y$ be a sesquilinear pairing between left $\mathcal{D}_Y$-modules. The pullback left $\mathcal{D}_X$-modules ${}_{\mathrm{D}}f^*\mathcal{M}', {}_{\mathrm{D}}f^*\mathcal{M}''$ are defined in Section 8.6.a, and are equal to the derived pullback modules, since f is smooth.

On the other hand, let η be a C^∞ form of maximal degree and compact support on X. It can be integrated along the fibers of f, to give rise, since f is smooth, to a C^∞ form $\int_f \eta$ of maximal degree and compact support on Y. This is a particular case of the pushforward of currents of maximal degree, as seen in Section 12.2.b: if φ is a C^∞ function on Y with compact support, we have

$$\Big\langle \int_f \eta, \varphi \Big\rangle = \int_X (\varphi \circ f) \cdot \eta.$$

Given a distribution u on Y, the pullback ${}_{\mathrm{D},\overline{\mathrm{D}}}f^* u$ is the distribution on X defined by

$$\langle \eta, {}_{\mathrm{D},\overline{\mathrm{D}}}f^* u\rangle := \Big\langle \int_f \eta, u \Big\rangle, \qquad \eta \in \mathcal{E}_{\mathrm{c}}^{n,n}(X).$$

***12.5.1. Definition* (Pullback of a sesquilinear pairing by a smooth morphism)**
The pullback ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s} : {}_{\mathrm{D}}f^*\mathcal{M}' \otimes_{\mathbb{C}} \overline{{}_{\mathrm{D}}f^*\mathcal{M}''} \to \mathfrak{Db}_X$ of a sesquilinear pairing between left $\mathcal{D}_X$-modules is defined as the morphism

$$\begin{aligned}\big(\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}'\big) \otimes_{\mathbb{C}} \big(\overline{\mathcal{O}_X \otimes_{f^{-1}\mathcal{O}_Y} f^{-1}\mathcal{M}''}\big) &\longrightarrow \mathfrak{Db}_X\\ (\varphi' \otimes m') \otimes (\overline{\varphi'' \otimes m''}) &\longmapsto \varphi' \cdot \overline{\varphi''} \cdot {}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}(m', \overline{m''})).\end{aligned}$$

The pullback of right sesquilinear pairings is obtained by the side-changing procedure, according to Remark 8.6.8:

$${}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}^{\mathrm{right}}) := ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}.$$

One checks that the above formulas define a sesquilinear pairing between ${}_{\mathrm{D}}f^*\mathcal{M}'$ and ${}_{\mathrm{D}}f^*\mathcal{M}''$.

***12.5.2. Lemma* (Pullback and Hermitian adjunction).** *The pullback by a smooth morphism commutes with Hermitian adjunction:*

$${}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}^*) = ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})^*.$$

***Proof*.** By definition and according to (12.3.3 $**$), it is enough to check the lemma for left sesquilinear pairings. We have

$$\begin{aligned}\big\langle \eta, {}_{\mathrm{D},\overline{\mathrm{D}}}f^*(\mathfrak{s}^*)(1\otimes m'', \overline{1\otimes m'})\big\rangle &= \Big\langle \int_f \eta, \mathfrak{s}^*(m'', \overline{m'})\Big\rangle = \Big\langle \int_f \eta, \overline{\mathfrak{s}(m', \overline{m''})}\Big\rangle\\ &= \overline{\Big\langle \overline{\int_f \eta}, \mathfrak{s}(m', \overline{m''})\Big\rangle} = \overline{\Big\langle \int_f \overline{\eta}, \mathfrak{s}(m', \overline{m''})\Big\rangle} \quad \text{by (12.2.10)}\\ &= \overline{\big\langle \overline{\eta}, ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})(1\otimes m', \overline{1\otimes m''})\big\rangle}\\ &= \big\langle \eta, ({}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s})^*(1\otimes m'', \overline{1\otimes m'})\big\rangle. \qquad \square\end{aligned}$$

***12.5.3. Example* (Pullback of $\mathfrak{s}_m$ by a smooth morphism).**

(1) Assume $\mathcal{M}' = \mathcal{M}'' = \mathcal{O}_Y$ and $\mathfrak{s} = \mathfrak{s}_m^{\mathrm{left}}$ (see Example 12.3.5). Then ${}_{\mathrm{D}}f^*\mathcal{O}_Y = \mathcal{O}_X$ and

$$\boxed{{}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}_m^{\mathrm{left}} = \mathfrak{s}_n^{\mathrm{left}}.}$$

(2) If $\mathcal{M}' = \mathcal{M}'' = \omega_Y$ and $\mathfrak{s} = \mathfrak{s}_m^{\mathrm{right}}$, then ${}_{\mathrm{D}}f^*\omega_Y = \omega_X$ and

$$\boxed{{}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}_m^{\mathrm{right}} = \mathfrak{s}_n^{\mathrm{right}}.}$$

12.5.b. Specialization of a sesquilinear pairing. Let $g : X \to \mathbb{C}$ be a holomorphic function on X and let $\mathcal{M}', \mathcal{M}''$ be $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g). Assume that $\mathfrak{s}$ is a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$ with values in $\mathfrak{C}_X$ (right case) or $\mathfrak{Db}_X$ (left case). We wish to define sesquilinear pairings between the $\mathcal{D}_X$-modules $\psi_{g,\lambda}\mathcal{M}'$ and $\psi_{g,\lambda}\mathcal{M}'$ with values in $\mathfrak{C}_X$ resp. $\mathfrak{Db}_X$.

We start with the case where g defines a smooth hypersurface $\iota_H : H \hookrightarrow X$, and $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable $\mathcal{D}_X$-modules along H, equipped with a sesquilinear pairing $\mathfrak{s}$. In order to define a sesquilinear pairing on nearby cycles, we will use a Mellin transform device by considering the residue of $\mathfrak{s}(m', \overline{m''})|g|^{2s}$ at various values of s, where g is some locally defined holomorphic function vanishing at order one on H.

It is important to notice that, while we need to restrict the category of coherent $\mathcal{D}_X$-modules in order to define nearby and vanishing cycles (i.e., to consider $\mathbb{R}$-*specializable* coherent $\mathcal{D}_X$-modules only), the specialization of a sesquilinear pairing between them does not need any new restriction: any sesquilinear pairing between such $\mathcal{D}_X$-modules can be specialized.

We assume that $\mathcal{M}', \mathcal{M}''$ are right $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along H, and we denote by $V_\bullet\mathcal{M}', V_\bullet\mathcal{M}''$ their V-filtration. Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing. Fix $x_o \in H$. For local sections m', m'' of $\mathcal{M}', \mathcal{M}''$ defined in some open neighbourhood $\mathrm{nb}_X(x_o)$ of x_o in X, the current $\mathfrak{s}(m', \overline{m''})$ of degree 0 has finite order on any compact subset of $\mathrm{nb}_X(x_o)$. Let us shrink the neighbourhood $\mathrm{nb}(X, x_o)$ so that $\mathfrak{s}(m', \overline{m''})$ has finite order p on $\mathrm{nb}(X, x_o)$. Let η be a test function with compact support in $\mathrm{nb}_X(x_o)$. For $2\operatorname{Re} s > p$, the function $x \mapsto |g(x)|^{2s}$ is C^p as well as its s-derivative $2s|g|^{2s}\log|g|$, so for every such s, the function

$$s \longmapsto \big\langle \mathfrak{s}(m', \overline{m''})|g|^{2s}, \eta\big\rangle := \big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\big\rangle$$

is well-defined and holomorphic on the half-plane $\{2\operatorname{Re} s > p\}$.

We also claim that, if we set $S_p = \{2\operatorname{Re} s > p\}$ and let η depend on s in such a way that $\eta \in \mathcal{C}^\infty_{\mathrm{c}}(W)$ for some $W \subset \mathrm{nb}(X, x_o) \times S_p$, then the correspondence $\eta \mapsto \big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\big\rangle$ defines a current depending continuously on S_p (see Section 12.2.a). This is obvious since derivatives up to order p of $|g|^{2s}\eta$ introduce polynomials of degree at most p in s.

12.5.4. Proposition. *Let $\mathcal{M}', \mathcal{M}'', \mathfrak{s}$ be as above. Let $x_o \in H$ and let $\alpha', \alpha'' \in \mathbb{R}$. There exist $L \geqslant 0$ and a finite set of real numbers γ satisfying*

$$\mathrm{gr}^\gamma_V \mathcal{M}'_{x_o}, \mathrm{gr}^\gamma_V \mathcal{M}''_{x_o} \neq 0, \quad \textit{and} \quad \gamma \leqslant \min(\alpha', \alpha''), \tag{12.5.4 $*$}$$

such that, for any sections $m' \in V_{\alpha'}\mathcal{M}'_{x_o}$ and $m'' \in V_{\alpha''}\mathcal{M}''_{x_o}$ defined on $\mathrm{nb}(X, x_o)$, and any test function η on $\mathrm{nb}(X, x_o)$, the function

$$s \longmapsto \big\langle \mathfrak{s}(m', \overline{m''})|g|^{2s}, \eta\big\rangle, \tag{12.5.4 $**$}$$

which is holomorphic on some half-plane $2\operatorname{Re} s > p$, extends as a meromorphic function on $\mathbb{C}_s$ of the form $h(s)\prod_\gamma \Gamma(s-\gamma)^L$, with $h(s) \in \mathcal{O}(\mathbb{C}_s)$.

Moreover, the correspondence $\eta \mapsto \prod_\gamma \Gamma(s-\gamma)^{-L}\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\rangle$ *defines a current depending continuously on* $S = \mathbb{C}_s$.

Proof. Let $b_{m'}(\mathrm{E})$ denote the Bernstein polynomial of m' (see Definition 9.3.4) and let $R(m')$ denote the set of its roots, so that we can write

$$b_{m'}(\mathrm{E}) = \prod_{\gamma\in R(m')} (\mathrm{E}-\gamma)^{\nu(\gamma)},$$

with $\nu(\gamma)$ bounded by the nilpotency index L of $\mathrm{E}-\gamma$. It is enough to prove that the product $\prod_{\gamma\in R(m')} \Gamma(s-\gamma)^{-\nu(\gamma)}$ of Γ factors can be used to make (12.5.4 $**$) an entire function (recall that the Γ function has no zeros and has simple poles at the non-positive integers, and no other poles). Indeed, arguing similarly for m'' and using that the set of roots $R(m'')$ of $b_{m''}(\mathrm{E})$ is real, one obtains that the product of Γ factors indexed by $R(m')\cap R(m'')$ can also be used to make (12.5.4 $**$) an entire function. It is then easy to check that Conditions (12.5.4 $*$) are satisfied by any $\gamma \in R(m')\cap R(m'')$.

We note first that, for every germ of operator $Q \in V_0\mathcal{D}_{X,x_o}$ and any test function η on $\mathrm{nb}_X(x_o)$, the function $Q\cdot(|g|^{2s}\eta)$ is C^p with compact support if $2\operatorname{Re} s > p$. Applying this to the Bernstein operator $Q = b_{m'}(\mathrm{E}) - P$ for m' (see Definition 9.3.4), one gets

$$\begin{aligned} 0 &= \big\langle \mathfrak{s}(m', \overline{m''})\cdot[b_{m'}(\mathrm{E}) - P], |g|^{2s}\eta\big\rangle \\ &= \big\langle \mathfrak{s}(m', \overline{m''}), [b_{m'}(\mathrm{E}) - P]\cdot(|g|^{2s}\eta)\big\rangle \\ &= b_{m'}(s)\big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\big\rangle + \big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}g\eta_1\big\rangle \end{aligned} \tag{12.5.5}$$

for some η_1, which is a polynomial in s with coefficients being C^∞ with compact support contained in that of η. As $|g|^{2s}g$ is C^p for $2\operatorname{Re} s + 1 > p$, we can argue by induction to show that, for every η and $k \in \mathbb{N}$,

$$s \longmapsto b_{m'}(s+k-1)\cdots b_{m'}(s)\big\langle \mathfrak{s}(m', \overline{m''})|g|^{2s}, \eta\big\rangle \tag{12.5.6}$$

extends as a holomorphic function on $\{s \mid 2\operatorname{Re} s > p-k\}$, and thus, letting $k \to \infty$,

$$s \longmapsto \prod_{\gamma\in R(m')} \Gamma(s-\gamma)^{-\nu(\gamma)}\cdot\big\langle \mathfrak{s}(m', \overline{m''})|g|^{2s}, \eta\big\rangle$$

extends as an entire function.

Let q denote the order of Q. Then (12.5.5) also shows that $b_{m'}(s)\big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\big\rangle$ defines a current of order $\leqslant p+q$ depending continuously on S_{p-1}, since the derivatives of order $\leqslant p$ of η_1 can be expressed in terms of derivatives of order $\leqslant p+q$ of η. By iterating this reasoning, we obtain the last part of the proposition. □

12.5.7. Remark. The previous proof also applies if we only assume that $\mathfrak{s}$ is $\mathcal{D}_{X,\overline{X}}$-linear *away from* H. Indeed, this implies that $\mathfrak{s}(m', \overline{m''})\cdot[b_{m'}(\mathrm{E}) - P]$ is supported on H, and (12.5.5) only holds for $\operatorname{Re} s$ big enough, maybe $\gg p$. Then (12.5.6) coincides with a holomorphic current of degree 0 defined on $\{s \mid 2\operatorname{Re} s > p-k\}$ only for $\operatorname{Re} s \gg 0$. But, by uniqueness of analytic extension, it coincides with it on $\operatorname{Re} s > p$.

12.5.8. Corollary. *With the assumptions of Proposition* 12.5.4*, assume moreover that* $\alpha' = \alpha'' =: \alpha$*. Let* $[m']$ *(resp.* $[m'']$*) be a germ of section of* $\mathrm{gr}_\alpha^V\mathcal{M}'$ *(resp.* $\mathrm{gr}_\alpha^V\mathcal{M}''$*) at* x_o*. Fix local liftings of* m', m'' *of* $[m'], [m'']$ *defined on* $\mathrm{nb}(X, x_o)$*. Then every polar coefficient at* $s = \alpha$ *of the meromorphic function* $s \mapsto \langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta\rangle$ *is the value of a well-defined current on* $H \cap \mathrm{nb}(X, x_o)$ *applied to the restriction* $\eta_{|H}$*. This current only depends on* $[m'], [m'']$*. It defines a sesquilinear pairing*

$$\mathrm{gr}_\alpha^V\mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{gr}_\alpha^V\mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

which does not depend on the choice of g *defining* H*.*

Proof. Any other local lifting of m' can be written as $m' + \mu'$, where μ' is a germ of section of $V_{<\alpha}\mathcal{M}'$. By the previous proposition, $\langle \mathfrak{s}(\mu', \overline{m''}), |g|^{2s}\eta\rangle$ is holomorphic at $s = \alpha$. If η vanishes on H, we have $\eta = t\eta_1 + \bar{t}\eta_2$ for some test functions η_1, η_2, and we conclude similarly. A similar argument can be used for the independence with respect to the choice of g: if $g' = g \cdot u$ where u is an invertible holomorphic function, then we apply the previous argument to $e^{s\log u}m', e^{s\log u}m''$ with the same function g. The last part of Proposition 12.5.4, together with Lemma 12.2.7, shows each polar coefficient is a current on $\mathrm{nb}(X, x_o)$.

Recall that $\mathrm{gr}_\alpha^V\mathcal{M}', \mathrm{gr}_\alpha^V\mathcal{M}''$ are $\mathcal{D}_H[\mathrm{E}]$-modules (see Remark 9.2.2). The $\mathcal{D}_{H,\overline{H}}$-linearity is a local statement on H, so we can further assume that $X \simeq H \times \Delta_t$, $g = t$ and choose η of the form $\eta_o \cdot \chi(t)$, where χ is a cut-off function on Δ_t. Then the $\mathcal{D}_{H,\overline{H}}$-linearity of the pairing given by a polar coefficient is clear. ☐

***12.5.9. Remark* (The left case).** Assume that $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable left $\mathcal{D}_X$-modules equipped with a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ between them. Up to replacing X with a neighbourhood of H such that g has no critical point on $X \smallsetminus H$, any test form η on X, we can write $\eta = \eta_g \wedge \frac{\mathrm{i}}{2\pi}(\mathrm{d}g \wedge \mathrm{d}\overline{g})$ for some relative differential form η_g on X. In particular, η_g restricts to a test form η_o on H. We consider the meromorphic function

$$s \longmapsto \langle |g|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\rangle,$$

and we link η to η_g by the previous relation. Then (left analogue of Corollary 12.5.8) the coefficients of the polar parts at α are the value of a distribution on H applied to η_o.

***12.5.10. Definition* (V-grading of a sesquilinear pairing).**

(1) (Left case) For every $\beta \in (-1, 0]$, the sesquilinear pairing

$$\mathrm{gr}_V^\beta\mathfrak{s} : \mathrm{gr}_V^\beta\mathcal{M}' \otimes \overline{\mathrm{gr}_V^\beta\mathcal{M}''} \longrightarrow \mathfrak{Db}_H$$

is well-defined by the formula

$$\langle \eta_o, \mathrm{gr}_V^\beta\mathfrak{s}([m'], \overline{[m'']})\rangle := \mathrm{Res}_{s=-\beta-1}\langle |g|^{2s}\eta, \mathfrak{s}(m', \overline{m''})\rangle, \tag{12.5.10 $*$}$$

where m', m'' are local liftings of $[m'], [m'']$ and η is any test form of maximal degree such that $\eta = \eta_1 \wedge \frac{\mathrm{i}}{2\pi}(\mathrm{d}g \wedge \mathrm{d}\overline{g})$ with $\eta_{1|H} = \eta_o$.

(2) (Right case) For $\alpha \in [-1, 0)$, the sesquilinear pairing

$$\mathrm{gr}_\alpha^V(\mathfrak{s}) : \mathrm{gr}_\alpha^V \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{gr}_\alpha^V \mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

is well-defined by the formula

$$([m'], \overline{[m'']}) \longmapsto \Big[\eta_o \mapsto \mathrm{Res}_{s=\alpha} \big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta \big\rangle \Big], \tag{12.5.10 $**$}$$

where m', m'' are local liftings of $[m'], [m'']$ and η is a test function such that $\eta_{|H} = \eta_o$.

***12.5.11. Lemma* (Side-changing).** *With the previous definition, we have*

$$\mathrm{gr}_\alpha^V \mathfrak{s}^{\mathrm{right}} = (\mathrm{gr}_V^\beta \mathfrak{s}^{\mathrm{left}})^{\mathrm{right}} \qquad (\alpha = -\beta - 1).$$

***Proof*.** Let η be a test function. We have

$$\big\langle \mathfrak{s}^{\mathrm{right}}(\omega' \otimes m', \overline{\omega'' \otimes m''}), |g|^{2s}\eta \big\rangle = \mathrm{Sgn}(n) \big\langle |g|^{2s}\eta\omega' \wedge \overline{\omega''}, \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}) \big\rangle,$$

hence, setting $\omega' = \omega'_o \wedge \mathrm{d}g$ and $\omega'' = \omega''_o \wedge \mathrm{d}g$, since $\mathrm{Sgn}(n-1) = (-1)^{n-1}\,\mathrm{Sgn}(n)\frac{2\pi}{\mathrm{i}}$,

$$\begin{aligned}
\mathrm{Res}_{s=\alpha} &\big\langle \mathfrak{s}^{\mathrm{right}}(\omega' \otimes m', \overline{\omega'' \otimes m''}), |g|^{2s}\eta \big\rangle \\
&= \mathrm{Sgn}(n)\,\mathrm{Res}_{s=-\beta-1} \big\langle |g|^{2s}\eta\,\omega' \wedge \overline{\omega''}, \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}) \big\rangle \\
&= (-1)^{n-1}\,\mathrm{Sgn}(n)\,\mathrm{Res}_{s=-\beta-1} \big\langle |g|^{2s}\eta\,\omega'_o \wedge \overline{\omega''_o} \wedge (\mathrm{d}g \wedge \overline{\mathrm{d}g}), \mathfrak{s}^{\mathrm{left}}(m', \overline{m''}) \big\rangle \\
&= \mathrm{Sgn}(n-1) \big\langle \eta_o\,\omega'_o \wedge \overline{\omega''_o}, \mathrm{gr}_V^\beta \mathfrak{s}^{\mathrm{left}}([m'], \overline{[m'']}) \big\rangle \\
&= \big\langle (\mathrm{gr}_V^\beta \mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}([\omega' \otimes m'], \overline{[\omega'' \otimes m'']}), \eta_o \big\rangle. \qquad \square
\end{aligned}$$

***12.5.12. Lemma* (N is self-adjoint).** *The nilpotent operator* $\mathrm{N} := -(\mathrm{E} - \beta)$ *is self-adjoint with respect to the pairing* (12.5.10 $*$)*, in the sense that*

$$\mathrm{gr}_V^\beta(\mathfrak{s})(\mathrm{N}[m'], \overline{[m'']}) = \mathrm{gr}_V^\beta(\mathfrak{s})([m'], \overline{\mathrm{N}[m'']}). \tag{12.5.12 $*$}$$

***Proof*.** The question is local, so we can assume $X = H \times \Delta_t$ and $\eta = \widetilde{\eta}_o \wedge \chi(t)\,\frac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \overline{\mathrm{d}t})$ with $\chi(t) \equiv 1$ near $t = 0$. Then the statement is a consequence of the following properties (recall that $[|t|^{2s}\chi(t)\,\frac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \overline{\mathrm{d}t})] \cdot (-t\partial_t) = \partial_t t(|t|^{2s}\chi(t))\,\frac{\mathrm{i}}{2\pi}(\mathrm{d}t \wedge \overline{\mathrm{d}t})$):

- β is real,
- $\bar{t}\partial_{\bar{t}}|t|^{2s} = t\partial_t|t|^{2s}$,
- $\bar{t}\partial_{\bar{t}}\chi(t)$ and $t\partial_t\chi(t)$ are zero in a neighbourhood of $t = 0$. $\square$

***12.5.13. Lemma* (Adjunction and V-grading).** *We have*

$$\mathrm{gr}_V^\beta(\mathfrak{s}^*) = (\mathrm{gr}_V^\beta \mathfrak{s})^*.$$

***Proof*.** Since β is real, we have $[\overline{m}] = \overline{[m]}$ in $\mathrm{gr}_V^\beta \mathcal{M}$ and we can compute the residue by assuming that s varies in $\mathbb{R}$. We compute using Definition 12.3.2(2):

$$\begin{aligned}
\langle \eta_o, \mathrm{gr}_V^\beta(\mathfrak{s}^*)([m''], \overline{[m']}) \rangle &= \mathrm{Res}_{s=-\beta-1} \big\langle |g|^{2s}\eta, \mathfrak{s}^*(m'', \overline{m'}) \big\rangle \\
&= \mathrm{Res}_{s=-\beta-1} \overline{\big\langle |g|^{2s}\overline{\eta}, \mathfrak{s}(m', \overline{m''}) \big\rangle} \\
&= \overline{\mathrm{Res}_{s=-\beta-1} \big\langle |g|^{2s}\overline{\eta}, \mathfrak{s}(m', \overline{m''}) \big\rangle} = \overline{\big\langle \overline{\eta}_o, \mathrm{gr}_V^\beta \mathfrak{s}([m'], \overline{[m'']}) \big\rangle} \\
&= \big\langle \eta_o, (\mathrm{gr}_V^\beta \mathfrak{s})^*([m''], \overline{[m']}) \big\rangle. \qquad \square
\end{aligned}$$

12.5.14. Remark **(Properties for right sesquilinear pairings).** Due to Lemma 12.5.11, the previous properties also hold for right sesquilinear pairings.

We now take up the setting of Exercise 9.25. Let $\iota : X \hookrightarrow X_1$ be a closed inclusion of complex manifolds, and let $g_1^{-1}(0) = H_1 \subset X_1$ be a smooth hypersurface such that $H := X \cap H_1$ is a smooth hypersurface of X, defined as the zero set of the smooth function $g = g_{1|X}$.

Let $\mathcal{M}', \mathcal{M}''$ be coherent right $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along H and let $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing between them. We deduce a sesquilinear pairing ${}_{\mathrm{T}}\iota_*\mathfrak{s}$ between ${}_{\mathrm{D}}\iota_*\mathcal{M}'$ and ${}_{\mathrm{D}}\iota_*\mathcal{M}''$. Let us denote by $\iota_H : H \hookrightarrow H_1$ the inclusion. We will consider the V-filtrations along H in X and H_1 in X_1.

12.5.15. Lemma **(Independence of the embedding).** *With these assumptions, we have for all $\alpha \in [-1, 0)$,*

$$\mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota_*\mathfrak{s}) = {}_{\mathrm{T}}\iota_{H*}(\mathrm{gr}_\alpha^V(\mathfrak{s})).$$

Proof. Recall that (Section 12.4.b), if η is any test function on X_1 and m', m'' are local sections of $\mathcal{M}', \mathcal{M}''$, so that $m' \otimes 1, m'' \otimes 1$ are local sections of $\iota_*\mathcal{M}'[\partial_x], \iota_*\mathcal{M}''[\partial_x]$, then

$$\langle {}_{\mathrm{T}}\iota_*\mathfrak{s}(m' \otimes 1, \overline{m'' \otimes 1}), \eta\rangle := \langle \mathfrak{s}(m', \overline{m''}), \eta_{|X}\rangle.$$

For m', m'' in $V_\alpha\mathcal{M}', V_\alpha\mathcal{M}''$, η a test function on X_1 as above, we thus have (arguing for $\operatorname{Re} s \gg 0$ first and then using analytic continuation),

$$\begin{aligned}\langle \mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota_*\mathfrak{s})([m' \otimes 1], \overline{[m'' \otimes 1]}), \eta_{|H_1}\rangle &= \mathrm{Res}_{s=\alpha}\langle {}_{\mathrm{T}}\iota_*\mathfrak{s}(m' \otimes 1, \overline{m'' \otimes 1}), |g_1|^{2s}\eta\rangle \\ &= \mathrm{Res}_{s=\alpha}\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta_{|X}\rangle \quad \text{by definition (12.4.6)} \\ &= \langle \mathrm{gr}_\alpha^V\mathfrak{s}([m'], \overline{[m'']}), \eta_{|H}\rangle \\ &= \langle ({}_{\mathrm{T}}\iota_*\mathrm{gr}_\alpha^V\mathfrak{s})([m' \otimes 1], \overline{[m'' \otimes 1]}), \eta_{|H_1}\rangle. \qquad \square\end{aligned}$$

We now consider the general case of nearby cycles along any holomorphic function g, for which the functor ψ_g is needed.

12.5.16. Definition **(Sesquilinear pairing on nearby cycles).** Let $g : X \to \mathbb{C}$ be any holomorphic function. Assume that $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable along (g). For a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes \overline{\mathcal{M}''} \to \mathfrak{C}_X$ resp. $\mathfrak{Db}_X$ and for every $\lambda \in \mathbb{S}^1$ and $\alpha \in [-1, 0)$ such that $\lambda = \exp(2\pi\mathsf{i}\,\alpha)$, we define

$$\boxed{\psi_{g,\lambda}\mathfrak{s} := \mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}) : \psi_{g,\lambda}\mathcal{M}' \otimes \overline{\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{C}_X} \tag{12.5.16 $*$}$$

resp. for every $\beta \in (-1, 0]$ such that $\lambda = \exp(-2\pi\mathsf{i}\,\beta)$,

$$\boxed{\psi_{g,\lambda}\mathfrak{s} := \mathrm{gr}_V^\beta({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}) : \psi_{g,\lambda}\mathcal{M}' \otimes \overline{\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{Db}_X.} \tag{12.5.16 $**$}$$

12.5.17. Properties of $\psi_{g,\lambda}\mathfrak{s}$**.** The following properties are obviously obtained from similar properties for $\mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota_*\mathfrak{s})$.

(1) $\psi_{g,\lambda}\mathfrak{s}(\mathrm{N}m', \overline{m''}) = \psi_{g,\lambda}\mathfrak{s}(m', \overline{\mathrm{N}m''})$ $\quad(m' \in \psi_{g,\lambda}\mathcal{M}'_{x_o},\ m'' \in \psi_{g,\lambda}\mathcal{M}''_{x_o})$.

(2) We have induced pairings

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathfrak{s} : \mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathcal{M}' \otimes \overline{\mathrm{gr}_{-\ell}^{\mathrm{M}}\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{C}_X \quad \text{resp. } \mathfrak{Db}_X,$$

and, for every $\ell \geqslant 0$,

$$\mathrm{P}_\ell\psi_{g,\lambda}\mathfrak{s} : \mathrm{P}_\ell\psi_{g,\lambda}\mathcal{M}' \otimes \overline{\mathrm{P}_\ell\psi_{g,\lambda}\mathcal{M}''} \longrightarrow \mathfrak{C}_X \quad \text{resp. } \mathfrak{Db}_X$$

is induced by $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathfrak{s}(\bullet, \overline{\mathrm{N}^\ell\bullet})$.

(3) $\psi_{g,\lambda}(\mathfrak{s}^*) = (\psi_{g,\lambda}\mathfrak{s})^*$, according to Section 12.4.b and Lemma 12.5.13.

(4) Recall that ${}_{\mathrm{D}}\iota_{g*}\mathcal{M} \simeq \mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t]$. Assume that $m' \otimes 1$ is a local section of $V_\alpha({}_{\mathrm{D}}\iota_{g*}\mathcal{M}')$ and $m'' \otimes 1$ is a local section of $V_\alpha({}_{\mathrm{D}}\iota_{g*}\mathcal{M}'')$ $(1 \leqslant \alpha < 0)$. In such a case, $\psi_{g,\lambda}\mathfrak{s}([m' \otimes 1], \overline{[m'' \otimes 1]})$ is given by a formula similar to (12.5.10 $**$), for any test function η on an open set of X where m', m'' are defined:

$$\big\langle \psi_{g,\lambda}\mathfrak{s}([m' \otimes 1], \overline{[m'' \otimes 1]}), \eta \big\rangle = \mathrm{Res}_{s=\alpha}\big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta \big\rangle. \tag{12.5.17 $*$}$$

Indeed, the left-hand term is equal to

$$\mathrm{Res}_{s=\alpha}\big\langle {}_{\mathrm{T}}\iota_{g*}\mathfrak{s}([m' \otimes 1], \overline{[m'' \otimes 1]}), |t|^{2s}\eta_1 \big\rangle$$

if $\eta_1(x,t)$ satisfies $\eta_1(x,0) = \eta$. This is also written as

$$\mathrm{Res}_{s=\alpha}\big\langle \mathfrak{s}(m', \overline{m''}), |g|^{2s}\eta_1(x, g(x)) \big\rangle$$

and by developing η_1 with respect to powers of $t, \bar{t}$, one checks by the same argument as that of Proposition 12.5.4 that only $\eta_1(x,0) = \eta(x)$ contributes to the residue.

12.5.c. Examples

12.5.18. Strictly non-characteristic restriction of a sesquilinear pairing. Let $\mathcal{M}', \mathcal{M}''$ be coherent left $\mathcal{D}_X$-modules such that the smooth hypersurface $\iota : H \hookrightarrow X$ is strictly non-characteristic for them (see Section 9.5.b). Then $\mathrm{gr}_V^\beta\mathcal{M}$ are zero except for $\beta \in \mathbb{N}$, and E acts by zero on $\mathrm{gr}_V^0\mathcal{M}$, so it is a $\mathcal{D}_H$-module even without assuming that H is defined by a global function (see Proposition 9.5.9). Let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing. Then $\mathrm{gr}_V^0\mathfrak{s}$, as defined by (12.5.10 $*$), is a sesquilinear pairing between $\mathrm{gr}_V^0\mathcal{M}' = \mathcal{M}'/t\mathcal{M}'$ and $\mathrm{gr}_V^0\mathcal{M}'' = \mathcal{M}''/t\mathcal{M}''$. We set

$${}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s} := \mathrm{gr}_V^0\mathfrak{s} : {}_{\mathrm{D}}\iota^*\mathcal{M}' \otimes_{\mathbb{C}} \overline{{}_{\mathrm{D}}\iota^*\mathcal{M}''} \longrightarrow \mathfrak{Db}_H\,.$$

12.5.19. Pullback in the smooth case. Assume that the left $\mathcal{D}_Y$-modules $\mathcal{M}', \mathcal{M}''$ are $\mathcal{O}_Y$-locally free of finite rank, and let $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{Db}_Y$ be a sesquilinear pairing. Recall (Lemma 12.3.6) that $\mathfrak{s}$ takes values in the sheaf of C^∞ functions. If $f : X \to Y$ is any holomorphic map between complex manifolds, the pullback $\mathcal{D}_X$-modules ${}_{\mathrm{D}}f^*\mathcal{M}', {}_{\mathrm{D}}f^*\mathcal{M}''$ are equal to the $\mathcal{O}_X$-pullbacks equipped with their natural pullback flat connection, hence are also $\mathcal{O}_X$-locally free of finite rank, and there is a natural pullback of $\mathfrak{s}$ as taking values in $\mathcal{C}_X^\infty$, that we denote by ${}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}$, defined such that

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}(1\otimes m', \overline{1\otimes m''}) = \mathfrak{s}(m',\overline{m''})\circ f. $$

For example, if $\mathcal{M}'=\mathcal{M}''=\mathcal{O}_Y$, then

$$ {}_{\mathrm{D},\overline{\mathrm{D}}}\iota^*\mathfrak{s}_m = \mathfrak{s}_n. $$

This definition is compatible with the general one when f is a smooth morphism (Section 12.5.1). We will check that it coincides with that of 12.5.18 when f is the inclusion $\iota: H\hookrightarrow X$ of a smooth hypersurface, and $\mathcal{M}',\mathcal{M}''$ are $\mathcal{O}_X$-locally free of finite rank.

Assume that $X=H\times\Delta_t$. Then $\mathcal{M}=V^0\mathcal{M}$, $\mathrm{gr}_V^\beta V\mathcal{M}=0$ for $\beta\notin\mathbb{N}$, and $\mathrm{gr}_V^0\mathcal{M}=\mathcal{M}/t\mathcal{M}$. Denoting by h'_o etc. the restriction to H, we obtain from Exercise 6.13, by choosing $\chi(t)=\widetilde{\chi}(|t|^2)$, the expected formula analogous to (12.3.7):

$$ (\mathrm{gr}_V^0\mathfrak{s})(\mu'_o\otimes h'_o, \overline{\mu''_o\otimes h''_o}) = (\mathfrak{s}^\nabla(\mu',\overline{\mu''})\cdot h'\overline{h''})_{|H}, $$

that is, $\mathrm{gr}_V^0(\mathfrak{s}^\nabla\cdot\mathfrak{s}_n)=\mathfrak{s}^\nabla_{|H}\cdot\mathfrak{s}_{n-1}$. Indeed, let η be a test form of maximal degree written as $\eta=\eta_t\wedge\chi(t)\,\frac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})$ (see Remark 12.5.9). Then

$$ \begin{aligned} \mathrm{Res}_{s=-1}&\big\langle\eta_t\wedge\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t}), \mathfrak{s}(h'\otimes\mu',\overline{\mu''\otimes h''})|t|^{2s}\big\rangle\\ &= \mathrm{Res}_{s=-1}\,\mathfrak{s}^\nabla(\mu',\overline{\mu''})_{|H}\int_X |t|^{2s}h'\overline{h''}\cdot\eta_t\wedge\chi(t)\,\tfrac{\mathrm{i}}{2\pi}(\mathrm{d}t\wedge\overline{\mathrm{d}t})\\ &= \mathfrak{s}^\nabla(\mu',\overline{\mu''})_{|H}\int_H h'_o\overline{h''_o}\,\eta_o \quad \text{(Exercise 6.13)}. \end{aligned} $$

12.5.20. Specialization along a strictly non-characteristic divisor with normal crossings

We take up the setting of Section 9.9.b and analyze how the isomorphisms (9.9.3 $*$) are compatible with sesquilinear pairings. Assume thus that $\mathcal{M}',\mathcal{M}''$ are non-characteristic along D_1,D_1,D_{12} and let $\mathfrak{s}:\mathcal{M}'\otimes\overline{\mathcal{M}''}\to\mathfrak{C}_X$ be a sesquilinear pairing between them. We work in the local setting of Proposition 9.9.3.

Complementing Lemma 9.9.2, we note that, if η_o is a C^∞ function on X,

(a) the two-variable Mellin transform $(s_1,s_2)\mapsto\langle\mathfrak{s}(m',\overline{m''}),|x_1|^{2s_1}|x_2|^{2s_2}\eta_o\rangle$ has only simple poles along the lines $s_1=-1-k_1$, $s_2=-1-k_2$, with $k_1,k_2\in\mathbb{N}$, and no other poles,

(b) $\mathrm{gr}_{-1}^{V^{(1)}}\mathrm{gr}_{-1}^{V^{(2)}}\mathfrak{s}=\mathrm{gr}_{-1}^{V^{(2)}}\mathrm{gr}_{-1}^{V^{(1)}}\mathfrak{s}$.

We will prove that, under the isomorphisms (9.9.3 $*$),

$$ \mathrm{P}_0\psi_{g,1}\mathfrak{s}=\mathrm{gr}_0^{\mathrm{M}}\psi_{g,1}\mathfrak{s}=\psi_{x_1,1}\mathfrak{s}\oplus\psi_{x_2,1}\mathfrak{s}, \tag{12.5.20 $*$} $$

$$ \mathrm{P}_1\psi_{g,1}\mathfrak{s}=\psi_{x_1,1}\psi_{x_2,1}\mathfrak{s}=\psi_{x_2,1}\psi_{x_1,1}\mathfrak{s}. \tag{12.5.20 $**$} $$

Let m' be a local section of $\mathcal{M}'$, let $m'\otimes 1$ its image in $\iota_{g*}\mathcal{M}'[\partial_t]=V_{-1}({}_{\mathrm{D}}\iota_{g*}\mathcal{M}')$ and let $[m'\otimes 1]$ its class in $\mathrm{gr}_{-1}^V({}_{\mathrm{D}}\iota_{g*}\mathcal{M}')$. Similar notation for m''. We have

$$ \begin{aligned} \big\langle(\mathrm{gr}_{-1}^V{}_{\mathrm{T}}\iota_{g*}\mathfrak{s})&([m'\otimes 1],\overline{[m''\otimes 1]}),\eta_o\big\rangle\\ &= \mathrm{Res}_{s=-1}\big\langle{}_{\mathrm{T}}\iota_{g*}\mathfrak{s}(m'\otimes 1,\overline{m''\otimes 1}),|t|^{2s}\eta_o\chi(t)\big\rangle\\ &= \mathrm{Res}_{s=-1}\big\langle\mathfrak{s}(m',\overline{m''}),|x_1x_2|^{2s}\eta_o\chi(g)\big\rangle \quad \text{(by (12.5.17 $*$))}. \end{aligned} $$

Let us consider the morphism $\varphi_1 : \mathcal{N}_1 = {}_{\mathrm{D}}\iota_{1*}\mathrm{gr}^{V^{(1)}}_{-1}\mathcal{M} \to \mathrm{gr}^{\mathrm{M}}_0\mathrm{gr}^V_{-1}({}_{\mathrm{D}}\iota_{g*}\mathcal{M})$ as described in (9.9.11). It sends $m\otimes 1$ to $mx_2\otimes 1$ ($m = m', m''$), so that

$$\big\langle(\mathrm{gr}^0_{\mathrm{M}}\mathrm{gr}^V_{-1}{}_{\mathrm{T}}\iota_{g*}\mathfrak{s})(\varphi_1[m'\otimes 1], \overline{\varphi_1[m''\otimes 1]}), \eta_o\big\rangle = \mathrm{Res}_{s=-1}\big\langle\mathfrak{s}(m'x_2, \overline{m''x_2}), |x_1x_2|^{2s}\eta_o\big\rangle.$$

On the other hand, we have

$$\begin{aligned}\langle({}_{\mathrm{D}}\iota_{1*}\mathrm{gr}^{V^{(1)}}_{-1}\mathfrak{s})([m'], \overline{[m'']}), \eta_o\rangle &= \mathrm{Res}_{s=-1}\langle\mathfrak{s}(m', \overline{m''}), |x_1|^{2s}\eta_o\rangle\\ &\overset{(*)}{=} \mathrm{Res}_{s=-1}\langle\mathfrak{s}(m', \overline{m''}), |x_1|^{2s}|x_2|^{2s+2}\eta_o\rangle\\ &= \mathrm{Res}_{s=-1}\langle\mathfrak{s}(m'x_2, \overline{m''x_2}), |x_1x_2|^{2s}\eta_o\rangle.\end{aligned}$$

The equality $(*)$ is obtained by using that the two-variable Mellin transform $\langle\mathfrak{s}(m', \overline{m''}), |x_1|^{2s_1}|x_2|^{2s_2+2}\eta_o\rangle$ has no pole along $s_2 = -1$. We argue similarly by changing the roles of x_1 and x_2, and we obtain (12.5.20 $*$) in this way, according to (12.5.16 $*$).

For (12.5.20 $**$), we wish to compute $\psi_{g,1}\mathfrak{s}([m'\otimes 1], \overline{\mathrm{N}[m''\otimes 1]})$ for

$$[m'\otimes 1] \in \mathrm{gr}^{\mathrm{M}}_1\mathrm{gr}^V_{-1}({}_{\mathrm{D}}\iota_{g*}\mathcal{M}'), \quad [m''\otimes 1] \in \mathrm{gr}^{\mathrm{M}}_1\mathrm{gr}^V_{-1}({}_{\mathrm{D}}\iota_{g*}\mathcal{M}'').$$

By using the isomorphisms (9.9.9) and (9.9.8), we find

$$\big\langle(\mathrm{gr}^V_{-1}{}_{\mathrm{T}}\iota_{g*}\mathfrak{s})([m'\otimes 1], \overline{\mathrm{N}[m''\otimes 1]}), \eta_o\big\rangle = \mathrm{Res}_{s=-1}(s+1)\big\langle\mathfrak{s}(m', \overline{m''}), |x_1x_2|^{2s}\eta_o\big\rangle,$$

since $\mathrm{N} = \partial_t t$. One notices that, for a meromorphic function $\varphi(s_1, s_2)$ on $\mathbb{C}^2$ as in (a) above, we have

$$\begin{aligned}\mathrm{Res}_{s=-1}(s+1)\varphi(s,s) &= \mathrm{Res}_{s_1=-1}\mathrm{Res}_{s_2=-1}\varphi(s_1, s_2)\\ &= \mathrm{Res}_{s_2=-1}\mathrm{Res}_{s_1=-1}\varphi(s_1, s_2).\end{aligned}$$

Then (12.5.20 $**$) follows.

12.5.21. Nearby cycles along a monomial function of a smooth $\mathcal{D}$-module

We take up the setting of Section 9.9.c. Since the question is local, we set $X = \mathbb{C}^n$. Let us assume that $\mathcal{M}', \mathcal{M}''$ are smooth left $\mathcal{D}_X$-modules and let $\mathfrak{s} : \mathcal{M}'\otimes\overline{\mathcal{M}''} \to \mathfrak{Db}_X$ be a sesquilinear pairing. We know that $\mathfrak{s}$ takes values in $\mathcal{C}^\infty_X$ (Lemma 12.3.6). In particular, the restriction ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_I^*\mathfrak{s}$ is a well-defined sesquilinear pairing between ${}_{\mathrm{D}}\iota_I^*\mathcal{M}'$ and ${}_{\mathrm{D}}\iota_I^*\mathcal{M}''$. We will show that, under the isomorphism of Proposition 9.9.12(3), we have

$$\mathrm{P}_\ell\psi_{g,1}\mathfrak{s} = \bigoplus_{J\in\mathfrak{J}_{\ell+1}} {}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_I^*\mathfrak{s}) \qquad (I = J^c). \tag{12.5.21 $*$}$$

We will restrict to the case where $\mathcal{M}' = \mathcal{M}'' = \mathcal{O}_X$ and $\mathfrak{s} = \mathfrak{s}_n$, the general case being similar (see Example 12.5.19). By the $\mathcal{D}$-linearity of the isomorphism in Lemma 9.9.21, it is enough to compute, for $J', J'' \in \mathfrak{J}_{\ell+1}$, $\mathrm{P}_\ell\psi_{g,1}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta, \overline{x_{I''}^{\mathbf{1}_{I''}}\delta})$ and to show that

- it is zero if $J' \neq J''$,
- if $J' = J'' = J$, it is equal to ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{I*}\mathfrak{s}_I(\delta_J, \overline{\delta_J})$, where $\mathfrak{s}_I = {}_{\mathrm{D},\overline{\mathrm{D}}}\iota_I^*\mathfrak{s}_n$ is the standard pairing on $\mathcal{O}_I$.

By definition, for a test form of maximal degree $\eta(x)$ and a cutoff function $\chi(t)$,

$$\begin{aligned}\big\langle\eta,\mathrm{P}_\ell\psi_{g,1}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle &= \big\langle\eta,\psi_{g,1}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{\mathrm{N}^\ell x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle\\ &= \mathrm{Res}_{s=-1}\big\langle\eta\chi(t)|t|^{2s}(\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}),\overline{(-t\partial_t)^\ell}{}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{g*}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle\\ &= \mathrm{Res}_{s=-1}(s+1)^\ell\big\langle\eta\chi(t)|t|^{2s}(\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}),{}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{g*}\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}}\delta,\overline{x_{I''}^{\mathbf{1}_{I''}}\delta})\big\rangle,\end{aligned}$$

where we have used, for the third equality, the property that the expression $\langle\eta\partial_{\overline{t}}\chi|t|^{2s}\cdots,\cdots\rangle$ is an entire function of s. By a computation already done, the last term can be expressed as

$$\mathrm{Res}_{s=-1}(s+1)^\ell\big\langle\eta\chi(g)|g|^{2s},\mathfrak{s}_n(x_{I'}^{\mathbf{1}_{I'}},\overline{x_{I''}^{\mathbf{1}_{I''}}})\big\rangle=\mathrm{Res}_{s=-1}(s+1)^\ell\int_{\Delta_r^n}x_{I'}^{\mathbf{1}_{I'}}\overline{x_{I''}^{\mathbf{1}_{I''}}}\chi(g)|g|^{2s}\eta,$$

where $\Delta_r\subset\mathbb{C}$ is a disc of radius $r>0$ large enough so that Δ_r^n contains the compact support of η. Let us forget $\chi(g)$, which plays no role, and let us assume that $p=n$, since the variables $x_{p+1},\dots,x_n$ do not play any important role here. Due to the formula (obtained by computing with polar coordinates)

$$\int_{\Delta_r}|t|^{2s}t^p\overline{t}^q\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}t\wedge\mathrm{d}\overline{t}=\begin{cases}0 & \text{if } p\neq q,\\ \dfrac{r^{2(s+p+1)}}{s+p+1} & \text{if } p=q,\end{cases}$$

we find that, for a monomial $\eta_{\boldsymbol{p},\boldsymbol{q}}=x^{\boldsymbol{p}}\overline{x}^{\boldsymbol{q}}\prod_i(\frac{\mathrm{i}}{2\pi}\,\mathrm{d}x_i\wedge\mathrm{d}\overline{x}_i)$ $(p_i,q_i\geqslant0)$,

$$\int_{\Delta_r^n}x_{I'}^{\mathbf{1}_{I'}}\overline{x_{I''}^{\mathbf{1}_{I''}}}|g|^{2s}\eta_{\boldsymbol{p},\boldsymbol{q}}=\begin{cases}0 & \text{if }\exists\, i,\ (\boldsymbol{p}+\mathbf{1}_{I'})_i\neq(\boldsymbol{q}+\mathbf{1}_{I''})_i,\\ \prod_i\dfrac{r^{2(s+1+(\boldsymbol{p}+\mathbf{1}_{I'})_i)}}{s+1+(\boldsymbol{p}+\mathbf{1}_{I'})_i} & \text{if }\forall\, i,\ (\boldsymbol{p}+\mathbf{1}_{I'})_i=(\boldsymbol{q}+\mathbf{1}_{I''})_i.\end{cases}$$

A pole at $s=-1$ can occur only if $J'=J''$, that we denote by J (hence $I'=I''$) and $p_J=q_J=0_J$. It is then of order $\ell+1$. We conclude that the residue above is nonzero only if $I'=I''$, that we denote by I, and if we set $\eta=\eta_I\prod_{j\in J}(\frac{\mathrm{i}}{2\pi}\,\mathrm{d}x_j\wedge\mathrm{d}\overline{x}_j)$, we obtain

$$\mathrm{Res}_{s=-1}(s+1)^\ell\int_{\Delta_r^n}x_{I'}^{\mathbf{1}_{I'}}\overline{x_{I''}^{\mathbf{1}_{I''}}}|g|^{2s}\eta=\int_{\Delta_r^I}\eta_{I|X_I},$$

where $X_I=\{x_j=0\mid\forall\, j\in J\}$. According to Example 12.4.8, this is nothing but $\langle\eta,({}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{I*}\mathfrak{s}_I)(\delta_J,\overline{\delta_J})\rangle$.

12.5.d. Sesquilinear pairing on vanishing cycles. If $\mathcal{M}',\mathcal{M}''$ are supported on $g^{-1}(0)$, the residue formulas (12.5.10 $*$) or (12.5.10 $**$) with kernel $|g|^{2s}$ return the value zero since any local section m of $\mathcal{M}$ is annihilated by some power of g and, for $\mathrm{Re}(s)\gg0$, the function (12.5.6) is identically zero (see Exercise 12.10). Therefore, these formulas do not lead to a definition of an interesting sesquilinear pairing

$$\phi_{t,1}\mathfrak{s}:\phi_{t,1}\mathcal{M}'\otimes\overline{\phi_{t,1}\mathcal{M}''}\longrightarrow\mathfrak{C}_X\quad\text{resp. }\mathfrak{Db}_X$$

for every $\mathcal{M}',\mathcal{M}''$ which are coherent and $\mathbb{R}$-specializable along $t=0$.

On the other hand, the Thom-Sebastiani formula for vanishing cycles (Section 11.8) led us to interpret vanishing cycles in terms of the algebraic microlocalized modules, by introducing the operator ∂_t^{-1} as a new variable θ. Since, by definition, θ acts in an invertible way on the microlocalized module, we could try to apply a residue formula with kernel $|\theta|^{2s}$ in order to define $\phi_{t,1}\mathfrak{s}$. Interpreting the variable $\tau = \theta^{-1}$ as the Fourier dual of the variable t, we will apply a residue formula with the kernel obtained by inverse Fourier transform from $|\theta|^{2(s-1)}$ (a similar shift by -1 having been already observed in Remark 11.8.3). We will use the properties of the functions $I_{\widehat{\chi}}(t,s)$ and $\widehat{I}_\chi(\tau,s)$ introduced in Exercises 7.21 and 7.22, in order to extend Definition 7.3.15 and the properties of $\phi_{t,1}\mathfrak{s}$ in dimension one.

12.5.22. The function $s\mapsto \big\langle \mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle$**.** We take up the setting and notation of Proposition 12.5.4 with right $\mathcal{D}_X$-modules $\mathcal{M}',\mathcal{M}''$. The properties of the functions $I_{\widehat{\chi},k,k}$ obtained in Exercise 7.21 enable us to apply arguments similar to those of Proposition 12.5.4 and Corollary 12.5.8 to obtain that, for $x_o\in X$, for local sections $m'\in V_0\mathcal{M}'_{x_o}$ and $m''\in V_0\mathcal{M}''_{x_o}$, and for any test function on $\mathrm{nb}(X,x_o)$, the function

$$s\longmapsto \big\langle \mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle$$

extends as a meromorphic function on the plane $\mathbb{C}$ with possible poles contained in $\mathbb{R}_{\leqslant 0}$. The correspondence

$$\eta_o\longmapsto \operatorname{Res}_{s=0}\big\langle \mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle,$$

for any test function η with $\eta_{|H}=\eta_o$, well-defines a current on H, which only depends on the classes $[m'],[m'']$ in $\mathrm{gr}_0^V\mathcal{M}',\mathrm{gr}_0^V\mathcal{M}''$. There is a left analogue, as in Remark 12.5.9.

We can now mimic Definition 12.5.10.

12.5.23. Definition **(V-grading of a sesquilinear pairing, continued)**

(1) (Left case) The sesquilinear pairing

$$\mathrm{gr}_V^{-1}\mathfrak{s}:\mathrm{gr}_V^{-1}\mathcal{M}'\otimes\overline{\mathrm{gr}_V^{-1}\mathcal{M}''}\longrightarrow \mathfrak{Db}_H$$

is well-defined by the formula

$$\big\langle \eta_o, \mathrm{gr}_V^{-1}\mathfrak{s}([m'],\overline{[m'']})\big\rangle := \operatorname{Res}_{s=0}\big\langle I_{\widehat{\chi}}(g,s)\eta, \mathfrak{s}(m',\overline{m''})\big\rangle, \tag{12.5.23 $*$}$$

where m',m'' are local liftings of $[m'],[m'']$ and η is any test form of maximal degree such that $\eta=\widetilde{\eta}_o\wedge\frac{\mathrm{i}}{2\pi}(\mathrm{d}g\wedge\mathrm{d}\overline{g})$ with $\widetilde{\eta}_{o|H}=\eta_o$.

(2) (Right case) The sesquilinear pairing

$$\mathrm{gr}_0^V(\mathfrak{s}):\mathrm{gr}_0^V\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathrm{gr}_0^V\mathcal{M}''}\longrightarrow \mathfrak{C}_H$$

is well-defined by the formula

$$([m'],\overline{[m'']})\longmapsto \Big[\eta_o\mapsto \operatorname{Res}_{s=0}\big\langle \mathfrak{s}(m',\overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle\Big], \tag{12.5.23 $**$}$$

where m',m'' are local liftings of $[m'],[m'']$ and η is a test function such that $\eta_{|H}=\eta_o$.

12.5.24. Some properties of $\mathrm{gr}_V^{-1}\mathfrak{s}$ ***and*** $\mathrm{gr}_0^V\mathfrak{s}$. The proof of Lemma 12.5.11 extends to the present case and shows that

$$\mathrm{gr}_0^V\mathfrak{s}^{\mathrm{right}} = (\mathrm{gr}_V^{-1}\mathfrak{s}^{\mathrm{left}})^{\mathrm{right}}.$$

The proofs of Lemmas 12.5.12, 12.5.13 and 12.5.15 also extend to the present case, due to the properties of $I_{\widehat{\chi}}$ given in Exercise 7.22. We conclude that N is self-adjoint for $\mathrm{gr}_V^{-1}\mathfrak{s}$ and $\mathrm{gr}_0^V\mathfrak{s}$, that the functors gr_V^{-1} and gr_0^V on sesquilinear forms commute with Hermitian adjunction and they do not depend on the embedding.

12.5.25. Definition **(Vanishing cycles of a sesquilinear pairing)**

Let $g : X \to \mathbb{C}$ be any holomorphic function and let $\mathcal{M}', \mathcal{M}''$ be $\mathbb{R}$-specializable. If $\mathfrak{s}$ is a sesquilinear pairing between them, we set

$$\phi_{g,1}\mathfrak{s} := \mathrm{gr}_V^{-1}({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}) \quad \text{resp. } \mathrm{gr}_0^V({}_{\mathrm{T}}\iota_{g*}\mathfrak{s}),$$

which is a sesquilinear pairing between $\phi_{g,1}\mathcal{M}'$ and $\phi_{g,1}\mathcal{M}''$.

12.5.26. Remark **(Properties of** $\phi_{g,1}\mathfrak{s}$**).** The properties of Remark 12.5.17 extend to similar properties for $\phi_{g,1}\mathfrak{s}$. In particular, 12.5.17(4) reads, for $m'\otimes 1 \in V_0({}_{\mathrm{D}}\iota_*\mathcal{M}')$ and $m''\otimes 1 \in V_0({}_{\mathrm{D}}\iota_*\mathcal{M}'')$,

$$\big\langle \phi_{g,1}\mathfrak{s}([m'\otimes 1], \overline{[m''\otimes 1]}), \eta\big\rangle = \mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), I_{\widehat{\chi}}(g,s)\eta\big\rangle. \tag{12.5.26 $*$}$$

12.5.27. Proposition **(Behaviour with respect to** can **and** var**)**

The following equalities holds for $[m']$ *in* $\psi_{g,1}\mathcal{M}'$ *and* $[m''] \in \phi_{g,1}\mathcal{M}''$*, resp.* $[m']$ *in* $\phi_{g,1}\mathcal{M}'$ *and* $[m''] \in \psi_{g,1}\mathcal{M}''$*:*

$$(\phi_{g,1}\mathfrak{s})(\mathrm{can}[m'], \overline{[m'']}) = -(\psi_{g,1}\mathfrak{s})([m'], \overline{\mathrm{var}[m'']})$$

resp.

$$(\phi_{g,1}\mathfrak{s})([m'], \overline{\mathrm{can}[m'']}) = -(\psi_{g,1}\mathfrak{s})(\mathrm{var}[m'], \overline{[m'']}).$$

Proof. Let us show the first equality in the right setting for example. We choose a test function η on $X\times\mathbb{C}$ of the form $\eta = \eta_o\chi(t)$, where η_o is C^∞ with compact support on X, χ is a cut-off function near $t=0$. We need to show

$$\begin{multline} -\mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), \partial_t\big(I_{\widehat{\chi}}(t,s)\chi(t)\big)\eta_o(x)\big\rangle \\ = \mathrm{Res}_{s=-1}\big\langle \mathfrak{s}(m', \overline{m''}), \bar t|t|^{2s}\chi(t)\eta_o(x)\big\rangle, \end{multline} \tag{12.5.28}$$

where $m' \in V_{-1}({}_{\mathrm{D}}\iota_{g*}\mathcal{M}')_{(x_o,0)}$ and $m'' \in V_0({}_{\mathrm{D}}\iota_{g*}\mathcal{M}'')_{(x_o,0)}$ are respective liftings of $[m']$ and $[m'']$, and we have written $\mathfrak{s}$ instead of ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_{g*}\mathfrak{s}$. Let us consider the left-hand side. By Exercise 7.21(6), the function $s \mapsto \big\langle \mathfrak{s}(m', \overline{m''}), I_{\widehat{\chi}}(t,s)\partial_t(\chi(t))\eta_o(x)\big\rangle$ is holomorphic. The left-hand side is thus equal to

$$-\mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), \partial_t(I_{\widehat{\chi}}(t,s))\chi(t)\eta_o(x)\big\rangle$$

and, using Exercise 7.21(7) and arguing for $\mathrm{Re}\, s > 1$ first, it is equal to

$$\mathrm{Res}_{s=0}\big\langle \mathfrak{s}(m', \overline{m''}), I_{\widehat{\chi},-1,0}(t,s)\chi(t)\eta_o(x)\big\rangle. \tag{12.5.29}$$

Let us denote by T the one-variable current with compact support (by definition of χ)

$$\mathcal{C}^\infty(\mathbb{C}) \ni \varphi(t) \longmapsto \langle \mathfrak{s}(m', \overline{m''}), \varphi(t)\chi(t)\eta_o\rangle.$$

Its Fourier transform $\mathcal{F}T := \langle T, e^{\overline{t\tau}-t\tau}\rangle$ is a C^∞ function of τ, which has moderate growth, as well as all its derivatives, when $\tau \to \infty$. The function whose residue is taken in (12.5.29) is then written as

$$\int \tau^{-1}\,|\tau|^{-2(s+1)}\,\widehat{\chi}(\tau)\mathcal{F}T(\tau)\,\tfrac{\mathrm{i}}{2\pi}(d\tau\wedge \mathrm{d}\overline{\tau}). \tag{12.5.30}$$

On the other hand, up to replacing χ with χ^2 in (12.5.28), which does not change the residue, as previously remarked, the function in the RHS of (12.5.28) is

$$\begin{aligned}\langle T, t|t|^{2s}\chi(t)\rangle &= \big\langle \mathcal{F}T\,\tfrac{\mathrm{i}}{2\pi}(d\tau\wedge\mathrm{d}\overline{\tau}), \mathcal{F}^{-1}(t|t|^{2s}\chi(t))\big\rangle\\ &= \int \widehat{I}_{\chi,1,0}(\tau,s)\cdot\mathcal{F}T(\tau)\,\tfrac{\mathrm{i}}{2\pi}(d\tau\wedge\mathrm{d}\overline{\tau}),\end{aligned} \tag{12.5.31}$$

where we have set

$$\widehat{I}_{\chi,k,\ell}(\tau,s) := \mathcal{F}^{-1}(|t|^{2s}t^k\overline{t}^\ell\chi(t)),$$

and the properties we need for the function $\widehat{I}_{\chi,1,0}(\tau,s)$ are made precise in Exercise 7.22. Using the function $\widehat{\chi}(\tau)$ as above, we conclude from Exercise 7.22(7) that the integral

$$\int \mathcal{F}T(\tau)\cdot\widehat{I}_{\chi,1,0}(\tau,s)\cdot(1-\widehat{\chi}(\tau))\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}\tau\wedge\mathrm{d}\overline{\tau} \tag{12.5.32}$$

is holomorphic with respect to s for $\operatorname{Re} s > -3/2$. It can thus be neglected when computing the residue at $s=-1$. The question reduces therefore to the comparison between $\widehat{I}_{\chi,1,0}(\tau,s)$ and $\tau^{-1}\,|\tau|^{-2(s+1)}$ when $\tau\to\infty$.

Let us set $\widehat{J}_{\chi,1,0}(\tau,s) = \tau\,|\tau|^{2(s+1)}\,\widehat{I}_{\chi,1,0}(\tau,s)$. Then, by (7.7.0 $**$), we have

$$\tau\,\frac{\partial\widehat{J}_{\chi,1,0}}{\partial\tau} = -\widehat{J}_{\partial\chi/\partial t,2,0},\quad \overline{\tau}\,\frac{\partial\widehat{J}_{\chi,1,0}}{\partial\overline{\tau}} = -\widehat{J}_{\partial\chi/\partial\overline{t},1,1},$$

and both functions $\widehat{J}_{\partial\chi/\partial t,2,0}$ and $\widehat{J}_{\partial\chi/\partial\overline{t},1,1}$ can be extended as C^∞ functions, infinitely flat at $\tau=\infty$ and holomorphic with respect to $s\in\mathbb{C}$.

12.5.33. Lemma. *For s in the strip $\operatorname{Re}(s+1)\in(-1,-1/4)$, the function $\tau\mapsto\widehat{J}_{\chi,1,0}(\tau,s)$ satisfies*

$$\lim_{\tau\to\infty}\widehat{J}_{\chi,1,0}(\tau,s) = -\frac{\Gamma(s+2)}{\Gamma(-s)}.$$

Proof. We can assume that χ is a C^∞ function of $|t|^2$. For simplicity, we assume that $\chi\equiv 1$ for $|t|\leqslant 1$. Then the limit of $\widehat{J}_{\chi,1,0}$ is also equal to the limit of the integral

$$J(\tau,s) = \int_{|t|\leqslant 1} e^{-\overline{t\tau}+t\tau}\,t\tau\,|t\tau|^{2(s+1)}\,\tfrac{\mathrm{i}}{2\pi}\,\frac{\mathrm{d}t}{t}\wedge\frac{\mathrm{d}\overline{t}}{\overline{t}}.$$

By a simple change of variables, we have

$$J(\tau,s) = \int_{|u|\leqslant|\tau|} e^{2i\operatorname{Im}u}u\,|u|^{2s}\,\tfrac{\mathrm{i}}{2\pi}\,\mathrm{d}u\wedge\mathrm{d}\overline{u}.$$

Using the Bessel function $J_{\pm 1}(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{-ix\sin y} e^{\pm iy} \mathrm{d}y$, we can write

$$\begin{aligned} J(\tau, s) &= 2\int_{\rho \leqslant |\tau|} J_{-1}(2\rho)\rho^{2(s+1)}\mathrm{d}\rho \\ &= -2^{-2(s+1)}\int_{\rho\leqslant 2|\tau|} J_1(\rho)\rho^{2(s+1)}\mathrm{d}\rho, \quad \text{as } J_1 = -J_{-1}. \end{aligned}$$

For $\mathrm{Re}(s+1) \in (-1, -1/4)$, the limit when $|\tau|$ of the previous integral is equal to $2^{2(s+1)}\Gamma(s+2)/\Gamma(-s)$ (see [**Wat22**, §13.24, p. 391]). $\square$

Let us set

$$\widehat{K}_\chi(\tau, s) = -\int_0^1 \big[\widehat{J}_{\partial\chi/\partial t,1,0}(\lambda\tau, s) + \widehat{J}_{\partial\chi/\partial\bar{t},0,1}(\lambda\tau, s)\big]\mathrm{d}\lambda.$$

Then $\widehat{K}_\chi$ is also C^∞, infinitely flat at $\tau = \infty$ and holomorphic with respect to $s \in \mathbb{C}$. According to Lemma 12.5.33, we can write, on the strip $\mathrm{Re}(s+1) \in (-1/2, -1/4)$,

$$\widehat{I}_{\chi,1,0}(\tau, s) = -\tau^{-1}\,|\tau|^{-2(s+1)}\,\frac{\Gamma(s+2)}{\Gamma(-s)} + K_\chi(\tau, s) \tag{12.5.34}$$

where $K_\chi(\tau, s) = -\tau^{-1}\,|\tau|^{-2(s+1)}\,\widehat{K}_\chi(\tau, s)$ is C^∞ on $\mathbb{C}\times\mathbb{C}$, infinitely flat at $\tau = \infty$ and holomorphic with respect to s. For any $p \geqslant 0$, let us apply $(\partial_\tau\partial_{\overline{\tau}})^p$ to the previous equality restricted to $\tau \neq 0$ (where both sides are C^∞ in τ and holomorphic with respect to s; preferably, multiply both sides by $\widehat{\chi}(\tau)$), to get, for s in the same strip,

$$\widehat{I}_{\chi,1,0}(\tau, s+p) = -\tau^{-1}\,|\tau|^{-2(s+p+1)}\,\frac{\Gamma(s+p+2)}{\Gamma(-s-p)} + (\partial_\tau\partial_{\overline{\tau}})^p K_\chi(\tau, s)$$

where the last term remains infinitely flat at $\tau = \infty$. It follows that (12.5.34) remains true on any strip $\mathrm{Re}(s+1) \in (p-1/2, p-1/4)$ with $p \geqslant 0$ and a function $K_\chi^{(p)}$ instead of K_χ.

Choose p such that the two the meromorphic functions considered in (12.5.28) are holomorphic on the strip $\mathrm{Re}(s+1) \in (p-1/2, p-1/4)$. The difference between $\Gamma(s+2)/\Gamma(-s)$ times the function in the LHS and the function in the RHS coincides, on this strip, with the restriction of a holomorphic function defined on the half-plane $\{s \mid \mathrm{Re}\, s > -3/2\}$ (taking into account (12.5.32) and $K_\chi^{(p)}$). It is then equal to it on this whole half-plane, hence has residue 0 at $s = -1$. $\square$

12.5.35. Corollary.

(1) *Assume that $\mathcal{M}', \mathcal{M}''$ are middle extensions along $g^{-1}(0)$ (see Definition 9.7.3), so that $\phi_{g,1}\mathcal{M} \simeq \mathrm{Im}[\mathrm{N} : \psi_{g,1}\mathcal{M} \to \psi_{g,1}\mathcal{M}]$ for $\mathcal{M} = \mathcal{M}', \mathcal{M}''$. Then, for $[\mu'] \in \phi_{g,1}\mathcal{M}'$ and $[\mu''] \in \phi_{g,1}\mathcal{M}''$, and $[\mu'] = \mathrm{N}[m']$, $[\mu''] = \mathrm{N}[m'']$, we have*

$$\phi_{g,1}\mathfrak{s}([\mu'], \overline{[\mu'']}) = -\psi_{g,1}\mathfrak{s}([m'], \overline{\mathrm{N}[m'']}) = -\psi_{g,1}\mathfrak{s}(\mathrm{N}[m'], \overline{[m'']}).$$

(2) *Assume that $\mathcal{M}', \mathcal{M}''$ are supported on $g^{-1}(0)$, so that $\mathcal{M} = \phi_{g,1}\mathcal{M}$ for $\mathcal{M} = \mathcal{M}', \mathcal{M}''$. Then $\mathfrak{s} = \phi_{g,1}\mathfrak{s}$.*

Proof.

(1) The second equality is due to Remark 12.5.17(1). The identification of $\phi_{g,1}\mathcal{M}$ with $\mathrm{Im}\,\mathrm{N}$ implies precisely that $[\mu'] = \mathrm{can}[m']$ and $[\mu''] = \mathrm{can}[m'']$. Then, according to Proposition 12.5.27,

$$\begin{aligned}\phi_{g,1}\mathfrak{s}([\mu'],\overline{[\mu'']}) &= \phi_{g,1}\mathfrak{s}(\mathrm{can}[m'],\overline{\mathrm{can}[m'']})\\ &= -\psi_{g,1}\mathfrak{s}([m'],\overline{\mathrm{var}\,\mathrm{can}[m'']}) = -\psi_{g,1}\mathfrak{s}([m'],\overline{\mathrm{N}[m'']}).\end{aligned}$$

(2) We can assume that $X = H\times\mathbb{C}$ and g is the projection $(x,t)\mapsto t$ (x is the variable in H) and we set $\iota : H\times\{0\}\hookrightarrow X$. We can assume that we are given $\mathcal{D}_H$-modules $\mathcal{M}'_o, \mathcal{M}''_o$ and a sesquilinear pairing between them. For local sections m'_o, m''_o, we have to identify $\mathrm{gr}^V_{-1}({}_{\mathrm{T}}\iota_*\mathfrak{s}_o)([m'_o\otimes 1],\overline{([m''_o\otimes 1]})$ with $\mathfrak{s}_o([m'_o],\overline{([m''_o]})$. Setting $\eta = \eta_o(x)\chi(t)$, we have, since $\chi(0)=1$,

$$\big\langle({}_{\mathrm{T}}\iota_*\mathfrak{s}_o)((m'_o\otimes 1),\overline{((m''_o\otimes 1)}),\eta_o I_{\widehat{\chi}}(t,s)\chi(t)\big\rangle = \big\langle\mathfrak{s}_o(m'_o,\overline{(m''_o}),\eta_o\big\rangle\cdot I_{\widehat{\chi}}(0,s).$$

The assertion follows since $\mathrm{Res}_{s=0}\, I_{\widehat{\chi}}(0,s) = 1$ (see Exercise 7.21(2)). $\square$

***12.5.36. Remark* (Kashiwara's equivalence and $\mathrm{gr}^V_0\,\mathfrak{s}$).** Let $H \xhookrightarrow{\iota} X$ be the inclusion of a smooth hypersurface (not necessarily defined by a global equation). If $\mathcal{M} = \mathcal{M}', \mathcal{M}''$ are coherent $\mathcal{D}_H$-modules, we have $V_0({}_{\mathrm{D}}\iota_*\mathcal{M}) = \mathcal{M}\otimes\mathbf{1}$ (notation of Section 12.4.b) and, for a sesquilinear pairing $\mathfrak{s}_H$ between $\mathcal{M}'$ and $\mathcal{M}''$, we recover $\mathfrak{s}_H$ from ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_*\mathfrak{s}_H$ by Formula (12.5.23 $**$) for any local equation of H, which therefore does not depend on the choice of such a local equation. We then denote $\mathfrak{s}_H = \mathrm{gr}^V_0({}_{\mathrm{T}}\iota_*\mathfrak{s}_H)$.

12.5.e. Localization of a sesquilinear pairing. Let $\mathfrak{s} : \mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''}\to\mathfrak{C}_X$ be a sesquilinear pairing between right $\mathcal{D}_X$-modules, and let D be an effective divisor in X. Recall that localization and dual localization are defined for $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along D and that we have natural morphisms (see Corollaries 11.3.8(2) and 11.4.13(2))

$$\mathcal{M}(!D)\xrightarrow{\mathrm{dloc}}\mathcal{M}\xrightarrow{\mathrm{loc}}\mathcal{M}(*D).$$

According to the results recalled above, $\mathfrak{s}$ defines a moderate sesquilinear pairing by localization:

$$\mathfrak{s}^{\mathrm{mod}\,D} : \mathcal{M}'(*D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(*D)}\longrightarrow\mathfrak{C}_X^{\mathrm{mod}\,D}.$$

Our aim is to refine it as a pairing taking values in $\mathfrak{C}_X$.

12.5.37. Proposition. *Assume that $\mathcal{M}', \mathcal{M}''$ are $\mathbb{R}$-specializable along D. Then $\mathfrak{s}^{\mathrm{mod}\,D}$ naturally induces sesquilinear pairings*

$$\begin{aligned}\mathfrak{s}^{(*D)} &: \mathcal{M}'(*D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(!D)}\longrightarrow\mathfrak{C}_X,\\ \mathfrak{s}^{(!D)} &: \mathcal{M}'(!D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(*D)}\longrightarrow\mathfrak{C}_X.\end{aligned}$$

Moreover, the second one is obtained by adjunction of the first one, that is,

$$\mathfrak{s}^{(*D)} = \big[\mathfrak{s}^{*(!D)}\big]^*.$$

Last, $\mathfrak{s}^{(*D)}$ and $\mathfrak{s}^{(!D)}$ are compatible with $\mathfrak{s}$, in the sense that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}'(*D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(!D)} \arrow[d, phantom] \arrow[rr, "\mathfrak{s}^{(*D)}"] & & \mathfrak{C}_X \arrow[d, equal] \\
\mathcal{M}'\otimes_{\mathbb{C}}\overline{\mathcal{M}''} \arrow[u, "\mathrm{loc}"] \arrow[rr, "\mathfrak{s}"] & & \mathfrak{C}_X \arrow[d, equal] \\
\mathcal{M}'(!D)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(*D)} \arrow[u, "\mathrm{dloc}"] \arrow[rr, "\mathfrak{s}^{(!D)}"] & & \mathfrak{C}_X
\end{tikzcd}$$

that is, for any local sections m', m'' of $\mathcal{M}', \mathcal{M}''$ and $m'_!, m''_!$ of $\mathcal{M}'(!D), \mathcal{M}''(!D)$, the following equalities hold:

$$\mathfrak{s}^{(*D)}(\mathrm{loc}(m'), \overline{m''_!}) = \mathfrak{s}(m', \overline{\mathrm{dloc}(m''_!)}) \quad \textit{and} \quad \mathfrak{s}^{(!D)}(m'_!, \overline{\mathrm{loc}(m'')}) = \mathfrak{s}(\mathrm{dloc}(m'_!), \overline{m''}).$$

***12.5.38. Remark* (Localization for left pairings).** The pairings $\mathfrak{s}^{(*D)}$ and $\mathfrak{s}^{(!D)}$ are defined by side-changing (12.3.3 $*$) from their right analogues.

Proof. The question is local, and we can reduce to the case where $D = H$ is smooth and $X = H\times\mathbb{C}$. The V-filtration is then well-defined for an $\mathbb{R}$-specializable $\mathcal{D}_X$-module. Since the morphisms $\mathcal{M}(!H)\to\mathcal{M}$ and $\mathcal{M}\to\mathcal{M}(*H)$ have kernels and cokernels supported in H, they induce isomorphisms between the $V_{<0}$ of these modules. In particular, the restriction of $\mathfrak{s}$ (hence of $\mathfrak{s}^{\mathrm{mod}\,H}$) to $V_{<0}\mathcal{M}'\otimes_{\mathbb{C}}\overline{V_{<0}\mathcal{M}''}$ takes values in $\mathfrak{C}_X$. We will construct $\mathfrak{s}^{(*H)}$, the case of $\mathfrak{s}^{(!H)}$ being similar.

For every $\ell\geqslant 1$, we first extend $\mathfrak{s}$ as a sesquilinear pairing

$$\mathfrak{s}_\ell : (V_{<0}\mathcal{M}'\cdot t^{-\ell})\otimes_{\mathbb{C}}\overline{V_{<0}\mathcal{M}''}\longrightarrow\mathfrak{C}_X\,.$$

We argue exactly as in the proof of Proposition 12.5.4 by extending, for every test function η on $\mathrm{nb}(x_o)$ and each local section m' of $V_{<0}\mathcal{M}'$ and m'' of $V_{<0}\mathcal{M}''$, the holomorphic function (for $\operatorname{Re}s\gg 0$)

$$s\longmapsto\big\langle\mathfrak{s}(m',\overline{m''})|t|^{2(s-\ell)}\bar{t}^{\ell},\eta\big\rangle$$

as a meromorphic function on $\mathbb{C}$, and by checking that it has no pole at $s=0$ since $m''\in V_{<0}\mathcal{M}''_{x_o}$. Taking the value of this function at $s=0$ gives the desired extension of $\mathfrak{s}$, since $|t|^{2(s-\ell)}\bar{t}^{\ell} = |t|^{2s}t^{-\ell}$. Moreover, one checks that $\mathfrak{s}_\ell$ restricts to $\mathfrak{s}_{\ell-1}$ on $(V_{<0}\mathcal{M}'\cdot t^{-\ell+1})\otimes_{\mathbb{C}}\overline{V_{<0}\mathcal{M}''}$, and thus defines a sesquilinear pairing

$$\mathfrak{s}^{(*H)} : \mathcal{M}'(*H)\otimes_{\mathbb{C}}\overline{V_{<0}\mathcal{M}''}\longrightarrow\mathfrak{C}_X\,.$$

This pairing can be extended in at most one way as a pairing

$$\mathfrak{s}^{(*H)} : \mathcal{M}'(*H)\otimes_{\mathbb{C}}\overline{\mathcal{M}''(!H)}\longrightarrow\mathfrak{C}_X,$$

due to the $\mathcal{D}_{\overline{X}}$-linearity and the equality $\mathcal{M}''(!H) = V_{<0}\mathcal{M}''\otimes_{V_0\mathcal{D}_X}\mathcal{D}_X$. However, since $\mathcal{D}_X$ is not locally free as a $V_0\mathcal{D}_X$-module, the existence of such an extension is not a priori obvious. Such an extension will exist near x_o if, for any finite family (m''_j) of elements of $V_{<0}\mathcal{M}''_{x_o}$, any finite family $(P_j)_j$ of germs of differential operators at x_o, and any $m'\in\mathcal{M}(*H)_{x_o}$, the condition $\sum_j m''_j\otimes P_j = 0$ implies $\sum_j\mathfrak{s}^{(*H)}(m',\overline{m''_j})\cdot\overline{P}_j = 0$.

This holds by definition if all P_j belong to $V_0\mathcal{D}_{X,x_o}$. Therefore, one can reduce to the case where $j=0,\dots,N$ and $P_j=\partial_t^j$.

We argue by induction on N, the case where $N=0$ being clear. We first claim that $m''_N\otimes\partial_t\in V_{<0}\mathcal{M}''(!H)$. Indeed, $\mathcal{M}''(!H)$ has the property that $\partial_t:\mathrm{gr}^V_\alpha\mathcal{M}''(!H)\to\mathrm{gr}^V_{\alpha+1}\mathcal{M}''(!H)$ is an isomorphism if $\alpha=-1$, and on the other hand it is an isomorphism for any other α (this holds for any $\mathbb{R}$-specializable coherent $\mathcal{D}_X$-module). This implies that

$$\partial_t^N: V_{<0}\mathcal{M}''(!H)/V_{<-1}\mathcal{M}''(!H)\longrightarrow V_{<N}\mathcal{M}''(!H)/V_{<N-1}\mathcal{M}''(!H)$$

is an isomorphism. Since

$$m''_N\otimes\partial_t^N=-\sum_{j=0}^{N-1}m''_j\otimes\partial_t^j\in V_{<N-1}\mathcal{M}''(!H)_{x_o},$$

we conclude that $m''_N\otimes 1\in V_{<-1}\mathcal{M}''(!H)_{x_o}$, hence the assertion.

