SMITH THEORY AND CYCLIC BASE CHANGE FUNCTORIALITY

TONY FENG

Abstract. Lafforgue and Genestier-Lafforgue have constructed the global and (semisimplified) local Langlands correspondences for arbitrary reductive groups over function fields. We establish various properties of these correspondences regarding functoriality for cyclic base change: For $\mathbf{Z}/p\mathbf{Z}$-extensions of global function fields, we prove the existence of base change for mod p automorphic forms on arbitrary reductive groups. For $\mathbf{Z}/p\mathbf{Z}$-extensions of local function fields, we construct a base change homomorphism for the mod p Bernstein center of any reductive group. We then use this to prove existence of local base change for mod p irreducible representation along $\mathbf{Z}/p\mathbf{Z}$-extensions, and that Tate cohomology realizes base change descent, verifying a function field version of a conjecture of Treumann-Venkatesh.

The proofs are based on equivariant localization arguments for the moduli spaces of shtukas. They also draw upon new tools from representation theory, including parity sheaves and Smith-Treumann theory. In particular, we use these to establish a categorification of the base change homomorphism for mod p spherical Hecke algebras, in a joint appendix with Gus Lonergan.

CONTENTS

1. INTRODUCTION

In this paper we prove several results on cyclic base change in the Langlands correspondence over function fields. To set the context for our results, let us recall some history. Global cyclic base change functoriality for reductive groups over number fields, established over many years in increasing generality by work of Saito [Sai77], Shintani [Shi79], Langlands [Lan80], Arthur-Clozel [AC89], Labesse [Lab99], Harris-Labesse [HL04] and others for cuspidal automorphic representations with characteristic zero coefficients (under some technical assumptions for general groups), is one of the major triumphs of Langlands' program thus far. In addition to its initial applications towards Artin's Conjecture, it plays a crucial role in much subsequent work, such as in automorphy lifting arguments following in the tradition of Wiles, Taylor, etc.

Our main progress in the present paper is on understanding cyclic base change in the *Local* Langlands correspondence, which was constructed (in a semisimplified form) for all reductive groups over local function fields by Genestier-Lafforgue [GL]. The proof (over number fields) of global cyclic base change is founded upon the twisted trace formula, a tool which does not seem to apply to our (local and mod p) context, and is in any case currently unavailable over function fields due to non-trivial analytic difficulties. We introduce a new strategy, which we use to prove the first general existence results for local base change of all irreducible representations of arbitrary reductive groups over local function fields. Furthermore, we establish descent theorems for cyclic base change that were conjectured by Treumann-Venkatesh; these are new even for specific groups such as GL_n where the full Local Langlands correspondence (hence in particular the existence of local base change) is already known. These advances involve the construction of a *base*

change homomorphism for Bernstein centers, as has been envisaged by Haines in the case of characteristic zero coefficients. En route to the local results we establish new global results as well: we prove the first general existence theorem for cyclic base change of mod p automorphic forms on arbitrary reductive groups over global *function fields*, again without any trace formula arguments. A major novelty of these results is their applicability to completely general groups and representations.

The proofs assemble a diverse selection of tools ranging from topology (particularly equivariant localization and Tate cohomology) to arithmetic geometry (of moduli stacks of shtukas) to p-adic groups (exploiting new constructions with Hecke algebras and Bernstein centers) to modular representation theory (using crucially the recent inventions of parity sheaves and Smith restriction).

We now proceed to give more precise descriptions of our results.

1.1. **Local results.** Genestier-Lafforgue have constructed a semi-simplified form of the Local Langlands correspondence over function fields [GL]. More precisely, let F_v be a function field of characteristic *not* equal to p and W_v the Weil group of F_v. Let k be an algebraic closure of $\mathbf{F}_p$.[1] For any reductive group G over F_v, [GL] constructs a map

$$\left\{ \begin{matrix} \text{irreducible admissible representations} \\ \pi \text{ of } G(F_v) \text{ over } k \end{matrix} \right\} / \sim \; \longrightarrow \; \left\{ \begin{matrix} \text{semi-simple } L\text{-parameters} \\ \rho_\pi \colon W_v \to {}^L G(k) \end{matrix} \right\} / \sim . \qquad (1.1)$$

Here ${}^L G$ is Langlands' L-group, regarded over k.

Langlands' principle of functoriality predicts that given two reductive groups H and G over F_v, and a map of L-groups $\phi \colon {}^L H \to {}^L G$, every L-packet of irreducible representations of $H(F_v)$ should admit a "transfer" to $G(F_v)$ compatible with ϕ. In this paper we are concerned with a specific type of functoriality: *base change functoriality*, arising from the case where H is any reductive group over F_v, and $G = \operatorname{Res}_{E_v/F_v}(H_{E_v})$ for a cyclic p-extension E_v/F_v. The relevant map $\phi_{\mathsf{BC}} \colon {}^L H \to {}^L G$ is characterized by the property that it is *admissible* and induces the diagonal embedding on their underlying identity connected components (i.e., the respective Langlands dual groups). We emphasize that it is crucial for our results that the degree of the extension coincides with the characteristic of our representations. In this situation, let us say that an irreducible representation Π of $G(F_v)$ is a *base change* of an irreducible representation π of $H(F_v)$ if $\phi_{\mathsf{BC}} \circ \rho_\pi \cong \rho_\Pi$.

Theorem 1.1 (Existence of local base change). *Let π be any irreducible representation of $H(F_v)$ over k. Then a base change of π to $G(F_v)$ exists.*

For $H = \operatorname{GL}_n$, a full Local Langlands correspondence has been established by Vignéras [Vig01], giving a much more precise result than Theorem 1.1. It seems reasonable to expect that Vignéras' methods could (eventually) be extended to some classical groups, after the stabilization of the twisted trace formula for automorphic forms over function fields is achieved. The novelty of Theorem 1.1 is that it applies uniformly to all reductive groups, and all irreducible representations. A motivation for Theorem 1.1 is a program of the author to compute explicitly the Genestier-Lafforgue parameters of the representations arising in [CO] which depends on the residue field being "large enough", so is convenient to be able to make an unramified base change.

We also prove a descent result for the above base change situation which was conjectured by Treumann-Venkatesh, and is new even for $H = \operatorname{GL}_n$ whenever $n > 1$. Let σ be a generator[2] of $\operatorname{Gal}(E_v/F_v)$; it acts on G and its induced action on $G(F_v) = H(E_v)$ is the Galois action. If the isomorphism class of a k-representation Π of $G(F_v)$ is preserved by σ, then it should come from base change. For any irreducible admissible representation Π of $G(F_v)$ whose isomorphism class is fixed by σ, there is a unique σ-action on Π compatible with the $G(F_v)$-action (Lemma 6.17). Hence we can form the *Tate cohomology* groups $T^0(\Pi)$, $T^1(\Pi)$ with respect to the σ-action (cf. §3.4), which retain actions of $H(F_v) = G(F_v)^\sigma$, and are conjecturally admissible $H(F_v)$-representations. We prove:

---

[1]In this paper our varieties are over fields of characteristic not equal to p while our coefficients are of characteristic p. This is to adhere to standard notational conventions for Smith theory; unfortunately, it is at odd with standard notational conventions in arithmetic geometry.

[2]The choice of generator is made for convenience of notation; all constructions involving it will be manifestly independent of the choice.

Theorem 1.2 (Tate cohomology realizes cyclic base change). *Assume p is an odd good prime[3] for $\widehat{G}$. Let Π be an irreducible representation of $G(F_v)$ whose isomorphism class is fixed by σ, and $\Pi^{(p)} := \Pi \otimes_{k,\mathrm{Frob}} k$ the Frobenius twist of Π. Let π be any irreducible admissible subquotient of $T^*(\Pi)$ as an $H(F_v)$-representation and $\rho_\pi \colon W_v \to {}^L H(k)$ be the corresponding L-parameter constructed by Genestier-Lafforgue. Then $\phi_{\mathsf{BC}} \circ \rho_\pi \cong \rho_{\Pi^{(p)}}$.*

This verifies, for the Genestier-Lafforgue construction of the semi-simplified Local Langlands correspondence, a Conjecture of Treumann-Venkatesh [TV16, Conjecture 6.3] that "Tate cohomology realizes functoriality". It had previously been proved for certain depth-zero supercuspidal representations of $\mathrm{GL}_n(F_v)$ by Ronchetti [Ron16], by direct calculation of the Tate cohomology and comparison to Vignéras' work. The difficulty of the calculations, even in those special cases, made them inaccessible to generalization. By contrast, our proof applies uniformly for all groups and all representations under only a very mild condition on p, and is completely conceptual; in particular, it avoids any computations with specific models of representations, for example as compact inductions.

Remark 1.3. In [BFH+], we will compute Tate cohomology for an interesting class of supercuspidal representations (of arbitrary depth) studied by Chan-Oi [CO], which provides many examples where Theorem 1.2 can be made very concrete.

We now proceed to describe our third main local result. Recall that the *Bernstein center (with coefficients in k) of $G(F_v)$*, denoted $\mathfrak{Z}(G)$, is the ring of endomorphisms of the identity functor on the category of smooth $G(F_v)$-representations (on k-vector spaces). Informally speaking, an element of $\mathfrak{Z}(G)$ is represented by a system of compatible endomorphisms of all smooth $G(F_v)$-representations (commuting with the $G(F_v)$-action). In particular, $\mathfrak{Z}(G)$ acts on any irreducible smooth $G(F_v)$-representation Π through a character $\chi_\Pi \colon \mathfrak{Z}(G) \to k$. Furthermore, the correspondence (1.1) turns out to assign isomorphic L-parameters to irreducible representations inducing the same character of $\mathfrak{Z}(G)$. The ideas used to establish the preceding theorems also allow us to construct a *base change homomorphism* between the Bernstein centers of $G(F_v)$ and $H(F_v)$ with the property detailed in the following Theorem.

Theorem 1.4 (Base change homomorphism for Bernstein centers). *Assume p is an odd good prime for $\widehat{G}$. Then there is a homomorphism*

$$\mathfrak{Z}(G) \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} \mathfrak{Z}(H)$$

such that for each irreducible $H(F_v)$-representation π, the character $\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}} \colon \mathfrak{Z}(G) \xrightarrow{\mathfrak{Z}_{\mathrm{TV}}} \mathfrak{Z}(H) \xrightarrow{\chi_\pi} k$ has the property that for any irreducible $G(F_v)$-representation Π on which $\mathfrak{Z}(G)$ acts through $\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}}$, there is an isomorphism of semi-simple L-parameters $\rho_\Pi \cong \phi_{\mathsf{BC}} \circ \rho_\pi$.

A base change homomorphism for Bernstein centers, with characteristic zero coefficients, has been sought by Haines [Hai14], and was constructed in some low-depth cases [Hai09, Hai12] (cf. also [Fen20] for the function field case). Haines also constructed a base change homomorphism for the *stable* Bernstein center of general groups, which in the case of GL_n coincides with the Bernstein center. Our Theorem 1.4 is somewhat different since it concerns characteristic p, but it is the first such construction that applies to general groups and depth. Its generality and provable connection to the Local Langlands correspondence make it rather new and compelling evidence for Haines' vision.

Remark 1.5. The construction of the map $\mathfrak{Z}_{\mathrm{TV}}$ applies equally well for local fields of characteristic 0 having residue characteristic distinct from p. However, our argument for proving that it has the "correct" effect in terms of the Local Langlands correspondence only works for function fields. The future work [Fen] aims to prove analogous results with respect to Fargues-Scholze's construction [FS] of the (semisimplified) local Langlands correspondence for arbitrary local fields.

1.2. **Global results.** Although our most striking progress is on the local Langlands correspondence, we also obtain new results in the global Langlands correspondence. In fact, the local results mentioned above are themselves deduced from analysis of Lafforgue's machine for constructing the global Langlands correspondence.

---

[3]Explicitly, this means that we require $p > 2$ if $\widehat{G}$ has simple factors of type A, B, C or D; $p > 3$ if $\widehat{G}$ has simple factors of type G_2, F_4, E_6, E_7; and $p > 5$ if $\widehat{G}$ has simple factors of type E_8.

Now let G be a reductive group over a global function field F, of characteristic *not* equal to p. Vincent Lafforgue has constructed in [Laf18, §13] a global "mod p" Langlands correspondence, decomposing the space of cuspidal automorphic functions $C_{\mathrm{cusp}}^\infty(G(F)\backslash G(\mathbf{A}_F), k)$ into summands indexed by semi-simple L-*parameters*, which are certain $\widehat{G}(k)$-conjugacy classes of continuous homomorphisms $\rho\colon \mathrm{Gal}(F^s/F) \to {}^L G(k)$. Work of Cong Xue [Xue20, Xuea, Xueb] extends Lafforgue's theory to the space of all compactly supported automorphic functions, $C_c^\infty(G(F)\backslash G(\mathbf{A}_F), k)$. See §5.2.4 for a more precise discussion. Let us call an L-parameter ρ *automorphic* if it arises from Lafforgue(-Xue)'s construction.

Let H be a reductive group over F, and $G = \mathrm{Res}_{E/F}(H_E)$ for a cyclic p-extension E/F. The relevant map $\phi_{\mathsf{BC}}\colon {}^L H \to {}^L G$ is the diagonal on the identity connected components.

Theorem 1.6 (Existence of global base change). *Assume p is an odd good prime for $\widehat{G}$. If $\rho\colon \mathrm{Gal}(F^s/F) \to {}^L H(k)$ is automorphic, then $\phi_{\mathsf{BC}} \circ \rho\colon \mathrm{Gal}(F^s/F) \to {}^L G(k)$ is automorphic.*

We comment on the relation of Theorem 1.6 to other base change theorems known in global contexts. To appreciate this it is important to highlight the distinction between "weak base change", which is determined Hecke eigensystems at almost all places of the global function, and "strong base change" as provided by Theorem 1.6, which concerns the entire L-parameter. These notions are equivalent for $H = \mathrm{GL}_n$, but for general groups "strong base change" is a strictly stronger notion. Indeed, Lafforgue's correspondence can assign different Langlands parameters to Hecke eigenfunctions with the same unramified eigensystem; in fact, it can even assign different parameters to different automorphic forms generating *isomorphic* automorphic representations[4], with examples occurring already for SL_n when $n \geq 3$ [Bla94, Lap99]. The reason for this is the failure of local conjugacy to imply global conjugacy; see [Laf18, §0.7] for more discussion of this phenomenon.

Remark 1.7. The distinction between weak and strong base change can be quite important in applications. For example, recent work of Sawin-Templier [ST21] shows that the Ramanujan Conjecture for cuspidal automorphic forms satisfying appropriate local conditions is implied by a strong form of cyclic base change, but weak base change does not suffice for their argument.

Our proof of Theorem 1.6 is inspired by work of Treumann-Venkatesh [TV16], which establishes existence of "weak base change" for the cohomology of locally symmetric spaces. The analogue of [TV16] in the function field context would guarantee the existence of a "weak base change" for mod p automorphic forms. The work of Treumann-Venkatesh is about Hecke operators, but in the function field context it is possible to go beyond Hecke operators to Lafforgue's *excursion operators*, and this is necessary to obtain "strong base change"; it is also what provides our handhold on the Local Langlands correspondence.

Over number fields, weak base change results with *characteristic zero coefficients* are known using the twisted trace formula, for all cuspidal automorphic representations of GL_n [AC89] or, on more general groups, cuspidal automorphic representations satisfying certain local conditions [Lab99]. Over function fields the analogous results are known for $H = \mathrm{GL}_n$ because the full global Langlands correspondence is already known in that case, again using the trace formula. But there are analytic difficulties in the theory of the twisted trace formula over function fields, which prevent parallel results from being known more generally. Instead, forthcoming work [BFH$^+$] will combine Theorem 1.6 with automorphy lifting theorems, generalizing those of [BHKT19], in order to obtain existence of cyclic order p base change for automorphic forms on split semisimple groups with characteristic 0 coefficients, for sufficiently large p and under a "large image" assumption (the latter is needed to make the notions of weak and strong base change coincide).

1.3. **Remarks on the proofs.** We emphasize at the outset that our arguments make no use of the traditional tool for analyzing cyclic base change, namely the twisted trace formula (which is in any case unavailable in our situation). Any serious discussion of the proofs of our main results would require an explanation of the construction of Lafforgue's and Genestier-Lafforgue's correspondences, in addition to a number of other ideas and definitions. To prevent this introduction from becoming overly technical, we confine ourselves to vague hints here.

The Genestier-Lafforgue correspondence is characterized by local-global compatibility, so the main input to the local results comes from an analysis of the global situation. The Global Langlands parametrization

---

[4]This implies that there is a difference between strong base change and an intermediate notion of base change which demands compatibility with local base change at *every* place (not just the unramified ones).

is extracted from the *cohomology of moduli stacks of shtukas*. This idea goes back to Drinfeld [Dri87], who introduced it to establish the global Langlands correspondence for GL_2 over function fields, later extended to GL_n by Laurent Lafforgue [Laf02]. For general groups, the role that Langlands' L-group $^L G$ should play presented a puzzle that was definitively resolved by Vincent Lafforgue: via the *Geometric Satake equivalence*, the category $\mathrm{Rep}(^L G)$ naturally indexes perverse sheaves that lives on the moduli stacks of G-shtukas, called Sht_G.

Summarizing roughly, the global Langlands correspondence involves two major inputs:

(1) A "topological" input, wherein the p-adic cohomology of spaces Sht_G supplies the source of interesting $\mathrm{Gal}(F^s/F)$-representations.

(2) A representation-theoretic input, wherein L-parameters into $^L G$ are extracted using that the coefficient sheaves for these cohomology groups are indexed functorially by $\mathrm{Rep}(^L G)$.

This will be explained more in §5. For now it is enough to appreciate that in order to produce a *functorial transfer* from H to G, we then need to address both of these aspects of Lafforgue's construction. More precisely, we need to:

(1) Show that cohomology classes on Sht_H can be "transferred" to cohomology classes on Sht_G.

(2) Give a geometric interpretation of the restriction functor $\mathrm{Rep}(^L G) \to \mathrm{Rep}(^L H)$ at the level of perverse sheaves.

The immediate difficulty of (1) is that in general there is not so much as a non-trivial map relating Sht_H and Sht_G. In the base change situation there is a natural map, but it is not even Hecke-equivariant, nor is it clear a priori that the map is not too destructive to cohomology groups. Ultimately, we solve (1) in our situation by looking at *Tate cohomology* instead of cohomology, and using a form of *equivariant localization* that relates the Tate cohomology of a space and its fixed points under a $\mathbf{Z}/p\mathbf{Z}$-action. Here we were inspired by work of Treumann-Venkatesh [TV16], where it was shown that such equivariant localization for locally symmetric spaces realized functoriality in that context.

For (2), the obvious difficulty in general is again that we are seeking to transport sheaves between two spaces that are not connected by any visible non-trivial geometric maps. In the base change situation there is a map, but the obvious functors it induces on sheaves do not come close to having the desired effect. In some sense, the problem is a categorified and local version of the problem in the previous paragraph. Our solution to this problem passes through certain "exotic" localizations of categories of sheaves called *Tate categories*, which can be seen as a categorification of Tate cohomology. The point is, vaguely speaking, that the desired relations of functoriality are satisfied in the relevant Tate categories. However, this does not interface well with Lafforgue's construction because localization to the Tate category does not interact well with the theory of perverse sheaves; our second main idea here is that this can be fixed by reworking the theory in terms of *parity sheaves* invented by Juteau-Mautner-Williamson [JMW14]. Here we were inspired by work of Leslie-Lonergan [LL], which used these tools to give a geometric interpretation of the Frobenius contraction functor in modular representation theory. (The key idea that parity sheaves play well with localization to the Tate category is also at the heart of recent work of Riche-Williamson [RW].) Ultimately, we are able to construct a "base change functor" that *categorifies* the base change homomorphism for spherical Hecke algebras, and which is suitable for input into the global setup. The construction is completed in the joint Appendix with Gus Lonergan.

To complete the proofs of the local results, we also need to exploit some new constructions with local Hecke algebras, in particular the base change homomorphism $\mathfrak{z}_{\mathrm{TV}}$ for Bernstein centers. A key insight in [TV16] is that the base change homomorphism for spherical Hecke algebras admits a more "geometric" description when the field extension is cyclic of order p and the coefficients also have characteristic p. We generalize this observation to the centers of higher depth Hecke algebras, and then to the Bernstein center, by a delicate analysis of Hecke algebras with respect to the subgroups coming from the Moy-Prasad filtration at a special vertex of the Bruhat-Tits building of G.

1.4. Organization of the paper. The outline of this paper is as follows.

In §2, we define *excursion algebras* and recall their relation to Langlands parameters. We explain functoriality from the perspective of excursion algebras.

In §3, we generalize the basic framework of *sheaf-theoretic Smith theory* from [Tre19, RW], which worked for topological spaces and finite type schemes respectively, to locally finite type schemes. This is needed because our spaces of interest are not of finite type. More specifically, we introduce the notion of Tate

categories, the Smith functor Psm and its properties, Tate cohomology, and explain the relation to classical equivariant localization theorems for $\mathbf{Z}/p\mathbf{Z}$-actions.

In §4, we recall the fundamentals of parity sheaves due to Juteau-Mautner-Williamson, and the analogous notion of "Tate-parity sheaves" due to Leslie-Lonergan. We explain how to combine these with the functor Psm to construct a *base change functor* for parity objects in the Satake category. This functor plays the categorified role of the base change homomorphism for Hecke algebras.

In §5, we prove a collection of global results, including Theorem 1.6. First we recall background on moduli spaces of shtukas and Lafforgue's global Langlands correspondence in terms of actions of the excursion algebra on the cohomology of shtukas. Then we establish certain equivariant localization isomorphisms for the Tate cohomology of shtukas in the setting of $\mathbf{Z}/p\mathbf{Z}$-base change, which gives relations between excursion operators in the context of functoriality. These are used later in the local applications, and Theorem 1.6 is also deduced as an application.

In §6 we prove our local results. We review the relevant aspects of the Genestier-Lafforgue correspondence. After analyzing the Brauer homomorphism for Hecke algebras with respect to subgroups arising from the Moy-Prasad filtration, we are able to construct the map $\mathfrak{Z}_{\mathrm{TV}}$ from Theorem 1.4, which we then establish using the global theory and local-global compatibility. Finally, we deduce Theorem 1.1 and Theorem 1.2.

1.6. **Notation.**

- (Coefficients) We let k be an algebraic closure of $\mathbf{F}_p$ (considered with the discrete topology).

  In general we will consider geometric objects over fields of characteristic $\neq p$, and étale sheaves over p-adically complete coefficients.
- (σ-actions) Throughout the paper, σ denotes a generator of a group isomorphic to $\mathbf{Z}/p\mathbf{Z}$. When we say that a widget has a "σ-action", what we mean is that the widget has an action of a cyclic group of order p with chosen generator σ.

  Let $N := 1 + \sigma + \ldots + \sigma^{p-1} \in \mathbf{Z}[\sigma]$. We will also denote by N the induced operation on any $\mathbf{Z}[\sigma]$-module.[5]

  If A is a ring or module for $\mathbf{Z}[\sigma]$, then A^σ denotes the σ-invariants in A.
- (Reductive groups) For us, reductive groups are connected by definition. The Langlands dual group $\widehat{G}$ is considered as a split reductive group over k. For our conventions on the L-group, see §2.1.

  For any group, $\mathbb{1}$ denotes the trivialization representation (with the group made clear by context).
- (Equivariant derived categories) If Y is a locally finite type stack and Λ is a coefficient ring in which the characteristic of Y is invertible, we let $D_c^b(Y; \Lambda)$ denote the usual bounded constructible derived category of étale sheaves over Λ.

  We shall also have occasional to consider larger categories of sheaves, where the constructibility condition is weakened. We let $D^b(Y; \Lambda)$ denote the bounded derived category of étale sheaves over Λ that are ind-constructible. In other words, it is the full subcategory of the (co-complete) category $D(Y; \Lambda)$, of ind-constructible étale sheaves over Λ, spanned by the bounded objects.

  If a (pro-)algebraic group Σ acts on Y, then we denote by $D_{c,\Sigma}^b(Y; \Lambda)$ or $D_c^b(X; \Lambda)^{B\Sigma}$ the Σ-equivariant bounded derived category of constructible sheaves with coefficients in Λ. We denote by $D_\Sigma^b(Y; \Lambda)$ or $D^b(X; \Lambda)^{B\Sigma}$ the analogous categories with the constructibility condition replaced by ind-constructibility, as above.

  When $\Lambda = k$ we may suppress it from the notation, writing instead $D_c^b(Y) := D_c^b(Y; k)$, etc.
- Functors between derived categories, e.g. $f_!, f_*, f^!, f^*$, will always denote the derived functors.

---

[5]This is to be contrasted with the operation Nm, which will mean $\mathrm{Nm}(a) = a * \sigma(a) * \ldots * \sigma^{p-1}(a)$ in the context where there is a monoidal operation $*$.

2. Functoriality and the excursion algebra

In this section we formalize the *abstract excursion algebra* $\mathrm{Exc}(\Gamma, {}^L G)$, a device used to decomposable a space into pieces indexed by Langlands parameters. This notion appears implicitly in [Laf18], but there it is the image[6] of the abstract excursion algebra in a certain endomorphism algebra which is emphasized.

Since we work with non-split groups, we first clarify in §2.1 our conventions regarding L-groups. This is a bit subtle, as one finds (at least) two natural versions of the L-group in the literature: the "algebraic L-group" ${}^L G^{\mathrm{alg}}$, following Langlands, and the "geometric L-group" ${}^L G^{\mathrm{geom}}$, derived from the Geometric Satake equivalence. The difference between them is parallel to the difference between L-algebraicity and C-algebraicity emphasized in [BG14].

We emphasize that the unadorned notation ${}^L G$ denotes the algebraic L-group, to be consistent with [Laf18], although the geometric L-group is really what appears more naturally in our arguments.

We introduce two explicit presentations for the excursion algebra in §2.2 and §2.4. The first presentation is more natural for making the connection to Langlands parameters, which we recall in 2.3. The second presentation is more amenable to constructing actions of the excursion algebra, which makes it more convenient for our purposes, and it is the only one that will be used in the sequel.

Finally in §2.5 we explain how functoriality is interpreted in terms of excursion algebras.

2.1. Conventions on L-groups and Langlands parameters.
For a reductive group G over a field $\mathbf{F}$ with separable closure $\mathbf{F}^s$, we regard its Langlands dual group $\widehat{G}$ as a split reductive group over k. The L-group is a certain semi-direct product ${}^L G = \widehat{G} \rtimes \mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$. Actually, in the case where $\mathbf{F}$ is a local field we shall instead work with the "Weil form" $\widehat{G} \rtimes \mathrm{Weil}(\mathbf{F}^s/\mathbf{F})$. (This is just for consistency with [GL]; because we consider representations over K, in our case it would make no difference to work with the Galois form.)

2.1.1. *Algebraic L-group.* In fact there are at least two conventions for the definition of the L-group. The one which is more traditionally used in the literature is what we shall call the *algebraic L-group*, denoted ${}^L G^{\mathrm{alg}}$, defined as in [TV16, §2.5]. The root datum $\Psi(G)$ of $G_{\mathbf{F}^s}$ determines a pinning for $\widehat{G}$, which in turns gives a splitting $\mathrm{Out}(\widehat{G}) \to \mathrm{Aut}(\widehat{G})$ and an identification $\mathrm{Aut}(\Psi(G)) \cong \mathrm{Out}(\widehat{G})$. The $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$-action on $\Psi(G)$ transports to an action $\mathrm{act}^{\mathrm{alg}}$ of $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$ on $\widehat{G}$, and we define ${}^L G^{\mathrm{alg}}$ to be the semidirect product

$$ {}^L G^{\mathrm{alg}} := \widehat{G} \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(\mathbf{F}^s/\mathbf{F}). $$

Since the action $\mathrm{act}^{\mathrm{alg}}$ factors through a finite quotient, we may regard ${}^L G^{\mathrm{alg}}$ as a pro-algebraic group over k.

2.1.2. *Geometric L-group.* We now make a different construction of the L-group, using the Tannakian theory, following [RZ15, Appendix A] and [Zhu17, §5.5]. We begin with the Geometric Satake equivalence,

$$ \mathrm{P}_{L^+ G_{\mathbf{F}^s}}(\mathrm{Gr}_{G,\mathbf{F}^s}; k) \cong \mathrm{Rep}_k(\widehat{G}). $$

The Galois group $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$ acts on $\mathrm{Gr}_{G,\mathbf{F}^s}$, inducing an action on the neutralized Tannakian category $(\mathrm{P}_{L^+ G_{\mathbf{F}^s}}(\mathrm{Gr}_{G,\mathbf{F}^s}; k), \underbrace{H^*(-)}_{\text{fiber functor}})$. By [RZ15, Lemma A.1] this in turn induces an action $\mathrm{act}^{\mathrm{geom}}$ of $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$ on $\widehat{G}_k$. We define

$$ {}^L G^{\mathrm{geom}} := \widehat{G}_k \rtimes_{\mathrm{act}^{\mathrm{geom}}} \mathrm{Gal}(\mathbf{F}^s/\mathbf{F}). $$

In the case at hand we shall see that $\mathrm{act}^{\mathrm{geom}}$ also factors through a finite quotient of $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$, so we may also regard ${}^L G^{\mathrm{geom}}$ as a pro-algebraic group.

2.1.3. *Relation between the two L-groups.* The relation between these two actions is as follows. We let ρ be the half sum of positive coroots of $\widehat{G}$, and we denote by $\rho \colon \mathbf{G}_m \to \widehat{G}_{\mathrm{ad}}$ the corresponding cocharacter. With $\mathrm{cyc}_p \colon \mathrm{Gal}(\mathbf{F}^s/\mathbf{F}) \to \mathbf{F}_p^\times$ denoting the mod p cyclotomic character, let χ denote the composite

$$ \mathrm{Gal}(\mathbf{F}^s/\mathbf{F}) \xrightarrow{\mathrm{cyc}_p} \mathbf{F}_p^\times \hookrightarrow k^\times \xrightarrow{\rho} \widehat{G}_{\mathrm{ad}}(k). $$

This induces a homomorphism $\mathrm{Ad}_\chi \colon \mathrm{Gal}(\mathbf{F}^s/\mathbf{F}) \to \mathrm{Aut}(\widehat{G})$.

Proposition 2.1. *We have* $\mathrm{act}^{\mathrm{geom}} = \mathrm{act}^{\mathrm{alg}} \circ \mathrm{Ad}_\chi$.

---

[6]This image is denoted $\mathcal{B}$ in [Laf18].

Proof. When $\widehat{G}$ is over $\mathbf{Q}_p$, this is [RZ15, Proposition 1.6]. More generally, it is established in [FS, §VI.11] over any p-adic ring. $\qquad\square$

Given a choice of lift $\widetilde{\chi}\colon \mathrm{Gal}(\mathbf{F}^s/\mathbf{F}) \to \widehat{G}(k)$ of χ, which could for example come from a square root of the mod p cyclotomic character, we get an isomorphism $^LG^{\mathrm{alg}} \xrightarrow{\sim} {}^LG^{\mathrm{geom}}$ by

$$(g,\gamma) \mapsto (g\widetilde{\chi}(\gamma^{-1}),\gamma). \tag{2.1}$$

By [Zhu17, Remark 5.5.8], a square root of the cyclotomic character exists whenever $\mathrm{char}(\mathbf{F}) > 0$. (However, in general it can happen that $^LG^{\mathrm{alg}}$ and $^LG^{\mathrm{geom}}$ are not isomorphic; for an example see [Zhu17, Example 5.5.9].)

At different points we will want to consider both versions of L-groups. If we write ^LG without a superscript, then by default we mean the algebraic L-group $^LG^{\mathrm{alg}}$.

2.1.4. *Representation categories.* For any Galois extension $\mathbf{F}'/\mathbf{F}$ such that $G_{\mathbf{F}'}$ is split, the analogous construction to §2.1.1 gives a "finite form" algebraic L-group $\widehat{G} \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(\mathbf{F}'/\mathbf{F})$. We define the category of (k-linear) algebraic representations of $^LG^{\mathrm{alg}}$ to be

$$\mathrm{Rep}_k(^LG^{\mathrm{alg}}) := \varinjlim_{\mathbf{F}'} \mathrm{Rep}_k(\widehat{G} \rtimes_{\mathrm{act}^{\mathrm{alg}}} \mathrm{Gal}(\mathbf{F}'/\mathbf{F})).$$

Let $\mathrm{Rep}_k(^LG^{\mathrm{geom}}) := \mathrm{Rep}_k(\widehat{G})^{B\,\mathrm{Gal}(\mathbf{F}^s/\mathbf{F}),\mathrm{geom}}$ denote the category of continuously $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$-equivariant objects in $\mathrm{Rep}_k(\widehat{G})$ with respect to the geometric action. The Geometric Satake equivalence induces by descent an equivalence

$$\mathrm{P}_{L^+G}(\mathrm{Gr}_G; k) \cong \mathrm{Rep}_k(\widehat{G})^{B\,\mathrm{Gal}(\mathbf{F}^s/\mathbf{F}),\mathrm{geom}} \tag{2.2}$$

where the action of $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$ on $\mathrm{Rep}_k(\widehat{G})$ on the RHS is via $\mathrm{act}^{\mathrm{geom}}$, and on the LHS, Gr_G is considered over $\mathbf{F}$. By definition, on the right side we take are taking objects on which $\mathrm{Gal}(\mathbf{F}^s/\mathbf{F})$ acts *continuously* with its Krull topology. Since k is algebraic over $\mathbf{F}_p$, *in this case* $\mathrm{Rep}_k(\widehat{G})^{B\,\mathrm{Gal}(\mathbf{F}^s/\mathbf{F}),\mathrm{geom}}$ can be identified with $\varinjlim_{\mathbf{F}'/\mathbf{F}} \mathrm{Rep}_k(\widehat{G})^{B\,\mathrm{Gal}(\mathbf{F}'/\mathbf{F}),\mathrm{geom}}$ where the limit runs over finite Galois extensions $\mathbf{F}'/\mathbf{F}$ over which the geometric action factors.

An isomorphism (2.1) gives an embedding $\mathrm{Rep}_k(^LG^{\mathrm{alg}}) \hookrightarrow \mathrm{Rep}_k(\widehat{G})^{\mathrm{Gal}(\mathbf{F}^s/\mathbf{F}),\mathrm{geom}}$, which as just remarked is an equivalence for our choice of k. See [RZ15, Proposition A.10] for a description of the essential image in general.

2.1.5. *L-parameters.*

Definition 2.2. Let Γ be a topological group and $\overline{\Gamma}$ be a quotient of Γ acting on $\widehat{G}$. A *L-parameter from Γ to $\widehat{G}(k) \rtimes \overline{\Gamma}$* is a $\widehat{G}(k)$-conjugacy class of continuous homomorphisms $\rho\colon \Gamma \to \widehat{G}(k) \rtimes \overline{\Gamma}$, which has the property that the composite map $\Gamma \to \widehat{G} \rtimes \Gamma \to \overline{\Gamma}$ is the given quotient $\Gamma \twoheadrightarrow \overline{\Gamma}$.

Equivalently, we may view ρ as an element of the continuous cohomology group $H^1_{\mathrm{cts}}(\Gamma, \widehat{G}(k))$, where the action of Γ on $\widehat{G}(k)$ is the given one (via $\Gamma \to \overline{\Gamma}$) in the semi-direct product.

We will consider L-parameters with $\widehat{G}(k) \rtimes \Gamma$ being either $^L\widehat{G}^{\mathrm{alg}}(k)$ or $^L\widehat{G}^{\mathrm{geom}}(k)$, and Γ being either $\mathrm{Gal}(F^s/F)$ for a global field F or $\mathrm{Weil}(F_v^s/F_v)$ for a local field F_v.

