Honors Single Variable Calculus 110.113

October 3, 2023

1 Equivalence Relation

Week 3 Reading: [5, Ch.3.5, Ch.4], *On the construction of* $\mathbb{Q}$*, see* [2, 2.4].

Learning Objectives

Last few lectures:

- Defined the natural numbers and sets axiomatically.
- Discussed how *cardinality* came up from "counting" sets.

This and next lecture:

- discuss equivalence relation.
- construct $\mathbb{Z}, \mathbb{Q}$. Extend addition and multiplication in this context.

1.1 Ordered pairs

We now describe a new mathematical object, we leave it as an exercise to see how this object can be can be constructed form axioms of set theory.

Axiom 1.1. If x, y are objects, there exists a mathematical object

$$(x, y)$$

denote the *ordered pair*. Two ordered pairs $(x, y) = (x', y')$ are equal iff $x = x'$ and $y = y'$.

Example

In sets:

- $\{1, 2\} = \{2, 1\}$

In ordered pairs

- $(1, 2) \neq (2, 1)$

Definition 1.2. Let X, Y be two sets. The *cartesian product* of X and Y is the set

$$X \times Y = \{(x, y) \,:\, x \in X, y \in Y\}$$

Currently, we can either put the existence of such a set as an axiom, or use the axioms of set theory, this is in hw.

Discussion

Let $n \in \mathbb{N}$. How can we generalize the above for an *ordered n-tuple* and *n-cartesian product?*

Pedagogy

As with construction quotient set, and function, we do not show how this can be derived from the axioms of set theory. We refer to the interested reader, [3, 7,8].

What is a relation? What kind of relations are there? We can make a mathematical interpretation using ordered pairs.

Definition 1.3. Given a set A, a *relation* on A is a subset R of $A \times A$. For $a, a' \in A$, We write

$$a \sim_R a'$$

if $(a, a') \in R$. We will drop the subscript for convenience. We say R is:

- *Reflexive* For all $a \in A$
$$a \sim a$$
- *Transitive.* For all $a, b, c \in A$,
$$a \sim b, b \sim c \Rightarrow a \sim c$$
- *Symmetric.* For all $a, b \in A$,
$$a \sim b \Leftrightarrow b \sim a$$

Discussion

What are example of each relations?

Often times, people do not describe the subset R, but describe it a relation *equivalently* as a rule: saying $a, b \in A$ are related if some property $P(a, b)$ is true. In short hand, one writes

$$a \sim b \text{ iff } \dots$$

Definition 1.4. Let R be an equivalence relation on A. Let $x \in A$, The *equivalence* class of x in A is the set of $y \in A$, such that $x \sim y$. We denote this as [1]

$$[x] := \{y \in A \,:\, x \sim y\}$$

An element in such an equivalence is called a *representative* of that class.

Definition 1.5. Quotient set. Given an equivalence relation R on a set A, the *quotient set* $A/\sim$ is the set of equivalence classes on A.

Example

Consider $\mathbb{N}$ and the equivalence relation that $a \sim b$ iff a and b have the same parity. [a]

- There are two equivalence classes: the odds and evens.
- For the odd class, a *representative*, or an element in the equivalence class, is 3.

[a] i.e. both or odd or even.

There is a relation between equivalence and partition of sets.

Definition 1.6. A *partition* of a set X is a collection ???

1.2 Integers

What are the integers? It consists of the natural numbers and the negative numbers. What is *subtraction*? We do not know yet. Can we define *negative* numbers without referencing minus sign? Yes, we can. Say

$$-1\text{is "}0-1\text{" is } (0,1)$$

Discussion

Let us say we define the integers as pairs (a, b) where $a, b \in \mathbb{N}$. Would this be our desired

$$\mathbb{Z} := \{\ldots, -1, 0, 1, \ldots\}$$

- How many -1s are there?

But we have a problem, there are multiple ways to express -1. Our system cannot have multiple -1s. What are other ways We can also have $1-2$, or the pair $(1, 2)$.

[1] It does not matter if we write $\{y \in A \,:\, y \sim x\}$ by symmetry condition.

