

REPRESENTATIONS OF REDUCTIVE GROUPS OVER LOCAL FIELDS

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[This chapter, especially the theory of asymptotics, is under construction and has not been proofread. Some statements may be slightly imprecise.]

In this chapter, we discuss representations of a group of the form $G(F)$, when F is a local (locally compact) field, and G is a reductive algebraic group over F . We treat the Archimedean and non-Archimedean cases in parallel, highlighting similarities. For economy of language, such a group will be called a “real or p -adic reductive group” — the p -adic case including non-Archimedean local fields in equal characteristic, $F = \mathbb{F}_q((t))$. The “real” case includes the case when $F = \mathbb{C}$ — notice that the complex structure plays no role in the representation theory, and we can think instead of $G(\mathbb{C})$ as $\text{Res}_{\mathbb{C}/\mathbb{R}} G(\mathbb{R})$. Everything in this chapter also applies to finite central extensions of reductive groups of the form $G(F)$, like the *metaplectic group*, which are not necessarily algebraic; however, the notation is mostly adapted to the algebraic case. When it is clear from the context, the group $G(F)$ will simply be denoted by G . If the word “reductive” is omitted, a “real group” will be a Lie group, and a “ p -adic group” will be a p -adic analytic group (although most

statements will be true for arbitrary totally disconnected, locally compact groups, in this case).

Remark 0.1. A common misunderstanding, when $G = G(\mathbb{C})$ is a complex group, and (π, V) is a smooth complex representation of G , is that the (complex) Lie algebra \mathfrak{g} acts by complex-linear endomorphisms on V ; it does not! Instead, G should be treated as a real Lie group; for any smooth, complex representation of a real Lie group, we have an action of the complexified Lie algebra $\mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ on V by complex-linear automorphisms.

1. Various categories of representations

1.1. Smooth and SF-representations. The notion of a continuous, in particular of a Banach representation of a topological group was introduced in Definition 2.1. We also introduced an F -representation (or Fréchet representation of moderate growth) in Definition 6.1, which is a Fréchet representation that is a countable limit of Banach representations.

Definition 1.2. A *smooth vector* in a representation (π, V) of a real or p -adic group, resp. an *analytic vector*, in the real case, is a vector $v \in V$ such that the action map $G \ni g \mapsto \pi(g)v \in V$ is smooth (resp., analytic).¹ In particular, in the real case, for a smooth vector v and any element D of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$, the element $\pi(D)v$ is defined.

The *space of smooth vectors* of a representation (π, V) is denoted by V^{∞} , and considered as a topological space, in the p -adic case with the direct limit topology over the subspaces V^J as J varies over open compact subgroups, and in the real case with the topology of convergence of all $\pi(D)v$, where D ranges over elements of the universal enveloping algebra $U(\mathfrak{g}_{\mathbb{C}})$.

A *smooth representation* (π, V) is a representation such that $V = V^{\infty}$ as topological vector spaces.

An *SF-representation*, or *smooth representation of moderate growth* of a real or p -adic group G is a smooth F -representation.

Lemma 1.3. *If V is a Fréchet representation of a Lie or p -adic group G , the subspace V^{∞} of smooth vectors is dense.*

Proof. By Proposition 3.3, the algebra $M_c^{\infty}(G)$ of smooth, compactly supported measures (= smooth, compactly supported functions times a Haar measure) acts on V . The image of the action is clearly in V^{∞} , and by an approximation of the identity, one sees that the image is dense. \square

A much stronger, and important, statement is true: the *Dixmier–Malliavin theorem* states that the image of the action of $M_c^{\infty}(G)$ is all of V^{∞} :

Theorem 1.4. *Let V be a Fréchet representation of a Lie group or p -adic group G . The action map*

$$M_c^{\infty}(G) \otimes V \rightarrow V^{\infty}$$

is surjective.

¹In the p -adic case, “smooth” means locally constant, so the definition is equivalent to requiring that v have an open stabilizer.

Notice that the tensor product here is not completed! The theorem means that every smooth vector can be written as a finite linear combinations of smooth, compactly supported measures acting on other vectors. (Also, without loss of generality, one might assume that $V = V^\infty$, if desired.)

Proof. The p -adic case is trivial, since every J -invariant vector (where J is an open compact subgroup) is fixed by the action of $e_J =$ the probability Haar measure on J . The real case is the theorem of Dixmier–Malliavin, see [DM78], or [Cas11]. \square

Remark 1.5. Outside of the realm of F -representations (Fréchet representations of moderate growth), the notion of smooth representation leads to counterintuitive examples, e.g., the space of distributions on a Lie group G is a smooth representation. We will only be considering smooth Fréchet representations of moderate growth from now on.

Lemma 1.6. *If V is an F -representation of a real group, then V^∞ is an SF -representation.*

Proof. First of all, notice that the topology on V^∞ is also given by a countable set of G -continuous seminorms: If ρ_n is a sequence of G -continuous seminorms on V , defining its topology, and we fix, for every $d \geq 0$, a basis $(D_{d,i})_i$ of the d -th filtered part of the universal enveloping algebra $U(\mathfrak{g}_\mathbb{C})$, then the seminorms $\rho_{d,n}(v) = \max_i \rho_n(D_{d,i}v)$ define the topology on V^∞ as n and d vary, and are G -continuous, because $\rho_{d,n}(gv) = \max_i \rho_n(g \cdot \text{Ad}(g^{-1})(D_{d,i})v) \ll \rho_{d,n}(v)$ (locally uniformly in G), since the adjoint representation preserves the filtration.

The content of the lemma, then, is that the topological vector space V^∞ is complete. One shows that the action map $g \mapsto \pi(g)v$ gives rise to a morphism $V^\infty \rightarrow C^\infty(G, V)$, where G acts by the right regular representation on $C^\infty(G, V)$, and that this is an isomorphism onto the closed subspace $C^\infty(G, V)^G$ of functions that are invariant under the simultaneous action: $g \cdot f(x) := \pi(g)f(g^{-1}x)$. \square

1.7. Unitary representations. Unitary representations have been introduced in §7. Their Plancherel decomposition was discussed in §8. Here, we will just add the uniqueness of the Plancherel decomposition, for reductive real or p -adic groups. [LATER]

1.8. (\mathfrak{g}, K) -modules. Topological representations of Lie groups do not form an abelian category. This is sometimes cumbersome; to make the theory more algebraic, we sometimes work with (\mathfrak{g}, K) -modules.

Definition 1.9. Let \mathfrak{g} be a complex Lie algebra, and H a Lie group, with an embedding $\mathfrak{h}_\mathbb{C} \hookrightarrow \mathfrak{g}$, and a representation $\text{Ad} : H \rightarrow \text{GL}(\mathfrak{g})$, extending the adjoint action on $\mathfrak{h}_\mathbb{C}$, whose differential coincides with the adjoint action of $\mathfrak{h} \subset \mathfrak{g}$. (For example, \mathfrak{g} is the complexified Lie algebra of a Lie group containing H .) A (\mathfrak{g}, K) -module is a vector space V with actions of both \mathfrak{g} and H , such that:

- (1) the action of H is locally finite;
- (2) the differential of the action of H coincides with the action of \mathfrak{h} , considered as a subalgebra of \mathfrak{g} ;
- (3) $h \cdot X \cdot h^{-1} \cdot v = \text{Ad}(h)(X) \cdot v$, for all $h \in H$, $X \in \mathfrak{g}$, $v \in V$.

This notion is most often (but not exclusively!) used when $H = K$ is a maximal compact subgroup of a Lie group G (with complexified Lie algebra \mathfrak{g}).

Lemma 1.10. *Let (π, V) be a representation of a Lie group G , and $H \subset G$ a subgroup. The subspace $V_{H\text{-fin}}$ of H -finite vectors is stable under the action of $\mathfrak{g}_{\mathbb{C}}$.*

Proof. For every $v \in V_{H\text{-fin}}$, the image of the action map $\mathfrak{g} \otimes \text{span}(Hv) \rightarrow V$ is finite-dimensional, and contains the element $h \cdot X \cdot v$ for all $X \in \mathfrak{g}$ and $h \in H$, since $h \cdot X \cdot v = \text{Ad}(h)(X) \cdot h \cdot v$. \square

Recall also from Theorem 4.4 that if $H = K$ is compact, and the representation is Fréchet, the space of K -finite vectors is dense.

Definition 1.11. Let G be a reductive Lie group, and $K \subset G$ a maximal subgroup; use \mathfrak{g} to denote the complexified Lie algebra of G . The (\mathfrak{g}, K) -module of a Fréchet representation (π, V) of G is the (\mathfrak{g}, K) -module $V_{K\text{-fin}}^{\infty}$ of K -finite smooth vectors in V .