Note that the algebraic Γ-action on $\widehat{G}(k)$ factors through a finite quotient $\Gamma \twoheadrightarrow \mathrm{Gal}(\mathbf{F}'/\mathbf{F})$. It is clear that L-parameters into $^LG^{\mathrm{alg}}(k)$ are in bijection (under restriction) with L-parameters into $\widehat{G}(k) \rtimes \mathrm{Gal}(\mathbf{F}'/\mathbf{F})$ for any such $\mathbf{F}'$; indeed, the set of all such L-parameters is identified with $H^1_{\mathrm{cts}}(\Gamma, \widehat{G}(k))$.

We say that a representation $\rho\colon \Gamma \to {}^LG^{\mathrm{alg}}(k)$ is *semisimple*[7] if the Zariski-closure of the image of ρ in $\widehat{G}(k) \rtimes \mathrm{Gal}(\mathbf{F}'/\mathbf{F})$, for any finite extension $\mathbf{F}'/\mathbf{F}$ over which the Γ-action factors, has reductive component group.

---

[7]Also called "completely reducible" in [BHKT19].

2.2. Presentation of the excursion algebra.

Let Γ be a group, which is either $\mathrm{Gal}(F^s/F)$ for a global field F or $\mathrm{Weil}(F^s/F)$ for a local field F. Let G be a reductive group over F and ${}^L G^{\mathrm{alg}}$ the algebraic L-group as defined in §2.1.1.

We will define the *excursion algebra* $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ to be the commutative algebra over k presented by explicit generators and relations given below. (The topology on Γ will not be relevant for the definition of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$.) For a more conceptual perspective see [Zhu, §2], wherein the excursion algebra is denoted $k[\mathcal{R}_{\Gamma, {}^L G^{\mathrm{alg}}//\widehat{G}}]$.

2.2.1. *Generators.* We define $\mathcal{O}({}^L G_k^{\mathrm{alg}}) := \varinjlim_{F'/F} \mathcal{O}(\widehat{G}_k \rtimes \mathrm{Gal}(F'/F))$ where the limit runs over finite extensions F'/F over which the Γ-action on $\widehat{G}_k$ factors.

Generators of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ will be denoted $S_{I,f,(\gamma_i)_{i \in I}}$, where the indexing set $(I, f, (\gamma_i)_{i \in I})$ consists of:

(i) I is a finite (possibly empty) set,
(ii) $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^I / \widehat{G}_k) := \mathcal{O}(({}^L G_k^{\mathrm{alg}})^I)^{\widehat{G}_k \times \widehat{G}_k}$, where the quotient is for the actions of $\widehat{G}_k$ by diagonal left and right translation, respectively, and
(iii) $\gamma_i \in \Gamma$ for each $i \in I$.

2.2.2. *Relations.* Next we describe the relations. (Compare [Laf18, §10].)

(i) $S_{\emptyset, f, *} = f(1_G)$, an element of $k \subset \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$.
(ii) The map $f \mapsto S_{I,f,(\gamma_i)_{i \in I}}$ is a k-algebra homomorphism in f, i.e.

$$S_{I, f+f', (\gamma_i)_{i \in I}} = S_{I, f, (\gamma_i)_{i \in I}} + S_{I, f', (\gamma_i)_{i \in I}},$$
$$S_{I, ff', (\gamma_i)_{i \in I}} = S_{I, f, (\gamma_i)_{i \in I}} \cdot S_{I, f', (\gamma_i)_{i \in I}},$$

and

$$S_{I, \lambda f, (\gamma_i)_{i \in I}} = \lambda S_{I, f, (\gamma_i)_{i \in I}} \quad \text{for all } \lambda \in k.$$

(iii) For all maps of finite sets $\zeta \colon I \to J$, all $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^I / \widehat{G}_k)$, and all $(\gamma_j)_{j \in J} \in \Gamma^J$, we have

$$S_{J, f^\zeta, (\gamma_j)_{j \in J}} = S_{I, f, (\gamma_{\zeta(i)})_{i \in I}}$$

where $f^\zeta \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^J / \widehat{G}_k)$ is defined by $f^\zeta((g_j)_{j \in J}) := f((g_{\zeta(i)})_{i \in I})$.

(iv) For all $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^I / \widehat{G}_k)$ and $(\gamma_i)_{i \in I}, (\gamma'_i)_{i \in I}, (\gamma''_i)_{i \in I} \in \Gamma^I$, we have

$$S_{I \sqcup I \sqcup I, \widetilde{f}, (\gamma_i)_{i \in I} \times (\gamma'_i)_{i \in I} \times (\gamma''_i)_{i \in I}} = S_{I, f, (\gamma_i (\gamma'_i)^{-1} \gamma''_i)_{i \in I}},$$

where $\widetilde{f} \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^{I \sqcup I \sqcup I} / \widehat{G}_k)$ is defined by

$$\widetilde{f}((g_i)_{i \in I} \times (g'_i)_{i \in I} \times (g''_i)_{i \in I}) = f((g_i(g'_i)^{-1} g''_i)_{i \in I}).$$

(v) If f is inflated from a function on Γ^I, then $S_{I, f, (\gamma_i)_{i \in I}}$ equals the scalar $f((\gamma_i)_{i \in I})$. More generally, if J is a subset of I and f is inflated from a function on $(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^J / \widehat{G}_k) \times \Gamma^{I \backslash J}$, then we have

$$S_{I, f, (\gamma_i)_{i \in I}} = S_{J, \check{f}, (\gamma_j)_{j \in J}}$$

where $\check{f}((g_j)_{j \in J}) := f((g_j)_{j \in J}, (\gamma_i)_{i \in I \backslash J})$. (Compare [Laf18, p. 164].)

Definition 2.3. The *excursion algebra* $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ is the k-algebra with generators and relations specified as above.

2.3. Constructing Galois representations.

The following result of Lafforgue (generalized to modular coefficients by Böckle-Harris-Khare-Thorne) explains how to obtain Langlands parameters from characters of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$.

Proposition 2.4 ([BHKT19, Theorem 4.5], [Laf18, §13]). *For any character $\nu \colon \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}}) \to k$, there is a semisimple L-parameter $\rho_\nu \colon \Gamma \to {}^L G^{\mathrm{alg}}(k)$ (for the discrete topology on Γ), unique up to conjugation by $\widehat{G}(k)$, which is characterized by the following condition:*
For all $n \in \mathbb{N}$, $f \in \mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}})^{n+1} / \widehat{G}_k)$, and $(\gamma_0, \ldots, \gamma_n) \in \Gamma^{n+1}$, we have

$$\nu(S_{\{0,\ldots,n\}, f, (\gamma_0, \gamma_1, \ldots, \gamma_n)}) = f((\rho_\nu(\gamma_0 \gamma_n), \rho_\nu(\gamma_1 \gamma_n), \ldots, \rho_\nu(\gamma_{n-1} \gamma_n), \rho_\nu(\gamma_n))). \tag{2.3}$$

Remark 2.5. See also [FS, Corollary VIII.4.3] for more perspectives on, and generalizations of, this statement.

Remark 2.6. In Proposition 2.4, the datum of ρ_ν up to conjugation is equivalent to that of a cohomology class $[\rho_\nu] \in H^1(\Gamma, \widehat{G}(k))$ where Γ is given the *discrete* topology.

2.4. **Another presentation for the excursion algebra.** We will now describe a second presentation of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$, following [Laf18, Lemma 0.31], which is more useful for constructing actions of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ in practice.

2.4.1. *Generators.* We take a set of generators indexed by tuples of data of the form $(I, W, x, \xi, (\gamma_i)_{i \in I})$, where:

  (i) I is a finite set,
  (ii) $W \in \mathrm{Rep}_k(({}^L G^{\mathrm{alg}})^I)$ (cf. §2.1.4),
 (iii) $x \in W$ is a vector invariant under the diagonal $\widehat{G}_k$-action,
 (iv) $\xi \in W^*$ is a functional invariant under the diagonal $\widehat{G}_k$-action,
  (v) $\gamma_i \in \Gamma$ for each i.

The corresponding generator of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ will be denoted by $S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}} \in \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$.

2.4.2. *Relations.* Next we describe the relations.

  (i) $S_{\emptyset, x, \xi, \{\}} = \langle x, \xi \rangle$, an element of $k \subset \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$.
  (ii) For any morphism of $({}^L G_k^{\mathrm{alg}})^I$-representations $u \colon W \to W'$ and functional $\xi' \in (W')^*$ invariant under the diagonal $\widehat{G}_k$-action, we have

$$S_{I, W, x, u^*(\xi'), (\gamma_i)_{i \in I}} = S_{I, W', u(x), \xi', (\gamma_i)_{i \in I}}, \tag{2.4}$$

     where $u^* \colon (W')^* \to W^*$ denotes the dual to u.
 (iii) For two tuples $(I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1})$ and $(I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2})$ as in §2.4.1, we have

$$S_{I_1 \sqcup I_2, W_1 \boxtimes W_2, x_1 \boxtimes x_2, \xi_1 \boxtimes \xi_2, (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} \circ S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \tag{2.5}$$

     Letting $\Delta \colon \mathbb{1} \to \mathbb{1} \oplus \mathbb{1}$ be the diagonal inclusion, and $\nabla \colon \mathbb{1} \oplus \mathbb{1} \to \mathbb{1}$ the addition map, we also have

$$S_{I_1 \sqcup I_2, W_1 \oplus W_2, (x_1 \oplus x_2) \circ \Delta, \nabla \circ (\xi_1 \oplus \xi_2), (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} + S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \tag{2.6}$$

     Furthermore, the assignment $(I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}) \mapsto S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}} \in \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ is k-linear in x and ξ.
 (iv) Let $\zeta \colon I \to J$ be a map of finite sets. Suppose $W \in \mathrm{Rep}(({}^L G)^I)$, $x \colon \mathbb{1} \to W|_{\Delta(\widehat{G})}$, $\xi \colon W|_{\Delta(\widehat{G})} \to \mathbb{1}$, and $(\gamma_j)_{j \in J} \in \Gamma^J$. Letting W^ζ be the restriction of W under the functor $\mathrm{Rep}(({}^L G)^I) \to \mathrm{Rep}(({}^L G)^J)$ induced by ζ, we have

$$S_{J, W^\zeta, x, \xi, (\gamma_j)_{j \in J}} = S_{I, W, x, \xi, (\gamma_{\zeta(i)})_{i \in I}}. \tag{2.7}$$

  (v) Letting $\delta_W \colon \mathbb{1} \to W \otimes W^*$ and $\mathrm{ev}_W \colon W^* \otimes W \to 1$ be the natural counit and unit, we have

$$S_{I, W, x, \xi, (\gamma_i (\gamma_i')^{-1} \gamma_i'')_{i \in I}} = S_{I \sqcup I \sqcup I, W \boxtimes W^* \boxtimes W, \delta_W \boxtimes x, \xi \boxtimes \mathrm{ev}_W, (\gamma_i)_{i \in I} \times (\gamma_i')_{i \in I} \times (\gamma_i'')_{i \in I}}. \tag{2.8}$$

 (vi) If W is inflated from a representation of $({}^L G^{\mathrm{alg}})^J \times \Gamma^{I \setminus J}$, then we have

$$S_{I, W, x, \xi, (\gamma_i)_{i \in I}} = S_{J, W|_{({}^L G^{\mathrm{alg}})^J}, ((1_j)_{j \in J}, (\gamma_i)_{i \in I \setminus J}) \cdot x, \xi, (\gamma_j)_{j \in J}}.$$

2.4.3. *Relation between the presentations.* The two presentations in §2.2 and §2.4 are related as follows. The generator $S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}}$ corresponds to $S_{I, f_{x, \xi}, (\gamma_i)_{i \in I}}$ where $f_{x, \xi}$ is the function on $({}^L G_k)^I$ given by $(g_i)_{i \in I} \mapsto \langle \xi, (g_i)_{i \in I} \cdot x \rangle$. The assumptions on ξ and x imply that $f_{x, \xi}$ is invariant under the left and right diagonal $\widehat{G}_k$-actions. The relations in §2.4.2 imply that $S_{I, W, x, \xi, (\gamma_i)_{i \in I}}$ depends only on $f_{x, \xi}$ (and not on the choice of x, ξ) [Laf18, Lemme 10.6].

2.5. **Functoriality for excursion algebras.** A homomorphism of L-groups $\phi\colon {}^L H^{\mathrm{alg}} \to {}^L G^{\mathrm{alg}}$ is *admissible* if it lies over the identity map on Γ, i.e. the diagram below commutes.

$$\begin{array}{ccc} {}^L H^{\mathrm{alg}} & \xrightarrow{\phi} & {}^L G^{\mathrm{alg}} \\ \downarrow & & \downarrow \\ \Gamma & \xLongequal{\mathrm{Id}} & \Gamma \end{array}$$

Lemma 2.7. *Let $\phi\colon {}^L H^{\mathrm{alg}} \to {}^L G^{\mathrm{alg}}$ be an admissible homomorphism. Then there is a homomorphism $\phi^*\colon \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}}) \to \mathrm{Exc}(\Gamma, {}^L H^{\mathrm{alg}})$ which on k-points sends $\rho \in H^1(\Gamma, \widehat{H}(k))$ to $\phi \circ \rho \in H^1(\Gamma, \widehat{G}(k))$.*

Proof. The map ϕ induces $\mathrm{Res}_\phi\colon \mathrm{Rep}_k({}^L G^{\mathrm{alg}}) \to \mathrm{Rep}_k({}^L H^{\mathrm{alg}})$. At the level of generators, the map ϕ^* sends

$$S_{V,x,\xi,\{\gamma\}_{i\in I}} \mapsto S_{\mathrm{Res}_\phi(V), \mathrm{Res}_\phi(x), \mathrm{Res}_\phi(\xi), \{\gamma_i\}_{i\in I}}.$$

We verify by inspection that this map sends relations to relations. To see that this indeed induces composition with ϕ at the level of Langlands parameters, use (2.3). $\square$

Definition 2.8 (Base change). In the base change situation, where H is a reductive group over F and $G = \mathrm{Res}_{E/F}(H_E)$, the relevant morphism of L-groups $\phi_{\mathsf{BC}}\colon {}^L H^{\mathrm{alg}} \to {}^L G^{\mathrm{alg}}$ is defined by the formula $(h,\gamma) \mapsto (\Delta(h), \gamma)$. In fact this same formula also defines the corresponding map of geometric L-groups $\phi_{\mathsf{BC}}^{\mathrm{geom}}\colon {}^L H^{\mathrm{geom}} \to {}^L G^{\mathrm{geom}}$, so $\phi_{\mathsf{BC}}^{\mathrm{geom}}$ and ϕ_{BC} are compatible with (2.1) if we use the same choice of square root of the cyclotomic character in the latter to define isomorphisms ${}^L H^{\mathrm{alg}} \approx {}^L H^{\mathrm{geom}}$ and ${}^L G^{\mathrm{alg}} \approx {}^L G^{\mathrm{geom}}$. We denote

$$\phi_{\mathsf{BC}}^*\colon \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}}) \to \mathrm{Exc}(\Gamma, {}^L H^{\mathrm{alg}})$$

the induced map of excursion algebras.

3. Smith theory in infinite type

Classical Smith theory concerns a type of equivariant localization that relates the mod p cohomology of a topological space with the mod p cohomology of its fixed points under a $\mathbf{Z}/p\mathbf{Z}$-action. Treumann proposed in [Tre19] that this could be understood in terms of a "sheaf-theoretic Smith theory" formalism, which he developed at least in the context of complex algebraic varieties in the analytic topology. An algebraic version of this theory was built in [RW] for p-adic étale sheaves on finite type schemes (over fields where p is invertible). We will need generalizations of this theory from finite type to *locally finite type*. This is because we will want to apply the theory to the moduli spaces of shtukas, which are of infinite type, but locally of finite type.

Let us comment on some of the technical issues that arise in doing so. Because the cohomology of locally finite type schemes is not necessarily finite-dimensional, already the basic formalism of constructible sheaves and perfect complexes from [Tre19, RW] does not apply. For example, we will have to enlarge the notion of "Tate category" to encompass the objects of interest.

We do not strive for the maximum possible generality, but our theory at least encompass all examples of interest that will show up in this paper. In particular, we will use tricks to avoid discussing Smith theory for *stacks*, which presents an interesting problem that could potentially refine our applications. For steps that are very similar to the case of finite type schemes as treated already in [RW], we will only sketch the proofs.

3.1. **The Tate category.** Let Λ be a p-adic coefficient ring; we will be interested in the cases where $\Lambda = k$ or $W(k)$. We will denote by $\Lambda[\sigma]$ the group ring of $\langle\sigma\rangle$ with coefficients in Λ. Our geometric objects will be over a field of characteristic $\neq p$ and we will consider Λ-adic sheaves.

For a separated, locally finite type scheme Y with a σ-action, there is an equivariant bounded derived category $D^b_\sigma(Y; \Lambda)$. We distinguish this from the equivariant bounded constructible derived category $D^b_{c,\sigma}(Y; \Lambda)$, a full subcategory of $D^b_\sigma(Y; \Lambda)$ that will also be of interest to us. If σ acts *trivially* on Y, then we have an equivalence of derived categories

$$D^b_\sigma(Y; \Lambda) \cong D^b(Y; \Lambda[\sigma]), \quad \text{and} \quad D^b_{c,\sigma}(Y; \Lambda) \cong D^b_c(Y; \Lambda[\sigma]). \tag{3.1}$$

We let $\mathrm{Perf}(Y; \Lambda[\sigma]) \subset D^b_c(Y^\sigma; \Lambda[\sigma])$ be the full subcategory consisting of complexes whose stalks at all geometric points of Y are perfect.

Definition 3.1. We define $\mathrm{Flat}^b(Y;\Lambda[\sigma]) \subset D^b(Y^\sigma;\Lambda[\sigma])$ to be the full subcategory consisting of bounded complexes whose stalks at all geometric points of Y are represented by bounded complexes of flat (but not necessarily finite) $\Lambda[\sigma]$-modules.

The following Lemma will not be used in this paper, but may help to clarify the nature of $\mathrm{Flat}^b(Y;k[\sigma])$. We thank Jesper Grodal for pointing out a gap in the original argument and also for suggesting its fix.

Lemma 3.2. *The subcategory* $\mathrm{Flat}^b(Y^\sigma;k[\sigma]) \subset D^b(Y^\sigma;k[\sigma])$ *coincides with the full subcategory of objects which locally have finite Tor-amplitude over* $k[\sigma]$.

Proof. For any commutative ring A, a complex of A-modules has finite Tor-amplitude if and only if it is represented by a bounded complex of flat A-modules [Sta20, Tag 08G1]. Furthermore, a complex has Tor-amplitude in $[a, b]$ if and only if all its stalks at geometric points have Tor-amplitude in $[a, b]$, by [Sta20, Tag 0DJJ]. This shows that any object of $D^b(Y^\sigma;k[\sigma])$ with locally finite Tor-amplitude over $k[\sigma]$ lies in $\mathrm{Flat}^b(Y^\sigma;k[\sigma])$.

The converse is subtler: we need to show that there is a "uniform" bound on the Tor-amplitude of the stalks. Pick a complex $\mathcal{K} \in \mathrm{Flat}^b(Y^\sigma;k[\sigma])$. Since $\mathcal{K}$ is bounded, locally on Y^σ it is represented by a bounded complex, say supported in degrees $[a, b]$. The key observation is: if $\mathcal{K} \in D^b(Y^\sigma;k[\sigma])$ is represented by a global complex concentrated in degrees $[a, b]$ and all its stalks at geometric points have finite Tor-amplitude, then in fact all of its stalks at geometric points have Tor-amplitude in $[a, b]$. Given this observation, we may conclude by using [Sta20, Tag 0DJJ].

Next we prove the observation. It suffices to show that if $\mathcal{K}$ is a complex over $k[\sigma]$ supported in degrees $[a, b]$ which has finite Tor-amplitude, then $\mathcal{K}$ has Tor-amplitude $[a, b]$. Since k is Artinian, $k[\sigma]$ is also Artinian. For a module over an Artinian local ring, the properties of being flat and projective coincide [Sta20, Tag 051E]. Therefore $\mathcal{K}$ also has finite projective dimension [Sta20, Tag 0A5M]. Furthermore, since $k[\sigma]$ has finitistic dimension zero, the projective amplitude of $\mathcal{K}$ lies in $[a, b]$. Hence $\mathcal{K}$ is represented by a complex of projective $k[\sigma]$-modules supported in degrees $[a, b]$, and therefore has Tor-dimension in $[a, b]$. $\square$

Definition 3.3. The *(constructible) Tate category of* Y (with respect to Λ) is the Verdier quotient category $D_c^b(Y;\Lambda[\sigma])/\mathrm{Perf}(Y;\Lambda[\sigma])$.

This is the construction considered under the name "Tate category" in [Tre19], at least when Y is a complex-analytic variety. According to [Tre19, Remark 4.1], the category $D_c^b(Y;\Lambda[\sigma])/\mathrm{Perf}(Y;\Lambda[\sigma])$ can be regarded as a derived category of perfect complexes over a certain "E_∞-ring spectrum" $\mathcal{T}_\Lambda$. So we will denote the corresponding Tate categories by $\mathrm{Perf}(Y;\mathcal{T}_\Lambda)$. For our purposes $\mathcal{T}_\Lambda$ can be thought of as just a notational device.

We will require the following enlargement of the constructible Tate category. We define the *(bounded ind-constructible) Tate category of* Y (with respect to Λ) to be the Verdier quotient category

$$\mathrm{Shv}(Y;\mathcal{T}_\Lambda) := D^b(Y;\Lambda[\sigma])/\mathrm{Flat}^b(Y;\Lambda[\sigma]).$$

We denote the tautological projection maps from $D_c^b(Y;\Lambda[\sigma])$ to $\mathrm{Perf}(Y;\mathcal{T}_\Lambda)$, and from $D^b(Y;\Lambda[\sigma])$ to $\mathrm{Shv}(Y;\mathcal{T}_\Lambda)$ by

$$\mathbb{T}\colon D^b(Y;\Lambda[\sigma]) \to \mathrm{Shv}(Y;\mathcal{T}_\Lambda), \quad \text{and} \quad \mathbb{T}\colon D_c^b(Y;\Lambda[\sigma]) \to \mathrm{Perf}(Y;\mathcal{T}_\Lambda).$$

Note that the inclusion $D_c^b(Y;\Lambda[\sigma]) \to D^b(Y;\Lambda[\sigma])$ carries $\mathrm{Perf}(Y;\Lambda[\sigma])$ into $\mathrm{Flat}^b(Y;\Lambda[\sigma])$ and so induces a functor

$$\mathrm{Perf}(Y;\mathcal{T}_\Lambda) \to \mathrm{Shv}(Y;\mathcal{T}_\Lambda), \tag{3.2}$$

which is conservative (because $\mathrm{Perf}(Y;\Lambda[\sigma]) \subset D_c^b(Y;\Lambda[\sigma])$ can also be characterized as the full subcategory of objects having finite Tor-amplitude over $\Lambda[\sigma]$).

Example 3.4 ([Tre19, Proposition 4.2]). The (bounded ind-constructible) Tate category over a point (meaning the spectrum of a separably closed field) is $D^b(\Lambda[\sigma])/\mathrm{Flat}(\Lambda[\sigma])$. In this category the shift-by-2 functor is isomorphic to the identity functor, as one sees by considering the nullhomotopic complex

$$0 \to V \to V \otimes \Lambda[\sigma] \xrightarrow{1-\sigma} V \otimes \Lambda[\sigma] \to V \to 0$$

whose middle two terms project to 0 in the Tate category.

3.2. **The Smith operation.** Let Y be a locally finite type scheme with a σ-action that is *admissible* in the sense of [SGA03, Exposé 5, Définition 1.7]. By [RW, Remark 2.3], this is automatic if Y is exhausted by quasi-projective schemes over a field. Then the "Smith operation" (cf. [Tre19, Definition 4.2]) is the functor

$$\text{Psm} := \mathbb{T} \circ i^* \colon D_{c,\sigma}^b(Y; \Lambda) \to \text{Perf}(Y^\sigma; \mathcal{T}_\Lambda) \tag{3.3}$$

defined as the composition of $i^* \colon D_{c,\sigma}^b(Y; \Lambda) \to D_{c,\sigma}^b(Y^\sigma; \Lambda) \overset{(3.1)}{\cong} D_c^b(Y^\sigma; \Lambda[\sigma])$ with the projection $\mathbb{T}$ to $\text{Perf}(Y^\sigma; \mathcal{T}_\Lambda)$.

We extend this definition to bounded ind-constructible Tate categories in the analogous manner, defining

$$\text{Psm} := \mathbb{T} \circ i^* \colon D_\sigma^b(Y; \Lambda) \to \text{Shv}(Y^\sigma; \mathcal{T}_\Lambda). \tag{3.4}$$

For $\mathcal{F} \in D_{c,\sigma}^b(Y; \Lambda)$, there is potential confusion about whether "$\text{Psm}(\mathcal{F})$" denotes the result of applying (3.3) or (3.4). But there is a natural isomorphism between the functors

$$D_{c,\sigma}^b(Y; \Lambda) \xrightarrow{\text{Psm}} \text{Perf}(Y^\sigma; \mathcal{T}_\Lambda) \xrightarrow{(3.2)} \text{Shv}(Y^\sigma; \mathcal{T}_\Lambda)$$

and

$$D_{c,\sigma}^b(Y; \Lambda) \xrightarrow{(3.2)} D_\sigma^b(Y; \Lambda) \xrightarrow{\text{Psm}} \text{Shv}(Y^\sigma; \mathcal{T}_\Lambda),$$

so the meaning is unambiguous once the ambient category is specified. When the distinction is important, we will take care to specify the ambient category.

The following properties are used to prove that our extended version of Psm retains the good behavior enjoyed by the constructible version.

Lemma 3.5. *Retain the notation and assumptions above. Assume that the σ-action on Y is free. Let $q \colon Y \to Y/\sigma$ denote the quotient (which exists as a map of schemes by admissibility of the σ-action on Y). Then for any $\mathcal{F} \in D^b(Y; \Lambda[\sigma])$, we have $q_* \mathcal{F} \in D^b(Y/\sigma; \Lambda[\sigma]) \in \text{Flat}^b(Y/\sigma; \Lambda[\sigma])$.*

Proof. The same argument as [RW, Lemma 2.4] works here. To summarize it: for any geometric point $\overline{y} \to Y/\sigma$, and $\overline{x} \to Y$ lifting it, we have

$$(q_* \mathcal{F})_{\overline{y}} \cong \mathcal{F}_{\overline{x}} \otimes_\Lambda \Lambda[\sigma],$$

which is visibly in $\text{Flat}^b(\overline{y}; \Lambda[\sigma])$. $\square$

Lemma 3.6. *Retain the notation and assumptions above. Let $U := Y \setminus Y^\sigma$ be the open complement of the σ-fixed locus of Y, and $j \colon U \hookrightarrow Y$ be its inclusion into Y. Then for any $\mathcal{F} \in D^b(Y; \Lambda[\sigma])$, and any geometric point $\overline{y}$ of Y^σ, the stalk $(Rj_* \mathcal{F})_{\overline{y}}$ lies in $\text{Flat}^b(\overline{y}; \Lambda[\sigma]) \subset D^b(\overline{y}; \Lambda[\sigma])$.*

Proof. A similar argument as in [RW, Proposition 2.6] works here. Since the map $q \colon Y \to Y/\sigma$ is totally ramified at $\overline{y}$, we have a σ-equivariant identification $(Rj_* \mathcal{F})_{\overline{y}} \cong (q_* Rj_* \mathcal{F})_{q(\overline{y})}$. Then by the commutativity of the diagram

$$\begin{array}{ccc} U & \overset{j}{\hooklongrightarrow} & Y \\ {\scriptstyle q_U}\downarrow & & \downarrow{\scriptstyle q} \\ U/\sigma & \overset{\overline{j}}{\hooklongrightarrow} & Y/\sigma \end{array}$$

we have $(q_* Rj_* \mathcal{F})_{q(\overline{y})} \cong (R\overline{j}_* q_{U*} \mathcal{F})_{q(\overline{y})}$. Now Lemma 3.5 implies that $q_{U*} \mathcal{F}$ has finite Tor-amplitude, and combining [Sta20, Tag 0F10] with [SGA73, Exposé XVII, Théorème 5.2.11] implies that $R\overline{j}_*$ preserves finiteness of Tor-amplitude, so their composition has locally finite Tor-amplitude. $\square$

The good properties of Psm come from the following Lemma, which was proved for finite type schemes in [RW, Lemma 3.5] (following [Tre19, Theorem 4.7] in the topological situation).

Lemma 3.7. *Retain the notation and assumptions above. Suppose Y is locally of finite type and let $i \colon Y^\sigma \hookrightarrow Y$. Then for any $\mathcal{F} \in D_\sigma^b(Y; \Lambda)$, the cone of $i^! \mathcal{F} \to i^* \mathcal{F}$ belongs to $\text{Flat}^b(Y^\sigma; \Lambda[\sigma])$.*

Proof. Consider the exact triangle $i_* i^! \mathcal{F} \to \mathcal{F} \to j_* j^* \mathcal{F}$ on Y. Applying i^* to it yields the exact triangle in $D^b(Y^\sigma; \Lambda[\sigma])$:

$$i^! \mathcal{F} \to i^* \mathcal{F} \to i^* j_* j^* \mathcal{F}.$$

By Lemma 3.6, $i^* j_* j^* \mathcal{F} \in \text{Flat}^b(Y^\sigma; \Lambda[\sigma])$. $\square$

Lemma 3.8. *Suppose* $f \colon Y \to Z$ *is a locally finite type and separated morphism between locally finite type schemes, of bounded dimension. Then* $Rf_! \colon D^b(Y; \Lambda[\sigma]) \to D^b(Z; \Lambda[\sigma])$ *carries* $\mathrm{Flat}^b(Y; \Lambda[\sigma])$ *to* $\mathrm{Flat}^b(Z; \Lambda[\sigma])$.

Proof. We may write Y as a filtered colimit of open subschemes Y_α of finite type. Then for $\mathcal{F} \in D^b(Y; \Lambda[\sigma])$, we have an identification of $Rf_! \mathcal{F}$ with the colimit over $Rf_!(\mathcal{F}|_{Y_\alpha})$. Since filtered colimits preserve flatness, we are reduced to the same statement in the finite type situation (where one can also replace "Flat^b" by Perf), which is obtained by combining [Sta20, Tag 0F10] and [SGA73, Exposé XVII, Théorème 5.2.11]. $\qquad\square$

Remark 3.9. Note that Lemma 3.8 would *not* have been true with "Flat^b" replaced by "Perf". This is why we need to consider ind-constructible sheaves when not in a finite type situation.

3.3. **Functors on Tate categories.** Let $f \colon Y \to S$ be a σ-equivariant locally finite type morphism of locally finite type schemes with admissible σ-action.

3.3.1. *Pullback.* Since $f^* \colon D^b_\sigma(S^\sigma; k) \to D^b_\sigma(Y^\sigma; k)$ preserves stalks, it preserves flat and perfect complexes, and so descends to the Tate category to induce $f^* \colon \mathrm{Shv}(S^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda)$ and $f^* \colon \mathrm{Perf}(S^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda)$.

3.3.2. *Proper pushforward.* In the situation of Lemma 3.8, $Rf_! \colon D^b(Y^\sigma; \Lambda[\sigma]) \to D^b(S^\sigma; \Lambda[\sigma])$ descends to

$$f_! \colon \mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Shv}(S^\sigma; \mathcal{T}_\Lambda).$$

Proposition 3.10. *Let* $f \colon Y \to S$ *be as in Lemma 3.8. Then the following diagram commutes:*

$$
\begin{array}{ccc}
D^b_\sigma(Y; \Lambda) & \xrightarrow{\;f_!\;} & D^b_\sigma(S; \Lambda) \\
\downarrow{\scriptstyle\mathrm{Psm}} & & \downarrow{\scriptstyle\mathrm{Psm}} \\
\mathrm{Shv}(Y^\sigma; \mathcal{T}_\Lambda) & \xrightarrow{\;f_!\;} & \mathrm{Shv}(S^\sigma; \mathcal{T}_\Lambda)
\end{array}
$$

Proof. We may as well replace S by S^σ and thus assume that the σ-action on S is trivial. Let $\mathcal{F} \in D^b_\sigma(Y; \Lambda)$. Denoting $i \colon Y^\sigma \hookrightarrow Y$ and j the inclusion of the open complement, we have a distinguished triangle in $D^b_\sigma(Y; \Lambda)$:

$$j_! j^* \mathcal{F} \to \mathcal{F} \to i_* i^* \mathcal{F}.$$

Abbreviate $f^\sigma := f \circ i \colon Y^\sigma \to S$. By definition σ acts freely on U, which implies that $f_! \circ (j_! j^* \mathcal{F}) \in \mathrm{Flat}^b(S; \Lambda[\sigma])$ by Lemma 3.8. Hence the cone of $f_! \mathcal{F} \to f^\sigma_!(i^* \mathcal{F})$ lies in $\mathrm{Flat}^b(S; \Lambda[\sigma])$, and therefore becomes 0 in $\mathrm{Shv}(S; \mathcal{T}_\Lambda)$. Hence we have

$$\mathbb{T}(f_! \mathcal{F}) \cong \mathbb{T}(f^\sigma_!(i^* \mathcal{F})) \cong f_! \, \mathrm{Psm}(\mathcal{F}) \in \mathrm{Shv}(S; \mathcal{T}_\Lambda),$$

which exactly expresses the desired commutativity. $\qquad\square$

3.4. **Tate cohomology.** For a $\Lambda[\sigma]$-module M, its *Tate cohomology groups* are

$$T^0(M) := \frac{M^\sigma}{N \cdot M}, \quad T^1(M) := \frac{\ker(N \colon M \to M)}{(1 - \sigma) \cdot M},$$

(Recall that $N := 1 + \sigma + \ldots + \sigma^{p-1}$.) We will generalize this to complexes and then sheaves.

3.4.1. *Tate cohomology of complexes.* Given a bounded-below complex of $\Lambda[\sigma]$-modules $C^\bullet$, we define its *Tate cohomology* as in [LL, §3.3]. Because of the importance of this notion for us, we will spell out some of the details.

The exact sequence

$$0 \to \Lambda \to \Lambda[\sigma] \xrightarrow{1-\sigma} \Lambda[\sigma] \to \Lambda \to 0$$

induces a morphism in the derived category of $\Lambda[\sigma]$-modules,

$$\Lambda \to \Lambda[2] \in D^b(\Lambda[\sigma]). \tag{3.5}$$

Consider the double complex below, where N denotes multiplication by $1 + \sigma + \ldots + \sigma^{p-1}$ (cf. §1.6)

$$(3.6)$$

We define $H^n(\epsilon^! C^\bullet)$ to be the nth cohomology group of the totalization of this double complex. We define $T^i(C^\bullet)$ to be $\varinjlim_n H^{i+2n}(\epsilon^! C^\bullet)$, where the transition maps are induced by (3.5).