Discussion

Now that we have our $\mathbb{Z}$, how do we define addition? [a]Can we leverage our understanding?

[a]What is addition abstractly? It is an operation $+ : X \times X \to X$.

Intuitively, the *integers* is an expression [2] of non-negative integers, (a, b), thought of as $a - b$. Two expressions (a, b) and (c, d) are the same if $a + d = b + c$. Formally

Definition 1.7. Let
$$R \subseteq (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N})$$
consists of all pairs (a, b) and (c, d) such that $a + d = b + c$. Equivalently,

$$R := \{(a, b), (c, d) \in (\mathbb{N} \times \mathbb{N}) \times (\mathbb{N} \times \mathbb{N}) \,:\, a + d = b + c\}$$

The *integers* is the set
$$\mathbb{Z} := \mathbb{N}^2 / \sim$$

Definition 1.8. Addition, multiplication. [5, 4.1.2] .

We can now finally define negation.

Definition 1.9. [5, 4.1.4].

Proposition 1.10. Algebraic properties. Let $x, y, z \in \mathbb{Z}$.

- Addition
  - Symmetric $x + y = y + x$.
  - Admits identity element.

1.3 Rational nunbers

Reading: [2, 2.4]. Be careful of the notation used! See 1.11.

Definition 1.11. The *rationals* is the set

$$\mathbb{Q} := \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}) / \sim$$

$$\mathbb{Z} \setminus \{0\} := \{n \in \mathbb{Z} \,:\, n \neq 0\}$$

where $(a, b) \sim (c, d)$ if and only if $ad = bc$. We will denote *the equivalence class* of pair (a, b) by $[a/b]$

[2]Rather than a pair, as an expression has multiple ways of presentation

Again, we need the notion of addition, multiplication, and negation.

Definition 1.12. Let $[a/b], [c/d] \in \mathbb{Q}$. Then

1. Addition:
$$[a/b] + [c/d] := [(ad + bc)/bd]$$
2. Multiplication
$$[a/b] \cdot [c/d] := [(ac)/(bd)]$$
3. Negation.
$$-[a/b] := [(-a)/b]$$

1.3.1 Is addition well-defined?

This subsection gives an extensive discussion of well-definess. The notation we use here is from 1.11. In 1. we *want* to define a function:

$$+ : \mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$$

which takes as input two equivalence class and outputs a new one. Let us consider two equivalence class

$$x := \left\{a'/b' \,:\, a'/b' \sim a/b\right\} \in \mathbb{Q}$$
$$y := \left\{c'/d' \,:\, c'/d' \sim c/d\right\} \in \mathbb{Q}$$

To add these two classes, we proceeded as follows:

1. We pick two representatives from each class, let us say a/b of x and c/d of y.
2. We define
$$x + y := [(ad + bc)/bd]$$

Why can't we say this is the definition of addition, yet? In the above description, $x + y$ can take *more than one possible value* - which is not a function!
For example, one could have chosen other pair of representatives, a'/b', and c'/d', and obtained $x + y$ as

$$[(a'd' + b'c')/b'd']$$

Thus, we have to check that

$$[(a'd' + b'c')/b'd'] = [(ad + bc)/bd]$$

To check this: by definition, this means we have to show:

$$bd(a'd' + b'c') = (ad + bc)b'd'$$

which is

$$bda'd' + bdb'c' = adb'd' + bcb'd' \tag{1}$$

Now $a'/b' \sim a/b$ and $c/d \sim c'/d'$ means $a'b = ab'$ and $cd' = c'd$, Now using commutativity in $\mathbb{Z}$, and the required two equalities for Eq. 1

$$bda'd' = a'bdd' =^{(a'b=ab')} ab'dd' = adb'd'$$

$$bdb'c' = c'dbb' =^{(cd'=c'd)} cd'bb' = bcb'd'$$

1.4 Order relation

Similarly, we can define also define order relation.

Definition 1.13. Let $x \in \mathbb{Q}$,

- x is *positive* iff $x = [a/b]$ where a, b are positive integers, we often denote positive integers as $\mathbb{Z}_{>0}$.
- x is *negative* iff $x = -y$ where y is some positive rational.