By induction, we thus have

$$\sum_{j=0}^{N-1}\mathfrak{s}^{(*H)}(m',\overline{m''_j})\cdot\partial_{\overline{t}}^j+\mathfrak{s}^{(*H)}(m',\overline{m''_N\otimes\partial_t})\cdot\partial_{\overline{t}}^{N-1}=0\in\mathfrak{C}_X\,.$$

It is therefore enough to check that, for $m'\in\mathcal{M}'(*H)_{x_o}$ and $m''\in V_{<-1}\mathcal{M}''_{x_o}$, we have

$$\mathfrak{s}^{(*H)}(m',\overline{m''\otimes\partial_t})=\mathfrak{s}^{(*H)}(m',\overline{m''})\cdot\partial_{\overline{t}}.$$

Notice now that $t:V_{<0}\mathcal{M}''_{x_o}\to V_{<-1}\mathcal{M}''_{x_o}$ is an isomorphism, hence $m''=n''t$ for some $n''\in V_{<0}\mathcal{M}''_{x_o}$. We thus have

$$\begin{aligned}\mathfrak{s}^{(*H)}(m',\overline{m''\otimes\partial_t})&=\mathfrak{s}^{(*H)}(m',\overline{n''t\otimes\partial_t})=\mathfrak{s}^{(*H)}(m',\overline{n''\otimes t\partial_t})\\&=\mathfrak{s}^{(*H)}(m',\overline{n''t\partial_t\otimes 1})=\mathfrak{s}^{(*H)}(m',\overline{n''\otimes 1})\cdot\overline{t}\partial_{\overline{t}}\\&=\mathfrak{s}^{(*H)}(m',\overline{n''t\otimes 1})\cdot\partial_{\overline{t}}=\mathfrak{s}^{(*H)}(m',\overline{m''\otimes 1})\cdot\partial_{\overline{t}}.\end{aligned}$$

The remaining assertions are straightforward, since $(\mathfrak{s}^*)^{\mathrm{mod}\,H}=(\mathfrak{s}^{\mathrm{mod}\,H})^*$. □

12.6. Compatibility between functors on sesquilinear pairings

12.6.a. Pushforward and specialization of sesquilinear pairings

Let $f:X\to Y$ be a holomorphic map between complex manifolds and let $h:Y\to\mathbb{C}$ be a holomorphic function. Set $g=h\circ f$. Let $\mathcal{M}',\mathcal{M}''$ be right $\mathcal{D}_X$-modules which are $\mathbb{R}$-specializable along (g). Let $\mathfrak{s}:\mathcal{M}'\otimes\overline{\mathcal{M}''}\to\mathfrak{C}_X$ be a sesquilinear pairing. Assume that f is proper on the support of $\mathcal{M}',\mathcal{M}''$. Recall that Theorem 9.8.8 implies:

- for every $k\in\mathbb{Z}$, ${}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ is $\mathbb{R}$-specializable along (h),
- for every $\alpha\in\mathbb{R}$, the natural morphism ${}_{\mathrm{D}}f_*^{(k)}V_\alpha\mathcal{M}\to{}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ is injective and its image is equal to $V_\alpha({}_{\mathrm{D}}f_*^{(k)}\mathcal{M})$.

12.6.1. Theorem. *With respect to the previous natural morphism, we have*

$${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\psi_{g,\lambda}\mathfrak{s}=\psi_{h,\lambda}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}),\quad {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\phi_{g,1}\mathfrak{s}=\phi_{h,1}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s})$$

Proof. We start with the case of a map $f\times\mathrm{Id}:X\times\mathbb{C}\to Y\times\mathbb{C}$ and we take for the function $h:Y\times\mathbb{C}\to\mathbb{C}$ the second projection. We assume that $\mathcal{M}',\mathcal{M}''$ are right $\mathcal{D}_{X\times\mathbb{C}}$-modules.

12.6.2. Lemma. *With these assumptions, for every $\alpha\in[-1,0]$ and $k\in\mathbb{Z}$,*

$$_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}(\mathrm{gr}_\alpha^V\mathfrak{s})=\mathrm{gr}_\alpha^V\big({}_{\mathrm{D},\overline{\mathrm{D}}}(f\times\mathrm{Id})_*^{(k,-k)}\mathfrak{s}\big).$$

Proof. Set $\beta=-\alpha-1$ and let

$$\begin{aligned}m'^k_\infty&\in\Gamma(U,f_*(\mathcal{E}^{n+1+k}_{X\times\mathbb{C}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}V^\beta\mathcal{M}'^{\mathrm{left}})),\\ m''^{-k}_\infty&\in\Gamma(U,f_*(\mathcal{E}^{n+1-k}_{X\times\mathbb{C}}\otimes_{\mathcal{O}_{X\times\mathbb{C}}}V^\beta\mathcal{M}''^{\mathrm{left}})).\end{aligned}$$

The cohomology classes $[m'^k_\infty]$ and $[m''^{-k}_\infty]$ can be regarded as sections of the modules $V_\alpha({}_{\mathrm{D}}f_*^{(k)}\mathcal{M}')\otimes_{\mathcal{O}_Y}\mathcal{C}^\infty_Y$ and $V_\alpha({}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}'')\otimes_{\mathcal{O}_Y}\mathcal{C}^\infty_Y$ respectively, according to the result recalled above. We can then compute with these classes. Let us also denote by $[m]_\alpha$ the class of $m\in V_\alpha$ modulo $V_{<\alpha}$. Let us assume $\alpha\in[-1,0)$ (the case of $\alpha=0$ is similar by using the function $I_{\widehat{\chi}}$). We have, for $\eta\in\mathcal{C}^\infty_Y(U)$,

$$\begin{aligned}\Big\langle\mathrm{gr}_\alpha^V\big({}_{\mathrm{D},\overline{\mathrm{D}}}(f\times\mathrm{Id})_*^{(k,-k)}\mathfrak{s}\big)([m'^k_\infty]_\alpha,\overline{[m''^{-k}_\infty]_\alpha}),\eta(y)\Big\rangle\qquad&\\ =\mathrm{Res}_{s=\alpha}\Big\langle\big({}_{\mathrm{D},\overline{\mathrm{D}}}(f\times\mathrm{Id})_*^{(k,-k)}\mathfrak{s}\big)([m'^k_\infty],\overline{[m''^{-k}_\infty]}),\eta(y)|t|^{2s}\chi(t)\Big\rangle&\\ =\mathrm{Res}_{s=\alpha}\Big\langle\mathfrak{s}(m'^k_\infty,\overline{m''^{-k}_\infty}),\eta\circ f(x)|t|^{2s}\chi(t)\Big\rangle&\\ =\Big\langle\mathrm{gr}_\alpha^V\mathfrak{s}((m'^k_\infty)_\alpha,\overline{(m''^{-k}_\infty)_\alpha}),\eta\circ f(x)\Big\rangle&\\ =\Big\langle\big({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathrm{gr}_\alpha^V\mathfrak{s}\big)\big([(m'^k_\infty)_\alpha],\overline{[(m''^{-k}_\infty)_\alpha]}\big),\eta(y)\Big\rangle&,\end{aligned}$$

and we obtain the desired equality since, as recalled, $[m^{\pm k}_\infty]_\alpha=[(m^{\pm k}_\infty)_\alpha]$ in $\Gamma(U,\mathrm{gr}_\alpha^V[{}_{\mathrm{D}}(f\times\mathrm{Id})_*^{(\pm k)}\mathcal{M}])=\Gamma(U,{}_{\mathrm{D}}f_*^{(\pm k)}\mathrm{gr}_\alpha^V\mathcal{M})$. □

We can now end the proof of Theorem 12.6.1. We have

$$\begin{aligned}{}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\psi_{g,\lambda}\mathfrak{s}&={}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathrm{gr}_\alpha^V({}_{\mathrm{T}}\iota^0_{g*}\mathfrak{s})\quad\text{(see (12.5.16 $*$))}\\ &=\mathrm{gr}_\alpha^V\big({}_{\mathrm{D},\overline{\mathrm{D}}}(f\times\mathrm{Id})_*^{(k,-k)}({}_{\mathrm{T}}\iota^0_{g*}\mathfrak{s})\big)\quad\text{(Lemma 12.6.2)}\\ &=\mathrm{gr}_\alpha^V\big({}_{\mathrm{D},\overline{\mathrm{D}}}\iota^0_{h*}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s})\big)\quad\text{(after (12.4.14 $*$))}\\ &=\psi_{h,\lambda}({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s}).\end{aligned}$$
□

12.6.b. Pushforward and localization of sesquilinear pairings

Let $f:X\to Y$ be a holomorphic map between complex manifolds, let D' be an effective divisor in X' and set $D=f^*(D')$. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along D. Then we have natural morphisms (see Corollary 11.7.1(1))

$$({}_{\mathrm{D}}f_*^{(k)}\mathcal{M})(!D')\longrightarrow{}_{\mathrm{D}}f_*^{(k)}\mathcal{M}(!D))\quad\text{and}\quad{}_{\mathrm{D}}f_*^{(k)}(\mathcal{M}(*D))\longrightarrow({}_{\mathrm{D}}f_*^{(k)}\mathcal{M})(*D').$$

12.6.3. Proposition. *With respect to the previous natural morphism, the sesquilinear pairings ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}(\mathfrak{s}^{(\star D)})$ and $({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s})^{(\star D')}$ coincide ($\star=*,!$).*

Proof. One first considers the naive localization, and recall that $\mathfrak{C}_X^{\mathrm{mod}\,D} = \mathfrak{C}_X(*D)$. One then easily checks that ${}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}(\mathfrak{s}(*D)) = ({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s})(*D')$ with values in $\mathfrak{C}_{X'}(*D')$. By definition and from the commutativity above, $({}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)}\mathfrak{s})^{(*D')}$ is the restriction of the latter to ${}_{\mathrm{D}}f_*^{(k)}(\mathcal{M}(*D)) \otimes \overline{{}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}(!D)}$, and the assertion follows for $\star = *$. The case $\star = !$ is similar. $\square$

12.7. The category $\widetilde{\mathcal{D}}$-Triples and its functors

We now come back to the filtered setting, and consider $\widetilde{\mathcal{D}}_X$-modules, with $\widetilde{\mathcal{D}}_X = R_F\mathcal{D}_X$. Given a $\widetilde{\mathcal{D}}_X$-module $\widetilde{\mathcal{M}}$, we denote by $\mathcal{M} := \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ the associated $\mathcal{D}_X$-module.

12.7.a. The category of $\widetilde{\mathcal{D}}$-triples. We extend to arbitrary dimensions the constructions of Section 7.4.a. The category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ is an abelian category, and possesses the basic functors we need for studying pure Hodge modules. For example, the pushforward functor will be denoted by ${}_{\mathrm{T}}f_*$, etc.

12.7.1. Definition. The category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ has

- objects consisting of triples $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$, where $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{D}}_X$-modules and $\mathfrak{s}$ is a sesquilinear pairing between $\mathcal{M}'$ and $\mathcal{M}''$ (with values in $\mathfrak{Db}_X$ in the left case, and in $\mathfrak{C}_X$ in the right case),
- morphisms $\varphi : \widetilde{\mathcal{T}}_1 \to \widetilde{\mathcal{T}}_2$ consisting of pairs $\varphi = (\varphi', \varphi'')$, where $\varphi' : \widetilde{\mathcal{M}}'_1 \to \widetilde{\mathcal{M}}'_2$ and $\varphi'' : \widetilde{\mathcal{M}}''_2 \to \widetilde{\mathcal{M}}''_1$ are $\widetilde{\mathcal{D}}_X$-linear, such that for all local sections m'_1 of $\mathcal{M}'_1$ and m''_1 of $\mathcal{M}''_2$,

$$\mathfrak{s}_1(m'_1, \overline{\varphi''(m''_2)}) = \mathfrak{s}_2(\varphi'(m'_1), \overline{m''_2}). \tag{12.7.1 $*$}$$

In particular, $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ is an abelian subcategory of $\mathsf{Mod}(\widetilde{\mathcal{D}}_X) \times \mathsf{Mod}(\widetilde{\mathcal{D}}_X)^{\mathrm{op}}$.

We say that an object $\widetilde{\mathcal{T}}$ of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ is *coherent*, resp. *strictly* $\mathbb{R}$*-specializable*, resp. *smooth*, if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{D}}_X$-coherent, resp. strictly $\mathbb{R}$-specializable, resp. $\widetilde{\mathcal{O}}_X$-locally free of finite rank. The *support* of $\widetilde{\mathcal{T}}$ is $\operatorname{Supp}\widetilde{\mathcal{T}} = \operatorname{Supp}\widetilde{\mathcal{M}}' \cup \operatorname{Supp}\widetilde{\mathcal{M}}''$.

12.7.2. Complexes in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$**.** A complex $(\widetilde{\mathcal{T}}^\bullet, d)$ consists of a graded object $\widetilde{\mathcal{T}}^\bullet = \bigoplus_k \widetilde{\mathcal{T}}^k$ of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ together with a differential $d : \widetilde{\mathcal{T}}^k \to \widetilde{\mathcal{T}}^{k+1}$ such that $d^2 = 0$. We write $\widetilde{\mathcal{T}}^k = (\widetilde{\mathcal{M}}'^k, \widetilde{\mathcal{M}}''^{-k}, \mathfrak{s}_k)$ and $d = (d', d'')$, so that $(\widetilde{\mathcal{M}}'^\bullet, d')$ and $(\widetilde{\mathcal{M}}''^\bullet, d'')$ are complexes and $\mathfrak{s}_k : \mathcal{M}'^k \otimes \overline{\mathcal{M}''^{-k}} \to \mathfrak{Db}_X$ (left case) satisfies $\mathfrak{s}_k(d'm'_{k-1}, \overline{m''_{-k}}) = \mathfrak{s}(m'_{k-1}, \overline{d''m''_{-k}})$.

12.7.3. Side-changing in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$**.** Let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be a *left* $\widetilde{\mathcal{D}}_X$-triple. We set

$$\widetilde{\mathcal{T}}^{\mathrm{right}} := (\widetilde{\mathcal{M}}'^{\mathrm{right}}, \widetilde{\mathcal{M}}''^{\mathrm{right}}, \mathfrak{s}^{\mathrm{right}}),$$

where $\mathfrak{s}^{\mathrm{right}}$ is defined by (12.3.3 $*$). The right-to-left side changing is defined correspondingly, so that the composition of both is the identity.

12.7.4. Hermitian duality. The Hermitian dual of an object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ is the object $\widetilde{\mathcal{T}}^* := (\widetilde{\mathcal{M}}'', \widetilde{\mathcal{M}}', \mathfrak{s}^*)$, where $\mathfrak{s}^*$ is the Hermitian adjoint sesquilinear pairing (see Definitions 12.3.1(2) and 12.3.2(2)). The Hermitian adjoint of a morphism $\varphi = (\varphi', \varphi'')$ is the morphism $\varphi^* := (\varphi'', \varphi')$. We clearly have $\widetilde{\mathcal{T}}^{**} = \widetilde{\mathcal{T}}$ and $\varphi^{**} = \varphi$.

12.7.5. Side-changing and Hermitian duality in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$. With the previous definitions, Hermitian duality commutes with side-changing, because of (12.3.3 $**$).

12.7.6. Hermitian dual of a graded triple. Let $\widetilde{\mathcal{T}}^\bullet = \bigoplus_k \widetilde{\mathcal{T}}^k$ be a graded object in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$. We write $\widetilde{\mathcal{T}}^k$ as $(\widetilde{\mathcal{M}}'^k, \widetilde{\mathcal{M}}''^{-k}, \mathfrak{s}_k)$. The Hermitian dual object is then

$$(\widetilde{\mathcal{T}}^*)^\bullet = \bigoplus_k (\widetilde{\mathcal{T}}^*)^k := \bigoplus_k (\widetilde{\mathcal{T}}^{-k})^*.$$

12.7.7. Tate twist. The definition of the twist follows the general definition 5.2.2(7), and the Tate twist is as in Notation 5.2.3:

$$\widetilde{\mathcal{T}}(\ell) = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})(\ell) := (\widetilde{\mathcal{M}}'(\ell), \widetilde{\mathcal{M}}''(-\ell), \mathfrak{s}).$$

12.7.8. Pre-polarization and Hermitian pairs. A pre-polarization of weight w of $\widetilde{\mathcal{T}}$ is an isomorphism $\mathrm{S} : \widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}^*(-w)$ which is Hermitian. Tate twist acts as $(\widetilde{\mathcal{T}}, \mathrm{S})(\ell) = (\widetilde{\mathcal{T}}(\ell), (-1)^\ell \mathrm{S})$. Any pre-polarized triple of weight w is isomorphic to a triple $(\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'(w), \mathcal{S})$ with pre-polarization $(\mathrm{Id}, \mathrm{Id})$. Hence, giving a pre-polarized triple $(\widetilde{\mathcal{T}}, \mathrm{S})$ of weight w is equivalent to giving the Hermitian pair $(\widetilde{\mathcal{M}}', \mathcal{S})$ and the weight w. Tate twist acts as

$$(\widetilde{\mathcal{M}}', \mathcal{S}, w)(\ell) = (\widetilde{\mathcal{M}}'(\ell), (-1)^\ell \mathcal{S}, w - 2\ell).$$

12.7.9. Two basic examples. Let us keep the notation of Examples 5.4.4 and 12.3.5.

(1) (Left case) The triple ${}_{\mathrm{H}}\widetilde{\mathcal{O}}_X = (\widetilde{\mathcal{O}}_X, \widetilde{\mathcal{O}}_X(n), \mathfrak{s}_n)$ is the smooth left triple with $\mathfrak{s}_n(1,1) = 1$. It satisfies $({}_{\mathrm{H}}\widetilde{\mathcal{O}}_X)^*(-n) = {}_{\mathrm{H}}\widetilde{\mathcal{O}}_X$.

(2) (Right case) The triple ${}_{\mathrm{H}}\widetilde{\omega}_X = (\widetilde{\omega}_X, \widetilde{\omega}_X(n), \mathfrak{s}_n)$ is the smooth right triple with $\mathfrak{s}_n(\omega', \overline{\omega''}) = \mathrm{Sgn}(n)(\omega' \wedge \overline{\omega''})$.

In both cases we have $\mathfrak{s}_n^* = \mathfrak{s}_n$.

12.7.10. Smooth triples. We say that $\widetilde{\mathcal{T}}$ is smooth if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{O}}_X$-locally free of finite rank. Then the corresponding sesquilinear pairing reads $\mathfrak{s}^{\nabla} \cdot \mathfrak{s}_n$ (see Lemma 12.3.6).

12.7.11. Lefschetz triples. The notion of Lefschetz structure $(\widetilde{\mathcal{T}}, \mathrm{N})$ in the abelian category $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$, or that of $\mathfrak{sl}_2$-structure $(\widetilde{\mathcal{T}}_\bullet, \mathrm{X}, \mathrm{Y})$, is obtained, as in Section 5.3. Using Hermitian duality in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$, we obtain as in Definition 5.3.2 the notion of Hermitian duality for a Lefschetz $\widetilde{\mathcal{D}}$-triple $(\widetilde{\mathcal{T}}, \mathrm{N})$. Therefore, if $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$, the nilpotent endomorphism $\mathrm{N} = (\mathrm{N}', \mathrm{N}'')$ consists of nilpotent endomorphisms

$$\mathrm{N}' : \widetilde{\mathcal{M}}' \longrightarrow \widetilde{\mathcal{M}}' \quad \text{and} \quad \mathrm{N}'' : \widetilde{\mathcal{M}}'' \longrightarrow \widetilde{\mathcal{M}}''$$

such that $\mathfrak{s}(\mathrm{N}' \bullet, \overline{\bullet}) = \mathfrak{s}(\bullet, \overline{\mathrm{N}'' \bullet})$ (see also Section 5.3.a). The notion of pre-polarization of weight w is defined as in Section 5.3.1.

12.7.b. Pullback, specialization and localization in $\widetilde{\mathcal{D}}$-Triple

12.7.12. Pullback by a smooth morphism. Let $f : X \to Y$ be a *smooth* holomorphic map of relative dimension p, as in Definition 12.5.1, and let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be a *left* $\widetilde{\mathcal{D}}_X$-triple. We set

$$
{}_{\mathrm{T}}f^*\widetilde{\mathcal{T}} = ({}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}', {}_{\mathrm{D}}f^*\widetilde{\mathcal{M}}''(p), {}_{\mathrm{D},\overline{\mathrm{D}}}f^*\mathfrak{s}).
$$

For a right $\widetilde{\mathcal{D}}_X$-triple, we use side-changing at the source and target to define ${}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}$, i.e.,

$$
{}_{\mathrm{T}}f^*(\widetilde{\mathcal{T}}^{\mathrm{right}}) := ({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}^{\mathrm{left}})^{\mathrm{right}}.
$$

If S is a pre-polarization of weight w of $\widetilde{\mathcal{T}}$, we regard $f^*\mathrm{S}$ as a pre-polarization of weight $w+p$ of ${}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}$ and we set

$$
{}_{\mathrm{T}}f^*(\widetilde{\mathcal{T}}, \mathrm{S}) = ({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}, (-1)^p f^*\mathrm{S}).
$$

12.7.13. Pullback of a smooth triple. Formulas similar to those in 12.7.12 hold if, instead of assuming f smooth and $\widetilde{\mathcal{T}}$ arbitrary, we assume f arbitrary but $\widetilde{\mathcal{T}}$ smooth.

12.7.14. Specialization in $\widetilde{\mathcal{D}}$-Triples. An object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-Triples(X) is said to be strictly $\mathbb{R}$-specializable along (g) if $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so. In a way similar to 7.4.2, we then define, for $\lambda \in \mathbb{S}^1$,

$$
\begin{aligned}
\psi_{g,\lambda}\widetilde{\mathcal{T}} &:= (\psi_{g,\lambda}\widetilde{\mathcal{M}}', \psi_{g,\lambda}\widetilde{\mathcal{M}}''(-1), \psi_{g,\lambda}\mathfrak{s}),\\
\phi_{g,1}\widetilde{\mathcal{T}} &:= (\phi_{g,1}\widetilde{\mathcal{M}}', \phi_{g,1}\widetilde{\mathcal{M}}'', \phi_{g,1}\mathfrak{s}).
\end{aligned}
\tag{12.7.14 $*$}
$$

Then $\psi_{g,\lambda}, \phi_{g,1}$ are functors from the full subcategory of strictly $\mathbb{R}$-specializable objects of $\widetilde{\mathcal{D}}$-Triples(X) to the category of objects supported on $g^{-1}(0)$. From Proposition 9.7.1 and Corollary 12.5.35(2) we deduce:

12.7.15. Proposition. *Assume $\widetilde{\mathcal{T}}$ is $\widetilde{\mathcal{D}}_X$-coherent and strictly $\mathbb{R}$-specializable along (g).*

(1) *For every $\lambda \in \mathbb{S}^1$, $\dim \operatorname{Supp} \psi_{g,\lambda}\widetilde{\mathcal{T}} < \dim \operatorname{Supp} \widetilde{\mathcal{T}}$.*
(2) *If $\operatorname{Supp} \widetilde{\mathcal{T}} \subset g^{-1}(0)$, then $\psi_{g,\lambda}\widetilde{\mathcal{T}} = 0$ for any $\lambda \in \mathbb{S}^1$ and $\widetilde{\mathcal{T}} \simeq \phi_{g,1}\widetilde{\mathcal{T}}$.* □

According to Remark 12.5.17(3) and Remark 12.5.26, these functors commute with Hermitian duality 12.7.4 as follows:

$$
\begin{aligned}
\psi_{g,\lambda}(\widetilde{\mathcal{T}}^*) &= (\psi_{g,\lambda}\widetilde{\mathcal{T}})^*(-1),\\
\phi_{g,1}(\widetilde{\mathcal{T}}^*) &= (\phi_{g,1}\widetilde{\mathcal{T}})^*.
\end{aligned}
\tag{12.7.15 $*$}
$$

If S is a pre-polarization of $\widetilde{\mathcal{T}}$ of weight w, then

- $\psi_{g,\lambda}\mathrm{S}$ is a pre-polarization of $\psi_{g,\lambda}\widetilde{\mathcal{T}}$ of weight $w-1$,
- $\phi_{g,1}\mathrm{S}$ is a pre-polarization of $\phi_{g,1}\widetilde{\mathcal{T}}$ of weight w,

and we set

$$
\begin{aligned}
\psi_{g,\lambda}(\widetilde{\mathcal{T}}, \mathrm{S}) &= (\psi_{g,\lambda}\widetilde{\mathcal{T}}, \psi_{g,\lambda}\mathrm{S}),\\
\phi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) &= (\phi_{g,1}\widetilde{\mathcal{T}}, \phi_{g,1}\mathrm{S}).
\end{aligned}
\tag{12.7.15 $**$}
$$

***12.7.16. Properties of* N, can *and* var.** The properties analogous to those of a Hodge-Lefschetz quiver explained in Section 5.3.6 also hold in the present setting, as follows from Remark 12.5.17(1), Remark 12.5.26 and Proposition 12.5.27. Let us denote by $\mathrm{N}', \mathrm{can}' \, \mathrm{var}'$ resp. $\mathrm{N}'', \mathrm{can}'' \, \mathrm{var}''$ the morphisms relative to $\widetilde{\mathcal{M}}'$ resp. $\widetilde{\mathcal{M}}''$. Then

(1) $\mathrm{N} := (\mathrm{N}', \mathrm{N}'')$ is a nilpotent morphism

$$\psi_{g,\lambda}\widetilde{\mathcal{T}} \longrightarrow \psi_{g,\lambda}\widetilde{\mathcal{T}}(-1) \quad \text{and} \quad \phi_{g,1}\widetilde{\mathcal{T}} \longrightarrow \phi_{g,1}\widetilde{\mathcal{T}}(-1),$$

(2) $\mathrm{can} = (\mathrm{can}', -\mathrm{var}'')$ is a morphism $\psi_{g,1}\widetilde{\mathcal{T}} \to \phi_{g,1}\widetilde{\mathcal{T}}$ commuting with the morphisms $\psi_{g,1}\varphi, \phi_{g,1}\varphi$ associated with any morphism φ between strictly $\mathbb{R}$-specializable objects of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$,

(3) $\mathrm{var} = (\mathrm{var}', -\mathrm{can}'')$ is a morphism $\phi_{g,1}\widetilde{\mathcal{T}} \to \psi_{g,1}\widetilde{\mathcal{T}}(-1)$ with the same commutation property as above,

(4) $\mathrm{N} = \mathrm{var} \circ \mathrm{can}$ on $\psi_{g,1}\widetilde{\mathcal{T}}$ and $\mathrm{can} \circ \mathrm{var}$ on $\phi_{g,1}\widetilde{\mathcal{T}}$, that is, on $\psi_{g,1}\widetilde{\mathcal{T}}$ for example, $\mathrm{N}' = \mathrm{var}' \circ \mathrm{can}'$ and $\mathrm{N}'' = \mathrm{var}'' \circ \mathrm{can}''$,

(5) $\mathrm{can}(\widetilde{\mathcal{T}})^* = -\mathrm{var}(\widetilde{\mathcal{T}}^*)$ and $\mathrm{var}(\widetilde{\mathcal{T}})^* = -\mathrm{can}(\widetilde{\mathcal{T}}^*)$, so that $\mathrm{N}(\widetilde{\mathcal{T}}^*) = \mathrm{N}(\widetilde{\mathcal{T}})^*$ (where $\mathrm{var}(\widetilde{\mathcal{T}})$, etc. means var relative to $\widetilde{\mathcal{T}}$, etc.),

(6) If S is a morphism $\widetilde{\mathcal{T}} \to \widetilde{\mathcal{T}}^*(-w)$ (e.g. a pre-polarization), then the following diagram commutes:

$$\begin{tikzcd}
\phi_{g,1}\widetilde{\mathcal{T}} \arrow[r, "\phi_{g,1}\mathrm{S}"] \arrow[d, "\mathrm{var}"'] & (\phi_{g,1}\widetilde{\mathcal{T}})^*(-w) \arrow[d, "-\mathrm{can}^*"] \\
\psi_{g,1}\widetilde{\mathcal{T}}(-1) \arrow[r, "\psi_{g,1}\mathrm{S}"] & (\psi_{g,1}\widetilde{\mathcal{T}})^*(-w)
\end{tikzcd}$$

This is seen by interpreting

- $(\phi_{g,1}\widetilde{\mathcal{T}})^*(-w)$ as $\phi_{g,1}(\widetilde{\mathcal{T}}^*(-w))$,
- $(\psi_{g,1}\widetilde{\mathcal{T}})^*(-w)$ as $\psi_{g,1}(\widetilde{\mathcal{T}}(-1)^*(-w))$
- and can^* as $-\mathrm{var}(\widetilde{\mathcal{T}}^*)$,

and by applying the commutation relations above to $\varphi = \mathrm{S}$.

In particular, $\operatorname{Im}\mathrm{N} = (\operatorname{Im}\mathrm{N}', \operatorname{Coim}\mathrm{N}'', \mathfrak{s}_{|\operatorname{Im}\mathrm{N}'\otimes\overline{\operatorname{Coim}\mathrm{N}''}})$. We also define the nearby/vanishing Lefschetz quiver of $\widetilde{\mathcal{T}}$ as the diagram:

$$\begin{tikzcd}
\psi_{g,1}\widetilde{\mathcal{T}} \arrow[r, bend left, "\mathrm{can}"] & \phi_{g,1}\widetilde{\mathcal{T}} \arrow[l, bend left, "\mathrm{var}", "(-1)"']
\end{tikzcd}$$

Notice also that Propositions 9.7.2 and 9.7.5 extend to the present setting, up to replacing “injective” with “monomorphism” and “onto” with “epimorphism”.

***12.7.17. The $\mathfrak{sl}_2$-triples attached to* $(\psi_{g,\lambda}\widetilde{\mathcal{T}}, \mathrm{N})$ *and* $(\phi_{g,1}\widetilde{\mathcal{T}}, \mathrm{N})$**

The monodromy filtration of N exists in the abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, and we have, according to Remark 12.5.17(2) (and similarly for $\phi_{g,1}\widetilde{\mathcal{T}}$ without the twist),

$$\begin{aligned}
\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}} &= (\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}', \mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}''(-1), \mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{g,\lambda}\mathfrak{s}),\\
\mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}} &= (\mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}', \mathrm{P}_{\ell}\psi_{g,\lambda}\widetilde{\mathcal{M}}''(-1), \mathrm{P}_{\ell}\psi_{g,\lambda}\mathfrak{s}) \quad (\ell \geqslant 0).
\end{aligned}$$

If $\mathrm{S} : (\widetilde{\mathcal{T}}, \mathrm{N}) \to (\widetilde{\mathcal{T}}, \mathrm{N})^*(-w) := (\widetilde{\mathcal{T}}^*, \mathrm{N}^*)(-w)$ is a pre-polarization of weight w, we define for any $\ell \geqslant 0$, as in Section 3.4.c, the morphisms

$$\begin{aligned}\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S} : \mathrm{P}_\ell\psi_{g,\lambda}\widetilde{\mathcal{T}} &\longrightarrow (\mathrm{P}_\ell\psi_{g,\lambda}\widetilde{\mathcal{T}})^*(-(w-1+\ell))\\ \mathrm{P}_\ell\phi_{g,1}\mathrm{S} : \mathrm{P}_\ell\phi_{g,1}\widetilde{\mathcal{T}} &\longrightarrow (\mathrm{P}_\ell\phi_{g,1}\widetilde{\mathcal{T}})^*(-(w+\ell)),\end{aligned}$$

which are pre-polarizations of respective weights $(w-1+\ell)$ and $(w+\ell)$, and we set

$$(12.7.17\,*)\qquad \begin{aligned}\mathrm{P}_\ell\psi_{g,\lambda}(\widetilde{\mathcal{T}}, \mathrm{S}) &= (\mathrm{P}_\ell\psi_{g,\lambda}\widetilde{\mathcal{T}}, (-1)^\ell\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}),\\ \mathrm{P}_\ell\phi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) &= (\mathrm{P}_\ell\phi_{g,1}\widetilde{\mathcal{T}}, (-1)^\ell\mathrm{P}_\ell\phi_{g,1}\mathrm{S}),\end{aligned}\qquad (\ell \geqslant 0,\ \text{see 3.2.11}).$$

12.7.18. Middle extension of a strictly $\mathbb{R}$-specializable $\widetilde{\mathcal{D}}$-triple. Assume that $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g). We say that it is a *middle extension along* (g) if $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so (see 9.7.3).

If $\widetilde{\mathcal{T}}$ is a middle extension along (g), then $\phi_{g,1}\widetilde{\mathcal{T}} \simeq \operatorname{Im}\mathrm{N}$ in the abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$.

12.7.19. S-decomposable $\widetilde{\mathcal{D}}$-triples. We say that a coherent $\widetilde{\mathcal{D}}$-triple $\widetilde{\mathcal{T}}$ is *S-decomposable along* (g) resp. *S-decomposable* if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so.

- If $\widetilde{\mathcal{T}}$ is S-decomposable along (g), it has a decomposition $\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_1 \oplus \widetilde{\mathcal{T}}_2$, where $\widetilde{\mathcal{T}}_1$ is a middle extension along (g) and $\widetilde{\mathcal{T}}_2$ is supported on $g^{-1}(0)$.
- If $\widetilde{\mathcal{T}}$ is S-decomposable, then $\widetilde{\mathcal{T}} = \bigoplus_i \widetilde{\mathcal{T}}_{S_i}$ with $\widetilde{\mathcal{T}}_{S_i}$ having pure support the irreducible closed analytic subset $S_i \subset X$ (see Proposition 12.3.9).

12.7.20. Proposition **(Criterion for S-decomposability along (g))**

Assume that $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g). Then $\widetilde{\mathcal{T}}$ is S-decomposable along (g) if and only if $\phi_{g,1}\widetilde{\mathcal{T}} = \operatorname{Im}\operatorname{can} \oplus \operatorname{Ker}\operatorname{var}$.

Proof. This follows from Proposition 9.7.5 and Lemma 12.3.10. $\square$

12.7.21. Non-characteristic restriction of a $\widetilde{\mathcal{D}}$-triple. Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ such that the smooth hypersurface $\iota : H \hookrightarrow X$ is strictly non-characteristic for its $\widetilde{\mathcal{D}}$-module components (see Section 9.5.b). Then $\mathrm{gr}_V^\beta\widetilde{\mathcal{T}}$ are zero except for $\beta \in \mathbb{N}$ and ${}_{\mathrm{T}}\iota^*\widetilde{\mathcal{T}} := (\mathrm{gr}_V^0\widetilde{\mathcal{M}}', \mathrm{gr}_V^0\widetilde{\mathcal{M}}''(-1), \mathrm{gr}_V^0\mathfrak{s})$ (see Section 12.5.18) is a well-defined object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(H)$, called the *non-characteristic restriction* of $\widetilde{\mathcal{T}}$ along H.

12.7.22. Properties along (g^r). If $\widetilde{\mathcal{T}}$ is an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is strictly $\mathbb{R}$-specializable along (g), a middle extension along (g), S-decomposable along (g), then it satisfies the corresponding properties along (g^r) for any $r \geqslant 2$. This follows from Proposition 9.9.1 and Exercise 12.12.

12.7.23. Specialization along a strictly non-characteristic divisor with normal crossings

We take up the setting of Sections 9.9.b and 12.5.20 with $g = x_1x_2$. As a consequence, we obtain the following property.

Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is holonomic, strictly non-characteristic with respect to D_1, D_2, D_{12}, hence strictly $\mathbb{R}$-specializable along (g). Then $\widetilde{\mathcal{T}}$ is a middle extension along (g) and we have $\psi_{g,\lambda}\widetilde{\mathcal{T}} = 0$ for $\lambda \neq 1$. Assume that $\widetilde{\mathcal{T}}$ is equipped with a pre-polarization S of weight w. Then there are isomorphisms

$$\begin{aligned} & \mathrm{P}_0\psi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) \simeq \psi_{x_1,1}(\widetilde{\mathcal{T}}, \mathrm{S}) \oplus \psi_{x_2,1}(\widetilde{\mathcal{T}}, \mathrm{S}), \\ (12.7.23\,*) \qquad & \mathrm{P}_1\psi_{g,1}(\widetilde{\mathcal{T}}, -\mathrm{S}) \simeq \psi_{x_1,1}\psi_{x_2,1}(\widetilde{\mathcal{T}}, \mathrm{S})(-1) = \psi_{x_2,1}\psi_{x_1,1}(\widetilde{\mathcal{T}}, \mathrm{S})(-1), \\ & \mathrm{P}_0\psi_{g,1}(\widetilde{\mathcal{T}}, \mathrm{S}) = 0 \quad \text{if } \ell \geqslant 2. \end{aligned}$$

Because of the pre-polarization, we can reduce the question to the case of a Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$ of weight w. Let us check the middle line for example. For the Hermitian pair, according to (9.9.3 $*$) and (12.5.20 $**$), we only need to check the sign of the pre-polarization. On the left-hand side, we introduce a minus sign (which is the sign that enters in front of $\mathrm{P}_1\mathrm{S}$ in Section 3.2.11), while on the right-hand side, the Tate twist (-1) introduces a minus sign, as wanted. Let us end by checking the weights. That of the left-hand side is, since $\ell = 1$, $1 + w - 1 = w$, while that of the right-hand side is $0 + w - 2 + 2 = w$, since it is equal to $\mathrm{P}_0\psi_{x_1,1}\mathrm{P}_0\psi_{x_2,1}(-1)(\bullet)$, so the weights also match.

12.7.24. Nearby cycles along a monomial function of a smooth $\widetilde{\mathcal{D}}$-module

We take up the setting of Sections 9.9.c and 12.5.21. As a consequence, we obtain the following property.

Let $\widetilde{\mathcal{T}}$ be a smooth object of $\widetilde{\mathcal{D}}$-Triples(X), where X is an open set in $\mathbb{C}^n$ with coordinates $x_1, \dots, x_n$, and set $g = x_1 \cdots x_p$. Recall that, for $\ell \leqslant p$, we denote by $\mathcal{J}_{\ell+1}$ the set of subsets $J \subset \{1, \dots, p\}$ with $\#J = \ell + 1$.

We have seen in Proposition 9.9.12 that $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable and a middle extension along (g) and the morphisms $\mathrm{N}, \mathrm{can}, \mathrm{var}$ are strict. Assume that $\widetilde{\mathcal{T}}$ is equipped with a pre-polarization S of weight w. Then, for each $\ell \geqslant 0$ there is an isomorphism

$$(12.7.24\,*) \qquad \mathrm{P}_\ell\psi_{g,1}(\widetilde{\mathcal{T}}, (-1)^\ell\mathrm{S}) \simeq \bigoplus_{J \in \mathcal{J}_{\ell+1}} {}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_I^*(\widetilde{\mathcal{T}}, \mathrm{S}))(-\ell) \qquad (I = J^{\mathrm{c}}).$$

The proof is similar to that of the previous example (one can identify ${}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_I^*(\widetilde{\mathcal{T}}, \mathrm{S})$ with the result of the iteration of $\psi_{x_j,1}$, for j varying in J, applied to $(\widetilde{\mathcal{T}}, \mathrm{S})$).

12.7.25. Localization and dual localization in $\widetilde{\mathcal{D}}$-Triples(X). Let D be an effective divisor in X and let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be an object of $\widetilde{\mathcal{D}}$-Triples(X) which is strictly $\mathbb{R}$-specializable and localizable along D (i.e., its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are so). If $D = (g)$, we then set

$$\begin{aligned} \widetilde{\mathcal{T}}[*D] &:= (\widetilde{\mathcal{M}}'[*D], \widetilde{\mathcal{M}}''[!D], \mathfrak{s}^{(*D)}), \\ \widetilde{\mathcal{T}}[!D] &:= (\widetilde{\mathcal{M}}'[!D], \widetilde{\mathcal{M}}''[*D], \mathfrak{s}^{(!D)}). \end{aligned}$$

These functors satisfy obvious identities with respect to Hermitian duality 12.7.4. By Proposition 12.5.37, there are natural morphisms

$$\widetilde{\mathcal{T}}[!D] \xrightarrow{\mathrm{dloc}} \widetilde{\mathcal{T}} \xrightarrow{\mathrm{loc}} \widetilde{\mathcal{T}}[*D].$$

If $\widetilde{\mathcal{T}}$ is a middle extension along D, then the first morphism is an epimorphism and the second one is a monomorphism.

On the other hand, if $D = H$ is smooth and strictly non-characteristic with respect to $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$, then (see Propositions 11.2.9 and 11.2.12) there are natural identifications $\operatorname{Ker} \operatorname{dloc} \simeq {}_{\mathrm{T}}\iota_{H*}({}_{\mathrm{T}}\iota_H^*\widetilde{\mathcal{T}})$ and ${}_{\mathrm{T}}\iota_{H*}({}_{\mathrm{T}}\iota_H^*\widetilde{\mathcal{T}})(-1) \simeq \operatorname{Coker} \operatorname{loc}$: they are obtained by means of the isomorphisms in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$

$$\operatorname{gr}_0^V(\operatorname{Coker}\operatorname{loc}) \simeq \operatorname{gr}_0^V(\widetilde{\mathcal{T}}[*H]) \xrightarrow[\sim]{\mathrm{var}} \operatorname{gr}_{-1}^V(\widetilde{\mathcal{T}}[*H])(-1) \simeq \operatorname{gr}_{-1}^V\widetilde{\mathcal{T}}(-1) = \widetilde{\mathcal{T}}_H(-1),$$
$$\operatorname{gr}_0^V(\operatorname{Ker}\operatorname{dloc}) \simeq \operatorname{gr}_0^V(\widetilde{\mathcal{T}}[!H]) \xleftarrow[\sim]{\mathrm{can}} \operatorname{gr}_{-1}^V(\widetilde{\mathcal{T}}[!H]) \simeq \operatorname{gr}_{-1}^V\widetilde{\mathcal{T}} = \widetilde{\mathcal{T}}_H.$$

In such a way, we have two exact sequences

$$0 \longrightarrow {}_{\mathrm{T}}\iota_{H*}({}_{\mathrm{T}}\iota_H^*\widetilde{\mathcal{T}}) \longrightarrow \widetilde{\mathcal{T}}[!H] \xrightarrow{\mathrm{dloc}} \widetilde{\mathcal{T}} \longrightarrow 0,$$
$$0 \longrightarrow \widetilde{\mathcal{T}} \xrightarrow{\mathrm{loc}} \widetilde{\mathcal{T}}[*H] \longrightarrow {}_{\mathrm{T}}\iota_{H*}({}_{\mathrm{T}}\iota_H^*\widetilde{\mathcal{T}})(-1) \longrightarrow 0. \tag{12.7.26}$$

12.7.c. Pushforward in the category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$

***12.7.27. Definition* (Proper pushforward).** Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)_{\mathrm{coh}}$ supported on S and let $f : X \to Y$ be a holomorphic map which is proper on S. Then the k-th pushforward ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ is the object

$${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}} := ({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}', {}_{\mathrm{D}}f_*^{(-k)}\widetilde{\mathcal{M}}'', {}_{\mathrm{T}}f_*^{(k,-k)}\mathfrak{s})$$

of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y)_{\mathrm{coh}}$. It satisfies (see (12.4.4 $*$))

$$({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}})^* = {}_{\mathrm{T}}f_*^{(-k)}(\widetilde{\mathcal{T}}^*). \tag{12.7.27 $*$}$$

It is convenient to consider the pushforward as a graded object ${}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}} = \bigoplus_k {}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$. Then a pre-polarization $\mathrm{S} : \widetilde{\mathcal{T}} \xrightarrow{\sim} \widetilde{\mathcal{T}}^*(-w)$ of weight w induces a pre-polarization

$${}_{\mathrm{T}}f_*^{(\bullet)}\mathrm{S} : {}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}} \xrightarrow{\sim} ({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}})^*(-w),$$

which is graded, by taking the grading considered in Section 12.7.6 for $({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}})^*$. More specifically, each ${}_{\mathrm{T}}f_*^{(k)}\mathrm{S}$ is an isomorphism ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}} \xrightarrow{\sim} ({}_{\mathrm{T}}f_*^{(-k)}\widetilde{\mathcal{T}})^*(-w)$.

If we represent $(\widetilde{\mathcal{T}}, \mathrm{S})$ by a Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$ of weight w with $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$, then ${}_{\mathrm{T}}f_*^{(\bullet)}(\widetilde{\mathcal{T}}, \mathrm{S}) = ({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{M}}, {}_{\mathrm{T}}f_*^{(\bullet)}\mathcal{S})$, where ${}_{\mathrm{T}}f_*^{(k)}\mathcal{S} := {}_{\mathrm{T}}f_*^{(k,-k)}\mathcal{S}$ pairs ${}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ and ${}_{\mathrm{D}}f_*^{(-k)}\overline{\mathcal{M}}$.

12.7.28. Kashiwara's equivalence. Let $\iota : Z \hookrightarrow X$ denote the inclusion of a closed submanifold. The the functor ${}_{\mathrm{T}}\iota_*$ from $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Z)$ into itself is fully faithful. Moreover, if $Z = H$ is smooth of codimension one in X, the functor ${}_{\mathrm{T}}\iota_*$ induces an equivalence between the full subcategory of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(H)$ whose objects are strict, and the full subcategory of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ whose objects are strictly $\mathbb{R}$-specializable along H and supported on H. An inverse functor is then $\widetilde{\mathcal{T}} \mapsto \operatorname{gr}_0^V\widetilde{\mathcal{T}}$ (see Remark 12.5.36 for $\operatorname{gr}_0^V\mathfrak{s}$). Indeed, this follows from Propositions 9.6.2 and 12.4.7.

12.7.29. The Lefschetz morphism for triples. From Definition 8.7.19 and (12.4.13 $*$) we can define the Lefschetz morphism attached to a real $(1,1)$-form η by the formula

$$\mathrm{X}_\eta = (\mathrm{X}'_\eta, \mathrm{X}''_\eta) : {}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}} \longrightarrow {}_{\mathrm{T}}f_*^{(k+2)}\widetilde{\mathcal{T}}(1).$$

It is functorial with respect to $\widetilde{\mathcal{T}}$ and satisfies $\mathrm{X}_\eta^* = \mathrm{X}_\eta$. Moreover, the graded object $({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}}, \mathrm{X}_\eta)$ is an $\mathfrak{sl}_2$-structure on the category $\mathsf{A} = \widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ with X_η corresponding to X, in the sense of Definition 3.3.3 (together with Remark 3.3.4 for the twist).

12.7.30. Adjunction and trace in the case of a finite morphism. We consider the setting of Example 8.7.30, with a finite morphism $f : X = \mathbb{C}^n \to Y = \mathbb{C}^n$ defined by $f_i(x_1, \dots, x_n) = x_i^{r_i}$, $i = 1, \dots, n$, and $r_i \geqslant 2$ for $i = 1, \dots, \ell$, and $r_i = 1$ for $i = \ell+1, \dots, n$. Furthermore, we assume that the object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y)$ is such that $\mathfrak{s}$ takes values in $\widetilde{\mathcal{C}}_Y^\infty$. We deduce from Examples 8.7.30 and 12.4.17 two morphisms

$$\widetilde{\mathrm{adj}}_f = (\mathrm{adj}'_f, \mathrm{Tr}''_f) : \widetilde{\mathcal{T}} \longrightarrow {}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}), \quad \widetilde{\mathrm{Tr}}_f = (\mathrm{Tr}'_f, \mathrm{adj}''_f) : {}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}}) \longrightarrow \widetilde{\mathcal{T}},$$

whose composition is the identity, making $\widetilde{\mathcal{T}}$ a direct summand of ${}_{\mathrm{T}}f_*^{(0)}({}_{\mathrm{T}}f^*\widetilde{\mathcal{T}})$ in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y)$.

12.7.31. Pushforward and specialization of $\widetilde{\mathcal{D}}$-triples. Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is coherent and strictly $\mathbb{R}$-specializable along $(g) = (g' \circ f)$, where $f : X \to Y$ is proper. Let $h : Y \to \mathbb{C}$ be a holomorphic function and set $g = h \circ f$. Let us assume that, for each k and λ, ${}_{\mathrm{T}}f_*^{(k)}(\psi_{g,\lambda}\widetilde{\mathcal{T}})$ and ${}_{\mathrm{T}}f_*^{(k)}(\phi_{g,1}\widetilde{\mathcal{T}})$ are *strict*.

It follows from Corollary 9.8.9 and Theorem 12.6.1 that there are natural isomorphisms of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$-Lefschetz structures

$$\begin{aligned}{}_{\mathrm{T}}f_*^{(k)}(\psi_{g,\lambda}\widetilde{\mathcal{T}}, \mathrm{N}) &\simeq \psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathcal{T}}, \mathrm{N})),\\ {}_{\mathrm{T}}f_*^{(k)}(\phi_{g,1}\widetilde{\mathcal{T}}, \mathrm{N}) &\simeq \phi_{h,1}({}_{\mathrm{T}}f_*^{(k)}(\widetilde{\mathcal{T}}, \mathrm{N})),\end{aligned}$$

and of of nearby/vanishing $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$-Lefschetz quivers

$${}_{\mathrm{T}}f_*^{(k)}\big((\psi_{g,1}\widetilde{\mathcal{T}}, \mathrm{N}), (\phi_{g,1}\widetilde{\mathcal{T}}, \mathrm{N}), \mathrm{can}, \mathrm{var}\big) \simeq \big((\psi_{g,1}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}), \mathrm{N}), (\phi_{g,1}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}), \mathrm{N}), \mathrm{can}, \mathrm{var}\big).$$

12.7.d. The pushforward functor as a cohomological functor

12.7.32. Complexes and double complexes in $\widetilde{\mathcal{D}}$-Triples. Let us start with a general remark which explains the introduction of a sign in Definition 12.4.3(2). Let $(\widetilde{K}'^\bullet, d')$ and $(\widetilde{K}''^\bullet, d'')$ be bounded complexes of right $\widetilde{\mathcal{D}}_Y$-modules and let $\mathfrak{s} : K'^\bullet \otimes_{\widetilde{\mathbb{C}}} \overline{K''^\bullet} \to \mathfrak{C}_Y$ be a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism to the right $\mathcal{D}_{Y,\overline{Y}}$-module $\mathfrak{C}_Y$ (i.e., a complex with $\mathfrak{C}_Y$ as its only nonzero term, placed in degree zero). We wish to transform this set of data to a complex in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y))$.

Let m'_k, resp. m''_ℓ, be a local section of K'^k, resp. K''^ℓ. The differential d of the simple complex associated to $K'^\bullet \otimes_{\widetilde{\mathbb{C}}} \overline{K''^\bullet}$ satisfies

$$d(m'_k \otimes \overline{m''_\ell}) = d'm'_k \otimes \overline{m''_\ell} + (-1)^k m'_k \otimes \overline{d''m''_\ell}.$$

Since $\mathfrak{s}$ is a morphism of complexes, it is compatible with d, and since the differential of the complex $\mathfrak{C}_Y$ is zero, we obtain the relation

$$\mathfrak{s}(d'm'_k, \overline{m''_\ell}) = (-1)^{k-1}\mathfrak{s}(m'_k, \overline{d''m''_\ell})$$

for every k, ℓ. Let $\mathfrak{s}_k : K'^k \otimes \overline{K''^{-k}} \to \mathfrak{C}_Y$ denote the pairing induced by $\mathfrak{s}$. The above relation implies that (recall that $\varepsilon(k) = (-1)^{k(k-1)/2}$)

$$(d', d'') : (\widetilde{K}'^k, \widetilde{K}''^{-k}, \varepsilon(k)\mathfrak{s}_k) \longrightarrow (\widetilde{K}'^{k+1}, \widetilde{K}''^{-k-1}, \varepsilon(k+1)\mathfrak{s}_{k+1})$$

is a morphism in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(Y)$. In this way we obtain a differential complex in $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y))$:

$$(\widetilde{K}^\bullet, d) = \bigoplus_k (\widetilde{K}^k, d), \quad \widetilde{K}^k = (\widetilde{K}'^k, \widetilde{K}''^{-k}, \varepsilon(k)\mathfrak{s}_k), \quad d = (d', d'').$$

For double complexes, the argument is similar. Given double complexes

$$\big((\widetilde{K}'^{i,j})_{i,j}, d'_1, d'_2\big), \quad \big((\widetilde{K}''^{k,\ell})_{k,\ell}, d''_1, d''_2\big)$$

of $\widetilde{\mathcal{D}}_Y$-modules and a sesquilinear pairing $\mathfrak{s}$ with values in $\mathfrak{C}_Y$ whose components are $\mathfrak{s}_{ij} : K'^{i,j} \otimes \overline{K''^{-i,-j}} \to \mathfrak{C}_Y$ and $\mathfrak{s}$ is zero from $K'^{i,j} \otimes \overline{K''^{k,\ell}}$ if $i + k \neq 0$ or $j + \ell \neq 0$, we obtain a double complex in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(Y)$ as

$$(\widetilde{K}^{\bullet,\bullet}, d_1, d_2) = \bigoplus_{i,j} (\widetilde{K}^{i,j}, d_1, d_2), \quad \widetilde{K}^{i,j} = (\widetilde{K}'^{i,j}, \widetilde{K}''^{-i,-j}, \varepsilon(i+j)\mathfrak{s}_{i,j}),$$
$$d_a = (d'_a, d''_a), \ a = 1, 2.$$

12.7.33. We interpret the functors ${}_{\mathrm{T}}f_*^{(k)}$ as cohomology functors of a pushforward functor ${}_{\mathrm{T}}f_* : \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)) \to \mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y))$. This will enable us to treat the Leray spectral sequence for the composition of maps. In order to do so, it is convenient to work with a flabby resolution of $\mathrm{Sp}_{X\to Y}(\widetilde{\mathcal{M}})$ ($\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$). We will use the resolution $\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{M}})$ by the relative C^∞ Spencer complex (Exercise 8.50). Recall that we set $\mathrm{Sp}_X^{\infty,k} = \bigoplus_\ell \widetilde{\Theta}_{X,\ell} \otimes \widetilde{\mathcal{E}}_X^{(0,k+\ell)}$.

Let $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ be an object of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$. We represent the pushforward complex of each component of $\widetilde{\mathcal{T}}$ as $\widetilde{K}^\bullet = \widetilde{K}'^\bullet, \widetilde{K}''^\bullet$ with

$$\widetilde{K}^\bullet = f_!(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}_X^{\infty,\bullet} \otimes_{f^{-1}\widetilde{\mathcal{O}}_Y} f^{-1}\widetilde{\mathcal{D}}_Y) \simeq f_!(\widetilde{\mathcal{M}} \otimes_{\widetilde{\mathcal{O}}_X} \mathrm{Sp}_X^{\infty,\bullet}) \otimes_{\widetilde{\mathcal{O}}_Y} \widetilde{\mathcal{D}}_Y, \quad \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'',$$

and the pushforward sesquilinear pairing as given by (12.4.5 $*$).

12.7.34. Definition. The pushforward triple ${}_{\mathrm{T}}f_*\widetilde{\mathcal{T}}$ is the complex whose k-th term is

$$\Big(K'^k, K''^{-k}, \varepsilon(k)(\textstyle\int f_* \mathrm{Sp}_Y^\infty(\mathfrak{s}))_k\Big).$$

and whose differentials are $(d', d'') = (f_*\widetilde{\delta}^\infty_{\widetilde{\mathcal{M}}',Y}, f_*\widetilde{\delta}^\infty_{\widetilde{\mathcal{M}}'',Y})$.

The following is then clear.

12.7.35. Lemma. *For each $k \in \mathbb{Z}$, the cohomology $\mathcal{H}^k({}_{\mathrm{T}}f_*\widetilde{\mathcal{T}})$ is isomorphic to ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ of Definition* 12.7.27. $\square$

We can now extend Corollary 8.7.27 to the case of the categories $\widetilde{\mathcal{D}}$-$\mathsf{Triples}$.

12.7.36. Corollary **(The Leray spectral sequence for the composition of maps)**

Let $f : X \to Y$ and $f' : Y \to Z$ be holomorphic maps and let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}$-Triples(X). We assume that $f' \circ f$ is proper on $S = \operatorname{Supp} \widetilde{\mathcal{T}}$ (hence so is f, and f' is proper on $f(\operatorname{Supp} \widetilde{\mathcal{T}})$). Then there exists a bounded spectral sequence with $E_2^{p,q} = {}_{\mathrm{T}}f'^{(p)}_*({}_{\mathrm{T}}f^{(q)}_* \widetilde{\mathcal{T}})$ which converges to ${}_{\mathrm{T}}(f' \circ f)^{(p+q)}_* \widetilde{\mathcal{T}}$.

The *Leray filtration* $\operatorname{Ler}^\bullet{}_{\mathrm{T}}(f' \circ f)^{(k)}_* \widetilde{\mathcal{T}}$ satisfies

$$\operatorname{gr}^p_{\operatorname{Ler}}\big({}_{\mathrm{T}}(f' \circ f)^{(p+q)}_* \widetilde{\mathcal{T}}\big) = E^{p,q}_\infty.$$

In particular, since $E^{p,q}_\infty$ is a subquotient of $E_2^{p,q}$, $\operatorname{gr}^p_{\operatorname{Ler}}\big({}_{\mathrm{T}}(f' \circ f)^{(k)}_* \widetilde{\mathcal{T}}\big)$ vanishes unless $p \in [-\dim Y, \dim Y]$ (see Remark 8.7.13).

Proof of Corollary **12.7.36.** ***Step 1.*** Arguing as in the proof of Theorem 8.7.22, we find a natural quasi-isomorphism

$$\operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}}) \otimes_{f^{-1}\mathcal{D}_{Y,\overline{Y}}} f^{-1} \operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}(\mathcal{D}_{Y,\overline{Y}}) \xrightarrow{\sim} \operatorname{Sp}_{X,\overline{X}\to Z,\overline{Z}}(\mathcal{D}_{X,\overline{X}}), \tag{12.7.37}$$

leading to a quasi-isomorphism

$$f'_*\Big(f_*\big(\mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big) \otimes_{\mathcal{D}_{Y,\overline{Y}}} \operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}\Big) \xrightarrow{\sim} f'_*\big(\mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Z,\overline{Z}}(\mathcal{D}_{X,\overline{X}})\big).$$

The integration morphism (12.2.12)

$$\textstyle\int^\bullet_f : f_*\big(\mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big) \longrightarrow \mathfrak{C}_{Y,f(S)}$$

can be composed with that for f' to yield

$$\textstyle\int^\bullet_{f'} \circ (\int^\bullet_f \otimes \mathrm{Id}) : f'_*\Big(f_*\big(\mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})\big) \otimes_{\mathcal{D}_{Y,\overline{Y}}} \operatorname{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}\Big) \longrightarrow \mathfrak{C}_Z\,.$$

On the other hand, we have the integration morphism

$$\textstyle\int^\bullet_{f'\circ f} : f'_*\big(\mathfrak{C}_{X,S} \otimes_{\mathcal{D}_{X,\overline{X}}} \operatorname{Sp}_{X,\overline{X}\to Z,\overline{Z}}(\mathcal{D}_{X,\overline{X}})\big) \longrightarrow \mathfrak{C}_Z\,.$$

We claim that, through the above quasi-isomorphism, both integration morphisms coincide. It is enough to prove that their restrictions to the degree-zero terms of the complexes coincide, on noting that these complexes have nonzero terms only on non-positive degrees. In degree zero, the inverse of the isomorphism (12.7.37) is induced by the natural morphism

$$\begin{aligned}&\mathcal{O}_{X,\overline{X}} \otimes_{(f'\circ f)^{-1}\mathcal{O}_{Z,\overline{Z}}} (f' \circ f)^{-1}\mathcal{D}_{Z,\overline{Z}}\\ &\quad\xrightarrow{\sim} (\mathcal{O}_{X,\overline{X}} \otimes_{f^{-1}\mathcal{O}_{Y,\overline{Y}}} f^{-1}\mathcal{D}_{Y,\overline{Y}}) \otimes_{f^{-1}\mathcal{D}_{Y,\overline{Y}}} (f^{-1}\mathcal{O}_{Y,\overline{Y}} \otimes_{(f'\circ f)^{-1}\mathcal{O}_{Z,\overline{Z}}} (f' \circ f)^{-1}\mathcal{D}_{Z,\overline{Z}})\end{aligned}$$

defined by $\varphi \otimes Q \mapsto (\varphi \otimes 1) \otimes (1 \otimes Q)$ (see Exercise 8.36).

We are thus led to checking that the following diagram commutes:

$$\begin{tikzcd}[column sep=small]
(f'\circ f)_*\,\mathfrak{C}_{X,S}\otimes_{\mathcal{O}_{Z,\overline{Z}}}\mathcal{D}_{Z,\overline{Z}} \arrow[rr] \arrow[dr, "\int_{f'\circ f}\otimes\mathrm{Id}"'] & & f'_*(f_*\,\mathfrak{C}_{X,S}\otimes_{\mathcal{O}_{Y,\overline{Y}}}\mathcal{D}_{Y,\overline{Y}})\otimes_{\mathcal{O}_{Z,\overline{Z}}}\mathcal{D}_{Z,\overline{Z}} \arrow[dl, "\int_{f'}\int_f\otimes\mathrm{Id}"] \\
& \mathfrak{C}_Z\otimes_{\mathcal{O}_{Z,\overline{Z}}}\mathcal{D}_{Z,\overline{Z}} &
\end{tikzcd}$$

This follows from the obvious commutation (Fubini) $\int_{f'\circ f}=\int_{f'}\int_f$ on currents. $\square$

***Proof of Corollary* 12.7.36. *Step 2*.** To compute the iterated pushforward ${}_{\mathrm{T}}f'_*({}_{\mathrm{T}}f_*(\widetilde{\mathfrak{T}}))$, we consider the double complexes

$$\begin{aligned}
\widetilde{K}'^{\bullet,\bullet}&=f'_*\Big[f_*\big(\widetilde{\mathcal{M}}'\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{\widetilde{\mathcal{D}}_Y}\mathrm{Sp}^\infty_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big]\\
\widetilde{K}''^{\bullet,\bullet}&=f'_*\Big[f_*\big(\widetilde{\mathcal{M}}''\otimes_{\widetilde{\mathcal{D}}_X}\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X)\big)\otimes_{\widetilde{\mathcal{D}}_Y}\mathrm{Sp}^\infty_{Y\to Z}(\widetilde{\mathcal{D}}_Y)\Big],
\end{aligned}$$

and the morphism defined from the sesquilinear pairing $\mathfrak{s}$ with values in the double complex

$$\begin{aligned}
&f'_*\Big[f_*\big(\mathfrak{C}_{X,S}\otimes_{\widetilde{\mathcal{D}}_{X,\overline{X}}}\mathrm{Sp}^\infty_{X,\overline{X}\to Y,\overline{Y}}(\widetilde{\mathcal{D}}_{X,\overline{X}})\big)\otimes_{\widetilde{\mathcal{D}}_{Y,\overline{Y}}}\mathrm{Sp}^\infty_{Y,\overline{Y}\to Z,\overline{Z}}(\widetilde{\mathcal{D}}_{Y,\overline{Y}})\Big]\\
&\qquad\qquad\simeq f'_*\Big[f_*\big(\mathfrak{C}_{X,S}\otimes_{\widetilde{\mathcal{D}}_{X,\overline{X}}}\mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\widetilde{\mathcal{D}}_{X,\overline{X}})\big)\otimes_{\widetilde{\mathcal{D}}_{Y,\overline{Y}}}\mathrm{Sp}_{Y,\overline{Y}\to Z,\overline{Z}}(\widetilde{\mathcal{D}}_{Y,\overline{Y}})\Big]
\end{aligned}$$

with $\mathrm{Sp}^\infty_{X,\overline{X}\to Y,\overline{Y}}(\widetilde{\mathcal{D}}_{X,\overline{X}}):=\mathrm{Sp}^\infty_{X\to Y}(\widetilde{\mathcal{D}}_X)\otimes_{\widetilde{\mathbb{C}}}\mathrm{Sp}^\infty_{\overline{X}\to\overline{Y}}(\widetilde{\mathcal{D}}_{\overline{X}})$. Composing with the double integration morphism in Step 1 yields a morphism $\mathfrak{s}:K'^{i,j}\otimes K''^{k,\ell}\to\mathfrak{C}_Z$ of double complexes, which is thus zero if $i+k\neq0$ or $j+\ell\neq0$. As explained in Section 12.7.32, we obtain a double complex in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(\widetilde{\mathcal{D}}_Z)$. The computation of Step 1 shows that the associated simple complex in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(\widetilde{\mathcal{D}}_Z)$ is quasi-isomorphic to the complex computing ${}_{\mathrm{T}}(f'\circ f)_*(\widetilde{\mathfrak{T}})$. We conclude that the spectral sequence of the double complex, with $E_2^{p,q}={}_{\mathrm{T}}f'^{(p)}_*({}_{\mathrm{T}}f^{(q)}_*\widetilde{\mathfrak{T}})$, converges to ${}_{\mathrm{T}}(f'\circ f)^{(p+q)}_*\widetilde{\mathfrak{T}}$. $\square$

***12.7.38. The restriction and Gysin morphisms in* $\widetilde{\mathcal{D}}$-Triples.** We take up the setting of Section 11.2.d, in which H is a smooth hypersurface in the compact manifold X. We consider an object $\widetilde{\mathfrak{T}}=(\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}'',\mathfrak{s})$ of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ and we assume that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}''$.

We denote by a_X the constant map on X. Recall that

$${}_{\mathrm{T}}a^{(k)}_{X*}\widetilde{\mathfrak{T}}=({}_{\mathrm{D}}a^{(k)}_{X*}\widetilde{\mathcal{M}}',a^{(-k)}_{X*}\widetilde{\mathcal{M}}'',{}_{\mathrm{T}}a^{(k,-k)}_{X*}\mathfrak{s}).$$

By applying ${}_{\mathrm{T}}a^{(\bullet)}_{X*}$ to the exact sequences (12.7.26) and noticing the identification ${}_{\mathrm{T}}a^{(\bullet)}_{X*}{}_{\mathrm{T}}\iota_{H*}\simeq{}_{\mathrm{T}}a^{(\bullet)}_{H*}$ following from (12.4.14 $*$) (a special case of Corollary 12.7.36), we define

$$\mathrm{restr}_H:{}_{\mathrm{D}}a^{(k)}_{X*}\widetilde{\mathfrak{T}}\longrightarrow{}_{\mathrm{D}}a^{(k+1)}_{H*}\widetilde{\mathfrak{T}}_H\quad\text{and}\quad\mathrm{Gys}_H:{}_{\mathrm{D}}a^{(-k-1)}_{H*}\widetilde{\mathfrak{T}}_H(-1)\longrightarrow{}_{\mathrm{D}}a^{(-k)}_{X*}\widetilde{\mathfrak{T}}$$

as the connecting morphisms in the corresponding long exact sequences in $\widetilde{\mathbb{C}}$-Triples.

12.7.39. Proposition. *We have a commutative diagram*

$$\begin{tikzcd}
& {}_{\mathrm{T}}a^{(k)}_{X*}\widetilde{\mathfrak{T}} \arrow[rr, "\mathrm{X}_{\mathcal{L}}"] \arrow[dr, "\mathrm{restr}_H"] & & {}_{\mathrm{T}}a^{(k+2)}_{X*}\widetilde{\mathfrak{T}}(1) \\
{}_{\mathrm{T}}a^{(k-1)}_{H*}\widetilde{\mathfrak{T}}_H(-1) \arrow[ur, "\mathrm{Gys}_H"] \arrow[rr, "\mathrm{X}_{\mathcal{L}}"] & & {}_{\mathrm{T}}a^{(k+1)}_{H*}\widetilde{\mathfrak{T}}_H \arrow[ur, "\mathrm{Gys}_H"'] &
\end{tikzcd}$$

Proof. We notice that the "prime" component of this diagram is the diagram of Proposition 11.2.18, while the "double-prime" component is the similar diagram after changing the exponents and the Tate twists to their opposite value, taking the arrows in the other direction and exchanging restr_H and Gys_H, and this is a diagram shifted from that of Proposition 11.2.18. Commutativity follows then from the commutativity proved in that proposition (since the sesquilinear pairing is not concerned for commutativity). $\square$

12.8. Exercises

Exercise 12.1 **(Pushforward of the sheaf of currents as a right $\mathcal{D}_{X,\overline{X}}$-module)**

Using the definition of Section 8.7.e, show that

$${}_{\mathrm{D},\overline{\mathrm{D}}}f_! \,\mathfrak{C}_X = f_!(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathrm{Sp}_{X,\overline{X}\to Y,\overline{Y}}(\mathcal{D}_{X,\overline{X}})).$$

Exercise 12.2. Assume that X is a closed submanifold of Y and denote by $\iota : X \hookrightarrow Y$ the embedding (which is a proper map). Denote by $\mathbf{1}$ the canonical section of $\mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}$. Show that the natural map

$${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_* \,\mathfrak{C}_X = \iota_*\big(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X,\overline{X}\to Y,\overline{Y}}\big) \longrightarrow \mathfrak{C}_Y, \quad u\otimes\mathbf{1} \longmapsto \int_\iota u$$

induces an isomorphism of the right $\mathcal{D}_{Y,\overline{Y}}$-module ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_* \,\mathfrak{C}_X$ with the submodule of $\mathfrak{C}_Y$ consisting of currents supported on X. [*Hint*: Use a local computation.]

For example, consider the case $\iota : X = X\times\{0\} \hookrightarrow X\times\mathbb{C}$, with coordinate t on $\mathbb{C}$ and identify ${}_{\mathrm{D},\overline{\mathrm{D}}}\iota_* \,\mathfrak{C}_X$ with $\iota_*\,\mathfrak{C}_X[\partial_t, \partial_{\overline{t}}]$.

Exercise 12.3. Extend the result of Exercise 8.53 to the case of right $\mathcal{D}_{X,\overline{X}}$-modules and show that the composed map

$$f_*\,\mathfrak{C}_X \longrightarrow {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_X \longrightarrow \mathfrak{C}_Y$$

is the integration of currents of Definition 12.2.9.

Exercise 12.4. Let $f : X\to Y$ be a holomorphic map and let $S\subset X$ be a closed subset on which f is proper.

(1) Define the sub-$\mathcal{D}_{X,\overline{X}}$-module $\mathfrak{C}_{X,S}$ of $\mathfrak{C}_X$ consisting of currents supported on S.

(2) Show that the integration of currents $\int_f$ induces a $\mathcal{D}_{Y,\overline{Y}}$-linear morphism of complexes

$$\int_f : {}_{\mathrm{D},\overline{\mathrm{D}}}f_*\,\mathfrak{C}_{X,S} \longrightarrow \mathfrak{Db}^{\bullet}_Y[2m] \otimes_{\mathcal{O}_{Y,\overline{Y}}} \mathcal{D}_{Y,\overline{Y}} \simeq \mathfrak{C}_Y\,.$$

[*Hint*: In Formula (12.2.9 $*$), let K be the compact support of η; choose a compact neighbourhood K' of $f^{-1}(K)\cap S$, and use a partition of the unity relative to the covering made by the complement of K' in X and the interior of K'.]

Exercise 12.5. If f is a a projection $X=Y\times T\to Y$ with $\dim T=p=n-m$, show that there exists a morphism

$$({12.5\,*})\qquad {}_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X\longrightarrow\mathfrak{C}_Y$$

which does not make use of the integration morphism (more precisely, it only uses integration of constant functions).

(1) Consider the morphism

$$\Theta_{X/Y,\overline{X}/\overline{Y},\bullet}:=\Theta_{X/Y,\bullet}\otimes_{\mathbb{C}}\Theta_{\overline{X}/\overline{Y},\bullet}.$$

and, following the same line as for (8.7.10 $*$), show that

$${}_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X\simeq\boldsymbol{R}f_!(\mathfrak{C}_X\otimes_{\mathcal{O}_{X,\overline{X}}}\Theta_{X/Y,\overline{X}/\overline{Y},\bullet}).$$

(2) By applying an argument similar to that of Exercise 8.26(1), prove that

$$\mathfrak{C}_X\otimes_{\mathcal{O}_{X,\overline{X}}}\Theta_{X/Y,\overline{X}/\overline{Y},\bullet}\simeq f^{-1}\mathcal{E}_Y^{m,m}\otimes_{f^{-1}\mathcal{C}^\infty_Y}\mathcal{E}^\bullet_{X/Y}\otimes_{\mathcal{C}^\infty_X}\mathfrak{Db}_X[2p].$$

(3) Note that a distribution on X annihilated by $\mathrm{d}_{X/Y}$ is locally a distribution on Y, and deduce that the complex $\mathcal{E}^\bullet_{X/Y}\otimes_{\mathcal{C}^\infty_X}\mathfrak{Db}_X$ is a resolution of $f^{-1}\,\mathfrak{Db}_Y$.

(4) Deduce an isomorphism

$$\mathfrak{C}_X\otimes_{\mathcal{O}_{X,\overline{X}}}\Theta_{X/Y,\overline{X}/\overline{Y},\bullet}\simeq f^{-1}\,\mathfrak{C}_Y[2p],$$

and thus

$${}_{\mathrm{D},\overline{\mathrm{D}}}f_!\,\mathfrak{C}_X\simeq\boldsymbol{R}f_!f^{-1}\,\mathfrak{C}_Y[2p].$$

(5) Since f is smooth of relative real dimension $2p$, there exists a natural morphism

$$\boldsymbol{R}f_!f^{-1}\,\mathfrak{C}_Y[2p]=\boldsymbol{R}f_!f^!\,\mathfrak{C}_Y\simeq\boldsymbol{R}f_!f^!\mathbb{C}_X\otimes_{\mathbb{C}_Y}\mathfrak{C}_Y\longrightarrow\mathfrak{C}_Y,$$

according to Verdier duality (see e.g. [**KS90**, Chap. 3]). Conclude the existence of (12.5 $*$).

(6) Compare with the morphism (12.2.12).

Exercise 12.6 **(Trace of a C^∞ function).** We consider the setting of Example 8.7.30 with the finite map $f:X=\mathbb{C}^n\to Y=\mathbb{C}^n$ defined by $f_i(x_1,\dots,x_n)=x_i^{r_i}$, $r_i\in\mathbb{N}^*$ and $r_i\geqslant 2\Leftrightarrow 1\leqslant i\leqslant\ell$. The goal of this exercise is to prove that the trace $\mathrm{Tr}_f(\varphi)$ of a C^∞ function φ on X is a C^∞ function on Y.[(2)]

(1) Use the Malgrange preparation theorem to show that the germ $\mathcal{C}^\infty_{X,0}$ is a module of finite type over $\mathcal{C}^\infty_{Y,0}$ generated by monomials $x^a\overline{x}^b$ with $0\leqslant a_i,b_i\leqslant r_i-1$ for all $i=1,\dots,n$.

(2) Show that $\mathrm{Tr}_f(x^a\overline{x}^b)=0$ if there exists i such that a_i-b_i is not a multiple of r_i.

(3) Show that, otherwise, $\mathrm{Tr}_f(x^a\overline{x}^b)$ is a monomial in $y_i,\overline{y}_i$, $i=1,\dots,n$.

[(2)] This property is specific to the finite map we consider; it would not be true for a general finite map; see [**Bar83**].

(4) Conclude that, for any C^∞ function φ on X, $\mathrm{Tr}_f(\varphi)$ is a C^∞ function on Y.

(5) Show that, for any test function φ on X, we have the equality

$$\int_X \varphi \cdot f^*(\mathrm{d}y \wedge \mathrm{d}\overline{y}) = \int_Y \mathrm{Tr}_f(\varphi) \cdot \mathrm{d}y \wedge \mathrm{d}\overline{y}.$$

[Hint: Use the Fubini theorem.]

Exercise 12.7 **(Trace of a C^∞ form of maximal degree and integral of currents)**

We keep the setting of Exercise 12.6. In analogy with the trace of holomorphic forms of maximal degree (Exercise 8.56), we define the trace of a form of maximal degree $h\,\mathrm{d}x \wedge \mathrm{d}\overline{x}$ as

$$\mathrm{Tr}_f(h\,\mathrm{d}x \wedge \mathrm{d}\overline{x}) := \frac{1}{\prod_i r_i^2} \cdot \frac{\mathrm{Tr}_f\big((\prod_i |x_i|^2)h\big)}{\prod_i |y_i|^2} \cdot \mathrm{d}y \wedge \mathrm{d}\overline{y}.$$

Show that $\mathrm{Tr}_f(h\,\mathrm{d}x \wedge \mathrm{d}\overline{x})$ is C^∞ on Y and is equal to the current $\int_f (h\,\mathrm{d}x \wedge \mathrm{d}\overline{x})$. [*Hint*: Use Exercise 12.6(5).]

Exercise 12.8 **(Kashiwara's equivalence).** We keep notation of Proposition 12.4.7. Let $\mathfrak{s} : {}_{\mathrm{D}}\iota_*\mathcal{M}' \otimes \overline{{}_{\mathrm{D}}\iota_*\mathcal{M}''} \to \mathfrak{C}_X$ be a sesquilinear pairing.

(1) Show that $\mathfrak{s}$ takes values in $\mathfrak{C}_{X,Z}$.

(2) Show that $\mathfrak{s}$ is determined by its values on $\iota_*(\mathcal{M}' \otimes \mathbf{1}) \otimes \overline{\iota_*(\mathcal{M}'' \otimes \mathbf{1})}$.

(3) Show that, for a test function η on X, $\langle \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})}), \eta\rangle$ only depends on $\eta_{|Z}$. [*Hint*: Write locally $X = Z \times \mathbb{C}_r$ with coordinates $x_1, \dots, x_r$ on $\mathbb{C}^r$ and use that $\mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})})\partial_{x_i} = \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})})\partial_{\overline{x}_i} = 0$.]

(4) Deduce that the correspondence $\eta_Z \mapsto \langle \mathfrak{s}((m' \otimes \mathbf{1}), \overline{(m'' \otimes \mathbf{1})}), \eta\rangle$, for some (or any) η with $\eta_{|Z} = \eta_Z$, defines a current of maximal degree on Z.

(5) Conclude the proof of Proposition 12.4.7.

Exercise 12.9 **(Pushforward of a sesquilinear pairing by a projection)**

Assume that f is the projection $X = Y \times T \to Y$. Set $p = \dim X - \dim Y = n - m$. In such a case, we have ${}_{\mathrm{D}}f_*\mathcal{M} = \boldsymbol{R}f_*(\mathcal{M} \otimes_{\mathcal{O}_X} \Theta_{X/Y,\bullet})$, according to (8.7.10 $*$). We assume that f is proper on $\operatorname{Supp}\mathcal{M}' \cap \operatorname{Supp}\mathcal{M}''$.

Let U be an open set in Y, and let

$$n'^k_\infty \in \Gamma(U, f_*(\mathcal{E}^{p+k}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}')), \quad n''^{-k}_\infty \in \Gamma(U, f_*(\mathcal{E}^{p-k}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}'')).$$

Then $f_*\mathfrak{s}(n'^k_\infty, \overline{n''^{-k}_\infty})$ is a section on U of $f_*\,\mathfrak{Db}^{p,p}_X$ and the integration of currents produces, together with a normalization similar to that of (4.2.17),

$$(-1)^{m(p+k)}\,\mathrm{Sgn}(p,k) \int_f f_*\mathfrak{s}(n'^k_\infty, \overline{n''^{-k}_\infty}) \in \Gamma(U, \mathfrak{Db}_Y). \tag{12.8.1}$$

Stokes formula implies that the pairing thus obtained

$$f_*(\mathcal{E}^{p+k}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes \overline{f_*(\mathcal{E}^{p-k}_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}'')} \longrightarrow \mathfrak{Db}_Y$$

vanishes on the image of the relative differential (on the left complex or on the right complex), and induces for each k therefore a pairing

$$\mathcal{H}^{p+k} f_*(\mathcal{E}^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}') \otimes \overline{\mathcal{H}^{p-k} f_*(\mathcal{E}^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}'')} \longrightarrow \mathfrak{Db}_Y,$$

which is easily seen to be left $\mathcal{D}_{Y,\overline{Y}}$-linear, and that we denote by

$$(12.8.2) \qquad {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)} : {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}' \otimes_{\mathbb{C}} \overline{{}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}''} \longrightarrow \mathfrak{Db}_Y\,.$$

By going from left to right at the source and the target, we obtain the corresponding pushforward sesquilinear pairing

$$(12.8.3) \qquad {}_{\mathrm{D},\overline{\mathrm{D}}}f_*^{(k,-k)} : {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}'^{\mathrm{right}} \otimes \overline{{}_{\mathrm{D}}f_*^{(-k)}\mathcal{M}''^{\mathrm{right}}} \longrightarrow \mathfrak{C}_Y\,.$$

Exercise 12.10. Show that, if $\mathcal{M}'$ or $\mathcal{M}''$ is supported on H, the right-hand side of (12.5.4 $**$) is always zero, and the residue formula (12.5.10 $**$) returns the value zero for every $\alpha \in \mathbb{R}$.