If $C^\bullet$ is bounded, the double complex (3.6) is eventually periodic, and $T^i(C^\bullet)$ can be computed as the ith cohomology group of the totalization of the double complex $\mathrm{Tate}(C^\bullet)$ below:

$$(3.7)$$

The formation of Tate cohomology descends to the derived category, so we can view Tate cohomology as a collection of functors

$$T^i \colon D^b(k[\sigma]) \to \mathrm{Vect}_{/k}.$$

The functors T^i are evidently 2-periodic, i.e. $T^i \cong T^{i+2}$. Now we specialize to the case where $\Lambda = k$. Note that by [Sta20, Tag 051E], a module over $k[\sigma]$ is flat if and only if it is free. Since Tate cohomology of free $k[\sigma]$-complexes vanishes (by inspection), this construction further factors through the Tate category, inducing

$$T^i \colon \mathrm{Shv}(\mathcal{T}_k) \to \mathrm{Vect}_{/k}.$$

Lemma 3.11. *Suppose $C^\bullet \in D^b(k[\sigma])$ is inflated from $D^b(k)$, i.e. σ acts trivially on $C^\bullet$. Then $T^* C^\bullet \cong H^*(C^\bullet) \otimes T^*(k)$, where k is equipped with the trivial σ-action in the formation of $T^*(k)$.*

Proof. In this case (3.7) decomposes as the tensor product of $C^\bullet$ and the Tate double complex for k; the result then follows from the Künneth theorem. $\square$

3.4.2. *The long exact sequence for Tate cohomology.* Given a distinguished triangle $\mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \in D^b(k[\sigma])$, we have a long exact sequence

$$\ldots T^{-1}\mathcal{F}'' \to T^0\mathcal{F}' \to T^0\mathcal{F} \to T^0\mathcal{F}'' \to T^1\mathcal{F}' \to T^1\mathcal{F} \to T^1\mathcal{F}'' \to T^2\mathcal{F}' \to \ldots$$

3.4.3. *Tate cohomology sheaves.* Let S be a locally finite type scheme, viewed with the trivial σ-action. Then $\mathrm{Shv}(S; \mathcal{T}_k)$ is defined. Given $\mathcal{F} \in \mathrm{Shv}(S; \mathcal{T}_k)$, we have by an analogous construction to (3.7) *Tate cohomology sheaves* $T^i\mathcal{F}$ on S, which are étale sheaves of $T^0(k) = k$-modules.

3.4.4. *Tate cohomology for a morphism.* Let $f\colon Y \to S$ be as in Lemma 3.8. For $\mathcal{F} \in D^b_\sigma(Y;k)$, we have $Rf_!\mathcal{F} \in D^b_\sigma(S;k)$. We will abbreviate $T^i(Y;\mathcal{F}) := T^i(Rf_!\mathcal{F})$, and call it the "relative Tate cohomology of Y with coefficients in $\mathcal{F}$".

3.4.5. *The Tate cohomology spectral sequence.* If $C^\bullet$ is bounded, then the double complex (3.7) induces a spectral sequence
$$E^{ij}_1 = H^j(C^\bullet) \implies T^{i+j}(C^\bullet).$$
The second page is $E^{ij}_2 = T^i(H^j(C^\bullet))$. Hence we find that the Tate cohomology of $C^\bullet$ has a filtration whose graded pieces are subquotients of the ordinary cohomology $H^j(C^\bullet)$.

3.4.6. *Equivariant localization.* We explain that Proposition 3.10 encompasses the classical equivariant localization theorems of "Smith theory", e.g., [Qui71, Theorem 4.2]. Let $f\colon Y \to S$ be a morphism over a separably closed field L, satisfying the conditions of Proposition 3.10, so that for $\mathcal{F} \in D^b_\sigma(Y;k)$ we have
$$\mathbb{T}(f_!\mathcal{F}) \cong (f|_{Y^\sigma})_!\operatorname{Psm}(\mathcal{F}) \in \operatorname{Shv}(S;\mathcal{T}_k).$$
In particular, taking $S = \operatorname{Spec} L$ (with the trivial action of σ) and then applying Tate cohomology, we obtain
$$T^i(Y;\mathcal{F}) \cong T^i(Y^\sigma;\operatorname{Psm}(\mathcal{F})). \tag{3.8}$$

4. Parity sheaves and the base change functor

We begin by indicating where this section is headed.

The Geometric Satake equivalence $\mathrm{P}_{L^+G}(\mathrm{Gr}_G;k) \cong \operatorname{Rep}_k(\widehat{G})$ provides the link between G and its Langlands dual group. In the situation of functoriality, we have a map $\widehat{H} \to \widehat{G}$ and we would like to describe the induced restriction operation $\operatorname{Rep}_k(\widehat{G}) \to \operatorname{Rep}_k(\widehat{H})$ on the other side of the Geometric Satake equivalence, as a geometric operation on perverse sheaves.

In the context of base change it is even the case that there is an embedding $\mathrm{Gr}_H \hookrightarrow \mathrm{Gr}_G$, and when seeking to describe functoriality it is natural to look to the Smith operation. (One motivation is that the papers [Tre19, TV16] verify that the *function-theoretic* Smith operation is indeed related to functoriality for Hecke algebras.) However, the Smith operation lands in a Tate category, and in Example 3.4 we saw that in the Tate category, the shift-by-2 functor is isomorphic to the identity functor. This makes it seem unlikely that one can capture the notion of "perverse sheaf" in the Tate category.

Juteau-Mautner-Williamson invented the theory of *parity sheaves*, which could be seen as a variant of perverse sheaves that seems to behave better in the setting of modular coefficients. Parity sheaves are cut out in the derived category by constraints on the parity of cohomological degrees, and can therefore make sense in a context where cohomological degrees are only defined modulo 2. The notion of *Tate-parity sheaves* was introduced in [LL] as an analog of parity sheaves for the Tate category, and was found to enjoy analogous properties.

After briefly reviewing the notions of parity and Tate-parity sheaves in §4.1 and §4.2, we will establish that the Smith operation respects the parity property, at least under certain conditions satisfied in our application of interest. Using "coefficient lifting" properties of parity sheaves, this will allow us to ultimately define a functor BC from parity sheaves on Gr_G to parity sheaves on Gr_H, which realizes base change functoriality on the geometric side. We note that in this section, we will only need the "constructible" version of Smith theory for schemes, and not the generalizations developed in §3.

4.1. Parity sheaves.
We begin with a quick review of the theory of parity sheaves. We will take coefficients in a ring Λ, which in our applications of interest will be either k or $\mathbb{O} := W(k)$.

Let Y be a stratified variety over a separably closed field of characteristic $\neq p$, with stratification $S = \{Y_\lambda\}$. For the theory to work, we need to assume that the (induced) stratification on Y is JMW, meaning:

- for any two local systems $\mathcal{L}, \mathcal{L}'$ on a stratum Y_λ, we have $\operatorname{Ext}^i(\mathcal{L}, \mathcal{L}')$ is free over Λ for all i, and vanishes when i is odd.

This holds for Kac-Moody flag varieties over separably closed fields, and in particular for affine flag varieties over separably closed fields [JMW14, §4.1].

Fix a *pariversity* $\dagger\colon S \to \mathbf{Z}/2\mathbf{Z}$. In this paper we will always take the *dimension pariversity* $\dagger(\lambda) := \dim Y_\lambda$ mod 2, so we will sometimes omit the pariversity from the discussion. Recall that [JMW14] define *even complexes* (with respect to the pariversity $\dagger$) to be those $\mathcal{F} \in D^b_S(Y;\Lambda)$ such that for all $i_\lambda\colon Y_\lambda \hookrightarrow Y$, for

$\lambda \in S$, $i_\lambda^* \mathcal{F}$ and $i_\lambda^! \mathcal{F}$ have cohomology sheaves supported in degrees congruent to $\dagger(\lambda)$ modulo 2, and *odd* complexes analogously. They define *parity complexes* to be direct sums of even and odd complexes. The full subcategory of (S-constructible) Tate-parity complexes (with coefficients in Λ) is denoted $\mathrm{Parity}_S(Y; \Lambda)$.

Theorem 4.1 ([JMW14, Theorem 2.12]). *Let $\mathcal{F}$ be an indecomposable parity complex. Then:*
- *$\mathcal{F}$ has irreducible support, which is therefore of the form $\overline{Y}_\lambda$ for some $\lambda \in \Lambda$,*
- *$i_\lambda^* \mathcal{F}$ is a shifted local system $\mathcal{L}[m]$, and*
- *Any indecomposable parity complex supported on $\overline{Y}_\lambda$ and extending $\mathcal{L}[m]$ is isomorphic to $\mathcal{F}$.*

A *parity* sheaf (with respect to $\dagger$) is an indecomposable parity complex (with respect to $\dagger$) with Y_λ the dense stratum in its support and extending $\mathcal{L}[\dim Y_\lambda]$. Given $\mathcal{L}[\dim Y_\lambda]$, it is not clear in general that a parity sheaf extending it extends exists. If it does exist, then Theorem 4.1 guarantees its uniqueness, and we denote it by $\mathcal{E}(\lambda, \mathcal{L})$. The existence is guaranteed for Gr_G with the usual stratification by L^+G-orbits; $\mathcal{E}(\lambda, \mathcal{L})$ is moreover L^+G-equivariant if p is not a torsion prime for G [JMW16, Theorem 1.4]. If $\mathcal{E}(\lambda, \mathcal{L})$ exists for all λ and $\mathcal{L}$, we will say that "all parity sheaves exist".

4.2. **Tate-parity sheaves.** As we have seen, the cohomological grading in the Tate category is only well-defined modulo 2, so it does not seem to make sense to talk about perverse sheaves in the Tate category. However, elements of the Tate category have Tate cohomology sheaves (§3.4.3), which are indexed by $\mathbf{Z}/2\mathbf{Z}$, so it *could* make sense to talk about an analog of parity sheaves in the Tate category. As [LL] observed, for this to work we must take coefficients in the *integral* version of the Tate category, meaning $\Lambda = \mathbb{O} = W(k)$, because then

$$\mathrm{Ext}^*_{\mathrm{Perf}(\mathcal{T}_\mathbb{O})}(\mathbb{T}(\mathbb{O}), \mathbb{T}(\mathbb{O})) = \bigoplus_{i \in \mathbf{Z}} k[2i] \tag{4.1}$$

is supported in even degrees. This is necessary for the assumption of non-vanishing odd Exts in the definition of the JMW stratification.

For a stratification S on Y, we define $\mathrm{Perf}_S(Y; \mathcal{T}_\mathbb{O}) \subset \mathrm{Perf}(Y; \mathcal{T}_\mathbb{O})$ to be the full subcategory generated by objects in $D_S^b(Y; \mathbb{O}[\sigma])$. Letting $\mathrm{Perf}_S(Y; \mathbb{O}[\sigma]) \subset \mathrm{Perf}(Y; \mathbb{O}[\sigma])$ be the full thick subcategory of S-constructible objects, we have by [LL, Corollary 4.7] that

$$D_S^b(Y; \mathbb{O}[\sigma]) / \mathrm{Perf}_S(Y; \mathbb{O}[\sigma]) \xrightarrow{\sim} \mathrm{Perf}_S(Y; \mathcal{T}_\mathbb{O}).$$

Definition 4.2 ([LL, Definition 5.3]). Let $\mathcal{F} \in \mathrm{Perf}_S(Y; \mathcal{T}_\mathbb{O})$. Fix a pariversity $\dagger \colon S \to \mathbf{Z}/2\mathbf{Z}$. Let $? \in \{*, !\}$.
(1) We say $\mathcal{F}$ is *?-Tate-even* (with respect to $\dagger$) if for each $\lambda \in S$, we have
$$T^{\dagger(\lambda)+1}(i_\lambda^? \mathcal{F}) = 0.$$
(2) We say $\mathcal{F}$ is *?-Tate-odd* (with respect to $\dagger$) if $\mathcal{F}[1]$ is *?-Tate-even*.
(3) We say $\mathcal{F}$ is *Tate-even* (resp. *Tate-odd*) if $\mathcal{F}$ is both $*$-Tate even (resp. odd) and $!$-Tate even (resp. odd).
(4) We say $\mathcal{F}$ is *Tate-parity complex* (with respect to $\dagger$) if it is isomorphic within $\mathrm{Perf}_S(Y; \mathcal{T}_\mathbb{O})$ to the direct sum of a Tate-even complex and a Tate-odd complex.[8]

The full subcategory of (S-equivariant) Tate-parity complexes (with coefficients in $\mathcal{T}_\mathbb{O}$) is denoted $\mathrm{Parity}_S(Y; \mathcal{T}_\mathbb{O})$.

Parallel to Theorem 4.1, we have the following result in this context:

Proposition 4.3 ([LL, Theorem 4.13]). *Let $\mathcal{F}$ be an indecomposable Tate-parity complex.*
(1) The support of $\mathcal{F}$ is of the form $\overline{Y}_\lambda$ for a unique stratum Y_λ.
(2) Suppose $\mathcal{G}$ and $\mathcal{F}$ are two indecomposable Tate-parity complexes such that $\mathrm{supp}(\mathcal{G}) = \mathrm{supp}(\mathcal{F})$. Letting $j_\lambda \colon Y_\lambda \hookrightarrow Y$ be the inclusion of the unique stratum open in this support, if $j_\lambda^* \mathcal{G} \cong j_\lambda^* \mathcal{F}$ then $\mathcal{G} \cong \mathcal{F}$.

Proof. The same argument as in [JMW14, Theorem 2.12] works. $\qquad\square$

We define $\epsilon_* \colon D_c^b(Y; \mathbb{O}) \to D_c^b(Y; \mathbb{O}[\sigma])$ for the inflation through the augmentation $\mathbb{O}[\sigma] \twoheadrightarrow \mathbb{O}$. Recall that $\mathbb{T} \colon D_c^b(Y; \mathbb{O}[\sigma]) \to \mathrm{Perf}(Y; \mathcal{T}_\mathbb{O})$ denotes projection to the Tate category. We are interested in Tate complexes that come from the composite functor

$$\mathbb{T}\epsilon_* \colon D_S^b(Y; \mathbb{O}) \xrightarrow{\epsilon_*} D_S^b(Y; \mathbb{O}[\sigma]) \xrightarrow{\mathbb{T}} \mathrm{Perf}_S(Y; \mathcal{T}_\mathbb{O}).$$

---

[8]This is to be distinguished from the (upcoming) notion of *Tate-parity sheaf*, which is more restrictive.

Definition 4.4. A *Tate-parity sheaf* $\mathcal{F} \in \text{Perf}_S(Y; \mathcal{T}_{\mathbb{O}})$ is an indecomposable Tate-parity complex with the property that its restriction to the unique stratum Y_λ which is dense in its support is of the form $\mathbb{T}\epsilon_*\mathcal{L}[\dim Y_\lambda]$ for an indecomposable Λ-free local system $\mathcal{L}$ on Y_λ. If such an $\mathcal{F}$ exists then it is unique, and we denote it by $\mathcal{E}_{\mathcal{T}}(\lambda, \mathcal{L})$.

If $\mathcal{E}_{\mathcal{T}}(\lambda, \mathcal{L})$ exists for all $\lambda \in S$ and all $\mathcal{L}$, we will say that "all Tate-parity sheaves exist" (for Y, S).

4.3. **Modular reduction.** We now explain that the functor $\mathbb{T}$ has good properties that one would expect from "base change of coefficients" functors for categories of sheaves in classical rings. We will suppression mention of the pariversity †.

Proposition 4.5 ([LL, Proposition 5.16, Theorem 5.17])**.**
   (1) If $\mathcal{F} \in D_S^b(X; \mathbb{O})$ is even/odd, then $\mathbb{T}\epsilon_*\mathcal{F} \in \text{Perf}_S(X; \mathcal{T}_{\mathbb{O}})$ is Tate-even/odd.
   (2) If the parity sheaf $\mathcal{E} = \mathcal{E}(\lambda, \mathcal{L})$ exists and satisfies $\text{Hom}_{D_S^b(Y; \mathbb{O})}(\mathcal{E}, \mathcal{E}[n]) = 0$ for all $n < 0$ (this holds for example if $\mathcal{E}$ is perverse[9]) then $\mathcal{E}_{\mathcal{T}}(\lambda, \mathcal{L})$ exists and we have

$$\mathbb{T}\epsilon_*\mathcal{E}(\lambda, \mathcal{L}) \cong \mathcal{E}_{\mathcal{T}}(\lambda, \mathcal{L}).$$

Remark 4.6. The Proposition (and its proof) are analogous to the following results of parity sheaves [JMW14, §2.5]. Let $\mathbb{F}$ denote the base change functor

$$\mathbb{F} = k \overset{L}{\otimes}_{\mathbb{O}} (-) \colon D_S^b(Y; \mathbb{O}) \to D_S^b(Y; k).$$

The functor $\mathbb{F}$ enjoys following properties.
   (1) $\mathcal{F} \in D_S^b(X; \mathbb{O})$ is a parity sheaf if and only if $\mathbb{F}(\mathcal{F}) \in D_S^b(X; k)$ is a parity sheaf.
   (2) If $\mathcal{E}(\lambda, \mathcal{L})$ exists, then $\mathcal{E}(\lambda, \mathbb{F}\mathcal{L})$ exists and we have

$$\mathbb{F}\mathcal{E}(\lambda, \mathcal{L}) \cong \mathcal{E}(\lambda, \mathbb{F}\mathcal{L}).$$

Proof of Proposition 4.5. We reproduce the proof from [LL] because it brings up certain ideas that will be needed later. The operation $\mathbb{T}\epsilon_*$ is compatible with formation of i_λ^* or $i_\lambda^!$. Hence to prove (1) we reduce to examining $T^i\epsilon_*\mathcal{L}$ for a local system $\mathcal{L}$ of free $\mathbb{O}$-modules, with the trivial σ-action. This reduces to the fact that the Tate cohomology of $\mathbb{O}$ is supported in even degrees, which is (4.1).

For (2), we just need to check that $\mathbb{T}\epsilon_*\mathcal{E}(\lambda, \mathcal{L})$ is indecomposable. Since $\text{Parity}_S(Y; \mathcal{T}_{\mathbb{O}})$ is Krull-Remak-Schmidt by [LL, Proposition 5.8], the endomorphism ring of $\mathbb{T}\epsilon_*\mathcal{E}(\lambda, \mathcal{L})$ is local. According to [LL, §4.6], for $\mathcal{F}, \mathcal{G} \in D_S^b(Y; \mathbb{O})$ we have

$$\text{Hom}_{\text{Perf}(Y; \mathcal{T}_\Lambda)}(\mathbb{T}\mathcal{F}, \mathbb{T}\mathcal{G}) \cong \bigoplus_{i \in \mathbf{Z}} \text{Hom}_{D_S^b(Y; k)}(\mathbb{F}\mathcal{F}, \mathbb{F}\mathcal{G}[2i]). \tag{4.2}$$

We apply this to $\mathcal{F} = \mathcal{G} = \epsilon_*\mathcal{E}(\lambda, \mathcal{L})$. Since $\mathcal{E}(\lambda, \mathcal{L})$ is indecomposable the subalgebra on the RHS of (4.2) indexed by $i = 0$ is local, and the assumption that the summands of (4.2) indexed by negative i vanish. This implies the desired locality of the graded algebra (4.2). $\qquad\square$

What we have seen can be summarized by the slogan:

> If all parity sheaves exist and have vanishing negative self-Exts, then all Tate-parity sheaves exist and $\mathbb{T} \circ \epsilon_*$ induces a bijection between parity sheaves and Tate-parity sheaves.

4.4. **The lifting functor.** We will now define a functor lifting Tate-parity sheaves to parity sheaves. In fact the preceding slogan already tells us what to do about objects, so we just need to specify what happens on morphisms.

Definition 4.7. A *normalized* (Tate-)parity complex is a direct sum of (Tate-)parity sheaves *with no shifts*. Hence, under our assumptions, an indecomposable (Tate-)parity complex is normalized if and only if its restriction to the dense open stratum in its support Y_λ is isomorphic to $\mathcal{L}[\dim Y_\lambda]$ (resp. $\mathbb{T}\epsilon_*\mathcal{L}[\dim Y_\lambda]$) for an indecomposable local system $\mathcal{L}$. We denote the full subcategory of normalized (Tate-)parity complexes by $\text{Parity}_S^0(Y; \mathbb{O}) \subset \text{Parity}_S(Y; \mathbb{O})$ (resp. $\text{Parity}_S^0(Y; \mathcal{T}_{\mathbb{O}}) \subset \text{Parity}_S(Y; \mathcal{T}_{\mathbb{O}})$), and called them the *categories of normalized (Tate-)parity sheaves.*

---

[9]In fact this is both necessary and sufficient by [MR18, Lemma 6.6], which we thank Simon Riche for pointing out to us.

Under the assumption that all parity sheaves exist and have vanishing negative self-Exts, we then have a "lifting functor" [LL, Theorem 5.19]

$$L \colon \mathrm{Parity}_S^0(Y; \mathcal{T}_{\mathbb{O}}) \to \mathrm{Parity}_S^0(Y; k)$$

sending $\mathcal{E}_{\mathcal{T}}(\lambda, \mathcal{L})$ to $\mathcal{E}(\lambda, \mathcal{L} \otimes_{\mathbb{O}} k)$ on objects, and on morphisms inducing projection to the summand indexed by $i = 0$ under identification (4.2). It can be thought of as an "intermediate" reduction between $\mathbb{O}$ and k in the sense that the following diagram commutes:

$$\begin{array}{ccc} \mathrm{Parity}_S^0(Y; \mathbb{O}) & \xrightarrow{\ \mathbb{T}\epsilon_*\ } & \mathrm{Parity}_S^0(Y; \mathcal{T}_{\mathbb{O}}) \\ & {}_{\mathbb{F}}\searrow & \downarrow{\scriptstyle L} \\ & & \mathrm{Parity}_S^0(Y; k) \end{array}$$

4.5. Parity sheaves on the affine Grassmannian and tilting modules.

We now consider the preceding theory in the context of the affine Grassmannian Gr_G over a separably closed field $\mathbf{F}$, with the stratification by L^+G-orbits. Since this is a special case of a Kac-Moody flag variety, the stratification is JMW by [JMW14, §4.1].

If p is a good prime for $\widehat{G}$, [MR18, Corollary 1.6] implies that all parity sheaves exist, and that all normalized parity sheaves are perverse. Therefore, the category of normalized parity sheaves corresponds under the Geometric Satake equivalence to some subcategory of $\mathrm{Rep}_k(\widehat{G})$, and it is natural to ask what this is. The answer is given in terms of *tilting modules* for $\widehat{G}$ (recall that these are the objects of $\mathrm{Rep}_k(\widehat{G})$ having both a filtration by standard objects, and a filtration by costandard objects). Let $\mathrm{Tilt}_k(\widehat{G}) \subset \mathrm{Rep}_k(\widehat{G})$ denote the full subcategory of tilting modules.

Theorem 4.8 ([MR18, Corollary 1.6]). *If p is good for G, then the Geometric Satake equivalence restricts to an equivalence*[10]

$$\mathrm{Parity}_{L^+G}^0(\mathrm{Gr}_G; k) \cong \mathrm{Tilt}_k(\widehat{G}).$$

Proof. The proof in [MR18] is written for the affine Grassmannian over the complex numbers but adapts to our situation with some small modifications. First, one takes Gr_G over $\mathbf{F}$ instead of over $\mathbf{C}$ as in [MR18]. The proof of Theorem 4.8 follows formally as in [MR18, §6.5] from an equivalence of categories, between the category of Iwahori-equivariant parity sheaves with coefficients in k on Gr_G, and the category of tilting objects in the heart of Bezrukavnikov's exotic t-structure on $\check{G} \times \mathbf{G}_m$-equivariant tilting objects on the Springer resolution $\widetilde{\mathcal{N}}$. This equivalence is in turn proved by a Soergel bimodule argument. The analysis of the "coherent side" in [MR18, §4,5] is literally the same as in our situation. The analysis of the "constructible side" in [MR18, §3] applies verbatim to Gr_G over $\mathbf{F}$ except at one point: in [MR18, Proof of Lemma 3.6, p.22] the property that "the map $\mathcal{O}(\mathfrak{t}^*/W \times \mathfrak{t}^*/W) \to H_{L^+G}^*(\mathrm{Gr}_G; \mathfrak{R})$ factors through $\mathcal{O}(\Delta)$" is proved using the "loop group presentation" of the complex affine Grassmannian; an alternate argument for this fact, that works in arbitrary characteristic, is provided in [Zhu17, Lemma 5.2.4]. $\square$

Remark 4.9. A much shorter argument for Theorem 4.8, but with a slightly worse bound on p, is given in [JMW16, Theorem 1.8].

We need a few facts about the representation theory of tilting modules. For our arithmetic applications, the key point is that there are "enough" tilting modules to generate the derived category of $\mathrm{Rep}_k(\widehat{G})$, as articulated by the Theorem below.

Theorem 4.10 ([BBM04]). *The subcategory $\mathrm{Tilt}_k(\widehat{G})$ generates the bounded derived category of $\mathrm{Rep}_k(\widehat{G})$. More precisely, the natural projection from the bounded homotopy category $K^b(\mathrm{Tilt}_k(\widehat{G}))$ to $D^b(\mathrm{Rep}_k(\widehat{G}))$ is an equivalence.*

Proof. This is a general statement in highest weight theory. A convenient reference is [Ric, Proposition 7.17]. $\square$

---

[10]Strictly speaking, the cited references employ the trivial pariversity instead of the dimension pariversity. Since dimensions of Schubert strata in Gr_G have constant parity on connected components, the trivial pariversity and dimension pariversity lead to the same notion of parity complexes in this case, so the only difference is in the notion of "normalization". We follow [LL] in the use of the dimension pariversity so that perverse sheaves are †-even.

4.6. **Base change functoriality for the Satake category.** We now consider a specific geometric situation relevant to Langlands functoriality for p-cyclic base change. Let $\mathbf{F}$ be a field of characteristic $\neq p$. We will consider reductive groups, and their affine Grassmannians, over $\mathbf{F}$.

4.6.1. *The base change setup.* We now specialize the situation a bit further: H is any reductive group over $\mathbf{F}$ and $G = H^p$. We let σ act on G by cyclic rotation, sending the ith factor to the $(i+1)$st (mod p) factor. Then it is clear that the stratification on Gr_G by L^+G-orbits induces by restriction the stratification on Gr_H by L^+H-orbits.

Evidently the "diagonal" embedding $H \hookrightarrow G$ realizes H as the fixed points of G under the automorphism σ. This map $H \hookrightarrow G$ also induces a diagonal map $\mathrm{Gr}_H \to \mathrm{Gr}_G$.

Lemma 4.11. *The diagonal map induces an isomorphism $\mathrm{Gr}_H \cong \mathrm{Gr}_G^\sigma$ as subfunctors of Gr_G.*

Proof. We have $\mathrm{Gr}_G \cong (\mathrm{Gr}_H)^p$, with σ acting by cyclic rotation of the factors, from which the claim is clear. □

Henceforth we assume that p is odd and good for $\widehat{G}$, so that the results of §4.5 apply.

Definition 4.12. We denote by
$$\mathrm{Res}_{\mathsf{BC}} \colon \mathrm{Tilt}_k(\widehat{G}) \to \mathrm{Tilt}_k(\widehat{H})$$
and
$$\mathrm{Res}_{\mathsf{BC}} \colon \mathrm{Rep}_k(\widehat{G}) \to \mathrm{Rep}_k(\widehat{H})$$
the restriction functors along the diagonal embedding $\widehat{H}_k \hookrightarrow \widehat{G}_k$.

We aim to give a "geometric" description of the corresponding functor under the Geometric Satake equivalence, $\mathrm{Parity}(\mathrm{Gr}_G; k) \to \mathrm{Parity}(\mathrm{Gr}_H; k)$, in terms of Smith theory. (Of course, one could give an "ad hoc" description using that $G = H^p$. The point is to define a functor that does not make reference to this, which will then generalize well, using descent, to the situation where $G = \mathrm{Res}_{\mathbf{E}/\mathbf{F}}(H)$ for a non-trivial field extension $\mathbf{E}/\mathbf{F}$.)

Definition 4.13. Given $\mathcal{F} \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G; k)$, we define
$$\mathrm{Nm}(\mathcal{F}) := \mathcal{F} * {}^\sigma\mathcal{F} * \ldots * {}^{\sigma^{p-1}}\mathcal{F} \in \mathrm{P}_{L^+G \rtimes \sigma}(\mathrm{Gr}_G; k),$$
equipped with the σ-equivariant structure coming from the commutativity constraint for $(\mathrm{P}_{L^+G}(\mathrm{Gr}_G; k), *)$:
$${}^\sigma\mathrm{Nm}(\mathcal{F}) = {}^\sigma\mathcal{F} * \ldots * {}^{\sigma^{p-1}}\mathcal{F} * \mathcal{F} \xrightarrow{\sim} \mathcal{F} * {}^\sigma\mathcal{F} * \ldots * {}^{\sigma^{p-1}}\mathcal{F} = \mathrm{Nm}(\mathcal{F}). \tag{4.3}$$
Using the realization functor $\mathrm{P}_{L^+G \rtimes \sigma}(\mathrm{Gr}_G; k) \to D_{L^+G \rtimes \sigma}(\mathrm{Gr}_G; k)$, we view $\mathrm{Nm}(\mathcal{F}) \in D_{L^+G \rtimes \sigma}(\mathrm{Gr}_G; k)$ (so that we may apply the Smith functor, for example). Equipping a general object of $D_{L^+G}(\mathrm{Gr}_G; k)$ with a σ-equivariant structure is much more involved than just specifying isomorphisms (4.3) (satisfying cocycle conditions), so we emphasize that we construct $\mathrm{Nm}(\mathcal{F})$ first as a σ-equivariant perverse sheaf, and then apply the realization functor to get a σ-equivariant object of $D_{L^+G}(\mathrm{Gr}_G; k)$.

Remark 4.14. In our applications we will assume that p is large enough so that all parity sheaves are perverse. The properties of being L^+G-constructible and L^+G-equivariant are equivalent for perverse sheaves on Gr_G. Therefore, we will not need to worry about any extra complications coming from the equivariance.

Lemma 4.15. *Let $i \colon \mathrm{Gr}_H \cong \mathrm{Gr}_G^\sigma \hookrightarrow \mathrm{Gr}_G$. For $\mathcal{F} \in D_{L^+G}^b(\mathrm{Gr}_G; \mathbb{O})$, regard $\mathrm{Nm}(\mathcal{F}) \in D_{L^+G \rtimes \sigma}^b(\mathrm{Gr}_G; \mathbb{O})$ as in Definition 4.13 above.*

(i) The stalks of $i^* \mathrm{Nm}(\mathcal{F})$ have Jordan-Hölder constituents being either trivial or free $\mathbb{O}[\sigma]$-modules.
(ii) The costalks of $i^! \mathrm{Nm}(\mathcal{F})$ have Jordan-Hölder constituents being either trivial or free $\mathbb{O}[\sigma]$-modules.

Proof. By filtering $\mathcal{F}$ into its Jordan-Hölder constituents, we may assume that $\mathcal{F}$ itself is simple. Any simple $L^+G \approx (L^+H)^p$-equivariant sheaf $\mathcal{F}$ on a stratum Gr_G^λ is of the form $\mathcal{F} \approx \mathcal{F}_1 \boxtimes \ldots \boxtimes \mathcal{F}_p$, since the stratum is a product of homogeneous spaces for (a finite type quotient of) L^+H. Then
$$\mathrm{Nm}(\mathcal{F}) \approx (\mathcal{F}_1 * \mathcal{F}_2 * \ldots * \mathcal{F}_p) \boxtimes (\mathcal{F}_2 * \ldots * \mathcal{F}_p * \mathcal{F}_1) \boxtimes \ldots \boxtimes (\mathcal{F}_p * \mathcal{F}_1 * \ldots * \mathcal{F}_{p-1}),$$
with σ acting by rotating the tensor factors, and the σ-equivariant structure coming from the commutativity constraint.

Write $\mathcal{F}' := \mathcal{F}_1 * \mathcal{F}_2 * \ldots * \mathcal{F}_p \in P_{L^+H}(\mathrm{Gr}_H; \mathbb{O})$. Since i may be identified with the diagonal embedding $\mathrm{Gr}_H \hookrightarrow \mathrm{Gr}_H^p$, we have $i^*(\mathrm{Nm}\,\mathcal{F}) \approx (\mathcal{F}')^{\otimes p}$, with σ-equivariant structure given by cyclic rotation of the tensor factors. In particular, the stalk of $i^*(\mathrm{Nm}\,\mathcal{F})$ at $x \in \mathrm{Gr}_H$ is the tensor-induction of the stalk of $\mathcal{F}'_x$ from $\mathbb{O}$ to $\mathbb{O}[\sigma]$. Hence it suffices to prove that any such tensor induction has Jordan-Hölder constituents being either trivial or free. This is verified by explicit inspection: choosing a basis for $\mathcal{F}'_x$, the induced basis of $(\mathcal{F}'_x)^{\otimes p}$ is grouped into either trivial or free orbits under the σ-action.

The argument for (ii) is completely analogous (alternatively, we could deduce it simply by applying Verdier duality to (i)). $\qquad\square$

4.6.2. *Smith theory for parity sheaves.* We return momentarily to the general setup for Smith theory: Y is a variety over $\mathbf{F}$ with an admissible σ-action and $Z = Y^\sigma$. We assume that $\mathbf{F}$ is separably closed and the stratification S on Y satisfies the JMW condition.

Proposition 4.16 (Variant of [LL, Theorem 6.3]). *Suppose $\mathcal{E} \in D^b_{S,\sigma}(Y; \mathbb{O})$ is a parity complex satisfying the condition:*

(*) *all $*$ and $!$-stalks of cohomology sheaves of $\mathcal{E}$ at fixed points $y \in Y$ have $\mathbb{O}[\sigma]$-module Jordan-Hölder constituents being trivial or free.*

Then $\mathrm{Psm}(\mathcal{E}) \in \mathrm{Perf}_S(Z; \mathcal{T}_{\mathbb{O}})$ is Tate-parity with respect to the induced stratification $Z_\lambda = Y_\lambda \cap Z$ and the induced pariversity $\dagger_Z(\lambda) := \dagger_Y(\lambda)$.

Proof. This theorem is closely related to Theorem 6.3 of [LL], except [LL, Theorem 6.3] imposes the stronger condition that the σ-action on all stalks is trivial. This is satisfied in their application (to the loop-rotation action), but not in ours, so we need to re-do the argument in the requisite generality.

Let $Z = Y^\sigma$ and take the induced stratification on Z. Let $i : Z \to Y$, $i^Y_\lambda : Y_\lambda \hookrightarrow Y$, $i^Z_\lambda : Z_\lambda \hookrightarrow Z$, $i^\lambda : Z_\lambda \hookrightarrow Y_\lambda$. Without loss of generality suppose $\mathcal{E}$ is an even complex on Y. We are given that $(i^Y_\lambda)^? \mathcal{E}$ has $\mathbb{O}$-free cohomology sheaves supported in degrees congruent to $\dagger_Y(\lambda)$ mod 2, where $? \in \{*, !\}$; we want to show that $(i^Z_\lambda)^? \mathrm{Psm}(\mathcal{E})$ has Tate-cohomology sheaves supported in degrees congruent to $\dagger_Z(\lambda)$ mod 2. Unraveling the definitions, we have

$$(i^Z_\lambda)^* \mathrm{Psm}(\mathcal{E}) = (i^Z_\lambda)^* \mathbb{T} i^* \mathcal{E} \cong \mathbb{T}(i^Z_\lambda)^* i^* \mathcal{E} \cong \mathbb{T}(i^\lambda)^* (i^Y_\lambda)^* \mathcal{E}.$$

Similarly, using Lemma 3.7 we have

$$(i^Z_\lambda)^! \mathrm{Psm}(\mathcal{E}) \cong \mathbb{T}(i^\lambda)^! (i^Y_\lambda)^! \mathcal{E}. \tag{4.4}$$

By hypothesis, $(i^Y_\lambda)^* \mathcal{E}$ has its cohomology sheaves supported in degrees congruent to $\dagger_Y(\lambda)$ (mod 2). Moreover, by assumption (*), all the stalks and costalks have Jordan-Hölder constituents being even shifts of either trivial or free $\mathbb{O}[\sigma]$-modules. So the stalks of $(i^\lambda)^* (i^Y_\lambda)^* \mathcal{E}$ are supported in degrees congruent to $\dagger_Y(\lambda)$ (mod 2), and we must verify that their Tate cohomology groups are also supported in degrees of a single parity.