With the notion of positive rationals[3] from def. 1.13, we can define order relation $<, \leq$ on $\mathbb{Q}$.

Definition 1.14. Let $x, y \in \mathbb{Q}$, then we denote

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is zero or positive.

Rational is sufficient to do much of algebra. However, we could not do *trigonometry*. One passes from a *discrete* system to a *continuous* system.

Discussion

What is something not in $\mathbb{Q}$?

Proposition 1.15. $\sqrt{2}$ is not rational.

Proof. ??? $\square$

[3]The same trick is used to define order in $\mathbb{N}, \mathbb{Z}, \mathbb{R}$

2 The real numbers

Week 3, Reading: [5, 5]*, notes by Todd, Cauchy's construction. Goldrei's textbook gives another construction of* $\mathbb{R}$ *using Dedekind cuts,* [2, 2.2].

Learning Objectives

We have defined $\mathbb{Q}$. To define $\mathbb{R}$.
- We use Cauchy sequence.

Pedagogy

We can define real numbers geometrically, adopted by Euclid, and mostly between 1500-1850, or as presented in [4]
- This ultimately leads to Dedekind's picture of how an irrational number sits among the rational.

2.1 Characterizing properties of $\mathbb{R}$: the completeness properrty

As with construction of $\mathbb{N}$, ultimately for $\mathbb{R}$, we are interested in the structural properties they have. The essential properties of $\mathbb{R}$ can be described by Thm. 2.1. If you have learned any algebra, this is also known as a complete ordered field.

Theorem 2.1. Properties of $\mathbb{R}$, this is a rehash of the list in [2, 2.3]. $\mathbb{R}$ is a set with

- operations $+$ and $\cdot$
- relations $=$ and $\leq$
- special elements $0, 1$ with $0 \neq 1$.

such that

1. $\leq$ is a reflexive and transitive relation.
2. $\leq$ behaves well under addition and multiplication : If $x \leq y$ and $z \geq 0$.
   - then $x + z \leq y + z$
   - $x \cdot z \leq y \cdot z$.
3. The operation $+$, def. is commutative and associative, admits inverses and admits identity 0. In other words:
   - Associativity: for all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.

- Commutativity: for all $x, y \in \mathbb{R}$, $x + y = y + x$.
- Admits inverse: for all $x \in \mathbb{R}$, there exists y such that

$$x + y = y + x = 0$$

- Admits identity 0: for all $x \in \mathbb{R}$,

$$x + 0 = 0 + x = x$$

4. The operation $\cdot$ is commutative and associative, admits inverses and identity 1:

5. Completeness: for any $A \subseteq \mathbb{R}$, $A \neq \emptyset$ which is bounded above has a least in upper bound *in* $\mathbb{R}$.

Proof. Properties of $+$ is left as homework. $\square$

Worthy of distinction is the last axiom.

Definition 2.2. A *partial order* on a set X, is a relation $\leq$ on X which is

- reflexive
- transitive: for all $a, b, c \in X$, if $a \leq b$, $b \leq c$, then $a \leq c$.
- antisymmetric: for all $a, b \in X$, $a \leq b$ and $b \leq a$ implies $a = b$.

Example

$(\mathbb{N}, \leq), (\mathbb{Q}, \leq), (\mathbb{Z}, \leq)$ are all partial orders. However $<$ is *not*.

Definition 2.3. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation. $M \in X$ is a *upper bound* iff for all $x \in E$, $x \leq M$.

Definition 2.4. Let $E \subseteq X$, where $(X, \leq)$ is a set with a relation. $M \in X$ is a *least upper bound for E* if

1. M is an upper bound for E.
2. any other upper bound M' on E must satisfy $M \leq M'$.

Example

Let us consider $(\mathbb{Q}, \leq)$. What is the order relation here? see 1.14. Discuss the upper bound and least upper bound for the following sets.

- $E := \{x \in \mathbb{Q} : x > 0\}$.
- $E := \left\{x \in \mathbb{Q} : x^2 < 2\right\}$
- $E := \emptyset$

2.2 Cauchy sequences

Let us start by constructing $\sqrt{2}$ using $\mathbb{Q}$. The idea is to represent such a number using sequence. All inequalities and numbers discussed in this section will be rationals.