Two representations V_1, V_2 are said to be *infinitesimally equivalent* if their (\mathfrak{g}, K) -modules are isomorphic.

Remark 1.12. Infinitesimal equivalence captures more of the essence of representation theory than isomorphisms of representations. For example, all Banach representations $L^p(\mathbb{R}^{\times})$ ($p \geq 1$) of the group \mathbb{R}^{\times} are infinitesimally equivalent, although they are not isomorphic as topological vector spaces. On the other hand, the “globalization” theorem of Casselman and Wallach [Cas89, Wal92, BK14] says that any finitely generated, *admissible* (see Definition 1.15) (\mathfrak{g}, K) -module admits a unique “globalization” to a smooth Fréchet representation of moderate growth. The proof of this theorem relies on the subrepresentation theorem (see Theorem 4.5), realizing irreducible (\mathfrak{g}, K) -modules as submodules of parabolically induced representations.

Lemma 1.13. *If V is a Fréchet representation of a reductive Lie group G , and $K \subset G$ a maximal subgroup, its (\mathfrak{g}, K) -module $V_{K\text{-fin}}^{\infty}$ is dense in V . In particular, if the (\mathfrak{g}, K) -module $V_{K\text{-fin}}^{\infty}$ is irreducible, so is V .*

Proof. This follows from Lemma 1.3 and Proposition 4.5. \square

The converse is true in the category of *admissible* representations (Theorem 1.20).

1.14. Admissibility.

Definition 1.15. A (\mathfrak{g}, K) -module (π, V) (in the real case), or a smooth K -module V (in the p -adic case) is called *admissible* if all irreducible representations of K appear with finite multiplicity, i.e., $\dim \text{Hom}_K(\tau, V) < \infty$ for every irreducible representation τ of K .

A (topological) representation (π, V) of a real or p -adic reductive group G is admissible if the (\mathfrak{g}, K) -module (resp. K -module, in the p -adic case) $V_{K\text{-fin}}^{\infty}$ is admissible. Here, K is any maximal compact subgroup of G , in the real case, and any compact open subgroup of G , in the p -adic case.

Remark 1.16. The property of being admissible, for a representation of G , does not depend on the choice of K ; indeed, in the real reductive case, all Cartan subgroups are conjugate, by Theorem 6.6. In the p -adic case, the independence follows from the lemma below.

Lemma 1.17. *In the p -adic case, a representation (π, V) is admissible if and only if, for every compact open $J \subset G$, we have $\dim V^J < \infty$.*

Proof. First of all, observe that $V_{K\text{-fin}}^\infty = V^\infty$ for every compact open $K \subset G$.

If a (smooth) irreducible representation τ of K appears with infinite multiplicity, then, obviously, $\dim V^J = \infty$ for all J with $\tau^J \neq 0$.

Vice versa, given K , for every open compact $J \subset K$, the set of (isomorphism classes of) irreducible representations τ of K with $\tau^J \neq 0$ is finite. Indeed, to prove this claim, we can replace J with the intersection of all its K -conjugates, which is still open and compact, but also normal. Then, if $\tau^J \neq 0$ and τ is irreducible, we have $\tau = \tau^J$, hence τ is an irreducible representation of the finite group K/J , and there are only finitely many such. Thus, admissibility according to Definition 1.15 implies that V^J is finite-dimensional, for every J . \square

Definition 1.18. The *contragredient* of a (\mathfrak{g}, K) -module V , in the real case, or a smooth G -representation V , in the p -adic case, is the (\mathfrak{g}, K) -module, resp. smooth G -representation $\tilde{V} := (V^*)_{K\text{-fin}}$ of K -finite vectors in the linear dual of V .

Lemma 1.19. *Assume that V is an admissible (\mathfrak{g}, K) -module V , in the real case, or an admissible smooth G -representation, in the p -adic case. Then, $\tilde{\tilde{V}} = V$.*

If V is irreducible, any automorphism of V (as a (\mathfrak{g}, K) -module, resp. as a G -representation) is scalar.

Proof. The module is a direct sum over its K -types, and those are finite-dimensional. The contragredient, as a representation of K , will be the direct sum of the duals, and any automorphism preserves the isotypic spaces. The result, now, follows easily from the finite-dimensional case. \square

The converse to Lemma 1.13 holds, for admissible representations:

Theorem 1.20. *If V is an irreducible admissible Fréchet representation of moderate growth of a reductive Lie group G , its (\mathfrak{g}, K) -module is irreducible, and all K -finite vectors are automatically analytic (in particular, smooth).*

Proof. First of all, since $V_{K\text{-fin}}^\infty$ is dense (Lemma 1.13) in V , it is also dense in $V_{K\text{-fin}}$. For any K -type τ , there is a measure μ_τ on K whose action on any Fréchet module is a projection onto the τ -isotypic component. Therefore, the τ -isotypic subspace $V^{\infty, \tau}$ is dense in V^τ . But the former is finite-dimensional, therefore the two coincide, i.e., every K -finite vector is smooth.

Suppose that $V_0 \subset V_{K\text{-fin}}$ is a nonzero (\mathfrak{g}, K) -submodule. We claim that the closure of V_0 is G -stable. This requires the “big hammer” of elliptic regularity to prove, so we only give a couple of steps, followed by references.

First, we notice that the action of the center $\mathfrak{z}(\mathfrak{g})$ of the universal enveloping algebra of the (complexified) Lie algebra \mathfrak{g} on $V_{K\text{-fin}}$ is locally finite: indeed, it preserves the finite-dimensional, K -isotypic subspaces.

Elliptic regularity, now, implies that all vectors in $V_{K\text{-fin}}$ are analytic; see [Wal88, 3.4.9].² And, the closure of a \mathfrak{g} -stable subspace of analytic vectors in V is stable under the identity component of G : simply apply the exponential map $\mathfrak{g}_\mathbb{R} \rightarrow G$, whose image generates the identity component.

Since V_0 is not only \mathfrak{g} -stable, but also K -stable, and K meets all connected components of G , V_0 is G -stable. Since V is irreducible, V_0 is dense. But, again,

²In Wallach’s book, the argument is formulated for representations on Hilbert spaces, but it holds verbatim for Banach spaces, and hence for Fréchet representations of moderate growth. Note that a function $G \rightarrow V$, where V is a Banach space, is (real) analytic iff it is *weakly* analytic, i.e., iff its composition with any continuous functional $v^* : V \rightarrow \mathbb{C}$ is analytic.

applying projectors to the K -types, this means that for any K -type τ , the τ -isotypic subspace V_0^τ is dense in V^τ . Since these spaces are finite-dimensional, $V_0^\tau = V^\tau$ for all τ , hence $V_0 = V_{K\text{-fin}}$. \square

2. Schwartz and Harish–Chandra Schwartz spaces

2.1. Schwartz space defined by a scale function. We follow [BK14, §2].

Definition 2.2. A *scale* on a locally compact group G is a function $s : G \rightarrow \mathbb{R}^+$ such that:

- s and s^{-1} are locally bounded,
- s is submultiplicative, i.e., $s(gh) \leq s(g)s(h)$ for all $g, h \in G$.

A scale function s' *dominates* a scale function s , if there exist positive constants C, N such that $s \leq Cs'^N$. They are *equivalent* if each dominates the other.

A *scale structure* on G is an equivalence class of scale functions.

In other words, a scale function is the exponential of a radial function, Definition 5.1.

In 5.2 we saw the “natural radial function” r_{nat} (and hence its exponential, the “natural scale function” s_{nat} , denoted $\|g\|$ there) of a compactly generated group; recall that $r_{\text{nat}}(g)$ counts how many times we need to multiply a compact generating neighborhood of the identity by itself in order to produce a set containing g .

Definition 2.3. Let G be a group equipped with a scale structure $[s]$ (Definition 2.2), with associated radial function $r = \log s$. The associated *Schwartz space* $\mathcal{S}_{[s]}(G)$ is the space of smooth vectors in the left and right regular F-representation on measures f on G which satisfy $f \cdot s^n \in L^1(G)$ for all $n \in \mathbb{N}$.

The *natural Schwarz space* $\mathcal{S}_{\text{nat}}(G)$ is the one defined by the class of natural scale functions.

Equivalently, the natural Schwartz space is the space of smooth vectors in the space of rapidly decaying measures of Definition 5.6.