For trivial $\mathbb{O}[\sigma]$-modules the odd Tate cohomology groups vanish by (4.1), while for free $\mathbb{O}[\sigma]$-modules all the Tate cohomology groups vanish. Hence for any $\mathbb{O}[\sigma]$ whose Jordan-Hölder constituents are all trivial or free, all odd Tate cohomology groups vanish by the long exact sequence for Tate cohomology (§3.4.2). This shows that the Tate cohomology sheaves of $(i^\lambda)^* (i^Y_\lambda)^* \mathcal{E}$ are supported in degrees congruent to $\dagger_Y(\lambda)$ (mod 2).

A completely analogous argument, using (4.4) instead, shows that $(i^\lambda)^! (i^Y_\lambda)^! \mathcal{E}$ also has Tate cohomology sheaves supported in degrees congruent to $\dagger_Y(\lambda)$ (mod 2). $\qquad\square$

For an $\mathbb{O}$-linear abelian category $\mathcal{C}$, with all Hom-spaces being free $\mathbb{O}$-modules, we denote by $\mathcal{C} \otimes_\mathbb{O} k$ the k-linear category obtained by tensoring all Hom-spaces with k over $\mathbb{O}$.

Lemma 4.17. *Suppose that all the strata Y_λ are simply connected and all parity sheaves $\mathcal{E}(\lambda, \mathcal{L})$ exist, for all $\lambda \in S$. Then we have that*

$$\mathrm{Parity}^0_{S,\sigma}(Y; \mathbb{O}) \otimes_\mathbb{O} k \xrightarrow{\sim} \mathrm{Parity}^0_{S,\sigma}(Y; k).$$

Proof. To see that the functor is well-defined, we note:

- The Hom-spaces of $\mathrm{Parity}^0_{S,\sigma}(Y; \mathbb{O})$ are all free $\mathbb{O}$-modules by [JMW14, Remark 2.7], so that the domain is well-defined.

- The functor lands in parity sheaves since the modular reduction of a $\mathbb{O}$-parity sheaf is a k-parity sheaf by Remark 4.6.

It is essentially surjective because every k-parity sheaf lifts to a $\mathbb{O}$-parity sheaf under our assumption that all parity sheaves exist and all strata are simply connected (which implies that all k-local systems on strata lift to $\mathbb{O}$, since they are trivial). The fact that the functor is fully faithful again follows from [JMW14, Remark 2.7]. $\qquad\square$

4.6.3. *The base change functor.* We return now to the base change setup of §4.6.1, with $\mathbf{F}$ separably closed. Let $\mathcal{F} \in \mathrm{Parity}^0_{L^+G}(\mathrm{Gr}_G; \mathbb{O})$. Then $\mathcal{F} \in \mathrm{P}_{L^+G}(\mathrm{Gr}_G; \mathbb{O})$ is perverse since p is good for $\widehat{G}$ (this is a part of Theorem 4.8), and $\mathrm{Nm}(\mathcal{F}) \in \mathrm{Parity}^0_{L^+G \rtimes \sigma}(\mathrm{Gr}_G; \mathbb{O})$ is a parity sheaf by [JMW16, Theorem 1.5]. Furthermore, the σ-equivariant structure on $\mathrm{Nm}(\mathcal{F})$ satisfies the assumption (*) of Proposition 4.16 by Lemma 4.15.

The Schubert cells of Gr_G are indexed by tuples $\lambda := (\lambda_1, \ldots, \lambda_p) \in X_*(G)^+$, with each $\lambda_i \in X_*(H)^+$, and we have

$$\begin{cases} \mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_H = \mathrm{Gr}_H^{\lambda_1} & \lambda = (\lambda_1, \ldots, \lambda_1), \\ \mathrm{Gr}_G^\lambda \cap \mathrm{Gr}_H = \emptyset & \text{otherwise.} \end{cases}$$

We claim that as long as $p > 2$, the induced pariversity coincides with the dimension pariversity on Gr_H, i.e., for $\lambda = (\lambda_1, \ldots, \lambda_p) \in X_*(G)^+$, we have

$$\dim \mathrm{Gr}_H^{\lambda_1} \equiv \dim \mathrm{Gr}_G^\lambda \pmod 2.$$

This will imply that:

(1) We may apply Proposition 4.16 to deduce that $\mathrm{Psm}(\mathrm{Nm}(\mathcal{F})) \in \mathrm{Parity}_{(L^+H)}(\mathrm{Gr}_H; \mathcal{T}_{\mathbb{O}})$ is Tate-parity with respect to the dimension pariversity on Gr_H.
(2) $\mathrm{Psm}(\mathrm{Nm}(\mathcal{F})) \in \mathrm{Parity}^0_{(L^+H)}(\mathrm{Gr}_H; \mathcal{T}_{\mathbb{O}})$, i.e., is normalized.

To prove the claim, we may focus on the case where $\lambda_1 = \ldots = \lambda_p$ or else the statement is vacuous. By [Zhu17, Proposition 2.1.5] we have $\dim \mathrm{Gr}_G^\lambda = \langle 2\rho_G, \lambda \rangle$. So we just have to verify that $\langle 2\rho_G, (\lambda_1, \ldots, \lambda_1) \rangle \equiv \langle 2\rho_H, \lambda_1 \rangle$ (mod 2). Indeed, $\rho_G = (\rho_H, \ldots, \rho_H)$, so $\langle 2\rho_G, (\lambda_1, \ldots, \lambda_1) \rangle = p\langle 2\rho_H, \lambda_1 \rangle$, and p is odd.

Thanks points (1) and (2) above, we can apply the lifting functor L to $\mathrm{Psm}(\mathrm{Nm}(\mathcal{F}))$. By Lemma 4.17, the composite functor $L \circ \mathrm{Psm} \circ \mathrm{Nm}$ factors uniquely through a functor $\mathrm{Parity}^0_{L^+G}(\mathrm{Gr}_G, k) \to \mathrm{Parity}^0_{L^+H}(\mathrm{Gr}_H; k)$.

Definition 4.18. We define

$$\mathsf{BC}^{(p)} \colon \mathrm{Parity}^0_{L^+G}(\mathrm{Gr}_G; k) \to \mathrm{Parity}^0_{L^+H}(\mathrm{Gr}_H; k)$$

to be the functor unique filling in the commutative diagram

$$\begin{array}{ccc} \mathrm{Parity}^0_{L^+G}(\mathrm{Gr}_G; \mathbb{O}) & \xrightarrow{\mathrm{Psm} \circ \mathrm{Nm}} & \mathrm{Parity}^0_{(L^+H)}(\mathrm{Gr}_H; \mathcal{T}_{\mathbb{O}}) \\ \downarrow{\scriptstyle \mathbb{F}} & & \downarrow{\scriptstyle L} \\ \mathrm{Parity}^0_{L^+G}(\mathrm{Gr}_G; k) & \dashrightarrow{\mathsf{BC}^{(p)}} & \mathrm{Parity}^0_{L^+H}(\mathrm{Gr}_H; k). \end{array} \tag{4.5}$$

One more step is required to obtain the desired base change functor. On a k-linear additive category there is an auto-equivalence Frob_p of the underlying category, which is the identity on objects and the Frobenius automorphism $(-) \otimes_{k, \mathrm{Frob}_p} k$ on morphisms. We define

$$\mathsf{BC} := \mathrm{Frob}_p^{-1} \circ \mathsf{BC}^{(p)} \colon \mathrm{Parity}^0_{L^+G}(\mathrm{Gr}_G; k) \to \mathrm{Parity}^0_{L^+H}(\mathrm{Gr}_H; k).$$

Remark 4.19. The construction of BC was motivated by a similar functor "LL" appearing in [LL, §6.2], which gives a partial geometric description of the Frobenius contraction functor. Another motivation was the "normalized Brauer homomorphism" of [TV16, §4.3], which our construction categorifies.

Theorem 4.20. *Let* $\mathrm{Res}_{\mathsf{BC}} \colon \mathrm{Rep}_k(\widehat{G}) \to \mathrm{Rep}_k(\widehat{H})$ *be restriction along the diagonal embedding. We also denote by* $\mathrm{Res}_{\mathsf{BC}}$ *the same functor restricted to the subcategories of tilting modules.*[11] *The following diagram*

---

[11]Note that it is not obvious that $\mathrm{Res}_{\mathsf{BC}}$ preserves the tilting property, but this follows from the non-trivial theorem (building on work of many authors – see the discussion around [JMW16, Theorem 1.2]) that tensor products of tilting modules are tilting.

commutes:

$$\begin{array}{ccc} \mathrm{Parity}^0(\mathrm{Gr}_G; k) & \xrightarrow{\ \mathsf{BC}\ } & \mathrm{Parity}^0(\mathrm{Gr}_H; k) \\ \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\ \mathrm{Tilt}_k(\widehat{G}) & \xrightarrow{\ \mathrm{Res}_{\mathsf{BC}}\ } & \mathrm{Tilt}_k(\widehat{H}) \end{array}$$

Proof sketch. The argument is given in Appendix A. For now let us just explain the key trick (which we learned from the proof of [LL, Theorem 7.3]): since Psm commutes with hyperbolic localization by §A.1, and the restriction functor to a maximal torus $\mathrm{Rep}(\widehat{H}) \to \mathrm{Rep}(T_{\widehat{H}})$ is faithful and injective on tilting objects, one can reduce to the case where H is a *torus*. In this case the functor can be computed explicitly, since the affine Grassmannian of a torus is simply a discrete set. $\square$

Assuming Theorem 4.20, let us give a few variants related to descent to a ground field which is not separably closed.

Suppose H base changed from some subfield $\mathbf{F}_0 \subset \mathbf{F}$, and $G = \mathrm{Res}_{\mathbf{E}_0/\mathbf{F}_0}(H_{\mathbf{E}_0})$ for some Galois extension $\mathbf{E}_0/\mathbf{F}_0$ with Galois group $\mathbf{Z}/p\mathbf{Z}$. Then $G_{\mathbf{F}} \approx (H_{\mathbf{F}})^p$ and $\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)$ acts on $H_{\mathbf{F}}, G_{\mathbf{F}}$ and therefore also on $\mathrm{Gr}_{H,\mathbf{F}}, \mathrm{Gr}_{G,\mathbf{F}}$.

Lemma 4.21 (Galois equivariance). *In the situation above, the functor*

$$\mathsf{BC} \colon \mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}}; k) \to \mathrm{Parity}^0(\mathrm{Gr}_{H,\mathbf{F}}; k)$$

is equivariant with respect to the action of $\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)$.

Proof. The constituent functors Nm, i^*, $\mathbb{T}$, and L are all $\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)$-equivariant, as is Frob_p^{-1}. It remains only to see that the dashed arrow in (4.5) is $\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)$-equivariant. This follows because $L \circ \mathrm{Psm} \circ \mathrm{Nm}$ and $\mathbb{F}$ both have this property, and $\mathbb{F}$ is essentially surjective and full. $\square$

4.6.4. *Equivariantization and Galois descent.* We refer to [DGNO10] for the theory of "equivariantization" and "de-equivariantization" of categories. Given a group Γ acting on categories $\mathcal{C}, \mathcal{D}$ and a Γ-equivariant functor $F \colon \mathcal{C} \to \mathcal{D}$, the Γ-*equivariantization* of F is the functor $F^{B\Gamma} \colon \mathcal{C}^{B\Gamma} \to \mathcal{D}^{B\Gamma}$. If $\mathcal{C}$ and $\mathcal{D}$ are derived categories of sheaves and F is induced by geometric operations that are Γ-equivariant, then the equivariantization construction exists for equivariant derived categories.

Thanks to Lemma 4.21, the equivariantization of BC induces

$$\mathsf{BC}^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)} \colon \mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}}; k)^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)} \to \mathrm{Parity}^0(\mathrm{Gr}_{H,\mathbf{F}}; k)^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)}.$$

We *define* $\mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}_0}; k) := \mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}}; k)^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)}$ and similarly for H (note that in §4.1 parity sheaves were only defined for varieties over separably closed fields, since the axioms of a JMW stratification would not otherwise be satisfied). We *define* $\mathrm{Tilt}_k({}^L G) := \mathrm{Tilt}_k(\widehat{G})^{\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0),\mathrm{geom}}$ and similarly for ${}^L H$. Then applying $\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)$-equivariantization to Theorem 4.20 yields:

Corollary 4.22. *The following diagram is commutative.*

$$\begin{array}{ccc} \mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}_0}; k) & \xrightarrow{\ \mathsf{BC}^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)}\ } & \mathrm{Parity}^0(\mathrm{Gr}_{H,\mathbf{F}_0}; k) \\ \Big\downarrow{\scriptstyle\sim} & & \Big\downarrow{\scriptstyle\sim} \\ \mathrm{Tilt}_k({}^L G) & \xrightarrow{\ \mathrm{Res}_{\mathsf{BC}}\ } & \mathrm{Tilt}_k({}^L H) \end{array}$$

Actually, what will be a bit more useful for us is the version below, where we first pass to a triangulated equivalence using Theorem 4.8 and Theorem 4.10, and then equivariantize with respect to the $\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)$-action.

Corollary 4.23. *The following diagram is commutative.*

$$\begin{array}{ccccccc} K^b(\mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}_0}; k)) & \longrightarrow & K^b(\mathrm{Parity}^0(\mathrm{Gr}_{G,\mathbf{F}}; k))^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)} & \xrightarrow{\sim} & D^b(\mathrm{Rep}_k(\widehat{G}))^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)} & \longleftarrow & D^b(\mathrm{Rep}_k({}^L G)) \\ \Big\downarrow{\scriptstyle \mathsf{BC}^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)}} & & \Big\downarrow{\scriptstyle \mathsf{BC}^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{\mathsf{BC}}} & & \Big\downarrow{\scriptstyle \mathrm{Res}_{\mathsf{BC}}} \\ K^b(\mathrm{Parity}^0(\mathrm{Gr}_{H,\mathbf{F}_0}; k)) & \longrightarrow & K^b(\mathrm{Parity}^0(\mathrm{Gr}_{H,\mathbf{F}}; k))^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)} & \xrightarrow{\sim} & D^b(\mathrm{Rep}_k(\widehat{H}))^{B\,\mathrm{Aut}(\mathbf{F}/\mathbf{F}_0)} & \longleftarrow & D^b(\mathrm{Rep}_k({}^L H)) \end{array}$$

5. On global base change

In this section we will apply the preceding theory to *moduli stacks of shtukas*, in the context of Lafforgue's construction of the global Langlands parametrization for function fields. In particular, we will prove Theorem 1.6, among other results.

We briefly review the relevant parts of Lafforgue's construction in §5.1 and §5.2. Then in §5.3, where we use a variant of Lafforgue's ideas to construct and analyze an action of the "σ-equivariant excursion algebra" on the Tate cohomology of moduli spaces of shtukas. In the situation of base change, equivariant localization mediates between the Tate cohomology of shtukas for G and for H, allowing us to relate certain excursion operators for the two groups. This is then used in §5.7 to establish the existence of base change for mod p automorphic forms; it will also be the crucial input for our local results in the next section.

5.1. **Moduli of shtukas.** We will use the theory of moduli stacks of shtukas, due to Drinfeld and generalized by Varshavsky. Here we very briefly recall the relevant definitions in order to set notation. More comprehensive references include [Var04] and [Laf18].

5.1.1. *Shtukas.* Fix a smooth projective curve X over a finite field $\mathbf{F}_\ell$ of characteristic $\neq p$. For a smooth affine group scheme $G \to X$ and a finite set I, the stack $\mathrm{Sht}_{G,I}$ has the following functor of points on $\mathbf{F}_\ell$-schemes S:

$$\mathrm{Sht}_{G,I} \colon S \mapsto \left\{ \begin{array}{c} (x_i)_{i \in I} \in X^I(S) \\ \mathcal{E} = \text{étale } G\text{-torsor over } X \times S \\ \varphi \colon \mathcal{E}|_{X \times S - \bigcup_{i \in I} \Gamma_{x_i}} \xrightarrow{\sim} {}^\tau \mathcal{E}|_{X \times S - \bigcup_{i \in I} \Gamma_{x_i}} \end{array} \right\},$$

where τ is the Frobenius Frob_ℓ on the S factor in $X \times S$, and ${}^\tau \mathcal{E}$ is the pullback of $\mathcal{E}$ under the map $1 \times \tau \colon X \times S \to X \times S$.

Geometrically, $\mathrm{Sht}_{G,I}$ has a Schubert stratification whose strata are Deligne-Mumford stacks locally of finite type. We regard it as an ind-(locally finite type) Deligne-Mumford stack.

5.1.2. *Hecke stack.* The Hecke stack $\mathrm{Hk}_{G,I}$ classifies

$$\mathrm{Hk}_{G,I} \colon S \mapsto \left\{ \begin{array}{c} (x_i)_{i \in I} \in X^I(S) \\ \mathcal{E}, \mathcal{E}' = \text{étale } G\text{-torsors over } X \times S \\ \varphi \colon \mathcal{E}|_{X \times S - \bigcup \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}'|_{X \times S - \bigcup \Gamma_{x_i}} \end{array} \right\}.$$

The Geometric Satake equivalence provides a functor $\mathrm{Rep}_k(({}^L G)^I) \to D(\mathrm{Hk}_{G,I}; k)$, which we normalize as in [Laf18, Theorem 0.9].

5.1.3. *Satake sheaves.* There is a map $\mathrm{Sht}_{G,I} \to \mathrm{Hk}_{G,I}$ sending $(\{x_i\}_{i \in I}, \mathcal{E}, \varphi)$ to $(\{x_i\}_{i \in I}, \mathcal{E}, {}^\tau \mathcal{E}, \varphi)$. Composing with the $*$-pullback through $\mathrm{Sht}_{G,I} \to \mathrm{Hk}_{G,I}$ induces a functor

$$\mathrm{Sat}^{\mathrm{geom}} \colon \mathrm{Rep}_k(\widehat{G}^I)^{B\,\mathrm{Gal}(F^s/F),\mathrm{geom}} \to D^b(\mathrm{Sht}_{G,I}; k).$$

Finally, we may identify $\mathrm{Rep}_k(({}^L G^{\mathrm{alg}})^I) \xrightarrow{\sim} \mathrm{Rep}_k(\widehat{G}^I)^{B\,\mathrm{Gal}(F^s/F),\mathrm{geom}}$ as in §2.1.4, giving a functor (cf. [Laf18, Theorem 0.11])

$$\mathrm{Sat} \colon \mathrm{Rep}_k(({}^L G^{\mathrm{alg}})^I) \to D^b(\mathrm{Sht}_{G,I}; k).$$

The Schubert stratification is defined by the support of the sheaves in the image of Sat, with the closure relations corresponding to the Bruhat order. (In particular, Sat lands in the derived category of sheaves constructible with respect to the Schubert stratification on $\mathrm{Sht}_{G,I}$.)

5.1.4. There is a map

$$\pi_I \colon \mathrm{Sht}_{G,I} \to X^I$$

projecting a tuple $(\{x_i\}_{i \in I}, \mathcal{E}, \varphi_i)$ to $\{x_i\}_{i \in I}$.

5.1.5. *Level structures.* For $D \subset X$ a finite-length subscheme, there are level covers $\mathrm{Sht}_{G,D,I} \to \mathrm{Sht}_{G,I}|_{(X-D)^I}$ which parametrize the additional datum of a trivialization of $\mathcal{E}$ over $S \times D$ compatible with τ and φ. Note that by definition, the "legs" $\{x_i\}_{i \in I} \in (X - D)(S)^I$ avoid D.

5.1.6. *Iterated shtukas.* Let $I_1, \ldots, I_r$ be a partition of I. We define $\mathrm{Sht}_{G,D,I}^{(I_1,\ldots,I_r)}$ (sometimes called a moduli stack of *iterated shtukas*) to be the stack

$$\mathrm{Sht}_{G,D,I}^{(I_1,\ldots,I_r)} \colon S \mapsto \left\{ \begin{array}{c} (x_i)_{i \in I} \in X^I(S) \\ \mathcal{E}_0, \ldots, \mathcal{E}_r = \text{étale } G\text{-torsors over } X \times S \\ \varphi_j \colon \mathcal{E}_{j-1}|_{X \times S - \bigcup_{i \in I_j} \Gamma_{x_i}} \xrightarrow{\sim} \mathcal{E}_j|_{X \times S - \bigcup_{i \in I_j} \Gamma_{x_i}} \quad j = 1, \ldots, r \\ \varphi \colon \mathcal{E}_r \xrightarrow{\sim} {}^\tau \mathcal{E}_0 \\ \upsilon = \text{level structure over } D \times S \end{array} \right\}.$$

There is a map $\nu \colon \mathrm{Sht}_{G,D,I}^{(I_1,\ldots,I_r)} \to \mathrm{Sht}_{G,D,I}$. A key property of this morphism is that it is *stratified small* (with respect to the Schubert stratification), which is a consequence of the same property of the convolution morphism for Beilinson-Drinfeld Grassmannians.

5.1.7. *Partial Frobenius.* There is a partial Frobenius $F_{I_1} \colon \mathrm{Sht}_{G,D,I}^{(I_1,I_2,\ldots,I_r)} \to \mathrm{Sht}_{G,D,I}^{(I_2,\ldots,I_r,I_1)}$ sending

$$x_i \mapsto \begin{cases} {}^\tau x_i & i \in I_1 \\ x_i & \text{otherwise} \end{cases}$$

$$(\mathcal{E}_0, \ldots, \mathcal{E}_r) \mapsto (\mathcal{E}_1, \ldots, \mathcal{E}_r, {}^\tau \mathcal{E}_0)$$

$$(\varphi_1, \ldots, \varphi_r) \mapsto (\varphi_2, \ldots, \varphi_r, {}^\tau \varphi_1).$$

It lies over the partial Frobenius Frob_{I_1} on X^I (applying Frob_ℓ to the coordinates indexed by $i \in I_1$), so that the diagram below is commutative (and cartesian up to radiciel maps):

$$\begin{array}{ccc} \mathrm{Sht}_{G,D,I}^{(I_1,\ldots,I_r)} & \xrightarrow{F_{I_1}} & \mathrm{Sht}_{G,D,I}^{(I_2,\ldots,I_r,I_1)} \\ \downarrow{\scriptstyle \nu} & & \downarrow{\scriptstyle \nu} \\ X^I & \xrightarrow{\mathrm{Frob}_{I_1}} & X^I \end{array} \tag{5.1}$$

5.1.8. *Base change setup.* We now consider the following "base change setup". Let F be the function field of X and H_F a reductive group over F. We choose a parahoric extension of H_F to a smooth affine group scheme H over X.

Let E/F be a cyclic extension of F having degree p, so E corresponds to the function field of a smooth projective curve X'. Define $G := \mathrm{Res}_{X'/X}(H_{X'})$, which is an affine group scheme over X with generic fiber $G_F \cong \mathrm{Res}_{E/F}(H_E)$. The group scheme $G \to X$ comes with an induced action of $\langle \sigma \rangle = \mathrm{Aut}(X'/X)$.

5.2. **Review of V. Lafforgue's global Langlands correspondence.** Write $\Gamma = \mathrm{Gal}(F^s/F)$. In [Laf18, §13], Lafforgue constructs an action of $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ on the space of cusp forms for G with coefficients in k. This has been improved by Cong Xue, who extended the action to all compactly supported functions ([Xuea, §7] for split G and [Xueb, §6] for all G).

We summarize the construction of the excursion action, as we shall make use of some of its internal aspects, and we also need to explain why it can be used to construct some excursion actions on Tate cohomology.

5.2.1. *Constructing actions of the excursion algebra.* We will explain an abstract setup that gives rise to actions of the excursion algebra.

Definition 5.1. Let A be a (not necessarily commutative) ring. A family of functors $H_I \colon \mathrm{Rep}_k(({}^L G)^I) \to \mathrm{Mod}_A(\Gamma^I)$, where I runs over (possibly empty) finite sets, is *admissible* if it satisfies the two conditions below.

(1) *(Compatibility with fusion)* For all $\zeta \colon I \to J$, there is a natural isomorphism χ_ζ between the functors $H_I \circ \mathrm{Res}_\zeta$ and $\mathrm{Res}_\zeta \circ H_J$ in the diagram:

$$\begin{array}{ccc} \mathrm{Rep}_k(({}^L G)^I) & \xrightarrow{H_I} & \mathrm{Mod}_A(\Gamma^I) \\ {\scriptstyle \mathrm{Res}_\zeta}\downarrow & \overset{\chi_\zeta}{\Longrightarrow} & \downarrow{\scriptstyle \mathrm{Res}_\zeta} \\ \mathrm{Rep}_k(({}^L G)^J) & \xrightarrow{H_J} & \mathrm{Mod}_A(\Gamma^J) \end{array} \tag{5.2}$$

(2) *(Compatibility with composition)* For $I' \xrightarrow{\zeta'} I \xrightarrow{\zeta} J$, we have $\chi_{\zeta \circ \zeta'} = \chi_\zeta \circ \chi_{\zeta'}$.

Construction 5.2. Let $\mathbb{1}$ denote the trivial representation of ^LG. Given an admissible family of functors $H_I \colon \operatorname{Rep}_k((^LG)^I) \to \operatorname{Mod}_A(\Gamma^I)$, we get an A-linear action of $\operatorname{Exc}(\Gamma, {}^LG)$ on $H_{\{0\}}(\mathbb{1})$ as follows.

For a tuple $(I, W, x, \xi, (\gamma_i)_{i \in I})$ we define an endomorphism, which gives the image of $S_{I,W,x,\xi,(\gamma_i)_{i\in I}}$ in $\operatorname{End}_A(H_{\{0\}}(\mathbb{1}))$, by the following composition:

$$H_{\{0\}}(\mathbb{1}) \xrightarrow{H_{\{0\}}(x)} H_{\{0\}}(W^\zeta) \xrightarrow[\sim]{\chi_\zeta} H_I(W) \xrightarrow{(\gamma_i)_{i\in I}} H_I(W) \xrightarrow[\sim]{\chi_\zeta^{-1}} H_{\{0\}}(W^\zeta) \xrightarrow{H_{\{0\}}(\xi)} H_{\{0\}}(\mathbb{1}).$$

From the assumptions of admissibility it is straightforward to check the relations in §2.4.2.

Remark 5.3. Note that it follows from admissibility that the A-module underlying $H_I(\mathbb{1})$ for any I is identified with $H_\emptyset(\mathbb{1})$ by $\chi_{\emptyset \to \{1\}}$. Proposition 2.4 then attaches a Galois representation to each generalized eigenvector for the $\operatorname{Exc}(\Gamma, {}^LG)$-action on $H_\emptyset(\mathbb{1})$. (Of course, such an eigenvector is not guaranteed to exist in general.)

5.2.2. *Excursion action on the cohomology of shtukas.* Let $\mathcal{H}_G$ be the Hecke algebra acting on $\operatorname{Sht}_{G,D}$; it is the tensor product of local Hecke algebras with the level structure dictated by D. For any finite set I, we have a map

$$R\pi_I \colon \operatorname{Sht}_{G,D,I} \to (X - D)^I$$

remembering the points of the curve indexed by I (which avoid D by definition). Let η^I denote the generic point of X^I and $\overline{\eta^I}$ the spectrum of an algebraic closure, viewed as a geometric generic point of X^I. When I is a singleton, we will just abbreviate these by η and $\overline{\eta}$.

We will define a family of functors indexed by finite sets I:

$$H_I \colon \operatorname{Rep}_k((^LG^{\operatorname{alg}})^I) \to \operatorname{Mod}_{\mathcal{H}_G}(\Gamma^I) \tag{5.3}$$

sending $V \in \operatorname{Rep}_k((^LG^{\operatorname{alg}})^I)$ to

$$R^0\pi_{I!}(\operatorname{Sht}_{G,D,I}|_{\overline{\eta^I}}; \operatorname{Sat}(V)). \tag{5.4}$$

Here and throughout, we use the *perverse* t-structure in formation of $R^0\pi_{I!}$. Note that a priori $H_I(V)$ has an action of $\pi_1(\eta^I, \overline{\eta^I})$, which maps[12] to Γ^I but neither injectively nor surjectively.

5.2.3.   We explain why the $\pi_1(\eta^I, \overline{\eta^I})$ extends canonically to an action of Γ^I. Assume I is non-empty, since otherwise there is nothing to prove. The Satake functor of §5.1.3 generalizes to a functor

$$\operatorname{Sat}^{(I_1,\ldots,I_r)} \colon \operatorname{Rep}_k((^LG)^I) \to D^b(\operatorname{Sht}_{G,D,I}^{(I_1,\ldots,I_r)}; k),$$

such that the map

$$\nu \colon \operatorname{Sht}_{G,D,I}^{(I_1,\ldots,I_r)} \to \operatorname{Sht}_{G,D,I}$$

has the property that $R\nu_! \operatorname{Sat}^{(I_1,\ldots,I_r)}(V) \cong \operatorname{Sat}(V)$. Furthermore, there are natural isomorphisms

$$F_{I_1}^* \operatorname{Sat}^{(I_1,I_2,\ldots,I_r)}(V) \cong \operatorname{Sat}^{(I_2,\ldots,I_r,I_1)}(V),$$

where F_{I_1} is the partial Frobenius from §5.1.7.

Write $I = \{1, \ldots, n\}$. Thanks to the above properties and (5.1), the partial Frobenius maps on $\operatorname{Sht}_{G,D,I}^{(\{1\},\ldots,\{n\})}$ then induce maps

$$\operatorname{Frob}_{\{1\}}^* H_I(V) \xrightarrow{\sim} H_I(V).$$

That equips $H_I(V)$ with the action of the larger group $\operatorname{FWeil}(\eta^I, \overline{\eta^I})$ that we now recall, summarizing [Laf18, Remarque 8.18]. Let F^I denote the function field of X^I, so $\eta^I = \operatorname{Spec} F^I$, and $\overline{F^I}$ an algebraic closure, so we may take $\overline{\eta^I} = \operatorname{Spec} \overline{F^I}$. Write $(F^I)^{\operatorname{perf}}$ for the perfect closure of F^I, and $\operatorname{Frob}_{\{i\}}$ for the "partial Frobenius" automorphism of $(F^I)^{\operatorname{perf}}$ induced by Frob_ℓ on the ith factor. We define

$$\operatorname{FWeil}(\eta^I, \overline{\eta^I}) := \{\gamma \in \operatorname{Aut}_{\overline{\mathbf{F}}_q}(\overline{F^I}) \colon \exists (n_i)_{i\in I} \in \mathbf{Z}^I \text{ such that } \gamma|_{(F^I)^{\operatorname{perf}}} = \prod_{i\in I}(\operatorname{Frob}_{\{i\}})^{n_i}\}.$$

Writing $\pi_1^{\operatorname{geom}}(\eta^I, \overline{\eta^I}) := \ker(\pi_1(\eta^I, \overline{\eta^I}) \xrightarrow{\deg} \widehat{\mathbf{Z}})$, this fits into an extension

$$0 \to \pi_1^{\operatorname{geom}}(\eta^I, \overline{\eta^I}) \to \operatorname{FWeil}(\eta^I, \overline{\eta^I}) \to \mathbf{Z}^I \to 0.$$

---

[12]The map is non-canonical: it depends on a choice of specialization as in [Laf18, Remark 8.18].

Fixing a specialization morphism $\overline{\eta^I} \rightsquigarrow \Delta(\eta^{\{1\}})$ induces a surjection

$$\mathrm{FWeil}(\eta^I, \overline{\eta^I}) \twoheadrightarrow \mathrm{Weil}(\eta, \overline{\eta})^I.$$

A form of Drinfeld's Lemma [Xuea, Lemma 7.4.2] is used to show that the action of $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$ on $H_I(V)$ factors through $\mathrm{Weil}(F^s/F)^I$; continuity considerations then imply that the action extends uniquely to one of Γ^I.

Example 5.4. Let us unravel

$$H_{\{1\}}(\mathbb{1}) = R^0 \pi_{\{1\}!}(\mathrm{Sht}_{G,D,\{1\}}|_{\overline{\eta^{\{1\}}}}; \mathrm{Sat}(\mathbb{1})). \tag{5.5}$$

By Remark 5.3 the underlying Hecke module of $H_{\{1\}}(\mathbb{1})$ is isomorphic to $H_\emptyset(\mathbb{1})$. According to [Laf18, Remarque 12.2], this is the space of compactly supported k-valued functions on the discrete groupoid

$$\mathrm{Bun}_{G,D}(\mathbf{F}_\ell) = \coprod_{\alpha \in \ker^1(F,G)} \left(G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F) / \prod_v K_v \right), \tag{5.6}$$

where G_α is the pure inner form of G corresponding to α, $K_v = G(\mathcal{O}_v)$ for $v \notin D$, and $K_v = \ker(G(\mathcal{O}_v) \to G_D)$.

The excursion action preserves the decomposition (5.20), and so gives an action of $\mathrm{Exc}(\Gamma, {}^L G)$ on each $H_c^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})_\alpha := C_c^\infty(G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F) / \prod_v K_v; k)$.

The family of functors H_I is admissible; this is an immediate consequence of the fact that Sat is already compatible compatible with composition and fusion. Hence Construction 5.2 applies to define an action of $\mathrm{Exc}(\Gamma, {}^L G)$ on $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$. Elements of the image of $\mathrm{Exc}(\Gamma, {}^L G)$ in $\mathrm{End}(C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k))$ are called "excursion operators".

5.2.4. *Xue's generalization.* Lafforgue defined an $\mathrm{Exc}(\Gamma, {}^L G)$-action on the finite-dimensional subspace of cuspidal functions $C_{\mathrm{cusp}}^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k) \subset C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$. This decomposes $C_{\mathrm{cusp}}^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$ into a direct sum of generalized eigenspaces under the action of $\mathrm{Exc}(\Gamma, {}^L G)$. Using Proposition 2.4, this decomposition corresponds to a parametrization by Langlands parameters.

Thanks to Xue's extension of the action to $\mathrm{Exc}(\Gamma, {}^L G) \curvearrowright C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$, it is meaningful to speak of Langlands parameters arising from $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$. However, since the excursion action does not stabilize any finite-dimensional subspaces of $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$ unless they are contained in the space of cusp forms, we must broaden what it means to have an L-parameter "come from" an automorphic function.

Definition 5.5. We say that an L-parameter $\rho \in H^1(\mathrm{Gal}(F^s/F), \widehat{G}(k))$ *arises from* $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$ if it arises via Proposition 2.4 from the $\mathrm{Exc}(\Gamma, {}^L G)$-action on some irreducible Hecke-subquotient of $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$; equivalently, if the corresponding maximal ideal $\mathfrak{m}_\rho \subset \mathrm{Exc}(\Gamma, {}^L G)$ is in the support of $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$ as an $\mathrm{Exc}(\Gamma, {}^L G)$-module.

We say will be called *automorphic* if it arises via Proposition 2.4 from $C_c^\infty(\mathrm{Bun}_{G,D}(\mathbf{F}_\ell); k)$ for some D; equivalently if the corresponding maximal ideal $\mathfrak{m}_\rho \subset \mathrm{Exc}(\Gamma, {}^L G)$ is in the support of $C_c^\infty(G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F); k)$ for some α.

5.3. **Excursion action on the Tate cohomology of shtukas.** For a category $\mathcal{C}$ with σ-action, we let $\mathcal{C}^{B\sigma}$ denote the category of σ-equivariant objects in $\mathcal{C}$. This comes equipped with a forgetful functor to $\mathcal{C}$.

5.3.1. *Tate cohomology of moduli of shtukas.* If σ acts on G, it induces an action $V \mapsto {}^\sigma V$ on $\mathrm{Rep}({}^L G)$.