Discussion

- If a "real" number x grows continually, but is bounded, does it approach a limiting value?

Definition 2.5. Let $m \in \mathbb{Z}$. A *sequence of rational numbers* denoted $(a_n)_{n=m}^{\infty}$ is a function

$$\{n \in \mathbb{Z} : n \geq m\} \to \mathbb{Q}$$

Discussion

Why don't we start the sequence at 0? We will see this when we discuss $\limsup$.

Definition 2.6. A sequence is $(x_n)_0^{\infty}$,

- *eventually ε-steady*, if exists some N such that for all $n, m \geq N$,

$$|x_n - x_m| < \varepsilon$$

- a *Cauchy sequence* iff for all $\varepsilon > 0$, $(x_n)_{n=0}^{\infty}$ is eventually ε-steady.

Example

Proofs using quantifiers. Prove for all positive rationals, ε, there exists a positive rational δ such that $\delta < \varepsilon$.
Mathematicians often translate this to notation

$$\forall \varepsilon \in \mathbb{Q}_{>0}, (\exists \delta \in \mathbb{Q}_{>0}, \delta < \varepsilon)$$

but this is up to taste.

Proof. ??? $\square$

Proposition 2.7. Prove that $(a_n)_{n=1}^{\infty} := (1/n)_{n=1}^{\infty}$ is a Cauchy sequence.

Proof. What do we show ??? $\forall \varepsilon > 0$

-

$\square$

Example

- $(n)_{n=0}^{\infty}, (\sqrt{n})_{n=0}^{\infty}$ are not Cauchy.

Discussion

We want to use a Cauchy sequence to represent the real numbers. However, two sequences can represent the same number. Consider

$$1.4, 1.41, 1.414, 1.4142, \ldots$$

$$1.5, 1.42, 1.4143, 1.41422, \ldots$$

Definition 2.8. Two sequences $(x_n)_{n=0}^{\infty}, (y_n)_{n=0}^{\infty}$ are *eventually ε-close.* if there exists some N, such that for all $n \geq N$,

$$|a_n - b_n| < \varepsilon$$

Discussion

Are the following two sequences Cauchy equivalent?

- $(10^{10}, 10^{1}000, 1, 1, \ldots)$ and $(1, 1, \ldots,)$

Definition 2.9. Let $\mathcal{C}$ denote the set of cauchy sequences.[4] Then we set

$$\mathbb{R} := \mathcal{C}/\sim$$

where $\sim$ is the equivalence relation that

$$(x_n)_{n=0}^{\infty} \sim (y_n)_{n=0}^{\infty} \text{ if and only if } (x_n)_{n=0}^{\infty} \text{ and } (y_n)_{n=0}^{\infty} \text{ are eventually } \varepsilon\text{-close}$$

We denote the equivalence of $(x_n)_{n=0}^{\infty}$ as $[(x_n)]$. Note that in [5], Tao denotes the class as $\text{LIM}_{n\to\infty} x_n$.

Definition 2.10. Let $x, y \in \mathbb{R}$. Choose two representatives[5], say $(x_n)_{n=0}^{\infty} \in x$ and $(y_n)_{n=0}^{\infty} \in y$, then

- the sum of x and y is defined as

$$x + y := [(x_n + y_n)_{n=0}^{\infty}]$$

  Addition is well-defined. [5, 5.3.6, 5.3.7].

- the product of x and y is defined as

$$x \cdot y := [(x_n \cdot y_n)_{n=0}^{\infty}]$$

Now we can define the order relation on $\mathbb{R}$, compare to def. 1.13

Definition 2.11. $x \in \mathbb{R}$ is

- *positive* iff there exists a positive rational $c \in \mathbb{Q}_{>0}$, and $(x_n)_{n=0}^{\infty} \in x$ such that $x_n \geq c$ for all $n \geq 1$.
- *negative* iff $-(x_n)_{n=0}^{\infty} := (-x_n)_{n=0}^{\infty}$ is positive.