Example 2.4. For the additive group $G = \mathbb{G}_a(F)$, we have $\mathcal{S}_{\text{nat}}(G) =$ the Schwartz space of smooth functions (times a Haar measure) which, together with their derivatives (in the real case), are of superexponential decay. On the other hand, for $G = \mathbb{G}_m(F)$, they coincide with smooth functions f (times a Haar measure) such that $f(x) \cdot |x|^n$ is bounded for all $n \in \mathbb{Z}$, and similarly for all derivatives (in the real case).

Remark 2.5. There is some clumsiness in trying to deal with the real and p -adic cases at the same time, which is due to the fact that the notion of “smooth” in the p -adic case is not quite analogous to that of “smooth” in the real case; for example, smooth vectors in an F-representation of a p -adic group do not produce Fréchet spaces. There is a notion of “almost smooth” vectors in the p -adic case, which is a better analogy to “smooth” in the real case, see [Sak13], but it is not very useful in practice. Because of the strong definition of smoothness (=local constancy), and because we tend to forget about the topology on spaces of smooth vectors of representations of p -adic groups, the “rapid decay” Schwartz spaces that we are defining here are not suitable for p -adic groups; in the next subsection, we will discuss algebraically defined Schwartz spaces using compactifications, where the definitions in the real and p -adic cases coincide, and produce compactly supported

functions/measures in the p -adic case. [But, note for the future: Maybe we can expand the notion of SF-representation to the p -adic case, to include the LF-spaces of smooth vectors in an F-representation; or, include a full discussion of “almost smooth” vectors, for the sake of uniformity.]

Proposition 2.6. *Let G be a real Lie group. The categories of smooth Fréchet representations of moderate growth of G , and of nondegenerate continuous algebra representations of $\mathcal{S}_{\text{nat}}(G)$ on Fréchet spaces, are equivalent.*

Proof. If (π, V) is any F-representation, the action of G extends to a continuous representation of the algebra of rapidly decaying measures by Proposition 5.7, in particular, to the natural Schwartz space.

A theorem of Dixmier and Malliavin [DM78] states that, if (π, V) is a smooth Fréchet representation of a real Lie group G , then the action map $M_c^\infty(G) \otimes V \rightarrow V$ is surjective. Hence, so is the map $\mathcal{S}_{\text{nat}}(G) \otimes V \rightarrow V$, i.e., V is nondegenerate.

Vice versa, if V is a nondegenerate continuous Fréchet $\mathcal{S}_{\text{nat}}(G)$ -module, that is, it is nondegenerate and the action map $\mathcal{S}_{\text{nat}}(G) \times V \rightarrow V$ is continuous, this action extends to the *projective tensor product* $\mathcal{S}_{\text{nat}}(G) \hat{\otimes}_\pi V \rightarrow V$, which is also a Fréchet space, and this gives a topological identification of V as a quotient of the projective tensor product. Quotients of SF-representations are SF-representations, see [BK14, Lemma 2.9 and Proposition 2.20] for more details. \square

2.7. Schwartz space of a semi-algebraic manifold. If G denotes the points of a linear algebraic group over a local field, we also define another scale function, that depends on the algebraic structure. (The same definition can be given for finite covers thereof, by passing to the algebraic quotient.)

Definition 2.8. Let G be a linear algebraic group, and fix a closed embedding $G \hookrightarrow \mathbb{A}^r$, with coordinates x_1, \dots, x_r . The corresponding *algebraic scale function* of $G(F)$, where F is a local field, is

$$s_{\text{alg}}(g) = \max_i |x_i(g)|.$$

It is easy to prove that any two algebraic scale functions are equivalent.

Lemma 2.9. *If G is a reductive group, the natural and algebraic scale functions on G are equivalent.*

Proof. The statement is easily seen to be true for a torus. For a general reductive group, it reduces to the case of tori by the Cartan decomposition $G = KA^+K$. \square

This leads to a notion of “algebraic Schwartz space” according to Definition 2.3, but in the p -adic case we would like a stricter definition that coincides with the space of compactly supported smooth measures. In this subsection, we will provide a uniform such for arbitrary real or p -adic (smooth) varieties (or semialgebraic spaces).

[Definition of Schwartz space $\mathcal{S}(X)$ on a Nash manifold X here. In the p -adic case, it coincides with $M_c^\infty(X)$. In particular, in the p -adic group case, $\mathcal{S}(G) = \mathcal{H}(G)$ = the Hecke algebra.]

Proposition 2.10. *For both real and p -adic reductive groups, there is an equivalence of categories between SF-representations (in the real case), or smooth representations without topology (in the p -adic case), and nondegenerate $\mathcal{S}(G)$ -modules.*

Proof. In the real case, this is just Proposition 2.6, together with the equivalence of natural and algebraic scale structures, Lemma 2.9.

In the p -adic case, the proof is similar (but simpler). Notice that the analogous statement holds, more generally, for any locally compact, totally disconnected group. \square

2.11. Harish-Chandra Schwartz space. We follow [Ber88]. We start by defining a notion of radial function for a homogeneous space of a locally compact group; everything in this section applies to a finite union of homogeneous spaces, as well.

Definition 2.12. Let X be a homogeneous space for a locally compact group G . A *radial function* is a locally bounded function $r_X : X \rightarrow \mathbb{R}_+$, such that:

- (1) for every $R \in \mathbb{R}_+$, the “ball” $B(R) = \{x \in X | r_X(x) \leq R\}$ is relatively compact in X ;
- (2) for any compact $\Omega \subset G$, there is a constant $C > 0$ such that $|r_X(gx) - r_X(x)| < C$.

We say that r_X, r'_X are two *equivalent radial functions* if there is a constant C such that $C^{-1}(1 + r_X) \leq (1 + r'_X) \leq C(1 + r_X)$.

We say that the space X is of *polynomial growth* (with respect to a given equivalence class of radial functions) if there is a $d \geq 0$ such that, for one (any) compact neighborhood Ω of the identity in G , and some positive constant C , the ball $B(R)$ can be covered by $\leq C(1 + R^d)$ orbits of Ω .

The equivalence class of *natural radial functions* on X is the equivalence class of the function $r_X(x) = \inf\{r(g)|x_0 \cdot g = x\}$, where r is a natural radial function on G , and x_0 is some fixed point on X .

Now we assume that X is a homogeneous real or p -adic manifold, of polynomial growth, under the action of a real or p -adic group G , or a finite union of such. [Fact, to be added: the space $X(F)$ of points of a spherical G -variety X over a local field F , under the action of the group $G(F)$, are such.]

Definition 2.13. Let X be a homogeneous real or p -adic manifold, under the action of a real or p -adic group G , of polynomial growth with respect to the natural scale. The *Harish-Chandra–Schwartz space* of X is the space $\mathcal{C}(X)$ of smooth vectors in the Fréchet space of half-densities f on X with $f \in \lim_{d \rightarrow 0} L^2(X, (1 + r)^d)$, where r is a natural scale function on X .

The notation $L^2(X, (1 + r)^d)$ stands for the Hilbert space of half-densities f with norm equal to the square root of $\int_X |f|^2 (1 + r)^d$.

Remark 2.14. The space $\mathcal{C}(X)$ is a nuclear Fréchet space, in the real case, and a countable direct limit over the nuclear Fréchet spaces of J -invariants, as J ranges over a basis of open compact subgroups, in the p -adic case.

Remark 2.15. If X has an invariant measure dx , or, more generally, a positive G -eigenmeasure with (positive) G -eigencharacter η , one can think of half-densities as functions, by dividing by $(dx)^{\frac{1}{2}}$, but the action of G on those functions is twisted by the square root of η , that is:

$$(2.15.1) \quad (g \cdot \Phi)(x) = \eta^{\frac{1}{2}}(g) \Phi(x \cdot g).$$

In other words, if $\mathcal{F}(X)$ denotes functions and $\mathcal{D}(X)$ denotes half-densities, division by $(dx)^{\frac{1}{2}}$ defines an equivariant isomorphism $\mathcal{D}(X) \xrightarrow{\sim} \mathcal{F}(X) \otimes \eta^{\frac{1}{2}}$.

For example, consider the pre-flag variety $X = U \backslash G_0$, where $U \subset P \subset G_0$ is the unipotent radical of a parabolic subgroup. Considering it as a homogeneous space for the product $G = L \times G_0$, where L is the Levi quotient of P , it possesses G -eigenmeasure, which is invariant under G_0 , but δ_P -equivariant under L , where δ_P is the modular character of P . Thus, half-densities on X can be identified (after a choice of such a measure, unique up to scalar), with functions on X , with the action of L on the latter twisted by δ_P .