Given a σ-equivariant representation $V \in \mathrm{Rep}_k(({}^L G^{\mathrm{alg}})^I)^{B\sigma}$, we can form $R\pi_{I!}(\mathrm{Sht}_G|_{\overline{\eta^I}}; \mathrm{Sat}(V))$ as above. The σ-equivariant structure on V equips this with a σ-equivariant structure; more formally, because Sat and π_I are σ-equivariant, $R\pi_{I!}(\mathrm{Sht}_G, \mathrm{Sat}(-))$ lifts to a functor $\mathrm{Rep}_k(({}^L G^{\mathrm{alg}})^I)^{B\sigma} \to D(X^I; k)^{B\sigma}$. Hence we can form $T^j(R\pi_{I!})(\mathrm{Sht}_{G,D,I}|_{\overline{\eta^I}}; \mathrm{Sat}(V))$, the Tate cohomology (§3.4) of $(R\pi_{I!})(\mathrm{Sht}_G|_{\eta^I}; \mathrm{Sat}(V))$; we shall always do this with respect to the *perverse* t-structure. To ease notation, we will abbreviate

$$T^j(\mathrm{Sht}_{G,D,I}; V) := T^j(R\pi_{I!})(\mathrm{Sht}_{G,D,I}|_{\overline{\eta^I}}; \mathrm{Sat}(V)). \tag{5.7}$$

Let us explain in what category we regard (5.7). Since $(R\pi_{I!})(\mathrm{Sht}_{G,D,I}|_{\overline{\eta^I}}; \mathrm{Sat}(V))$ has commuting actions of $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$ and the Hecke algebra $\mathcal{H}_G$ (the former commuting with the σ-action), its Tate cohomology has commuting actions of $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$ and of $T^0(\mathcal{H}_G)$, where Tate cohomology is formed with respect to

the σ-action. We regard (5.7) as a $T^0(\mathcal{H}_G)[\mathrm{FWeil}(\eta^I, \overline{\eta^I})]$-module, a priori. (Later we will see that the FWeil$(\eta^I, \overline{\eta^I})$-action factors uniquely through a $\pi_1(\eta, \overline{\eta})^I$-action, and it will be natural to regard (5.7) as a $T^0(\mathcal{H}_G)[\pi_1(\eta, \overline{\eta})^I]$-module.)

Using Lemma 5.9 and Lemma 3.11 we deduce the following simple but important identity: if σ acts trivially on Sht_H and $\mathcal{F}$, then

$$T^*(R\pi_{I!})(\mathrm{Sht}_{H,D,I}|_{\overline{\eta^I}}; \mathcal{F}) \cong R^*\pi_{I!}(\mathrm{Sht}_{H,D,I}|_{\overline{\eta^I}}; \mathcal{F}) \otimes T^*(k). \tag{5.8}$$

5.3.2. σ-equivariant excursion algebra.

Definition 5.6. We define the $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ to be the algebra on generators $S_{V,x,\xi,(\gamma_i)_{i \in I}}$ where

- $V \in \mathrm{Rep}_k(({}^L G)^I)^{B\sigma}$,
- $x: \mathbb{1} \to V|_{\Delta(\widehat{G})}$ and $\xi: V|_{\Delta(\widehat{G})} \to \mathbb{1}$ are σ-equivariant morphisms of $\widehat{G}$-representations, and
- $(\gamma_i)_{i \in I} \subset \Gamma^I$,

with the following relations.

(i) $S_{\emptyset,x,\xi,*} = \langle x, \xi \rangle$.

(ii) For any σ-equivariant morphism of σ-equivariant $({}^L G)^I$-representations $u: W \to W'$ and functional $\xi' \in (W')^*$ invariant under the diagonal $\widehat{G} \rtimes \sigma$-action, we have

$$S_{I,W,x,{}^t u(\xi'),(\gamma_i)_{i \in I}} = S_{I,W',u(x),\xi',(\gamma_i)_{i \in I}}, \tag{5.9}$$

where ${}^t u: (W')^* \to W^*$ denotes the dual to u.

(iii) For two tuples $(I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1})$ and $(I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2})$ as above, we have

$$S_{I_1 \sqcup I_2, W_1 \boxtimes W_2, x_1 \boxtimes x_2, \xi_1 \boxtimes \xi_2, (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} \circ S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \tag{5.10}$$

Letting $\Delta: \mathbb{1} \to \mathbb{1} \oplus \mathbb{1}$ be the diagonal inclusion, and $\nabla: \mathbb{1} \oplus \mathbb{1} \to \mathbb{1}$ the addition map, we also have

$$S_{I_1 \sqcup I_2, W_1 \oplus W_2, (x_1 \oplus x_2) \circ \Delta, \nabla \circ (\xi_1 \oplus \xi_2), (\gamma_i^1)_{i \in I_1} \times (\gamma_i^2)_{i \in I_2}} = S_{I_1, W_1, x_1, \xi_1, (\gamma_i^1)_{i \in I_1}} + S_{I_2, W_2, x_2, \xi_2, (\gamma_i^2)_{i \in I_2}}. \tag{5.11}$$

Furthermore, the assignment $(I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}) \mapsto S_{I, \boxtimes_{i \in I} V_i, x, \xi, (\gamma_i)_{i \in I}} \in \mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})^{B\sigma}$ is k-linear in x and ξ.

(iv) Let $\zeta: I \to J$ be a map of finite sets. Suppose $W \in \mathrm{Rep}(({}^L G)^I)^{B\sigma}$, $x: \mathbb{1} \to W|_{\Delta(\widehat{G})}$, $\xi: W|_{\Delta(\widehat{G})} \to \mathbb{1}$, and $(\gamma_j)_{j \in J} \in \Gamma^J$. Letting W^ζ be the restriction of W under the functor $\mathrm{Rep}(({}^L G)^I)^{B\sigma} \to \mathrm{Rep}(({}^L G)^J)^{B\sigma}$ induced by ζ, we have

$$S_{J, W^\zeta, x, \xi, (\gamma_j)_{j \in J}} = S_{I, W, x, \xi, (\gamma_{\zeta(i)})_{i \in I}}. \tag{5.12}$$

(v) Letting $\delta_W: \mathbb{1} \to W \otimes W^*$ and $\mathrm{ev}_W: W^* \otimes W \to 1$ be the natural counit and unit, we have

$$S_{I, W, x, \xi, (\gamma_i (\gamma_i')^{-1} \gamma_i'')_{i \in I}} = S_{I \sqcup I \sqcup I, W \boxtimes W^* \boxtimes W, \delta_W \boxtimes x, \xi \boxtimes \mathrm{ev}_W, (\gamma_i)_{i \in I} \times (\gamma_i')_{i \in I} \times (\gamma_i'')_{i \in I}}. \tag{5.13}$$

(vi) If W is inflated from a representation of $({}^L G^{\mathrm{alg}})^J \times \Gamma^{I \setminus J}$, then we have

$$S_{I, W, x, \xi, (\gamma_i)_{i \in I}} = S_{J, W|_{({}^L G^{\mathrm{alg}})^J}, ((1_j)_{j \in J}, (\gamma_i)_{i \in I \setminus J}) \cdot x, \xi, (\gamma_j)_{j \in J}}.$$

In short, $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ has the same type of generators and relations as in §2.4, but all data must be σ-equivariant.

Remark 5.7 (σ-action on the excursion algebra). Since σ acts on G, it acts on $\mathrm{Exc}(\Gamma, {}^L G^{\mathrm{alg}})$ by transport of structure. Concretely, we have

$$\sigma \cdot S_{V,x,\xi,(\gamma_i)_{i \in I}} = S_{\sigma(V), \sigma(x), \sigma(\xi), (\gamma_i)_{i \in I}}. \tag{5.14}$$
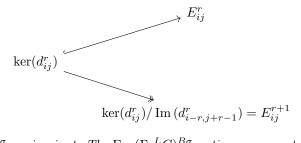
There is an obvious map $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma} \to \mathrm{Exc}(\Gamma, {}^L G)$ sending $S_{V,x,\xi,(\gamma_i)_{i \in I}} \in \mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ to the element with the same name in $\mathrm{Exc}(\Gamma, {}^L G)$.

It seems natural to ask if this map is injective and identifies $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ with the σ-invariants on $\mathrm{Exc}(\Gamma, {}^L G)^\sigma \subset \mathrm{Exc}(\Gamma, {}^L G)$. We believe this is true at least in characteristic 0.

Lemma 5.8. *Let D be any non-empty divisor on X. Recall the Tate cohomology spectral sequence (§3.4.5),*

$$E_{ij}^1 = H_c^j(\mathrm{Sht}_{G,D,\emptyset}; \mathrm{Sat}(\mathbb{1})) \implies T^{i+j}(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}).$$

(i) *There is an* $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$*-action on each page of the Tate cohomology spectral sequence* $E_{ij}^r \implies T^*(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})$*, such that the diagrams*

$$
\begin{array}{ccc}
 & & E_{ij}^r \\
 & \nearrow & \\
\ker(d_{ij}^r) & & \\
 & \searrow & \\
 & & \ker(d_{ij}^r)/\operatorname{Im}(d_{i-r,j+r-1}^r) = E_{ij}^{r+1}
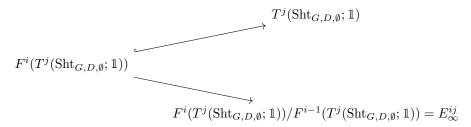\end{array}
$$

are all $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$*-equivariant. The* $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$*-action on every term for $r \geq 1$ factors through the map* $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma} \to \mathrm{Exc}(\Gamma, {}^L G)$ *from Remark 5.7.*

(ii) *There is an* $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$*-action on* $T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})$*, which preserves the (increasing) filtration* $F^\bullet T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})$ *induced by the Tate cohomology spectral sequence 3.4.5, so that it descends to an action on each* $F^i(T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}))$ *and* E_∞^{ij} *making the maps in the diagram below* $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$*-equivariant:*

$$
\begin{array}{ccc}
 & & T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}) \\
 & \nearrow & \\
F^i(T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})) & & \\
 & \searrow & \\
 & & F^i(T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}))/F^{i-1}(T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})) = E_\infty^{ij}
\end{array}
$$

where the action on E_∞^{ij} *is the same as in (i).*

Proof. For (i), we define an action of $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ on $E_1^{ij} = H_c^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})$ to be inflated via Lafforgue-Xue's action via $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma} \to \mathrm{Exc}(\Gamma, {}^L G)$. Since the action of $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ commutes with σ, it commutes with each differential in the Tate spectral sequence and so descends to each E_r^{ij}.

(ii) We begin by constructing the action. We will define a family of functors $T_I^j \colon \mathrm{Rep}(({}^L G)^I)^{B\sigma} \to \mathrm{Rep}_{T^0 \mathcal{H}_G}(\Gamma^I)$ which is compatible with composition and fusion. From this, the action of $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ is defined as in Construction 5.2. We set

$$T_I^j(V) := T^j(\mathrm{Sht}_{G,D,I}; V)$$

regarded a priori as a $T^0(\mathcal{H}_G)[\mathrm{FWeil}(\eta^I, \overline{\eta^I})]$-module (via the legs indexed by I). The compatibility with fusion and composition follow formally from these same properties of the functor Sat. The extension of the natural $\pi_1(\eta^I, \overline{\eta^I})$-action to an $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$-action using partial Frobenius is the same as in §5.2. The only issue is to check that the $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$-action on $T_I^j(V)$ factors through $\pi_1(\eta, \overline{\eta})^I$. This follows from the same argument as for the ordinary cohomology of $\mathrm{Sht}_{G,D,I}$ [Xueb, Proposition 9.1.2], noting that the step [Xueb, §8.3.11] in the proof of the Eichler-Shimura relations in the ordinary cohomology [Xueb, Lemma 8.3.7] formally implies the same relations in Tate cohomology.

By design, the resulting action of $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ commutes with σ, and hence all the maps in the Tate double complex (3.7). In particular, it preserves the filtration, hence descends to each subquotient $F^i(T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}))/F^{i-1}(T^j(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1})) = E_\infty^{ij}$. Furthermore, the action on E_∞^{ij} is compatible with the one in (i) by the Tate spectral sequences abutting to each $T^j(\mathrm{Sht}_{G,D,I}; V)$, which step-by-step identify the respective maps in Construction 5.2 constituting the excursion operators for each action. $\qed$

5.4. **Analysis of fixed points.**

We study the σ-fixed points of $\mathrm{Sht}_{G,D,I}$, in anticipation of applying the theory of §3 to it.

According to [Var04, Proposition 2.16] (stated there for split G, but valid for all G by the same argument), $\mathrm{Sht}_{G,D,I}$ is exhausted by open substacks $\mathrm{Sht}_{G,D,I}^{\leq P}$ as P runs over Harder-Narasimhan (HN) polygons for G.

The open substack is determined by the Cartesian square

$$
\begin{array}{ccc}
\mathrm{Sht}^{\leq P}_{G,D,I} & \hookrightarrow & \mathrm{Sht}_{G,D,I} \\
\downarrow & & \downarrow \\
\mathrm{Bun}^{\leq P}_{G} & \hookrightarrow & \mathrm{Bun}_{G}
\end{array}
$$

Furthermore, for fixed P the Deligne-Mumford stack $\mathrm{Sht}^{\leq P}_{G,D,I}$ can be presented as a quotient of a quasi-projective scheme by a finite group; for any closed point $x_0 \in X$, the quasi-projective scheme can be taken to be $\mathrm{Sht}^{\leq P}_{G,D+nx_0,I}$ for sufficiently large n relative to P, and the group is then the automorphisms of the level structure. The same applies for the variants $\mathrm{Sht}^{(I_1,\ldots,I_r)}_{G,D,I}$.

We fix the following notation below. Let P be an HN polygon for H, and let $\widetilde{P}$ be the induced HN polygon for G. Then we have a Cartesian square

$$
\begin{array}{ccc}
\mathrm{Bun}^{\leq P}_{\widetilde{H}} & \longrightarrow & \mathrm{Bun}_{H} \\
\downarrow & & \downarrow \\
\mathrm{Bun}^{\leq \widetilde{P}}_{\widetilde{G}} & \longrightarrow & \mathrm{Bun}_{G}
\end{array}
$$

which induces the Cartesian square

$$
\begin{array}{ccc}
\mathrm{Sht}^{(I_1,\ldots,I_r),\leq P}_{H,D,I} & \longrightarrow & \mathrm{Sht}^{(I_1,\ldots,I_r)}_{H,D,I} \\
\downarrow & & \downarrow \\
\mathrm{Sht}^{(I_1,\ldots,I_r),\leq \widetilde{P}}_{G,D,I} & \longrightarrow & \mathrm{Sht}^{(I_1,\ldots,I_r)}_{G,D,I}
\end{array}
\qquad (5.15)
$$

Lemma 5.9. *If n is sufficiently large so that $\mathrm{Sht}^{(I_1,\ldots,I_r),\leq P}_{H,D+nx_0,I}$ and $\mathrm{Sht}^{(I_1,\ldots,I_r),\leq \widetilde{P}}_{G,D+nx_0,I}$ are representable by schemes, then the diagonal map $H \to G$ induces an isomorphism*

$$
\mathrm{Sht}^{(I_1,\ldots,I_r),\leq P}_{H,D+nx_0,I} \xrightarrow{\sim} (\mathrm{Sht}^{(I_1,\ldots,I_r),\leq \widetilde{P}}_{G,D+nx_0,I})^\sigma.
$$

Proof. For notational convenience we just treat the case of non-iterated shtukas, $\mathrm{Sht}_{G,D,I}$; the general case is essentially the same but with cumbersome extra notation.

There is an obvious map in one direction, $\mathrm{Sht}^{\leq P}_{H,D+nx_0,I} \to (\mathrm{Sht}^{\leq \widetilde{P}}_{G,D+nx_0,I})^\sigma$. We will construct the inverse.

Notate the S-points of $\mathrm{Sht}^{\leq \widetilde{P}}_{G,D+nx_0,I}$ as the set $\{(\{x_i\}_{i\in I}, \mathcal{E}, \varphi, \upsilon)\}$. For any S, there is an equivalence of categories between $\mathrm{Res}_{X'/X}(H)$-torsors on X_S and H-torsors on X'_S, which we denote $\mathcal{E} \mapsto \mathcal{E}'$. The datum of a σ-fixed point of $\mathrm{Bun}_{G,D}$ translates under the above equivalence to the datum of an H-torsor $\mathcal{E}'$ on X'_S together with an isomorphism $h \colon \mathcal{E}' \xrightarrow{\sim} \sigma^* \mathcal{E}'$. We claim that, since the point $(\{x_i\}_{i\in I}, \mathcal{E}, \varphi, \upsilon)$ has no non-trivial automorphisms, such an isomorphism automatically satisfies the cocycle condition, hence is equivalent to a descent datum from $\mathcal{E}'$ to an H-torsor over X_S. Furthermore, the map φ and level structure υ will similarly descend uniquely.

Let $\mathrm{Nm}(h) := \sigma^{p-1}h \circ \ldots \circ \sigma h \circ h \colon \mathcal{E}' \to \mathcal{E}'$. The claim amounts to checking that $\mathrm{Nm}(h)$ is the identity automorphism of $\mathcal{E}'$. By definition, it corresponds to some automorphism of $\mathcal{E}$ compatible with φ and the level structure υ. But by assumption, this datum had no non-trivial automorphisms, so $\mathrm{Nm}(h)$ can only be the identity automorphism.

This constructs a map $\mathrm{Sht}_{H,D+nx_0,I} \leftarrow (\mathrm{Sht}^{\leq \widetilde{P}}_{G,D+nx_0,I})^\sigma$ which is manifestly a one-sided inverse; we conclude by using (5.15) to see that it lands in $\mathrm{Sht}^{\leq P}_{H,D+nx_0,I}$. $\qquad\square$

5.5. Cohomology at infinite level.
We will use Lemma 5.9 to apply Smith theory. However, the excursion action does not stabilize the piece of cohomology coming from bounding the HN polygon, so we need to let P and n both go "off to infinity".

Definition 5.10. Fix a closed point $x_0 \in X$ and consider the system of Deligne-Mumford stacks, $\{\mathrm{Sht}_{H,D+nx_0,I}^{\leq P}\}$ as n and P vary. For $V \in \mathrm{Rep}((^L H^{\mathrm{alg}})^I)$, we *define*

$$R\pi_{I!}(\mathrm{Sht}_{H,D+\infty x_0,I}\,|_{\overline{\eta^I}}; \mathrm{Sat}(V)) = \varinjlim_{n,P} R\pi_{I!}(\mathrm{Sht}_{H,D+nx_0,I}^{\leq P}\,|_{\overline{\eta^I}}; \mathrm{Sat}(V))$$

where the maps in the P variable are the covariant maps induced by open embeddings, while the maps in the n variable are the contravariant maps induced by pullback. Note that the colimit is *filtered* because both indexing posets are filtered.

Remark 5.11. As explained above, for any fixed P, and all sufficiently large n depending on P, $\mathrm{Sht}_{H,D+nx_0,I}^{\leq P}$ is representable by a scheme. Hence, the subposet of indices (n, P) for which $\mathrm{Sht}_{H,D+nx_0,I}^{\leq P}$ is representable by a scheme is cofinal, so $R\pi_{I!}(\mathrm{Sht}_{H,D+\infty x_0,I}\,|_{\overline{\eta^I}}; \mathrm{Sat}(V))$ is naturally isomorphic to the colimit taken along this subposet.

Furthermore, note that for any cofinal subposet of HN polygons P for H, the induced HN polygons $\widetilde{P}$ form a cofinal poset for G.

Definition 5.12. Fix a closed point $x_0 \in X$ and $V \in \mathrm{Rep}_k((^L H^{\mathrm{alg}})^I)^{B\sigma}$. We define

$$T^j(\mathrm{Sht}_{H,D+\infty x_0,I}; V) := T^j(R\pi_{I!})(\mathrm{Sht}_{H,D+\infty x_0,I}\,|_{\overline{\eta^I}}; \mathrm{Sat}(V)).$$

We note that $R\pi_{I!}(\mathrm{Sht}_{H,D+\infty x_0,I}\,|_{\overline{\eta^I}}; \mathrm{Sat}(V))$ is bounded, since the dimension of the support of $\mathrm{Sat}(V)$ on each $\mathrm{Sht}_{H,D+nx_0,I}^{\leq P}\,|_{\overline{\eta^I}}$ is uniformly bounded for all n, P.

5.6. **Equivariant localization for excursion operators.** We define $\mathrm{Nm}\colon \mathrm{Rep}_k((^L G)^I) \to \mathrm{Rep}_k((^L G)^I)^{B\sigma}$ to be the functor taking a representation V to $V \otimes_k {}^{\sigma}V \otimes_k \ldots \otimes_k {}^{\sigma^{p-1}}V$, with the σ-equivariant structure

$$^{\sigma}\mathrm{Nm}(V) = {}^{\sigma}V \otimes_k {}^{\sigma^2}V \otimes_k \ldots \otimes_k {}^{\sigma^{p-1}}V \otimes_k V \xrightarrow{\sim} V \otimes_k {}^{\sigma}V \otimes_k \ldots \otimes_k {}^{\sigma^{p-1}}V = \mathrm{Nm}(V)$$

given by the commutativity constraint for tensor products. It corresponds under Geometric Satake to Definition 4.13. Given $h\colon V \to V' \in \mathrm{Rep}_k((^L G)^I)$, we set

$$\mathrm{Nm}(h) := h \otimes {}^{\sigma}h \otimes \ldots \otimes {}^{\sigma^{p-1}}h \colon \mathrm{Nm}(V) \to \mathrm{Nm}(V').$$

Note that Nm is *not* an additive functor, nor is it even k-linear. We linearize it by defining $\mathrm{Nm}^{(p^{-1})} := \mathrm{Frob}_p^{-1} \circ \mathrm{Nm}$, where (as in §4.6.3) Frob_p^{-1} is the identity on objects and on morphisms it is $(-) \otimes_{k,\mathrm{Frob}_p^{-1}} k$. Then $\mathrm{Nm}^{(p^{-1})}\colon \mathrm{Rep}_k((^L G)^I) \to \mathrm{Rep}_k((^L G)^I)^{B\sigma}$ is k-linear, although still not additive.

For $V \in \mathrm{Rep}_k((^L G)^I)$, we denote by $N \cdot V$ the σ-equivariant representation $V \oplus {}^{\sigma}V \oplus \ldots \oplus {}^{\sigma^{p-1}}V$, with σ-equivariant structure

$$^{\sigma}(N \cdot V) = {}^{\sigma}V \oplus {}^{\sigma^2}V \oplus \ldots \oplus {}^{\sigma^{p-1}}V \oplus V \xrightarrow{\sim} V \oplus {}^{\sigma}V \oplus \ldots \oplus {}^{\sigma^{p-1}}V = (N \cdot V)$$

given by the commutativity constraint for direct sums. For $h\colon V \to V' \in \mathrm{Rep}_k((^L G)^I)$, we denote by $N \cdot h\colon N \cdot V \to N \cdot V'$ the σ-equivariant map $h \oplus {}^{\sigma}h \oplus \ldots \oplus {}^{\sigma^{p-1}}h$. Let $\Delta_p\colon \mathbb{1} \to \mathbb{1}^{\oplus p}$ denote the diagonal map and $\nabla_p\colon \mathbb{1}^{\oplus p} \to \mathbb{1}$ denote the sum map.

Our goal in this subsection is to prove the theorem below.

Theorem 5.13. *Fixed a closed point x_0 on X and let D be any divisor on X.*

(i) The action of $S_{\mathrm{Nm}^{(p^{-1})}(V),\mathrm{Nm}^{(p^{-1})}(x),\mathrm{Nm}^{(p^{-1})}(\xi),(\gamma_i)_{i \in I}}$ on $T^*(\mathrm{Sht}_{G,D+\infty x_0}; \mathbb{1})$ is identified with the action of $S_{\mathrm{Res}_{\mathsf{BC}}(V),x,\xi,(\gamma_i)_{i \in I}}$ on $T^*(\mathrm{Sht}_{H,D+\infty x_0}; \mathbb{1})$.

(ii) The action of $S_{N \cdot V,(N \cdot x) \circ \Delta_p,\nabla_p \circ (N \cdot \xi),(\gamma_i)_{i \in I}}$ on $T^*(\mathrm{Sht}_{G,D+\infty x_0}; \mathbb{1})$ is 0.

This will be established in several steps. The heart of the matter is the following equivariant localization isomorphism.

Proposition 5.14. *Fixed a closed point x_0 on X and let D be any divisor on X. Let G be a reductive group over F. We equip $\mathrm{Tilt}_k(\widehat{G}^I)$ and $\mathrm{Rep}_k(\widehat{G}^I)$ with the $\pi_1(\eta^I, \overline{\eta^I})$-actions coming from the geometric action of $\pi_1(\eta^I, \overline{\eta^I})$ on $\widehat{G}^I$ (§2.1.2).*

(i) For each $j \in \{0,1\}$, there is a natural isomorphism of $\pi_1(\eta^I, \overline{\eta^I})$-equivariant functors $\mathrm{Rep}_k(\widehat{G}^I) \to \mathrm{Vect}_{/k}$,

$$T^j(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathrm{Nm}^{(p^{-1})}(V)) \cong T^j(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Sat}(\mathrm{Res}_{\mathsf{BC}}(V))), \quad V \in \mathrm{Rep}_k(\widehat{G}^I).$$

(ii) For each $j \in \{0,1\}$, there is a natural isomorphism of functors in $V \in \mathrm{Rep}_k(({}^L G)^I) \to \mathrm{Mod}_k(\pi(\eta, \overline{\eta})^I)$,

$$T^*(\mathrm{Sht}_{G,D+\infty x_0, I}; \mathrm{Nm}^{(p^{-1})}(V)) \cong T^*(\mathrm{Sht}_{H,D+\infty x_0, I}; \mathrm{Res}_{\mathsf{BC}}(V)), \quad V \in \mathrm{Rep}_k(({}^L G)^I), \tag{5.16}$$

which is compatible with fusion and composition.

Proof. (i) We first establish the statement on the subcategory of tilting modules: for $j \in \{0,1\}$, there is a natural isomorphism of $\pi_1(\eta^I, \overline{\eta^I})$-equivariant functors $\mathrm{Tilt}_k(\widehat{G}^I) \to \mathrm{Vect}_{/k}$,

$$T^j(\mathrm{Sht}_{G,D+\infty x_0, I}; \mathrm{Nm}^{(p^{-1})}(V)) \cong T^j(\mathrm{Sht}_{H,D+\infty x_0, I}; \mathrm{Sat}(\mathrm{Res}_{\mathsf{BC}}(V))), \quad V \in \mathrm{Tilt}_k(\widehat{G}^I). \tag{5.17}$$

The commutative diagram

$$\begin{array}{ccc} \mathrm{Sht}_{H,I} & \longrightarrow & \mathrm{Hk}_{H,I} \\ \downarrow & & \downarrow \\ \mathrm{Sht}_{G,I} & \longrightarrow & \mathrm{Hk}_{G,I} \end{array}$$

induces a natural isomorphism between the two composite pullback functors $D(\mathrm{Hk}_{G,I}\mid_{\overline{\eta^I}}; k) \to D(\mathrm{Sht}_{G,I}\mid_{\overline{\eta^I}}; k) \to D(\mathrm{Sht}_{H,I}\mid_{\overline{\eta^I}}; k)$ and $D(\mathrm{Hk}_{G,I}\mid_{\overline{\eta^I}}; k) \to D(\mathrm{Hk}_{H,I}\mid_{\overline{\eta^I}}; k) \to D(\mathrm{Sht}_{H,I}\mid_{\overline{\eta^I}}; k)$ equivariant for the $\pi_1(\eta^I, \overline{\eta^I})$-actions. The same discussion applies with any level structure and HN truncation.

Using Lemma 5.9 to identify the σ-fixed points of $\mathrm{Sht}_{G,D+\infty x_0, I}$, this in turn induces $\pi_1(\eta^I, \overline{\eta^I})$-equivariant natural isomorphisms between the following two functors $\mathrm{Tilt}_k(\widehat{G}^I) \to \mathrm{Shv}(\mathrm{Sht}_{H,D+nx_0, I}\mid_{\overline{\eta^I}}; \mathcal{T}_k)$.

- The functor sending $V \in \mathrm{Tilt}_k(\widehat{G}^I)$ to $\mathrm{Frob}_p^{-1} \circ \mathrm{Psm}(\mathrm{Nm}(\mathrm{Sat}(V))) \in \mathrm{Shv}(\mathrm{Sht}_{H,D+nx_0, I}\mid_{\overline{\eta^I}}; \mathcal{T}_k)$ (here again, Frob_p^{-1} is the automorphism of the k-linear category of sheaves on $\mathrm{Sht}_{H,D,I}$ obtained by applying $(-) \otimes_{k, \mathrm{Frob}_p^{-1}} k$ to spaces of morphisms).
- The functor sending V to $\mathbb{T}\mathrm{Sat}(\mathrm{Res}_{\mathsf{BC}}(V)) \in \mathrm{Shv}(\mathrm{Sht}_{H,D+nx_0, I}\mid_{\overline{\eta^I}}; \mathcal{T}_k)$.

For any HN polygon P for H, inducing the HN polygon $\widetilde{P}$ for G, and n sufficiently large so that that the moduli spaces are schemes and Lemma 5.9 applies, we may apply the equivariant localization results from §3.4.6 to obtain:

$$T^j(\mathrm{Sht}_{G,D+nx_0, I}^{\leq \widetilde{P}}; \mathrm{Nm}^{(p^{-1})}(V)) := T^j(\mathrm{Sht}_{G,D+nx_0, I}^{\leq \widetilde{P}}; \mathrm{Sat}(\mathrm{Nm}^{(p^{-1})}(V)))$$

$$\text{Lemma 5.9 and (3.8)} \implies \cong T^j(\mathrm{Sht}_{H,D+nx_0, I}^{\leq P}; \mathrm{Frob}_p^{-1} \circ \mathrm{Psm}(\mathrm{Nm}(\mathrm{Sat}(V))))$$

$$\text{Theorem 4.20} \implies \cong T^j(\mathrm{Sht}_{H,D+nx_0, I}^{\leq P}; \mathrm{Sat}(\mathrm{Res}_{\mathsf{BC}}(V))). \tag{5.18}$$

Moreover, these isomorphisms are natural in V, and all functors involved are $\pi_1(\eta^I, \overline{\eta^I})$-equivariant: the only case in which equivariance is not immediate is established in Lemma 4.21. We conclude by taking the colimit of these isomorphisms along such n and P, using that they form a cofinal poset by Remark 5.11.

Next we bootstrap from $\mathrm{Tilt}_k(\widehat{G}^I)$ to $\mathrm{Rep}_k(\widehat{G}^I)$. The functor on the RHS of (5.17) is evidently additive, and extends uniquely to a $\pi_1(\eta^I, \overline{\eta^I})$-equivariant triangulated functor

$$(\mathscr{T}_2^j)^{\mathrm{de-eq}} \colon K^b(\mathrm{Tilt}_k(\widehat{G}^I)) \to \mathrm{Vect}_{/k}.$$

Therefore the same holds for the functor on the LHS of (5.17): it extends uniquely to a $\pi_1(\eta^I, \overline{\eta^I})$-equivariant triangulated functor

$$(\mathscr{T}_1^j)^{\mathrm{de-eq}} \colon K^b(\mathrm{Tilt}_k(\widehat{G}^I)) \to \mathrm{Vect}_{/k}$$

and there is a natural isomorphism $(\mathscr{T}_1^j)^{\mathrm{de-eq}} \cong (\mathscr{T}_2^j)^{\mathrm{de-eq}}$.

By Theorem 4.10, we have $K^b(\mathrm{Tilt}_k(\widehat{G}^I)) \cong D^b(\mathrm{Rep}_k(\widehat{G}^I))$, so these extensions may be viewed as defined on $D^b(\mathrm{Rep}_k(\widehat{G}^I))$. Then restricting the isomorphism $(\mathscr{T}_1^j)^{\mathrm{de-eq}} \cong (\mathscr{T}_2^j)^{\mathrm{de-eq}}$ along $\mathrm{Rep}_k(\widehat{G}^I) \to D^b(\mathrm{Rep}_k(\widehat{G}^I))$ completes the proof of (i).

(ii) Since the $\mathrm{FWeil}(\eta^I, \overline{\eta^I})$-actions on $T^*(\mathrm{Sht}_{G,D+\infty x_0}; \mathrm{Nm}^{(p^{-1})}(V))$ and on $T^*(\mathrm{Sht}_{H,D+\infty x_0}; \mathrm{Res}_{\mathsf{BC}}(V))$ are determined by their respective $\pi_1(\eta^I, \overline{\eta^I})$-actions plus partial Frobenius morphisms, we can and will focus on these two equivariance structures separately, starting with the $\pi_1(\eta^I, \overline{\eta^I})$-actions.

Equivariantizing the natural isomorphism $(\mathscr{T}_1^j)^{\mathrm{de-eq}} \cong (\mathscr{T}_2^j)^{\mathrm{de-eq}} \colon D^b(\mathrm{Rep}_k(\widehat{G}^I)) \to \mathrm{Vect}_{/k}$ from (i) with respect to the $\pi_1(\eta^I, \overline{\eta^I})$-actions yields a natural isomorphism of functors

$$\mathscr{T}_1^j \cong \mathscr{T}_2^j \colon D^b(\mathrm{Rep}_k(\widehat{G}^I))^{B\pi_1(\eta^I, \overline{\eta^I})} \to \mathrm{Mod}_k(\pi_1(\eta^I, \overline{\eta^I})).$$

The restriction of $\mathscr{T}_1^j$ along $\mathrm{Rep}_k(({}^L G)^I) \cong \mathrm{Rep}_k(\widehat{G}^I)^{B\pi_1(\eta^I, \overline{\eta^I})} \to D^b(\mathrm{Rep}_k(\widehat{G}^I))^{B\pi_1(\eta^I, \overline{\eta^I})}$ recovers the LHS of (5.16), and the restriction of $\mathscr{T}_2^j$ along the same composition recovers the RHS of (5.16). This provides the desired natural isomorphism.

Finally, we check the compatibility with partial Frobenius. We want to show that the diagram

$$
\begin{array}{ccc}
F_{\{1\}}^* T^j(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathrm{Nm}^{(p^{-1})}(V)) & \overset{\sim}{\longrightarrow} & T^j(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathrm{Nm}^{(p^{-1})}(V)) \\
\downarrow\sim & & \downarrow\sim \\
F_{\{1\}}^* T^j(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Res}_{\mathsf{BC}}(V)) & \overset{\sim}{\longrightarrow} & T^j(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Res}_{\mathsf{BC}}(V))
\end{array}
\tag{5.19}
$$

commutes, where the vertical isomorphisms (as $\pi_1(\eta^I, \overline{\eta^I})$-modules) have just been established. By Lemma 5.9, there is a cofinal system of n, P, P' such that applying σ-fixed points to the diagram

$$F_{\{1\}} \colon \mathrm{Sht}_{G,D+nx_0,I}^{(\{1\},\{2\},\dots,\{r\}),\leq \widetilde{P}} \to \mathrm{Sht}_{G,D,I}^{(\{2\},\dots,\{r\},\{1\}),\leq \widetilde{P}'}$$

yields the diagram

$$F_{\{1\}} \colon \mathrm{Sht}_{H,D+nx_0,I}^{(\{1\},\{2\},\dots,\{r\}),\leq P} \to \mathrm{Sht}_{H,D+nx_0,I}^{(\{2\},\dots,\{r\},\{1\}),\leq P'}.$$

(The need for P' arises because $F_{\{1\}}$ does not preserve HN polygons.) This implies that the isomorphisms (5.18) are compatible with the maps $F_{\{1\}}^*$. Taking the filtered colimit along such n, P, P' completes the proof. □

Proof of Theorem 5.13. (i) Proposition 5.14(ii) gives a chain of compatible identifications

$$
\begin{array}{ccccccc}
T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathbb{1}) & \xrightarrow{\mathrm{Nm}^{(p^{-1})}(x)} & T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathrm{Nm}^{(p^{-1})}(V)) & \xrightarrow{(\gamma_i)_{i\in I}} & T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathrm{Nm}^{(p^{-1})}(V)) & \xrightarrow{\mathrm{Nm}^{(p^{-1})}(\xi)} & T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathbb{1}) \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathbb{1}) & \xrightarrow{x} & T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Res}_{\mathsf{BC}}(V)) & \xrightarrow{(\gamma_i)_{i\in I}} & T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Res}_{\mathsf{BC}}(V)) & \xrightarrow{\xi} & T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathbb{1})
\end{array}
$$

The operator $S_{\mathrm{Nm}^{(p^{-1})}(V),\mathrm{Nm}^{(p^{-1})}(x),\mathrm{Nm}^{(p^{-1})}(\xi),(\gamma_i)_{i\in I}}$ on $T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathbb{1})$ is obtained by tracing along the upper row, while the operator $S_{\mathrm{Res}_{\mathsf{BC}}(V),x,\xi,(\gamma_i)_{i\in I}}$ on $T^*(\mathrm{Sht}_H; \mathbb{1})$ is obtained by tracing along the lower row. Hence they coincide under the vertical identifications.