Definition 2.12. Let $x, y \in \mathbb{R}$, we say

- $x > y$ iff $x - y$ is positive.
- $x \geq y$ iff $x - y$ is positive or $x = y$.

[4] This is a subset of $\mathbb{Q}^{\mathbb{N}}$.

[5] an element of the equivalence class

3 More on Sequences

Reading: [5, 6].
Previously, we have worked with Cauchy sequences of rational numbers, see def 2.6, these were used to define $\mathbb{R}$. Now let us work with Cauchy sequences of real numbers:

Definition 3.1. A sequence $(x_n)_0^\infty$ of real numbers, i.e. a map $\mathbb{N} \to \mathbb{R}$, is

- *eventually ε-steady*, if exists some N such that for all $n, m \geq N$,

$$|x_n - x_m| < \varepsilon$$

- a *Cauchy sequence* iff for all $\varepsilon > 0$, $(x_n)_{n=0}^\infty$ is eventually ε-steady.

Learning Objectives

- Understand the notion of supremum and infima.
- Note that all convergent sequence is bounded, but is the bounded sequences convergent? This is the monotone convergence theorem. [5, 6.3.8].

We have the following hierarchy.

$$\{\text{Convergent}\} \Rightarrow \{\text{Cauchy}\} \Rightarrow \{\text{Bounded}\}$$

But is the converse true?

Theorem 3.2. Let $(a_n)_{n=0}^\infty$

Now that we have defined $\mathbb{R}$, we will review again the notion of convergence. We can slowly increase our level of "closeness" of *a sequence* to *a point* via these three definitions.

Definition 3.3. Let $x \in \mathbb{R}$.

1. Let $\varepsilon \in \mathbb{R}_{>0}$. $(a_n)_{n=0}^\infty = \{a_0, a_1, \ldots\}$ is ε-*adherent* to x if exists $N \in \mathbb{N}$ st. $|a_N - x| < \varepsilon$.

2. Let $\varepsilon \in \mathbb{R}_{>0}$ we say $(a_n)_{n=0}^\infty$, is *ε-close to x* if $|a_n - x| < \varepsilon$ for all $n \geq 0$.

3. Let $\varepsilon \in \mathbb{R}_{>0}$ we say $(a_n)_{n=0}^\infty$ is *eventually ε-close to x* if there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
$$|a_n - x| < \varepsilon$$

Discussion

Consider our favourite sequence of 1.

$$0.9, 0.99, 0.999$$

- What are choices of x that satisfy 1?

Definition 3.4. A sequence $(a_n)_{n=0}^{\infty}$ of rationals *converges to* x iff it is eventually ε convergence to x for all $\varepsilon \in \mathbb{Q}_{>0}$.

Discussion

- In 1. what if $n = 0$? For instance

$$1, 0, 0, 0, 0, 0, \ldots$$

  is ε close to 1. This wouldn't be a nice definition of the sequence "converging to x".

- In 2. This may be too much of demand? What about the sequence

$$1, 1/2, 1/3, \ldots, 1/n, \ldots$$

Proposition 3.5. Uniqueness of limits of sequences. [5, 6.1.7].

3.1 Extending the number system

We will begin by defining the *suprema* and *infima* of sets. To make our life easier, we define the extended real number system.

Definition 3.6. The *extended number system* consists of

$$\bar{\mathbb{R}} := \{-\infty\} \cup \mathbb{R} \cup \{\infty\}$$

Let $x, y, z \in \bar{\mathbb{R}}$. Define the order relation, 1.3 $x \leq y$ if and only if one of the following holds.

1. If $x, y \in \mathbb{R}$, $x \leq y$.
2. $x = -\infty$
3. $y = \infty$.

Thus, we artificially add in new terms.

- We do not include any operations. This can be dangerous. Of course, this can be done: say we can demand :

$$c + (+\infty) = (+\infty) + c := +\infty \quad \forall c \in \mathbb{R}$$

$$c + (-\infty) = (-\infty) + c =: -\infty \quad \forall c \in \mathbb{R}$$

  but requires a lot of care.

- We can define order and negation.

This is a common practice for mathematics, in order for one to make better statements.