Definition 2.16. The space of *tempered half-densities* on X is the dual of the topological vector space $\mathcal{C}(X)$. If a G -eigenmeasure dx on X is chosen (always to be taken G -invariant, if possible), the dual of the space $(dx)^{-\frac{1}{2}}\mathcal{C}(X)$ of Harish-Chandra–Schwartz functions is the space of *tempered measures*, and the dual of the space $(dx)^{\frac{1}{2}}\mathcal{C}(X)$ of Harish-Chandra–Schwartz measures is the space of *tempered generalized functions*.

The space of *tempered smooth half-densities* (and, correspondingly, functions or measures in the presence of an eigenmeasure) is the space of smooth vectors in the contragredient of the F -representation $\bigcap_d L^2(X, (1+r)^d)$ (Definition 6.5), that is, in the direct limit of Hilbert spaces $\lim_{d \rightarrow \infty} L^2(X, (1+r)^d)$.

Here is the main result of [Ber88]:

Theorem 2.17. *The inclusion $\mathcal{C}(X) \hookrightarrow L^2(X)$ is fine; that is, for any morphism from $L^2(X)$ to a direct integral $H = \int H_z \mu(z)$ of Hilbert spaces, the composition $\mathcal{C}(X) \rightarrow H$ is pointwise defined (Definition 8.12).*

Proof. This is [Ber88, Theorem 3.2], applied to the setting of [Ber88, §3.5, 3.7]. \square

Definition 2.18. An admissible smooth representation π of a real or p -adic reductive group is called *tempered* if its matrix coefficients are tempered, i.e., have image in the space $C_{\text{temp}}^\infty(G)$ of smooth, tempered functions (Definition 2.16).

More generally, if X is a homogeneous G -space of polynomial growth, a morphism $\ell : \pi \rightarrow C^\infty(X)$ is called *tempered* if the image lies in $C_{\text{temp}}^\infty(X)$.

3. Asymptotics

3.1. General setup. When $X = H \backslash G$ is a homogeneous G -space, and π a smooth representation of G , a morphism $m : \pi \rightarrow C^\infty(X)$ is sometimes called a *generalized matrix coefficient*; the reason is that any such morphism is equivalent (by Frobenius reciprocity) to an H -invariant functional ℓ , so $m(v)(x) = \langle \pi(g)v, \ell \rangle$ is a “matrix coefficient”, where the covector ℓ is allowed to be non-smooth. In this section, we compare generalized matrix coefficients of certain representations of G on a spherical variety X , with generalized matrix coefficients on the boundary degenerations.

There are similarities, but also differences, between the real and p -adic cases. The main difference, in the real case, is that we need to restrict to admissible modules. (A general theory of asymptotics for smooth representations would be very desirable, but has not yet been developed! The naive translation of statements from the p -adic to the real case does not hold, in general.)

For the remainder of this section, G is a real or p -adic reductive group, and K is a maximal compact subgroup, if G is real. We compare generalized matrix coefficients on X and X_Θ by choosing some reasonable (but noncanonical) identification of the spaces “close to infinity”:

Definition 3.2. Let Z be the closure of a G -orbit in a toroidal embedding \bar{X} of X . An *approximate exponential map* is an analytic map $\phi : U_Z \rightarrow \bar{X}(F)$, where U_Z is a neighborhood of Z in the F -points of the normal bundle $N_Z \bar{X}$, with the property that the partial differential of ϕ induces the identity between on the normal bundle, and ϕ maps the intersection of every G -orbit with U_Z to the corresponding G -orbit on \bar{X} . The *exponential bundle* $\text{Exp}_Z \bar{X}$ over Z is the fiber bundle of germs, over Z , of approximate exponential maps.

Note that $\text{Exp}_Z \bar{X}$ is a torsor for the group bundle $\text{Exp}_Z N_Z \bar{X}$ of germs of approximate exponential maps from the normal bundle to itself (defined the same way).

Proposition 3.3. *Assume that F is non-Archimedean. Using the notation of Definition 3.2, let $\phi : U_Z \rightarrow \bar{X}(F)$ be an approximate exponential map for some orbit closure $Z \subset \bar{X}$. Then, given an open compact subgroup $J \subset G$, there is a J -invariant neighborhood $U'_Z \subset U_Z$ of Z , with J -invariant image $U'_X \subset \bar{X}(F)$, such that ϕ descends to a bijection: $U'_Z/J \rightarrow U'_X/J$. Moreover, any two approximate exponential maps descend to the same bijection, if the neighborhood U'_Z is taken sufficiently small.*

Proof. See [SV17, Proposition 4.3.1]. The reader is encouraged to check it directly in the baby case of $\bar{X} = \mathbb{A}^1$, $Z = \{0\}$, $G = \mathbb{G}_m$. \square

Now, the normal bundle to Z contains some open G -orbit, which we have called the boundary degeneration; let's denote it by X_Θ . This proposition implies that, for any J -invariant functions f, f_Θ on X and X_Θ , respectively, there is a well-defined notion of the functions being asymptotically equal:

Definition 3.4. Assume that F is non-Archimedean. Let X be a spherical variety, and X_Θ an asymptotic cone thereof, obtained as the open G -orbit in the normal bundle of some orbit Z in a toroidal embedding. If $f \in C^\infty(X)$, $f_\Theta \in C^\infty(X_\Theta)$, we say that f is *asymptotically equal* to f_Θ , written $f \sim f_\Theta$, if there is an approximate exponential map $\phi : U_Z \rightarrow \bar{X}(F)$ (Definition 3.2), where U_Z is a neighborhood of Z , such that, after possibly replacing U_Z by a smaller neighborhood, $\phi^* f|_{U_Z} = f_\Theta|_{U_Z}$.

Notice that, by Proposition 3.3, this notion does not depend on the choice of approximate exponential.

In the real case, things are finer, since smooth functions are not locally constant. Therefore, any such attempt to identify f and f_Θ will depend on the choice of approximate exponential. Instead of looking at arbitrary smooth functions, here, we will restrict our attention to “functions that look like generalized characters” (of the tori A_Θ) at infinity—we will call such functions “asymptotically finite”. The following baby example captures the essence of such functions:

Example 3.5. Let $\bar{X} = \mathbb{A}^1 \supset X = \mathbb{A}^1 \setminus \{0\}$, over $F = \mathbb{R}$. Let $Z = \{0\}$; then, $N_Z \bar{X} = \mathbb{A}^1$. Here, we want to think of \bar{X} simply as a variety (without a group action), while $N_Z \bar{X}$ has a \mathbb{G}_m -action. Any analytic map $\phi : U_Z \rightarrow \mathbb{R}$, where U_Z is a neighborhood of zero, fixing zero and inducing the identity on its tangent space, is an asymptotic exponential. Explicitly, such a ϕ is given by a power series of the form $\phi(x) = x + \sum_{n=2}^{\infty} a_n x^n$, convergent within some radius.

An “asymptotically finite” function f on X is a function with the property that $\phi^* f = \sum_{\lambda} f_{\lambda} \cdot h_{\lambda}$, a finite sum indexed by characters of the multiplicative group,

where f_λ is a generalized G_m -eigenfunction with generalized eigencharacter λ , and $h_\lambda \in C^\infty(U_Z)$. The reader should check [exercise!] that this notion does not depend on the choice of approximate exponential ϕ .

Definition 3.6. Let F be real or non-Archimedean, and let X be a spherical variety over F . An *asymptotically finite* function on X is a smooth function f with the property that, for some toroidal compactification \bar{X} , in a neighborhood of any point $z \in \bar{X}$ (belonging to a G -orbit Z whose normal bundle is the boundary degeneration X_Z), and for any approximate exponential ϕ defined in a neighborhood U of z , the function $\phi^* f$, restricted to a neighborhood $U' \subset U$ of z , is equal to

$$(3.6.1) \quad \sum_{\lambda} f_{\lambda} \cdot h_{\lambda},$$

a finite sum indexed by characters of A_Z , where f_{λ} is a generalized A_Z -eigenfunction with generalized eigencharacter λ , and $h_{\lambda} \in C^\infty(U')$.

We let $\text{Fin}_Z(\bar{X})$ denote the bundle of germs, over a G -orbit Z , of asymptotically finite functions defined in a neighborhood of Z in \bar{X} , and call the image (germ) of such a function f in $\text{Fin}_Z(\bar{X})$ the *asymptotic expansion* of f at Z . Equivalently, if $\text{Fin}_Z(N_Z \bar{X})$ denotes the space of germs, at Z , of functions of the form (3.6.1) defined in a neighborhood of Z in $N_Z \bar{X}$, the asymptotic expansion of f is the induced map

$$\text{Exp}_Z(\bar{X}) \rightarrow \text{Fin}_Z(N_Z X)$$

from germs of approximate exponential functions (see Definition 3.2), which is equivariant for the group bundle $\text{Exp}_Z(N_Z \bar{X})$.