***Exercise 12.11* (see Remark 9.4.9).** Show that $\mathrm{gr}^V_\alpha(\mathfrak{s})$ induces pairings ($\ell \in \mathbb{Z}$):

$$\mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^V_\alpha(\mathfrak{s}) := \mathrm{gr}^{\mathrm{M}}_\ell \mathrm{gr}^V_\alpha \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{gr}^{\mathrm{M}}_{-\ell} \mathrm{gr}^V_\alpha \mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

and, for $\ell \geqslant 0$,
$$\mathrm{P}_\ell \mathrm{gr}^V_\alpha(\mathfrak{s}) := \mathrm{P}_\ell \mathrm{gr}^V_\alpha \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathrm{P}_\ell \mathrm{gr}^V_\alpha \mathcal{M}''} \longrightarrow \mathfrak{C}_H$$

by composing with N^ℓ on any side.

Exercise 12.12. In the setting of Proposition 9.9.1, show that, with respect to the corresponding isomorphisms,

$$\psi_{h,\lambda}\mathfrak{s} = \psi_{g,\lambda^r}\mathfrak{s} \quad \text{and} \quad \phi_{h,1}\mathfrak{s} = \phi_{g,1}\mathfrak{s}.$$

[*Hint*: Use (12.5.17 $*$) and (12.5.26 $*$).]

12.9. Comments

Complex conjugation of a locally constant sheaf of $\mathbb{C}$-vector spaces can be defined in a straightforward way by considering the conjugate vector space of each fiber. Complex conjugation of a constructible complex of $\mathbb{C}$-vector spaces can be defined similarly, and the complex conjugate of a constructible complex remains a constructible complex. Assume that this complex takes the form of the de Rham complex of a holonomic $\mathcal{D}_X$-module $\mathcal{M}$. Is it possible to define another holonomic $\mathcal{D}_X$-module $\mathcal{N}$ whose de Rham complex is the complex conjugate of that of $\mathcal{M}$? A solution to this question has been given by M. Kashiwara [**Kas86a**] when $\mathcal{M}$ is a regular holonomic $\mathcal{D}_X$-module, and by T. Mochizuki [**Moc11b**, §4.4] (see also [**Sab00**, §II.3.1]) for any holonomic $\mathcal{D}_X$-module. However, the idea of M. Kashiwara is that it is easier to find $\mathcal{N}$ whose de Rham complex is the *Verdier dual* of the complex conjugate of the de Rham complex of $\mathcal{M}$. Namely, $\mathcal{N}$ is defined as the complex conjugate (in the sense of passing from $\mathcal{D}_{\overline{X}}$-modules to $\mathcal{D}_X$-modules) of the $\mathcal{D}_{\overline{X}}$-module $\mathcal{H}om_{\mathcal{D}_X}(\mathcal{M}, \mathfrak{Db}_X)$. In other words, when working with $\mathcal{D}_X$-modules, it is easier to handle the *Hermitian dual* $\mathcal{D}_X$-module than the conjugate $\mathcal{D}_X$-module, since the duality functor is not needed.

This explains why, when considering complex Hodge structures and having in mind the extension to $\mathcal{D}_X$-modules, instead of considering pairs of vector spaces together with an isomorphism of one space with the complex conjugate of the other one, we consider pairs of vector spaces together with an isomorphism of one space with the Hermitian dual of the other one, that is, pairs of vector spaces together with a non-degenerate sesquilinear pairing between them. Furthermore, the notion of non-degeneracy is difficult to manipulate under various operations on $\mathcal{D}_X$-modules, and this explains why this property is relaxed in the definition of the category $\mathcal{D}$-$\mathsf{Triples}$.

The idea of considering poles of Mellin transforms with kernel $|f|^{2s}$ for some holomorphic function f, in order to analyze its nearby cycles and the monodromy on them, goes back to the work of D. Barlet ([**Bar82**] and the subsequent works [**Bar84, Bar85, Bar86, BM87, BM89**]). It has been instrumental in order to define nearby cycles in the theory of twistor $\mathcal{D}$-modules ([**Sab05**]), where the idea of taking a residue of such Mellin transforms has been introduced. Many aspects of this theory have then been much improved in the works of T. Mochizuki [**Moc11a, Moc15**] and we have taken advantage of these improvements in the presentation of this chapter.

CHAPTER 13

$\mathcal{D}$-MODULES OF NORMAL CROSSING TYPE

Summary. This chapter, although somewhat technical, is nevertheless essential to understand the behaviour of Hodge modules when the singularities form a normal crossing divisor. It analyzes the compatibility properties on a given $\mathbb{R}$-specializable $\mathcal{D}$-module with respect to various functions, when these functions form part of a coordinate system. The results of this chapter will therefore be of a local nature.

13.1. Introduction

13.1.1. Notation. In this chapter, the setting is as follows. The space $X = \Delta^n$ is a polydisc in $\mathbb{C}^n$ with analytic coordinates $x_1, \dots, x_n$, we fix $\ell \leqslant n$ and we denote by D the divisor $\{x_1 \cdots x_\ell = 0\}$. We also denote by D_i $(i = 1, \dots, \ell)$ the smooth components of D and by $D_{(\ell)}$ their intersection $D_1 \cap \cdots \cap D_\ell$. We will shorten the notation $\mathbb{C}[x_1, \dots, x_\ell]$ into $\mathbb{C}[x]$ and $\mathbb{C}[x_1, \dots, x_\ell]\langle \partial_{x_1}, \dots, \partial_{x_\ell}\rangle$ into $\mathbb{C}[x]\langle \partial_x\rangle$. We will set $I = \{1, \dots, \ell\}$.

Given a non-constant monomial function vanishing on D at most, that we denote by $g = x^{\boldsymbol{a}} = x_1^{a_1} \cdots x_\ell^{a_\ell}$ ($a_i \geqslant 0$ for $i \in I$ and $a_i > 0$ for some i), we denote by $I_g \subset I$ the non-empty set of $i \in I$ such that $a_i \neq 0$.

We will mainly consider *right* $\mathcal{D}$-modules.

13.1.2. Simplifying assumptions. All over this section, we will consider the simple case where $\ell = n$, that is, $D_{(\ell)}$ is reduced to the origin in Δ^n, in order to make the computations clearer. We then have $I = \{1, \dots, n\}$. The general case $\ell \neq n$ brings up objects which are $\mathcal{O}_{D_{(\ell)}}$-locally free and the adaptation is straightforward.

In higher dimensions, the theory of vector bundles on Δ with meromorphic integrable connections with poles along D starts with the simplest objects, namely those with regular singularities [**Del70**]. One first extends naturally these objects as locally free $\mathcal{O}_\Delta(*D)$-modules with integrable connection and the regularity property amounts to the existence of locally free $\mathcal{O}_\Delta$-module of maximal rank on which the connection has logarithmic poles. The category of such objects is equivalent to that of locally constant sheaves on $\Delta \smallsetminus D$, that is, of finite dimensional representations of $\pi_1(\Delta \smallsetminus D) \simeq \mathbb{Z}^n$. These objects behave like products of meromorphic bundles with connection having a regular singularity in dimension 1. We say that these objects are *of normal crossing type.*

Our first aim is to extend this notion to other holonomic $\mathcal{D}_\Delta$-modules. We mainly have in mind the middle extension of such meromorphic connections. In terms of general $\mathcal{D}$-module theory—that we will not use—we could characterize such $\mathcal{D}$-modules as the regular holonomic $\mathcal{D}$-modules whose characteristic variety is adapted to the natural stratification of the divisor D. In other words, these are the simplest objects in higher dimension.

We can settle the problem as follows. Let $\mathcal{M}$ be a coherent $\mathcal{D}$-module on Δ. Assume that $\mathcal{M}$ is $\mathbb{R}$-specializable along each component D_i of D. How do the various V-filtrations interact? The notion of *normal crossing type* aims at reflecting that these V-filtrations behave independently, i.e., without any interaction. In other words, the transversality property of the components of D is extended to the transversality property of the V-filtrations. This is first explained in Section 13.2.a for the simpler "algebraic case" and then in Section 13.3.a for the general holomorphic case.

Sesquilinear pairings between coherent $\mathcal{D}$-modules of normal crossing type have then a simple expression in terms of *basic distributions or currents* (Section 13.5).

When thinking in terms of characteristic varieties, one can expect that the notion of "normal crossing type" is stable with respect to taking nearby or vanishing cycles along a monomial function in the given coordinates. However, obtaining an explicit expression of the various monodromies in terms of the original ones leads to a delicate combinatorial computation, which is achieved in Section 13.2.b for the simpler "algebraic case" and then in Section 13.3.b for the general holomorphic case.

We are mainly interested in the previous results in the presence of an F-filtration and, for a coherently F-filtered $\mathcal{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$, we will express the independence of the V-filtrations in the presence of $F_\bullet\mathcal{M}$. By looking in dimension 1, one first realizes that $(\mathcal{M}, F_\bullet\mathcal{M})$ should be $\mathbb{R}$-specializable along any component D_i of D. But adding an F-filtration to the picture also leads us to take much care of the behaviour of this filtration with respect to the various V-filtrations along the components D_i of the divisor D. The compatibility property (Definition 10.3.9), is essential in order to have a reasonable control on various operations on these filtered $\mathcal{D}$-modules.

An important question, given a filtered $\mathcal{D}$-module $(\mathcal{M}, F_\bullet\mathcal{M})$ such that $\mathcal{M}$ is of normal crossing type along D, is to have an effective criterion on the F-filtration for $(\mathcal{M}, F_\bullet\mathcal{M})$ to be of filtered normal crossing type. We give such a criterion in terms of *parabolic bundles* (Section 13.4.c).

The following notation will be used.

13.1.3. Notation. The V-filtration (of $\mathcal{D}_X$, $\widetilde{\mathcal{D}}_X$ or a module over these sheaves of rings) along D_i will be denote by $V_\bullet^{(i)}$, where $\bullet$ runs in $\mathbb{Z}$ or $A+\mathbb{Z}$. We will then set (when the simplifying assumption 13.1.2 holds)

$$V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\widetilde{\mathcal{D}}_X := \bigcap_{i=1}^n V_{\alpha_i}^{(i)}\widetilde{\mathcal{D}}_X, \quad \boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_n).$$

Our main objective in this chapter is to compute the nearby cycles of such filtered $\mathcal{D}$-modules along a monomial function $g = x^{\boldsymbol{a}}$ (with respect to coordinates adapted to D). It is stated as follows, where the still undefined notions will be explained with details below.

13.1.4. Theorem. *Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. Assume that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a middle extension along each D_i $(i\in I)$. Then $(\mathcal{M}, F_\bullet\mathcal{M})$ is $\mathbb{R}$-specializable and a middle extension along (g). Moreover, for every $\lambda \in \mathbb{S}^1$, $(\psi_{g,\lambda}\mathcal{M}, F_\bullet\psi_{g,\lambda}\mathcal{M})$ is of normal crossing type along D.*

13.2. Algebraic normal crossing type

13.2.a. $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type. In this section, we consider the algebraic setting where we replace the sheaf $\mathcal{D}_X$ with the ring $\mathbb{C}[x]\langle\partial_x\rangle$ and correspondingly (right) $\mathcal{D}_X$-modules with (right) $\mathbb{C}[x]\langle\partial_x\rangle$-modules, that we denote by a capital letter like M.

Let us consider, for every $\boldsymbol{\alpha} \in \mathbb{R}^n$, the subspace $M_{\boldsymbol{\alpha}}$ of M defined by

$$M_{\boldsymbol{\alpha}} = \bigcap_{i\in I} \bigcup_k \operatorname{Ker}(x_i\partial_{x_i} - \alpha_i)^k.$$

This is a $\mathbb{C}$-vector subspace of M. The endomorphism $x_i\partial_{x_i}$ acting on $M_{\boldsymbol{\alpha}}$ will be denoted by E_i and $(x_i\partial_{x_i} - \alpha_i)$ by N_i. The family $(\mathrm{N}_1, \dots, \mathrm{N}_n)$ forms a commuting family of endomorphisms of $M_{\boldsymbol{\alpha}}$, giving $M_{\boldsymbol{\alpha}}$ a natural $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n]$-module structure, and every element of $M_{\boldsymbol{\alpha}}$ is annihilated by some power of each N_i. Moreover, for $i \in I$, the morphism $x_i : M \to M$ (resp. $\partial_{x_i} : M \to M$) induces a $\mathbb{C}$-linear morphism $x_i : M_{\boldsymbol{\alpha}} \to M_{\boldsymbol{\alpha}-\mathbf{1}_i}$ (resp. $\partial_{x_i} : M_{\boldsymbol{\alpha}} \to M_{\boldsymbol{\alpha}+\mathbf{1}_i}$). For each fixed $\boldsymbol{\alpha} \in \mathbb{R}^n$, we have

$$M_{\boldsymbol{\alpha}} \cap \Big(\sum_{\boldsymbol{\alpha}'\neq\boldsymbol{\alpha}} M_{\boldsymbol{\alpha}'}\Big) = 0 \quad \text{in } M.$$

Indeed, for $m = \sum_{\boldsymbol{\alpha}'\neq\boldsymbol{\alpha}} m_{\boldsymbol{\alpha}'}$, if $m \in M_{\boldsymbol{\alpha}}$, then $m - \sum_{\alpha'_1=\alpha_1} m_{\boldsymbol{\alpha}'}$ is annihilated by some power of $x_1\partial_{x_1} - \alpha_1$ and by a polynomial $\prod_{\alpha'_1\neq\alpha_1}(x_1\partial_{x_1} - \alpha'_1)^{k_{\alpha'_1}}$, hence is zero, so we can restrict the sum above to $\alpha'_1 = \alpha_1$. Arguing similarly for $i = 2, \dots, n$ gives finally $m = 0$. It follows that

$$M' := \bigoplus_{\boldsymbol{\alpha}\in\mathbb{R}^n} M_{\boldsymbol{\alpha}} \subset M \tag{13.2.1}$$

is a $\mathbb{C}[x]\langle\partial_x\rangle$-submodule of M. The actions of x_i and ∂_{x_i} is treated in Exercise 13.1.

13.2.2. Definition. Let M be a $\mathbb{C}[x]\langle\partial_x\rangle$-module. We say that M is *of normal crossing type along D* if the following properties are satisfied.

(a) There exists a finite subset $\boldsymbol{A} \subset [-1, 0)^n$, called the set of *exponents of M*, such that $M_{\boldsymbol{\alpha}} = 0$ for $\boldsymbol{\alpha} \notin \boldsymbol{A} + \mathbb{Z}^n$.

(b) Each $M_{\boldsymbol{\alpha}}$ $(\boldsymbol{\alpha} \in \mathbb{R}^n)$ is finite-dimensional.

(c) The natural inclusion (13.2.1) is an equality.

13.2.3. Remark. For every $\boldsymbol{\alpha} \in \boldsymbol{A}$, let us set

$$M_{\boldsymbol{\alpha}+\mathbb{Z}^n} = \bigoplus_{\boldsymbol{k}\in\mathbb{Z}^n} M_{\boldsymbol{\alpha}+\boldsymbol{k}},$$

so that $M = \bigoplus_{\boldsymbol{\alpha}\in\boldsymbol{A}} M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$. Then $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is a $\mathbb{C}[x]\langle\partial_x\rangle$-module. In such a way, M is the direct sum of $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type having a single exponent.

The category of $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type along D is, by definition, the full subcategory of that of $\mathbb{C}[x]\langle\partial_x\rangle$-modules whose objects are of normal crossing type along D.

13.2.4. Proposition. *Every morphism between $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type along D is graded with respect to the decomposition* (13.2.1)*, and the category of $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type along D is abelian.*

Proof. By $\mathbb{C}[x]\langle\partial_x\rangle$-linearity and using Bézout's theorem, one checks that any morphism $\varphi : M_1 \to M_2$ sends $M_{1,\boldsymbol{\alpha}}$ to $M_{2,\boldsymbol{\alpha}}$, and has no component from $M_{1,\boldsymbol{\alpha}}$ to $M_{2,\boldsymbol{\beta}}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$. $\square$

13.2.5. Proposition **(Description by quivers).** *Let us fix $\boldsymbol{\alpha} \in [-1,0)^n$ and let us set $I(\boldsymbol{\alpha}) = \{i \in I \mid \alpha_i = -1\}$. Then the category of $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type with exponent $\boldsymbol{\alpha}$, that is, of the form $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$, is equivalent to the category of $I(\boldsymbol{\alpha})$-quivers having the vertex $M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ equipped with its $\mathbb{C}[\mathrm{N}_1,\dots,\mathrm{N}_n]$-module structure at the place $\boldsymbol{k} \in \{0,1\}^{I(\boldsymbol{\alpha})}$ and arrows*

$$\begin{aligned} \mathrm{can}_i &: M_{\boldsymbol{\alpha}+\boldsymbol{k}} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_i}, \\ \mathrm{var}_i &: M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_i} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}}, \end{aligned} \qquad \textit{if } k_i = 0,$$

subject to the conditions

$$\begin{cases} \mathrm{var}_i \circ \mathrm{can}_i = \mathrm{N}_i : M_{\boldsymbol{\alpha}+\boldsymbol{k}} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}}, \\ \mathrm{can}_i \circ \mathrm{var}_i = \mathrm{N}_i : M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_i} \longrightarrow M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_i}, \end{cases} \qquad \textit{if } k_i = 0.$$

(It is understood that if $I(\boldsymbol{\alpha}) = \varnothing$, then the quiver has only one vertex and no arrows.)

Proof. It is straightforward, by using that, for $\boldsymbol{k} \in \mathbb{Z}^n$, $\partial_{x_i} : M_{\boldsymbol{\alpha}+\boldsymbol{k}} \to M_{\boldsymbol{\alpha}+\boldsymbol{k}+\mathbf{1}_i}$ is an isomorphism if $i \notin I(\boldsymbol{\alpha})$ or $i \in I(\boldsymbol{\alpha})$ and $k_i \geqslant 0$, while $x_i : M_{\boldsymbol{\alpha}+\boldsymbol{k}} \to M_{\boldsymbol{\alpha}+\boldsymbol{k}-\mathbf{1}_i}$ is an isomorphism if $i \notin I(\boldsymbol{\alpha})$ or $i \in I(\boldsymbol{\alpha})$ and $k_i \leqslant -1$. $\square$

13.2.6. Remark. In order not to specify a given exponent of a $\mathbb{C}[x]\langle\partial_x\rangle$-module of normal crossing type along D, it is convenient to define the quiver with vertices indexed by $\{0,1\}^I$ instead of $\{0,1\}^{I(\boldsymbol{\alpha})}$. We use the convention that, for a fixed $\boldsymbol{\alpha} \in [-1,0)^n$ and for $i \notin I(\boldsymbol{\alpha})$, $\mathrm{var}_i = \mathrm{Id}$ and $\mathrm{can}_i = \alpha_i \,\mathrm{Id} + \mathrm{N}_i/\,2\pi\mathrm{i} = \mathrm{E}_i$ (hence both are isomorphisms). Then the category of $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type along D is equivalent to the category of such quivers.

13.2.7. Definition. We say that M is *dual localized* (resp. a *middle extension*) along D_{i_o}, that we denote by $M = M(!D_{i_o})$ (resp. $M = M(!{*}D_{i_o})$) if can_{i_o} is bijective (resp. can_{i_o} is onto and var_{i_o} is injective).

13.2.8. Definition. We say that M is a *middle extension along* $D_{i\in I}$ if, for each $i\in I$, every can_i is onto and every var_i is injective.

See Exercises 13.4–13.6.

***13.2.9. Example* (The simple case).** Let M be a $\mathbb{C}[x]\langle\partial_x\rangle$-module of normal crossing type along D which is *simple* (i.e., has no non-trivial such sub or quotient module). By the previous exercise, it must be a middle extension with support along $D_{i\in I}$. Moreover, every nonzero vertex of its quiver has dimension 1, so that E_i acts as α_i on $M_{\boldsymbol{\alpha}}$ and N_i acts by zero.

***13.2.10. Remark* (Suppressing the simplifying assumptions 13.1.2)**

If $\ell<n$, every $M_{\boldsymbol{\alpha}}$ ($\boldsymbol{\alpha}\in\mathbb{R}^\ell$) has to be assumed $\mathcal{O}_{D_{(\ell)}}$-coherent in Definition 13.2.2(b). Since it is a $\mathcal{D}_{D_{(\ell)}}$-module, it must be $\mathcal{O}_{D_{(\ell)}}$-locally free of finite rank. All the previous results extend in a straightforward way to this setting by replacing $\mathbb{C}[x]$ with $\mathcal{O}_{D_{(\ell)}}[x]$ (where $x:=(x_1,\dots,x_\ell)$) and $\mathbb{C}[x]\langle\partial_x\rangle$ with $\mathcal{D}_{D_{(\ell)}}[x]\langle\partial_x\rangle$.

13.2.b. Nearby cycles for $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type

We continue to refer implicitly to Notation 13.1.1 and the simplifying assumptions 13.1.2.

13.2.11. Notation. The indices for which $a_i=0$ do not play an important role. Let us denote by $I_g=I(\boldsymbol{a}):=\{i\mid a_i\neq0\}\subset\{1,\dots,n\}$ the complementary subset, $\boldsymbol{a}'=(a_i)_{i\in I_g}$ and $n'=\#I_g$. Accordingly, we decompose the set of variables $(x_1,\dots,x_n)$ as (x',x''), with $x'=(x_i)_{i\in I_g}$.

In this section, we consider the variant of Theorem 13.1.4 where we forget the filtration $F_\bullet$ and where we consider the case of $\mathbb{C}[x]\langle\partial_x\rangle$-modules, with the notation of Section 13.2.a. The proof will be done by giving an explicit expression of the V-filtration of ${}_{\mathrm{D}}\iota_{g*}M$ with respect to t, as well as its associated graded modules. The proof will also make precise the set of jumping indices of the V-filtration (see Remark 13.2.42(2)).

In order to simplify the notation, we will set $N={}_{\mathrm{D}}\iota_{g*}M\simeq M[\partial_t]$, which is a $\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle$-module. According to (8.7.7 $*$), the action of $\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle$ is as follows:

$$
\begin{aligned}
(m\otimes\partial_t^k)\cdot\partial_t&=m\otimes\partial_t^{k+1}\\
(m\otimes1)\cdot\partial_{x_i}&=m\partial_{x_i}\otimes1-(a_imx^{\boldsymbol{a}-\mathbf{1}_i})\otimes\partial_t\\
(m\otimes1)\cdot f(x,t)&=mf(x,x^{\boldsymbol{a}})\otimes1.
\end{aligned}
\tag{13.2.12}
$$

As a consequence, for $i\in I_g$ we have

$$
(m\otimes1)\cdot t\partial_t=(mx^{\boldsymbol{a}}\otimes1)\cdot\partial_t=\frac{1}{a_i}\big[(mx_i\partial_{x_i}\otimes1)-(m\otimes1)x_i\partial_{x_i}\big].
\tag{13.2.13}
$$

13.2.14. Notation. In order to distinguish between the action of $x_i\partial_{x_i}$ trivially coming from that on M and the action $x_i\partial_{x_i}$ on N, it will be convenient to denote by S_i the first one, defined by

$$
(m\otimes\partial_t^k)\cdot\mathrm{S}_i=(mx_i\partial_{x_i})\otimes\partial_t^k.
$$

Then we can rewrite S_i as

$$(m\otimes\partial_t^k)\cdot\mathrm{S}_i=(m\otimes 1)\cdot(x_i\partial_{x_i}+a_it\partial_t)\partial_t^k=(m\otimes\partial_t^k)\cdot(x_i\partial_{x_i}+a_i(t\partial_t-k)),$$

a formula that can also be read

$$(m\otimes\partial_t^k)\cdot x_i\partial_{x_i}=(m\otimes\partial_t^k)\cdot(\mathrm{S}_i-a_it\partial_t+a_ik). \tag{13.2.15}$$

Note that N is naturally graded: $N=\bigoplus_{\boldsymbol{\alpha},k}M_{\boldsymbol{\alpha}}\otimes\partial_t^k$.

***Proof of Theorem* 13.1.4*, Step* 1*:* $\mathbb{R}$*-specializability of* N *along* (t)**

13.2.16. Proposition. *The $\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle$-module N is $\mathbb{R}$-specializable along (t). Moreover, $N=N[!{*}t]$.*

We will show the $\mathbb{R}$-specializability by making explicit a V-filtration of N. To get started, consider the following simple example.

13.2.17. Example. Let us fix $\gamma\in\mathbb{R}$. Assume we know that ${}^\gamma N:=\mathrm{gr}_\gamma^VN$ is of normal crossing type along D. Suppose that for some $m\in M_{\boldsymbol{\alpha}}$ and some $k\geqslant 0$, the section $m\otimes\partial_t^k$ belongs to $V_\gamma N$, and that its projection to gr_γ^VN is nonzero and happens to lie in the subspace

$${}^\gamma N_{\boldsymbol{\beta}}:=(\mathrm{gr}_\gamma^VN)_{\boldsymbol{\beta}}.$$

In this situation, $\gamma,\boldsymbol{\alpha},\boldsymbol{\beta}$, and k are related. Indeed, the identity in (13.2.15) shows that

$$(m\otimes\partial_t^k)\cdot\big((x_i\partial_{x_i}-\beta_i)-(\mathrm{S}_i-\alpha_i)+a_i(\mathrm{E}-\gamma)\big)=(m\otimes\partial_t^k)\cdot\big(\alpha_i-\beta_i-a_i(\gamma-k)\big).$$

By assumption, $\mathrm{E}-\gamma=t\partial_t-\gamma$ and $x_i\partial_{x_i}-\alpha_i=x_i\partial_{x_i}-\beta_i$ both act nilpotently on ${}^\gamma N_{\boldsymbol{\beta}}$; since $\mathrm{S}_i-\alpha_i$ acts nilpotently on $M_{\boldsymbol{\alpha}}\otimes\partial_t^k$, the conclusion is that $\boldsymbol{\alpha}=\boldsymbol{\beta}+(\gamma-k)\boldsymbol{a}$. Thus we expect elements of $M_{\boldsymbol{\beta}+(\gamma-k)\boldsymbol{a}}\otimes\partial_t^k$ to contribute to the subspace ${}^\gamma N_{\boldsymbol{\beta}}$.

This computation motivates the following definition (see Notation 13.1.3).

13.2.18. Definition. For $\gamma<0$, we set

$$V_\gamma N=(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M\otimes 1)\cdot\mathbb{C}[x]\langle\partial_x\rangle=\sum_{\boldsymbol{k}\in\mathbb{N}^n}(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M\otimes 1)\cdot\partial_x^{\boldsymbol{k}}. \tag{13.2.18 $*$}$$

For every $\gamma\in[-1,0)$ and $j\geqslant 1$, we define inductively

$$V_{\gamma+j}N=V_\gamma N\cdot\partial_t^j+V_{<\gamma+j}N. \tag{13.2.18 $**$}$$

Note that the latter formula is natural if we expect that N is a middle extension along (t).

13.2.19. Lemma. *The filtration $V_\bullet N$ is a Kashiwara-Malgrange filtration for N.*

We first need to check that (13.2.18 $*$) and (13.2.18 $**$) define a V-filtration.

13.2.20. Lemma. *For every $\gamma\in\mathbb{R}$, $V_\gamma N$ is a $V_0(\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle)$-module that satisfies*

$$V_\gamma N\cdot t\subset V_{\gamma-1}N,\quad V_{\gamma-1}N\cdot\partial_t+V_{<\gamma}N\subset V_\gamma N,$$

with equality in the first inclusion if $\gamma<0$ and in the second one if $\gamma>0$.

Proof. Assume first that $\gamma < 0$. By definition, $V_\gamma N$ is a $\mathbb{C}[x]\langle\partial_x\rangle$-module, so it remains to prove stability by the actions of t and $t\partial_t$. On the one hand,

$$(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1)\cdot t = V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}Mx^{\boldsymbol{a}} \otimes 1 = V_{(\gamma-1)\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1,$$

hence

$$V_\gamma N \cdot t = V_{\gamma-1}N \subset V_\gamma N.$$

On the other hand, (13.2.13) shows that, for any $i \in I_g$, we have

$$(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1)\cdot t\partial_t \subset V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1 + (V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}Mx_i \otimes 1)\partial_{x_i}. \tag{13.2.21}$$

We conclude that the statements of the lemma hold for $\gamma < 0$. Moreover, the stability of $V_\gamma N$ by $t\partial_t$ given by (13.2.21) implies, for $\gamma < 0$:

$$V_\gamma N = (V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1)\cdot \mathbb{C}[x]\langle\partial_x\rangle[t\partial_t]. \tag{13.2.22}$$

The assertions for $\gamma \geqslant 0$ follow then easily from Definition (13.2.18 $**$). ☐

13.2.23. Remark. Note that, for $i \notin I_g$ (Notation 13.2.11), we have $\gamma a_i = 0$ and

$$\sum_{k_i \geqslant 0}(V_0^{(i)}M \otimes 1)\partial_{x_i}^{k_i} = \sum_{k_i \geqslant 0}(V_0^{(i)}M\partial_{x_i}^{k_i} \otimes 1) = M \otimes 1.$$

As a consequence, for $\gamma < 0$, (13.2.18 $*$) can be simplified as follows (see Notation 13.2.11):

$$V_\gamma N = \sum_{\boldsymbol{k}' \in \mathbb{N}^{n'}}(V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}M \otimes 1)\cdot \partial_{x'}^{\boldsymbol{k}'}. \tag{13.2.24}$$

In order to analyze $V_\gamma N$, it will be useful to obtain a natural presentation as a $\mathbb{C}[x', x'', t]\langle\partial_{x'}, \partial_{x''}, t\partial_t\rangle$-module. For that purpose, we recall that $V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}M$ is a $\mathbb{C}[x', x'']\langle x'\partial_{x'}, \partial_{x''}\rangle$-module. We forget for a moment the action of $\mathbb{C}[x']\langle x'\partial_{x'}\rangle$ and consider

$$K_\gamma := V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}M \otimes_{\mathbb{C}} \mathbb{C}\langle\partial_{x'}, t\partial_t\rangle.$$

It is naturally equipped with an action of $\mathbb{C}[x', x'', t]\langle\partial_{x'}, \partial_{x''}, t\partial_t\rangle$, by imposing the relations $m \otimes x_i = mx_i \otimes 1$ ($i \in I_g$) and $m \otimes t = mx^{\boldsymbol{a}'} \otimes 1$, and the usual commutation rules. Formula (13.2.24) induces a surjective morphism of $\mathbb{C}[x', x'', t]\langle\partial_{x'}, \partial_{x''}, t\partial_t\rangle$-modules:

$$K_\gamma \longrightarrow V_\gamma N$$

sending any $mx_i\partial_{x_i} \otimes 1 - m \otimes x_i\partial_{x_i} - (m \otimes 1)a_i t\partial_t$ to zero ($i \in I_g$), according to (13.2.13).

***Proof of Lemma* 13.2.19 *and Proposition* 13.2.16.** Let us start with $\gamma < 0$. Since $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M$ has finite type over $\mathbb{C}[x]$, Formula (13.2.18 $*$) implies that $V_\gamma N$ has finite type over $\mathbb{C}[x]\langle\partial_x\rangle$, and a fortiori over $V_0(\mathbb{C}[x,t]\langle\partial_x, \partial_t\rangle)$.

In order to show that some power of $(t\partial_t - \gamma)$ sends $V_\gamma N$ to $V_{<\gamma}N$ we first notice that a power of $\mathrm{S}_i - \gamma a_i$ does so for every $i = 1, \dots, n$. It is thus enough to check that $\prod_{i\in I_g}(\mathrm{S}_i - a_i t\partial_t)$ sends $(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1)$ into $V_{\gamma'}N$ for some $\gamma' < \gamma$. (Indeed, this will imply that some power of $(t\partial_t - \gamma)$ sends $(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M \otimes 1)$ into $V_{\gamma'}N$, and (13.2.18 $*$) enables us to conclude.)

We have $\gamma\boldsymbol{a}-\mathbf{1}_{I_g}\leqslant\gamma'\boldsymbol{a}$ for some $\gamma'<\gamma$, so $(V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M\otimes 1)\cdot\prod_{i\in I_g}x_i\subset(V^{(\boldsymbol{n})}_{\gamma'\boldsymbol{a}}M\otimes 1)$, and thus, by (13.2.18 $*$),

$$(V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M\otimes 1)\cdot\prod_{i\in I_g}x_i\partial_{x_i}=(V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M\otimes 1)\cdot\prod_{i\in I_g}x_i\prod_{i\in I_g}\partial_{x_i}\subset V_{\gamma'}N.$$

Therefore,

$$(V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M\otimes 1)\cdot\prod_{i\in I_g}(\mathrm{S}_i-a_it\partial_t)\subset V_{\gamma'}N.$$

In order to conclude that N is $\mathbb{R}$-specializable along (t) and that $V_\bullet N$ is its Kashiwara-Malgrange filtration along (t), it only remains to be proved that $N=\bigcup_\gamma V_\gamma N$, and so it is enough to prove that any element of $M\otimes 1$ belongs to some $V_\gamma N$. Let us consider a component $M_{\boldsymbol{\beta}}\otimes 1$ with $\boldsymbol{\beta}\in\mathbb{R}^n$, that we write $\boldsymbol{\beta}=\boldsymbol{\alpha}+\boldsymbol{k}_+-\boldsymbol{k}_-$, $\boldsymbol{\alpha}\in\boldsymbol{A}$, $\boldsymbol{k}_+,\boldsymbol{k}_-\in\mathbb{N}^n$ with disjoint support. The middle extension property of M implies that $x^{\boldsymbol{k}_-}\partial_x^{\boldsymbol{k}_+}:M_{\boldsymbol{\alpha}}\to M_{\boldsymbol{\beta}}$ is onto. We can thus use iteratively (13.2.12) to write any element of $M_{\boldsymbol{\beta}}$ as a sum of terms $(\mu_k\otimes 1)\cdot\partial_t^k$ ($k\geqslant 0$), where the components of each μ_k in the decomposition (13.3.1) only involve indices in $(\mathbb{R}_{<0})^n$, and therefore belongs to $V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M$ for some $\gamma<0$.

Let us end by proving that N is a middle extension along (t). We first remark that t acts injectively on N: if we consider the filtration $G_\bullet N$ by the degree in ∂_t, then the action of t on $\mathrm{gr}^GN\simeq M[\tau]$ is equal to the induced action of $x^{\boldsymbol{a}}$ on $M[\tau]$, hence is injective by the assumption on M; a fortiori, the action of t on N is injective. We thus have $N\subset N[*t]$. By Definition 13.2.18 and the exhaustivity of $V_\bullet N$ proved above, N is the image of $V_{<0}N\otimes\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle$ in $N[*t]$. This is nothing but $N[!*t]$ (see Definition 11.5.2 and Definition 11.4.6). $\square$

***Proof of Theorem* 13.1.4, *Step* 2*: normal crossing type of* $\mathrm{gr}^V_\gamma N$.** We aim at proving that each $\mathrm{gr}^V_\gamma N$ ($\gamma\in[-1,0)$) is of normal crossing type along D, and at making explicit the summands. We now fix such a γ, and set ${}^\gamma\! N=\mathrm{gr}^V_\gamma N$ for the remaining part of the proof. Let $\boldsymbol{\beta}\in\mathbb{R}^n$. Let us define ${}^\gamma\! N_{\boldsymbol{\beta}}$ by the formula

$${}^\gamma\! N_{\boldsymbol{\beta}}=\bigcap_{i=1}^n\bigcup_k\mathrm{Ker}(x_i\partial_{x_i}-\beta_i)^k,$$

where we regard each $(x_i\partial_{x_i}-\beta_i)^k$ as acting on ${}^\gamma\! N$ through its action on N given by (13.2.15). We then denote by N_i the action of $x_i\partial_{x_i}-\beta_i$ on ${}^\gamma\! N_{\boldsymbol{\beta}}$ (and, as usual, by E, resp. N, the action of $t\partial_t$, resp. $t\partial_t-\gamma$ on ${}^\gamma\! N$ and ${}^\gamma\! N_{\boldsymbol{\beta}}$). By using Bézout's theorem, one checks that ${}^\gamma\! N_{\boldsymbol{\beta}}$ intersects only at zero any sum of submodules ${}^\gamma\! N_{\boldsymbol{\beta}'}$, where $\boldsymbol{\beta}'$ runs in a finite set not containing $\boldsymbol{\beta}$, so we will only need to check the finite dimensionality of each ${}^\gamma\! N_{\boldsymbol{\beta}}$ and the existence of a decomposition ${}^\gamma\! N\simeq\sum_{\boldsymbol{\beta}}{}^\gamma\! N_{\boldsymbol{\beta}}$ (hence ${}^\gamma\! N\simeq\bigoplus_{\boldsymbol{\beta}}{}^\gamma\! N_{\boldsymbol{\beta}}$). This will be done at the next step, and we start by modifying the expression of ${}^\gamma\! N_{\boldsymbol{\beta}}$.

13.2.25. Lemma. *For every $\boldsymbol{\beta}\in\mathbb{R}^n$, ${}^\gamma\! N_{\boldsymbol{\beta}}$ is the image of $V_\gamma N\cap\big(\bigoplus_j M_{\boldsymbol{\beta}+(\gamma-j)\boldsymbol{a}}\otimes\partial_t^j\big)$ in ${}^\gamma\! N$.*

Proof. Let us consider an arbitrary element of $V_\gamma N$, expressed as a finite sum

$$\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n}\sum_{j\in\mathbb{N}} m_{\boldsymbol{\alpha},j}\otimes\partial_t^j,$$

with $m_{\boldsymbol{\alpha},j}\in M_{\boldsymbol{\alpha}}$. Assume that its image in $\mathrm{gr}^V_\gamma N={}^\gamma N$ belongs to ${}^\gamma N_{\boldsymbol{\beta}}$, i.e.,

$$\Big(\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n}\sum_{j\in\mathbb{N}} m_{\boldsymbol{\alpha},j}\otimes\partial_t^j\Big)\cdot(x_i\partial_{x_i}-\beta_i)^k\in V_{<\gamma}N$$

for every $i\in I$ and some $k\gg 0$. Our aim is to prove that, modulo $V_{<\gamma}N$, only those terms with $\boldsymbol{\alpha}=\boldsymbol{\beta}+(\gamma-j)\boldsymbol{a}$ matter.

13.2.26. Lemma. *In the situation considered above, one has*

$$\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n}\sum_{j\in\mathbb{N}} m_{\boldsymbol{\alpha},j}\otimes\partial_t^j=\sum_{j\in\mathbb{N}} m_{\boldsymbol{\beta}+(\gamma-j)\boldsymbol{a}}\otimes\partial_t^j \mod V_{<\gamma}N.$$

Proof. Let us start with an elementary lemma of linear algebra.

13.2.27. Lemma. *Let T be an endomorphism of a complex vector space V, and $W\subset V$ a linear subspace with $TW\subset W$. Suppose that $v_1,\dots,v_k\in V$ satisfy*

$$T^\mu(v_1+\cdots+v_k)\in W$$

for some $\mu\geqslant 0$. If there are pairwise distinct complex numbers $\lambda_1,\dots,\lambda_k$ with $v_h\in E_{\lambda_h}(T)$, then one has $\lambda_h v_h\in W$ for every $h=1,\dots,k$.

Proof. Choose a sufficiently large integer $\mu\in\mathbb{N}$ such that $(T-\lambda_h)^\mu v_h=0$ for $h=1,\dots,k$, and such that $T^\mu(v_1+\cdots+v_k)\in W$. Assume that $\lambda_k\neq 0$. Setting $Q(T)=T^\mu(T-\lambda_1)^\mu\cdots(T-\lambda_{k-1})^\mu$, we have by assumption

$$Q(T)(v_1+\cdots+v_k)\in W$$

The left-hand side equals $Q(T)v_k$. Since $Q(T)$ and $T-\lambda_k$ are coprime, Bézout's theorem implies that $v_k\in W$. At this point, we are done by induction. □

We now go back to the proof of Lemma 13.2.26. Let us consider an element as in the lemma. As we have seen before,

$$(m_{\boldsymbol{\alpha},j}\otimes\partial_t^j)\cdot\big((x_i\partial_{x_i}-\beta_i)+a_i(t\partial_t-\gamma)\big)=(m_{\boldsymbol{\alpha},j}\otimes\partial_t^j)\cdot\big(\mathrm{S}_i-\beta_i-a_i(\gamma-j)\big),$$

and since some power of $t\partial_t-\gamma$ also send this element in $V_{<\gamma}N$, we may conclude that

$$\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n}\sum_{j\in\mathbb{N}}\Big(m_{\boldsymbol{\alpha},j}\otimes\partial_t^j\cdot\big(\mathrm{S}_i-\beta_i-a_i(\gamma-j)\big)^k\Big)\in V_{<\gamma}N \tag{13.2.28}$$

for every $i\in I$ and $k\gg 0$.

In order to apply Lemma 13.2.27 to our situation, let us set $V=N$ and $W=V_{<\gamma}N$, and for a fixed choice of $i=1,\dots,n$, let us consider the endomorphism

$$T_i=(x_i\partial_{x_i}-\beta_i)+a_i(t\partial_t-\gamma);$$

Evidently, $T_iW\subset W$. Since we have

$$T_i(m_{\boldsymbol{\alpha},j}\otimes\partial_t^j)=(m_{\boldsymbol{\alpha},j}\otimes\partial_t^j)\cdot\big((\mathrm{S}_i-\alpha_i)+\alpha_i-\beta_i-a_i(\gamma-j)\big),$$

it is clear that $m_{\boldsymbol{\alpha},j}\otimes\partial_t^j$ is annihilated by a large power of $T_i-(\alpha_i-\beta_i-a_i(\gamma-j))$. Grouping terms according to the value of $\alpha_i-\beta_i-a_i(\gamma-j)$, we obtain

$$\sum_{\boldsymbol{\alpha}\in\mathbb{R}^n}\sum_{j\in\mathbb{N}}m_{\boldsymbol{\alpha},j}\otimes\partial_t^j=v_1+\cdots+v_k$$

with $v_k\in E_{\lambda_k}(T_i)$ and $\lambda_1,\dots,\lambda_k\in\mathbb{R}$ are pairwise distinct. According to Lemma 13.2.27, we have $v_h\in W$ whenever $\lambda_h\neq 0$; what this means is that the sum of all $m_{\boldsymbol{\alpha},j}\otimes\partial_t^j$ with $\alpha_i-\beta_i-a_i(\gamma-j)\neq 0$ belongs to $V_{<\gamma}N$. After subtracting this sum from our original element, we may therefore assume that $\alpha_i=\beta_i-a_i(\gamma-j)$ for every term. We obtain the asserted congruence by performing this procedure for $T_1,\dots,T_n$. This ends the proof of Lemma 13.2.26 and at the same time that of Lemma 13.2.25. $\square$

Proof of Theorem* 13.1.4*, Step* 3*: computation of nearby cycles. Suppose now that $\gamma<0$ and $\beta_1,\dots,\beta_n\leqslant 0$, that we shall abbreviate as $\boldsymbol{\beta}\leqslant 0$. Let $j\in\mathbb{N}$. We observe that

$$a_i\neq 0\Longrightarrow\beta_i+(\gamma-j)a_i=(\beta_i+\gamma a_i)-ja_i<-ja_i.$$

Given a vector $m_j\in M_{\boldsymbol{\beta}+(\gamma-j)\boldsymbol{a}}$, this means that m_j is divisible by $x_i^{ja_i}$. Consequently, $m_j=mx^{j\boldsymbol{a}}$ for a unique m in $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$, and therefore

$$m_j\otimes\partial_t^j=(m\otimes 1)\cdot t^j\partial_t^j$$

is a linear combination of $(m\otimes 1)(t\partial_t)^k$ for $k=1,\dots,j$. Since $m\otimes 1\in V_\gamma N$ and $V_\gamma N$ is stable by $t\partial_t$, we conclude that

$$\bigoplus_j M_{\boldsymbol{\beta}+(\gamma-j)\boldsymbol{a}}\otimes\partial_t^j=M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[t\partial_t]\subset V_\gamma N,$$

and, by Lemma 13.2.25, ${}^{\gamma}N_{\boldsymbol{\beta}}$ is the image of $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[t\partial_t]\bmod V_{<\gamma}N$. Let us consider E as a new variable and let us equip $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}]:=M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\otimes_{\mathbb{C}}\mathbb{C}[\mathrm{E}]$ with the $\mathbb{C}[\mathrm{N}_1,\dots,\mathrm{N}_n,\mathrm{N}]$-module structure such that N_i acts by $\mathrm{S}_i-\beta_i-a_i\,\mathrm{E}$ and N acts by $\mathrm{E}-\gamma$ (see (13.2.15)), and let us equip ${}^{\gamma}N_{\boldsymbol{\beta}}$ with its natural $\mathbb{C}[\mathrm{N}_1,\dots,\mathrm{N}_n,\mathrm{N}]$-module structure (see §13.2.a). We thus have proved the following result.

13.2.29. Proposition. *We have a surjective $\mathbb{C}[\mathrm{N}_1,\dots,\mathrm{N}_n,\mathrm{N}]$-linear morphism*

$$M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}]\longrightarrow{}^{\gamma}N_{\boldsymbol{\beta}}$$

that takes $m\otimes\mathrm{E}^k$ to the class of $m\otimes(t\partial_t)^k\in V_\gamma N$ modulo $V_{<\gamma}N$. $\square$

13.2.30. Corollary. *We have ${}^{\gamma}N=\bigoplus_{\boldsymbol{\beta}}{}^{\gamma}N_{\boldsymbol{\beta}}$.*

Proof. We have seen in the beginning of Step 2 that it is enough to prove ${}^{\gamma}N=\sum_{\boldsymbol{\beta}}{}^{\gamma}N_{\boldsymbol{\beta}}$. Let us set ${}^{\gamma}N_{\leqslant 0}=\bigoplus_{\boldsymbol{\beta}\leqslant 0}{}^{\gamma}N_{\boldsymbol{\beta}}$. It is enough to check that ${}^{\gamma}N=\sum_{\boldsymbol{k}}{}^{\gamma}N_{\leqslant 0}\partial_x^{\boldsymbol{k}}$. We have seen that $V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]\subset V_\gamma N$ and has image equal to ${}^{\gamma}N_{\leqslant 0}$. Then $\sum_{\boldsymbol{k}}{}^{\gamma}N_{\leqslant 0}\partial_x^{\boldsymbol{k}}$ contains the image of $\sum_{\boldsymbol{k}}(V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M\otimes 1)\partial_x^{\boldsymbol{k}}$, which is equal to $V_\gamma N$, by (13.2.18 $*$). $\square$

13.2.31. Remark. At this point, it can be clearer to write $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}] = M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]$ and to consider the latter space as a free $\mathbb{C}[\mathrm{N}]$-module, where N is considered as a new variable. The action of N_i on $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$ induces an action denoted by $\mathrm{N}_i \otimes 1$ on $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]$, and we define the action of N_i on ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ as that induced by $\mathrm{N}_i \otimes 1 - a_i\mathrm{N}$.

In order to have an explicit expression of ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ ($\boldsymbol{\beta} \leqslant 0$) and eventually prove its finite dimensionality, it remains to find the kernel of the morphism in Proposition 13.2.29. To do that, we introduce the set

$$I_g(\boldsymbol{\beta}) = \{i \in I \mid a_i \neq 0 \text{ and } \beta_i = 0\} \subset I_g.$$

Given $m \in M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$, we have $\big(m \prod_{i\in I_g(\boldsymbol{\beta})} x_i\big) \otimes 1 = m \otimes t \in V_{<\gamma}N$ and therefore also

$$(m \otimes 1) \textstyle\prod_{i\in I_g(\boldsymbol{\beta})} x_i\partial_{x_i} = (m \otimes 1) \cdot \prod_{i\in I_g(\boldsymbol{\beta})} (\mathrm{S}_i - a_i t\partial_t) \in V_{<\gamma}N.$$

In this way, we obtain a large collection of elements in the kernel.

13.2.32. Corollary. *If $\gamma < 0$ and $\boldsymbol{\beta} \leqslant 0$, ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ is isomorphic to the cokernel of the injective morphism*

$$\varphi_{\boldsymbol{\beta}} := \prod_{i\in I_g(\boldsymbol{\beta})} (\mathrm{S}_i/a_i - \mathrm{E}) \in \mathrm{End}(M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}]), \tag{13.2.32 $*$}$$

or equivalently

$$\varphi_{\boldsymbol{\beta}} := \prod_{i\in I_g(\boldsymbol{\beta})} ((\mathrm{N}_i \otimes 1)/a_i - \mathrm{N}) \in \mathrm{End}(M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]). \tag{13.2.32 $**$}$$

13.2.33. Remark. We have assumed in Theorem 13.1.4 that M is a middle extension along the normal crossing divisor $D_{i\in I}$. However, the previous expression shows that, for $\gamma < 0$ and $\boldsymbol{\beta} \leqslant 0$, ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ only depends on the $M_{\boldsymbol{\alpha}}$'s with $\alpha_i < 0$ if $i \in I_g$. For such a γ, we conclude that $\mathrm{gr}^V_\gamma N$ only depends on the localized module $M(*g)$.

Moreover, by definition, the action of N_i (resp. N) on ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ is that induced by $\mathrm{N}_i \otimes 1 - a_i\mathrm{N}$ (resp. N). We thus find that $\prod_{i\in I_g} \mathrm{N}_i$ acts by zero on ${}^{\gamma}\!N_{\boldsymbol{\beta}}$.

13.2.34. Corollary. *If $\gamma < 0$ and $\boldsymbol{\beta} \leqslant 0$, ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ is finite-dimensional.*

Proof. Set $b = |I_g(\boldsymbol{\beta})|$. Corollary 13.2.32 implies that the natural $\mathbb{C}$-linear morphism

$$\bigoplus_{k=0}^{b-1} M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\,\mathrm{E}^k \longrightarrow {}^{\gamma}\!N_{\boldsymbol{\beta}}$$

is an isomorphism. Since every $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$ is finite-dimensional, we obtain the desired assertion. $\square$

Note also that the action of E on ${}^{\gamma}\!N_{\boldsymbol{\beta}}$, and thus that of $\mathrm{N} = \mathrm{E} - \gamma$ is easily described on this expression:

$$m\,\mathrm{E}^k \cdot \mathrm{E} = \begin{cases} m\,\mathrm{E}^{k+1} & \text{if } k < b-1, \\ m\big[\,\mathrm{E}^b - \prod_{i\in I_g(\boldsymbol{\beta})}(\mathrm{E} - \mathrm{S}_i/a_i)\big] & \text{if } k = b-1. \end{cases}$$

***Proof of Corollary* 13.2.32.** Injectivity of $\varphi_{\boldsymbol{\beta}}$ is clear by considering the effect of $\varphi_{\boldsymbol{\beta}}$ on the term of highest degree with respect to E. On the other hand, we already know that every element of ${}^{\gamma}N_{\boldsymbol{\beta}}$ is the image of some $m = \sum_k (m_k \otimes 1)\,\mathrm{E}^k$ with $m_k \in M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$ for every k. If we expand this using $\mathrm{E} = t\partial_t$, we find

$$m \in \bigoplus_{j\in\mathbb{N}} M_{\boldsymbol{\beta}+(\gamma-j)\boldsymbol{a}} \otimes \partial_t^j. \tag{13.2.35}$$

Now suppose that m actually lies in $V_{<\gamma}N$. It can then be written as (see (13.2.24))

$$m = \sum_{\substack{\boldsymbol{\alpha}\in\mathbb{R}^n\\ \boldsymbol{k}\in\mathbb{N}^{I_g}}} (m_{\boldsymbol{\alpha},\boldsymbol{k}} \otimes 1)\partial_{x'}^{\boldsymbol{k}}, \tag{13.2.36}$$

where $m_{\boldsymbol{\alpha},\boldsymbol{k}} \in M_{\boldsymbol{\alpha}}$ satisfies $\alpha_i < \gamma a_i$ whenever $a_i \neq 0$. If we expand the expression $(m_{\boldsymbol{\alpha},\boldsymbol{k}}\otimes 1)\partial_{x'}^{\boldsymbol{k}}$ according to (13.2.12), all the terms that appear belong to $M_{\boldsymbol{\alpha}+\boldsymbol{k}-j\boldsymbol{a}}\otimes\partial_t^j$ for some $j \leqslant |\boldsymbol{k}|$ (we identify $\boldsymbol{k}$ with $(\boldsymbol{k},0) \in \mathbb{Z}^n$). Comparing with (13.2.35), we can therefore discard those summands in (13.2.36) with $\boldsymbol{\alpha}+\boldsymbol{k} \neq \boldsymbol{\beta}+\gamma\boldsymbol{a}$ without changing the value of the sum. The sum in (13.2.36) is thus simply indexed by those $\boldsymbol{k} \in \mathbb{N}^{I_g}$ such that $k_i > \beta_i$ for all $i \in I_g$ and the index $\boldsymbol{\alpha}$ is replaced with $\boldsymbol{\beta}+\gamma\boldsymbol{a}-\boldsymbol{k}$.

Now, if $a_i \neq 0$ then $\alpha_i = (\beta_i+\gamma a_i) - k_i < -k_i$ and so $m_{\boldsymbol{\alpha},\boldsymbol{k}}$ is divisible by $x_i^{k_i}$. This means that we can write

$$m_{\boldsymbol{\alpha},\boldsymbol{k}} = m'_{\boldsymbol{k}} x'^{\boldsymbol{k}}$$

for some $m'_{\boldsymbol{k}} \in M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$. Therefore, (13.2.36) reads

$$m = \sum_{\substack{\boldsymbol{k}\in\mathbb{N}^{I_g}\\ k_i>\beta_i\,\forall\, i\in I_g}} (m'_{\boldsymbol{k}} \otimes 1)x'^{\boldsymbol{k}}\partial_{x'}^{\boldsymbol{k}}, \quad m'_{\boldsymbol{k}} \in M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}.$$

If $m'_{\boldsymbol{k}} \neq 0$, then $k_i \geqslant 1$ for $i \in I_g(\boldsymbol{\beta})$ (since $\beta_i = 0$), and consequently, $x'^{\boldsymbol{k}}\partial_{x'}^{\boldsymbol{k}}$ is forced to be a multiple of

$$\prod_{i\in I_g(\boldsymbol{\beta})} x_i\partial_{x_i} = \prod_{i\in I_g(\boldsymbol{\beta})} (\mathrm{S}_i - a_i\,\mathrm{E}).$$

As a consequence,

$$\begin{aligned}
m \in \sum_{\boldsymbol{\ell}\in\mathbb{N}^{I_g}} (M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}} \otimes 1)x'^{\boldsymbol{\ell}}\partial_{x'}^{\boldsymbol{\ell}} \cdot &\prod_{i\in I_g(\boldsymbol{\beta})} (\mathrm{S}_i - a_i t\partial_t)\\
&= \sum_{\boldsymbol{\ell}\in\mathbb{N}^{I_g}} (M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}} \otimes 1)(\mathrm{S} - \boldsymbol{a}t\partial_t)^{\boldsymbol{\ell}} \cdot \prod_{i\in I_g(\boldsymbol{\beta})} (\mathrm{S}_i - a_i t\partial_t)\\
&\subset M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}] \cdot \prod_{i\in I_g(\boldsymbol{\beta})} (\mathrm{S}_i - a_i\,\mathrm{E}). \qquad \square
\end{aligned}$$

We end this section by giving the explicit description of the quiver of ${}^{\gamma}N = \mathrm{gr}_\gamma^V N$ for $\gamma < 0$ (see Proposition 13.2.5). We thus consider the vector spaces ${}^{\gamma}N_{\boldsymbol{\beta}}$ for

$\boldsymbol{\beta} \in [-1, 0]^n$, and the morphisms

$$\begin{tikzcd} {}^{\gamma}\!N_{\boldsymbol{\beta}-\mathbf{1}_i} \arrow[r, bend left, "\mathrm{can}_i(\boldsymbol{\beta})"] & {}^{\gamma}\!N_{\boldsymbol{\beta}} \arrow[l, bend left, "\mathrm{var}_i(\boldsymbol{\beta})"] \end{tikzcd} \tag{13.2.37}$$

for every i such that $\beta_i = 0$. We know from that Corollary 13.2.32 that ${}^{\gamma}\!N_{\boldsymbol{\beta}} \neq 0$ only if $\beta_i = 0$ for some $i \in I_g$ (i.e., such that $a_i \neq 0$). Moreover, the description of ${}^{\gamma}\!N_{\boldsymbol{\beta}}$ given in this corollary enables one to define a natural quiver as follows.

(1) If $i \notin I_g$ and $\beta_i = 0$, we also have $(\boldsymbol{\beta} + \gamma\boldsymbol{a})_i = 0$, and we will see that the diagram

$$\begin{tikzcd} M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}-\mathbf{1}_i}[\mathrm{N}] \arrow[r, bend left, "\mathrm{can}_i \otimes 1"] & M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[l, bend left, "\mathrm{var}_i \otimes 1"] \end{tikzcd}$$

commutes with $\varphi_{\boldsymbol{\beta}}$ (which only involves indices $j \in I_g$), inducing therefore in a natural way a diagram

$$\begin{tikzcd} {}^{\gamma}\!N_{\boldsymbol{\beta}-\mathbf{1}_i} \arrow[r, bend left, "\mathrm{c}_i(\boldsymbol{\beta})"] & {}^{\gamma}\!N_{\boldsymbol{\beta}} \arrow[l, bend left, "\mathrm{v}_i(\boldsymbol{\beta})"] \end{tikzcd}$$

We notice moreover that the middle extension property for M is preserved for this diagram, that is, $\mathrm{c}_i(\boldsymbol{\beta})$ remains surjective and $\mathrm{v}_i(\boldsymbol{\beta})$ remains injective.

(2) If $i \in I_g$, we set $\varphi_{\mathbf{1}_i} = (\mathrm{N}_i \otimes 1)/a_i - \mathrm{N}$ so that, with obvious notation, $\varphi_{\boldsymbol{\beta}} = \varphi_{\mathbf{1}_i}\varphi_{\boldsymbol{\beta}-\mathbf{1}_i} = \varphi_{\boldsymbol{\beta}-\mathbf{1}_i}\varphi_{\mathbf{1}_i}$, and we can regard $\varphi_{\boldsymbol{\beta}}, \varphi_{\mathbf{1}_i}, \varphi_{\boldsymbol{\beta}-\mathbf{1}_i}$ as acting (injectively) both on $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]$ and $M_{\boldsymbol{\beta}-\mathbf{1}_i+\gamma\boldsymbol{a}}[\mathrm{N}]$. Moreover, the multiplication by x_i, which is an isomorphism $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}} \xrightarrow{\sim} M_{\boldsymbol{\beta}-\mathbf{1}_i+\gamma\boldsymbol{a}}$, is such that $x_i \otimes 1$ commutes with $\varphi_{\boldsymbol{\beta}-\mathbf{1}_i}$. In such a way, we can regard ${}^{\gamma}\!N_{\boldsymbol{\beta}-\mathbf{1}_i}$ as $\operatorname{Coker} \varphi_{\boldsymbol{\beta}-\mathbf{1}_i}$ acting on $M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]$. We can then define c_i and v_i as naturally induced by the following commutative diagrams:

$$\begin{tikzcd} M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, "\varphi_{\boldsymbol{\beta}-\mathbf{1}_i}"] \arrow[d, equal] & M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, two heads] \arrow[d, "\varphi_{\mathbf{1}_i}"'] & {}^{\gamma}\!N_{\boldsymbol{\beta}-\mathbf{1}_i} \arrow[d, "\mathrm{c}_i(\boldsymbol{\beta})"'] \\ M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, "\varphi_{\boldsymbol{\beta}}"'] & M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, two heads] & {}^{\gamma}\!N_{\boldsymbol{\beta}} \end{tikzcd} \quad \text{resp.} \quad \begin{tikzcd} M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, "\varphi_{\boldsymbol{\beta}-\mathbf{1}_i}"] & M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, two heads] \arrow[d, equal] & {}^{\gamma}\!N_{\boldsymbol{\beta}-\mathbf{1}_i} \\ M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, "\varphi_{\boldsymbol{\beta}}"'] \arrow[u, "\varphi_{\mathbf{1}_i}"] & M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}] \arrow[r, two heads] & {}^{\gamma}\!N_{\boldsymbol{\beta}} \arrow[u, "\mathrm{v}_i(\boldsymbol{\beta})"'] \end{tikzcd}$$

In other words, $\mathrm{c}_i(\boldsymbol{\beta})$ is the natural morphism

$$M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]/\operatorname{Im} \varphi_{\boldsymbol{\beta}-\mathbf{1}_i} \xrightarrow{\varphi_{\mathbf{1}_i}} M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]/\operatorname{Im} \varphi_{\boldsymbol{\beta}},$$

and $\mathrm{v}_i(\boldsymbol{\beta})$ is the natural morphism induced by the inclusion $\operatorname{Im} \varphi_{\boldsymbol{\beta}} \subset \operatorname{Im} \varphi_{\boldsymbol{\beta}-\mathbf{1}_i}$:

$$M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]/\operatorname{Im} \varphi_{\boldsymbol{\beta}} \longrightarrow M_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{N}]/\operatorname{Im} \varphi_{\boldsymbol{\beta}-\mathbf{1}_i}.$$

We note that $\mathrm{v}_i(\boldsymbol{\beta})$ is *surjective*. Moreover,

13.2.38. Proposition. *For $\gamma<0$, the quiver of $\mathrm{gr}^V_\gamma N$ has vertices ${}^\gamma N_{\boldsymbol{\beta}}=\mathrm{Coker}\,\varphi_{\boldsymbol{\beta}}$ for $\boldsymbol{\beta}\in[-1,0]^n$ such that*

(1) *$\boldsymbol{\beta}=\boldsymbol{\alpha}-\gamma\boldsymbol{a}$ for some $\boldsymbol{\alpha}\in A+\mathbb{Z}$,*
(2) *$\beta_i=0$ for some $i\in I_g$.*

It is isomorphic to the quiver defined by the morphisms $\mathrm{c}_i(\boldsymbol{\beta}),\mathrm{v}_i(\boldsymbol{\beta})$ as described above.

13.2.c. More on the structure of the nearby cycles. We keep the notation and assumptions of Theorem 13.1.4 in the present setting (no filtration, $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type) and we will make more precise the $\mathbb{C}[x]\langle\partial_x\rangle$-module structure of ${}^\gamma N:=\mathrm{gr}^V_\gamma N$. The general principle is that the graded object $\mathrm{gr}^{\mathrm{M}}_\bullet{}^\gamma N$ with respect to the monodromy filtration of the nilpotent endomorphism $\mathrm{N}=\mathrm{E}-\gamma$ should be simpler to understand, and enough for the purpose of Hodge theory, and moreover it is completely determined by the primitive modules $\mathrm{P}_k{}^\gamma N$, by means of the Lefschetz decomposition. We first exhibit the simplification brought by grading with respect to a suitable finite filtration $U_\bullet$, and we will next consider the monodromy filtration.

Structure of $V_0^{(\boldsymbol{n})}{}^\gamma N$. We consider E as a new variable and we set $V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]=V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M\otimes_{\mathbb{C}}\mathbb{C}[\mathrm{E}]$. We equip $V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]$ with the following twisted $\mathbb{C}[x]\langle x\partial_x\rangle[\mathrm{E}]$-structure, compatible with (13.2.12):

$$\begin{aligned}(m\otimes\mathrm{E}^k)\cdot f(x)&=mf(x)\otimes\mathrm{E}^k,\\(m\otimes\mathrm{E}^k)\cdot\mathrm{E}&=m\otimes\mathrm{E}^{k+1},\\(m\otimes\mathrm{E}^k)\cdot x_i\partial_{x_i}&=mx_i\partial_{x_i}\otimes\mathrm{E}^k-a_im\otimes\mathrm{E}^{k+1}.\end{aligned}$$

If $\boldsymbol{\alpha}$ is such that $M_{\boldsymbol{\alpha}}\neq0$ (see Definition 13.2.2), we set

$$I_g(\gamma,\boldsymbol{\alpha})=\{i\in I_g\mid\alpha_i=\gamma a_i\}.$$

Corollary 13.2.32 provides us with a presentation of $V_0^{(\boldsymbol{n})}({}^\gamma N)$ as the cokernel of

$$\varphi=\bigoplus_{\boldsymbol{\alpha}\leqslant\gamma\boldsymbol{a}}\varphi_{\boldsymbol{\alpha}}:\bigoplus_{\boldsymbol{\alpha}\leqslant\gamma\boldsymbol{a}}M_{\boldsymbol{\alpha}}[\mathrm{E}]\longrightarrow\bigoplus_{\boldsymbol{\alpha}\leqslant\gamma\boldsymbol{a}}M_{\boldsymbol{\alpha}}[\mathrm{E}],\tag{13.2.39}$$

with

$$\varphi_{\boldsymbol{\alpha}}:=\prod_{i\in I_g(\gamma,\boldsymbol{\alpha})}(\mathrm{S}_i/a_i-\mathrm{E}).$$

This morphism is $\mathbb{C}\langle x\partial_x,\mathrm{E}\rangle$-linear, and we have $V_0^{(\boldsymbol{n})}({}^\gamma N)\simeq\mathrm{Coker}\,\varphi$ as a $\mathbb{C}\langle x\partial_x,\mathrm{E}\rangle$-module. Moreover, Corollary 13.2.32 also implies that, for every $i=1,\dots,n$, the V-filtration of ${}^\gamma N$ in the direction of D_i is determined by the formula

$$V^{(i)}_{\beta_i}({}^\gamma N)\cap V_0^{(\boldsymbol{n})}({}^\gamma N)=\mathrm{image}\big((V^{(i)}_{\beta_i+\gamma a_i}V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M)[\mathrm{E}]\big),\quad\text{for }\beta_i\leqslant0.\tag{13.2.40}$$

However, since x_i does not commute with S_i, the morphism φ is not $\mathbb{C}[x]\langle x\partial_x\rangle[\mathrm{E}]$-linear.

We will recover $\mathbb{C}[x]\langle x\partial_x\rangle[\mathrm{E}]$-linearity after grading by a suitable finite filtration $U_\bullet$. For $\boldsymbol{\alpha}\leqslant\gamma\boldsymbol{a}$ fixed, and for $j\in I_g(\gamma,\boldsymbol{\alpha})$, we have

$$\Big(\prod_{i\in I_g(\gamma,\boldsymbol{\alpha})}x_i\partial_{x_i}\Big)\cdot x_j-x_j\cdot\Big(\prod_{i\in I_g(\gamma,\boldsymbol{\alpha})}x_i\partial_{x_i}\Big)=x_j\cdot\prod_{\substack{i\in I_g(\gamma,\boldsymbol{\alpha})\\ i\neq j}}x_i\partial_{x_i},$$

and the right-hand term sends $M_{\boldsymbol{\alpha}}[\mathrm{E}]$ in $M_{\boldsymbol{\alpha}-\mathbf{1}_j}[\mathrm{E}]$. Moreover, for $j\in I_g(\gamma,\boldsymbol{\alpha})$, the equality $\#I_g(\gamma,\boldsymbol{\alpha}-\mathbf{1}_j)=\#I_g(\gamma,\boldsymbol{\alpha})-1$ holds true. We are thus led to define the finite increasing filtration by $\mathbb{C}[x]\langle x\partial_x\rangle[\mathrm{E}]$-submodules

$$U_kV^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]=\bigoplus_{\substack{\boldsymbol{\alpha}\leqslant\gamma\boldsymbol{a}\\ \#I_g(\gamma,\boldsymbol{\alpha})\leqslant k}}M_{\boldsymbol{\alpha}}[\mathrm{E}].$$

Grading with respect to $U_\bullet$ has the only effect of killing the action of x_j on $M_{\boldsymbol{\alpha}}[\mathrm{E}]$ for $j\in I_g(\gamma,\boldsymbol{\alpha})$. Moreover, the image filtration $U_\bullet V_0^{(\boldsymbol{n})}{}^\gamma N$ is nothing but the filtration

$$U_kV_0^{(\boldsymbol{n})}{}^\gamma N=\bigoplus_{\substack{\boldsymbol{\beta}\leqslant 0\\ \#I_g(\boldsymbol{\beta})\leqslant k}}{}^\gamma N_{\boldsymbol{\beta}}.$$

Every $\mathbb{C}[x]\langle x\partial_x\rangle[\mathrm{E}]$-module $\mathrm{gr}_k^UV_0^{(\boldsymbol{n})}{}^\gamma N$ is equal to the direct sum of its submodules $(\mathrm{gr}_k^UV_0^{(\boldsymbol{n})}{}^\gamma N)_J$ with

$$(\mathrm{gr}_k^UV_0^{(\boldsymbol{n})}{}^\gamma N)_J=\bigoplus_{\boldsymbol{\beta}\mid I_g(\boldsymbol{\beta})=J}{}^\gamma N_{\boldsymbol{\beta}},\quad\text{for } J\subset I_g \text{ with } |J|=k,$$

and x_j acts by zero on $(\mathrm{gr}_k^UV_0^{(\boldsymbol{n})}{}^\gamma N)_J$ for $j\in J$. In other words, $(\mathrm{gr}_k^UV_0^{(\boldsymbol{n})}{}^\gamma N)_J$ is supported on $\bigcap_{i\in J}D_i$.

***13.2.41. Proposition* (Structure of $\mathrm{gr}^UV_0^{(\boldsymbol{n})}({}^\gamma N)$).** *The morphism φ is strictly compatible with the filtration $U_\bullet V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]$ and we have an exact sequence*

$$0\longrightarrow\mathrm{gr}^UV^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]\xrightarrow{\ \mathrm{gr}^U\varphi\ }\mathrm{gr}^UV^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}M[\mathrm{E}]\longrightarrow\mathrm{gr}^UV_0^{(\boldsymbol{n})}({}^\gamma N)\longrightarrow 0.$$

Proof. This is obvious since φ is graded as a $\mathbb{C}[x\partial_x,\mathrm{E}]$-linear morphism. □

13.2.42. Remarks.

(1) The computation shows that $\mathrm{gr}^UV_0^{(\boldsymbol{n})}({}^\gamma N)$, hence $V_0^{(\boldsymbol{n})}({}^\gamma N)$, hence ${}^\gamma N$, is supported by the divisor of g, as expected of course.

(2) As a consequence, we can also determine the negative *jumping indices* of the V-filtration of N. Let $\boldsymbol{A}\subset[-1,0)^n$ be the finite set of exponents of M (see Definition 13.2.2). Let us fix $\gamma<0$. Then $\mathrm{gr}_\gamma^VN\neq0$ if and only if ${}^\gamma N_{\boldsymbol{\beta}}\neq0$ for some $\boldsymbol{\beta}\leqslant0$, that is, $I_g(\boldsymbol{\beta})\neq\varnothing$, that is, $\gamma a_i\in\alpha_i-\mathbb{N}$ for some $\boldsymbol{\alpha}\in\boldsymbol{A}$ and some $i\in I_g$. We conclude the set of negative jumping indices is the set

$$\bigcup_{i\in I_g}\frac{1}{a_i}(\alpha_i-\mathbb{N}),\quad\boldsymbol{\alpha}\in\boldsymbol{A}.$$

13.2.d. A simple example with monodromy filtration. We illustrate the previous general results on a simple example, where we can give more details on the monodromy filtration on the nearby cycles ${}^{\gamma}N = \mathrm{gr}^V_{\gamma} N$ ($\gamma < 0$).

13.2.43. Assumption. M is simple, that is, all properties of Example 13.2.9 are satisfied. In particular the set of exponents is reduced to one element $\boldsymbol{\alpha} \in [-1,0)^n$, $\mathrm{rk}\, M_{\boldsymbol{\alpha}} = 1$ and $M_{\boldsymbol{\alpha}+\boldsymbol{k}} = 0$ if $\alpha_i = -1$ and $k_i \geqslant 1$.

Let us summarize the results already obtained in the present setting.

(i) The set of negative jumping indices γ for the V-filtration is $\bigcup_{i\in I_g} \frac{1}{a_i}(\alpha_i - \mathbb{N})$. For such a a jumping index $\gamma\in[-1,0)$, we will set $J(\boldsymbol{\alpha},\gamma) := \{i\in I_g \mid \alpha_i \equiv \gamma a_i \bmod \mathbb{Z}\}$ and $k_\gamma = \#J(\boldsymbol{\alpha},\gamma) - 1$. Moreover, for $\boldsymbol{j} \in \{0,1\}^I$, we set

$$\|\boldsymbol{j}\| := \sum_{i\in J(\boldsymbol{\alpha},\gamma)} j_i = \#\{i \in J(\boldsymbol{\alpha},\gamma) \mid j_i = 1\}.$$

(ii) Such a jumping index $\gamma \in [-1,0)$ being fixed, and setting ${}^{\gamma}N = \mathrm{gr}^V_{\gamma} N$, the only possible $\boldsymbol{\beta} \leqslant 0$ such that ${}^{\gamma}N_{\boldsymbol{\beta}} \neq 0$ are of the form $\boldsymbol{\beta} = \boldsymbol{\alpha} - \gamma\boldsymbol{a} - \boldsymbol{k}$ for suitable $\boldsymbol{k} \in \mathbb{N}^n$: for all $i \in \{1,\dots,n\}$ we have $\alpha_i - \gamma a_i > \alpha_i \geqslant -1$, so $\alpha_i - \gamma a_i - k_i > 0$ if $k_i \leqslant -1$. (Note also that some components of $\boldsymbol{\alpha} - \gamma\boldsymbol{a}$ can be > 0). There exists a unique $\boldsymbol{k}^o \in \mathbb{N}^n$ such that

$$\begin{cases} k_i^o = 0 & \text{if } i \notin I_g,\\ \beta_i^o := \alpha_i - \gamma a_i - k_i^o \in [-1,0) & \text{if } i \in I_g,\end{cases}$$

and we set $\boldsymbol{\beta}^o = \boldsymbol{\alpha}^o - \gamma\boldsymbol{a}$, with $\boldsymbol{\alpha}^o := \boldsymbol{\alpha} - \boldsymbol{k}^o$, so that in particular $\beta_i^o = \alpha_i^o$ if $i \notin I_g$. Then ${}^{\gamma}N$ has a single exponent, equal to $\boldsymbol{\beta}^o$, and its quiver (see Remark 13.2.6) has vertices ${}^{\gamma}N_{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} = \boldsymbol{\beta}^o + \boldsymbol{j}$, $\boldsymbol{j} \in \{0,1\}^I$. The corresponding $\boldsymbol{\beta} + \gamma\boldsymbol{a}$ is then equal to $\boldsymbol{\alpha}^o + \boldsymbol{j}$. We note that

$$J(\boldsymbol{\alpha},\gamma) = \{i \in I_g \mid \beta_i^o = -1\} \quad\text{and}\quad I_g(\boldsymbol{\beta}^o + \boldsymbol{j}) = \{i \in J(\boldsymbol{\alpha},\gamma) \mid j_i = 1\}.$$

(iii) The action of $x_i\partial_{x_i}$ on $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}$ (with $\boldsymbol{\alpha}^o$ and $\boldsymbol{j}$ as above), that we have denoted by S_i above, is the multiplication by the constant $\alpha_i^o + j_i$. On the other hand, the action of $x_i\partial_{x_i}$ on $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{E}]$ is by $\mathrm{S}_i - a_i\,\mathrm{E}$. We note that, for $i \in I_g(\boldsymbol{\beta}^o + \boldsymbol{j})$, we have $\beta_i^o + j_i = 0$, hence $\beta_i^o = -1$ and $j_i = 1$, so $(\alpha_i^o + j_i)/a_i = \gamma$. Proposition 13.2.29 describes ${}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$ as the cokernel of $(\mathrm{E} - \gamma)^{\|\boldsymbol{j}\|}$ acting on $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{E}]$.

Let us consider the operator $\mathrm{N} = 2\pi\mathrm{i}(\mathrm{E} - \gamma)$, and identify in a natural way $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{E}]$ with $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{N}]$. Then, as a $\mathbb{C}[\mathrm{N}]$-module, we have

$${}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}} = M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{N}]/\operatorname{Im}\mathrm{N}^{\|\boldsymbol{j}\|}.$$

In other words, ${}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$ is a Jordan block of size $\|\boldsymbol{j}\|$ with respect to N. In particular, ${}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}} = 0$ for any $\boldsymbol{j}$ all of whose components on $J(\boldsymbol{\alpha},\gamma)$ are zero. The action of $\mathrm{N}_i = x_i\partial_{x_i} - (\beta_i^o + j_i)$ on ${}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$, which is induced from that on $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{E}]$, is by $-a_i\mathrm{N}$. As a consequence, the primitive part $\mathrm{P}_k({}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}})$ is zero if $k \neq \|\boldsymbol{j}\| - 1$ and has dimension 1 if $k = \|\boldsymbol{j}\| - 1$. We then denote by $\mathrm{P}({}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}})$ this primitive part.

We conclude that for $k \in \mathbb{N}$, the quiver of $\mathrm{P}_k{}^{\gamma}N$ is zero if $k > k_\gamma$, and otherwise has vertices $\mathrm{P}({}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}})$ for $\boldsymbol{j} \in \{0,1\}^I$ such that $\|\boldsymbol{j}\| = k+1$.

(iv) Let now us describe the var_i arrows in the quiver of ${}^{\gamma}N$. The action of x_i on ${}^{\gamma}N$ is induced by that on M so, if $j_i = 1$, $x_i : {}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}} \to {}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i}$ it is the morphism

$$x_i \otimes 1 : M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{N}]/\operatorname{Im}\mathrm{N}^{\|\boldsymbol{j}\|} \longrightarrow M_{\boldsymbol{\alpha}^o+\boldsymbol{j}-\mathbf{1}_i}[\mathrm{N}]/\operatorname{Im}\mathrm{N}^{\|\boldsymbol{j}-\mathbf{1}_i\|}.$$

Therefore,

(a) if $i \notin J(\boldsymbol{\alpha},\gamma)$, we have $\|\boldsymbol{j}-\mathbf{1}_i\| = \|\boldsymbol{j}\|$ and $x_i : {}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}} \to {}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i}$ is injective, since $x_i : M_{\boldsymbol{\alpha}^o+\boldsymbol{j}} \to M_{\boldsymbol{\alpha}^o+\boldsymbol{j}-\mathbf{1}_i}$ by our assumption of middle extension on M. For the same reason, $x_i : \mathrm{P}{}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}} \to \mathrm{P}{}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i}$ is injective.

(b) If $i \in J(\boldsymbol{\alpha},\gamma)$, then x_i induces zero on the N-primitive part $\mathrm{P}{}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$, so var_i is zero on the quiver of $\mathrm{P}_k{}^{\gamma}N$ for every k.