Definition 3.7. Negaion of reals.

Example

What is the supremum of the set

- $$\{0, 1, 2, 3, 4, 5, \ldots\}$$
- $$\{1 - 2, 3, -4, 5, -6, \ldots\}$$

Definition 3.8. [Least upper bound] Let $E \subseteq \bar{\mathbb{R}}$. Then $\sup E$, the least upper bound [5, 6.2.6] is defined by the following rule:

- Let $E \subseteq \mathbb{R}$. So $\infty, -\infty \notin E$.
- If $\infty \in E$.

We can define the infimum without the use of another definition.

Definition 3.9. We let

$$\inf E := -\sup(-E)$$

$$E := \{-x \ : , x \in E\}$$

In many cases we have *two limits*.

Example

Let E be negative integers.

$$\inf(E) = -\sup(-E) = -\infty$$

Discussion

Consider the sequence

$$1.1, -1.01, 1.001, -1.0001, 1.0001, \ldots \quad (*)$$

What two limits do you see? It is a combination of two sequences:

- $1.1, 1.001, 1.0001, 1.00001, \ldots$.
- $-1.01, -1.0001, -1.000001, \ldots$.

Definition 3.10. Let $(a_n)_{n=m}^{\infty}$ be a sequence. Then set

$$a_N^+ := \sup\left[(a_n)_{n=N}^{\infty}\right]$$

$$\limsup_n a_n := \inf\left[(a_N^+)_{N=m}^{\infty}\right]$$

Example

In (*)

- $(a_n^+) = (a_0^+, a_1^+, \ldots)$ is the sequence

$$1.1, 1.01, 1.001$$

Proposition 3.11. Properties of limsup and liminf.

Homework for week 4

Due: Week 5, Wednesday. You will select 3 problems to be graded.
References: [2, 2], [5, 5].

You are free to assume anything you know about $\mathbb{Q}$. The problem on Dedekind construction is one problem it self. It has extended number of points not because of its difficulty, but because of its length.

Problems

1. (**) Prove that the relation defined in def. 2.9, is an equivalence relation.
2. Review the definition of addition on $\mathbb{R}$, **??**. Prove that addition, $+$, on $\mathbb{R}$ satisfies properties from 2.1. That is, prove :
   - Associativity: for all $x, y, z \in \mathbb{R}$, $x + (y + z) = (x + y) + z$.
   - Commutativity: for all $x, y \in \mathbb{R}$, $x + y = y + x$.
   - Admits identity 0: for all $x \in \mathbb{R}$,

     $$x + 0 = 0 + x = x$$
3. (*) Review the definition of multiplication on $\mathbb{R}$, def. **??** Prove that any $x \in \mathbb{R}$ where $x \neq 0$ [6] admits a multiplicative inverse y, i.e. exists $y \in \mathbb{R}$ such that

   $$x \cdot y = y \cdot x = 1$$
4. Let $E \subseteq \mathbb{Q}$. Prove that under the order relation $\leq$, least upper bound is unique if exists
5. (**) Here we discuss some conditions to see whether a sequence of rationals $(a_n)_{n=0}^{\infty}$ is Cauchy:
   (a) Suppose that for all $n \in \mathbb{N}$,

   $$|a_{n+1} - a_n| < 2^{-n}$$

   prove that (a_n) is Cauchy.

   (b) if we replace the condition in a. as

   $$|a_{n+1} - a_n| < 1/(n+1)$$

   for all $n \in \mathbb{N}$, give an example where (a_n) is not Cauchy.

[6] here $0 := (0)_{n=0}^{\infty}$ is the Cauchy sequence consisting of 0s

6. (***) How can we construct $\sqrt{2}$ using Cauchy sequence? Consider the following three sequence $(a_n), (b_n), (x_n)$ defined as follows

$$a_0 = 1, b_0 = 2$$

for each $n \geq 0$,

$$x_n = \frac{1}{2}(a_n + b_n)$$

$$a_{n+1} = \begin{cases} x_n & x_n^2 < 2 \\ a_n & \text{otherwise} \end{cases}$$

$$b_{n+1} = \begin{cases} b_n & x_n^2 < 2 \\ x_n & \text{otherwise} \end{cases}$$

   (a) Prove that all sequences are Cauchy.