The characters λ in an expansion (3.6.1) will always be assumed to be such that no quotient of two of them extends to a smooth function on U' . Under that assumption, the *dominant term* of an asymptotically finite function of the form (3.6.1) is the sum $f_Z = \sum_{\lambda} f_{\lambda} \in C^\infty(U')$; when U' contains the entire orbit of z , f_{λ} extends uniquely as a generalized A_Z -eigenfunction to X_Z , and we will consider the dominant term as a function on X_Z . (This depends on the orbit of z , not just the isomorphism class of X_Z !) We write $f \sim f_Z$ to indicate that f_Z is the dominant term of f .

Remark 3.7. Notice that, in the non-Archimedean case, the functions h_{λ} in the asymptotic expansion (3.6.1) are not needed, since they are constant in a neighborhood of z ; hence, an asymptotically finite function is exactly equal to an A_Z -eigenfunction in a neighborhood of z .

Lemma 3.8. *The dominant term f_Z of an asymptotically finite function along a G -orbit is independent of the choice of an approximate exponential function used to define it.*

Proof. [Easy; will be added.] □

In the real case, asymptotically finite functions with respect to a given compactification \bar{X} , set E of “exponents” λ , and bounded degree for the generalized eigenfunctions f_{λ} have a natural structure of a Fréchet space. [Details are left to the reader, for now.]

Remark 3.9. The following is expected to be true for every spherical variety:

Expected theorem:

Let X denote the points of a homogeneous spherical G -variety, and let X_Θ be a boundary degeneration.

If π is any smooth representation of G , in the p -adic case, and an admissible SF representation of G , in the real case, then for any morphism $\ell : \pi \rightarrow C^\infty(X)$, there is a unique morphism $\ell_\Theta : \pi \rightarrow C^\infty(X_\Theta)$, such that $\ell(v) \sim \ell_\Theta(v)$ for all $v \in \pi$. (In particular, in the admissible case, $\ell(v)$ is asymptotically finite.)

In fact, one can make a stronger statements, where the neighborhood of infinity, or the rate of convergence of asymptotic expansions, is determined by a compact open subset by which v is invariant, resp. a continuous seminorm of v . This theorem has not appeared in the literature in complete generality. In the next subsections we will formulate (some of) the cases that are known.

3.10. Asymptotics in the non-Archimedean case.

Theorem 3.11. *Let X denote the points of a homogeneous spherical G -variety over a non-Archimedean field, and let X_Θ be a boundary degeneration. Under the following assumptions:*

- G is split and X is of wavefront type (see [SV17, §2.1]), OR
- X is symmetric,

the following is true: There is a unique morphism

$$e_\Theta : \mathcal{S}(X_\Theta) \rightarrow \mathcal{S}(X)$$

with the property that, whenever X_Θ is realized in the normal bundle of an orbit Z in a smooth toroidal compactification of X , and ϕ is an approximate exponential map (Definition 3.2), for every open compact subgroup J there is a J -stable neighborhood U'_Z of Z as in Proposition 3.3— in particular, ϕ induces a bijection $U'_Z/J = U'_X/J$, where U'_X is the image of U'_Z in X — such that, for $f \in \mathcal{S}(U'_Z)^J$, $e_\Theta(f) = \phi_*(f)$, its pushforward to U'_X/J through this identification.

In particular, the adjoint morphism $e_\Theta^* : C^\infty(X) \rightarrow C^\infty(X_\Theta)$ has the property that $e_\Theta^* f|_{U'_Z} = \phi^* f|_{U'_Z}$, for every $f \in C^\infty(X)^J$.

The theorem is expected to hold without these assumptions on X .

In particular, if $\ell : \pi \rightarrow C^\infty(X)$ is any morphism of smooth representations, we obtain the asymptotic morphism ℓ_Θ of the “Expected Theorem” of Remark 3.9 as $\ell_\Theta = e_\Theta^* \circ \ell$.

Proof. See [SV17, Theorem 5.1.1] and [Del18, Theorem 1]. □

3.12. Asymptotics in the real case.

Theorem 3.13. (1) *Let $X = H$, a (connected) reductive group over \mathbb{R} , under the $G = H \times H$ -action. Let τ be an admissible smooth Fréchet representation of moderate growth of H , and $\tilde{\tau}$ its contragredient. Then, for every class P of parabolics in H , there exists a finite set E of A_P -exponents and a degree d , depending on τ , such that all matrix coefficients*

$$f_{v,\tilde{v}}(g) := \langle \tau(g)v, \tilde{v} \rangle$$

are asymptotically finite with exponents $\lambda \in E$ and degree bounded by d in a neighborhood of P -infinity, and the map from $\tau \hat{\otimes} \tilde{\tau}$ to the corresponding Fréchet space $\text{Fin}_P^{E,d}$ of asymptotic expansions is continuous.

In particular, considering only leading terms, there is a morphism $\ell_P : \tau \hat{\otimes} \tilde{\tau} \rightarrow C^\infty(X_P)$ such that $f_{v,\tilde{v}} \sim \ell_P(v \otimes \tilde{v})$ in a neighborhood of P -infinity.

Moreover, if $\ell_P = 0$ (i.e., the matrix coefficients of τ are of rapid decay), for any P , then $\tau = 0$.

- (2) Let X be any real spherical variety for a reductive group G , and π an admissible representation with a tempered morphism $\ell : \pi \rightarrow C_{\text{temp}}^\infty(X)$, and let X_Θ denote a boundary degeneration, identified with the open G -orbit in the normal bundle of some orbit in a toroidal compactification. Then, there exists a tempered morphism $\ell_\Theta : \pi \rightarrow C_{\text{temp}}^\infty(X_\Theta)$, an A_Θ -eigenfunction h on X_Θ with real positive eigencharacter which is < 1 on $\exp(\mathfrak{a}_\Theta^+)$, and a continuous seminorm q , such that, for any approximate exponential map ϕ , $|\phi^*\ell(v) - \ell_\Theta(v)| \leq h \cdot q(v)$ in a neighborhood of Θ -infinity.

Proof. For the group case, see [Wal88, 4.4]. For the tempered case, see [DKS19]. \square

Definition 3.14. Let τ be an arbitrary smooth representation of a p -adic reductive group H , or an admissible smooth representation of moderate growth of a real reductive group H . For every class P of parabolics in H , let H_P be the corresponding boundary degeneration. The *asymptotic matrix coefficient* morphism associated to P is the morphism

$$m_P : \tau \hat{\otimes} \tau \rightarrow C^\infty(H_P),$$

where $m_P = \ell_P$ in the notation of Theorem 3.13, in the real case, and $m_P = e_P^* \circ m$, where m is the matrix coefficient map, and $e_P^* : C^\infty(H) \rightarrow C^\infty(H_P)$ is the asymptotics map of Theorem 3.11, in the p -adic case.

4. Consequences of the asymptotics

4.1. Supercuspidals.

Proposition 4.2. For an admissible smooth representation of a real or p -adic Lie group H , the following are equivalent:

- (1) The matrix coefficients of τ are of rapid decay (in the real case) or compactly supported (in the p -adic case) modulo the center.
- (2) The asymptotic matrix coefficient morphisms m_P (Definition 3.14) are zero for every class P of proper parabolics in H .

In particular, in the real case, if the matrix coefficients are of rapid decay modulo the center, then $\tau = 0$.

Proof. Follows immediately from Theorems 3.11 and Theorem 3.13, together with the fact that, in the real case, if the asymptotic expansion at infinity is zero, then the function is of rapid decay (modulo center). \square

Definition 4.3. Let H be a p -adic reductive group. An irreducible admissible representation τ of H is called *supercuspidal* if its matrix coefficients are compactly supported modulo the center.