(v) We consider the can_i arrows in the quiver of ${}^{\gamma}N$. So we consider ∂_{x_i} : ${}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i} \to {}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$ with $j_i = 1$.

(a) If $i \notin I_g$, then the action of ∂_{x_i} on $M_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i}[\mathrm{N}]$ is simply induced from that on $M_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i}$ (see (13.2.12)), hence can_i is onto since $\|\boldsymbol{j}-\mathbf{1}_i\| = \|\boldsymbol{j}\|$ and by our assumption of middle extension on M. The same property holds for every $\mathrm{P}_k{}^{\gamma}N$.

(b) If $i \in I_g \smallsetminus J(\boldsymbol{\alpha},\gamma)$, then $\beta_i^o \in (-1,0)$ and can_i is an isomorphism by our convention (Remark 13.2.6). The same property holds for every $\mathrm{P}_k{}^{\gamma}N$.

(c) If $i \in J(\boldsymbol{\alpha},\gamma)$, then for a given $k \in \mathbb{N}$, either $\mathrm{P}_k{}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$ or $\mathrm{P}_k{}^{\gamma}N_{\boldsymbol{\beta}^o+\boldsymbol{j}-\mathbf{1}_i}$ is zero, so can_i is zero on the quiver of $\mathrm{P}_k{}^{\gamma}N$ for every k.

Summarizing the discussion, let us emphasize the consequences on the primitive parts $\mathrm{P}_k\mathrm{gr}_\gamma^V N$.

13.2.44. Corollary. *The $\mathbb{C}[x]\langle\partial_x\rangle$-module $\mathrm{P}_k\mathrm{gr}_\gamma^V N$ vanishes for $k > k_\gamma$ and the support of $\mathrm{P}_k\mathrm{gr}_\gamma^V N$ has codimension k if $k \leqslant k_\gamma$. More precisely, if $k \leqslant k_\gamma$, then*

$$\mathrm{P}_k\mathrm{gr}_\gamma^V N = \bigoplus_{\substack{J\subset J(\boldsymbol{\alpha},\gamma)\\ \#J=k}} (\mathrm{P}_k\mathrm{gr}_\gamma^V N)_J,$$

where each $(\mathrm{P}_k\mathrm{gr}_\gamma^V N)_J$ is supported on $D_J := \bigcap_{i\in J} D_i$ and, when regarded as a $\mathbb{C}[x_J]\langle\partial_{x_J}\rangle$-module, it is of normal crossing type along the divisor induced by $\bigcup_{i\notin J} D_i$ and the corresponding quiver is isomorphic to the $(I \smallsetminus J)$-quiver of M. In particular, $\mathrm{P}_{k_\gamma}\mathrm{gr}_\gamma^V N$ is a middle extension with support along $D_{i\in I}$. □

The general case. How much of the previous discussion remains valid in the general case of a $\mathbb{C}[x]\langle\partial_x\rangle$-module M of normal crossing type which is a middle extension along $D_{i\in I}$? First, we can assume that the set $\boldsymbol{A}$ of exponents of M is reduced to a single element $\boldsymbol{\alpha} \in [-1,0)^n$ since M is the direct sum of such modules (see Remark 13.3.8(2)). Therefore, Properties (i) and (ii) of the simple case still hold.

However, in Property (iii), we have to take into account the nilpotent part $\mathrm{S}_i^{\mathrm{nilp}}$ of the action of $x_i\partial_{x_i}$ on $M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}$. We can then describe ${}^\gamma\!N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$ as

$$^\gamma\!N_{\boldsymbol{\beta}^o+\boldsymbol{j}} = M_{\boldsymbol{\alpha}^o+\boldsymbol{j}}[\mathrm{N}]/\operatorname{Im}\Big(\textstyle\prod_{i\in I_g(\boldsymbol{\beta}^o+\boldsymbol{j})}(\mathrm{N}-\mathrm{S}_i^{\mathrm{nilp}}/a_i)\Big).$$

In Property (iv) (resp. (v)), the statements (iva) (resp. (va) and (vb)) remain true, but it is not clear how to compute the primitive parts of ${}^\gamma\!N_{\boldsymbol{\beta}^o+\boldsymbol{j}}$, and therefore (ivb) and (vc) do not extend in a simple way.

13.3. Normal crossing type

In this section, we treat the case of $\mathcal{D}_X$-modules without taking care of a filtration.

13.3.a. Coherent $\mathcal{D}_X$-modules of normal crossing type. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. In order to express the normal crossing property for V-filtrations, we introduce for every $\boldsymbol{\alpha}\in\mathbb{R}^n$ the sub-space $M_{\boldsymbol{\alpha}}$ of $\mathcal{M}$ defined by

$$M_{\boldsymbol{\alpha}} = \bigcap_{i\in I}\bigcup_k \operatorname{Ker}(x_i\partial_{x_i}-\alpha_i)^k.$$

This is a $\mathbb{C}$-vector subspace of $\mathcal{M}$, which is contained in $V^{(1)}_{\alpha_1}\mathcal{M}\cap\cdots\cap V^{(n)}_{\alpha_n}\mathcal{M}$ if $\mathcal{M}$ is $\mathbb{R}$-specializable along each component D_i of D and we have $\mathbb{C}$-linear morphisms $x_i : M_{\boldsymbol{\alpha}}\to M_{\boldsymbol{\alpha}-\boldsymbol{1}_i}$ (resp. $\partial_{x_i} : M_{\boldsymbol{\alpha}}\to M_{\boldsymbol{\alpha}+\boldsymbol{1}_i}$) as in the algebraic setting. Arguing as for $\mathbb{C}[x]\langle\partial_x\rangle$-modules,

$$M := \bigoplus_{\boldsymbol{\alpha}\in\mathbb{R}^\ell} M_{\boldsymbol{\alpha}} \tag{13.3.1}$$

is a $\mathbb{C}[x]\langle\partial_x\rangle$-submodule of $\mathcal{M}$, and there is a natural morphism

$$M\otimes_{\mathbb{C}[x]\langle\partial_x\rangle}\mathcal{D}_X \longrightarrow \mathcal{M}, \tag{13.3.2}$$

which is injective since $\mathcal{D}_X$ is $\mathbb{C}[x]\langle\partial_x\rangle$-flat (because $\mathcal{O}_X$ is $\mathbb{C}[x]$-flat).

13.3.3. Definition. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module. We say that $\mathcal{M}$ is *of normal crossing type along D* if the following properties are satisfied.

(a) The $\mathbb{C}[x]\langle\partial_x\rangle$-submodule M is of normal crossing type along D (Definition 13.2.2).

(b) The natural morphism (13.3.2) is an isomorphism.

In the next proposition, we use Notation 13.1.3.

13.3.4. Proposition. *Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module which is of normal crossing type along D. Then the following properties are satisfied.*

(1) *$\mathcal{M}$ is $\mathbb{R}$-specializable along D_i ($i\in I$), giving rise to V-filtrations $V^{(i)}_\bullet\mathcal{M}$. In particular, all properties of Definition* 9.3.14 *hold for each filtration $V^{(i)}_\bullet\mathcal{M}$.*

(2) *The V-filtrations $V^{(i)}_\bullet\mathcal{M}$ ($i\in I$) are compatible, in the sense of Definition* 10.3.9 *(see also Theorem* 10.3.11*).*

(3) *For $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_n)\in\mathbb{R}^n$, $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ is a $V^{(\boldsymbol{n})}_0\mathcal{D}_X$-module which is $\mathcal{O}_X$-coherent, and $\mathcal{O}_X$-locally free if $\alpha_i<0$ for all $i\in I$.*

(4) *For any multi-index* $\boldsymbol{\alpha} \in \mathbb{R}^n$, *the natural morphism of* $\mathbb{C}[\mathrm{N}_1, \dots, \mathrm{N}_n]$*-modules*

$$M_{\boldsymbol{\alpha}} \longrightarrow \operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}} \mathcal{M} := \operatorname{gr}_{\alpha_1}^{V^{(1)}} \cdots \operatorname{gr}_{\alpha_\ell}^{V^{(\ell)}} \mathcal{M}$$

is an isomorphism (see Remark 10.3.15 *for the multi-grading).*

13.3.5. Caveat. In order to apply Definition 10.3.9, one should regard $V_{\bullet}^{(i)}\mathcal{M}$ as a filtration indexed by $\mathbb{Z}$, by numbering the sequence of real numbers α_i such that $\operatorname{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M} \neq 0$. See also the setup in Section 10.7.a. Setting

$$V_{<\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M} := \sum_{\substack{\boldsymbol{\beta} \leqslant \boldsymbol{\alpha} \\ \boldsymbol{\beta} \neq \boldsymbol{\alpha}}} V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M},$$

the compatibility implies $\operatorname{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M} = V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}/V_{<\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}$.

***Proof of Proposition* 13.3.4.**

(1) By Exercise 13.1, $M_{\leqslant \boldsymbol{\alpha}} := \bigoplus_{\boldsymbol{\alpha}' \leqslant \boldsymbol{\alpha}} M_{\boldsymbol{\alpha}'}$ is a $\mathbb{C}[x]\langle x\partial_x\rangle$-module which is of finite type over $\mathbb{C}[x]$, and $\mathbb{C}[x]$-free if $\alpha_i < 0$ for all $i \in I$. The definition of V-filtrations along the hypersurfaces $x_i = 0$ extend in an obvious way to this algebraic case (which in fact was first considered by Bernstein for the definition of the Bernstein polynomial). One checks that $V_{\alpha_i}^{(i)}M := \bigoplus_{\boldsymbol{\alpha}' \mid \alpha'_i \leqslant \alpha_i} M_{\boldsymbol{\alpha}'}$ satisfies the characteristic properties of the $V^{(i)}$-filtration of M, and thus so does

$$V_{\alpha_i}^{(i)}\mathcal{M} = V_{\alpha_i}^{(i)}M \otimes_{V_0^{(i)}\mathbb{C}[x]\langle\partial_x\rangle} V_0^{(i)}\mathcal{D}_X,$$

for $\mathcal{M}$. In such a way, we get the $\mathbb{R}$-specializability of $\mathcal{M}$ along D_i.

(2) With the previous definition of $V_{\alpha_i}^{(i)}M$, we have $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}M = M_{\leqslant \boldsymbol{\alpha}}$. Set $\boldsymbol{\alpha} = (\boldsymbol{\alpha}_I, \boldsymbol{\alpha}_J, \boldsymbol{\alpha}_K)$ and choose $\boldsymbol{\alpha}'_I \leqslant \boldsymbol{\alpha}_I$ and $\boldsymbol{\alpha}'_J \leqslant \boldsymbol{\alpha}_J$. The compatibility property amounts to complete the star in any diagram as below in order to produce exact sequences:

$$\begin{tikzcd}
\dfrac{M_{\leqslant(\boldsymbol{\alpha}'_I,\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_K)}}{M_{\leqslant(\boldsymbol{\alpha}'_I,\boldsymbol{\alpha}'_J,\boldsymbol{\alpha}_K)}} \arrow[r] & \dfrac{M_{\leqslant(\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_K)}}{M_{\leqslant(\boldsymbol{\alpha}_I,\boldsymbol{\alpha}'_J,\boldsymbol{\alpha}_K)}} \arrow[r] & \star \\
M_{\leqslant(\boldsymbol{\alpha}'_I,\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_K)} \arrow[r] \arrow[u] & M_{\leqslant(\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_K)} \arrow[r] \arrow[u] & \dfrac{M_{\leqslant(\boldsymbol{\alpha}_I,\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_K)}}{M_{\leqslant(\boldsymbol{\alpha}'_I,\boldsymbol{\alpha}_J,\boldsymbol{\alpha}_K)}} \arrow[u] \\
M_{\leqslant(\boldsymbol{\alpha}'_I,\boldsymbol{\alpha}'_J,\boldsymbol{\alpha}_K)} \arrow[r] \arrow[u] & M_{\leqslant(\boldsymbol{\alpha}_I,\boldsymbol{\alpha}'_J,\boldsymbol{\alpha}_K)} \arrow[r] \arrow[u] & \dfrac{M_{\leqslant(\boldsymbol{\alpha}_I,\boldsymbol{\alpha}'_J,\boldsymbol{\alpha}_K)}}{M_{\leqslant(\boldsymbol{\alpha}'_I,\boldsymbol{\alpha}'_J,\boldsymbol{\alpha}_K)}} \arrow[u]
\end{tikzcd}$$

The order $\leqslant$ is the partial natural order on $\mathbb{R}^n$: $\boldsymbol{\alpha}' \leqslant \boldsymbol{\alpha} \iff \alpha'_i \leqslant \alpha_i$, $\forall\, i$. Then

$$\star = \bigoplus_{\substack{\boldsymbol{\alpha}'_I \nleqslant \boldsymbol{\alpha}''_I \leqslant \boldsymbol{\alpha}_I \\ \boldsymbol{\alpha}'_J \nleqslant \boldsymbol{\alpha}''_J \leqslant \boldsymbol{\alpha}_J \\ \boldsymbol{\alpha}''_K \leqslant \boldsymbol{\alpha}_K}} M_{\boldsymbol{\alpha}''}$$

is a natural choice in order to complete the diagram.

By flatness of $V_0^{(\boldsymbol{n})}\mathcal{D}_X$ over $V_0^{(\boldsymbol{n})}\mathbb{C}[x]\langle\partial_x\rangle$, the similar diagram for $\mathcal{M}$ is obtained by tensoring by $V_0^{(\boldsymbol{n})}\mathcal{D}_X$, and is thus also exact, leading to the compatibility property of $V_\bullet^{(i)}\mathcal{M}$ $(i \in I)$.

(3) The argument above reduces the proof of (3) to the case of M, which has been obtained in (1).

(4) This is now obvious from the previous description, since $\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M} = \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}} M$. ☐

The morphisms between $\mathcal{D}_X$-modules of normal crossing type can also be regarded as being of normal crossing type, as follows from the next proposition.

Let $\varphi : \mathcal{M}_1 \to \mathcal{M}_2$ be a morphism between coherent $\mathcal{D}_X$-modules of normal crossing type. Then φ is compatible with the V-filtrations $V_\bullet^{(i)}$, and for every $\boldsymbol{\alpha} \in \mathbb{R}^n$, its multi-graded components $\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}_1 \to \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}_2$ do not depend on the order of grading (according to the compatibility of the V-filtrations and Remark 10.3.16). We denote this morphism by $\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\varphi$. On the other hand, regarding $M_{\boldsymbol{\alpha}}$ as an $\mathbb{C}$-submodule of $\mathcal{M}$, we notice that φ sends $M_{1,\boldsymbol{\alpha}}$ to $M_{2,\boldsymbol{\alpha}}$, due to the $\mathcal{D}$-linearity, and has no component from $M_{1,\boldsymbol{\alpha}}$ to $M_{2,\boldsymbol{\beta}}$ if $\boldsymbol{\beta} \neq \boldsymbol{\alpha}$. We denote by $\varphi_{\boldsymbol{\alpha}}$ the induced morphism $M_{1,\boldsymbol{\alpha}} \to M_{2,\boldsymbol{\alpha}}$. The following is now obvious.

13.3.6. Proposition. *With respect to the isomorphism $M_{\boldsymbol{\alpha}} \xrightarrow{\sim} \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ of Proposition* 13.3.4(4), *$\varphi_{\boldsymbol{\alpha}}$ coincides with $\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\varphi$.* ☐

13.3.7. Corollary. *The category of $\mathcal{D}_X$-modules of normal crossing type along D is abelian.*

Proof. Each φ_α is $\mathbb{C}$-linear, hence its kernel and cokernel are also finite-dimensional. ☐

13.3.8. Remarks.

(1) If $\mathcal{M}$ is of normal crossing type along D, then for any $i \in I$ and any $\alpha_i \in \mathbb{R}$, $\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$ is of normal crossing type on $(D_i, \bigcup_{j\neq i} D_j)$ and $V_\bullet^{(j)}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$ is the filtration naturally induced by $V_\bullet^{(j)}\mathcal{M}$ on $\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$, that is,

$$V_\bullet^{(j)}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M} = \frac{V_\bullet^{(j)}\mathcal{M} \cap V_{\alpha_i}^{(i)}\mathcal{M}}{V_\bullet^{(j)}\mathcal{M} \cap V_{<\alpha_i}^{(i)}\mathcal{M}}.$$

Indeed, due to the isomorphism (13.3.2), it is enough to prove the result for the multi-graded module $M := \mathrm{gr}^{V^{(\boldsymbol{n})}}\mathcal{M}$, for which all assertions are clear.

(2) We deduce from (13.3.2) a decomposition $\mathcal{M} = \bigoplus_{\boldsymbol{\alpha}\in\boldsymbol{A}} \mathcal{M}_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ similar to that of Remark 13.2.3. We have

$$V_{\boldsymbol{\alpha}+\boldsymbol{m}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^n} = \bigoplus_{\boldsymbol{n}\leqslant\boldsymbol{m}} M_{\boldsymbol{\alpha}+\boldsymbol{n}}.$$

It follows that, for $\boldsymbol{\alpha} \in \boldsymbol{A}$ (so that $\alpha_i < 0$ for all i), we have

$$V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})} M_{\boldsymbol{\alpha}+\mathbb{Z}^n} = M_{\boldsymbol{\alpha}} \otimes_{\mathbb{C}} \mathbb{C}[x],$$

and we conclude that $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is $\mathbb{C}[x]$-locally free of finite rank. It follows then easily that the same property holds for $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}-\boldsymbol{k}} M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ for every $\boldsymbol{k} \in \mathbb{N}^n$, and that $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}+\boldsymbol{k}} M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is of finite type over $\mathbb{C}[x]$ for every $\boldsymbol{k} \in \mathbb{Z}^n$. From (13.3.2) we conclude that $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M}$ is $\mathcal{O}_X$-coherent for every $\boldsymbol{\alpha} \in \mathbb{R}^n$ and is $\mathcal{O}_X$-locally free in the neighbourhood of the origin for $\boldsymbol{\alpha} \in (-\infty, 0)^n$. In the latter case, we can thus regard $(V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M})^{\mathrm{left}}$ as an $\mathcal{O}_X$-locally free module of finite rank equipped with a flat D-logarithmic connection. Moreover, for any $\boldsymbol{\alpha} \in \mathbb{R}^n$, $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M}_{X \smallsetminus D}$ is $\mathcal{O}_{X \smallsetminus D}$ locally free, and more precisely $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M}(*D)$ is $\mathcal{O}_X(*D)$-locally free.

Behaviour with respect to localization, dual localization and middle extension

Let us fix $i \in I$ and set $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', \alpha_i)$. By $\mathbb{R}$-specializability along D_i we have isomorphisms

$$x_i : V^{(i)}_{\alpha_i} \mathcal{M} \xrightarrow{\sim} V^{(i)}_{\alpha_i - 1} \mathcal{M},\ (\alpha_i < 0) \quad \text{and} \quad \partial_{x_i} : \mathrm{gr}^{V^{(i)}}_{\alpha_i} \mathcal{M} \xrightarrow{\sim} \mathrm{gr}^{V^{(i)}}_{\alpha_i+1} \mathcal{M},\ (\alpha_i > -1).$$

One checks on M, and then on $\mathcal{M}$ due to (13.3.2), that they induce isomorphisms

$$\begin{aligned} x_i &: V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{M} \xrightarrow{\sim} V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}-\boldsymbol{1}_i} \mathcal{M},\ (\alpha_i < 0) \\ \partial_{x_i} &: V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \mathrm{gr}^{V^{(i)}}_{\alpha_i} \mathcal{M} \xrightarrow{\sim} V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \mathrm{gr}^{V^{(i)}}_{\alpha_i+1} \mathcal{M},\ (\alpha_i > -1). \end{aligned} \tag{13.3.9}$$

The following lemma shows that the localization (resp. dual localization, resp. minimal extension) property along one component D_{i_o} of D is compatible the other filtrations $V^{(i)}$.

13.3.10. Lemma. *Assume that $\mathcal{M}$ is of normal crossing type along D. Let us fix $i_o \in I$ and let us set $n' = n - 1$, corresponding to forgetting i_o. Then, for every $\boldsymbol{\alpha}' \in \mathbb{R}^{n'}$, each of the following properties*

$$\begin{aligned} \mathrm{can}_{i_o} &: V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \mathrm{gr}^{V^{(i_o)}}_{-1} \mathcal{M} \longrightarrow V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \mathrm{gr}^{V^{(i_o)}}_{0} \mathcal{M} \quad \textit{is onto, resp. bijective,} \\ \mathrm{var}_{i_o} &: V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \mathrm{gr}^{V^{(i_o)}}_{0} \mathcal{M} \longrightarrow V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \mathrm{gr}^{V^{(i_o)}}_{-1} \mathcal{M} \quad \textit{is injective, resp. bijective,} \end{aligned}$$

holds as soon as it holds when forgetting $V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'}$.

Proof. We first work with M. Setting $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', \alpha_{i_o})$, the morphism $x_{i_o} : \mathrm{gr}^{V^{(i_o)}}_0 M \to \mathrm{gr}^{V^{(i_o)}}_{-1} M$ decomposes as the direct sum of morphisms $x_{i_o} : M_{\boldsymbol{\alpha}',0} \to M_{\boldsymbol{\alpha}',-1}$, and similarly for $\partial_{x_{i_o}}$. Therefore var_{i_o} is injective (resp. bijective) or can_{i_o} is surjective (resp. bijective) if and only if each $\boldsymbol{\alpha}'$-component is so. This implies the lemma for M. One concludes that the lemma holds for $\mathcal{M}$ by flat tensorization. □

By a similar argument, considering M first, we obtain:

13.3.11. Lemma. *Let $\mathcal{M}$ be a coherent module of normal crossing type along D. Let us fix $i_o \in I$. Then $\mathcal{M}(*D_{i_o})$, $\mathcal{M}(!D_{i_o})$, $\mathcal{M}(!{*}D_{i_o})$ are of normal crossing type along D.* □

13.3.12. Remark. It is now easy to show that the two possible definitions of $M(!D_{i_o})$ and $M(!{*}D_{i_o})$ (see Definition 13.2.7) coincide.

13.3.13. Definition. We say that $\mathcal{M}$ is a middle extension along $D_{i\in I}$ if the corresponding M is a middle extension in the sense of Definition 13.2.8.

***13.3.14. Remark* (Suppressing the simplifying assumptions 13.1.2)**
If $\ell < n$, we apply the same changes as in Remark 13.2.10. All the previous results extend in a straightforward way to this setting.

13.3.b. Nearby cycles for coherent $\mathcal{D}$-modules of normal crossing type

We now consider Theorem 13.1.4 in the analytic setting, but we forget the filtration. Given $\mathcal{M}$ of normal crossing type along D as in §13.3.a, we denote by M the associated $\mathbb{C}[x]\langle\partial_x\rangle$-module, so that $\mathcal{M} = \mathcal{D}_X \otimes_{\mathbb{C}[x]\langle\partial_x\rangle} M$. In order to simplify the notation, we will set $\mathcal{N} := {}_{\mathrm{D}}\iota_{g*}\mathcal{M}$. Then, using the notation of §13.2.b, we have $\mathcal{N} = \mathcal{D}_{X\times\mathbb{C}} \otimes_{\mathbb{C}[x,t]\langle\partial_x,\partial_t\rangle} N$ and $\mathrm{gr}_\gamma^V\mathcal{N} = \mathcal{D}_X \otimes_{\mathbb{C}[x]\langle\partial_x\rangle} \mathrm{gr}_\gamma^V N$. As a consequence from the results proved for M and N in §13.2.b, we obtain that Theorem 13.1.4 holds for $\mathcal{M}$. The results of §13.2.c also extend to $\mathcal{M}$ and $\mathcal{N}$ in a straightforward way. Let us end this subsection by adapting Proposition 13.2.41 to $\mathcal{M}$. Let us fix $\gamma \in [-1, 0)$ and take up the notation ${}^\gamma\mathcal{N} = \mathrm{gr}_\gamma^V\mathcal{N}$ like in Example 13.2.17.

***Structure of* $V_0^{(\boldsymbol{n})}{}^\gamma\mathcal{N}$.** We have

$$V_0^{(\boldsymbol{n})}{}^\gamma\mathcal{N} = \mathcal{O}_X \otimes_{\mathbb{C}[x]\langle\partial_x\rangle} V_0^{(\boldsymbol{n})}{}^\gamma N = \mathcal{O}_X \otimes_{\mathbb{C}[x]\langle\partial_x\rangle} \Big(\sum_{\boldsymbol{\beta}\leqslant 0} {}^\gamma N_{\boldsymbol{\beta}}\Big).$$

The $\mathcal{O}_X$-module $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] := V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} \otimes_{\mathbb{C}} \mathbb{C}[\mathrm{E}] = \mathcal{O}_X \otimes_{\mathbb{C}[x]\langle\partial_x\rangle} V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})} M[\mathrm{E}]$ is equipped with an induced action of $V_0^{(\boldsymbol{n})}\mathcal{D}_X$ (and the obvious action of E, see below), and we have a surjective $V_0^{(\boldsymbol{n})}\mathcal{D}_X$-linear morphism

$$V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] \longrightarrow V_0^{(\boldsymbol{n})}\mathcal{N}. \tag{13.3.15}$$

In the analytic setting, the filtration $U_\bullet$ is defined by analytification of that on $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M[\mathrm{E}]$ and $V_0^{(\boldsymbol{n})}{}^\gamma N$ and the analytification of $\mathrm{gr}^U\varphi$ gives rise to a $V_0^{(\boldsymbol{n})}\mathcal{D}_X$-linear presentation for each $k \geqslant 0$

$$\mathrm{gr}_k^U V_0^{(\boldsymbol{n})}\mathcal{N} = \mathrm{Coker}\Big[\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] \xrightarrow{\varphi_k} \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]\Big], \tag{13.3.16}$$

and φ_k is injective.

The filtration $U_\bullet V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ (and then $U_\bullet V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$) can be defined in terms of $\mathcal{M}$ only. For $J \subset I_g$, let us denote $J^{\mathrm{c}} := I_g \smallsetminus J$ and $I_g^{\mathrm{c}} := I \smallsetminus I_g$, so that $I = J^{\mathrm{c}} \sqcup J \sqcup I_g^{\mathrm{c}}$. Let us decompose correspondingly $\boldsymbol{a} = (\boldsymbol{a}_{J^{\mathrm{c}}}, \boldsymbol{a}_J, 0_{I_g^{\mathrm{c}}})$ and $\boldsymbol{n} = (\boldsymbol{n}_{J^{\mathrm{c}}}, \boldsymbol{n}_J, \boldsymbol{n}_{I_g^{\mathrm{c}}})$. Then, by considering first $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}M$, one checks that

$$U_k V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} = \sum_{\substack{J\subset I_g\\ \#J\leqslant k}} V_{<\gamma\boldsymbol{a}_{J^{\mathrm{c}}}}^{(\boldsymbol{n}_{J^{\mathrm{c}}})} V_{\gamma\boldsymbol{a}_J}^{(\boldsymbol{n}_J)} V_0^{(\boldsymbol{n}_{I_g^{\mathrm{c}}})}\mathcal{M}, \quad \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} = \bigoplus_{\substack{J\subset I_g\\ \#J=k}} V_{<\gamma\boldsymbol{a}_{J^{\mathrm{c}}}}^{(\boldsymbol{n}_{J^{\mathrm{c}}})} V_0^{(\boldsymbol{n}_{I_g^{\mathrm{c}}})} \mathrm{gr}_{\gamma\boldsymbol{a}_J}^{V^{(\boldsymbol{n}_J)}}\mathcal{M}.$$

We can give an interpretation of the filtration $U_\bullet V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}$ as a convolution of filtrations, as in Exercise 10.11(2). Let us define the following filtrations on $V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}$, for $i\in I$:

$$(13.3.17)\qquad \begin{aligned} U^{(i)}_{-1}V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}&=0,\\ U^{(i)}_{0}V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}&=\begin{cases}V^{(i)}_{<\gamma a_i}V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M} & \text{if } i\in I_g,\\ V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M} & \text{if } i\notin I_g,\end{cases}\\ U^{(i)}_{1}V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}&=V^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}.\end{aligned}$$

Then

$$(13.3.18)\qquad U_kV^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}=(U^{(1)}_\bullet\star\cdots\star U^{(n)}_\bullet)_kV^{(\boldsymbol{n})}_{\gamma\boldsymbol{a}}\mathcal{M}.$$

13.4. Filtered normal crossing type

13.4.a. Coherent filtrations of normal crossing type. We now extend the notion of "normal crossing type" to filtered coherent $\mathcal{D}$-modules. Of course the underlying $\mathcal{D}$-module should be of normal crossing type, but the isomorphism (13.3.2), together with the decomposition (13.3.1), is not expected to hold at the filtered level. This would be a too strong condition. On the other hand, the properties in Proposition 13.3.4 can be naturally extended to the filtered case. We keep the simplifying assumptions 13.1.2.

13.4.1. Definition. Let $(\mathcal{M},F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module. We say that $(\mathcal{M},F_\bullet\mathcal{M})$ is of *normal crossing type along D* if

(1) $\mathcal{M}$ is of normal crossing type along D (see Definition 13.3.3),

(2) $(\mathcal{M},F_\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable along D_i for every component D_i of D (see Section 10.6),

(3) the filtrations $(F_\bullet\mathcal{M},V^{(1)}_\bullet\mathcal{M},\dots,V^{(n)}_\bullet\mathcal{M})$ are compatible (see Definition 10.3.9).

13.4.2. Remarks.

(a) Condition (3) implies that $\mathrm{gr}^F_p\mathrm{gr}^{V^{(n)}}_{\boldsymbol{\alpha}}\mathcal{M}$ does not depend on the way $\mathrm{gr}^{V^{(n)}}_{\boldsymbol{\alpha}}\mathcal{M}$ is computed.

(b) Note that (2) implies 13.3.4(1) for $\mathcal{M}$, and similarly (3) implies 13.3.4(2). So the condition that $\mathcal{M}$ is of normal crossing type along D only adds the existence of the isomorphism (13.3.2).

(c) Let us recall that $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ is $\mathcal{O}_X$-coherent for every $\boldsymbol{\alpha}\in\mathbb{R}^n$ (see Remark 13.3.8(2)). Since $F_p\mathcal{M}$ is $\mathcal{O}_X$-coherent, it follows that $F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}:=F_p\mathcal{M}\cap V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ (see §10.6) and $\mathrm{gr}^F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ are also $\mathcal{O}_X$-coherent and therefore the filtration $F_\bullet V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ is locally finite, hence is a coherent $F_\bullet V^{(\boldsymbol{n})}_0\mathcal{D}_X$-filtration.

(d) Moreover, each $\mathrm{gr}^F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ is $\mathcal{O}_X$-locally free if $\alpha_i<0$ for all $i\in I$. Indeed, the family $(F_p\mathcal{M},V^{(1)}_{\alpha_1+k_1}\mathcal{M},\dots,V^{(n)}_{\alpha_n+k_n}\mathcal{M})$ ($p\in\mathbb{Z}$, $k_1,\dots,k_n\in\mathbb{N}$) is a compatible family; the $\mathcal{O}_X$-coherent sheaf $\mathrm{gr}^F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ has generic rank (on its support)

$\leqslant \dim \mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}/(x_1,\dots,x_n)$; but

$$\sum_p \dim \mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}/(x_1,\dots,x_n) = \dim V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}/(x_1,\dots,x_n)$$
$$= \mathrm{rk}\, V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M} = \sum_p \mathrm{rk}\,\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M},$$

so in fact $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}/(x_1,\dots,x_n)$ has dimension equal to the generic rank of $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}$. As a consequence, $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}$ is $\mathcal{O}_X$-locally free.

(e) Since each $\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ is finite dimensional, the induced filtration $F_\bullet \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ is finite, and there exists a (non-canonical) splitting compatible with $F_\bullet$:

$$F_p\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M} \simeq \bigoplus_{q\leqslant p} \mathrm{gr}_q^F \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}.$$

(f) There are a priori two ways for defining the filtration $F_\bullet M_{\boldsymbol{\alpha}}$, namely, either by inducing it on $M_{\boldsymbol{\alpha}} \subset \mathcal{M}$, or by inducing it on $\mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ and transport it by means of the isomorphism $M_{\boldsymbol{\alpha}} \xrightarrow{\sim} \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$. We always consider the latter one. The filtration $F_\bullet\mathcal{M}$ is a priori not isomorphic to $\bigoplus_\alpha F_\bullet \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ by means of the isomorphism $\mathcal{M} \simeq \bigoplus_\alpha \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$ induced by 13.3.4(4) and (13.3.2). Using the compatibility of the filtrations, we have

$$F_pM_{\boldsymbol{\alpha}} = M_{\boldsymbol{\alpha}} \cap (F_pV_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M} + V_{<\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}) \subset \mathcal{M}.$$

The graded filtered module $(\bigoplus_{\boldsymbol{\alpha}} M_{\boldsymbol{\alpha}}, \bigoplus_{\boldsymbol{\alpha}} F_\bullet M_{\boldsymbol{\alpha}})$ is obviously of normal crossing type if $(\mathcal{M}, F_\bullet\mathcal{M})$ is so.

Inductive arguments below will make use of the following lemma.

13.4.3. Lemma. *Assume that* $(\mathcal{M}, F_\bullet\mathcal{M})$ *is of normal crossing type along* D. *Then for any* $i \in I$ *and any* $\alpha_i \in \mathbb{R}$, $(\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}, F_\bullet\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M})$ *is of normal crossing type on* $(D_i, \bigcup_{j\neq i} D_j)$, *where* $F_\bullet\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$ *is the filtration naturally induced by* $F_\bullet\mathcal{M}$ *on* $\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$.

Proof. We know by Remark 13.3.8(1) that $\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$ is of normal crossing type on $(D_i, \bigcup_{j\neq i} D_j)$, and that the filtrations $V_\bullet^{(j)}$ on $\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}$ are naturally induced by $V_\bullet^{(j)}\mathcal{M}$. It follows that $(F_\bullet\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}, (V_\bullet^{(j)}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M})_{j\neq i})$ are compatible (see Remark 10.3.10). We know, by Proposition 10.8.3, $(\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}, F_\bullet\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M})$ is coherent as a filtered $\mathcal{D}_{D_i}$-module. Note also that, setting $\boldsymbol{\alpha}' = (\alpha_j)_{j\neq i}$ and $\boldsymbol{n}' = (j)_{j\neq i}$, we have

$$\mathrm{gr}_p^F \mathrm{gr}_{\boldsymbol{\alpha}'}^{V^{(\boldsymbol{n}')}} \mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M} = \mathrm{gr}_p^F \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}}\mathcal{M}$$

(since, by the compatibility property, we can take graded objects in any order).

It remains to showing the strict $\mathbb{R}$-specializability property, namely,

$$x_j : F_pV_{\alpha_j}^{(j)}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M} \xrightarrow{\sim} F_pV_{\alpha_j-1}^{(j)}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}, \quad \forall\, p, \forall\, j\neq i, \forall\, \alpha_j < 0,$$
$$\partial_{x_j} : F_p\mathrm{gr}_{\alpha_j}^{V^{(j)}}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M} \xrightarrow{\sim} F_{p+1}\mathrm{gr}_{\alpha_j+1}^{V^{(j)}}\mathrm{gr}_{\alpha_i}^{V^{(i)}}\mathcal{M}, \quad \forall\, p, \forall\, j\neq i, \forall\, \alpha_j > -1.$$

Let us first show that, by applying $\mathrm{gr}_{\alpha_i}^{V^{(i)}}$, we get isomorphisms

$$x_j : \mathrm{gr}_{\alpha_i}^{V^{(i)}} F_p V_{\alpha_j}^{(j)} \mathcal{M} \xrightarrow{\sim} \mathrm{gr}_{\alpha_i}^{V^{(i)}} F_p V_{\alpha_j - 1}^{(j)} \mathcal{M}, \quad \forall\, p, \forall\, j \neq i, \forall\, \alpha_j < 0, \tag{13.4.4}$$

$$\partial_{x_j} : \mathrm{gr}_{\alpha_i}^{V^{(i)}} F_p \mathrm{gr}_{\alpha_j}^{V^{(j)}} \mathcal{M} \xrightarrow{\sim} \mathrm{gr}_{\alpha_i}^{V^{(i)}} F_{p+1} \mathrm{gr}_{\alpha_j+1}^{V^{(j)}} \mathcal{M}, \quad \forall\, p, \forall\, j \neq i, \forall\, \alpha_j > -1. \tag{13.4.5}$$

By the strict $\mathbb{R}$-specializability of $(\mathcal{M}, F_\bullet\mathcal{M})$ along D_j and since $\mathcal{M}$ is of normal crossing type, we have isomorphisms

$$F_p V_{\alpha_j}^{(j)} \mathcal{M} \xrightarrow[\sim]{x_j} F_p V_{\alpha_j-1}^{(j)} \mathcal{M}, \qquad \begin{cases} V_{\alpha_i}^{(i)} V_{\alpha_j}^{(j)} \mathcal{M} \\ V_{<\alpha_i}^{(i)} V_{\alpha_j}^{(j)} \mathcal{M} \end{cases} \xrightarrow[\sim]{x_j} \begin{cases} V_{\alpha_i}^{(i)} V_{\alpha_j-1}^{(j)} \mathcal{M} \\ V_{<\alpha_i}^{(i)} V_{\alpha_j-1}^{(j)} \mathcal{M}, \end{cases}$$

hence isomorphisms

$$\begin{cases} V_{\alpha_i}^{(i)} F_p V_{\alpha_j}^{(j)} \mathcal{M} \\ V_{<\alpha_i}^{(i)} F_p V_{\alpha_j}^{(j)} \mathcal{M} \end{cases} \xrightarrow[\sim]{x_j} \begin{cases} V_{\alpha_i}^{(i)} F_p V_{\alpha_j-1}^{(j)} \mathcal{M} \\ V_{<\alpha_i}^{(i)} F_p V_{\alpha_j-1}^{(j)} \mathcal{M}, \end{cases}$$

and thus the isomorphisms (13.4.4). We argue similarly for the isomorphisms (13.4.5). Now, the desired assertion follows from the compatibility property (3) which enables us to switch $F_p V_{\alpha_j}^{(j)}$ or $F_p \mathrm{gr}_{\alpha_j}^{V^{(j)}}$ with $\mathrm{gr}_{\alpha_i}^{V^{(i)}}$. □

By the same argument as above, setting $\boldsymbol{\alpha} = (\boldsymbol{\alpha}', \alpha_j)$ and $n' = n - 1$, the filtered analogue of (13.3.9) holds (any $\boldsymbol{\alpha}' \in \mathbb{R}^{n'}$, $p \in \mathbb{Z}$):

$$\begin{aligned} F_p V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} V_{\alpha_j}^{(j)} \mathcal{M} &\xrightarrow[\sim]{x_j} F_p V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} V_{\alpha_j-1}^{(j)} \mathcal{M} \quad \text{if } \alpha_j < 0, \\ F_p V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} \mathrm{gr}_{\alpha_j}^{V^{(j)}} \mathcal{M} &\xrightarrow[\sim]{\partial_{x_j}} F_{p+1} V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} \mathrm{gr}_{\alpha_j+1}^{V^{(j)}} \mathcal{M} \quad \text{if } \alpha_j > -1. \end{aligned} \tag{13.4.6}$$

The following lemma is similar to Lemma 13.3.10, but weaker when considering surjectivity for can_{i_o}.

13.4.7. Lemma. *Assume that $(\mathcal{M}, F_\bullet\mathcal{M})$ is of normal crossing type along D. Let us fix $i_o \in I$ and let us set $n' = n - 1$, corresponding to forgetting i_o. Then, for every $\boldsymbol{\alpha}' \in \mathbb{R}^{n'}$, each of the following properties*

$$\begin{aligned} &\mathrm{can}_{i_o} : F_p V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} \mathrm{gr}_{-1}^{V^{(i_o)}} \mathcal{M} \longrightarrow F_{p+1} V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} \mathrm{gr}_0^{V^{(i_o)}} \mathcal{M} \quad \textit{is bijective,} \\ &\mathrm{var}_{i_o} : F_p V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} \mathrm{gr}_0^{V^{(i_o)}} \mathcal{M} \longrightarrow F_p V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')} \mathrm{gr}_{-1}^{V^{(i_o)}} \mathcal{M} \quad \textit{is} \begin{cases} \textit{injective,} \\ \textit{resp. bijective,} \end{cases} \end{aligned} \tag{13.4.7 $*$}$$

holds for all p as soon as it holds when forgetting $V_{\boldsymbol{\alpha}'}^{(\boldsymbol{n}')}$. □

13.4.8. Remark. As a consequence, if var_{i_o} is injective, then the first line of (13.4.6) with $j = i_o$ also holds for $\alpha_j = 0$. That the lemma does not a priori hold when can_{i_o} is onto leads to the definition below.

***13.4.9. Definition* (Middle extension along $D_{i\in I}$).** Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. We say that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a *middle extension along $D_{i\in I}$* if $\mathcal{M}$ is a middle extension independently along each

D_i ($i \in I$) and moreover, for each $i_o \in I$, and every $\boldsymbol{\alpha}' \in \mathbb{R}^{n'}$ (equivalently, every $\boldsymbol{\alpha}' \in [-1,0]^{n'}$),

$$\operatorname{can}_{i_o} : F_p V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \operatorname{gr}^{V^{(i_o)}}_{-1} \mathcal{M} \longrightarrow F_{p+1} V^{(\boldsymbol{n}')}_{\boldsymbol{\alpha}'} \operatorname{gr}^{V^{(i_o)}}_{0} \mathcal{M} \quad \text{is onto, } \forall\, p.$$

If $n = 1$ this notion is equivalent to that of Definition 9.7.3, but if $n \geqslant 2$ it is a priori stronger than the condition of filtered middle extension along each D_i independently (see Definition 10.6.1).

13.4.b. A complement on compatible filtrations. We will make more explicit the general notion of compatible filtrations in the case of x_i-adic filtrations on a coherent $\mathcal{O}_X$-module. For such a module $\mathcal{E}$, assume we are given, for each $i = 1, \dots, n$, an increasing filtration $V^{(i)}_\bullet \mathcal{E}$ indexed by $[-1, 0)$ by coherent submodules, such that $\mathcal{E} = \bigcup_{\alpha_i \in [-1,0)} V^{(i)}_{\alpha_i} \mathcal{E}$ and the set of jumps $A_i \subset [-1, 0)$ is finite. We extend the filtration as a filtration indexed by $A_i + \mathbb{Z}$ by setting

$$V^{(i)}_{\alpha_i + k} \mathcal{E} = \begin{cases} x_i^k V^{(i)}_{\alpha_i} \mathcal{E} & \text{if } k \leqslant 0, \\ V^{(i)}_{\alpha_i} \mathcal{E} & \text{if } k \geqslant 0. \end{cases}$$

We define as above $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{E} = \bigcap_i V^{(i)}_{\alpha_i} \mathcal{E}$.

13.4.10. Example **(Rank-one objects).** Assume that $\mathcal{E}$ is $\mathcal{O}_X$-locally free of rank 1. Then, for each i, A_i is reduced to one element $\alpha_i^o \in [-1, 0)$ and we have for any $\boldsymbol{\alpha} \in \prod_i (A_i + \mathbb{Z})$

$$V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{E} = \mathcal{E}\big(\textstyle\sum_{i \mid \alpha_i \leqslant \alpha_i^o} [\alpha_i - \alpha_i^o] D_i\big).$$

We claim that the family $(V^{(1)}_\bullet \mathcal{E}, \dots, V^{(n)}_\bullet \mathcal{E})$ is compatible. Indeed, the multi-Rees module $R_V \mathcal{E}$ (see Section 10.3.b) reads

$$\bigoplus_{\boldsymbol{k} \in \mathbb{Z}^n} x^{\boldsymbol{k}} \mathcal{E} z^{-\boldsymbol{k}}, \quad \text{with } x_i^{k_i} := 1 \text{ if } k_i \leqslant 0.$$

We have to check that each permutation of $(z_1, \dots, z_n)$ is a regular sequence on $R_{V^{(\boldsymbol{n})}} \mathcal{E}$. This is obtained by induction, noticing that z_i is injective on $R_{V^{(\boldsymbol{n})}} \mathcal{E}$ and $R_{V^{(\boldsymbol{n})}} \mathcal{E} / z_i R_{V^{(\boldsymbol{n})}} \mathcal{E}$ is the Rees module of $\operatorname{gr}^{V^{(i)}}_\bullet \mathcal{E}$ equipped with the similar filtrations $V^{(j)}_\bullet \operatorname{gr}^{V^{(i)}}_\bullet \mathcal{E}$ ($j \neq i$).

13.4.11. Proposition. *Let $\mathcal{E}$ be a coherent $\mathcal{O}_X$-module and let $(V^{(1)}_\bullet \mathcal{E}, \dots, V^{(n)}_\bullet \mathcal{E})$ be filtrations as defined above. Let us assume that, for each $\boldsymbol{\alpha} \in \prod_i (A_i + \mathbb{Z})$,*

(1) *the $\mathcal{O}_X$-module $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{E}$ is locally free,*
(2) *if $\alpha_i < 0$, then $x_i V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}} \mathcal{E} = V^{(\boldsymbol{n})}_{\boldsymbol{\alpha} - \mathbf{1}_i} \mathcal{E}$.*

Then the filtrations $(V^{(1)}_\bullet \mathcal{E}, \dots, V^{(n)}_\bullet \mathcal{E})$ are compatible.

Proof. Note that the assumption implies that $\mathcal{E}$ itself is $\mathcal{O}_X$-locally free. The multi-Rees module $R_{V^{(\boldsymbol{n})}} \mathcal{E}$ is the direct sum over $\boldsymbol{\alpha}^o \in \prod_i A_i$ of multi-Rees modules associated with the multi-filtrations $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}^o + \mathbb{Z}} \mathcal{E}$. To check its $\mathbb{C}[z_1, \dots, z_n]$-flatness, it is enough to check that of each summand. We can therefore assume that $\prod_i A_i = \{\boldsymbol{\alpha}^o\}$. We then simply write $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}^o+\boldsymbol{k}}\mathcal{E} = V^{(\boldsymbol{n})}_{\boldsymbol{k}}\mathcal{E}$. By (2),

$$V^{(\boldsymbol{n})}_{\boldsymbol{k}}\mathcal{E} = \mathcal{E}(\textstyle\sum_{i|k_i\leqslant 0} k_iD_i)$$

and we argue as in the example to conclude. □

13.4.c. A criterion for filtered normal crossing type. It is easier to deal with coherent $\mathcal{O}$-modules instead of coherent $\mathcal{D}$-modules. Our aim is to deduce properties on $F_\bullet\mathcal{M}$ from properties on $F_\bullet\mathcal{M}_0$ with $\mathcal{M}_0 := V^{(\boldsymbol{n})}_0\mathcal{M}$ and, in the case of a middle extension, from $F_\bullet\mathcal{M}_{<0}$. We first explain which properties should be expected on the latter $\mathcal{O}$-module, in order to recover the normal crossing property of $(\mathcal{M}, F_\bullet\mathcal{M})$ from them. We will then give a criterion to check whether they are satisfied.

***13.4.12. Proposition* (Properties of $F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$).** *Let $(\mathcal{M}, F_\bullet\mathcal{M})$ be a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D. Set $\mathcal{M}_0 := V^{(\boldsymbol{n})}_0\mathcal{M}$. For $\alpha\in\mathbb{R}^n$, let us set $F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M} := F_p\mathcal{M}\cap V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$. Then*

(1) *$F_\bullet V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ is a coherent $F_\bullet V^{(\boldsymbol{n})}_0\mathcal{D}_X$-filtration.*

(2) *The filtrations $(F_\bullet\mathcal{M}_0, V^{(1)}_\bullet\mathcal{M}_0,\dots,V^{(n)}_\bullet\mathcal{M}_0)$ are compatible and*

$$F_p\mathcal{M} = \sum_{q\geqslant 0}(F_{p-q}\mathcal{M}_0)\cdot F_q\mathcal{D}_X.$$

Proof. The compatibility property of the filtrations on $\mathcal{M}_0$ clearly follows from that on $\mathcal{M}$, as noted in Remark 10.3.10(2). By the same argument we have compatibility for the family of filtrations on each $V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ ($\boldsymbol{\alpha}\in\mathbb{R}^n$).

It remains to justify the expression for $F_p\mathcal{M}$. We have seen in the proof of Lemma 13.4.3 that, for $\boldsymbol{k}\geqslant 0$ and any $i\in I$, setting $\boldsymbol{k} = (\boldsymbol{k}', k_i)$, we have an isomorphism

$$\partial_{x_i} : F_{p-1}V^{(\boldsymbol{n}')}_{\boldsymbol{k}'}\mathrm{gr}^{V^{(i)}}_{k_i}\mathcal{M} \xrightarrow{\ \sim\ } F_pV^{(\boldsymbol{n}')}_{\boldsymbol{k}'}\mathrm{gr}^{V^{(i)}}_{k_i+1}\mathcal{M},$$

and thus

$$F_pV^{(\boldsymbol{n})}_{\boldsymbol{k}+\mathbf{1}_i}\mathcal{M} = F_{p-1}V^{(\boldsymbol{n})}_{\boldsymbol{k}}\mathcal{M}\cdot\partial_{x_i} + F_pV^{(\boldsymbol{n})}_{\boldsymbol{k}}\mathcal{M},$$

which proves (2) by an easy induction. □

The property 13.4.12(2) can be made more precise. For $\boldsymbol{\alpha}\in[-1,0]^n$ and $p\in\mathbb{Z}$, let $E_{\boldsymbol{\alpha},p}$ be a finite $\mathbb{C}$-vector space of sections of $F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}$ whose image in $\mathrm{gr}^F_p\mathrm{gr}^{V^{(\boldsymbol{n})}}_{\boldsymbol{\alpha}}\mathcal{M}$ is an $\mathbb{C}$-basis of sections of this free $\mathbb{C}$-module. Given any $\boldsymbol{\gamma}\in\mathbb{R}^n$, we decompose it as $(\boldsymbol{\gamma}', 0, \boldsymbol{\gamma}'')$, where each component γ_i of $\boldsymbol{\gamma}'$ (resp. $\boldsymbol{\gamma}''$) satisfies $\gamma_i<0$ (resp. $\gamma_i>0$). When $\boldsymbol{\gamma}$ is fixed, any $\boldsymbol{\alpha}\in[-1,0]^n$ decomposes correspondingly as $(\boldsymbol{\alpha}', \boldsymbol{\alpha}^o, \boldsymbol{\alpha}'')$, of respective size n', n^o, n''.

13.4.13. Proposition. *With these assumptions and notation, for every $\boldsymbol{\gamma}\in\mathbb{R}^n$ and $p\in\mathbb{Z}$, we have*

$$F_pV^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M} = \sum_{\boldsymbol{\alpha}''\in(-1,0]^{n''}}\ \sum_{\substack{\boldsymbol{b}''\\ \forall i,\, b_i+\alpha_i\leqslant\gamma_i}} F_{p-|\boldsymbol{b}''|}V^{(\boldsymbol{n})}_{(\boldsymbol{\gamma}',0,\boldsymbol{\alpha}'')}\mathcal{M}\cdot\partial_x^{\boldsymbol{b}''},$$

and, for every $\boldsymbol{\alpha}'' \in (-1,0]^{n''}$,

$$F_q V^{(\boldsymbol{n})}_{(\boldsymbol{\gamma}',0,\alpha'')}\mathcal{M} = \sum_{\boldsymbol{\alpha}'\in[-1,0)^{n'}} E_{(\boldsymbol{\alpha}',0,\boldsymbol{\alpha}''),q}\cdot x^{\boldsymbol{a}'}\mathcal{O}_X,$$

where $\boldsymbol{a}'$ *has the indices of* $\boldsymbol{\gamma}'$ *and for each such index* i, $\alpha_i - a_i \leqslant \gamma_i$, *that is,*

$$a_i = \begin{cases} -[\gamma_i]-1 & \text{if } \alpha_i \leqslant \gamma_i - [\gamma_i]-1\\ -[\gamma_i] & \text{if } \alpha_i > \gamma_i - [\gamma_i]-1.\end{cases}$$

Proof. The first equality is obtained by induction from the second line of (13.4.6), and the second equality comes from the first line of (13.4.6). □

13.4.14. Remark **(The case of a middle extension along $D_{i\in I}$)**

In that case (Definition 13.4.9), Proposition 13.4.12 holds with the replacement of $\mathcal{M}_0$ with $\mathcal{M}_{<0} := V^{\boldsymbol{n}}_{<0}\mathcal{M} = \bigcap_{i\in I} V^{(i)}_{<0}\mathcal{M}$, and Proposition 13.4.13 reads as follows. We now decompose $\boldsymbol{\gamma}$ as $(\boldsymbol{\gamma}',\boldsymbol{\gamma}'')$, where each component γ_i of $\boldsymbol{\gamma}'$ (resp. $\boldsymbol{\gamma}''$) satisfies $\gamma_i < 0$ (resp. $\gamma_i \geqslant 0$). Then

$$F_p V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M} = \sum_{\boldsymbol{\alpha}\in[-1,0)^n} \sum_{\substack{\boldsymbol{b}''\\ \forall\, i,\, b_i+\alpha_i\leqslant\gamma_i}} E_{\boldsymbol{\alpha},p-|\boldsymbol{b}''|}\cdot x^{\boldsymbol{a}'}\partial_x^{\boldsymbol{b}''}\mathcal{O}_X,$$

where $\boldsymbol{a}'$ is as in Proposition 13.4.13.

The compatibility property of the filtrations $(F_\bullet\mathcal{M}, V^{(1)}_\bullet\mathcal{M},\dots,V^{(n)}_\bullet\mathcal{M})$ can be checked on $V^{(\boldsymbol{n})}_0\mathcal{M}$, as asserted by the proposition below.

13.4.15. Proposition **(From $\mathcal{M}_0$ to $\mathcal{M}$).** *Let* $\mathcal{M}$ *be a coherent* $\mathcal{D}_X$*-module of normal crossing type along* D. *Set* $\mathcal{M}_0 := V^{(\boldsymbol{n})}_0\mathcal{M}$. *Denote by* $V^{(i)}_\bullet\mathcal{M}_0$ *the filtration naturally induced by* $V^{(i)}_\bullet\mathcal{M}$ *and let* $F_\bullet\mathcal{M}_0$ *be any coherent* $F_\bullet V^{(\boldsymbol{n})}_0\mathcal{D}_X$*-filtration such that* $(F_\bullet\mathcal{M}_0, V^{(1)}_\bullet\mathcal{M}_0,\dots,V^{(n)}_\bullet\mathcal{M}_0)$ *are compatible filtrations and that* $(\mathcal{M}_0,F_\bullet\mathcal{M}_0)$ *is strictly* $\mathbb{R}$*-specializable along each* D_i, *in the sense that* $x_iF_pV^{(i)}_{\alpha_i}\mathcal{M}_0 = F_pV^{(i)}_{\alpha_i-1}\mathcal{M}_0$ *for every* i *and* $\alpha_i<0$, *and* ∂_{x_i} *sends* $F_pV^{(i)}_{-1}\mathcal{M}_0$ *to* $F_{p+1}V^{(i)}_0\mathcal{M}_0$. *Set*

$$F_p\mathcal{M} := \sum_{q\geqslant 0}(F_{p-q}\mathcal{M}_0)\cdot F_q\mathcal{D}_X.$$

Then

(1) $(\mathcal{M},F_\bullet\mathcal{M})$ *is strictly* $\mathbb{R}$*-specializable along each* D_i, *and for* $\alpha\in[-1,0]^n$,

$$F_pV^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}_0 := F_p\mathcal{M}_0\cap V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}_0 = F_p\mathcal{M}\cap V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}_0,$$

(2) *and* $(F_\bullet\mathcal{M}, V^{(1)}_\bullet\mathcal{M},\dots,V^{(n)}_\bullet\mathcal{M})$ *are compatible filtrations.*

Proof. For every $\boldsymbol{\gamma}\in\mathbb{R}^n$, we define

$$G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}) := \sum_{\substack{\boldsymbol{\alpha}\leqslant 0,\,\boldsymbol{j}\geqslant 0\\ \boldsymbol{\alpha}+\boldsymbol{j}\leqslant\boldsymbol{\gamma}}} F_{p-|\boldsymbol{j}|}V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}}. \tag{13.4.16}$$

For example, we have $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}) = F_pV^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}$ if $\boldsymbol{\gamma} \leqslant 0$, i.e., $\gamma_i \leqslant 0$ for all i. Similarly, if $\boldsymbol{\gamma}' = (\gamma_i)_{i|\gamma_i>0}$ denotes the "positive part" of $\boldsymbol{\gamma}$ and γ_- the non-positive part, we have, with obvious notation,

$$G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}) := \sum_{\substack{\boldsymbol{\alpha}'\leqslant 0,\,\boldsymbol{j}'\geqslant 0\\ \boldsymbol{\alpha}'+\boldsymbol{j}'\leqslant\boldsymbol{\gamma}'}} F_{p-|\boldsymbol{j}'|}V^{(\boldsymbol{n})}_{(\boldsymbol{\alpha}',\boldsymbol{\gamma}_-)}\mathcal{M}\cdot\partial_{x'}^{\boldsymbol{j}'}. \tag{13.4.17}$$

Let us note that

$$\varinjlim_{\boldsymbol{\gamma}} G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}) = \sum_{\boldsymbol{\alpha}\leqslant 0,\,\boldsymbol{j}\geqslant 0} F_{p-|\boldsymbol{j}|}V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}} = \sum_{\boldsymbol{j}\geqslant 0} F_{p-|\boldsymbol{j}|}\mathcal{M}_0\cdot\partial_x^{\boldsymbol{j}} =: F_p\mathcal{M}.$$

We set $V^{(i)}_{\bullet}V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M} = V^{(i)}_{\bullet}\mathcal{M}\cap V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}$. We will prove the following properties.

(a) Let $\boldsymbol{\beta} < \boldsymbol{\gamma}$ (i.e., $\beta_i \leqslant \gamma_i$ for all i and $\boldsymbol{\beta} \neq \boldsymbol{\gamma}$). Then $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}) \cap V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M} = G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M})$.

(b) $(G_\bullet(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}), V^{(1)}_\bullet V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M},\dots,V^{(n)}_\bullet V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M})$ are compatible filtrations,

(c) the following inclusion is $(n+1)$-strict:

$$(V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M}, G_\bullet(V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M}), (V^{(i)}_\bullet V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M})_{i\in I}) \hookrightarrow (V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}, G_\bullet(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}), (V^{(i)}_\bullet V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M})_{i\in I}).$$

Let us indicate how to obtain 13.4.15 from (a)–(c). Strict $\mathbb{R}$-specializability of $(\mathcal{M}, F_\bullet\mathcal{M})$ along D_i amounts to

$$F_{p+1}\mathcal{M}\cap V^{(i)}_{\beta_i+1}\mathcal{M} \subset (F_p\mathcal{M}\cap V^{(i)}_{\beta_i}\mathcal{M})\cdot\partial_{x_i} + V^{(i)}_{<\beta_i+1}\mathcal{M}) \quad \text{if } \beta_i > -1.$$

By taking inductive limit on $\boldsymbol{\gamma} > 0$ (i.e., $\boldsymbol{\gamma} = \boldsymbol{\gamma}'$) in (a), we obtain

$$F_p\mathcal{M}\cap V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M} = G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M})$$

for every $\boldsymbol{\beta}$, and taking $\beta_k \gg 0$ for $k \neq i$ gives

$$F_p\mathcal{M}\cap V^{(i)}_{\beta_i}\mathcal{M} = \sum_{\substack{\alpha_i\leqslant 0,\,\boldsymbol{j}\geqslant 0\\ \alpha_i+j_i\leqslant\beta_i}} F_{p-|\boldsymbol{j}|}V^{(i)}_{\alpha_i}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}},$$

and thus, if $\beta_i > -1$,

$$F_{p+1}\mathcal{M}\cap V^{(i)}_{\beta_i+1}\mathcal{M} = (F_p\mathcal{M}\cap V^{(i)}_{\beta_i}\mathcal{M})\cdot\partial_{x_i} + \sum_{\substack{\alpha_i\leqslant 0,\,\boldsymbol{j}\geqslant 0\\ j_i=0}} F_{p+1-|\boldsymbol{j}|}V^{(i)}_{\alpha_i}\mathcal{M}\cdot\partial_x^{\boldsymbol{j}},$$

hence the desired strict $\mathbb{R}$-specializability, since $V^{(i)}_{\alpha_i}\mathcal{M}\subset V^{(i)}_{<\beta_i+1}\mathcal{M}$. The other assertions in 13.4.15 are also obtained by taking the inductive limit on $\boldsymbol{\gamma}$. We also note that (a) and (b) for $\boldsymbol{\gamma}$ imply (c) for $\boldsymbol{\gamma}$, according to Example 10.3.20. Conversely, (c) for $\boldsymbol{\gamma}$ implies (a) for $\boldsymbol{\gamma}$.

Let us first exemplify the proof of (a) and (b) in the case $n = 1$. Condition (b) is empty. For (a), we can assume $\gamma > 0$, and it is enough, by an easy induction on $\gamma - \beta$, to prove $G_p(V^{(1)}_\gamma\mathcal{M})\cap V^{(1)}_{<\gamma}\mathcal{M} = G_p(V^{(1)}_{<\gamma}\mathcal{M})$. For that purpose, we notice that

$$G_p(V^{(1)}_\gamma\mathcal{M}) = G_p(V^{(1)}_{<\gamma}\mathcal{M}) + F_{p-j}V^{(1)}_\alpha\mathcal{M}\cdot\partial_{x_1}^j,$$

where j is such that $\gamma - j \in (-1, 0]$ and $\alpha := \gamma - j$. Hence

$$G_p(V^{(1)}_\gamma\mathcal{M})\cap V^{(1)}_{<\gamma}\mathcal{M} = G_p(V^{(1)}_{<\gamma}\mathcal{M}) + \big(F_{p-j}V^{(1)}_\alpha\mathcal{M}\cdot\partial_{x_1}^j \cap V^{(1)}_{<\gamma}\mathcal{M}\big).$$

Now, by the strictly $\mathbb{R}$-specializable property, $F_{p-j}V_\alpha^{(1)}\mathcal{M}\cdot\partial_{x_1}^j\cap V_{<\gamma}^{(1)}\mathcal{M}=F_{p-j}V_{<\alpha}^{(1)}\mathcal{M}\cdot\partial_{x_1}^j$, so we obtain (a) in this case.

We will prove (a)–(c) by induction on the lexicographically ordered pair $(n,|\boldsymbol{\gamma}'|)$. The case $n=1$ is treated above, so we can assume $n\geqslant 2$. Moreover, if $|\boldsymbol{\gamma}'|=0$, i.e., if $\boldsymbol{\gamma}\leqslant 0$, there is nothing to prove. Assume that $\gamma_1>0$ and let $\alpha_1\in(-1,0]$ be such that $j_1:=\gamma_1-\alpha_1$ is an integer. We also set $\boldsymbol{\gamma}=(\gamma_1,\boldsymbol{\gamma}'')$ and $n''=n-1$.

In order to prove (a), we can argue by decreasing induction on $\boldsymbol{\beta}$, and we are reduced to the case where $\boldsymbol{\beta}$ is the predecessor in one direction, say k, of $\boldsymbol{\gamma}$, that is, $\beta_i=\gamma_i$ for $i\neq k$ and β_k is the predecessor of γ_k. Assume first that $\gamma_k>0$, so we can also assume $k=1$. We then have

$$G_p(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M})=G_{p-1}(V_{\boldsymbol{\gamma}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M})\cdot\partial_{x_1}+G_p(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}),$$

and we are reduced to proving

$$G_{p-1}(V_{\boldsymbol{\gamma}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M})\cdot\partial_{x_1}\cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}\subset G_p(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}).$$

Since $\gamma_1>0$ and $\mathcal{M}$ is of normal crossing type, we have an isomorphism

$$\partial_{x_1}:V_{\boldsymbol{\gamma}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M}/V_{\boldsymbol{\beta}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M}=V_{\widetilde{\boldsymbol{\gamma}}}^{(\widetilde{\boldsymbol{n}})}\operatorname{gr}_{\gamma_1-1}^{V^{(1)}}\mathcal{M}\xrightarrow{\sim}V_{\widetilde{\boldsymbol{\gamma}}}^{(\widetilde{\boldsymbol{n}})}\operatorname{gr}_{\gamma_1}^{V^{(1)}}\mathcal{M}=V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M}/V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}$$

which sends surjectively, hence bijectively, the image of $G_{p-1}(V_{\boldsymbol{\gamma}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M})$ to that of $G_p(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M})$. It follows that

$$G_{p-1}(V_{\boldsymbol{\gamma}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M})\cdot\partial_{x_1}\cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}=\big[G_{p-1}(V_{\boldsymbol{\gamma}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M})\cap V_{\boldsymbol{\beta}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M}\big]\cdot\partial_{x_1}.$$

By the inductive assumption on $\boldsymbol{\gamma}$, the latter term is contained in $G_{p-1}(V_{\boldsymbol{\beta}-\mathbf{1}_1}^{(\boldsymbol{n})}\mathcal{M})\cdot\partial_{x_1}$, hence in $G_p(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M})$.

For induction purpose, let us consider $\mathcal{M}^{(1)}:=\operatorname{gr}_{\alpha_1}^{V^{(1)}}\mathcal{M}$, which is a $\mathcal{D}_{D_1}$-module of normal crossing type for which the $V^{(i)}$-filtrations $(i=2,\dots,n)$ are those naturally induced by the $V^{(i)}$-filtrations on $\mathcal{M}$. We set $\mathcal{M}_0^{(1)}=V_0^{(\boldsymbol{n}'')}\mathcal{M}^{(1)}$, that we equip with the naturally induced filtration $F_\bullet\mathcal{M}_0^{(1)}$. By Remark 10.3.10(1), the family $(F_\bullet\mathcal{M}_0^{(1)},(V_\bullet^{(i)}\mathcal{M}_0^{(1)})_{2\leqslant i\leqslant n})$ is compatible. The inductive assumption on n implies that(a)–(c) hold for $V_{\boldsymbol{\gamma}''}^{(\boldsymbol{n}'')}\mathcal{M}^{(1)}$. Note that $V_{\boldsymbol{\gamma}''}^{(\boldsymbol{n}'')}\mathcal{M}^{(1)}=V_{(\alpha_1,\boldsymbol{\gamma}'')}^{(\boldsymbol{n})}\mathcal{M}/V_{(<\alpha_1,\boldsymbol{\gamma}'')}^{(\boldsymbol{n})}\mathcal{M}$.

We now claim that $G_p(V_{\boldsymbol{\gamma}''}^{(\boldsymbol{n}'')}\mathcal{M}^{(1)})$ is the filtration induced by $G_p(V_{(\alpha_1,\boldsymbol{\gamma}'')}^{(\boldsymbol{n})}\mathcal{M})$. Indeed, this follows from the expression (13.4.17), which does not involve the variable x_1. Now, $\partial_{x_1}^{j_1}$ induces an isomorphism $\mathcal{M}^{(1)}\xrightarrow{\sim}\operatorname{gr}_{\gamma_1}^{V^{(1)}}\mathcal{M}$, which is strictly compatible with the filtrations induced by $V_\bullet^{(i)}\mathcal{M}$. On the other hand, the filtration induced by $G_p(V_{(\alpha_1,\boldsymbol{\gamma}')}^{(\boldsymbol{n})}\mathcal{M})$ is sent surjectively (hence bijectively) onto that induced by $G_p(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M})$. By induction on n, (a)–(c) hold for $\operatorname{gr}_{\gamma_1}^{V^{(1)}}\mathcal{M}$, and the filtrations are those induced by the filtrations on $V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M}$.

Let us now assume that $\gamma_k\leqslant 0$. To prove $G_p(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M})\cap V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}=G_p(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M})$ for all p, it is enough to prove $G_p(V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M})\cap G_{p+1}(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M})=G_p(V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M})$ for all p, and (replacing p with $p-1$), this amounts to proving for all p the injectivity of

$$\operatorname{gr}_p^G V_{\boldsymbol{\beta}}^{(\boldsymbol{n})}\mathcal{M}\longrightarrow\operatorname{gr}_p^G V_{\boldsymbol{\gamma}}^{(\boldsymbol{n})}\mathcal{M}.$$

Set $\boldsymbol{\gamma} = (\gamma_1, \boldsymbol{\gamma}'')$, $\widetilde{\boldsymbol{\gamma}} = ({<}\gamma_1, \boldsymbol{\gamma}'')$, $\boldsymbol{\beta} = (\gamma_1, \boldsymbol{\beta}'')$ and $\widetilde{\boldsymbol{\beta}} = ({<}\gamma_1, \boldsymbol{\beta}'')$, with $\boldsymbol{\beta}'' = (\gamma_2, \dots, {<}\gamma_k, \dots, \gamma_n)$. By the inductive assumption on n and $\boldsymbol{\gamma}'$, we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{gr}_p^G V^{(\boldsymbol{n})}_{\widetilde{\boldsymbol{\beta}}}\mathcal{M} \arrow[r] \arrow[d, hook] & \mathrm{gr}_p^G V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M} \arrow[r] \arrow[d] & \mathrm{gr}_p^G V^{(\boldsymbol{n}'')}_{\boldsymbol{\beta}''}\mathrm{gr}^{(1)}_{\gamma_1}\mathcal{M} \arrow[r] \arrow[d, hook] & 0 \\
0 \arrow[r] & \mathrm{gr}_p^G V^{(\boldsymbol{n})}_{\widetilde{\boldsymbol{\gamma}}}\mathcal{M} \arrow[r] & \mathrm{gr}_p^G V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M} \arrow[r] & \mathrm{gr}_p^G V^{(\boldsymbol{n}'')}_{\boldsymbol{\gamma}''}\mathrm{gr}^{(1)}_{\gamma_1}\mathcal{M} \arrow[r] & 0
\end{tikzcd}$$

where the horizontal sequences are exact (by the first part of the proof of (a)) and both extreme vertical arrows are injective (because $|\widetilde{\boldsymbol{\gamma}}'| < |\boldsymbol{\gamma}'|$ for the left one, and $n'' < n$ for the right one). We conclude that the middle vertical arrow is injective, which finishes the proof of (a).

Let us now prove (b) and let us come back to the case where β_1 is the predecessor of $\gamma_1 > 0$. We have seen that $G_p(V^{(\boldsymbol{n}'')}_{\boldsymbol{\gamma}''}\mathrm{gr}^{(1)}_{\gamma_1}\mathcal{M})$ is the filtration induced by $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M})$, so the previous injective morphism can be completed for all p into the exact sequence

$$0 \longrightarrow \mathrm{gr}_p^G V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M} \longrightarrow \mathrm{gr}_p^G V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M} \longrightarrow \mathrm{gr}_p^G V^{(\boldsymbol{n}'')}_{\boldsymbol{\gamma}''}\mathrm{gr}^{(1)}_{\gamma_1}\mathcal{M} \longrightarrow 0,$$

hence, since $G_\bullet$ is bounded below,

$$0 \longrightarrow G_p V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M} \longrightarrow G_p V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M} \longrightarrow G_p V^{(\boldsymbol{n}'')}_{\boldsymbol{\gamma}''}\mathrm{gr}^{(1)}_{\gamma_1}\mathcal{M} \longrightarrow 0.$$

The inductive assumption implies that (b) holds for $V^{(\boldsymbol{n})}_{\boldsymbol{\beta}}\mathcal{M}$ and for $V^{(\boldsymbol{n}'')}_{\boldsymbol{\gamma}''}\mathrm{gr}^{(1)}_{\gamma_1}\mathcal{M}$. We can now apply Exercise 10.11(3a) to conclude that (b) holds for $V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M}$. □

***13.4.18. Remark* (The case of a middle extension along $D_{i\in I}$)**

Assume moreover that, in Proposition 13.4.15, $\mathcal{M}$ is a middle extension along each D_i ($i \in I$). Then we can replace everywhere $\mathcal{M}_0$ with $\mathcal{M}_{<0} := \bigcap_{i\in I} V^{(i)}_{<0}\mathcal{M}$ and we can moreover conclude that $(\mathcal{M}, F_\bullet\mathcal{M})$ is a middle extension along $D_{i\in I}$ (Definition 13.4.9). In the proof, we modify the definition of $G_p(V^{(\boldsymbol{n})}_{\boldsymbol{\gamma}}\mathcal{M})$ for $\boldsymbol{\gamma} \geqslant -1$, by summing over $\boldsymbol{\alpha} \in [-1, 0)^n$.

A useful example. Let $\mathcal{M}$ be a $\mathcal{D}$-module of normal crossing type which is a middle extension along each D_i ($i \in I$) and let us consider the locally free $\mathcal{O}$-module $\mathcal{M}_{<0} = V^{(\boldsymbol{n})}_{<\boldsymbol{0}}\mathcal{M}$, equipped with the induced filtrations $V^{(i)}_\bullet\mathcal{M}_{<0}$ (which are thus compatible). For $\boldsymbol{\alpha} < \boldsymbol{0}$, we have

$$V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M}_{<0} := \bigcap_i V^{(i)}_{\alpha_i}\mathcal{M}_{<0} = V^{(\boldsymbol{n})}_{\boldsymbol{\alpha}}\mathcal{M} \cap \mathcal{M}_{<0}.$$

Let $F_\bullet\mathcal{M}_{|X\smallsetminus D}$ be a coherent (finite) $\mathcal{D}$-filtration such that each $\mathrm{gr}^F_p\mathcal{M}_{|X\smallsetminus D}$ is $\mathcal{O}$-locally free and let us set

$$F_\bullet\mathcal{M}_{<0} = j_* F_\bullet\mathcal{M}_{|X\smallsetminus D} \cap \mathcal{M}_{<0}$$

and

$$F_p\mathcal{M} = \sum_{q\geqslant 0} F_{p-q}\mathcal{M}_{<0} \cdot F_q\mathcal{D}_X.$$

13.4.19. Proposition. *With these assumptions, let us moreover assume that each* $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}_{<0}$ *is* $\mathcal{O}$*-locally free. Then the filtered* $\mathcal{D}$*-module* $(\mathcal{M}, F_\bullet\mathcal{M})$ *is of normal crossing type and a middle extension along* $D_{i\in I}$*.*

Proof. We consider the filtrations $F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)}$ on $\mathcal{M}_{<0}$. Except possibly compatibility, they satisfy the assumptions of Proposition 13.4.15 in the setting of Remark 13.4.18. It remains to prove that they are compatible. We will use the criterion in term of flatness of Theorem 10.3.11, and more precisely the criterion in terms of regular sequences of Lemma 10.3.13 together with the criteria of Exercise 10.10.

We consider the multi-Rees module $R_{FV}\mathcal{M}_{<0}$, which is a $\mathbb{C}[z_0, z_1, \dots, z_n]$-module. Exercise 10.10(2b) shows that it is flat if any subsequence of $z_0, z_1, \dots, z_n$ is regular.

• If the subsequence does not contain z_o, then we apply Proposition 13.4.11 with $\mathcal{E} = F_p\mathcal{M}_{<0}$ for each p. The assumption of freeness of each $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}_{<0}$ implies that of $V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}F_p\mathcal{M}_{<0}$, so 13.4.11(1) is satisfied. 13.4.11(2) is also satisfied according to the definition of $F_p\mathcal{M}_{<0}$.

• If the subsequence contains z_o, we are considering flatness for $R_V\mathrm{gr}^F\mathcal{M}_{<0}$. We apply Proposition 13.4.11 once more, now with $\mathcal{E} = \mathrm{gr}_p^F\mathcal{M}_{<0}$ for each p, and freeness of each $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{n})}\mathcal{M}_{<0}$ implies that 13.4.11(1) is satisfied. Similarly, 13.4.11(2) is also satisfied according to the definition of $F_p\mathcal{M}_{<0}$. $\square$

13.4.d. Nearby cycles for coherently F-filtered $\mathcal{D}$-modules of normal crossing type

We now take up the proof of Theorem 13.1.4 in the filtered case, and we set $(\mathcal{N}, F_\bullet\mathcal{N}) = {}_{\mathrm{D}}\iota_{g*}(\mathcal{M}, F_\bullet\mathcal{M})$, or equivalently $\mathcal{N} = {}_{\mathrm{D}}\iota_{g*}\mathcal{M}$.

Step 1***: improvement of*** (13.2.18 $*$)**.** We first aim at improving Formula (13.2.18 $*$) (extended to $\mathcal{N} := \mathcal{O}_X \otimes_{\mathbb{C}} \mathbb{C}[x]$). As usual, we set $F_pV_\gamma := F_p \cap V_\gamma$.

13.4.20. Lemma. *For* $\gamma < 0$ *and any* $p \in \mathbb{Z}$*, we have*

$$F_pV_\gamma\mathcal{N} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} (F_{p-|\boldsymbol{k}|}V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\otimes 1)\cdot\partial_x^{\boldsymbol{k}}.$$

***Proof of Lemma* 13.4.20.** We first simplify the right-hand side by only taking into account indices $i \in I_g$, i.e., for which $a_i \neq 0$, as in Remark 13.2.23, from which we keep the notation. We set $\mathbb{N}^n = \mathbb{N}^{n'} \times \mathbb{N}^{n''}$ with $n' = \#I_g$ and $n'' = n - n'$. We claim that

$$(13.4.21)\quad \sum_{\boldsymbol{k}\in\mathbb{N}^n} (F_{p-|\boldsymbol{k}|}V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\otimes 1)\cdot\partial_x^{\boldsymbol{k}} = F'_pV_\gamma\mathcal{N} := \sum_{\boldsymbol{k}'\in\mathbb{N}^{n'}} (F_{p-|\boldsymbol{k}'|}V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M}\otimes 1)\cdot\partial_x^{\boldsymbol{k}'}.$$

By the second line in (13.4.6), arguing as for the proof of Proposition 13.4.12(2), we have

$$F_q\mathcal{M} = \sum_{\boldsymbol{k}''\in\mathbb{N}^{n''}} F_{q-|\boldsymbol{k}''|}V_{\boldsymbol{k}''}^{(\boldsymbol{n}'')}\mathcal{M}\cdot\partial_x^{\boldsymbol{k}''}.$$

Therefore, summing first on $\boldsymbol{k}''$ in the right-hand side of Lemma 13.4.20, and using that $(m\otimes 1)\partial_{x_i} = m\partial_{x_i}\otimes 1$ for $i \notin I_g$, we get the desired assertion.

The assertion of the lemma amounts thus to

$$F_p\mathcal{N}\cap V_\gamma\mathcal{N}=F'_pV_\gamma\mathcal{N}\quad(\gamma<0,\,p\in\mathbb{Z}),$$

and an easy computation shows that it is equivalent to the injectivity of

$$\operatorname{gr}^{F'}V_\gamma\mathcal{N}\longrightarrow\operatorname{gr}^F\mathcal{N}.\tag{13.4.22}$$

In a way similar to what is done in Remark 13.2.23, we set

$$\mathcal{K}_\gamma=V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}\langle\partial_{x'},\mathrm{E}\rangle,$$

equipped with an action of $V_0\mathcal{D}_{X\times\mathbb{C}}$. For $i\in I_g$, let us set

$$\delta_i=\operatorname{Id}x_i\partial_{x_i}\otimes1-(\operatorname{Id}\otimes1)x_i\partial_{x_i}-(\operatorname{Id}\otimes1)a_i\,\mathrm{E}\in\operatorname{End}(\mathcal{K}_\gamma),$$

and let us consider the Koszul complex $\mathcal{K}^\bullet_\gamma:=(\mathcal{K}_\gamma,(\delta_i)_{\in I_g})$. We have a natural morphism

$$\mathcal{K}_\gamma\longrightarrow\mathcal{N}=\mathcal{M}[\partial_t]$$

sending $m\otimes\partial_{x'}^{\boldsymbol{k}'}$ to $(m\otimes1)\partial_{x'}^{\boldsymbol{k}'}$ and $m\otimes a_i\,\mathrm{E}$ to $mx^{\boldsymbol{a}}\otimes\partial_t$. By Remark 13.2.23 and since δ_i vanishes on $\mathcal{N}$ ($i\in I_g$), the previous morphism factorizes through surjective morphisms ($\gamma<0$)

$$\mathcal{K}_\gamma\longrightarrow H^{n'}(\mathcal{K}^\bullet_\gamma,(\delta_i)_{\in I_g})\longrightarrow V_\gamma\mathcal{N}.\tag{13.4.23}$$

Let us filter $\mathcal{K}_\gamma$ by

$$F_p\mathcal{K}_\gamma:=\sum_j F_{p-j}V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes_{\mathbb{C}}F_j\mathbb{C}\langle\partial_{x'},\mathrm{E}\rangle,$$

where $F_j\mathbb{C}\langle\partial_{x'},\mathrm{E}\rangle$ is the filtration by the degree in $(\partial_{x'},\mathrm{E})$. We will prove the following two assertions which immediately imply the injectivity of (13.4.22):

(a) The natural morphism $\operatorname{gr}^F\mathcal{K}_\gamma\to\operatorname{gr}^{F'}V_\gamma\mathcal{N}$ is onto.