   (b) Prove that all sequences are Cauchy equivalent.

   (c) Prove $[(a_n)_{n=0}^{\infty}] \cdot [(a_n)_{n=0}^{\infty}] = 2$.

7. Show that a Cauchy sequence is bounded.

4 Continuity

Week5, Reading [5, 9.3].
Previously we have been dealing with sequences, 3.

Learning Objectives

In the next two lectures:

- Understand the underlying algebra
- State the Intermediate Value Theorem.

Define the exponential function exp, or $e^{(-)}$. To do this we need.

- Continuity.
- Formal power series.

4.1 Subsets in analysis

Reading: [5, 9.1].
In analysis, we often work with certain subsets of $\mathbb{R}$. To define these, we need to know the partial order $\leq$ on $\mathbb{R}$, see def. 2.12.

Definition 4.1. Let $a, b \in \mathbb{R}$.

- We define the closed interval.
$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$
- The *half open* intervals
$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\} \quad (a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$
- The open intervals
$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

Example

What is

- $(2, 2)$
- $[2, 2)$
- $(4, 3)$.
- $[3, 3]$.

Definition 4.2. Sequences of real numbers. Same as 2.5 but with $\mathbb{R}$ instead of $\mathbb{Q}$.

Definition 4.3. Same as 3.4 but with real sequences and converging to real number.

Proposition 4.4. Uniqueness of limits. [5, 6.1.7].

4.2 Working with real valued functions

In this section we study real valued functions

$$f : \mathbb{R} \to \mathbb{R}$$

Example

1. Characteristic functions. Important for measure theory.

2. Polynomial functions.

We will denote the collection of functions from $\mathbb{R}$ to $\mathbb{R}$,

$$\mathrm{Cts}(\mathbb{R}, \mathbb{R}) \subset \mathrm{Fct}(\mathbb{R}, \mathbb{R})$$

Whenever you have a collection of objects you can always ask what structure does this have?

Definition 4.5. [5, 9.2.1] Structure on $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$. This is what algebraist refer as *composition rings.*

1. Composition.

2. Multiplication.

3. Addition.

4. Negation.

Except the compositional structure, all such structures exists on *function algebras*, that is sets of the form $\mathrm{Fct}(X, \mathbb{R})$ for X any set. For example, when $X = \mathbb{N}$,

$$\mathrm{Fct}(\mathbb{N}, \mathbb{R}) = \{(x_n)_{n=0}^{\infty} : x_n \in \mathbb{R}\}$$

This space of functions is the set of real sequences starting at 0. The goal now is to study $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$ generalizing

$$\mathrm{Fct}(\mathbb{N}, \mathbb{R})$$

Discussion

Which of the following are true?

1. $(f + g) \circ h = (f \circ h) + (g \circ h)$.

2. $(f + g) \cdot h = (f \cdot h) + (g \cdot h)$.

In the realm of geometry, there is a duality between spaces and their algebra of functions, [1].

In the context of sequences, we were able to make sense of "limit" to a point, "∞"

$$\lim_{n\to\infty} x_n = L$$

[7] Similarly, in the context $\mathrm{Fct}(\mathbb{R},\mathbb{R})$ we would like to consider points $a \in \mathbb{R}$, where we can write

$$\lim_{x\to a} f(x) = L$$

Then to study $f : \mathbb{R} \to \mathbb{R}$, it would be helpful to steady $f|_X : X \to \mathbb{R}$ for $X \subseteq \mathbb{R}$ of subsets that are intervals.

Definition 4.6. The restriction operation: let $E \subseteq X \subseteq \mathbb{R}$ be subsets of $\mathbb{R}$. The restriction map is defined as

$$\mathrm{Fct}(X,\mathbb{R}) \to \mathrm{Fct}(E,\mathbb{R})$$

$$f \mapsto f|_E$$

where $f|_E(x) := f(x)$.

4.3 Limiting value of functions

Reading, [5, 9.3]. We know what it means for a sequence to converge. Now we understand what it means for a function defined on an *interval* to converge.