4.4. The subrepresentation theorem.

Theorem 4.5. *Any irreducible admissible representation τ of a real reductive group H , is infinitesimally equivalent to a submodule of an irreducible representation induced from a minimal parabolic; that is, there exists an irreducible (finite-dimensional, necessarily) representation σ of the Levi quotient L of the minimal parabolic subgroup P of H , and an embedding of (\mathfrak{g}, K) -modules $\tau_{K\text{-fin}} \hookrightarrow I_P(\sigma)_{K\text{-fin}}$, where $I_P(\sigma) = \text{Ind}_P^H(\sigma \delta_P^{\frac{1}{2}})$ is the (normalized) induced representation.*

Proof. This relies on the statement of Theorem 3.13, that the asymptotics of matrix coefficients in any direction have to be nontrivial. In particular, for the minimal direction we have a non-zero map, which by irreducibility has to be an embedding, $\tau \otimes \tilde{\tau} \rightarrow C^\infty(H_P) = I_{P \times P^-} C^\infty(L)$, whose image consists of A_P -finite functions. By projecting to an A_P -eigenquotient of the image, we may assume that the image is in an eigenspace, with respect to some character χ of A_P . Notice that L/A_P is compact; hence, the space $C^\infty(L/A_P, \chi)$ has a dense subspace of L -finite vectors, which are spanned by matrix coefficients of irreducible representations. Thus, restricting to K -finite vectors, there is a morphism (necessarily an embedding) of (\mathfrak{g}, K) -modules $(\tau \otimes \tilde{\tau})_{K \times K\text{-fin}} \hookrightarrow I_{P \times P^-}(\sigma \otimes \tilde{\sigma})_{K \times K\text{-fin}} = I_P(\sigma)_{K\text{-fin}} \otimes I_{P^-}(\tilde{\sigma})_{K\text{-fin}}$, for some irreducible representation σ of L , and by fixing a vector in $\tilde{\tau}_{K\text{-fin}}$, we get the embedding claimed in the theorem. \square

5. The Langlands classification

Definition 5.1. Let G be a reductive real or p -adic group, let $P \rightarrow L$ be a parabolic subgroup with its Levi quotient, and let $\nu : L \rightarrow \mathbb{C}^\times$ be a character. We will say that ν is *P -dominant* if $\log(\nu) \in \mathfrak{a}_P^{*,+}$, and *strictly P -dominant* if $\log(\nu) \in \mathfrak{a}_P^{*,+}$. Here, $\mathfrak{a}_P^* = \text{Hom}(L, \mathbb{G}_m) \otimes \mathbb{R}$, \log is the map that sends the absolute value of an algebraic character to its image in \mathfrak{a}_P^* , and $\mathfrak{a}_P^{*,+}$, $\mathfrak{a}_P^{*,+}$ are those characters which are non-negative (resp. strictly positive) on coroots $\tilde{\alpha}$ corresponding to roots in the unipotent radical of P , i.e., $|\nu(e^{\tilde{\alpha}}(x))| = |x|^\epsilon$ for some $\epsilon \geq 0$ (resp. $\epsilon > 0$).

Equivalently, \mathfrak{a}_P^* is identified with a subspace of \mathfrak{a}^* (spanned by the F -rational characters of the universal Cartan), and $\mathfrak{a}_P^{*,+}$ (resp. $\mathfrak{a}_P^{*,+}$) is just the corresponding wall (resp., relative interior of the wall) of the dominant Weyl chamber.

Theorem 5.2 (The Langlands quotient theorem). *Let G be a reductive real or p -adic group, let $P \rightarrow L$ be a parabolic subgroup with its Levi quotient, and let τ be an irreducible tempered representation of L . For any character $\nu : L \rightarrow \mathbb{C}^\times$ which is strictly P -dominant, the (normalized) induced representation $I_P^G(\tau\nu)$ has a unique irreducible quotient $\pi_{P,\tau\nu}$, and every irreducible representation of G is of this form, for a unique (up to conjugacy) pair $(P, \tau\nu)$. Moreover, $\pi_{P,\tau\nu}$ is the image of the standard intertwining operator $M_{P^-|P}(\tau\nu) : I_P(\tau\nu) \rightarrow I_{P^-}(\tau\nu)$.*

Proof. [Later] \square

Example 5.3. The trivial representation, for a quasisplit group, is equal to $\pi_{B,\delta^{\frac{1}{2}}}$, where B is a Borel subgroup, and δ is its modular character.

Remark 5.4. The Langlands quotient theorem reduces the classification of irreducible representations to the case of irreducible tempered representations, offering an invaluable link between the “smooth” and the “ L^2 theory/Plancherel formula” of irreducible representations. It is also supposed to be compatible with the parametrization provided by the local Langlands conjecture: If $\phi_\tau : \Gamma \rightarrow {}^L L$ and

$\phi_\nu : \Gamma \rightarrow {}^L L$ are Langlands parameters for τ and ν (where Γ , here, denotes the appropriate version of the Weil, or Weil–Deligne group), then $\phi_\tau \cdot \phi_\nu : \Gamma \rightarrow {}^L L \hookrightarrow {}^L G$ is a Langlands parameter for $\pi_{P,\tau\nu}$. (Notice that ϕ_τ and ϕ_ν commute, because ν is a character.)

For example, the Langlands parameter (or rather, its projection to \check{G}) of the trivial representation of a quasisplit group is given by $\Gamma \rightarrow \mathbb{C}^\times \rightarrow \check{G}$, where $\Gamma \rightarrow \mathbb{C}^\times$ is the “cyclotomic”/absolute value character, and $\mathbb{C}^\times \rightarrow \check{G}$ is given by $e^{2\rho} : \mathbb{G}_m \rightarrow \check{A} \subset \check{G}$ (where \check{A} is the dual of the universal Cartan).

6. The Satake isomorphism

Definition 6.1. A reductive group over a local non-Archimedean field F is said to be *unramified* if it is quasisplit, and splits over an unramified extension.

Proposition 6.2. *For a connected reductive group G over F , the following are equivalent:*

- (1) G is unramified (Definition 6.1);
- (2) G admits a reductive model of the ring of integers \mathfrak{o} (i.e., a smooth model with connected reductive geometric fibers).

Moreover, the integral model over \mathfrak{o} is unique up to $G_{\text{ad}}(F)$ -conjugacy, that is, for any two reductive \mathfrak{o} -groups $\mathcal{G}_1, \mathcal{G}_2$ with general fiber identified with G , there is an isomorphism $\mathcal{G}_1 \simeq \mathcal{G}_2$ that restricts to an inner automorphism (over F) on G .

[This proposition should be moved to the chapter on algebraic groups.]

Proof. For the direction from the second to the first, see [Con14, Corollary 5.2.14]. The opposite direction follows from the classification in terms of root data with Galois actions. [To be added.] For the uniqueness, see [Con14, Theorem 7.2.16]. \square

Definition 6.3. A hyperspecial subgroup of $G(F)$, where G is an unramified connected reductive group over F , is a subgroup of the form $K = \mathcal{G}(\mathfrak{o})$, where \mathcal{G} is a reductive integral model.

Hyperspecial subgroups are unique, up to conjugacy, for adjoint groups, as follows from the uniqueness statement of Proposition 6.2. This does not need to be true when $G(F)$ does not surject onto $G_{\text{ad}}(F)$.

Proposition 6.4. *A hyperspecial subgroup (Definition 6.3) is maximal.*

Proof. [Omitted for now.] \square

From now on, G will denote $G(F)$. Fix a hyperspecial subgroup $K = \mathcal{G}(\mathfrak{o})$, corresponding to an \mathfrak{o} -model \mathcal{G} , and consider the *integral* unramified (“spherical”) Hecke algebra $\mathcal{H}(G, K)$ of \mathbb{Z} -valued, K -biinvariant functions on G . If we consider them as functions on the discrete space G/K , using the counting measure on this space we can identify them as measures, and this defines their convolution and, more generally, their action on the K -invariant vectors of any representation V (with arbitrary coefficients!). Explicitly, if $v \in V^K$, the characteristic function of a double coset KgK acts as

$$1_{KgK} \cdot v = \sum_{\gamma \in [KgK/K]} \gamma \cdot v.$$

The goal of this section is to establish the integral Satake isomorphism. For this purpose, let \mathcal{Y} be “the” full pre-flag variety of \mathcal{G} over \mathfrak{o} , $\mathcal{Y} \simeq \mathcal{N} \backslash \mathcal{G}$, where \mathcal{N} is the unipotent radical of a Borel subgroup. We do not really choose a Borel subgroup, but the choice of integral model matters, as it endows \mathcal{Y} with a distinguished K -orbit, equal to $\mathcal{Y}(\mathfrak{o})$, that will serve as our base point. As for the group, we will use Y etc. to denote F -points. We let $T \supset T_0$ denote the universal Cartan $T = B/N$, and its maximal compact subgroup $T_0 = \mathcal{T}(\mathfrak{o})$; we reserve the letter A for the maximal split torus in T .

We let $\mathcal{S}(Y/K)$ denote the space of \mathbb{Z} -valued, compactly supported, K -invariant functions on \mathcal{S} . It is a module for $\mathcal{H}(G, K)$ (under the right action of G on Y) and for T (under the “left” action of T on Y). To be clear, the action of an element $t \in T$ on functions is defined as translation by t , not t^{-1} , and it is not normalized by any modular character — which is not defined over \mathbb{Z} : $(t \cdot f)(y) = f(ty)$; this way, the center of G acts the same, whether it is considered as a subgroup of G or of T .