(b) The natural morphism $H^{n'}\operatorname{gr}^F\mathcal{K}^\bullet_\gamma\to\operatorname{gr}^F\mathcal{N}$ is injective.

Proof of (a). By (13.4.23) $F_p\mathcal{K}_\gamma$ surjects onto $F'_pV_\gamma\mathcal{N}$ (defined by (13.4.21)): this is already true if we start from the submodule $\sum_j F_{p-j}V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes_{\mathbb{C}}F_j\mathbb{C}\langle\partial_{x'}\rangle$ of $F_p\mathcal{K}_\gamma$ (by forgetting E), so it suffices to notice that $F_{p-1}V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes_{\mathcal{O}_X}\mathrm{E}$ is sent to $F_pV^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes1+(F_{p-1}V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes1)\partial_{x_i}$, which follows from Formula (13.2.13).

Proof of (b). In order to manipulate the filtration $F_\bullet\mathcal{K}_\gamma$ and its graded objects, it is convenient to introduce the auxiliary filtration

$$G_q\mathcal{K}_\gamma:=V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}\otimes_{\mathbb{C}}F_q\mathbb{C}\langle\partial_{x'},\mathrm{E}\rangle,$$

and correspondingly,

$$G_p\mathcal{N}=\bigoplus_{j\leqslant p}\mathcal{M}\otimes\partial_t^j$$

which induces in a natural way a filtration on $\operatorname{gr}^F\mathcal{N}$, so that it is sufficient to prove the injectivity of

$$\operatorname{gr}^GH^{n'}\operatorname{gr}^F\mathcal{K}^\bullet_\gamma\longrightarrow\operatorname{gr}^G\operatorname{gr}^F\mathcal{N}.$$

We will prove

(c) The complex $\mathrm{gr}^G\mathrm{gr}^F\mathcal{K}_\gamma^\bullet$ has nonzero cohomology in degree n' at most.

From (c) one deduces that $H^{n'}G_{j-1}\mathrm{gr}^F\mathcal{K}_\gamma^\bullet \to H^{n'}G_j\mathrm{gr}^F\mathcal{K}_\gamma^\bullet$ is injective for every j, and therefore

$$\mathrm{gr}^G H^{n'}\mathrm{gr}^F\mathcal{K}_\gamma^\bullet = H^{n'}\mathrm{gr}^G\mathrm{gr}^F\mathcal{K}_\gamma^\bullet = H^{n'}\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\gamma^\bullet,$$

so it is enough to prove the injectivity of

$$H^{n'}\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\gamma^\bullet \longrightarrow \mathrm{gr}^F\mathrm{gr}^G\mathcal{N}. \tag{13.4.24}$$

On the one hand, we have

$$F_p\mathrm{gr}_q^G\mathcal{K}_\gamma = F_{p-q}V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\xi',\mathrm{E}]_q,$$

where $\mathbb{C}[\xi',\mathrm{E}]_q$ consists of polynomials of degree q in $\xi' = (\xi_i)_{i\in I_g}$ (class of ∂_{x_i}) and E (still denoting the class of E), and thus[(1)]

$$\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\gamma = \mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M}\otimes_{\mathbb{C}}\mathbb{C}[\xi',\mathrm{E}].$$

The bi-graded endomorphism corresponding to δ_i reads $-\mathrm{Id}\otimes(x_i\xi_i + a_i\,\mathrm{E})$. On the other hand, $\mathrm{gr}^G\mathcal{N} = \mathcal{M}[\tau]$, where τ is the class of ∂_t, and $\mathrm{gr}^F\mathrm{gr}^G\mathcal{N} = (\mathrm{gr}^F\mathcal{M})[\tau]$. The morphism $\mathrm{gr}^F\mathrm{gr}^G\mathcal{K}_\gamma \to \mathrm{gr}^F\mathrm{gr}^G\mathcal{N}$ is the morphism

$$(\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})[\xi',\mathrm{E}] \longrightarrow (\mathrm{gr}^F\mathcal{M})[\tau]$$

induced by the natural morphism $\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M} \to \mathrm{gr}^F\mathcal{M}$ and sending ξ_i to $\partial g/\partial x_i\cdot\tau$ and E to $g\cdot\tau$. It factorizes through the inclusion $(\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})[\tau] \to (\mathrm{gr}^F\mathcal{M})[\tau]$. Let us also recall that the localization morphism $\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M} \to (\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})(g^{-1})$ is injective (first line of (13.4.6)).

13.4.25. Assertion. *The sequence* $(x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g}$ *is a regular sequence on*

$$\Big((\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})(g^{-1})/(\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})\Big)[\xi',\mathrm{E}].$$

It is easy to check that $(x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g}$ is a regular sequence on the localized module $(\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})(g^{-1})[\xi',\mathrm{E}]$, since one is reduced to consider the sequence $(\xi_i + a_i\,\mathrm{E}/x_i)_{i\in I_g}$. The assertion implies that $(x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g}$ is also a regular sequence on $\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M}[\xi',\mathrm{E}]$, which in turn implies (c) above. Let us check that it also implies the injectivity of (13.4.24). We wish to prove the injectivity of

$$\begin{aligned}(\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})[\xi',\mathrm{E}]/(x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g} &\longrightarrow (\mathrm{gr}^F V_{\gamma\boldsymbol{a}'}^{(\boldsymbol{n}')}\mathcal{M})[\tau]\\ \xi_i \longmapsto \partial g/\partial x_i\cdot\tau,&\quad \mathrm{E}\longmapsto g\cdot\tau.\end{aligned}\tag{13.4.26}$$

It is easy to see that its localization by g is an isomorphism. It is therefore enough to prove that the localization morphism for the left-hand side of (13.4.26) is injective. This in turn follows from the assertion.

[(1)]In the following, we do not make precise the bi-grading of the objects and how the isomorphisms are bi-graded, as it is straightforward.

In order to end the proof of Lemma 13.4.20, we are left with proving the assertion. Since

$$g^k : (\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})g^{-k}/(\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})g^{-k+1} \longrightarrow (\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})/(\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})g, \quad k \geqslant 0$$

is an isomorphism, an easy induction reduces to proving that $(x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g}$ is a regular sequence on $\big((\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})/(\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})g\big)[\xi', \mathrm{E}]$. It is therefore enough to prove that $\big((x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g}, g\big)$ is a regular sequence on $(\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M})[\xi', \mathrm{E}]$.

Let us set $X = X' \times X''$, where X'' has coordinates x_i with $i \in I'_g := I \smallsetminus I_g$ and has dimension n''. First, $\big((x_i\xi_i + a_i\,\mathrm{E})_{i\in I_g}, g\big)$ is a regular sequence on $\mathcal{O}_{X'}[\xi', \mathrm{E}]$, since one computes easily that the zero set of the corresponding ideal has codimension $n' + 1$ in $X' \times \mathbb{A}^{n'+1}$. It is thus enough to prove that $\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}$ is $\mathcal{O}_{X'}$-flat.

Let us recall that, for $\boldsymbol{\beta}'' \in \mathbb{R}^{n''}$ such that $\beta_i < 0$ for all $i \in I_g^{\mathrm{c}}$, $\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'} V^{(\boldsymbol{n}'')}_{\boldsymbol{\beta}''}\mathcal{M}$ is $\mathcal{O}_X$-locally free (see Remark 13.4.2(d)), hence $\mathcal{O}_{X'}$-flat. By using the $\mathcal{O}_{X'}$-linear isomorphisms ∂_{x_i} $(i \in I_g^{\mathrm{c}})$ as in the second line of (13.4.6), one finds inductively (by using the compatibility 13.4.1(3)) that $\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'} V^{(\boldsymbol{n}'')}_{\boldsymbol{\beta}''}\mathcal{M}$ is $\mathcal{O}_{X'}$-flat for any $\boldsymbol{\beta}''$. Taking the inductive limit for $\boldsymbol{\beta}'' \to \infty$, one obtains the $\mathcal{O}_{X'}$-flatness of $\mathrm{gr}^F V^{(\boldsymbol{n}')}_{\gamma\boldsymbol{a}'}\mathcal{M}$. This ends the proof of Lemma 13.4.20. $\square$

Step* 2*: strict* $\mathbb{R}$*-specializability properties for* $\gamma < 0$*. As in Lemma 13.2.19, we deduce from Lemma 13.4.20 that $t : F_pV_\gamma\mathcal{N} \xrightarrow{\sim} F_pV_{\gamma-1}\mathcal{N}$ for $\gamma < 0$ and any p, as required by Proposition 9.3.20(a).

Step* 3*: strict* $\mathbb{R}$*-specializability and middle extension properties for* $\gamma \geqslant 0$*. We aim at proving that, for $\gamma \geqslant 0$ and any $p \in \mathbb{Z}$,

$$F_pV_\gamma\mathcal{N} := F_p\mathcal{N} \cap V_\gamma\mathcal{N} = (F_p\mathcal{N} \cap V_{<\gamma}\mathcal{N}) + (F_{p-1}V_{\gamma-1}\mathcal{N}) \cdot \partial_t.$$

By definition, $F_p\mathcal{N} = \bigoplus_{k\geqslant 0} F_{p-k}\mathcal{M} \otimes \partial_t^k$. On the other hand,

$$F_p\mathcal{M} = \sum_{\boldsymbol{k}\in\mathbb{N}^n} F_{p-|\boldsymbol{k}|} V^{(\boldsymbol{n})}_{<0}\mathcal{M} \cdot \partial_x^{\boldsymbol{k}},$$

according to Proposition 13.4.12(2) and Remark 13.4.14. Then, if $m = \sum_{k\geqslant 0} m_k \otimes \partial_t^k$ belongs to $F_p\mathcal{N} \cap V_\gamma\mathcal{N}$, and if we set $m_0 = \sum_{\boldsymbol{k}} m_{0,\boldsymbol{k}}\partial_x^{\boldsymbol{k}}$ with $m_{0,\boldsymbol{k}} \in F_{p-|\boldsymbol{k}|}V^{(\boldsymbol{n})}_{<0}\mathcal{M}$, the first line of (13.2.12) shows that

$$m = m' + \sum_{\boldsymbol{k}} (m_{0,\boldsymbol{k}} \otimes 1)\partial_x^{\boldsymbol{k}}, \qquad \begin{cases} m' \in F_p\mathcal{N} \cap V_\gamma\mathcal{N} \cap \mathcal{N} \cdot \partial_t, \\ \sum_{\boldsymbol{k}} (m_{0,\boldsymbol{k}} \otimes 1)\partial_x^{\boldsymbol{k}} \in F_pV_{<0}\mathcal{N}. \end{cases}$$

Now, by definition, $F_p\mathcal{N} \cap \mathcal{N} \cdot \partial_t = F_{p-1}\mathcal{N} \cdot \partial_t$. Moreover, since $\partial_t : \mathrm{gr}_\beta^V\mathcal{N} \to \mathrm{gr}_{\beta+1}\mathcal{N}$ is injective for $\beta \neq -1$, we deduce easily that, for $\gamma \geqslant 0$, $V_\gamma\mathcal{N} \cap \mathcal{N} \cdot \partial_t = V_{\gamma-1}\mathcal{N} \cdot \partial_t$. In conclusion,

$$F_p\mathcal{N} \cap V_\gamma\mathcal{N} \cap \mathcal{N} \cdot \partial_t = (F_{p-1}\mathcal{N} \cdot \partial_t) \cap (V_{\gamma-1}\mathcal{N} \cdot \partial_t) = (F_{p-1}\mathcal{N} \cap V_{\gamma-1}\mathcal{N}) \cdot \partial_t,$$

where the latter equality follows from the injectivity of ∂_t on $\mathcal{N}$, and so

$$F_pV_\gamma\mathcal{N} \subset (F_p\mathcal{N} \cap V_{<0}\mathcal{N}) + (F_{p-1}V_{\gamma-1}\mathcal{N}) \cdot \partial_t,$$

as wanted.

Step* 4*: normal crossing type properties. Let us fix $\gamma \in [-1,0)$ and take up the notation ${}^\gamma\mathcal{N} = \mathrm{gr}^V_\gamma\mathcal{N}$ like in Example 13.2.17. By Theorem 13.1.4 without filtration, we know that ${}^\gamma\mathcal{N}$ is a coherent $\mathcal{D}_X$-module of normal crossing type along D. We wish to show that this result also holds with filtration, namely that $(\mathrm{gr}^V_\gamma\mathcal{N}, F_\bullet\mathrm{gr}^V_\gamma\mathcal{N})$ is a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type along D (Definition 13.4.1).

The formula given in Lemma 13.4.20 implies that, setting ${}^\gamma\mathcal{N}_{\leqslant 0} := V_0^{(\boldsymbol{n})}{}^\gamma\mathcal{N}$ as in Proposition 13.4.12,

$$F_p{}^\gamma\mathcal{N} = \sum_{q\geqslant 0}(F_{p-q}{}^\gamma\mathcal{N}_{\leqslant 0})\cdot F_q\mathcal{D}_X,$$

so Proposition 13.4.15 reduces to proving the following properties:

(a) $({}^\gamma\mathcal{N}_{\leqslant 0}, F_\bullet{}^\gamma\mathcal{N}_{\leqslant 0})$ is strictly $\mathbb{R}$-specializable along every component D_i of D (as defined in the proposition),

(b) the filtrations $(F_\bullet{}^\gamma\mathcal{N}_{\leqslant 0}, V_\bullet^{(1)}{}^\gamma\mathcal{N}_{\leqslant 0}, \dots, V_\bullet^{(n)}{}^\gamma\mathcal{N}_{\leqslant 0})$ are compatible (where we have set $V_\bullet^{(i)}{}^\gamma\mathcal{N}_{\leqslant 0} := V_\bullet^{(i)}{}^\gamma\mathcal{N} \cap {}^\gamma\mathcal{N}_{\leqslant 0}$),

(c) each $\mathrm{gr}^F_p\mathrm{gr}^{V^{(n)}}_{\boldsymbol{\alpha}}{}^\gamma\mathcal{N}_{\leqslant 0}$ ($p\in\mathbb{Z}$, $\boldsymbol{\alpha}\in[-1,0]^n$) is $\mathbb{C}$-locally free.

***Proof of* (b).** We will use the presentation (13.3.16) and it will be easier to define and analyze the filtrations on $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$. In a natural way we set

$$F_p(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]) := \sum_{q\geqslant 0} F_{p-q}V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\otimes \mathrm{E}^q,$$

$$V_{\beta_i}^{(i)}(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]) := (V_{\beta_i+\gamma a_i}^{(i)}\mathcal{M}\cap V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})[\mathrm{E}] \quad (\beta_i \leqslant 0).$$

Claim 1. *The filtrations $F_\bullet{}^\gamma\mathcal{N}_{\leqslant 0}, V_\bullet^{(i)}{}^\gamma\mathcal{N}_{\leqslant 0}$ are respectively the images of the filtrations above by the morphism* (13.3.15) $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] \to {}^\gamma\mathcal{N}_{\leqslant 0}$.

Proof. For the filtrations $V^{(i)}$, this has been seen in (13.2.40). We have seen (and used) that $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] = \sum_{j\geqslant 0} V_{(\gamma-j)\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\otimes\partial_t^j \subset \mathcal{N}$. Therefore,

$$F_p\mathcal{N}\cap V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] = \sum_{j\geqslant 0} F_{p-j}V_{(\gamma-j)\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\otimes\partial_t^j \subset \mathcal{N}.$$

Since $F_{p-j}V_{(\gamma-j)\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} = F_{p-j}V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\cdot x^{j\boldsymbol{a}}$, we conclude that

$$F_p\mathcal{N}\cap V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] = \sum_{j\geqslant 0} F_{p-j}V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\otimes t^j\partial_t^j = F_p(V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]). \qquad \square$$

Claim 2. *The family $(F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ of filtrations of $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$ is compatible.*

Proof. This is true if we replace this family by the family $(G_\bullet, F'_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$, where $G_\bullet$ is the filtration by the degree in E and $F'_\bullet$ is $(F_\bullet V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})[\mathrm{E}]$, due to the compatibility on $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$. Now, $F_\bullet$ being the convolution of $F'_\bullet$ and $G_\bullet$, we can apply Exercise 10.11. $\square$

Let K be the kernel of the surjective $(V_0\mathcal{D}_X)[\mathrm{E}]$-linear morphism $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] \to V_0{}^\gamma\mathcal{N}$.

Claim 3. *The family induced by $(F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ on K is compatible and the inclusion $K \hookrightarrow V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$ is $(n+1)$-strict.*

This claim implies the compatibility of $(F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ on ${}^\gamma\mathcal{N}_{\leqslant 0}$ as in (b) above. We will use the criterion of Lemma 10.3.21.

We first work on the graded objects with respect to the filtration $U_\bullet$ (which takes the role of F^0) and with the induced family $(F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$. Obviously, Conditions (a) and (b) of this lemma are satisfied. We are thus reduced to prove the claim for $\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$ for every k.

Let us fix $k \geqslant 0$. According to (13.3.16), $\mathrm{gr}_k^U K$ is the image of the injective morphism φ_k, that we regard as an $(n+1)$-filtered morphism of degree k with respect to $F_\bullet$. We will apply once more Lemma 10.3.21 to $\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$, where now the filtration F^0 is the filtration $G_\bullet$ of $\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$ by the degree in E. It is obvious that φ_k is G-strict. Moreover, $\mathrm{gr}_q^G \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] = \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}\cdot \mathrm{E}^q$ and $\mathrm{gr}^G\varphi_k$ is simply the multiplication by E^k. Moreover, $V_\bullet^{(i)}(\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]) = V_\bullet^{(i)}(\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})[\mathrm{E}]$, so

$$V_\bullet^{(i)}(\mathrm{gr}_q^G \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]) = V_\bullet^{(i)}(\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) \cdot \mathrm{E}^q\,.$$

On the other hand, the filtration F induced on $\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$ is still equal to the convolution of the filtration $F'_\bullet \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$ induced by $F'_\bullet V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] = (F_\bullet V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M})[\mathrm{E}]$, and the filtration $G_\bullet$. Therefore,

$$F_p \mathrm{gr}_q^G \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] = (F_{p-q}\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}) \cdot \mathrm{E}^q\,.$$

It is then clear that $\mathrm{gr}^G\varphi_k = \cdot\mathrm{E}^k$ is $(n+1)$-strict on every $\mathrm{gr}_q^G \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$. Last, let us check compatibility of the induced family $(F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ on $\mathrm{gr}_q^G \mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}]$. It amounts to that on $\mathrm{gr}_k^U V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$. For that purpose, we remark that the family of filtrations $(U_\bullet, F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ is compatible on $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$. Indeed, the family without $U_\bullet$ is compatible, and the filtration $U_\bullet$ can be expressed as a convolution of filtrations whose terms are terms of the $V^{(i)}$-filtrations, by (13.3.18). Exercise 10.11 applies then as in Claim 2. As a consequence, we obtain the desired compatibility (see Remark 10.3.10(1), or use the flatness criterion). □

Proof of (a). We know that $x_i : F_p V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M} \to F_p V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}$ has image $F_p V_{\gamma\boldsymbol{a}-\boldsymbol{1}_i}^{(\boldsymbol{n})}\mathcal{M}$. By Claim 1 and the $(n+1)$-strictness of $V_{\gamma\boldsymbol{a}}^{(\boldsymbol{n})}\mathcal{M}[\mathrm{E}] \to {}^\gamma\mathcal{N}_{\leqslant 0}$, the same property holds for $F_p {}^\gamma\mathcal{N}_{\leqslant 0} = F_p V_0^{(\boldsymbol{n})\gamma}\mathcal{N}$. That ∂_{x_i} sends $F_p V_{-1}^{(i)\gamma}\mathcal{N}_{\leqslant 0}$ into $F_{p+1} V_0^{(i)\gamma}\mathcal{N}_{\leqslant 0}$ is clear. □

Proof of (c). By the same argument as in the last part of the proof of (b), the family of filtrations $(U_\bullet, F_\bullet, V_\bullet^{(1)}, \dots, V_\bullet^{(n)})$ is compatible on ${}^\gamma\mathcal{N}_{\leqslant 0}$. As a consequence, grading with respect to $F, V^{(i)}, U$ can be made in any order, and it is enough to prove the $\mathbb{C}$-local freeness of $\mathrm{gr}_p^F \mathrm{gr}_{\boldsymbol{\alpha}}^{V^{(\boldsymbol{n})}} \mathrm{gr}_k^U {}^\gamma\mathcal{N}_{\leqslant 0}$ for every k. This is obtained as in the last part of the proof of (b).

This ends the proof of Theorem 13.1.4. □

13.4.e. A simple example. We take up the simple example of Section 13.2.d and, together with Assumption 13.2.43, we assume that

13.4.27. Assumption. $(\mathcal{M}, F_\bullet\mathcal{M})$ is a coherently F-filtered $\mathcal{D}_X$-module of normal crossing type, such that $F_p\mathcal{M} = M_{\boldsymbol{\alpha}} \cdot F_p\mathcal{D}_X$ for all p. Let us decompose any $\boldsymbol{k} \in \mathbb{Z}^n$ as $\boldsymbol{k} = \boldsymbol{k}^+ - \boldsymbol{k}^-$ with $\boldsymbol{k}^+, \boldsymbol{k}^- \in \mathbb{N}^n$ with disjoint support. Then (and according to Assumption13.2.43), considering $M = M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ with the graded filtration

$$F_{-1}M = 0, \quad F_pM = \bigoplus_{\substack{\boldsymbol{k}\in\mathbb{Z}^n\\ |\boldsymbol{k}^+|\leqslant p}} M_{\boldsymbol{\alpha}+\boldsymbol{k}} \quad (p \geqslant 0),$$

we have $F_p\mathcal{M} = \mathcal{O}_X \otimes_{\mathbb{C}} F_pM$.

13.4.28. Theorem. *Under the previous assumptions, the following properties hold for every $\lambda \in \mathbb{S}^1$:*

(1) *for every $k \geqslant 1$,*

$$\mathrm{N}^k : (\mathrm{gr}^{\mathrm{M}}_k \psi_{g,\lambda}\mathcal{M}, F_\bullet \mathrm{gr}^{\mathrm{M}}_k \psi_{g,\lambda}\mathcal{M}) \xrightarrow{\ \sim\ } (\mathrm{gr}^{\mathrm{M}}_{-k} \psi_{g,\lambda}\mathcal{M}, F_\bullet \mathrm{gr}^{\mathrm{M}}_{-k} \psi_{g,\lambda}\mathcal{M})(-k)$$

is a strict isomorphism,

(2) *For every $k \geqslant 0$, the morphism*

$$\mathrm{gr}^{\mathrm{M}} \mathrm{can}_t : (\mathrm{P}_{k+1}\psi_{g,1}\mathcal{M}, F_\bullet \mathrm{P}_{k+1}\psi_{g,1}\mathcal{M}) \longrightarrow (\mathrm{P}_k\phi_{g,1}\mathcal{M}, F_\bullet \mathrm{P}_k\phi_{g,1}\mathcal{M})(-1)$$

is an isomorphism.

13.4.29. Remark. Using the formalism of $\widetilde{\mathcal{D}}_X$-modules as in Chapter 9, we set $\widetilde{\mathcal{M}} = R_F\mathcal{M}$. This $\widetilde{\mathcal{D}}_X$-module is strictly specializable along (g) and is a middle extension along g, as follows from the results of Section 13.4.d. Then the first statement that N^k is a strict isomorphism is equivalent, according to Proposition 9.4.10, to the property that every $\mathrm{gr}^{\mathrm{M}}_k \mathrm{gr}^V_\gamma \widetilde{\mathcal{M}}$ is strict ($k \in \mathbb{Z}$), equivalently so is every primitive part $\mathrm{P}_k \mathrm{gr}^V_\gamma \widetilde{\mathcal{M}}$.

Proof. It is not difficult to check that the description (i)–(v) of Section 13.2.c extends with the filtration F to a description of $\mathrm{P}\mathrm{gr}^V_\gamma \mathcal{N}$, since this amounts to taking into account the degree in N only. The first point of the theorem follows.

For the second point, we have seen that, using the language of $\widetilde{\mathcal{D}}_X$-modules, $\mathcal{N}$ is strictly $\mathbb{R}$-specializable along (t) and is a middle extension as such. The morphism can_t is then isomorphic to $\mathrm{N} : \mathrm{gr}^V_{-1}\mathcal{N} \to \mathrm{Im}\,\mathrm{N}$ and the desired isomorphism follows from Proposition 3.1.11(c). $\square$

13.5. Sesquilinear pairings of normal crossing type

In this section, we take up the setting of Chapter 13. We aim at computing the behaviour of a sesquilinear pairing with respect to the nearby cycle functor with respect to a monomial function, and we will make it even more explicit in the example of a simple coherently F-filtered $\mathcal{D}$-module of normal crossing type.

13.5.a. Basic distributions. The results of §7.3.a in dimension 1 extend in a straightforward way to Δ^n. We will present them in the same context of left $\mathcal{D}$-modules. We continue using the simplifying assumptions 13.1.2.

13.5.1. Proposition. *Fix $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in [-1,\infty)^n$ and $k \in \mathbb{N}$, and suppose a current $u \in \mathfrak{C}(\Delta^n) = \mathfrak{Db}^{n,n}(\Delta^n)$ solves the system of equations*

$$(x_i\partial_{x_i} - \beta'_i)^k u = (\overline{x}_i\partial_{\overline{x}_i} - \beta''_i)^k u = \partial_{x_j} u = \partial_{\overline{x}_j} u = 0 \quad (i \in I,\ j \notin I). \tag{13.5.1 $*$}$$

for an integer $k \geqslant 0$.

(a) *If $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in (-1,\infty)^n$, we have $u = 0$ unless $\boldsymbol{\beta}' - \boldsymbol{\beta}'' \in \mathbb{Z}^n$.*

(b) *If $\boldsymbol{\beta}' = \boldsymbol{\beta}'' = \boldsymbol{\beta}$, then, up to shrinking Δ^n, u is a $\mathbb{C}$-linear combination of the basic distributions*

$$u_{\boldsymbol{\beta},\boldsymbol{p}} = \prod_{\substack{i\in I\\ \beta_i>-1}} |x_i|^{2\beta_i} \frac{\mathrm{L}(x_i)^{p_i}}{p_i!} \prod_{\substack{i\in I\\ \beta_i=-1}} \partial_{x_i}\partial_{\overline{x}_i} \frac{\mathrm{L}(x_i)^{p_i+1}}{(p_i+1)!}, \tag{13.5.1 $**$}$$

where $0 \leqslant p_1, \dots, p_n \leqslant k-1$. These distributions are $\mathbb{C}$-linearly independent.

Proof. Assume first $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in (-1,\infty)^n$. If $\operatorname{Supp} u \subset D$, then $x^{\boldsymbol{m}}u = 0$ for some $\boldsymbol{m} \in \mathbb{N}^n$ and, arguing as in the proof of Proposition 7.3.2, we find $u = 0$.

Otherwise, set $x_i = e^{\xi_i}$ and pullback u as $\widetilde{u}$ on the product of half-planes $\operatorname{Re}\xi_i > 0$. Set $v = e^{-\boldsymbol{\alpha\xi}}e^{-\boldsymbol{\beta}\overline{\boldsymbol{\xi}}}\widetilde{u}$. Then v is annihilated by $(\partial_{\xi_i}\partial_{\overline{\xi}_i})^k$ for every $i = 1,\dots,n$ —therefore by a suitable power of the n-Laplacian $\sum_i \partial_{\xi_i}\partial_{\overline{\xi}_i}$— and a suitable $k \geqslant 1$, and by ∂_{x_j} and $\partial_{\overline{x}_j}$, that we will now forget. By the regularity of the Laplacian, v is C^∞ and, arguing with respect to each variable as in Proposition 7.3.2, we find that v is a polynomial $P(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})$ and thus $\widetilde{u} = e^{\boldsymbol{\alpha\xi}}e^{\boldsymbol{\beta}\overline{\boldsymbol{\xi}}}P(\boldsymbol{\xi},\overline{\boldsymbol{\xi}})$. We now conclude (a), as well as (b) for $\boldsymbol{\beta}', \boldsymbol{\beta}'' \in [-1,\infty)^n$, as in dimension 1.

In the general case, we will argue by induction on $\#\{i\in I \mid \beta_i = -1\}$, assumed to be $\geqslant 1$. Let $I' = \{i\in I \mid \beta_i > -1\}$, $I'' \cup \{i_o\} = \{i\in I \mid \beta_i = -1\}$. Set $\boldsymbol{\beta} = (\boldsymbol{\beta}', -1, -1_{i_o})$, $\widetilde{\boldsymbol{\beta}} = \boldsymbol{\beta} + \mathbf{1}_{i_o} = (\boldsymbol{\beta}', -1, 0_{i_o})$ and let us decompose correspondingly $\boldsymbol{p} \in \mathbb{N}^n$ as $\boldsymbol{p} = (\boldsymbol{p}', \boldsymbol{p}'', p_o)$. By induction we find

$$|x_{i_o}|^2 u = \sum_{\boldsymbol{p}} c_{\boldsymbol{p}',\boldsymbol{p}'',p_o+2} \cdot u_{\widetilde{\boldsymbol{\beta}},\boldsymbol{p}}, \quad c_{\boldsymbol{p}} \in \mathbb{C},$$

for $p_i = 0, \dots k-1$ $(i = 1,\dots,n)$, and this is also written as

$$|x_{i_o}|^2 \partial_{x_{i_o}}\partial_{\overline{x}_{i_o}} \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\widetilde{\boldsymbol{\beta}},\boldsymbol{q}},$$

with $q_i = 0, \dots k-1$ for $i \neq i_o$ and $q_o = 2, \dots, k+1$. Let us set

$$v = u - \partial_{x_{i_o}}\partial_{\overline{x}_{i_o}} \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\widetilde{\boldsymbol{\beta}},\boldsymbol{q}},$$

so that $|x_{i_o}|^2 v = 0$. A computation similar to that in §7.3.a shows that the basic distributions $u_{\widetilde{\boldsymbol{\beta}},\boldsymbol{q}}$ satisfy the equations (13.5.1 $*$) (with respect to the parameter $\boldsymbol{\beta}$) except when $q_o = k+1$, in which case we find

$$(\partial_{x_o} x_{i_o})^k \partial_{x_{i_o}}\partial_{\overline{x}_{i_o}} u_{\widetilde{\boldsymbol{\beta}},\boldsymbol{q}',\boldsymbol{q}'',k+1} = (-1)^{k+1} u_{(\boldsymbol{\beta}',-1),(\boldsymbol{q}',\boldsymbol{q}'')}\delta_o(x_{i_o}),$$

and similarly when applying $(\partial_{\overline{x}_o}\overline{x}_{i_o})^k$, where $\delta_o(x_{i_o})$ is the distribution δ_o in the variable x_{i_o} (see Exercise 7.19) and where we have set, for a distribution w depending on the variables $\neq x_{i_o}$, and for a test form η of maximal degree, written as $\eta = \eta_o^{(i_o)} \wedge \frac{\mathrm{i}}{2\pi}(\mathrm{d}x_{i_o} \wedge \mathrm{d}\overline{x}_{i_o})$,

$$\langle \eta, w \cdot \delta_o(x_{i_o})\rangle := \langle \eta^{(i)}_{o|D_{i_o}}, w\rangle.$$

On the other hand, according to Exercise 12.2 and as in Proposition 7.3.3, the equation $|x_{i_o}|^2 v = 0$ implies

$$v = v_0\delta_o(x_{i_o}) + \sum_{j\geqslant 0}\big(\partial^j_{x_{i_o}}(v'_j\delta_o(x_{i_o})) + (\partial^j_{\overline{x}_{i_o}}(v''_j\delta_o(x_{i_o}))\big),$$

where v_0, v'_j, v''_j are sections of $\mathfrak{Db}_{D_{i_o}}$ on a possibly smaller Δ^{n-1}. Applying $(\partial_{x_{i_o}}x_{i_o})^k$ and its conjugate to

$$u = \partial_{x_{i_o}}\partial_{\overline{x}_{i_o}}\sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\widetilde{\boldsymbol{\beta}},\boldsymbol{q}} + v_0\delta_o(x_{i_o}) + \sum_{j\geqslant 0}\big(\partial^j_{x_{i_o}}(v'_j\delta_o(x_{i_o})) + (\partial^j_{\overline{x}_{i_o}}(v''_j\delta_o(x_{i_o}))\big)$$

gives

$$0 = (-1)^{k+1}c_{\boldsymbol{q}',\boldsymbol{q}'',k+1}\cdot u_{(\boldsymbol{\beta}',0),(\boldsymbol{q}',\boldsymbol{q}'')}\delta_o(x_{i_o}) + \sum_{j\geqslant 1}(-j)^k\partial^j_{x_{i_o}}(v'_j\delta_o(x_{i_o})),$$
$$0 = (-1)^{k+1}c_{\boldsymbol{q}',\boldsymbol{q}'',k+1}\cdot u_{(\boldsymbol{\beta}',0),(\boldsymbol{q}',\boldsymbol{q}'')}\delta_o(x_{i_o}) + \sum_{j\geqslant 1}(-j)^k\partial^j_{\overline{x}_{i_o}}(v''_j\delta_o(x_{i_o})).$$

By the uniqueness of the decomposition in $\mathfrak{Db}_{D_{i_o}}[\partial_{x_{i_o}}, \partial_{\overline{x}_{i_o}}]$, we conclude that

$$c_{\boldsymbol{q}',\boldsymbol{q}'',k+1} = 0, \qquad v'_j = v''_j = 0 \quad (j \geqslant 1),$$

and finally $u = \sum_{\boldsymbol{q}} c_{\boldsymbol{q}} u_{\boldsymbol{\beta},\boldsymbol{q}} + v_0\delta_o(x_{i_o})$, up to changing the notation for $c_{\boldsymbol{q}}$ in order that q_i varies in $0, \dots, k-1$ for all i. Now, v_0 has to satisfy Equations (13.5.1 $*$) on D_{i_o}, so has a decomposition on the basic distributions (13.5.1 $**$) on D_{i_o} by the inductive assumption, and we express $v_0\delta_o(x_{i_o}$ as a basic distribution by using the formula proved in Exercise 7.19 with respect to the variable x_{i_o}. □

13.5.b. Sesquilinear pairings between holonomic $\mathcal{D}_X$-modules of normal crossing type

We make explicit the expression of a sesquilinear pairing between holonomic $\mathcal{D}_X$-modules of normal crossing type, by extending to higher dimensions Proposition 7.3.6. Due to the simplifying assumptions 13.1.2, the modules $M^{\boldsymbol{\beta}}$ considered below are finite dimensional $\mathbb{C}$-vector spaces.

13.5.2. Proposition. *Let $\mathfrak{s}$ be a sesquilinear pairing between $\mathcal{M}', \mathcal{M}''$ of normal crossing type.*

(1) *The induced pairing $\mathfrak{s} : M'^{\boldsymbol{\beta}'} \otimes \overline{M''^{\boldsymbol{\beta}''}} \to \mathfrak{Db}_{\Delta^n}$ vanishes if $\boldsymbol{\beta}' - \boldsymbol{\beta}'' \notin \mathbb{Z}^n$.*

(2) *If $m' \in M'^{\boldsymbol{\beta}}$ and $m'' \in M''^{\boldsymbol{\beta}}$ with $\boldsymbol{\beta} \geqslant -1$, then the induced pairing $\mathfrak{s}(\boldsymbol{\beta})(m', \overline{m''})$ is a $\mathbb{C}$-linear combination of the basic distributions $u_{\boldsymbol{\beta},\boldsymbol{p}}$ ($\boldsymbol{p} \in \mathbb{N}^n$).* □

As in dimension 1, we find a decomposition

$$\mathfrak{s}(\boldsymbol{\beta}) = \sum_{\boldsymbol{p}\in\mathbb{N}^n} \mathfrak{s}^{(\beta)} g_{\boldsymbol{p}} \cdot u_{\boldsymbol{\beta},\boldsymbol{p}},$$

where $\mathfrak{s}^{(\beta)} g_{\boldsymbol{p}} : M'^{\boldsymbol{\beta}} \otimes_{\mathbb{C}} M''^{\boldsymbol{\beta}} \to \mathbb{C}$ is a sesquilinear pairing (between finite-dimensional $\mathbb{C}$-vector spaces) and, setting $\mathfrak{s}^{\boldsymbol{\beta}} = \mathfrak{s}^{(\beta)} g_0$, we can write in a symbolic way

$$\mathfrak{s}(\boldsymbol{\beta})(m', \overline{m''}) = \prod_{i\mid\beta_i=-1} \partial_{x_i}\partial_{\overline{x}_i}\, \mathfrak{s}^{(\beta)} g\Big(\prod_{i\mid\beta_i>-1} |x_i|^{2(\beta_i \operatorname{Id} - \mathrm{N}_i)} \prod_{i\mid\beta_i=-1} \frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}\, m', \overline{m''}\Big),$$

where $\mathrm{N}_i = -(x_i\partial_i - \beta_i)$. As a corollary we obtain:

13.5.3. Corollary. *With the assumptions of the proposition, we have*

$$x_i\partial_{x_i}\, \mathfrak{s}(m', \overline{m''}) = \overline{x_i}\partial_{\overline{x_i}}\, \mathfrak{s}(m', \overline{m''}). \qquad \square$$

Notice also that the same property holds for $-(x_i\partial_{x_i} - \beta_i)$ since β_i is real. Therefore, with respect to the nilpotent operator N_i, $\mathfrak{s} : M'^{\boldsymbol{\beta}} \otimes \overline{M''^{\boldsymbol{\beta}}} \to \mathfrak{Db}_X$ satisfies

$$\mathfrak{s}(\mathrm{N}_i m', \overline{m''}) = \mathfrak{s}(m', \overline{\mathrm{N}_i m''}).$$

13.5.4. Remark. In the context of right $\mathcal{D}$-modules, we consider currents instead of distributions. We denote by Ω_n the (n,n)-form $\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_n\wedge\mathrm{d}\overline{x}_1\wedge\cdots\wedge\mathrm{d}\overline{x}_n$, that we also abbreviate by $\mathrm{d}x\wedge\mathrm{d}\overline{x}$. In order to state similar results, we set $\alpha = -\beta-1$ and we consider the basic currents $\Omega_n u_{\boldsymbol{\beta},\boldsymbol{p}}$. Given a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_{\Delta^n}$, the induced pairing $\mathfrak{s} : M'_{\boldsymbol{\alpha}'} \otimes \overline{M''_{\boldsymbol{\alpha}''}} \to \mathfrak{C}_{\Delta^n}$ vanishes if $\boldsymbol{\alpha}' - \boldsymbol{\alpha}'' \notin \mathbb{Z}^n$, and for $m' \in M'_{\boldsymbol{\alpha}}$ and $m'' \in M''_{\boldsymbol{\alpha}}$ with $\boldsymbol{\alpha} \leqslant 0$, the induced pairing $\mathfrak{s}(\boldsymbol{\alpha})(m', \overline{m''})$ can be written as

$$\mathfrak{s}(\boldsymbol{\alpha})(m', \overline{m''}) = \Omega_n \mathfrak{s}_{\boldsymbol{\alpha}}\Big(m' \prod_{i\mid\alpha_i<0} |x_i|^{-2(1+\alpha_i+\mathrm{N}_i)} \prod_{i\mid\alpha_i=0} \frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}, \overline{m''}\Big) \cdot \prod_{i\mid\alpha_i=0} \partial_{x_i}\partial_{\overline{x}_i},$$

where $\mathrm{N}_i = (x_i\partial_i - \alpha_i)$. Similarly, N_i is self-adjoint with respect to $\mathfrak{s}$.

13.5.c. Induced sesquilinear pairing on nearby cycles. We now consider the setting of Section 13.3.b and switch back to the right setting. Suppose we have a sesquilinear pairing $\mathfrak{s} : \mathcal{M}' \otimes_{\mathbb{C}} \overline{\mathcal{M}''} \to \mathfrak{C}_{\Delta^n}$. We still denote by $\mathfrak{s}$ the pushforward sesquilinear pairing $\mathcal{N}' \otimes \overline{\mathcal{N}''} \to \mathfrak{C}_{\Delta^{n+1}}$ by the inclusion defined by the graph of $g(x) = x^{\boldsymbol{a}}$.

The purpose of this section is to find a formula (see Proposition 13.5.5 below) for the induced pairing

$$\mathrm{gr}^V_\gamma \mathfrak{s} : \mathrm{gr}^V_\gamma \mathcal{N}' \otimes \overline{\mathrm{gr}^V_\gamma \mathcal{N}''} \longrightarrow \mathfrak{C}_{\Delta^n}$$

for $\gamma \in [-1, 0)$, as defined by (12.5.10 $**$), that we fix below. Since we already know that $\mathrm{gr}^V_\gamma \mathcal{N}', \mathrm{gr}^V_\gamma \mathcal{N}''$ are of normal crossing type, $\mathrm{gr}^V_\gamma \mathfrak{s}$ is uniquely determined by the pairings

$${}^\gamma\mathfrak{s}_{\boldsymbol{\beta}} : {}^\gamma N'_{\boldsymbol{\beta}} \otimes \overline{{}^\gamma N''_{\boldsymbol{\beta}}} \longrightarrow \mathbb{C}$$

for $\boldsymbol{\beta} \leqslant 0$. What we have to do then is to derive a formula for ${}^\gamma\mathfrak{s}_{\boldsymbol{\beta}}$ in terms of the original pairing $\mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$.

13.5.5. Proposition. *We have*

$$ {}^{\gamma}\mathfrak{s}_{\boldsymbol{\beta}}\Big(\sum_{j\geqslant 0} n'_j\mathrm{N}^j, \overline{\sum_{k\geqslant 0} n''_k\mathrm{N}^k}\Big) = \mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\Big(\mathrm{Res}_{s=0}\, m' \sum_{j,k\in\mathbb{N}}\prod_{i\in I_g(\boldsymbol{\beta})} \frac{s^{j+k}}{\mathrm{N}_i - a_i s}, \overline{m''}\Big). $$

Proof. Fix $m' \in M'_{\boldsymbol{\beta}+\gamma\boldsymbol{a}} \subset M'_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}]$ and $m'' \in M''_{\boldsymbol{\beta}+\gamma\boldsymbol{a}} \subset M''_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}[\mathrm{E}]$, and let us consider their images n', n'' by the morphism in Proposition 13.2.29. The induced pairing is given by the formula, for $\eta_o \in C^\infty_{\mathrm{c}}(\Delta^n)$ and a cut-off function $\chi \in C^\infty_{\mathrm{c}}(\Delta)$ (see (12.5.10 $**$))

$$ \begin{aligned} \langle {}^{\gamma}\mathfrak{s}_{\boldsymbol{\beta}}(n', \overline{n''}), \eta_o\rangle &= \mathrm{Res}_{s=\gamma}\langle \mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}(m'\otimes 1, \overline{m''\otimes 1}), \eta_o|t|^{2s}\chi(t)\rangle \\ &= \mathrm{Res}_{s=\gamma}\langle \mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}(m', \overline{m''}), \eta_o|g|^{2s}\chi(g)\rangle. \end{aligned} $$

If we set $\mathrm{N} = \mathrm{E}-\gamma$, any element of ${}^{\gamma}N'_{\boldsymbol{\beta}}$ can be expanded as $\sum_j n'_j\mathrm{N}^j$ where n'_j is in the image of $M'_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$, and similarly with $M''_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}$, and we find

$$ \begin{aligned} (13.5.6)\quad &\Big\langle {}^{\gamma}\mathfrak{s}_{\boldsymbol{\beta}}\Big(\sum_{j\geqslant 0} n'_j\mathrm{N}^j, \overline{\sum_{k\geqslant 0} n''_k\mathrm{N}^k}\Big), \eta_o\Big\rangle \\ &\qquad = \mathrm{Res}_{s=\gamma}\Big((s-\gamma)^{j+k}\sum_{j,k\in\mathbb{N}}\big\langle \mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}(m'_j, \overline{m''_k}), \eta_o|g|^{2s}\chi(g)\big\rangle\Big). \end{aligned} $$

Using the symbolic notation from above, the current $\mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}(m', \overline{m''})$ is equal to

$$ \Omega_n\mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\Big(m'\prod_{i|\beta_i+\gamma a_i<0}|x_i|^{-2(1+\beta_i+\gamma a_i+\mathrm{N}_i)}\prod_{i|\beta_i=a_i=0}\frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}, \overline{m''}\Big)\cdot\prod_{i|\beta_i=a_i=0}\partial_{x_i}\partial_{\overline{x}_i}. $$

The factor $\chi(g)$ does not affect the residue, and $|g|^{2s} = |x|^{2\boldsymbol{a}s}$. If we now define $F(s)$ as the result of pairing the current

$$ \Omega_n\mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\Big(\prod_{i|\beta_i+\gamma a_i<0}|x_i|^{2a_is-2(1+\beta_i+N_i)}\prod_{i|\beta_i=a_i=0}\frac{|x_i|^{-2N_i}-1}{N_i}\, m', \overline{m''}\Big) $$

against the test function $\prod_{i|\beta_i=a_i=0}\partial_{x_i}\partial_{\overline{x}_i}\eta_o(x)$, then $F(s)$ is holomorphic on the half-space $\mathrm{Re}\, s > 0$, and

$$ \langle {}^{\gamma}\mathfrak{s}_{\boldsymbol{\beta}}(n', \overline{n''}), \eta_o\rangle = \mathrm{Res}_{s=0} F(s). $$

Recall the notation $I_g = \{i\in I \mid a_i\neq 0\}$ and $I_g(\boldsymbol{\beta}) = \{i\in I_g \mid \beta_i = 0\}$. Looking at

$$ \prod_{i\in I_g(\boldsymbol{\beta})}|x_i|^{2a_is-2-2\mathrm{N}_i}\prod_{i\in I_g\smallsetminus I_g(\boldsymbol{\beta})}|x_i|^{2a_is-2(1+\beta_i)-2\mathrm{N}_i}\prod_{i|\beta_i=a_i=0}\frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}, $$

we notice that the second factor is holomorphic near $s=0$; the problem is therefore the behavior of the first factor near $s=0$. To understand what is going on, we apply integration by parts, in the form of the identity (6.8.6 $**$); the result is that $F(s)$ is equal to the pairing between the current

$$ \Omega_n\mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\Big(\prod_{i\in I_g(\boldsymbol{\beta})}\frac{|x_i|^{2a_is-2\mathrm{N}_i}-1}{\mathrm{N}_i-a_is)^2}\prod_{i|\beta_i<0}|x_i|^{2a_is-2(1+\beta_i+\mathrm{N}_i)}\prod_{i|\beta_i=a_i=0}\frac{|x_i|^{-2\mathrm{N}_i}-1}{\mathrm{N}_i}\, m', \overline{m''}\Big) $$

and the test function

$$\prod_{i|\beta_i=0} \partial_{x_i}\partial_{\overline{x}_i}\eta_o(x).$$

The new function is meromorphic on a half-space of the form $\operatorname{Re} s > -\varepsilon$, with a unique pole of some order at the point $s = 0$. We know a priori that $\operatorname{Res}_{s=0} F(s)$ can be expanded into a linear combination of $\langle u_{\boldsymbol{\beta},\boldsymbol{p}}, \eta_o\rangle$ for certain $\boldsymbol{p} \in \mathbb{N}^n$, and that ${}^\gamma\mathfrak{s}_{\boldsymbol{\beta}}(n', \overline{n''})$ is the coefficient of $u_{\boldsymbol{\beta},0}$ in this expansion; here

$$u_{\boldsymbol{\beta},0} = \Big[\Omega_n \prod_{i|\beta_i<0} |x|^{-2(1+\beta_i)} \prod_{i\in I_g(\boldsymbol{\beta})} \mathrm{L}(x_i)\Big] \cdot \partial_{x_i}\partial_{\overline{x}_i}.$$

Throwing away all the terms that cannot contribute to $\langle u_{\boldsymbol{\beta},0}, \eta_o\rangle$, we eventually arrive at the formula

$${}^\gamma\mathfrak{s}_{\boldsymbol{\beta}}(n', \overline{n''}) = \mathfrak{s}_{\boldsymbol{\beta}+\gamma\boldsymbol{a}}\Big(\operatorname{Res}_{s=0} m' \prod_{i\in I_g(\boldsymbol{\beta})} \frac{1}{\mathrm{N}_i - a_i s}, \overline{m''}\Big),$$

where the residue simply means here the coefficient of $1/s$. In particular, we have ${}^\gamma\mathfrak{s}_{\boldsymbol{\beta}}(n', \overline{n''}) = 0$ if $\#I_g(\boldsymbol{\beta}) \geqslant 2$. By means of (13.5.6), we obtain the desired result. $\square$

In the setting of this chapter (see Section 13.1), we consider the category $\mathcal{D}$-$\mathsf{Triples}(X)$.

13.5.7. Definition **(Triples of $\widetilde{\mathcal{D}}_X$-modules of normal crossing type)**
We say that an object $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ is of normal crossing type along D if its components $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are strict and the corresponding filtered $\mathcal{D}_X$-modules $(\mathcal{M}', F_\bullet\mathcal{M}'), (\mathcal{M}'', F_\bullet\mathcal{M}'')$ are of normal crossing type along D.

Then Theorem 13.1.4 extends in an obvious way to triples of normal crossing type along D.

13.6. Exercises

Exercise 13.1. Show that $x_i : M_{\boldsymbol{\alpha}} \to M_{\boldsymbol{\alpha}-\mathbf{1}_i}$ is an isomorphism if $\alpha_i < 0$ and $\partial_{x_i} : M_{\boldsymbol{\alpha}} \to M_{\boldsymbol{\alpha}+\mathbf{1}_i}$ is an isomorphism if $\alpha_i > -1$.

Exercise 13.2. Show that a $\mathbb{C}[x]\langle\partial_x\rangle$-module of normal crossing type is of finite type over $\mathbb{C}[x]\langle\partial_x\rangle$. Moreover, show that $M_{\leqslant\boldsymbol{\alpha}} := \bigoplus_{\boldsymbol{\alpha}'\leqslant\boldsymbol{\alpha}} M_{\boldsymbol{\alpha}'}$ is a $\mathbb{C}[x]\langle x\partial_x\rangle$-module which is of finite type over $\mathbb{C}[x]$, and $\mathbb{C}[x]$-free if $\alpha_i < 0$ for all $i \in I$

Exercise 13.3. Let $i_o \in I$ and let $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ be a $\mathbb{C}[x]\langle\partial_x\rangle$-module of normal crossing type with the single exponent $\boldsymbol{\alpha} \in [-1, 0)^n$.

(1) Show that $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is supported on D_{i_o} if and only if $\alpha_{i_o} = -1$ and, for $\boldsymbol{k} \in \mathbb{Z}^n$, $M_{\boldsymbol{\alpha}+\boldsymbol{k}} = 0$ if $k_{i_o} \leqslant 0$, that is, if and only if $i_o \in I(\boldsymbol{\alpha})$ and, setting $\boldsymbol{k} = (\boldsymbol{k}', k_{i_o})$, every vertex $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$ of the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ is zero.

(2) Show that $M_{\boldsymbol{\alpha}+\mathbb{Z}^n} = M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(*D_{i_o})$, i.e., x_{i_o} acts in a bijective way on $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$, if and only if $i_o \notin I(\boldsymbol{\alpha})$ or $i_o \in I(\boldsymbol{\alpha})$ and var_{i_o} is an isomorphism.

(3) Show that the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(*D_{i_o})$ is that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ if $i_o \notin I(\boldsymbol{\alpha})$ and, otherwise, setting $\boldsymbol{k} = (\boldsymbol{k}', k_{i_o})$, is isomorphic to the quiver is obtained from that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ by replacing $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',1)}$ with $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$, var_{i_o} with Id and can_{i_o} with N_{i_o}.

Let now M be any $\mathbb{C}[x]\langle\partial_x\rangle$-module of normal crossing type along D, and consider its quiver as in Remark 13.2.6.

(4) Show that M is supported on D_{i_o} if and only if, for any exponent $\boldsymbol{\alpha} \in [-1,0)^n$, we have $\alpha_{i_o} = -1$ and every vertex of the quiver with index $\boldsymbol{k} \in \{0,1\}^n$ satisfying $k_{i_o} = 0$ vanishes.

(5) Show that $M = M(*D_{i_o})$ if and only if var_{i_o} is bijective.

Exercise 13.4. Relate the notions of Definitions 13.2.7 and 13.2.8 to that of Sections 11.4 and 11.5 by considering Proposition 11.4.7 (in the case of $\mathcal{D}_X$-modules).

Exercise 13.5. Define the endofunctors $(!D_{i_o})$, $(!*D_{i_o})$ of the category of $\mathbb{C}[x]\langle\partial_x\rangle$-modules of normal crossing type along D in such a way that the quiver of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(!D_{i_o})$, resp. $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}(!*D_{i_o})$ is that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ if $i_o \notin I(\boldsymbol{\alpha})$ and, otherwise, setting $\boldsymbol{k} = (\boldsymbol{k}', k_{i_o})$, the quiver is obtained from that of $M_{\boldsymbol{\alpha}+\mathbb{Z}^n}$ by replacing

- $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}$ with $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',1)}$, var_{i_o} with N_{i_o} and can_{i_o} with Id,
- resp. $M_{\boldsymbol{\alpha}+(\boldsymbol{k}',1)}$ with $\mathrm{image}[\mathrm{N}_{i_o} : M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)} \to M_{\boldsymbol{\alpha}+(\boldsymbol{k}',0)}]$, var_{i_o} with the natural inclusion and can_{i_o} with N_{i_o}.

Show that there is a natural morphism $M(!D_{i_o}) \to M(*D_{i_o})$ whose image is $M(!*D_{i_o})$.

Exercise 13.6. Say that M *middle extension with support along* $D_{i\in I}$ if, for each $i \in I$, either the source of every can_i is zero, or every can_i is onto and every var_i is injective. In other words, we accept $\mathbb{C}[x]\langle\partial_x\rangle$-modules supported on the intersection of some components of D, which are middle extension along any of the other components.

Show that any $\mathbb{C}[x]\langle\partial_x\rangle$-module M of normal crossing type along D is a successive extension of such $\mathbb{C}[x]\langle\partial_x\rangle$-modules which are middle extensions with support along $D_{i\in I}$.

Exercise 13.7. Let $\mathcal{M}$ be a coherent $\mathcal{D}_X$-module of normal crossing type along D. Show that $\mathcal{M}$ is a successive extension of $\mathcal{D}_X$-modules of normal crossing type along D, each of which being moreover a middle extension with support along $D_{i\in I}$. [*Hint*: Use Exercise 13.6.]

13.7. Comments

This chapter is intended to be an expanded version of the part of Section 3 in **[Sai90]** which is concerned only with filtered $\mathcal{D}$-modules. As already explained, we do not refer to perverse sheaves, so the perverse sheaf version, which is present in loc. cit., is not relevant here. Nevertheless, the content of §13.3.a is much inspired by it.

APPENDIX. SIGN CONVENTIONS FOR HODGE MODULES

A.1. General principles

In this appendix, we explain how one can arrive at the correct sign conventions for polarized Hodge modules. This is a bit of a detective story, fortunately with a happy ending. Finding the correct signs looks difficult at the beginning, because there are many places in the theory where one might have to choose a sign factor, and it is not clear that all those choices can be made consistently. For example, should there be a sign in the conversion between left and right $\mathcal{D}$-modules? What are the correct signs to use for direct images? For nearby and vanishing cycles? For the duality functor?

Before going into any details, we think it may be helpful to list a few general principles that have turned out to be useful in the solution:

(1) Make all definitions in such a way that they do not depend on the choice of the imaginary unit $\mathsf{i} = \sqrt{-1}$.

(2) Make all constructions compatible with closed embeddings, and therefore independent of the choice of ambient complex manifold.

(3) In particular, work consistently with right $\mathcal{D}$-modules and currents (instead of with left $\mathcal{D}$-modules and distributions).

(4) When defining a current, choose the sign in such a way that the resulting current is positive, if possible.

(5) Use Deligne's Koszul sign rule for graded objects. Under this rule, switching two quantities x and y produces a sign factor of $(-1)^{\deg x \deg y}$.

A.1.1. Example. The integral over a complex manifold X depends on the orientation; the orientation is induced by the standard orientation on $\mathbb{C}$, in which $1, \mathsf{i}$ is a positively oriented basis over $\mathbb{R}$. To make the integral independent of the choice of i, it is better to work with the expression

$$\frac{1}{(2\pi\mathsf{i})^{\dim X}} \int_X$$

instead. Similarly, the Lefschetz operator $L_\omega \alpha = \omega \wedge \alpha$ on the cohomology of a compact Kähler manifold depends on the choice of i, because the Kähler form ω is minus the imaginary part of the Kähler metric. It is therefore better to work with the operator $(2\pi\mathsf{i})L_\omega$ instead.

A.1.2. Example. The fundamental group of the punctured disk

$$\Delta^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}$$

is naturally the group $\mathbb{Z}(1) = (2\pi\mathsf{i})\mathbb{Z}$. Indeed, independently of the choice of i, the universal covering space of the punctured disk is $\exp\colon \mathbb{H} \to \Delta^*$, where

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\}$$

is the *left* halfplane. The group $\mathbb{Z}(1)$ acts on this space by translations.

A.1.3. Example. Polarizations are defined as Hermitian pairings with values in the sheaf of currents. The following collection of basic currents on the unit disk Δ plays an important role in the theory. Define $\mathrm{L}(t) = -\log|t|^2$, with a minus sign to make the function positive on Δ^*. For $\alpha < 0$ and $p \in \mathbb{N}$, the formula

$$\big\langle C_{\alpha,p}, \varphi \big\rangle = \frac{\varepsilon(2)}{2\pi\mathsf{i}} \int_\Delta \frac{\mathrm{L}(t)^p}{p!} |t|^{-2(1+\alpha)} \varphi \, \mathrm{d}t \wedge \mathrm{d}\bar{t} \tag{A.1.4}$$

defines a current on Δ. The factor $2\pi\mathsf{i}$ makes the current independent of the choice of $\mathsf{i} = \sqrt{-1}$, and the sign factor $\varepsilon(2) = -1$ makes it positive, as suggested by the general principles above. For different values of $p \in \mathbb{N}$, the basic currents are related by the identity

$$C_{\alpha,p}(t\partial_t - \alpha) = C_{\alpha,p}(\bar{t}\partial_{\bar{t}} - \alpha) = C_{\alpha,p-1},$$

which can be proved using integration by parts. The delta function

$$\big\langle \delta_0, \varphi \big\rangle = \varphi(0)$$

can be expressed in terms of the basic currents as

$$\delta_0 = -C_{-1,1}\partial_t\partial_{\bar{t}}; \tag{A.1.5}$$

the proof is again by integration by parts.

A.2. Hodge structures and polarizations

The first place where a sign factor appears is in the definition of complex Hodge structures. Let H be a finite-dimensional complex vector space. Recall that a *Hodge structure* of weight k on H is a decomposition

$$H = \bigoplus_{p+q=k} H^{p,q}. \tag{A.2.1}$$

A *polarization* of H is a Hermitian form

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C},$$

with the following two properties:

(a) The decomposition in (A.2.1) is orthogonal with respect to S.
(b) The Hermitian form $c_k(-1)^q\mathrm{S}$ is positive definite on the subspace $H^{p,q}$.

In this definition, c_k is a sign factor depending on the weight of the Hodge structure. We will see below that there are only two choices: either $c_k = (-1)^k$, which is the convention used in classical Hodge theory; or $c_k = 1$, which is the convention used in Saito's work. We will find that $c_k = 1$ is indeed the correct choice for the theory of Hodge modules, but we shall give all formulas with c_k for the time being, so as not to prejudge the issue.

A.2.2. Example. On $\mathbb{C} = \mathbb{C}^{0,0}$, the natural Hermitian form is $\mathrm{S}(a, b) = a\bar{b}$. If we want this to be a polarization, we have to use $c_0 = 1$.

A.2.3. Example. If H is a Hodge structure of weight k, then the conjugate vector space $\overline{H}$ inherits a Hodge structure of weight k, with Hodge decomposition

$$\overline{H}^{p,q} = \overline{H^{q,p}}.$$

The *Tate twist* $H(n)$ is the Hodge structure of weight $k - 2n$ with

$$H(n)^{p,q} = H^{p+n,q+n}.$$

The first condition in the definition of a polarization is equivalent to saying that

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}(-k)$$

is a morphism of Hodge structures of weight $2k$.

A.2.4. Example. Let $H_{\mathbb{R}}$ be a finite-dimensional real vector space. Cattani, Kaplan, and Schmid define a *real Hodge structure* of weight k to be a decomposition

$$H = \mathbb{C} \otimes_{\mathbb{R}} H_{\mathbb{R}} = \bigoplus_{p+q=k} H^{p,q}$$

with the property that $\overline{H^{p,q}} = H^{q,p}$. They say that a bilinear pairing

$$\mathrm{Q}_{\mathbb{R}}\colon H_{\mathbb{R}} \otimes_{\mathbb{R}} H_{\mathbb{R}} \longrightarrow \mathbb{R}$$

is a *polarization* if the following conditions are satisfied: $\mathrm{Q}_{\mathbb{R}}$ is $(-1)^k$-symmetric; the Hodge decomposition is orthogonal with respect to $\mathrm{Q}_{\mathbb{R}}$; and $\mathrm{Q}_{\mathbb{R}}(i^{p-q}v, \overline{v}) > 0$ for every nonzero $v \in H^{p,q}$. In that case, the Hermitian pairing

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}, \quad \mathrm{S}(v, w) = c_k(-1)^k \cdot (2\pi\mathrm{i})^{-k}\mathrm{Q}_{\mathbb{R}}(v, \overline{w})$$

is a polarization in our sense. Indeed, for nonzero $v \in H^{p,q}$, one gets

$$c_k(-1)^q \cdot \mathrm{S}(v, v) = c_k(-1)^k i^k i^{p-q} \cdot \mathrm{S}(v, v) = (2\pi)^{-k}\mathrm{Q}_{\mathbb{R}}(i^{p-q}v, \overline{v}) > 0.$$

One can interpret the factor $(2\pi\mathrm{i})^{-k}$ as saying that $\mathrm{Q}_{\mathbb{R}}\colon H_{\mathbb{R}} \otimes_{\mathbb{R}} H_{\mathbb{R}} \to \mathbb{R}(-k)$ is a morphism of real Hodge structures of weight $2k$; in classical Hodge theory, it is therefore more natural to take $c_k = (-1)^k$.

A.3. Cohomology of compact Kähler manifolds

We can pin down some of the signs by working out what happens for the cohomology of compact Kähler manifolds. Let X be a compact Kähler manifold of dimension n. For each $k \in \{0, 1, \ldots, 2n\}$, the k-th cohomology has a Hodge structure of weight k, with Hodge decomposition

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X).$$

A choice of Kähler metric h determines a polarization of the Hodge structure; it also determines the Lefschetz operator, which makes the direct sum of all cohomology groups into a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Our goal will be to describe all this information as concisely as possible.

Note. The advantage of this example – and the reason for putting it at the beginning of our analysis – is that there are no choices involved in constructing a positive definite pairing. Indeed, the Kähler metric induces a positive definite Hermitian inner product on the space of harmonic k-forms, hence on $H^k(X, \mathbb{C})$. All we have to do is figure out what signs appear when we compare this inner product to the pairing given by wedge product and integration over X.

Fix a choice of $\mathsf{i} = \sqrt{-1}$. The Kähler form $\omega = -\operatorname{Im} h \in A^2(X, \mathbb{R})$ and its cohomology class $[\omega] \in H^2(X, \mathbb{R})$ depend on the choice of i, because the imaginary part $\operatorname{Im}\colon \mathbb{C} \to \mathbb{R}$ does. The choice of i endows the two-dimensional real vector space $\mathbb{C}$ with an orientation, by declaring that $1, \mathsf{i}$ is a positively-oriented basis; the induced orientation on X has the property that

$$\int_X \frac{\omega^n}{n!} = \operatorname{vol}(X) > 0.$$

We can remove the dependence on the choice of i by defining $\mathbb{R}(1) = 2\pi\mathsf{i} \cdot \mathbb{R} \subseteq \mathbb{C}$, and working with the closed two-form $2\pi\mathsf{i}\,\omega \in A^2(X, \mathbb{R}(1))$; its cohomology class is $[2\pi\mathsf{i}\,\omega] \in H^2(X, \mathbb{R}(1))$. Instead of the usual integral, we use

$$\frac{1}{(2\pi\mathsf{i})^n} \int_X : A^{2n}(X, \mathbb{C}) \longrightarrow \mathbb{C}.$$

Now all terms in the identity

$$\frac{1}{(2\pi\mathsf{i})^n} \int_X \frac{(2\pi\mathsf{i}\,\omega)^n}{n!} = \operatorname{vol}(X)$$

are independent of the choice of i.

A.3.1. Example. On $\mathbb{P}^1$, with the Fubini-Study metric, one has

$$2\pi\mathsf{i}\,\omega_{\mathrm{FS}} = c_1\big(\mathcal{O}_{\mathbb{P}^1}(1)\big) \in H^2\big(\mathbb{P}^1, \mathbb{Z}(1)\big),$$

and the volume comes out to

$$\operatorname{vol}(\mathbb{P}^1) = \frac{1}{2\pi\mathsf{i}} \int_{\mathbb{P}^1} 2\pi\mathsf{i}\,\omega_{\mathrm{FS}} = 1.$$

This is the reason for including the factor 2π into the definition. Some of the formulas below would look nicer without the 2π, but we shall keep it for the sake of tradition.

Let $A^k(X) = A^k(X, \mathbb{C})$ be the space of smooth complex-valued k-forms. The Kähler metric h induces on $A^k(X)$ a Hermitian inner product

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\overline{\beta},$$

where $*\colon A^k(X) \to A^{2n-k}(X)$ is the Hodge $*$-operator. Like the integral, the Hodge $*$-operator depends on the orientation, whereas the inner product only depends on the Kähler metric h. We define the *Lefschetz operator*

$$L_\omega \colon A^\bullet(X) \longrightarrow A^{\bullet+2}(X)$$

by the formula $L_\omega \alpha = \omega \wedge \alpha$, and its adjoint

$$\Lambda_\omega \colon A^\bullet(X) \longrightarrow A^{\bullet-2}(X)$$

by the formula $\langle L_\omega \alpha, \beta \rangle = \langle \alpha, \Lambda_\omega \beta \rangle$. The main tool for describing the polarization is the following result, known as *Weil's identity.*

A.3.2. Proposition. *If $\alpha \in A^{p,q}(X)$ is primitive, in the sense that $\Lambda_\omega \alpha = 0$, then*

$$*\alpha = \mathsf{i}^{q-p}\, \varepsilon(k)\, \frac{L_\omega^{n-k}}{(n-k)!}\, \alpha, \tag{A.3.3}$$

where $\varepsilon(k) = (-1)^{k(k-1)/2}$ and $k = p + q$.

We can use Weil's identity to express the Hodge $*$-operator in terms of representation theory. The complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ has the standard basis

$$\mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

In the complex Lie group $\mathrm{SL}_2(\mathbb{C})$, consider the *Weil element*

$$\mathrm{w} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = e^{\mathrm{X}} e^{-\mathrm{Y}} e^{\mathrm{X}}.$$

It has the property that $\mathrm{w}^{-1} = -\mathrm{w}$, and under the adjoint action of $\mathrm{SL}_2(\mathbb{C})$ on its Lie algebra, one has the identities

$$\mathrm{wHw}^{-1} = -\mathrm{H}, \quad \mathrm{wXw}^{-1} = -\mathrm{Y}, \quad \mathrm{wYw}^{-1} = -\mathrm{X}.$$

From this, one deduces that $e^{\mathrm{X}} = \mathrm{w}e^{-\mathrm{X}}e^{\mathrm{Y}} = e^{\mathrm{Y}}\mathrm{w}e^{\mathrm{Y}}$, which gives another way to remember the formula for w.

The (infinite-dimensional) vector space

$$A^\bullet(X) = \bigoplus_{k=0}^{2n} A^k(X)$$

becomes a representation of $\mathfrak{sl}_2(\mathbb{C})$ if we set

$$\mathrm{X} = 2\pi\mathsf{i}\; L_\omega \quad \text{and} \quad \mathrm{Y} = (2\pi\mathsf{i})^{-1}\Lambda_\omega,$$

and let H act as multiplication by $k - n$ on the subspace $A^k(X)$. The reason for this (non-standard) definition is that it makes the representation not depend on the choice of i. It is easy to see how the Weil element w acts on primitive forms. Suppose that $\alpha \in A^{n-k}(X)$ satisfies $\mathrm{Y}\alpha = 0$. Then $\mathrm{w}\alpha \in A^{n+k}(X)$, and if we expand both sides of the identity

$$e^{\mathrm{X}}\alpha = e^{\mathrm{Y}}\mathrm{w}e^{\mathrm{Y}}\alpha = e^{\mathrm{Y}}\mathrm{w}\alpha$$

into power series, and compare terms in degree $n + k$, we get

$$\mathrm{w}\alpha = \frac{\mathrm{X}^k}{k!}\alpha.$$

This formula is the reason for using w (instead of the otherwise equivalent w^{-1}): there is no sign on the right-hand side.

Note. One should be careful: the element

$$\mathrm{w}^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

acts on $A^k(X)$ as $(-1)^{k-n}$, and *not* just as -1.

We deduce the following generalization of Weil's identity, which shows again how the Hodge $*$-operator depends on the choice of i.

A.3.4. Proposition. *For every $\alpha \in A^{p,q}(X)$, one has*

$$* \, \alpha = \frac{1}{(2\pi\mathsf{i})^n} \cdot (-1)^q \, \varepsilon(p+q)(2\pi)^{p+q} \cdot \mathrm{w}\alpha.$$

Proof. Suppose first that $\mathrm{Y}\alpha = 0$. Setting $k = p + q$, we have

$$\mathrm{w}\alpha = \frac{\mathrm{X}^{n-k}}{(n-k)!}\,\alpha.$$

On the other hand, Weil's identity (A.3.3) becomes

$$*\alpha = i^{q-p}\,\varepsilon(k) \cdot (2\pi\mathsf{i})^{k-n}\frac{\mathrm{X}^{n-k}}{(n-k)!}\,\alpha = (2\pi\mathsf{i})^{-n} \cdot (-1)^q\,\varepsilon(k)(2\pi)^k \cdot \mathrm{w}\alpha,$$

as claimed. The general case follows by using the relations

$$* \, \mathrm{X} = -(2\pi)^2 \, \mathrm{Y} \, * \quad \text{and} \quad \mathrm{wX} = -\mathrm{Yw},$$

the Lefschetz decomposition for α, and the identity $\varepsilon(k+2) = -\,\varepsilon(k)$. ☐

Now we can easily derive the Hodge-Riemann bilinear relations. Suppose that $\alpha, \beta \in A^{p,q}(X)$, and set $k = p + q$. Then

$$\alpha \wedge *\overline{\beta} = \alpha \wedge \overline{(*\beta)} = \frac{1}{(2\pi\mathsf{i})^n} \cdot (-1)^{n+q}\,\varepsilon(k)(2\pi)^k \cdot \alpha \wedge \overline{(\mathrm{w}\beta)}.$$

If we put this into the formula for the inner product, we get

$$\langle \alpha, \beta \rangle = \int_X \alpha \wedge *\overline{\beta} = (-1)^{n+q}\,\varepsilon(k)(2\pi)^k \cdot \frac{1}{(2\pi\mathsf{i})^n} \int_X \alpha \wedge \overline{(\mathrm{w}\beta)}. \tag{A.3.5}$$

According to our definition, this means that the Hermitian pairing

$$(\alpha, \beta) \longmapsto (-1)^n c_k \; \varepsilon(k) \cdot \frac{1}{(2\pi \mathsf{i})^n} \int_X \alpha \wedge \overline{(\mathrm{w}\beta)},$$

polarizes the Hodge structure on $H^k(X, \mathbb{C})$.

It turns out that there is a much more concise way of describing the polarization. Let us set $H_k = H^{n+k}(X, \mathbb{C})$; this has a Hodge structure of weight $n + k$, and its weight with respect to the action by H is equal to k. Also set

$$H = \bigoplus_{k \in \mathbb{Z}} H_k,$$

with the induced action by the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the Lie group $\mathrm{SL}_2(\mathbb{C})$. For each $k \in \{-n, \ldots, n\}$, we have a sesquilinear pairing

$$\text{(A.3.6)} \qquad \mathrm{S}_k \colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C}, \quad \mathrm{S}_k(\alpha, \beta) = b_k \cdot \frac{1}{(2\pi \mathsf{i})^n} \int_X \alpha \wedge \overline{\beta}.$$

Here b_k is a sign factor; our goal will be to choose b_k in such a way that all the formulas become as simple as possible. We can put all of the S_k together into one big sesquilinear pairing

$$\mathrm{S} \colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}, \quad \mathrm{S}_{|H_k \otimes_{\mathbb{C}} \overline{H_\ell}} = \begin{cases} \mathrm{S}_k & \text{if } \ell = -k, \\ 0 & \text{otherwise.} \end{cases}$$

The following two identities can be checked with a brief calculation:

$$\overline{\mathrm{S}_k(\alpha, \beta)} = (-1)^k b_{-k} b_k \cdot \mathrm{S}_{-k}(\beta, \alpha) \quad \text{and} \quad \mathrm{S}_k(\mathrm{H}\alpha, \beta) = -\mathrm{S}_k(\alpha, \mathrm{H}\beta)$$

for $\alpha \in H_k$ and $\beta \in H_{-k}$. Since $\mathrm{X} = 2\pi \mathsf{i} \; L_\omega$, it is also not hard to show that

$$\mathrm{S}_{k+2}(\mathrm{X}\alpha, \beta) = -b_k b_{k+2} \cdot \mathrm{S}_k(\alpha, \mathrm{X}\beta)$$

for $\alpha \in H_k$ and $\beta \in H_{-(k+2)}$. A slightly longer calculation, based on Proposition A.3.4, is required to prove the identity

$$\mathrm{S}_k(\alpha, \mathrm{w}\beta) = (-1)^k b_{-k} b_k \cdot \mathrm{S}_{-k}(\mathrm{w}\alpha, \beta)$$

for every $\alpha, \beta \in H_k$. Now the fact that $\mathrm{Y} = -\mathrm{w}\mathrm{X}\mathrm{w}^{-1}$ can be used to deduce the following more surprising identity:

$$\mathrm{S}_{k-2}(\mathrm{Y}\alpha, \beta) = -b_k b_{k-2} \cdot \mathrm{S}_k(\alpha, \mathrm{Y}\beta)$$

for every $\alpha \in H_k$ and every $\beta \in H_{-(k-2)}$. We write "surprising" because it is not at all clear, at first glance, that one can move the adjoint Λ_ω of the Lefschetz operator from one factor of the integral to the other.

Clearly, we should require $b_{k+2} = -b_k$ for every $k \in \mathbb{Z}$, in order to eliminate all the sign factors from the above formulas. Let us restate the resulting identities in terms of the sesquilinear pairing $\mathrm{S} \colon H \otimes_{\mathbb{C}} \overline{H} \to \mathbb{C}$: first, S is Hermitian symmetric; second, one has the four identities

$$\begin{aligned}
\mathrm{S}\circ(\mathrm{H}\otimes\mathrm{Id}) &= -\mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{H}),\\
\mathrm{S}\circ(\mathrm{X}\otimes\mathrm{Id}) &= \mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{X}),\\
\mathrm{S}\circ(\mathrm{Y}\otimes\mathrm{Id}) &= \mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{Y}),\\
\mathrm{S}\circ(\mathrm{w}\otimes\mathrm{Id}) &= \mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{w}).
\end{aligned}\tag{A.3.7}$$

Now suppose that $\alpha,\beta\in A^{p,q}(X)$ are harmonic forms. It will be convenient to define $k=(p+q)-n$, so that $[\alpha],[\beta]\in H_k$. Then

$$\mathrm{S}_k(\alpha,\mathrm{w}\beta)=b_k\cdot\frac{1}{(2\pi\mathsf{i})^n}\int_X\alpha\wedge\overline{(\mathrm{w}\beta)}.$$

Going back to (A.3.5), we can rewrite this in the form

$$\mathrm{S}_k(\alpha,\mathrm{w}\beta)=(-1)^q\cdot b_k(-1)^n\,\varepsilon(n+k)\cdot\frac{\langle\alpha,\beta\rangle}{(2\pi)^{n+k}}.$$

The conclusion is H_k has a Hodge structure of weight $n+k$, which is polarized by the Hermitian form $\mathrm{S}_k\circ(\mathrm{Id}\otimes\mathrm{w})$, provided that

$$b_k=(-1)^n\,\varepsilon(n+k)c_{n+k}.\tag{A.3.8}$$

Note. Recall that $b_{k+2}=-b_k$. Since $\varepsilon(k+2)=-\varepsilon(k)$, it follows that $c_{k+2}=c_k$ for every $k\in\mathbb{Z}$; together with the normalization $c_0=1$, this leaves the two values $c_k=1$ and $c_k=(-1)^k$ as the only possibilities. We will see below that $c_k=1$ is the better choice for the theory of Hodge modules.

Let us summarize our findings. Setting $H_k=H^{n+k}(X,\mathbb{C})$, the vector space

$$H=\bigoplus_{k\in\mathbb{Z}}H_k$$

is a representation of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. Each weight space H_k has a Hodge structure of weight $n+k$, and the two operators

$$\mathrm{X}\colon H_k\longrightarrow H_{k+2}(1)\quad\text{and}\quad\mathrm{Y}\colon H_k\longrightarrow H_{k-2}(-1)$$

are morphisms of Hodge structure. All of these Hodge structures are simultaneously polarized by the Hermitian form $\mathrm{S}\circ(\mathrm{Id}\otimes\mathrm{w})$, where $\mathrm{S}\colon H\otimes_{\mathbb{C}}\overline{H}\to\mathbb{C}$ is assembled from the individual sesquilinear pairings

$$\mathrm{S}_k\colon H_k\otimes_{\mathbb{C}}\overline{H_{-k}}\longrightarrow\mathbb{C},\quad \mathrm{S}_k(\alpha,\beta)=(-1)^nc_{n+k}\,\varepsilon(n+k)\cdot\frac{1}{(2\pi\mathsf{i})^n}\int_X\alpha\wedge\overline{\beta},$$

and satisfies the identities in (A.3.7).

A.4. $\mathfrak{sl}_2$-Hodge structures and polarizations

The cohomology of a compact Kähler manifold is both a representation of $\mathfrak{sl}_2(\mathbb{C})$ and a direct sum of polarized Hodge structures, in a compatible way. Since the same kind of structure also appears in the analysis of polarized variations of Hodge structure on the punctured disk, it will be useful to give it a name.