(ii) By Lemma 5.9 and §3.8 we have a chain of compatible identifications

$$
\begin{array}{ccccccc}
T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathbb{1}) & \xrightarrow{(N\cdot x)\circ\Delta_p} & T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; N\cdot V) & \xrightarrow{(\gamma_i)_{i\in I}} & T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; N\cdot V) & \xrightarrow{\nabla_p\circ(N\cdot\xi)} & T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathbb{1}) \\
\downarrow\sim & & \downarrow\sim & & \downarrow\sim & & \downarrow\sim \\
T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathbb{1}) & \xrightarrow{(N\cdot x)\circ\Delta_p} & T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Psm}(N\cdot V)) & \xrightarrow{(\gamma_i)_{i\in I}} & T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Psm}(N\cdot V)) & \xrightarrow{\nabla_p\circ(N\cdot\xi)} & T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathbb{1})
\end{array}
$$

The operator $S_{N\cdot V,(N\cdot x)\circ\Delta_p,\nabla_p\circ(N\cdot\xi),(\gamma_i)_{i\in I}}$ on $T^*(\mathrm{Sht}_{G,D+\infty x_0,I}; \mathbb{1})$ is obtained by tracing along the upper row. But the stalks and costalks of $N\cdot\mathrm{Sat}(V)|_{\mathrm{Gr}_H}$ are all complexes of induced $k[\sigma]$-modules, and in fact they are perfect complexes over $k[\sigma]$. Hence $\mathrm{Psm}(N\cdot V)$ is equivalent to 0 in the Tate category of $\mathrm{Sht}_{H,D,I}$ for all D, so $T^*(\mathrm{Sht}_{H,D+\infty x_0,I}; \mathrm{Psm}(N\cdot V)) = 0$. Therefore the endomorphism in question factors through the zero map, hence is itself zero. □

5.7. Applications to base change for automorphic forms.

In §5.2 we described Lafforgue's action of $\mathrm{Exc}(\Gamma, {}^L G)$ on $H_\emptyset(\mathbb{1})$. By (5.6), we have

$$H_\emptyset(\mathbb{1}) = \bigoplus_{\alpha \in \ker^1(F,G)} C_c^\infty(G_\alpha(F) \backslash G_\alpha(\mathbf{A}_F) / \prod_v K_v; k).$$

Here $\ker^1(F,G) := \ker(H^1(F,G) \to \prod_v H^1(F_v,G))$ is the isomorphism class of the generic fiber of the G-torsor. More generally, this defines a decomposition

$$\mathrm{Sht}_{G,D,I} = \coprod_{\alpha \in \ker^1(F,G)} (\mathrm{Sht}_{G,D,I})_\alpha \tag{5.20}$$

according to the isomorphism class of the generic fiber of $\mathcal{E}$.

In the base change situation, the "diagonal embedding" map $\phi\colon H \to G$ induces a map $\phi_*\colon \ker^1(F, H) \to \ker^1(F, G)$, compatible with the map $\mathrm{Bun}_{H,D}(\mathbf{F}_\ell) \to \mathrm{Bun}_{G,D}(\mathbf{F}_\ell)$. Theorem 1.6 is evidently implied by the theorem below, whose proof occupies this subsection.

Theorem 5.15. *Fix any closed point $x_0 \in X$ and any divisor D on X. Let $[\rho] \in H^1(\Gamma, \widehat{H}(k))$ be an L-parameter arising from the action of $\mathrm{Exc}(\Gamma, {}^L H)$ on $H^0_c(\mathrm{Sht}_{H, D+\infty x_0, I}; \mathrm{Sat}(\mathbb{1}))_\alpha$ in the sense of §5.2.4. Then the image of $[\rho]$ in $H^1(\Gamma, \widehat{G}(k))$ arises in the action of $\mathrm{Exc}(\Gamma, {}^L G)$ on $H^0_c(\mathrm{Sht}_{G, D+\infty x_0, I}; \mathrm{Sat}(\mathbb{1}))_{\phi(\alpha)}$ in the sense of §5.2.4.*

We establish some preliminaries in preparation for the proof.

Definition 5.16 (The Tate diagonal)**.** For a commutative algebra A in characteristic p with σ-action, we denote by $N \cdot A$ the subset consisting of elements of the form $(1 + \sigma + \ldots + \sigma^{p-1})a$ for $a \in A$. One easily checks that $N \cdot A$ is an ideal in A^σ.

We denote by $\mathrm{Nm}\colon A \to A^\sigma$ the set map sending $a \mapsto a \cdot \sigma(a) \cdot \ldots \cdot \sigma^{p-1}(a)$. It is multiplicative but not additive. It is an exercise to verify that the composition of Nm with the quotient $A^\sigma \twoheadrightarrow A^\sigma/N \cdot A$ is an algebra homomorphism, which we call the *Tate diagonal homomorphism* $\Delta^p\colon A \to T^0(A)$.

Lemma 5.17. *Let A be a commutative ring over $\mathbf{F}_p$ with a σ-action. Let $A' \subset A^\sigma$ be a subring containing $\mathrm{Nm}(A)$ and $N \cdot A$. (Since $N \cdot A$ is an ideal in A^σ, it is also an ideal in any such A'.) Any character $\chi\colon A' \to k$ factoring through $A'/N \cdot A$ extends uniquely to a character $\widetilde{\chi}\colon A \to k$, which is given by*

$$\widetilde{\chi}(a) = \chi(\mathrm{Nm}(a))^{1/p}. \tag{5.21}$$

Proof. The same proof as that of [TV16, §3.4] works, but since our situation is a little more general we reproduce it. One easily checks that the given formula (5.21) works (it is a ring homomorphism since k is in characteristic p, and it clearly extends χ).

Next we check that it is the unique extension. Note that σ acts on characters of A by pre-composition; we denote this action by $\widetilde{\chi} \mapsto \sigma \cdot \widetilde{\chi}$. Clearly (5.21) is the unique σ-fixed extension, so we will show that any extension $\widetilde{\chi}'$ must be σ-fixed. Indeed, since any extension $\widetilde{\chi}'$ is trivial on $N \cdot A$ by the assumption that χ factors through A'/NA, we have

$$\sum_{i=0}^{p-1} \sigma^i \cdot \widetilde{\chi}' = 0.$$

By linear independence of characters [Sta20, Tag 0CKK] we must have $\sigma^i \cdot \widetilde{\chi}' = \widetilde{\chi}'$ for all i, i.e. $\widetilde{\chi}'$ is σ-fixed. $\qquad\square$

Lemma 5.18. *Inside $\mathrm{Exc}(\Gamma, {}^L G)$ we have*

$$\mathrm{Nm}(S_{V, x, \xi, (\gamma_i)_{i \in I}}) = S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}}$$

and

$$N \cdot S_{V, x, \xi, (\gamma_i)_{i \in I}} = S_{N \cdot V, (N \cdot x) \circ \Delta_p, \nabla_p \circ (N \cdot \xi), (\gamma_i)_{i \in I}}.$$

Proof. The first equality follows from repeated application of the relations (2.7), (2.5) and the explicit description of the σ-action in (5.14). The second equality follows from repeated application of relations (2.7), (2.6) and the explicit description of the σ-action in (5.14). $\qquad\square$

Note that $\mathrm{Exc}(\Gamma, {}^L H)$ has an $\mathbf{F}_p$-structure coming via the presentation §2.2 from the $\mathbf{F}_p$-structure on $\mathcal{O}(\widehat{G}_k \backslash ({}^L G_k^{\mathrm{alg}}) / \widehat{G}_k)$ given by $\mathcal{O}(\widehat{G}_{\mathbf{F}_p} \backslash ({}^L G_{\mathbf{F}_p}^{\mathrm{alg}}) / \widehat{G}_{\mathbf{F}_p})$. Hence $\mathrm{Aut}(k)$ acts by semi-linear automorphisms on $\mathrm{Exc}(\Gamma, {}^L H)$. By inspecting the comparison between the presentations in §2.2 and §2.4, we see that the Frobenius automorphism $\mathrm{Frob}_p \in \mathrm{Aut}(k)$ sends

$$S_{\mathrm{Nm}^{(p-1)}(V), \mathrm{Nm}^{(p-1)}(x), \mathrm{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}} \mapsto S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}}. \tag{5.22}$$

Definition 5.19. Let $\mathrm{Exc}(\Gamma, {}^L G)' \subset \mathrm{Exc}(\Gamma, {}^L G)$ be the subalgebra generated by $N \cdot \mathrm{Exc}(\Gamma, {}^L G)$ and all elements of the form $\mathrm{Nm}(S_{V, x, \xi, (\gamma_i)_{i \in I}}) = S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}}$ (the equality by Lemma 5.18).

Proof of Theorem 5.15. The L-parameter $[\rho] \in H^1(\Gamma, \widehat{H}(k))$ corresponds to a character $\chi_\rho \colon \mathrm{Exc}(\Gamma, {}^L H) \to k$ under Proposition 2.4. The assumption that χ_ρ appears in the action of $\mathrm{Exc}(\Gamma, {}^L H)$ on $H^0_c(\mathrm{Sht}_{H, D + \infty x_0, \emptyset}; \mathbb{1})_\alpha$ implies that there is a vector v_ρ in a subquotient Q of $H^0_c(\mathrm{Sht}_{H, D + \infty x_0, I}; \mathbb{1})_\alpha$ on which $S \in \mathrm{Exc}(\Gamma, {}^L H)$ acts as multiplication by $\chi_\rho(S) \in k$. Since the action of $\mathrm{Exc}(\Gamma, {}^L H)$ on $H^0_c(\mathrm{Sht}_{H, D + \infty x_0, \emptyset}; \mathbb{1})_\alpha$ is defined over $\mathbf{F}_p \subset k$, the semi-linear actions of $\mathrm{Aut}(k)$ on $\mathrm{Exc}(\Gamma, {}^L H)$ and $H^0_c(\mathrm{Sht}_{H, D + \infty x_0, \emptyset}; \mathbb{1})_\alpha$ satisfy

$$\gamma(S \cdot v) = \gamma(S) \cdot \gamma(v), \quad \text{for all } \gamma \in \mathrm{Aut}(k), \ S \in \mathrm{Exc}(\Gamma, {}^L H), \ v \in H^0_c(\mathrm{Sht}_{H, D + \infty x_0, \emptyset}; \mathbb{1})_\alpha. \tag{5.23}$$

In particular, taking $\gamma = \mathrm{Frob}_p \in \mathrm{Aut}(k)$, we see that $\mathrm{Frob}_p(S) \in \mathrm{Exc}(\Gamma, {}^L H)$ acts $v_\rho^{(p)} := \mathrm{Frob}_p(v_\rho) \in \mathrm{Frob}_p(Q)$ as multiplication by $\chi_\rho(S)^p$.

The decomposition (5.20) induces a compatible direct sum decomposition of Tate cohomology and the Tate spectral sequence, and we denote by a subscript $\alpha \in \ker^1(F, G)$ the summand indexed by α. By (5.8), this eigenvector $v_\rho^{(p)}$ maps to a *non-zero* $\overline{v}_\rho^{(p)}$ in some $\mathrm{Exc}(\Gamma, {}^L H)$-equivariant finite-dimensional subquotient of $(E^{ij}_\infty)_\alpha$, and the latter is itself a subquotient of $T^*(\mathrm{Sht}_{H, D + \infty x_0, I}; \mathbb{1})_\alpha$. By Lemma 5.8, $\overline{v}_\rho^{(p)}$ is also an eigenvector for $\mathrm{Exc}(\Gamma, {}^L H)$ with the same eigensystem as $v_\rho^{(p)}$, namely $(\chi_\rho)^p$.

By Theorem 5.13 and Lemma 5.8, $\mathrm{Exc}(\Gamma, {}^L G)^{B\sigma}$ acts on $\overline{v}_\rho$ with eigensystem

$$S_{\mathrm{Nm}^{(p-1)}(V), \mathrm{Nm}^{(p-1)}(x), \mathrm{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}} \cdot \overline{v}_\rho = \chi_\rho(S_{\mathrm{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}) \overline{v}_\rho,$$

and (using Lemma 5.18) $N \cdot S$ acts by 0 for any $S \in \mathrm{Exc}(\Gamma, {}^L G)$. Let $\mathrm{Exc}(\Gamma, {}^L G)' \subset \mathrm{Exc}(\Gamma, {}^L G)$ be the subalgebra defined in Definition 5.19. Since $\mathrm{Frob}_p \in \mathrm{Aut}(k)$ takes $S_{\mathrm{Nm}^{(p-1)}(V), \mathrm{Nm}^{(p-1)}(x), \mathrm{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}}$ to $S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}}$ by (5.22), (5.23) implies that $\overline{v}_\rho^{(p)}$ is an eigenvector for $\mathrm{Exc}(\Gamma, {}^L G)'$ with eigensystem $\chi_\rho' \colon \mathrm{Exc}(\Gamma, {}^L G)' \to k$ given by

$$S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}} \mapsto \chi_\rho(S_{\mathrm{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}})^p, \tag{5.24}$$
$$N \cdot S \mapsto 0.$$

This defines a certain maximal ideal $\mathfrak{m}_\rho$ of $\mathrm{Exc}(\Gamma, {}^L G)'$, and the existence of $\overline{v}_\rho^{(p)}$ implies that $\mathfrak{m}_\rho$ appears in the support of $(E^{ij}_\infty)_\alpha$ as an $\mathrm{Exc}(\Gamma, {}^L G)'$-module. By Lemma 5.8, $(E^{ij}_\infty)_\alpha$ is an $\mathrm{Exc}(\Gamma, {}^L G)'$-module subquotient of $H^0_c(\mathrm{Sht}_{G, D + \infty x_0, I}; \mathbb{1})_{\phi(\alpha)}$, so $\mathfrak{m}_\rho$ is also in the support of $H^*_c(\mathrm{Sht}_{G, D + \infty x_0, \emptyset}; \mathbb{1})_{\phi(\alpha)}$ as an $\mathrm{Exc}(\Gamma, {}^L G)'$-module.

Therefore, there is a maximal ideal of $\mathrm{Exc}(\Gamma, {}^L G)$ lying over $\mathfrak{m}_\rho$ which is in the support of $\mathrm{Exc}(\Gamma, {}^L G)$ as an $\mathrm{Exc}(\Gamma, {}^L G)$-module. Now, Lemma 5.17 and Lemma 5.18 imply that there is a *unique* extension of $\mathfrak{m}_\rho$ to a maximal ideal of $\mathrm{Exc}(\Gamma, {}^L G)$, corresponding to the unique extension of χ_ρ' to a character of $\mathrm{Exc}(\Gamma, {}^L G)$ given by:

$$S_{V, x, \xi, (\gamma_i)_{i \in I}} \mapsto \chi_\rho'(S_{\mathrm{Nm}(V), \mathrm{Nm}(x), \mathrm{Nm}(\xi), (\gamma_i)_{i \in I}})^{1/p} = \chi_\rho(S_{\mathrm{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}).$$

This is precisely the composition $\chi \circ \phi^*_{\mathrm{BC}}$, as desired. $\qquad\square$

6. On local base change

In this section we will prove the main local results mentioned in the Introduction. We begin by reviewing the relevant aspects of the Genestier-Lafforgue correspondence in §6.1. Its key property is local-global compatibility, which will allow us to leverage the global results proved in the preceding section.

After that we embark on the construction of the map $\mathfrak{Z}_{\mathrm{TV}}$ from Theorem 1.4. Its definition does not require any geometry, and works equally well over local fields of characteristic zero (and residue characteristic different from p), but requires a fair amount of technical preliminaries on Hecke algebras, which we establish in §6.2. Then we review the Brauer homomorphism in §6.3, which is needed to finally construct $\mathfrak{Z}_{\mathrm{TV}}$ and prove Theorem 1.4. We then use it (and intermediate results established along the way) to prove Theorem 1.2 in §6.5 and Theorem 1.1 in §6.6.

6.1. **Review of the Genestier-Lafforgue correspondence.** Let F_v be a local function field with ring of integers $\mathcal{O}_v$ and residue characteristic $\ell \neq p$. Let W_v be the Weil group of F_v. Let G be a reductive group over F_v and denote $G_v := G(F_v)$. In [GL, Théorèm 8.1], Genestier-Lafforgue construct a map

$$\left\{ \begin{array}{c} \text{irreducible admissible representations} \\ \pi \text{ of } G_v \text{ over } k \end{array} \right\} / \sim \ \longrightarrow \ \left\{ \begin{array}{c} \text{semi-simple } L\text{-parameters} \\ \rho_\pi \colon W_v \to {}^L G(k) \end{array} \right\} / \sim .$$

which is characterized by local-global compatibility with Lafforgue's Global Langlands correspondence.

We briefly summarize the aspects of the Genestier-Lafforgue correspondence that we will need.

6.1.1. *The Bernstein center.* We begin by recalling the formalism of the Bernstein center [Ber84]. Let $K \subset G_v$ be an open compact subgroup. The *Hecke algebra of G with respect to K with coefficients in Λ* is

$$\mathcal{H}(G, K; \Lambda) := C_c(K \backslash G_v / K; \Lambda),$$

the compactly supported functions on $K \backslash G_v / K$ valued in Λ. This forms an algebra under convolution, where we use (for all K) the left Haar measure on G_v for which $G(\mathcal{O}_v)$ has volume 1. We let $\mathfrak{Z}(G, K; \Lambda) := Z(\mathcal{H}(G, K; \Lambda))$ be the center of $\mathcal{H}(G, K)$. The *Bernstein center of G (with coefficients in Λ)* is

$$\mathfrak{Z}(G; \Lambda) := \varprojlim_K \mathfrak{Z}(G, K; \Lambda),$$

where the transition maps to $\mathfrak{Z}(G, K; \Lambda)$ are given by convolution with $\mathbb{1}_K$, the normalized indicator function of K (which is the unit of $\mathcal{H}(G, K; \Lambda)$) viewed as an idempotent in the full Hecke algebra of compactly supported smooth functions on G_v.

If $\Lambda = k$, we abbreviate $\mathcal{H}(G, K) := \mathcal{H}(G, K; k)$, $\mathfrak{Z}(G, K) := \mathfrak{Z}(G, K; k)$, and $\mathfrak{Z}(G) := \mathfrak{Z}(G; k)$.

The ring $\mathfrak{Z}(G)$ has another interpretation as the ring of endomorphisms of the identity functor of the category of smooth k-representations of G_v. Explicitly, smoothness of Π implies that $\Pi = \bigcup_{\text{open compact } K \subset G_v} \Pi^K$, and $\mathfrak{Z}(G, K)$ acts on Π^K as an $\mathcal{H}(G, K)$-module; this assembles into action of $\mathfrak{Z}(G)$ on Π. In particular, any irreducible admissible representation Π of G_v induces a character of $\mathfrak{Z}(G)$.

6.1.2. *Action of the excursion algebra.* The main result of [GL] is the construction of a homomorphism

$$Z_G \colon \operatorname{Exc}(W_v, {}^L G) \to \mathfrak{Z}(G). \tag{6.1}$$

Let $x \in \mathcal{B}(G/F_v)$ be a point of the Bruhat-Tits building of G, and use it to extend G to a parahoric group scheme over $\mathcal{O}_v$. (Some reminders on Bruhat-Tits theory will appear in §6.2.2.) For $r \geq 0$, let $K_r := G(F_v)_{x,r}$; this is an open compact subgroup of F_v. Let $U_r := K_r^{\operatorname{Gal}(E_v/F_v)} = K_r \cap H_v$. We write $Z_{G,r} \colon \operatorname{Exc}(W_v, {}^L G) \to \mathfrak{Z}(G, K_r)$ for the composition of Z_G with the tautological projection to $\mathcal{H}(G, K_r)$, and similarly $Z_{H,r} \colon \operatorname{Exc}(W_v, {}^L H) \to \mathfrak{Z}(H, U_r)$.

We will shortly give a characterization of (6.1). First let us indicate how (6.1) defines the correspondence $\Pi \mapsto \rho_\Pi$. An irreducible admissible Π induces a character of $\mathfrak{Z}(G)$, as discussed above. Composing with Z_G then gives a character of $\operatorname{Exc}(W_v, {}^L G)$, which by Proposition 2.4 gives a semisimple Langlands parameter $\rho_\Pi \in H^1(W_v, \widehat{G}(k))$.

6.1.3. *Local-global compatibility.* Choose a smooth projective curve X over $\mathbf{F}_\ell$ and a point $v \in X$ so that $X_v = \operatorname{Spec} \mathcal{O}_v$, and G extends to a reductive group over the generic point of X. Then choose a further extension of G to a parahoric group scheme over all of X, such that $G/\mathcal{O}_v$ is the parahoric group scheme corresponding to the chosen point $x \in \mathcal{B}(G/F_v)$.

The map (6.1) is characterized by local-global compatibility with the global excursion action. The idea is that for $(\gamma_i)_{i \in I} \subset W_v^I$, the action of the the excursion operator $S_{I,f,(\gamma_i)_{i \in I}}$ on $H_c^0(\operatorname{Sht}_{G,D,\emptyset}; \mathbb{1})$ is local at v, i.e. it acts through a Hecke operator for G_v. Moreover, it commutes with other Hecke operators because all excursion operators commute with all Hecke operators, hence it must actually be in the center of the relevant Hecke algebra. This idea is affirmed by the Proposition below.

Proposition 6.1 (Genestier-Lafforgue Prop 1.3)**.** *Let $r \geq 0$ be an integer and $D := rv + D^v$ a divisor on X, with D^v supported away from v. For $(\gamma_i)_{i \in I} \subset W_v^I$, the operator $S_{I,f,(\gamma_i)_{i \in I}}$ acts on $H_c^0(\operatorname{Sht}_{G,D,\emptyset}; \mathbb{1})$ as convolution by $Z_{G,r}(S_{I,f,(\gamma_i)_{i \in I}}) \in \mathfrak{Z}(G, K_r)$.*

Remark 6.2. By [GL, Lemme 1.4], for large enough D^v the action of $Z_{G,r}(S_{I,f,(\gamma_i)_{i \in I}})$ on $H_c^0(\operatorname{Sht}_{G,D,\emptyset}; \mathbb{1})$ is faithful. Therefore, Proposition 6.1 certainly characterizes the map (6.1). What is not clear is that the resulting $Z_{G,r}(S_{I,f,(\gamma_i)_{i \in I}})$ is independent of choices (of the global curve, or the integral model of the affine group scheme). In [GL] this is established by giving a purely local construction of (6.1) in terms of "restricted shtukas", but for our purposes it will be enough to accept Proposition 6.1 as a black box.

6.2. **Preliminary results on Hecke algebras.** We next establish some technical lemmas which aid to study the properties of the Brauer homomorphism. The only result that will be needed in later subsections is Corollary 6.4.

6.2.1. *Assumptions.* In this subsection, we allow F_v to be any local field (including one of characteristic zero) of residue characteristic $\ell \neq p$. Let E_v/F_v be a finite Galois assumption such that $\mathrm{Gal}(E_v/F_v)$ has order coprime to ℓ. We let H be any (connected) reductive group over F_v and $G := \mathrm{Res}_{E_v/F_v}(H_{E_v})$. We abbreviate $H_v = H(F_v)$ and $G_v = G(F_v) = H(E_v)$.

6.2.2. *Reminders on Bruhat-Tits theory.* First we recall some relevant facts from Bruhat-Tits theory, originally developed in [BT72] but explained in the form used below in [KP][13].

Let $\mathcal{B}(H/F)$ be the Bruhat-Tits building of H/F_v and $x \in \mathcal{B}(H/F_v)$. Associated to x there is the parahoric group $H(F_v)_x^0 := H(F_v)_{x,0}$, along with its decreasing filtration $H(F_v)_{x,r}$ for $r \geq 0$. The subgroup $H(F)_{x,0+} := \bigcup_{s>0} H(F)_{x,s}$ is pro-ℓ.

We record some descent properties: if E_v/F_v is unramified then $H(E_v)_{x,r}^{\mathrm{Gal}(E_v/F_v)} = H(F_v)_{x,r}$, and if E_v/F_v is tamely ramified then $H(E_v)_{x,r}^{\mathrm{Gal}(E_v/F_v)} \supset H(F_v)_{x,r}$ [KP, §11].

We shall make use of the *Cartan decomposition* [BT72, §4]. We summarize the description in [KP, §3.2]. Let S be a maximal k-split torus of H, and $Z = Z_H(S)$. Referring to [KP, (3.2.0.1)] for undefined notation, we have a subset

$$\mathcal{Z} := \{z \in Z(F_v): \alpha(\omega_Z(z)) \geq 0 \text{ for all } \alpha \in \Phi^+\} \subset Z(F_v). \tag{6.2}$$

According to [KP, Theorem 3.2.1], for a *special vertex* x in the apartment of S, we have

(1) $H(F_v) = H(F_v)_{x,0} \cdot \mathcal{Z} \cdot H(F_v)_{x,0}$, and
(2) for $z, z' \in \mathcal{Z}$, $H(F_v)_{x,0} z H(F_v)_{x,0} = H(F_v)_x^0 z' H(F_v)_x^0$ if and only if $z'(z^{-1}) \in Z(F_v)^0$,

where for a Levi subgroup $M \subset G$, the subgroup $M(F_v)^0 \subset M(F_v)$ is the kernel of the Kottwitz homomorphism $\kappa_M \colon M(F_v) \to \pi_1(M)_I^{\mathrm{Gal}(\breve{F}_v/F_v)}$; it may be defined alternatively as in [KP, Definition 1.12.29]. We have $Z(F_v)^0 \subset H(F_v)_{x,0}$.

6.2.3. *Maps of double coset spaces.* Following the notation of §6.2.1, let S_G be a maximal F_v-split torus of G, and define $Z_E := Z_G(S_G)$, and $\mathcal{Z}_E \subset Z_E(F_v)$ as in (6.2). Let S_F be a maximal F_v-split torus of H contained in S_G, $Z_F = Z_H(S_H)$, and $\mathcal{Z}_F \subset Z_F(F_v)$ as in (6.2). Let $x \in \mathcal{B}(G/F_v)$ be a special vertex in the apartment of S_G. We will abbreviate $K_r := G(F_v)_{x,r}$ and $U_r = K_r^{\mathrm{Gal}(E_v/F_v)} \supset H(F_v)_{x,r}$. The goal of this subsection is to prove the following.

Proposition 6.3. *If $r \geq 0$, then the map $U_r \backslash H_v / U_r \to K_r \backslash G_v / K_r$ is injective.*

In the following Corollary, we let $\mathrm{Gal}(E_v/F_v)$ act on $G_v = H(E_v)$ by the natural Galois action, which induces an action on $\mathcal{H}(G, K_r; \Lambda)$.

Corollary 6.4. *Suppose $r \geq 0$. Then the restriction map $\mathcal{H}(G, K_r; \Lambda)^{\mathrm{Gal}(E_v/F_v)} \to \mathcal{H}(H, U_r; \Lambda)$ is surjective (as a map of Λ-modules).*

Proposition 6.3 will be proved in several steps.

Lemma 6.5. *The map $U_0 \backslash H_v / U_r \to K_0 \backslash G_v / K_r$ is injective.*

Proof. We first handle the case $r = 0$, which will seen to be a consequence of Cartan decomposition. Since $U_0 \supset H(F_v)_{x,0}$, the Cartan decomposition implies that double cosets $U_0 \backslash G_v / U_0$ are represented by $z \in \mathcal{Z}_F$. If $z_1, z_2 \in \mathcal{Z}_F$ are such that $K_0 z_1 K_0 = K_0 z_2 K_0$, then $z_2 z_1^{-1} \in Z_E(F_v)^0 \subset K_0$. On the other hand, clearly $z_2 z_1^{-1} \in H(F_v)$, so we conclude that $z_2 z_1^{-1} \in K_0 \cap H_v = U_0$. Therefore, $U_0 z_1 U_0 = U_0 z_2 U_0$, and the case $r = 0$ is concluded.

We now assume going forward that $r > 0$. By Cartan decomposition, any double coset in $U_0 \backslash H_v / U_r$ is represented by an element of the from zh for $z \in \mathcal{Z}_F$ and $h \in U_0$. Suppose $z_1 h_1$ and $z_2 h_2$, for $z_i \in \mathcal{Z}_F$ and $h_i \in U_0$, represent the same double coset in $K_0 \backslash G_v / K_r$. Then they represent the same double coset in $K_0 \backslash G_v / K_0$, and hence by the case $r = 0$ they represent the same double coset in $U_0 \backslash H_v / U_0$. Therefore $z_1^{-1} z_2 \in U_0$, so by adjusting the choice of representative we may therefore assume that $z_1 = z_2 = z$.

The assumption that $K_0 z h_1 K_r = K_0 z h_2 K_r$ is equivalent to the existence of $k \in K_0, k' \in K_r$ such that

$$kzh_1 k' = zh_2, \tag{6.3}$$

---

[13]This wonderful manuscript is not yet publicly available, but may be obtained from the authors.

Element	Lies in Group
k	K_0
k', k'', k_r	K_r
z_1, z_2, z	$\mathcal{Z}_F$
h_1, h_2, h, u, u''	U_0
u''	U_r

FIGURE 1. Table of notation (keeping track of which elements lie in which group) used in the proof of Lemma 6.5.

and we want to show that such an equation exists with $k \in U_0, k' \in U_r$. Since U_0 normalizes $U_r \ni k'$, we may rearrange (6.3) as

$$kzk''h_1 = zh_2 \quad \text{with } k'' = h_1 k' h_1^{-1} \in K_r. \tag{6.4}$$

Letting $h = h_2 h_1^{-1} \in U_0$, (6.4) is equivalent to $kzk'' = zh$, which we may rewrite as

$$z^{-1}kzk'' = h. \tag{6.5}$$

Since $h \in U_0 \subset K_0$ and $k'' \in K_0$, (6.5) implies that $z^{-1}kz \in K_0$. Considering (6.5) modulo K_r on the right, we have that $z^{-1}kzK_r = hK_r$. We would like to find $u \in U_0$ such that $z^{-1}uzK_r = z^{-1}kzK_r$. Since K_r is pro-ℓ because $r > 0$, and $\mathrm{Gal}(E_v/F_v)$ is by assumption of order prime to ℓ, we have that $H^1(\mathrm{Gal}(E_v/F_v), K_r') = 0$ for every subgroup $K_r' \subset K_r$. Applying this with $K_r' = z^{-1}K_0 z \cap K_r$, the long exact sequence for group cohomology of $\mathrm{Gal}(E_v/F_v)$ implies that in the diagram below, the image of red consists of all of the $\mathrm{Gal}(E_v/F_v)$-invariants of the image of red$'$.

$$
\begin{array}{ccccc}
(z^{-1}K_0 z \cap K_r)^{\mathrm{Gal}(E_v/F_v)} & \hookrightarrow & (z^{-1}K_0 z \cap K_0)^{\mathrm{Gal}(E_v/F_v)} & \xrightarrow{\mathrm{red}'} & (K_0/K_r)^{\mathrm{Gal}(E_v/F_v)} \\
\| & & \| & & \| \\
z^{-1}U_0 z \cap U_r & \hookrightarrow & z^{-1}U_0 z \cap U_0 & \xrightarrow{\mathrm{red}} & U_0/U_r
\end{array}
$$

Since $h \in U_0$, (6.5) shows that $\mathrm{red}'(z^{-1}kz)$ is indeed fixed by $\mathrm{Gal}(E_v/F_v)$, so there exists $u \in U_0 \cap zU_r z^{-1}$ such that $z^{-1}kzK_r = z^{-1}uzK_r$. Plugging this in above, we find that there exists $k_r \in K_r$ such that

$$z^{-1}uzk_r k'' = h. \tag{6.6}$$

Setting $u'' := k_r k'' \in K_r$, (6.6) implies that $u'' = (z^{-1}uz)^{-1}h \in U_0$, so $u'' \in K_r^{\mathrm{Gal}(E_v/F_v)} = U_r$. Substituting back into (6.6) yields $z^{-1}uzu'' = h_2 h_1^{-1}$, which we can rewrite as $zh_2 = uzu''h_1 = uzh_1(h_1^{-1}u''h_1)$ with $u \in U_0$ and $h_1^{-1}u''h_1 \in U_r$. This shows that zh_2 and zh_1 represent the same double cosets in $U_0\backslash H/U_r$. $\square$

Proof of Proposition 6.3. By Cartan decomposition, any two double cosets of $U_r\backslash H/U_r$ are represented by $h_1 z_1 h_1'$ and $h_2 z_2 h_2'$ with $z_i \in \mathcal{Z}_F$ and $h_i, h_i' \in U_0$. Suppose that these represent the same double cosets in $K_r\backslash G_v/K_r$. Then they represent the same double cosets in $K_0\backslash G_v/K_r$, hence by Lemma 6.5 they represent the same double cosets in $U_0\backslash H_v/U_r$. So, after adjusting the choice of h_1', we may assume that

$$U_0 z_1 h_1' = U_0 z_2 h_2'. \tag{6.7}$$

In particular, $U_0 z_1 U_0 = U_0 z_2 U_0$, hence by Cartan decomposition we have $z_1 = z^\circ z_2$ for $z^\circ \in Z_F(F_v)^0 \subset U_0$, and by absorbing z° into h_1 we may assume that $z_1 = z_2 = z \in \mathcal{Z}_F$. Putting this back into (6.7), we find that $h' := h_2'(h_1')^{-1}$ lies in $U_0 \cap z^{-1}U_0 z$.

Element	Lies in Group
$k_1, k_1', k_1'', k_1''', k_r$	K_r
z_1, z_2, z	$\mathcal{Z}_F$
h_1, h_2, h_1', h_2', h	U_0
h'	$U_0 \cap z^{-1}U_0 z$
u', u''	U_r

FIGURE 2. Table of notation (keeping track of which elements lie in which group) used in the proof of Proposition 6.3.

Now, the assumption that $h_1 z_1 h'_1$ and $h_2 z_2 h'_2$ represent the same double cosets in $K_r \backslash G_v / K_r$ is equivalent to the existence of $k_1, k'_1 \in K_r$ such that

$$k_1 h_1 z h'_1 k'_1 = h_2 z h'_2. \tag{6.8}$$

Since $h_i \in U_0$ which normalizes K_r, we have $h_1^{-1} k_1 h_1 = k''_1 \in K_r$, and $(h'_1)^{-1} k'_1 h'_1 = k'''_1 \in K_r$. Substituting this into (6.8), we may rewrite it as

$$k''_1 z k'''_1 z^{-1} = h z h' z^{-1} \tag{6.9}$$

with $h := (h_1^{-1} h_2)$ and $h' = (h'_2 (h'_1)^{-1})$. Since $h \in U_0$ and $z h' z^{-1} \in U_0 \subset K_0$, and $k''_1 \in K_r \subset K_0$, (6.9) tells us that $z k'''_1 z^{-1} \in K_0$.