Lemma 6.5. *Every element of $\mathcal{S}(Y/K)$ is T_0 -invariant, hence the action of T factors through the quotient T/T_0 ; in particular, we have an action of the Hecke algebra $\mathcal{H}(T, T_0)$. Under this action, $\mathcal{S}(Y/K)$ is a free module of rank one, and the element $e_0 = 1_{\mathcal{Y}(\mathfrak{o})}$ is a generator.*

Proof. The Iwasawa decomposition [needs to be added] $G = NTK$ shows that $G = \bigsqcup_{t \in T/T_0} NtK$, and in particular:

- every $N \backslash G/K$ -coset is left invariant by T_0 ;
- the group T/T_0 acts simply transitively on the cosets.

One of these cosets is equal (modulo N) to $\mathcal{Y}(\mathfrak{o})$. □

Let $\Lambda = T/T_0$. We start with the Satake isomorphism for tori:

Proposition 6.6. *Let T be an unramified torus over F , and let $A \subset T$ be the maximal split subtorus. If $\varpi \in F$ is a uniformizer, the map $\Lambda := X_*(A) \ni \lambda \mapsto \lambda(\varpi) \in A(F)$ descends to an isomorphism $X_*(A) \simeq T/T_0$.*

Moreover, let \tilde{T} be the dual torus to T , understood as a group scheme over \mathbb{Z} , with an action of the unramified Galois group $\Gamma = \langle \sigma \rangle$, where σ denotes the Frobenius element. Then, the dual \check{A} of A is the maximal torus quotient of \tilde{T} where Γ acts trivially, and the natural maps induce isomorphisms of algebras

$$(6.6.1) \quad \mathbb{Z}[\tilde{T}\sigma]^{\tilde{T}} = \mathbb{Z}[\check{A}] = \mathbb{Z}[\Lambda] = \mathcal{H}(T, T_0),$$

where the “coset” $\tilde{T}\sigma$ is the space \tilde{T} equipped with the σ -twisted conjugation of \tilde{T} , $x\sigma \cdot t = (t \cdot {}^\sigma t^{-1})x\sigma$ (and the notation $\mathbb{Z}[\cdot]$ is used both for group rings and coordinate rings — it should be clear which is which).

Proof. Consider the quotient of algebraic group schemes over \mathfrak{o} :

$$1 \rightarrow \mathcal{A} \rightarrow \mathcal{T} \rightarrow \mathcal{T}/\mathcal{A} \rightarrow 1.$$

Since A is split, this induces (by Hilbert 90) surjections at the level of F - and \mathbb{F}_q -points, hence (by smoothness) at the level of \mathfrak{o} -points. On the other hand, T/A is anisotropic, and $T/A(F) = \mathcal{T}/\mathcal{A}(\mathfrak{o})$. Therefore, $T(F) = A(F)\mathcal{T}(\mathfrak{o})$, and since $A(F) \cap \mathcal{T}(\mathfrak{o}) = A(\mathfrak{o})$, and $A(F)/A(\mathfrak{o}) = \Lambda$, this shows the bijection $\Lambda \xrightarrow{\sim} T/T_0$.

For the coordinate rings, if Λ' is the F -cocharacter group of T , then the torus A is spanned by the images of Galois-stable cocharacters, hence $\Lambda = (\Lambda')^\Gamma$. On

the dual side, the embedding $\Lambda \hookrightarrow \Lambda'$ induces a morphism of dual tori $\check{T} \rightarrow \check{A}$, which identifies \check{A} with the quotient of \check{T} by the subtorus of all elements of the form $(t \cdot {}^\sigma t^{-1})$, $t \in \check{T}$, hence $\mathbb{Z}[\check{T}\sigma]^{\check{T}} = \mathbb{Z}[\check{A}] = \mathbb{Z}[\Lambda]$. \square

Now denote by t_λ a representative for $\lambda \in \Lambda = T/T_0$ in T , and let $\delta_\lambda = 1_{Nt_\lambda K} = t_\lambda^{-1} \cdot 1_{\mathcal{Y}(\mathfrak{o})}$; thus, the elements δ_λ form a basis for $\mathcal{S}(Y/K)$. By Lemma 6.5, the action map $\mathcal{H}(T, T_0) \ni h \mapsto h \cdot 1_{\mathcal{Y}(\mathfrak{o})}$ identifies the spaces $\mathcal{S}(Y/K)$ and $\mathcal{H}(T, T_0) \simeq \mathbb{Z}[\Lambda]$; notice, however, that the characteristic function $t_\lambda T_0$ in $\mathcal{H}(T, T_0)$ corresponds to $\delta_{-\lambda}$.

Theorem 6.7 (Satake isomorphism). *Let $S : \mathcal{H}(G, K) \rightarrow \mathcal{H}(T, T_0)$ be the map given by the action map $\mathcal{H}(G, K) \ni h \mapsto h \cdot 1_{\mathcal{Y}(\mathfrak{o})} \in \mathcal{S}(Y/K)$ and the identification of $\mathcal{S}(Y/K)$ with $\mathcal{H}(T, T_0)$ (again through the analogous action map), i.e., $S(h)$ is that element of $\mathcal{H}(T, T_0)$ such that $h \cdot 1_{\mathcal{Y}(\mathfrak{o})} = S(h) \cdot 1_{\mathcal{Y}(\mathfrak{o})}$. Then S gives rise to an isomorphism of algebras:*

$$(6.7.1) \quad \mathcal{H}(G, K) \simeq \mathcal{H}(T, T_0)_{\mathbb{Z}} \cap \mathcal{H}(T, T_0)_{\mathbb{Q}}^{W, \bullet} = \mathbb{Z}[\Lambda] \cap \mathbb{Q}[\Lambda]^{W, \bullet},$$

where the \bullet -action of the (relative) Weyl group W is defined by $(w \bullet f)(t_\lambda) = q^{\langle \lambda, \rho - w\rho \rangle} f(t_{w^{-1}\lambda})$ (where q is the degree of the residue field).

Explicitly, the element $h \in \mathcal{H}(G, K)$, is mapped to the element of $\mathcal{H}(T, T_0)$ whose value at t is equal to

$$(6.7.2) \quad h \cdot 1_{\mathcal{Y}(\mathfrak{o})}(Nt^{-1}) = \int_G h(g) 1_{NK}(Nt^{-1}g) dg = \int_N h(tn) dn,$$

where the Haar measure on N gives volume 1 to $\mathcal{N}(\mathfrak{o})$. The way that the Satake isomorphism is usually defined in the literature is through the equation (6.7.2), multiplied by $\delta_B(t)^{\frac{1}{2}}$ (where δ_B is the modular character of the Borel subgroup), in order to replace the \bullet -action of W by the usual action of W — but this modification is not defined over \mathbb{Z} .

Notice also that, the characteristic function of $t_\lambda T_0$ in $\mathcal{H}(T, T_0)$ corresponds to the element $\delta_{-\lambda}$ of $\mathcal{S}(Y/K)$, image of the action map will be invariant under the following Weyl group action on $\mathcal{S}(Y/K)_{\mathbb{Q}}$:

$$(6.7.3) \quad w \bullet \delta_\lambda = q^{\langle \lambda, \rho - w^{-1}\rho \rangle} \delta_{w\lambda}.$$

Proof. First of all, we notice that the map S is a homomorphism of algebras, because the actions of G and T on Y commute, and $\mathcal{H}(T, T_0)$ is abelian: if $h_i \in \mathcal{H}(G, K)$ and $h'_i \in \mathcal{H}(T, T_0)$ are such that $h_i \cdot 1_{\mathcal{Y}(\mathfrak{o})} = h'_i \cdot 1_{\mathcal{Y}(\mathfrak{o})}$, then $(h_1 h_2) \cdot 1_{\mathcal{Y}(\mathfrak{o})} = h_1 \cdot (h_2 \cdot 1_{\mathcal{Y}(\mathfrak{o})}) = h_1(h'_2 \cdot 1_{\mathcal{Y}(\mathfrak{o})}) = h'_2(h_1 \cdot 1_{\mathcal{Y}(\mathfrak{o})}) = (h'_2 h'_1) \cdot 1_{\mathcal{Y}(\mathfrak{o})} = (h'_1 h'_2) \cdot 1_{\mathcal{Y}(\mathfrak{o})}$.

Next, we claim that the image lies in $\mathbb{Q}[\Lambda]^{W, \bullet}$. We will present two proofs for that, one of them only for the split case [but it can be generalized — to do].