Definition 4.7. Converging function. Let $X \subset \mathbb{R}$ be an interval. We discuss $\mathrm{Fct}(X,\mathbb{R})$.

1. ε-closeness. $f \in \mathrm{Fct}(X,\mathbb{R})$ is ε close if for all $x \in X$,

$$|f(x) - L| < \varepsilon$$

2. [5, 9.3.3]. $f \in \mathrm{Fct}(X,\mathbb{R})$ is *local ε-close to L at a* iff there exists $\delta > 0$ such that

   (a) some interval $(x-\delta, x+\delta) \subset X$

   (b) $f|_{(x-\delta,x+\delta)}$ is ε-close to L.

3. Let $L \in \mathbb{R}$, and $a \in X$, then we say f *converges to L as x approaches a*, if forall $\varepsilon \in \mathbb{R}_{>0}$, f is local ε-close to L at a. In which case we denote

$$\lim_{x\to a} f(x) = L$$

[7] in fact, this is the limit of $\mathbb{N}$, when phrased correctly.

Example

In 1. Let $f(x) = x^2$.

- 4-close to 2?

- 1-close to 1?

$g(x) = x^3$. $g_1 := g|_{[0,1]}$ and $g_2 := g|_{[1,2]}$.

- 4-close to 2?

- 1-close to 1?

	Sequences (x_n)	f converging to L at a.
	$\mathbb{N}$	$X \subset \mathbb{R}$ contains a
ε-close	$\forall n \in \mathbb{N}$ $\lvert x_n - L\rvert < \varepsilon$.	$\forall x \in X, \lvert f(x) - L\rvert < \varepsilon$.
ev'/local ε-close	$\exists N$, for all $n \geq N$ $\lvert x_n - L\rvert < \varepsilon$	$\exists \delta > 0$, $\lvert f(x) - L\rvert < \varepsilon$, $\forall x \in (a - \delta, a + \delta)$.
Converges	$\forall \varepsilon > 0$, (x_n) is ev' ε-close	$\forall \varepsilon > 0$, (x_n) is local ε-close

4.4 Continuous functions

Definition 4.8. Let $X \subset \mathbb{R}$ be an open interval. $f : X \to \mathbb{R}$ is continuous at $x_0 \in X$ if for all $\varepsilon > 0$ there exists $\delta > 0$...?

We will consider three fundamental results in continuity of functions, [4, 7].

Homework for week 5

Due: Week 6, Friday. We will select 4 problems to be graded.

1. Which of the following are true on $\mathrm{Fct}(\mathbb{R}, \mathbb{R})$: let $f, g, h \in \mathrm{Fct}(\mathbb{R}, \mathbb{R})$:

   (a) Composition $\circ$ is associativity :

   $$f \circ (g \circ h) = (f \circ g) \circ h$$

   (b) Composition distributes over multiplications:

   $$(f \cdot g) \circ h = (f \circ h) \cdot (g \circ h)$$

   (c) Composition distributes over addition:

   $$(f + g) \circ h = f \circ h + g \circ h$$

2. Let (x_n) be a sequence of real numbers. Let $x_1 = 2$,

   $$x_{n+1} = \frac{1}{2}x_n + \frac{1}{x_n}$$

   show that x_n limits to a number L where $L^2 = 2$.

3. Prove 3.5.

4. Let $a < b$, and $f : [a, b] \to \mathbb{R}$ be a continuous and strictly monotone function. Then f restricts to a bijection $f : [a, b] \to [f(a), f(b)]$. Show that f^{-1} is also continuous and strictly monotone.

5. Prove that $f(x) = |x|^3$ is twice differentiable in $\mathbb{R}$ but not three times. (First prove that $f^{(2)}(x) = 6|x|$.

References

[1] John Baez, *Isbell duality* (2022).

[2] Derek Goldrei, *Propositional and predicate calculus: A model of argument*, 2005.

[3] Paul R. Halmos, *Naive set theory*, 1961.

[4] Michael Spivak, *Calculus, 4th edition.*

[5] Terence Tao, *Analysis I, 4th edition*, 2022.