A.4.1. Definition. An $\mathfrak{sl}_2$*-Hodge structure* on a finite-dimensional complex vector space H is a representation of $\mathfrak{sl}_2(\mathbb{C})$ on H with the following properties:

(a) Each weight space $H_k = E_k(\mathrm{H})$ has a Hodge structure of weight $n+k$; the integer n is called the *(central) weight* of the $\mathfrak{sl}_2$-Hodge structure.

(b) The two operators

$$\mathrm{X}\colon H_k \longrightarrow H_{k+2}(1) \quad \text{and} \quad \mathrm{Y}\colon H_k \longrightarrow H_{k-2}(-1)$$

are morphisms of Hodge structure.

Equivalently, an $\mathfrak{sl}_2$-Hodge structure of weight n is a bigraded vector space

$$H = \bigoplus_{p,q\in\mathbb{Z}} H^{p,q}$$

that is simultaneously a representation of $\mathfrak{sl}_2(\mathbb{C})$, in a way that is compatible with the bigrading. This means that

$$\mathrm{X}\colon H^{p,q} \longrightarrow H^{p+1,q+1} \quad \text{and} \quad \mathrm{Y}\colon H^{p,q} \longrightarrow H^{p-1,q-1},$$

and that H acts on the subspace $H^{p,q}$ as multiplication by the integer $(p+q)-n$. This makes each of the weight spaces

$$H_k = \bigoplus_{p+q=n+k} H^{p,q}$$

into a Hodge structure of weight $n+k$. In this abstract setting, we can again define the *Weil element*

$$\mathrm{w} = e^{\mathrm{X}} e^{-\mathrm{Y}} e^{\mathrm{X}} \in \mathrm{GL}(H).$$

The Weil element induces isomorphisms $\mathrm{w}\colon H_k \to H_{-k}$ among opposite weight spaces, due to the fact that $\mathrm{wHw}^{-1} = -\mathrm{H}$.

A.4.2. Lemma. *If H is an $\mathfrak{sl}_2$-Hodge structure, then $\mathrm{w}\colon H_k \to H_{-k}(-k)$ is an isomorphism of Hodge structures (of weight $n+k$).*

Proof. We first prove an auxiliary formula. Suppose that $b \in H_{-\ell}$ is primitive, in the sense that $\mathrm{Y}b = 0$ (and therefore $\ell \geqslant 0$). From $\mathrm{w}e^{-\mathrm{X}} = e^{\mathrm{X}}e^{-\mathrm{Y}}$, we get $\mathrm{w}e^{-\mathrm{X}}b = e^{\mathrm{X}}b$, and after expanding and comparing terms in degree $\ell - 2j$, also

$$\mathrm{w}\frac{\mathrm{X}^j}{j!}b = (-1)^j \frac{\mathrm{X}^{\ell-j}}{(\ell-j)!}b. \tag{A.4.3}$$

Now any $a \in H_k$ has a unique Lefschetz decomposition

$$a = \sum_{j\geqslant \max(k,0)} \frac{\mathrm{X}^j}{j!} a_j,$$

where $a_j \in H_{k-2j}$ satisfies $Ya_j = 0$. Here we only need to consider $j \geqslant k$ in the sum because $\mathrm{X}^{2j-k+1}a_j = 0$, which implies that $\mathrm{X}^j a_j = 0$ for $j < k$. Suppose further that $a \in H^{p,q}$, where $p+q = n+k$. Then $\mathrm{X}^i a_j \in H^{p+i,q+i}$, and by descending induction on $j \geqslant \max(k,0)$, we deduce that

$$a_j \in H^{p-j,q-j}.$$

In other words, the Lefschetz decomposition holds in the category of Hodge structures. We can now check what happens when we apply w. Using (A.4.3),

$$\mathrm{w}a = \sum_{j \geqslant \max(k,0)} \mathrm{w}\frac{\mathrm{X}^j}{j!} a_j = \sum_{j \geqslant \max(k,0)} (-1)^j \frac{\mathrm{X}^{j-k}}{(j-k)!} a_j \in H^{p-k,q-k},$$

and so w is a morphism of Hodge structures. Since w is bijective, it must be an isomorphism of Hodge structures, as claimed. □

We define polarizations of $\mathfrak{sl}_2$-Hodge structures by analogy with the case of compact Kähler manifolds.

A.4.4. Definition. A *polarization* of an $\mathfrak{sl}_2$-Hodge structure H is a Hermitian form

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}$$

that satisfies the four identities

$$\begin{aligned}
\mathrm{S} \circ (\mathrm{H} \otimes \mathrm{Id}) &= -\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{H}),\\
\mathrm{S} \circ (\mathrm{X} \otimes \mathrm{Id}) &= \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{X}),\\
\mathrm{S} \circ (\mathrm{Y} \otimes \mathrm{Id}) &= \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{Y}),\\
\mathrm{S} \circ (\mathrm{w} \otimes \mathrm{Id}) &= \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{w}),
\end{aligned}$$

such that $\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{w})$ polarizes the Hodge structure of weight $n+k$ on each H_k.

The relation $\mathrm{S} \circ (\mathrm{H} \otimes \mathrm{Id}) = -\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{H})$ implies that

$$\mathrm{S}_{|H_k \otimes_{\mathbb{C}} \overline{H_\ell}} = \begin{cases} \mathrm{S}_k & \text{if } \ell = -k,\\ 0 & \text{otherwise.} \end{cases}$$

and so S is actually given by a collection of sesquilinear pairings

$$\mathrm{S}_k\colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C},$$

exactly as in the previous section.

A.4.5. Example. With the exception of positivity, all the conditions in the definition have a nice functorial interpretation. The conjugate complex vector space $\overline{H}$ is again an $\mathfrak{sl}_2$-Hodge structure of weight n: the action of H is unchanged, but X and Y act with an extra minus sign. This sign change is dictated by the geometric case, where $\mathrm{X} = 2\pi\mathsf{i}\, L_\omega$ and $\mathrm{Y} = (2\pi i)^{-1}\Lambda_\omega$. Likewise, if H' and H'' are $\mathfrak{sl}_2$-Hodge structures of weights n' and n'', then the tensor product $H' \otimes_{\mathbb{C}} H''$ is naturally an $\mathfrak{sl}_2$-Hodge structure of weight $n' + n''$: to be precise,

$$\bigl(H' \otimes_{\mathbb{C}} H''\bigr)_k = \bigoplus_{i+j=k} H'_i \otimes_{\mathbb{C}} H''_j,$$

and the $\mathfrak{sl}_2(\mathbb{C})$-action is given by the usual formulas

$$\begin{aligned}
\mathrm{X}(v' \otimes v'') &= \mathrm{X}v' \otimes v'' + v' \otimes \mathrm{X}v'',\\
\mathrm{Y}(v' \otimes v'') &= \mathrm{Y}v' \otimes v'' + v' \otimes \mathrm{Y}v'',\\
\mathrm{H}(v' \otimes v'') &= \mathrm{H}v' \otimes v'' + v' \otimes \mathrm{H}v''.
\end{aligned}$$

Lastly, we can turn $\mathbb{C}(-n)$ into an $\mathfrak{sl}_2$-Hodge structure of weight $2n$ by letting $\mathfrak{sl}_2(\mathbb{C})$ act trivially. Then all the identities in Definition A.4.4 can be summarized in one line by saying that the Hermitian form

$$\mathrm{S}\colon H \otimes_{\mathbb{C}} \overline{H} \longrightarrow \mathbb{C}(-n)$$

is a morphism of $\mathfrak{sl}_2$-Hodge structures of central weight $2n$. This shows that the choice of the sign factor b_k in (A.3.8) is the only natural one.

A.5. Pairings on $\mathcal{D}$-modules

Let us return to the cohomology of compact Kähler manifolds, in particular, to the formula for the sesquilinear pairing

$$\mathrm{S}_k\colon H_k \otimes_{\mathbb{C}} \overline{H_{-k}} \longrightarrow \mathbb{C}, \quad \mathrm{S}_k(\alpha,\beta) = (-1)^n c_{n+k}\,\varepsilon(n+k)\cdot \frac{1}{(2\pi\mathsf{i})^n}\int_X \alpha\wedge\overline{\beta}.$$

The sign factor $(-1)^n c_{n+k}\,\varepsilon(n+k)$ in this formula represents an interesting puzzle, whose solution is another important step in finding the correct sign conventions for Hodge modules, especially for direct images.

Recall that $H_k = H^{n+k}(X,\mathbb{C})$ is isomorphic to the $(n+k)$-th hypercohomology of the holomorphic de Rham complex $\mathrm{DR}(\mathcal{O}_X)$; this is the complex

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\ \mathrm{d}\ } \Omega^1_X \xrightarrow{\ \mathrm{d}\ } \cdots \xrightarrow{\ \mathrm{d}\ } \Omega^n_X \longrightarrow 0,$$

which naturally lives in degrees $0,\dots,n$. Equivalently, H_k is the k-th hypercohomology of the shifted de Rham complex $\mathrm{DR}(\mathcal{O}_X)[n]$; under the Koszul sign rule, the differential in the complex $\mathrm{DR}(\mathcal{O}_X)[n]$ has to be $(-1)^n d$.

Now the left $\mathcal{D}_X$-module $\mathcal{O}_X$ comes with a natural Hermitian pairing, given by taking two local sections $f,g\in\mathcal{O}_X$ to the product $f\overline{g}$. What should the corresponding pairing on the right $\mathcal{D}_X$-module ω_X be? The correct answer to this question turns out to be

$$\text{(A.5.1)}\qquad \mathrm{S}_X : \omega_X \otimes_{\mathbb{C}} \overline{\omega_X} \longrightarrow \mathfrak{C}_X, \quad \bigl\langle \mathrm{S}_X(\omega',\omega''),\varphi\bigr\rangle = \frac{\varepsilon(n+1)}{(2\pi\mathsf{i})^n}\int_X \varphi\cdot\omega'\wedge\overline{\omega''},$$

where $\mathfrak{C}_X$ is the sheaf of currents of maximal degree, and $\varepsilon(k) = (-1)^{k(k-1)/2}$. Note that S_X is a morphism of right $\mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_{\overline{X}}$-modules. It is also Hermitian symmetric and, with the sign factor $\varepsilon(n+1)$ in front, positive definite: if the test function φ is real-valued and nonnegative, then

$$\bigl\langle \mathrm{S}_X(\omega,\omega),\varphi\bigr\rangle \geqslant 0,$$

and equality for every φ implies that $\omega = 0$. Following the general principle that currents should be defined to be positive where possible, this is clearly the most natural choice for the pairing on ω_X.

Note. With this definition of S_X, the induced pairing on the space $H^{n,0}(X) = H^0(X,\omega_X)$ is already positive definite. If we do not want to add any additional sign factors, then we need to use $c_k = 1$ and not $c_k = (-1)^k$; in other words, in a polarized Hodge structure, the sign of the polarization on the subspace $H^{p,q}$ should be $(-1)^q$. We will see below that this choice works well in all cases.

Back to the puzzle of the sign factor $(-1)^n\,\varepsilon(n+k)$. We have

$$(-1)^n\,\varepsilon(n+k) = (-1)^n\,\varepsilon(n)\,\varepsilon(k)(-1)^{nk} = \varepsilon(n+1)\,\varepsilon(k)(-1)^{nk},$$

which means that we can write the sesquilinear pairing from above as

$$\mathrm{S}_k(\alpha,\beta) = \varepsilon(k)\cdot(-1)^{nk}\cdot\frac{\varepsilon(n+1)}{(2\pi\mathsf{i})^n}\int_X \alpha\wedge\overline{\beta}. \tag{A.5.2}$$

The third factor is consistent with the pairing S_X on the right $\mathcal{D}_X$-module ω_X, and the first factor $\varepsilon(k)$ only depends on the degree of the cohomology (which is what we need to get a pairing that is embedding-independent); the question is where the extra factor of $(-1)^{nk}$ comes from. Deligne gave a technical answer in a letter to Saito (in terms of tensor products and shifts of complexes), but a more natural answer in our setting is that it is caused by the conversion between right and left $\mathcal{D}$-modules. Namely, in order to convert the natural pairing on the *right* $\mathcal{D}_X$-module ω_X into a pairing on the de Rham complex of the *left* $\mathcal{D}_X$-module $\mathcal{O}_X$, some sign changes are needed, and these sign changes nicely account for the factor $(-1)^{nk}$ in the above formula.

Since this is an important issue, we shall spend the remainder of this section going through the details. To begin with, we describe a naive way for getting a pairing on cohomology, in the setting of right $\mathcal{D}$-modules. Let $\mathcal{M}$ be a right $\mathcal{D}_X$-module, and suppose that we have a flat Hermitian pairing

$$\mathrm{S}\colon \mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}} \longrightarrow \mathfrak{C}_X\,.$$

We use the notation $\mathrm{Sp}_X(\mathcal{M})$ for the Spencer complex

$$0\longrightarrow \mathcal{M}\otimes_{\mathcal{O}_X}\wedge^n\Theta_X \xrightarrow{\ \delta\ } \cdots \xrightarrow{\ \delta\ } \mathcal{M}\otimes_{\mathcal{O}_X}\Theta_X \xrightarrow{\ \delta\ } \mathcal{M}\longrightarrow 0,$$

which naturally lives in degrees $-n,\dots,0$. The formula for the differential is

$$\delta(m\otimes\partial_J) = \sum_{i=1}^{p}(-1)^{i-1}(m\partial_{j_i})\otimes\partial_{J\setminus\{j_i\}},$$

where, given an ordered index set $J=\{j_1,\dots,j_p\}$ with $j_1<\cdots<j_p$, we set

$$\partial_J = \partial_{j_1}\wedge\cdots\wedge\partial_{j_p}\quad\text{and}\quad \mathrm{d}x_J = \mathrm{d}x_{j_1}\wedge\cdots\wedge\mathrm{d}x_{j_p}$$

Here and elsewhere, we always stick to the Koszul sign rule: on the i-th term in the sum, we need to commute ∂_{j_i} past $(i-1)$ other vector fields, hence the sign of $(-1)^{i-1}$. We are going to write out all the formulas involving signs in what follows, to be sure that everything works out correctly.

Now for the definition of the naive pairing. The tensor product

$$\mathrm{Sp}_X(\mathcal{M})\otimes_{\mathbb{C}}\overline{\mathrm{Sp}_X(\mathcal{M})}$$

is naturally a double complex, with term in bidegree $(-p,-q)$ given by

$$\Big(\mathcal{M}\otimes_{\mathcal{O}_X}\wedge^p\Theta_X\Big)\otimes_{\mathbb{C}}\Big(\overline{\mathcal{M}}\otimes_{\mathcal{O}_{\overline{X}}}\wedge^q\Theta_{\overline{X}}\Big)\cong\Big(\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\Big)\otimes_{\mathcal{O}_{X,\overline{X}}}\Big(\wedge^p\Theta_X\otimes_{\mathbb{C}}\wedge^q\Theta_{\overline{X}}\Big).$$

Here $\mathcal{O}_{X,\overline{X}}$ is a convenient shorthand for the sheaf of algebras $\mathcal{O}_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}$. The associated simple complex, with Deligne's sign rule for the differential, lives in degrees $-2n,\dots,0$, and its term in degree $-k$ is

$$\big(\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\big)\otimes_{\mathcal{O}_{X,\overline{X}}}\wedge^k\Theta_{X,\overline{X}}.$$

To simplify the notation, we have introduced the additional sheaf

$$\Theta_{X,\overline{X}}=\big(\Theta_X\otimes_{\mathbb{C}}\mathcal{O}_{\overline{X}}\big)\oplus\big(\mathcal{O}_X\otimes_{\mathbb{C}}\Theta_{\overline{X}}\big),$$

which is locally free of rank $2n$ over $\mathcal{O}_{X,\overline{X}}$; in the formula above, the wedge product is over $\mathcal{O}_{X,\overline{X}}$. We denote the associated simple complex by the symbol

$$\mathrm{Sp}_{X,\overline{X}}\big(\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\big),$$

because the formula for the differential is exactly the same as in the usual Spencer complex, but where $\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}$ is now considered as a right module over $\mathcal{D}_{X,\overline{X}}=\mathcal{D}_X\otimes_{\mathbb{C}}\mathcal{D}_{\overline{X}}$, and where Θ_X is replaced by $\Theta_{X,\overline{X}}$.

A.5.3. Example. Indeed, say we have a local section

$$m'\otimes m''\otimes\partial_J\otimes\overline{\partial}_K,$$

with $|J|=p$ and $|K|=q$; it lives in bidegree $(-p,-q)$ in the double complex, and in degree $-(p+q)$ in the associated simple complex. Under Deligne's sign conventions, the differential of the simple complex takes this element to

$$\sum_{i=1}^{p}(-1)^{i-1}(m'\partial_{j_i})\otimes m''\otimes\partial_{J\setminus\{j_i\}}\otimes\overline{\partial}_K \\ +(-1)^p\sum_{i=1}^{q}(-1)^{i-1}m'\otimes(m''\overline{\partial}_{k_i})\otimes\partial_J\otimes\overline{\partial}_{K\setminus\{k_i\}}.$$

But this is exactly the image of $m'\otimes m''\otimes\partial_J\wedge\overline{\partial}_K$ under the differential of the Spencer complex, and so the notation $\mathrm{Sp}_{X,\overline{X}}(\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}})$ is justified.

Since our Hermitian pairing

$$\mathrm{S}\colon\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\longrightarrow\mathfrak{C}_X$$

is a morphism of right $\mathcal{D}_{X,\overline{X}}$-modules, it induces a morphism of complexes

$$\mathrm{Sp}_{X,\overline{X}}\big(\mathcal{M}\otimes_{\mathbb{C}}\overline{\mathcal{M}}\big)\longrightarrow\mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X),$$

simply by applying S termwise. The net result is that we have a morphism

$$\mathrm{Sp}_X(\mathcal{M})\otimes_{\mathbb{C}}\overline{\mathrm{Sp}_X(\mathcal{M})}\longrightarrow\mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X).\tag{A.5.4}$$

The Poincaré lemma for distributions implies that the complex $\mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X)$ is a fine resolution of the constant sheaf $\mathbb{C}[2n]$. So the morphism in (A.5.4) induces, without any further work, sesquilinear pairings

$$H^k\big(X, \mathrm{Sp}_X(\mathcal{M})\big) \otimes_{\mathbb{C}} \overline{H^{-k}\big(X, \mathrm{Sp}_X(\mathcal{M})\big)} \longrightarrow H^{2n}(X, \mathbb{C}) \cong \mathbb{C}$$

on the level of cohomology. In fact, one can be more precise about the identification between $H^{2n}(X, \mathbb{C})$ and $\mathbb{C}$: the isomorphism

$$H^0\big(X, \mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X)\big) \xrightarrow{\cong} \mathbb{C}$$

is given by evaluating currents on the constant test function 1.

Now we can formulate the answer to the puzzle in a more precise way: the Hermitian pairing S on the right $\mathcal{D}_X$-module ω_X induces naive pairings between the cohomology spaces

$$H_k \cong H^k\big(X, \mathrm{Sp}_X(\omega_X)\big),$$

and the claim is that this procedure explains the mysterious factor $(-1)^{nk}$ in (A.5.2). To understand why, we need to work through the conversion between the Spencer complex $\mathrm{Sp}_X(\omega_X)$ and the (shifted) de Rham complex $\mathrm{DR}(\mathcal{O}_X)[n]$. That is to say, we need to a formula for the pairing on the de Rham complex, induced by the naive pairing

$$\mathrm{Sp}_X(\omega_X) \otimes_{\mathbb{C}} \overline{\mathrm{Sp}_X(\omega_X)} \longrightarrow \mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X)$$

under the isomorphism between the de Rham complex and the Spencer complex. As before, it is important to use the Koszul sign rule consistently.

To keep the notation simple, let us suppose more generally that $\mathcal{N}$ is any left $\mathcal{D}_X$-module. Its de Rham complex is the complex

$$0 \longrightarrow \mathcal{N} \xrightarrow{\nabla} \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{N} \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^n_X \otimes_{\mathcal{O}_X} \mathcal{N} \longrightarrow 0,$$

which naturally lives in degrees $0, \dots, n$. We shall insist on using the notation $\mathrm{DR}_X(\mathcal{N})[n]$ for the shifted de Rham complex, as a reminder that the differential in this complex is $(-1)^n\nabla$. Concretely, the formula for the differential is

$$\mathrm{d}x_J \otimes m \longmapsto (-1)^n(-1)^{|J|} \sum_{j=1}^n \mathrm{d}x_J \wedge \mathrm{d}x_j \otimes (\partial_j m),$$

where the $(-1)^{|J|}$ comes from the fact that we had to move the differential in the complex (which has degree 1) past the form $\mathrm{d}x_J$.

A.5.5. Lemma. *The shifted de Rham complex of $\mathcal{N}$ is isomorphic to the Spencer complex of the associated right $\mathcal{D}_X$-module $\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}$.*

Proof. Since it matters in what follows, let us carefully write down the exact formulas for the morphism of complexes

$$\mathrm{Sp}_X(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \longrightarrow \mathrm{DR}_X(\mathcal{N})[n].$$

They are determined by the condition that, in degree zero, we want the morphism $\omega_X \otimes_{\mathcal{O}_X} \mathcal{N} \to \Omega_X^n \otimes_{\mathcal{O}_X} \mathcal{N}$ to be the identity. This forces us to define

$$(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{O}_X} \wedge^p \Theta_X \longrightarrow \Omega_X^{n-p} \otimes_{\mathcal{O}_X} \mathcal{N}$$

by the following rule:

$$\omega \otimes m \otimes \partial_J \longmapsto (-1)^{(n-j_1)+\cdots+(n-j_p)} \mathrm{d}x_{J^c} \otimes m \tag{A.5.6}$$

Here $J = \{j_1, \dots, j_p\}$ is an ordered index set, $J^c = \{1, \dots, n\} \setminus J$ is the complement, with the natural ordering, and $\omega = \mathrm{d}x_1 \wedge \cdots \wedge \mathrm{d}x_n$. The sign factor is explained by the fact that we have to move ∂_{j_1} past $\mathrm{d}x_{j_1+1}, \dots, \mathrm{d}x_n$, before we can contract it against $\mathrm{d}x_{j_1}$, causing a factor of $(-1)^{n-j_1}$ to appear, and so on.

Let us verify that (A.5.6) really defines a morphism of complexes: each square

$$\begin{tikzcd}
(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{O}_X} \wedge^p \Theta_X \arrow[r] \arrow[d, "\delta"'] & \Omega_X^{n-p} \otimes_{\mathcal{O}_X} \mathcal{N} \arrow[d, "(-1)^n \nabla"] \\
(\omega_X \otimes_{\mathcal{O}_X} \mathcal{N}) \otimes_{\mathcal{O}_X} \wedge^{p-1} \Theta_X \arrow[r] & \Omega_X^{n-p+1} \otimes_{\mathcal{O}_X} \mathcal{N}
\end{tikzcd}$$

commutes. Starting from $\omega \otimes m \otimes \partial_J$ with $|J| = p$, and going along the arrows on the top and right, we obtain

$$\begin{aligned}
&(-1)^{(n-j_1)+\cdots+(n-j_p)}(-1)^n(-1)^{|J^c|} \sum_{j=1}^n \mathrm{d}x_{J^c} \wedge \mathrm{d}x_j \otimes (\partial_j m) \\
&\qquad = (-1)^{(n+1)p}(-1)^{j_1+\cdots+j_p} \sum_{i=1}^p \mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i} \otimes (\partial_{j_i} m).
\end{aligned} \tag{A.5.7}$$

Going along the arrow on the left, we obtain

$$\sum_{i=1}^p (-1)^{i-1}(\omega \otimes m)\partial_{j_i} \otimes \partial_{J\setminus\{j_i\}} = \sum_{i=1}^p (-1)^i \omega \otimes (\partial_{j_i} m) \otimes \partial_{J\setminus\{j_i\}},$$

and the arrow on the bottom turns this into

$$\begin{aligned}
&(-1)^{(n-j_1)+\cdots+(n-j_p)} \sum_{i=1}^p (-1)^i (-1)^{n-j_i} \mathrm{d}x_{(J\setminus\{j_i\})^c} \otimes (\partial_{j_i} m) \\
&= (-1)^{np}(-1)^{j_1+\cdots+j_p} \sum_{i=1}^p (-1)^p \mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i} \otimes (\partial_{j_i} m).
\end{aligned} \tag{A.5.8}$$

The point is that $\mathrm{d}x_{(J\setminus\{j_i\})^c} = (-1)^{(p-i)+(n-j_i)} \mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i}$, because putting the expression $\mathrm{d}x_{J^c} \wedge \mathrm{d}x_{j_i}$ into the correct order requires moving $\mathrm{d}x_{j_i}$ past a form of degree $(n-j_i)-(p-i)$. In any case, the two expressions in (A.5.7) and (A.5.8) are equal, and so we do have a morphism of complexes. $\square$

For the same reason, we have an isomorphism of complexes

$$\mathrm{Sp}_{X,\overline{X}}(\mathfrak{C}_X) \longrightarrow \mathrm{DR}_{X,\overline{X}}(\mathfrak{Db}_X)[2n],$$

where $\mathfrak{Db}_X$ is the sheaf of distributions on X, considered as a left module over $\mathcal{D}_{X,\overline{X}}$, and where the (shifted) de Rham complex is defined in the same way as for $\mathcal{D}_X$-modules, but using the wedge powers of the locally free $\mathcal{O}_{X,\overline{X}}$-module

$$\Omega^1_{X,\overline{X}} = \bigl(\Omega^1_X \otimes_{\mathbb{C}} \mathcal{O}_{\overline{X}}\bigr) \oplus \bigl(\mathcal{O}_X \otimes_{\mathbb{C}} \Omega^1_{\overline{X}}\bigr),$$

and the differential $(-1)^{2n}\nabla$. Concretely, the morphism of complexes is defined on the terms in degree $-k$, which are

$$\mathfrak{C}_X \otimes_{\mathcal{O}_{X,\overline{X}}} \wedge^k \Theta_{X,\overline{X}} \longrightarrow \Omega^{2n-k}_{X,\overline{X}} \otimes_{\mathcal{O}_{X,\overline{X}}} \mathfrak{Db}_X,$$

by the following formula (dictated by the Koszul sign rule): write a given current locally as $D\omega \wedge \overline{\omega}$, for a unique distribution D; then

$$\begin{aligned} &\bigl(D\omega \wedge \overline{\omega}\bigr) \otimes \partial_J \wedge \overline{\partial}_K \longmapsto \\ &\qquad (-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)}(-1)^{nq} \mathrm{d}x_{J^c} \wedge \mathrm{d}\overline{x}_{K^c} \otimes D \end{aligned} \tag{A.5.9}$$

where $|J| = p$ and $|K| = q$, and $p+q = k$. The sign factor is again explained by the number of swaps that are needed to move everything into the right place, which is $(2n-j_1)+\cdots+(2n-j_p)+(n-k_1)+\cdots+(n-k_q)$.

We can now derive a formula for the induced pairing

$$\mathrm{DR}_X(\mathcal{O}_X)[n] \otimes_{\mathbb{C}} \overline{\mathrm{DR}_X(\mathcal{O}_X)[n]} \longrightarrow \mathrm{DR}_{X,\overline{X}}(\mathfrak{Db}_X). \tag{A.5.10}$$

Take two local sections $\alpha = \mathrm{d}x_{J^c}$ and $\beta = \mathrm{d}x_{K^c}$, where $|J| = p$ and $|K| = q$. Under the isomorphism $\mathrm{DR}_X(\mathcal{O}_X)[n] \cong \mathrm{Sp}_X(\omega_X)$ in Lemma A.5.5, the holomorphic $(n-p)$-form α goes to

$$(-1)^{np}(-1)^{j_1+\cdots+j_p} \cdot \omega \otimes \partial_J,$$

and the holomorphic $(n-q)$-form β goes to

$$(-1)^{nq}(-1)^{k_1+\cdots+k_q} \cdot \omega \otimes \partial_K.$$

The naive pairing on $\mathrm{Sp}_X(\omega_X)$ takes those two sections to

$$(-1)^{n(p+q)}(-1)^{(j_1+\cdots+j_p)+(k_1+\cdots+k_q)} S(\omega,\omega) \otimes \partial_J \wedge \overline{\partial}_K,$$

where S is defined in (A.5.1). Now $\mathrm{S}(\omega,\omega) = D\omega \wedge \overline{\omega}$, where D is the distribution

$$D = \frac{\varepsilon(n+1)}{(2\pi\mathrm{i})^n} \int_X \in H^0(X, \mathfrak{Db}_X).$$

Under the isomorphism in (A.5.9), the section from above therefore goes to

$$(-1)^{np} \mathrm{d}x_{J^c} \wedge \mathrm{d}\overline{x}_{K^c} \otimes D = (-1)^{n(\deg\alpha - n)} \alpha \wedge \overline{\beta} \otimes D.$$

The formula we have just derived also works for smooth forms. In other words, the same formula can be used to extend (A.5.10) to a pairing on the de Rham complex of smooth forms (which is the usual Dolbeault resolution used to compute cohomology). The resulting pairings on cohomology

$$H^{n+k}(X,\mathbb{C}) \otimes_{\mathbb{C}} \overline{H^{n-k}(X,\mathbb{C})} \longrightarrow \mathbb{C}$$

are of course given by the same formula

$$(\alpha, \beta) \longmapsto (-1)^{n(\deg \alpha - n)} \frac{\varepsilon(n+1)}{(2\pi \mathrm{i})^n} \int_X \alpha \wedge \overline{\beta}.$$

Since $\deg \alpha = n+k$, we have succeeded in explaining the mysterious sign factor $(-1)^{nk}$ in (A.5.2), in a very natural way!

Let us summarize the result of this rather lengthy computation. If we define the Hermitian pairing S_X on the right $\mathcal{D}_X$-module ω_X as in (A.5.1), and if we use the naive pairing on the Spencer complex, we obtain a collection of pairings

$$\mathrm{S}_k \colon H^k\big(X, \mathrm{Sp}_X(\omega_X)\big) \otimes_{\mathbb{C}} \overline{H^{-k}\big(X, \mathrm{Sp}_X(\omega_X)\big)} \longrightarrow \mathbb{C},$$

with all signs dictated by the Koszul sign rule alone. The conclusion is then that these pairings polarize the $\mathfrak{sl}_2$-Hodge structure of weight $n = \dim X$ on the graded vector space

$$\bigoplus_{k \in \mathbb{Z}} H^k\big(X, \mathrm{Sp}_X(\omega_X)\big),$$

provided that we multiply the k-th pairing S_k by the factor $\varepsilon(k)$. This is good news, because it describes the $\mathfrak{sl}_2$-Hodge structure and its polarization in a way that does not mention the dimension of the compact Kähler manifold X, a crucial point if we want a theory that is independent of the choice of ambient complex manifold.

A.6. Direct images

It is now an easy matter to figure out the sign conventions for direct images. Since every morphism between complex manifolds factors into a closed embedding followed by a projection, we only need to consider those two cases.

The first case is that of a closed embedding $i \colon X \hookrightarrow Y$. Suppose that $\mathcal{M}$ is a coherent right $\mathcal{D}_X$-module, and $\mathrm{S} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ a Hermitian pairing. Let

$$\mathcal{D}_{X \to Y} = \mathcal{O}_X \otimes_{i^{-1}\mathcal{O}_Y} i^{-1}\mathcal{D}_Y$$

be the transfer module, which is a $(\mathcal{D}_X, i^{-1}\mathcal{D}_Y)$-bimodule. The direct image

$$i_+\mathcal{M} = i_*\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}\big)$$

is a coherent right $\mathcal{D}_Y$-module. There is an induced Hermitian pairing

$$i_+\mathrm{S} \colon i_+\mathcal{M} \otimes_{\mathbb{C}} \overline{i_+\mathcal{M}} \longrightarrow \mathfrak{C}_Y,$$

that can be described in a coordinate-free way as follows. Since the tensor product over $\mathbb{C}$ is exact, we have a natural isomorphism

$$\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}\big) \otimes_{\mathbb{C}} \overline{\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X \to Y}\big)} \cong \big(\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}}\big) \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X \to Y, \overline{X} \to \overline{Y}},$$

where $\mathcal{D}_{X,\overline{X}} = \mathcal{D}_X \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X}}$ and $\mathcal{D}_{X \to Y, \overline{X} \to \overline{Y}} = \mathcal{D}_{X \to Y} \otimes_{\mathbb{C}} \mathcal{D}_{\overline{X} \to \overline{Y}}$. Applying the sheaf theoretic direct image i_*, and composing with $\mathrm{S} \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$, we get

$$i_+\mathcal{M} \otimes_{\mathbb{C}} \overline{i_+\mathcal{M}} \longrightarrow i_*\big(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X \to Y, \overline{X} \to \overline{Y}}\big).$$

Pushforward of currents defines a morphism $i_* \mathfrak{C}_X \to \mathfrak{C}_Y$, according to the rule

$$\langle i_* C, \varphi \rangle = \langle C, i^* \varphi \rangle.$$

From this, we obtain another natural morphism

$$i_*\big(\mathfrak{C}_X \otimes_{\mathcal{D}_{X,\overline{X}}} \mathcal{D}_{X\to Y,\overline{X}\to\overline{Y}}\big) \longrightarrow \mathfrak{C}_Y, \quad C \otimes (f \otimes P) \otimes (\bar{g} \otimes \bar{Q}) \longmapsto i_*(Cf\bar{g}) \cdot P\bar{Q}.$$

After composing the two morphisms, we arrive at the desired Hermitian pairing

$$i_+\mathrm{S}\colon i_+\mathcal{M} \otimes_{\mathbb{C}} \overline{i_+\mathcal{M}} \longrightarrow \mathfrak{C}_Y .$$

All of the currents in the image are supported on X; for two sections in the subsheaf $i_*\mathcal{M}$, the current is just obtained by pushforward from X to Y, but in general, the construction involves some derivatives in directions normal to Y.

The second case is that of a projection $f\colon X \to Y$, say with $X = F \times Y$ and $f = p_2$. Let $\mathcal{M}$ be a coherent right $\mathcal{D}_X$-module, and $\mathrm{S}\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ a Hermitian pairing. The direct image

$$f_+\mathcal{M} = \mathbf{R}f_*\big(\mathcal{M} \otimes_{\mathcal{D}_X} \mathcal{D}_{X\to Y}\big)$$

is computed using the relative Spencer complex $\mathrm{Sp}_f(\mathcal{M})$. This is the complex

$$0 \longrightarrow \mathcal{M} \otimes_{p_1^{-1}\mathcal{O}_F} \wedge^r p_1^{-1}\Theta_F \xrightarrow{\delta} \cdots \xrightarrow{\delta} \mathcal{M} \otimes_{p_1^{-1}\mathcal{O}_F} p_1^{-1}\Theta_F \xrightarrow{\delta} \mathcal{M} \longrightarrow 0,$$

which naturally lives in degrees $-r, \dots, 0$, where $r = \dim F$; the formula for the differential is the same as in the absolute case. By a similar construction as in the previous section, we obtain a naive pairing on the complex $\mathrm{Sp}_f(\mathcal{M})$, which we may write by analogy with the absolute case as

$$\mathrm{Sp}_f(\mathcal{M}) \otimes_{\mathbb{C}} \overline{\mathrm{Sp}_f(\mathcal{M})} \longrightarrow \mathrm{Sp}_{f,\overline{f}}(\mathfrak{C}_X).$$

Here the complex on the right-hand side lives in degrees $-2r, \dots 0$, and with similar notation as in the previous section, the term in degree $-k$ looks like

$$\mathfrak{C}_X \otimes_{p_1^{-1}\mathcal{O}_{F,\overline{F}}} \wedge^k p_1^{-1}\Theta_{F,\overline{F}}.$$

By a relative version of the Poincaré lemma for distributions, this complex is a fine resolution of $f^{-1}\mathfrak{C}_Y[2r]$, and and so we obtain induced sesquilinear pairings

$$\mathrm{S}_k\colon R^k f_* \,\mathrm{Sp}_f(\mathcal{M}) \otimes_{\mathbb{C}} \overline{R^{-k} f_* \,\mathrm{Sp}_f(\mathcal{M})} \longrightarrow \mathfrak{C}_Y .$$

The isomorphism $R^{2r} f^{-1} \mathfrak{C}_Y \cong \mathfrak{C}_Y$ is given, in terms of the explicit fine resolution from above, simply by pushforward of currents.

Now for the general case. Suppose that $f\colon X \to Y$ is a holomorphic mapping between two complex manifolds. Let $\mathcal{M}$ be a coherent right $\mathcal{D}_X$-module, and $\mathrm{S}\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_X$ a Hermitian pairing. Suppose that f is proper, or at least proper on the support of $\mathcal{M}$. By factoring f as

$$X \xrightarrow{i} X \times Y \xrightarrow{p_2} Y$$

and applying the two constructions from above, we obtain a collection of induced sesquilinear pairings

$$\mathrm{S}_k\colon \mathcal{H}^k f_+\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{H}^{-k} f_+\mathcal{M}} \longrightarrow \mathfrak{C}_Y$$

The sign conventions are then easy to state: for each $k \in \mathbb{Z}$, we should multiply the naive pairing S_k by the factor $\varepsilon(k) = (-1)^{k(k-1)/2}$. In the special case of a closed embedding, this means that we simply use the pairing $i_+\mathrm{S}$ induced by pushforward of currents (because $\varepsilon(0) = 1$). This convention is suggested by the analysis in the previous section. The direct image theorem for polarized Hodge modules then takes the following form:

A.6.1. Theorem. *Let $f\colon X \to Y$ be a projective morphism between two complex manifolds. Let $M \in \mathsf{HM}(X, w)$ be a polarized Hodge module of weight w. Then*

$$\bigoplus_{k\in\mathbb{Z}} H^k f_* M$$

is a polarized $\mathfrak{sl}_2$-Hodge module of weight w; here $\mathrm{X} \in \mathfrak{sl}_2(\mathbb{C})$ acts as $2\pi\mathsf{i}\, L_\omega$, and the polarization is given by the sesquilinear pairings $\varepsilon(k)\mathrm{S}_k$ from above.

A.7. Variations of Hodge structure and polarizations

Recall that a *variation of Hodge structure* of weight n on a complex manifold X is a smooth vector bundle E with a decomposition into smooth subbundles

$$E = \bigoplus_{p+q=n} E^{p,q},$$

and a flat connection $d\colon A^0(X, E) \to A^1(X, E)$ that maps the space of sections $A^0(X, E^{p,q})$ of the subbundle $E^{p,q}$ into the direct sum

$$A^{1,0}(X, E^{p,q}) \oplus A^{1,0}(X, E^{p-1,q+1}) \oplus A^{0,1}(X, E^{p,q}) \oplus A^{0,1}(X, E^{p+1,q-1}).$$

Note that we are describing the connection in terms of its action on the space of smooth sections of E; equivalently, we could consider d as a morphism from the sheaf of smooth sections of E to the sheaf of smooth 1-forms with coefficients in E. Lastly, a *polarization* of E is a Hermitian form

$$\mathrm{S}_E\colon A^0(X, E) \otimes_{A^0(X)} \overline{A^0(X, E)} \longrightarrow A^0(X)$$

that is flat with respect to d, and whose restriction to each fiber E_x polarizes the Hodge structure of weight n on the vector space

$$E_x = \bigoplus_{p+q=n} E_x^{p,q}.$$

From the polarization, we obtain a smooth Hermitian metric h_E on the bundle E, called the *Hodge metric*, by setting

$$h_E(v, w) = c_n \sum_{p+q=n} (-1)^q \mathrm{S}_E(v^{p,q}, w^{p,q}).$$

Most of the results about variations of Hodge structure are, directly or indirectly, statements about the Hodge metric.

If we decompose the connection by type as $d = d' + d''$, then the $(0,1)$-part d'' gives E the structure of a holomorphic vector bundle that we denote by the symbol $\mathcal{E}$, and the $(1,0)$-part d' defines a flat holomorphic connection

$$\nabla \colon \mathcal{E} \longrightarrow \Omega_X^1 \otimes_{\mathcal{O}_X} \mathcal{E}.$$

The condition above says that the Hodge bundles

$$F^p E = E^{p,q} \oplus E^{p+1,q-1} \oplus E^{p+2,q-2} \oplus \cdots$$

have the structure of holomorphic subbundles $F^p \mathcal{E} \subseteq \mathcal{E}$, and that the holomorphic connection ∇ satisfies the Griffiths transversality condition

$$\nabla(F^p\mathcal{E}) \subseteq \Omega_X^1 \otimes_{\mathcal{O}_X} F^{p-1}\mathcal{E}. \tag{A.7.1}$$

The process for converting a polarized variation of Hodge structure of weight n into a polarized Hodge module of weight $n + \dim X$ is as follows. First, consider the associated right $\mathcal{D}_X$-module

$$\mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} \mathcal{E},$$

with the action by vector fields defined in terms of the connection as

$$(\omega \otimes s) \cdot \xi = (\omega \cdot \xi) \otimes s - \omega \otimes \nabla_\xi s.$$

The Hodge filtration on $\mathcal{E}$ defines an increasing filtration

$$F_\bullet \mathcal{M} = \omega_X \otimes_{\mathcal{O}_X} F^{-\bullet - \dim X}\mathcal{E},$$

which is compatible with the $\mathcal{D}_X$-module structure because of (A.7.1).

Note. The shift by $\dim X$ is necessary in order to make the isomorphism in Lemma A.5.5 between the Spencer complex $\mathrm{Sp}_X(\mathcal{M})$ and the shifted de Rham complex $\mathrm{DR}_X(\mathcal{E})$ into a *filtered* isomorphism.

Finally, we should define the Hermitian pairing

$$\mathrm{S}_M \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \longrightarrow \mathfrak{C}_X$$

by the following formula, suggested by (A.5.1):

$$\big\langle \mathrm{S}_M(\omega' \otimes s', \omega'' \otimes s''), \varphi \big\rangle = \frac{\varepsilon(n+1)}{(2\pi \mathrm{i})^n} \int_X \varphi \cdot \mathrm{S}_E(s', s'')\, \omega' \wedge \overline{\omega''} \tag{A.7.2}$$

Assuming that X is compact, and that p is the largest integer such that $F^p\mathcal{E} \neq 0$, the induced pairing on the space $H^0(X, \omega_X \otimes F^p\mathcal{E})$ is then $c_n(-1)^{n-p}$-positive definite with this definition.

A.8. Degenerating variations of Hodge structure

In this section, we are going to check our sign conventions against another real-world example: polarized variations of Hodge structure on the punctured disk. This is another instance where $\mathfrak{sl}_2$-Hodge structures appear, and we will see that the sign conventions we have developed so far also work nicely in this case.

Using the notation from the previous section, let us consider a polarized variation of Hodge structure E of weight n on the punctured unit disk

$$\Delta^* = \{t \in \mathbb{C} \mid 0 < |t| < 1\}.$$

In order to have a fixed reference frame, we introduce the complex vector space V of all multivalued flat section of (E, d); equivalently, these are the flat sections of the pullback $\exp^* E$ to the universal covering space $\exp\colon \mathbb{H} \to \Delta^*$. Note that the universal covering space of Δ^* is naturally the *left* half plane

$$\mathbb{H} = \{z \in \mathbb{C} \mid \operatorname{Re} z < 0\},$$

and the group of deck transformations is

$$\mathbb{Z}(1) = \{z \in \mathbb{C} \mid e^z = 1\} = (2\pi \mathsf{i})\mathbb{Z} \subseteq \mathbb{C}.$$

Translation by elements of $\mathbb{Z}(1)$ defines a group homomorphism

$$\rho\colon \mathbb{Z}(1) \longrightarrow \mathrm{GL}(V);$$

to be specific, Schmid's convention is that $\rho(\zeta)$ takes a flat section $v(z)$ of the bundle $\exp^* E$ to the flat section $v(z - \zeta)$. In particular, we have the (positively oriented) monodromy transformation

$$T = \rho(2\pi \mathsf{i}) \in \mathrm{GL}(V),$$

which depends on the choice of $i = \sqrt{-1}$. If we write its Jordan decomposition in the form

$$T = T_s \cdot e^{2\pi \mathsf{i}\, N},$$

with $T_s \in \mathrm{GL}(V)$ semisimple, $N \in \mathrm{End}(V)$ nilpotent, and $[T_s, N] = 0$, then N is independent of the choice of i. The polarization induces a Hermitian pairing

$$\mathrm{S}\colon V \otimes_{\mathbb{C}} \overline{V} \longrightarrow \mathbb{C},$$

and since T preserves the pairing, one easily checks that

$$\mathrm{S} \circ (T_s \otimes T_s) = \mathrm{S} \quad \text{and} \quad \mathrm{S} \circ (N \otimes \mathrm{Id}) = \mathrm{S} \circ (\mathrm{Id} \otimes N).$$

According to the *monodromy theorem*, all eigenvalues of the monodromy transformation T have absolute value 1. After fixing an interval $[\alpha, \alpha + 1) \subseteq \mathbb{R}$, we can therefore write the semisimple operator T_s uniquely as

$$T_s = e^{2\pi \mathsf{i}\, S_\alpha},$$

where $S_\alpha \in \mathrm{End}(V)$ is semisimple with real eigenvalues contained in $[\alpha, \alpha + 1)$. We then have $T = e^{2\pi \mathsf{i}(S_\alpha + N)}$, and the operator $S_\alpha + N$ in the exponent does not depend on the choice of i.

A.8.1. Example. The definition of the monodromy operator appears unmotivated – why not use $v(z+\zeta)$ instead? – but the operator $S_\alpha+N$ does have a natural interpretation in terms of the connection. Let $\widetilde{\mathcal{E}}^\alpha$ be the canonical extension of $(\mathcal{E},\nabla)$, characterized by the property that ∇ extends to a logarithmic connection

$$\nabla\colon \widetilde{\mathcal{E}}^\alpha \longrightarrow \Omega^1_\Delta(\log 0)\otimes_{\mathcal{O}} \widetilde{\mathcal{E}}^\alpha$$

whose residue at the origin

$$R_\alpha = \mathrm{Res}_{t=0}(\nabla) \in \mathrm{End}\big(\widetilde{\mathcal{E}}^\alpha_{|0}\big)$$

has eigenvalues in the interval $[\alpha,\alpha+1)$. There is a distinguished trivialization

$$\mathcal{O}_\Delta \otimes_{\mathbb{C}} \widetilde{\mathcal{E}}^\alpha_{|0} \cong \widetilde{\mathcal{E}}^\alpha,$$

depending only on the choice of coordinate t on the disk, with the property that

$$\nabla(1\otimes v) = \frac{\mathrm{d}t}{t}\otimes R_\alpha v, \quad \text{for } v\in \widetilde{\mathcal{E}}^\alpha_{|0}.$$

After pulling everything back to the universal covering space $\mathbb{H}$, we obtain

$$\nabla(1\otimes v) = dz\otimes R_\alpha v,$$

where ∇ denotes the induced flat holomorphic connection on the pullback of $\mathcal{E}$. A brief computation shows that the expression

$$\sigma_v(z) = e^{-zR_\alpha}(1\otimes v) = \sum_{j=0}^\infty \frac{(-1)^j}{j!} z^j \otimes R_\alpha^j v$$

defines a global section of $\exp^*\mathcal{E}$ that is annihilated by ∇. This sets up an isomorphism between $\widetilde{\mathcal{E}}^\alpha_{|0}$ and the space of multivalued flat sections of $(\mathcal{E},\nabla)$, and so we can describe the canonical extension as

$$\widetilde{\mathcal{E}}^\alpha \cong \mathcal{O}_\Delta\otimes_{\mathbb{C}} V.$$

With this identification, the monodromy transformation is $T=e^{2\pi\mathsf{i}\,R_\alpha}$, because

$$\sigma_v(z-2\pi\mathsf{i}) = e^{2\pi\mathsf{i}\,R_\alpha}\sigma_v(z).$$

It follows that the operator $S_\alpha+N=R_\alpha$ is exactly the residue of the logarithmic connection on $\widetilde{\mathcal{E}}^\alpha$.

The main result is that the vector space V has an $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the Hermitian pairing S. This is not entirely canonical, though, because the representation of $\mathfrak{sl}_2(\mathbb{C})$ depends on the choice of a splitting for the weight filtration. First, recall that the nilpotent operator $N\in\mathrm{End}(V)$ determines the *monodromy weight filtration* $W_\bullet$, which is the unique increasing filtration with $NW_\ell\subseteq W_{\ell-2}$ for all $\ell\in\mathbb{Z}$, such that

$$N^\ell\colon \mathrm{gr}^W_\ell \longrightarrow \mathrm{gr}^W_{-\ell}$$

is an isomorphism for every $\ell\geqslant 0$. The weight filtration governs the asymptotic behavior of the Hodge metric, in the sense that

$$v\in W_\ell\setminus W_{\ell-1} \quad\Longleftrightarrow\quad h(v,v)\sim |\mathrm{Re}\, z|^\ell,$$

at least as long as $|\operatorname{Im} z|$ stays bounded. These asymptotic formulas for the norm of multivalued flat sections are known as the "Hodge norm estimates". Looking at these formulas, a natural idea is to rescale the Hodge metric, in order even out the different powers of $|\operatorname{Re} z|$. For that purpose, we have to choose a *splitting* for the weight filtration. By this, we mean a semisimple operator $H \in \operatorname{End}(V)$ with integer eigenvalues, such that

$$W_\ell = E_\ell(H) \oplus W_{\ell-1} \quad \text{and} \quad [H, N] = -2N.$$

In addition, we can easily arrange that $\mathrm{S} \circ (H \otimes \mathrm{Id}) + \mathrm{S} \circ (\mathrm{Id} \otimes H) = 0$ and that $[H, T_s] = 0$; note that, even with these extra conditions, the splitting H is far from unique in general. For $v \in E_\ell(H)$, we now have

$$e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v = |\operatorname{Re} z|^{-\frac{\ell}{2}} v.$$

It turns out that rescaling by the operator $e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H}$ not only removes the singular behavior of the Hodge metric, but it also makes the family of polarized Hodge structures of weight n converge to a limit.

To describe the convergence, we need to introduce two additional pieces of notation. The first is the *period domain* D. The points of D parametrize all possible Hodge structures

$$V = \bigoplus_{p+q=n} V^{p,q}$$

of weight n on the vector space V that are polarized by the Hermitian form S and have the appropriate set of Hodge numbers $\dim V^{p,q}$. The polarization being fixed, a Hodge structure is uniquely determined by its Hodge filtration

$$F^p V = V^{p,q} \oplus V^{p+1,q-1} \oplus V^{p+2,q-2} \oplus \cdots,$$

and this makes D a subset of the "compact dual" $\check{D}$, the space of all decreasing filtrations on V by subspaces of the appropriate dimensions $\dim F^p V$. The compact dual $\check{D}$ is a compact complex manifold, and a homegeneous space for the complex Lie group $\mathrm{GL}(V)$; the period domain D is an open subset, and a homogeneous space for the real Lie group

$$G = \{g \in \mathrm{GL}(V) \mid \mathrm{S} \circ (g \otimes g) = \mathrm{S}\}.$$

The polarized variation of Hodge structure E determines a *period mapping*

$$\Phi \colon \mathbb{H} \longrightarrow D,$$

where $\Phi(z)$ is the Hodge structure on V induced by the isomorphism $V \cong E_{e^z}$.

Note. For clarity, we are going to use the notation

$$V = \bigoplus_{p+q=n} V^{p,q}_{\Phi(z)}$$

for the Hodge decomposition in the Hodge structure $\Phi(z)$, and $\Phi^p(z)$ for the subspaces in the Hodge filtration. We also write

$$\langle v, w \rangle_{\Phi(z)} = c_n \sum_{p+q=n} (-1)^q \mathrm{S}(v^{p,q}, w^{p,q}) = h_E(v, w)(z)$$

for the resulting Hermitian inner product on V. The action by the real group G works in such a way that $\langle gv, gw\rangle_{g\Phi(z)} = \langle v, w\rangle_{\Phi(z)}$.

Since $T \in G$, the definition of the monodromy operator implies that

$$\Phi(z + 2\pi \mathsf{i}) = T \cdot \Phi(z),$$

This means that the expression $e^{-z(S_\alpha+N)}\Phi(z)$ is invariant under translation by $2\pi\mathsf{i}$, and so it descends to a holomorphic mapping

$$\Psi_\alpha \colon \Delta^* \longrightarrow \check{D}, \quad \Psi_\alpha(e^z) = e^{-z(S_\alpha+N)}\Phi(z).$$

The following result is known as the "nilpotent orbit theorem".

A.8.2. Theorem. *The holomorphic mapping Ψ_α extends over the origin, and the limiting value $\Psi_\alpha(0) \in \check{D}$ satisfies $N\Psi_\alpha^p(0) \subseteq \Psi_\alpha^{p-1}(0)$ for all $p \in \mathbb{Z}$.*

Note. An equivalent formulation is that the Hodge bundles $F^p\mathcal{E}$ extend to holomorphic subbundles $F^p\widetilde{\mathcal{E}}^\alpha$ of the canonical extension. Under the isomorphism

$$\widetilde{\mathcal{E}}^\alpha_{|0} \cong V$$

with the space of multivalued flat sections, the filtration $\Psi_\alpha(0)$ is then simply the filtration induced by these subbundles,

$$F^p\widetilde{\mathcal{E}}^\alpha_{|0} \cong \Psi_\alpha^p(0),$$

and the second half of the nilpotent orbit theorem is asserting that the residue R_α maps the subspace $F^p\widetilde{\mathcal{E}}^\alpha_{|0}$ into the subspace $F^{p-1}\widetilde{\mathcal{E}}^\alpha_{|0}$.

Now we are ready to discuss the convergence properties of the period mapping. As suggested above, we consider the *rescaled period mapping*

$$\hat{\Phi}_H \colon \mathbb{H} \longrightarrow D, \quad \hat{\Phi}_H(z) = e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-\frac{1}{2}(z-\bar{z})(S_\alpha+N)}\Phi(z).$$

Since both exponential factors belong to the real group G, the rescaled period mapping still takes values in the period domain D. It is also invariant under translation by $2\pi\mathsf{i}$, and for any multivalued flat section $v \in V$, the expression

$$\|v\|^2_{\hat{\Phi}_H(z)} = \left\| e^{\frac{1}{2}(z-\bar{z})(S_\alpha+N)} e^{-\frac{1}{2}\log|\operatorname{Re} z|\, H} v \right\|_{\Phi(z)}$$

remains bounded as $\operatorname{Re} z \to -\infty$ (due to the Hodge norm estimates). The nice thing is that this rescaling also makes the polarized Hodge structures converge.

A.8.3. Theorem. *The rescaled period mapping $\hat{\Phi}_H$ converges to a limit*

$$e^{-N}F_H = \lim_{\operatorname{Re} z \to -\infty} \hat{\Phi}_H(z) \in D.$$

Moreover, the filtration $F_H \in \check{D}$ has the property that, for all $p \in \mathbb{Z}$,

$$NF_H^p \subseteq F_H^{p-1}, \quad HF_H^p \subseteq F_H^p, \quad \text{and} \quad T_sF_H^p \subseteq F_H^p.$$

The filtration F_H in the statement of the theorem is obtained from the filtration $\Psi_\alpha(0)$ in the nilpotent orbit theorem in two steps. One can check that

$$\hat{\Phi}_H(z) = e^{-N} \cdot e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z| S_\alpha} \Psi_\alpha(e^z),$$

and since $\Psi_\alpha(e^z)$ converges to its limit $\Psi_\alpha(0)$ at a rate $|e^z| = e^{|\operatorname{Re} z|}$, this gives

$$F_H = \lim_{|\operatorname{Re} z| \to \infty} e^{\frac{1}{2}\log|\operatorname{Re} z|\, H} e^{-|\operatorname{Re} z| S_\alpha} \Psi_\alpha(0). \tag{A.8.4}$$

Let us briefly digress on the effect of the two exponential factors, since this may be helpful for understanding where the filtration F_H comes from. Suppose for a moment that $S \in \operatorname{End}(V)$ is an arbitrary semisimple endomorphism with real eigenvalues $\alpha_1 < \alpha_2 < \cdots < \alpha_r$. Then for any filtration $F \in \check{D}$, the limit

$$F_S = \lim_{x \to \infty} e^{xS} F \in \check{D}$$

exists and is compatible with S, in the sense that $SF_S^p \subseteq F_S^p$ for all $p \in \mathbb{Z}$. The effect of the limit can be understood concretely as follows. Consider the filtration by increasing eigenvalues of S, with terms

$$G_j = E_{\alpha_1}(S) \oplus \cdots \oplus E_{\alpha_j}(S).$$

The filtration F induces a filtration on each subquotient G_j/G_{j-1}, and under the obvious isomorphism $E_{\alpha_j}(S) \cong G_j/G_{j-1}$, we have

$$F_S^p \cap E_{\alpha_j}(S) \cong (F^p \cap G_j + G_{j-1})/G_{j-1}.$$

In the specific case in (A.8.4) that we care about, this means:

(1) The effect of the exponential factor $e^{-|\operatorname{Re} z| S_\alpha}$ is to produce a filtration

$$F_{\lim} = \lim_{x \to \infty} e^{-xS_\alpha} F \in \check{D} \tag{A.8.5}$$

that is compatible with the eigenspace decomposition of the semisimple operator $T_s = e^{2\pi i\, S_\alpha}$. Because of the minus sign in the exponent, the relevant filtration is by *decreasing* eigenvalues of S_α.

(2) The effect of the exponential factor $e^{\frac{1}{2}\log|\operatorname{Re} z|\, H}$ is to produce a filtration

$$F_H = \lim_{x \to \infty} e^{\frac{1}{2}\log x\, H} F_{\lim} \in \check{D}$$

that is also compatible with the eigenspace decomposition of the semisimple operator H. The relevant filtration is the monodromy weight filtration $W_\bullet$, which is exactly the filtration by increasing eigenvalues of H.

The fact that $e^{-N} F_H \in D$ is a polarized Hodge structure of weight n implies, after some linear algebra, that the filtration F_H is the Hodge filtration of a polarized $\mathfrak{sl}_2$-Hodge structure of weight n. We now describe the relevant objects. Because of the relation $[H, N] = -2N$, the two operators $H, N \in \operatorname{End}(V)$ are part of a representation of $\mathfrak{sl}_2(\mathbb{C})$. With respect to the standard basis

$$\mathrm{H} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \mathrm{X} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \mathrm{Y} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

we let $\mathrm{H} \in \mathfrak{sl}_2(\mathbb{C})$ act as the semisimple operator $H \in \operatorname{End}(V)$, and we let $\mathrm{Y} \in \mathfrak{sl}_2(\mathbb{C})$ act as the nilpotent operator $-N$. By construction, the semisimple part T_s of the monodromy transformation commutes with the action by $\mathfrak{sl}_2(\mathbb{C})$.

Note. The minus sign in $\mathrm{Y} = -N$ is important; we shall justify in a minute why it has to be there and why it is the natural choice.

To match our earlier notation, let us write

$$V_\ell = E_\ell(\mathrm{H}) \cong \operatorname{gr}_\ell^W V$$

for the weight spaces of the semisimple operator H. Recall that the Hermitian form $\mathrm{S} \colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ has the property that

$$\mathrm{S} \circ (\mathrm{H} \otimes \mathrm{Id}) = -\mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{H}) \quad \text{and} \quad \mathrm{S} \circ (\mathrm{Y} \otimes \mathrm{Id}) = \mathrm{S} \circ (\mathrm{Id} \otimes \mathrm{Y}),$$

as required by (A.3.7). The main result is then the following.

A.8.6. Theorem. *With notation as above, the space of multivalued flat sections*

$$V = \bigoplus_{\ell \in \mathbb{Z}} V_\ell$$

becomes an $\mathfrak{sl}_2$-Hodge structure of weight n, polarized by the Hermitian form S. *Its Hodge filtration is the filtration F_H, in the sense that*

$$F_H^p \cap V_\ell = V_\ell^{p,\ell-p} \oplus V_\ell^{p+1,\ell-(p+1)} \oplus V_\ell^{p+2,\ell-(p+2)} \oplus \cdots$$

for all integers $p, \ell \in \mathbb{Z}$. Moreover, the operator $T_s \in \operatorname{End}(V)$ is an endomorphism of the polarized $\mathfrak{sl}_2$-Hodge structure.

A.8.7. Example. Here is a simple example that shows the sign conventions at work. Consider the standard representation of $\mathfrak{sl}_2(\mathbb{C})$ on the vector space $V = \mathbb{C}^2$, with the standard Hermitian form

$$\mathsf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

If we set $F^1 = \mathbb{C}(1,0)$ and $F^0 = \mathbb{C}^2$, then $e^{\mathrm{Y}} F$ is the Hodge filtration of a polarized Hodge structure of weight 1: the Hodge decomposition is

$$V = V^{1,0} \oplus V^{0,1} = \mathbb{C}(1,1) \oplus \mathbb{C}(1,-1),$$

and S is clearly positive on the first subspace and negative on the second one (in agreement with our convention that $c_n = 1$). On the other hand, the Weil element $\mathrm{w} \in \mathrm{SL}_2(\mathbb{C})$ satisfies

$$\mathrm{w}(0,1) = (1,0) \quad \text{and} \quad \mathrm{w}(1,0) = -(0,1),$$

and so we do get a polarized $\mathfrak{sl}_2$-Hodge structure of weight 1, with

$$V_1 = V_1^{1,1} = \mathbb{C}(1,0) \quad \text{and} \quad V_{-1} = V_{-1}^{0,0} = \mathbb{C}(0,1),$$

since for example $\mathrm{S}\big((0,1), \mathrm{w}(0,1)\big) = 2$. Note that the signs do not work out properly if we use the Hodge filtration $e^{-\mathrm{Y}} F$ instead; this is one reason why it is necessary to define the $\mathfrak{sl}_2(\mathbb{C})$-representation using $\mathrm{Y} = -N$.

A.9. Hodge modules on the unit disk

Before we turn to the sign conventions for nearby and vanishing cycles, it may be useful to summarize the results of the previous section in the language of Hodge modules. The polarized variation of Hodge structure E of weight n on Δ^* determines a polarized Hodge module $M \in \mathsf{HM}(\Delta, n+1)$, with pure support Δ. Let us denote by $(\mathcal{M}, F_\bullet\mathcal{M})$ its underlying filtered $\mathcal{D}_\Delta$-module, and by $\mathrm{S}_M \colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \to \mathfrak{C}_\Delta$ the Hermitian pairing giving the polarization.

We briefly review the construction of M. The various canonical extensions $\widetilde{\mathcal{E}}^\alpha$ and $\widetilde{\mathcal{E}}^{>\alpha}$ embed into Deligne's meromorphic extension $\widetilde{\mathcal{E}}$, which is naturally a left $\mathcal{D}_\Delta$-module, with ∂_t acting through the logarithmic connection. The subsheaves $\widetilde{\mathcal{E}}^\alpha$ define a decreasing filtration on $\widetilde{\mathcal{E}}$, and

$$\widetilde{\mathcal{E}}^\alpha/\widetilde{\mathcal{E}}^{\alpha+1} = \widetilde{\mathcal{E}}^\alpha/t\widetilde{\mathcal{E}}^\alpha \widetilde{\mathcal{E}}^\alpha_{|0} \cong V.$$

Under this isomorphism, the operator $t\partial_t$ goes to the residue $R_\alpha = S_\alpha + N$ of the logarithmic connection. The corresponding right $\mathcal{D}_\Delta$-module $\omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}$ has a unique maximal submodule with pure support Δ, namely

$$\mathcal{M} = \big(\omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}^{>-1}\big) \cdot \mathcal{D}_\Delta \subseteq \omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}.$$

For $\alpha < 0$, the V-filtration with respect to $t = 0$ is given by the formula

$$V_\alpha\mathcal{M} = \omega_\Delta \otimes_{\mathcal{O}_\Delta} \widetilde{\mathcal{E}}^{-(\alpha+1)}.$$

In particular, this leads to a canonical isomorphism

$$V_\alpha\mathcal{M}/V_{\alpha-1}\mathcal{M} = V_\alpha\mathcal{M}/V_\alpha\mathcal{M} \cdot t \cong \widetilde{\mathcal{E}}^{-(\alpha+1)}_{|0} \cong V,$$

under which right multiplication by $t\partial_t$ becomes left multiplication by $-\partial_t t = -(t\partial_t + 1)$, hence goes to the operator $-(R_{-(\alpha+1)} + \mathrm{Id})$. Moreover, the induced filtration $V_\bullet\mathcal{M}/V_{\alpha-1}\mathcal{M}$ becomes, on the vector space V, the filtration by *decreasing* eigenvalues of $S_{-(\alpha+1)}$. For $\alpha < 0$, this gives

$$\mathrm{gr}^V_\alpha\mathcal{M} \cong E_{e^{-2\pi i \alpha}}(T_s), \tag{A.9.1}$$

and under this isomorphism, the nilpotent operator $t\partial_t - \alpha$ on the left-hand side corresponds to the nilpotent operator $\mathrm{Y} = -N$ on the right-hand side (which is therefore the natural choice for the $\mathfrak{sl}_2(\mathbb{C})$-representation).

Note. This is another instance of the general principle that one can arrive at the correct signs simply by working consistently with right $\mathcal{D}$-modules.

The filtration $F_\bullet\mathcal{M}$ is constructed in such a way that

$$F_pV_\alpha\mathcal{M} = F_p\mathcal{M} \cap V_\alpha\mathcal{M} = \omega_\Delta \otimes_{\mathcal{O}_\Delta} F^{-p-1}\widetilde{\mathcal{E}}^{-\alpha-1}$$

for $\alpha < 0$. It induces a filtration on $\mathrm{gr}^V_\alpha\mathcal{M}$, with terms

$$F_p\mathrm{gr}^V_\alpha\mathcal{M} = \big(F_pV_\alpha\mathcal{M} + V_{<\alpha}\mathcal{M}\big)/V_{<\alpha}\mathcal{M}.$$

Since the V-filtration corresponds, on the vector space V, to the filtration by decreasing eigenvalues of $S_{-(\alpha+1)}$, this matches up nicely with our earlier discussion: under the isomorphism in (A.9.1), the filtration $F_\bullet \mathrm{gr}_\alpha^V \mathcal{M}$ becomes the limiting Hodge filtration $F_{\lim}^{-\bullet-1}$, defined in (A.8.5). Consequently, after choosing a splitting $H \in \mathrm{End}(V)$ for the weight filtration, the induced filtration on

$$\mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} \cong E_\ell(H) \cap E_{e^{-2\pi \mathsf{i}\alpha}}(T_s)$$

is precisely the filtration $F_H^{-\bullet-1}$. We can therefore restate Theorem A.8.6 by saying that, for each $\alpha \in [-1,0)$, the graded vector space

$$\mathrm{gr}^W \mathrm{gr}_\alpha^V \mathcal{M} = \bigoplus_{\ell \in \mathbb{Z}} \mathrm{gr}_\ell^W \mathrm{gr}_\alpha^V \mathcal{M} \tag{A.9.2}$$

has an $\mathfrak{sl}_2$-Hodge structure of central weight n; here the representation by $\mathfrak{sl}_2(\mathbb{C})$ is defined by letting Y act as $t\partial_t - \alpha$, and the Hodge filtration is the filtration

$$F_{-\bullet-1}\mathrm{gr}^W \mathrm{gr}_\alpha^V \mathcal{M}$$

induced by the filtration $F_\bullet \mathcal{M}$.

Note. This formulation of Theorem A.8.6 does not require choosing a splitting for the weight filtration (because it is a result about the associated graded object).

Since it is instructive, let us also review how to recover the polarization on the $\mathfrak{sl}_2$-Hodge structure from the Hermitian pairing S_M on the $\mathcal{D}$-module $\mathcal{M}$. Recall from above that we have a preferred trivialization

$$\widetilde{\mathcal{E}}^{>-1} \cong \mathcal{O}_\Delta \otimes_{\mathbb{C}} V$$

for the canonical extension. In this frame, the polarization S_E on the variation of Hodge structure takes the form

$$\mathrm{S}_E(1 \otimes v', 1 \otimes v'') = \sum_{\beta \in (-1,0]} \sum_{j=0}^{\infty} |t|^{2\beta} \mathrm{L}(t)^j \cdot \frac{(-1)^j}{j!} \mathrm{S}\big(v'_\beta, N^j v''_\beta\big).$$

Note that the expression on the right-hand side is locally integrable precisely for $\beta > -1$. The Hermitian pairing on $\mathcal{M}$ is defined in such a way that, on the subsheaf $V_{<0}\mathcal{M} \cong \omega_\Delta \otimes_{\mathbb{C}} V$, one has

$$\big\langle \mathrm{S}_M(\mathrm{d}t \otimes v', \mathrm{d}t \otimes v''), \varphi \big\rangle = \frac{\varepsilon(2)}{2\pi \mathsf{i}} \int_\Delta \varphi \cdot \mathrm{S}_E(1 \otimes v', 1 \otimes v'') \mathrm{d}t \wedge \mathrm{d}\bar{t}.$$

The constants in this formula are of course dictated by (A.7.2). In terms of the basic currents $C_{\alpha,p}$ from (A.1.4), the definition of the pairing reads

$$\mathrm{S}_M(\mathrm{d}t \otimes v', \mathrm{d}t \otimes v'') = \sum_{\beta \in (-1,0]} \sum_{j=0}^{\infty} (-1)^j \mathrm{S}\big(v'_\beta, N^j v''_\beta\big) \cdot C_{-(\beta+1),j}, \tag{A.9.3}$$

From this asymptotic expansion, we can recover the restriction of the Hermitian pairing $\mathrm{S}\colon V \otimes_{\mathbb{C}} \overline{V} \to \mathbb{C}$ to the subspace

$$\mathrm{gr}_\alpha^V \mathcal{M} \cong E_{e^{-2\pi \mathsf{i}\alpha}}(T_s)$$

by taking the coefficient of the basic current $C_{\alpha,0}$; here $\alpha = -(\beta+1) \in [-1,0)$.

Note. There are no additional signs in this description; this is due to our principle of defining currents to be positive where possible.

So the conclusion is that $\mathfrak{sl}_2$-Hodge structure on (A.9.2) is polarized by the Hermitian pairing that we get by taking the coefficient of the basic current $C_{\alpha,0}$ in the asymptotic expansion of the pairing S_M. One can extract this coefficient, without writing down the asymptotic expansion, by using the Mellin transform.

A.9.4. Example. Another useful example is the direct image of a polarized Hodge structure H under the closed embedding $i\colon \{0\} \hookrightarrow \Delta$. If the weight of H is equal to n, then $i_* H \in \mathsf{HM}(\Delta, n)$. Using the notation from above, let us write

$$\mathrm{S}_0\colon i_+H \otimes_{\mathbb{C}} \overline{i_+H} \longrightarrow \mathfrak{C}_\Delta$$

for the induced Hermitian pairing. For two vectors $h', h'' \in H$, we have

$$\mathrm{S}_0(h', h'') = \mathrm{S}(h', h'') \cdot \delta_0,$$

where δ_0 is the delta function. So in this case, we can recover the polarization on H from the Hermitian pairing on i_+H as the coefficient in front of δ_0.

A.10. Nearby and vanishing cycles

In this section, we discuss the sign conventions for nearby and vanishing cycles, taking the example in the previous section as a model. Let us begin with a brief review of the general construction and its properties. Fix a complex manifold X. On the product $X \times \mathbb{C}$, we have the holomorphic function $t\colon X \times \mathbb{C} \to \mathbb{C}$, and the corresponding holomorphic vector field ∂_t. Suppose that $M \in \mathsf{HM}(X \times \mathbb{C}, w)$ is a polarized Hodge module of weight w on the product $X \times \mathbb{C}$. As usual, we denote by $(\mathcal{M}, F_\bullet\mathcal{M})$ the underlying filtered right $\mathcal{D}_{X\times\mathbb{C}}$-module, and by

$$\mathrm{S}_M\colon \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathcal{M}} \longrightarrow \mathfrak{C}_{X\times\mathbb{C}}\,.$$

the Hermitian pairing giving the polarization. Lastly, we use the notation $V_\bullet\mathcal{M}$ for the V-filtration on $\mathcal{M}$ relative to $t = 0$.

(1) For every $\alpha \in [-1, 0)$, one has the *nearby cycles* $\psi_{t,\lambda}M$ for the eigenvalue $\lambda = e^{-2\pi i\alpha}$. This is an object on X. The underlying filtered $\mathcal{D}_X$-module

$$\bigl(\mathrm{gr}_\alpha^V\mathcal{M}, F_{\bullet-1}\mathrm{gr}_\alpha^V\mathcal{M}\bigr)$$

comes with a nilpotent operator $N_\alpha = t\partial_t - \alpha$ and a Hermitian pairing

$$\mathrm{S}_\alpha\colon \mathrm{gr}_\alpha^V\mathcal{M} \otimes_{\mathbb{C}} \overline{\mathrm{gr}_\alpha^V\mathcal{M}} \longrightarrow \mathfrak{C}_X, \quad \mathrm{S}_\alpha \circ (N_\alpha \otimes \mathrm{Id}) = \mathrm{S}_\alpha \circ (\mathrm{Id} \otimes N_\alpha).$$

If we denote by $W_\bullet$ the weight filtration of N_α, then

$$\mathrm{gr}^W(\psi_{t,\lambda}M) = \bigoplus_{\ell\in\mathbb{Z}} \mathrm{gr}_\ell^W(\psi_{t,\lambda}M)$$

is a polarized $\mathfrak{sl}_2$-Hodge module of weight $w-1$; the element $\mathrm{Y} \in \mathfrak{sl}_2(\mathbb{C})$ acts as $N_\alpha = t\partial_t - \alpha$, and the polarization is induced by S_α.

(2) For $\alpha = 0$, one has the *unipotent vanishing cycles* $\phi_{t,1}M$. This is again an object on X. The underlying filtered $\mathcal{D}_X$-module

$$\big(\mathrm{gr}_0^V \mathcal{M}, F_\bullet \mathrm{gr}_0^V \mathcal{M}\big)$$

comes with a nilpotent operator $N_0 = t\partial_t$ and a Hermitian pairing

$$\mathrm{S}_0 \colon \mathrm{gr}_0^V \mathcal{M} \otimes_{\mathbb{C}} \overline{\mathrm{gr}_0^V \mathcal{M}} \longrightarrow \mathfrak{C}_X, \quad \mathrm{S}_0 \circ (N_0 \otimes \mathrm{Id}) = \mathrm{S}_0 \circ (\mathrm{Id} \otimes N_0).$$

If we denote by $W_\bullet$ the weight filtration of N_0, then

$$\mathrm{gr}^W(\phi_{t,1}M) = \bigoplus_{\ell \in \mathbb{Z}} \mathrm{gr}_\ell^W \phi_{t,1}M$$

is a polarized $\mathfrak{sl}_2$-Hodge module of weight w; the element $\mathrm{Y} \in \mathfrak{sl}_2(\mathbb{C})$ acts as $N_0 = t\partial_t$, and the polarization is induced by S_0.

Note that the Hodge filtration and the weight of the $\mathfrak{sl}_2$-Hodge module are different in both cases; this is forced on us by the following two examples:

A.10.1. Example. A polarized variation of Hodge structure of weight n on the punctured disk Δ^* gives rise to a polarized Hodge module $M \in \mathsf{HM}(\Delta, n+1)$, with $F_\bullet \mathcal{M} = \omega_\Delta \otimes_{\mathcal{O}_\Delta} F^{-\bullet-1}\mathcal{E}$. In this case, $\psi_{t,\lambda}M \cong E_\lambda(T_s)$, and we have seen in the previous section that $\mathrm{gr}^W(\psi_{t,\lambda}M)$ is a polarized $\mathfrak{sl}_2$-Hodge structure of weight $n = (n+1) - 1$. To get back the correct Hodge filtration, we also need to undo the shift that is built into the definition of $F_\bullet \mathcal{M}$.

A.10.2. Example. A polarized Hodge structure H of weight n gives rise to a polarized Hodge module $i_* H \in \mathsf{HM}(\Delta, n)$, where $i \colon \{0\} \hookrightarrow \Delta$ is the embedding of the origin. In this case, $\phi_{t,1}(i_* H) \cong H$ clearly has weight n, and there is no shift in the Hodge filtration.

This is the general picture, but we still need to figure what signs to use in the construction of the pairings S_α. Let us begin by treating the nearby cycles, because that case is slightly easier to explain. Fix a real number $\alpha \in [-1, 0)$. Consider local sections $m', m'' \in V_\alpha \mathcal{M}$ and the current $\mathrm{S}_M(m', m'') \in \mathfrak{C}_{X \times \mathbb{C}}$. Ideally, $\mathrm{S}_M(m', m'')$ would have an asymptotic expansion in t, in terms of the basic currents from (A.1.4), and the coefficient in front of $C_{\alpha,0}$ would be a current on X that could be used to define the pairing between $[m'], [m''] \in \mathrm{gr}_\alpha^V \mathcal{M}$. Fortunately, we can accomplish the same thing, without having the asymptotic expansion, by working with Mellin transforms.

More precisely, suppose that $m', m'' \in H^0(U, V_\alpha \mathcal{M})$. Let $\varphi(x)$ be a test function on X, and let $\eta(t)$ be a cutoff function on $\mathbb{C}$, such that the product $\eta(t)\varphi(x)$ has compact support inside U. The Mellin transform

$$F_{m',m''}(s) = \big\langle \mathrm{S}_M(m', m''), |t|^{2s} \eta(t) \varphi(x) \big\rangle$$

is holomorphic for $\mathrm{Re}\, s \gg 0$, and has a meromorphic extension to $\mathbb{C}$ with poles contained in the interval $(-\infty, \alpha]$. One can show that the residue at $s = \alpha$ depends continuously on φ, and that the formula

$$\big\langle \mathrm{S}_\alpha\big([m'],[m'']\big),\varphi\big\rangle = \mathrm{Res}_{s=\alpha}\big\langle \mathrm{S}_M(m',m''),|t|^{2s}\eta(t)\varphi(x)\big\rangle$$

defines the desired Hermitian pairing S_α. Let us check in several examples that this definition (with no extra sign factors) is the correct one.

A.10.3. Example. The first example explains how the Mellin transform can be used to pick up individual terms in a (hypothetical) asymptotic expansion. On the unit disk Δ, fix a test function $\varphi(t)$. Because the function $|t|^{2s-2} = e^{-(s-1)\mathrm{L}(t)}$ is locally integrable for $\mathrm{Re}\, s > 0$, the expression

$$F(s) = \frac{\varepsilon(2)}{2\pi \mathsf{i}}\int_{\mathbb{C}} |t|^{2s-2}\varphi \mathrm{d}t\wedge \mathrm{d}\bar t = \big\langle C_{-1,0},|t|^{2s-2}\varphi\big\rangle$$

defines a holomorphic function on the halfplane $\mathrm{Re}\, s > 0$. To understand its behavior near $s = 0$, one can use integration by parts to prove the identity

$$s^2F(s) = \frac{\varepsilon(2)}{2\pi \mathsf{i}}\int_{\Delta} |t|^{2s}\,\frac{\partial^2\varphi}{\partial t\partial \bar t}\,\mathrm{d}t\wedge \mathrm{d}\bar t,$$

valid for $\mathrm{Re}\, s > 0$. The function on the right-hand side is holomorphic for $\mathrm{Re}\, s > -1$, and so $F(s)$ extends to a meromorphic function on this larger halfplane. From the power series expansion of the exponential function, we get

$$s^2F(s) = \sum_{j=0}^{\infty}(-1)^j s^j\,\frac{\varepsilon(2)}{2\pi \mathsf{i}}\int_{\Delta}\frac{\mathrm{L}(t)^j}{j!}\,\frac{\partial^2\varphi}{\partial t\partial \bar t}\mathrm{d}t\wedge \mathrm{d}\bar t = \sum_{j=1}^{\infty}(-1)^j s^j\big\langle C_{-1,j}\partial_t\partial_{\bar t},\varphi\big\rangle.$$

Using the identity $\delta_0 = -C_{-1,1}\partial_t\partial_{\bar t}$, we can rewrite this as

$$F(s) = \frac{\varphi(0)}{s} + \sum_{j=0}^{\infty}(-1)^j s^j\big\langle C_{-1,j+2}\partial_t\partial_{\bar t},\varphi\big\rangle. \tag{A.10.4}$$

Differentiating under the integral sign p times gives

$$\big\langle C_{-1,p},|t|^{2s-2}\varphi\big\rangle = \frac{\varepsilon(2)}{2\pi \mathsf{i}}\int_{\mathbb{C}} |t|^{2s-2}\frac{\mathrm{L}(t)^p}{p!}\varphi \mathrm{d}t\wedge \mathrm{d}\bar t = \frac{(-1)^p}{p!}F^{(p)}(s) \equiv \frac{\varphi(0)}{s^{p+1}}$$

modulo entire functions. Consequently, the Mellin transform of the basic current $C_{-1,p}$ has a pole of order exactly $p+1$ at the point $s = 0$; the residue is $\varphi(0)$ for $p = 0$, and trivial for $p \geqslant 1$.