The first proof, following [Car79], uses the explicit expression (6.7.2) of the Satake transform as an integral, and replaces it by an orbital integral for the action of G on itself by conjugacy. Namely, choose a lift of the quotient map $B \rightarrow T$, thus identifying T as a subtorus of B . If $t \in T$ is a *regular* element (i.e., has trivial stabilizer under the action of the Weyl group), then we have the following formula:

$$\int_N h(tn) dn = |\det(\text{Ad}_{\mathfrak{n}}(t^{-1}) - 1)| \int_N f(n^{-1}tn) dn = |\det(\text{Ad}_{\mathfrak{n}}(t^{-1}) - 1)| \int_{T \backslash G} f(g^{-1}tg) dg,$$

where $\text{Ad}_{\mathfrak{n}}$ denotes the left adjoint action of T on the Lie algebra \mathfrak{n} , and the invariant measure on $T \backslash G$ is normalized so that the total measure of K -orbits represented by

$T \backslash TN(\mathfrak{o})$ is 1. This formula follows easily by considering the map $N \rightarrow tN$ given by $n \mapsto n^{-1}tn$, and representing the measures as absolute values of volume forms; at the last step, one uses the K -invariance of H to represent $T \backslash G$ by NK .

Hence, for $w \in W$ and t a regular element in T (any class in T/T_0 has such representatives),

$$\begin{aligned} \frac{\int_N h(n^w t) dn}{\int_N h(nt) dn} &= \frac{|\det(\text{Ad}_n(wt^{-1}) - 1)|}{|\det(\text{Ad}_n(t^{-1}) - 1)|} = \left| \prod_{\alpha > 0} \frac{1 - e^{-w^{-1}\alpha}(t)}{1 - e^{-\alpha}(t)} \right| \\ &= \left| \prod_{\alpha > 0, w\alpha < 0} \frac{1 - e^{\alpha}(t)}{1 - e^{-\alpha}(t)} \right| = \left| \prod_{\alpha > 0, w\alpha < 0} e^{\alpha}(t) \right| = |e^{\rho - w\rho}(t)|, \end{aligned}$$

which amounts to the stated invariance property. (We have not assumed G to be split, for this calculation: the terms inside the absolute values are algebraic functions, and therefore it is valid to manipulate them over the algebraic closure — where all roots are defined.)

[Another proof with Fourier transforms to be added.]

Finally, we prove that the map $\mathcal{H}(G, K) \rightarrow \mathbb{Z}[\Lambda] \cap \mathbb{Q}[\Lambda]^{W, \bullet}$ is an isomorphism. We will argue by identifying the space on the right (call it M) as a subspace of $\mathcal{S}(Y/K)$ — the \bullet -action of W is given by (6.7.3). Notice that M is a free \mathbb{Z} -module with generators m_λ , indexed by dominant cocharacters λ into A , given by $m_\lambda = \sum_{\lambda' \sim \lambda} q_{\lambda'} \delta_{\lambda'}$, where $\lambda' \sim \lambda$ means that $\lambda' = w\lambda$ for some $w \in W$, and in this case we set $q_{\lambda'} = q^{\langle \lambda, \rho - w^{-1}\rho \rangle}$. We define a filtration of this module with respect to the partial ordering by coroots, $\mu \geq \lambda$ if $\mu - \lambda$ is a sum of positive coroots. Similarly, $\mathcal{H}(G, K)$ has a basis consisting of the characteristic functions of the cosets $Kt_{-\lambda}K$ (with λ dominant, again), and we use it to define a filtration of $\mathcal{H}(G, K)$ indexed by dominant weights. We claim that the map $\mathcal{H}(G, K) \rightarrow M$ respects these filtrations:

$$F^\lambda \mathcal{H}(G, K) \rightarrow F^\lambda M,$$

and that the generator of the λ -th graded piece, represented by the function $1_{Kt_\lambda K}$, maps to the generator of the λ -th graded piece, represented by m_λ . These statements follow from the following fundamental fact:

For λ dominant, we have

$$(6.7.4) \quad Kt_\lambda K \subset \bigcup_{\mu \leq \lambda} Nt_\mu \cdot K,$$

$$\text{and } Kt_\lambda K \cap Nt_\lambda K = \mathcal{N}(\mathfrak{o})t_\lambda K.$$

[The proof of this will be added together with the proof of the Cartan and Iwasawa decompositions.]

This implies that $1_{Kt_{-\lambda}K} \cdot \delta_0 = \delta_\lambda + \sum_{\mu < \lambda} c_{\mu, \lambda} \delta_\mu$ for some coefficients $c_{\mu, \lambda} \in \mathbb{N}$. We leave it to the reader to check that this is equivalent to the claim. \square

7. Langlands parameters

Let G be a (connected) reductive group over a local field F . We will write G for $G(F)$. The L -group and the C -group of G have been defined in Section ???. We denote by Γ_F the Galois group of F (of a fixed separable extension³), and by \mathcal{W}_F its

³See Remark 4.3: it is better not to fix a separable extension, and to translate these definitions to sheaves over the étale site of F .

Weil group. For definitions, see [Tat79]. We only remind here that the Weil group comes with isomorphisms $\mathcal{W}_F/\mathcal{W}_E = \Gamma_F/\Gamma_E = \text{Hom}(E, F^s)$ for every separable extension E of F , and $\mathcal{W}_F^{ab} \xrightarrow{\sim} F^\times$, compatible with the isomorphism $\Gamma_F \xrightarrow{\sim} \widehat{F^\times}$ (profinite completion) of class field theory. As in [Tat79], we will normalize the isomorphism of class field theory so that a Frobenius element maps to the inverse of a uniformizer, i.e., a *geometric Frobenius* element maps to a uniformizer. In particular, we have a norm map $|\bullet| : \mathcal{W}_F \rightarrow F^\times \rightarrow \mathbb{R}_+^\times$, sending a Frobenius element to q : the degree of the residue field.

We also remind of the modification of the Weil group that is needed in order to pass from l -adic to complex representations:

Definition 7.1. Let F be a non-Archimedean field. The *Weil–Deligne group* \mathcal{W}'_F is the semidirect product $\mathcal{W}_F \ltimes \mathbb{G}_a$, with $w x w^{-1} = |w|x$ for $w \in \mathcal{W}_F$ and $x \in \mathbb{G}_a$. A *representation of the Weil–Deligne group* over a field E of characteristic zero is a pair (ρ, N) consisting of a representation of \mathcal{W}_F with open kernel on a finite-dimensional vector space V over E , and a nilpotent endomorphism N of V , satisfying $\rho(w)N\rho(w)^{-1} = |w|N$.

The “open kernel” condition is the important one here; it makes irrelevant the topology of $\text{GL}_E(V)$. The Weil–Deligne group is a convenient way to de-topologize the l -adic representations of the Weil group that show up “in nature” (in étale cohomology), and translate them among different l ’s, or to the complex numbers:

Proposition 7.2. *Let l be a prime different from the residual characteristic p of (a non-Archimedean field) F , and let E be a finite extension of \mathbb{Q}_l . There is a canonical bijection between isomorphism classes of (continuous) finite-dimensional E -representations $\phi : \mathcal{W}_F \rightarrow \text{GL}(V)$ and representations (ρ, N) of the Weil–Deligne group over E (Definition 7.1), characterized by the property that*

$$(7.2.1) \quad \phi(\Phi\sigma) = \rho(\Phi\sigma) \exp(t_l(\sigma)N),$$

for some Frobenius element $\Phi \in \mathcal{W}_F$, any element σ of the inertia subgroup, and t_l a choice of isomorphism of the pro- l -quotient of (tame) inertia with \mathbb{Z}_l .

Recall that the tame inertia quotient is generated by n -th roots of a uniformizer, for $(n, p) = 1$, and is isomorphic (up to a choice of topological generator) to $\widehat{\mathbb{Z}}^p = \prod_{l \neq p} \mathbb{Z}_l$.

Proof. See [Tat79, §4.2] for references. \square

Definition 7.3. A *Langlands parameter* into the L -group of G is a morphism $\mathcal{W}'_F \rightarrow {}^L G$ over Γ .

The local Langlands conjecture posits the existence of a canonical finite-to-one map:

$$\{\text{irreducible admissible representations of } G\} / \sim \rightarrow \{\text{Langlands parameters into } {}^L G\} / \sim.$$

8. Other chapters

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| (1) Introduction | (3) Representations of compact groups |
| (2) Basic Representation Theory | |

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| (4) Lie groups and Lie algebras: general properties | (10) Representations of reductive groups over local fields |
| (5) Structure of finite-dimensional Lie algebras | (11) Plancherel formula: reduction to discrete spectra |
| (6) Verma modules | (12) Construction of discrete series |
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