Consider (6.9) modulo the left action of K_r. We claim that there exists some $u' \in U_r$ such that $K_r z k'''_1 z^{-1} = K_r z u' z^{-1}$. Since K_r is pro-ℓ because $r > 0$ and $\mathrm{Gal}(E_v/F_v)$ is by assumption of order prime to ℓ, we have that $H^1(\mathrm{Gal}(E_v/F_v), K'_r) = 0$ for every subgroup $K'_r \subset K_r$. Applying this with $K'_r = z^{-1} K_r z \cap K_r$, the long exact sequence for group cohomology implies that in the diagram below, the image of red consists of all of the $\mathrm{Gal}(E_v/F_v)$-invariants of the image of red$'$.

$$
\begin{array}{ccccc}
(z K_r z^{-1} \cap K_r)^{\mathrm{Gal}(E_v/F_v)} & \hookrightarrow & (z K_r z^{-1} \cap K_0)^{\mathrm{Gal}(E_v/F_v)} & \xrightarrow{\mathrm{red}'} & (K_r \backslash K_0)^{\mathrm{Gal}(E_v/F_v)} \\
\| & & \| & & \| \\
z U_r z^{-1} \cap U_r & \hookrightarrow & z U_r z^{-1} \cap U_0 & \xrightarrow{\quad \mathrm{red} \quad} & U_r \backslash U_0
\end{array}
$$

Since $z h' z^{-1} \in U_0$, (6.9) implies that $\mathrm{red}'(z k'''_1 z^{-1}) \in K_r \backslash K_0$ is fixed by $\mathrm{Gal}(E_v/F_v)$. Hence there exist $u' \in U_r \cap z^{-1} U_0 z$ and $k_r \in K_r$ such that

$$k_r z k'''_1 z^{-1} = z u' z^{-1} \tag{6.10}$$

Substituting (6.10) into (6.9) yields

$$k''_1 k_r^{-1} z u' z^{-1} = h z h' z^{-1}. \tag{6.11}$$

The RHS of (6.11) lies in U_0 and $z u' z^{-1} \in U_0$, so $u'' := k''_1 k_r^{-1} \in U_0 \cap K_r = U_r$. Putting this back into (6.11) yields $u'' z u' = h_1^{-1} h_2 z h'_2 (h'_1)^{-1}$. This may then be re-arranged to

$$(h_1 u'' h_1^{-1}) h_1 z h'_1 ((h'_1)^{-1} u' h'_1) = h_2 z h'_2$$

which, since $h_1, h'_1 \in U_0$ normalize $U_r \ni u', u''$, shows what we wanted. $\qquad \square$

6.3. The Brauer homomorphism.
We introduce the notion of the Brauer homomorphism from [TV16], whose utility for our purpose is to capture the relationship between Π and its Tate cohomology from the perspective of Hecke algebras.

6.3.1. *Assumptions.* In this subsection we allow F_v to be any local field (including one of characteristic zero) of residue characteristic $\ell \neq p$. We assume that $\mathrm{Gal}(E_v/F_v)$ is cyclic of order p, and we let $\sigma \in \mathrm{Gal}(E_v/F_v)$ be a generator. We let H be any (connected) reductive group over F_v and $G := \mathrm{Res}_{E_v/F_v}(H_{E_v})$. Subgroups $K_r \subset G_v$, $U_r \subset H_v$ are defined as in §6.2.3.

6.3.2. *The (un-normalized) Brauer homomorphism.* Let $K \subset G_v$ be an open compact subgroup, and let $U := K^\sigma \subset H_v$. We say that $K \subset G_v$ is a *plain subgroup* if $(G_v/K)^\sigma = H_v/U$.

We can view $\mathcal{H}(G, K)$ as the ring of G_v-invariant (for the diagonal action) functions on $(G_v/K) \times (G_v/K)$ under convolution.

Lemma 6.6. *If $K \subset G_v$ is a plain subgroup, then the restriction map*

$$\mathcal{H}(G, K)^\sigma = \mathrm{Fun}_{G_v, c}((G_v/K) \times (G_v/K), k)^\sigma \tag{6.12}$$

$$\xrightarrow{restrict} \mathrm{Fun}_{H_v, c}((H_v/U) \times (H_v/U), k) = \mathcal{H}(H_v, U)$$

is an algebra homomorphism.

Proof. What we must verify is that for $x, z \in H_v/U$, and $f, g \in \mathcal{H}(G, K)^\sigma$, we have

$$\sum_{y \in G_v/K} f(x, y) g(y, z) = \sum_{y \in H_v/U} f(x, y) g(y, z). \tag{6.13}$$

Since f and g are σ-invariant, we have

$$f(x,y) = f(\sigma x, \sigma y) = f(x, \sigma y) \text{ and } g(y,z) = g(\sigma y, \sigma z) = g(\sigma y, z).$$

If $y \notin H_v/U$, then the plain-ness assumption implies that y is not fixed by σ. Therefore the contribution from the orbit of σ on y to (6.13) is a multiple of p, which is 0 in k. $\square$

The map of Lemma 6.6 was introduced in [TV16, §4] and called the *(un-normalized) Brauer homomorphism*. We denote it

$$\mathrm{Br}\colon \mathcal{H}(G,K)^\sigma \to \mathcal{H}(H,U).$$

Lemma 6.7. *If $K \subset G(F_v)_{x,0+}$ for any $x \in \mathcal{B}(G/F_v)$, then K is plain.*

Proof. By the long exact sequence for group cohomology, the plain-ness of $K \subset G_v$ is equivalent to condition that the map on non-abelian cohomology $H^1(\langle\sigma\rangle; K) \to H^1(\langle\sigma\rangle; G_v)$ has trivial fiber over the trivial class. But since $G(F_v)_{x,0+}$ is pro-ℓ, all its subgroups are acyclic for $H^1(\langle\sigma\rangle, -)$ as σ has order p. Therefore $H^1(\langle\sigma\rangle, K)$ vanishes for all such $K \subset G(F_v)_{x,0+}$. $\square$

Lemma 6.8 (Relation to the Brauer homomorphism). *Assume $K \subset G_v$ is plain. Suppose Π is a σ-fixed representation of G_v. Then the map of Tate cohomology groups $T^*(\Pi^K) \to T^*(\Pi)$ lands in the U-invariants, and for any $h \in \mathcal{H}(G,K)^\sigma$ we have the commutative diagram below.*

$$
\begin{array}{ccc}
T^*(\Pi^K) & \longrightarrow & T^*(\Pi)^U \\
\downarrow{\scriptstyle T^0 h} & & \downarrow{\scriptstyle \mathrm{Br}(h)} \\
T^*(\Pi^K) & \longrightarrow & T^*(\Pi)^U
\end{array}
$$

(Here $T^0 h$ is the element of $T^0(\mathcal{H}(G,K))$ represented by h.)

Proof. This is [TV16, §6.2]; it follows from a direct computation similar to the proof of Lemma 6.6. $\square$

6.3.3. *Treumann-Venkatesh homomorphism.* If we take $K = K_r$ as in Corollary 6.4, then the Brauer homomorphism $\mathrm{Br}\colon \mathcal{H}(G,K_r)^\sigma \to \mathcal{H}(H,U_r)$ is a surjective algebra homomorphism, hence induces a map on centers

$$Z(\mathrm{Br})\colon Z(\mathcal{H}(G,K_r)^\sigma) \to \mathfrak{Z}(H,U_r). \tag{6.14}$$

It is evident from the definition that $Z(\mathrm{Br})$ through the quotient $Z(\mathcal{H}(G,K_r)^\sigma)/N \cdot \mathfrak{Z}(G,K_r)$.

Since $\mathfrak{Z}(G,K_r)$ is commutative, it has a Tate diagonal homomorphism (Definition 5.16) $\mathfrak{Z}(G,K_r) \xrightarrow{\Delta^p} T^0(\mathfrak{Z}(G,K_r))$. Since $Z(\mathcal{H}(G,K_r))^\sigma \subset Z(\mathcal{H}(G,K_r)^\sigma)$, we may compose with $Z(\mathrm{Br})$ to obtain a map

$$\mathfrak{Z}(G,K_r) \xrightarrow{\Delta^p} T^0(\mathfrak{Z}(G,K_r)) \xrightarrow{Z(\mathrm{Br})} \mathfrak{Z}(H,U_r). \tag{6.15}$$

Note however that it is not k-linear, since Δ^p is Frob-semilinear over k.

Definition 6.9. We define *Treumann-Venkatesh homomorphism* $\mathfrak{Z}_{\mathrm{TV},r}$ to be the linearization of (6.15), i.e. the (unique) homomorphism fitting into the commutative diagram

$$
\begin{array}{c}
\mathfrak{Z}(G,K_r) \\
\downarrow{\scriptstyle \sim} \qquad \searrow^{Z(\mathrm{Br})\circ\Delta^p} \\
\mathfrak{Z}(G,K_r) \otimes_{k,\mathrm{Frob}} k \xrightarrow{\mathfrak{Z}_{\mathrm{TV},r}} \mathfrak{Z}(H,U_r)
\end{array}
$$

Remark 6.10. Definition 6.9 is not considered in [TV16], but it is inspired by the definition of the *normalized Brauer homomorphism* in [TV16, §4.3].

6.4. **The base change homomorphism for Bernstein centers.** For now, assumptions are as in §6.3.1. For $s > r$, so that $K_s \subset K_r$, we have a map $e_G^{s \to r} \colon \mathfrak{Z}(G, K_s) \to \mathfrak{Z}(G, K_r)$ given by convolution with $\mathbb{1}_{K_r}$, the indicator function of K_r normalized to be idempotent. (Technically $e_G^{s \to r}$ also depends on the point $x \in \mathcal{B}(G/F_v)$ used to define the K_r, but we suppress this from our notation.) Similarly, we have $e_H^{s \to r} \colon \mathfrak{Z}(H, U_s) \to \mathfrak{Z}(H, U_r)$ given by convolution with $\mathbb{1}_{U_r}$.

Lemma 6.11. *The diagram*

$$
\begin{array}{ccc}
\mathfrak{Z}(G, K_s) & \xrightarrow{\mathfrak{Z}_{\mathrm{TV},s}} & \mathfrak{Z}(H, U_s) \\
\downarrow{\scriptstyle e_G^{s \to r}} & & \downarrow{\scriptstyle e_H^{s \to r}} \\
\mathfrak{Z}(G, K_r) & \xrightarrow{\mathfrak{Z}_{\mathrm{TV},r}} & \mathfrak{Z}(H, U_r)
\end{array}
$$

commutes.

Proof. Observe that $\mathbb{1}_{K_r}$ is an idempotent and $\mathrm{Br}(\mathbb{1}_{K_r}) = \mathbb{1}_{U_r}$. Then the commutativity follows by direct computation, using Lemma 6.6. $\qquad\square$

Definition 6.12 (Base change homomorphism for Bernstein centers)**.** We define the map $\mathfrak{Z}_{\mathrm{TV}}$ by:

$$
\mathfrak{Z}_{\mathrm{TV}} := \varprojlim_r \mathfrak{Z}_{\mathrm{TV},r} \colon \mathfrak{Z}(G) \cong \varprojlim_r \mathfrak{Z}(G, K_r) \to \varprojlim_r \mathfrak{Z}(H, U_r) \cong \mathfrak{Z}(H).
$$

Definition 6.12 is well-defined over local fields of any residue characteristic $\ell \neq p$, but in this paper we will only prove properties of it for local function fields. Hence, **for the rest of the paper, we assume that F_v is a local *function* field**. The rest of this subsection shall be devoted to the proof of Theorem 1.4.

6.4.1. The maps

$$
\mathrm{Exc}(W_v, {}^L G) \xrightarrow{Z_{G,r}} \mathfrak{Z}(G, K_r) \to \mathrm{End}_{\mathcal{H}_G}(H_c^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}))
$$

induce upon applying Tate cohomology,

$$
T^0 \mathrm{Exc}(W_v, {}^L G) \xrightarrow{T^0 Z_{G,r}} T^0 \mathfrak{Z}(G, K_r) \to \mathrm{End}_{T^0 \mathcal{H}_G}(T^0(H_c^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}))).
$$

Fix a closed point x_0 on X distinct from v. For each integer r, we will impose level structure along $D := rv + \infty x_0$, interpreted as in §5.5. By Remark 6.2, the map $\mathfrak{Z}(G, K_r) \to \mathrm{End}_{\mathcal{H}_G}(H_c^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}))$ is injective.

6.4.2. Theorem 5.13 implies that under the identification $T^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}) \cong T^0(\mathrm{Sht}_{H,D,\emptyset}; \mathbb{1})$, we have

$$
\begin{pmatrix} \text{the action of} \\ S_{\mathrm{Nm}^{(p-1)}(V), \mathrm{Nm}^{(p-1)}(x), \mathrm{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}} \\ \text{on } T^0(\mathrm{Sht}_{G,D,\emptyset}; \mathbb{1}) \end{pmatrix} = \begin{pmatrix} \text{the action of } S_{\mathrm{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}} \\ \text{on } T^0(\mathrm{Sht}_{H,D,\emptyset}; \mathbb{1}) \end{pmatrix}.
$$

The latter action factors through the action of $S_{\mathrm{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}$ on $H_c^0(\mathrm{Sht}_{H,D,\emptyset}; \mathbb{1})$, because (5.8) implies that $T^0(\mathrm{Sht}_{H,D,\emptyset}; \mathbb{1}) \cong H_c^0(\mathrm{Sht}_{H,D,\emptyset}; \mathbb{1})$.

6.4.3. For any set S, we let $k[S]$ denote the k-vector space of k-valued functions on S.

Now suppose $\widetilde{S}$ is a set with an action of $G_v \rtimes \langle \sigma \rangle$, on which an open compact subgroup $K \subset G_v$ acts freely. Then for $S := \widetilde{S}/K$, there is a natural action of $\mathcal{H}(G, K)$ on $k[S]$ since we may view $\mathcal{H}(G, K) = \mathrm{Hom}_{G_v}(k[G_v/K], k[G_v/K])$ and $k[S] = \mathrm{Hom}_{G_v}(k[G_v/K], k[\widetilde{S}])$. This induces an action of $T^0(\mathcal{H}(G, K))$ on $T^0(k[S]) \cong k[S^\sigma]$, and then by inflation an action of $\mathcal{H}(G, K)^\sigma$ on $k[S^\sigma]$.

By the same mechanism, for $U := K^\sigma$ there is an induced action of $\mathcal{H}(H, U)$ on $k[\widetilde{S}^\sigma/K^\sigma] = k[\widetilde{S}^\sigma/U]$.

Lemma 6.13. *Assume $K \subset G_v$ is a plain subgroup. Then $k[\widetilde{S}^\sigma/U]$ is a $\mathcal{H}(G, K)^\sigma$-direct summand of $k[S^\sigma]$, and for all $h \in \mathcal{H}(G, K)^\sigma$ we have*

$$
\Big(\text{the action of } h \text{ on } k[\widetilde{S}^\sigma/U]\Big) = \Big(\text{the action of } \mathrm{Br}(h) \in \mathcal{H}(H, U) \text{ on } k[\widetilde{S}^\sigma/U]\Big).
$$

Proof. See [TV16, equation (4.2.2)]. $\qquad\square$

From §6.4.1 we have the diagram

$$
\begin{array}{ccc}
T^0 \operatorname{Exc}(W_v, {}^LG) \xrightarrow{\;Z_{G,r}\;} T^0 \mathfrak{Z}(G, K_r) \longrightarrow \operatorname{End}_{T^0 \mathcal{H}_G}(T^0(\operatorname{Sht}_{G,D,\emptyset}; \mathbb{1})) \\
\Big\downarrow {\scriptstyle Z(\mathrm{Br})} \qquad\qquad \Big\downarrow \\
\operatorname{Exc}(W_v, {}^LH) \xrightarrow{\;Z_{H,r}\;} \mathfrak{Z}(H, U_r) \lhook\joinrel\longrightarrow \operatorname{End}_{\mathcal{H}_H}(T^0(\operatorname{Sht}_{H,D,\emptyset}; \mathbb{1}))
\end{array}
\tag{6.16}
$$

Corollary 6.14. *Let D_0 be a non-empty divisor on X. For all $r \geq 1$, the action of $z \in T^0 \mathfrak{Z}(G, K_r)$ on $T^0(\operatorname{Sht}_{G,D_0,\emptyset}; \mathbb{1})$ in (6.16) agrees with the action of $Z(\mathrm{Br})(z)$ on $T^0(\operatorname{Sht}_{H,D_0,\emptyset}; \mathbb{1})$ in (6.16) under the identification $T^0(\operatorname{Sht}_{G,D_0,\emptyset}; \mathbb{1}) \cong T^0(\operatorname{Sht}_{H,D_0,\emptyset}; \mathbb{1})$ from §3.4.6. In other words, the square in diagram (6.16) commutes.*

Proof. Each $\operatorname{Sht}_{G,D_0,\emptyset}$ is a discrete groupoid with finite stabilizers. Furthermore, we claim that (since D_0 is non-empty), the automorphisms of $\operatorname{Sht}_{G,D,\emptyset}$ are finite unipotent groups, which therefore have no cohomology. Indeed, for any G-torsor $\mathcal{E}$ on X and any point $v \in X$, we have a restriction map

$$
\operatorname{Aut}(\mathcal{E}) \xrightarrow{\;\mathrm{ev}_v\;} \operatorname{Aut}(\mathcal{E}|_v).
$$

The kernel of ev_v is pro-ℓ, since $\operatorname{Aut}(\mathcal{E})$ embeds into the group of automorphisms of $\mathcal{E}$ restricted to a formal disk around v, which is $G(\mathcal{O}_v)$, and the kernel of the evaluation map $G(\mathcal{O}_v) \to G$ is pro-ℓ.

Hence we may apply the preceding discussion with $S := [\operatorname{Sht}_{G,D_0,\emptyset}]$ the *set* of isomorphism classes in $\operatorname{Sht}_{G,D_0,\emptyset}$, and $\widetilde{S} := [\operatorname{Sht}_{G,\infty v+D_0,\emptyset}] := \varprojlim_{j \geq 0} [\operatorname{Sht}_{G,(r+j)v+D_0,\emptyset}]$. Then $k[S]$ is identified with the functions on $[\operatorname{Sht}_{G,rv+D_0,\emptyset}]$, and Lemma 5.9 plus §3.4.6 identify $k[\widetilde{S}^\sigma/K^\sigma]$ with the functions on $[\operatorname{Sht}_{H,rv+D_0,\emptyset}]$.

The assertions for compactly supported functions then follows by duality. $\qquad\square$

Corollary 6.15. *For all $r \geq 1$, for all $\{V, x, \xi, (\gamma_i)_{i \in I}\}$ as in §2.4, Br sends*

$$
Z_{G,r}(S_{\operatorname{Nm}^{(p^{-1})}(V), \operatorname{Nm}^{(p^{-1})}(x), \operatorname{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}) \in \mathfrak{Z}(G, K_r) \subset \mathcal{H}(G, K_r)
$$

$$
\xrightarrow{\;\mathrm{Br}\;} Z_{H,r}(S_{\operatorname{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}) \in \mathfrak{Z}(H, U_r) \subset \mathcal{H}(H, U_r).
$$

Proof. The equality from §6.4.2 shows that

$$
\left(
\begin{array}{c}
\text{the image of} \\
\underbrace{S_{\operatorname{Nm}^{(p^{-1})}(V), \operatorname{Nm}^{(p^{-1})}(x), \operatorname{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}}_{\in T^0 \operatorname{Exc}(W_v, {}^LG)} \\
\text{in } \operatorname{End}_{\mathcal{H}_H}(T^0(\operatorname{Sht}_{H,mv+\infty x_0,\emptyset}; \mathbb{1})) \\
\text{via (6.16)}
\end{array}
\right)
=
\left(
\begin{array}{c}
\text{the image of} \\
S_{\operatorname{Res}_{\mathrm{BC}}(V), x, \xi, (\gamma_i)_{i \in I}} \in \operatorname{Exc}(W_v, {}^LH) \\
\text{in } \operatorname{End}_{\mathcal{H}_H}(T^0(\operatorname{Sht}_{H,mv+\infty x_0,\emptyset}; \mathbb{1})) \\
\text{via (6.16)}
\end{array}
\right).
\tag{6.17}
$$

On the other hand, Corollary 6.14 shows that the left hand side of (6.17) agrees with the image of $\mathrm{Br}(Z_{G,m}(S_{\operatorname{Nm}^{(p^{-1})}(V), \operatorname{Nm}^{(p^{-1})}(x), \operatorname{Nm}^{(p^{-1})}(\xi), (\gamma_i)_{i \in I}}))$ via (6.16), for all $m \geq 1$. We conclude by using injectivity of $\mathfrak{Z}(H, U_r) \hookrightarrow \operatorname{End}_{\mathcal{H}_H}(T^*(\operatorname{Sht}_{H,D,\emptyset}; \mathbb{1}))$ in (6.16).

$\qquad\square$

6.4.4.   In §6.4.1 – §6.4.3 we have not used the results of §6.2. The latter section, in particular Corollary 6.4, is used only to see that $\mathfrak{Z}_{\mathrm{TV}}$ lands in the Bernstein center, which will be invoked presently. Recall that in Definition 2.8 we have defined a map $\phi_{\mathrm{BC}} \colon {}^LH \to {}^LG$ over k.

Corollary 6.16. *The following diagram commutes:*

$$
\begin{array}{ccc}
\operatorname{Exc}(W_v, {}^LG) & \xrightarrow{\;\phi_{\mathrm{BC}}^*\;} & \operatorname{Exc}(W_v, {}^LH) \\
\Big\downarrow {\scriptstyle Z_G} & & \Big\downarrow {\scriptstyle Z_H} \\
\mathfrak{Z}(G) & \xrightarrow{\;\mathfrak{Z}_{\mathrm{TV}}\;} & \mathfrak{Z}(H)
\end{array}
\tag{6.18}
$$

Proof. The commutativity of the diagram

$$
\begin{array}{ccc}
\operatorname{Exc}(W_v, {}^LG) & \xrightarrow{\;\Delta^p\;} & T^0 \operatorname{Exc}(W_v, {}^LG) \\
\Big\downarrow {\scriptstyle Z_G} & & \Big\downarrow {\scriptstyle T^0(Z_G)} \\
\mathfrak{Z}(G) & \xrightarrow{\;\Delta^p\;} & T^0 \mathfrak{Z}(G)
\end{array}
$$

implies that $\mathrm{Br}(Z) \circ \Delta^p \circ Z_G = \mathrm{Br}(Z) \circ T^0(Z_G) \circ \Delta^p$. Hence $\mathfrak{Z}_{\mathrm{TV}} \circ Z_G$ is the linearization of $\mathrm{Br}(Z) \circ T^0(Z_G) \circ \Delta^p$.

By Lemma 5.18, the Tate diagonal $\Delta^p \colon \mathrm{Exc}(W_v, {}^L G) \to T^0(\mathrm{Exc}(W_v, {}^L G))$ sends

$$S_{V,x,\xi,(\gamma_i)_{i \in I}} \xrightarrow{\Delta^p} S_{\mathrm{Nm}(V),\mathrm{Nm}(x),\mathrm{Nm}(\xi),(\gamma_i)_{i \in I}}.$$

It linearization therefore sends $S_{V,x,\xi,(\gamma_i)_{i \in I}}$ to $S_{\mathrm{Nm}^{(p-1)}(V),\mathrm{Nm}^{(p-1)}(x),\mathrm{Nm}^{(p-1)}(\xi),(\gamma_i)_{i \in I}}$. Applying Corollary 6.15 with $r \to \infty$, we have

$$\mathrm{Br}(Z) \circ Z_G(S_{\mathrm{Nm}^{(p-1)}(V),\mathrm{Nm}^{(p-1)}(x),\mathrm{Nm}^{(p-1)}(\xi),(\gamma_i)_{i \in I}}) = Z_H(S_{\mathrm{Res}_{\mathsf{BC}}(V),x,\xi,(\gamma_i)_{i \in I}})$$
$$= Z_H(\phi_{\mathsf{BC}}^*(S_{V,x,\xi,(\gamma_i)_{i \in I}})).$$

Therefore the linearization of $\mathrm{Br}(Z) \circ T^0(Z_G) \circ \Delta^p$ agrees with $Z_H \circ \phi_{\mathsf{BC}}^*$. $\qquad\square$

Completion of the proof of Theorem 1.4. Let π be an irreducible representation of H_v and $\chi_\pi \colon \mathfrak{Z}(H) \to k$ the induced character. By the definition of the Genestier-Lafforgue parametrization, ρ_π corresponds to $\chi_\pi \circ Z_G$ via Proposition 2.4. Then (6.18) implies that $\chi_\pi \circ \mathfrak{Z}_{\mathrm{TV}} \circ Z_G = \chi_\pi \circ Z_G \circ \phi_{\mathsf{BC}}^*$ is associated to the L-parameter $\phi_{\mathsf{BC}} \circ \rho_\pi$. $\qquad\square$

6.5. **The Treumann-Venkatesh Conjecture.** In this subsection we will prove Theorem 1.2. We begin by formulating the Treumann-Venkatesh Conjecture precisely in this setting. (The original phrasing of [TV16] is in terms of a hypothetical Local Langlands correspondence.)

6.5.1. *Assumptions.* In this subsection the assumptions are as in §6.3, and we furthermore assume F_v is a local function field. We note, however, that the formulation of all the statements in §6.12 makes sense for any local field F_v of residue field $\ell \neq p$, with a suitable replacement for the Genestier-Lafforgue correspondence, and that all our arguments in this subsection apply if those statements are true for F_v.

6.5.2. *Formulation of the Conjecture.* Let Π be an irreducible admissible representation of G_v over k. Let Π^σ be the representation of G_v obtained by composing Π with $\sigma \colon G_v \to G_v$. We say that Π is σ-*fixed* if $\Pi \approx \Pi^\sigma$ as G_v-representations.

Lemma 6.17 ([TV16, Proposition 6.1]). *If Π is σ-fixed, then the G_v-action on Π extends uniquely to an action of $G_v \rtimes \langle \sigma \rangle$.*

Using Lemma 6.17 we can form the Tate cohomology groups $T^0(\Pi)$ and $T^1(\Pi)$ with respect to the σ-action, which are then representations of H_v. Treumann-Venkatesh conjecture that they are in fact admissible representations of H_v, but we do not prove or use this.

Definition 6.18 (Linkage). An irreducible admissible representation π of H_v is *linked* with an irreducible admissible representation Π of G_v if $\pi^{(p)}$ appears in $T^0(\Pi)$ or $T^1(\Pi)$, where $\pi^{(p)}$ is the Frobenius twist

$$\pi^{(p)} := \pi \otimes_{k,\mathrm{Frob}} k.$$

Conjecture 6.19 ([TV16, Conjecture 6.3]). *If π is linked to Π, then π base changes to Π.*

Example 6.20. The need for the Frobenius twist can be seen in a simple example. Suppose $G = H^p$ and σ acts by cyclic permutation. Then G^σ is the diagonal copy of H. In this case a representation π of H_v should transfer to $\pi^{\boxtimes p}$ of G_v. And indeed,

$$T^0(\pi^{\boxtimes p}) = \frac{\ker(1 - \sigma \mid \pi^{\boxtimes p})}{N \cdot \pi^{\boxtimes p}} \cong \pi^{(p)}.$$

Remark 6.21. Conjecture 6.19 is highly non-trivial even for groups such as GL_n where the full Local Langlands correspondence, hence in particular existence of cyclic base change, is already known. In fact, the main result of [Ron16] is a special case of the conjecture, for depth-zero supercuspidal representations of GL_n compactly induced from cuspidal Deligne-Lusztig representations. Despite the very explicit nature of the Local Langlands Correspondence for such representations, the proof in *loc. cit.* involves rather hefty calculations, which were not amenable to generalization.

Our proof of Conjecture 6.19 (when p is odd and good for $\widehat{G}$) is conceptual and applies to all representations, without using any explicit models such as models for supercuspidal representations as compact inductions. Furthermore, the unramified and tamely ramified base change are handled completely differently in [Ron16], whereas our proof will be completely uniform in the field extension, the reductive group, and the irreducible representation.

Theorem 6.22. *Assume p is an odd good prime for $\widehat{G}$. Let Π be an irreducible admissible representation of G_v and let*

$$\chi_{\Pi^{(p)}} \colon \operatorname{Exc}(W_v, {}^L G) \to k$$

the associated character of $\Pi^{(p)}$. Form $T^*(\Pi) := T^*(\langle \sigma \rangle, \Pi)$, viewed as a smooth H_v-representation. Then for any irreducible character $\chi \colon \operatorname{Exc}(W_v, {}^L H) \to k$ appearing in the action on $T^*(\Pi)$ via $Z_H \colon \operatorname{Exc}(W_v, {}^L H) \to \mathfrak{Z}(H)$, the composite character

$$\operatorname{Exc}(W_v, {}^L G) \xrightarrow{\phi_{\mathsf{BC}}^*} \operatorname{Exc}(W_v, {}^L H) \xrightarrow{\chi} k$$

agrees with $\chi_{\Pi^{(p)}}$.

It is clear that Theorem 6.22 implies Theorem 1.2.

Proof. Let Π be a representation of G_v. Then $\mathfrak{Z}(G)$ acts G_v-equivariantly on Π, inducing an H_v-equivariant action of $\mathfrak{Z}(G)^\sigma$ on $T^*(\Pi)$. In particular, as Z_G maps the image of $\operatorname{Exc}(W_v, {}^L G)^{B\sigma} \to \operatorname{Exc}(W_v, {}^L G)$ (cf. Remark 5.7) into $\mathfrak{Z}(G)^\sigma$, we get an H_v-equivariant action of $\operatorname{Exc}(W_v, {}^L G)^{B\sigma}$ on $T^*(\Pi)$.

By Lemma 6.7, K_r is plain as soon as $r \geq 1$. Taking the (filtered) colimit over r in Lemma 6.8 with $K = K_r$, we find that for all $S \in \operatorname{Exc}(W_v, {}^L G)^{B\sigma}$, we have

$$\begin{pmatrix} \text{the action on } T^*(\Pi) \text{ of} \\ Z_G(S) \end{pmatrix} = \begin{pmatrix} \text{the action on } T^*(\Pi) \text{ of} \\ \operatorname{Br}(Z_G(S)) \end{pmatrix}.$$

In other words, the diagram below commutes:

$$
\begin{array}{ccc}
\mathfrak{Z}(G) & \longrightarrow & \operatorname{End}_{G_v}(\Pi) \\
\downarrow{\scriptstyle Z(\operatorname{Br})} & & \downarrow \\
\mathfrak{Z}(H) & \longrightarrow & \operatorname{End}_{H_v}(T^*\Pi)
\end{array}
\tag{6.19}
$$

On the other hand, taking the inverse limit over r in Corollary 6.15 yields that

$$\operatorname{Br}(Z_G(S_{\operatorname{Nm}^{(p-1)}(V), \operatorname{Nm}^{(p-1)}(x), \operatorname{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}})) = Z_H(S_{\operatorname{Res}_{\mathsf{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}). \tag{6.20}$$

Combining (6.19) and (6.20) shows that

$$\begin{pmatrix} \text{the action on } T^*(\Pi) \text{ of} \\ Z_G(S_{\operatorname{Nm}^{(p-1)}(V), \operatorname{Nm}^{(p-1)}(x), \operatorname{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}}) \end{pmatrix} = \begin{pmatrix} \text{the action on } T^*(\Pi) \text{ of} \\ Z_H(S_{\operatorname{Res}_{\mathsf{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}) \end{pmatrix}. \tag{6.21}$$

From now on, assume Π is an irreducible (smooth) representation of G_v. Then $\operatorname{End}_{G_v}(\Pi) \cong k$ (by Schur's Lemma applied to the Hecke action on the invariants of Π for every compact open subgroup of G_v). The L-parameter attached to Π corresponds under Proposition 2.4 to the character

$$\chi_\Pi \colon \operatorname{Exc}(W_v, {}^L G) \xrightarrow{Z_G} \mathfrak{Z}(G) \to \operatorname{End}_{G_v}(\Pi) \cong k.$$

This induces

$$T^0 \chi_\Pi \colon T^0 \operatorname{Exc}(W_v, {}^L G) \xrightarrow{T^0 Z_G} T^0 \mathfrak{Z}(G) \to T^0 \operatorname{End}_{G_v}(\Pi) \cong k.$$

Let ι denote the natural map $T^0 \operatorname{End}_{G_v}(\Pi) \to \operatorname{End}_{H_v}(T^*\Pi)$. We also consider the homomorphism

$$\chi_{T^0\Pi} \colon \operatorname{Exc}(W_v, {}^L H) \xrightarrow{Z_H} \mathfrak{Z}(H) \to \operatorname{End}_{H_v}(T^0\Pi).$$

We have just seen in (6.21) that

$$\iota \circ T^0 \chi_\Pi (S_{\operatorname{Nm}^{(p-1)}(V), \operatorname{Nm}^{(p-1)}(x), \operatorname{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}}) = \chi_{T^0\Pi}(S_{\operatorname{Res}_{\mathsf{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}). \tag{6.22}$$

Note that the fact that the right hand side of (6.22) lies in k is already non-obvious. In particular, (6.22) implies that for any irreducible subquotient π of $T^0\Pi$, we have

$$
\begin{aligned}
\chi_\pi(S_{\operatorname{Res}_{\mathsf{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}) &= \chi_{T^0\Pi}(S_{\operatorname{Res}_{\mathsf{BC}}(V), x, \xi, (\gamma_i)_{i \in I}}) \\
&= (T^0 \chi_\Pi)(S_{\operatorname{Nm}^{(p-1)}(V), \operatorname{Nm}^{(p-1)}(x), \operatorname{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}}) \\
&= \chi_\Pi(S_{\operatorname{Nm}^{(p-1)}(V), \operatorname{Nm}^{(p-1)}(x), \operatorname{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}}).
\end{aligned}
\tag{6.23}
$$

Let χ_Π be the character giving the action of $\operatorname{Exc}(W_v, {}^L G)$ on an irreducible G_v-representation Π. Then by (5.22), the character $\chi_{\Pi^{(p)}}$ giving the action of $\operatorname{Exc}(W_v, {}^L G)$ on $\Pi^{(p)} := \Pi \otimes_{k, \operatorname{Frob}_p} k$ satisfies

$$\chi_{\Pi^{(p)}}(S_{\operatorname{Nm}(V), \operatorname{Nm}(x), \operatorname{Nm}(\xi), (\gamma_i)_{i \in I}}) = \chi_\Pi(S_{\operatorname{Nm}^{(p-1)}(V), \operatorname{Nm}^{(p-1)}(x), \operatorname{Nm}^{(p-1)}(\xi), (\gamma_i)_{i \in I}})^p. \tag{6.24}$$

By Lemma 5.17 and Lemma 5.18, we have

$$
\begin{aligned}
\chi_{\Pi^{(p)}}(S_{V,x,\xi,(\gamma_i)_{i\in I}}) &= \chi_{\Pi^{(p)}}(S_{\mathrm{Nm}(V),\mathrm{Nm}(x),\mathrm{Nm}(\xi),(\gamma_i)_{i\in I}})^{1/p} \\
[(6.24) \implies] &= \chi_{\Pi}(S_{\mathrm{Nm}^{(p-1)}(V),\mathrm{Nm}^{(p-1)}(x),\mathrm{Nm}^{(p-1)}(\xi),(\gamma_i)_{i\in I}}) \\
[(6.23) \implies] &= \chi_{\pi}(S_{\mathrm{Res}_{\mathsf{BC}}(V),x,\xi,(\gamma_i)_{i\in I}}) \\
&= \chi_{\pi} \circ \phi_{\mathsf{BC}}^*(S_{V,x,\xi,(\gamma_i)_{i\in I}}).
\end{aligned}
$$

This shows that $\chi_{\Pi^{(p)}} = \chi_{\pi} \circ \phi_{\mathsf{BC}}^*$ for any irreducible subquotient π of $T^*(\Pi)$, which completes the proof.