A.10.5. Example. Now let us go back to polarized variations of Hodge structure on Δ^*, and compute the nearby cycles with respect to $t = 0$, using the notation from the previous section. Let $v', v'' \in E_{e^{-2\pi \mathsf{i}\alpha}}(T_s)$ be two multivalued flat sections, for some $\alpha \in [-1, 0)$. The formula for the pairing in (A.9.3) shows that

$$\mathrm{S}_M(\mathrm{d}t\otimes v',\mathrm{d}t\otimes v'') = \sum_{j=0}^{\infty}(-1)^j\mathrm{S}\big(v',N^jv''\big)\cdot C_{\alpha,j}.$$

According to the calculations in the preceding example, the Mellin transform

$$\big\langle \mathrm{S}_M(\mathrm{d}t\otimes v',\mathrm{d}t\otimes v''),|t|^{2s}\varphi(t)\big\rangle$$

is holomorphic on the halfplane $\operatorname{Re} s > \alpha$, and the polar part at $s = \alpha$ equals

$$\sum_{j=0}^{\infty} (-1)^j \mathrm{S}\big(v', N^j v''\big) \frac{\varphi(0)}{(s-\alpha)^{j+1}}.$$

In particular, the residue

$$\operatorname{Res}_{s=\alpha} \big\langle \mathrm{S}_M(\mathrm{d}t \otimes v', \mathrm{d}t \otimes v''), |t|^{2s} \varphi(t) \big\rangle = \mathrm{S}(v', v'') \cdot \varphi(0)$$

recovers the restriction of S to the eigenspace $E_{e^{-2\pi i \alpha}}(T_s)$; we saw in the previous section that this pairing gives the polarization on the $\mathfrak{sl}_2$-Hodge structure.

Now we turn to the unipotent nearby cycles, which are the boundary case $\alpha = 0$. The general idea is the same, but the construction needs to be modified slightly. As before, let $m', m'' \in V_0\mathcal{M}$ be two local sections, and consider the current $\mathrm{S}_M(m', m'')$. In the hypothetical asymptotic expansion of $\mathrm{S}_M(m', m'')$, we should take the coefficient of the delta function δ_0; recall that

$$\langle \delta_0, \varphi \rangle = \varphi(0).$$

The problem is that the Mellin transform of the delta function is trivial, and so a small trick is required. It is based on the identity

$$\delta_0 = -C_{-1,1} \partial_t \partial_{\bar{t}},$$

that has already appeared in the example above. Because ∂_t and $\partial_{\bar{t}}$ are surjective on the level of currents, we can extract the term with δ_0 from the hypothetical asymptotic expansion by writing our current in the form $-T \partial_t \partial_{\bar{t}}$, and then looking at the Mellin transform of T.

To make this precise, let $m', m'' \in H^0(U, V_0\mathcal{M})$ be two sections. Choose a current $T_{m',m''} \in H^0(U, \mathfrak{C}_{X \times \mathbb{C}})$ with the property that

$$\mathrm{S}_M(m', m'') = -T_{m',m''} \partial_t \partial_{\bar{t}};$$

such a current always exists, and is unique up to adding harmonic functions. With $\varphi(x)$ and $\eta(t)$ as above, the Mellin transform

$$G_{m',m''}(s) = \big\langle T_{m',m''}, |t|^{2s-2} \eta(t) \varphi(x) \big\rangle$$

is holomorphic for $\operatorname{Re} s \gg 0$, and extends to a meromorphic function on $\mathbb{C}$ with poles contained in the interval $(-\infty, 0]$. Integration by parts shows that, modulo entire functions, one has

$$F_{m',m''}(s) \equiv -s^2 G_{m',m''}(s),$$

and so the the quantity of interest is now the coefficient in front of $1/s^2$, hence the residue of $sG_{m',m''}(s)$ at $s = 0$. This observation suggests defining the Hermitian pairing S_0 by the formula

$$\big\langle \mathrm{S}_0\big([m'], [m'']\big), \varphi \big\rangle = \operatorname{Res}_{s=0} \big\langle T_{m',m''}, s \cdot |t|^{2s-2} \eta(t) \varphi(x) \big\rangle.$$

The following example explains why this definition is the correct one.

A.10.6. Example. For direct images along the closed embedding $i\colon X \hookrightarrow X \times \mathbb{C}$, we recover the pairing on the original $\mathcal{D}_X$-module. Indeed, suppose $\mathcal{N}$ is a coherent right $\mathcal{D}_X$-module, and $\mathrm{S}_N : \mathcal{N} \otimes_{\mathbb{C}} \overline{\mathcal{N}} \to \mathfrak{C}_X$ a Hermitian pairing. Then

$$\mathcal{M} = i_+\mathcal{N} \cong \mathcal{N} \otimes_{\mathbb{C}} \mathbb{C}[\partial_t], \quad \mathrm{S}_M = i_+\mathrm{S}_N,$$

and under this isomorphism, we have $V_0\mathcal{M} \cong \mathcal{N} \otimes 1$, hence $\mathrm{gr}_0^V\mathcal{M} \cong \mathcal{N}$. For two local sections $n', n'' \in \mathcal{N}$, the current

$$\mathrm{S}_M(n' \otimes 1, n'' \otimes 1) = i_*\mathrm{S}_N(n', n'')$$

is a multiple of δ_0; under the isomorphism $\mathrm{gr}_0^V\mathcal{M} \cong \mathcal{N}$, the construction above therefore recovers the original pairing: $\mathrm{S}_0 = \mathrm{S}_N$

We close this section with a brief discussion of the sign conventions for the "canonical morphism" and the "variation morphism",

$$\mathrm{can}\colon \psi_{t,1}M \longrightarrow \phi_{t,1}M \quad \text{and} \quad \mathrm{var}\colon \phi_{t,1}M \longrightarrow \psi_{t,1}M(-1).$$

The underlying morphisms of filtered $\mathcal{D}_X$-modules are the obvious ones:

$$\begin{aligned}
\mathrm{can}\colon \big(\mathrm{gr}_{-1}^V\mathcal{M}, F_{\bullet-1}\mathrm{gr}_{-1}^V\mathcal{M}\big) &\longrightarrow \big(\mathrm{gr}_0^V\mathcal{M}, F_{\bullet}\mathrm{gr}_0^V\mathcal{M}\big), \quad \mathrm{can}(m) = m\partial_t\\
\mathrm{var}\colon \big(\mathrm{gr}_0^V\mathcal{M}, F_{\bullet}\mathrm{gr}_0^V\mathcal{M}\big) &\longrightarrow \big(\mathrm{gr}_{-1}^V\mathcal{M}, F_{\bullet}\mathrm{gr}_{-1}^V\mathcal{M}\big), \quad \mathrm{var}(m) = mt
\end{aligned}$$

In particular, we have $\mathrm{can} \circ \mathrm{var} = N_0$ and $\mathrm{var} \circ \mathrm{can} = N_{-1}$. In the proof that every polarized Hodge module admits a decomposition by pure support, the following identity for can and var plays a crucial role:

$$\mathrm{S}_{-1} \circ (\mathrm{var} \otimes \mathrm{Id}) + \mathrm{S}_0 \circ (\mathrm{Id} \otimes \mathrm{can}) = 0 \tag{A.10.7}$$

With our construction of the pairings S_{-1} and S_0, this identity is easily proved using integration by parts. It can be shown that both can and var reduce the index in the weight filtration by 1; consequently,

$$\mathrm{can}\colon \mathrm{gr}_\ell^W(\psi_{t,1}M) \longrightarrow \mathrm{gr}_{\ell-1}^W(\phi_{t,1}M)$$

is a morphism of Hodge structures of weight $w + \ell - 1$, and

$$\mathrm{var}\colon \mathrm{gr}_\ell^W(\phi_{t,1}M) \longrightarrow \mathrm{gr}_{\ell-1}^W(\psi_{t,1}M)(-1)$$

is a morphism of Hodge structures of weight $w + \ell$ (for every $\ell \in \mathbb{Z}$).

PART III

POLARIZABLE HODGE MODULES

CHAPTER 14

POLARIZABLE HODGE MODULES AND THEIR DIRECT IMAGES

Summary. This chapter contains the definition of polarizable Hodge modules. The actual presentation justifies the introduction of the language of triples. The main properties are abelianity and semi-simplicity of the category of polarizable pure Hodge modules of weight w. It is convenient to also introduce polarizable Hodge-Lefschetz modules, as they appear in many intermediate steps of various proofs, due to the very definition of a polarizable Hodge module. We also give the proof of one of the two main important results concerning polarizable Hodge modules, namely, the decomposition theorem. The proof of the structure theorem will be given in Chapter 15. Here, we will use the machinery of filtered $\mathcal{D}$-module theory and sesquilinear pairings to reduce the proof to the case of the map from a compact Riemann surface to a point, that we have analyzed in Chapter 7, according to the results of Schmid and Zucker developed in Chapter 6. This strategy justifies the somewhat complicated and recursive definition of the category $\mathsf{pHM}(X, w)$ of polarizable Hodge modules.

14.1. Introduction

Polarizable Hodge modules on a complex analytic manifold X are supposed to play the role of polarizable Hodge structures with a multi-dimensional parameter. These objects can acquire singularities. The way each characteristic property of a Hodge structure is translated in higher dimension of the parameter space is given by the table below.

dimension 0	dimension $n \geqslant 1$
$\mathcal{H}$ a $\mathbb{C}$-vector space	$\mathcal{M}$ a holonomic $\mathcal{D}$-module
$F^\bullet\mathcal{H}$ a filtration	$F_\bullet\mathcal{M}$ a coherent filtration
$\mathscr{H} = R_F\mathcal{H}$	$\mathscr{M} = R_F\mathcal{M}$
$H = (\mathscr{H}', \mathscr{H}'', \mathfrak{s})$ a graded triple of $\mathbb{C}[z]$-vector spaces	$M = (\mathscr{M}', \mathscr{M}'', \mathfrak{s})$ a graded triple of $R_F\mathcal{D}$-modules
$\mathrm{S} : H \to H^*(-w)$ a polarization	$\mathrm{S} : M \to M^*(-w)$ a polarization

Why choosing holonomic $\mathcal{D}$-modules as analogues of $\mathbb{C}$-vector space? The reason is that the category of holonomic $\mathcal{D}$-modules is Artinian, that is, any holonomic $\mathcal{D}$-module has finite length (locally on the underlying manifold). A related reason is that its de Rham complex has constructible cohomology, generalizing the notion of local system attached to a flat bundle. Moreover, the property of holonomicity is preserved by various operations (proper pushforward, pullback by a holomorphic map), and the nearby/vanishing cycle theory (the V-filtration) is well-defined for holonomic $\mathcal{D}$-modules without any other assumption, so that the issue concerning nearby/vanishing cycles of filtered holonomic $\mathcal{D}$-modules only comes from the filtration.

In order to define the Hodge properties, we use the same method as in dimension 1 (see Chapter 7):

• as in Section 7.4.a, we work in the ambient abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, which has been defined in Section 12.7;

• the definition of the category $\mathsf{pHM}(X, w)$ of polarizable Hodge modules of weight w is obtained by induction on the dimension of the support of the triples entering the definition; contrary to dimension 1, many steps may be needed before reaching the case of polarizable Hodge structures.

The definition of a polarizable Hodge module can look frightening: in order to check that an object $M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ belongs to $\mathsf{pHM}(X, w)$, we have to consider in an inductive way nearby cycles with respect to *all* germs of holomorphic functions.

That the category of polarizable Hodge modules is non-empty is a non trivial fact. Already, it is not obvious at all that $\mathcal{O}_X$ underlies a polarizable Hodge module when $\dim X \geqslant 2$. The reason is that the definition involves considering nearby and vanishing cycles along *any* germ of holomorphic function, whose singularities can be arbitrarily complicated. In dimension 1, holomorphic functions are just powers of coordinates, and this explains why the property is easier to check. The higher-dimensional case will be proved in Theorem 14.6.1.

The question should however be considered the other way round. Once we know at least one polarizable Hodge module (e.g. a polarizable variation of Hodge structure, according to Theorem 14.6.1), we automatically know an infinity of them, by considering (monodromy-graded) nearby or vanishing cycles with respect to *any* holomorphic function and pushforward by any projective morphism, by the Hodge-Saito theorem 14.3.1.

In the same vein, due to this inductive definition, the proof of many properties of polarizable Hodge modules can be done by induction on the dimension of the support, and this reduces to checking the property for polarizable Hodge structures.

14.2. Definition and first properties of polarizable Hodge modules

The notion of a polarizable Hodge module will be defined by induction on the dimension of the support, and we will make extensive use of the properties of the abelian category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ introduced in Section 12.7, in particular the definitions relative to nearby/vanishing cycles (Section 12.7.14). We mimic the definitions in dimension 1.

14.2.1. Definition **(of a polarizable Hodge module of weight w)**

The category $\mathsf{pHM}(X,w)$ of polarizable Hodge modules of weight w on X is the full subcategory of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ whose objects $\widetilde{\mathcal{T}}$ are holonomic and for which there exists a morphism $\mathrm{S}:\widetilde{\mathcal{T}}\to\widetilde{\mathcal{T}}^*(-w)$ such that $(\widetilde{\mathcal{T}},\mathrm{S})$ is a polarized Hodge module of weight w on X in the sense of Definition 14.2.2 below.

We will denote by M a triple which is a polarizable Hodge module and by $\mathsf{pHM}(X,w)$ the full subcategory of the category of holonomic $\widetilde{\mathcal{D}}_X$-triples whose objects are polarizable Hodge modules of weight w. Objects of $\mathsf{pHM}(X,w)$ can be represented either by left or right triples, by using the corresponding definition for the functors in the left or right case. The definition below has to be understood in an inductive way, with respect to the dimension of the support of a triple.

14.2.2. Definition **(of a polarized Hodge module of weight w)**

Let $\widetilde{\mathcal{T}}$ be an object of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$ which is holonomic, and let $\mathrm{S}:\mathcal{T}\to\widetilde{\mathcal{T}}^*(-w)$ be a morphism ($w\in\mathbb{Z}$).

(0) If $\dim\operatorname{Supp}\widetilde{\mathcal{T}}=0$ and ι denotes the inclusion $\operatorname{Supp}\widetilde{\mathcal{T}}\hookrightarrow X$, we say that $(\widetilde{\mathcal{T}},\mathrm{S})$ is a *polarized Hodge module of weight* w on X if

$$(\widetilde{\mathcal{T}},\mathrm{S})\simeq\bigoplus_{x\in\operatorname{Supp}\widetilde{\mathcal{T}}}{}_{\mathrm{T}}\iota_*(H_x,\mathrm{S}_x),$$

where each (H_x,S_x) is a polarized Hodge structure of weight w.

($>$0) For $d\geqslant 1$, assume we have defined polarized Hodge module of weight w having support of dimension $<d$. and let $(\widetilde{\mathcal{T}},\mathrm{S})$ be such that $\dim\operatorname{Supp}\widetilde{\mathcal{T}}=d$. We say that $(\widetilde{\mathcal{T}},\mathrm{S})$ is a *polarized Hodge module of weight* w on X if $\widetilde{\mathcal{T}}$ is *strict* and for any open set $U\subset X$ and any holomorphic function $g:U\to\mathbb{C}$,

$(1)_g$ $\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g);

$(2)_g$ if moreover $g^{-1}(0)\cap\operatorname{Supp}\widetilde{\mathcal{T}}$ has everywhere codimension 1 in $\operatorname{Supp}\widetilde{\mathcal{T}}$, then for every $\ell\geqslant 0$ and every $\lambda\in\mathbb{S}^1$,

(a) $\mathrm{P}_\ell\psi_{g,\lambda}(\widetilde{\mathcal{T}},\mathrm{S})$ is a polarized Hodge module of weight $w+\ell-1$ on U,

(b) $\mathrm{P}_\ell\phi_{g,1}(\widetilde{\mathcal{T}},\mathrm{S})$ is a polarized Hodge module of weight $w+\ell$ on U.

(See (12.7.17 $*$) for the objects considered in $(2)_g$.) Note that, by the strictness assumption, $\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}''$ correspond to coherently F-filtered holonomic $\mathcal{D}_X$-modules $(\mathcal{M}',F_\bullet\mathcal{M}')$ and $(\mathcal{M}'',F_\bullet\mathcal{M}'')$.

14.2.3. Remarks. Let us already emphasize some properties that will be proved in Theorem 14.2.17 below, or are a consequence of this theorem.

(1) The restriction on g in $(2)_g$ can be relaxed, and in fact $(2)_g$ holds for any g.

(2) The morphism S, that we call a *polarization of* $\widetilde{\mathcal{T}}$ is in fact a pre-polarization of weight w of the triple $\widetilde{\mathcal{T}}$, that is, a Hermitian isomorphism.

(3) If Properties 14.2.2$(1)_g$ and $(2)_g$ are satisfied, then so are 14.2.2$(1)_{g^r}$ and $(2)_{g^r}$ for any $r\geqslant 2$. This follows from Section 12.7.22.

(4) If $(\widetilde{\mathcal{T}},\mathrm{S})$ satisfies $(1)_g$, $(2\mathrm{a})_g$ and is a middle extension along (g), then it also satisfies $(2\mathrm{b})_g$. This follows from the vanishing cycle theorem 14.2.22.

14.2.a. First properties of $\mathsf{pHM}(X,w)$

14.2.4. Hermitian duality. Hermitian duality in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ exchanges $\mathsf{pHM}(X,w)$ with $\mathsf{pHM}(X,-w)^{\mathrm{op}}$.

14.2.5. Tate twist. The Tate twist (ℓ) in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ sends the category $\mathsf{pHM}(X,w)$ to $\mathsf{pHM}(X,w+2\ell)$. More precisely, if S is a polarization of M, then $(-1)^\ell\mathrm{S}$ is a polarization of $M(\ell)$.

14.2.6. Strictness of N. We also note that, for an object M of $\mathsf{pHM}(X,w)$ and for any function $g:U\to\mathbb{C}$ such that $g^{-1}(0)\cap\operatorname{Supp}M$ has everywhere codimension 1 in $\operatorname{Supp}M$, the morphism N is strict on $\psi_{g,\lambda}M$ and $\phi_{g,1}M$: this follows from Proposition 9.4.10. We will relax below the restriction on g.

14.2.7. Stability by direct sums and isomorphisms. The category $\mathsf{pHM}(X,w)$ is stable by direct sums in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$: this is clear for polarizable Hodge structures of weight w in the category $\widetilde{\mathbb{C}}\text{-}\mathsf{Triples}$ (see Section 5.2), and the general case follows by induction on the dimension of the support. Similarly, we obtain that any object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ which is isomorphic of an object of $\mathsf{pHM}(X,w)$ is an object of $\mathsf{pHM}(X,w)$.

14.2.8. Stability by direct summands. The category $\mathsf{pHM}(X,w)$ is stable by direct summand in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$. More precisely, if $\widetilde{\mathcal{T}}_1\oplus\widetilde{\mathcal{T}}_2=M$ is in $\mathsf{pHM}(X,w)$ and if S is a polarization of M, then $\widetilde{\mathcal{T}}_1,\widetilde{\mathcal{T}}_2$ are in $\mathsf{pHM}(X,w)$ and S induces a polarization on each of them. Indeed, the property of coherence and holonomicity restricts to direct summands, as well as strictness and the property of strict $\mathbb{R}$-specializability along any g (see Exercise 9.22(1)). We then argue by induction on the dimension of the support, the case of dimension zero reducing to Lemma 5.2.8 and Exercise 2.12(1). If the support has dimension $\geqslant 1$, let S_1 the morphism $\widetilde{\mathcal{T}}_1\to\widetilde{\mathcal{T}}_1^*(-w)$ induced by S. Then, for any g such that $g^{-1}(0)\cap\operatorname{Supp}M$ has everywhere codimension 1 in $\operatorname{Supp}M$, $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}$ induces $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}_1$ on $\mathrm{P}_\ell\psi_{g,\lambda}M_1$, and this is a polarization by the induction hypothesis. A similar property holds for $\phi_{g,1}$, showing that $(\widetilde{\mathcal{T}}_1,\mathrm{S}_1)$ satisfies $(2)_g$.

14.2.9. Proposition **(Kashiwara's equivalence).** *Let $Z\xhookrightarrow{\iota}X$ be a closed analytic submanifold of the analytic manifold X. The functor ${}_{\mathrm{T}}\iota_*$ induces an equivalence between* $\mathsf{pHM}(Z,w)$ *and* $\mathsf{pHM}_Z(X,w)$ *(objects supported on Z).*

Proof. Full faithfulness follows from Section 12.7.28. It follows that essential surjectivity is a local question, and more precisely, if essential surjectivity holds locally for polarized objects (M,S), it holds globally. In the local setting, we can argue by induction and assume that $Z=H$ is a smooth hypersurface. Then Proposition 9.6.6 (and its obvious variant for sesquilinear pairings, hence for objects in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$), implies the assertion by induction on $\dim X$. $\square$

14.2.10. Proposition **(Generic structure of polarizable Hodge modules)**

Let M be an object of $\mathsf{pHM}(X,w)$ *with support on an irreducible closed analytic set $S\xhookrightarrow{\iota}X$. Then there exists an open dense set $S^o\subset S$ and a smooth Hodge triple H*

of weight w on S^o, such that $M_{|S^o} = {}_{\mathrm{T}}\iota_* H$. In particular, if $S = X$, then $M_{|X^o}$ is a smooth Hodge triple of weight w.

Note that we use Definition 5.4.7 for a smooth Hodge triple, in order to have an object similar to the object M. By definition, it corresponds to a polarizable variation of Hodge structure of weight $w - \dim S$ on S^o.

Proof. Set $M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ and let S be a polarization. We first restrict to the smooth locus of S and apply Kashiwara's equivalence 14.2.9 to reduce to the case when $S = X$. By Corollary 9.7.13 (that we can apply since M is strict), there exists a dense open subset X^o of X such that $\widetilde{\mathcal{M}}'_{|X^o}$ and $\widetilde{\mathcal{M}}''_{|X^o}$ are $\widetilde{\mathcal{O}}_{X^o}$-locally free of finite rank. Then $\mathfrak{s}_{|X^o}$ takes values in $\widetilde{\mathcal{C}}^\infty_{|X^o}$ (see Lemma 12.3.6). We now restrict to X^o and argue by induction on $\dim X$. It will be convenient to use the left setting.

Let t be a local coordinate and set $H = \{t = 0\}$. We have seen in the proof of Proposition 9.7.10 that $\mathrm{gr}_V^0 \widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}/t\widetilde{\mathcal{M}}$ for $\widetilde{\mathcal{M}} = \widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$. After Remark 12.5.18 and Example 12.5.19, $\mathrm{gr}_V^0 \mathfrak{s}$ is the restriction of $\mathfrak{s}$ to $t = 0$ as a C^∞ function. We conclude that $\psi_{t,1} M$ is the pushforward ${}_{\mathrm{T}}\iota_*^{-1} M_{|t=0}$. It is also pure of weight $w - 1$ since N is easily seen to be zero. Therefore, $M_{|t=0}$ is pure of weight $w - 1$ and, by induction on $\dim X$, is a smooth Hodge triple of weight $w - 1$. Since this holds for any H and since $\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}''$ are $\widetilde{\mathcal{O}}_X$-locally free, it is clear that M is a smooth Hodge triple of weight w. A similar argument shows that S is a polarization of this smooth Hodge triple. □

14.2.11. Caveat. At this point, we do not know the converse property that a polarizable smooth Hodge triple of weight w on X is an object of $\mathsf{pHM}(X, w)$, since we have not checked that $(2)_g$ holds for *any* nonzero g for such a triple. This will be done in Theorem 14.6.1.

14.2.b. Abelianity and the S-decomposition theorem

Before proving the main properties of $\mathsf{pHM}(X, w)$, we introduce other categories which will prove useful at some intermediate steps.

14.2.12. The category of W-filtered Hodge modules. As a first approximation of the category of mixed Hodge modules, we consider the category $\mathsf{WHM}(X)$: this is the full subcategory of $\mathsf{W}\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ (see Section 2.6.b) such that, for each object $(\widetilde{\mathcal{T}}, W_\bullet \widetilde{\mathcal{T}})$, the graded object $\mathrm{gr}_\ell^W \widetilde{\mathcal{T}}$ belongs to $\mathsf{pHM}(X, \ell)$. We denote the objects of $\mathsf{WHM}(X)$ as $(M, W_\bullet M)$. We can regard each $\mathsf{pHM}(X, w)$ as a full subcategory of $\mathsf{WHM}(X)$ by considering on M the filtration $W_\bullet$ which jumps at w only.

14.2.13. The category of polarizable Hodge-Lefschetz modules. We also consider the category $\mathsf{pHLM}(X, w)$ of *polarizable Hodge-Lefschetz modules with central weight w*. An object in this category consists of a Lefschetz triple $(\widetilde{\mathcal{T}}, \mathrm{N})$ (see Section 12.7.11), that is, $\widetilde{\mathcal{T}}$ is an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ and N is a nilpotent endomorphism of $\widetilde{\mathcal{T}}$, such that there exists a pre-polarization $\mathrm{S} : (\widetilde{\mathcal{T}}, \mathrm{N}) \to (\widetilde{\mathcal{T}}, \mathrm{N})^*(-w)$ of weight w satisfying (see Section 5.3)

• $(\mathrm{P}_\ell\widetilde{\mathfrak{T}}, (-1)^\ell \mathrm{P}_\ell \mathrm{S})$ is a polarized Hodge module of weight $w+\ell$ for every $\ell \geqslant 0$, where $\mathrm{P}_\ell\mathrm{S}$ is the morphism $\mathrm{P}_\ell\mathrm{S} : \mathrm{P}_\ell\widetilde{\mathfrak{T}} \to (\mathrm{P}_\ell\widetilde{\mathfrak{T}})^*(-(w+\ell))$ defined in a way similar to that of Sections 3.2.11 and 3.4.c.

We denote an object of $\mathsf{pHLM}(X,w)$ as (M,N) and we also say that $(M,\mathrm{N},\mathrm{S})$ is a *polarized Hodge-Lefschetz module with central weight* w. From the Lefschetz decomposition, we deduce that, setting $W_kM := \mathrm{M}_{k-w}M$ (i.e., $\mathrm{M}_\ell = W_{w+\ell}$), $(M, W_\bullet)$ is an object of $\mathsf{WHM}(X)$ (but $\mathsf{pHLM}(X,w)$ is not a full subcategory of $\mathsf{WHM}(X)$, since morphisms have to commute with N).

14.2.14. Caveat. We do not claim that objects and morphisms in $\mathsf{WHM}(X)$ or $\mathsf{pHLM}(X,w)$ are strictly specializable along any (g). On the other hand, objects and morphisms in the graded category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ defined below are so, since they are graded with respect to the weight or monodromy filtration.

14.2.15. The category of polarizable Hodge-Lefschetz quivers. In a way similar to that of Definition 3.4.19, we also define the notion of *polarized/polarizable Hodge-Lefschetz quiver with central weight* w, starting from a Lefschetz quiver in $\widetilde{\mathfrak{D}}\text{-}\mathsf{Triples}(X)$ (defined in a way similar to what is done in Section 5.3.6): such an object consists of the data $\big((M,\mathrm{N},\mathrm{S}), (M_1,\mathrm{N}_1,\mathrm{S}_1), \mathrm{c}, \mathrm{v}\big)$, where the first terms are polarized Hodge-Lefschetz modules of weight $w-1$ and w respectively, and $\mathrm{c} : M \to M_1$ and $\mathrm{v} : M_1 \to M(-1)$ are morphisms in $\widetilde{\mathfrak{D}}\text{-}\mathsf{Triples}(X)$ such that $\mathrm{v}\circ\mathrm{c} = \mathrm{N}$, $\mathrm{c}\circ\mathrm{v} = \mathrm{N}_1$ and the following diagram commutes (see (3.2.14)):

$$\begin{array}{ccc} M_1 & \xrightarrow{\ \mathrm{S}_1\ } & M_1^*(-w) \\ {\scriptstyle \mathrm{v}}\big\downarrow & & \big\downarrow{\scriptstyle -\mathrm{c}^*} \\ M(-1) & \xrightarrow{\ \mathrm{S}\ } & M^*(-w) \end{array}$$

The corresponding category is denoted by $\mathsf{pHLQ}(X,w)$.

We can rephrase Condition $(2)_g$ of Theorem 14.2.2 as follows:

$(2')_g$ if moreover $g^{-1}(0)\cap \operatorname{Supp}\widetilde{\mathfrak{T}}$ has everywhere codimension 1 in $\operatorname{Supp}\widetilde{\mathfrak{T}}$, then for every $\ell \geqslant 0$ and every $\lambda\in\mathbb{S}^1$,

(a) for each $\lambda\in\mathbb{S}^1\smallsetminus\{1\}$, $(\psi_{g,\lambda}\widetilde{\mathfrak{T}}, \mathrm{N}, \psi_{g,\lambda}\mathrm{S})$ is an object of $\mathsf{pHLM}(X,w-1)$,

(b) the set of data $\big((\psi_{g,1}\widetilde{\mathfrak{T}}, \psi_{g,1}\mathrm{S}), (\phi_{g,1}\widetilde{\mathfrak{T}}, \phi_{g,1}\mathrm{S}), \mathrm{can}, \mathrm{var}\big)$ is a polarized object of $\mathsf{pHLQ}(X,w)$.

Indeed, the only properties which need a check are those for can and var, and they have been proved in 12.7.16.

14.2.16. The category of polarizable $\mathfrak{sl}_2$-Hodge modules. The category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ of *polarizable* $\mathfrak{sl}_2$*-Hodge modules with central weight* w consists of objects of $\mathsf{pHLM}(X,w)$ which are *graded* with respect to their monodromy filtration $\mathrm{M}_\bullet$. Morphisms should also be graded. A polarization of an object $(M_\bullet,\mathrm{N})$ of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ is by definition a polarization of $(M_\bullet,\mathrm{N})$ as an object of $\mathsf{pHLM}(X,w)$ which is *graded.* This is not a restrictive condition since the conditions on the polarization S in $\mathsf{pHLM}(X,w)$ concern $\mathrm{gr}\mathrm{S}$ (see Section 3.4.c).

Therefore, given an object (M,N) of $\mathsf{pHLM}(X,w)$, the graded object $(\mathrm{gr}^{\mathrm{M}}_{\bullet}M,\mathrm{grN})$ is an object of $\mathsf{p\mathfrak{sl}_2HM}(X,w)$ and, conversely, any object of $\mathsf{p\mathfrak{sl}_2HM}(X,w)$ is an object of $\mathsf{pHLM}(X,w)$ (by forgetting the grading). On the other hand, morphisms in $\mathsf{p\mathfrak{sl}_2HM}(X,w)$ are graded with respect the given grading. So, by definition, there is a functor gr^{M} from $\mathsf{pHLM}(X,w)$ to $\mathsf{p\mathfrak{sl}_2HM}(X,w)$. We denote by $(M_\bullet,\rho)$ an object of $\mathsf{p\mathfrak{sl}_2HM}(X,w)$, where ρ is meant for the corresponding $\mathfrak{sl}_2$-representation with $\rho(\mathrm{H})$ defined by means of the grading and $\rho(\mathrm{Y})=\mathrm{N}$.

One can set $\mathrm{H}=\ell\,\mathrm{Id}$ on M_ℓ and, due to Proposition 3.1.6 applied with the category $\mathsf{A}=\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, one can extend uniquely $\mathrm{Y}=\mathrm{N},\mathrm{H}$ as an $\mathfrak{sl}_2$-triple $\mathrm{X},\mathrm{Y},\mathrm{H}$. Then X induces morphisms $M_\ell\to M_{\ell+2}(1)$ in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$.

On the other hand, given a finitely $\mathbb{Z}$-graded object $M_\bullet$ of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ endowed with an endomorphism X which satisfies the Lefschetz property, there is a unique action of Y defining a representation ρ of $\mathfrak{sl}_2$ on $M_\bullet$ such that H is defined by means of the grading. We then say that $(M_\bullet,\mathrm{X})$ is an object of $\mathsf{p\mathfrak{sl}_2HM}(X,w)$ if $(M_\bullet,\rho)$ is so.

We set $(M^*)_\ell=(M_{-\ell})^*$. Then $M^*(-w)$ is also an object of $\mathsf{p\mathfrak{sl}_2HM}(X,w)$. By definition, a polarization S of M is a (graded, by definition) morphism $\mathrm{S}:M\to M^*(-w)$ such that $(-1)^\ell\mathrm{P}_\ell\mathrm{S}$ is a polarization of $\mathrm{P}_\ell M$ for every $\ell\geqslant 0$.

14.2.17. Theorem **(Main properties of polarizable Hodge modules)**

(1) *Any object* $M=(\widetilde{\mathcal{M}}',\widetilde{\mathcal{M}}'',\mathfrak{s})$ *of* $\mathsf{pHM}(X,w)$ *is S-decomposable in* $\mathsf{pHM}(X,w)$, *and the components of the pure support of* $\widetilde{\mathcal{M}}'$ *and* $\widetilde{\mathcal{M}}''$ *are the same.*

(2) *There is no nonzero morphism (in* $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$*) from an object in* $\mathsf{pHM}(X,w_1)$ *to an object in* $\mathsf{pHM}(X,w_2)$ *if* $w_1>w_2$.

(3) *Property* 14.2.2(2)$_g$ *holds without any restriction on* g.

(4) *The category* $\mathsf{pHM}(X,w)$ *is abelian. Any morphism is strict and strictly specializable along any* (g).

(5) *Any polarization of an object of* $\mathsf{pHM}(X,w)$ *or* $\mathsf{pHLM}(X,w)$ *is a Hermitian isomorphism (i.e., a pre-polarization of weight* w *of the corresponding triple).*

(6) *If* M_1 *is a subobject of* M *in* $\mathsf{pHM}(X,w)$, *then it is a direct summand and a polarization* S *of* M *induces a polarization on each summand.*

(7) *The category* $\mathsf{p\mathfrak{sl}_2HM}(X,w)$ *is abelian. Any morphism is strict and strictly specializable along any* (g). *Any sub-object of an object* $(M_\bullet,\rho)$ *in* $\mathsf{p\mathfrak{sl}_2HM}(X,w)$ *is a direct summand and a polarization of* $(M_\bullet,\rho)$ *induces a polarization on it.*

(8) *The category* $\mathsf{WHM}(X)$ *is abelian, and any morphism is strict and strictly compatible with* $W_\bullet$.

(9) *The category* $\mathsf{pHLM}(X,w)$ *is abelian. Any morphism is strict and strictly compatible with the monodromy filtration* $\mathrm{M}_\bullet$.

(10) *Any polarizable Hodge-Lefschetz quiver* $\big(M,M_1,\mathrm{c},\mathrm{v}\big)$ *with central weight* w *satisfies* $(M_1,\mathrm{N}_1)=\operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}$ *in* $\mathsf{pHLM}(X,w)$.

Let us emphasize some direct consequences of the theorem.

14.2.18. Notation. If $S \subset X$ is a closed irreducible analytic subset, we denote by $\mathsf{pHM}_S(X,w)$ the full sub-category of $\mathsf{pHM}(X,w)$ whose objects have pure support S. By the S-decomposition property 14.2.17(1), Any object of $\mathsf{pHM}(X,w)$ resp. any morphism between objects of $\mathsf{pHM}(X,w)$ decomposes as the direct sum of objects resp. morphisms in of $\mathsf{pHM}_{S_i}(X,w)$ for a suitable locally finite family of closed irreducible analytic subsets $S_i \subset X$.

14.2.19. Corollary. *Given two objects M_1, M_2 in $\mathsf{pHM}(X,w)$, any morphism between them (as objects of $\widetilde{\mathfrak{D}}$-$\mathsf{Triples}(X)$) has kernel, image and cokernel in $\mathsf{pHM}(X,w)$; and a corresponding statement for $\mathsf{pHLM}(X,w)$ and $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$.* $\square$

***14.2.20. Corollary* (S-decomposition theorem and semi-simplicity for $\mathsf{pHM}(X,w)$)**

(1) *Each object M decomposes uniquely into the direct sum of objects in $\mathsf{pHM}(X,w)$ having pure support a closed irreducible analytic subset of X.*

(2) *The category $\mathsf{pHM}(X,w)$ is semi-simple (all objects are semi-simple and morphisms between simple objects are zero or isomorphisms).*

(3) *The category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ is semi-simple.* $\square$

14.2.21. Corollary. *If M is an object of $\mathsf{pHM}(X,w)$ with polarization S, then for every open subset $U \subset X$ and every holomorphic function $g : U \to \mathbb{C}$,*

(1) *for every $\ell \geqslant 1$, $\mathrm{N}^\ell : \psi_{g,\lambda}M \to \psi_{g,\lambda}M(-\ell)$ and $\phi_{g,1}M \to \phi_{g,1}M(-\ell)$ are strict and strictly shift $\mathrm{M}_\bullet(\mathrm{N})$ by 2ℓ, and a similar property holds for grN^ℓ,*

(2) $\mathrm{can} : \psi_{g,1}M \to \phi_{g,1}M$ *and* $\mathrm{var} : \phi_{g,1}M \to \psi_{g,1}M(-1)$ *are strict and strictly shift $\mathrm{M}_\bullet$ by 1.* $\square$

***14.2.22. Corollary* (The vanishing cycle theorem).** *Let $(M,\mathrm{N},\mathrm{S})$ be a polarized object of $\mathsf{pHLM}(X,w-1)$. Let us endow $(\mathrm{Im}\,\mathrm{N}, \mathrm{N}_{|\mathrm{Im}\,\mathrm{N}})$ with the morphism*

$$\mathrm{S}_1 : (\mathrm{Im}\,\mathrm{N}, \mathrm{N}_{|\mathrm{Im}\,\mathrm{N}}) \longrightarrow (\mathrm{Im}\,\mathrm{N}, \mathrm{N}_{|\mathrm{Im}\,\mathrm{N}})(-w)$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Im}\,\mathrm{N} \arrow[r, "\mathrm{S}_1"] \arrow[d, "\mathrm{incl.}"'] & (\mathrm{Im}\,\mathrm{N})^*(-w) \arrow[d, "\mathrm{N}^*"] \\
M(-1) \arrow[r, "\mathrm{S}"] & M^*(-w)
\end{tikzcd}$$

Then $(\mathrm{Im}\,\mathrm{N}, \mathrm{N}_{|\mathrm{Im}\,\mathrm{N}}, \mathrm{S}_1)$ a polarized object of $\mathsf{pHLM}(X,w)$.

Proof. We use the same argument as in the proof of Proposition 3.4.20. Strictness of $\mathrm{can} = \mathrm{N} : M \to \mathrm{Im}\,\mathrm{N}$, $\mathrm{var} = \mathrm{incl.} : \mathrm{Im}\,\mathrm{N} \to M(-1)$, $\mathrm{can}^* = \mathrm{N}^*$, and S (according to 14.2.17(9)) enables us to reduce the problem to the graded case. We note that, arguing as in (3.2.16), for $\ell \geqslant 0$, the isomorphism $\mathrm{can} : \mathrm{P}_{\ell+1}M \xrightarrow{\sim} \mathrm{P}_\ell M_1$ transports the polarization $(-1)^{\ell+1}\mathrm{P}_{\ell+1}\mathrm{S}$ to $(-1)^\ell \mathrm{P}_\ell \mathrm{S}_1$. $\square$

14.2.23. Corollary. *Given any morphism $\varphi : M_1 \to M_2$ between objects of $\mathsf{pHM}(X,w)$ and any germ g of holomorphic function on X, then, the specialized morphisms $\psi_{g,\lambda}\varphi$ ($\lambda \in \mathbb{S}^1$) and $\phi_{g,1}\varphi$ are strictly compatible with the monodromy filtration $\mathrm{M}_\bullet$ and,* *for every $\ell \in \mathbb{Z}$, $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$ (and similarly $\mathrm{gr}_\ell^{\mathrm{M}}\phi_{g,1}\varphi$) decomposes with respect to the Lefschetz decomposition, i.e.,*

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi = \begin{cases} \bigoplus_{k\geqslant 0} \mathrm{N}^k \mathrm{P}_{\ell+2k}\psi_{g,\lambda}\varphi & (\ell \geqslant 0), \\ \bigoplus_{k\geqslant 0} \mathrm{N}^{k-\ell} \mathrm{P}_{-\ell+2k}\psi_{g,\lambda}\varphi & (\ell \leqslant 0). \end{cases}$$

In particular we have

$$\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\,\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$$

and similarly for Coker*, where, on the left side, the filtration* $\mathrm{M}_\bullet$ *is that induced naturally by* $\mathrm{M}_\bullet\psi_{g,\lambda}M_1$ *or, equivalently, the monodromy filtration of* N *acting on* $\psi_{g,\lambda}\,\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\psi_{g,\lambda}\varphi$. □

14.2.24. Corollary. *If M is in $\mathsf{pHM}(X,w)$, then the Lefschetz decomposition for $\mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M$ ($\lambda \in \mathbb{S}^1$) resp. $\mathrm{gr}_\ell^{\mathrm{M}}\phi_{g,1}M$ holds in $\mathsf{pHM}(X,w-1+\ell)$ resp. $\mathsf{pHM}(X,w+\ell)$.*

Proof. Indeed, $\mathrm{N} : \mathrm{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}M \to \mathrm{gr}_{\ell-2}^{\mathrm{M}}\psi_{g,\lambda}M(-1)$ is a morphism in the category $\mathsf{pHM}(X,w-1+\ell)$, which is abelian, so the primitive part is an object of this category, and therefore each term of the Lefschetz decomposition is also an object of this category. □

Similarly to Proposition 7.4.9, we can simplify the data of a polarizable Hodge module.

14.2.25. Proposition **(Simplified form for an object of $\mathsf{pHM}(X,w)$ or $\mathsf{pHLM}(X,w)$)**
Any object M of $\mathsf{pHM}(X,w)$ resp. (M,N) of $\mathsf{pHLM}(X,w)$, resp. $(M_\bullet,\mathrm{N})$ of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$, is isomorphic to an object of the form

$$\big((\mathcal{M},F^\bullet\mathcal{M}),(\mathcal{M},F^\bullet\mathcal{M})(w),\mathcal{S}\big)$$

(resp. ...) such that $\mathcal{S}^* = \mathcal{S}$ and with polarization $(\mathrm{Id},\mathrm{Id}) : M \to M^*(-w)$. □

We call the data $\big((\mathcal{M},F^\bullet\mathcal{M}),\mathcal{S}\big)$ a *Hodge-Hermitian pair of weight w* (resp. *Hodge-Lefschetz Hermitian pair with central weight w*, resp. $\mathfrak{sl}_2$*-Hodge Hermitian pair with central weight w*) if the corresponding triple $\big((\mathcal{M},F^\bullet\mathcal{M}),(\mathcal{M},F^\bullet\mathcal{M})(w),\mathcal{S}\big)$ with polarization $(\mathrm{Id},\mathrm{Id})$ is polarized Hodge module of weight w (resp. ...).

14.2.26. Example **(of filtered Hermitian pairs).** We consider the following corresponding filtered Hermitian pairs (see Example 12.3.5)

$${}_{\mathrm{H}}\mathcal{O}_X := ((\mathcal{O}_X,F_\bullet\mathcal{O}_X),\mathfrak{s}_n^{\mathrm{left}}),\quad {}_{\mathrm{H}}\omega_X := ((\omega_X,F_\bullet\omega_X),\mathfrak{s}_n^{\mathrm{right}}).$$

We will prove in Theorem 14.6.1 that they are Hodge-Hermitian pairs of weight n. The case where $n=1$ is a consequence of the results in Chapter 7 (see Exercise 14.1).

***Proof of Theorem* 14.2.17.** It is done by induction on the dimension of the support. By the point (0) in Definition 14.2.2, the categories of objects with support equal to a point as considered in the theorem are equivalent to the corresponding categories for $X = \mathrm{pt}$. In such a case, the assertions of the theorem are proved in Chapters 2 and 3.

We will thus fix $d \geqslant 1$ and assume that the assertions are proved for the subcategories consisting of objects having support of dimension $< d$, in order to prove them when the dimension of the support of M is d.

$(10)_{<d} \Rightarrow (1)_d$. Let $x_o \in \operatorname{Supp} M$ and let g be a germ of holomorphic function at x_o such that $g^{-1}(0) \cap \operatorname{Supp} M$ has everywhere codimension 1 in M. By Condition $14.2.13(2')_g$, the nearby/vanishing quiver of M along (g) satisfies the assumption of $(10)_{<d}$, hence its conclusion, so M is S-decomposable along (g) in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$. By (14.2.8), the summands also belong to $\mathsf{pHM}(X, w)$. This proves S-decomposability in $\mathsf{pHM}(X, w)$.

We assume that there is a pure component S' of $\widetilde{\mathcal{M}}'$ which is not a pure component of $\widetilde{\mathcal{M}}''$. Then we have an summand $(\widetilde{\mathcal{M}}'_{S'}, 0, 0)$ of M in $\mathsf{pHM}(X, w)$, according to the previous argument. We wish to show that $\widetilde{\mathcal{M}}'_{S'} = 0$, and it is enough, by the condition of the pure support, to show the vanishing on the smooth locus of S'. We can thus reduce to the case where $S' = X$, according to Proposition 9.7.10.

We now argue by induction on $\dim X$, the case $\dim X = 0$ reducing to the case of Hodge structures, which is easy. Let t be a local coordinate on X. Arguing as in Corollary 9.7.11, one checks that $\widetilde{\mathcal{M}}'_X / t\widetilde{\mathcal{M}}'_X = \psi_{t,1}\widetilde{\mathcal{M}}'_X$, and that $\psi_{t,\lambda}\widetilde{\mathcal{M}}'_X = 0$ for $\lambda \in \mathbb{S}^1 \smallsetminus \{1\}$, as well as $\phi_{t,1}\widetilde{\mathcal{M}}'_X = 0$. It follows that $\mathrm{N} = 0$, so $\psi_{t,1}\widetilde{\mathcal{M}}'_X$ is S-decomposable, according to Condition $14.2.2(2)_t$. By induction, the object $\psi_{t,1}(\widetilde{\mathcal{M}}'_X, 0, 0)$ is zero. Hence $\widetilde{\mathcal{M}}'_X / t\widetilde{\mathcal{M}}'_X = 0$, and by applying Nakayama's lemma as in Corollary 9.7.11, we obtain $\widetilde{\mathcal{M}}_X = 0$.

$(1)_d \Rightarrow (2)_d$. Since any morphism between S-decomposable objects decomposes correspondingly, it is enough to consider a morphism $\varphi : M_1 \to M_2$ between polarizable Hodge modules of respective weights w_1, w_2 having pure support. Since the result is clear for polarizable variations of Hodge structure (see Proposition 2.5.5(2)), it follows from Proposition 14.2.10 that the support of $\operatorname{Im}\varphi$ is strictly smaller than S. By definition of the pure support (see Definition 9.7.9), this implies that $\operatorname{Im}\varphi = 0$.

$(1)_d \Rightarrow (3)_d$. The question is local at x_o and by assumption we can assume that M_{x_o} has pure support a closed irreducible subset $S_{x_o} \subset X_{x_o}$. Let $g : X_{x_o} \to \mathbb{C}$ be a germ of holomorphic function. If g is non-constant on S_{x_o}, it satisfies the constraint in Definition $14.2.2(2)_g$. Otherwise, $\operatorname{Supp} M_{x_o} \subset |g^{-1}(0)|$ and Proposition 12.7.15 implies that $M_{x_o} = \phi_{g,1} M_{x_o}$ (and similarly $\mathrm{S} = \phi_{g,1}\mathrm{S}$). Moreover, $\psi_{g,\lambda} M_{x_o} = 0$ for any $\lambda \in \mathbb{S}^1$, and $\mathrm{N} = 0$. Hence, if M_{x_o} is an object of $\mathsf{pHM}((X, x_o), w)$, $14.2.2(2)_g$ obviously holds.

$(4)_{<d}$, $(6)_{<d}$ & $(8)_{<d} \Rightarrow (4)_d$. The question is local. Let $\varphi : M_1 \to M_2$ be a morphism in $\mathsf{pHM}(X, w)$ between objects having support in dimension d. Then, by $(8)_{<d}$ applied to $\psi_{g,\lambda}\varphi, \phi_{g,1}\varphi$, for any germ g satisfying the constraint of Definition $14.2.2(2)_g$, $\varphi : M_1 \to M_2$ is strictly $\mathbb{R}$-specializable along (g) and Corollary 10.8.6 implies that it is strict. Moreover, $\psi_{g,\lambda}\varphi$ and $\phi_{t,1}\varphi$ are strict with respect to the monodromy filtrations, since these are weight filtrations up to a shift.

It remains to check that $\operatorname{Ker}\varphi, \operatorname{Im}\varphi; \operatorname{Coker}\varphi$ belong to $\mathsf{pHM}(X,w)$. Let us check this for $\operatorname{Ker}\varphi$ for example. It follows from the M-strictness above that

$$\operatorname{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\operatorname{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\varphi$$

and thus, for any $\ell \geqslant 0$, $\mathrm{P}_\ell\psi_{g,\lambda}\operatorname{Ker}\varphi = \operatorname{Ker}\mathrm{P}_\ell\psi_{g,\lambda}\varphi$. Since $\mathrm{P}_\ell\psi_{g,\lambda}\varphi$ is a morphism in $\mathsf{pHM}(X, w-1+\ell)$ between objects having support in dimension $< d$, $(4)_{<d}$ implies that $\operatorname{Ker}\mathrm{P}_\ell\psi_{g,\lambda}\varphi$ is an object of $\mathsf{pHM}(X, w-1+\ell)$ and, according to $(6)_{<d}$, is a direct summand of $\mathrm{P}_\ell\psi_{g,\lambda}M_1$. If S is a polarization of M_1, let S_φ denote the morphism induced by S on $\operatorname{Ker}\varphi$. On the one hand, the morphism induced by $-\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}$ on $\operatorname{Ker}\mathrm{P}_\ell\psi_{g,\lambda}\varphi$ is a polarization, according to 14.2.8. On the other hand, it is equal to $-\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}_\varphi$. We can argue similarly with $\phi_{g,1}$, by assumption on g. This shows that $(\operatorname{Ker}\varphi, \mathrm{S}_\varphi)$ satisfies $14.2.2(2)_g$.

$(4)_d$ & $(5)_{<d} \Rightarrow (5)_d$. A polarization S of M is a morphism $M \to M^*(-w)$, hence it is strict and strictly specializable along any (g). Let g be a holomorphic function such that $g^{-1}(0)\cap\operatorname{Supp}M$ has everywhere codimension 1 in $\operatorname{Supp}M$. $(5)_{<d}$ implies that $\mathrm{P}_\ell\psi_{g,\lambda}\mathrm{S}$ and $\mathrm{P}_\ell\phi_{g,1}\mathrm{S}$ are isomorphisms for every $\ell \geqslant 0$, which implies the same property for $\operatorname{gr}_\ell^{\mathrm{M}}\psi_{g,\lambda}\mathrm{S}$ and $\operatorname{gr}_\ell^{\mathrm{M}}\phi_{g,1}\mathrm{S}$ and thus for $\psi_{g,\lambda}\mathrm{S}$ and $\phi_{g,1}\mathrm{S}$. By strict $\mathbb{R}$-specializability, $\psi_{g,\lambda}$ and $\phi_{g,1}$ commute with taking Ker and Coker on S. We conclude that $\psi_{g,\lambda}\operatorname{Ker}\mathrm{S} = 0$ and $\phi_{g,1}\operatorname{Ker}\mathrm{S} = 0$, and similarly with Coker. Since $\operatorname{Ker}\mathrm{S}$ and $\operatorname{Coker}\mathrm{S}$ are in $\mathsf{pHM}(X,w)$ by $(4)_d$, we can apply to them the regularity property along (g) of Corollary 10.8.5, which implies they both are zero.

That S is Hermitian is obtained similarly by applying the argument to $\operatorname{Im}(\mathrm{S}-\mathrm{S}^*)$.

$(1)_d \Rightarrow (6)_d$. A polarization of M decomposes with respect to the S-decomposition of M, and it is clear that it induces a polarization on each summand. We can thus restrict to considering objects M with pure support a closed irreducible analytic subset S of X.

If $\dim S = 0$, we apply Exercise 2.12. If $\dim S \geqslant 1$, we consider the exact sequences (defining S_1)

$$\begin{tikzcd}
0 & M_1^*(-w) \arrow[l] & M^*(-w) \arrow[l, "i^*"'] & M_2^*(-w) \arrow[l] & 0 \arrow[l] \\
0 \arrow[r] & M_1 \arrow[r, "i"] \arrow[u, "\mathrm{S}_1"] & M \arrow[r] \arrow[u, "\mathrm{S}", "\wr"'] & M_2 \arrow[r] & 0
\end{tikzcd}$$

where M_2 is the cokernel, in the abelian category $\mathsf{pHM}(X,w)$, of $M_1 \hookrightarrow M$. We first show that S_1 is an isomorphism. It is enough to prove it on an open dense subset S^o of S. By Kashiwara's equivalence 14.2.9 and the generic structure 14.2.10, we are reduced to considering the case of polarizable variations of Hodge structure, which is follows from Exercise 4.2. We conclude that we have a projection $p = \mathrm{S}_1^{-1}\circ i^*\circ\mathrm{S} : M \to M_1$ such that $p\circ i = \mathrm{Id}$, and a decomposition $M = M_1 \oplus \mathrm{S}^{-1}M_2^*(-w)$. By construction, S splits correspondingly, and it is then clear that each summand is a polarization.

$(4)_d$ & $(6)_d \Rightarrow (7)_d$. Abelianity and strictness resp. strict $\mathbb{R}$-specializability of morphisms follow from $(4)_d$ in a straightforward way by the grading property. In the same way, $(6)_d$ implies the similar property for $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$.

$(4)_d$ & $(2)_d \Rightarrow (8)_d$. We note first that, since objects of $\mathsf{pHM}(X,w)$ are strict, Lemma 5.1.9(1) implies that the $\widetilde{\mathcal{D}}_X$-modules which are components of an object in $\mathsf{WHM}_{\leqslant d}(X)$ are strict. According to $(2)_d$ and Proposition 2.6.3, $(4)_d$ implies that the category $\mathsf{WHM}_{\leqslant d}(X)$ is abelian and that morphisms are strictly compatible with W. Using Lemma 5.1.9(2), we conclude that all morphisms are strict.

$(8)_d \Rightarrow (9)_d$. Since $\mathsf{pHLM}(X,w)$ is a subcategory of $\mathsf{WHM}(X)$ with the weight filtration given by the shifted monodromy filtration, strictness of morphisms and strict compatibility with $W_\bullet$ follow from $(8)_d$.

$(7)_d$, $(8)_d$ & $(10)_{<d} \Rightarrow (10)_d$. Since c,v are morphisms in $\mathsf{WHM}(X)$, they are strictly compatible with the weight filtration, due to $(8)_d$, hence strictly shift by -1 the monodromy filtrations. We then denote by $\mathrm{gr}\,\mathrm{c},\mathrm{gr}\,\mathrm{v}$ the corresponding morphisms, graded of degree -1 with respect to $\mathrm{M}_\bullet$. We then have $\mathrm{gr}^{\mathrm{M}}\operatorname{Im}\mathrm{c} = \operatorname{Im}\mathrm{gr}\,\mathrm{c}$ and $\mathrm{gr}^{\mathrm{M}}\operatorname{Ker}\mathrm{v} = \operatorname{Ker}\mathrm{gr}\,\mathrm{v}$. Moreover, the natural morphism $\operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}\to M_1$ is strict with respect to the weight filtration, hence to the monodromy filtrations. It follows that, if the graded morphism $\operatorname{Im}\mathrm{gr}\,\mathrm{c}\oplus\operatorname{Ker}\mathrm{gr}\,\mathrm{v}\to\mathrm{gr}^{\mathrm{M}}M_1$ is an isomorphism, then $M_1=\operatorname{Im}\mathrm{c}\oplus\operatorname{Ker}\mathrm{v}$, as wanted. We are therefore reduced to proving the assertion in the category of polarizable graded Hodge-Lefschetz quivers.

In such a case, $M,M_1,\mathrm{c},\mathrm{v}$ are strict and strictly $\mathbb{R}$-specializable along any (g), according to $(7)_d$, and by the regularity property (Corollary 10.8.5), it is enough to prove locally, for any holomorphic germ g, the decompositions

$$\begin{aligned}\psi_{g,\lambda}M_1 &= \operatorname{Im}\psi_{g,\lambda}\mathrm{c}\oplus\operatorname{Ker}\psi_{g,\lambda}\mathrm{v},\quad \forall\,\lambda\in\mathbb{S}^1,\\ \phi_{g,1}M_1 &= \operatorname{Im}\phi_{g,1}\mathrm{c}\oplus\operatorname{Ker}\phi_{g,1}\mathrm{v}.\end{aligned}$$

Let us argue with $\phi_{g,1}$ for example. Recall that $M=\bigoplus_\ell M_\ell$ and $M_1=\bigoplus_\ell M_{1,\ell}$, with $M_\ell\in\mathsf{pHM}(X,w-1+\ell)$ and $M_{1,\ell-1}\in\mathsf{pHM}(X,w+\ell-1)$, and that $\phi_{g,1}\mathrm{c}$ is a morphism $\phi_{g,1}M_\ell\to\phi_{g,1}M_{1,\ell-1}$. It is strictly compatible with the weight filtration on these spaces, which is nothing but $\mathrm{M}_{w+\ell-1+\bullet}(\mathrm{N}_g)$, hence with the monodromy filtration of N_g. The same argument holds for v. It is thus enough to prove

$$\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}M_{1,\ell-1} = \operatorname{Im}\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}\mathrm{c}\oplus\operatorname{Ker}\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}\mathrm{v}.$$

We can therefore apply $(10)_{<d}$ to the quiver

$$(\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}M_\ell,\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}M_{1,\ell-1},\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}\mathrm{c},\mathrm{gr}^{\mathrm{M}}_j\phi_{g,1}\mathrm{v}),$$

with central weight $w+\ell-1+j$. □

14.2.27. The category of polarizable $\mathfrak{sl}_2^{\mathrm{k}}$-Hodge modules

In the presence of k commuting nilpotent endomorphisms, we can extend the definition of the category $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(X,w)$ of $\mathfrak{sl}_2$-Hodge modules to that of the category $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X,w)$ of $\mathfrak{sl}_2^{\mathrm{k}}$-Hodge modules. The objects of $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X,w)$ are $\mathbb{Z}^k$-graded polarizable Hodge modules $M=\bigoplus_{\boldsymbol{\ell}\in\mathbb{Z}^k}M_{\boldsymbol{\ell}}$ such that

- for each $\boldsymbol{\ell}$, $M_{\boldsymbol{\ell}}$ is an object in $\mathsf{pHM}\big(X, w+\sum_i \ell_i\big)$,
- M is endowed with actions ρ_i of $\mathfrak{sl}_2$ ($i=1,\dots,k$) such that, for each i, $\mathrm{H}_i = \ell_i \operatorname{Id}$ on $M_{\boldsymbol{\ell}}$ and $\mathrm{Y}_i : M_{\boldsymbol{\ell}} \to M_{\boldsymbol{\ell}-2\mathbf{1}_i}(-1)$, $\mathrm{X}_i : M_{\boldsymbol{\ell}} \to M_{\boldsymbol{\ell}+2\mathbf{1}_i}(1)$ satisfy the isomorphism property for an $\mathfrak{sl}_2$-Hodge module, so that there is a Lefschetz k-decomposition (argue as in Exercise 3.9),
- M can be endowed with a *polarization* S, that is, a (multi) graded morphism $\mathrm{S} : M \to M^*(-w)$ (i.e., S sends $M_{\boldsymbol{\ell}}$ to $M^*_{\boldsymbol{\ell}}(-w) = (M_{-\boldsymbol{\ell}})^*(-w)$), such that each $\mathrm{Y}_i, \mathrm{X}_i$ is skew-adjoint with respect to S (i.e., S is a morphism $(M, \mathrm{Y}) \to (M, \mathrm{Y})^*(-w)$) and that, for every $\boldsymbol{\ell} = (\ell_1, \dots, \ell_k)$ with non-negative components, the induced morphism (see Section 3.4.c)

$$\mathrm{X}_1^{*\ell_1} \cdots \mathrm{X}_k^{*\ell_k} \circ \mathrm{S} : M_{-\boldsymbol{\ell}} \longrightarrow (M_{-\boldsymbol{\ell}})^*(-w-\boldsymbol{\ell})$$

induces a polarization of the object $\mathrm{P}_{-\boldsymbol{\ell}} M_{-\boldsymbol{\ell}} := \bigcap_{i=1}^k \operatorname{Ker} \mathrm{X}_i^{\ell_i+1}$ of $\mathsf{HM}(X, w-\boldsymbol{\ell})$. (One can also use the Y_i's or use alternatively Y_i's and X_j's with an obvious modification of the twists and the signs, e.g. $(-\mathrm{Y}_1)^{*\ell_1} \cdots (-\mathrm{Y}_k)^{*\ell_k} \circ \mathrm{S}$ should induce a polarization on $\mathrm{P}_{\boldsymbol{\ell}} M_{\boldsymbol{\ell}}$.)

Morphisms should be compatible with the $\mathfrak{sl}_2^{\mathrm{k}}$-structure, hence k-graded of k-degree zero. The category is abelian, and any morphism is strict and strictly $\mathbb{R}$-specializable (this is proved as 14.2.17(7)).

14.2.28. Lemma. *Let $(M, \mathrm{X}, \mathrm{Y}, \mathrm{H})$ be an object of the category $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X, w)$ and let g be a germ of holomorphic function. Then, for every $\lambda \in \mathbb{S}^1$, the graded nearby cycle object $\big(\mathrm{gr}^{\mathrm{M}}_{\bullet} \psi_{g,\lambda} M, (\mathrm{gr}^{\mathrm{M}}_{\bullet} \psi_{g,\lambda} \mathrm{Y}, \mathrm{N}_g)\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2^{k+1}\mathsf{HM}\big(X, w-1\big)$ and for each $\boldsymbol{\ell} \in \mathbb{Z}^k$ and $\ell \in \mathbb{Z}$, $\mathrm{P}_{\boldsymbol{\ell}} \mathrm{gr}^{\mathrm{M}}_{\ell} \psi_{g,\lambda} M_{\boldsymbol{j}} = \mathrm{gr}^{\mathrm{M}}_{\ell} \psi_{g,\lambda} \mathrm{P}_{\boldsymbol{\ell}} M_{\boldsymbol{j}}$, where $\mathrm{P}_{\boldsymbol{\ell}}$ denotes the multi-primitive part. A similar statement holds with $\phi_{g,1}$ and $\mathsf{p}\mathfrak{sl}_2^{k+1}\mathsf{HM}\big(X, w\big)$.*

Proof. The lemma is a direct consequence of the strict compatibility of $\psi_{t,\lambda}\mathrm{X}_i, \psi_{t,\lambda}\mathrm{Y}_i$ with the monodromy filtration $\mathrm{M}(\mathrm{N}_g)$, as follows from 14.2.17(9) applied to the morphisms $\mathrm{X}_i, \mathrm{Y}_i$. $\square$

14.2.29. Lemma. *The category $\mathsf{p}\mathfrak{sl}_2^k\mathsf{HM}(X, w)$ has an inductive definition as in Definition 14.2.2. Furthermore, Properties 14.2.17(5)–(7) hold for this category.*

Proof. This directly follows from the commutativity of $\mathrm{P}_{\boldsymbol{\ell}}$ and $\mathrm{gr}^{\mathrm{M}}_{\ell} \psi_{g,\lambda}$ and $\mathrm{gr}^{\mathrm{M}}_{\ell} \phi_{g,1}$ shown in Lemma 14.2.28. $\square$

14.3. Introduction to the direct image theorem

The theory of polarizable Hodge modules was developed in order to give an analytic proof, relying on Hodge theory, of the decomposition theorem of the pushforward by a projective morphism of the intersection complex attached to a local system underlying a polarizable variation of Hodge structure. Two questions arise in this context:

• to relate polarizable variations of Hodge structure on a smooth analytic Zariski open subset of a complex analytic set with a polarizable Hodge module on a complex manifold containing this analytic set as a closed analytic subset (the structure theorem),

• to prove the Hodge-Saito (i.e., direct image) theorem for the pushforward by a projective morphism of a polarizable Hodge module.

Recall Definition 12.7.27 for the pushforward functor in the category $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, and the corresponding definition of the pushforward of a pre-polarization S. In particular, we consider the pushforward ${}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{T}}$ as a graded object in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(Y)$. The Hodge-Saito theorem describes the behaviour by projective pushforward of an object of $\mathsf{pHM}(X,w)$. The case of the constant map $X \to \mathrm{pt}$ and of the Hodge module ${}_{\mathrm{H}}\mathcal{O}_X$ corresponds to the results of Section 2.4.

The proof of the Hodge-Saito theorem is obtained by reducing to the case of a constant map, by using the nearby cycle functor and its compatibility with pushforward. In the case of the constant map, one can reduce to the case where the Hodge module is a polarizable variation of Hodge structure on the complement of a normal crossing divisor in a complex manifold by using Hironaka's theorem on resolution of singularities, and the decomposition theorem already proved (by induction) for the resolution morphism. One can use a Lefschetz pencil to apply an inductive process, after having blown up the base locus of the pencil. In such a way, one is reduced to the case of the constant map on a smooth projective curve, where one can apply the Hodge-Saito theorem 7.4.19.

Another approach in the case of a constant map would make full use of the higher dimensional analogues of the results proved in Chapter 6 for polarized variations of Hodge structure, but this would need to include in the inductive process the structure theorem for polarizable Hodge modules in the normal crossing case.

The Hodge-Saito theorem enables us to give a proof of a simple case of the structure theorem, namely, that a variation of Hodge structure of weight w on a complex manifold X is a polarizable Hodge module of weight $w + \dim X$. It is indeed difficult to check the behaviour along an arbitrary holomorphic function g (e.g. strict $\mathbb{R}$-specializability), but the case where the function is a monomial can be reduced to the case where the function is a product of coordinates, and in that case Example 12.7.24 provides the result by induction on the dimension. The pushforward theorem 12.7.31 enables us to obtain the result for an arbitrary holomorphic function, according to Hironaka's resolution of singularities of holomorphic functions.

***14.3.1. Theorem* (Hodge-Saito theorem).** *Let $f : X \to Y$ be a projective morphism between complex analytic manifolds and let M be a polarizable Hodge module of weight w on X. Let $\mathcal{L}$ be an ample line bundle on X and let $\mathrm{X}_{\mathcal{L}} = (2\pi\mathsf{i})\mathrm{L}_{\mathcal{L}}$ be the corresponding Lefschetz operator. Then $\big({}_{\mathrm{T}}f_*^{(\bullet)}M, \mathrm{X}_{\mathcal{L}}\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(Y,w)$.*

A special case of the Hodge-Saito theorem is the case where f is a closed embedding, which is a consequence of Kashiwara's equivalence 14.2.9.

Let us make explicit this statement. Let us choose a polarization S on $M = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$. It induces an isomorphism $\widetilde{\mathcal{M}}'' \simeq \widetilde{\mathcal{M}}'(w)$ and we can assume that M corresponds to a Hodge-Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$, i.e., $M = (\widetilde{\mathcal{M}}, \widetilde{\mathcal{M}}(w), \mathcal{S})$ with polarization $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$.

(a) ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$, regarded as an object of $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\widetilde{\mathcal{D}}_Y)$, is strict, that is, for every k, ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is a strict $\widetilde{\mathcal{D}}_Y$-module. Moreover, ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is S-decomposable.

(b) Each ${}_{\mathrm{T}}f_*^{(k)}M$ is a polarizable Hodge module of weight $w+k$ on Y.

(c) (*Relative hard Lefschetz theorem*) For each $k \geqslant 0$, the Lefschetz operator $\mathrm{X}_{\mathcal{L}}$ induces isomorphisms in $\mathsf{pHM}(Y, w+k)$:

$$\mathrm{X}_{\mathcal{L}}^k : {}_{\mathrm{T}}f_*^{(-k)}M \xrightarrow{\sim} {}_{\mathrm{T}}f_*^{(k)}M(k),$$

so that $\big({}_{\mathrm{T}}f_*^{(\bullet)}M, \mathrm{X}_{\mathcal{L}}\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2\mathsf{HM}(Y, w)$, that is, a graded Hodge-Lefschetz Hermitian pair with central weight w.

(d) The object $({}_{\mathrm{T}}f_*^{(\bullet)}\widetilde{\mathcal{M}}, {}_{\mathrm{T}}f_*^{(\bullet)}\mathcal{S}), \mathrm{X}_{\mathcal{L}})$ (see Section 12.4.a for ${}_{\mathrm{T}}f_*\mathcal{S}$) is an $\mathfrak{sl}_2$-Hodge-Hermitian pair.

One of the most notable consequence of the Hodge-Saito theorem is the decomposition theorem.

***14.3.2. Theorem* (Decomposition theorem).** *Let $f : X \to Y$ be a projective morphism of complex manifolds. Let $\widetilde{\mathcal{M}}$ be a $\widetilde{\mathcal{D}}_X$-module underlying a polarizable Hodge module. Then the complex ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\widetilde{\mathcal{D}}_Y)$ decomposes (in a non-canonical way) as $\bigoplus_k {}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}[-k]$. Similarly, if $\mathcal{M} = \widetilde{\mathcal{M}}/(z-1)\widetilde{\mathcal{M}}$ is the underlying $\mathcal{D}_X$-module, then there exists a (non canonical) decomposition ${}_{\mathrm{D}}f_*\mathcal{M} \simeq \bigoplus_k {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}[-k]$ in $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathcal{D}_Y)$.*

Proof. This is a direct consequence of Deligne's criterion 3.3.8 for a spectral sequence to degenerate at E_2. We apply this theorem to ${}_{\mathrm{D}}f_*\widetilde{\mathcal{M}}$ as an object of $\mathsf{D}^{\mathrm{b}}(\widetilde{\mathcal{D}}_Y)$, by using the Hard Lefschetz theorem furnished by the Hodge-Saito theorem. $\square$

***14.3.3. Sketch of the proof of Theorem* 14.3.1.** That holonomicity is preserved by proper pushforward is recalled in Remark 8.8.25. We will now focus on the other properties defining a polarizable Hodge module. The proof of Theorem 14.3.1 is done by induction on the pair

$$(n, m) = (\dim \operatorname{Supp} M, \dim \operatorname{Supp} {}_{\mathrm{T}}f_*M)$$

ordered lexicographically. Note that the pairs occurring satisfy $0 \leqslant m \leqslant n$.

(a) In the case where $n = 0$, the assertion of Theorem 14.3.1 is easily obtained: we can assume that M is supported on a point x_o, hence is equal the pushforward by $\iota : \{x_o\} \hookrightarrow X$ of a polarizable Hodge structure, and ${}_{\mathrm{T}}f_*M$ is equal to this Hodge structure.

(b) In the case where $\dim X = 1$ with X smooth, and $m = 0$, it is straightforward to reduce to the case where X is also connected, so that f factorizes as $X \to \mathrm{pt} \hookrightarrow Y$. As already remarked for the case of a closed embedding, we are left with considering the case of the constant map $a_X : X \to \mathrm{pt}$ from a compact Riemann surface, which has been treated in Chapter 7 (see Corollary 7.4.14 and the Hodge-Saito theorem 7.4.19 in dimension 1, i.e., the Hodge-Zucker theorem 6.11.1).

Both (a) and (b) provide the property [$(14.3.1)_{(\leqslant 1,0)}$ with $\operatorname{Supp} M$ smooth].

(c) $(14.3.1)_{(n,m)} \Rightarrow (14.3.1)_{(n+1,m+1)}$ is proved in Section 14.4. In such a case, the behaviour of f_*M with respect to nearby and vanishing cycles for a function g on the base is controlled by the behaviour of M with respect to nearby and vanishing cycles for the function $g \circ f$ on the source, plus a good behaviour of these by the pushforward ${}_{\mathrm{T}}f_*$ relying on 12.7.31. The main point is provided by Proposition 14.4.2.

(d) $(14.3.1)_{(\leqslant n-1,0)}$ & [$(14.3.1)_{(\leqslant 1,0)}$ with $\operatorname{Supp} M$ smooth] $\Rightarrow (14.3.1)_{(n,0)}$ for $n \geqslant 1$ is proved in Section 14.5 by using the method of Lefschetz pencils. In this case, f is the constant map and we factor it through a map to $\mathbb{P}^1$ (up to taking a blowing-up along the axis of the pencil). If such a blow-up is not needed, i.e., a factorization of f exists, the proof relies on the analysis of the corresponding Leray spectral sequence. The general case follows the same strategy.

Conclusion. Let us check that the statements (a)–(d) lead to the proof of Theorem 14.3.1.

Given a pair $(n,m) \in \mathbb{N}^2$ with $m \leqslant n$, let us assume that the theorem is proved for every pair $(n',m') < (n,m)$. If $m \geqslant 1$, (c) gives the theorem by induction since $(n-1, m-1) < (n,m)$. We can thus assume that $m = 0$. By (a), it is enough to consider the case $n \geqslant 1$. Then (d), together with (a) and (b), reduces the proof to that of $(14.3.1)_{(n-1,0)}$, which is also true by induction. $\square$

14.4. Behaviour of the Hodge module properties by projective pushforward

In this section we fix n and we assume that $(14.3.1)_{(n',m')}$ holds for any $n' \leqslant n$ and any $m' \leqslant n'$. We aim at proving that $(14.3.1)_{(n+1,m+1)}$ holds for any $m \leqslant n$.

Let $f : X \to Y$ be a projective morphism between complex manifolds, let h be a holomorphic function on Y and set $g = h \circ f : X \to \mathbb{C}$. Let $\mathcal{L}$ be a relatively ample line bundle on X. In other words, we choose a relative embedding

(14.4.1)
$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr, "f"'] \arrow[drr, bend right, "g"'] & Y \times \mathbb{P}^N \arrow[d] & \\ & Y \arrow[r, "h"] & \widetilde{\mathbb{C}} \end{tikzcd}$$

so that $\mathcal{L}$ comes by pullback from an ample line bundle on $\mathbb{P}^N$. We aim at proving that the properties 14.2.2(1) and (2) relative to the given g are preserved (in some sense) under pushforward by f under weak assumptions on (M, S), and a support condition that allows the application of the induction hypothesis $(14.3.1)_{(n,m)}$.

14.4.2. Proposition. *Let* $\widetilde{\mathcal{T}} = (\widetilde{\mathcal{M}}', \widetilde{\mathcal{M}}'', \mathfrak{s})$ *be a an object of* $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)_{\mathrm{hol}}$ *and let* S *be a pre-polarization of* $\widetilde{\mathcal{T}}$ *of weight* w*. We assume*

(a) $\dim(\operatorname{Supp}\widetilde{\mathcal{T}} \cap g^{-1}(0)) \leqslant n$,

(b) $(\widetilde{\mathcal{T}}, \mathrm{S})$ *satisfies* 14.2.2(1)$_g$ *and* (2)$_g$*. In other words, we assume that the objects* $(\mathrm{gr}^{\mathrm{M}}_{\bullet}\psi_{g,\lambda}\widetilde{\mathcal{T}}, \mathrm{grN}, \mathrm{gr}_{\bullet}\psi_{g,\lambda}\mathrm{S})$ *and* $(\mathrm{gr}^{\mathrm{M}}_{\bullet}\phi_{g,1}\widetilde{\mathcal{T}}, \mathrm{grN}, \mathrm{gr}_{\bullet}\phi_{g,1}\mathrm{S})$ *are respectively polarized* $\mathfrak{sl}_2$*-Hodge triples with central weight* $w-1$ *and* w *(*grN *of type* Y *in both cases and denoted* Y_g*).*

Then, if Theorem 14.3.1 *holds for pairs* (n', m') *with* $n' \leqslant n$*, the following holds.*

(1) ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ *is strictly* $\mathbb{R}$*-specializable and S-decomposable along* (h) *for every* $k \in \mathbb{Z}$*.*

(2) $\big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}), (\mathrm{X}_{\mathcal{L}}, \mathrm{Y}_g), \mathrm{gr}^{\mathrm{M}}_{\ell}\psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}\mathrm{S})\big)$ *is a polarized bi-*$\mathfrak{sl}_2$ *Hodge triple with central weight* $w-1$*.*

(3) $\big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{\ell}\phi_{h,1}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}), (\mathrm{X}_{\mathcal{L}}, \mathrm{Y}_g), \mathrm{gr}^{\mathrm{M}}_{\ell}\phi_{h,1}({}_{\mathrm{T}}f_*^{(k)}\mathrm{S})\big)$ *is a polarized bi-*$\mathfrak{sl}_2$ *Hodge triple with central weight* w*.*

Before giving the proof of this proposition, we will introduce the technical tools that are needed for it.

14.4.a. bi-$\mathfrak{sl}_2$ Hodge modules

14.4.3. Proposition. *The conclusions of Propositions* 3.2.26 *and* 3.2.27 *remain valid for polarizable bi-*$\mathfrak{sl}_2$ *Hodge modules.*

Proof.

(1) Let us start with Proposition 3.2.27. Let $\big((M_{\boldsymbol{j}\in\mathbb{Z}^2}, \rho_1, \rho_2\big)$ be an object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X, w)$ with a polarization S. We assume that it comes equipped with a bi-graded differential $d : M_{\bullet} \to M_{\bullet-(1,1)}(-1)$ which commutes with Y_1 and Y_2 and is self-adjoint with respect to S. In particular, d is strict and strictly specializable and we have, for any germ g of holomorphic function, any $\lambda \in \mathbb{S}^1$ and any $\ell \geqslant 0$,

$$\mathrm{P}_{\ell}\psi_{g,\lambda}(\operatorname{Ker} d/\operatorname{Im} d) = \operatorname{Ker}(\mathrm{P}_{\ell}\psi_{g,\lambda}d)/\operatorname{Im}(\mathrm{P}_{\ell}\psi_{g,\lambda}d)$$

(see Corollary 14.2.23). By induction on the dimension of the support, we can assert that $\big(\mathrm{P}_{\ell}\psi_{g,\lambda}(\operatorname{Ker} d/\operatorname{Im} d), \mathrm{P}_{\ell}\psi_{g,\lambda}\rho_1, \mathrm{P}_{\ell}\psi_{g,\lambda}\rho_2\big)$ is an object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X, w-1+\ell)$ with polarization $\mathrm{P}_{\ell}\psi_{g,\lambda}\mathrm{S}$, and we conclude with Lemma 14.2.29. The case where the dimension of the support is zero is obtained from Proposition 3.2.27.

(2) The analogue of Proposition 3.2.26 is proved similarly. $\square$

***14.4.4. Corollary* (Degeneration of a spectral sequence).** *Let* $(\widetilde{\mathcal{T}}^{\bullet}, d)$ *be a bounded complex in* $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$*, with* $d : \widetilde{\mathcal{T}}^j \to \widetilde{\mathcal{T}}^{j+1}(1)$ *and* $d \circ d = 0$*. Let us assume that it is equipped with the following data:*

(a) *a morphism of complexes* $\mathrm{S} : (\widetilde{\mathcal{T}}^{\bullet}, d) \to (\widetilde{\mathcal{T}}^{\bullet}, d)^*(-w)$ *which is* $(-1)^w$*-Hermitian, that is, for every* k*, a morphism* $\mathrm{S} : \widetilde{\mathcal{T}}^k \to (\widetilde{\mathcal{T}}^{-k})^*(-w)$ *which is compatible with* d *and* d^**, and such that* $\mathrm{S}^* = (-1)^w\mathrm{S}$*,*

(b) *a morphism* $\mathrm{X}' : (\widetilde{\mathcal{T}}^{\bullet}, d) \to (\widetilde{\mathcal{T}}^{\bullet+2}(1), d)$ *which is self-adjoint with respect to* S*,*

(c) *a morphism* $\mathrm{N} : (\widetilde{\mathfrak{T}}^\bullet, d) \to (\widetilde{\mathfrak{T}}^\bullet(-1), d)$ *which is nilpotent, commutes with* X', *and self-adjoint with respect to* S, *with monodromy filtration of* $\mathrm{M}_\bullet(\mathrm{N})$.

Let us consider the spectral sequence associated to the filtered complex $(\mathrm{M}_{-\ell}\widetilde{\mathfrak{T}}^\bullet, d)$ *with* $E_1^{\ell,j-\ell} = H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet$. *We set* $\mathrm{Y} = \mathrm{gr}\mathrm{N}$. *We assume that*

$$\bigoplus_{j,\ell}\Big(E_1^{\ell,j-\ell} = H^j(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet), (H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}', H^j\mathrm{Y}), H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$. *Then*

(1) *the spectral sequence degenerates at* E_2,

(2) *the filtration* $W_\ell H^j(\widetilde{\mathfrak{T}}^\bullet) := \mathrm{image}[H^j(\mathrm{M}_\ell\widetilde{\mathfrak{T}}^\bullet) \to H^j(\widetilde{\mathfrak{T}}^\bullet)]$ *is equal to the monodromy filtration* $\mathrm{M}_\bullet H^j(\widetilde{\mathfrak{T}}^\bullet)$ *associated to* $H^j\mathrm{N} : H^j(\widetilde{\mathfrak{T}}^\bullet) \to H^j(\widetilde{\mathfrak{T}}^\bullet)$,

(3) *the object*

$$\bigoplus_{j,\ell}\Big(\mathrm{gr}^{\mathrm{M}}_{-\ell}H^j(\widetilde{\mathfrak{T}}^\bullet), (\mathrm{gr}^{\mathrm{M}}_{-\ell}H^j\mathrm{X}', \mathrm{gr}H^j\mathrm{N}), \mathrm{gr}^{\mathrm{M}}_{-\ell}H^j\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$.

Proof. Let us first make clear the statement. Since d and X' commute with N, d and X' are compatible with the monodromy filtration $\mathrm{M}_\bullet(\mathrm{N})$, hence for each ℓ we have a graded complex $(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet, d)$, and X' induces for every ℓ a morphism $\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}' : (\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet, d) \to (\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^{\bullet+2}(1), d)$, and thus a morphism $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}' : E_1^{\ell,j-\ell} \to E_1^{\ell,j+2-\ell}(1)$. Similarly, $H^j\mathrm{Y}$ is a morphism $E_1^{\ell,j-\ell} \to E_1^{\ell+2,j-\ell-2}(-1)$. We consider the bi-grading such that $E_1^{\ell,j-\ell}$ is in bi-degree (j,ℓ).

The differential $d_1 : H^j(\mathrm{gr}^{\mathrm{M}}_{-\ell}\widetilde{\mathfrak{T}}^\bullet) \to H^{j+1}(\mathrm{gr}^{\mathrm{M}}_{-\ell-1}\widetilde{\mathfrak{T}}^\bullet)(1)$ is a morphism in $\mathsf{HM}(X, w+j-\ell)$ that commutes with $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{X}'$ and $H^j\mathrm{Y}$. We will check below that d_1 is self-adjoint with respect to $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$. From the analogue of Proposition 3.2.27 (see Proposition 14.4.3), we deduce that $\bigoplus_{j,\ell}E_2^{\ell,j-\ell}$ is part of an object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(X,w)$. Now, one shows inductively that, for $r \geqslant 2$, $d_r : E_2^{\ell,j-\ell} \to E_2^{\ell+r,j-\ell-r+1}$ is a morphism of pure Hodge modules, the source having weight $w+j-\ell$ and the target $w+j-\ell-r+1 < w+j-\ell$ and thus, by applying 14.2.17(2), that $d_r = 0$. This proves 14.4.4(1).

In order to prove (2), we notice that, due to the degeneration property above, we have an identification

$$\mathrm{gr}^W_\ell H^j(\widetilde{\mathfrak{T}}^\bullet) \simeq E_2^{\ell,j-\ell},$$

and the action of $\mathrm{gr}\mathrm{N}$ on the left-hand side is that induced by $H^j\mathrm{Y}$ on the right-hand side. By the $\mathfrak{sl}_2$ property of E_2 relative to $H^j\mathrm{Y}$, we deduce that $\mathrm{gr}\mathrm{N}$ satisfies the Lefschetz property on $\mathrm{gr}^W_\bullet H^j(\widetilde{\mathfrak{T}}^\bullet)$. In other words, (2) holds.

Last, due to the above identification, (3) amounts to the bi-$\mathfrak{sl}_2$ Hodge property of E_2. $\square$

Proof that d_1 ***is self-adjoint.*** We regard $\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$ as a morphism $\mathrm{gr}^{\mathrm{M}}_{-\ell}M^j \to (\mathrm{gr}^{\mathrm{M}}_\ell M^{-j})^*$. It is compatible with d and d^* on these complexes, since N commutes with d. Then, $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$ is a morphism $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}M^\bullet \to (H^{-j}\mathrm{gr}^{\mathrm{M}}_\ell M^{-\bullet})^*$. Since d_1 is obtained by a standard formula from d on the filtered complex, the equality $\mathrm{S}\circ d = d^*\circ \mathrm{S}$ implies $H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}\circ d_1 = (d_1)^*\circ H^j\mathrm{gr}^{\mathrm{M}}_{-\ell}\mathrm{S}$. $\square$

14.4.b. Proof of Proposition 14.4.2 and of 14.3.3(c)

***Proof of Proposition* 14.4.2.** One of the points to understand is the way to pass from properties of ${}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}}$ to properties of $\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{h,\lambda}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}})$, and similarly with $\phi_{g,1}$. Although we know that $\psi_{t,\lambda}({}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}})$ is isomorphic to ${}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}}$ if the latter is strict, according to 12.7.31, we have to check the strictness property. Moreover, we are left with the question of passing from ${}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}$ to $\mathrm{gr}^{\mathrm{M}}_{-\ell}{}_{\mathrm{T}}f_*^{(k)}$. Here, we do not have a commutation property, but we will use Corollary 14.4.4 to analyze the corresponding spectral sequence. At this point, the existence of a polarization is essential. The S-decomposability is not obvious either, and the polarization also plays an essential role for proving it.

Since we assume that Theorem 14.3.1 holds for objects in $\mathsf{pHM}_{\leqslant n}(X)$ and since $\dim(\operatorname{Supp}\widetilde{\mathcal{T}}\cap g^{-1}(0))\leqslant n$, we deduce that, for every $\lambda\in\mathbb{S}^1$,

$$\Big(\bigoplus_{k,\ell}{}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\widetilde{\mathcal{T}},(\mathrm{X}_{\mathcal{L}},{}_{\mathrm{T}}f_*^{(k)}\mathrm{grN}),{}_{\mathrm{T}}f_*^{(k)}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{g,\lambda}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(Y,w-1)$ if we keep here the grading convention used in Corollary 14.4.4. This corollary implies that

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}{}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}},(\mathrm{X}_{\mathcal{L}},\mathrm{gr}\,{}_{\mathrm{T}}f_*^{(k)}\mathrm{N}),\mathrm{gr}^{\mathrm{M}}_{-\ell}{}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(Y,w-1)$. In particular, each $\mathrm{gr}^{\mathrm{M}}_{-\ell}{}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}}$ is strict, and therefore so is ${}_{\mathrm{T}}f_*^{(k)}\psi_{g,\lambda}\widetilde{\mathcal{T}}$. We argue similarly for $\phi_{g,1}$.

We can now apply Corollary 9.8.9 to conclude that ${}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}}$ is strictly $\mathbb{R}$-specializable along (g) for every k. We also conclude from 12.7.31 that

$$(\psi_{h,\lambda}\,{}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}},\mathrm{N})={}_{\mathrm{T}}f_*^{(k)}(\psi_{g,\lambda}\widetilde{\mathcal{T}},\mathrm{N}),\qquad(\phi_{h,1}\,{}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}},\mathrm{N})={}_{\mathrm{T}}f_*^{(k)}(\phi_{g,1}\widetilde{\mathcal{T}},\mathrm{N}).$$

We have thus proved that

$$\Big(\bigoplus_{k,\ell}\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{h,\lambda}\,{}_{\mathrm{T}}f_*^{(k)}\widetilde{\mathcal{T}},(\mathrm{X}_{\mathcal{L}},\mathrm{grN}),\mathrm{gr}^{\mathrm{M}}_{-\ell}\psi_{h,\lambda}\,{}_{\mathrm{T}}f_*^{(k)}\mathrm{S}\Big)$$

is a polarized object of $\mathsf{p}\mathfrak{sl}_2^2\mathsf{HM}(Y,w-1)$, and a corresponding assertion for $\phi_{h,1}$. $\square$

Proof of* 14.3.3(c)*, i.e., $(14.3.1)_{(n,m)}\Rightarrow(14.3.1)_{(n+1,m+1)}$. Let $f:X\to Y$ be a projective morphism and let (M,S) be a polarized object of $\mathsf{pHM}_S(X,w)$, where S is an irreducible analytic subset of X of dimension $n+1$. We can assume that (M,S) is represented as a Hodge-Hermitian pair $(\widetilde{\mathcal{M}},\mathcal{S})$ of weight w, and we will omit $\mathrm{S}=(\mathrm{Id},\mathrm{Id})$ in the notation. Assume that $f(S)$ has dimension $m+1$ and that $(14.3.1)_{(n,m)}$ holds. Since Theorem 14.3.1 is a local statement on Y, we can work in an open neighbourhood of a point $y_o\in f(S)$, that we can take as small as needed. By the S-decomposability of $(\widetilde{\mathcal{M}},\mathcal{S})$ on X, we can therefore assume that S and $f(S)$ are irreducible when restricted to a fundamental basis of neighborhoods of $f^{-1}(y_o)$ and y_o respectively.

Let h be a holomorphic function on some $\mathrm{nb}(y_o)$ and set $g = h \circ f$. We distinguish two cases. We note that strictness of ${}_{\mathrm{T}}f_*^{(k)}M$ on $\mathrm{nb}(y_o)$ is obtained by choosing any h as in Case (1) below.

(1) $h^{-1}(0) \cap f(S)$ has codimension 1 in $f(S)$. Then $g^{-1}(0) \cap S$ has codimension 1 in S. We can thus apply Proposition 14.4.2. It follows that each ${}_{\mathrm{T}}f_*^{(k)}M$ is strict and satisfies 14.2.2(1)$_h$ and (2)$_h$.

(2) The function h vanish identically on the closed irreducible subset $f(S) \cap \mathrm{nb}(y_o)$ of $\mathrm{nb}(y_o)$. We now omit referring to $\mathrm{nb}(y_o)$. We denote by

$$\iota_g : X \hookrightarrow X \times \mathbb{C}_t \quad \text{and} \quad \iota : X \times \{0\} \hookrightarrow X \times \mathbb{C}_t$$

the respective graph and trivial inclusions, and similarly on Y. The only property to be checked relative to h is that ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (h), that is, 14.2.2(1)$_h$: indeed, in such a case, Proposition 12.7.15 implies $\phi_{g,1}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ and $\psi_{g,\lambda}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}) = 0$ for any $\lambda \in \mathbb{S}^1$, so 14.2.2(2)$_h$ is trivially satisfied. Since ${}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}$ is strict, ${}_{\mathrm{D}}\iota_*({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}})$ is strictly $\mathbb{R}$-specializable along (t) and it is enough to prove

$${}_{\mathrm{D}}\iota_{h*}({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}\iota_*({}_{\mathrm{D}}f_*^{(k)}\widetilde{\mathcal{M}}) \quad \forall\, k.$$

The left-hand term is equal to ${}_{\mathrm{D}}f_*^{(k)}\,{}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}}$, if we still denote by f the map $f \times \mathrm{Id}_{\mathbb{C}}$. Similarly the right-hand term is equal to ${}_{\mathrm{D}}f_*^{(k)}\,{}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$, with obvious abuse of notation. Since $g \equiv 0$ on S and $\widetilde{\mathcal{M}}$ is assumed to be strictly $\mathbb{R}$-specializable along (g), we have ${}_{\mathrm{D}}\iota_{g*}\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota_*\widetilde{\mathcal{M}}$, hence the desired assertion. $\square$

14.5. End of the proof of the Hodge-Saito theorem

Recall that we wish to prove

(d) $(14.3.1)_{(\leqslant n-1,0)}$ & $[(14.3.1)_{(1,0)}$ with $\operatorname{Supp} M$ smooth$] \Longrightarrow (14.3.1)_{(n,0)}$ for $n \geqslant 1$.

We thus fix $n \geqslant 1$ in this section and assume that both properties of the left term hold true. It follows then from the results of Section 14.4 that $(14.3.1)_{(\leqslant n,m)}$ is true for any $m \geqslant 1$. As already noticed in the case 14.3.3(b), we only have to consider the case of the constant map $a_X : X \to \mathrm{pt}$.

Let (M, S) be a polarized Hodge module of weight w on a smooth complex projective variety X and let $\mathcal{L}$ be an ample line bundle on X. We can assume that $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$ and consider the Hodge-Hermitian pair $(\widetilde{\mathcal{M}}, \mathcal{S})$ for M as in Proposition 14.2.25. We can also assume that M has pure support S, which is an irreducible closed n-dimensional algebraic subset of X ($n \geqslant 1$). It is not restrictive to assume that $\mathcal{L}$ is very ample, so that, by Kashiwara's equivalence (Proposition 14.2.9), we can further assume that $X = \mathbb{P}^N$ and $\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)$.

14.5.a. The case where X maps to a curve

In order to emphasize the main steps, we start with the simpler case where we assume that there exists a morphism f from X to a curve C which is non-constant on the irreducible pure support $S = \operatorname{Supp} M$, that we decompose as in (14.4.1) with

$Y = C$. We use the corresponding notations for the ample line bundles $\mathcal{L}$ on $\mathbb{P}^N$ and $\mathcal{L}'$ on C. We decompose the constant map a_X on the projective manifold X of dimension n as $X \xrightarrow{f} C \xrightarrow{a_C} \mathrm{pt}$ and we consider the Leray spectral sequence for this decomposition (see Corollary 12.7.36).

Our induction hypothesis implies that Theorem 14.3.1 holds for both maps f and a_C: indeed, $(14.3.1)_{(\leqslant n,1)}$ holds true, and thus $\big({}_{\mathrm{T}}f_*^{(\bullet)}(M,\mathrm{S}),\mathrm{X}_{\mathcal{L}}\big)$ is a polarized $\mathfrak{sl}_2$-Hodge module with central weight w; furthermore, by $(14.3.1)_{(1,0)}$, the push-forward $\big({}_{\mathrm{T}}a_{C*}^{(\bullet)}({}_{\mathrm{T}}f_*^{(\bullet)}(M,\mathrm{S})),\mathrm{X}_{\mathcal{L}},\mathrm{X}_{\mathcal{L}'}\big)$ by the constant map a_C on the curve C is a polarized bi-$\mathfrak{sl}_2$ Hodge structure with central weight w. We are thus led to analyzing the Leray spectral sequence in order to get that $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}(M,\mathrm{S}),(\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'})\big)$ is a polarized $\mathfrak{sl}_2$-Hodge structure.

According to Corollary 12.7.36, there exists a spectral sequence in $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(\mathrm{pt})$ whose E_2 term is $E_2^{p,q} = {}_{\mathrm{T}}a_{C*}^{(p)}({}_{\mathrm{T}}f_*^{(q)}M)$. Since $\dim C = 1$, we have $E_2^{p,q} = 0$ unless $p = -1, 0, 1$. By our induction hypothesis, we can apply the decomposition theorem 14.3.2 to f and (M,S), and the spectral sequence degenerates at E_2.

Furthermore, our induction hypothesis implies that $((E_2^{\bullet,\bullet},\mathrm{S}),\mathrm{X}_{\widetilde{\mathcal{L}}},\mathrm{X}_{\widetilde{\mathcal{L}}'})$ is a polarized $\mathfrak{sl}_2^2$-Hodge structure with central weight w. We set $E_2^k = \bigoplus_{p+q=k} E_2^{p,q}$. We apply Proposition 3.2.26 to deduce a polarized $\mathfrak{sl}_2$-Hodge structure $((E_2^{\bullet},\mathrm{S}),\mathrm{X}_{\widetilde{\mathcal{L}}}+\mathrm{X}_{\widetilde{\mathcal{L}}'})$ with central weight w. It follows that $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}M,(\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'})\big)$ has a filtration $\mathrm{Ler}^{\bullet}$ (the Leray filtration attached to the spectral sequence) whose graded term is an $\mathfrak{sl}_2$-Hodge structure polarized by the pre-polarization induced from ${}_{\mathrm{T}}a_{X*}^{(\bullet)}\mathrm{S}$. From this property one deduces at once that $\big({}_{\mathrm{T}}a_{X*}^{(\bullet)}M,(\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'})\big)$ is an $\mathfrak{sl}_2$-Hodge structure of central weight w and that S induces a pre-polarization of it. However, at this step, we cannot assert that S is a polarization (i.e., that the positivity property holds), since it is only a successive extension of polarizations.

In order to overcome this difficulty, we will make use of the criterion provided by Theorem 3.2.20, which relies on the weak Lefschetz property. Since we have at our disposal a pre-polarization, we will work with the Hodge Hermitian pair $(\widetilde{\mathcal{M}},\mathcal{S},w)$ attached to (M,S) (see Proposition 14.2.25).

The operator $\mathrm{X}_{\mathcal{L}}+\mathrm{X}_{\mathcal{L}'}$ is the Lefschetz operator attached to the ample line bundle $\mathcal{L}\boxtimes\mathcal{L}'$ and, up to multiplying $\mathrm{X}_{\mathcal{L}\boxtimes\mathcal{L}'}$ by some positive integer, we can assume that it is very ample. It defines an embedding $X \hookrightarrow \mathbb{P}^{N'}$ and its restriction to X takes the form $\widetilde{\mathcal{O}}_X(H)$ for a general hyperplane of $\mathbb{P}^{N'}$ that we can assume to be non-characteristic with respect $\widetilde{\mathcal{M}}$. Since, by Definition $14.2.2(1)_g$ for any local equation g of H, $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along H, it follows from Proposition 9.5.9 that H is strictly non-characteristic with respect to $\widetilde{\mathcal{M}}$.

For each $k\in\mathbb{Z}$, we consider the X-$\mathfrak{sl}_2$-Hodge quiver with center $w-1$, with

$$\begin{aligned}
(H_\bullet,\mathrm{X}) &= ({}_{\mathrm{T}}a_{X*}^{(\bullet)}(\widetilde{\mathcal{M}},\mathcal{S},w),\mathrm{X}_{\mathcal{L}\boxtimes\mathcal{L}'}),\\
(G_\bullet,\mathrm{X}) &= ({}_{\mathrm{T}}a_{H*}^{(\bullet+1)}(\widetilde{\mathcal{M}}_H,\mathcal{S}_H,w-1),\mathrm{X}_{\mathcal{L}\boxtimes\mathcal{L}'}),\\
\mathrm{can} &= \mathrm{restr}_H,\quad \mathrm{var} = \mathrm{Gys}_H\,.
\end{aligned}$$

Our induction hypothesis yields that $(G_\bullet, \mathrm{X})$ is a polarized $\mathfrak{sl}_2$-Hodge-Hermitian pair of weight $w-1$. Furthermore, $\mathrm{Gys}_H : {}_{\mathrm{D}}a_{H*}^{(k)}\widetilde{\mathcal{M}}_H \to {}_{\mathrm{D}}a_{X*}^{(k+1)}\widetilde{\mathcal{M}}(1)$, being induced by a morphism of Hodge structures, is a strict morphism for each k. We can therefore apply the criterion of Proposition 11.2.26 to deduce that the X-$\mathfrak{sl}_2$-Hodge quiver $((H_\bullet, \mathrm{X}), (G_\bullet, \mathrm{X}), \mathrm{can}, \mathrm{var})$ satisfies the weak Lefschetz property. According to Theorem 3.2.20, we are left with proving the positivity of $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}\mathrm{S})$ on $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ in order to deduce the desired positivity property for the pre-polarization of $\bigl({}_{\mathrm{T}}a_{X*}^{(\bullet)}M, (\mathrm{X}_{\mathcal{L}} + \mathrm{X}_{\mathcal{L}'})\bigr)$.

14.5.1. Lemma. *The pure Hodge structure* $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ *is a Hodge sub-structure of* ${}_{\mathrm{T}}a_{C*}^{(0)}\bigl(\mathrm{P}(f_*^{(0)}M_H)\bigr)$ *and the pre-polarization* $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}\mathrm{S})$ *is induced by* ${}_{\mathrm{T}}a_{C*}^{(0)}\bigl(\mathrm{P}(f_*^{(0)}\mathrm{S}_H)\bigr)$.

The proof of (d) is achieved with this lemma, according to Exercise 2.12. $\square$

***Proof of Lemma* 14.5.1.** We can assume that S and S_H are of the form $(\mathrm{Id}, \mathrm{Id})$, so that we only need to show the first part.

We first check that $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) \subset \mathrm{Ler}^0({}_{\mathrm{T}}a_{X*}^{(0)}M)$. Due to the weak Lefschetz property, we have $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) = \mathrm{Ker}[\mathrm{restr}_H : {}_{\mathrm{T}}a_{X*}^{(0)}M \to {}_{\mathrm{T}}a_{H*}^{(1)}M_H]$ (see Remark 3.1.14(2)). Since the Leray filtration has only three terms $0 \subset \mathrm{Ler}^1 \subset \mathrm{Ler}^0 \subset \mathrm{Ler}^{-1}$, we are reduced to showing that $\mathrm{gr}_{\mathrm{Ler}}^{-1}\mathrm{restr}_H$ is injective. This is the restriction morphism ${}_{\mathrm{T}}a_{C*}^{(-1)}({}_{\mathrm{T}}f_*^{(1)}M) \to {}_{\mathrm{T}}a_{C*}^{(-1)}({}_{\mathrm{T}}f_*^{(2)}M_H)$ induced by the restriction morphism ${}_{\mathrm{T}}f_*^{(1)}M \to {}_{\mathrm{T}}f_*^{(2)}M_H$ relative to f: the latter is the connecting morphism in the long exact sequence in $\mathsf{pHM}(C)$ obtained as in (11.2.17) by applying ${}_{\mathrm{T}}f_*$ to the exact sequence (12.7.26). Since any morphism in $\mathsf{pHM}(C)$ is strict, we can apply the criterion for the weak Lefschetz property in Proposition 11.2.26 and deduce that this morphism is an isomorphism, hence in particular the desired injectivity.

We then claim that it is enough to check that $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ does not intersect $\mathrm{Ler}^1({}_{\mathrm{T}}a_{X*}^{(0)}M)$, hence injects into $\mathrm{gr}_{\mathrm{Ler}}^0({}_{\mathrm{T}}a_{X*}^{(0)}M) = {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(0)}M)$. Indeed, having proved this, we note that the action of $\mathrm{X}_{\mathcal{L}'}$ on this space is zero since ${}_{\mathrm{T}}a_{C*}^{(2)}(\bullet) = 0$. Therefore,

$$\begin{aligned}\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) &\subset \mathrm{Ker}\bigl[(\mathrm{X}_{\mathcal{L}} + \mathrm{X}_{\mathcal{L}'}) : \mathrm{gr}_{\mathrm{Ler}}^0({}_{\mathrm{T}}a_{X*}^{(0)}M) \longrightarrow \mathrm{gr}_{\mathrm{Ler}}^0({}_{\mathrm{T}}a_{X*}^{(2)}M)\bigr]\\ &= \mathrm{Ker}\bigl[{}_{\mathrm{T}}a_{C*}^{(0)}(\mathrm{X}_{\mathcal{L}}) : {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(0)}M) \longrightarrow {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(2)}M)\bigr].\end{aligned}$$

Due to the Lefschetz decomposition of ${}_{\mathrm{T}}f_*^{(\bullet)}M$ with respect to $\mathrm{X}_{\mathcal{L}}$, ${}_{\mathrm{T}}f_*^{(0)}M$ decomposes as $\mathrm{P}({}_{\mathrm{T}}f_*^{(0)}M) \oplus \mathrm{X}_{\mathcal{L}}({}_{\mathrm{T}}f_*^{(-2)}M)$, and $\mathrm{X}_{\mathcal{L}} : \mathrm{X}_{\mathcal{L}}({}_{\mathrm{T}}f_*^{(-2)}M) \to {}_{\mathrm{T}}f_*^{(2)}M$ is an isomorphism. Then ${}_{\mathrm{T}}a_{C*}^{(0)}(\mathrm{X}_{\mathcal{L}}) : {}_{\mathrm{T}}a_{C*}^{(0)}\bigl(\mathrm{X}_{\mathcal{L}}({}_{\mathrm{T}}f_*^{(-2)}M)\bigr) \to {}_{\mathrm{T}}a_{C*}^{(0)}({}_{\mathrm{T}}f_*^{(2)}M)$ is also an isomorphism, hence $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M)$ does not intersect its source, that is, $\mathrm{P}({}_{\mathrm{T}}a_{X*}^{(0)}M) \subset {}_{\mathrm{T}}a_{C*}^{(0)}\bigl(\mathrm{P}({}_{\mathrm{T}}f_*^{(0)}M)\bigr)$, which is the desired inclusion.

For the claim, we have $\mathrm{Ler}^1({}_{\mathrm{T}}a_{X*}^{(0)}M) = \mathrm{gr}_{\mathrm{Ler}}^1({}_{\mathrm{T}}a_{X*}^{(0)}M) = {}_{\mathrm{T}}a_{C*}^{(1)}({}_{\mathrm{T}}f_*^{(-1)}M)$, and the action of $\mathrm{X}_{\mathcal{L}} + \mathrm{X}_{\mathcal{L}'}$ reduces to that of $\mathrm{X}_{\mathcal{L}}$. By the Lefschetz decomposition of ${}_{\mathrm{T}}f_*^{(\bullet)}M$ with respect to $\mathrm{X}_{\mathcal{L}}$, the morphism $\mathrm{X}_{\mathcal{L}} : {}_{\mathrm{T}}f_*^{(-1)}M \to {}_{\mathrm{T}}f_*^{(1)}M$ is an isomorphism, hence so is the morphism ${}_{\mathrm{T}}a^{(1)}_{C*}(\mathrm{X}_{\mathcal{L}}) : {}_{\mathrm{T}}a^{(1)}_{C*}({}_{\mathrm{T}}f^{(-1)}_{*}M) \to {}_{\mathrm{T}}a^{(1)}_{C*}({}_{\mathrm{T}}f^{(1)}_{*}M)$. It follows that $\mathrm{Ler}^1({}_{\mathrm{T}}a^{(0)}_{X*}M) \cap \mathrm{P}({}_{\mathrm{T}}a^{(0)}_{X*}M) = 0$. $\square$

14.5.b. The general case. In general however, we do not have such a decomposition $X \xrightarrow{f} C \xrightarrow{a_C} \mathrm{pt}$ of the constant map as in Section 14.5.a, and the usual trick is to consider a Lefschetz pencil instead, a procedure that introduces a supplementary complication due to the base locus of the pencil, that we can choose as generic as we want nevertheless.

Let us choose a pencil of hyperplanes in $X = \mathbb{P}^N$ with axis $A \simeq \mathbb{P}^{N-2}$. It defines a map $X \smallsetminus A \to \mathbb{P}^1$, whose graph is contained in $(X \smallsetminus A) \times \mathbb{P}^1$. Let X_A be the closure of this graph in $X \times \mathbb{P}^1$ with projection π to X, and let A_A be the pullback $\pi^{-1}(A)$. By definition, X_A is the blow-up space of X along the axis A of the pencil, and A_A is a smooth divisor in it. We have the following commutative diagram:

$$
\begin{tikzcd}
A \times \mathbb{P}^1 \arrow[r, hook] \arrow[d, equal] & X \times \mathbb{P}^1 \arrow[d, bend left=20, "p"'] & \\
A_A \arrow[r, hook] \arrow[d] & X_A \arrow[u, hook, "\iota"'] \arrow[r, "f"'] \arrow[d, "\pi"] & \mathbb{P}^1 \arrow[d, "a_{\mathbb{P}^1}"] \\
A \arrow[r, hook] & X \arrow[r, "a_X"] & \mathrm{pt}
\end{tikzcd}
\tag{14.5.2}
$$

The restriction of π to any fiber $f^{-1}(t)$ is an isomorphism onto the corresponding hyperplane in X and, conversely, the pullback by π of this hyperplane is the union of $f^{-1}(t)$ and $A_A = A \times \mathbb{P}^1$, whose intersection $f^{-1}(t) \cap A_A = A \times \{t\}$ is transversal. Similarly, the pullback $\pi^{-1}S$ of the support S of M consists of the union of the strict transform S_A of S by π, i.e., the blow-up space of S along the ideal $\mathcal{I}_A\mathcal{O}_S$, and $(A \cap S) \times \mathbb{P}^1$.

We set $\mathcal{L}' = \mathcal{O}_{\mathbb{P}^1}(1)$, and we consider the ample line bundle $\mathcal{L} \otimes \mathcal{L}'$ on $X \times \mathbb{P}^1$. We will simply denote by X, X' the Lefschetz operators $\mathrm{X}_{\mathcal{L}}, \mathrm{X}_{\mathcal{L}'}$, so that $\mathrm{X} + \mathrm{X}'$ is the Lefschetz operator that is to be considered on $X \times \mathbb{P}^1$.

We consider the pullback $({}_{\mathrm{T}}\pi^*M, \mathrm{S})$. Although we cannot assert, at this stage of the theory, that it is a polarized Hodge module, we will prove that it enjoys a similar behaviour along the divisors $(f - t)$ when t varies in $\mathbb{P}^1$. This will enable us, by decomposing a_{X_A} as $a_{\mathbb{P}^1} \circ f$, to obtain for ${}_{\mathrm{T}}a_{X_A*}({}_{\mathrm{T}}\pi^*M, \mathrm{S})$ the same results as in the simple case 14.5.a.

On the other hand, we consider the decomposition of a_{X_A} as $a_X \circ \pi$. We will show (with the induction hypothesis at hand) that the pushforward ${}_{\mathrm{T}}\pi_*({}_{\mathrm{T}}\pi^*M)$ decomposes as the direct sum of its cohomology objects, and that M is a direct summand of ${}_{\mathrm{T}}\pi^{(0)}_*({}_{\mathrm{T}}\pi^*M)$. It follows that $\big(\bigoplus_k {}_{\mathrm{T}}a^{(k)}_{X*}M, \mathrm{X}\big)$ is a direct summand of $\big(\bigoplus_k {}_{\mathrm{T}}a^{(k)}_{X_A*}({}_{\mathrm{T}}\pi^*M), \mathrm{X} + \mathrm{X}'\big)$. Then, according to the previous step, the result follows from stability of polarizable $\mathfrak{sl}_2$-Hodge modules by direct summand (see Lemma 5.2.8 and Exercise 2.12(1), as already used in 14.2.8).

The detailed proof will take various steps.

Step 1. We define ${}_{\mathrm{T}}\pi^*$ as the composition ${}_{\mathrm{T}}\iota^* \circ {}_{\mathrm{T}}p^*$. This first step aims at showing that, under a non-characteristic condition,

- *the pullback* ${}_{\mathrm{T}}\pi^*(M, \mathrm{S})$ *is well-defined, is strict and satisfies* 14.2.2(1)$_{f-t}$ *and* (2)$_{f-t}$, *for every* $t \in \mathbb{P}^1$.

The smooth pullback ${}_{\mathrm{T}}p^*M$ is well-defined as an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X \times \mathbb{P}^1)$ (see Section 12.7.12). In order to define ${}_{\mathrm{T}}\iota^*({}_{\mathrm{T}}p^*M)$, we will prove strict $\mathbb{R}$-specializability of ${}_{\mathrm{T}}p^*M$ along the graph $\iota(X_A)$. Note however that we do not know that the pullback ${}_{\mathrm{T}}p^*M$ satisfies Hodge properties along *every* germ of holomorphic function on $X \times \mathbb{P}^1$. Non-characteristic properties obtained by choosing the axis of the pencil generic enough will help us to overcome this difficulty.

More precisely, let us choose the pencil generic enough so that the axis A of the pencil is *non-characteristic* with respect to $\widetilde{\mathcal{M}}$ (see Section 9.5.b). If the characteristic variety of $\widetilde{\mathcal{M}}$ is contained in $\Lambda \times \mathbb{C}_z$ with Λ Lagrangian in T^*X, there exists a complex stratification of the support of of $\widetilde{\mathcal{M}}$ by locally closed sub-manifolds S_i^o with analytic closure S_i, such that $\Lambda \subset \bigsqcup_i T^*_{S_i^o}X$. Then A is chosen to be transversal to every stratum S_i^o. In particular, since the axis A has codimension two, it does not intersect any zero- and one-dimensional stratum. Moreover, for every i, the blow-up S_{iA} of S_i contains $(A \cap S_i) \times \mathbb{P}^1$. This implies that $S_{iA} = \pi^{-1}(S_i)$.

14.5.3. Lemma.

(1) *The inclusion* $\iota : X_A \hookrightarrow X \times \mathbb{P}^1$ *is strictly non-characteristic with respect to* ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$.

(2) *We have* $\boldsymbol{L}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$.

(3) *The* $\widetilde{\mathcal{D}}_{X_A}$*-module* ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ *is holonomic, strict and strictly* $\mathbb{R}$*-specializable along each divisor* $(f - t)$.

According to this lemma, the pullback functor ${}_{\mathrm{T}}\iota^*$ is defined as in Section 12.7.21.

Proof.

(1) We first prove the non-characteristic property. We postpone the proof of strict $\mathbb{R}$-specializability after the proof of (3). Since p is a projection, the characteristic variety of ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is contained in the union of the sets $T^*_{S_i \times \mathbb{P}^1}(X \times \mathbb{P}^1) \times \mathbb{C}_z$.

- Away from $A_A = A \times \mathbb{P}^1$, ι is the graph inclusion of a map to $\mathbb{P}^1$ and, in a local setting, we are reduced to proving the claim for the inclusion $\iota : U = U \times \{0\} \hookrightarrow U \times \mathbb{C}$ and the projection $p : U \times \mathbb{C} \to \mathbb{C}$, where the claim is obviously true.
- Let us now consider the neighbourhood of a point of $A_A = A \times \mathbb{P}^1$ in $X \times \mathbb{P}^1$. Since A is non-characteristic with respect to each S_i, so is A_A with respect to each $S_i \times \mathbb{P}^1$ — and therefore so is X_A near any point of A_A, since in such a point the space $T^*_{X_A}(X \times \mathbb{P}^1)$ is contained in $T^*_{A_A}(X \times \mathbb{P}^1)$. The non-characteristic property is then also true along A_A.

(2) We now prove that $L^k{}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}} = 0$ for $k \neq 0$. Since X_A is of codimension 1 in $X \times \mathbb{P}^1$, this amounts to the property that ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ has no local section supported on X_A. Notice that ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is strict, since $\widetilde{\mathcal{O}}_{X \times \mathbb{P}^1}$ is $\widetilde{\mathcal{O}}_X$-flat. Any coherent $\widetilde{\mathcal{D}}_{X \times \mathbb{P}^1}$-submodule of ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is then also strict, and it is supported on X_A if and only if the associated

$\mathcal{D}_{X\times\mathbb{P}^1}$-module is so. But such a coherent $\mathcal{D}_{X\times\mathbb{P}^1}$-module is a submodule of ${}_{\mathrm{D}}\pi^*\mathcal{M}$, hence is holonomic with characteristic variety contained in $\Lambda \times T^*_{\mathbb{P}^1}\mathbb{P}^1$. This cannot be the characteristic variety of a holonomic $\mathcal{D}_{X\times\mathbb{P}^1}$-module with support on X_A.

(3) Note that, as a consequence of Theorem 9.5.6, the characteristic variety of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}} = {}_{\mathrm{D}}\iota^*{}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ is contained in the union of sets $(T^*_{S_{iA}}X_A) \times \mathbb{C}_z$. Hence ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is holonomic.

We also claim that, for every $t \in \mathbb{P}^1$, the inclusion $A \times \{t\} \hookrightarrow X_A$ is non-characteristic with respect to ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$. Indeed, by the choice of A, for every S_i as above, the intersection of $T^*_{A\times\{t\}}(X \times \mathbb{P}^1)$ with $T^*_{S_i\times\mathbb{P}^1}(X \times \mathbb{P}^1)$ is contained in the zero-section of $T^*(X \times \mathbb{P}^1)$. As we have $T^*_{A\times\{t\}}(X \times \mathbb{P}^1) = (T^*\iota)^{-1}(T^*_{A\times\{t\}}X_A)$, it follows that $T^*_{A\times\{t\}}X_A \cap T^*_{S_{iA}}X_A \subset T^*_{X_A}X_A$.

This implies that, for every $t \in \mathbb{P}^1$, the inclusion $f^{-1}(t) \hookrightarrow X_A$ is non-characteristic for ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ near any point $(x_o, t) \in A \times \{t\}$ since $A \times \{t\}$ is contained in $f^{-1}(t)$.

Let us fix a point $x_o \in A$ and let $g = 0$ be a local equation of the hyperplane $f = t$ of X near x_o. We will prove strict $\mathbb{R}$-specializability of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ along $(f-t)$ and we will identify $\psi_{f-t}({}_{\mathrm{T}}\pi^*M)$ near $(x_o, t) \in A_A$ with $\psi_g M$.

Since f is smooth, we can locally consider good V-filtrations along $(f-t)$ in order to compute $\psi_{f-t}({}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}})$. Arguing as in the beginning of the proof of Proposition 9.5.9, one obtains that ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is specializable along $f = t$ and that there exists a good V-filtration for which $\mathrm{gr}^V_{-1}({}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}) = {}_{\mathrm{D}}\iota^*_{f^{-1}(t)}({}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}})$. The latter module is equal to ${}_{\mathrm{D}}\iota^*_{g^{-1}(0)}\widetilde{\mathcal{M}}$, which itself is equal to $\psi_{g,1}\widetilde{\mathcal{M}}$, as $\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along (g) according to 14.2.2$(1)_g$; it follows that ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $(f-t)$, hence strictly non-characteristic along $(f-t)$ (see Proposition 9.5.9(2)).

(1) Let us end the proof of the first statement. The strict $\mathbb{R}$-specializability property for ${}_{\mathrm{D}}p^*\widetilde{\mathcal{M}}$ amounts to strictness of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$. A local section of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ which is of z-torsion is supported on $A \times \mathbb{P}^1$ since $\widetilde{\mathcal{M}}$ is strict. It is thus a local section of the coherent submodule of ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ supported on the divisor $(f-t)$, for every t. Since ${}_{\mathrm{D}}\pi^*\widetilde{\mathcal{M}}$ is strictly $\mathbb{R}$-specializable along $(f-t)$ by (3), this submodule is strict, according to Exercise 9.24. $\square$

End of the proof of Step 1. A similar argument is used to identify the sesquilinear pairings. The identification of the pre-polarizations S is straightforward, as they all both equal to $(\mathrm{Id}, \mathrm{Id})$.

Using the identification above near the axis, and the properties assumed for (M, S) on and out of the axis, we get all properties asserted for ${}_{\mathrm{T}}\pi^*(M, \mathrm{S})$ with respect to $(f-t)$ for any t. This concludes the first step. $\square$

Step 2. Let us set $(M_A, \mathrm{S}_A) = {}_{\mathrm{T}}\pi^*(M, \mathrm{S})$ that we consider as a pre-polarized object of $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X_A)$ since we do not yet know that it is a polarized Hodge module of weight w. Nevertheless, we aim at showing that, for the constant map $a_{X_A} : X_A \to \mathrm{pt}$ and the object (M_A, S_A),

- $\big({}_{\mathrm{T}}a^{(\bullet)}_{X_A*}(M_A, \mathrm{S}_A), \mathrm{X} + \mathrm{X}'\big)$ *is a polarized* $\mathfrak{sl}_2$*-Hodge structure of weight* w.

The support of M_A is $\pi^{-1}S$, which is equal to the blow-up S_A of S as we have seen above, and the fibers of $f_{|S_A}$ all have dimension $n-1$ ($n=\dim S$). According to Step 1 and to Assumption $(14.3.1)_{(n-1,0)}$, the assumptions of Proposition 14.4.2 are satisfied by the pre-polarized triple (M_A,S_A), and the conclusion of this proposition yields that $\big({}_{\mathrm{T}}f_*^{(\bullet)}\big((M_A,\mathrm{S}_A),\mathrm{X}\big)$ is a polarized $\mathfrak{sl}_2$-Hodge module of weight w. From this point, the arguments developed for the simple case of Section 14.5.a apply with no change to the present situation, and they yield the desired assertion.

Step 3. We now prove that

• *the pushforward* ${}_{\mathrm{T}}\pi_*^{(0)}(M_A)$ *decomposes as a direct sum in* $\widetilde{\mathcal{D}}$-$\mathsf{Triples}(X)$, *one summand being* M.

Let us first check that this is a local statement on X. If such a decomposition exists locally, then ${}_{\mathrm{T}}\pi_*^{(0)}(M_A)=M\oplus M_1$ locally, with M_1 supported on A. We need to prove that this decomposition is unique, in order to glue it along X (along A in fact, since π is an isomorphism away from A). Let g be a local equation for the hyperplane $f=t$ near a point $x_o\in A$. We claim that ${}_{\mathrm{D}}\pi_*^{(0)}(\widetilde{\mathcal{M}}_A)$ is strictly $\mathbb{R}$-specializable along (g). Indeed, we have seen in Step 1 that $\widetilde{\mathcal{M}}_A$ is strictly $\mathbb{R}$-specializable along $(f-t)$ and we have identified locally $\psi_{f-t}(\widetilde{\mathcal{M}}_A)$ with $\psi_g\widetilde{\mathcal{M}}$ (and we have a strict non-characteristic property, so that $\phi_{f-t,1}(\widetilde{\mathcal{M}}_A)$ is zero). We have also used that $\pi:\{f=t\}\to\{g=0\}$ is an isomorphism. By the pushforward theorem 9.8.8 or 10.6.4, we conclude that ${}_{\mathrm{D}}\pi_*^{(0)}(\widetilde{\mathcal{M}}_A)$ is strictly $\mathbb{R}$-specializable along (g). Since $\widetilde{\mathcal{M}}$ has pure support S, if ${}_{\mathrm{D}}\pi_*^{(0)}(\widetilde{\mathcal{M}}_A)$ decomposes locally as $\widetilde{\mathcal{M}}\oplus\widetilde{\mathcal{M}}_1$ with $\widetilde{\mathcal{M}}_1$ supported in A, hence in $\{g=0\}$, we can apply Proposition 9.7.2 to conclude that there does not exist any non-zero morphism $\widetilde{\mathcal{M}}\to\widetilde{\mathcal{M}}_1$ and $\widetilde{\mathcal{M}}_1\to\widetilde{\mathcal{M}}$, and thus the local decomposition of $\widetilde{\mathcal{M}}$ is unique. Similarly, according to Lemma 12.3.10, any sesquilinear pairing between $\mathcal{M}$ and $\mathcal{M}_1$ is zero, hence ${}_{\mathrm{D},\overline{\mathrm{D}}}\pi_*^{(0)}({}_{\mathrm{D},\overline{\mathrm{D}}}\pi^*\mathfrak{s})$ decomposes uniquely as $\mathfrak{s}\oplus\mathfrak{s}_1$.

Let us then consider the local statement near (x_o,t_o), that we can assume to belong to $A\times\mathbb{P}^1$, as π is an isomorphism outside of A. Let g be a local equation of a hyperplane containing A.

We claim that $\widetilde{\mathcal{M}}_A$ is strictly non-characteristic along both components of $g\circ\pi=0$ and their intersection. The components consist of

• the germ at x_o of the hyperplane $f=t_o$ containing A, for which the assertion has been proved in Step 1,

• the germ at (x_o,t_o) of $A\times\mathbb{P}^1$; by considering the left square in (14.5.2), the assertion follows from the property that $\widetilde{\mathcal{M}}$ is strictly non-characteristic along A, since $A_A\to A$ is smooth;

• the germ at (x_o,t_o) of $A\times\{t_o\}$, for which we apply the same argument as the previous one.

We can therefore apply the results of Section 12.7.23 together with Remark 14.2.3(4). They show that (M_A,S_A) satisfies $14.2.2(1)_{g\circ\pi}$ and $(2)_{g\circ\pi}$.

Arguing as in Proposition 14.4.2 (this is permissible due to the inductive hypothesis $(14.3.1)_{(\leqslant(n-1),0)}$, as the fibers of $\pi:S_A\to S$ have dimension $\leqslant n-1$), we conclude that $\big(\bigoplus_k {}_{\mathrm{T}}\pi_*^{(k)}(M_A), \mathrm{X}'\big)$ is strict and satisfies 14.2.2(1)$_g$ and (2)$_g$ in the sense of Lemma 14.2.29. We can cover A by finitely many open sets where the previous argument applies.

Let us set $M_0 := {}_{\mathrm{T}}\pi_*^{(0)}(M_A)$. We note that, as $\mathrm{X}'^2 = 0$, $M_0 = \mathrm{P}'_0{}_{\mathrm{T}}\pi_*^{(0)}(M_A)$ is strict and satisfies 14.2.2(1)$_g$ and (2)$_g$. By applying 14.2.17(10) to the quiver $(\psi_{g,1}M_0, \phi_{g,1}M_0, \mathrm{c}, \mathrm{v})$ and arguing as in the proof of $((10)_{<d} \Rightarrow (1)_d)$, we find that M_0 is S-decomposable along (g). We will identify M with a direct summand of it.

Let us set $M_0 = (\widetilde{\mathcal{M}}_0, \mathcal{S}_0)$. It decomposes therefore as $M_1 \oplus M_2$, with M_2 supported on $g^{-1}(0)$ and M_1 being a middle extension along (g). By Proposition 8.7.29, there is an adjunction morphism $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_0$. This morphism is an isomorphism away from A, hence from $g^{-1}(0)$, and is injective, as $\widetilde{\mathcal{M}}$ has no coherent submodule supported on $g^{-1}(0)$. Its image is thus contained in $\widetilde{\mathcal{M}}_1$.

At this point, we cannot assert that the image is equal to $\widetilde{\mathcal{M}}_1$, since the middle extension property 9.7.2(2) of $\widetilde{\mathcal{M}}_1$ only implies the vanishing of some quotient modules, and not all of them a priori. Nevertheless, the morphism $\mathcal{M} \to \mathcal{M}_1$ between the underlying $\mathcal{D}_X$-modules is an isomorphism (since no restriction occurs in 9.7.2(2) for $\mathcal{D}_X$-modules). It follows then from Proposition 12.3.8 applied to any germ of hyperplane containing A that $\mathcal{S} = \mathcal{S}_1$. It also follows that the cokernel of $\widetilde{\mathcal{M}} \to \widetilde{\mathcal{M}}_1$ is of z-torsion.

We thus have a monomorphism of Hermitian pairs $M \to M_1$. It is strictly $\mathbb{R}$-specializable along (g), since the associated nearby and vanishing cycle morphisms are morphisms in $\mathsf{pHLM}(X, w-1)$ or $\mathsf{pHLM}(X, w)$. Therefore, this morphism is strict, according to Corollary 10.8.6. The cokernel, being strict and of z-torsion, must then vanish, and $M \simeq M_1$, as wanted.

Step 4. As X' vanishes on M, we conclude from Step 3 that $({}_{\mathrm{T}}a_{X,*}M, \mathrm{N})$ is a direct summand of $({}_{\mathrm{T}}a_{X_A*}M_A, \mathrm{X}+\mathrm{X}')$. From Step 2 and **[Del68]** we have a (non canonical) decomposition ${}_{\mathrm{T}}a_{X_A*}M_A \simeq \bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}(M_A)[-k]$. Therefore, this decomposition can be chosen to induce a decomposition ${}_{\mathrm{T}}a_{X,*}M \simeq \bigoplus_k {}_{\mathrm{T}}a_{X,*}^{(k)}(M)[-k]$. In particular, $\big(\bigoplus_k {}_{\mathrm{T}}a_{X,*}^{(k)}(M), \mathrm{X}\big)$ is an $\mathfrak{sl}_2$-Hodge structure with central weight w, being a direct summand of the $\mathfrak{sl}_2$-Hodge structure $\big(\bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}(M_A), \mathrm{X}+\mathrm{X}'\big)$ with central weight w.

Step 5. It remains to show the polarization property. By the result of Step 2, $\big(\bigoplus_k {}_{\mathrm{T}}a_{X_A*}^{(k)}(M_A), \mathrm{X}+\mathrm{X}'\big)$ is polarized by $(\mathrm{Id}, \mathrm{Id})$, which induces the desired prepolarization on $\big(\bigoplus_k {}_{\mathrm{T}}a_{X,*}^{(k)}(M), \mathrm{X}\big)$. That the latter is a polarization is a particular case of Proposition 3.4.18(2). This concludes the proof of 14.3.3(d), hence that of the Hodge-Saito theorem 14.3.1. $\square$

14.6. Variations of Hodge structure are Hodge modules

The first non trivial example of a polarizable Hodge module is given by a polarizable variation of Hodge structure. The following theorem is a partial converse of Proposition 14.2.10.

14.6.1. Theorem. *Let X be a complex manifold of dimension n and let H be a smooth Hodge triple of weight w, that is, a polarizable variation of pure Hodge structure of weight $w-n$ (see Definition 5.4.7). Then H is a polarizable Hodge module of weight w.*

From Theorem 14.6.1 and the Hodge-Saito theorem 14.3.1, we deduce:

14.6.2. Corollary. *Let $(\mathcal{H}, \nabla)$ be vector bundle with connection on X underlying a variation of polarizable Hodge structure of weight w. Then its direct image (in the category of $\widetilde{\mathcal{D}}$-modules) by a projective morphism $f : X \to Y$ decomposes non-canonically in $\mathsf{D}^{\mathrm{b}}(\mathcal{D}_Y)$*

$$
{}_{\mathrm{D}}f_*(\mathcal{H}, \nabla) \simeq \bigoplus_k {}_{\mathrm{D}}f_*^{(k)}(\mathcal{H}, \nabla),
$$

and each ${}_{\mathrm{D}}f_*^{(k)}(\mathcal{H}, \nabla)$ underlies a polarizable Hodge module of weight $w + \dim X + k$. □

***Proof of Theorem* 14.6.1.** This assertion is not trivially satisfied since one has to check in an iterative way that nearby cycles and vanishing cycles along *any* germ of holomorphic function are polarizable Hodge modules. We assume that the polarization is $\mathrm{S} = (\mathrm{Id}, \mathrm{Id})$, i.e., we realize H as a Hermitian pair $(\widetilde{\mathcal{H}}, \mathcal{S})$.

We first note that $14.2.2(1)_g$ and $(2)_g$ hold for $(H, (\mathrm{Id}, \mathrm{Id}))$ if g is a local coordinate on X. According to Remark 14.2.3(3), these properties also hold when g is a power of a local coordinate on X. As a consequence, the assertion of the theorem holds if $\dim X = 1$.

If $\dim X \geqslant 2$, the proof is by induction on $\dim X$. We thus assume that the theorem holds for $\dim X < n$ ($n \geqslant 2$), and we assume $\dim X = n$. We wish to prove that, for any germ of holomorphic function g on X, $14.2.2(1)_g$ and $(2)_g$ hold for $(H, (\mathrm{Id}, \mathrm{Id}))$.

Step 1: reduction to the case where $D := (g)$ is a normal crossing divisor

We assume that $14.2.2(1)_g$ and $(2)_g$ hold for $(H, (\mathrm{Id}, \mathrm{Id}))$ if g defines a normal crossing divisor in X. Let us then take any germ g on X centered at $x \in X$. We simply denote by X the germ (X, x) and by D the germ of the reduced divisor defined by g. Let $f : X' \to X$ be a projective modification which is an isomorphism $X' \smallsetminus f^{-1}(D) \to X \smallsetminus D$ such that $g' := g \circ f$ defines a normal crossing divisor D' in X'.

The pullback $(H', (\mathrm{Id}, \mathrm{Id})) := {}_{\mathrm{T}}f^*(H, (\mathrm{Id}, \mathrm{Id}))$ is also a polarized variation of pure Hodge structure of weight $w-n$ (see 12.7.13) and is strict as an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X')$. Furthermore, by our assumption, $14.2.2(1)_{g'}$ and $(2)_{g'}$ hold for $(H', (\mathrm{Id}, \mathrm{Id}))$. It follows from Proposition 14.4.2 that $({}_{\mathrm{T}}f_*^{(0)}H', (\mathrm{Id}, \mathrm{Id}))$ satisfies $14.2.2(1)_g$ and $(2)_g$, that it is S-decomposable along (g) as an object of $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$ and Corollary 10.8.8 yields that it is strict.

Let us denote by $(H'_0, (\mathrm{Id}, \mathrm{Id}))$ the component of $({}_{\mathrm{T}}f_*^{(0)}H', (\mathrm{Id}, \mathrm{Id}))$ with pure support X. It also satisfies $14.2.2(1)_g$ and $(2)_g$, is strict, and is a middle extension along (g). It corresponds to a coherently filtered $\mathcal{D}_X$-module $(\mathcal{H}'_0, F_\bullet\mathcal{H}'_0)$. We will show that $(H'_0, (\mathrm{Id}, \mathrm{Id}))$ is isomorphic to $(H, (\mathrm{Id}, \mathrm{Id}))$, concluding thereby the first step.

We start with identifying the $\widetilde{\mathcal{D}}_X$-module components. Composing the adjunction morphism $\widetilde{\mathcal{H}} \to {}_{\mathrm{D}}f_*^{(0)}\widetilde{\mathcal{H}}'$ of Proposition 8.7.29 with the projection (coming from the

S-decomposition) ${}_{\mathrm{D}}f_*^{(0)}\widetilde{\mathcal{H}}' \to \widetilde{\mathcal{H}}'_0$ yields a morphism $\widetilde{\mathcal{H}} \to \widetilde{\mathcal{H}}'_0$ which is an isomorphism on the complement of D. Since $\widetilde{\mathcal{H}}$ is $\widetilde{\mathcal{O}}_X$-locally free, this morphism is injective. On the other hand, $\mathcal{H}'_0$ is a middle extension along (g) (Example 11.5.3 and Remark 11.5.5). Therefore, $\mathcal{H} \to \mathcal{H}'_0$ is an isomorphism (see Exercise 9.41(1)).

What about the Hodge filtrations? We know that the morphism $F_p\mathcal{H} \to F_p\mathcal{H}'_0 =: F'_p\mathcal{H}$ is injective and is an isomorphism on $X \smallsetminus D$, so $F'_p\mathcal{H}/F_p\mathcal{H}$ is supported in D. On the other hand, $\mathcal{H}/F_p\mathcal{H}$ is $\mathcal{O}_X$-locally free, being a successive extension of $\mathcal{O}_X$-locally free modules $\mathrm{gr}_q^F\mathcal{H}$. Since we have an inclusion $F'_p\mathcal{H}/F_p\mathcal{H} \hookrightarrow \mathcal{H}/F_p\mathcal{H}$, it follows that $F'_p\mathcal{H}/F_p\mathcal{H} = 0$, that is, $F'_p\mathcal{H} = F_p\mathcal{H}$, as desired.

What about the sesquilinear pairing $\mathcal{S}$ on $\mathcal{H}$ and $\mathcal{S}'$ on $\mathcal{H}'_0 \simeq \mathcal{H}$? Both take values in $\mathcal{C}_X^\infty$ (Lemma 12.3.6) and coincide on $X \smallsetminus D$, hence they coincide. □

Step 2: reduction to the case where (g) ***is a reduced normal crossing divisor.*** According to Step 1, we can assume that g is a monomial $x_1^{r_1}\cdots x_\ell^{r_\ell}$ in a local coordinate system $(x_1,\dots,x_n)$. We still denote by X the corresponding local coordinate chart. There exists a multi-cyclic ramified covering $f : X' \to X$ such that $h := g\circ f$ is a power of a product of local coordinates $(x'_1\cdots x'_\ell)^r$. Set $h' = x'_1\cdots x'_\ell$ and let us assume that $14.2.2(1)_{h'}$ and $(2)_{h'}$ hold for $(H', (\mathrm{Id},\mathrm{Id})) := {}_{\mathrm{T}}f^*(H,(\mathrm{Id},\mathrm{Id}))$. Then $14.2.2(1)_h$ and $(2)_h$ hold for $(H',(\mathrm{Id},\mathrm{Id})))$, according to Remark 14.2.3(3). We wish to prove that $14.2.2(1)_g$ and $(2)_g$ hold for $(H,(\mathrm{Id},\mathrm{Id}))$. We argue in a way similar to that of Step 1 and take the same notation. In particular, $(H'_0,(\mathrm{Id},\mathrm{Id}))$ is the component of ${}_{\mathrm{T}}f_*^{(0)}(H',(\mathrm{Id},\mathrm{Id})))$ with pure support X, and $14.2.2(1)_g$ and $(2)_g$ hold for it.

According to 12.7.30, $(H,(\mathrm{Id},\mathrm{Id}))$ is a direct summand of $(H'_0,(\mathrm{Id},\mathrm{Id}))$. Since $14.2.2(1)_g$ and $(2)_g$ are stable by direct summand in $\widetilde{\mathcal{D}}\text{-}\mathsf{Triples}(X)$, as follows from 14.2.8, they hold for $(H,(\mathrm{Id},\mathrm{Id}))$. □

Step 3: case where (g) ***is a reduced normal crossing divisor.*** Assume now that g is a product of distinct coordinates of a local coordinate system. We are thus in the setting of Example 12.7.24. We then know that H is strictly $\mathbb{R}$-specializable and a middle extension along (g), so we only need to check $14.2.2(2\mathrm{a})_g$, according to Remark 14.2.3(4).

We are therefore led to showing that the right-hand side in (12.7.24 $*$) is a polarized Hodge module of weight $w+\ell-1$ ($\ell \geqslant 0$), where each ι_I occurring there is the inclusion of a codimension $(\ell+1)$ submanifold in X. By induction on $\dim X$, each variation ${}_{\mathrm{T}}\iota_I^*(H,\mathrm{S})$ for $J = I^{\mathrm{c}} \in \mathcal{J}_{\ell+1}$ is a polarized Hodge module of weight $w-(\ell+1)$, since its support has dimension $n-(\ell+1)$. Hence, by Kashiwara's equivalence of Proposition 14.2.9, $({}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_I^*H),(\mathrm{Id},\mathrm{Id}))$ is a polarized Hodge module of weight $w-(\ell+1)$, and $({}_{\mathrm{T}}\iota_{I*}({}_{\mathrm{T}}\iota_I^*H),(\mathrm{Id},\mathrm{Id}))(-\ell)$ is a polarized Hodge module of weight $w+\ell-1$. □

14.7. Exercises

Exercise 14.1. Show that if the conditions in Definition 14.2.2 hold for a function g, they hold for g^r for any $r \in \mathbb{N}^*$. [*Hint*: Use the example of Section 9.9.a.] Conclude that, if $n = 1$, Definition 14.2.2 reduces to Definition 7.4.7.

14.8. Comments

The relation between Hodge theory and the theory of nearby or vanishing cycles in dimension bigger than one starts with the work of Steenbrink [**Ste76, Ste77**]. It concerns 1-parameter families of projective varieties, regarded as proper functions from a complex manifold to a disc. A canonical Hodge structure is constructed on the cohomology of the nearby fiber of a singular fiber of the family by means of replacing the special fiber with a divisor with normal crossings and by computing the nearby or vanishing cohomology in terms of a logarithmic de Rham complex, in order to apply Deligne's method in [**Del71b**]. This gives a geometric construction of Schmid's limit mixed Hodge structure in the case of a variation of geometric origin. The need of passing from the assumption of unipotent monodromy, as used in the work of Schmid [**Sch73**] to the assumption of quasi-unipotent monodromy is justified by this geometric setting. This leads Steenbrink [**Ste77**] to developing the notion of logarithmic de Rham complex in the setting of V-manifolds. Steenbrink also obtains, as a consequence of this construction, the local invariant cycle theorem and the Clemens-Schmid exact sequence. We can regard this work as the localization of Hodge theory in the analytic neighbourhood of a projective variety.

The work of Varchenko [**Var82**] and others on asymptotic Hodge theory has localized even more Hodge theory. This work is concerned with an isolated singularity of a germ of holomorphic function and it constructs a Hodge-Lefschetz structure on the space of vanishing cycles of this function, by taking advantage that the vanishing cycles are supported at the isolated singularity, which is trivially a projective variety. The construction of Varchenko has been later analyzed in terms of $\mathcal{D}$-modules by Pham [**Pha83**], Saito [**Sai83b, Sai83a, Sai84, Sai85**] and Scherk-Steenbrink [**SS85**]. It is then natural to consider the cohomology of the vanishing cycle sheaf of a holomorphic function on a complex manifold whose critical locus is projective, but possibly not the special fiber of the function, and to ask for a mixed Hodge structure on it.

The theory of polarizable Hodge modules, as developed by Saito in [**Sai88**], emphasizes the local aspect of Hodge theory, by constructing a category defined by local properties in a way similar, but much more complicated, to the definition of a the category of variations of Hodge structure. It can then answer the question above. This idea has proved very efficient, eventually allowing to use the formalism of Grothendieck's six operations in Hodge theory. Many standard cohomological results, like the Clemens-Schmid exact sequence and the local invariant cycle theorem, can be read in this functorial way.

The definition of complex Hodge modules as developed here, not relying on a $\mathbb{Q}$-structure and on the notion of a perverse sheaf, is inspired by the extension of the notion of polarizable Hodge module to twistor theory, as envisioned by Simpson [**Sim97**], and achieved by Sabbah [**Sab05**] and Mochizuki [**Moc07, Moc15**], although the way the sesquilinear pairing is used on both theories is not exactly the same. We refer to the comments of Chapter 12 for the idea of using sesquilinear pairings in the framework of holonomic $\mathcal{D}$-modules.

CHAPTER 15

THE STRUCTURE THEOREM FOR POLARIZABLE HODGE MODULES

Summary. Under work.

15.1. Introduction

15.2. Polarized Hodge modules in the normal crossing case

Let X be a complex manifold and let $D = \bigcup_{i\in I} D_i$ be a reduced divisor with normal crossings. Let (M, S) be a polarized Hodge module with pure support X and singularities on D, so that $(M, \mathrm{S})_{|X\smallsetminus D}$ is a polarized variation of Hodge structure.

15.2.1. Theorem. *With these assumptions, the filtered $\mathcal{D}_X$-module $(\mathcal{M}, F^\bullet\mathcal{M})$ underlying M is of normal crossing type and a middle extension along $D_{i\in I}$ (Definitions* 13.4.1 *and* 13.4.9*).*

We first check the property for $\mathcal{M}$.

15.2.2. Lemma. *With these assumptions, the underlying $\mathcal{D}_X$-module $\mathcal{M}$ is of normal crossing type (Definition* 13.3.3*) and a middle extension along $D_{i\in I}$ (Definition* 13.2.8*).*

Let us recall the local setting of Chapter 13. The space X is a polydisc in $\mathbb{C}^n$ with analytic coordinates $x_1, \dots, x_n$, we fix $\ell \leqslant n$ and we denote by D the divisor $\{x_1 \cdots x_\ell = 0\}$. We also denote by D_i ($i \in I$) the smooth components of D and by $D_{(\ell)}$ their intersection $D_1 \cap \cdots \cap D_\ell$. We will shorten the notation $\mathcal{O}_{D_{(\ell)}}[x_1, \dots, x_\ell]$ into $\mathcal{O}_{D_{(\ell)}}[x]$ and $\mathcal{D}_{D_{(\ell)}}[x_1, \dots, x_\ell]\langle\partial_{x_1}, \dots, \partial_{x_\ell}\rangle$ into $\mathcal{D}_{D_{(\ell)}}[x]\langle\partial_x\rangle$.

***Proof of Lemma* 15.2.2.** Since $\mathcal{M}$ is holonomic, and smooth on $X \smallsetminus D$, $\mathcal{M}(*D)$ is a coherent $\mathcal{O}_X(*D)$-module, according to Example 11.3.12.

On the smooth open subset of D, we can apply the same argument as for Proposition 7.4.12 and conclude that for each p, we have the equality

$$F_p\mathcal{M} \cap V_{<0}\mathcal{M} = (j_*j^{-1}F_p\mathcal{M}) \cap V_{<0}\mathcal{M}.$$

In particular, for $p \gg 0$ we obtain that $V_{<0}\mathcal{M} = F_p\mathcal{M} \cap V_{<0}\mathcal{M}$ is $\mathcal{O}_X$-coherent. This means that the $\mathcal{O}_X(*D)$-module with flat connection $\mathcal{M}(*D)$ has regular singularities along the smooth open subset of D. It follows from [**Del70**, Th. 4.1 p. 88] that $\mathcal{M}(*D)$

is $\mathcal{O}_X(*D)$-locally free and has regular singularities along D, so $\mathcal{M}(*D)$ is of normal crossing type along D. Moreover, $\mathcal{M}$ is its middle extension along $D_{i\in I}$, hence is also of normal crossing type. $\square$

15.2.3. Lemma. *Assume there exists a coherent filtered $\mathcal{D}_X$-module $(\mathcal{M}', F_\bullet\mathcal{M}')$ of filtered normal crossing type along D and a middle extension along $D_{i\in I}$ such that $j^{-1}(\mathcal{M}, F_\bullet\mathcal{M}) \simeq j^{-1}(\mathcal{M}', F_\bullet\mathcal{M}')$. Then $(\mathcal{M}, F_\bullet\mathcal{M}) \simeq (\mathcal{M}', F_\bullet\mathcal{M}')$.*

This lemma reduces the proof of Theorem 15.2.1 to the construction of $(\mathcal{M}', F_\bullet\mathcal{M}')$.

Proof. By Lemma 15.2.2 we have $\mathcal{M} \simeq \mathcal{M}'$, so we identify these modules, and we set $F_\bullet\mathcal{M}' = F'_\bullet\mathcal{M}$. Let g be a defining equation for D and let ι_g be the corresponding graph embedding. Then $(\mathcal{M}, F'_\bullet\mathcal{M})$ is strictly $\mathbb{R}$-specializable along (g) and a middle extension along $D_{i\in I}$, according to Theorem 13.1.4. By definition, the same property holds for $(\mathcal{M}, F_\bullet\mathcal{M})$. Applying Remark 10.6.2 to $F_{\bullet\mathrm{D}}\iota_{g*}\mathcal{M}$ and $F'_{\bullet\mathrm{D}}\iota_{g*}\mathcal{M}$ leads to $F_{\bullet\mathrm{D}}\iota_{g*}\mathcal{M} = F'_{\bullet\mathrm{D}}\iota_{g*}\mathcal{M}$, hence $F_\bullet\mathcal{M} = F'_\bullet\mathcal{M}$. $\square$

According to Proposition 13.4.19, the proof of Theorem 15.2.1 will be achieved if we prove the higher dimensional analogue of Theorem 6.7.3:

15.2.4. Theorem. *If $(M, \mathrm{S})_{X\smallsetminus D}$ is a polarized variation of Hodge structure on $X \smallsetminus D$, and if we set $F_p\mathcal{M}_{<0} = j_*F_p\mathcal{M}_{|X\smallsetminus D}$, then for each $\boldsymbol{\alpha} < \mathbf{0}$ the sheaves $\mathrm{gr}_p^F V_{\boldsymbol{\alpha}}^{(\boldsymbol{\ell})}\mathcal{M}$ are (coherent and) locally free $\mathcal{O}_X$-modules.*

15.3. The structure, decomposition and semi-simplicity theorems

15.3.a. The structure theorem. This is the converse of Proposition 14.2.10. Let X be a complex manifold and let Z be an irreducible closed analytic subset of X. Let $\mathsf{VHS}_{\mathrm{gen}}(Z, w)$ the category of "generically defined variations of Hodge structure of weight w on Z".

We say that a pair (Z^o, H) consisting of a smooth Zariski-dense open subset Z^o of Z and of a variation of Hodge structure H of weight w on Z^o is equivalent to a pair (Z'^o, H') if H and H' coincide on $Z^o \cap Z'^o$. An object of $\mathsf{VHS}_{\mathrm{gen}}(Z, w)$ is such an equivalence class. Note that it has a maximal representative (by considering the union of the domains of all the representatives). A morphism between objects of $\mathsf{VHS}_{\mathrm{gen}}(Z, w)$ is defined similarly.

We also denote by $\mathsf{pVHS}_{\mathrm{gen}}(Z, w)$ the full subcategory of $\mathsf{VHS}_{\mathrm{gen}}(Z, w)$ consisting of objects which are polarizable, i.e., have a polarizable representative.

By Proposition 14.2.10, there is a restriction functor

$$\mathsf{pHM}_Z(X, w - \operatorname{codim} Z) \longmapsto \mathsf{pVHS}_{\mathrm{gen}}(X, w).$$

***15.3.1. Theorem* (Structure theorem).** *Under these assumptions, the restriction functor $\mathsf{pHM}_Z(X, w - \operatorname{codim} Z) \mapsto \mathsf{pVHS}_{\mathrm{gen}}(X, w)$ is an equivalence of categories.*

Since each polarizable Hodge module has a unique decomposition with respect to the irreducible components of its pure support, the structure theorem gives a complete description of the category $\mathsf{pHM}(X, w)$.

The structure theorem enables us to prove that the pullback by a holomorphic map of complex manifolds of a polarizable Hodge module remains a polarizable Hodge module. This statement is not obvious: for the constant map $f : X \to \mathrm{pt}$, the pullback ${}_{\mathrm{T}}f^*{}_{\mathrm{T}}\mathbb{C}_{\mathrm{pt}}$ is ${}_{\mathrm{H}}\mathcal{O}_X$.

***Sketch for the structure theorem* 15.3.1.** We first notice that the restriction functor $\mathsf{pHM}_Z(X, w) \to \mathsf{pVHS}_{\mathrm{gen}}(Z, w - \operatorname{codim} Z)$ is faithful. Indeed, let M_1, M_2 be objects of $\mathsf{pHM}_Z(X, w)$ and let $\varphi, \varphi' : M_1 \to M_2$ be morphisms between them, which coincide on some Z^o. Then the image of $\varphi - \varphi'$ is an object of $\mathsf{pHM}(X, w)$, according to Corollary 14.2.19, and is supported on $Z \smallsetminus Z^o$, hence is zero according to the definition of the pure support.

Due to the faithfulness, we note that the question is local: for fullness, if a morphism between the restriction to some Z^o of two polarized Hodge modules locally extends on Z, then it globally extends by uniqueness of the extension; for essential surjectivity, we note that two local extensions as polarized Hodge modules of a polarized variation of Hodge structure coincide, by extending the identity morphism on some Z^o according to local fullness, and we can thus glue local extensions into a global one.

For the essential surjectivity we start from a polarized variation of Hodge structure on some smooth Zariski-dense open subset $Z^o \subset Z$. We choose a projective morphism $f : Z' \to X$ with Z' smooth and connected, such that f is an isomorphism $Z'^o := f^{-1}(Z^o) \to Z^o$, and such that $Z' \smallsetminus Z'^o$ is a divisor with normal crossing. Assuming we have extended the variation on Z'^o as a polarized Hodge module on Z' with pure support Z', we apply to the latter the direct image theorem 14.3.1 for f, and get the desired polarized Hodge module as the component of this direct image ${}_{\mathrm{T}}f_*^0$ having pure support Z. We argue similarly for the fullness: if any morphism defined on some Z^o can be extended as a morphism between the extended objects on Z', we push it forward by f and restrict it as a morphism between the corresponding components.

We are thus reduced to the case where $Z = X$ and the variation exists on $X^o := X \smallsetminus D$, where D is a divisor with normal crossings. Moreover, the question is local. By using the asymptotic theory of variations of Hodge structure we construct coherent $\mathcal{D}_X$-modules $\mathcal{M}', \mathcal{M}''$ and we prove that the sesquilinear pairing $\mathfrak{s}^o$ takes values in the sheaf of moderate distributions along D.

15.3.b. Semi-simplicity and decomposition of projective pushforwards

Any polarizable Hodge module of weight w is semi-simple in the category $\mathsf{pHM}(X, w)$ (Corollary 14.2.20). If X is a projective complex manifold, semi-simplicity also holds for the underlying holonomic $\mathcal{D}_X$-module, that is, the analogue of Theorem 4.3.3 holds for polarizable Hodge modules.

***15.3.2. Theorem* (Semi-simplicity).** *Assume X is projective. Let (M,S) be a polarized Hodge module of weight w (so that $\mathcal{M}'\simeq\mathcal{M}''$ by means of a polarization). Then the underlying $\mathcal{D}_X$-module $\mathcal{M}$ is semi-simple. Furthermore, any simple component $\mathcal{M}_\alpha$ of $\mathcal{M}$ underlies a unique (up to equivalence) polarized Hodge module $(M_\alpha,\mathrm{S}_\alpha)$ of the same weight w and there exists a polarized Hodge structure $(H^o_\alpha,\mathrm{S}^o_\alpha)$ of weight 0 such that $(M,\mathrm{S})\simeq\bigoplus_\alpha((H^o_\alpha,\mathrm{S}^o_\alpha)\otimes(M_\alpha,\mathrm{S}_\alpha))$.*

(See Section 4.3.c for the notion of equivalence.)

Proof. By the S-decomposition theorem (Corollary 14.2.20), we can assume that M has pure support an irreducible variety $Z\subset X$. If $\dim Z=0$, the result is clear by Definition 14.2.2(0). If $\dim Z\geqslant 1$, the restriction of (M,S) to a suitable smooth Zariski dense open subset Z^o of Z is a polarized variation of Hodge structure of weight $w-\dim Z$ (Proposition 14.2.10). The equivalence provided by the structure theorem 15.3.1 implies that it is enough to prove the result for smooth polarized Hodge modules on Z^o.

• If $\dim Z^o=1$, the underlying local system is semi-simple (Corollary 6.4.2) and each simple component underlies a unique (up to equivalence) polarized variation of Hodge structure of weight $w-\dim Z^o$ and the polarized variation $(M,\mathrm{S})_{|Z^o}$ is recovered by the formula of Theorem 4.3.13(2), according to Theorem 6.14.17.

• If $\dim Z^o\geqslant 2$, we fix a projective embedding of Z. The Zariski-Lefschetz theorem [**HL85**, Th. 1.1.3(ii)] implies that, for a generic hyperplane H, the inclusion $H\cap Z^o\hookrightarrow Z^o$ induces a *surjective* morphism of fundamental groups. By induction and according to Remark 4.3.2(2), we conclude that the local system underlying $M_{|Z^o}$ is semi-simple.

In order to obtain the last statement of the theorem, we will apply the same argument as for Theorem 4.3.13(2). For that purpose, we need to know that the space of global sections of a polarized variation of Hodge structure on Z^o is a polarized Hodge structure. Since we have the choice of a compactification Z^o, we can assume that Z is smooth and $D=Z\smallsetminus Z^o$ is a divisor with normal crossings. By the structure theorem, the variation extends as a polarizable Hodge module M with pure support Z. By the Hodge-Saito theorem 14.3.1 applied to the constant map $a_Z:Z\to\mathrm{pt}$, $\boldsymbol{H}^{-\dim Z}{}_{\mathrm{H}}a_{Z*}M$ is a polarizable Hodge structure. Its underlying vector space is $\boldsymbol{H}^{-n}(Z,{}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})=H^0(Z,\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}))$ (since all differentials d_r ($r\geqslant 2$) in the spectral sequence starting with $E_2^{i,j}=H^i(Z,\mathcal{H}^j({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}))$ vanish on $E_2^{0,-n}$). We are thus left with proving $H^0(Z,\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}))=H^0(Z^o,\underline{\mathcal{H}})$, with $\underline{\mathcal{H}}=\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})_{|Z^o}$. This amounts to the equality of sheaves $\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})=j_*\underline{\mathcal{H}}$, where $j:Z^o\hookrightarrow Z$ denotes the inclusion, and this is a local question in the neighbourhood of each point of D.

By Lemma 15.2.2, the $\mathcal{D}_X$-module $\mathcal{M}$ is of normal crossing type and a middle extension along $D_{i\in I}$. Let us work in the local setting with the simplifying assumption 13.1.2. We will show the equality of germs $\mathcal{H}^{-n}({}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M})_0=(j_*\underline{\mathcal{H}})_0$. The germ at 0 of the de Rham complex ${}^{\mathrm{p}}\mathrm{DR}\,\mathcal{M}$ is the simple complex associated to the n-complex having vertices equal to $\bigoplus_{\boldsymbol{\alpha}\in[-1,0)^n} M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ with $\boldsymbol{k}\in\{0,1\}^n$ and arrows in the i-th direction induced by ∂_{x_i}. The latter is an isomorphism on each $M_{\boldsymbol{\alpha}+\boldsymbol{k}}$ with $\alpha_i\neq -1$ and $k_i=0$. This complex is thus isomorphic to its subcomplex with vertices $M_{-\mathbf{1}+\boldsymbol{k}}$, so that ∂_{x_i} reads can_i, and $\mathcal{H}^0(\mathrm{DR}\,\mathcal{M})_0=\bigcap_i \operatorname{Ker}\mathrm{can}\, i\subset M_{-\mathbf{1}}$. A similar analysis shows that $(j_*\underline{\mathcal{H}})_0=\bigcap_i \operatorname{Ker}\mathrm{N}_i\subset M_{-\mathbf{1}}$. Recall now that $\mathcal{M}$ is a middle extension along $D_{i\in I}$. This means that can_i is onto and var_i is injective, so $\operatorname{Ker}\mathrm{N}_i=\operatorname{Ker}\mathrm{can}_i$, and this concludes the proof. $\square$

When both X and Y are projective, we can combine Theorems 14.3.2, 14.3.1 and 15.3.2 to obtain:

15.3.3. Corollary. *Let $f:X\to Y$ be a morphism between projective complex manifolds and let $\mathcal{M}$ be a semi-simple holonomic $\mathcal{D}_X$-module underlying a polarizable Hodge module. Then ${}_{\mathrm{D}}f_*\mathcal{M}$ decomposes non-canonically as $\bigoplus_k {}_{\mathrm{D}}f_*^{(k)}\mathcal{M}[-k]$, and each ${}_{\mathrm{D}}f_*^{(k)}\mathcal{M}$ is itself a semi-simple holonomic $\mathcal{D}_Y$-module.* $\square$

15.4. Comments

Here come the references to the existing work which has been the source of inspiration for this chapter.

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