$\square$

6.6. Local mod p cyclic base change.

In this subsection, we will prove Theorem 1.1. Assumptions are as in §6.5. We note, however, that the formulation of all the statements in §6.12 makes sense for any local field F_v of residue field $\ell \neq p$, with a suitable replacement for the Genestier-Lafforgue correspondence, and that all our arguments in this subsection apply if those statements are true for F_v.

6.6.1. *Formulation of local base change.* We begin by formulating a precise notion of local base change.

Definition 6.23. Let π be an irreducible admissible representation of H_v over k, and Π be an irreducible admissible representation of G_v over k. We say that π *base changes to* Π if $\rho_\Pi \cong \phi_{\mathsf{BC}} \circ \rho_\pi \in H^1(W_v, \widehat{G}(k))$.

This definition is an approximation to the notion of base change for L-packets. An L-packet for H_v should be said to base change to an L-packet for G_v if the corresponding L-parameters are related by ϕ_{BC}. A more refined version of Definition 6.23 would declare π to base change to Π if the L-packet of π base changes to the L-packet of Π, but we lack a definition of L-packets for general groups and representations; therefore, we use the fibers of the Genestier-Lafforgue correspondence as a substitute for L-packets.

6.6.2. *Finiteness conditions on Hecke algebras.* We will use the following recent result of Dat-Helm-Kurinczuk-Moss. We are keeping the running assumption that p differs from the residue characteristic of F_v.

Theorem 6.24 ([DHKM])**.** *For every $x \in \mathcal{B}(G/F_v)$ and every $r \geq 0$, and $K_r := G(F_v)_{x,r}$, the Hecke algebra $\mathcal{H}(G, K_r)$ is finite over its center $\mathfrak{Z}(G, K_r)$, which is itself a finitely generated algebra over k.*

Remark 6.25. The paper [DHKM] proves a much stronger result, where coefficients are allowed to be an arbitrary $\mathbf{Z}_p$-algebra. The analogous result with coefficients in a characteristic zero field is an old result of Bernstein. The case where p is banal for G was known to experts to follow in a similar manner from work of Vignéras, although it is not explicitly written down in the literature.

6.6.3. *Existence of local base change.* Fix $x \in \mathcal{B}(G/F_v)$, and let $K_r := G(F_v)_{x,r}$ and $U_r := K_r^\sigma$. We prove the following theorem, which in particular implies Theorem 1.1.

Theorem 6.26. *Suppose p is an odd good prime for $\widehat{G}$. Let π be an irreducible representation of H_v over k, having non-zero U_r-fixed vectors, with L-parameter $\rho_\pi \in H^1(W_v, \widehat{H}(k))$. Then there is an irreducible representation Π of G_v over k, having non-zero K_r-fixed vectors, such that $\rho_\Pi \cong \phi_{\mathsf{BC}} \circ \rho_\pi$.*

Proof. By hypothesis, we have a non-zero algebra homomorphism $\mathcal{H}(H, U_r) \to \mathrm{End}(\pi^{U_r})$, which has the property that the composite homomorphism

$$
\mathrm{Exc}(W_v, {}^L H) \xrightarrow{Z_{H,r}} \mathfrak{Z}(H, U_r) \to \mathcal{H}(H, U_r) \to \mathrm{End}(\pi^{U_r})
$$

has kernel the maximal ideal $\mathfrak{m}_\pi \subset \mathrm{Exc}(W_v, {}^L H)$ corresponding to the L-parameter ρ_π. Let $\chi_\pi \colon \mathrm{Exc}(W_v, {}^L H) \to k$ be the corresponding character. By Corollary 6.15, the Brauer homomorphism $\mathrm{Br} \colon \mathcal{H}(G, K_r)^\sigma \to \mathcal{H}(H, U_r)$

fits into a commutative diagram[14]

$$
\begin{array}{ccccccc}
\mathrm{Exc}(W_v, {}^L H) & \xrightarrow{Z_{H,r}} & \mathfrak{Z}(H, U_r) & \longrightarrow & \mathcal{H}(H, U_r) & \hookrightarrow & \mathrm{End}(\pi^{U_r}) \\
 & & & & \uparrow{\scriptstyle\mathrm{Br}} & & \\
\mathrm{Exc}(W_v, {}^L G)' & \xrightarrow{Z_{G,r}} & \mathfrak{Z}(G, K_r)^\sigma & \hookrightarrow & \mathcal{H}(G, K_r)^\sigma & & \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathrm{Exc}(W_v, {}^L G) & \xrightarrow{Z_{G,r}} & \mathfrak{Z}(G, K_r) & \hookrightarrow & \mathcal{H}(G, K_r) & &
\end{array}
$$

where $\mathrm{Exc}(W_v, {}^L G)' \subset \mathrm{Exc}(W_v, {}^L G)$ is the subalgebra of Definition 5.19. Let $\mathfrak{m}'_\pi \subset \mathrm{Exc}(W_v, {}^L G)'$ be the maximal ideal which is the kernel of the map $\mathrm{Exc}(W_v, {}^L G)' \to \mathrm{End}(\pi^{U_r})$ obtained by tracing through the diagram above. Lemma 5.17 implies that the corresponding homomorphism $\chi'_\pi \colon \mathrm{Exc}(W_v, {}^L G)' \to k$ has a *unique* extension to a character $\mathrm{Exc}(W_v, {}^L G) \to k$, which by Corollary 6.15 is $\chi_\pi \circ \phi^*_{\mathsf{BC}}$.

Let $\mathfrak{n}'_\pi$ be the ideal of $\mathfrak{Z}(G, K_r)^\sigma$ generated by $\mathfrak{m}'_\pi$ via the map $Z_{G,r} \colon \mathrm{Exc}(W_v, {}^L G)' \to \mathfrak{Z}(G, K_r)^\sigma$, and $\mathfrak{n}_\pi$ be its extension to $\mathfrak{Z}(G, K_r)$. Note that $\mathfrak{n}_\pi \cap \mathrm{Exc}(W_v, {}^L G)$ contains $\mathfrak{m}'_\pi$, and so the only maximal ideal containing it is the one corresponding to the character $\chi_\pi \circ \phi^*_{\mathsf{BC}}$. The previous paragraph implies that the localization of $\mathcal{H}(G, K_r)^\sigma$ at $\mathfrak{m}'_\pi$ as an $\mathrm{Exc}(W_v, {}^L G)'$-module is non-zero, so the localization of $\mathcal{H}(G, K_r)^\sigma$ at $\mathfrak{n}'_\pi$ as a $\mathfrak{Z}(G, K_r)^\sigma$-module is also non-zero. By Theorem 6.24 and the Artin-Tate Lemma, $\mathfrak{Z}(G, K_r)$ is finite over $\mathfrak{Z}(G, K_r)^\sigma$ and then $\mathcal{H}(G, K_r)$ is finite over $\mathfrak{Z}(G, K_r)^\sigma$. Since the localization of $\mathcal{H}(G, K_r)^\sigma \subset \mathcal{H}(G, K_r)$ at $\mathfrak{n}'_\pi$ as a $\mathfrak{Z}(G, K_r)^\sigma$-module is non-zero, the localization of $\mathcal{H}(G, K_r)$ at $\mathfrak{n}_\pi$ as a $\mathfrak{Z}(G, K_r)$-module is non-zero, so Nakayama's Lemma implies that the left $\mathcal{H}(G, K_r)$-module quotient $\mathcal{H}(G, K_r)/\mathcal{H}(G, K_r)\mathfrak{n}_\pi$ is finite-dimensional and non-zero. By design, the only maximal ideal in its support over $\mathrm{Exc}(W_v, {}^L G)$ is $\ker(\chi_\pi \circ \phi^*_{\mathsf{BC}})$, so there is an irreducible $\mathcal{H}(G, K_r)$-subquotient Ξ of $\mathcal{H}(G, K_r)/\mathcal{H}(G, K_r)\mathfrak{n}_\pi$ on which $\mathrm{Exc}(W_v, {}^L G)$ acts through $\phi^*_{\mathsf{BC}} \circ \chi_\pi$.

To conclude, we produce an irreducible G_v-module quotient Π of $C_c^\infty(G_v/K_r)$ whose $\mathrm{Exc}(W_v, {}^L G)$-action factors through $\phi^*_{\mathsf{BC}} \circ \chi_\pi$. In the preceding paragraph we exhibited an irreducible $\mathcal{H}(G, K_r)$-module Ξ on which the $\mathfrak{Z}(G, K_r)$-action factors over $\mathfrak{n}_\pi$. Let Π be any irreducible G_v-quotient of $C_c^\infty(G_v/K_r) \otimes_{\mathcal{H}(G, K_r)} \Xi$ (which is guaranteed to exist because $C_c^\infty(G_v/K_r) \otimes_{\mathcal{H}(G, K_r)} \Xi$ is finitely generated as a G_v-representation). Then we have

$$
\mathrm{Hom}_{G_v}(C_c^\infty(G_v/K_r) \otimes_{\mathcal{H}(G, K_r)} \Xi, \Pi) \cong \mathrm{Hom}_{\mathcal{H}(G, K_r)}(\Xi, \Pi^{K_r}).
$$

Since the map from Ξ to Π^{K_r} is non-zero by design, this forces Π^{K_r} (an irreducible $\mathcal{H}(G, K_r)$-module) to have the same central character as Ξ. Therefore the action of $\mathrm{Exc}(W_v, {}^L G)$ on Π is through the character $\phi^*_{\mathsf{BC}} \circ \chi_\pi$. $\qquad\qquad\square$

Remark 6.27. The proof of Theorem 6.26 goes through with a weaker hypothesis than Conjecture **??**: it suffices to assume the existence of *some* vertex x, and an infinite sequence of $r \to \infty$, for which the Hecke algebra $\mathcal{H}(G, K_r)$ is finite over its center $\mathfrak{Z}(G, K_r)$, and the latter is a finitely generated algebra over k. However, this weaker assumption does not suffice to prove the depth estimate in the following Remark, and in practice it does not seem that the weaker assumption is easier to verify.

Remark 6.28 (Depth estimates). For applications it is useful to have control of the *depth* of the base change. The proof of Theorem 6.26 implies an estimate on the depth, which we now spell out. Recall from [MP96] that the *depth* of an irreducible representation Π of G_v is the minimal r such that for some $x \in \mathcal{B}(G/F_v)$, $\Pi^{G(F_v)_{x,r+}} \neq 0$. Let us emphasize that the definition of the Moy-Prasad filtration $\{G(F_v)_{x,r}\}$ is normalized so that $F_v^\times$ has value group $\mathbf{Z}$.

Let π be an irreducible representation of H of depth r, and let $x \in \mathcal{B}(H/F_v)$ such that $\pi^{H(F_v)_{x,r+}} \neq 0$. First assume that E_v/F_v is *unramified*. Then $\mathcal{B}(H/F_v) = \mathcal{B}(G/F_v)^{\mathrm{Gal}(E_v/F_v)}$, and we have $H(F_v)_{x,r+} = G(F_v)_{x,r+}^{\mathrm{Gal}(E_v/F_v)}$ [KP, §7]. The proof of Theorem 6.26 shows that there exists a local base change Π of π such that $\Pi^{G(F_v)_{x,r+}} \neq 0$, so that $\mathrm{depth}(\Pi) \leq \mathrm{depth}(\pi)$. (The proof does not use Corollary 6.4 or the Treumann-Venkatesh homomorphism.)

---

[14]Since we are not assuming that x is a special vertex, we cannot invoke Corollary 6.4 to say that Br is surjective, so it may not induce a map of centers.

Next suppose E_v/F_v is *tamely ramified*. By [Pra20] we still have $\mathcal{B}(H/F_v) = \mathcal{B}(G/F_v)^{\mathrm{Gal}(E_v/F_v)}$, and by [KP, Proposition 13.9.2] we have $(G(F_v)_{x,r+})^{\mathrm{Gal}(E_v/F_v)} = H(F_v)_{x,r+}$ for all $r \geq 0$. Hence in this case, the proof of Theorem 6.26 shows that there exists a local base change Π of π such that $\Pi^{G(F_v)_{x,r+}} \neq 0$, so that $\mathrm{depth}(\Pi) \leq \mathrm{depth}(\pi)$. Let us caution, however, that if we regard $x \in \mathcal{B}(H/E_v)$ instead of $\mathcal{B}(G/F_v)$ and Π as a representation of $H(E_v)$ instead of $G(F_v)$, then it is natural to define the Moy-Prasad filtration $H(E_v)_{x,r}$ so that $E_v^\times$ has value group $\mathbf{Z}$, for which $H(E_v)_{x,pr} = G(F_v)_{x,r}$. Hence, in this normalization our estimate would instead be "$\mathrm{depth}(\Pi) \leq p \cdot \mathrm{depth}(\pi)$".

In both cases, the inequalities we obtain are expected to be optimal [AL10].

APPENDIX A. THE BASE CHANGE FUNCTOR REALIZES LANGLANDS FUNCTORIALITY
by TONY FENG AND GUS LONERGAN

In this section we prove Theorem 4.20.

First we recall some general properties of Smith theory for *schemes*.

A.1. **Recollections on Smith theory for schemes.** The Tate category for schemes enjoys a robust 6-functor formalism (observed in the topological case in [Tre19, §4.3], and proved for schemes in [RW, §2,3]). Let us recall the statements for later use. Let $f\colon Y \to S$ be a σ-equivariant morphism of varieties with admissible σ-action, over a field of characteristic $\ell \neq p$. Let Λ be a p-adic ring of coefficients; we are most interested in $\Lambda \in \{W(k), k\}$.

- The pullback functor $f^*\colon D_c^b(S^\sigma; \Lambda[\sigma]) \to D_c^b(Y^\sigma; \Lambda[\sigma])$ descends to
$$f^*\colon \mathrm{Perf}(S^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda).$$
  The proper pushforward $f_!\colon D_c^b(Y^\sigma; \Lambda[\sigma]) \to D_c^b(S^\sigma; \Lambda[\sigma])$ descends to
$$f_!\colon \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Perf}(S^\sigma; \mathcal{T}_\Lambda).$$

- As Verdier duality $\mathbb{D}_Y\colon D_{c,\sigma}^b(Y^\sigma; \Lambda) \to D_{c,\sigma}^b(Y^\sigma; \Lambda)$ preserves $\mathrm{Perf}(Y^\sigma; \Lambda[\sigma])$, it descends to the Tate category to define
$$\mathbb{D}_Y\colon \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda).$$

Using this, we may define the operations
$$f^! := \mathbb{D}_Y \circ f^* \circ \mathbb{D}_Y \colon \mathrm{Perf}(S^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda)$$
and
$$f_* := \mathbb{D}_Y \circ f_! \circ \mathbb{D}_Y \colon \mathrm{Perf}(Y^\sigma; \mathcal{T}_\Lambda) \to \mathrm{Perf}(S^\sigma; \mathcal{T}_\Lambda).$$

We now list some properties which could be remembered under the slogan, "the Smith operation commutes with all operations" (cf. [Tre19, §4.4]).

A.1.1. *Compatibility with pullback.* If f satisfies the assumptions above, then the following diagrams commute:

$$
\begin{array}{ccc}
D_{c,\sigma}^b(Y;\Lambda) & \xleftarrow{\ f^*\ } & D_{c,\sigma}^b(S;\Lambda) \\
\downarrow{\scriptstyle\mathrm{Psm}} & & \downarrow{\scriptstyle\mathrm{Psm}} \\
\mathrm{Perf}(Y^\sigma;\mathcal{T}_\Lambda) & \xleftarrow{\ f^*\ } & \mathrm{Perf}(S^\sigma;\mathcal{T}_\Lambda)
\end{array}
\qquad
\begin{array}{ccc}
D_{c,\sigma}^b(Y;\Lambda) & \xleftarrow{\ f^!\ } & D_{c,\sigma}^b(S;\Lambda) \\
\downarrow{\scriptstyle\mathrm{Psm}} & & \downarrow{\scriptstyle\mathrm{Psm}} \\
\mathrm{Perf}(Y^\sigma;\mathcal{T}_\Lambda) & \xleftarrow{\ f^!\ } & \mathrm{Perf}(S^\sigma;\mathcal{T}_\Lambda)
\end{array}
$$

The proof for the first square is formal; from the second it follows immediately from the first plus [RW, Lemma 3.5], whose proof is the same as that for Lemma 3.7.

A.1.2. *Compatibility with pushforward.* If f satisfies the assumptions above, then the following diagrams commute:

$$
\begin{array}{ccc}
D_{c,\sigma}^b(Y;\Lambda) & \xrightarrow{\ f_*\ } & D_{c,\sigma}^b(S;\Lambda) \\
\downarrow{\scriptstyle\mathrm{Psm}} & & \downarrow{\scriptstyle\mathrm{Psm}} \\
\mathrm{Perf}(Y^\sigma;\mathcal{T}_\Lambda) & \xrightarrow{\ f_*\ } & \mathrm{Perf}(S^\sigma;\mathcal{T}_\Lambda)
\end{array}
\qquad
\begin{array}{ccc}
D_{c,\sigma}^b(Y;\Lambda) & \xrightarrow{\ f_!\ } & D_{c,\sigma}^b(S;\Lambda) \\
\downarrow{\scriptstyle\mathrm{Psm}} & & \downarrow{\scriptstyle\mathrm{Psm}} \\
\mathrm{Perf}(Y^\sigma;\mathcal{T}_\Lambda) & \xrightarrow{\ f_!\ } & \mathrm{Perf}(S^\sigma;\mathcal{T}_\Lambda)
\end{array}
$$

The proof for the second diagram is the same as that of Proposition 3.10. Then the commutativity of the first diagram follows by applying Verdier duality and using Lemma 3.7.

A.2. **Setup for the proof of Theorem 4.20.** We keep the setup of §4.6.1: H is any reductive group over a separably closed field $\mathbf{F}$ of characteristic $\neq p$, and $G = H^p$. We let σ act on G by cyclic rotation, sending the ith factor to the $(i+1)$st (mod p) factor.

A.3. **Proof of linearity.** We first prove that BC is additive, i.e., we exhibit a natural isomorphism $\mathsf{BC}(\mathcal{F} \oplus \mathcal{F}') \cong \mathsf{BC}(\mathcal{F}) \oplus \mathsf{BC}(\mathcal{F}')$. We have

$$\mathrm{Nm}(\mathcal{F} \oplus \mathcal{F}') = (\mathcal{F} \oplus \mathcal{F}') * (^{\sigma}\mathcal{F} \oplus {}^{\sigma}\mathcal{F}') * \ldots * (^{\sigma^{p-1}}\mathcal{F} \oplus {}^{\sigma^{p-1}}\mathcal{F}')$$
$$\cong \mathrm{Nm}(\mathcal{F}) \oplus \mathrm{Nm}(\mathcal{F}') \oplus (\text{direct sum of free } \sigma\text{-orbits}).$$

Therefore, the restrictions of $\mathrm{Nm}(\mathcal{F} \oplus \mathcal{F}')$ and $\mathrm{Nm}(\mathcal{F}) \oplus \mathrm{Nm}(\mathcal{F}')$ to Y^{σ} differ by a perfect complex of $\mathbb{O}[\sigma]$-modules, and hence project to isomorphic objects in the Tate category $\mathrm{Shv}(Y^{\sigma}; \mathcal{T}_{\mathbb{O}})$. This shows that $\mathrm{Psm} \circ \mathrm{Nm}$ is additive. We conclude by using that the modular reduction functor $\mathbb{F}$ and the lifting functor L are both additive. $\qquad\square$

A.4. **Reduction to the case of a torus.** Let T_H be a maximal torus of H. Recall that the restriction functor $\mathrm{Rep}(\widehat{H}) \to \mathrm{Rep}(T_{\widehat{H}})$ is intertwined under the Geometric Satake equivalence with the *hyperbolic localization* functor [BR18, §5.3].

Since $*/!$-restriction and $*/!$-pushforward all commute with Psm by §A.1, the hyperbolic localization functor commutes with Psm. As the restriction functor $\mathrm{Rep}(\widehat{H}) \to \mathrm{Rep}(T_{\widehat{H}})$ is faithful and injective on tilting objects (i.e. "tilting modules are determined by their characters") by [Don93, p. 46], it suffices to prove Theorem 4.20 in the special case where H is a *torus*.

A.5. **Proof in the case of a torus.** Finally, we examine the case when H is a torus. Since the theorem is compatible with products, we can even reduce to the case $H = \mathbf{G}_m$. For $H = \mathbf{G}_m$ the underlying reduced scheme of Gr_H is a disjoint union of points labeled by the integers.

The irreducible algebraic representations of $\widehat{H}$ are indexed by $n \in \mathbf{Z}$, with $V_n \in \mathrm{Rep}(\widehat{H})$ corresponding to the constant sheaf supported on the component Gr_H^n labeled by n. The irreducible algebraic representations of $\widehat{G}$ are then labeled by p-tuples of integers $(n_1, \ldots, n_p) \in \mathbf{Z}^p$. By the linearity of BC established in §A.3 and the complete reducibility of algebraic representations of tori, we may assume that $\mathcal{F}$ is irreducible, say $\mathcal{F} = \mathcal{F}(n_1, \ldots, n_p)$ is the constant sheaf supported on $\mathrm{Gr}_G^{(n_1, \ldots, n_p)}$. Then the σ-equivariant sheaf $\mathrm{Nm}(\mathcal{F})$ is the constant sheaf $\underline{k}$ supported on the component $\mathrm{Gr}_G^{(n_1 + \ldots + n_p, \ldots, n_1 + \ldots + n_p)}$. Its restriction to the diagonal copy of Gr_H is the constant sheaf with value k supported on $\mathrm{Gr}_H^{n_1 + \ldots + n_p}$. This is already an indecomposable k-parity sheaf, which tautologically lifts its own image in the Tate category. Hence we have shown that

$$\underline{k}_{\mathrm{Gr}_H^{n_1 + \ldots + n_p}} = \mathsf{BC}^{(p)}(V_{n_1, \ldots, n_p}).$$

And indeed, this is precisely the sheaf which corresponds under geometric Satake to $\mathrm{Res}_{\mathsf{BC}}(V_{n_1} \boxtimes V_{n_2} \boxtimes \ldots \boxtimes V_{n_p}) \cong V_{n_1 + n_2 + \ldots + n_p} \in \mathrm{Rep}(\widehat{H})$. This confirms the commutativity of the diagram

$$\begin{array}{ccc} \mathrm{Parity}^0(\mathrm{Gr}_G; k) & \xrightarrow{\mathsf{BC}} & \mathrm{Parity}^0(\mathrm{Gr}_H; k) \\ \downarrow{\sim} & & \downarrow{\sim} \\ \mathrm{Tilt}_k(\widehat{G}) & \xrightarrow{\mathrm{Res}_{\mathsf{BC}}} & \mathrm{Tilt}_k(\widehat{H}) \end{array}$$

at the level of objects. Our final step is to verify the commutativity on morphisms. Since (as H is a torus) the categories involved are all semi-simple, the commutativity at the level of morphisms reduces to examining a scalar endomorphism of the simple object $\mathcal{F}$ above, which corresponds to the simple representation $V_{n_1, \ldots, n_p}$. The restriction functor $\mathrm{Res}_{\mathsf{BC}}$ is k-linear, so what we have to check is that BC sends multiplication by λ on $\mathcal{F}$ to multiplication by λ on $\mathsf{BC}(\mathcal{F})$. Now, multiplication by λ on $\mathcal{F}$ is sent under Nm to multiplication by λ^p on $\mathrm{Nm}(\mathcal{F})$, which restricts to multiplication by λ^p on $\mathsf{BC}^{(p)}(\mathcal{F})$. Then the inverse Frobenius twist Frob_p^{-1} sends it to multiplication by λ, so $\mathsf{BC} := \mathrm{Frob}_p^{-1} \circ \mathsf{BC}^{(p)}$ behaves as desired. $\qquad\square$

References

[AC89] James Arthur and Laurent Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies, vol. 120, Princeton University Press, Princeton, NJ, 1989.

[AL10] Jeffrey D. Adler and Joshua M. Lansky, *Depth-zero base change for ramified* U(2, 1), Trans. Amer. Math. Soc. **362** (2010), no. 10, 5569–5599.

[BBM04] A. Beilinson, R. Bezrukavnikov, and I. Mirković, *Tilting exercises*, Mosc. Math. J. **4** (2004), no. 3, 547–557, 782.

[Ber84] J. N. Bernstein, *Le "centre" de Bernstein*, Representations of reductive groups over a local field, Travaux en Cours, Hermann, Paris, 1984, Edited by P. Deligne, pp. 1–32.

[BFH+] Gebhard Böckle, Tony Feng, Michael Harris, Chandrashekhar Khare, and Jack A. Thorne, *Cyclic base change for cuspidal automorphic representations over function fields*, In preparation.

[BG14] Kevin Buzzard and Toby Gee, *The conjectural connections between automorphic representations and Galois representations*, Automorphic forms and Galois representations. Vol. 1, London Math. Soc. Lecture Note Ser., vol. 414, Cambridge Univ. Press, Cambridge, 2014, pp. 135–187.

[BHKT19] Gebhard Böckle, Michael Harris, Chandrashekhar Khare, and Jack A. Thorne, *Ĝ-local systems on smooth projective curves are potentially automorphic*, Acta Math. **223** (2019), no. 1, 1–111.

[Bla94] Don Blasius, *On multiplicities for* SL(n), Israel J. Math. **88** (1994), no. 1-3, 237–251.

[BR18] Pierre Baumann and Simon Riche, *Notes on the geometric Satake equivalence*, Relative aspects in representation theory, Langlands functoriality and automorphic forms, Lecture Notes in Math., vol. 2221, Springer, Cham, 2018, pp. 1–134.

[BT72] F. Bruhat and J. Tits, *Groupes réductifs sur un corps local*, Inst. Hautes Études Sci. Publ. Math. (1972), no. 41, 5–251.

[CO] Charlotte Chan and Masao Oi, *Geometric L-packets of Howe-unramified toral supercuspidal representations*, Available at `https://arxiv.org/pdf/2105.06341.pdf`.

[Dat09] Jean-François Dat, *Finitude pour les représentations lisses de groupes p-adiques*, J. Inst. Math. Jussieu **8** (2009), no. 2, 261–333.

[DGNO10] Vladimir Drinfeld, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik, *On braided fusion categories. I*, Selecta Math. (N.S.) **16** (2010), no. 1, 1–119.

[DHKM] Jean-François Dat, David Helm, Robert Kurinczuk, and Gilbert Moss, *The local Langlands conjecture in families I: the banal case*, In preparation.

[Don93] Stephen Donkin, *On tilting modules for algebraic groups*, Math. Z. **212** (1993), no. 1, 39–60.

[Dri87] V. G. Drinfel'd, *Cohomology of compactified moduli varieties of F-sheaves of rank* 2, Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) **162** (1987), no. Avtomorfn. Funkts. i Teor. Chisel. III, 107–158, 189.

[Fen] Tony Feng, *On mod ℓ base change in the Fargues-Scholze correspondence*, In preparation.

[Fen20] _____, *Nearby cycles of parahoric shtukas, and a fundamental lemma for base change*, Selecta Math. (N.S.) **26** (2020), no. 2, Paper No. 21, 59.

[Fin21] Jessica Fintzen, *Types for tame p-adic groups*, Ann. of Math. (2) **193** (2021), no. 1, 303–346.

[FS] Laurent Fargues and Peter Scholze, *Geometrization of the local Langlands correspondence*, Available at `https://arxiv.org/pdf/2102.13459.pdf`.

[GL] Alain Genestier and Vincent Lafforgue, *Chtoucas restreints pour les groupes réductifs et paramétrisation de Langlands locale*, Available at `https://arxiv.org/abs/1709.00978`.

[Hai09] Thomas J. Haines, *The base change fundamental lemma for central elements in parahoric Hecke algebras*, Duke Math. J. **149** (2009), no. 3, 569–643.

[Hai12] _____, *Base change for Bernstein centers of depth zero principal series blocks*, Ann. Sci. Éc. Norm. Supér. (4) **45** (2012), no. 5, 681–718 (2013).

[Hai14] _____, *The stable Bernstein center and test functions for Shimura varieties*, Automorphic forms and Galois representations. Vol. 2, London Math. Soc. Lecture Note Ser., vol. 415, Cambridge Univ. Press, Cambridge, 2014, pp. 118–186.

[HL04] Michael Harris and Jean-Pierre Labesse, *Conditional base change for unitary groups*, Asian J. Math. **8** (2004), no. 4, 653–683.

[JMW14] Daniel Juteau, Carl Mautner, and Geordie Williamson, *Parity sheaves*, J. Amer. Math. Soc. **27** (2014), no. 4, 1169–1212.

[JMW16] _____, *Parity sheaves and tilting modules*, Ann. Sci. Éc. Norm. Supér. (4) **49** (2016), no. 2, 257–275.

[KP] Tasho Kaletha and Gopal Prasad, *Bruhat-Tits theory: a new approach*, In preparation.

[Lab99] Jean-Pierre Labesse, *Cohomologie, stabilisation et changement de base*, Astérisque (1999), no. 257, vi+161, Appendix A by Laurent Clozel and Labesse, and Appendix B by Lawrence Breen.

[Laf02] Laurent Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands*, Invent. Math. **147** (2002), no. 1, 1–241.

[Laf18] Vincent Lafforgue, *Chtoucas pour les groupes réductifs et paramétrisation de Langlands globale*, J. Amer. Math. Soc. **31** (2018), no. 3, 719–891.

[Lan80] Robert P. Langlands, *Base change for* GL(2), Annals of Mathematics Studies, No. 96, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1980.

[Lap99] Erez M. Lapid, *Some results on multiplicities for* SL(n), Israel J. Math. **112** (1999), 157–186.

[LL] Spencer Leslie and Gus Lonergan, *Parity sheaves and Smith Theory*, Available at `https://arxiv.org/abs/1708.08174`.

[MP96]   Allen Moy and Gopal Prasad, *Jacquet functors and unrefined minimal K-types*, Comment. Math. Helv. **71** (1996), no. 1, 98–121. MR 1371680

[MR18]   Carl Mautner and Simon Riche, *Exotic tilting sheaves, parity sheaves on affine Grassmannians, and the Mirković-Vilonen conjecture*, J. Eur. Math. Soc. (JEMS) **20** (2018), no. 9, 2259–2332.

[Pra20]   Gopal Prasad, *Finite group actions on reductive groups and buildings and tamely-ramified descent in Bruhat-Tits theory*, Amer. J. Math. **142** (2020), no. 4, 1239–1267.

[Qui71]   Daniel Quillen, *The spectrum of an equivariant cohomology ring. I, II*, Ann. of Math. (2) **94** (1971), 549–572; ibid. (2) 94 (1971), 573–602.

[Ric]   Simon Riche, *Geometric representation theory in positive characteristic*, Habilitation thesis, available at https://tel.archives-ouvertes.fr/tel-01431526/document.

[Ron16]   Niccolò Ronchetti, *Local base change via Tate cohomology*, Represent. Theory **20** (2016), 263–294.

[RW]   Simon Riche and Geordie Williamson, *Smith-Treumann theory and the linkage principle*, Available at https://arxiv.org/pdf/2003.08522.pdf.

[RZ15]   Timo Richarz and Xinwen Zhu, *Appendix, the geometric Satake correspondence for ramified groups*, Ann. Sci. Éc. Norm. Supér. (4) **48** (2015), no. 2, 409–451.

[Sai77]   Hiroshi Saito, *Automorphic forms and algebraic extensions of number fields*, Sūgaku **29** (1977), no. 1, 28–38.

[SGA73]   *Théorie des topos et cohomologie étale des schémas. Tome 3*, Lecture Notes in Mathematics, Vol. 305, Springer-Verlag, Berlin-New York, 1973, Séminaire de Géométrie Algébrique du Bois-Marie 1963–1964 (SGA 4), Dirigé par M. Artin, A. Grothendieck et J. L. Verdier. Avec la collaboration de P. Deligne et B. Saint-Donat.

[SGA03]   *Revêtales étales and fundamental group (SGA 1)*, Mathematical Documents (Paris) [Mathematical Documents (Paris)], vol. 3, Société Mathématique de France, Paris, 2003, Séminaire de géomé triealgé brick du Bois Marie 1960–61. [Algebraic Geometry Seminar of Bois Marie 1960-61], Directed by A. Grothendieck, With two papers by M. Raynaud, Updated and annotated reprint of the 1971 original [Reading Notes in Math., 224, Springer, Berlin; MR0354651 (50 # 7129)].

[Shi79]   Takuro Shintani, *On liftings of holomorphic cusp forms*, Automorphic forms, representations and *L*-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 97–110.

[ST21]   Will Sawin and Nicolas Templier, *On the Ramanujan conjecture for automorphic forms over function fields I. Geometry*, J. Amer. Math. Soc. **34** (2021), no. 3, 653–746.

[Sta20]   The Stacks Project Authors, *Stacks Project*, https://stacks.math.columbia.edu, 2020.

[Tre19]   David Treumann, *Smith theory and geometric Hecke algebras*, Math. Ann. **375** (2019), no. 1-2, 595–628.

[TV16]   David Treumann and Akshay Venkatesh, *Functoriality, Smith theory, and the Brauer homomorphism*, Ann. of Math. (2) **183** (2016), no. 1, 177–228.

[Var04]   Yakov Varshavsky, *Moduli spaces of principal F-bundles*, Selecta Math. (N.S.) **10** (2004), no. 1, 131–166.

[Vig94]   Marie-France Vignéras, *Banal characteristic for reductive p-adic groups*, J. Number Theory **47** (1994), no. 3, 378–397.

[Vig96]   ———, *Représentations l-modulaires d'un groupe réductif p-adique avec $l \neq p$*, Progress in Mathematics, vol. 137, Birkhäuser Boston, Inc., Boston, MA, 1996.

[Vig97]   ———, *Cohomology of sheaves on the building and R-representations*, Invent. Math. **127** (1997), no. 2, 349–373.

[Vig01]   ———, *Correspondance de Langlands semi-simple pour $GL(n, F)$ modulo $\ell \neq p$*, Invent. Math. **144** (2001), no. 1, 177–223.

[Xuea]   Cong Xue, *Cohomology with integral coefficients of stacks of shtukas*, Available at https://arxiv.org/pdf/2001.05805.pdf.

[Xueb]   ———, *Smoothness of the cohomology of stacks of shtukas*, Available at https://arxiv.org/pdf/2012.12833.pdf, but in this paper we cite a forthcoming version that has not yet been uploaded.

[Xue20]   ———, *Finiteness of cohomology groups of stacks of shtukas as modules over Hecke algebras, and applications*, Épijournal de Géométrie Algébrique **4** (2020), no. 6, 1–42.

[Zhu]   Xinwen Zhu, *Coherent sheaves on the stack of langlands parameters*, Preprint available at https://arxiv.org/abs/2008.02998.

[Zhu17]   ———, *An introduction to affine Grassmannians and the geometric Satake equivalence*, Geometry of moduli spaces and representation theory, IAS/Park City Math. Ser., vol. 24, Amer. Math. Soc., Providence, RI, 2017, pp. 59–154.